All Exact Solutions of Non-Abelian Vortices
from Yang-Mills Instantons

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Abstract

We successfully exhaust the complete set of exact solutions of non-Abelian vortices in a quiver
gauge theory, that is, the $S[\text{U}(N) \times \text{U}(N)]$ gauge theory with a bi-fundamental scalar field on a
hyperbolic plane with a certain curvature, from $SO(3)$-invariant $SU(2N)$ Yang-Mills instanton
solutions. This work provides, for the first time, exact non-Abelian vortex solutions. We establish
the ADHM construction for non-Abelian vortices and identify all the moduli parameters and the
complete moduli space.
I. INTRODUCTION

Since the discovery of non-Abelian vortices [1–3], they have been studied extensively [4–6]. While they are a natural extension of Abelian vortices [7, 8] appearing in conventional superconductors, their analogues also appear in high-density quantum chromodynamics (QCD) showing color superconductivity [9]. In supersymmetric gauge theories, they are Bogomol’nyi-Prasad-Sommerfield (BPS) solitons [10] and are stable not only classically but also perturbatively and non-perturbatively. BPS non-Abelian vortices serve as an elegant tool to demonstrate [11] the coincidence of BPS spectra in four-dimensional gauge theories and two-dimensional sigma models [12]. Non-Abelian vortices also play prominent roles as instantons in non-perturbative dynamics of gauge theories in lower dimensions, similar to the role of Yang-Mills instantons [13] in four dimensions; the non-perturbative partition function has been extensively studied by the vortex counting in $N=(2,2)$ supersymmetric gauge theories in two dimensions [14], similar to the instanton counting in four dimensions [15].

However, vortex equations are not integrable even in the BPS limit [10], and explicit solutions and the moduli space metric are not available. This is in contrast to the case of the self-dual equations for Yang-Mills instantons, for which the well-known Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction is available [17]. Thus far, some efforts to obtain the moduli space have been made. The moduli space of non-Abelian vortices was determined without the moduli space metric by solving half of the BPS equations [3]. The moduli space metric was obtained implicitly with a matrix function satisfying a differential equation [18]. The asymptotic metric for well-separated vortices was obtained [19] to study the low-energy scattering [20]. The metrics were also obtained on submanifolds for the coincidence limit [21] and on the symmetry orbits [22]. However, the full moduli space is far beyond our reach because of the non-integrability.

Nevertheless, with changes to the geometry, the situation can become totally different. The BPS Abelian vortex equations on the hyperbolic plane $H^2$ of curvature $-1/2$ are integrable [23], and a general formula for the exact moduli space metric has been obtained [24]. The integrability is a consequence of the fact that these vortices are obtained as a dimensional reduction from $SO(3)$-symmetric Yang-Mills instantons on flat space $\mathbb{R}^4$ [23]. More generally, the vortex equations on Riemann surfaces $\Sigma$ are integrable, when they are
obtained from self-dual Yang-Mills equations on $\Sigma \times S^2$ \cite{25}. Recently, hyperbolic vortices have been studied extensively \cite{26}. In particular, BPS non-Abelian vortex equations in a quiver gauge theory, \textit{i.e.}, $S[U(N) \times U(N)]$ gauge theory coupled with a bi-fundamental scalar field on a hyperbolic space were obtained from $SO(3)$-symmetric $SU(2N)$ Yang-Mills instantons in a previous study \cite{27}. However, that study only considered embedding of the Abelian vortex solutions into the diagonal $U(1)^N$ subgroup. The same vortices on a flat space were also studied in another work \cite{28}. However, there remain open question on whether these vortices have non-trivial orientational moduli or what the complete set of solutions is.

In this Letter, we construct, for the first time, a complete set of all the exact solutions of non-Abelian vortices in the $S[U(N) \times U(N)]$ gauge theory with a bi-fundamental scalar field on a hyperbolic plane with a certain curvature. We use $SO(3)$-invariant $SU(2N)$ Yang-Mills instanton solutions. We also establish the ADHM construction for non-Abelian vortices and identify all the moduli parameters and the complete moduli space.

