Rigid and gauge Noether symmetries for constrained systems

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We develop the general theory of Noether symmetries for constrained systems, that is, systems
that are described by singular Lagrangians. In our derivation, the Dirac bracket structure with
respect to the primary constraints appears naturally and plays an important role in the characteri-
zation of the conserved quantities associated to these Noether symmetries. The issue of projectability
of these symmetries from tangent space to phase space is fully analyzed, and we give a geometrical
interpretation of the projectability conditions in terms of a relation between the Noether conserved
quantity in tangent space and the presymplectic form defined on it. We also examine the enlarged
formalism that results from taking the Lagrange multipliers as new dynamical variables; we find the
equation that characterizes the Noether symmetries in this formalism, and we also prove that the
standard formulation is a particular case of the enlarged one. The algebra of generators for Noether
symmetries is discussed in both the Hamiltonian and Lagrangian formalisms. We find that a fre-
quent source for the appearance of open algebras is the fact that the transformations of momenta in
phase space and tangent space only coincide on shell. Our results apply with no distinction to rigid
and gauge symmetries; for the latter case we give a general proof of existence of Noether gauge sym-
metries for theories with first and second class constraints that do not exhibit tertiary constraints in
the stabilization algorithm. Among some examples that illustrate our results, we study the Noether
gauge symmetries of the Abelian Chern-Simons theory in $2n+1$ dimensions. An interesting feature
of this example is that its primary constraints can only be identified after the determination of the
secondary constraint. The example is worked out retaining all the original set of variables.

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1. INTRODUCTION

Variational principles and symmetries are two of the most fundamental organizing concepts in theoretical physics. One can add to this list the gauge principle. Noether symmetries, that is, continuous transformations that leave the action invariant up to boundary terms, constitute a link between the first two principles. In the case of a given gauge theory, some of the existing Noether transformations exhibit special features (the presence of Noether identities, arbitrary functions, etc.) that are associated with the redundancy in the description of degrees of freedom and the structure of constraints that characterize these theories. In a gauge theory there are two types of Noether symmetries, rigid and gauge. The first ones are physical symmetries with standard conserved quantities; the second, the gauge symmetries, are unphysical, and their associated conserved quantities vanish on-shell. We will deal simultaneously with both types of symmetries.

Noether transformations provide for a wide class of symmetries of the equations of motion at the classical level. They also become, in the absence of anomalies, the symmetries of the quantum systems, and they are neatly displayed in the path integral formulation. But despite of the relevance of these symmetries for physical systems, many aspects of their characterization as well as the characterization of their conserved quantities, either in the tangent space or in the phase space of some configuration space, have been insufficiently studied. There is at least one reason: It is neither immediate nor trivial to extend the results that one can obtain for regular theories (theories with no gauge invariances) to theories described through singular Lagrangians (thus having room for gauge invariance). Our aim is to contribute to such a study.

We will first focus in the first place (more general cases will be dealt with later) on infinitesimal Noether transformations of the type \( \delta q(q, \dot{q}; t) \) in the tangent bundle \( TQ \) of some configuration space \( Q \), extended to include the independent variable (time) to \( TQ \times R \), for some theory whose dynamics is described by a first order variational principle based on a time-independent Lagrangian \( L(q, \dot{q}) \). Our transformations act only on the dependent variables \( q \); this is general enough because any infinitesimal transformation which also acts on the independent variable \( t \) (or the space-time variables in field theory) can be brought to this form. The defining property of these transformations is that they leave the Lagrangian invariant up to a total time derivative:

\[
\delta L = \frac{d F}{d t}.
\]  

(In the case of field theory the total time derivative is substituted by a total space-time divergence) We use the language most common to physics. In a more mathematically oriented language, see [1], \( \delta L \) would be understood as the action on \( L \) of the prolongation of the vector field that generates the transformation. Except for an infinitesimal parameter, \( \delta q \) is the characteristic of such a vector field.

Equation (1.1) guarantees that \( \delta q \) maps solutions of the equations of motion into solutions, because the equations of motion remain invariant. Associated with this Noether transformation there is always a conserved quantity,

\[
G_L = (\partial L/\partial \dot{q})\delta q - F.
\]

If the Lagrangian is non-singular, that is, if \( \det [\partial L/\partial \dot{q} \partial \dot{q}] \neq 0 \), then velocities are mapped one-to-one to canonical momenta, and the conserved quantity \( G_L \) becomes in phase space the canonical generator, \( G \), acting through the Poisson bracket, of the transformation \( \delta q \). However, the most common case in theoretical physics is the case when \( L \) is singular. It is only under this circumstance that the phenomena of gauge freedom may occur. For a singular \( L \), \( G_L \) is still a projectable quantity [2], that is, it may be brought to the phase space as a function \( G \). This function \( G \) is now uniquely defined only up to the addition of linear combinations of the primary constraints. In contrast with the non-gauge case, the fact of being a canonical conserved quantity does not guarantee that \( G \), or any of its equivalent functions whose pull-back to tangent space is \( G_L \), generates \( \delta q \) or even a Noether transformation. It is not even guaranteed that \( \delta q \) is a transformation projectable to phase space.

Our study will clarify in what sense, and to what extent, we may consider \( G \) as the generator of the original transformation \( \delta q \). We give a general characterization of the functions \( G \) in phase space that are associated with a Lagrangian Noether transformation in the case of gauge theories, and we show the general construction, made out of \( G \), of these transformations. We prove that the projectability to phase space of \( \delta q \) is related to a geometric requirement that \( G_L \) must satisfy. When this requirement is met, we show that \( \delta q \) is canonically generated.

Our procedure shows that there is a natural generalization to a framework, the enlarged formalism, where the dynamics is defined through the canonical Lagrangian. In this enlarged formalism the Lagrange multipliers associated to the primary Hamiltonian constraints become new independent variables. Then we can define generalized Noether transformations depending on the Lagrange multipliers and their time derivatives at any finite order. We make contact here with the formulation in [3,4]. It turns out that the results in [3] can be understood as an application of our formulation in order to generate Noether gauge transformations for systems with only first class constraints in a systematic way.

Our regularity assumptions are standard: We consider that the Hessian matrix of the Lagrangian with respect to the velocities has constant rank. We also assume that the primary constraints may be split into first and second class on the primary constraint surface. Some of our results, particularly in section 4, will not depend on this second assumption.
The paper is organized as follows: In section 2 we introduce some known results on the dynamics of Lagrangian and Hamiltonian systems that will be needed in the following sections. In section 3 we develop the general theory of Noether symmetries for gauge systems. In particular we characterize the conserved quantities in phase space associated with these symmetries and we give the method to retrieve the original Noether transformation out of its conserved quantity. It is remarkable that the Dirac bracket structure plays a natural role in this context. We also relate the projectability conditions for the Noether transformation to a property of the Lagrangian Noether conserved quantity $G_L$ that relates to the presymplectic form in tangent space. We also see that a projectable Noether transformation always becomes a canonical transformation in phase space.

In section 4 we introduce the enlarged formalism with the canonical Lagrangian. The definition of Noether transformations is then generalized by allowing $\delta g$ to depend on the Lagrange multipliers \[2\] and we prove the equivalence with the former formulation when this dependence does not contain derivatives of the Lagrange multipliers. In section 5 some properties of the algebras of transformations and generators are exhibited. In particular we show that the closure under Poisson bracket of the algebra of generators in phase space does not necessarily guarantee the closure of the algebra of the infinitesimal transformations in configuration space. This fact is relevant for the application of BRST methods \[3\]. In section 6 we distinguish between rigid and gauge Noether symmetries, and extend some theorems concerning the existence of canonical gauge Noether transformations to systems with first and second class constraints with only one step in the stabilization algorithm. Finally, we present some examples to illustrate our results and a short appendix devoted to the concept of auxiliary variables that will be used in section 4. Of particular interest is example 4, devoted to the analysis of the Noether gauge symmetries of the Abelian Chern-Simons theory in $2n + 1$ dimensions. The example is worked out retaining all the original set of variables. It is interesting to remark that this example violates one of our regularity assumptions, but we show that it can be still accommodated within our formalism.

Our results are local. They apply to systems with a finite number of degrees of freedom as well as to field theories. We use, for simplicity, the language of mechanics. DeWitt’s condensed index notation \[4\] translate our results directly to field theory as long as the boundary conditions allow for the elimination of surface terms.

We rely on previous work, particularly \[5] and references therein, and so we must first summarize some of the results of these papers before proceeding to subsequent developments.

### II. ELEMENTS OF LAGRANGIAN AND HAMILTONIAN DYNAMICS FOR GAUGE THEORIES

The Euler-Lagrange functional derivative of a first order Lagrangian $L(q, \dot{q})$, is

$$[L]_i := \alpha_i - W_{is} \dot{q}^s,$$

where

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \quad \text{and} \quad \alpha_i := -\frac{\partial^2 L}{\partial \dot{q}^i \partial q^s} \dot{q}^s + \frac{\partial L}{\partial \dot{q}^i}.$$

We consider the general case where the Hessian $W = (W_{ij})$ may be a singular matrix \[11\]. We assume that its rank is constant in the region of tangent space of our interest. If $W$ is singular, there exists a kernel for the pull-back $\mathcal{F}^*L$ of the Legendre map $\mathcal{F}L$ from configuration-space $TQ$ (the tangent bundle $TQ$ of the configuration space $Q$) to phase space $T^*Q$ (the cotangent bundle). This kernel is spanned by the vector fields

$$\Gamma_\mu = \gamma^i_\mu \frac{\partial}{\partial \dot{q}^i},$$

where $\gamma^i_\mu$ span a basis for the null vectors of $W_{ij}$. \[1\] A function $g(q, \dot{q})$ is projectable to phase space if and only if

$$\Gamma_\mu g = 0.$$

Notice that the Lagrangian equations of motion $[L]_i = 0$ imply the primary Lagrangian constraints

$$\chi_\mu := (\alpha_i \gamma^i_\mu) = 0.$$

The time evolution for a gauge theory is not unique until the gauge freedom has been removed, for example, by way of some gauge fixing. This is reflected in the ambiguities present in the Lagrangian time-evolution differential operator \[3\].

$$X_L := \frac{\partial}{\partial \mu} + \dot{q}^i \frac{\partial}{\partial q^i} + a^s(q, \dot{q}) \frac{\partial}{\partial \dot{q}^s} + \eta^\mu \Gamma_\mu =: X_0 + \eta^\mu \Gamma_\mu,$$

where $a^s$ are functions which are determined by the formalism, and $\eta^\mu$ are arbitrary functions. These arbitrary functions express the gauge freedom of the time-evolution

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1 Notice that if the phase space is enlarged by introducing the Lagrange multipliers as new dynamical variables, the accordingly enlarged Legendre map will now become invertible. Therefore we expect that the problems related with projectability can be overcome in this enlarged formalism. We will consider this issue in section 4.
operating. Notice that projectable quantities have a well defined unambiguous dynamics. The tangency of \( X_L \) to the primary Lagrangian constraint surface, defined by \( \lambda_\mu = 0 \), may lead to new constraints and to the determination of some of the functions \( \eta^\mu \). At this point, new tangency requirements may occur \([10,11]\).

It is not necessary to use the Hamiltonian technique to find the \( \Gamma_\mu \), but it does facilitate the calculation:

\[
\gamma^\mu = \mathcal{F}L^* \left( \frac{\partial \phi_\mu}{\partial p_i} \right),
\]

where the \( \phi_\mu \) are the Hamiltonian primary constraints. They satisfy by definition \( \mathcal{F}L^* \phi_\mu = 0 \).

These constraints \( \phi_\mu \) span a basis for the ideal of functions in \( T^*Q \) that vanish on the image of the Legendre map \( \mathcal{F}L \). We take as an assumption that these constraints may be split into first class, \( \phi_{\mu_0} \), and second class, \( \phi_{\mu_1} \), satisfying

\[
\{ -,- \} = pc, \quad \det |\{ \phi_{\mu_1}, \phi_{\nu_1} \}| \neq 0,
\]

where \( \{-,-\} \) is the Poisson Bracket structure and \( pc \) stands for a generic linear combination of the primary constraints.

The Lagrangian energy, \( E_L(q, \dot{q}) := \dot{q}^i (\partial L/\partial \dot{q}^i) - L \), is a function projectable to phase space. The canonical Hamiltonian \( H_c(q, p) \), which is only uniquely defined up to primary constraints, is defined such that its pullback to tangent space is the Lagrangian energy:

\[
\mathcal{F}L^* H_c = E_L.
\]

To connect the Lagrangian and Hamiltonian dynamics it is convenient to write down the two following identities \([7]\):

\[
\dot{q}^i = \mathcal{F}L^* \left( \frac{\partial H_c}{\partial p_i} \right) + v^\mu (q, \dot{q}) \mathcal{F}L^* \left( \frac{\partial \phi_\mu}{\partial p_i} \right),
\]

\[
\frac{\partial L}{\partial \dot{q}^i} = -\mathcal{F}L^* \left( \frac{\partial H_c}{\partial q^i} \right) - v^\mu (q, \dot{q}) \mathcal{F}L^* \left( \frac{\partial \phi_\mu}{\partial q^i} \right);
\]

where the functions \( v^\mu \) are determined so as to render the first relation an identity. Notice the important relation

\[
\Gamma_\mu v^\nu = \delta^\nu_\mu,
\]

which stems from applying \( \Gamma_\mu \) to the first identity and taking into account that

\[
\Gamma_\mu \circ \mathcal{F}L^* = 0,
\]

where \( \circ \) denotes the composition operation.

The Hamiltonian time evolution vector field is given by

\[
X_H := \frac{\partial}{\partial t} + \{ -, H_c \} + \lambda^\nu \{ -, \phi_\mu \},
\]

where \( \lambda^\nu \) are arbitrary functions of time. However, they are determined as non-projectable functions in tangent space: They are the functions \( v^\mu (q, \dot{q}) \) implicitly defined by equations (2.5). These variables \( \lambda^\nu \) are Lagrange multipliers.

