The characteristic classes and Weyl invariants of Spinor groups

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Abstract

Based on a pair of cohomology operations on so called δ2–formal spaces, we construct the integral cohomologies of the classifying spaces of the Lie groups Spin(n) and Spinc(n).

As applications, we introduce characteristic classes for the reduced topological KSpin theory, determine the ring of integral Weyl invariants of the group Spin(n), and generalize the classical Rokhlin Theorem.

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1 Introduction

The spin group Spin(n) is the universal cover of the special orthogonal group SO(n). The spinc group Spinc(n) is the central extension Spin(n) ×Z2 U(1) of SO(n) by the circle group U(1). In the first part of the paper we introduce a pair F = (α, γ) of secondary cohomology operations, which is applied to construct the integral cohomology rings of the classifying spaces BSpin(n) and BSpin(n).

The mod 2 cohomology of the space BSpin(n) has been determined by Borel [3] for n ≤ 10, and completed by Quillen in [31]. Concerning the integral cohomology H∗(BSpin(n)) partial results are known. In [36, Theorem (1.2)] Thomas calculated the cohomology H∗(BSpin(n)) in the stable range n = ∞, but the result was subject to the choice of two sequences {Φi}, {Ψi} of indeterminacies. Another inspiring approach is due to Benson and Wood [6]. By computing with the Weyl invariants a presentation of the ring H∗(BSpin(n)) is formulated in [6, Theorem 11.1], where explicit generators and relations are absent. Granted with the cohomology operation F = (α, γ) these uncertainties are clarified in our approach, see Remarks 8.6 and 9.6.

Knowing the integral cohomology of the classifying space BG of a Lie group G has direct consequences in geometry and representation theory. Precisely,

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assume that a minimal system \( \{ q_1, \ldots, q_n \} \) of generators of the ring \( H^*(B_G) \) has been specified. One can introduce the characteristic classes for a principle \( G \)-bundle \( \xi \) over a space \( X \) by setting

\[
q_r(\xi) := f_\xi^*(q_r) \in H^*(X), \quad 1 \leq r \leq m,
\]

where \( f_\xi : X \to B_G \) is the classifying map of the bundle \( \xi \); one obtains also the basic Weyl invariants of the group \( G \) as (see Lemma 8.1)

\[
d_r := B^r_*(q_r) \in H^*(B_T)^W, \quad 1 \leq r \leq m,
\]

where \( T \) is a maximal torus on \( G \), and the map \( B_T : B_T \to B_G \) is induced by the inclusion \( T \subset G \). For the classical groups \( G = U(n), SO(n) \) and \( Sp(n) \) these stories have been completed by the 1950’s [29, §4, §14, §15]. In the second part of the paper we implement these topics for the case \( G = Spin(n) \).

2 Statements of the main results

For a topological space \( X \) let \( Sq^h \) be the Steenrod squares on the mod 2 cohomology algebra \( H^*(X; \mathbb{Z}_2) \), and denote by \( \delta_m \) the Bockstein from the mod \( m \) cohomology \( H^*(X; \mathbb{Z}_m) \) to the integral cohomology \( H^{*+1}(X) \). For the homomorphisms of coefficients groups

\[
\theta : \mathbb{Z}_2 \to \mathbb{Z}_4 \text{ by } \theta(1) = 2, \text{ and } \rho_m : \mathbb{Z} \to \mathbb{Z}_m \text{ by } \rho_m(1) = 1,
\]

the same notion are applied to denote their induced maps on the cohomologies.

Let \( B : H^{2r}(X; \mathbb{Z}_2) \to H^{4r}(X; \mathbb{Z}_4) \) be the Pontryagin square [9]. For an even degree class \( u \in H^{2r}(X; \mathbb{Z}_2) \) there holds the following universal relation

\[
(2.1) \quad \delta_2(u \cup u) = 2\delta_4 B(u) \in H^{4r+1}(X) \text{ (see (3.1)).}
\]

**Definition 2.1.** The space \( X \) is called \( \delta_2 \)-formal if the relation \( \delta_2(u \cup u) = 0 \) holds for every \( u \in H^{2r}(X; \mathbb{Z}_2) \), \( r \geq 1 \).

It follows from (2.1) that, if \( X \) is a space whose integral cohomologies \( H^*(X) \) in degrees \( 4r + 1 \) have no torsion element of order 4, then \( X \) is \( \delta_2 \)-formal.

In particular, all the 1-connected Lie groups, the classifying spaces \( BSO(n) \), \( BSpin(n) \) and \( BSpin^c(n) \), are examples of \( \delta_2 \)-formal spaces.

To introduce the promised operations we notice that the Bockstein operator \( Sq^1 = \rho_2 \circ \delta_2 \) defines the following decomposition on the \( \mathbb{Z}_2 \) space \( H^*(X; \mathbb{Z}_2) \):

\[
H^*(X; \mathbb{Z}_2) = \ker Sq^1 \oplus S^2_1(X) \text{ with } S^2_1(X) = H^*(X; \mathbb{Z}_2)/\ker Sq^1.
\]

**Theorem A.** Let \( X \) be a \( \delta_2 \)-formal space. There exists a unique pair

\[
F : H^{2r}(X; \mathbb{Z}_2) \to H^{4r}(X; \mathbb{Z}_2) \times S^{4r}(X; \mathbb{Z}_2)
\]

of cohomological operations, written \( F(u) = (\alpha(u), \gamma(u)) \), \( u \in H^{2r}(X; \mathbb{Z}_2) \), that is subject to the following three properties

i) \( \alpha(u) \in \text{Im} \rho_4 \);

ii) \( B(u) = \alpha(u) + \theta(\gamma(u)) \);

iii) \( Sq^1(\gamma(u)) = Sq^{2r} Sq^1(u) + u \cup Sq^1(u) \).
The uniqueness assertion in Theorem A implies that the properties i), ii) and iii) can be taken as an axiomatic definition of the operation \( F \). In particular, since \( S^1 \) injects on \( S^2(X) \) while \( \gamma(u) \in S^2(X) \), the operator \( \gamma \) is determined by the equation iii) alone. Its iteration gives rise to the following notion.

**Definition 2.2.** For an even degree cohomology class \( u \in H^{2r}(X; \mathbb{Z}_2) \) of a \( \delta_2 \)-formal space \( X \), the sequence \( \{ u^{(0)}, u^{(1)}, u^{(2)}, \ldots \} \) of cohomology classes defined by \( u^{(0)} = u \), \( u^{(k+1)} = \gamma(u^{(k)}) \), is called the derived sequence of the class \( u \).

**Example 2.3.** For the \( \delta_2 \)-formal space \( \mathcal{B}_{SO(n)} \) let \( w_i \) be the \( i \)-th Stiefel–Whitney class of the canonical real \( n \)-bundle on \( \mathcal{B}_{SO(n)} \). Then

\[
H^*(\mathcal{B}_{SO(n)}; \mathbb{Z}_2) = \mathbb{Z}_2[w_2, \ldots, w_n].
\]

Since the \( Sq^k \) action on \( H^*(\mathcal{B}_{SO(n)}; \mathbb{Z}_2) \) is completely determined by the Wu-formula and the Cartan product formula, for a given class \( u \in H^{2r}(\mathcal{B}_{SO(n)}; \mathbb{Z}_2) \) one can solve the equation iii) in Theorem A to get \( \gamma(u) \) using the coefficients comparison method. In particular, for \( u = w_2 \) we obtain

\[
\begin{align*}
  w_2^{(1)} &= w_4, \\
  w_2^{(2)} &= w_8 + w_2w_6, \\
  w_2^{(3)} &= w_{16} + w_2w_{14} + w_4w_{12} + w_8w_{10} + w_2^2(w_{12} + w_2w_{10} + w_4w_8) + w_4w_6^2 + w_3(w_6 + w_2w_4) + w_2w_7,
\end{align*}
\]

and in general, if \( 2^k \leq n \), that

\[
w_2^{(k)} = w_{2^k} + w_2w_{2^k-2} + \cdots + w_{2^k-1}w_{2^k-1} + \text{higher terms}.
\]

In comparison with the solution [30] to the Peterson’s hit problem for the algebra \( H^*(\mathcal{B}_{SO(n)}; \mathbb{Z}_2) \) over the Steenrod algebra, above formulae reveals a remarkable phenomenon about the operator \( \gamma \): modulo the decomposable elements the derived sequence of \( w_2 \) consists of the 2-power Stiefel–Whitney classes:

\[
(2.2) \quad \{ w_2^{(0)}, w_2^{(1)}, w_2^{(2)}, \ldots \} \equiv \{ w_2, w_4, \ldots, w_{2^k}, 0, \ldots \}, l(n) = [\ln n]. \]

Property i) of Theorem A guaranties that the operator \( \alpha \) always admits an integral lift \( f_\alpha : H^{2r}(X; \mathbb{Z}_2) \to H^{4r}(X) \) (i.e. \( \alpha = \rho_4 \circ f_\alpha \)). In the case \( X = \mathcal{B}_{SO(n)} \) of our interest a canonical choice of such a \( f_\alpha \) can be easily formulated. Recall from Brown [17] and Feshbach [17] that the integral cohomology ring of the classifying space \( \mathcal{B}_{SO(n)} \) is

\[
(2.3) \quad H^*(\mathcal{B}_{SO(n)}) = \begin{cases} 
  \mathbb{Z}[p_1, p_2, \ldots, p_{2^k}, e_n] \oplus \tau(\mathcal{B}_{SO(n)}) & \text{if } n \text{ is even;} \\
  \mathbb{Z}[p_1, p_2, \ldots, p_{2^k-1}] \oplus \tau(\mathcal{B}_{SO(n)}) & \text{if } n \text{ is odd},
\end{cases}
\]

where \( \tau(X) \) denotes the torsion ideal of the cohomology \( H^*(X) \), \( 2\tau(\mathcal{B}_{SO(n)}) = 0 \), and where \( p_i \) (resp. \( e_n \)) is the \( i \)-th Pontryagin class (resp. the Euler class) of the canonical real \( n \)-bundle on \( \mathcal{B}_{SO(n)} \). In term of (2.3) we define the operation \( f : H^*(\mathcal{B}_{SO(n)}; \mathbb{Z}_2) \to H^*(\mathcal{B}_{SO(n)}) \) by the following practical rules:
There exists a unique sequence
\[ \{ q_r, r \geq 0 \} \] of integral cohomology classes on \( B_{\text{Spin}^c(n)} \), deg \( q_r = 2^{r+1} \), that satisfies the following system
\begin{align*}
\text{i}) & \quad q_0 = \iota^*(x), \\
\text{ii}) & \quad \rho_2(q_r) = \pi^*w_2^{(r)}; \\
\text{iii}) & \quad 2q_{r+1} + q_r^2 = \pi^*f(w_2^{(r)}), \quad r \geq 0.
\end{align*}

Based on Theorem A we shall show that

**Theorem B.** The pair \( (f, \gamma) \) of operations on \( H^*(B_{\text{SO}(n)}; \mathbb{Z}_2) \) satisfies, for any \( u \in S^*_2(B_{\text{SO}(n)}) \), that
\begin{align*}
i) & \quad \mathcal{B}(u) = \rho_4(f(u)) + \theta(\gamma(u)), \\
\text{ii}) & \quad Sq^4(\gamma(u)) = Sq^{2r} Sq^4(u) + u \cup Sq^1(u).
\end{align*}

**Application 2.4.** For the element \( u = w_{2r} \in S^*_2(B_{\text{SO}(n)}) \) we have \( f(w_{2r}) = p_r \) by the definition of \( f \). Solving the equation ii) of Theorem B by coefficients comparison yields that \( \gamma(w_{2r}) = w_{4r} + w_2w_{4r-2} + \cdots + w_{2r-2}w_{2r+2} \). Substituting these into formula i) of Theorem B we obtain
\[ \mathcal{B}(w_{2r}) = \rho_4(p_r) + \theta(w_{4r} + w_2w_{4r-2} + \cdots + w_{2r-2}w_{2r+2}). \]

This formula implies that the mod 4 reductions of the Pontryagin classes of a manifold are homotopy invariants. It was obtained by W.T. Wu in [10] by computing with the Schubert cell decomposition on \( B_{\text{SO}(n)} \). S.S. Chern suggested a different approach which was implemented by Thomas in [37]. Concerning this topic property i) of Theorem B may be called the generalized Wu–formula. □

The classifying spaces \( B_{\text{Spin}(n)} \) and \( B_{\text{Spin}^c(n)} \) fit into the fibered sequences
\begin{align*}
\text{(2.5)} & \quad \mathbb{C}P^\infty \xrightarrow{\iota} B_{\text{Spin}^c(n)} \xrightarrow{\pi} B_{\text{SO}(n)} \quad \text{and} \\
\text{(2.6)} & \quad U(1) \rightarrow B_{\text{Spin}(n)} \xrightarrow{\pi} B_{\text{Spin}^c(n)} \xrightarrow{\iota} \mathbb{C}P^\infty,
\end{align*}
where the maps \( \pi \) and \( \iota \) are induced by the obvious epimorphisms
\[ \text{Spin}(n) \times \mathbb{Z}_2 U(1) \rightarrow SO(n) \quad \text{and} \quad \text{Spin}(n) \times \mathbb{Z}_2 U(1) \rightarrow U(1), \]
respectively. On the other hand let \( \{ w_2, w_2^{(1)}, w_2^{(2)}, \cdots \} \subset S^*_2(B_{\text{SO}(n)}) \) be the derived sequence of the second Stiefel Whitney class \( w_2 \). Applying the operation \( f \) of Theorem B gives rise to the sequence \( \{ f(w_2), f(w_2^{(1)}), f(w_2^{(2)}), \cdots \} \) of integral cohomology classes on \( B_{\text{SO}(n)} \). By examining the \( \pi^* \) images of these two sequences in the cohomologies of \( B_{\text{Spin}^c(n)} \) we shall show the following result, where \( x \) denotes the Euler class of the Hopf line bundle \( \lambda \) on \( \mathbb{C}P^\infty \).

**Theorem C.** There exists a unique sequence \( \{ q_r, r \geq 0 \} \) of integral cohomology classes on \( B_{\text{Spin}^c(n)} \), deg \( q_r = 2^{r+1} \), that satisfies the following system
\begin{align*}
\text{i}) & \quad q_0 = \iota^*(x), \\
\text{ii}) & \quad \rho_2(q_r) = \pi^*w_2^{(r)}; \\
\text{iii}) & \quad 2q_{r+1} + q_r^2 = \pi^*f(w_2^{(r)}), \quad r \geq 0.
\end{align*}
For an integer $n \geq 3$ we set $h(n) = \lceil \frac{n-1}{2} \rceil$ and let $\theta_n \in H^{2(n)+1}(B_{\text{Spin}^c(n)})$ be the Euler class of the complex bundle $\xi_n$ associated to the complex spin presentation of the group $\text{Spin}^c(n)$ [22, Theorem 3.5]. Regarding the cohomology $H^*(B_{\text{Spin}^c(n)})$ as a module over its subring $\pi^* H^*(BSO(n))$ we present the ring $H^*(B_{\text{Spin}^c(n)})$ by the unique sequence $\{q_r, r \geq 0\}$ obtained by Theorem C, together with the Euler class $\theta_n$.

