Factorial Schur Functions and the Yang-Baxter Equation

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April 28, 2013

Abstract. Factorial Schur functions are generalizations of Schur functions that have, in addition to the usual variables, a second family of “shift” parameters. We show that a factorial Schur function times a deformation of the Weyl denominator may be expressed as the partition function of a particular statistical-mechanical system (six vertex model). The proof is based on the Yang-Baxter equation. There is a deformation parameter $t$ which may be specialized in different ways. If $t = -1$, then we recover the expression of the factorial Schur function as a ratio of alternating polynomials. If $t = 0$, we recover the description as a sum over tableaux. If $t = \infty$ we recover a description of Lascoux that was previously considered by McNamara. We also are able to prove using the Yang-Baxter equation the asymptotic symmetry of the factorial Schur functions in the shift parameters. Finally, we give a proof using our methods of the dual Cauchy identity for factorial Schur functions. Thus using our methods we are able to give thematic proofs of many of the properties of factorial Schur functions.

1 Introduction

Factorial Schur functions are generalizations of ordinary Schur functions $s_\lambda(z) = s_\lambda(z_1,\cdots,z_n)$ for which a surprising amount of the classical theory remains valid. In addition to the spectral parameters $z = (z_1,\cdots,z_n)$ and the partition $\lambda$ they involve a set $\alpha = (\alpha_1,\alpha_2,\alpha_3,\cdots)$ of shifts that can be arbitrary complex numbers, and are denoted $s_\lambda(z|\alpha)$. In the original discovery of Biedenharn and Louck [2], only the special case where $\alpha_n = 1-n$ was considered. Their original motivation, inspired by questions from mathematical physics, was to the decomposition of tensor products of representations with using particular bases.

Other early foundational papers are Chen and Louck [3], who gave new foundations based on divided difference operators, and Goulden and Hamel [7] where the
analogy between Schur functions and factorial Schur functions was further developed.
In particular they gave a Jacobi-Trudi identity. See Louck [15] for further historical remarks.
Biedenharn and Louck (in the special case $\alpha_n = 1 - n$) defined $s_\lambda(z|\alpha)$ to be a sum over Gelfand-Tsetlin patterns, and this definition extends to the general case. Translated into the equivalent language of tableaux, their definition is equivalent to (7) below. It was noticed independently by Macdonald [16] and by Goulden and Greene [6] that one could generalize the factorial Schur functions of Biedenharn and Louck by making use of an arbitrary set $\alpha$ of shifts. Macdonald observed an alternative definition of the factorial Schur functions as a ratio of two alternating polynomials, generalizing the Weyl character formula. This definition is (2) below.

Both Macdonald and Goulden and Green also noticed a relationship with what are called supersymmetric Schur functions. These are Schur functions in two sets of variables, $z = (z_1, z_2, \cdots)$ and $w = (w_1, w_2, \cdots)$. They are called Hook Schur functions by Berele and Regev [1] and Schur-Littlewood functions by Bump and Gamburd [4]. They are defined in terms of the ordinary Schur functions by

$$s_\lambda(z\|w) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(z)s_\nu'(w),$$

where $c^\lambda_{\mu\nu}$ is the Littlewood-Richardson coefficient, $\mu$ and $\nu$ run through partitions and $\nu'$ is the conjugate partition. The relationship between the factorial Schur functions and the supersymmetric Schur functions is this: although the $s_\lambda(z|\alpha)$ are symmetric in the $z_i$ they are not symmetric in the $\alpha_i$. Nevertheless, as the number $n$ of the spectral parameters $z_i$ tends to infinity, they become symmetric in the $\alpha_i$ in a certain precise sense, and in the limit, they stabilize. Thus in a suitable sense

$$\lim_{n \to \infty} s_\lambda(z|\alpha) = s_\lambda(z\|\alpha).$$

Another important variant of the factorial Schur functions are the shifted Schur functions that were proposed by Olshanskii, and developed by Okounkov and Olshanskii [23], [22]. Denoted $s^*_\lambda(x_1, \cdots, x_n)$, they are essentially the same as the factorial Schur functions of Biedenharn and Louck, but incorporate shifts in the parameters so that they are no longer symmetric in the usual sense, but at least satisfy the stability property $s^*_\lambda(x_1, \cdots, x_n) = s^*_\lambda(x_1, \cdots, x_n, 0)$. These were applied to the representation theory of the infinite symmetric group.

Molev and Sagan [20] give various useful results for factorial Schur functions, including a Littlewood-Richardson rule. A further Littlewood-Richardson rule was found by Kreiman [11]. Knutson and Tao [10] show that factorial Schur functions
correspond to Schubert classes in the equivariant cohomology of Grassmanians. See also Mihalcea [19] and Ikeda and Naruse [9].

Tokuyama [24] gave a formula for Schur functions that depends on a parameter $t$. This formula may be regarded as a deformation of the Weyl character formula. It was shown by Hamel and King [8] that Tokuyama’s formula could be generalized and reformulated as the evaluation of the partition function for a statistical system based on the six-vertex model in the free-fermionic regime. Brubaker, Bump and Friedberg [3] gave further generalizations of the results of Hamel and King, with new proofs based on the Yang-Baxter equation. More specifically, they showed that the six-vertex model in the free-fermionic regime satisfies a parametrized Yang-Baxter equation with nonabelian parameter group $\text{GL}(2) \times \text{GL}(1)$. Using this, they showed how to give statistical-mechanical systems whose partition functions were Schur functions times a deformation of the Weyl denominator. This result generalizes the results of Tokuyama and of Hamel and King.

Lascoux [12], [13] found six-vertex model representations of Grassmannian Schubert polynomials. A proof of this representation based on the Yang-Baxter equation was subsequently found by McNamara [18].

We will give a simultaneous generalization of [3] and of the representations of Lascoux and of McNamara. As in [3] we will consider partition functions of statistical-mechanical systems in the free-fermionic regime. A significant difference between this paper and that was that in [3] the Boltzmann weights were constant along the rows, depending mainly on the choice of a spectral parameter $z_i$. Now we will consider systems in which we assign a spectral parameter $z_i$ to each row, but also a shift parameter $\alpha_j$ to each column. Furthermore, we will make use of a deformation parameter $t$ that applies to the entire system.

