CONNECTED SUM OF CR MANIFOLDS WITH POSITIVE CR YAMABE CONSTANT

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Abstract. Suppose $M_1$ and $M_2$ are 3-dimensional closed (compact without boundary) CR manifolds with positive CR Yamabe constant. In this note, we show that the connected sum of $M_1$ and $M_2$ also admits a CR structure with positive CR Yamabe constant.

1. Introduction

In Riemannian geometry, the scalar curvature is the simplest curvature invariant of a Riemannian manifold. It was shown by Gromov-Lawson in [5] and independently by Schoen-Yau in [9] that the connected sum of two closed (that is, compact without boundary) manifolds of positive scalar curvature has a metric of positive scalar curvature. It was also shown by Schoen-Yau in [9] that the connected sum of two closed conformally flat manifolds of positive scalar curvature has a conformally flat metric of positive scalar curvature (see Corollary 5 in [9]). In view of the similarity between the scalar curvature in Riemannian geometry and the Tanaka-Webster scalar curvature in CR geometry, it would be natural to ask if the corresponding results hold for the Tanaka-Webster scalar curvature. It is the purpose of this note to answer this question.

For basic materials in CR geometry and pseudohermitian geometry, we refer the readers to [4], [6], [8] or [10], and the references therein. Let $(M, J)$ be a closed, strictly pseudoconvex CR manifold of dimension $2n + 1$. For a contact form $\theta$, one can define subgradient $\nabla_b$ and the Tanaka-Webster scalar curvature $R$ or $R_{J, \theta}$ on the pseudohermitian manifold $(M, J, \theta)$. Take the volume form $dV_\theta := \theta \wedge (d\theta)^n$. We define the CR Yamabe constant $\lambda(M, J)$ (or
\( \lambda(M) \) if \( J \) is clear in the context) as follows: (see [6])

\[
\lambda(M, J) = \inf_{u > 0} \frac{E_\theta(u)}{\left( \int_M u^{2 + \frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}}},
\]

where

\[
E_\theta(u) = \int_M \left( (2 + \frac{2}{n})|\nabla_b u|^2 + Ru^2 \right) dV_\theta.
\]

Similar to the Riemannian case, one can show that \( \lambda(M, J) > 0 \) if and only if there exists a contact form \( \tilde{\theta} \) conformal to \( \theta \) such that the Tanaka-Webster scalar curvature of \( \tilde{\theta} \) is positive.

In [1], the first and the second authors proved the following theorem, which is the CR version of Schoen-Yau’s result mentioned above (see also [7] for a different proof by O. Kobayashi).

**Theorem 1.1.** ([1]) Suppose \((M_1, J_1)\) and \((M_2, J_2)\) are two closed, spherical CR manifolds of dimension \(2n+1\) with \( \lambda(M_k, J_k) > 0 \) for \( k = 1, 2 \). Then their connected sum \( M_1 \# M_2 \) admits a spherical CR structure \( J \) with \( \lambda(M_1 \# M_2, J) > 0 \).

The idea of the proof of Theorem 1.1 was motivated by the work of O. Kobayashi in [7]. More precisely, we fix a point \( p_j \in M_j \) for \( j = 1, 2 \). We first take off two small balls around \( p_1 \) and \( p_2 \). Since \( M_j \) are spherical, we can attach the Heisenberg cylinder in each of punched neighborhood of \( p_j \). We then glue two Heisenberg cylinders together to get a spherical CR manifold.

In this note, we continue our study on the Tanaka-Webster scalar curvature of connected sum on CR manifolds. In particular, we prove the following theorem, which can be viewed as the analogous result of Gromov-Lawson and of Schoen-Yau mentioned above.

**Theorem A.** Suppose \((M_1, J_1)\) and \((M_2, J_2)\) are two 3-dimensional closed CR manifolds with \( \lambda(M_k, J_k) > 0 \) for \( k = 1, 2 \). Then their connected sum \( M_1 \# M_2 \) admits a CR structure \( J \) with \( \lambda(M_1 \# M_2, J) > 0 \).
Note that the above argument for the spherical case cannot be applied directly, since we cannot attach the Heisenberg cylinder to the punched neighborhood of a point. However, in this paper, we will mainly construct a new CR structure which outside a ball is the given CR structure and is spherical in a neighborhood contained in the ball. In addition, we can construct such a CR structure such that its Yamabe constant is as close as possible to the one of the given CR structure. Hence, together with Theorem 1.1, we obtain Theorem A.

