Statistical inference of random graphs with a surrogate likelihood function

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Abstract

Spectral estimators have been broadly applied to statistical network analysis but they do not incorporate the likelihood information of the network sampling model. This paper proposes a novel surrogate likelihood function for statistical inference of a class of popular network models referred to as random dot product graphs. In contrast to the structurally complicated exact likelihood function, the surrogate likelihood function has a separable structure and is log-concave yet approximates the exact likelihood function well. From the frequentist perspective, we study the maximum surrogate likelihood estimator and establish the accompanying theory. We show its existence, uniqueness, large sample properties, and that it improves upon the baseline spectral estimator with a smaller sum of squared errors. A computationally convenient stochastic gradient descent algorithm is designed for finding the maximum surrogate likelihood estimator in practice. From the Bayesian perspective, we establish the Bernstein–von Mises theorem of the posterior distribution with the surrogate likelihood function and show that the resulting credible sets have the correct frequentist coverage. The empirical performance of the proposed surrogate-likelihood-based methods is validated through the analyses of simulation examples and a real-world Wikipedia graph dataset. An R package implementing the proposed computation algorithms is publicly available at https://fangzheng-xie.github.io/./materials/lgraph_0.1.0.tar.gz.

Keywords: Bernstein–von Mises theorem; Maximum surrogate likelihood estimation; Random dot product graph; Stochastic gradient descent.

1 Introduction

In the contemporary world of data science, network data are pervasive in a broad range of applications such as sociology (Lacetera et al., 2016; Young and Scheinerman, 2007), econometrics (Mele, 2017; Mele et al., 2022), and neuroscience (Tang et al., 2019). Statistical network analysis is also an interdisciplinary area of research connected with many other fields, including computer science, machine learning, combinatorics, applied mathematics, and physics. To model and analyze network data, various random graph models have

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been proposed in the literature, including the Erdős-Rényi graph (Erdős et al., 1960), the stochastic block model (Holland et al., 1983), and the latent space model (Hoff et al., 2002).

In this paper, we focus on random dot product graphs (Young and Scheinerman, 2007), a class of random graph models that are popular due to its simple architecture and flexibility. On one hand, the edge probability matrix of a random dot product graph has a low-rank structure, which motivates, among others, the use of spectral methods in statistical network analysis. On the other hand, the random dot product graph model is quite flexible because it not only encompasses the popularly used stochastic block models (Holland et al., 1983; Abbe et al., 2016) and their offspring (Airoldi et al., 2008; Binkiewicz et al., 2017; Lyzinski et al., 2017; Sengupta and Chen, 2018), but can also approximate general latent position graphs when the rank of the edge probability matrix grows with the number of vertices at a certain rate (Gao et al., 2015; Tang et al., 2013).

Because the adjacency matrix has a low expected rank, spectral quantities such as the leading eigenvectors of the adjacency matrix and those of the normalized Laplacian matrix, have been extensively used for low-rank random graph inference. In particular, it is well known that the rows of these eigenvectors encode the cluster membership information when the underlying graph is generated from a stochastic block model (Abbe et al., 2020; Lyzinski et al., 2014; Lei and Rinaldo, 2015; Rohe et al., 2011; Sussman et al., 2012). There has been substantial recent development on the theory for spectral methods and the corresponding subsequent inference tasks in random dot product graphs. For an incomplete list of reference, see Athreya et al. (2016); Sussman et al. (2014); Sarkar and Bickel (2015); Tang and Priebe (2018); Tang et al. (2013, 2017a,b). The readers are also referred to the survey paper Athreya et al. (2017) for a review of the recent advances in this topic.

It has been pointed out (Xie and Xu, 2020, 2021; Xie, 2022) that, although the spectral methods for random dot product graphs have gained marvelous success and been broadly applied, the Bernoulli likelihood information contained in the graph distribution has been neglected. This observation has motivated the development of likelihood-based inference for random dot product graphs. Xie and Xu (2020) proposed a fully Bayesian approach for estimating the latent positions in random dot product graphs, referred to as posterior spectral embedding, and established its global minimax optimality. Xie and Xu (2021) proposed a novel one-step procedure, which lead to a one-step estimator that took advantage of the Bernoulli likelihood information of the sampling model through the score function and the Fisher information matrix, to estimate random dot product graphs from the frequentist perspective. There, the authors further established the asymptotic efficiency of the one-step estimator and its smaller asymptotic sum of squared errors compared to that of the spectral estimators. Later, Tang et al. (2022) applied the idea of the one-step refinement of spectral methods to stochastic block models when the block probability matrix is rank deficient.

Despite the success of the one-step estimator, a central question regarding likelihood-based inference for random dot product graphs remains open: What is the behavior of the frequentist maximum likelihood estimator? Also, a related question is: What is the behavior of the Bayes estimator? Efforts attempting to address these two questions aim to gain deeper insight into the likelihood-based inference for random dot product graphs from the frequentist and the Bayesian perspective, respectively. These two questions are also closely related through the Bernstein–von Mises phenomenon (see, for example, Section 10.2 in Van der Vaart, 2000). Here, the major technical barrier is the complicated structure of the parameter space
for the latent positions. In this paper, we partially answer the aforementioned two questions by resorting to a cleverly-designed surrogate likelihood function that simplifies the parameter space enormously. Our work features the following novel contributions: Firstly, the surrogate likelihood function has a separable structure, is log-concave, and the associated parameter space for the latent positions is a convex relaxation of the original latent space. These features greatly facilitate both the theoretical analyses and the related practical computations. Secondly, we establish the existence, uniqueness, and the asymptotic efficiency of the frequentist maximum surrogate likelihood estimator under the minimal sparsity condition. In particular, similar to the one-step estimator, the maximum surrogate likelihood estimator improves upon the baseline spectral estimators with a smaller sum of squared errors. Thirdly, we design a computationally efficient stochastic gradient descent algorithm for the maximum surrogate likelihood estimator with adaptive step sizes. Fourthly, regarding the Bayes procedure, we establish the Bernstein–von Mises theorem for the posterior distribution with the surrogate likelihood function and show that the resulting credible sets have the correct frequentist coverage probabilities.

The remaining part of the article is structured as follows. In Section 2, we review the background of random dot product graphs and introduce the surrogate likelihood function. Section 3 elaborates on the theoretical properties and the computational algorithm of the maximum surrogate likelihood estimation. In Section 4, we establish the large sample properties of the Bayes procedure with the surrogate likelihood function. Section 5 demonstrates the empirical performance of the proposed methods through simulation examples and the analysis of a real-world Wikipedia network dataset. We conclude the paper with a discussion in Section 6.

Notations: Let \([n]\) denote the set of consecutive integers from 1 to \(n\): \([n] = \{1, \ldots, n\}\). The symbol \(\lesssim\) means an inequality up to a constant depending on \(\delta\), that is, \(a \lesssim b\) if \(a \leq C_\delta b\) for some constant \(C_\delta > 0\) depending on \(\delta\); a similar definition also applies to the symbol \(\gtrsim\). The notation \(\|x\|\) denotes the Euclidean norm of a vector \(x = [x_1, \ldots, x_d]^T \in \mathbb{R}^d\), that is, \(\|x\| = (\sum_{k=1}^d x_k^2)^{1/2}\). The \(d \times d\) identity matrix is denoted as \(I_d\). The notation \(O(n, d) = \{U \in \mathbb{R}^{n \times d} : U^T U = I_d\}\) denotes the set of all orthonormal \(d\)-frames in \(\mathbb{R}^n\), where \(d \leq n\), and we write \(O(d) = O(d,d)\). For a matrix \(X = [x_{ik}]_{n \times d}\), \(\sigma_k(X)\) denotes its \(k\)th largest singular value, and when \(X\) is square and symmetric, \(\lambda_k(X)\) denotes its \(k\)th largest eigenvalue in magnitude. Matrix norms with following definitions are used: the spectral norm \(\|X\|_2 = \sigma_1(X)\), the Frobenius norm \(\|X\|_F = (\sum_{i=1}^n \sum_{k=1}^d x_{ik}^2)^{1/2}\), the matrix infinity norm \(\|X\|_{\infty} = \max_{i \in [n]} \sum_{k=1}^d |x_{ik}|\), and the two-to-infinity norm \(\|X\|_{2 \to \infty} = \max_{i \in [n]} (\sum_{k=1}^d x_{ik}^2)^{1/2}\). In particular, these norm notations apply to any Euclidean vector \(x \in \mathbb{R}^d\) viewed as a \(d \times 1\) matrix. Given two symmetric positive semidefinite matrices \(A,B\) of the same dimension, we write \(A \preceq B\) (\(A \succeq B\), respectively) if \(B - A\) (\(A - B\), respectively) is positive semidefinite.

## 2 Background and the Surrogate Likelihood

### 2.1 Background on random dot product graphs

We begin by briefly reviewing the background on random dot product graphs and adjacency spectral embedding. Consider a graph with \(n\) vertices labeled as \([n] = \{1, \ldots, n\}\). Let \(\mathcal{X}\) be a subset of \(\mathbb{R}^d\) such that \(x_1^T x_2 \in (0,1)\) for all \(x_1, x_2 \in \mathcal{X}\), where \(d\) is fixed and \(d \leq n\), and let \(\rho_n \in (0,1]\) be a sparsity factor. Each ver-
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are deterministic parameters to be estimated. Another slightly different modeling approach is to consider \( x_1, \ldots, x_n \) as independent and identically distributed latent random variables (see, for example, Athreya et al., 2016; Tang et al., 2017b; Tang and Priebe, 2018). This random formulation of the latent positions introduces implicit homogeneity and is connected to the infinite exchangeable random graphs (Janson and Diaconis, 2008). The same homogeneity condition was retained in Xie and Xu (2021) using a Glivenko–Cantelli type condition when \( x_1, \ldots, x_n \) are deterministic. The latter Glivenko–Cantelli type condition is also relaxed in the current work as we only require that \( \sigma_d(X) > 0 \) (see Remark 2 below).

Remark 2 (Nonidentifiability). The latent position matrix \( X \) is not uniquely identified in the following two senses. Firstly, any low-rank positive semidefinite connection probability matrix \( P = XX^T \) can have different factorizations because for any orthogonal matrix \( W \in \mathbb{O}(d) \), \( XX^T = (XW)(XW)^T \). Secondly, for any \( d' > d \) and any latent position matrix \( X \in \mathbb{R}^d \), there exists another matrix \( X' \in \mathbb{R}^{d'} \) such that \( XX^T = X'(X')^T \). The latter source of non-identifiability can be removed by requiring that \( \sigma_d(X) > 0 \), while the former source is inevitable without further constraints. Thus, any estimator of the latent position matrix \( X \) can only recover it up to an orthogonal transformation.

Example 1 (Stochastic block model). Random dot product graphs have connections with the popular stochastic block model (Holland et al., 1983). Consider a graph with \( n \) vertices that are partitioned into \( K \) communities, where \( K \) is assumed to be much smaller than \( n \). Let \( \tau : [n] \rightarrow [K] \) be a cluster assignment function that assigns each vertex to a unique community. Let \( B = [B_{kl}]_{K \times K} \in \{0, 1\}^{K \times K} \) be a symmetric probability matrix and \( A_{ij} \) be the binary indicator of the existence of an edge between vertices \( i \) and \( j \). Then the stochastic block model specifies that \( A_{ij} \sim \text{Bernoulli}(B_{\tau(i)\tau(j)}) \) independently for all \( i, j \in [n], i < j \), and \( A_{ij} = A_{ji} \) for all \( i > j \). By converting the community assignment to a matrix \( Z = [1\{\tau(i) = k\}]_{n \times K} \), we see that the expected adjacency matrix \( XZBZ^T \) is symmetric and of low rank. Furthermore, if \( B \) is positive semidefinite with rank \( d \leq K \) and can be factorized as \( B = VV^T \) for a \( K \times d \) matrix \( V \), then \( A \) can be seen as an adjacency matrix generated by the random dot product graph with latent position matrix \( X = ZV \), that is, \( A \sim \text{RDPG}(ZV) \).

Motivated by the low-rank structure of random dot product graphs, Sussman et al. (2012) proposed to estimate the latent position matrix \( X \) by solving the least squares problem \( \hat{X} = \arg \min_X \in \mathbb{R}^{n \times d} \| A - XX^T \|_F^2 \). The interpretation is that \( \hat{X} \) can be viewed as the projection of the data matrix \( A \) onto the space of all \( n \times n \) rank-\( d \) positive semidefinite matrices with regard to the Frobenius norm distance. The solution \( \hat{X} \) is referred to as the adjacency spectral embedding of \( A \) into \( \mathbb{R}^d \), and can be computed as the matrix of eigenvectors associated with the top \( d \) eigenvalues of \( A \), scaled by the square roots of the corresponding
spectral embedding: There exists a which is immaterial. This approximation step is motivated by the uniform consistency of the adjacency
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remaining
The motivation is that the exact likelihood function has a complicated structure, bringing challenges for
developing the theory of the maximum likelihood estimation. The difficulty partially comes from the fact
that the random dot product graph model belongs to a curved exponential family, and the theory of the
maximum likelihood estimation is much more difficult in curved exponential families than in the canonical
ones (see, for example, Section 2.3 in Bickel and Doksum, 2007).

To distinguish a generic latent position \( x_i \in \mathbb{R}^d \) and its true value associated with the data generating
distribution, let \( x_{0i} \) denote the ground truth of \( x_i \), \( i \in [n] \), and \( X_0 = [x_{01}, \ldots, \ x_{0n}]^T \). Let us begin by considering
the log-likelihood function of a single \( x_i \), when the remaining latent positions \( (x_{0j})_{j \neq i} \) are accessible:

\[
\ell_{0in}(x_i) = \sum_{j \neq i} \left\{ A_{ij} \log(\rho_n x_i^T x_{0j}) + (1 - A_{ij}) \log(1 - \rho_n x_i^T x_{0j}) \right\} + \left\{ A_{ii} \log(\rho_n x_i^T x_i) + (1 - A_{ii}) \log(1 - \rho_n x_i^T x_i) \right\}.
\]  

(2.1)

We refer to \( \ell_{0in}(x_i) \) in (2.1) as the oracle log-likelihood function because it requires the true values of the
remaining \( x_j \)'s with \( j \neq i \). Theorem 2 in Xie and Xu (2021) established the consistency and asymptotic
normality of the maximizer of the oracle log-likelihood function \( \ell_{0in}(x_i) \) in (2.1). Nevertheless, the oracle
log-likelihood is not computable because \( (x_{0j}) \) are not accessible in practice. Following the idea in Xie and
Xu (2021), we consider replacing the unknown latent positions by the corresponding rows of the adjacency
spectral embedding. Formally, let \( \tilde{x}_j \) be the \( j \)th row of the adjacency spectral embedding \( \tilde{X} \), \( j \in [n] \). Then
we obtain the following approximation to the oracle log-likelihood:

\[
\ell_{0in}(x_i) \approx \sum_{j=1}^n \left\{ A_{ij} \log(\rho_n^{1/2} x_i^T \tilde{x}_j) + (1 - A_{ij}) \log(1 - \rho_n^{1/2} x_i^T \tilde{x}_j) \right\}.
\]  

(2.2)

Note that the last term in \( \ell_{0in} \) is replaced by \( A_{ii} \log(\rho_n^{1/2} x_i^T \tilde{x}_i) + (1 - A_{ii}) \log(1 - \rho_n^{1/2} x_i^T \tilde{x}_i) \) for convenience,
which is immaterial. This approximation step is motivated by the uniform consistency of the adjacency
spectral embedding: There exists a \( d \times d \) orthogonal \( W \) such that \( \| \tilde{X} W - \rho_n^{1/2} X_0 \|_2 = O(\sqrt{\log n}/n) \) with
high probability (Lyzinski et al., 2014; Xie, 2022).

With the approximation in (2.2), the constraints for the latent position \( x_i \) become a system of linear
inequalities: \( 0 < \rho_n^{1/2} x_i^T \tilde{x}_j < 1 \) for all \( j \in [n] \). Geometrically, these constraints correspond to a convex
polyhedron. Namely, given any vector \( x_i \in \mathbb{R}^d \), checking whether \( x_i \) is in such a convex polyhedron requires
\( O(n) \) operations, so that the relevant computation could be cumbersome. We now resolve this issue by
applying a quadratic Taylor approximation to the terms \( \log(\rho_n^{1/2} x_i^T \tilde{x}_j) \) and relax the parameter space for \( x_i \).
Here we can drop the sparsity factor \( \rho \) without loss of generality. Formally, write \( g_{ij}(x_i) = A_{ij} \log(x_i^T \hat{x}_j) \). Then a quadratic Taylor approximation to \( g_{ij} \) at \( x_i = \bar{x}_i \) leads to

\[
g_{ij}(x_i) = g_{ij}(\bar{x}_i) + \frac{A_{ij}\bar{x}_i^T(x_i - \bar{x}_i)}{\bar{x}_i^T \bar{x}_j} - \frac{A_{ij}(x_i - \bar{x}_i)^T \bar{x}_j \bar{x}_i^T (x_i - \bar{x}_i)}{2(\bar{x}_i^T \bar{x}_j)^2} + \text{remainder}. \tag{2.3}
\]

Meanwhile, it is also conceivable that

\[
\sum_{j=1}^{n} \frac{A_{ij}}{2(\bar{x}_i^T \bar{x}_j)^2} (x_i - \bar{x}_i)^T \bar{x}_j \bar{x}_i^T (x_i - \bar{x}_i) \approx \sum_{j=1}^{n} \frac{1}{2\bar{x}_i^T \bar{x}_j} (x_i - \bar{x}_i)^T \bar{x}_j \bar{x}_i^T (x_i - \bar{x}_i) \tag{2.4}
\]

because \( \mathbb{E}_0(A_{ij}) = \rho_n x_i^T x_{0j} \approx x_i^T \bar{x}_j \). Hence, ignoring the constant terms that are free of \( x_i \), combining the approximations in (2.2), (2.3), and (2.4) leads to the following surrogate log-likelihood function

\[
\tilde{\ell}_{in}(x_i) = \sum_{j=1}^{n} \left\{ \frac{A_{ij}\bar{x}_i^T x_i}{\bar{x}_i^T \bar{x}_j} + \bar{x}_j^T x_i - \frac{1}{2\bar{x}_i^T \bar{x}_j} x_i^T \bar{x}_j \bar{x}_i^T x_i + (1 - A_{ij}) \log(1 - x_i^T \bar{x}_j) \right\}. \tag{2.5}
\]

Therefore, by Cauchy–Schwarz inequality, the the parameter space for \( x_i \) associated with the surrogate likelihood of vertex \( i \) can be taken as the unit ball \( \{x_i \in \mathbb{R}^d : \|x_i\| \leq 1 \} \) for all \( i \in [n] \) when \( \max_{j \in [n]} \|\bar{x}_j\|_2 < 1 \) (which holds with high probability). Consequently, we relax the original complicated parameter space \( \{x_i \in \mathbb{R}^d : 0 < x_i^T \bar{x}_j < 1, j \in [n] \} \) to a simple unit ball, which is much more tractable to work with. Moreover, the surrogate likelihood function has a separable structure because \( \tilde{\ell}_{in}(x_i) \) does not involve \( x_j \) for \( j \neq i \). This convenience enables parallelization when related computation is requested. In addition, a simple algebra shows that the surrogate likelihood function is log-concave, a highly desired feature when optimization and Monte Carlo sampling are needed.

### 2.3 Comparison with the one-step estimator

Recently, Xie and Xu (2021) proposed a one-step estimator \( \hat{X}^{OS} = [\hat{x}_1^{OS}, \ldots, \hat{x}_n^{OS}]^T \) for random dot product graphs that improves upon the adjacency spectral embedding:

\[
\hat{x}_i^{OS} = \bar{x}_i + \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{\bar{x}_j \bar{x}_j^T}{\bar{x}_i^T \bar{x}_j (1 - \bar{x}_i^T \bar{x}_j)} \right\}^{-1} \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{(A_{ij} - \bar{x}_i^T \bar{x}_j) \bar{x}_j}{\bar{x}_i^T \bar{x}_j (1 - \bar{x}_i^T \bar{x}_j)} \right\}, \quad i \in [n]. \tag{2.6}
\]

The one-step estimator originates from a one-step updating scheme of the Newton-Raphson method for maximizing the log-likelihood function with the initial guess being the adjacency spectral embedding (see, e.g., Section 5.7 in Van der Vaart, 2000). It is clear from the construction that the one-step estimator takes advantage of the likelihood information of the sampling distribution through the Fisher information matrix and the score function.