Using the Yang-Baxter equation, we will show that the partition function may be expressed as the product of a factor (depending on $t$) that may be recognized as a deformation of the Weyl denominator, times the factorial Schur function $s_\lambda(z|\alpha)$. This is thus a Tokuyama-type formula for the factorial Schur functions. By specializing $t$ in various ways, we will obtain different formulas for the factorial Schur functions. Taking $t = 0$, we obtain the tableau definition. Taking $t = -1$, we obtain the representation as a ratio of alternating polynomials, which was Macdonald’s generalization of the Weyl character formula. Next, when $t = \infty$ we recover the representation of Lascoux and McNamara. Although we do not prove the supersymmetric limit [1], we will at least prove the key fact that the $s_\lambda(z|\alpha)$ are asymptotically symmetric in the $\alpha_j$ as the number $n$ of spectral parameters $z_i$ tends to infinity. We will obtain this by another application of the Yang-Baxter equation. We also give a proof of the dual Cauchy identity for factorial Schur functions using our methods.
This work was supported in part by JSPS Research Fellowship for Young Scientists and by NSF grant DMS-0652817 and DMS-1001079.

2 Factorial Schur functions

A good primary reference for factorial Schur functions is Macdonald [16]. They are also in Macdonald [17], Ex. 20 in Section I.3 on p.54.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of complex numbers. If $z \in \mathbb{C}$ let

$$(z|\alpha)^r = (z + \alpha_1) \cdots (z + \alpha_r).$$

Macdonald [16] gives two formulas for factorial Schur functions that we will also prove to be equivalent by our methods. Let $\mu = (\mu_1, \cdots, \mu_n)$ where the $\mu_i$ are nonnegative integers. Let $z_1, \cdots, z_n$ be nonzero complex numbers which we will call spectral parameters. Define

$A_\mu(z|\alpha) = \det((z_i|\alpha)^{\mu_j})_{i,j}$

where $1 \leq i, j \leq n$ in the determinant. We will also use the notation

$z^\mu = \prod_i z_i^\mu_i.$

Let $\delta = (n-1, n-2, \cdots, 0)$ and let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition of length at most $n$. Define

$s_\lambda(z|\alpha) = \frac{A_{\lambda+\delta}(z|\alpha)}{A_\delta(z|\alpha)}.$

The denominator here is actually independent of $\alpha$ and is given by the Weyl denominator formula:

$A_\delta(z|\alpha) = \prod_{i < j} (z_i - z_j)$

Indeed, it is an alternating polynomial in the $z_i$ of the same degree as the right-hand side, and so the ratio is a polynomial in $\alpha$ that is independent of $z_i$. To see that the ratio is independent of $\alpha$, one may compare the coefficients of $z^\delta$ on both sides of (3).

**Theorem 1** The ratio $s_\lambda(z|\alpha)$ is a symmetric polynomial in $z_1, \cdots, z_n$. 

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We will prove this at the end of Section 3.

Macdonald also gives a combinatorial formula for the factorial Schur function as a sum over semi-standard Young tableaux of shape $\lambda$ in $\{1, 2, \cdots, n\}$. Let $T$ be such a tableau. If $i, j$ are given such that $j \leq \lambda_i$, let $T(i, j)$ be the entry in the $i$-th row and $j$-th column of $T$. Let

$$T^*(i, j) = T(i, j) + j - i,$$

and define

$$(z|\alpha)^T = \prod_{(i, j)} (z_{T(i,j)} + \alpha_{T^*(i,j)}).$$

For example, if $\lambda = (4, 2, 0)$ and $n = 3$, then we might have

$$T = \begin{array}{ccc}
1 & 1 & 1
\end{array}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
2
\end{array},
T^* = \begin{array}{ccc}
1 & 2 & 3
\end{array}
\begin{array}{c}
6
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array},$$

and

$$(z|\alpha)^T = (z_1 + \alpha_1)(z_1 + \alpha_2)(z_1 + \alpha_3)(z_3 + \alpha_6)(z_2 + \alpha_1)(z_2 + \alpha_2).$$

**Theorem 2** Let $\lambda$ be a partition. Then

$$s_\lambda(z|\alpha) = \sum_T (z|\alpha)^T,$$

where the sum is over semistandard Young tableaux with shape $\lambda$ in $1, 2, 3, \cdots, n$.

We will prove this in Section 4. If $\lambda$ is a partition then $\lambda'$ will denote the conjugate partition whose Young diagram is the transpose of that of $\lambda$.

Let $\mu$ be a partition, and let $\alpha_\mu$ denote the sequence

$$\alpha_\mu = (\alpha_{\mu_1+n}, \alpha_{\mu_2+n-1}, \cdots, \alpha_{\mu_n+1}).$$

**Theorem 3** (Vanishing Theorem) We have

$$s_\lambda(-\alpha_\mu|\alpha) = \begin{cases} 0 & \text{if } \lambda \nsubseteq \mu, \\
\prod_{(i,j)\in\lambda}(\alpha_{n-i+\lambda_i+1} - \alpha_{n-\lambda'_{i+1}}) & \text{if } \lambda = \mu.
\end{cases}$$

See Okounkov [21] Section 2.4 and Molev and Sagan [20]. We will prove this in Section 5.
3 Yang-Baxter equation

We review the six-vertex model and a case of the Yang-Baxter equation from [3]. We will consider a planar graph. Each vertex $v$ is assumed to have exactly four edges adjacent to it. Interior edges adjoin two vertices, and around the boundary of the graph we allow boundary edges that adjoin only a single vertex. Every vertex has six numbers $a_1(v), a_2(v), b_1(v), b_2(v), c_1(v), c_2(v)$ assigned to it. These are called the Boltzmann weights at the vertex. Once we have specified the graph, the Boltzmann weights at the vertices, and the boundary spins, we have specified a statistical system $\mathcal{G}$.

A state $s$ of the system will be an assignment of spins to the interior edges. Given a state of the system, every edge, boundary or interior, has a spin assigned to it. Then every vertex will have a definite configuration of spins on its four adjacent edges, and we assume these to be in one of the two orientations listed in (8). Then let $\beta_s(v)$ equal $a_1(v), a_2(v), b_1(v), b_2(v), c_1(v)$ or $c_2(v)$ depending on the configuration of spins on the adjacent edges. If $v$ does not appear in the table, the weight is zero.

| $a_1(v)$ | $a_2(v)$ | $b_1(v)$ | $b_2(v)$ | $c_1(v)$ | $c_2(v)$ |
|----------|----------|----------|----------|----------|----------|

The Boltzmann weight of the state $\beta(s)$ is the product $\prod_v \beta_s(v)$ of the Boltzmann weights at every vertex. We only need to consider configurations in which the spins adjacent to each vertex are in one of the configurations from the above table; if this is true, the state is called admissible. A state that is not admissible has Boltzmann weight zero.

The partition function $Z(\mathcal{G})$ is $\sum_s \beta(s)$, the sum of the Boltzmann weights of the states. We may either include or exclude the inadmissible states from this sum, since they have Boltzmann weight zero.

