TWISTED YANGIANS AND FINITE W-ALGEBRAS

JONATHAN BROWN

Abstract. We construct an explicit set of generators for the finite W-algebras associated to nilpotent matrices in the symplectic or orthogonal Lie algebras whose Jordan blocks are all of the same size. We use these generators to show that such finite W-algebras are quotients of twisted Yangians.

1. INTRODUCTION AND NOTATION

There has been renewed interest recently in the study of finite W-algebras associated to nilpotent orbits in semisimple Lie algebras; see e.g. [P1] [P2] [GC] [DK] [BGK] [Lo]. The goal of this paper is to show that the finite W-algebras associated to nilpotent matrices in the symplectic or orthogonal Lie algebras whose Jordan blocks are all of the same size are homomorphic images of Olshanski’s twisted Yangians from [O] [MNO]. Results along these lines were first obtained by Ragoucy [R] by a different approach, although Ragoucy was primarily concerned with the classical case, i.e. the commutative Poisson algebras that arise from the algebras considered here on passing to their associated graded algebras. One new discovery in the present paper is the following crossover phenomenon: when the Jordan blocks are of even size, the finite W-algebra arising from an orthogonal Lie algebra is a quotient of the twisted Yangian associated to a symplectic Lie algebra and vice versa. In [BK2], Brundan and Kleshchev proved an analogous result relating the finite W-algebras associated to arbitrary nilpotent elements in type A to quotients of so-called shifted Yangians. This paper is an attempt to adapt some of their methods to types B, C and D, specifically, the techniques from [BK2] §12 dealing with nilpotent matrices whose Jordan blocks have the same size.

We begin by fixing explicit matrix realizations for the classical Lie algebras. For any integer \(n \geq 1\), we will label the rows and columns of \(n \times n\) matrices by the ordered index set

\[ I_n = \{-n+1, -n+3, \ldots, n-1\}. \]

Let \(\mathfrak{g}_n = \mathfrak{gl}_n(\mathbb{C})\) with standard basis given by the matrix units \(\{e_{i,j} \mid i, j \in I_n\}\). Let \(J_n^+\) be the \(n \times n\) matrix with \((i, j)\) entry equal to \(\delta_{i,-j}\), and set

\[ \mathfrak{g}_n^+ = \mathfrak{so}_n(\mathbb{C}) = \{x \in \mathfrak{g}_n \mid x^T J_n^+ + J_n^+ x = 0\}, \]

where \(x^T\) denotes the usual transpose of an \(n \times n\) matrix. Assuming in addition that \(n\) is even, let \(J_n^-\) be the \(n \times n\) matrix with \((i, j)\) entry equal to \(\delta_{i,-j}\) if \(j > 0\) and \(-\delta_{i,-j}\) if \(j < 0\), and set

\[ \mathfrak{g}_n^- = \mathfrak{sp}_n(\mathbb{C}) = \{x \in \mathfrak{g}_n \mid x^T J_n^- + J_n^- x = 0\}. \]

We adopt the following conventions regarding signs. For \(i \in I_n\), define \(\hat{i} \in \mathbb{Z}/2\) by

\[ \hat{i} = \begin{cases} 0 & \text{if } i \geq 0; \\ 1 & \text{if } i < 0. \end{cases} \]

(1.1)
We will often identify a sign $\epsilon = \pm$ with the integer $\pm 1$ when writing formulae. For example, $\epsilon^i$ denotes 1 if $\epsilon = +$ or $i = 0$, and it denotes $-1$ if $\epsilon = -$ and $i = 1$. With this notation, $g_n^\phi$ is spanned by the matrices \{ $e_{i,j} - \epsilon^{i+j}e_{-j,-i}$ \mid \ j, i \in \mathbb{I}_n \}.

For the remainder of the article, we fix integers $n,l \geq 1$ and signs $\epsilon, \phi \in \{ \pm \}$, assuming that $\phi = \epsilon$ if $l$ is odd, $\phi = -\epsilon$ if $l$ is even, and $\phi = +$ if $n$ is odd. We will show that the finite $W$-algebra $W_{n,l}^\epsilon$ constructed from a nilpotent matrix of Jordan type $(l^n)$ in the Lie algebra $g_{nl}^\phi$ is the level $l$ quotient of the twisted Yangian $Y_n^\phi$ associated to the Lie algebra $g_n^\phi$.

First consider the finite $W$-algebra side. Let $g = g_{nl}^\epsilon$ and $f_{a,b} = e_{a,b} - \epsilon^{a+b}e_{-b,-a}$, so $g$ is spanned by the matrices \{ $f_{a,b} \mid a, b \in \mathbb{I}_{nl}$ \}. Up to isomorphism, the finite $W$-algebra to be defined shortly only depends on $g$ and the Jordan type $(l^n)$. However we need to fix an explicit choice of coordinates so that we can be absolutely explicit about the isomorphism in the main theorem below. We do this by introducing an $n \times l$ rectangular array of boxes, labeling rows in order from top to bottom by the index set $\mathbb{I}_n$ and columns in order from left to right by the index set $\mathbb{I}_l$. Also label the individual boxes in the array with the elements of the set $\mathbb{I}_{nl}$. For $a \in \mathbb{I}_{nl}$ we let row($a$) and col($a$) denote the row and column numbers of the box in which $a$ appears. We require that the boxes are labeled skew-symmetrically in the sense that row($a$) = row($b$) and col($b$) = col($a$). If $\epsilon = -$ we require in addition that $a > 0$ either if col($a$) $> 0$ or if col($a$) = 0 and row($a$) $> 0$; this additional restriction streamlines some of the signs appearing in formulae below, notably (1.11). For example, if $n = 3, l = 2$ and $\epsilon = -, \phi = +$, one could pick the labeling

\[
\begin{array}{ccc}
-5 & 1 \\
-3 & 3 \\
-1 & 5 \\
\end{array}
\]

and get that row(1) = $-2$ and col(1) = 1. We remark that the above arrays are a special case of the pyramids introduced by Elashvili and Kac in [EK]; see also [BG].

Having made these choices, we let $e \in g$ denote the following nilpotent matrix of Jordan type $(l^n)$:

\[
e = \sum_{a, b \in \mathbb{I}_{nl}} f_{a, b} + \sum_{\substack{a, b \in \mathbb{I}_{nl} \\ \text{row}(a) = \text{row}(b) \\ \text{col}(a) + 2 = \text{col}(b) \geq 2}} f_{a, b} + \sum_{\substack{a, b \in \mathbb{I}_{nl} \\ \text{row}(a) = \text{row}(b) > 0 \\ \text{col}(a) + 2 = \text{col}(b) = 1}} f_{a, b} + \sum_{\substack{a, b \in \mathbb{I}_{nl} \\ \text{row}(a) = \text{row}(b) = 0 \\ \text{col}(a) + 2 = \text{col}(b) = 1}} \frac{1}{2} f_{a, b}.
\]

In the above example, $e = f_{-1, 5} + \frac{1}{2} f_{-3, 3} = e_{-1, 5} + e_{-5, 1} + e_{-3, 3}$. Also define an even grading

\[
g = \bigoplus_{r \in \mathbb{Z}} g(r)
\]

with $e \in g(2)$ by declaring that deg($f_{a, b}$) = col($b$) - col($a$). Note this grading coincides with the grading obtained by embedding $e$ into an $\mathfrak{sl}_2$-triple $(e, h, f)$ and considering the ad $h$-eigenspace decomposition of $g$. Let $p = \bigoplus_{r \geq 0} g(r)$ and $m = \bigoplus_{r < 0} g(r)$. Define $\chi : m \to \mathbb{C}$ by $x \mapsto \frac{1}{2} \text{tr}(ex)$. An explicit calculation using the formula for the nilpotent matrix $e$ recorded above shows that

\[
\chi(f_{a, b}) = -\epsilon^{a+b}\chi(f_{-b,-a}) = 1
\]
if \( \text{row}(a) = \text{row}(b), \text{col}(a) = \text{col}(b) + 2 \) and either \( \text{col}(a) \geq 2 \) or \( \text{col}(a) = 1, \text{row}(a) \geq 0 \); all other \( f_{a,b} \in \mathfrak{m} \) satisfy \( \chi(f_{a,b}) = 0 \). Let \( I \) be the left ideal of the universal enveloping algebra \( U(\mathfrak{g}) \) generated by the elements \{\( x - \chi(x) \mid x \in \mathfrak{m} \)\}. By the PBW theorem, we have that

\[
U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I.
\]

Define \( \text{pr} : U(\mathfrak{g}) \to U(\mathfrak{p}) \) to be the projection along this direct sum decomposition. Finally the finite \( W \)-algebra associated to \( e \) is the subalgebra

\[
W^{e}_{n,l} = \{ u \in U(\mathfrak{p}) \mid \text{pr}(\{ x, u \}) = 0 \text{ for all } x \in \mathfrak{m} \}.
\]

We refer the reader to the introduction of [BK2], where the relationship between this definition (which is essentially the setup of [Ly]) and the more general setup of [PL, GG] is explained in detail.

To make the connection between \( W^{e}_{n,l} \) and the twisted Yangians, we exploit a shifted version of the Miura transform, which we define as follows. Let \( \mathfrak{h} = \mathfrak{g}(0) \) be the Levi factor of \( \mathfrak{p} \) coming from the grading. It is helpful to bear in mind that there is an isomorphism

\[
\mathfrak{h} \cong \begin{cases} 
\mathfrak{g}^0_n & \text{if } l = 2m; \\
\mathfrak{g}^e_n \oplus \mathfrak{g}^0_n & \text{if } l = 2m + 1.
\end{cases}
\]

Although we never need this explicitly, we note for completeness that this isomorphism maps \( f_{a,b} \in \mathfrak{h} \) to \( f_{\text{row}(a),\text{row}(b)} \in \mathfrak{g}^e_n \) if \( \text{col}(a) = \text{col}(b) = 0 \) or to \( e_{\text{row}(a),\text{row}(b)} \) in the \( \left\lfloor \frac{\text{col}(a)}{2} \right\rfloor \)th copy of \( \mathfrak{g}_n \) if \( \text{col}(a) = \text{col}(b) > 0 \). For \( q \in \mathcal{I}_l \), let

\[
\rho_q = \begin{cases} 
(nq - \epsilon)/2 & \text{if } q > 0; \\
(nq + \epsilon)/2 & \text{if } q < 0; \\
0 & \text{if } q = 0.
\end{cases}
\]

Let \( \eta \) be the automorphism of \( U(\mathfrak{h}) \) defined on generators by \( \eta(f_{a,b}) = f_{a,b} - \delta_{a,b}\rho_{\text{col}(a)} \). Let \( \xi : U(\mathfrak{p}) \to U(\mathfrak{h}) \) be the algebra homomorphism induced by the natural projection \( \mathfrak{p} \to \mathfrak{h} \). The Miura transform \( \mu : U(\mathfrak{p}) \to U(\mathfrak{h}) \) is the composite map

\[
\mu = \eta \circ \xi.
\]

By [Ly] §2.3 (or Theorem 3.4 below) the restriction of \( \mu \) to \( W^{e}_{n,l} \) is injective.

