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Scheme invariants in $\phi^4$ theory in four dimensions

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We provide an analysis of the structure of renormalization scheme invariants for the case of $\phi^4$ theory, relevant in four dimensions. We give a complete discussion of the invariants, up to four loops, and include some partial results at five loops, showing that there are considerably more invariants than one might naively have expected. We also show that one-vertex reducible contributions may consistently be omitted in a well-defined class of schemes, which of course includes MS.

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I. INTRODUCTION

Beyond leading order, it is well-known that the values of $\beta$-function coefficients are scheme-dependent, i.e., depend on the renormalization scheme. On the other hand, one would expect that statements with physical meaning should be expressible in a scheme-independent way. A notable recent example is the issue of the existence of an $a$ function; i.e., a function that generates the $\beta$ functions through a gradient-flow equation. For this to be feasible, the $\beta$-function coefficients must satisfy a set of consistency conditions, which must clearly be scheme invariant; as this has been verified for various field theories in three [1–3], four [4], and six [5] dimensions. The number of scheme-independent combinations at each loop order would naively be expected to be given by the difference of the number of $\beta$-function coefficients and the number of independent variations of coefficients; however, the number of independent invariants actually found is considerably larger. This may be understood in a pragmatic way, in terms of the structure of the expressions for the scheme changes of the coefficients; however, a possibly deeper insight is afforded by Hopf algebra considerations. A general discussion of scheme dependence, with a particular focus on one-particle irreducible (1PI) structures, was recently given in Ref. [6], and here, the study of scheme-invariant combinations was initiated with reference to the $\mathcal{N} = 1$ scalar-fermion theory.

The present paper is to be seen as a companion to a forthcoming article [7] where the ideas of scheme invariance and the relation to Hopf algebra will be explored in general and also exemplified for the case of $\phi^3$ theory in six dimensions; our purpose here is to extend the discussion to $\phi^4$ theory in four dimensions. We shall summarize results of Ref. [7], where it is necessary to render the present discussions self-contained. An additional complication in $\phi^3$ theory is due to the existence of one-vertex reducible (1VR) graphs. These are one-particle irreducible (1PI) graphs that may be separated into two distinct portions by severing a vertex. They have no simple poles when using minimal subtraction and dimensional regularization, and hence, a vanishing $\beta$-function coefficient in this scheme. It would be convenient to be able to omit these coefficients from our considerations. Indeed, we shall show that although we may, if desired, include such coefficients, we may also consistently confine our attention to a well-defined subset of schemes in which these coefficients are absent.

The structure of the paper is as follows: in Sec. II, we introduce the $\phi^4$ theory and give the results at one, two, and three loops. Section III contains our main results, namely the full set of four-loop scheme invariants and a partial five-loop calculation. In Sec. IV, we show that one may straightforwardly restrict attention to a set of renormalization schemes in which 1VR contributions are absent. In Sec. V, we set our results for scheme invariants within the Hopf algebra framework. Finally, we summarize our results and give pointers to future work in the Conclusion. Some general theory, which is developed in detail in Ref. [7] and which underpins our work, is summarized in Appendix A. Appendix B lists

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some Hopf algebraic cocommutative coproducts that arise in Sec. V but were too complex for inclusion in the main text. Finally, in Appendix C, we show how to express scheme changes in terms of differential operators acting on the $\beta$-function coefficients.

II. ONE, TWO, AND THREE LOOP CALCULATIONS

In this section, we establish our notation and obtain the invariants up to three loop order (the first nontrivial case for $\phi^4$ theory). We consider the action

$$S = \int d^dx \left( \frac{1}{2} \partial_{\mu} \phi^i \partial^\mu \phi^i - \frac{1}{2} m^2 \phi^i \phi^i - \frac{1}{4!} g_{ijkl} \phi^i \phi^j \phi^k \phi^l \right)$$

for the case $d = 4$, which corresponds to a renormalizable theory. The anomalous dimension $\gamma_{ij}$ may be expressed as a series of two-point 1PI diagrams with 4-point vertices connected by internal lines representing the contractions of couplings. Up to three loops, we have

$$\beta = \tilde{\beta} + S_4 \gamma$$

where $\gamma$ is attached to each external line. Up to three loops, the contributions to $\tilde{\beta}$ are given by

For later convenience we introduce the notation that $g_{3a}^j$ is the graph corresponding to $c_{3a}$, and $g_{3}^f$ is the graph corresponding to $d_2$, etc. We note that in Eq. (4) the graph $g_{3f}^j$ is primitive in that it has no divergent subgraph.

Changes of the renormalization scheme are well-known to be equivalent to redefinitions of the coupling, which may be parametrized as [6]

$$g_{ijkl} = (g + f(g))_{mnpq} C_{mi} C_{nj} C_{pk} C_{ql}$$

where here and elsewhere we suppress indices as far as possible. We consistently neglect contributions from “snail” diagrams in which a bubble is attached to a propagator. Such contributions do not arise in minimal subtraction, and they will not be generated by redefinitions if the redefinitions themselves do not include such diagrams. The $\beta$-function $\beta_{ijkl}$ may then be decomposed into 1PI pieces together with one-particle reducible pieces determined by the anomalous dimension, in the form:

$$\beta = \tilde{\beta} + S_4 \gamma$$

with $\tilde{\beta}$ denoting the 1PI contributions and $S_4$ the sum over the four terms where $\gamma$ is attached to each external line. Up to three loops, the contributions to $\tilde{\beta}$ are given by

For later convenience we introduce the notation that $g_{3a}^j$ is the graph corresponding to $c_{3a}$, and $g_{3}^f$ is the graph corresponding to $d_2$, etc. We note that in Eq. (4) the graph $g_{3f}^j$ is primitive in that it has no divergent subgraph.

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$$g_{ijkl} = (g + f(g))_{mnpq} C_{mi} C_{nj} C_{pk} C_{ql}$$

where

$$C(g) = (1 - 2c(g))^{\frac{1}{2}}.$$
where $c_X$ and $d_X$ represent coefficients of generic diagrams in series such as Eqs. (4), (2) respectively. As explained in the Appendix (which in turn is a summary of the discussion in Ref. [7]), it is useful to parametrize the scheme change by $v$ defined implicitly by Eq. (A4). We assume that $v$ is parametrized in a similar way to Eqs. (5), (6), with analogues of $f(g)$, $c(g)$ given by similar diagrammatic series to those for the $\beta$-function and anomalous dimension, but with $c_X \rightarrow \delta_X$ and $d_X \rightarrow \epsilon_X$.

At one and two loops, we have

$$\delta c_1 = \delta d_1 = \delta c_2 = \delta c_{2R} = \delta d_2 = 0. \quad (8)$$

At three loops, we find using Eqs. (A10) and (A11)

$$\delta c_{3a} = 2X_{1,2R}^{ij} + 2X_{2,1R}^{ij}, \quad \delta c_{3b} = 2X_{2,1}^{ij},$$
$$\delta c_{3c} = 2X_{1,2}^{ij} + 2X_{2,1}^{ij}, \quad \delta c_{3d} = 2X_{1,2}^{ij},$$
$$\delta c_{3e} = 0, \quad \delta c_{3f} = 0,$$
$$\delta c_{3aR} = X_{1,2R}^{ij}, \quad \delta c_{3bR} = 2X_{1,2R}^{ij}, \quad \delta d_3 = 6X_{1,2R}^{ij}. \quad (9)$$

Here,

$$X_{X,Y}^{ij} = c_X \delta y - \delta_X c_Y,$$
$$X_{X,Y}^{ij} = d_X \delta y - \epsilon_X c_Y, \quad (10)$$

with corresponding definitions for $X_{X,Y}^{ij}$, $X_{X,Y}^{ij}$ when needed. We see from Eq. (A10) that the coefficients appearing in $X_{2,1}^{ij}$ etc., should in principle be “hatted” quantities defined according to Eq. (A11); but at this level, there is no distinction between the two, i.e., $\hat{c}_1 = c_1$, $\hat{c}_2 = c_2$, $\hat{d}_2 = d_2$. Note that $c_{3e}$ and $c_{3f}$ are individually invariant—which in the case of $c_{3f}$ follows immediately from the fact that it corresponds to a primitive graph. In deriving invariant combinations of coefficients, it is important to note that

$$X_{X,Y}^{ij} = -X_{Y,X}^{ij}, \quad X_{X,Y}^{ij} = -X_{Y,X}^{ij}, \quad X_{X,Y}^{ij} = -X_{Y,X}^{ij}. \quad (11)$$

We now start the search for these invariant combinations of coefficients at lowest (three-loop) order. A priori, since at this order there are nine three-loop coefficients and five variations $\delta_1$, $\delta_5$, $\delta_2$, $\delta_3$, $\delta_4$, one’s naive expectation would be $9 - 5 = 4$ invariants. However, the variations on the right-hand side of Eq. (9) are expressed in terms of only three independent quantities, $X_{1,2}^{ij}$, $X_{2,1}^{ij}$, and $X_{1,2R}^{ij}$, and so, in fact, we should have $9 - 3 = 6$ independent invariant combinations of three-loop coefficients. Indeed, we easily find from Eq. (9) that

$$I_1^{(3)} = c_{3a} + c_{3d} - 2c_{3aR}, \quad I_2^{(3)} = 2c_{3aR} - c_{3bR},$$
$$I_3^{(3)} = c_{3a} + c_{3c}, \quad I_4^{(3)} = 3c_{3b} + d_3. \quad (12)$$

are four independent invariant combinations (making a total of six invariants with the individually invariant $c_{3e}$ and $c_{3f}$).

### III. THE FOUR AND FIVE LOOP CALCULATIONS

In this section, we comprehensively examine the issue of scheme invariants at four loops and partially (due to increased calculational complexity) at five loops. The full list of four loop diagrams was presented in Ref. [8]. The anomalous dimension is given at this order by

$$2\gamma^{(4)} = d_{4a} + d_{4b} + d_{4c} + d_{4d}, \quad (13)$$

while the 1PI part of the $\beta$ function will be parametrized as...
\[ \tilde{\beta}^{(4)} = S_3 \left( c_{4a} + c_{4b} + c_{4c} \right) + S_6 \left( c_{4d} + c_{4e} + c_{4f} + c_{4g} + c_{4h} + c_{4i} \right) + S_{12} \left( c_{4j} + c_{4k} + c_{4l} + c_{4m} + c_{4n} + c_{4o} + c_{4p} \right) + c_{4q} S_6 + c_{4r} S_{24} + c_{4s} \]

for the one-vertex irreducible coefficients,

\[
\begin{align*}
\delta c_{4a} &= 4 \hat{x}_{1,3e}^{ij} + 4 \hat{x}_{3,1}^{ij} + 4 \hat{x}_{2,2}^{ij}, \\
\delta c_{4b} &= -\delta c_{4f} = 2 \hat{x}_{1,3a}^{ij} + 2 \hat{x}_{3,1}^{ij}, \\
\delta c_{4c} &= 6 \hat{x}_{1,3b}^{ij} + 2 \hat{x}_{1,3}^{ij}, \\
\delta c_{4d} &= 2 \hat{x}_{1,3a}^{ij} + 2 \hat{x}_{3,1}^{ij} + 2 \hat{x}_{3d,1}^{ij}, \\
\delta c_{4e} &= 2 \hat{x}_{1,3b}^{ij} + 2 \hat{x}_{2,2}^{ij}, \\
\delta c_{4g} &= 3 \hat{x}_{1,3a}^{ij} + 2 \hat{x}_{3,1}^{ij} + 2 \hat{x}_{2,2}^{ij}, \\
\delta c_{4h} &= \delta c_{4i} = 2 \hat{x}_{2,2}^{ij} + 2 \hat{x}_{1,3e}^{ij}, \\
\delta c_{4j} &= \delta c_{4k} = \hat{x}_{1,3b}^{ij} + \hat{x}_{2,2}^{ij}, \\
\delta c_{4l} &= 2 \hat{x}_{1,3c}^{ij} + \hat{x}_{3,1}^{ij} + 2 \hat{x}_{3b,1}^{ij}, \\
\delta c_{4m} &= \delta c_{4n} = 2 \hat{x}_{3,1}^{ij} + 2 \hat{x}_{2,2}^{ij}, \\
\delta c_{4o} &= \hat{x}_{1,3c}^{ij} + 2 \hat{x}_{1,3e}^{ij} + \hat{x}_{2,2}^{ij}, \\
\delta c_{4p} &= -\delta c_{4q} = \hat{x}_{1,3f}^{ij}, \\
\delta c_{4r} &= 2 \hat{x}_{1,3d}^{ij} + \hat{x}_{1,3e}^{ij},
\end{align*}
\]

for the anomalous dimension coefficients. At this level, in contrast to the earlier three-loop calculation, we do need to distinguish “hatted” from “unhattted” quantities. The \( \hat{x}^{ij} \) quantities are defined by
\[ \hat{X}_{X,Y}^{ij} = \hat{c}_X \delta_Y - \hat{c}_Y \delta_X. \] (18)

In other words as for \( X^{ij} \) in Eq. (10) but with the \( \beta \)-function quantities \( c_{X,Y} \) replaced by hatted quantities \( \hat{c}_{X,Y} \). Similar definitions apply to \( X^{\lambda\lambda} \), etc., but with \( d_{X,Y} \) replaced by hatted quantities \( \hat{d}_{X,Y} \) where relevant. Here, again \( \hat{c}_1 = c_1 \), \( \hat{c}_2 = c_2 \), \( \hat{d}_2 = d_2 \), while the quantities \( \hat{c}_{3a} \) etc. are defined by

\[ \hat{c}_{3a} = c_{3a} + \frac{1}{2} \delta c_{3a}, \] (19)

with \( \delta c_{3a} \) as defined as in Eq. (9), and similar expressions for \( \hat{c}_{3b} \) etc., and also \( \hat{d}_3 \). The additional terms in the hatted quantities derive from the first Lie derivative term on the right-hand side of Eq. (A10).

