A NEW DETERMINANT FOR THE $Q$-ENUMERATION OF ALTERNATING SIGN MATRICES

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Abstract. Fischer provided a new type of binomial determinant for the number of alternating sign matrices involving the third root of unity. In this paper we prove that her formula, when replacing the third root of unity by an indeterminate $q$, is actually the $(2+q+q^{-1})$-enumeration of alternating sign matrices. By evaluating a generalisation of this determinant we are able to reprove a conjecture of Mills, Robbins and Rumsey stating that the $Q$-enumeration is a product of two polynomials in $Q$. Further we provide a closed product formula for the generalised determinant in the $0$, $1$-, $2$- and $3$-enumeration case, leading to a new proof of the $1$-, $2$- and $3$-enumeration of alternating sign matrices, and a factorisation in the $4$-enumeration case. Finally we relate the $1$-enumeration of our generalised determinant to the determinant evaluations of Ciucu, Eisenkölbl, Krattenthaler and Zare, which counts weighted cyclically symmetric lozenge tilings of a hexagon with a triangular hole and is a generalisation of a famous result by Andrews.

1. Introduction

An alternating sign matrix (or short ASM) of size $n$ is an $n \times n$ matrix with entries $-1$, $0$ or $1$ such that all row and column sums are equal to $1$ and the non-zero entries alternate in each row and column, see Figure 1 left. These matrices were introduced by Robbins and Rumsey \cite{16} and arose from generalising the determinant to the so-called $\lambda$-determinant. Together with Mills \cite{15} they conjectured the enumeration formula for the number of ASMs of size $n$

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$  

This was remarkable since there existed already a different family of combinatorial objects, descending plane partitions (or DPPs), which were proven by Andrews \cite{1} to have the same enumeration formula, but which are of a very different nature. In \cite{14} Mills, Robbins and Rumsey conjectured this very enumeration formula for another family of combinatorial objects, namely totally symmetric self complementary plane partitions (or TSSCPPs), which was proven by Andrews \cite{2}.

The first proof of the ASM Theorem was found by Zeilberger \cite{17}, who related a constant term formula for ASMs to a constant term formula for TSSCPPs. Shortly later Kuperberg \cite{13} presented a proof by using a different approach. He used the fact that there exists an easy bijection between ASMs and the six-vertex model, a model in statistical mechanics. This technique was later used as a standard method to prove enumeration formulas of various symmetry classes of ASMs. In \cite{15} Mills Robbins and Rumsey also proved a formula for the $2$-enumeration and conjectured one for the $3$-enumeration of ASMs, where the $Q$-enumeration of ASMs is a weighted enumeration of ASMs with each ASM having the weight of $Q$ to the power of the number of its $-1$’s. Kuperberg \cite{13} could prove with his methods the $3$-enumeration and also that the $Q$-enumeration of ASMs is a product of two polynomials in $Q$ which was also conjectured in \cite{15} Conjecture 4.

Roughly a century later, Fischer \cite{6} provided a new proof of the ASM Theorem which was independent of the first two methods. The proof is based on the bijection between ASMs and

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monotone triangles and the operator formula \([5, \text{Theorem 1}]\) which enumerates monotone triangles. In 2016 Fischer \([8]\) presented a concise version of her original proof which, except of relying at one place on the Lindström-Gessel-Viennot Theorem, is self-contained.

A monotone triangle with \(n\) rows is a triangular array \((a_{i,j})_{1 \leq j \leq i \leq n}\) of integers of the following form
\[
\begin{array}{cccccc}
  & a_{1,1} & & & & \\
  & a_{2,1} & a_{2,2} & & & \\
  & a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n-1} & \\
  & a_{n,1} & a_{n,2} & \ldots & a_{n,3} & \\
\end{array}
\]
whose entries are weakly increasing along north-east \(a_{i+1,j} \leq a_{i,j}\) and south-east diagonals \(a_{i,j} \leq a_{i+1,j+1}\) and strictly increasing along rows \(a_{i,j} < a_{i,j+1}\). For an example see Figure 1 right. Given an ASM, the \(i\)-th row from top of its corresponding monotone triangle records the columns of the ASM with positive partial column-sum of the top \(i\) rows. It is easy to see that this yields a bijection between ASMs of size \(n\) and monotone triangles with bottom row \(1,2,\ldots,n\). We assign to a monotone triangle a weight \(Q^\Sigma\) where \(\Sigma\) is the number of entries \(a_{i,j}\) of the triangle such that \(a_{i+1,j} < a_{i,j} < a_{i+1,j+1}\). For ASMs this weight is exactly \(Q\) to the power of the number of \(-1\)'s.

The core of this paper is the following theorem.

**Theorem 1.** The \((2+q+q^{-1})\)-enumeration of alternating sign matrices is equal to

\[
d_{n,k}(x,q) := \det_{1 \leq i,j \leq n} \left( \frac{(x+i+j-2)!}{(j-1)!} \frac{1 - (-q)^{i+j+k}}{1+q} \right),
\]

for \(x = 0\) and \(k = 1\).

The above determinant appeared first in \([10]\) for \(k = 1\) and was shown by Fischer to count the number of ASMs for \(x = 0\), \(k = 1\) and \(q\) being a primitive third root of unity. By introducing the variable \(x\), which was suggested in \([10]\), and the integer parameter \(k\) in the determinant, we are able to write \(d_{n,k}(x,q)\) as a closed product formula for arbitrary \(k\) and \(q\) being a primitive second root of unity (Theorem \([14]\), third root of unity (Theorem \([15]\)), fourth root of unity (Theorem \([17]\)) or sixth root of unity (Theorem \([19]\)), which was conjectured for \(k = 1\) in \([10]\). For \(q = 1\) we provide in Theorem \([21]\) a factorisation of the determinant as a polynomial in \(x\). Contrary to other known determinantal formulas for the \(Q\)-enumeration of ASMs, for which the evaluation is rather complicated, the evaluation of the determinant considered in this paper turns out to be very easy and thus leading immediately to the the known formulas for the 0-,1-,2- and 3- enumeration of alternating sign matrices by setting \(x = 0\).

In Theorem \([10]\) we prove a general factorisation result which states that the determinant \(d_{n,k}(x,q)\) factors for arbitrary \(q\) into a power of \(q\), a polynomial \(p_{n,k}(x) \in \mathbb{Q}[x]\) which factorises into linear factors and a polynomial \(f_{n,k}(x,q) \in \mathbb{Q}[x,q]\) which is given recursively. Theorem \([14]\) implies further that all linear factors in \(x\) of \(d_{n,k}(x,q)\) are covered in \(p_{n,k}(x)\). For \(k = 1\), the above implies that the determinant \(d_{n,k}(x,q)\) can be written as a product of two Laurent
polynomials in $q$ with coefficients in $\mathbb{Q}[x]$, which was conjectured in [10]. As a direct consequence we obtain that the generating function of ASMs with respect to the number of $-1$ is a product of two polynomials in $Q = q^1 + 2 + q$, which was conjectured in [15, Conjecture 4] and first proven in [13].

