Instantons and radial excitations in attractive Bose-Einstein condensates

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In a series of experiments, a nonuniform Bose-Einstein condensate (BEC) of \(^7\)Li atoms was formed and proved metastable for atom numbers \(N < N_c\), with \(N_c \approx 1300\), in agreement with theoretical predictions. Due to attraction between atoms, such condensate is bound to collapse when \(N > N_c\), but it may collapse also for \(N < N_c\) via quantum or thermal tunneling. Within the mean-field approach, BEC is described by one wave function. Our aim in this work was to find the exact mean-field description of the quantum tunneling in this simplest conceivable many-body system. Our additional motivation is the recently demonstrated ability to control the interaction of \(^85\)Rb atoms in BEC, which opens a new perspective to systematic experimental checks on quantum tunneling.

Up to now, studies of the quantum tunneling of BEC relied on assuming gaussian wave functions, at least in assigning the mass parameter \(\hbar\). Strictly speaking, once the mean field equation is specified, there is no place for such an assumption: This equation, taking a form of the nonlinear field equation, by itself determines the dynamics. To find solutions which correspond to quantum decay we use the method of instantons, i.e. fields evolving in imaginary time, carried over to mean-field theories of many-body systems. It gives the decay rate \(\Gamma = Ae^{-S}\), where the exponent \(S\) is the action for the optimal mean-field instanton, called bounce. We find the exact instantons for spherical BEC by first transforming the original imaginary-time mean-field equation to an equation for the condensate density, and then solving it numerically. Having done that we can check the gaussian ansatz.

The real-time version of the obtained equation encompasses finite amplitude collective radial oscillations of BEC. Applying quantization rule we find energies of radial eigenmodes. By using imaginary-time dynamics we also find periodic instantons determining decay exponents of these breathing modes. In this way, the collective dynamics of an attractive spherical condensate close to instability is obtained from the mean field equation.

We assume that the dynamics of BEC is governed by the time-dependent Gross-Pitaevskii equation (GPE)

\[
i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + (V_{\text{trap}} + g | \psi |^2) \psi,
\]

where \(V_{\text{trap}}\) is the static trap potential and \(g = 4\pi \hbar^2 a/m\), with \(a\) the s-wave scattering length and \(m\) the atomic mass. The wave function is normalized as \(\int d^3r \ | \psi(r,t) |^2 = N\), with \(N\) the total number of atoms in the condensate.

In the present work we consider a harmonic, spherically symmetric trap \(V_{\text{trap}} = \frac{1}{2}m\omega_0^2 r^2\). This suggests choosing \(d_0 = \sqrt{\hbar/m\omega_0}\) as the unit of length, \(1/\omega_0\) as the unit of time and \(\hbar\omega_0\) as the unit of energy. We also change the normalization of the wave function to unity, \(4\pi \int \ | \psi(r,t) |^2 r^2 dr = 1\). To simplify equations we work with the function \(\phi(r,t) = r\psi(r,t)\), for which the GPE reads

\[
\frac{1}{i} \frac{\partial \phi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial r^2} + \left( \frac{1}{2} r^2 + K \rho_0^2 \right) \phi,
\]

where \(K = 4\pi Na/d_0\), and the generalised density \(\rho(r,t) = r^2 | \psi(r,t) |^2\). For a stationary state, \(\phi(r,t) = \phi(r) exp(-i\epsilon t)\), \(\epsilon\) being the single-particle energy or chemical potential. For each \(N < N_c\) there are two stationary states, one metastable and another unstable, at the top of the energy barrier (cf Fig. 1).
Quantum tunneling of BEC is described by a specific solution to the equation
\[
\frac{\partial \phi}{\partial \tau} - \frac{1}{2} \frac{\partial^2 \phi}{\partial r^2} + (\frac{1}{r})^2 + K \frac{\rho}{r^2} \phi = \epsilon \phi
\] (3)