In Section 2.2, we have shown the derivation of the surrogate log-likelihood function by applying a quadratic Taylor approximation to the logarithm function \( \log(x_i^T \bar{x}_j) \). We now show that the same approximation treatment applied to the entire function in (2.2) results in an approximate log-likelihood function
whose maximizer is exactly the one-step estimator. Formally, applying a second-order Taylor expansion to the term \( \log(1 - x_i^T \tilde{x}_j) \) at \( x_i = \tilde{x}_i \) yields

\[
\log \left( \frac{1 - x_i^T \tilde{x}_j}{1 - \tilde{x}_i^T \tilde{x}_j} \right) = -\frac{\tilde{x}_j^T (x_i - \tilde{x}_i)}{1 - \tilde{x}_i^T \tilde{x}_j} - \frac{(x_i - \tilde{x}_i)^T \tilde{x}_j \tilde{x}_j^T (x_i - \tilde{x}_i)}{2(1 - \tilde{x}_i^T \tilde{x}_j)^2} + \text{remainder}. \tag{2.7}
\]

Following the idea in (2.4), we can also conceive the following approximation:

\[
\sum_{j=1}^{n} \frac{(1 - A_{ij})(x_i - \tilde{x}_i)^T \tilde{x}_j \tilde{x}_j^T (x_i - \tilde{x}_i)}{2(1 - \tilde{x}_i^T \tilde{x}_j)^2} \approx \sum_{j=1}^{n} \frac{(x_i - \tilde{x}_i)^T \tilde{x}_j \tilde{x}_j^T (x_i - \tilde{x}_i)}{2(1 - \tilde{x}_i^T \tilde{x}_j)}. \tag{2.8}
\]

We thus obtain the following quadratic approximation to (2.2) modulo a constant term from (2.5), (2.7), and (2.8):

\[
\ell_{\text{in}}^{(\text{OS})}(x_i) = \sum_{j=1}^{n} \frac{(A_{ij} - \tilde{x}_j^T \tilde{x}_j)(x_i - \tilde{x}_i)}{\tilde{x}_i^T \tilde{x}_j(1 - \tilde{x}_i^T \tilde{x}_j)} - \sum_{j=1}^{n} \frac{(x_i - \tilde{x}_i)^T \tilde{x}_j \tilde{x}_j^T (x_i - \tilde{x}_i)}{2\tilde{x}_i^T \tilde{x}_j(1 - \tilde{x}_i^T \tilde{x}_j)}. \tag{2.9}
\]

Then a simple algebra shows that the one-step estimator \( \hat{x}_i^{\text{(OS)}} \) maximizes \( \ell_{\text{in}}^{(\text{OS})} \) defined in (2.9).

Clearly, the surrogate log-likelihood function in (2.5) is constructed by applying a Taylor expansion to the term \( \log(x_i^T \tilde{x}_j) \), whereas the one-step estimator is obtained by applying the Taylor expansion to the entire function. Thus, intuitively, the surrogate likelihood retains more likelihood information than the one-step procedure does. Below, we visualize this heuristic using a toy numerical example.

**Example 2.** Consider the following random dot product graph model. Let \( n = 300 \), \( (t_i)_{i=1}^{n} \) be equidistant points over \([0, 1]\), \( x_{0i} = 0.2 + 0.6 \sin(\pi t_i), i \in [n] \), and \( X_0 = [x_{01}, \ldots, x_{0n}]^T \in \mathbb{R}^{n \times 1} \). Suppose \( A \sim \text{RDPG}(X_0) \) and we focus on the likelihood function for \( x_i \) with \( i = 100 \). Figure 1 visualizes the comparison among the oracle log-likelihood \( \ell_{\text{in}}(x_i) \), the surrogate log-likelihood \( \ell_{\text{in}}^{(\text{OS})}(x_i) \), and the approximate log-likelihood \( \ell_{\text{in}}^{(\text{OS})}(x_i) \) associated with the one-step estimator. The constant terms of these functions have been added to make them comparable. The vertical lines mark the maximizers of the three functions, respectively. It is visually clear that the maximizer of the surrogate log-likelihood estimate is closer to that of the oracle log-likelihood than the one-step estimator is, suggesting that the maximum surrogate likelihood estimator may outperform the one-step estimator in some practical finite sample problems.

## 3 Maximum Surrogate Likelihood Estimation

### 3.1 Theoretical properties

This subsection elaborates on the theoretical properties of the frequentist inference with the surrogate likelihood. Below, Theorem 3.1 establishes the existence and uniqueness of the maximum surrogate likelihood estimator.

**Theorem 3.1.** Suppose \( A \sim \text{RDPG}(\rho_n^{1/2} X_0) \) and \( (\log n)/(np_n) \to 0 \) as \( n \to \infty \). Assume \( \lambda_d(X_0^T X_0/n) \geq \lambda \) for some constant \( \lambda > 0 \) for all \( n > d \), and \( \min_{i,j \in [n]}(x_{0i}^T x_{0j}, 1 - x_{0i}^T x_{0j}) \geq \delta \) for some constant...
Figure 1: Comparison among the oracle log-likelihood function $\ell_{0in}(x_i)$, the surrogate log-likelihood function $\tilde{\ell}_{in}(x_i)$, and the approximate log-likelihood function $\tilde{\ell}_{in}^{(OS)}(x_i)$ associated with the one-step estimator. The three vertical lines mark the one-step estimate, the maximum surrogate likelihood estimate, and the oracle maximum likelihood estimate, respectively.

Let $x_i \in [n]$ be a fixed vertex and consider the maximum surrogate likelihood estimator $\hat{x}_i = \arg\max_{x_i: \|x_i\|_2 \leq 1} \tilde{\ell}_{in}(x_i)$. Then for any $c > 0$, there exists some constant $N_{c,\delta,\lambda} \in \mathbb{N}$ depending on $c, \delta, \lambda$ such that $\mathbb{P}_0(\hat{x}_i \text{ exists and is unique}) \geq 1 - n^{-c}$ for all $n \geq N_{c,\delta,\lambda}$.

Let $G_{0in} = (1/n) \sum_{j=1}^{n} x_{ij}^T x_{ij}^T \{x_{ij}^T x_{ij} (1 - x_{ij}^T x_{ij})\}^{-1}$ be the Fisher information matrix with regard to the latent position $x_i$. Theorem 3.2 below, which is one of the main results in this article, establishes the large sample properties of the maximum surrogate likelihood estimator.

**Theorem 3.2.** Suppose the conditions of Theorem 3.1 hold and the embedding dimension $d$ is fixed. For each $i \in [n]$, let $\hat{x}_i = \arg\max_{x_i: \|x_i\|_2 \leq 1} \tilde{\ell}_{in}(x_i)$ be the maximum surrogate likelihood estimator. Then there exists an orthogonal matrix $W \in \mathbb{O}(d)$ that depends on $n$, such that for each $i \in [n]$,

$$\sqrt{n} G_{0in}^{1/2} (W^T \hat{x}_i - \rho_n^{1/2} x_0) \to N_d(0_d, I_d)$$

in distribution. Furthermore, if $(\log n)^4/(n \rho_n) \to 0$, then

$$\|\hat{X}W - \rho_n^{1/2} X_0\|_F^2 - \frac{1}{n} \sum_{i=1}^{n} \text{tr}(G_{0in}^{-1}) \to 0$$

in probability, where $\hat{X} = [\hat{x}_1, \ldots, \hat{x}_n]^T$.

**Remark 3 (Sparsity condition).** The sparsity condition that $(\log n)/(n \rho_n) \to 0$ required in Theorem 3.1 and in the asymptotic normality of Theorem 3.2 is minimal in the following sense. It is well known that the random adjacency matrix $A$ no longer concentrates around its expected value $\mathbb{E}_0(A)$ when $(\log n)/(n \rho_n) \to \infty$.
Remark 4 (Comparison with the adjacency spectral embedding and the one-step estimator). Athreya et al. (2016), Tang and Priebe (2018), and Xie and Xu (2021) have established the large sample properties of the adjacency spectral embedding and the one-step estimator as the following. Let \( \tilde{\text{adjacency spectral embedding}} \) and the one-step estimator be independent \( \tilde{\text{Unif}}(1/n) \) random variables, where \( \Sigma_{in} \) satisfies \( \Sigma_{in} \succeq G_{0in}^{-1} \). Theorem 3.2 thus suggests that the maximum surrogate likelihood estimator improves upon the adjacency spectral embedding and is (first-order) asymptotically equivalent to the one-step estimator. This phenomenon is also known as the local efficiency (Xie and Xu, 2021) because the asymptotic covariance matrix for a single latent position \( \tilde{x}_i \) is the same as that of the oracle maximum likelihood estimator.

3.2 Computation details

This subsection discusses the detailed algorithm for computing the maximum surrogate likelihood estimator. For a given vertex \( i \in [n] \), the estimator \( \tilde{x}_i = \arg \max_{x_i} \tilde{\ell}_{in}(x_i) \) can be computed separately for each vertex \( i \in [n] \). Thus, it is sufficient to design an algorithm for solving the optimization problem \( \max_{\|x_i\|_2 \leq 1} (1/n) \tilde{\ell}_{in}(x_i) \). Observe that the objective function \( (1/n) \tilde{\ell}_{in}(x_i) \) is concave and can be written in a sample average fashion, which motivates us to adopt the stochastic gradient descent algorithm (Robbins and Monro, 1951). Let \( j_1, \ldots, j_s \) be independent \( \tilde{\text{Unif}}(1, \ldots, n) \) random variables, where \( s \in \{1, \ldots, n\} \) is the so-called batch size, and for any \( j \in [n] \), let

\[
m_i(x_i, j) = \frac{A_{ij} \tilde{x}_j^T x_i}{p_{ij}} + \tilde{x}_j^T x_i - \frac{1}{2p_{ij}} x_i^T x_i + (1 - A_{ij}) \log(1 - x_i^T x_j).
\]

It is clear that for each \( j_k, k \in [s] \), \( m_i(x_i, j_k) \) can be viewed as a noisy measurement of the objective function \( (1/n) \tilde{\ell}_{in}(x_i) \) because \( (1/n) \tilde{\ell}_{in}(x_i) = E_{j_k} \{ m_i(x_i, j_k) \} \). Then given a sequence of step sizes \( \{ \alpha_t \}_{t \geq 1} \) and an initial guess \( \tilde{x}_i^{(0)} \), the stochastic gradient descent algorithm generates a sequence of iterates \( \{ \tilde{x}_i^{(t)} \}_{t \geq 1} \) using the updating scheme

\[
\tilde{x}_i^{(t+1)} = \tilde{x}_i^{(t)} + \alpha_t \sum_{k=1}^{s} \frac{\partial m_i}{\partial x}(\tilde{x}_i^{(t)}, j_k^{(t)}),
\]

(3.1)
where \( \{j_1^{(t)}, \ldots, j_s^{(t)}\}_{t \geq 1} \) are independent copies of \( (j_1, \ldots, j_s) \). The advantage of the stochastic gradient descent method over the classical gradient descent algorithm is that, with a comparatively small batch size \( s \), one only needs to compute \( s \) gradient measurements of \( m_i(x_i, j) \) rather than all the gradient measurements of \( \{m_i(x_i, j)\}_{j=1}^n \). This computational convenience is especially desired when the network contains large number of vertices. To implement the algorithm with adaptive step sizes, we follow the suggestion given by Duchi et al. (2011) and Li and Orabona (2019) and take

\[
\alpha_t = a_0 \left( b_0 + \sum_{l=1}^{t-1} \left| \frac{1}{s} \sum_{k=1}^{s} \frac{\partial m_i(x_i^{(l)}, j_k^{(l)})}{\partial x_i} \right|_2 \right)^\frac{\epsilon}{(\epsilon+1/2)},
\]

where \( a_0, b_0 > 0 \) and \( 0 < \epsilon \leq 1/2 \) are constants.

The key difference between our algorithm and the standard stochastic gradient descent algorithm is that the feasible region \( \{x_i \in \mathbb{R}^d : \|x_i\| \leq 1\} \) is compact. Therefore, whenever \( \|\tilde{x}_i^{(t+1)}\| \) stays outside the feasible region, one repeats step-halving procedures until \( \|\tilde{x}_i^{(t+1)}\| \leq 1 \). Below, Theorem 3.3 shows the convergence of such a step-halving stochastic gradient descent with adaptive step sizes.

**Theorem 3.3.** Let the vertex \( i \in [n] \) be fixed and suppose \((1/n)\tilde{\ell}_\text{in}(x_i)\) is well-defined. Assume that \( \tilde{x}_i = \arg \max_{x_i : \|x_i\| \leq 1} (1/n)\tilde{\ell}_\text{in}(x_i) \) lies in the interior of \( \{x_i \in \mathbb{R}^d : \|x_i\| \leq 1\} \). Then the sequence of iterates \( \{\tilde{x}_i^{(t)}\}_{t \geq 1} \) generated by (3.1) with step sizes \( \{\alpha_t\}_{t \geq 1} \) given by (3.2) and step-halving converges to \( \tilde{x}_i \) almost surely with regard to the distribution of \( \{\tilde{\ell}_\text{in}(x_i)\}_{t \geq 1} \).

**Remark 5.** The surrogate log-likelihood function \( \tilde{\ell}_\text{in}(x_i) \) is well-defined only when \( x_i^T \tilde{x}_j < 1 \) for all \( j \in [n] \) because of the logarithm terms \( \{\log(1-x_i^T \tilde{x}_j)\}_{j=1}^n \). For sufficiently large \( n \), the constraint is satisfied by requiring that \( \|x_i\|_2 \leq 1 \) since the adjacency spectral embedding \( \tilde{X} = [\tilde{x}_1, \ldots, \tilde{x}_n]^T \) satisfies \( \max_{j \in [n]} \|\tilde{x}_j\|_2 < 1 \) with high probability. However, this requirement may not hold in certain finite sample problems, in which case the surrogate log-likelihood function \( \tilde{\ell}_\text{in}(x_i) \) is no longer well-defined. This numerical issue can be practically addressed by the following smooth concatenation technique. Roughly speaking, for a fixed \( j \in [n] \), when \( 1 - x_i^T \tilde{x}_j \) drops below a small threshold, we replace the objective function \((1/n)\tilde{\ell}_\text{in}(x_j)\) by a quadratic function such that the two pieces of functions are concatenated smoothly. Formally, let \( \tau > 0 \) be a small threshold and define

\[
h_i(x_i, j) = \begin{cases} m_i(x_i, j), & \text{if } 1 - x_i^T \tilde{x}_j \geq \tau, \\ \alpha_{ij} (x_i^T \tilde{x}_j)^2 + \beta_{ij} (x_i^T \tilde{x}_j) + \gamma_{ij}, & \text{if } 1 - x_i^T \tilde{x}_j < \tau, \end{cases}
\]

for each \( j \in [n] \), where \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) are coefficients such that \( h_i(\cdot, j) \) is twice continuously differentiable. Then the objective function \((1/n)\tilde{\ell}_\text{in}(x_i)\) can be replaced by \((1/n)\sum_{j=1}^n h_i(x_i, j)\) and the aforementioned stochastic gradient descent algorithm applies with \( \partial m_i(x_i, j) / \partial x_i \) replaced by \( \partial h_i(x_i, j) / \partial x_i \).

## 4 Bayesian Estimation With Surrogate Likelihood

This section explores Bayesian estimation of random dot product graphs with the proposed surrogate likelihood. Although Xie and Xu (2020) has established the minimax optimality of the Bayesian random dot
product graph model with the exact likelihood, the asymptotic shape of the posterior distribution is yet to be characterized because of the complicated structure of the exact likelihood function. In contrast, thanks to the separable and log-concave properties of the surrogate likelihood, we are able to completely characterize the asymptotic posterior distribution of the latent positions with the exact likelihood replaced by the surrogate.

Formally, for any fixed vertex \( i \in [n] \) and a prior distribution \( \pi(\cdot) \) supported on \( \{ x \in \mathbb{R}^d : \| x \|_2 \leq 1 \} \), the posterior distribution of \( x_i \) given \( A \) with the surrogate log-likelihood function \( \tilde{\ell}_{in}(x_i) \) can be written as

\[
\pi_{in}(x_i \mid A) = \frac{\exp(\tilde{\ell}_{in}(x_i))\pi(x_i)}{\int \exp(\tilde{\ell}_{in}(x_i))\pi(x_i)dx_i}.
\]  

(4.1)

Then the joint posterior density of the entire latent position matrix \( X = [x_1, \ldots, x_n]^T \) is taken as the product \( \pi_n(X \mid A) = \prod_{i=1}^{n} \pi_{in}(x_i \mid A) \) because the surrogate log-likelihood function is separable across different vertices.

When the exact likelihood function is not available or intractable for analysis or computation, the idea of using a general statistical criterion function to replace the likelihood in the Bayes formula is not entirely new, among which an influential work is Chernozhukov and Hong (2003). There have also been several recent works addressing the large sample properties of the so-called quasi-posterior or Gibbs posterior distributions (Kleijn and van der Vaart, 2012; Miller, 2021; Syring and Martin, 2018, 2022). One key difference is that unlike the well-specified exact posterior distributions, the frequentist coverage of the credible sets of the quasi-posterior distributions may not agree with their credibility level (Kleijn and van der Vaart, 2012).

Below, we show that, with the surrogate likelihood, the posterior distribution produces credible sets that quasi-posterior distributions may not agree with their credibility level (Kleijn and van der Vaart, 2012). One key difference is that the surrogate log-likelihood function is separable across different vertices.

**Theorem 4.1.** Suppose the conditions of Theorem 3.1 hold and the embedding dimension \( d \) is fixed. Let \( \pi(\cdot) \) be a prior density satisfying \( c \leq \pi(x_i) \leq C \) and \( |\pi(x) - \pi(y)| \leq C'||x - y||_2 \) for any \( x, y \) with \( ||x||_2, ||y||_2 \leq 1 \) for some constants \( 0 < c, C, C' < \infty \). Let \( W \) be the \( d \times d \) orthogonal matrix in Theorem 3.2. For any fixed vertex \( i \in [n] \), let \( \hat{x}_i = \arg\max_{x_i : \|x_i\|_2 \leq 1} \tilde{\ell}_{in}(x_i), \ t = \sqrt{n}W^T(x_i - \hat{x}_i), \) and \( \pi_{in}^*(t \mid A) \) be the density of \( t \) induced from (4.1). Then for any \( \alpha > 0 \),

\[
\max_{i \in [n]} \int \left( 1 + ||t||_2^2 \right) \pi_{in}(t \mid A) - \det(2\pi G_{in}^{-1})^{-1/2}e^{-t^TG_{in}t/2} \ dt \to 0 \quad (4.2)
\]

in probability.

Below, Corollary 4.1 discusses the effect of Theorem 4.1 on subsequent inference. In particular, it shows that for each vertex \( i \in [n] \), the posterior mean has the same asymptotic distribution as the maximum surrogate likelihood estimator, and the asymptotic level-\( \alpha \) credible set has the correct frequentist coverage probability.

**Corollary 4.1.** Suppose the conditions of Theorem 4.1 hold. For any vertex \( i \in [n] \), write \( x_{i}^* = \int x_i \pi_{in}(x_i \mid A)dx_i \) and \( \Sigma_{in}^* = \int (x_i - x_{i}^*)(x_i - x_{i}^*)^T \pi_{in}(x_i \mid A)dx_i \) as the posterior mean and the posterior covariance matrix of \( x_i \), respectively. Let \( q_{1-\alpha} \) be the \( (1-\alpha) \) quantile of the \( \chi_d^2 \) distribution and \( C_{in}(\alpha) = \{ x_i : (x_i - x_{i}^*)^T(\Sigma_{in}^*)^{-1}(x_i - x_{i}^*) \leq q_{1-\alpha} \} \) be the asymptotic \( (1-\alpha) \)-credible set for \( x_i \), where \( W \in O(d) \) is given
in Theorem 3.2. Then
\[ \sqrt{n}G_{0n}^{1/2}(W^TX_i^* - \rho_n^{1/2}X_{0i}) \rightarrow N_d(0_d, I_d) \]
in distribution, and
\[ P_0\{\rho_n^{1/2}Wx_i \in C_n(\alpha)\} \rightarrow 1 - \alpha. \]
Furthermore, if \((\log n)^4/(n\rho_n) \to 0\), then
\[ \|X^*W - \rho_n^{1/2}X_0\|_F^2 - \frac{1}{n} \sum_{i=1}^{n} \text{tr}(G_{0in}^{-1}) \to 0 \]
in probability, where \(X^* = [x_1^*, ..., x_n^*]^T\).

In practice, the posterior distribution based on the surrogate likelihood can be computed using a standard Metropolis–Hastings algorithm with parallelization over the vertices \(i \in [n]\). The detailed algorithm is provided in the Supplementary Material. Note that in practice, we can also apply the smooth concatenation technique discussed in Remark 5 to the posterior computation by simply replacing the surrogate log-likelihood function \(\tilde{\ell}_n(x_i)\) in the Bayes formula (4.1) by \(\sum_{j=1}^{n} h_i(x_i, j)\) defined in (3.3).

5 Numerical Examples

5.1 A latent curve example

In this subsection, we study the empirical performance of the proposed estimation procedures through a simulated random dot product graph example, where the latent positions are generated from a one-dimensional curve. Consider a random dot product graph with \(n\) vertices and latent dimension \(d = 1\). For each vertex \(i \in [n]\), the latent position \(x_{0i}\) for the \(i\)th vertex is set to \(x_{0i} = 0.8\sin\{\pi(i - 1)/(n - 1)\} + 0.1\). Let \(X_0 = [x_{01}, ..., x_{0n}]^T\), \(n = 1000\). Given \(A \sim \text{RDPG}(X_0)\), we consider the following four estimation procedures for \(X_0\): the adjacency spectral embedding, the one-step estimate, the maximum surrogate likelihood estimate obtained using the step-halving stochastic gradient descent algorithm, and the Bayes estimate with the surrogate likelihood. For the Bayes estimate, the Metropolis–Hastings sampler is implemented with parallelization over vertices \(i \in [n]\), and each Markov chain contains 1000 burn-in iterations and 200 post-burn-in samples with a thinning of 5. The posterior mean is taken as the point estimate. The convergence diagnostics of the Markov chains are provided in the Supplementary Material, showing no signs of non-convergence.