Surprisingly the determinant $d_{n,k}(x, q)$ is connected to the famous determinant by Andrews [1] and its generalisation by Ciucu, Eisenkölbl, Krattenthaler and Zare [4] in the following way

$$d_{n,3-k}(x, q^2) = q^{-n} \det_{1 \leq i,j \leq n} \left( \begin{array}{cc} x + i + j - 2 \\ j - 1 \end{array} \right) + q^k \delta_{i,j},$$

where $q$ is a sixth root of unity. This fact was first conjectured for $k = 2, 4$ in [10]. This is remarkable because of two reasons. Firstly in the above cases the evaluation of the determinants have a closed product formula. This fact is very easy to prove for the left hand side of (1) by using the Desnanot-Jacobi identity which is also known as the Condensation method. For the right hand side this method is however not applicable and the proof relies on the method of identification of factors. Secondly the determinant by Ciucu et al. is a weighted enumeration of cyclically symmetric lozenge tilings of hexagons with a triangular hole, these are a “one parameter generalisations” of DPPs, where $q$ corresponds to the weight and $x$ is the side length of the hole. Very recently Fischer proved in [9] that this determinant also enumerates alternating sign trapezoids, which are one parameter generalisations of ASTs. Hence the determinant $d_{n,k}(x, q)$ suggests a one parameter refinement for ASMs and might be of help in finding one.

The paper is structured in the following way. In Section 2 we follow the steps of [10, 11] and prove Theorem 1. Section 3 contains a description of the factorisation of the determinant $d_{n,1}(x, q)$ and prove Theorem 1. Section 4 we state the product formulas for $d_{n,k}(x, q)$ for $q$ being a primitive second, third, fourth or sixth root of unity and present a factorisation if $q = 1$. This leads to new proofs for the 0-,1-,2- and 3-enumeration of ASMs. Finally in Section 5 we relate the determinant $d_{n,k}(x, q)$ to the Andrews determinant and its generalisation by Ciucu, Eisenkölbl, Krattenthaler and Zare. The paper ends by an appendix containing a list of specialisations of $d_{n,k}(x, q)$ which turn out to be known enumeration formulas.

2. A determinantal formula for the number of ASMs

For the rest of this paper we fix the following notations,

$E_x(f)(x) := f(x + 1)$  
$\Delta_x := E_x - \text{Id}$ \hspace{2cm} \text{shift operator,}$

$\overline{\Delta}_x := \text{Id} - E^{-1}_x$ \hspace{2cm} \text{forward difference,}$

$\underline{\Delta}_x := \text{Id} - E^{-1}_x$ \hspace{2cm} \text{backward difference.}$

We denote by $\mathcal{A}_x f(x_1, \ldots, x_n)$ the antisymmetriser of $f$ with respect to $x_1, \ldots, x_n$ which is defined as

$$\mathcal{A}_x f(x_1, \ldots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn} \sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Further we are using at various points the multi-index notion, i.e., a bold variable $\mathbf{x}$ refers to a vector $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}^a$ is defined as $\mathbf{x}^a := \prod_{i=1}^n x_i^{a_i}$.

This section is used to deduce the determinantal expression for the $Q$-enumeration of ASMs. It is a revision of results in [11] and generalises results in [10]. The starting point is a weighted version of the operator formula for monotone triangles.

**Theorem 2** ([4, Theorem 1]). The generating function of monotone triangles with bottom row $k = (k_1, \ldots, k_n)$ with respect to the $Q$-weight is given by evaluating the polynomial

$$M_n(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq n} (Q \text{Id} + (Q - 1) \overline{\Delta}_x x_i + \Delta_x x_j + \Delta_x \overline{\Delta}_x x_j) \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i},$$

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Theorem 4 ([11, Theorem 16]). A determinant; the theorem is a variation by Fischer of Equation (D.3) in [12].

Lemma 5. Where

Proposition 3 ([11, Proposition 10.1]). The number of monotone triangles with bottom row \( k = (k_1, \ldots, k_n) \) with respect to the \( Q \)-weight is the constant term of

\[
\text{det}_{x_1, \ldots, x_n} \left( \prod_{i=1}^{n} (1 + x_i)^{k_i} \prod_{1 \leq i < j \leq n} \left( Q + (Q - 1)x_i + x_j + x_ix_j \right) \right) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1}.
\]

The following theorem allows us to transfer the antisymmetriser in the above proposition into a determinant; the theorem is a variation by Fischer of Equation (D.3) in [12].

Theorem 4 ([11, Theorem 16]). Let \( f(x, y) = qx - q^{-1}y \) and \( h(x, y) = x - y \), then holds

\[
\text{det}_{w_1, \ldots, w_n} \left( \prod_{1 \leq i < j \leq n} f(w_i, w_j) \prod_{1 \leq i < j \leq n} h(w_i, y_i) \right) = \prod_{1 \leq i < j \leq n} h(y_j, y_i).
\]

By setting \( w_i = \frac{x_i + q^{-1}}{x_i + q} \), \( q = 2 + q^{-1} \) and taking the limit of \( y_i \to 1 \) for all \( 1 \leq i \leq n \), the above theorem becomes

\[
\frac{(-1)^{\frac{n(n+1)}{2}}}{(q - q^{-1})^\frac{n(n+3)}{2}} \text{det}_{x_1, \ldots, x_n} \left( \prod_{1 \leq i < j \leq n} (Q + (Q - 1)x_i + x_j + x_ix_j) \prod_{i=1}^{n} (x_i + 1 + q)^2 \right)
\]

\[
= q^n \lim_{y_1, \ldots, y_n \to 1} \text{det} \left( \frac{1}{y_j - x_i + 1 + q^{-1}} \right) \prod_{1 \leq i < j \leq n} (y_j - y_i)^{-1}.
\]

Hence, by Proposition 3 the \((2 + q + q^{-1})\)-enumeration of ASMs is given by

\[
\text{CT}_{x_1, \ldots, x_n} \left( (-1)^{\frac{n(n+1)}{2}} q^n (q - q^{-1})^{\frac{n(n+3)}{2}} \prod_{i=1}^{n} (1 + x_i)^{n+1} (x_i + 1 + q)^{-2} \right)
\]

\[
\times \lim_{y_1, \ldots, y_n \to 1} \text{det} \left( \frac{1}{y_j - x_i + 1 + q^{-1}} \right) \prod_{1 \leq i < j \leq n} (y_j - y_i)^{-1}.
\]

Using the partial fraction decomposition \( \frac{1}{(y-a)(y-b)} = \frac{1}{a-b} \left( \frac{1}{y-a} - \frac{1}{y-b} \right) \) we can rewrite the determinant in (2) as

\[
(1 - q^2)^{-n} \prod_{i=1}^{n} (x_i + 1 + q)^2 (x_i + 1 + q^{-1})^{-1}
\]

\[
\text{det} \left( \frac{1}{y_j (x_i + 1 + q) - (x_i + 1 + q^{-1}) - q^2 (x_i + 1 + q^{-1})} \right) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1}.
\]

The following lemma allows us to evaluate the limit in (2).