**Theorem D.** The cohomology ring $H^*(B_{\text{Spin}^c(n)})$ has the presentation

\begin{equation}
H^*(B_{\text{Spin}^c(n)}) = \pi^* H^*(BSO(n)) \otimes \mathbb{Z}[q_0, q_1, \cdots, q_{h(n)-1}, \theta_n]/R_n,
\end{equation}

where $R_n$ denotes the ideal generated by the following elements

i) $2q_{r+1} + q_r^2 - \pi^* f(w_2^{(r)}) \otimes 1$;

ii) $\pi^* \delta_2(z) \otimes q_r - \pi^* \delta_2(z \cup w_2^{(r)}) \otimes 1$, $0 \leq r \leq h(n) - 2$;

iii) $(-1)^{h(n)} \cdot 4 \cdot \theta_n + q_{h(n)-1}^2 - \alpha_n$,

where $z \in H^*(BSO(n), \mathbb{Z}_2)$, and where $\alpha_n \in \pi^* H^+(BSO(n)) \otimes \Delta(q_0, q_1, \cdots, q_{h(n)-1})$ is an element which will be specified in Lemma 5.3.

Rigorously the relations i), ii) and iii) of Theorem D should be

\begin{align*}
2 \otimes q_{r+1} + 1 \otimes q_r^2 - \pi^* f(w_2^{(r)}) \otimes 1, \\
\pi^* \delta_2(z) \otimes q_r - \pi^* \delta_2(z \cup w_2^{(r)}) \otimes 1, \\
(-1)^{h(n)} \cdot 4 \otimes \theta_n + 1 \otimes q_{h(n)-1}^2 - \alpha_n
\end{align*}

respectively. We hope that the current less strict expressions simplify the presentation, and will adopt the idea in a few places applicable.

The remaining sections are so arranged. Sections §3–5 are devoted, respectively, to show Theorems A–D. In particular, a concise additive presentation of the cohomology $H^*(B_{\text{Spin}^c(n)})$ is obtained in Theorem 5.2. The calculation is extended in Section §6 to obtain a similar presentation of the ring $H^*(B_{\text{Spin}(n)})$ in Theorem D′.

Sections §7 to §10 are devoted to the applications of Theorem D′. In particular, we determine in §8 the ring of integral Weyl invariants of the group $\text{Spin}(n)$; introduce in §9 the Spin characteristic classes for the spin vector bundles. To show the usage of the Spin characteristic classes in spin geometry generalizations of the classical Rokhlin theorem are discussed in section §10.

### 3 The proofs of Theorems A and B

For a mod 2 cohomology class $u \in H^{2r}(X; \mathbb{Z}_2)$ of a CW complex $X$ the Pontryagin square $B(u) \in H^{4r}(X; \mathbb{Z}_4)$ can be defined by the formula [9]

\[ B(u) \equiv \rho_4(\tilde{u} \cup_0 \tilde{u} + \delta(\tilde{u})) \cup_1 \tilde{u}, \]

where $\tilde{u}$ is an integral lift of $u$ in the cochain complex $C^*(X; \mathbb{Z})$ associated to $X$, and $\cup_i$ denotes the $i^{th}$ cup product on $C^*(X; \mathbb{Z})$. Based on this formula a cochain level calculation verifies the following universal relations
(3.1) \( \delta_2(u \cup u) = 2\delta_4(B(u)) \) in \( H^{4r+1}(X) \);
(3.2) \( \rho_2 \delta_4 B(u) = Sq^r Sq^1 u + u \cup Sq^1 u \) in \( H^{4r+1}(X; \mathbb{Z}_2) \).

**Proof of Theorem A.** Let \( X \) be a \( \delta_2 \) formal space. For each \( u \in H^{2r}(X; \mathbb{Z}_2) \) the property \( \delta_2(u \cup u) = 0 \) implies that \( \delta_4(B(u)) \in \text{Im} \delta_2 \) by (3.1). In view of the isomorphism
\[
\delta_2 : S^2_2(X) := H^*(X; \mathbb{Z}_2)/\ker Sq^1 \cong \text{Im} \delta_2
\]
there exists a unique element \( u_1 \in S^2_2(X) \) so that
\[
(3.3) \quad \delta_2(u_1) = \delta_4(B(u)).
\]
We can now formulate the desired operation
\[
F = (\gamma, \alpha) : H^{2r}(X; \mathbb{Z}_2) \to S^2_2(X) \otimes H^{4r}(X; \mathbb{Z}_4)
\]
by setting \( \gamma(u) := u_1 \) and \( \alpha(u) := B(u) - \theta(\gamma(u)) \).

Applying \( \rho_2 \) to both sides of (3.3) one gets by (3.2) that
\[
Sq^1 \gamma(u) = Sq^r Sq^1 u + u \cup Sq^1 u.
\]
Moreover, from \( \delta_4 \circ \theta = \delta_2 \) and by (3.3) we get
\[
\delta_4 \alpha(u) = \delta_4(B(u) - \theta(\gamma(u))) = \delta_4(B(u)) - \delta_2(\gamma(u)) = 0,
\]
implying \( \alpha(u) \in \text{Im} \rho_1 \).

Summarizing, the pair \( F = (\gamma, \alpha) \) of operators fulfills the properties i), ii) and iii) of Theorem A, whose uniqueness comes directly from its definition. \( \Box \)

**Proof of Theorem B.** Let \( f : H^*(B_{SO(n)}; \mathbb{Z}_2) \to H^*(B_{SO(n)}) \) be the map entailed in (2.4). It has been shown by Thomas [36, Lemma (3.9)] that for any \( u \in S^2_2(B_{SO(n)}) \) there exists a unique element \( v \in S^2_2(B_{SO(n)}) \) such that
\[
(3.5) \quad B(u) = \rho_4(f(u)) + \theta(v).
\]
It suffices to show that \( v = \gamma(u) \).

Applying the operations \( \alpha, \gamma \) to the class \( u \in S^2_2(B_{SO(n)}) \) we get by Theorem A the system
\[
\begin{align*}
a) \quad B(u) &= \rho_4(\alpha'(u)) + \theta(\gamma(u)), \\
b) \quad Sq^1(\gamma(u)) &= Sq^2n Sq^1 u + u \cup Sq^1 u.
\end{align*}
\]
where \( \alpha'(u) \in H^*(B_{SO(n)}) \) is an integral lift of \( \alpha(u) \). Since
\[
\rho_2(\alpha'(u)) = \rho_2(f(u)) = \rho_2(B(u)) = u \cup u
\]
there exists an integral class \( w \in H^*(B_{SO(n)}) \) such that \( \alpha'(u) = f(u) + 2w \). Substitute this into a) to get
\[
B(u) = \rho_4(f(u)) + \theta(\rho_2(w) + \gamma(u)).
\]
Comparing this with (3.5) one gets
\[
v \equiv \rho_2(w) + \gamma(u) \mod \text{Im} Sq^1.
\]
Applying \( Sq^1 \) to both sides yields \( Sq^1 v = Sq^1 \gamma(u) \). Since \( v, \gamma(u) \in S^2_2(B_{SO(n)}) \) while \( Sq^1 \) injects on \( S^2_2(B_{SO(n)}) \), one obtains \( v = \gamma(u) \), completing the proof. \( \Box \)
4 The proof of Theorem C

Denote by $\mathbb{Z}_0$ the field of rationals and let $\rho_0$ be the cohomology homomorphism induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}_0$. As in Theorem C we set $q_0 := \iota^*(x)$, where $x$ is the Euler class of the Hopf line bundle $\lambda$ on $\mathbb{C}P^\infty$.

Lemma 4.1. If either $p = 0$ or $p \geq 3$ is a prime, the cohomology $H^*(B_{Spin^c(n)}; \mathbb{Z}_p)$ is the polynomial algebra on the generators

\[(4.1) \rho_p(q_0), \rho_p(\pi^*p_1), \ldots, \rho_p(\pi^*p_{[n-1]}), \text{ and } \rho_p(\pi^*e_n) \text{ if } n \equiv 0 \mod 2.\]

In addition, the map

$$\rho : H^*(B_{Spin^c(n)}) \to H^*(B_{Spin^c(n)}; \mathbb{Z}_0) \times H^*(B_{Spin^c(n)}; \mathbb{Z}_2)$$

by $\rho(z) = (\rho_0(z), \rho_2(z))$, $z \in H^*(B_{Spin^c(n)})$, injects.

Proof. Since the composition $\iota \circ i : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ (see (2.5) and (2.6)) is of degree 2, the class $i^*(q_0) = 2x$ generates the algebra $H^*(\mathbb{C}P^\infty; \mathbb{Z}_p) = \mathbb{Z}_p[x]$ for all $p \neq 2$. We obtain the first assertion by the Leray–Hirsch Theorem \[23\] p. 231 for the fibration $(B_{Spin^c(n)}, \pi, B_{SO(n)})$ with $\mathbb{Z}_p$ coefficients.

According to Borel and Hirzebruch \[\text{[5, 30.6.]}\] if $X$ is a space with $2\tau(X) = 0$, then the map $\rho : H^*(X) \to H^*(X; \mathbb{Z}_0) \times H^*(X; \mathbb{Z}_2)$ by $\rho(z) = (\rho_0(z), \rho_2(z))$ injects. The second assertion follows from $2\tau(B_{Spin^c(n)}) = 0$ due to Harada and Kono \[22\] Theorem 3.7]. □

Let $x_r := (\rho_2(x))^2^{r-1} \in H^2 (\mathbb{C}P^\infty; \mathbb{Z}_2)$. Since the Borel transgression $\sigma$ in the fibration $(B_{Spin^c(n)}, \pi, B_{SO(n)})$ commutes with the Steenrod squares, we get by the standard relation $Sq^2 x_r = x_{r+1}$ on $H^*(\mathbb{C}P^\infty; \mathbb{Z}_2)$ that

\[(4.2) \sigma(x_{r+1}) = Sq^2 \sigma(x_r) \in H^*(B_{SO(n)}; \mathbb{Z}_2), r \geq 0.\]

On the other hand let $\{w_2, w_2(1), \cdots\}$ be the derived sequence of $w_2$.

Lemma 4.2. Let $J_{n,r}$ be the ideal on $H^*(B_{SO(n)}; \mathbb{Z}_2)$ generated by the sequence $\sigma(x_1), \cdots, \sigma(x_r)$ for $r \geq 1$, and set $J_{n,0} = \{0\}$. Then

\[(4.3) Sq^1(w^{(r)}_2) = \sigma(x_{r+1}) + \beta_r \text{ with } \beta_r \in J_{n,r}, r \geq 0.\]

Proof. If $r = 0$ then formula (4.3) is verified by $\sigma(x_1) = w_3 = Sq^1(w_2)$. Suppose next that (4.3) holds for some $r \geq 0$. That is

$$Sq^1(w^{(r)}_2) = \sigma(x_{r+1}) + \beta_r \in J_{n,r}.$$ 

With $w^{(r+1)}_2 = \gamma(w^{(r)}_2)$ we get by formula ii) of Theorem B that

$$Sq^1(w^{(r+1)}_2) = Sq^2 Sq^1(w^{(r)}_2) + w^{(r)}_2 \cup Sq^1 w^{(r)}_2$$

$$= Sq^2 (\sigma(x_{r+1}) + \beta_r) + w^{(r)}_2 \cup (\sigma(x_{r+1}) + \beta_r)$$

$$= \sigma(x_{r+2}) + Sq^2 \beta_r + w^{(r)}_2 \cup \sigma(x_{r+1}) + w^{(r)}_2 \cup \beta_r \text{ (by (4.2))}. $$
The inductive procedure is completed by taking

\[ \beta_{r+1} := Sq^{2r+1} \beta_r + w_2^{(r)} \cup \sigma(x_{r+1}) + w_2^{(r)} \cup \beta_r \]

Consider the exact ladder of the Bockstein sequences induced by \( \pi \)

\[ \cdots \rightarrow H^k(B_{SO(n)}) \xrightarrow{\rho_2} H^k(B_{SO(n)}; \mathbb{Z}_2) \xrightarrow{\delta_3} H^{k+1}(B_{SO(n)}) \rightarrow \cdots \]

\[ \cdots \rightarrow H^k(B_{Spin^c(n)}) \xrightarrow{\rho_2} H^k(B_{Spin^c(n)}; \mathbb{Z}_2) \xrightarrow{\delta_3} H^{k+1}(B_{Spin^c(n)}) \rightarrow \cdots \]

Since \( \pi^* \circ \sigma = 0 \) we get from formula (4.3) that

\[ Sq^1(\pi^*(w_2^{(r)})) = \pi^* Sq^1(w_2^{(r)}) = 0, \quad r \geq 0, \]

implying that the torsion elements \( \delta_2(\pi^*(w_2^{(r)})) \in H^*(B_{Spin^c(n)}) \) are divisible by 2. Since the space \( B_{Spin^c(n)} \) is \( \delta_2 \) formal we obtain further that \( \delta_2(\pi^*(w_2^{(r)})) = 0 \). By the exactness of the bottom sequence there exist integral classes \( q_r^i \in H^*(B_{Spin^c(n)}) \) such that

(4.4) \( \rho_2(q_r^i) = \pi^*(w_2^{(r)}), \quad r \geq 0. \)

In particular

\[ B(\pi^*(w_2^{(r)})) = \rho_4(q_r^i \cup q_r^i), \quad \theta(\pi^* w_2^{(r+1)}) = 2 \rho_4(q_{r+1}^i). \]

Therefore, for \( u = w_2^{(r)} \) applying \( \pi^* \) to the relation i) of Theorem B gives rise to the relation on \( H^*(B_{Spin^c(n)}; \mathbb{Z}_4) \)

\[ \rho_4(q_r^i \cup q_r^i) = \rho_4(\pi^* f(w_2^{(r)})) + 2 \rho_4(q_{r+1}^i), \]

implying that there exists an integral class \( v_{r+1} \in H^*(B_{Spin^c(n)}) \) so that

(4.5) \( 2q_{r+1}^i + q_r^i \cup q_r^i = \pi^* f(w_2^{(r)}) + 4v_{r+1}. \)

**Proof of Theorem C.** Since the class \( q_0 = c^*(x) \) generates \( H^2(B_{Spin^c(n)}) = \mathbb{Z} \) with \( \rho_2(q_0) = \pi^*(w_2) \) we can take in (4.4) that \( q_0^i = q_0 \). Next, define in term of (4.5) that \( q_1 := q_1^i - 2v_1 \). Then, for the case \( r = 1 \) the relations ii) and iii) of Theorem C are verified by

\[ \rho_2(q_1) = \rho_2(q_1^i) = \pi^*(w_2^{(1)}) \] (by (4.4));

\[ 2q_1 + q_0 \cup q_0 = \pi^* f(w_2^{(0)}) \] (by (4.5))

respectively.

Assume next that a sequence \( q_0, \ldots, q_r \) of classes satisfying the properties i), ii) and iii) of Theorem C has been obtained for some \( r \geq 1 \). Take in (4.4) that \( q_r^i = q_r \) and define in term of (4.5) that \( q_{r+1} := q_{r+1}^i - 2v_{r+1} \). Then by (4.4) and (4.5)

\[ \rho_2(q_{r+1}) = \pi^*(w_2^{(r+1)}), \quad 2q_{r+1} + q_r \cup q_r = \pi^* f(w_2^{(r)}). \]
This completes the inductive construction of a sequence \{q_r, r \geq 0\} fulfilling the system in Theorem C.