Now again we look for invariants at this order. Note that \( c_{4m}, c_{4n}, c_{4s}, d_{4a} \) are individually invariant—which again, in the case of \( c_{4s} \), follows immediately from the fact that it corresponds to a primitive graph. There are 30 four-loop coefficients whose variations are given in Eqs. (15)–(17); and there are 18 variations up to the three-loop level, namely \( \delta_{c_{3a} - 3\lambda^3}, \delta_{c_{3b} \lambda^3 R}, e_3, \delta_1, c_1, \delta_2, c_2, \delta_3, \delta_3, c_3 \). We would therefore naively expect \( 30 - 18 = 12 \) invariants. However, the variations on the right-hand sides of Eqs. (15)–(17) are expressed in terms of only 12 independent \( X/\hat{X} \) combinations, and therefore, the correct expectation is \( 30 - 12 = 18 \) invariants. Indeed, together with the four individually invariant coefficients \( c_{4m}, c_{4n}, c_{4s}, d_{4a} \), we find the following 14 linear invariant combinations:

\[ I_{1}^{(4)L} = c_{4h} - c_{4l}, \]
\[ I_{2}^{(4)L} = c_{4h} + c_{4f}, \]
\[ I_{3}^{(4)L} = c_{4d} + 2c_{4f} + 2c_{4l}, \]
\[ I_{4}^{(4)L} = c_{4l} + 2c_{4o} - 2c_{4r} - c_{4bR} + 2c_{4yR}, \]
\[ I_{5}^{(4)L} = c_{4c} + 3c_{4e} + d_{4c}, \]
\[ I_{6}^{(4)L} = c_{4d} + c_{4f} - c_{4k} + c_{4r} - c_{4hR}, \]
\[ I_{7}^{(4)L} = c_{4b} - c_{4d} + c_{4g} - c_{4o} + c_{4cR} + \frac{1}{2} c_{4yR}, \]
\[ I_{8}^{(4)L} = c_{4h} - c_{4k} + 2c_{4o} - c_{4r} - c_{4bR} + c_{4yR}, \]
\[ I_{9}^{(4)L} = 3c_{4c} + 6c_{4j} + 4d_{4c} - 2d_{4d}, \]
\[ I_{10}^{(4)L} = c_{4p} + c_{4q}, \]
\[ I_{11}^{(4)L} = 2d_{4b} + 3d_{4c} - 3d_{4d} + 6c_{4eR}, \]
\[ I_{12}^{(4)L} = 4c_{4cR} - 4c_{4hR} + c_{4bR}, \]
\[ I_{13}^{(4)L} = c_{4BR} - 2c_{4dR}, \]
\[ I_{14}^{(4)L} = c_{4cR} - c_{4hR} + c_{4yR} - c_{4fR}, \] (20)

We call these 18 invariants “linear.” We also find three “quadratic” invariants:

\[ I_{1}^{(4)Q} = c_{1}(2d_{4c} - d_{4d}) + 3c_{2}c_{3b} + 3d_{2}c_{3d}, \]
\[ I_{2}^{(4)Q} = c_{1}c_{4eR} - c_{2R}c_{3b} - d_{2}c_{3bR}, \]
\[ I_{3}^{(4)Q} = c_{1}(c_{4dR} - c_{4yR}) + c_{2}c_{3aR} - c_{2R}c_{3d}, \] (21)

which are a consequence of the relations

\[ c_2 X_{1,2}^{ij} - d_2 X_{1,2}^{ij} = c_1 X_{2,2}^{ij}, \]
\[ c_2 X_{1,2}^{ij} - d_2 X_{1,2}^{ij} = c_1 X_{2,2}^{ij}, \]
\[ c_2 X_{2,2}^{ij} - d_2 X_{2,2}^{ij} = c_1 X_{2,2}^{ij}, \] (22)

respectively. Altogether we have found 21 invariants, considerably more than (in fact almost double) the 12 that might naively have been expected.

We note that one may derive a fourth identity

\[ c_2 X_{2,2}^{ij} + d_2 X_{2,2}^{ij} = c_2 X_{2,2}^{ij}, \] (23)

which leads to an invariant

\[ I_{4}^{(4)Q} = d_{2}(c_{4bR} - 2c_{4yR}) + 2c_{2}c_{4eR} + \frac{2}{3} c_{2R}(2d_{4c} - d_{4d}); \] (24)

but in fact, Eq. (23) may be derived from linear combinations of the identities in Eq. (22) and correspondingly \( I_{4}^{(4)Q} \) is a linear combination of invariants already found in Eqs. (20) and (21).

We now proceed to a very partial five-loop calculation. The number of diagrams at five loops is dauntingly high, so we have not undertaken a complete calculation of all the invariants. A natural place to start is with the five-loop anomalous dimension, which has only 11 terms:
We find from Eqs. (A10) and (A11) that the variations of the coefficients in Eq. (25) are given by
\[
\begin{align*}
\delta d_{s1} &= 2\hat{\delta X}^{ij}_{3^0,2}, \\
\delta d_{s2} &= 12\hat{\delta X}^{ij}_{3e2} + 4\hat{\delta X}^{ij}_{14c} + 4\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s3} &= 6\hat{\delta X}^{ij}_{3e,2} + 4\hat{\delta X}^{ij}_{14d} + 4\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s4} &= 6\hat{\delta X}^{ij}_{14e} + 3\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s5} &= 6\hat{\delta X}^{ij}_{3e2} + 4\hat{\delta X}^{ij}_{14c} + 2\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s6} &= 4\hat{\delta X}^{ij}_{14a} + 3\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s7} &= 6\hat{\delta X}^{ij}_{3d2} + 3\hat{\delta X}^{ij}_{14c} + 2\hat{\delta X}^{ij}_{14d} + 2\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s8} &= 6\hat{\delta X}^{ij}_{3a,2} + 3\hat{\delta X}^{ij}_{3br,2} + 2\hat{\delta X}^{ij}_{14a} + 2\hat{\delta X}^{ij}_{14d} + 3\hat{\delta X}^{ij}_{2,3}, \\
\delta d_{s9} &= \hat{\delta X}^{ij}_{3b2} + 2\hat{\delta X}^{ij}_{14a} + 2\hat{\delta X}^{ij}_{2,3}. \\
\end{align*}
\]

The hatted \(X\)-type terms are defined in a similar manner to Eq. (18), i.e., by replacing \(\beta\)-function quantities \(c_{X,Y}\) and \(d_{X,Y}\) in Eq. (10) by hatted quantities \(\hat{c}_{X,Y}\), and \(\hat{d}_{X,Y}\). The hatted coefficients are, in turn, defined in terms of the corresponding unhatted quantities in a manner similar to Eq. (19). However, in the case of four-loop anomalous dimension coefficients, we need to define
\[
\hat{d}_{ab} = d_{ab} + \frac{1}{2} \beta d_{ab},
\]
where \(\beta d_{ab}\) (and similarly \(\beta d_{4ab}\)) are defined as in Eq. (17), but with hatted replaced by tilded quantities, namely
\[
\delta d_{ab} = 3\hat{\delta X}^{ij}_{1,3} + 6\hat{\delta X}^{ij}_{2,3}. 
\]
\(\hat{\delta X}^{ij}_{1,3}\) is defined as for \(X^{ij}_{1,3}\) but with \(d_1\) replaced by \(\hat{d}_1\). This in turn is defined by a similar equation to Eq. (19), but with \(\frac{1}{2} \rightarrow \frac{1}{3}\), i.e.

\[
\tilde{\delta}_3 = d_3 + \frac{1}{3} \delta d_3,
\]

with \(\delta d_3\) as in Eq. (9). This appears rather complicated, but simply reflects the nested structure of Eq. (A11). This feature has not been apparent in our calculations until now, because there the terms quadratic in \(L\) have not hitherto contributed.

However, it proves impossible to construct an invariant combination purely of anomalous dimension coefficients, and in fact, we need to include some 1VR four-point contributions, depicted below:

![Diagram](https://example.com/diagram.png)

The variations of the corresponding coefficients are given by
\[
\begin{align*}
\delta c_{5aR} &= \hat{\delta X}^{ij}_{1,4e} + 2\hat{\delta X}^{ij}_{2,3aR}, \\
\delta c_{5bR} &= 2\hat{\delta X}^{ij}_{1,4e} + 2\hat{\delta X}^{ij}_{2,3bR}, \\
\delta c_{5cR} &= 6\hat{\delta X}^{ij}_{1,4e} + 2\hat{\delta X}^{ij}_{2,3R},
\end{align*}
\]

where the hatted quantities are again defined in a similar way to Eq. (18). Note that [as we see in Eq. (16)] the variation \(\delta c_{4eR}\) is expressed in terms of unhatted quantities, so there is no need to invoke the modified \(\beta\) here. Naively, no linear invariant constructed purely from the coefficients in Eqs. (25) and (31) would be expected—there are 16 independent variations in Eq. (26) and only 14 coefficients. However, it turns out that there are three unexpected relations among the invariance conditions, resulting in just one five-loop linear invariant formed using only anomalous dimension and 1VR coefficients, namely
In addition, we also find several quadratic invariants, namely

\begin{align}
I_1^{(5)Q} &= c_1 d_{5a} + 2 d_2 c_{4p} + c_{3b} c_{5f}, \\
I_2^{(5)Q} &= 2 c_1 (d_{5g} - d_{5k}) - d_2 c_{4e} + d_3 c_{3b}, \\
I_3^{(5)Q} &= c_1 (2 c_{5bh} - c_{5br}) - 2 d_2 (2 c_{4ar} - c_{4r}) - c_{3b} (2 c_{3aR} - c_{3bR}), \\
I_4^{(5)Q} &= c_1 (d_{5d} - 3 d_{5g}) - \frac{1}{2} d_2 J - \frac{1}{2} d_2^2, \\
I_5^{(5)Q} &= c_1 (3 c_{5br} - c_{5cR}) + \frac{1}{2} c_{2R} J + 6 d_2 (c_{4ar} - c_{4cR}) + d_3 c_{3bR}, \\
I_6^{(5)Q} &= c_1 (d_{5c} + 2 d_{5e} - 2 d_{5h}) - 6 d_2 (c_{4b} - c_{4d} - 2 c_{4ar} + 2 c_{4cR}) + c_{2R} J - 2 d_3 (c_{3c} - c_{3d}), \\
I_7^{(5)Q} &= c_1 (d_{5b} - 2 d_{5e}) + 3 d_2 (c_{4l} - 2 c_{4o} + 2 c_{4r}) + (c_2 - c_{2R}) J + 6 c_{3b} c_{3e} + 2 c_{3c} d_3, \\
I_8^{(5)Q} &= c_1 (d_{5e} + d_{5f} - 2 d_{5i}) - 3 d_2 (c_{4b} - 3 c_{4ar} + 2 c_{4cR}) + \frac{1}{4} (5 c_{2R} - 4 c_2) J + d_3 (c_{3a} - c_{3c} - c_{3ar} + c_{3br}).
\end{align}

where \( J \) denotes the frequently occurring combination defined by

\[ J = 2 c_{4e} + 3 c_{4e} - 6 c_{4j}. \]

These owe their existence to relations like

\[ c_1 \hat{X}_{3a,2}^{\hat{g}} + d_2 \hat{X}_{3a,1}^{\hat{g}} + \hat{c}_{3a} \hat{X}_{2,1}^{\hat{g}} = 0, \tag{35} \]

together with similar relations for \( 3b \sim 3f, 3aR, 3bR; \) together with

\begin{align}
c_1 \hat{X}_{2,3}^{\hat{g}} + d_2 \hat{X}_{1,2}^{\hat{g}} + c_2 \hat{X}_{3,1}^{\hat{g}} &= 0, \\
c_1 \hat{X}_{2,3}^{\hat{g}} + d_2 \hat{X}_{1,2}^{\hat{g}} + c_2 \hat{X}_{3,1}^{\hat{g}} &= 0, \\
c_1 \hat{X}_{2,3}^{\hat{g}} + d_2 \hat{X}_{1,2}^{\hat{g}} + c_2 \hat{X}_{3,1}^{\hat{g}} &= 0. \tag{36} \end{align}

The number of invariants is as expected, since the 11 relations of the form Eqs. (35) and (36) reduce the effective number of independent variations from 16 to 5, yielding 14 - 5 = 9 invariants (both quadratic and linear).

