Morrey estimates for a class of elliptic equations with drift term

Abstract: We consider the following boundary value problem
\[
\begin{cases}
-\text{div}[M(x)\nabla u - E(x)u] = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), with \( N > 2 \), \( M : \Omega \to \mathbb{R}^{N^2} \) is a symmetric matrix, \( E(x) \) and \( f(x) \) are respectively a vector field and function both belonging to suitable Morrey spaces and we study the corresponding regularity of \( u \) and \( Du \).

Keywords: Morrey regularity, higher differentiability, drift term

MSC: 35J25, 35B65

1 Introduction

This paper is devoted to the study of the regularity of a weak solution \( u \) of the following homogeneous Dirichlet problem
\[
\begin{cases}
-\text{div}[M(x)\nabla u - E(x)u] = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]  
(1.1)
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), with \( N > 2 \), \( M : \Omega \to \mathbb{R}^{N^2} \) is a matrix with measurable bounded entries \( M_{ij}(x) \) satisfying the standard ellipticity condition, \( E(x) \) and \( f(x) \) are respectively a vector field and function both belonging to suitable Morrey spaces to be specified later on.

The study of the above problem goes back to the papers \([53, 54]\) by G. Stampacchia and it presents a difficulty due to the noncoercivity of the differential operator \( u \to -\text{div}[M(x)\nabla u - E(x)u] \). In the quoted papers the existence and uniqueness of a weak solution \( u \in W^{1,2}_0(\Omega) \) have been proved assuming that
\[
|E| \in L^N(\Omega), \quad \|E\|_{L^N(\Omega)} \text{ sufficiently small,}
\]  
(1.2)
and
\[
f \in L^{\frac{2N}{N-2}}(\Omega).
\]  
(1.3)

Moreover, the following regularity results hold:

(i) if \( f \in L^m(\Omega) \), \( \frac{2N}{N+2} \leq m < \frac{N}{2} \), then \( u \in L^{m^{**}}(\Omega) \), \( m^{**} = \frac{Nm}{N-2m} \);

(ii) if \( f \in L^m(\Omega) \), \( m > \frac{N}{2} \) and \( |E| \in L^r(\Omega) \), with \( r > N \), then \( u \in L^\infty(\Omega) \).

---

G. R. Cirmi, Dipartimento di Matematica e Informatica, Viale A. Doria 6 – 95125 Catania, Italy, E-mail: cirmi@dmi.unict.it

S. D’Asero, Dipartimento di Matematica e Informatica, Viale A. Doria 6 – 95125 Catania, Italy, E-mail: dasero@dmi.unict.it

*Corresponding Author: S. Leonardi, Dipartimento di Matematica e Informatica, Viale A. Doria 6 – 95125 Catania, Italy, E-mail: leonardi@dmi.unict.it

Open Access. © 2019 G. R. Cirmi et al., published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 License.
Later, using a nonlinear approach and exploiting techniques issued from those of G. Stampacchia, L. Boccardo in [3] retrieved the previous results without the smallness condition in (1.2).

The aforementioned results are stated in a global fashion and they have been obtained when the data belong to usual Lebesgue spaces. On the other hand, G. Stampacchia in [54] also studies local properties of the solution of the problem (1.1). In particular, under the hypotheses stated in (ii), he proves that a $W^{1,2}$-solution of the problem (1.1) is locally bounded and locally Hölder continuous (1).

So that, naturally it raises the question of studying the problem in the setting of Morrey spaces, in order to try to obtain results similar to those obtained by S. Campanato in [6], for example. As far as we know the only available results in this framework are contained in the paper [21], where Hölder continuity and a Morrey estimate of a weak solution $u$ have been proved. The technique used is essentially based on the representation formula of the solution but it doesn't allow to obtain any Morrey estimate of the gradient of the solution.

In this paper we suppose that the right–hand side $f$ and the vector field $E$ belong to suitable Morrey spaces and we recover the gradient estimate of a solution $u$ in the corresponding Morrey space, so to retrieve the regularity theory at the "gradient level" as in Campanato’s work.

Moreover, we also weaken the assumptions on the data when dealing with the existence of bounded weak solutions.

Namely, under the following minimal assumptions
\[ f \in L^{1,\lambda}(\Omega) \quad \text{and} \quad |E| \in L^{2,\lambda}(\Omega) \quad \text{with} \quad N - 2 < \lambda < N, \]
we will prove that there exists a weak solution $u \in W^{1,2}_{0}(\Omega)$ such that $u \in L^{\infty}(\Omega)$ and $|Du| \in L^{2,\lambda_0}(\Omega)$, for some $\lambda_0 \in ]N - 2, \lambda[$. The latter information brings back the Hölder continuity of $u$ (see Theorem 3.2 below). Moreover, if $f$ has an higher summability exponent and a lower Morrey exponent, that is
\[ f \in L^{2k, \frac{\lambda N}{N + k}}(\Omega) \quad \text{with} \quad 0 < \lambda < N - 2, \]
and
\[ |E| \in L^{2,\mu}(\Omega) \quad \text{with} \quad N - 2 < \mu < N, \]
then any weak solution $u$ of problem (1.1) satisfies $|Du| \in L^{2,\lambda}(\Omega)$ (see Theorem 3.7 below).

At last, Morrey estimate obtained for $|Du|$ allows us to extend to the problem (1.1) the Calderon–Zygmund theory introduced in the paper [50] by G. Mingione (see also [2]).

In the framework of regularity theory of weak solutions the reader can also refer to the following papers
[7–10, 16, 22, 23, 27–30, 32–36, 43, 45, 51, 52].

\section{Main notations, functions spaces and auxiliary lemmas}

In this section, for reader’s convenience, we recall some useful properties of functions spaces and some lemmas that we are going to exploit.

In the sequel, $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N > 2$, with a sufficiently smooth boundary $\partial \Omega$.

**Definition 2.1 (Morrey Space).** Let $p \geq 1$ and $0 \leq \lambda < N$. $L^{p,\lambda}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ such that
\[ \sup_{x_0 \in \Omega, 0 < R \leq d_\Omega} R^{-\lambda} \int_{\Omega \setminus B(x_0, R)} |u|^p \, dx < +\infty \]
where $B(x_0, R)$ is the ball centered at $x_0$ with radius $R$ and $d_\Omega$ is the diameter of $\Omega$.

**Remark 2.2.** We list below some well-known features of Morrey spaces which will be tacitly used throughout the paper.

---

1 Further local properties of solutions depending on local properties of the data are object of study in the forthcoming paper [11].
for some 
and moreover 
if \( p \geq q \) and \( \frac{N - \lambda}{p} \leq \frac{N - \mu}{q} \) then \( L^{p, \lambda}(\Omega) \hookrightarrow L^{q, \mu}(\Omega) \); 
if \( \lambda > N - p \), with \( p > 1 \), then \( L^{p, \lambda}(\Omega) \subset W^{-1, p}(\Omega) \); 
if \( u \in W^{1, p}(\Omega) \) and \( |Du| \in L^{p, \lambda}(\Omega) \), with \( \lambda > N - p \), then \( u \in C^{0, \mu}(\Omega) \) with \( \mu = 1 - \frac{N - \lambda}{p} \).

**Definition 2.3 (Fractional Sobolev Space).** Let \( t \in [0, 1] \) and \( p \geq 1 \). \( W^{t, p}(\Omega) \) is the space of all functions \( u \in L^{p}(\Omega) \) such that

\[
\| u \|_{W^{t, p}(\Omega)} = \| u \|_{p} + \| u \|_{t, p, \Omega} < +\infty
\]

where

\[
[u]_{t, p, \Omega} = \begin{cases} 
\left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p}}{|x - y|^{N + tp}} \, dx \, dy \right)^{\frac{1}{t}} & \text{if } t < 1 \\
\| Du \|_{L^{p}(\Omega)} & \text{if } t = 1.
\end{cases}
\]

We will then need the following technical lemma.

**Lemma 2.4 ([6]).** Let \( \varphi(t) \) be a non–negative and nondecreasing function defined in \([0, \delta]\) such that for every couple \( \rho, R (0 < \rho < R \leq \delta) \) the following inequality

\[
\varphi(\rho) \leq A \left( \frac{\rho}{R} \right)^{\alpha} \varphi(R) + B R^{\beta}
\]

holds with \( A, B, \alpha \) and \( \beta \) positive constants, \( \alpha - \beta > 0 \). Then there exists a positive constant \( C \) depending on \( \lambda \) and \( \alpha - \beta \) such that for every couple \( \rho, R (0 < \rho < R \leq \delta) \), the following inequality

\[
\varphi(\rho) \leq A \left( \frac{\rho}{R} \right)^{\beta} \varphi(R) + C B p^{\beta}
\]

holds.

Next, we state the following Sobolev like embedding Lemma due to D. Adams.