To see the uniqueness of the sequence \{q_r, r \geq 0\} so obtained we make use of the injection \rho in Lemma 4.1. Note that the properties i) and iii) of Theorem C suffices to decide the \rho-image of \rho as \rho(q_r) = (q_r, w_2^{(r)}), r \geq 1, where \rho is the unique rational polynomial in the generators in (4.1) defined recurrently by the relation iii) as

\[
\rho_0(q_1) = g_1 := \frac{1}{2}(\rho_0(\pi p_1) - \rho_0(q_0^2)) \\
\rho_0(q_r) = g_r := \frac{1}{2}(\rho_0(\pi f(w_2^{(r-1)})) - q_r^{2}) \geq 2.
\]

These imply that, if \{q'_r, 1 \leq r \} is a second sequence satisfying properties i), ii) and iii) of Theorem C, then \rho'_r = \rho_r, completing the proof of Theorem C. □

Let \(H^*(B_{SO(n)})\) be the subring of \(H^*(B_{SO(n)})\) consisting of elements in the positive degrees. Since \(\iota^* \circ \pi^* = 0\) on \(H^+(B_{SO(n)})\) the relation ii) of Theorem C implies that

\[
2\iota^*(q_{r+1}) = -\iota^*(q_r)^2, r \geq 0.
\]

Inputting \(\iota^*(q_0) = 2\xi\) we get by an induction on \(r\) that

\[
(4.6) \quad \iota^*(q_r) = (-1)^r \cdot 2^r \cdot x^{2r}, \quad r \geq 0.
\]

We conclude this section with the some information required by showing Theorem D. For a set \{y_1, \cdots, y_r\} of graded elements denoted by \(\Delta(y_1, \cdots, y_r)\) the graded free \(\mathbb{Z}\) module with the basis \(\{1, y_1, y_2, \cdots y_k, 1 \leq i_1 < \cdots < i_k \leq r\}\) (i.e. the square free monomials in the \(y_1, \cdots , y_r\)). For a prime \(p\) we write \(\Delta_p(y_1, \cdots , y_r)\) to simplify the tensoring \(\mathbb{Z}_p \otimes \Delta(y_1, \cdots , y_r)\).

If \(\{a_1, \cdots , a_r\}\) is a finite sequence of a ring \(A\) denote by \(\langle a_1, \cdots , a_r \rangle\) the ideal generated by \(a_1, \cdots , a_r\), and write \(A/\langle a_1, \cdots , a_r \rangle\) for the quotient algebra.

For a topological space \(X\) let \(H^*_B(X)\) be the Bockstein cohomology of \(X\) (i.e. the kernel modulo the image of \(\beta = Sq^1\) on \(H^*(X; \mathbb{Z}_2)\)). As a familiar example we have, in accordance to \(n = 2k + 1\) or \(n = 2k\), that

\[
H^*_B(B_{SO(n)}) = \mathbb{Z}_2[w_2^2, \cdots , w_{2k}^2] \text{ or } \mathbb{Z}_2[w_2^2, \cdots , w_{2(k-1)}^2, w_{2k}].
\]

As in Theorem D let \(\theta_n\) be the Euler class of the bundle \(\xi_n\) on \(B_{Spin^c(n)}\).

**Lemma 4.3.** For \(X = B_{Spin^c(n)}\) let \{q_r, r \geq 0\} be the unique sequence of cohomology classes on \(X\) obtained by Theorem C. Then the algebras \(H^*(X; \mathbb{Z}_2)\), \(H^*_B(X)\) and \(\tau(X)\) have the following presentations

\[
(4.7) \quad H^*(B_{Spin^c(n)}; \mathbb{Z}_2) = \pi^* H^*(B_{SO(n)}; \mathbb{Z}_2) \otimes \mathbb{Z}_2[\theta_n] \\
(4.8) \quad H^*_B(B_{Spin^c(n)}) = \pi^* H^*_B(B_{SO(n)}) \otimes \Delta_2(q_0, \cdots , q_{(n-1)}) \otimes \mathbb{Z}_2[\theta_n] \\
(4.9) \quad \tau(B_{Spin^c(n)}) = \pi^* \tau(B_{SO(n)}) \otimes \mathbb{Z}_2[\theta_n],
\]

respectively, where the map \(\pi\) induces the isomorphisms

\[
H^*(B_{SO(n)}; \mathbb{Z}_2)/ (\sigma(x_1), \cdots , \sigma(x_{(n-1)})) \cong \pi^* H^*(B_{SO(n)}; \mathbb{Z}_2) \\
H^*_B(B_{SO(n)}) \cong \pi^* H^*_B(B_{SO(n)}),
\]

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and where, with \( \mathbb{Z}_2 \) coefficients being understood, we can write \( q_r \) and \( \theta_n \) in places of \( \rho_2(q_r) \) and \( \rho_2(\theta_n) \).

**Proof.** Formula (4.7) has been shown by Harada and Kono \[22, Theorem 3.5\].

In view of (4.7) the operation \( \beta \) on \( H^*(B_{Spin^c(n)}; \mathbb{Z}_2) \) is determined by

\[
\beta(\pi^*(w_{2r})) = \pi^*(w_{2r+1}), \quad \beta(\pi^*(w_{2r+1})) = 0,
\]

implying that \( H_\beta^*(B_{Spin^c(n)}) \) is a module over the algebra \( \pi^*H_\beta^*(B_{SO(n)}) \). On the other hand, one has by Theorem C the canonical classes \( \rho_2(q_r) = \pi^*w_2^r \in H_\beta^*(B_{Spin^c(n)}) \) whose squares satisfy the relation

\[
\rho_2(q_r)^2 = \rho_2(\pi^*f(w_2^r)) \in \pi^*H_\beta^*(B_{SO(n)}), \quad r \geq 0.
\]

Therefore, the inclusions \( \rho_2(q_r), \rho_2(\theta_n) \in H_\beta^*(B_{Spin^c(n)}) \) extends to the well defined homomorphism

\[
\pi^*H_\beta^*(B_{SO(n)}) \otimes \Delta_2(q_0, \ldots, q_{h(n)-1}) \otimes \mathbb{Z}_2[\theta_n] \to H_\beta^*(B_{Spin^c(n)}).
\]

The method initiated by Borel and Hirzebruch in \[5, \S 30.4, \S 30.5\], and illustrated by Kono \[25\] or Benson and Wood \[6, \S 9\] in their calculation of \( H_\beta^*(B_{Spin^c(n)}) \), is applicable to show that this map is an isomorphism, verifying (4.8).

Finally, with \( 2\tau(B_{SO(n)}) = 2\tau(B_{Spin^c(n)}) = 0 \) we have the commutative diagram

\[
\begin{array}{ccc}
0 & \to & \tau(B_{SO(n)}) \xrightarrow{\pi^*} H^*(B_{SO(n)}; \mathbb{Z}_2) \\
\pi^* & \downarrow & \pi^* \\
0 & \to & \tau(B_{Spin^c(n)}) \xrightarrow{\pi^*} H^*(B_{Spin^c(n)}; \mathbb{Z}_2)
\end{array}
\]

in which both \( \rho_2 \) inject. Since \( \tau(B_{Spin^c(n)}) \) is an ideal the map

\[
h : \pi^*\tau(B_{SO(n)}) \otimes \mathbb{Z}_2[\theta_n] \to \tau(B_{Spin^c(n)})
\]

by \( h(\pi^*x \otimes \theta_n) = \pi^*x \cup \theta_n, \quad x \in \tau(B_{SO(n)}) \), is well defined. For the formula (4.9) it suffices to show that \( h \) is an isomorphism.

By (4.7) the composition \( \rho_2 \circ h \) injects, hence \( h \) injects, too. On the other hand, for any \( x \in \tau(B_{Spin^c(n)}) \) there exists \( y \in H^*(B_{Spin^c(n)}; \mathbb{Z}_2) \) so that \( \delta_2(y) = x \). By (4.7) we can write

\[
y = \pi^*(y_0) + \pi^*(y_1) \cup \theta_n + \cdots + \pi^*(y_r) \cup (\theta_n)^r, \quad y_i \in H^*(B_{SO(n)}; \mathbb{Z}_2)
\]

to get

\[
x = \pi^*(\delta_2(y_0)) + \pi^*(\delta_2(y_1)) \cup \theta_n + \cdots + \pi^*(\delta_2(y_r)) \cup (\theta_n)^r,
\]

implying that the map \( h \) also surjects. This completes the proof. \( \Box \)
5 The cohomology \( H^*(B_{\text{Spin}^c(n)}) \)

It would be convenient for us to begin with an additive approximation to
the graded group \( H^*(B_{\text{Spin}^c(n)}) \). Let \( \{q_r, r \geq 0\} \) be the unique sequence on
\( H^*(B_{\text{Spin}^c(n)}) \) obtained by Theorem C, and consider the graded module over
the ring \( \pi^* H^*(B_{SO(n)}) \)

\[
(5.1) \quad A := \pi^* H^*(B_{SO(n)}) \otimes \Delta(q_0, \cdots, q_{b(n)-1}) \otimes \mathbb{Z}[\theta_n].
\]

Denote by \( \Delta^+(q_0, \cdots, q_{b(n)-1}) \) the subgroup of \( \Delta(q_0, \cdots, q_{b(n)-1}) \) consisting
of the elements with positive degrees. In view of the decomposition (2.3) on
\( H^*(B_{SO(n)}) \), as well as \( \Delta = \mathbb{Z} \oplus \Delta^+ \), the group \( A \) has the direct summand

\[
A_1 := \pi^* \tau(B_{SO(n)}) \otimes \Delta^+(q_0, \cdots, q_{b(n)-1}) \otimes \mathbb{Z}[\theta_n].
\]

Let \( A_2 \) be the complement of \( A_1 \) in \( A \), and let \( \varphi : A \rightarrow H^*(B_{\text{Spin}^c(n)}) \) be the
module map induced by the inclusions \( \theta_n, q_r \in H^*(B_{\text{Spin}^c(n)}) \).

**Lemma 5.1.** The map \( \varphi \) restricts to an isomorphism \( A_2 \rightarrow H^*(B_{\text{Spin}^c(n)}) \),
hence fits into the short exact sequence

\[
(5.2) \quad 0 \rightarrow A_1 \xrightarrow{\delta} A \xrightarrow{\varphi} H^*(B_{\text{Spin}^c(n)}) \rightarrow 0.
\]

**Proof.** Let \( \psi := \varphi \mid A_2 \). For a prime \( p \) denote by \( \psi_p \) the induced map
\[
\psi \otimes \text{id} : A_2 \otimes \mathbb{Z}_p \rightarrow H^*(B_{\text{Spin}^c(n)}) \otimes \mathbb{Z}_p.
\]

As the torsion elements of both \( A_2 \) and \( H^*(B_{\text{Spin}^c(n)}) \) are killed by multiplying 2
it suffices to show that \( \psi_p \) is an isomorphism for every \( p \). Since \( 2\tau(B_{\text{Spin}^c(n)}) = 0 \)
we already have

\[
(5.3) \quad H^*(B_{\text{Spin}^c(n)}) \otimes \mathbb{Z}_2 = H^*_\beta(B_{\text{Spin}^c(n)}) \oplus \tau(B_{\text{Spin}^c(n)});
\]

\[
(5.4) \quad H^*(B_{\text{Spin}^c(n)}) \otimes \mathbb{Z}_p = H^*(B_{\text{Spin}^c(n)}; \mathbb{Z}_p) \text{ if } p \neq 2.
\]

The remaining calculation is divided by the parity of the prime \( p \).

By (2.3) we have \( H^*(B_{SO(n)}) \otimes \mathbb{Z}_2 = H^*_\beta(B_{SO(n)}) \oplus \tau(B_{SO(n)}) \). Accordingly,
by the formulae (5.1), (4.8) and (4.9) we can write
\[
A \otimes \mathbb{Z}_2 = A_1 \otimes \mathbb{Z}_2 + H^*_3(B_{\text{Spin}^c(n)}) \oplus \tau(B_{\text{Spin}^c(n)}).
\]
to get
\[
A_2 \otimes \mathbb{Z}_2 = H^*_\beta(B_{\text{Spin}^c(n)}) \oplus \tau(B_{\text{Spin}^c(n)}).
\]
Comparing this with (5.3) one finds that \( \psi_2 \) is an isomorphism.

Assume next that \( p \neq 2 \). With \( 2\tau(B_{SO(n)}) = 0 \) we have
\[
H^*(B_{SO(n)}) \otimes \mathbb{Z}_p = H^*(B_{SO(n)}; \mathbb{Z}_p) \text{ and } A_1 \otimes \mathbb{Z}_p = 0.
\]
It follows then from \( A = A_1 \oplus A_2 \) and (5.1) that
\[
A_2 \otimes \mathbb{Z}_p = A \otimes \mathbb{Z}_p = \pi^* H^*(B_{SO(n)}; \mathbb{Z}_p) \otimes \Delta(q_0, \cdots, q_{b(n)-1}) \otimes \mathbb{Z}_p[\theta_n].
\]
Since, by (4.6) and by $i^*(\xi_n) = 2^k(n) \cdot \lambda$,

$$i^*(q_r) = 2 \cdot (-1)^r \cdot x^{2^r}$$

and $i^*(\theta_n) = x^{2^k(n)}$,

the composition

$$\Delta_p(q_0, \cdots, q_{h(n) - 1}) \otimes \mathbb{Z}[\theta_n] \xrightarrow{\psi_p} H^*(B_{\text{Spin}^c(n)}; \mathbb{Z}_p) \xrightarrow{i^*} H^*(\mathbb{C}P^\infty; \mathbb{Z}_p)$$

is an isomorphism of $\mathbb{Z}_p$ spaces. The Leray–Hirsch Theorem [23, p.231] for the fibration $(B_{\text{Spin}^c(n)}, \pi, B_{SO(n)})$ over $\mathbb{Z}_p$ now implies that

$$A_2 \otimes \mathbb{Z}_p = \pi^*H^*(B_{SO(n)}; \mathbb{Z}_p) \otimes H^*(\mathbb{C}P^\infty; \mathbb{Z}_p) = H^*(B_{\text{Spin}^c(n)}; \mathbb{Z}_p).$$

Comparing this with (5.4) one sees that $\psi_p$ is also an isomorphism. □

It should be noted that the map $\kappa$ in (5.2) is not the obvious inclusion $A_1 \subset A$. To clarify this map consider a typical element

$$\pi^*\delta_2(x) \otimes q_r, x \in H^*(B_{SO(n)}; \mathbb{Z}_2),$$

of the subgroup $A_1$. By the property i) of Theorem C we have

$$(5.5) \quad \varphi(\pi^*\delta_2(x) \otimes q_r) = \pi^*\delta_2(x) \cup q_r = \pi^*\delta_2(x \cup w_2(r)).$$

In general, for a multi-index $I = (i_1, \cdots, i_k)$ with $0 \leq i_1 < \cdots < i_k \leq h(n) - 1$, repeatedly applying (5.5) yields

$$\varphi(\pi^*\delta_2(x) \otimes q_I) = \pi^*\delta_2(x \cup w_2(I)), x \in H^*(B_{SO(n)}; \mathbb{Z}_2),$$

where $q_I = q_{i_1} \cup \cdots \cup q_{i_k}$, $w_2(I) = w_2^{(i_1)} \cup \cdots \cup w_2^{(i_k)}$. This shows that

**Theorem 5.2.** The cohomology $H^*(B_{\text{Spin}^c(n)})$ has the additive presentation

$$(5.6) \quad H^*(B_{\text{Spin}^c(n)}) \cong \pi^*H^*(B_{SO(n)}) \otimes \Delta(q_0, \cdots, q_{h(n) - 1}) \otimes \mathbb{Z}[\theta_n]/\text{Im} \kappa,$$

where for $y = \pi^*\delta_2(x) \otimes q_I \otimes (\theta_n)^s \in A_1$ with $x \in H^*(B_{SO(n)}; \mathbb{Z}_2)$ and $s \geq 0$

$$\kappa(y) = y - \pi^*\delta_2(x \cup w_2(I)) \otimes (\theta_n)^s. □$$

By virtue of (5.6) we can now specify a crucial relation on the ring $H^*(B_{\text{Spin}^c(n)})$ that relates the class $q_{h(n) - 1}$ with the Euler $\theta_n$. Let $H^+(B_{SO(n)}$) be the subring of $H^*(B_{SO(n)})$ consisting of the elements in the positive degrees.