Note added: While we were writing this paper, we were informed that Dietrich [3] has obtained the same result for all dimensions. In the case dim $M \geq 5$, since $M$ is embeddable, Dietrich can construct such a CR structure in terms of defining functions. This method is not available in 3-dimensional case. The difficulty is that not every 3-dimensional CR structure is embeddable. He provided another approach to deal with the 3-dimensional case. On the contrary, in this paper, we use the deformation tensor to construct explicit CR structures we need in 3-dimensional case. In the higher dimensional situation, the difficulty of our approach is that we do not know if our CR structures in the construction are integrable.

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2. Basic Material

For basic material in $CR$ and pseudohermitian geometry, we refer the reader to [4], [6], [8] or [10]. Let $(M^3, J, \theta)$ be a pseudohermitian manifold. In [10], S. Webster showed that there is a natural connection in the bundle $\xi_{1,0}$ of all CR holomorphic vectors adapted to the pseudohermitian structure $(J, \theta)$. To define the connection, choose an orthonormal admissible coframe $\{\theta^1\}$ and dual frame $\{Z_1\}$ for $\xi_{1,0}$. Webster showed that there are uniquely
determined 1-forms $\theta^1, \tau^1$ on $M$ satisfying the following structure equations

$$
\begin{align*}
  d\theta^1 &= \theta^1 \wedge \theta^1 + \theta \wedge \tau^1, \\
  0 &= \theta^1 + \theta^1, \\
  0 &= \tau^1 \wedge \theta^1,
\end{align*}
$$

(2.1)

in which $\tau^1 = \tau^1$. The forms $\theta^1, \tau^1$ are called the pseudohermitian connection form and torsion form, respectively. Recall that the Heisenberg group $H_1$ is the space $\mathbb{R}^3$ endowed with the group multiplication

$$(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_1x_2 - x_1y_2),$$

which is a 3-dimensional Lie group. The space of all left invariant vector fields is spanned by the following three vector fields:

$$
\begin{align*}
  \hat{e}_1 &= \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \\
  \hat{e}_2 &= \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \quad \text{and} \quad T = \frac{\partial}{\partial z}.
\end{align*}
$$

The standard contact bundle on $H_1$ is the subbundle $\hat{\xi}$ of the tangent bundle $TH_1$, which is spanned by $\hat{e}_1$ and $\hat{e}_2$. It can also be equivalently defined as the kernel of the contact form

$$
\Theta = dz + xdy - ydx.
$$

The CR structure on $H_1$ is the endomorphism $\hat{J} : \hat{\xi} \rightarrow \hat{\xi}$ defined by

$$
\hat{J}(\hat{e}_1) = \hat{e}_2 \quad \text{and} \quad \hat{J}(\hat{e}_2) = -\hat{e}_1.
$$

One can view $H_1$ as a pseudohermitian manifold with the standard pseudohermitian structure $(\hat{J}, \hat{\xi})$. In the Heisenberg group $H_1$, relative to the standard left invariant frame $\hat{Z}_1 = \frac{1}{2}(\hat{e}_1 - i\hat{e}_2)$ (dual coframe is $\hat{\theta}^1 = dx + idy$), it is easy to see that both forms $\theta^1$ and $\tau^1$ vanish.
2.1. The deformation tensor. Suppose \( J \) is a CR structure compatible with \( \Theta \) in the sense: it is defined on \( \xi \) and \( d\Theta(X, JX) > 0 \) for any nonzero vector \( X \in \xi \). Let \( Z_1 \) be a CR holomorphic vector field relative to \( J \). We express it as

\[
Z_1 = a_1^1 \tilde{Z}_1 + b_1^1 \bar{Z}_1,
\]

for some function \( a_1^1, b_1^1 \). We compute

\[
Z_1 \wedge Z_1 = (|a_1^1|^2 - |b_1^1|^2) \tilde{Z}_1 \wedge \bar{Z}_1.
\]

The compatibility of \( J \) with \( \Theta \) implies that \( |a_1^1|^2 > |b_1^1|^2 \). In particular, we have \( a_1^1 \neq 0 \).

