On $L^p$-theory for parabolic and elliptic integro-differential equations with scalable operators in the whole space

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Abstract Elliptic and parabolic integro-differential model problems are considered in the whole space. By verifying Hörmander condition, the existence and uniqueness is proved in $L^p$-spaces of functions whose regularity is defined by a scalable, possibly nonsymmetric, Levy measure. Some rough probability density function estimates of the associated Levy process are used as well.

Keywords Non-local parabolic and elliptic integro-differential equations · Lévy processes

Mathematics Subject Classification 45K05 · 60J75 · 35B65

1 Introduction

Let $\sigma \in (0, 2)$ and $\mathcal{A}^\sigma$ be the class of all nonnegative measures $\pi$ on $\mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$ such that $\int |y|^2 \wedge 1 d\pi < \infty$ and

$$\sigma = \inf \left\{ \alpha < 2 : \int_{|y| \leq 1} |y|^\alpha d\pi < \infty \right\}.$$

In addition we assume that for $\pi \in \mathcal{A}^\sigma$,

$$\int_{|y| > 1} |y| d\pi < \infty \text{ if } \sigma \in (1, 2),$$
\[
\int_{R < |y| \leq R'} y d\pi = 0 \text{ if } \sigma = 1 \text{ for all } 0 < R < R' < \infty.
\]

In this paper we consider the parabolic Cauchy problem with \( \lambda \geq 0 \)
\[
\partial_t u(t, x) = Lu(t, x) - \lambda u(t, x) + f(t, x) \text{ in } E = [0, T] \times \mathbb{R}^d, \\
u(0, x) = 0,
\]
and the elliptic problem with \( \lambda > 0 \),
\[
\lambda u(x) - Lu(x) = g(x), \quad x \in \mathbb{R}^d,
\]
with \( \lambda \geq 0 \) and \( \lambda > 0 \) respectively, and integro-differential operator
\[
L \varphi(x) = L^\sigma \varphi(x) = \int [\varphi(x+y) - \varphi(x) - \chi_\sigma(y) \cdot \nabla \varphi(x)] \pi(dy), \varphi \in C_0^\infty(\mathbb{R}^d),
\]
where \( \pi \in \mathfrak{A}^\sigma, \chi_\sigma(y) = 0 \text{ if } \sigma \in [0, 1), \chi_\sigma(y) = 1_{\{|y|\leq 1\}}(y) \text{ if } \sigma = 1 \) and \( \chi_\sigma(y) = 1 \text{ if } \sigma \in (1, 2) \). The symbol of \( L \) is
\[
\psi(\xi) = \psi^\pi(\xi) = \int \left[ e^{i\xi \cdot y} - 1 - i \chi_\sigma(y) \xi \cdot y \right] \pi(dy), \xi \in \mathbb{R}^d.
\]
Note that \( \pi(dy) = dy/|y|^{d+\sigma} \in \mathfrak{A}^\sigma \) and, in this case, \( L = L^\sigma = c_0(\sigma, d)(-\Delta)^{\sigma/2} \), where \((-\Delta)^{\sigma/2}\) is a fractional Laplacian. Let \( \pi_0 \in \mathfrak{A}^\sigma \) and
\[
c_1 \left| \psi_0(\xi) \right| \leq \left| \psi^\pi(\xi) \right| \leq c_2 \left| \psi_0(\xi) \right|, \xi \in \mathbb{R}^d,
\]
for some \( 0 < c_1 \leq c_2 \). Given \( \pi_0 \in \mathfrak{A}^\sigma, p \in [1, \infty) \), we denote \( H^\pi_0(\mathbb{R}^d) \) (resp. \( \mathcal{H}^\pi_0(E) \)) the closure in \( L^p(\mathbb{R}^d) \) (resp. \( L^p(E) \)) of \( C^\infty_0(\mathbb{R}^d) \) (resp. \( C^\infty_0(E) \)) with respect to the norm
\[
|f|_{\pi_0, p} = |f|_{L^p(\mathbb{R}^d)} + |L^\pi_0 f|_{L^p(\mathbb{R}^d)}, \text{ resp. } |g|_{\pi_0, p} = |g|_{L^p(E)} + |L^\pi_0 g|_{L^p(E)}.
\]
If \( f_n \in C^\infty_0(\mathbb{R}^d), f_n \to f \) and \( L^\pi f_n \to g \) in \( L^p(\mathbb{R}^d) \) we denote \( g = L^\pi f \).

In this note, under certain “scalability” assumptions (see Assumption D(\( \kappa, l \)) below), we prove the existence and uniqueness of (1.1) and (1.2) in \( \mathcal{H}^\pi_0(E) \) (resp. \( H^\pi_0(\mathbb{R}^d) \)). Moreover the following estimates hold:
\[
|u|_{\mathcal{H}^\pi_0} \leq C |f|_{L^p(E)} , \quad |u|_{H^\pi_0} \leq C |f|_{L^p(\mathbb{R}^d)} \cdot
\]
The symbol \( \psi^\pi(\xi) \) is not smooth in \( \xi \) and the standard Fourier multiplier results do not apply in this case. In order to prove (1.4), we associate to \( L^\pi \) a family of balls and verify Hörmander condition (see Theorem 5 and (4.37) below) for it, and apply
Calderon–Zygmund theorem. As an example, we consider $\pi \in \mathcal{A}^\sigma$ defined in radial and angular coordinates $r = |y|$, $w = y/r$, as

$$
\pi (\Gamma) = \int_0^\infty \int_{|w|=1} \chi_\Gamma (rw) a (r, w) j (r) r^{d-1} S (dw) \, dr, \quad \Gamma \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
$$

(1.5)

where $S (dw)$ is a finite measure on the unit sphere on $\mathbb{R}^d$. In [10], the parabolic equation (1.1) was considered, with $\pi$ in the form (1.5) with $a = 1$, $j (r) = r^{-d-\sigma}$, and such that

$$
\int_0^\infty \int_{|w|=1} \chi_\Gamma (rw) r^{-1-\sigma} \rho_0 (w) S (dw) \, dr \leq \pi (\Gamma) = \int_0^\infty \int_{|w|=1} \chi_\Gamma (rw) r^{-1-\sigma} a (r, w) S (dw) \, dr
$$

$$
\leq \int_0^\infty \int_{|w|=1} \chi_\Gamma (rw) r^{-1-\sigma} S (dw) \, dr, \quad \pi \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
$$

and (1.3) holds with $\psi^{(\sigma)} (t) = |\xi|^\sigma$, $\xi \in \mathbb{R}^d$. In this case, $\mathcal{H}_{p}^{\sigma} (E) = \mathcal{H}^{\sigma}_{p} (E)$ is the fractional Sobolev space. The solution estimate (1.4) for (1.1) was derived in [10], using $L^\infty - \text{BMO}$ type estimate. In [4], the elliptic problem (1.2) was studied for $\pi$ in the form (1.5) with $S (dw) = dw$ being a Lebesgue measure on the unit sphere in $\mathbb{R}^d$, with $0 < c_1 \leq a \leq c_2$, and a set of technical assumptions on $j (r)$. The inequality (1.4) for (1.2) was obtained using sharp function estimate based on the solution Hölder norm estimate (following the idea in [1]), where (1.2) was considered in $\mathcal{H}^{\sigma} (\mathbb{R}^d)$ with $\pi$ as in (1.5) with $j (r) = r^{-d-\sigma}$ and $0 < c_1 \leq a \leq c_2$.

The note is organized as follows. In Sect. 2, the main theorem is stated, and an example of the form (1.5) considered. In Sect. 3, the essential technical results are presented. The main theorem is proved in Sect. 4.

2 Notation and main results

Denote $E = [0, T] \times \mathbb{R}^d$ with $T < \infty$, $\mathcal{N} = \{0, 1, 2, \ldots \}$, $\mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$. If $x, y \in \mathbb{R}^d$, we write

$$
x \cdot y = \sum_{i=1}^d x_i y_i, \quad |x| = (x \cdot x)^{1/2}.
$$

For a function $u (t, x)$ on $E$, we denote its partial derivatives by $\partial_t u (t, x) = \partial u/\partial t, \partial_i u = \partial u/\partial x_i$, and $D^\gamma u = \partial^{\gamma_1} u/\partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}$, where multiindex $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$, $\nabla u = (\partial_1 u, \ldots, \partial_d u)$ denotes the gradient of $u$ with respect to $x$. For $k \in \mathcal{N}$, we denote $D^k u = (\partial^\gamma u)_{|\gamma|=k}$.

Let $L_p (T) = L_p (E)$ is the space of $p -$integrable functions with norm, $p \geq 1$,

$$
|f|_{L_p (E)} = \left( \int_0^T \int |f (t, x)|^p \, dx \, dt \right)^{1/p}, \quad |f|_{L_\infty (E)} = \text{ess sup}_{(t, x) \in E} |f (t, x)|.
$$
Similar space of functions on \( \mathbb{R}^d \) is denoted by \( L_p(\mathbb{R}^d) \).

Let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space of smooth real valued rapidly decreasing functions. For \( s \in \mathbb{N} \), we define the Sobolev space \( H^s_p(\mathbb{R}^d) \) (resp. \( H^s_p(E) \)) as closure of \( C_0^\infty(\mathbb{R}^d) \) (resp. \( C_0^\infty(E) \)) with respect to the norm

\[
|f|_{n,p} = \sum_{|\beta| \leq n} |D^\beta f|_{L_p(\mathbb{R}^d)}, \quad \text{resp.} \quad |g|_{n,p} = \sum_{|\beta| \leq n} |D^\beta g|_{L_p(E)}.
\]

For \( s \in (0, 2) \) and \( v \in C_0^\infty(\mathbb{R}^d) \), we define the fractional Laplacian

\[
\partial^s v(x) = \int \nabla^s_y v(x) \frac{dy}{|y|^{d+s}}, x \in \mathbb{R}^d,
\]

where

\[
\nabla^s_y v(x) = v(x + y) - v(x) - (\nabla v(x), y) \chi_{\sigma}(y)
\]

with \( \chi_{\sigma}(y) = 1_{|y| \leq 1}1_{\sigma=1} + 1_{|\sigma \in (1,2)} \) is the integrand in the definition of \( L^\pi \).

Given \( \pi_0 \in \mathfrak{A}, \ p \in [1, \infty) \), we denote \( H^{\pi_0}_p(\mathbb{R}^d) \) (resp. \( H^{\pi_0}_p(E) \)) the closure in \( L_p(\mathbb{R}^d) \) (resp. \( L_p(E) \)) of \( C_0^\infty(\mathbb{R}^d) \) (resp. \( C_0^\infty(E) \)) with respect to the norm

\[
|f|_{\pi_0,p} = |f|_{L_p(\mathbb{R}^d)} + |L^{\pi_0} f|_{L_p(\mathbb{R}^d)}, \quad \text{resp.} \quad |g|_{\pi_0,p} = |g|_{L_p(E)} + |L^{\pi_0} g|_{L_p(E)}.
\]

If \( f_n \in C_0^\infty(\mathbb{R}^d), f_n \to f \) and \( L^\pi f_n \to g \) in \( L_p(\mathbb{R}^d) \) we denote \( g = L^\pi f \). Notice that \( f_n \to 0, \ L^\pi f_n \to h \) in \( L_p(\mathbb{R}^d) \) implies that \( h = 0 \). Indeed,

\[
\int \varphi L^\pi f_n = \int f_n L^{\pi^*} \varphi \to 0, \varphi \in C_0^\infty(\mathbb{R}^d),
\]

where \( \pi^*(\Gamma) = \pi(-\Gamma), \ \Gamma \in \mathcal{B}(\mathbb{R}^d) \), i.e. \( \pi^* \in \mathfrak{A} \) as well. Note that if \( \pi \in \mathfrak{A} \), then for any \( f \in C_0^\infty(\mathbb{R}^d) \),

\[
|L^\pi f|_{L_p(\mathbb{R}^d)} \leq \left| \int_{|y| \leq 1} \cdots \right|_{L_p(\mathbb{R}^d)} + \left| \int_{|y| > 1} \cdots \right|_{L_p(\mathbb{R}^d)} \leq C |f|_{2,p},
\]

that is \( H^2_p(\mathbb{R}^d) \subseteq H^\pi_p(\mathbb{R}^d) \) and the embedding is continuous. The same holds for \( H^2_p(E) \subseteq H^\pi_p(E) \).

We denote \( \mathfrak{A} = \cup_{\sigma \in (0, 2)} \mathfrak{A}^\sigma \).

We denote Fourier transform and its inverse

\[
\mathcal{F} v(\xi) = \hat{v}(\xi) = \int v(x) e^{-i2\pi x \cdot \xi} dx, \xi \in \mathbb{R}^d,
\]

\[
\mathcal{F}^{-1} v(x) = \int v(\xi) e^{i2\pi x \cdot \xi} d\xi, x \in \mathbb{R}^d, v \in \mathcal{S}(\mathbb{R}^d).
\]
We denote $C^\infty_b (E)$ the space of bounded infinitely differentiable in $x$ functions whose derivatives are bounded.

$C = C (\cdot, \ldots, \cdot)$ denotes constants depending only on quantities appearing in parentheses. In a given context the same letter is (generally) used to denote different constants depending on the same set of arguments.

We also introduce an auxiliary Levy measure $\mu_0$ on $\mathbb{R}^d_0$ such that the following assumption holds.

**Assumption A_0 (σ).** Let $\mu^0 \in \mathfrak{A}, \chi_{\{|y| \leq 1\}} \mu^0 (dy) = \mu^0 (dy)$, and

$$
\int |y| \mu^0 (dy) + \int |\xi|^2 [1 + \xi (\xi)]^{d+3} \exp \{-\phi_0 (\xi)\} d\xi \leq N_0 \text{ if } \sigma \in (0, 1),
$$

$$
\int |y|^2 \mu^0 (dy) + \int |\xi|^4 [1 + \xi (\xi)]^{d+3} \exp \{-\phi_0 (\xi)\} d\xi \leq N_0 \text{ if } \sigma \in [1, 2),
$$

where

$$
\phi_0 (\xi) = \int_{|y| \leq 1} [1 - \cos (2\pi \xi \cdot y)] \mu^0 (dy),
$$

$$
\xi (\xi) = \int_{|y| \leq 1} \chi_\sigma (|y| [\{|y| \leq 1\} \wedge 1]) \mu^0 (dy), \xi \in \mathbb{R}^d.
$$

In addition, we assume that for any $\xi \in S_{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$,

$$
\int_{|y| \leq 1} |\xi \cdot y|^2 \mu^0 (dy) \geq c_1 > 0.
$$

For $\pi \in \mathfrak{A} = \bigcup_{\sigma \in (0, 2)} \mathfrak{A}_\sigma$ and $R > 0$, we denote

$$
\pi_R (\Gamma) = \int \chi_\Gamma (y/R) \pi (dy), \Gamma \in \mathcal{B} (\mathbb{R}^d_0).
$$

**Definition 1** We say that a continuous function $\kappa : (0, \infty) \to (0, \infty)$ is a scaling function if $\lim_{R \to 0} \kappa (R) = 0$, $\lim_{R \to \infty} \kappa (R) = \infty$ and there is a nondecreasing continuous function $l (\epsilon) , \epsilon > 0$, such that $\lim_{\epsilon \to 0} l (\epsilon) = 0$ and

$$
\kappa (\epsilon r) \leq l (\epsilon) \kappa (r), r > 0, \epsilon > 0.
$$

We call $l (\epsilon) , \epsilon > 0$, a scaling factor of $\kappa$.

For a scaling function $\kappa$ with a scaling factor $l$ and $\pi \in \mathfrak{A}_\sigma$ we introduce the following

**Assumption D(κ, l).** (i) For every $R > 0$,

$$
\tilde{\pi}_R (dy) = \kappa (R) \pi_R (dy) \geq 1_{\{|y| \leq 1\}} \mu^0 (dy),
$$

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with \( \mu^0 = \mu^{0;\pi} \) satisfying Assumption A\(_0\) (\( \sigma \)). If \( \sigma = 1 \) we, in addition assume that
\[
\int_{R < |y| \leq R'} \gamma \mu^0 (dy) = 0 \text{ for any } 0 < R < R' \leq 1. \text{ Here } \tilde{\pi}_R (dy) = \kappa (R) \pi_R (dy).
\]

(ii) There exist \( \alpha_1 \) and \( \alpha_2 \) and a constant \( N > 0 \) such that
\[
\int_{|z| \leq 1} |z|^{\alpha_1} \tilde{\pi}_R (dz) + \int_{|z| > 1} |z|^{\alpha_2} \tilde{\pi}_R (dz) \leq N \forall R > 0,
\]
where \( \alpha_1, \alpha_2 \in (0, 1) \) if \( \sigma \in (0, 1); \) \( \alpha_1, \alpha_2 \in (1, 2) \) if \( \sigma \in (1, 2); \) \( \alpha_1 \in (1, 2) \) and \( \alpha_2 \in [0, 1) \) if \( \sigma = 1 \).