In the absence of a complete calculation, one may estimate the total number of invariants which will be found at five loops. The five-loop \( \beta \) function was calculated in Ref. [9], and it contained contributions from 124 1PI 5-loop 4-point diagrams and 11 5-loop 2-point anomalous dimension diagrams, making 135 coefficients in total. \(^1\) There are 67 independent variations at five loops, implying a naive expectation of 135 - 67 = 68 linear invariants. On the other hand, there are 57 five-loop \( X \)-type terms [some of which of course appear in Eq. (26)], which, following the argument explained at four loops, implies an actual total of 135 - 57 = 78 linear invariants. But furthermore, there are altogether 27 identities of the form Eqs. (35) and (36), constructed from the one one-loop quantity, the three two-loop quantities, and the nine three-loop quantities. This implies an additional 27 quadratic invariants, making 105 invariants in total. As at four loops, there are considerably more invariants than might have been expected. One may also speculate on the possible existence of higher-order invariants based on higher-order Jacobi-style identities.

### IV. ONE-VERTEX REDUCIBLE GRAPHS

In this section, we briefly discuss the issue of \( \beta \)-function contributions from one-particle reducible (1VR) graphs. It is well-known that no such contributions arise using minimal subtraction within dimensional regularization (MS), as may easily be established by consideration of the diagram-by-diagram subtraction process. It would be convenient if when considering scheme redefinitions, one could restrict attention to schemes that have the same feature. In fact, if we start from a scheme such as \( \overline{\text{MS}} \) in which the \( \beta \)-function coefficients corresponding to four-point 1VR graphs \( G_R \) are zero, i.e., \( c_{G_R} = 0 \), it is clear from Eqs. (A10) and (A11) that the simple conditions

\[ \delta_{G_R} = 0 \tag{37} \]

will ensure that the redefined coefficients will also satisfy \( c_{G_R} = 0 \). This relies on the fact that for \( L, L' \) loop graphs \( G, G' \), with \( L + L' \geq 3 \), if (in the notation of the Appendix) \( L \cap G' \) contains 1VR graphs, then at least one of \( G \) or \( G' \) must itself be 1VR. We therefore have a simple all-orders prescription given by Eq. (37) for defining schemes with no 1VR contributions.

The redefined coupling as given by Eqs. (5) and (6) turns out to adopt a simple form when \( c_{G_R} = \delta_{G_R} = 0 \). We assume that \( f(g), c(g) \) in Eqs. (5) and (6) are given by similar diagrammatic series to those for the \( \beta \) function and anomalous dimension, but with \( c_X = \delta_X \) and \( d_X = \epsilon_X \). At one loop, we simply find \( \delta_1 = \delta_1 \). At two loops, we find

\[ \delta_2 = \delta_2 + \delta_1^2, \]

\[ \delta_{2R} = \delta_{2R} + \delta_1^2, \tag{38} \]

so that the condition for 1VI graphs is

\(^1\) The six-loop \( \beta \) function was recently computed in Ref. [10].

\(^2\) There are no 1VR two-point graphs, and therefore, there is no need to impose \( c_{G_R} = 0 \).
\[ c_{2R} = 0, \quad \tilde{\delta}_{2R} = \tilde{\delta}_1^2. \]  \hfill (39)

At three loops,
\[
\begin{align*}
\tilde{\delta}_{3a} &= \delta_{3d} + \delta_1(\delta_2 + 2\delta_{2R}) + \frac{2}{3}\delta_1^3, \\
\tilde{\delta}_{3b} &= \delta_{3b} + \delta_1\epsilon_2, \\
\tilde{\delta}_{3c} &= \delta_{3c} + \delta_1(\delta_2 + 2\delta_{2R}) + \frac{2}{3}\delta_1^3, \\
\tilde{\delta}_{3d} &= \delta_{3d} + \delta_1\delta_2 + \frac{2}{3}\delta_1^3, \\
\tilde{\delta}_{3e} &= \delta_{3e} + 2\delta_1\delta_2 + \frac{2}{3}\delta_1^3, \\
\tilde{\delta}_{3f} &= \delta_{3bR} + \frac{5}{2}\delta_1\delta_{2R} + \delta_1^3, \\
\tilde{\delta}_{3g} &= \delta_{3bR} + \frac{1}{2}(\delta_2 + 2\delta_{2R}) + \delta_1^3, \\
\tilde{\delta}_{3h} &= \epsilon_3 = \epsilon_3 + 3\delta_1\epsilon_2. \\
\end{align*}
\hfill (40)

It is easy to confirm using Eq. (38) that \( \delta_{2R} = \delta_{3aR} = \delta_{3bR} = 0 \) corresponds to
\[
\tilde{\delta}_{3aR} = \tilde{\delta}_1^1, \quad \tilde{\delta}_{3bR} = \tilde{\delta}_1\tilde{\delta}_2. \hfill (41)
\]

The emerging pattern is clear; the value for \( \tilde{\delta}_{G_{R}} \) is the product of the \( \tilde{\delta} \) s for its 1VI subgraphs. At four loops, we find
\[
\begin{align*}
\tilde{\delta}_{4aR} &= \delta_{4aR} + 3\delta_1\delta_{3aR} + \frac{3}{2}\delta_1^2\delta_{2R} + \frac{13}{3}\delta_1^2\delta_{2R} + \delta_1^1, \\
\tilde{\delta}_{4bR} &= \delta_{4bR} + 2\delta_1\delta_{3bR} + \delta_2 + \frac{4}{3}\delta_1\delta_{2R} + 2\delta_1^2\delta_2 + \delta_1^4, \\
\tilde{\delta}_{4cR} &= \delta_{4cR} + \frac{3}{2}\delta_1\delta_{3aR} + \frac{3}{3}\delta_1^2\delta_{2R} + \frac{2}{3}\delta_1^2\delta_{2R} + \delta_1^4, \\
\tilde{\delta}_{4dR} &= \delta_{4dR} + \delta_1\delta_{3c} + \delta_1\delta_{3bR} + \delta_2 + \delta_1^2\delta_2 + \frac{5}{3}\delta_1^2\delta_{2R} + \frac{2}{3}\delta_1^4, \\
\tilde{\delta}_{4eR} &= \delta_1\delta_{3b} + 2\epsilon_2\delta_{2R} + \delta_1^2\epsilon_2, \\
\tilde{\delta}_{4fR} &= \delta_{4fR} + \delta_1\delta_{3a} + \delta_1\delta_{3dR} + \frac{1}{2}\delta_1\delta_{3bR} + \delta_1^2\delta_2 + \frac{2}{3}\delta_1^3\delta_{2R} + \frac{2}{3}\delta_1^4, \\
\tilde{\delta}_{4gR} &= \delta_{4gR} + \frac{1}{2}\delta_1\delta_{3e} + \frac{2}{3}\delta_1^2\delta_2 + \frac{2}{3}\delta_1^2\delta_{2R} + \frac{2}{3}\delta_1^4. \\
\end{align*}
\hfill (42)

Using Eqs. (38) and (40), we find that \( \delta_{G_{R}} = 0 \) up to this level corresponds to taking
\[
\begin{align*}
\tilde{\delta}_{4aR} &= \tilde{\delta}_1^1, \quad \tilde{\delta}_{4bR} = \tilde{\delta}_2^2, \quad \tilde{\delta}_{4cR} = \tilde{\delta}_1^2\tilde{\delta}_2, \quad \tilde{\delta}_{4dR} = \tilde{\delta}_1\tilde{\delta}_3, \quad \tilde{\delta}_{4eR} = \tilde{\delta}_1\tilde{\delta}_2, \quad \tilde{\delta}_{4fR} = \tilde{\delta}_1\tilde{\delta}_3, \quad \tilde{\delta}_{4gR} = \tilde{\delta}_1\tilde{\delta}_3. \\
\end{align*}
\hfill (43)

so that each four-loop 1VR \( \delta \) is the product of the \( \tilde{\delta} \) s for its 1VI subgraphs, as expected. It seems highly likely that this simple pattern persists to all orders, but we have not been able to construct a proof.

When considering the scheme invariants, we can therefore restrict ourselves to those schemes with \( c_{G_{R}} = 0 \). The counting of invariants is then slightly different. Upon setting \( c_{3aR} = c_{3bR} = 0 \) in Eq. (12), there are then just three invariant combinations, namely, \( I_1^{(3)} = c_{3a} + c_{3d}, I_2^{(3)} \), and \( I_3^{(3)} \). We have lost two coefficients (\( c_{3aR} \) and \( c_{3bR} \)) and one independent variation \( X_{1i,2k} \), and so we expect to lose \( 2 - 1 = 1 \) invariants.

The pattern is similar at four loops; if we impose Eq. (41), then we have \( \delta c_{4aR} - 4\epsilon_0 = 0 \), and so we can consistently set \( c_{4aR} - 4\epsilon_0 = 0 \) in Eq. (20). We now have 23 coefficients and the 14 variations \( \tilde{\delta}_{3aR} - 3f, \tilde{\delta}_3, \tilde{\delta}_1^1, \tilde{\delta}_1^2, \tilde{\delta}_1\tilde{\delta}_2, \tilde{\delta}_2, \tilde{\delta}_2, \tilde{\delta}_1^1, \tilde{\delta}_1 \), leading to a naive expectation of 23 + 14 = 9 invariants. On the other hand, out of the original 18 linear invariants in Eq. (20), we are left with 11 linear invariant combinations, plus the four individual invariants, making 15. Again, this is as anticipated, since we have lost the seven coefficients \( c_{4aR} - 4\epsilon_0 \) and the four independent variations \( X_{1i,3aR}, X_{1i,3bR}, X_{2i,2R}, \) and \( X_{2i,2R} \), so we lose \( 7 - 4 = 3 \) invariant linear combinations. Furthermore, it is clear that in the 1VI case only one of the identities in Eq. (22) remains, and consequently, only one of the quadratic invariants in Eq. (21) survives. The total number of invariants is therefore 16; once again, almost double the naively expected number.

Finally, we can consistently set \( c_{5aR} = c_{5bR} = c_{5cR} = 0 \) in Eq. (32), to obtain a invariant constructed solely from anomalous dimension coefficients
\[ I_1^{(5)} = d_{5k} - 2d_{5c} - 2d_{5e} - 2d_{5f} + 4d_{5j}. \]  \hfill (44)

V. RELATION WITH HOPF ALGEBRA

Scheme invariants may be described graphically by adopting and extending rules described by Panzer [11] using the Hopf algebra coproduct \( \Delta : G \rightarrow G \otimes G \), where \( G \) is the vector space spanned by the set of connected 1PI superficially divergent graphs and the disconnected products of such graphs. The action of the coproduct \( \Delta \) on a Feynman graph \( g \in G \) is defined by
\[ \Delta g = \sum_i g_i \otimes g_i \quad \forall \text{ subgraphs } g_i \subset g, g_i, g \in G, \]
\[ g_i \neq 1, g, \text{ otherwise } \Delta g = \emptyset. \]  \hfill (45)

