MEAN STAIRCASE OF THE RIEMANN ZEROS: A COMMENT ON THE LAMBERT W FUNCTION AND AN ALGEBRAIC ASPECT

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Abstract. In this note we discuss explicitly the structure of two simple set of zeros which are associated with the mean staircase emerging from the zeta function and we specify a solution using the Lambert W function. The argument of it may then be set equal to a special $N \times N$ classical matrix (for every $N$) related to the Hamiltonian of the Mehta-Dyson model. In this way we specify a function of an hermitean operator whose eigenvalues are the “trivial zeros” on the critical line. The first set of trivial zeros is defined by the relations $\text{Im} \left( \zeta \left( \frac{1}{2} + i \cdot t \right) \right) = 0 \land \text{Re} \left( \zeta \left( \frac{1}{2} + i \cdot t \right) \right) \neq 0$ and viceversa for the second set. (To distinguish from the usual trivial zeros $s = \rho + i \cdot t = -2n$, $n \geq 1$ integer)

This (heuristic, non rigorous) research note is dedicated to the international Swiss-Italian mathematician and physicist Professor Dr. Sergio Albeverio on the occasion of his seventieth birthday; a friend and for years the scientific director of Cerfim (Research Center for Mathematics and Physics of Locarno), situated opposite the "Rivellino".

1. INTRODUCTION: A SEARCH FOR AN HERMITEAN OPERATOR ASSOCIATED WITH THE RIEMANN ZETA FUNCTION

There is much interest in understanding the complexity related to the Riemann Hypothesis and concerned with the location and the structure of the non trivial zeros of the Riemann zeta function $\zeta(s)$ where $s = \rho + i \cdot t$ is the complex variable. Following a suggestion of Hilbert and Polya, in recent years many efforts have been devoted to a possible construction of an hermitean operator having as eigenvalues the imaginary parts $t_n$ of the non trivial zeros of $\zeta$ ($\zeta$ being meromorphic, the zeros are countable). These are given by the solutions of the equation $\zeta \left( \rho_n + i \cdot t_n \right) = 0$, $n = 1, 2, \ldots$. If $\rho_n = \frac{1}{2}$ for all $n$, then all the zeros lie on the critical line (the Riemann Hypothesis is true); the program is then to find an hermitean “operator” $T$ such that $T \cdot \varphi_n = t_n \cdot \varphi_n$ in some appropriate space ($\varphi_n$ would be the $n^{th}$ eigenvector of $T$). There are today many strategies in the direction of constructing such an operator and in the sequel we will comment on some (among many others)
very stimulating works on the subject. In [1], Pitkänen’s heuristic work goes in the
direction of constructing orthogonality relations between eigenfunctions of a non
hermitean operator related to the superconformal symmetries; a different operator
than the one just mentioned has also been proposed in [2] by Castro, Granik and
Mahecha in terms of the Jacobi Teta series and an orthogonal relation among its
eigenfunctions has also been found. In the rigorous work by Elizalde et.al [3] some
problems with those approaches have been pointed out. In a work of some years ago
Julia [4] proposed a fermionic version of the zeta function which should be related
to the partition function of a system of p-adic oscillators in thermal equilibrium. In
two others pioneering works of these years, Berry and Keating [5, 6] proposed an
interesting heuristic operator to study the energy levels $t_n$ (the imaginary parts of
the non trivial zeros of the zeta function). The proposed Hamiltonian has a very
simple form given, on a dense domain, by: $H = p \cdot x + \frac{1}{2}$, where

\begin{equation}
  p = \left( \frac{1}{\sqrt{i}} \right) \frac{\partial}{\partial x}
\end{equation}

