We propose a model of time evolution of quantum objects which unites the unitary evolution and the measurement procedures. The model allows to treat the time on equal footing with other dynamical variables.

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I. INTRODUCTION

One of the main goals of quantum theory formalism is a description of time evolution of quantum systems. Usually the evolution is regarded as a two step procedure which includes reversible, unitary evolution and irreversible "measurement" handled by the so called projection or reduction postulate. There are some models which come across attempts to unify both, qualitatively different steps as for example [1, 2, 6, 8, 10, 14, 22, 37, 38]. However some of the major questions are not satisfactory solved so far.

The main directions of the investigations are related to a few major modifications and interpretations of quantum theory like consistent histories approaches [8, 18, 35], modal interpretations [34, 38], quantum jumps and collapse ideas [2, 9, 10] and different types of models based on stochastic equations [11, 12, 13, 14, 15, 16, 17].

The measurement process is treated differently depending on interpretation of quantum mechanics. There is no place here to discuss all the possible interpretations. A good, but because of very dynamic development of the field, not quite up-to-date review can be found in [19, 36], see also [20, 21] and references therein.

The fundamental part of description of measurement seems to be the projection postulate which, from the orthodox point of view, determines the state of the system just after measurement. However, this procedure leads to inconsistencies with the Schrödinger equation. In other words, the usual description of a process of quantum time evolution requires a coexistence of two different evolution mechanisms. This leads to additional problems like the time duration and dynamics of the "wave function collapse" [11, 23, 26], a general problem of physical reality before and after measurement [8, 4, 5] and non–local–like behavior in case of spatially separated quantum subsystems [27].

This short introduction is, of course, far from being complete, but we can conclude that the problems mentioned above are still opened and their explanation requires probably fundamental changes of the evolution postulates of quantum theory.

The goal of this paper is to propose an alternative model of quantum evolution. In the following, we work in a nearly standard framework of quantum mechanics replacing only the unitary time evolution and projection postulate by a projection evolution (PEv) idea. It means that we reject the Schrödinger equation as the fundamental equation of motion. It can be recovered as a special case of the projection evolution.

The proposed evolution procedure unites both the unitary evolution and the process of measurement. Within this idea the evolution is only a sequence of some projections made randomly by Nature, and with a system dependent probability distribution. The sequence of projections is dependent on structure of the physical system under consideration. The procedure can be extended to a POVM scheme [19], as well.

The main difference among our proposal and other major models is that the PEv formalism allows to treat time on equal footing with the spatial variables. The time is not a parameter but a dynamical variable. This feature makes the model hardly comparable with the ones in which the physical time is the evolution parameter.

Nevertheless, some aspects of the PEv idea are similar to well known models. In this sense, this idea lies somewhere 'between' the decoherent histories approach [8] and some modal interpretations (see pp.: 108, 179, 199, 253 in)
In fact, in the decoherent histories approach the paths of evolution are determined partially by a family of decompositions of unity. On the other hand, the preferred basis can define such decomposition in the modal theories. These decompositions of unity describe the physical, basic properties of quantum systems and play a role the evolution operator within the PEv approach.

One can also find some formal similarities of the projection evolution to the analysis of the quantum Zeno and anti-Zeno effects e.g., [28]. However, the basic mechanism we propose does not lead to a sequence of subsequent measurements only, but to random choices of the next states of the system, where the probability distribution is a function dependent on previous states of the system.

The idea of the PEv can be treated as a special case of more general concept:

1. The states of a quantum system are described by either density operators or the appropriate functionals, and the term “quantum evolution” is understood as a sequence of state changes ordered by the the Nature.

   This idea requires introducing of an ordering parameter which orders the subsequent changes of the state of a physical system. This parameter, denoted by \( \tau \) will be called the evolution parameter.

   The above statements requires additional explanations. Usually changes are related to the time parameter. In general, in our approach, there is no direct relation between both, the time and state changes. We suggest that the natural change of the state of a physical is the primitive notion. The space and time are the additional structures of our Universe. They are dynamical variables in the theory.

   The standard non–relativistic quantum mechanics can be considered as approximation in which both the evolution parameter and the physical time are in one–to–one correspondence. In this case, the time is not a dynamical variable, but a parameter which can be identified with the evolution parameter of the PEv.

   Both, the evolution parameter and the physical time can be approximately related one to another for the physical systems described by the states which are strongly localized in the time variable. This can explain the successes of the traditional approach but it leads also to some well known paradoxes. We will consider this type of quantum systems in the the section containing a few examples.

   Many problems like time–energy uncertainty principle, time of arrival, life–time problems, the Young type experiments for time–slits [24] and similar phenomena suggest that the physical time should be the dynamical variable not a parameter. It is much more natural assumption.

   In addition, the relativistic quantum mechanics, in fact, requires that both notions the evolution parameter and the time variable should be different. Note, that mixing of the space and time coordinates by the Lorentz transformations results in the undetermined status of transformed coordinates. They are neither parameters nor variables. The same one can say even about Galilean transformations.

   We claim that the physical time should be treated on the same footing as the other spatial coordinates. It should be also the dynamical variable which spans the additional dimension in the physical space. The physical states should determine localization of the system in space and time according to quantum rules.

   We suppose that the evolution parameter \( \tau \) is a common parameter for the whole Universe. In this sense it parameterizes a kind of the ”internal clock” of the Universe. Because \( \tau \) is the parameter it cannot be affected by physical interactions (the physical time can !)

   For simplicity we assume that \( \tau \) is a real number parameter.

2. There exists a World Lottery mechanism (chooser) which chooses randomly the next state of quantum system in a way dependent on physical structure of this quantum system.

   The realization of the chooser is based on the so called selected collapse projection postulate (see [23] and references therein). Our chooser is not driven by the random noise but it is a kind of the quantum state dependent World Lottery.

   The evolution parameter \( \tau \) is an internal parameter of the ”World Lottery” mechanism. For every \( \tau \) the Nature draws lots to get next state of the Universe.

   It means, that the projection evolution (PEv ) can be considered as a stochastic process (in respect to \( \tau \), instead of time) determining, at each step of evolution, a physical state of the system represented by a density operator \( \rho \).

3. The probability distribution for the chooser is determined, in some way, by the previous states of the quantum system.
4. The evolutions of a larger quantum system and its subsystems have to be consistent.

Because we assume that the evolution parameter $\tau$ is common for our Universe, the Universe itself changes states as a whole. It means that the effective evolution for a given subsystem should be induced by the global evolution.

The description of the global evolution (for the whole Universe) is in practice not available, and in fact, we are able to write down only evolution operators for larger or smaller subsystems. These effective operators have to satisfy some scaling conditions. In the next section a proposal for scaling condition is discussed. However, this is still an open problem.

Because we expect that such fundamental mechanism as evolution of quantum system should be a rather simple physical law, consistent with known results of quantum mechanics, we propose a rather natural realization of this idea in the next paragraph.

The approach is observer-free. We do not need neither to split the physical world into observer and the physical objects which have to be observed nor to introduce a mind, which allows to choose the state after observation, as is required in some approaches (collapse of states is the most natural process in the PEv).

The PEv do not require also splitting of the Universe into classical and quantum worlds.

We do not need also any strange assumptions about our "knowledge in which way" e.g., the particle passes through the slits.

These results are a consequence of a single principle postulated within the PEv approach.

II. PROJECTION EVOLUTION

Let us consider a quantum system described by states represented by the quantum density operators $\rho$. In the following by $\tau$ we denote the evolution parameter which orders the evolution of the Universe. In addition, for each $\tau$ we define a family of projectors which gives an orthogonal resolution of unity i.e., roughly speaking, for each $\tau$ they fulfill the following conditions:

$$E(\tau;\nu) E(\tau;\nu') = \delta_{\nu,\nu'} E(\tau;\nu)$$

$$\sum_\nu E(\tau;\nu) = \mathbb{I},$$

where $\mathbb{I}$ denotes the unit operator.