II. HYPERBOLIC VORTICES FROM $SO(3)$-INVARIANT INSTANTONS

A. $S[U(N) \times U(N)]$ vortices on a hyperbolic plane

We consider a hyperbolic plane $\mathbb{H}^2$ as the upper half plane with a complex coordinate $z$, with $r \equiv \text{Im } z > 0$, endowed with the metric

$$g_{zz} = -\frac{2R^2}{(z - \bar{z})^2} = \frac{R^2}{2r^2}. \quad (1)$$

The constant $R$ is related to the scalar curvature $-1/R^2$. See Appendix A.

Let us consider the $U(N) \times U(N)$ gauge theory with gauge fields $A_z(z, \bar{z})$ and $\tilde{A}_z(z, \bar{z})$, coupled with a single bi-fundamental Higgs field $H(z, \bar{z})$. Note that the overall $U(1)$ gauge group is trivial, and hence, the actual gauge group is $S[U(N) \times U(N)]$. For simplicity, we take the gauge coupling $g$ to be common for all gauge groups. The covariant derivative is $\mathcal{D}_z H = (\partial_z + iA_z - iH \bar{A}_z)$. In this setup, the action of our model is expressed as

$$S = v^2 \int d^2x \sigma \text{tr} \left[ \frac{1}{\sigma^2} |F_{zz}|^2 + \frac{1}{\sigma^2} |\tilde{F}_{zz}|^2 + \frac{2}{\sigma} |\mathcal{D}_z H|^2 + \frac{2}{\sigma} |\mathcal{D}_\bar{z} H|^2 \right.$$

$$\left. + \frac{\lambda}{4} (HH^\dagger - 1_N)^2 + \frac{\lambda}{4} (H^\dagger H - 1_N)^2 \right]. \quad (2)$$
where the Higgs field $H$ is rescaled so that its vacuum expectation value $v$ becomes the overall constant of the Lagrangian. The function $\sigma$ is the rescaled hyperbolic metric defined by

$$\sigma \equiv \frac{g^2v^2}{2}g_{zz} = \frac{g^2v^2R^2}{4r^2}. \quad (3)$$

In this Letter, we consider the critical coupling (the BPS limit) $\lambda^4 = 1$ and the "integrable" case:

$$R = \frac{1}{gv} \quad \iff \quad \sigma = \frac{1}{4r^2}. \quad (4)$$

The action in Eq. (2) can be rewritten in the following form:

$$E = v^2 \int d^2x \left[ \sigma \left| i\sigma^{-1}F_{zz} + HH^\dagger - 1_N \right|^2 + \sigma \left| i\sigma^{-1}\tilde{F}_{zz} - H^\dagger H + 1_N \right|^2 + 4|D_z H|^2 \right. \nonumber$$

$$\left. + 2D_z(D_z H H^\dagger) - 2D_z(D_z H H^\dagger) + 2i(F_{zz} - \tilde{F}_{zz}) \right]. \quad (5)$$

Since the covariant derivative of the scalar field $D_\mu H$ should vanish at the boundary of the hyperbolic plane, the lower bound of the action is given by

$$S \geq -v^2 \int_{\mathbb{H}} \text{tr} (F - \tilde{F}). \quad (6)$$

This Bogomol’nyi bound is saturated if the following BPS vortex equations are satisfied

$$0 = D_z H, \quad (7)$$

$$0 = i\sigma^{-1}F_{zz} + HH^\dagger - 1_N, \quad (8)$$

$$0 = i\sigma^{-1}\tilde{F}_{zz} - H^\dagger H + 1_N. \quad (9)$$

**B. SU(2N) instanton to $S[U(N) \times U(N)]$ vortices**

Here, we consider the $SO(3)$-rotationally-invariant $SU(2N)$ Yang-Mills instantons in four-dimensional Euclidean space $\mathbb{R}^4$. The $SO(3)$ action on $\mathbb{R}^4$ rotates the coordinates $(x_1, x_2, x_3)$ and leaves the $x^4$-axis as fixed, while the $SO(3)$ orbit of a point is $S^2$, as shown in Fig. 1. We obtain an upper-half plane $\mathbb{H}^2$ by an $S^2$-dimensional reduction from $\mathbb{R}^4$ with the $SO(3)$ fixed line (the $x^4$-axis) removed. Since the classical pure Yang-Mills theory is conformally invariant, the conformal equivalence, $\mathbb{R}^4 - \mathbb{R} \sim \mathbb{H}^2 \times S^2$, implies that the $SO(3)$-invariant $SU(2)$ instantons are reduced by an $S^2$-dimensional reduction to $U(1)$ Abelian-Higgs vortices.
FIG. 1: We consider the $SO(3)$ action on $(x^1, x^2, x^3)$. The $x^4$ axis is a fixed line. The grey region is an upper-half plane or a hyperbolic surface $\mathbb{H}^2$.

on a hyperbolic plane $\mathbb{H}^2$ with a specific curvature [23]. Here, we extend this relation to the non-Abelian case [27].