The requirement of tangency of \( X_H \) to the primary Hamiltonian constraint surface, defined by \( \phi_\mu \), may lead to new constraints and to the determination of some of the functions \( \lambda^\nu \). This is the setting of the stabilization algorithm in phase space, that runs parallel (see \([7]\)) to the corresponding algorithm in tangent space. Details on the relationship between the functions \( \eta^\mu \) in \( TQ \) and the functions \( \lambda^\nu \) in \( T^*Q \) are given in \([12]\).

With the identities (2.3) and (2.4), we can relate the time evolution in the Hamiltonian and the Lagrangian formalisms for any function \( f(q, p, t) \). In \([7]\) an evolution operator \( K \) is defined that gives the time evolution of a function \( f \) in \( T^*Q \times R \) as a function in \( TQ \times R \),

\[
K f := \mathcal{F}L \frac{\partial f}{\partial t} + \dot{q} \mathcal{F}L^* \frac{\partial f}{\partial q} + \frac{\partial L}{\partial q} \mathcal{F}L^* \frac{\partial f}{\partial p}.
\]

This operator \( K \) is fully studied in \([7]\). Notice the immediate result

\[
K \phi_\mu = \chi_\mu,
\]

which is deduced by recalling that \( \mathcal{F}L^* \phi_\mu = 0 \). Using the identities (2.7) and (2.8), we can get a new expression for \( K f \),

\[
K f = \mathcal{F}L^* \frac{\partial f}{\partial t} + \mathcal{F}L^* \{ f, H_c \} + v^\mu \mathcal{F}L^* \{ f, \phi_\mu \}.
\]

With this new expression, application of the vector fields \( \Gamma_\mu \) to (2.11) gives the following result \([7]\):

\[
\Gamma_\mu \chi_\nu = \mathcal{F}L^* \{ \phi_\nu, \phi_\mu \}.
\]

Now we are ready to relate the primary Lagrangian constraints to the secondary Hamiltonian constraints and the canonical determination of some arbitrary functions of the Hamiltonian dynamics. To this end, let us first apply (2.10) to \( f = \phi_{\mu_0} \),

\[
\chi_{\mu_0} = K \phi_{\mu_0} = \mathcal{F}L^* \{ \phi_{\mu_0}, H_c \} + v^\mu \mathcal{F}L^* \{ \phi_{\mu_0}, \phi_\mu \}
\]

\[
= \mathcal{F}L^* \{ \phi_{\mu_0}, H_c \},
\]

where we have used (2.4). Then, if we define the secondary Hamiltonian constraints as

\[
\phi^1_{\mu_0} := \{ \phi_{\mu_0}, H \},
\]
their pullback to tangent space gives a subset of the primary Lagrangian constraints:

\[ \chi_{\mu_0} = \mathcal{F}L^* \phi_{\mu_0}^0. \]  

Next, applying \((2.10)\) to \(f = \phi_{\mu_1}\), we find

\[ \chi_{\mu_1} = K \phi_{\mu_1} = \mathcal{F}L^* \{ \phi_{\mu_1}, H \} + v^* \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \} \]

\[ = \mathcal{F}L^* \{ \phi_{\mu_1}, H \} + v^* \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \}. \]  

The stabilization of the second class constraints fixes

\[ \chi_{\lambda} \]

nutance that every Lagrangian constraints may give some redundancy, for there is no guar-

mary Lagrangian constraints

their pullback to tangent space gives a subset of the primary Lagrangian constraints:

\[ \chi_{\mu_0} = \mathcal{F}L^* \phi_{\mu_0}^0. \]  

Next, applying \((2.10)\) to \(f = \phi_{\mu_1}\), we find

\[ \chi_{\mu_1} = K \phi_{\mu_1} = \mathcal{F}L^* \{ \phi_{\mu_1}, H \} + v^* \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \} \]

\[ = \mathcal{F}L^* \{ \phi_{\mu_1}, H \} + v^* \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \}. \]  

The stabilization of the second class constraints fixes

\[ \chi_{\lambda} \]

The arbitrary functions \(\lambda^\nu_1\) become determined as canonical functions \(\lambda^\nu_c=0\) through

\[ 0 = \{ \phi_{\mu_1}, H_c \} + \lambda^\nu_c \{ \phi_{\mu_1}, \phi_\nu \}. \]  

Then, we can put together \((2.16)\) and \((2.17)\) to get:

\[ \chi_{\mu_1} = (v^\nu - \mathcal{F}L^* \lambda^\nu_1) \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \}. \]  

Notice that the first class and second class classification of the primary constraints has an effect in the tangent space formalism: it classifies the primary Lagrangian constraints into two sets according to their projectability to phase space. Some of the Lagrangian constraints, \(\chi_{\mu_0}\), are projectable, as shown by \((2.15)\), whereas the rest, the constraints \(\chi_{\mu_1}\), are non-projectable, as we can verify by applying \((2.7)\) and \((2.8)\) to \((2.18)\). Notice also that the constraints of the type \(\chi_{\mu_0}\), or some of them, may vanish identically, whereas the constraints of the type \(\chi_{\mu_1}\) are all independent.

This finishes the summary of results that will be used in the next sections.

III. NOETHER SYMMETRIES

A. General theory of Noether symmetries in tangent space and in phase space

We start with an infinitesimal Noether Lagrangian symmetry on \(TQ \times R\), that is, an infinitesimal \(\delta q(q, \dot{q}; t)\) such that

\[ \delta L = \frac{dF}{dt}, \]

with

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \dot{q} \frac{\partial}{\partial \dot{q}} + \ldots, \]

\[ \chi_{\mu_0} = \mathcal{F}L^* \phi_{\mu_0}^0. \]  

Next, applying \((2.10)\) to \(f = \phi_{\mu_1}\), we find

\[ \chi_{\mu_1} = K \phi_{\mu_1} = \mathcal{F}L^* \{ \phi_{\mu_1}, H \} + v^* \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \} \]

\[ = \mathcal{F}L^* \{ \phi_{\mu_1}, H \} + v^* \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \}. \]  

The stabilization of the second class constraints fixes

\[ \chi_{\lambda} \]

\(\lambda^\nu_1\) become determined as canonical functions \(\lambda^\nu_c=0\) through

\[ 0 = \{ \phi_{\mu_1}, H_c \} + \lambda^\nu_c \{ \phi_{\mu_1}, \phi_\nu \}. \]  

Then, we can put together \((2.16)\) and \((2.17)\) to get:

\[ \chi_{\mu_1} = (v^\nu - \mathcal{F}L^* \lambda^\nu_1) \mathcal{F}L^* \{ \phi_{\mu_1}, \phi_\nu \}. \]  

Notice that the first class and second class classification of the primary constraints has an effect in the tangent space formalism: it classifies the primary Lagrangian constraints into two sets according to their projectability to phase space. Some of the Lagrangian constraints, \(\chi_{\mu_0}\), are projectable, as shown by \((2.15)\), whereas the rest, the constraints \(\chi_{\mu_1}\), are non-projectable, as we can verify by applying \((2.7)\) and \((2.8)\) to \((2.18)\). Notice also that the constraints of the type \(\chi_{\mu_0}\), or some of them, may vanish identically, whereas the constraints of the type \(\chi_{\mu_1}\) are all independent.

This finishes the summary of results that will be used in the next sections.

We contract \((3.3)\) with a null vector \(\gamma^i\) to find that

\[ \Gamma_\mu G_L = 0. \]  

It follows that \(G_L\) is projectable to a function \(G\) in \(T^*Q\); that is, it is the pullback of a function (not necessarily unique) in \(T^*Q\):

\[ G_L = \mathcal{F}L^* (G). \]

This important property, valid for any conserved quantity associated with an infinitesimal Noether symmetry of the type considered here, was first pointed out in [2]. Observe that \(G\) is determined up to the addition of linear combinations of the primary constraints. Substitution of this result in \((3.3)\) gives

\[ W_is \left( \delta q^i - \mathcal{F}L^* \left( \frac{\partial G}{\partial p_i} \right) \right) = 0, \]

and so the parentheses enclose a null vector of \(W\):

\[ \delta q^i = \mathcal{F}L^* \left( \frac{\partial G}{\partial p_i} \right) - \sum \mu r^\mu \gamma^i_\mu, \]

for some \(r^\mu(q, \dot{q}; t)\).

Our aim is to get a complete characterization of the canonical generator \(G\) in phase space. We start by defining

\[ \hat{p}_i = \frac{\partial L}{\partial \dot{q}^i}. \]

Notice that if \(W_is\) is invertible we can deduce from \((3.3)\) the form of the associated Noether transformation,

\[ \delta q^i = W_is \frac{\partial G_L}{\partial \dot{q}^i}, \]

where \(W_is\) denotes the inverse to \(W_is\).
After subtraction from (3.2) of the piece containing \( \ddot{q} \), we obtain

\[
\left( \frac{\partial L}{\partial q^i} - \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \right) \left( FL^* (\frac{\partial G}{\partial p_i}) - \sum_{\mu} r^\mu \gamma_{\mu}^i \right) + \dot{q}^i \frac{\partial}{\partial q^i} FL^* (G) + FL^* \left( \frac{\partial G}{\partial t} \right) = 0, \tag{3.7}
\]

which simplifies to

\[
\frac{\partial L}{\partial q^i} FL^* (\frac{\partial G}{\partial p_i}) + \dot{q}^i FL^* (\frac{\partial G}{\partial q^i}) + FL^* \left( \frac{\partial G}{\partial t} \right) = r^\mu \chi_{\mu}. \tag{3.8}
\]

Substitution of the two identities (2.15) and (2.18) into (3.9). We get

\[
FL^* \left( \frac{\partial G}{\partial t} \right) + FL^* \{G, H_c\} + v^\nu FL^* \{G, \phi_\mu\} = r^\mu \chi_{\mu}. \tag{3.9}
\]

Notice that (3.9) is invariant under

\[
r^\mu \rightarrow r^\mu + s^\mu, \quad \text{where} \quad s^\mu = A^\mu_{\nu} \chi_{\nu} + s_0^\mu, \tag{3.10}
\]

with \( A \) an arbitrary antisymmetric matrix with functions in \( TQ \times R \) as components, and with \( s_0^\mu \) any function in \( TQ \times R \) not vanishing at the Lagrangian primary constraint surface and such that \( s_0^\mu \chi_{\nu} \) is identically zero. Such \( s_0^\mu \) only exist \([13]\) for theories whose projectable primary Lagrangian constraints are not all independent.

These indeterminacies for \( r^\mu \) in (3.9) entail an irrelevant (trivial) change in the \( \delta q^i \) of (3.9):

\[
\delta q^i \rightarrow \delta_i q^i = \delta q^i - a_i^{ij} \left[ L \right]_{ij} - s_0^\mu \gamma_{\mu}^i, \tag{3.11}
\]

where \( a_i^{ij} = \gamma_{\mu}^i A^\mu_{\nu} \gamma_{\nu}^j \). This result is already obvious in (3.2) because \( \left[ L \right]_{ij} \delta q^i = \left[ L \right]_{ij} \delta q^j \).

Now we will obtain a purely Hamiltonian characterization of \( G \). In doing so the relevance of the Dirac bracket structure will be emphasized. Let us introduce \( v^\nu \) of (2.18) into (3.9). We get

\[
FL^* \left( \frac{\partial G}{\partial t} \right) + FL^* \{G, H_c\} + v^\nu FL^* \{G, \phi_\mu\} \]

\[
+ (FL^* \chi_{\nu}^i + M^\mu_{\nu} \chi_{\mu}) FL^* \{G, \phi_\nu\} = r^\mu \chi_{\mu}, \tag{3.12}
\]

where \( M = (M^\mu_{\nu}) \) is defined as the matrix inverse of the Poisson bracket matrix of the primary second class constraints. Define

\[
r^\mu_1 := r^\mu - FL^* \{G, \phi_\nu\} M^\nu_{\mu} \]

and

\[
r^\mu_0 := r^\mu_0. \]

We get

\[
FL^* \left( \frac{\partial G}{\partial t} \right) + FL^* \{G, H_c\} + v^\nu FL^* \{G, \phi_\mu\} \]

\[
+ FL^* \chi_{\nu}^i FL^* \{G, \phi_\nu\} = r^\mu \chi_{\mu}. \tag{3.13}
\]

However, \( \chi_{\nu}^i \) is determined from (2.17) as

\[
\chi_{\nu}^i = - M^\mu_{\nu} \{\phi_\nu, H_c\}. \tag{3.14}
\]

Introduction of this result into (3.13) produces the natural appearance of the Dirac bracket \([10]\):

\[
FL^* \left( \frac{\partial G}{\partial t} \right) + FL^* \{G, H_c\} + v^\nu FL^* \{G, \phi_\mu\} = r^\mu \chi_{\mu}, \tag{3.14}
\]

where the Dirac bracket is defined, at this stage of the stabilization algorithm, by

\[
\{A, B\} := \{A, B\} - \{A, \phi_\nu\} M^\mu_{\nu} \{\phi_\nu, B\}. \tag{3.15}
\]

Now apply \( \Gamma_{v_0} \) to (3.14). Using (2.7) and (2.8) we arrive at

\[
FL^* \{G, \phi_\mu\} = (\Gamma_{v_0} v^\nu) \chi_{\nu}, \tag{3.15}
\]

where we have also used (2.13) and (2.4) to get \( \Gamma_{v_0} v^\nu = 0 \). But since the left side of (3.15) is a projectable function, so must be the right. However, only the constraints \( \chi_{\nu} \) are projectable. Indeed, according to (2.13), they are the pullback of the secondary Hamiltonian constraints. In conclusion, we can write that

\[
\{G, \phi_\mu\} = sc + pc, \tag{3.16}
\]

where \( sc \) (pc) stands for a linear combination of secondary (primary) Hamiltonian constraints. Introduce this result in (3.14); the same reasoning yields

\[
\frac{\partial G}{\partial t} + \{G, H_c\}^* = sc + pc. \tag{3.17}
\]

Notice that (3.16) can be equivalently written as

\[
\{G, \phi_\mu\}^* = sc + pc. \tag{3.18}
\]

Let us remark that the secondary constraints in the right sides of (3.16) and (3.17) are the stabilization of the primary first class constraints, as defined in (2.13). Therefore we are not only saying that the left sides of (3.16) and (3.17) must vanish on the surface defined by the primary plus secondary constraints. We are saying something more restrictive, because some of the secondary constraints, as obtained through the stabilization of the primary first class constraints according to (2.14), may be ineffective (that is, such that their gradient also vanishes in the constraint surface). In such a case, the left sides of (3.16) and (3.17) must reflect this fact. Particular examples of this behavior can be found in [16].