Lemma 5 ([11, Eq. (43)-(47)]). Let \( f(x, y) = \sum_{i,j \geq 0} c_{i,j} x^i y^j \) be a formal power series in \( x \) and \( y \), then holds

\[
\lim_{y_1, \ldots, y_n \to y} \frac{\text{det} \left( f(x_i, y_j) \right)_{1 \leq i < j \leq n}}{\text{det} \left( (x_j - x_i)(y_j - y_i) \right)_{1 \leq i < j \leq n}} = \text{det} \left( [u v^j] f(x + u, y + v) \right)_{0 \leq i, j \leq n-1},
\]

where \([u v^j] f(x + u, y + v)\) denotes the coefficient of \( u^i v^j \) in \( f(x + u, y + v) \).

\( x = k \).

Following [11], we can rewrite the operator formula as a constant term expression.

Proposition 3 ([11, Proposition 10.1]). The number of monotone triangles with bottom row \( k = (k_1, \ldots, k_n) \) with respect to the \( Q \)-weight is the constant term of
By the above lemma we need to calculate the coefficient of $x^iy^j$ in
\[
\frac{1}{(y + 1)(x + 1 + q) - (x + 1 + q^{-1})} - \frac{1}{(y + 1)(x + 1 + q) - q^2(x + 1 + q^{-1})}.
\] (3)

Using the geometric series expansion in $x$ and $y$ of (3), we obtain for the coefficient of $x^iy^j$
\[
\binom{j}{i}(1 + q)^{-i-1}(q - 1)^{-j-1}(-1)^j q^{i+1} - \sum_{k=i}^{i+j} (-1)^k \binom{k}{j} \binom{j}{i-k+j} \frac{(1 + q)^{k-i-j-1}}{(1 - q)^{j+1}}.
\]

Putting the above together, it follows that the $(2 + q + q^{-1})$-enumeration of ASMs of size $n$ is given by
\[
(1 + q)^{-n} q^{-\binom{n}{2}} \det_{0 \leq i,j \leq n-1} \left( \binom{j}{i} (-1)^i q^{i+1} + \sum_{k=0}^{n-1} \binom{k+i}{j} \binom{j}{k} (-1 - q)^{k+i-j} \right).
\]

Finally we use the following identity which is due to Fischer [10] and can be proven using basic properties of the binomial coefficient
\[
\binom{j}{i} (-1)^i q^{i+1} + \sum_{k=0}^{n-1} \binom{k+i}{j} \binom{j}{k} (-1 - q)^{k+i-j} = \sum_{k=0}^{n-1} (-1)^i \binom{i}{k} (-1 - q)^{k+i-j} (q^{i+1}(-1)^k + q^k(-1)^j).
\]

Hence the $(2 + q + q^{-1})$-enumeration of ASMs is
\[
det \left( \left( \binom{i+j}{j} (-1)^{i+j} \right)_{0 \leq i,j \leq n-1} \right) \times \left( \binom{i+j}{j} \frac{1 - (-q)^{j-i+1}}{1 + q} \right)_{0 \leq i,j \leq n-1} = \det_{1 \leq i,j \leq n} \left( \binom{i+j-2}{j-1} \frac{1 - (-q)^{j-i+1}}{1 + q} \right),
\] (4)

which proves Theorem 1.

3. A GENERALISED $(2 + q + q^{-1})$-ENUMERATION

We introduce to the determinant in (4) a variable $x$, as suggested in [10], and further a parameter $k \in \mathbb{Z}$. The determinant of our interest is then $d_{n,k}(x,q) := \det(D_{n,k}(x,q))$, where $D_{n,k}(x,q)$ is defined by
\[
D_{n,k}(x,q) := \left( \binom{x+i+j-2}{j-1} \frac{1 - (-q)^{j-i+k}}{1 + q} \right)_{1 \leq i,j \leq n}.
\]

The evaluation of this determinant will generally follow two steps. The first step is to guess a formula for $d_{n,k}(x,q)$ using a computer algebra system. The second step is to use induction and the Desnanot-Jacobi Theorem, which is sometimes also called condensation method.

**Theorem 6** (Desnanot-Jacobi). Let $n$ be a positive integer, $A$ an $n \times n$ matrix and denote by $A^{i_1,\dotsc,i_k}_{j_1,\dotsc,j_k}$ the submatrix of $A$ in which the $i_1,\dotsc,i_k$-th rows and $j_1,\dotsc,j_k$-th columns are omitted. Then holds
\[
det A \det A^{i_1,n}_{j_1,n} = \det A^n_{1} - \det A^n_{1} \det A^n_{1}.
\]
Deleting the first or last row and the first or last column of the matrix $D_{n,k}(x,q)$ can be expressed as follows

\[
\det(D_{n,k}(x,q)_{1}) = \det(D_{n-1,k}(x+2,q)) \left(\frac{x+n}{n-1}\right),
\]

\[
\det(D_{n,k}(x,q)^{n}_{n}) = \det(D_{n-1,k}(x,q)),
\]

\[
\det(D_{n,k}(x,q)_{1,n}) = \det(D_{n-2,k}(x+2,q)) \left(\frac{x+n-1}{n-2}\right),
\]

\[
\det(D_{n,k}(x,q)^{1}_{1}) = \det(D_{n-1,k+1}(x+1,q)),
\]

\[
\det(D_{n,k}(x,q)^{n}_{n}) = \det(D_{n-1,k-1}(x+1,q)) \left(\frac{x+n-1}{n-1}\right).
\]

We will prove it in the case of $D_{n,k}(x,q)_{1}$, the other cases follow analogously. By definition we have

\[
\det(D_{n,k}(x,q)_{1}) = \frac{\det\left(\begin{array}{c}
(x+i+j-2) \\ j-1
\end{array}\right) 1 - (-q)^{j-i+k}}{1+q} = \frac{\det\left(\begin{array}{c}
(x+i+j) \\ j
\end{array}\right) 1 - (-q)^{j-i+k}}{1+q}.
\]