First, let us consider the generators of the $SU(2N)$ gauge group, which are invariant under the diagonal group of the spatial rotation $SO(3)$ and $SU(2) \subset SU(2N)$ generated by $1_N \otimes \sigma_i$. It is convenient to take the following basis for the $SU(2)$-invariant generators

$$\Lambda = T \otimes P, \quad \tilde{\Lambda} = \tilde{T} \otimes \tilde{P}, \quad (10)$$

where $T$ and $\tilde{T}$ are $N$-by-$N$ Hermitian matrices that can be viewed as the generators of $S[U(N) \times U(N)]$. The 2-by-2 matrices $P$ and $\tilde{P}$ are the projection operators defined by

$$P \equiv \frac{1_2 - \hat{x}_i \sigma_i}{2}, \quad \tilde{P} \equiv \frac{1_2 + \hat{x}_i \sigma_i}{2}, \quad (11)$$

where $\sigma_i$ ($i = 1, 2, 3$) denote the Pauli matrices. We define the complex coordinate on $\mathbb{H}^2$ by $z \equiv x^4 + ir$ with $r \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, whereas a unit vector for $S^2$ is denoted by $\hat{x}_i \equiv x_i/r$.

The general $SU(2)$-invariant gauge one-form on $\mathbb{R}^4$ takes the form

$$A_{4d} = A \otimes P + \tilde{A} \otimes \tilde{P} - \frac{1}{2}(H - 1_N) \otimes \omega - \frac{1}{2}(H^\dagger - 1_N) \otimes \omega^\dagger, \quad (12)$$

where $\omega$ is the $SU(2)$-invariant one-form on $S^2$ defined by

$$\omega \equiv iP\sigma_i d\hat{x}_i = i\sigma_i d\hat{x}_i \tilde{P} = \frac{i}{2r} (\delta_{ij} - \hat{x}_i \hat{x}_j - i\epsilon_{ijk} \hat{x}_k) \sigma_j dx_i. \quad (13)$$
As we will see, the one-forms $A$ and $\tilde{A}$ can be interpreted as the gauge fields on $\mathbb{H}^2$. For the gauge field (12), the field strength $F_{4d} = dA_{4d} + iA_{4d} \wedge A_{4d}$ is given by

$$F_{4d} = F \otimes P + \tilde{F} \otimes \tilde{P} + \frac{1}{2} DH \otimes \omega + \frac{1}{2} D H^\dagger \otimes \omega^\dagger - \frac{i}{4} (H H^\dagger - 1_N) \otimes \omega^\dagger \wedge \omega - \frac{i}{4} (H^\dagger H - 1_N) \otimes \omega \wedge \omega^\dagger. \tag{14}$$

Substituting this field strength into the Yang-Mills action and integrating over $S^2$, we find that the 4d Yang-Mills action reduces to the action of the 2d $U(N) \times U(N)$ gauge theory for the integrable case (4)

$$\frac{1}{g_4^2} \int S^2 \text{Tr}[F_{4d} \wedge *F_{4d}] = \{\text{Eq. (2) with Eq. (1)}\}, \quad v^2 = \frac{4\pi}{g_4^2}, \tag{15}$$

where we have used the following relations

$$* (dz \wedge d\bar{z}) \otimes 1_2 = i^2 (\omega^\dagger \wedge \omega - \omega \wedge \omega^\dagger), \tag{16}$$

$$* (dz \wedge \omega) = -dz \wedge \omega, \tag{17}$$

$$* (d\bar{z} \wedge \omega) = d\bar{z} \wedge \omega, \tag{18}$$

$$\int_{S^2} \frac{i}{2} \text{tr}[\omega \wedge \omega^\dagger] = 4\pi. \tag{19}$$

Similarly, the topological charge of instantons in 4d reduces to that of vortices in 2d

$$- \frac{1}{g_{4d}^2} \int_{\mathbb{R}^4} \text{tr} [F \wedge *F] = -v^2 \int_{\mathbb{H}^2} (F - \tilde{F}). \tag{20}$$

This implies that the anti-self-dual equation, $F_{4d} = -*F_{4d}$, for Yang-Mills instantons reduces to the BPS vortex equations (7), (8), and (9) on a hyperbolic plane $\mathbb{H}^2$ for the integrable case (4). Therefore, we can use $SU(2N)$ instanton solutions to obtain $S[U(N) \times U(N)]$ vortex solutions.