The performance of the above estimates is investigated via the following two objectives: The recovery of the latent position matrix \(X_0\); The empirical coverage probabilities of the vertex-wise confidence intervals based on the maximum surrogate likelihood estimate and credible intervals based on the Bayes estimate. Specifically, for the first objective, given a generic estimate \(\bar{X}\) for \(X_0\), we use the sum of squared errors \(\inf_{W \in \{\pm 1\}} \|XW - X_0\|_F^2\) as the evaluation metric. For the second objective, we compute the vertex-wise asymptotic 95% frequentist confidence intervals and Bayesian credible intervals. The vertex-wise 95% confidence intervals based on the maximum surrogate likelihood estimate are computed as follows: Denote the
$1 - \alpha/2$ quantile of the standard normal distribution by $z_{1-\alpha/2}$. Then by Theorem 3.2, for each $i \in [n]$, the $(1 - \alpha)$ confidence interval for $x_{0i}$ is $(|\hat{x}_i| - \{n\hat{G}(\hat{x}_i)\}^{-1/2}z_{1-\alpha/2}, |\hat{x}_i| + \{n\hat{G}(\hat{x}_i)\}^{-1/2}z_{1-\alpha/2})$, where $\hat{G}_m(\hat{x}_i) = (1/n) \sum_{j=1}^n \hat{x}_j(1 - \hat{x}_i\hat{x}_j)^{-1}$ is the plug-in estimate of the asymptotic variance. The vertex-wise 95% credible intervals based on the posterior distribution with the surrogate likelihood function can be obtained directly from the Metropolis–Hastings samples. The same numerical experiment is repeated for 1000 Monte Carlo replicates.

Table 1: The average sum of squared errors and their standard errors for Section 5.1. ASE, adjacency spectral embedding; OSE, one-step estimate; MSLE, maximum surrogate likelihood estimate computed by stochastic gradient descent; BE, Bayes estimate computed by Markov chain Monte Carlo sampling.

| Estimate                  | ASE  | OSE  | MSLE | BE   |
|---------------------------|------|------|------|------|
| Sum of squared errors     | 0.4707 | 0.4592 | 0.4596 | 0.4608 |
| Standard errors for sum of squared errors | 0.0216 | 0.0209 | 0.0209 | 0.0210 |

For the first objective, the sum of squared errors of the estimates are shown in Table 1. We can see that the sum of squared errors of the adjacency spectral embedding is comparatively larger than those of the remaining competitors, while the likelihood-based estimates have smaller sum of squared errors. This phenomenon empirically validate the conclusion that the likelihood-based estimates, namely, the one-step estimate, the maximum surrogate likelihood estimate, and the Bayes estimate, improve upon the the spectral-based adjacency spectral embedding.

For the second objective, Fig. 2 (a) and (b) visualize the empirical coverage probabilities of the vertex-wise 95% confidence intervals based on the maximum surrogate likelihood estimate and the vertex-wise 95% Bayesian credible intervals constructed from the Metropolis–Hastings samples, respectively, where the red horizontal lines mark the 95% nominal coverage probability.
5.2 A stochastic block model example

We now consider a stochastic block model in the context of a random dot product graph. The latent dimension is $d = 2$, the number of communities is $K = 5$, and the unique latent positions are $v_1 = [0.3, 0.3]^T$, $v_2 = [0.5, 0.5]^T$, $v_3 = [0.7, 0.7]^T$, $v_4 = [0.3, 0.7]^T$, and $v_5 = [0.7, 0.3]^T$. The cluster assignments of the vertices $(z_i)_{i=1}^n$ are drawn from a categorical distribution with probability vector $[1/K, \ldots, 1/K]^T$ and we set $x_{0i} = v_{z_i}$, $i \in [n]$. Note that $v_3$ is very close to the boundary of the parameter space. Let $X_0 = [x_{01}, \ldots, x_{0n}]^T$ and suppose an adjacency matrix $A$ is generated from $RDPG(X_0)$. The number of vertices $n$ is 2000.

We consider the performance of the same estimates as in Section 5.1 given a realization $A \sim RDPG(X_0)$: the adjacency spectral embedding, the one-step estimate, the maximum surrogate likelihood estimate and the Bayesian estimation with the surrogate likelihood. For the maximum surrogate likelihood estimate, we implement the step-halving stochastic gradient descent algorithm with the batch size set to $s = 500$ and $s = n$ (giving rise to the classical gradient descent algorithm) to compare the computational costs. The setup of the Metropolis–Hastings sampler for the Bayesian estimation is the same as in Section 5.1, and the convergence diagnostics are provided in the Supplementary Material. We take the posterior mean as the point estimate as before. The same experiment is repeated for 1000 independent Monte Carlo replicates.

Similar to Section 5.1, given a generic estimate $\bar{X}$, we compute the sum of squared errors of the estimates $\inf_{W \in O(2)} \|\bar{X}W - X_0\|_F^2$ to measure the estimation accuracy. These results are summarized in Table 2. We see that the one-step estimate is numerically unstable because $v_3$ is close to the boundary of the parameter space. Overall, the Bayes estimate outperforms the other competitors with the least errors, while the adjacency spectral embedding and the maximum surrogate likelihood estimate have similar performance in terms of the estimation error. This phenomenon suggests that, when some latent positions are close to the boundary of the parameter space, the Bayesian estimation method based on the Markov chain Monte Carlo sampler is numerically more stable than the optimization-based frequentist adjacency spectral embedding and the maximum surrogate likelihood estimate.

The computation times of a single experiment for different estimation procedures are reported in Table 2. We see that the adjacency spectral embedding and the one-step estimate are faster to compute, whereas the maximum surrogate likelihood estimate obtained through the classical gradient descent algorithm and the Bayes estimate are more computationally expensive. We also observe that the stochastic gradient descent algorithm is significantly faster than the classical gradient descent algorithm for finding the maximum surrogate likelihood estimate and it well balances between the computational efficiency and the estimation accuracy.

5.3 Analysis of Wikipedia Graph Dataset

In this section, we apply the proposed surrogate likelihood estimation methods to a real-world Wikipedia graph dataset. The network data is structured as follows: The vertices represent 1382 Wikipedia articles that are connected to the article named Algebraic Geometry within two hyperlinks, and an edge is assigned to link two articles if they are connected by a hyperlink. Besides the network itself, each Wikipedia article is also assigned with one of the following six class labels: people, places, dates, things, math and category.
Table 2: Numerical results for Section 5.2: The average sum of squared errors, their standard errors, the computation time of a single experiment (in seconds), and their standard errors. ASE, adjacency spectral embedding; OSE, one-step estimate; MSLE-SGD, maximum surrogate likelihood estimate computed by the stochastic gradient descent with batch size being 500; MSLE-GD, maximum surrogate likelihood estimate computed by the classical gradient descent; BE, Bayes estimate computed by Markov chain Monte Carlo sampling.

| Estimate                  | ASE   | OSE   | MSLE-SGD | MSLE-GD | BE    |
|---------------------------|-------|-------|----------|---------|-------|
| Sum of squared errors     | 8.570 | 31.646| 8.510    | 8.513   | 7.970 |
| Standard errors for the sum of squared errors | 0.250 | 18.242| 0.250    | 0.250   | 0.378 |
| Computation time (seconds)| 0.240 | 0.425 | 9.920    | 18.028  | 98.524|
| Standard errors for the computation time | 0.044 | 0.058 | 1.611    | 1.405   | 0.947 |

Table 3: Numerical results of Wikipedia graph data: Rand indices between the class labels and the clustering results based on the four estimates, across embedding dimensions \(d\) from 1 to 10, respectively. ASE, adjacency spectral embedding; OSE, one-step estimate; MSLE, maximum surrogate likelihood estimate; BE, Bayes estimate with the surrogate likelihood function.

| \(d\) | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-------|------|------|------|------|------|------|------|------|------|------|
| ASE   | 0.7446 | 0.7201 | 0.7213 | 0.7228 | 0.7313 | 0.7348 | 0.7364 | 0.7209 | 0.7231 | 0.7147 |
| OSE   | 0.6973 | 0.7056 | 0.7236 | 0.7278 | 0.7353 | 0.7385 | 0.7391 | 0.7409 | 0.7435 | 0.7399 |
| MSLE  | 0.7234 | 0.7114 | 0.7262 | 0.7357 | 0.7388 | 0.7441 | 0.7415 | 0.7444 | 0.7424 | 0.7468 |
| BE    | 0.7175 | 0.7146 | 0.7239 | 0.7349 | 0.7352 | 0.7423 | 0.7432 | 0.7435 | 0.7438 | 0.7453 |

The dataset is publicly available at at [http://www.cis.jhu.edu/~parky/Data/data.html](http://www.cis.jhu.edu/~parky/Data/data.html).

The goal is to study the clustering accuracy using different estimates when the embedding dimension varies. Given a selected embedding dimension \(d \geq 1\), we consider the following four estimates: the adjacency spectral embedding, the one-step estimate, the maximum surrogate likelihood estimate computed using the step-halving stochastic gradient descent algorithm, and the Bayes estimate based on the surrogate likelihood (we consider the posterior mean as the point estimate). Unlike the scenarios in the simulated examples in Sections 5.1 and 5.2, for this real dataset, the underlying ground truth of the latent positions is unknown. Rather, only the class labels of the vertices are available to us. To this end, we follow the suggestion in Tang and Priebe (2018) and apply the Gaussian-mixture-model-based clustering to the aforementioned four estimates. Namely, these estimates are regarded as the input for learning the clustering structure of the Wikipedia article network. We report the clustering accuracy using the Rand index (Rand, 1971) as the evaluation metric.

The Rand indices of the clustering results using different estimates across different embedding dimensions \(d \in \{1, 2, \ldots, 10\}\) are shown in Table 3. On one hand, we can see that when \(d \leq 2\), the adjacency spectral embedding yields better clustering accuracy with a higher Rand index value than the remaining competitors. On the other hand, as the embedding dimension \(d\) increases from 2 to 10, the maximum surrogate likelihood estimate and the Bayes estimate with the surrogate likelihood outperform the other two competitors. A plausible explanation of this phenomenon could be that the eigenvectors of the adjacency matrix with smaller eigenvalues are noisier than the top two eigenvectors, but this source of noise is reduced through the additional information introduced by the surrogate likelihood function.
6 Discussion

In Section 2.3, we have seen that the one-step estimator also corresponds to the maximizer of an approximate likelihood function, but it has a worse approximation quality than the proposed surrogate likelihood near the oracle maximum likelihood estimator. Surprisingly, under a framework of generalized estimating equations proposed by Xie and Wu (2022), the gradients of both the surrogate log-likelihood function and the approximate log-likelihood function associated with the one-step estimator can be viewed as some generalized estimating equations that take advantage of the likelihood function information. This intuition conforms to the fact that the estimators based on approximation of likelihood are asymptotically equivalent up to the first order. However, we have also found in some finite sample problems that the maximum surrogate likelihood estimator outperforms the one-step estimator. This difference may be caused by the difference in higher order terms, which is an interesting topic that we defer to future research.

Supplementary material

This supplementary material contains the proofs of the main results of the manuscript, the proof of the convergence of the step-halving stochastic gradient descent algorithm, additional implementation details including the algorithm details, an additional simulation example, and the convergence diagnostics of the Metropolis–Hastings sampler.
Supplementary Material for “Statistical inference of random graphs with a surrogate likelihood”

Abstract

This supplementary material contains the proofs of the main results of the manuscript, the proof of the convergence of the stochastic gradient descent, and additional implementation details, including the algorithm details, an additional simulation example, and the convergence diagnostics of the Metropolis–Hastings sampler.

A Preliminary Results

Lemma A.1. Let $\mathbf{A} \sim \text{RDPG}(\rho_n^{1/2} \mathbf{X}_0)$ with $n \rho_n \gtrsim \log n$. Denote $\Delta_n = (1/n) \mathbf{X}_0^{T} \mathbf{X}_0$. Assume $\lambda_d(\Delta_n) \geq \lambda$ for some constant $\lambda > 0$ for all sufficiently large $n$, and $\min_{i,j \in [n]} (\mathbf{X}_0^{T} \mathbf{X}_{ij}, 1 - \mathbf{X}_0^{T} \mathbf{X}_{ij}) \geq \delta$ for some constant $\delta > 0$. Then for all $c > 0$, there exists some constant $N_{c,\lambda} \in \mathbb{N}_+$ depending on $c$, $\lambda$, such that for all $n \geq N_{c,\lambda}$,

$$
\| \mathbf{X} - \rho_n^{1/2} \mathbf{X}_0 \|_{2 \rightarrow \infty} \lesssim_{c,\lambda} \sqrt{\frac{\log n}{n}},
$$

with probability at least $1 - n^{-c}$.

Proof. Denote $\kappa(\Delta_n) = \lambda_1(\Delta_n)/\lambda_d(\Delta_n)$. By Corollary 4.1 in Xie (2022), for all $c > 0$, we can pick a constant $N_c \in \mathbb{N}_+$ such that for all $n \geq N_c$, with probability at least $1 - n^{-c}$,

$$
\| \mathbf{X} - \rho_n^{1/2} \mathbf{X}_0 \|_{2 \rightarrow \infty} \lesssim_{c} \frac{\| \mathbf{U_p} \|_{2 \rightarrow \infty}}{(n \rho_n)^{1/2} \lambda_d(\Delta_n)^2} \max \left\{ \frac{(\log n)^{1/2}}{\lambda_d(\Delta_n)^2}, \frac{\kappa(\Delta_n)}{\lambda_d(\Delta_n)^2}, \frac{\log n}{\lambda_d(\Delta_n)^{1/2}} \right\}
$$

Observe that $\lambda_d(\Delta_n)$ is lower bounded by a constant $\lambda > 0$ for sufficiently large $n$, and $\lambda_1(\Delta_n) \leq (1/n)\|\mathbf{X}_0\|_F^2 \leq 1$. Also note that

$$
\| \mathbf{U_p} \|_{2 \rightarrow \infty} \leq \| \rho_n^{1/2} \mathbf{X}_0 \|_{2 \rightarrow \infty} \| \mathbf{S_p}^{-1/2} \|_2 \leq \sqrt{\frac{\rho_n}{n \rho_n \lambda_d(\Delta_n)}} \leq \frac{1}{\sqrt{n \lambda}}.
$$

Therefore, by the fact that $(\log n)/(n \rho_n)$ is bounded, we can pick a constant $N_{c,\lambda} \in \mathbb{N}_+$ depending on $c$, $\lambda$, such that for all $n \geq N_{c,\lambda}$, with probability at least $1 - n^{-c}$,

$$
\| \mathbf{X} - \rho_n^{1/2} \mathbf{X}_0 \|_{2 \rightarrow \infty} \lesssim_{c} \frac{\| \mathbf{U_p} \|_{2 \rightarrow \infty}}{(n \rho_n)^{1/2} \lambda^2} \cdot \frac{\log n}{\lambda_d(\Delta_n)^{1/2}} \leq \frac{\log n}{n \lambda}. \quad \Box
$$

Lemma A.2 (Some frequently used results). Suppose $\mathbf{A} \sim \text{RDPG}(\rho_n^{1/2} \mathbf{X}_0)$ and assume the conditions of Theorem 3.1 hold. Denote $\tilde{p}_{ij} = \tilde{x}_i^T \tilde{x}_j$, $i, j \in [n]$. Then for any $c > 0$, there exists a constant $N_{c,\delta,\lambda} \in \mathbb{N}_+$
depending on \( c, \delta, \lambda \) such that for all \( n \geq N_{c,\delta,\lambda} \), the following hold with probability at least \( 1 - n^{-c} \):

\[
\max_{j \in [n]} \| \bar{x}_j \|_2 \leq \rho_n \left( 1 - \frac{\delta}{2} \right),
\]

\[
\max_{i,j \in [n]} | \bar{p}_{ij} - \rho_n x_{0i}^T x_{0j} | \lesssim_{c,\lambda} \rho_n^{1/2} \sqrt{\log \frac{n}{n}},
\]

\[
\frac{\rho_n \delta}{2} \leq \min_{i,j \in [n]} \bar{p}_{ij} \leq \max_{i,j \in [n]} \bar{p}_{ij} \leq \rho_n \left( 1 - \frac{\delta}{2} \right),
\]

\[
\max_{j \in [n]} \| W^T \bar{x}_j x_j^T W - \rho_n x_{0j} x_{0j}^T \|_2 \lesssim_{c,\lambda} \rho_n^{1/2} \sqrt{\log \frac{n}{n}}.
\]

Proof: For the first result, by Lemma A.1 and the condition that \( \frac{\log n}{n^{p_n}} \to 0 \), we can pick a constant \( N_{c,\delta,\lambda} \in \mathbb{N} \) depending on \( c, \delta, \lambda \) such that for all \( n \geq N_{c,\delta,\lambda} \), with probability at least \( 1 - n^{-c} \),

\[
\| \bar{X} W - \rho_n^{1/2} X_0 \|_{2 \to \infty} = \max_{j \in [n]} \| W^T \bar{x}_j - \rho_n^{1/2} x_{0j} \|_2 \leq \rho_n^{1/2} \left( 1 - \frac{\delta}{2} - \sqrt{1 - \delta} \right).
\]

This is because \((1 - \delta/2)^2 = 1 - \delta + \delta^2/4 > 1 - \delta\). Then

\[
\max_{j \in [n]} \| \bar{x}_j \|_2 \leq \max_{j \in [n]} \| W^T \bar{x}_j - \rho_n^{1/2} x_{0j} \|_2 + \max_{j \in [n]} \| \rho_n^{1/2} x_{0j} \|_2 \\
\leq \rho_n^{1/2} \left( 1 - \frac{\delta}{2} - \sqrt{1 - \delta} \right) + \rho_n^{1/2} \sqrt{1 - \delta}.
\]

For the second result, over the same event as above, we have

\[
\max_{i,j \in [n]} | \bar{p}_{ij} - \rho_n x_{0i}^T x_{0j} | \leq \max_{i,j \in [n]} | \bar{x}_i^T W (W^T \bar{x}_j - \rho_n^{1/2} x_{0j}) | + \max_{i,j \in [n]} | (W^T \bar{x}_i - \rho_n^{1/2} x_{0i})^T \rho_n^{1/2} x_{0j} | \\
\leq (\max_{j \in [n]} \| \bar{x}_j \|_2 + \rho_n^{1/2}) \| \bar{X} W - \rho_n^{1/2} X_0 \|_{2 \to \infty} \lesssim_{c,\lambda} \rho_n^{1/2} \sqrt{\log \frac{n}{n}}.
\]

For the third result, over the same event as above, we have

\[
\max_{i,j \in [n]} \bar{p}_{ij} \leq \max_{i,j \in [n]} | \bar{p}_{ij} - \rho_n x_{0i}^T x_{0j} | + \max_{i,j \in [n]} \rho_n x_{0i}^T x_{0j} \leq C_{c,\lambda} \rho_n^{1/2} \sqrt{\log \frac{n}{n}} + \rho_n (1 - \delta).
\]

Since \( \frac{\log n}{n^{p_n}} \to 0 \) and \( \max_{i,j \in [n]} x_{0i}^T x_{0j} \leq 1 - \delta \), we can pick a (possibly larger) constant \( N_{c,\delta,\lambda} \) such that \( C_{c,\lambda} \sqrt{\log \frac{n}{n^{p_n}}} \leq \delta/2 \) for all \( n \geq N_{c,\delta,\lambda} \). Then

\[
\max_{i,j \in [n]} \bar{p}_{ij} \leq \rho_n (1 - \frac{\delta}{2}).
\]

Similarly,

\[
\min_{i,j \in [n]} \bar{p}_{ij} \geq \min_{i,j \in [n]} | \bar{p}_{ij} - \rho_n x_{0i}^T x_{0j} | - \max_{i,j \in [n]} \rho_n x_{0i}^T x_{0j} \geq \rho_n \delta - C_{c,\lambda} \rho_n^{1/2} \sqrt{\log \frac{n}{n}} \geq \frac{\rho_n \delta}{2}.
\]
For the fourth one, over the same event as above, we have

\[
\max_{j \in [n]} \| W^T \tilde{x}_j \tilde{x}_j^T W - \rho_n x_{0j} x_{0j}^T \|_2 \\
\leq \max_{j \in [n]} \| W^T \tilde{x}_j (x_j W - \rho_n^{-1/2} x_{0j}^T) \|_2 + \max_{j \in [n]} \| (W^T \tilde{x}_j - \rho_n^{-1/2} x_{0j}) \rho_n^{-1/2} x_{0j}^T \|_2 \\
\leq (\max_{j \in [n]} \| x_j \|_2 + \rho_n^{1/2}) \| \tilde{X} W - \rho_n^{1/2} X_0 \|_2 \rightarrow \infty \lesssim_{c, \delta, \lambda} \rho_n^{1/2} \sqrt{\frac{\log n}{n}}.
\]

**Lemma A.3** (Concentration of Hessian matrices). Suppose \( A \sim \text{RDPG}(\rho_n^{1/2} X_0) \) and assume the conditions of Theorem 3.1 hold. Denote \( \tilde{p}_{ij} = \tilde{x}_i^T \tilde{x}_j, i, j \in [n] \) and let \( \epsilon > 0 \) be sufficiently small. Then for any \( c > 0 \), there exists a constant \( N_{c, \delta, \lambda} \in \mathbb{N}_+ \) depending on \( c, \delta, \lambda \) such that for all \( n \geq N_{c, \delta, \lambda} \), the following hold with probability at least \( 1 - n^{-c} \):

\[
\max_{i \in [n]} \sup_{x_i: \| W^T x_i - \rho_n x_{0i} \|_2 \leq \epsilon} \left\| \frac{1}{n} \sum_{j=1}^n \left( \frac{1}{\tilde{p}_{ij}} + \frac{1 - A_{ij}}{(1 - \tilde{x}_i^T \tilde{x}_j)^2} \right) W^T \tilde{x}_j \tilde{x}_j^T W - \frac{1}{n} \sum_{j=1}^n x_{0j}^T x_{0j} \right\|_2 \\
\lesssim_{c, \delta, \lambda} \rho_n^{1/2} \epsilon_n + \sqrt{\frac{\log n}{n}}.
\]

**Proof.** For simplicity of notation, denote \( p_{0ij} = \rho_n x_{0i}^T x_{0j} \). The large probability bounds below are with regard to \( n \geq N_{c, \delta, \lambda} \) for some large constant \( N_{c, \delta, \lambda} \) depending on \( c, \delta, \lambda \).

