The formation of optical frequency combs (OFCs) in high-Q driven microresonators with Kerr type nonlinearity has attracted considerable attention in the past ten years [1, 2]. Recent works show that dispersive cavities with quadratic nonlinearities may provide an alternative to Kerr cavities for the generation of frequency combs [3–5]. The main advantages of quadratic OFCs are the reduced pump power requirements and the possibility of comb generation in spectral regions that are separate from that of the pump laser frequency. For example, optical parameter oscillators (OPOs) may allow for the efficient generation of OFCs in the mid-infrared using near-infrared continuous wave (CW) sources. It was recently shown that modulation instability (MI) induces pattern and frequency comb formation in degenerate OPOs [6]. While promising, it is still unclear whether solitary waves exist in that configuration.

In this letter, we propose the locking of domain walls (DWs), also called wave fronts and switching-waves, as an alternative mechanism to MI for the generation of OFCs in doubly resonant cavity configurations. Such a system can be described by an infinite map for the slowly varying envelopes of the fields $A_m(z,t)$ and $B_m(z,t)$, of the electric field

$$E_m(z,t) = \text{Re} \left[ A_m(z,t)e^{ik_1z-\omega_0t} + B_m(z,t)e^{ik_2z-2\omega_0t} \right],$$

centered at frequencies $\omega_0$ and $2\omega_0$, respectively. Propagation of these cavity fields over the $m^{th}$ round trip is governed by the evolution equations:

$$\partial_t A_m = -\left( \frac{\alpha_{c1}}{2} + i \frac{k''}{2} \partial^2 \right) A_m + i \kappa B_m \bar{A}_m e^{-i\Delta z},$$

$$\partial_t B_m = -\left( \frac{\alpha_{c2}}{2} + \Delta k' \partial_{\tau} + i \frac{k''}{2} \partial^2 \right) B_m + i \kappa A_m^2 e^{i\Delta k z},$$

where $\tau = t - z/v_g$, and $v_g = 1/k_1'$ is the group velocity of the fundamental field $A_0$. Here $z \in [0, L_c]$ with $L_c$ the length of the cavity; $\alpha_{c1,2}$ describe linear propagation losses for the fields $A_m$ and $B_m$; $k'' = d^2 k/d\omega^2|_{\omega_0,2\omega_0}$ are the group velocity dispersion (GVD) coefficients; $\Delta k = 2k(\omega_0) - k(2\omega_0)$ is the wave-vector mismatch at the degeneracy point; $\Delta k' = dk/d\omega|_{2\omega_0} - dk/d\omega|_{\omega_0}$ is the corresponding group-velocity mismatch, or rate of temporal walk-off; and the nonlinear coupling strength

$$\alpha_{c1,2}$$

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In this letter we theoretically investigate the formation of localized temporal dissipative structures, and their corresponding frequency combs in doubly resonant dispersive optical parametric oscillators. We derive a nonlocal mean field model, and show that domain wall locking allows for the formation of stable coherent optical frequency combs.

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\( \kappa \propto \chi^{(2)} \) is normalized such that \( |A_m|^2, |B_m|^2 \) and \( |B_{in}|^2 \) are measured in Watts. Furthermore, the intracavity fields \( A_{m+1}(0, \tau) \) and \( B_{m+1}(0, \tau) \) at the beginning of the \((m+1)\)th round-trip are related to the fields at the end of the \(m\)th round-trip by

\[
A_{m+1}(0, \tau) = \sqrt{1 - T_1} A_m(L_c, \tau)e^{-i \delta_1} \quad (3a)
\]

\[
B_{m+1}(0, \tau) = \sqrt{1 - T_2} B_m(L_c, \tau)e^{-i \delta_2} + \sqrt{T_2} B_{in} \quad (3b)
\]

