Measure Concentration and the Topology of Positively-Curved Riemannian Manifolds

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Abstract

In this paper, I shall demonstrate that sufficiently high-dimensional closed positively-curved Riemannian manifolds are either diffeomorphic to a spherical space form, or isometric to a locally compact rank one symmetric space. This surprising classification of positively-curved Riemannian manifolds results from combining the concentration of measures of Grassmannians with Brendle-Schoen pointwise (weakly)-$1/4$-pinching Theorem. A direct corollary of the main result within this paper is the answering of the long standing Hopf Conjecture in sufficiently high dimensions.

1 Introduction

The concentration of measure phenomenon, discovered by P.Levy in [15] and re-explored by V.Milman in the Seventies (mainly in [17]) is the key to a new discipline in mathematics known as high dimensional geometry, or, the geometry of probability (or perhaps even some other names). It is a key with which to study the geometry of high-dimensional convex bodies of $\mathbb{R}^n$, as well as to introduce probabilistic methods into convex geometry (K.Ball’s survey [1] on this matter is definitely worth a look). The main idea is that a good function $f$ (say Lipschitz for example) defined on a sufficiently high-dimensional sphere is concentrated (or takes as its value) around a single point (called the median) of $f$. The measure theoretic interpretation of this phenomena states that the normalised Riemannian measure of the sphere is concentrated around a hyper-sphere, which means more precisely that the measure of a small neighborhood of a hyper-sphere (of a sufficiently high dimensional sphere) is very close to 1. These all seem qualitative. Let us now give a quantitative version of what was just stated:

Theorem 1 (Levy-Milman) Let $\mu$ be the normalised Riemannian measure defined on the canonical Riemannian sphere $S^n$. Let $f : S^n \to \mathbb{R}$ be a 1-Lipshitz function. Let $\varepsilon > 0$. Then, there exists a point $m \in \mathbb{R}$ such that:

$$\mu(\{|f - m| \leq \varepsilon\}) \geq 1 - 2e^{-(n-1)\varepsilon^2/2},$$

where $|f - m| \leq \varepsilon$ is the pre-image of $[m - \varepsilon, m + \varepsilon]$ under the map $f$, i.e.

$$f^{-1}([m - \varepsilon, m + \varepsilon]).$$

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Remark: The sharp value of the concentration inequality (see [9]) is equal to:
\[
\frac{\int_0^\varepsilon (\cos(t))^{n-1} dt}{\int_0^{\pi/2} (\cos(t))^{n-1} dt}.
\]

It is by employing the concentration of measure phenomenon (precisely Theorem 1) that V. Milman was able to give a proof of the Dvoretzky Theorem in [18]. Of course, the sphere is not the only metric-measure space on which the measure has the concentration property. In [14], there is a detailed presentation of the concentration phenomenon and the spaces on which this property holds.

In this paper, I shall show that the strength of measure concentration goes beyond convex geometry, and can be applied to prove topological properties of Riemannian manifolds. The idea is to use a concentration theorem (such as Theorem 1) for the Grassmanians and combine it with a deep characterisation theorem of Brendle-Schoen in [3]. We shall see this in detail in Chapters 3 and 5. Before, however, a few words on the geometry/topology of positively-curved Riemannian manifolds:

We concern ourselves with closed (compact and boundary-less) Riemannian manifolds of positive curvature. By curvature, we mean the sectional curvature. The sectional curvature of a Riemannian manifold $M$ can be defined (geometrically) as a function which takes a couple $(m, p)$ (where $m \in M$ and $p \in G(2, n)$, a 2-plane in the tangent space $T_m(M)$) and gives the Gauss (or the usual) curvature of the image of $p$ by the application $exp_m$. So basically, the sectional curvature at each point of the manifold is a function on a 2-Grassmanian $G(2, n)$. This simple definition of sectional curvature is sufficient in this paper, since we will not do any Riemannian geometry related to this geometric invariant- apart from using results of Brendle-Schoen (presented in [3] and surveyed in [4]). Now, if we say that a manifold is positively-curved, we mean that for every $(m, p)$, the sectional curvature function gives a positive value. A great deal of Riemannian geometry is trying to find or construct examples of positively-curved Riemannian manifolds. Unfortunately, we do not have many examples available, and we do not have canonical procedures which could provide us with more examples. (See [25], [26], [23] and [4] for excellent surveys on this matter, as well as the presentation of all the examples available so far on the class of positively-curved Riemannian manifolds).

The result of this paper will be surprising in this matter because I show, in fact, that after a certain dimension, we simply do not have any examples other than the usual (or “God given” as it was said by Ziller in [26]) symmetric Riemannian manifolds. The main theorem of this paper is:

**Theorem 2** Let $M^n$ be an $n$-dimensional closed Riemannian manifold of (strict) positive curvature. An integer $N \in \mathbb{N}$ exists such that for every $n \geq N$, $M^n$ is either diffeomorphic to a spherical space form or isometric to a locally compact rank one symmetric space.