Here, \( g/g_i \) denotes the graph obtained from \( g \) by contracting each connected 1PI graph in the subgraph to a single vertex, or a single line if the connected 1PI graph has two external lines. Further details and a general discussion will be presented in Ref. [7], but this brief overview is sufficient for our present purposes. The invariants of Eqs. (20), (21), and (32) should correspond to combinations of graphs with a symmetric, or cocommutative, coproduct, following the general results of Ref. [7]. In this section, we verify this by explicit calculation. First, we readily derive the following useful results: At three loops
\[ \Delta(g_1^{4a}) = g_1^1 \otimes g_3^{3a} + 2g_3^{3c} \otimes g_1^1 + g_2^2 \otimes g_2^{2R}, \]
\[ \Delta(g_1^{4b}) = 2g_1^1 \otimes g_3^{3a} + 2g_3^{3c} \otimes g_1^1 + g_2^2 \otimes g_2^{2R}, \]
\[ \Delta(g_1^{4c}) = 2g_1^1 \otimes g_3^{3b} + g_2^2 \otimes g_1^1, \]
\[ \Delta(g_1^{4d}) = g_1^1 \otimes g_3^{3a} + g_1^1 \otimes g_3^{3b} + g_3^{3d} \otimes g_1^1 + g_1^2 \otimes g_2^2 + (g_1^1)^2 \otimes g_2^{2R} + g_1^1 g_2^2 \otimes g_1^1, \]
\[ \Delta(g_1^{4e}) = g_1^2 \otimes g_2^2 + g_1^1 \otimes g_3^{3a} + g_3^{3c} \otimes g_1^1 + (g_1^1)^2 \otimes g_2^2 + g_1^1 g_2^2 \otimes g_1^1. \] (46)

and at four loops we have for the four-point graphs

\[ \Delta(g_1^{3aR}) = 3g_1^1 \otimes g_2^{2R} + 2g_2^{2R} \otimes g_1^1 \]
\[ \Delta(g_1^{3bR}) = g_1^1 \otimes g_2^{2R} + g_1^1 \otimes g_2^2 + g_1^1 \otimes g_3^{3a} + (g_1^1)^2 \otimes g_2^2 + g_1^1 g_2^2 \otimes g_1^1, \]

(47)

and for the two-point graphs,
\[ 
\Delta(g_{4a}^{5a}) = g_{r}^{2} \otimes g_{r}^{2}, \\
\Delta(g_{4b}^{5b}) = 3g_{r}^{1} \otimes g_{r}^{3} + 2g_{r}^{2R} \otimes g_{r}^{2} + (g_{r}^{1})^{2} \otimes g_{r}^{2}, \\
\Delta(g_{r}^{4c}) = g_{r}^{1} \otimes g_{r}^{3} + 2g_{r}^{2} \otimes g_{r}^{2}, \\
\Delta(g_{r}^{4d}) = 2g_{r}^{1} \otimes g_{r}^{3} + 2g_{r}^{2} \otimes g_{r}^{2} + (g_{r}^{1})^{2} \otimes g_{r}^{2}. 
\]

At five loops, the basic coproducts are

\[ 
\begin{align*}
\Delta(g_{5}^{3a}) &= g_{r}^{3} \otimes g_{r}^{2}, \\
\Delta(g_{5}^{3b}) &= 2g_{r}^{3c} \otimes g_{r}^{2} + g_{r}^{3} \otimes g_{r}^{3} + g_{r}^{1} \otimes g_{r}^{3}, \\
\Delta(g_{5}^{3c}) &= 2g_{r}^{3c} \otimes g_{r}^{2} + 2g_{r}^{3} \otimes g_{r}^{3} + 2g_{r}^{1} \otimes g_{r}^{3} + (g_{r}^{1})^{2} \otimes g_{r}^{3} + 2g_{r}^{1}g_{r}^{2} \otimes g_{r}^{2}, \\
\Delta(g_{5}^{3d}) &= g_{r}^{3} \otimes g_{r}^{3} + 2g_{r}^{1} \otimes g_{r}^{3}, \\
\Delta(g_{5}^{3e}) &= 2g_{r}^{3c} \otimes g_{r}^{2} + 2g_{r}^{3} \otimes g_{r}^{3} + 2g_{r}^{1} \otimes g_{r}^{3}, \\
\Delta(g_{5}^{3f}) &= 2g_{r}^{3R} \otimes g_{r}^{2} + 2g_{r}^{3} \otimes g_{r}^{3} + 2g_{r}^{1} \otimes g_{r}^{3}, \\
\Delta(g_{5}^{3g}) &= g_{r}^{3} \otimes g_{r}^{3} + 2g_{r}^{1} \otimes g_{r}^{3} + 2g_{r}^{1}g_{r}^{2} \otimes g_{r}^{2}, \\
\Delta(g_{5}^{3h}) &= 2g_{r}^{3c} \otimes g_{r}^{2} + 2g_{r}^{3R} \otimes g_{r}^{2} + 2g_{r}^{3} \otimes g_{r}^{3} + 2g_{r}^{1} \otimes g_{r}^{3} + (g_{r}^{1})^{2} \otimes g_{r}^{3} + 2g_{r}^{1}g_{r}^{2} \otimes g_{r}^{2}.
\end{align*} 
\]

At three loops, the coproducts for \( g_{3}^{3c} \) and \( g_{3}^{3f} \) are comcomutative and zero, respectively, corresponding to the individual invariance of \( c_{3c} \) and \( c_{3f} \). Corresponding to the invariants in Eq. (12), we have the following combinations with comcomutative coproducts:

\[ 
\begin{align*}
\Delta(g_{3}^{3a} + g_{3}^{3d} - g_{3}^{3AR}) &= 2g_{r}^{1} \otimes g_{r}^{2} - 2g_{r}^{1} \otimes g_{r}^{2R}, \\
\Delta(g_{3}^{3AR} - g_{3}^{3BR}) &= 2g_{r}^{1} \otimes g_{r}^{2R} - g_{r}^{1} \otimes g_{r}^{2}, \\
\Delta(g_{3}^{3a} + g_{3}^{3c}) &= 2g_{r}^{1} \otimes g_{r}^{2} + g_{r}^{1} \otimes g_{r}^{2R}, \\
\Delta(2g_{3}^{3b} + g_{3}^{3}) &= 2g_{r}^{1} \otimes g_{r}^{2}, 
\end{align*} 
\]

where

\[ G_{1} \otimes G_{2} = G_{1} \otimes G_{2} + G_{2} \otimes G_{1}. \]

The scheme-invariant combination of RG coefficients corresponding to a combination of graphs \( \sum_{i} \alpha_{i}g_{i}^{3} + \sum_{j} \beta_{j}g_{j}^{3} \) with a comcomutative coproduct is [7]

\[ \sum_{i} \alpha_{i}S_{i}c_{i} + \sum_{j} \beta_{j}S_{j}d_{j}, \]

where \( S_{i} \) are the symmetry factors for the four-point graphs, and \( S_{j} \), those for the two-point graphs. The relevant symmetry factors at this loop order are given by

\[ 
\begin{align*}
S_{3f} &= 1, \\
S_{3c} &= 2, \\
S_{3a} &= S_{3c} = 3d = S_{3bR} = 4, \\
S_{3b} &= 6, \\
S_{3aR} &= 8.
\end{align*} 
\]

So for instance,

\[ g_{3}^{3a} + g_{3}^{3d} - g_{3}^{3AR} \rightarrow 4c_{3a} + 4c_{3d} - 8c_{3aR}. \]

which agrees with \( I_{i}^{(3)} \) in Eq. (12) up to an overall factor.

At four loops, the coproducts for \( g_{4}^{4m} \), \( g_{4}^{4a} \) and \( g_{4}^{4d} \) are comcomutative, and that for \( g_{4}^{4s} \) is zero, corresponding to the individual invariance of \( c_{4m} \), \( c_{4a} \), \( c_{4s} \), and \( d_{4a} \). Corresponding to the invariants in Eq. (20), we have the following combinations with comcomutative coproducts:

\[ 065011-10 \]
Δ(gα4b − gα4l) = C(4)L,  \\
Δ(gα4b + 2gα4f) = C(4)S,  \\
Δ(gα4a + gα4f + gα4l) = C(4)L,  \\
Δ(gα4l + gα4o − 2gα4r − gα4bR + 2gα4gR − gα4gc) = C(4)L,  \\
Δ(gα4c + 2gα4e + gα4c) = C(4)L,  \\
Δ(gα4d + gα4f − gα4k + gα4r − gα4bR) = C(4)L,  \\
Δ(gα4b − gα4d + gα4o − gα4r − gα4bR + gα4gR) = C(4)L,  \\
Δ(gα4b − gα4k + gα4o − gα4r − gα4bR + gα4gR) = C(4)L,  \\
\Delta(gα4e + gα4j + 2gα4e3 + 1 2 gα41gα43) = C(4)L,  \\
\Delta(gα4g + gα44) = C(4)L,  \\
Δ(gα4b + 3gα4c − 3gα4d + 2gα4kR + gα4l) = C(4)L,  \\
Δ(gα4dR + gα4bR − 2gα4cR) = C(4)L,  \\
Δ(gα4bR − gα4hR + gα41gα43c) = C(4)L,  \\
Δ(2gα4bR − gα4cR + gα4fR − 2gα4gR − gα41gα43a) = C(4)L.  \\

Here, rather than give explicit expressions on the right-hand side, we use C(4)L ∈ G ⊗ G to denote l-loop cocommutative coproducts corresponding to linear invariants. Since their exact form is not especially significant, we relagate the full expressions to Appendix B. The noteworthy new feature here is the necessity sometimes to add quadratic terms, of course with no counterpart in the original linear invariants of Eq. (20), on the left-hand side in order to obtain cocommutative results. The need for this is explained in general in Ref. [7].

Corresponding to the quadratic invariants in Eq. (21), we have

\[ \Delta(2gα1gα4e − gα1gα4d + gα2gα3d) \]
\[ + 2gα2gα3b − (gα1)2gα3b) = C(4)Q, \]
\[ \Delta(gα1gα4R − gα2gα3b − gα2gα3b) = C(4)Q, \]
\[ \Delta(gα1(2gα4dR − 2gα4bR + 2gα2gα3R − gα2gα3d) = C(4)Q. \]

Here, we see the need for additional cubic terms on the left-hand side, in addition to the quadratic terms corresponding to those in the invariant. The relevant graph combination corresponding to the additional invariant in Eq. (24) may be derived from those already given, and hence, it is not displayed here. Here, we use C(4)L ∈ G ⊗ G to denote l-loop cocommutative coproducts corresponding to quadratic invariants. The coefficients of the linear invariants in Eq. (20) may be obtained from the linear terms on the left-hand side of Eq. (54) by substitutions similar to those described at three loops after Eq. (50). Likewise, the coefficients of the quadratic invariants in Eq. (21) may be obtained from the quadratic terms on the left-hand side of Eq. (55) by similar substitutions. Here, the relevant symmetry factors are given by

\[ S_{4s} = 1, \quad S_1 = S_2 = S_{4a} = S_{4m} = S_{4n} = S_{4p} = S_{4q} = 2, \]
\[ S_{2R} = S_{4b} = S_{4q} = S_{4c} = S_{4d} = S_{4f} = S_{4h} = S_{4l} \]
\[ = S_{4k} = S_{4l} = S_{4r} = S_{4c} = S_{4d} = 4, \]
\[ S_5 = 3, \quad S_{6c} = S_{6l} = S_{6j} = S_{6a} = 12, \]
\[ S_{4cR} = S_{4dR} = S_{4fR} = S_{4b} = S_{4q} = S_{4o} = S_{4b} = 8, \quad S_{4aR} = 16. \]

(56)

Together with those in Eq. (52). We also find corresponding to Eq. (32)

\[ \Delta(4gα1gα5b − 4gα1gα5e − 2gα1gα5r − gα1gα5f + 4gα5j − 2gα5aR + 4gα5bR) \]
\[ + gα1gα5cR + gα1gα1gα4c − 4gα1gα3b + 2gα1gα2gα3b) = C(5)L. \]

(57)

Corresponding to the quadratic invariants in Eq. (33), we find
Δ[g_1^5 g_3^{5a} + g_7^2 g_3^{4p} + g_3^6 g_3^{4f}] = C_1^{(5)Q},
Δ[g_1^1 (g_7^5 - 2 g_3^5)] - g_7^2 g_3^{4c} + g_3^2 g_3^{4b}] = C_2^{(5)Q},
Δ[g_1^1 (g_3^{5aR} - g_3^{5bR}) + g_3^2 (-g_3^{4aR} + g_3^{4cR}) + g_3^3 (-g_3^{3aR} + g_3^{3bR})] = C_3^{(5)Q},
Δ[g_1^1 (g_3^5d - g_3^5g) - 2 g_3^2 G_J + 1/2 (g_3^3)^2 + g_1^1 g_3^2 g_3^3] = C_4^{(5)Q},
Δ[g_1^1 (2 g_3^{5bR} - g_3^{5cR}) + g_3^2 g_3 G_J + g_1^2 (g_3^{4aR} - 2 g_3^{4cR}) + g_3^3 (g_3^{3bR} + (g_3^1)^2 (g_3^{4j} - g_3^{4e} - g_3^{4f} + g_3^1 g_3^{2R} g_3^{3b}]) = C_5^{(5)Q},
Δ[g_1^1 (g_3^5c + g_3^5e - 2 g_3^5h) - g_3^2 (g_3^{ab} - 2 g_3^{ak} - 2 g_3^{ac} + 2 g_3^{ae} + 2 g_3^{aR} + 2 g_3^{ecR} + g_3^{2R} G_J)]
+ (g_3^2 - 2 g_3^{2R}) g_1^1 g_3^2 - g_3^1 g_3^2 g_3^3 + (g_3^1)^2 g_3^4] = C_6^{(5)Q},
Δ[g_1^1 (2 g_3^5f - 4 g_3^5i) - g_3^2 (2 g_3^{ab} - 3 g_3^{ak} + 4 g_3^{ac})]
+(5 g_3^{2R} - 8 g_3^2) G_J + g_1^2 (2 g_3^{3a} - 2 g_3^{3c} - g_3^{3aR} + 2 g_3^{3bR}) + (g_3^2 - 2 g_3^{2R}) g_1^1 g_3^2 - g_3^1 g_3^2 g_3^3 + (g_3^1)^2 g_3^4 = C_8^{(5)Q},
(58)

where

G_J = g_3^{4c} + g_3^{4e} - g_3^{4j}
(59)

corresponds to J defined in Eq. (34). The invariants of Eqs. (32) and (33) may be recovered from Eqs. (57) and (58) as before. Here, the relevant symmetry factors [in addition to those in Eqs. (52) and (56)] are

