Functional determinants for a single scalar field with negative mass squared are evaluated on homogeneous lens spaces. For example, on even order spaces, the Hartle–Hawking wavefunction oscillates about its zeros with increasing amplitude as the (imaginary) mass increases. I also present results for the binary tetrahedral, octahedral and icosahedral factors of the three–sphere. The final answer is given as a quadrature and some graphs are drawn. In the technical evaluation of the infinite sums, the explicit form of the degeneracies is not needed.
1. Introduction.

This paper should be considered as a technical continuation of an earlier communication, [1], which contains a calculation of the functional determinant on a sphere for a single scalar field with negative mass squared. This is relevant for some aspects of the conjectured dS/CFT correspondance, in particular the ‘equality’ of the bulk Hartle–Hawking wave function and the CFT partition function suggested by Maldacena, [2], [3].

My interest now is on factored spheres such as lens spaces and lunes. The latter possess conical singularities at the fixed points of the group action and some relevant numerics have already been given in [4] for the case of positive mass squared. Lens spaces were dealt with in [5] and this paper extends this to negative mass squared.

Negative mass squared means negative eigenvalues of the propagating operator leading to a little awkwardness computing the $\zeta$–function, which is how I approach the determinants. A further calculational point is that the degeneracies on lens spaces, for example, are more complicated and one has to work harder to reduce the $\zeta$–functions, if used directly, e.g. [6]. The method employed in [5], however, mostly avoids explicit introduction of the degeneracies, but ends up with a quadrature.

2. The $\zeta$–function on lens spaces

I restrict attention to the three–sphere. Although the basic equations have been given in [5] I have to repeat a few here. For any homogeneous quotient, $S^3/(1 \times \Gamma)$, it is sufficient to compute for lens spaces, $\Gamma = \mathbb{Z}_q$.

Following on from [1], where earlier references are given, the scalar eigenvalues are now taken to be $l^2 - \alpha^2$, $l = 1, 2, \ldots$, with total degeneracies, $D_l(q)$. I need the associated $\zeta$–function,

$$Z(s, q, \alpha) = \sum_l \frac{D_l(q)}{(l^2 - \alpha^2)s}.$$  \hspace{1cm} (1)

The value $\alpha = 1/2$ gives the eigenvalues for the Laplacian conformal in three dimensions, and $\alpha = 1$ gives minimal coupling. I am here more interested in $\alpha$ real and greater than 1/2.

One, minor point, is that for even $q$, $l$ must be odd, and so $D_{2l}(q)$ will be zero. This follows essentially by lifting from an orbifolded two–sphere. I will not treat odd and even $q$ separately.
3. Zero modes and the determinant

The problem with negative mass squared is an infrared one occasioned by zero modes that cause non–convergence as \( \tau \to \infty \). I treat this difficulty in a utilitarian fashion by separating off the zero/negative modes and modifying the generating function. This leads to a convergent integral. I will not give the general formulation but deal with the three–sphere.

The modes with \( l = 1, 2, \ldots, [\alpha] \) will be either zero or negative ones. The former occur when \( l = \alpha \) and \( \alpha = [\alpha] \), the rest being negative. The degeneracies of these \([\alpha]\) modes are give by the first \([\alpha]\) terms in the expansion, in \( t \), of the generating function given later.

I split the \( \zeta \)–function, (1), by extracting a finite number, \( N \), of the early modes in the following way,

\[
Z(s, q, \alpha) = \sum_{l=1}^{N} \frac{D_l(q)}{(l^2 - \alpha^2)^s} + \sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^s} = \sum_{l=1}^{N} \frac{D_l(q)}{(l^2 - \alpha^2)^s} + Z_N(s, q, \alpha),
\]

where I have defined a subtracted \( \zeta \)–function, \( Z_N(s, q, \alpha) \),

\[
Z_N(s, q, \alpha) = \sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^s}.
\]

For IR convergence, \( N \) must be larger than or equal to \([\alpha]\), the number of negative modes. For a given \( N \), the expression will be valid for all \( \alpha \) up to \([\alpha]\) equal to \( N \). The overlapping values, for different \( N \), provide a check of the numerics.