**Remark:** In section 3, I will explain what these manifolds (given by Theorem 2) are.

We said that the class of positively-curved Riemannian manifolds does not contain many examples. But we also have very few theorems characterising the topology of positively-curved Riemannian manifolds. We do have the famous Hopf Conjecture and all the work which has been done concerning this conjecture, but almost all of these works consider a great deal of symmetry that the manifolds are required to have (I used almost since I haven’t read every bit of progress
made on the Hopf Conjecture). The formulation of the Hopf Conjecture is simple and is as follows:

**Conjecture 1.1 (Hopf)** A compact even dimensional positively-curved Riemannian manifold has positive Euler characteristic.

One could ask the same (for every dimension) in the class of non-negatively curved Riemannian manifolds, by changing the positive Euler characteristic to non-negative.

The **Euler characteristic** of a finite CW-complex (or simplicial complex) is equal to \( \sum_n (-1)^n c_n \), where \( c_n \) is the number of \( n \)-dimensional cells. It is a topological invariant meaning that it remains constant under homeomorphism between topological spaces. See [11] for more details.

See the surveys [25], [20], [23] and [3] in order to find more information on the progress made on the Hopf Conjecture, as well as the references within these surveys.

An immediate corollary of Theorem 2 is a positive answer to the Hopf Conjecture in dimensions higher than \( N \). We shall prove this later on in Chapter 5.

This paper is organised as follows: the next section concerns a quick recap of the concentration of measure phenomenon. Section 3 quickly reviews some definitions and results needed from Riemannian and Alexandrov Geometry. In Section 4, I provide an overview of the geometry of the Grassmanians, and the concentration property of the canonical measures defined on Grassmanians will be explained and announced. Section 5 is the final section of this paper in which I present the proof of our main Theorem 2 as well as the Hopf Conjecture in high dimensions. Several remarks and questions related to this subject will also be presented.

## 2 Background on Concentration of Measure Phenomenon

The concentration of measure on the canonical sphere (and more precisely Theorem 1) is a consequence of the isoperimetric problem on this space. In order to understand this link (which will become useful later on) I should begin by giving a few definitions:

**Definition 2.1 (Metric-Measure Space)** The triple \((X, d, \mu)\) is called a Metric-Measure (or mm)-space if \(X\) is a metric space (supposed to be complete with countable basis), \(d\) is the distance defined by the metric on \(X\) and \(\mu\) is a Borel \(\sigma\)-finite measure defined on \(X\). If \(\mu\) is a probability measure, we call the triple a probability-metric (or pm) space.

**Definition 2.2 (Tubular Neighborhood)** Let \((X, d, \mu)\) be a mm-space, \(Y\) a subspace of \(X\), and let \(\varepsilon > 0\). The \(\varepsilon\)-neighborhood of \(Y\) is defined and denoted by:

\[ Y + \varepsilon = \{ x \in X | d(x, Y) \leq \varepsilon \}, \]

where

\[ d(x, Y) = \inf_{y \in Y} d(x, y). \]

In order to study the isoperimetric problem on a general metric-measure space, it is useful to define the following function:

**Definition 2.3 (Isoperimetric Function)** Let \((X, d, \mu)\) be a pm-space. The isoperimetric function of this space is defined on \(\mathbb{R}_+\) by

\[ \alpha_X(r) = \sup \{ 1 - \mu(A + r) | A \subset X, \mu(A) \geq \frac{1}{2} \}. \]
Remark: Requiring $\mu(A) \geq \frac{1}{2}$ is a matter of choice. We could ask for a number ($<1$) other than 1/2.

In order to study the concentration properties on a probability-metric space, it is useful to define:

**Definition 2.4 (The Concentration Profile)** Let $(X, d, \mu)$ be a pm-space. The concentration profile of $(X, d, \mu)$ is the smallest function $\pi$ on $\mathbb{R}_+$ such that for every $\varepsilon > 0$ and for every 1-Lipschitz function defined on $X$, there exists a $m \in \mathbb{R}$ such that

$$\mu(\{|f - m| > \varepsilon\}) \leq \pi(\varepsilon).$$

We would like to link the isoperimetric function of a space to its concentration profile. This is achieved by the following:

**Proposition 3** Isoperimetry $\Rightarrow$ Concentration. In particular for all $\varepsilon > 0$ we have:

$$\pi_X(\varepsilon) \leq 2 \alpha_X(\varepsilon).$$

The proof of Proposition 3 will be presented in the proof of (more general) Proposition 4 of this chapter.

Proposition 3 is very important since it asserts that in order to achieve a satisfying concentration property on a pm-space, it is sufficient to study its isoperimetric properties. The isoperimetric problem has been studied on many spaces (for example many Riemannian manifolds) and this is how one can, for example, prove a theorem similar to Theorem 1. Our first goal in this paper is to provide a concentration theorem for the Grassmanians, and by applying Proposition 3 to suitable Riemannian manifolds, we shall see in Chapter 4 that indeed we can achieve a satisfying concentration theorem for the Grassmanians.

