Steady state conditions predictions of time-dependent infiltration problems: an LTDRM with a predictor-corrector scheme approach

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Abstract. Time-dependent infiltration problems from periodic irrigation channels with root-water uptake processes, which is modelled using a Richards equation, are studied. The problems is solved numerically. To solve the problems, a set of transformations and a Laplace Transform Dual Reciprocity Method (LTDRM) with a predictor corrector scheme are employed. Finally, employing Gaver-Stehfest formula and diffusivity factor, the numerical solutions of these problems are obtained. Predictions of time needed to achieve steady state conditions for different homogeneous soils are presented and discussed.

1. Introduction
Many works on steady water infiltration from irrigation channels or furrows were conducted by researchers. Such researchers are Batu [1], Gardner [2], and Solekhudin [3, 4]. Barry and Parlange [5], Basha [6], Clements and Lobo [7], and Zahroh [8] studied time-dependent infiltration cases. In the study of time dependent infiltration problems, results are given at some dimensionless times. For instance, Clements and Lobo studied non steady infiltration problems without sink term, and presented dimensionless Matric Flux Potential (MFP) at six different dimensionless time levels [7]. Solekhudin and Ang investigated non steady infiltration problems with sink term [9]. The values of dimensionless MFP are also obtained at six different dimensionless time levels. However, solutions at actual times are not discussed.

To obtain solutions at actual times, soil water diffusivity needs to be determined. According to Chaudari and Somawanshi, soil water diffusivity is influenced by sodium absorption ratio (SAR) and total electrolyte concentration (TEC) [10]. It is assumed that TEC and SAR are in appropriate proportion for the suitability of water irrigation [10]. In this paper, we examine predictions of steady state conditions of time-dependent infiltration problems from irrigation furrows with water absorption in three homogeneous porous media. The porous media in this study is soil.

2. Methods
We consider problems involving infiltration in three distinct homogeneous soil types. The homogeneous soils considered in this study are clay-loam, clay soils, and silt-loam. Periodic furrows are constructed on the surface of land or soil. The cross sectional geometry of the furrow is a trapezium. It is assumed that the furrows are filled with water, and kept filled all the time. On the middle of the beddings, a row
of plants are planted. It is also assumed that the furrows are very long and there are very large number of furrows.

In order to solve the problems, we consider a system of Cartesian coordinate $OXYZ$. The positive direction of $Z$-axis leads downward. The geometry of the channels and root zone is assumed to be not vary along $Y$-axis. For every unit length of furrow (1 cm), the furrow has surface area of 100 cm$^2$. Two adjacent rows of crops have a distance of 200 cm. The land surface width related to the process of transpiration, the depth and the width of the root zone are denoted by $2L$, $Z_m$ and $2X_m$ respectively. The geometry of the problem is illustrated in Figure 1.

From the problems description, the region to consider is a semi-infinite region. This region is notated by $R$, which is bounded by $0 \leq X \leq 100$ cm and $Z \geq 0$ as a consequence of the problems symmetry. This region is bounded by curve $C = C_1 \cup C_2 \cup C_3 \cup C_4$ as illustrated in Figure 2. It is assumed that the fluxes along the surface of channels are constant $v_0$. No fluxes flow across the soil surface outside the furrows. The fluxes across the planes $X = 0$ and $X = L + D$ are zero. Following Batu, we assume $\lim_{Z \to \infty} \frac{\partial \theta}{\partial X} = 0$ and $\lim_{Z \to \infty} \frac{\partial \theta}{\partial Z} = 0$ [1].

