Equidistributions of Sign Patterns of the Liouville Function and Normal Numbers

N. A. Carella

Abstract

The equidistribution of the double sign patterns of the Liouville function $\lambda$ is proved unconditionally. As application, it is shown that the computable real number

$$\sum_{n \geq 1} \frac{1 + \lambda(n)}{2^n}$$

is a simply normal number in base 4.

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1 Introduction

Let \( t \neq 0 \) be a fixed integer and let
\[
(\lambda(n) = \pm 1, \lambda(n + t) = \pm 1) = (\pm, \pm)
\]
denotes the corresponding double-sign patterns of the Liouville function \( \lambda : \mathbb{N} \rightarrow \{-1, 1\} \), defined in (2.1). This note proposes the following equidistribution results for the double-sign patterns, these include new results, and simpler proofs of the current available results in the literature.

**Theorem 1.1.** Let \( x \) be a large number, and let \( t \neq 0 \) be a fixed integer. Then, the double-sign patterns ++, +−, −+, and −− of the Liouville pair \( \lambda(n), \lambda(n + t) \) are equidistributed on the interval \([1, x]\). In particular, each double-sign pattern has the natural density
\[
\delta^\pm_\lambda(t) = \frac{1}{4}.
\]

The application to normal numbers is the following.

**Theorem 1.2.** Let \( \lambda : \mathbb{N} \rightarrow \{-1, 1\} \) be the Liouville function. Then, the computable real number
\[
\sum_{n \geq 1} \frac{1 + \lambda(n)}{2^n} = 1.162324637623929785959733583622409170 \ldots,
\]
is a simply normal number in base \( b = 4 \). In particular, the 4-adic expansion
\[
\sum_{n \geq 0} \frac{1 + \lambda(n)}{4^n} = 1.02212032012320013232002110332223010021 \ldots,
\]
contains infinitely many digit 0, infinitely many digit 1, infinitely many digit 2, and infinitely many digit 3.

The essential foundational materials are covered in Section 2 to Section 3, and the Appendix 6. The proof of the equidistribution of the double sign patterns of the Liouville function in Theorem 1.1 appears in Section 5.1. The proof Theorem 1.2 appears in Section 5.2.
2 Sign Patterns Characteristic Functions for the Liouville Function

The formulas for the characteristic functions of several sign-patterns of the Liouville function are developed in this section.

2.1 Single-Sign Patterns-Liouville Function

The Liouville function \( \lambda : \mathbb{N} \to \{-1, 1\} \) is defined by

\[
\lambda(n) = (-1)^{v_1 + v_2 + \cdots + v_w},
\]

where \( n = p_1^{v_1} p_2^{v_2} \cdots p_w^{v_w} \), the \( p_i \geq 2 \) are primes, and \( v_i \geq 1 \) are integers.

**Lemma 2.1.** If \( \lambda(n) \in \{-1, 1\} \) is the Liouville function, then,

\[
\lambda^\pm(n) = \frac{1 \pm \lambda(n)}{2} = \begin{cases} 
1 & \text{if } \lambda(n) = \pm 1, \\
0 & \text{if } \lambda(n) \neq \pm 1,
\end{cases}
\]

of the subset of integers

\[
\mathcal{N}_\lambda^\pm = \{ n \geq 1 : \lambda(n) = \pm 1 \}.
\]

2.2 Double-Sign Patterns-Liouville Function

The analysis of single-sign pattern characteristic functions is extended here to the double-sign patterns

\[
\lambda(n) = \pm 1 \quad \text{and} \quad \lambda(n + t) = \pm 1,
\]

where \( t \neq 0 \) is a small integer, and \( n \geq 1 \) is an integer. Some of the research on multiple sign patterns appear in [7], [12], [21, Corollary 1.7], [17], [19], and similar literature.

