\[
L^2 \text{ ESTIMATES OF POINCARÉ-LELONG EQUATIONS ON CONVEX DOMAINS IN } \mathbb{C}^n
\]

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Abstract. In this paper, we prove the existence of solutions of the Poincaré-Lelong equation \(\sqrt{-1} \partial \overline{\partial} u = f\) on a strictly convex bounded domain \(\Omega \subset \mathbb{C}^n (n \geq 1)\), where \(f\) is a \(d\)-closed \((1,1)\) form and is in the weighted Hilbert space \(L^2_{(1,1)}(\Omega, e^{-\varphi})\). The novelty of this paper is to apply a weighted \(L^2\) version of Poincaré Lemma for real 2-forms, and then apply Hörmander’s \(L^2\) solutions for Cauchy-Riemann equations.

1. Introduction

In this paper, a continuation of \([1]\), we will study the Poincaré-Lelong equation \(\sqrt{-1} \partial \overline{\partial} u = f\) in a weighted Hilbert space on a strictly convex bounded domain \(\Omega \subset \mathbb{C}^n (n \geq 1)\). Using a weighted \(L^2\) version of Poincaré Lemma for real forms, we obtain the existence of solutions with the norm estimate. More precisely, we prove the following theorem.

Main Theorem. Let \(\Omega\) be a strictly convex bounded domain in \(\mathbb{C}^n\). Let \(\varphi\) be a nonnegative smooth function on \(\Omega\) such that
\[
\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k \geq c|\xi|^2 \quad \text{for all } \xi = (\xi_1, \cdots, \xi_{2n}) \in \mathbb{R}^{2n},
\]
where \(c > 0\) is a constant. Then, for each \((1,1)\) form \(f\) in the weighted Hilbert space \(L^2_{(1,1)}(\Omega, e^{-\varphi})\) with \(\partial f = \overline{\partial} f = 0\), there exists a solution \(u\) in \(L^2(\Omega, e^{-\varphi})\) solving the Poincaré-Lelong equation
\[
\sqrt{-1} \partial \overline{\partial} u = f
\]
in \(\Omega\), in the sense of distributions, with the norm estimate
\[
\int_{\Omega} |u|^2 e^{-\varphi} \leq \frac{8}{c^2} \int_{\Omega} |f|^2 e^{-\varphi}.
\]

Corollary. For each \((1,1)\) form \(f\) in \(L^2_{(1,1)}(\Omega)\) with \(\partial f = \overline{\partial} f = 0\), there exists a solution \(u\) in \(L^2(\Omega)\) solving the Poincaré-Lelong equation \(\sqrt{-1} \partial \overline{\partial} u = f\) in \(\Omega\), in the sense of distributions, with the \(L^2\) norm estimate
\[
\int_{\Omega} |u|^2 \leq c_{\Omega} \int_{\Omega} |f|^2,
\]
where \(c_{\Omega} > 0\) is a constant depended on the diameter of \(\Omega\).
For the Poincaré-Lelong equation \( \sqrt{-1} \partial \overline{\partial} u = f \), P. Lelong \([2]\) studied it in connection with questions on entire functions, and showed, unexpectedly, that with suitable restrictions on the growth of \( f \), the equation could be reduced to solving the more familiar equation \( \frac{1}{4} \Delta u = \text{trace}(f) \) (Poisson equation). Mok, Siu and Yau \([3]\) studied the equation on a complete Kähler manifold and obtained important applications to questions on when a (noncompact) Kähler manifold is biholomorphically equivalent to \( \mathbb{C}^n \). Andersson \([4]\) studied the Poincaré-Lelong equation for smooth forms in the unit ball using integral representations. Recently, Chen \([5]\) obtained solutions of the equation when \( f \) is assumed to be a smooth \((1,1)\) \( d \)-closed form with compact support in \( \mathbb{C}^n \), and he applied his result to prove a version of Hartog’s extension theorem for pluriharmonic functions (for related results see also \([6]\), \([7]\) and \([8]\)).

Recently, we \([1]\) studied the Poincaré-Lelong equation in the whole space \( \mathbb{C}^n \), and proved the existence of (global) solutions in the weighted Hilbert space with Gaussian measure as follows.

**Theorem.** For each \((1,1)\) form \( f \) in the weighted Hilbert space \( L^2_{(1,1)}(\mathbb{C}^n, e^{-|z|^2}) \) with \( \partial f = \overline{\partial} f = 0 \), there exists a solution \( u \) in \( L^2(\mathbb{C}^n, e^{-|z|^2}) \) solving the Poincaré-Lelong equation

\[
\sqrt{-1} \partial \overline{\partial} u = f
\]

in \( \mathbb{C}^n \), in the sense of distributions, with the norm estimate

\[
\int_{\mathbb{C}^n} |u|^2 e^{-|z|^2} \leq 2 \int_{\mathbb{C}^n} |f|^2 e^{-|z|^2}.
\]

As a matter of fact, the key idea of the proof of the main theorem is quite similar to that of the theorem above. First we convert the \((1,1)\) form \( f \) to a real 2-form. Second for the real 2-form, we apply a weighted \( L^2 \) version of Poincaré Lemma that we shall give a detailed proof. At last, we apply Hörmander’s \( L^2 \) solutions for the Cauchy-Riemann equations. Since the domain considered has a smooth boundary, we have to study carefully about the adjoint of the \( \partial \overline{\partial} \) operator following the approach of Berndtsson \([9]\), who essentially gave the second proof of the Hörmander’s theorem for \( \overline{\partial} \).

It was Berndtsson \([10]\), who first studied the \( d \)-equation for real 1-forms with morse function weights and pointed out that the Hörmander’s \( L^2 \) method could be used for the \( d \)-equation in convex domains and with a convex weight function. Since our proof of the main theorem depends significantly on a weighted \( L^2 \) version of Poincaré Lemma (below), and the classical Poincaré Lemma would not provide a \( L^2 \) estimates for \( d \)-equation, we decide to include a detailed proof of Poincaré Lemma. In addition, the proof will provide a specific constant that we shall use in the main theorem.

**Poincaré Lemma.** (A weighted \( L^2 \) version for \( p + 1 \)-forms) Let \( N \geq 1 \) be an integer and \( G \) be a strictly convex bounded domain in \( \mathbb{R}^N \). Let \( \varphi \) be a nonnegative smooth function on
$G$ such that
\[ \sum_{j,k=1}^{N} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k \geq c |\xi|^2 \] for all $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$,

where $c$ is a positive constant. Let $p$ be an integer with $0 \leq p \leq N - 1$. Then, for each $f$, a $d$-closed $p+1$-form in the weighted Hilbert space $L^2_{p+1}(G, e^{-\varphi})$, there exists a solution $p$-form $u$ in $L^2_p(G, e^{-\varphi})$ solving equation
\[ du = f \]
in $G$, in the sense of distributions, with the norm estimate
\[ \int_G |u|^2 e^{-\varphi} \leq \frac{1}{c(p+1)} \int_G |f|^2 e^{-\varphi}. \]

In this paper, Section 2 and 3 are for Poincaré Lemma; Section 4 and 5 are for the main theorem.