Since the $i$-th row is divisible by $(x+i+1)$ we can pull it out for all $1 \leq i \leq n-1$. Further we pull out $j^{-1}$ from the $j$-th row for $1 \leq j \leq n-1$ and obtain

\[
\det(D_{n,k}(x,q)_{1}) = \prod_{i=1}^{n} \frac{(x+i+1)}{i} \frac{\det\left(\begin{array}{c}
(x+i+j) \\ j
\end{array}\right) 1 - (-q)^{j-i+k}}{1+q} = \left(\frac{x+n}{n-1}\right) \det(D_{n-1,k}(x+2,q)).
\]

By applying (5) to Theorem 6 we obtain

\[
(n-1)d_{n,k}(x,q)d_{n-2,k}(x+2,q) = (x+n)d_{n-1,k}(x,q)d_{n-1,k}(x+2,q) - (x+1)d_{n-1,k+1}(x+1,q)d_{n-1,k-1}(x+1,q).
\] (6)

We define for the rest of this section $a_{j}$ for $j \in \mathbb{Z}$ as

\[
a_{j} = \frac{1 - (-q)^{j}}{1+q} = \begin{cases} 
\sum_{i=0}^{j-1}(-q)^{i} & j > 0, \\
0 & j = 0, \\
-\sum_{i=1}^{-j}(-q)^{-i} & j < 0.
\end{cases}
\]

The entries of the matrix $D_{n,k}(x,q)$ are polynomials in $x$ and Laurent polynomials in $q$ and hence the same is true for the determinant $d_{n,k}(x,q)$, i.e., $d_{n,k}(x,q) \in \mathbb{Q}[q,q^{-1},x]$.

**Lemma 7.** Let $n,k$ be positive integers, then the determinant $d_{n,k}(x,q)$ is as an element in $\mathbb{Q}[q,q^{-1},x]$ divisible by

\[
\prod_{l=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (x+k+2l+1).
\]

**Proof.** Let $l$ be a non-negative integer. The $(i,j)$-th entry of $D_{n,k}(x,q)$ is divisible by $(x+k+2l+1)$ if $i \leq k+2l+1$ and $i+j \geq k+2l+3$ or if $i = j+k$. Now choose $l$ with $0 \leq l \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ and set $x = -k-2l-1$. Then the $(i,j)$-th entry of $D_{n,k}(x,q)$ is equal to 0 for all $k+i+1 \leq i \leq k+2l+1 \leq n$ and $j \geq l+1$. Therefore the $(k+l+1)$-st row up to the $(k+2l+1)$-st row are linear dependent and the determinant is henceforth equal to 0. \hfill $\square$

**Lemma 8.** Let $n,k$ be non-negative integers, then the following identity holds

\[
d_{n,-k}(x,q) = (-1)^{n(k-1)}q^{-nk}d_{n,k}(x,q).
\]
Proof. Let $\sigma \in S_n$ be a permutation, then holds
\[
\prod_{i=1}^{n} \left( x + i + \sigma(i) - 2 \right) = \prod_{i=1}^{n} \frac{(x + i + \sigma(i) - 2)!}{(\sigma(i) - 1)!(x + 1 - i)!}
= \prod_{i=1}^{n} \frac{(x + i + \sigma(i) - 2)!}{(i - 1)!(x + 1 - \sigma(i))!}
= \prod_{i=1}^{n} \left( x + \sigma(i) + i - 2 \right).
\]
This implies that we can replace the binomial coefficient $\binom{x+j-2}{j-1}$ in the determinant $d_{n,k}(x,q)$ with $\binom{x+i+j-2}{i-1}$ without changing the determinant. By using this fact and pulling out the factor $-(-q)^j$ from the $j$-th column and $-(-q)^{i-k}$ from the $i$-th row for all rows and columns, we obtain
\[
d_{n,-k}(x,q) = \det_{1 \leq i,j \leq n} \left( \binom{x + i + j - 2}{j - 1} \frac{1 - (-q)^{j-i-k}}{1 + q} \right)
= \prod_{i=1}^{n} (-q)^{-i-k} \prod_{j=1}^{n} (-1)^{j} \det_{1 \leq i,j \leq n} \left( \binom{x + i + j - 2}{i - 1} \frac{1 - (-q)^{i+k-j+1}}{1 + q} \right)
= (-1)^{n(k-1)} q^{-nk} \det(D_{n,k}(x,q)^T) = (-1)^{n(k-1)} q^{-nk} d_{n,k}(x,q).
\]

\[\square\]

Corollary 9. Let $n$ be an odd positive integer, then holds $d_{n,0}(x,q) = 0$.

With the above two lemmas at hand we can prove the following structural theorem.

**Theorem 10.** The determinant $d_{n,k}(x,q)$ has the form
\[
d_{n,k}(x,q) = q^{c_q(n,k)} p_{n,k}(x) f_{n,k}(x,q),
\]
with
\[
p_{n,k}(x) = \prod_{i=1}^{n-1} \left| \frac{x}{i!} \right| \prod_{i=0}^{n-|k|-1} (x + |k| + 2i + 1),
\]
\[
c_q(n,k) = \begin{cases} 0 & k > 0, n \leq k, \\ nk & k < 0, n \leq -k, \\ -\sum_{i=1}^{n-|k|} \left| \frac{x}{i} \right| & \text{otherwise,} \end{cases}
\]
and $f_{n,k}(x,q)$ being a polynomial in $x$ and $q$ satisfying for positive $k$ the recursions
\[
f_{n,-k}(x,q) = (-1)^n f_{n,k}(x,q),
\]
\[
f_{2n,0}(x,q)f_{2n-2,0}(x+2,q) = -f_{2n-1,1}(x+1,q)^2,\]
\[
f_{2n,0}(x,q)f_{2n,0}(x+2,q) = f_{2n,1}(x+1,q)^2,
\]
\[
f_{n,k}(x,q) = \frac{1}{f_{n-2,k}(x+2,q) \left( \frac{n-1}{2} \right)^{|n/2|+1}} \times \left( (x+n)^{|n/2|+(2n+k+1)} (q(x+n+1))^{n \in 2N+k+2} f_{n-1,k}(x,q)f_{n-1,k}(x+2,q) - (x+1)f_{n-1,k-1}(x+1,q)f_{n-1,k+1}(x+1,q) \right),
\]
where $[\text{statement}]$ is 1 if the statement is true and 0 otherwise.

