EXCHANGE OPERATORS AND EXTENDED HEISENBERG ALGEBRA
FOR THE THREE-BODY CALOGERO-MARCHIORO-WOLFES
PROBLEM

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Abstract
The exchange operator formalism previously introduced for the Calogero problem
is extended to the three-body Calogero-Marchioro-Wolfes one. In the absence of
oscillator potential, the Hamiltonian of the latter is interpreted as a free particle
Hamiltonian, expressed in terms of generalized momenta. In the presence of oscillator
potential, it is regarded as a free modified boson Hamiltonian. The modified boson
operators are shown to belong to a $D_6$-extended Heisenberg algebra. A proof of
complete integrability is also provided.
1 Introduction

A long time ago, Calogero [1] solved the Schrödinger equation for three particles in one dimension, interacting pairwise via harmonic and inverse square potentials. Later, Wolfes [2] extended Calogero’s method to the case where there is an additional three-body potential of a special form. The same problem in the absence of harmonic potential was also studied by Calogero and Marchioro [3]. Other exactly solvable many-body problems were then introduced, analyzed from the viewpoint of classical or quantum integrability, and shown to be related to root systems of Lie algebras [4].

A breakthrough in the study of integrable models occurred some three years ago when Brink et al [5] and Polychronakos [6] independently introduced an exchange operator formalism, leading to covariant derivatives, known in the mathematical literature as Dunkl operators [7], and to an $S_N$-extended Heisenberg algebra [8, 9]. In terms of the latter, the $N$-body quantum-mechanical Calogero model can indeed be interpreted as a model of free modified oscillators. Such an approach emphasizes the relations between the Calogero problem and fractional statistics [10] and is connected with the spin generalization of the former (see e.g. Ref. [11] and references quoted therein).

Recently, there has been a renewed interest in the three-body Calogero-Marchioro-Wolfes (CMW) problem [2, 3] and some other related three-particle problems including a three-body potential. Khare and Bhaduri [12] indeed showed that they can be solved by using supersymmetric quantum-mechanical techniques. Their approach is based upon Calogero’s original method [1], wherein after eliminating the centre-of-mass motion, the three-body problem is mapped to that of a particle on a plane and the corresponding Schrödinger equation is separated into a radial and an angular equations.

The purpose of the present letter is to show that the exchange operator formalism of Brink et al [5] and Polychronakos [6] can be extended to the CMW problem, thereby leading to a further generalization of the Heisenberg algebra. Our starting point will be another supersymmetric approach to the three-body problem, wherein the latter is mapped to that of a particle in three-dimensional space and use is made of the Andrianov et al generalization of supersymmetric quantum mechanics for multidimensional Hamiltonians [13].
2 Supersymmetric Approach to the CMW Problem

The three-particle Hamiltonian of the CMW problem is given by \[2, 3\]

\[
H = \sum_{i=1}^{3} \left(-\partial_i^2 + \omega^2 x_i^2\right) + g \sum_{i \neq j}^{3} \frac{1}{(x_i - x_j)^2} + 3f \sum_{i \neq j \neq k}^{3} \frac{1}{(x_i + x_j - 2x_k)^2}, \tag{2.1}
\]

where \(x_i, i = 1, 2, 3\), denote the particle coordinates, \(\partial_i \equiv \partial/\partial x_i\), and the inequalities \(g > -1/4\), and \(f > -1/4\) are assumed to be valid to prevent collapse. Let \(x_{ij} \equiv x_i - x_j, i \neq j\), and \(y_{ij} \equiv x_i + x_j - 2x_k, i \neq j \neq k \neq i\), where in the latter we suppressed index \(k\) as it is entirely determined by \(i\) and \(j\).

Since for singular potentials crossing is not allowed, in the case of distinguishable particles the wave functions in different sectors of configuration space are disconnected. We shall therefore restrict the particle coordinates to the ranges \(x_1 > x_2 > x_3\) if \(g \neq 0, f = 0\), \(x_1 > x_3, x_2 > x_3, |x_1 - x_2| < x_1 - x_3, |x_1 - x_2| < x_2 - x_3\) if \(g = 0, f \neq 0\), and \(x_1 > x_2 > x_3, x_1 - x_2 < x_2 - x_3\) if \(g \neq 0, f \neq 0\). In the case of indistinguishable particles, there is an additional symmetry requirement, which we shall not review here as it was discussed in detail in Refs. \[2\] and \[3\].