C. $SU(2N)$ instantons from the ADHM construction

In order to construct $SO(3)$-invariant instanton solutions, we use the ADHM construction [17]. Let $B_1$ and $B_2$ be $k \times k$ complex matrices and $I$ and $J$ be $k \times 2N$ and $2N \times k$ complex matrices, respectively. Then, “the zero-dimensional Dirac operator” is defined by

$$\nabla^\dagger = \begin{pmatrix} I \\ J^\dagger \end{pmatrix} \begin{pmatrix} z_2 - B_2 & z_1 - B_1 \\ -(\bar{z}_1 - B_1^\dagger) & \bar{z}_2 - B_2^\dagger \end{pmatrix}. \tag{21}$$
where we have defined $z_1 \equiv ix_1 + x_2$, $z_2 \equiv x_4 + ix_3$. Now, let us consider the following $SO(3)$ action on the ADHM data $(B_i, I, J)$

$$\nabla^\dagger \rightarrow g \nabla^\dagger h, \quad g = 1_k \otimes U^\dagger, \quad h = \begin{pmatrix} 1_N \otimes U \\ 1_k \otimes U \end{pmatrix},$$

(22)

where $U$ is an arbitrary $SU(2)$ matrix. The most general ADHM data $(B_i, I, J)$ which are invariant under the $SO(3)$ transformation take the following forms (see Appendix B)

$$B_1 = B_1^\dagger = 0, \quad B_2 = B_2^\dagger = T, \quad \begin{pmatrix} I \\ J^\dagger \end{pmatrix} = \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix} = \psi \otimes 1_2,$$

(23)

where $T$ is an arbitrary $k$-by-$k$ Hermitian matrix and $\psi$ is an arbitrary $k$-by-$N$ matrix. We can show that the $SO(3)$-invariant ADHM data automatically satisfy the following ADHM equations.

$$0 = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \quad 0 = [B_1, B_2] + IJ.$$

(24)

More generally, the ADHM equations are satisfied if $B_1$ and $B_2$ are diagonal and $(I, J^\dagger)$ take the form (23). In such a case, the operator $\nabla^\dagger$ is given by

$$\nabla^\dagger = \begin{pmatrix} \psi \otimes 1_2 & (x^\mu - T^\mu) \otimes \bar{e}_\mu \end{pmatrix},$$

(25)

where $e_\mu = (-i\sigma_i, 1_2)$ and $\bar{e}_\mu = (i\sigma_i, 1_2)$, and we have taken $B_1 = iT_1 + T_2$ and $B_2 = iT_3 + T_4$ with $k \times k$ mutually commuting Hermitian matrices $T_\mu$. For notational simplicity, first, we deal with the case of the mutually commuting matrices $T_\mu$ and then return to the $SO(3)$-invariant case by setting $T_i = T$ and $T_i = 0$ ($i = 1, 2, 3$).

For the operator $\nabla^\dagger$ of the form (25), the zero modes $V$, which are a $(2N + 2k) \times 2N$ complex matrix satisfying the equation

$$\nabla^\dagger V = 0,$$

(26)

are found to be

$$V = \begin{pmatrix} 1_N \otimes 1_2 \\ -(x^\nu - T^\nu) [(x^\mu - T^\mu)^2]^{-1} \psi \otimes e_\nu \end{pmatrix} (S^\dagger)^{-1} \otimes 1_2).$$

(27)

