Statistical Constraints

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Abstract. We introduce statistical constraints, a declarative modelling tool that links statistics and constraint programming. We discuss two novel statistical constraints and some associated filtering algorithms. Finally, we illustrate applications to standard problems encountered in statistics and to a novel inspection scheduling problem in which the aim is to find inspection plans featuring desirable statistical properties.

1 INTRODUCTION

Informally speaking, a statistical constraint exploits statistical inference to determine what assignments satisfy a given statistical property at a prescribed significance level. For instance, a statistical constraint may be used to determine, for a given distribution, what values for one or more of its parameters, e.g. the mean, are consistent with a given set of samples. Alternatively, it may be used to determine what sets of samples are compatible with one or more hypothetical distributions. In this work, we introduce the first two examples of statistical constraints embedding two well-known statistical tests: the $t$-test and the Kolmogorov-Smirnov test. We discuss for the first time filtering algorithms for statistical constraints and applications spanning from standard problems encountered in statistics to a novel inspection scheduling problem in which the aim is to find inspection plans featuring desirable statistical properties.

2 FORMAL BACKGROUND

In this section we introduce the relevant formal background.

2.1 Statistical inference

A probability space, as introduced in [5], is a mathematical tool that aims at modelling a real-world experiment consisting of outcomes that occur randomly. As such it is described by a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ denotes the sample space — i.e. the set of all possible outcomes of the experiment; $\mathcal{F}$ denotes the sigma-algebra on $\Omega$ — i.e. the set of all possible events on the sample space, where an event is a set that includes zero or more outcomes; and $\mathbb{P}$ denotes the probability measure — i.e. a function $\mathbb{P}: \mathcal{F} \rightarrow [0, 1]$ returning the probability of each possible event. A random variable $\omega$ is an $\mathcal{F}$-measurable function $\omega: \Omega \rightarrow \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ mapping its sample space to the set of all real numbers. Given $\omega$, we can ask questions such as “what is the probability that $\omega$ is less than or equal to element $s \in \mathbb{R}$.” This is clearly the probability of event

$\{ o : \omega(o) \leq s \} \in \mathcal{F}$, which is often written as $F_\omega(s) = \Pr(\omega \leq s)$, where $F_\omega(s)$ is the cumulative distribution function (CDF) of $\omega$.

A multivariate random variable is a random vector $(\omega_1, \ldots, \omega_n)^T$, where $^T$ denotes the “transpose” operator. A random vector may be used to represent an experiment repeated $n$ times, i.e. a sampling, where each replica $i$ generates a sample $\omega_i'$ and the result of the experiment is the vector of samples $(\omega_1', \ldots, \omega_n')^T$.

Consider a multivariate random variable defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{D}$ be a set of possible CDFs on the sample space $\Omega$. In what follows, we adopt the following definition of a statistical model [6].

Definition 1 A statistical model is a pair $(\mathcal{D}, \Omega)$.

Let $\mathcal{D}$ denote the set of all possible CDFs on $\Omega$. Consider a finite-dimensional parameter set $\Theta$ together with a function $g: \Theta \rightarrow \mathcal{D}$, which assigns to each parameter point $\theta \in \Theta$ a CDF $F_\theta$ on $\Omega$.

Definition 2 A parametric statistical model is a triple $(\Theta, g, \Omega)$.

Definition 3 A non-parametric statistical model is a pair $(\mathcal{D}, \Omega)$.

Note that there are also semi-parametric models, which however for the sake of brevity we do not cover in the following discussion.

Consider now the outcome $o \in \Omega$ of an experiment. Statistics operates under the assumption that there is a distinct element $d \in \mathcal{D}$ that generates the observed data $o$. The aim of statistical inference is then to determine which element(s) are likely to be the one generating the data. A widely adopted method to carry out statistical inference is hypothesis testing.

In hypothesis testing the statistician selects a significance level $\alpha$ and formulates a null hypothesis, e.g. “element $d \in \mathcal{D}$ has generated the observed data,” and an alternative hypothesis, e.g. “another element in $\mathcal{D}/d$ has generated the observed data.” Depending on the type of hypothesis formulated, she must then select a suitable statistical test and derive the distribution of the associated test statistic under the null hypothesis. By using this distribution, one determines the probability $p_o$ of obtaining a test statistic at least as extreme as the one associated with outcome $o$, i.e. the “$p$-value”.

