Seiberg dualities and the 3d/4d connection

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Abstract

We discuss the degeneration limits of $d = 4$ superconformal indices that relate Seiberg duality for the $d = 4 \, \mathcal{N} = 1$ SQCD theory to Aharony and Giveon-Kutasov dualities for $d = 3 \, \mathcal{N} = 2$ SQCD theories. On a mathematical level we argue that this 3d/4d connection entails a new set of non-standard degeneration identities between hyperbolic hypergeometric integrals. On a physical level we propose that such degeneration formulae provide a new route to the still illusive Seiberg dualities for $d = 3 \, \mathcal{N} = 2$ SQCD theories with $SU(N)$ gauge group.
1. Degeneration schemes of partition functions and the 3d/4d connection

Quantum field theories (QFTs) related by reduction on a spatial $S^1$ frequently exhibit similar properties, e.g. similarities in duality and spontaneous supersymmetry breaking patterns. One can try to trace the quantum dynamics of the compactified theory as a function of the compactification radius [1], but this is typically hard.

The superconformal indices (SCIs) of $d = 4$ QFTs provide an interesting new perspective on such relations. Under an $S^1$ reduction the SCI, which is a partition function on $S^3 \times S^1$, reduces to the $S^3$ partition function of a three dimensional QFT [2-5]. For generic gauge theories with a known UV Lagrangian description there is a standard prescription for the computation of $d = 4$ SCIs and $S^3$ partition functions. The $d = 4$ SCIs are expressed in terms of elliptic hypergeometric integrals [6] and the $S^3$ partition functions in terms of hyperbolic hypergeometric integrals [7-10]. Original work on the mathematics of the elliptic hypergeometric integrals was performed in [11,12] (see [13] for a review). A lengthy treatise on hyperbolic hypergeometric integrals, whose notation we will follow closely, is [14]. The first paper to describe the reduction from elliptic to hyperbolic hypergeometric integrals was [15].

In this framework, a field theory duality in four dimensions translates to a corresponding duality transformation property of elliptic hypergeometric integrals. The subsequent reduction of this transformation to hyperbolic hypergeometric integrals implies a corresponding field theory duality in three dimensions. It is believed that every duality in four dimensions [16] descends in this manner to a duality in three dimensions [2].

In practice, the descent between a four dimensional and a three dimensional duality identity is not just a single $S^1$ reduction of the four dimensional SCI but a sequence of reductions whose purpose is to remove constraining conditions on external parameters, e.g. real masses, and/or add extra parameters like Fayet-Iliopoulos (FI) terms and Chern-Simons (CS) interactions. The latter steps are crucial at the end of the process when we read off the specifics of the three dimensional duality from the corresponding form of the duality transformation properties of hyperbolic hypergeometric integrals. Examples of such reductions in a field theory context have been provided in [17,2].

The mathematical implementation of these steps relies on specific degeneration schemes between elliptic and/or hyperbolic hypergeometric integrals. Such schemes have been studied in the mathematics literature in [18,14] and have been implemented in [17,2,13,22] to demonstrate certain $d = 3$ dualities on the level of $S^3$ partition functions.
In this note we will argue that the generic reduction between a $d = 4$ and a $d = 3$ duality involves more general degeneration schemes with qualitatively new features whose study is both physically and mathematically interesting.

For concreteness, in this paper we will focus on the example of $d = 4$ Seiberg duality [23] and its reduction to $d = 3$ Aharony [24] and Giveon-Kutasov dualities [25]. The known route to the integral identities implied by the matching of $S^3$ partition functions in Aharony/Giveon-Kutasov dualities proceeds along the lines of the following degeneration scheme. The starting point is Seiberg duality for the $d = 4 \mathcal{N} = 1$ SQCD theory with gauge group $Sp(2N)$ (also known as the Intriligator-Pouliot duality [26]), and the corresponding transformation properties of SCIs, which were proven in [27], are of the BC type. The $S^1$ degeneration of these identities becomes a duality transformation property of the so-called $I_{BC}$ top level integral. A subsequent degeneration scheme [14] that reduces the $I_{BC}$ top level integral to the $S^3$ partition functions of $\mathcal{N} = 2$ SQCD and Chern-Simons SQCD theories with gauge group $U(N)$ allows the derivation of the transformation properties implied by Aharony/Giveon-Kutasov dualities.

All the reductions involved in this particular degeneration scheme share the following (technically convenient) features: ($i$) they keep the number of integration variables invariant, and ($ii$) they can be derived by exchanging the integral with the degeneration limits. In what follows we will call such reductions ‘standard’. We will argue that there are also more involved reductions that do not obey ($i$) and ($ii$). We will call the latter ‘non-standard reductions’.

We notice that the starting point of the above scheme is not Seiberg duality for the $d = 4 \mathcal{N} = 1$ SQCD theory with $SU(N)$ gauge group. Since there is a direct $S^1$ reduction between the $d = 4 \mathcal{N} = 1$ and $d = 3 \mathcal{N} = 2$ SQCD theories with unitary gauge group it is physically more interesting to find a degeneration scheme between the partition functions of these theories. This entails a more direct connection between the duality transformation properties of the $d = 4 \mathcal{N} = 1$ $SU(N)$ SQCD theory [27,14], and the transformation properties of hyperbolic hypergeometric integrals required by Aharony/Giveon-Kutasov dualities [14]. Our main goal will be to discuss explicitly how this connection is implemented and what mathematical properties it requires. We will find that by gauging the baryon symmetry of the $d = 4$ SQCD theory we recover the Aharony/Giveon-Kutasov dualities for $U(N) \mathcal{N} = 2$ SQCD theories. Without gauging the baryon symmetry of the four dimensional theory we obtain a mathematically concrete route towards a long-suspected
3d Seiberg duality for $SU(N) \, \mathcal{N} = 2$ SQCD theories. The latter cannot be derived using the degeneration scheme that starts with the $d = 4$ $Sp(2N)$ Intriligator-Pouliot duality.