The operators $E(\tau;\nu)$ should represent the essential constraints of the physical system, responsible for its time evolution. The Hamiltonian plays the same role in traditional approach. In this sense, they play a role of the evolution operators because they force changes of the physical system which they describe. The indices $\nu$ represent here the sets of quantum numbers labelling the projection operators uniquely.

Instead of usual unitary evolution we postulate that the new state $\rho(\tau;\nu)$ of the physical system, at the evolution parameter $\tau$, is created from the previous state $\rho(\tau - dt;\nu')$ by randomly chosen, projection $E(\tau;\nu)$. We will show a few examples how to construct such families of projectors, in next section.

The projectors $E(\tau;\nu)$ are chosen randomly according to the following probability distribution:

$$\text{Prob}(\tau;\nu) = \text{Tr}[E(\tau;\nu)\rho(\tau - dt;\nu')E(\tau;\nu)],$$

where $\nu$ runs over the sets of quantum numbers describing the system at the evolution parameter $\tau$ and $\nu'$ is the set of quantum numbers which describes the previous state. It means, that the Nature applies the 'projection postulate' to get a new state from the previous one:

$$\rho(\tau;\nu) = \frac{E(\tau;\nu)\rho(\tau - dt;\nu')E(\tau;\nu)}{\text{Tr}[E(\tau;\nu)\rho(\tau - dt;\nu')E(\tau;\nu)]}. \tag{3}$$

To simplify our notation let us imagine that the projections $E(\tau;\nu)$ are constant functions of the evolution parameter on the intervals $\tau_n, \tau_{n+1}$, where $\tau_0 < \tau_1 < \tau_2 < \tau_n, \ldots$. Using this notation the idea of PEv leads to the following recurrence equation for states of the system under consideration:

$$\rho(\tau_{n+1};\nu_{n+1}) = \frac{E(\tau_{n+1};\nu_{n+1})\rho(\tau_n;\nu_n)E(\tau_{n+1};\nu_{n+1})}{\text{Tr}[E(\tau_{n+1};\nu_{n+1})\rho(\tau_n;\nu_n)E(\tau_{n+1};\nu_{n+1})]} \tag{4}.$$
We do not need any "environment"; the same procedure should be applied for microscopic and macroscopic systems. The formula (2) is a conservation of "purity" of states. If the previous state is pure, the next one is also a pure state. This is described by the Nature and can be written as:

$$ρ_{A,τ} = \text{Tr}_A(ρ_W).$$

To have consistent description both $W$ and $A$ one needs to fulfill the following obvious condition (scaling property):

$$\text{Tr}_A \frac{Ε(W,τ;ν)ρ_W(τ - dτ)Ε(W,τ;ν)}{Ε(W,τ;ν)ρ_W(τ - dτ)Ε(W,τ;ν)} = \frac{Ε(A,τ;μ)ρ_A(τ - dτ)Ε(A,τ;μ)}{Ε(A,τ;μ)ρ_A(τ - dτ)Ε(A,τ;μ)},$$

where $Ε(A,τ;μ)$ are required effective PEv operators for the subsystem $A$ with the appropriate sets of quantum numbers $μ$ corresponding, in some way, to the quantum numbers $ν$ of the larger system $W$. It means that drawing of lots required by the PEv approach should be correlated within the connected parts of the Universe.

In principle, having the state $ρ_W$ and $Ε(W,τ;ν)$ for the Universe the condition (5) should allow obtaining the appropriate states and the PEv operators for required subsystems. Till now, we have no proof for uniqueness of solutions of the equations (5) in respect to $Ε(A,τ;μ)$.

The general equation (4) suggests that the process of evolution can follow various paths $ν_0 → ν_1 → ν_2 → ... → ν_n$ considered as the series of projections chosen randomly at the evolution parameters $τ_0, τ_1, ... , τ_n$. Then one can ask about the conditional probability of choosing of the state $ρ(τ_n;ν_n)$ under condition, that the previous states chosen by the Nature are described by $ν_0 → ν_1 → ν_2 → ... → ν_{n-1}$.

At this point, one needs to observe that the PEv generates a set of "quantum histories" consisted of products of projection operators similar to those in (4). However, the histories are ordered in respect to the evolution parameter, not by the physical time. In this case there is no algebra of histories because the only available "histories" are those defined by the PEv operator which does not allow for logical operations on the evolution paths. For example, in notation of (3), the negation of the compound "history" like $Y = F_1(τ_1) ∩ F_2(τ_2)$ has no meaning in PEv, because PEv represents a step by step stochastic process and $1 - Y$ cannot be ordered by the evolution parameter. The same one can say about the other logical operations on the paths.

According to the PEv idea, the main physical interest is not in the products of projections only, but also in series of states which they produce. More precisely, the system is fully described at the evolution parameter $τ$ if we have the state and the last projection operator responsible for generating this state from the previous one – a kind of history of the system is a sequence of pairs of quantum states and the appropriate projections.

The probability of finding a given path of evolution at the evolution parameter $τ$ can be calculated using equations (4) and (2) and can be written as:

$$\text{Prob}(τ = τ_n; ν_0, ν_1, ν_2, ... , ν_n) = \text{Tr} \left[ Ε(τ_n; ν_n)Ε(τ_{n-1}; ν_{n-1})...Ε(τ_1; ν_1)Ε(τ_0; ν_0)ρ_0 \right]
$$

$$Ε(τ_0; ν_0)Ε(τ_1; ν_1)...Ε(τ_{n-1}; ν_{n-1})Ε(τ_n; ν_n)].$$

In this way we have defined the evolution of quantum systems as a kind of a stochastic game played by Nature. This game is not parameterized by the physical time but the evolution parameter. Within this idea we treat this game as a fundamental process – a Law of Nature.

### III. EXAMPLES

In this section we show three simple examples of description in terms of the projection evolution. The most important seems to be a reproducing of the Schrödinger–like evolution. It is the purpose of the next subsection in which we show how to derive the Schrödinger evolution using a rather general method of generating operator. The other examples concern a toy model of Mach-Zehnder interferometer and the evolution of a particle passing through the device.

#### A. The harmonic oscillator

Let us introduce a set of mutually commuting hermitian operators $\hat{W}(τ;1), \hat{W}(τ;2), ... , \hat{W}(τ;N)$ which describes the essential properties of the system within the space–time and other degrees of freedom. It is important to notice
that, due to the spectral theorem, there exists a common decomposition of unity for all the operators $\hat{W}(\tau, k)$, $k = 1, 2, \ldots, N$. This resolution of unity should give the PEv operators. The operators $\hat{W}(\tau, 1), \hat{W}(\tau, 2), \ldots, \hat{W}(\tau, N)$ will be called the evolution generating operators.

Let us consider the 3-D harmonic oscillator, but in the non-relativistic space–time. For this purpose we introduce the single particle state space (without spin) $\mathcal{K} = L^2(R^4)$ in the four dimensional real space $R^4$. The scalar product in $\mathcal{K}$ is defined as:

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{R^4} dt d^3 \vec{x} \Psi_1(t, \vec{x})^* \Psi_2(t, \vec{x})$$  \hspace{1cm} (7)

and it contains the integration over the time. It means that the time is dynamical variable.