We have seen that knowledge of the isoperimetric function gives direct knowledge of the concentration profile. There is another invariant located somewhere in between the isoperimetry and the concentration which is called the 1-waist. The *waists* are defined first in [8] and studied in [16].

**Definition 2.5 (1-waist)** Let $(X, d, \mu)$ be a metric-measure space. The 1-waist of $X$ is the largest function $w$ defined on $\mathbb{R}_+$ such that for every continuous function $f : X \to \mathbb{R}$, there exists a point $m \in \mathbb{R}$ such that for every $\varepsilon > 0$ we have:

$$\mu(f^{-1}(m) + \varepsilon) \geq w(\varepsilon).$$

**Remark:** Remember that $f^{-1}(m) + \varepsilon$ is not necessarily equal to $\{|f - m| \leq \varepsilon\}$, meaning that the left-hand side in the concentration formula calculates something different from the left-hand side in the waist formula.

The following proposition (studied in detail in [16]) explains why the waist function is located in between the isoperimetry and the concentration:

**Proposition 4** Let $X$ be a metric-measure space. Assume that all of the balls in $X$ are connected. Then

Isoperimetry $\Rightarrow$ 1-waist $\Rightarrow$ Concentration. In particular for every $\varepsilon > 0$ we have:

$$1 - 2 \alpha_X(\varepsilon) \leq w(\varepsilon) \leq 1 - \pi_X(\varepsilon).$$
Proof of Proposition 4

First, observe that this proposition contains (and hence is more general than) Proposition 3.

We begin by proving the implication:

\[ 1 - \text{waist} \Rightarrow \text{Concentration}. \]

We assume to know a bound on the function 1-waist of the metric-measure space \( X \). Let \( f : X \to \mathbb{R} \) be a 1-Lipschitz function. Since Lipschitz functions are continuous, we can apply the definition of the 1-waist. Therefore a point \( m \in \mathbb{R} \) exists such that for every \( \varepsilon > 0 \) we have:

\[ \mu(f^{-1}(m) + \varepsilon) \geq w_X(\varepsilon). \]

Furthermore, since \( f \) is assumed to be a 1-Lipschitz map, we have:

\[ f^{-1}(m) + \varepsilon \subset f^{-1}([m - \varepsilon, m + \varepsilon]), \]

and hence in conclusion:

\[
\begin{align*}
\mu(f^{-1}([m - \varepsilon, m + \varepsilon]) & \geq \mu(f^{-1}(m) + \varepsilon) \\
& \geq w_X(\varepsilon).
\end{align*}
\]

This proves the 1-waist implies the concentration and more particularly that:

\[ w_X(\varepsilon) \leq 1 - \pi_X(\varepsilon). \]

We now prove the less trivial implication:

\[ \text{Isoperimetry} \Rightarrow 1 - \text{waist}. \]

Let \( f : X \to \mathbb{R} \) be a continuous function. Let \( m \) be the median of this function i.e. \( \mu(f \geq m) \geq \frac{1}{2} \) and \( \mu(f \leq m) \geq \frac{1}{2} \). The median exists but doesn’t have to be unique.

We define the six sets \( A = \{f \leq m\} + \varepsilon, B = \{f \geq m\} + \varepsilon, C = \{f < m\}, D = \{f > m\}, E = f^{-1}(m) + \varepsilon \text{ and } F = f^{-1}(m). \) It is straightforward to check that

\[
\begin{align*}
A & \subset C \cup F \cup (E - C \cup F) \\
B & \subset D \cup F \cup (E - D \cup F) \\
X & = C \cup D \cup F.
\end{align*}
\]

Indeed, if \( x \in A \), a \( y \in X \) exists such that \( f(y) \leq m \) and \( d(x, y) < \varepsilon \). If \( f(x) \leq m \) then \( x \in C \cup F \). If \( f(x) > m \), since \( B(x, \varepsilon) \) is assumed to be connected, a \( z \in B(x, \varepsilon) \) exists such that \( f(z) = m \), hence \( x \in E \). The proof is similar for \( B \). Hence we have:

\[
\begin{align*}
\mu(E - C \cup F) & \geq \mu(A) - \mu(C) - \mu(F) \\
\mu(E - D \cup F) & \geq \mu(B) - \mu(D) - \mu(F) \\
\mu(E - F) & \geq \mu(A) + \mu(B) - (\mu(C) + \mu(D) + \mu(F)) - \mu(F) \\
\mu(E) & = \mu(E - F) + \mu(F) \geq \mu(A) + \mu(B) - 1.
\end{align*}
\]

The isoperimetric inequality implies that for every \( \varepsilon > 0 \)

\[
\begin{align*}
\mu(\{f \leq m\} + \varepsilon) & \geq 1 - \alpha_X(\varepsilon) \\
\mu(\{f \geq m\} + \varepsilon) & \geq 1 - \alpha_X(\varepsilon),
\end{align*}
\]

