A Direct Proof of the 2nd Atiyah-Sutcliffe Conjecture for Convex Quadrilaterals

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Abstract

We present a direct proof of the second conjecture made by M. Atiyah and P. Sutcliffe for the case of convex quadrilaterals. Unlike previous work on this conjecture, our proof does not require any computer aided computations. The new proof relies on a new geometric inequality proved recently by the author.

Keywords: Atiyah determinant ; Atiyah-Sutcliffe conjectures

1 Introduction

While studying the spin statistics theorem using classical quantum theory, M. V. Berry and J. M. Robbins came across the following purely geometric question in $\mathbb{R}^3$: Does there exist a continuous function from the set of $n$ distinct points in $\mathbb{R}^3$ to the flag manifold $U(n)/T^n$ which is compatible with the action of the symmetric group? In his attempt to answering their question, M. Atiyah gave a very natural and elegant solution whose validity depends on the nonvanishing of a certain determinant function. Although another construction was given by Atiyah in answering the question of Berry and Robbins in the positive, the nonvanishing of the determinant given
in the original construction remained an unresolved conjecture. Numerical evidence of the validity of the conjecture were provided by Atiyah and Sutcliffe in their paper [7] where they added two new conjectures which imply the first one. All three conjectures have remained open since then except for a few verifications on special configurations of points [4], [5], [6].

The construction of the determinant begins with \( n \) distinct points in \( \mathbb{R}^3 \). By choosing a preferred axis in \( \mathbb{R}^3 \) we identify it with \( \mathbb{R} \times \mathbb{C} \) and denote our \( n \) points by \( P_j = (a_j, z_j) \) for \( j = 1, \ldots, n \). We consider the vector \( V_{jk} = (a_k - a_j, z_k - z_j) \) to be the vector from \( P_j \) to \( P_k \) and use the Hopf map \( h : \mathbb{C}^2 \rightarrow \mathbb{R}^3 \) given by \( h(z, w) = (((|z|^2 - |w|^2)/2, zw) \) to lift \( V_{jk} \) from \( \mathbb{R}^3 \) to \( \mathbb{C}^2 \).

If we denote also \( a_k - a_j \) by \( a_{jk} \), \( z_k - z_j \) by \( z_{jk} \) and set \( d_{jk} = a_{jk} + \sqrt{a_{jk}^2 + |z_{jk}|^2} \) we see that the lift of \( V_{jk} \) is given by

\[
h^{-1}(V_{jk}) = e^{i\theta} \frac{1}{\sqrt{d_{jk}}}(d_{jk}, z_{jk}) \quad ; \quad \theta \in \mathbb{R}, \ i = \sqrt{-1}
\]

Since \( h^{-1}(V_{jk}) \) does not uniquely depend on \( V_{jk} \) we follow the normalization of Atiyah by choosing \( (-\overline{w}, \overline{z}) \) as a lift (under \( h \)) for \( V_{kj} \) whenever \( (z, w) \) is a lift for \( V_{jk} \) (see [2]). We let \( C_k \) be the symmetric tensor product of \( h^{-1}(V_{kj}) \) for all \( j \neq k \). The determinant of the \( n \times n \) matrix whose \( k \)th column is \( C_k \) will be called the Atiyah determinant and will be denoted by \( At(P_1, \ldots, P_n) \) or just \( At \) if the points are already specified. The determinant function \( At \) is invariant under rotations and translations and gets conjugated under plane reflections. The first conjecture (C1) of Atiyah is \( At \neq 0 \) while the second Atiyah-Sutcliffe conjecture (C2) is \( At \geq \prod_{j<k} (2||P_jP_k||) \), and the third one (C3) is

\[
At^{n-2}(P_1, \ldots, P_n) \geq \prod_{j=1}^{n} At(P_1, \ldots, P_{j-1}, P_{j+1}, P_n)
\]

for all distinct points \( P_1, \ldots, P_n \) of \( \mathbb{R}^3 \). It is clear from the statement of these conjectures that (C3) \( \implies \) (C2) \( \implies \) (C1).

Most of the methods applied in previous work on this determinant considered special cases of configurations of points (see for example [4], [6]) or used some sort of computer aid in their computations. The main work on the general four-point case was done by M. Eastwood and P. Norbury in [5] where they proved the first conjecture of Atiyah to be true for \( n = 4 \) points.
in space. Others followed their formula using Maple (see [8]) and proved the three conjectures with computer aid. Our paper provides the first direct proof of the second Atiyah-Sutcliffe conjecture for four points forming a convex quadrilateral. The proof we present here does not rely on any computer calculations, but rather on a geometric inequality proved by the author in a previous paper [9]. We hope that this new approach will pave the way for generalizations that may solve the full conjecture.