**Lemma 2.2.** Let \( t \neq 0 \) be small fixed integer, and let \( \lambda(n) \in \{-1, 1\} \) be the Liouville function. Then,

\[
\lambda^{\pm \pm}(t) = \left( \frac{1 \pm \lambda(n)}{2} \right) \left( \frac{1 \pm \lambda(n + t)}{2} \right)
\]

\[
= \begin{cases} 
1 & \text{if } \lambda(n) = \pm 1, \mu(n + t) = \pm 1, \\
0 & \text{if } \lambda(n) \neq \pm 1, \mu(n + t) \neq \pm 1,
\end{cases}
\]

are the characteristic functions of the subset of integers

\[
\mathcal{N}_\lambda^{\pm \pm}(t) = \{ n \geq 1 : \lambda(n) = \pm 1, \lambda(n + t) = \pm 1 \}.
\]
2.3 \( k \)-Signs Patterns Liouville Characteristic Functions

The general sign patterns of a vector of values of Liouville functions is considered here. A slight change in notation to simplify the formulas is introduced in this subsection.

Let \( k \geq 1 \) be an integer. Define the integer \( k \)-tuple

\[
a = (a_1, a_2, \ldots, a_k),
\]

(2.7)

where \( 0 \leq a_1 < a_2 < \cdots < a_k \leq x \), and the \( k \)-sign pattern

\[
\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k),
\]

(2.8)

where \( \epsilon_i \in \{-1, 1\} \). The same principle used for single-sign and double-sign patterns is extended to the general \( k \)-sign patterns characteristic function of \( k \)-tuple of Liouville function values

\[
(\lambda(n+a_1), \lambda(n+a_2), \ldots, \lambda(n+a_k)) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k).
\]

(2.9)

**Lemma 2.3.** Let \( n \in \mathbb{N} \) be an integer, and let \( \lambda(n) \in \{-1, 1\} \) be the Liouville function. If \( a \) is an integer \( k \)-tuple, and \( \epsilon \) is a \( k \)-sign pattern, then,

\[
\lambda(a, \epsilon, n) = \prod_{0 \leq i < k} \left( \frac{1 \pm \lambda(n + a_i)}{2} \right)
\]

(2.10)

\[
= \begin{cases} 
1 & \text{if } \lambda(n + a_1) = \epsilon_1, \ldots, \lambda(n + a_k) = \epsilon_k, \\
0 & \text{if } \lambda(n + a_1) \neq \epsilon_1, \ldots, \lambda(n + a_k) \neq \epsilon_k,
\end{cases}
\]

is the characteristic functions of the subset of integers

\[
N_{\lambda}(a, \epsilon) = \{ n \geq 1 : \lambda(n) = \epsilon_1, \ldots, \lambda(n + k - 1) = \epsilon_k \}.
\]

(2.11)

3 Sign Patterns Counting Functions for the Liouville Function

The analysis of the single-sign patterns for the Liouville and Mobius functions are well known, but are included here as a reference.

3.1 Single-Sign Patterns-Liouville Counting Function

The single-sign pattern counting functions \( N_{\lambda}^+(x) = \# \{ n \leq x : \lambda(n) = 1 \} \) and \( N_{\lambda}^-(x) = \# \{ n \leq x : \lambda(n) = -1 \} \) of the Liouville function over the integers have the simplest analysis.
Lemma 3.1. If $\lambda(n) \in \{-1, 1\}$ is the Liouville function, then,

$$N^+_{\lambda}(x) = \frac{x}{2} + O \left( xe^{-c\sqrt{\log x}} \right),$$

and

$$N^-_{\lambda}(x) = \frac{x}{2} + O \left( xe^{-c\sqrt{\log x}} \right),$$

where $c > 0$ is an absolute constant.

Proof. Utilize the characteristic function, Lemma 2.1, to express the counting function in the form

$$N^+_{\lambda}(x) = \sum_{n \leq x, \lambda(n) = 1} 1 = \sum_{n \leq x} \left( \frac{1 + \lambda(n)}{2} \right) = \frac{x}{2} + O \left( xe^{-c\sqrt{\log x}} \right),$$

respectively. The error terms follow from Theorem 6.1. The other case has a similar proof.

In terms of the single-sign pattern counting functions, the summatory function has the asymptotic formula

$$N_{\lambda}(x) = \sum_{n \leq x} \lambda(n) = N^+_{\lambda}(x) - N^-_{\lambda}(x) = O \left( xe^{-c\sqrt{\log x}} \right).$$

(3.2)

Basically, it is a different form of the Prime Number Theorem

$$\pi(x) = \text{li}(x) + O \left( xe^{-c\sqrt{\log x}} \right),$$

(3.3)

where $\text{li}(x) = \int_2^x (\log t)^{-1} dt$ is the logarithm integral, and $c > 0$ is an absolute constant, see [3, Eq. 27.12.5], [4, Theorem 3.10], et alii.