Now we turn our attention to the twisted Yangian \( Y_n^{\phi} \), recalling that \( \phi = -\epsilon \) if \( l \) is even and \( \phi = \epsilon \) if \( l \) is odd. By definition, \( Y_n^{\phi} \) is a subalgebra of the Yangian \( Y_n \). The latter is a certain Hopf algebra over \( \mathbb{C} \) with countably many generators \( \{ T^{(r)}_{i,j} \mid i, j \in \mathcal{I}_n, r \in \mathbb{Z}_{>0} \} \); see e.g. [MNO, §1] for the precise relations. Letting

\[
T_{i,j}(u) = \sum_{r \geq 0} T^{(r)}_{i,j} u^{-r} \in Y_n[[u^{-1}]]
\]

where \( T^{(0)}_{i,j} = \delta_{i,j} \), the comultiplication \( \Delta : Y_n \to Y_n \otimes Y_n \) is defined by the formula

\[
\Delta(T_{i,j}(u)) = \sum_{k \in \mathcal{I}_n} T_{i,k}(u) \otimes T_{k,j}(u).
\]

This and subsequent formulae involving generating functions should be interpreted by equating coefficients of the indeterminate \( u \) on both sides of equations, as discussed in
detail in [MNO] §1. By [MNO] §3.4, there exists an automorphism \( \tau : Y_n \to Y_n \) of order 2 defined by
\[
\tau(T_{i,j}(u)) = \phi^{j+i}T_{-j,-i}(-u).
\]
We define the twisted Yangian \( Y_n^\phi \) to be the subalgebra of \( Y_n \) generated by the elements \( \{ S^{(r)}_{i,j} \mid i, j \in \mathcal{I}_n, r \in \mathbb{Z}_{>0} \} \) coming from the expansion
\[
S_{i,j}(u) = \sum_{r \geq 0} S^{(r)}_{i,j} u^{-r} = \sum_{k \in \mathbb{Z}_n} \tau(T_{i,k}(u))T_{k,j}(u) \in Y_n[[u^{-1}]].
\] (1.8)
This is not the same embedding of \( Y_n^\phi \) into \( Y_n \) as used in [MNO] §3; we have twisted the embedding there by the automorphism \( \tau \). Because of this and the fact that \( \tau \) is a coalgebra antiautomorphism of \( Y_n \), we get from [MNO] §4.17 that the restriction of \( \Delta \) to \( Y_n^\phi \) has image contained in \( Y_n^\phi \otimes Y_n^\phi \) and
\[
\Delta(S_{i,j}(u)) = \sum_{h,k \in \mathbb{Z}_n} S_{h,k}(u) \otimes \tau(T_{i,k}(u))T_{h,j}(u). \] (1.9)

We let \( \Delta^{(m)} : Y_n \to Y_n^{\otimes (m+1)} \) denote the \( m \)th iterated comultiplication. The preceding formula shows that it maps \( Y_n^\phi \) into \( Y_n^\phi \otimes Y_n^{\otimes m} \).

By [MNO] §1.16 there is an evaluation homomorphism \( Y_n \to U(\mathfrak{g}_n) \). In view of this and (1.4), we obtain for every \( 0 < p \in \mathcal{I}_l \) a homomorphism
\[
ev_p : Y_n \to U(\mathfrak{h}), \quad T_{i,j}(u) \mapsto \delta_{i,j} + u^{-1} f_{a,b},
\] (1.10)
where \( a, b \in \mathcal{I}_nl \) are defined from \( \text{row}(a) = i, \text{row}(b) = j \) and \( \text{col}(a) = \text{col}(b) = p \). The image of this map is contained in the subalgebra of \( U(\mathfrak{h}) \) generated by the \( [p/2] \)th copy of \( \mathfrak{g}_n \) from the decomposition (1.4). There is also an evaluation homomorphism \( Y_n^\phi \to U(\mathfrak{g}_n^\phi) \) defined in [MNO] §3.11. If we assume that \( l \) is odd (so \( \epsilon = \phi \)), we can therefore define another homomorphism
\[
ev_0 : Y_n^\phi \to U(\mathfrak{h}), \quad S_{i,j}(u) \mapsto \delta_{i,j} + (u + \frac{\phi}{2})^{-1} f_{a,b},
\] (1.11)
where \( \text{row}(a) = i, \text{row}(b) = j \) and \( \text{col}(a) = \text{col}(b) = 0 \); if \( \epsilon = - \) this depends on our convention for labeling boxes as specified above. The image of this map is contained in the subalgebra of \( U(\mathfrak{h}) \) generated by the subalgebra \( \mathfrak{g}_n^\epsilon \) in the decomposition (1.4).

Putting all these things together, we deduce that there is a homomorphism
\[
\kappa_l : Y_n^\phi \to U(\mathfrak{h})
\]
defined by
\[
\kappa_l = \begin{cases} 
\ev_1 \otimes \ev_3 \otimes \cdots \otimes \ev_{l-1} \circ \Delta^{(m)} & \text{if } l = 2m + 2; \\
\ev_0 \otimes \ev_2 \otimes \cdots \otimes \ev_{l-1} \circ \Delta^{(m)} & \text{if } l = 2m + 1,
\end{cases}
\] (1.12)
where \( \otimes \) indicates composition with the natural multiplication in \( U(\mathfrak{h}) \). We define the twisted Yangian of level \( l \) to be the image of this map. Now we are ready to state the main theorem of the article.

**Theorem 1.1.** \( \mu(W_{n,l}) = \kappa_l(Y_n^\phi) \).
We will show moreover that the kernel of \( \kappa_l \) is generated by the elements

\[
\left\{ \begin{array}{l}
S_{i,j}^{(r)} & i, j \in \mathcal{I}_n, r > l \\
S_{i,j}^{(r)} + \frac{\phi}{2} S_{i,j}^{(r-1)} & i, j \in \mathcal{I}_n, r > l
\end{array} \right\}
\] if \( l \) is even;

\[
\left\{ \begin{array}{l}
S_{i,j}^{(r)} & i, j \in \mathcal{I}_n, r > l
\end{array} \right\}
\] if \( l \) is odd.

(1.13)

Since \( W_{n,l}^\epsilon \cong \mu(W_{n,l}^\epsilon) \) by injectivity of the Miura transform, and a full set of relations between the generators \( S_{i,j}^{(r)} \) of \( Y_n^\phi \) are known by [MNO, §3.8], this means that we have found a full set of generators and relations for the finite \( W \)-algebra \( W_{n,l}^\epsilon \).

The key step in our proof of Theorem 1.1 is a remarkable explicit formula for the generators of \( W_{n,l}^\epsilon \) corresponding to the elements \( S_{i,j}^{(r)} \) of \( Y_n^\phi \). In the remainder of the introduction, we want to explain this formula. Given \( i, j \in \mathcal{I}_n \) and \( p, q \in \mathcal{I}_l \), let \( a, b \) be the elements of \( \mathcal{I}_n \) such that \( \text{col}(a) = p \), \( \text{col}(b) = q \), \( \text{row}(a) = i \), and \( \text{row}(b) = j \).

Define a linear map \( s_{i,j} : g_l \to g \) by setting

\[
s_{i,j}(e_{p,q}) = \phi^{i+j} f_{a,b}.
\]

(1.14)

Let \( M_n \) denote the algebra of \( n \times n \) matrices over \( \mathbb{C} \), with rows and columns labeled by the index set \( \mathcal{I}_n \) as usual, and let \( T(g_l) \) be the tensor algebra on the vector space \( g_l \). Let

\[
s : T(g_l) \to M_n \otimes U(g)
\]

(1.15)

be the algebra homomorphism that maps a generator \( x \in g_l \) to \( \sum_{i,j \in \mathcal{I}_n} e_{i,j} \otimes s_{i,j}(x) \).

This in turn defines linear maps

\[
s_{i,j} : T(g_l) \to U(g),
\]

such that

\[
s(x) = \sum_{i,j \in \mathcal{I}_n} e_{i,j} \otimes s_{i,j}(x)
\]

for every \( x \in T(g_l) \). Note for any \( x, y \in T(g_l) \) that

\[
s_{i,j}(xy) = \sum_{k \in \mathcal{I}_n} s_{i,k}(x)s_{k,j}(y)
\]

(1.16)

and also \( s_{i,j}(1) = \delta_{i,j} \).

If \( A \) is an \( l \times l \) matrix with entries in some ring, we define its row determinant \( \text{rdet} A \) to be the usual Laplace expansion of determinant, but keeping the (not necessarily commuting) monomials that arise in row order; see e.g. [BK2, (12.5)]. For \( q \in \mathcal{I}_l \) and an indeterminate \( u \), let

\[
u_q = u + e_{q,q} + \rho_q \in T(g_l)[u],
\]

recalling the definition of \( \rho_q \) from (1.5). Define \( \Omega(u) \) to be the \( l \times l \) matrix with entries in \( T(g_l)[u] \) whose \((p,q)\) entry for \( p,q \in \mathcal{I}_l \) is equal to

\[
\Omega(u)_{p,q} = \begin{cases} 
  e_{p,q} & \text{if } p < q; \\
  u_q & \text{if } p = q; \\
  -1 & \text{if } p = q + 2 < 0; \\
  -\phi & \text{if } p = q + 2 = 0; \\
  1 & \text{if } p = q + 2 > 0; \\
  0 & \text{if } p > q + 2.
\end{cases}
\]

(1.17)
For example, if \( l = 4 \) then
\[
\Omega(u) = \begin{pmatrix}
  u_{-3} & e_{-3, -1} & e_{-3, 1} & e_{-3, 3} \\
  -1 & u_{-1} & e_{-1, 1} & e_{-1, 3} \\
  0 & 1 & u_1 & e_{1, 3} \\
  0 & 0 & 1 & u_3
\end{pmatrix}.
\]
If \( l \) is odd we also need the \( l \times l \) matrix \( \Omega(u) \) defined by
\[
\Omega(u)_{p,q} = \begin{cases}
  \Omega(u)_{p,q} & \text{if } p \neq 0 \text{ or } q \neq 0; \\
  e_{0,0} & \text{if } p = q = 0.
\end{cases}
\]
For example, if \( l = 5 \) then
\[
\Omega(u) = \begin{pmatrix}
  u_{-4} & e_{-4, -2} & e_{-4, 0} & e_{-4, 2} & e_{-4, 4} \\
  -1 & u_{-2} & e_{-2, 0} & e_{-2, 2} & e_{-2, 4} \\
  0 & -\phi & u_0 & e_{0, 2} & e_{0, 4} \\
  0 & 0 & 1 & u_2 & e_{2, 4} \\
  0 & 0 & 0 & 1 & u_4
\end{pmatrix},
\]
\[
\bar{\Omega}(u) = \begin{pmatrix}
  u_{-4} & e_{-4, -2} & e_{-4, 0} & e_{-4, 2} & e_{-4, 4} \\
  -1 & u_{-2} & e_{-2, 0} & e_{-2, 2} & e_{-2, 4} \\
  0 & -\bar{\phi} & u_0 & e_{0, 2} & e_{0, 4} \\
  0 & 0 & 1 & u_2 & e_{2, 4} \\
  0 & 0 & 0 & 1 & u_4
\end{pmatrix}.
\]

Then we let
\[
\omega(u) = \sum_{r=-\infty}^{l} \omega_{l-r} u^r = \begin{cases}
  \text{rdet } \Omega(u) & \text{if } l \text{ is even;} \\
  \text{rdet } \Omega(u) + \sum_{r=1}^{\infty} (-2\phi u)^{-r} \text{rdet } \bar{\Omega}(u) & \text{if } l \text{ is odd.}
\end{cases}
\]
This defines elements \( \omega_i \in T(\mathfrak{g}_l) \), hence elements \( s_{i,j}(\omega_r) \in U(\mathfrak{g}) \) for \( i, j \in \mathcal{I}_n \) and \( r \geq 1 \). It is obvious from the definition that each \( s_{i,j}(\omega_r) \) actually belongs to \( U(\mathfrak{p}) \).

**Theorem 1.2.** The elements \( \{ s_{i,j}(\omega_r) \mid i, j \in \mathcal{I}_n, r \geq 1 \} \) generate the subalgebra \( W_{n,l}^\epsilon \).
Moreover, \( \mu(s_{i,j}(\omega_r)) = \kappa_l(S^{(r)}_{i,j}) \).

