L_p REGULARITY FOR CONVOLUTION OPERATOR EQUATIONS IN BANACH SPACES

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ABSTRACT. Here we utilize operator–valued L_q → L_p Fourier multiplier theorems to establish lower bound estimates for large class of elliptic integro-differential equations in \( \mathbb{R}^d \). Moreover, we investigate separability properties of parabolic convolution operator equations that arise in heat conduction problems in materials with fading memory. Finally, we give some remarks on optimal regularity of elliptic differential equations and Cauchy problem for parabolic equations.

1. Introduction, notations and background

After pioneering works of Herbert Amann [1] and Lutz Weis [26] on operator-valued Fourier multiplier theorems (OFMT), theory of differential–operator equations (DOEs) in Banach valued function spaces is improved significantly. Many researchers applied them in the investigation of different classes of equations especially in maximal L_p and \( B^{p,r}_s \) regularity for parabolic and elliptic DOE. The exposition of FMT, their applications and some related references can be found in [1], [3], [5], [8 − 11], and [26]. For the references concerning FMT in periodic function spaces, optimal regularity results for convolution operator equations (COE) and delay DOE see e.g. [12-14], [18], [21] and reference therein.

Here we shall first extend the well known FMT in [27] and [23]. Then we will establish lower bound estimates for elliptic type integro-differential operators of form

\[ Lu = \sum_{k,j=1}^d c_{kj} \frac{\partial^2 u(x)}{\partial x_k \partial x_j} + \sum_{k,j=1}^d a_{kj} \ast \frac{\partial^2 u(x)}{\partial x_k \partial x_j} + b_0 Au + b_1 \ast Au \quad (1.1) \]

where \( c_{kj}, b_0 \in \mathbb{C}, a_{kj}, b_1 \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \) and \( A \) is a possible unbounded operator in a Banach space \( E \). Particularly, we show the following Sobolev type estimates

\[ \|u\|_{L_p(\mathbb{R}^d, E)} \leq C \|Lu\|_{L_q(\mathbb{R}^d, E)} \quad (1.2) \]

for exponents \( 1 < q \leq p < \infty \) satisfying gap condition

\[ \frac{1}{q} - \frac{1}{p} \leq \frac{2}{d} \]

Next we will prove separability for parabolic COE

\[ Lu = a_0 u' + a_1 \ast u' + b_0 Au + b_1 \ast Au = f \quad (1.3) \]

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where \( f \in X = L_p(R, E) \), \( a_0, b_0 \in C \), \( a_1, b_1 \in S'(R, C) \) and \( A \) is a possible unbounded operator in a Banach space \( E \). Our aim is to obtain the following coercive estimate for (1.3)

\[
\|u'\|_X + \|a_1 \ast u'\|_X + \|Au\|_X + \|b_1 \ast Au\|_X \leq C \|f\|_X.
\]

Note that, model problems for (1.3) are heat conduction problems with fading memory, population dynamic problems, etc. These problems can be inferred from (1.3) by choosing \( A \) as a second order elliptic differential operator and \( E = L_r(R^2) \) along with appropriate boundary conditions. For physical interpretations and for detailed information about problems with fading memory see e.g. [15] and [19].

In the last section we will give some remarks on well-known results concerning parabolic and elliptic DOE.

Let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \), where \( \alpha_i \) are integers. An \( E \)-valued generalized function \( D^\alpha f \) is called a generalized derivative in the sense of Schwartz distributions, if the equality

\[
\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle
\]

holds for all \( \varphi \in S \).

The Fourier transform \( F : S(X) \to S(X) \) is defined by

\[
(Ff)(t) \equiv \hat{f}(t) = \int_{\mathbb{R}^N} \exp(-its)f(s)ds
\]

is an isomorphism whose inverse is given by

\[
(F^{-1}f)(t) \equiv \hat{f}(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(its)f(s)ds,
\]

where \( f \in S(X) \) and \( t \in \mathbb{R}^N \). It is clear that

\[
F(D^\alpha_x f) = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n} \hat{f}, \quad D^\alpha_x(F(f)) = F((ix_n)^{\alpha_1} \cdots (ix_n)^{\alpha_n} f)
\]

for all \( f \in S^d(R^n; E) \).

Let \( C \) be a set of complex numbers and

\[
S_\varphi = \{\xi; \; \xi \in C, \; |\arg\xi| \leq \varphi\}, \; 0 \leq \varphi < \pi.
\]

Suppose \( E_1 \) and \( E_2 \) are two Banach spaces. \( B(E_1, E_2) \) will denote the space of all bounded linear operators from \( E_1 \) to \( E_2 \).

A linear operator \( A \) is said to be \( \varphi \)-positive in a Banach space \( E \), with bound \( M \) if \( D(A) \) is dense in \( E \) and

\[
\|(A + \lambda I)^{-1}\|_{B(E)} \leq M (1 + |\lambda|)^{-1}
\]

for all \( \lambda \in S_\varphi \), with \( \varphi \in [0, \pi) \), where \( M \) is a positive constant and \( I \) is identity operator in \( E \).

