On Two Complementary Types of Directional Derivative and Flow Field Specification in Classical Field Theory

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We discuss a general definition of directional derivative of any tensor flow field and its practical applications in physics. It is shown that both Lagrangian and Eulerian descriptions as complementary types of flow field specifications adopted in modern theoretical hydrodynamics, imply two complementary types of directional derivatives as corresponding mathematical constructions. One of them is the Euler substantive derivative useful only in the context of initial Cauchy problem and the other, called here as the local directional derivative, arises only in the context of so-called final Cauchy problem. The choice between Lagrangian and Eulerian specifications is demonstrated to be equivalent to the choice between space-time with Euclidean and Minkowski metric for any flow field domain, respectively. Mathematical consideration is developed within the framework of congruencies for general 4-dimensional differentiable manifold. The analytical expression for local directional derivative is formulated in form of a theorem. Although the consideration is developed for one-component (scalar) flow field, it can be easily generalized to any tensor field. Some implications of the local directional derivative concept for the classical theory of fields are also explored.

I. INTRODUCTION

Physical background and motivation for Euler’s mathematical construction known as substantive (or material) derivative becomes manifest in the way how the whole flow field may be specified in hydrodynamics. Usually, two complementary types of specifications (or representations) are thought to suffice in order to provide a general description of the flow field kinematics. The first, Lagrangian specification is based on identifying individual elements or bits of fluid domain. This idea associates a fluid motion with a geometrical transformation $H_t$ on the closure $\Omega_0$ such that the set $H_t\Omega_0$ represents the same individual bit of fluid at time $t$. This representation is valid only if the identification can be maintained by some kind of labelling usually denoting the initial position at instant $t_0$. The second, Eulerian specification was conceived as dissociated from identification of individual bits of fluid, only making use of the flow quantities as functions of local position in space at each instant of time during the motion. This specification results especially useful for hydrodynamics of liquids and electromagnetic field description in which any attempt of Lagrangian identification is impossible. Thus, in Euler’s approach the velocity vector field is a primary notion. Assigning a velocity vector to each point of the fluid domain, one obtains the system of ordinary differential equations and their solutions as integral curves, intimately related to the given velocity vector field. The Eulerian representation provides a time parameterization of the curve in local coordinate system as a differentiable mapping from an open set of $R^3$ into $R^3$.

Both complementary types of specifications (Eulerian and Lagrangian), generally speaking, different and certainly useful in complementary contexts, can be made mathematically equivalent under special conditions. In terms of modern notation, they result equivalent within the formulation of the initial Cauchy problem for an ordinary differential equation for velocity field. It gives a kind of dictionary for translating from one specification to the other.

In this respect it is interesting to note that from the very birth of theoretical hydrodynamics as an independent body of mathematical knowledge, the conventional formulation of the directional derivative of flow field quantities tacitly implies the equivalence with the Lagrangian specification related to the initial Cauchy problem. Thus, according to this type of specification, Euler’s substantial (or material) derivative $\frac{D}{Dt}$ describes the rate of time variation of $f$-property of fluid element on its path from one to the other point of space. As Euler himself coined it two and a half centuries ago \[\frac{D}{Dt}\], his mathematical construction described the rate of time variation of material properties following the motion of the fluid.

On the other hand, the conventional definition of the Lie derivative on general differentiable manifolds, also admits the same interpretation related to the initial Cauchy problem within the framework of congruencies for a given parameterization $\{\lambda_i\}$. When comparing scalars, vectors or tensors at different points $\{\lambda_i\}$ and $\{\lambda_i + \Delta\lambda_i\}$ on a certain integral curve, those entities under comparison are Lie (or invariably) dragged back from $\{\lambda_i + \Delta\lambda_i\}$ to the point $\{\lambda_i\}$. This gives a unique difference and hence a unique derivative as the limit of the difference between the values of scalars, vectors or tensors at different points on a manifold. A notion of a Lie invariably dragged function along a congruence $\lambda_i$ is essential in the conventional definition of the directional (or Lie’s) derivative $\frac{D}{D\lambda}$ along a flow (or general vector) field. It defines the rule necessary to compare values of a given mathematical entity at two different points.

The differential operator $\frac{D}{D\lambda}$ is a tangent vector to the curve $r(\lambda_i)$ on a manifold. This association of the concept of a directional derivative from the classical analysis allows to maintain a visual picture on the Lie derivative as a tangent vector that generates a kind of a motion along
the curve \( r(\lambda_i) \). In fact, when a differentiable manifold is an ordinary Euclidean domain both Euler’s and Lie’s derivatives coincide.

As we shall discuss in this paper, both Eulerian and Lagrangian complementary flow field specifications which are traditionally related to the initial Cauchy problem do not cover all possible situations. It motivates a definition of complementary mathematical object denoted here as a local directional derivative. As a counter-part of the conventional approach, it appears in this paper formulated within the so-called final Cauchy problem and describes the time variation of \( f \)-property at some fixed point in a local coordinate system. It allows the Euler description of the flow field to be dissociated from any need of identification of individual bits of fluid and, hence, to be considered properly as function of a position at every instant of time.