2 The Planar Case

In this section, we explore the Atiyah determinant for four planar points and present it in standard form. First note that $At$ is a real number when all points lie in a plane. This is because a reflection in their plane leaves the points fixed since $A\bar{t} = At$. This was already mentioned in [2] and will also be seen here concretely from the expansion of its standard form. Moreover, after applying a solid motion, we may assume our points are in $\{0\} \times \mathbb{C}$ so that all $a_{jk}$ values become zeros and our points can be written as $(0, z_1), \ldots, (0, z_4)$. Based on that, the construction of the Atiyah determinant is done as follows:

When the first point is considered as an observer of the three other points we obtain $(0, z_{12}), (0, z_{13}), (0, z_{14})$ and these lift under the Hopf map $h$ to

$$\frac{1}{\sqrt{r_{12}}}(r_{12}, \overline{z}_{12}), \frac{1}{\sqrt{r_{13}}}(r_{13}, \overline{z}_{13}), \frac{1}{\sqrt{r_{14}}}(r_{14}, \overline{z}_{14})$$

where $r_{jk}$ denotes $|z_{jk}|$.

We do the same thing when $(0, z_2)$ is a vision point and get the vectors $(0, z_{21}), (0, z_{23}), (0, z_{24})$ whose lifts are

$$\frac{1}{\sqrt{r_{21}}}(z_{21}, r_{21}), \frac{1}{\sqrt{r_{23}}}(z_{23}, r_{23}), \frac{1}{\sqrt{r_{24}}}(r_{24}, z_{24}).$$

Similarly, the lifts corresponding for the vision points $(0, z_3)$ and $(0, z_4)$ are

$$\frac{1}{\sqrt{r_{31}}}(z_{31}, r_{31}), \frac{1}{\sqrt{r_{32}}}(z_{32}, r_{32}), \frac{1}{\sqrt{r_{34}}}(r_{34}, z_{34}).$$

and
\[ \frac{1}{\sqrt{r_{41}}} (z_{41}, r_{41}), \quad \frac{1}{\sqrt{r_{42}}} (z_{42}, r_{42}), \quad \frac{1}{\sqrt{r_{43}}} (z_{43}, r_{43}). \]

respectively.

Using \( u_{jk} = \frac{z_{jk}}{r_{jk}} \) to be the direction of \( z_{jk} \) we see that the lifts can be written as:

\[ \sqrt{r_{12}} (1, \bar{u}_{12}), \quad \sqrt{r_{13}} (1, \bar{u}_{13}), \quad \sqrt{r_{14}} (1, \bar{u}_{14}) \]

\[ \sqrt{r_{21}} u_{21} (1, \bar{u}_{21}), \quad \sqrt{r_{23}} (1, \bar{u}_{23}), \quad \sqrt{r_{24}} (1, \bar{u}_{24}) \]

\[ \sqrt{r_{31}} u_{31} (1, \bar{u}_{31}), \quad \sqrt{r_{32}} u_{32} (1, \bar{u}_{32}), \quad \sqrt{r_{34}} (1, \bar{u}_{34}) \]

\[ \sqrt{r_{41}} u_{41} (1, \bar{u}_{41}), \quad \sqrt{r_{42}} u_{42} (1, \bar{u}_{42}), \quad \sqrt{r_{43}} u_{43} (1, \bar{u}_{43}) \]

The symmetric tensor product of the vectors in each of the lines above gives us:

\[ \sqrt{r_{12} r_{13} r_{14}} (1, \bar{u}_{12} + \bar{u}_{13} + \bar{u}_{14}, \quad \bar{u}_{12} \bar{u}_{13} + \bar{u}_{12} \bar{u}_{14} + \bar{u}_{13} \bar{u}_{14}, \quad \bar{u}_{12} \bar{u}_{13} \bar{u}_{14}) \]

\[ \sqrt{r_{21} r_{23} r_{24}} u_{21} (1, \bar{u}_{21} + \bar{u}_{23} + \bar{u}_{24}, \quad \bar{u}_{21} \bar{u}_{23} + \bar{u}_{21} \bar{u}_{24} + \bar{u}_{23} \bar{u}_{24}, \quad \bar{u}_{21} \bar{u}_{23} \bar{u}_{24}) \]