We have arrived at the following result: The necessary and sufficient condition for a function \( G \in TQ \times R \) to be a Noether canonical conserved quantity, that is, such that its pullback to \( TQ \times R \) satisfies equation (3.13), is that \( G \) satisfies equations (3.16) and (3.17).

This result generalizes to systems with gauge freedom the standard definition of a Noether conserved quantity in phase space.
B. Getting $\delta q$ from $G$

Equations (3.14) and (3.17) express the most general condition for a canonical conserved quantity to be Noether, that is, such that $G_L = \mathcal{F}L^*G$ satisfies (3.15), for some $\delta q^i(q, q; t)$. In fact, if $G$ satisfies (3.14) and (3.17), then $\delta q^i$ can be obtained, except, of course, for the arbitrariness described in (3.11), from $G$ as follows. Let us first rewrite (3.16) and (3.17) using a notation for the coefficients in the secondary constraints:

$$\frac{\partial G}{\partial t} + \{G, H_c\}^* = A^\mu_0 \phi^1_{\mu_0} + pc, \quad (3.19)$$

$$\{G, \phi_{\mu_0}\} = B^\mu_0 \phi^1_{\mu_0} + pc. \quad (3.20)$$

Notice that only the coefficients $B^\mu_0$ are invariant under the changes of $G$ allowed by the addition of arbitrary linear combinations of the primary constraints. Comparing (3.19) and (3.20) with (3.14) and (3.15), we identify a set of solutions for $r^{\mu_0}$,

$$r^{\mu_0} = \mathcal{F}L^*A^{\mu_0} + v^{\mu_0} \mathcal{F}L^*B^\mu_0, \quad (3.21)$$

that give

$$r^{\mu_1} = 0,$$

and

$$r^{\mu_1} = \mathcal{F}L^*\{G, \phi_{\nu_1}\}M^{\nu_1}. \quad (3.22)$$

The general solution for the $r^{\mu}$ may be obtained using (3.10).

Using (3.21) and (3.22) in (3.3) we find the result we were looking for,

$$\delta q^i = \mathcal{F}L^*\{q^i, G\}^* - (\mathcal{F}L^*A^{\mu_0} + v^{\mu_0} \mathcal{F}L^*B^\mu_0)\gamma^i_{\nu_0}. \quad (3.23)$$

Notice again the natural appearance of the Dirac Bracket.

Up to now, $G$ is any function in $T^*Q \times R$ whose pullback to $TQ \times R$ is $G_L$. We have obtained therefore the following results: First, we have in (3.16) and (3.17) the conditions for a function $G$ in phase space to be associated with a Lagrangian Noether transformation. Next, this transformation is entirely recovered through (3.23), up to the addition of trivial pieces of the type described in (3.11).

Notice that in general there are obstructions that prevent $\delta q^i$ from being canonically generated. As we will see now, the quantities $A^{\mu_0}$ are readily absorbed through a redefinition of $G$. Indeed, the quantities $B^\mu_0$ are the true obstructions to projectability: they cannot be absorbed in $G$ by the changes allowed because $G$ is only determined up to primary constraints. We define

$$G' = G - A^{\mu_0} \phi_{\mu_0}. \quad (3.24)$$

and then

$$G'^* = G' - \{G', \phi_{\mu_1}\}M^{\mu_1\nu_1} \phi_{\nu_1}. \quad (3.25)$$

($G'^*$ is the “starred” function defined [10] for $G'$; it allows to “put the star within the bracket” and to continue with the Poisson bracket instead of the Dirac bracket: for any $f$, $\{-, f\}^* = \{-, f^*\} + pc$) Conditions (3.19) and (3.20) are then modified to

$$\frac{\partial G'^*}{\partial t} + \{G'^*, H_c\} = pc, \quad (3.26)$$

$$\{G'^*, \phi_{\mu_0}\} = B^\mu_0 \phi^1_{\mu_0} + pc, \quad (3.27)$$

$$\{G'^*, \phi_{\mu_1}\} = pc. \quad (3.28)$$

The Noether transformation $\delta q^i$ then becomes

$$\delta q^i = \mathcal{F}L^*\{q^i, G'^*\} - v^{\mu_0} (\mathcal{F}L^*B^\mu_0)\gamma^i_{\nu_0}. \quad (3.29)$$

We also learn from this last expression that the conditions for the projectability of the transformation $\delta q^i$ are equivalent to the conditions for $\delta q^i$ to be canonically generated. That is: A Noether transformation in phase space is always a canonical transformation.

Notice that all primary constraints satisfy (3.19) and (3.20) as does $G$. In the case of the primary second class constraints $\phi_{\mu_1}$, $A^{\mu_0}$ and $B^\mu_0$ vanish, and $\delta q^i$ in (3.23) is just zero; and so this case is uninteresting. In the case of the primary first class constraints $\phi_{\mu_0}$, an interesting case is when $\{\phi_{\mu_0}, H_c\} = pc$, that is, when $A^{\mu_0}$ is zero. Then $\epsilon(t)\phi_{\mu_0}$ is a gauge generator for arbitrary $\epsilon$. Other cases associated with $\phi_{\mu_0}$ must include [18–21] a “chain” of first class secondary, tertiary, etc., constraints as well.

C. Geometrical interpretation of the projectability conditions

It is possible to give a geometrical meaning to the projectability conditions for the Noether transformations from tangent space to phase space. Observe in (3.23), or (3.29), that it is the presence of the matrix $B = (B^\mu_0)$ that prevents this transformation $\delta q^i$ from being projectable to phase space. Technically, the condition of projectability for at least one of the transformations $\delta q^i$ among the set given by (3.11) is that

$$\mathcal{F}L^*(B^\mu_0 \phi^1_{\mu_0}) = 0, \quad \forall \mu_0, \quad (3.30)$$

In the case of gauge symmetries, a redefinition of the arbitrary functions may help to achieve projectability, see section 6.2.
because this makes the sc term in the right side of (3.20), or (3.27), a pc term. In such case (3.20), (3.27), and (3.28) become
\[
\frac{\partial G^*}{\partial t} + \{G^*, H_c\} = pc, \quad (3.31)
\]
\[
\{G^*, \phi_\mu\} = pc. \quad (3.32)
\]
Equations (3.31) and (3.32) are the conditions obtained in [8] to define a projectable Noether transformation \(\delta q^i = \mathcal{L}^* \{q^i, G^*\}\).

There is an elegant way to rephrase the conditions that make the Lagrangian Noether transformation projectable to phase space. Consider the kernel of the presymplectic form \(\omega_L\) in tangent space. This presymplectic form is defined as the pullback of the standard symplectic form in phase space, that is,
\[
\omega_L := dq^i \wedge d\left(\frac{\partial L}{\partial \dot{q}^i}\right).
\]
A basis for its kernel is provided [17] by the vector fields
\[
\Gamma_\mu = \gamma^j_\mu \frac{\partial}{\partial q^j},
\]
and
\[
\Delta_{\mu_0} = \gamma^j_{\mu_0} \frac{\partial}{\partial q^j} + \beta^j_{\mu_0} \frac{\partial}{\partial \dot{q}^j}, \quad (3.34)
\]
with \(\beta^j_{\mu_0}\) given by
\[
\beta^j_{\mu_0} = K \frac{\partial \phi_{\mu_0}}{\partial p_j} - \mathcal{L}^* \left(\frac{\partial \phi_{\mu_0}}{\partial \dot{p}_j}\right). \quad (3.35)
\]
It is also shown in [17] that, for any function \(f(q, p; t)\) on \(T^*Q \times R\), the following property holds:
\[
\Delta_{\mu_0} (\mathcal{L}^* f) = \mathcal{L}^* \{f, \phi_{\mu_0}\}. \quad (3.36)
\]
Let us now apply the basis vectors of this kernel to our Noether conserved quantity \(G_L\). We have (3.4),
\[
\Gamma_\mu G_L = 0,
\]
and also
\[
\Delta_{\mu_0} G_L = \Delta_{\mu_0} (\mathcal{L}^* G) = \mathcal{L}^* \{G, \phi_{\mu_0}\} = \mathcal{L}^* (B^i_{\mu_0} \phi^1_{\mu_0}), \quad (3.37)
\]
where (3.36) and (3.20) have been used. Recall that the conditions of projectability for \(\delta q^i\) are equations (3.30). We can write therefore the projectability conditions for \(\delta q^i\) as
\[
\Gamma_\mu G_L = 0, \quad \Delta_{\mu_0} G_L = 0. \quad (3.38)
\]
Therefore: The necessary and sufficient condition for a Lagrangian Noether conserved function \(G_L\) to be associated through (3.4) with a transformation \(\delta q^i\) that is projectable to phase space is that \(G_L\) must give zero when acted upon by the vector fields in the kernel of the presymplectic form in tangent space.

Notice, as a consequence, that if all primary constraints are second class (at the primary level), then the basis of this kernel is simply \(\Gamma_\mu\), and therefore all Noether transformations may be chosen to be projectable.

D. Conditions for the Noether conserved quantity, revisited

Equations (3.10) and (3.17) display the necessary and sufficient conditions for a constant of motion \(G\) in phase space to be associated with a Noether transformation (either projectable or not) in tangent space. What is the rationale for these conditions? Of course they say that \(G\) is indeed a constant of motion. But there are constants of motion that only satisfy less restrictive conditions that \(G\) does, because in (3.10) and (3.17) use has not been made of the full stabilization algorithm –to exhibit all the constraints of the theory–, but only the first step. Why is it that, for a constant of motion to be Noether, the specific conditions (3.10) and (3.17) must be met? Now we will give an independent argument to explain these results.

As we have already seen, the relation (3.2)
\[
[L]_i \delta q^i + \frac{dG_L}{dt} = 0
\]
is telling us that \(G_L\) is a constant of motion for the dynamics defined by \([L]_i = 0\). In (3.2), \(q, \dot{q}, \ddot{q}\) play the role of independent variables. This means that only the primary Lagrangian constraints \(\lambda_{\mu_0} = 0\) are taken into account, for they are the only constraints algebraically included in \([L]_i = 0\). The correspondence of these constraints with the Hamiltonian constraints has been discussed in section 2. The primary Hamiltonian constraints, \(\phi_\mu = 0\), are non-dynamical in the sense that they do not appear as a consequence of the equations of motion but only define the image in \(T^*Q\) of the Legendre map. Recalling section 2 they classify into first class and second class. The first step of the Hamiltonian stabilization algorithm will determine some arbitrary functions \(\lambda^\mu_\nu\) (the Lagrange multipliers of the primary second class constraints) as canonical functions \(\lambda^\mu_\nu\), and some secondary constraints \(\phi^1_{\mu_0} = \{\phi_{\mu_0}, H_c\}\). As we have seen in (2.13), the pullback of the secondary constraints gives a subset of the primary Lagrangian constraints. The rest of the primary Lagrangian constraints, as displayed in (2.18), come from comparing the canonical determination \(\lambda^\mu_\nu\) of the functions \(\lambda^\mu_\nu\) from their determination \(\phi^1_{\mu_0}\) in tangent space.

It is therefore clear what the status of the Noether canonical quantity \(G\) must be: It must be a constant
of motion for the dynamical operator that is obtained after the first step of the stabilization algorithm has been performed. We can translate this result into equations. The Hamiltonian dynamics after this first step is given by an evolution operator that, using the Dirac bracket, has the form:

\[ X^1_H := \frac{\partial}{\partial t} + \{ - , H_c \}^* + \lambda^{\mu_0} \{ - , \phi_{\mu_0} \} \quad (3.39) \]

with \( \lambda^{\mu_0} \) arbitrary functions. It is irrelevant to write a Poisson bracket or a Dirac bracket for the term with \( \phi_{\mu_0} \). The dynamics given by \( X^1_H \) is tangent to the Hamiltonian primary constraint surface but not necessarily to the secondary constraint surface defined by \( \phi_{\mu_0} = 0 \). This is parallel to the fact that the Lagrangian dynamics is not necessarily tangent to the primary Lagrangian constraint surface, defined by \( \chi_\mu = 0 \). Since at this stage of the stabilization algorithm we have only information on the primary and secondary constraints, under the action of the dynamics the constant of motion \( G \) must satisfy

\[ X^1_H(G) = \frac{\partial G}{\partial t} + \{ G, H_c \}^* + \lambda^{\mu_0} \{ G, \phi_{\mu_0} \} = sc + pc, \]

but since the functions \( \lambda^{\mu_0} \) are, also at this stage, completely arbitrary, this relation splits into

\[ \frac{\partial G}{\partial t} + \{ G, H_c \}^* = sc + pc, \]

and

\[ \{ G, \phi_{\mu_0} \} = sc + pc, \]

which are exactly the conditions \( (3.16) \) and \( (3.17) \), found in the previous section. We have argued, therefore, that these conditions are the correct characterization of the canonical constants of motion \( G \) associated with Lagrangian Noether transformations.