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and in conclusion
\[
\mu(f^{-1}(m) + \varepsilon) \geq \mu(\{f \leq m\} + \varepsilon) + \mu(\{f \geq m\} + \varepsilon) - 1 \\
\geq 2 - 2\alpha_X(\varepsilon) - 1 \\
\geq 1 - 2\alpha_X(\varepsilon).
\]
Hence the implication
\[
w_X(\varepsilon) \geq 1 - 2\alpha_X(\varepsilon),
\]
is proved.

\[ \square \]

3 A Review of the Geometry of Positively Curved Riemannian Manifolds

The curvature is the most important geometric invariant of a Riemannian manifold. It has a very long history. In dimension 2, there is a very pleasing geometric interpretation of the curvature leading to the Gauss curvature. In higher dimensions things get much more complicated. Every geometric property related to curvature is inscribed on a complicated tensor called the Riemann curvature. The Riemann curvature at each point of the manifold is a multilinear function \( R : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R} \). It has certain symmetries, namely:
\[
R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Z,W,X,Y),
\]
and the first Bianchi identity:
\[
R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0
\]
for every \( X,Y,Z,W \in T_pM \). In the introduction, I provided a definition of sectional curvature which was quite rough. The sectional curvature of a Riemannian manifold can be defined from its Riemann curvature tensor as follows: given any point \( m \in M \) and any \( p \in G(2,n) \), the Grassmanian of the tangent space at the point \( m \), the sectional curvature at \( p \) is defined by:
\[
K(p) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2-g(X,Y)^2},
\]
where \( g \) is the metric tensor of the Riemannian manifold \( M \) and \( \{X,Y\} \) is a basis of \( p \). This definition is independent of the choice of basis for the 2-plane \( p \) and choosing \( X,Y \) orthonormal, one would have:
\[
K(p) = R(X,Y,X,Y).
\]
Note that the sectional curvatures at each point of \( M \) determine the Riemann curvature tensor.

By contracting the Riemann Curvature with respect to the metric, we obtain the Ricci and scalar curvatures of a Riemannian manifold \( (M,g) \). More precisely:
\[
Ric(X,Y) = \sum_{k=1}^n R(X,e_k,Y,e_k),
\]
and

\[ Scal = \sum_{k=1}^{n} Ric(e_k, e_k), \]

where \( X, Y \) are arbitrary vectors in \( T_p M \) and \( \{ e_1, \ldots, e_n \} \) is an orthonormal basis of the tangent space \( T_p M \).

Sometimes, it is helpful to study several classes of Riemannian manifolds together. It is then very useful to define a metric, describing the distance between two Riemannian manifolds. To understand this distance, first recall the following:

**Definition 3.1 (Hausdorff Distance)** Let \( (X, d) \) be a metric space. Let \( Y_1, Y_2 \) be two sub-spaces. The Hausdorff distance between \( Y_1 \) and \( Y_2 \) is defined to be:

\[ d_X(Y_1, Y_2) = \inf_{\varepsilon > 0} \{ Y_1 \subset Y_2 + \varepsilon, Y_2 \subset Y_1 + \varepsilon \}. \]

**Remark:** The Hausdorff distance defines a complete metric on the set of all compact subsets of a (fixed) complete metric space.

Using the above definition, one can define a distance between (any) two abstract metric spaces as follows:

**Definition 3.2 (Gromov-Hausdorff Distance)** Let \( (X, d) \) and \( (Y, \delta) \) be two metric spaces. The Gromov-Hausdorff distance between \( (X, d) \) and \( (Y, \delta) \), denoted by \( d_{GH}((X, d), (Y, \delta)) \) is defined to be:

\[ d_{GH}(X, Y) = \inf_{d_Z(X, Y)} \{ f_1 : X \to Z, f_2 : Y \to Z \}, \]

where \( Z \) is a metric space on which \( X \) and \( Y \) are mapped isometrically via \( f_1 \) and \( f_2 \).

**Remark** The Gromov-Hausdorff distance defines a complete metric on the set of isometry classes of compact metric spaces.

**Definition 3.3 (Limit of a Sequence of Riemannian Manifolds)** Let \( \{ M_i \}_{i=1}^{\infty} \) be a sequence of Riemannian manifolds.

\[ \lim_{i \to \infty} M_i = M_{\infty}, \]

if

\[ \lim_{i \to \infty} d_{GH}(M_i, M_{\infty}) = 0 \]

Recall the following:

**Theorem 5 (Alexandrov-Toponogov)** (See figure below) Let \( M \) be a Riemannian manifold with \( K \geq \kappa \). Let \( S^n(\kappa) \) be a sphere of constant curvature equal to \( \kappa \). Let \( x, y, z, v, w \in M \) and let \( x', y', z', v', w' \in S^n(\kappa) \). Assume \( d(x, y) = d(x', y') \), \( d(x, z) = d(x', z') \), \( d(y, z) = d(y', z') \), \( d(x, v) = d(x', v') \), \( d(x, w) = d(x', w') \). Assume also \( v \in xy, w \in xz, v' \in x'y' \) and \( w' \in x'z' \), where \( ab \) stands for (a) geodesic segment from \( a \) to \( b \). Then we have:

\[ d(v, w) \geq d(v', w'). \]
Triangle Comparison

For a proof of Theorem 5, consult [21] or [6].

