Conservative Force Fields in Nonextensive Kinetic Theory

J.A.S. Lima*, J.R.Bezerra†, and R.Silva‡

Università Federal do Rio Grande do Norte,
Departamento de Física, Caixa Postal 1641,
59072-970 Natal, RN, Brazil

(March 22, 2022)

Abstract

We investigate the nonextensive $q$-distribution function for a gas in the presence of an external field of force possessing a potential $U(r)$. In the case of a dilute gas, we show that the power law distribution including the potential energy factor term can rigorously be deduced based on kinetic theoretical arguments. This result is significant as a preliminary to the discussion of long range interactions according to nonextensive thermostatistics and the underlying kinetic theory. As an application, the historical problem of the unbounded isothermal planetary atmospheres is rediscussed. It is found that the maximum height for the equilibrium atmosphere is $z_{\text{max}} = K_B T/mg(1-q)$. In the extensive limit, the exponential Boltzmann factor is recovered and the length of the atmosphere becomes infinite.

Keywords: Nonextensive statistics; Kinetic theory; Power laws

*limajas@dfte.ufrn.br
†zero@dfte.ufrn.br
‡raimundo@dfte.ufrn.br
In gas dynamics it is very important to investigate how the molecular motion is modified by force-fields different from those exerted by the containing vessel or even by the other particles of the gas. Particular examples are ions in an external magnetic field and a gas in the earth’s gravitational field [1,2].

As widely known, a classical gas under steady state conditions and immersed in a conservative force field, \( \mathbf{F} = -\nabla U(\mathbf{r}) \), is described by a distribution function that differs from the Maxwellian velocity distribution by an extra exponential factor involving the potential energy. In this case, the total equilibrium distribution function reads

\[
f(\mathbf{r}, v) = n_o \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{\frac{1}{2}mv^2 + U(\mathbf{r})}{k_B T} \right),
\]

where \( m \) is the mass of the particles, \( T \) is the temperature and \( n_o \) is the particle number density in the absence of the external force field. In addition, since this distribution function is normalized, it is easy to see that the number density is given by

\[
n(\mathbf{r}) = n_o \exp \left[ \frac{-U(\mathbf{r})}{k_B T} \right],
\]

where the factor, \( \exp[-U(\mathbf{r})/k_B T] \), which is responsible for the inhomogeneity of \( f(\mathbf{r}, v) \), is usually called the Boltzmann factor. Expression (1) follows naturally from an integration of the collisionless Boltzmann’s equation

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}(\mathbf{r}) \cdot \partial f}{m} = 0,
\]

when stationary conditions are adopted along with the assumption that the total distribution can be factored

\[
f(\mathbf{r}, v) = f_0(v) \chi(\mathbf{r}),
\]

where \( f_0(v) \) represents the Maxwell equilibrium distribution function, and \( \chi(\mathbf{r}) \) is a scalar function of \( \mathbf{r} \). As one may show, after a simple normalization, the resulting expression for \( \chi(\mathbf{r}) \) is exactly the Boltzmann factor for the potential energy of the external field, namely:

\[
\chi(\mathbf{r}) = \exp \left[ \frac{-U(\mathbf{r})}{k_B T} \right].
\]
and combining this result with equation (4) the Boltzmann stationary distribution (1) is readily obtained.

On the other hand, recent efforts on the kinetic foundations of the \( q \)-nonextensive statistics proposed by Tsallis [3,4] lead to an equilibrium velocity distribution of the form [5,6] 

\[
f_0(v) = B_q \left[ 1 - (1 - q) \frac{mv^2}{2k_B T} \right]^{\frac{1}{1-q}}.
\] (6)

