Summation of Series Defined by Counting Blocks of Digits

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Abstract
We discuss the summation of certain series defined by counting blocks of digits in the $B$-ary expansion of an integer. For example, if $s_2(n)$ denotes the sum of the base-2 digits of $n$, we show that \[ \sum_{n \geq 1} \frac{s_2(n)}{2n(2n+1)} = \frac{\gamma + \log \frac{4}{\pi}}{2}. \] We recover this previous result of Sondow and provide several generalizations.

MSC: 11A63, 11Y60.

1 Introduction

A classical series with rational terms, known as Vacca’s series [17] or in an equivalent integral form as Catalan’s integral [1] (see also [6] and [16]), evaluates to Euler’s constant $\gamma$:

\[ \gamma = \sum_{n \geq 1} \frac{(-1)^n}{n} \left\lfloor \frac{\log n}{\log 2} \right\rfloor = \int_0^1 \frac{1}{1+x} \sum_{n \geq 1} x^{2^n-1} dx. \]

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In a recent paper [15] Sondow gave the following two formulas:

\[
\gamma^\pm = \sum_{n \geq 1} \frac{N_1(n) \pm N_0(n)}{2n(2n+1)}
\]

where \(\gamma^+ = \gamma\) is the Euler constant, \(\gamma^- = \log \frac{4}{\pi}\) is the “alternating Euler constant” [14], and \(N_1(n)\) (resp. \(N_0(n)\)) is the number of 1’s (resp. 0’s) in the binary expansion of the integer \(n\). The series for \(\gamma^+ = \gamma\) is equivalent to Vacca’s. The formulas for \(\gamma^\pm\) show in particular that

\[
\sum_{n \geq 1} \frac{s_2(n)}{2n(2n+1)} = \frac{\gamma + \log \frac{4}{\pi}}{2}
\]

where \(s_2(n)\) is the sum of the binary digits of the integer \(n\).

This last formula reminds us of one of the problems posed at the 1981 Putnam competition [9]: Determine whether or not

\[
\exp \left( \sum_{n \geq 1} \frac{s_2(n)}{n(n+1)} \right)
\]

is a rational number. In fact, \(\sum \frac{s_2(n)}{n(n+1)} = 2 \log 2\). A generalization was proven by Shallit [13], where the base 2 is replaced by any integer base \(B \geq 2\). A more general result, where the sum of digits is replaced by the function \(N_{w,B}(n)\), which counts the number of occurrences of the block \(w\) in the \(B\)-ary expansion of the integer \(n\), was given by Allouche and Shallit [2].

The purpose of the present paper is to show that the result of [15] cited above can be deduced from a general lemma in [2]. Furthermore, we sum the series

\[
\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)} \quad \text{and} \quad \sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)}
\]

thus generalizing Corollary 1 in [15] and a series for Euler’s constant in [5, 1, 11] (dated February 1967, August 1967, February 1968), respectively. Finally, we indicate some generalizations of our results, including an extension to base \(B > 2\), and a method for giving alternate proofs without using the general lemma from [2].

2 A general lemma

The first lemma in this section is taken from [2]; for completeness we recall the proof. We also give two classical results presented as lemmas, together with a new result (Lemma 4).

We start with some definitions. Let \(B \geq 2\) be an integer. Let \(w\) be a word on the alphabet of digits \(\{0, 1, \cdots, B-1\}\) (that is, \(w\) is a finite block of digits). We denote by \(N_{w,B}(n)\) the number of (possibly overlapping) occurrences of \(w\) in the \(B\)-ary expansion of an integer \(n > 0\), and we set \(N_{w,B}(0) = 0\).
Given \( w \) as above, we denote by \(|w|\) the length of the word \( w \) (i.e., if \( w = d_1d_2\cdots d_k \), then \(|w| = k\)). Denote by \( w^j \) the concatenation of \( j \) copies of the word \( w \).

Given \( w \) and \( B \) as above, we denote by \( v_B(w) \) the value of \( w \) when \( w \) is interpreted as the base \( B \)-expansion (possibly with leading 0’s) of an integer.