\( E(A^\theta) \) denotes the space \( D(A^\theta) \) with graphical norm

\[
\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^\frac{1}{p}, \; 1 \leq p < \infty, \; -\infty < \theta < \infty.
\]

We indicate mixed derivative in the following form

\[
D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}, \quad D_k^{\alpha} = \left(\frac{\partial}{\partial x_k}\right)^{\alpha}
\]
Suppose $\Omega \subset \mathbb{R}^n$. Then $W^1_p(\Omega; E_0, E)$ is a space of functions $u \in L_p(\Omega; E_0)$ such that $D_k^j u = \frac{\partial^j u}{\partial x_k^j} \in L_p(\Omega; E)$ and

$$
\|u\|_{W^1_p(\Omega; E_0, E)} = \|u\|_{L_p(\Omega; E_0)} + \sum_{k=1}^n \|D_k^j u\|_{L_p(\Omega; E)} < \infty.
$$

For $E_0 = E$ the space $W^1_p(\Omega; E_0, E)$ will be denoted by $W^1_p(\Omega; E)$.

2. $L_q \to L_p$ FMT

In this section we shall study scalar-valued FMT from $L_q(X)$ to $L_p(X)$ for $1 < q \leq p < \infty$. Let us first introduce some basic definitions and facts.

**Definition 2.0.** A Banach space $X$ is called UMD space if $X$-valued martingale difference sequences are unconditional in $L_p(R^d; X)$ for $p \in (1, \infty)$, i.e. there exists a positive constant $C_p$ such that for any martingale $\{f_k, k \in \mathbb{N}_0\}$ (see [6, § 5.17]), any choice of signs $\{\varepsilon_k, k \in \mathbb{N}\} \in \{-1, 1\}$ and $N \in \mathbb{N}$

$$
\left\| f_0 + \sum_{k=1}^N \varepsilon_k (f_k - f_{k-1}) \right\|_{L_p(\Omega, \Sigma, \mu, X)} \leq C_p \left\| f_N \right\|_{L_p(\Omega, \Sigma, \mu, X)}.
$$

It is shown in [2] and [4] that a Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} \, dy
$$

is bounded in the space $L_p(R, X)$, $p \in (1, \infty)$ for only those spaces $X$, which possess the UMD property. UMD spaces include e.g. $L_p$, $l_p$ spaces and Lorentz spaces $L_{pq}$, $p, q \in (1, \infty)$.

**Definition 2.1.** Let $X$ and $Y$ be Banach spaces. A family of operators $\tau \subset B(X, Y)$ is called $R$-bounded (see e.g. [5] and [11]) if there is a positive constant $C$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \tau$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables $r_j$ on a probability space $(\Omega, \Sigma, \mu)$ the inequality

$$
\left\| \sum_{j=1}^N r_j T_j x_j \right\|_{L_p(\Omega, Y)} \leq C \left\| \sum_{j=1}^N r_j x_j \right\|_{L_p(\Omega, X)},
$$

is valid. The smallest such $C$ is called $R$-bound of $\tau$, we denote it by $R_p(\tau)$.

Let us note that wide classes of classical operators are $R$-bounded. (see [10] and reference therein). The basic properties of $R$-boundedness are collected in the recent monograph of Denk et al. [5]. For the reader’s convenience, we present some results from [5].

(a) The definition of $R$-boundedness is independent of $p \in [1, \infty)$.

(b) If $\tau \subset B(X, Y)$ is $R$-bounded then it is uniformly bounded with

$$
\sup \{\|T\| : T \in \tau\} \leq R_p(\tau).
$$

(c) If $X$ and $Y$ are Hilbert spaces, $\tau \subset B(X, Y)$ is $R$-bounded $\iff \tau$ is uniformly bounded.
(d) Let $X, Y$ be Banach spaces and $\tau_1, \tau_2 \subset B(X,Y)$ be $R$-bounded. Then
$$\tau_1 + \tau_2 = \{T + S : T \in \tau_1, S \in \tau_2\}$$
is $R$-bounded as well, and $R_p(\tau_1 + \tau_2) \leq R_p(\tau_1) + R_p(\tau_2)$.

(e) Let $X, Y, Z$ be Banach spaces and $\tau_1 \subset B(X,Y)$ and $\tau_2 \subset B(Y,Z)$ be $R$-bounded. Then
$$\tau_1 \tau_2 = \{ST : T \in \tau_1, S \in \tau_2\}$$
is $R$-bounded as well, and $R_p(\tau_1 \tau_2) \leq R_p(\tau_1)R_p(\tau_2)$.

One of the most important tools in $R$-boundedness is the contraction principle of Kahane. We shall frequently apply it in the next sections.

[5, Lemma 3.5.] Let $X$ be a Banach spaces, $n \in N$, $x_j \in X$, $r_j$ independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \Sigma, \mu)$ and $\alpha_j, \beta_j \in \mathbb{C}$ such that $|\alpha_j| \leq |\beta_j|$, for each $j = 1, \ldots, N$. Then
$$\left\| \sum_{j=1}^{N} \alpha_j r_j x_j \right\|_{L_p(\Omega,X)} \leq 2 \left\| \sum_{j=1}^{N} \beta_j r_j x_j \right\|_{L_p(\Omega,X)}.$$The constant 2 can be omitted in case $\alpha_j$ and $\beta_j$ are real.

**Theorem 2.2.** Let $X$ be an UMD space and $1 < q \leq p < \infty$. If for a bounded function $\psi : R^d \setminus \{0\} \to \mathbb{C}$
$$\sup \left\{ |\xi|^{\alpha} |d(\frac{1}{q} - \frac{1}{p})| D^\alpha \psi (\xi), \xi \in R^d \setminus \{0\} \right\} < \infty \tag{2.1}$$
then
$$\|T_\psi f\|_{L_p(R^d; X)} = \left\| F^{-1} \left[ \hat{\psi} \hat{f} \right] \right\|_{L_p(R^d; X)} \leq C \| f \|_{L_q(R^d; X)}$$
for all $f \in S(R^d, X)$.

