Strong backward uniqueness for sublinear parabolic equations

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Abstract. In this paper, we establish strong backward uniqueness for solutions to sublinear parabolic equations of the type (1.1). The proof of our main result Theorem 1.3 is achieved by means of a new Carleman estimate and a Weiss type monotonicity formula that are tailored for such parabolic sublinear operators.

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1. Introduction and the statement of the main result

The purpose of this work is to establish strong backward uniqueness for parabolic sublinear equations of the type

\[
\text{div}(A(x,t)\nabla v) + v_t + Wv + h((x,t),v) = 0 \quad \text{in } \mathbb{R}^n \times [0,T),
\]

where

\[
\|W\|_{L^\infty} \leq M,
\]

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and the matrix $A$ is symmetric, uniformly elliptic and satisfies
\begin{equation}
\lambda |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n.
\end{equation}
Furthermore, we assume that,
\begin{equation}
|\nabla a_{ij}(x,t)| \leq \frac{M}{1 + |x|}, |\partial_t a_{ij}(x,t)| \leq M.
\end{equation}
On the sublinear term $h$, we assume the following.
\begin{align}
h((x,t), 0) &= 0, \\
H((x,t), s) &= \int_0^s h(X, s) ds, \\
0 &< sh\left((x,t), s \right) \leq qH\left((x,t), s \right) \quad \text{for some } q \in [1,2), \\
H((x,t), s) &\geq \varepsilon_0 \quad \text{for all } |s| > 1 \text{ and some } \varepsilon_0 > 0, \\
|\nabla_s H| &\leq \frac{C_0}{1 + |x|} H, \\
|\partial_t H| &\leq C_0 H, \\
h\left((x,t), s \right) &\leq C_0 \sum_{i=1}^m |s|^{p_i - 1} \quad \text{for } p_s \in [1,2) \text{ and some } C_0 > 0.
\end{align}
We note that from (1.5) it follows that given $L > 0$, there exists $c_0 = c_0(L) > 0$, such that
\begin{equation}
H\left((x,t), s \right) \geq c_0 |s|^q \quad \text{for } |s| < L.
\end{equation}
A prototypical $h$ satisfying (1.5) is given by
\begin{equation}
h((x,t), v) = \sum_{i=1}^t c_i(x,t)|v|^{p_i - 2}v,
\end{equation}
where for each $i$, $p_i \in [1,2)$, $0 < k_0 < c_i < k_1$, $|\nabla_x c_i| < \frac{C_0}{1 + |x|}$ and $|\partial_t c_i| < C_0$ for some $k_0, k_1$ and $C_0$. In this case, we can take $q = \max\{p_i\}$. In order to put things in the right perspective, we note that motivated by the study of nonlinear eigenvalue problems as well as the analysis of corresponding nodal domains as in [22] and also because of certain connections to porous media type equations (as in [27]), Soave and Weth in [25] established weak unique continuation for equations of the type
\begin{equation}
\text{div}(A(x)\nabla v) + h(x,v) + Wv = 0.
\end{equation}
Such equations are modeled on
\begin{equation}
-\Delta v = |v|^{p-2}v.
\end{equation}
Note that the study of strong unique continuation for (1.8) cannot be reduced to that for
\begin{equation}
-\Delta + W
\end{equation}
where the known results apply because in this case, $W = |v|^{p-2}$ need not be in $L^p$ for any $p$ near the zero set of $v$ as $p \in (1,2)$ and the known results
require $W \in L^{n/2}$ (see [13]). In fact such sublinear equations have their intrinsic difficulties and this is also partly visible from the fact that the sign assumption on the sublinearity $h$ in (1.5) is crucial because otherwise unique continuation fails. This later fact follows from a counterexample in [25] where it is shown that unique continuation is not true for

$$\Delta v = |v|^{p-2}v, \quad p \in (1, 2).$$

(1.9)

In [25], the authors adapted the frequency function approach of Garofalo and Lin as in [10]. The question of strong unique continuation for such sublinear equations was then later addressed by Ruland in [23] via new Carleman estimates for such sublinear elliptic operators. Such a result was generalized to degenerate Baouendi–Grushin type operators by one of us with Garofalo and Manna in [4]. We also refer to the interesting work of Soave and Terracini in [24], where the authors study the following two phase membrane problem

$$-\Delta v = \lambda_+ (v^+)^{q-1} - \lambda_- (v^-)^{q-1}, \quad \text{where } \lambda_+, \lambda_- > 0, q \in [1, 2)$$

(1.10)