2. Preliminary for Poincaré Lemma

Let $N \geq 1$ and $p \geq 0$ be integers. For multiindex $I = (i_1, \cdots, i_p)$, where $i_1, \cdots, i_p$ are integers between 1 and $N$, define $|I| = p$ and $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Let $G$ be a strictly convex bounded domain in $\mathbb{R}^N$. In general, a $p$-form $f$ on $G$ is a formal combination
\[ f = \sum_{|I|=p} f_I dx^I, \]
where $\sum'$ implies that the summation is performed only over strictly increasing multi-indices and $f_I : G \to \mathbb{R}$ is a real-valued function for each $I$. For $p$-forms $f$ and $g$, we denote their pointwise scalar product by
\[ f \cdot g = \sum_{|I|=p} f_I g_I. \]

Let $\varphi$ be a nonnegative smooth function on $\overline{G}$ and the weighted Hilbert space for $p$-forms
\[ L^2_{p}(G, e^{-\varphi}) = \{ f = \sum_{|I|=p} f_I dx^I \mid f_I \in L^2_{\text{loc}}(G); \int_G |f|^2 e^{-\varphi} < +\infty \}, \]
where $|f|^2 = f \cdot f$. We denote the weighted inner product for $f, g \in L^2_{p}(G, e^{-\varphi})$ by
\[ \langle f, g \rangle_{L^2_{p}(G, e^{-\varphi})} = \int_G f \cdot g e^{-\varphi}, \]
and the weighted norm of \( f \in L^2_p(G, e^{-\varphi}) \) by \( \|f\|_{L^2_p(G, e^{-\varphi})} = \sqrt{\langle f, f \rangle_{L^2_p(G, e^{-\varphi})}} \). In particular, we denote
\[
L^2_0(G, e^{-\varphi}) = L^2(G, e^{-\varphi}) = \{ f \mid f \in L^2_{\text{loc}}(G); \int_G |f|^2 e^{-\varphi} < +\infty \}.
\]

For simplicity, we will write \( L^2_p(e^{-\varphi}) \) for \( L^2_p(G, e^{-\varphi}) \) in Section 2 and 3, since we only deal with \( G \) in these sections.

Let \( \mathcal{D}_p \) denote the set of \( p \)-forms whose coefficients are smooth functions with compact support in \( G \). For \( p \)-form \( u = \sum_{|I|=p} u_I dx^I \in L^2_p(e^{-\varphi}) \), in the sense of distributions, the differential of \( u \) is that:
\[
du = \sum_{|I|=p} \sum_{j=1}^N \frac{\partial u_I}{\partial x_j} dx_j \wedge dx^I,
\]
and for \( p + 1 \)-form \( f \in L^2_{p+1}(e^{-\varphi}) \), we say that \( f \) is the differential \( du \), written \( du = f \), provided
\[
\int_G du \cdot \alpha = \int_G f \cdot \alpha
\]
for all test forms \( \alpha = \sum_{|I|=p+1} \alpha_I dx^I \in \mathcal{D}_{p+1} \). By the definition of \( \mathcal{D}_p \), the operator \( d : \mathcal{D}_p \to \mathcal{D}_{p+1} \) is well defined. We now extend the definition of the operator \( d \) by allowing it to act on any \( u \in L^2_p(e^{-\varphi}) \) such that \( du \), in the sense of distributions, lies in \( L^2_{p+1}(e^{-\varphi}) \). Then we obtain a closed, densely defined operator
\[
T : L^2_p(e^{-\varphi}) \to L^2_{p+1}(e^{-\varphi}),
\]
where the domain of \( T \) is
\[
\text{Dom}(T) = \{ u \in L^2_p(e^{-\varphi}) \mid du \in L^2_{p+1}(e^{-\varphi}) \}.
\]

Now we consider the Hilbert space adjoint of \( T \):
\[
T^* : L^2_{p+1}(e^{-\varphi}) \to L^2_p(e^{-\varphi}).
\]
Let \( \text{Dom}(T^*) \) be the domain of \( T^* \) and \( \alpha \in L^2_{p+1}(e^{-\varphi}) \). By functional analysis, we say that \( \alpha \in \text{Dom}(T^*) \) if there exists a constant \( c = c(\alpha) > 0 \) such that
\[
|\langle Tu, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})}| \leq c\|u\|_{L^2_p(e^{-\varphi})}
\]
for all \( u \in \text{Dom}(T) \). This definition is equivalent to that \( \alpha \in \text{Dom}(T^*) \) if and only if there exists \( v \in L^2_p(e^{-\varphi}) \) such that
\[
\langle u, v \rangle_{L^2_p(e^{-\varphi})} = \langle Tu, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})}
\]
for all \( u \in \text{Dom}(T) \). Note that \( v \) is unique. We set \( v = T^* \alpha \). Then \( T^* : \text{Dom}(T^*) \to L^2_p(e^{-\varphi}) \) is a linear operator and satisfies
\[
\langle u, T^* \alpha \rangle_{L^2_p(e^{-\varphi})} = \langle Tu, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})}
\]
for all \( u \in \text{Dom}(T) \) and \( \alpha \in \text{Dom}(T^*) \).
for all \( u \in \text{Dom}(T) \), \( \alpha \in \text{Dom}(T^*) \). It is well-known that \( T^* \) is again a closed, densely defined operator.

In order to compute \( T^* \), we first compute \( T_{\text{formal}}^* \), the formal adjoint of \( T \), which is defined using only test forms, i.e., we demand

\[
\langle Tu, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} = \langle u, T_{\text{formal}}^* \alpha \rangle_{L^2_{p}(e^{-\varphi})}
\]

for \( u \in \text{Dom}(T) \) and \( \alpha \in \mathcal{D}_{p+1} \). For \( \alpha = \sum'_{|J|=p+1} \alpha_J dx^J \), if \( J_1 \) is a permutation of \( J \), we write \( \alpha_{J_1} = \epsilon_{J_1}^J \alpha_J \), where \( \epsilon_{J_1}^J \) is the signature of the permutation (for example, the signature is \(-1\) if only two indices are interchanged). In particular, a term \( \alpha_{jI} = 0 \) if \( j \in I \), where \( I = (i_1, \ldots, i_p) \) and \( jI = (j, i_1, \ldots, i_p) \). Then by (1) and integration by parts, the left side of (3) is given by

\[
\langle Tu, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} = \int_G du \cdot \alpha e^{-\varphi}
= \int_G \sum'_{|I|=p} \sum_{j=1}^N \frac{\partial u_I}{\partial x_j} \alpha_{jI} e^{-\varphi}
= -\int_G \sum'_{|I|=p} \sum_{j=1}^N u_I \frac{\partial (\alpha_{jI} e^{-\varphi})}{\partial x_j}
= \int_G \left( \sum'_{|I|=p} u_I \left( -e^{\varphi} \sum_{j=1}^N \frac{\partial (\alpha_{jI} e^{-\varphi})}{\partial x_j} \right) \right) e^{-\varphi}
= \int_G \left( \sum'_{|I|=p} u_I A_I \right) e^{-\varphi},
\]

where

\[
A_I = -e^{\varphi} \sum_{j=1}^N \frac{\partial (\alpha_{jI} e^{-\varphi})}{\partial x_j}.
\]

For example, if \( p = 1 \), then

\[
A_I = A_i = -e^{\varphi} \left( \sum_{1 \leq j < i} \frac{\partial (\alpha_{ji} e^{-\varphi})}{\partial x_j} - \sum_{i < j \leq N} \frac{\partial (\alpha_{ij} e^{-\varphi})}{\partial x_j} \right).
\]

Note that \( \alpha \in \mathcal{D}_{p+1} \) in (4). Then \( A_I \) is a smooth function with compact support in \( G \), so \( \sum'_{|I|=p} A_I dx^I \in \mathcal{D}_p \subset L^2_p(e^{-\varphi}) \). Thus, the formal adjoint

\[
T_{\text{formal}}^* \alpha = \sum'_{|I|=p} A_I dx^I,
\]
where $A_I$ is as [1]. This implies that $\mathcal{D}_{p+1} \subset \text{Dom}(T^*)$.