**Remark 1** The occurrences of a given word in the \( B \)-ary expansion of the integer \( n \) may overlap. For example, \( N_{11,2}(7) = 2 \).

If the word \( w \) begins with 0, but \( v_B(w) \neq 0 \), then in computing \( N_{w,B}(n) \) we assume that the \( B \)-ary expansion of \( n \) starts with an arbitrarily long prefix of 0’s. If \( v_B(w) = 0 \) we use the usual \( B \)-ary expansion of \( n \) without leading zeros. For example, \( N_{011,2}(3) = 1 \) (write 3 in base 2 as 0\( \cdots \)011) and \( N_{0,2}(2) = 1 \).

**Lemma 1** (\cite{2}) Fix an integer \( B \geq 2 \), and let \( w \) be a non-empty word on the alphabet \( \{0, 1, \cdots, B-1\} \). If \( f : \mathbb{N} \to \mathbb{C} \) is a function with the property that \( \sum_{n \geq 1} |f(n)| \log n < \infty \), then

\[
\sum_{n \geq 1} N_{w,B}(n) \left( f(n) - \sum_{0 \leq j < B} f(Bn + j) \right) = \sum f(B^{\lfloor |w|/B \rfloor}n + v_B(w)),
\]

where the last summation is over \( n \geq 1 \) if \( w = 0^j \) for some \( j \geq 1 \), and over \( n \geq 0 \) otherwise.

**Proof.** (See \cite{2}.) As \( N_{w,B}(n) \leq \lfloor \frac{\log n}{\log B} \rfloor + 1 \), all series \( \sum N_{w,B}(un + v)f(un + v) \), where \( u \) and \( v \) are nonnegative integers, are absolutely convergent. Let \( \ell \) be the last digit of \( w \), and let \( g := B^{\lfloor |w|/B \rfloor - 1} \). Then

\[
\sum_{n \geq 0} N_{w,B}(n)f(Bn + \ell) = \sum_{0 \leq k < g} \sum_{n \geq 0} N_{w,B}(gn + k)f(Bgn + Bk + \ell)
\]

and

\[
\sum_{n \geq 0} N_{w,B}(Bn + \ell)f(Bn + \ell) = \sum_{0 \leq k < g} \sum_{n \geq 0} N_{w,B}(Bgn + Bk + \ell)f(Bgn + Bk + \ell).
\]

Now, if either \( n \neq 0 \) or \( v_B(w) \neq 0 \), then for \( k = 0, 1, \ldots, g-1 \) we have

\[
N_{w,B}(Bgn + Bk + \ell) - N_{w,B}(gn + k) = \begin{cases} 
1, & \text{if } k = \lfloor \frac{v_B(w)}{B} \rfloor; \\
0, & \text{otherwise}.
\end{cases}
\]

On the other hand, if \( n = 0 \) and \( v_B(w) = 0 \) (hence \( \ell = 0 \)), then the difference equals 0 for every \( k \in \{0, 1, \cdots, g-1\} \). Hence

\[
\sum_{n \geq 0} N_{w,B}(Bn + \ell)f(Bn + \ell) - \sum_{n \geq 0} N_{w,B}(n)f(Bn + \ell) = \sum f\left(Bgn + B\lfloor \frac{v_B(w)}{B} \rfloor + \ell\right)
\]

\[
= \sum f(B^{\lfloor |w|/B \rfloor}n + v_B(w)) \quad (*),
\]
the last two summations being over \(n \geq 0\) if \(w\) is not of the form \(0^j\), and over \(n \geq 1\) if \(w = 0^j\) for some \(j \geq 1\). We then write

\[
\sum_{n \geq 0} N_{w,B}(n) f(n) = \sum_{0 \leq j < B} \sum_{n \geq 0} N_{w,B}(Bn + j) f(Bn + j)
\]

\[
= \sum_{j \in [0,B) \setminus \{\ell\}} \sum_{n \geq 0} N_{w,B}(Bn + j) f(Bn + j) + \sum_{n \geq 0} N_{w,B}(Bn + \ell) f(Bn + \ell)
\]

which together with (*) gives

\[
\sum_{n \geq 0} N_{w,B}(n) \left( f(n) - \sum_{0 \leq j < B} f(Bn + j) \right) = \sum f(\lfloor w \rfloor n + v_B(w)).
\]