Define

\[
\phi = (a_1^1)^{-1} b_1^1,
\]

where \( a_1^1, b_1^1 \) is the conjugate of \( a_1^1, b_1^1 \), respectively. We call \( \phi \) the deformation tensor of \( J \) (note that \( \phi \) depends on frames. It behaves as a tensor when changing frames. For notational simplicity, we suppress its tensor indices). The compatibility thus implies that \( |\phi| < 1 \). It is easy to see that any CR anti-holomorphic vector field \( Z_1 \) has the form \( Z_1 = a_1^1(\tilde{Z}_1 + \phi \bar{Z}_1) \), for some function \( a_1^1 \). Conversely, any function \( \phi \) with \( |\phi| < 1 \) defines a CR structure \( J \) compatible with \( \Theta \) by regarding \( Z_1 = a_1^1(\tilde{Z}_1 + \phi \bar{Z}_1) \) as its corresponding CR anti-holomorphic vector field.

3. Proof

Let \( (M^3, J, \theta) \) be a pseudohermitian manifold. To prove Theorem A, first we would like to construct a sequence of pseudohermitian structures \( \{(J_i, \theta_i)\} \) such that \( \{(J_i, \theta_i)\} \) converges to \( (J, \theta) \) in \( C^0 \) and the corresponding Webster curvature \( R_i \) also converges to the Webster curvature \( R \) of \( (J, \theta) \) in \( C^0 \). In addition, each CR structure \( J_i \) is CR spherical around \( p \in M \).

We construct such a sequence as follows:

For each \( p \in M \), there exists a neighborhood \( U \) of \( p \in M \) which is contactomorphic to a neighborhood \( V \) of \( 0 \in H_1 \). Let \( \Phi : U \rightarrow V \) be thus a contactomorphism and \( \Phi(p) = 0 \), we
identify $U$ with $V$ under $\Phi$. Then, on $U$ (or on $V$), the CR structure $J$ can be represented by a deformation tensor $\phi$ with $|\phi| < 1$ such that $\hat{Z}_1 + \phi\bar{Z}_1$ is a CR anti-holomorphic vector field. In addition, it is easy to see that one can take a contactomorphism $\Phi$ with $\Phi(p) = 0$ such that the deformation function $\phi$ satisfies $\phi(0) = 0$ and $\phi_1(0) = \phi_1(0) = 0$, where $\phi_1 = \bar{Z}_1\phi$ and $\phi_1 = \bar{Z}_1\phi$. Thus, we can assume, without loss of generality, that

$$\theta|_U = \Theta,$$

$$\phi(0) = \phi_1(0) = \phi_1(0) = 0.$$  

(3.1)

Relative to the contact form $\Theta$,

$$Z_1 = \left(\frac{1}{1 - |\phi|^2}\right)^{1/2} (\bar{Z}_1 + \phi\bar{Z}_1),$$

(3.2)

is a unit vector field. The dual coframe is

$$\theta^1 = \left(\frac{1}{1 - |\phi|^2}\right)^{1/2} (\bar{\theta}^1 - \phi\bar{\theta}^1).$$

(3.3)

**Proposition 3.1.** Let $\theta^1$ and $\tau^1 = A^1\bar{\theta}^1$ be the pseudohermitian connection form and torsion form relative to $\theta^1$, respectively. Then we have

$$A^1 = -\frac{\phi_0}{1 - |\phi|^2};$$

$$\theta^1 = -d\ln\left(\frac{1}{1 - |\phi|^2}\right)^{1/2} + \left[\frac{\bar{\phi}\phi_0}{1 - |\phi|^2}\right] \Theta$$

$$+ \left[\frac{\phi_1 + \bar{\phi}\phi_1}{1 - |\phi|^2} + \bar{\phi}_1\right] \theta^1$$

$$- \frac{\phi_1 + \bar{\phi}_1}{1 - |\phi|^2} + |\phi|^2 \bar{\hat{Z}}_1 \left(\frac{1}{1 - |\phi|^2}\right) + \hat{Z}_1 \left(\frac{1}{1 - |\phi|^2}\right) \bar{\theta}^1,$$

(3.4)

where all the derivatives are computed on $H_1$; for example, $\phi_1 = \bar{Z}_1\phi$, $\phi_0 = T\phi$, and so on.