and established strong unique continuation property as well as a regularity result for the nodal domains. The key object in their analysis was a new monotonicity formula for a 2-parameter family of Weiss type functionals first introduced by Weiss in his seminal work [29] in the context of classical obstacle problem. See also the very recent preprint [26] for related results on fractional sublinear problems. The space like strong unique continuation for such backward parabolic sublinear operators as in (1.1) has been obtained more recently by one of us with Manna in [5] by a generalization of the Carleman estimates in [8,9] to the sublinear situation. More precisely in [5] it is shown that if a solution $u$ to (1.1) in $B_1 \times [0, T)$ vanishes to infinite order in the sense of Definition 1.11 at $(0,0)$, then $u(\cdot, 0) \equiv 0$. We note that such a space like vanishing result is optimal for local solutions in view of an example due to Frank Jones in [12] (see Remark 1.5 below). In this paper, we complement such a result by establishing a strong backward uniqueness result in this framework, i.e. we show that global solutions to (1.1) in $\mathbb{R}^n \times [0, T)$ under some growth assumptions at infinity vanish identically in $\mathbb{R}^n \times [0, T)$ provided they vanish to infinite order at a point. Our main result Theorem 1.3 also constitutes the sublinear counterpart of the backward uniqueness result recently obtained by Wu and Zhang in [30] for linear equations with similar structural assumptions (see also [31]). Similar to that in [5], the proof of our result relies on a careful generalization of the Carleman estimate in [30] to the sublinear case. Also when the principal part $A = I$, we obtain a strong backward uniqueness result when the structure condition (1.5) holds for $p_i, q$ in the bigger range $[1, 2)$ (see Theorem 1.3 ii) below). This is achieved by means of a new parabolic Weiss type monotonicity formula which generalizes the one in the elliptic case due to Soave and Terracini in [24]. As the reader will see, the proof of both the Carleman estimate as well as the Weiss type monotonicity is made possible by a combination of several non-trivial geometric facts which thanks to the specific structure of the sublinearity, beautifully combine. Moreover compared to that in [5], some new challenges appear in the proof of the Carleman estimate (2.7) below because
the weights involved are different from that in [5] and consequently our proof crucially relies on new inequalities in such weighted spaces. This constitutes one of the novelties of our work. We also believe that the new parabolic Weiss type monotonicity that we obtain in this framework also has an independent interest. Before stating our main result, we now define the relevant notions of vanishing to infinite order.

**Definition 1.1.** We say that a function \( u \) vanishes to infinite order in space at some \((x_0, t_0)\) if

\[
\text{given } k > 0, \text{ there exists } C_k > 0 \text{ such that } |u(x, t_0)| \leq C_k |x - x_0|^k \text{ as } x \to x_0. \tag{1.11}
\]

Likewise, vanishing to infinite order in space-time is defined as follows.

**Definition 1.2.** We say that a function \( u \) vanishes to infinite order in space-time at some \((x_0, t_0)\) if

\[
\text{given } k > 0, \text{ there exists } C_k > 0 \text{ such that } |u(x, t)| \leq C_k (|x - x_0|^2 + |t - t_0|^{1/2})^k \text{ as } (x, t) \to (x_0, t_0). \tag{1.12}
\]

Our main result can now be stated as follows.

**Theorem 1.3.** Let \( v \) be a solution to the backward parabolic sublinear equation (1.1) in \( \mathbb{R}^n \times [0, 1] \). Then the following backward uniqueness results hold.

(i) Assume that \( v \) satisfies the following Tychonoff type growth assumption

\[
|v(x, t)| \leq Ne^{N|x|^2} \tag{1.13}
\]

for some \( N > 0 \). Also assume that the matrix \( A \) satisfies the derivative bounds as in (1.4), the potential \( W \) satisfies the bound in (1.2) and the structure condition (1.5) holds for \( p_i \)'s and \( q \in (1, 2) \). Now if \( v \) vanishes to infinite order in the sense of Definition 1.1 at \((0, 0)\), then \( v \equiv 0 \).

(ii) Let \( A = I \), \( v \) be bounded and assume that \( h \) satisfies the structure conditions in (1.5) for \( p_i \)'s and \( q \in [1, 2) \). Now if \( v \) vanishes to infinite order in the sense of Definition 1.2 at \((0, 0)\), then \( v \equiv 0 \).

Before proceeding further, we make a couple of important remarks.

**Remark 1.4.** We note that it is shown in [5] that the two notions of vanishing order (1.1) and (1.2) coincide when \( v \) solves (1.1) with \( h \) satisfying (1.5) for \( p_i, q \in (1, 2) \). Over here we would like to mention that in the linear case, i.e. when \( p_i = q = 2 \), the equivalence of the two notions was earlier established in [3]. We also observe that unlike Theorem 1.3 (i), the result in Theorem 1.3 (ii) covers the end point case \( q = 1 \) when \( A = I \). The proof of this later fact is based on a new Weiss type monotonicity which works for any generic sublinearity satisfying (1.5) and thus provides an alternate proof of the backward uniqueness result in (i) when the principal part is the Laplacian. A typical scenario where the result in Theorem 1.3 (ii) applies is say in the case of a parabolic two phase membrane problem of the type,

\[
-\Delta v = v_t + \lambda_+(v^+)^{q-1} - \lambda_-(v^-)^{q-1}, \quad \text{where } \lambda_+, \lambda_- > 0, q \in [1, 2). \]
Over here, we note that when $q = 1$, $(v^+)^{q-1} := \chi_{\{v > 0\}}$ and $(v^-)^{q-1} := \chi_{\{v < 0\}}$. It remains to be seen whether Theorem 1.3 (ii) continues to be valid for more general $A$ satisfying (1.4). In this regard, we would however like to mention that to the best of our knowledge, Weiss type monotonicity is not known even for linear parabolic equations with variable coefficient principal part satisfying (1.4).

**Remark 1.5.** We would also like to remark that a growth condition of the type (1.13) is needed for backward uniqueness to hold. This follows from a counterexample due to Frank Jones in [12] where it is shown that there exists a non-trivial unbounded caloric function that is supported in a time strip of the type $\mathbb{R}^n \times (t_1, t_2)$. Also from an example as in [31], it follows that the decay assumption on the derivatives of the principal part as in (1.4) is somewhat optimal as well. We note that such a decay is related to the exponential growth rate of the solution as in (1.13).

We finally note that the subject of strong unique continuation and backward uniqueness has a long history and several ramifications by now. We refer the reader to [1,2,6–11,13–16,18–21,28,32] and one can find other references therein.

The paper is organized as follows. In Sect. 2, we introduce some basic notations and also gather some known results that are relevant to our present work. In Sect. 3, we prove our new Carleman estimate and a Weiss type monotonicity and consequently establish our backward uniqueness results.