Here, $S$ is an $N$-by-$N$ matrix determined from the orthogonality condition

$$V^\dagger V = 1_N,$$

(28)
or equivalently

\[ SS^\dagger = 1_N + \psi^\dagger [(x^\mu - T^\mu)^2]^{-1} \psi, \] (29)

where we have used the identity \( \bar{e}_\mu e_\nu + \bar{e}_\nu e_\mu = 2\delta_{\mu\nu}1_2 \). From the matrix \( V \), the instanton solutions can be explicitly given by

\[ A_{4d} = -iV^\dagger dV = -\frac{i}{2}S^{-1}\partial^\mu S \otimes (\delta_{\mu\nu}1_2 - i\eta_{\mu\nu}^{(+)}dx^\nu) + (h.c.), \] (30)

where \( \eta_{\mu\nu}^{(+)} \) is the self-dual 't Hooft tensor defined by \( \eta_{\mu\nu}^{(+)} = \frac{1}{2i}(\bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu) \). This solution can be viewed as a generalization of the 't Hooft's multi-instanton configuration for the SU(2) gauge group.

For our purpose, we impose the SO(3) invariance by setting \( T_4 = T \) and \( T_i = 0 \ (i = 1, 2, 3) \). In this case, Eq. (29) indicates that matrix \( S \) is independent of the coordinates of \( S^2 \). Thus, the solutions become

\[ A_{4d} = i \left[ -S^{-1}\partial_\bar{z}S \otimes P + \partial_\bar{z}S^\dagger S^{\dagger-1} \otimes \bar{P} \right] d\bar{z} - i\rho (S^{-1}\partial_zS + \partial_zS^\dagger S^{\dagger-1}) \otimes \omega + (h.c.), \] (31)

where we have used

\[
(\delta_{\mu\nu}1_2 - i\eta_{\mu\nu}^{(+)}dx^\nu\partial_\mu = (\bar{e}_\mu dx^\mu)e^\nu\partial_\nu \\
= 2(dzP + r\omega)\partial_z + 2(d\bar{z}\bar{P} - r\omega^\dagger)\partial_{\bar{z}} + \text{derivatives on } S^2. \] (32)

III. ALL EXACT \( S[U(N) \times U(N)] \) VORTEX SOLUTIONS

A. Exact solutions

Comparing Eq. (12) with Eq. (31), we can obtain the vortex solutions \( A_\alpha \) and \( H \). Let \( T \) be a \( k \times k \) Hermitian matrix and \( \psi \) be a \( k \times N \) complex matrix, made of the respective moduli parameters. The general form of the vortex solution is

\[ A_\alpha = -iW^\dagger \partial_\alpha W, \quad \tilde{A}_\alpha = -i\tilde{W}^\dagger \partial_\alpha \tilde{W}, \quad H = W^\dagger \tilde{W}, \quad (\alpha = z, \bar{z}), \] (33)

where \( W \) and \( \tilde{W} \) are \( (N + k) \times k \) matrices, given by

\[
W \equiv \begin{pmatrix} 1_N \\ (\bar{z} - T)^{-1} \psi \end{pmatrix} S^{\dagger-1}, \quad \tilde{W} \equiv \begin{pmatrix} 1_N \\ (z - T)^{-1} \psi \end{pmatrix} S^{\dagger-1} \] (34)
with $S(z, \bar{z})$ satisfying

$$SS^\dagger = 1_N + \psi^\dagger [(z - T)(\bar{z} - T)]^{-1} \psi.$$ \hspace{1cm} (35)

Here, $S$ is the same matrix as the one for instantons in Eq. (27); condition (35) originates from Eq. (29) with the identification $z = x^4 + ir$ and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

**B. The ADHM construction for vortices**

From the fact that the $(N + k) \times k$ matrices $W$ and $\tilde{W}$ in Eq. (34) are analogous to the $2(N + k) \times 2k$ matrix $V$ in Eq. (27) for the ADHM construction of instantons, the solutions can be recast into the ADHM form. In fact, for a given ADHM date $(T, \psi)$, the matrices $W$ and $\tilde{W}$ are solution of the “Dirac equations”

$$\nabla^\dagger_v W = 0, \hspace{1cm} \tilde{\nabla}^\dagger_v \tilde{W} = 0,$$ \hspace{1cm} (36)

where the “Dirac operators” for the vortices are given by

$$\nabla^\dagger_v \equiv \begin{pmatrix} \psi & T - \bar{z} \end{pmatrix}, \hspace{1cm} \tilde{\nabla}^\dagger_v \equiv \begin{pmatrix} \psi & T - z \end{pmatrix}.$$ \hspace{1cm} (37)

These Dirac operators are analogous to those of instantons in Eq. (26). Condition (35) is equivalent to the orthogonality conditions for matrices $W$ and $\tilde{W}$:

$$W^\dagger W = 1_N, \hspace{1cm} \tilde{W}^\dagger \tilde{W} = 1_N.$$ \hspace{1cm} (38)

These conditions are also counterparts of Eq. (28) for instantons.