In this expression the \( q \)-parameter quantifies the nonadditivity property of the associated gas entropy whose main effect at the level of the distribution function is to replace the standard Gaussian form by a power law. The quantity \( B_q \) is a \( q \)-dependent normalization constant whose expression for \( 1/3 < q \leq 1 \) is given by

\[
B_q = n(1 - q)^{1/2} \frac{5 - 3q}{2} \frac{3 - q}{2} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left(\frac{m}{2\pi k_B T}\right)^{3/2},
\] (7)

which reduces to the Maxwellian result in the limit \( q = 1 \). As explained in Ref. [5], the above power law distribution can be deduced from two simple requirements: (i) isotropy of the velocity space, and (ii) a suitable nonextensive generalization of the Maxwell factorizability condition [7], or equivalently, the assumption that \( F(v) = f(v_x)f(v_y)f(v_z) \). For \( q > 0 \), the above distribution function satisfies a generalized H-theorem, and its reverse has also been proved, that is, the collisional equilibrium is given by the Tsallis’ \( q \)-nonextensive velocity distribution [6].

In the last few years, several applications of this equilibrium power law velocity distribution have been done in many disparate branches of physics [8-14]. It is worth notice that the widely spread belief that Tsallis equilibrium distribution cannot be applied to Hamiltonian systems seems to be a profound misunderstanding on the foundations of statistical mechanics and kinetic theory. Indeed, the BG canonical ensemble approach is valid only for sufficiently short-range interactions. It fails when gravitational or unscreened Coulombian fields are present, that is, to the class of systems where the usually assumed entropy additivity postulate is not valid. In other words, when the dynamics plays a nontrivial role, the system does not relaxes to the BG distribution, but they evolve to a non-Gaussian ve-
locity distribution which can be fitted by the monoparametric class of Tsallis’ nonextensive statistics (for a more detailed discussion see [15,16]).

In this letter we investigate how the potential energy term can rigorously be introduced in the \( q \)-distribution with basis on the kinetic approach. More precisely, it is shown that an analytical expression for the equilibrium distribution of a dilute gas under the action of a conservative field may also be calculated as a stationary solution of the collisionless Boltzmann equation. This result is significant as a preliminary to the discussion of long range interactions according to nonextensive thermostatistics and the underlying kinetic theory. As an illustration we discuss the case of planetary atmospheres. It is found that the problem of an infinite isothermal atmosphere is naturally solved in this extended framework. For a given value of \( q < 1 \), the extension of the atmosphere becomes finite and its length is uniquely determined by the temperature and mass of the molecular components.

Let us now consider a spatially inhomogeneous dilute gas supposed in equilibrium at temperature \( T \). It is immersed in a conservative external field in such a way that \( f(r,v)d^3vd^3r \) is the number of particles with velocity lying within a volume element \( d^3v \) about \( v \) and positions lying within a volume element \( d^3r \) around \( r \). In this case, we see from (7) that the stationary Boltzmann equation can be rewritten as

\[
\mathbf{v} \cdot \nabla_r f - \frac{1}{m} \nabla_r U \cdot \nabla_v f = 0.
\] (8)

In order to introduce the nonextensive effects we first recall that the factorizability condition cannot be applied in this extended framework. This means that the standard starting assumption (see equation (4)) must be somewhat modified. In the spirit of the \( q \)-nonextensive formalism, a consistent \( q \)-generalization of (4) is

\[
f(r,v) = B_q e_q \left[ \ln_q \left( \frac{f_0}{B_q} \right) + \ln_q \chi(r) \right],
\] (9)

where \( B_q \) is the constant expression given by (7) which has been introduced for mathematical convenience, and the functions \( q \)-exp and \( q \)-log, \( e_q(f), \ln_q(f) \), are defined by

\[
e_q(f) = [1 + (1 - q)f]^{1/1-q},
\] (10)
Note that in the limit $q \to 1$ the above identities reproduce the usual properties of the exponential and logarithm functions. In addition, $e_q(\ln_q f) = \ln_q(e_q(f)) = f$, and as should be expected from (9), the factored decomposition (4) is readily recovered in the extensive limit. In what follows, the properties of $q$-exponential and $q$-log differentiation

\[ \frac{d \ln_q f}{dx} = f^{-q} \frac{df}{dx}, \]