### 2. Notations and preliminaries

A generic point $(x, t)$ in space time $\mathbb{R}^n \times (0, \infty)$ will be denoted by $X$. For notational convenience, $\nabla f$ and $\text{div} f$ will respectively refer to the quantities $\nabla_x f$ and $\text{div}_x f$ of a given function $f$. The partial derivative in $t$ will be denoted by $\partial_t f$ and also by $f_t$. The partial derivative $\partial_{x_i} f$ will be denoted by $f_{x_i}$. We indicate with $C^\infty_0(\Omega)$ the set of compactly supported smooth functions in the region $\Omega$ in space-time. Also for $H$ as in (1.5), $\partial_t H$ or $H_t$ will denote the derivative with respect to the variable $t$ of the function,

$$t \rightarrow H((x, t), v)$$

where $x$ and $v$ are treated as constants. Likewise, $\nabla_x H$ will denote the derivative with respect to the variable $x$ of the function

$$x \rightarrow H((x, t), v)$$

with $t$ and $v$ being constants.

We now state the relevant Rellich type identity from [30] which is a slight generalization of the one in [8,9]. See Lemma 3.1 in [30].
Lemma 2.1. Suppose $F$ is differentiable, $F_0$ and $G$ are twice differentiable and $G > 0$. Then, the following identity holds for any $u \in C_0^\infty([R^n \times [0, T])$,

$$
\frac{1}{2} \int_{R^n \times [0, T]} M_0 u^2 G dx dt + \int_{R^n \times [0, T]} \left( 2 DG + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) A \right) \nabla u \cdot \nabla u G dx dt
$$

- $\int_{R^n \times [0, T]} u A \nabla u \cdot \nabla (F - F_0) G dx dt$

$$
= 2 \int_{R^n \times [0, T]} Lu(Pu - Lu) G dx dt + \int_{R^n} A \nabla u \cdot \nabla u G dx |_0^T + \frac{1}{2} \int_{R^n} u^2 FG dx |_0^T,
$$

where

$$
\tilde{\Delta} = \text{div}(A(x) \nabla)
$$

$Pu = \text{div}(A(x, t) \nabla u) + u_t$

$Lu = u_t - A \nabla u \cdot \nabla \log G + \frac{F}{2} u$

$M_0 = \partial_t F + F \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) + \tilde{\Delta} F_0 - A \nabla (F - F_0) \cdot \nabla \log G,$

and $D_{ij}^G = A^{ik} \partial_{kl}(\log G) a^{ij} + \frac{\partial_t (\log G)}{2} (a^{ik} \partial_k a^{ij} + a^{jk} \partial_k a^{li} - a^{kl} \partial_k a^{ij}) + \frac{1}{2} \partial_t a^{ij}$

Similar to that in [30], we now let

$$
G = e^{\gamma (t^{-\kappa} - 1)} - \frac{b(x)^2 + K}{t^2}, \quad (K \text{ large enough to be chosen later})
$$

where $< x > = \sqrt{1 + |x|^2}$. Then we observe that

$$
\frac{\partial_t G - \tilde{\Delta} G}{G} = \frac{b \langle x \rangle^2}{t^2} - 4b^2 a^{ij} x_i x_j + K + \frac{2b (a^{ii} + \partial_k a^{kl} x_i)}{t} - 2\gamma \frac{K}{t^{K+1}}.
$$

Likewise, we define

$$
F = \frac{b \langle x \rangle^2}{t^2} - 4b^2 a^{ij} x_i x_j + K + \frac{2ba^{ii} - d}{t} - 2\gamma \frac{K}{t^{K+1}},
$$

$d$ will be chosen in terms of $K$ as in Lemma 2.2 below

(2.4)

$$
F_0 = \frac{b \langle x \rangle^2}{t^2} - 4b^2 a^{ij}_\varepsilon x_i x_j + K + \frac{2ba^{ii}_\varepsilon - d}{t} - 2\gamma K \frac{K}{t^{K+1}},
$$

where $a^{ij}_\varepsilon$ is the $\varepsilon$ mollification of $a^{ij}$ corresponding to $\varepsilon = 1/2$ (see lemma 3.2 in [30]). We also need the following inequalities from [30] corresponding to these choices of $G, F$ and $F_0$( see lemma 3.3 in [30]).

Lemma 2.2. Set $b = \frac{1}{8\lambda}$ and $d = \frac{K}{4}$. Then for $K \geq K_0(n, \Lambda, \lambda, M)$, we have

$$
2DG + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) A \geq \frac{\lambda K}{8t} I_n
$$

$$
\partial_t F + F \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) \geq \frac{b K \langle x \rangle^2}{16t^3}
$$
\[ |\tilde{\Delta}F_0| \leq \frac{C \langle x \rangle^2}{t^2} \]
\[ |\nabla (F - F_0)| \leq \frac{C \langle x \rangle}{t^2} \]

where \( C \) depends on \( n, \lambda, \Lambda, M \) and is independent of \( K \).

Lemma 2.2 in particular implies that the following estimate holds for the quantity on the left hand side of the identity in (2.1).

