Is Sharma-Mittal entropy really a step beyond Tsallis and Rényi entropies?

E. Aktürk
Department of Physics Engineering, University of Hacettepe, 06800, Ankara, Turkey

G. B. Bağıcı
Department of Physics, University of North Texas, P.O. Box 311427, Denton, TX 76203-1427, USA

R. Sever
Department of Physics, Middle East Technical University, 06800, Ankara, Turkey

(Dated: February 6, 2008)

We studied the Sharma-Mittal relative entropy and showed that its physical meaning is the free energy difference between the off-equilibrium and equilibrium distributions. Unfortunately, Sharma-Mittal relative entropy may acquire this physical interpretation only in the limiting case when both parameters approach to 1 in which case it approaches Kullback-Leibler entropy. We also note that this is exactly how Rényi relative entropy behaves in the thermostatistical framework thereby suggesting that Sharma-Mittal entropy must be thought to be a step beyond not both Tsallis and Rényi entropies but rather only as a generalization of Rényi entropy from a thermostatistical point of view. Lastly, we note that neither of them conforms to the Shore-Johnson theorem which is satisfied by Kullback-Leibler entropy and one of the Tsallis relative entropies.

PACS numbers: PACS: 05.70.-a; 05.70.Ce; 05.70.Ln
Keywords: Sharma-Mittal entropy, Tsallis entropy, Rényi entropy, relative entropy, entropy maximization, free energy, Shore-Johnson theorem

I. INTRODUCTION

Recently there has been a growing interest in generalized entropies such as Tsallis [1] and Rényi [2] entropies in the context of a generalized thermostatistics. For example, Tsallis entropy has been applied to nonlinear diffusion equations [3] and Fokker-Planck systems [4, 5] while Rényi entropy has been studied for its inverse power law equilibrium distribution [6] and has been shown to satisfy the zeroth law of thermodynamics [7]. It has been observed that an unifying entropy exists which seems to generalize both of these entropies [8, 9, 10]. This two-parametric entropy is called Sharma-Mittal entropy [11]. It generalizes Tsallis, Rényi and Boltzmann-Gibbs (BG) entropies since its two parameters generate these entropies in limiting cases. Our aim in this paper is to shed some light on this new entropy measure from a thermostatistical point of view. It is organized as follows: In Section II, the review of the physical meaning of BG relative entropy i.e., Kullback-Leibler entropy will be given. It will be seen that it provides a measure of free energy differences between the off-equilibrium and equilibrium distributions. In Section III, we will briefly mention Sharma-Mittal entropy and some of its important properties. In Section IV, we will study the relative entropy associated with this entropy and show that it can be given a physical meaning in a generalized thermostatistical framework in terms of free energy differences only when Sharma-Mittal relative entropy reduces to Kullback-Leibler entropy. Its connection with Rényi relative entropy will be investigated and shown to behave in a similar manner. Lastly, in Section V, we will see that Sharma-Mittal entropy does not conform to Shore-Johnson theorem sharing also this feature with Rényi relative entropy. We will summarize the conclusions in Section VI.

II. THE PHYSICAL MEANING OF KULLBACK-LEIBLER ENTROPY

The relative entropy is an important concept whose uses range from the numerical analysis of protein sequences [12], pricing models in the market [13], to medical decision making [14]. In this paper, we will study it from a thermosta-
tistical point of view. In order to study the physical meaning of any relative entropy in a thermostatistical framework, one has first to obtain the equilibrium distribution associated with the entropy of that particular thermostatistics. This can be done by maximizing BG entropy subject to some constraints following the well known recipe of entropy maximization. BG entropy reads

\[ S_{BG}(p) = -\sum_{i}^{W} p_i \ln p_i, \]  

(1)

where \( p_i \) is the probability of the system in the ith microstate, \( W \) is the total number of the configurations of the system. Note that Boltzmann constant \( k \) is taken to be equal to one throughout the paper. Let us assume that the internal energy function is given by \( U = \sum \varepsilon_i p_i \) where \( \varepsilon_i \) denotes the energy of the ith microstate. In order to get the equilibrium distribution associated with BG entropy, we maximize the following functional

\[ \Phi(p) = -\sum_{i}^{W} p_i \ln p_i - \alpha \sum_{i}^{W} p_i - \beta \sum_{i}^{W} \varepsilon_i p_i, \]  

