Analysis of the periodic superconducting CPW transmission line with stepped-impedance for on-chip microwave filter applications

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Abstract. This paper investigates the filter properties of a superconducting coplanar waveguide (CPW) periodically loaded with stepped-impedances. The unit cell of the stepped-impedance filter is a cascade of two transmission lines (TLs) with different characteristic impedances. This impedance difference is obtained by step changing on the width of the center strip of the CPW. We use the ABCD matrix representation together with the Floquet analysis to derive the dispersion relation of the infinitely long periodic structure. Scattering matrix analysis along with impedance calculation is performed for an arbitrary number of unit cells. Complex propagation constant, scattering parameters and the input impedance of the structure are expressed in a closed-form equations. The results display the low-pass filtering property. Also, the effect of changing geometrical parameters on the cut off frequency, rejection bandwidth and attenuation in rejection band of such a filter are shown and discussed in this paper.

1. Introduction
Over the past fifteen years, High Temperature Superconductor (HTS) filters have come into widespread use in cellular and satellite telecommunication systems and indeed it has become the major commercial application of superconductor technology in the microwave industry [1],[2],[3]. The most significant improvement achieved by the use of superconductor thin-films is in the frequency selectivity of microwave filters which is determined by the steepness of the filter skirts and out-of-band rejection [1]. Due to the ultra-low dissipation and distortion in superconductor materials, HTS filters have demonstrated extensively high quality factors [4],[5] and dramatic improvement in insertion loss [6]. They enhance the signal-to-noise ratio and offer extremely reliable products with a small-size [7], lightweight and high-performance design for the wide variety of applications such as at the front-end receivers in wireless communication systems [8], Josephson parametric amplifier [9], [10] and read-out circuits around the superconducting qubits for Quantum information processing [11], [12].

Moreover, the existence of a discrete set of pass-bands separated by stop-bands in microwave periodic structures has offered the opportunity of using them for filter purposes [13],[14]. The stepped-impedance filter is one of the most popular and simplest type of distributed planar filter [14],[15],[16], which implemented over a planar Transmission Line (TL) with alternative sections of very high and very low characteristic impedance lines. The center strip of this structure periodically consists of narrow and wide segments to realize a low-pass stepped impedance filter. Such structures
are appropriate for many applications in communication circuits because of easy fabrication and low cost.

The method that is employed for the analysis of this structure is the ABCD matrix representation of each unit cell and then application of the Fouquet theorem to conduct the dispersion analysis. The step change in the width of the center strip is captured by a shunt capacitor in this analysis to have a more rigorous model. In practice, where the number of unit cells is limited, scattering matrix is employed to describe the microwave characteristics of this structure. The results obtained by this analysis is a good approximation for the infinitely long (open) periodic structure, as demonstrated in this paper.

This paper is organized as follows. Section 2 contains a brief analysis of an open periodically loaded CPW transmission line, where we use the ABCD matrix and the Floquet theorem to characterize the behavior of the wave propagation that finally leads to the derivation of a closed form equation for the propagation constant. Impedance calculation and s-parameter analysis are also carried out for the finite periodic structure. In addition, the CAD tool (Computer Aided Design) that was developed for calculating the distributed circuit parameters of the superconducting CPW is briefly described in this section. Numerical results and discussions are presented in Section 3, followed by conclusions in Section 4.

2. Periodic superconducting CPW transmission lines transmission line theory

2.1. Structure

Fig. 1(a) shows a unit cell of a periodically loaded CPW transmission line with stepped-impedance. Each unit cell is comprised of two superconducting transmission lines. In this figure, the width of the center conductor for transmission lines 1 and 2 are \( s_1 \) and \( s_2 \), respectively, the spacing between the center conductor and the ground plane conductor for transmission lines 1 and 2 are \( w_1 \) and \( w_2 \), respectively. The height of the dielectric spacer is \( h \), and thickness of the conductor plates is \( t \).

![Fig. 1. (a) A unit cell of periodically loaded CPW, (b) Modeling a step discontinuity by a capacitance.](image)

As can be seen in Fig. 1(a), the two transmission lines of a unit cell are connected together. At the connection point (junction point), a step discontinuity (coplanar step discontinuity) exists, resulting in sharp edges, which causes electrical charge accumulation at the corner. Therefore, the coplanar step discontinuity can be modelled by a capacitor \( C_s \) [17], as shown in Fig. 1(b).