**Lemma 2.5 ([1]).** Let \( m \) be a positive Radon measure supported in \( \Omega \), \( 1 < p < N \), \( q > p \) and \( \sigma = q \left( \frac{N}{p} - 1 \right) \). Assume that there exists \( M > 0 \) such that

\[
m(B(x_{0}, r)) \leq Mr^{\sigma} \quad \text{for all } x_{0} \in \mathbb{R}^{N} \text{ and } r > 0.
\]

Then, there exists a positive constant \( H \) depending on \( p, N, q, M \) such that

\[
\left( \int_{\Omega} |u|^{q} \, dm \right)^{\frac{1}{q}} \leq H \| u \|_{W^{1, p}(\Omega)}
\]

for every \( u \in W^{1, p}_{0}(\Omega) \).

Next lemma concerns the product of two functions belonging to Morrey spaces.

**Lemma 2.6 ([21]).** Let \( N - 2 < \mu < N \), \( E \in L^{2, \mu}(\Omega) \) and \( u \in L^{2, \mu+2}(\Omega) \) such that \( |Du| \in L^{2, \nu}(\Omega) \) for some \( \nu \in [0, N - 2] \). Then

\[
E u \in L^{2, \mu+\nu-2}(\Omega)
\]

and moreover

\[
\| Eu \|_{L^{2, \mu+\nu-2}(\Omega)} \leq C \| E \|_{L^{2, \mu}(\Omega)} \left( \| Du \|_{L^{2, \nu}(\Omega)} + \| u \|_{L^{2, \nu}(\Omega)} \right)
\]

for some \( C > 0 \) independent of \( u \) and \( E \).
Finally, the last result we state is a Sobolev-Morrey embedding Lemma.

**Lemma 2.7 (cfr. [14], [15]).** Assume that $\partial \Omega \in C^1$. Let $u \in W^{1,2}_0(\Omega)$ such that $Du \in L^{2,\nu}(\Omega)$ with $\nu \in ]0, N - 2[$. Then

$$u \in L^{2,\frac{N}{2-\nu}}(\Omega)$$

where

$$\frac{1}{2\nu} = \frac{1}{2} - \frac{1}{N-\nu}$$

and moreover there exists a positive constant $C$ depending on $N, \nu$ such that

$$\|u\|_{L^{2,\frac{N}{2-\nu}}(\Omega)} \leq C \|Du\|_{L^{2,\nu}(\Omega)}.$$  

### 3 Statement of the main results

Let $M : \Omega \to \mathbb{R}^{N^2}$ be a matrix with measurable entries $M_{ij}$ such that

$$a|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta, \quad \text{for a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N$$

(3.4)

We will initially consider the problem (1.1) under the following assumptions on the data

$$f \in L^1(\Omega), \quad |E| \in L^2(\Omega).$$

**Definition 3.1.** By a weak solution of the problem (1.1) we mean a function $u$ such that

$$\begin{cases}
    u \in W^{1,2}_0(\Omega), \\
    |Eu| \in L^2(\Omega)
\end{cases}
$$

for all $\varphi \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$.

Our first result concerns the existence and uniqueness of a bounded solution of the problem (1.1) with data $E$ and $f$ belonging to Morrey spaces having a large Morrey exponent but a lower summability exponent.

**Theorem 3.2.** Assume that (3.4) holds and

$$f \in L^{1,\lambda}(\Omega),$$

(3.6)

$$|E| \in L^{2,\lambda}(\Omega),$$

(3.7)

with $\lambda \in ]N - 2, N[$. Then there exists a unique weak solution $u$ of the problem (1.1) such that

$$u \in L^\infty(\Omega)$$

and

$$\|u\|_{L^\infty(\Omega)} \leq c_1$$

for some positive constant $c_1$ depending on $\alpha, \beta, N, \Omega, \|E\|_{L^{2,\lambda}(\Omega)}, \|f\|_{L^{1,\lambda}(\Omega)}$.

Moreover, there exists $\lambda_0 \in ]N - 2, \lambda[$ such that

$$|Du| \in L^{2,\lambda_0}(\Omega)$$

and

$$u \in C^{0,1 - \frac{N-\lambda_0}{2}}(\Omega)$$
with corresponding norms estimates
\[\|Du\|_{L^{1,\lambda_0}(\Omega)} \leq c_2,\] (3.8)
\[|u|_{C^{0, \frac{\lambda}{2}}(\Omega)}^{1,\lambda_0} \leq c_3,\] (3.9)
where \(c_2, c_3\) are two positive constants depending on \(c_1\) and \(\|u\|_{W^{1,2}_0(\Omega)}\).

**Remark 3.3.** It is worth noticing that, with respect to the results in \([3, 20, 21]\), the boundedness of the solution \(u\) and its Hölder’s continuity have been obtained under weaker assumptions on data (see Remark 2.2).

Namely, in \([3]\) Theorem 5.6 or in \([20]\) Theorem 4.1 and pag. 203, the boundedness of \(u\) is obtained under the hypotheses
\[|E| \in L^m(\Omega), \text{ with } m > N, \ f \in L^r(\Omega), \text{ with } r > N/2;\]
while in \([21]\) Theorem 5.2, Hölder continuity of \(u\) is achieved under the hypotheses
\[|E| \in L^{2,\lambda}(\Omega), \ f \in L^{\frac{2N}{N-\lambda}}(\Omega), \text{ with } \lambda \in ]N-2, N[.\]

**Remark 3.4.** We stress that, for some constants \(c_4, c_5 > 0\), one has
\[|Eu| \leq c_4 |E|^{\frac{\lambda}{2}} + c_5 |u|^{\frac{\lambda}{2}},\]
so that our lower order term lies in the framework of “controlli limite” as described in Campanato’s book \([6]\) pages 122 and 125 (see, in particular, Osservazione 4.1).

For this problem, also in the nonlinear setting, it is proven an \(L^p\)–estimate for \(|Du|\) under the stronger assumption \(|E|^{N/2} \in L^p(\Omega), p > 2\) (see Theorem 4.III, pag. 125 in \([6]\)).

**Remark 3.5.** We observe that if \(|E| = 0\) then our result is consistent with the classical Morrey–Campanato regularity result proved for \(Du\) in Theorem 8.V, pag. 92 of \([6]\), where it assumed that
\[f \in L^{\frac{2N}{N+\lambda}}(\Omega), \quad 0 < \lambda < N.\]

Indeed, by embedding properties of Morrey spaces (see Remark 2.2) one has
\[L^{\frac{2N}{N+\lambda}}(\Omega) \hookrightarrow L^{1, \frac{N-2+\lambda}{2}}(\Omega)\]
where, if \(\lambda > N - 2,\)
\[\frac{N - 2 + \lambda}{2} > N - 2.\]
Consequently our result improves the previous one, at least in the case when \(\lambda > N - 2\) as stated in the following

**Corollary 3.6.** Assume (3.4), \(f \in L^{\frac{2N}{N+\lambda}}(\Omega), \lambda \in ]N - 2, N[\) and \(|E| \in L^{2, \frac{N-2+\lambda}{2}}(\Omega)\).

Then there exists a unique weak solution \(u\) of the problem (1.1) satisfying (3.8) and (3.9).

If we assume
\[f \in L^{\frac{2N}{N+\lambda}}(\Omega), \quad 0 < \lambda < N - 2\] (3.10)
then
\[f \in L^{1, \frac{N-2+\lambda}{2}}(\Omega) \text{ with } \frac{N-2+\lambda}{2} < N - 2\]
and we cannot expect any bounded solution $u$ of the problem (1.1). However, a regularity result similar to Campanato’s one (see [6] pag. 91) can be proved for weak solutions of problem (1.1).

We point out that, in this case, since the datum $f$ has higher integrability (i.e. equal to the duality exponent $\frac{N}{N-2}$) by a weak solution of problem (1.1) we mean a function $u$ such that

$$
\begin{cases}
  u \in W^{1,2}_0(\Omega), & |E u| \in L^2(\Omega) \\
  \int_{\Omega} M(x) Du D\varphi \, dx = \int_{\Omega} E u D\varphi \, dx + \int_{\Omega} f \varphi \, dx \quad \forall \varphi \in W^{1,2}_0(\Omega).
\end{cases}
$$

(3.11)

**Theorem 3.7.** Assume that hypotheses (3.4), (3.10) and

$$
|E| \in L^{2,\mu}(\Omega) \quad N - 2 < \mu < N
$$

(3.12)

hold and let $u$ be a weak solution of the problem (1.1). Then,

i) $Du \in L^{2,\Lambda}(\Omega)$

ii) $u \in L^{2,1,\Lambda}(\Omega)$ where

$$
\frac{1}{\Sigma_1} = \frac{1}{2} - \frac{1}{N-\Lambda}
$$

with corresponding norms estimates.