in one dimension. As explained by the authors, the difficulty is then to define
appropriate spaces and boundary conditions to properly determine $p$ and $H$ as her-
mitean operators. In such an approach the heuristic appearance of “instantons” is
also discussed. In another important work Bump et al. [7] introduced a local Rie-
mann Hypothesis and proved in particular that the Mellin transform of the Hermite
polynomials (associated with the usual quantum mechanical harmonic oscillator)
contain as a factor a polynomial $p_n(s)$, corresponding to the $n$-energy eigenstate
of the oscillator, whose zeros are exactly located on the critical line $\sigma = \frac{1}{2}$. The
relation of the polynomials $p_n(s)$ with some truncated approximation of the en-
tire function $\xi(s)$ (the Xi function), related to the Riemann zeta function seems to
be still lacking. Others important mathematical results concerning the non trivial
Riemann zeros, have been obtained by many leading specialist (see among others
the work by Connes [8] and the work by Albeverio and Cebulla [9]). For a recent
work on the xp hamiltonian see G. Sierra [10].

Let us also mention that for the nontrivial zeros of zeta an interesting equation
has been proposed originally by Berry and Keating in [5]. In fact, remembering the
definition of $\xi(s)$ where $\zeta(s)$ is the Riemann zeta function, given by:

\begin{equation}
  \xi(s) = \frac{1}{2} \cdot s \cdot (s - 1) \cdot \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)
\end{equation}

the equation for possible zeros of $\xi$ proposed in [5] is given by:

\begin{equation}
  \pi^{-\frac{s}{2}} \cdot \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = 0
\end{equation}

As stated by the authors, Eq. [3] could be considered as a “quantization condition”.
Unfortunately, as mentioned in [5], Eq. [3] possesses complex zeros and so can not
be used to provide an hermitean operator which would generate the non trivial
zeros of $\zeta$. The content of our note is concerned with the “mean staircase” of the
Riemann zeros: we first specify the two set of trivial zeros on the critical line related
to it and point out an explicit construction using the Lambert W function; then we
introduce a specific argument (a $n \times n$ hermitean matrix $H$, describing a discrete
harmonic oscillator with creation and annihilation “operators” $a$ and $a^*$ such that
\[ [a, a^*] = -2 \] into the Lambert W function. We obtain then, for the trivial zeros, the goal that the “Polya-Hilbert program” has for the non trivial zeros.

2. THE MEAN STAIRCASE OF THE RIEMANN ZEROS AND THE TRIVIAL ZEROS ON THE CRITICAL LINE ASSOCIATED WITH IT

Let \( \zeta(s) \) be the \( \xi \) function given by \[2\]. If \( N(t) \) denotes the number of zeros of \( \zeta \) in the critical strip of height smaller or equal to \( t \), and if \( S(t) \equiv \frac{1}{\pi} \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) \), then \[11\]

\[
N(t) = \langle N(t) \rangle + S(t) + O \left( \frac{1}{t} \right),
\]

where

\[
\langle N(t) \rangle = \frac{t}{2\pi} \cdot \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) + \frac{7}{8} \]

\( \langle N(t) \rangle \), the “bulk contribution” to \( N \), is called the “mean staircase of the zeros” (cfr. \[11\]). The fluctuations of the number of zeros around the mean staircase, are given by the function \( S(t) \). It is known \[11\] that \( S(t) = O(\ln t) \) without assuming RH while, assuming RH is true, it is known that \( S(t) = O \left( \frac{\ln \ln t}{\ln t} \right) \). At this point, since our remark has mainly to do with \( \langle N(t) \rangle \), we will set \( S(t) = 0 \) in Eq.(4).