Analogous reduction schemes can be implemented for more general $d = 4$ Seiberg dualities [16]. The descent between Kutasov [28] and Brodie [29] dualities with adjoint and fundamental matter to their $d = 3$ descendants [30,31] is an example. We will not discuss explicitly this possibility in this paper.

2. From $d = 4$ Seiberg duality to $d = 3$ Aharony/Giveon-Kutasov duality

2.1. The superconformal index of $d = 4$ $\mathcal{N} = 1$ SQCD

The SCI of the $d = 4 \, \mathcal{N} = 1$ SQCD theory with $N_f$ pairs of quark supermultiplets in the (anti)fundamental representation of the gauge group $SU(N_c)$ is [32,16] (for the precise conventions used here see also eqs. (4.6), (4.7) of the review [16]):

$$I_{E}^{(SU)}(N_c, N_f; s; t) = \frac{(p;p)^{N_c-1}(q;q)^{N_c-1}}{N_c!}$$

$$\int_{\Gamma_{Nc-1}} \frac{dz_j}{2\pi i z_j} \prod_{j=1}^{N_c} \frac{1}{\prod_{a=1}^{N_c} \prod_{j=1}^{N_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q) \prod_{1 \leq i < j \leq N_c} \Gamma_e(z_i z_j^{-1}, z_i^{-1} z_j; p, q)}$$

$$\prod_{j=1}^{N_c} z_j$$

for the electric description, and

$$I_{M}^{(SU)}(\tilde{N}_c, N_f; s; t) = \frac{(p;p)^{\tilde{N}_c-1}(q;q)^{\tilde{N}_c-1}}{N_c!} \prod_{a,b=1}^{N_f} \Gamma_e(s_a t_b^{-1}; p, q)$$

$$\int_{\Gamma_{\tilde{N}_c-1}} \frac{dz_j}{2\pi i z_j} \prod_{j=1}^{\tilde{N}_c} \frac{1}{\prod_{a=1}^{\tilde{N}_c} \prod_{j=1}^{\tilde{N}_c} \Gamma_e(S^{\frac{1}{N_c}} s_a^{-1} z_j, T^{\frac{1}{N_c}} t_a z_j^{-1}; p, q) \prod_{1 \leq i < j \leq \tilde{N}_c} \Gamma_e(z_i z_j^{-1}, z_i^{-1} z_j; p, q)}$$

$$\prod_{j=1}^{\tilde{N}_c} z_j$$

for the magnetic description.

We make a short parenthesis to explain the notation. The rank of the dual gauge group will be denoted as

$$\tilde{N}_c = N_f - N_c .$$

$(z, p)_\infty$ is the $q$-Pochhammer symbol (thus $(p, p)_\infty$ is equivalent to the Euler function $\phi(p)$) and $\Gamma_e(z; p, q)$ the elliptic $\Gamma_e$-function (we refer the reader to [16,14] for precise definitions...
and references to the original literature). We use the common convention $\Gamma_e(z_1, z_2; p, q) = \Gamma_e(z_1; p, q)\Gamma_e(z_2; p, q)$. In the expressions (2.1), (2.2) the external vector parameters $s = (s_1, \ldots, s_{N_f})$ and $t = (t_1, \ldots, t_{N_f})$ denote (renormalized) fugacities of the global flavor group, whereas $p, q$ are fugacities related to the U(1) $R$-symmetry of the theory (further details are available in the review [16]). The fugacities obey the balancing conditions

$$S := \prod_{a=1}^{N_f} s_a = (pq)^{N_f r_Q}, \quad T := \prod_{a=1}^{N_f} t_a = (pq)^{-N_f r_{\tilde{Q}}}$$  \hspace{1cm} (2.3)

where

$$r_Q = \frac{\tilde{N}_c}{2N_f} + x, \quad r_{\tilde{Q}} = \frac{\tilde{N}_c}{2N_f} - x$$  \hspace{1cm} (2.4)

are the $R$-charges of the fundamental and antifundamental multiplets $Q, \tilde{Q}$. $x$ captures the effects of a baryon $U(1)_B$ fugacity.

Seiberg duality implies the following mathematical identity

$$I_E^{(SU)}(N_c, N_f; s; t) = I_M^{(SU)}(\tilde{N}_c, N_f; s; t).$$  \hspace{1cm} (2.5)

It was shown by Dolan and Osborn [5] that this identity coincides with the $A_n \leftrightarrow A_m$ root systems symmetry transformation established by Rains in [27].