(iii) Let \( \gamma (t) = \inf (s > 0 : l (s) > t), t > 0. \) With \( I_1 = \{ t > 0 : \gamma (t) \leq 1 \} \), \( I_2 = \{ t > 0 : \gamma (t) > 1 \} \) we have
\[
\int_{I_1} [t \gamma (t)^{-\alpha_1} + 1_{\sigma \in (1, 2)} \gamma (t)^{-1}] dt \leq N_1 < \infty,
\]
and
\[
\int_{I_2} [\gamma (t)^{-(1 + \alpha_2)} + \gamma (t)^{-2 \alpha_2}] dt \leq N_1 < \infty.
\]

The main result of this paper for (1.2) is

**Theorem 1** Let \( p \in (1, \infty), \pi_0, \pi \in \mathfrak{A}^{\sigma}, \lambda > 0. \) Assume there is a scaling function \( \kappa \) such that \( D(\kappa, l) \) hold for both, \( \pi \) and \( \pi_0. \)

Then for each \( f \in L_p (\mathbb{R}^d) \) there is a unique \( u \in H^\pi_{\pi_0} (\mathbb{R}^d) \) solving (1.2). Moreover, there is \( C = C (d, p, \kappa, l, N_0, N, N_1, c_1) \) such that
\[
|L^{\pi_0} u|_{L_p (\mathbb{R}^d)} \leq C |f|_{L_p (\mathbb{R}^d)},
\]
\[
|u|_{L_p (\mathbb{R}^d)} \leq \frac{1}{\lambda} |f|_{L_p (\mathbb{R}^d)}.
\]

The main result for (1.1) is

**Theorem 2** Let \( p \in (1, \infty), \pi_0, \pi \in \mathfrak{A}^{\sigma}. \) Assume there is a scaling function \( \kappa \) such that \( D(\kappa, l) \) hold for both, \( \pi \) and \( \pi_0. \)

Then for each \( f \in L_p (E) \) there is a unique \( u \in H^\pi_{\pi_0} (E) \) solving (1.1). Moreover, there is \( C = C (d, p, \kappa, l, N_0, N, N_1, c_1) \) such that
\[
|L^{\pi_0} u|_{L_p (E)} \leq C |f|_{L_p (E)},
\]
\[
|u|_{L_p (E)} \leq \left( \frac{1}{\lambda} \wedge T \right) |f|_{L_p (E)}.
\]

**Remark 1** Assumption \( D(\kappa, l) \) holds for both, \( \pi, \pi_0, \) means that \( \kappa, l, \) and the parameters \( \alpha_1, \alpha_2, N, N_1, N_0, c_1 \) are the same.
2.1 Example

Let \( \mu (dt) \) be a measure on \((0, \infty)\) such that \( \int_0^\infty (1 \wedge t) \mu (dt) < \infty \), and let

\[
\phi (r) = \int_0^\infty (1 - e^{-r^t}) \mu (dt), \quad r \geq 0,
\]
be a Bernstein function (see \([4,5]\)). Let

\[
j (r) = \int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp \left( -\frac{r^2}{4t} \right) \mu (dt), \quad r > 0.
\]

We consider \( \pi \in \mathcal{A} = \cup_{\sigma \in (0,2)} \mathcal{A}^\sigma \) defined in radial and angular coordinates \( r = |y| \), \( w = y/r \), as

\[
\pi (\Gamma) = \int_0^\infty \int_{|w|=1} \chi_{\Gamma} (r w) a (r, w) j (r) r^{d-1} S (d w) dr, \quad \Gamma \in \mathcal{B}\left( \mathbb{R}^d_0 \right),
\]

where \( S (d w) \) is a finite measure on the unite sphere on \( \mathbb{R}^d \). If \( S (d w) = d w \) is the
Lebesgue measure on the unit sphere, then

\[
\pi (\Gamma) = \pi^{J,a} (\Gamma) = \int_{\mathbb{R}^d} \chi_{\Gamma} (y) a (|y|, y/|y|) J (y) dy, \quad \Gamma \in \mathcal{B}\left( \mathbb{R}^d_0 \right),
\]

where \( J (y) = j (|y|) \), \( y \in \mathbb{R}^d \). Let \( \pi_0 = \pi^{J,1} \), i.e.,

\[
\pi_0 (\Gamma) = \int_{\mathbb{R}^d} \chi_{\Gamma} (y) J (y) dy, \quad \Gamma \in \mathcal{B}\left( \mathbb{R}^d_0 \right).
\]

We assume

**H.** (i) There is \( N > 0 \) so that

\[
N^{-1} \phi \left( r^{-2} \right) r^{-d} \leq j (r) \leq N \phi \left( r^{-2} \right) r^{-d}, \quad r > 0.
\]

(ii) There are \( 0 < \delta_1 \leq \delta_2 < 1 \) and \( N > 0 \) so that for \( 0 < r \leq R \)

\[
N^{-1} \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\phi (R)}{\phi (r)} \leq N \left( \frac{R}{r} \right)^{\delta_2}.
\]

**G.** There is \( \rho_0 (w) \geq 0 \), \( |w| = 1 \), such that \( \rho_0 (w) \leq a (r, w) \leq 1, \quad r > 0, \quad |w| = 1 \), and for all \( |\xi| = 1 \),

\[
\int_{|w|=1} |\xi \cdot w|^2 \rho_0 (w) S (d w) \geq c > 0
\]
for some $c > 0$.

For example, in [4,5] among others the following specific Bernstein functions satisfying $H$ are listed:

1. $\phi(r) = \sum_{i=1}^{n} r^{\alpha_i}$, $\alpha_i \in (0, 1)$, $i = 1, \ldots, n$;
2. $\phi(r) = (r + r^{\alpha})^{\beta}$, $\alpha, \beta \in (0, 1)$;
3. $\phi(r) = r^{\alpha} \ln(1 + r)^{\beta}$, $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha)$;
4. $\phi(r) = \left[\ln(\cosh \sqrt{r})\right]^{\alpha}$, $\alpha \in (0, 1)$.

The following statement holds.

**Remark 2** Let $\pi, \pi_0$ be given by (2.1) and (2.2). Assume $H$ and $G$ hold.

(a) If $2\delta_1 > 1$, then Theorems 2 (resp. 1) hold in $H_{\pi_0}^0(E)$ (resp. $H_{\pi_0}^0(R^d)$).
(b) If $2\delta_2 < 1$ and $2\delta_1 > \delta_2$, then Theorems 2 (resp. 1) hold in $H_{\pi_0}^0(E)$ (resp. $H_{\pi_0}^0(R^d)$).

**Proof** We verify that the assumptions of Theorems 2 and 1 hold. Indeed, $H$ implies that there are $0 < c \leq C$ so that

$$cr^{-d-2\delta_1} \leq j(r) \leq Cr^{-d-2\delta_2}, r \leq 1,$$

$$cr^{-d-2\delta_2} \leq j(r) \leq Cr^{-d-2\delta_1}, r > 1.$$

Hence $2\delta_1 \leq \sigma \leq 2\delta_2$. In this case $\kappa(R) = j(R)^{-1} R^{-d}, R > 0$, is a scaling function: $\kappa(\varepsilon R) \leq l(\varepsilon) \kappa(R), \varepsilon, R > 0$, with

$$l(\varepsilon) = \begin{cases} C_1 \varepsilon^{2\delta_1} & \text{if } \varepsilon \leq 1, \\ C_1 \varepsilon^{2\delta_2} & \text{if } \varepsilon > 1 \end{cases}$$

for some $C_1 > 0$. Hence

$$\gamma(t) = l^{-1}(t) = \begin{cases} C_1^{-1/2\delta_1} t^{1/2\delta_1} & \text{if } t \leq C_1, \\ C_1^{-1/2\delta_2} t^{1/2\delta_2} & \text{if } t > C_1. \end{cases}$$

We see easily that $\alpha_1$ is any number $> 2\delta_2$ and $\alpha_2$ is any number $< 2\delta_1$. The measure $\mu^0$ for $\pi$ is

$$\mu^0(dy) = \mu^{0,\pi}(dy) = c_1 \int \chi_{dy}(rw) \chi_{(r \leq 1)} r^{-1-2\delta_1} \rho_0(w) S(dw) dr;$$

and $\mu^0$ for $\pi_0$ is

$$\mu^0(dy) = \mu^{0,\pi_0}(dy) = c'_1 \int \chi_{dy}(rw) \chi_{(r \leq 1)} r^{-1-2\delta_1} dw dr$$

with some $c_1, c'_1$. Integrability conditions D(k, l)(iii) easily follow from a) or b).
3 Auxiliary results

In this section we present some auxiliary results.

3.1 Some $L_p$ estimates

We start with the following observation.

**Remark 3** If $\pi \in A^\sigma$, then for any $f \in C^\infty_0 (\mathbb{R}^d)$,

$$|L^\pi f|_{L^p(\mathbb{R}^d)} \leq \left| \int_{|y| \leq 1} \cdots \right|_{L^p(\mathbb{R}^d)} + \left| \int_{|y| > 1} \cdots \right|_{L^p(\mathbb{R}^d)} \leq C |f|_{2,p}.$$ 

Hence $H^2_p (\mathbb{R}^d) \subseteq H^\pi_p (\mathbb{R}^d)$ and the embedding is continuous. The same holds for $H^2_p (E) \subseteq H^\pi_p (E)$.

We will use the following equality for Sobolev norm estimates.

**Lemma 1** (Lemma 2.1 in [6]) For $\alpha \in (0,1)$ and $u \in S(\mathbb{R}^d)$,

$$u(x + y) - u(x) = C \int k^{(\alpha)}(y,z) \partial^\alpha u(x - z) dz,$$

where the constant $C = C(\alpha, d)$ and

$$k^{(\alpha)}(z, y) = |z + y|^{-d+\alpha} - |z|^{-d+\alpha},$$

$$\partial^\alpha u(x) = \int [u(x + y) - u(x)] \frac{dy}{|y|^{d+\alpha}}, x \in \mathbb{R}^d.$$ 

Moreover, there is a constant $C = C(\alpha, d)$ such that for each $y \in \mathbb{R}^d$

$$\int |k^{(\alpha)}(z, y)| dz \leq C |y|^\alpha.$$ 

**Corollary 1** Let $\alpha \in (0,1]$, $p \geq 1$. Then (denoting $\partial^1 = \nabla$),

(i) for $y \in \mathbb{R}^d$,

$$|\partial^\alpha u|_{L^p(\mathbb{R}^d)} \leq C |u|_{H^1_p(\mathbb{R}^d)}, u \in S \left( \mathbb{R}^d \right);$$

$$|u(y + y) - u|_{L^p(\mathbb{R}^d)} \leq C |\partial^\alpha u|_{L^p(\mathbb{R}^d)} |y|^\alpha,$$

$$|u(y + y) - u - y \cdot \nabla u|_{L^p(\mathbb{R}^d)} \leq C |\partial^\alpha \nabla u|_{L^p(\mathbb{R}^d)} |y|^{1+\alpha},$$

$$u \in S \left( \mathbb{R}^d \right).$$

(ii) for any $\varepsilon > 0$,

$$\partial^\alpha [u(\varepsilon \cdot)] = \varepsilon^\alpha (\partial^\alpha u)(\varepsilon x), \partial^\alpha \nabla [u(\varepsilon \cdot)] = \varepsilon^{1+\alpha} (\partial^\alpha \nabla u)(\varepsilon x), x \in \mathbb{R}^d.$$
Proof Let $u \in \mathcal{S}(\mathbb{R}^d)$. Obviously,

$$u(x+y) - u(x) = \int_0^1 \nabla u(x+s y) \cdot y \, ds, \quad x, y \in \mathbb{R}^d. \quad (3.5)$$

Since for $x \in \mathbb{R}^d$,

$$|\partial^\alpha u(x)| \leq \int_{|y| \leq 1} \int_0^1 |y| \, |\nabla u(x+s y)| \, ds \, \frac{dy}{|y|^{d+\alpha}} + \int_{|y| > 1} (|u(x+y)| + |u(x)|) \, \frac{dy}{|y|^{d+\alpha}}$$

(3.2) follows. Applying generalized Minkowski inequality to (3.1) and (3.5), we derive easily (3.3). Similarly, using

$$u(x+y) - u(x) - y \cdot \nabla u(x) = \int_0^1 y \cdot [\nabla u(x+s y) - \nabla u(x)] \, ds$$

and (3.3) we derive (3.4).

Changing the variable of integration, for $\alpha \in (0, 1)$ we have

$$\partial^\alpha [u(\varepsilon \cdot)](x) = \varepsilon^\alpha \int [u(\varepsilon x + y) - u(\varepsilon x)] \, \frac{dy}{|y|^{d+\alpha}} = \varepsilon^\alpha (\partial^\alpha u)(\varepsilon x), \quad x \in \mathbb{R}^d. \quad \square$$

Corollary 2 Let $\pi \in \mathfrak{K}^\sigma$ and

$$\int_{|z| \leq 1} |z|^{\alpha_1} \pi(dz) + \int_{|z| > 1} |z|^{\alpha_2} \pi(dz) \leq N,$$

where $\alpha_1, \alpha_2 \in (0, 1)$ if $\sigma \in (0, 1)$; $\alpha_1, \alpha_2 \in (1, 2)$ if $\sigma \in (1, 2)$; $\alpha_1 \in (1, 2)$ and $\alpha_2 \in (0, 1)$ if $\sigma = 1$.

Then there is a constant $C = C(d, N)$ such that for any $v \in \mathcal{S}(\mathbb{R}^d)$, (denoting $\partial^\gamma = \nabla$ if $\gamma = 1$),

$$|L^\pi v|_{L_1} \leq C \left( |\partial^{\alpha_1} v|_{L_1} + |v|_{L_1} \right),$$

$$|L^\pi v|_{L_1} \leq C \left( |\nabla v|_{L_1} + |\partial^{\alpha_2} v|_{L_1} \right),$$

if $\sigma \in (0, 1)$;

$$|L^\pi v|_{L_1} \leq C \left( |\partial^{\alpha_1-1} \nabla v|_{L_1} + |v|_{L_1} \right),$$

$$|L^\pi v|_{L_1} \leq C \left( |D^2 v|_{L_1} + |\partial^{\alpha_2} v|_{L_1} \right),$$

if $\sigma \in (1, 2)$.
if $\sigma = 1$:

$$
|L^\pi v|_{L_1} \leq C \left( \left| \partial^{\alpha_1-1} v \right|_{L_1} + |\nabla v|_{L_1} \right),
$$

$$
|L^\pi v|_{L_1} \leq C \left( \left| D^2 v \right|_{L_1} + \left| \partial^{\alpha_2-1} v \right|_{L_1} \right)
$$

if $\sigma \in (1, 2)$.

Proof By Corollary 1, there is a constant $C = C(d)$ such that for $|y| \leq 1$

$$
|v (\cdot + y) - v - \chi_\sigma (y) y \cdot \nabla v|_{L_1} \leq C \left\{ \begin{array}{ll}
|\partial^{\alpha_1} v|_{L_1} |y|^{\alpha_1} & \text{if } \sigma \in (0, 1), \\
|\partial^{\alpha_1-1} v|_{L_1} |y|^{\alpha_1} & \text{if } \sigma \in [1, 2)
\end{array} \right.
$$

and for $|y| > 1$,

$$
|v (\cdot + y) - v - \chi_\sigma (y) y \cdot \nabla v|_{L_1} \leq C \left\{ \begin{array}{ll}
2 |v|_{L_1} & \text{if } \sigma \in (0, 1), \\
2 |\nabla v|_{L_1} |y|^{\alpha_2} & \text{if } \sigma \in (1, 2).
\end{array} \right.
$$

On the other hand, for $|y| \leq 1$,

$$
|v (\cdot + y) - v - \chi_\sigma (y) y \cdot \nabla v|_{L_1} \leq C \left\{ \begin{array}{ll}
|\nabla v|_{L_1} |y|^{\alpha_1} & \text{if } \sigma \in (0, 1), \\
|D^2 v|_{L_1} |y|^{\alpha_1} & \text{if } \sigma \in [1, 2),
\end{array} \right.
$$

and for $|y| > 1$,

$$
|v (\cdot + y) - v - \chi_\sigma (y) y \cdot \nabla v|_{L_1} \leq C \left\{ \begin{array}{ll}
|\partial^{\alpha_2} v|_{L_1} |y|^{\alpha_2} & \text{if } \sigma \in (0, 1), \\
|\partial^{\alpha_2-1} v|_{L_1} |y|^{\alpha_2} & \text{if } \sigma \in (1, 2).
\end{array} \right.
$$

The statement follows. \hfill \Box

In addition, the following holds.

**Lemma 2** For any $\beta \in [0, 1], a \geq 0, |z| \leq 1$ and $u \in S (\mathbb{R}^d)$,

$$
\int_{|x| \geq a} |u(x + z) - u(x)| \, dx \\
\leq 2^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |u(x)| \, dx \right)^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |\nabla u(x)| \, dx \right)^{\beta} |z|^{\beta},
$$

Proof Let $u \in S (\mathbb{R}^d)$. For $\beta \in [0, 1], x, z \in \mathbb{R}^d$,

$$
|u(x + z) - u(x)| \leq |u(x + z) - u(x)|^{1-\beta} \left( \int_0^1 |\nabla u(x + sz)| \, ds \right)^{\beta} |z|^{\beta}.
$$
By Hölder inequality, for $|z| \leq 1$,
\[
\int_{|x| \geq a} |u(x + z) - u(x)| \, dx \\
\leq \int_{|x| \geq a} |u(x + z) - u(x)|^{1-\beta} \left( \int_0^1 |\nabla u(x + sz)| \, ds \right)^{\beta} \, dx |z|^\beta \\
\leq \left( \int_{|x| \geq a} |u(x + z) - u(x)| \, dx \right)^{1-\beta} \left( \int_0^1 \int_{|x| \geq a} |\nabla u(x + sz)| \, dx \, ds \right)^{\beta} |z|^\beta \\
\leq \left( 2 \int_{|x| \geq (a-1) \vee 0} |u(x)| \, dx \right)^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |\nabla u(x)| \, dx \right)^{\beta} |z|^\beta.
\]
\[
\int_{|x| \geq a} |u(x + z) - u(x) - z \cdot \nabla u(x)| \, dx \\
\leq 2^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |\nabla u(x)| \, dx \right)^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |D^2 u(x)| \, dx \right)^{\beta} |z|^{1+\beta}.
\]

**Corollary 3** For any $\beta \in [0, 1]$, $a \geq 0$, $|z| \leq 1$ and $u \in S(\mathbb{R}^d)$,
\[
\int_{|x| \geq a} |u(x + z) - u(x) - z \cdot \nabla u(x)| \, dx \\
\leq 2^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |\nabla u(x)| \, dx \right)^{1-\beta} \left( \int_{|x| \geq (a-1) \vee 0} |D^2 u(x)| \, dx \right)^{\beta} |z|^{1+\beta}.
\]

**Proof** For $\beta \in [0, 1]$, $x, z \in \mathbb{R}^d$, $|z| \leq 1$, and
\[
|u(x + z) - u(x) - z \cdot \nabla u(x)| \leq \int_0^1 |\nabla u(x + sz) - \nabla u(x)| \, ds |z|
\]
and the claim follows by Lemma 2. \qed

### 3.2 Density estimates

We start with the following simple statement about the existence of a probability density function (pdf).

**Lemma 3** Let $\mu^0$ be a nonnegative measure on $\mathbb{R}_0^d$ such that $\chi_{|y| \leq 1}\mu^0(dy) = \mu^0(dy)$ and
\[
\int |y| \, d\mu^0 \leq K_0 \text{ if } \sigma \in (0, 1), \\
\int |y|^2 \, d\mu^0 \leq K_0 \text{ if } \sigma \in [1, 2).
\]

Let $\eta$ be a r.v. such that
\[
E e^{i2\pi \xi \cdot \eta} = \exp \{ \psi_0(\xi) \}, \xi \in \mathbb{R}^d, \tag{3.6}
\]

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where
\[ \psi_0(\xi) = \int \left[ e^{i2\pi \xi \cdot y} - 1 - \chi_\sigma(y) i2\pi \xi \cdot y \right] \mu^0(dy), \xi \in \mathbb{R}^d. \]

Assume \( n \geq 0 \) and
\[
\int |\xi|^n [1 + \chi_\sigma(y) |y|] d\xi \leq K_0,
\]
where \( \phi_0(\xi) = \text{Re} \psi_0(\xi), \xi \in \mathbb{R}^d \) and
\[
\chi_\sigma(y) |y| [(|\xi| |y|) \wedge 1] \mu^0(dy), \xi \in \mathbb{R}^d.
\]

Then \( \eta \) has a pdf \( p_0(x), x \in \mathbb{R}^d \), such that
\[
\sup_x |\partial^\beta p_0(x)| + \int (1 + |x|^2) |\partial^\beta p_0(x)| dx \leq C \forall |\beta| \leq n
\]
for some \( C = C(d, K_0) \).