obtained from (2) by a transition to imaginary time, \(t \rightarrow -i \tau\)\(^\text{[4][13]}\). Now, the density \(\rho(r, \tau) = \phi(r, -\tau)^* \phi(r, \tau)\) as \(\phi(r, t)^* \) is replaced by \(\phi(r, -\tau)^* \) upon \(t \rightarrow -i \tau\). This makes Eq.(3) nonlocal in time. Bounce has to satisfy the boundary conditions 1) periodicity, \(\phi(r, \tau_p/2) = \phi(r, -\tau_p/2) = \phi_0(r) = r \psi_0(r)\), with \(\psi_0(r)\) the amplitude of the metastable state, and 2) barrier penetration, i.e. \(\phi(r, \tau = 0)\) has to be some state of BEC at the other side of the potential barrier. Eq. (3) preserves both the normalization \(4\pi \int d\rho(r, \tau) = 1\) and the energy
\[
E = 4\pi \int_0^\infty dr \left\{ \frac{1}{2} \frac{\partial \phi(-\tau)^*}{\partial r} \frac{\partial \phi(\tau)}{\partial r} + \frac{1}{2} \rho r^2 + K \frac{\rho^2}{2 r^2} \right\},
\] (4)

with \(E = N \mathcal{E}\) the energy of the metastable state. For a bounce starting from the metastable state the period \(\tau_p\) extends to infinity \(\mathcal{E}\). The decay exponent reads \(10\).

Since Eq.(3) is real and the boundary value \(\phi_0(r)\) may be taken real, we assume real \(\phi(r, \tau)\) in the following.

Now the point is to transform the nonlocal in time instanton equation (3) into an evolution equation for the condensate density. A transformation of the real-time GPE (3) to a fluid-dynamic form provides an analogy, but is conceptually simpler.

The bounce equation (3), with the boundary conditions specified above, splits into two time-local equations for the time-even density \(\rho\) and the time-odd current \(j(r, \tau) = -1/2(\phi(-\tau) \partial_\tau \phi(\tau) - \phi(\tau) \partial_\tau \phi(-\tau))\)
\[
\frac{\partial \rho}{\partial \tau} + \frac{\partial j}{\partial r} = 0,
\] (6)

\[
\frac{\partial j}{\partial \tau} + \frac{1}{4} \frac{\partial^3 \rho}{\partial r^3} - \frac{\partial \Theta}{\partial r} - \rho \frac{\partial}{\partial r} \left( r K \frac{\partial f}{\partial r} \right) = 0,
\] (7)

where the kinetic energy density \(\Theta = \partial_\tau \phi(-\tau) \partial_\tau \phi(\tau) = \frac{1}{4} (\partial_\tau \rho)^2 - j^2 / \rho\). When \(\rho\) is non-negative (which is very plausible in the present case, but not granted in general as \(\rho(r, \tau) = \phi(r, \tau) \phi(-r, \tau) \neq \phi(r, \tau)^2\), one can define a regular, time-odd function \(\chi = -\frac{1}{2} (\ln |\phi(\tau)| - \ln |\phi(-\tau)|)\) which allows decomposition \(\phi = \sqrt{\rho} e^{-\chi}\). From this, the fluid-dynamic representation of (3) follows, with the velocity field \(\partial \chi / \partial r = j / \rho\). However, even for arbitrary \(\rho\), one can eliminate \(j\) from Eqs.(6),(7), which is more convenient. Introducing \(f(r, \tau) = \int_0^r \rho(r', \tau) dr'\), so that \(j = -\partial f / \partial \tau\) and \(\rho = \partial f / \partial r\), we automatically fulfill (3), and (3) transforms to the equation for the primitive of the bounce density, basic for the imaginary-time dynamics of spherical BEC-s:
\[
\frac{\partial^2 f}{\partial \tau^2} - \frac{1}{4} \frac{\partial^4 f}{\partial r^4} + \frac{\partial}{\partial r} \left( \frac{1}{2} \frac{\partial^2 f}{\partial r^2} - \frac{(\partial f)^2}{2 \partial r^2} \right) + \frac{\partial f}{\partial r} [r + K \frac{\partial}{\partial r} \left( \frac{\partial f}{\partial r} \right)] = 0.
\] (8)

Notice, that the finite amplitude oscillations of BEC around the metastable minimum are described by the real-time version of Eq.(3), in which \(\partial^2 f\) and \((\partial f)^2\) are replaced by \(-\partial^2 f\) and \(-\partial f f\), respectively.