(2)

where \( \alpha \) and \( \beta \) are Lagrange multipliers related to normalization and internal energy constraints respectively. Equating the derivative of the functional to zero, we obtain

\[ \frac{\delta \Phi(p)}{\delta p_i} = -\ln \tilde{p}_i - 1 - \alpha - \beta \varepsilon_i = 0. \]  

(3)

Tilde denotes the equilibrium distribution obtained by the maximization of BG entropy. By multiplying Eq. (3) by \( \tilde{p}_i \) and summing over \( i \), using the normalization and internal energy constraints, we have

\[ \alpha + 1 = \tilde{S}_{BG} - \beta \tilde{U}. \]  

(4)

Substitution of Eq. (4) into Eq. (3) results in the following equilibrium distribution

\[ \tilde{p}_i = e^{-\tilde{S}_{BG} e^{\beta \tilde{U}}} e^{-\beta \varepsilon_i}. \]  

(5)

The relative entropy in ordinary BG case is called Kullback-Leibler (K-L) entropy \([15]\). It reads

\[ K[p||q] \equiv \sum_{i} p_i \ln(p_i/q_i). \]  

(6)

Note that it is a convex function of \( p_i \), always non-negative and equal to zero if and only if \( p = q \). If we now use the equilibrium distribution \( \tilde{p} \) as the reference distribution in K-L entropy, we can write

\[ K[p||\tilde{p}] = \sum_{i} p_i \ln(p_i/\tilde{p}_i). \]  

(7)

The equation above can be rewritten as

\[ K[p||\tilde{p}] = -S_{BG} - \sum_{i} p_i \ln \tilde{p}_i. \]  

(8)

We then insert the equilibrium distribution given by Eq. (5) in the equation above to find

\[ K[p||\tilde{p}] = -S_{BG} - \sum_{i} p_i (-\tilde{S}_{BG} + \beta \tilde{U} - \beta \varepsilon_i). \]  

(9)
Taking care of the effect of summation, we have

\[ K[p||\tilde{p}] = -S_{BG} + \tilde{S}_{BG} - \beta \tilde{U} + \beta U, \]  

which can be cast into the form

\[ K[p||\tilde{p}] = \beta (F - \tilde{F}). \]  

The free energy term is given as usual by

\[ F = U - S_{BG}/\beta. \]

The result above shows us that the physical meaning of the K-L entropy is nothing but the difference of the off-equilibrium and equilibrium free energies when the reference distribution is taken to be the equilibrium distribution given by Eq. (5) above. This result can be used, for example, to study equilibrium fluctuations or non-equilibrium relaxation of polymer chains [16].

III. SHARMA-MITTAL ENTROPY

In this Section, we will briefly review the Sharma-Mittal entropy and some of its important properties from a generalized thermostatistical point of view. The Sharma-Mittal entropy [11] is given by

\[ S_{SM}(p) = \frac{1}{1 - r} \left[ \left( \sum_i p_i^q \right)^{\frac{1}{q-r}} - 1 \right]. \]  

In the limit \( r \to 1 \), Sharma-Mittal entropy becomes Rényi entropy [2] which is

\[ S_R(p) = \frac{1}{1 - q} \ln \left( \sum_i p_i^q \right), \]  

while for \( r \to q \), it is Tsallis entropy [1] given by

\[ S_T(p) = \frac{\sum_i p_i^q - 1}{1 - q}. \]

In the limiting case when both parameters approach 1, we recover the ordinary Boltzmann-Gibbs (BG) entropy which reads

\[ S_{BG}(p) = -\sum_i p_i \ln p_i. \]

For two statistically independent systems given by probability distributions \( p_i \) and \( p'_k \), Sharma-Mittal entropy satisfies [9]

\[ S_{SM}(\{p_i p'_k\}) = S_{SM}(\{p_i\}) + S_{SM}(\{p'_k\}) + (1 - r)S_{SM}(\{p_i\})S_{SM}(\{p'_k\}). \]  

This relation is important in order to see how Sharma-Mittal measure includes both Tsallis and Rényi entropies as its limiting cases. When we take \( r = q \), it becomes the relation satisfied by Tsallis entropy and shows its nonextensivity. On the other hand, when \( r = 1 \), we have the extensive property which relates to Rényi entropy. In this sense, Sharma-Mittal entropy includes both extensive and nonextensive features in it.