Since \(N_{w,B}(0) = 0\), the proof is complete. \(\square\)

Now let \(\Gamma\) be the usual gamma function, let \(\Psi := \Gamma'/\Gamma\) be the logarithmic derivative of the gamma function, let \(\zeta(s)\) be the Riemann zeta function, let \(\zeta(s, x) := \sum_{n \geq 0}(n + x)^{-s}\) be the Hurwitz zeta function, and let \(\gamma\) denote Euler’s constant.

**Lemma 2** If \(a\) and \(b\) are positive real numbers, then

\[
\sum_{n \geq 1} \left( \frac{1}{an} - \frac{1}{an + b} \right) = \frac{1}{b} + \frac{\gamma + \Psi(b/a)}{a}.
\]

**Proof.** We write

\[
\sum_{n \geq 1} \left( \frac{1}{an} - \frac{1}{an + b} \right) = \lim_{s \to 1^+} \sum_{n \geq 1} \left( \frac{1}{(an)^s} - \frac{1}{(an + b)^s} \right) = \frac{1}{a} \lim_{s \to 1^+} \sum_{n \geq 1} \left( \frac{1}{(n + \frac{b}{a})^s} \right)
\]

\[
= \frac{1}{b} + \frac{1}{a} \lim_{s \to 1^+} \left( \zeta(s) - \zeta\left( s, \frac{b}{a} \right) + \left( \frac{a}{b} \right)^s \right)
\]

\[
= \frac{1}{b} + \frac{1}{a} \left( \gamma + \frac{\Gamma'(b/a)}{\Gamma(b/a)} \right) = \frac{1}{b} + \frac{\gamma + \Psi(b/a)}{a}
\]

(see for example [18, p. 271]). \(\square\)

**Lemma 3** For \(x > 0\) we have

\[
\sum_{r \geq 1} \left( \frac{x}{r} - \log\left(1 + \frac{x}{r}\right) \right) = \log x + \gamma x + \log \Gamma(x).
\]

**Proof.** Take the logarithm of the Weierstraß product for \(1/\Gamma(x)\) (see, for example, [8, Section 1.1] or [18, Section 12.1]). \(\square\)
The next lemma in this section is the last step before proving our theorems.

**Lemma 4** Let $a$ and $b$ be positive real numbers. Then

\[
\sum_{n \geq 1} \left( \frac{1}{an} - \log \frac{an + 1}{an} \right) = \log \Gamma \left( \frac{1}{a} \right) + \frac{\gamma}{a} - \log a
\]

and

\[
\sum_{n \geq 0} \left( \frac{1}{an + b} - \log \frac{an + b + 1}{an + b} \right) = \log \Gamma \left( \frac{b + 1}{a} \right) - \log \Gamma \left( \frac{b}{a} \right) - \frac{\Psi(b/a)}{a}.
\]

**Proof.** The proof is straightforward. The first formula follows directly from Lemma 3. To prove the second, write the $n$th term of the series for $n \geq 1$ as the following sum of $n$th terms of three absolutely convergent series:

\[
\frac{1}{an + b} - \frac{1}{an} - \frac{b}{an} + \log \left( 1 + \frac{b}{an} \right) + \frac{b + 1}{an} - \log \left( 1 + \frac{b + 1}{an} \right);
\]

then use Lemmas 2 and 3. □

3 Two theorems

In this section we give two theorems that are consequences of Lemma 1, and that generalize results in [15] and [5, 1, 11].

**Theorem 1** Let $w$ be a non-empty word on the alphabet \{0, 1\}, and let $\Psi$ denote the logarithmic derivative of the Gamma function.