S_{5a} = 1, S_{5b} = 2, S_{5c} = S_{5e} = 4, S_{5cR} = S_{5d} = S_{5e} = S_{5j} = S_{5f} = 8,
S_{5bR} = S_{5k} = 12, S_{5j} = 16, S_{5aR} = S_{5g} = 24.
(60)

VI. a-Function Considerations

A good deal of effort has been invested in recent years [12–15] in the search for an a-theorem, a generalization of Zamolodchikov’s two-dimensional c-theorem [16] to four dimensions (or indeed to other dimensions higher than two [17–21]). From our point of view, as mentioned in the introduction, the crucial development is the demonstration that the β functions in theories in four and in six dimensions obey a gradient flow equation similar to one that plays a critical role in the derivation of the c-theorem [22–25]. These gradient flow equations often place constraints relating the β-function coefficients, as has been shown for four-dimensional gauge theories [4] and six-dimensional φ^4 theories [5] (similar gradient flows have been demonstrated in three dimensions [1–3], though here, the theoretical underpinning has not yet been provided). Our purpose in this section is to apply the same considerations to our four-dimensional φ^4 theory, where we are able to confirm our results using the explicit calculations available to a high loop order. We start by presenting the basic theoretical background in general notation in the interests of clarity and brevity. For a theory with couplings g^4, the corresponding β functions are defined by

β^4 = μ d/μ g^4,
(61)

where μ is a mass scale (in practice usually the standard dimensional regularization mass scale). The essential conclusion of Refs. [23,24] is the existence of a function A, such that

∂_J A = T_{JJ} β^J
(62)

where ∂_J = ∂/∂ g^J and

T_{JJ} = G_{JJ} + ∂_J W_J - ∂_J W_I
(63)

with G_{JJ} symmetric. The function A is invariant up to

A → A + g_I β^I β^J,
(64)

where g_I is an arbitrary symmetric matrix. At lowest order, we have an a function given by

A^{(4)} = A_1^{(4)}
(65)

In general, for a theory with a symmetry, the β function should be replaced by a “generalized” β function [24]. It was shown by explicit calculation in Ref. [26] that the difference between the two becomes nontrivial at three loops for a fermion-scalar theory in four dimensions. However, for a pure scalar theory, we do not expect any distinction until five loops, which is beyond our interests in this section.
and Eq. (62) simply implies
\[ 3A_1^{(4)} = 3c_1 \Rightarrow A_1^{(4)} = c_1 \tag{66} \]

At the next order, we have
\[ 4A_1^{(5)} = 2d_2, \]
\[ 4A_2^{(5)} = 3c_2 R + c_1 T^{(4)}, \]
\[ 4A_3^{(5)} = 6c_2 + 2c_1 T^{(4)}. \tag{67} \]

Here, \( T^{(4)} \) represents the coefficient of the single fourth-order metric term. The figure below displays this structure by showing its contraction with a \( d g \) (represented by a cross) and a \( \beta^{(1)} \) (represented by a diamond).

In Eq. (68), there are two equations and three unknowns resulting in one residual free parameter. This corresponds to the invariance under
\[ A_2^{(5)} \rightarrow A_2^{(5)} + 3g^{(3)} c_1^2, \quad A_3^{(5)} \rightarrow A_3^{(5)} + 6g^{(3)} c_1^2, \]
\[ T^{(4)} \rightarrow T^{(4)} + 12g^{(3)} c_1 \tag{70} \]
reflecting the freedom described by Eq. (64) at lowest order (with \( g_{IJ} = g^{(3)} \delta_{IJ}, g^{(3)} \) arbitrary). The six-loop \( a \)-function is given by

and the seven associated five-loop metric contributions are depicted below, with the same conventions as for \( T^{(4)} \) earlier.

We now find from Eq. (62)
\[ A_1^{(6)} = 3c_{3b} + d_2 T^{(4)}, \]
\[ 2A_1^{(6)} = 2d_3 + 3c_1 (T_6^{(5)} + T_7^{(5)}), \]
\[ 2A_1^{(6)} = 3c_1 T_5^{(5)} + d_2 T^{(4)}, \]
\[ 5A_2^{(6)} = 3c_3 f, \]
\[ A_3^{(6)} = 6c_{3d} + 2c_1 T_2^{(5)}, \]
\[ 4A_3^{(6)} = 12c_{3e} + 2c_1 T_3^{(5)} + 4c_2 T^{(4)}, \]
\[ 5A_4^{(6)} = 3c_{3aR} + c_1 (T_4^{(5)} + T_1^{(5)}) + c_{2R} T^{(4)}, \]
\[ 2A_5^{(6)} = 6c_{3c} + c_1 T_3^{(5)} + 2c_{2R} T^{(4)}, \]
\[ 2A_5^{(6)} = 6c_{3bR} + c_1 (T_2^{(5)} + 2T_4^{(5)}) + c_2 T^{(4)}, \]
\[ A_6^{(6)} = 3c_{3a} + 2c_1 T_1^{(5)} + c_2 T^{(4)}. \] (73)

The solution of Eq. (73) is then

\[ A_1^{(6)} = -\frac{9}{8} + \frac{1}{6} T^{(4)}, \]
\[ A_2^{(6)} = \frac{36}{5}, \]
\[ A_3^{(6)} = \frac{51}{5} - 2T^{(4)} + 4A_4^{(6)}, \]
\[ A_4^{(6)} = \frac{27}{10} - T^{(4)} + 4A_4^{(6)}, \]
\[ T_1^{(5)} = \frac{3}{5} + 2A_4^{(6)}, \]
\[ T_2^{(5)} = \frac{33}{5} - T^{(4)} + 2A_4^{(6)}, \]
\[ T_3^{(5)} = \frac{42}{5} - 2T^{(4)} + 8A_4^{(6)}, \]
\[ T_4^{(5)} = -\frac{3}{5} + 3A_4^{(6)}, \]
\[ T_5^{(5)} = -\frac{3}{4} + \frac{1}{18} T^{(4)}, \]
\[ T_6^{(5)} + T_7^{(5)} = -\frac{2}{3} + \frac{1}{9} T^{(4)}. \] (76)

Here, we have ten equations for 12 unknowns, resulting in two free parameters. This corresponds to the lower-order invariance, together with the invariance under

\[ A_3^{(6)} \rightarrow A_3^{(6)} + 4g^{(4)}, \]
\[ A_4^{(6)} \rightarrow A_4^{(6)} + g^{(4)}, \]
\[ A_5^{(6)} \rightarrow A_5^{(6)} + 4g^{(4)}, \]
\[ T_4^{(5)} \rightarrow T_4^{(5)} + 3g^{(4)}, \]
\[ T_1^{(5)} \rightarrow T_1^{(5)} + 2g^{(4)}, \]
\[ T_2^{(5)} \rightarrow T_2^{(5)} + 2g^{(4)}, \]
\[ T_3^{(5)} \rightarrow T_3^{(5)} + 8g^{(4)}, \] (77)
reflecting the freedom under

\[ A \rightarrow A + g^{(4)} \beta_{ijkl} \beta_{ijmn} g_{klmn}, \] (78)

with \( g^{(4)} \) arbitrary. Finally, the seven-loop \( a \) function is parametrized as
These seven-loop vacuum diagrams were given in Fig. 6 of Ref. [27], and we have retained their ordering (similarly, the five and six loop vacuum diagrams were depicted in their Figs. 4 and 5, respectively). Since there are 24 six-loop metric contributions, we have introduced a compact notation to avoid depicting them all individually. Equation (80) shows the six-loop vacuum diagrams, seen already in Eq. (71), but now with some vertices labeled. We introduce the notation $T^{(6)}_{nxy}$ to denote a metric contribution where the vertices $x, y$ in diagram $n$ correspond to the $I, J$ indices, respectively, of a contribution to $T^{(6)}_{IJ}$. The labeling shown is sufficient to cover all of the independent possibilities.

The number of $T$-type contributions is the number of distinct ways of selecting an ordered pair of vertices from the diagrams shown in (80), namely 24. At this order, Eq. (62) implies

$$4A_1^{(7)} = \frac{3}{2} d_2(T_6^{(5)} + T_7^{(5)}) + 2d_{4a},$$
$$2A_1^{(7)} = \frac{3}{2} d_2 T_5^{(5)},$$
$$6A_2^{(7)} = \frac{1}{2} d_2 (T_4^{(5)} + T_6^{(5)} + T_7^{(5)}),$$
$$2A_3^{(7)} = c_c T_{1ab}^{(6)} + c_{3b}(T_4^{(5)} + 2T_7^{(5)}) + 6c_{4eR},$$
$$2A_3^{(7)} = c_1 T_{1ab}^{(6)} + T_{1ce}^{(6)} + T_{1cd}^{(6)} + 3c_{2R}(T_6^{(5)} + T_7^{(5)}) + 2d_{4b},$$
$$2A_3^{(7)} = c_1 T_{1be}^{(6)} + c_1 T_{1ba}^{(6)} + 3c_{2R} T_5^{(5)} + d_2 T_4^{(5)}.$$
\[
6A_4^{(7)} = c_1 T_{3ad}^{(6)} + 2c_3 T_{3}^{(4)} + 6c_{4i} \\
6A_5^{(7)} = c_{4s} \\
2A_6^{(7)} = 2c_1 T_{1ab}^{(6)} + \frac{1}{2} d_2 (2T_{2}^{(5)} + T_{3}^{(5)}) + 12c_{4j} \\
2A_6^{(7)} = 2c_1 (T_{1cb}^{(6)} + T_{1ce}^{(6)}) + 3c_2 (T_{6}^{(5)} + T_{7}^{(5)}) + 2d_{4d} \\
2A_6^{(7)} = 2c_1 T_{1bc}^{(6)} + 3c_2 T_{5}^{(5)} + \frac{1}{2} d_2 T_{3}^{(5)} \\
2A_7^{(7)} = 2c_{3b} T^{(4)} + d_2 T_{3}^{(5)} + 6c_{4e} \\
2A_7^{(7)} = 2c_1 T_{1cd}^{(6)} + 3c_2 (T_{6}^{(5)} + T_{7}^{(5)}) + 2d_{4c} \\
2A_7^{(7)} = 2c_1 T_{1ba}^{(6)} + 3c_2 T_{5}^{(5)} + d_2 T_{2}^{(5)} \\
4A_8^{(7)} = 3c_1 (T_{1cd}^{(6)} + T_{1bc}^{(6)} + T_{1bd}^{(6)}) + d_3 T^{(4)} \\
2A_8^{(7)} = 3c_1 T_{1ac}^{(6)} + d_3 T^{(4)} + 3c_{4c} \\
4A_9^{(7)} = 3c_1 T_{2ab}^{(6)} + 12c_{4p} \\
2A_9^{(7)} = 6c_{4q} + c_{3f} T^{(4)} \\
6A_{10}^{(7)} = c_1 (T_{4ad}^{(6)} + T_{4ac}^{(6)}) + c_{2R} (T_{4}^{(5)} + T_{3}^{(5)}) + c_{3aR} T^{(4)} + 3c_{4aR} \\
4A_{11}^{(7)} = c_1 (T_{5cd}^{(6)} + T_{5bc}^{(6)}) + 2c_{2R} T_{4}^{(5)} + c_{3} T^{(4)} + 6c_{4dR} \\
2A_{11}^{(7)} = c_1 T_{5ac}^{(6)} + 2c_{2R} T_{1}^{(5)} + c_{3} T^{(4)} + 3c_{4b} \\
4A_{12}^{(7)} = c_1 (2T_{5cb}^{(6)} + T_{3ad}^{(6)}) + 2c_2 T_{3}^{(5)} + 4c_{3bR} T^{(4)} + 12c_{4k} \\
2A_{12}^{(7)} = 2c_1 (T_{5be}^{(6)} + T_{5ab}^{(6)}) + d_2 T_{5}^{(4)} + 2c_{3d} T^{(4)} + 6c_{4d} \\
6A_{13}^{(7)} = c_1 T_{2ad}^{(6)} + 2c_{3d} T^{(4)} + 6c_{4b} \\
4A_{14}^{(7)} = 2c_1 T_{5ca}^{(6)} + c_{2} T_{3}^{(5)} + 2c_{3a} T^{(4)} + 6c_{4f} \\
2A_{14}^{(7)} = 2c_1 T_{5ba}^{(6)} + c_2 T_{2}^{(5)} + 3c_{4bR} \\
2A_{15}^{(7)} = c_1 (T_{5cb}^{(6)} + T_{5ac}^{(6)}) + c_{2R} T_{3}^{(5)} + 2c_{3aR} T^{(4)} + 6c_{4g} \\
2A_{15}^{(7)} = c_1 (T_{5cb}^{(6)} + T_{5ba}^{(6)} + 2T_{4ab}^{(6)}) + c_{2} T_{4}^{(5)} + c_{2R} T_{2}^{(5)} + c_{3bR} T^{(4)} + 6c_{4eR} \\
2A_{15}^{(7)} = c_1 (T_{5ab}^{(6)} + T_{4ac}^{(6)}) + 2c_2 T_{5}^{(5)} + 2T_{1}^{(5)} + c_{3a} T^{(4)} + c_{3bR} T^{(4)} + 6c_{4fR} \\
2A_{16}^{(7)} = c_1 (T_{3ab}^{(6)} + T_{3ac}^{(6)}) + 2c_{2R} T_{3}^{(5)} + 4c_{3c} T^{(4)} + 12c_{4d} \\
2A_{16}^{(7)} = c_1 (2T_{5bc}^{(6)} + T_{3ad}^{(6)}) + 4c_2 T_{5}^{(5)} + 2c_{3a} T^{(4)} + 12c_{4gR} \\
A_{16}^{(7)} = c_1 (2T_{5cd}^{(6)} + T_{3da}^{(6)}) + c_{2R} T_{2}^{(5)} + 12c_{4a} \\
A_{16}^{(7)} = 2c_1 T_{5ac}^{(6)} + 4c_2 T_{1}^{(5)} + 2c_{3d} T^{(4)} + 3c_{4a} \\
2A_{17}^{(7)} = c_1 (T_{3bc}^{(6)} + T_{3da}^{(6)}) + 4c_2 T_{2}^{(5)} + 24c_{4r} \\
2A_{17}^{(7)} = 2c_1 T_{3ac}^{(6)} + 2c_2 T_{3}^{(5)} + 4c_{3a} T^{(4)} + 12c_{4m} \\
2A_{17}^{(7)} = 2c_1 T_{3ab}^{(6)} + 2c_2 T_{3}^{(5)} + 4c_{3a} T^{(4)} + 12c_{4n}. \quad (81)
\]
The counting of unknowns is now slightly more subtle; we shall explain this in some detail since the solution of Eqs. 81 leads to constraints on the $\beta$-function coefficients, and we would like to be sure that we have obtained the correct number of these. There are 36 four-loop structures (including IPR structures, which cannot contribute to the $\beta$-function and hence must be set to zero) leading to the 36 equations in Eq. 81; and there are 17 A coefficients [as shown in Eq. (79)] and 24 $T$ coefficients at this order.

