Complex Hadamard matrices attached to even orthogonal schemes of class 4

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Abstract

A complex Hadamard matrix is a square matrix $W$ with complex entries of absolute value 1 satisfying $WW^* = nI$, where $*$ stands for the Hermitian transpose and $I$ is the identity matrix of order $n$. In this paper, we give constructions of complex Hadamard matrices in the Bose–Mesner algebra of a certain 4-class symmetric association scheme. Moreover, we determine the Nomura algebras to show that the resulting matrices are not decomposable into nontrivial generalized tensor products.

1 Introduction

A complex Hadamard matrix is a square matrix $W$ with complex entries of absolute value 1 satisfying $WW^* = nI$, where $*$ stands for the Hermitian transpose and $I$ is the identity matrix of order $n$. They are the natural generalization of real Hadamard matrices. Complex Hadamard matrices appear frequently in various branches of mathematics and quantum physics.

A type-II matrix, or an inverse orthogonal matrix, is a square matrix $W$ with nonzero complex entries satisfying $WW^{(-)}\top = nI$, where $W^{(-)}$ denotes the entrywise inverse of $W$. Obviously, a complex Hadamard matrix is a type-II matrix.

In [7], we gave a method to find a complex Hadamard matrix in the Bose–Mesner algebra of a symmetric association scheme. Applying this result, we classified complex Hadamard matrices in the Bose–Mesner algebra of a certain 3-class association scheme. In this paper, we construct certain complex Hadamard matrices in the Bose–Mesner algebra of a 4-class association scheme $(X, \{R_i\}_{i=0}^4)$ with the first eigenmatrix:

$$P = \begin{bmatrix}
1 & \frac{1}{2}(q - 2)q^{2m-1} & \frac{1}{2}q^{2m} & q(q^{2m-2} - 1) & q - 2 \\
1 & \frac{1}{2}(q - 2)q^{m-1} & \frac{1}{2}q^m & -(q - 1)(q^{m-1} + 1) & q - 2 \\
1 & -(q - 2)q^{m-1} & \frac{1}{2}q^m & -(q - 1)(q^{m-1} - 1) & q - 2 \\
1 & \frac{1}{2}q^m & -\frac{1}{2}q^m & 0 & -1 \\
1 & -\frac{1}{2}q^m & \frac{1}{2}q^m & 0 & -1
\end{bmatrix}, \quad (1)$$

where $q$ and $m$ are positive integers with $q \geq 4$ and $m \geq 2$. Then $|X| = q^{2m} - 1$, $R_4$ is a disconnected relation, and $R_2$ defines a strongly regular graph. If $q$ is a power of 2, an even orthogonal scheme is an example of an association scheme with the first eigenmatrix $\mathbb{P}$ (see [3] Chapter 12.1). If $m = 1$, then $R_3 = \emptyset$, and this scheme reduces to an even orthogonal scheme of class 3 which we considered in [7].
For a type-II matrix $W \in M_X(\mathbb{C})$ and $a, b \in X$, we define column vectors $Y_{ab}$ by setting

$$(Y_{ab})_x = \frac{W_{xa}}{W_{xb}} \quad (x \in X).$$

The Nomura algebra $N(W)$ of $W$ is the algebra of matrices in $M_n(\mathbb{C})$ such that $Y_{ab}$ is an eigenvector for all $a, b \in X$. It is shown in [9, Theorem 1] that the Nomura algebra is a Bose–Mesner algebra.

Throughout this paper, we denote by $X = (X, \{R_i\}_{i=0}^4)$ a symmetric association scheme with the first eigenmatrix (1). Let $A_0, A_1, A_2, A_3, A_4$ be the adjacency matrices of $X$. Let $w_0 = 1, w_1, w_2, w_3, w_4$ be nonzero complex numbers, and set

$$W = \sum_{j=0}^{4} w_j A_j,$$  \hspace{1cm} (2)

$$a_{i,j} = \frac{w_i}{w_j} + \frac{w_j}{w_i} \quad (0 \leq i < j \leq 4).$$  \hspace{1cm} (3)

The main purpose of this paper is to prove the following:

**Theorem 1.** Assume that

$$w_4 = 1.$$  \hspace{1cm} (4)

(i) Assume $w_1 = 1$. Then, the matrix $W$ in (2) is a complex Hadamard matrix if and only if

$$w_2^2 + \frac{2(q^{2m} - 2)}{q^{2m}}w_2 + 1 = 0 \quad \text{and} \quad w_3 = 1.$$  \hspace{1cm} (5)

(ii) Assume

$$a_{0,1} = \frac{2(q^{4m-2} - (q + 2)q^{2m-1} + 2)}{(q^{2m-1} + q - 2)q^{2m-1}}.$$  \hspace{1cm} (6)

Then, the matrix $W$ in (2) is a complex Hadamard matrix if and only if

$$w_2 = -\frac{(q - 1)q^{2m-1}w_1 + q^{2m-1} + q - 2}{(q^{2m-1} - 1)q},$$

$$w_3 = 1.$$  \hspace{1cm} (7)

**Theorem 2.** Let $W$ be a complex Hadamard matrix given in (i) and (ii) of Theorem 1. The algebra $N(W)$ coincides with the linear span of $I$ and $J$. In particular, $W$ is not equivalent to a nontrivial generalized tensor product.