3.2 Double-Sign Patterns-Liouville Counting Function

The counting functions for the single-sign patterns $\lambda(n) = 1$ and $\lambda(n) = -1$ are extended to the counting functions for the double-sign patterns

$$(\lambda(n), \lambda(n + t)) = (\pm 1, \pm 1).$$

(3.4)
The double-sign patterns counting functions are defined by

\[ N_{\lambda}^{++}(t, x) = \sum_{n \leq x, \lambda(n)=1, \lambda(n+t)=1} 1 = \sum_{n \leq x, n \in \mathbb{N}^{++}} 1, \]  
(3.5)

\[ N_{\lambda}^{+-}(t, x) = \sum_{n \leq x, \lambda(n)=1, \lambda(n+t)=-1} 1 = \sum_{n \leq x, n \in \mathbb{N}^{+-}} 1, \]  
(3.6)

\[ N_{\lambda}^{-+}(t, x) = \sum_{n \leq x, \lambda(n)=-1, \lambda(n+t)=1} 1 = \sum_{n \leq x, p \in \mathbb{N}^{-+}} 1, \]  
(3.7)

\[ N_{\lambda}^{--}(t, x) = \sum_{n \leq x, \lambda(n)=-1, \lambda(n+t)=-1} 1 = \sum_{n \leq x, p \in \mathbb{N}^{--}} 1. \]  
(3.8)

The double-sign patterns counting functions (3.5) to (3.8) are precisely the counting functions of the subsets of integers

1. \( N_{\lambda}^{++}(t) \subset \mathbb{N} \),
2. \( N_{\lambda}^{+-}(t) \subset \mathbb{N} \),
3. \( N_{\lambda}^{-+}(t) \subset \mathbb{N} \),
4. \( N_{\lambda}^{--}(t) \subset \mathbb{N} \),
defined in (2.6).

**Lemma 3.2.** Let \( x \) be a large number, and let \( t \neq 0 \) be a fixed integer. If \( \lambda: \mathbb{Z} \rightarrow \{-1, 1\} \) is the Liouville function, then,

\[ N_{\lambda}^{\pm \pm}(t, x) = \frac{x}{4} + O \left( \frac{x}{(\log \log x)^{1/2-\varepsilon}} \right), \]

where \( \varepsilon > 0 \) is an arbitrary small number.

**Proof.** Without loss in generality, consider the pattern \( (\lambda(n), \lambda(n+t)) = (+1, +1) \). Now, use Lemma 2.2 to express the double-sign pattern counting function as

\[ 4N_{\lambda}^{++}(t, x) = \sum_{n \leq x} \lambda^{++}(t, n) \]
\[ = \sum_{n \leq x} (1 + \lambda(n)) (1 + \lambda(n+t)) \]
\[ = \sum_{n \leq x} (1 + \lambda(n) + \lambda(n+t) + \lambda(n)\lambda(n+t)) \]  
(3.9)
\[ = \sum_{n \leq x} 1 + \sum_{n \leq x} \lambda(n) + \sum_{n \leq x} \lambda(n+t) + \sum_{n \leq x} \lambda(n)\lambda(n+t) \]
\[ \geq 0. \]
The first three finite sums on the last line have the following evaluations or estimates.

1. \[ \sum_{n \leq x} 1 = x, \]

2. \[ \sum_{n \leq x} \lambda(n) = O \left( x e^{-c \sqrt{\log x}} \right), \] see Theorem 6.1.

3. \[ \sum_{n \leq x} \lambda(n + t) = O \left( x e^{-c \sqrt{\log x}} \right), \] see Theorem 6.1,

4. \[ \sum_{n \leq x} \lambda(n) \lambda(n + t) = O \left( \frac{x}{(\log \log x)^{1/2 - \varepsilon}} \right), \] see Theorem 6.4,

where \([x]\) is the largest integer function, \(c > 0\) is an absolute constant, and \(\varepsilon > 0\). Summing these evaluations or estimates verifies the claim for \(\mathcal{N}^{++}_\lambda(t, x) \geq 0\). The verifications for the next three double-sign pattern counting functions \(\mathcal{N}^{+-}_\lambda(t, x) \geq 0, \mathcal{N}^{-+}_\lambda(t, x) \geq 0,\) and \(\mathcal{N}^{--}_\lambda(t, x) \geq 0\) are similar.