**Lemma 5.3.** There exists an element

$$\alpha_n \in \pi^*H^+(B_{SO(n)}) \otimes \Delta(q_0, \cdots, q_{h(n) - 1}), \quad \deg \alpha_n = 2(h(n) + 1),$$

so that the following relation holds on $H^*(B_{\text{Spin}^c(n)})$

$$(5.7) \quad (-1)^{h(n)} \cdot 4 \cdot \theta_n + q_{h(n) - 1}^2 = \varphi(\alpha_n).$$
Proof. Since in degree $2^{h(n)+1}$ the quotient of $A$ by its subgroup $\pi^*H^*(B_{SO(n)}) \otimes \Delta(q_0, \cdots, q_{h(n)-1})$ is isomorphic to $\mathbb{Z}$ with the basis $\{\theta_n\}$, formula (5.6) implies that the class $q_{h(n)}(n) \in H^*(B_{Spin^c(n)})$ admits the expression

$$q_{h(n)} = k \cdot \varphi(\beta_n)$$

for some $\beta_n \in \pi^*H^*(B_{SO(n)}) \otimes \Delta(q_0, \cdots, q_{h(n)-1})$

and integer $k \in \mathbb{Z}$. Applying $i^*$ to this equality we obtain by (4.6), as well as the relation $i^* \circ \pi^* = 0$ on $H^*(B_{SO(n)})$, that

$$(-1)^{h(n)} \cdot 2 \cdot x^{\delta(n)} = k \cdot x^{\delta(n)}$$

on $H^{2^{h(n)+1}}(\mathbb{C}P^\infty)$,

showing that $k = (-1)^{h(n)} \cdot 2$. On the other hand, by the relation iii) of Theorem C we have

$$2q_{h(n)} + q_{h(n)-1}^2 = \pi^*f(w_2^{(h(n)-1)}) \in H^*(B_{SO(n)}).$$

We obtain (5.7) by taking $\alpha_n = \pi^*f(w_2^{(h(n)-1)}) \otimes 1 - 2 \cdot \beta_{n}$.

Proof of Theorem D. Set $\tilde{A} := \pi^*H^*(B_{SO(n)}) \otimes \mathbb{Z}[q_0, q_1, \cdots, q_{h(n)-1}, \theta_n]$. The inclusions $q_r, \theta_n \in H^*(B_{Spin^c(n)})$ extend to a ring map $\tilde{\varphi} : \tilde{A} \to H^*(B_{Spin^c(n)})$ which satisfies, by the properties ii) and iii) of Theorem C, that

(5.8) $\tilde{\varphi}(2q_{r+1} + q_r^2 - \pi^*f(w_2^{(r)})) = 0, \ 0 \leq r \leq h(n) - 2$;

(5.9) $\tilde{\varphi}(\pi^*\delta_2(x) \otimes q_r - \pi^*\delta_2(x \cup w_2^{(r)})) = 0$ (see (5.4)).

In addition, we have by the relation (5.7) that

(5.10) $\tilde{\varphi}(-1)^{h(n)} \cdot 4 \cdot \theta_n + q_{h(n)-1}^2 - \alpha_n = 0$.

Therefore $R_n \subseteq \ker \tilde{\varphi}$, where $R_n$ is the ideal on $\tilde{A}$ stated in Theorem D. It remains to show that the map $\tilde{\varphi}$, when factoring through the quotient $\tilde{A} \mod R_n$, yields an additive isomorphism onto $H^*(B_{Spin^c(n)})$.

Putting in transparent form the relations (5.8) and (5.10) tell that

$$q_r^2 = -2q_{r+1} + \pi^*f(w_2^{(r)}), \ 0 \leq r \leq h(n) - 2;$$

$$q_{h(n)-1} = (-1)^{h(n)+1} \cdot 4 \cdot \theta_n + \alpha_n.$$

These are just the relations required to express every monomial $q_{r_0}^{i_0}q_{r_1}^{i_1} \cdots q_{r_{h(n)-1}}^{i_{h(n)-1}} \in \tilde{A}$ with $r_i \geq 2$ for some $0 \leq i \leq h(n) - 1$ as a $\pi^*H^*(B_{SO(n)}) \otimes \mathbb{Z}[\theta_n]$-linear combination of the base elements

$$\{1, q_i, q_i^2 \cdots q_i^k, 0 \leq i_1 < \cdots < i_k \leq h(n) - 1\}$$

of $\Delta(q_0, \cdots, q_{h(n)-1})$ (i.e. as an element of the subgroup $A \subset \tilde{A}$), showing that

(5.11) $\tilde{A} \mod L_n \cong A$,

where $L_n$ denotes the ideal on $\tilde{A}$ generated by the relations (5.8) and (5.10). Furthermore, the proof of Theorem 5.2 indicates that the remaining relation (5.9) is exactly requested to formulate $\ker \kappa$ to yield the desired isomorphism

$$\tilde{A} \mod R_n \cong A \mod \ker \kappa \cong H^*(B_{Spin^c(n)}),$$

where the first isomorphism comes from (5.11). This completes the proof. □
6 The cohomology $H^*(B_{Spin(n)})$

In sections §4–5 we have concentrated to the calculation in the space $B_{Spin^c(n)}$. Indeed, an analogous procedure works well to formulate the ring $H^*(B_{Spin(n)})$. Consider the fibration induced by the covering $Spin(n) \to SO(n)$

\[
(6.1) \quad \mathbb{R}P^\infty \xrightarrow{i} B_{Spin(n)} \xrightarrow{\pi} B_{SO(n)}.
\]

**Theorem C’.** There exists a unique sequence $\{\overline{\eta}_r, r \geq 1\}$ of integral cohomology classes on $B_{Spin(n)}$, deg $\overline{\eta}_r = 2^r + 1$, that satisfies the following system

1. $\rho_2(\overline{\eta}_r) = \pi^* w_2^{(r)}$;
2. $2\overline{\eta}_1 = \pi^* p_1, 2\overline{\eta}_{r+1} + \overline{\eta}_r = \pi^* f(w_2^{(r)}), r \geq 1$.

**Proof.** Note that $\overline{\eta} = \pi \circ \psi$, where $\psi : B_{Spin(n)} \to B_{Spin^c(n)}$ the $U(1)$ fibration in (2.6). The existence of a sequence $\{\overline{\eta}_r, r \geq 0\}$ fulfilling the properties i) and ii) is obtained from ii) and iii) of Theorem C by setting $\overline{\eta}_r := \psi^*(q_r)$. Note that $\overline{\eta}_0 = 0$ as $q_0$ belongs to the image of the transgression in the $U(1)$ fibration $(B_{Spin(n)}, \psi, B_{Spin^c(n)})$ in (2.6).

For the uniqueness of the sequence $\{\overline{\eta}_r, r \geq 1\}$ so obtained we unitize the map $\rho$ in Lemma 4.1 for the space $B_{Spin(n)}$. With $2\tau(B_{Spin(n)}) = 0$ the map $\rho$ injects by [5] 30.6. On the other hand, the relations i) and ii) suffice to express the image $\rho(\overline{\eta}_r)$, $r \geq 1$. This competes the proof. ∎

Let $\eta_n$ be the oriented real vector bundle on $B_{Spin(n)}$ associated to the real spin representation of the group $Spin(n)$ [1], and denote by $\overline{\theta}_n \in H^n(B_{Spin(n)})$ its Euler class, where

\[ k(n) = \left[ \frac{n-1}{2} \right] \text{ if } n \mod 8 \neq 0, 1, 7, \text{ or } \left[ \frac{n-1}{2} \right] - 1 \text{ otherwise.} \]

It is known that ([6] Theorem 8.3, [31] Theorem 6.5]):

**Lemma 6.1.** The maps $\psi$ in (2.6) and $i$ in (6.1) satisfy, respectively, that

\[
(6.2) \quad \psi^*(\theta_n) = \overline{\theta}_n \text{ or } \overline{\theta}_n^2, \text{ in accordance to } n \mod 8 \neq 0, 1, 7, \text{ or otherwise.}
\]

\[
(6.3) \quad i^*(\overline{\theta}_n) = z^{k(n)+1}, \text{ where } z \text{ is the generator of } H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2. \quad ∎
\]

Let $\{\overline{\eta}_r, r \geq 1\}$ be the unique sequence obtained by Theorem C’. Denote by $\overline{\eta}$ the transgression in the fibration (6.1), and set $z_{r+1} := z^{2^r} \in H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$. With these convention the counterpart of Lemma 4.3 is:

**Lemma 6.2.** For $X = B_{Spin(n)}$ the algebras $H^*(X; \mathbb{Z}_2)$, $H^*_s(X)$ and $\tau(X)$ have the following presentations

\[
(6.4) \quad H^*(B_{Spin(n)}; \mathbb{Z}_2) = \pi^* H^*(B_{SO(n)}; \mathbb{Z}_2) \otimes \mathbb{Z}_2[\overline{\theta}_n]
\]

\[
(6.5) \quad H^*_s(B_{Spin(n)}) = \pi^* H^*_s(B_{SO(n)}) \otimes \Delta_2(\overline{\eta}_1, \cdots, \overline{\eta}_{k(n)-1}) \otimes \mathbb{Z}_2[\overline{\theta}_n]
\]

\[
(6.6) \quad \tau(B_{Spin(n)}) = \pi^* \tau(B_{SO(n)}) \otimes \mathbb{Z}_2[\overline{\theta}_n],
\]

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respectively, where the map $\overline{\pi}$ in (6.1) induces the isomorphisms

$$H^*(BSO(n); \mathbb{Z}_2)/\{w_2, \overline{\pi}(z_1), \ldots, \overline{\pi}(z_{k(n)-1})\} \cong \overline{\pi} H^*(BSO(n); \mathbb{Z}_2)$$

$$H^*_\beta(BSO(n))/\{w_2^\beta\} \cong \overline{\pi} H^*_\beta(BSO(n)).$$

**Proof.** Formula (6.4) is due to Quillen [31, Theorem 6.5], who showed in particular that the sequence $\{w_2, \overline{\pi}(z_1), \ldots, \overline{\pi}(z_{k(n)-1})\}$ is regular on $H^*(BSO(n); \mathbb{Z}_2)$.

The formula (6.5) agrees essentially with the one on $H^*_\beta(BSpin(n))$ implicitly given by Benson and Wood in [30] Lemma 9.1. Note that with $\mathbb{Z}_2$ coefficients being understood the class $\overline{\tau}_f$ in (6.5) is identical to $\overline{\rho}_2(\overline{\tau}_f) = \overline{\pi} w_2^{(r)}$, where the class $w_2^{(r)}$ play the same role as the the element $s_{r+2}$ in [30] Lemma 9.1.

From $2\tau(BSpin(n)) = 2\tau(BSO(n)) = 0$ we get the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \tau(BSO(n)) \\
\pi^* \downarrow & & \pi^* \downarrow \\
0 & \rightarrow & \tau(BSpin(n))
\end{array}
$$

by which formula (6.6) can be deduced from (6.4), by the same method used in the proof of formula (4.9) of Lemma 4.3.□

To formulate an additive presentation of $H^*(BSpin(n))$ by its subring $\overline{\pi} H^*(BSO(n))$, as well as the classes $\overline{\pi}_f$ and $\overline{\pi}_n$, we must take care of the first relation $\overline{\pi}_1 = 2\overline{\pi} p_1$ in ii) of Theorem C'. To eliminate this overlap from further consideration we introduce the subring of $H^*(BSO(n))$ (compare with (2.3))

$$\overline{\pi}'(BSO(n)) := \left\{ \begin{array}{l}
\mathbb{Z}[p_2, \ldots, p_{r+1}], \epsilon_n \oplus \tau(BSO(n)) \text{ if } n \text{ is even}; \\
\mathbb{Z}[p_2, \ldots, p_{r+1}] \oplus \tau(BSO(n)) \text{ if } n \text{ is odd.}
\end{array} \right.$$  

Combining this with $\overline{\pi}_f$ and $\overline{\pi}_n$ we formulate the graded group

$$A := \overline{\pi}' H^*(BSO(n)) \otimes \Delta(\overline{\pi}_1, \ldots, \overline{\pi}_{k(n)-1}) \otimes \mathbb{Z}[\overline{\pi}_n].$$

and let $\varphi : A \rightarrow H^*(BSpin(n))$ be the map induced by the inclusions $\overline{\pi}_n, \overline{\pi}_f \in H^*(BSpin(n))$. As is clear the group $A$ has the direct summand

$$A_1 := \overline{\pi}' \tau(BSO(n)) \otimes \Delta^+(\overline{\pi}_1, \ldots, \overline{\pi}_{k(n)-1}) \otimes \mathbb{Z}[\overline{\pi}_n].$$

The same arguments used in the proofs of Lemma 5.1 and Theorem 5.2 are applicable to show that

**Lemma 6.3.** The map $\varphi$ fits into the short exact sequence

$$0 \rightarrow A_1 \xrightarrow{\kappa} A \xrightarrow{\varphi} H^*(BSpin(n)) \rightarrow 0$$

where for $y = \overline{\pi} \delta_2(x) \otimes \overline{\pi}_1 \otimes (\overline{\pi}_n)^s \in A_1$ with $x \in H^*(BSO(n); \mathbb{Z}_2)$ and $s \geq 0$

$$\kappa(y) = y - \overline{\pi} \delta_2(x \cup w_2^{(r)}) \otimes (\overline{\pi}_n)^s.$$ 

Consider the class $\overline{\pi}_{k(n)} \in H^*(BSpin(n))$. Since in degree $2^{k(n)+1}$ the quotient of $A$ by its subgroup $\overline{\pi}' H^*(BSO(n)) \otimes \Delta(\overline{\pi}_1, \ldots, \overline{\pi}_{k(n)-1})$ is isomorphic to $\mathbb{Z}$ with the basis $\{\overline{\pi}_n\}$, we get by Lemma 6.3 the expression
These maps fit into the commutative diagram

\[ \begin{array}{ccc}
\text{inclusions} & & \\
\lambda_i & \rightarrow & \lambda_i \cap \text{Spin}
\end{array} \]

in (6.1) fails to be helpful to evaluate the constant \( \kappa \). Nevertheless, resorting to the canonical inclusion \( U(k) \subset \text{Spin}(2k) \) we shall show in Lemma 7.5 that 

\( \kappa = (-1)^{k(n)-1} \cdot 2 \). Thus, combining (6.7) with the relation 

\[ 2 \varphi_{k(n)} + \varphi_{k(n)-1}^2 = \varphi(\varphi_{n}) \]

by Theorem C’ yields

**Lemma 6.4.** There exists an \( \varphi_n \in \pi^* H^+(B_{SO(n)}) \otimes \Delta(\varphi_1, \cdots, \varphi_{k(n)-1}) \) so that 

\[ (-1)^{k(n)-1} \cdot 4 \cdot \varphi_n + \varphi_{k(n)-1}^2 = \varphi(\varphi_n). \]

Granted with Theorem C’, together with Lemmas 6.3 and 6.4, the same arguments as that used in the proof of Theorem D is applicable to obtain the following result, whose proof can therefore be omitted.