The hardest part of the proof is to show that each \( s_{i,j}(\omega_r) \) belongs to \( W_{n,l}^\epsilon \). This is established by a lengthy calculation which we postpone until §4. In §2 we study the twisted Yangian of level \( l \), in particular proving a PBW theorem for this algebra and computing the kernel of \( \kappa_l \) as mentioned above. We also check that \( \mu(s_{i,j}(\omega_r)) = \kappa_l(S^{(r)}_{i,j}) \). Then in §3 we complete the proofs of Theorems 1.1 and 1.2. At the same time we obtain a direct proof of the injectivity of the Miura transform in this case.

In subsequent work, we will combine the results of this article with work of Molev [M] to deduce the classification of finite dimension irreducible representations of the finite \( W \)-algebras \( W_{n,l}^\epsilon \); we expect this will allow us to verify [BGK] Conjecture 5.2] in this case. It seems possible that the connection to finite \( W \)-algebras could also be used to derive explicit character formulae for the finite dimensional irreducible representations of twisted Yangians, as was done in type A in [BK3, §8].
Acknowledgments. The author would like to thank Jonathan Brundan for suggesting this problem and for his generous advice while writing this article, and Alexander Molev for some helpful comments.

2. The twisted Yangian of level \( l \)

Continuing with notation from the introduction, we begin this section by giving a different description of the map \( \kappa_l : Y^\phi_n \to U(\mathfrak{h}) \) from (1.12). Let

\[
T(u) = \sum_{i,j \in I_n} e_{i,j} \otimes T_{i,j}(u) \in M_n \otimes Y_n[[u^{-1}]],
\]

\[
S(u) = \sum_{i,j \in I_n} e_{i,j} \otimes S_{i,j}(u) \in M_n \otimes Y^\phi_n[[u^{-1}]].
\]

For a linear map \( f : V \to W \), we use the same notation \( f \) for the induced map \( \text{id} \otimes f : M_n \otimes V \to M_n \otimes W \). Thinking of elements of \( M_n \otimes V \) (resp. \( M_n \otimes W \)) as \( n \times n \) matrices with entries in \( V \) (resp. \( W \)), this is just the linear map obtained by applying \( f \) simultaneously to all matrix entries. We extend (1.10) by defining a homomorphism \( \text{ev}_p : Y_n \to U(\mathfrak{h}) \) for \( 0 < p \in I_l \) by setting

\[
\text{ev}_p = \text{ev}_p \circ \tau.
\]

(2.1)

Since the images of \( \text{ev}_p \) and \( \text{ev}_q \) commute for \( p \neq q \), it is then the case by (1.12), (1.7), (1.8), and (1.9) that

\[
\kappa_l(S(u)) =
\begin{cases}
\text{ev}_{1-l}(T(u)) \cdots \text{ev}_{1}(T(u)) \text{ev}_{1}(T(u)) \cdots \text{ev}_{l-1}(T(u)) & \text{if } l \text{ is even}; \\
\text{ev}_{1-l}(T(u)) \cdots \text{ev}_{2}(T(u)) \text{ev}_{0}(S(u)) \text{ev}_{2}(T(u)) \cdots \text{ev}_{l-1}(T(u)) & \text{if } l \text{ is odd},
\end{cases}
\]

(2.2)

where the product on the right hand side is in the algebra \( M_n \otimes U(\mathfrak{h})[[u^{-1}]] \).

For any \( 0 \neq p \in I_l \), (2.1), (1.10), and the labeling convention for boxes implies that

\[
\text{ev}_p(T_{i,j}(u)) = \delta_{i,j} + u^{-1} \phi^{i+j} f_{a,b},
\]

where \( a, b \in I_{nl} \) satisfy \( \text{row}(a) = i, \text{row}(b) = j \) and \( \text{col}(a) = \text{col}(b) = p \). Hence in the notation (1.14) we have that

\[
\text{ev}_p(T_{i,j}(u)) = \delta_{i,j} + u^{-1} s_{i,j}(e_{p,p}).
\]

Also (1.11) is equivalent to

\[
\text{ev}_0(S_{i,j}(u)) = \delta_{i,j} + (u + \frac{\phi}{2})^{-1} s_{i,j}(e_{0,0}) = \delta_{i,j} + \sum_{r=0}^{\infty} (-2\phi)^{-r} u^{-1-r} s_{i,j}(e_{0,0}).
\]

Using the more sophisticated notation (1.15), we deduce that

\[
u \text{ev}_p(T(u)) = s(u + e_{p,p}),
\]

\[
u \text{ev}_0(S(u)) = s(u + e_{0,0}) + \sum_{r=1}^{\infty} (-2\phi u)^{-r} s(e_{0,0}).
\]

Hence (2.2) is equivalent to the equation

\[
u^l \kappa_l(S(u)) = s((u + e_{1-l,1-l}) \cdots (u + e_{-1,-1})(u + e_{1,1}) \cdots (u + e_{l-1,l-1}))
\]

(2.3)
Theorem 2.1. \( u^l \kappa_l(S(u)) = \mu(s(\omega(u))) \).

Proof. The Miura transform (1.6) satisfies \( \mu(s(u_p)) = s(u + e_{p,p}) \) and \( \mu(s(e_{p,q})) = 0 \) if \( p < q \). So recalling the matrices \( \Omega(u) \) and \( \bar{\Omega}(u) \) from (1.17) and (1.18) we get that

\[
\mu(s(\text{rdet} \, \Omega(u))) = s((u + e_{1-l,1-l}) \cdots (u + e_{1,1-l})),
\]

and

\[
\mu(s(\text{rdet} \, \bar{\Omega}(u))) = s((u + e_{1-l,1-l}) \cdots (u + e_{-2,-2}) e_{0,0} (u + e_{2,2}) \cdots (u + e_{l-1,l-1})).
\]

The theorem follows on comparing (1.19), (2.3) and (2.4). \( \Box \)

The goal now is to prove a PBW theorem for the twisted Yangian of level \( l \), \( \kappa_l(Y_n^\phi) \). We will need the following elementary lemma, which is established in the proof of [BK1] Theorem 3.1.

Lemma 2.2. Let \( X \) be the variety of tuples \( (A_1, A_3, \ldots, A_{l-1}) \) of \( n \times n \) matrices. Let \( x_{i,j}^{[r]} \in \mathbb{C}[X] \) be the coordinate function picking out the \( (i,j) \) entry of \( A_r \). Let \( Y \) be the variety of tuples \( (B_1, \ldots, B_l) \) of \( n \times n \) matrices. Let \( y_{i,j}^{[r]} \in \mathbb{C}[Y] \) be the coordinate function picking out the \( (i,j) \) entry of \( B_r \). Define

\[
\theta : X \to Y, \quad (A_1, \ldots, A_{l-1}) \mapsto (B_1, \ldots, B_l)
\]

where

\[
B_r = \sum_{p_1, \ldots, p_r \in \mathbb{N}} \sum_{p_1 < \cdots < p_r} A_{p_1} A_{p_2} \cdots A_{p_r},
\]

that is, \( B_r \) is the \( r \)th elementary symmetric function in the matrices \( (A_1, \ldots, A_{l-1}) \). Then the comorphism \( \theta^* : \mathbb{C}[Y] \to \mathbb{C}[X] \) satisfies

\[
\theta^*(y_{i,j}^{[r]}) = \sum_{i_1, \ldots, i_r \in \mathbb{N}} x_{i_1,j_1}^{[p_1]} x_{i_2,j_2}^{[p_2]} \cdots x_{i_r,j_r}^{[p_r]}
\]

Moreover the derivative \( d\theta_x : T_x(X) \to T_{\theta(x)}(Y) \) is an isomorphism for any point \( x = (c_{l-1} I_n, \ldots, c_{l-1} I_n) \) such that \( c_{l-1}, \ldots, c_{l-1} \) are pairwise distinct scalars.
We observe by (2.5) for \( i, j \in \mathcal{I}_n \) that

\[
\begin{align*}
\kappa_l(S_{i,j}^{(r)}) &= 0 & \text{if } l \text{ is even and } r > l; \\
\kappa_l(S_{i,j}^{(r)}) &= -\frac{1}{2} \kappa_l(S_{i,j}^{(r-1)}) & \text{if } l \text{ is odd and } r > l.
\end{align*}
\]

(2.6)

Following \cite{MNO} §3.14, we say \((i, j, r)\) is admissible if \( i, j \in \mathcal{I}_n, 1 \leq r \leq l \), and

\[
\begin{align*}
i + j &\leq 0 \text{ if } \phi = + \text{ and } r \text{ is even}; \\
i + j &< 0 \text{ if } \phi = + \text{ and } r \text{ is odd}; \\
i + j &< 0 \text{ if } \phi = - \text{ and } r \text{ is even}; \\
i + j &\leq 0 \text{ if } \phi = - \text{ and } r \text{ is odd}.
\end{align*}
\]

Now consider the standard filtration on \( U(\mathfrak{h}) \) defined by declaring that each \( x \in \mathfrak{h} \) is in degree 1. This induces a filtration on the subalgebra \( \kappa_l(Y_n^{\phi}) \) so that \( \text{gr } \kappa_l(Y_n^{\phi}) \) is a subalgebra of \( \text{gr } U(\mathfrak{h}) \). Note by (2.5) that each \( \kappa_l(S_{i,j}^{(r)}) \) belongs to the filtered degree \( r \) component of \( U(\mathfrak{h}) \).

**Theorem 2.3.** The elements \( \{ \text{gr}_r \kappa_l(S_{i,j}^{(r)}) \mid (i, j, r) \text{ is admissible} \} \) are algebraically independent generators for the commutative algebra \( \text{gr } \kappa_l(Y_n^{\phi}) \). Hence the monomials in the elements \( \{ \kappa_l(S_{i,j}^{(r)}) \mid (i, j, r) \text{ is admissible} \} \) taken in some fixed order form a basis for \( \kappa_l(Y_n^{\phi}) \).

**Proof.** As in \cite{MNO} (3.6.4)], we have for all \( i, j \in \mathcal{I}_n \) the following relation in \( Y_n^{\phi}[[u^{-1}]] \):

\[
\phi^{i+j} S_{-j,-i}(-u) = S_{i,j}(u) + \phi S_{i,j}(u) - S_{i,j}(-u).
\]

(2.7)

By (2.6) and (2.7) monomials in the elements \( \{ \text{gr}_r \kappa_l(S_{i,j}^{(r)}) \mid (i, j, r) \text{ is admissible} \} \) taken in some fixed order generate \( \text{gr } \kappa_l(Y_n^{\phi}) \), so it suffices to prove they are algebraically independent. Let notation be as in Lemma 2.2. Let \( V \) be the closed subspace of \( X \) defined by the ideal \( I \) generated by

\[
\left\{ x_{i,j}^{[r]} + \phi^{i+j} x_{-j,-i}^{[-r]} \mid i, j \in \mathcal{I}_n, r \in \mathcal{I}_l \right\}.
\]

As \( \mathfrak{h} \) is the vector space spanned by \( \{ s_{i,j}(e_{p,p}) \mid i, j \in \mathcal{I}_n, p \in \mathcal{I}_l \} \) subject only to the relations \( s_{i,j}(e_{p,p}) = -\phi^{i+j} s_{-j,-i}(e_{-p,-p}) \), we can identify \( \text{gr } U(\mathfrak{h}) \) with \( \mathbb{C}[V] \), by declaring that \( \text{gr}_1 s_{i,j}(e_{p,p}) = x_{i,j}^{[p]} + I \).

