On the Turán density of \{1, 3\}-Hypergraphs

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Abstract

In this paper, we consider the Turán problems on \{1, 3\}-hypergraphs. We prove that a \{1, 3\}-hypergraph is degenerate if and only if it’s \(H^{1,3}_5\)-colorable, where \(H^{1,3}_5\) is a hypergraph with vertex set \(V = [5]\) and edge set \(E = \{(2), (3), (1, 2, 4), (1, 3, 5), (1, 4, 5)\}\). Using this result, we further prove that for any finite set \(R\) of distinct positive integers, except the case \(R = \{1, 2\}\), there always exist non-trivial degenerate \(R\)-graphs.

We also compute the Turán densities of some small \{1, 3\}-hypergraphs.

Keywords: Turán density, non-uniform hypergraph, degenerate \(R\)-graph.

1 Background

Turán theory is an important and active area in the extremal combinatorics. In 1941, Turán [10] determined the graph with maximum number of edges among all simple graphs on \(n\) vertices that doesn’t contain the complete graph \(K_\ell\) as a sub-graph. For any \(\epsilon > 0\), the Turán density \(\pi(H)\) of a graph \(H\) is the least number \(\alpha\) so that any large graph with edge density \((\alpha + \epsilon)\) will always contain a sub-graph isomorphic to \(H\). Erdős-Simonovits-Stone theorem [1, 2] determined the Turán densities of all non-bipartite graphs. (The Turán density of any bipartite graph is always 0. Those are called the degenerate graphs.)

Turán problems on uniform hypergraphs have been actively studied for many decades. However, on non-uniform hypergraphs, these problems are rarely considered. Johnston and Lu [4] established the framework of the Turán theory for non-uniform hypergraphs. A hypergraph \(H = (V, E)\) consists of a vertex set \(V\) and an edge set \(E \subseteq 2^V\). Here the edges of \(E\) could have different cardinalities. The set of all the cardinalities of edges in \(H\) is denoted by \(R(H)\), the set of edge types. In this paper, we will fix a finite set \(R\) of positive integers and consider all simple hypergraphs \(H\) with \(R(H) \subseteq R\), which are called \(R\)-hypergraphs (or \(R\)-graphs, for short). We say a hypergraph is simple if there is at most one edge

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we have the Turán densities of the arguments of Katona-Nemetz-Simonovits theorem [6]. They completely classified the density of Lu and Johnston [4]. Given a family of hypergraphs \( H \), we denote \( H^R_n \) as the complete hypergraph on \( n \) vertices with edge set \( \bigcup_{i \in R} \binom{[n]}{i} \). We say \( H' \) is a sub-graph of \( H \), denoted by \( H' \subseteq H \), if there exists a 1-1 map \( f : V(H') \to V(H) \) so that \( f(e) \in E(H) \) for any \( e \in E(H') \). A necessary condition for \( H' \subseteq H \) is \( R(H') \subseteq R(H) \). A chain \( C^R \) is a special \( R \)-graph containing exactly one edge of each size such that any pair of these edges are comparable under inclusion relation.

To measure the edge density of a non-uniform hypergraph, we use the Lubell function, which is the expected number of edges in the hypergraph hit by a random full chain [4]. For a non-uniform hypergraph \( G \) on \( n \) vertices, the Lubell function of \( G \) is defined by

\[
h_n(G) := \sum_{e \in E(G)} \frac{1}{\binom{n}{|e|}} = \sum_{r \in R(G)} \frac{|E(G^r)|}{\binom{n}{r}}.
\]

Given a family of hypergraphs \( \mathcal{H} \) with common set of edge types \( R \), we say \( G \) is \( \mathcal{H} \)-free if \( G \) doesn’t contain any member of \( \mathcal{H} \) as a sub-graph. Let \( \pi_n(\mathcal{H}) \) be the maximum edge density of any \( \mathcal{H} \)-free \( R \)-graph on \( n \) vertices. The Turán density of \( \mathcal{H} \) is defined to be:

\[
\pi(\mathcal{H}) = \lim_{n \to \infty} \pi_n(\mathcal{H}) = \lim_{n \to \infty} \max \{ h_n(G) : |v(G)| = n, G \subseteq K^R_n, \text{and } G \text{ is } \mathcal{H} \text{-free} \}.
\]

A hypergraph \( G := G^R_n \) is extremal with respect to the family \( \mathcal{H} \) if \( G \) is \( \mathcal{H} \)-free and \( h_n(G) \) is maximized.

Lu and Johnston [4] proved that this limit always exists by a simple average argument of Katona-Nemetz-Simonovits theorem [6]. They completely classified the Turán densities of \( \{1, 2\} \)-graphs.

**Theorem 1** (Lu and Johnston [4]). For any hypergraph \( H \) with \( R(H) = \{1, 2\} \), we have

\[
\pi(H) = \begin{cases} 
2 - \frac{1}{\chi(H^2)} & \text{if } H^2 \text{ is not bipartite;} \\
\frac{4}{9} & \text{if } H^2 \text{ is bipartite and } \min\{k : \overline{P}_{2k} \subseteq H\} = 1; \\
\frac{4}{9} & \text{if } H^2 \text{ is bipartite and } \min\{k : \overline{P}_{2k} \subseteq H\} \geq 2; \\
1 & \text{if } H^2 \text{ is bipartite and } \overline{P}_{2k} \not\subseteq H \text{ for any } k \geq 1.
\end{cases}
\]

where \( H^2 \in H \) is the graph with all edges of cardinality 2, \( \overline{P}_{2k} \) is a closed path of length 2k, and \( \chi(H^2) \) is the chromatic number of \( H^2 \).

It is trivial that \( \pi(H) \leq |R(H)| \) and it is easy to see that \( \pi(H) \geq |R(H)| - 1 \), since we can take an \((|R(H)| - 1)\)-complete hypergraph \( K^{|R(H)|-1}_n \) without the
appearance of \( H \). We are interested in these \( R \)-graphs with the smallest Turán density.

**Definition 1** (Degenerate hypergraphs). A hypergraph \( H \) is called degenerate if \( \pi(H) = |R(H)| - 1 \).

What do the degenerate \( R \)-graphs look like? For the special case \( R = \{r\} \), Erdős [3] showed that an \( r \)-uniform hypergraph \( H \) is degenerate if and only if it is \( r \)-partite, that is, a sub-graph of a blow-up of a single edge of cardinality \( r \). As a natural extension of a single edge, the chain \( C_R \) for any set \( R \) is degenerate. Thus every sub-graph of a blow-up of a chain is also degenerate. We say a degenerate \( R \)-graph is trivial if it is a sub-graph of a blow-up of the chain \( C_R \). For \( R = \{1, 2\} \), by Theorem 1 all degenerate \( \{1, 2\} \)-graphs are trivial. However, a nontrivial degenerate \( \{2, 3\} \)-graph is found in [4]. It indicates that this question is more intrigue for other \( R \)-graphs.