Proof By Proposition 2.5, Chapter I in \[8\], \( \eta \) has a continuous bounded density
\[
p_0(x) = \int e^{-i2\pi x \cdot \xi} \exp \{\psi_0(\xi)\} d\xi
\]
if
\[
\int \exp \{\phi_0(\xi)\} d\xi < \infty.
\]

The assumption (3.7) implies that for any multiindex \( |\beta| \leq n \),
\[
\partial^\beta p_0(x) = \int e^{-i2\pi x \cdot \xi} (-i2\pi \xi)^\beta \exp \{\psi_0(\xi)\} d\xi, x \in \mathbb{R}^d,
\]
is a bounded continuous function. The function \( (1 + |x|^2) \partial^\beta p_0 \) is integrable if
\[
( -i2\pi x_j )^{d+1} (-i2\pi x_k) \partial^\beta p_0(x)
\]
\[
= \int \partial_{\xi_j}^{d+1} \partial_{\xi_k}^2 [e^{-i2\pi x \cdot \xi}] (-i2\pi \xi)^\beta \exp \{\psi_0(\xi)\} d\xi
\]
\[
= (-1)^{d+3} \int e^{-i2\pi x \cdot \xi} \partial_{\xi_j}^{d+1} \partial_{\xi_k}^2 [(-i2\pi \xi)^\beta \exp \{\psi_0(\xi)\}] d\xi
\]
is bounded for all \( j, k \). Since \( \partial^{\mu} \psi_0(\xi) \) is bounded for \( |\mu| \geq 2 \) and
\[
|\nabla \psi_0(\xi)| \leq C (1 + \chi_\sigma(\xi)), \xi \in \mathbb{R}^d,
\]
the boundedness of (3.9) follows from assumption (3.7). Therefore \( p_0(x) \) has \( n \) bounded continuous derivatives and for any multiindex \( |\beta| \leq n \),

\[
\int \left( 1 + |x|^2 \right) \left| \partial^\beta p_0(x) \right| \, dx \leq C
\]

with \( C = C(d, K_0) \).

We will need the following tail estimate.

**Lemma 4** Let \( \pi \in \mathfrak{A}^\sigma \). Assume

\[
\int_{|z| \leq 1} |z|^{\alpha_1} \, \pi(dz) + \int_{|z| > 1} |z|^{\alpha_2} \, \pi(dz) \leq N,
\]

where \( \alpha_1, \alpha_2 \in (0, 1) \) if \( \sigma \in (0, 1) \); \( \alpha_1, \alpha_2 \in (1, 2) \) if \( \sigma \in (1, 2) \); \( \alpha_1 \in (1, 2) \) and \( \alpha_2 \in [0, 1) \) if \( \sigma = 1 \). Let \( \zeta_t \) be the Levy process such that

\[
\mathbf{E} e^{i2\pi \xi \cdot \zeta_t} = \exp\{\psi(\xi) \, t\}, \; t \geq 0,
\]

with

\[
\psi(\xi) = \int \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\sigma(y) \, y \cdot \xi \right] \, d(\pi - \bar{\pi}), \; \xi \in \mathbb{R}^d,
\]

where \( \bar{\pi} \) is a measure on \( \mathbb{R}^d \setminus \{0\} \) and \( 0 \leq \bar{\pi} \leq \pi \). Let \( t > 0 \) and \( \mathcal{L}_t(dy) \) be the distribution measure of \( \zeta_t \) on \( \mathbb{R}^d \). Then for each \( \delta > 0 \) there is a constant \( C = C(\delta, N) \) such that

\[
\mathcal{L}_t(|y| > \delta) \leq Ct.
\]

**Proof** Let \( \Pi = \pi - \bar{\pi} \). Recall (see Corollary 4.19 in [3])

\[
\zeta_t = \int_0^t \int \chi_\sigma(y)q(ds, dy) + \int_0^t \int (1 - \chi_\sigma(y))p(ds, dy), \; t \geq 0,
\]

\( p(ds, dy) \) is Poisson point measure with

\[
\mathbf{E} p(ds, dy) = \Pi(dy) \, ds, \quad q(ds, dy) = p(ds, dy) - \Pi(dy) \, ds.
\]

Now, \( \zeta_t = \tilde{\zeta}_t + \tilde{\zeta}_t \) with

\[
\tilde{\zeta}_t = \int_0^t \int_{|y| \leq 1} \chi_\sigma(y)q(ds, dy) + \int_0^t \int_{|y| \leq 1} (1 - \chi_\sigma(y))p(ds, dy),
\]

\[
\tilde{\zeta}_t = \int_0^t \int_{|y| > 1} \chi_\sigma(y)q(ds, dy) + \int_0^t \int_{|y| > 1} (1 - \chi_\sigma(y))p(ds, dy),
\]

\( t \geq 0 \).
Case 1: $\sigma \in (0, 1)$. In this case (3.11) holds with $\alpha_1, \alpha_2 \in (0, 1]$. Then
\[
\left| \tilde{\zeta}_t \right|^{\alpha_1} = \sum_{s \leq t} \left| [ \tilde{\zeta}_s + \Delta \tilde{\zeta}_s ]^{\alpha_1} - \left| \tilde{\zeta}_s \right|^{\alpha_1} \right| \leq \sum_{s \leq t} \left| \Delta \tilde{\zeta}_s \right|^{\alpha_1},
\]
and
\[
E \left| \tilde{\zeta}_t \right|^{\alpha_1} \leq t \int \left| y \right|^{\alpha_1} \Pi(dy) \leq t \int \left| y \right|^{\alpha_1} \pi(dy) \leq Nt.
\]
Similarly, $E \left| \tilde{\zeta}_t \right|^{\alpha_2} \leq Nt$.

Case 2: $\sigma \in (1, 2)$. In this case, $\alpha_1, \alpha_2 \in (1, 2]$. Then
\[
E[\tilde{\zeta}_t^2] = \int \left| y \right|^{2} \Pi(dy) t \leq \int \left| y \right|^{2} \pi(dy) t \leq Nt,
\]
and
\[
E \left| \tilde{\zeta}_t \right|^{\alpha_2} \leq 2t \int \left| y \right|^{\alpha_2} \Pi(dy) \leq 2t \int \left| y \right|^{\alpha_2} \pi(dy) \leq Nt.
\]

Case 3: $\sigma = 1$. In this case, $\alpha_1 \in (1, 2]$ and $\alpha_2 \in [0, 1)$. Similarly as above, we find that
\[
E[\tilde{\zeta}_t^2] = t \int \left| y \right|^{2} \Pi(dy) \leq t \int \left| y \right|^{2} \pi(dy) \leq Nt,
\]
and
\[
E \left[ \tilde{\zeta}_t \right|^{\alpha_2} \leq Nt.
\]

The claim follows by Chebyshev inequality. For instance, in Case 1
\[
P \left( |\zeta_t| > \delta \right) \leq P \left( \left| \tilde{\zeta}_t \right|^{\alpha_1} > \left( \frac{\delta}{2} \right)^{\alpha_1} \right) + P \left( \left| \tilde{\zeta}_t \right|^{\alpha_2} > \left( \frac{\delta}{2} \right)^{\alpha_2} \right).
\]

The statement is proved. \qed

Let $\pi \in \mathfrak{p}$ and $p(dt, dy)$ be a Poisson point measure on $[0, \infty) \times \mathbb{R}_0^d$ such that $Ep(dt, dy) = \pi(dy) dt$. Let $q(dt, dy) = p(dt, dy) - \pi(dy) dt$. We associate to $L^\pi$ the stochastic process with independent increments
\[
Z_t = Z_t^\pi = \int_0^t \int \chi_\sigma(y)q(ds, dy) + \int_0^t \int (1 - \chi_\sigma(y))yp(ds, dy), t \geq 0. \tag{3.13}
\]
By Ito formula,
\[
Ee^{i2\pi \xi \cdot Z_t^\pi} = \exp \left\{ \psi^\pi (\xi) t \right\}, t \geq 0, \xi \in \mathbb{R}^d, \tag{3.14}
\]
where
\[
\psi^\pi (\xi) := \int \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi y \cdot \xi \chi_\sigma (y) \right] \pi(dy).
\]
Let \( \kappa (R) , R > 0 \), be a scaling function, \( Z_t = Z_t^R \) be the stochastic process with independent increments associated with \( \tilde{\pi}_R = \kappa (R) \pi_R \), i.e.,

\[
\mathbb{E} e^{i2\pi \xi \cdot Z_t^R} = \exp \left\{ \psi_{\tilde{\pi}_R} (\xi) t \right\}
\]

with

\[
\psi_{\tilde{\pi}_R} (\xi) = \int \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\sigma (y) \cdot \xi \right] d\tilde{\pi}_R, \xi \in \mathbb{R}^d.
\]

Note \( Z_t^R \) and \( R^{-1} Z_t^{\pi_{\kappa(R)}} \), \( t > 0 \), have the same distribution.

**Lemma 5** Let \( \pi \in \mathcal{A}^\sigma , \kappa \) be a scaling function with scaling factor 1. Assume

\[
\tilde{\pi}_R (dy) = \kappa (R) \pi_R (dy) \geq 1_{|y| \leq 1} \mu^0 (dy)
\]

with \( \mu^0 \) satisfying the assumptions of Lemma 3 (in particular, (3.7) with \( n \geq 0 \) and the constant \( K_0 \)), and let

\[
\psi_0 (\xi) = \int \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\sigma (y) \xi \cdot y \right] \mu^0 (dy), \xi \in \mathbb{R}^d.
\]

(a) For each \( t > 0, R > 0 \), we have \( Z_t^R = \eta_t + \eta'_t \) (in distribution), \( \eta_t \) and \( \eta'_t \) are independent with

\[
\mathbb{E} e^{i2\pi \xi \cdot \eta_t} = \exp \{ \psi_0 (\xi \cdot \gamma (t)) \}, \xi \in \mathbb{R}^d.
\]

and \( \mu^0_{\gamma(t)^{-1}} \leq t\tilde{\pi}_R \), where \( \gamma (t) = l^{-1} (t) = \inf (s : l (s) > t) \). Moreover, \( \eta_t = \gamma (t) \eta \) (in distribution), where \( \eta \) is a r.v. in Lemma 3.

(b) For every \( t > 0, R > 0 \), the process \( Z_t^R \) (equivalently \( R^{-1} Z_t^{\pi_{\kappa(R)}} \)) has a bounded continuous probability density function

\[
p^R (t, x) = \gamma (t)^{-d} \int p_0 \left( \frac{x-y}{\gamma (t)} \right) P_{t,R} (dy), x \in \mathbb{R}^d,
\]

where \( P_{t,R} (dy) \) is the distribution measure of \( \eta'_t \) on \( \mathbb{R}^d \) and \( p_0 \) is pdf of \( \eta \). Moreover, \( p^R (t, x) \) has \( n \) bounded continuous derivatives such that for any multiindex \( |\beta| \leq n \),

\[
\int \left| \partial^\beta p^R (t, x) \right| dx \leq \gamma (t)^{-|\beta|} \int \left| \partial^\beta p_0 (x) \right| dx,
\]

\[
\sup_{x \in \mathbb{R}^d} \left| \partial^\beta p^R (t, x) \right| \leq \gamma (t)^{-d-|\beta|} \sup_{x} \left| \partial^\beta p_0 (x) \right| ,
\]

and for any \( \alpha \in (0, 1) \) such that \( |\beta| + \alpha < n \)

\[
\int \left| \partial^\alpha \partial^\beta p^R (t, x) \right| dx \leq \gamma (t)^{-|\beta|-\alpha} \int \left| \partial^\alpha \partial^\beta p_0 (x) \right| dx.
\]

(3.16)
where $\alpha_1, \alpha_2 \in (0, 1)$ if $\sigma \in (0, 1)$; $\alpha_1, \alpha_2 \in (1, 2]$ if $\sigma \in (1, 2)$; $\alpha_1 \in (1, 2]$ and $\alpha_2 \in [0, 1)$ if $\sigma = 1$. Then for each $a > 0$ there is $C = C (a, \alpha, N, K_0, n)$ such that for any multiindex $|\beta| \leq n, R > 0, t > 0$,

$$
\int_{|x| > a} \left| \partial^\beta p^R (t, x) \right| \, dx \leq C \left( \gamma (t)^{2-|\beta|} + t \gamma (t)^{-|\beta|} \right).
$$

Proof (a) Let $R > 0, t > 0$. Since $l \left( l^{-1} (t) \right) = t$, we have $\kappa (R) t \geq \kappa \left( R l^{-1} (t) \right) = \kappa \left( R \gamma (t) \right)$. Hence

$$
\tilde{\pi}_R t \geq \kappa \left( R \gamma (t) \right) \pi_{R \gamma (t) / \gamma (t)} \geq \mu^0_{\gamma (t)^{-1}},
$$

(3.18)

and $\mu^0_{\gamma (t)^{-1}} (dy) = \mu^0 \left( \gamma (t)^{-1} dy \right)$ is the Levy measure of a random variable, denoted $\eta_t$, such that (3.15) holds. Let

$$
\psi^R_\ell (\xi) = \psi_0 (\ell \gamma (t)) + \psi' (\xi), \xi \in \mathbb{R}^d.
$$

The inequality (3.18) implies that $\psi' = \psi_\Pi^R$ with $\Pi_t = \tilde{\pi}_R t - \mu^0_{\gamma (t)^{-1}}$. Let $\eta'_t$ be a random variable independent of $\eta_t$ with characteristic function $\exp \left[ \psi' (\xi) \right]$. Obviously the distribution of $Z^R_t$ coincides with the distribution of the sum $\eta_t + \eta'_t$. If $\eta$ is a r.v. with characteristic function (3.6), then $\eta_t = \gamma (t) \eta$ in distribution.

(b) Note that $\phi_0 (\xi) = \text{Re} \psi_0 (\xi), \xi \in \mathbb{R}^d$. Let $t > 0$. By part (a), $Z^R_t = \gamma (t) \eta + \eta'_t$ (in distribution), $\eta$ and $\eta'_t$ are independent. The pdf of $\eta'_{\gamma (t)}$ is

$$
p_0 (t, x) = \gamma (t)^{-d} p_0 (x / \gamma (t)), x \in \mathbb{R}^d.
$$

(3.19)

Let $P_{t, R} (dy)$ be the distribution measure of $\eta'_t$ on $\mathbb{R}^d$. Since $\eta_{\gamma (t)}$ and $\eta'_t$ are independent, $Z^R_t$ has a density

$$
p^R (t, x) = \int p_0 (t, x - y) P_{t, R} (dy), x \in \mathbb{R}^d.
$$

According to (3.19) (see (3.10) as well), for any $|\beta| \leq n$,

$$
\partial^\beta p^R (t, x) = \int \partial^\beta p_0 (t, x - y) P_{t, R} (dy),
$$

and, according to Lemma 3,

$$
\sup_{x, R} \left| \partial^\beta p^R (t, x) \right| \leq \gamma (t)^{-d-|\beta|} \sup_{x} \left| \partial^\beta p_0 (x) \right| < \infty.
$$
\[ \int |\partial^\beta p^R(t, x)| \, dx \leq \gamma(t)^{-|\beta|} \int |\partial^\beta p_0(x)| \, dx < \infty. \quad (3.20) \]

Similarly, see Corollary 1, (3.16) follows.

(c) Fix \( s > 0 \). Let \( \rho_R = \tilde{\pi}_R - s^{-1} \mu_0^{0, y(s)^{-1}} \). Let \( Y_t \) be the Levy process corresponding to \( \rho_R \), i.e.
\[ \mathbb{E} e^{i2\pi Y_t; \xi} = \exp \{ \psi(\xi) t \}, t \geq 0, \xi \in \mathbb{R}^d, \]
with \( \psi(\xi) = \int \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\sigma(y) \cdot \xi \right] d\rho_R, \xi \in \mathbb{R}^d. \)

By Lemma 4 there is \( C = C(N, a) \) such that
\[ P(|Y_t| > a) \leq Ct, t \geq 0; \]
in particular for \( t = s \),
\[ P(|\eta'_s| > a) = P(|Y_s| > a) \leq Cs. \]

Since \( s \) is arbitrary, there is \( C = C(N, a) \) such that
\[ P_{s,R}(|y| > a) = P(|\eta'_s| > a) \leq Cs, s \geq 0. \quad (3.21) \]

Let \( a > 0, |\beta| \leq n \). Then according to (3.21),
\[ \int_{|x| > a} |\partial^\beta p^R(t, x)| \, dx = \gamma(t)^{-d - |\beta|} \int_{|x| > a} \left| \int (\partial^\beta p_0) \left( \frac{x - y}{\gamma(t)} \right) P_{t,R}(dy) \right| \, dx \]
\[ \leq \int \int_{|x - y| > a/2} \cdots + \int \int_{|y| > a/2} \cdots \]
\[ \leq C \left[ \gamma(t)^{2 - |\beta|} \int |x|^2 |\partial^\beta p_0(x)| \, dx + t \gamma(t)^{-|\beta|} \left| \partial^\beta p_0 \right|_{L_1} \right]. \]

We will need some estimates involving the operators \( L^\pi \).