Let \( W \) be the closed subspace of \( Y \) defined by the ideal \( J \) generated by

\[
\left\{ y_{i,j}^{[r]} - (-1)^r \phi^{i+j} y_{-j,-i}^{[r]} \mid i, j \in \mathcal{I}_n, r = 1, \ldots, l \right\}.
\]
We claim that \( \theta(V) \subseteq W \), i.e. \( \theta^*(J) \subseteq I \). To see this note that
\[
\theta^*(y_{i,j}^{[r]}) = \sum_{i_1, \ldots, i_{r-1} \in I_n} x_{i_1}^{[p_1]} x_{i_2}^{[p_2]} \cdots x_{i_{r-1}}^{[p_{r-1}]} \cdot \theta^*(\sum_{i_{r-1} \in I_n} x_{i_{r-1}}^{[p_{r-1}]} \cdot \cdots \cdot x_{i_1}^{[p_1]}) \mod (I)
\]
\[
\equiv (-1)^{r} \phi^{i+j} \sum_{i_1, \ldots, i_{r-1} \in I_n} x_{-i_1}^{[p_{r-1}]} \cdots x_{-i_{r-1}}^{[p_1]} \mod (I)
\]
\[
\equiv (-1)^{r} \phi^{i+j} \theta^*(y_{i,j}^{[r]}) \mod (I).
\]
Hence \( \theta^*(y_{i,j}^{[r]}) - (-1)^{r} \phi^{i+j} y_{i,j}^{[r]} \in I \).

Choose \( x = (c_{1-\ell} I_n, \ldots, c_{l-1} I_n) \in X \) so that \( c_{1-\ell}, \ldots, c_{l-1} \) are pairwise distinct and \( c_{\ell} + c_{-\ell} = 0 \). Then \( x \) belongs to \( V \). Now apply Lemma \( \ref{lemma2} \) to deduce that \( d\theta_x : T_x(V) \rightarrow T_{\theta(x)}(W) \) is injective. An easy calculation shows that \( \dim V = \dim W, \) hence \( d\theta_x : T_x(V) \rightarrow T_{\theta(x)}(W) \) is an isomorphism. By \( \ref{cor:4.3.6(i)} \) this implies that \( \theta^* : C[W] \rightarrow C[V] = \text{gr} U(\mathfrak{h}) \) is injective. As \( C[W] \) is freely generated by the elements \( \{ y_{i,j}^{[r]} \mid (i,j,r) \text{ is admissible} \} \), we deduce that the elements \( \{ \theta^*(y_{i,j}^{[r]}) \mid (i,j,r) \text{ is admissible} \} \) are algebraically independent too. It remains to observe by applying \( \text{gr}_r \) to \( \ref{equation2.5} \) and using \( \ref{lemma1.16} \) that
\[
\text{gr}_r \kappa_l(S_{i,j}^{(r)}) = \sum_{i_1, \ldots, i_{r-1} \in I_n} x_{i_1}^{[p_1]} x_{i_2}^{[p_2]} \cdots x_{i_{r-1}}^{[p_{r-1}]} = \theta^*(y_{i,j}^{[r]}).
\]

\( \square \)

**Corollary 2.4.** The elements
\[
\left\{ S_{i,j}^{(r)} \mid i,j \in I_n, r > \ell \right\} \quad \text{if } l \text{ is even;}
\]
\[
\left\{ S_{i,j}^{(r)} + \phi S_{i,j}^{(r-1)} \mid i,j \in I_n, r > \ell \right\} \quad \text{if } l \text{ is odd}
\]
generate the kernel of \( \kappa_l \).

**Proof.** Let \( I \) denote the two-sided ideal of \( Y_0^\phi \) generated by the elements listed in \( \ref{equation2.8} \). It is obvious that \( \kappa_l \) induces a map \( \tilde{\kappa}_l : Y_0^\phi / I \rightarrow \kappa_l(Y_0^\phi) \). Since \( Y_0^\phi / I \) is spanned by the set of all monomials in the elements \( \{ S_{i,j}^{(r)} + I \mid (i,j,r) \text{ is admissible} \} \) taken in some fixed order by \( \ref{MNO} \) \( \S 3.14 \), and the images of these monomials are linearly independent in \( \kappa_l(Y_0^\phi) \) by Theorem \( \ref{thm:2.3} \), we deduce that \( \tilde{\kappa}_l \) is an isomorphism. \( \square \)

We also obtain a new proof of the PBW theorem for twisted Yangians, different from the one in \( \ref{MNO} \) \( \S 3 \).

**Corollary 2.5.** The set of all monomials in the elements \( \{ S_{i,j}^{(r)} \mid (i,j,r) \text{ is admissible} \} \) taken in some fixed order forms a basis for \( Y_0^\phi \).

**Proof.** It is clear from \( \ref{equation2.7} \) that such monomials span \( Y_0^\phi \). The fact that they are linearly independent follows from Theorem \( \ref{thm:2.3} \) by taking sufficiently large \( \ell \). \( \square \)
3. The finite $W$-algebra

In §4 below we will prove the following theorem:

**Theorem 3.1.** For $i, j \in I_n$ and $r \geq 1$, the element $s_{i,j}^{(\omega_r)}$ belongs to $W_{n,l}^\varepsilon$.

In the remainder of this section we explain how to deduce the main results formulated in the introduction from this theorem.

The finite $W$-algebra $W_{n,l}^\varepsilon$ possesses two natural filtrations. The first of these, the *Kazhdan filtration*, is the filtration on $W_{n,l}^\varepsilon$ induced by the filtration on $U(g)$ generated by declaring that each element $x \in g(r)$ in the grading (1.2) is of degree $r/2 + 1$. The fundamental *PBW theorem* for finite $W$-algebras asserts that the associated graded algebra $\text{gr} W_{n,l}^\varepsilon$ under the Kazhdan filtration is isomorphic to the coordinate algebra of the Slodowy slice at $e$; see e.g. [GG, Theorem 4.1].

The second important filtration, called the *good filtration* in [BGK], is defined as follows. The grading (1.2) induces a non-negative grading on $U(p)$. Although $W_{n,l}^\varepsilon$ is not a graded subalgebra of $U(p)$, this grading on $U(p)$ still induces a filtration on $W_{n,l}^\varepsilon$ with respect to which the associated graded algebra $\text{gr} W_{n,l}^\varepsilon$ is naturally identified with a graded subalgebra of $U(p)$. The fundamental result about the good filtration, which is a consequence of the PBW theorem and [P2, (2.1.2)], is that

$$\text{gr} W_{n,l}^\varepsilon = U(e)$$

(3.1)

as subalgebras of $U(p)$, where $e$ denotes the centralizer of $e$ in $g$; see also [BGK, Theorem 3.5]. The element $s_{i,j}^{(\omega_{r+1})}$ belongs to the subspace of elements of degree $r$ in the good filtration, and we have that $s_{i,j}^{(\omega_{r+1})} \in W_{n,l}^\varepsilon$, by Theorem 3.1. So it makes sense to define

$$f_{i,j;r} = \text{gr} R s_{i,j}^{(\omega_{r+1})} \in U(e)$$

(3.2)

for $r \geq 0$. Explicitly, we have that

$$f_{i,j;r} = \sum_{p,q \in I_l \atop q-p = 2r} \alpha_{p,q} s_{i,j}(e_{p,q})$$

(3.3)

where

$$\alpha_{p,q} = \begin{cases} 1 & \text{if } q < 0; \\ \phi(-1)^{q/2} & \text{if } p < 0 \text{ and } q \geq 0 \text{ and } l \text{ is odd}; \\ (-1)^{(q+1)/2} & \text{if } p < 0 \text{ and } q > 0 \text{ and } l \text{ is even}; \\ (-1)^{(q-p)/2} & \text{if } p \geq 0. \end{cases}$$

This formula comes from the fact that the monomial $e_{p,q}$ where $q-p = 2r$ occurs in $\text{rdet} \Omega(u)$ as a coefficient of $u^{l-(r+1)}$ (and thus in $\omega_{r+1}$) because of the element $\sigma = (p, q, q-2, \ldots, p+2)$ in the symmetric group on $I_l$. Now $\alpha_{p,q} = \text{sgn}(\sigma) \ast N$, where $N$ is the number of $-1$'s strictly below and strictly to the left of $e_{p,q}$ in the matrix $\Omega(u)$.

So (3.3) shows that each $f_{i,j;r} \in U(e)$ is an element of $g$, hence belongs to $g_e$.

**Lemma 3.2.** The elements $\{f_{i,j;r} \mid (i,j,r+1) \text{ is admissible}\}$ form a basis for $g_e$.
Proof. We have already observed that each $f_{i,j,r}$ belongs to $\mathfrak{g}_e$. By \[J, \S3.2\], the dimension of $\mathfrak{g}_e$ is
\[
\begin{cases} 
  n^2l/2 & \text{if } l \text{ is even;} \\
  (n^2l - n\epsilon)/2 & \text{if } l \text{ is odd.}
\end{cases}
\]
An easy calculation shows that this is the same as the number of admissible triples. Now it just remains to show that the elements $f_{i,j,r}$ for all admissible $(i, j, r + 1)$ are linearly independent. This is easy to see on noting that all these elements are non-zero, which follows by computing some explicit matrix coefficients. \[\square\]

Theorem 3.3. The elements $\{s_{i,j}(\omega_r) \mid (i, j, r) \text{ is admissible}\}$ generate $W_n^\epsilon$. 

Proof. By (3.1), (3.2) and Lemma 3.2 the elements $\{\text{gr'}_r s_{i,j}(\omega_{r+1}) \mid (i, j, r + 1) \text{ is admissible}\}$ generate $\text{gr'}W_n^\epsilon$, the associated graded algebra in the good filtration. The theorem follows from this statement by induction on the filtration. \[\square\]

Theorems 1.1 and 1.2 from the introduction follow from Theorems 3.1, 3.3 and 2.1. Finally we include a proof of the following theorem, which is originally due to \[LY, \text{Corollary 2.3.2}\] in a more general setting.

Theorem 3.4. The Miura transform $\mu : W_n^\epsilon \rightarrow U(\mathfrak{h})$ from (1.6) is injective.