We shall study the relation \( N(t) = \langle N(t) \rangle + O \left( \frac{1}{t} \right) \). The two sets (which we call here “trivial zeros on the critical line”) of interest are defined by the above mean staircase as follows. The first set is given by the zeros of \( \text{Im} \zeta \left( \frac{1}{2} + it \right) \) alone i.e. such that \( \text{Re} \zeta \left( \frac{1}{2} + it \right) \neq 0 \) (the first set of trivial zeros on the critical line). The second set is given by the zeros of \( \text{Re} \zeta \left( \frac{1}{2} + it \right) \), such that \( \text{Im} \zeta \left( \frac{1}{2} + it \right) \neq 0 \). For the first set:

\[
\text{Im} \zeta \left( \frac{1}{2} + it_n^* \right) = 0 \land \text{Re} \zeta \left( \frac{1}{2} + it_n^* \right) \neq 0
\]

Then since \( \text{Im} \zeta \left( \frac{1}{2} + it_n^* \right) = 0 \) we have that \( \pi^{-1} \arg \zeta \left( \frac{1}{2} + it_n^* \right) = \pi^{-1} (-\pi n) \) and \( \pi^{-1} \arg \xi = 0 \) is given by those \( t_n^* \) such that

\[
N(t_n^*) \equiv \langle N(t_n^*) \rangle = \frac{t_n^*}{2\pi} \cdot \left( \ln \left( \frac{t_n^*}{2\pi} \right) - 1 \right) + \frac{7}{8} = n, n \text{ integer } \geq 1
\]

at large values of \( t \) or \( n \). So the nonlinear equations to be solved which should give the values where only \( \text{Im} \zeta \left( \frac{1}{2} + it \right) \) vanishes, i.e. \( \{t_n^*\} \), is given by:

\[
\langle N(t_n^*) \rangle = n, n \text{ integer } \geq 1
\]

while for the second set

\[
\langle N(t_n^*) \rangle = n - \frac{1}{2}, n \text{ integer } \geq 1
\]

(The first set has been known for a long time and constitutes the Gram points, \( \sin(\theta) = 0 \), where \( \theta \) is the phase of the \( \zeta \) function, while for the second set one has \( \cos(\theta) = 0 \).)

We note, the values of interest are given by the abscissa of the intersection points between the staircase (Eq.(5)) and the two functions \( \pi^{-1} \arg \zeta \left( \frac{1}{2} + it \right) \) and \( \pi^{-1} \arg \xi \left( \frac{1}{2} + it \right) - \frac{1}{2} \). The plot of Fig 1 illustrates the situation for some low lying zeros. The values for \( t_n^* \) lie mostly in between the exact value of the Riemann zeros \( t_{n-1} \) and \( t_n \), but it is known that the Gram law fails for the first time at \( t = 282.4 \).
(“first istanton” according to [5]). The solution of the above equation which gives $t^*_n, t^{**}_n$ using a very special function (the LambertW function, see [12]) is given below.

![Plot](image.png)

**Figure 1.** The plot of $\langle N(t) \rangle$ (continuous curve), of $N(t)$ (full stair) and $N(t) - \frac{1}{2}$ (intermittent stair)

3. **AN EXACT SOLUTION FOR THE SEQUENCE $t^*_n$ AND $t^{**}_n$**

The equation corresponding to (8), may be written in the form

$$\left( \frac{t}{2\pi e} \right)^{\frac{1}{2\pi e}} = e^{\frac{n-\frac{1}{2}}{e}}$$

and the equation corresponding to (9) in the form

$$\left( \frac{t}{2\pi e} \right)^{\frac{1}{2\pi e}} = e^{\frac{n-\frac{1}{2}-\frac{1}{2}}{e}}$$

so that introducing the new variables $x = \exp\left(\frac{n-\frac{1}{2}}{e}\right)$ resp. $x = \exp\left(\frac{n-\frac{1}{2}-\frac{1}{2}}{e}\right)$ we obtain the equation (from (10) and (11), $x > 0$)

$$W(x) \cdot \exp(W(x)) = x$$

The function $W(x)$ is called the Lambert W function and has been studied extensively in these recent years. In fact such an equation appears in many fields of science. In particular the use of such a function has appeared in the study of the wave equation in the double-well Dirac delta function model or in the solution of a jet fuel problem. See [12] for an important work on the subject. Moreover the Lambert W function appears also in combinatorics as the generating function of trees and as explained in [12] the W function has many applications, even if the presence of the W function often goes unrecognized.