In this paper, we will give a necessary and sufficient condition for the degenerate \( \{1, 3\} \)-graphs. Given two graphs \( G \) and \( H \), we say \( G \) is \( H \)-colorable if and only if there exists a hypergraph homomorphism \( f \) from \( G \) to \( H \). (see Definition 3). And we have

**Theorem 2.** A \( \{1, 3\} \)-hypergraph is degenerate if and only if it’s \( H_{\{1, 3\}}^{\{1,3\}} \)-colorable, where \( H_{\{1,3\}}^{\{1,3\}} \) is a hypergraph with vertex set \( V = \{5\} \) and edge set

\[
E = \{\{2\}, \{3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}.
\]

Using this non-trivial degenerate \( \{1, 3\} \)-graph, we prove the following result.

**Theorem 3.** Let \( R \) be a set of distinct positive integers with \( |R| \geq 2 \) and \( R \neq \{1, 2\} \). Then a non-trivial degenerate \( R \)-graph always exists.

We then continue to study the non-degenerate \( \{1, 3\} \)-graphs. Let \( K_{3\bullet\bullet}^* \) be a \( \{1, 3\} \)-graph with edges \( \{\{1\}, \{2\}, \{1, 2, 3\}\} \), we have

**Theorem 4.** For any \( K_{3\bullet\bullet}^* \)-colorable \( \{1, 3\} \)-graph \( H \), we have

1. if \( K_{3\bullet\bullet}^* \not\subseteq H \), \( H \) must be \( H_{\{1,3\}}^{\{1,3\}} \)-colorable, then \( \pi(H) = 1 \);
2. if \( K_{3\bullet\bullet}^* \subseteq H \), then \( \pi(H) = \pi(K_{3\bullet\bullet}^*) = 1 + \frac{\sqrt{3}}{3} \).

A result following Theorem 4 indicates a break for the Turán density of \( \{1, 3\} \)-graphs:

**Corollary 1.** Let \( \alpha \) be a real value in \( [1, \frac{4}{3} + \frac{\sqrt{3}}{3}] \). For any \( \{1, 3\} \)-graph \( H \) with \( \pi(H) \leq \alpha \), it must be the case that \( \pi(H) = 1 \).

We also obtain the Turán densities of some 3-partite \( \{1, 3\} \)-graphs, the results are shown in Section 4.

The paper is organized as follows. In section 2 we introduce some notations and lemmas for non-uniform hypergraphs. In section 3 we will prove the Theorem 2, Theorem 4 and Corollary 1. In section 4, we determine the Turán densities of some 3-partite \( \{1, 3\} \)-graphs. In section 5, we prove Theorem 3.
2 Notation and lemmas

In this section, we introduce some notations and lemmas for \( R \)-graphs and then for the \{1,3\}-graphs. We call an edge of cardinality \( i \) as an \( i \)-edge, for each \( i \in R \). For convenience, we call a vertex that forms a 1-edge as “black vertex”, otherwise, “white vertex”. We use notations of form \( H^*_n \) to represent a hypergraph on \( n \) vertices that contains only one “black vertex”, similarly, \( H^*_n \) represents a hypergraph on \( n \) vertices that contains two “black vertices”, and so on. To simplify our notations for \{1,3\}-graphs, we use form of \( abc \) to denote the edge \{a,b,c\}.

For a fixed set \( R = \{ k_1, k_2, \ldots, k_r \} \), with \( (k_1 < k_2 < \ldots < k_r) \), \( R \)-flag is an \( R \)-graph containing exactly one edge of each size. The chain \( C^R \) is the special \( R \)-flag with the edge set \( E(C^R) = \{ [k_1], [k_2], \ldots, [k_r] \} \), where \( [k_i] \) is the set of all positive integers from 1 to \( k_i \) for each \( i \in [r] \). For \( R = \{ 1,3 \} \), the chain \( C^{(1,3)} = \{ 1,123 \} \). For any \( R \)-flag \( L \), we have \( \pi(L) = |R| - 1 \) (see [4]). Thus the chain \( C^{(1,3)} \) is a degenerate \{1,3\}-graph.

The following definitions and lemmas on non-uniform hypergraphs are generalized from uniform hypergraphs.

**Definition 2 (Blow-up hypergraphs).**[4] For any hypergraph \( H \) on \( n \) vertices and positive integers \( s_1, s_2, \ldots, s_n \), the blow-up of \( H \) is a new hypergraph \( (V, E) \), denoted by \( H_n(s_1, s_2, \ldots, s_n) \), satisfying

- \( V := \bigsqcup_{i=1}^n V_i \), where \( |V_i| = s_i \),
- \( E := \bigcup_{F \in E(H)} \prod_{i \in F} V_i \).

When \( s_1 = s_2 = \ldots = s_n = s \), we simply write it as \( H(s) \).

The blow-up operation does not change the Turán density.

**Theorem 5 (Blow-up Families).**[4] Let \( \mathcal{H} \) be a finite family of hypergraphs and let \( s \geq 2 \). Then \( \pi(\mathcal{H}(s)) = \pi(\mathcal{H}) \).

A direct corollary of Theorem 5 is the following result.

**Theorem 6 (Squeeze Theorem).**[4] Let \( H \) be any hypergraph. If there exists a hypergraph \( H' \) and an integer \( s \geq 2 \) such that \( H' \subseteq H \subseteq H'(s) \), then \( \pi(H) = \pi(H') \).

It is easy to generalize the concepts of homomorphisms and \( H \)-coloring to general \( R \)-graphs.

**Definition 3.** Given two \( R \)-graphs \( G \) and \( H \), a hypergraph homomorphism is a vertex map \( f : V(G) \to V(H) \) such that, if \( \{v_1, \ldots, v_r\} \in E(G) \) then \( \{f(v_1), \ldots, f(v_r)\} \in E(H) \), for all \( r \in R \).

**Definition 4.** A hypergraph \( G \) is called \( H \)-colorable if and only if there exists a homomorphism from \( G \) to \( H \).

Note that, if there exists a homomorphism from \( G \) to \( H \), then \( G \) is isomorphic to a sub-graph of a blow-up of \( H \). Thus we have:

**Lemma 1.** If \( G \) is \( H \)-colorable, then \( \pi(G) \leq \pi(H) \).
2.1 $R$-graphs with loops, blow-up, and Lagrangian

A loop edge is a multiset of vertices. Sometimes we need to enlarge the concept of $R$-graphs to $R$-graphs with loops. For example, consider a $\{1, 3\}$-graph $H_1$ with the edge set $\{x, xyy, yyy\}$. Here $xyy$ is a loop edge with vertex $x$ occurring once and vertex $y$ twice. In general, a loop edge $e = x_1^{m_1} \cdots x_i^{m_i}$ consists of $m_1$ copies of vertex $x_1$, $m_2$ copies of vertex $x_2$, and so on. For a loop edge $e = x_1^{m_1} \cdots x_l^{m_l}$, the cardinality of $e$ is $|e| = \sum_i m_i$. We also define a multinomial coefficient $c_e$ to be

$$c_e := \binom{|e|}{m_1, m_2, \ldots, m_l} = \frac{|e|!}{m_1!m_2! \cdots m_l!}.$$  

**Definition 5.** The polynomial form of an $R$-graph $H$ with loops on $n$ vertices, denoted by $\lambda(H, \vec{x})$ with $\vec{x} = (x_1, x_2, \ldots, x_n)$ is defined as

$$\lambda(H, \vec{x}) := \sum_{e \in E(H)} c_e \prod_{i \in e} x_i.$$  

The Lagrangian of $H$, denoted by $\lambda(H)$, is the maximum value of the polynomial $\lambda(H, \vec{x})$ over the simplex $S_n = \{(x_1, x_2, \ldots, x_n) \in [0, 1]^n : \sum_{i=1}^n x_i = 1\}$.