An alternative, global approach to quantum tunneling is to look for a minimum of action (3) under the condition of constant energy (4) \(\mathcal{E}\) and norm. Indeed, for a regular \(\chi\), i.e. positive \(\rho\), \(S = 4\pi N \int_{-\tau_p/2}^{\tau_p/2} d\tau \int_0^\infty dr \rho^2 / 2\). Using Eq.(4) and introducing an observable \(Q\) uniquely labelling states along the barrier, explicitly \(Q(\tau) = (r^2)^2 / N = 4\pi \int_0^\infty dr \rho r^2\), we obtain the following functional
\[
S[f] = 2N \int_{Q(0)}^{Q(\tau_p/2)} dQ \sqrt{2B(Q)(V(Q) - \mathcal{E})}
\] (9)

which is minimized by the primitive of the bounce density (note, that \(Q(0) < Q(\tau_p/2)\) for BEC). The potential energy \(V(Q) = V[\rho(Q)]\),
\[ V[\rho] = 4\pi \int_0^\infty dr \left[ \frac{(\partial_r \rho)^2}{8\rho} + \frac{1}{2} \partial_r \rho^2 + \frac{K \rho^2}{2\rho^2} \right] \]

and the effective mass parameter \( B(Q) = B[f(Q)] \)

\[ B(Q) = 4\pi \int_0^\infty dr \frac{(\partial_r f)^2}{\rho} \]

are both the functionals of \( f \). Note that Eq. (10) is invariant with respect to a change of the controlling variable, as for any other such variable \( q \), \( B(q) = B(q)(dQ/dq)^2 \). The energy conservation (3) implies that for bounce \( \dot{Q}^2 = 2(V(Q) - \mathcal{E})/B(Q) \), with \( \dot{Q} = dQ/d\tau \), and therefore \( \dot{Q} = \frac{d}{d\tau}[(V(Q) - \mathcal{E})/B(Q)] \).

### III. RESULTS AND DISCUSSION

We have solved Eq. (3) using the variable \( Q \) rather than \( \tau \). An initial sequence of densities \( \rho_s(r, Q_i) \), with 30-50 \( Q_i \) points covering the barrier region, was constructed by minimizing \( V[\rho] \) under the constraint \( Q = Q_i \). These constrained stationary densities were then improved upon iteratively. The details of \( \rho^{1/2}/r \), the counterpart of \( \psi \) of Eq. (3), are obtained more precisely when the \( r^2 \) behaviour near \( r = 0 \) and the harmonic oscillator asymptotics at infinity are factored out in Eq. (3). In numerical work, we use a function \( \alpha(r, \tau) \) such that \( \rho(r, \tau) = r^2 e^{-r^2} e^{2\alpha(r, \tau)} \), and properly express the term \( \partial_{Q} f - \partial_{\rho}(\partial_{r} f)^2/\rho \) - see Appendix.

We also performed the minimization of the functional (3) treating \( \rho(r_i, Q_i) \) as independent variables. It turns out that \( \rho(r, \tau) \) thus obtained do not fulfill Eq. (3) accurately, but the accuracy in \( S \) is better than 0.1%.

The numerical results have been obtained with physical data on \( ^7\text{Li} \) adopted after the most accurate treatment up to date (3). These values give \( K = -5.74 \times 10^{-3} \times N \), and we obtain the critical value \( K_c \) between \(-7.2249 \) and \(-7.2255 \) (\( N_c \) between 1258.7 and 1258.8).