Notice also that in the same way that \( (3.16) \) and \( (3.17) \) are related to \( X^1_H \), that is, the dynamics after the first step in the stabilization algorithm, the conditions \( (3.32) \) and \( (3.33) \) that ensure the projectability of a Noether transformation are related to the evolution operator \( X^1_H \) defined in equation \( (2.9) \), that describes the dynamics before the first step in the stabilization algorithm.

**E. The transformation of momenta**

Up to now, we have been using Hamiltonian techniques to characterize the Noether conserved quantity in phase space, but we have only considered the transformation of the configuration variables, \( q \), as in \( (3.29) \).

\[ \delta q^i = \mathcal{F}L^* \{ q^i, G^* \} - \nu^{\mu_0} \mathcal{F}L^* \{ B^\mu_{\mu_0} \{ q^i, \phi_{\mu_0} \} \}. \quad (3.40) \]

Now we will explore the transformation for the momentum variables. Using the tools introduced in section 2, we can compute \( \delta \hat{p} \), for \( \hat{p} \) defined in \( (3.4) \). The result is

\[ \delta \hat{p}_i = \mathcal{F}L^* \{ p_i, G^* \} - \nu^{\mu_0} \mathcal{F}L^* \{ B^\mu_{\mu_0} \{ p_i, \phi_{\mu_0} \} \} + \{ L \}_j \frac{\partial \delta q^j}{\partial q^i}. \quad (3.41) \]

The last piece vanishes on shell, that is, when the equations of motion are satisfied, so we have an interesting parallelism between \( (3.40) \) and \( (3.41) \). This new piece \( \{ L \}_j \frac{\partial \delta q^j}{\partial q^i} \) will play an important role when we consider in section 5 the commutation algebra of transformations in configuration space as compared to the Poisson algebra of generators in phase space.

Equations \( (3.40) \) and \( (3.41) \) suggest an enlargement of the formalism: Replace the functions \( \nu^{\mu_0} \) by a set of independent variables \( \lambda^{\mu_0} \), the Lagrange multipliers, with vanishing Poisson bracket with the canonical variables, and define

\[ G^c := G^* - \lambda^{\mu_0} B^\mu_{\mu_0} \phi_{\mu_0}. \]

Then,

\[ \delta q^i = (\mathcal{F}L^* \{ q^i, G^c \})_{\lambda=\nu}, \quad \delta \hat{p}_i = (\mathcal{F}L^* \{ p_i, G^c \})_{\lambda=\nu} + \{ L \}_j \frac{\partial \delta q^j}{\partial q^i}. \]

Since in \( \delta \hat{p}_i \) the second time derivatives of the configuration variables \( q^i \) are absorbed within a piece that vanishes on shell, it is natural to introduce in this enlarged space of variables \( q, p \) and \( \lambda \) the Noether transformations of the canonical variables as

\[ \delta q^i = \{ q^i, G^c \}, \quad \delta p_i = \{ p_i, G^c \}, \]

thus putting the transformations of \( q \)’s and \( p \)’s on the same footing. This is the starting point for the next section.

**IV. THE ENLARGED FORMALISM**

It is useful in many respects to reformulate the action principle with the canonical Lagrangian,

\[ L_c(q, p, \lambda; \dot{q}, \dot{p}, \dot{\lambda}) := p_i \dot{q}^i - H_c(q, p) - \lambda^\mu \phi_{\mu}(q, p). \]

The new configuration space for \( L_c \) is the old phase space enlarged with the Lagrange multipliers \( \lambda^\mu \) as new independent variables. We use “enlarged” instead of “extended” to avoid any confusion with the “extended” Dirac Hamiltonian dynamics, where all final first class constraints, primary, secondary, etc., are added to the Hamiltonian with independent Lagrange multipliers (in the usual Dirac’s theory, which is always equivalent to the Lagrangian formulation, only the final primary first class constraints are added to the Hamiltonian.) The dynamics given by \( L_c \) is nothing but the constrained
Dirac Hamiltonian dynamics for a system with canonical Hamiltonian $H_c$ and a number of primary constraints $\phi^\mu$.

This formulation has the advantage that all constraints are holonomic, that is, defined in configuration space and that the Poisson bracket structure defined in the old phase space is still available. Inspired by the results of the last section, we will look for Noether transformations for $L_c$ that may depend on the Lagrange multipliers and their time derivatives to any finite order and that are canonically generated for the variables $q^i$ and $p_i$ ($\lambda, \lambda, \ldots$ will have a vanishing Poisson bracket with these variables). Let us establish the conditions for a function $G^c(q, p, \lambda, \dot{\lambda}, \ldots; t)$ to be a Noether generator, under the definitions

$$\delta_c q^i = \{q^i, G^c\}, \quad \delta_c p_i = \{p_i, G^c\},$$

and with $\delta_c \lambda^\mu$ to be determined below.

Compute $\delta_c L_c$,

$$\delta_c L_c = \delta_c p_i \dot{q}^i + \frac{d}{dt}(p_i \delta_c q^i) - \dot{p}_i \delta_c q^i - \delta_c H_c$$

$$- \lambda^\mu \delta_c \phi^\mu - (\delta_c \lambda^\mu) \phi^\mu$$

$$= \frac{d}{dt}(p_i \delta_c q^i) + \{p_i, G^c\} \dot{q}^i - \{q^i, G^c\} \dot{p}_i$$

$$- \{H_c, G^c\} - \lambda^\mu \{\phi^\mu, G^c\} - (\delta_c \lambda^\mu) \phi^\mu$$

$$= \frac{d}{dt}(p_i \delta_c q^i - q^i \dot{p}_i \partial G^c \partial \dot{q}^i - \dot{p}_i \partial G^c \partial p_i - \{H_c, G^c\}$$

$$- \lambda^\mu \{\phi^\mu, G^c\} - (\delta_c \lambda^\mu) \phi^\mu$$

$$= \frac{d}{dt}(p_i \delta_c q^i - G^c - \frac{DG^c}{Dt} + \{G^c, H_D\} - \delta_c \lambda^\mu) \phi^\mu,$$

where we have defined the Dirac Hamiltonian

$$H_D = H_c + \lambda^\mu \phi^\mu.$$  \hspace{2cm} (4.2)

We have also introduced the notation $[3]

$$\frac{DG^c}{Dt} := \partial G^c \partial t + \lambda^\mu \partial G^c \partial \lambda + \lambda^\mu \partial G^c \partial \lambda,$$

and the total time derivative

$$\frac{dG^c}{dt} := \frac{DG^c}{Dt} + \dot{q} \frac{\partial G^c}{\partial q} + \dot{q} \frac{\partial G^c}{\partial \dot{q}}.$$  

If we require

$$\frac{DG^c}{Dt} + \{G^c, H_D\} = pc,$$  \hspace{2cm} (4.3)

and if we represent this combination $pc$ of primary constraints as $pc = C^\mu \phi^\mu$, then the definition

$$\delta_c \lambda^\mu = C^\mu,$$  \hspace{2cm} (4.4)

makes $\delta_c L_c = \frac{d}{dt}(p \dot{q} - G)$, that is, a Noether transformation for the enlarged formalism. Equation (4.3) is important: it is the equation characterizing the generators of Noether transformations in the enlarged formalism. It applies both to rigid and gauge Noether symmetries, and encodes in a compact way the theoretical setting of the results given in [3] (see also [4]) to find an algorithm to produce gauge generators for theories with only first class constraints. We have arrived at the following result: The necessary and sufficient condition for a function $G^c(q, p, \lambda, \dot{\lambda}, \ldots; t)$ to generate through (4.3) a Noether symmetry in the enlarged formalism is that $G^c$ must fulfill equation (4.3).

Note that this result has been obtained with no assumptions concerning the first and second class structure of the primary constraints. Note also that in (4.3) only the primary constraints are relevant.

A. Back to the original Lagrangian

We may wonder whether these Noether symmetries for $L_c$ are also Noether symmetries for $L$. The answer is yes, and the proof goes as follows.

First, we must show how to obtain the original Lagrangian $L$ from the canonical Lagrangian $L_c$. Consider the equations of motion $[L_c] = 0$ for $p$ and $\lambda$,

$$\dot{q}^i - \frac{\partial H_c}{\partial p_i} + \lambda^\mu \frac{\partial \phi^\mu}{\partial p_i} = 0$$

$$\phi^\mu = 0.$$  

These equations may be used to obtain $p$ and $\lambda$ in terms of the variables $q$ and $\dot{q}$. We find

$$p_i = \dot{p}_\lambda(q, \dot{q}), \quad \lambda^\mu = v^\mu(q, \dot{q}),$$

where the functions $v^\mu$ are those defined in (2.3) and the functions $\dot{p}_\lambda$ are yet to be interpreted. Then the original Lagrangian $L$ is retrieved as

$$L(q, \dot{q}) = L_c|_{(p_i = \dot{p}_\lambda, \lambda = v)},$$

and it satisfies that $\partial L/\partial \dot{q}^i = \dot{p}_i$. This is the interpretation for the functions $\dot{p}_\lambda$.

What we have just described is the standard method to re-obtain $L$ by using the information provided by $H_c$ and the primary constraints $\phi^\mu$. This method, when rephrased, as we do here, in terms of the canonical Lagrangian, is just a reduction procedure from $L_c$ to $L$ that can be independently justified [5]. We prove in the Appendix that it is legitimate to substitute within the Lagrangian ($L_c$ in our case) the auxiliary variables, that is, the variables ($p$ and $\lambda$ in our case) that can be isolated by using their own equations of motion.

Now, given a generalized Noether transformation $\delta_c \dot{q}^i$ associated, according to (4.1), with a constant of motion
\(G^c\), we can readily prove that \(\delta q^i := \left(\delta \lambda \theta^i \right)_{\theta=\lambda=v}\) (\(v = v\) includes, obviously, \(v = \dot{\theta} = (\partial v / \partial q) + \dot{\mu} (\partial v / \partial \dot{q})\), and so on for \(\lambda\), etc.) defines a Noether transformation for \(L\). Indeed we have, by definition,
\[
[L_c] q \delta q + [L_c] p \delta p + [L_c] \lambda \delta \lambda + \frac{dG^c}{dt} = 0;
\]
\([L\lambda] q\), \([L\lambda] p\), and \([L\lambda] \lambda\) stand for the Euler-Lagrange derivatives of \(L_c\) with respect to the coordinates \(q^i\), \(p_i\) and \(\lambda^i\), respectively) then, realizing that \([L\lambda] p\) and \([L\lambda] \lambda\) are identically zero, we find
\[
([L\lambda] q \delta q)_{\theta=\lambda=v} + \frac{dG^c}{dt} = 0,
\]
where \(G_L := G^c\) \(\theta=\lambda=v\). But, since \(\lambda\) and \(p\) are auxiliary variables for the enlarged formalism, we have
\[
[L_c] q |_{\theta=\lambda=v} = [L] q
\]
and therefore,
\[
[L] \delta q + \frac{dG_L}{dt} = 0,
\]
which proves our assertion.

B. Equivalence between the enlarged and the standard formalisms

First, we will prove that all our results of section 3 can be translated to the enlarged formalism. To this end, take \(G^*\) satisfying (3.26), (3.27), and (3.28). Then the definition
\[
G^c := G^* - \lambda^i B_{\mu^i}^0 \phi_{0^i}
\]
(4.5)
makes \(G^c\) satisfy (4.3). (The proof is straightforward)

Next we will prove the reverse in the only case for which such a proof makes sense, namely, the case of a Noether generator \(G^c\) in the enlarged formalism that only depends on \(\lambda\) but not on its time derivatives (in section 3 we considered \(\delta q\) and \(G_L\) defined in \(TQ \times R\)). Let \(G^c\) \((q, p, \lambda; t)\) be such a Noether generator, fulfilling the requirements (4.3). The coefficient of \(\dot{\lambda}\) in (4.3) will therefore satisfy
\[
\frac{\partial G^c}{\partial \lambda} = pc,
\]
which implies the following form for \(G^c\):
\[
G^c(q, p, \lambda; t) = G_0(q, p) + G_{\mu^0}^c(q, p, \lambda; t) \phi_{0^i}(q, p).
\]
The Lagrangian conserved quantity \(G_L\) is then
\[
G_L := G^c |_{\theta=\lambda=v} = G_0 |_{\theta=\lambda=v} = \mathcal{N} \star G_0,
\]
and the infinitesimal Noether transformation for \(L\) is
\[
\delta q^i := \left(\delta q^i \right)_{\theta=\lambda=v} = \frac{\partial G_0}{\partial p} + G_{\mu^0}^c(q, p, \lambda; t) |_{\theta=\lambda=v} \gamma^i_{\mu^0},
\]
which satisfies, according to the results of the previous subsection, \([L] \delta q + dG_L/dt = 0\).

Since \(G_L\) is the pullback of \(G_0\), we conclude, owing to the results of the previous subsection and those of section 3 (under the assumption of the existence of the splitting of the primary constraints into first and second class), that \(G_0\) satisfies (4.19) and (4.21) with some \(A^\mu^0\) and \(B_{\mu^0}^0\). Now, following the same route as we did in section 3.2 from equation (3.23) to (3.29), we can define \(G^*_0\), out of \(G_0\), such that
\[
\delta_1 q^i := \{q^i, G^*_0\} - \psi^\mu \left(\mathcal{N} \star B_{\mu^0}^0, \chi_{\nu^i}\right)
\]
satisfies \([L] \delta_1 q + dG_L/dt = 0\). Subtracting this equation from the equation satisfied by \(\delta q^i\) above gives
\[
[L] \delta q^i - \delta_1 q^i = 0.
\]

The coefficient of \(\tilde{q}\) in (4.6) tells us that \(\delta q^i - \delta_1 q^i = \tilde{q}\mu\) for some functions \(\tilde{q}\). The rest of the equation dictates that \(\tilde{q}\chi_{\nu^i} = 0\). Thus the difference between \(\delta q^i\) and \(\delta_1 q^i\) is entirely due to the indeterminacies already displayed in (4.11).