2.2. Transmission Line Model of the Structure

All of the equations that have been used for calculating the propagation constant of each transmission line of the unit cell and investigating its dispersion behavior are based on the distributed circuit parameters of the superconducting CPW transmission line. For calculating distributed circuit parameters, it is necessary to develop a CAD application. Indeed, there are various configurations for coplanar waveguides [18], and in this work we study the “conventional CPW” where the ground plane is of semi-infinite extent on either side. For a superconducting CPW transmission line, its shunt capacitance and conductance per unit length formulas are taken from [18], as it does not change for both TLs with superconductor or normal conductors [19]. For calculating the capacitance, it is needed
to use the complete elliptical integral of the first kind as it is demonstrated in [18]. Since the CPW transmission line is an inhomogeneous transmission line, the effective dielectric constant has to be defined with the consideration of thickness of the superconducting plates as well as dispersion behaviour of the line, as explained in [18], [20], [21]. The kinetic inductance per unit length for superconducting CPW is given in [22] and [23]. By calculating the real and imaginary parts of the surface impedance associated with superconductor plates, the series ohmic resistance per unit length of the line is found [13].

2.3. Floquet Analysis of the Structure

The unit cell of the structure shown in Fig. 1 consists of a cascade connection of three successive sections. Therefore, the ABCD matrix representation is chosen to model each section, as the whole unit cell can be mathematically described by multiplying the ABCD matrices of the individual sections. The transmission line model of each unit cell is illustrated in Fig. 2.

![Fig. 2. A unit cell’s equivalent transmission line model (s1 > s2).](image)

From a theoretical standpoint, the analysis of an open periodic structure, which is made by the infinite repetition of the unit cell, can be dramatically simplified by applying the Floquet theorem [13], [14]. Applying this theorem, the closed-form dispersion relation for the propagation constant is found by

\[
\gamma = \alpha + j \beta = \frac{1}{l} \cosh^{-1} \left\{ \cosh \gamma_1 l_1 \cosh \gamma_2 l_2 + 0.5 \left( \frac{Z_{01}}{Z_{02}} + \frac{Z_{02}}{Z_{01}} \right) \sinh \gamma_1 l_1 \sinh \gamma_2 l_2 \right\} + j 0.5 \omega C_s (Z_{01} \sinh \gamma_1 l_1 \cosh \gamma_2 l_2 + Z_{02} \cosh \gamma_1 l_1 \sinh \gamma_2 l_2) \tag{1}
\]

where \( \alpha \) (NP/m), \( \beta \) (rad/m), \( \omega \) (rad/s) and \( C_s \) are the attenuation constant, phase constant, angular frequency and the coupling capacitance associated with this structure. In equation (1), \( Z_{01}, \gamma_1, l_1, Z_{02}, \gamma_2, \) and \( l_2 \) are the characteristic impedance, the propagation constant, and the length of transmission lines 1 and 2, respectively, as clearly illustrated in Fig. 2. It is also important to note that the spatial period of the structure is \( l = l_1 + l_2 \).

2.4. Impedance calculation

Floquet analysis in the subsection 2.3 yields the dispersion relation which contains essential information about the wave propagation through the periodic structure. As this structure is to be connected to other peripheral microwave circuits, other source of data such as reflection and transmission can be obtained by its input impedance. For the case of infinitely long periodic structure, the input impedance seen from any interface of two subsequent unit cells can be found by following quadratic equations

\[
C Z_{in}^2 + (D - A) Z_{in} - B = 0 \tag{2}
\]

where

\[
A = \cosh(\gamma_1 l_1) \cosh(\gamma_2 l_2) + j \omega C_s Z_{01} \sinh(\gamma_1 l_1) \cosh(\gamma_2 l_2) + \frac{Z_{01}}{Z_{02}} \sinh(\gamma_1 l_1) \sinh(\gamma_2 l_2) \tag{3}
\]
\[ B = Z_{02} \cosh(y_1 l_1) \sinh(y_2 l_2) + j \omega C Z_0 Z_{02} \sinh(y_1 l_1) \sinh(y_2 l_2) + Z_{01} \sinh(y_1 l_1) \cosh(y_2 l_2) \]  
\[ C = \frac{1}{Z_{01}} \sinh(y_1 l_1) \cosh(y_2 l_2) + j \omega C \cosh(y_1 l_1) \cosh(y_2 l_2) + \frac{1}{Z_{02}} \cosh(y_1 l_1) \sinh(y_2 l_2) \]  
\[ D = \frac{Z_{02}}{Z_{01}} \sinh(y_1 l_1) \sinh(y_2 l_2) + j \omega C Z_{02} \cosh(y_1 l_1) \sinh(y_2 l_2) + \cosh(y_1 l_1) \cosh(y_2 l_2) \].