One feature of Sharma-Mittal entropy is that it fails to be concave [8]. The concavity entails thermodynamic stability and can be defined as follows: Consider probability distributions \( P = \{p_1, p_2, ..., p_N\} \) and \( P' = \{p'_1, p'_2, ..., p'_N\} \). Let us also define an intermediate probability distribution given by \( P'' = \{p''_1, p''_2, ..., p''_N\} \) where

\[ p''_i = \mu p_i + (1 - \mu)p'_i, \forall \mu \in [0, 1]. \]
Now, $S(P)$ is said to be concave if and only if

$$S(P'') \geq \mu S(P) + (1 - \mu) S(P').$$  \hfill (18)

It is worth remark that BG entropy and Tsallis entropy is concave whereas Rényi entropy is not concave for all values of parameter $q$ \cite{17}.

Another feature of Sharma-Mittal entropy is that it is not Lesche-stable \cite{18}. Lesche stability checks the stability of the entropy functional under arbitrary small variations of the probabilities. It can be stated in a more rigorous way by defining the deformation of the probability distribution as

$$\|p - p'\| = \sum_i |p_i - p'_i| < \delta \varepsilon, \forall \delta \varepsilon > 0.$$  \hfill (19)

Then, $S(P)$ is Lesche-stable if

$$\Delta = \left| \frac{S(P) - S(P')}{S_{\text{max}}} \right| < \varepsilon, \forall \varepsilon > 0,$$  \hfill (20)

where $S_{\text{max}}$ is the maximum value that $S$ can attain over all microstates. It has been shown that BG and Tsallis entropy is always Lesche-stable whereas Rényi entropy is unstable for all $q \neq 1$ \cite{17, 18}.

Lastly, it may be remarked that Sharma-Mittal entropy does not lead to finite entropy production per unit time whereas BG and Tsallis entropies do \cite{8}.

IV. SHARMA-MITTAL RELATIVE ENTROPY AND FREE ENERGY

In order to study the physical meaning of Sharma-Mittal relative entropy in thermostatistical framework as we did with BG entropy in Section II, we have to obtain the equilibrium distribution corresponding to Sharma-Mittal entropy. For this purpose, we begin by maximizing the following associated functional

$$\Phi_{SM}(p) = \frac{1}{1 - q} \left[ \left( \sum_i p_i^q \right)^{\frac{1}{1 - q}} - 1 \right] - \alpha \sum_i p_i - \beta \sum_i \varepsilon_i p_i^q.$$  \hfill (21)

We take the derivative of the functional and equal it to zero in order to obtain the following

$$\frac{\delta \Phi_{SM}(p)}{\delta p_i} = \frac{q}{1 - q} \bar{p}_i^{q - 1} \left( \sum_i \bar{p}_i q \right)^{\frac{1}{1 - q}} - \alpha - \beta^* q \bar{p}_i^{q - 1} (\varepsilon_i - \tilde{U}) = 0,$$  \hfill (22)

where $\beta^*$ is given by

$$\beta^* = \frac{\beta}{\left( \sum_i \bar{p}_i^q \right)}.$$  \hfill (23)

Multiplying Eq. (22) by $\bar{p}_i$ and summing over the index $i$, we obtain

$$\alpha = \frac{q}{1 - q} \left( \sum_i \bar{p}_i q \right)^{\frac{1}{1 - q}}.$$  \hfill (24)

Note that tilde shows that the distribution is calculated at equilibrium. Substituting this explicit expression of $\alpha$ into Eq.(22), we calculate the associated equilibrium distribution \cite{9} as

$$\bar{p}_i = \left( \frac{1}{\sum p_i^q} \right)^{\frac{1}{1 - q}} \left[ 1 - (1 - q)\beta^* (\varepsilon_i - \tilde{U}) \right]^{\frac{1}{1 - q}}.$$  \hfill (25)
where $\beta^{**}$ is given by

$$\beta^{**} = \frac{\beta^*}{(\sum_i \tilde{p}_i^q)^{\frac{1}{1-q}}} = \frac{\beta}{(\sum_i \tilde{p}_i^q)^{\frac{1}{1-q}}}.$$  \quad (26)

From Eq.(12), we see that

$$\left( \sum_i p_i^q \right)^{\frac{1}{1-q}} = 1 + (1 - r)S_{SM},$$  \quad (27)

which also holds for equilibrium distribution given by Eq.(25)

$$\left( \sum_i \tilde{p}_i^q \right)^{\frac{1}{1-q}} = 1 + (1 - r)\tilde{S}_{SM}.$$  \quad (28)