where \( T_1 \) and \( T_2 \) are the power transmission coefficients, \( \omega_0 \) and \( 2\omega_0 \) of the coupler used to inject the CW field \( B_{in} \), and \( \delta_1 = (\omega_j - \omega_{e,j}) t_R \) with \( j = 1, 2 \) represents the phase detuning of the intracavity field \( A_m \) \((B_m)\) for the cavity resonance frequency \( \omega_{e,1} \) \((\omega_{e,2})\) closest to \( \omega_1 = \omega_0 \) \((\omega_2 = 2\omega_0)\) over one round-trip time \( t_R \) of \( A_m \). In what follows we set \( \Delta k = 0, \delta_2 = 2\delta_1 \). We use the map (23) to numerically explore the natural dynamics of the system. Figs. (1a) and (b) show the final steady OFCs for \( A_m \) and \( B_m \) obtained from the evolution of an initial noisy background after a sufficient number of round-trips (\( m = 2 \times 10^4 \)). These combs correspond to the temporal dissipative structures shown in panels (b) and (c)[top], where \( \text{Re}[A_m] \) and \( \text{Re}[B_m] \) are plotted in blue and red, respectively. The evolution of the fields after each round-trip is shown in the bottom sub-panels of (c)-(d). Both fields exhibit a constant temporal drift of about 0.0014 ps per round-trip. Looking to the \( A_m \) field [see Fig. (1c), top], we can identify a sequence of DWs connecting two different continuous wave states, forming a disordered stationary state. This particular solution consists of seven LSs \((\text{see LS}_1 - \text{LS}_7)\) of different widths and separations between them. At the location of each of the DWs in the \( A_m \) field, one finds pulses in the \( B_m \) field [see Figure 1(d), top]. Furthermore, one can use each such LS \((\text{see for instance LS}_1)\) as a new initial condition for the map, and find that each LS is also a localized steady state solution of the system. Moreover, when starting with a noisy background and the same parameter set, we find that different realizations of the noise can lead to many different sequences of such pulses, revealing multistability of LSs. As far as we know, the existence of this type of structures has not yet been reported in the context of quadratic dispersive cavities. Therefore, it is important to elucidate their formation and properties.

Assuming that the resonator exhibits high finesse, that both fields do not vary significantly over a single trip (i.e., the combined effects of nonlinearity and dispersion are weak), and following Refs. [5, 17, 18] one can reduce Eqs. (2) and (3), to two coupled mean-field equations:

\[
\partial_t A = -(1+i\Delta_1)A - i\beta_1 \partial^2_{\tau} A + iB\tilde{A} \quad (4a)
\]

\[
\partial_t B = -(\alpha + i\Delta_2)B - (d\partial_{\tau} + i\beta_2 \partial^2_{\tau}) B + iA^2 + S \quad (4b)
\]

where \( A(t, \tau) \) and \( B(t, \tau) \) are the normalized cavity field envelopes at \( z = 0 \) defined as \( A(t = mt_R, \tau) = \kappa L_e A_m(z = 0, \tau) / \alpha_1 \) and \( B(t = mt_R, \tau) = \kappa L_e B_m(z = 0, \tau) / \alpha_1 \), and the index \( m \) has been replaced by the slow-time variable \( t = mt_R \). Here the normalized variables and parameters are \( \alpha = \alpha_2 / \alpha_1 \), where \( \alpha_2 = (T_{1.2} + \omega_{e,1}L_e / \omega_0)^2, \Delta_1 = \delta_1 / \alpha_1, d = \Delta k' \sqrt{2L_e / \alpha_1 |k'_{c2}|}, S = B_m \sqrt{2\kappa L_e / \alpha^2_1, \beta_1 = \text{sign}(k'_{c2}), \beta_2 = k''_{c2} / |k'_{c2}|} \), and \( \tau' = \tau / \sqrt{2\alpha_1 / |k'_{c2}|L_e} \).

Interestingly, we find through numerical inspection that, for a large range of parameters, \( B \) evolves slowly in \( t \). Thus, we can make the approximation that the term \( \partial_t B \) can be neglected in Eq. (4b). Under this observation we can further simplify Eqs. (4) to a single mean-field model, as done in Refs. [18, 19]. To do so, from Eq. (4b), one can obtain an expression of \( B \) as a function of \( A^2 \) and \( S \), that once inserted in Eq. (4a) gives:

\[
\partial_t A = -(1+i\Delta_1)A - i\beta_1 \partial^2_{\tau} A - A(A^2 \otimes J) + \rho A \quad (5)
\]

with \( \otimes \) denoting the convolution with the nonlocal kernel

\[
J(\tau') = \frac{1 + \tilde{\Delta}_2}{2\pi} \int_{-\infty}^{\infty} e^{-i\gamma \Omega - \eta \Omega^2} d\Omega, \quad (6)
\]