However, $T_{1cb}^{(6)}$ and $T_{1cd}^{(6)}$ only appear in the combination $T_{1bc}^{(6)} + T_{1ce}^{(6)}$, and $T_{1bd}^{(6)}$, and $T_{1cd}^{(6)}$ only appear in the combination $T_{1bc}^{(6)} + T_{1bd}^{(6)} + T_{1cd}^{(6)}$, furthermore, there are two invariances, under shifts among $T_{2bc}^{(6)}$, $T_{3da}^{(6)}$, $T_{5cd}^{(6)}$, $T_{5ec}^{(6)}$, and among $T_{4ab}^{(6)}$, $T_{4ac}^{(6)}$, $T_{5ab}^{(6)}$, $T_{5be}^{(6)}$. Therefore, there is a total of $17 + 26 - 5 = 38$ unknowns at this order. The lower-order metric coefficients $T_{1}^{(5)} - T_{7}^{(5)}$ get determined in Eq. (76) up to two unknowns, resulting in 40 unknowns in total. There are seven five-loop vacuum diagrams, which can contribute to the freedom in Eq. (64) [the diagrams appearing in (72) but with insertions of $\beta^{(1)}$ replacing the diamonds and crosses], but two of these give the same contribution. There is also one four-loop vacuum diagram contributing to the freedom in Eq. (64) [the one appearing in (69) but with insertions of $\beta^{(1)}$, $\beta^{(2)}$ replacing the diamond and cross, respectively]. Finally there is a three-loop vacuum diagram corresponding to the freedom in Eq. (70). Therefore, the number of unknowns that are solved for is only $40 - 6 - 1 - 1 = 32$. This implies that $36 - 32 = 4$ of the 36 equations must remain as constraints. Indeed, after solving the equations we find the constraints

$$\begin{align*}
2c_1d_{4a} - d_2I_4^{(3)} & = 0, \\
2c_1(I_{11}^{(4)} - I_{15}^{(4)} - 3I_{16}^{(4)}) + 3d_2\left(I_2^{(3)} - I_3^{(3)} + \frac{1}{2}c_3\right) + 2(c_2 - c_2R)I_4^{(3)} & = 0, \\
2c_1(I_{11}^{(4)} - I_9^{(4)}) + 4(c_2 - c_2R)I_4^{(3)} - 3d_2(2I_2^{(3)} - c_3) & = 0, \\
& - c_1(2I_2^{(4)} - I_3^{(4)} + I_4^{(4)} + I_1^{(4)} + \frac{1}{2}c_4n + \frac{1}{2}c_4d) + (c_2 - c_2R)(2I_2^{(3)} - c_3) & = 0.
\end{align*}$$

(82)

We note that as is to be expected, these constraints may be expressed in terms of the invariants defined in Eqs. (12), (20), and (21). At four loops (again extracted from Ref. [8]) the coefficients are

$$\begin{align*}
c_{4a} & = \frac{1}{3}(6\zeta_3 - 11), & c_{4b} & = 1 - \zeta_3, & c_{4c} & = \frac{7}{12}, \\
c_{4d} & = \frac{1}{2}, & c_{4e} & = \frac{121}{144}, & c_{4f} & = 1 - 2\zeta_3, \\
c_{4g} & = c_{4o} = \frac{1}{4}(2\zeta_3 - 1), & c_{4h} & = c_{4i} = \frac{1}{6}(5 - 6\zeta_3), \\
c_{4j} & = \frac{5}{6}, & c_{4k} & = -\frac{37}{288}, & c_{4l} & = \frac{2}{3}, \\
c_{4m} & = 4\zeta_3 - 5, & c_{4n} & = -5, \\
c_{4p} & = 3(\zeta_3 - 2\zeta_5), & c_{4q} & = -3(2\zeta_3 + \zeta_4), & c_{4r} & = -40\zeta_5, \\
c_{4s} & = \frac{5}{48}, & d_{4b} & = -\frac{5}{32}, & d_{4c} & = \frac{13}{48}, & d_{4d} & = \frac{2}{3},
\end{align*}$$

(83)

with $c_{4m} = \cdots = c_{4r} = 0$, and we may easily check that the values in Eqs. (74), (75), and (83) satisfy the constraints in Eq. (82).

We refrain from giving the values of the $a$ coefficients in the general case. However, an interesting special case is that of a symmetric $T_{JJ}$. It turns out that we can impose symmetry on $T_{JJ}$ up to this order without needing to impose any further constraints on the $\beta$-function coefficients. The $a$-function coefficients are then
We have also considered the construction of the algebra approach to renormalization, each invariant is invariants, which would be missed in a naive counting. In higher than might be expected from a naive counting. Inconsistent with general expectations, though considerably particular, we have derived the full set of scheme invariants be analyzed within a compact and efficient framework. In

\[ T_{1bc} = \frac{1}{120} - \frac{3}{8} T(4) + A_4(6), \]
\[ T_{1ac} = \frac{39}{200} - \frac{3}{8} T(4) + \frac{2}{3} A_4(6), \]
\[ T_{1ab} = \frac{371}{240} - \frac{3}{8} T(4) + \frac{1}{2} A_4(6), \]
\[ T_{1cd} = \frac{607}{120} - \frac{5}{12} T(4) + \frac{1}{3} A_4(6), \]
\[ T_{1be} = \frac{1}{5} - \frac{3}{8} T(4) + \frac{2}{3} A_4(6), \]
\[ T_{2ab} = -72 \xi_3, \]
\[ T_{3bc} = \frac{284}{5} + 48 \xi_3 + 3 T(4) + 24 A_4(6) - 24 A_{10}^{(7)} + 16 A_{11}^{(7)}, \]
\[ T_{3ab} = \frac{122}{5} + 30 \xi_3 + 6 T(4) - 8 A_4(6) + 4 A_{11}^{(7)}, \]
\[ T_{3ac} = T_{3ab} - 24 \xi_3, \]
\[ T_{3ad} = \frac{4}{5} - 18 \xi_3 + 7 T(4) - 36 A_4(6) + 24 A_{10}^{(7)} - 12 A_{11}^{(7)}, \]
\[ T_{4ab} = \frac{27}{20} - \frac{9}{4} \xi_3 - \frac{1}{2} T(4) + 3 A_4(6) + \frac{3}{2} A_4^{(7)}, \]
\[ T_{4ac} = -T_{4ab} + 6 A_{10}^{(7)}, \]
\[ T_{5cd} = -\frac{39}{5} + 12 \xi_3 - T(4) + 14 A_4(6) - 12 A_{10}^{(7)} + 8 A_{11}^{(7)}, \]
\[ T_{5bc} = \frac{39}{5} - 12 \xi_3 + \frac{3}{2} T(4) - 14 A_4^{(6)} + 12 A_{10}^{(7)} - 4 A_{11}^{(7)}, \]
\[ T_{5ac} = -3 + 3 \xi_3 + \frac{1}{2} T(4) + 2 A_4^{(7)}, \]
\[ T_{6be} = \frac{21}{5} - 9 \xi_3 + \frac{5}{2} T(4) - 16 A_4^{(6)} + 12 A_{10}^{(7)} - 6 A_{11}^{(7)}, \]
\[ T_{5ha} = \frac{6}{5} - \frac{3}{2} \xi_3 + \frac{1}{2} T(4) - A_4^{(6)} + A_4^{(7)}. \]

We see that the effect of imposing symmetry has been to reduce the freedom in the $a$-function coefficients from the original six parameters to two.

**VII. CONCLUSIONS**

We have shown how scheme changes in $\phi^4$ theory may be analyzed within a compact and efficient framework. In particular, we have derived the full set of scheme invariants up to four loop order and shown that their number is consistent with general expectations, though considerably higher than might be expected from a naive counting. In particular, we have identified the existence of quadratic invariants, which would be missed in a naive counting. Furthermore, we have shown that in the context of the Hopf algebra approach to renormalization, each invariant is associated with a cocommutative combination of graphs. We have also considered the construction of the $a$ function generating the $\beta$ functions up to four-loop order via a gradient flow equation. In particular, we have analyzed the consistency conditions, which guarantee this construction, again showing that their number is as expected, and furthermore, as expected, they may be expressed in terms of linear combinations of the scheme invariants. Finally, we have considered one-vertex reducible diagrams and shown that there is a natural family of schemes in which these do not contribute to the $\beta$ function.

Future work might explore the Hopf algebra connection further. Furthermore, at higher orders than we have yet considered, there might be the possibility of cubic and higher order invariants. The extension of the analysis presented here to gauge theories might present additional challenges.