**Remark 3.8.** We point out that the existence and uniqueness of a weak solution of the problem (1.1) is ensured by the additional assumption $|E| \in L^{N}(\Omega)$ (see [20], [3] and [4]).

We recall that, being $N \geq 3$, in general it is not true that $L^{2,\mu}(\Omega) \hookrightarrow L^{N}(\Omega)$.

**Remark 3.9.** The above theorem improves Theorem 5.1 from [21] where it is proved that $u \in L^{p,\Lambda}(\Omega)$ for any $p \in [1, 2\Lambda]$. Moreover our result provides as well information on the gradient of the solution.

**Remark 3.10.** Observe that if $|E| = 0$ then we retrieve the result of Theorem 8.V, pag. 92 of [6].

Finally we state a Theorem on the fractional differentiability of $Du$. For the sake of brevity we will focus only on the case of lower Morrey exponent. The same calculations can be repeated also in the previous case $\lambda > N - 2$ (see [50], Theorem 1.10 for the case $|E| = 0$).

**Theorem 3.11.** Assume that hypotheses (3.4), (3.10), (3.12) and

$$
M_{ij} \in C^{0,\eta}(\Omega), \quad 0 < \eta \leq 1
$$

hold. Let $u \in W^1_0(\Omega)$ be a weak solution $u$ of (1.1). Then

$$
Du \in W^{1,2}_{\text{loc}}(\Omega)
$$

(13.13)

for every $t \in [0, \eta \delta]$ and for every $\delta \in [0, \min \{1, \frac{1}{\delta}\}]$.

Moreover, for every couple of open subset $\Omega’ \subset \subset \Omega$ there exists a constant $c_6 = c_6(\alpha, \beta, N, \Omega, \|E\|_{L^{2,\mu}(\Omega)}, \|f\|_{L^{\frac{2N}{N-2},\frac{1}{\lambda}}(\Omega)}, \frac{\eta}{\lambda})$, independent of $u$, such that

$$
|Du|^2_{W^{1,2}(\Omega’)} \leq c_6 \left[ \int_{\Omega’} |Du|^2 \, dx + \|Du\|^2_{L^{2,\lambda}(\Omega’)} \right].
$$

(3.14)
Remark 3.12. It is worthwhile to observe that, in the case when \( M_{ij} \), \( |E| \in C^{0,1}(\Omega) \) and \( f \in L^2(\Omega) \subset L^{\frac{2N}{N+2}}(\Omega) \), the weak solution \( u \in W^{2,2}_{loc}(\Omega) \) (see e.g. [24] pag. 183).

A result similar to the aforementioned one is also present in [6], Theorem 1.1 pag. 167, where it is proved that if \( M_{ij} \in C^1(\Omega) \), \( |E| = 0 \) and \( f \in L^{\frac{2N}{N+2}}(\Omega) \) then \( u \in W^{2,2}(\Omega) \).

Previously, in [5] Theorem 10.1 pag. 346, it was proved that if \( M_{ij} \in C^1(\Omega) \), \( |E| = 0 \) and \( f \in L^{2,\lambda}(\Omega) \), with \( 0 < \lambda < N (\Omega) \), then \( D^2 u \in L^{2,\lambda}(\Omega) \).

Finally we notice that, at least formally, if \( \lambda = 0 \) then no fractional differentiability property of \( D\phi \) seems to be achievable.

Further details can also be found in [11–13, 17–19, 25, 26, 31, 37–42, 44, 46–49].

4 Proofs of Theorems 3.2, 3.7 and 3.11.

Proof of Theorem 3.2.
The proof will be performed in several steps.

Step 1 (Global boundedness)
For every \( n \in \mathbb{N} \) and for a.e. \( x \in \Omega \), let us introduce the bounded functions

\[
 f_n(x) = \frac{f(x)}{1 + \frac{1}{n} |f(x)|} \quad \text{and} \quad E_n(x) = \frac{E(x)}{1 + \frac{1}{n} |E(x)|}.\]

Note that

\[
 f_n \to f \quad \text{strongly in } L^1(\Omega),
\]

with

\[
 |f_n|_{L^1(\Omega)} \leq |f|_{L^1(\Omega)},
\]

and

\[
 E_n \to E \quad \text{strongly in } L^2(\Omega),
\]

with

\[
 |E_n|_{L^2(\Omega)} \leq |E|_{L^2(\Omega)}.
\]

We consider the following approximating problems

\[
 \begin{cases}
 u_n \in W^{1,2}_0(\Omega) \\
 - \text{div} (M(x)Du_n) = - \text{div} \left( E_n(x) \frac{u_n}{1 + \frac{1}{n} |u_n|} \right) + f_n(x) \quad \text{in } \Omega.
\end{cases}
\]

(4.17)

Thanks to Schauder fixed point Theorem, for every fixed \( n \in \mathbb{N} \), there exists a weak solution \( u_n \) of the problem (4.17) i.e.

\[
 \begin{cases}
 u_n \in W^{1,2}_0(\Omega), \\
 \int_{\Omega} M(x)Du_n D\varphi \, d x = \int_{\Omega} E_n \frac{u_n}{1 + \frac{1}{n} |u_n|} D\varphi \, d x + \int_{\Omega} f_n \varphi \, d x,
\end{cases}
\]

for every \( \varphi \in W^{1,2}_0(\Omega) \).

Moreover, due to the boundedness of the functions \( E_n(x) \frac{u_n}{1 + \frac{1}{n} |u_n|} \) and \( f_n \), every \( u_n \) is bounded (see [53], [54]).

Next, we prove that the sequence \( \{u_n\} \) is uniformly bounded in \( L^\infty(\Omega) \).

\[ \begin{align}
 \text{Recall that } L^{2,\lambda}(\Omega) & \hookrightarrow L^{\frac{2N}{N+2}}(\Omega) \hookrightarrow L^{\frac{2N}{N-\lambda}}(\Omega) \quad \text{if } \lambda > 0.
\end{align} \]
Let (see [3])

$$\psi(s) = \begin{cases} 
0 & \text{if } |s| \leq k \\
\frac{s^+}{1+n} - \frac{k}{1+n} & \text{if } s > k, \\
\frac{s^-}{1+n} + \frac{k}{1+n} & \text{if } s < -k 
\end{cases}$$

and take $\varphi = \psi(u_n)$ as test function in the weak formulation (4.18). Using Young's inequality and taking into account that $|\psi(u_n)| \leq 1$, we have

$$\int_{\Omega} M(x)|Du_n|^2 |\psi'(u_n)| \, dx \leq \int_{|u_n| \geq k} |E_n| |u_n| |Du_n| |\psi'(u_n)| \, dx + \int_{\Omega} |f_n||\psi(u_n)| \, dx$$

$$\leq \int_{|u_n| \geq k} |E| |Du_n| \, dx + \int_{|u_n| \geq k} |f| \, dx$$

$$\leq \frac{a}{2} \int_{|u_n| \geq k} \frac{|Du_n|^2}{(1 + |u_n|)^2} \, dx + \frac{1}{2\alpha} \int_{|u_n| \geq k} |E|^2 \, dx + \int_{|u_n| \geq k} |f| \, dx$$

(4.19)

Exploiting (3.4) from (4.19) we obtain

$$\frac{a}{2} \int_{|u_n| \geq k} \frac{|Du_n|^2}{(1 + |u_n|)^2} \, dx \leq \frac{1}{2\alpha} \int_{|u_n| \geq k} |E|^2 \, dx + \int_{|u_n| \geq k} |f| \, dx,$$

which implies (for $k = e^{h} - 1$)

$$\frac{a}{2} \int_{\log(1 + |u_n|) > h} |D \log(1 + |u_n|)|^2 \, dx \leq \int_{\log(1 + |u_n|) > h} \left[ \frac{1}{2\alpha} |E|^2 + |f| \right] \, dx.$$  

(4.20)

For any measurable set $A \subseteq \Omega$, the function

$$m(A) := \int_A \left( |E|^2 + |f| \right) \, dx,$$

is a positive Radon measure and, moreover,

$$m(B_r) \leq C r^\lambda \left( ||f||_{L^{3,1}(A)} + ||E||_{L^{3,1}} \right)$$

(4.21)

for every $r > 0$.