The reason for the assumption (4) is as follows: Calculating the conditions under which the matrix (2) becomes a complex Hadamard matrix experimentally for small $q$ and $m$, we find that (4) is fulfilled, or

(iii) $a_{0,4} = 2(q^{2m} - 6)/(q^{2m} - 4)$, or

(iv) $a_{0,4}$ is a zero of a polynomial of degree 9.
For the case (iii) with $m = 2, 3$, we have $w_1 = w_3 = w_4$. Therefore, this case reduces to the case in which the matrix $W$ given in (2) belongs to the Bose–Mesner algebra of the strongly regular graph defined by $R_2$. However, it seems to be difficult to prove $w_1 = w_3 = w_4$ for arbitrary $m \geq 4$.

For the case (iv), we verified that the polynomial in $a_{0,4}$ of degree 9 is an irreducible polynomial for $m = 2, \ldots, 9$ and $q = 2^s$ with $2 \leq s \leq 10000$. However, it seems difficult to determine the polynomial of degree 9 satisfied by $a_{0,4}$ in general. For example, for $(q, m) = (4, 2)$, if the matrix $[2]$ is a complex Hadamard matrix, then $a_{0,4}$ is a zero of the polynomial

$$p(x) = x^9 - \frac{235721}{1785} x^8 - \frac{1795726593}{62475} x^7 + \frac{33219815829811}{937125} x^6 - \frac{12554318926285933}{4685625} x^5 + \frac{29740292638491103}{312375} x^4 - \frac{696525696876795217}{187425} x^3 + \frac{851886544261448041}{37485} x^2 - \frac{12458391943976136}{833} x + \frac{30888835313436500}{3}.$$

It can be shown that $p(x)$ has only one real root in $(-2, 2)$ by using Sturm’s theorem. Then, by using Lemmas 1 and 2 below, there exist $w_1, w_2, w_3, w_4$ such that (2) is a complex Hadamard matrix.

Under the hypothesis of (4), we find that $a_{0,1} = 2$ or $a_{0,1}$ is given by (5), or

(v) $a_{0,1}$ is a zero of a polynomial of degree 4.

It seems to be difficult to determine $w_1, w_2, w_3$ for the case (vi) for arbitrary $q$ and $m$. For example, for $(q, m) = (4, 2)$, if the matrix $[2]$ is a complex Hadamard matrix, then $w_1, w_2, w_3$ are given by the following:

$$w_1^2 + \frac{21s - 7140 \pm 85t}{176} w_1 + 1 = 0,$$
$$w_2 = -\frac{64(w_1^2 - 1)}{127w_1 + 64a_{0,2}},$$
$$w_3 = \frac{90(w_1^2 - 1)}{90a_{1,3}w_1 - 4s + 1117},$$
$$a_{0,2} = \frac{43s - 14620 \pm 85t}{352},$$
$$a_{1,3} = \frac{21s - 1848 \mp (4s + 1253)t}{2640},$$
$$s = \sqrt{104899},$$
$$t^2 = \frac{8s - 2591}{3}.$$

2 Preliminaries

We define a polynomial in three indeterminates $X, Y, Z$ as follows:

$$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$$
We define a polynomial in six indeterminates $X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}$ as follows:

$$h(X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}) = \det \begin{bmatrix} 2 & X_{0,1} & X_{0,2} \\ X_{0,1} & 2 & X_{1,2} \\ X_{0,3} & X_{1,3} & X_{2,3} \end{bmatrix}.$$ 

For a finite set $N$ and a positive integer $k$, we denote by $\binom{N}{k}$ the collection of all $k$-element subsets of $N$.