For distinguishable particles, the unnormalized ground-state wave function of Hamiltonian \(2.1\), corresponding to the eigenvalue

\[
E_0 = 3\omega(2\kappa + 1) \quad \text{if} \ g \neq 0, f = 0,
\]

\[
= 3\omega(2\lambda + 1) \quad \text{if} \ g = 0, f \neq 0,
\]

\[
= 3\omega(2\kappa + 2\lambda + 1) \quad \text{if} \ g \neq 0, f \neq 0, \tag{2.2}
\]

is

\[
\psi_0(x) = \exp\left(-\frac{1}{2\omega} \sum_i x_i^2\right) |x_{12}x_{23}x_{31}|^\kappa \quad \text{if} \ g \neq 0, f = 0,
\]

\[
= \exp\left(-\frac{1}{2\omega} \sum_i x_i^2\right) |y_{12}y_{23}y_{31}|^\lambda \quad \text{if} \ g = 0, f \neq 0,
\]

\[
= \exp\left(-\frac{1}{2\omega} \sum_i x_i^2\right) |x_{12}x_{23}x_{31}|^\kappa |y_{12}y_{23}y_{31}|^\lambda \quad \text{if} \ g \neq 0, f \neq 0. \tag{2.3}
\]
where $\kappa \equiv \frac{1}{2}(1 + \sqrt{1 + 4g})$, $\lambda \equiv \frac{1}{2}(1 + \sqrt{1 + 4f})$ (implying $g = \kappa(\kappa - 1)$, $f = \lambda(\lambda - 1)$).

In terms of the function $\chi(x) = -\ln \psi_0(x)$, one can construct six differential operators $Q^\pm = \mp \partial_i + \partial_i \chi$, $i = 1, 2, 3$, whose explicit form is given by

$$Q^\pm = \mp \partial_i + \omega x_i - \kappa \sum_{j \neq i} \frac{1}{x_{ij}} \text{ if } g \neq 0, f = 0,$$

$$\mp \partial_i + \omega x_i - \lambda \left( \sum_{j \neq i} \frac{1}{y_{ij}} - \sum_{i \neq j \neq k \neq i} \frac{1}{y_{jk}} \right) \text{ if } g = 0, f \neq 0,$$

$$\mp \partial_i + \omega x_i - \kappa \sum_{j \neq i} \frac{1}{x_{ij}} - \lambda \left( \sum_{j \neq i} \frac{1}{y_{ij}} - \sum_{i \neq j \neq k \neq i} \frac{1}{y_{jk}} \right) \text{ if } g \neq 0, f \neq 0. \quad (2.4)$$

It can be easily shown [13] that $H - E_0$ can be regarded as the $H^{(0)}$ component of a supersymmetric Hamiltonian

$$\hat{H} = \begin{pmatrix} H^{(0)} & 0 & 0 & 0 \\ 0 & H^{(1)} & 0 & 0 \\ 0 & 0 & H^{(2)} & 0 \\ 0 & 0 & 0 & H^{(3)} \end{pmatrix} \quad (2.5)$$

with supercharge operators

$$\hat{Q}^+ = \begin{pmatrix} Q^+_{0,1} & 0 & 0 & 0 \\ 0 & Q^+_{1,2} & 0 & 0 \\ 0 & 0 & Q^+_{2,3} & 0 \end{pmatrix}, \quad \hat{Q}^- = (\hat{Q}^+)^\dagger = \begin{pmatrix} 0 & Q^-_{1,0} & 0 & 0 \\ 0 & 0 & Q^-_{2,1} & 0 \\ 0 & 0 & 0 & Q^-_{3,2} \end{pmatrix}, \quad (2.6)$$

i.e., $\hat{H}, \hat{Q}^+, \hat{Q}^-$ generate the supersymmetric quantum-mechanical algebra $\text{sqm}(2)$,

$$\{ \hat{Q}^+, \hat{Q}^- \} = \hat{H}, \quad \{ \hat{Q}^+, \hat{Q}^+ \} = \{ \hat{Q}^-, \hat{Q}^- \} = 0, \quad [\hat{Q}^+, \hat{H}] = [\hat{Q}^-, \hat{H}] = 0. \quad (2.7)$$

In [2, 6], $Q^+_{0,1, 1,2, 2,3}$ (resp. $Q^-_{0,1, 2,3}$, $Q^-_{1,0, 2,3}$) denote $3 \times 1$, $3 \times 3$, $1 \times 3$ (resp. $1 \times 3$, $3 \times 3$, $3 \times 1$) matrices, whose elements are $Q^\pm_i, P^+_{ij} = \epsilon_{ijk} Q^-_k, Q^-_{ij}$ (resp. $Q^\pm_i, P^-_{ij} = \epsilon_{ijk} Q^+_{kj}, Q^\pm_i$), where $\epsilon_{ijk}$ is the antisymmetric tensor and there is a summation over dummy indices.