By gauging the baryon symmetry $U(1)_B$ it is not difficult to derive the $U(N)$ version of the identity (2.5)

$$I_E^{(U)}(N_c, N_f; s; t) = I_M^{(U)}(\tilde{N}_c, N_f; s; t)$$  \hspace{1cm} (2.6)

where

$$I_E^{(U)}(N_c, N_f; s; t) = \frac{(p; p)_{N_c}^{N_c-1}(q; q)_{N_c-1}^{N_c-1}}{N_c!} \int_{T^{N_c}} \prod_{j=1}^{N_c} dz_j \prod_{a=1}^{N_f} \prod_{j=1}^{N_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)
\int_{T^{\tilde{N}_c}} \prod_{j=1}^{\tilde{N}_c} dz_j \prod_{a=1}^{\tilde{N}_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)
\int_{T^{N_c}} \prod_{j=1}^{N_c} dz_j \prod_{a=1}^{N_f} \prod_{j=1}^{N_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)
\int_{T^{\tilde{N}_c}} \prod_{j=1}^{\tilde{N}_c} dz_j \prod_{a=1}^{\tilde{N}_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)$$

$$= \int_{S^1} \frac{dx}{2\pi i x} I_E^{(SU)}(N_c, N_f; x^{-1} s; x^{-1} t),$$  \hspace{1cm} (2.7)

and

$$I_M^{(U)}(\tilde{N}_c, N_f; s; t) = \frac{(p; p)_{\tilde{N}_c}^{\tilde{N}_c-1}(q; q)_{\tilde{N}_c-1}^{\tilde{N}_c-1}}{\tilde{N}_c!} \int_{T^{\tilde{N}_c}} \prod_{j=1}^{\tilde{N}_c} dz_j \prod_{a=1}^{\tilde{N}_c} \prod_{j=1}^{\tilde{N}_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)
\int_{T^{N_c}} \prod_{j=1}^{N_c} dz_j \prod_{a=1}^{N_f} \prod_{j=1}^{N_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)
\int_{T^{\tilde{N}_c}} \prod_{j=1}^{\tilde{N}_c} dz_j \prod_{a=1}^{\tilde{N}_c} \prod_{j=1}^{\tilde{N}_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)
\int_{T^{N_c}} \prod_{j=1}^{N_c} dz_j \prod_{a=1}^{N_f} \prod_{j=1}^{N_c} \Gamma_e(s_a z_j, t_a^{-1} z_j^{-1}; p, q)$$

$$= \int_{S^1} \frac{dx}{2\pi i x} I_M^{(SU)}(\tilde{N}_c, N_f; x^{-1} s; x^{-1} t).$$  \hspace{1cm} (2.8)
In what follows we consider a degeneration scheme based on the $U(N)$ identity (2.6). We will return to the reduction of the $SU(N)$ identity (2.5) in section 4.

2.2. First degeneration: the $S^1$ reduction

Following standard procedure we set
\[
p = e^{2\pi i \omega_1}, \quad q = e^{2\pi i \omega_2}, \quad s_a = e^{2\pi i \nu_a}, \quad t_a = e^{-2\pi i \nu_a}, \quad z_j = e^{2\pi i u_j},
\]
where $i, j = 1, \ldots, N_c, a = 1, \ldots, N_f$, and take the degeneration limit $v \to 0$. In this limit the elliptic $\Gamma_e$-functions reduce to hyperbolic $\Gamma_h$-functions and by exchanging limit and integral we obtain the degeneration formulae
\[
\lim_{v \to 0} I_{E}^{(U)}(N_c, N_f; s, t) = \sqrt{-v^2 \omega_1 \omega_2} e^{\frac{\pi i \omega(N_f^2 + 1)}{6v^2 \omega_1 \omega_2}} J_{N_c(N_f,N_f),0}(\mu; \nu; 0), \tag{2.10}
\]
\[
\lim_{v \to 0} I_{M}^{(U)}(\tilde{N}_c, N_f; s, t) = \sqrt{-v^2 \omega_1 \omega_2} e^{\frac{\pi i \omega(N_f^2 + 1)}{6v^2 \omega_1 \omega_2}} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b; \omega_1, \omega_2)
\]
\[
J_{N_c(N_f,N_f),0}(\omega - \nu; \omega - \mu; 0), \tag{2.11}
\]
where $J_{n,(s_1,s_2),t}$ is the function
\[
J_{n,(s_1,s_2),t}(\mu; \nu; 2\lambda) = \frac{1}{n!} \int \prod_{j=1}^{n} \left( \frac{du_j}{\sqrt{-\omega_1 \omega_2}} e^{\frac{2\pi i \lambda u_j}{\omega_1 \omega_2}} e^{\frac{\pi i u_j^2}{\omega_1 \omega_2}} \right) \prod_{j=1}^{n} \prod_{a=1}^{s_1} \Gamma_h(\mu_a - u_j; \omega_1, \omega_2) \prod_{b=1}^{s_2} \Gamma_h(\nu_b + u_j; \omega_1, \omega_2) \prod_{1 \leq i < j \leq n} \Gamma_h(u_i - u_j; \omega_1, \omega_2) \Gamma_h(u_j - u_i; \omega_1, \omega_2) \tag{2.12}
\]

The contour of the integral, which goes from $\Re u = -\infty$ to $\Re u = +\infty$, is chosen appropriately to avoid the poles of the $\Gamma_h$-functions (see [14] for further details).

The function $J_{N_c(N_f,N_f),0}(\mu; \nu; 0)$ that appears in the first degeneration formula (2.10) expresses the partition function of the $d = 3 \mathcal{N} = 2$ SQCD theory with gauge group $U(N_c)$. When the parameters $\omega_1, \omega_2$ are chosen to have the form
\[
\omega_1 = \mathbf{i} b, \quad \omega_2 = \mathbf{i} b^{-1}, \quad b \in \mathbb{R}_+
\]
this is a partition function on the squashed $S^3$ with squashing parameter $b$ [10]. The parameters $\mu_a, \nu_a$ are related to the real masses $m_a, \tilde{m}_a$ and $m_A$ (understood as background values of scalars for external vector multiplets of $SU(N_f)_L$, $SU(N_f)_R$, and $U(1)_A$ respectively) by the following relation
\[
\mu_a = \tilde{m}_a + m_A + \omega R_Q, \quad \nu_a = -m_a + m_A + \omega R_Q, \quad \sum_{a=1}^{N_f} m_a = \sum_{a=1}^{N_f} \tilde{m}_a = 0. \tag{2.14}
\]
$R_Q$ is the $U(1)_R$ charge of the $d = 3$ theory and
\[ \omega := \frac{\omega_1 + \omega_2}{2}. \]