If at the vertex $v$ we have

$$a_1(v)a_2(v) + b_1(v)b_2(v) - c_1(v)c_2(v) = 0,$$

the vertex is called free-fermionic. We will only consider systems that are free-fermionic at every vertex.
Brubaker, Bump and Friedberg proved the following version of the Yang-Baxter equation.

**Theorem 4** [3, Theorem 3] Let $v, w$ be vertices with free-fermionic Boltzmann weights. Define another type of vertex $u$ with

\[
\begin{align*}
    a_1(u) &= a_1(v)a_2(w) + b_2(v)b_1(w), \\
    a_2(u) &= b_1(v)b_2(w) + a_2(v)a_1(w), \\
    b_1(u) &= b_1(v)a_2(w) - a_2(v)b_1(w), \\
    b_2(u) &= -a_1(v)b_2(w) + b_2(v)a_1(w), \\
    c_1(u) &= c_1(v)c_2(w), \\
    c_2(u) &= c_2(v)c_1(w).
\end{align*}
\]

Then for any assignment of edge spins $\varepsilon_i \in \{\pm\}$ ($i = 1, 2, 3, 4, 5, 6$) the following two configurations have the same partition function:

Note that by the definition of the partition function, the interior edge spins (labeled $\nu, \gamma, \mu$ and $\delta, \psi, \phi$) are summed over, while the boundary edge spins, labeled $\varepsilon_i$ are invariant. In order to obtain this from Theorem 3 of [3] one replaces the R-matrix $\pi(R)$ in the notation of that paper by a constant multiple.

Now let us consider two types of Boltzmann weights. Let $z_1, \ldots, z_n$ be spectral parameters, $\alpha_1, \alpha_2, \alpha_3, \ldots$ another sequence of complex numbers, and $t$ another
parameter. If $1 \leq i, k \leq n$ and if $j \geq 0$ are integers, we will use the following weights.

| $v_{\Gamma}(i, j, t)$ | 1 | $z_i - t\alpha_j$ | $t$ | $z_i + \alpha_j$ | $z_i(t + 1)$ | 1 |
|-----------------------|---|------------------|----|------------------|-------------|---|
| $v_{\Gamma}(i, k, t)$ | $i$ | $k$ | $i$ | $k$ | $i$ | $k$ | $i$ | $k$ | $i$ | $k$ | $i$ | $k$ |
| $t z_i + z_k$ | $t z_k + z_i$ | $t(z_k - z_i)$ | $z_i - z_k$ | $(t + 1) z_i$ | $(t + 1) z_k$ |

**Lemma 1** We may take $u = v_{\Gamma}(i, k, t)$, $v = v_{\Gamma}(i, j, t)$ and $w = v_{\Gamma}(k, j, t)$ in Theorem 4.

**Proof** The relation $a_1(u) = a_1(v)a_2(w) + b_2(v)b_1(w)$ becomes

$$t z_i + z_k = 1 \cdot (z_k - t\alpha_j) + (z_i + \alpha_j)t,$$

and all the other relations are checked the same way. □

We will make use of two special $n$-tuples of integers, namely

$$\rho = (n, \cdots, 3, 2, 1), \quad \delta = (n - 1, n - 2, \cdots, 2, 1, 0).$$

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition, so $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Let us consider a lattice with $n$ rows and $n + \lambda_1$ columns. We will index the rows from 1 to $n$ and the columns from 1 to $n + \lambda_1$, in reverse order. We put the vertex $v_{\Gamma}(i, j, t)$ at the vertex in the $i$ row and $j$ column.

We impose the following boundary edge spins. On the left and bottom boundaries, every edge is labeled $+$. On the right boundary, every edge is labeled $-$. On the top, we label the edges indexed by elements of form $\lambda_j + n - j + 1$ for $j = 1, 2, \cdots, n$ with a $-$ sign. These are the entries in $\lambda + \rho$. The remaining columns we label with a $+$ sign.

For example, suppose that $n = 3$ and that $\lambda = (5, 4, 1)$, so $\lambda + \rho = (8, 6, 2)$. Since $\lambda_j + n - j + 1$ has the values 8, 6 and 2, we put $-$ in these columns. We label the
vertex \( v_T(i, j, t) \) in the \( i \) row and \( j \) column by \( ij \) as in the following diagram:

\[
\begin{array}{cccccccccccc}
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
\end{array}
\]

Let this system be called \( \mathfrak{S}^\Gamma_{\lambda,t} \).

We recall that a Gelfand-Tsetlin pattern is an array

\[
\mathfrak{T} = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{22} & & & p_{2n} \\
& \ddots & & \\
p_{nn} & & &
\end{pmatrix}
\]

in which the rows are interleaving partitions. The pattern is strict if each row is strongly dominant, meaning that \( p_{ii} > p_{i,i+1} > \cdots > p_{nn} \).

A staircase is a semistandard Young tableau of shape \( (\lambda_1 + n, \lambda_1 + n - 1, \ldots, \lambda_1)' \) filled with numbers from \( \{1, 2, \ldots, \lambda_1 + n\} \) with the additional condition that the diagonals are weakly decreasing in the south-east direction when written in the French notation.

An example of a staircase for \( \lambda = (5, 4, 1) \) is

\[
\begin{array}{cccc}
8 & 7 & 8 & \\
6 & 7 & 8 & \\
5 & 6 & 6 & 7 & \\
4 & 5 & 5 & 5 & \\
3 & 3 & 3 & 4 & \\
2 & 2 & 2 & 3 & \\
1 & 1 & 1 & 1 & \\
\end{array}
\]
**Proposition 1** Let $\lambda$ be a partition. There are natural bijections between the following three sets of combinatorial objects

1. States of the six vertex model $\mathcal{G}_{\lambda,t}^\Gamma$.
2. Strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$, and
3. Staircases whose rightmost column consists of the integers between 1 and $\lambda_1 + n$ that are not in $\lambda + \rho$.

**Proof** Suppose we start with a state of the six vertex model $\mathcal{G}_{\lambda,t}^\Gamma$. Let us record the locations of all minus signs that live on vertical edges. Between rows $k$ and $k + 1$, there are exactly $n - k$ such minus signs for each $k$. Placing the column numbers of the locations of these minus signs into a triangular array gives a strict Gelfand-Tsetlin pattern with top row $\lambda + \rho$.