Now we define the evolution generating operators as:

$$\hat{W}(\tau; 1) = i\hbar \frac{\partial}{\partial t} - \hat{H}$$

$$\hat{W}(\tau; 2) = \hat{L}^2,$$

$$\hat{W}(\tau; 3) = \hat{M},$$  \hspace{1cm} (8)

where $\hat{H}$ denotes the usual harmonic oscillator Hamiltonian, $\hat{L}^2$ is the square of the angular momentum operator and $\hat{M}$ denotes the third component of the angular momentum operator.

All the operators $\hat{W}(\tau; k)$, $k = 1, 2, 3$ are Hermitian and they are the subject of the spectral theorem. The first operator $\hat{W}(\tau; 1)$ has the continuous spectrum, the second one and the third have the discrete spectrum. It allows to write the common spectral measure as a product of spectral measures of these three operators:

$$dM(w, l, m) = dM_{\hat{W}(\tau; 1)}(w) M_{\hat{W}(\tau; 2)}(l) M_{\hat{W}(\tau; 1)}(m),$$  \hspace{1cm} (9)

where $dM_{\hat{W}(\tau; 1)}(w)$ denotes the spectral measure for $\hat{W}(\tau; 1)$, $M_{\hat{W}(\tau; 2)}(l)$, the decomposition of unity for $\hat{L}^2$ operator and $M_{\hat{W}(\tau; 1)}(m)$ the corresponding decomposition of unity for $\hat{M}$.

The suggested evolution operator should be of the following form:

$$\Phi_{nlm}(\tau; w, l, m) = dM_{\hat{W}(\tau; 1)}(w) M_{\hat{W}(\tau; 2)}(l) M_{\hat{W}(\tau; 1)}(m),$$  \hspace{1cm} (10)

and, in fact, it is independent of evolution parameter $\tau$. As usually, there are known difficulties for the continuous spectrum case. These problems can be solved in many ways e.g., by making use of the smeared out operators method \cite{32}.

The projection evolution operator $\Phi_{nlm}$ leads to the evolution–parameter independent evolution. In this case, the first random choice decides about the state of the system which do not evolve further with increasing $\tau$. For every $\tau$ the state chosen by the Nature is of the following form:

$$\Phi_{nlm}(\tau; t, \vec{x}) = \sum_n c_{nlm} e^{-\frac{i}{\hbar}(E_n(\tau)+w)t} \phi_{nlm}(\vec{x}) = e^{-\frac{i}{\hbar}wt} \sum_n c_{nlm} e^{-\frac{i}{\hbar}E_n(\tau)t} \phi_{nlm}(\vec{x}),$$  \hspace{1cm} (11)

where $c_{nlm}$ are numerical coefficients. The vectors $\phi_{nlm}$ are the common eigenvectors of the three operators: the harmonic oscillator Hamiltonian $\hat{H}$, $\hat{L}^2$ and $\hat{M}$ with the eigenvalues $E_n$, $l(l+1)$ and $m$, respectively. The "only" difference between the standard approach and PEv is that the functions (11) are functions of time, where time is a dynamical variable. It means that such states are not localized in time.

As it was mentioned above, the vectors (11) can be normalized to the Dirac–delta distribution. However, in reality one can expect the states in the form of wave packets:
The states $|13\rangle$ correspond exactly to the traditional Schrödinger type solutions for the harmonic oscillator. However, one needs to realize that within the PEv these solutions represent the states which are localized in the fixed time $t'$. It means, that one needs to have some additional interactions to "shift" these states in time. This is what we are doing in the traditional unitary evolution approach. Within the standard quantum mechanics one assumes implicitly, that there exists a "time machine" which evolves the state perfectly localized in time to later moments. This mechanics describes the state vectors projected onto the subsequent moments of time loosing possible non–local time correlations.

On the other hand, the state $|12\rangle$ describes a full time behavior of the system as a single vector which allows to calculate typical statistical characteristics like averages, variations and other statistical moments. For example, the average time position of the harmonic oscillator described by the wave packets of the form $|15\rangle$ can be obtained in the usual way by the integration:

$$
<t> = \int_{R} dt \int_{R} dx \int_{R} dw \int_{R} dw' a(w')^{*} a(w) \Phi_{w'l'm}(\tau; t, \vec{x})^{*} t \Phi_{w'm}(\tau; t, \vec{x})
$$

(14)

Obviously, this example can be generalized to any arbitrary system which one can describe in the traditional way.

B. Beam splitter – toy models

In this paragraph we consider a very simplified Mach-Zehnder interferometer. The aim of the considerations is to show the temporal behavior of the interferometer in two models.

The first model does not correspond to the Schrödinger type motion because the beam splitters localize the particle. The second model is based on more traditional thinking about the beam splitters as some unitary devices. In both cases we get the similar results which differs mainly in temporal part of the description.

The main disadvantage of both models is that they do not show, in general possible, spreading in time of states of the system. Because of this, within these toy models, the time of arrival of a particle to the detector is a sharp number, without random distribution. However, this type of models can be analyzed on the elementary level and they are able to show some interesting features of the formalism.

Toy Model I: Beam splitter which localizes events.

The Mach-Zehnder interferometer shown in $4$ allows to show the quantum interference. The particles are produced in the source $Z$. The first beam splitter $D_{1}$ splits an incoming beam into two separate channels. Next, there is a phase shifter $PS$ and the second beam splitter $D_{2}$. $D_{1}$ and $D_{2}$ denote the detectors which detect the particle in the first or the second output channel. However, for our purpose we consider a simplified version of the Mach-Zehnder interferometer. It is shown in the figure $2$. It is possible to simulate, to some extend, the Mach–Zehnder interferometer shown in the figure $4$ for example, it happens for the $\frac{1}{2}$–spin particles or even polarized photons.

In the following we are using the toy model (like in sect. 2.5 of $5$) in which the space is discrete. However, in our approach the time is also a dynamical variable and we have to use not one–dimensional space but two–dimensional space–time for which both the time $t$ and the coordinate $x$ takes on only a finite numbers of integer values:

$$
-T_{a} \leq t \leq T_{b}
$$

$$
-X_{a} \leq x \leq X_{b}
$$

(15)

Like in $5$ we assume the "periodic boundary conditions" for both the time and the space coordinates, so that last sites for $t$ and $x$ are adjacent to the first sites. In this way, our space–time $X=T \times X$ is the Cartesian product of two discrete "circles". First one representing the time $T$ consists of $T_{a} + T_{b} + 1$ points and the second one $X$ contains $X_{a} + X_{b} + 1$ space coordinate points, respectively. In addition, we assume an extra two–valued variable $\zeta = a, b$ which determines two possible channels in which each space–time point can be located (see Ch. 12 of $5$).

The space $K$ of quantum states consists of complex functions:

$$
\psi : X \times \{a, b\} \rightarrow \mathbb{C}.
$$

(16)

The scalar product in the state space $K$ is:

$$
\langle \psi_{1} | \psi_{2} \rangle = \sum_{t', x', \zeta} \psi_{1}(t', x', \zeta)^{*} \psi_{2}(t', x', \zeta).
$$

(17)

Again, it is very important to keep in mind, that time is not a parameter but a dynamical variable. It does not enumerate the subsequent events and doesn’t describe the evolution of the system in this sense. More, it is even
possible that two subsequent, ordered by the evolution parameter $\tau$, events occur in the inverse order in time. In this way we can have so called backward causation required for quantum world by some authors [31].

To construct the projection evolution operator we have to analyze the physical structure (in required approximation) of devices shown in the diagrams.