Let now $v_n(x) = \log(1 + |u_n(x)|)$ and set

$$G_k(v_n) = v_n - \max \{-k, \min \{v_n, k\} \} \in W^{1,2}_0(\Omega).$$

Due to Lemma 2.5, applied with $\sigma = \lambda$ and $q = \lambda - \frac{2}{N-2}$ (note that $q > 2$ since $\lambda > N-2$), and by virtue of (4.20), (4.21) we deduce the inequality

$$\left( \int_{\Omega} |G_k(v_n)|^q m \right)^{\frac{1}{q}} \leq c_7 \int_{A(k)} |DG_k(v_n)|^2 \, dx = c_7 \int_{A(k)} |D \log(1 + |u_n|)|^2 \, dx \leq c_8 m(A(k))$$

where $A(k) = \{ x \in \Omega : v_n(x) > k \}$, which in turn implies

$$\left( h - k \right)^q \cdot m(A(h)) = \int_{A(h)} |h - k|^q m \leq \int_{\Omega} |G_k(v_n)|^q m \leq c_9 \left[ m(A(k)) \right]^\frac{1}{2}$$

(4.22)

for every $h > k$. 
Thanks to the inequality (4.22) and a well–known Stampacchia’s Lemma we obtain that
\[
\log(1 + |u_n(x)|) \leq L \quad \text{a.e. in } \Omega,
\]
with \( L \) independent of \( n \), and so
\[
|u_n|_{L^\infty(\Omega)} \leq M_1, \quad \forall n \in \mathbb{N}
\]
(4.23)
where the constant \( M_1 \) is independent of \( n \).

Now, choosing \( \varphi = u_n \) in (4.18) and taking into account (3.4), we deduce
\[
\alpha \int_\Omega |Du_n|^2 \, dx \leq \int_\Omega |E_n| \frac{|u_n|^2}{1 + \frac{1}{2}|u_n|} \, dx + \int_\Omega |f_n| |u_n| \, dx \leq C(M_1) \int_\Omega [|E| + |f|] \, dx.
\]
(4.24)
Therefore, for every \( n \in \mathbb{N} \)
\[
|u_n|_{W^{1,2}_0(\Omega)} \leq M_2,
\]
(4.25)
where \( M_2 \) is a positive constant independent of \( n \in \mathbb{N} \).

**Step 2 (Local Morrey regularity)**

In a fixed ball \( B_R \subset \subset \Omega \) we write
\[
u_n(x) = v_n(x) + w_n(x),
\]
where \( w_n \in W^{1,2}_0(B_R) \) is the unique usual weak solution of the problem
\[
\begin{cases}
-\text{div}(M(x) Dw_n) = -\text{div} \left( E_n(x) \frac{u_n}{1 + \frac{1}{2}|u_n|} \right) + f_n(x) & \text{in } B_R \\
w_n = 0 & \text{on } \partial B_R.
\end{cases}
\]
(4.26)
and \( \nu_n \in W^{1,2}(B_R) \) is the usual weak solution of the problem
\[
\begin{cases}
\text{div}(M(x) D\nu_n) = 0 & \text{in } B_R \\
\nu_n = u_n & \text{on } \partial B_R.
\end{cases}
\]
(4.27)

The existence and uniqueness of the solution \( w_n \) of the problem (4.26) is ensured by the Lax–Milgram Theorem.

Moreover the classical Maximum Principle and the boundedness of \( u_n \) provide us
\[
|\nu_n|_{L^\infty(B_R)} \leq |u_n|_{L^\infty(\Omega)}
\]
and consequently
\[
|w_n|_{L^\infty(B_R)} \leq |u_n|_{L^\infty(B_R)} + |\nu_n|_{L^\infty(B_R)} \leq 2|u_n|_{L^\infty(\Omega)}.
\]

Now we choose \( w_n \) as test function in the weak formulation of the problem (4.26) and we use Young’s inequality, (3.4) and the boundedness of \( u_n \) and \( w_n \) we obtain
\[
\alpha \int_{B_R} |Dw_n|^2 \, dx \leq \int_{B_R} |E| |u_n| |Dw_n| \, dx + \int_{B_R} |f| |w_n| \, dx
\]
\[
\leq \frac{\alpha}{2} \int_{B_R} |Dw_n|^2 \, dx + \frac{1}{2\alpha} |u_n|_{L^\infty(\Omega)}^2 \int_{B_R} |E|^2 \, dx + |w_n|_{L^\infty(B_R)} \int_{B_R} |f| \, dx.
\]
(4.28)
Thus, the above inequality and hypotheses (3.6) and (3.7) imply
\[
\int_{B_R} |Dw_n|^2 \, dx \leq c(\alpha, M_1, |E|_{L^2(\Omega)}, |f|_{L^1(\Omega)}) R^4.
\]
(4.29)
On the other hand, it is well known that \( v_N \) satisfies the so-called Saint Venant’s principle (see [6] pag. 91, Theorem 8.1V), that is, there exist two constants \( c_{10} = c_{10}(\alpha, \beta, N) > 0 \) and \( y = y(\beta, N) \in ]0, 1[ \) such that

\[
\int_{B_R} |Dv_N|^2 \, dx \leq c_{10} \left( \frac{P}{R} \right)^{N-2+2y} \int_{B_R} |Dv_n|^2 \, dx, \quad \forall 0 < \rho \leq R. \tag{4.30}
\]

From (4.29) and (4.30) we deduce

\[
\int_{B_R} |Du_n|^2 \, dx \leq c_{11} \left( \frac{P}{R} \right)^{N-2+2y} \int_{B_R} |Dv_n|^2 \, dx + c_{12} \int_{B_R} |Dw_n|^2 \, dx \leq c_{13} \left( \frac{P}{R} \right)^{N-2+2y} \int_{B_R} |Du_n|^2 \, dx \leq c_{14} \left( \frac{P}{R} \right)^{N-2+2y} \int_{B_R} |Du_n|^2 \, dx \leq c_{15} \left( \frac{P}{R} \right)^{N-2+2y} \int_{B_R} |Du_n|^2 \, dx + c_{16} R^4. \tag{4.31}
\]

Taking into account (4.31) and using Lemma 2.4, we obtain

\[
\int_{B_R} |Du_n|^2 \, dx \leq c_{15} (M_1, M_2, ||E||_{L^{1,1}(\Omega)}, ||f||_{L^{1,1}(\Omega)}) R^{\lambda_0}. \tag{4.32}
\]

where \( \lambda_0 = \lambda \) if \( \lambda \in ]N - 2, N - 2 + 2y[ \), otherwise \( \lambda_0 \) is arbitrarily chosen in \( ]N - 2, N - 2 + 2y[ \).

Consequently,

\[
Du_n \in L^{2,\lambda_0}_{loc}(\Omega) \quad \text{and} \quad u_n \in C^{0,\tau}_{loc}(\Omega) \quad \text{with} \quad \tau = 1 - \frac{N - \lambda_0}{2}.
\]

Now the linearity of the equation (4.17), the convergences (4.15) and (4.16), allow us to apply a standard limiting procedure as in [3] to the problems (4.18). Indeed, thanks to the estimates (4.23) and (4.25) up to a subsequence, still denoted by \( \{u_n\}, \{u_n\} \) converges weakly in \( W^{1,2}_0(\Omega) \) and a.e. in \( \Omega \) to a function \( u \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \) such that

\[
\int_{\Omega} M(x)DuD\varphi \, dx = \int_{\Omega} EuD\varphi \, dx + \int_{\Omega} f\varphi \, dx, \quad \forall \varphi \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega).
\]

Moreover, by the lower semicontinuity of norms, taking the limit as \( n \to +\infty \) in the estimate (4.32), \( Du \) satisfies

\[
\int_{B_R} |Du|^2 \, dx \leq c_{15} (M_1, M_2, ||E||_{L^{1,1}(\Omega)}, ||f||_{L^{1,1}(\Omega)}) R^{\lambda_0}. \tag{4.33}
\]

which implies

\[
Du \in L^{2,\lambda_0}_{loc}(\Omega) \quad \text{and} \quad u \in C^{0,\tau}_{loc}(\Omega) \quad \text{with} \quad \tau = 1 - \frac{N - \lambda_0}{2}.
\]

Finally, we remark that the uniqueness of \( u \) is a consequence of Theorem 6.1 of [3], where only the assumption \( ||E|| \in L^2(\Omega) \) was used.

**Step 3** (Boundary Morrey regularity)

Now we prove the regularity of \( |Du| \) up to the boundary of \( \Omega \). For this purpose, following the idea of G. M. Troianiello [55] we will deduce boundary and then global regularity from the previous interior result through an extension technique and successive standard "flattening and covering" arguments.

This technique is illustrated in Lemma 2.18 and in Theorem 2.19 of the cited book. We will reproduce here the main steps for reader’s convenience.

We denote a vector of \( \mathbb{R}^N \) by \( x = (x_1, \cdots, x_{N-1}, x_N) \equiv (x', x_N) \).