**Proof.** One can check directly that $\theta^1$ and $A^1$ in (3.4) satisfy the structure equations (2.1). And by uniqueness, we complete the proof. □
Proposition 3.2. Let $R^{\phi,\Theta}$ and $\Delta_{b}^{\phi,\Theta}$ be the Webster curvature and (negative) sub-laplacian of $(J,\Theta)$ on $U$, respectively. Then we have

$$R^{\phi,\Theta} = -\hat{Z}_1 \left[ \frac{\phi_1 + \bar{\phi} \phi_1}{1 - |\phi|^2} + \phi \hat{Z}_1 \left( \frac{1}{1 - |\phi|^2} \right) + \hat{Z}_1 \left( \frac{1}{1 - |\phi|^2} \right) \right]$$

$$+ \frac{\hat{\phi} \phi_0}{1 - |\phi|^2}.$$  (3.5)

and

$$\Delta_{b}^{\phi,\Theta} u = \left[ \frac{1 + |\phi|^2}{1 - |\phi|^2} \right] \hat{\Delta}_b u - \left[ \frac{2\phi}{1 - |\phi|^2} \right] u_{11} - \left[ \frac{2\phi}{1 - |\phi|^2} \right] u_{11}$$

$$- \left[ \frac{2\bar{\phi}_1 + |\phi|^2}{1 - |\phi|^2} + 2\phi \hat{Z}_1 \left( \frac{1}{1 - |\phi|^2} \right) + (1 + |\phi|^2) \hat{Z}_1 \left( \frac{1}{1 - |\phi|^2} \right) \right] u_1$$

$$- \left[ \frac{2\phi_1 + |\phi|^2}{1 - |\phi|^2} + (1 + |\phi|^2) \hat{Z}_1 \left( \frac{1}{1 - |\phi|^2} \right) + 2\phi \hat{Z}_1 \left( \frac{1}{1 - |\phi|^2} \right) \right] u_1.$$  (3.6)

Proof. Recall that Webster showed that $d\theta_1^1$ can be written

$$d\theta_1^1 = R\theta^1 \wedge \theta^1, \text{ mod } \theta,$$  (3.7)

where $R$ is the Webster curvature. Since $\theta^1$ is an unit coframe, we have $\theta^1 \wedge \theta^1 = -id\Theta = \hat{\theta}^1 \wedge \hat{\theta}^1$. On the other hand, $d\hat{\theta}^1 = 0$. Therefore we have (3.5) immediately from (3.4). For (3.6), recall that

$$\Delta_{b}^{\phi,\Theta} u = -\left( Z_1 Z_1 u - \theta_1^1 (Z_1 Z_1 u) \right) + \text{conjugate},$$  (3.8)

and (3.6) is just a straightforward computation in terms of (3.4). \(\square\)

Remark 3.3. The first author and I H. Tsai deduced a more general formula for the Webster curvature (see (4.6) in [2]).

If we consider the new contact form $\theta^u = u^2 \Theta$ then, on $U$, we have the transformation law of the Webster curvatures (for the details, see [4, 6])

$$R^{\phi,\theta^u} = u^{-3} (4\Delta_{b}^{\phi,\Theta} u + R^{\phi,\Theta} u),$$  (3.9)
where $R^{\phi,\theta}$ is the Webster curvature with respect to $(J, u^2 \Theta)$.

On the other hand, the standard CR structure of the Heisenberg group on $U$ is represented by the zero deformation function $\phi \equiv 0$, which is CR spherical. Let $u$ be a positive function in a neighborhood of 0 such that $u(0) = 1$, $(\bar{Z}_1 u)(0) = (\bar{Z}_1 u)(0) = 0$ and

$$R^{0,\theta}(0) = R^{\phi,\Theta}(0).$$

Here

$$R^{0,\theta} = u^{-3}(4 \Delta_b u).$$

To prove Theorem A, we also need the following lemma which is a standard result in the literature (see [3, 7]).

**Lemma 3.4.** For any $\delta > 0$, there is a nonnegative function $\chi_\delta \in C^\infty(R)$ such that

(i) $0 \leq \chi_\delta \leq 1$, $\chi_\delta(t) \equiv 1$ in a neighborhood of 0 and $\chi_\delta(t) \equiv 0$ for $|t| \geq \delta$;

(ii) $|\chi'_\delta(t)| \leq \delta t^{-1}$ and $|\chi''_\delta(t)| \leq \delta t^{-2}$ for all $t$.

3.1. **Construction of a sequence** $(J^\delta, \theta^\delta)$. To construct a sequence of pseudohermitian structures we describe in the beginning of this section, we re-formulate (3.5) and (3.6) as what we need.