Let us assume that the source $Z$ produces the particles localized in both the time $t$ and the position $x_Z$ i.e., in the states $| t, x_Z, \sigma \rangle$ which are defined as the "eigenfunctions" of time, position and channel operators. Other words, the states $| t, x, \sigma \rangle$ represent the functions localizing the particle in point $(t, x)$ of the space–time and in the channel $\sigma$:

$$| t, x, \sigma \rangle \rightarrow \theta_{t x \sigma}(t', x', \zeta) = \delta_{t t'} \delta_{x x'} \delta_{\sigma \zeta}.$$  \hspace{1cm} (18)

Note that $t, x$ and $\sigma = a, b$ are here the quantum numbers but $t', x'$ and $\zeta$ denote the corresponding variables.

Starting from the source, the particle is created in one of the possible states $| t, x_Z, \sigma = a \rangle$ i.e., at a randomly chosen time $t$, at fixed place $x_Z$ and in the $a$ channel. There are also possible other, less interesting processes which are represented here by a set of projection operators denoted by $\mathcal{OP}(Z; \nu)$ (other processes). Together with the creation of particle they complete a resolution of unity. At this moment we do not need to analyze them more carefully.

The assumptions allow to write the following projection evolution operator for the source:

$$\mathcal{E}_Z(\nu) = \begin{cases} | t, x_Z, a \rangle \langle t, x_Z, a | & \text{for } \nu = t \in T \\ \mathcal{OP}(Z; \nu) & \text{otherwise} \end{cases} \hspace{1cm} (19)$$

Note that the operator is independent of $\tau$. 

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**FIG. 1:** Mach-Zehnder interferometer

**FIG. 2:** One–dimensional Mach–Zehnder interferometer
As the next step we have to construct the toy model of free evolution. This type of evolution is required to describe behavior of the particle outside of devices i.e., in the free space–time.

In our toy model (I) we assume the free evolution of projective type but which resembles the Schrödinger’s type of evolution. For this purpose we define the unitary transformation $S$ (see the toy models in [8]):

$$\hat{S}|x,\sigma\rangle = |x + 1,\sigma\rangle,$$

(20)

where the ket $|x,\sigma\rangle$ represents the function $\xi_{x,\sigma}(x',\zeta)$, see [11]. The operator $\hat{S}$ is the unitary operator within the space spanned by the kets $|x,\sigma\rangle$ i.e., spanned by the functions $\xi_{x,\sigma}(x',\zeta)$.

The last property allows to define the set of vectors which allow to define the Schrödinger’s type of evolution. These vectors will be denoted by:

$$|\phi_{l,x,\sigma}\rangle \rightarrow |\phi_{l,x,\sigma}(t',x',\zeta) = \chi_l(t')\hat{S}(t')\xi_{x,\sigma}(x',\zeta),$$

(21)

where $\hat{S}(t') = \hat{S}^t$ is the local (it is a function of time!) unitary operator in [20]. We assume that the set of functions $\chi_l$, where $l$ is an integer, give an orthonormal basis in the space of all functions dependent only on time:

$$\langle\chi_{l'}|\chi_l\rangle = \sum_l \chi_{l'}(t')^* \chi_l(t') = \delta_{l,l'}.$$  

(22)

The functions $\chi_l$ determine the main part of the amplitude describing the fact that the particle can be localized in time (see remarks below the eq. [21]).

It implies that the functions $|\phi_{l,x,\sigma}\rangle$ form an orthonormal basis in the space of states $\mathcal{K}$.

Because these vectors are independent of the evolution parameter, the projection evolution operator for free evolution can be considered as a single random choice (like in the first example):

$$\mathbb{E}_F(\nu) = \sum_{x,\sigma} |\phi_{\nu,x,\sigma}\rangle \langle \phi_{\nu,x,\sigma}|, \text{ where } \nu \in \mathbb{Z}. $$

(23)

Observation: The action of the operator $\mathbb{E}_F$ on a vector state $\psi$ gives a Schrödinger–type function:

$$\mathbb{E}_F(\nu)\psi(t',x',\zeta) = \sum_{x,\sigma} \langle\phi_{\nu,x,\sigma}|\psi\rangle \phi_{\nu,x,\sigma}(t',x',\zeta) = \sum_{x,\sigma} \langle\phi_{\nu,x,\sigma}|\psi\rangle \chi_{\nu}(t') \xi_{x+\nu,\sigma}(x',\zeta).$$

(24)

The Nature chooses by chance (with the distribution determined by the last state) the actual state labelled by $\nu$, which characterizes mainly the potential localization of the particle in the physical time. In this model, time and space positions are treated on the same footing. It means that there are possible situation when particle cannot be found in some regions of time. However, this situation cannot be interpreted in the usual way that such particle does not exist. Existence or non–existence of the particle is determined here by the total state considered in the full space–time.

Next device in our system is the beam splitter which can be thought as a device with two input channels and two output channels. In this model we assume that the beam splitters are able to localize particles in the space–time. It means that they are not the unitary–type devices.

To construct the appropriate operators, let us define a family of vectors which allows to “send” a particle into both channels simultaneously:

$$|bs_1;t x_{BS}\epsilon\rangle = \cos \epsilon |t x_{BS} a\rangle + \sin \epsilon \left[ \frac{1}{\sqrt{2}} \left(|t x_{BS} + 1 b\rangle + |t x_{BS} + 1 a\rangle\right) \right]$$

$$|bs_2;t x_{BS}\epsilon\rangle = -\sin \epsilon \left[ \frac{1}{\sqrt{2}} \left(-|t x_{BS} + 1 b\rangle + |t x_{BS} + 1 a\rangle\right) \right] + \cos \epsilon |t x_{BS} b\rangle$$

$$|bs_3;t x_{BS}\epsilon\rangle = -\sin \epsilon |t x_{BS} a\rangle + \cos \epsilon \left[ \frac{1}{\sqrt{2}} \left(|t x_{BS} + 1 b\rangle + |t x_{BS} + 1 a\rangle\right) \right]$$

$$|bs_4;t x_{BS}\epsilon\rangle = \cos \epsilon \left[ \frac{1}{\sqrt{2}} \left(-|t x_{BS} + 1 b\rangle + |t x_{BS} + 1 a\rangle\right) \right] + \sin \epsilon |t x_{BS} b\rangle$$

(25)

As the beam splitter has two input (entrance) channels, its projection evolution operator should take into account these two possibilities and can be written as the resolution of unity as follows:

$$\mathbb{E}_{BSn}(\tau;\nu) = \begin{cases} 
|bs_k;t x_{BSn},\epsilon\rangle \langle bs_k;t x_{BSn},\epsilon| + |bs_{k+1};t x_{BSn},\epsilon\rangle \langle bs_{k+1};t x_{BSn},\epsilon| & \text{for } \nu = (k,t), \ k = 1,3; t \in T; \\
\mathcal{O}(BSn,\tau;\nu) & \text{otherwise}
\end{cases}$$

(26)
where $BSn = BS1, BS2$ denotes the first or the second beam splitter. The beam splitters are placed in different positions $x_{BS1}$ and $x_{BS2}$.

The parameter $\varepsilon$ is changing continuously from 0 to $\frac{\pi}{2}$, while the particle is inside the device. The parameter $\varepsilon = \varepsilon(\tau)$ is a monotonic, increasing function of the evolution parameter. More detailed structure of the function $\varepsilon = \varepsilon(\tau)$ is not needed.