\[ \square \]

### 4 Equidistribution of Liouville Sign Patterns

#### 4.1 Equidistribution of Single-Sign Patterns

The counting functions are defined by

\[ \mathcal{N}^{+}_\lambda(x) = \#\{n \leq x : \lambda(n) = 1\} \] (4.1)

and

\[ \mathcal{N}^{-}_\lambda(x) = \#\{n \leq x : \lambda(n) = -1\}. \] (4.2)

The corresponding densities functions are defined by

\[ \delta^+_\lambda = \lim_{x \to \infty} \frac{\mathcal{N}^{+}_\lambda(x)}{x} \quad \text{and} \quad \delta^-_\lambda = \lim_{x \to \infty} \frac{\mathcal{N}^{-}_\lambda(x)}{x}. \] (4.3)

**Theorem 4.1.** If \(x\) is a large number, then the single-sign patterns of the Liouville function are equidistributed on the interval \([1, x]\). Specifically,

\[ \delta^+_\lambda = \frac{1}{2} \quad \text{and} \quad \delta^-_\lambda = \frac{1}{2}. \]

**Proof.** Utilize Lemma 3.1 to evaluate the limits in (4.3). \[ \square \]
The nontrivial result for the summatory Liouville function
\[ \sum_{n \leq x} \lambda(n) = N_+^\lambda(x) - N_-^\lambda(x) = O \left( x e^{-c \sqrt{\log x}} \right) \] (4.4)
has no main term. It vanishes because the number of single-sign patterns
\( N_+^\lambda(x) \) and \( N_-^\lambda(x) \) have the same main terms, see Lemma 3.1. This is implied
by the equidistribution of the single sign patterns.

### 4.2 Equidistribution of Double-Sign Patterns

The equidistribution results for single-sign patterns are extended to equidistribu-
tion of the double-sign patterns.

Recall that the double sign counting function is defined by
\[ N_{\pm \pm}^\lambda(t, x) = \# \{ n \leq x : \lambda(n) = \pm 1, \lambda(n + t) = \pm 1 \}, \] (4.5)
and the natural density of a double-sign pattern is defined by
\[ \delta_{\pm \pm}^\lambda(t) = \lim_{x \to \infty} \frac{\# \{ n \leq x : \lambda(n) = \pm 1, \lambda(n + t) = \pm 1 \}}{x}. \] (4.6)

**Proof of Theorem 1.1.** For large \( x \), the asymptotic formula for \( N_{\pm \pm}^\lambda(t, x) > 0 \) is proved in Lemma 3.2. Next, compute the limit of the proportion of
double-sign pattern
\[ \delta_{\pm \pm}^\lambda(t) = \lim_{x \to \infty} \frac{\# \{ n \leq x : \lambda(n) = \pm 1, \lambda(n + t) = \pm 1 \}}{x} \]
\[ = \lim_{x \to \infty} \frac{[x] + O \left( x (\log \log x)^{-1/2+\varepsilon} \right)}{4x} \]
\[ = \frac{1}{4}. \]

This proves that the double-sign patterns ++, +−, −+, and −− are equidis-
tributed on the interval \([1, x]\) as \( x \to \infty \).

**Example 4.1.** Let \( t = 1 \). By Theorem 1.1, in any sufficiently large interval
\([1, x]\), the number of any double-sign pattern \( \lambda(n) = \pm 1, \lambda(n + 1) = \pm 1 \) is
\[ N_{\pm \pm}^\lambda(t, x) = \delta_{\pm \pm}^\lambda(t)x + o(x) = \frac{1}{4} x + O \left( \frac{x}{(\log \log x)^{1/2-\varepsilon}} \right). \] (4.7)
The numerical data for \( x = 10^5 \), shows that the actual value of the autocorrelation function is
\[
\sum_{n \leq x} \lambda(n)\lambda(n+1) = 68,
\] (4.8)
and the actual values of the double-sign pattern counting functions are tabulated below.