II. EULER’S DIRECTIONAL DERIVATIVE AND INITIAL CAUCHY PROBLEM

As the first task, let us explore the relationship between the Eulerian and Lagrangian specifications within the conventional definition of the directional derivative. To simplify our analysis, the further discussion will be based on the consideration of one-component (scalar) fluid moving in a three-dimensional Euclidean domain. We denote by \( f(r, t) \) some regular function in an arbitrary space-time coordinate system. For instance, in fluid dynamics or elasticity theory it could be a density of some physical medium \( \rho \) in Cartesian coordinates.

If in Lagrangian specification a geometrical transformation \( H_t \) represents a mapping of the closure \( \bar{\Omega} \) onto \( H_t \bar{\Omega} \) for the same individual bit of fluid at time \( t \), then \( H_t \) also represents the function \( \bar{\Omega} \):

\[
r = H_t r_0 = r(r_0, t) \tag{2.1}
\]

where points \( r \) and \( r_0 \) denote the position-vector of the fluid identifiable point-particle at time \( t \) and initial time \( t_0 \), respectively. Thus, the velocity vector field on the domain is defined as:

\[
v = \frac{\partial}{\partial t} r(r_0, t) \tag{2.2}
\]

where \( r_0 \) is fixed.

In the context of Eulerian specification, there is no explicit consideration of the function \( r = r(r_0, t) \). The primary notion is the velocity field

\[
\frac{dr}{dt} = v(r, t) \tag{2.3}
\]

as function of position in space \( (r) \) and time \( (t) \) on a fluid domain. Based on this convention, it is clear that both specifications become mathematically equivalent when an initial condition \( r_0 = r(t_0) \) for Cauchy problem is added to the ordinary differential equation (2.3). It gives a rule for translating from one specification to the other. If any quantity has the Eulerian representation \( f(r, t) \), its Lagrangian representation is \( g(r_0, t) \):\n
\[
g(r_0, t) = f(r(r_0, t), t) \tag{2.4}
\]

and, therefore, Euler’s material or substantive derivative is conventionally defined as \( \bar{\Omega} \):

\[
D f = \frac{\partial}{\partial t} (v, \nabla) f = \frac{\partial}{\partial t} g(r_0, t) \tag{2.5}
\]

where \( r_0 \) is fixed.

The differential operator \( \frac{D}{Dt} \) has meaning only when applied to flow field variables as functions of \( (r(t, t)) \) and gives the definition of directional derivative as a time derivative following the motion of the fluid in the direction of its velocity field \( v \):

\[
\frac{D}{Dt} f(r(t, t)) = \lim_{t \to 0} \frac{1}{t} [f(r(t_0 + t), t_0 + t) - f(r(t_0), t_0)] \tag{2.6}
\]

Here, according to the standard definition \( \bar{\Omega} \) related to the initial Cauchy problem \( r(t_0) = r_0 \), both values \( f_0 = f(r(t_0), t_0) \) and \( f = f(r(t_0 + t), t_0 + t) \) represent the \( f \)-property at two different points of space \( r_0 \) and \( r_0 + \Delta r \), respectively.

Thus, in this type of Eulerian of specification, one can reconstruct the property \( f \) of any identified bit of fluid at a new position \( r(t_0 + dt) = r_0 + dv \) and instant \( t_0 + dt \), based only on the knowledge of partial time derivative \( \frac{\partial f}{\partial t} \) and local distributions of the gradient \( \nabla f \) and velocity field \( v \) in local coordinate system:

\[
f = f_0 + \left( \frac{\partial f}{\partial t} + (v, \nabla f) \right) dt \tag{2.7}
\]

The function \( f \) has, generally speaking, explicit as well as implicit (through \( r(t) \)) time dependencies and, therefore, may be defined on a 4-dimensional space-time manifold with no metric known \( a \ priori \). In this respect, the Lie derivative, as a particularly interesting generalization of \( \bar{\Omega} \) (or (2.6)), will be convenient for further considerations, since it provides a necessary framework on manifolds without metric. The differential equation \( \bar{\Omega} \) will define a congruence or \( t \)-parameterized set of integral world-lines filling a 4-dimensional manifold:

\[
\frac{dx^i}{dt} = V^i; \quad x^i(t) = x_0^i + \int_{t_0}^{t_0 + t} V^i dt \tag{2.8}
\]

where \( x = (x_0, x^1, x^2, x^3) = (t, r) \); \( V = (1, v) \); \( x_0 = x(t_0) \) and, for our convenience, we leave for the time variable \( x^0 \).
its original denomination \( t \). Upper indices are used for the coordinate functions \( x^i(t) \) so that the 1-forms will satisfy the index conventions of modern differential geometry.