Proof. Note that $\mu$ is a filtered map with respect to the Kazhdan filtration on $W_n^\epsilon$ and the standard filtration on $U(\mathfrak{h})$. We actually show that the associated graded map $\text{gr} \mu : \text{gr} W_n^\epsilon \rightarrow \text{gr} U(\mathfrak{h})$ is injective, which implies the theorem. Each $s_{i,j}(\omega_r)$ is in degree $r$ under the Kazhdan filtration and $\kappa_l(S_{i,j})$ is in degree $r$ under the standard filtration on $U(\mathfrak{h})$. Moreover Theorem 2.1 shows that $\mu(s_{i,j}(\omega_r)) = \kappa_l(S_{i,j})$, hence $(\text{gr} \mu)(\text{gr} s_{i,j}(\omega_r)) = \text{gr} \kappa_l(S_{i,j})$. So by Theorem 2.3 and the PBW theorem for $W_n^\epsilon$ we deduce that $\text{gr} \mu : \text{gr} W_n^\epsilon \rightarrow \text{gr} U(\mathfrak{h})$ is injective. \[\square\]

4. Proof of invariance

In this section we prove Theorem 3.4. We need to show for $i, j \in \mathcal{I}_n$ and $r \geq 1$ that
\[
\text{pr}([x, s_{i,j}(\omega_r)]) = 0
\]
for all $x \in \mathfrak{m}$. Since $\mathfrak{m}$ is generated by the elements
\[
\{s_{i,j}(e_{q+2,q}) \mid i, j \in \mathcal{I}_n, q \in \mathcal{I}_n, -1 \leq q < l - 1\},
\]
we just need to consider the actions of these elements on each $s_{i,j}(\omega_r)$. Actually we work in terms of the generating series $s_{i,j}(\omega(u))$ from (1.19), and we use the natural extension of $\text{pr}$ to a homomorphism $\text{pr} : U(\mathfrak{g})[u] \rightarrow U(\mathfrak{p})[u]$. As the calculations are lengthy, we break them up into a series of lemmas. Throughout the section we will set \[
\tilde{i} = \begin{cases} 
  0 & \text{if } i \leq 0; \\
  1 & \text{if } i > 0.
\end{cases}
\]
Lemma 4.1. Let $y_1, \ldots , y_m \in g_l$. Let $i, j, h, k \in \mathcal{I}_n$. Let $p, q \in \mathcal{I}_l$. Then

$$[s_{i,j}(e_{p,q}), s_{h,k}(y_1 \otimes \cdots \otimes y_m)]$$

$$= \sum_{t=1}^{m} s_{h,j}(y_1 \otimes \cdots \otimes y_{t-1}) s_{i,k}(e_{p,q} y_t \otimes y_{t+1} \otimes \cdots \otimes y_m)$$

$$- \sum_{t=1}^{m} s_{h,j}(y_1 \otimes \cdots \otimes y_{t-1} \otimes y e_{p,q}) s_{i,k}(y_{t+1} \otimes \cdots \otimes y_m)$$

$$+ \gamma \left( - \sum_{t=1}^{m} s_{h,-i}(y_1 \otimes \cdots \otimes y_{t-1}) s_{-j,k}(e_{-q,-p} y_t \otimes y_{t+1} \otimes \cdots \otimes y_m) \\
+ \sum_{t=1}^{m} s_{h,-i}(y_1 \otimes \cdots \otimes y_{t-1} \otimes y e_{-q,-p}) s_{-j,k}(y_{t+1} \otimes \cdots \otimes y_m) \right)$$

where

$$\gamma = \begin{cases} 
\phi^{i\hat{p} + j\hat{q} + \hat{y} e_{\hat{p} + \hat{q}}} & \text{if } p, q \neq 0; \\
\phi^{\hat{q} + \hat{y} e_{\hat{p} + \hat{q}}} & \text{if } p = 0, q \neq 0; \\
\phi^{i\hat{p} + j\hat{q} e_{\hat{j}}} & \text{if } p \neq 0, q = 0; \\
\hat{c}^{i + j} & \text{if } p, q = 0,
\end{cases} \tag{4.3}$$

and $e_{p,q} y_t, y e_{p,q}, e_{-q,-p} y_t$, and $y e_{-q,-p}$ denote matrix multiplication in $M_l$.

Proof. First note that for $a, b, c, d \in \mathcal{I}_n$,

$$[f_{a,b}, f_{c,d}] = [e_{a,b} - e^{\hat{a} + h} e_{-b,-a}, e_{c,d} - e^{\hat{c} + d} e_{-d,-c}]$$

$$= \delta_{c,b} e_{a,d} - \delta_{b,-d} e^{\hat{a} + h} e_{a,-c} - \delta_{a,-c} e^{\hat{a} + d} e_{-b,d} + \delta_{a,d} e^{\hat{b} + \hat{c}} e_{-b,-c}$$

$$- \delta_{a,d} e^{\hat{a} + d} e_{-d,b} + \delta_{a,c} e^{\hat{a} + \hat{b}} e_{c,a} - \delta_{a,c} e^{\hat{a} + \hat{b}} e_{c,-a}$$

$$= \delta_{c,b} e_{a,d} - \delta_{a,d} e^{\hat{a} + h} e_{-a,f_{b,d}} + \delta_{-b,d} e_{f_{c,-a}}$$

Thus for $v, w \in \mathcal{I}_l$ and $a, b, c, d$ such that $row(a) = i, col(a) = p, row(b) = j, col(b) = q, row(c) = h, col(c) = v, row(d) = k, col(d) = w$ we have that

$$[s_{i,j}(e_{p,q}), s_{h,k}(e_{v,w})]$$

$$= [\phi^{i\hat{p} + j\hat{q} + \hat{y} w} f_{a,b}, \phi^{\hat{h} + \hat{v} + k\hat{w}} f_{c,d}]$$

$$= \phi^{i\hat{p} + j\hat{q} + \hat{h} + \hat{v} + \hat{w}} (\delta_{c,b} f_{a,d} - \delta_{a,d} f_{c,b} + \epsilon^{\hat{a} + \hat{b}} (-\delta_{a,-f_-d b} + \delta_{-b,d f_{c,-a}}))$$

$$= \phi^{i\hat{p} + j\hat{q} + \hat{h} + \hat{v} + \hat{w}} (\phi^{\hat{p} + \hat{q} + \hat{w}} s_{h,j}(1) s_{i,k}(e_{p,q} e_{v,w}) - \phi^{\hat{h} + \hat{v} + \hat{w}} s_{h,j}(e_{v,w} e_{p,q}) s_{i,k}(1))$$

$$+ \epsilon^{\hat{a} + \hat{b}} (\phi^{\hat{p} + \hat{q} + \hat{w}} s_{h,-i}(1) s_{-j,k}(e_{-q,-p} e_{v,w}) + \phi^{\hat{h} + \hat{v} + \hat{w}} s_{h,-i}(e_{v,w} e_{-q,-p}) s_{-j,k}(1))$$

$$= s_{h,j}(1) s_{i,k}(e_{p,q} e_{v,w}) - s_{h,j}(e_{v,w} e_{p,q}) s_{i,k}(1)$$

$$+ \gamma (-s_{h,-i}(1) s_{-j,k}(e_{-q,-p} e_{v,w}) + s_{h,-i}(e_{v,w} e_{-q,-p}) s_{-j,k}(1)),$$

on noting that the $\epsilon$ term in $\gamma$ equals $\epsilon^{\hat{a} + \hat{b}}$ due to the labeling convention specified in the introduction. Now the linearity of $s$ implies the lemma holds for $m = 1$ and any $y_1 \in g_l$, and the lemma follows from induction on $m$. \hfill \Box

For $p, q \in \mathcal{I}_l$, let $\Omega_{p,q}(u)$ and $\tilde{\Omega}_{p,q}(u)$ denote the square submatrices of $\Omega(u)$ and $\tilde{\Omega}(u)$, respectively, with rows and columns indexed by $\{p, p + 2, \ldots, q\}$. 

Lemma 4.2. For each $i, j \in \mathcal{I}_n$ and for $q \in \mathcal{I}_l$ such that $q \geq 0$,

\[
\begin{pmatrix}
\Pr \left[s_{i,j} \mid \text{rdet} \left(\begin{array}{cccc}
\varepsilon_{q+2,q} & \varepsilon_{q+2,q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
1 & \nu_{q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
0 & 1 & \nu_{q+4} & \cdots & \varepsilon_{q+4,l-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{l-1}
\end{array}\right) \right) \right) \\
= (u + \rho_{q+2} - n)s_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)) \\
= (u + \rho_{q+2} - n)s_{i,j}(\text{rdet} \tilde{\Omega}_{q+4,l-1}(u)).
\]  

Proof. By (1.3) for any $f, g \in \mathcal{I}_n$, \( \Pr(s_{f,g}(\varepsilon_{q+2,q})) = \delta_{f,g} = s_{f,g}(1) \). So

\[
\begin{pmatrix}
\Pr \left[s_{i,j} \mid \text{rdet} \left(\begin{array}{cccc}
\varepsilon_{q+2,q} & \varepsilon_{q+2,q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
1 & \nu_{q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
0 & 1 & \nu_{q+4} & \cdots & \varepsilon_{q+4,l-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{l-1}
\end{array}\right) \right) \\
= s_{i,j} \left[ \begin{array}{c}
1 \\
1 \\
0 \\
\vdots \\
0
\end{array} \right] \\
= s_{i,j} \left[ \begin{array}{c}
\varepsilon_{q+2,q+2} \\
\nu_{q+2} \\
\nu_{q+4} \\
\vdots \\
\nu_{l-1}
\end{array} \right]
\]  

\[
+ \sum_{m \in \mathcal{I}_n} \Pr([s_{i,m}(\varepsilon_{q+2,q}), s_{m,j}(\text{rdet} \Omega_{q+2,l-1}(u))]).
\]  

(4.4)

Since $\nu_{q+2} = \varepsilon_{q+2,q+2} + u + \rho_{q+2}$, doing the obvious row operation gives that

\[
\begin{pmatrix}
1 & \varepsilon_{q+2,q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
1 & \nu_{q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
0 & 1 & \nu_{q+4} & \cdots & \varepsilon_{q+4,l-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{l-1}
\end{pmatrix}
\]  

\[
= \text{rdet} \left(\begin{array}{cccc}
0 & -(u + \rho_{q+2}) & 0 & \cdots & 0 \\
1 & \nu_{q+2} & \varepsilon_{q+2,q+4} & \cdots & \varepsilon_{q+2,l-1} \\
0 & 1 & \nu_{q+4} & \cdots & \varepsilon_{q+4,l-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{l-1}
\end{array}\right) \\
= (u + \rho_{q+2}) \text{rdet} \Omega_{q+4,l-1}(u)
\]  

(4.5)

Next we apply Lemma (1.1) to get that

\[
[s_{i,m}(\varepsilon_{q+2,q}), s_{m,j}(\text{rdet} \Omega_{q+2,l-1}(u))] = -s_{m,m}(\varepsilon_{q+2,q})s_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)).
\]

By (1.3) \( \Pr(s_{m,m}(\varepsilon_{q+2,q})) = 1 \), so

\[
\Pr([s_{i,m}(\varepsilon_{q+2,q}), s_{m,j}(\text{rdet} \Omega_{q+2,l-1}(u))]) = -s_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)).
\]  

(4.6)
Combining (4.5) and (4.6) into (4.4) gives that

\[
\text{pr} \left( s_{i,j} \begin{pmatrix} \text{rdet} & & & & \\
1 & u_{q+2} & e_{q+2,q+2} & \cdots & e_{q+2,1-1} \\
0 & 1 & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u_{l-1} \\
\end{pmatrix} \right) 
= (u + \rho_{q+2})s_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)) - ns_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)) 
= (u + \rho_{q+2} - n)s_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)) 
= (u + \rho_{q+2} - n)s_{i,j}(\text{rdet} \Omega_{q+4,l-1}(u)) 
\text{since } \Omega_{q+4,l-1}(u) = \Omega_{q+4,l-1}(u) \text{ because } q \geq 0 \text{ by assumption.} \]

Lemma 4.3. For each \(i, j, h, k \in \mathcal{I}_n\), for \(q \in \mathcal{I}_l\) such that \(q > 0\), and for \(p \in \mathcal{I}_l\) such that \(-q < p < q\),

\[
\text{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet} \Omega_{p,l-1}(u))]) = 0,
\]

and

\[
\text{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet} \Omega_{p,l-1}(u))]) = 0.
\]

Proof. We shall prove the result for \(\Omega(u)\), but note that an identical proof holds for \(\bar{\Omega}(u)\). We compute using Lemma 4.1 to get that

\[
[s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet} \Omega_{p,l-1}(u)))] = A - B,
\]

where

\[
A = s_{h,j}(\text{rdet} \Omega_{p,q-2}(u))s_{i,k} \begin{pmatrix} \text{rdet} & & & & \\
1 & u_{q+2} & e_{q+2,q+2} & \cdots & e_{q+2,1-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u_{l-1} \\
\end{pmatrix},
\]

and

\[
B = s_{h,j} \begin{pmatrix} \text{rdet} & & & & \\
0 & 0 & 0 & \cdots & u_{q+2,q} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & u_{q} & e_{q,q} \\
0 & 0 & 1 & e_{q+2,q} \\
\end{pmatrix} s_{i,k}(\text{rdet} \Omega_{q+4,l-1}(u)).
\]

We will be more explicit how to calculate \(B\), \(A\) is computed in a similar manner. Let \(S(X)\) denote the symmetric group on a set \(X\). Let \(M = \Omega_{p,l-1}(u)\). By definition,

\[
\text{rdet } M = \sum_{\sigma \in S(p,p+2,\ldots,l-1)} M_{p,\sigma(p)}M_{p+2,\sigma(p+2)} \cdots M_{l-1,\sigma(l-1)}.
\]

All of the monomials in \(B\) come from the second sum in Lemma 4.1 (all the monomials \(A\) come from the first sum in Lemma 4.1 and the last two sums from that Lemma in the calculation of \([s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet} \Omega_{p,l-1}(u)))]\) are zero). Furthermore every term in this sum is zero except for those coming from \(s_{h,k}\) applied to monomials in \(\text{rdet } M\) which contain \(e_{v,q+2}\) for some \(v \in \mathcal{I}_l, v \leq q + 2\). Now since the only nonzero terms of \(M\) below the diagonal are scalars occurring immediately below the diagonal, if \(\sigma \in S(p,p+2,\ldots,l-1)\) contributes a nonzero term to the second sum of Lemma 4.1 then \(\sigma \in S(p,p+1,\ldots,q+2) \times S(q+4,q+6,\ldots,l-1)\). Thus the sum of the
terms of \([s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet } \Omega_{p,l-1}(u))]\) which come from the second sum in Lemma 4.2.