If this probability is less than $\alpha$, this means that the observed result is highly unlikely under the null hypothesis, and the statistician should therefore “reject the null hypothesis.” Conversely, if this probability is greater or equal to $\alpha$, the evidence collected is insufficient to support a conclusion against the null hypothesis, hence we say that one “fails to reject the null hypothesis.”

In what follows, we will survey two widely adopted tests [12]. A parametric test: the Student’s $t$-test [13]; and a non-parametric one: the Kolmogorov-Smirnov test [4,14]. These two tests are relevant in the context of the following discussion.

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2.1.1 Student’s t-test

A t-test is any statistical hypothesis test in which the test statistic follows a Student’s t distribution if the null hypothesis is supported.

The classic one-sample t-test compares the mean of a sample to a specified mean. We consider the null hypothesis \( H_0 \) that “the population mean is equal to a specified value \( \mu \).” The test statistic is

\[
t = \frac{\bar{x} - \mu}{s / \sqrt{n}}
\]

where \( \bar{x} \) is the sample mean, \( s \) is the sample standard deviation and \( n \) is the sample size. Since Student’s t distribution is symmetric, \( H_0 \) is rejected if \( Pr(x > t|H_0) < \alpha/2 \) or \( Pr(x < t|H_0) < \alpha/2 \) that is

\[
\mu < \bar{x} + \frac{s}{\sqrt{n}}T^{-1}_{n-1}(\alpha/2) \quad \text{or} \quad \mu > \bar{x} - \frac{s}{\sqrt{n}}T^{-1}_{n-1}(\alpha/2)
\]

where \( T^{-1}_{n-1} \) denotes the inverse Student’s t distribution with \( n-1 \) degrees of freedom.

The two-sample t-test compares means \( \mu_1 \) and \( \mu_2 \) of two populations. We consider the special case in which sample sizes are different, but variance is assumed to be equal for the two populations. The test statistic is

\[
t = \frac{\bar{x}_1 - \bar{x}_2}{s} = \sqrt{\frac{\frac{s^2}{n_1} + \frac{s^2}{n_2}}{n_1 + n_2 - 2}} \cdot \frac{n_1 n_2}{n_1 + n_2} - 2(\alpha) \sqrt{\frac{\frac{1}{n_1} + \frac{1}{n_2}}{n_1 n_2}} \geq 0
\]

Null hypothesis such as \( \mu_1 > \mu_2 \) or \( \mu_1 \neq \mu_2 \) are tested in a similar fashion.

2.1.2 Kolmogorov-Smirnov test

The one-sample Kolmogorov-Smirnov (KS) test is a non-parametric test used to compare a sample with a reference CDF defined on a continuous support under the null hypothesis \( H_0 \) that the sample is drawn from such reference distribution.

Consider a sample \( s = (\omega_1, \ldots, \omega_n)^T \) and assume that \( \omega_i \), for \( i = 1, \ldots, n \) are independent and identically distributed random variables. The empirical CDF \( F_s \) is defined as

\[F_s(x) = \frac{1}{n} \sum_{i=1}^{n} I(\omega_i \leq x)\]

where the indicator function \( I(\omega_i \leq x) \) is 1 if \( \omega_i \leq x \) and 0 otherwise. For a target CDF \( F \), let

\[d_s^+ = \sqrt{n} \sup_{x \in S} F_s(x) - F(x) \quad \text{and} \quad d_s^- = \sqrt{n} \sup_{x \in S} F(x) - F_s(x)\]

the KS statistic is

\[d_s = \max(d_s^+, d_s^-)\]

where \( \sup_{x \in \mathbb{Z}} \) is the supremum of the set of distances between the empirical and the target CDFs. Under the null hypothesis, \( d_s \) converges to the Kolmogorov distribution. Therefore, the null hypothesis is rejected if \( Pr(x > d_s|H_0) < \alpha \), that is \( 1 - K(d_s) < \alpha \), where \( K(t) \) is the CDF of the Kolmogorov distribution, which can be numerically approximated \([8],[13]\).

The single-tailed one-sample KS test can be used to determine if the sample is drawn from a distribution that has first-order stochastic dominance over the reference distribution — i.e. \( F_s \leq F(x) \) for all \( x \in S \) and with a strict inequality at some \( x \) — in which case the relevant test statistic is \( d_s^n \); or vice-versa, in which case the relevant test statistic is \( d_s^m \).

Note that the inverse Kolmogorov distribution \( K^{-1} \) can be employed to set a confidence band around \( F \). Let \( d_a = K^{-1}(1-\alpha) \), then with probability 1 - \( \alpha \) a band of \( \pm d_a \) around \( F \) will entirely contain the empirical CDF \( F_s \).