**Lemma 6** Let \( \pi_0 \in \mathfrak{K}^\sigma, \kappa \) be a scaling function with scaling factor 1, and \( D(\kappa, l) \)
(i)–(ii) hold for \( \pi_0 \). Let \( \pi \in \mathfrak{K}^\sigma \) be such that
\[ \int_{|z| \leq 1} |z|^\alpha_1 \tilde{\pi}_R(dz) + \int_{|z| > 1} |z|^\alpha_2 \tilde{\pi}_R(dz) \leq N \forall R > 0, \]
for some \( N > 0 \), where \( \tilde{\pi}_R(dy) = \kappa(R) \pi_R(dy) \), and \( \alpha_1, \alpha_2 \) are exponents in assumption \( D(\kappa, l) \) for \( \pi_0 \). Let \( R > 0 \) and \( p^R(t, x), x \in \mathbb{R}^d \), be pdf of \( R^{-1}Z_{\pi(R)^{-1}}(\kappa(R), t) \) (see Lemma 5), \( \gamma(t) = l^{-1}(t), t > 0, \) and \( L^{0,R} \) be the operator corresponding to Levy measure \( \kappa(R) \pi_0(Rdy) \). Then there is \( C = C(d, N_0, N) \) such that
\[ \int_{|x| > 2} \left| L_{\tilde{\pi}R} p^R (t, x) \right| \, dx \leq C \left[ 1 + \sigma_{\in (1, 2)} \gamma (t)^{-1} + \gamma (t)^{-\alpha_1} \left( \gamma (t)^2 + t \right) \right] , \] 

(3.22)

for all \( t > 0 \);

(b) denoting \( I_2 = \{ t > 0 : \gamma (t) > 1 \} \),

\[ \left| L_{\tilde{\pi}R} \nabla p^R (t, \cdot) \right|_{L^1} \leq C \gamma (t)^{(1+\alpha_2)} , t \in I_2 \] 

(3.23)

(c) \[ \left| L_{\tilde{\pi}R} L_{0.}^R p^R (t, \cdot) \right|_{L^1} \leq C \gamma (t)^{-2\alpha_2} , t \in I_2. \] 

(3.24)

**Proof** (a) For any \( t > 0 \),

\[ \int_{|x| > 2} \left| L_{\tilde{\pi}R} p^R (t, x) \right| \, dx \leq \int \int_{|x| > 2} \left| \nabla_{\tilde{\pi}R} p^R (t, x) \right| \, dx \, d\tilde{\pi}_R (dz) . \]

Now, for \( |z| \leq 1 \), by Lemmas 2 and 5,

\[ \int_{|x| > 2} \left| \nabla_{\tilde{\pi}R} p^R (t, x) \right| \, dx \leq C |z|^{\alpha_1} \left( \int_{|x| > 1} \left| p^R (t, x) \right| \, dx \right)^{1-\alpha_1} \left( \int_{|x| > 1} \left| \nabla p^R (t, x) \right| \, dx \right)^{\alpha_1} \]

\[ \leq C |z|^{\alpha_1} \left[ \gamma (t)^2 + t \right] \gamma (t)^{-\alpha_1} \text{ if } \sigma \in (0, 1) ; \]

and by Corollary 3 and Lemma 5,

\[ \int_{|x| > 2} \left| \nabla_{\tilde{\pi}R} p^R (t, x) \right| \, dx \leq C |z|^{\alpha_1} \left( \int_{|x| > 1} \left| \nabla p^R (t, x) \right| \, dx \right)^{2-\alpha_1} \left( \int_{|x| > 1} \left| \nabla^2 p^R (t, x) \right| \, dx \right)^{\alpha_1-1} \]

\[ \leq C |z|^{\alpha_1} \left[ t + \gamma (t)^2 \right] \gamma (t)^{-\alpha_1} \text{ if } \sigma \in [1, 2) . \]

For \( |z| > 1 \),

\[ \int_{|x| > 2} \left| \nabla_{\tilde{\pi}R} p^R (t, x) \right| \, dx \leq 2 \text{ if } \sigma \in (0, 1] , \]

and

\[ \int_{|x| > 2} \left| \nabla_{\tilde{\pi}R} p^R (t, x) \right| \, dx \leq C \left( 1 + |z| \int_{|x| > 2} \left| \nabla p^R (t, x) \right| \, dx \right) \text{ if } \sigma \in (1, 2) . \]

Hence

\[ \int \int_{|x| > 2} \left| \nabla_{\tilde{\pi}R} p^R (t, x) \right| \, dx \, d\tilde{\pi}_R (dz) \leq C \left[ 1 + \gamma (t)^{-\alpha_1} \left( \gamma (t)^2 + t \right) \right] \]
if $\sigma \in (0, 1]$, and by Lemma 5(b),
\[
\int \int_{|x|>2} |\nabla_x^\sigma p_R (t, x)| \, dx \pi_R (dz) \leq C \gamma (t)^{-\alpha_1} \left( \gamma (t)^2 + t \right) + C \left( 1 + \gamma (t)^{-1} \right)
\]
if $\sigma \in (1, 2)$.

(b) Let $t \in I_2$. By Corollary 2 and Lemma 5(b),
\[
\left| \tilde{L}^R \nabla p_R (t, \cdot) \right|_{L^1} \leq C \left( \left| D^2 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2} \nabla p_R (t, \cdot) \right|_{L^1} \right) \\
\leq C \left[ \gamma (t)^{-2} + \gamma (t)^{-(1+\alpha_2)} \right] \leq C \gamma (t)^{-(1+\alpha_2)}
\]
if $\sigma \in (0, 1)$. Similarly,
\[
\left| L^R \nabla p_R (t, \cdot) \right|_{L^1} \leq C \left[ \gamma (t)^{-3} + \gamma (t)^{-(1+\alpha_2)} \right] \leq C \gamma (t)^{-(1+\alpha_2)}
\]
if $\sigma = 1$, and
\[
\left| L^R \nabla p_R (t, \cdot) \right|_{L^1} \leq C \left( \left| D^3 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2} D^2 p_R (t, \cdot) \right|_{L^1} \right) \\
\leq C \left[ \gamma (t)^{-3} + \gamma (t)^{-(1+\alpha_2)} \right] \leq C \gamma (t)^{-(1+\alpha_2)}
\]
if $\sigma \in (1, 2)$.

(c) Let $t \in I_2$, i.e., $\gamma (t) > 1$. By Corollary 2,
\[
\left| \tilde{L}^R L^{0,R} p_R (t, \cdot) \right|_{L^1} \leq C \left( \left| L^{0,R} \nabla p_R (t, \cdot) \right|_{L^1} + \left| L^{0,R} \partial^{\alpha_2} p_R (t, \cdot) \right|_{L^1} \right) \\
\leq C \left( \left| D^2 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2} \nabla p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2} \partial^{\alpha_2} p_R (t, \cdot) \right|_{L^1} \right)
\]
if $\sigma \in (0, 1)$;
\[
\left| \tilde{L}^R L^{0,R} p_R (t, \cdot) \right|_{L^1} \leq C \left( \left| L^{0,R} D^2 p_R (t, \cdot) \right|_{L^1} + \left| L^{0,R} \partial^{\alpha_2} p_R (t, \cdot) \right|_{L^1} \right) \\
\leq C \left( \left| D^4 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2} D^2 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2} \partial^{\alpha_2} p_R (t, \cdot) \right|_{L^1} \right)
\]
if $\sigma = 1$;
\[
\left| \tilde{L}^R L^{0,R} p_R (t, \cdot) \right|_{L^1} \leq C \left( \left| L^{0,R} D^2 p_R (t, \cdot) \right|_{L^1} + \left| L^{0,R} \partial^{\alpha_2-1} \nabla p_R (t, \cdot) \right|_{L^1} \right) \\
\leq C \left( \left| D^4 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2-1} D^3 p_R (t, \cdot) \right|_{L^1} + \left| \partial^{\alpha_2-1} \partial^{\alpha_2-1} D^2 p_R (t, \cdot) \right|_{L^1} \right)
\]
if $\sigma \in (1, 2)$. The estimate (3.24) follows by Lemma 5.

\[\square\]

3.3 Estimates of $\psi^\pi$

We present now some properties of the functions $\psi^\pi(\xi), \xi \in \mathbb{R}^d$, with $\pi \in \mathfrak{A}^\sigma$.

**Lemma 7** Let $\pi \in \mathfrak{A}^\sigma$ and $\kappa (R), R > 0$, be a scaling function, and $\tilde{\pi}_R (dy) = \kappa (R) \pi (Rdy)$.

(a) Assume there is $N_2 > 0$ so that

\[
\int (|y| \wedge 1) \tilde{\pi}_R (dy) \leq N_2 \text{ if } \sigma \in (0, 1),
\]

\[
\int (|y|^2 \wedge 1) \tilde{\pi}_R (dy) \leq N_2 \text{ if } \sigma = 1,
\]

\[
\int_{|y| \leq 1} |y|^2 \tilde{\pi}_R (dy) + \int_{|y| > 1} |y| \tilde{\pi}_R (dy) \leq N_2 \text{ if } \sigma \in (1, 2)
\]

(3.25)

for any $R > 0$. Then there is a constant $C_1$ so that for all $\xi \in \mathbb{R}^d$,

\[
\int [1 - \cos(2\pi \xi \cdot y)] \pi (dy) \leq C_1 N_2 \kappa \left( |\xi|^{-1} \right)^{-1},
\]

\[
\int |\sin(2\pi \xi \cdot y) - 2\pi \chi_\sigma (y) \xi \cdot y| \pi (dy) \leq C_1 N_2 \kappa \left( |\xi|^{-1} \right)^{-1},
\]

assuming $\kappa \left( |\xi|^{-1} \right)^{-1} = 0$ if $\xi = 0$.

(b) Assume there is a $n_1 > 0$ such that

\[
\int |\xi \cdot y|^2 \tilde{\pi}_R (dy) \geq n_1,
\]

(3.26)

for all $R > 0$ and $\xi \in S_{d-1} = \{ \xi \in \mathbb{R}^d : |\xi| = 1 \}$. Then there is a constant $c_2 = c_2 (l) > 0$ such that

\[
\int [1 - \cos(2\pi \xi \cdot y)] \pi (dy) \geq c_2 n_1 \kappa \left( |\xi|^{-1} \right)^{-1}
\]

for all $\xi \in \mathbb{R}^d$, assuming $\kappa \left( |\xi|^{-1} \right)^{-1} = 0$ if $\xi = 0$.

**Proof** The following simple trigonometric estimates hold:

\[
|\sin x - x| \leq \frac{|x|^3}{6}, \quad 1 - \cos x \leq \frac{1}{2} x^2, \quad x \in \mathbb{R},
\]

\[
1 - \cos x \geq \frac{x^2}{\pi} \text{ if } |x| \leq \pi/2.
\]

(3.27)
(a) Let $\xi \neq 0$. Denoting $\hat{\xi} = \xi / |\xi|$, and using (3.27),

$$
\int \left| 1 - \cos \left( 2\pi \hat{\xi} \cdot y \right) \right| \pi \left( dy \right) = \kappa \left( |\xi|^{-1} \right)^{-1} \int \left| 1 - \cos \left( 2\pi \hat{\xi} \cdot y \right) \right| \tilde{\pi}_{|\xi|^{-1}} \left( dy \right)
\leq \kappa \left( |\xi|^{-1} \right)^{-1} 2\pi^2 \int \left( |y|^2 + 1 \right) \tilde{\pi}_{|\xi|^{-1}} \left( dy \right),
$$

and there is $C_1$ so that

$$
\int \left| \sin \left( 2\pi \hat{\xi} \cdot y \right) - 2\pi \chi_{\sigma} \left( y \right) \hat{\xi} \cdot y \right| \pi \left( dy \right)
= \kappa \left( |\xi|^{-1} \right)^{-1} \int \left| \sin \left( 2\pi \hat{\xi} \cdot y \right) - 2\pi \chi_{\sigma} \left( y \right) \hat{\xi} \cdot y \right| \tilde{\pi}_{|\xi|^{-1}} \left( dy \right)
\leq C_1 N_2 \kappa \left( |\xi|^{-1} \right)^{-1}
$$

for all $\xi \in \mathbb{R}^d$.

(b) By (3.27), for all $\xi \in \mathbb{R}^d$,

$$
\int \left[ 1 - \cos \left( 2\pi \hat{\xi} \cdot y \right) \right] \pi \left( dy \right)
= \int \left[ 1 - \cos \left( 2\pi \hat{\xi} \cdot y \right) \right] \tilde{\pi}_{|\xi|^{-1}} \left( dy \right)
\geq \int_{|y| \leq \frac{1}{4}} 4\pi \left| \hat{\xi} \cdot y \right|^2 \tilde{\pi}_{|\xi|^{-1}} \left( dy \right)
= 4^{-1} \int_{|y| \leq 1} \pi \left| \hat{\xi} \cdot 4y \right|^2 \tilde{\pi}_{|\xi|^{-1}} \left( dy \right)
= 4^{-1} \kappa \left( |4\xi|^{-1} \right)^{-1} \int_{|y| \leq 1} \pi \left| \hat{\xi} \cdot y \right|^2 \tilde{\pi}_{|\xi|^{-1}} \left( dy \right)
\geq n_1 4^{-1} \pi \kappa \left( |\xi|^{-1} \right)^{-1} \frac{\kappa \left( |\xi|^{-1} \right)}{\kappa \left( |4\xi|^{-1} \right)}.
$$

Let $l$ be a scaling factor of $\kappa$, i.e., $\kappa \left( \varepsilon R \right) \leq l \left( \varepsilon \right) \kappa \left( R \right)$, $\varepsilon, R > 0$. Then

$$
\frac{\kappa \left( |\xi|^{-1} \right)}{\kappa \left( |4\xi|^{-1} \right)} \geq \frac{1}{l \left( 4^{-1} \right)} > 0 \forall \xi \in \mathbb{R}^d.
$$

The claim follows. \hfill \Box

For $\pi \in \mathcal{H}^\sigma$, let

$$
\tilde{\psi} \left( \xi \right) = \tilde{\psi}^\pi \left( \xi \right) = \int \left[ \cos \left( 2\pi \hat{\xi} y \right) - 1 \right] \pi \left( dy \right) = \text{Re} \ \psi^\pi \left( \xi \right),
\varphi \left( \xi \right) = \varphi^\pi \left( \xi \right) = \int \left[ \sin \left( 2\pi \hat{\xi} \cdot y \right) - 2\pi \chi_{\sigma} \left( y \right) \hat{\xi} \cdot y \right] \pi \left( dy \right) = \text{Im} \ \psi^\pi \left( \xi \right),
$$
ξ ∈ R^d. Note ψ^π*(ξ) = ψ^π(−ξ), ξ ∈ R^d, where π*(dy) = π(−dy). An obvious consequence of Lemma 7 is the following

Corollary 4 Let κ be a scaling function with scaling factor l and both assumptions, (3.25) and (3.26), of Lemma 7 hold for π ∈ M^c. Then there is constant c = c(n_1, N_2, l) > 0 such that

\[ c |\psi^\pi(\xi)| \leq |\tilde{\psi}^\pi(\xi)| \leq |\psi^\pi(\xi)|, \xi \in R^d, \]

and \(|\psi^\pi(\xi)| \leq c^{-1} |\tilde{\psi}^\pi(\xi)|, \xi \in R^d.\)

Note that it implies

\[ c |\psi^\pi*(\xi)| \leq |\tilde{\psi}^\pi(\xi)| = |\tilde{\psi}^\pi*(\xi)|, \xi \in R^d. \]

4 Proof of the main results

In this section we prove the main results in three steps. First we prove the existence and uniqueness of classical solutions for smooth input functions. Then we derive \(|u|_{L^p}^0\)-norm estimates with constants independent of the regularity of the input function. Finally, continuity estimate of \(L^\pi\) with respect to \(|u|_{L^p}^0\)-norm allows to pass to the limit and derive the results for the input function \(f \in L^p\).

4.1 Existence and uniqueness for smooth input functions

For \(E = [0, T] \times R^d\), we denote by \(\tilde{C}^\infty(E)\) the space of all measurable functions \(f\) on \(E\) such that for any multiindex \(\gamma \in N_0^d\) and for all \(1 \leq p < \infty\)

\[ \sup_{(t,x) \in E} |D^\gamma f (t, x)| + \sup_{t \in [0, T]} |D^\gamma f (t, \cdot)|_{L^p(R^d)} < \infty. \]

Similarly, let \(\tilde{C}^\infty(R^d)\) be the space of all measurable functions \(f\) on \(R^d\) such that for any multiindex \(\gamma \in N_0^d\) and for all \(1 \leq p < \infty\)

\[ \sup_{x \in R^d} |D^\gamma f (x)| + |D^\gamma f|_{L^p(R^d)} < \infty. \]

Next we suppose that \(f \in \tilde{C}^\infty(E)\) and derive some estimates for the solution.

Lemma 8 Let \(f \in \tilde{C}^\infty(E)\) then there is unique \(u \in \tilde{C}^\infty(E)\) solving (1.1). Moreover,

\[ u(t, x) = \int_0^t e^{-\lambda(t-s)} E f(s, x + Z^\pi_{t-s}) \, ds, (t, x) \in E, \quad (4.1) \]

and for \(p \in [1, \infty]\) and any multiindex \(\gamma \in N_0^d\),
\[
|D^y u|_{L^p(E)} \leq \rho_\lambda |D^y f|_{L^p(E)}, \quad (4.2)
\]
\[
|D^y u(t)|_{L^p(\mathbb{R}^d)} \leq \int_0^t |D^y f(s)|_{L^p(\mathbb{R}^d)} \, ds, \quad t \geq 0.
\]  
(4.3)

where \( \rho_\lambda = (1/\lambda) \wedge T. \)

**Proof** Denote \( Z_t = Z_t^\pi, t \geq 0. \) **Uniqueness.** Let \( u_1, u_2 \in \tilde{C}_\infty^\infty(E) \) solve (1.1) and \( u = u_1 - u_2. \) Then \( u \) solve (1.1) with \( f = 0. \) Let \( t \in (0, T]. \) Consider smooth function
\[
F(s, y) = e^{\lambda(t-s)} u(t-s, y), \quad 0 \leq s \leq t, \quad y \in \mathbb{R}^d.
\]

Note that for \((s, y) \in [0, t] \times \mathbb{R}^d, \)
\[
F(t, y) = 0, \quad F(0, y) = e^{\lambda t} u(t, y), \\
\partial_y F(s, y) = e^{\lambda(t-s)} [-\partial_t u(t-s, y) - \lambda u(t-s, y)].
\]

By Ito formula for \( F(s, x + Z_s) = e^{\lambda(t-s)} u(t-s, x + Z_s), \) \( 0 \leq s \leq t, \) we have
\[
-e^{\lambda t} u(t, x) = E \int_0^t e^{\lambda(t-s)} [-\partial_t + L^\pi - \lambda] u(t-s, x + Z_s) \, ds = 0.
\]

Hence \( u(t, x) = 0 \) for all \((t, x) \in E. \)

**Existence** Let \( f \in \tilde{C}_\infty^\infty(E). \) Set
\[
u(t, x) = \int_0^t e^{-\lambda(t-s)} E f(s, x + Z_{t-s}^\pi) \, ds, \quad (t, x) \in E.
\]

Then
\[
D^y u(t, x) = \int_0^t e^{-\lambda(t-s)} E D^y f(s, x + Z_{t-s}^\pi) \, ds, \quad (t, x) \in E,
\]
and (4.3) follows.

By Hölder inequality for \( \lambda > 0, \)
\[
|D^y u|_{L^p(T)}^p \leq \lambda^{-p} \int_0^T \int_0^t e^{-\lambda(t-s)} |D^y f(s, \cdot)|_{L^p(\mathbb{R}^d)}^p \, ds \, dt \leq \lambda^{-p} |D^y f|_{L^p(T)}^p.
\]

By Hölder inequality for \( \lambda \geq 0, \)
\[
|D^y u|_{L^p(T)}^p \leq \int_0^T t^{p-1} \int_0^t |D^y f(s, \cdot)|_{L^p(\mathbb{R}^d)}^p \, ds \, dt \leq \frac{T_p}{p} |D^y f|_{L^p(T)}^p.
\]

We fix \( s \in [0, T], x \in \mathbb{R}^d, \) and applying Ito formula with \( e^{-\lambda x} f(s, x + \cdot) \) and \( Z_r, \) \( 0 \leq r \leq t-s, \) we have
\[
e^{-\lambda(t-s)} f(s, x + Z_{t-s})
\]
\[
= f(s, x) + \int_0^t \int_{R_0} e^{-\lambda r} \left[ f(s, x + Z_{r-} + y) - f(s, x + Z_{r-}) \right] q(dr, dy) \\
+ \int_0^t e^{-\lambda r} \left( L^\pi - \lambda \right) f(s, x + Z_{r}) dr
\]

Taking expectation on both sides, and integrating with respect to \( s \), we obtain by Fubini’s theorem for each \((t, x) \in E, \)

\[
\int_0^t e^{-\lambda(t-s)} E f(s, x + Z_{t-s}) ds \\
= \int_0^t f(s, x) ds + \int_0^t \int_0^{t-s} e^{-\lambda r} E \left[ L^\pi - \lambda \right] f(s, x + Z_{r}) dr ds \\
= \int_0^t f(s, x) ds + \int_0^t \int_s^t e^{-\lambda(r-s)} E \left[ L^\pi - \lambda \right] f(s, x + Z_{r-s}) dr ds \\
= \int_0^t f(s, x) ds + \int_0^t \int_0^r e^{-\lambda(r-s)} E \left[ L^\pi - \lambda \right] f(s, x + Z_{r-s}) ds dr.
\]

Since for each \((r, x) \in E \)

\[
\int_0^r e^{-\lambda(r-s)} E L^\pi f(s, x + Z_{r-s}) ds = L^\pi u(r, x),
\]

it follows that for each \((t, x) \in E, \)

\[
u(t, x) = \int_0^t f(s, x) ds + \int_0^t \left[ L^\pi u(r, x) - \lambda u(r, x) \right] dr.
\]

The statement follows. \( \Box \)

Similarly we can handle the problem (1.2).