The Lambert W function has many complex branches; of interest here is the principal branch of W which is analytic at $x = 0$. So, the solution of (8, 9) is given
by

\begin{equation}
\tag{13}
t^*_n = 2\pi e \cdot \exp \left( W \left( \frac{n - \frac{7}{8}}{e} \right) \right)
\end{equation}

\begin{equation}
\tag{14}
t^{**}_n = 2\pi e \cdot \exp \left( W \left( \frac{n - \frac{1}{2} - \frac{7}{8}}{e} \right) \right)
\end{equation}

We have thus specified, with the help of the LambertW function, the sequences \( \{ t^*_n \} \) resp. \( \{ t^{**}_n \} \), which are the zeros of \( \text{Im} \left( \zeta \left( \frac{1}{2} + it \right) \right) \) such that \( \text{Re} \left( \zeta \left( \frac{1}{2} + it \right) \right) \neq 0 \) resp. \( \text{Re} \left( \zeta \left( \frac{1}{2} + it \right) \right) \neq 0 \).

It should be noted here that in Eq.(8), \( n \), which would correspond to the exact value of a true zero value \( t_n \) (non trivial zero) of the \( \zeta \) function would not be an integer \( n \) or \( n - \frac{1}{2} \) since we have replaced in Eq. (5) \( S(t) \) by zero. For the first few low zeros (the true zeros), it may be observed numerically that the corresponding values, let say \( n^* \), are randomly distributed mostly between two consecutive integers, but the mean values are nearby the integers plus \( \frac{1}{2} \). A calculation with some zeros gives a mean value of 0.49 instead of 0.5. So, in average it seems that the behavior of the true zeroes \( t_n \) “follows” more the pattern of the set \( t^*_n \). In the similar way the zeros of the first set, i.e. \( t_n^* \), lie mostly in between two non trivial zeros of \( \zeta \) but of course it is known that there are very complicated phenomena associated with the chaotic behavior of the non trivial zeros of the Riemann \( \zeta \) function.

As an example, the first of the istantons, corresponding to \( n = 126 \), cited above, is located at the value of \( t = 282.44 \ldots \). On the table below we give the values of \( t^*_n \) and of \( t_n \) of a true zero around \( t = 280 \).

| \( t_n \) | \( t^*_n \) |
|---|---|
| 126 | 279.22925 |
| 126 | 280.80246 |
| 127 | 282.4547596 |
| 127 | 282.4651147 |
| 128 | 283.211185 |
| 128 | 284.1045158 |
| 129 | 284.8359639 |

**Table 1.**

From those numerical computations we see that two consecutive zeros of \( \text{Im}(\zeta) \) alone are followed by two consecutive true zeros, that is \( t^*_{127} \) anticipates \( t_{127} \). The difference between the two subsequent \( t \) values is very small and given by \( \Delta t = 0.0103 \). The phase change is given by \( it\pi \) as illustrated on the plot of \( \text{Im} \left( \ln \left( \zeta \left( \frac{1}{2} + it \right) \right) \right) \) (step curve) and that of \( \text{Im} \left( \zeta \left( \frac{1}{2} + it \right) \right) \).

For the first 500 energy levels, that is for values of \( t \) from 0 to \( t = 811.184 \ldots \) (level number \( n = 500 \)), it may be seen that there are 13 istantons (in the language of [5]), all with a Maslov phase change of \( +it\pi \) or of \( -it\pi \). The width is usually small but it is larger for the istanton located at \( t = 650.66 \) (\( n \) corresponding to 379), where this time \( \Delta t = 0.31 \ldots \). Returning now to the trivial zeros (the two sets \( t^*_n \), \( t^{**}_n \) defined above), we note the elementar relation which follow from (8) and (9), and given by:

\begin{equation}
\tag{15}
\frac{t^*_n + t^*_{n+1}}{2} = t^{**}_{n+\frac{1}{2}}
\end{equation}
and
\[ \frac{t_{n+\frac{1}{2}}^{**} + t_{n+\frac{1}{2}}^{**}}{2} = t_n^{*} \]
Eq. (16) say that the zeros of the real part alone are obtained by those of the imaginary part alone by simple average and viceversa. The two sequences are regularly spaced and the mean distance between two trivial zeros at the height \( t \), as the mean staircase indicates (Eq. (5)), is given approximatively by:
\[ \frac{t}{\langle N(t) \rangle} = \frac{2\pi}{\log \left( \frac{1}{2\pi} \right)} = \frac{2\pi}{\log(n)} \]
for \( t \) and \( n \) large

Before proposing an hermitean operator for the sequences of the trivial zeros it is important to investigate a possible “quantization condition” for the non trivial zeros. For this we start with the Riemann symmetry of the \( \zeta \) function.

From the exact relation for the \( \xi \)-function given by:
\[ \xi(s) = \frac{1}{2} \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) s(s-1) = \xi(1-s) = \frac{1}{2} \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s)(1-s)(1-s-1) \]
\( s \in \mathbb{C} \), we have that
\[ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s) \]
In equation (19) we limit ourselves to consider the values \( s = \rho + it = \frac{1}{2} \uparrow + it \), \( t \in \mathbb{R} \), and thus \( 1-s = \frac{1}{2} \uparrow - it \); moreover we are interested in high values of \( t \) so
that we may use the Stirling’s formula for the Gamma function given by:

\begin{equation}
\Gamma(x) \approx (2\pi)^{\frac{1}{2}} x^{\frac{1}{2}} e^{-x}
\end{equation}

as \( x \to \infty \). From (19) and (20) we then obtain (asymptotically for \( t \to \infty \))

\begin{align*}
\exp \left( i\pi \left( \frac{t}{2\pi} \right) \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) - \frac{1}{8} + i \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) \right) &= \\
\exp \left( -i\pi \left( \frac{t}{2\pi} \right) \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) - \frac{1}{8} + i \arg \left( \zeta \left( \frac{1}{2} - it \right) \right) \right)
\end{align*}

Since

\begin{align*}
\exp \left( i \arg \left( \zeta \left( \frac{1}{2} \downarrow + it \right) \right) \right) &= \exp \left( i \arg \left( \zeta \left( \frac{1}{2} \downarrow + it \right) \right) + i\pi \right) \\
&= -\exp \left( i \arg \left( \zeta \left( \frac{1}{2} \downarrow + it \right) \right) \right)
\end{align*}

we then have, taking the limit \( \rho = \frac{1}{2} \downarrow = \frac{1}{2} \), that:

\begin{equation}
\cos(\Psi) = 0 \quad \text{where} \quad \Psi = t \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) - \frac{\pi}{8} + \arg \left( \zeta \left( \frac{1}{2} + it \right) \right)
\end{equation}

Thus \( \Psi = \pi \left( n + \frac{1}{2} \right) \). We then obtain:

\begin{align*}
\frac{t}{2\pi} \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) - \frac{1}{8} + \frac{1}{\pi} \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) &= n - \frac{1}{2}
\end{align*}

hence

\begin{equation}
\frac{t}{2\pi} \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) + \frac{7}{8} + \frac{1}{\pi} \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) = n + \frac{1}{2}
\end{equation}

Eq. (22) may be seen as an approximate “quantum condition” for the true Riemann zeros, but it is only a consequence of the Riemann symmetry (Eq. (19)). In fact, if in equation (22) we neglect the last term \( \arg(\zeta) \), then (22) has as a solution the second set of trivial zeros \( \{t_n^*\} \). It is true, as remarked by Berry and Keating, that their Eq.(3) has complex zeros which are not the Riemann zeros, but it should be remarked that if in Eq.(3) we set \( \Re(s) = \frac{1}{2} \) then Eq.(3) reduces to Eq.(22) without the fluctuation term \( \arg(\zeta) \); so the solution of Berry and Keating Eq.(3) for \( \Re(s) = \frac{1}{2} \) is the same as the second set of trivial zeros \( \{t_n^*\} \) we have specified.