The two-sample KS test compares two sample populations \( s_1 \) and \( s_2 \) under the null hypothesis \( H_0 \) that these samples are drawn from the same distribution. In this case, let

\[d_s^+ = \sqrt{\frac{n_1 n_2}{n_1 + n_2} \sup_{x \in S} F_{s_1}(x) - F_{s_2}(x)}\]

\[d_s^- = \sqrt{\frac{n_1 n_2}{n_1 + n_2} \sup_{x \in S} F_{s_2}(x) - F_{s_1}(x)}\]

the test statistic is

\[d_s = \max(d_s^+, d_s^-)\]

Finally, also in this case it is possible to perform single-tailed tests using test statistics \( d_s^m \) or \( d_s^m \) to determine if one of the samples is drawn from a distribution that stochastically dominates the one from which the other sample is drawn.

2.2 Constraint programming

A Constraint Satisfaction Problem (CSP) is a triple \((V, C, D)\), where \( V \) is a set of decision variables, \( D \) is a function mapping each element of \( V \) to a domain of potential values, and \( C \) is a set of constraints stating allowed combinations of values for subsets of variables in \( V \). A solution to a CSP is an assignment of variables to values in their respective domains such that all of the constraints are satisfied. The constraints used in constraint programming are of various kinds: e.g. logic constraints, linear constraints, and global constraints \([7]\). A global constraint captures a relation among a non-fixed number of variables. Constraints typically embed dedicated filtering algorithms able to remove provably infeasible or suboptimal values from the domains of the decision variables that are constrained and, therefore, to enforce some degree of consistency, e.g. arc consistency, bound consistency \([7]\) or generalised arc consistency. A constraint is generalised arc consistent if and only if, when a variable is assigned any of the values in its domain, there exist compatible values in the domains of all the other variables in the constraint. Filtering algorithms are repeatedly called until no more values are pruned. This process is called constraint propagation. In addition to constraints and filtering algorithms, constraint solvers also feature a heuristic search engine, e.g. a backtracking algorithm. During search, the constraint solver explores partial assignments and exploits filtering algorithms in order to proactively prune parts of the search space that cannot lead to a feasible or to an optimal solution.
3 STATISTICAL CONSTRAINTS

Definition 4 A statistical constraint is a constraint that embeds a parametric or a non-parametric statistical model and a statistical test with significance level $\alpha$ that is used to determine which assignments satisfy the constraint.

A parametric statistical constraint $c$ takes the general form $c(T, g, O, \alpha)$; where $T$ and $O$ are sets of decision variables and $g$ is a function as defined in Section 2.1. Let $T \equiv \{t_1, \ldots, t_{|T|}\}$, then $\Theta = D(t_1) \times \ldots \times D(t_{|T|})$. Furthermore, let $O \equiv \{o_1, \ldots, o_\Omega\}$, then $\Omega = D(o_1) \times \ldots \times D(o_\Omega)$. An assignment is consistent with respect to $c$ if the statistical test fails to reject the associated null hypothesis, e.g. "$F_0$ generated $o_1, \ldots, o_\Omega$" at significance level $\alpha$.

A non-parametric statistical constraint $c$ takes the general form $c(O_1, \ldots, O_k, \alpha)$, where $O_1, \ldots, O_k$ are sets of decision variables. Let $O_i \equiv \{o_{i1}, \ldots, o_{i\Omega_i}\}$, then $\Omega = \bigcup_{i=1}^k D(o_{i1}) \times \ldots \times D(o_{i\Omega_i})$. An assignment is consistent with respect to $c$ if the statistical test fails to reject the associated null hypothesis, e.g. "$o_{i1}, \ldots, o_{i\Omega_i}, \ldots, o_{i1}, \ldots, o_{i\Omega_i}$ are drawn from the same distribution," at significance level $\alpha$.

In contrast to classical statistical testing, the sampled populations are here represented by means of sets of decision variables. This paves the way to a number of novel applications.

We now introduce a number of parametric and non-parametric statistical constraints.

3.1 Parametric statistical constraints

In this section we introduce two parametric statistical constraints: the Student’s $t$ test constraint and the Kolmogorov-Smirnov constraint.