In the sense of distributions, the formal adjoint $T_{\text{formal}}^*\alpha$ is actually well-defined for all $\alpha \in L_{p+1}^2(e^{-\varphi})$ as $\varphi$ is smooth. We claim that

$$ T^*\alpha = T_{\text{formal}}^*\alpha \quad \text{for all } \alpha \in \text{Dom}(T^*). $$

(6)

Indeed, if $\alpha \in \text{Dom}(T^*)$, then by [2] and $\mathcal{D}_p \in \text{Dom}(T)$, we have for all $u \in \mathcal{D}_p$,

$$ \int_G u \cdot T^*\alpha e^{-\varphi} = \langle u, T^*\alpha \rangle_{L_p^2(e^{-\varphi})} = \langle Tu, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})} = \int_G u \cdot T_{\text{formal}}^*\alpha e^{-\varphi}. $$

3. PROOF OF POINCARÉ LEMMA

We first prove some lemmas. Let $0 \leq p \leq N - 1$.

**Lemma 3.1.** For each $p + 1$-form $f$ in $L_{p+1}^2(e^{-\varphi})$, there exists a solution $p$-form $u$ in $L_p^2(e^{-\varphi})$ solving the equation

$$ du = f $$

in $G$, in the sense of distributions with the norm estimate

$$ \|u\|_{L_p^2(e^{-\varphi})} \leq c $$

if and only if

$$ \langle f, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})} \leq c \|T^*\alpha\|_{L_p^2(e^{-\varphi})}^2 \quad \text{for all } \alpha \in \mathcal{D}_{p+1}. $$

*Proof.* (Necessity) For all $\alpha \in \mathcal{D}_{p+1}$, from the definition of $T^*$ and the Cauchy-Schwarz inequality, we have

$$ |\langle f, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})}|^2 = |\langle Tu, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})}|^2 = \left| \langle u, T^*\alpha \rangle_{L_p^2(e^{-\varphi})} \right|^2 \leq \|u\|_{L_p^2(e^{-\varphi})} \|T^*\alpha\|_{L_p^2(e^{-\varphi})}^2. $$

Note that $\|u\|_{L_p^2(e^{-\varphi})} \leq c$. Then $|\langle f, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})}|^2 \leq c \|T^*\alpha\|_{L_p^2(e^{-\varphi})}^2$.

(Sufficiency) Consider the subspace

$$ E = \{ T^*\alpha \mid \alpha \in \mathcal{D}_{p+1} \} \subset L_p^2(e^{-\varphi}). $$

Define a linear functional $L_f : E \to \mathbb{R}$ by

$$ L_f(T^*\alpha) = \langle f, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})}. $$

Since

$$ |L_f(T^*\alpha)| = \left| \langle f, \alpha \rangle_{L_{p+1}^2(e^{-\varphi})} \right| \leq \sqrt{c} \|T^*\alpha\|_{L_p^2(e^{-\varphi})}, $$

then $L_f$ is a bounded functional on $E$. So by Hahn-Banach’s extension theorem, $L_f$ can be extended to a linear functional $\widetilde{L}_f$ on $L_p^2(e^{-\varphi})$ such that

$$ |\widetilde{L}_f(g)| \leq \sqrt{c} \|g\|_{L_p^2(e^{-\varphi})} \quad \text{for all } g \in L_p^2(e^{-\varphi}). $$

Using the Riesz representation theorem for $\widetilde{L}_f$, there exists a unique $u_0 \in L_p^2(e^{-\varphi})$ such that

$$ \widetilde{L}_f(g) = \langle u_0, g \rangle_{L_p^2(e^{-\varphi})} \quad \text{for all } g \in L_p^2(e^{-\varphi}). $$

(7)
Now we prove \( du_0 = f \). For all \( \alpha \in \mathcal{D}_{p+1} \), apply \( g = T^* \alpha \) in (8). Then
\[
\widetilde{L}_f (T^* \alpha) = \langle u_0, T^* \alpha \rangle_{L^2_p(e^{-\varphi})} = \langle T u_0, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})}.
\]
Note that
\[
\widetilde{L}_f (T^* \alpha) = L_f (T^* \alpha) = \langle f, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})}.
\]
Therefore,
\[
\langle T u_0, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} = \langle f, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} \quad \text{for all} \quad \alpha \in \mathcal{D}_{p+1}.
\]
Thus, \( T u_0 = f \), i.e., \( du_0 = f \).

Next we give a bound for the norm of \( u_0 \). Let \( g = u_0 \) in (7) and (8). Then we have
\[
\| u_0 \|_{L^2_p(e^{-\varphi})}^2 = \left| \langle u_0, u_0 \rangle_{L^2_p(e^{-\varphi})} \right| = \left| \widetilde{L}_f (u_0) \right| \leq \sqrt{c} \| u_0 \|_{L^2_p(e^{-\varphi})}.
\]
Therefore, \( \| u_0 \|_{L^2_p(e^{-\varphi})} \leq c. \)

Let \( u = u_0 \). Then the lemma is proved. \( \square \)

From Lemma 3.1 for proving Poincaré Lemma, we only need to prove that for all \( \alpha \in \mathcal{D}_{p+1} \),
\[
\| \langle f, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} \|_{L^2_p(e^{-\varphi})}^2 \leq \frac{1}{c(p+1)} \| T^* \alpha \|_{L^2_p(e^{-\varphi})}^2,
\]
where \( \varphi, p, f, c \), are as Poincaré Lemma. For this purpose, we prove the following lemmas.