**Figure 1.** Periodic trapezoidal furrows with rows of crops.

**Figure 2.** Boundary of the problem.

### 2.1. Basic equations

Mathematical model of time-dependent infiltration with water absorption by crops is a Richards equation, which is formulated as

$$\frac{\partial \theta}{\partial T} = \frac{\partial}{\partial X} \left( K(\psi) \frac{\partial \psi}{\partial X} \right) + \frac{\partial}{\partial Z} \left( K(\psi) \frac{\partial \psi}{\partial Z} \right) - \frac{\partial K(\psi)}{\partial Z} - S(X, Z, \psi),$$  \hspace{1cm} (1)

where $\theta$ is moisture content in the soil, $\psi$ is the suction potential, $K$ is the hydraulic conductivity, and $S$ is the root-water uptake function. The root-water uptake function is as that in the study conducted by Utset et. al. [11], given by

$$S(X, Z, \psi) = \gamma(\psi)S_{\text{max}}(X, Z).$$  \hspace{1cm} (2)

Here, $\gamma \in [0, 1]$ is the water stress response function, and $S_{\text{max}}$ is maximum root uptake defined by

$$S_{\text{max}}(X, Z) = \frac{L_\ell \beta(X, Z)T_{\text{pot}}}{\int_0^{Z_m} \int_{L}^{L+D} \beta(X, Z) \text{d}X \text{d}Z}. $$  \hspace{1cm} (3)

Here $\beta(X, Z)$ is the distribution of spatial root-water uptake, and $T_{\text{pot}}$ is potential transpiration. Using the hydraulic conductivity $K$ proposed by Gardner [2].
\[ K(\psi) = K_S e^{a\psi}, \]  

where \( K_S \) is the saturated hydraulic conductivity and \( \alpha \) is an empirical parameter of soil, and matrix flux potential or MFP defined as

\[ \Theta(X, Z, T) = \int_{-\infty}^{\psi} K(s) \, ds, \]  

Equation (1) can be written as

\[ \frac{\partial \Theta}{\partial T} = \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Z^2} - a \frac{\partial \Theta}{\partial Z} - S(X, Z, \psi). \]  

As discussed by Solekhudin and Ang [9], Equation (6) can be transformed to

\[ \frac{1}{D(\theta)} \frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Z^2} - \theta - \gamma^*(\theta)S_m^*(x, z)e^{-z}, \]  

where

\[ \gamma^*(\theta) = \gamma \left( \frac{1}{\alpha} \ln \left( \frac{\alpha v_0 L \theta e^x}{\pi K_S} \right) \right), \]  

and \( S_m^*(x, z) \) is similar to that in Solekhudin’s study [3], that is

\[ S_m^*(x, z) = \frac{2\pi}{aL} \int_0^{z_m} \int_a^b \beta^*(x, z) \, dx \, dz, \]  

Here

\[ \beta^*(x, z) = \left( 1 - \frac{b - x}{x_m} \right) \left( 1 - \frac{z}{z_m} \right) \exp \left\{ \frac{2}{\alpha} \left[ \frac{p_x}{x_m} |x^* - (b - x)| + \frac{p_z}{z_m} |z^* - z| \right] \right\}, \]  

and

\[ l_t = \frac{\alpha}{2} L_t; \; x_m = \frac{\alpha}{2} x_m; \; z_m = \frac{\alpha}{2} z_m; \; x^* = \frac{\alpha}{2} x^*; \; z^* = \frac{\alpha}{2} Z^*; \]  

\[ p_x = \frac{\alpha}{2} P_x; \; p_z = \frac{\alpha}{2} P_z; \; a = \frac{\alpha}{2} L; \; b = \frac{\alpha}{2} (L + D). \]
2.2. An LTDRM with predictor-corrector scheme