(a) If $v_2(w) = 0$, then

\[
\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n + 1)} = \log \Gamma \left( \frac{1}{2|w|} \right) + \frac{\gamma}{2|w|} - |w| \log 2.
\]

(b) If $v_2(w) \neq 0$, then

\[
\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n + 1)} = \log \Gamma \left( \frac{v_2(w) + 1}{2|w|} \right) - \log \Gamma \left( \frac{v_2(w)}{2|w|} \right) - \frac{1}{2|w|} \Psi \left( \frac{v_2(w)}{2|w|} \right).
\]

**Proof.** Let

\[
A_n := \frac{1}{n} - \log \frac{n + 1}{n}
\]

for $n \geq 1$. Noting that $A_n - A_{2n} - A_{2n+1} = \frac{1}{2n(2n+1)}$, the theorem follows from Lemma 1 with $B = 2$, and $f(n) := A_n$ for $n \geq 1$, together with Lemma 4. □
Example 1 Taking $w = 0$ and $w = 1$, and recalling that $\Gamma(1/2) = \sqrt{\pi}$ and $\Psi(1/2) = -\gamma - 2 \log 2$ by Gauß’s theorem (see for example [8] p. 19 or [10] p. 94), we get
\[
\sum_{n\geq 1} \frac{N_{0,2}(n)}{2n(2n+1)} = \frac{1}{2} \log \pi + \frac{\gamma}{2} - \log 2
\]
and
\[
\sum_{n\geq 1} \frac{s_2(n)}{2n(2n+1)} = \sum_{n\geq 1} \frac{N_{1,2}(n)}{2n(2n+1)} = -\frac{1}{2} \log \pi + \frac{\gamma}{2} + \log 2.
\]
These equalities imply the formulas in the Introduction:
\[
\sum_{n\geq 1} \frac{N_{1,2}(n) \pm N_{0,2}(n)}{2n(2n+1)} = \gamma^\pm
\]
where (following the notations of [15]) $\gamma^+ := \gamma$ and $\gamma^- := \log \frac{4}{\pi}$, which is Corollary 1 of [15].

Remark 2 The formulas in Theorem 1 are analogous to those in [2, p. 25]. The analogy becomes more striking if one uses Gauß’s theorem to write all expressions of the form $\Psi(\frac{x}{2})$, with $x$ a rational number in $(0, 1]$ using only trigonometric functions, logarithms, and Euler’s constant.

Theorem 2 Let $w$ be a non-empty word on the alphabet $\{0, 1\}$.

(a) If $v_2(w) = 0$, then
\[
\sum_{n\geq 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)} = \log \Gamma \left( \frac{1}{2|w|} \right) + \frac{\gamma}{2|w|+1} - |w| \log 2 - \frac{1}{2|w|+1} \Psi \left( \frac{1}{2|w|} \right) - \frac{1}{2}.
\]

(b) If $v_2(w) \neq 0$, then
\[
\sum_{n\geq 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)} = \log \Gamma \left( \frac{v_2(w) + 1}{2|w|} \right) - \log \Gamma \left( \frac{v_2(w)}{2|w|} \right) - \frac{1}{2|w|+1} \left( \Psi \left( \frac{v_2(w)}{2|w|} \right) + \Psi \left( \frac{v_2(w) + 1}{2|w|} \right) \right).
\]

Proof. Noting that $\frac{1}{2n(2n+1)} - \frac{1}{4} \cdot \frac{1}{n(n+1)} = \frac{1}{2n(2n+1)(2n+2)}$, it suffices to use Theorem 1 and the following result, deduced from [2] top of p. 26 in the case $B = 2$.