3.1.1 Student’s $t$ test constraint

Consider statistical constraint

$$t\text{-test}_c^\alpha(O, m)$$

where $O \equiv \{o_1, \ldots, o_n\}$ is a set of decision variables each of which represents a sample $\omega_i$; $m$ is a decision variable representing the mean of the random variable that generated the sample. Parameter $\alpha \in (0, 1)$ is the significance level; and parameter $w \in \{\leq, \geq, =\}$ identifies the type of statistical test that should be employed; e.g. "$\geq\" refers to a single-tailed one-sample KS test that determines if the distribution originating the sample has first-order stochastic dominance over exponential $(\alpha, =\" refers to a two-tailed one-sample KS test that determines if the sample is likely to have been originated by exponential $(\alpha, =\)$, etc.

An assignment $\bar{o}_1, \ldots, \bar{o}_n, \bar{\lambda}$ satisfies $t\text{-test}_c^\alpha$ if and only if a one-sample $t$ test fails to reject the null hypothesis identified by $w$; e.g. if $w = \"\geq\", \text{then the null hypothesis is "}\geq\text{"}\) of the random variable originating the sample is less than or equal to $m$: "$\geq\" refers to a two-tailed Student’s $t$ test that determines if the mean of the random variable originating the sample is equal to $m$, etc.

An assignment $\bar{o}_1, \ldots, \bar{o}_n, \bar{\lambda}$ satisfies $t\text{-test}_c^\alpha$ if and only if a one-sample Student’s $t$ test fails to reject the null hypothesis identified by $w$; e.g. if $w = \"\geq\", \text{then the null hypothesis is "}\geq\text{"}\) of the random variable originating the sample is equal to $\bar{m}$.

The statistical constraint just presented can be seen as a special case of

$$t\text{-test}_c^\alpha(O_1, O_2)$$

in which the set $O_2$ contains a single decision variable, i.e. $m$. However, in general $O_2$ is defined as $O_2 \equiv \{o_{m+1}, \ldots, o_n\}$. In this case, an assignment $\bar{o}_1, \ldots, \bar{o}_m$ satisfies $t\text{-test}_c^\alpha$ if and only if a two-sample Student’s $t$ test fails to reject the null hypothesis identified by $w$; e.g. if $w = \"\geq\", \text{then the null hypothesis is "}\geq\text{"}\) of the random variable originating population $\bar{o}_1, \ldots, \bar{o}_n$ is equal to that of the random variable generating population $\bar{o}_{m+1}, \ldots, \bar{o}_m$.

Note that $t\text{-test}_c^\alpha$ is equivalent to enforcing both $t\text{-test}_c^{\leq}$ and $t\text{-test}_c^{\geq}$; and that $t\text{-test}_c^{=} \text{is the complement of } t\text{-test}_c^{\geq}$.

We conjecture that filtering $t\text{-test}_c^{\leq}$ and $t\text{-test}_c^{\geq}$ is NP-hard. We leave the proof of this result and the development of effective filtering strategies as future research directions.

3.1.2 Parametric Kolmogorov-Smirnov constraint

Consider statistical constraint

$$KS\text{-test}_c^\alpha(O, \omega, \alpha)$$

where $O \equiv \{o_1, \ldots, o_n\}$ is a set of decision variables each of which represents a sample $\omega_i$; $\lambda$ is a decision variable representing the rate of the exponential distribution. Note that exponential $(\alpha,\lambda)$ may be, in principle, replaced with any other parameterised distribution. However, due to its relevance in the context of the following discussion, in this section we will limit our attention to the exponential distribution. Once more, parameter $\alpha \in (0, 1)$ is the significance level; and parameter $w \in \{\leq, \geq, =\}$ identifies the type of statistical test that should be employed; e.g. "$\geq\" refers to a single-tailed one-sample KS test that determines if the distribution originating the sample has first-order stochastic dominance over exponential $(\alpha, =\" refers to a two-tailed one-sample KS test that determines if the sample is likely to have been originated by exponential $(\alpha, =\)$, etc.

An assignment $\bar{o}_1, \ldots, \bar{o}_n, \bar{\lambda}$ satisfies $KS\text{-test}_c^\alpha$ if and only if a one-sample KS test fails to reject the null hypothesis identified by $w$; e.g. if $w = \"=\", \text{then the null hypothesis is "}\text{population } \bar{o}_1, \ldots, \bar{o}_n \text{ has been sampled from an exponential } (\alpha, =\)\)".