**Lemma 3.2.** Let \( G = \{ x \in \mathbb{R}^N \mid \rho(x) < 0 \} \) be a convex bounded domain, where \( \rho \) is a smooth defining function. Suppose \( \alpha = \sum_{|J|=p+1} \alpha_j dx^J \) is a smooth \( p+1 \)-form on \( \overline{G} \), and that \( \alpha \in \text{Dom}(T^*) \). Then for any strictly increasing multiindex \( I \) with \( |I| = p \), we have
\[
\sum_{j,k=1}^N \alpha_{kl} \frac{\partial \alpha_{jl}}{\partial x_k} \frac{\partial \rho}{\partial x_j} = -\sum_{j,k=1}^N \alpha_{jl} \alpha_{kl} \frac{\partial^2 \rho}{\partial x_j \partial x_k}.
\]

**Proof.** Let \( u \) be a smooth \( p \)-form on \( \overline{G} \). Then using integration by parts, we have
\[
\langle T u, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} = \int_G du \cdot \alpha e^{-\varphi}
\]
\[
= \int_G \sum_{|I|=p} \sum_{j=1}^N \frac{\partial u_I}{\partial x_j} \alpha_{jl} e^{-\varphi}
\]
\[
= -\int_G \sum_{|I|=p} \sum_{j=1}^N u_I \frac{\partial (\alpha_{jl} e^{-\varphi})}{\partial x_j} + \int_{\partial G} \sum_{|I|=p} \sum_{j=1}^N u_I \alpha_{jl} e^{-\varphi} \frac{\partial \rho}{\partial x_j} \frac{dS}{|\partial \rho|}.
\]

Note that
\[
\langle T u, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} = \langle u, T^* \alpha \rangle_{L^2_p(e^{-\varphi})}
\]
and

\[- \int_G \sum_{|I|=p} \sum_{j=1}^N u_I \frac{\partial (\alpha_{jI}e^{-\varphi})}{\partial x_j} = \langle u, T^*\alpha \rangle_{L^p_\beta(e^{-\varphi})}.\]

Then

\[\int_{\partial G} \sum_{|I|=p} \sum_{j=1}^N u_I \alpha_{jI} e^{-\varphi} \frac{\partial \rho}{\partial x_j} \frac{dS}{|\partial \rho|} = 0\]

for any \(u\). Thus, for any strictly increasing multiindex \(I\) with \(|I| = p\), we have

\[\sum_{j=1}^N \alpha_{jI} \frac{\partial \rho}{\partial x_j} = 0 \text{ on } \partial G.\]

For the multiindex \(I\) above, let \(F_I = \sum_{j=1}^N \alpha_{jI} \frac{\partial \rho}{\partial x_j}\) and \(L_I = \sum_{k=1}^N \alpha_{kI} \frac{\partial \rho}{\partial x_k}\), a tangential differential operator. Then on \(\partial G\), we have

\[0 = L_I(F_I) = \sum_{k=1}^N \alpha_{kI} \frac{\partial}{\partial x_k} \left( \sum_{j=1}^N \alpha_{jI} \frac{\partial \rho}{\partial x_j} \right)\]

\[= \sum_{j,k=1}^N \alpha_{jI} \frac{\partial}{\partial x_k} \left( \frac{\partial \rho}{\partial x_j} \right) + \sum_{j,k=1}^N \alpha_{jI} \alpha_{kI} \frac{\partial^2 \rho}{\partial x_j \partial x_k}.\]

Therefore, the lemma is proved. \(\square\)

**Lemma 3.3.** Let \(\alpha = \sum_{|J|=p+1}^\prime \alpha_J d\mathbf{x}^J\). Then

\[|d\alpha|^2 = \sum_{|J|=p+1}^\prime \sum_{j=1}^N \left| \frac{\partial \alpha_J}{\partial x_j} \right|^2 - \sum_{|I|=p+1}^\prime \sum_{k=1}^N \frac{\partial \alpha_{kI}}{\partial x_j} \frac{\partial \alpha_{jI}}{\partial x_k}.\]  \(\text{(9)}\)

**Proof.** Note that \(d\alpha = 0\) when \(p + 1 = N\). When \(p + 1 < N\),

\[d\alpha = \sum_{|J|=p+1}^\prime \sum_{j=1}^N \frac{\partial \alpha_J}{\partial x_j} dx_j \wedge dx^J\]

\[= \sum_{|J|=p+1}^\prime \sum_{j \in J} \frac{\partial \alpha_J}{\partial x_j} \epsilon_{jJ} \left( \sum_{J_{jJ}} \epsilon_{J_{jJ}} \right) dx^J\]

\[= \sum_{|M|=p+2}^\prime \left( \sum_{j \in M} \frac{\partial \alpha_M}{\partial x_j} \epsilon_{M} \right) dx^M.\]
where \((jJ)'\) is the permutation of \(jJ\) such that \((jJ)'\) is a strictly increasing multiindex, \(\epsilon^{ij}_{(jJ)'}\) is the signature of the permutation and \(M^j\) is the increasing multiindex with \(j\) removed from \(M\). Then we prove the lemma by two cases.

Case 1: \(p + 1 = N\). Recall that \(\alpha_{jI} = 0\) if \(j \in I\). Then for the second term on the right side of (9), we have

\[
\sum_{|I|=N-1} \sum_{j \notin I} \left| \frac{\partial \alpha_{jI}}{\partial x_j} \right|^2 = \sum_{|I|=N-1} \sum_{j \notin I} \left| \frac{\partial \alpha_{(jI)'}}{\partial x_j} \right|^2 = \sum_{|I|=N} \sum_{j=1}^N \left| \frac{\partial \alpha_{jI}}{\partial x_j} \right|^2 ,
\]

which is the same as the first term on the right side of (9). Then (9) is proved for Case 1.

Case 2: \(p + 1 < N\). We have

\[
|d\alpha|^2 = \sum_{|M|=p+2} \left( \sum_{j \in M} \frac{\partial \alpha_{Mj}}{\partial x_j} \epsilon^{jM}_{M} \right)^2
\]

\[
= \sum_{|M|=p+2} \sum_{j \in M} \left| \frac{\partial \alpha_{Mj}}{\partial x_j} \right|^2 + \sum_{|M|=p+2} \sum_{j \in M} \sum_{k \in M \setminus j} \frac{\partial \alpha_{Mj}}{\partial x_j} \frac{\partial \alpha_{Mk}}{\partial x_k} \epsilon^{jM}_{M} \epsilon^{kM}_{M}
\]

\[
= \sum_{|J|=p+1} \sum_{j \notin J} \left| \frac{\partial \alpha_{J}}{\partial x_j} \right|^2 + \sum_{|I|=p} \sum_{j \in I \setminus \{k\}} \frac{\partial \alpha_{(ki)'} \partial \alpha_{(ji)'} \epsilon^{j(ki)'}_{(j(ki)')} \epsilon^{k(ji)'}_{(k(ji)')}}{\partial x_j} \partial x_k \epsilon^{j(ki)'}_{j(ki)'} \epsilon^{k(ji)'}_{k(ji)'}
\]

\[
= \sum_{|J|=p+1} \sum_{j \notin J} \left| \frac{\partial \alpha_{J}}{\partial x_j} \right|^2 - \sum_{|I|=p} \sum_{j \in I \setminus \{k\}} \frac{\partial \alpha_{kI} \partial \alpha_{jI} \epsilon^{kI}_{jI} \epsilon^{jI}_{kI}}{\partial x_j} \partial x_k .
\]

Note that

\[
\sum_{|J|=p+1} \sum_{j \in J} \left| \frac{\partial \alpha_{J}}{\partial x_j} \right|^2 = \sum_{|I|=p} \sum_{j \notin I} \left| \frac{\partial \alpha_{(jI)'}}{\partial x_j} \right|^2 = \sum_{|I|=p} \sum_{j \notin I} \left| \frac{\partial \alpha_{jI}}{\partial x_j} \right|^2 .
\]
Then

\[
|d\alpha|^2 = \left( \sum_{|J| = p+1}^t \sum_{j \notin J} |\partial \alpha_j / \partial x_j|^2 + \sum_{|I| = p+1}^t \sum_{j \in I} |\partial \alpha_j / \partial x_j|^2 \right) - \left( \sum_{|I| = p}^t \sum_{j \notin I} |\partial \alpha_j / \partial x_j|^2 + \sum_{|I| = p}^t \sum_{j \notin I} \partial \alpha_{ij} / \partial x_j \partial \alpha_l / \partial x_k \right)
\]