Equation (2) has two roots such that one of them has a negative real part which should be discarded.

2.5. Scattering parameter

When the number of unit cells is limited to \( N \), the structure reduced to the finite periodic structure and it can be viewed by cascading multiple identical units with two boundaries at the two ends of the network, as illustrated in Fig. 3.

![Fig. 3. General representation of a finite periodic structure in microwave.](image)

The \( s \)-parameters of \( s_{11} \) and \( s_{21} \) associated with the entire structure shown in Fig. 3. are expressed in a tidy close-form relations

\[ s_{11} = \frac{\left( A + \frac{B}{Z_{\text{ref}}} - C Z_{\text{ref}} - D \right) U_{N-1}}{\left( A + \frac{B}{Z_{\text{ref}}} + C Z_{\text{ref}} + D \right) U_{N-1} - 2 U_{N-2}} \]  
\[ s_{21} = \frac{\left( A + \frac{B}{Z_{\text{ref}}} + C Z_{\text{ref}} + D \right) U_{N-1} - 2 U_{N-2}}{2} \]

where \( A, B, C \) and \( D \) are computed based on equations (3) to (6), \( Z_{\text{ref}} \) is the reference impedance which is usually chosen 50 \( \Omega \) and \( U_{N-1} \) is a parameter defined by

\[ U_N = \frac{\sinh[(N + 1)\gamma l]}{\sinh(\gamma l)}. \]

In equation (9), \( l \) is a period of the periodic TL shown in Fig. 2 and \( \gamma \) is defined by

\[ \gamma l = \cosh^{-1} \left( \frac{A + D}{2} \right) \]

where \( A \) and \( D \) are those in equation (3) and (6).
3. Numerical Results

The periodic stepped-impedance superconducting CPW was simulated with the following materials and parameters. The dielectric material is MgO with the relative dielectric constant of $\varepsilon_r = 9.8$, the height of the dielectric spacer is $h = 500 \mu m$, the spacing between two ground planes is $b = s + 2w = 150 \mu m$, and the thickness of the ground plane conductor is $t = 600 \text{ nm}$. Superconductor plates are made of DyBaCuO which is an HTS material [24], [25] with the critical temperature $T_c = 88.5 \text{ K}$, the penetration depth at zero temperature $\lambda(0) = 165 \text{ nm}$, and the DC conductivity of a normal channel $\sigma_0 = 5 \times 10^4 \text{ S/m}$. Liquid nitrogen is used to cool down the structure, so the temperature is held at $T = 77 \text{ K}$. Hence, a TL with characteristic impedance of $Z_0 = 50 \Omega$ can be achieved by dimensions of $s = 80 \mu m$ and $w = 35 \mu m$, as shown in Fig. 4. Also, the period of the structure is chosen to be $l = 3 \text{ mm}$ in our simulations to have the first cut-off frequency around 20 GHz. We define a new parameter $a$, called the grating ratio which is $a = l_t / (l_1 + l_2)$, to observe the effect of length variations of each transmission line segment on dispersion. Fig. 4 demonstrates that decreasing or increasing the aspect ratio ($s/b$) leads to a high or low impedance CPW, respectively. Thus, we designed 6 TLs for our simulations with dimensions shown in Table 1 in order to compare their combination to achieve a high quality stepped-impedance filter with two subsequent low and high impedance TLs.

![Fig. 4. Characteristic impedance of a b = 150 μm CPW as a function of aspect ratio.](image)

| Transmission Line Segment with $b = 150 \mu m$ | $s$ ($\mu m$) | $w$ ($\mu m$) | $Z_0$ ($\Omega$) |
|---------------------------------------------|---------------|---------------|-----------------|
| TL1                                         | 80            | 35            | 50              |
| TL2                                         | 10            | 70            | 110             |
| TL3                                         | 20            | 65            | 90              |
| TL4                                         | 40            | 55            | 70              |
| TL5                                         | 100           | 25            | 42              |
| TL6                                         | 130           | 10            | 32              |
The dispersion diagram or variation of phase constant verses frequency ($\beta$ v.s. $\omega$) associated with the main mode of propagation (TEM mode) is depicted in Fig. 5 for grating ratios of $\alpha = 0.5$ and $\alpha = 0.2$. This periodic stepped-impedance TL has the unit cell that composed of TL1 and TL3 described in Table 1. Band gap formation in this periodic stepped-impedance TL, which occurs at cut-off frequencies of 18.96 GHz and 38.77 GHz for $\alpha = 0.2$ TL, can be clearly seen in this figure. This result can be verified by computing the Bragg frequency given by $f_n = nV_p/2l$, where $f_n$ is the band-edge of the gap, $V_p$ is the phase velocity, $l$ is the length of the unit cell (spatial period) and $n$ is an integer number starting from 1. The value of phase velocity can be obtained by dividing $\omega$ by $\beta$ on the dispersion diagram of Fig. 5. The values of the phase velocities for different grating ratios are written on the Fig. 5. For the case of $\alpha = 0.2$, the value of the phase velocity is $11.4 \times 10^7$ m/s, resulting in the sequence of Bragg frequencies 19 GHz and 38 GHz, as marked in Fig. 5.