For any $R$-graph $H$ (with possible loops), one can construct the family of $H$-colorable $R$-graph by blowing up $H$ in a certain way. The Lagrangian of $H$ is the maximum edge density of the $H$-colorable $R$-graphs that one can get in this way. This definition of Lagrangian is the same as the one in [5]; but differs from the classical Lagrangian for $r$-uniform hypergraphs such as in [8] by a constant multiplicative factor. This is not essential. This is a special case of more general Lagrangian of non-uniform hypergraphs introduced by Peng-Wu-Yao [9].

**Construction A:** Consider a $\{1, 3\}$-graph (with loops) $H_A$ on two vertices $\{x, y\}$ with edges $\{x, xyy, yyy\}$. The polynomial form of $H_A$ is

$$\lambda(H_A, \vec{x}) = x_1 + 3x_1x_2^2 + x_2^3.$$  

It can be shown that $\lambda(H, \vec{x})$ reaches the maximum $1 + \frac{\sqrt{3}}{18}$ over the simplex $S_2 = \{(x_1, x_2) \in [0, 1]^2, x_1 + x_2 = 1\}$ at $x_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Thus, we have

$$\lambda(H_A) = 1 + \frac{\sqrt{3}}{18} \approx 1.096225.$$  

A $\{1, 3\}$-graph $G_A$ on $n$ vertices is generated by blowing-up $H_A$ as follows: set a vertex partition $V(G_A) = X \cup Y$ with $|X| \approx (\frac{1}{2} - \frac{\sqrt{3}}{6})n$ such that all 1-edges are in $X$ (drawn by a black point), and all 3-edges are either formed by three vertices in $Y$ or by one vertex in $X$ plus two vertices in $Y$. In another words,

$$E(G_A) = \binom{X}{1} \cup \binom{X}{1} \times \binom{Y}{2} \cup \binom{Y}{3}.$$
We have
\[
h_n(G_A) = \frac{|X|}{n} + \frac{|X|(|Y|)}{2} + \binom{|Y|}{3} \left( \frac{n}{3} \right)
\]
\[= \lambda(H_A, \overline{x}) + O\left( \frac{1}{n} \right).
\]
\[= \lambda(H_A) + O\left( \frac{1}{n} \right).
\]
Here \( \overline{x} = \left( \frac{|X|}{n}, \frac{|Y|}{n} \right) = (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6}) \).

\[G_A: \quad \text{max } h_n(G_A) = 1 + \frac{\sqrt{3}}{18} + o_n(1), \text{ reached at } |X| = (\frac{1}{2} - \frac{\sqrt{3}}{6})n.
\]

**Construction B:** Let \( H_B \) be a general \( \{1, 3\} \)-graph on three vertices \( \{a, b, c\} \) with the edge set \( \{a, b, abc\} \). We have
\[
\lambda(H_B, \overline{x}) = x_1 + x_2 + 6x_1x_2x_3.
\]
It is easy to check
\[
\lambda(H_B) = \frac{4}{9} + \frac{\sqrt{3}}{3} \approx 1.021794714,
\]
which is reached at \( x_1 = x_2 = \frac{1 + \sqrt{3}}{6}, \text{ and } x_3 = \frac{2 - \sqrt{3}}{3} \). A \( \{1, 3\} \)-graph \( G_B \) on \( n \) vertices is generated by blowing-up \( H_B \) as follows: set a vertex partition \( V(G_B) = A \cup B \cup C \). All 1-edges are in \( A \) and \( B \) (drawn by black points), all 3-edges are formed by exactly one vertex in each partition. We have
\[
E(G_B) = \left( \begin{array}{l} A \end{array} \right) \cup \left( \begin{array}{l} B \end{array} \right) \cup \left( \begin{array}{l} A \end{array} \right) \times \left( \begin{array}{l} B \end{array} \right) \times \left( \begin{array}{l} C \end{array} \right).
\]
Note that \( G_B \) is \( H_B \)-colorable. Thus \( h_n(G_B) = \lambda(H_B) + O\left( \frac{1}{n} \right) \).

\[G_B: \quad \text{max } h_n(G_B) = \frac{4}{9} + \frac{\sqrt{3}}{3} + o_n(1), \text{ reached at } |A| = |B| = \left( \frac{1 + \sqrt{3}}{6} \right) n.
\]
2.2 Product of two $R$-graphs

Let’s define the product of $R$-graphs (with loops):

**Definition 6.** For any two general $R$-graphs $H_1$ and $H_2$ with vertices set $V_1$ and $V_2$ respectively, we define the product of $H_1$ and $H_2$, which is denoted by $H_1 \times H_2 = (V, E)$, where

$$V = V_1 \times V_2, \ E = \bigcup_{r \in R} E(H_1^r) \times E(H_2^r),$$

the $E(H_i^r)$ denotes the set of all edges of cardinality $r$ in $H_i$ for $i = 1, 2$. Here $E(H_1^r) \times E(H_2^r)$ consists of all products of $e \times \sigma f$, where $\sigma = (\sigma(1), \ldots, \sigma(r))$ takes over all permutations of $[r]$. For example, given $e = \{v_1, \ldots, v_r\} \subseteq E(H_1)$, $f = \{u_1, \ldots, u_r\} \subseteq E(H_2)$, then $e \times \sigma f = \{(v_1, u_{\sigma(1)}), (v_2, u_{\sigma(2)}), \ldots, (v_r, u_{\sigma(r)})\}$ is an edge in $E(H_1^r) \times E(H_2^r)$.

**Example 1.** The product of two $\{1, 3\}$-graphs $H_A$ and $H_B$ is given below. Let $ax$ stand for $(a, x)$, similar for other labels, then the vertex set is $V(H_A \times H_B) = \{ax, ay, bx, by, cx, cy\}$ and the edge set is $E(H_A \times H_B) = \{(ax), \{bx\}, \{cy, bx, ay\}, \{cy, ay, by\}, \{cy, by, ax\}, \{cx, ay, by\}\}$.

![Diagram](image)

**Lemma 2.** For any two $R$-graphs $H_1$ and $H_2$, if hypergraph $H$ is $H_1$ and $H_2$-colorable, then it’s $(H_1 \times H_2)$-colorable.