If \( y = (y', 0) \) we define

\[
B'_\rho(y) = \{ x \in B(y, \rho) : x_N > 0 \},
\]

\[
B''_\rho(y) = \{ x \in B(y, \rho) : x_N = 0 \}.
\]
Fixed $R_1 > 0$, let $M'$ be a symmetric matrix with bounded coefficients $M'_{ij}$, $E'$ be a vector field and $f'$ be a function defined in $\Omega = B_{R_1}(y)$. We begin by investigating a solution of the problem

\[
\begin{align*}
    u' &\in W^{1,2}(B_{R_1}(y)) \\
    u'|_{\partial B_{R_1}(y)} &\equiv 0 \\
    \int_{B_{R_1}(y)} M'(x) Du' D\varphi \, dx &= \int_{B_{R_1}(y)} E'(x) u' D\varphi \, dx + \int_{B_{R_1}(y)} f' \varphi \, dx
\end{align*}
\]

for all $\varphi \in W^{1,2}_0(B_{R_1}(y)) \cap L^\infty(B_{R_1}(y))$.

We state the following

**Lemma 4.1.** Assume that $M'$, $E'$ and $f'$ satisfy respectively (3.4), (3.6) and (3.7) with $\Omega = B_{R_1}$. Let $u$ be a solution of problem (4.34).

Then, for every $R \in [0, R_1]$, we have

\[
    |Du'| \in L^{2,\lambda_0}(B_R^d)
\]

and there exists a positive constant $c_{16}$ depending only on $\alpha, \beta, N, R_1, \|u'\|_{L^\infty(\Omega)}, \|E\|_{L^{2,1}(\Omega)}, \|f\|_{L^{1,1}(\Omega)}$ such that

\[
    \|Du'\|_{L^{2,\lambda_0}(B_R^d)} \leq c_{16}.
\]

**Proof.** We extend the functions $M'_{ij}(x)$, $E'_{ij}(x)$, $f'$ and $u'$ a.e. to $B_{R_1}(y)$ by setting

\[
\begin{align*}
    \overline{M}_{ij}(x', x_N) &= \begin{cases} M'_{ij}(x', x_N) & \text{if } x_N > 0 \\ -M'_{ij}(x', -x_N) & \text{if } x_N < 0 \end{cases} \\
    \overline{M}_{ij}(x', x_N) &= \begin{cases} M'_{ij}(x', x_N) & \text{if } x_N > 0 \\ -M'_{ij}(x', -x_N) & \text{if } x_N < 0 \end{cases} \\
    \overline{E}_{ij}(x', x_N) &= \begin{cases} E'_{ij}(x', x_N) & \text{if } x_N > 0 \\ -E'_{ij}(x', -x_N) & \text{if } x_N < 0 \end{cases}
\end{align*}
\]

for all the remaining values of $i, j$,

\[
\begin{align*}
    \overline{f}(x', x_N) &= \begin{cases} f'(x', x_N) & \text{if } x_N > 0 \\ -f'(x', -x_N) & \text{if } x_N < 0 \end{cases}
\end{align*}
\]

and finally

\[
\begin{align*}
    \overline{u}(x', x_N) &= \begin{cases} u'(x', x_N) & \text{if } x_N > 0 \\ -u'(x', -x_N) & \text{if } x_N < 0 \end{cases}
\end{align*}
\]

Note that functions $\overline{M}_{ij}$, $\overline{E}$ and $\overline{f}$ satisfy the assumption (3.4), (3.7) and (3.6) in $B_{R_1}(y)$, respectively and $\overline{u} \in W^{1,2}_0(B_{R_1}(y))$.

---

3 $\lambda_0$ is the number introduced in Step 2.
Given a function \( v(x) \), with \( x = (x', x_N) \in B_{R_i}(y) \), we set

\[
\tilde{v}(x', x_N) = v(x', -x_N)
\]

Fixed a function \( v \in C^1_0(B_{R_i}(y)) \), we note that \( v - \tilde{v} \in C^1_0(B_{R_i}^+(y)) \). Therefore, simple calculations show that

\[
\int_{B_{R_i}(y)} M(x) D\tilde{v} \, dx - \int_{B_{R_i}^+(y)} E(x) Dv \, dx = \int_{B_{R_i}^+(y)} M'(x) Du' D(v - \tilde{v}) \, dx - \int_{B_{R_i}^+(y)} E'(x) u' D(v - \tilde{v}) \, dx
\]

\[
= \int_{B_{R_i}^+(y)} f'(v - \tilde{v}) \, dx = \int_{B_{R_i}^+(y)} \tilde{f} v \, dx
\]

and by density argument function \( \tilde{\pi} \) is solution of the problem \( (1.1) \) in \( \Omega = B_{R_i}(y) \) with \( M', E' \) and \( f' \) replaced by \( \tilde{M}, \tilde{E}, \tilde{f} \). Therefore \( \tilde{\pi} \) verifies \((4.33)\) and the Lemma follows by changing back the coordinates.

**Step 4** (Global Morrey regularity)

Now we can prove the global Morrey regularity.

Since \( \partial \Omega \subset C^1 \), for each \( \tilde{y} \in \partial \Omega \) there is a ball \( B_{R_i}(\tilde{y}) \) and a \( C^1(B_{R_i}(\tilde{y})) \)-diffeomorphism \( \Lambda : B_{R_i}(\tilde{y}) \to B_{\alpha_1 R_i}(0) \), which straighten \( \partial \Omega \cap B_{R_i}(\tilde{y}) \) and such that

1. \( \Lambda(y) = 0 \)
2. \( B_{\alpha_1 R_i}(0) \subset \Lambda(B_{R_i}(\tilde{y}) \cap \Omega) \subset B_{\alpha_2 R_i}(0) \) for some \( 0 < \alpha_1 \leq \alpha_2 \).

Put \( R_1 = \alpha_1 R_0 \). If \( z \in B_{R_1}^+(0) \equiv B_{R_i}^+ \), we set

\[
M_i(z) = M_{hk}(\Lambda^{-1}(z)) \frac{\partial \Lambda_i}{\partial y_h}(\Lambda^{-1}(z)) \frac{\partial \Lambda_i}{\partial y_k}(\Lambda^{-1}(z)) J(z)
\]

\[
E_i'(z) = E_i(\Lambda^{-1}(z)) \frac{\partial \Lambda_i}{\partial y_h}(\Lambda^{-1}(z)) J(z)
\]

\[
f'(z) = f(\Lambda^{-1}(z)) J(z)
\]

\[
u'(z) = u(\Lambda^{-1}(z)) J(z)
\]

where \( z = \Lambda(y) \), \( y = \Lambda^{-1}(z) \) and \( J(z) \) denotes the absolute value of the Jacobian determinant of \( \Lambda^{-1} \) at \( z \).

Let us observe that \( M_i' \) belong to \( L^{\infty}(B_{R_i}^+) \), \( E_i' \) belong to \( L^{N}(B_{R_i}^+) \cap L^{2/s}(B_{R_i}^+) \).

Moreover from the definition \((4.37)\), it follows that

\[
M_i' J \xi_i J \xi_j = M_{hk} \frac{\partial \Lambda_i}{\partial y_h} \frac{\partial \Lambda_j}{\partial y_k} \xi_i \xi_j J \geq \alpha \left( \sum_{h=1}^{N} \left( \sum_{i=1}^{N} \frac{\partial \Lambda_i}{\partial y_h} \xi_i \right)^2 \right) \min_{B_{R_i}^+} J \geq \alpha |\xi|^2
\]

for all \( \xi \in \mathbb{R}^N \).

Thus, a change of variables in \((1.1)\) yields

\[
\begin{aligned}
&\begin{cases}
    u' \in W^{1,2}(B_{R_i}^+)
    \\
    u'_{|B_{R_i}^+}(0) = 0
  \end{cases}

&\int_{B_{R_i}^+} M'(z) Du' D\varphi \, dz = \int_{B_{R_i}^+} E'(z) u' D\varphi \, dz + \int_{B_{R_i}^+} f' \varphi \, dz
\end{aligned}
\]

\[
\text{for all } \varphi \in W^{1,2}_0(B_{R_i}^+) \cap L^{\infty}(B_{R_i}^+).
\]

To \((4.38)\) we apply Lemma 4.1 and thus we get the membership of \( Du' \) in \( L^{2,1}(B_{R_i}^+) \), \( 0 < R < R_1 \), with norm estimate \((4.35)\).

We extend \( Du' \) a.e. to \( B_R \) setting \( Du'(x', x_N) = Du'(x', -x_N) \) if \( x_N < 0 \). Thus \( Du' \in L^{2,1}(B_R) \).
As a consequence, since for some \( r > 0 \) \( B_r(\overline{y}) \cap \Omega \subset A^{-1}(B^r_0) \), the vector–function \( Du'(A(y)) \), \( y \in B_r(\overline{y}) \) belongs to \( L^{2,1}(B_r(\overline{y})) \) that is, by the chain rule, \( Du \) belongs to \( L^{2,1}(\Omega \cap B_r(\overline{y})) \); moreover, by virtue of \((4.37)\) and \((4.35)\),

\[
||Du||_{L^{2,1}(B_r(\overline{y}) \cap \Omega)}^2 \leq c.
\]

Because \( \partial \Omega \) is compact, there is a finite number of balls such as \( B_r(\overline{y}) \), say \( B^1, B^2, ..., B^m \), which cover \( \partial \Omega \). Moreover, there exists an open set \( \Omega \setminus \bigcup_{i=1}^m B_i \subset H_0 \subset \Omega \) such that \( H_0, B^1, B^2, ..., B^m \) cover \( \Omega \).