**Proposition 3.5.** Let $F = F(\phi) = \left(\frac{1}{1 - |\phi|^2}\right)^{\frac{1}{2}}$. We have

$$R^{\phi,\Theta} = -F^2(\phi_{11} + \bar{\phi}_{11}) + \sum_{a,b \in \{1,1\}} P_{ab} \phi_{ab} + \sum_{a,b \in \{1,1\}} Q_{ab} \bar{\phi}_{ab} + P,$$

$$\Delta_b^{\phi,\Theta} u = F^2 \Delta_b u + \sum_{a,b \in \{1,1\}} S_{ab} u_{ab} + Su_1 + \bar{S} u_1,$$

where $P_{ab}, Q_{ab}, P, S_{ab}$ and $S$ are all polynomials in $F, \phi, \bar{\phi}, \phi_1, \bar{\phi}_1, \phi_{11}, \bar{\phi}_{11}$ such that

$$P_{ab}(0) = Q_{ab}(0) = P(0) = S_{ab}(0) = S(0) = 0.$$
Since $F(0) = 1$, condition (3.13) means that each polynomial does not include monomial terms $F^k$ for some nonnegative integer $k$.

Now, for each $\delta$, we define a pseudohermitian structure $(J^\delta, \theta^\delta)$ by

$$
\phi^\delta = (1 - \chi_\delta(\rho))\phi, \quad \text{which is the deformation function of } J^\delta;
$$

$$
\theta^\delta = (1 - \chi_\delta(\rho))\Theta + \chi_\delta(\rho)(\theta^u), \quad \theta^u = u^2\Theta,
$$

$$
= (v^\delta)\Theta, \quad (v^\delta)^2 = 1 + \chi_\delta(u^2 - 1).
$$

(3.14)

It is easy to see that $(\phi^\delta, \theta^\delta) = (\phi, \Theta)$ outside the $\delta$-ball $B(\delta)$ centered at 0, and $(\phi^\delta, \theta^\delta) = (0, \theta^u)$ in a neighborhood of 0. Moreover, notice that

$$
R^{0, \theta^u}(0) = R^{\phi, \Theta}(0),
$$

(3.15)

$$
\phi(0) = \phi_1(0) = \phi_1(0) = 0,
$$

$$
u(0) = 1, \quad u_1(0) = u_1(0) = 0.
$$

Define

$$
F^\delta = \left(\frac{1}{1 - |\phi^\delta|^2}\right)^{\frac{1}{2}}.
$$

Since $|\phi^\delta| \leq |\phi|$, we have $1 \leq |F^\delta| \leq |F|$, and hence $|F^\delta|$ has an uniform bound. By a direct calculation, we have $(v^\delta)^2 - u^2 = (1 - \chi_\delta)(1 - u^2)$ which implies

$$
u^2 - |u^2 - 1| \leq (v^\delta)^2 \leq u^2 + |u^2 - 1|.
$$

(3.16)

This implies that $v^\delta$ has an uniform bound. For $a, b \in \{1, \bar{1}\}$,

$$
\phi^\delta_a = -(\chi_\delta)_a\phi + (1 - \chi_\delta)\phi_a
$$

$$
\phi^\delta_{ab} = -(\chi_\delta)_{ab}\phi - (\chi_\delta)_a\phi_b - (\chi_\delta)_b\phi_a + (1 - \chi_\delta)\phi_{ab}.
$$

(3.17)
Formulae (3.15), (3.17), together with Lemma 3.4, show that $|\phi_{ab}|$ has an uniform upper bound for each $a, b \in \{1, \bar{1}\}$. We also compute

\begin{align*}
(v^\delta)_a &= \frac{1}{2} \frac{(\chi\delta)_a(u^2 - 1) + \chi\delta(u^2)_a}{v^\delta}, \\
(v^\delta)_{ab} &= \frac{1}{2} \frac{(\chi\delta)_{ab}(u^2 - 1) + (\chi\delta)_a(u^2)_b + (\chi\delta)_b(u^2)_a + \chi\delta(u^2)_{ab}}{v^\delta} \\
&\quad - \frac{1}{4} \frac{((\chi\delta)_a(u^2 - 1) + \chi\delta(u^2)_a)((\chi\delta)_b(u^2 - 1) + \chi\delta(u^2)_b)}{(v^\delta)^3}.
\end{align*}

(3.18)

For the same reason, formulae (3.15), (3.18) together with Lemma 3.4 show that $|(v^\delta)_a|, |(v^\delta)_{ab}|$ has an uniform upper bound for each $a, b \in \{1, \bar{1}\}$.