**Lemma 2.3.** With \( G, F, F_0 \) as in Lemma 2.2, we have that the following integral estimate holds for all \( K \) sufficiently large and \( u \in C^\infty_0 (\mathbb{R}^n \times (0, T)) \),

\[
\frac{1}{2} \int_{\mathbb{R}^n \times (0, T)} M_0 u^2 G dx dt + \int_{\mathbb{R}^n \times (0, T)} \left( 2D_G + \left( \frac{\partial_t G - \tilde{\Delta}G}{G} - F \right) A \right) \nabla u \cdot \nabla u G dx dt
\]
\[ - \int_{\mathbb{R}^n \times (0, T)} u A \nabla u \cdot \nabla (F - F_0) G dx dt \geq CK \left( \int_{\mathbb{R}^n \times (0, T)} \frac{u^2 \langle x \rangle^2}{t^3} G dx dt + \int_{\mathbb{R}^n \times (0, T)} \frac{|
abla u|^2}{t} G dx dt \right) \]  

for some universal \( C > 0 \).

**Proof.** This can be deduced from the estimates (41)–(44) in [30] which in turns relies on the inequalities in Lemma 2.2 above. \( \square \)

We also need the following real analysis lemma (see lemma 3.3 in [8]).

**Lemma 2.4.** Given \( m > 0 \), \( \exists C_m \) such that for all \( y \geq 0 \) and \( 0 < \epsilon < 1 \),

\[ y^m e^{-y} \leq C_m \left[ \epsilon + \left( \log \left( \frac{1}{\epsilon} \right) \right)^m e^{-y} \right] \]

We now state our main Carleman estimate which is needed to prove the backward uniqueness result.

**Theorem 2.5.** Let \( u \in C^\infty_0 (\mathbb{R}^n \times (0, T)) \) be a solution of

\[ \text{div}(A(x, t) \nabla u) + u_t + h((x, t), u) + Wu = g \]  

where \( h \) satisfies the structure conditions in (1.5). Then the following estimate holds with \( G \) as in (2.2) for some universal \( C > 0 \),

\[
K \int (u^2 + |
abla u|^2) G dx dt + \gamma K \int \frac{HG}{t^{K+1}} dx dt \leq C \left( \int He^{-2\gamma} - \frac{b \langle x \rangle^2 + K}{t^{K+1}} dx dt + \int g^2 G dx dt \right)
\]  

where \( K, \gamma \) are large enough depending only on \( n, \lambda, \Lambda, p_i, q, M, T \).
3. Proof of the main results

We first establish the Carleman estimate as in (2.7) which is needed to prove Theorem 1.3 (i).

Proof of Theorem 2.5. Let \( G, F, F_0 \) be as in Sect. 2. We start by applying the identity in Lemma 2.1 with \( u \) as in the hypothesis. Before proceeding further, we remark that in all the subsequent integrals, the measure \( dxdt \) will be omitted. By using the equation (2.6) satisfied by \( u \), we note that the corresponding right hand side in (2.1) equals

\[
2 \int Lu(Pu - Lu) G
\]

\[
= 2 \int \left( u_t - A \nabla u \cdot \nabla \log G + \frac{F}{2} \right) \left( g - Wu - h(X, u) \right) G - 2 \int (Lu)^2 G
\]

\[
\leq 2 \int g^2 G dx dt + 2M \int u^2 G - 2 \int \left( u_t - A \nabla u \cdot \nabla \log G + \frac{F}{2} \right) h(X, u) G
\]

(3.1)

Now by integrating by parts, the last integral in (3.1) (which we denote by \( I_1 \)) can be expressed as

\[
I_1 = -2 \int \left( (H(X, u))_t - \partial_t H \right) - A \nabla H(X, u) \cdot \nabla \log G \]

\[
= 2 \int H(X, u) (G_t - \tilde{\Delta} G) - \int F u h(X, u) G + 2 \int (\partial_t H - A \nabla_x H \cdot \nabla \log G) G
\]

(3.2)

We note that in (3.2) above, we rewrote \( A \nabla u \cdot \nabla \log G h(X, u) \) as

\[
A \nabla u \cdot \nabla \log G h(X, u) = A \nabla H(X, u) \cdot \nabla \log G \leq A \nabla_x H \cdot \nabla \log G
\]

Then from (2.3) and the expression of \( F \) as in (2.4), we observe that \( I_1 \) can be further rewritten as

\[
I_1 = 2 \int (\partial_t H - A \nabla_x H \cdot \nabla \log G) G
\]

\[
+ 2 \int \left( H(X, u) - \frac{uh(X, u)}{2} \right) \left( b(x)^{2} - 4b^2a_{ij}x_i x_j + K \right) \frac{G}{t^2}
\]

\[
+ 2 \int \left( H(X, u) - \frac{uh(X, u)}{2} \right) \frac{2ba_{ij}x_i x_j G}{t} - 2 \int \left( H(X, u) - \frac{uh(X, u)}{2} \right) \frac{2\gamma K}{t^{K+1}} G
\]

\[
+ 2 \int \frac{H(X, u)}{t} \partial_{k} a^{k l} x_j G + \int \frac{uh(X, u)}{t} d_{k} G
\]

(3.3)

Now using the bounds on \( \nabla_x H \) and \( \partial_t H \) as in (1.5), the bounds for \( \nabla_x a^{ij} \) as in (1.4) and also that \( uh(X, u) \leq qH(X, u) \) for some \( q \in (1, 2) \), we obtain from (3.3) that \( I_1 \) can be estimated from above in the following way,

\[
I_1 \leq C \int \frac{HG}{t} + C \int \frac{H G (x)^2}{t^2} + C \int \frac{K}{t^2} HG + (q - 2) \int \frac{HG}{t^{K+1}} (2\gamma K)
\]

(3.4)
where $C$ is some universal constant independent of $K$. We note that the last integral in (3.4) comes from estimating the integral $2 \int \left( H(X, u) - \frac{uh(X,u)}{2} \right) \frac{2\gamma K}{t^{K+1}} G$ in (3.3) in the following way,

$$2 \int \left( H(X, u) - \frac{uh(X,u)}{2} \right) \frac{2\gamma K}{t^{K+1}} G \geq (2 - q) \int \frac{HG}{t^{K+1}} (2\gamma K).$$

We would like to mention that this is precisely the place where we use the specific structure of the sublinearity $h$ and as one would see, the fact that $q < 2$ would be crucially exploited subsequently.