If \( \{g_i\}_{i=0, 1, ..., m} \) is a partition of the unity relative to the above covering then it turns out

\[
||Du||_{L^{2,1}(\Omega)} \leq c_{18} \left[ ||Du||_{L^{2,1}(H_0)} + \sum_{i=1}^m ||Du||_{L^{2,1}(B^i(\overline{y}) \cap \Omega)} \right].
\]

Then \((3.8)\) follows from \((4.40)\) by joining together \((4.39)\) and the interior estimate \((4.32)\) of Step 2.

\(\square\)

**Proof of Theorem 3.7.**

Let \( u \in W^{1,2}_0(\Omega) \) be a weak solution of the problem \((1.1)\) in the weak sense \((3.11)\). As we have done in the proof of the previous theorem we fix \( B_R \subset \subset \Omega \) and we set

\[
u(x) = w(x) + v(x), \quad x \in B_R,
\]

where \( w \in W^{1,2}_0(B_R) \) and \( v \in W^{1,2}(B_R) \) are respectively solutions of the following boundary problems

\[
\begin{aligned}
&-\text{div}(M(x)Dw) = -\text{div}(E(x)u) + f \quad \text{in } B_R, \\
&w = 0 \quad \text{on } \partial B_R,
\end{aligned}
\]

and

\[
\begin{aligned}
&\text{div}(M(x)Dv) = 0 \quad \text{in } B_R, \\
&v = u \quad \text{on } \partial B_R.
\end{aligned}
\]

Choosing \( w \) as test function in the weak formulation of the problem \((4.41)\) and using hypothesis \((3.4)\), by standard calculations, we obtain

\[
a \int_{B_R} |Dw|^2 \, dx + \frac{1}{2a} \int_{B_R} |Eu|^2 \, dx + a \int_{B_R} |Dw|^2 \, dx + C(\sigma) \left( \int_{B_R} |f|^\frac{2^*}{2^* - 2} \, dx \right)^{\frac{N+2}{N}} + \sigma S \int_{B_R} |Dv|^2 \, dx, \quad \forall \varepsilon, \sigma > 0.
\]

\(\text{(4.43)}\)

where \( S \) is the Sobolev’s constant and \( \sigma \) is a positive constant chosen sufficiently small in order to get

\[
\int_{B_R} |Dw|^2 \, dx \leq c_{19} \int_{B_R} |Eu|^2 \, dx + c_{20} \left( \int_{B_R} |f|^\frac{2^*}{2^* - 2} \, dx \right)^{\frac{N+2}{N}}.
\]

\(\text{(4.44)}\)

As in \((4.30)\) the solution \( v \) of the problem \((4.27)\) satisfies the Saint Venaint’s principle; therefore, by \((4.44)\), we deduce for every \( 0 < \rho < R \)

\[
\int_{B_\rho} |Du|^2 \, dx = c_{21} \int_{B_\rho} |Dv|^2 \, dx + c_{22} \int_{B_\rho} |Dv|^2 \, dx
\]

\[
\leq c_{23} \left( \frac{\rho}{R} \right)^{N-2+2y} \int_{B_R} |Dv|^2 \, dx + c_{24} \int_{B_R} |Eu|^2 \, dx + c_{25} \left( \int_{B_R} |f|^\frac{2^*}{2^* - 2} \, dx \right)^{\frac{N+2}{N}}
\]

\[
\leq c_{26} \left( \frac{\rho}{R} \right)^{N-2+2y} \int_{B_\rho} |Du|^2 \, dx + c_{27} \int_{B_\rho} |Eu|^2 \, dx + c_{28} \left( \int_{B_\rho} |f|^\frac{2^*}{2^* - 2} \, dx \right)^{\frac{N+2}{N}}.
\]

\(\text{(4.45)}\)
We point out that \( u \in L^2(\Omega) \subset L^{2,2}(\Omega) \). Consequently, by virtue of Lemma 2.6
\[
Eu \in L^{2,\mu_0}(\Omega) \quad \text{with} \quad \mu_0 = \mu - N + 2
\]
and from (4.45) we obtain
\[
\int_{B_R} |Du|^2 \, dx \leq c_{26} \left( \frac{p}{R} \right)^{N-2+2\gamma} \int_{B_R} |Du|^2 \, dx + c_{27} \|Eu\|^2_{L^{2,\mu_0}(\Omega)} R^{\mu_0} + c_{28} \|f\|^2_{L^{\frac{2N}{N+2\gamma}}(\Omega)} R^1
\]
\[
\leq c_{26} \left( \frac{p}{R} \right)^{N-2+2\gamma} \int_{B_R} |Du|^2 \, dx + c_{29} \left( \|Eu\|_{L^{2,\mu_0}(\Omega)} \right)^2 \left( \|f\|_{L^{\frac{2N}{N+2\gamma}}(\Omega)} \right) R^{\mu_1} (4.46)
\]
where
\[
\mu_1 = \min \{ \mu_0, \lambda \} < N - 2.
\]

Now, iterating the above inequality (see Campanato (cfr. [6])), we establish
\[
|Du| \in L^{2,\mu_1}_{\text{loc}}(\Omega).
\]
and
\[
||Du||^2_{L^{2,\mu_1}(\Omega)} \leq c_{30} \left[ ||Du||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2\gamma}}(\Omega)} \right] \quad (4.48)
\]
with \( c_{30} \) a positive constant independent of \( u, E \) and \( f \).

Next, using the extension technique and the successive standard “flattening and covering” arguments as we have done in Steps 3 and 4 of proof of Theorem 3.2, we can prove the regularity of \( Du \) up to the boundary of \( \Omega \), with the norm estimate
\[
||Du||^2_{L^{2,\mu_1}(\Omega)} \leq c_{31} \left[ ||Du||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2\gamma}}(\Omega)} \right]. \quad (4.49)
\]

Now, we compare \( \mu_0 \) with \( \lambda \). If \( \mu_0 \geq \lambda \) then \( \mu_1 = \lambda \) and the thesis follows.
Otherwise \( \mu_1 = \mu_0 \) and we can apply Lemma 2.7 to the function \( u \in W^{1,2}_0(\Omega) \); since \( Du \in L^{2,\mu_1}(\Omega) \) we obtain
\[
u \in L^{2,\mu_1,\mu_1}(\Omega) \quad \text{where} \quad \frac{1}{2\mu_1} = \frac{1}{2} - \frac{1}{N-\mu_1}.
\]
But \( L^{2,\mu_1,\mu_1}(\Omega) \) is embedded into \( L^{2,2+\mu_1}(\Omega) \) therefore a new application of Lemma 2.6 gives us
\[
Eu \in L^{2,\mu_0+\mu_1}(\Omega)
\]
with the norm estimate
\[
||Eu||_{L^{2,\mu_0+\mu_1}(\Omega)} \leq c_{32} ||Eu||_{L^{2,\mu_1}(\Omega)} \left( ||Du||_{L^{2,\mu_1}(\Omega)} + ||u||_{L^{2,2+\mu_1}(\Omega)} \right) \quad (4.50)
\]
for some \( c_{32} > 0 \) independent of \( u \) and \( E \).

Using in (4.45) the improved norm estimate (4.50) of \( |Eu| \), we have
\[
\int_{B_R} |Du|^2 \, dx \leq c_{26} \left( \frac{p}{R} \right)^{N-2+2\gamma} \int_{B_R} |Du|^2 \, dx + c_{33} ||Eu||^2_{L^{2,\mu_0+\mu_1}(\Omega)} R^{\mu_0+\mu_1} + c_{34} \|f\|^2_{L^{\frac{2N}{N+2\gamma}}(\Omega)} R^1
\]
\[
\leq c_{26} \left( \frac{p}{R} \right)^{N-2+2\gamma} \int_{B_R} |Du|^2 \, dx + c_{35} \left( ||Eu||_{L^{2,\mu_0+\mu_1}(\Omega)} \right)^2 \left( ||f||_{L^{\frac{2N}{N+2\gamma}}(\Omega)} \right) R^{\mu_2} (4.51)
\]
where
\[
\mu_2 = \min \{ \mu_0 + \mu_1, \lambda \} < N - 2.
\]
As usually Lemma 2.4 provides us
\[
Du \in L^{2,\mu_1}_{\text{loc}}(\Omega)
\]
and as in the previous step we deduce the regularity of $Du$ up to the boundary of $\Omega$, that is

$$Du \in L^{2,\mu_1}(\Omega)$$

and the corresponding estimate

$$||Du||^2_{L^{2,\mu_1}(\Omega)} \leq c_{36} \left[ ||Du||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0+\eta_1}(\Omega)} + ||f||^2_{L^{2,N-\frac{1}{2}}(\Omega)} \right]$$

(4.52)

where $c_{36} > 0$ is a constant independent of $u$, $E$ and $f$.