| \( \lambda(n) \) | \( \lambda(n+1) \) | Actual Count | Expected \( \mathcal{N}^{\pm\pm}_\lambda(1, x) \) |
|-----------------|-----------------|--------------|---------------------|
| +1              | +1              | 99492/4      | 100000/4 + o(x)     |
| +1              | −1              | 99932/4      | 100000/4 + o(x)     |
| −1              | +1              | 99932/4      | 100000/4 + o(x)     |
| −1              | −1              | 100644/4     | 100000/4 + o(x)     |

Given the small scale of this experiment, \( x = 10^5 \), the actual data fits the prediction very well. The tiny differences among the actual values, (in third column), and the prediction by the double-sign pattern counting functions \( \mathcal{N}^{\pm\pm}_\lambda(1, x) \) seem to be properties of the races between the different subsets of integers \( \mathcal{N}^{\pm\pm}_\lambda(t) \) attached to the double-sign patterns, see (2.6). For an introduction to the literature in comparative number theory, prime number races, and similar topics, see [5], et cetera.

5 Applications to Normal Numbers I

The earliest study was centered on the distributions of the digits in the decimal expansions of algebraic irrational numbers such as \( \sqrt{2} = 1.414213562\ldots \). This problem is known as the Borel conjecture.

The digits \( w_n \in \{0, 1, 2, \ldots, b-1\} \) in the \( b \)-adic expansion \( \alpha = \sum_{n \geq 0} w_n b^{-n} \) of a normal number are random, and uniformly distributed.

**Definition 5.1.** Let \( b > 1 \) be an integer base. A real number \( \alpha \in (0, 1) \) is said to be **simply normal in base** \( b \) if each digit \( w \in \{0, 1, 2, \ldots, b-1\} \) in its \( b \)-adic expansion \( \alpha = \sum_{n \geq 0} w_n b^{-n} \) occurs with probability
\[
P(w) = \frac{1}{b}.
\]

**Definition 5.2.** Let \( b > 1 \) be an integer base, and let \( k \geq 1 \) be an integer. A real number \( \alpha \in (0, 1) \) is said to be **normal in base** \( b \) if every sequence of digits \( w = w_0 w_1 \cdots w_{k-1} \in \{0, 1, 2, \ldots, b-1\}^k \) occurs in its \( b \)-adic expansion \( \alpha = \sum_{n \geq 0} w_n b^{-n} \) with probability
\[
P(w) = \frac{1}{b^k}.
\]
5.1 A Result for Simply Normal in Base $b = 2$

The 2-adic digits of the real number $\beta = 0.w_1w_2w_3 \cdots$ are generated by the map

$$n \mapsto w_n = \frac{1 + \lambda(n)}{2} \in \{0, 1\}. \quad (5.1)$$

The single digit pattern counting function is defined by

$$N_\beta(a, x) = \#\{w_n = a : n \leq x\}, \quad (5.2)$$

and the natural digit density is defined by

$$p_\beta(a) = \lim_{x \to \infty} \frac{N_\beta(a, x)}{x} = \lim_{x \to \infty} \frac{N_\pm(x)}{x} = \delta_\pm. \quad (5.3)$$

**Theorem 5.1.** Let $\lambda : \mathbb{N} \to \{-1, 1\}$ be the Liouville function. Then, the computable real number

$$\beta = \sum_{n \geq 1} \frac{1+\lambda(n)}{2^n} = 1.16232463762392978595979733583622409170 \ldots,$$

is simply normal number in base 2.

**Proof.** By Theorem 4.1, the precise probabilities of the values of the single digit pattern are the followings.

1. $p_\beta(w_n = 0) = \delta^- = \frac{1}{2}$, since $0 = \frac{1+\lambda(n)}{2}$ with $\lambda(n) = -1$,
2. $p_\beta(w_n = 1) = \delta^+ = \frac{1}{2}$, since $1 = \frac{1+\lambda(n)}{2}$ with $\lambda(n) = 1$.