We then note that since $q < 2$, therefore for $\gamma$ sufficiently large (depending on $C, T$ and $2 - q$) and for $K > 2$, we can ensure that

$$C \int \frac{HG}{t} + C \int \frac{K}{t^2} HG + (q - 2) \int \frac{HG}{t^{K+1}} (2\gamma K) \leq (q - 2) \int \frac{HG}{t^{K+1}} \gamma K \quad (3.5)$$

Now the remaining term

$$C \int \frac{HG \langle x \rangle^2}{t^2}$$

is estimated using Lemma 2.4 in the following way.

$$\int \frac{HG \langle x \rangle^2}{t^2} \leq C \left( \int \frac{H}{t} e^{-2\gamma - \frac{b \langle x \rangle^2 + K}{t}} + 2\gamma \int \frac{H}{t^{K+1}} G \right) \quad (3.6)$$

We note that (3.6) follows by an application of the inequality in Lemma 2.4 with $\epsilon = e^{-2\gamma(t-K)}, m = 1$ and $y = \frac{b \langle x \rangle^2}{t}$. Using (3.5) and (3.6) in (3.4), we obtain for some other constant $C$ that the following holds,

$$I_1 \leq C \left( \int \frac{H}{t} e^{-2\gamma - \frac{b \langle x \rangle^2 + K}{t}} + 2\gamma \int \frac{H}{t^{K+1}} G \right) + (q - 2) \int \frac{HG}{t^{K+1}} \gamma K \quad (3.7)$$

Consequently, if $K$ is chosen large enough, then we can guarantee that,

$$2C\gamma \int \frac{H}{t^{K+1}} G + (q - 2) \int \frac{HG}{t^{K+1}} \gamma K \leq \left( \frac{q}{2} - 1 \right) \int \frac{HG}{t^{K+1}} \gamma K \quad (3.8)$$

Thus by using using (3.8) in (3.7), we obtain

$$I_1 \leq C \int \frac{H}{t} e^{-2\gamma - \frac{b \langle x \rangle^2 + K}{t}} + \left( \frac{q}{2} - 1 \right) \int \frac{HG}{t^{K+1}} \gamma K \quad (3.9)$$

Also using Lemma 2.3, we note that the left hand side in the corresponding identity in Lemma 2.1 can be bounded from below by

$$C_1 K \int (u^2 + |\nabla u|^2) G dx \quad (3.10)$$

for some universal $C_1 > 0$. Finally from (3.1), (3.9) and (3.10), we observe that if $K$ is additionally chosen large enough such that

$$C_1 K \geq 4M$$
then the following integral in (3.1),

$$2M \int u^2 G$$

can be absorbed in the left hand side and consequently the desired estimate as claimed in (2.7) follows. \hfill \Box

**Remark 3.1.** We cannot stress enough the crucial role of the inequality (3.6) and its interplay with the sublinear structure that allows us to establish the estimate (2.7).

With the Carleman estimate as in (2.7) in our hands, we now proceed with the proof of Theorem 1.3 (i).

**Proof of Theorem 1.3 (i).** First we note that it follows from the space like strong unique continuation result as in Theorem 1.1 in [5] that \( v(\cdot, 0) \equiv 0 \). Moreover from Step 2 in the proof of Theorem 1.1 in [5], it also follows that \( v \) vanishes to infinite order in time at \( t = 0 \). Furthermore from the classical regularity theory as in [17], we have that \( \nabla^2 u, u_t \in L^p_{loc} \) for all \( p < \infty \). Thus for \( t < 0 \), if we extend \( v \) by 0, the principal coefficients \( a^{ij} \) by \( a^{ij}(x, 0) \) and \( W \) by 0, we note that the extended \( v \) continues to solve an equation of the type (1.1) with similar structural assumptions. Then by using rescaling and translation of the type,

$$\tilde{v}(x, t) = v(rx, r^2(t - 1/2))$$

for \( r \) sufficiently small, we can ensure that

$$|\tilde{v}(x, t)| \leq Ce^{|x|^2}. \quad (3.11)$$

Subsequently we let \( \tilde{v} \) be our new \( v \) and thus we are reduced to the situation where \( v \equiv 0 \) for \( t \leq 1/2 \). Now let \( \eta \) be a function of \( t \) defined as

$$\eta(t) \equiv \begin{cases} 1 & \text{for } t < 3/4 \\ 0 & \text{for } t > 7/8 \end{cases} \quad (3.12)$$

By a standard limiting argument (using cut-offs in space) as in the proof of Lemma 2.1 in [30], thanks to the Tychonoff type bounds in (3.11) (which ensures integrability of the integrals in (2.7) as \( |x| \to \infty \) corresponding to the weight \( G \) as in (2.2)), the Carleman estimate in (2.7) continues to be valid for \( u = \eta v \). Then we note that using (1.1), we have that \( u \) solves (2.6) with

$$g = \eta v_t + h(X, u) - \eta h(X, v)$$

From the definition of \( \eta \) above, it follows that

$$g \equiv 0 \text{ for } t \leq 3/4 \text{ and } |g| \leq C(|v| + \sum |v|^{p_i-1}) \quad (3.13)$$