**Theorem D’**. The ring \( H^*(B_{Spin(n)}) \) has the presentation

\[ H^*(B_{Spin(n)}) = \pi^* H^*(B_{SO(n)}) \otimes \mathbb{Z}[\varphi_1, \cdots, \varphi_{k(n)-1}, \varphi_n]/K_n, \]

in which \( K_n \) denotes the ideal generated by the following elements

i) \( \varphi_{k-1} - \varphi^* p_1, \varphi_{k+1} + \varphi^* f(w^r_2), 1 \leq r \leq k(n) - 2; \)

ii) \( \varphi^* \delta_2(y) \cup \varphi_r - \varphi^* \delta_2(y \cup w^r_2), 1 \leq r \leq k(n) - 2; \)

iii) \( (-1)^{k(n)-1} \cdot 4 \cdot \varphi_n + \varphi_{k(n)-1}^2 - \varphi_n, \)

where \( y \in H^*(B_{SO(n)}, \mathbb{Z}_2), \varphi_n \in \pi^* H^+(B_{SO(n)}) \otimes \Delta(\varphi_1, \cdots, \varphi_{k(n)-1}). \)

\[ \square \]

7 The subgroup \( U(k) \subset \text{Spin}(2k) \)

The unitary group \( U(k) \) of order \( k \) is contained in \( SO(2k) \) as the centralizer of the circle subgroup \( \text{diag}\{z, \cdots, z\} \) with \( k \) copies \( z \in SO(2) \). It is also embedded in \( \text{Spin}(2k) \) as the centralizer of the circle subgroup (see [8], p.173, [14])

\[ U(1) = \{ \prod_{1 \leq i \leq k} (\cos t + \sin t \cdot e_{2i-1} e_{2i}) \in \text{Spin}(2k), t \in \mathbb{R} \}. \]

Let \( \lambda_0 : U(k) \to SO(2k) \) and \( \lambda : U(k) \to \text{Spin}(2k) \) be the corresponding inclusions, respectively, and define \( \lambda^c \) to be the homomorphism

\[ \lambda^c = \lambda \times_{\mathbb{Z}_2} \text{id} : U^c(k) = U(k) \times_{\mathbb{Z}_2} U(1) \to \text{Spin}^c(2k). \]

These maps fit into the commutative diagram

\[ \begin{array}{ccc}
U(k) & \xrightarrow{\alpha^c} & U^c(k) \\
\downarrow \lambda & & \downarrow \lambda^c \\
\text{Spin}(2k) & \xrightarrow{\phi} & \text{Spin}^c(2k) \end{array} \]

\[ \text{and} \]

\[ \begin{array}{ccc}
U(k) & \xrightarrow{\alpha^c} & U^c(k) \\
\downarrow \lambda & & \downarrow \lambda^c \\
\text{Spin}(2k) & \xrightarrow{\phi} & \text{Spin}^c(2k) \end{array} \]

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where \( a \) (resp. \( a' \)) denotes the kernel of the epimorphism

\[
\text{Spin}(2k) \times \mathbb{Z}_2 \ U(1) \to U(1) \quad \text{(resp.} \ U(k) \times \mathbb{Z}_2 \ U(1) \to U(1)\),
\]

and where \( b \) (resp. \( b' \)) is the obvious epimorphism

\[
\text{Spin}(2k) \times \mathbb{Z}_2 \ U(1) \to SO(2k) \quad \text{(resp.} \ U(k) \times \mathbb{Z}_2 \ U(1) \to U(k)\).
\]

In addition, it is easy to see that

Lemma 7.1. The composition \( b \circ a \) is the universal covering on \( SO(2k) \).

The composition \( b' \circ a' \) is the two sheets covering of \( U(k) \) by its self with associated mod 2 Euler class the generator of \( H^1(U(k); \mathbb{Z}_2) = \mathbb{Z}_2 \).

The homomorphisms \( \lambda \) and \( \lambda^c \) are particularly relevant to the understanding of the relationship between spinors and complex structure (see [2] p.10). In this section we determine their induced maps on the cohomologies of \( B_{\text{Spin}^c(2k)} \) and \( B_{\text{Spin}(2k)} \). The result will bring us a direct recurrence to produce the integral Weyl invariants of the group \( \text{Spin}(n) \).

7.1. Conventions. For a homomorphism \( h : H \to G \) of Lie groups denote by \( B_h : B_H \to B_G \) its induced map on the classifying spaces. The diagram (7.1) then gives rise to the commutative diagram of ring maps

\[
\begin{array}{ccc}
H^*(B_{U(k)}) & \xrightarrow{b^*_r} & H^*(B_{U'(k)}) \\
B_{\lambda^c}^r \uparrow & & B_{\lambda^c}^r \uparrow \\
H^*(B_{Spin(2k)}) & \xrightarrow{\psi^*} & H^*(B_{Spin^c(2k)}) \\
& & B^r_{\lambda^c} \uparrow
\end{array}
\]

where \( \psi = B_a \) and \( \pi = B_b \) in the notation of (2.5) and (2.6). By the naturality of characteristic classes, and for the sake to simplify notation, we shall adopt the following convention throughout this section. Let \( 1 + c_1 + \cdots + c_k \) be the total Chern class of the canonical complex bundle on \( B_{U(k)} \), and recall that \( H^*(B_{U(k)}) = \mathbb{Z}[c_1, \ldots, c_k] \).

a) We shall use \( c_r \in H^*(B_{U'(k)}) \) to simplify \( B^r_a(c_r) \).

b) Let \( p_r \in H^*(B_{SO(2k)}) \) be the \( r^{th} \) Pontryagin class if \( r < k \), and be \( e_{2k} \) if \( r = k \). This notion \( p_r \) will be reserved for both \( \pi^* p_r \in H^*(B_{\text{Spin}^c(2k)}) \) and \( \psi^* \pi^* p_r \in H^*(B_{\text{Spin}(2k)}) \), as well as the polynomials

\[
B_{\lambda^c}(p_r) \in H^*(B_{U(k)}) \text{ and } B_{\lambda^c}(p_r) \in H^*(B_{U'(k)})
\]

whose canonical expression is

\[
p_r = c_r^2 - 2 c_{r-1} c_{r+1} + \cdots + 2(-1)^{r-1} c_{2r-1} e_{2r} + 2(-1)^r c_{2r} \quad \text{(} [29] \text{ p.177)}
\]

where \( c_i = 0 \) if \( i > k \).

c) The notion \( e_{2k} \) for the Euler class of the canonical bundle on \( B_{SO(2k)} \) will also be used to denote either \( \pi^* e_{2k} \in H^*(B_{\text{Spin}^c(2k)}) \) or \( \psi^* \pi^* e_{2k} \in H^*(B_{\text{Spin}(2k)}) \). These conventions a), b) and c) above will not cause confusion, as in any circumstance the cohomologies or the homomorphisms involved will be clearly stated.

7.2. The cohomology \( H^*(B_{U'(k)}) \). To formulate the ring \( H^*(B_{U'(k)}) \) we note that the group \( U^c(k) \) has two obvious 1 dimensional unitary representations
whose Euler classes are $c_1$ and $B^*_U(q_0)$, respectively, where $q_0 = e^*(x)$ as in Theorem C. By the commutivity of the second diagram in (7.2) we have

$$\rho_2(B^*_U(q_0) - c_1) = B^*_U \circ B'_U(w_2)) - B'_U \circ B^*_U(w_2) = 0,$$

implying that the difference $B^*_U(q_0) - c_1 \in H^2(B_U(x))$ is divisible by 2. This brings us the integral class

$$y := \frac{1}{2}(B^*_U(q_0) - c_1) \in H^2(B_U(x))$$

by which the following result is transparent by Lemma 7.1.

**Lemma 7.2.** We have $H^*(B_U(x)) = \mathbb{Z}[y,c_1,\cdots,c_k]$ so that

i) $B^*_U(x) = c_r, 1 \leq r \leq k$;

ii) $B^*_U(z) = c_1,2c_1$ or $c_r$ for $z = y,c_1$ or $c_r$ with $r \geq 2$;

iii) $B^*_U(q_0) = 2y - c_1 \square$

In view of the formula $H^*(B_U(U);\mathbb{Z}_2) = \mathbb{Z}_2[\rho_2(y),\rho_2(c_1),\cdots,\rho_2(c_k)]$ by Lemma 7.2 we introduce the operation

$$R : H^{2k}(B_U(U);\mathbb{Z}_2) \rightarrow H^{4k}(B_U(U))$$

by the rules

$$(7.4) \quad R(u) := \begin{cases} 0 \text{ if } \rho_2(y) \text{ is a factor of } u; \\ p_1 \cdots p_s, \text{ if } u = \rho_2(c_1) \cdots \rho_2(c_s) \text{ is a monomial}; \\ R(u_1) + \cdots + R(u_j) \text{ if } u = u_1 + \cdots + u_j, \end{cases}$$

where $u_i$'s are distinct monomials in the $\rho_2(c_1),\cdots,\rho_2(c_k)$. The operation $R$ is not additive, but with $\rho_2(p_r) \equiv \rho_2(c_r)^2$ by (7.3) it satisfies that

$$\rho_2(R \circ \rho_2(u) - u^2) = 0, \ u \in H^*(B_U(U)).$$

Since the ring $H^*(B_U(U))$ is torsion free the formula

$$(7.5) \quad d(u) := \frac{1}{2}(R \circ \rho_2(u) - u^2), \ u \in H^*(B_U(U))$$

indicates a well defined operation $d : H^{2n}(B_U(U)) \rightarrow H^{4n}(B_U(U))$.

**Definition 7.3.** For a polynomial $u \in H^{2n}(B_U(U))$ the sequence $\{u_{(0)}, u_{(1)}, \cdots\}$ on $H^*(B_U(U))$ defined inductively by $u_{(0)} = u$ and $u_{(k+1)} = d(u_{(k)})$ will be called the derived sequence of the initial $u$. $\square$

Alternatively, without referring to the operation $d$ we have that

**Corollary 7.4.** For any $u \in H^*(B_U(U))$ the derive sequence $\{u_{(0)}, u_{(1)}, \cdots\}$ is characterized uniquely by the recurrence

$$u_{(0)} = u, \ 2u_{(r+1)} + u_{(r)}^2 = R \circ \rho_2(u_{(r)}), \ r \geq 0. \square$$
**Example 7.5.** The formulae (7.4) and (7.5) indicate an effective recurrence to produce the two sequences \( \{u_r\}, r \geq 0 \) and \( \{R \circ \rho_2(u_r)\}, r \geq 0 \) of polynomials from the initial one \( u \in H^\ast(B_{U(2)}) \) in the order illustrated below:

\[
\begin{align*}
\mathbf{u} & \; \Rightarrow \; u_{(1)} \; \Rightarrow \; u_{(2)} \\
R \circ \rho_2 & \; \Rightarrow \; R \circ \rho_2(u_{(1)}) \; \Rightarrow \; R \circ \rho_2(u_{(2)})
\end{align*}
\]

As a relevant example we take \( u = 2y - c_1 \in H^2(B_{U(2)}) \). For the small values \( r = 1, 2, 3 \) the polynomials \( u_r \) and \( R \circ \rho_2(u_r) \) so obtained are tabulated below:

| \( r \) | \( u_r \) | \( f_r := R \circ \rho_2(u_r) \) |
|---|---|---|
| 0 | \(-c_1 + 2y\) | \( p_1 \) |
| 1 | \(-c_2 + 2yc_1 - 2y^2\) | \( p_2 \) |
| 2 | \( c_3 - c_1 c_2 - 2y^2 c_1^2 - 2y c_1 c_2 + 2y^2 c_2 - 2y c_1 c_2 + 4y c_1^2 \) | \( p_4 + p_1 p_3 \) |
| 3 | \( c_4 - c_1 c_2 c_3 - 2y^2 c_1 c_2 - 2y^2 c_1 c_2 + 2y^2 c_2 - 2y^2 c_2 - 2y^2 c_2 + 2y^2 c_2 + 2y^2 c_1^2 \) | \( p_4^2 + p_2 p_4 + p_1 p_3 + p_1 p_5 + p_4 + p_2 p_5 + p_3 p_5 \) |

Table 1. The basic \( Spin^\ast(n) \) Weyl invariants \( u_r \) and their switch functions \( f_r \)

where, to emphasize that the elements \( R \circ \rho_2(u_r) \in H^\ast(B_{U(2)}) \) are in fact polynomials in the Pontryagin classes, we have written \( f_r(p_1, \ldots, p_k) \) in addition to \( R \circ \rho_2(u_r) \). For the latter convenience we call \( f_r \) the switch function from \( u_{r-1} \) to \( u_r \). It can be shown that

\[(7.6) \quad f_r(p_1, \ldots, p_k) \in (p_2, \ldots, p_k), \quad r > 1,
\]

where \( (p_2, \ldots, p_k) \) denotes the ideal of \( H^\ast(B_{U(2)}) \) generated by \( p_2, p_3, \ldots, p_k \).

### 7.3. The ring map \( B_{\chi} \): \( H^\ast(B_{Spin^\ast(2k)}) \to H^\ast(B_{U(2)}) \)

Granted with the conventions in 7.1 partial information on the ring map \( B_{\chi} \) has already known. Precisely we have

\[
B_{\chi}(c_{2k}) = c_k, \quad B_{\chi}(p_r) = p_r, \quad B_{\chi}(\tau(B_{Spin^\ast(n)})) = 0,
\]

where the third relation follows from the fact that the ring \( H^\ast(B_{U(2)}) \) is torsion free. In addition, as the generator \( \theta_n \in H^\ast(B_{Spin^\ast(n)}) \) (see in Theorem D) is the Euler class of the the complex spin presentation of the group \( Spin^\ast(n) \), the element \( B_{\chi}(\theta_n) \) is well understood in representation theory. Summarizing, to determine the ring map \( B_{\chi} \), it suffices to express the sequence

\[
\{B_{\chi}(q_r), 0 \leq r \leq h(n) - 1\}
\]

as explicit polynomials in \( H^\ast(B_{U(2)}) \). This is implemented in the next result.

**Theorem 7.6.** The sequence \( \{B_{\chi}(q_r), 0 \leq r \leq h(n) - 1\} \) on \( H^\ast(B_{U(2)}) \) is the first \( h(n) \) terms of the derived sequence \( \{u(1), u(2), \ldots\} \) of the initial polynomial \( u = c_1 - 2y \in H^2(B_{U(2)}) \) (see in Example 7.5).