We need to check the existence of the inverse of $\Delta^\dagger \Delta$ in the ADHM construction for instantons, and therefore there should be a corresponding condition for the operators in Eq. (37) for vortices. However we will not study it in more detail in this paper and leave it as a future problem.

**C. Moduli space**

Here, we discuss the moduli parameters encoded in solutions (33), (34), and (35) and identify the moduli space. The solutions have the following redundancy in the moduli data $(T, \psi)$:

$$T \to UTU^{-1}, \hspace{1cm} \psi \to U \psi, \hspace{1cm} U \in U(k).$$ \hspace{1cm} (39)
They can be fixed as

\[ T = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{pmatrix}, \quad \psi = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k1} & \rho_{k2} & \cdots & \rho_{kN} \end{pmatrix}, \quad t_i \in \mathbb{R}, \quad \rho_{ij} \in \mathbb{C}. \quad (40) \]

The remaining \( U(1)^k \) redundancy, \( \rho_{ij} \to \exp(i\theta_i)\rho_{ij} \), can be fixed as \( \rho_{ii} \in \mathbb{R} \). Therefore, the dimension of the \( S[U(N) \times U(N)] \) vortex moduli space is \( 1/4 \) of that of the (framed) \( SU(2N) \) instanton moduli space:

\[ \dim_{\mathbb{R}} \mathcal{M}_{k,N}^{\text{vortex}} = 2kN. \quad (41) \]

If we further divide the moduli space by the \( SU(N) \) (global) gauge symmetry \( \psi \to \psi g, \ g \in SU(N) \), the dimension of the moduli space becomes

\[ \dim_{\mathbb{R}}[\mathcal{M}_{k,N}^{\text{vortex}}/SU(N)] = \begin{cases} 
2kN - N^2 + 1 & \text{for } k > N \\
k^2 + 1 & \text{for } k \leq N 
\end{cases}. \quad (42) \]

To determine the physical meaning of the moduli parameters, let us calculate

\[ \det H \propto \det \left( 1_N + \psi \psi^\dagger (z1_k - T)^{-2}\psi \right) \]
\[ \propto \det \left( 1_k + (z1_k - T)^{-2}\psi \psi^\dagger \right) \]
\[ \propto \det \left( (z1_k - T)^2 + \psi \psi^\dagger \right). \quad (43) \]

Now, let us define a \( k \times k \) complex matrix \( Z \) by

\[ Z \equiv T + iR \quad (44) \]

with a \( k \times k \) Hermitian matrix \( R \) satisfying

\[ i[T, R] + \psi \psi^\dagger = R^2. \quad (45) \]

Hence, Eq. (43) can be rewritten as

\[ \det H \propto \det (z - Z)(z - Z^\dagger). \quad (46) \]

Since the unbroken gauge symmetry becomes larger inside the vortex cores, the zeros of \( \det H \) can be interpreted as the vortex positions. Therefore, Eq. (46) implies that the
The eigenvalues of $Z$ are the vortex positions. It is interesting to see that Eq. (45) is a remnant of the D-term condition of a Kähler quotient construction of the vortex moduli space. We thus obtain

$$\mathcal{M}_{k,N} \simeq \left\{ (Z, \psi) \mid \frac{1}{2} [Z^\dagger, Z] + \psi \psi^\dagger = R^2 \right\} / U(k),$$

(47)

where the $U(k)$ action is $Z \to UZU^{-1}$, $\psi \to U\psi$, and $R \to URU^{-1}$, as in Eq. (39). This is analogous to the Kähler quotient for $U(N)$ vortices on flat space $\mathbb{C}$. The moduli $Z$ represent the vortex positions, and the moduli $\psi$ can be identified as the orientational moduli, which are $k$ copies of $\mathbb{C}P^{N-1}$ for the separated vortices. The difference between our hyperbolic case and the flat case is that the right hand side of the D-term condition is $R^2$ in our case while it is just $(4\pi/g^2)1_k$ for the flat case. Since the eigenvalues of $R$ are the vortex positions in the $r$ coordinate, our D-term condition can be interpreted as a result of a position-dependent gauge coupling, as can be inferred from Eqs. (2) and (3). Note that the Kähler quotient in Eq. (47) does not give the correct moduli space metric, as in the case of the flat space.