With these conventions the balancing conditions (2.3) reduce to ((2.14) sets $x = 0$)
\[ \sum_{a=1}^{N_f} \mu_a = \sum_{a=1}^{N_f} \nu_a = N_f (m_A + \omega R_Q) = \tilde{N}_c \omega. \tag{2.15} \]

Similarly, the second degeneration formula (2.11) expresses the (squashed) $S^3$ partition function of a $d = 3$ $U(N_c)$ SYM theory. If this limit captures correctly the reduction to Aharony duality, then the partition function on the rhs of eq. (2.11) ought to be the partition function of the magnetic dual of the $d = 3 \mathcal{N} = 2 U(N_c)$ SQCD theory. This is indeed the case. Combining the $d = 4$ duality transformation property (2.6) with the degeneration formulae (2.10), (2.11) we obtain the identity
\[ J_{N_c,(N_f,N_f),0}(\mu;\nu;0) = \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b; \omega_1, \omega_2) \tilde{J}_{\tilde{N}_c,(N_f,N_f),0}(\omega - \nu; \omega - \mu;0), \tag{2.16} \]

which is a special case of eq. (5.5.21) in Theorem 5.5.11 of Ref. [14] that expresses Aharony duality. To the best of our knowledge this particular derivation of the identity (2.16) has not appeared in the literature before.

In the next subsection we will see that the balancing condition (2.15), which was inherited from four dimensions, trivializes the contribution of the gauge-singlet chiral superfields $V_{\pm}$ that are part of the magnetic description of the $d = 3 \mathcal{N} = 2$ SQCD theory and makes them invisible in the degeneration formula (2.11). Hence, without relaxing the conditions (2.15), it is impossible to read off the complete matter content of the magnetic theory. The general form of the identities implied by Aharony duality can be obtained by further degeneration limits that remove the conditions (2.15).

2.3. Second degeneration: removal of the balancing conditions and FI terms

The second degeneration step removes two pairs of quark supermultiplets by sending two pairs of real masses with opposite signs to infinity. In order to obtain a final theory with $N_f$ quark supermultiplets we start from the identity (2.16) renaming
\[ N_f \rightarrow N_f + 2, \quad \tilde{N}_c \rightarrow \tilde{N}_c + 2. \tag{2.17} \]
We keep the definition $\tilde{N}_c = N_f - N_c$ unchanged. In the resulting expression we set

$$\mu_{N_f+1} = \xi_1 + \alpha S, \quad \nu_{N_f+1} = \zeta_1 - \alpha S,$$  \hspace{1cm} (2.18)

$$\mu_{N_f+2} = \xi_2 - \alpha S, \quad \nu_{N_f+2} = \zeta_2 + \alpha S$$  \hspace{1cm} (2.19)

and eventually take the limit $S \to +\infty$. $\alpha$ is a pure phase chosen in a manner that allows to perform the ensuing standard reductions by exchanging limits and integrals \cite{[14]}.

With this ansatz the balancing conditions (2.15) become

$$\sum_{a=1}^{N_f} \mu_a + \xi_1 + \xi_2 = \sum_{a=1}^{N_f} \nu_a + \zeta_1 + \zeta_2 = N_f (m_A + \omega R_Q) = \tilde{N}_c \omega$$  \hspace{1cm} (2.20)

freeding the parameters $\mu_a, \nu_a$ ($a = 1, \ldots, N_f$) from any constraints. It will be convenient to define an additional parameter

$$\lambda := -\xi_1 - \zeta_1 + \tilde{N}_c \omega - \frac{1}{2} \sum_{a=1}^{N_f} (\mu_a + \nu_a)$$  \hspace{1cm} (2.21)

in terms of which we obtain the expressions

$$\xi_1 + \zeta_1 = -\lambda + \tilde{N}_c \omega - \frac{1}{2} \sum_{a=1}^{N_f} (\mu_a + \nu_a), \quad \xi_2 + \zeta_2 = \lambda + \tilde{N}_c \omega - \frac{1}{2} \sum_{a=1}^{N_f} (\mu_a + \nu_a).$$  \hspace{1cm} (2.22)

Applying the standard reduction identity

$$\lim_{S \to \infty} J_{n,(s_1+2,s_2+2),t}(\mu, \xi_1 + \alpha S, \xi_2 - \alpha S; \nu, \zeta_1 - \alpha S, \zeta_2 + \alpha S; 2\lambda + \xi_1 + \zeta_1 - \xi_2 - \zeta_2)$$

$$\quad \text{e}^{\frac{\pi i n}{\omega_1 \omega_2}} (\xi_1 - \alpha S - \omega)^2 + (\xi_2 - \alpha S - \omega)^2 - (\xi_1 + \alpha S - \omega)^2 - (\xi_2 + \alpha S - \omega)^2)$$

$$= J_{n,(s_1,s_2),\lambda}(\mu, \nu; 2\lambda)$$  \hspace{1cm} (2.23)

to the case at hand

$$t = 0, \quad n = N_c, \quad s_1 = s_2 = N_f$$  \hspace{1cm} (2.24)