**Proof.** By definition, there exist two graph homomorphisms $f_1 : V(H) \rightarrow V(H_1)$ and $f_2 : V(H) \rightarrow V(H_2)$. Note that $H$ could be an $R$-graph. Then for any $r \in R$, if edge $e = \{v_1, \ldots, v_r\} \subseteq E(H)$, we have

$$f_1(e) = \{f_1(v_1), \ldots, f_1(v_r)\} \subseteq E(H_1)$$

and

$$f_2(e) = \{f_2(v_1), \ldots, f_2(v_r)\} \subseteq E(H_2).$$

Define a map $f := f_1 \times f_2$ from $V(H)$ to $V(H_1) \times V(H_2)$, such that $f(v) = (f_1(v), f_2(v))$. Then we have

$$f(e) = \{(f_1(v_1), f_2(v_1)), \ldots, (f_1(v_r), f_2(v_r))\} \subseteq f_1(e) \times f_2(e) \subseteq E(H_1 \times H_2).$$

Thus the map $f$ takes edges in $H$ to edges in $H_1 \times H_2$, it is a graph homomorphism. Therefore, $H$ is $(H_1 \times H_2)$-colorable. \[\square\]
3 Proof of Theorem 2 and Theorem 4

3.1 Proof of Theorem 2

For \( \{1, 3\}\)-graph \( H \), we have \( 1 \leq \pi(H) \leq 2 \). Observe that the product of two general \( R \)-graphs could be an \( R \)-graph. This is very useful in determining the Turán density. A degenerate \( R \)-graph \( H \) must be \( G \)-colorable for any \( R \)-graph \( G \) with \( \lambda(G) > 1 \). By Lemma 2, it must be colorable by the product of these \( R \)-graphs. In this section, we will characterize the degenerate \( \{1, 3\}\)-graph. Let's consider \( H_5^{(1,3)} \), the 4-vertex \( \{1, 3\}\)-graph with edge set \{2, 3, 124, 135, 145\}.

We first prove the following lemma.

**Lemma 3.** Any degenerate \( \{1, 3\}\)-graph is \( H_5^{(1,3)} \)-colorable.

**Proof.** Observe that a degenerate \( \{1, 3\}\)-graph \( H \) must be contained in \( G_A \) and \( G_B \). Equivalently, \( H \) is both \( H_A \) and \( H_B \)-colorable, then it must be colorable by the product \( H_A \times H_B \). We define a map \( f : V(H_A \times H_B) \rightarrow [5] \) such that:

\[
\begin{align*}
f(cx) &= f(cy) = 1, \\
f(bx) &= 2, \\
f(ax) &= 3, \\
f(ay) &= 4, \\
f(by) &= 5.
\end{align*}
\]

Obviously, \( f \) is a graph homomorphism from \( H_A \times H_B \) to \( H_5^{(1,3)} \). The result follows. \( \square \)

Let \( K_{3}^{\bullet \bullet} \) be a \( \{1, 3\}\)-graph on 3 vertices with edges \{1, 2, 123\}, and \( G_4^* \) be a \( \{1, 3\}\)-graph on 4 vertices with edges \{1, 123, 134, 234\}.

**Remark 1.** \( K_{3}^{\bullet \bullet} \) is not contained in \( G_A \) whose edge density reaches \( 1 + \frac{\sqrt{3}}{18} \), and \( G_4^* \) is not contained in \( G_B \) whose edge density reaches \( \frac{4}{9} + \frac{\sqrt{3}}{3} \). Thus both \( K_{3}^{\bullet \bullet} \) and \( G_4^* \) are non-degenerate \( \{1, 3\}\)-graphs.

**Lemma 4.** \( \pi(\{K_{3}^{\bullet \bullet}, G_4^*\}) = 1 \).

**Proof.** For any positive integer \( n \), let \( G \) be a \( \{K_{3}^{\bullet \bullet}, G_4^*\}\)-free \( \{1, 3\}\)-graph on \( n \) vertices. Denote \( S \) as the set of all 1-edges of \( G \), i.e. \( S = \{v \in V(G) : \{v\} \in E(G)\} \), and let \( |S| = xn \) for some \( x \in (0, 1) \). Let \( \overline{S} \) be the complement of \( S \), i.e. \( \overline{S} = V(G) \setminus S \), then \( |\overline{S}| = (1 - x)n \).

Denote \( E(G^3) \) as the set of all 3-edges of \( G \). To forbidden \( K_{3}^{\bullet \bullet} \), there is at most one black vertex in any 3-edges of \( G \), thus we have

\[
E(G^3) \subseteq \binom{S}{1} \times \binom{\overline{S}}{2} \cup \binom{\overline{S}}{3}.
\]
We consider the 3-edges of $G$ in edge set $\binom{S}{1} \times \binom{S}{2}$. Define $y$ as the average edge density of such 3-edges in $G$. Thus

$$y = \frac{|E(G^3) \cap (S \times \binom{S}{2})|}{|S| \times \binom{|S|}{2}}.$$  

Note that there exists one vertex $s_0 \in S$ such that $|C(s_0)| \geq y \times \binom{|S|}{2}$, where $C(s_0)$ is the set of 3-edges that contain the black vertex $s_0$. For any vertex $u \in S$, define

$$W_u := \{v \in S | s_0 u v \in E(G)\}.$$  

We then have

$$\sum_{u \in S} |W_u| \leq |S| \times (|S| - 1)$$  

and

$$|C(s_0)| = \frac{1}{2} \sum_{u \in S} |W_u|,$$

which implies

$$\sum_{u \in S} |W_u| \geq 2y \times \binom{|S|}{2}.$$  

To forbidden $G_4^\bullet$, if $s_0 uv, s_0 uk \in E(G)$, then $uvk \not\in E(G)$. Since for each $u \in \overline{S}$, there are $\binom{|W_u|}{2}$ pair of vertices each can form a 3-edge with $u$, we need to remove these edges in $\binom{\overline{S}}{3}$. Let $N$ be the number of 3-edges in $\binom{\overline{S}}{3}$ but not in $G$, by Cauchy-Schwarz inequality, we have

$$N \geq \frac{1}{3} \sum_{u \in S} \binom{|W_u|}{2} \geq \frac{1}{6} \frac{1}{|S|} (\sum_{u \in S} |W_u|)^2 - \frac{1}{6} \sum_{u \in S} |W_u| \geq \frac{1}{6} y^2 |S|^3 - \frac{1}{6} |S| \times (|S| - 1).$$  

Thus we have

$$h_n(G) \leq x + y \times 3x \times (1 - x)^2 + (1 - x)^3 - y^2(1 - x)^3 + o_n(1).$$  

When $x \leq \frac{2}{5}$, the above expression reaches the maximum value when $y = \frac{3}{2} \frac{x}{1-x} \leq 1$. When $x \geq \frac{2}{5}$, the above expression reaches the maximum value when $y = 1$. Thus we obtain

$$h_n(G) \leq \begin{cases} x + (1 - x)^3 + \frac{9}{16} x^2 (1 - x) & \text{for } x \leq \frac{2}{5}; \\ x + 3x(1 - x)^2 & \text{for } x \geq \frac{2}{5}. \end{cases}$$
When $x \leq \frac{2}{7}$, by solving $f(x) = x + (1-x)^3 + \frac{2}{7}x^2 (1-x) \leq 1$, we get $x \leq \frac{8}{13}$. This always holds since $x \leq \frac{2}{7} \leq \frac{8}{13}$. When $x \geq \frac{2}{7}$, $g(x) = x + 3x(1-x)^2 \leq 1$ is equivalent to $3x(1-x) \leq 1$ which is always true. Thus in both cases, we have $h_n(G) \leq 1 + o_n(1)$, implies that $\pi(\{K^*_3, G^*_1\}) = \lim_{n \to \infty} h_n(G) = 1$. The proof is complete.