LTDRM method is proposed by Zhu et al [12]. The method has been applied to solve various problems involving time-dependent cases [13, 14, 15]. As discussed by Yun [15], in order to apply an LTDRM, Equation (9) is recast to

\[
\kappa(\varepsilon, \zeta) \varphi(\xi, \eta, t) = \iint_R Y(x, z; \varepsilon, \xi, \zeta) \left[ \frac{\partial \varphi}{\partial t} + \varphi + \gamma^* (\varphi) S_m(x, z) e^{-z} \right] dx dz + \iint_C \left[ \varphi(x, z) \frac{\partial}{\partial n} (Y(x, z; \varepsilon, \xi, \zeta)) - Y(x, z; \varepsilon, \xi, \zeta) \frac{\partial}{\partial n} (\varphi(x, z)) \right] ds, \quad (11)
\]

where \( Y(x, z; \varepsilon, \xi, \zeta) \) is the fundamental solution of two-dimensional Laplace equation,

\[
\kappa(\varepsilon, \zeta) = \begin{cases} \frac{1}{2}, & (\xi, \eta) \text{ on smooth part of } C, \\ 1, & (\xi, \eta) \text{ on region } R. \end{cases}
\]

Assuming \( \gamma^* (\varphi) \) constant [11], and then employing Laplace transform

\[
\phi(x, z, s) = \int_0^\infty e^{-st} \varphi(x, z, t) dt, \quad (12)
\]

with

\[
\varphi(x, z, 0) = 0, \quad (13)
\]
on Equation (11), we have

\[
\lambda(\xi, \eta) \phi(\xi, \eta, s) = \iint_R Y(x, z; \varepsilon, \xi, \zeta) \left[ \frac{e^{-z}}{s} \gamma^* (\varphi) S_m(x, z) + (1 + s) \phi(x, z, s) \right] dx dz + \iint_C \left[ \phi(x, z, s) \frac{\partial}{\partial n} (Y(x, z; \varepsilon, \xi, \zeta)) - Y(x, z; \varepsilon, \xi, \zeta) \frac{\partial}{\partial n} (\phi(x, z, s)) \right] ds. \quad (14)
\]

Equation (14) is the integro-differential equation of

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = (1 + s) \phi + \frac{e^{-z}}{s} \gamma^* (\varphi) S_m(x, z). \quad (15)
\]

To apply the LTDRM, the domain’s boundary is required to be a simple closed curve. Hence, we need an imposed boundary to supersede \( z \to \infty \). The imposed boundary is \( z = c, c > 0 \), denoted by \( C \). Problems boundary conditions described in the preceding section in terms of \( \phi \) are

\[
\frac{\partial \phi}{\partial n} = \phi n_2 + \frac{2\pi}{\alpha L s} e^{-z}, \quad \text{on } C_1 \quad (16)
\]

\[
\frac{\partial \phi}{\partial n} = -\phi, \quad \text{on } C_2 \text{ and } C_5 \quad (17)
\]

\[
\frac{\partial \phi}{\partial n} = 0, \quad \text{on } C_3 \text{ and } C_4. \quad (18)
\]

Equation (15) with respect to Boundary conditions (16) - (18) is then solved numerically following an LTDRM procedure. The detail of the method is presented in [9].
Using the LTDRM, numerical results of this problem are obtained in the Laplace domain. Hence, the dimensionless MFP numerical value is determined by employing inverse of Laplace transform. One of Laplace transform inverse with good accuracy is Gaver-Stehfest formula. The formula is

\[ q(x, z, t) \approx \frac{\log 2}{t} \sum_{q=1}^{2^Q} K_q \phi(x, z, s_q), \]

where

\[ s_q = \frac{q \log 2}{t}, K_q = (-1)^{q+q} \sum_{i=(q+1)/2}^{\min(q, Q)} \frac{l^q(2l)!}{(Q-l)! (l-1)! (q-l)! (2l-q)!} \]

and \( Q \) is a positive integer.

3. Results and discussion

This section explains the numerical result of time-dependent infiltration with water uptake at three homogenous soils. The hydraulic properties of three homogenous soils is based on Warrick’s study given as \( \alpha = 1.9 \times 10^{-2} \text{ cm}^{-1} \) and \( K = 6.24 \text{ cm/day} \) for Clay-Loam soil, \( \alpha = 2 \times 10^{-2} \text{ cm}^{-1} \) and \( K = 10.8 \text{ cm/day} \) for Silt-Loam soil, \( \alpha = 8 \times 10^{-3} \text{ cm}^{-1} \) and \( K = 4.8 \text{ cm/day} \) for Clay soil [16].