We show the first conclusion first. Write

\[
\left\| \frac{1}{n} \sum_{j=1}^n \left( \frac{1}{\tilde{p}_{ij}} + \frac{1 - A_{ij}}{(1 - \tilde{x}_i^T \tilde{x}_j)^2} \right) W^T \tilde{x}_j \tilde{x}_j^T W - \frac{1}{n} \sum_{j=1}^n x_{0j}^T x_{0j} \right\|_2 \\
\leq \left\| \frac{1}{n} \sum_{j=1}^n (1 - A_{ij}) \left\{ \frac{1}{(1 - \tilde{x}_i^T \tilde{x}_j)^2} - \frac{1}{1 - p_{0ij}} \right\} W^T \tilde{x}_j \tilde{x}_j^T W \right\|_2 \\
+ \left\| \frac{1}{n} \sum_{j=1}^n A_{ij} - p_{0ij} \left( W^T \tilde{x}_j \tilde{x}_j^T W - \rho_n x_{0j} x_{0j}^T \right) \right\|_2 \\
+ \left\| \frac{1}{n} \sum_{j=1}^n \left( \frac{1}{\tilde{p}_{ij}} - \frac{1}{p_{0ij}} \right) W^T \tilde{x}_j \tilde{x}_j^T W \right\|_2 + \left\| \frac{1}{n} \sum_{j=1}^n \frac{W^T \tilde{x}_j \tilde{x}_j^T W - \rho_n x_{0j} x_{0j}^T}{p_{0ij} (1 - p_{0ij})} \right\|_2.
\]

For the first term, with probability at least \( 1 - n^{-c} \),

\[
\max_{i \in [n]} \sup_{x_i: \| W^T x_i - \rho_n^{1/2} x_{0i} \|_2 \leq \epsilon} \left\| \frac{1}{n} \sum_{j=1}^n (1 - A_{ij}) \left\{ \frac{1}{(1 - \tilde{x}_i^T \tilde{x}_j)^2} - \frac{1}{1 - p_{0ij}} \right\} W^T \tilde{x}_j \tilde{x}_j^T W \right\|_2
\]
we see that where by Lemma A.2 and the result that \( \| \cdot \| \) Schwarz inequality, and in the fourth inequality Lemma A.1 and Lemma A.2.

For the third term, for a typical \((k, t)\) entry, by Bernstein’s inequality and a union bound over \(i \in [n]\), for any \(t > 0\),

\[
\mathbb{P} \left\{ \max_{i \in [n]} \left| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - p_{0ij}) \frac{\rho_n x_{0jk} x_{0jl}}{(1 - p_{0ij})^2} \right| \geq t \right\} \leq 2n \exp \left\{ \frac{-n^2 t^2}{6 \sum_{j=1}^{n} \frac{\rho_n^2 x_{0jk}^2 x_{0jl}^2}{(1 - p_{0ij})^2} + 2 \max_{j \in [n]} \frac{\rho_n x_{0jk} x_{0jl}}{1 - p_{0ij}} \frac{\rho_n}{n} t} \right\} \\
\leq 2n \exp \left\{ -K_\delta \frac{n t^2}{\rho_n^2 + \rho_n t} \right\},
\]

where \(K_\delta > 0\) is a constant depending on \(\delta\). Taking \(t = C \sqrt{(\rho_n^2 \log n) / n}\) for an appropriate constant \(C > 0\), we see that

\[
\max_{i \in [n]} \left| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - p_{0ij}) \frac{\rho_n x_{0jk} x_{0jl}}{(1 - p_{0ij})^2} \right| \leq C_\epsilon \delta \sqrt{\frac{\rho_n^2 \log n}{n}}.
\]
with probability at least $1 - n^{-c}$. Since $d$ is fixed (it implicitly depends on $\lambda$), we have

$$
\max_{i \in [n]} \left\| \frac{1}{n} \sum_{j=1}^{n} \left( A_{ij} - \hat{p}_{ij} \right) \rho_{n} x_{0i} x_{0j}^{\top} \right\|_2^2 \lesssim_{c,\delta,\lambda} \rho_{n}^{2} \sqrt{\log \frac{n}{\rho_{n}}}
$$

with probability at least $1 - n^{-c}$.

For the fourth term, with probability at least $1 - n^{-c}$,

$$
\max_{i \in [n]} \left\| \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{\hat{p}_{ij}} - \frac{1}{\tilde{p}_{ij}} \right) \mathbf{W}^{\top} \tilde{x}_{j} \tilde{x}_{j}^{\top} \mathbf{W} \right\|_2 \leq \max_{i,j \in [n]} \frac{\left| \tilde{p}_{ij} - p_{ij} \right|}{\hat{p}_{ij} \tilde{p}_{ij}} \| \tilde{x}_{j} \|_2^2 \lesssim_{c,\delta,\lambda} \sqrt{\log \frac{n}{\rho_{n}}}
$$

by Lemma A.2.

For the fifth term, with probability at least $1 - n^{-c}$,

$$
\max_{i \in [n]} \left\| \frac{1}{n} \sum_{j=1}^{n} \frac{\mathbf{W}^{\top} \tilde{x}_{j} \tilde{x}_{j}^{\top} \mathbf{W} - \rho_{n} x_{0i} x_{0j}^{\top}}{p_{ij}(1 - \hat{p}_{ij})} \right\|_2 \lesssim_{c,\delta,\lambda} \frac{\rho_{n}^{-1} \max_{j \in [n]} \| \mathbf{W}^{\top} \tilde{x}_{j} \tilde{x}_{j}^{\top} \mathbf{W} - \rho_{n} x_{0i} x_{0j}^{\top} \|_2}{\sqrt{\log \frac{n}{\rho_{n}}}}
$$

by Lemma A.2. So the first conclusion is shown by combining the above five bounds.

Next, we show the second conclusion. Write

$$
\left\| \frac{1}{n} \sum_{j=1}^{n} \left( A_{ij} - \hat{p}_{ij} \right) \tilde{x}_{j} \tilde{x}_{j}^{\top} \right\|_2 = \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \rho_{n} x_{0i} x_{0j}^{\top}) \rho_{n} x_{0i} x_{0j}^{\top} \right\|_2 + \left\| \frac{1}{n} \sum_{j=1}^{n} (\hat{p}_{ij} - \rho_{n} x_{0i} x_{0j}^{\top}) \rho_{n} x_{0i} x_{0j}^{\top} \right\|_2
$$

The first term is $O(\rho_{n}^{2} \sqrt{\log \frac{n}{\rho_{n}}})$ with probability at least $1 - n^{-c}$ as previously shown.

For the second term, with probability at least $1 - n^{-c}$,

$$
\left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \hat{p}_{ij}) \left( \rho_{n} x_{0i} x_{0j}^{\top} \right) \left( \frac{\mathbf{W}^{\top} \tilde{x}_{j} \tilde{x}_{j}^{\top} \mathbf{W}}{(1 - \hat{p}_{ij})^2} - \frac{\mathbf{W}^{\top} \tilde{x}_{j} \tilde{x}_{j}^{\top} \mathbf{W}}{(1 - \tilde{p}_{ij})^2} \right) \right\|_2
$$

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constant \( v > 0 \) obtained by replacing the (Z)
adjacency matrices, (Theorem 6.7 in Boucheron et al. (2013))
Lemma A.5
relies on the following logarithmic Sobolev concentration inequality:
A.6 in Tang et al. (2017a) and we include the proof here only for the sake of completeness. The key idea
The proof of Lemma A.4 here is almost identical to those of Lemma D.5 in Xie and Xu (2021) and Lemma
Lemma A.4.
by Lemma A.2. So the second conclusion is shown by combining the above three bounds.
\[ \| A - \rho_n X_0 X_0^T \|_\infty \]
\[ \leq \frac{1}{n} \sum_{j=1}^{n} \left| A_{ij} - p_{0ij} \right| \left\{ \left\| \rho_n x_0 j x_0^T (1 - p_{0ij})^2 W \right\|_2 + \left\| W^T x_j x_j^T W \right\|_2 \right\} \]
\[ \leq \frac{1}{n} \| A - \rho_n X_0 X_0^T \|_\infty \]
\[ \times \max_{i,j\in[n]} \left\{ \left\| \rho_n x_0 j x_0^T - W^T x_j x_j^T W \right\|_2 + \left\| (p_{i,j} - p_{0ij})(2 - \tilde{p}_{i,j} - p_{0ij}) \right\|_2 \right\} \]
\[ \lesssim_{c,\lambda} \rho_n \frac{1}{n} \rho_n \sqrt{\frac{\log n}{n} + \rho_n \sqrt{\frac{\log n}{n}}} \]
\[ \lesssim_{c,\lambda} \rho_n \sqrt{\frac{\log n}{n}} \]
by Cauchy–Schwarz inequality, Lemma A.2, and \( \| A - \rho_n X_0 X_0^T \|_\infty \leq \| A \|_\infty + \| \rho_n X_0 X_0^T \|_\infty \lesssim n \rho_n \) with
probability at least \( 1 - n^{-c} \).
For the third term, with probability at least \( 1 - n^{-c} \),
\[ \left\| \frac{1}{n} \sum_{j=1}^{n} (p_{i,j} - \rho_n x_0 j x_0^T (1 - \tilde{p}_{i,j})^2 W \right\|_2 \]
\[ \lesssim_{c,\lambda} \rho_n \sqrt{\frac{\log n}{n}} \]
by Lemma A.2. So the second conclusion is shown by combining the above three bounds.
\[ \square \]
Lemma A.4. Suppose \( A \sim \text{RDPG}(\rho_n^{1/2} x_0) \) and assume the conditions of Theorem 3.2 hold. Denote
\[ Z = Z(A) = \sum_{i=1}^{n} \left\| \frac{1}{n \rho_n^{1/2}} \sum_{j=1}^{n} \left( A_{ij} - \rho_n x_0 j x_0^T x_0 j (1 - \rho_n x_0 j x_0^T x_0 j) \right) \right\|_2. \]
Then \( Z = \mathbb{E}_0 Z + a_{0n}(1) \).
Proof. Denote
\[ \gamma_{ij} = \frac{G_{0n^{-1}} x_0 j}{n \rho_n^{1/2} x_0 i x_0 j (1 - \rho_n x_0 i x_0 j)}, \quad i, j \in [n]. \]
The proof of Lemma A.4 here is almost identical to those of Lemma D.5 in Xie and Xu (2021) and Lemma
A.6 in Tang et al. (2017a) and we include the proof here only for the sake of completeness. The key idea
relies on the following logarithmic Sobolev concentration inequality:
Lemma A.5 (Theorem 6.7 in Boucheron et al. (2013)). Let \( A, A' \in \{0, 1\}^{n \times n} \) be two symmetric random
adjacency matrices, \( Z = Z(A) \) be a measurable function of of \( A \). Denote by \( A^{(k,l)} \) the adjacency matrix
obtained by replacing the \((k,l)\) and \((l,k)\) entries of \( A \) by those of \( A' \), and \( Z_{kl} = Z(A^{(k,l)}) \). If there exists a
constant \( v > 0 \) such that
\[ \mathbb{P} \left( \sum_{k \leq l} (Z - Z_{kl})^2 > v \right) \leq \eta, \]
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then for all \( t > 0 \), \( \Pr(|Z - \mathbb{E}(Z)| > t) \leq 2\exp\{-t^2/(2\nu)\} + \eta.

Let \( A' \) be another symmetric random adjacency matrix. Denote by \( A^{(kl)} \) the adjacency matrix obtained by replacing the \((k,l)\) and \((l,k)\) entries of \( A \) by those of \( A' \) and \( Z_{kl} = Z(A^{(kl)}) \) Observe that \( A \) and \( A^{(kl)} \) are the same except for the \((k,l)\) and \((l,k)\) entries. Also note that \( A \) and \( A' \) are binary random matrices.

Therefore, when \( Z - Z_{kl} \neq 0 \),

\[
(A_{kl} - A'_{kl})(Z - Z_{kl}) = C_{1kl} + C_{2kl} + c_{kl},
\]

where

\[
C_{1kl} = 2 \sum_{a=1}^{n} (A_{ka} - \rho_n x_{0k}^T x_{0a})\gamma_{kl}^T \gamma_{ka}, \\
C_{2kl} = 2 \sum_{a=1}^{n} (A_{la} - \rho_n x_{0l}^T x_{0a})\gamma_{kl}^T \gamma_{la},
\]

and \( c_{kl} = (1 - 2\rho_n x_{0k}^T x_{0l}) (\|\gamma_{kl}\|_2^2 + \|\gamma_{lk}\|_2^2) \). Observe that

\[
\sup_{i,j \in [n]} \|\gamma_{ij}\|_2 \lesssim 1/n\rho_n^{1/2}.
\]

It follows that

\[
\sum_{k \leq l} \mathbb{E}_0(C_{1kl}^2) = 4 \sum_{k \leq l} \sum_{a=1}^{n} \mathbb{E}_0((A_{ka} - \rho_n x_{0k}^T x_{0a})(A_{kb} - \rho_n x_{0k}^T x_{0b})) (\gamma_{kl}^T \gamma_{ka})(\gamma_{kl}^T \gamma_{kb})
\]

\[
= 4 \sum_{k \leq l} \sum_{a=1}^{n} \mathbb{E}_0((A_{ka} - \rho_n x_{0k}^T x_{0a})^2)(\gamma_{kl}^T \gamma_{ka})^2
\]

\[
\leq 4 \sum_{k \leq l} \sum_{a=1}^{n} \rho_n x_{0k}^T x_{0a}(1 - \rho_n x_{0k}^T x_{0a})\|\gamma_{kl}\|_2^2 \|\gamma_{ka}\|_2^2 \lesssim 1/n\rho_n,
\]

\[
\sum_{k \leq l} \mathbb{E}_0(C_{2kl}^2) = 4 \sum_{k \leq l} \sum_{a=1}^{n} \mathbb{E}_0((A_{la} - \rho_n x_{0l}^T x_{0a})^2)(\gamma_{lk}^T \gamma_{la})^2 \lesssim 1/n\rho_n,
\]

\[
\sum_{k \leq l} \mathbb{E}_0(c_{kl}^2) \leq 12 \sum_{k \leq l} (1 - 2\rho_n x_{0k}^T x_{0l})^2 (\|\gamma_{kl}\|_2^4 + \|\gamma_{lk}\|_2^4)
\]

\[
+ 12 \sum_{k \leq l} \mathbb{E}_0((A_{kl} - \rho_n x_{0k}^T x_{0l})^2)(\|\gamma_{kl}\|_2^4 + \|\gamma_{lk}\|_2^4) \lesssim 1/n^2\rho_n^2 + \rho_n^2 n^2 \rho_n^2 \lesssim 1/n^2\rho_n^2.
\]

Namely \( \mathbb{E}_0(\sum_{k \leq l}(Z - Z_{kl})^2) \lesssim (n\rho_n)^{-1} \). Therefore, by Markov’s inequality,

\[
\mathbb{P}_0\left\{ \sum_{k \leq l}(Z - Z_{kl})^2 > \frac{1}{\log n} \right\} \lesssim \frac{\log n}{n\rho_n} \to 0.
\]
Invoking Lemma A.5, we obtain that
\[ P_0 \left( |Z - E_0 Z| > \epsilon \right) \lesssim d \exp \left( -\frac{1}{2} \epsilon^2 \log n \right) + \frac{\log n}{n \rho_n} \to 0 \]
for all \( \epsilon > 0 \). The proof is thus completed. \( \square \)

**Theorem A.6** (Theorem 4.7 in Xie, 2022). Suppose \( \mathbf{A} \sim \text{RDPG}(\rho_n^{1/2} \mathbf{X}_0) \) and assume the conditions of Theorem 3.1 hold. Define the one-step estimator \( \hat{x}_i^{(OS)} \) by
\[
\hat{x}_i^{(OS)} = \hat{x}_i + \left( \frac{1}{n} \sum_{j=1}^{n} \frac{\bar{x}_j \bar{x}_j^T}{\tilde{p}_{ij}(1 - \tilde{p}_{ij})} \right)^{-1} \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{(A_{ij} - \tilde{p}_{ij}) \hat{x}_j}{\tilde{p}_{ij}(1 - \tilde{p}_{ij})} \right\}.
\]
Then
\[
G_{0in}^{1/2} (W^T \hat{x}_i^{(OS)} - \rho_n^{1/2} \mathbf{x}_{0i}) = \frac{1}{n \rho_n} \sum_{j=1}^{n} \frac{(A_{ij} - \rho_n \mathbf{x}_{0i}^T \mathbf{x}_{0j}) G_{0in}^{-1/2} \mathbf{x}_{0j}}{\mathbf{x}_{0j} \mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^T \mathbf{x}_{0j})} + r_{in}^{(OS)},
\]
where
\[
G_{0in} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{x}_{0j} \mathbf{x}_{0j}^T \mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^T \mathbf{x}_{0j}),
\]
and for all \( c > 0 \), there exists a constant \( N_{c,\delta,\lambda} \in \mathbb{N}_+ \) depending on \( c, \delta, \lambda \), such that for all \( n \geq N_{c,\delta,\lambda} \), with probability at least \( 1 - (n \rho_n)^{-c} \), \( \| r_{in}^{(OS)} \|_2 \lesssim (\log(n \rho_n))^2 / (n \rho_n^{1/2}) \). Furthermore,
\[
\sqrt{n} G_{0in}^{1/2} (W^T \hat{x}_i^{(OS)} - \rho_n^{1/2} \mathbf{x}_{0i}) \overset{d}{\to} N_d(0_d, \mathbf{I}_d),
\]

**B Proofs of the Main Results**

**B.1 Proof of Theorem 3.1**

**Proof.** ■ We first prove existence. For any \( c > 0 \), there exists \( N_{c,\delta,\lambda} \in \mathbb{N}_+ \) such that
\[
\sup_{\| \mathbf{x}_i \| \leq 2} \max_{j \in [n]} |\mathbf{x}_i^T \hat{x}_j| \leq \max_{j \in [n]} \| \hat{x}_j \|_2 \leq \rho_n (1 - \frac{\delta}{2}) < 1
\]
with probability at least \( 1 - n^{-c} \), where the first inequality follows from Cauchy–Schwarz inequality, the second from Lemma A.2. By definition of \( \hat{M}_{in}(\mathbf{x}_i) \), it is continuous over the closed unit ball \( \{ \mathbf{x}_i \in \mathbb{R}^d : \| \mathbf{x}_i \|_2 \leq 1 \} \) over this event. Hence the maximizer \( \hat{x}_i \) of \( \hat{M}_{in}(\mathbf{x}_i) \) exists with probability at least \( 1 - n^{-c} \).

■ Next we prove uniqueness. By definition, with probability at least \( 1 - n^{-c} \), \( \hat{M}_{in}(\mathbf{x}_i) \) is twice continuously differentiable, with
\[
-\frac{\partial \hat{M}_{in}(\mathbf{x}_i) \mathbf{x}_i}{\partial \mathbf{x}_i \partial \mathbf{x}_i^T} = \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{1}{\tilde{p}_{ij}} + \frac{1 - A_{ij}}{(1 - \mathbf{x}_j^T \hat{x}_j)^2} \right\} \hat{x}_j \hat{x}_j^T \geq \frac{1}{n} \sum_{j=1}^{n} \frac{\bar{x}_j \bar{x}_j^T}{\tilde{p}_{ij}} \geq \frac{1}{n \rho_n} \sum_{j=1}^{n} \frac{\bar{x}_j \bar{x}_j^T}{\tilde{p}_{ij}} \geq \frac{1}{n \rho_n} \sigma_d(\bar{x})^2 \mathbf{I}_d.
\]
By Theorem 5.2 in Lei and Rinaldo (2015) and Weyl’s inequality, there exists a constant depending on \( c \), such that with probability at least \( 1 - n^{-c} \),

\[
\sigma_d(\bar{X})^2 = \lambda_d(A) \geq \frac{1}{2} \lambda_d \left( \rho_n X_0 X_0^T \right) = \frac{1}{2} n \rho_n \lambda_d \left( \frac{1}{n} X_0 X_0^T \right) \geq \frac{1}{2} n \rho_n \lambda > 0.
\]

Therefore, for any \( c > 0 \), there exists \( N_{c,\delta,\lambda} \in \mathbb{N}_+ \) such that for all \( n \geq N_{c,\delta,\lambda} \), with probability at least \( 1 - n^{-c} \), \( \tilde{M}_{in}(x_i) \) is strictly concave. Hence it has a unique maximizer \( \hat{x}_i \).