This proves that, except for these irrelevant indeterminacies, that are inherent to the formalism, the Noether transformations in the tangent bundle that are obtained through the methods of section 3 are the same than those obtained within the enlarged formalism when we restrict ourselves to dependences on the Lagrange multipliers that do not include their time derivatives. Moreover, the enlarged formalism provides us with conditions to be satisfied by a general function \(G^c(q, p, \lambda, \dot{\lambda}, \dot{\mu}; t)\) in order to generate a Noether transformation in the \(n\)-th tangent bundle.

V. THE ALGEBRA OF TRANSFORMATIONS AND GENERATORS

First consider the simplest case, when the Noether transformations are defined in tangent space and are projectable to phase space. This means that formulas (4.31) and (3.32) apply in this case. Consider two functions \(G_1\) and \(G_2\) defined in \(T^q Q \times R\), which satisfy
\[
\frac{\partial G_r}{\partial t} + \{G_r, H_c\} = pc,
\]
and
\[
\{G_r, \phi_{\mu^i}\} = pc,
\]
for \(r = 1, 2\). Then it is straightforward to prove that \(G_{(2,1)} := \{G_2, G_1\}\) also satisfies the same equations (4.1) and (4.2). \(G_{(2,1)}\) is a Noether conserved quantity and a generator of Noether transformations. Does
$G_{(2,1)}$ generate the commutator transformation $[\delta_1, \delta_2]q^i$? The answer in general is no. Let us distinguish between Lagrangian transformations $\delta^L$ and Hamiltonian transformations $\delta^H$. If we define $\delta^H f := \{f, G_r\}$ for any $f \in T^*Q \times R$, then

$$\delta^H q^i = FL^* (\delta^H q^i),$$

and according to (3.4),

$$\delta^L p_i = FL^* (\delta^H p_i) + [L] \frac{\partial \delta_L q^i}{\partial q^i}.$$

Now we can compare the Poisson bracket of canonical generators with the commutator of Lagrangian transformations. The Jacobi identity implies, for the commutator of the Hamiltonian transformations,

$$[\delta^H_1, \delta^H_2]q^i = \{q^i, \{G_2, G_1\}\}.$$

Instead, the commutator of the Lagrangian transformations of the configuration variables becomes

$$[\delta^L_1, \delta^L_2]q^i = FL^* (\{q^i, \{G_2, G_1\}\})$$

$$- [L] W_{ij}(\frac{\partial^2 G_1}{\partial q_j \partial p_l} \frac{\partial^2 G_2}{\partial q_l \partial p_j} - \frac{\partial^2 G_1}{\partial q_l \partial p_j} \frac{\partial^2 G_2}{\partial q_j \partial p_l}).$$

The second term in the right side of equation (5.3) is an antisymmetric combination of the equations of motion. (5.4) displays an open algebra structure for the commutator of the transformations of the configuration variables, that is, an algebra that only closes on shell. More specifically, even if we have, at the canonical level, a closed Poisson algebra structure, for instance a Lie algebra of generators, when we turn to configuration space, the commutation algebra of the transformations will develop a term which is an antisymmetric combination of the equations of motion. Unless one of the generators involved, $G_1, G_2$, is at most linear in the momenta, in which case its second partial derivatives in (5.3) vanish, these open algebra terms are almost unavoidable. The general rule is that a sufficient condition for the commutation relations of the projectable Noether transformations in configuration space to coincide with the Poisson bracket relations of their generators in phase space is that at least all but one of the generators are linear in the momenta. This result was already noted in (23). An interesting example of a set of generators satisfying this sufficient condition is furnished by the canonical formulation of general relativity, that is, the ADM formalism (23).

The gauge algebra plays a fundamental role concerning the quantization of gauge theories, as shown by the BRST methods. In particular, the field-antifield method (3), that works in the space of field configurations, is bound to exhibit, in a certain number of cases, open algebra structures that originate in the type of phenomena here discussed.

Next, we consider the case of Noether transformations $\delta q$ of the type (3.24), that is, transformations defined in $TQ \times R$ that are not projectable to $T^*Q \times R$. It is convenient to work in the framework of the enlarged formalism of section 4, that is, we consider $\delta q^i := (\delta, q^i)|_{(\rho = \phi, \lambda = \nu)}$, with $\delta q^i = (q^i, G_c)$, where $G_c$ as defined in (1.3), satisfies (1.3). If we have two such transformations, their corresponding Noether conserved quantities, $G^c_1$ and $G^c_2$, will satisfy equation (1.3)

$$DG^c_r + \{G^c_r, H_D\} = C^c_{\mu}(\phi_{\mu}),$$

for certain functions $C^c_{\mu}$ ($r = 1, 2$). It is easy to check that $\{G^c_2, G^c_1\}$ does not satisfy (1.3) in general. The reason is that the variation of the Lagrange multipliers has not been taken into account. When (1.3) is included, we find

$$C^c_{(2,1)} := \{G^c_2, G^c_1\} - (C^c_{1\mu} B_{0\mu} - C^c_{2\mu} B_{1\mu}) \phi_{0\mu}.$$

It is immediate to check that $C^c_{(2,1)}$ satisfies (1.3). Notice that $C^c_{(2,1)}$ contains a new dependence in $\lambda^\nu$ that comes from the quantities $C^c_{\mu\nu}$. The structure of this dependence may be displayed by noting that $C^c_{\mu\nu} = \lambda^\nu B_{\mu 0} + \ldots$, therefore

$$G^c_{(2,1)} := \{G^c_2, G^c_1\} - \left(\lambda^\nu (B_{1\mu 0} B_{\mu 20} - B_{2\mu 0} B_{\mu 10}) + \ldots\right) \phi_{\sigma 0}.$$

This, unless the matrices $B_1$ and $B_2$ commute, the new generator in the enlarged formalism $C^c_{(2,1)}$ will not be associated with a transformation of the type (3.29) (because of this new dependence on $\lambda^\nu$). Now $C^c_{(2,1)}$ is a generalized generator of Noether transformations defined within the framework of the enlarged formalism of section 4, that is, satisfying (1.3).

Using the two Lagrangian Noether conserved quantities,

$$G^L_1 := FL^* G^c_1, \quad G^L_2 := FL^* G^c_2,$$

we can get the third Lagrangian Noether conserved quantity

$$G^L_{(2,1)} := \{G^c_{(2,1)}\}|_{(\rho = \phi, \lambda = \nu)}$$

$$= FL^* (\{G^c_2, G^c_1\})|_{(\lambda = \nu)}$$

$$= FL^* (G^c_{2, G^c_1})$$

$$- \phi_{\nu} FL^* (B_{1\nu 0} B_{\nu 20} - B_{2\nu 0} B_{\nu 10}) \lambda_{\sigma 0},$$

but notice that $G^L_{(2,1)}$ is not in general projectable to phase space because of its dependence on the functions $\phi_{\nu}$. The associated transformation is generated by (5.4) and in general will exhibit dependences in the accelerations, contained in $\lambda^\nu$. Indeed it is projectable if the matrices $B_1$ and $B_2$ commute.
VI. NOETHER TRANSFORMATIONS, RIGID AND GAUGE

Up to now our discussion has been completely general and it applies to both rigid and gauge infinitesimal Noether transformations. Rigid, that is, global Noether transformations are physical symmetries of the system. Instead, gauge, that is, local Noether transformations are unphysical and describe a redundancy of the true degrees of freedom of the system. Gauge transformations depend on arbitrary functions and are constructed from first class constraints. Here we state some considerations and results about both types of symmetries.

A. Noether Rigid transformations

The conserved quantity $G$ in $T^*Q \times R$ associated with a rigid Noether transformation initially defined in $TQ \times R$ is a solution of our general conditions (3.10) and (3.11). We know that (3.11) contains information as to whether the associated Noether transformation is projectable to phase space or not. In case it is not, in order to get a projectable Noether transformation one may try the following. Let us add to $G$ a generator, say $\tilde{G}$, of a gauge transformation where the values of the arbitrary functions have been previously fixed as constants. Since the generators of gauge transformations are combinations of constraints (see next subsection), the value on shell of the conserved quantity remains unaltered. It is then possible that a convenient choice of $\tilde{G}$ makes the Noether transformation generated by $G + \tilde{G}$ projectable.

We will not discuss the question of the existence of rigid Noether symmetries. This problem is closely related with the integrability of the dynamical system at hand. A typical example can be constructed in the case of autonomous systems, where the energy is a constant of motion. In field theory, the explicit independence of the Lagrangian with respect to the space-time variables leads to the existence of a conserved energy-momentum tensor. Here also the associated Noether transformation can be non-projectable, and the resulting energy-momentum tensor may not be gauge invariant. In some cases, though, by adding the appropriate gauge generator, it is possible to construct a projectable Noether symmetry whose associated energy-momentum tensor is gauge invariant.

As an example of a rigid transformation that is non-projectable, consider the case of an autonomous system with canonical Hamiltonian $H_c$. In the place of $G$, $H_c$ itself satisfies (3.10) and (3.11) with $A^{\mu_0} = 0$ and $B_{\nu_0}^{\mu_0} = -\delta_{\nu_0}^{\mu_0}$ (indices $\mu_0$ are only available if there are first class constraints at the canonical primary level). Hence $H_c$ generates a Noether transformation that is not projectable to phase space, an exception being made, of course, in the case when all the primary constraints are second class among themselves. The fact that the matrix $B_{\nu_0}^{\mu_0}$ is different from zero prevents the associated $\delta_0$ from being projectable. Let us be more specific. Considering the infinitesimal conserved quantity $G = \delta t H_c$, with $\delta t$ an infinitesimal parameter, we have, according to (3.23),

$$\delta q^j = FL^*\{q^j, \delta t H_c\} + \delta t v^{\mu_0}\gamma^j_{\mu_0},$$

where we have used the values for $A^{\mu_0}$ and $B_{\nu_0}^{\mu_0}$ determined above. Using the definition of the Dirac bracket, we can write $\delta q^j$ as

$$\delta q^j = FL^*\{q^j, \delta t H_c\} + \delta t FL^*\delta_q\gamma^j + \delta t v^{\mu_0}\gamma^j_{\mu_0} + \delta t (FL^*\{q^j, H_c\} + +v^{\mu_0}\gamma^j_{\mu_0} - M^{\mu_1\mu_2}c^j{\gamma^j_{\mu_1}}) = \delta t(q^j + [L, b^j]),$$

where we have used the identity (2.3), and where $b^j$ stands for the antisymmetric quantity $b^{ij} = \gamma_{\mu_1}M^{\mu_1\mu_2}\gamma_{\mu_2}$. So, except for a trivial piece, the transformation $\delta q^j$ is just $\delta q^j = \delta t q^j$, that is, the infinitesimal time translation, as it must be.

A lesson may be drawn from this rather simple exploration: The existence of a rigid Noether conserved quantity does not guarantee that its associated Noether transformation is projectable to the canonical formalism.

In connection with the results of the preceding section, we may also note in this example that, since the matrix $B_{\nu_0}^{\mu_0}$ associated to the Hamiltonian is a multiple of the identity, this matrix will commute with any other matrix $B_{1\nu_0}^{\mu_0}$ associated with another non-projectable Noether transformation. That is, the second term in the right side of (6.1) and (6.6) will vanish. In particular, taking $G^L_1$ and $G^L_2$ from (5.3), and letting $G^L_{2(1)} := FL^*H_c$, then it turns out that $G^L_{2(1)}$ in (6.6) is

$$G^L_{2(1)} = \frac{\partial G^L}{\partial t},$$

This is an expected result for any time independent Lagrangian: The explicit time derivative of a conserved Noether quantity is also a conserved Noether quantity.

B. Noether Gauge transformations. Existence theorems

The generators of Noether gauge transformations are combinations of first class constraints (with respect to the final constraint surface). They have the general form $\hat{G} = \epsilon_\alpha G^\alpha_1 + \epsilon_\alpha G^\alpha_2 + \ldots$,
that depends on some combinations $G_1 \beta$, $G_2 \beta$,... of first class constraints, and the arbitrary functions $\epsilon_\alpha$ and their time derivatives. To construct these generators for a given theory, it is convenient to solve first the full stabilization algorithm, in order to determine the first class constraints needed in $G$.

The presence of the arbitrary functions $\epsilon_\alpha$ in the generators of Noether gauge transformations makes them very versatile, for we can redefine these arbitrary functions with changes of the type

$$\epsilon_\alpha = f_\alpha^\beta \eta_\beta,$$

with $f_\alpha^\beta(q, p, t)$ a given set of functions, and $\eta_\beta$ playing now the role of new arbitrary functions. This redefinition of the arbitrary functions amounts to a change of the basis of constraints used in the expansion of the generators. The usefulness of such changes of basis is twofold: on one side, they modify the algebraic structure of the generators of the gauge group [24], and hence one can pass from an "open" algebra to a "closed" one, or one can even end up with an "abelianized" algebra. On the other side, they may help to make the transformations projectable to phase space. An interesting example of this last application is that of the gauge group of diffeomorphism-induced transformations in generally covariant theories with a metric [25]: In order to have these transformations projectable to phase space it is compulsory that the original arbitrary functions of the spacetime diffeomorphisms include some precise dependence on the lapse and shift functions (components of the metric in a $3+1$ decomposition).

In general, with regard to gauge transformations, we must distinguish the case where all constraints are first class from the others. In this case, it is proven in [27] that, under some standard regularity conditions (constancy of the rank of the Hessian matrix and absence of ineffective constraints), Noether canonical gauge transformations do exist and with the right number to describe all the gauge freedom available to the system, that is, the number of final first class primary constraints. As a matter of fact, the proof in [27] is not completely general but is only valid for cases with at most quaternary constraints (three steps of the stabilization algorithm being sufficient), but the proof is easily extended to cover the general case.