Definition 3.4 (Alexandrov Spaces) A length space of finite dimension is said to be an Alexandrov space with $K \geq \kappa$ if it satisfies the conclusion of Alexandrov-Topogonov’s Theorem 5.

Remark: A length structure in a space is roughly a class of paths with which we can assign their length. A length space is then a metric space by which the metric can be obtained from the distance function associated to the length structure (see [5] for more details).

For understanding the convergence of (some class of) Riemannian manifolds, it is useful to remember the following:

Theorem 6 (Gromov’s Compactness Theorem) The set of Alexandrov spaces with $K \geq \kappa > 0$ and (Hausdorff) dimension $\leq n$ is compact (in the Gromov-Hausdorff topology).

and additionally:

Proposition 7 Let $X_i$ be a sequence of compact length spaces and let $X = \lim_{i \to \infty} X_i$. If $X_i$ are Alexandrov spaces with $K \geq \kappa$ then $X$ is also an Alexandrov space with $K \geq \kappa$.

For the above materials, I recommend to read the following: [9], [7], [2], [5] and [21].

Definition 3.5 A Riemannian manifold $M$ is said to be weakly $\delta$-pinched in the pointwise sense if

$$0 \leq \delta K(p_1) \leq K(p_2),$$

for every points $m \in M$ and all $p_1, p_2 \in G(2, n)$, where $G(2, n)$ is the 2-Grassmanian of the tangent space $T_m(M)$. If the strict inequality holds, we say that $M$ is strictly $\delta$-pinched in the pointwise sense.

First, let’s recall the following theorem:

Theorem 8 (Brendle-Schoen) Let $(M, g)$ be a compact Riemannian manifold which is weakly 1/4-pinched in the sense of definition 3.5. Then $M$ is either diffeomorphic to a spherical space form or is isometric to a locally compact rank one symmetric space.
The proof of Theorem 8 is presented in [3].

Let me explain here what the manifolds that occur in Theorem 8 are:

We all know the canonical Riemannian sphere $\mathbb{S}^n$, which is a compact space of constant curvature everywhere equal to 1. In 1926, H. Hopf proved (see [12] and [13]) that a compact, simply connected Riemannian manifold with constant sectional curvature 1 is necessarily isometric to the canonical $\mathbb{S}^n$. If we drop the simple connectivity property on a compact Riemannian manifold and we ask to classify all those with constant sectional curvature, then we get Riemannian manifolds isometric to a quotient $\mathbb{S}^n/G$, with $G$ a finite group of isometries acting freely on $\mathbb{S}^n$. These are the manifolds known as spherical space forms, and are all classified by Wolf in [24]. For $n$, an even integer, the projective space $\mathbb{R}P^n$ and $\mathbb{S}^n$ are the only spherical space forms, but if $n$ is odd there are infinite examples of spherical space forms.

The class of compact rank one symmetric space includes $\mathbb{S}^n$ and $\mathbb{R}P^n$ but there are other examples which are not spherical space forms. The complex projective space $\mathbb{C}P^n$, the quaternionic projective space $\mathbb{H}P^n$, and the Cayley projective plane of dimension 16 are the other examples. A manifold is a locally compact rank one symmetric space if it is covered by a compact rank one symmetric space. This is all we need to know, and in fact the main Theorem 2 of this paper states that after a certain dimension, these are in fact all the compact manifolds of positive curvature.

The largest dimension for a manifold of strictly positive curvature (which is not a spherical space form nor a locally compact rank one symmetric space) so far was given by Wallach in [22] and has a dimension equal to 24.

4 The Grassmanians as Metric-Measure Spaces

Before doing anything interesting, we need to understand Grassmanians as metric-measure spaces. $\mathbb{R}^n$ is the usual Euclidean space enhanced with its Lebesgue measure. For $1 \leq k \leq n$, the Grassmanian $G(k,n)$ is the space of $k$-planes of $\mathbb{R}^n$, containing the origin. We would like to put a metric on this space (hence give it a topological structure). For any $A, B \in G(k,n)$, the natural distance between them is defined to be the Hausdorff distance of the unit spheres contained in $A$ and $B$, i.e.

$$d(A, B) = \sup\{d(x, \mathbb{S}^{n-1} \cap A) \mid x \in \mathbb{S}^{n-1} \cap B\}.$$ 

The orthogonal group $O(n)$ acts on $\mathbb{R}^n$ and the metric defined on $G(k,n)$ is invariant under this action. Furthermore, the action of $O(n)$ is transitive on $G(k,n)$, therefore a unique Borel probability measure $\mu_{n,k}$ on $G(k,n)$ exists (the Haar measure on the group $G(k,n)$). The measure $\mu_{n,k}$ is obviously invariant under the action of the orthogonal group. There is an agreeable way to see and understand this measure, and it is via the standard Gaussian measure $\gamma_n$ defined on $\mathbb{R}^n$. For every measurable subset $A \subset \mathbb{R}^n$, recall the (standard) Gaussian measure of $A$ which is defined by

$$\gamma_n(A) = \frac{1}{(2\pi)^{n/2}} \int_A e^{-|x|^2/2} dx.$$ 

Consider the product of Gaussian measures on $k$ copies of $\mathbb{R}^n$, and for any $k$-couple of points, consider the map which sends them to the span of these points (hence a $k$-plane in $\mathbb{R}^n$ and a point of $G(k,n)$). Then one can see the measure $\mu_{k,n}$ as the push-forward of the product of Gaussian measures under the above map.