(12)

\[ \frac{de_q(f)}{dx} = e_q^q(f) \frac{df}{dx}, \]

(13)

will also be extensively used. In particular, for the total $q$-distribution (9), we obtain

\[ \nabla_v f = -e_q^q(x) \frac{mv}{k_B T}. \]

(14)

Now, substituting $\nabla_r f$ and $\nabla_v f$ into the stationary Boltzmann equation (8), and performing the elementary calculations one obtains

\[ \chi^{-q} \nabla \chi \cdot d\mathbf{r} = -\frac{1}{k_B T} \nabla U \cdot d\mathbf{r} \]

(15)

the solution of which is

\[ \chi(\mathbf{r}) = e_q\left(-\frac{U(\mathbf{r})}{k_B T} + C\right), \]

(16)

where $C$ is an arbitrary constant.

Now, inserting (16) into (9), and integrating the result in the velocity space it follows that

\[ \int B_q e_q \left[ \ln_q \left( \frac{f_0}{B_q} \right) - \frac{U}{k_B T} + C \right] d^3 v = n(\mathbf{r}). \]

(17)

Finally, by substituting the expression of $f_0$ and considering a region where $U(\mathbf{r}) = 0$, one finds

\[ B_q \int e_q \left(-\frac{mv^2}{2k_B T} + C\right) d^3 v = n_0, \]

(18)
and from normalization condition, \( n_0 = \int f_0(v)dv \), it follows that the unique allowed value for the integration constant is \( C = 0 \). Consequently, (16) becomes

\[
\chi(r) = e_q \left[ \frac{U(r)}{k_B T} \right],
\]

which is the \( q \)-generalized Boltzmann factor.

Finally, inserting this result into (9), we obtain the complete \( q \)-distribution function in the presence of an external field

\[
f(r, v) = B_q \left[ 1 - (1 - q) \left( \frac{mv^2}{2k_BT} + \frac{U(r)}{k_BT} \right) \right]^{1/(1-q)} \equiv B_q e_q (-E/k_BT), \tag{20}
\]

where \( E \) is the total energy of the particles. It thus follows that a generalized \( q \)-exp factor for Tsallis’ nonextensive thermostatistics can exactly be deduced if the standard approach is slightly modified. The particle number density, \( n(r) = \int f(r, v)dv \), also becomes a function of the position given by

\[
n(r) = n_0 \left[ 1 - (1 - q) \frac{U(r)}{k_BT} \right]^{(5-3q)/(2(1-q))} \equiv n_0 [e_q (-U/k_BT)]^{5-3q} , \tag{21}
\]

and as should be expected, in the extensive limit \( (q \to 1) \), the standard exponential Maxwellian expressions for \( f(r, v) \) and \( n(r) \) are readily recovered.

At this point it is interesting to analyze some application of the above result. We recall that the most popular problem of a gas in a force-field is the planetary atmosphere. In the standard simplified treatment, the temperature is uniform and the tree-dimensional motion occurs under the action of a constant gravitational field \( g \) along the z-direction. Thus, each molecule has a potential energy, \( U(z) = mgz \), so that the concentration of particles, or equivalently, the gas density \( (\rho = nm) \) is given by the familiar barometric formula (see Eq. (2)).

\[
\rho(z) = \rho_0 \exp \left[ -\frac{mgz}{k_BT} \right], \tag{22}
\]

with the pressure obeying a similar exponential law in virtue of the standard equation of state for a perfect gas. The simplest conclusion based on the above expression is that an
ideal isothermal atmosphere would extend to indefinite distances from the planet surface. This result disagree completely with the existing observations even by considering that the upper-most part of the earth atmosphere is nearly in thermodynamic equilibrium. Since there is no a natural boundary in this Boltzmann like picture, a possible solution to this problem is to assume that the atmospheres of the planets are actually finite, although being only in a pseudo-equilibrium state with the particles continuously streaming off into space at a very slow rate (evaporation).