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**APPENDIX A: GENERAL RESULTS**

For a theory with couplings $g^I$, the corresponding $\beta$ functions are defined by

\[ \beta^I(g) = \mu \frac{d}{d\mu} g^I, \]

and the $\beta$ functions in a new renormalization group scheme defined by $g'^I(g)$ are given by

\[ \beta'^I(g') = \beta(g)g'^I, \]

where for any vector $V$ in coupling space,

\[ V_g = V^j \frac{\partial}{\partial g^j}. \]

We choose to parametrize the redefined coupling as

\[ g' = e^{\nu g} g. \]

We then find using the easily proved result

\[ f(e^{\nu h} g) = e^{\nu s} f(h) \]

that

\[ \beta'(g) = e^{-\nu s} \beta(g) e^{\nu s} g. \]

Then using

\[ [v_g, V_g] = (L_v V)_g, \quad L_v V = v_g V - V_g v, \]

and

\[ \beta'(g) = e^{-\nu s} \beta(g) e^{\nu s} g. \]
together with

$$e^A Be^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \cdots$$  \hspace{1cm} (A8)

we find

$$\beta'(g) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{L}_n^* \beta(g).$$  \hspace{1cm} (A9)

For our purposes, it is useful to use this result in the form

$$\delta \beta(g) = \beta'(g) - \beta(g) = - \mathcal{L}_n^* \beta,$$  \hspace{1cm} (A10)

where

$$\hat{\beta} = \beta - \frac{1}{2!} \mathcal{L}_1^* \beta + \frac{1}{3!} \mathcal{L}_2^* \beta + \cdots$$

$$= \beta - \frac{1}{2} \mathcal{L}_1^* \left( \beta - \frac{1}{3} \mathcal{L}_2^* (\beta - \cdots) \right)$$  \hspace{1cm} (A11)

### APPENDIX B: SYMMETRIC HOPF COPRODUCT

In this Appendix, we give the full results for the cocommutative expressions on the right-hand sides of Eqs. (54), (55), (57), and (58). For the combinations corresponding to four-loop linear invariants in Eq. (54), we have

\[
\begin{align*}
C^{(4)\xi}_1 &= 0, \\
C^{(4)\xi}_2 &= 2g_1^1 \otimes s_1 g_3^a + 2g_1^1 \otimes g_3^3 + 4g_3^2 \otimes g_3^2 + g_3^{2R} \otimes g_3^{2R}, \\
C^{(4)\xi}_3 &= 2g_1^1 \otimes g_3^a g_3^3 + 2g_2^1 \otimes g_3^a g_3^3 + g_3^{2R} \otimes g_3^2 + 2g_2^2 \otimes g_3^2, \\
C^{(4)\xi}_4 &= 2g_2^1 \otimes g_3^a g_3^3 + 4g_2^2 \otimes g_3^3 - 2g_3^2 \otimes g_3^2 - 2g_1^1 \otimes g_3^a g_3^3 - (g_3^1)^2 \otimes g_3^2, \\
C^{(4)\xi}_5 &= 2g_1^1 \otimes g_3^3b + g_1^1 \otimes g_3^3 + 2g_2^2 \otimes g_3^2, \\
C^{(4)\xi}_6 &= g_1^1 \otimes g_3^a g_3^3 - g_1^1 \otimes g_3^{3R} + g_2^2 \otimes g_3^2, \\
C^{(4)\xi}_7 &= 2g_2^1 \otimes g_3^{3c} - 2g_3^2 \otimes g_3^3 + g_1^1 \otimes g_3^3 + g_3^{3R} + g_3^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^a g_3^3 + g_2^2 \otimes g_3^2, \\
C^{(4)\xi}_8 &= g_2^2 \otimes g_3^2 - 4g_2^2 \otimes g_3^2 - 2g_1^1 \otimes g_3^a g_3^3 + g_1^1 \otimes g_3^3 + g_2^2 \otimes g_3^2, \\
C^{(4)\xi}_9 &= 2g_2^2 \otimes g_3^2 + g_2^1 \otimes g_3^3 + g_1^1 \otimes g_3^3 + g_2^1 \otimes g_3^2, \\
C^{(4)\xi}_{10} &= g_1^1 \otimes g_3^3f, \\
C^{(4)\xi}_{11} &= 2g_1^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^3 + g_1^1 \otimes g_3^3 + 2g_1^1 \otimes g_3^2, \\
C^{(4)\xi}_{12} &= 2g_1^1 \otimes g_3^3 + 2g_1^1 \otimes g_3^3 + 2g_1^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^2 + 2g_1^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^2, \\
C^{(4)\xi}_{13} &= 2g_2^2 \otimes g_3^2 - 2g_2^2 \otimes g_3^2 + 2g_2^2 \otimes g_3^2 + (g_2^1)^2 \otimes g_3^2, \\
C^{(4)\xi}_{14} &= 4g_2^2 \otimes g_3^2 - 2g_2^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^3 - g_1^1 \otimes g_3^3 + g_1^1 \otimes g_3^2 - 2g_1^1 \otimes g_3^2 + g_3^2. 
\end{align*}
\]  \hspace{1cm} (B1)

For the combinations corresponding to four-loop quadratic invariants in Eq. (55), we have

\[
\begin{align*}
C^{(4)\xi}_1 &= 2g_1^1 \otimes g_3^4 - g_1^1 \otimes g_3^4 + g_3^2 \otimes g_3^3 + 2g_2^2 \otimes g_3^3 + 2g_1^1 \otimes g_3^3 + g_2^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^2, \\
C^{(4)\xi}_2 &= g_3^1 \otimes g_3^4 + g_3^2 \otimes g_3^2 + g_3^3 \otimes g_3^3 - g_3^2 \otimes g_3^2 + 2g_1^1 \otimes g_3^3 + g_3^2 \otimes g_3^2, \\
C^{(4)\xi}_3 &= 2g_1^1 \otimes g_3^4 - g_2^2 \otimes g_3^3 + g_3^2 \otimes g_3^2 + (g_1^1)^2 \otimes g_3^2, \\
C^{(4)\xi}_4 &= g_1^1 \otimes g_3^4 + (g_1^1)^2 \otimes g_3^2, \\
C^{(4)\xi}_5 &= 2g_1^1 \otimes g_3^3 - 2g_1^1 \otimes g_3^3 + 2g_1^1 \otimes g_3^2 - 2g_1^1 \otimes g_3^2 + 5(g_1^1)^2 \otimes g_3^2. 
\end{align*}
\]  \hspace{1cm} (B2)

For the combination corresponding to the five-loop invariant in Eq. (57), we have

$$
\begin{align*}
C^{(5)\xi}_1 &= -2g_1^2 \otimes g_3^3 + 4g_1^2 \otimes g_3^3 + g_3^3 - 2g_1^1 \otimes g_3^3 + g_3^2 \otimes g_3^2 - 4g_1^1 \otimes g_3^2 + 2g_1^1 \otimes g_3^2, \\
&\quad + (g_1^1)^2 \otimes g_3^2 + g_1^1 \otimes g_3^3. 
\end{align*}
\]  \hspace{1cm} (B3)
Finally, for the combinations corresponding to five-loop quadratic invariants in Eq. (58), we have

\[ C^{(5)}_1 = g_1 \otimes g_4 \delta^a + g_4 \otimes g_4 \delta^b + g_3 \otimes g_1 \delta^f + g_1 \otimes g_1 \delta^f, \]

\[ C^{(5)}_2 = g_1 \otimes (g_5 \delta^a - g_5 \delta^b) + g_4 \otimes g_4 \delta^c + g_1 \otimes g_1 \delta^c - (g_1 \otimes 2g_4 \delta^b) \otimes g_1 \delta^f + 2g_4 \otimes g_1 \delta^f, \]

\[ C^{(5)}_3 = g_1 \otimes (g_5 \delta^a - g_5 \delta^b) + g_4 \otimes g_4 \delta^c + g_1 \otimes g_1 \delta^c - (g_1 \otimes 2g_4 \delta^b) \otimes g_1 \delta^f + 2g_4 \otimes g_1 \delta^f, \]

\[ C^{(5)}_4 = g_1 \otimes (g_7 \delta^d - g_7 \delta^e) - 2g_7 \otimes G_7 - g_7 \otimes g_6 \otimes g_4 \delta^e - g_4 \otimes g_1 \delta^e + 2g_7 \otimes g_1 \delta^e, \]

\[ C^{(5)}_5 = g_1 \otimes (g_7 \delta^c + g_7 \delta^e - 2g_7 \delta^b) - g_7 \otimes g_4 \delta^c + g_4 \otimes g_1 \delta^c - (g_4 \otimes 2g_7 \delta^b) \otimes g_1 \delta^e + 2g_7 \otimes g_1 \delta^e, \]

\[ C^{(5)}_6 = g_1 \otimes (g_7 \delta^c + g_7 \delta^e - 2g_7 \delta^b) - g_7 \otimes g_4 \delta^c + g_4 \otimes g_1 \delta^c - (g_4 \otimes 2g_7 \delta^b) \otimes g_1 \delta^e + 2g_7 \otimes g_1 \delta^e, \]

\[ C^{(5)}_7 = g_1 \otimes (g_7 \delta^c + g_7 \delta^e - 2g_7 \delta^b) - g_7 \otimes g_4 \delta^c + g_4 \otimes g_1 \delta^c - (g_4 \otimes 2g_7 \delta^b) \otimes g_1 \delta^e + 2g_7 \otimes g_1 \delta^e, \]

\[ C^{(5)}_8 = g_1 \otimes (g_7 \delta^c + g_7 \delta^e - 2g_7 \delta^b) - g_7 \otimes g_4 \delta^c + g_4 \otimes g_1 \delta^c - (g_4 \otimes 2g_7 \delta^b) \otimes g_1 \delta^e + 2g_7 \otimes g_1 \delta^e, \]

\[ \text{Eq. (58)} \]
APPENDIX C: DIFFERENTIAL OPERATORS FOR SCHEME CHANGES

Following the general considerations of Ref. [7], we may define differential operators

\[ Y = \sum_{l,s} (\delta_{l,s} Y^l_{sfs} + \epsilon_{l,s} Y^l_{rsf}), \tag{C1} \]

where

\[ Y^l_{sfs} = \sum_{l,r} (c_{l,r} D^{l,r}_{sfs} + d_{l,r} D^{l,r}_{sfs}), \]
\[ Y^l_{rsf} = \sum_{l,r} (c_{l,r} D^{l,r}_{rsf} + d_{l,r} D^{l,r}_{rsf}), \tag{C2} \]

which generate scheme changes according to

\[ \{c_{l,r}, d_{l,r}\} \rightarrow \exp(Y)\{c_{l,r}, d_{l,r}\}. \tag{C3} \]

Here, \{r, s\} label the \( \beta \) or \( \gamma \) function coefficients at each loop order \{l, l\}. The operators \( D^{l,r}_{sfs} \), etc., satisfy

\[ D^{l,r}_{sfs} = -D^{l,s}_{rfs}, \quad D^{l,r}_{rsf} = -D^{r,s}_{lfs}, \quad D^{l,r}_{sfs} = -D^{r,s}_{lfs}. \tag{C4} \]

Scheme invariants are then determined as polynomial functions \( F(\{c_{l,r}, d_{l,r}\}) \), such that

\[ Y^l_{sfs} F = Y^l_{rsf} F = 0 \tag{C5} \]

for all \( l, r \). In the case of \( \phi^4 \) theory, we find at the lowest order

\[ D^{l,2} = -2 \frac{\partial}{\partial c_3} + 2 \frac{\partial}{\partial c_3 c} + 2 \frac{\partial}{\partial c_3 d}, \]
\[ D^{l,2R} = 2 \frac{\partial}{\partial c_3 a} - 2 \frac{\partial}{\partial c_3 c} + 2 \frac{\partial}{\partial c_3 R} + 2 \frac{\partial}{\partial c_3 b}, \]
\[ D^{l,2} = -2 \frac{\partial}{\partial c_3 b} + 6 \frac{\partial}{\partial d_3}, \tag{C6} \]

and at the next-to-leading order

\[ D^{l,3a} = 4 \frac{\partial}{\partial c_4 a} + 2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 d} - 2 \frac{\partial}{\partial c_4 f}, \]
\[ D^{l,3b} = 6 \frac{\partial}{\partial c_4 a} - 2 \frac{\partial}{\partial c_4 c} + \frac{\partial}{\partial c_4 j}, \]
\[ D^{l,3c} = -2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 f} + 2 \frac{\partial}{\partial c_4 g} + 2 \frac{\partial}{\partial c_4 h} + \frac{\partial}{\partial c_4 i}, \]
\[ D^{l,3d} = -2 \frac{\partial}{\partial c_4 d} + 2 \frac{\partial}{\partial c_4 h} + 2 \frac{\partial}{\partial c_4 i} + 2 \frac{\partial}{\partial c_4 k} + 2 \frac{\partial}{\partial c_4 l}, \]
\[ D^{l,3e} = -4 \frac{\partial}{\partial c_4 a} + 2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 c} + 2 \frac{\partial}{\partial c_4 d} + 2 \frac{\partial}{\partial c_4 e} + 2 \frac{\partial}{\partial c_4 f} \]
\[ D^{l,3f} = \frac{\partial}{\partial c_4 a} - \frac{\partial}{\partial c_4 b}, \]
\[ D^{l,3aR} = -2 \frac{\partial}{\partial c_4 a} + 2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 c} + 2 \frac{\partial}{\partial c_4 d} + \frac{\partial}{\partial c_4 f}, \]
\[ D^{l,3bR} = 2 \frac{\partial}{\partial c_4 a} - 2 \frac{\partial}{\partial c_4 b} + 4 \frac{\partial}{\partial c_4 c} + \frac{\partial}{\partial c_4 d} + 2 \frac{\partial}{\partial c_4 d}, \]
\[ D^{l,3c} = -2 \frac{\partial}{\partial c_4 b} + 3 \frac{\partial}{\partial c_4 d} + 4 \frac{\partial}{\partial c_4 d} + 4 \frac{\partial}{\partial c_4 d}, \]
\[ D^{l,3d} = 4 \frac{\partial}{\partial c_4 a} - 2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 d} + 2 \frac{\partial}{\partial c_4 e} + 2 \frac{\partial}{\partial c_4 f} + 2 \frac{\partial}{\partial c_4 g} + 2 \frac{\partial}{\partial c_4 h} + 2 \frac{\partial}{\partial c_4 i} + 2 \frac{\partial}{\partial c_4 j}, \]
\[ D^{l,3e} = -2 \frac{\partial}{\partial c_4 a} + 2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 c} + 2 \frac{\partial}{\partial c_4 d} + 2 \frac{\partial}{\partial c_4 e} + 2 \frac{\partial}{\partial c_4 f} \]
\[ D^{l,3f} = -2 \frac{\partial}{\partial c_4 a} + 2 \frac{\partial}{\partial c_4 b} + 2 \frac{\partial}{\partial c_4 c} + 2 \frac{\partial}{\partial c_4 d} + 2 \frac{\partial}{\partial c_4 e} + 2 \frac{\partial}{\partial c_4 f} \]

Note that, here we suppress the label \( r \) in the case of the one-loop \( \beta \) function and the two-loop \( \gamma \) function, where there is only one coefficient.