**Proof.** The main idea is to apply $L_p \to L_p$ FMT along with Sobolev embedding and to use nice properties of a function $s(t) = |t|^{d(\frac{1}{q} - \frac{1}{p})}$. Since $\psi$ satisfies the general Miklin’s condition i.e.
$$\sup \left\{ |\xi|^{\alpha} |D^\alpha \psi (\xi), \xi \in R^d \setminus \{0\} \right\} \leq C,$
s $\cdot \psi$ satisfies classical one i.e.
$$\sup \left\{ |\xi|^{\alpha} |D^\alpha (s\psi) (\xi), \xi \in R^d \setminus \{0\} \right\} \leq C.$$

Therefore applying the Sobolev embedding theorem and by using the definition of homogeneous Sobolev spaces (in the sense of Riesz potentials) we get desired result:
$$\left\| F^{-1} \left[ \hat{\psi} \hat{f} \right] \right\|_{L_p(R^d; X)} \leq K \left\| F^{-1} \left[ \hat{s} \hat{\psi} \hat{f} \right] \right\|_{W_p^d(\frac{1}{q} - \frac{1}{p})(R^d; X)}$$
$$= K \left\| F^{-1} \left[ |t|^{d(\frac{1}{q} - \frac{1}{p})} \psi (\cdot) \hat{f} (\cdot) \right] \right\|_{L_q(R^d; X)}$$
$$= K \left\| F^{-1} \left[ (s\psi) \hat{f} (\cdot) \right] \right\|_{L_q(R^d; X)} \leq C \| f \|_{L_q(R^d; X)}.$$
Theorem 2.3. Let $X$ and $Y$ be UMD spaces and $1 < q \leq p < \infty$. If for a bounded function $M : R^d \setminus \{0\} \rightarrow B(X,Y)$,
\begin{equation}
R \left\{ |x|^{\alpha + d\left(\frac{1}{\gamma} - 1\right)} D^\alpha M(x), \ x \in R^d \setminus \{0\}, \ (\alpha \leq 1, \ldots, 1) \right\} < \infty \tag{2.2}
\end{equation}
then
\[ ||T_M f||_{L_p(R^d, Y)} = \left\| F^{-1} \left[ M(\cdot)\hat{f}(\cdot) \right] \right\|_{L_p(R^d, Y)} \leq C ||f||_{L_q(R^d, X)} \]
for all $f \in S(R^d, X)$.

3. Sobolev type estimates for (1.1)

Let us consider the second order elliptic integro-differential equation (1.1) in $R^d$. Here we characterize conditions on coefficients of (1.1) so that they imply Sobolev type estimate (1.2).

Since we utilize Fourier integral methods we naturally impose conditions on symbols. Therefore, to avoid contradictions due to Riemann-Lebesgue lemma we make some auxiliary assumptions along with certain regularity and ellipticity conditions on coefficients of $L$.

Definition 3.0. Let $E$ be a Banach space and $D(A)$ dense in $E$. A $\varphi$-positive operator $A$ is said to be $R$-positive if the following set
\[ \{(1 + \xi)(A + \xi)^{-1} : \xi \in S_{\varphi}, \varphi \in [0, \pi)\} \]
is $R$-bounded.

In what follows $R$ will denote the set of real numbers excluding zero i.e. $R^d = R^d \setminus \{0\}$.

Condition 3.1. Suppose the following are satisfied:
1. $c_{kj}, b_0 \in C, a_{k1}, b_1 \in S'(R^d, C), a_{k1}, b_1 \in C^d(\hat{R}^d, C)$ and $C_b = \inf_{\xi \in R^d} \left| |b_0 + \hat{b}_1(\xi)| \right| > 0$;
2. There exists a constant $C$ such that
\[ |N(\xi)| = \sum_{k,j=1}^d (c_{kj} + \hat{a}_{kj}(\xi)) \xi_k \xi_j \geq C |\xi|^2; \]
3. $\eta(\xi) = \frac{N(\xi)}{\hat{b}_1(\xi) + b_0} \in S_\varphi$ for $\varphi \in [0, \pi)$;
4. there are some constants $C_i$ such that for all $\xi \in \hat{R}^d$
\[ |\xi| |\beta| \left| \frac{\partial^{[\beta]} a_{kj}(\xi)}{\partial \xi_1^{\beta_1} \partial \xi_2^{\beta_2} \cdots \partial \xi_d^{\beta_d}} \hat{a}_{kj}(\xi) \right| \leq C_0 \text{ for all } k, j = 1, 2, \ldots, d, \]
\[ |\xi| |\beta| \left| \frac{\partial^{[\beta]} b_{1}(\xi)}{\partial \xi_1^{\beta_1} \partial \xi_2^{\beta_2} \cdots \partial \xi_d^{\beta_d}} \hat{b}_1(\xi) \right| \leq C_1, \]
where $\beta_i \in \{0, 1\}$ and $0 \leq |\beta| \leq d$. 
Theorem 3.2. Suppose $E$ is an UMD space and Condition 3.1 holds. Let $A$ be an $R$-positive operator in $E$ with $0 \leq \varphi < \pi$. Then, (1.1) satisfies Sobolev type estimate (1.2) for exponents $1 < q \leq p < \infty$ satisfying gap condition

$$
\frac{1}{q} - \frac{1}{p} \leq \frac{2}{d}.
$$

To prove our main result we will need the following preliminary lemmas.