Below the plots of the left hand side of Eq.(22), with and without the term \( \arg(\zeta) \). As an illustration, we may observe on the plot the first istanton discussed above and the second one. In fact the maximum of the function which gives \( t^* \) (Eq. (22) without the term \( \arg(\zeta) \)) is outside the plot of the step function given by (22) (the true function). This is visible on the plot near \( t = 282 \) and near \( t = 296 \) (the second istanton). This conclude our remark on (3) and Eq.(22). In the next section, we shall construct an hermitean operator whose eigenvalues are the trivial zeros of the zeta function on the critical line.

Now in (13) and (14) the value of a trivial zero \( \{t_n^* \text{ or } t_n^{**}\} \) is given through his index \( n \) by means of the Lambert W function so that such zeros are related in a non linear way to the integers \( n \), i.e. in principle to the spectrum of an harmonic oscillator. So, for the trivial zeros, no boundary condition is needed here, since they are obtained by means of (13) and (14) in the large \( t \) limit. At this moment we are free to introduce a hermitean matrix which may generates the trivial zeros.
4. An Hermitean operator (matrix) associated with the mean staircase (trivial zeros) of the Riemann Zeta function

As remarked above, in (13) and (14) the only “quantal number” is the index $n$ of the trivial zeros and the construction may be given using a hermitean $n \times n$ matrix $H$, for any $n$, at our disposal and related to the classical one dimensional many body system whose fluctuation spectrum around the equilibrium positions is that of the harmonic oscillator. In fact, the one dimensional Mehta-Dyson model of random matrices (which may be seen as a classical Coulomb system with $n$ particles) has, at low temperature an energy fluctuation spectrum given by the integers and it is possible to introduce classical annihilation and creation operators, as studied in [13] (a short discussion is presented in the Appendix). The matrix elements of the associated hermitean matrix are then functions of the zeros of the Hermite polynomials; in this case we do not have a Hilbert space and no Schrödinger Equation will be associated with the Lambert W function. Another direction, i.e. that of introducing a Schrödinger Equation to describe the trivial zeros may in principle be obtained as an application of the results given by G. Nash [14]: this because for large $n$, as it is known, [13] and [14] give the behavior ([11], pag. 214) related to the asymptotic behavior of the LambertW function:

\[ t_n = \frac{2\pi n}{\ln(n)}, n \to \infty \]  

and thus the spectrum appears in fact as a one where the associated Schrödinger Equation contains a Gaussian type of potential [14]. Here we will consider the matrix formulation: the point may seem to be somewhat artificial but the hermitean matrix we will use (specified in the Appendix) is related to the Mehta-Dyson model, the “starting point” of the random matrix theory. To do this, we begin to write...
in a slightly different form using the Stirling formula for the Gamma function of real argument given by:

\[ \Gamma(x) = (2\pi)^{\frac{1}{2}} x^{\frac{1}{2}} e^{-x} \text{ as } x \to \infty \]

We then have that, as \( t \to \infty \),

\[
\ln \left( \Gamma \left( \frac{t}{2\pi e} + \frac{1}{2} \right) \right) = \frac{t}{2\pi e} \ln \left( \frac{t}{2\pi e} - 1 \right) + \frac{7}{8} + \frac{1}{2} \ln (2\pi) - \frac{7}{8} = n^* + \frac{1}{2} \ln (2\pi) - \frac{7}{8} = n^* + \theta
\]

where \( \theta = \frac{1}{2} \ln (2\pi) - \frac{7}{8} \).