\[ \sqrt{r_{31} r_{32} r_{34}} u_{31} u_{32} (1, \bar{u}_{31} + \bar{u}_{32} + \bar{u}_{34}, \quad \bar{u}_{31} \bar{u}_{32} + \bar{u}_{31} \bar{u}_{34} + \bar{u}_{32} \bar{u}_{34}, \quad \bar{u}_{31} \bar{u}_{32} \bar{u}_{34}) \]

\[ \sqrt{r_{41} r_{42} r_{43}} u_{41} u_{42} u_{43} (1, \bar{u}_{41} + \bar{u}_{42} + \bar{u}_{43}, \quad \bar{u}_{41} \bar{u}_{42} + \bar{u}_{41} \bar{u}_{43} + \bar{u}_{42} \bar{u}_{43}, \quad \bar{u}_{41} \bar{u}_{42} \bar{u}_{43}) \]

Consequently, if we define the angular part of \( At \) to be \( At_{ang} \) given by the product \( u_{12} u_{13} u_{14} u_{23} u_{24} u_{34} \) times the determinant
then the Atiyah determinant will be \( At = PA_{\text{ang}} \) where \( P = r_{12}r_{13}r_{14}r_{23}r_{24}r_{34} \) is the product of the lengths of all sides. The decomposition \( At = PA_{\text{ang}} \) will be called the standard form of the Atiyah determinant for four planar points. It is also natural to call \( P \) the scalar part of \( At \) since the rest of the product \( A_{\text{ang}} \) is the angular part of \( At \).

### 3 The Convex Case

Let us consider the terms of the Atiyah determinant corresponding to the permutation \((1, 2, 3, 4)\) in the expansion of \( At \). This corresponds to the product of the main diagonal entries:

\[

(\overline{u}_{21} + \overline{u}_{23} + \overline{u}_{24})(\overline{u}_{31} \overline{u}_{32} + \overline{u}_{31} \overline{u}_{34} + \overline{u}_{32} \overline{u}_{34})\overline{u}_{41} \overline{u}_{42} \overline{u}_{43}

\]

multiplied with \( r_{12}r_{13}r_{14}r_{23}r_{24}r_{34}u_{12}u_{13}u_{14}u_{23}u_{24}u_{34} \). We decompose this expression as an angular part \((1 + \overline{u}_{21} \overline{u}_{23} + \overline{u}_{21} \overline{u}_{24})(1 + \overline{u}_{31} \overline{u}_{34} + \overline{u}_{32} \overline{u}_{34})\) multiplied with the scalar part \( P = r_{12}r_{13}r_{14}r_{23}r_{24}r_{34} \). The angular part, expressed with respect to the angles appearing in figure 1, can be written as

\[

(1 + e^{i\gamma_2} + e^{i\beta_2})(1 + e^{i\alpha_3} + e^{i\gamma_3})

\]

This presentation of the terms of the Atiyah determinant can be done in a similar way for any permutation of the set \( \{1, 2, 3, 4\} \). For example, the terms corresponding to the permutation \((1, 4, 3, 2)\) can also be written as the scalar part \( P = r_{12}r_{13}r_{14}r_{23}r_{24}r_{34} \) multiplied with the angular part \((1 + e^{-i\gamma_4} + e^{-i\alpha_4})(1 + e^{-i\beta_3} + e^{-i\gamma_3})\). Taking all terms of the Atiyah determinant into account requires considering all 24 permutations. Let us first start with the terms of the identity permutation: Expanding
the angular part corresponding for the identity permutation yields a linear part given by $1 + e^{i\gamma_2} + e^{i\beta_2} + e^{i\gamma_3} + e^{i\alpha_3}$ and a quadratic part given by $e^{i(\alpha_2+\gamma_3)} + e^{i(\beta_2+\gamma_3)} + e^{i(\gamma_2+\alpha_3)} + e^{i(\beta_2+\alpha_3)}$. We need to collect all similar terms from all permutations to get a general formula for the Atiyah determinant:

The Linear Terms: These terms can be summarized as follows: We have 24 ones, one from each permutation. We also have all angles of the quadrilateral appearing with an equal number of times as $e^{i\theta}$, where $\theta$ is any one of the $\alpha_j$'s, $\beta_j$'s or $\gamma_j$'s, and $j = 1, 2, 3, 4$. It is easy to see that conjugate terms appear from permutations of the form $(a, b, c, d)$ and their reverse order $(d, c, b, a)$, which confirms that the determinant is a real number. This allows us to see that each linear term $e^{i\theta}$ appears 4 times and each $e^{-i\theta}$ appears 4 times. Accordingly, the linear terms in the expansion of the Atiyah determinant are

$$24 + 8 \sum \cos \theta_k$$

where $\theta_k$ runs over the 12 angles $\alpha_j$, $\beta_j$, and $\gamma_j$, where $j = 1, 2, 3, 4$. 
The Quadratic Terms: Listing all quadratic terms based on their corresponding permutations gives us:

\[(1,2,3,4): \ e^{i(\gamma_2+\gamma_3)} + e^{i(\beta_2+\gamma_3)} + e^{i(\gamma_2+\alpha_3)} + e^{i(\beta_2+\alpha_3)}\]
\[(1,2,4,3): \ e^{i(\beta_2-\beta_4)} + e^{i(\gamma_2-\beta_4)} + e^{i(\beta_2-\gamma_4)} + e^{i(\gamma_2-\gamma_4)}\]
\[(1,3,2,4): \ e^{i(-\beta_3+\alpha_2)} + e^{i(\alpha_3-\alpha_2)} + e^{i(-\beta_3+\beta_2)} + e^{i(\alpha_3+\beta_2)}\]
\[(1,3,4,2): \ e^{i(\alpha_3+\beta_4)} + e^{i(-\beta_3+\beta_4)} + e^{i(\alpha_3-\alpha_4)} + e^{i(-\beta_3-\alpha_4)}\]
\[(1,4,2,3): \ e^{i(-\alpha_4+\alpha_2)} + e^{i(-\gamma_4+\alpha_2)} + e^{i(-\alpha_4+\gamma_2)} + e^{i(-\gamma_4+\gamma_2)}\]
\[(1,4,3,2): \ e^{i(-\gamma_4-\gamma_3)} + e^{i(-\alpha_4-\gamma_3)} + e^{i(-\gamma_4-\beta_3)} + e^{i(-\alpha_4-\beta_3)}\]
\[(2,1,3,4): \ e^{i(-\alpha_1+\alpha_3)} + e^{i(-\gamma_1+\alpha_3)} + e^{i(-\alpha_1+\gamma_3)} + e^{i(-\gamma_1+\gamma_3)}\]
\[(2,1,4,3): \ e^{i(-\gamma_1-\gamma_4)} + e^{i(-\alpha_1-\gamma_4)} + e^{i(-\gamma_1-\beta_4)} + e^{i(-\alpha_1-\beta_4)}\]
\[(3,1,2,4): \ e^{i(\alpha_1+\beta_2)} + e^{i(-\beta_1+\beta_2)} + e^{i(\alpha_1-\alpha_2)} + e^{i(-\beta_1-\alpha_2)}\]
\[(3,1,4,2): \ e^{i(-\beta_1-\alpha_4)} + e^{i(\alpha_1-\alpha_4)} + e^{i(-\beta_1+\beta_4)} + e^{i(\alpha_1+\beta_4)}\]
\[(4,1,2,3): \ e^{i(\gamma_1+\gamma_2)} + e^{i(\beta_1+\gamma_2)} + e^{i(\gamma_1+\alpha_2)} + e^{i(\beta_1+\alpha_2)}\]
\[(4,1,3,2): \ e^{i(\beta_1-\beta_3)} + e^{i(\gamma_1-\beta_3)} + e^{i(\beta_1-\gamma_3)} + e^{i(\gamma_1-\gamma_3)}\]

The other 12 permutations are nothing but the conjugates of the ones appearing above. Accordingly, adding all terms would result in a sum of twice the cosines of the angles appearing in this list. We note that all the terms appearing in the first column will cancel out. For example, \(\cos(\gamma_2 + \gamma_3) + \cos(\alpha_3 - \alpha_1) = 0\). This can be seen to apply to all terms of the first column. On the other hand, the terms of the fourth column appear twice in the list (including the conjugates).