The potential energy \( V(Q) \) from the bounce solutions (Fig.1), nearly identical with \( V[\rho_s(Q)] \) for constrained stationary \( \rho_s(r, Q_i) \), is very flat between \( Q(0) \) and \( Q(\tau_p/2) \) for \( N \approx N_c \). For smaller \( N \), it becomes quite peaked around the summit, and its fall on the side of small \( Q \) becomes very steep. All the obtained bounce solutions result from the small adjustment of the initial densities. For larger \( N_c - N \) (and increasing \( \dot{Q}^2 \) and \( \ddot{Q} \) terms, cf Eq. (4)) this adjustment becomes gradually more difficult. We could not obtain the exact solution of Eq. (4) (or even a constrained stationary \( \rho_s(r, Q_i) \) for small \( Q_i \)) for \( N \leq 1200 \).

The mass parameters \( B(Q) \) from various instanton solutions (Fig.2) are nearly identical which shows that there exists a universal inertia for the radial collective motion of BEC close to instability. This may be understood as a consequence of a nearly static character of solutions \( f(r, Q) \) in the limited range of \( N(K) \) values of interest: \( f \) depends weakly on \( N \) close to critical \( N_c \).

For a gaussian density with a variable width \( b(r) \), \( \rho(r, \tau) = \pi^{-3/2}b^{3/2} \exp(-r^2/b^2) \), the current \( j = b(r)/b \rho \) (Eq. (5)), and the mass parameter \( B(b) = 4\pi b^2 \int_{-\infty}^{\infty} r^2 \rho dr = 3/2 \) (cf Ref. [3]). Hence, as for the gaussian density \( Q = 3b^2/2 \), one has \( 4QB(b) = B(\sqrt{\mathcal{E}}) = 1 \), as used in Ref. [3]. As seen in Fig.2, this is a fair assumption near the metastable minima, much worse though for smaller \( Q \) around the barrier’s summit. The error in \( S \) due to the gaussian value of \( B \) amounts to 3-4% in the cases studied.

The bounce ”amplitude”, \( \sqrt{\mathcal{E}}/r \) (Fig.3), differs from the constrained stationary values \( \sqrt{\mathcal{E}}/r \) mostly near \( r = 0 \) and for small \( Q \), by up to 1%. Decay exponents calculated with the initial densities \( \rho_s(r, Q_i) \) are only up to 0.3% larger than the exact ones.

Next we turn to collective radial oscillations of BEC and their decay rates. Quantum tunneling from the excited state with energy \( \hbar\omega_n \) may be treated by solving Eq. (3) as before, except that now the period \( \tau_p \) must be finite and \( N \mathcal{E} \) replaced by \( N \mathcal{E}_n = N \mathcal{E} + \hbar\omega_n \). One has to fix the boundary values \( \rho(r, \tau_p/2) \) and the energies \( \hbar\omega_n \).

The energies \( \hbar\omega_n \) of the excited quantum states follow from the quantization condition \( S = 2n\pi \) [14], where \( S = 2N \int_{Q(0)/2}^{Q(t_p/2)} dQ \sqrt{2B(Q)(\mathcal{E}_n - V(Q))} \), cf Eq. (3), with the period \( t_p \) depending on \( \hbar\omega_n \). Essentially, the imaginary-time boundary value \( \rho(r, \tau_p/2) \) should match the real-time oscillatory solution \( \rho(r, t_p/2) \) at the turning point \( Q \) at which \( V(Q) = \mathcal{E}_n \).

We could not solve the real-time version of Eq. (3) as accurately as for instantons. However, since the action \( S \) is insensitive to the details of solution, the lowest quantized \( \hbar\omega_n \) are quite accurate. Their comparison to frequencies of the small amplitude oscillations, obtained from the GPE linearized around \( \psi_0 \) (so called Bogolyubov spectrum, see [10]), is interesting. The first radial excitation, \( \hbar\omega_1 \), is nearly equal, but little smaller than, the lowest Bogolyubov mode corresponding to the small amplitude limit, except that it may not exist for a too shallow \( V(Q) \), like for \( N = 1255 \). The higher modes, \( \hbar\omega_n, n = 2, 3, ..., \) come out roughly at the multiples of \( \hbar\omega_1 \) (Table 1), and thus...
represent nearly harmonic spectrum of collective radial oscillations. Its \( n = 2, 3, 4 \) states are lower than the second Bogolyubov mode, lying slightly above \( 4\hbar \omega_0 \).