The Sharma-Mittal divergence is given by [19]

$$I_{SM}[p || \tilde{p}] = \frac{1}{r-1} \left[ \left( \sum_i p_i^q \tilde{p}_i^{1-q} \right)^{\frac{1}{1-q}} - 1 \right].$$  \quad (29)

We now substitute equilibrium distribution in Eq. (25) as the reference distribution into Sharma-Mittal divergence given above and obtain

$$I_{SM}[p || \tilde{p}] = \frac{1}{r-1} \left[ \left( \sum_i p_i^q(1 + (1 - r)\tilde{S}_{SM}) \right)^{\frac{1}{1-q}} \left( 1 - (1 - q)\beta^{**}(\tilde{\varepsilon}_i - \tilde{U}) \right)^{\frac{1}{1-q}} - 1 \right].$$  \quad (30)

Using Eq.(27), it can be put into a more appropriate form which is

$$I_{SM}[p || \tilde{p}] = \frac{1}{r-1} \left[ \left( \frac{1 + (1 - r)S_{SM}}{1 + (1 - r)\tilde{S}_{SM}} \right)^{\frac{1}{1-q}} \left( 1 - (1 - q)\beta^{**}(U - \tilde{U}) \right)^{\frac{1}{1-q}} - 1 \right].$$  \quad (31)

The expression above is very different than Eq. (11). It cannot be written in terms of free energy differences. Indeed, this can be achieved only by taking the limit $q$ approaches 1 first

$$I_{SM}[p || \tilde{p}] = \frac{1}{r-1} \left[ \left( \frac{1 + (1 - r)S_{SM}}{1 + (1 - r)\tilde{S}_{SM}} \right)^{\frac{1}{1-q}} e^{\beta^{**}(\tilde{U} - U)(1-r)} - 1 \right].$$  \quad (32)

where the internal energy functions and $\beta^{**}$ must be calculated at $q = 1$ and then considering the limit of the above expression as $r$ goes to 1, which in turn gives

$$K[p || \tilde{p}] = \beta(F - \tilde{F}).$$  \quad (33)

This shows that the physical meaning of Sharma-Mittal divergence is the difference between the off-equilibrium free energy and equilibrium free energy when the reference distribution is taken to be the equilibrium distribution obtained from the maximization of Sharma-Mittal entropy only in the limiting case when both parameters approach to 1 in which case it approaches Kullback-Leibler entropy.

This negative result above is the one exactly mimicked by the Rényi relative entropy [20]. It reads

$$I_q[p || r] = \frac{1}{q-1} \ln(\sum_i p_i^{q} r_i^{1-q}).$$  \quad (34)
Note that this definition of Rényi relative entropy is always non-negative and equal to zero if and only if \( p = r \). It also reduces to K-L entropy as the parameter \( q \) approaches 1. Let us write the associated functional where internal energy constraint is given in terms of escort probabilities i.e., \( U = \sum_j \varepsilon_j p_j^q \), thereby yielding

\[
\Phi_R(p) = \frac{1}{1-q} \ln \left( \sum_i p_i^q \right) - \alpha \sum_i p_i - \beta \frac{\sum_j \varepsilon_j p_j^q}{\sum_i p_i^q}.
\]  

We again take the derivative of the functional and equate it to zero in order to obtain the following

\[
\frac{\delta \Phi_R(p)}{\delta p_i} = \frac{q}{1-q} \frac{p_i^{q-1}}{\sum_j p_j^q} - \alpha - \beta^* q p_i^{q-1} (\varepsilon_i - \bar{U}) = 0,
\]

where \( \beta^* \) is given by

\[
\beta^* = \frac{\beta}{\sum_j p_j^q}.
\]

Multiplying Eq. (36) by \( \bar{p}_i \) and summing over the index \( i \), we find

\[
\alpha = \frac{q}{1-q}.
\]

Note that tilde again denotes that the distribution is calculated at equilibrium. Substituting this explicit expression of \( \alpha \) back into Eq. (36), Rényi equilibrium distribution reads

\[
\bar{p}_i = \left( 1 \frac{1}{e(1-q)S_R} - (1-q)\beta^* (\varepsilon_i - \bar{U}) \right)^{1/(1-q)}.
\]