Now by applying the Carleman estimate (2.7) to \( u \) we obtain,

$$K \int (\eta v)^2 + |\nabla(\eta v)|^2 G dx dt \leq C \int g^2 G dx dt$$
+ \int \frac{H(X, \eta v)}{t} e^{-\frac{b(\xi)^2+K}{t} - 2\gamma} dx dt \tag{3.14}

Now let \( l \in (1/2, 3/4) \). Then by minorizing the integral on the left hand side in (3.14) over the region \( \{ \frac{1}{2} \leq t \leq \frac{3}{4} \} \) and by using (3.13) we deduce that the following holds,

\[
K \int_{\frac{1}{2} \leq t \leq \frac{3}{4}} (v^2 + |\nabla v|^2)G dx dt 
\leq C \int_0^1 \frac{H(X, \eta v)}{t} e^{-\frac{b(\xi)^2+K}{t} - 2\gamma} dx dt + C \int_{\frac{3}{4} \leq t \leq 1} g^2 G dx dt \tag{3.15}
\]

Continuing further we obtain,

\[
\int_{\frac{1}{2} \leq t \leq l} (v^2 + |\nabla v|^2)G dx dt \leq C \int_{\frac{1}{2} \leq t \leq 1} |v|^p G dx + C \int_{\frac{3}{4} \leq t \leq 1} g^2 G dx dt
\]

Dividing both sides of (3.16) by \( e^{2\gamma(l-K-1)} \) we find again by using (3.13),

\[
\int_{\frac{1}{2} \leq t \leq l} (v^2 + |\nabla v|^2) e^{-\frac{b(\xi)^2+K}{t}} dx dt \leq C e^{-2\gamma l-K} \int_{\frac{1}{2} \leq t \leq 1} \sum |v|^p e^{-\frac{b(\xi)^2+K}{t}} dx dt 
+ e^{2\gamma((\frac{3}{4})-K-1)} C \int_{\frac{3}{4} \leq t \leq 1} (|v|^2 + \sum |v|^{2(p_i-1)}) e^{-\frac{b(\xi)^2+K}{t}} dx dt
\]

Now letting \( \gamma \to \infty \) in (3.17), we find that the right hand side goes to 0 and thus we can assert that \( v(\cdot, t) \equiv 0 \) for \( \frac{1}{2} \leq t \leq l \). Now by going back to the original \( v \) by scaling back, we obtain that \( v(\cdot, t) \equiv 0 \) for \( 0 \leq t \leq \kappa \) for some \( \kappa > 0 \) universal. Therefore in this way we can spread the zero set and hence conclude that \( v \equiv 0 \). \( \square \)

**Proof of Theorem 1.3 (ii).** As previously mentioned in the introduction, the proof of this result is via a geometric variational approach based on a Weiss type monotonicity formula which is tailor-made for the sublinear problem. The proof is divided into two steps.

**Step 1: A Weiss type monotonicity formula.** With \( G = \frac{1}{|t|^\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \) [note that this \( G \) is different from one in (2.2)], following [21, 24], we let

\[
H(R) = \int_{t=R^2} v^2 G dx 
\]

\[
I(R) = R^2 \int_{t=R^2} |\nabla v|^2 G dx - 2R^2 \int_{t=R^2} H(X, v) G dx
\]

\[
\frac{1}{2} \frac{d}{dt} I(R) \leq \int_{t=R^2} (v^2 + |\nabla v|^2) e^{-\frac{b(\xi)^2+K}{t} - 2\gamma} dx dt
\]

Now let \( l \in (1/2, 3/4) \). Then by minorizing the integral on the left hand side in (3.14) over the region \( \{ \frac{1}{2} \leq t \leq \frac{3}{4} \} \) and by using (3.13) we deduce that the following holds,
\[ W_\gamma(R) = \frac{I(R)}{R^{2\gamma}} - \frac{\gamma}{2R^{2\gamma}} H(R) \]

We now make the following claim.

**Claim:** For \( \gamma \) sufficiently large depending also on the \( L^\infty \) norm of \( v \), we have that

\[ W'_\gamma(R) \geq 0 \text{ for a.e. } R \in (0,1). \quad (3.19) \]

In order to establish the claim, we compute the derivatives of the quantities involved in the expression of \( W_\gamma \). We first compute \( H' \). In all the computations below, we will be using the Einstein’s notation of summation of repeated indices. By integrating by parts and by using Eq. (1.1), we note that

\[ H'(R) = \int 2vv_t(2R)Gdx + \int v^2G_t(2R)dx \quad (3.20) \]

\[ = 4R \int vv_tGdx + 2R \int v^2\Delta Gdx \]

\[ = 4R \int v[-\Delta v - h(X,v) - Wv]Gdx - 2R \int \nabla(v^2) \cdot \nabla Gdx \]

\[ = 4R \int (\nabla(vG)) \cdot \nabla vdx - 4R \int vh(X,v)Gdx \]

\[ - 4R \int Wv^2Gdx - 2R \int \nabla(v^2) \cdot \nabla Gdx \]

\[ = 4R \int \nabla |v|^2Gdx - 4R \int vh(X,v)Gdx - 4R \int Wv^2Gdx \]

Also

\[ I(R) = R^2 \int |\nabla v|^2Gdx - 2R^2 \int H(X,v)Gdx = I_1 + I_2 \quad (3.21) \]