(a) If $v_2(w) = 0$, then
\[
\sum_{n\geq 1} \frac{N_{w,2}(n)}{n(n+1)} = \frac{1}{2|w|-1} \left( \Psi \left( \frac{1}{2|w|} \right) + \gamma + 2|w| \right).
\]

(b) If $v_2(w) \neq 0$, then
\[
\sum_{n\geq 1} \frac{N_{w,2}(n)}{n(n+1)} = \frac{1}{2|w|-1} \left( \Psi \left( \frac{v_2(w) + 1}{2|w|} \right) - \Psi \left( \frac{v_2(w)}{2|w|} \right) \right).
\]

\[\square\]
Example 2 Taking \( w = 0 \) and \( w = 1 \), we get

\[
\sum_{n\geq 1} \frac{N_{0.2}(n)}{2n(2n+1)(2n+2)} = \frac{1}{2} \log \pi + \frac{\gamma}{2} - \frac{1}{2} \log 2 - \frac{1}{2}
\]

and

\[
\sum_{n\geq 1} \frac{s_2(n)}{2n(2n+1)(2n+2)} = \sum_{n\geq 1} \frac{N_{1.2}(n)}{2n(2n+1)(2n+2)} = -\frac{1}{2} \log \pi + \frac{\gamma}{2} + \frac{1}{2} \log 2.
\]

Hence

\[
\sum_{n\geq 1} \frac{N_{1.2}(n) \pm N_{0.2}(n)}{2n(2n+1)(2n+2)} = \delta^\pm
\]

where \( \delta^+ := \gamma - \frac{1}{2} \) and \( \delta^- := \frac{1}{2} - \log \frac{\pi}{2} \), which are respectively a formula given in \([5, 1, 11]\) and a seemingly new companion formula.

Remark 3 As mentioned, all expressions of the form \( \Psi(x) \), with \( x \) a rational number in \((0, 1]\), can be written using only trigonometric functions, logarithms, and Euler’s constant.

4 Generalizations

Several extensions or generalizations of our results are possible. We give some of them in this section.

4.1 Variation on \( A_n \)

Instead of applying Lemma 1 with \( f(n) = A_n = \frac{1}{n} - \log \frac{n+1}{n} \) for \( n \geq 1 \), we could replace \( A_n \) with

\[
A_n^{(k)} := \frac{1}{n+k} - \log \frac{n+1}{n}
\]

for \( n \geq 1 \), where \( k \) is a nonnegative integer. Defining the (rational) function \( Q^{(k)}(n) \) by

\[
Q^{(k)}(n) := A_n^{(k)} - A_{2n}^{(k)} - A_{2n+1}^{(k)}
\]

and noting that summing \( \sum_{n\geq 1} A_{an+b}^{(k)} \) boils down to summing \( \sum_{n\geq 1} \left( \frac{1}{an+b+k} - \frac{1}{an+b} \right) \), which as in the proof of Lemma 2 involves the Hurwitz zeta function, we obtain explicit formulas for the sum of the series \( \sum_{n\geq 1} N_{w.2}(n)Q^{(k)}(n) \).

4.2 Extension to base \( B > 2 \)

Lemma 1 has been used above only for base \( B = 2 \). There are applications to other bases in \([2]\). We also note that the relation among the \( A_n \)’s,

\[
A_n = \frac{1}{2n(2n+1)} + A_{2n} + A_{2n+1}
\]
for \( n \geq 1 \), can be generalized to base \( B \). Namely,

\[
A_n = Q(n, B) + R(n, B)
\]

where

\[
Q(n, B) := \frac{1}{Bn(Bn + 1)} + \frac{2}{Bn(Bn + 2)} + \cdots + \frac{B - 1}{Bn(Bn + B - 1)}
\]

and

\[
R(n, B) := A_{Bn} + A_{Bn+1} + \cdots + A_{Bn+B-1}.
\]

This allows us to use Lemmas \( \square \) and \( \square \) to sum, for example, the series

\[
\sum_{n \geq 1} N_{w,3}(n) \frac{9n + 4}{3n(3n + 1)(3n + 2)},
\]

since

\[
Q(n, 3) = A_n - A_{3n} - A_{3n+1} - A_{3n+2} = \frac{9n + 4}{3n(3n + 1)(3n + 2)}.
\]

### 4.3 Weighted \( A_n \)'s

In this section we consider a weighted form of the \( A_n \)'s. First we need to study a relation between sequences of real numbers.