Hence, $(G(k,n), d, \mu_{k,n})$ is now a pleasant metric-measure space upon which we would like to present a concentration theorem similar to Theorem 1.
Theorem 9 Let the metric-measure space \((G(k,n), d, \mu_{k,n})\) be defined as above. For every 1-Lipschitz functions \(f: G(k,n) \to \mathbb{R}\), a point \(m \in \mathbb{R}\) exists such that for every \(\varepsilon > 0\), we have:

\[
\mu_{k,n}(|f - m| \leq \varepsilon) \geq 1 - 2e^{-(n-1)\varepsilon^2/8},
\]

where \(|f - m| \leq \varepsilon\) is the pre-image of \([m - \varepsilon, m + \varepsilon]\) under the map \(f\).

Proof of Theorem 9:

Let us first recall an important comparison tool related to the isoperimetry of Riemannian manifolds with a lower bound on their Ricci curvature. Recall the definition of the Ricci curvature from the previous section.

Theorem 10 (Levy-Gromov) Let \((M, g)\) be a compact connected Riemannian manifold of dimension \(n \geq 2\). We normalise the volume on \(M\) and see it as a pm-space. Let

\[
R = \inf_{X \in TM} \text{Ric}(X, X),
\]

suppose \(R > 0\). Let \(S^n(\kappa)\) be the Riemannian sphere such that its Ricci curvature is equal to \(R\). (Basically this sphere is the boundary of a ball of radius \(\sqrt{\frac{n-1}{R}}\)). Then

\[
\pi_M \geq \pi_{S^n(\kappa)}.
\]

This theorem is proved in the Appendix of the book [9] (one may also look in the Appendix of the book [19]). An important corollary of the above theorem is the following:

Corollary 4.1 Let \((M, g)\) be a compact connected Riemannian manifold of dimension \(n \geq 2\). Normalise the volume and see \(M\) as a pm-space. Let \(f: M \to \mathbb{R}\) be a 1-Lipschitz function on \(M\). Then, \(m \in \mathbb{R}\) exists such that for every \(\varepsilon > 0\) we have:

\[
\mu(|f - m| \leq \varepsilon) \geq 1 - 2e^{-R\varepsilon^2/2},
\]

where \(R\) is defined by:

\[
R = \inf_{X \in TM} \text{Ricci}(X, X),
\]

and is supposed to be strictly positive.

Proof of Corollary 4.1

For the sphere \(S^n(\kappa)\), which is the boundary of a Euclidean ball of radius \(R > 0\) equipped with the normalised Riemannian measure \(\mu\), the concentration profile satisfies:

\[
\pi_{S^n(\kappa)}(\varepsilon) \leq e^{-(n-1)\varepsilon^2/2R^2},
\]

applying Proposition 4 and Levy-Gromov Theorem 10 the proof of Corollary 4.1 follows.
We now apply Corollary 4.1 to a special manifold of which we know the value

\[ R = \inf_{X \in TM} \text{Ricci}(X, X). \]

Let \( SO(n) \) be the special orthogonal group, seen as a Lie group equipped with its normalised Haar measure. We know (one can confirm this in classical textbooks on Lie groups and in [2], [6] and [21]) that for this metric-measure space, we have:

\[ R = \frac{n - 1}{4}. \]

We shall require another useful proposition which maps the concentration profile under Lipschitz mappings of metric-measure spaces:

**Proposition 11** Let \( \phi \) be a Lipschitz map between two metric spaces \((X, d)\) and \((Y, \delta)\):

\[ \phi : X \to Y, \]

such that for every \( x, x' \in X \) we have:

\[ \delta(\phi(x), \phi(x')) \leq ||\phi||_{Lip} d(x, x'), \]

where \( ||\phi||_{Lip} \) is the Lipschitz constant of the map \( \phi \). Let \((X, d)\) be a probability-metric space equipped with a probability measure \( \mu \) and denote by \( \mu_{\phi} \) the push-forward of \( \mu \) by \( \phi \), seen as a probability measure on \((Y, \delta)\). Then for every \( \varepsilon > 0 \), we have:

\[ \pi_Y(\varepsilon) \leq \pi_X(\varepsilon ||\phi||_{Lip}^{-1}). \]

In particular \( \pi_Y \leq \pi_X \) if \( \phi : X \to Y \) is a 1-Lipschitz map.