Then (9) is proved for Case 2. □

**Lemma 3.4.** Let \( G = \{ x \in \mathbb{R}^N \mid \rho(x) < 0 \} \) be a convex bounded domain, where \( \rho \) is a smooth defining function. Suppose \( \alpha = \sum_{|J| = p+1}^t \alpha_j dx^j \) is a smooth \( p+1 \)-form on \( G \), and that \( \alpha \in \text{Dom}(T^*) \). Then

\[
\|T^* \alpha\|^2_{L^2_\rho(e^{-\varphi})} + \|d\alpha\|^2_{L^2_{p+2}(e^{-\varphi})} = \int_G \left( \sum_{|I| = p+1}^t \sum_{j \notin I} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \alpha_{ij} \alpha_{kl} e^{-\varphi} \right. \\
+ \left. \sum_{|I| = p}^t \sum_{j \notin I} \frac{\partial \alpha_j}{\partial x_j} e^{-\varphi} + \int_G \left( \sum_{|I| = p+1}^t \sum_{j \notin I} \alpha_{ij} \alpha_{kl} \frac{\partial^2 \rho}{\partial x_j \partial x_k} e^{-\varphi} \right) dS \right),
\]

(10)

In particular, if \( G \) is a strictly convex bounded domain, and for \( \varphi \), there exists a constant \( c > 0 \) such that

\[
\sum_{j,k=1}^N \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k \geq c|\xi|^2 \text{ for all } \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N.
\]

Then

\[
\|T^* \alpha\|^2_{L^2_\rho(e^{-\varphi})} + \|d\alpha\|^2_{L^2_{p+2}(e^{-\varphi})} \geq c(p+1)\|\alpha\|^2_{L^2_{p+1}(e^{-\varphi})}.
\]

(11)

**Proof.** Consider the expression

\[
Q = \|T^* \alpha\|^2_{L^2_\rho(e^{-\varphi})} = \langle T^* \alpha, T^* \alpha \rangle_{L^2_\rho(e^{-\varphi})} = \langle TT^* \alpha, \alpha \rangle_{L^2_\rho(e^{-\varphi})}.
\]

(12)

By (4)-(6), we have

\[
TT^* \alpha = d(T^* \alpha) = d \left( \sum_{|I| = p}^t A_I dx^I \right) = \sum_{|I| = p}^t \sum_{k=1}^N \frac{\partial A_I}{\partial x_k} dx_k \wedge dx^I,
\]

where \( A_I \) is as (4). Let

\[
\delta_j = e^\varphi \frac{\partial}{\partial x_j} e^{-\varphi} = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j}.
\]
Then

$$A_I = - \sum_{j=1}^{N} \delta_j \alpha_{jI}. $$

Observe that

$$\delta_j \frac{\partial}{\partial x_k} = \frac{\partial^2}{\partial x_j \partial x_k} - \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_k}. $$

We have

$$\frac{\partial}{\partial x_k} \delta_j = \frac{\partial^2}{\partial x_j \partial x_k} - \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_k} = \delta_j \frac{\partial}{\partial x_k} - \varphi_{jk}. $$

Here $\varphi_{jk} = \frac{\partial^2 \varphi}{\partial x_j \partial x_k}$. So for $1 \leq k \leq N$, we have

$$\frac{\partial A_I}{\partial x_k} = - \sum_{j=1}^{N} \left( \frac{\partial}{\partial x_k} \delta_j \right) \alpha_{jI} = \sum_{j=1}^{N} \left( \varphi_{kj} \alpha_{jI} - \delta_j \frac{\partial \alpha_{jI}}{\partial x_k} \right). $$

Then by (12), we have

$$Q = \int_G T T^* \alpha \cdot \alpha e^{-\varphi}$$

$$= \int_G \sum_{|I|=p} \sum_{j=1}^{N} \frac{\partial A_I}{\partial x_k} \alpha_{kI} e^{-\varphi}$$

$$= \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} \varphi_{kj} \alpha_{jI} \alpha_{kI} e^{-\varphi} + \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} (-1) \left( \delta_j \frac{\partial \alpha_{jI}}{\partial x_k} \right) \alpha_{kI} e^{-\varphi}$$

$$= Q_1 + Q_2. \quad (13)$$

For $Q_2$, by Lemma 3.2 and 3.3 we obtain that

$$Q_2 = \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} (-1) \frac{\partial}{\partial x_j} \left( e^{-\varphi} \frac{\partial \alpha_{jI}}{\partial x_k} \right) \alpha_{kI}$$

$$= \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} \frac{\partial \alpha_{kI}}{\partial x_j} \frac{\partial \alpha_{jI}}{\partial x_k} e^{-\varphi} - \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} \alpha_{kI} \frac{\partial \alpha_{jI}}{\partial x_k} e^{-\varphi} \frac{\partial \rho}{\partial x_j} dS$$

$$= \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} \frac{\partial \alpha_{kI}}{\partial x_j} \frac{\partial \alpha_{jI}}{\partial x_k} e^{-\varphi} + \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} \alpha_{jI} \alpha_{kI} \frac{\partial^2 \rho}{\partial x_j \partial x_k} e^{-\varphi} \frac{dS}{\rho}$$

$$= \int_G \sum_{|I|=p+1} \sum_{j=1}^{N} \left( \frac{\partial \alpha_{jI}}{\partial x_j} \right)^2 e^{-\varphi} - \|\alpha\|^2_{L^2_{p+2}(e^{-\varphi})} + \int_G \sum_{|I|=p} \sum_{j,k=1}^{N} \alpha_{jI} \alpha_{kI} \frac{\partial^2 \rho}{\partial x_j \partial x_k} e^{-\varphi} \frac{dS}{\rho}. \quad (14)$$

Then (10) is proved by (12), (13) and (14).
Now we prove (11). Observe that
\[
\sum_{i=1}^t \sum_{j=1}^N \varphi jk \alpha j \alpha k I \geq \sum_{i=1}^t \sum_{j=1}^N \varphi jk \alpha j I^2 = c \sum_{i=1}^t \sum_{j=1}^N \alpha j I^2 = c \sum_{j=1}^t \sum_{j=1}^N \alpha f j^2 = c(p+1)|\alpha|^2.
\]
Then for the first term on the right side of (11),
\[
\int_{G} \sum_{i=1}^t \sum_{j=1}^N \varphi jk \alpha j \alpha k I^2 e^{-\varphi} \geq \int_{G} c(p+1)|\alpha|^2 e^{-\varphi} = c(p+1)\|\alpha\|_{L^2_p}^2(e^{-\varphi}).
\]
Since \(G\) is a strictly convex bounded domain, we have
\[
\sum_{j=1}^N \frac{\partial^2 \rho}{\partial x_j \partial x_k} \xi_j \xi_k \geq \tilde{c} |\xi|^2 \quad \text{for all} \quad \xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N,
\]
where \(\tilde{c}\) is a positive valued function in \(G\). Then the last term on the right side of (11) is nonnegative. Note that the second terms on the right side of (11) is always nonnegative. Thus, (11) is proved.