Given a strict Gelfand-Tsetlin pattern $\mathfrak{T} = (t_{ij})$ with top row $\lambda + \rho$, we construct a staircase whose rightmost column is missing $\lambda + \rho$ in the following manner: We fill column $j + 1$ with integers $u_1, \ldots, u_{n-j+\lambda_1}$ such that

$$\{1, 2, \ldots, n + \lambda_1\} = \{u_1, \ldots, u_{n-j+\lambda_1}\} \cup \{t_{n+1-j,1}, \ldots, t_{n+1-j,j-1}\}.$$

It is easily checked that these maps give the desired bijections. □

We give an example. As before let $n = 3$ and $\lambda = (5, 4, 1)$. Here is an admissible state:

![Six vertex model state](image)

Then the entries in the $i$-th row of $\mathfrak{T}$ are to be the columns $j$ in which a $-$ appears above the $(i,j)$ vertex. Therefore

$$\mathfrak{T} = \begin{pmatrix} 8 & 6 & 2 \\ 7 & 4 & \\ 4 & \end{pmatrix}.$$
Taking the complements of the rows (including a fourth empty row) gives the following sets of numbers:

\[
\begin{align*}
7 & 5 & 4 & 3 & 1 \\
8 & 6 & 5 & 3 & 2 & 1 \\
8 & 7 & 6 & 5 & 3 & 2 & 1 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{align*}
\]

so the corresponding staircase is \( \begin{array} {c c c c c}
 & & & & \\
 & & & & \\
 & & & & \\
 & & & & \\
\end{array} \) \( \begin{array} {c c c c c}
1 & 2 & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{array} \)\( \) \( \begin{array} {c c c c c c c c c}
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
\end{array} \).

**Proposition 2** The function

\[
\prod_{i>j} (t z_j + z_i) \quad Z(\mathcal{E}^\Gamma_{\lambda,t}),
\]

(13)

is symmetric under permutations of the \( z_i \).

**Proof** Let \( \mathcal{G} = \mathcal{E}^\Gamma_{\lambda,t} \). It is sufficient to show that (13) is invariant under the interchange of \( z_i \) and \( z_{i+1} \). The factors in front are permuted by this interchange with one exception, which is that \( t z_i + z_{i+1} \) is turned into \( t z_{i+1} + z_i \). Therefore we want to show that \( (t z_i + z_{i+1}) Z(\mathcal{G}) = (t z_{i+1} + z_i) Z(\mathcal{G}') \) where \( \mathcal{G}' \) is the system obtained from \( \mathcal{G} \) by interchanging the weights in the \( i, i+1 \) rows. Now consider the modified system obtained from \( \mathcal{G} \) by attaching \( v_{\Gamma \Gamma}(i, i+1, t) \) to the left of the \( i, i+1 \) rows. For example, if \( i = 1 \), this results in the following system:

Referring to (8), there is only one admissible configuration for the two interior edges adjoining the new vertex, namely both must be +, and so the Boltzmann weight of this vertex will be \( t z_i + z_{i+1} \). Thus the partition function of this new system equals \( (t z_i + z_{i+1}) Z(\mathcal{G}) \). Applying the Yang-Baxter equation repeatedly, this equals the partition function of the system obtained from \( \mathcal{G}' \) by adding \( v_{\Gamma \Gamma}(i, i+1, t) \) to the
right of the $i, i+1$ rows, that is, $(tz_{i+1} + z_i)Z(\mathcal{G}')$, as required. This proves that (13) is symmetric. □

**Theorem 5** We have

$$Z(\mathcal{G}^\Gamma_{\lambda,t}) = \left[ \prod_{i<j} (tz_j + z_i) \right] s_\lambda(z|\alpha).$$

**Proof** We show that the ratio

$$\frac{Z(\mathcal{G}^\Gamma_{\lambda,t})}{\prod_{i<j} (tz_j + z_i)} \quad (14)$$

is a polynomial in the $z_i$, and that it is independent of $t$. Observe that (13) is an element of the polynomial ring $\mathbb{C}[z_1, \ldots, z_n, t]$, which is a unique factorization domain. It is clearly divisible by $tz_j + z_i$ when $i > j$, and since it is symmetric, it is therefore divisible by all $tz_j + z_i$ with $i \neq j$. These are coprime, and therefore it is divisible by their product, in other words (14) is a polynomial. We note that the numerator and the denominator have the same degree in $t$, namely $\frac{1}{2}n(n-1)$. For the denominator this is clear and for the numerator, we note that each term is a monomial whose degree is the number of vertices with a $-$ sign on the vertical edge below. This is the number of vertical edges labeled $-$ excluding those at the top, that is, the number of entries in the Gelfand-Tsetlin pattern in Lemma 1 excluding the first row. Therefore each term in the sum $Z(\mathcal{G}^\Gamma_{\lambda,t})$ is a monomial of degree $\frac{1}{2}n(n-1)$ and so the sum has at most this degree; therefore the ratio (14) is a polynomial of degree 0 in $t$, that is, independent of $t$.

To evaluate it, we may choose $t$ at will. We take $t = -1$.

Let us show that if the pattern occurs with nonzero Boltzmann weight, then every row of the pattern (except the top row) is obtained from the row above it by discarding one element. Let $\mu$ and $\nu$ be two partitions that occur as consecutive rows in this Gelfand-Tsetlin pattern:

$$\left\{ \begin{array}{cccccc} \mu_1 & \mu_2 & \cdots & \cdots & \cdots & \mu_{n-i+1} \\ \nu_1 & \nu_2 & \cdots & \cdots & \nu_{n-i} \\ \end{array} \right\},$$

where $\mu_k$ are the column numbers of the vertices $(i, \mu_k)$ that have a $-$ sign on the edge above the vertex, and $\nu_k$ are the column numbers of the vertices $(i, \nu_k)$ that have a $-$ sign on the edge below it. Because $t = -1$, the pattern

\[ \begin{array}{cccc} \oplus & \oplus & \oplus & \oplus \\ \oplus & \oplus & \oplus & \oplus \\ \end{array} \]
does not occur, or else the Boltzmann weight is zero, and the term may be discarded. Therefore every \(-\) sign below the vertex in the \(i\)-th row must be matched with a \(-\) above the vertex. It follows that every \(\nu_i\) equals either \(\mu_i\) or \(\mu_i + 1\). Thus the partition \(\nu\) is obtained from \(\mu\) by discarding one element.