**Lemma 9** Let \( g \in \tilde{C}^\infty(R^d) \) then there is unique \( u \in \tilde{C}^\infty(R^d) \) solving (1.2). Moreover,

\[
u(x) = \int_0^\infty e^{-\lambda t} E g(x + Z^\pi_t) dt, x \in R^d, \quad (4.4)
\]

and for \( p \in [1, \infty) \) and any multiindex \( \gamma \in N_0^d, \)

\[
|D^\gamma u|_{L_p(R^d)} \leq (1/\lambda) |D^\gamma g|_{L_p(R^d)} . \quad (4.5)
\]

**Proof** Denote \( Z_t = Z^\pi_t, t \geq 0. \) Uniqueness. Let \( u_1, u_2 \in \tilde{C}^\infty(R^d) \) solve (1.2) and \( u = u_1 - u_2. \) Then \( u \) solve (1.2) with \( g = 0. \) Let \( x \in R^d. \) By Ito formula for \( e^{-\lambda t} u(x + Z_t), 0 \leq s \leq t, \) we have

\[
e^{-\lambda t} Eu(x + Z_t) - u(x) = E \int_0^t e^{-\lambda s} \left[ L^\pi u(x + Z_s) - \lambda u(x + Z_s) \right] ds = 0.
\]
Passing to the limit as $t \to \infty$ we obtain that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

**Existence** Let $g \in \tilde{C}^\infty(\mathbb{R}^d)$. Set

$$u(x) = \int_0^\infty e^{-\lambda t} E g(x + Z^\pi_t) dt, \ x \in \mathbb{R}^d.$$ 

By direct estimate using Hölder inequality, as in Lemma 8, (4.5) readily follows, i.e. $u \in \tilde{C}^\infty(\mathbb{R}^d)$. We fix $x \in \mathbb{R}^d$, and applying Ito formula with $e^{-\lambda t} g(x + Z_t), \ 0 \leq t,$ we have for all $t > 0, x \in \mathbb{R}^d$.

$$e^{-\lambda t} E g(x + Z_t) = g(x) + \int_0^t e^{-\lambda s} E (L^\pi - \lambda) g(x + Z_s) ds$$

Passing to the limit as $t \to \infty$ we have

$$L^\pi u(x) - \lambda u(x) + g(x) = 0, x \in \mathbb{R}^d.$$ 

The statement follows. \hfill \Box

In addition, we have the following obvious consequence of Lemma 9.

**Corollary 5** Let $g \in \tilde{C}^\infty(\mathbb{R}^d)$ and

$$u(x) = \int_0^\infty e^{-\lambda t} E g(x + Z^\pi_t) dt, x \in \mathbb{R}^d.$$ 

Then $u \in \tilde{C}^\infty(\mathbb{R}^d)$, and

$$g(x) = (\lambda - L^\pi) \int_0^\infty e^{-\lambda t} E g(x + Z^\pi_t) dt$$

$$= \int_0^\infty e^{-\lambda t} E (\lambda - L^\pi) g(x + Z^\pi_t) dt, x \in \mathbb{R}^d.$$ 

**Proof** Indeed, by Lemma 9,

$$-g(x) = (L^\pi - \lambda) u(x) = (L^\pi - \lambda) \int_0^\infty e^{-\lambda t} E g(x + Z^\pi_t) dt$$

$$= \int_0^\infty e^{-\lambda t} E (L^\pi - \lambda) g(x + Z^\pi_t) dt, x \in \mathbb{R}^d.$$ \hfill \Box

**4.2 $L_2$-estimates**

We derive first some $L_2$-estimates independent of the regularity of $f$. Given $\pi \in \mathcal{A}^\sigma$, let

$$\tilde{\psi} = \tilde{\psi}^\pi = \text{Re} \, \psi^\pi (\xi) = \int [\cos (2\pi y \cdot \xi) - 1] \pi (dy), \xi \in \mathbb{R}^d.$$
For $v \in \tilde{C}^\infty (\mathbf{R}^d)$ (resp. $v \in \tilde{C}^\infty (E)$) we define

$$|v|_{\tilde{\psi},2} = \left| \mathcal{F}^{-1} \tilde{\psi} \hat{v} \right|_{L^2(\mathbf{R}^d)}$$

(resp. $|v|_{\tilde{\psi},2;E} = \left| \mathcal{F}^{-1} \tilde{\psi} \hat{v} \right|_{L^2(E)}$);

in the case $v \in \tilde{C}^\infty (E)$, $\hat{v}$ denotes Fourier transform in $x$.

**Lemma 10** Let $\pi \in \mathfrak{A}^\sigma$.
(a) Let $f \in \tilde{C}^\infty (E)$ and $u \in \tilde{C}^\infty (E)$ be the unique solution to (1.1), defined in Lemma 8. Then

$$|u|_{\tilde{\psi},2;E} \leq |f|_{L^2(E)}.$$  \hspace{1cm} (4.6)

Moreover,

$$|u|_{L^2(E)} \leq \left( \lambda^{-1} \wedge T \right) |f|_{L^2(E)}.$$  \hspace{1cm} (4.7)

and for $t \geq 0$,

$$\int_0^t \int \int [u (s, x + y) - u (s, x)]^2 \pi (dy) \, dx \, ds \leq \left( \int_0^t |f (s)|_{L^2(\mathbf{R}^d)} \, ds \right)^2.$$  \hspace{1cm} (4.8)

(b) Let $f \in \tilde{C}^\infty (\mathbf{R}^d)$ and $v \in \tilde{C}^\infty (\mathbf{R}^d)$ be the unique solution to (1.2), defined in Lemma 9. Then

$$|v|_{\tilde{\psi},2} \leq |f|_{L^2(\mathbf{R}^d)}.$$  \hspace{1cm} (4.9)

Moreover,

$$|v|_{L^2(\mathbf{R}^d)} \leq \left( 1 / \lambda \right) |f|_{L^2(\mathbf{R}^d)},$$  \hspace{1cm} (4.10)

and

$$\int \int [v (x + y) - v (x)]^2 \pi (dy) \, dx \leq \frac{2}{\lambda} |f|_{L^2(\mathbf{R}^d)}^2.$$  \hspace{1cm} (4.11)

**Proof** (a) Let $f \in \tilde{C}^\infty (E)$. Taking Fourier transform in $x$ in the representation (4.1) we find that

$$\hat{u} (t, \xi) = \int_0^t e^{-\lambda (t - s)} \exp \{ \psi (\xi) (t - s) \} \hat{f} (s, \xi) \, ds, \,(t, \xi) \in E.$$  

Hence, by Hölder inequality, for any $(t, \xi) \in E$,

$$\left| \tilde{\psi} (\xi) \hat{u} (t, \xi) \right|^2 \leq \left| \tilde{\psi} (\xi) \right|^2 \int_0^t \exp \{ \tilde{\psi} (\xi) (t - s) \} \left| \hat{f} (s, \xi) \right|^2 \, ds,$$

and, by Fubini theorem,
The inequality (4.7) is derived in Lemma 8. We derive the remaining inequalities using chain rule and integrating. For any \((t, x) \in E\),

\[
u(t, x)^2 = 2 \int_0^t \nu(s, x) \left[ L^\pi \nu(s, x) - \lambda \nu(s, x) + f(s, x) \right] ds. \tag{4.11}
\]

Now, for any \((s, x), y \in \mathbb{R}^d\),

\[
2 \nu(s, x) \left[ \nu(s, x + y) - \nu(s, x) - \chi_\sigma(y) \cdot \nabla \nu(s, x) \right] \\
\quad = - \left[ \nu(s, x + y) - \nu(s, x) \right]^2 \\
\quad + \nu(s, x + y)^2 - \nu(s, x)^2 - \chi_\sigma(y) \cdot \nabla[\nu(s, x)^2]. \tag{4.12}
\]

Using (4.12) and integrating both sides of (4.11) in \(x\), we have for any \(t \in [0, T]\),

\[
\|\nu(t)\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \int \left[ \nu(s, x + y) - \nu(s, x) \right]^2 \pi(dy) dx ds \\
\quad \leq 2 \int_0^t \int f(s, x) \nu(s, x) dx ds \leq 2 \int_0^t |f(s)|_{L^2(\mathbb{R}^d)} |\nu(s)|_{L^2(\mathbb{R}^d)} ds \\
\quad \leq \left( \int_0^t |f(s)|_{L^2(\mathbb{R}^d)} ds \right)^2
\]

and (4.8) follows.

(b) Let \(f \in \tilde{C}^\infty(\mathbb{R}^d)\). Taking Fourier transform in (4.4) we find that

\[
h(\xi) = \int_0^\infty e^{-\lambda t} \exp \left\{ \psi^\pi(\xi) t \right\} \hat{f}(\xi) dt, \xi \in \mathbb{R}^d.
\]

Hence for any \(\xi \in \mathbb{R}^d\),

\[
\left| \tilde{\psi}^\pi(\xi) \hat{h}(\xi) \right|^2 \leq \left| \tilde{\psi}^\pi(\xi) \right| \int_0^\infty \exp \left\{ \tilde{\psi}^\pi(\xi) t \right\} \left| \hat{f}(\xi) \right|^2 dt \leq \left| \hat{f}(\xi) \right|^2,
\]

and

\[
\left| \tilde{\psi}^\pi \hat{v} \right|_{L^2(\mathbb{R}^d)}^2 \leq \left| \hat{f} \right|_{L^2(\mathbb{R}^d)}^2.
\]

Inequality (4.9) was derived in Lemma 9. Multiplying both sides of (1.2) by \(2v\) and integrating as in the part a), we have

\[
\int \int [v(x + y) - v(x)]^2 \pi(dy) dx \\
\quad \leq 2 |v|_{L^2(\mathbb{R}^d)} |f|_{L^2(\mathbb{R}^d)} \leq \frac{2}{\lambda} |f|_{L^2(\mathbb{R}^d)}^2.
\]

\(\square\)
Remark 4 For \( v \in \tilde{C}^\infty (\mathbb{R}^d) \), by Plancherel’s equality,

\[
\int \int [v(x + y) - v(x)]^2 \pi (dy) \, dx = \int \int |e^{2\pi i \xi \cdot y} - 1|^2 \pi (dy) |\hat{v}(\xi)|^2 \, d\xi
= 2 \int \left( -\tilde{\psi} \pi (\xi) \right) |\hat{v}(\xi)|^2 \, d\xi
= 2 \left| \sqrt{-\tilde{\psi} \hat{v}} \right|_{L^2(\mathbb{R}^d)}^2 = 2 |v|_{(-\tilde{\psi})^{1/2}}^2,
\]

where \( |v|_{(-\tilde{\psi})^{1/2}}^2 = \left| \mathcal{F}^{-1} \sqrt{-\tilde{\psi} \hat{v}} \right|_{L^2(\mathbb{R}^d)}. \)

4.3 Continuity of \( L^\pi \) and proof of Theorem 1

We show first the \( L^2 \) continuity as follows.

Lemma 11 Let \( \pi, \pi_0 \in \mathfrak{A}^\sigma \). Assume there is \( C_0 > 0 \) so that

\[
|\psi_\pi (\xi)| \leq C_0 |\psi_{\pi_0} (\xi)|, \xi \in \mathbb{R}^d.
\]

Then

\[
|L^\pi \varphi|_{L^2(\mathbb{R}^d)} \leq C_0 |L^{\pi_0} \varphi|_{L^2(\mathbb{R}^d)}, \varphi \in \tilde{C}^\infty (\mathbb{R}^d).
\]

Proof Let \( \varphi \in \tilde{C}^\infty (\mathbb{R}^d) \). By Plancherel equality,

\[
|L^\pi \varphi|_{L^2(\mathbb{R}^d)} = |\psi_\pi \varphi|_{L^2(\mathbb{R}^d)} \leq C_0 |\psi_{\pi_0} \hat{\varphi}|_{L^2(\mathbb{R}^d)} = C_0 |L^{\pi_0} \varphi|_{L^2(\mathbb{R}^d)}.
\]

\( \square \)

Let \( \pi_0 \in \mathfrak{A}^\sigma, \kappa \) be a scaling function with a scaling factor \( l \) and \( D(\kappa, l)(i)-(ii) \) hold for \( \pi_0 \). Let \( \gamma(t) = \inf \{ s > 0 : l(s) > t \}, t > 0 \). According to Lemma 5, for each \( t > 0 \), the associated Levy process \( Z_{t}^{\pi_0} \) has a bounded probability density function \( p^{\pi_0}(t, x), x \in \mathbb{R}^d \). Also, \( \int |\nabla p^{\pi_0}(t, x)| \, dx \leq C \gamma(t)^{-1}, t > 0 \).

Define for \( v \in \tilde{C}^\infty (\mathbb{R}^d), \lambda > 0, \varepsilon \geq 0, \)

\[
K^\varepsilon_\lambda v(x) = \int_\varepsilon^\infty e^{-\lambda t} E v(x + Z_t^{\pi_0}) \, dt = \int_\varepsilon^\infty \int e^{-\lambda t} v(x + y) p^{\pi_0}(t, y) \, dy \, dt
= \int v(x - y) \int_\varepsilon^\infty e^{-\lambda t} p^{\pi_0}(t, y) \, dt \, dy
= \int v(y) \int_\varepsilon^\infty e^{-\lambda t} p^{\pi_0}(t, x - y) \, dt \, dy, x \in \mathbb{R}^d.
\]

Let

\[
T^\varepsilon_\lambda v(x) = L^\pi K^\varepsilon_\lambda v(x) = \int L^\pi v(x - y) \int_\varepsilon^\infty e^{-\lambda t} p^{\pi_0}(t, y) \, dt \, dy
\]

\( \square \) Springer
\[ = \int_{-\infty}^{\infty} e^{-\lambda t} \int L^\pi p^{\pi_0^*}(t, x - y) v(y) \, dy \, dt \]

\[ = \int m_{\lambda}^\varepsilon (x - y) v(y) \, dy, \quad v \in \tilde{C}^\infty \left( \mathbb{R}^d \right), \quad (4.13) \]

with

\[ m^\varepsilon (x) = m_{\lambda}^\varepsilon (x) = \int_{-\infty}^{\infty} e^{-\lambda t} L^\pi p^{\pi_0^*}(t, x) \, dt, \quad x \in \mathbb{R}^d. \]

Note that according to Lemma 5(b) and Corollary 2,

\[ \int |m_{\lambda}^\varepsilon | (x) \, dx < \infty. \]

**Lemma 12** Let \( \pi_0 \in \mathcal{A}_\sigma, \kappa \) be a scaling function with scaling factor 1, and \( D(\kappa, l) \) hold for \( \pi_0 \). Let \( \pi \in \mathcal{A}_\sigma \) be such that

\[ \int_{|z| \leq 1} |z|^{\alpha_1} \tilde{\pi}_R(dz) + \int_{|z| > 1} |z|^{\alpha_2} \tilde{\pi}_R(dz) \leq N \forall R > 0, \]

where \( \tilde{\pi}_R (dy) = \kappa (R) \pi_R (dy) \), and \( \alpha_1, \alpha_2 \) are exponents in assumption \( D(\kappa, l) \) for \( \pi_0 \).

Then for each \( p \in (1, \infty) \) there is a constant \( C = C (d, p, \kappa, l, N_0, N, N_1, c_1) \) such that

\[ |T^\varepsilon_{\lambda} v|_{L^p} \leq C |v|_{L^p}, \quad v \in \tilde{C}^\infty \left( \mathbb{R}^d \right). \]

**Proof** 1. First we prove the statement for \( p = 2 \). Observe that

\[ \hat{m}^\varepsilon_{\lambda} (\xi) = \psi^\pi (\xi) \int_{-\infty}^{\infty} \exp \left\{ \psi^{\pi_0^*} (\xi) t - \lambda t \right\} dt, \xi \in \mathbb{R}^d. \]

Hence by Lemma 7 and Corollary 4 there is \( C = C (N, c_1) \) such that

\[ |m^\varepsilon_{\lambda} (\xi)| \leq C \left| \psi^{\pi_0^*} (\xi) \right| \int_{0}^{\infty} \exp \left\{ \tilde{\psi}^{\pi_0^*} (\xi) t - \lambda t \right\} dt \leq C \]

for all \( \xi \in \mathbb{R}^d \), and

\[ |T^\varepsilon_{\lambda} v|_{L^2} \leq C |v|_{L^2}, \quad v \in \tilde{C}^\infty \left( \mathbb{R}^d \right). \]

2. Since we already have an \( L^2 \)-estimate, according to Theorem 3 of Chapter I in [9], it suffices to show that

\[ \int_{|x| \geq |s|} |m^\varepsilon_{\lambda} (x - s) - m^\varepsilon_{\lambda} (x)| \, dx \leq C, \quad \forall s \neq 0. \quad (4.14) \]
Let \( s \neq 0, R = |s| \). Changing the variable in (4.14), we see that we have to prove that

\[
R^d \int_{|x| \geq 3} |m^e_{\lambda}(R(x - \hat{s})) - m^e_{\lambda}(Rx)| \, dx \leq C, \quad |\hat{s}| = 1, \quad R > 0.
\]

(4.15)

Let \( p^{*R} \) be the pdf corresponding to the Levy measure \( \kappa(R) \pi^*_0 (R \, dy) \), and let \( L^R = L^{\pi_R} \) with \( \pi_R = \kappa(R) \pi (R \, dy) \). For \( R > 0 \), changing the variables of integration, we have

\[
m^e_{\lambda}(Rx) = \int_0^\infty \int e^{-\lambda t} (\nabla_y^e p^{*0}_e)(t, Rx) \pi (dy) \, dt
\]

\[
= \int_0^\infty \int e^{-\lambda t} (\nabla^e_R p^{*0}_e)(\kappa(R)t, Rx) \pi_R (dy) \, dt
\]

\[
= \int_0^\infty e^{-\lambda t} R^{-d} (L^{*R}) (t, x) \, dt, \quad x \in \mathbb{R}.
\]

Thus, in order to prove (4.15), it is enough to show that

\[
\int_{|x| > 3} \int_0^\infty \left| L^R p^{*R} (t, x - \hat{s}) - L^R p^{*R} (t, x) \right| \, dt \, dx
\]

\[
\leq C, \quad |\hat{s}| = 1, \quad R > 0,
\]

(4.16)

with \( C = C (d, \rho, \kappa, l, N_0, N, N_1) \).