**Proof of Theorem 4.** Note that $H^{(1,3)}_5$ is $K^*_3$ and $G^*_1$-colorable respectively, we have $\pi(H^{(1,3)}_5) \leq \pi(\{K^*_3, G^*_1\})$. By Lemma 2 and Lemma 4 the result follows.

### 3.2 Proof of Theorem 4

In this subsection, we will consider the non-degenerate $\{1, 3\}$-graphs. In particularly, we consider the non-degenerate 3-partite $\{1, 3\}$-graphs. A hypergraph is called 3-partite if its vertex set $V$ can be partitioned into 3 different classes $V_1, V_2, V_3$ such that every edge intersects each class in exactly one vertex. A 3-partite $\{1, 3\}$-graph is $K^*_3$-colorable, where $K^*_3$ is a $\{1, 3\}$-graph on 3 vertices with edge set $\{1, 2, 3, 123\}$.

So far we know that the chain $C^{(1,3)} = \{1, 123\}$ is 3-partite and it is degenerate, while a slightly larger 3-partite $\{1, 3\}$-graph $K^*_3 = \{1, 2, 123\}$ is not degenerate. We have $\pi(K^*_3) \geq 1 + \sqrt[3]{3}$ since it’s not contained in the $G_A$. Now we are ready to prove Theorem 4 and Corollary 4.

**Proof of Theorem 4.** For Item 1: Since $H$ satisfies $K^*_3 \subseteq H \subseteq K^*_3(s)$ for $s \geq 2$, then $H$ is $K^*_3$-colorable, which means there exists a vertex partition $V(H) = V_1 \cup V_2 \cup V_3$ so that $H$ is 3-partite and the level-graph $H^d$ only appears in at most two vertex partitions (say $V_2$ and $V_3$). Since $H$ does not contain $K^*_3$ as sub-graph, then each edge of the level-graph $H^d$ can only intersect one vertex in $V_1$, plus one white (black) vertex in $V_2$ and one black (white) vertex in $V_3$, or intersect one vertex in $V_1$ plus two white vertices in $V_2$ and $V_3$. Let $f$ be a map such that $f(v) = 1$ if $v \in V_1$, $f(v) = 2$ if $v$ is a black vertex in $V_2$, $f(v) = 5$ if $v$ is a white vertex in $V_2$, and $f(v) = 3$ if $v$ is a black vertex in $V_3$, $f(v) = 4$ if $v$ is a white vertex in $V_3$. One can check that $f$ is a hypergraph homomorphism from $H$ to $H^{(1,3)}_5$. Thus $H$ is $H^{(1,3)}_5$-colorable, we have $\pi(H) = 1$.

For Item 2: Since $K^*_3 \subseteq H \subseteq K^*_3(s)$ for $s \geq 2$, by Theorem 6 we have $\pi(H) = \pi(K^*_3)$. For any $K^*_3$-free $\{1, 3\}$-graph $G$ on $n$ vertices, let $X$ be the set of all 1-edges in $G$, and $Y \subseteq V(G)$ be the complement of $X$. On one hand, since it is $K^*_3$-free, there is no 3-edge of form $\binom{X}{3}$ or form $\binom{X}{2} \times \binom{Y}{1}$. Thus $G$ is $H_A$-colorable. Therefore $\lim_{n \to \infty} h_n(G) \leq \lambda(H_A) = 1 + \sqrt[3]{3}$. On the other hand, the construction $G_A$ is $K^*_3$-free. We have $\pi(K^*_3) \geq \lambda(H_A) = 1 + \sqrt[3]{3}$. Thus, we have $\pi(H) = \pi(K^*_3) = 1 + \sqrt[3]{3}$.

**Proof of Corollary 4.** Let $H$ be any $\{1, 3\}$-graph with $\pi(H) < \lambda(H_B) = \frac{3}{2} + \sqrt[3]{3}$. Then $H$ must be $H_B$-colorable, hence $K^*_3$-colorable. By Item 2 of Theorem 4 $\pi(H)$ is either $1$ or $1 + \sqrt[3]{3}$, thus we must have $\pi(H) = 1$.  

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4 The 3-partite \{1,3\}-graphs

In previous section, all \{1,3\}-graphs we studied are 3-partite. In this section, we continue to study the Turán densities of 3-partite \{1,3\}-graphs.

Lemma 5. Let \(H\) be a 3-partite \{1,3\}-graph such that \(K_3^{***} \subseteq H\), where \(K_3^{***} = \{1,2,3,123\}\). Then \(\pi(H) = 1 + \frac{2\sqrt{3}}{9}\).

Proof. Since any 3-partite \{1,3\}-graph is \(K_3^{***}\)-colorable, we only need to prove \(\pi(K_3^{***}) = 1 + \frac{2\sqrt{3}}{9}\).

On one hand, consider an extremal \(K_3^{***}\)-free \{1,3\}-graph \(G_n\). Let \(X\) be the set vertices of 1-edges in \(G_n\). Projecting all the vertices in \(X\) into a single vertex \(x\) and all the vertices not in \(X\) into a single vertex \(y\), we get an \{1,3\}-graph (with loops) \(H_C\): where \(E(H_C) = \{x, xxy, xyy, yyy\}\). This projection is a hypergraph homomorphism from \(G_n\) to \(H_C\) since \(G\) is \(K_3^{***}\)-free. Thus \(G\) is \(H_c\)-colorable. In particular, we have

\[
\pi(K_3^{***}) = \lim_{n \to \infty} h_n(G_n) \leq \lambda(H_C) = 1 + \frac{2\sqrt{3}}{9}.
\]

On the other hand, any blow-up of \(H_c\) does not contain the sub-graph \(K_3^{***}\). The blow-up graph \(G_C\) has the maximal edge density \(1 + \frac{2\sqrt{3}}{9}\):

\[
G_C: h_n(G_C) = 1 + \frac{2\sqrt{3}}{9} + o_n(1), \text{ at } |X| = \frac{2\sqrt{3}}{3} n.
\]

Hence, \(\pi(K_3^{***}) = \lambda(H_c) = 1 + \frac{2\sqrt{3}}{9}\).