The \( Y^l_{sfs} \) and \( Y^l_{rsf} \), defined according to Eq. (C2), satisfy the commutation relations

\[ [Y^{l1}, Y^{l2}] = -2Y^{l3a} + 2Y^{l3c} + 2Y^{l3e}, \]
\[ [Y^{l1}, Y^{l2R}] = 2Y^{l3a} - 2Y^{l3c} + Y^{l3aR} + 2Y^{l3bR}, \]
\[ [Y^{l1}, Y^{l2}] = -2Y^{l3b} + 6Y^{l3}, \tag{C8} \]

and

\[ [Y^{l1}, Y^{l3a}] = 4Y^{l4a} + 2Y^{l4b} + 2Y^{l4d} - 2Y^{l4f}, \]
\[ [Y^{l1}, Y^{l3b}] = 6Y^{l4a} - 2Y^{l4b} + 2Y^{l4f}, \]
\[ [Y^{l1}, Y^{l3c}] = -2Y^{l4b} + 2Y^{l4f} + 3Y^{l4g} + 2Y^{l4k} + Y^{l4o}, \]
\[ [Y^{l1}, Y^{l3d}] = -2Y^{l4d} - 2Y^{l4h} - 2Y^{l4i} + 2Y^{l4a} + 2Y^{l4k}, \]
\[ [Y^{l1}, Y^{l3e}] = -4Y^{l4a} + 2Y^{l4b} + 2Y^{l4i} + 2Y^{l4k} + 2Y^{l4l} + Y^{l4r}, \]
\[ [Y^{l1}, Y^{l3f}] = Y^{l4b} - Y^{l4q}, \]
\[ [Y^{l1}, Y^{l3aR}] = -2Y^{l4b} + 2Y^{l4aR} + 2Y^{l4cR} + 2Y^{l4fR}, \]
\[ [Y^{l1}, Y^{l3bR}] = 2Y^{l4dR} - 4Y^{l4b} + 4Y^{l4h} + Y^{l4eR} + 2Y^{l4dR} + 2Y^{l4fR} - Y^{l4fR}, \]
\[ [Y^{l2}, Y^{l3}] = -4Y^{l4a} + 3Y^{l4b} + 2Y^{l4c} + 4Y^{l4d}, \]
\[ [Y^{l2}, Y^{l3R}] = 4Y^{l4b} - 2Y^{l4g} - 2Y^{l4l} - Y^{l4o} + 2Y^{l4gR} + 2Y^{l4fR}, \]
\[ [Y^{l2}, Y^{l3}] = -2Y^{l4b} - 6Y^{l4c} + 6Y^{l4d}. \tag{C9} \]
Note that, the structure constants appearing in Eqs. (C8) and (C9) are the same as those in Eqs. (C6) and (C7), which is a consequence of the Jacobi identities following from the associativity of the graph insertion process as described in Ref. [7]. At the following order, we have

\[
\mathcal{D}^{3a} \gamma^2 = 3 \frac{\partial}{\partial d_{5i}}, \\
\mathcal{D}^{3b} \gamma^2 = 3 \frac{\partial}{\partial d_{5k}}, \\
\mathcal{D}^{3c} \gamma^2 = 6 \frac{\partial}{\partial d_{5e}} + 3 \frac{\partial}{\partial d_{5j}}, \\
\mathcal{D}^{3d} \gamma^2 = 6 \frac{\partial}{\partial d_{5h}}, \\
\mathcal{D}^{3e} \gamma^2 = 12 \frac{\partial}{\partial d_{5b}} + 6 \frac{\partial}{\partial d_{5e}} + 3 \frac{\partial}{\partial d_{5h}}, \\
\mathcal{D}^{3f} \gamma^2 = 2 \frac{\partial}{\partial d_{5a}}, \\
\mathcal{D}^{3aR} \gamma^2 = 6 \frac{\partial}{\partial d_{5f}} - 2 \frac{\partial}{\partial c_{5aR}}, \\
\mathcal{D}^{3bR} \gamma^2 = 3 \frac{\partial}{\partial d_{5j}} + 3 \frac{\partial}{\partial d_{5j}} - 2 \frac{\partial}{\partial c_{5bR}}, \\
\mathcal{D}^{3cR} \gamma^2 = 6 \frac{\partial}{\partial d_{5d}} + 2 \frac{\partial}{\partial d_{5e}} + 2 \frac{\partial}{\partial d_{5k}}, \\
\mathcal{D}^{3dR} \gamma^2 = 4 \frac{\partial}{\partial d_{5j}} + 2 \frac{\partial}{\partial d_{5e}} + 2 \frac{\partial}{\partial d_{5k}} + \frac{\partial}{\partial d_{5j}}, \\
\mathcal{D}^{3eR} \gamma^2 = 4 \frac{\partial}{\partial d_{5f}} + 2 \frac{\partial}{\partial d_{5e}} + 2 \frac{\partial}{\partial d_{5k}} + \frac{\partial}{\partial d_{5f}}, \\
\mathcal{D}^{3fR} \gamma^2 = \frac{\partial}{\partial c_{5aR}} + 6 \frac{\partial}{\partial c_{5bR}} + 2 \frac{\partial}{\partial c_{5cR}},
\]

with, correspondingly, the commutation relations

\[
[Y^{3a}, Y^{r2}] = 3Y^{5i}, \\
[Y^{3b}, Y^{r2}] = 3Y^{5k}, \\
[Y^{3c}, Y^{r2}] = 6Y^{5e} + 3Y^{5j}, \\
[Y^{3d}, Y^{r2}] = 6Y^{5h}, \\
[Y^{3e}, Y^{r2}] = 12Y^{5b} + 6Y^{5d} + 3Y^{5h}, \\
[Y^{3f}, Y^{r2}] = 2Y^{5a}, \\
[Y^{3aR}, Y^{r2}] = 6Y^{5f} - 2Y^{5aR}, \\
[Y^{3bR}, Y^{r2}] = 3Y^{5i} + 3Y^{5j} - 2Y^{5bR}, \\
[Y^{3cR}, Y^{r2}] = 6Y^{5d} + 2Y^{5g} + 2Y^{5k}, \\
[Y^{3dR}, Y^{r2}] = 4Y^{5f} + 2Y^{5i} + 2Y^{5j}, \\
[Y^{3eR}, Y^{r2}] = 4Y^{5b} + 2Y^{5e} + 2Y^{5h} + 2Y^{5i}, \\
[Y^{3fR}, Y^{r2}] = Y^{5c} + 2Y^{5h} + 2Y^{5j}, \\
[Y^{3a}, Y^{r2}] = Y^{5c} + 2Y^{5f} + 2Y^{5j}, \\
[Y^{3b}, Y^{r2}] = Y^{5c} + 2Y^{5f} + 2Y^{5j}, \\
[Y^{3c}, Y^{r2}] = Y^{5c} + 2Y^{5f} + 2Y^{5j}, \\
[Y^{3d}, Y^{r2}] = Y^{5c} + 2Y^{5f} + 2Y^{5j} - 2Y^{5cR}.
\]

It is readily verified using Eqs. (C2), (C6), (C7), and (C10) that the linear and quadratic invariants constructed in previous sections satisfy Eq. (C5).

[1] I. Jack, D. R. T. Jones, and C. Poole, Gradient flows in three dimensions, J. High Energy Phys. 09 (2015) 061.
[2] J. A. Gracey, I. Jack, C. Poole, and Y. Schröder, a-Function for \( N = 2 \) supersymmetric gauge theories in three dimensions, Phys. Rev. D 95, 025005 (2017).
[3] I. Jack and C. Poole, The a-function in three dimensions: Beyond the leading order, Phys. Rev. D 95, 025010 (2017).
[4] I. Jack and C. Poole, The a-function for gauge theories, J. High Energy Phys. 01 (2015) 138.
[5] J. A. Gracey, I. Jack, and C. Poole, The a-function in six dimensions, J. High Energy Phys. 01 (2016) 174.
[6] I. Jack and H. Osborn, Scheme dependence and multiple couplings, arXiv:1606.02571.
[7] I. Jack and H. Osborn, RG scheme invariants for multiple couplings (to be published).
[8] D. I. Kazakov, O. V. Tarasov, and A. A. Vladimirov, Calculation of critical exponents by quantum field theory methods, Sov. Phys. JETP 50, 521 (1979).
[9] H. Kleinert and V. Schulte-Frohlinde, Critical Properties of \( \phi^4 \)-Theories (World Scientific, Singapore, 2001).
[10] M. V. Kompaniets and E. Panzer, Minimally subtracted six loop renormalization of \( O(n) \)-symmetric \( \phi^4 \) theory and critical exponents, Phys. Rev. D 96, 036016 (2017).
[11] E. Panzer, Feynman integrals and hyperlogarithms, arXiv:1506.07243.
[12] J. L. Cardy, Is there a $c$-theorem in four dimensions?, Phys. Lett. B 215, 749 (1988).
[13] Z. Komargodski and A. Schwimmer, On renormalization group flows in four dimensions, J. High Energy Phys. 12 (2011) 099.
[14] Z. Komargodski, The constraints of conformal symmetry on RG flows, J. High Energy Phys. 07 (2012) 069.
[15] M. A. Luty, J. Polchinski, and R. Rattazzi, The $\alpha$-theorem and the asymptotics of 4D quantum field theory, J. High Energy Phys. 01 (2013) 152.
[16] A. B. Zamolodchikov, Irreversibility of the flux of the renormalization Group in a 2D Field Theory, JETP Lett. 43, 730 (1986).
[17] H. Elvang, D. Z. Freedman, L. Y. Hung, M. Kiermaier, R. C. Myers, and S. Theisen, On renormalization group flows and the $\alpha$-theorem in 6d, J. High Energy Phys. 10 (2012) 011.
[18] B. Grinstein, A. Stergiou, and D. Stone, Consequences of Weyl consistency conditions, J. High Energy Phys. 11 (2013) 195.
[19] B. Grinstein, A. Stergiou, D. Stone, and M. Zhong, A Challenge to the $\alpha$-Theorem in Six Dimensions, Phys. Rev. Lett. 113, 231602 (2014).
[20] B. Grinstein, A. Stergiou, D. Stone, and M. Zhong, Two-loop renormalization of multiflavor $\phi^4$ theory in six dimensions and the trace anomaly, Phys. Rev. D 92, 045013 (2015).
[21] H. Osborn and A. Stergiou, Structures on the conformal manifold in six-dimensional theories, J. High Energy Phys. 04 (2015) 157.
[22] H. Osborn, Derivation of a four-dimensional $c$-theorem, Phys. Lett. B 222, 97 (1989).
[23] I. Jack and H. Osborn, Analogs for the $c$-theorem for four-dimensional renormalizable field theories, Nucl. Phys. B343, 647 (1990).
[24] I. Jack and H. Osborn, Constraints on RG flow for four-dimensional quantum field theories, Nucl. Phys. B883, 425 (2014).
[25] H. Osborn, Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories, Nucl. Phys. B363, 486 (1991).
[26] J. F. Fortin, B. Grinstein, and A. Stergiou, Scale without conformal invariance at three loops, J. High Energy Phys. 08 (2012) 085.
[27] I. R. Klebanov and G. Tarnopolsky, On large $N$ limit of symmetric traceless tensor models, J. High Energy Phys. 10 (2017) 037.