Using the quantized oscillation energies we have found periodic instantons and calculated decay exponents for some low collective radial excitations. Such instanton with a period \( \tau_p \) is quite similar to the so-called "thermon" which describes thermal decay from the metastable ground state at the temperature \( kT = \hbar/\tau_p \). This similarity is easily understood: Thermal decay goes via thermal excitation of quantum states above the metastable ground state and successive tunneling from them through the smaller barrier. At higher temperatures, details of the excitation spectrum and tunneling become irrelevant and the thermal tunneling rate \( \exp(-(E_b - E)/(\hbar kT)) \), with \( E_b = N\mathcal{E}_b \) the barrier energy, replaces a thermal mixture of rates \( \exp(-E_i) \) from a few lowest excited states at small temperature. The critical temperature \( T_c \) at which this happens is usually determined as the one at which \( \Gamma_{\text{thermal}}(T_c) = \Gamma_{\text{thermon}}(T_c) \).

Numerical results are collected in Table 1. Those in columns 2-4 depend only on \( K \) and relate to all BEC-s with attractive interaction. Decay rates \( \Gamma \) from the metastable \(^7\)Li BEC were calculated for \( \omega_0 = 908.41 \text{ s}^{-1} \), assuming prefactor \( \Gamma/\omega_0 \) \( = (\omega_1/\omega_0)(15S/2\pi)^{1/2} \), with \( \omega_1 \) from the small amplitude limit. The exact prefactor is difficult to calculate, but it must be of the same order as in Table 1. Obtained values of \( \Gamma \) are very close to those of Ref. [7], where the effect of too small \( B \)-s is fortuitously reduced by half by the interpolation overestimating \( V(Q) \).

The decrease of the decay exponents \( S_n \) with the mode number \( n \) and their comparison to exponents \( S \) for the ground state are seen in Table 1, col. 5. For example, quantum decay of the \( n = 2 \) radial mode in BEC with \( N = 1245 \) is nearly as quick as for the ground state of the \( N = 1255 \) BEC. The crossover temperature \( T_c \) at which the thermal decay begins to dominate over the quantum tunneling is approximately, neglecting prefactors, as \( kT_c/(\hbar \omega_0) = (E_b - E)/S \), with \( E_b = N\mathcal{E}_b \) the barrier energy, \( E = N\mathcal{E} \) for the metastable state, and \( E = N\mathcal{E}_n \) for the excited state. For the latter, the meaning of critical temperature is extended in analogy with that for the ground state: Suppose the condensate in the oscillation state \( n \) and ask at which temperature \( T_c(n) \) the thermal rate exponent equals the instanton action \( S_n \). These ratios (Table 1, column 3) show that for presently measurable \( \Gamma \) and \( \omega_0 = 908.41 \text{ s}^{-1} \), corresponding to \( T = 6.94 \text{ nK} \), \( T_c \approx 1\text{nK} \), both for ground state and collective radial oscillations. We notice, that for all finite-\( \tau_p \) instantons describing decay out of the \( n \)-th collective radial excitation, \( h/\tau_p < T_c(n) \) so, at their own thermon "temperature", they dominate over the thermal decay.

The quantity \( S/N \) (Table 1, column 4) shows \( (N_c - N)^2 \) behaviour close to \( N_c \), with \( \xi = 5/4 \). For larger \( N_c - N \) the effective exponent slightly decreases to \( \xi \approx 1.2 \). Consider now two different attractive BEC-s with critical particle numbers \( N_c \) and \( N_c' \). The same decay exponent \( S \) is obtained for such (not too large) \( N_c - N \) and \( N_c' - N' \) which satisfy the relation \( N_c' - N' = (N_c - N)/N_c' - 1/\xi \), with \( 1 - 1/\xi \approx 1/5 \). Thus, e.g. we obtain \( S = 9.44 \) for \( N_c' - N' \approx 12 \) in BEC with \( N_c' = 6294 \), and for \( N_c' - N' = 6 - 7 \) in BEC with \( N_c' = 251.76 \), cf Table 1.