An immediate corollary to the above proposition is:

**Corollary 4.2** Let \( X \) be a topological group equipped with a (left-) translation invariant metric \( d \) and \( Y \) is a quotient \( X/G \) equipped with the quotient metric:

\[ \delta(y, y') = \inf \{ d(x, x') | \phi(x) = y, \phi(x') = y' \}. \]

Then \( \phi : X \to X/G \) is a 1-Lipschitz map and if additionally we assume that \( X \) is a probability-metric space and \( X/G \) is equipped with the push-forward measure from \( \phi \), then:

\[ \pi_{X/G} \leq \pi_X. \]

### 4.1 End Proof of Theorem 9

Since \( G(k, n) \) is a quotient of \( SO(n) \), then \( G(k, n) \) equipped with the normalised Haar measure \( \mu_{k,n} \) satisfies the assumptions of Corollary 4.2 and hence:

\[ \pi_{G(k, n)} \leq \pi_{SO(n)}. \]

Thus the proof of Theorem 9 follows. \( \square \)
5 Proof of Theorem 2

Let \((M, g)\) be a compact Riemannian manifold of positive curvature. Let \(\text{dim}(M) = n\). Let the Grassmanian \(G(2, n)\) be enhanced with its metric-measure structure as defined in Section 4. For every point \(m \in M\), the 2-Grassmanian \((G(2, n))\) of the tangent space \(T_m M\) has this structure. Let \(0 < d = \text{diam}(G(2, n))\) be the diameter of this space, and consider it fixed from this point on. By the compactness of \(M\) and the Grassmanians \(G(2, n)\), a \(k > 0\) exists such that for every point \(m \in M\) and every two-plane \(p \in G(2, n)\), we have

\[ K_m(p) \geq k. \]

There is a rescaling procedure for Riemannian manifolds \((M, g)\) which is simply mapping \(g \rightarrow \lambda g\). In this case, it is not hard to see that the new Riemannian manifold \((M, \lambda g)\) satisfies:

\[ K_{\lambda g} = \lambda^{1/2} K_g, \]

the diameter behaves as follows:

\[ \text{diam}(M, \lambda g) = \lambda^{1/2} \text{diam}(M, g), \]

and finally, for the volume, we have:

\[ \text{Vol}(M, \lambda g) = \lambda^{n/2} \text{Vol}(M, g). \]

Hence, by the observation made above, taking \(\lambda > 0\) by the rescaling map \(g \rightarrow \lambda g\) for \(\lambda > 0\), without loss of generality, we can assume that \(k = \frac{5d}{6}\).

It is clear by the definition of the sectional curvature that this function is smooth. Thus, the sectional curvature function (at each point of \(M\)) can be seen as a Lipschitz function. We would like to apply the concentration Theorem 9 appropriately in order to get a bound on the dimension of \(M\), such that (after this critical dimension) we are sure that at each point of \(M\), the sectional curvature function will be weakly 1/4-pinched in the pointwise sense of Brendle-Schoen. If we want to have the pointwise 1/4-pinching property everywhere on \(M\), the sectional function at each point of \(M\) has to take a value in the interval \([x - \varepsilon, x + \varepsilon]\), where \(x\) (the median of the sectional curvature) is supposed to satisfy \(x \geq \frac{5d}{6}\). Hence by taking

\[ \varepsilon = \frac{d}{2}, \]

we are certain that at every point \(m \in M\), the sectional curvature is weakly 1/4-(pointwise) pinched. This observation already provides a lower bound for the dimension, which can be calculated as follows:

Applying the concentration Theorem 9 we know that for every \(\varepsilon > 0\) we have:

\[ \mu_{2,n}(|K - x| \leq \varepsilon) \geq 1 - 2e^{-(n-1)\varepsilon^2/8}, \]

where \(K\) is the sectional curvature (at a certain point of \(M\)). If we have

\[ e^{-(n-1)\varepsilon^2/8} < 1/2, \]

then we are sure that a point in \(G(2, n)\) exists such that its image lies inside the interval \([x - \varepsilon, x + \varepsilon]\). The above argument indeed holds for every point \(m \in M\) (but for a different median \(x\) and the same \(\varepsilon\)).
Manipulating the inequality (1), we get:

\[ n > \frac{8 \ln 2}{\varepsilon^2} + 1 \]
\[ > \frac{16 \ln 2}{d^2} + 1 \]
\[ = N_1. \]

Now we are certain that if \( n \geq N_1 \), for every point on the Grassmanian \( G(2, n) \) the value of the sectional curvature lies in the interval \([x - \varepsilon, x + \varepsilon]\) (thanks to the value of \( \varepsilon \) we have chosen).