Lemma 3.3. Let $A$ be an $R$-positive operator in $E$ and assume Condition 3.1 holds. Then the following set

$$
\left\{ |\xi|^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \sigma(\xi); \; \xi \in \hat{R}^d \right\}
$$

is $R$-bounded where

$$
\sigma(\xi) = \frac{1}{b_1(\xi) + b_0} (A + \eta(\xi))^{-1}.
$$

Proof. Since $\eta(\xi) \in S_c$ for $\varphi \in [0, \pi)$, from [7, Lemma 2.3] there exist $K > 0$ independent of $\xi$ so that

$$
|1 + \eta(\xi)|^{-1} \leq K(1 + |\eta(\xi)|)^{-1}.
$$

Therefore, for all $\xi \in \hat{R}^d$ we have uniform estimate

$$
\frac{|\xi|^{d\left(\frac{1}{q} - \frac{1}{p}\right)}}{b_1(\xi) + b_0} \leq K \frac{|\xi|^{d\left(\frac{1}{q} - \frac{1}{p}\right)}}{|1 + \eta(\xi)|} \leq K \frac{|\xi|^{d\left(\frac{1}{q} - \frac{1}{p}\right)}}{b_1(\xi) + b_0 + |N(\xi)|} \leq \frac{K}{C}.
$$

Now let us define families of operators

$$
\tau_1 = \left\{ T_j = |\xi_j|^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \sigma(\xi_j); \; \xi_j \in \hat{R}^d \right\}
$$

and

$$
\tau_2 = \left\{ S_j = (1 + \eta(\xi_j)) (A + \eta(\xi_j))^{-1}; \; \xi_j \in \hat{R}^d \right\}.
$$

Taking into consideration $R$-positivity of $A$, applying assumptions of Condition 3.1 and Kahane’s contraction principle [7, Lemma 3.5] we get desired result:

$$
\left\| \sum_{j=1}^{N} r_j T_j x_j \right\|_X = \left\| \sum_{j=1}^{N} r_j \frac{|\xi_j|^{d\left(\frac{1}{q} - \frac{1}{p}\right)}}{b_1(\xi_j) + b_0(1 + \eta(\xi_j))} S_j x_j \right\|_X \leq 2 \frac{K}{C} \sum_{j=1}^{N} r_j S_j x_j \leq 2 \frac{K}{C} R_p(\tau_2) \sum_{j=1}^{N} r_j x_j \right\|_X.
$$

where $X = L_p((0, 1), E)$. Hence

$$
R_p(\tau_1) \leq 2 K R_p(\tau_2).
$$

In the next lemmas we will estimate $R$-bounds of partial derivatives of $\sigma(\xi)$.

Lemma 3.4. Let $A$ be an $R$-positive operator in $E$ and assume Condition 3.1 holds. Then, the following set

$$
\left\{ |\xi|^{1+d\left(\frac{1}{q} - \frac{1}{p}\right)} \frac{\partial}{\partial \xi_i} \sigma(\xi), \; \xi \in \hat{R}^d \right\}
$$

be $R$-positive operator in $E$ and assume Condition 3.1 holds. Then, the following set

$$
\left\{ |\xi|^{1+d\left(\frac{1}{q} - \frac{1}{p}\right)} \frac{\partial}{\partial \xi_i} \sigma(\xi), \; \xi \in \hat{R}^d \right\}
$$
is $R$-bounded.

**Proof.** It clear to see that first derivative of $\sigma(\xi)$ consist of 3 terms namely

$$\sigma_1(\xi) = \frac{-\frac{\partial b_1(\xi)}{\partial \xi_i}}{b_1(\xi) + b_0} (A + \eta(\xi))^{-1},$$

$$\sigma_2(\xi) = \left(\frac{1}{b_1(\xi) + b_0}\right)^3 \frac{\partial N(\xi)}{\partial \xi_i} b_1(\xi) N(\xi) (A + \eta(\xi))^{-2},$$

$$\sigma_3(\xi) = \left(\frac{1}{b_1(\xi) + b_0}\right)^3 \frac{\partial b_1(\xi)}{\partial \xi_i} N(\xi) (A + \eta(\xi))^{-2}.$$  

For the sake of simplicity we will only estimate $R$-bound of the set

$$\left\{ |\xi|^{1+d\left(\frac{q}{2} - \frac{1}{p}\right)} \sigma_1(\xi) : \xi \in \hat{R}^d \right\},$$

Define a family of operators

$$\tau = \left\{ T_j = |\xi|^{1+d\left(\frac{q}{2} - \frac{1}{p}\right)} \sigma_1(\xi^j) : \xi^j \in \hat{R}^d \right\}.$$  

Making use of Condition 3.1 and $R$-positivity of $A$ we get

$$\left\| \sum_{j=1}^{N} r_j T_j x_j \right\|_X \leq 2 \frac{K C_1}{C C_b} \left\| \sum_{j=1}^{N} r_j S_j x_j \right\|_X \leq 2 \frac{K C_1}{C C_b} R_p(\tau_2) \left\| \sum_{j=1}^{N} r_j x_j \right\|_X,$$

which implies

$$R_p(\tau) \leq 2 \frac{K C_1}{C C_b} R_p(\tau_2).$$

In a similar fashion one can also prove

$$R \left\{ |\xi|^{1+d\left(\frac{q}{2} - \frac{1}{p}\right)} \sigma_j(\xi) : \xi \in \hat{R}^d \right\} \leq M.$$  

Hence we get assertion of the lemma. $\blacksquare$

The next result is generalization of Lemma 3.4. We will omit the prove since it analogously follows from previous results.