Let \( I_1 = \{ t > 0 : \gamma(t) \leq 1 \} \), \( I_2 = \{ t > 0 : \gamma(t) > 1 \} \). Now,

\[
\int_{|x| > 3} \int_0^\infty \left| L^R p^{*R} (t, x - \hat{s}) - L^R p^{*R} (t, x) \right| \, dt \, dx
\]

\[
\leq \int_{|x| > 3} \int_{I_1} \left| L^R p^{*R} (t, x - \hat{s}) - L^R p^{*R} (t, x) \right| \, dt \, dx
\]

\[
+ \int_{|x| > 3} \int_{I_2} \left| L^R p^{*R} (t, x - \hat{s}) - L^R p^{*R} (t, x) \right| \, dt \, dx
\]

\[= A_1 + A_2.\]

By Lemma 6,

\[
A_1 = \int_{|x| > 3} \int_{I_1} \left| L^R p^{*R} (t, x - \hat{s}) - L^R p^{*R} (t, x) \right| \, dt \, dx
\]

\[
\leq 2 \int_{|x| > 2} \int_{I_1} \left| L^R p^{*R} (t, x) \right| \, dt \, dx
\]

\[
\leq C \int_{I_1} \left( 1 + t \gamma(t)^{-\alpha_1} + 1_{\sigma \in (1,2)} \gamma(t)^{-1} \right) \, dt \leq C.
\]
By Fubini theorem,

\[ A_2 = \int_{|x|>3} \int_{I_2} \left| L^R p^* (t, x - \hat{s}) - L^R p^* (t, x) \right| dt dx \]

\[ \leq \int_{|x|>3} \int_{I_2} \int_0^1 \left| L^R \nabla p^* (t, x - r\hat{s}) \right| dr dt dx \]

\[ \leq \int_{I_2} \int_{|x|>2} \left| L^R \nabla p^* (t, x) \right| dx dt \leq \int_{I_2} \left| L^R \nabla p^* (t, \cdot) \right|_{L^1} dt. \]

By Lemma 6,

\[ \int_{I_2} \left| L^R \nabla p^* (t, \cdot) \right|_{L^1} dt \leq C \int_{I_2} \gamma (t)^{-(1+\alpha_2)} dt \leq C. \]

The statement is proved. \( \square \)

**Corollary 6** Let \( \pi_0 \in \mathfrak{A}^\sigma, \kappa \) be a scaling function with scaling factor 1, and \( D(\kappa, l) \) hold for \( \pi_0 \). Let \( \pi \in \mathfrak{A}^\sigma \) be such that

\[ \int |z|^{\alpha_1} \tilde{\pi}_R (dz) + \int |z|^{\alpha_2} \tilde{\pi}_R (dz) \leq N \forall R > 0, \]

where \( \tilde{\pi}_R (dy) = \kappa (R) \pi_R (dy) \), and \( \alpha_1, \alpha_2 \) are exponents in assumption \( D(\kappa, l) \) for \( \pi_0 \). Let \( v \in \bar{C}^\infty (\mathbb{R}^d) \) and \( u \in \bar{C}^\infty (\mathbb{R}^d) \) be the unique solution to

\[ (L^{\pi_0} - \lambda) u = v \text{ in } \mathbb{R}^d. \tag{4.17} \]

Then for each \( p \in (1, \infty) \) there is a constant \( C = C (d, p, \kappa, l, N_0, N, N_1, c_1) \) such that

\[ \left| L^\pi u \right|_{L^p} \leq C \left| v \right|_{L^p}. \]

**Proof** Let \( v \in \bar{C}^\infty (\mathbb{R}^d), \lambda > 0 \). There is a unique \( u \in \bar{C}^\infty (\mathbb{R}^d) \) solving (4.17). According to Lemma 9,

\[ L^\pi u(x) = \int_0^\infty e^{-\lambda t} E L^\pi v \left( x + Z_{\pi_0}^t \right) dt, x \in \mathbb{R}^d. \]

By (4.13),

\[ T^\pi_\lambda v (x) = \int L^\pi v(x - y) \int_0^\infty e^{-\lambda t} p_{\pi_0}^0 (t, y) dt dy \]

\[ = \int_{\mathbb{R}^d} e^{-\lambda t} E L^\pi v \left( x + Z_{\pi_0}^t \right) dt, x \in \mathbb{R}^d. \tag{4.18} \]
By Lemma 12, for each \( p \in (1, \infty) \) there is a constant \( C = C (d, p, \kappa, l, N_0, N, N_1, c_1) \) such that

\[
|T^\varepsilon_k v|_{L_p} \leq C |v|_{L_p}.
\]

(4.19)

Passing to the limit in (4.19) and (4.18) as \( \varepsilon \to 0 \), we have

\[
|L^\pi u|_{L_p} \leq C |v|_{L_p}.
\]

□

Now we prove the continuity of \( L^\pi \)-norm.

**Proposition 1** Let \( \pi_0 \in \mathcal{A}^\sigma, \kappa \) be a scaling function with scaling factor \( l \), and \( D(\kappa, l) \) hold for \( \pi_0 \). Let \( \pi \in \mathcal{A}^\sigma \) be such that

\[
\int_{|z| \leq 1} |z|^\alpha_1 \tilde{\pi}_R (dz) + \int_{|z| > 1} |z|^\alpha_2 \tilde{\pi}_R (dz) \leq N \forall R > 0,
\]

where \( \tilde{\pi}_R (dy) = \kappa (R) \pi_R (dy) \), and \( \alpha_1, \alpha_2 \) are exponents in assumption \( D(\kappa, l) \) for \( \pi_0 \).

Then for each \( p \in (1, \infty) \) there is a constant \( C = C (d, p, \kappa, l, N_0, N, N_1, c_1) \) such that

\[
|L^\pi f|_{L_p} \leq C |L^{\pi_0} f|_{L_p}, \quad f \in \tilde{C}^\infty (\mathbb{R}^d).
\]

**Proof** Let \( f \in \tilde{C}^\infty (\mathbb{R}^d), \lambda > 0 \). Then \( v = (L^{\pi_0} - \lambda) f \in \tilde{C}^\infty (\mathbb{R}^d) \), and, by Corollary 6, there is \( C = C (d, p, \kappa, \mu^0, N_2, N, C_0) \) such that

\[
|L^\pi f|_{L_p} \leq C |v|_{L_p} = C |(L^{\pi_0} - \lambda) f|_{L_p}.
\]

Since \( C \) does not depend on \( \lambda > 0 \), the statement follows. □

**4.3.1 Proof of Theorem 1**

**Existence** Let \( f \in L^p (\mathbb{R}^d) \). There is a sequence \( f_n \in C_0^\infty (\mathbb{R}^d) \) such that \( f_n \to f \) in \( L^p \). For each \( n \), there is unique \( u_n \in \tilde{C}^\infty (\mathbb{R}^d) \) solving (1.2). Hence

\[
(L^\pi - \lambda) (u_n - u_m) = f_n - f_m.
\]

By Corollary 6 and Lemma 9,

\[
|L^{\pi_0} (u_n - u_m)|_{L_p} \leq C |f_n - f_m|_{L_p} \to 0,
\]

\[
|u_n - u_m|_{L_p} \leq \frac{1}{\lambda} |f_n - f_m|_{L_p} \to 0,
\]

as \( n, m \to \infty \). Hence there is \( u \in H^{\pi_0}_p \) so that \( u_n \to u \) in \( H^{\pi_0}_p \). Using Proposition 1, we can pass to the limit in \( (L^\pi - \lambda) u_n = f_n \) as \( n \to \infty \). Obviously, \( (L^\pi - \lambda) u = f \) in \( L^p \).
Uniqueness. Assume \( u_1, u_2 \in H^\pi_p (\mathbb{R}^d) \) solve (1.2). Then \( u = u_1 - u_2 \in H^\pi_p \) solves \((L^\pi - \lambda) u = 0\), i.e. \( \forall \varphi \in \tilde{C}^\infty (\mathbb{R}^d) \)

\[
\int \varphi (L^\pi - \lambda) u = \int u (L^\pi - \lambda) \varphi dx = 0
\]

According to Lemma 9, \( \int uf dx = 0 \forall f \in \tilde{C}^\infty (\mathbb{R}^d) \). Hence \( u = 0 \) a.e. The statement is proved.

4.4 Proof of Theorem 2

Let \( f \in \tilde{C}^\infty (E) \) and \( u \in \tilde{C}^\infty (E) \) be the solution to

\[
\begin{align*}
\partial_t u &= L^\pi u - \lambda u + f \text{ in } E, \\
u (0, \cdot) &= 0.
\end{align*}
\] (4.20)

By Lemma 5, the associated process \( Z^\pi_t \) has a density function \( p^\pi (t, x), x \in \mathbb{R}^d \). Then \( p^{\pi^*} (t, x) = p^\pi (t, -x), x \in \mathbb{R}^d \), is pdf of \( Z^{\pi^*}_t \). By Lemma 8,

\[
\begin{align*}
u (t, x) &= \int_0^t e^{-\lambda (t-s)} E f (s, x + Z^\pi_{t-s}) ds \\
&= \int_0^t \int e^{-\lambda (t-s)} f (s, x-y) p^{\pi^*} (t-s, y) dy ds \\
&= \int_0^t \int e^{-\lambda (t-s)} p^{\pi^*} (t-s, x-y) f (s, y) dy ds,
\end{align*}
\]

and

\[
\begin{align*}
L^{\pi_0} u (t, x) &= \int_0^t e^{-\lambda (t-s)} EL^{\pi_0} f (s, x + Z^\pi_{t-s}) ds \\
&= \int_0^t \int e^{-\lambda (t-s)} L^{\pi_0} f (s, x-y) p^{\pi^*} (t-s, y) dy ds \\
&= \int_0^t \int e^{-\lambda (t-s)} (L^{\pi_0} p^{\pi^*}) (t-s, x-y) f (s, y) dy ds. (4.21)
\end{align*}
\]

Hence

\[
L^{\pi_0} u (t, x) = \int_{-\infty}^{\infty} \int e^{-\lambda (t-s)} K (t-s, x-y) f (s, y) dy ds dy,
\]

where \( f (s, y) = \chi_{[0, \infty)} (s) \tilde{f} (s, y), (s, y) \in \mathbb{R}^{d+1} \),

\[
K (t, x) = L^{\pi_0} p^{\pi^*} (t, x) \chi_{[0, \infty)} (t), (t, x) \in \mathbb{R}^{d+1}.
\]
Let $\varepsilon \in (0, 1)$,
\[
K^\varepsilon (t, x) = L^{\pi_0} p^{\pi^*} (t, x) \chi_{[\varepsilon, \infty)} (t), (t, x) \in \mathbb{R}^{d+1},
\]
and consider for $h \in C_0^\infty (\mathbb{R}^{d+1})$,
\[
T_\lambda^\varepsilon h (t, x) = \int_{-\infty}^{t-\varepsilon} \int e^{-\lambda (t-s)} p^{\pi^*} (t-s, x-y) L^{\pi_0} h (s, y) \, dy \, ds
\]
\[
= \int_{-\infty}^{\infty} \int K^\varepsilon_\lambda (t-s, x-y) h (s, y) \, dy \, ds, (t, x) \in \mathbb{R}^{d+1},
\]
where $K^\varepsilon_\lambda (t, x) = e^{-\lambda t} K^\varepsilon (t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}$.

**Claim 1** For each $p \in (1, \infty)$ there is a constant $C = C (d, p, l, N, \eta, N_0, N_1, \kappa)$ such that
\[
|T_\lambda^\varepsilon h|_{L^p} \leq C |h|_{L^p}, h \in C_0^\infty (\mathbb{R}^{d+1}),
\]
here $L_p = L_p (\mathbb{R}^{d+1})$.

**Proof** We will apply Calderon–Zygmund theorem (see Theorem 5 in “Appendix”).
First we will prove the estimate in $L^2$. Then, according to Theorem 5, the proof reduces to verification of Hörmander condition (see below).

1. We start with $p = 2$. Obviously,
\[
|T_\lambda^\varepsilon h (t, \xi)| \leq \int_{-\infty}^{t} e^{-\lambda (t-s)} |\psi^{\pi_0} (\xi)| \exp \left| \Re \psi^{\pi^*} (\xi) (t-s) \right| |\hat{h} (s, \xi)| \, ds,
\]
$(t, \xi) \in E$. Hence by Hölder inequality, Fubini theorem, Lemma 7 and Corollary 4, we have $|\psi^{\pi_0} (\xi)| \leq C |\Re \psi^{\pi^*} (\xi)|, \xi \in \mathbb{R}^d$, for some $C = C (N, \kappa, l)$ and
\[
|T_\lambda^\varepsilon h|^2_{L^2} \leq C |h|_{L^2},
\]
i.e., (4.23) follows (cf. the proof of Lemma 10).

2. We prove (4.23) for $p \in (1, 2)$ using a version of Calderon–Zygmund theorem (Theorem 5 in Appendix). Let $Q$ be the collection of sets $Q_\delta = Q_\delta (t, x) = (t - \kappa (\delta), t + \kappa (\delta)) \times B_\delta (x), (t, x) \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}, \delta > 0$.

Note
(i) $(t, x) \notin (s - \kappa (c\delta), s + \kappa (c\delta)) \times B_\delta (y) \iff (t-s, x-y) \notin Q_{c\delta} (0) \iff |t-s| \geq \kappa (c\delta)$ or $|x-y| \geq c\delta$;
(ii) $(\tilde{s}, \tilde{y}) \in Q_\delta (s, y) \iff (\tilde{s} - s, \tilde{y} - y) \in Q_\delta (0) \iff |s - \tilde{s}| < \kappa (\delta)$ and $|y - \tilde{y}| < \delta$.

According to Theorem 5, it is enough to show
\[
\int \chi_{Q_{c\delta} (0)} (t-s, x-y) \left| K^\varepsilon_\lambda (t-\tilde{s}, x-\tilde{y}) - K^\varepsilon_\lambda (t-s, x-y) \right| \, dx \, dt \leq A
\]
(4.24)
for all \((\tilde{s}, \tilde{y}) \in Q_{\delta}(s, y) \iff (\tilde{s} - s, \tilde{y} - y) \in Q_{\delta}(0)\). Equivalently, we have to prove that

\[
\int \chi_{Q_{\delta}(0)^c}(t, x) \left| K^{x}_{\tilde{s}}(t - \tilde{s}, x - \tilde{y}) - K^{x}_{\lambda}(t, x) \right| \, dx dt
\]

\[
= \delta^d \kappa(\delta) \int \chi_{Q_{\delta}(0)^c}(\kappa(\delta) t, \delta x) \left| K^{x,\delta}_{\lambda}(t - \tilde{s}, x - \tilde{y}) - K^{x,\delta}_{\lambda}(t, x) \right| \, dx dt
\]

\[
\leq A \forall |\tilde{s}| \leq 1, |\tilde{y}| \leq 1,
\]

where

\[
K^{x,\delta}_{\lambda}(t, x) = K^{x}_{\lambda}(\kappa(\delta) t, \delta x), (t, x) \in \mathbb{R}^{d+1}.
\]

Fix \(c > 4\) such that \((\tilde{c} - 1, \infty) \subseteq I_2\) with \(\tilde{c} = l(1/c)^{-1} > 3. \) Let \(G = (\tilde{c}, \tilde{c}) \times B_{\epsilon}(0)\). Since \(\chi_G(t, x) \leq \chi_{Q_{\delta}(0)}(\kappa(\delta) t, \delta x)\), it is enough to prove that

\[
\delta^d \kappa(\delta) \int_{\mathbb{R}^{d+1}} \chi_{G^c}(t, x) \left| K^{x,\delta}_{\lambda}(t - \tilde{s}, x - \tilde{y}) - K^{x,\delta}_{\lambda}(t, x) \right| \, dx dt \leq A \quad (4.25)
\]

for all \(|\tilde{s}| \leq 1, |\tilde{y}| \leq 1\). Since

\[
\delta^d \kappa(\delta) K^{x,\delta}_{\lambda}(t, x) = e^{-\lambda \kappa(\delta)t} L^{0,\delta} p^{\tilde{s}}_{\delta}(t, x) \chi(\epsilon/\kappa(\delta), \infty)(t), (t, x) \in \mathbb{R}^{d+1},
\]

with \(L^{0,\delta} = L^{\tilde{c},0,\delta}, \tilde{p}_{0,\delta}(dy) = \kappa(\delta) \pi_0(\delta dy), \tilde{p}^{\tilde{s}}_{\delta}(dy) = \kappa(\delta) \pi^*(\delta dy)\), we rewrite (4.25) as

\[
B = \int_{\mathbb{R}^{d+1}} \chi_{G^c}(t, x) \left| e^{-\lambda \kappa(\delta)(t - \tilde{s})} L^{0,\delta} p^{\tilde{s}}_{\delta}(t - \tilde{s}, x - \tilde{y}) \chi(\epsilon/\kappa(\delta), \infty)(t - \tilde{s}) \right| \, dx dt
\]

\[
\leq A \quad (4.26)
\]

Since \(G^c \subseteq \{(t, x) : |t| \leq \tilde{c}, |x| \geq 3\} \cup \{(t, x) : |t| \geq \tilde{c}\} = G_1 \cup G_2, \)

\[
B \leq \int_{G_1} \cdots + \int_{G_2} \cdots = B_1 + B_2.
\]

By Lemma 6(a),

\[
B_1 \leq 2 \int_{0}^{\tilde{c}+1} \int_{|x| > 2} \left| L^{0,\delta} p^{\tilde{s}}_{\delta}(t, x) \right| \, dt dx
\]

\[
\leq C \int_{0}^{\tilde{c}+1} \left( 1 + t \gamma(t)^{-\alpha_1} + \sigma_{1,2,\gamma}(t)^{-1} \right) dt.
\]

**Estimate of \(B_2\)** We have

\[
B_2 \leq \int_{\tilde{c}-1}^{\infty} \int_{0}^{\infty} \left| L^{0,\delta} p^{\tilde{s}}_{\delta}(t, x) \right| \left| \chi(\epsilon/\kappa(\delta), \infty)(t) - \chi(\epsilon/\kappa(\delta), \infty)(t - \tilde{s}) \right| \, dx dt
\]
\[ + \int_{\tilde{c}}^{\infty} \int |e^{-\lambda \kappa(\delta)(t-\hat{s})} L^{0,\delta} p^\hat{s}_t (t-\hat{s}, x - \hat{y}) - e^{-\lambda \kappa(\delta)t} L^{0,\delta} p^\hat{s}_t (t, x)| \, dx \, dt = B_{21} + B_{22}. \]