In summary, the equations for the condensate density, describing both the real- and imaginary-time dynamics of spherical BEC, were formulated and the exact instanton solutions were found numerically, also for collective radial excitations. The determined mass parameter (Fig. 2) deviates from the gaussian ansatz, but the calculated decay exponents for the metastable states agree well with Ref. [11]. It follows from Fig. 2 that the exact mass parameter may be more relevant to the behind-the-barrier collapse phase, i.e. for smaller \( Q \). The quantized energies of collective finite amplitude radial vibrations form nearly harmonic (slightly compressed) spectrum with \( \omega_n \approx n\omega_1 \), where \( \omega_1 \) is slightly lower than the lowest Bogolyubov mode. The \( n = 2, 3, 4 \) collective oscillation states lie lower than the second Bogolyubov mode. Any excitation (thermal or otherwise, e.g. by modulation of the trapping oscillator frequency) of these collective vibrations must lead to a faster decay of the condensate, as the Table 1 shows.

If quantum tunneling is not to be overshadowed by thermal decay, the experiments should proceed at low \( T \) and/or large \( \omega_0 \). Since theoretical results are more certain away from \( N_c \), where the exponent \( S \) dominates decay, one should probe a range of moderate \( S \), giving observable, but not too large \( \Gamma \) (perhaps, by discarding prompt collapses). The corresponding range of \( N_c - N \) values depends on \( N_c \) as \( N_c^{1/5} \).

**APPENDIX: NUMERICAL METHODS**

We have \( \rho(s, \tau) = se^{-s}e^{2\alpha(s, \tau)} \) with \( s = r^2 \). The stationary GPE leads to the equation for \( \alpha(s) \)

\[
2s \frac{d^2 \alpha}{ds^2} + \left( \frac{d\alpha}{ds} \right)^2 - \frac{d\alpha}{ds} + 3 \frac{d\alpha}{ds} - Ke^{2\alpha - s} + \beta = 0,
\]

(A1)

with \( \beta = \epsilon - 3/2 \). For large \( s \), \( \alpha \approx \frac{\beta}{2} \ln s \). The normalization of \( \phi \) implies the next two terms

\[
\alpha(s) = \frac{\beta}{2} \ln s + C - \frac{\beta(\beta - 1)}{4s} + O\left( \frac{1}{s^2} \right).
\]

(A2)

A solution regular at \( s = 0 \) must fulfill
\[
\frac{d\alpha}{ds}(0) = \frac{1}{3}(3/2 - \epsilon + Ke^{2\alpha(0)}).
\]  
(A3)

These boundary conditions suggest a method of solution: For a given \( K \) we assume some \( \epsilon \) and \( C \) and, starting from the asymptotic values (A2) at large \( s \), integrate Eq.(A1) to \( s = 0 \). We check Eq.(A3) and the normalization and correct \( \epsilon \) and \( C \) until we fulfil both.

By using a new variable \( \bar{v} = \frac{\partial f}{\partial \rho}/(r\rho) \) and factoring out \( 2r\rho \), we transform Eq.(8) to a form

\[
\frac{1}{2} \ddot{Q} \bar{v} + \dot{Q}^2 \left( \frac{1}{2} \frac{\partial \bar{v}}{\partial Q} - \frac{\partial \alpha}{\partial Q} \bar{v} + \bar{v}^2 [1 + s(2 \frac{\partial \alpha}{\partial s} - 1)] \right) = \frac{\partial R}{\partial s}.
\]  
(A4)