If the velocity vector field \( V \) is \( C^\infty \), the coordinate transformation \( \mathcal{L}_\xi \) is a diffeomorphism, forming part of a one-parameter Lie group. Let us denote this transformation as \( F_t : (x_0 \to x(x_0,t)) \), which defines the mapping of \( f(x_0) = f(t_0, r(t)) \) along the congruence (called also as Lie dragging \([2]\)) into a new function \( f(x_0 + x) = f(t_0 + t, r(t) + t) \). A Lie dragging of scalar field has a simple geometrical interpretation in Lagrangian specification of the fluid field: \( F_t \) transforms the \( f \)-property of the identified fluid element at \( x_0 \) according the rule \([3]\):

\[
(F_t f)(x_0) = f(F_t(x_0)) = f(x_0 + x) \tag{2.9}
\]

into the \( f \)-property of the same fluid element at \( x_0 + x \). This interpretation also concerns the analytic expression of the Lie derivative \( L_V \) along the vector field \( V = (1, \mathbf{v}) \):

\[
L_V f = \frac{d}{dt} F_t f |_{t=0} = \lim_{t \to 0} \frac{1}{t} [f(x_0 + x) - f(x_0)] \tag{2.10}
\]

where \( x(t_0) = 0 \). The concept of a Lie invariably dragged function along a congruence is used in this conventional definition (and implicitly in \([2]\)): in fact, the quantity \( f(x_0 + x) \) is invariably dragged along the congruence from \( x_0 + x \) back to \( x_0 \), since in general tensor calculus it has no clear meaning to compare both values \( f(x_0) \) and \( f(x_0 + x) \) at different points in a manifold without metric. Thus, \([2.10]\) gives a unique difference and therefore a unique derivative. In fact, when \( t \) is too small, the mapping \( F_t : (x_0 \to x(x_0,t)) \) has an explicit form:

\[
F_t x_0 = x^i(x_0, t) = x_0^i + t V^i(x_0) + o(t) \tag{2.11}
\]

which gives analytic expression for the Lie derivative:

\[
L_V f = \frac{d}{dt} F_t f = V^i \frac{\partial f}{\partial x^i} \tag{2.12}
\]

Important that both traditional definitions for Euler’s and Lie’s directional derivatives, respectively, turn out to be defined entirely in the spirit of original Lagrangian specification, i.e. when a fluid element or a point on a congruence are constantly identified in a local coordinate system. It explains why in an ordinary Euclidean domain the Lie mathematical construction takes a familiar form of Euler’s directional derivative \([3]\):

\[
L_V f = V^i \frac{\partial f}{\partial x^i} = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) f = \frac{Df}{Dt} \tag{2.13}
\]

It is known as a full derivative along the vector filed \( V = (1, \mathbf{v}) \) and in hydrodynamics it also has numerous applications for the description of the motion of macroscopic individual bodies of fluid.

In this respect, let us now consider a time variation of the fluid \( f \)-content in a macroscopic 3-dimensional space domain \( V \) moving with the fluid:

\[
\frac{d}{dt} \int_V f dV = \frac{d}{dt} \int_{\delta V(t)} f dV = \int_{\delta V(t)} \frac{d}{dt} (f dV) \tag{2.14}
\]

where, for our convenience, the macroscopic volume \( V(t) \) is represented as the sum of individual microscopic volumes \( \delta V(t) \), i.e. \( V = \sum \delta V \).

A geometrical transformation \( H_t \) describes the evolution of the whole volume \( V(t) = H_t V_0 \) as well as its individual bits \( \delta V(t) = H_t \delta V_0 \). The assumption that the domain \( V \) and all \( \delta V \) move with the fluid, means that there is no flux of \( f \) through the common bounding surface of the fluid domain \( \partial V \) and bounding surfaces of all individual elements \( \partial \delta V \). It imposes the condition that normal components of the relative fluid velocity field are zero at every point of all such surfaces moving with the fluid. By the chain rule we obtain from \([2.14]\):

\[
\frac{d}{dt} \int_V f dV = \int_V \frac{Df}{Dt} dV + \sum_{\delta V(t)} f \frac{d}{dt} \delta V(t) \tag{2.15}
\]

The time variation of a microscopic volume \( \delta V \) is a result of movement of each point of the bounding surface \( \partial \delta V \) and it is described by the divergence theorem \([5]\):

\[
\frac{d}{dt} \delta V(t) = \int_{\partial \delta V} (\mathbf{n}, \mathbf{u}) dS = \int_{\delta V} (\nabla, \mathbf{u}) dV \tag{2.16}
\]

where \( \mathbf{n} \) is a unit vector normal to the surface \( \partial \delta V \) and \( \mathbf{u} \) is the local velocity field of surface points in the coordinate system attached to \( \partial V \). Since the divergence does not depend on the choice of coordinate basis, we can write \( (\nabla, \mathbf{u}) = (\nabla, \mathbf{v}) \) for the flow velocity field \( \mathbf{v} \) defined in the main coordinate system at rest. Hence the local rate of expansion or dilation of a microscopic volume element \( \delta V \) is:

\[
\frac{d}{dt} \delta V(t) = (\nabla, \mathbf{v}) \delta V \tag{2.17}
\]

When this is substituted into \([2.15]\), one gets the result of the Convection Theorem:

\[
\frac{d}{dt} \int_V f dV = \int_V \left( \frac{Df}{Dt} + f (\nabla, \mathbf{v}) \right) dV \tag{2.18}
\]