The continuous series of projections, while $\varepsilon$ is changing from 0 to $\frac{\pi}{2}$, lead to the following effective projection evolution operator:

$$E_{BSn}^{\text{eff}}(\nu) = \begin{cases} \frac{1}{\sqrt{2}} \left( |t, x_{BS} + 1, b\rangle + |t, x_{BS} + 1, a\rangle \right) \langle t, x_{BS}, a| \quad \text{for } \nu = (1, t), t \in T \\ \frac{1}{\sqrt{2}} \left( -|t, x_{BS} + 1, b\rangle + |t, x_{BS} + 1, a\rangle \right) \langle t, x_{BS}, b| \quad \text{for } \nu = (1, t), t \in T \\ \frac{1}{\sqrt{2}} |t, x_{BS}, a\rangle \left( |t, x_{BS} + 1, b\rangle + |t, x_{BS} + 1, a\rangle \right) \langle t, x_{BS}, b| \quad \text{for } \nu = (3, t), t \in T \end{cases}$$

(27)

where $n = 1, 2$ denotes the first or the second beam splitter. As the label for this effective step of evolution one can choose any evolution parameter $\tau_{PS}$ chosen from the interval of the evolution parameter required to perform the evolution of the particle within the phase shifter.

In case of the first beam splitter each of the input channels $a$ and $b$ is transformed into a superposition of both output channels $a$ and $b$. Next the particle passes the phase shifter described below and there is the second beam splitter where two input channels $a$ and $b$ are again transformed into the proper superposition of two output channels $a$ and $b$.

The phase shifter in the one dimensional model as presented in Fig. 2 acts on two channels $a$ and $b$ simultaneously. To construct its evolution operator one can use the same idea as in the construction of (26) and (27). For this purpose we define a family of the following vectors:

$$|ps_1; t, x_{PS}, a, \varepsilon\rangle = \cos \varepsilon |t, x_{PS}, a\rangle + \sin \varepsilon e^{i\phi_a} |t + 1, x_{PS}, a\rangle$$

$$|ps_2; t, x_{PS}, a, \varepsilon\rangle = -\sin \varepsilon |t, x_{PS}, a\rangle + \cos \varepsilon e^{i\phi_a} |t + 1, x_{PS}, a\rangle,$$

(28)

where $\alpha = a, b$ labels both channels and $x_{PS}$ denotes the position of the phase shifter.

The evolution operator for $PS$ is presented here by the continuous family of the following projections:

$$E_{PS}(\tau; \nu) = \begin{cases} \sum_{\alpha} \langle ps_k; t, x_{PS}, a, \varepsilon| ps_k; t, x_{PS}, a, \varepsilon\rangle \quad \text{for } \nu = (k, t), k = 1, 2, t \in T \\ \text{OP}(PS, \tau; \nu) \quad \text{otherwise} \end{cases}$$

(29)

where the parameter $\varepsilon$ is changing continuously from 0 to $\frac{\pi}{2}$, while the particle is inside the device. The parameter $\varepsilon = \varepsilon(\tau)$ is a monotonic, increasing function of the evolution parameter.

The proper effective projection evolution operator for the phase shifter can be created in the same way as in case of the beam splitter:

$$E_{PS}^{\text{eff}}(\nu) = \begin{cases} e^{i\phi_a} |t + 1, x_{PS}, a\rangle \langle t, x_{PS}, a| + e^{i\phi_b} |t + 1, x_{PS}, b\rangle \langle t, x_{PS}, b| \quad \text{for } \nu = (1, t), t \in T \\ -e^{-i\phi_a} |t, x_{PS}, a\rangle \langle t + 1, x_{PS}, a| - e^{-i\phi_b} |t, x_{PS}, b\rangle \langle t + 1, x_{PS}, b| \quad \text{for } \nu = (2, t), t \in T \end{cases}$$

(30)

As the label for this effective step of evolution one can choose any evolution parameter $\tau_{PS}$ chosen from the interval of the evolution parameters required to perform the evolution of the particle within the phase shifter.

The detectors are the last devices we have to describe. We assume that the first detector is placed in $x_{Da}$ and it is able to detect particle in the channel $a$ and the second detector placed in $x_{Db}$ detects the particle in the channel $b$. We assume also that both detectors localize particle in the space and time. These assumptions determine uniquely the form of the projection evolution operator for detectors:

$$E_{D}(\nu) = \begin{cases} |t, x_{Da}, a\rangle \langle t, x_{Da}, a| \quad \text{for } \nu = (t, a), t \in T \\ |t, x_{Db}, b\rangle \langle t, x_{Db}, b| \quad \text{for } \nu = (t, b), t \in T \\ \text{OP}(D; \nu) \quad \text{otherwise}. \end{cases}$$

(31)

Of course, the process of evolution is described by one evolution operator collected from all of the above. They
describe the subsequent events enumerated by the evolution parameter $\tau$:

$$
\mathbf{E}(\tau; \nu) = \begin{cases} 
\mathbf{E}_Z(\nu) & \text{for } \tau = 0, \\
\mathbf{E}_F(\nu) & \text{for } 0 < \tau < \tau_{BS1} \\
\mathbf{E}_{BS1}(\tau; \nu) & \text{for } \tau_{BS1} \leq \tau < \tau_{BS1} \\
\mathbf{E}_F(\nu) & \text{for } \tau_{BS1} \leq \tau < \tau_{PS} \\
\mathbf{E}_{PS}(\tau; \nu) & \text{for } \tau_{PS} \leq \tau < \tau_{PS}' \\
\mathbf{E}_F(\nu) & \text{for } \tau_{PS}' \leq \tau < \tau_{BS2} \\
\mathbf{E}_{BS2}(\tau; \nu) & \text{for } \tau_{BS2} \leq \tau < \tau_D \\
\mathbf{E}_F(\nu) & \text{for } \tau_D \leq \tau < \tau_D \\
\mathbf{E}_D(\nu) & \text{for } \tau = \tau_D. 
\end{cases}
$$

The first line describes the emission of a particle in the source $Z$. The second line gives the free motion between the source and the first beam splitter. It can be labelled by any evolution parameter $\tau_{F1} \in (0, \tau_{BS1})$ because the projection of the source is again the same projection. The third line describes the beam splitter and it can be replaced by the effective operator assuming the evolution parameter equal e.g., to $\tau_{BS1}$. In fact, one can choose any evolution parameter from the interval $[\tau_{BS1}, \tau_{BS1}]$. Next, we have free motion between the beam splitter and the phase shifter. For the phase shifter, described by the next line, one can also introduces the effective operator and the evolution parameter $\tau_{PS}$. Next two lines describe the second beam splitter which can be interpreted in terms of the effective operators at the evolution parameter $\tau_{BS2}$ and two detectors which detect, or not, the particle at the evolution parameter $\tau_D$.

In the following we are not interested in full analysis of all possibilities given by the projection evolution operators. We want to find the states and the appropriate probabilities of the particle detected at the evolution parameter $\tau_D$.

According to general rules of $P_{Ev}$ the interesting states can be calculated as:

$$
\rho(\tau_D; \nu_Z, \nu_{F1}, \nu_{BS1}, \nu_{F2}, \nu_{PS}, \nu_{F3}, \nu_{BS2}, \nu_{F4}, \nu_D) = \frac{\mathbf{E}_D(\nu_D) \mathbf{E}_F(\nu_{F4}) \mathbf{E}^{ff}_{BS2}(\nu_{BS2}) \mathbf{E}_F(\nu_{F3}) \mathbf{E}^{ff}_{PS}(\nu_{PS}) \mathbf{E}_F(\nu_{F2}) \mathbf{E}^{ff}_{BS1}(\nu_{BS1}) \mathbf{E}_F(\nu_{F1}) \mathbf{E}_Z(\nu_Z) \rho_0 \mathbf{E}_Z(\nu_Z)}{\text{Tr} \left[ \mathbf{E}_D(\nu_D) \mathbf{E}_F(\nu_{F4}) \mathbf{E}^{ff}_{BS2}(\nu_{BS2}) \mathbf{E}_F(\nu_{F3}) \mathbf{E}^{ff}_{PS}(\nu_{PS}) \mathbf{E}_F(\nu_{F2}) \mathbf{E}^{ff}_{BS1}(\nu_{BS1}) \mathbf{E}_F(\nu_{F1}) \mathbf{E}_Z(\nu_Z) \rho_0 \mathbf{E}_Z(\nu_Z) \right]}
$$

The most interesting path of the evolution is given by the following sequence:

$$
h = (\nu_Z = t_Z, \nu_{F1} = 1, \nu_{BS1} = t_{BS1}, \nu_{F2} = \nu_{PS} = 1, \nu_{PS} = 1, \nu_{BS2} = 1, \nu_{F3} = 1, \nu_{F4} = \nu_D = (t_D, \sigma_D)).
$$

Obviously, the whole path is chosen randomly in step by step procedure, as described in $P_{Ev}$.