### B.2 Proof of Theorem 3.2

**Proof.** We first establish the following consistency result: For any \( c > 0 \), there exists some constant \( N_{c,\delta,\lambda} \in \mathbb{N}_+ \) depending on \( c, \delta, \lambda \) such that for all \( n \geq N_{c,\delta,\lambda} \in \mathbb{N}_+ \), there exists an orthogonal matrix \( W \in \mathcal{O}(d) \), such that with probability at least \( 1 - n^{-c} \),

\[
\max_{i \in [n]} \| W^T \tilde{x}_i - \rho_n^{1/2} x_{0i} \|_2 \leq c, \delta, \lambda \{ \log n/(n \rho_n) \}^{1/2}.
\]

Define \( \tilde{M}_{in}(x_i) = (1/n) \tilde{t}_{in}(x_i) \) and the population counterpart of \( \tilde{M}_{in}(x_i) \) as

\[
M_{in}(x_i) = \frac{1}{n} \sum_{j=1}^n \left\{ 2 \rho_n^{1/2} x_i^T x_{0j} - \frac{x_i^T x_{0j} x_i^T x_{0j}}{2x_{0j}^T x_{0j}} + (1 - \rho_n x_{0j}^T x_{0j}) \log(1 - \rho_n^{1/2} x_i^T x_{0j}) \right\}.
\]

Simple calculation shows that

\[
\frac{\partial M_{in}}{\partial x_i}(x_i) = \frac{1}{n} \sum_{j=1}^n \rho_n^{1/2} x_{0j}^T \left( \rho_n^{1/2} x_{0j} - x_i \right) \left\{ \frac{1}{\rho_n x_{0j}^T x_{0j}} + \frac{1}{1 - \rho_n^{1/2} x_i^T x_{0j}} \right\} \rho_n^{1/2} x_{0j},
\]

\[
\frac{\partial^2 M_{in}}{\partial x_i \partial x_i}(x_i) = -\frac{1}{n} \sum_{j=1}^n \left\{ \frac{1}{\rho_n x_{0j}^T x_{0j}} + \frac{1 - \rho_n x_{0j}^T x_{0j}}{(1 - \rho_n^{1/2} x_i^T x_{0j})^2} \right\} \rho_n x_{0j} x_i^T x_{0j},
\]

and

\[
\frac{\partial \tilde{M}_{in}}{\partial x_i}(x_i) = \frac{1}{n} \sum_{j=1}^n (A_{ij} - x_i^T \tilde{x}_j) \left\{ \frac{1}{\tilde{p}_{ij}} + \frac{1}{1 - x_i \tilde{x}_j} \right\} \tilde{x}_j,
\]

\[
\frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i}(x_i) = -\frac{1}{n} \sum_{j=1}^n \left\{ \frac{1}{\tilde{p}_{ij}} + \frac{1 - A_{ij}}{(1 - x_i \tilde{x}_j)^2} \right\} \tilde{x}_j \tilde{x}_j^T.
\]

For simplicity of notation, in what follows the large probability bounds are with regard to \( n \geq N_{c,\delta,\lambda} \) for some large constant \( N_{c,\delta,\lambda} \) depending on \( c, \delta, \lambda \).

**Claim I (identifiability):** For all \( \epsilon > 0 \),

\[
\inf_{\| x_i - \rho_n^{1/2} x_{0i} \|_2 \geq \epsilon} \left\| \frac{\partial M_{in}}{\partial x_i}(x_i) \right\|_2 \geq \lambda \epsilon > \left\| \frac{\partial M_{in}}{\partial x_i}(\rho_n^{1/2} x_{0i}) \right\|_2 = 0.
\]
Claim II (uniform convergence): With probability at least $1 - n^{-c}$,

$$
\max_{\mathbf{x} \in [n]} \sup_{\|\mathbf{x}\|_2 \leq 1} \left\| W^T \frac{\partial M_{in}(\mathbf{Wx_i})}{\partial x_i} - \frac{\partial M_{in}(\mathbf{x_i})}{\partial x_i} \right\|_2 \lesssim c, \lambda \sqrt{\frac{\log n}{n \rho_n}}
$$

Now we show Claim I. It is obvious that $\frac{\partial M_{in}}{\partial x_i} (\rho_n^{1/2} \mathbf{x}_0) = 0_d$. Because $\rho_n \leq 1$, $\|\mathbf{x}\|_2 \leq 1$, and $\max_{j \in [n]} \|\mathbf{x}_0\|_2 \leq 1$, we have

$$
- \frac{\partial^2 M_{in}}{\partial x_i \partial x_j^T} (\mathbf{x}_i) \geq \frac{1}{n} \sum_{j=1}^{n} \mathbf{x}_j^T \mathbf{x}_0 \mathbf{x}_j \geq \frac{1}{n} \sum_{j=1}^{n} \mathbf{x}_0 \mathbf{x}_j^T \mathbf{x}_0 \geq \lambda_d \left( \frac{1}{n} \mathbf{x}_0^T \mathbf{x}_0 \right) 1_d \geq \lambda 1_d,
$$

which implies that $M_{in}(\mathbf{x}_i)$ is strictly concave with $\rho_n^{1/2} \mathbf{x}_0$ as a unique maximizer. By Taylor’s theorem,

$$
\frac{\partial M_{in}}{\partial x_i} (\mathbf{x}_i) = \frac{\partial^2 M_{in}}{\partial x_i \partial x_i^T} (\mathbf{x}_i) (\mathbf{x}_i - \rho_n^{1/2} \mathbf{x}_0),
$$

where $\mathbf{x}_i = \theta \rho_n^{1/2} \mathbf{x}_0 + (1 - \theta) \mathbf{x}_i$ for some $\theta \in [0, 1]$. It follows that

$$
\left\| \frac{\partial M_{in}}{\partial x_i} (\mathbf{x}_i) \right\|_2 ^2 \geq 2 \lambda \left( \frac{1}{n} \mathbf{x}_0^T \mathbf{x}_0 \right) \left\| \mathbf{x}_i - \rho_n^{1/2} \mathbf{x}_0 \right\|_2 ^2 \geq 2 \lambda \left\| \mathbf{x}_i - \rho_n^{1/2} \mathbf{x}_0 \right\|_2 ^2,
$$

so

$$
\inf_{\|\mathbf{x}_i - \rho_n^{1/2} \mathbf{x}_0\|_2 \geq \epsilon} \left\| \frac{\partial M_{in}}{\partial x_i} (\mathbf{x}_i) \right\|_2 ^2 \geq \epsilon \lambda. \text{ Thus Claim I is shown. Now we show Claim II. By triangle inequality,}
$$

$$
\left\| W^T \frac{\partial M_{in}}{\partial x_i} (\mathbf{Wx_i}) - \frac{\partial M_{in}}{\partial x_i} (\mathbf{x_i}) \right\|_2 
\leq \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \rho_n \mathbf{x}_0^T \mathbf{x}_0) \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \rho_n^{1/2} \mathbf{x}_0 \right\|_2
\leq \left\| \frac{1}{n} \sum_{j=1}^{n} \left( \mathbf{x}_j^T W^T \mathbf{x}_j - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0 \right) \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \rho_n^{1/2} \mathbf{x}_0 \right\|_2
\leq \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \mathbf{x}_i^T W^T \mathbf{x}_j) \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \mathbf{x}_j \right\|_2
\leq \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \mathbf{x}_i^T W^T \mathbf{x}_j) \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \mathbf{x}_j \right\|_2
\leq \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \mathbf{x}_i^T W^T \mathbf{x}_j) \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \mathbf{x}_j \right\|_2
$$

For the second term,

$$
\max_{\mathbf{x}_i \in [n]} \sup_{\|\mathbf{x}_i\|_2 \leq 1} \left\| \frac{1}{n} \sum_{j=1}^{n} \left( \mathbf{x}_j^T W^T \mathbf{x}_j - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0 \right) \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \rho_n^{1/2} \mathbf{x}_0 \right\|_2
\leq \max_{\mathbf{x}_i \in [n]} \sup_{\|\mathbf{x}_i\|_2 \leq 1} \frac{1}{n} \sum_{j=1}^{n} \left\| W^T \mathbf{x}_j - \rho_n \mathbf{x}_0 \right\|_2 \left\| \mathbf{x}_i \right\|_2 \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \rho_n^{1/2} \left\| \mathbf{x}_0 \right\|_2
\leq \frac{1}{n} \sum_{j=1}^{n} \left\| W^T \mathbf{x}_j - \rho_n \mathbf{x}_0 \right\|_2 \left\| \mathbf{x}_i \right\|_2 \left( \frac{1}{\rho_n \mathbf{x}_0^T \mathbf{x}_0} + \frac{1}{1 - \rho_n^{1/2} \mathbf{x}_i^T \mathbf{x}_0} \right) \rho_n^{1/2} \left\| \mathbf{x}_0 \right\|_2
$$
\[
\leq \delta \left\| \tilde{X}W - \rho_n^{1/2}X_0 \right\|_{2 \rightarrow \infty} \rho_n^{-1/2}
\]

\[
\lesssim_c \delta \sqrt{\frac{\log n}{n \rho_n}}
\]

with probability at least \(1 - n^{-c}\). For the third term,

\[
\max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - x_i^T \tilde{W}^\top \tilde{x}_j) \right\|_2
\]

\[
\leq \max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \frac{1}{n} \sum_{j=1}^{n} (A_{ij} + \|x_i\|_2 \|\tilde{x}_j\|_2)
\]

\[
\times \left( \frac{1}{\rho_{ij}} \rho_n \|x_0^T x_0\| + \frac{\|W^\top \tilde{x}_j - \rho_n^2 x_0\|_2 \|x_i\|_2}{1 - \rho_n^2 \|x_0^T x_0\|} \right) \|\tilde{x}_j\|_2
\]

\[
\lesssim_c, \delta, \lambda \max_{i \in [n]} \frac{1}{n} \sum_{j=1}^{n} (A_{ij} + \rho_n^{1/2}) \left( \rho_n^{-3/2} \sqrt{\frac{\log n}{n}} + \sqrt{\frac{\log n}{n}} \right) \rho_n^{1/2}
\]

\[
\lesssim_c, \delta, \lambda \left( \frac{1}{n} \|A\|_\infty + \rho_n^{1/2} \right) \rho_n^{-1} \sqrt{\frac{\log n}{n}}
\]

\[
\lesssim_c, \delta, \lambda \sqrt{\frac{\log n}{n \rho_n}}
\]

with probability at least \(1 - n^{-c}\), where the second inequality follows from Lemma A.2, and the last one from \(\|A\|_\infty \lesssim_c n \rho_n\) with probability at least \(1 - n^{-c}\), which follows from Bernstein’s inequality and triangle inequality.

For the fourth term,

\[
\max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - x_i^T \tilde{W}^\top \tilde{x}_j) \left( \frac{1}{\rho_n \|x_0^T x_0\|} + \frac{1}{1 - \rho_n^2 \|x_0^T x_0\|} \right) (W^\top \tilde{x}_j - \rho_n^2 x_0) \right\|_2
\]

\[
\leq \max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \frac{1}{n} \sum_{j=1}^{n} (A_{ij} + \|x_i\|_2 \|\tilde{x}_j\|_2) \left( \frac{1}{\rho_n \|x_0^T x_0\|} + \frac{1}{1 - \rho_n^2 \|x_0^T x_0\|} \right) \|W^\top \tilde{x}_j - \rho_n^2 x_0\|_2
\]

\[
\lesssim_c, \delta, \lambda \max_{i \in [n]} \frac{1}{n} \sum_{j=1}^{n} (A_{ij} + \rho_n^{1/2}) \rho_n^{-1} \left\| \tilde{X}W - \rho_n^{1/2}X_0 \right\|_{2 \rightarrow \infty}
\]

\[
\lesssim_c, \delta, \lambda \left( \frac{1}{n} \|A\|_\infty + \rho_n^{1/2} \right) \rho_n^{-1} \sqrt{\frac{\log n}{n}} \lesssim_c, \delta, \lambda \sqrt{\frac{\log n}{n \rho_n}}
\]

with probability at least \(1 - n^{-c}\).

In order to bound the first term, a maximal inequality is required. We use the results in Chapter 8 of
Kosorok (2008). Define a stochastic process on \( \{ \mathbf{y} \in \mathbb{R}^d : \| \mathbf{y} \|_2 \leq 1 \} \) for each \( k \in [d] \),

\[
J_{\text{ink}}(\mathbf{y}) = \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \rho_n \mathbf{x}_{0i}^\top \mathbf{x}_{0j}) \left( \frac{1}{\rho_n \mathbf{x}_{0i}^\top \mathbf{x}_{0j}} + \frac{1}{1 - \rho_n^{1/2} \mathbf{y}^\top \mathbf{x}_{0j}} \right) \rho_n^{1/2} x_{0jk}.
\]

Then for any \( \mathbf{y}, \mathbf{y}' \) with \( \| \mathbf{y} \|_2 \leq 1, \| \mathbf{y}' \|_2 \leq 1 \),

\[
|J_{\text{ink}}(\mathbf{y}) - J_{\text{ink}}(\mathbf{y}')| = \left| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \rho_n \mathbf{x}_{0i}^\top \mathbf{x}_{0j}) \frac{\rho_n^{1/2} \mathbf{x}_{0j}^\top (\mathbf{y} - \mathbf{y}')}{(1 - \rho_n^{1/2} \mathbf{y}'^\top \mathbf{x}_{0j})^2} \rho_n^{1/2} x_{0jk} \right|,
\]

where \( \overline{y} = \theta \mathbf{y} + (1 - \theta) \mathbf{y}' \) for some \( \theta \in [0, 1] \). By Hoeffding’s inequality,

\[
\mathbb{P}\{ |J_{\text{ink}}(\mathbf{y}) - J_{\text{ink}}(\mathbf{y}')| \geq t \} \leq 2 \exp \left\{ - \frac{2n^2 t^2}{\sum_{j=1}^{n} (\rho_n^{1/2} \mathbf{x}_{0j}^\top (\mathbf{y} - \mathbf{y}'))^2 \rho_n x_{0jk}^2 / (1 - \rho_n^{1/2} \mathbf{y}'^\top \mathbf{x}_{0j})^4} \right\}
\]

\[
\leq 2 \exp \left\{ - \frac{n t^2}{C_\delta \rho_n^2 \| \mathbf{y} - \mathbf{y}' \|_2^2} \right\},
\]

where \( C_\delta > 0 \) is a constant depending on \( \delta \), which indicates that \( J_{\text{ink}}(\mathbf{y}) \) is a sub-Gaussian process on \( \{ \mathbf{y} \in \mathbb{R}^d : \| \mathbf{y} \|_2 \leq 1 \} \) with respect to the metric \( d_n(\mathbf{y}, \mathbf{y}') = \| \mathbf{y} - \mathbf{y}' \|_2 \sqrt{C_\delta \rho_n^2} / n \). The metric entropy of the metric space \( \{ \mathbf{y} \in \mathbb{R}^d : \| \mathbf{y} \|_2 \leq 1 \}, d_n \) can be bounded by

\[
\log D(\epsilon, \{ \mathbf{y} \in \mathbb{R}^d : \| \mathbf{y} \|_2 \leq 1 \}, d_n) \leq d \log \left( \frac{K_\delta}{\epsilon} \sqrt{\frac{\rho_n^2}{n}} \right),
\]

where \( K_\delta \) is a constant depending on \( \delta \). Recall that the \( \psi_2 \)-Orlicz norm (sub-Gaussian norm) of a random variable \( X \) is defined as

\[
\|X\|_{\psi_2} = \inf \left\{ c > 0 : \mathbb{E} \psi_2 \left( \frac{X}{c} \right) \leq 1 \right\},
\]

where \( \psi_2(x) = e^{x^2} - 1 \) (see Chapter 8 of Kosorok, 2008).

By Theorem 8.4 in Kosorok (2008),

\[
\left\| \sup_{\| \mathbf{y} \|_2 \leq 1} J_{\text{ink}}(\mathbf{y}) \right\|_{\psi_2} \leq \int_0^2 \sqrt{\frac{4 \psi_2}{\pi \epsilon^2}} \log D(\epsilon, \{ \mathbf{y} \in \mathbb{R}^d : \| \mathbf{y} \|_2 \leq 1 \}, d_n) d\epsilon
\]

\[
\leq \int_0^2 \sqrt{\frac{4 \psi_2}{\pi \epsilon^2}} \left( \log \left( \frac{K_\delta}{\epsilon} \sqrt{\frac{\rho_n^2}{n}} \right) \right) d\epsilon
\]

\[
= \int_{K_\delta}^\infty K_\delta \sqrt{\frac{4 \psi_2}{\pi \epsilon^2}} \sqrt{ue^{-u}} du \lesssim_{\delta, \lambda} \sqrt{\frac{\rho_n^2}{n}},
\]

where we note that \( d \) depends on \( \lambda \) implicitly. Then by Lemma 8.1 in Kosorok (2008) and a union bound
over \( i \in [n] \), \( \max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} |J_{ink}(x_i)| \lesssim c,\delta,\lambda \sqrt{(\rho_n^2 \log n)/n} \) with probability at least \( 1 - n^{-c} \). So

\[
\max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \left\| \frac{1}{n} \sum_{j=1}^{n} (A_{ij} - \rho_n x_i^T x_{0j}) \left( \frac{1}{\rho_n x_i^T x_{0j}} + \frac{1}{1 - \rho_n x_i^T x_{0j}} \right) \right\|_2 \\
\leq \max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \sum_{k=1}^{d} |J_{ink}(x_i)| \lesssim c,\delta,\lambda \rho_n \sqrt{\log n \over n}
\]

with probability at least \( 1 - n^{-c} \). Thus Claim II is shown.

By Theorem 3.1, \( \hat{x}_i \) is the unique zero of \( \|\partial M_{in}/\partial x_i(x_i)\|_2 \) with probability at least \( 1 - n^{-c} \). Now

\[
\max_{i \in [n]} \left( \left\| \partial M_{in} (W^T \hat{x}_i) \right\|_2 - \left\| \partial M_{in} (\rho_{1/2} x_{0i}) \right\|_2 \right) \\
\leq \max_{i \in [n]} \left( \left\| \partial M_{in} (W^T \hat{x}_i) \right\|_2 - \left\| \partial \tilde{M}_{in} (\hat{x}_i) \right\|_2 \right) \\
+ \max_{i \in [n]} \left( \left\| \partial \tilde{M}_{in} (W \rho_{1/2} x_{0i}) \right\|_2 - \left\| \partial \tilde{M}_{in} (\rho_{1/2} x_{0i}) \right\|_2 \right) \\
\leq 2 \max_{i \in [n]} \sup_{\|x_i\|_2 \leq 1} \left\| W^T \partial \tilde{M}_{in} (W x_i) - \partial \tilde{M}_{in} (x_i) \right\|_2 \lesssim c,\delta,\lambda \sqrt{\log n \over n \rho_n},
\]

where the first inequality follows from \( \hat{x}_i \) being the unique zero of \( \|\partial \tilde{M}_{in}/\partial x_i(x_i)\|_2 \) with probability at least \( 1 - n^{-c} \), the second inequality from triangle inequality, and the third inequality from Claim II.

By Claim I, take \( \epsilon = K_{c,\delta,\lambda} \sqrt{(\log n)/(n \rho_n)} \), we have

\[
\max_{i \in [n]} \left\| W^T \hat{x}_i - \rho_{1/2} x_{0i} \right\|_2 \lesssim c,\delta,\lambda \sqrt{\log n \over n \rho_n}
\]

with probability at least \( 1 - n^{-c} \).