How does this basic prediction is modified when an equilibrium $q$-distribution is assumed? The nonextensive answer to this question is immediate. The new distribution itself is given by (see (20))

$$f(r, v) = B_q \left[ 1 - (1 - q) \left( \frac{mv^2}{2k_B T} + \frac{mgz}{k_B T} \right) \right]^{1/(1-q)},$$

while from (21), the matter density assumes the form

$$\rho(z) = \rho_0 \left[ 1 - (1 - q) \frac{mgz}{k_B T} \right]^{(5-3q)/2(1-q)}.$$  

For comparison, in Fig. 1 we plot the gas density as a function of $z$ for some values of the $q$-parameter including the Boltzmannian case. Note that the standard exponential curve ($q = 1$) is replaced by the characteristic power law behavior of Tsallis’ nonextensive framework. In a more realistic treatment, as in the case of mixed atmospheres, the above distribution (23)-(25) may separately be applied to each component. As in the Boltzmann case, heavier gases are more concentrated near the surface of the earth than the lighter ones. In particular, this means that at greater heights the fraction of oxygen is smaller than helium as compared to the corresponding fractions at the surface. Such a result remains true as long as the maximum height allowed for a given component is not attained. In fact, for values of $q$ smaller than unity, the positiviness of the argument appearing in the power law implies that (24) exhibits a “thermal cut-off” to the allowed values of the $z$-coordinate which defines the length of the atmosphere. It thus follows that the maximum height of the atmosphere (or more generally, the length of any isothermal layer) is
\[
  z_{\text{max}} = \frac{k_B T}{m g (1 - q)} .
\]  

(25)

We see that for given values of the temperature and \( q < 1 \), the maximum height associated to a specific component is inversely proportional to its molecular mass. In table 1, we display the predicted values of \( z_{\text{max}} \) for a temperature \( T = 298,15 K \) and two different kind of gases (Oxygen and Hydrogen). In figure 1, one may see two different plots showing the finiteness of the atmosphere for values of \( q \) smaller than unity. Note that \( z_{\text{max}} \) increases without limit when \( q \to 1 \) recovering the Maxwellian result. As one may check, the predicted atmosphere is infinite for all values of \( q \geq 1 \) since the matter density, as given by equation (24), does not vanish at any finite height.

The above solution to the isothermal atmosphere with infinite length resembles a similar solution proposed in galactic dynamics to the collisionless stellar isothermal sphere. In the latter case, the Maxwell-Boltzmann distribution also predicts an infinite system whose integrated density profile yields an infinite total galactic mass. This theoretical problem was also solved with help of the nonextensive power law distribution [8]. In the case of planetary atmosphere, the nonextensive prediction of a finite length, as well as that heavier components are associated to a smaller maximum height could be tested by experiments using a mass spectrograph in atmospheric balloons.
TABLES

TABLE I. Limits to the height of the atmosphere

| Gas (\( T = 298,15 \, K \)) | \( q \) | \( z_{\text{max}} \) (Km) |
|-------------------------------|-------|-----------------------------|
| Hydrogen:.................... | 1.0   | \( \infty \)                |
|                               | 0.9   | 1239                        |
|                               | 0.8   | 619.5                       |
|                               | 0.7   | 413                         |
| Oxygen:....................... | 1.0   | \( \infty \)                |
|                               | 0.9   | 77.4                        |
|                               | 0.8   | 33.7                        |
|                               | 0.7   | 25.8                        |
FIG. 1. Maximum height of the isothermal atmosphere. The gas density as a function of $z$ for some selected values of the $q$ parameter. In the left panel, the solid line ($q = 1$) is the exponential curve for an infinite atmosphere predicted by the standard Boltzmann distribution. For values of $q < 1$ all curves decrease rapidly toward the maximum value of $z$ which is a physical cut-off present in the power law behavior. The right panel displays the logarithm of the gas density versus $z$ making more clear the existence of this cut-off. Note that the standard curve ($q = 1$) is a straight line going to infinite. However, for values of $q < 1$, there is a knee causing an intersection of the curves at finite values of the $z$ axis.