Iterating the previous procedure and setting for every $k = 1, 2, \ldots$

$$\mu_k = \min \{ \mu_0 + \mu_{k-1}, \lambda \} < N - 2,$$

(4.53)

it follows

i) $Eu \in L^{2,\mu_0+\mu_1}(\Omega)$, with the norm estimate

$$||Eu||_{L^{2,\mu_0+\mu_1}(\Omega)} \leq C_k ||E||^{\frac{k}{\nu}}_{L^{\nu}(\Omega)} \left[ ||Du||_{L^{2,\mu_k}(\Omega)} + ||u||_{L^{2,\mu_k}(\Omega)} \right]$$

for some constant $C_k > 0$ independent of $u$ and $E$;

ii) $\int_{B_k} |Du|^2 \, dx \leq c_{26} \left( \frac{P}{R} \right)^{N-2+2\gamma} \int_{B_k} |Du|^2 \, dx + c_{37} \left( ||Eu||^2_{L^{2,\mu_0+\eta_1}(\Omega)} + ||f||^2_{L^{2,N-\frac{1}{2}}(\Omega)} \right) R^{\mu_{k+1}}$

iii) $Du \in L^{2,\mu_0+\mu_1}(\Omega)$, with norm estimate

$$||Du||^2_{L^{2,\mu_0+\mu_1}(\Omega)} \leq c_{38} \left[ ||Du||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0+\eta_1}(\Omega)} + ||f||^2_{L^{2,N-\frac{1}{2}}(\Omega)} \right]$$

for some constant $c_{38} > 0$ independent of $u$, $E$ and $f$.

After a finite number of steps we will have $\mu_0 + \mu_k > \lambda$, that is $\mu_{k+1} = \lambda$, and so $Du \in L^{2,\lambda}(\Omega)$.

Finally, a further application of Lemma 2.7 provides us

$$u \in L^{2,\lambda}(\Omega)$$

with $\frac{1}{z_3} = \frac{1}{2} - \frac{1}{N-3}$.

Proof of Theorem 3.11.

We will proceed as in the proof of Theorem 4 of [16].

Let $B \subset\subset \Omega$ be a ball of radius $R$ and let $\tilde{B}$ be the enlarged ball of radius $32R$. We shall denote by $Q_{inn}(B)$ and $Q_{out}(B)$ the largest and the smallest cubes, concentric to $B$ and with sides parallel to the coordinate axes, contained in $B$ and containing $B$ respectively. If we put

$$Q_{inn} = Q_{inn}(B), \quad Q_{out} = Q_{out}(B)$$

and

$$\tilde{Q}_{inn} = Q_{inn}(\tilde{B}), \quad \tilde{Q}_{out} = Q_{out}(\tilde{B}),$$

we have the following inclusions

$$Q_{inn} \subset B \subset\subset 4B \subset\subset \tilde{Q}_{inn} \subset \tilde{B} \subset \tilde{Q}_{out} \quad \text{4.54}.$$ 

Let $\Omega'$ and $\Omega''$ be a couple of open subset such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and $x_0 \in \Omega'$. For any $r \in ]0, 1[$ (that will be chosen later) we fix $h \in \mathbb{R}$ with $0 < |h| < \min \{ 1, d(\Omega', \partial\Omega'') \}$ such that, denoted with $B = B(x_0, |h|^r)$ the ball centered in $x_0$ and with radius $|h|^r$, the outer cube of $B$, $\tilde{Q}_{out}$ is included in $\Omega''$. 

4 With $KB$ we denote the ball with radius $KR$, $K \in \mathbb{N}$. 
Let \( v \in W^{1,2}(\hat{B}) \) be the unique weak solution to the problem

\[
\begin{cases}
\text{div}(M(x)Dv) = 0 & \text{in } \hat{B} \\
v = u & \text{on } \partial \hat{B},
\end{cases}
\]

(4.55)

and let \( v_0 \in W^{1,2}(8B) \) be the unique weak solution to the problem

\[
\begin{cases}
\text{div}(M(x_0)Dv_0) = 0 & \text{in } 8B \\
v_0 = v & \text{on } \partial 8B.
\end{cases}
\]

(4.56)

Then we have

\[
\int_{\hat{B}} |\tau_{ih}(Du)|^2 \, dx \leq c_{39} \left[ \int_{B} |\tau_{ih}(Dv_0)|^2 \, dx + \int_{\hat{B}} |Du - Dv|^2 \, dx + \int_{8B} |Dv - Dv_0|^2 \, dx \right]
\]

(4.57)

The first term on the right-hand side of (4.57) can be estimated as

\[
\int_{B} |\tau_{ih}(Dv_0)|^2 \, dx \leq c_{40} |h|^{2(1-\tau)} \int_{8B} |Dv_0 - z_0|^2 \, dx,
\]

(4.58)

for all \( z_0 \in \mathbb{R}^n \) (see [50]).

The third term on the right-hand side of (4.57) can be estimated as

\[
\int_{8B} |Dv - Dv_0|^2 \, dx \leq c_{41} |h|^{2\tau} \int_{\hat{B}} |Du|^2 \, dx
\]

(4.59)

(see [50]).

Finally, we estimate

\[
\int_{\hat{B}} |Dv - Dv_0|^2 \, dx.
\]

Let us observe that the function \( w = v - u \in W^{1,2}_0(\hat{B}) \) is the weak solution to the equation

\[
\text{div}[M(x)D(w + u)] = 0 \quad \text{in } \hat{B},
\]

(4.60)

whence, by assumption (3.4), we deduce

\[
\begin{align*}
\int_{\hat{B}} |Dv - Dv_0|^2 \, dx & = \int_{\hat{B}} |Dw|^2 \, dx \leq \frac{1}{\alpha} \int_{\hat{B}} |M(x)||Dw| \, dx \leq \frac{\beta}{\alpha} \int_{\hat{B}} |Du||Dw| \, dx \\
& \leq c_{42} \left[ \varepsilon \int_{\hat{B}} |D(u - v)|^2 \, dx + C(\varepsilon) \int_{\hat{B}} |Du|^2 \, dx \right]
\end{align*}
\]

(4.61)

with \( \varepsilon, C(\varepsilon) \) positive constants independent of the radius of \( \hat{B} \).

In turn, for a sufficiently small \( \varepsilon \), inequality (4.61) yields

\[
\int_{\hat{B}} |Dv - Dv_0|^2 \, dx \leq c_{43} \int_{\hat{B}} |Du|^2 \, dx.
\]

(4.62)

From this point on, we gather together inequalities (4.57), (4.58), (4.59) and (4.62) and we can argue as in the proof of Theorem 4 in [16], exploiting the method introduced in [50].

\[ \square \]

**Acknowledgments:** This work has been supported by Piano della Ricerca 2016-2018–linea di intervento 2: "Metodi variazionali ed equazioni differenziali".
References