We next establish the asymptotic normality. We utilize the asymptotic normality of the one-step estimator \( \hat{x}_i^{(OS)} \) (Theorem A.6) to establish the asymptotic normality of the maximum surrogate likelihood estimator \( \hat{x}_i \). By the previous part of the theorem, we know that with probability at least \( 1 - n^{-c} \), \( \hat{x}_i \) is in the interior of the closed unit ball \( B(0_d, 1) = \{ x \in \mathbb{R}^d : \| x \| \leq 1 \} \). For each \( k \in [d] \), we apply Taylor’s theorem to \( (\partial \tilde{M}_{in})/(\partial x_{ik})(\hat{x}_i) = 0 \) at \( x_i = \hat{x}_i \) to obtain

\[
0 = \partial \tilde{M}_{in} (\hat{x}_i) = \partial \tilde{M}_{in} (\hat{x}_i) + \partial \tilde{M}_{in} (\hat{x}_i) (\hat{x}_i - \hat{x}_i) \\
+ \frac{1}{2} (\hat{x}_i - \hat{x}_i)^T \partial^2 \tilde{M}_{in} (\hat{x}_i) (\hat{x}_i - \hat{x}_i),
\]

where
where $\mathbf{x}_i = \theta \bar{\mathbf{x}}_i + (1 - \theta) \mathbf{x}_i$ for some $\theta \in [0, 1]$. It is easy to compute

$$
\frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_i^T} \frac{\partial \widetilde{M}_{in}}{\partial x_{ik}} (\mathbf{x}_i) = -\frac{2}{n} \sum_{j=1}^{n} \frac{(1 - A_{ij}) \bar{\mathbf{x}}_{jk} \bar{\mathbf{x}}_j^T}{\left(1 - \mathbf{x}_i^T \mathbf{x}_j \right)^3},
$$

then

$$
\sup_{\|\mathbf{x}_i\|_2 \leq 1} \left\| \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_i^T} \frac{\partial \widetilde{M}_{in}}{\partial x_{ik}} (\mathbf{x}_i) \right\|_2 = \sup_{\|\mathbf{x}_i\|_2 \leq 1} \left\| \frac{2}{n} \bar{\mathbf{x}}_i^T \text{diag}\left\{ \frac{1 - A_{i1}}{(1 - \mathbf{x}_i^T \mathbf{x}_j)^3} \ldots, \frac{1 - A_{in}}{(1 - \mathbf{x}_i^T \mathbf{x}_n)^3} \right\} \bar{\mathbf{x}} \right\|_2 
\lesssim_{\delta} \frac{1}{n} \bar{\mathbf{x}}_i^T \bar{\mathbf{x}} \leq \frac{1}{n} \|\mathbf{A}\|_2 \leq \frac{1}{n} (\|\mathbf{A} - \mathbf{P}\|_2 + \|\mathbf{P}\|_2) \lesssim c \rho_n,
$$

where in the last inequality we applied the fact that $\|\mathbf{A} - \mathbf{P}\|_2 \lesssim c \sqrt{n} \rho_n$ with probability at least $1 - n^{-c}$ (Theorem 5.2 in Lei and Rinaldo, 2015). By Lemma A.1 and the previous part of the theorem, with probability at least $1 - n^{-c}$,

$$
\|\hat{\mathbf{x}}_i - \bar{\mathbf{x}}_i\|_2 \leq \|\mathbf{W}^T \hat{\mathbf{x}}_i - \rho_n^{1/2} \mathbf{x}_0\|_2 + \|\mathbf{W}^T \bar{\mathbf{x}}_i - \rho_n^{1/2} \mathbf{x}_0\|_2 \lesssim_{c, \delta} \sqrt{\log n} \frac{\rho_n}{n^2}.\]

So the Taylor expansion of $(\partial \widetilde{M}_{in})/(\partial \mathbf{x}_i)$ mentioned above can be written as

$$
- \left( \frac{\partial^2 \widetilde{M}_{in}}{\partial \mathbf{x}_i \partial \mathbf{x}_i^T}(\bar{\mathbf{x}}_i) + \mathbf{R}_{in1} \right) (\bar{\mathbf{x}}_i - \hat{\mathbf{x}}_i) = \frac{\partial \widetilde{M}_{in}}{\partial \mathbf{x}_i}(\bar{\mathbf{x}}_i),
$$

where $\mathbf{R}_{in1} \in \mathbb{R}^{d \times d}$ is a random matrix with $\|\mathbf{R}_{in1}\|_2 \lesssim_{c, \delta} \rho_n^{1/2} \sqrt{(\log n)/n}$ with probability at least $1 - n^{-c}$.

By definition of $\widetilde{M}_{in}(\mathbf{x}_i)$ and Lemma A.3,

$$
\left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\bar{p}_{ij}(1 - \bar{p}_{ij})} \bar{\mathbf{x}}_j \bar{\mathbf{x}}_j^T + \mathbf{R}_{in2} \right) (\bar{\mathbf{x}}_i - \hat{\mathbf{x}}_i) = \frac{1}{n} \sum_{j=1}^{n} \frac{A_{ij} - \bar{p}_{ij}}{\bar{p}_{ij}(1 - \bar{p}_{ij})} \bar{\mathbf{x}}_j,
$$

where $\mathbf{R}_{in2} \in \mathbb{R}^{d \times d}$ is a random matrix with $\|\mathbf{R}_{in2}\|_2 \lesssim_{c, \delta, \lambda} \rho_n^{1/2} \sqrt{(\log n)/n}$ with probability at least $1 - n^{-c}$ and $\bar{p}_{ij} = \bar{\mathbf{x}}_i^T \bar{\mathbf{x}}_j$, $i, j \in [n]$.

Denote $\mathbf{G}_{in} = \frac{1}{n} \sum_{j=1}^{n} \frac{\bar{\mathbf{x}}_j \bar{\mathbf{x}}_j^T}{\bar{p}_{ij}(1 - \bar{p}_{ij})}$. Similarly as in the proof of Theorem 3.1,

$$
\begin{align*}
\frac{\lambda}{2} &\leq \frac{1}{n \rho_n} \lambda_d(\mathbf{A}) = \lambda_d \left( \frac{1}{n \rho_n} \bar{\mathbf{x}}_i^T \bar{\mathbf{x}} \right) \leq \lambda_d(\mathbf{G}_{in}) \leq \lambda_1(\mathbf{G}_{in}) \\
&\lesssim_{\delta} \lambda_1 \left( \frac{1}{n \rho_n} \bar{\mathbf{x}}_i^T \bar{\mathbf{x}} \right) = \frac{1}{n \rho_n} \lambda_1(\mathbf{A}) \lesssim c 1,
\end{align*}
$$

i.e., $\mathbf{G}_{in}$ is finite and positive definite with probability at least $1 - n^{-c}$.

Now write

$$
\hat{\mathbf{x}}_i - \bar{\mathbf{x}}_i = \left( \mathbf{G}_{in} + \mathbf{R}_{in2} \right)^{-1} \frac{1}{n} \sum_{j=1}^{n} \frac{A_{ij} - \bar{p}_{ij}}{\bar{p}_{ij}(1 - \bar{p}_{ij})} \bar{\mathbf{x}}_j
$$
\[
\begin{align*}
(\mathbf{I}_d + \mathbf{G}_{in}^{-1} \mathbf{R}_{in2})^{-1} \mathbf{G}_{in}^{-1} \frac{1}{n} \sum_{j=1}^{n} \mathbf{A}_{ij} - \hat{\mathbf{P}}_{ij}(1 - \hat{\mathbf{P}}_{ij}) \tilde{x}_j &= \sum_{m=0}^{\infty} (-\mathbf{G}_{in}^{-1} \mathbf{R}_{in2})^m (\mathbf{x}_i^{(OS)} - \bar{x}_i) \\
&= (\mathbf{x}_i^{(OS)} - \bar{x}_i) + \sum_{m=1}^{\infty} (-\mathbf{G}_{in}^{-1} \mathbf{R}_{in2})^m (\mathbf{x}_i^{(OS)} - \bar{x}_i),
\end{align*}
\]
then
\[
\|\tilde{x}_i - \mathbf{x}_i^{(OS)}\|_2 \leq \sum_{m=1}^{\infty} \|\mathbf{G}_{in}^{-1}\|_2^m \|\mathbf{R}_{in2}\|_2^m \|\mathbf{x}_i^{(OS)} - \bar{x}_i\|_2
\]
\[
= \frac{\|\mathbf{G}_{in}^{-1}\|_2 \|\mathbf{R}_{in2}\|_2}{1 - \|\mathbf{G}_{in}^{-1}\|_2 \|\mathbf{R}_{in2}\|_2} \|\mathbf{x}_i^{(OS)} - \bar{x}_i\|_2
\]
\[
\lesssim_{c,d,\lambda} n^{1/2} \frac{\sqrt{\log n}}{n} \left( \|\mathbf{W}^T \mathbf{x}_i^{(OS)} - \rho_n^{1/2} \mathbf{x}_{0i}\|_2 + \|\mathbf{X} - \rho_n^{1/2} \mathbf{X}_0\|_2 \right). \tag{B.1}
\]
Note that
\[
\lambda \leq \lambda_d \left( \frac{1}{n} \mathbf{x}_0^T \mathbf{x}_0 \right) \leq \lambda_d (\mathbf{G}_{0in}) \leq \lambda_1 (\mathbf{G}_{0in}) \leq \frac{1}{\delta^2} \lambda_1 \left( \frac{1}{n} \mathbf{x}_0^T \mathbf{x}_0 \right) \leq \frac{1}{\delta^2},
\]
i.e., \( \mathbf{G}_{0in} \) is positive definite with eigenvalues bounded away from 0 and \( \infty \). By Theorem A.6 and Bernstein’s inequality, \( \|\mathbf{W}^T \mathbf{x}_i^{(OS)} - \rho_n^{1/2} \mathbf{x}_{0i}\|_2 \lesssim_{c,d,\lambda} \sqrt{\frac{\log n}{n}} \) with probability at least \( 1 - (n \rho_n)^{-c} \). By Lemma A.1, \( \|\mathbf{X} - \rho_n^{1/2} \mathbf{X}_0\|_2 \lesssim_{c,d,\lambda} \sqrt{\frac{\log n}{n}} \) with probability at least \( 1 - n^{-c} \). So \( \|\tilde{x}_i - \mathbf{x}_i^{(OS)}\|_2 \lesssim_{c,d,\lambda} \rho_n^{1/2} \frac{\log n}{n} \) with probability at least \( 1 - (n \rho_n)^{-c} \). By Theorem A.6 and Slutsky’s theorem, we have
\[
\sqrt{n} \mathbf{G}_{0in}^{1/2} \left( \mathbf{W}^T \mathbf{x}_i - \rho_n^{1/2} \mathbf{x}_{0i} \right) \xrightarrow{d} N(\mathbf{0}_d, \mathbf{I}_d),
\]
and
\[
\mathbf{G}_{0in}^{1/2} (\mathbf{W}^T \mathbf{x}_i - \rho_n^{1/2} \mathbf{x}_{0i}) = \frac{1}{n \rho_n} \frac{1}{2^{1/2}} \sum_{j=1}^{n} \left( (\mathbf{A}_{ij} - \rho_n \mathbf{x}_{0i}^T \mathbf{x}_{0j}) \mathbf{G}_{0in}^{-1/2} \mathbf{x}_{0j} \right) \mathbf{x}_{0i} \mathbf{x}_{0j} (1 - \rho_n \mathbf{x}_{0i}^T \mathbf{x}_{0j}) + \mathbf{r}_{in},
\]
where
\[
\|\mathbf{r}_{in}\|_2 \lesssim_{c,d,\lambda} \rho_n^{1/2} \frac{\log n}{n} + \frac{1}{\sqrt{n}} \sqrt{\frac{(\log(n \rho_n))^4}{n \rho_n}}
\]
with probability at least \( 1 - (n \rho_n)^{-c} \).

We finally show the convergence of the sum of squares errors, that is
\[
\|\mathbf{X} - \rho_n^{1/2} \mathbf{X}_0\|_2^2 - \frac{1}{n} \sum_{i=1}^{n} \text{tr}(\mathbf{G}_{0in}^{-1}) \xrightarrow{p} 0.
\]
By the previous result, we have
\begin{align*}
\|\tilde{X}W - \rho_n^{1/2} X_0\|_F^2 &= \sum_{i=1}^n \left\| \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j} + G_{0in}^{-1/2} r_{in} \right\|_2^2 \\
&= \sum_{i=1}^n \left\| \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j} + G_{0in}^{-1/2} r_{in} \right\|_2^2 + \sum_{i=1}^n \| G_{0in}^{-1/2} r_{in} \|_2^2 \\
&\quad + 2 \sum_{i=1}^n \left\langle \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j} x_{0i}^T, G_{0in}^{-1/2} r_{in} \right\rangle.
\end{align*}

By Lemma A.4, the first term equals
\begin{align*}
\sum_{i=1}^n E_0 \left\| \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j} \right\|_2^2 &+ o_P(1) \\
&= \frac{1}{n^2\rho_n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n E_0 \left( (A_{ia} - \rho_n x_{0i}^T x_{0a}) (A_{ib} - \rho_n x_{0i}^T x_{0b}) \right) x_{0j}^T G_{0in}^{-2} x_{0j} + o_P(1) \\
&= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \left\langle x_{0j}^T G_{0in}^{-2} x_{0j}, x_{0i}^T x_{0j} (1 - \rho_n x_{0i}^T x_{0j}) \right\rangle + o_P(1) \\
&= \frac{1}{n} \sum_{i=1}^n \text{tr} \left\{ \frac{1}{n} \sum_{j=1}^n x_{0j}^T x_{0j} (1 - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-2} \right\} + o_P(1) \\
&= \frac{1}{n} \sum_{i=1}^n \text{tr}(G_{0in}^{-1}) + o_P(1).
\end{align*}

For the second term, by Theorem 4.7 in Xie (2022), we have
\begin{align*}
\sum_{i=1}^n \| r_{in} \|_2^2 &\leq n \max_{i \in [n]} \| r_{in} \|_2^2 \lesssim c_{\varepsilon, \delta} \frac{\rho_n (\log n)^2}{n} + \frac{(\log n)^4}{n\rho_n}
\end{align*}
with probability at least $1 - n^{-c}$ for all $n \geq N_{c,\delta,\lambda}$, so $\sum_{i=1}^n \| r_{in} \|_2^2 = o_P(1)$ by the condition that $(\log n)^4 = o(n\rho_n)$. For the third term, by Cauchy–Schwarz inequality, we have
\begin{align*}
&\left\| \sum_{i=1}^n \left\langle \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j}, G_{0in}^{-1/2} r_{in} \right\rangle \right\|_2 \\
&\leq \left\| \sum_{i=1}^n \left\langle \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j}, x_{0i}^T x_{0j} (1 - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1/2} r_{in} \right\rangle \right\|_2 \left\| G_{0in}^{-1/2} r_{in} \right\|_2 \\
&\leq \left\{ \sum_{i=1}^n \left\| \frac{1}{n\rho_n} \sum_{j=1}^n (A_{ij} - \rho_n x_{0i}^T x_{0j}) G_{0in}^{-1} x_{0j} \right\|_2^2 \right\}^{1/2} \left\{ \sum_{i=1}^n \left\| G_{0in}^{-1/2} r_{in} \right\|_2^2 \right\}^{1/2}.
\end{align*}
\[ = O_{P_0}(1) \times o_{P_0}(1) = o_{P_0}(1). \]

Hence, we conclude that
\[
\| \hat{X} - \rho_n^{1/2}X_0 \|^2 \|_F = \frac{1}{n} \sum_{i=1}^n \text{tr}(G_{0i}^{-1}) + o_{P_0}(1).
\]

\[ \Box \]

**B.3 Proof of Theorem 4.1**

Proof. Similar to the earlier proofs, the large probability bounds below are with regard to \( n \geq N_{c, \delta, \lambda} \) for some large constant \( N_{c, \delta, \lambda} \) depending on \( c, \delta, \lambda \). By definition, \( t = \sqrt{n}W^T(x_i - \hat{x}_i) \), then \( x_i = \hat{x}_i + Wt/\sqrt{n} \).

Denote the parameter space of \( t \) by \( \hat{\Theta}_n = \{ t \in \mathbb{R}^d : \| \hat{x}_i + Wt/\sqrt{n} \|_2 \leq 1 \} \). Denote the normalizing constant by
\[
d_{in} = \int_{\mathbb{R}^d} \exp \left\{ n\hat{M}_{in}(\hat{x}_i + \frac{Wt}{\sqrt{n}}) - n\hat{M}_{in}(\hat{x}_i) \right\} \pi(\hat{x}_i + \frac{Wt}{\sqrt{n}})I(t \in \hat{\Theta}_n)dt.
\]

By definition,
\[
\hat{\pi}^*_n(t | A) = \frac{1}{d_{in}} \exp \left\{ n\hat{M}_{in}(\hat{x}_i + \frac{Wt}{\sqrt{n}}) - n\hat{M}_{in}(\hat{x}_i) \right\} \pi(\hat{x}_i + \frac{Wt}{\sqrt{n}})I(t \in \hat{\Theta}_n).
\]

It is sufficient to show that
\[
\max_{i \in [n]} \int_{\mathbb{R}^d} (1 + \| t \|^2_2) \left| \exp \left\{ n\hat{M}_{in}(\hat{x}_i + \frac{Wt}{\sqrt{n}}) - n\hat{M}_{in}(\hat{x}_i) \right\} \pi(\hat{x}_i + \frac{Wt}{\sqrt{n}})I(t \in \hat{\Theta}_n)
\]
\[
- e^{-\frac{1}{2}t^T G_{0i} t} \left( \rho_n^{1/2} WX_0 \right) \left| dt \right| = o_{P_0}(1).
\]

To see this, note that (4.2) in the manuscript can be rewritten as
\[
\max_{i \in [n]} \left| \frac{1}{d_{in}} \int_{\mathbb{R}^d} (1 + \| t \|^2_2) \left| \exp \left\{ n\hat{M}_{in}(\hat{x}_i + \frac{Wt}{\sqrt{n}}) - n\hat{M}_{in}(\hat{x}_i) \right\} \pi(\hat{x}_i + \frac{Wt}{\sqrt{n}})I(t \in \hat{\Theta}_n)
\]
\[
- e^{-\frac{1}{2}t^T G_{0i} t} \left( \rho_n^{1/2} WX_0 \right) \right| dt
\]
\[
\leq \max_{i \in [n]} \left| \frac{1}{d_{in}} \int_{\mathbb{R}^d} (1 + \| t \|^2_2) \left| \exp \left\{ n\hat{M}_{in}(\hat{x}_i + \frac{Wt}{\sqrt{n}}) - n\hat{M}_{in}(\hat{x}_i) \right\} \pi(\hat{x}_i + \frac{Wt}{\sqrt{n}})I(t \in \hat{\Theta}_n)
\]
\[
- e^{-\frac{1}{2}t^T G_{0i} t} \pi(\hat{x}_i + \frac{Wt}{\sqrt{n}}) \right| dt
\]
\[
+ \max_{i \in [n]} \left| \frac{\pi(W \hat{X}_0^{1/2} X_0)}{d_{in}} \right| - \text{det}(2\pi G_{0i}^{-1})^{-1/2} \int_{\mathbb{R}^d} (1 + \| t \|^2_2) e^{-\frac{1}{2}t^T G_{0i} t}dt.
\]

Since (B.2) implies that \( \max_{i \in [n]} |d_{in} - \text{det}(2\pi G_{0i}^{-1})^{-1/2} \pi(W \hat{X}_0^{1/2} X_0)| = o_{P_0}(1) \) (by taking \( \alpha = 0 \)), it can be seen that (B.2) implies that the two terms on the right hand side of the previous display are \( o_{P_0}(1) \). Hence,
we are left with establishing (B.2).

Let \( \{\eta_n\}_{n=1}^{\infty} \) be a sequence to be determined later with \( 0 < \eta_n \to \infty \) and consider the following partition of \( \mathbb{R}^d \):

\[
\mathcal{A}_1 = \{ t \in \tilde{\Theta}_{in} : ||t||_2 \leq \eta_n \}, \quad \mathcal{A}_2 = \{ t \in \tilde{\Theta}_{in} : ||t||_2 > \eta_n \}, \quad \mathcal{A}_3 = \tilde{\Theta}_{in}^c.
\]

We first consider the integral of (B.2) over \( \mathcal{A}_3 \). By definition of \( \mathbb{I}(t \in \tilde{\Theta}_{in}) \), the integral over \( \mathcal{A}_3 \) can be bounded by

\[
\max_{i \in [n]} \int_{\mathcal{A}_3} (1 + ||t||_2^\alpha) e^{-\frac{1}{2} t^T (\rho_n^{-1/2} G_{0,n}) t} \pi t dt \\
\leq \int_{\mathcal{A}_3} (1 + ||t||_2^\alpha) e^{-\min_{i \in [n]} \lambda_d(G_{0,n}) ||t||_2/2} \pi t dt \quad (B.3)
\]

\[
\leq \int_{\mathcal{A}_3} (1 + ||t||_2^\alpha) e^{-\lambda ||t||_2^2} \pi t dt \to 0,
\]

since \( \mathcal{A}_3 \) is shrinking to empty set and \( \min_{i \in [n]} ||G_{0,n}|| \geq \lambda \) has been shown in the proof of Theorem 3.2 (see display (B.1)). We next consider the integral of (B.2) over \( \mathcal{A}_2 \). Define the event

\[
\mathcal{E}_{2n} = \left\{ A : \max_{i \in [n]} \max_{||x||_2 \leq 1} s^T \frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i^T} (\tilde{x}_i) s \leq -\frac{\lambda}{2} ||s||_2^2 \quad \forall s \in \mathbb{R}^d \right\}.
\]

Note that by Lemma A.2, Theorem 5.2 in Lei and Rinaldo (2015), and Weyl’s inequality, with probability at least \( 1 - n^{-c} \),

\[
\min_{i \in [n]} \min_{||x||_2 \leq 1} s^T \left( -\frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i^T} (\tilde{x}_i) \right) s = \min_{i \in [n]} \min_{||x||_2 \leq 1} s^T \left( \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{1}{p_{ij}} + \frac{1 - A_{ij}}{(1 - x_i^T \tilde{x}_j)^2} \right\} \tilde{x}_j \tilde{x}_j^T \right) s \\
\geq \frac{1}{\max_{i,j \in [n]} p_{ij}} \frac{1}{n} \sum_{j=1}^{n} s^T \tilde{x}_j \tilde{x}_j^T s \geq \frac{1}{n \rho_n} s^T \tilde{X}^T \tilde{X} s \geq \frac{1}{n \rho_n} \lambda_d(A) ||s||_2 \geq \frac{\lambda}{2} ||s||_2^2.
\]