we deduce the limit

$$\lim_{S \to \infty} J_{N_c,(N_f+2,N_f+2),0}(\mu, \xi_1 + \alpha S, \xi_2 - \alpha S; \nu, \zeta_1 - \alpha S, \zeta_2 + \alpha S; 0)$$

$$\text{e}^{\frac{\pi i N_c}{\omega_1 \omega_2}} (\xi_1 - \alpha S - \omega)^2 + (\xi_2 - \alpha S - \omega)^2 - (\xi_1 + \alpha S - \omega)^2 - (\xi_2 + \alpha S - \omega)^2)$$  \hspace{1cm} (2.25)

$$= J_{N_c,(N_f,N_f),0}(\mu, \nu; 2\lambda).$$
The rhs of this equation expresses the (squashed) \( S^3 \) partition function of the electric description of the \( d = 3, \mathcal{N} = 2 \) \( U(N_c) \) SQCD theory with FI term \( \lambda \) and no restrictions on the real mass parameters \( \mu_a, \nu_a \). This explains how the degeneration limit \( S \to \infty \) acts on the lhs of the duality relation (2.16).

The effect of the limit on the magnetic side of the duality follows by inserting the transformation (2.16) into the lhs of the reduction formula (2.25)

\[
\lim_{S \to \infty} \left( \prod_{a,b=1}^{N_f+2} \Gamma_h(\mu_a + \nu_b; \omega_1, \omega_2) \right) J_{\tilde{N}_c, (N_f+2,N_f+2),0} \omega - \nu, \omega - \zeta_1 + \alpha S, \omega - \zeta_2 - \alpha S; \omega - \mu, \omega - \xi_1 - \alpha S, \omega - \xi_2 + \alpha S; 0 \right) \]

\[
\frac{\pi \lambda \omega_1 \omega_2}{\omega_{1,2}} \sum_{a=1}^{N_f} \left( \mu_a - \nu_a \right) .
\]

(2.26)

We have denoted the result of this limit by using a function \( Z_M \). It is clear that \( Z_M \) cannot be obtained with the application of the standard reduction formula (2.23). That formula would reduce to the function \( J_{\tilde{N}_c, (N_f,N_f),0} \) keeping the number of integration variables invariant. This is in direct contradiction with the basic duality formula \( \tilde{N}_c = N_f - N_c \), which is already apparent from the duality identity (2.16). We conclude that the degeneration limit (2.26) is mathematically more involved than the standard one in (2.23) and cannot be obtained by exchanging limits and integrals. We will call such degeneration limits ‘non-standard’ to distinguish them from the standard ones that play a prominent role in the degeneration schemes of Ref. [14]. Unfortunately, we are not aware of an efficient computational method for such limits, but we will have more to say about them in the next section.

It is mathematically interesting that the alternate degeneration scheme of Ref. [14] allows us to bypass this complicated reduction formula and derive the function \( Z_M \) by using a significantly different scheme based only on standard reductions. The result,

\[1\) There is no a priori reason to anticipate that this alternate route will be a generic possibility. We expect non-standard reductions, like the one above, to be one of the main steps in general reductions of \( d = 4 \) dualities to \( d = 3 \) dualities. The example of \( SU(N) \) dualities in section 4 appears to be an illustration of this statement.
which follows from eq. (5.5.21) in Theorem 5.5.11 of [14], determines $Z_M$ as the dual of the rhs of eq. (2.25)

$$Z_M(\tilde{N}_c, N_f; \mu; \nu; \lambda) = \Gamma_h \left( (\tilde{N}_c + 1) \omega - \frac{1}{2} \sum_{a=1}^{N_f} (\mu_a + \nu_a) \pm \lambda \right) \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b)$$

$$J_{\tilde{N}_c,(N_f,N_f),0}(\omega - \nu; \omega - \mu; -2\lambda) .$$

The first $\Gamma_h$ factor on the rhs of this equation captures the contribution of the gauge-singlet multiplets $V_\pm$. This factor is invisible in the special case of the balancing condition (2.15) since

$$\Gamma_h^2 \left( (\tilde{N}_c + 1) \omega - \frac{1}{2} \sum_{a=1}^{N_f} (\mu_a + \nu_a) \right) = \Gamma_h^2(\omega) = 1 .$$

The second term, which is a product of $\Gamma_h$-functions, captures the contribution of the $N^2_f$ gauge-singlet meson superfields of the dual description. The contribution of the dual gauge fields and quarks comes into the last factor $J_{\tilde{N}_c,(N_f,N_f),0}$.

### 2.4. Third degeneration: Chern-Simons interactions

There is a standard third reduction which corresponds to integrating out real masses with the same sign. This operation introduces the Chern-Simons interaction. The resulting Chern-Simons-matter theories exhibit the Giveon-Kutasov duality [25]. Since this is a well known standard step we will not discuss it explicitly here. For completeness and later convenience we list the (squashed) $S^3$ partition functions for the electric and magnetic descriptions of the $d = 3 \ N = 2 \ U(N_c)$ Chern-Simons theory at level $k$ coupled to $N_f$ pairs of (anti)fundamental supermultiplets, and the duality transformation property that relates them. Without loss of generality we assume that the level $k$ is positive.