**Lemma 3.5.** Assume $A$ is an $R$-positive operator in $E$ and Condition 3.1 holds. Then, the following set

$$\left\{ |\xi|^{\alpha_1+d\left(\frac{q}{2} - \frac{1}{p}\right)} D^{\alpha} \sigma(\xi), \xi \in \hat{R}^d, \alpha \leq \text{comp.wise } (1,\ldots,1) \right\}$$

is $R$-bounded.

**Proof of Theorem 3.2.** Suppose $Lu = f$ for some $f \in L_q(R^d;E)$. Taking into consideration Condition 3.1 and applying the Fourier transform to both side
of equation we get

\[ u(x) = F^{-1} \left[ \frac{1}{\hat{b}_1(\xi) + b_0} \left( A + \frac{N(\xi)}{\hat{b}_1(\xi) + b_0} \right)^{-1} \hat{f}(\xi) \right]. \]

Now it is easy to see that (1.2) is equivalent to \( L_q(R^d; E) \rightarrow L_p(R^d; E) \) boundedness of above Fourier multiplier operator. Since the operator-valued multiplier function

\[ m(\xi) = \frac{1}{\hat{b}_1(\xi) + b_0} \left( A + \frac{N(\xi)}{\hat{b}_1(\xi) + b_0} \right)^{-1} \]

satisfies assumptions of Theorem 2.3 we complete the proof.

4. Optimal regular Parabolic COE

Due to its nice applications, many researchers investigated (1.3) in various function spaces. The Besov space regularity for (1.3) is studied in [1] and the Holder space \((C^\alpha\text{ with } 0 < \alpha < 1)\) case is presented in [14]. Moreover, maximal regularity results for (1.3) in different periodic function spaces can be found in very recent paper [12].

Here we study the same problem in \( L_p(R; E) \) under some natural assumptions on coefficients. The main tool we implement here will be FMT of Weis [26]. First we state our assumptions:

**Condition 4.1.** Suppose the following are satisfied:

1. \( a_0, b_0 \in C, a_1, b_1 \in S'(R, C), \hat{a}_1, \hat{b}_1 \in C^1(\hat{R}, C) \) and

\[ \lim_{|\xi| \rightarrow \infty} \inf_{|\eta| \leq 1} |a_0 + \hat{a}_1(\xi)| > 0 \text{ and } \inf_{\xi \in \hat{R}} \left| b_0 + \hat{b}_1(\xi) \right| = C_b > 0; \]

2. \( \frac{i\xi (\hat{a}_1(\xi) + a_0)}{\hat{b}_1(\xi) + b_0} \in S_\varphi \text{ for } \varphi \in [0, \pi ]; \)

3. There are constants \( C_i \) such that

\[ |\hat{a}_1(\xi)| \leq C_0, \quad \left| \frac{d}{d\xi} \hat{a}_1(\xi) \right| \leq C_1 \]

\[ |\hat{b}_1(\xi)| \leq C_2, \quad \left| \frac{d}{d\xi} \hat{b}_1(\xi) \right| \leq C_3, \text{ for all } \xi \in \hat{R}. \]

**Theorem 4.2.** Assume \( E \) is an UMD space and Condition 4.1 holds. Let \( A \) be an \( R \)-positive operator in \( E \) with \( 0 \leq \varphi < \pi \) and \( 1 < p < \infty \). Then, the equation (1.3) has a unique solution \( u \in W^1_p(R, E(A); E) \) satisfying (1.4).

**Proof.** Taking into consideration Condition 4.1 and applying the Fourier transform to both side of (1.3) we get

\[ u(x) = F^{-1} \left[ \mu(\xi) (A + \eta(\xi)\eta)\xi \right]^{-1} \hat{f}(\xi) \]

where

\[ \eta(\xi) = i\xi (\hat{a}_1(\xi) + a_0) \mu(\xi) \]
and

\[ \mu(\xi) = \frac{1}{b_1(\xi) + b_0}. \]

Since

\[ \|u\|_{L^p(R_+,E)} = \left\| F^{-1} \left[ \mu(\xi) (A + \eta(\xi))^{-1} \hat{f}(\xi) \right] \right\|_{L^p(R_+,E)} \]

\[ \|u\|_{L^p(R_+,E)} = \left\| F^{-1} \left[ i\xi \mu(\xi) (A + \eta(\xi))^{-1} \hat{f}(\xi) \right] \right\|_{L^p(R_+,E)} \]

\[ \|a_1 * u\|_{L^p(R_+,E)} = \left\| F^{-1} \left[ i\xi \hat{a}_1(\xi) \mu(\xi) (A + \eta(\xi))^{-1} \hat{f}(\xi) \right] \right\|_{L^p(R_+,E)} \]

\[ \|Au\|_{L^p(R_+,E)} = \left\| F^{-1} \left[ \mu(\xi) A (A + \eta(\xi))^{-1} \hat{f}(\xi) \right] \right\|_{L^p(R_+,E)} \]

and

\[ \|b_1 * Au\|_{L^p(R_+,E)} = \left\| F^{-1} \left[ \hat{b}_1(\xi) \mu(\xi) A (A + \eta(\xi))^{-1} \hat{f}(\xi) \right] \right\|_{L^p(R_+,E)} \]

it suffices to show

\[ m_0(\xi) = \mu(\xi) (A + \eta(\xi))^{-1}, \quad m_1(\xi) = i\xi \mu(\xi) (A + \eta(\xi))^{-1} \]

\[ m_2(\xi) = i\xi \hat{a}_1(\xi) \mu(\xi) (A + \eta(\xi))^{-1}, \quad m_3(\xi) = \mu(\xi) A (A + \eta(\xi))^{-1} \]

and

\[ m_4(\xi) = \hat{b}_1(\xi) \mu(\xi) A (A + \eta(\xi))^{-1} \]

are Fourier multipliers in \( L^p(R,E) \). Therefore, we will prove in several steps that \( m_i(\xi) \) satisfy (2.2) for \( p = q \) and \( d = 1 \).