Proof. Since $d_{n,k}(x,q)$ is a polynomial in $x$ and a Laurent polynomial in $q$ we can write $d_{n,k}(x,q)$ as in (7) where $f_{n,k}(x,q)$ is a rational function in $x$ and $q$. In Lemma 7 we proved that $\frac{d_{n,k}(x,q)}{p_{n,k}(x)}$ is a polynomial in $x$. 

As already stated, we write $d_{n,k}(x,q)$ as

$$d_{n,k}(x,q) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \left( \frac{x + i + \sigma(i) - 2}{\sigma(i) - 1} \right)^{a_k + \sigma(i) - i},$$

(12)

with

$$a_j = \begin{cases} \sum_{i=0}^{j-1}(-q)^i & j > 0, \\ 0 & j = 0, \\ -\sum_{i=1}^{-j}(-q)^{-i} & j < 0. \end{cases}$$

Let $\sigma \in \mathfrak{S}_n$. The exponent of the smallest power of $q$ that appears in the summand associated to $\sigma$ in (12) is

$$\sum_{i:k+\sigma(i) - i < 0} (k + \sigma(i) - i).$$

(13)

It is an easy proof for the reader to show that the minimum of (13) for all $\sigma \in \mathfrak{S}_n$ is exactly $c_q(n,k)$. Hence $q^{-c_q(n,k)}d_{n,k}(x,q)$ is a polynomial in $q$ implying that $f_{n,k}(x,q)$ is a polynomial in $x$ and $q$.

It remains to prove that $f_{n,k}$ satisfies the equations (9) – (11), where (9) is a direct consequence of Lemma 8. For the others we use induction on $n$. Setting $k = 0$ and using (6) and the induction hypothesis immediately implies (9) and (10). Now let $k \geq 1$. Using the induction hypothesis, (6) becomes

$$d_{n,k}(x,q)q^{c_q(n-2,k)} \prod_{i=1}^{n-3} \left( \frac{i}{q^i} \right) \prod_{i=0}^{\frac{n-k}{2} - 1} (x + k + 2i + 3)f_{n-2,k}(x + 2, q)$$

$$= \prod_{i=1}^{n-2} \left( \frac{i}{q^i} \right)^2 \frac{1}{(n-1)}$$

$$\times \left( (x + n)q^{2c_q(n-1,k)} \prod_{i=0}^{\frac{n-k}{2} - 1} (x + k + 2i + 1)(x + k + 2i + 3)f_{n-1,k}(x, q)f_{n-1,k}(x + 2, q)$$

$$- (x + 1)q^{c_q(n-1,k-1)+c_q(n-1,k+1)} \prod_{i=0}^{\frac{n-k}{2} - 1} (x + k + 2i + 1)$$

$$\times \prod_{i=0}^{\frac{n-k}{2} - 1} (x + k + 2i + 3)f_{n-1,k-1}(x + 1, q)f_{n-1,k+1}(x + 1, q) \right).$$

After cancellation we obtain

$$d_{n,k}(x,q)f_{n-2,k}(x + 2, q) = \frac{q^{c_q(n,k)}p_{n,k}(x)}{(n-1)!}$$

$$\times \left( (x + n)^{\lfloor n\in\mathbb{N}\setminus(2N+k+1)\rfloor}(x + n + 1)^{\lfloor n\in\mathbb{N}\setminus(2N+k+2)\rfloor}q^{\lfloor n\in\mathbb{N}\setminus(2N+k+2)\rfloor}f_{n-1,k}(x, q)f_{n-1,k}(x + 2, q)$$

$$- (x + 1)f_{n-1,k-1}(x + 1, q)f_{n-1,k+1}(x + 1, q) \right).$$

By replacing $d_{n,k}(x,q)$ by $q^{c_q(n,k)}p_{n,k}(x)f_{n,k}(x,q)$, this implies the last recursion (11). \qed

Computer experiments suggest that $f_{n,k}(x,q)$ is a polynomial with integer coefficients and that the leading coefficient is either 1 or $-1$, where we order the monomials $x^aq^b$ with respect to the lexicographic order of $(a,b)$. We can not prove this, but we can prove the following related statement.
Proposition 11. The leading coefficient of $f_{n,1}(x,q)$ is 1, where we order the monomials $x^aq^b$ with respect to the reverse lexicographic order of $(a,b)$.

Proof. In order to prove this we first calculate the leading coefficient of the highest power of $q$ in $d_{n,1}(x,q)$ which is a polynomial in $x$ and then calculate the leading coefficient of this polynomial.

We will need the determinantal evaluations

$$\det_{1 \leq i,j \leq n} \left( \frac{(j-1)!}{(a+j-1)!} \prod_{i=1}^a (x+j-i), \left( (-1)^{i+j}(x+i+j-2) \right) \right) \prod_{j=1}^n (a+j-1)! \prod_{i=1}^a (x+j-i),$$

which are variations of the Vandermonde determinant evaluation. As already stated in (12) the determinant $d_{n,1}(x,q)$ can be written as

$$d_{n,1}(x,q) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n (x+i+\sigma(i)-2)_{\sigma(i)-1} a_{1+\sigma(i)-i}.$$ (15)

Let $\sigma \in \mathcal{S}_n$ and $i_1, \ldots, i_l$ be the rows such that $\sigma(i) - i < 0$, then the following is the exponent of the highest $q$ power that appears in the summand associated to $\sigma$ in (15)

$$-l + \sum_{i : \sigma(i) - i > 0} (\sigma(i) - i) = -l - \sum_{j=1}^l (i_j - \sigma(i_j)) \leq -l + l(n - l).$$

It is obvious that there exists a $\sigma \in \mathcal{S}_n$ such that the above inequality is sharp. The maximal exponent is reached for $l = \frac{n-1}{2}$ if $n \equiv 1 \mod 2$ or in the two cases $l = \frac{n-2}{2}$ or $l = \frac{n}{2}$ for $n \equiv 0 \mod 2$. First let $n \equiv 1 \mod 2$ and hence $l = \frac{n-1}{2}$. The maximal $q$ power is reached for all $\sigma \in \mathcal{S}_n$ with $i_j = j$ and $\sigma(i_j) \leq l$ for all $1 \leq j \leq l$. Hence the coefficient of the maximal $q$ power in $d_{n,1}(x,q)$ can be written as

$$\det_{1 \leq i,j \leq l} \left( \left( x+n-l+i+j-2 \right) \right) \prod_{i=1}^{n-l} (i-1)! \prod_{i=1}^l (l+i-1)!.$$ (16)

By (13) the leading coefficient of the above determinants as polynomials in $x$ is equal to

$$\prod_{i=1}^{n-l} \frac{(i-1)!}{(l+i-1)!},$$

By simple manipulations and using $l = \frac{n-1}{2}$ this transforms to

$$\prod_{i=1}^{n-1} \frac{\left( \frac{n}{2} \right)!}{\left( \frac{n}{2} \right)!},$$

which proves the claim for odd $n$.