More detailed and rigorous demonstrations can be found, for instance, in \([2, 3]\). We only need the way it was reasoned as well as its interpretations for further discussions.
To finish this Section, we conclude that the formula (2.13) for Euler’s derivative \( \frac{\partial f}{\partial t} \) (as well as its integral counter-part \( \frac{\partial \int f}{\partial t} \)) provide the rule for reconstruction of the \( f \)-property (or the \( f \)-content) at a new position \( \mathbf{r} = \mathbf{r}_0 + \mathbf{d} \mathbf{r} \) (or in a new domain \( V(t) = H_i V_0 \)), generally speaking, different from the initial one. This hydro-dynamics interpretation will become useful in the next Section in order to contrast a complementary specification of a flow field which provides the reconstruction of the \( f \)-property inside \( \delta V_0 \) is necessary as it was in (2.15) for the reliable use of \( \frac{\partial f}{\partial t} \). The integrand in the left-hand side of (2.22) is not a simple multivariable function \( f(\mathbf{r}, t) \) but a rather different mathematical entity \( f(\mathbf{r}(t), t) \), i.e. flow field quantity. This feature makes it difficult the straightforward application of the partial time derivative \( \frac{\partial f}{\partial t} \) meaningful for a fixed space variable \( \mathbf{r} \) as function of a \textit{frozen} velocity field \( (\mathbf{v} = 0) \). Therefore, the use of the partial time derivative does not seem to be fully justified in the conventional approach. This circumstance was also critically pointed out in \( \text{[6], [7]}. \)

To clarify the situation, let us go through the same mathematical construction as was used before for Euler’s and Lie’s derivatives, i.e we shall define an appropriate specification of the flow field in this case. The terminology of ordinary differential equations theory, combined with notions from classical analysis, enables us to give a useful and compact definition of the left-hand side of (2.22):

Let \( \mathbf{r}_0 \) be a fixed point of the closure \( V_0 \) and let the path of some identified elementary bit of fluid (that at some earlier instant \( t_0 \) passed through a certain point of space \( \mathbf{r}^*(t_0) \)) lie at present instant \( t_0 + t \) on the position of the fixed point of space \( \mathbf{r}_0 \). Then the full time derivative is understood as the limit of the difference between the values of the volume \( f \)-content at different instants \( t_0 \) and \( t_0 + t \). This requirement provides a framework necessary to derive analytic expression for the left-hand side of (2.22):

\[
\frac{d}{dt} \int_{V_0} f(\mathbf{r}(t), t)dV = \lim_{t \to 0} \frac{1}{t} \int_{V_0} [f(\mathbf{r}(t_0 + t), t_0 + t) - f(\mathbf{r}(t_0), t_0)]dV
\]

where \( \mathbf{r}(t_0) = \mathbf{r}_0 \). Both values \( f(\mathbf{r}(t_0), t_0) \) and \( f(\mathbf{r}(t_0 + t), t_0 + t) \) represent the \( f \)-property at the same point of space \( \mathbf{r}_0 \).

Based on this convention we note that \( \mathbf{r}(t_0 + t) = \mathbf{r}_0 \) should lie at the end of the integral curve:

\[
\mathbf{r}_0 = \mathbf{r}^*(t_0) + \int_{t_0}^{t_0 + t} \mathbf{v}(\mathbf{r}, t)dt
\]

Extrapolation on a set of integral curves that fill our domain is straightforward. The path of every such curve has to end at some fixed point of the closure \( V_0 \). All of them are solutions of initial Cauchy problems for the first-order differential equation:

\[
\frac{d\mathbf{r}}{dt} = \mathbf{v}; \quad \mathbf{r}(t_0) = \mathbf{r}^* \in V^*
\]

In Lagrangian specification it represents a geometrical transformation (or mapping) \( H_i \) of the original 3-dimensional domain \( V^* \) at instant \( t_0 \) onto \( H_i V^* = V_0 \) at
instant $t_0 + t$. If the inverse mapping $H^{-1}$ is single-valued, then $V^* = H^{-1}V_0$ can be regarded as a reconstruction of initial conditions from the knowledge of the final domain $V_0$, i.e. $V^*$ becomes dependent on $t$ parameter. Similarly to (2.1), for an individual bit of fluid, $H_t$ defines the function:

$$\mathbf{r}_0 = H_t\mathbf{r}^* = \mathbf{r}(\mathbf{r}^*, t); \quad \mathbf{r}_0 \in V_0 \quad (3.7)$$

where $\mathbf{r}_0$ is fixed and belongs to $V_0$.

By analogy with the initial Cauchy problem for (2.3), it may be called as a final Cauchy problem for (3.6). If any flow field quantity has the Eulerian representation $f(\mathbf{r}, t)$, its Lagrangian representation $g^*$ in the context of the final Cauchy problem will be:

$$g^*(\mathbf{r}_0, t) = f(\mathbf{r}(\mathbf{r}_0, t), t) \quad (3.8)$$

and, therefore, the partial time derivative $\frac{\partial g^*}{\partial t}$ (with the fixed $\mathbf{r}_0$) will define some new mathematical construction:

$$\frac{\partial}{\partial t} g^*(\mathbf{r}_0, t) = D^* f \quad (3.9)$$

Note that the differential operator $\frac{D^*}{\partial t}$ makes sense only when applied to flow field variables as functions of the final Cauchy problem and will be called here as the local directional derivative by analogy with the definition of $\frac{D}{\partial t}$ within the framework of classical analysis:

$$\frac{D^* f}{\partial t} = \lim_{t \to 0} \frac{1}{t} [f(\mathbf{r}(t_0 + t), t_0 + t) - f(\mathbf{r}(t_0), t_0)] \quad (3.10)$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and we denote by $\mathbf{r}(t_0 + t)$ the final point $\mathbf{r}_t$ of the integral curve (3.4).