Differentiating (2) gives the quantity under investigation,

\[
Z'(0, q, \alpha) = Z_N'(0, q, \alpha) - \sum_{l=1}^{N} D_l(q) \log |(l^2 - \alpha^2)| + i\pi \sum_{l=1}^{[\alpha]} D_l(q),
\]

the negative modes contributing the imaginary part, which is undetermined up to integer multiples of \( 2\pi i \). The determinant is therefore unambiguous.

For the moment, I concentrate on the more important real part, which I continue to denote by \( Z'(0, q, \alpha) \), and now require the first term on the right–hand side.
Character theory, *cf* [7] eqn. (17), allows a closed form for the generating function for the \( D_l(q) \),

\[
F(t, q) \equiv \sum_{l=1}^{\infty} D_l(q) t^l = \sum_{l=1}^{\infty} l d_l(q) t^l = t \frac{d}{dt} \sum_{l=1}^{\infty} d_l(q) t^l = t \frac{d}{dt} \frac{t(1 + t^q)}{(1 - t^2)(1 - t^q)}.
\]

(5)

The procedure I employ for dealing with the negative modes is an extension of the one in [5] for minimal coupling. The basic idea, when finding \( Z'(0, q, \alpha) \), is to work with a resolvent convergent at \( s = 0 \) which is obtained by differentiating the \( \zeta \)-function in the usual way,

\[
\left( \frac{d}{d\alpha^2} \right)^2 Z_N(s, q, \alpha) = s(s + 1) \sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^{s+2}},
\]

which converges at \( s = 0 \), and so,

\[
\left( \frac{d}{d\alpha^2} \right)^2 Z_N(0, q, \alpha) = \sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^2}.
\]

(6)

Integrating this twice delivers the required quantity, \( Z'_N(0, q, \alpha) \). The two necessary constants of integration can be determined from values at some reference point. In [5] I chose \( \alpha = 0 \) since,

\[
Z_N(s, q, 0) = Z(s, q, 0) - \sum_{l=1}^{N} \frac{D_l(q)}{l^2 s},
\]

(8)

is known, [7]. For later use I note the special values,

\[
Z_N(1, q, 0) = Z(1, q, 0) - \sum_{l=1}^{N} \frac{D_l(q)}{l^2} \equiv Z(1, q, 0) - A_N(q)
\]

(9)

and

\[
Z_N(0, q, 0) = Z'(0, q, 0) + \sum_{l=1}^{N} D_l(q) \log l^2 \equiv Z'(0, q, 0) + B_N(q),
\]

(10)

where \( A_N \) and \( B_N \) are calculable constants.
Next a Bessel Laplace transform is used to reexpress the right–hand side in (7) as a convergent integral,

$$\sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^2} = \sqrt{\pi} \int_0^\infty \tau \mathcal{K}_N(\tau, q) \left(\frac{\tau}{2\alpha}\right)^{3/2} I_{3/2}(\alpha \tau).$$

(11)

where $\mathcal{K}_N(\tau, q)$ is the subtracted cylinder kernel for the square root of the propagating operator with $\alpha = 0$,

$$\mathcal{K}_N(\tau, q) = \sum_{l=N+1}^{\infty} D_l(q) e^{-l\tau}. \quad \text{(12)}$$

Apart from a change of variable ($t = e^{-\tau}$) this is the (subtracted) degeneracy generating function,

$$\mathcal{F}_N(t, q) \equiv \sum_{l=N+1}^{\infty} D_l(q) t^l$$

$$= t \frac{d}{dt} \text{Rem}_N \frac{t(1 + t^q)}{(1 - t^2)(1 - t^q)}$$

$$= t \frac{d}{dt} \text{Rem}_N G(t, q) \equiv t \frac{d}{dt} \mathcal{G}_N(t, q), \quad \text{(13)}$$

where Rem$_N$ stands for the remainder after the removal of the first $N$ terms in the Taylor expansion in $t$.

I can rewrite (12) and (13) as,

$$\mathcal{K}_N(\tau, q) = -\frac{d}{d\tau} \mathcal{H}_N(\tau, q),$$

where $\mathcal{H}_N(\tau, q) \equiv \mathcal{G}_N(e^{-\tau}, q)$ for convenience.