Following the same steps as before, we then substitute equilibrium distribution above as the reference distribution into the Rényi relative entropy given by Eq. (34) and get

\[
I_q[p||\bar{p}] = \frac{1}{q-1} \ln \left( \sum_i p_i^q \left( \frac{1}{e(1-q)S_R} - (1-q)\beta^* (\varepsilon_i - \bar{U}) \right) \right).
\]

It can be put into a more appropriate form which is

\[
I_q[p||\bar{p}] = \frac{1}{q-1} \ln \left( e(1-q)(S_R - \bar{S}_R) - (1-q)\beta^{**}(U - \bar{U}) \right),
\]

where \( \beta^{**} \) is given by

\[
\beta^{**} = \frac{\beta}{\sum_j p_j^q}.
\]

Making Taylor series expansion about \( q = 1 \) for the exponential term within the parentheses and keeping the first two terms only, we have

\[
I_q[p||\bar{p}] = \frac{1}{q-1} \ln \left( 1 + (1-q)(S_{BG} - \bar{S}_{BG}) - (1-q)\beta^{**}(U - \bar{U}) \right).
\]
Arranging the terms as follows

\[ I_q[p\|\bar{p}] = \frac{1}{q-1} \ln[1 + (1 - q)((S_{BG} - \bar{S}_{BG}) - \beta^*(U - \bar{U}))] \]  

(44)

and making Taylor expansion about \( q = 1 \) again but this time to the logarithmic term, we obtain

\[ I_q[p\|\bar{p}] = \frac{1}{(q-1)}(1 - q)((S_{BG} - \bar{S}_{BG}) - \beta(U - \bar{U})) \],  

(45)

which can be written as

\[ I_q[p\|\bar{p}] = \beta(U - S_{BG}/\beta) - (\bar{U} - \bar{S}_{BG}/\beta). \]  

(46)

This is nothing but the free energy differences since it can be rewritten as

\[ I_q[p\|\bar{p}] = \beta(F - \bar{F}), \]  

(47)

where free energy expressions are given by exactly as in the BG case. This is exactly the same expression obtained in Section II by using BG entropy and K-L entropy. It should be noted that the first Taylor expansion turned the Rényi entropies into BG entropies while second Taylor expansion turned the Lagrange multiplier and internal energy functions into their corresponding BG values [21].

This shows that Sharma-Mittal relative entropy behaves exactly in the same way as Rényi relative entropy when one considers them in terms of their physical meanings in a generalized thermostatistical framework.

V. SHARMA-MITTAL RELATIVE ENTROPY AND SHORE-JOHNSON THEOREM

At this point, it is important to remember Shore-Johnson theorem (see Refs. [22, 23] for details). According to it, any relative entropy \( J[p\|r] \) with the prior \( r_i \) and posterior \( p_i \) which satisfies five very general axioms, must be of the form

\[ J[p\|r] = \sum_i p_i h(p_i/r_i), \]  

(48)

for some function \( h(x) \). These axioms are listed as

1. Axiom of Uniqueness: If the same problem is solved twice, then the same answer is expected to result both times.

2. Axiom of Invariance: The same answer is expected when the same problem is solved in two different coordinate systems, in which the posteriors in the two systems should be related by the coordinate transformation.

3. Axiom of System Independence: It should not matter whether one accounts for independent information about independent systems separately in terms of their marginal distributions or in terms of the joint distribution.

4. Axiom of Subset Independence: It should not matter whether one treats independent subsets of the states of the systems in terms of their separate conditional distributions or in terms of the joint distribution.

5. Axiom of Expansibility: In the absence of new information, the prior should not be changed.

Ordinary relative entropy i.e., K-L entropy is in accordance with Shore-Johnson theorem since we can find a function \( h(x) \) which allows us to write K-L entropy as Eq. (48) requires. This function \( h(x) \) is nothing but natural logarithm indeed. In the case of Rényi relative entropy [20] given by Eq. (34), we see that it cannot be cast into a form which will conform to the Shore-Johnson theorem for any function \( h(x) \). Inspection of Sharma-Mittal relative entropy shows
that it shares also this feature of Rényi relative entropy. In other words, both fails to conform to Shore-Johnson theorem. On the other hand, the nonextensive counterpart of relative entropy given by

\[ I_{T_{\text{Tsallis}}}^q[p||r] = \frac{1}{1-q} \left[ 1 - \sum_i p_i^q r_i^{1-q} \right] \]  

(49)

is seen to conform to Shore-Johnson theorem when the function \( h(x) \) is taken to be the the negative of the \( q \)-logarithm function defined by \( \ln_q(x) = \frac{x^{1-q} - 1}{q} \) but with argument \( x \) replaced by \( 1/x \) [23]. Therefore, in the case of nonextensive thermostatistics, one has a relative entropy which conforms to Shore-Johnson theorem as K-L entropy in BG thermostatistics does.