The variation of attenuation constant (distortion analysis) and input impedance (resonance study) of an infinite periodic stepped-impedance TL over frequency in the first Brillouin zone are depicted in Fig. 6 for different geometries with grating ratio of $\alpha = 0.5$. Comparison between curves in this figure reveals that the stop-band in the phase constant occurs exactly where the attenuation constant dramatically increases due to the constitutive reflection at Bragg frequencies or where the input impedance rapidly decreases at resonant frequencies. Interesting point is that all band-edge, Bragg and resonant frequencies in phase constant, attenuation constant and input impedance diagram, respectively, coincide with the same frequency which is referred to as a cut-off frequency. The effect of changing $s_2$, and in turn $w_2$, on the attenuation in the rejection band and the bandwidth of the stop-band are shown in Fig. 6 which displays variations in the attenuation constant versus frequency.

![Fig. 5. Phase constant of a traveling wave through the periodic superconducting stepped-impedance CPW.](image-url)

In addition, comparison between curves in Fig. 6 and looking up their corresponding characteristic impedance ($Z_0$) in Table 1, demonstrates this fact that the quality of the filter, i.e. more rejection in stop-band, can be improved as large discrepancy exists in the $Z_0$ of the two TL segments in a unit cell. Furthermore, larger discrepancies in $Z_0$ leads to the larger bandwidth as seen in the Fig. 6.
Fig. 6. Attenuation constant and input impedance of the periodic superconducting stepped-impedance CPW.

Fig. 7. The amplitude of s₁₁ and s₂₁ for the N=8 number of unit cells in stepped-impedance low-pass filter. l=3 mm, a = 0.5, Z₁ref₁ = 60 Ω (for s₁=80 μm) and Z₂ref₁ = 70 Ω (for s₂=130 μm).

Now we consider the finite periodic stepped-impedance filter with 8 number of subsequent unit cells. The grating ratio is 0.5, and the reference impedance as well as load impedance are chosen to be equal to the input impedance of the filter. Fig. 7 displays the variation of the amplitudes of s-parameters s₁₁ and s₂₁ in terms of dB over frequencies. According to this figure, where the incident
signal stopped to propagate ($s_{21} = 0$), it is completely reflected back ($s_{11} = 1$). Also, Fig. 7 verified the fact that the combination of a higher-$Z_0$ TL segment with a lower-$Z_0$ TL segment results in more attenuation. Fig. 8 draws a comparison between the stop-bands obtained by the simulation of an infinite and finite periodic stepped-impedance CPW for grating ratio 0.5 and 0.2. As seen, the pass- and stop-bands are exactly fallen in the same intervals on frequency axes; therefore, having enough number of unit cells is a good implementation for an open microwave periodic structure. Moreover, like the stop-band (band-gap) width in Fig. 5, the maximum reflection, and therefore maximum attenuation level is achieved when the grating ratio is $a = 0.5$, as seen in Fig. 8, when there is enough number of unit cells.

Fig. 8. Comparison between the stop-bands obtained by looking at attenuation constant, input impedance and s-parameter of the periodic structure. $l=3 \text{ mm}, a = 0.5, Z_{\text{ref}} = 60 \Omega$.

4. Conclusion
We studied the characteristics and behavior of a microwave signal inside a periodic superconducting stepped-impedance CPW. We developed a CAD tool to calculate the circuit parameters of a superconducting CPW. We used ABCD matrix representation to model all microwave components in the unit cell. The Floquet theorem is applied to the infinite periodic structure to find the dispersion diagram of the structure. The input impedance of such a filter is calculated. If the number of unit cells in this distributed filter is limited the scattering matrix analysis is employed. All results are expressed in closed-form equations. Numerical results indicate that the profile of the phase constant, attenuation constant, input impedance and s-parameters have the same pass-band and stop-band and are completely matched with the concept of Bragg frequency. Also, the results of the finite case approaches the results of infinite periodic structure when there is enough number of unit cells.
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