Then an integration by parts (the endpoint contributions vanish) $^2$ gives, as before, [5],

$$\sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^2} = \sqrt{\pi} \int_0^{\infty} \tau \mathcal{H}_N(\tau, q) \frac{d}{d\tau} \left(\frac{\tau}{2\alpha}\right)^{3/2} I_{3/2}(\alpha \tau)$$

$$= \sqrt{\pi} \int_0^{\infty} \tau \mathcal{H}_N(\tau, q) \left(\frac{\tau}{2\alpha}\right)^{3/2} I_{1/2}(\alpha \tau)$$

$$= \frac{1}{2} \int_0^{\infty} \tau \mathcal{H}_N(\tau, q) \frac{\sinh \alpha \tau}{\alpha}. \quad \text{(14)}$$

$^2$ This is not an essential manoeuvre.
From now on the calculation is identical to that in [5] except that (9) and (10) have to be used when finding the constants of integration. A first integration with respect to $\alpha^2$ is now easy to perform, yielding,

$$\frac{d}{d\alpha^2} Z_N'(0, q, \alpha) = \int_0^\infty d\tau \mathcal{H}_N(\tau, q)(\cosh \alpha \tau - 1) + \frac{d}{d\alpha^2} Z_N'(0, q, \alpha) \bigg|_{\alpha=0}. \quad (15)$$

The constant of integration (the final term) is found from the standard relation,

$$\frac{d}{d\alpha^2} Z_N(s, q, \alpha) = s \sum_{l=N+1}^{\infty} \frac{D_l(q)}{(l^2 - \alpha^2)^{s+1}} = s Z_N(s + 1, q, \alpha), \quad (16)$$

and one obtains for (15),

$$\frac{d}{d\alpha^2} Z_N(0, q, \alpha) = \int_0^\infty d\tau \mathcal{H}_N(\tau, q)(\cosh \alpha \tau - 1) + Z_N(1, q, 0)$$

$$= \int_0^\infty d\tau \mathcal{H}_N(\tau, q)(\cosh \alpha \tau - 1) + Z(1, q, 0) - A_N(q),$$

using (9). I find the $Z$ value later.

A final integration with respect to $\alpha^2$ is required, and produces,

$$Z_N'(0, q, \alpha) = \int_0^\infty \frac{d\tau}{\tau^2} \mathcal{H}_N(\tau, q)
(2\alpha \tau \sinh \alpha \tau - 2 \cosh \alpha \tau + 2 - \alpha^2 \tau^2) + \alpha^2 (Z(1, q, 0) - A_N(q)) + Z'(0, q, 0) + B_N(q), \quad (17)$$

again using (9) and (10). The $Z$ quantities on the right–hand side can be found numerically from the method in [4]. I copy the result here,

$$Z'(0, q, \alpha) = \int_0^\infty dx \Re \left( \frac{q \sinh \tau + \cosh \tau \sinh q \tau}{2 \tau \sinh^2 \tau \sinh^2 q \tau/2} \right) \cosh \alpha \tau, \quad 0 \geq \Re \alpha < 1. \quad (18)$$

Or the contour technique of [8] allows the values of $Z(n, q, \alpha) \ (n \in \mathbb{Z}_+)$ to be obtained easily. For example, [5],

$$Z(1, q, \alpha) = \frac{1}{2} \int_0^\infty dx \Re \frac{\coth(q \tau/2) \cosh \alpha \tau}{\sinh \tau}, \quad \tau = x + iy, \quad 0 < y < 4\pi/q. \quad (19)$$

Finally, therefore, from (4), the real part of the total derivative is,

$$Z'(0, q, \alpha) = \int_0^\infty \frac{d\tau}{\tau^2} \mathcal{H}_N(\tau, q)
(2\alpha \tau \sinh \alpha \tau - 2 \cosh \alpha \tau + 2 - \alpha^2 \tau^2) + \alpha^2 (Z(1, q, 0) - A_N(q)) + Z'(0, q, 0) + \sum_{l=1}^{N} D_l(q) \log \frac{l^2}{|\alpha^2 - l^2|}, \quad (20)$$
in which everything can be calculated. Only for \( A_N \) and the last term does one need a finite number of individual degeneracies, but these are easily computed.