Again by using (1.1) and by integrating by parts, we find for a.e. \( R \),

\[ I'_1(R) = 2R \int |\nabla v|^2Gdx + R^2 \int 2vv_t(2R)Gdx + R^2 \int |\nabla v|^2G_t(2R)dx \quad (3.22) \]

\[ = 2R \int |\nabla v|^2Gdx + 4R^3 \int vv_tGdx + 2R^3 \int |\nabla v|^2\Delta Gdx \]

\[ = 2R \int |\nabla v|^2Gdx - 4R^3 \int (\Delta vG + \nabla v \cdot \nabla G) v_tdx - 2R^3 \int \nabla(|\nabla |^2v) \cdot \nabla Gdx \]

\[ = 2R \int |\nabla v|^2Gdx - 4R^3 \int [(v_t - h(X,v) - Wv)G + \nabla v \cdot \nabla G] v_tdx \]

\[ + 2R^3 \int 2v_i v_j \frac{x_j}{2t}Gdx \quad \text{(note that } \nabla G = -\frac{x}{2t}G) \]

\[ = 2R \int |\nabla v|^2Gdx - 4R^3 \int ((v_t - h(X,v) - Wv)G + \nabla v \cdot \nabla G) v_tdx \]

\[ + 2R \int (v_i(v_j x_j) \frac{x_i}{2t} - v_i^2)Gdx - 2R \int \Delta vv_j x_j Gdx \]

\[ = -4R^3 \int ((v_t - h(X,v) - Wv)G + \nabla v \cdot \nabla G) v_tdx \]
\[ +4R^3 \int v_i \frac{x_i}{2t} v_j \frac{x_j}{2t} Gdx - 2R \int (-v_t - h(X,v) - Wv)v_j x_j Gdx \]
\[ = 4R^3 \int (v_t + \nabla v \cdot \frac{x}{2t})^2 Gdx + 4R^3 \int (h(X,v) + Wv)\nabla v \cdot \frac{x}{2t} Gdx \]
\[ + 4R^3 \int ((h(X,v) + Wv)v_t) Gdx \]
\[ = 4R^3 \int (v_t + \nabla v \cdot \frac{x}{2t} + \frac{1}{2}Wv)^2 Gdx - R^3 \int W^2 v^2 Gdx \]
\[ + 4R^3 \int (h(X,v))(\nabla v \cdot \frac{x}{2t} + v_t) Gdx \]

We note that in the intermediate computations for \( I_1'(R) \) in (3.22) above, the term \( v_{it} \) appears and the solution \( v \) may not possess that much of Sobolev regularity in general. However such formal computations can be justified by approximating \( v \) by solutions to smoother problems and a limiting argument which crucially uses the fact that \( v_t, \nabla v, \nabla^2 v \in L^2_{loc} \) with universal bounds depending on the \( L^\infty \) norm of \( v \). Likewise

\[ I_2'(R) = -4R \int H(X,v) Gdx - 4R^3 \int H_t(X,v) Gdx \]
\[ - 4R^3 \int h(X,v)v_t Gdx - 4R^3 \int H(X,v) G_t dx \quad (3.23) \]
\[ = -4R \int H(X,v) Gdx - 4R^3 \int H_t(X,v) Gdx \]
\[ - 4R^3 \int h(X,v)v_t Gdx - 4R^3 \int h(X,v) \nabla v \cdot \frac{x}{2t} Gdx \]
\[ - 4R^3 \int \nabla x H \cdot \frac{x}{2t} Gdx \]

Thus from (3.22) and (3.23) we observe that the integral \( 4R^3 \int_{t - R^2}^t (h(X,v))(\nabla v \cdot \frac{x}{2t} + v_t) Gdx \) gets cancelled in the expression of \( I'(R) \). Therefore using (3.20), (3.22) and (3.23) we obtain,