Similar results are obtained within the enlarged formalism of section 4 for theories with first class constraints and satisfying the same regularity conditions. In such case, as proven in [3], generalized gauge transformations, depending upon the variables $\lambda, \dot{\lambda}, \ddot{\lambda}, ...$, always exist. The freedom to choose, in this case, the basis for the primary constraints, has the price of the appearance of dependence on the Lagrange multipliers and their derivatives. A careful choice of the basis for the primary constraints allows for solutions for the gauge generators independent of $\lambda$, in agreement with the results of [20].

In case second class constraints are present, the theorems in [24] prove that there are still canonical gauge transformations that map solutions of the equations of motion into other solutions, and in the right number, but there is no guarantee that the action is conserved up to boundary terms. The difficulty to get a proof for the existence of canonical Noether gauge symmetries in this general case lies in the structure of the stabilization algorithm, as we now discuss.

In [24] the authors claim to have solved in full generality the problem of existence of gauge transformations for theories containing second class constraints. In that paper it is taken for granted (formulas (9) and (11) of [27]) that a basis for the first class constraints exists, $\Phi_m^\alpha$; where $\alpha$ numbers the level of the stabilization algorithm: primary ($\alpha = 1$), secondary ($\alpha = 2$), etc., $m_\alpha$ numbers the constraints in the level $\alpha$ such that, together with a first class canonical Hamiltonian, $H_{FC}$, the following relations hold (the notation is the same as in [24,28]):

$$\{\Phi_m^\alpha, H_{FC}\} = g_{\alpha \beta} m_\alpha m_\beta, \Phi_m^\beta,$$

$$\{\Phi_m^\alpha, \Phi_m^\beta \} = f_{\alpha \beta \gamma} m_\alpha m_\beta m_\gamma \Phi_m^\gamma.$$  

(6.2)

But this assumption is not proven, neither in [27] nor in a preceding paper [23]. In principle one could think that (6.2) is a simple consequence of the fact, first proved by Dirac [11], that the Poisson bracket between first class objects is also first class. This is true, of course, but the contents of (6.2) is much more restrictive. Indeed, one must take into account that any product of two secondary constraints is also first class, and therefore one has, if $\Psi$ stands generically for a secondary constraint,

$$\{\Phi_m^\alpha, H_{FC}\} = g_{\alpha \beta} m_\alpha m_\beta \Phi_m^\beta + O(\Psi)^2,$$

$$\{\Phi_m^\alpha, \Phi_m^\beta \} = f_{\alpha \beta \gamma} m_\alpha m_\beta m_\gamma \Phi_m^\gamma + O(\Psi)^2.$$  

(6.3)

where $O(\Psi)^2$ stands for any piece quadratic in the secondary constraints. It is not difficult to get rid of these quadratic pieces for the Poisson bracket of the $\alpha$-level first class constraints with the first class Hamiltonian, simply by defining the $(\alpha + 1)$-level first class constraints as the results of these Poisson bracket and disregarding the redundant constraints that may result. But then we cannot prevent the Poisson bracket of first class constraints from developing quadratic pieces in the secondary constraints. Or vice-versa, we can write the constraints, if they are all effective, in a new basis such that all are canonical variables, a "Darboux" basis. In such case, the Poisson bracket between first class constraints has no quadratic pieces -it just vanishes- but nothing prevents these quadratic pieces from being present in the Poisson bracket of the first class constraints with the first class Hamiltonian.

Since the assumption (6.2) seems instrumental in obtaining the existence theorems for Noether gauge transformations, we must therefore assert that there has not yet been produced a general proof of the existence of canonical Noether gauge symmetries for systems with first as well as second class constraints. Nevertheless, we
are going to prove in the next subsection a theorem of existence for canonical Noether gauge transformations for general theories having only primary and secondary constraints, that is, theories whose stabilization algorithm has only one step. Since most of the physical cases, like general relativity or Yang-Mills theories, fall into this case, we can say that our results, though incomplete, do have some interest.

C. Existence of canonical Noether gauge transformations for theories with one-step stabilization algorithm

Consider a theory satisfying our standard regularity conditions, with canonical Hamiltonian \( H_c \), and with a set of primary constraints. We classify them into first and second class. Next we look for the secondary constraints, which we will suppose to be effective. They introduce new restrictions to the primary constraint surface, so that they are defined up to the addition of primary constraints. And suppose, also, that there are no more levels (tertiary,...) of constraints. Some of the secondary constraints make second class some of the former first class primary constraints. The rest of the secondary constraints, chosen in a convenient basis, will be first class. By changing the basis for the primary and secondary constraints and performing a subsequent canonical transformation, we can express all the constraints in a “Darboux” basis,

\[
\begin{align*}
\text{Primary} : & P_1, \ldots, P_m, P_{m+1}, \ldots, P_n, P_{n+1}, \ldots, P_{n+r}, \\
\text{Secondary} : & P'_1, \ldots, P'_l, Q_{n+1}, \ldots, Q_{n+r}.
\end{align*}
\]

\(P_1, \ldots, P_m\) are the final first class primary constraints, \(P_{m+1}, \ldots, P_n\) are the former first class primary constraints that become second class when the secondary constraints are introduced. \(P_{n+1}, \ldots, P_{n+r}, Q_{n+1}, \ldots, Q_{n+r}\) are couples of canonical variables corresponding to the primary second class constraints. \(P'_1, \ldots, P'_l\), with \(l \leq m\), are the secondary first class constraints, and \(Q_{m+1}, \ldots, Q_n\) are the secondary second class constraints that make the primary subset \(P_{m+1}, \ldots, P_n\) second class.

Since the canonical Hamiltonian is only determined up to primary constraints, we can, without any lost of generality, set to zero in \( H_c \) the variables corresponding to the primary constraints. Notice that, since the Poisson bracket between first class objects is also first class, we will have, for this new \( H_c \),

\[
\{ P_i, H_c \} = A^a_i P^a + D_{ii}^a Q_s Q_t,
\]

for some matrices (of functions) \( A^a_i \) and \( D_{ii}^a \); \( i = 1, \ldots, m, a = 1, \ldots, l, s, t = m+1, \ldots, n \). \( A^a_i \) is a maximum rank matrix. If \( l < m \), \( A^a_i \) has \( m-l \) null vectors \( C^a_{\sigma} \), \( C^a_{\sigma} A^a_i = 0, \sigma = l+1, \ldots, m \).

Having done all these preliminaries, finding \( m \) independent canonical Noether gauge quantities enclosed in \( G, G = \epsilon^i(t)G_i, i = 1, \ldots, m, \) with \( \epsilon^i \) arbitrary functions, becomes trivial. They are,

\[
G_a = P'_a, a = 1, \ldots, l; \quad G_\sigma = C^a_\sigma P_i, \sigma = l+1, \ldots, m.
\]

It is trivial to check (3.19) and (3.20) for all these quantities. The coefficients \( A^{ij}_{\mu} \) in (3.19) vanish for \( G_\sigma \) but not necessarily for \( G_a \). Since \( B^a_{\mu} = 0 \) for all \( G_a, G_\sigma \), we know by (3.38) that their associated Noether transformations are canonical.

Thus we have arrived at the following result: Any theory with only primary and secondary constraints (all effective) exhibits a basis of independent canonical Noether gauge generators in a number that equals the number of final primary first class constraints.

This proof of existence of Noether gauge transformations cannot be easily generalized to theories whose stabilization algorithm has more than one step. Of course one can prepare the constraints in a “Darboux” basis, where conditions (3.20) can be readily met. But the unsolved problem is to find the right number of objects, combinations of first class constraints, satisfying (3.19). This is still an open problem.

VII. EXAMPLES

A. Example 1: Dirac Hamiltonian

For any time independent first order Lagrangian, the Dirac Hamiltonian \( H_D \) (4.2), which is also time independent, satisfies the condition (4.3) to be a generator of Noether transformations. These transformations, \( \delta q = \{ q, \delta H_D \} \) become, in tangent space,

\[
\delta q^i := (\delta_c q^i)|_{(p=\tilde{p}, \lambda=\tilde{\lambda})} = \delta t (F \mathcal{L}^\star \{ q^i, H_c \} + v^\mu \mathcal{L}^\star (\{ q^i, \phi_\mu \})) = \delta t (K q^i) = i^i_\sigma \delta t.
\]

This is an expected result that was already discussed in the context of non-projectable rigid Noether symmetries in subsection 6.1.

B. Example 2: Presence of terms quadratic in the constraints

Having theorems that guarantee the existence of gauge Noether transformations is not the same as finding them in practice. Writing the constraints in a “Darboux” basis may prove cumbersome, and in many cases it is advantageous to circumvent these procedures and to obtain the Noether conserved quantities from simpler considerations. Here we consider an example exhibiting first and second class constraints. It illustrates some special
features that are absent in the case with only first class constraints. Our Hamiltonian is
\[ H_c = \frac{1}{2}(p_1^2 + p_2^2), \]
where \( p_1 \) and \( p_2 \) are vectors in Minkowski space. The primary constraints are the following scalar products
\[ \phi_1 = (p_1, x_2) = 0, \quad \phi_2 = (p_2, x_2) = 0, \]
where \( x_2 \) is the vector whose components are the canonical coordinates conjugate to those of \( p_2 \). The corresponding Lagrangian is
\[ L = \frac{1}{2}(x_1^2 - \frac{(\dot{x}_1 x_2)^2}{x_2^2}) + \frac{1}{2}(\dot{x}_2^2 - \frac{(\dot{x}_2 x_2)^2}{x_2^2}). \]
Both constraints are first class on the primary surface, and \( \{\phi_1, \phi_2\} = 0 \). Their stabilization gives the secondary constraints:
\[ \xi_1 = \{\phi_1, H_c\} = (p_1, p_2), \quad \xi_2 = \{\phi_2, H_c\} = p_2^2. \]
No more constraints appear. Notice that \( \{\phi_1, \xi_1\} = p_1^2 \).

Therefore, if we restrict ourselves to a region in phase space with \( p_1^2 \neq 0 \), the constraints \( \phi_1 \) and \( \xi_1 \) become second class. For a correct definition of \( L \), we will also assume that our region satisfies \( x_2^2 \neq 0 \).

The obvious candidate for a conserved quantity \( G \) associated with a Noether gauge transformation is the secondary constraint \( \xi_2 \), which is a final first class constraint. In fact, \( G = \epsilon(t)\xi_2 \), with \( \epsilon(t) \) an arbitrary function, satisfies (3.19) and (3.20) with \( A_1 = A^2 = 0 \) and \( B_1 = B_2^2 = -2 \). Thus, we have at hand the identification of the Noether transformation \( \delta q \) associated with \( G \), just by using (3.23). Since \( FL^*B_{\mu_0}^\mu \neq 0 \), we know by (3.33) that \( \delta q \) is not projectable. Our experience with theories with only first class constraints tells us that to get a projectable \( \delta q \) we may try to replace \( \epsilon(t) \) by a function \( \epsilon = \eta(t)f(q, p) \) with \( \eta \) arbitrary and \( f \) to be determined so as to render \( B_{\mu_0}^\mu = 0 \). This substitution amounts to a change of basis for the secondary first class constraint. But this substitution does not work in our case. The subtle point is that the change of basis \( \xi_2 \to f\xi_2 \) is not general enough, that is, we have not fully used the freedom to modify \( G \). In fact, the square of the secondary class constraint \( \xi_1 \) is first class (because any ineffective constraint is). Therefore, we may add to \( G \) a term linear in \( (\xi_1)^2 \), that is, we can take \( G = \eta(t)(f\xi_2 + g(\xi_1)^2) \).

Now we can find simple solutions for \( f \) and \( g \) that make \( B_{\mu_0}^\mu = 0 \), for instance
\[ G = \eta(t)(x_1^2 p_1^2 \xi_2 - x_2^2 (\xi_1)^2). \]

The values of \( A_1 \) and \( A^2 \), according to (3.19), for this \( G \), may be used to obtain, by means of (3.24), the canonical generator \( G^\epsilon \) of a projectable \( \delta q \). We get
\[ G^\epsilon = \eta(t)(x_1^2 p_1^2 \xi_2 - x_2^2 (\xi_1)^2) + \eta(t)(x_2^2 p_1^2 \xi_2 - x_2^2 \xi_1 \phi_1). \]
Notice that all terms use constraints that are first class, though some are the product of final second class constraints. Had we not considered these quadratic pieces, we would have not a canonical generator of a projectable Noether gauge transformation. Obviously these special features disappear if we use a “Darboux” basis for the constraints, but it is harder to get this basis than to proceed along our lines.

C. Example 3: A non-projectable Noether transformation

As an example which exhibits a non-projectable Noether transformation, we take the Lagrangian
\[ L = \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{2}q_3^2 + \frac{1}{2}q_4^2 + q_3 q_5, \]
which differs from the example presented in [27] by a simple canonical transformation. The primary constraints are \( p_4 = 0 \) and \( p_5 = 0 \). The canonical Hamiltonian is
\[ H_c = \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 - \frac{1}{2}q_3^2 - \frac{1}{2}q_4^2 - q_3 q_5, \]
while the stabilization of \( p_4 = 0 \) gives rise to the chain of first class constraints
\[ \phi_1 = p_4, \quad \phi_2 = -\frac{1}{2}q_2^2, \quad \phi_3 = -q_2 p_2, \quad \phi_4 = -q_2^2 - q_4 p_2^2, \]
Using the Dirac bracket we can eliminate the canonical pairs \((q_3, p_3)\) and \((q_5, p_5)\) though it is not necessary. A solution to our fundamental conditions (3.19) and (3.20) is
\[ G = \phi_2(\bar{\epsilon} - 4q_4 \epsilon) - \phi_3 \bar{\epsilon} + \phi_4 \epsilon, \]
with \( A = \bar{\epsilon} - 4q_4 \epsilon \) and \( B = -4 \epsilon \). Since \( B \neq 0 \) the generator has an associated non-projectable Noether symmetry. It is easy to show that in this case it is not possible to use a redefinition of the constraint surface to render the corresponding transformation projectable. Notice that, in particular, the transformation \( \delta q_4 = -A + 4q_4 \epsilon \) is not canonically generated. Nevertheless, we can use the enlarged formalism of section 4 to define a canonical generator \( G_\epsilon \) of this non-projectable Noether symmetry. To accomplish this we promote the Lagrange multiplier \( \lambda \) associated to the primary first class constraint \( p_4 = 0 \) to a dynamical variable — that is, we work in the enlarged formalism — and use the relation (4.3) to define a canonical generator
\[ G^\epsilon = G + (4 \lambda \epsilon - A)p_4. \]
This canonical generator satisfies equation (4.3) (the Dirac Hamiltonian is $H_D = H_c + \lambda p_4$). The canonical transformation $\delta q_4$ generated by $G^c$, $\delta q_4 = 4\epsilon \lambda - A$, coincides with the previous non-canonical symmetry upon the substitution $\lambda = q_4, p_4 = 0$ that comes from the equations of motion for $p_4$ and $\lambda$, respectively. This is an expected result according to the general arguments introduced in section 4.