Although the $\mathbb{C}P^{N-1}$ orientational moduli for a single vortex can be absorbed by a global gauge transformation, the relative orientations change the physical quantities. Let us consider a coincident vortex configuration in the $N = k = 2$ case. If we set the matrices $T$ and $\psi$ as

$$T = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \psi = \begin{pmatrix} \sqrt{r_0^2 + a^2} & 0 \\ 2a & \sqrt{r_0^2 - 3a^2} \end{pmatrix}, \quad a \in \left[ 0, \frac{r_0}{\sqrt{3}} \right],$$

(48)

the matrix $R$ is solved as

$$R = \begin{pmatrix} r_0 & \frac{r_0 + ia}{\sqrt{r_0^2 + a^2}} \\ \frac{r_0 - ia}{\sqrt{r_0^2 + a^2}} & r_0 \end{pmatrix}.$$ 

(49)

In this setting, the matrix $Z = T + iR$ has the degenerate eigenvalue $ir_0$; hence, the two vortices are coincident. Since the vortex position is independent of $a$, $a$ parameterizes the internal orientation of the vortices. Indeed, we can see from Eq. (48) that $(T, \psi)$ reduce to two copies of the data for an Abelian vortex at $a = 0$ while $(T, \psi)$ become identical to the data of two vortices in the Abelian case at $a = r_0/\sqrt{3}$. We can confirm that the parameter $a$ is physical by observing the trace of the magnetic flux $F_{\bar{z}z} = -\bar{F}_{z\bar{z}}$.

$$i\sigma^{-1} \text{tr} F_{\bar{z}z} = 8r^2(a^2 + r_0^2) \left[ \frac{1}{\{ |z|^2 + r_0^2 + a(z - \bar{z}) \}^2} + \frac{1}{\{ |\bar{z}|^2 + r_0^2 - a(z - \bar{z}) \}^2} \right].$$

(50)
IV. SUMMARY AND DISCUSSION

In summary, we have constructed a complete set of all the exact solutions of non-Abelian vortices in the \( S[U(N) \times U(N)] \) gauge theory with a bi-fundamental scalar field on a hyperbolic plane with a certain curvature, by using \( SO(3) \)-invariant \( SU(2N) \) Yang-Mills instanton solutions. We also have established the ADHM construction for non-Abelian vortices. We further identified the complete moduli space of \( k \) vortices, consisting of the moduli parameters encoded in the \( k \times k \) matrix \( Z \) for the position moduli and the \( k \times N \) matrix \( \psi \) for the orientational moduli. We have found the Kähler quotient for the moduli space, whose complex dimension is \( kN \) as in the flat case.

Future works on related topics will include studies on the index theorem of vortices in quiver gauge theories, the explicit moduli space metric, and low-energy dynamics of vortices; and an extension to arbitrary gauge groups \([29]\), particularly \( SO(N) \) and \( USp(2N) \) \([30]\), from the Yang-Mills instantons with corresponding groups. Since the hyperbolic surface is topologically equivalent to the flat space, quantum dynamics such as the vortex counting should be studied on the hyperbolic surface. In the case of \( N = 1 \), our model reduces to the Abelian-Higgs model \([23]\), in which the vortex equation is reduced to the Liouville equation. This implies the presence of a non-Abelian generalization of the Liouville equation.

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Appendix A: Hyperbolic plane

The hyperbolic plane is a subspace in \( \mathbb{R}^{2,1} \) given by

\[
X_1^2 + X_2^2 - X_3^2 = -R^2. \tag{A1}
\]
The solution is parameterized by \( \phi \in [0, 2\pi), \rho \in \mathbb{R}_{\geq 0} \) as
\[
X_1 = R \cos \phi \sinh \rho, \quad X_2 = R \sin \phi \sinh \rho, \quad X_3 = R \cosh \rho.
\] (A2)