**Proof.** It is easy to verify that the operation \( R \) on \( H^\ast(B_{U(2)}; \mathbb{Z}_2) \) is compatible with the operation \( f \) on \( H^\ast(B_{SO(2k)}; \mathbb{Z}_2) \) in Theorem B in the following sense:
On the other hand, by virtue of the second diagram in (7.2), applying the ring map $B^*_\lambda$ to the relation ii) and iii) of Theorem C gives rise to the relations

$$\rho_2 \circ B^*_\lambda(q_r) = B^*_\lambda \circ \pi^*(w^{(r)}_2) \text{ on } H^*(B_{U^r(k)}; \mathbb{Z}_2),$$

and

$$2B^*_\lambda(q_{r+1}) + B^*_\lambda(q_r)^2 = B^*_\lambda \circ \pi^* f(w^{(r)}_2), \ r \geq 0, \text{ on } H^*(B_{U^r(k)}),$$

respectively. Setting $u_{r+1} := B^*_\lambda(q_r)$ these imply that

$$2u_{r+1} + u_r^2 = B^*_\lambda \circ \pi^* f(w^{(r)}_2) \text{ (by (7.9))},$$

$$= R \circ B^*_\lambda \circ \pi^* f(w^{(r)}_2) \text{ (by (7.7))},$$

$$= R \circ \rho_2(u_r) \text{ (by (7.8))}.$$  

With $u_1 = 2g - c_1$ by iii) of Lemma 7.2, Corollary 7.5 concludes that the sequence \{u_r, r \geq 0\} is the derived sequence of the polynomial $u = 2g - c_1$.

### 7.4. The ring map $B^*_\lambda : H^*(B_{Spin(2k)}) \to H^*(B_{U^r(k)})$.

As in Example 7.5 let $u(r) = u(r)(g_1, c_1, \cdots, c_k) \in H^*(B_{U^r(k)}), r \geq 0,$ be the derived sequence of the initial polynomial $u = c_1 - 2g \in H^2(B_{U(k)})$. Applying the ring map $B^*_\lambda$ we get by ii) of Lemma 7.2 the following polynomials in $H^*(B_{U(k)})$

$$(7.10) \ g_r := B^*_\lambda(p_r) = p_r + 2(-1)^{r-1}c_1c_{2r-1} \text{ (see (7.3) for } p_r);$$

$$(7.11) \ \alpha_r := B^*_\lambda(u(r)) = u(r)(c_1, 2c_1, c_2, \cdots, c_k).$$

On the other hand, according to Theorem D’ and by the convention in 7.1, the ring $H^*(B_{Spin(2k)})$ is generated multiplicatively by the elements

$$\{p_r, 1 \leq r \leq k - 1\}; \{\overline{q}_r, 1 \leq r \leq h(n) - 1\}; e_{2k}, \overline{a}_n,$$

together with the torsion ideal $\tau(B_{Spin^e(n)})$. Thus, we obtain from Theorem 7.6, the relation $\overline{q}_r = \psi^*(q_r)$ by Theorem C’, as well as the commutivity of the first diagram in (7.2), the following result.

**Theorem 7.7.** The map $B^*_\lambda$ is determined by

i) $B^*_\lambda(p_r) = g_r, 1 \leq r \leq k - 1;$

ii) $B^*_\lambda(\overline{q}_r) = \alpha_r, 1 \leq r \leq h(n) - 1,$

and by $B^*_\lambda(e_{2k}) = c_k, B^*_\lambda(\tau(B_{Spin^e(n)})) = 0.$

**Example 7.8.** By (7.10) and (7.11) the sequence $\{\alpha_r, r \geq 1\}$ on $H^*(B_{U(k)})$ satisfies the following recurrence relations

$$\alpha_1 = -c_2 + 2c_1^2, \ \alpha_{r+1} = \frac{1}{2}(f_r(g_1, \cdots, g_k) - \alpha_r^2).$$

It enables one to evaluate the sequence from the switch polynomials $f_r(p_1, \cdots, p_k)$ obtained by Example 7.5 without referring to (7.11). For the values $1 \leq r \leq 3$ the polynomials so obtained are tabulated below.

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For a Remark 7.10.

respectively, the correspondences between homotopy sets

Spin

is also of geometric interests [10, 11]: the fiber manifold Spin(2k)/U(k) agrees with the Grassmanian of complex structures on the 2k dimensional Euclidean space \( \mathbb{R}^{2k} \); can be identified with the classifying space of the complex k bundles whose real (resp. Spin) reductions are trivial; and is also the space of minimal geodesics joining \( I_{2k} \) to \(-I_{2k}\) on SO(2k).

\[ r \begin{array}{c}
1 & \alpha_r \\
2 & -c_2 + 2c_4 \\
3 & c_4 - 2c_2 + 2c_2^2 - 2c_4^2 \\
\end{array} \]

Table 2. The basic Spin(n) Weyl invariants.

7.5. We conclude this section with the following result, which has been used in the proof of Lemma 6.4.

**Lemma 7.9.** The constant \( \kappa \) in the formula (6.7) is \((-1)^{k(n)-1} \cdot 2\).

**Proof.** Let \( D \) be the ideal on \( H^*(B_U(k)) = \mathbb{Z}[c_1, \ldots, c_k] \) generated by the \( c_r \) with \( r \geq 2 \), and consider the ring map \( e: H^*(B_U(k)) \to \mathbb{Z} \) defined by

\[ e(c_1) = 1, \; e(c_r) = 0, \; 2 \leq r \leq k. \]

Then

i) \( u = e(u) \cdot c_1^k + h(u) \) with \( h(u) \in D \) for any \( u \in H^{2k}(B_U(k)) \);

ii) \( e(u) = 0 \) if and only if \( u \in D \).

In particular, applying \( e \) to the relations

\[ 2\alpha_{r+1} + \alpha_r^2 = f_r(g_1, \ldots, g_{k-1}) \]

on \( H^*(B_U(k)) \) we get from \( e(\alpha_1) = 2 \) and \( f_r(g_1, \ldots, g_{k-1}) \in D \) by (7.6) that

\[ (7.14) \; e(\alpha_r) = (-1)^{r-1} \cdot 2. \]

On the other hand, since \( B^*_\lambda(\overline{\theta}_n) \) is the Euler class of the induced bundle \( B^*_\lambda\eta_n \) on \( B_U(k) \) we have

\[ (7.15) \; e(B^*_\lambda(\overline{\theta}_n)) = 1 \text{ (i.e. } B^*_\lambda(\overline{\theta}_n) = c_1^{k(n)} + \alpha \text{ for some } \alpha \in D). \]

Thus, applying the ring map \( e \circ B^*_\lambda \) to the equation (6.7) and noting that

\[ B^*_\lambda(\varphi(\overline{\theta}_n)) \in D \text{ (since } \overline{\theta}_n \in \prod \overline{\theta}(B_{SO(n)}(2k)) \otimes \Delta(\overline{\theta}_1, \ldots, \overline{\theta}_{k(n)-1})), \]

we obtain that \((-1)^{k(n)-1} \cdot 2 = \kappa\), completing the proof. \( \Box \)

**Remark 7.10.** For a CW complex \( X \) the maps \( B_{\lambda r} \), \( B_\lambda \) and \( B_\lambda^r \) induce, respectively, the correspondences between homotopy sets

\[ B_{\lambda r}: [X, B_U(k)] \to [X, B_{SO(2k)}] \] by \( B_{\lambda r}[g] = [B_{\lambda r} \circ g], \]

\[ B_{\lambda}: [X, B_U(k)] \to [X, B_{Spin(2k)}] \] by \( B_{\lambda}[g] = [B_{\lambda} \circ g], \]

\[ B_{\lambda^r}: [X, B_U(k)] \to [X, B_{Spin^r(2k)}] \] by \( B_{\lambda^r}[g] = [B_{\lambda^r} \circ g], \]

where \( B_{\lambda r} \) corresponds to the operation of real reduction on the complex bundles \([29]\) p.155. Likewise, the maps \( B_\lambda \) and \( B_{\lambda^r} \) can be regarded as the spin reduction and the spin^r reduction of complex bundles, respectively.

The map \( B_\lambda \) fits into the fibration \( Spin(2k)/U(k) \hookrightarrow B_U(k) \to B_{Spin(2k)} \) that is also of geometric interests [10][11]: the fiber manifold \( Spin(2k)/U(k) \) agrees with the Grassmanian of complex structures on the 2k dimensional Euclidean space \( \mathbb{R}^{2k} \); can be identified with the classifying space of the complex k bundles whose real (resp. Spin) reductions are trivial; and is also the space of minimal geodesics joining \( I_{2k} \) to \(-I_{2k}\) on SO(2k). \( \Box \)

21
The Weyl invariants of the group $Spin(n)$

Let $G$ be a compact connected Lie group with a maximal torus $T$, the Weyl group $W = N_G(T)/T$. The canonical $W$ action on $T$ induces an action on the integral cohomology $H^*(B_T)$. Denote by $H^*(B_T)^W$ the subring consisting of all the $W$ invariants, and let $B_t : B_T \to B_G$ be induced by the inclusion $t : T \to G$. A classical result of Borel [4] states that

**Lemma 8.1.** The ring map $B_t^*: H^*(B_G) \to H^*(B_T)$ annihilates the torsion ideal $\tau(B_G)$, and induces an injection $H^*(B_G)/\tau(B_G) \to H^*(B_T)^W$. □

The problem of presenting the ring $H^*(B_T)^W$ by explicit generators and relations is a classical topic of the representation theory [20, 39]. A closely related problem in topology is to decide the subring $\text{Im } B_t^* \subseteq H^*(B_T)^W$. For the matrix groups $G = U(n)$, $Sp(n)$ and $SO(n)$ one has $\text{Im } B_t^* = H^*(B_T)^W$, and the solutions to both problems are well known [18, §3. Examples]. For the case $G = Spin(n)$ the problems have been studied by Borel, Feshbach, Totaro, Benson and Wood [11, 16, 18, 55].

For an integer $n > 3$ we set $k = [n/2]$, and let $h$ be the inclusion of the diagonal subgroup $T = U(1) \times \cdots \times U(1)$ ($k$ copies) into $U(k)$. Then a convenient maximal torus on $Spin(n)$ is

$$t = \lambda \circ h : T \to U(k) \to Spin(2k) \subseteq Spin(n),$$

Furthermore, with respect to the canonical presentation $H^*(B_T) = \mathbb{Z}[x_1, \ldots, x_k]$, deg $x_i = 2$, the ring map $B_h^*: H^*(B_U(k)) \to H^*(B_T)$ is given by

$$B_h^*(e_r) = e_r(x_1, \ldots, x_k), 1 \leq r \leq k,$$

where $e_r$ is the $r^{th}$ elementary symmetric function in the $x_1, \ldots, x_k$.

It follows that the map $B_h^*$ carries $H^*(B_U(k))$ isomorphically onto the subring $\text{Sym}[x_1, \ldots, x_k] \subseteq H^*(B_T)$ of symmetric functions, while $B_h^*$ induces an injection from the quotient ring $H^*(B_{Spin(n)})/\tau(B_{Spin(n)})$ into $H^*(B_U(k))$ by Theorem 8.1. Thus, applying the ring map $B_h^*$ to the formula (6.8) of $H^*(B_{Spin(n)})$ we conclude by Theorem 7.7 that

**Theorem 8.2.** The subring $\text{Im } B_t^*$ has the presentations:

(8.1) $\text{Im } B_t^* = \left\{ \frac{Z_1(g_1, \ldots, g_k, \alpha_1, \ldots, \alpha_{k(n)-1}, B_n^*(\bar{\theta}_n))/D_n}{Z_2(g_1, \ldots, g_k, \alpha_1, \ldots, \alpha_{k(n)-1}, B_n^*(\bar{\theta}_n))/D_n} : n = 2k; \right.$

where $D_n$ denotes the ideal generated by the following relations

i) $2\alpha_1 - g_1, 2\alpha_{r+1} + \alpha_r - f_r(g_1, \ldots, g_k), 1 \leq r \leq k(n) - 2,$

ii) $(-1)^{k(n)-1}.4 \cdot B_n^*(\bar{\theta}_n) + \alpha_{k(n)-1} = \beta_n = B_n^*(\bar{\theta}_n) \in \text{Im } B_t^*.$

where $\text{Im } B_t^*$ denotes the subring generated by $g_1, \ldots, g_{k(n)-1}, \alpha_1, \ldots, \alpha_{k(n)-2},$ and $\alpha_k$ if $n = 2k$. □

**Corollary 8.3.** The quotient group $\text{Im } B_t^{2k(n)} / \text{Im } B_t^{2k(n)}$ is isomorphic to $\mathbb{Z}_2$ with generator $\alpha_{k(n)-1}$. □
For $G = \text{Spin}(n)$ the extension problem from $\text{Im} \ B^*_l$ to $H^*(B_T)^W$ is a subtle one \cite{13}, and has been solved by Benson and Wood \cite{3}. Attempt to bring the relevant calculations taking place in \cite{6} into our context gives rise to the following result.

**Lemma 8.4.** Assume that $G = \text{Spin}(n)$ with $n \geq 6$.

i) If $n \neq 3, 4, 5 \mod 8$ the ring $H^*(B_T)^W$ is isomorphic to $\text{Im} \ B^*_l$.

ii) If $n = 3, 4, 5 \mod 8$ the ring $H^*(B_T)^W$ is generated by its subring $\text{Im} \ B^*_l$, together with an invariant $\omega_{k(n)-1} \in H^{2k(n)}(B_T)^W$ that is related to the known invariants $B^*_X(\overline{g}_n)$ and $\alpha_{k(n)-1}$ via the relations

\[ a) \ \omega^2_{k(n)-1} = B^*_X(\overline{g}_n); \]

\[ b) \ 2\omega_{k(n)-1} - \alpha_{k(n)-1} = l_n, \quad l_n \in \text{Im} \ B^*_l. \]

**Proof.** In term of the classes $\eta_j, \mu_j \in H^*(B_T)$ introduced in \cite{6} §4 we set

\[ \omega_{k(n)-1} := \eta_{k(n)-1} \text{ or } \mu_{k(n)-1} \text{ if } n \equiv 3, 5 \mod 8 \text{ or } n \equiv 4 \mod 8. \]

Then \cite{6} Proposition 4.1] shows that

\[ \omega_{k(n)-1} \in H^*(B_T)^W \text{ and } \omega_{k(n)-1}^2 = B^*_X(\overline{g}_n), \]

where in the notation of \cite{6} Table 2

\[ B^*_X(\overline{g}_n) = \eta_{k(n)} \text{ or } \mu_{k(n)} \text{ if } n \equiv 3, 5 \mod 8 \text{ or } n \equiv 4 \mod 8. \]

Now, all the statements of the lemma can be verified by comparing \cite{6} Theorem 7.1 with \cite{6} Theorem 10.2, except for the relation b).