First we show that \( S_i = \left\{ m_i(\xi) : \xi \in \hat{R} \right\} \) are \( R \)-bounded sets.

**Lemma 4.3.** Let \( E \) be an UMD space and \( A \) be an \( R \)-positive operator in \( E \) with \( 0 \leq \varphi < \pi \). If Condition 4.1 holds then \( S_i = \left\{ m_i(\xi) : \xi \in \hat{R} \right\} \) are \( R \)-bounded sets.

**Proof.** As in Lemma 3.3 for all \( \xi \in \hat{R} \) we have uniform estimates

\[ \frac{\|\mu(\xi)\|_{1+\eta(\xi)}}{1+\eta(\xi)} \leq \frac{1}{\hat{b}_1(\xi) + b_0 + |i\xi(\hat{a}_1(\xi) + a_0)|} \leq \frac{1}{C_b}, \]

\[ \frac{|\xi||\mu(\xi)|}{1+\eta(\xi)} \leq \frac{|i\xi|}{|\hat{a}_1(\xi) + a_0||i\xi| + C_b} \leq K, \]

\[ \frac{|\xi||\mu(\xi)||\hat{a}_1(\xi)|}{1+\eta(\xi)} \leq KC_0, \]

\[ \frac{|\mu(\xi)||\eta(\xi)|}{1+\eta(\xi)} \leq \frac{1}{C_b} \]

and

\[ \left| \hat{b}_1(\xi) \right| \left| \mu(\xi) \right| \frac{|\eta(\xi)|}{1+\eta(\xi)} \leq \frac{C_2}{C_b}. \]

Now let us define families of operators

\[ \tau_i = \left\{ T^i_j = m_i(\xi^j); \xi^j \in \hat{R} \right\} \text{ for } i = 0, \ldots, 4 \]

and

\[ \tau = \left\{ S_j = (1 + \eta(\xi^j))(A + \eta(\xi^j))^{-1}; \xi^j \in \hat{R} \right\}. \]
Taking into consideration $R$-positivity of $A$, applying assumptions of Condition 3.1 and Kahane’s contraction principle [7, Lemma 3.5] we get

\[ \left\| \sum_{j=1}^{N} r_j T_j^0 x_j \right\|_X \leq \frac{2}{C_b} R_p(\tau) \left\| \sum_{j=1}^{N} r_j x_j \right\|_X, \]

\[ \left\| \sum_{j=1}^{N} r_j T_j^1 x_j \right\|_X \leq 2K R_p(\tau) \left\| \sum_{j=1}^{N} r_j x_j \right\|_X \]

and

\[ \left\| \sum_{j=1}^{N} r_j T_j^2 x_j \right\|_X \leq 2KC_0 R_p(\tau) \left\| \sum_{j=1}^{N} r_j x_j \right\|_X \]

which implies

\[ R_p(\tau_0) \leq \frac{2}{C_b} R_p(\tau), \quad R_p(\tau_1) \leq 2K R_p(\tau) \text{ and } R_p(\tau_2) \leq 2KC_0 R_p(\tau). \]

Finally, in view of resolvent properties of positive operators and again by [7, Lemma 3.5] we deduce

\[ R_p(\tau_3) = R \left\{ \mu(\xi) \left[ I - \eta(\xi) \left( A + \eta(\xi) \right)^{-1} \right] : \xi \in \hat{R} \right\} \]

\[ \leq R \left\{ \mu(\xi) I : \xi \in \hat{R} \right\} + R \left\{ \mu(\xi) \eta(\xi) \left( A + \eta(\xi) \right)^{-1} \right\} \]

\[ \leq \frac{1}{C_b} + \frac{2}{C_b} R_p(\tau) \]

and

\[ R_p(\tau_4) \leq \frac{C_2}{C_b} + \frac{2C_2}{C_b} R_p(\tau). \]

Next we will estimate derivatives of operator valued functions $m_i(\xi)$.

**Lemma 4.4.** Let $E$ be an UMD space and $A$ be an $R$-positive operator in $E$ with $0 \leq \varphi < \pi$. If Condition 4.1 holds then $S_i = \left\{ \xi \frac{d}{d\xi} m_i(\xi) : \xi \in \hat{R} \right\}$ are $R$-bounded sets.