where \( \tilde{\Delta}_2 = \Delta_2 / \alpha, \gamma = d / \alpha, \eta = \beta_2 / \alpha \). The term \( A^2 \otimes J \) introduces a nonlocal nonlinear coupling between the
A (HSS) solutions of Eq. (5), namely the CWs correspond to the homogeneous steady state and that are related by the transformation. We found that these initial temporal profiles quickly obtained with the infinite map (2)-(3), such as those in Fig. 1, as initial condition in the two models (4) and (5). For large enough detuning, ∆ > 1, such that ˜∆1 + ∆1 − 1, only the A− branch exists, and bifurcates supercritically from a pitchfork bifurcation at a pump strength ρ = 1. Moreover, LSs can form through the locking of these DWs. To illustrate this mechanism, let us first consider the presence of β1 ˜∆2 only. A natural way for doing so is to neglect walk-off and the GVD of B, so that γ = 0, and η = 0. In this case, the convolution A2 ⊗ J reduces to (1 − i ˜∆2)A2 and Eq. (5) becomes
\[ \partial_t A = -(1 + i\Delta_1)A - i\beta_1 \partial_x^2 A - (1 - i\Delta_2)|A|^2 A + \rho A, \] which is a simpler version of the well known parametrically forced Ginzburg-Landau equation with 2:1 resonance [22]. A similar equation can be also derived in the framework of singly resonant diffractive OPOs [16]. Notice that the HSS solutions of this equation are also given by Eq. (7). In diffusive systems that are not bounded periodically, DWs connecting the equivalent states −A+ and A+ can exist in both supercritical and subcritical regimes [14]. Fig. 2(b) shows two examples of a DW in the subcritical regime for (Δ1, ρ) = (−2, 4), one showing an upwards connection of −A+ to A+ (top), and the other showing a downwards connection from A+ to −A+ (bottom). We refer to these two types of DWs as having an opposite polarity. Furthermore, they are invariant under the simultaneous transformations τ → −τ, and A → −A, and they are stationary, i.e. they are Ising fronts [22].

In order for DWs of opposite polarity to form stable connections, it is of critical importance to consider the way that the wave fronts approach the HSS asymptotically. In the linear regime, this approach can be described by A(τ′) = A+ + ae−τ′ + c.c., where |a| ≪ 1 and λ = Q + iK, with Q and K real numbers. These eigenvalues λ depend on the control parameters of the system, and can be obtained by studying the system of linearized ordinary differential equations for the perturbations, derived through direct substitution of the ansatz into Eq. (5) [16] [22]. In the linear regime (i.e. far from the DW core), the overall shape of the DW oscillatory tail that approaches the HSS is determined by the leading eigenvalue λ0 = Q0 + iK0, which is the eigenvalue with the real part closest to zero, as all the other directions are damped faster. An example of an oscillatory tail (K0 ̸= 0) can be seen in detail in the close-up view of Fig. 2(b). If λ0 is purely real, the front approaches the HSS monotonically.

In a periodic system like ours, single isolated DWs do not exist: to satisfy the boundary conditions they must necessarily always come in pairs. Two DWs of opposite polarity exhibit a particle-like interaction force whose strength decays exponentially with their temporal separation D [see Fig. 2(e)], as described by ∂t D ~

FIG. 2: In (a) we show the HSS solution of Eq. (5) in the subcritical regime for Δ1 = −2. Panel (b) shows an example of two DW solutions of Eq. (5) with different polarities for (Δ1, ρ) = (−2, 4). For the same parameters, panels (c)-(e) show different LSs formed through the locking of the DWs at different separations. Here (γ, η) = (0, 0), points of the fast variable τ [23, 21]. The normalized field reads A = Ae−iψ/ 2\sqrt{\alpha(1 + \Delta_2)} with \psi = π/4 + atan(Δ2)/2, and the normalized driving field amplitude is ρ = S/(\alpha\sqrt{1 + \Delta_2^2}). With this approximation, the B field is dynamically slaved to the A field, and it is explicitly given by B = -(A ⊗ J + ρ)e−itan(Δ2). Without loss of generality we will consider the normal GVD regime (β1 = 1), and α = 1, such that Δ2 = 2Δ1.

We have used a variety of steady state solutions obtained with the infinite map [21, 22], such as those in Fig. 1 as initial condition in the two models [14] and [15]. We found that these initial temporal profiles quickly converge, and that the converged solutions were almost identical in all models. This confirms the validity of both mean field models [14] and [15].

Based on our observations in Fig. 1(c) [top], we anticipate that the LSs are composed of two DWs connecting different CWs that coexist for the same parameter values and that are related by the transformation A → −A. The CWs correspond to the homogeneous steady state (HSS) solutions of Eq. (5), namely A0 = 0, and the two branches of solutions A = A± satisfying
\[ |A|^2 = \frac{(\Delta_1 \Delta_2 - 1) ± \sqrt{(1 + \Delta_2)^2 - (\Delta_2 + \Delta_1)^2}}{1 + \Delta_2^2}, \] with A± = |A±|e±iφ±, and φ± = acos[(|A±|^2 + 1)/ρ]/2. Here we only consider DWs between −A+ and A±, as done in Refs. [15] [16]. For large enough detuning, Δ1 > 1/Δ2, only the A+ branch exists, and bifurcates supercritically from a pitchfork bifurcation at a pump strength ρ = \sqrt{1 + Δ_2^2}. However, if Δ1 < 1/Δ2, A− emerges subcritically (and therefore unstably), and stabilizes at saddle-node SN1, at ρ_l = (Δ_2 + Δ_1)/\sqrt{1 + Δ_2^2}, where it merges with A±. We will focus on the subcritical regime, whose bifurcation diagram is shown in Fig. 2(a).
FIG. 3: Bifurcation diagram showing the influence of walk-off $\gamma$ on a single bump LS in the absence of $\eta$ (the GVD of $B$). Here the width $D$ and speed $v$ of the LS are shown as a function of $\gamma$. The labels (a)-(d) correspond to the LSs profiles shown in the subpanels. Here $(\Delta_1, \rho) = (-2, 4)$.