The metric is given by
\[
ds^2 = dX_1^2 + dX_2^2 - dX_3^2 = R^2 (d\rho^2 + \sinh^2 \rho \, d\phi^2).
\] (A3)

which gives a constant scalar curvature \(-1/R^2\). The hyperbolic plane can be parametrized by a complex coordinate \( y \) in an unit disc as
\[
y = \tanh \frac{\rho}{2} e^{i\phi}, \quad |y| < 1.
\] (A4)

Then, the metric becomes
\[
ds^2 = 2g_{yy} dy d\bar{y} = \frac{4R^2}{(1 - |y|^2)^2} dy d\bar{y}.
\] (A5)

An upper-half plane is also used to parameterize the hyperbolic plane, where a complex coordinate \( z \) is given by
\[
z = \frac{y + i}{1 + iy}, \quad \text{Im} \, z > 0.
\] (A6)

In terms of this coordinate the metric becomes
\[
ds^2 = R^2 \frac{dz d\bar{z}}{\text{Im} \, z}.
\] (A7)

The \( SO(2, 1) \) isometry of \( \mathbb{R}^{2,1} \) acts on \( z \) as
\[
z \to \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}.
\] (A8)

**Appendix B: \( SO(3) \)-invariant ADHM data**

In this section, we show that the most general \( SO(3) \)-invariant ADHM data takes the form of Eq. (23). First, let us rewrite the pair of the matrices \((I, J^i)\) as
\[
\begin{pmatrix}
I \\
J^i
\end{pmatrix}
= \begin{pmatrix}
\phi_4 + i\phi_3 & i\phi_1 + \phi_2 \\
i\phi_1 - \phi_2 & \phi_4 - i\phi_3
\end{pmatrix}
= \phi_i \otimes \sigma_i + \phi_4 \otimes 1_2,
\] (B1)

where \( \phi_i \) (\( i = 1, 2, 3 \)) and \( \phi_4 \) are arbitrary \( k \)-by-\( N \) matrices. The \( SO(3) \) transformations rotate \( \phi_i \) and \( \phi_4 \) as
\[
\phi_i \to R_{i}^{\,j} \phi_j, \quad \phi_4 \to \phi_4, \quad R^T R = 1_3.
\] (B2)
This transformation can be canceled if there exists a $U(k)$ gauge transformation such that

$$U\phi_i = R^j_i\phi_j, \quad U\phi_4 = \phi_4, \quad U \in U(k). \quad \text{(B3)}$$

This implies that the matrix $U$ must be in a direct sum of several triplet and singlet representations: $\phi_4 = 0$ for the triplet part while $\phi_i = 0$ for the singlet part. However, we can show that the triplet part does not satisfy the ADHM equations

$$\frac{1}{2}\epsilon_{ijk}[T_j, T_k] + [T_i, T_4] + \frac{1}{2}\epsilon_{ijk}(\phi_j\phi_k^\dagger - \phi_k\phi_j^\dagger) + \phi_i\phi_4^\dagger - \phi_4\phi_i^\dagger = 0. \quad \text{(B4)}$$

For simplicity, let us consider the case of a single triplet representation ($k = 3$). The explicit form of the invariant data is given by

$$T_{i \gamma} = i a \epsilon_{i \gamma r s}, \quad (T_4)_{r s} = b \delta_{r s}, \quad (\phi_i)_{r A} = \delta_{ri} \psi_A, \quad \phi_4 = 0, \quad \text{(B5)}$$

where $a, b \in \mathbb{R}$ and $\psi_A \in \mathbb{C}$ ($A = 1, \cdots , N$) are arbitrary parameters. Then, Eq. (B4) reduces to

$$\epsilon_{i \gamma r s}(a^2 + |\psi_A|^2) = 0. \quad \text{(B6)}$$

This allows only a trivial solution $a = \psi_A = 0$ for which the operator $\Delta^\dagger \Delta$ is not invertible at $x^\mu = (0, 0, 0, b)$. Therefore, there is no solution to the ADHM equations for the triplet representation. Similarly, we can in general show that there is no $SO(3)$-invariant ADHM data in a direct sum representation containing the triplet representation. Namely, the most general form of the $SO(3)$-invariant data should be in a direct sum of the singlet representations:

$$T_i = \phi_i = 0 \quad (i = 1, 2, 3), \quad T_4 = T, \quad \phi_4 = \psi, \quad \text{(B7)}$$

where $T$ and $\psi$ are arbitrary $k$-by-$k$ and $k$-by-$N$ matrices, respectively.

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