Then by Lemma 5(b) and Corollary 2,

\[ B_{21} \leq \int_{\tilde{c}}^{\infty} \left( \frac{\pi(\pi+1)}{\pi(\pi-1)} \right)^{1/2} \int |L^{0,\delta} p^\hat{s}_t (t, x)| \, dx \, dt \leq \int_{\tilde{c}}^{\infty} C \, dt \leq C. \]

Since

\[ e^{-\lambda \kappa(\delta)(t-\hat{s})} L^{0,\delta} p^\hat{s}_t (t-\hat{s}, x - \hat{y}) - e^{-\lambda \kappa(\delta)t} L^{0,\delta} p^\hat{s}_t (t, x) = \int_{0}^{1} [-\lambda \kappa(\delta)] e^{-\lambda \kappa(\delta)(t-r\hat{s})} L^{0,\delta} p^\hat{s}_t (t - r\hat{s}, x - r\hat{y}) + e^{-\lambda \kappa(\delta)(t-r\hat{s})} L^{0,\delta} \partial_t p^\hat{s}_t (t - r\hat{s}, x - r\hat{y}) (\hat{s}) \, dr \]

\[ + \int_{0}^{1} e^{-\lambda \kappa(\delta)(t-r\hat{s})} \hat{y} \cdot L^{0,\delta} \nabla p^\hat{s}_t (t - r\hat{s}, x - r\hat{y}) \, dr \]

we have

\[ B_{22} \leq \int_{\tilde{c}}^{\infty} \int_{0}^{1} \lambda \kappa(\delta) e^{-\lambda \kappa(\delta)(t-r\hat{s})} |L^{0,\delta} p^\hat{s}_t (t - r\hat{s}, x - r\hat{y})| \, dr \, dx \, dt \]

\[ + \int_{\tilde{c}}^{\infty} \int_{0}^{1} |L^{0,\delta} \partial_t p^\hat{s}_t (t - rs, x - r\hat{y})| \, dr \, dx \, dt \]

\[ + \int_{\tilde{c}}^{\infty} \int_{0}^{1} |L^{0,\delta} \nabla p^\hat{s}_t (t - r\hat{s}, x - r\hat{y})| \, dr \, dx \, dt = b_1 + b_2 + b_3. \]

By Lemma 5(b),

\[ b_1 \leq \int_{\tilde{c}-1}^{\infty} \int \lambda \kappa(\delta) e^{-\lambda \kappa(\delta)t} |L^{0,\delta} p^\hat{s}_t (t, x)| \, dx \, dt \leq C. \]

Since \( \partial_t p^\hat{s}_t (t, x) = L^{\hat{s}} p^\hat{s}_t (t, x), t > 0 \), we have by Lemma 6(c),

\[ b_2 = \int_{\tilde{c}}^{\infty} \int_{0}^{1} |L^{0,\delta} L^{\hat{s}} p^\hat{s}_t (t - r\hat{s}, x - r\hat{y})| \, dr \, dx \, dt \leq \int_{\tilde{c}-1}^{\infty} \int_{0}^{\infty} |L^{0,\delta} L^{\hat{s}} p^\hat{s}_t (t, x)| \, dt \, dx \leq C \int_{\tilde{c}-1}^{\infty} \gamma(t)^{-\alpha_5} \, dt. \]
Finally, by Lemma 6(b),

\[
    b_3 \leq \int_{\varepsilon-1}^{\infty} \int |L^{0,\delta} \nabla p^\pi_\delta (t, x)| \, dx \, dt \\
    \leq C \int_{\varepsilon-1}^{\infty} y(t)^{\alpha_2} \, dt.
\]

Claim is proved for \( p \in (1, 2) \) by Theorem 5.

3. We prove the statement for \( p > 2 \) in a standard way (by duality argument). First note that

\[
    L^{\pi_0} p^\pi_\pi (t, x) = L^{\pi_0} p^\pi (t, x), (t, x) \in \mathbb{R}^{d+1},
\]

and let

\[
    \tilde{K}^\varepsilon (t, x) = K^\varepsilon (t, -x) = L^{\pi_0^\varepsilon} p^\pi (t, x) \chi_{\mathbb{R}_+^d} (t, x) \in \mathbb{R}^{d+1},
\]

and

\[
    \tilde{T}_\varepsilon^\lambda g (s, y) = \int e^{-\lambda(s-t)} \tilde{K}^\varepsilon (s-t, y-x) g(t, x) \, dt \, dx, (s, y) \in \mathbb{R}^{d+1}.
\]

Let \( 1/p + 1/q = 1 \), \( h, g \in C_0^\infty (\mathbb{R}^{d+1}) \). Then, denoting \( \tilde{g} (t, x) = g (-t, x), (t, x) \in \mathbb{R}^{d+1} \), we have (by Fubini theorem and changing the variable of integration)

\[
    \int T_\varepsilon^\lambda h (t, x) g(t, x) \, dt \, dx \\
    = \int \int e^{-\lambda(t-s)} K^\varepsilon (t-s, x-y) h(s, y) \, ds \, dy \, g(t, x) \, dt \, dx \\
    = \int \int e^{-\lambda(s-t)} \tilde{K}^\varepsilon (s-t, y-x) g(-t, x) \, dt \, dx \, h(-s, y) \, ds \, dy \\
    = \int \tilde{T}_\varepsilon^\lambda \tilde{g} (s, y) h(-s, y) \, ds \, dy,
\]

and (4.23) holds for \( \tilde{T}_\varepsilon^\lambda \) and \( q \in (1, 2) \) (see Corollary 4). Hence by Hölder inequality,

\[
    \left\| \int T_\varepsilon^\lambda h (t, x) g(t, x) \, dt \, dx \right\| \leq \left\| \tilde{T}_\varepsilon^\lambda \tilde{g} \right\|_{L_q} |h|_{L_p} \leq C \left\| g \right\|_{L_q} |h|_{L_p},
\]

and (4.23) holds for \( p > 2 \) as well, that is for all \( p \in (1, \infty) \). The claim is proved. \( \square \)

Now, we see that for \( h(t, x) = \chi_{[0, T]} (t) f(t, x) \) with \( f \in \tilde{C} (E) \),

\[
    T_\varepsilon^\lambda h (t, x) = \int_0^{t-\varepsilon} e^{-\lambda(t-s)} E L^{\pi_0^\varepsilon} f (s, x + Z^\pi_{t-s}) \, ds, (t, x) \in E. \tag{4.27}
\]
If \( u \in \tilde{C}^\infty (E) \) solves (4.20), then
\[
L^{\pi_0} u (t, x) = \int_0^t e^{-\lambda (t-s)} E L^{\pi_0} f (s, x + Z^{\pi}_{t-s}) ds, \tag{4.28}
\]
and \( |T^{\varepsilon}_\lambda h - L^{\pi_0} u|_{L^p} \to 0 \) as \( \varepsilon \to 0 \). Since the constant \( C = C (d, p, l, N, N_0, N_1, c_1) \) in the above Claim 1 does not depend on \( \varepsilon \), passing to the limit in it we get the estimate
\[
|L^{\pi_0} u|_{L^p} \leq C |f|_{L^p}. \tag{4.29}
\]

We finish the proof of Theorem 2 the same way as the proof of Theorem 1. Let \( f \in L^p (E) \). There is a sequence \( f_n \in \tilde{C}^\infty (E) \) such that \( f_n \to f \) in \( L^p (E) \). For each \( n \), there is unique \( u_n \in \tilde{C}^\infty (E) \) solving (1.1). Hence
\[
\partial_t (u_n - u_m) = (L^{\pi} - \lambda) (u_n - u_m) + f_n - f_m.
\]
By (4.29) and Lemma 8,
\[
\left| L^{\pi_0} (u_n - u_m) \right|_{L^p (E)} \leq C |f_n - f_m|_{L^p (E)} \to 0,
\]
\[
|u_n - u_m|_{L^p (E)} \leq \left( \frac{1}{\lambda} \wedge T \right) |f_n - f_m|_{L^p (E)} \to 0,
\]
\[
|u_n (t) - u_m (t)|_{L^p (\mathbb{R}^d)} \leq |f_n - f_m|_{L^p (E)} \forall t \in [0, T], \tag{4.30}
\]
as \( n, m \to \infty \). Hence there is \( u \in \mathcal{H}^{\pi_0}_{L^p} \) so that \( u_n \to u \) in \( \mathcal{H}^{\pi_0}_{L^p} \). Moreover,
\[
\sup_{t \leq T} |u_n (t) - u (t)|_{L^p (\mathbb{R}^d)} \to 0, \tag{4.31}
\]
and, according to Proposition 1,
\[
\left| L^{\pi} f \right|_{L^p (E)} \leq C \left| L^{\pi_0} f \right|_{L^p (E)}, \quad f \in \tilde{C}^\infty (E). \tag{4.32}
\]
Hence (see (4.30)–(4.32)) we can pass to the limit in the equation
\[
u_n (t) = \int_0^t [L^{\pi} u_n (s) - \lambda u_n (s) + f_n (s)] ds, 0 \leq t \leq T. \tag{4.33}
\]
Obviously, (4.33) holds for \( u \) and \( f \). We proved the existence part of Theorem 2.

Uniqueness Assume \( u_1, u_2 \in \mathcal{H}^{\pi_0}_{L^p} \) solve (1.1). Then \( u = u_1 - u_2 \in \mathcal{H}^{\pi_0}_{L^p} \) solves (4.20) with \( f = 0 \). Now, let \( \varphi \in \tilde{C}^\infty (E) \), and \( \tilde{\varphi} (t, x) = \varphi (T - t, x), (t, x) \in E \). By Lemma 8, there is unique \( \tilde{v} \in \tilde{C}^\infty (E) \) solving (4.20) with \( f = \tilde{\varphi} \) and \( \pi^+ \) instead of \( \pi \). Let \( v (t, x) = \tilde{v} (T - t, x), (t, x) \in E \). Then \( \partial_t v + L^{\pi^+} v - \lambda v + \varphi = 0 \) in \( E \) and \( v (T) = v (T, \cdot) = 0 \). Integrating by parts,
\[
\int_E \varphi u = \int_E u \left( -\partial_t v - L^{\pi^+} v + \lambda v \right)
\]
\begin{equation*}
\int_E v \left( \partial_t u - L\pi u + \lambda u \right) = 0.
\end{equation*}

Hence \( \int_E u \varphi \, dt \, dx = 0 \, \forall \varphi \in \tilde{C}\infty (E) \). Hence \( u = 0 \) a.e. Theorem 2 is proved.

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**Appendix**

Given a function \( \kappa : (0, \infty) \rightarrow (0, \infty) \), consider the collection \( \mathbb{Q} \) of sets \( Q_\delta = Q_\delta (t, x) = (t - \kappa (\delta), t + \kappa (\delta)) \times B_\delta (x) \), \((t, x) \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}, \delta > 0. \) The volume \(|Q_\delta (t, x)| = c_0 \kappa (\delta) \delta^d \). We will need the following assumptions.

**A1** \( \kappa \) is continuous, \( \lim_{\delta \rightarrow 0} \kappa (\delta) = 0 \) and \( \lim_{\delta \rightarrow \infty} \kappa (\delta) = \infty. \)

**A2** There is a nondecreasing continuous function \( l (\varepsilon), \varepsilon > 0 \), such that \( \lim_{\varepsilon \rightarrow 0} l (\varepsilon) = 0 \) and 

\begin{equation*}
\kappa (\varepsilon r) \leq l (\varepsilon) \kappa (r), r > 0, \varepsilon > 0.
\end{equation*}

Since \( Q_\delta (t, x) \) not “exactly” increases in \( \delta \), we present the basic estimates involving maximal functions based on the system \( \mathbb{Q} = \{ Q_\delta \}. \)

**Vitali lemma, maximal functions**

We start with engulfing property.

**Lemma 13** Let **A2** hold. If \( Q_\delta (t, x) \cap Q_{\delta'} (r, z) \neq \emptyset \) with \( \delta' \leq \delta \), then there is \( K_0 \geq 3 \) such that \( Q_{K_0 \delta} (t, x) \) contains both \( Q_\delta (t, x) \) and \( Q_{\delta'} (r, z), \) and

\begin{equation*}
|Q_\delta (t, x)| \leq |Q_{K_0 \delta} (t, x)| \leq K_0 l (K_0) |Q_\delta (t, x)|.
\end{equation*}

**Proof** Let \((s, y) \in Q_\delta (t, x) \cap Q_{\delta'} (r, z) \) with \( \delta' \leq \delta \). If \((r', z') \in Q_{\delta'} (r, z)\), then \(|z' - x| \leq 3\delta\), and using **A2**,

\begin{equation*}
|r' - t| \leq |r' - r| + |r - s| + |s - t| \leq 2\kappa (\delta') + \kappa (\delta) \leq [2l (1) + 1] \kappa (\delta).
\end{equation*}

We choose \( K_0 \geq 3 \) so that \([2l (1) + 1] l (1/K_0) \leq 1 \). By **A2**,

\begin{equation*}
[2l (1) + 1] \kappa (\delta) \leq [2l (1) + 1] l (1/K_0) \kappa (K_0 \delta) \leq \kappa (K_0 \delta).
\end{equation*}

Hence \( Q_{\delta'} (r, z) \subseteq Q_{K_0 \delta} (t, x) \) and, obviously, \( Q_\delta (t, x) \subseteq Q_{K_0 \delta} (t, x) \). Also,

\begin{equation*}
|Q_{K_0 \delta} (t, x)| = c_0 K_0^d \delta^d \kappa (K_0 \delta) \leq c_0 K_0^d \delta^d l (K_0) \kappa (\delta) = K_0^d l (K_0) |Q_\delta (t, x)|.
\end{equation*}

\( \square \)

Now, following 3.1 in [9], we prove Vitali covering lemma.
Lemma 14 Let \( E \subseteq \mathbb{R} \times \mathbb{R}^d \) be a union of a finite collection \( \{ Q' \} \) of sets from the system \( \{ Q_\delta (t, x) : (t, x) \in \mathbb{R}^{d+1}, \delta > 0 \} \) and \( A2 \) hold.

There is a positive \( c = \frac{1}{K_0^d (K_0)} \) and a disjoint subcollection \( \{ Q^k = Q_{\delta_k} (t_k, x_k) , \ 1 \leq k \leq m \} \) such that

\[
\sum_{k=1}^{m} |Q^k| \geq c \ |E| .
\]

**Proof** Let \( Q^1 = Q_{\delta_1} (t_1, x_1) \) be the set of the collection \( \{ Q' \} \) with maximal \( \delta \). Let \( Q^2 = Q_{\delta_2} (t_2, x_2) \) be the set with maximal \( \delta \) among remaining sets in \( \{ Q' \} \) that do not intersect \( Q^1 \). According to Lemma 13, \( Q_{K_0 \delta_1} (t_1, x_1) \) contains \( Q^1 \) and all \( Q_\delta \) in \( \{ Q' \} \) that intersect \( Q^1 \) and such that \( \delta \leq \delta_1 \). Continuing we get \( Q_{K_0 \delta_k} (t_k, x_k) \) containing \( Q^k = Q_{\delta_k} (t_k, x_k) \) and all \( Q_\delta \) in \( \{ Q' \} \) that intersect \( Q^k \) and such that \( \delta \leq \delta_k \). So we obtain a finite disjoint subcollection \( \{ Q^k = Q_{\delta_k} (t_k, x_k) , 1 \leq k \leq m \} \) such that \( \bigcup_{k=1}^{m} Q_{K_0 \delta_k} (t_k, x_k) \supseteq Q_{\delta} \) for any \( Q_{\delta} \) in \( \{ Q' \} \). Hence \( \bigcup_{k=1}^{m} Q_{K_0 \delta_k} (t_k, x_k) \supseteq E \), and by Lemma 13,

\[
|E| \leq \sum_{k=1}^{m} |Q_{K_0 \delta_k}| \leq K_0^d (K_0) \sum_{k=1}^{m} |Q^k| .
\]

\( \square \)

**Remark 5** The statement of the Lemma 14 still holds if instead of \( A2 \) we assume that there is a constant \( C \) so that \( C \kappa (\delta) \geq \kappa (\delta') \) whenever \( \delta \geq \delta' \).

Following [9], for a locally integrable function \( f (t, x) \) on \( \mathbb{R}^{d+1} \) we define

\[
(A_\delta f) (t, x) = \frac{1}{|Q_\delta (t, x)|} \int_{Q_\delta (t, x)} f (s, y) dsdy, (t, x) \in \mathbb{R} \times \mathbb{R}^d , \delta > 0
\]

and the maximal function of \( f \) by

\[
\mathcal{M} f (t, x) = \sup_{\delta > 0} (A_\delta |f|) (t, x), (t, x) \in \mathbb{R}^{d+1} .
\]

We use collection \( \mathcal{Q} \) to define a larger, noncentered maximal function of \( f \), as

\[
\tilde{\mathcal{M}} f (t, x) = \sup_{(t, x) \in \mathcal{Q}} \frac{1}{|Q|} \int_{Q} |f (s, y) |dsdy, (t, x) \in \mathbb{R}^{d+1},
\]

where sup is taken over all \( Q \in \mathcal{Q} \) that contain \( (t, x) \).

**Remark 6** Let \( A2 \) hold and \( K_0 \) be a constant in Lemma 13 . For a locally integrable \( f \) on \( \mathbb{R}^{d+1} \),

\[
\mathcal{M} f \leq \tilde{\mathcal{M}} f \leq \frac{1}{K_0^d (K_0)} \mathcal{M} f .
\]
Indeed, if \((t, x) \in Q' = Q_\delta(t', x')\), then by Lemma 13

\[
\frac{1}{|Q'|} \int_{Q'} |f| \leq \frac{K_0^d l(K_0)}{|Q_{K_0\delta}(t, x)|} \int_{Q_{K_0\delta}(t, x)} |f|.
\]

Note \(\widetilde{M}f\) is lower semicontinuous.

Theorem 1.3.1 in [9] holds for \(Q\) (we sketch its proof).

**Theorem 3** Let \(A2\) hold and \(f\) be measurable function on \(\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d\).

(a) If \(f \in L_p, 1 \leq p \leq \infty\), then \(Mf\) is finite a.e.

(b) If \(f \in L_1\), then for every \(\alpha > 0\),

\[
|\{Mf(t, x) > \alpha\}| \leq \frac{c}{\alpha} \int |f| dt dx.
\]

(c) If \(f \in L_p, 1 < p \leq \infty\), then \(Mf \in L_p\) and

\[
|Mf|_{L_p} \leq N_p |f|_{L_p},
\]

where \(N_p\) depends only on \(p, l\) and \(K_0\).