Unfortunately, the dimension \( N_1 \) still doesn’t solve our problem. The issue here is that we assumed the sectional curvature functions all to be 1-Lipschitz. Of course, by their smoothness we know they are \( \text{Lipschitz} \) functions, but nothing tells that they are one-Lipschitz. We have to somehow remedy this issue. It is resolved as follows:

Define the following

\[ \mathcal{M}_n = \{ \text{dim}(M) = n, K \geq \frac{5d}{6} \}. \]

(We could make the class \( \mathcal{M}_n \) narrower using a theorem of Grove-Shiohama in [10], by taking an upper bound for the diameter). We now define a sequence \( \{k_n\}_{n=2}^{+\infty} \) as follows: For every \( n \geq 2 \) define:

\[ k_n = \sup_{M_n \in \mathcal{M}_n} \left( \sup_{m \in M_n} \|K_m\|_{\text{Lip}} \right), \]

where \( \|K_m\|_{\text{Lip}} \) is the Lipschitz constant of the sectional curvature function at the point \( m \in M_n \).

Since we are dealing with compact manifolds, and (again) since the 2-Grassmanian is a compact manifold, we have

\[ \sup_{m \in M_n} \|K_m\|_{\text{Lip}} \neq +\infty. \]

If a sequence \( \{M_i\}_{i=1}^{+\infty} \) of Riemannian manifolds (where for every \( i \geq 1, M_i \in \mathcal{M}_n \)) converges to a Riemannian manifold of the same dimension the sequence

\[ k_i(M_i) = \sup_{m \in M_i} \|K_m\|_{\text{Lip}}, \]

converges to a positive number. Additionally (according to Gromov’s (pre)-compactness Theorem [6]) if the sequence \( \{M_i\}_{i=1}^{+\infty} \) collapses (for \( i \) large enough) on almost every point of \( G(2, n) \), the sectional curvature function remains constant. Hence, the sequence \( k_i(M_i) \) converges to a non-negative number. The sequence \( k_n \) is well-defined. However, according to the concentration Theorem [9] when \( n \to \infty \) (asymptotically) every Lipschitz function (and hence differentiable function) on the 2-Grassmanian \( G(2, n) \), becomes a constant function for which the Lipschitz constant is equal to zero. Therefore:

\[ \lim_{n \to \infty} k_n = 0. \]

Now take \( N_2 \in \mathbb{N} \) such that for every \( n \geq N_2 \), we have \( k_n \leq 1 \). Set

\[ N = \max\{N_1, N_2\}. \]

For every Riemannian manifold \( M \) of dimension \( \geq N \), Theorem [8] may be applied in order to determine the topology of \( M \).

This ends the proof of our main Theorem [2].
Theorem 2 suggests the following:

**Definition 5.1 (Critical Dimension)** The critical dimension of the class of compact positively-curved Riemannian manifolds (denoted by $n_{crc}$) is the smallest dimension after which the topology of such manifolds are either a spherical space form or a locally compact rank 1-symmetric space. The existence of this number is provided by Theorem 2.

*Remark:* We already know that $n_{crc} \geq 25$. This is due to the existence of the Wallace manifold in dimension 24:

$$W^{24} = F_4/\text{Spin}(8),$$

which is a fibration over the Cayley plane $Ca\mathbb{P}^2$, and having as fibers the 8-dimensional sphere $S^8$. This is a manifold with positive-curvature which is neither a spherical space form nor a locally compact rank one symmetric space.

A direct corollary to Theorem 2 is the following:

**Corollary 5.1** The Hopf Conjecture 1.1 is true for every $n \geq n_{crc}$.

*Proof of Corollary 5.1*

It is not hard to compute the Euler characteristic of compact rank 1 symmetric spaces. In fact, as shown by the above remark, since we know $n_{crc} \geq 25$, we only have to consider the compact symmetric spaces, which are $S^n$, $CP^n$ and $HP^n$. We denote the Euler characteristic of a manifold $M$ by $\chi(M)$. Studying their (co)homology, we easily can deduct that:

$$\chi(S^n) = 1 + (-1)^n.$$  

Hence for $n = 2k$, we get that the Euler characteristic of the spheres are equal to 2 (which is $> 0$ as suggested by the Hopf Conjecture). For the complex projective space we have:

$$\chi(CP^n) = n + 1 \quad > 0.$$ 

And for the quaternionic projective space we also have:

$$\chi(HP^n) = n + 1 \quad > 0.$$ 

A manifold which is a (finite) cover over one of those shown above will have positive Euler characteristic. Indeed a covering does not change the *sign* of the Euler characteristic. This ends the proof of Corollary 5.1.

*Questions and Remarks:*

- Is $n_{crc} = 25$?
• Can one use the 1-waist invariant $w$ instead of the concentration in order to get a bound for the critical dimension $n_{\text{cric}}$?

• It is clear that $k_n$ (the sequence defined during the proof of Theorem [2]) depends on the lower bound of the sectional curvature. In order to estimate $n_{\text{cric}}$ sharply, one way would be to study the sequence $k_n$. A sharp estimate on $k_n$ leads to a sharp estimate of $n_{\text{cric}}$. My calculations above were far from sharp.

• Once again, we witnessed that geometry in higher dimensions is easier than in lower dimensions. Studying positively-curved Riemannian manifolds in lower dimensions is much, much harder and also much more technical. Take a look at [20] for an example of this difficulty in lower dimensions.

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