**Proposition 3.6.** The sequence $\{(\phi^\delta, \theta^\delta)\}$ converges to $(\phi, \Theta)$ in $C^0$. The corresponding Webster curvature $R^{\phi^\delta, \theta^\delta}$ also converges to $R^{\phi, \Theta}$ in $C^0$.

**Proof.** From the construction of $(\phi^\delta, \theta^\delta)$, and noting that $\phi(0) = 0$ and $u(0) = 1$, it is apparent that $\{(\phi^\delta, \theta^\delta)\}$ converges to $(\phi, \Theta)$ in $C^0$. Therefore we only need to show that $R^{\phi^\delta, \theta^\delta}$ converges to $R^{\phi, \Theta}$ in $C^0$.

Let $v^\delta$ be the positive function such that $\theta^\delta = (v^\delta)^2\Theta$, i.e., $(v^\delta)^2 = 1 + \chi\delta(u^2 - 1)$. From the transformation law of Webster curvature, we have

\begin{align*}
|R^{\phi^\delta, \theta^\delta} - R^{\phi, \Theta}| &= \left| \frac{4 \Delta_b^{\phi^\delta, \Theta} v^\delta}{(v^\delta)^3} + \frac{R^{\phi^\delta, \Theta}}{(v^\delta)^2} - R^{\phi, \Theta} \right| \\
&\leq \left| \frac{4 \Delta_b^{\phi^\delta, \Theta} v^\delta}{(v^\delta)^3} + \frac{R^{\phi^\delta, \Theta}}{(v^\delta)^2} - R^{\phi, \Theta}(0) \right| + \left| R^{\phi, \Theta}(0) - R^{\phi^\delta, \Theta} \right|.
\end{align*}

(3.19)

Notice that $v^\delta$ has an uniform bound and all $|\phi_{ab}|, |(v^\delta)_a|, |(v^\delta)_{ab}|$ and $|F^\delta|$ has an uniform upper bound. Using formulae (3.12) with $\phi, u$ replaced by $\phi^\delta, v^\delta$ respectively, together with
by (3.15), (3.16) and Lemma 3.4, we have

\[
\left| \frac{4\Delta_{b}^\varphi_{\Theta, v^\delta}}{(v^\delta)^3} + \frac{R_{\varphi_{\Theta, v^\delta}}}{(v^\delta)^2} - R_{\varphi_{\Theta, 0}} \right| \leq (F^\delta)^2 \left( \frac{4\Delta_{b}^\varphi_{\Theta, v^\delta}}{(v^\delta)^3} - \frac{(\bar{\phi}_{\Omega}^\delta + \phi_{11}^\delta)}{(v^\delta)^2} - R_{\varphi_{\Theta, 0}} \right) + C\delta,
\]

(3.19) 

\[
\leq C \left| \frac{4\Delta_{b}^\varphi_{\Theta, v^\delta}}{(v^\delta)^3} - \frac{(\bar{\phi}_{\Omega}^\delta + \phi_{11}^\delta)}{(v^\delta)^2} - R_{\varphi_{\Theta, 0}} \right| + C\delta, \text{ by (3.18) and Lemma 3.4.}
\]

(3.20) 

\[
\leq C \left| \frac{4\Delta_{b}^\varphi_{\Theta, v^\delta}}{(v^\delta)^3} - (1 - \chi_{\delta})(\bar{\phi}_{\Omega}^\delta + \phi_{11}) - R_{\varphi_{\Theta, 0}} \right| + C\delta,
\]

(3.19) 

\[
\leq C \left| \frac{4\Delta_{b}^\varphi_{\Theta, v^\delta}}{(v^\delta)^3} + \chi_{\delta}(\bar{\phi}_{\Omega}^\delta + \phi_{11}) \right| + \left| -\phi_{\Omega}^\delta + \phi_{11}^\delta - R_{\varphi_{\Theta, 0}} \right| + C\delta,
\]

(3.20) 

\[
\leq C\delta, \text{ by (3.11), (3.11) and (3.12).}
\]

for some positive constant \( C \). Combining (3.19) and (3.20), we complete the proof of the proposition. \( \square \)

3.2. **Proof of Theorem A.** To prove Theorem A, it suffices to prove the following proposition.