\[ e^{-Q_0D}\cos(K_0D) \]  

The stationary solutions of this equation, $D_n = 2\pi n/K_0$ (with $n \in \mathbb{N}$), correspond to the locatons at which two DWs can lock to each other, and form a LS of width $D \approx D_n$. Due to the oscillatory nature of this force, the interaction of DWs alternates between attraction and repulsion, as does the stability of the stationary separations. Thus, for a given set of parameters, multiple LSs of different widths can coexist, as illustrated in Fig. 2(c)-(e). These LSs only form when the DWs approach the HSS in an oscillatory way ($K_0 \neq 0$). In contrast, when the tails are monotonic ($K_0 = 0$), two DWs attract each other, and move towards each other until they annihilate one another in a process called coarsening [23].

When either the GVD of the $B$ field, $\eta$, and/or the walk-off, $\gamma$, are present, the nonlinear nonlinearity must be taken into account, and one has to consider the more general Eq. (5). In this model, DW solutions still exist, although their shape and symmetry properties are modified, since nonlinear nonlinear coupling can significantly alter the eigenvalues $\lambda$ [20,21]. Nevertheless, the mechanism of DW locking remains the same for solutions of Eq. (5), and similar LSs can be found. To illustrate this, we explore the influence of the walk-off $\gamma$ on the LSs that we found using the local model [see Fig. 2]. The walk-off breaks the left/right symmetry ($\tau \rightarrow -\tau$), and the LSs now drift at a constant velocity $v$ proportional to $\gamma$. Considering a change of coordinates to a moving reference frame (i.e., $\tau' \rightarrow \tau' - vt$) and using a numerical continuation algorithm, based on a Newton-Raphson solver, the LSs and their velocity $v$ can be continuously tracked in the parameter $\gamma$. In this way, the bifurcation diagram in Fig. 3 was constructed, where we tracked two LSs of different widths [i.e. the solutions shown in Fig. 2(c) and Fig. 2(d)]. The diagram at the top of Fig. 3 shows how the width (blue) and the velocity $v$ (red) of these LS solutions change with increasing walk-off $\gamma$. The solid lines corresponds to the continuation of a narrow LS [Fig. 2(c)], while the dashed lines show the changes of a wider LS [Fig. 2(d)]. Changes in the profiles of the LSs are illustrated in the subpanels (a)-(f). In both cases, the width of the LSs increases monotonically with $\gamma$, due to an increase in the wavelength of the oscillatory tails of the DWs [see the profiles plotted in Fig. 3(a)-(f)]. The velocities of both LSs are approximately equal, indicating that the velocity depends mostly on the strength of the walk-off, and not on the shape and width of the structures. The LS velocity, initially negative, becomes positive at $\gamma \approx 2$, and then sharply increases until reaching a peak value $v \approx 2$, after which it decreases monotonically and saturates for large values of the walk-off $\gamma$. For large walk-off, the LSs of Fig. 3 closely resemble the structures that we found by simulating the full map, as shown in Fig. 1. The fact that this type of LSs exists for very large values of walk-off can have a big practical advantage for OFC generation since no dispersion engineering of the resonator is needed other than for the phase-matching. By incorporating both walk-off $\gamma$ and GVD $\eta$, we can then retrieve exactly the LSs in Fig. 1. In the absence of $\gamma$, the presence of $\eta$ induces modifications in the tails of the LSs. However, for the parameter values considered in this work, $\gamma$ dominates and the effect of $\eta$ is almost negligible. The influence of the different control parameters of the system on the bifurcation structure of the LSs can be quite complex and their study is beyond the scope of this paper.

In summary, we have studied the formation of temporal LSs in doubly resonant dispersive OPOs in the presence of walk-off. Using an infinite map for the slowly varying envelopes of the OPO fields, we showed that a random sequence of LSs of different widths is formed naturally. We then derived a nonlocal mean-field model with a nonlocal nonlinear coupling that describes the dynamics of these OPO cavities, in which we confirmed the existence of LSs. They are formed through the locking of DWs connecting two equivalent homogeneous states, which could be explained more easily in the local case. Afterwards, using continuation techniques, we studied how these LSs are altered by walk-off and GVD of the pump field $B$. Remarkably, the formation of this type of LSs does not depend on modulational instabilities. In the frequency domain, LSs correspond to coherent frequency combs formed around both $\omega_0$ and $2\omega_0$. We expect that these structures may play a significant role for future integrated ultra-broadband frequency comb generation.

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