The electric and magnetic partition functions have respectively the following forms

$$Z_E(\tilde{N}_c, N_f, k; \mu; \nu; \lambda) = J_{\tilde{N}_c,(N_f,N_f),2k}(\mu; \nu; 2\lambda) ,$$

$$Z_M(\tilde{N}_c, N_f, k; \mu; \nu; \lambda) = \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b) J_{\tilde{N}_c,(N_f,N_f),-2k}(\omega - \nu; \omega - \mu; -2\lambda) .$$

$\lambda$ denotes again a FI term and $\mu, \nu$ are vectors of real mass parameters. The Giveon-Kutasov duality requires the transformation property

$$Z_E(\tilde{N}_c, N_f, k; \mu; \nu; \lambda) = e^{i\theta(N_c, N_f, k; \mu; \nu; 2\lambda)} Z_M(\tilde{N}_c, N_f, k; \mu; \nu; \lambda)$$
where
\[
e^{i\vartheta(N_c, N_f; k; \mu; \nu; \lambda)} := e^{\frac{\pi(\omega_1^2 + \omega_2^2)(k^2 + 2)}{24\omega_1\omega_2}} e^{-\frac{\pi}{4\omega_1\omega_2}(\lambda^2 + 2k\omega^2(\tilde{N}_c - N_c))}
\]
\[
e^{-\frac{\pi}{4\omega_1\omega_2}(\lambda(\sum_a \nu_a - \sum_a \mu_a) + 2k\omega(2N_f\omega - \sum_a \mu_a - \sum_a \nu_a))}.
\]

Eq. (2.31) is the last identity of Theorem 5.5.11 in [14] as was already noticed in [21].

3. A lesson from Giveon-Kutasov duality tests

In the original work on Seiberg duality in $\mathcal{N} = 2$ Chern-Simons-matter theories [25], several checks were performed on the duality using a D-brane setup of D3, D5, NS5 and $(1, k)$ fivebrane bound states. One of these checks aimed to verify that the duality is consistent with the limit where equal masses with opposite sign for a quark pair $(Q_1, \tilde{Q}_1)$ are sent to infinity removing the respective supermultiplets. Since this limit is a clean example of the non-standard reduction that we discussed in the previous section, it will be instructive to consider it here in more detail from the partition function point of view.

An interesting feature of this reduction, that was also noted in [25], is the fact that it involves two separate supersymmetric vacua. In other words, the reduction can be performed in two different ways. In vacuum 1, $N_c \rightarrow \bar{N}_c$, $N_f \rightarrow N_f - 1$ on the electric side; on the magnetic side $\tilde{N}_c \rightarrow \tilde{N}_c - 1$, $N_f \rightarrow N_f - 1$. In vacuum 2, $N_c \rightarrow \bar{N}_c - 1$, $N_f \rightarrow N_f - 1$ on the electric side, and $\tilde{N}_c \rightarrow \bar{N}_c$, $N_f \rightarrow N_f - 1$ on the magnetic side. We notice that there is a possibility of two different types of reductions: a standard one that keeps the rank of the gauge group (equivalently, the number of integration variables in the $S^3$ partition function) invariant, and a non-standard one that changes the rank of the gauge group.

In the D-brane interpretation of the duality, [23], the following branes in $\mathbb{R}^{1,9}$ participate

\[
\begin{align*}
0 & 1 2 3 4 5 6 7 8 9 \\
1 \text{ NS5} : & \bullet \bullet \bullet \bullet \bullet \\
1 (1, k) : & \bullet \bullet \circ \circ \bullet \bullet \\
N_c \text{ D3} : & \bullet \bullet \bullet \bullet \\
N_f \text{ D5} : & \bullet \bullet \bullet \bullet
\end{align*}
\]
The circles ◦ indicate that the brane is oriented along a line in the (37) plane. Giving equal and opposite real masses to a quark pair corresponds to moving the corresponding D5 brane away from the D3 branes in the 3-direction. In vacuum 1 the D3 branes continue to stretch between the NS5 brane and the (1, k) fivebrane. In vacuum 2 the D3 branes break on the displaced D5 brane.

On the level of the $S^3$ partition functions (2.29), (2.30) we set

$$\mu_1 = \xi + \alpha S, \quad \nu_1 = \zeta - \alpha S$$

and eventually take the limit $S \to +\infty$. This is a slightly simpler version of the reductions (2.18), (2.19) in subsection 2.3. In the absence of balancing conditions it is now possible to consider the limit on a single quark-antiquark supermultiplet pair.

3.1. Vacuum 1

We perform the standard reduction on the electric side

$$\lim_{S \to \infty} J_{N_c, (N_f, N_f), 2k}(\mu, \xi + \alpha S; \nu, \zeta - \alpha S; 2\lambda + \xi + \zeta - 2\omega)e^{\frac{\pi N_c}{2} ((\xi - \alpha S - \omega)^2 - (\xi + \alpha S - \omega)^2)} = J_{N_c, (N_f-1, N_f-1), 2k}(\mu; \nu; 2\lambda).$$

This degeneration formula is the second formula in Proposition 5.3.24 of Ref. [14] for $\tau = \omega$. It holds with certain assumptions on the external parameters $(\mu, \nu)$, $(\xi, \zeta)$, and $\varphi = \text{arg}(\alpha)$, which are listed in [14]. The assumption we want to single out is the assumption on $\varphi$

$$\varphi \in \left[\varphi_-, \varphi_+ + \frac{\varphi_- + \varphi_+}{2}\right] \cap (\varphi_\omega - \pi, \varphi_\omega),$$

where

$$\varphi_\omega = \text{arg}(\omega), \quad \varphi_+ = \max(\text{arg}(\omega_1), \text{arg}(\omega_2)), \quad \varphi_- = \min(\text{arg}(\omega_1), \text{arg}(\omega_2)).$$

In the case of physical interest (2.13)

$$\varphi_- = \varphi_+ = \varphi_\omega = \frac{\pi}{2} \quad \text{and} \quad \varphi \in \left(-\frac{\pi}{2}, 0\right).$$