To find the analytic expression for $\frac{D^*}{\partial t}$, let us use a consideration similar to the applied in the previous Section for the definition of $\frac{D}{\partial t}$. The differential equation (3.6) will define a congruence or $t$-parameterized set of integral world-lines filling a 4-dimensional space-time manifold:

$$\frac{dx^i}{dt} = V^i; \quad x^i(t) = x^i + \int_{t_0}^{t_0 + t} V^i dt \quad (3.11)$$

where $x^* = (t_0, \mathbf{r}^*)$ and $x = (t_0 + t, \mathbf{r}_t)$.

This transformation, which we denote as $G_t$: $(x^* \rightarrow x_0 = x(x^*, t))$, defines the mapping of $f(x^*)$ along the congruence into a new function $f(x_0)$. In Lagrangian specification of the flow field it has an obvious geometrical interpretation: $G_t$ transforms the $f$ property of the identified fluid element at $x^*$ according the rule:

$$(G_t f)(x^*) = f(G_t(x_*)) = f(x_0) \quad (3.12)$$

The inconvenience of this description is that we are now at the local coordinate system as a function of $\mathbf{r}^*(t)$. Since the integration in (3.11) is effected over the fixed domain $V_0$, we choose a local coordinate system attached to $V_0$ by means of coordinate transformation:

$$\mathbf{r}^*(t) = \mathbf{r}_0 - \int_{t_0}^{t_0 + t} \mathbf{v} (\mathbf{r}, t) dt \quad (3.13)$$

or in 4-dimensional notations:

$$\frac{dx_i^*}{dt} = V^i; \quad x_i^*(t) = x_i^0 + \int_{t_0}^{t_0 + t} V^i dt \quad (3.14)$$

where $V^* = (1, -\mathbf{v})$. It defines an equivalent mapping which we denote as $G^*_t$: $(x_0 \rightarrow x^* = x(x_0, t))$.

Note that $G^*_t$ is the same mapping $G_t$ but defined in the local coordinate system attached to $V_0$. More precisely, it means that the final Cauchy problem (3.6) admits an equivalent formulation as an initial Cauchy problem (3.10) (i.e. $G^*_t$ is not the inverse transformation $(G_t)^{-1}$, since the course of time is not changed on the opposite one).

Thus, when $t$ is too small, the transformation $G^*_t$ has an explicit form:

$$G^*_t x_0 = x^i(x_0, t) = x_i^0 + t V^i(x_0) + o(t) \quad (3.15)$$

which gives analytic expression for the Lie derivative along the congruence $V^*$ at a local coordinate system:

$$L_{V^*} f = \frac{d}{dt} f(G_t^* x_0) = V^i \frac{\partial f}{\partial x^i} \quad (3.16)$$

In an ordinary Euclidean domain this mathematical construction takes the following expression:

$$L_{\nabla^*} f = (\frac{\partial}{\partial t} - (\mathbf{v}, \nabla)) f = \frac{D^* f}{D^* t} \quad (3.17)$$

Now, noting that the integrand in the right-hand side of the equation (3.4) is $\frac{D^*}{D^* t}$ according to the definition (3.10), we can proceed to the formulation of our result (3.17) as a theorem proven in an ordinary Euclidean domain:

**Theorem 1 (Local Convection Theorem):** Let $\mathbf{v}$ be a vector field generating a fluid flow through a fixed 3-dimensional domain $V_0$ and if $f(\mathbf{r}, t) \in C^1(\overline{V_0})$, then

$$\frac{d}{dt} \int_{V_0} f dV = \int_{V_0} \left(\frac{\partial}{\partial t} - (\mathbf{v}, \nabla)\right) f dV \quad (3.18)$$

where $dV$ denotes the fixed volume element.
This result formulated within the framework of final Cauchy problems could be regarded as a complementary counter-part of the Convection Theorem considered within the framework of initial Cauchy problems.

In Eulerian specification, the formula admits a clear hydrodynamic interpretation: \( \frac{Df}{Dt} \) provides the rule for reconstruction of the \( f \)-property at a fixed point of space \( r_0 \) at instant \( t_0 + dt \), based only on the knowledge of partial time derivative \( \frac{\partial f}{\partial t} \) and local distributions of gradient \( \nabla f \) and velocity field \( v \) in the vicinity of \( r_0 \):

\[
f = f_0 + \left( \frac{\partial f}{\partial t} - v \cdot \nabla f \right) dt \tag{3.19}
\]

Generally speaking, this type of Eulerian specification of the flow field does not imply any sort of identification of fluid elements and hence ought to be complementary to the original Lagrangian approach. In fact, it considers the rate of time variation of \( f \)-property locally, at fixed position of space. Whereas the Euler derivative complementarily describes the rate of time variation following the motion of the fluid.