**Proof** (b) Let \(E_\alpha = \{\widetilde{M}f(t, x) > \alpha\}\) and \(E \subseteq E_\alpha\) be any compact subset. Since \(\widetilde{M}f\) is lower semicontinuous, \(E_\alpha\) is open. By definition of \(\widetilde{M}f\) for each \((t, x) \in E\), there is \(Q \in Q\) so that \((t, x) \in Q\) and

\[
|Q| \leq \frac{1}{\alpha} \int_Q |f|.
\]

Since \(E\) is compact there exist a finite number \(Q_{\delta_1}(t_1, x_1), \ldots, Q_{\delta_n}(t_n, x_n) \in Q\) so that \(E \subseteq \bigcup_{j=1}^n Q_{\delta_j}(t_j, x_j)\). By Lemma 14, there is a subcovering of disjoint sets \(Q^1, \ldots, Q^m\) so that

\[
|E| \leq \sum_{k=1}^m |Q^k| \leq \frac{c}{\alpha} \sum_{k=1}^m \int_{Q^k} |f| \leq \frac{c}{\alpha} \int |f|.
\]

with \(c = K_0^d l(K_0)\). Taking sup over all such compacts \(E\) we get (b).

(c) Let \(f_1 = f \chi_{\{|f| > \alpha/2\}}\). Note that \(\widetilde{M}f \leq \widetilde{M}f_1 + \frac{\alpha}{2}\). Hence by part (b)

\[
\left|\{\widetilde{M}f > \alpha/2\}\right| \leq \left|\{\widetilde{M}f_1 > \alpha/2\}\right| \leq \frac{2c}{\alpha} \int_{|f| > \alpha/2} |f| dm.
\]

On the other hand,

\[
\int (\widetilde{M}f)^p = p \int_0^\infty \left|\{\widetilde{M}f > \alpha\}\right| \alpha^{p-1} d\alpha
\]
\[
\leq 2c_p \int_0^{2|f|} \alpha^{p-2} d\alpha |f| = c_p \cdot \frac{p}{p-1} \int |f|^p.
\]

**Corollary 7** Let \( f \in L_1 \). Then

\[
\lim_{\delta \to 0} A_\delta f(t, x) = f(t, x) \text{ a.e.}
\]

and \(|f(t, x)| \leq M f(t, x)\) a.e. Moreover, for every \( \alpha > 0 \),

\[
\left| \left\{ M f(t, x) > \alpha \right\} \right| \leq \frac{2c}{\alpha} \int_{\{M f(t, x) > \alpha/2\}} |f| dm,
\]

where \( c \) is a constant in Theorem 3.

**Proof** Let \( f \in L_1, \varepsilon > 0 \). There is \( g \in C_c(\mathbb{R}^{d+1}) \) so that \(|f - g|_{L_1} \leq \varepsilon\). Let \( \eta > 0 \). Since \( g \) is uniformly continuous, for all \((t, x)\)

\[
|A_\delta g(t, x) - g(t, x)| \leq \frac{1}{|Q_\delta(t, x)|} \int_{Q_\delta(t, x)} |g(s, y) - g(t, x)| ds dy \leq \eta
\]

if \( \delta \leq \delta_0 \) for some \( \delta_0 > 0 \). Hence \( \sup_{t, x} |A_\delta g(t, x) - g(t, x)| \to 0 \) as \( \delta \to 0 \). Now for \((t, x) \in \mathbb{R}^{d+1}, \)

\[
\limsup_{\delta \to 0} |A_\delta f(t, x) - f(t, x)|
\]

\[
\leq \limsup_{\delta \to 0} |A_\delta f(t, x) - A_\delta g(t, x)| + |g(t, x) - f(t, x)|
\]

\[
\leq \mathcal{M}(f - g)(t, x) + |g(t, x) - f(t, x)|.
\]

Hence for any \( \alpha > 0 \), by Theorem 3,

\[
\left| \left\{ \limsup_{\delta \to 0} |A_\delta f(t, x) - f(t, x)| > \alpha \right\} \right|
\]

\[
\leq |\mathcal{M}(f - g) > \alpha/2| + ||g - f| > \alpha/2||
\]

\[
\leq \frac{2c\varepsilon}{\alpha} + \frac{2\varepsilon}{\alpha}.
\]

Since \( \varepsilon \) and \( \alpha \) are arbitrary, it follows that \( \limsup_{\delta \to 0} |A_\delta f(t, x) - f(t, x)| = 0 \) a.e. Hence for almost all \((t, x)\)

\[
|f(t, x)| = \left| \lim_{\delta \to 0} A_\delta f(t, x) \right| \leq \lim_{\delta \to 0} \frac{1}{|Q_\delta(t, x)|} \int_{Q_\delta(t, x)} |f(t, y)| dt dy.
\]
Finally, for \( f_1 = f \chi_{[|f| > \alpha/2]} \) we have \( \tilde{M}f \leq \tilde{M}f_1 + \frac{q}{2} \), and by Theorem 3(b),
\[
\left| \left\{ \tilde{M}f > \alpha \right\} \right| \leq \left| \left\{ \tilde{M}f_1 > \alpha/2 \right\} \right| \leq \frac{2c}{\alpha} \int_{|f| > \alpha/2} |f| \leq \frac{2c}{\alpha} \int_{\tilde{M}f > \alpha/2} |f|.
\]

\( \square \)

Calderon–Zygmund decomposition

Assume \( A_1, A_2 \) hold. Let \( F \subseteq \mathbb{R} \times \mathbb{R}^d \) be closed and \( O = F^c = \mathbb{R}^{d+1} \setminus F \). For \((t, x) \in O\), let
\[
D(t, x) = \inf \{ \delta > 0 : Q_\delta(t, x) \cap F \neq \emptyset \}.
\]
For each \((t, x) \in O\), \( D(t, x) \in (0, \infty) \). Let \( K_0 \) be a constant in Lemma 13. We fix \( A > 1 \) so that \( l(1/A) < 1 \) and \( \varepsilon > 0 \) so that \( l(2K_0\varepsilon) < 1 \), \( \varepsilon \leq \frac{1}{4AK_0} < 1 \). Then, denoting \( D = D(t, x) \), we have
\[
\kappa(\varepsilon D) \leq l(2\varepsilon)\kappa(D/2) \leq \kappa\left(\frac{D}{2}\right), \quad \kappa(\varepsilon D) \leq l(\varepsilon)\kappa(D) \leq \kappa(D),
\]
\[
\kappa(D) \leq l(1/A)\kappa(AD) < \kappa(AD),
\]
and
\[
\kappa(\varepsilon D) \leq l(2K_0\varepsilon)\kappa(D/2K_0) \leq \kappa(D/2K_0).
\]
Consider the covering \( Q_{\varepsilon D(t,x)}(t, x), (t, x) \in O \), of \( O \). Let
\[
Q^k = Q_{\varepsilon D(t_k,x_k)}(t_k, x_k), k \geq 1,
\]
be its maximal disjoint subcollection: for any \( Q_{\varepsilon D(t,x)}(t, x) \) there is \( k \) so that \( Q_{\varepsilon D(t,x)}(t, x) \cap Q^k \neq \emptyset \). Let
\[
Q^{*k} = Q_{D(t_k,x_k)/2}(t_k, x_k), Q^{**k} = Q_{AD(t_k,x_k)}(t_k, x_k).
\]
Note that \( Q^k \subseteq Q^{*k} \subseteq Q_{D(t_k,x_k)}(t_k, x_k) \subseteq O \), \( Q^{**k} \cap F \neq \emptyset \). We will show that \( \bigcup_k Q^{*k} = O \). Let \((t, x) \in O \) and \( Q_{\varepsilon D(t_k,x_k)}(t_k, x_k) \cap Q_{\varepsilon D(t,x)}(t, x) \neq \emptyset \) for some \( k \). Since
\[
Q_{\varepsilon D(t_k,x_k)}(t_k, x_k) \subseteq Q_{D(t_k,x_k)}(t_k, x_k) \subseteq Q_{AD(t_k,x_k)}(t_k, x_k),
\]
\[
Q_{\varepsilon D(t,x)}(t, x) \subseteq Q_{D(t,x)/(2K_0)},
\]
it follows that
\[
Q_{D(t,x)/(2K_0)}(t, x) \cap Q_{AD(t_k,x_k)}(t_k, x_k) \neq \emptyset.
\]
We show by contradiction that $AD(t_k, x_k) \geq D(t, x) / 2K_0$. If not so, then $AD(t_k, x_k) < D(t, x) / 2K_0$, and, by Lemma 13, $Q_{D(t, x) / 2K_0}(t, x)$ and $Q_{AD(t_k, x_k)}(t_k, x_k)$ are contained in $Q_{D(t, x)/2}(t, x) \subseteq O$: a contradiction to $Q_{AD(t_k, x_k)}(t_k, x_k) \cap F \neq \emptyset$. Therefore $AD(t_k, x_k) \geq D(t, x) / 2K_0$ and $2AK_0 \varepsilon D(t_k, x_k) \geq \varepsilon D(t, x)$. Now, $Q_{\varepsilon D(t_k, x_k)}(t_k, x_k) \subseteq Q_{2AK_0 \varepsilon D(t_k, x_k)}(t_k, x_k)$ and $Q_{\varepsilon D(t, x)}(t, x) \neq \emptyset$. Hence by Lemma 13, $Q_{\varepsilon D(t, x)}$ is contained in $Q_{2AK_0 \varepsilon D(t_k, x_k)}(t_k, x_k)$. Since $2AK_0^2 \varepsilon \leq \frac{1}{2K_0}$, it follows by Lemma 13 that

$$Q_{\varepsilon D(t, x)}(t, x) \subseteq Q_{2AK_0 \varepsilon D(t_k, x_k)}(t_k, x_k) \subseteq Q_{D(t_k, x_k)/2}(t_k, x_k) = Q^{\star k}.$$ 

So we proved the following statement.

**Lemma 15** (cf. Lemma 2 in Chapter I, 3.2 of [9]) Assume $A1, A2$ hold. Given a closed nonempty $F$, there are sequences $Q^k$, $Q^{\star k}$ and $Q^{\star \star k}$ in $Q$ having the same center but with radius expanded by the same factor $c_{1}^{\star\star} > c_1^* > c_1$ so that $Q^k \subseteq Q^{\star k} \subseteq Q^{\star \star k}$ (all of them are of the form $Q_k \subseteq Q^{\star k} \subseteq Q^{\star \star k}$) and

(a) the sets $Q^k$ are disjoint.
(b) $\bigcup_k Q^{\star k} = O = F^c$.
(c) $Q^{\star \star k} \cap F \neq \emptyset$ for each $k$.

**Remark 7** Assume $A1, A2$ hold and $Q^k \subseteq Q^{\star k} \subseteq Q^{\star \star k}$ be the sequences in $Q$ from Lemma 15. It is easy to find a sequence of disjoint measurable sets $C^k$ so that $Q^k \subseteq C^k \subseteq Q^{\star k}$ and $\bigcup_k C^k = O$. For example (see Remark, p. 15, in [9]),

$$C^k = Q^{\star k} \cap \left( \bigcup_{j < k} C^j \right)^c \cap \left( \bigcup_{j > k} Q^j \right)^c.$$ 

Now we derive Calderon–Zygmund decomposition for $Q$.

**Theorem 4** (cf. Theorem 2 in Chapter I, 4.1 of [9]) Assume $A1, A2$ hold. Let $f \in L^1(R \times R^d)$, $\alpha > 0$ and $O_{\alpha} = \{f | Mf > \alpha\}$. Consider the sets $Q^k \subseteq C^k \subseteq Q^{\star k} \subseteq O_{\alpha}$ of Lemma 15 and Remark 7 associated to $O_{\alpha}$.

There is a decomposition $f = g + b$ with

$$g(t, x) = \begin{cases} f(t, x) & \text{if } (t, x) \notin O_{\alpha}, \\ \frac{1}{|C^k|} \int_{C^k} f & \text{if } (t, x) \in C^k, k \geq 1, \end{cases}$$

and with $b = \sum_k b_k$, where

$$b_k = \chi_{C^k} \left[ f(x) - \frac{1}{|C^k|} \int_{C^k} f \right], k \geq 1,$$

(note $C^k$ are disjoint, $\bigcup_k C^k = O_{\alpha}$). Also,

(i) $|g(t, x)| \leq c\alpha$ for a.e. $x$. 

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(ii) \( \text{support}(b_k) \subseteq Q^*k \),

\[
\int b_k = 0 \text{ and } \int |b_k| \leq c\alpha |Q^*k| .
\]

(iii) \( \sum_k |Q^*k| \leq \frac{c}{\alpha} \int |f| . \)

Proof. The set \( O_\alpha = \{ \tilde{M} f > \alpha \} \) is open. We can apply Lemma 15 and Remark 7 to it and consider the sets \( Q^k \subseteq C^k \subseteq Q^*k \subseteq O_\alpha \) with \( C^k \) disjoint and \( \cup_k C^k = O_\alpha \).

Define \( g \) by (4.34). Hence \( f = g + \sum_k b_k \) with \( b_k \) given by (4.35). Obviously

\[
\sum_k |Q^k| \leq |O_\alpha| .
\]

(i) By Corollary 7, \( |f(t, x)| \leq \alpha \) a.e. on \( O^c_\alpha = \{ \tilde{M} f(t, x) \leq \alpha \} \). Hence: so \( |g(x)| \leq \alpha \) a.e. on \( O^c_\alpha \). On the other hand, if \( Q^{**k} \in \mathbb{Q} \) is the sequence of Lemma 15, then

\[
\frac{1}{|Q^{**k}|} \int_{Q^{**k}} |f| \leq \alpha
\]

because \( Q^{**k} \cap O^c_\alpha \neq \emptyset \) and \( \tilde{M} f(t, x) \leq \alpha \) on \( O^c_\alpha \) (the definition of \( \tilde{M} \) implies it).

Since \( |Q^k| \leq |C^k| \leq |Q^*k| \leq |Q^{**k}| \leq l \left( \frac{c_*}{c_1} \right) |Q^k| \) and \( C^k \subseteq Q^{**k} \), it follows that

\[
|g| \leq c\alpha .
\]

(ii) Only inequality is not trivial:

\[
\int |b_k| \leq 2 \int_{C^k} |f| \leq 2 |Q^{**k}| \frac{1}{|Q^{**k}|} \int_{Q^{**k}} |f| \leq c\alpha |Q^k| .
\]

(iii) We have

\[
|O_\alpha| = |\{ \tilde{M} f(t, x) > \alpha \}| \geq \sum_k |Q^k| \geq \tilde{c} \sum_k |Q^*k|
\]

and the inequality follows by Theorem 3. \( \square \)

\( L_p \)-estimates

Let

\[
(T f)(t, x) = \int_{\mathbb{R}^{d+1}} K(t, x, s, y) f(s, y) \, ds \, dy , (t, x) \in \mathbb{R}^{d+1},
\]

where \( K \) is measurable and for almost all \((t, x) \in \mathbb{R}^{d+1}\) the function \( K(t, x, \cdot) \) is integrable for all \( f \in C_0^\infty(\mathbb{R}^{d+1}) \). We assume that \( T \) is bounded on \( L_q \):

\[
|T f|_{L_q} \leq C |f|_{L_q} , f \in L_q .
\]
In addition, we assume that Hörmander condition holds: there are constants \( c > 1, A > 0 \) so that for any \( Q_\delta \in \mathbb{Q} \),

\[
\int_{\mathbb{R}^{d+1}\setminus Q_\delta(s,y)} |K(t,x,\bar{s},\bar{y}) - K(t,x,s,y)| \, dx dt \leq A, (\bar{s}, \bar{y}) \in Q_\delta(s,y) .
\] (4.37)

**Theorem 5** (cf. Theorem 3 of Chapter I, 5.1 in [9]) Let \( A_1, A_2 \) (4.36) and (4.37) hold. Then \( T \) is bounded in \( L_p \)-norm on \( L_p \cap L_q \) if \( 1 < p < q \). More precisely,

\[
|T(f)|_{L_p} \leq A_p \, |f|_{L_p}, \, f \in L_p \cap L_q, \, 1 < p < q ,
\]

where \( A_p \) depends only on the constant \( A \) and \( p \).

**Proof** By Marcinkiewicz interpolation theorem (see [9]), it is enough to prove that

\[
m(|Tf| > \alpha) \leq \frac{A'}{\alpha} \int |f| \, dx dt , \, f \in L_1 \cap L_q , \, \alpha > 0 ,
\]

where \( A' \) depends on \( A \).

For a large constant \( c' \) (to be determined) we estimate \( m(\cup_n Q_n^{**}) \leq \sum_n m(Q_n^{**}) \leq c \sum_n m(Q_n) \leq \frac{c}{\alpha} \int |f| \).

It is enough to show that

\[
\left| \{|Tg| > (c'/2) \alpha \} \right| + \left| \{|Tb| > (c'/2) \alpha \} \right| \leq A'/\alpha \int |f| \, dx .
\]

First notice that \( g \in L_q \). Indeed (recall \( \cup_k Q_k^* = \cup_k C_k \)),

\[
\int |g|^q = \int (\cup_k Q_k^*)^c |g|^q + \int \cup_k C_k |g|^q \leq c \alpha^{q-1} \int |f|
\]

because

\[
\int (\cup_k Q_k^*)^c |g|^q \leq \alpha^{q-1} \int (\cup_k Q_k^*)^c |f| ,
\]

\[
\int \cup_k Q_k^* |g|^q \leq c \alpha^q \sum_k |Q_k^*| \leq c \alpha^{q-1} \int |f| .
\]

By Chebyshev inequality,

\[
\left| \{|Tg| > (c'/2) \alpha \} \right| \leq \left( \frac{c' \alpha}{2} \right)^{-q} |Tg|_L_q^q \leq \left( \frac{c' \alpha}{2} \right)^{-q} A^q |g|_{L_q}^q
\]

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\[
\leq c \left( \frac{c' \alpha}{2} \right)^{-q} A^q \alpha^{q-1} \int \left| f \right| \leq A' \frac{\alpha}{\alpha} \left| f \right|_{L^1},
\]

and

\[
\int_{(\bigcup_k Q^+_k)^c} \left| T b \right| \leq \sum_k \int_{(Q^+_k)^c} \left| T b_k \right|.
\]

Let \((s_k, y_k)\) be the center of \(Q^+_k\) (and \(Q^+_{k^*}\)). Since for \(x \notin Q^+_k\), we have, denoting \(f_k = 1/\left| C_k \right| \int_{C_k} f\),

\[
T b_k = \int_{C_k} K(t, x, s, y) \left[ f(s, y) - f_k \right] ds dy
\]

\[
= \int_{C_k} \left[ K(t, x, s, y) - K(t, x, s_k, y_k) \right] \left[ f(s, y) - f_k \right] ds dy,
\]

and

\[
\int_{(Q^+_k)^c} \left| T b_k \right| dtdx \leq \int_{(Q^+_k)^c} \left| b_k \right| ds dy \sup_{(s, y) \in Q^+_k} \int_{(Q^+_{k^*})^c} \left| K(t, x, s, y) - K(t, x, s_k, y_k) \right| dtdx \leq c A \alpha \left| Q^+_k \right|.
\]

Hence

\[
\int_{(\bigcup_k Q^+_k)^c} \left| T b \right| \leq c A \alpha \sum_k \left| Q^+_k \right| \leq c A \int \left| f \right|.
\]

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