The constraint (3.4) restricts the direction along which we take the limit and is instrumental when we exchange the limit and integral to derive the degeneration formula (3.3).
The corresponding action on the magnetic side follows from (3.3) with the use of the transformation identity (2.31) on both sides of the equation
\[
\lim_{S \to \infty} \left[ e^{i\theta(N_c,N_f,k;\mu,\xi+\alpha S;\nu,\zeta-\alpha S;2\lambda+\xi+\zeta-2\omega)} e^{\frac{\pi i N_e}{2}((\xi-\alpha S-\omega)^2-(\xi+\alpha S-\omega)^2)} \right] 
\]
\[
Z_M \left( \tilde{N}_c, N_f, k; \mu, \xi + \alpha S; \nu, \zeta - \alpha S; \lambda + \frac{1}{2}(\xi + \zeta) \right) 
= e^{i\theta(N_c,N_f-1,k;\mu,\nu;2\lambda)} Z_M(\tilde{N}_c - 1, N_f - 1, k; \mu, \nu; \lambda) 
\]
giving the expected reduction of the rank of the dual gauge group \( \tilde{N}_c \to \tilde{N}_c - 1 \).

Formula (3.7) is a clean example of what we call a non-standard reduction. Notice, that as soon as we assume the validity of (3.4) in order to implement (3.3) on the electric side, we no longer have the option of a standard reduction, where we exchange limit and integral, on the magnetic side. Indeed, that option on the magnetic side (3.7) would require taking in addition
\[
\varphi \in \left( \frac{\varphi + \varphi - \pi}{2}, \varphi_+ \right) \cap (\varphi_-, \varphi_+) \quad (3.8)
\]
which has zero intersection with (3.4). Hence, we are forced uniquely by duality to reduce on the magnetic side along the lines of eq. (3.7).

3.2. Vacuum 2

In this case we make a different choice. We adopt (3.8) and perform a standard reduction on the magnetic side
\[
\lim_{S \to \infty} \left[ e^{\frac{\pi i N_e}{2}((\xi+\alpha S)^2-(\xi-\alpha S)^2)} \prod_{a=2}^{N_f} e^{\frac{\pi i}{2}((\xi+\alpha S+\nu_a)^2-(\xi-\alpha S+\mu_a)^2)} \right] 
\]
\[
Z_M \left( \tilde{N}_c, N_f, k; \mu, \xi + \alpha S; \nu, \zeta - \alpha S; \lambda + \frac{1}{2}(\xi + \zeta) \right) 
= \Gamma_h(\xi + \zeta) Z_M(\tilde{N}_c, N_f - 1, k; \mu, \nu; \lambda) .
\]

Then, combining this formula with the duality relation we obtain a new non-standard reduction on the electric side
\[
\lim_{S \to \infty} \left[ e^{\frac{\pi i N_e}{2}((\xi+\alpha S)^2-(\xi-\alpha S)^2)} \prod_{a=2}^{N_f} e^{\frac{\pi i}{2}((\xi+\alpha S+\nu_a)^2-(\xi-\alpha S+\mu_a)^2)} \right] 
\]
\[
e^{-i\theta(N_c,N_f,k;\mu,\xi+\alpha S;\nu,\zeta-\alpha S;2\lambda+\xi+\zeta)} Z_E \left( N_c, N_f, k; \mu, \xi + \alpha S; \nu, \zeta - \alpha S; \lambda + \frac{1}{2}(\xi + \zeta) \right) 
= e^{-i\theta(N_c-1,N_f-1,k;\mu,\nu;2\lambda)} \Gamma_h(\xi + \zeta) Z_E(N_c - 1, N_f - 1, k; \mu, \nu; \lambda) .
\]
The factor $\Gamma_h(\xi + \zeta)$ also comes out in accordance with the D-brane picture. As we mentioned above, in vacuum 2 the D3 branes break on the displaced D5 brane. Hence, as the D5 moves away along $x^3$, half of the broken D3 stretches between the NS5 and the D5, while the other half stretches between the D5 and the $(1, k)$ fivebrane. The latter half gives rise to a meson supermultiplet degree of freedom that decouples from the rest of the theory. The extra factor $\Gamma_h(\xi + \zeta)$ in the rhs of eqs. (3.9)-(3.10) accounts correctly for this additional decoupled degree of freedom.

4. Towards 3d Seiberg duality with $SU(N)$ gauge group

By gauging the baryon symmetry $U(1)_B$ of the $d = 4 \mathcal{N} = 1$ SQCD theory and then reducing its SCI we recovered the well known partition function identities required by the Aharony and Giveon-Kutasov dualities for the $d = 3 \mathcal{N} = 2$ SQCD theories with gauge group $U(N_c)$. Similar dualities for the $d = 3 \mathcal{N} = 2$ SQCD theory with gauge group $SU(N_c)$ (in the presence or absence of Chern-Simons interactions) have long been suspected to exist, but a viable proposal has not been proposed so far. The general philosophy of this work suggests the following approach to this problem.