The components of $\hat{H}$ can be expressed in terms of such matrices as

$$H^{(n)} = H^{(n)} + H^{(n)}_{\omega}, \quad H^{(n)} = Q^+_n Q^-_{n-1}, \quad H^{(n)} = Q^-_n Q^+_n. \quad (2.8)$$
where \( n = 0, 1, 2, 3 \), and \( Q_{-1,0}^+ = Q_{0,-1}^- = Q_{3,4}^+ = Q_{4,3}^- = 0 \). Apart from some additive constants, \( H^{(0)} = Q_i^+ Q_i^- \), and \( H^{(3)} = Q_i^+ Q_i^- \) are given by (2.1) with \( g = \kappa(\kappa - 1) \), \( f = \lambda(\lambda - 1) \), and \( g = \kappa(\kappa + 1) \), \( f = \lambda(\lambda + 1) \) respectively, while the remaining two components \( H^{(1)} \) and \( H^{(2)} \) have the form of Schrödinger operators with matrix potentials. The operators (2.8) satisfy the intertwining relations

\[ H'(n+1)Q_{n,n+1}^- = Q_{n+1,n}^- H'(n+1), \quad Q_{n+1,n}^- H'(n) = H'^{(n)} Q_{n,n+1}^-, \]  

and similar relations with \( H'^{(n+1)} \) and \( H'^{(n)} \) replaced by \( H^{(n+1)} \) and \( H^{(n)} \), respectively. Eq. (2.9) shows that all discrete eigenvalues of \( H'^{(n)} \) and \( H'^{(n+1)} \) (or, equivalently, \( H^{(n)} \) and \( H^{(n+1)} \)) but one are the same and that their eigenfunctions are connected by the operators \( Q_{n,n+1}^+, Q_{n+1,n}^- \).

### 3 Generalized Momenta for the CMW Problem without Oscillator Potential

In the absence of harmonic oscillator and three-body potentials, i.e., for \( \omega = f = 0 \), the operators \( Q_i^- \), defined in (2.4), can be converted into covariant derivatives \( D_i \) by inserting particle permutation operators \( K_{ij} \), obeying

\[ K_{ij} = K_{ji} = K_{ij}^\dagger, \quad K_{ij}^2 = 1, \quad K_{ij} K_{jk} = K_{jk} K_{ki} = K_{ki} K_{ij}, \]  

and

\[ K_{ij} x_j = x_i K_{ij}, \quad K_{ij} x_k = x_k K_{ij}, \]  

for all \( i \neq j \neq k \neq i \). The operators \( (3.3) \)

\[ D_i = \partial_i - \kappa \sum_{j \neq i} \frac{1}{x_{ij}} K_{ij}, \quad i = 1, 2, 3, \]  

indeed satisfy the relations

\[ K_{ij} D_j = D_i K_{ij}, \quad K_{ij} D_k = D_k K_{ij} \quad (k \neq i, j), \quad D_i^\dagger = D_i, \quad [D_i, D_j] = 0, \]  

and

\[ - \sum_i D_i^2 = - \sum_i \partial_i^2 + \sum_{ij} \frac{1}{x_{ij}^2} \kappa(\kappa - K_{ij}). \]  

5
From (3.7), it results that in those subspaces of Hilbert space wherein $K_{ij} = +1$ or $-1$ for any $i, j$, the Calogero Hamiltonian without oscillator potential, corresponding to the parameter value $\kappa$ or $\kappa + 1$ respectively, can be regarded as a free particle Hamiltonian expressed in terms of the generalized momenta $\pi_i = -iD_i$.

We now plan to show that this exchange operator formalism can be easily extended to the case where the three-body potential is present. For such purpose, we note that from (3.2)

$$K_{ij} x_{ij} = -x_{ij}K_{ij}, \quad K_{ij} x_{jk} = -x_{ki}K_{ij}, \quad K_{ij} R = RK_{ij},$$

where $i \neq j \neq k \neq i$, and $R = (x_1 + x_2 + x_3)/3$ denotes the centre-of-mass coordinate. As the dependence of Hamiltonian (2.1) upon $x_{ij}$ and $y_{ij}$ is similar, let us introduce some operators $L_{ij}$ satisfying properties analogous to (3.1) and (3.6), where in the latter $y_{ij}$ is substituted for $x_{ij}$,

$$L_{ij} = L_{ji}, \quad L_{ij}^2 = 1, \quad L_{ij} L_{jk} = L_{jk} L_{ki} = L_{ki} L_{ij},$$

$$L_{ij} y_{ij} = -y_{ij} L_{ij}, \quad L_{ij} y_{jk} = -y_{ki} L_{ij}, \quad L_{ij} R = RL_{ij}.$$  