VI. RESULTS AND DISCUSSIONS

The Sharma-Mittal entropy seems to generalize both Rényi and Tsallis entropies through the adjustment of its two parameters. We have investigated whether this interpretation is plausible from the thermostatistical point of view using the associated Sharma-Mittal relative entropy. The relative entropy is an important concept and has many applications in diverse fields such as quantum information theory [24], biophysics [12] and finance [13]. Its physical meaning in ordinary thermostatistics is the difference of free energies associated with equilibrium and off-equilibrium distributions. In this paper, we have shown that a similar result can be obtained in the case of Sharma-Mittal entropy but only when it reduces to K-L entropy, rendering the use of Sharma-Mittal relative entropy redundant in this generalized thermostatistical framework. We also observe that this is exactly how Rényi relative entropy behaves when it is subject to same kind of calculation. Another negative feature which Sharma-Mittal relative entropy has in common with Rényi entropy is that associated relative entropies violate the Shore-Johnson theorem which is satisfied by Kullback-Leibler entropy and one of the Tsallis relative entropies. Considering all these negative results common to both of them including the failure of concavity and stability, we believe that Sharma-Mittal entropy must be thought to be a step beyond not both Tsallis and Rényi entropies but rather as a generalization of Rényi entropy from a thermostatistical point of view although the explicit form of Sharma-Mittal entropy suggests that it has both of these entropies in its content when we only consider the limiting values of its parameters.

[1] C. Tsallis, J.Stat. Phys. 52, 479 (1988).
[2] A. Rényi, On measures of entropy and information, in: Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, vol. 1, University California Press, Berkeley, 1961, pp. 547-561.
[3] A. R. Plastino, M. Casas, A. Plastino, Physica A 280 (2000) 289.
[4] D.A. Stariolo, Phys. Lett. A 185 (1993) 262.
[5] T. D. Frank, A. Daffertshofer, Physica A 292 (2001) 392.
[6] E. K. Lenzi, R. S. Mendes, L. R. da Silva, Physica A 280, 337 (2000).
[7] A. S. Parvan, T. S. Biró, Phys. Lett. A 340, 375 (2005).
[8] Marco Masi, Phys. Lett. A 383, 217 (2005).
[9] T. D. Frank, A. R. Plastino, Eur. Phys. J. B 30, 543 (2002).
[10] T. D. Frank, A. Daffertshofer, Physica A, 285, 351 (2000).
[11] B. D. Sharma, D. P. Mittal, J. Math. Sci. 10, 28 (1975).
[12] J. Gorodkin, L. J. Heyer, S. Brunak and G. D. Stormo, Comput. Appl. Biosci. 13, 583 (1997).
[13] Marco Avellaneda, Int. J. of Theor and Appl. Finance 1, 447 (1998).
[14] William A. Benish, Medical Decision Making 19, 202 (1999).
[15] R. Gray, Entropy and Information Entropy, Springer-Verlag, New York, 1990.
[16] H. Qian, Phys. Rev. E 63, 042103 (2001).
[17] S. Abe, Phys. Rev. E 66, 046134 (2002).
[18] B. Lesche, J. Stat. Phys. 27, 419 (1982).
[19] Marco Masi, Generalized information-entropy measures and Fisher information, arXiv:cond-mat/0611300 (21 Nov 2006).
[20] P. Harremoës, Physica A 365, 57 (2006).
[21] G. B. Bagci, The Physical Meaning of Rényi Relative Entropies, arXiv:cond-mat/0703008 (1 Mar 2007).
[22] J. E. Shore, R. W. Johnson, IEEE Transactions on Information Theory IT-26, 26 (1980); IT-27, 472 (1981); IT-29, 942 (1983).
[23] S. Abe, G. B. Bagci, Phys. Rev. E 71, 016139 (2005).
[24] V. Vedral, Rev. Mod. Phys. 74, 197 (2002).