**Proof.** For the sake of simplicity we shall prove only for $S_0$. The other cases can be proved analogously with the help of above techniques. Taking derivative of $m_0$ and applying similar ideas as in Lemma 4.3 we get desired result:

\[ R \left\{ \xi \frac{d}{d\xi} m_0(\xi) \right\} \leq R \left\{ -\frac{\xi \frac{d}{d\xi} b_1(\xi)}{b_1(\xi) + b_0} \mu(\xi) \left( A + \eta(\xi) \right)^{-1} \right\} + R \left\{ \eta'(\xi) \xi \mu(\xi) \left( A + \eta(\xi) \right)^{-2} \right\} \]

\[ \leq \frac{2}{C_b} R_p(\tau) \left[ C_3 + K \left( C_0 + |a_0| + \frac{C_3(C_0 + |a_0|)}{C_b} + C_1 \right) \right] < \infty. \]

**Corollary 4.5.** Let $E$ be an UMD space and $A$ be an $R$-positive operator in $E$ with $0 \leq \varphi < \pi$. If Condition 4.1 holds then $m_i(\xi)$ are Fourier multipliers in $L_p(R, E)$ for $1 < p < \infty$. 
Example 4.6. As an application of our main result we can give a heat conduction problem in materials with fading memory. Really, choosing \( E = L_q(\Omega) \), \( A = -\frac{d^2}{dt^2} + c \), \( a_1(t) = e^{-m|t|} \), \( b_1(t) = e^{-k|t|} \) in (1.3) we obtain the following integro-differential equation

\[
\partial_t u + \int_{-\infty}^\infty e^{-m|t-s|}\partial_t u(s, x)ds =
\]

\[
= f(t, x) + (\partial_{xx} - c)u + \int_{-\infty}^\infty e^{-k|t-s|}(\partial_{xx} - c)u(s, x)ds,
\]

where \( f \in X = L_q(R; L_q(\Omega)) = L_{p,q}(R \times \Omega) \), \( c, m, k > 0 \) and \( \partial \Omega \) is a sufficiently smooth boundary. Since all assumptions of the Condition 4.1 are satisfied, the above equation has a unique solution \( u \in W^{(1,2)}(R \times \Omega) = \{ u(t, x) \in X : \partial_t u, \partial_{xx}u \in X, \text{ and } u(t, x)_{x \in \partial \Omega} = 0 \} \)

satisfying coercive estimate

\[
\| \partial_t u \|_X + \left\| e^{-m|\cdot|} \ast \partial_t u \right\|_X + \| \partial_{xx}u \|_X + \left\| e^{-k|\cdot|} \ast \partial_{xx}u \right\|_X \leq C \| f \|_X.
\]

5. Remarks on Parabolic and Elliptic DOE

In recent years Lutz Weis [26] and Herbert Amann [1] established maximal \( L_p(E) \) and \( B^{s}_{q,r}(E) \) regularity for abstract cauchy problem

\[
u'(t) + Au = f(t)
\]

\[\quad u(0) = 0 \tag{5.1}\]

Here based on obtained FMT, we shall give some remarks on (5.1).

Remark 5.1. Let \( E \) be an UMD space. Suppose \( A \) is an \( R \)-positive operator in \( E \) i.e.

\[
R \left\{ (1 + \lambda)R(\lambda, A) : \lambda \in S_\phi \text{ for } \frac{\pi}{2} < \phi < \pi \right\} < \infty \tag{5.2}
\]

Then for each \( f \in L_q(R^+; E) \), (5.1) has a unique solution

\[
u \in \bigcap_{q < \theta < \infty} L_q(R^+; E) \cap W^1_q(R^+, E(A), E) \tag{5.3}
\]

satisfying coercive estimate

\[
\| u \|_{W^1_q(R^+, E(A), E)} + \| u \|_{L_q(R^+; E)} \leq C \| f \|_{L_q(R^+; E)}.
\]

Since \( A \) is a generator of bounded analytic semigroup \( T_t \), solutions of (5.1) can be represented in the form of

\[
u(t) = \int_0^t T_{t-s} f(s) \, ds
\]
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\[ T_{t-s} = T(t-s) = e^{-A(t-s)}. \]

Therefore,
\[ u'(t) = f(t) - \int_0^t AT_{t-s} f(s) \, ds. \]

and
\[ Au(t) = \int_0^t AT_{t-s} f(s) \, ds. \]

Now, it is easy to see that maximal \( L_q(R^+; E) \) to \( L_\theta(R^+; E) \) regularity of \((5.1)\) is equivalent to the boundedness of operator
\[
Kf(t) = (AT) * f = \int_{-\infty}^{\infty} AT_{t-s}(f(s))ds = F^{-1} \left[ (AT)^\wedge (\cdot) \hat{f}(\cdot) \right]
\]
where
\[
AT(t) = \begin{cases} AT_t \text{ for } t > 0 \\ 0 \text{ for } t \leq 0 \end{cases}
\]