Since \(\rho_{q+2} - n = \rho_q\), by Lemma 4.2

\[ \text{pr}(A) = (u + \rho_q)s_{h,j}(\text{rdet } \Omega_{p,q-2}(u))s_{i,k}(\text{rdet } \Omega_{q+4,I-1}(u)). \]

By (1.3) for any \(f, g \in I_n, \text{pr}_x(s_{f,g}(e_{q+2,q})) = \delta_{f,g} = s_{f,g}(1)\). So the obvious column operation gives that

\[ \text{pr}(B) = s_{h,j}(\text{rdet } \Omega_{q+4,l-1}(u)) \]

\[ \begin{pmatrix}
  u_p & \ldots & e_{p,q} & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & u_q & -(u + \rho_q) \\
  0 & \ldots & 1 & 0
\end{pmatrix}
\]

\[ = (u + \rho_q)s_{h,j}(\text{rdet } \Omega_{p,q-2}(u))s_{i,k}(\text{rdet } \Omega_{q+4,I-1}(u)). \]

The lemma now follows. \(\square\)

**Lemma 4.4.** For each \(i, j, h, k \in I_n\) and for \(q \in I_l\) so that \(q > 0\),

\[ \text{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet } \Omega(u))]) = 0 \]

and

\[ \text{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet } \Omega(u))]) = 0. \]

**Proof.** We shall prove the result for \(\Omega(u)\), but note that an identical proof holds for \(\tilde{\Omega}(u)\). We compute using Lemma 4.1 to get that

\[ [s_{i,j}(e_{q+2,q}), s_{h,k}(\text{rdet } \Omega(u))] = A - B + \phi^3 + \frac{1}{3}(-C + D), \]

where

\[ A = s_{h,j}(\text{rdet } \Omega_{1-I,q-2}(u))s_{i,k}(\text{rdet } \Omega_{q+4,l-1}(u)) \]

\[ \begin{pmatrix}
  u_{1-l} & \ldots & e_{1-l,q} & e_{1-l,q} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & u_q & e_{q} \\
  0 & \ldots & 1 & e_{q+2,q}
\end{pmatrix}\]

\[ \begin{pmatrix}
  e_{q+2,q} & e_{q+2,q+2} & \ldots & e_{q+2,l-1} \\
  1 & u_{q+2} & \ldots & e_{q+2,l-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & u_{l-1}
\end{pmatrix}\]

\[ B = s_{h,j}(\text{rdet } \Omega_{q+4,l-1}(u))s_{i,k}(\text{rdet } \Omega_{1-l,q-2}(u)) \]

\[ \begin{pmatrix}
  e_{q-2, e_{q-2,q-2}} & \ldots & e_{q-2,l-1} \\
  -1 & u_{q-2} & \ldots & e_{q-2,l-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & u_{l-1}
\end{pmatrix}\]

\[ C = s_{h,-i}(\text{rdet } \Omega_{1-l,q-4}(u))s_{j,k}(\text{rdet } \Omega_{q+4,l-1}(u)) \]

\[ \begin{pmatrix}
  u_{1-l} & \ldots & e_{1-l,q-2} & e_{1-l,q-2} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & u_{q-2} & e_{q-2,q-2} \\
  0 & \ldots & -1 & e_{q-2,q-2}
\end{pmatrix}\]

and

\[ D = s_{h,-i}(\text{rdet } \Omega_{q+4,l-1}(u))s_{j,k}(\text{rdet } \Omega_{1-l,q-4}(u)) \]

By Lemma 4.2

\[ \text{pr}(A) = (u + \rho_q)s_{h,j}(\text{rdet } \Omega_{1-l,q-2}(u))s_{i,k}(\text{rdet } \Omega_{q+4,I-1}(u)). \]
The obvious column operation gives that
\[
\text{pr}(B) = s_{h,j} \begin{pmatrix}
u_{1-l} & \ldots & e_{1-l,q} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & u_q & -(u + \rho_q) \\
0 & \ldots & 1 & 0
\end{pmatrix} s_{i,k} \left( \det \Omega_{q+4,l-1}(u) \right)
\]
\[= (u + \rho_q)s_{h,j} \left( \det \Omega_{1-l,q-2}(u) \right) s_{i,k} \left( \det \Omega_{q+4,l-1}(u) \right).
\]
Hence \(\text{pr}(A - B) = 0\).

Since by (1.3) \(\text{pr}(s_{f,g}(e_{-q,-q-2})) = -\delta_{f,g} = s_{f,g}(-1)\) for any \(f, g \in \mathcal{I}_n\), we have that
\[
\text{pr}(C) = s_{h,-i} \left( \det \Omega_{1-l,-q+4}(u) \right) s_{-j,k} \begin{pmatrix}
0 & -e_{-q,-q} & \ldots & 0 \\
-1 & u_{-q} & \ldots & e_{-q,l-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & u_{l-1}
\end{pmatrix}
\]
\[= s_{h,-i} \left( \det \Omega_{1-l,-q+4}(u) \right) \text{pr}([s_{-j,m}(e_{-q,-q-2}), s_{m,k} \left( \det \Omega_{-q,l-1}(u) \right)]).
\]
(4.7)

The obvious row operation gives that
\[
s_{-j,k} \begin{pmatrix}
-1 & e_{-q,-q} & \ldots & e_{-q,l-1} \\
-1 & u_{-q} & \ldots & e_{-q,l-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & u_{l-1}
\end{pmatrix}
\]
\[= s_{-j,k} \begin{pmatrix}
0 & -(u + \rho_{-q}) & \ldots & 0 \\
-1 & u_{-q} & \ldots & e_{-q,l-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & u_{l-1}
\end{pmatrix}
\]
\[= -(u + \rho_{-q})s_{-j,k} \left( \det \Omega_{-q+2,l-1}(u) \right).
\]
(4.8)

Next we compute using Lemma 4.1 to get that
\[
[s_{-j,m}(e_{-q,-q-2}), s_{m,k} \left( \det \Omega_{-q,l-1}(u) \right)]
\]
\[= -s_{m,m}(e_{-q,-q-2})s_{-j,k} \left( \det \Omega_{-q+2,l-1}(u) \right) - A' + B',
\]
where
\[
A' = \phi^{\bar{j}+\bar{m}} s_{m,j} \left( \det \Omega_{-q,q-2}(u) \right) s_{-m,k} \begin{pmatrix}
e_{q+2,q} & e_{q+2,q+2} & \ldots & e_{q+2,l-1} \\
1 & u_{q+2} & \ldots & e_{q+2,l-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & u_{l-1}
\end{pmatrix}
\]
and
\[
B' = \phi^{\bar{j}+\bar{m}} s_{m,j} \begin{pmatrix}
u_{q} & \ldots & e_{q-q} & e_{-q,q} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & u_{q} & e_{q-q} \\
0 & \ldots & 1 & e_{q+2,q}
\end{pmatrix} s_{-m,k} \left( \det \Omega_{q+4,l-1}(u) \right).
\]
By Lemma 4.2
\[ \text{pr}(A') = \phi^{j \to m}(u + \rho q)s_{m,j}(\text{rdet } \Omega_{-q,q-2}(u))s_{-m,k}(\text{rdet } \Omega_{q+4,l-1}(u)). \]

The usual column operation gives that
\[ \text{pr}(B') = \phi^{j \to m}s_{m,j} \left( \text{rdet} \begin{pmatrix} u_{-q} & \cdots & e_{-q,q} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & u_q & -(u + \rho q) \\ 0 & \cdots & 1 & 0 \end{pmatrix} \right) s_{-m,k}(\text{rdet } \Omega_{q+4,l-1}(u)) \]
\[ = \phi^{j \to m}(u + \rho q)s_{m,j}(\text{rdet } \Omega_{-q,q-2}(u))s_{-m,k}(\text{rdet } \Omega_{q+4,l-1}(u)). \]

Thus \( \text{pr}(-A' + B') = 0. \)

By Lemma 4.3 we have that \([s_{m,m}(e_{-q,q-2}), s_{-j,k}(\text{rdet } \Omega_{-q+2,l-1}(u))] = 0. \) Now since \( \text{pr}(s_{m,m}(e_{-q,q-2})) = -1, \) we get that
\[ \text{pr}(s_{m,m}(e_{-q,q-2})s_{-j,k}(\text{rdet } \Omega_{-q+2,l-1}(u))) = -s_{-j,k}(\text{rdet } \Omega_{-q+2,l-1}(u)). \]
So
\[ \text{pr}([s_{-j,m}(e_{-q,q-2})s_{m,k}(\text{rdet } \Omega_{q,l-1}(u))] = s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)). \quad (4.9) \]

By combining (4.8) and (4.9) we get that
\[ \text{pr}(C) = -(u + \rho q)s_{h,-i}(\text{rdet } \Omega_{1-l,-q-4}(u))s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)) \]
\[ + ns_{h,-i}(\text{rdet } \Omega_{1-l,-q-4}(u))s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)) \]
\[ = -(u + \rho q)s_{h,-i}(\text{rdet } \Omega_{1-l,-q-4}(u))s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)). \]

Finally, we need to apply \( \text{pr} \) to \( D. \) By (4.3) \( \text{pr}(s_{f,g}(e_{-q,q-2})) = -\delta_{f,g} = s_{f,g}(-1) \) for any \( f, g \in \mathcal{I}_n. \) By Lemma 4.3 \( s_{m,-i}(e_{-q,q-2}) \) commutes with \( s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)). \)
So the usual column operation gives that
\[ \text{pr}(D) = s_{h,-i} \left( \text{rdet} \begin{pmatrix} u_{1-l} & \cdots & e_{1-l,-q-2} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & u_{-q-2} & -(u + \rho_{-q-2}) \\ 0 & \cdots & -1 & -1 \end{pmatrix} \right) \times s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)) \]
\[ = -(u + \rho_{-q-2})s_{h,-i}(\text{rdet } \Omega_{1-l,-q-4}(u))s_{-j,k}(\text{rdet } \Omega_{q+2,l-1}(u)). \]
Thus \( \text{pr}(-C + D) = 0. \)

Lemma 4.5. Suppose that \( l \) is even. For each \( i, j, h, k \in \mathcal{I}_n \)
\[ \text{pr}([s_{i,j}(e_{i,1}), s_{h,k}(\text{rdet } \Omega_{1,1}(u))]) = 0. \]