As a conclusion, we call attention for some simple physical situations where the power law distribution (24) can play an interesting role. The first example is the question concerning the escape of molecules to infinity. This phenomenon seems to be more easily understood in this extended framework since the power law distribution naturally defines a limiting boundary frontier where the atmosphere terminates (see Table 1 and Fig. 1). We recall that in the standard treatment, the rate of loss to infinity is based on the assumption that the upper part of the atmosphere extends in isothermal equilibrium only until a certain height.
In other words, unlike in the $q$-nonextensive framework, there is a cut-off introduced by hand in the infinity isothermal atmosphere predicted by the Boltzmann approach. A more quantitative analysis of evaporation taking into account the earth curvature will be presented in a forthcoming communication.

Other examples of forces, like the ones involving giroscopic terms may also be trivially added to the power law distribution (24). For rotating frames with constant angular velocity, for instance, the effect is just to add the term $-1/2m\omega^2l^2$ to the potential energy, where $l$ is the perpendicular distance from the axis of rotation. As usual, this term simulates a slight change in the potential energy due to gravity [17].

Finally, we also remark that for colloidal particles in solutions, the gravity acts on the particles which are buoyed up by the liquid in which they are suspended. As a consequence, instead of the weight $mg$ the net force is $mg(\rho_p - \rho_l)/\rho_p$, where $\rho_p$ and $\rho_l$ are the particles and liquid densities, respectively. In this case, the resulting nonextensive expression for $n(z)$ could experimentally be tested by repeating the experiments with colloidal suspensions as originally done by Perrin (see [18] and references there in). Naturally, the inclusion of other physical effects leading to independent probes of the $q$-statistics take even greater interest.

Acknowledgments: The authors are grateful to Odylio Aguiar for helpful discussions. This work was supported by Pronex/FINEP (No. 41.96.0908.00), Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq and CAPES (Brazilian Research Agencies).
REFERENCES

[1] Kerson Huang, *Statistical Mechanics* John Wiley & Sons, (1987).

[2] N. A. Krall and Trivelpiece, Principles of Plasma Physics, McGraw-Hill, Kogakusha (1973).

[3] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[4] See http.tsallis.cat.cbpf.br/biblio.htm for an extensive list of references.

[5] R. Silva, A. R. Plastino, and J. A. S. Lima, Phys. Lett. A **249**, 401 (1998).

[6] J. A. S. Lima, R. Silva, and A. R. Plastino, Phys. Rev. Lett. **86**, 2938 (2001).

[7] J. C. Maxwell, Philos. Mag. Ser. 4, **20**, 21 (1860).

[8] A. R. Plastino and A. Plastino, Phys. Lett. A **174**, 384 (1993).

[9] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati and C. Tsallis, Astroph. Lett. and Comm., **35**, 449 (1998).

[10] A. R. Plastino and J. A. S. Lima, Phys. Lett. A **260**, 46 (1999).

[11] M. Buiatti, P. Grigolini, and A. Montagnini, Phys. Rev. Lett. **82**, 3383 (1999).

[12] J. A. S. Lima, R. Silva, and Janilo Santos, Phys. Rev. E **61**, 3260 (2000).

[13] R.S. Mendes and C. Tsallis, Phys Lett A **285**, 273 (2001).

[14] G. Kaniadakis, Physica A **296**, 405 (2001).

[15] V. Latora et al., Phys. Rev. E **64** 056134 (2001); Physica A **305** 129 (2002).

[16] C. Tsallis, E. Borges, F. Baldovin, Physica A **305**, 1 (2002).

[17] L. D. Landau and E. M. Lifshitz, Mechanics, Pergamon Press, 1976.

[18] J. Perrin, Atoms, Constable & Co., London, 1920.