[1] D. Adams, *A note on Riesz potentials*, Duke Math. J., 42, pp.765–778 (1975).
[2] B. Avelin, T. Kuusi, G. Mingione, *Nonlinear Calderón–Zygmund theory in the limiting case*, Arch. for Rat. Mech. and Anal., 227 (2018), pp. 663–714.
[3] L. Boccardo, *Some developments on Dirichlet problems with discontinuous coefficients*, Boll. Un. Mat. Italiana (9) 2, n. 1, (2009), pp. 285–297.
[4] L. Boccardo, S. Buccheri, G. R. Cirmi *Two linear noncoercive Dirichlet problems in duality*, Milan J. Math.,(2018), doi.org/10.1007/s00032-018-0281-5.
[5] S. Campanato, *Equazioni ellittiche del II° ordine e spazi L^2,1*, Ann. Mat. Pura e Appl., 69, (1965).
[6] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all’interno*. Quaderni SNS, Pisa, 1980.
[7] P. Cianci, G. R. Cirmi, S. D’Asero and S. Leonardi, *Morrey estimates for solutions of singular quadratic non linear equations*, Annali di Matematica Pura e Applicata, 196, n. 5, (2017).
[8] G. R. Cirmi, S. D’Asero and S. Leonardi, *Fourth-order nonlinear elliptic equations with lower order term and natural growth conditions*, Nonlinear Analysis: TMA, 108, (2014), pp. 66-86.
[9] G. R. Cirmi, S. D’Asero and S. Leonardi, *Gradient estimate for solutions of a class of nonlinear elliptic equations below the duality exponent*, AIP Conference Proceedings, 1863, (2017).
[10] G. R. Cirmi, S. D’Asero and S. Leonardi, *Gradient estimate for solutions of nonlinear singular elliptic equations below the duality exponent*, Math. Meth. in Appl. Sci., 41, n. 1, (2018).
[11] G. R. Cirmi, S. D’Asero, S. Leonardi, M. M. Porzio, *Local regularity results for solutions of linear elliptic equations with drift term*, Advances in Calculus of Variations, https://doi.org/10.1515/acv-2019-0048.
[12] G. R. Cirmi and S. Leonardi, *Higher differentiability for solutions of linear elliptic systems with measure data*, Discrete and Continuous Dynamical Systems, vol. 26, n. 1, (2010), pp. 89–104.
[13] G. R. Cirmi, S. Leonardi, and J. Stará, *Regularity results for the gradient of solutions of a class of linear elliptic systems with L^{1,\lambda} data*, Nonlinear Analysis: Theory, Methods Applications, 68 n. 12, (2008), pp. 3609–3624.
[14] G. R. Cirmi, S. Leonardi, *Regularity results for the gradient of solutions of linear elliptic equations with L^{1,\lambda} data*, Annali di Mat. Pura e Appl., (4) vol. 185, (2006).
[15] G. R. Cirmi, S. Leonardi, *Regularity results for solutions of nonlinear elliptic equations with L^{1,\lambda} data*, Nonlinear Anal. TMA, Vol 69, N.1, pp. 230–244 (2008).
[16] G. R. Cirmi and S. Leonardi, *Higher differentiability for the solutions of nonlinear elliptic systems with lower order terms and L^{1,\lambda} data*, Ann. Mat. Pura e Appl., (4), 193 n. 1, (2014).
[17] S. D’Asero, *On Harnack inequality for degenerate nonlinear higher-order elliptic equations*, Appl. Anal. 85, no. 8, pp. 971–985 (2006).
[18] S. D’Asero, *On removability of the isolated singularity for solutions of high-order elliptic equations*, Complex Var. Elliptic Equ. 55 , no. 5-6, pp. 525–536 (2010).
[19] S. D’Asero and D.V. Larin, *Degenerate nonlinear higher-order elliptic problems in domains with fine-grained boundary*, Nonlinear Analysis T. M. A., 64, pp. 788–825 (2006).
[20] T. Del Vecchio, R. M. Posteraro, *An existence result for nonlinear and noncoercive problems*, Nonlinear Analysis, T. M. A, vol. 31, n. 1/2, (1998), pp. 191–206
[21] G. Di Fazio, *On Dirichlet problem in Morrey spaces*, Differential Integral Equations 6, no. 2 (1993), pp. 383–391.
[22] P. Drabek, F. Nicolosi, K. Schmitt, *Semilinear boundary value problems at resonance with general nonlinearities*, Differential and Integral Equations, 5, Issue 2, (1992), pp. 339–355.
[23] L. Esposito, F. Leonetti and G. Mingione, *Sharp regularity for functional with (p, q)-growth*, J. Differential Equations, 204, (2004), pp.5-55.
[24] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, II Edit., Springer Verlag 1983.
[25] W. Hao, S. Leonardo and M. Steinbauer, *Examples of discontinuous divergence-free solutions to elliptic variational problems*, Comm. Math. Univ. Carolinae 36, 3 (1995).
[26] W. Hao, S. Leonardo and J. Nečas, *An example of irregular solution to a nonlinear Euler-Lagrange elliptic system with real analytic coefficients*, Annali Scuola Normale Superiore Pisa (IV) XXIII, 1 (1996).
[27] A. Kovalevsky, F. Nicolosi, *On regularity up to the boundary of solutions to degenerate nonlinear elliptic high-order equations* Nonlinear Analysis, Theory, Methods and Applications, 40, Issue 1, (2000), pp. 365–379
[28] J. Kristensen and G. Mingione, *The singular set of ω-minima*, Archive for Rational Mechanics & Analysis 177, (2005), pp. 93–114.
[29] J. Kristensen and G. Mingione, *The singular set of minima of integral functionals*, Archive for Rational Mechanics & Analysis 180, (2006), pp. 331–398.
[30] J. Kristensen and G. Mingione, *Boundary regularity in variational problems*, Archive for Rational Mechanics & Analysis 198, (2010), pp. 369–455.
[31] A. Kufner and S. Leonardi, *Solvability of degenerate elliptic boundary value problems: another approach*, Mathematica Bohemica 119, no. 3, pp. 255–274 (1994).
[32] T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal. **262**, (2012), pp. 4205–4269.

[33] T. Kuusi and G. Mingione, *Linear potentials in nonlinear potential theory*, Archive for Rational Mechanics & Analysis **207**, (2013), pp. 215–246.

[34] T. Kuusi and G. Mingione, *Guide to nonlinear potential estimates*, Bull. Math. Sci. **4** (2014), pp. 1–82.

[35] T. Kuusi and G. Mingione, *The Wolff gradient bound for degenerate parabolic equations*, J. Europ. Math. Soc. (JEMS) **16**, (2014), pp. 835–892.

[36] T. Kuusi and G. Mingione, *Vectorial nonlinear potential theory*, J. Europ. Math. Soc. **20** (2018), pp. 929–1004.

[37] S. Leonardi, *On constants of some regularity theorems. De Giorgi’s type counterexample*, Mathematische Nachrichten **192**, n. 1, pp. 191–204 (1998).

[38] S. Leonardi, *Remarks on the regularity of solutions of elliptic systems*, Applied nonlinear analysis, pp. 325–344, Kluwer/Plenum, New York, 1999.

[39] S. Leonardi, *Weighted Miranda-Talenti inequality and applications to equations with discontinuous coefficients*, Commentationes Mathematicae Universitatis Carolinae, **43** n. 1, (2002), pp. 43–60.

[40] S. Leonardi, *A maximum principle for linear elliptic systems with discontinuous coefficients*, Comment. Math. Univ. Carolin. **45**, n. 3, (2004), pp. 457–474.

[41] S. Leonardi, *Gradient estimates below duality exponent for a class of linear elliptic systems*, Nonlinear Differ. Equ. Appl., **18**, n. 3, (2011).

[42] S. Leonardi, *Fractional differentiability for solutions of a class of parabolic systems with $L^{1,\theta}$ data*, Nonlinear Analysis: TMA, **95**, (2014), pp. 530–542.

[43] S. Leonardi, *Morrey estimates for some classes of elliptic equations with a lower order term*, Nonlinear Analysis TMA, **177**, part. B, (2018), pp. 611–627.

[44] S. Leonardi, J. Kottas and J. Stará, *Hölder regularity of the solutions of some classes of elliptic systems in convex non smooth domains*, Nonlinear Anal. TMA **60**, (5) (2005).

[45] S. Leonardi, F. Leonetti, C. Pignotti, E. Rocha, V. Staicu, *Maximum principles for some quasilinear elliptic systems*, Nonlinear Analysis TMA, https://doi.org/10.1016/j.na.2018.11.004.

[46] S. Leonardi and J. Stará, *Regularity results for the gradient of solutions of linear elliptic systems with VMO–coefficients and $L^1$ data*, Forum Math. **22**, 5 (2010).

[47] S. Leonardi and J. Stará, *Regularity up to the boundary for the gradient of solutions of linear elliptic systems with VMO coefficients and $L^{1,\lambda}$ data*, Complex Variables and Elliptic Equ., vol. **55**, n. 6, (2011).

[48] S. Leonardi and J. Stará, *Regularity results for solutions of a class of parabolic systems with measure data*, Nonlinear Analysis **75**, (2012), pp. 2069-2089.

[49] S. Leonardi and J. Stará, *Higher differentiability for solutions of a class of parabolic systems with $L^{1,\theta}$–data*, The Quarterly J. of Math., **66**, n. 2, (2015), pp. 659-676.

[50] G. Mingione, *The Calderón-Zygmund theory for elliptic problems with measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), Vol. VI, (2007), pp. 195–261.

[51] G. Mingione, *Gradient estimates below the duality exponent*, Math. Ann. **346**, (2010), pp. 571–627.

[52] G. Mingione, *Gradient potential estimates*, J. Europ. Math. Soc. (JEMS), Vol. **13**, (2011), pp. 459–486.

[53] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier, Grenoble, Vol. **15**, 1 (1965).

[54] G. Stampacchia, *Équations elliptiques du second ordre à coefficients discontinus*, Université de Montréal, (1966).

[55] G. M. Troianiello *Elliptic Differential Equations and Obstacle Problems*, The University Series in Mathematics. Plenum Press, New York, 1987.