Now let $n = 2a$, then the maximal $q$ power is obtained for $l = a$ or $l = a - 1$. Analogously to the above case we can write the coefficient of the highest power of $q$ as

$$\det_{1 \leq i,j \leq a} \left( \left( \begin{array}{c} 0 \\ x+a+i+j-2 \end{array} \right) \text{ otherwise,} \right) \det_{1 \leq i,j \leq a} \left( \left( (-1)^{i+j+a}(x+i+j+a-2) \right) \right)$$

$$+ \det_{1 \leq i,j \leq a-1} \left( x+a+i+j-1 \right) \prod_{j=1}^a \left( \left( \begin{array}{c} 0 \\ x+i+j+a-3 \end{array} \right) \text{ otherwise,} \right).$$
We can rewrite this by using Laplace expansion
\[
\frac{\det_{1 \leq i,j \leq a} \left( x + a + i + j - 2 \right)}{\det_{1 \leq i,j \leq a} \left( j - 1 \right)} \frac{\det_{1 \leq i,j \leq a} \left( (-1)^{i+j+a} \left( x + i + j + a - 2 \right) \right)}{\det_{1 \leq i,j \leq a} \left( a + j - 1 \right)} + (-1)^a \frac{\det_{1 \leq i,j \leq a-1} \left( x + a + i + j - 1 \right)}{\det_{1 \leq i,j \leq a-1} \left( j - 1 \right)} \frac{\det_{1 \leq i,j \leq a+1} \left( (-1)^{i+j+a} \left( x + i + j + a - 2 \right) \right)}{\det_{1 \leq i,j \leq a+1} \left( a + j - 1 \right)} - (-1)^a \frac{\det_{1 \leq i,j \leq a-1} \left( x + a + i + j - 1 \right)}{\det_{1 \leq i,j \leq a-1} \left( j - 1 \right)} \frac{\det_{1 \leq i,j \leq a} \left( (-1)^{i+j+a} \left( x + i + j + a - 2 \right) \right)}{\det_{1 \leq i,j \leq a} \left( a + j - 1 \right)}.
\]

The second and fourth line cancel each other and the degree of \( x \) is by (11) in the first line \( a^2 \) and for the third line \( a^2 - 1 \). Hence the leading coefficient is
\[
\prod_{i=1}^{a} \frac{(i-1)!}{(a+i-1)!} = \prod_{i=1}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor!}{i!},
\]
where the equality is an easy transformation.

One could extend the above proof to the calculation of the leading coefficient of \( f_{n,k}(x,q) \) for arbitrary positive \( k \) where we order the monomials \( x^a q^b \) with respect to the reverse lexicographic order of \( (a, b) \). However the leading coefficient of \( f_{n,k}(x,q) \) will not be equal to \pm 1 for \( k \neq 1 \).

It seems that the polynomial \( f_{n,k}(x,q) \) is for general \( k \) irreducible over \( \mathbb{Q}[x,q] \). While this appears to be impossible to prove, Theorem 14 implies that \( p_{n,k}(x) \) is maximal in \( f_{n,k}(x,q) \), i.e., there exists no non-trivial polynomial \( p'(x) \in \mathbb{Q}[x] \) dividing \( f_{n,k}(x,q) \). For small values of \( k \) on the other side we could prove a closed product formula of \( f_{n,k}(x,q) \) which was already suggested to exist in [10].

**Proposition 12.** The function \( f_{n,k}(x,q) \) has in the special case for \( k = 0, 1 \) the form
\[
f_{n,0}(x,q) = \begin{cases} 0 & n \equiv 1 \text{ mod } 2, \\ (-1)^{\frac{n}{2}} F_{\frac{n}{2}}(x + 1, q)^2 & n \equiv 0 \text{ mod } 2, \end{cases}
\]
\[
f_{n,1}(x,q) = F_{\left\lfloor \frac{n+1}{2} \right\rfloor}(x,q) F_{\left\lfloor \frac{n}{2} \right\rfloor}(x+2,q),
\]
where \( F_n(x,q) \) is a polynomial in \( x \) and \( q \) over \( \mathbb{Q} \).

**Proof.** Corollary 3 and the equations 9 and 10 imply the statement.

For \( k = 2 \) computer experiments still suggest a decomposition of \( f_{n,k}(x,q) \) into two or three factors, depending on the parity of \( n \), whereas for \( k \geq 3 \) the polynomial \( f_{n,k}(x,q) \) seems to be irreducible over \( \mathbb{Q}[x,q] \). In order to prove the factorisation for \( k = 2 \) one would need to show that the rational function
\[
\frac{q(x + 2n + 1)(x + 2n + 2) F_{n}(x,q) F_{n}(x+4,q) - n F_{n+1}(x,q) F_{n-1}(x+4,q)}{F_{n}(x+2,q)}
\]
is a polynomial in \( x \) and \( q \). Further if one could guess the resulting polynomial, one would obtain a recursion for \( F_n(x,q) \).

The following corollary is a direct consequence of Theorem 10 and the above proposition.
Corollary 13. Set $Q = 2 + q + q^{-1}$ and define $\tilde{p}_n(q)$ as the Laurent polynomial

\[
\tilde{p}_{2n}(q) := 2q^{-\left\lfloor \frac{n}{2} \right\rfloor} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{(i+1)_i} F_n(0, q),
\]

\[
\tilde{p}_{2n+1}(q) := \frac{1}{2} q^{-\left\lfloor \frac{n}{2} \right\rfloor} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (i)_i F_n(2, q),
\]

where $(x)_j := x(x+1) \cdots (x+j-1)$ is the Pochhammer symbol. Then the $Q$-enumeration of ASMs $A_n(Q)$ is given by

\[
A_{2n}(Q) = 2\tilde{p}_{2n}(q)\tilde{p}_{2n+1}(q),
\]

\[
A_{2n+1}(Q) = \tilde{p}_{2n+1}(q)\tilde{p}_{2n+2}(q).
\]

It is an easy proof to the reader that the Laurent polynomials $\tilde{p}_n(q)$ are actually polynomials in $Q$. The above corollary was actually conjectured in [15, Conjecture 4] and was first proven in [13].