This shows that \( \mathbb{P}_0(\mathcal{E}_{2n}) \geq 1 - n^{-c} \) for all \( n \geq N_{c, \delta, \lambda} \). By Taylor’s expansion, for any \( t \in \tilde{\Theta}_{in} \), we have

\[
n \tilde{M}_{in}(\tilde{x}_i + \frac{W_t}{\sqrt{n}}) - n \tilde{M}_{in}(\tilde{x}_i) = \frac{1}{2} t^T \frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i^T} (\tilde{x}_i) W_t, \quad (B.4)
\]

where \( \tilde{x}_i = \tilde{x}_i + \theta_i W_t/\sqrt{n} \) for some \( \theta_i \in [0, 1] \) because the gradient of \( \tilde{M}_{in} \) evaluated at \( x_i = \tilde{x}_i \) is zero by definition of the maximum surrogate likelihood estimator \( \tilde{x}_i \). Over this event, the integral of (B.2) over \( \mathcal{A}_2 \) can be upper bounded by

\[
\max_{i \in [n]} \int_{\mathcal{A}_2} (1 + ||t||_2^\alpha) \exp \left\{ \frac{1}{2} t^T W_t^T \frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i^T} (\tilde{x}_i) W_t \right\} \pi(\tilde{x}_i + \frac{W_t}{\sqrt{n}}) dt \\
+ \max_{i \in [n]} \int_{\mathcal{A}_2} (1 + ||t||_2^\alpha) e^{-t^T (G_{0,n})^{1/2} \pi t ||W_n||} dt
\]

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\[
\leq C \int_{\mathcal{A}_2} (1 + \|t\|^2) \exp \left\{ \max_{i \in [n]} \max_{\|x\|_2 \leq 1} \frac{1}{2} t^T W^T \frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i^T} (\tilde{x}_i) W t \right\} dt \\
+ \max_{i \in [n]} C \int_{\mathcal{A}_2} (1 + \|t\|^2) e^{-\frac{1}{2} t^T G_{0,n} t} dt \\
\leq 2C \int_{\|t\|_2 > \eta_n} (1 + \|t\|^2) e^{-\frac{1}{2} \lambda \|t\|^2} dt.
\]

Denote the last line of the above display by \( \epsilon_{2n} \), then \( \epsilon_{2n} \to 0 \) because \( \eta_n \to \infty \). It follows that

\[
\mathbb{P}_0 \left\{ \max_{i \in [n]} \int_{\mathcal{A}_2} (1 + \|t\|^2) \exp \left\{ n \widetilde{M}_{in}(\tilde{x}_i + \frac{W t}{\sqrt{n}}) - n \widetilde{M}_{in}(\tilde{x}_i) \right\} \pi(\tilde{x}_i + \frac{W t}{\sqrt{n}}) \mathbb{I}(t \in \tilde{\Theta}_n) \\
- e^{-\frac{1}{2} t^T G_{0,n} t} \pi(\tilde{x}_i) \right\} dt \geq \epsilon_{2n} \right\} \leq n^{-c}
\]

for all \( n \geq N_{redc,\delta,\lambda} \). Hence,

\[
\max_{i \in [n]} \int_{\mathcal{A}_2} (1 + \|t\|^2) \exp \left\{ n \widetilde{M}_{in}(\tilde{x}_i + \frac{W t}{\sqrt{n}}) - n \widetilde{M}_{in}(\tilde{x}_i) \right\} \pi(\tilde{x}_i + \frac{W t}{\sqrt{n}}) \mathbb{I}(t \in \tilde{\Theta}_n) \\
- e^{-\frac{1}{2} t^T G_{0,n} t} \pi(\tilde{x}_i) \right\} dt \to 0.
\]

We next consider the integral of (B.2) over \( \mathcal{A}_1 \). Take \( \eta_n = \min \{ (n \rho_n / \log n)^{1/8}, \sqrt{\log n} / \rho_n \} \). Recall that \( t = \sqrt{n} W^T (x_i - \tilde{x}_i) \), and

\[
\max_{i \in [n]} \| W^T x_i - \rho_n^{1/2} x_0 \|_2 \lesssim_{c,\delta,\lambda} \sqrt{\log n} / n \rho_n
\]

with probability at least \( 1 - n^{-c} \) because \( \eta_n / \sqrt{n} \leq \sqrt{\log n} / (n \rho_n) \), which also implies that there exists a constant \( C_{c,\delta,\lambda} > 0 \) (possibly depending on \( c, \delta, \lambda \)), such that

\[
\{ x_i : \| t \|_2 \leq \eta_n \} \subset \left\{ x_i : \| W^T x_i - \rho_n^{1/2} x_0 \|_2 \leq C_{c,\delta,\lambda} \sqrt{\log n} / n \rho_n \right\}
\]

with probability at least \( 1 - n^{-c} \). Define the event

\[
\mathcal{E}_{1n} = \left\{ A : \max_{i \in [n]} \sup_{x : \| t \|_2 \leq \eta_n} \left\| W^T \frac{\partial^2 \tilde{M}_{in}}{\partial x_i \partial x_i^T} (x_i) W + G_{0,in} \right\|_2 \leq K_{c,\delta,\lambda} \sqrt{\log n} / n \rho_n \right\}
\]

\[
\cap \left\{ A : \max_{i \in [n]} \| W^T x_i - \rho_n^{1/2} x_0 \|_2 \leq K_{c,\delta,\lambda} \sqrt{\log n} / n \rho_n \right\}.
\]

for an appropriate constant \( K_{c,\delta,\lambda} \) depending on \( c, \delta, \lambda \). By Lemma A.3, one can select \( K_{c,\delta,\lambda} \) such that \( \mathbb{P}_0(\mathcal{E}_{1n}) \geq 1 - n^{-c} \) for all \( n \geq N_{c,\delta,\lambda} \). Then over the event \( \mathcal{E}_{1n} \), by Taylor’s expansion (B.4) and the
mean-value theorem applied to the exponential function, we have

\[
\max_{i \in [n]} \int_{A_1} (1 + \|t\|_2^2) \left| \exp \left\{ n \tilde{M}_m (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) - n \tilde{M}_m (\tilde{x}_i) \right\} \pi (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) \mathbb{1} (t \in \hat{\Theta}_m) \right| dt \\
- e^{-\frac{1}{2} t^T G_{0,m} t} \pi (\rho_n^{1/2} Wx_0) \right| dt \\
= \max_{i \in [n]} \int_{A_1} (1 + \|t\|_2^2) \left| \exp \left\{ \frac{1}{2} t^T W^T \frac{\partial^2 \tilde{M}_m}{\partial x_i \partial x_i^T} (\tilde{x}_i) W t \right\} \pi (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) \right| dt \\
- e^{-\frac{1}{2} t^T G_{0,m} t} \pi (\rho_n^{1/2} Wx_0) \right| dt \\
= \max_{i \in [n]} \int_{A_1} (1 + \|t\|_2^2) \left| \exp \left\{ \frac{1}{2} t^T \left( W^T \frac{\partial^2 \tilde{M}_m}{\partial x_i \partial x_i^T} (\tilde{x}_i) W + G_{0,m} \right) t \right\} \pi (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) \right| dt \\
- e^{-\frac{1}{2} t^T G_{0,m} t} \pi (\rho_n^{1/2} Wx_0) \right| dt \\
\leq \max_{i \in [n]} \int_{A_1} (1 + \|t\|_2^2) \left\{ \left| \exp \left\{ \frac{1}{2} t^T \left( W^T \frac{\partial^2 \tilde{M}_m}{\partial x_i \partial x_i^T} (\tilde{x}_i) W + G_{0,m} \right) t \right\} \right| - 1 \right\} \left( \exp \left\{ \frac{1}{2} t^T G_{0,m} t \right\} \right) e^{-\frac{1}{2} t^T G_{0,m} t} \pi (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) \right| dt \\
\leq \left( \exp \left\{ \frac{1}{2} K_{c,\delta,\lambda} \frac{1}{n^{1/2}} \eta_n \right\} \frac{1}{2} K_{c,\delta,\lambda} \frac{1}{n^{1/2}} \eta_n \right) \sup_{i \in [n]} \left( \frac{1}{n^{1/2}} \eta_n \right) \left( 1 - \frac{\pi (\rho_n^{1/2} Wx_0)}{\pi (x_i)} \right) \right) \times \int e^{-\lambda \|t\|_2^2} dt.
\]

Denote the last form of the above display by \( \epsilon_{1n} \). It is obvious that \( \exp \left\{ \frac{1}{2} K_{c,\delta,\lambda} \frac{1}{n^{1/2}} \eta_n \right\} \rightarrow 1 \) (since \( \eta_n = \frac{(n \rho_n \log n)^{1/2}}{\lambda} \)). By the assumptions on \( \pi (x_i) \), \( \max_{i \in [n]} \sup_{x_i \|W^T x_i - \rho_n^{1/2} x_0\|_2 \leq \delta, \lambda} \left| 1 - \frac{\pi (\rho_n^{1/2} Wx_0)}{\pi (x_i)} \right| \rightarrow 0 \). It follows that \( \epsilon_{1n} \rightarrow 0 \) as \( n \rightarrow \infty \), and

\[
\mathbb{P}_0 \left\{ \max_{i \in [n]} \int_{A_1} (1 + \|t\|_2^2) \left| \exp \left\{ n \tilde{M}_m (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) - n \tilde{M}_m (\tilde{x}_i) \right\} \pi (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) \mathbb{1} (t \in \hat{\Theta}_m) \right| dt \geq \epsilon_{1n} \right\} \leq n^{-c},
\]

for all \( n \geq N_{c,\delta,\lambda} \). Hence,

\[
\max_{i \in [n]} \int_{A_1} (1 + \|t\|_2^2) \left| \exp \left\{ n \tilde{M}_m (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) - n \tilde{M}_m (\tilde{x}_i) \right\} \pi (\tilde{x}_i + \frac{Wt}{\sqrt{n}}) \mathbb{1} (t \in \hat{\Theta}_m) \right| dt \rightarrow 0.
\]

(B.6)
The proof of (B.2) is completed by combining (B.3), (B.5), and (B.6).

### B.4 Proof of Corollary 4.1

*Proof.* We first show the convergence of the mean and covariance of \( \tilde{\pi}_n^*(t \mid A) \), which is a direct consequence of Theorem 4.1:

\[
\max_{i \in [n]} \left\| \int t \tilde{\pi}_n^*(t \mid A) dt \right\|_2 = \max_{i \in [n]} \left\| \int t \tilde{\pi}_n^*(t \mid A) dt - \int t e^{-t^T G_{0in} t/2} dt \right\|_2 \\
\leq \max_{i \in [n]} \left\| \int t^2 \tilde{\pi}_n^*(t \mid A) dt - \int t^2 e^{-t^T G_{0in} t/2} dt \right\|_2 \\
\leq \max_{i \in [n]} \left\| \int \left( t^2 - \frac{e^{-t^T G_{0in} t/2}}{\det(2\pi G_{0in}^{-1})^{1/2}} \right) dt \right\|_2 \Rightarrow 0,
\]

and

\[
\max_{i \in [n]} \left\| \int t^T \tilde{\pi}_n^*(t \mid A) dt - G_{0in}^{-1} \right\|_2 = \max_{i \in [n]} \left\| \int t^T \tilde{\pi}_n^*(t \mid A) dt - \int t^T e^{-t^T G_{0in} t/2} dt \right\|_2 \\
\leq \max_{i \in [n]} \left\| \int t^2 \tilde{\pi}_n^*(t \mid A) dt - \int t^2 e^{-t^T G_{0in} t/2} dt \right\|_2 \\
\leq \max_{i \in [n]} \left\| \int \left( t^2 - \frac{e^{-t^T G_{0in} t/2}}{\det(2\pi G_{0in}^{-1})^{1/2}} \right) dt \right\|_2 \Rightarrow 0.
\]

Now

\[
\max_{i \in [n]} \left\| \sqrt{n}(x_i^* - \tilde{x}_i) \right\|_2 = \max_{i \in [n]} \left\| \int \sqrt{n}(x_i - \tilde{x}_i) \tilde{\pi}_n(x_i \mid A) dx_i \right\|_2 \\
= \max_{i \in [n]} \left\| \int t \tilde{\pi}_n^*(t \mid A) dt \right\|_2 \\
= o_p(1),
\]

then by Theorem 3.2 and Slutsky’s Theorem, \( \sqrt{n} G_{0in}^{1/2}(W^T x_i^* - \rho_n^{1/2} x_{0i}) \overset{d}{\to} N_d(0_d, I_d) \). Also,

\[
\max_{i \in [n]} \left\| n W^T \Sigma_{0in}^* W - G_{0in}^{-1} \right\|_2 \\
= \max_{i \in [n]} \left\| n W^T (x_i^* - x_i^*) (x_i - x_i^*)^T \tilde{\pi}_n(x_i \mid A) dx_i - G_{0in}^{-1} \right\|_2 \\
= \max_{i \in [n]} \left\| n W^T (x_i - \tilde{x}_i + \tilde{x}_i - x_i^*) (x_i - \tilde{x}_i + \tilde{x}_i - x_i^*)^T \tilde{\pi}_n(x_i \mid A) dx_i - G_{0in}^{-1} \right\|_2 \\
\leq \max_{i \in [n]} \left\| \int t^T \tilde{\pi}_n^*(t \mid A) dt - G_{0in}^{-1} \right\|_2 + o_p(1) \\
= o_p(1).
\]

Note that \( G_{0in} \) is finite and positive definite. By continuous mapping theorem,

\[
(\rho_n^{1/2} W x_i - x_i^*)^T (\Sigma_{0in}^*)^{-1} (\rho_n^{1/2} W x_i - x_i^*) \overset{d}{\to} \chi^2_d,
\]

so \( P_0\{(\rho_n^{1/2} W x_i - x_i^*)^T (\Sigma_{0in}^*)^{-1} (\rho_n^{1/2} W x_i - x_i^*) \leq q_{1-\alpha} \} \to 1 - \alpha. \)

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We now focus on the last assertion. By the previous proof, we know that \( \max_{i \in [n]} \| x^* - \hat{x}_i \|_2^2 = o_P(1/n) \).

It follows directly that

\[
\| X^* - \hat{X} \|_F^2 = \sum_{i=1}^n \| x^* - \hat{x}_i \|_2^2 \leq n \max_{i \in [n]} \| x^* - \hat{x}_i \|_2^2 = o_P(1).
\]

Therefore, by Theorem 3.2 and Cauchy–Schwarz inequality, we have

\[
\| X^* W - \rho_n^{1/2} X_0 \|_F^2 = \| \hat{X} W - \rho_n^{1/2} X_0 \|_F^2 + \| X^* W - \hat{X} W \|_F^2 \\
+ 2 \langle \hat{X} W - \rho_n^{1/2} X_0, X^* W - \hat{X} W \rangle_F \\
= \frac{1}{n} \sum_{i=1}^n \text{tr}(G_{0in}^{-1}) + o_P(1) + O \left( \| \hat{X} W - \rho_n^{1/2} X_0 \|_F \| X^* W - \hat{X} W \|_F \right) \\
= \frac{1}{n} \sum_{i=1}^n \text{tr}(G_{0in}^{-1}) + o_P(1),
\]

where \( \langle \cdot, \cdot \rangle_F \) denotes the Frobenius inner product between matrices. The proof is thus completed. \( \square \)

C  Proof of the Convergence of the Stochastic Gradient Descent

**Lemma C.1** (Lemma A.5 in Mairal, 2013). Let \((a_t)_{t \geq 1}, (b_t)_{t \geq 1}\) be two non-negative real sequences. Assume that \( \sum_{t=1}^\infty a_t b_t \) converges and \( \sum_{t=1}^\infty a_t \) diverges, and \( |b_{t+1} - b_t| \leq K a_t \) for some constant \( K \geq 0 \). Then \( b_t \) converges to 0.

**Lemma C.2** (Lemma 2 in Li and Orabona, 2019). Let \( a_0 > 0, a_i \geq 0, i = 1, \ldots, T \) and \( \beta > 1 \). Then

\[
\sum_{t=1}^T \frac{a_t}{(a_0 + \sum_{i=1}^t a_i)^\beta} < \frac{1}{(\beta-1)a_0^{\beta-1}}.
\]

**Lemma C.3** (Lemma 3 in Li and Orabona, 2019). Let \( f : \mathcal{X} \subset \mathbb{R}^d \to \mathbb{R} \) be twice continuously differentiable whose minimum is attained at \( x = x^* \) and suppose there exists a constant \( L > 0 \), such that for all \( x, y \in \mathcal{X} \),

\[
\left\| \frac{\partial f}{\partial x}(x) - \frac{\partial f}{\partial x}(y) \right\|_2 \leq L \| x - y \|_2.
\]

Suppose \( g(x, z) \) is a function of a random vector \( z \), such that \( \mathbb{E}_z g(x, z) = \partial f(x)/\partial x \). Let \( (z_t)_{t \geq 1} \) be a sequence of independent and identically distributed (i.i.d.) copies of \( z \). Consider a sequence of iterates \( x^{(t)} \) generated by

\[
x^{(t+1)} = x^{(t)} - H_t g(x^{(t)}, z_t),
\]

where \( H_t \in \mathbb{R}^{d \times d} \) is a step-size matrix for the \( t \)th iteration. Then the sequence \((x^{(t)})_{t \geq 1}\) satisfies the following inequality:

\[
\mathbb{E}_{z_1, \ldots, z_N} \left[ \sum_{t=1}^N \left\langle \frac{\partial f}{\partial x}(x^{(t)}), H_t \frac{\partial f}{\partial x}(x^{(t)}) \right\rangle \right]
\]
of Theorem 3.3. The proof is similar to Theorem 1 in Li and Orabona (2019), with some slight modifications. In the setting here, the expectation is taken with respect to the randomness of the stochastic gradient descent conditioned on the adjacency matrix, that is, the data and the ASE are viewed as deterministic. Here, we suppress the subscript \( i \in [n] \) and use \( x^{(t)} \) to denote the \( t \)th iterate in the optimization, and \( \bar{x} \) the maximizer of the average surrogate log-likelihood function \( \tilde{M}_{in}(x) := (1/n)\bar{\ell}_{in}(x) \).

For the surrogate log-likelihood function, by the computation of the gradient and Hessian of \( \tilde{M}_{in} \) in the proof of Theorem 3.2, they are bounded over \( \{x_i : \|x_i\|_2 \leq 1\} \) when \( \max_j \|\bar{x}_j\|_2 < 1 \). So both \( \tilde{M}_{in}(x) \) and its gradient are Lipschitz in \( \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\} \) by the mean value theorem. Let \( C_1 \) and \( C_2 \) be the Lipschitz constants for \( \tilde{M}_{in}(x) \) and its gradient, respectively. In the context of Section 3.2, the random vector \( z_t \) corresponds to the randomly generated indices \( (j_1^{(t)}, \ldots, j_s^{(t)}) \) in a single iteration of the mini-batch SGD algorithm, and \( g(x^{(t)}, z_t) \) takes the form

\[
g(x^{(t)}, z_t) = \frac{1}{s} \sum_{k=1}^{s} \frac{\partial m_i}{\partial x}(x^{(t)}, j_k^{(t)}).
\]

It is clear that \( \mathbb{E}_{x_t} g(x^{(t)}, z_t) \) coincides with the gradient of \( \tilde{M}_{in}(x^{(t)}) \). Also, for the stochastic gradient \( g(x^{(t)}, z_t) \), it is easy to see that \( \|g(x^{(t)}, z_t) - \nabla \tilde{M}_{in}(x^{(t)})\|_2 \leq C_3 \) for all \( x^{(t)} \in B(0_d, 1) \).

Observe that

\[
\sum_{t=1}^{\infty} \|\alpha_t g(x^{(t)}, z_t)\|_2^2 = \sum_{t=1}^{\infty} \alpha_{t+1}^2 \|g(x^{(t)}, z_t)\|_2^2 + \sum_{t=1}^{\infty} (\alpha_t^2 - \alpha_{t+1}^2) \|g(x^{(t)}, z_t)\|_2^2 \\
\leq \frac{\alpha_0^2}{2b_0^2} + \max_{t \geq 1} \|g(x^{(t)}, z_t)\|_2^2 \sum_{t=1}^{\infty} \alpha_t^2 - \alpha_{t+1}^2 \\
\leq \frac{\alpha_0^2}{2b_0^2} + \max_{t \geq 1} \|g(x^{(t)}, z_t)\|_2^2 \alpha_1^2 \\
\leq \frac{\alpha_0^2}{2b_0^2} + 2\alpha_1^2 \max_{t \geq 1} \left( \|\nabla \tilde{M}_{in}(x^{(t)})\|_2^2 + \|\nabla \tilde{M}_{in}(x^{(t)}) - g(x^{(t)}, z_t)\|_2^2 \right) \\
\leq \frac{\alpha_0^2}{2b_0^2} + \frac{2\alpha_1^2}{b_0 + 2\epsilon} \left( C_1^2 + C_2^2 \right) < \infty,
\]

where in the first inequality we have used Lemma C.2, in the third one the elementary inequality \( \|x + y\|_2^2 \leq 2\|x\|_2^2 + 2\|y\|_2^2 \). Therefore, for any \( m \in \mathbb{N}_+ \), by Cauchy–Schwarz inequality, we have

\[
\|x^{(N+m)} - x^{(N)}\|_2^2 = \left\| \sum_{t=N}^{N+m-1} x^{(t+1)} - x^{(t)} \right\|_2^2 \leq \sum_{t=N}^{N+m-1} \|x^{(t+1)} - x^{(t)}\|_2^2 \\
\leq \sum_{t=N}^{N+m-1} \|\alpha_{t} g(x^{(t)}, z_t)\|_2^2,
\]

and the previous infinite sum being finite implies that \( \lim_{N \to \infty} \|x^{(N+m)} - x^{(N)}\|_2 = 0 \) a.s., that is, \( \{x^{(t)}\}_t \)
forms a Cauchy sequence, and thus converges to some point \( x^* \in B(0, 1) \text{ a.s.} \). Note that \( x^* \) is a random variable with respect to the randomness of \( z_t \). Next we need to show that \( x^* \) is indeed the maximizer of the surrogate log-likelihood function.