For the proof of Poincaré Lemma, we need the following density lemma since the elements in \(\text{Dom}(T^*) \cap \text{Dom}(S)\) are not necessarily smooth forms in general, and in the lemmas above, the computation is all based on the smooth elements.

**Lemma 3.5.** Let \(f \in \text{Dom}(T^*) \cap \text{Dom}(S)\). Then there exists a sequence \(\{f_\nu\}\) of smooth \(p+1\)-forms on \(G\), such that \(f_\nu \in \text{Dom}(T^*) \cap \text{Dom}(S), f_\nu \rightarrow f\) in \(L^2_p\), \(T^*f_\nu \rightarrow T^*f\) in \(L^2_p(e^{-\varphi})\) and \(Sf_\nu \rightarrow Sf\) in \(L^2_{p+2}(e^{-\varphi})\).

The proof of this lemma would be, in principle, similar to Berndtsson’s [9] (Proposition 1.5.3) for his proof of the Hörmander’s theorem for \(\overline{\partial}\), which is rather technical and non-trivial. We feel that if we had included the proof, it would have made this paper rather long. For interested readers, refer for the proof of Proposition 1.5.3 in [9].

Now we give the proof of Poincaré Lemma.

**Proof.** Let \(N = \{f \mid f \in L^2_{p+1}(e^{-\varphi}), df = 0\}\), which is a closed subspace of \(L^2_{p+1}(e^{-\varphi})\). For each \(\alpha \in \mathcal{D}_{p+1}\), clearly \(\alpha \in L^2_{p+1}(e^{-\varphi})\), so we can decompose \(\alpha = \alpha^1 + \alpha^2\), where \(\alpha^1\) lies in \(N\) and \(\alpha^2\) is orthogonal to \(N\). This implies that \(\alpha^2\) is orthogonal to any form \(Tu\), since \(Tu \in N\). So by the definition of \(\text{Dom}(T^*)\), we see that \(\alpha^2\) lies in the domain of \(T^*\) and \(T^*\alpha^2 = 0\). Since \(\alpha\) lies in the domain of \(T^*\), it follows that \(T^*\alpha = T^*\alpha^1\).

Note that \(\alpha^1 \in \text{Dom}(T^*) \cap \text{Dom}(S)\). Then by Lemma 3.3, there exists a sequence \(\{\alpha_\nu\}\), which are smooth \(p+1\)-forms on \(G\), such that \(\alpha_\nu \in \text{Dom}(T^*) \cap \text{Dom}(S), \alpha_\nu \rightarrow \alpha^1\) in \(L^2_{p+1}(e^{-\varphi})\), \(T^*\alpha_\nu \rightarrow T^*\alpha^1\) in \(L^2_p(e^{-\varphi})\), and \(S\alpha_\nu \rightarrow S\alpha^1\) in \(L^2_{p+2}(e^{-\varphi})\).

For \(\alpha_\nu\), by Lemma 3.4, we have
\[
\|T^*\alpha_\nu\|_{L^2_p(e^{-\varphi})}^2 + \|S\alpha_\nu\|_{L^2_{p+2}(e^{-\varphi})}^2 \geq c(p+1)\|\alpha_\nu\|_{L^2_{p+1}(e^{-\varphi})}^2.
\]
Let $\nu \to +\infty$, so
\[ \|T^* \alpha^1\|_{L^2_p(e^{-\varphi})}^2 + \|S\alpha^1\|_{L^2_{p+2}(e^{-\varphi})}^2 \geq c(p+1)\|\alpha^1\|_{L^2_{p+1}(e^{-\varphi})}^2, \]
which means that
\[ \|T^* \alpha^1\|_{L^2_p(e^{-\varphi})}^2 \geq c(p+1)\|\alpha^1\|_{L^2_{p+1}(e^{-\varphi})}^2 \]
since $S\alpha^1 = 0$.

By the Cauchy-Schwarz inequality, we have
\[ \left| \langle f, \alpha^1 \rangle_{L^2_{p+1}(e^{-\varphi})} \right|^2 \leq \|f\|_{L^2_{p+1}(e^{-\varphi})}^2 \|\alpha^1\|_{L^2_{p+1}(e^{-\varphi})}^2 \]
\[ = \left( \frac{1}{c(p+1)} \|f\|_{L^2_{p+1}(e^{-\varphi})}^2 \right) \left( c(p+1) \|\alpha^1\|_{L^2_{p+1}(e^{-\varphi})}^2 \right) \]
\[ \leq \left( \frac{1}{c(p+1)} \|f\|_{L^2_{p+1}(e^{-\varphi})}^2 \right) \|T^* \alpha^1\|_{L^2_p(e^{-\varphi})}^2. \]

Let $\hat{c} = \frac{1}{c(p+1)} \|f\|_{L^2_{p+1}(e^{-\varphi})}^2$. Then
\[ \left| \langle f, \alpha^1 \rangle_{L^2_{p+1}(e^{-\varphi})} \right|^2 \leq \hat{c} \|T^* \alpha^1\|_{L^2_p(e^{-\varphi})}^2 \]
for all $\alpha \in D_{p+1}$.

Note that $f \in N$. Thus,
\[ \left| \langle f, \alpha \rangle_{L^2_{p+1}(e^{-\varphi})} \right|^2 = \left| \langle f, \alpha^1 \rangle_{L^2_{p+1}(e^{-\varphi})} \right|^2 \leq \hat{c} \|T^* \alpha^1\|_{L^2_p(e^{-\varphi})}^2 = \hat{c} \|T^* \alpha\|_{L^2_p(e^{-\varphi})}^2. \]

By Lemma 3.1, there exists a solution $u \in L^2_p(e^{-\varphi})$ solving the equation
\[ du = f \]
in $G$ with the norm estimate $\|u\|_{L^2_p(e^{-\varphi})} \leq \hat{c}$, i.e.,
\[ \int_G |u|^2 e^{-\varphi} \leq \frac{1}{c(p+1)} \int_G |f|^2 e^{-\varphi}. \]