suitable for both instantons and oscillations if \( \dot{Q}^2 = 2(V(Q) - \xi_\ast)/B(Q) \) is understood. \( R[\alpha] \) stands for the l.h.s. of Eq.(A1). Let us call the l.h.s. of Eq.(A4) \( F \). If \( F = 0 \) (no time dependence) we recover stationary solutions of Eq.(A1). For instantons, we solve Eq.(A4) iteratively. Having a set of \( \alpha(s,Q_i) \) we calculate \( F_i = F(s,Q_i) \). For each \( F_i \) we solve (A4) as the ordinary differential equation in \( s \) to obtain new \( \alpha(s,Q_i) \). The method, as for the stationary case, is to adjust the asymptotic form (A2) to the proper regularity condition at \( s = 0 \). The new and old \( F_i \)-s are combined to provide initial \( F_i \)-s for the next iteration. With a careful modification of \( F_i \)-s this iteration leads to the self-consistency, i.e. \( F_i(old) = F_i(new) \). The initial densities \( \rho_s(s,Q_i) \) are obtained using the constrained imaginary-time step Hartree procedure. We use the Runge-Kutta-Merson procedure for integration of Eqs.(A4). Energies are calculated using a mesh of 128 equidistant points, \( r/d_0 = 0 - 4.5 \), and the cubic spline interpolation for derivatives. We have checked that doubling the mesh density does not change results in any appreciable manner.
Table 1 - Energies, crossover temperatures, decay exponents and rates for metastable and radially excited states of the $^7$Li BEC. Results from bounce solutions (* from the functional minimization).

| $N$  | $\omega_n/\omega_0$ | $\bar{g}^2/\bar{g}_{0}$ | $S \times 10^3$ | $\Gamma [s^{-1}]$ | $\Gamma [s^{-1}]$ |
|------|----------------------|--------------------------|------------------|------------------|------------------|
| 1255 | 0.130                | 2.6744                   | 3.356            | 4.305            | 1.36 $\cdot 10^2$ |
| 1250 | 0.165                | 7.5520                   | 9.440            | 4.756            | 3.43 $\cdot 10^{-1}$ |
| n=1  | 0.967                | 2.5244                   | 3.156            |                  |                  |
| 1245 | 0.189                | 13.097                   | 16.306           | 6.87             | 5.17 $\cdot 10^{-4}$ |
| n=1  | 1.086                | 7.6934                   | 9.578            |                  |                  |
| n=2  | 2.112                | 3.5636                   | 4.437            |                  |                  |
| 1240 | 0.207                | 19.082                   | 23.662           | 8.80             | 4.23 $\cdot 10^{-7}$ |
| n=1  | 1.169                | 13.436                   | 16.661           |                  |                  |
| n=2  | 2.292                | 9.0346                   | 11.203           |                  |                  |
| n=3  | 3.373                | 5.1464                   | 6.382            |                  |                  |
| 1230 | 0.238                | 31.980                   | 39.335           | 12.31            | 9.24 $\cdot 10^{-14}$ |
| n=1  | 1.268                | 26.084                   | 32.083           |                  |                  |
| n=2  | 2.518                | 21.347                   | 26.257           |                  |                  |
| n=3  | 3.748                | 17.076                   | 21.004           |                  |                  |
| 1200 | 0.305                | 75.40                    | 90.490           | 21.12            | 9.63 $\cdot 10^{-3b}$ |

Figure captions

Fig. 1 Potential energy $E(Q) = NV(Q)$ of BEC (in $\bar{\hbar}\omega_0$) for various $N < N_c$. ($Q$ in units $d_0^2$)

Fig. 2 Mass parameters $B(\sqrt{Q}) = 4QB(Q)$ from various instanton solutions, overlayed in one picture. For gaussians, $4QB(Q)=1$.

Fig. 3 Bounce penetrates the barrier practically in a finite $\tau$ (in units $\omega_0^{-1}$). The metastable density $\rho(r, \pm \infty)$ is nearly equal to $\rho(r, \tau = 3.43)$ shown.
$4 \ Q \ B(Q)$
$\rho^{1/2}(r, \tau)/r$

$N=1240$