Both types of directional derivatives \( \frac{Df}{dt} \) and \( \frac{D^*f}{dt} \) can be analyzed in terms of 1-forms or real-valued functions of vectors in 4-dimensional manifolds:

\[
\omega = (\omega_i) = \left( \frac{\partial f}{\partial x^i} \right) \tag{3.20}
\]

where \( i = 0, 1, 2, 3 \) and \( \left( \frac{\partial}{\partial x} \right) = \left( \frac{\partial}{\partial t}, \nabla f \right) \) in an ordinary Euclidean domain.

Now we point out that in tensor algebra the set \( \{ \omega_i V^j \} \) are components of a linear operator or \( \left( \begin{array}{cc} 1 & 1 \\ \end{array} \right) \) tensor. The formation of a scalar \( \omega(V) \) is called the contraction of the 1-form \( \omega \) with the vector \( V \) and it is an alternative representation of directional derivatives:

\[
\frac{Df}{Dt} = \omega_i V^i; \quad \frac{D^*f}{Dt} = \omega_i V^{*i} \tag{3.21}
\]

The contraction of diagonal components of the tensor \( \omega_i V^j \) is independent of the basis. Importantly, this law shows that both types of directional derivatives \( \frac{Df}{dt} \) and \( \frac{D^*f}{dt} \) are invariant and do not depend on the choice of a local coordinate system. On the other hand, it is also the property of scalar product in manifolds with metric. The metric tensor maps 1-forms into vectors in a 1-1 manner. This pairing is usually written as:

\[
\omega_i = g_{ij} \omega^j; \quad V^i = g^{ij} V_j \tag{3.22}
\]

Therefore, from the point of view of tensor algebra, can be considered as a scalar product in a 4-dimensional manifold with metric:

\[
\frac{Df}{Dt} = g_{ij} \omega^i V^j; \quad \frac{D^*f}{Dt} = g_{ij} \omega^i V^{*j} \tag{3.23}
\]

where \( g_{ij} = \delta_{ij} \) is the Euclidean metric tensor. A Minkowski metric is also consistently singled out for local directional derivative \( \frac{D^*f}{dt} \):

\[
\frac{D^*f}{dt} = g_{ij} \omega^i V^{*i} = g_{ij} \omega^i V^i \tag{3.24}
\]

where \( V^* = (1, -v) \); \( g^*_{ij} = \text{diag}(1, -1, -1, -1) \) is indefinite or Minkowski metric tensor.

Another consequence of this form is that it gives orthonormal bases for space-time manifolds (previously introduced with no metric known a priori). For Lagrangian flow field specification, a basis is Cartesian and a transformation matrix \( \Lambda_c \) from one such basis to another is orthogonal matrix:

\[
\Lambda_c^T = \Lambda_c^{-1}; \quad 'g_{ij} = \Lambda_c^{-1} \delta_{ij} \Lambda_c \tag{3.25}
\]

These matrices \( \Lambda_c \) form the symmetry group \( O(4) \).

Likewise, for Eulerian specification a Minkowski metric picks out a preferred set of bases known as pseudo-Euclidean or Lorentz bases. A transformation matrix \( \Lambda_L \) from one Lorentz basis to another satisfies:

\[
\Lambda_L^T = \Lambda_L^{-1}; \quad 'g_{ij} = \Lambda_L^{-1} g_{ij} \Lambda_L \tag{3.26}
\]

\( \Lambda_L \) is called a Lorentz transformation and belongs to the Lorentz group \( L(4) \) or \( O(3,1) \).

The point that needs to be emphasized here is the remarkable circumstance of Euler’s specification in evoking the Minkowski metric without any previous postulation. In other words, consistent mathematical description of fluids is perfectly compatible with the Lorentz symmetry group. This fact was not seriously considered in theoretical hydrodynamics until now.

The Galilean group as one of subgroups of \( O(4) \), is commonly used in modern classical mechanics in flat space-time manifolds. This is not surprising in view that all classical mechanics laws are written in Lagrangian specification by constant identification of mechanical objects and within the formulation of initial Cauchy problem for equations of motion. It was therefore natural to admit that space-time in classical mechanics has a Galileian group symmetry. The Special Relativity postulation of Lorentz group symmetry on mechanics is not trivial, having in mind the complementary character of Lagrangian and Euler’s descriptions. Perhaps it can explain a paradoxical nature of some conclusions in relativistic mechanics but it overcomes the scope of this work and will be considered elsewhere.

Thus, from the point of view of flow field specifications, both kinds of directional derivatives are complementary.
and equally valid but should be used in different contexts. Euler’s derivative has therefore a more narrow framework of applicability in the classical field theory than it was supposed. In what follows we will confine our attention on some example from the classical field theory.

IV. LOCAL DIRECTIONAL DERIVATIVE IN CLASSICAL FIELD THEORY

Let us now consider the description of the conservation of the fluid $f$-content in an arbitrary 3-dimensional space domain. If the volume $V$ moves with the fluid, the Convection Theorem (2.18) written in differential form states that the $f$-content is conserved when the total time derivative is zero:

$$\frac{Df}{Dt} + f(\nabla, v) = \frac{\partial f}{\partial t} + \nabla(fv) = 0 \quad (4.1)$$

In particular, when the velocity field $v$ is locally zero, it represents the continuity equation of any elastic medium locally at rest:

$$\frac{\partial f}{\partial t} + f(\nabla, v) = 0 \quad (4.2)$$

where the extra term $(v, \nabla f)$ due to the fluid movement has disappeared.