Thus introducing the operator \( T = T(H) \) whose eigenvalues should be the trivial energy levels (for the first as well as for the second set defined by (13) and (14) as well as \( H \), the hermitean matrix given in the Appendix and related to the Mehta-Dyson model, we may write the following heuristic matrix equation:

\[
\Gamma \left( \frac{t}{2\pi e} + \frac{1}{2} \right) = e^{H+\theta}
\]

where \( I \) is the unit matrix. Eq. (26) is the equation for \( T \), giving the trivial zeros. The inversion of this formula (if it is possible to take it) yields heuristically:

\[
T = T(H) = 2\pi \left( \Gamma^{-1} \left( e^{H+\theta} \right) - \frac{I}{2} \right)
\]

To conclude, if \( H\varphi_n = (n + \frac{1}{2}) \varphi_n \), where \( \varphi_n \) is the \( n \)th eigenfunction of \( H \), then

\[
T\varphi_n = 2\pi \left( \Gamma^{-1} \left( e^{H+\theta} \right) - \frac{I}{2} \right) \varphi_n = 2\pi \left( \Gamma^{-1} \left( e^{n^*+\theta} \right) - \frac{1}{2} \right) \varphi_n = t_n \varphi_n
\]

where \( t_n = t_n^* \) resp. \( t_n^{**} \). (in that latter case with \( \theta \) lowered by 1/2).

Of course Eq. (26) for the operator \( T \) is more appealing than (13, 14) (where \( n \) is replaced by \( H \) and \( t_n \) resp. \( t_n^* \) are replaced by \( T \)) due to the combinatorial nature of the Gamma function, but the eigenvalues of the operators are the same in the “thermodynamic limit”, \( t \to \infty \).

Remark: If one consider the usual map \( z \to 1 - \frac{1}{2} \) then the critical line \( s = \frac{1}{2} + it \) \( (t \in \mathbb{R}) \) is mapped onto the unit circle \( |z| = 1 \); the two sets of trivial zeros \( \{t_n^*\} \) and \( \{t_n^{**}\} \) have as accumulation point \( z = 1 \) (as \( n \to \infty \)), which is the same accumulation point for the real zeros of the \( \zeta \) function given by \( \bar{z}_n = 1 - \frac{1}{2n} = 1 + \frac{1}{2n} \), as \( n \to \infty \) (see Fig 4).

Neglecting the real zeros \( \{z_n\} \), Fig. 4 illustrate by means of two sets of trivial zeros \( \{t_n^*\} \) and \( \{t_n^{**}\} \) the Lee-Yang Theorem for the zeros of the partition function for some general spin lattice system studied in statistical mechanics. If RH is true, then all non trivial zeros of \( \zeta(s) \) shall be located at the same circle \( |z| = 1 \), with \( z = 1 \) as accumulation point.

5. Conclusion

In this note we have specified an operator equation for the operator \( T \), having as eigenvalues the trivial zeros of the \( \zeta \) function on the critical line (13 and 14); such zeros are the so called trivial zeros (mean staircases), and are related in a strong non linear way to the eigenvalues of a discrete harmonic oscillator described by \( H \) in the large \( t \) limit. The introduction of \( H \) may seem to be artificial: nevertheless
the two sets of trivial zeros ($\{t^*_n\}$ and $\{t'^*_n\}$) are the eigenvalues of $T(H)$. In a subsequent note we will present a study of another sequence of zeros possibly more connected with the true Riemann zeros.

**Update**

Very recently G. Sierra and P.K. Townsend [15] introduced and studied an interesting physical model (a charged particle in the plane in presence of an electrical and a magnetic potential). In particular, the lowest Landau level is connected with the smoothed counting function that gives the average number of zeros, i.e. the staircase which here we have studied, by means of a classical one-dimension model of $N$ interacting charged particles.