Proof. Since \( l \) is even, \( \epsilon = -\phi, \) so in all cases by (4.3) we have that for all \( f, g \in \mathcal{I}_n \)
\[ \text{pr}(s_{f,g}(e_{1,1})) = \delta_{f,g} = s_{f,g}(1). \quad (4.10) \]

We compute using Lemma 4.1 to get that
\[ [s_{i,j}(e_{i,1}), s_{h,k}(\text{rdet } \Omega_{1,1}(u))] = A - B + \phi^{j \to l}(-C + D), \]
where

\[
A = s_{h,j}(\text{rdet } \Omega_{1-l,-3}(u)) s_{i,k} \left( \begin{pmatrix} e_{1,-1} & e_{1,1} & \cdots & e_{1,l-1} \\ 1 & u_1 & \cdots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right),
\]

\[
B = s_{h,j} \left( \text{rdet } \begin{pmatrix} u_{1-l} & e_{1-l,1} & e_{1-l,-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_{l-1} \\ 0 & \cdots & 1 \end{pmatrix} \right) s_{i,k}(\text{rdet } \Omega_{3,l-1}(u)),
\]

\[
C = s_{h,-i}(\text{rdet } \Omega_{1-l,-3}(u)) s_{-j,k} \left( \begin{pmatrix} e_{1,-1} & e_{1,1} & \cdots & e_{1,l-1} \\ 1 & u_1 & \cdots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right),
\]

and

\[
D = s_{h,-i} \left( \text{rdet } \begin{pmatrix} u_{1-l} & e_{1-l,1} & e_{1-l,-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_{l-1} \\ 0 & \cdots & 1 \end{pmatrix} \right) s_{-j,k}(\text{rdet } \Omega_{3,l-1}(u)).
\]

Consider A first. Note that

\[
\text{pr} \left( s_{i,k} \left( \text{rdet } \begin{pmatrix} e_{1,-1} & e_{1,1} & \cdots & e_{1,l-1} \\ 1 & u_1 & \cdots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right) \right)
= s_{i,k} \left( \text{rdet } \begin{pmatrix} 1 & e_{1,1} & \cdots & e_{1,l-1} \\ 1 & u_1 & \cdots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right)
+ \sum_{m \in \mathcal{I}_n} \text{pr}(s_{i,m}(e_{1,-1}), s_{m,k}(\text{rdet } \Omega_{1,l-1}(u))).
\]

The obvious row operation gives that

\[
s_{i,k} \left( \text{rdet } \begin{pmatrix} 1 & e_{1,1} & \cdots & e_{1,l-1} \\ 1 & u_1 & \cdots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right) = s_{i,k} \left( \text{rdet } \begin{pmatrix} 0 & -(u + \rho_1) & 0 & \cdots & 0 \\ 1 & u_1 & e_{1,3} & \cdots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right)
= (u + \rho_1)s_{i,k}(\text{rdet } \Omega_{3,l-1}(u)).
\]
Next consider the terms \( \Pr([s_{i,m}(e_{1,-1}), s_{m,k}(\det \Omega_{1,l-1}(u))] \) from (4.11). We calculate using Lemma 4.1 to get that
\[
[s_{i,m}(e_{1,-1}), s_{m,k}(\det \Omega_{1,l-1}(u))]
\] shows that
\[
=-s_{m,m}(e_{1,-1})s_{i,k}(\det \Omega_{3,l-1}(u)) + \phi^{i+m} \epsilon s_{m,-i}(e_{1,-1})s_{m,k}(\det \Omega_{3,l-1}(u)).
\]
So
\[
\Pr([s_{i,m}(e_{1,-1}), s_{m,k}(\det \Omega_{1,l-1}(u))])
\] equals
\[
=-s_{i,k}(\det \Omega_{3,l-1}(u)) + \phi^{i+m} \epsilon s_{m,-i}(\det \Omega_{3,l-1}(u)).
\] (4.13)
So by combining (4.13) and (4.12) in (4.11) we get that
\[
\Pr(A) = (u + \rho_1)s_{h,j}(\det \Omega_{1,l-3}(u))s_{i,k}(\det \Omega_{3,l-1}(u)) - ns_{h,j}(\det \Omega_{1,l-3}(u))s_{i,k}(\det \Omega_{3,l-1}(u)) + \epsilon s_{h,j}(\det \Omega_{1,l-3}(u))s_{i,k}(\det \Omega_{3,l-1}(u)) = (u + \rho_1)s_{h,j}(\det \Omega_{1,l-3}(u))s_{i,k}(\det \Omega_{3,l-1}(u)),
\] (4.14)
since \( \rho_1 - n + \epsilon = \rho_{-1} \).
Next we consider \( B \). The usual column operation gives that
\[
\Pr(B) = s_{h,j} \left( \begin{array}{ccc}
0 & \ldots & e_{1-l,1} \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1
\end{array} \right)
\begin{array}{l}
\det \left( \begin{array}{ccc}
u_{1-l} & \ldots & e_{1-l,1} \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1
\end{array} \right)
\end{array}
\right) s_{i,k}(\det \Omega_{3,l-1}(u))
\] equals
\[
= s_{h,j} \left( \begin{array}{ccc}
0 & \ldots & e_{1-l,1} \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1
\end{array} \right)
\begin{array}{l}
\det \left( \begin{array}{ccc}
u_{1-l} & \ldots & e_{1-l,1} \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1
\end{array} \right)
\end{array}
\right) s_{i,k}(\det \Omega_{3,l-1}(u))
\] equals
\[
= (u + \rho_{-1})s_{h,j}(\det \Omega_{1,l-3}(u))s_{i,k}(\det \Omega_{3,l-1}(u)).
\] (4.15)
So by (4.14) and (4.15), \( \Pr(A - B) = 0 \).
Next consider \( C \). Since \( C \) is nearly identical to \( A \), an argument nearly identical to that used for \( A \) shows that
\[
\Pr(C) = (u + \rho_{-1})s_{h,-i}(\det \Omega_{1,l-3}(u))s_{j,k}(\det \Omega_{3,l-1}(u)).
\]
Since \( D \) is nearly identical to \( B \), an argument nearly identical to that used for \( B \) shows that
\[
\Pr(D) = (u + \rho_{-1})s_{h,-i}(\det \Omega_{1,l-3}(u))s_{j,k}(\det \Omega_{3,l-1}(u)).
\]
So \( \Pr(-C + D) = 0 \).

\begin{lemma}
Suppose that \( l \) is odd. For \( i, j, h, k \in I_n \),
\[
\Pr([s_{i,j}(e_{2,0}), s_{h,k}(\det \Omega(u))])
\] equals
\[
= \phi/2s_{h,j}(\det \Omega_{1-l,2}(u))s_{i,k}(\det \Omega_{4,l-1}(u)) + \phi^{i+j+1}/2s_{h,-i}(\det \Omega_{1-l,4}(u))s_{j,k}(\det \Omega_{4,l-1}(u)) - \phi^{i+j}/2s_{h,-i}(\det \Omega_{1-l,4}(u))s_{j,k}(\det \Omega_{2,l-1}(u)).
\]
\end{lemma}
and
\[
\text{pr}([s_{i,j}(e_{2,0}), s_{h,k}(\text{rdet } \Omega(u))])
\]
\[
= (u + \phi/2)s_{h,j}(\text{rdet } \Omega_{1-l-2}(u))s_{i,k}(\text{rdet } \Omega_{4,l-1}(u))
\]
\[
+ \phi^i+j(u + \phi/2)s_{h,-i}(\text{rdet } \Omega_{1-l-4}(u))s_{-j,k}(\text{rdet } \Omega_{4,l-1}(u))
\]
\[
- \phi^{i+j+1}(u + \phi/2)s_{h,-i}(\text{rdet } \Omega_{1-l-4}(u))s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)).
\]

**Proof.** Since \(l\) is odd, \(\epsilon = \phi\). We compute using Lemma 4.1 to get that
\[
[s_{i,j}(e_{2,0}), s_{h,k}(\text{rdet } \Omega(u))] = A - B + \phi^{i+j}(-C + D),
\]
and
\[
[s_{i,j}(e_{2,0}), s_{h,k}(\text{rdet } \Omega(u))] = \bar{A} - \bar{B} + \phi^{i+j}(-\bar{C} + \bar{D}),
\]
where
\[
A = \bar{A} = s_{h,j}(\text{rdet } \Omega_{1-l-2}(u))s_{i,k}(\text{rdet }\begin{pmatrix} e_{2,0} & e_{2,2} & \cdots & e_{2,l-1} \\ 1 & u_2 & \cdots & e_{2,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix}),
\]
\[
B = s_{h,j}(\text{rdet }\begin{pmatrix} u_{1-l} & \cdots & e_{1-l,0} & e_{1-l,0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e_{0,0} + u & e_{0,0} \\ 0 & \cdots & 1 & e_{2,0} \end{pmatrix})s_{i,k}(\text{rdet } \Omega_{4,l-1}(u)),
\]
\[
\bar{B} = s_{h,j}(\text{rdet }\begin{pmatrix} u_{1-l} & \cdots & e_{1-l,0} & e_{1-l,0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e_{0,0} & e_{0,0} \\ 0 & \cdots & 1 & e_{2,0} \end{pmatrix})s_{i,k}(\text{rdet } \Omega_{4,l-1}(u)),
\]
\[
C = s_{h,-i}(\text{rdet } \Omega_{1-l-4}(u))s_{-j,k}(\text{rdet }\begin{pmatrix} e_{0,-2} & e_{0,0} & \cdots & e_{0,l-1} \\ -\phi & e_{0,0} + u & \cdots & e_{0,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix}),
\]
\[
\bar{C} = s_{h,-i}(\text{rdet } \Omega_{1-l-4}(u))s_{-j,k}(\text{rdet }\begin{pmatrix} e_{0,-2} & e_{0,0} & \cdots & e_{0,l-1} \\ -\phi & e_{0,0} & \cdots & e_{0,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{l-1} \end{pmatrix}),
\]
and
\[
D = \bar{D} = s_{h,-i}(\text{rdet }\begin{pmatrix} u_{1-l} & \cdots & e_{1-l,-2} & e_{1-l,-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & u_{2} & e_{2,-2} \\ 0 & \cdots & -\phi & e_{0,-2} \end{pmatrix})s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)).
\]
By Lemma 4.2, we have

\[ pr(A) = pr(\bar{A}) = (u + \rho_2 - n)s_{h,j}(r\det \Omega_{1-l,-2}(u))s_{i,k}(r\det \Omega_{4,1-1}(u)) \]

\[ = (u - \phi/2)s_{h,j}(r\det \Omega_{1-l,-2}(u))s_{i,k}(r\det \Omega_{4,1-1}(u)). \]

By Lemma 4.2 for any \( f, g \in \mathcal{I}_n \), \( pr(s_{f,g}(e_{2,0})) = \delta_{f,g} = s_{f,g}(1) \). So the obvious column operation gives that

\[ pr(B) = s_{h,j}(r\det \Omega_{1-l,-2}(u))s_{i,k}(r\det \Omega_{4,1-1}(u)) \]

and

\[ pr(\bar{B}) = 0. \]

So

\[ pr(A - B) = -\phi/2s_{h,j}(r\det \Omega_{1-l,-2}(u))s_{i,k}(r\det \Omega_{4,1-1}(u)), \quad (4.16) \]

and

\[ pr(\bar{A} - \bar{B}) = (u - \phi/2)s_{h,j}(r\det \Omega_{1-l,-2}(u))s_{i,k}(r\det \Omega_{4,1-1}(u)). \quad (4.17) \]