By \cite{6} Corollary 7.2] there exists a polynomial $l_n \in \text{Im} \ B^*_l$ so that

\[ (8.2) \ 2\omega_{k(n)-1} - q_{k(n)-1} = l_n, \]

where $q_{k(n)-1}$ is a term of the sequence \{$q_r, r \geq 1$\} on $H^*(B_T)^W$ constructed in the proof of \cite{6} Proposition 3.3], which generates also the quotient group

\[ \text{Im} \ B^*_l \cong \mathbb{Z}_2 \]

by properties ii) and iii) of \cite{6} Theorem 10.2]. By Lemma 8.3 we can replace the class $q_{k(n)-1}$ in (8.2) by $\alpha_{k(n)-1}$ to obtain b), completing the proof.$\Box$

Assume that $n = 3, 4, 5 \mod 8$. The relations a) and b) in Lemma 8.3 imply that the generators $B^*_X(\overline{g}_n)$ and $\alpha_{k(n)-1}$ of the subring $\text{Im} \ B^*_l \subseteq H^*(B_T)^W$ can be expressed as polynomials in the elements

\[ \omega_{k(n)-1}, g_2, \cdots, g_{\frac{n-1}{2}}, \alpha_1, \cdots, \alpha_{k(n)-2}, \text{ and } c_k \text{ if } n = 2k. \]

In addition, combining the relation b) with the relation on $\text{Im} \ B^*_l$

\[ 2\alpha_{k(n)-1} + \alpha_{k(n)-2}^2 = f_{k(n)-2}(g_1, \cdots, g_k) \text{ (by Theorem 8.2)} \]

one gets
\[ 4\omega_{k(n)-1} - \alpha_{k(n)-2}^2 = \varepsilon_n \]

for some \( \varepsilon_n \in \text{Im} B^*_\tau \). Thus, we obtain by Theorems 8.2 and Lemma 8.4 that

**Theorem 8.5.** Assume that \( G = \text{Spin}(n) \) with \( n \geq 6 \). The ring \( H^*(B^*_\tau)^W \) is isomorphic to \( \text{Im} B^*_1 \) with the following exceptions:

i) \( \mathbb{Z}[g_1, \cdots, g_k, \alpha_1, \cdots, \alpha_{k(n)-2}, \omega_{k(n)-1}]/D_n \) if \( n \equiv 3, 5 \mod 8 \);

ii) \( \mathbb{Z}[g_1, \cdots, g_{k-1}, c_k, \alpha_1, \cdots, \alpha_{k(n)-2}, \omega_{k(n)-1}]/D_n \) if \( n \equiv 4 \mod 8 \),

where \( D_n \) denotes the ideal generated by the following elements

\[
2\alpha_1 - g_1, \ 2\alpha_{r+1} + \alpha_r^2 - f_r(g_1, \cdots, g_k), \ 1 \leq r \leq k(n) - 3, \\
4 \cdot \omega_{k(n)-1} - \alpha_{k(n)-2}^2 - \varepsilon_n, \ \varepsilon_n \in \text{Im} B^*_\tau.
\]

**Remarks 8.6.** Theorems 8.2 and 8.4 present both of the rings \( \text{Im} B^*_1 \) and \( H^*(B^*_\tau)^W \) by explicit generators and relations, where formulae of the invariants \( \omega_{k(n)-1} \) and \( B^*_1(\overline{g}_n) \) can be found in [6]. In addition, for the small values \( n \leq 5 \) the ring \( H^*(B^*_\tau)^W \) is given in [6] Remark 7.3.

In [6] Theorem 7.1 Benson and Wood obtained a presentation of the ring \( H^*(B^*_\tau)^W \) without specifying the relations among the relevant generators. In our results the relations

\[
2\alpha_1 - g_1, \ 2\alpha_{r+1} + \alpha_r^2 - f_r(g_1, \cdots, g_k), \ 1 \leq r \leq k(n) - 2
\]

stem from Theorem C, which indicate a direct recurrence to produce the sequence \( \{\alpha_1, \alpha_2, \cdots\} \) of invariants from the initial one \( \alpha_1 = -c_2 + 2c_1^2 \).

9 The Spin characteristic classes

For the groups \( SO = \cup_{n=2}^\infty SO(n) \) and \( \text{Spin} = \cup_{n=2}^\infty \text{Spin}(n) \) in the stable range we have by formulae (2.3) and (6.8) that

\[
H^*(B_{SO}) = \mathbb{Z}[p_1, p_2, \cdots] \oplus 2 \cdot \tau(B_{SO}) = 0, \\
H^*(B_{Spin}) = \overline{\pi}^* H^*(B_{SO}) \oplus \mathbb{Z}[\overline{\tau}_1, \overline{\tau}_2, \cdots]/K^\infty,
\]

respectively. In term of the formulae we introduce the sequence \( \{Q_k, \ k \geq 1\} \), \( \deg Q_k = 4k \), of integral cohomology classes of \( B_{Spin} \) by setting

\[
Q_k := \bigg\{ \begin{array}{ll}
\overline{\pi}^* p_k & \text{if } k > 1 \text{ is not a power of } 2; \\
\overline{\tau}_r & \text{if } k = 2^r, \ r \geq 0.
\end{array}
\]

Then the relation ii) of Theorem C’ implies the formulae

\[
2Q_1 = \overline{\pi}^* p_1, \ 2Q_2 + Q^2_{2r-1} = \overline{\pi}^* f(w_2^{(r)}),
\]

where by Example 2.3 and the definition (2.4) of \( f \)

\[
f(w_2^{(r)}) = p_{2r} + p_1 p_{2r-2} + \cdots + p_{2r-1} p_{2r-1} + \text{ higher terms}.
\]

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These imply that every 2–power Pontryagin class $\pi^*p_{2^r}$ can be expressed as a polynomial in the elements $Q_k, k \geq 1$, plus some torsion element. Thus, we can eliminate both

$$\pi^*p_{2^r} \text{ and } 2Q_{2^r} + Q_{2^r-1}^2 = \pi^*f(w_2)$$

form the sets of generators and relations to obtain the following result from Theorem C', Theorem D', as well as the formula (6.6) of $\tau(B_{\text{Spin}}(n))$.

**Theorem 9.1.** The integral cohomology of $B_{\text{Spin}}$ has the presentation

\[(9.1) \ H^*(B_{\text{Spin}}) = \mathbb{Z}[Q_1, Q_2, Q_3, \cdots] \oplus \pi^*\tau(B_{\text{SO}}), \text{ deg } Q_k = 4k,\]

where the generators $Q_k$ are characterized uniquely by the following properties:

i) if $k > 1$ is not a power of 2, then $Q_k = \pi p_k$;

ii) if $k = 2^r$ with $r \geq 0$ then

$$\rho_2(Q_k) = \pi (w_2^{(k+1)}),$$

$$2Q_1 = \pi (p_1), \ 2Q_{2k} + Q_k^2 = \pi f(w_2^{(k+1)}), \ k \geq 1,$$

In addition, the product between the two summands in (9.1) is given by

$$Q_k \cup \pi^*(\delta_2(x)) = \begin{cases} \pi (\delta_2(x \cup w_2)) & \text{if } k > 1 \text{ is not a power of } 2; \\ \pi (\delta_2(x \cup w_2^{(k+1)})) & \text{if } k = 2^r. \end{cases} \Box$$

For a connected CW–complex $X$ denote by $\tilde{K}_{\text{SO}}(X)$ or $\tilde{K}_{\text{Spin}}(X)$ the reduced topological $K$–theories of the stable equivalence classes $[\xi]$ of the oriented or spin vector bundles $\xi$ over $X$, respectively. These theories admit the homotopy interpretations

$$\tilde{K}_{\text{SO}}(X) = [X, B_{\text{SO}}] \text{ and } \tilde{K}_{\text{Spin}}(X) = [X, B_{\text{Spin}}],$$

respectively. Furthermore, the operation of Whitney sums furnishes these sets with the structure of abelian groups that fit into the exact sequence

$$H^1(X; \mathbb{Z}_2) \to \tilde{K}_{\text{Spin}}(X) \xrightarrow{\tau^*} \tilde{K}_{\text{SO}}(X) \xrightarrow{w_2} H^2(X; \mathbb{Z}_2),$$

where $w_2$ evaluates on a class $[\xi] \in \tilde{K}_{\text{SO}}(X)$ the second Stiefel–Whitney class $w_2(\xi)$ of the oriented bundle $\xi$, and where $\pi[g] = [\pi \circ g]$.

In the formula (9.1) of the ring $H^*(B_{\text{Spin}})$ the torsion ideal $\pi^*\tau(B_{\text{SO}})$ contributes nothing essentially new, because it is fashioned from the ideal $\tau(B_{\text{SO}})$ which is well understood. It is the generators $\{Q_k, k \geq 1\}$ of the free part, together with their uniqueness property, are just what requested for us to introduce the spin characteristic classes for the $\tilde{K}_{\text{Spin}}$ theory.

**Definition 9.2.** For a spin vector bundle $\xi$ over a connected CW–complex $X$ let $f_\xi : X \to B_{\text{Spin}}$ be the classifying map of the stable equivalent class $[\xi] \in \tilde{K}_{\text{Spin}}(X)$. The cohomology class

$$q_k(\xi) := f_\xi(Q_k) \in H^{4k}(X), \ k \geq 1$$

(resp. the sum $q(\xi) := 1 + q_1(\xi) + q_2(\xi) + \cdots \in H^*(X)$)
is called the $k^{th}$ Spin characteristic class (resp. the total Spin characteristic class) of the bundle $\xi$. □

By the naturality of the pull back construction on vector bundles, the spin characteristic classes so defined possesses the naturality property:

**Corollary 9.3.** For any map $g : Y \to X$ between CW–complexes one has

$$q_k(g^*\xi) := g^*q_k(\xi), \quad k \geq 1.$$

In particular, the generator $Q_{2r}$ of $H^*(B_{Spin})$ restricts to the generator $\overline{\eta}_r$ of $H^*(B_{Spin}(n))$ (see (6.8)) via the inclusion $B_{Spin(n)} \subset B_{Spin}$.

For a spin vector bundle $\xi$ over $X$ its total Stiefel Whitney class and Pontryagin class

$$w(\xi) = 1 + w_3(\xi) + w_4(\xi) + \cdots \in H^*(X; \mathbb{Z}_2)$$

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \cdots \in H^*(X)$$

are also defined. Since $H^*(B_{SO}; \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, \cdots]$ is a free polynomial algebra, for each $u \in H^*(B_{SO}; \mathbb{Z}_2)$ we can formulate the cohomology classes

$$u(\xi) := u|_{w_k = w_k(\xi)} \in H^*(X; \mathbb{Z}_2).$$

$$f(u)(\xi) := f(u)|_{p_i = p_i(\xi), \delta_2(Sq^{2r}w_{2r+1}) = \delta_2(Sq^{2r}w_{2r+1}(\xi))} \in H^*(X).$$

of the space $X$, where $f : H^*(B_{SO}; \mathbb{Z}_2) \to H^*(B_{SO})$ is the operation (2.4) in the stable range. Alternatively, assuming that the spin bundle $\xi$ has the classifying map $g_\xi : X \to B_{Spin}$, then, as an oriented real bundle, $\xi$ has the classifying map $\overline{\eta} \circ g_\xi$, implying that

$$u(\xi) = (\overline{\eta} \circ g_\xi)^*(u), \quad f(u)(\xi) = (\overline{\eta} \circ g_\xi)^*(f(u)).$$

Therefore, applying the induced cohomology ring map $g_\xi^*$ to the relations i) and ii) of Theorem 9.1 yields the following result.

**Corollary 9.4.** For a spin bundle $\xi$ over a CW complex $X$ the three classes characteristic classes $w(\xi)$, $p(\xi)$ and $q(\xi)$ are subject to the following constraints

i) $p_2(q_k(\xi)) = w_2^{(r)}(\xi)$ or $w_2k(\xi)^2$ if $k = 2^r$ or $k \neq 2^r$;

ii) $q_k(\xi) = p_k(\xi)$ if $k \neq 2^r$; $2q_{2k}(\xi) + q_k(\xi)^2 = f(w_2^{(r)}(\xi))$ if $k = 2^r$. □

Concerning results of Corollary 9.4 two cautions are called for.

i) the sequence $\{w_2^{(r)}(\xi), r \geq 1\}$ on $H^*(X; \mathbb{Z}_2)$ can not be regarded as the derived sequence of $w_2(\xi)(= 0)$, because the base space $X$ may not be $\delta_2$ formal and therefore, the operation $\gamma$ in Theorem 1 may not be applicable to $X$.

ii) if the cohomology of $X$ is torsion free, the properties i) and ii) of Corollary 9.4 characterize the Spin classes $\{q_r(\xi), r \geq 1\}$ uniquely. However, this fails if the cohomology of $X$ has torsion elements of order 2. As being pointed out by Thomas [39] that, using the Bott divisibility theorem, one can construct on the
Moor space \( M(\mathbb{Z}_4; 2^r) \) a spin vector bundle \( \eta_r \) for which \( p(\eta_r) = 1 \), \( w(\eta_r) = 1 \) but \( q(\eta_r) \neq 1 \).

**Remark 9.6.** The idea of Spin characteristic classes was initiated by Thomas in [36], who described the ring \( H^*(B\text{Spin}) \) using a sequence \( \{Q_j\} \) of generators that is subject to two sequences \( \{\Phi_j\} \) and \( \{\Psi_j\} \) of indeterminacies. Comparing Theorem 9.1 with Thomas’s result [36, Theorem (1.2)] one sees that these ambiguities can be clarified by taking

\[
\Phi_{4j} = f(w_2^{(r)}) - \pi^*p_j \quad \text{and} \quad \Psi_{4j} = w_2^{(r)} - \pi^*w_{2j}, \ j = 2^r. \]

\[\square\]

### 10 Rokhlin type formulae in spin geometry

For a smooth manifold \( M \) the total Pontryagin characteristic class \( p(M) \) is defined to be that of the tangent bundle \( TM \) on \( M \), and will be denoted by

\[
p(M) := p(TM) = 1 + p_1 + \cdots + p_k, \quad k = \left\lfloor \frac{\dim M}{4} \right\rfloor.
\]

Similarly, if \( M \) is spin (i.e. \( w_2(M) = 0 \)) its total Spin characteristic class is defined, and will be denoted by

\[
q(M) := 1 + q_1 + \cdots + q_k, \quad k = \left\lfloor \frac{\dim M}{4} \right\rfloor.
\]

In addition, ignoring the torsion elements, the relations in ii) of Corollary 9.4 enable us to express the Pontryagin classes by the Spin classes, such as

\[
(10.1) \quad p_1 = 2q_1, \quad p_2 = 2q_2 + q_1^2, \quad p_3 = q_3, \quad p_4 = 2q_4 + q_2^2 - 2q_1q_3, \quad \cdots.
\]

In the topological approaches to spin geometry the Spin characteristic classes appears more convenient to use than the Pontryagin classes. In this section we provide such evidences motivated by the classical Rokhlin Theorem [32, 19]. Further examples are devoted to the subsequent works [13, 14].