D. Example 4: Abelian Chern-Simons field theory in $2n + 1$ dimensions

Here we present, as a field theory example, the generic Abelian Chern-Simons theory in $2n + 1$ dimensions [1,2]. We use some results of [2] and, mostly, their notation. Our treatment differs from [3] in that we retain all the variables, including $A_0$.

Abelian Chern-Simons theory has only primary and secondary constraints; therefore, in principle, a Noether gauge generator can be constructed. One must be aware that in some field theories the basis of constraints to achieve this goal can be involved. Indeed, there are cases where the Noether generator is a non-local function, as in the canonical formulation of the electromagnetic duality transformation (see for example [3]). In our case, however, our procedures will not meet these difficulties.

The Lagrangian density is

$$L^{2n+1}_{CS} = \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n}} A_{\mu_0} F_{\mu_0 \mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}},$$

where the greek indices $\mu$ run from 0 to $2n$, $F_{\mu \nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\epsilon^{\mu_1 \mu_2 \cdots \mu_{2n}}$ is totally antisymmetric with $\epsilon^{012 \cdots 2n} = 1$. The Noether gauge invariance for $L^{2n+1}_{CS}$ is given by

$$\delta A_\mu = -\partial_\mu \epsilon + F_{\mu \nu} \eta^\nu,$$

where $\epsilon$ and $\eta^\mu$ are infinitesimal arbitrary functions (the usual diffeomorphisms are recovered by taking $\epsilon = A_0 \eta^\nu$).

Eventhough there is no metric, we take, as it is customary, the $x^0$ coordinate as the time evolution parameter. Since $L^{2n+1}_{CS}$ does not depend on $A_0$, it is convenient to identify in (7.5) the terms containing $A_0$. The Lagrangian is equivalently written as (the latin indices $i$ run from 1 to $2n$)

$$L^{2n+1}_{CS} = A_0 K + (\partial_0 A_i - \partial_i A_0) l^i,$$

with

$$K := \epsilon^{0i_1 \cdots i_{2n}} F_{i_1 i_2} \cdots F_{i_{2n-1} i_{2n}},$$

and

$$l^i := 2n \epsilon^{0i_1 \cdots i_{2n}} A_j F_{i_1 i_3} \cdots F_{i_{2n-1} i_{2n}}.$$

(Notice that $\partial_0 l^i = n K$) We can go to the Hamiltonian formulation. The Lagrangian momenta are

$$\dot{p}^0 := \frac{\partial L^{2n+1}_{CS}}{\partial A_0} = 0,$$

$$\dot{p}^i := \frac{\partial L^{2n+1}_{CS}}{\partial A_i} = l^i,$$

whereby we can read off the primary Hamiltonian constraints

$$\phi^0 := p^0 = 0, \quad \phi^i := p^i - l^i = 0. \quad (7.8)$$

(These equalities to zero are weak equalities in the Dirac sense: they are part of the equations of motion) The canonical Hamiltonian is determined only up to terms linear in the primary constraints. A simple choice is

$$H_c = -A_0 (K + \partial_i p^i). \quad (7.9)$$

(This Hamiltonian differs from the one taken in [2] by terms linear in the primary constraints). At this point, the dynamics is determined by the vector field in phase space

$$X_H = \frac{\partial}{\partial \theta^i} + \{ -, H_c \} + \lambda_\mu \{ -, \phi^\mu \}. \quad (7.10)$$

Let us obtain, for future use, the Lagrangian determination $v_\mu$ of the Lagrange multipliers $\lambda_\mu$, as discussed in section 2. Since the time derivatives of the configuration variables $A_\mu$ are

$$\dot{A}_0 = X_H (A_0) = \lambda_0, \quad \dot{A}_i = X_H (A_i) = \dot{\theta}_i A_0 + \lambda_i,$$

we find

$$v_0 = \dot{A}_0, \quad v_i = F_{0i}. \quad (7.11)$$

These functions $v_0, v_i$ are clearly non-projectable to phase space because the Hessian matrix for $L^{2n+1}_{CS}$ vanishes identically (the Lagrangian is linear in the velocities) and therefore the vectors $\Gamma_\mu$ in (3.3) are now $\Gamma_\mu = -\frac{\partial}{\partial \lambda_\mu}$.

We have the following Poisson brackets:

$$\{ \phi^0, \phi^i \} = 0,$$

$$\{ \phi^i, \phi^j \} = -2n (n + 1) \epsilon^{0i j_1 \cdots j_{2n}} F_{i j_1 j_2} \cdots F_{i j_{2n-1} j_{2n}} =: \Omega^{ij},$$

$$\{ \phi^0, H_c \} = K + \partial_\mu p^\mu =: \psi^0,$$

$$\{ \phi^i, H_c \} = 0. \quad (7.12)$$

The stabilization algorithm for the primary constraint $\phi^0 = 0$ gives the secondary constraint $\psi^0 = 0$. The other primary constraints give

$$0 = \phi^i = X_H \phi^i = \Omega^{ij} \lambda_j. \quad (7.13)$$

To continue, it is necessary to know the rank of the matrix $\Omega^{ij}$. The identity

$$\Omega^{ij} F_{jk} = (n + 1) \delta^i_k K, \quad (7.14)$$
is telling us, taking into account that

\[(n + 1)K = \psi^0 - \partial_i \phi^i,\]

that neither \(\Omega^{ij}\) nor \(F^{i}_{jk}\) can have rank \(2n\), on shell. From now on we will work in the region of configuration space where the theory is generic \(\Omega^{ij}\), that is, \(F^{i}_{jk}\) has the maximum rank compatible with the equations of motion. This maximum rank is \(2n - 2\). We take \(\Sigma^{ij}\) coordinates \(x^i = x^\alpha,\ x^p; \alpha = 1, 2; p = 3, \ldots, 2n\), such that the maximum rank condition is attained by \(F^{pq}\), that is, \(\text{det}(F_{pq}) \neq 0\).

Define \(F^{pq}\) as the matrix inverse to \(F^{pq}\). Equation \((\ref{eq:7.14})\) implies the on shell (that is, when \(K = 0\)) relationship

\[\Omega^{ip} = -\Omega^{ij} F^{jq} F^{pq},\]

that shows that the maximum rank for \(\Omega^{ij}\) is \(2\), on shell. Indeed this rank is attained by \(\Omega^{ij}\). Defining its inverse matrix as \(\Omega^{ij}\), we can write, using \((\ref{eq:7.14})\), the identities

\[F^{pq} F^{qr} + \Omega^{ij} \Omega^{ip} = 0,\]
\[\Omega^{ij} - \Omega^{pq} \Omega^{ip} \Omega^{jq} = (n + 1) F^{pq} K \]

that will be used below.

The preceding analysis shows that, being the theory generic, the only primary second class constraints are \(\phi^1\) and \(\phi^2\). The Dirac bracket is then defined as

\[\{-, -\} = \{-, -\} - \{-, \phi^0\} \Omega^{ij} \{\phi^j, -\}.\]

The Lagrangian multipliers \(\lambda_1\) and \(\lambda_2\) become determined by \((\ref{eq:7.13})\). Recalling \((\ref{eq:3.33})\), the new evolution operator is

\[\mathbf{X}^1_H = \frac{\partial}{\partial t} + \{-, H_c\} + \lambda_p \{-, \phi^0\} + \lambda_0 \{-, \phi^0\} + \cdots.\]

Our next task is to find the Noether gauge generators. Our Chern-Simons theory exhibits a special circumstance which is worth mentioning: In contrast to the standard case developed in section 3, our primary first class constraints only close (to exhibit their first class condition) when the secondary constraint is also taken into account. This is due to the fact that in the determination of the rank of \(\Omega^{ij}\) (in the generic case) use has been made of the secondary constraint. Nevertheless, since the closure of the primary first class constraints only involves the secondary constraint our theory still applies, as we will see. Using the second identity in \((\ref{eq:7.13})\) we compute this closure property. It reads:

\[(\phi^p, \phi^q)^* = (n + 1) F^{pq} K = F^{pq} \psi^0 + pc.\]

Now we can look for the gauge generators. It is clear that \(\epsilon_0 \psi^0\) and \(\epsilon_p \phi^p\), with \(\epsilon_0, \epsilon_p\) arbitrary functions, satisfy the requirements for the function \(G\) in \((\ref{eq:3.19})\) and \((\ref{eq:3.20})\). For \(\epsilon_0 \psi^0\) the coefficient \(A\) in \((\ref{eq:3.14})\) is \(\epsilon_0\) and the coefficient \(B\) in \((\ref{eq:3.20})\) vanishes. For \(\epsilon_p \phi^p\) the coefficient \(A\) vanishes and \(B^i_p = \epsilon_p F^{ip}\). Thus the transformations generated by \(\epsilon_p \phi^p\) are not projectable. This non-projectability suggests us to work in the enlarged formalism. Let

\[G := \epsilon_0 \psi^0 + \epsilon_p \phi^p,\]

then, the quantity \(G^{\ast}\) in \((\ref{eq:3.25})\) becomes \(\phi^{0\ast} = \phi^0, \psi^{0\ast} = \psi^0\):

\[G^{\ast} = \epsilon_0 \psi^0 - \epsilon_0 \phi^0 + \epsilon_p \phi^p\]
\[= \epsilon_0 \psi^0 - \epsilon_0 \phi^0 + \epsilon_p (\phi^p - \Omega^{pq} \Omega^{pq} \phi^q),\]
\[= \epsilon_0 \psi^0 - \epsilon_0 \phi^0 + \epsilon_p (\phi^p + F^{pq} \phi^q),\]

where we have used the first identity in \((\ref{eq:7.17})\). Finally, the generator \(G^{c}\) in \((\ref{eq:4.3})\) is

\[G^{c} = \epsilon_0 \psi^0 - \epsilon_0 \phi^0 + \epsilon_p (\phi^p + F^{pq} \phi^q + \lambda_q F^{pq} \phi^0),\]

and it is straightforward to prove that \(G^{c}\) satisfies equation \((\ref{eq:4.13})\). It is convenient to redefine the arbitrary functions \(\epsilon_p\) as \(\epsilon_p = F^{pq} \eta^q\). Then

\[G^{c} = \epsilon_0 \psi^0 - \epsilon_0 \phi^0 + \eta^q (F^{pq} \phi^p + F^{pq} \phi^q + \lambda_q F^{pq} \phi^0),\]

Recalling the Lagrangian determination \((\ref{eq:7.11})\) for \(\lambda_q\), we can write the gauge transformations generated by \(G^{c}\), \((\ref{eq:4.3})\),

\[\delta A_0 = \{A_0, G^{c}\}_{(\lambda = 0)} = -\epsilon_0 + F_0 q \eta^q,\]
\[\delta A_\beta = \{A_\beta, G^{c}\}_{(\lambda = 0)} = -\partial_\beta \epsilon_0 + F_\beta q \eta^q,\]
\[\delta A_p = \{A_p, G^{c}\}_{(\lambda = 0)} = -\partial_p \epsilon_0 + F_{pq} \eta^q,\]

or, in a more compact way,

\[\delta A_{\mu} = -\partial_{\mu} \epsilon_0 + F_{\mu q} \eta^q.\]

The time diffeomorphism and the diffeomorphisms in the \(x^1\) and \(x^2\) directions are hidden in \((\ref{eq:7.21})\). To find them, one must consider the Lagrangian equations of motion,

\[\Omega^{ij} F_{0ij} = 0.\]

(They can be read off from \((\ref{eq:7.13})\) after using the lagrangian determination \((\ref{eq:7.11})\) for \(\lambda\)). Since a basis for the null vectors for \(\Omega^{ij}\) is already spanned by \(F_{ij},\ p = 3, 4, \ldots, 2n\), we conclude that there exist matrices \(N^q_{\alpha}\), \(N^p_{\alpha}\) such that, on shell,

\[F_{\alpha \mu} = N^q_{\alpha} F_{q \mu}, \quad F_{0 \mu} = N^q_0 F_{q \mu}.\]

Then, since \(\eta^q\) are arbitrary functions, they can be expressed as

\[\eta^q = \rho^q + N^q_{\alpha} \rho^\alpha + N^q_0 \rho^0.\]
with $\rho^0, \rho^2, \rho^3$ arbitrary functions. Therefore, on shell
\[
\delta A_{\mu} = -\partial_{\mu} \epsilon_0 + F_{\mu\nu} \rho^\nu. \tag{7.23}
\]