Each digit has the same probability

$$p_\beta(w_n = 0) = p_\beta(w_n = 1) = \frac{1}{2}. \quad (5.4)$$

Therefore, by Definition 5.1, the real number $\beta$ is simply normal number. \qed

5.2 A Result for Simply Normal in Base $b = 4$

The approach used to prove simply normality in base $b = 2$ is here to verify a result for simply normality in base $b = 4$. The 4-adic expansion of the number under analysis has the form

$$\sum_{n \geq 1} \frac{1+\lambda(n)}{2^n} = 1.16232463762392978595979733583622409170$$

$$= \sum_{n \geq 1} w_n b^n. \quad (5.5)$$
The \( n \)th digit is defined by
\[
    w_n = \frac{1 + \lambda(n)}{2} + \left( \frac{1 + \lambda(n + 1)}{2} \right) \cdot 2. \tag{5.6}
\]

This is generated by the double sign patterns
\[
    \left( \frac{1 + \lambda(n)}{2}, \frac{1 + \lambda(n + 1)}{2} \right). \tag{5.7}
\]

**Proof of Theorem 1.2.** By Theorem 1.1, the precise probabilities for the double digit patterns are the followings.

1. \((0,0) = \left( \frac{1 + \lambda(n)}{2}, \frac{1 + \lambda(n + 1)}{2} \right)\), \( p_{\beta}(w_n = 0) = \delta_{\lambda}^-(1) = \frac{1}{4} \),
2. \((0,1) = \left( \frac{1 + \lambda(n)}{2}, \frac{1 + \lambda(n + 1)}{2} \right)\), \( p_{\beta}(w_n = 1) = \delta_{\lambda}^+(1) = \frac{1}{4} \),
3. \((1,0) = \left( \frac{1 + \lambda(n)}{2}, \frac{1 + \lambda(n + 1)}{2} \right)\), \( p_{\beta}(w_n = 2) = \delta_{\lambda}^+(1) = \frac{1}{4} \),
4. \((1,1) = \left( \frac{1 + \lambda(n)}{2}, \frac{1 + \lambda(n + 1)}{2} \right)\), \( p_{\beta}(w_n = 3) = \delta_{\lambda}^+(1) = \frac{1}{4} \).

Each of the 4 double digit patterns (5.6), equivalently each digit \( w_n \in \{0,1,2,3\} \), has the same probability
\[
    p_{\beta}(w_n = 0) = p_{\beta}(w_n = 1) = p_{\beta}(w_n = 2) = p_{\beta}(w_n = 3) = \frac{1}{4}. \tag{5.8}
\]

Therefore, by Definition 5.2, the real number (5.5) is simply normal number in base \( b = 4 \).

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### 6 Appendix A: Basic Results for the Liouville Function

Some standard results required in the proofs of the equidistributions of the sign patterns of the Liouville function are recorded in this section.
6.1 Average Orders of Liouville Functions

**Theorem 6.1.** If \( \lambda : \mathbb{N} \rightarrow \{ -1, 1 \} \) is the Liouville function, then, for any large number \( x \), the following statements are true.

(i) \( \sum_{n \leq x} \lambda(n) = \zeta(1/2)x^{1/2} + O\left(xe^{-c\sqrt{\log x}}\right) \), unconditionally,

(ii) \( \sum_{n \leq x} \frac{\lambda(n)}{n} = O\left(e^{-c\sqrt{\log x}}\right) \), unconditionally,

where \( c > 0 \) is an absolute constant.

The most recent research on the summatory Liouville function seems to be [16].

6.2 Twisted Exponential Sums

One of the earliest result for exponential sum with multiplicative coefficients is stated below.

**Theorem 6.2.** ([2]) If \( \alpha \neq 0 \) is a real number, and \( c > 0 \) is an arbitrary constant, then

(i) \( \sup_{\alpha \in \mathbb{R}} \sum_{n \leq x} \lambda(n)e^{i2\pi\alpha n} < \frac{c_3 x}{(\log x)^c} \), unconditionally,

(ii) \( \sup_{\alpha \in \mathbb{R}} \sum_{n \leq x} \frac{\lambda(n)}{n} e^{i2\pi\alpha n} < \frac{c_4}{(\log x)^c} \), unconditionally,

where \( c_3 = c_3(c) > 0 \) and \( c_4 = c_4(c) > 0 \) are constants depending on \( c \), as the number \( x \to \infty \).

Advanced, and recent works on these exponential sums with multiplicative coefficients, and the more general exponential sums

\[
\sum_{n \leq x} f(n)e^{i2\pi\alpha n} \quad (6.1)
\]

where \( f : \mathbb{N} \to \mathbb{C} \) is a function, are covered in [18], [10], [1], [15], et alii.
6.3 Logarithm Average and Arithmetic Average Connection

The connection between the logarithm average
\[ \sum_{n \leq x} \frac{f(n)}{n} \]  
(6.2)
and the arithmetic average
\[ \sum_{n \leq x} f(n) \]  
(6.3)
of an arithmetic function \( f : \mathbb{N} \rightarrow \mathbb{C} \) is important in partial summations. The required error term to compute the arithmetic average (6.3) directly from the logarithm average (6.2) is explained in [6, Section 2.12], see also [6, Exercise 2.12].