In the case when the fluid moves through a volume $V_0$ fixed in local coordinate system at rest, a mathematical restriction on conservation immediately leads to the well-established integro-differential form of continuity equation:

$$\frac{d}{dt} \int_{V_0} f dv = -\int_{S_0} f(v \cdot dS) = -\int_{V_0} \nabla(fv) dV \quad (4.3)$$

Note that here both sides of the equation are obviously independent on the choice of a particular coordinate basis.

As we already mentioned earlier in the previous Section, it is commonly thought that, in this case, the total time derivative can be substituted in the integrand by the partial derivative, giving place to the conventional form of continuity equation in the reference system at rest:

$$\frac{\partial f}{\partial t} = -\nabla(fv) \quad (4.4)$$

The circumstance that it coincides with the expression (4.4) derived for the volume in motion, is mainly attributed to the cross-verification of the standard differential form of continuity equation. Nevertheless, it is strange to contemplate that the differential equation (4.4) does not possess the symmetry properties of its original integral counter-part (13). The left-hand side of (4.4) becomes manifestly dependent on the choice of a coordinate basis which, generally speaking, leads to a more narrow group of symmetries. Let us see whether results of the previous Section may help to clarify the situation.

In fact, implementation of Euler’s type of flow field specification for the left-hand side of (4.4) in the framework of the final Cauchy problem changes the character of the integrand expression. If it is considered as the local directional derivative of $f$-property, the continuity equation (4.3) takes the following differential form:

$$\frac{D^* f}{D^* t} = \frac{\partial f}{\partial t} - (v, \nabla f) = -\nabla(fv) \quad (4.5)$$

that coincides with (12). The right-hand side of (4.5) as a divergence and the left-hand side as the local directional derivative $D^*$ do not dependent on the choice of a coordinate basis. It means that this differential form of the continuity equation has the symmetry properties of its original integral counter-part (13).

A brief comment is worthy in this respect. Why the traditional approach based on the Convection Theorem gives a different result (14)? Certainly, it is correct but it has a non-invariant extra term $(v, \nabla f)$ due to the fact that the description is effected in the reference system at rest for the domain following the motion of the fluid. These shortcomings of the direct application of the Convection Theorem was not appreciated until now. If an observer moves with the fluid, this Theorem gives the equation (12) without an extra term and with the symmetry properties of the original integral equation (13). On the other hand, one could logically ask why all numerical simulations based on the standard differential form of the continuity equation (4.4) do not lead to incorrect predictions? The answer is the following: traditional time discretization schemes for the partial time derivative of flow field quantities (see, for instance, [11]) treat it as if it were the total time derivative.

Another interesting task would be an application of the local derivative concept to the integral form of Maxwell’s equations. Two of them contain the full time derivative over volume integrals and are known as induction laws for electric $E$ and magnetic $H$ vector fields, respectively, in the local frame of reference at rest:

$$\int_C (H, dl) = \frac{4\pi}{c} \int_S (j, dS) + \frac{1}{c} \frac{d}{dt} \int_V (\nabla, E) dV \quad (4.6)$$

$$\int_C (E, dl) = \frac{1}{c} \frac{d}{dt} \int_V (\nabla, H) dV \quad (4.7)$$

Straightforward application of (3.18) in this case is hindered by a priori unknown nature of the velocity vector field for electric and magnetic field components. At this
stage, only quasistatic approximation can admit a reliable application of the local directional derivative concept. In fact, the Special Relativity firmly established that electromagnetic field components of uniformly moving single charge do not depend explicitly on time parameter \( t \). In other words, \( \mathbf{E} \) and \( \mathbf{H} \) are thought to be rigidly attached to the charged particle and uniformly move with it. This is one of the consequences of the Relativity Principle. Thus, if the charge velocity \( \mathbf{v}_q \) is known, the velocity vector field \( \mathbf{v} \) for quasistatic components of electric and magnetic field is also defined in the closure \( V \). Applying the result of the Theorem (4.14), we can rewrite (4.6)-(4.7) in a more convenient form:

\[
\int_C (\mathbf{H}, dl) = \frac{4\pi}{c} \int_S (j, dS) + \frac{1}{c} \int_V (\frac{\partial}{\partial t} - (\mathbf{v}, \nabla)) \nabla \mathbf{E} dV \\
(4.8)
\]

\[
\int_C (\mathbf{E}, dl) = -\frac{1}{c} \int_V (\frac{\partial}{\partial t} - (\mathbf{v}, \nabla)) \nabla \mathbf{H} dV \\
(4.9)
\]

where \( \mathbf{v} = \mathbf{v}_q \) is the instantaneous velocity field in the closure \( V \).