**Proposition 3.7.** Let \( \lambda(M, J^\delta) \) is the Yamabe constant with respect to \( J^\delta \), we have

\[
(3.21) \quad \lim_{\delta \to 0} \lambda(M, J^\delta) = \lambda(M, J).
\]

**Proof.** Recall that we have constructed a sequence \((J^\delta, \theta^\delta)\) which converges to \((J, \theta)\) in \( C^0 \). In addition, \((J^\delta, \theta^\delta) = (J, \theta) \) outside the ball \( B(\delta) \), and

\[
\phi^\delta = (1 - \chi(\rho))\phi, \quad \theta^\delta = (v^\delta)^2\Theta, \text{ on } B(\delta).
\]

Notice that we have chosen the contact form \( \theta \) such that \( \theta|_{B(\delta)} = \Theta \). Let \( dV = \theta \wedge d\theta \) and \( dV^\delta = \theta^\delta \wedge d\theta^\delta = (v^\delta)^2 dV \). And let \( R = R_{J, \theta} = R_{\phi, \Theta} \) and \( R^\delta = R_{J^\delta, \theta^\delta} = R_{\phi^\delta, \theta^\delta} \). Since
\((J^\delta, \theta^\delta) \to (J, \theta)\) in \(C^0\), it is easy to see that for any \(\varepsilon > 0, \varepsilon \ll 1\), if \(\delta\) is small enough then we have

\[|v^\delta|_{\pm} - 1| \leq \varepsilon, \quad |R - R^\delta| \leq \varepsilon,\]

and hence \(|R^\delta|\) has an uniform bound. Next we need the following lemma

**Lemma 3.8.** Given \(\varepsilon > 0\), if \(\delta\) is small enough then we have

\[
(3.22) \quad \frac{1}{(1 + \varepsilon)}|\nabla_b^\delta u|_{\delta}^2 \leq |\nabla_b^\delta u|^2 \leq (1 + \varepsilon)|\nabla_b^\delta u|_{\delta}^2
\]

**Proof of Lemma 3.8.** First we have

\[
|\nabla_b^\delta u|^2 = 2F^2((1 + |\phi|^2)|u_1|^2 + \phi(u_1)^2 + \bar{\phi}(u_1)^2)
\]

\[
|\nabla_b^\delta u|_{\delta}^2 = 2(F^\delta)^2((1 + |\phi^\delta|^2)|u_1|^2 + \phi^\delta(u_1)^2 + \bar{\phi}(u_1)^2)(v^\delta)^{-2}
\]

Therefore, on \(u_1 \neq 0\), we have

\[
\frac{|\nabla_b^\delta u|^2}{|\nabla_b^\delta u|_{\delta}^2} = (v^\delta)^2 \frac{F^2((1 + |\phi|^2) + \phi(u_1)^2 + \bar{\phi}(u_1)^2)}{(F^\delta)^2((1 + |\phi^\delta|^2) + \phi^\delta(u_1)^2 + \bar{\phi}(u_1)^2)}
\]

\[
\leq (v^\delta)^2 \frac{F^2(1 + |\phi|^2 + 2|\phi|)}{(F^\delta)^2(1 + |\phi^\delta|^2 - 2|\phi^\delta|)}
\]

\[
= (v^\delta)^2 \frac{F^2(1 + |\phi|)^2}{(F^\delta)^2(1 - |\phi^\delta|)^2}
\]

\[
\leq (v^\delta)^2 \left(\frac{1 + |\phi|}{1 - |\phi|}\right)^2 \leq (1 + \varepsilon),
\]

if \(\delta\) is small enough (since \(\phi(0) = 0\)). Similarly, we have

\[
\frac{|\nabla_b^\delta u|_{\delta}^2}{|\nabla_b^\delta u|^2} \leq (1 + \varepsilon).
\]

We have thus completed the proof of Lemma 3.8. \(\square\)