4. The 0-, 1-, 2-, 3- and 4-Enumeration of ASMs

In the following we prove factorisations of the determinant $d_{n,k}(x, q)$ where $q$ is a primitive first, second, third, fourth or sixth root of unity. As a consequence of these factorisations we obtain the known formulas for the 1-, 2- and 3-enumeration of ASMs. The following table shows the connection between the specialisation of $q$ and the weighted enumeration of ASMs.

| Enumeration | $q$ | Notes |
|-------------|-----|-------|
| 0-enumeration | $q = -1$ | (primitive second root of unity), |
| 1-enumeration | $q = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ | (primitive third root of unity), |
| 2-enumeration | $q = \pm i$ | (primitive fourth root of unity), |
| 3-enumeration | $q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ | (primitive sixth root of unity), |
| 4-enumeration | $q = 1$ | (primitive first root of unity). |

The factorisations in the following theorems can be proven (except for $q = -1$) by induction on $n$ together with [6] and Corollary [9] and cancellation of terms. The case $q = -1$ however is proven solely by using row manipulations. We remind the reader of the definition of the Pochhammer symbol $(x)_j := x(x+1) \cdots (x+j-1)$.

Theorem 14. For $q = -1$ holds

\[
d_{n,1}(x, -1) = \left( 2 \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \right) !! \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (x+2i).
\]

Proof. The limit of $d_{n,1}(x, q)$ for $q$ to $-1$ is

\[
\lim_{q \to -1} d_{n,k}(x, q) = \left( \frac{x+i+j-2}{j-1} \right)_{1 \leq i,j \leq n}.
\]

The following identity of matrices can be shown by using a variant of the Chu–Vandermonde identity.

\[
\begin{align*}
&\left((-1)^{i+j} \binom{i-1}{j-1}\right)_{1 \leq i,j \leq n} \times \left( \frac{x+i+j-2}{j-1} \right)_{1 \leq i,j \leq n} \\
&= \left( \delta_{i,j} - \delta_{i,j+1} \left( \frac{j}{x+j} \right)_{j \equiv 0(2)} \right)_{1 \leq i,j \leq n} \times \left( \frac{x+j-1}{j-i} \right)_{j \equiv 1(2)} \left( (x+j)_{j \equiv 0(2)} \right)_{1 \leq i,j \leq n}.
\end{align*}
\]

The closed product formula of $d_{n,1}(x, -1)$ is implied by taking the determinant on both sides. □
Theorem 15. Let \( q \) be a primitive third root of unity, then holds

\[
d_{n, 6k+1}(x, q) = 2 \left[ \frac{x}{2} \right] \left[ \frac{n+1}{2} \right] \prod_{i=1}^{n} \left[ \frac{i-1}{2} \right] \prod_{i=0}^{\infty} \left( \frac{x}{2} + 3i \right) \left( \frac{x}{2} + 3i + 3 \right) \frac{n}{2} \left[ \frac{n}{2} + i \right] \prod_{i=0}^{\infty} \left( \frac{x}{2} + n - i \right) \left( \frac{x}{2} + n - i + \frac{1}{2} \right) \left[ \frac{n}{2} - i \right],
\]

\[
d_{n, 6k+2}(x, q) = 2 \left[ \frac{x}{2} \right] \left[ \frac{n+2}{2} \right] (1-q)^n \prod_{i=1}^{n} \left( \frac{x}{2} + n - i \right) \left[ \frac{n}{2} - i \right] \prod_{i=0}^{\infty} \left( \frac{x}{2} + 3i + \frac{3}{2} \right) \left( \frac{x}{2} + 3i + 5 \right) \left[ \frac{n+3}{2} - i \right],
\]

\[
d_{n, 6k+3}(x, q) = (-q)^n 2 \left[ \frac{x}{2} \right] \left[ \frac{n+1}{2} \right] \prod_{i=1}^{n} \left( \frac{x}{2} + n - i \right) \left[ \frac{n}{2} - i \right] \prod_{i=0}^{\infty} \left( \frac{x}{2} + 3i \right) \left( \frac{x}{2} + 3i + 7 \right) \left[ \frac{n+3}{2} - i \right],
\]

\[
d_{n, 6k+4}(x, q) = q^{-n} d_{n, 6k+2}(x, q),
\]

\[
d_{n, 6k+5}(x, q) = q^{-n} d_{n, 6k+1}(x, q),
\]

\[
d_{2n+1, 6k}(x, q) = 0,
\]

\[
d_{2n, 6k}(x, q) = q^{2n^2} \prod_{i=1}^{n} \left( \frac{x}{2} + 3i + 1 \right) \left( \frac{x}{2} + 3i + 7 \right) n-2i \left( \frac{x}{2} + 2n - i \right)^2 n-2i-1.
\]

Corollary 16. The number \( A_n \) of ASMs of size \( n \) is given by

\[
A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.
\]

Proof. Reorder the terms of the product formula of \( d_{n,1}(0, q) \), where \( q \) is a primitive third root of unity. \( \square \)
Theorem 17. Let $q$ be a primitive fourth root of unity. Then the following holds

\[ d_{2n+1,4k}(x, q) = 0, \]

\[ d_{2n,4k}(x, q) = 2^n \prod_{i=1}^{2n-1} \frac{[\frac{i}{2}]}{i!} q^n \prod_{i=1}^{n} \left( \frac{x}{2} + 2i + \frac{1}{2} \right) \prod_{i=0}^{n-1} \left( \frac{x}{2} + 2i + \frac{1}{2} \right)^{2n-4i+1}, \]

\[ d_{n,4k+1}(x, q) = 2 \prod_{i=1}^{n} \frac{[\frac{i}{2}]}{i!} q^n \prod_{i=1}^{\frac{n-1}{2}} \left( \frac{x}{2} + i \right)^{n-2i+1}, \]

\[ d_{n,4k+2}(x, q) = \prod_{i=1}^{n} \left( \frac{x}{2} + i + \frac{1}{2} \right)^{\frac{n}{2}} \prod_{i=1}^{\frac{n}{2}} \left( \frac{x}{2} + i + \frac{1}{2} \right), \]

\[ d_{n,4k+3}(x, q) = (-q)^n d_{n,4k+1}(x, q). \]

Corollary 18. The 2-enumeration $A_n(2)$ of ASMs of size $n$ is given by

\[ A_n(2) = 2^{\binom{n}{2}}. \]

Proof. Reorder the terms of the product formula of $d_{n,1}(0, q)$, where $q$ is a primitive fourth root of unity. \qed

Theorem 19. Let $q$ be a primitive sixth root of unity and $k$ an integer. Then the following holds

\[ d_{2n+1,3k}(x, q) = 0, \]

\[ d_{2n,3k}(x, q) = c(2n) \prod_{i=0}^{n} (x + 1 + 3i) \prod_{i=1}^{n} (x + 3i)^{2(n-i)}, \]

\[ d_{n,3k+1}(x, q) = c(n) \prod_{i=0}^{\frac{n}{2}} (x + 2 + 3i)^{n-2i}, \]

\[ d_{n,3k+2}(x, q) = q^n d_{n,3k+1}(x, q), \]

with

\[ c(n) = \begin{cases} \frac{3}{3} \prod_{i=0}^{\frac{n}{2}} \frac{\frac{3}{2}}{i!} & \text{n is even,} \\ \frac{3}{3} \prod_{i=0}^{\frac{n}{2}} \frac{\frac{3}{2}}{i!} & \text{otherwise.} \end{cases} \]