The theorem is proved. $\square$

4. Preliminary for the main theorem

Let $n \geq 1$ be an integer and $\Omega$ be a strictly convex bounded domain in $\mathbb{C}^n$. Let $\varphi$ be a nonnegative smooth function on $\Omega$ and the weighted Hilbert space
\[ L^2(\Omega, e^{-\varphi}) = \{ u : \Omega \to \mathbb{C} | u \in L^2_{loc}(\Omega); \int_\Omega |u|^2 e^{-\varphi} < +\infty \}. \]
We denote the weighted inner product for $u, v \in L^2(\Omega, e^{-\varphi})$ by
\[ \langle u, v \rangle_{L^2(\Omega, e^{-\varphi})} = \int_\Omega u \overline{v} e^{-\varphi}, \]
and the weighted norm of $u \in L^2(\Omega, e^{-\varphi})$ by $\|u\|_{L^2(\Omega, e^{-\varphi})} = \sqrt{\langle u, u \rangle_{L^2(\Omega, e^{-\varphi})}}$. 
In general, a \((1, 1)\) form \(f\) on \(\Omega\) is a formal combination
\[
f = \sum_{i,j=1}^{n} f_{ij} dz_i \wedge d\overline{z}_j,
\]
where \(f_{ij} : \Omega \to \mathbb{C}\) is a function for \(1 \leq i, j \leq n\). For \((1, 1)\) forms \(f\) and \(g\), we denote their pointwise scalar product by
\[
f \cdot \overline{g} = \sum_{i,j=1}^{n} f_{ij} \overline{g}_{ij}.
\]
We also consider the weighted Hilbert space for \((1, 1)\) forms
\[
L^2_{(1,1)}(\Omega, e^{-\varphi}) = \{ f = \sum_{i,j=1}^{n} f_{ij} dz_i \wedge d\overline{z}_j \mid f_{ij} \in L^2_{\text{loc}}(\Omega); \int_{\Omega} |f|^2 e^{-\varphi} < +\infty \},
\]
where \(|f|^2 = f \cdot \overline{f}\). We denote the weighted inner product for \(f, g \in L^2_{(1,1)}(\Omega, e^{-\varphi})\) by
\[
\langle f, g \rangle_{L^2_{(1,1)}(\Omega, e^{-\varphi})} = \int_{\Omega} f \cdot \overline{g} e^{-\varphi},
\]
and the weighted norm of \(f \in L^2_{(1,1)}(\Omega, e^{-\varphi})\) by
\[
\|f\|_{L^2_{(1,1)}(\Omega, e^{-\varphi})} = \sqrt{\langle f, f \rangle_{L^2_{(1,1)}(\Omega, e^{-\varphi})}}.
\]

For the conversion between complex and real forms, we need the following lemmas, which can be verified by simple computations.

**Lemma 4.1.** Let \(f = \sum_{i,j=1}^{n} (A_{ij} + \sqrt{-1} B_{ij}) dz_i \wedge d\overline{z}_j\) be any real \((1, 1)\) form (i.e. \(f = \overline{f}\)), where \(A_{ij}\) and \(B_{ij}\) are real-valued functions. Then \(A_{ij} = -A_{ji}, B_{ij} = B_{ji}\) and \(f\) can be decomposed to a real 2-form
\[
f = 2 \left( \sum_{1 \leq i < j \leq n} A_{ij} dx_i \wedge dx_j + \sum_{1 \leq i < j \leq n} A_{ij} dy_i \wedge dy_j + \sum_{i,j=1}^{n} B_{ij} dx_i \wedge dy_j \right).
\]

**Lemma 4.2.** Let \(v = \sum_{j=1}^{2n} v_j dx_j\) be any real 1-form, where \(v_j\) are real-valued functions. Let \(z_j = x_{2j-1} + \sqrt{-1} x_{2j}\). Then \(v\) can be decomposed to
\[
v = v^{1,0} + v^{0,1},
\]
where \(v^{1,0} = \sum_{j=1}^{n} \left( \frac{1}{2} v_{2j-1} + \frac{1}{2\sqrt{-1}} v_{2j} \right) d\overline{z}_j\) is a \((1, 0)\) form and \(v^{0,1} = \sum_{j=1}^{n} \left( \frac{1}{2} v_{2j-1} - \frac{1}{2\sqrt{-1}} v_{2j} \right) d\overline{z}_j\) is a \((0, 1)\) form.

5. Proof of the main theorem

First we give three lemmas. They are all well-known and can be simply verified by virtue of the definition of distributions.

**Lemma 5.1.** If \(u \in L^2(\Omega, e^{-\varphi})\) and \(\overline{\partial u} \in L^2_{(1,1)}(\Omega, e^{-\varphi})\). Then \(\partial u = \overline{\partial u}\), where \(\partial u\) and \(\overline{\partial u}\) are in the sense of distributions.
Lemma 5.2. If \( u \in L^2(\Omega, e^{-\varphi}) \). Then \( \overline{\partial} u = -\partial \overline{\partial} u \) in the sense of distributions.

Remark 5.1. In the lemma, it is crucial that \( \overline{\partial} u \) and \( \partial \overline{\partial} u \) are both forms. Otherwise, when \( n = 1 \), \( \overline{\partial} u = \partial \overline{\partial} u \) if \( \overline{\partial} u = \partial \overline{\partial} u = \frac{\partial^2 u}{\partial x \partial \overline{z}} \) are as weak derivatives.

Lemma 5.3. Let \( u \in L^2(\Omega, e^{-\varphi}) \). If \( \overline{\partial} u \in L^2_{0,1}(\Omega, e^{-\varphi}) \), then \( \overline{\partial} \overline{u} = \partial (\overline{\partial} u) \) in the sense of distributions. If \( \partial u \in L^2_{0,1}(\Omega, e^{-\varphi}) \), then \( \overline{\partial} \overline{u} = \overline{\partial} (\overline{\partial} u) \) in the sense of distributions.

To prove the main theorem, we also need the following simple version of the Hörmander’s theorem [1] (page 92, Lemma 4.4.1).

Hörmander’s theorem. (A simple version for \((0,1)\) forms) Let \( \Omega \) be a pseudoconvex open set in \( \mathbb{C}^n \). Let \( \varphi \) be a real-valued smooth function in \( \Omega \) such that
\[
\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x_j \partial \overline{z}_k} \omega_j \overline{\omega}_k \geq c |\omega|^2 \quad \text{for all } \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n,
\]
where \( c > 0 \) is a constant. For each \( f \in L^2(\Omega, e^{-\varphi}) \) such that \( \overline{\partial} f = 0 \), there exists a solution \( u \) in \( L^2(\Omega, e^{-\varphi}) \) solving equation \( \overline{\partial} u = f \) in \( \Omega \), in the sense of distributions, with the norm estimate
\[
\int_{\Omega} |u|^2 e^{-\varphi} \leq \frac{2}{c} \int_{\Omega} |f|^2 e^{-\varphi}.
\]

In order to apply the Hörmander’s theorem above, we need the following results that convert real convexity to plurisubharmonicity, which is well-known, but for which a brief proof is provided.

Lemma 5.4. Let \( \varphi \) be a smooth function in a domain in \( \mathbb{R}^{2n} \). Let \( \xi = (\xi_1, \ldots, \xi_{2n}) \in \mathbb{R}^{2n} \) and \( x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \). For \( 1 \leq j \leq n \), let \( \omega_j = \xi_j + \sqrt{-1} \xi_{j+n} \) and \( z_j = x_j + \sqrt{-1} x_{j+n} \). Then
\[
\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k = \sum_{j,k=1}^{n} \left( \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \overline{\omega}_k + 2 \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \overline{\omega}_k + \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \overline{\omega}_k \right).
\]

Proof. Consider \( \varphi \) in real variables. Let \( \phi(t) = \varphi(x + t \xi) \). Then the second derivative of \( \phi \) at 0 is
\[
\phi''(0) = \sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k.
\]
Consider the same function \( \phi \) in complex variables as \( \phi(t) = \varphi(z + tw) \). Then the second derivative of \( \phi \) at 0 is
\[
\phi''(0) = \sum_{j,k=1}^{n} \left( \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \overline{\omega}_k + 2 \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \overline{\omega}_k + \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \overline{\omega}_k \right).
\]
Thus the lemma is proved. \( \square \)
From the lemma above, we have the following lemma, which we need to use.