Let \(\mu_k\) be the element of \(\mu\) that is not in \(\nu\). It is easy to see that in the horizontal edges in the \(i\)-th row, we have a \(-\) sign to the right of the \(\mu_k\)-th column and \(+\) to the left. Since \(t = \lambda\), the Boltzmann weights of the two patterns:

\[
\begin{array}{c}
\text{Both have the same Boltzmann weight } z_i + \alpha_j. \\
\text{We have one such contribution from every column to the right of } \mu_k \text{ and these contribute}
\end{array}
\]

\[
\prod_{j=1}^{\mu_k-1} (z_i + \alpha_j) = (z_i | \alpha)^{\mu_k-1}.
\]

For each column \(j = \mu_l\) with \(l < k\) we have a pattern

\[
\begin{array}{c}
\text{and these contribute } -1. \text{ Therefore the product of the Boltzmann weights for this row is}
\end{array}
\]

\[
(-1)^{k-1} (z_i | \alpha)^{\mu_k-1}.
\]

Since between the \(i\)-th row \(\mu\) and the \((i + 1)\)-st row \(\nu\) of the Gelfand-Testlin pattern one element is discarded, there is some permutation \(\sigma\) of \(\{1, 2, 3, \ldots, n\}\) such that \(\nu\) is obtained by dropping the \(\sigma(i)\)-th element of \(\mu\). In other words,

\[
\mu_k - 1 = (\lambda + \rho)_{\sigma(i)} - 1 = (\lambda + \delta)_{\sigma(i)}
\]

and we conclude that

\[
Z(\mathcal{S}^\Gamma_{\lambda,-1}) = \sum_{\sigma \in S_n} \pm \prod_i (z_i | \alpha)^{(\lambda+\delta)_{\sigma(i)}}.
\]
The sign may be determined by taking all $\alpha_i = 0$. The term corresponding to $\sigma$ is then $\pm \prod_i z^{(\lambda+\rho)_{\sigma(i)}}$. Because the ratio \((14)\) is symmetric, and since (when $t = -1$) the denominator is antisymmetric, we have

$$Z(G^{\Gamma}_{\lambda,-1}) = \pm \sum_{\sigma \in S_n} (-1)^{l(\sigma)} (z_{\alpha})^{(\lambda+\rho)_{\sigma(i)}} = \pm A_{\lambda+\delta}(z|\alpha).$$

Thus using (3) we see that \((14)\) equals $s_{\lambda}(z|\alpha)$. To evaluate the sign, we may take $t = 0$ and all $z_i = 1$. Then the partition function is a sum of positive terms, and $s_{\lambda}(1, \ldots, 1)$ is positive, proving that \((14)\) equals $s_{\lambda}(z|\alpha)$.

**Proof of Theorem 1** Theorem 1 follows from Proposition 2 and Theorem 5.

**4 Proof of Theorem 2**

When $t = 0$, the Boltzmann weight for \[\begin{array}{c} \otimes \\
\end{array}\] is zero, so we are limited to states omitting this configuration. If $\mathcal{F} = \mathcal{F}_{\lambda+\rho}$ is the Gelfand-Tsetlin pattern corresponding to this state, and if the entries of $\mathcal{F}$ in are denoted $p_{ik}$ as in \([11]\), it is easy to see that the equality $p_{i-1,k-1} = p_{i,k}$ would cause this configuration to appear at the $i,j$ position, where $j = p_{i,k}$. Therefore $p_{i-1,k-1} > p_{i,k}$. This inequality implies that we may obtain another Gelfand-Tsetlin pattern $\mathcal{F}_{\lambda}$ with top row $\lambda$ by subtracting $\rho_{n-i+1} = (n - i + 1, n - i, \ldots, 1)$ from the $i$-th row of $\mathcal{F}_{\lambda+\rho}$.

Consider the example of $\lambda = (4, 2, 0)$ with $n = 3$. Then

if $\mathcal{F}_{\lambda+\rho} = \begin{bmatrix} 7 & 4 & 1 \\ 5 & 3 \\ 4 \end{bmatrix}$ then $\mathcal{F}_{\lambda} = \begin{bmatrix} 4 & 2 & 0 \\ 3 & 2 \\ 3 \end{bmatrix}$.

We associate with $\mathcal{F}_\lambda$ a tableau $T(\mathcal{F}_\lambda)$ of shape $\lambda$. In this tableau, removing all boxes labeled $\begin{array}{c} n \end{array}$ from the diagram produces a tableau whose shape is the second row of $\mathcal{F}_\lambda$. Then removing boxes labeled $\begin{array}{c} n-1 \end{array}$ produces a tableau whose shape is the third row of $\mathcal{F}_\lambda$, and so forth. Thus in the example, $T(\mathcal{F}_\lambda)$ is the tableau $T$ in \([6]\). Let $w_0$ denote the long element of the Weyl group $S_n$, which is the permutation $i \mapsto n + 1 - i$. 

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Proposition 3 Let $t = 0$, and let $s = s(\Sigma_{\lambda+\rho})$ be the state corresponding to a special Gelfand-Tsetlin pattern $\Sigma_{\lambda+\rho}$. With $\Sigma_{\lambda}$ as above and $T = T(\Sigma_{\lambda})$ we have

$$ w_0 \left( \prod_{v \in s} \beta_s(v) \right) = z^{w_0(\rho)}(z|a)^T. \quad (15) $$

Before describing the proof, let us give an example. With $\Sigma_{\lambda+\rho}$ as above, the state $s(\Sigma_{\lambda+\rho})$ is

The locations labeled $\circ$ produce powers of $z_i$, and the locations labeled $\bullet$ produce shifts of the form $z_i + \alpha_j$. The weight of this state is

$$ \prod_{v \in s} \beta_s(v) = z_1^2 z_2(z_1 + \alpha_6)(z_2 + \alpha_2)(z_3 + \alpha_1)(z_3 + \alpha_2)(z_3 + \alpha_1). $$

Applying $w_0$ interchanges $z_i \leftrightarrow z_{4-i}$. The factor $z_1^2 z_2$ becomes $z_3^2 z_2 = z^{w_0(\rho)}$, and the terms that remain agree with

$$ \prod_{(i,j)} (z_{T(i,j)} + \alpha_{T^*(i,j)}). $$

Proof We are using the following Boltzmann weights.

| Gamma | Ice | Boltzmann weight |
|-------|-----|------------------|
| $k_j$ | $k_j$ | 1                |
| $k_j$ | $k_j$ | $z_i$            |
|       | $k_j$ | 0                |
|       | $k_j$ | $z_i + \alpha_j$ |
|       | $k_j$ | $z_i$            |
|       | $k_j$ | 1                |
We have contributions of $z_i$ from vertices that have $-$ on the vertical edge below, and there is one of these for each entry in the Gelfand-Tsetlin pattern. These contribute a factor of $z^\rho$. Applying $w_0$ to $z$ leads to $z^{w_0(\rho)}$.

Considering the contribution from the $(i,j)$ vertex, there will be a factor of $z_i + \alpha_j$ when the vertex has the configuration . In the above example, the locations are labeled by $\bullet$.