**Lemma 5** Let \( (r_n)_{n \geq 1} \) and \( (R_i)_{i \geq 1} \) be sequences of real numbers. Set \( r_0 := 0 \) and \( R_0 := 0 \). Then the following two properties are equivalent:

1. for \( i \geq 1 \)
   \[
   R_i = \sum_{k \geq 0} r_{\lfloor \frac{i}{2^k} \rfloor} = r_i + r_{\lfloor \frac{i}{2} \rfloor} + r_{\lfloor \frac{i}{4} \rfloor} + \cdots 
   \]
   (note that this is actually a finite sum);

2. for \( n \geq 1 \)
   \[
   r_n = R_n - R_{\lfloor \frac{n}{2} \rfloor}.
   \]

**Proof.** The implication (1) \( \Rightarrow \) (2) is easily seen by considering the cases \( n \) even and \( n \) odd. Likewise, for (2) \( \Rightarrow \) (1) take \( i \) even and \( i \) odd. \( \square \)

**Remark 4** See \([\square] \) Theorem 9\] for more about this relation.

**Theorem 3** Assume that \( r_1, r_2, \ldots \) and \( R_1, R_2, \ldots \) are real numbers related as in Lemma 5. Then the series \( \sum |r_n|n^{-2} \) converges if and only if the series \( \sum |R_i|i^{-2} \) converges, and in this case we have

\[
S := \sum_{n \geq 1} r_n \left( \frac{1}{n} - \log \frac{n + 1}{n} \right) = \sum_{i \geq 1} \frac{R_i}{2i(2i + 1)}
\]
Proof. First note that if the series \( \sum |R_i| i^{-2} \) converges, then so does the series \( \sum |r_n| n^{-2} \); use the expression for \( r_n \) in terms of the \( R_i \)'s in Lemma 5. Now suppose that the series \( \sum |r_n| n^{-2} \) converges. As before, let \( A_n := \frac{1}{n} - \log \frac{n+1}{n} \). Then \( 0 < A_n < \frac{1}{n} - \frac{1}{n+1} \). This implies that the series \( S := \sum r_n \left( \frac{1}{n} - \log \frac{n+1}{n} \right) \) is absolutely convergent. Now

\[
A_n = \frac{1}{2n(2n+1)} + A_{2n} + A_{2n+1}
\]

implies

\[
A_n = \frac{1}{2n(2n+1)} + \frac{1}{4n(4n+1)} + \frac{1}{(4n+2)(4n+3)} + A_{4n} + A_{4n+1} + A_{4n+2} + A_{4n+3}.
\]

Hence, repeating \( K \) times,

\[
A_n = \sum_{1 \leq k \leq K} \sum_{0 \leq m < 2^{k-1}} \frac{1}{(2^kn + 2m)(2^kn + 2m + 1)} + \sum_{0 \leq q < 2K} A_{2^K + q}.
\]

Using the bounds \( 0 < A_n < \frac{1}{n} - \frac{1}{n+1} \) and telescoping, the last sum is less than \( 2^{-K} \). Letting \( K \) tend to infinity, we obtain the (rapidly convergent) series

\[
A_n = \sum_{k \geq 1} \sum_{0 \leq m < 2^k-1} \frac{1}{(2^kn + 2m)(2^kn + 2m + 1)}.
\]