We set \( X = 50 \text{ cm} \), which is half of that reported by Vrugt et al. [17], as the width of root zone in this study is half of that in Vrugt et al.’s study. We also chose \( L = X \). The fitted parameters are \( X = 0 \text{ cm}, P = 1.00, Z = 20 \text{ cm} \), and \( P = 1.00 \). Potential transpiration in this study is set to be \( 0.4 \text{ cm/day} \) [4]. The relation between root-water stress response function, \( \gamma \) and suction potential, \( \psi \), is the same as that reported by Utset et al [11]. Corresponding values of TEC and SAR to determine \( D(\theta) \) are \( 5 \text{ meq l}^{-1} \) and \( 5 \text{ mmol l}^{-1} \), respectively. These values are presented in [8]. The values of \( D \) for clay-loam, silt-loam, and clay are \( 3.3 \times 10^6 e^{36.4 \theta_v} \text{ cm}^2 \text{ min}^{-1} \), \( 0.048 \times e^{21.9 \theta_v} \text{ cm}^2 \text{ min}^{-1} \), \( 4.39 \times 10^8 e^{38.3 \theta_v} \text{ cm}^2 \text{ min}^{-1} \), respectively. Here, \( \theta_v \) is the saturated volumetric water content of soils [10].

To implement the LTDRM, we set \( N = 407, M = 899 \) for Clay and \( N = 401, M = 892 \) for Clay-Loam and Silt-Loam. These numbers are chosen after performing several computational experiments. Some of the results are presented in Figure 3 – Figure 5.

**Figure 3.** Matrix flux potential of Clay-Loam at some fixed values of \( X \) for \( 0 \leq Z \leq 300 \text{ cm} \).
Figure 4. Matrix flux potential of Clay at some fixed values of $X$ for $0 \leq Z \leq 300$ cm.

Figure 5. Matrix flux potential of Silt-Loam at some fixed values of $X$ for $0 \leq Z \leq 300$ cm.

Figure 3 – Figure 5 show the graphs of MFP for Clay-Loam, Clay and Silt-Loam at $X = 10$ cm, $X = 30$ cm, $X = 50$ cm, and $X = 90$ cm along $0 \leq Z \leq 300$ cm at various times. Values of MFP are evaluated for each soil at different time levels. For Clay-Loam, values of MFP are evaluated at 1.25, 2, 3, 4 and 5 hours. Values of MFP are evaluated at 1, 2, 3, and 4 hours for Silt-Loam and at 2, 3, 4, 5, 6, 7, 8 and 9 hours for Clay. The graphs show that the values of MFP increasing from time to time until reaching the steady condition.

Figure 3 shows that for Clay-Loam, MFP reaches steady state at about 5 hours after the infiltration process. From Figures 4 and 5, we can observe that for Clay and Silt-Loam, the steady state achieved after 9 hours and 4 hours, respectively. These mean that among the three types of homogeneous soils, Silt-Loam gives fastest time to reach conditions of steady. On the other hand, Clay gives latest time to gain such conditions. These results imply that coarser soil types reach steady state conditions faster than those finer.

It can be seen that values of MFP increase significantly from $T = 1.25$ hours to $T = 2$ hours for Clay-Loam, from $T = 1$ hour to $T = 2$ hours for Silt-Loam, and from $T = 2$ hours to $T = 4$ hours for Clay.
Afterwards, the increases in MFP decrease gradually until reaching steady state conditions. These mean that the increase in MFP at the beginning of infiltration tends to be more quickly than those at other times. In other words, the increases in MFP are lower from time to time. It can also be seen that values MFP decrease as X increases in soil levels less than a certain depth of soil. For instance, the values of MFP along X = 10 cm are higher than those along X = 30 cm. After a certain depth of soil, given a fixed values of Z, values of MFP are the same.