Let us restrict \(H\) to be 3-partite but containing no \(K_3^{***}\) as sub-graph. One can check such \(H\) must be \(H_6^{\{1,3\}}\)-colorable, where

\[
H_6^{\{1,3\}} = \{1,2,3,124,145,135,236,246,356,456\}\]

\(H_6^{\{1,3\}}\) is not contained in \(G_D\) with maximal edge density \(\frac{4}{3}\).

\[
H_6^{\{1,3\}}: h_n(H_6^{\{1,3\}}) = \frac{4}{3} + o_n(1), \text{ at } |X| = \frac{4}{3} n.
\]

\[
G_D: h_n(G_D) = \frac{4}{3} + o_n(1), \text{ at } |X| = \frac{4}{3} n.
\]
So far we couldn’t determine the upper bound of $\pi(H^{(1,3)_6})$. We leave this open. Let’s turn our attention to the sub-graphs of $H^{(1,3)_6}$ and we aim to determine their Turán densities. Now let us first consider two sub-graphs of $H^{(1,3)_6}$:

$$H^*_5 = \{1, 2, 3, 124, 145, 135\}$$

and

$$H^*_6 = \{1, 2, 3, 124, 135, 236\}.$$  

For both of them, the Turán density is greater than $1 + \frac{\sqrt{3}}{9}$, since they are not contained in $G_E$ and $G_F$ respectively ($\lim_{n \to \infty} h_n(G_E) = \lim_{n \to \infty} h_n(G_F)$).

$G_E$: $h_n(G_E) = \frac{2 + \sqrt{3}}{9} + o_n(1)$, at $|X| = (\frac{3 + \sqrt{3}}{6})n$.

$G_F$: $h_n(G_F) = \frac{2 + \sqrt{3}}{9} + o_n(1)$, at $|X| = |Y| = (\frac{3 - \sqrt{3}}{6})n$.

To calculate the upper bounds of $\pi(H^{*}_5)$, we need the following lemma.

**Lemma 6.** Let $H^{***}_4 = \{1, 2, 123, 124, 134\}$, then $\pi(\{K^{***}_3, H^{***}_4\}) = 1 + \frac{\sqrt{3}}{9}$.

**Proof.** To see the lower bound, observe that both $H^{***}_4$ and $K^{***}_3$ are not contained in $G_E$.

To see the upper bound, let $G$ represent a $\{K^{***}_3, H^{***}_4\}$-free graph on $n$ vertices, let $X \subseteq V(G)$ be the set of all 1-edges of $G$, $|X| = xn$ for some real $x \in (0, 1)$, and let $Y = V(G) \setminus X$, then $|Y| = (1 - x)n$. To forbidden $K^{***}_3$, there is no 3-edge of form $\binom{X}{3}$. Let $y$ be the density of 3-edges in $G$ among all edges of form $\binom{X}{2} \times \binom{Y}{1}$. For any pair of vertices $(i,j)$ in $X$, denote $d_{ij}$ as the number of vertices $k \in Y$ so that $\{ijk\} \in E(G)$. Then

$$y = \frac{\sum_{(i,j) \in \binom{X}{2}} d_{ij}}{\binom{\binom{X}{2}}{1} \times \binom{Y}{1}}.$$  

To forbidden $H^{***}_4$, for each pair of vertices $(i,j) \in \binom{X}{2}$, if $ijk$ and $ijl$ are in $E(G)$, neither $kli$ nor $klj$ can be contained in $E(G)$. Thus for every pair of $(i,j)$, the number of 3-edges not shown in $G$ is at least $2\binom{d_{ij}}{3}$. Let $M$ be the total number of 3-edges of form $\binom{X}{1} \times \binom{Y}{2}$ not shown in $G$, then by Cauchy-Schwarz.
inequality, we have

\[
M \geq \frac{\sum_{i,j \in X} 2^{d_{ij}}}{|X|} \\
\geq \left( \frac{\sum_{i,j \in X} d_{ij}}{\binom{|X|}{2}} \right)^2 - \frac{\sum_{i,j \in X} d_{ij}}{xn} \\
\geq \frac{1}{2}y^2x(1-x)^2n^3 - \frac{1}{2}yx(1-x)n^2.
\]

Thus

\[h_n(G) \leq x + (1 - x)^3 + 3x^2(1-x)y + 3x(1-x)^2 - 3x(1-x)^2y^2 + o_n(1).
\]

A simple calculation can show that \(h_n(G)\) achieves maximum value at \(y = 1\), which implies that for any positive integer \(n\), any extremal \(1,3\)-graph of \(\{K_3^{***}, H_4^{***}\}\) is \(H_k\)-colorable where \(E(H_k) = \{x, xxy, yyy\}\). Therefore, \(\pi(\{K_3^{***}, H_4^{***}\}) = \lim_{n \to \infty} h_n(G_n) \leq \lambda(H_k) = 1 + \frac{\sqrt{3}}{9}\). The result follows. \(\square\)

**Lemma 7.** \(\pi(H_5^*) = 1 + \frac{\sqrt{3}}{9}\).

**Proof.** On one hand, \(H_5^*\) is not contained in \(G_E\), then \(\pi(H_5^*) \geq 1 + \frac{\sqrt{3}}{9}\). On the other hand, \(H_5^*\) is \(K_3^{***}\) and \(H_4^{***}\)-colorable, thus \(\pi(H_5^*) \leq \pi(\{K_3^{***}, H_4^{***}\}) \leq 1 + \frac{\sqrt{3}}{9}\). The result follows. \(\square\)

**Corollary 2.** The proper sub-graphs of \(H_5^*\) can be classified to two different sets: either the sub-graph contains \(K_3^{***}\) and is \(K_3^{***}\)-colorable, in this case the Turán density is \(1 + \frac{\sqrt{3}}{18}\); or the sub-graph does not contain \(K_3^{***}\), then it is \(H_5^{1,3}\)-colorable, in this case the Turán density is 1.

To calculate the upper bounds of \(\pi(H_5^*)\), we need the following lemma.

**Lemma 8.** Let \(H_4^{***} = \{1, 2, 3, 124, 134, 234\}\), then \(\pi(\{K_3^{***}, H_4^{***}\}) = 1 + \frac{\sqrt{3}}{9}\).

**Proof.** To see the lower bound, observe that both \(K_3^{***}\) and \(H_4^{***}\) are not contained in \(G_F\). To see the upper bound, let \(G\) represent a \(\{K_3^{***}, H_4^{***}\}\)-free graph on \(n\) vertices, let \(X \subseteq V(G)\) be the set of all 1-edges of \(G\), \(|X| = xn\) for some real \(x \in (0, 1)\), let \(Y = V(G) \setminus X\), then \(|Y| = (1-x)n\). To forbidden \(K_3^{***}\), there is no 3-edge of form \(\binom{X}{3}\).