We have recovered the complete set of gauge transformations (7.3) that in fact are true Noether transformations, on and off shell. Obviously they are not all independent: the time and the $x^1, x^2$ diffeomorphisms can be obtained either by using the arbitrary functions $\rho^2, \rho^3$ or as a byproduct of the $x^0$ diffeomorphisms, as we have just seen. The reason for this is the fact that the dynamics introduces relations between the components of the curvature tensor $F_{\mu\nu}$ that produce a redundancy in the action on shell of the algebra of diffeomorphisms. Thus, the gauge algebra is reducible on shell. This is a well known fact that, in the language of BRST methods [3], implies the introduction of “ghosts for ghosts” in the formalism. Indeed, the transformations (7.23) are reducible on shell because, if we take
\[
\delta A_{\mu} = R_{\mu\nu} \rho^\nu, \tag{7.24}
\]
with $R_{\mu\nu} = F_{\mu\nu}$, then there exist matrices $Z'_{\alpha}, Z''_{\alpha}$, with
\[
Z'_{\alpha} = (0, \delta_{\alpha}^2, -N_{\alpha}^p), \quad Z''_{\alpha} = (1, 0, -N_{\alpha}^p)
\]
such that, according to (7.23), the following relations hold on shell,
\[
Z'_{\alpha} R_{\mu\nu} = 0, \quad Z''_{\alpha} R_{\mu\nu} = 0.
\]

This redundancy has been expressed in a non covariant form. A full covariant treatment can be given by noticing that the covariant form of the equations of motion (including the constraint $K = 0$) is
\[
Q^\mu := \epsilon^{\mu\nu_1\ldots\nu_{2n}} F_{\nu_1\nu_2} \ldots F_{\nu_{2n-1}\nu_{2n}} = 0.
\]
( $Q^0 = 0$ is $K = 0$, $Q^i = 0$ is $\Omega^{ij} F_{0j} = 0$) If we define
\[
\Omega^{\mu\nu\rho} := \epsilon^{\mu\nu\rho\mu_1\mu_2\ldots\mu_{2n}} F_{\mu_1\mu_2} \ldots F_{\mu_{2n-1}\mu_{2n}},
\]
the following identity holds:
\[
\Omega^{\mu\nu\rho} F_{\rho\sigma} = \frac{1}{2n} (\delta^\mu_\nu Q^\rho - \delta^\nu_\rho Q^\mu).
\]
Then, the covariant expression for the reducibility of the gauge transformations (7.24) is expressed, on shell, as
\[
\Omega^{\mu\nu\rho} R_{\sigma\rho} = 0.
\]
This expression is in its turn reducible because, on shell,
\[
F_{\rho\nu} \Omega^{\mu\nu\rho} = 0,
\]
and the tower of reducibility continues indefinitely. This fact was already noticed in [3]. Since in our approach we retain all the variables (that is, including $A_0$), our description is fully covariant.

1. Counting the degrees of freedom

A) In tangent space.
Here we present an alternative counting [2] of the degrees of freedom for the abelian Chern-Simons theory. Since most of our Noether gauge transformations are not projectable, we will study the gauge fixing procedure in the Lagrangian formalism as discussed in [3]. Here we will ignore the fact that the Lagrangian is linear in the velocities and we will apply standard machinery. First we count the Lagrangian constraints in the usual sense, that is, constraints involving configuration and velocity variables. They are $K = 0, K = 0$ and $\Omega^{ij} F_{0j} = 0$. This amounts for a total of $2n + 2$ Lagrangian constraints. Now we must introduce new constraints in order to eliminate the gauge freedom. The independent gauge degrees of freedom are described by the arbitrary functions $\epsilon_0$ and $\eta^p$ in (7.21); these functions must be fixed by introducing the gauge fixing constraints. This means we must introduce $2(n - 1) + 1$ new constraints to fix the dynamics. At this point there is still a redundancy in the setting of the initial conditions (residual gauge freedom, see [3] for details) because, at any given time $t_0$, the function $\epsilon_0$ is an independent function and one can verify there are still gauge motions that preserve the gauge fixing constraints. A new gauge fixing constraint is necessary. Thus, we have introduced a total of $2(n - 1) + 2 = 2n$ gauge fixing constraints. The final number of constraints is, therefore, $4n + 2$. This number equals the dimension (in the sense of field theory) of the tangent space. There are no local degrees of freedom.

B) In configuration space.
Since the Lagrangian equations of motion are linear in the velocities, the real setting of the initial conditions is in configuration space. So let us do things better: We will only consider constraints in configuration space (we will call them $cs$-constraints). The theory only provides with one such a constraint, namely $K = 0$.

Using the first relation in (7.13) (we always consider the theory to be generic), the equations of motion,
\[
\Omega^{ij} F_{0j} = 0,
\]
can be written as
\[
F_{0\alpha} = F_{0\eta} F^{\eta\rho} F_{\rho\alpha}.
\]
Thus, $F_{0\eta}$ and $A_0$ are arbitrary. Defining (to coincide with the notation used in the Hamiltonian analysis)
\[
\lambda_0 := \dot{A}_0, \quad \lambda_\rho := \dot{f}_\rho - \partial_\rho A_0 = F_{0\rho},
\]
we can express the evolutionary vector field in a way that encodes all the arbitrariness through these functions $\lambda_0, \lambda_\rho$:
\[
X_L = \frac{\partial}{\partial t} + \lambda_0 \frac{\delta}{\delta A_0} + (\lambda_\rho + \partial_\rho A_0) \frac{\delta}{\delta A_\rho} + (\partial_\alpha A_0 + \lambda_\rho \eta^{\rho\eta} F_{\rho\eta}) \frac{\delta}{\delta A_\alpha}.
\]
As usual, the gauge fixing procedure has two steps: fixing the arbitrariness in the dynamics and fixing the redundancy in the initial conditions. To fix the dynamics we need to introduce as many gauge fixing cs-constraints as arbitrary functions in $X_L$, that is, $2(n-1)+1$ cs-constraints. Once the dynamics is fixed we must consider the residual gauge freedom that is still available in the setting of the initial conditions. The same argument used before makes it necessary an additional gauge fixing constraint. Once the dynamics is fixed we must consider the residual gauge freedom that is still available in the setting of the initial conditions. This argument used before makes it necessary an additional gauge fixing constraint. Now the gauge freedom has been completely eliminated. We end up with $2n$ gauge fixing cs-constraints to be added to the original cs-constraint $K=0$. The final number of cs-constraints, $2n+1$, equals the dimension of configuration space. Again, there are no local degrees of freedom.

E. Example 5: Pure electromagnetism

In this example we consider the construction of the gauge symmetry in pure electromagnetism. From the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

we get a canonical Hamiltonian,

$$H_c = \int dx \left[ \frac{1}{2} (\pi^2 + B^2) + \pi \cdot \nabla A_0 \right],$$

with $B^k = -\frac{1}{2} \epsilon^{kij} F_{ij}$, and a primary constraint (coming from the definition of the momenta) $\pi_0 = 0$. Stability of this constraint under the Hamiltonian dynamics leads to the secondary constraint $\dot{\pi}_0 = \{\pi_0, H_c\} = \nabla \cdot \pi = 0$, and no more constraints arise. Both constraints are first class, so we can directly apply the existence theorem of subsection 6.3 to claim that

$$G = \int dx \left[ \epsilon(x, t) \nabla \cdot \pi \right]$$

is a Noether conserved quantity satisfying (3.16) and (3.17), with $\epsilon$ an arbitrary function. Coefficients in (3.19) and (3.20) are, in this case, $A = \dot{\epsilon}$, $B = 0$. The transformation (3.23) becomes

$$\delta A_\mu = -\partial_\mu \epsilon,$$

which is canonically generated by

$$G' = G - A_\pi_0 = -\int dx \pi^\nu \partial_\mu \epsilon.$$

This gauge generator, $G'$, satisfies, by construction, (3.31) and (3.32).

VIII. CONCLUSIONS

In this paper we give results concerning the formulation of Noether symmetries, either in the tangent space or in the phase space for some configuration space of a dynamical system that may contain gauge freedom.

The conserved quantities associated with the Noether symmetries are characterized in the phase space, and we show the role played by the Dirac bracket structure for this characterization. We also give a geometric property that must be satisfied by a Noether conserved quantity in the tangent space in order that the associated Noether transformation be projectable to phase space. In such case, the Noether transformation becomes a canonical transformation.

We introduce the enlarged formalism (that includes the Lagrange multipliers as independent variables) as a wider framework to deal with generalized Noether symmetries. A general formula is obtained that fully characterizes the conserved quantities associated to these symmetries. We also show how to bring these Noether symmetries in the enlarged formalism to the original Lagrangian.

The algebra for the set of Noether transformations is also discussed, and it is pointed out that the closure of the algebra of generators under the Poisson bracket in phase space does not guarantee the closure of the commutator of transformations in configuration space. This means that sometimes the presence of an open algebra structure will be just an artifact of the pullback from the phase space description to the description in configuration space.

Noether transformations in general can be rigid or gauge. We discuss some aspects of both types of symmetries and, in the gauge case, we give a general proof of the existence of the right number of independent Noether gauge transformations for theories with first and second class constraints with a stabilization algorithm that does not exhibit tertiary constraints. It is still an open problem to prove the existence of the right number of independent Noether gauge transformations for general theories with an undetermined number of steps in the stabilization algorithm.

The generality of the enlarged formalism suggests us that it is the most general framework to deal with Noether symmetries, that is, that any Noether symmetry can be cast into the general formula (4.3). Work is in progress to prove this assertion (34).

We present some examples to illustrate the application of our results. Of particular interest are examples 2 and 4. The first because it exhibits a generator of Noether gauge transformations that must include terms that are quadratic in the constraints, which is uncommon. The second because it is an example of a field theory that has a relevant role in the modern developments in quantum field theory.

We have worked under standard regularity assumptions: the constancy of the rank of the Hessian matrix
and the existence of a splitting of the primary constraints into first and second class on the primary constraint surface. Under these assumptions, here we list our main results as presented in the text (except for result 5, that only needs the first assumption):

1. The necessary and sufficient condition for a function $G \in T^*Q \times R$ to be a Noether canonical conserved quantity, that is, such that its pullback $G_L$ to $TQ \times R$ satisfies

$$[L, \delta q^i] + \frac{dG_L}{dt} = 0,$$

for some $\delta q$, is that $G$ must satisfy

$$\{G, \phi_{\mu_0}\} = sc + pc, \quad \frac{\partial G}{\partial t} + \{G, H_c\}^* = sc + pc$$

where $pc$ (sc) stands generically for primary (secondary) constraints.

(Section 3, subsection 3.1)

2. The Noether transformation is reconstructed from $G$ through

$$\delta q^i = FL^* \{q^i, G\}^* - (FL^* A^{\mu_0} + \nu^\mu_0 FL^* B_{\mu_0}^{\nu_0}) \nu^\nu_0,$$

where $B_{\mu_0}^{\nu_0}$ and $A^{\mu_0}$ are the coefficients of the sc terms in the preceding expression.

(Section 3, subsection 3.2)

3. A Noether transformation in phase space is always a canonical transformation.

(Section 3, subsection 3.2)

4. The necessary and sufficient condition for a Lagrangian Noether conserved function $G_L$ to be associated through

$$[L, \delta q^i] + \frac{dG_L}{dt} = 0,$$

with a transformation $\delta q^i$ that is projectable to phase space is that $G_L$ must give zero when acted upon by the vector fields in the kernel of the presymplectic form in tangent space.

(Section 3, subsection 3.3)

5. The necessary and sufficient condition for a function $G^c(q, p, \lambda, \dot{\lambda}, \ldots; t)$ to generate through

$$\delta q^i = \{q^i, G^c\}, \quad \delta p_i = \{p_i, G^c\},$$

a Noether symmetry in the enlarged formalism is that $G^c$ must satisfy

$$\frac{D G^c}{Dt} + \{G^c, H_D\} = pc$$

where $pc$ stands generically for primary constraints and

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \lambda^\mu \frac{\partial}{\partial \lambda} + \dot{\lambda}^\mu \frac{\partial}{\partial \lambda} + \ldots,$$

(Section 4)

6. A sufficient condition for the commutation relations of the projectable Noether transformations in configuration space to coincide with the Poisson bracket relations of their generators in phase space is that at least all but one of the generators are linear in the momenta.

(Section 5)

7. The existence of a rigid Noether conserved quantity does not guarantee that its associated Noether transformation is projectable to the canonical formalism.

(Section 6, subsection 6.1)

8. Any theory with only primary and secondary constraints (all effective) exhibits a basis of independent canonical Noether gauge generators in a number that equals the number of final primary first class constraints.

(Section 6, subsection 6.3)

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Appendix: Auxiliary Variables

Consider a configuration space locally described by the coordinates $x_a, y_j$. Suppose a general Lagrangian of the form

$$L(x_a, y_j; \dot{x}_a, \dot{y}_j),$$

and suppose that the $y$ variables are auxiliary variables, that is, their equations of motion allow for the isolation of these variables in terms of $x$ and $\dot{x}$,

$$[L]_y = 0 \iff y = \tilde{y}(x, \dot{x}).$$
The components of the two dimensional metric in the Polyakov string \( \tilde{M} \) are examples of such type of variables. Then define the reduced Lagrangian \( L_r \) as

\[
L_r(x, \dot{x}, \ddot{x}) = L|_{y=\dot{y}(x,\dot{x})},
\]

We will prove that \( L_r \) gives the right equations of motion for the remaining variables \( x \).

There is the following relationship between the equations of motion for \( L_r \) and those for \( L \),

\[
[L_r]_x = [L]_x|_{y=\dot{y}(x,\dot{x})} + \frac{\partial \tilde{y}}{\partial x} [L]_{y=\dot{y}(x,\dot{x})} - \frac{d}{dt} \left( \frac{\partial \tilde{y}}{\partial x} [L]_{y=\dot{y}(x,\dot{x})} \right),
\]

but since

\[
[L]_{y=\dot{y}(x,\dot{x})} = 0
\]

identically, we conclude that

\[
[L_r]_x = [L]_x|_{y=\dot{y}(x,\dot{x})}.
\]

This proves that the dynamics for the reduced variables \( x \) is governed by the reduced Lagrangian \( L_r \).

When we apply this result to section 4, the Lagrangian \( L_r \), from which we start the reduction procedure does not depend on the velocities \( \dot{y} \). The \( y \) variables are, in the notations of section 4, \( p \) and \( \lambda \).

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