Corollary 20. The 3-enumeration $A_n(3)$ of ASMs of size $n$ is given by

\[ A_{2n+1}(3) = 3^{\binom{n+1}{2}} \prod_{i=1}^{n} \frac{(3i - 1)^2}{(n + i)!}, \]

\[ A_{2n}(3) = 3^{n-1} (3n - 1)! (n - 1)! \frac{(2n - 1)!}{(2n)!} A_{2n-1}(3). \]

Proof. Reorder the terms of the product formula of $d_{n,1}(0, q)$, where $q$ is a primitive sixth root of unity. \qed

Theorem 21. For $q = 1$ holds

\[ d_{n,2k+1}(x, 1) = \prod_{i=2}^{\frac{n}{2}} (2i - 1)^{-\frac{n}{2} + 1} \prod_{i=1}^{n} (x + 2i) p_{n}(x) p_{n-1}(x), \]

\[ d_{2n,2k}(x, 1) = (-1)^n \prod_{i=2}^{n} (2i - 1)^{-\frac{n}{2} + 1} \prod_{i=1}^{n} (x + 2i - 1) p_{2n}(x - 1)^2, \]

\[ d_{2n+1,2k}(x, 1) = 0, \]
where \( p_n(x) \) is a polynomial in \( x \) satisfying the following recursion

\[
p_1(x) = 1, \\
p_3(x) = 2x + 5, \\
p_{2n}(x) = p_{2n-1}(x + 2), \\
p_{2n+1}(x) = \frac{(x + 2n + 1)(x + 2n + 2)p_{2n-1}(x)p_{2n-1}(x + 4) - (x + 1)(x + 2)p_{2n-1}(x + 2)^2}{2np_{2n-3}(x + 4)}.
\]

Computer experiments suggest that the polynomials \( p_n(x) \) are irreducible over \( \mathbb{Q} \).

5. CONNECTIONS TO THE ANDREWS DETERMINANT

In \cite{10} Fischer conjectured that \( d_{n,1}(x, q) \) is connected to the determinant

\[
\det_{1 \leq i,j \leq n} \left( \begin{array}{c} x + i + j - 2 \\ j - 1 \end{array} \right) + q \delta_{i,j}.
\]

(16)

It was shown by Andrews \cite{11} that for \( q = 1 \) the above determinant has a closed product formula and that it counts for \( x = 2 \) the number of descending plane partitions (DPPs) with parts less than \( n \). In \cite{13}, it was shown by Ciucu et al. that the above determinant is equal to the weighted enumeration of cyclically symmetric lozenge tilings of a hexagon with side lengths \( n, n, n, n + x, n, n + x \) and a central hole of side length \( x \); these objects generalise DPPs. Further they proved that the evaluation of the determinant can be expressed by a closed product formula if \( q \) is a sixth root of unity. Comparing the factorisations of \( d_{n,k}(x, q) \) and of the above determinant implies

\[
d_{n,3-k}(x, q^2) = q^{-n} \det_{1 \leq i,j \leq n} \left( \begin{array}{c} x + i + j - 2 \\ j - 1 \end{array} \right) + q^k \delta_{i,j},
\]

where \( q \) is a primitive sixth root of unity. This implies that all factorisations of the determinant in (16) are covered by the determinant \( d_{n,k}(x, q) \), where \( q \) is a third root of unity.

Very recently it was shown in \cite{9} by Fischer that for \( q = 1 \) the determinant in (116) also enumerates \( (n, x) \)-alternating sign trapezoids, which generalise alternating sign triangles (ASTs). This implies that not only DPPs and ASTs (and hence ASMs) are equinumerous but also a “one parameter generalisation” of DPPs and ASTs. An interpretation of \( x \) in \( d_{n,k}(x, q) \), e.g. in form of a one parameter generalisation of ASMs, would be very interesting and could give important insight on the nature of the equinumerosity between ASMs, DPPs and ASTs. Such possible interpretations of \( x \) will be studied in a forthcoming work.

APPENDIX A. AN OVERVIEW OF ENUMERATION FORMULAS CONNECTED TO \( d_{n,k}(x, q) \)

It turns out that some specialisations of \( x, q, k \) in \( d_{n,k}(x, q) \) are known enumeration formulas. The following is a summary of the specialisations we found

\[
d_{n,1}(0, \zeta_3) = \zeta_6^{n-1} d_{n-1,3}(2, \zeta_3) = \#(\text{ASMs of size } n),
\]

(17)

\[
\zeta_6^{n-1} d_{n-1,3}(0, \zeta_3) = \#(\text{cyclic symmetric plane partitions in an } n - \text{cube})
= \#(\text{half turn symmetric ASMs of size } 2n)/\#(\text{ASMs of size } n),
\]

(18)

\[
\zeta_6^{n-1} d_{2(n-1),2}(1, \zeta_3) = \frac{n+1}{\sqrt{-3}} d_{2n-1,2}(-1, \zeta_3) = \#(\text{ASMs with two U-turn sides of size } 4n),
\]

(19)

\[
\frac{n+1}{\sqrt{-3}} d_{2n-1,2}(-2, \zeta_3) = \#(\text{quarter turn symmetric ASMs of size } 4n),
\]

(20)
A NEW DETERMINANT FOR THE $Q$-ENUMERATION OF ASMS

\[ d_{n,1}(0, \zeta_4) = (2\text{-}enumeration \text{ of ASMs of order } n) \]

\[ = \#(\text{perfect matchings of an order } n \text{Atzec diamond}) \]

\[ = \#(\text{Gelfand-Tsetlin patterns with bottom row } 1, 2, \ldots, n) = 2(n + 1), \]

\[ \zeta_4^n d_{2n,2}(1, \zeta_4) = \frac{\zeta_8^{2n+1}}{\sqrt{2}} d_{2n+1,2}(-1, \zeta_4) = 4n^2, \]

\[ \frac{\zeta_8^{2n-1}}{\sqrt{2}} d_{2n-1,2}(1, \zeta_4) = 4(n+1), \]

\[ d_{n,1}(0, \zeta_6) = \#(3\text{-}enumeration \text{ of ASMs of size } n), \]

where \( \zeta_l \) denotes the \( l \)-th root of unity \( \zeta_l = e^{2\pi i l} \).

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