Let \(y\) be the density of 3-edges in \(G\) among all edges of form \(\binom{X}{2} \times \binom{Y}{1}\). For each \(i \in Y\), let \(D_i = \{\{j, k\} \in \binom{X}{2} | ijk \in E(G)\}\), denote \(d_i = |D_i|\). Then

\[y = \frac{\sum_{i \in Y} d_i}{\binom{X}{2} \times \binom{Y}{1}}.\]

Suppose \(y > \frac{1}{2}\), then there exists \(i \in Y\), such that \(d_i > \frac{1}{2}\binom{|X|}{2}\). By the fact that the Turán density of a triangle graph is \(\frac{1}{2}\), there must exist a triple \(\{j, k, l\} \in \binom{X}{3}\)
such that \( \{ijk, ijl, ikl\} \subseteq E(G) \), which is a copy of \( H_4^{***} \), a contradiction. Note that the existence of 3-edge of form \( (i, j, k) \) or \( (j, k, l) \) does not result in an occurrence of \( H_4^{***} \) or \( K_3^{***} \) in \( G \). Thus we can take all such edges. Thus we have \( y \leq \frac{1}{2} \), then

\[
h_n(G) \leq x + (1 - x)^3 + \frac{3}{2}x^2(1 - x) + 3x(1 - x)^2 + o_n(1),
\]

which achieves the maximum \( 1 + \frac{\sqrt{3}}{9} \) at \( x = 1 - \frac{\sqrt{3}}{3} \).

Hence, we have \( \pi(\{K_3^{***}, H_4^{***}\}) = \lim_{n \to \infty} h_n(G_n) \leq 1 + \frac{\sqrt{3}}{9} \). The result follows. \( \square \)

**Lemma 9.** \( \pi(H_6^*) = 1 + \frac{\sqrt{3}}{18} \).

**Proof.** On one hand, \( H_6^* \) is not contained in \( G_F \), then \( \pi(H_6^*) \geq 1 + \frac{\sqrt{3}}{9} \). On the other hand, \( H_6^* \) is \( K_3^{***} \) and \( H_4^{***} \)-colorable, thus \( \pi(H_6^*) \leq \pi(\{K_3^{***}, H_4^{***}\}) \leq 1 + \frac{\sqrt{3}}{9} \). The result follows. \( \square \)

Since we are considering all sub-graphs of \( H_6^{1,3} \), we start this by looking at the larger sub-graphs then the smaller sub-graphs. Using above two lemmas, we are able to determine the Turán density for a list of sub-graphs of \( H_6^{1,3} \). Now we let \( H \) be a sub-graph of \( H_6^{1,3} \), we have:

1. If \( H \) is \( K_3^{***} \)-colorable, thus \( \pi(H) = 1 \) or \( \pi(H) = 1 + \frac{\sqrt{3}}{18} \).
2. If \( H \) is not above case, then \( H \) must contain all 1-edges: 1, 2, 3, and none of them is isolated. Then we have several different cases:

   (a) Suppose \( H \) is obtained from \( H_6^{1,3} \) by removing one 3-edge consisting of two black vertices and one white vertex (say 236 or equivalence), then one can check \( H \) is \( H_5^* \)-colorable. Note that \( H_5^* \subseteq H \), by Lemmas we have \( \pi(H) = 1 + \frac{\sqrt{3}}{9} \). Similarly, for any sub-graph \( H' \) of \( H \), if \( H' \) contains \( H_5^* \) or a \( H_5^* \)-colorable graph as sub-graph, then \( \pi(H') = 1 + \frac{\sqrt{3}}{9} \). If \( H' \) is not above case, by trial and error, there is only one situation: \( H' \) contains the following sub-graph \( H^* \) (or its equivalence) which is not contained in \( G_E \). Thus \( \pi(H') = 1 + \frac{\sqrt{3}}{9} \).

   \[
   \begin{array}{c}
   1 \\
   \bullet
   \\
   4 \\
   5 \\
   \\
   2 \\
   6 \\
   H^*
   \\
   3
   \end{array}
   \]

   (b) Let \( H \) be sub-graph of \( H_6^{1,3} \) by removing edges 145, 246, 456 and 356, the resulting graph is \( H_6^* \). By Lemma \( \pi(H) = \pi(H_6^*) = 1 + \frac{\sqrt{3}}{9} \).
3. The following graphs $H_6^{(1,3)}$, $H_6^a$, $H_6^b$, $H_6^c$, $H_6^d$ and $H_6^e$ are unsolved. We conjecture that the extremal configuration of $H_6^{(1,3)}$ and $H_6^b$ is Construction $G_D$: thus we conjecture $\pi(H_6^{(1,3)}) = \pi(H_6^b) = \frac{4}{3}$.

5 Non-trivial degenerate $R$-graphs

Recall that a degenerate $R$-graph $H$ is trivial if it is contained in a blow-up of the chain $C^R$, otherwise, we say $H$ is non-trivial. In this section, we will show that except the case $R = \{1, 2\}$, there always exist non-trivial degenerate $R$-graphs for any finite set $R$ of at least two distinct positive integers. We will use the suspension operations on hypergraphs.

**Definition 7.** [4] The suspension of a hypergraph $H$, denoted by $S(H)$, is the hypergraph with $V = V(H) \cup \{v\}$ where $\{v\}$ is a new vertex not in $V(H)$, and the edge set $E = \{e \cup \{v\} : e \in E(H)\}$. We write $S^t(H)$ to denote the hypergraph obtained by iterating the suspension operation $t$-times, i.e. $S^2(H) = S(S(H))$ and $S^3(H) = S(S(S(H)))$, etc.

The relationship between $\pi(H)$ and $\pi(S(H))$ was investigated in [4].

**Proposition 1.** [4] For any family of hypergraphs $\mathcal{H}$ we have that $\pi(S(\mathcal{H})) \leq \pi(\mathcal{H})$.

Given a general set $R$ and positive integer $t$, we denote $(R + t)$ as the set obtained from $R$ by adding $t$ to each element of $R$. Note that if the $R$-graph $H$ is not contained in a blow-up of chain $C^R$, then $S^t(H)$ is not contained in a blow-up of the chain $C^{(R+t)}$. Thus we have the following fact:

**Corollary 3.** Let $H$ be a non-trivial degenerate $R$-graph, let $t$ be any positive integer. Then the $t$-times suspension $S^t(H)$ is a non-trivial degenerate $(R + t)$-graph.
Lemma 10. Given a positive integer \( t \geq 2 \), and a \( \{1,t\} \)-graph \( H \), let \( T(H) \) be the \( \{1,t+1\} \)-graph obtained from \( H \) by adding a new vertex \( v \notin V(H) \) such that \( V = V(H) \cup \{v\} \), \( T(H)^1 = H^1 \) and \( T(H)^{1+t} = \{e \cup \{v\} : e \in E(H^1)\} \). Then we have \( \pi(T(H)) \leq \pi(H) \).