Reducing the SCI (2.1) of the four dimensional SQCD theory without gauging the baryon symmetry we obtain the (squashed) $S^3$ partition function of the $d = 3 \mathcal{N} = 2$ SQCD theory with gauge group $SU(N_c)$ (see also Theorem 4.6 of [18])

$$\lim_{v \to 0} I_E^{SU}(N_c, N_f; s; t; p, q) = e^{\frac{\pi \omega}{6(\omega^2+1)}} \tilde{J}_{N_c,(N_f,N_f),0}(\mu; \nu)$$ (4.1)

where we have defined

$$\tilde{J}_{N_c,(N_f,N_f),0}(\mu; \nu) =$$

$$\frac{1}{N_c!} \int \prod_{j=1}^{N_c-1} du_j \prod_{a=1}^{N_f} \prod_{j=1}^{N_c} \Gamma_h(\mu_a - u_j; \omega_1, \omega_2) \Gamma_h(\nu_a + u_j; \omega_1, \omega_2) \prod_{1 \leq i < j \leq N_c} \Gamma_h(u_i - u_j; \omega_1, \omega_2) \Gamma_h(u_j - u_i; \omega_1, \omega_2) \sum_{j=1}^{N_c} u_j = 0.$$ (4.2)

A similar reduction on the magnetic description of the four dimensional theory gives

$$\lim_{v \to 0} I_M^{SU}(\bar{N}_c, N_f; s; t; p, q) = e^{\frac{\pi \omega}{6(\omega^2+1)}} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b; \omega_1, \omega_2) \tilde{J}_{\bar{N}_c,(N_f,N_f),0}(\omega - \mu; \omega - \nu).$$ (4.3)
Consequently, the duality identity (2.5) implies

\[ \tilde{J}_{N_c,(N_f,N_f),0}(\mu;\nu) = \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b; \omega_1, \omega_2) \tilde{J}_{N_c,(N_f,N_f),0}(\omega - \mu; \omega - \nu) . \]  

(4.4)

The balancing conditions (2.3) reduce to

\[ ST^{-1} = (pq)^\tilde{N}_c \Rightarrow \sum_{a=1}^{N_f} (\mu_a + \nu_a) = 2 \tilde{N}_c \omega . \]  

(4.5)

This condition does not allow a straightforward field theory interpretation of the duality relation (4.4). A second degeneration step based on the ansatz (2.18), (2.19) can be applied on both sides to remove the balancing condition and lead to the \( SU(N_c) \) version of Aharony duality. On the electric side this is a straightforward standard reduction. On the magnetic side this is a non-standard reduction. Unfortunately, the absence of an efficient computational method for such reductions hinders the completion of this exercise. The result would allow us to determine the (squashed) \( S^3 \) partition function of the magnetic theory from which its full matter content can be determined. A third standard reduction that sends equal real masses of the same sign to infinity can then be used to determine the \( SU(N_c) \) version of the Giveon-Kutasov duality.

5. Other features of degeneration schemes

In this paper we discussed explicitly on the level of SCIIs and \( S^3 \) partition functions how the standard Seiberg duality for the four dimensional \( \mathcal{N} = 1 \) SQCD theory reduces in three dimensions to Aharony and Giveon-Kutasov dualities for \( \mathcal{N} = 2 \) SQCD theories. On a mathematical level we argued that this reduction entails a set of non-standard degeneration identities which cannot be determined by exchanging limits and integrals. An efficient method for the computation of such identities remains an open problem. On a physical level we proposed that such degenerations can be used to determine the precise properties of the still illusive Aharony and Giveon-Kutasov dualities for \( d = 3 \mathcal{N} = 2 \) SQCD theories with \( SU(N) \) gauge group. Analogous reduction schemes can be envisioned for generic \( d = 4 \) Seiberg dualities [16].

In a general 3d/4d connection the degeneration formula

\[ \lim_{v \to 0} I = Z \]  

(5.1)
between a four dimensional SCI $I$ and a three dimensional partition function $Z$ can be useful in relating also other properties of the four and three dimensional theories. An example that deserves further study has to do with spontaneous supersymmetry breaking.

If $I$ is zero in a certain regime (independent of fugacities, but dependent on parameters like $N_c, N_f$ etc.), then according to (5.1) $Z$ will also be zero in that regime. In [33] we conjectured that zeros of $Z$ are related to spontaneous supersymmetry breaking in the three dimensional theory. The opposite argument may not be true, namely it is not apriori obvious from eq. (5.1) whether $Z = 0$ implies $I = 0$.

In the specific SQCD example of this note the following spontaneous supersymmetry breaking patterns occur. The four dimensional $\mathcal{N} = 1$ SQCD theory exhibits spontaneous supersymmetry breaking for $N_f < N_c$. One can readily check (using properties of the elliptic $\Gamma_e$-functions) that the SCI index vanishes when $N_f < N_c$ and is non-zero otherwise. By reduction the same property carries over to the $S^3$ partition function of the $\mathcal{N} = 2$ SQCD theory without CS interactions (for $N_f < N_c$) and the $S^3$ partition function of the $\mathcal{N} = 2$ SQCD theory with CS interactions (for $N_f + k < N_c$ and CS level $k$). Without CS interactions it is indeed known that a dynamically generated superpotential lifts the space of supersymmetric vacua for $N_f < N_c - 1$. The case $N_f = N_c - 1$ is a bit more tricky, as was already pointed out in [22]. In that case there is a smooth moduli space of supersymmetric vacua. The vanishing of the hyperbolic hypergeometric integral expression based on the standard UV description of the theory does not reflect this fact, presumably because of accidental symmetries. This subtlety, however, does not appear when we make further reductions to obtain Chern-Simons-matter theories. In that case spontaneous supersymmetry breaking occurs precisely when the UV description-based $S^3$ partition function vanishes.

We believe that such similarities in spontaneous supersymmetry breaking patterns between four and three dimensional theories related by $S^1$ reductions are naturally explained by degeneration formulae of the type (5.1) in the manner outlined above. Note that the precise mechanism of spontaneous breaking of supersymmetry in general depends on the number of spacetime dimensions. The details of this connection deserve further study.

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