**Note added**

Very recently Schumayer et al [16] constructed (in particular) the Quantum mechanical potential for $\xi(s)$ zeros, with the first 200 energy eigenvalues (non trivial zeros). It is expected that the same form of a quantum mechanical potential would appear using only the two sets of zeroes we have discussed in this note. For the construction of an Hamiltonian whose spectrum coincides with the primes, see also the recent work of S. Sekatskii [17].
The hermitean matrix associated with the Mehta-Dyson model, \( H \): discrete annihilation and creation operators associated to \( H \) whose spectrum is given by the set of integers \((1, 2, \ldots, n)\), for any finite \( n \).

In Refs.\[13, 18\] it was studied the one dimensional Mehta-Dyson model defined by the Hamiltonian
\[
E = N \sum_{i=0}^{N} \left( \frac{1}{2} y_i^2 - \sum_{i<j<N} \log(|y_i - y_j|) \right),
\]
where \( y_i \) is the position of the \( i \)-th particle on the line; then the fluctuation around the equilibrium positions (these are given by the zeros of the Hermite polynomials of degree \( N \), where \( N \) is the number of particles on the line, for every finite \( N \)), i.e., the harmonic fluctuation spectrum is given by the eigenvalues of the hermitean \( N \times N \) real matrix whose elements are given by:
\[
\begin{align*}
H_{ij} &= -\frac{1}{|x_i - x_j|^2} \quad i \neq j \\
H_{ij} &= 1 + \sum_{k \neq i} \frac{1}{|x_i - x_k|^2} \quad i = j
\end{align*}
\]
i, \( j = 1, \ldots, N \), where now the \( x_i \) are the “equilibrium positions” i.e. the zeros of the Hermite polynomials of degree \( N \).

The spectrum of \( H \) is given exactly by the integers \((1, 2, \ldots, N)\) for every finite \( N \) and the eigenfunctions are given in terms of the Mehta-Dyson polynomials of order 1 up to \( N \). The Hamiltonian describing the harmonic fluctuations takes then the form \[13\]:
\[
H = N \cdot I - \frac{1}{2} aa^*
\]
where \( I \) is the unit matrix of order \( N \) and \( a, a^* \), are the discrete annihilation and creation operators (matrices of order \( N \times N \)) which satisfy the commutator relation \([a, a^*] = -2\).

Moreover \([H, a^*] = a^* \) and \([H, a] = -a\).

If \( X_k \) is the \( k \)-ten eigenvector of \( H \) with eigenvalue the integer \( k \), one has:
\[ a^* X_{k+1} = X_{k+2} \]
and
\[ a X_{k+1} = 2(N - k) X_k \]
Explicitly, if \( X_k = (\varphi_{1k}(x_1), \ldots, \varphi_{Nk}(x_N)) \) is the \( k \)-ten eigenvector, where \( \varphi_k(x) \) is the \( k \)-th Mehta-Dyson polynomial of argument \( x \), then
\[
\begin{align*}
a \varphi_{k+1}(x_1) &= \frac{d}{dx_1}(\varphi_{k+1}(x_1)) = \sum_{i \neq 1}^{N} \frac{\varphi_{k+1}(x_1) - \varphi_{k+1}(x_i)}{(x_1 - x_i)} \\
\text{and} \quad a^* \varphi_{k+1}(x_1) &= \left(2x_1 - \frac{d}{dx_1}\right) \varphi_{k+1}(x_1)
\end{align*}
\]
a and \( a^* \) are the two discrete annihilation and creation operators of \( H \).

\( H \) as above with \( N = n \) may be used to give the first \( n \) trivial zeros of the first set in \[13\] i.e. \( t_1^* \ldots t_n^* \) while \( H + \frac{1}{2} \) may be used for obtaining the first \( n \) trivial zeros of the second set in \[14\] i.e. \( t_1^{**} \ldots t_n^{**} \) in the discussion on the mean staircases given in Section 4 above.
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