Hence,

$$L_{ij} x_i = (2R - x_j) L_{ij}, \quad L_{ij} x_k = (2R - x_k) L_{ij} \quad (k \neq i, j),$$

and $L_{ij}$ can be written as

$$L_{ij} = K_{ij} I_r = I_r K_{ij}, \quad I_r = I_r^\dagger, \quad I_r^2 = 1,$$

where

$$I_r x_i = (2R - x_i) I_r, \quad I_r x_{ij} = -x_{ij} I_r, \quad I_r y_{ij} = -y_{ij} I_r, \quad I_r R = RI_r.$$  

The new operator $I_r$ is therefore the inversion operator in relative-coordinate space. Together with $K_{ij}$, it generates a group of order twelve, which is the direct product of the symmetric group $S_3$ and the group of order two $\{1, I_r\}$, and is isomorphic to the dihedral group $D_6$.

For simplicity’s sake, from now on we shall work in the centre-of-mass coordinate system and therefore set $R \equiv 0$ in (3.6)–(3.10). As a consequence, any coordinate $x_i$ may
be replaced by $-\sum_{j \neq i} x_j$. This substitution will play an important role in some of the subsequent calculations.

By inserting the operators $L_{ij}$ into $Q_i^-$, the operators (3.3) are generalized into

$$D_i = \partial_i - \kappa \sum_{j \neq i} \frac{1}{x_{ij}} K_{ij} - \lambda \left( \sum_{j \neq i} \frac{1}{y_{ij}} L_{ij} - \sum_{j,k} \frac{1}{y_{jk}} L_{jk} \right), \quad (3.11)$$

whenever $g \neq 0$, and $f \neq 0$. For $g = 0$, and $f \neq 0$, the second term on the right-hand side of (3.11) is not present. After some algebra, one finds that the operators (3.11) still satisfy Eq. (3.4), and that in addition

$$I_r D_i = -D_i I_r, \quad L_{ij} D_i = -D_j L_{ij}, \quad L_{ij} D_k = -D_k L_{ij} \quad (k \neq i, j), \quad (3.12)$$

and

$$-\sum_i D_i^2 = -\sum_i \partial_i^2 + \sum_{i,j} \frac{1}{x_{ij}^2} \kappa (\kappa - K_{ij}) + 3 \sum_{i,j} \frac{1}{y_{ij}^2} \lambda (\lambda - L_{ij}). \quad (3.13)$$

The operators $I_n = \sum_i \pi_i^2$, where $\pi_i = -i D_i$, $i = 1, 2, 3$, are generalized momenta, commute with one another, and are left invariant under $D_6$. Hence, their projection in the subspaces of Hilbert space characterized by $(K_{ij}, L_{ij}) = (1, 1), (1, -1), (-1, 1), \text{ or } (-1, -1)$, also commute. In those subspaces, $I_1$ becomes the CMW Hamiltonian corresponding to parameter values $(\kappa, \lambda), (\kappa, \lambda + 1), (\kappa + 1, \lambda)$, or $(\kappa + 1, \lambda + 1)$ respectively, while $I_2$ and $I_3$ become the integrals of motion of such Hamiltonian.

Note that operators similar to (3.11) were independently derived by Buchstaber et al [14] in a work on generalized Dunkl operators. In their approach, use is made of the root system and the Weyl group of the semi-simple Lie algebra $G_2$, to which the CMW Hamiltonian is known to be related [4]. Such a Weyl group is just the dihedral group $D_6$ hereabove considered.
4 Modified Boson Operators for the CMW Problem with Oscillator Potential

When the oscillator potential is present in Hamiltonian (2.1), it is advantageous to introduce modified boson creation and annihilation operators, defined by

$$a_i^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_i - D_i), \quad a_i = (a_i^\dagger)^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_i + D_i).$$

By using (3.2) and (3.8), it can be easily shown that they satisfy the commutation relations

$$[a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0,$$

$$[a_i, a_i^\dagger] = 1 + \kappa \sum_{j \neq i} K_{ij} + \frac{\lambda}{3} \left( \sum_{j \neq i} L_{ij} + 2 \sum_{j, k \neq i} L_{jk} \right),$$

$$[a_i, a_j^\dagger] = -\kappa K_{ij} + \frac{\lambda}{3} \left( L_{ij} - 2 \sum_{k \neq i, j} (L_{ik} + L_{jk}) \right), \quad i \neq j,$$

and the exchange relations

$$K_{ij}a_j = a_i K_{ij}, \quad K_{ij}a_k = a_k K_{ij} \quad (k \neq i, j), \quad I_r a_i = -a_i I_r.$$