4. Conclusion
Problems involving time-dependent infiltration in three different types of homogeneous soils with water absorption by plants have been solved using a numerical method. The numerical method is Laplace transform dual reciprocity method with a scheme of predictor-corrector. Applying the method, MFP values at different times are obtained. Furthermore, estimations of time to reach conditions of steady are determined. The results show that the time required to attain steady state condition for one soil type is different from that for other soil types. The time needed has strong relation to the roughness or the softness of soil. In coarse-textured soils, water spread more quickly than that in fine-textured soils. Therefore, steady state conditions for coarser-textured soils are achieved faster in than those in finer-textured soils.

5. References
[1] Batu V 1978 Steady infiltration from single and periodic strip sources Soil Science Society America Journal 42 544-549.
[2] Gardner WR 1958 Some steady state solutions of the unsaturated moisture flow equation with application to evaporation from a water table Soil Science 85 228-232.
[3] Solekhudin I 2016 Water infiltration from periodic trapezoidal channels with different types of root-water uptake Far East Journal of Mathematical Sciences 100 2029-2040.
[4] Solekhudin I 2017 Suction potential and water absorption from periodic channels in a homogeneous soil with different root uptakes Advances and Applications in Fluid Mechanics 20 127-139.
[5] Barry DA and Parlange JY 1996 Infiltration subject to time-dependent surface ponding: exact results and correspondence to solute transport with nonlinear reaction Subsurface-Water Hydrology 33-48.
[6] Basha HA 1999 Multidimensional linearized nonsteady infiltration with prescribed boundary conditions at the soil surface Water Resources Research 35 75-83.
[7] Clements DL and Lobo M 2010 A BEM for time dependent infiltration from an irrigation channel Engineering Analysis with Boundary Elements 34 1100-1104.
[8] Zahroh M 2016 A dual reciprocity boundary element method (DRBEM) for time-dependent infiltration from periodic channels with root-water uptake at different types of homogeneous soils (Yogyakarta: Master Thesis, Department of Mathematics, Universitas Gadjah Mada).
[9] Solekhudin I and Ang KC 2015 A Laplace transform DRBEM with a predictor-corrector scheme for time-dependent infiltration from periodic channels with root-water uptake Engineering Analysis with Boundary Elements 50 141-147.
[10] Chaudhari SK and Somawanshi RB 2004 Unsaturated flow of different quality irrigation waters through Clay, Clay Loam and Silt Loam soils and its dependence on soil and solution parameters Agricultural Water Management 64 69-90.
[11] Utset A, Ruiz ME, Garcia J and Feddes RA 2000 A SWACROP-based potato root water-uptake function as determined under tropical conditions, Potato Research 43 19-29.
[12] Zhu SP, Satravaha P and Lu X 1994 Solving linear diffusion equations with the dual reciprocity method in Laplace space Engineering Analysis with Boundary Elements 13 1-10.
[13] Satravaha P and Zhu SP 1997 An application of the LTDRM to transient diffusion problems with nonlinear material properties and nonlinear boundary condition Applied Mathematics Computation 87 127-160.
[14] Ang WT 2002 A Laplace transformation dual-reciprocity boundary element method for a class of two-dimensional microscale thermal problems *Engineering Computations* 19 467-478.

[15] Yun BI 2012 A Laplace transform dual-reciprocity boundary element method for axisymmetric elastodynamic problems *World Academic Science, Engineering and Technology* 66 302-308.

[16] Warrick AW 2002 *Soil Physics Companion* (Washington DC: CRC Press).

[17] Vrugt JA, Hopmans JW and Simunek J 2001 Calibration of a Two-Dimensional Root Water Uptake Model *Soil Science Society of America Journal* 65 1027-1037.