Let $\Sigma_{\lambda+\rho}$ be the Gelfand-Tsetlin pattern \([11]\). Let $\Sigma_\lambda$ and $T = T(\Sigma_\lambda)$ be as described above, and let $T^*$ be as in \([14]\). Let the entries in $\Sigma_{\lambda+\rho}$ and $\Sigma_\lambda$ be denoted $p_{i,j}$ and $q_{i,j}$, with the indexing as in \([11]\). Thus $q_{i,j} = p_{i,j} - n + j - 1$. We will make the convention that $q_{i,n+1} = p_{i,n+1} = 0$. The condition for $\bullet$ in the $(i,j)$ position of the state is that for some $k$ with $i \leq k \leq n$ we have $p_{i+1,k+1} < j < p_{i,k}$. Translating this in terms of the $q_{i,j}$, the condition becomes $q_{i+1,k+1} + n - k + 1 \leq j < q_{i,k} + n - k$. The effect of $w_0$ is to interchange $z_i \leftrightarrow z_{n-i+1}$, and therefore

$$w_0 \left( \prod_{v \in S} \beta_s(v) \right) = \prod_{i=1}^n \prod_{k=1}^n \prod_{j=q_{i+1,k+1}+n-k+1}^{q_{i,k}+n-k} (z_{n+1-i} + \alpha_j)$$

On the other hand, in the tableau $T$, the location of the entries equal to $n+1-i$ in the $(k+1-i)$-th row is between columns $q_{i+1,k+1}+n-k+1$ and $q_{i,k}$, and if $j$ is one of these columns then

$$T(k+1-j,i) = n+1-i, \quad T^*(k+1-j,i) = n+j-k.$$ 

Therefore in the notation \([5]\) we have

$$(z|\alpha)^T = \prod_{i=1}^n \prod_{k=0}^n \prod_{j=q_{i+1,k+1}+1}^{q_{i,k}} (z_{n+1-i} + \alpha_{n+j-k})$$

This equals \([16]\) and the proof is complete. \(\Box\)

We now give the proof of Theorem 2. Summing over states, the last Proposition implies that

$$Z_\lambda(w_0(z), \alpha, 0) = z^{w_0(\rho)} \sum_T (z|a)^T.$$ 

Since $s_\lambda^T(z,\alpha,0) = s_\lambda^T(w_0(z),\alpha,0)$ we have

$$s_\lambda^T(z,\alpha,0) = \frac{Z_\lambda(w_0(z),\alpha,0)}{\prod_{i>j} w_0(z)_{ij}} = \frac{z^{w_0(\rho)} \sum_T (z|a)^T}{z^{w_0(\rho)} \sum_T (z|a)^T} = \sum_T (z|a)^T,$$

and the statement follows.
5 The limit as \( t \) tends to infinity

In this section, we examine the behaviour of our six-vertex model as we send the parameter \( t \) to infinity. The first result of this section may be construed as a rederivation of a theorem of Lascoux [13, Theorem 1]. However the approach we take is to interpret it as giving us a proof of the equivalence of factorial Schur functions and double Schubert polynomials for Grassmannian permutations. We also obtain the vanishing theorem for factorial Schur functions (Theorem 3).

We start with two simple lemmas. To state these, we describe the six admissible arrangements of edge spins around a vertex as types \( a, a_2, b, b_1, c, c_1 \) respectively, when reading from left to right in the diagram (8).

**Lemma 2** In each state, the total number of sites of type \( a_2, b_1 \) and \( c_1 \) is equal to \( n(n-1)/2 \).

**Proof** This number is equal to the number of minus signs located in the interior of a vertical string. □

Let \( \mu \) be the partition \( (\lambda + \rho)' \).

**Lemma 3** In each state, the number of occurrences of \( a_2, b_2 \) and \( c_1 \) patterns in the \( i \)-th column is equal to \( \mu_i \).

**Proof** This number is equal to the number of minus signs located on a horizontal string between the strings labelled \( i \) and \( i + 1 \). This count is known since for any rectangle, knowing the boundary conditions on the top, bottom and rightmost sides determines the number of such spins along the leftmost edge. □

As a consequence of Lemma 2, the result of taking the limit as \( t \to \infty \) can be interpreted as taking the leading degree term in \( t \) for each of the Boltzmann weights \( v_T(i,j,t) \) as in Section 3. Thus with the set of Boltzmann weights \((1, -\alpha_j, 1, z_i + \alpha_j, z_i, 1)\) and corresponding partition function \( Z(\mathfrak{C}_\lambda, \infty) \), we have

\[
Z(\mathfrak{C}_\lambda, \infty) = z^\rho s_\lambda(z|\alpha).
\]

By Lemma 3, if we consider our ice model with the series of Boltzmann weights in the following diagram1, then the corresponding partition function is given by
Let us denote the partition function for this set of Boltzmann weights by $Z(\mathfrak{S}_{\lambda,\infty}' (z|\alpha))$. Thus we obtain the result of [13, Theorem 1.1],

$$Z(\mathfrak{S}_{\lambda,\infty}' (z|\alpha)) = z^\rho \frac{\prod_{i+j \leq n+m} (x_i - y_j)}{(-\alpha)^{\lambda+\rho}} s_{\lambda}(z|\alpha). \quad (17)$$

We now pause to introduce the notions of double Schubert polynomials and Grassmannian permutations so that we can make the connection to [13, Theorem 1] precise.

A permutation is Grassmannian if it has a unique (right) descent. Let $n$ and $m$ be given and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition such that $\lambda_1 \leq m$. Then there is associated with $\lambda$ a Grassmannian permutation $w_\lambda \in S_{n+m}$. This is the permutation such that

$$w_\lambda(i) = \begin{cases} 
\lambda_{n+1-i} + i & \text{if } i \leq n, \\
\lambda_n - i & \text{if } i > n,
\end{cases}$$

where $\lambda'$ is the conjugate partition. This has $w_\lambda(n+1) < w_\lambda(n)$ and no other descent.