Next we consider \( pr(C) \). Since \( \epsilon = \phi \), in all cases we have by (1.3) that for any \( f, g \in \mathcal{I}_n \), \( pr(s_{f,g}(e_{0,-2})) = -\phi \delta_{f,g} = s_{f,g}(-\phi) \). So we have that

\[ pr(C) = s_{h,-i}(r\det \Omega_{1-l,-4}(u))s_{-j,k}(r\det \Omega_{4,1-4}(u)) \]

\[ + \sum_{m \in \mathcal{I}_n} s_{h,-i}(r\det \Omega_{1-l,-4}(u))pr([s_{-j,m}(e_{0,-2}), s_{m,k}(r\det \Omega_{0,l-1}(u))]). \quad (4.18) \]

The obvious row operation gives that

\[ s_{-j,k}(r\det \Omega_{4,1-4}(u)) \]

\[ = s_{-j,k}(r\det \Omega_{2,l-1}(u)). \quad (4.19) \]
Next we consider the terms \([s_{-j,m}(e_{0,-2}), s_{m,k}(\text{rdet } \Omega_{0,l-1}(u))]\) from (4.18). By applying Lemma 4.1, we compute that

\[
[s_{-j,m}(e_{0,-2}), s_{m,k}(\text{rdet } \Omega_{0,l-1}(u))]
= -s_{m,m}(e_{0,-2})s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u))
- \phi^{j+1+m} \delta_{m,j} s_{-m,k} \left( \begin{array}{cccc}
 e_{2,0} & e_{2,2} & \cdots & e_{2,l-1} \\
 1 & u_2 & \cdots & e_{2,l-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & u_{l-1}
\end{array} \right)
+ \phi^{j+1+m} s_{m,j} \left( \begin{array}{c}
 e_{0,0} + u \\
 e_{0,0} \\
 e_{2,0}
\end{array} \right) s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.20)

We need to apply \(\text{pr}\) to each term of this expression. First we use Lemma 4.1 again to get that

\[
\text{pr}(s_{m,m}(e_{0,-2})s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)))
= -\phi s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)) + \text{pr}([s_{m,m}(e_{0,-2}), s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u))])
= -\phi s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)) + \phi^{j+1+m} \text{pr}(s_{-j,-m}(e_{2,0})s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)))
= -\phi s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)) + \phi \delta_{j,m} s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.21)

Next by applying Lemma 4.2 we have that

\[
\text{pr} \left( s_{-m,k} \left( \begin{array}{cccc}
 e_{2,0} & e_{2,2} & \cdots & e_{2,l-1} \\
 1 & u_2 & \cdots & e_{2,l-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & u_{l-1}
\end{array} \right) \right) = (u + \rho_2 - n) s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u))
= (u - \phi/2) s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.22)

Next note that

\[
\text{pr}( s_{m,j} \left( \begin{array}{c}
 e_{0,0} + u \\
 e_{0,0} \\
 e_{2,0}
\end{array} \right) s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)))
= u \delta_{m,j} s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.23)

So by combining (4.21), (4.22), and (4.23) in (4.20) we get that

\[
\text{pr}([s_{-j,m}(e_{0,-2}), s_{m,k}(\text{rdet } \Omega_{0,l-1}(u))])
= \phi s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u)) - \phi \delta_{j,m} s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u))
- \phi^{j+1+m} \delta_{m,j} (u - \phi/2) s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u))
+ \phi^{j+1+m} u \delta_{m,j} s_{-m,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.24)
So by combining (4.19) and (4.24) in (4.18) we get that
\[
\text{pr}(C) = -\phi u s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u))
+ \phi n s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{-j,k}(\text{rdet } \Omega_{2,l-1}(u))
- \phi s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{4,l-1}(u))
- \phi^{j+1+j}(u - \phi/2)s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{4,l-1}(u))
+ \phi^{j+1+j} u s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{4,l-1}(u))
= -\phi(u - n)s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{2,l-1}(u))
- \phi/2 s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.25)

For the last equality we use that \(\phi^{j+1} = \phi\), since \(j\) cannot be zero if \(\phi = -1\).

A very similar calculation shows that
\[
\text{pr}(\tilde{C}) = \phi n s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{2,l-1}(u))
- (u + \phi/2)s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.26)

Finally we must calculate \(\text{pr}(D)\). Note that
\[
\text{pr}(D) = \text{pr}(\tilde{D})
= s_{h,-i} \left( \begin{array}{ccc}
u_{1-l} & \ldots & e_{1-l-2} \\
\vdots & \ddots & \vdots \\
0 & \ldots & u_{-2}
\end{array} \right)
\begin{pmatrix} 1 \\
0 \\
0
\end{pmatrix}
\left( \begin{array}{cc}
u_{1-l} & e_{1-l-2} \phi \\
0 & e_{2-2} \phi \\
0 & -\phi & -\phi
\end{array} \right)
\right)
\] (4.27)

The obvious column operation gives that
\[
\left( \begin{array}{ccc}
u_{1-l} & \ldots & e_{1-l-2} \\
\vdots & \ddots & \vdots \\
0 & \ldots & u_{-2}
\end{array} \right)
\left( \begin{array}{cc}
u_{1-l} & e_{1-l-2} \phi \\
0 & e_{2-2} \phi \\
0 & -\phi & -\phi
\end{array} \right)
= s_{h,-i} \left( \begin{array}{ccc}
u_{1-l} & \ldots & e_{1-l-2} \\
\vdots & \ddots & \vdots \\
0 & \ldots & u_{-2}
\end{array} \right)
\begin{pmatrix} 1 \\
0 \\
0
\end{pmatrix}
\left( \begin{array}{ccc}
u_{1-l} & e_{1-l-2} \\
0 & e_{2-2} \\
0 & -\phi
\end{array} \right)
\] (4.28)

Next we consider the terms \(\text{pr}([s_{m,-i}(e_{0,-2}), s_{j,k}(\text{rdet } \Omega_{2,l-1}(u))])\) from (4.27). We compute using Lemma 4.11 to get that
\[
\text{pr}([s_{m,-i}(e_{0,-2}), s_{j,k}(\text{rdet } \Omega_{2,l-1}(u))]) = \phi^{m+1+i} \text{pr}([s_{j,-m}(e_{2,0}), s_{i,k}(\text{rdet } \Omega_{4,l-1}(u))])
= \phi^{m+1+i} \delta_{j,m} s_{i,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.29)

So by combining (4.28) and (4.29) in (4.27) we have that
\[
\text{pr}(D) = \text{pr}(\tilde{D}) = -\phi(u + \rho_{-2})s_{h,-i}(\text{rdet } \Omega_{1,l-4}(u))s_{j,k}(\text{rdet } \Omega_{2,l-1}(u))
+ \phi^{j+1+i}s_{h,j}(\text{rdet } \Omega_{1,l-4}(u))s_{i,k}(\text{rdet } \Omega_{4,l-1}(u)).
\] (4.30)
So by \((4.15)\), \((4.25)\), and \((4.30)\) we have that
\[
\text{pr}(A - B + \phi^{i+j}(-C + D)) = \phi/2s_{h,j}(\text{rdet}\, \Omega_{1-t,-2}(u)s_{i,k}(\text{rdet}\, \Omega_{4,t-1}(u)) \\
+ \phi^{i+j+1}/2s_{h,-i}(\text{rdet}\, \Omega_{1-t,-4}(u))s_{j,k}(\text{rdet}\, \Omega_{4,t-1}(u)) \\
- \phi^{i+j}/2s_{h,-i}(\text{rdet}\, \Omega_{1-t,-4}(u))s_{j,k}(\text{rdet}\, \Omega_{2,t-1}(u)).
\]

By \((4.17)\), \((4.26)\), and \((4.30)\) we have that
\[
\text{pr}(A - B + \phi^{i+j}(-C + D)) \\
= (u + \phi/2)s_{h,j}(\text{rdet}\, \Omega_{1-t,-2}(u))s_{i,k}(\text{rdet}\, \Omega_{4,t-1}(u)) \\
+ \phi^{i+j}(u + \phi/2)s_{h,-i}(\text{rdet}\, \Omega_{1-t,-4}(u))s_{j,k}(\text{rdet}\, \Omega_{4,t-1}(u)) \\
- \phi^{i+j+1}(u + \phi/2)s_{h,-i}(\text{rdet}\, \Omega_{1-t,-4}(u))s_{j,k}(\text{rdet}\, \Omega_{2,t-1}(u)).
\]

\(\square\)

Now we can prove Theorem 3.1. We need to show that the equation \((4.1)\) holds for all elements \(x\) lying in the generating set \((4.2)\) for \(m\). This follows from Lemmas 4.4, 4.5 and 4.6 using the definition of \(\omega(u)\) from \((1.19)\).

References

[BG] J. Brundan and S. Goodwin, Good grading polytopes, Proc. London Math. Soc. 94 (2007), 155–180; \texttt{math.QA/0510205}

[BGK] J. Brundan, S. Goodwin and A. Kleshchev, Highest weight theory for finite W-algebras, preprint.

[BK1] J. Brundan and A. Kleshchev, Parabolic presentations of the Yangian \(Y(\mathfrak{gl}_n)\), Commun. Math. Phys. 254 (2005) 191–220; \texttt{math.QA/0407011}

[BK2] J. Brundan and A. Kleshchev, Shifted Yangians and finite W-algebras, Adv. Math. 200 (2006), 136–195; \texttt{math.QA/0407012}

[BK3] J. Brundan and A. Kleshchev, Representations of shifted Yangians and finite W-algebras, to appear in Mem. Amer. Math. Soc.; \texttt{arXiv:math.RT/0508003}

[DK] A. De Sole and V. Kac, Finite vs affine W-algebras, Jpn. J. Math. 1 (2006), 137–261; \texttt{math-ph/0511055}

[EK] P. Emaashvili and V. Kac, Classification of good gradings of simple Lie algebras, Amer. Math. Soc. Transl. 213 (2005), 85–104; \texttt{math-ph/0312030}

[GG] W. L. Gan and V. Ginzburg, Quantization of Slodowy slices, Int. Math. Res. Notices 5 (2002) 243–255; \texttt{math.RT/0105225}

[J] J. C. Jantzen, Nilpotent orbits in representation theory, Prog. Math. 228 (2004).

[Lo] I. Losev, Quantized symplectic actions and W-algebras; \texttt{math.RT/0707.3108}

[Ly] T. E. Lynch, Generalized Whittaker vectors and representation theory, Ph.D. Thesis, MIT, Cambridge, MA, 1979

[M] A. Molev, Finite-dimensional irreducible representations of twisted Yangians, J. Math. Phys. 39 (1998), 5559-5600; \texttt{arXiv:q-alg/9711022}

[MNO] A. Molev, M. Nazarov and G. Olshanskii, Yangians and classical Lie algebras, Russian Math. Surveys 51 (1996), 205–282.

[O] G. Olshanski, Twisted Yangians and infinite-dimensional classical Lie algebras, in “Quantum Groups (Leningrad, 1990)”, Lecture Notes in Math. 1510, Springer, 1992, pp. 103–120.

[P1] A. Premet, Special transverse slices and their enveloping algebras, Advances Math. 170 (2002), 1–55.

[P2] A. Premet, Enveloping algebras of Slodowy slices and the Joseph ideal, J. Eur. Math. Soc. 9 (2007), in press; \texttt{math.RT/0504343}

[R] E. Ragoucy, Twisted Yangians and folded W-algebras, Int. J. Mod. Phys. A 16 13 (2001), 2411-2433; \texttt{math.QA/0012182}

[S] T. A. Springer, Linear algebraic groups, Birkhäuser, second edition, 1998.
Department of Mathematics, University of Oregon, Eugene, OR 97403.
E-mail address: jbrown8@uoregon.edu