**Example 10.1.** The \( n \) dimensional complex projective space \( \mathbb{C}P^n \) is spin if and only if \( n \) is odd. The \( n \) dimensional quaternion projective space \( \mathbb{H}P^n \) is 3 connected, hence is always spin. Let \( x \in H^2(\mathbb{C}P^n) \) and \( z \in H^4(\mathbb{H}P^n) \) be the Euler classes of the canonical complex line bundle on \( \mathbb{C}P^n \) and the canonical quaternion line bundle on \( \mathbb{H}P^n \), respectively. For \( M = \mathbb{C}P^n \) or \( \mathbb{H}P^n \) and for small values of \( n \), we tabulate both \( p(M) \) and \( q(M) \) in the table below

| \( M \)  | \( p(M) \)  | \( q(M) \)  |
|--------|-------------|-------------|
| \( \mathbb{C}P^3 \) | 1 + 4x^2   | 1 + 2x^2    |
| \( \mathbb{C}P^5 \) | 1 + 6x^2 + 15x^4 | 1 + 3x^4 + 3x^4 |
| \( \mathbb{H}P^2 \) | 1 + 2z + 7z^2 | 1 + z + 3z^2  |
| \( \mathbb{H}P^4 \) | 1 + 4z + 12z^2 + 8z^4 | 1 + 2z + 4z^2 + 8z^4 |

where we notice that the polynomials \( q(M) \) are always simpler than \( p(M) \). \( \square \)

In general, let \( M^{4m} \) be a \( 4m \) dimensional smooth manifold. Recall that the \( \widetilde{A} \) genus \( \alpha_m \) and the \( L \) genus \( \tau_m \) of \( M^{4m} \) are polynomials in the Pontryagin class \( p_1, \ldots, p_m \) with the forms
\[ \alpha_m = a_m \cdot p_m + l_m(p_1, \cdots, p_{m-1}) \] and
\[ \tau_m = b_m \cdot p_m + k_m(p_1, \cdots, p_{m-1}) \text{ [29 §19]}, \]

where \( a_m \) and \( b_m \) are non-zero rationals, and where \( l_m \) and \( k_m \) are certain polynomials in \( p_1, \cdots, p_{m-1} \) of homogeneous degree 4\( m \). This allows us to eliminate \( p_m \) to obtain the expression of \( \tau_m \) without involving \( p_m \)

\[ (10.2) \quad \tau_m = \frac{b_m}{a_m} (\alpha_m - l_m(p_1, \cdots, p_{m-1})) + k_m(p_1, \cdots, p_{m-1}). \]

For the geometric implications of the polynomials \( \alpha_m \) and \( \tau_m \) we recall the following classical results.

**Lemma 10.2 (Hirzebruch [29, p.224])** The \( L \)-genus \( \tau_m \) of \( M^{4m} \) equals to the signature \( \sigma_M \) of the intersection form on \( H^{2m}(M^{4m}). \)

**Lemma 10.3 (Borel–Hirzebruch)** If the manifold \( M^{4m} \) is spin then its \( \hat{A} \)

**Lemma 10.4 (Gromov, Lawson and Stolz [26, 33]).** If \( M^{4m} \) is a simply connected spin manifold with \( m > 1 \), then \( M^{4m} \) admits a metric with positive scalar curvature if and only if \( \alpha_m = 0. \)

Precisely, for \( 1 \leq m \leq 4 \) the polynomials \( \alpha_m \) and \( \tau_m \) are, respectively,

\[ \begin{align*}
\alpha_1 &= -\frac{1}{77}p_1; \\
\alpha_2 &= \frac{1}{2739}(-4p_2 + 7p_1^2); \\
\alpha_3 &= \frac{1}{378629}(-16p_3 + 44p_2p_1 - 31p_1^3); \\
\alpha_4 &= \frac{1}{378629}(-192p_4 + 512 \cdot p_1p_3 + 208p_2^2 - 904p_1^3p_2 + 381p_1^4),
\end{align*} \]

and

\[ \begin{align*}
\tau_1 &= \frac{1}{7}p_1; \\
\tau_2 &= \frac{1}{378629}(7p_2 - p_1^2); \\
\tau_3 &= \frac{1}{378629}(62p_3 - 13p_2p_1 + 2p_1^3); \\
\tau_4 &= \frac{1}{378629}(381p_4 - 71 \cdot p_1p_3 - 19p_2^2 + 22p_1^3p_2 - 3p_1^4).
\end{align*} \]

Assume now that our manifold \( M \) is spin (i.e. \( w_2(M) = 0 \)). Then the formulae in (10.1) is applicable to replace the Pontryagin classes \( p_1, \cdots, p_{m-1} \) in (10.2) to yield the following surprisingly simple expressions of the signature \( \sigma_M = \tau_m \) (by Lemma 10.2) by the Spin characteristic classes.

**Theorem 10.5.** In accordance to \( m = 1, 2, 3 \) and 4, the signature \( \sigma_M \) of a smooth spin manifold \( M^{4m} \) satisfy, respectively, that

\[ \begin{align*}
\sigma_M &= -2^4 \cdot \alpha_1; \\
\sigma_M &= q_1^2 - 2^5 \cdot (2^3 - 1) \cdot \alpha_2; \\
\sigma_M &= \frac{2}{3} (q_1q_2 - q_1^3) - 2^7 \cdot (2^3 - 1) \cdot \alpha_3; \\
\sigma_M &= \frac{2}{3} (q_1q_4 + \frac{1}{3} q_2^2 - 2^3 q_1^2q_2 + \frac{17}{3} \cdot q_1^4 - 2^9 (2^7 - 1) \cdot \alpha_4. \quad \square
\end{align*} \]
Example 10.6. For a smooth spin manifold $M$ one has by Theorem 10.5 and Lemma 10.3 that

i) if $\dim M = 4$, then $\sigma_M \equiv 0 \mod 2^4$;

ii) if $\dim M = 8$, then $\sigma_M - q_1^2 \equiv 0 \mod 2^5 \cdot (2^3 - 1)$.

iii) if $\dim M = 12$, then $\sigma_M - \frac{2}{3}(q_1 q_2 - q_1^3) \equiv 0 \mod 2^8 \cdot (2^5 - 1)$,

where assertion i) is primarily due to Rokhlin \[32, 19\]. For this reason the formulae in Theorem 10.5 may be called the Rokhlin type formulae of spin manifolds. □

Example 10.7. The string group $\text{String}(n)$ is the 3–connected cover of $\text{Spin}(n)$. It follows that a spin manifold is string \[34\] if and only if its first spin characteristic class $q_1$ vanishes. In this case the second spin characteristic class $q_2$ can be shown to be divisible by 3 \[27\]. For a smooth string manifold $M$ we have by Lemma 10.3 and Theorem 10.5 the following Rokhlin type formulae.

i) if $\dim M = 8$, then $\sigma_M \equiv 0 \mod 2^5 \cdot (2^3 - 1)$.

ii) if $\dim M = 12$, then $\sigma_M \equiv 0 \mod 2^8 \cdot (2^5 - 1)$.

iii) if $\dim M = 16$, then $\sigma_M - \frac{1}{3}(q_1^2) \equiv 0 \mod 2^9 \cdot (2^7 - 1)$.

Moreover, if $M$ is a simply connected, then $M$ admits a metric with positive scalar curvature if and only if

$$\sigma_M = 0, 0 \text{ or } \frac{1}{3}(q_2)^2$$

in accordance to $\dim M = 8, 12$ or 16 by Lemma 10.4. □

The formulae in Theorem 10.5 are also useful to study the existence of smooth structures on certain triangulable manifolds. To provide such examples we need the following notation.

Definition 10.8. For a unimodular symmetric integral matrix $A = (a_{ij})_{n \times n}$ of rank $n$, and a sequence $b = (b_1, \cdots, b_n)$ of integers with length $n$, the pair $(A, b)$ is called a Wall pair if the following congruences are satisfied

$$a_{ii} \equiv b_i \mod 2, 1 \leq i \leq n. \quad (10.3)$$

Let $D^8$ be the unit disk on the Euclidean space $\mathbb{R}^8$. The following result is due to C.T.C. Wall \[38\].

Theorem 10.9. For each Wall pair $(A, b)$ with $A = (a_{ij})_{n \times n}$ and $b = (b_1, \cdots, b_n)$, there exists a closed 8 dimensional topological manifold $M$ that satisfies the following properties

i) $M$ admits a decomposition $M = W \cup_h D^8$, where $W$ is a 3 connected smooth manifold with boundary $\partial W$ a homotopy 7–sphere, and $h : \partial W \to \partial D^8$ is a homeomorphism;

ii) there is a basis $\{x_1, \cdots, x_n\}$ on $H^4(M)$ so that $x_i \cup x_j = a_{i,j} \omega_M$, where $\omega_M \in H^8(M)$ is an orientation class on $M$;

iii) the first Spin characteristic class $q_1$ of $M$ is well defined (in view of i)), and is determined by $b$ as $q_1 = b_1x_1 + \cdots + b_n x_n \in H^4(M)$. 29
Furthermore, if \((A', b')\) is a second Wall pair, then the associated manifold \(M'\) is combinatorially homeomorphic to \(M\) (in the sense of [38]) if and only if there exists an integer matrix \(P = (p_{i,j})_{n \times n}\) so that
\[
P^{-1}AP = A' \text{ and } bP = b'.
\]
□

Remark 10.10. In [38] Wall classified the combinatorial homeomorphism types of all the \((n - 1)\) connected 2\(n\) dimensional manifolds \(M\) that are smooth off one point \(o \in M\). The result in Theorem 10.9 corresponds to the case \(n = 4\).

It is known that for a 4–dimensional real vector bundle \(\xi\) on the 4 dimensional sphere \(S^4\) the difference \(2e(\xi) - p_1(\xi)\) (resp. \(e(\xi) - q_1(\xi)\)) is divisible by 4 (resp. by 2), where \(e(\xi)\) is the Euler class of \(\xi\) [29 Lemma 20.10]. In Theorem 10.9 the necessity of the Wall condition (10.3) is governed by the following geometric fact. According to Haefliger [21], for the manifold \(W\) in i) of Theorem 10.9, there exist \(n\) smooth embeddings \(\iota_i : S^4 \to W\), \(1 \leq i \leq n\), so that the Kronecker dual of the cycle classes \(\iota_i^*[S^4] \in H_4(M)\) is the basis \(\{x_1, \cdots, x_n\}\) on \(H^4(M)\). Then, the matrix \(A\) is the intersection form on \(H^4(M)\) corresponding to the basis, while the normal bundle \(\gamma_i\) of the embedding \(\iota_i\) is related to the pair \((A, b)\) by the relations
\[
(e(\gamma_i), q_1(\gamma_i)) = (a_{ii} \cdot \omega, b_i \cdot \omega), \quad 1 \leq i \leq n,
\]
where \(\omega\) is the orientation class on \(S^4\) that corresponds \(x_i\) via \(\iota_i\).

For an 8–dimensional manifold \(M\) associated to a Wall pair \((A, b)\) properties ii) and iii) of Theorem 10.9 imply, respectively, that
\[
\sigma_M = \text{sign}(A) \quad \text{and} \quad q_1(M)^2 = bAb^T,
\]
where \(b^T\) denotes the transpose of the row vector \(b\).

Concerning the manifold \(M\) associated to a Wall pair \((A, b)\) a natural question is whether there exists a smooth structure that extends the given one on \(W\). For the special case \(A = (1)_{1 \times 1}\) this problem has been studied by Milnor [28], Eells and Kuiper [15, §6] in their calculation on the group \(\Theta_7\) of homotopy 7 spheres.

Theorem 10.11. Let \(M^8\) be the manifold associated to a Wall pair \((A, b)\). There exists a smooth structure on \(M^8\) extending the one on \(W\) if and only if
\[
\text{(10.5) } \text{sign}(A) \equiv bAb^T \pmod{2^5 \cdot (2^3 - 1)}.
\]

Proof. The necessity of (10.5) comes from ii) of Example 10.6, while the sufficiency is verified by computing with the Eells-Kuiper \(\mu\) invariant [15 formula (11)] of the boundary \(\partial W\), which in our notation reads
\[
\mu(\partial W) \equiv \frac{4bAb^T - 4\text{sign}(A)}{2 \cdot (2^3 - 1)} \equiv \frac{bAb^T - \text{sign}(A)}{2 \cdot (2^3 - 1)} \pmod{1}.
\]

Theorem 10.11 has several notable consequences. A theorem of Kervaire states that there exist a 10 dimensional manifold which do not admit any smooth structure [24]. Eells and Kuiper provided further examples which have the same cohomology ring as that of the projective plane [16].

Corollary 10.12. If \((A, b)\) is a Wall pair so that

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\[ \text{sign}(A) \neq bAb^* \text{ mod } 2^5 \cdot (2^3 - 1), \]
then the corresponding manifold \( M \) does not admit any smooth structure. \( \square \)

Conversely, for those \( M \) which do admit smooth structures, their total Spin characteristic class \( q(M) \) can be determined completely.

**Corollary 10.13.** If the manifold \( M \) associated to a Wall pair \((A, b)\) admits a smooth structure, then its total Spin characteristic class is

\[ (10.6) \quad q(M) = 1 + (b_1 x_1 + \cdots + b_1 x_n) + \frac{3(15 \cdot \text{sign}(A) - bAb^*)}{2(2^5 - 1)} \cdot \omega_M, \]

where \( 2 \cdot (2^3 - 1) \) divides \( 15 \cdot \text{sign}(A) - bAb^* \) by ii) of Example 10.6.

**Proof.** With \( \tau_2 = \sigma_M = \text{sign}(A) \) and \( p_2^2 = 4bAb^* \) the formula \( \tau_2 = \frac{7p_2 - p_2^2}{3 \cdot 5} \)
implies

\[ p_2 = \frac{45 \cdot \text{sign}(A) + 4bAb^*}{7} \cdot \omega_M. \]

From \( p_2 = 2q_2 + q_1^2 \) by (10.1) we get \( q_2 = \frac{3(15 \cdot \text{sign}(A) - bAb^*)}{2(2^5 - 1)} \cdot \omega_M. \) \( \square \)

In view of the formula of \( q_2 \) given in (10.6) Lemma 10.4 implies that:

**Corollary 10.14.** For a manifold \( M \) associated to a Wall pair \((A, b)\) the following statements are equivalent:

i) \( M \) is smooth and admits a metric with positive scalar curvature

ii) \( \text{sign}(A) = bAb^* \);

iii) \( q_2 = 3\text{sign}(A) \cdot \omega_M. \) \( \square \)

**Example 10.15.** Corollary 10.14 reduces the problem of finding all the 3 connected 8 dimensional smooth manifolds that have a metric with positive scalar curvature to the arithmetic problem of finding those Wall pairs \((A, b)\) satisfying the quadratic equation ii) above.

Consider a Wall pair \((A, b)\) with \( A \) the identity matrix of rank \( n \), and with \( b = (2k_1 + 1, \cdots, 2k_n + 1), k_i \in \mathbb{Z} \) (see (10.3)). The equation ii) is

\[ n = (2k_1 + 1)^2 + \cdots + (2k_n + 1)^2. \]

It implies that the corresponding manifold \( M \) is smooth and admits a metric with positive scalar curvature, if and only if \( M \) is combinatorially homeomorphic to \( \mathbb{H}P^2 \# \cdots \# \mathbb{H}P^2 \), the connected sum of \( n \) copies of the projective plane \( \mathbb{H}P^2 \).

Consider next a Wall pair \((A, b)\) with \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( b = (2k_1, 2k_2), k_i \in \mathbb{Z} \) (see (10.3)). The equation ii) turns to be \( k_1 \cdot k_2 = 0 \). It implies that the corresponding manifold \( M \) is smooth and admits a metric with positive scalar curvature, if and only if \( M \) is combinatorially homeomorphic to the spherical bundle \( S(\xi) \) of a 5 dimensional Euclidean bundle \( \xi \) over \( S^4 \). Such manifolds \( S(\xi) \) are classified by the homotopy group \( \pi_3(SO(5)) \).

About this topic general cases will be studied in the sequel work \( [14] \). \( \square \)
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