Because the final state is a pure state, it can be immediately written as:

$$
\rho(\tau_D; \nu_Z, \nu_{F1}, \nu_{BS1}, \nu_{F2}, \nu_{PS}, \nu_{F3}, \nu_{BS2}, \nu_{F4}, \nu_D) = |t_D, x_{D_{r_D}}, \sigma_D)\rangle\langle t_D, x_{D_{r_D}}, \sigma_D|.
$$

The distribution of times $t_D$ of detection of the particle in the channel $\sigma_D$ can be calculated as the conditional probability:

$$
\text{Prob}(\tau_D; \nu_Z, \nu_{F1}, \nu_{BS1}, \nu_{F2}, \nu_{PS}, \nu_{F3}, \nu_{BS2}, \nu_{F4}, \nu_D) = \text{Tr} \left[ \mathbf{E}_D(\nu_D) \mathbf{E}_F(\nu_{F4}) \mathbf{E}^{ff}_{BS2}(\nu_{BS2}) \mathbf{E}_F(\nu_{F3}) \mathbf{E}^{ff}_{PS}(\nu_{PS}) \mathbf{E}_F(\nu_{F2}) \mathbf{E}^{ff}_{BS1}(\nu_{BS1}) \mathbf{E}_F(\nu_{F1}) \mathbf{E}_Z(\nu_Z) \rho_0 \mathbf{E}_Z(\nu_Z) \right] \mathbf{E}_F(\nu_{F1}) \mathbf{E}^{ff}_{BS2}(\nu_{BS2}) \mathbf{E}_F(\nu_{F2}) \mathbf{E}^{ff}_{PS}(\nu_{PS}) \mathbf{E}_F(\nu_{F3}) \mathbf{E}^{ff}_{BS1}(\nu_{BS1}) \mathbf{E}_F(\nu_{F4}) \mathbf{E}_D(\nu_D) \right].
$$

The conditional probability gives the probability of detection of the particle in the state labelled by $\nu_D$ under the conditions that the particle is created in $\nu_Z$ an then evolves through $\nu_{F1}, \nu_{BS1}, \nu_{F2}, \nu_{PS}, \nu_{F3}, \nu_{BS2}$ and $\nu_{F4}$.

Now assuming the initial state is the pure state denoted by:

$$
\rho_0 = |\psi_0\rangle\langle \psi_0|,
$$

the particle will be created from the source with the probability $|\langle t_Z, x_Z, a|\psi_0\rangle|^2$ and after the creation, the state of the particle is

$$
|\psi_Z\rangle = |t_Z, x_Z, a\rangle.
$$
Let us calculate a scalar product which will be useful in further considerations:

$$\langle \phi_{\nu,\sigma}|t_Z, x_Z, a \rangle = \sum_{t', x', \zeta} \chi_{\nu}(t') \xi_{\nu}(x', \zeta) \delta_{z_z}(x', \zeta) = \chi_{\nu}(t) \xi_{\nu}(x + t, \zeta). \quad (38)$$

After the particle is created from the source, it undergoes a free evolution. Making use of (24) and (38) one can write the state of the particle as:

$$|\psi_{F1}\rangle = \mathcal{N} \xi_{F}(\nu_{F1})|t_Z, x_Z, a\rangle = \mathcal{N} \sum_{x, \sigma} \langle \phi_{\nu_{F1}, x, \sigma}|t_Z, x_Z, a\rangle = |\phi_{\nu_{F1}, x_Z - t_Z, a}\rangle. \quad (39)$$

The probability that the particle is found in such a state, under the condition that it was in the state (37), is equal $\chi_{\nu_{F1}}(t_Z)^2$. The coefficient $\mathcal{N}$ is the normalization factor which is not needed for further considerations.

Next we can derive a state of the particle in the next step of the evolution, i.e. in the beam splitter $BS_1$. Once again we calculate explicitly the scalar product

$$\langle \chi_{\nu,\sigma}|t_{BS_1}, x_{BS_1}, \sigma|\psi_{F1}\rangle = \sum_{t', x', \zeta} \delta_{\nu_{BS_1}}(x', \zeta) \chi_{\nu_{F1}}(t') \xi_{\nu_{BS_1} + (t' - t_Z), \sigma}(x', \zeta) = \chi_{\nu_{F1}}(t_{BS_1}) \xi_{\nu_{BS_1} + (t_{BS_1} - t_Z), \sigma}(x_{BS_1}, a). \quad (40)$$

This scalar product is not equal zero, for the time $t = t_{BS_1}$ only, i.e. it must satisfy the condition $\chi_{\nu_{F1}}(t_{BS_1})^2 \delta_{x_{BS_1} + (t_{BS_1} - t_Z), \sigma} \neq 0$. It allows to write the state of the particle (for this step of the evolution) as:

$$|\psi_{BS_1}\rangle = \frac{1}{\sqrt{2}} \left(|t_{BS_1}, x_{BS_1} + 1, b\rangle + |t_{BS_1}, x_{BS_1} + 1, a\rangle\right). \quad (41)$$

In the similar way one can find the state in the phase shifter $PS$ after a free evolution $F_2$. It is chosen by the Nature, with the conditional probability $|\chi_{\nu_{PS}}(t_{BS_1})|^2 |\chi_{\nu_{PS}}(t_{PS})|^2 \delta_{x_{BS_1} + (t_{BS_1} - t_{PS}), \sigma}$ in the following form:

$$|\psi_{PS}\rangle = \frac{1}{\sqrt{2}} \left(e^{i\phi_b}|t_{PS} + 1, x_{PS}, b\rangle + e^{i\phi_a}|t_{PS} + 1, x_{PS}, a\rangle\right). \quad (42)$$

Similarly, the state of the particle in the second beam splitter $BS_2$, after a free evolution $F_3$, with the conditional probability $|\chi_{\nu_{PS}}(t_{PS})|^2 |\chi_{\nu_{BS_2}}(t_{BS_2})|^2 \delta_{x_{PS} + (t_{BS_2} - t_{PS}), \sigma}$ is represented by

$$|\psi_{BS_2}\rangle = -\frac{1}{2} \left(e^{i\phi_b} + e^{i\phi_a}|t_{BS_2}, x_{BS_2} + 1, a\rangle - \frac{1}{2} \left(e^{i\phi_b} - e^{i\phi_a}|t_{BS_2}, x_{BS_2} + 1, b\rangle. \quad (43)$$

Finally, according to (44), the state in the detector $D$ after a free evolution $F_4$ is:

$$|\psi_D\rangle = |t_D, x_{D, a} \rangle. \quad (44)$$

The corresponding conditional probabilities of finding the particle in the output channels $a$ and $b$ are:

$$|\chi_{\nu_{PS}}(t_{BS_2})|^2 |\chi_{\nu_{PS}}(t_{D})|^2 \delta_{x_{BS_2} + (t_{D} - t_{BS_2}), \sigma_{D, a}} \cos^2 \frac{\phi_a - \phi_b}{2} \quad (45)$$

and

$$|\chi_{\nu_{PS}}(t_{BS_2})|^2 |\chi_{\nu_{PS}}(t_{D})|^2 \delta_{x_{BS_2} + (t_{D} - t_{BS_2}), \sigma_{D, b}} \sin^2 \frac{\phi_a - \phi_b}{2}, \quad (46)$$

respectively.