Summarizing all terms can be done as follows: the multiplicity-one terms can be put in 6 families: \(2\cos(\alpha_j - \alpha_{j+1})\), \(2\cos(\beta_j - \beta_{j+1})\), \(2\cos(\gamma_j + \alpha_{j+1})\), \(2\cos(\gamma_{j+1} + \beta_j)\), \(2\cos(\gamma_j - \alpha_{j+1})\), \(2\cos(\gamma_j - \beta_{j+2})\), where \(j = 1, 2, 3, 4\), modulo 4, and the multiplicity-two terms can be listed as follows: \(4\cos(\alpha_1 + \beta_4)\), \(4\cos(\alpha_2 + \beta_1)\), \(4\cos(\alpha_3 + \beta_2)\), \(4\cos(\alpha_4 + \beta_3)\), \(4\cos(\gamma_1 - \gamma_3)\), and \(4\cos(\gamma_2 - \gamma_4)\). Contemplating this list, we realize that these terms can be grouped based on vertices facing opposite triangles in the following manner:

\[\cos(\alpha_j - \alpha_{j+1})\]: where \(\alpha_j\) is at vertex \(z_j\) and \(\alpha_{j+1}\) is in the opposite triangle
\[\cos(\beta_j - \beta_{j-1})\]: where \(\beta_j\) is at vertex \(z_j\) and \(\beta_{j-1}\) is in the opposite triangle
\[\cos(\gamma_j + \alpha_{j+1})\]: where \(\gamma_j\) is at vertex \(z_j\) and \(\alpha_{j+1}\) is in the opposite triangle
\[ \cos(\gamma_j + \beta_{j-1}) \]: where \( \gamma_j \) is at vertex \( z_j \) and \( \beta_{j-1} \) is in the opposite triangle
\[ \cos(\alpha_j - \gamma_{j+2}) \]: where \( \alpha_j \) is at vertex \( z_j \) and \( \gamma_{j+2} \) is in the opposite triangle
\[ \cos(\beta_j - \gamma_{j+2}) \]: where \( \beta_j \) is at vertex \( z_j \) and \( \gamma_{j+2} \) is in the opposite triangle

in addition to the multiplicity-two terms:
\[ \cos(\alpha_1 + \beta_4) \]: where \( \alpha_1 \) is at vertex \( z_1 \) and \( \beta_4 \) is in the opposite triangle
or \( \beta_4 \) is at vertex \( z_4 \) and \( \alpha_1 \) is in the opposite triangle
\[ \cos(\alpha_2 + \beta_1) \]: where \( \alpha_2 \) is at vertex \( z_2 \) and \( \beta_1 \) is in the opposite triangle
or \( \beta_1 \) is at vertex \( z_1 \) and \( \alpha_2 \) is in the opposite triangle
\[ \cos(\alpha_3 + \beta_2) \]: where \( \alpha_3 \) is at vertex \( z_3 \) and \( \beta_2 \) is in the opposite triangle
or \( \beta_2 \) is at vertex \( z_2 \) and \( \alpha_3 \) is in the opposite triangle
\[ \cos(\alpha_4 + \beta_3) \]: where \( \alpha_4 \) is at vertex \( z_4 \) and \( \beta_3 \) is in the opposite triangle
or \( \beta_3 \) is at vertex \( z_3 \) and \( \alpha_4 \) is in the opposite triangle
\[ \cos(\gamma_1 - \gamma_3) \]: where \( \gamma_1 \) is at vertex \( z_1 \) and \( \gamma_3 \) is in the opposite triangle
or \( \gamma_3 \) is at vertex \( z_3 \) and \( \gamma_1 \) is in the opposite triangle
\[ \cos(\gamma_2 - \gamma_4) \]: where \( \gamma_2 \) is at vertex \( z_2 \) and \( \gamma_4 \) is in the opposite triangle
or \( \gamma_4 \) is at vertex \( z_4 \) and \( \gamma_2 \) is in the opposite triangle

For example, let us consider the vertex \( z_1 \) facing the triangle \( z_2 z_3 z_4 \). The angle \( \alpha_1 \) at the vertex \( z_1 \) is coupled with each of the angles of the triangle \( z_2 z_3 z_4 \) in the terms: \( \cos(\alpha_1 - \alpha_2), \cos(\alpha_1 - \gamma_3), \) and \( \cos(\alpha_1 + \beta_4) \). Also, the angle \( \beta_1 \) at the vertex \( z_1 \) is coupled with each of the angles of the triangle \( z_2 z_3 z_4 \) in the terms: \( \cos(\beta_1 + \alpha_2), \cos(\beta_1 - \gamma_3), \) and \( \cos(\beta_1 - \beta_4) \). Finally, the angle \( \gamma_1 \) is coupled with each of the angles of the triangle \( z_2 z_3 z_4 \) in the terms: \( \cos(\gamma_1 + \alpha_2), \cos(\gamma_1 - \gamma_3), \) and \( \cos(\gamma_1 + \beta_4) \).