Substituting into the sum defining \( S \) yields the double series

\[
S = \sum_{n \geq 1} \sum_{k \geq 1} \sum_{0 \leq m < 2^{k-1}} \frac{r_n}{(2^kn + 2m)(2^kn + 2m + 1)},
\]

which converges absolutely. Thus we may collect terms with the same denominator, and we arrive at the series

\[
S = \sum_{i \geq 1} \frac{R_i'}{2i(2i+1)},
\]

where

\[
R_i' := \sum_{n \in \mathcal{E}_i} r_n
\]

with \( \mathcal{E}_i := \{ n \geq 1, \exists k \geq 1, \exists m \in [0, 2^{k-1}), 2^{k-1}n + m = i \} \). On the one hand, this proves that the series \( \sum \frac{R_i'}{2i(2i+1)} \) is absolutely convergent (hence the series \( \sum |R_i'| i^{-2} \) is convergent). On the other hand, \( R_i' \) can also be written as

\[
R_i' := \sum_{1 \leq k \leq \frac{i-1}{2^{k-1}}} \sum_{\frac{i-1}{2^{k-1}} \leq m < \frac{i-1}{2^{k-1}}+1} r_n = \sum_{k \geq 0} R_{\lfloor \frac{i}{2^k} \rfloor}.
\]

(recall that we have set \( r_0 := 0 \)). Thus \( R_i' = R_i \) by the hypothesis, and the proof is complete. \( \square \)
Example 3  Theorem 3 yields in particular the series for $\gamma$ and $\log \frac{4}{\pi}$ in the Introduction, in Example 1 and in [15, Corollary 1]. Namely, If $r_1 = r_2 = \cdots = 1$, then the series defining $S$ sums to $\gamma$ from Lemma 4 and the formula defining $R'_i = R_i$ reduces to $R_i = \lfloor \frac{\log 2i}{\log 2} \rfloor = N_{1,2}(i) + N_{0,2}(i)$.

If $r_n = (-1)^{n-1}$, then $S = \log \frac{4}{\pi}$ (see [14] or decompose $S$ into $\sum \text{ (odd terms)} - \sum \text{ (even terms)}$ and apply Lemma 4), and the formula defining $R'_i = R_i$ implies $R_i = N_{1,2}(i) - N_{0,2}(i)$. To see this equality, first note that if it holds for $i \geq 1$, then using Lemma 5 and looking at the cases $n$ odd and $n$ even,

$$r_n = R_n - R_{\lfloor \frac{n}{2} \rfloor} = (-1)^{n-1}$$

for $n \geq 1$ (compare [15, Lemma 2]). Now recall that properties (1) and (2) in Lemma 5 are equivalent.

Remark 5  Example 3 shows that it is possible to deduce the formula

$$\sum_{n \geq 1} \frac{N_{1,2}(n) - N_{0,2}(n)}{2n(2n+1)} = \log \frac{4}{\pi}$$

from Theorem 3 and Lemma 4 without using Lemma 4. This yields a proof of the formula that is different from those in [15] and Example 1. Similar reasoning applies for any ultimately periodic sequence $(r_n)_{n \geq 1}$. In particular, it is not hard to see that the relations giving $r_{2n}$ and $r_{2n+1}$ in terms of the $R_i$'s imply that the sequence $(r_n)_{n \geq 1}$ is periodic whenever $R_i = N_{w,2}(i)$ for some fixed $w$ and for every $i \geq 1$. Hence Theorem 1 can be deduced from Theorem 3 and Lemma 4 (along with the method for decomposing series employed in Example 3), without using Lemma 4. In the same vein, the generalization in Section 4.2 can be proved using a generalization of Theorem 3 to base $B$, together with Lemma 4.

5  Future directions

Lemma 1 is the main tool for summing series in [2] and in the present paper. It might be possible to use the lemma to obtain the base $B$ accelerated series for Euler's constant in [15, Theorem 2], and to sum more general series with $N_{w,B}(n)$. On the other hand, it might also be possible to extend the results of [2] and the present paper, and sum series where $(N_{w,B}(n))_{n \geq 1}$ is replaced by a more general integer sequence $(a_n)_{n \geq 1}$, using the decomposition in [12] of a sequence $(a_n)_{n \geq 1}$ into a (possibly infinite) linear combination of block-counting sequences $(N_{w,B}(n))_{n \geq 1}$ (see also [3]). Of course, since this may replace a series with an infinite sum, for the method to work the new series must be summable in closed form.

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