Observation: if there were no phase shifter on the evolution path, the second detector would never detect the particle, which is in agreement with the experimental observation.

Making use of the expression (37) one can write the total, conditional probability of detecting the particle at $t_D$, in either $a$ or $b$ channel:

$$\text{Prob}(t_D; t_Z, \nu_{BS_1}, \nu_{BS_2}, \nu_{PS}, \nu_{F3}, \nu_{F4}, \nu_{D}) = |\langle t_Z, x_Z, a|\psi_0\rangle|^2 \chi_{\nu_{F1}}(t_Z)^2 \chi_{\nu_{F2}}(t_{BS_1})^2 \chi_{\nu_{PS}}(t_{PS})^2 \chi_{\nu_{F3}}(t_{BS_2})^2 \chi_{\nu_{F4}}(t_{D})^2 \delta_{x_{BS_2} + (t_{D} - t_{BS_2}), \sigma_{D, a}} \cos^2 \frac{\phi_a - \phi_b}{2} + \delta_{x_{BS_2} + (t_{D} - t_{BS_2}), \sigma_{D, b}} \sin^2 \frac{\phi_a - \phi_b}{2}. \quad (47)$$
The probability \( |\psi_D; \sigma_D \rangle \) can be nonzero only for \( t_D = t_Z + (x_D - x_Z) \). It implies that the final state \(|\psi_D \rangle\) is chosen by the Nature, with the probability \(|\psi_D \rangle\), as one of two vectors:

\[
|\psi_D; \sigma_D \rangle = \begin{cases} 
|t_D = t_Z + (x_D - x_Z), x_{D_a}, a \rangle & \text{for } \sigma_D = a, \\
|t_D = t_Z + (x_D - x_Z), x_{D_b}, b \rangle & \text{for } \sigma_D = b 
\end{cases}
\]  

(48)

Because the physical time is here a dynamical variable one can construct different observables characterizing time behavior of our system. One of the most interesting is the observable which measure a possibility of finding the particle at given time in the channel \( \sigma \). The corresponding decomposition of unity can be written as:

\[
M_T(t, \sigma) = \sum_x |t, x, \sigma \rangle \langle t, x, \sigma|,
\]

(49)

where \( t \in \mathbb{R} \) and \( \sigma = a, b \). The conditional probability of finding the particle at given time in the channel \( \sigma \), while the particle is in a state \( \rho \), can be calculated according to the usual formula:

\[
\text{Prob}(\rho; t, \sigma) = \text{Tr}(M_T(t, \sigma) \rho).
\]

(50)

In our, very simple model the conditional probability \( |\psi_D \rangle \) calculated for the state \(|\psi_D \rangle\) is non–zero only if \( t = t_D \) and \( \sigma = \sigma_D \). In this case the probability is equal to one for \( t_D = t_Z + (x_D - x_Z) \). It means that \( t_D \) can be interpreted as time of detection. In more realistic model the final state can have quite complicated distribution in time.

Full probability of detection of the particle which passed the evolution path \( h \) (shown above the equation (52)) is the product of all the conditional probabilities along the evolution path and the conditional probability given by (50).

In this toy model the states of the particle are localized in time by all devices. The only exception is free evolution which gives states spread over the time.

All the moments which appear in this evolution process like \( t_Z \), which represents the time of particle emission from the source, \( t_{BS1}, t_{PS}, t_{BS2} \) and \( t_D \) which denote the times of arrival to the appropriate devices, are random variables. In addition, their probability distributions are not independent e.g., the probability of choosing by the "World Lottery" the time of arrival to the first beam splitter \( t_{BS1} \) is dependent on time of emission from the source \( t_Z \), and so on. These are special features of the model considered above.

As an example let us calculate the average time at which the particle can be localized in \( x_{BS1} \) (the first beam splitter) under the condition that the particle was emitted from the source at the fixed time \( t_Z \).

In general, for any evolution parameter \( \tau_F \), where \( 0 < \tau_F \leq \tau_{BS1} \), the state can be obtained by two-step evolution according to the evolution operator \( |\psi_D \rangle \). It can be written as:

\[
\rho(\tau_F; t_Z, \nu_F) \sum_{t_1, t_2} \chi_{\nu_F}(t_1) \chi_{\nu_F}(t_2)^* |t_1, x_Z - t_Z, a \rangle \langle t_2, x_Z - t_Z, a|.
\]

(51)

To calculate the average time of finding the particle in the place where is the located the first beam splitter, we have to define the appropriate operators. First, we define the decomposition of unity which is able to determine the position of particle in the space–time but independently of channels:

\[
M_G(t, x) = \sum_{\sigma} |tx\sigma \rangle \langle tx\sigma|,
\]

(52)

The observable which measure the time at fixed coordinate point \( x \) can be thus defined as:

\[
\mathcal{T}(x) = \sum_t t M_G(t, x).
\]

(53)

The average time of finding the particle at \( x_{BS1} \) can be calculated according to the usual rules as:

\[
\langle \mathcal{T}(x) \rangle = \text{Tr}(\mathcal{T}(x) \rho(\tau_F; t_Z, \nu_F))
\]

\[
= \sum_{t'} t' |\chi_{\nu_F}(t')|^2 \delta_{x_Z - t_Z + t'_{BS1}}
\]

\[
= (x_{BS1} - x_Z + t_Z) |\chi_{\nu_F}(x_{BS1} - x_Z + t_Z)|^2.
\]

(54)

This time can be also called the time of arrival because the particle is emitted at the fixed time \( t_Z \) and the difference \( \langle \mathcal{T}(x) \rangle - t_Z \) should give the average time of fly between the source and the first beam splitter.
We obtain the expected result, though with the additional factor \(|\chi_\nu_f(x_{BS1} - x_Z + t_Z)|^2\). The time of arrival is proportional to distance between the source and the beam splitter and it is what we expected, because in this very simple model there is no spreading of "wave packets" over the space. The additional coefficient is equal to the probability of finding the particle at a given point of time axis. In the case of Schrödinger equation, as an equation of motion (implemented by the projection operators like those for free motion, see (21, 23)), the functions \(|\chi_\nu(t)| = 1\).

Taking into account that times of emissions \(t_Z\) of particles, in principle, are also random variables, the probability that a particle being in the initial state \(|\psi_0\rangle\), can be localized at a given time in \(x_{BS1}\) is the product of \(|\langle t_Z, x_Z, a|\psi_0\rangle|^2\) and \(\text{Tr}(M_G(t, x_{BS1})\rho(\tau_f; t_Z, \nu_f))\). This allows to derive the average time \(<t_{ar}>\) of finding the particle at \(x_{BS1}\) averaged, in addition, over the emission times \(t_Z\) as:

\[
<t_{ar}> = \sum_{t, t_Z} t |\langle t_Z, x_Z, a|\psi_0\rangle|^2 \text{Tr}(M_G(t, x_{BS1})\rho(\tau_f; t_Z, \nu_f)) = \sum_{t, t_Z} |\langle t_Z, x_Z, a|\psi_0\rangle|^2 |\chi_\nu_f(x_{BS1} - x_Z + t_Z)1^2 (x_{BS1} - x_Z + t_Z) .
\]

(55)

Toy Model II: Beam splitter of unitary–type

As the second model we consider the same one–dimensional Mach–Zehnder interferometer as in the Fig. 2. We assume the same space of states as in the Model I.