Collecting the product of cosines \( \cos \alpha \cos \beta \) from \( \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \) for each of the quadratic terms we obtain the sum of products

\[
2 \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) \left( \cos \alpha_{j+1} + \cos \gamma_{j+2} + \cos \beta_{j+3} \right)
\]

where all indices are taken modulo four. Here, we see that \( \cos \alpha_j + \cos \beta_j + \cos \gamma_j \) is the sum of cosines of all angles at the vertex \( z_j \) whereas \( \cos \alpha_{j+1} + \cos \gamma_{j+2} + \cos \beta_{j+3} \) is the sum of cosines of all angles of the opposite triangle.

To accommodate for the remaining product of sines \( \mp \sin \alpha \sin \beta \) from \( \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \), let us call the sum of all products of
two sines (together with their signs) to be the expression $E$. Based on that we have proved the following theorem:

**Theorem 1**: The Atiyah determinant can be written as $r_{12}r_{13}r_{14}r_{23}r_{24}r_{34}$ times the angular part

$$24 + 8 \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j)$$

$$+ 2 \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) (\cos \alpha_{j+1} + \cos \gamma_{j+2} + \cos \beta_{j+3})$$

$$+ E$$

Let us call the first sum $S_1 = \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j)$ and the second sum $S_2 = \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) (\cos \alpha_{j+1} + \cos \gamma_{j+2} + \cos \beta_{j+3})$

**Corollary 1**:

$$S_1 = \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) \geq 4$$

Proof: This proof relies on the identity

$$\cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} ; \quad \alpha + \beta + \gamma = \pi$$

Accordingly, we regroup the terms in our summation $S_1$ in such a way that angles of the same triangle are grouped together to obtain

$$S_1 = \sum_{j=1}^{4} (\cos \alpha_{j+1} + \cos \gamma_{j+2} + \cos \beta_{j+3})$$
\[
\begin{align*}
&= \sum_{j=1}^{4} \left( 1 + 4 \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2} \right) \\
&= 4 + 4 \sum_{j=1}^{4} \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2}
\end{align*}
\]

which is obviously greater than or equal to 4 since all other terms are non-negative.

**Corollary 2 :**

\[8S_1 + 2S_2 \geq 40\]

Proof: First, we collect from corollary 1 the simplified expression of \(S_1\) and write it as

\[8S_1 = 32 + 32 \sum_{j=1}^{4} \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2}\]

We now turn our attention to \(S_2\). Here we use the identity (*) again on the second factors of the sum \(S_2\) to write it as

\[S_2 = \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) (1 + 4 \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2})\]

Consequently,

\[2S_2 = 2 \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) + 8 \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2}\]

The first line of \(2S_2\) can also be simplified since angles can be regrouped into four groups, each corresponding to one of the four triangles of the quadrilateral, so we write as
Accordingly, $8S_1 + 2S_2$ is equal to
\[= 40 + 40 \sum_{j=1}^{4} \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2} + 8 \sum_{j=1}^{4} (\cos \alpha_j + \cos \beta_j + \cos \gamma_j) \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2}\]

which finally gives us that
\[8S_1 + 2S_2 = 40 + 8 \sum_{j=1}^{4} (5 + \cos \alpha_j + \cos \beta_j + \cos \gamma_j) \sin \frac{\alpha_{j+1}}{2} \sin \frac{\gamma_{j+2}}{2} \sin \frac{\beta_{j+3}}{2}\]

Since $5 + \cos \alpha_j + \cos \beta_j + \cos \gamma_j \geq 0$, we can easily see that $8S_1 + 2S_2 \geq 40$ which proves the second corollary.