**Lemma 5.5.** Let \( \varphi \) be a real-valued smooth function in a domain in \( \mathbb{R}^{2n} \) such that

\[
\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k \geq c |\xi|^2 \quad \text{for all} \quad \xi = (\xi_1, \cdots, \xi_{2n}) \in \mathbb{R}^{2n},
\]

where \( c > 0 \) is a constant. For \( 1 \leq j \leq n \), let \( z_j = x_j + \sqrt{-1} x_{j+n} \). Then

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \geq \frac{c}{2} |\omega|^2 + \left| \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \right| \quad \text{for all} \quad \omega = (\omega_1, \cdots, \omega_n) \in \mathbb{C}^n.
\]

**Proof.** By Lemma 5.4, we have

\[
\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k = 2 \sum_{j,k=1}^{n} \left( \text{Re} \left( \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \right) + \frac{\partial \varphi}{\partial z_j} \omega_j \right),
\]

where \( \omega_j = \xi_j + \sqrt{-1} \xi_{j+n} \) for \( 1 \leq j \leq n \). Then for all \( \omega = (\omega_1, \cdots, \omega_n) \in \mathbb{C}^n \),

\[
2 \sum_{j,k=1}^{n} \left( \text{Re} \left( \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \right) + \frac{\partial \varphi}{\partial z_j} \omega_j \right) \geq c |\omega|^2,
\]

i.e.,

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \geq \frac{c}{2} |\omega|^2 - \text{Re} \left( \sum_{j,k=1}^{n} \frac{\partial \varphi}{\partial z_j} \omega_j \right).
\]

For any fixed \( \omega \in \mathbb{C}^n \), replace \( \omega \) by \( e^{\sqrt{-1} \theta} \omega \) in the above formula, where \( \theta \) is a real number such that

\[
-\text{Re} \left( e^{2\sqrt{-1} \theta} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \right) = \left| \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \right|.
\]

Then we have

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \omega_j \omega_k \geq \frac{c}{2} |\omega|^2 + \left| \sum_{j,k=1}^{n} \frac{\partial \varphi}{\partial z_j} \omega_j \right|.
\]

Thus, the lemma is proved. \( \square \)

Finally, we are ready to give the proof of the main theorem.

**Proof.** First we prove the theorem for the case that \( f \) is a real \((1,1)\) form. Observe that \( \sqrt{-1} \partial \overline{\partial} \) is a real operator by Lemma 5.2.

For real \((1,1)\) form \( f \in L^2_{(1,1)}(\Omega, e^{-\varphi}) \), by Lemma 4.4, \( f \) can be seen as a real 2-form and \( f \in L^2_{(2)}(\Omega, e^{-\varphi}) \). Then by Poincaré Lemma \((p = 1)\), there exists \( v \in L^1_{(1)}(\Omega, e^{-\varphi}) \) such that

\[
dv = f \tag{15}
\]
with
\[ \|v\|^2_{L^2_2(\Omega, e^{-\varphi})} \leq \frac{1}{2c} \|f\|^2_{L^2_2(\Omega, e^{-\varphi})}. \] (16)

For this \( v \), by Lemma 4.2 we have the decomposition in complex forms
\[
v = v^{1.0} + v^{0.1},
\] (17)
where \( v^{1.0} \in L^2_{1,0}(\Omega, e^{-\varphi}) \), \( v^{0.1} \in L^2_{0,1}(\Omega, e^{-\varphi}) \), \( v^{1.0} = v^{1.0} \) and \( v^{0.1} = v^{0.1} \). By (15) and (17), we have
\[
f = (\partial + \overline{\partial})(v^{1.0} + v^{0.1}) = \partial v^{1.0} + \overline{\partial} v^{0.1} + \overline{\partial} v^{1.0} + \overline{\partial} v^{0.1}.
\] (18)

Note that \( \overline{\partial} v^{1.0} \) is a \((2, 0)\) form, \( \overline{\partial} v^{0.1} \) is a \((0, 2)\) form and \( f \) can be seen as a \((1, 1)\) form. So from (18), we have \( \partial v^{1.0} = 0 \), \( \overline{\partial} v^{0.1} = 0 \) and
\[
\partial v^{0.1} + \overline{\partial} v^{1.0} = f.
\] (19)

For \( v^{0.1} \), by Lemma 5.5 and the Hörmander’s theorem (replace \( c \) by \( \frac{\xi}{2} \)), there exists \( w \in L^2(\Omega, e^{-\varphi}) \) such that
\[
\overline{\partial} w = v^{0.1},
\] (20)
with
\[
\int_\Omega |w|^2 e^{-\varphi} \leq \frac{4}{c} \int_\Omega |v^{0.1}|^2 e^{-\varphi}.
\] (21)

So for \( w \), by Lemma 5.1 and \( v^{0.1} = v^{1.0} \), we have
\[
\overline{\partial} w = \overline{\partial} w = v^{1.0}.
\] (22)

Then by Lemma 4.2, 5.2, 5.3, (16), and (19)-(22), we obtain
\[
\overline{\partial} (w - \overline{w}) = \overline{\partial} \overline{w} - \overline{\partial} \overline{w} = \overline{\partial} \overline{w} + \overline{\partial} \overline{w} = \partial (\overline{\partial} w) + \overline{\partial} (\overline{\partial} w) = \partial v^{0.1} + \overline{\partial} v^{1.0} = f,
\] (23)
with
\[
\int_\Omega |w - \overline{w}|^2 e^{-\varphi} \leq 4 \int_\Omega |w|^2 e^{-\varphi} \leq \frac{16}{c} \int_\Omega |v^{0.1}|^2 e^{-\varphi} = \frac{4}{c} \|v\|^2_{L^2_2(\Omega, e^{-\varphi})} \leq \frac{2}{c^2} \|f\|^2_{L^2_2(\Omega, e^{-\varphi})}.
\] (24)

Let \( u = -\sqrt{-1} (w - \overline{w}) \). Note that \( \|f\|^2_{L^2_2(\Omega, e^{-\varphi})} = 4 \|f\|^2_{L^2_2(\Omega, e^{-\varphi})} \). Then \( u \in L^2(\Omega, e^{-\varphi}) \) and
\[
\sqrt{-1} \overline{\partial} u = f \quad \text{with} \quad \|u\|^2_{L^2_2(\Omega, e^{-\varphi})} \leq \frac{8}{c^2} \|f\|^2_{L^2_2(\Omega, e^{-\varphi})}.
\]
So the theorem is proved for the case that \( f \) is a real \((1, 1)\) form.

Now we prove the theorem for the case that \( f \) is not a real \((1, 1)\) form. Write \( f = f_1 + \sqrt{-1} f_2 \), where \( f_1 = \frac{1}{2} (f + \overline{f}) \) and \( f_2 = \frac{1}{2 \sqrt{-1}} (f - \overline{f}) \). Then \( f_1 \) and \( f_2 \) are real 2-forms. Apply twice the same way above and then the theorem is proved. \[\square\]
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