Lemma 6.1. Let \( t \neq 0 \) be a small integer, and let \( x \geq 1 \) be a large number. If the logarithm average \( A(x) = \sum_{n \leq x} \lambda(n)\lambda(n+t)n^{-1} = O(\log x)(\log \log x)^{-1/2} \), then the arithmetic average
\[ \sum_{n \leq x} \lambda(n)\lambda(n+t) < \frac{x}{(\log \log x)^{1/2-\varepsilon}}, \]
where \( \varepsilon > 0 \) is a small number.

Proof. Assume \( B(z) = \sum_{n \leq z} \lambda(n)\lambda(n+t) \geq z(\log \log z)^{-1/2+\varepsilon} \). Then,
\[ \frac{\log x}{(\log \log x)^{1/2}} \gg \sum_{n \leq x} \frac{\lambda(n)\lambda(n+t)}{n} \]  
(6.4)
\[ = \int_1^x \frac{1}{z} dB(z) \]
\[ = \int_1^x \frac{B(z)}{z} dB(z) \]
\[ = \frac{B(x)}{x} + \int_1^x \frac{B(z)}{z^2} dz. \]
Since the integral
\[ \int_1^x \frac{B(z)}{z^2} dz = \int_2^x \frac{1}{z(\log \log z)^{1/2-\varepsilon}} dz \gg \frac{\log x}{(\log \log x)^{1/2-\varepsilon}}, \]  
(6.5)
for sufficiently large \( x \geq 1 \), the assumption is false. Hence, it implies that \( B(x) < x(\log \log x)^{-1/2+\varepsilon} \). \( \square \)
A more general result is worked out in the next result.

**Lemma 6.2.** If $x$ is a large number, and $0 \leq a_1 < a_2 < \ldots < a_k$ is an integer $k$-tuple, then a nontrivial logarithmic average

$$\sum_{n \leq x} \frac{\lambda(n + a_1)\lambda(n + a_2) \cdots \lambda(n + a_k)}{n} = o(\log x).$$

implies a nontrivial arithmetic average

$$\sum_{n \leq x} \lambda(n + a_1)\lambda(n + a_2) \cdots \lambda(n + a_k) = o(x).$$

**Proof.** Suppose that the arithmetic average is trivial, that is,

$$B(z) = \sum_{n \leq z} \lambda(n + a_1)\lambda(n + a_2) \cdots \lambda(n + a_k) = cz + o(z), \quad (6.6)$$

where $c > 0$ is a constant. Then,

$$o(\log x) = \sum_{n \leq x} \frac{\lambda(n + a_1)\lambda(n + a_2) \cdots \lambda(n + a_k)}{n} \quad (6.7)$$

$$= \int_1^x \frac{1}{z} dB(z)$$

$$= \frac{B(x)}{x} + \int_1^x \frac{B(z)}{z^2} dz.$$

Substituting and evaluating the integral yield

$$o(\log x) = \frac{B(x)}{x} + \int_1^x \frac{B(z)}{z^2} dz \quad (6.8)$$

$$= c + o(1) + c \log x + o(\log x),$$

for all sufficiently large $x \geq 1$. Clearly, the assumption (6.6) is false. Hence, it implies that $B(x) = o(x)$.

### 6.4 Autocorrelation Functions

The current estimate of the logarithmic average order of a product of two shifted Liouville functions. The result has the following asymptotic formula.

**Theorem 6.3.** Let $\lambda : \mathbb{N} \to \{ -1, 1 \}$ be the Liouville function, and let $x$ be a large number. If $t \neq 0$ is a fixed integer, then

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n + t)}{n} = O\left( \frac{\log x}{\sqrt{\log \log x}} \right).$$
The proof appears in Theorem 6.4 or [8, Corollary 2]. This improves the estimate $O((\log x)(\log \log \log x)^{-c})$, where $c > 0$ is a constant, described in [21, p. 5].