In order to show
\[
\|u\|_{L_\theta(R^+; E)} \leq C \|f\|_{L_q(R^+; E)},
\]
and
\[
\|u\|_{W^1_q(R^+, E(A), E)} \leq C \|f\|_{L_q(R^+; E)}, \text{ for } 1 < q \leq \theta < \infty
\]
it suffices to prove
\[
m_0(t) = (T)^\wedge (t) = R(it, A)
\]
and
\[
m_1(t) = (AT)^\wedge (t) = AR(it, A)(t) = itR(it, A) - I
\]
are Fourier multipliers. Hence, we have to show
\[
R \left\{ |t|^{\frac{1}{q} - \frac{1}{\theta}} m_i(t) : t \in R \setminus \{0\} \right\} \leq C_1 \tag{5.4}
\]
and
\[
R \left\{ |t|^{1+\frac{\theta}{q} - \frac{1}{\theta}} \frac{d}{dt} m_i(t) : t \in R \setminus \{0\} \right\} \leq C_2, \tag{5.5}
\]
for \( i = 0, 1 \). For the function \( m_0 \), \( (5.4) \) and \( (5.5) \) hold for each \( \theta \) satisfying \( 1 < \theta \leq \infty \) due to \((5.2)\). However for the second function we have
\[
\left\| \frac{d}{dt} m_1 \right\| \leq C \|t\|^{1+\frac{\theta}{q} - \frac{1}{\theta}} \|R(it, A)\| \leq C \frac{|t|^{1+\frac{\theta}{q} - \frac{1}{\theta}}}{1+|t|}
\]
which implies that the right hand side is unbounded whenever \( q < \theta \). Thus \((5.4)\) and \( (5.5) \) do not hold for \( m_1 \) unless \( q = \theta \). Eventually, \((5.1)\) has a unique solution
\[
u \in \bigcap_{q<\theta<\infty} L_\theta(R^+; E) \cap W^1_q(R^+, E(A), E)
\]
satisfying the coercive estimate \((5.3)\). \[ \square \]
and the following coercive estimate holds
\[ u''(t) + Au = f(t) \] (5.6)
has a unique solution
\[ u \in \bigcap_{q<\theta<\infty} W^1_q(R; E) \cap W^2_q(R; E(A), E) \]
satisfying coercive estimate
\[ \|u\|_{W^2_q(R; E(A), E)} + \|u\|_{W^1_q(R; E)} \leq C \|f\|_{L_q(R; E)}. \]
Applying Fourier transform to equation (5.6), we obtain
\[ [\xi^2 + A] \hat{u}(\xi) = \hat{f}(\xi). \]
Since \( A \) is \( R \)-positive, solutions of (5.6) can be represented in the following form
\[ u(x) = F^{-1}[A + \xi^2]^{-1} \hat{f}. \] (5.7)
By using (5.7), we get
\[ \|u\|_{L_q(R; E)} = \left\| F^{-1} \left[ \xi (A + \xi^2)^{-1} \right] \hat{f} \right\|_{L_q(R; E)} \]
\[ \|Au\|_{L_q(R; E)} = \left\| F^{-1} \left[ A(A + \xi^2)^{-1} \right] \hat{f} \right\|_{L_q(R; E)} \]
\[ \|u''\|_{L_q(R; E)} = \left\| F^{-1} [\xi^2(A + \xi^2)^{-1} \hat{f}] \right\|_{L_q(R; E)}. \]
Therefore, it suffices to show operator–functions
\[ \sigma_0(\xi) = [A + \xi^2]^{-1}; \quad \sigma_1(\xi) = (A + \xi^2)^{-1}; \quad \sigma_2(\xi) = \xi^2(A + \xi^2)^{-1} \]
and
\[ \sigma_3(\xi) = A[A + \xi^2]^{-1} = I - \xi^2[A + \xi^2]^{-1} \]
are Fourier multipliers. Applying similar techniques as in the Remark 5.1 one can easily show that \( \sigma_j \) are FMT.

**Example 5.3.** As an application of the Remark 5.1 we can give a mixed problem for infinite system of diffusion equations i.e.
\[ \frac{\partial u_k(t, x)}{\partial t} - (\Delta + c) u_k(t, x) = f_k(t, x), \quad \text{for} \ k = 0, 1, \ldots . \] (5.8)
\[ u_k(0, x) = 0, \ u_k(t, x)|_{x \in \partial G} = 0 \]
We assume \( c > 0, \ 1 < q < \infty, \ p \in (1, \infty), \ E = L^p(G;l_p), \ A = -(\Delta + c) \) and \( E(A) = W^2_p(G;l_p) \). Here \( G \subseteq \mathbb{R}^n \) and \( u \in W^2_p(G;l_p) \) is assumed to satisfy a boundary condition \( u|_{\partial G} = 0 \). Then, for each \( f \in L_q(R^+; E) \) (5.8) has a unique solution
\[ u \in \bigcap_{q<\theta<\infty} L^\theta(R^+; L^p(G;l_p)) \cap W^1_q(R^+, E(A), E) \]
and the following coercive estimate holds
\[ \|u\|_{W^1_q(R^+, E(A), E)} + \|u\|_{L^\theta(R^+; L^p(G;l_p))} \leq C \|f\|_{L_q(R^+; L^p(G;l_p))}. \]
Remark 5.4. The FMT play an important role in the study of embedding theorems. For instance, under certain abstract conditions on a Banach space $E$, author proved in [20] that

$$D^\alpha : W^s_p(\Omega; E(A; E) \rightarrow L^q(\Omega; E(A^{1-s}))$$

is continuous for $x = \frac{\alpha + \frac{1}{p} - \frac{1}{q}}{t} \leq 1$, and the Gagliardo-Nirenberg type sharp estimate

$$\|D^\alpha u\|_{L^q(A(x) \cap (\Omega,E))} \leq h^\mu \|u\|_{W^s_p(\Omega; E(A; E))} + h^{-(1-\mu)} \|u\|_{L^p(\Omega; E)},$$

holds for all $u \in W^s_p(\Omega; E(A; E))$, $1 < p < q < \infty$, $0 < \mu \leq 1-x$ and $0 < h \leq h_0 < \infty$. However, using the same techniques as in [20] and applying Theorem 2.3 one can remove this combined assumption on $E$. Note that these embedding theorems play key role in the theory of DOE, especially in estimation of lower order terms in DOE of type e.g.

$$-u''(t) + A_1(t)u'(t) + Au(t) = f(t)$$

where $A_1(t)$ is a variable and generally unbounded operator. More general form of above equation and the embedding theorems are studied in [21-22].

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