The main difference between both models is that in the second one we treat both beam splitters and phase shifter as a single device. It is described by the evolution operator similar to the description of free motion in the first model.

The source of the particles we describe exactly by the same decomposition of unity as in the Model I, see (10).

On the other hand, both beam splitters and the phase shifter we describe by the operator:

\[
E_{DEV}(\nu) = \sum_{t, x, \zeta} |\phi_{\nu, x, \sigma}(t', x', \zeta)\rangle\langle\phi_{\nu, x, \sigma}(t', x', \zeta)| , \text{ where } \nu \in \mathbb{Z} ,
\]

(56)

where \(\phi_{\nu, x, \sigma}\) is given by the equation (21), but with another operator \(\hat{S}(t) = \hat{S}_t\). The operator \(\hat{S}\) describes all the three devices at once, see (8):

\[
\hat{S}|x, \sigma\rangle = \begin{cases} 
\frac{1}{\sqrt{2}} (|x_{BSK} + 1, b\rangle + |x_{BSK} + 1, a\rangle) & \text{for } x = x_{BSK}, \sigma = a \text{ and } k = 1, 2, \\
\frac{1}{\sqrt{2}} (-|x_{BSK} + 1, b\rangle + |x_{BSK} + 1, a\rangle) & \text{for } x = x_{BSK}, \sigma = b \text{ and } k = 1, 2, \\
e^{i\phi_x} |x_{PS} + 1, \sigma\rangle & \text{for } x = x_{PS}, \sigma = a, b \text{ and } , \\
|x + 1, \sigma\rangle & \text{otherwise.}
\end{cases}
\]

(57)

It means that a single drawing of lots by the Nature decides about the evolution through all three devices. In this case, there are no intermediate times like \(t_{BS1}, t_{PS}\) and \(t_{PS2}\) which are effects of subsequent steps of the evolution.

The last step of the evolution is again determined by the detectors and it is described by the operator (31).

In this case the projection evolution operator is shorter and can be written as

\[
E(\tau; \nu) = \begin{cases} 
E_{\nu}(\nu) & \text{for } \tau = 0, \\
E_{DEV}(\nu) & \text{for } 0 < \tau < \tau_D \\
E_D(\nu) & \text{for } \tau = \tau_D.
\end{cases}
\]

(58)

In this model there are only 3 steps of the evolution. The system starts from the emission of the particle, passes all the three devices (the first beam splitter, the phase shifter and the second beam splitter) and ends at the detectors:

\[
\rho(\tau_D; \nu_Z, \nu_{DEV}, \nu_D) = \frac{E_D(\nu_D)E_{DEV}(\nu_{DEV})E_{Z}(\nu_Z)\rho_0E_{Z}(\nu_Z)E_{DEV}(\nu_{DEV})E_D(\nu_D)}{\text{Tr}[E_D(\nu_D)E_{DEV}(\nu_{DEV})E_{Z}(\nu_Z)\rho_0E_{Z}(\nu_Z)E_{DEV}(\nu_{DEV})E_D(\nu_D)]}
\]

(59)

The probability distribution of finding the particle at the time \(t_D\), in a given output channel, assuming the same initial state (39), when it follows a given evolution path \(h = (\nu_Z = t_Z, \nu_{DEV}, \nu_D = (t_D, \sigma_D))\) is

\[
\text{Prob}(\tau_D; \nu_Z, \nu_{DEV}, \nu_D) = \text{Tr}[E_D(\nu_D)E_{DEV}(\nu_{DEV})E_{Z}(\nu_Z)\rho_0E_{Z}(\nu_Z)E_{DEV}(\nu_{DEV})E_D(\nu_D)] = \langle \langle t_Z, x_Z, a|\psi_0\rangle|^2 |\chi_{\nu_{DEV}}(t_Z)|^2 |\chi_{\nu_{DEV}}(t_D)|^2 |(x_D, \sigma_D)\hat{S}^\Delta|x_Z, a|^2
\]

(60)
where \( \Delta t = t_D - t_Z \) denotes the difference between the time of emission of the particle and the time of its detection. Let us explicitly write the action of the unitary operator \( S^{\Delta t} \) onto the state \( |x_Z a\rangle \):

\[
S^{\Delta t}|x_Z a\rangle = \frac{1}{\sqrt{2}}(|x_Z + \Delta t, a\rangle + |x_Z + \Delta t, b\rangle) \\
\frac{1}{\sqrt{2}}(e^{i\Phi_b}|x_Z + \Delta t, b\rangle + e^{i\Phi_a}|x_Z + \Delta t, a\rangle) \\
\frac{1}{2}((-e^{i\Phi_b} + e^{i\Phi_a})|x_Z + \Delta t, b\rangle + (e^{i\Phi_b} + e^{i\Phi_a})|x_Z + \Delta t, a\rangle)
\]

for \( 0 \leq \Delta t < x_{BS1} - x_Z \)

for \( x_{BS1} - x_Z \leq \Delta t < x_{PS} - x_Z \)

for \( x_{PS} - x_Z \leq \Delta t < x_{BS2} - x_Z \)

for \( \Delta t \geq x_{BS2} - x_Z \)

Using above expression, the total conditional probability (60) can be easily rewritten as:

\[
\text{Prob} \left( \tau_D; \nu_Z, \nu_{DEV}, \nu_D \right) = |\langle t_Z, x_Z, a|\psi_0\rangle|^2 \left| \chi_{\nu_{DEV}}(t_Z) \right|^2 \left| \chi_{\nu_{DEV}}(t_D) \right|^2
\]

\[
\left[ \delta_{x_Z+(t_D-t_Z),x_D,a} \cos^2 \frac{\phi_a - \phi_b}{2} + \delta_{x_Z+(t_D-t_Z),x_D,b} \sin^2 \frac{\phi_a - \phi_b}{2} \right].
\]

The result is very similar to those from the Model I, see [47], especially for the case when the functions \( \chi_{\nu}(t) \approx 1 \). As in the previous case, the case between the emission of the particle by the source and its detection is determined by distance between the source and the detector. The main difference between both models is in timing of evolution steps. In the Model II, besides of \( t_Z \) and \( t_D \) there are no intermediate times (which are random variables) and additional factors in probabilities related to them. Lack of spreading in time is determined here by the "toy" form of the operator \( S \) [57].

In this notes we have compared only two models of the Mach-Zehnder interferometer within the projection evolution approach. In practice, they give nearly the same results of the evolution. However, it seems that up to date experiments show that, the beam splitters and phase shifters are closer to the second model than to the first one. In both models the physical time is a dynamical variable which is changing in rather primitive (nearly no smearing in time) way. Because of this, they cannot show all features of dynamics of time which is in principle a random variable in the evolution process.

IV. SUMMARY

In this paper, we propose a fundamental mechanism of quantum evolution based on the idea of "natural measurements" performed by the Nature at each step of changes (evolution) of the physical systems.

This scheme leads to unification of unitary evolution and measurements handled by the so-called projection postulate.

As a special case, our postulate allows to reproduce the Schrödinger equation and go beyond it. In this way one can reproduce not only the Schrödinger equation but also other quantum equations of motions e.g., the relativistic equations.

Obviously, the PEv postulates require many tests, yet. However, it seems that for each physical system one can find an appropriate set of projection operators which allows to apply the PEv postulate to get physical states. A possible method to construct the PEv operators is the method of generating operators.

It is also important to notice that the PEv postulate gives the unique opportunity to treat space and time on equal footing. This feature can open some new directions of development of quantum theory.

In the preprint we left many open questions like the problem of time operator, the time–energy uncertainty relation, problem of interactions (potentials) in time variable and many others.

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