Eqs. (3.1), (3.3), (4.2), and (4.3) define a $D_6$-extended Heisenberg algebra. Whenever $f = 0$ and $g \neq 0$, the terms proportional to $\lambda$ on the right-hand side of (4.2) disappear, so that the algebra reduces to the $S_3$-extended one considered in Refs. [5, 6, 8], where $S_3 = \{1, K_{12}, K_{23}, K_{31}, K_{12}K_{23}, K_{23}K_{12}\}$. Whenever $g = 0$ and $f \neq 0$, the terms proportional to $\kappa$ are not present on the right-hand side of (4.2), so that the algebra becomes a $D_3$-extended Heisenberg algebra, where $D_3 = \{1, L_{12}, L_{23}, L_{31}, L_{12}L_{23}, L_{23}L_{12}\}$. Note that although $D_3$ and $S_3$ are isomorphic, their action on the creation and annihilation operators is different, thus giving rise to different extended algebras.

From (4.1) and (3.13), we obtain

$$\omega \sum_i \{a_i^\dagger, a_i\} = \sum_i (-D_i^2 + \omega^2 x_i^2)$$

$$= -\sum_i \partial_i^2 + \omega^2 \sum_i x_i^2 + \sum_{i \neq j} \frac{1}{x_{ij}^2} \kappa(\kappa - K_{ij}) + 3 \sum_{i \neq j} \frac{1}{y_{ij}^2} \lambda(\kappa - L_{ij}).$$
Hence, the CMW Hamiltonian with oscillator potential corresponding to the parameter values \((\kappa, \lambda), (\kappa, \lambda + 1), (\kappa + 1, \lambda),\) or \((\kappa + 1, \lambda + 1),\) can be regarded as a free modified boson Hamiltonian.

To show the existence of conserved quantities, let us consider the operators

\[
I_n = \sum_i h_i^n, \quad h_i = \frac{1}{2} \{a_i^\dagger, a_i\}, \quad n = 1, 2, 3.
\]  

(4.5)

From the defining relations of the \(D_6\)-extended Heisenberg algebra, it can be shown that the operators \(h_i\) obey the commutation relations

\[
[h_i, h_j] = \frac{1}{4} \left( \kappa^2 + \lambda^2 - 2\kappa\lambda I_r \right) \sum_{k \neq i, j} (K_{ij} - K_{ik}) K_{jk}, \quad i \neq j.
\]  

(4.6)

It results that \(I_1\) commutes with any \(h_i\), hence with \(I_2\) and \(I_3\).

From (4.6), it also follows that

\[
[I_2, h_i] = \sum_j \left( h_j[h_j, h_i] + [h_j, h_i]h_j \right)
= \frac{1}{4} \left( \kappa^2 + \lambda^2 - 2\kappa\lambda I_r \right) \sum_{j,k \neq i, j} \left[ (h_i - h_j)K_{ij} + (h_i - h_k)K_{ik} \right] K_{jk},
\]  

(4.7)

from which we obtain

\[
[I_2, h_i^3] = [I_2, h_i]h_i^2 + h_i[I_2, h_i]h_i + h_i^2[I_2, h_i]
= \frac{1}{4} \left( \kappa^2 + \lambda^2 - 2\kappa\lambda I_r \right) \sum_{j,k \neq i, j} \left[ (h_i^3 - h_j^3) K_{ij} + (h_i^3 - h_k^3) K_{ik} \right] K_{jk}.
\]  

(4.8)

It is then straightforward to prove that \(I_2\) also commutes with \(I_3\).

So we did obtain three mutually commuting conserved quantities \(I_n, n = 1, 2, 3\). Since the latter are invariant under \(D_6\), their projections in the subspaces characterized by \(K_{ij}\) and \(L_{ij}\) equal to +1 or −1 do still commute with one another. In those subspaces, \(I_1\) becomes proportional to the corresponding CMW Hamiltonian, while \(I_2\) and \(I_3\) become the integrals of motion of the latter.
5 Conclusion

In the present letter, we did show that the exchange operator formalism previously introduced for the Calogero problem can be extended to the three-body CMW one, thus providing us with an easy proof of complete quantum integrability for the latter. In the analysis of the problem, we were led to consider a $D_6$-extended Heisenberg algebra whenever two and three-body interactions are both present, and a $D_3$-extended one whenever only the latter is taken into account.

Whether a similar approach can be used for some other Hamiltonians containing three-body interactions remains an open question, to which we hope to come back in a near future.

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