In contrast to the $t\text{-test}_c^\alpha$ constraint, because of the structure of test statistics $d_c^{\leq}$ and $d_c^{\geq}$, KS-test $c^\alpha$ is monotonic — i.e. it satisfies Definition 9 in [17] — and bound consistency can be enforced using standard propagation strategies. In Algorithm [8] we present a bound propagation algorithm for parametric KS-test $c^\alpha$ when the target CDF $F(x)$ is exponential with rate $\lambda, i.e. mean 1/\lambda$; sup($D(x)$) and inf($D(x)$) denote the supremum and the infimum of the domain of decision variable $x$, respectively. Note the KS test at lines [1] and [8].

Propagation for parametric KS-test $c^\alpha$ is based on test statistic $d_c^{\leq}$ and follows a similar logic. Also in this case KS-test $c^\alpha$ is equivalent to enforcing both KS-test $c^{\leq}$ and KS-test $c^{\geq}$; KS-test $c^{=}$ is the complement of KS-test $c^\alpha$.

3.2 Non-parametric statistical constraint

In this section we introduce a non-parametric version of the Kolmogorov-Smirnov constraint.

3.2.1 Non-parametric Kolmogorov-Smirnov constraint

Consider statistical constraint

$$KS\text{-test}_c^\alpha(O_1, O_2)$$

where $O_1 \equiv \{o_1, \ldots, o_n\}$ and $O_2 \equiv \{o_{m+1}, \ldots, o_n\}$ are sets of decision variables representing a sample $\omega_i$; once more, parameter $\alpha \in (0, 1)$ is the significance level and parameter $w \in \{\leq, \geq, =\}$ identifies the type of statistical test that should be employed; e.g. "$\geq\" refers to a single-tailed two-sample KS test that determines if the distribution originating sample $O_1$ has first-order stochastic dominance over the distribution originating sample $O_2$; "$=\" refers to a two-tailed two-sample KS test that determines if the two samples have been originated by the same distribution, etc.
In this section we discuss a number of applications for the statistical testing. The first problem is parametric, while the second is non-parametric. Constraints are employed to solve classical problems in hypothesis testing. The first problem is parametric, while the second is non-parametric.

The first application is a standard t-test on the mean of a sample population. Given a significance level $\alpha = 0.05$ and a sample population $\{8, 14, 6, 12, 9, 10, 9, 10, 5\}$, we are interested in finding out the mean of the random variable originating the sample. This task can be accomplished via a CSP such as the one in Fig. 4. After propagating constraint (1), the domain of $m$ reduces to $\{8, 9, 10, 11\}$, so with significance level $\alpha = 0.05$ we reject the null hypothesis that the mean of the sample population is outside this range. Despite the fact that in this work we do not discuss a filtering strategy for the $t$-test constraint, in this specific instance we were able to propagate this constraint due to the fact that all decision variables $o_i$ were ground. In general the domain of these variables may not be a singleton. In the next example we illustrate this case.

Consider the CSP in Fig. 4. There is a reference population $O_1$ in which all decision variables are ground, this choice is made for illustrative purposes and in general variables in $O_1$ may feature larger domains. Decision variables in population $O_2$ feature non-singleton domains. The problem is basically that of finding a subset of the cartesian product $D(o_{11}) \times \ldots \times D(o_{20})$ such that for all elements
Figures 2 and 3 provide empirical CDFs of populations that are likely to be generated from the same random variable that originated population $O_1$. The CDFs are shown in black and the null hypothesis is rejected if the empirical CDF is not fully contained within the confidence bands of an exponential distribution.

In this case we force the interval between two consecutive inspections to be less or equal to $M$ and we also make sure that the last inspection is carried out during the last month of the year (constraint 3).

### 4.2 Inspection scheduling

We introduce the following inspection scheduling problem. There are 10 units to be inspected 25 times each over a planning horizon comprising 365 days. An inspection lasts 1 day and requires 1 inspector. There are 5 inspectors in total that can carry out inspections at any given day. The average rate of inspection $\lambda$ should be 1 inspection every 5 days. However, there is a further requirement that inter arrival times between subsequent inspections at the same unit of inspection should be approximately exponentially distributed — in particular, if the null hypothesis that intervals between inspections follows an exponential($\lambda$) is rejected at significance level $\alpha = 0.1$ then the associated plan should be classified as infeasible.