By Lemma C.3, taking the limit \( T \to \infty \) and exchanging the expectation and the limits due to non-negative terms, we have

\[
\mathbb{E} \left[ \sum_{t=1}^{\infty} \alpha_t \left\| \nabla \tilde{M}_{in}(x^{(t)}) \right\|_2^2 \right] \leq \tilde{M}_{in}(x^*) - \tilde{M}_{in}(x_1) + \frac{C_2}{2} \mathbb{E} \left[ \sum_{t=1}^{\infty} \left\| \alpha_t g(x^{(t)}, z_t) \right\|_2^2 \right].
\]

With the right hand side being finite, we have

\[
\sum_{t=1}^{\infty} \alpha_t \left\| \nabla \tilde{M}_{in}(x^{(t)}) \right\|_2^2 < \infty.
\]

Observe that by definition,

\[
\sup_{z_t, x^{(t)}} \left\| \alpha_t g(x^{(t)}, z_t) \right\|_2 \leq \frac{a_0}{(b_0)^{1/2+\epsilon}} \sup_{z_t, x^{(t)}} \left\| g(x^{(t)}, z_t) \right\|_2 < \infty,
\]

that is, the updating of the iterate is bounded. By assumption, the MSLE \( \tilde{x} \) is in the interior of the feasible region. So there exists an integer \( m^* \) such that for all \( t \in \mathbb{N}_+ \), the number of times that step-halving in the algorithm is called is no greater than \( m^* \). This implies that

\[
\frac{1}{m^*} a_0 \left[ b_0 + \sum_{i=1}^{t-1} \left\| g(x^{(t)}, z_t) \right\|_2^2 \right]^{-(1/2+\epsilon)} \leq \alpha_t \leq a_0 \left[ b_0 + \sum_{i=1}^{t-1} \left\| g(x^{(t)}, z_t) \right\|_2^2 \right]^{-(1/2+\epsilon)}
\]

for all \( t \in \mathbb{N}_+ \), which further implies that

\[
\sum_{t=1}^{\infty} \alpha_t \geq \frac{1}{m^*} \sum_{t=1}^{\infty} a_0 \left[ b_0 + \sum_{i=1}^{t-1} \left\| g(x^{(t)}, z_t) \right\|_2^2 \right]^{-(1/2+\epsilon)}
\]

\[
\geq \frac{1}{m^*} \sum_{t=1}^{\infty} a_0 \left[ b_0 + 2(t-1)(C_1^2 + C_3^2) \right]^{-(1/2+\epsilon)} = \infty.
\]

Using the fact that both \( \tilde{M}_{in}(x) \) and \( \nabla \tilde{M}_{in}(x) \) are Lipschitz, we also have

\[
\left\| \nabla \tilde{M}_{in}(x^{(t+1)}) - \nabla \tilde{M}_{in}(x^{(t)}) \right\|_2^2 \\
= \left( \left\| \nabla \tilde{M}_{in}(x^{(t+1)}) \right\|_2 + \left\| \nabla \tilde{M}_{in}(x^{(t)}) \right\|_2 \right) \cdot \left\| \nabla \tilde{M}_{in}(x^{(t+1)}) - \nabla \tilde{M}_{in}(x^{(t)}) \right\|_2 \\
\leq 2C_1C_2 \left\| x^{(t+1)} - x^{(t)} \right\|_2 = 2C_1C_2 \alpha_t \left\| g(x^{(t)}, z_t) \right\|_2 \leq 2C_1C_2(C_1 + C_3) \alpha_t.
\]

Hence, we can use Lemma C.1 to obtain that \( \lim_{t \to \infty} \left\| \nabla \tilde{M}_{in}(x^{(t)}) \right\|_2 = 0 \text{ a.s.} \). The continuity of \( \nabla M_{in}(x) \) implies that \( x^{(t)} \to \tilde{x} \) a.s..
D Additional implementation details

D.1 Additional details of the algorithms

This subsection provides the detailed Metropolis–Hastings sampler for computing the joint posterior distribution \( \pi_n(X | A) \) using the surrogate likelihood function. For each \( i \in [n] \), we use the normal random walk truncated in the unit ball as the proposal distribution, with the covariance matrix being the inverse of

\[
n\hat{G}_{in} = \sum_{j=1}^{n} \frac{\bar{x}_j \bar{x}_j^T}{\bar{x}_i^T \bar{x}_j (1 - \bar{x}_i^T \bar{x}_j)}.
\]

The above covariance matrix is the plug-in estimator of the asymptotic covariance matrix of the Bernstein–von Mises limit distribution. Below, we provide the detailed Metropolis–Hastings sampler in the algorithm below. The computation of the posterior distribution of the entire latent position matrix \( X \) can be done by a parallelization over \( i \in [n] \).

**Algorithm 1** Metropolis–Hastings sampler for computing the posterior distribution of \( X \).

**Input:** The adjacency matrix \( A = [A_{ij}]_{n \times n} \);
- The embedding dimension \( d \);
- The tuning parameter \( \sigma \);
- Number of burn-in iterations \( B \);
- Number of post-burn-in samples \( n_{mc} \);
- Thinning size \( b \).

Compute the spectral decomposition of the adjacency matrix

\[ A = \sum_{i=1}^{n} \hat{\lambda}_i \hat{u}_i \hat{u}_i^T, \] where \( |\hat{\lambda}_1| \geq \ldots \geq |\hat{\lambda}_n| \), and \( \hat{u}_i^T \hat{u}_j = 1(i = j) \) for all \( i,j \in [n] \).

Compute the adjacency spectral embedding

\[ \bar{X} = \hat{X}^{(ASE)} = [\hat{u}_1, \ldots, \hat{u}_d] \cdot \text{diag}(|\hat{\lambda}_1|^{1/2}, \ldots, |\hat{\lambda}_d|^{1/2}) \]

and write \( \bar{X} = [\bar{x}_1, \ldots, \bar{x}_n]^T \in \mathbb{R}^{n \times d} \). Denote \( \bar{p}_{ij} = \bar{x}_i^T \bar{x}_j \) for all \( i,j \in [n] \).

For \( i = 1 \) to \( n \)

Initialize \( x_i^{(1)} = \bar{x}_i \).

For \( t = 2 \) to \( B + n_{mc} \times b \)

Generate \( x_i^{(t)} \sim N \left( \bar{x}_i, \sigma^2 \hat{G}_{in}^{-1} / n \right) \cdot 1(||x_i||_2 < 1) \).

Generate \( \alpha_t \sim \text{Unif}(0,1) \).

If \( \log \alpha_t < \bar{\ell}_m(x_i^{(t)}) - \bar{\ell}_m(x_i^{(t)}) + \log \pi(x_i^{(t)}) - \log \pi(x_i^{(t)}) \)

Set \( x_i^{(t+1)} \leftarrow x_i^{(t)} \); Else

Set \( x_i^{(t+1)} \leftarrow x_i^{(t)} \).

End If

End For

Output \( X^{(B+1+b \times N)} \) for \( N = 1, 2, \ldots, \lceil (n_{mc} - 1) / b \rceil \), where \( X^{(t)} = [x_1^{(t)}, \ldots, x_n^{(t)}]^T \).
D.2 An additional simulation example

In this subsection, we provide an additional simulation example to explore the performance of the proposed estimation methods in a comparatively small sample regime with \( n = 30 \) to supplement the example in Section 5.1. The simulation setup is the same as that in Section 5.1 but we set the number of vertices to \( n = 30 \). Here, we focus on the performance of different estimates using the sum of square errors as the evaluation metric. Besides the four estimates considered in Section 5.1 of the manuscript (the adjacency spectral embedding, the one-step estimate, the maximum surrogate likelihood estimate, and the Bayes estimate with the surrogate likelihood), we also consider the maximum likelihood estimate. Note that although the theory of the maximum likelihood estimation is still open, it is always possible to find a local maximizer of the likelihood function using any optimization toolkit. Here we use the \texttt{R} built-in \texttt{optim} function in practice. We repeat the same numerical experiment for 1000 independent Monte Carlo replicates and visualize the boxplots of the sum of squares errors in Figure 3.

![Boxplot of sum of square errors for different estimates](image_url)

Figure 3: Numerical results for Section D.2 with \( n = 30 \): The boxplot of the sum of square errors for the adjacency spectral embedding (ASE), the one-step estimate (OSE), the maximum surrogate likelihood estimate (MSLE), the maximum likelihood estimate (MLE), and the Bayes estimate (GBE).

We can see that in this small sample regime, the maximum surrogate likelihood estimate and the one-step estimate do not outperform the baseline adjacency spectral embedding and the maximum likelihood estimate, while the Bayes estimate has the least sum of squares errors. This observation shows the potential advantage of the Bayesian estimation procedure based on the Markov chain Monte Carlo sampling algorithm over the classical optimization-based estimation methods for finite-sample problems in practice.

D.3 Convergence diagnostics of the Metropolis–Hastings sampler

In this subsection, we provide some convergence diagnostics of Metropolis–Hastings sampler. Specifically, we choose one realization of the simulated data in the case of the stochastic block model with \( d = 2 \) and
\( n = 2000 \) (Section 5.2 of the manuscript). The parameters of this random dot product graph are the entries of a \( 2000 \times 2 \) matrix, so we get \( 2000 \times 2 = 4000 \) Markov chains as the output of Metropolis–Hastings sampler. The total number of iterations in one Markov chain is 2000, where we discard the first 2000 as burn-in and apply a thinning of 5 to the rest, resulting in a chain of length 200. To diagnose convergence, we use \texttt{coda::heidel.diag()} in R, which uses the Cramer–von Mises statistic to test the null hypothesis that the sampled values come from a stationary distribution.

Below, Fig. 4 presents the numerical diagnostics results. From the histogram of the 4000 \( p \)-values from the output of \texttt{coda::heidel.diag()} applied to the 4000 Markov chains, we see that there are very few \( p \)-values that are less than 0.05 (only 36 among the 4000 \( p \)-values in this trial). Furthermore, with different trials of Metropolis–Hastings sampler, the specific parameters which give the small \( p \)-values are different. So we can say that the occurrence of some small \( p \)-values is very likely due to the randomness in the data and in the Metropolis–Hastings sampler. A histogram of the accept rates from the Metropolis–Hastings algorithm of the 2000 vertices is provided as well. To investigate more closely, the trace plot and auto-correlation function (ACF) plot of the second coordinate of the 808th vertex which gives a \( p \)-value smaller than 0.05 in this trial are provided. We can see that although it gives a small \( p \)-value, the trace plot and the ACF plot of the Metropolis–Hastings sample are not too abnormal.

Next, we investigate the convergence of the Metropolis–Hastings sampler in the Wikipedia graph dataset (Section 5.3 of the manuscript). For each \( d \), there are \( 1382 \times d \) parameters to estimate, so we get \( 1382 \times d \) Markov chains as the output of Metropolis–Hastings sampler. The total number of iterations in one Metropolis–Hastings sampler is \( 4000(2d + 1) \), where we discard the first half as burn-in and apply a thinning of \( 4d \), resulting in a chain of length slightly more than 1000.

For \( d = 1, \ldots, 15 \), the histograms of 1382 accept rates and of \( 1382 \times d \) \( p \)-values are provided in the upper and lower panel of Fig. 6, respectively.

To investigate more closely, the trace plots and autocorrelation function (ACF) plots of two chains which give \( p \)-values smaller than 0.05 are provided, as in Fig. 7.
Figure 4: Convergence diagnostics for the simulation example in Section 5.2 of the manuscript. Top left panel: histogram of 4000 p-values. Top right panel: histogram of 2000 accept rates. Bottom left panel: Trace plot of a parameter whose Metropolis–Hastings sample gives a p-value less than 0.05. Bottom right panel: ACF plot of a parameter whose Metropolis–Hastings sample gives a p-value less than 0.05.
Figure 5: Convergence diagnostics for the Wikipedia graph data example in Section 5.3 of the manuscript: Histograms of accept rates, where the horizontal axis represents accept rates and the vertical axis represents counts.
Figure 6: Convergence diagnostics for the Wikipedia graph data example in Section 5.3 of the manuscript. Top panel: histograms of accept rates, where the horizontal axis represents accept rates and the vertical axis represents counts. Bottom panel: Histograms of p-values, where the horizontal axis represents p-values and the vertical axis represents counts.
Figure 7: Convergence diagnostics for the Wikipedia graph data example in Section 5.3 of the manuscript. Top left panel: Trace plot of the Markov chain of the first coordinate of the 354th vertex with p-value = 0.0019, $d = 11$. Top right panel: ACF plot of the Markov chain of the first coordinate of the 354th vertex, $d = 11$. Bottom left panel: Trace plot of the Markov chain of the tenth coordinate of the 14th vertex with p-value = 0.0004, $d = 11$. Bottom right panel: ACF plot of the Markov chain of the tenth coordinate of the 14th vertex, $d = 11$. 
References

Abbe, E., Bandeira, A. S., and Hall, G. (2016). Exact recovery in the stochastic block model. *IEEE Transactions on Information Theory*, 62(1):471–487.

Abbe, E., Fan, J., Wang, K., and Zhong, Y. (2020). Entrywise eigenvector analysis of random matrices with low expected rank. *The Annals of Statistics*, 48(3):1452 – 1474.

Airoldi, E. M., Blei, D., Fienberg, S., and Xing, E. (2008). Mixed membership stochastic blockmodels. In Koller, D., Schuurmans, D., Bengio, Y., and Bottou, L., editors, *Advances in Neural Information Processing Systems*, volume 21. Curran Associates, Inc.

Athreya, A., Fishkind, D. E., Levin, K., Lyzinski, V., Park, Y., Qin, Y., Sussman, D. L., Tang, M., Vogelstein, J. T., and Priebe, C. E. (2017). Statistical inference on random dot product graphs: A survey. *Journal of Machine Learning Research*, 18.

Athreya, A., Priebe, C. E., Tang, M., Lyzinski, V., Marchette, D. J., and Sussman, D. L. (2016). A limit theorem for scaled eigenvectors of random dot product graphs. *Sankhya A*, 78(1):1–18.

Bickel, P. J. and Doksum, K. A. (2007). *Mathematical statistics: basic ideas and selected topics*. Pearson Prentice Hall.

Binkiewicz, N., Vogelstein, J. T., and Rohe, K. (2017). Covariate-assisted spectral clustering. *Biometrika*, 104(2):361–377.

Boucheron, S., Lugosi, G., and Massart, P. (2013). *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press.

Chernozhukov, V. and Hong, H. (2003). An mcim approach to classical estimation. *Journal of Econometrics*, 115(2):293–346.

Duchi, J., Hazan, E., and Singer, Y. (2011). Adaptive subgradient methods for online learning and stochastic optimization. *Journal of machine learning research*, 12(7).

Eckart, C. and Young, G. (1936). The approximation of one matrix by another of lower rank. *Psychometrika*, 1(3):211–218.

Erdős, P., Rényi, A., et al. (1960). On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5(1):17–60.

Gao, C., Lu, Y., and Zhou, H. H. (2015). Rate-optimal graphon estimation. *The Annals of Statistics*, 43(6):2624 – 2652.

Hoff, P. D., Raftery, A. E., and Handcock, M. S. (2002). Latent space approaches to social network analysis. *Journal of the American Statistical Association*, 97(460):1090–1098.
Holland, P. W., Laskey, K. B., and Leinhardt, S. (1983). Stochastic blockmodels: First steps. *Social Networks*, 5(2):109–137.

Janson, S. and Diaconis, P. (2008). Graph limits and exchangeable random graphs. *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, 28:33–61.

Kleijn, B. and van der Vaart, A. (2012). The Bernstein-Von-Mises theorem under misspecification. *Electronic Journal of Statistics*, 6(none):354 – 381.

Kosorok, M. R. (2008). *Introduction to Empirical Processes and Semiparametric Inference*. Springer New York, New York, NY.

Lacetera, N., Macis, M., and Mele, A. (2016). Viral altruism? charitable giving and social contagion in online networks. *Sociological Science*, 3(11):202–238.

Lei, J. and Rinaldo, A. (2015). Consistency of spectral clustering in stochastic block models. *The Annals of Statistics*, 43(1):215 – 237.

Li, X. and Orabona, F. (2019). On the convergence of stochastic gradient descent with adaptive stepsizes. In Chaudhuri, K. and Sugiyama, M., editors, *Proceedings of the Twenty-Second International Conference on Artificial Intelligence and Statistics*, volume 89 of *Proceedings of Machine Learning Research*, pages 983–992. PMLR.

Lyzinski, V., Sussman, D. L., Tang, M., Athreya, A., and Priebe, C. E. (2014). Perfect clustering for stochastic blockmodel graphs via adjacency spectral embedding. *Electronic Journal of Statistics*, 8(2):2905 – 2922.

Lyzinski, V., Tang, M., Athreya, A., Park, Y., and Priebe, C. E. (2017). Community detection and classification in hierarchical stochastic blockmodels. *IEEE Transactions on Network Science and Engineering*, 4(1):13–26.

Mairal, J. (2013). Stochastic majorization-minimization algorithms for large-scale optimization. In Burges, C., Bottou, L., Welling, M., Ghahramani, Z., and Weinberger, K., editors, *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc.

Mele, A. (2017). A structural model of dense network formation. *Econometrica*, 85(3):825–850.

Mele, A., Hao, L., Cape, J., and Priebe, C. E. (2022). Spectral estimation of large stochastic blockmodels with discrete nodal covariates. *Journal of Business & Economic Statistics*, accepted for publication.

Miller, J. W. (2021). Asymptotic normality, concentration, and coverage of generalized posteriors. *Journal of Machine Learning Research*, 22(168):1–53.

Rand, W. M. (1971). Objective criteria for the evaluation of clustering methods. *Journal of the American Statistical Association*, 66(336):846–850.
Robbins, H. and Monro, S. (1951). A stochastic approximation method. *The Annals of Mathematical Statistics*, 22(3):400–407.

Rohe, K., Chatterjee, S., and Yu, B. (2011). Spectral clustering and the high-dimensional stochastic blockmodel. *The Annals of Statistics*, 39(4):1878 – 1915.

Sarkar, P. and Bickel, P. J. (2015). Role of normalization in spectral clustering for stochastic blockmodels. *The Annals of Statistics*, 43(3):962 – 990.

Sengupta, S. and Chen, Y. (2018). A block model for node popularity in networks with community structure. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(2):365–386.

Sussman, D. L., Tang, M., Fishkind, D. E., and Priebe, C. E. (2012). A consistent adjacency spectral embedding for stochastic blockmodel graphs. *Journal of the American Statistical Association*, 107(499):1119–1128.

Sussman, D. L., Tang, M., and Priebe, C. E. (2014). Consistent latent position estimation and vertex classification for random dot product graphs. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 36(1):48–57.

Syring, N. and Martin, R. (2018). Calibrating general posterior credible regions. *Biometrika*, 106(2):479–486.

Syring, N. and Martin, R. (2022). Gibbs posterior concentration rates under sub-exponential type losses. *Bernoulli*, accepted for publication.

Tang, M., Athreya, A., Sussman, D. L., Lyzinski, V., Park, Y., and Priebe, C. E. (2017a). A semiparametric two-sample hypothesis testing problem for random graphs. *Journal of Computational and Graphical Statistics*, 26(2):344–354.

Tang, M., Cape, J., and Priebe, C. E. (2022). Asymptotically efficient estimators for stochastic blockmodels: The naive MLE, the rank-constrained MLE, and the spectral estimator. *Bernoulli*, 28(2):1049 – 1073.

Tang, M. and Priebe, C. E. (2018). Limit theorems for eigenvectors of the normalized laplacian for random graphs. *The Annals of Statistics*, 46(5):2360–2415.

Tang, M., Sussman, D. L., and Priebe, C. E. (2013). Universally consistent vertex classification for latent positions graphs. *The Annals of Statistics*, 41(3):1406–1430.

Tang, R., Ketcha, M., Badea, A., Calabrese, E. D., Margulies, D. S., Vogelstein, J. T., Priebe, C. E., and Sussman, D. L. (2019). Connectome smoothing via low-rank approximations. *IEEE Transactions on Medical Imaging*, 38(6):1446–1456.

Van der Vaart, A. W. (2000). *Asymptotic statistics*, volume 3. Cambridge university press.
Xie, F. (2022). Entrywise limit theorems of eigenvectors for signal-plus-noise matrix models with weak signals. *arXiv preprint:2106.09840*.

Xie, F. and Wu, D. (2022). Eigenvector-assisted statistical inference for signal-plus-noise matrix models. *arXiv preprint:2203.16688*.

Xie, F. and Xu, Y. (2020). Optimal Bayesian estimation for random dot product graphs. *Biometrika*, 107(4):875–889.

Xie, F. and Xu, Y. (2021). Efficient estimation for random dot product graphs via a one-step procedure. *Journal of the American Statistical Association*.

Young, S. J. and Scheinerman, E. R. (2007). Random dot product graph models for social networks. In Bonato, A. and Chung, F. R. K., editors, *Algorithms and Models for the Web-Graph*, pages 138–149, Berlin, Heidelberg. Springer Berlin Heidelberg.