Based on these two corollaries, we can prove the second Atiyah-Sutcliffe conjecture for a convex quadrilateral if we can prove that $E \geq 0$. This will finish the proof. To do that, we will prove that $E$ is nothing but the expression $4(E_{12} + E_{23} + E_{34} + E_{14} - E_{13} - E_{24})$ which was proved to be non-negative in [9]. Before we start comparing our work with [9], note that the side-lengths in our quadrilateral were denoted in [9] as follows: $r_{23} = a$, $r_{13} = b$, $r_{12} = c$, $r_{14} = d$, $r_{24} = e$, and $r_{34} = f$. Let us proceed by regrouping the terms in the following way:

Consider each side of the quadrilateral separately. For example, let us consider the products of sines corresponding to the side $z_3z_4$. This side belong to two triangles, namely $z_2z_3z_4$ and $z_1z_3z_4$. When $z_1$ is facing triangle $z_2z_3z_4$, the terms corresponding for $z_3z_4$ are all the terms corresponding for the angle $\beta_1$ at $z_1$ combined with the angles of the triangle $z_2z_3z_4$. These are:
\[
\cos(\beta_1 - \beta_4), \cos(\beta_1 - \gamma_3), \text{ and } \cos(\beta_1 + \alpha_2). \]
By doing a similar choice of the terms when \(z_2\) is facing triangle \(z_1z_3z_4\), we see that the terms corresponding for the side \(z_3z_4\) are: \(\cos(\alpha_2 - \alpha_3), \cos(\alpha_2 - \gamma_4), \text{ and } \cos(\alpha_2 + \beta_1)\). When we collect the product of two sines from these terms as was formed in \(E\) we obtain

\[
\sin \beta_1 \sin \beta_4 + \sin \beta_1 \sin \gamma_3 + \sin \alpha_2 \sin \alpha_3 + \sin \alpha_2 \sin \gamma_4 - 2 \sin \alpha_2 \sin \beta_1
\]

Multiplying this with the scalar part \(abcdef\), we note that we can write this expression in terms of the areas of the triangles as follows:

\[
(abcdef) \sin \beta_1 \sin \beta_4 = 4ac(\frac{1}{2}bd \sin \beta_1)(\frac{1}{2}ef \sin \beta_4) = 4acA_{134}A_{234}
\]
\[
(abcdef) \sin \beta_1 \sin \gamma_3 = 4ce(\frac{1}{2}bd \sin \beta_1)(\frac{1}{2}af \sin \gamma_3) = 4ceA_{134}A_{234}
\]
\[
(abcdef) \sin \alpha_2 \sin \alpha_3 = 4cd(\frac{1}{2}ae \sin \alpha_2)(\frac{1}{2}bf \sin \alpha_3) = 4cdA_{134}A_{234}
\]
\[
(abcdef) \sin \alpha_2 \sin \gamma_4 = 4bc(\frac{1}{2}ae \sin \alpha_2)(\frac{1}{2}df \sin \gamma_4) = 4bcA_{134}A_{234}
\]
\[
(abcdef) \sin \alpha_2 \sin \beta_1 = 4cf(\frac{1}{2}ae \sin \alpha_2)(\frac{1}{2}bd \sin \beta_1) = 4cfA_{134}A_{234}
\]

Accordingly, when factorizing \(4A_{134}A_{234}\), the terms corresponding for the side \(z_3z_4\) can be written as

\[
4A_{134}A_{234}c(a + e + d + b - 2f)
\]

This is precisely the quantity \(4E_{34}\) as defined in [9]. Repeating this computation for each of the sides of the quadrilateral, we find that:

The products of sines corresponding to the side \(z_1z_2\) is \(4A_{123}A_{124}f(a + b + e + d - 2c)\) which is \(4E_{12}\) as defined in [9].

The products of sines corresponding to the side \(z_1z_3\) is \(-4A_{123}A_{134}e(a + c + d + f - 2b)\) which is \(-4E_{13}\) as defined in [9].

The products of sines corresponding to the side \(z_1z_4\) is \(4A_{124}A_{134}a(c + e + b + f - 2d)\) which is \(4E_{14}\) as defined in [9].

The products of sines corresponding to the side \(z_2z_3\) is \(4A_{123}A_{234}d(c + b + e + f - 2a)\) which is \(4E_{23}\) as defined in [9].
The products of sines corresponding to the side $z_2z_4$ is $-4A_{124}A_{234}b(c + d + f + a - 2e)$ which is $-4E_{24}$ as defined in [9].

Consequently, requiring $E \geq 0$ is the same as requiring that

$$E_{12} + E_{23} + E_{34} + E_{14} - E_{13} - E_{24} \geq 0$$

This is what we proved in [9]. With that accomplished, we can see that the proof of the second Atiyah-Sutcliffe conjecture for a convex quadrilateral is now complete.

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