Let $x_1, \ldots, x_{n+m}$ and $y_1, \ldots, y_{n+m}$ be parameters. We define the divided difference operators as follows. If $1 \leq i < n+m$ and $f$ is a function of the $x_i$, let

$$\partial_i f(x_1, \ldots, x_{n+m}) = \frac{f - s_i f}{x_i - x_{i+1}}$$

where $s_i f$ is the function obtained by interchanging $x_i$ and $x_{i+1}$. Then if $w \in S_{n+m}$, let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression of $w$ as a product of simple reflections. Then let $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$. This is well defined since the divided difference operators $\partial_i$ satisfy the braid relations. Let $w_0$ be the long element of $S_{n+m}$. Then the double Schubert polynomials, which were defined by Lascoux and Schützenberger [14] are given by

$$\mathcal{S}_w(x,y) = \partial_w^{-1} w_0 \left( \prod_{i+j \leq n+m} (x_i - y_j) \right).$$
**Theorem 6** The factorial Schur functions are equal to double Schubert polynomials for Grassmannian permutations. More precisely,

\[ \mathcal{G}_{w,\lambda}(x, y) = s_\lambda(x|y) \]

Although this comparison result is well-known, it does not appear to have a good reference in the literature. One place where the relationship between factorial Schur functions and double Schubert polynomials for Grassmannian permutations is discussed is in Mihalcea [19], Remark 2 in Section 5.

**Proof** The proof is a comparison of (17) with [13, Theorem 1]. To translate the left-hand side in (17) into the staircase language of Lascoux we use the bijection in Proposition 1. In Theorem 1 of [13] if Lascoux’s \( x \) is our \( z \) and his \( y \) is our \( -\alpha \), then the left-hand side of his identity is exactly the partition function on the left-hand side of our (17). The monomial \( x^\rho y^{-(\tilde{\omega})} \) on the right-hand side of his identity equals the monomial \( z^\rho (-\alpha)^{-(\lambda + \rho)^\prime} \) on the right-hand side of (17). Lascoux’ \( X_{\tilde{\nu},\nu} \) is the double Schubert polynomial \( S_{w,\lambda} \). The statement follows. \( \square \)

To conclude this section, we shall use this description to give a proof of the characteristic vanishing property of Schur functions. In lieu of (17) above, the following result is clearly equivalent to Theorem 3.

**Theorem 7** For two partitions \( \lambda \) and \( \mu \) of at most \( n \) parts, we have

\[ Z(S_{\Gamma,\infty}^\prime(-\alpha_\mu|\alpha)) = 0 \quad \text{unless} \quad \lambda \subset \mu, \]

\[ Z(S_{\Gamma,\infty}^\prime(-\alpha_\lambda|\alpha)) = \prod_{(i,j) \in \lambda} \left( \frac{\alpha_{n+1-i+\lambda_i}}{\alpha_{n-\lambda'_j+j}} - 1 \right). \]

**Proof** Fix a state, and assume that this state gives a non-zero contribution to the partition function \( Z(S_{\Gamma,\infty}^\prime(-\alpha_\mu|\alpha)) \). Under the bijection between states of square ice and strict Gelfand-Tsetlin patterns, let \( k_i \) be the rightmost entry in the \( i \)-th row of the corresponding Gelfand-Tsetlin pattern. We shall prove by descending induction on \( i \) the inequality

\[ n + 1 - i + \mu_i \geq k_i. \]

For any \( j \) such that \( k_{i+1} < j < k_i \), there is a factor \( (x_i/\alpha_j - 1) \) in the Boltzmann weight of this state. We have the inequality \( n+1-i+\mu_i > n+1-(i+1)+\mu_{i+1} \geq k_{i+1} \) by our inductive hypothesis. Since \( (\alpha_\mu)_i = \alpha_{n+1-i+\mu_i} \), in order for this state to give a non-zero contribution to \( Z(S_{\Gamma,\infty}^\prime(-\alpha_\mu|\alpha)) \), we must have that \( n + 1 - i + \mu_i \geq k_i \), as required.
Note that for all \( i \), we have \( k_i \geq n + 1 - i + \lambda_i \). Hence \( \mu_i \geq \lambda_i \) for all \( i \), showing that \( \mu \supset \lambda \) as required, proving the first part of the theorem.

To compute \( Z(\mathfrak{S}_{\lambda,\infty}(-\alpha_\lambda|\alpha)) \), notice that the above argument shows that there is only one state which gives a non-zero contribution to the sum. Under the bijection with Gelfand-Tsetlin patterns, this is the state with \( p_{i,j} = p_{1,j} \) for all \( i,j \). The formula for \( Z(\mathfrak{S}_{\lambda,\infty}(-\alpha_\lambda|\alpha)) \) is now immediate. \( \square \)

6 Asymptotic Symmetry

Macdonald [16] shows that the factorial Schur functions are \textit{asymptotically symmetric} in the \( \alpha_i \) as the number of spectral parameters \( z_i \) increases. To formulate this property, let \( \sigma \) be a permutation of the parameters \( \alpha = (\alpha_1, \alpha_2, \cdots) \) such that \( \sigma(\alpha_j) = \alpha_j \) for all but finitely many \( j \). Then we will show that if the number of spectral parameters \( z_i \) is sufficiently large, then \( s_\lambda(z|\sigma\alpha) = s_\lambda(z|\alpha) \). How large \( n \) must be depend on both \( \lambda \) and on the permutation \( \sigma \).

We will give a proof of this symmetry property of factorial Schur functions using the Yang-Baxter equation. In the following theorem we will use the following Boltzmann weights:

| \( v \) | \( 1 \) | \( z_i - t\alpha \) | \( t \) | \( z_i + \alpha \) | \( z_i(t + 1) \) | \( 1 \) |
| \( w \) | \( 1 \) | \( z_i - t\beta \) | \( t \) | \( z_i + \beta \) | \( z_i(t + 1) \) | \( 1 \) |
| \( u \) | \( 1 \) | \( 1 \) | \( \beta - \alpha \) | \( 0 \) | \( 1 \) | \( 1 \) |

**Theorem 8** With the above Boltzmann weights, and with \( \epsilon_1, \cdots, \epsilon_6 \) fixed signs \( \pm \),
the following two systems have the same partition function.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure}
\caption{Diagram illustrating the two systems with the same partition function.}
\end{figure}

**Proof** This may be deduced from Theorem 3 of [3] with a little work. By rotating the diagram 90° and changing the signs of the horizontal edges, this becomes a case of that result. The R-matrix is not the matrix for \( u \) given above, but a constant multiple. We leave the details to the reader. \( \square \)

Let \( \sigma_i \) be the map on sequences \( \alpha = (\alpha_1, \alpha_2, \cdots) \) that interchanges \( \alpha_i \) and \( \alpha_{i+1} \). Let \( \lambda \) be a partition. We will show that sometimes:

\[ s_\lambda(z|\alpha) = s_\lambda(z|\sigma_i\alpha) \]

and sometimes

\[ s_\lambda(z|\alpha) = s_\lambda(z|\sigma_i\alpha) + s_\mu(z|\sigma_i\alpha)(\alpha_{i+1} - \alpha_i), \]

where \( \mu \) is another partition. The next Proposition gives a precise statement distinguishing between the two cases (18) and (19).