Proof. Let \( n \) be a positive integer and \( G = (V,E) \) be an extremal \( T(H) \)-free \( \{1,t+1\} \)-graph on \( n \) vertices. We have \( \pi_n(T(H)) = h_n(G) \). Denote \( E_i \) as the set of \( i \)-edges of \( G \), for \( i = 1, t+1 \). For any vertex \( v \in V(G) \), denote \( G_v \) as the hypergraph obtained from \( G \) with the vertex set \( V(G_v) = V \setminus \{v\} \) and the edge sets \( E(G_v) = E_{v,1} \cup E_{v,t} \), where \( E_{v,1} = \{ u \in V(G_v) : u \in E_1 \} \) and \( E_{v,t} = \{ \{u_1, \ldots, u_t\} : \{v,u_1, \ldots, u_t\} \in E_{t+1} \} \). Observe that \( G_v \) is an \( H \)-free \( \{1,t\} \)-graph on \( n - 1 \) vertices. Thus \( h_{n-1}(G_v) \leq \pi_{n-1}(H) \).

Since

\[
|E_1| = \frac{1}{n-1} \sum_{v \in V(G)} |E_{v,1}| \quad \text{and} \quad |E_{t+1}| = \frac{1}{(t+1)} \sum_{v \in V(G)} |E_{v,t}|,
\]

then

\[
h_n(G) = \frac{|E_1|}{n} + \frac{|E_{t+1}|}{n+1} = \sum_{v \in V(G)} \frac{|E_{v,1}|}{(n-1)(n)} + \sum_{v \in V(G)} \frac{|E_{v,t}|}{(t+1)(n+1)}
\]

\[
= \frac{1}{n} \sum_{v \in V(G)} \left( \frac{|E_{v,1}|}{n-1} + \frac{|E_{v,t}|}{n+1} \right) = \frac{1}{n} \sum_{v \in V(G)} h_{n-1}(G_v)
\]

\[
\leq \pi_{n-1}(H).
\]

Thus \( \pi(T(H)) = \lim_{n \to \infty} \pi_n(T(H)) = \lim_{n \to \infty} h_n(G) \leq \pi(H) \). \( \square \)

Lemma 11. Let \( R \) be a set of two distinct positive integers, \( R \neq \{1,2\} \). Then there exist non-trivial degenerate \( R \)-graphs.

Proof. By Corollary \( \mathbb{K} \), for every positive integer \( k \), one can take the suspension of \( H_5^{1,3} \)-\( k \)-times, the resulting graph \( S^k(H_5^{1,3}) \) is a non-trivial degenerate \( \{1+k, 3+k\} \)-graph. Thus there are non-trivial degenerate hypergraphs of edge types: \( \{1,3\}, \{2,4\}, \{3,5\}, \ldots, \{k,k+2\}, \ldots \)

In \( \mathbb{H} \), the authors found a non-trivial degenerate \( \{2,3\} \)-graph: \( H_3^{2,3} = \{12,13,234\} \). Similarly, by Corollary \( \mathbb{K} \) there are non-trivial degenerate hypergraphs of edge types: \( \{2,3\}, \{3,4\}, \{4,5\}, \ldots, \{k,k+1\}, \ldots \)

Using Lemma \( \mathbb{H} \) on \( H_5^{1,3} \), there are non-trivial degenerate hypergraphs of edge types: \( \{1,4\}, \{1,5\}, \ldots, \{1,t\}, \ldots \), for integer \( t \geq 4 \). For each of these non-trivial degenerate \( \{1+t\} \)-graphs, applying Corollary \( \mathbb{K} \) there are non-trivial degenerate hypergraphs of edge types: \( \{2,1+t\}, \{3,2+t\}, \ldots, \{k,k-1+t\}, \ldots \).
To summarize, for each integer \( k \geq 2 \) and each integer \( t \geq 3 \), we have non-trivial degenerate hypergraphs of edge types \( \{1, t\}, \{k, k+1\}, \{k, k+2\}, \{k, k+t\} \), which cover all sets of two distinct positive integers, except \( \{1, 2\} \).

**Lemma 12.** Let \( R \) be a set of distinct positive integers with \( |R| \geq 2 \) and \( 1 \not\in R \). If there exist non-trivial degenerate \( R \)-graphs, then there exist non-trivial degenerate \( \{1\} \cup R \)-graphs.

**Proof.** For each \( R \) stated in the lemma, let \( H \) be the non-trivial degenerate \( R \)-graph. Let \( H' \) be the disjoint union of \( H \) with a single 1-edge \( v \not\in H \). Clearly, \( H' \) is not contained in a blow-up of chain \( C^{(1)\cup R} \). We will prove that \( H' \) is also degenerate.

Let \( n \) be a positive integer and \( G = (V, E) \) be an extremal \( H' \)-free \( \{1\} \cup R \)-graph on \( n \) vertices. We have \( \pi_n(H') = h_n(G) \). Denote \( E_i \) as the set of \( i \)-edges of \( G \), for each \( i \in \{1\} \cup R \). For any 1-edge \( v \in E_1 \), consider the sub-graph \( G_v \) of \( G \) by removing all 1-edges (keep the vertices of these 1-edges in \( G_v \)). Then the vertex set \( V(G_v) = V \), set of \( i \)-edges \( E_i(G_v) = E_i(G) \) for each \( i \in R \). Then we have

\[
|E_i(G)| = \frac{1}{|E_1|} \sum_{v \in E_1} |E_i(G_v)|, \forall i \in R.
\]

Observe that \( G_v \) is an \( H \)-free \( R \)-graph on \( n \) vertices, so \( \pi_n(H) \geq h_n(G_v) \).

Then we have

\[
h_n(G) = \sum_{i \in \{1\} \cup R} \frac{|E_i|}{\binom{n}{i}} = \frac{|E_1|}{\binom{n}{1}} + \sum_{i \in R} \sum_{v \in E_1} \frac{|E_i(G_v)|}{|E_1| \binom{n}{i}}
\]

\[
\leq 1 + \frac{1}{|E_1|} \sum_{v \in E_1} h_n(G_v)
\]

\[
\leq 1 + \pi_n(H).
\]

Thus \( \pi(H') = \lim_{n \to \infty} \pi_n(H') = \lim_{n \to \infty} h_n(G) \leq 1 + \pi(H) = |R| \), then \( \pi(H') = |R| \). Therefore, \( H' \) is a non-trivial degenerate \( \{1\} \cup R \)-graph.

**Proof of Theorem 3** Using the non-trivial degenerate \( R \)-graph for \( R \) stated in Lemma 11, we obtain non-trivial degenerate \( R \)-graphs for \( |R| = 3 \) and \( 1 \in R \). Apply Corollary 3, we then obtain all non-trivial degenerate \( R \)-graphs for \( |R| = 3 \). Repeatedly apply Lemma 12 and Corollary 3, we can obtain all \( R \)-graphs for \( |R| \geq 4 \), the result follows.

We conjecture that for any set \( R \), there exists an \( R \)-graph \( H^R \) such that if \( G^R \) is \( R \)-degenerate if and only if \( G^R \) is \( H^R \)-colorable. This conjecture is true for the case \( R = \{r\} \) with \( r \geq 2 \) and \( R = \{1, 2\} \) and is confirmed for \( R = \{1, 3\} \) in this paper.
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