The previous result is sufficient to derive a weak form of the arithmetic average order a product of two shifted Liouville functions.

**Theorem 6.4.** Let $\lambda : \mathbb{N} \rightarrow \{-1, 1\}$ be the Liouville function, and let $x$ be a large number. If $t \neq 0$ is a fixed integer, then

$$\sum_{n \leq x} \lambda(n)\lambda(n + t) = O\left(\frac{\log x}{(\log \log x)^{1/2 - \varepsilon}}\right),$$

where $\varepsilon > 0$ is a small number.

**Proof.** This follows from Lemma 6.1. \qed

The general estimate of the logarithmic average order of a product of an even number of shifted Liouville functions remains an open problem. However, logarithmic average order of a product of an odd number of shifted Liouville functions has a nontrivial result.

**Theorem 6.5. ([22, Theorem 1.1]).** Let $k \geq 1$ be an odd natural number, and let $a_1, \ldots, a_k, b_1, \ldots, b_k$ be natural numbers. Then,

$$\sum_{n \leq x} \frac{\lambda(a_1n + b_1)\lambda(a_2n + b_2) \cdots \lambda(a_kn + b_k)}{n} = o(\log x),$$

as $x \rightarrow \infty$.

**Theorem 6.6.** Let $k = 2m + 1 \geq 1$ be an odd integer, and let $0 \leq a_1, \ldots, a_k$ be an integer $k$-tuple. Then,

$$\sum_{n \leq x} \lambda(n + a_1)\lambda(n + a_2) \cdots \lambda(n + a_k) = o(x),$$

as $x \rightarrow \infty$.

**Proof.** Set $a_1 = a_2 = \cdots = a_k = 1$, and let $0 \leq b_1 < b_2 < \cdots < b_k$ be an integer $k$-tuple. Then, this follows from Lemma 6.2 and Theorem 6.5. \qed
6.5 Distribution Functions

For any fixed integer $k > 4$, the $k$-tuples $\mu(n+a_1), \mu(n+a_2), \ldots, \mu(n+a_k)$ of Mobius values are not random, but pseudorandom or quasirandom. For example, the $k$-tuple

$$-1, -1, -1, \mu(n+a_4), \mu(n+a_5), \ldots, \mu(n+a_k),$$

and infinitely many other similarly structured $k$-tuples are not possible. This property seems to preempt the effect of the Linear Independence Conjecture on the summatory Mobius function. Assuming, the LI, the limits

$$\liminf_{x \to \infty} \frac{\sum_{n \leq x} \mu(n)}{x^{1/2}} = -\infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{\sum_{n \leq x} \mu(n)}{x^{1/2}} = \infty$$

were proved in [11], and refinements in [14]. This conjecture seems to imply that $\sum_{n \leq x} \mu(n) = O(x^{1/2+\varepsilon})$, where $\varepsilon > 0$. But, the Simple Zero Conjecture seems to imply that $\sum_{n \leq x} \mu(n) = O(x^{1/2})$, see [20, Theorem 14.29] for details. The numerical data are given in [13] is not conclusive, and it supports many different conjectures on the Mertens sum.

For any fixed integer $k > 4$, the $k$-tuples $\lambda(n+a_1), \lambda(n+a_2), \ldots, \lambda(n+a_k)$ of Liouville function values appears to be random, there are no known obstacles. For example, the $k$-tuple

$$-1, -1, -1, -1, \lambda(n+a_4), \lambda(n+a_5), \ldots, \lambda(n+a_k),$$

and infinitely many other similarly structured $k$-tuples are possible. Thus, the law of iterated logarithm for sequences of independent random variables seem to imply that

$$\liminf_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n)}{\sqrt{2x \log \log x}} = -\infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n)}{\sqrt{2x \log \log x}} = \infty.$$

In synopsis, the Linear Independence Conjecture does seem to apply to the summatory Liouville function. In particular, $\sum_{n \leq x} \lambda(n) = \zeta(1/2)x^{1/2} + O(x^{1/2+\varepsilon})$, where $\varepsilon > 0$.

The information in (6.9), (6.10), and the information in (6.11), (6.12) seems to imply that the random or pseudorandom variables

$$L(x) = \sum_{n \leq x} \lambda(n) \quad \text{and} \quad M(x) = \sum_{n \leq x} \mu(n)$$

have different distribution functions.
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