**Proposition 4** (i) Suppose that \( i+1 \in \lambda + \rho \) but that \( i \notin \lambda + \rho \). Let \( \mu \) be the partition characterized by the condition that \( \mu + \rho \) is obtained from \( \lambda + \rho \) by replacing the unique entry equal to \( i+1 \) by \( i \). Then (19) holds with this \( \mu \).

(ii) If either \( i+1 \notin \lambda + \rho \) or \( i \in \lambda + \rho \) then (18) holds.

For example, suppose that \( \lambda = (3, 1) \) and \( n = 5 \). Then \( \lambda + \rho = (8, 5, 3, 2, 1) \). If \( i = 4 \), then \( i+1 = 5 \in \lambda + \rho \) but \( i \notin \lambda + \rho \), so (19) holds with \( \mu + \rho = (8, 4, 3, 2, 1) \), and so \( \mu = (3) \).

**Proof** We will use Theorem 8 with \( \alpha = \alpha_i \) and \( \beta = \alpha_{i+1} \). The parameter \( t \) may be arbitrary for the following argument. We first take \( i = n \) and attach the vertex \( u \) below the \( i \) and \( i+1 \) columns, arriving at a configuration like this one in the case...
$n = 3, \lambda = (4, 2, 1)$.

There is only one legal configuration for the signs of the edges between $u$ and the two edges above it, which in the example connect with $(3, 5)$ and $(3, 4)$: these must both be $+$. The Boltzmann weight at $u$ in this configuration is unchanged, and the partition function of this system equals that of $\mathfrak{S}_\lambda^\Gamma$. After applying the Yang-Baxter equation, we arrive at a configuration with the $u$ vertex above the top row, as follows:

Now if we are in case (i), the signs of the two edges are $-, +$ then there are two legal configurations for the vertex, and separating the contribution these we obtain

$$Z(\mathfrak{S}_\lambda^\Gamma) = (\sigma_i Z(\mathfrak{S}_\lambda^\Gamma)) + (\beta - \alpha) (\sigma_i Z(\mathfrak{S}_\mu^\Gamma)).$$
If we are in case (ii), there is only one legal configuration, so $Z(\mathcal{G}_{\lambda,t}) = (\sigma_i Z(\mathcal{G}_{\lambda,t}))$.

\[ \Box \]

Corollary 1 Let $\lambda$ be a partition, and let $l$ be the length of $\lambda$. If $n \geq l + i$ then $s_\lambda(z|a) = s_\lambda(z|\sigma_i a)$.

Proof For if $l$ is the length of the partition $\lambda$, then the top edge in column $j$ is a $-$ sign when $j \leq n - l$, and therefore we are in case (i).

This Corollary implies Macdonald’s observation that the factorial Schur functions are asymptotically symmetric in the $\alpha_i$.

7 The Dual Cauchy Identity

Let $m$ and $n$ be positive integers. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \leq m$, we define a new partition $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_m)$ by

$$\hat{\lambda}_i = |\{ j \mid \lambda_j \leq m - i \}|.$$ 

After a reflection, it is possible to fit the Young diagrams of $\lambda$ and $\hat{\lambda}$ into a rectangle.

We shall prove the following identity, known as the dual Cauchy identity.

Theorem 9 (Dual Cauchy Identity) For two finite alphabets of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, we have

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j) = \sum_\lambda s_\lambda(x|\alpha)s_\hat{\lambda}(y|\alpha).$$

The sum is over all partitions $\lambda$ with at most $n$ parts and with $\lambda_1 \leq m$.

Proof The proof will consist of computing the partition function of a particular six vertex model in two different ways. We will use the weights $v_\Gamma(i, j, t)$ introduced in Section 3 with $t = 1$ and for our spectral parameters $(z_1, \ldots, z_{m+n})$, we will take the sequence $(y_m, \ldots, y_1, x_1, \ldots, x_n)$.

As for the size of the six-vertex model we shall use and the boundary conditions, we take a $(m + n) \times (m + n)$ square array, with positive spins on the left and lower edges, and negative spins on the upper and right edges. By Theorem 5 (with $\lambda = 0$), the partition function of this array is

$$\prod_{i<j} (y_i + y_j) \prod_{i,j} (y_i + x_j) \prod_{i<j} (x_i + x_j).$$
We shall partition the set of all states according to the set of spins that occur between the rows with spectral parameters \( y_1 \) and \( x_1 \). Such an arrangement of spins corresponds to a partition \( \lambda \) in the usual way, i.e. the negative spins are in the columns labelled \( \lambda_i + n - i + 1 \). In this manner we can write our partition function as a sum

\[
\sum_\lambda Z^\text{top}_\lambda Z^\text{bottom}_\lambda.
\]

Here \( Z^\text{top}_\lambda \) is the partition function of the system with \( m \) rows, \( m + n \) columns, spectral parameters \( y_m, \ldots, y_1 \) and boundary conditions of positive spins on the left, negative spins on the top and right, and the \( \lambda \) boundary condition spins on the bottom. And \( Z^\text{bottom}_\lambda \) is the partition function of the system with \( n \) rows, \( m + n \) columns, spectral parameters \( x_1, \ldots, x_n \) and the usual boundary conditions for the partition \( \lambda \).

By Theorem 5, we have

\[
Z^\text{bottom}_\lambda = \prod_{i<j} (x_i + x_j)s_\lambda(x|\alpha).
\]

It remains to identify \( Z^\text{top}_\lambda \). To do this, we perform the following operation to the top part of our system. We flip all spins that lie on a vertical strand, and then reflect the system about a horizontal axis. As a consequence, we have changed the six-vertex system that produces the partition function \( Z^\text{top}_\lambda \) into a system with more familiar boundary conditions, namely with positive spins along the left and bottom, negative spins along the right hand side, and along the top row we have negative spins in those columns labelled \( \hat{\lambda}_i + m - i + 1 \).

The Boltzmann weights for this transformed system are (recalling that \( t = 1 \))

| Gamma Ice | Boltzmann weight |
|------------|------------------|
| \( \gamma_{ij} \) | \( y_i + \alpha_j \) |
| \( \gamma_{ij} \) | 1 |
| \( \gamma_{ij} \) | \( y_i - \alpha_j \) |
| \( \gamma_{ij} \) | 2\( y_i \) |
| \( \gamma_{ij} \) | 1 |

Now again we use Theorem 5 to conclude that

\[
Z^\text{top}_\lambda = \prod_{i<j} (y_i + y_j)s_\hat{\lambda}(y|\alpha).
\]

Comparing the two expressions for the partition function of this square system completes the proof. \( \square \)
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