This problem can be modelled via the cumulative constraint (1) as shown in Fig. 4 where $s_k$, $e_k$ and $t_k$ are the start time, end time and duration of inspection $k$; finally $c_k$ is the number of inspectors required to carry out an inspection. The memoryless property of the inspection plan can be ensured by introducing decision variables $i_{u,j-1}$ that model the interval between inspection $j$ and inspection $j-1$ at unit of inspection $u$ (constraint 4). Then, for each unit of inspection $u$ we enforce a statistical constraint KS-test($\alpha$, exponential($\lambda$)), where $O_u$ is the list of intervals between inspections at unit of inspection $u$. Note that it is possible to introduce side constraints in this case we force the interval between two consecutive inspections to be less or equal to $M$ and we also make sure that the last inspection is carried out during the last month of the year (constraint 3).

In Fig. 5 we illustrate a feasible inspection plan for the 10 units of assessment over a 365 days horizon. In Fig. 6 we show that the inspection plan for unit of assessment 1 — first from the bottom in Fig. 5 — satisfies the statistical constraint. In fact, the empirical CDF of the intervals between inspections (dashed stepwise function) is fully contained within the confidence bands of an exponential($\lambda$) distribution (dashed function) at significance level $\alpha$. 

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### Parameters:

- $U = 10$ Units to be inspected
- $I = 25$ Inspections per unit
- $H = 365$ Periods in the planning horizon
- $D = 1$ Duration of an inspection
- $M = 36$ Max interval between two inspections
- $C = 1$ Inspectors required for an inspection
- $m = 5$ Inspectors available
- $\lambda = 1/5$ Inspection rate

### Constraints:

1. cumulative($s, e, t, c, m$) for all $u \in 1, \ldots, U$
2. KS-test($\alpha$, exponential($\lambda$)) for all $u \in 1, \ldots, U$ and $j \in 2, \ldots, I$
3. $i_{u,j-1} = s_{u,j-1} - s_{u,j-1} - 1$
4. $s_{u,j-1} > s_{u,j-1}$

### Decision variables:

- $s_k \in \{1, \ldots, H\}, \quad \forall k \in 1, \ldots, I - U$
- $e_k \in \{1, \ldots, H\}, \quad \forall k \in 1, \ldots, I - U$
- $t_k \in [D, 1], \quad \forall k \in 1, \ldots, I - U$
- $c_k \in C, \quad \forall k \in 1, \ldots, I - U$
- $i_{u,j-1} \in \{0, \ldots, M\}, \quad \forall u \in 1, \ldots, U$ and $\forall j \in 2, \ldots, I$
- $O_u \equiv \{i_{u,1}, \ldots, i_{u,I-1}\}, \forall u \in 1, \ldots, U$
Figure 6. Empirical CDF of intervals (in days) between inspections for unit of assessment 1

4.3 Further application areas

The techniques discussed in this work may be used in the context of classical problems encountered in statistics \( [12] \), e.g. regression analysis, distribution fitting, etc. In other words, one may look for solutions to a CSP that fit a given set of populations or distributions. In addition, as seen in the case of inspection scheduling, statistical constraints may be used to address the inverse problem of designing sampling plans that feature specific statistical properties; such analysis may be applied in the context of design of experiments \( [3] \) or quality management \( [4] \). Further applications may be devised in the context of supply chain coordination. For instance, one may identify replenishment plans featuring desirable statistical properties, e.g. obtain a production schedule in which the ordering process, while meeting other technical constraints, mimics a given stochastic process, e.g. Poisson(\( \lambda \)); this information may then be passed upstream to suppliers to ensure effective coordination without committing to a replenishment plan fixed a priori or to a specific replenishment policy. Finally, note that the techniques here presented generalise the discussion in \( [11] \), in which statistical inference is applied in the context of stochastic constraint satisfaction to identify approximate solutions featuring given statistical properties. However, it should be emphasised the fact that stochastic constraint programming \( [10] \) works with decision and random variables over a set of decision stages; random variable distributions are assumed to be known. Statistical constraints instead operate under the assumption that distribution of random variables is only partially specified (parametric statistical constraints) or not specified at all (non-parametric statistical constraints); furthermore, statistical constraints do not model explicitly random variables, they model instead sample populations as decision variables for the first time.

5 CONCLUSION

Statistical constraints represent a bridge that links statistics and constraint programming for the first time in the literature. The declarative nature of constraint programming offers a unique opportunity to exploit statistical inference in order to identify sets of assignments featuring specific statistical properties. Beside introducing the first two examples of statistical constraints, this work discusses filtering algorithms and applications spanning from standard problems encountered in statistics to a novel inspection scheduling problem in which the aim is to find inspection plans featuring desirable statistical properties.

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