Since the motion is uniform, all partial derivatives vanish from (4.6)-(4.9). Applying a well-known expression for a general vector field \( \mathbf{A} \):

\[
(\mathbf{v}, \nabla) \mathbf{A} = \mathbf{v}(\nabla \mathbf{A}) - [\nabla, \mathbf{v}, \mathbf{A}] \\
(4.10)
\]

and reducing the volume \( V \) to zero, we arrive to the well-established relationship between quasistatic magnetic and electric field strength of an uniformly moving charge from the point of view of a local inertial reference system \( \mathbf{E} \):

\[
\mathbf{H} = \frac{1}{c} [\mathbf{v}, \mathbf{E}]; \quad \mathbf{E} = -\frac{1}{c} [\mathbf{v}, \mathbf{H}] \\
(4.11)
\]

It is worth stressing that \textit{a priori} no relativity principle was needed in deriving these transformation rules for electric and magnetic field components. The term proportional to \( -\frac{1}{c} (\mathbf{v}, \nabla) \) can be considered as convective displacement current \( \mathbf{j} \) by analogy with Maxwell’s displacement current proportional to \( \frac{\partial}{\partial t} \mathbf{E} \). Note that the integral form of Maxwell’s equations (4.14) written in Euler’s specification is now compatible with the charge conservation law (4.15) also represented in Euler’s specification.

On the other hand, the Lorentz and Ampere force concepts are manifestly valid quasi-static approximations (4.11) and therefore are inclosed into integral form of Maxwell’s field equations on a basic level. It means that there may be no need to postulate them separately as it was done in Maxwell-Lorentz microscopic electron theory and remains accepted at present. Nevertheless, any full analysis of these issues comes out of the scope of the present consideration and will be given elsewhere.

V. CONCLUSIONS

We attempted to consider a logical background and structure of useful mathematical constructions which are traditionally based on both Eulerian and Lagrangian flow field representations, complementary to each other. This account provides a mathematical method that justifies the definition of a complementary counter-part for Euler’s directional derivative which is called here as the local directional derivative.

The point that needs to be emphasized is the complementary character of the above introduced concept. By no means it substitutes the Euler mathematical construction. By contrary, it is shown that both types of directional derivatives are equally valid but should be used in different contexts. In fact, Euler’s substantive derivative arises in the context of initial Cauchy problems and therefore becomes useful within the framework of the Lagrangian type of description of flow field quantities. Likewise, it is possible to define a complementary framework of so-called final Cauchy problems appropriate for the Euler flow field specification as a function of position in space and in time for fluid domain. From the point of view of the classical theory of fields it means a more narrow framework of applicability for Euler’s derivative than it was thought.

The analytic expression for the local directional derivative is formulated in form of a theorem analogous and complementary to the \textit{Convection Theorem} well-established in theoretical hydrodynamics. One of its interesting conclusions is that the choice between Lagrangian and Eulerian types of flow field representation is equivalent to the choice between space-time manifolds with Euclidean and Minkowski metric, respectively. Therefore, the consistent mathematical description of kinematics of fluids in Eulerian representation results compatible with the Lorentz group symmetry \( L(4) \). In fact, it could be understood as complementary to the \( O(4) \) group symmetry compatible with the Lagrangian representation.

On the other hand, the definition of the mathematical construction complementary to the traditional one, helps to get a deeper insight on the cross-verification of several partial differential equations obtained from their well-established integral counter-parts in classical theory of fields.

Although the consideration in this work was developed for one-component (scalar) flow field, the notion of the local directional derivative can be easily generalized on Lie’s derivatives for any general tensor field on differentiable manifolds. Both types of Lie’s derivative will correspond to both complementary types of specifications.

In place of concluding remark let us give asserting and encouraging words of a great mathematician. Gauss once wrote in his letter to Bessel (quoted form \[12\]): \textit{...One should never forget that the function of complex variable}, like all mathematical constructions, are only our own creations, and that when the definition with
which one begins ceases to make sense, one should not ask, what is, but what is convenient to assume in order that it remain significant...”.

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[1] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Vol. 2 (Oxford University Press, 1972)
[2] R.E. Meyer, *Introduction to Mathematical Fluid Dynamics* (Wiley, 1972)
[3] B. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge University Press, 1980)
[4] L.D. Landau and E.M. Lifshitz, *Classical Theory of Fields* (Pergamon, 1985)
[5] B. Doubrovine, S. Novikov and A. Fomenko, *Modern Geometry*, Vol. 1 (Ed. Mir, Moscow, 1982)
[6] A.E. Chubykalo, R.A. Flores, J.A. Perez, *Proceedings of the International Congress 'Lorentz Group, CPT and Neutrino’, Zacatecas University (Mexico)*, 384 (1997)
[7] A.E. Chubykalo and R. Alvarado-Flores, *Hadronic Journal*, 25 159 (2002)
[8] G.K. Batchelor, *Introduction to Fluid Dynamics* (Cambridge University Press, 1967)
[9] A.J. Chorin and J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer-Verlag, 1993)
[10] A.E. Chubykalo and R. Smirnov-Rueda, *Modern Physics Letters A*, 12(1) 1 (1997)
[11] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (Cambridge University Press, 1998)
[12] M. Kline, *Mathematics, the Lost of Certainty* (New York, Oxford University Press, 1980)