Further, according to (13), the subtracted generating function is given by,

\[
\mathcal{H}_N(\tau, q) = \text{Rem}_N \left. \frac{t(1 + t^q)}{(1 - t^2)(1 - t^q)} \right|_{t = e^{-\tau}},
\]

which evaluates to a ratio of polynomials.

4. The numerics

Fig. 1. lens space logdet. Imaginary mass. Full sphere

![Graph 1](image)

Fig. 2. lens space logdet. Imaginary mass.

![Graph 2](image)
I plot just logdet which equals \(3 \log |\Psi_{HH}|\) on the dS/CFT correspondence. The figures show the variation of this with the parameter, \(\alpha\), occurring in my analysis related to the one, \(\sigma\), favoured in [2,3] by \(\alpha^2 = 1/4 - \sigma\).

For comparison, Fig.1 is for the full sphere (\(q = 1\)). Fig.2 is for odd \(q = 3, 5, 7\) and Fig.3 shows even \(q = 2, 4, 6\). The negative divergences are due to the logarithmic behaviour near the zero modes. These divergences become zeros of the Hartle–Hawking wave function. In between, \(|\Psi_{HH}|\) has a single maximum which generally increases with increasing \(\alpha\), except that for odd \(q\) there is a preliminary decrease. Apart from the full sphere, the degeneracy at \(l = 2\) is zero.

---

3 I normalise the wavefunction so that this is true.
For amusement, Fig. 4 shows the results for the homogeneous tetrahedral (T'), octahedral (O') and icosahedral (Y') factors of the three–sphere.

From (4) the wavefunction itself will change sign at a zero mode when the degeneracy there is odd.

The structure of the degeneracy, \( D(l, q) \), is as follows. For odd \( q \), for \( l = 1 \) to \( q - 1 \), the \( l \)-even \( D(l, q) \) are zero and the \( l \)-odd ones equal \( l \). The next one, \( l = q \), is \( D(q, q) = q \). Thereafter the values alternate between even and odd, and must be calculated. Even \( q \) is somewhat similar except that all the \( l \)-even values are zero and the non–zero ones are odd. Again, for \( l = 1, 3, \ldots, q - 1 \), \( D(l, q) = l \), the same as for odd \( q \). An interpretation of this is that as \( q \) tends to infinity the factored circle disappears leaving just the two sphere with degeneracy \( 2l + 1 \).

One sees that for even \( q \) the wavefunction oscillates through its zeros while for odd \( q \) it does this for a while and then alternates between changing signs and having extrema at its zeros.

5. Discussion

The increasing nature of the maxima in \(|\Psi_{HH}|\) is taken in [3] as a sign of a possible instability. There is no indication that use of homogeneous lens spaces alters this. The calculation for the inhomogeneous case is harder but would provide a different deformation.

Because of the non–trivial topology, twisted fields can be introduced. If real, these amount to a sign and only exist for even \( q \). For complex fields, there are more possibilities, classified by the \( q \)th roots of unity, corresponding to discrete \( U(1) \) fluxes, [9]. The severity of the analysis is only slightly increased and might be given at another time.

As a spectral exercise, the calculation can be repeated for spinor and vector fields. The necessary ingredients are in [7].

The same method can be applied with only small modifications to the case of lunes and will be expounded in a companion communication.
References.

1. J.S. Dowker, *Massive sphere determinants*, ArXiv:1404:0986.
2. Anninos, D., Denef, F. and Harlow, D. *Phys. Rev.* **D88** (2013) 084049.
3. Anninos, D., Denef, F., Constantinidis, G. and Shaghoulian, E. *Higher Spin de Sitter Holography from Functional Determinants*, ArXiv:1305.6321.
4. Dowker, J.S. *J. Phys.A:Math.Theor.* **46** (2013) 2254.
5. Dowker, J.S. *J. Phys.A:Math.Theor.* **46** (2013) 285202.
6. Radičević, D. *Singlet Vector Models on Lens Spaces*, ArXiv:1210:0255.
7. Dowker, J.S. *Class. Quant. Grav.* **21** (2004) 4247.
8. Candelas, P. and Weinberg, S. *Nucl. Phys.* **B237** (1984) 397.
9. Dowker, J.S. and Banach, R. *J. Phys.* **A11** (1978) 2255.