Now for each \(\delta\), there exists a function \(u^\delta\) such that

\[
\int_M (u^\delta)^4 dV^\delta = 1,
\]

\[
\lambda(M, J^\delta) \leq E_{\theta^\delta}(u^\delta) \leq \lambda(M, J^\delta) + \varepsilon,
\]
With these materials above, we make the following estimate

\[
E_\theta(u^\delta) = \int_M (4|\nabla_bu^\delta|^2 + R(u^\delta)^2) \, dV
\]

\[
= E_{\theta^\delta}(u^\delta) + 4 \left( \int_M |\nabla_bu^\delta|^2 \, dV - \int_M |\nabla^\delta_bu^\delta|_\delta^2 \, dV^\delta \right)
\]

\[
+ \left( \int_M R(u^\delta)^2 \, dV - \int_M R^\delta(u^\delta)^2 \, dV^\delta \right),
\]

where, by Hölder inequality,

\[
\int_M R(u^\delta)^2 \, dV - \int_M R^\delta(u^\delta)^2 \, dV^\delta
\]

\[
= \int_M R(u^\delta)^2 (dV - dV^\delta) + \int_M (R - R^\delta)(u^\delta)^2 \, dV^\delta
\]

\[
= \int_M R(u^\delta)^2 ((v^\delta)^{-4} - 1) \, dV^\delta + \int_M (R - R^\delta)(u^\delta)^2 \, dV^\delta
\]

\[
\leq C\varepsilon, \quad \text{for some positive constant } C,
\]

and

\[
\int_M |\nabla_bu^\delta|^2 \, dV - \int_M |\nabla^\delta_bu^\delta|_\delta^2 \, dV^\delta
\]

\[
= \int_M (|\nabla_bu^\delta|^2(v^\delta)^{-4} - |\nabla^\delta_bu^\delta|_\delta^2) \, dV^\delta
\]

\[
\leq \int_M ((1 + \varepsilon)|\nabla^\delta_bu^\delta|_\delta^2(v^\delta)^{-4} - |\nabla^\delta_bu^\delta|_\delta^2) \, dV^\delta
\]

\[
= \int_M |\nabla^\delta_bu^\delta|_\delta^2 \left( [(v^\delta)^{-4} - 1] + \varepsilon(v^\delta)^{-4} \right) \, dV^\delta
\]

\[
\leq C\varepsilon E_{\theta^\delta}(u^\delta), \quad \text{for some } C > 0,
\]

since \( |\int_M R^\delta(u^\delta)^2 \, dV^\delta| \) has an uniform upper bound. Substituting (3.24) and (3.25) into (3.23), we obtain

\[
E_\theta(u^\delta) \leq (1 + C\varepsilon) E_{\theta^\delta}(u^\delta) + C\varepsilon
\]

(3.26)
Similarly, we have

\[ E_{\theta}(u^0) \leq (1 + C\varepsilon)E_{\theta}(u^0) + C\varepsilon, \]

\[ \leq (1 + C\varepsilon)(\lambda(M, J) + \varepsilon) + C\varepsilon \]

(3.27)

where \( u^0 \) is a function such that

\[ \int_M (u^0)^4 dV = 1, \]

\[ \lambda(M, J) \leq E_{\theta}(u^0) \leq \lambda(M, J) + \varepsilon. \]

Since \( \int_M (u^0)^4 dV = 1 \), we have

\[ 1 - \varepsilon \leq \int_M (u^0)^4 dV^\delta \leq 1 + \varepsilon. \]

(3.28)

Similarly we have

\[ 1 - \varepsilon \leq \int_M (u^\delta)^4 dV \leq 1 + \varepsilon. \]

(3.29)

By means of (3.26), (3.27), (3.28) and (3.29), we get

\[ \frac{(1 - \varepsilon)\lambda(M, J) - \varepsilon(1 + C\varepsilon) + C\varepsilon}{1 + C\varepsilon} \leq \lambda(M, J^\delta) \leq \frac{(1 + C\varepsilon)(\lambda(M, J) + \varepsilon) + C\varepsilon}{1 - \varepsilon} \]

(3.30)

This completes the proof of Proposition 3.7.

Therefore, if \( \delta \) is small enough, then \( \lambda(M, J^\delta) \) is positive. Notice that each \( J^\delta \) is spherical around a point. To complete the proof of Theorem A, we choose \( \delta_1, \delta_2 \) so that both Yamabe constants \( \lambda(M_1, J_1^{\delta_1}) \) and \( \lambda(M_2, J_2^{\delta_2}) \) are positive. Then using the argument in [1] (see the paragraph after Theorem 1.1) to glue \( M_1 \) and \( M_2 \) by a Heisenberg cylinder, we get a CR structure on the connected sum \( M_1 \# M_2 \) with positive Yamabe constant.
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