\[ W'_\gamma(R) = \frac{I'(R)}{R^{2\gamma}} - \frac{2\gamma I(R)}{R^{2\gamma + 1}} - \frac{\gamma}{2R^{2\gamma}} H'(R) + \frac{\gamma^2}{R^{2\gamma + 1}} H(R) \quad (3.24) \]
\[ = \frac{4R^3}{R^{2\gamma}} \int_{t - R^2}^t (v_t + \nabla v \cdot \frac{x}{2t} + \frac{1}{2}Wv)^2 Gdx \]
\[ - \frac{R^3}{R^{2\gamma}} \int_{t - R^2}^t W^2 v^2 Gdx - \frac{4R}{R^{2\gamma}} \int_{t - R^2}^t H(X,v) Gdx \]
\[ - \frac{4R^3}{R^{2\gamma}} \int_{t - R^2}^t H_t Gdx - \frac{4R^3}{R^{2\gamma}} \int_{t - R^2}^t \nabla x H \cdot \frac{x}{2t} Gdx - \frac{2\gamma R^2}{R^{2\gamma + 1}} \int_{t - R^2}^t |\nabla v|^2 Gdx \]
\[ + \frac{4\gamma R^2}{R^{2\gamma + 1}} \int_{t - R^2}^t H(X,v) Gdx - \frac{4\gamma R}{2R^{2\gamma}} \int_{t - R^2}^t |\nabla v|^2 Gdx \]
\[ + \frac{4\gamma R}{2R^{2\gamma}} \int_{t - R^2}^t vh(X,v) Gdx \]
\[ + \frac{4\gamma R}{2R^{2\gamma}} \int_{t - R^2}^t W v^2 Gdx + \frac{\gamma^2}{R^{2\gamma + 1}} \int_{t - R^2}^t v^2 Gdx \]
Now again from (1.1) and divergence theorem, we note that the integral \( \int_{t=R^2} |\nabla v|^2 Gdx \) can be rewritten as
\[
\int_{t=R^2} |\nabla v|^2 Gdx = \int_{t=R^2} (v_t + Wv + h(X, v) + \nabla v \cdot \frac{x}{2t})vGdx \quad (3.25)
\]
Using (3.25) in (3.24), we find
\[
W'_\gamma (R) = \frac{1}{R^{2\gamma - 1}} \int_{t=R^2} \left[ [(2R)(v_t + \nabla v \cdot \frac{x}{2t} + \frac{1}{2} Wv)]^2 + (\frac{\gamma v}{R})^2 \right] Gdx
\]
\[
+ \frac{4\gamma}{R^{2\gamma - 1}} \int_{t=R^2} (-v_t - Wv - h(X, v) - \nabla v \cdot \frac{x}{2t})vGdx
\]
\[
- \frac{R^3}{R^{2\gamma}} \int_{t=R^2} W^2 v^2 Gdx - \frac{4R}{R^{2\gamma}} \int_{t=R^2} H(X, v)Gdx
\]
\[
- \frac{4R^3}{R^{2\gamma}} \int_{t=R^2} H_t Gdx - \frac{4R^3}{R^{2\gamma}} \int_{t=R^2} \nabla_x H \cdot \frac{x}{2t} Gdx
\]
\[
+ \frac{4\gamma R^2}{R^{2\gamma + 1}} \int_{t=R^2} H(X, v)Gdx + \frac{4\gamma R}{2R^{2\gamma}} \int_{t=R^2} vh(X, v)Gdx
\]
\[
+ \frac{2\gamma R^2}{R^{2\gamma + 1}} \int_{t=R^2} W^2 vGdx
\]
\[
= \frac{1}{R^{2\gamma - 1}} \int_{t=R^2} \left[ (2R)(v_t + \nabla v \cdot \frac{x}{2t} + \frac{1}{2} Wv) - \frac{\gamma v}{R} \right]^2 Gdx
\]
\[
- \frac{R^3}{R^{2\gamma}} \int_{t=R^2} W^2 v^2 Gdx
\]
\[
- \frac{4R}{R^{2\gamma}} \int_{t=R^2} H(X, v)Gdx
\]
\[
- \frac{4R^3}{R^{2\gamma}} \int_{t=R^2} H_t Gdx - \frac{4R^3}{R^{2\gamma}} \int_{t=R^2} \nabla_x H \cdot \frac{x}{2t} Gdx
\]
\[
+ \frac{4\gamma R^2}{R^{2\gamma + 1}} \int_{t=R^2} H(X, v)Gdx - \frac{2\gamma}{R^{2\gamma - 1}} \int_{t=R^2} vh(X, v)Gdx
\]
Now using the bounds for \( H_t \) and \( \nabla_x H \) as in (1.5), it follows that for \( \gamma \) sufficiently large depending on the bounds in (1.5) and (1.2), we have that for some universal \( C > 0 \),
\[
W'_\gamma (R) \geq - \frac{C}{R^{2\gamma - 1}} \int_{t=R^2} v^2 Gdx + \frac{4\gamma}{R^{2\gamma - 1}} \int_{t=R^2} H(X, v)Gdx \quad (3.26)
\]
\[
- \frac{2\gamma}{R^{2\gamma - 1}} \int_{t=R^2} vh(X, v)Gdx - \frac{C}{R^{2\gamma - 1}} \int_{t=R^2} H(X, v)Gdx
\]
Then by using \( vh(X, v) \leq qH(X, v) \), we obtain from (3.26) that the following holds,
\[
W'_\gamma (R) \geq - \frac{C}{R^{2\gamma - 1}} \int_{t=R^2} v^2 Gdx + \frac{1}{R^{2\gamma - 1}} \int_{t=R^2} (2\gamma (2 - q) - C)H(X, v)Gdx \quad (3.27)
\]
Now since \( q < 2 \). therefore by choosing \( \gamma \) large enough, we can ensure that

\[
W'_\gamma(R) \geq \frac{1}{R^{2\gamma-1}} \int_{t=R^2} (C_1 \gamma H(x,v) - Cv^2)Gdx
\]  

(3.28)

At this point, we use the fact that since \( v \) is bounded, therefore from (1.6) it follows that

\[
H(x,v) \geq c_0 |v|^q
\]  

(3.29)

where \( c_0 \) depends on the \( L^\infty \) norm of \( v \) (this is precisely where we use the boundedness of \( v \)).

Using (3.29) in (3.28), we deduce that the following holds for a new constant \( C_2 \) depending also on \( c_0 \),

\[
W'_\gamma(R) \geq \frac{1}{R^{2\gamma-1}} \int_{t=R^2} |v|^q (C_2 \gamma - C|v|^{2-q})Gdx
\]  

(3.30)

Now since \( v \) is bounded, therefore by choosing \( \gamma \) sufficiently large depending also on the \( L^\infty \) norm of \( v \), we can ensure that

\[
C_2 \gamma - C|v|^{2-q} \geq 0
\]  

(3.31)

and thus we can assert that the Weiss type monotonicity as claimed in (3.19) holds.

**Step 2: Conclusion.** Assume on the contrary that \( v \) is not identically zero in \( \mathbb{R}^n \times (0,1) \). Then there exists \( R > 0 \) such that \( v(\cdot, R^2) \neq 0 \). We now choose \( \gamma > 0 \) large enough such that simultaneously both (3.19) as well as

\[
W_\gamma(0+) < 0
\]

hold. Then from the monotonicity of \( W_\gamma \), we must have that \( W_\gamma(0+) < 0 \). However since \( v \) vanishes to infinite order at \( (0,0) \) in the sense of (1.2), it follows from the expression of \( W_\gamma \) as in (3.18) that \( W_\gamma(0+) \geq 0 \) (see for instance Lemma 1.4 in [21]). This leads to a contradiction and thus finishes the proof of the Theorem. \( \square \)

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