**Lemma 1** ([7, Lemma 4]). Let $N = \{0, 1, \ldots, d\}$, $N_3 = \binom{N}{3}$ and $N_4 = \binom{N}{4}$. Let $a_{i,j}$ $(0 \leq i, j \leq d, \ i \neq j)$ be complex numbers satisfying

$$a_{i,j} = a_{j,i} \quad (0 \leq i < j \leq d), \quad (7)$$

$$g(a_{i,j}, a_{j,k}, a_{i,k}) = 0 \quad (\{i, j, k\} \in N_3), \quad (8)$$

$$h(a_{i,j}, a_{i,k}, a_{i,\ell}, a_{j,k}, a_{j,\ell}, a_{k,\ell}) = 0 \quad (\{i, j, k, \ell\} \in N_4). \quad (9)$$

**Assume**

$$a_{i_0,i_1} \neq \pm 2 \quad \text{for some} \ i_0, i_1 \ \text{with} \ 0 \leq i_0 < i_1 \leq d. \quad (10)$$

Let $w_{i_0}, w_{i_1}$ be nonzero complex numbers satisfying

$$\frac{w_{i_0}}{w_{i_1}} + \frac{w_{i_1}}{w_{i_0}} = a_{i_0,i_1}. \quad (11)$$

Then for complex numbers $w_i$ $(0 \leq i \leq d, \ i \neq i_0, i_1)$, the following are equivalent:

(i) for all $i, j$ with $0 \leq i, j \leq d$ and $i \neq j$,

$$\frac{w_j}{w_i} + \frac{w_i}{w_j} = a_{i,j} \quad (12)$$

(ii) for all $i, j$ with $0 \leq i \leq d, \ i \neq i_0, i_1$,

$$w_i = \frac{w_{i_1}^2 - w_{i_0}^2}{a_{i_1,i_0}w_{i_1} - a_{i_0,i}w_{i_0}}. \quad (13)$$

Moreover, if one of the two equivalent conditions (i), (ii) is satisfied, $a_{i,j}$ $(0 \leq i < j \leq d)$ are all real and

$$-2 < a_{i_0,i_1} < 2, \quad (14)$$

then $|w_i| = |w_j|$ for $0 \leq i < j \leq d$.

We let $\mathcal{A}$ denote a symmetric Bose–Mesner algebra with adjacency matrices $A_0, A_1, \ldots, A_d$. Let $n$ be the size of the matrices $A_i$, and we denote by

$$P = (P_{i,j})_{0 \leq i \leq d} \quad (0 \leq j \leq d)$$

the first eigenmatrix of $\mathcal{A}$. Then the adjacency matrices are expressed as

$$A_j = \sum_{i=0}^{d} P_{i,j}E_i \quad (j = 0, 1, \ldots, d),$$

where $E_0 = \frac{1}{n}I, E_1, \ldots, E_d$ are the primitive idempotents of $\mathcal{A}$.

Let $w_0, w_1, \ldots, w_d$ be nonzero complex numbers, and set

$$W = \sum_{j=0}^{d} w_j A_j \in \mathcal{A}. \quad (15)$$
Lemma 2 ([7, Lemma 7]). Let $X_{i,j}$ $(0 \leq i < j \leq d)$ be indeterminates and let $e_k$ be the polynomial defined by

$$e_k = \sum_{0 \leq i < j \leq d} P_{k,i}P_{k,j}X_{i,j} + \sum_{i=0}^{d} P_{k,i}^2 - n \quad (k = 1, \ldots, d).$$

Let $a_{i,j}$ $(0 \leq i, j \leq d, i \neq j)$ and $w_i$ $(0 \leq i \leq d)$ be complex numbers. Assume that $w_i \neq 0$ for all $i$ with $0 \leq i \leq d$ and that (12) holds. Then the following statements are equivalent:

(i) the matrix $W$ given by (15) is a type-II matrix,

(ii) $(a_{i,j})_{0 \leq i < j \leq d}$ is a common zero of $e_k$ $(k = 1, \ldots, d)$.

Moreover, if one of the two equivalent conditions (i), (ii) is satisfied, $a_{i,j} \in \mathbb{R}$ for all $i, j$ with $0 \leq i < j \leq d$, and (14) holds for some $i_0, i_1$ with $0 \leq i_0 < i_1 \leq d$, then $W$ is a scalar multiple of a complex Hadamard matrix.

We now describe the proof of Theorem [1] briefly. Let $A_0, A_1, A_2, A_3, A_4$ be the adjacency matrices of an association scheme $X$ with the first eigenmatrix (1). Let $w_0 = 1, w_1, w_2, w_3, w_4$ be nonzero complex numbers, and $W$ be the matrix defined by (2). For $i, j \in \{0, 1, 2, 3, 4\}$, define $a_{i,j}$ by (3). We write

$$\alpha = (a_{0,1}, a_{0,2}, a_{0,3}, a_{0,4}, a_{1,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,4})$$

for brevity. Consider the polynomial ring

$$R = \mathbb{C}[X_{0,1}, X_{0,2}, X_{0,3}, X_{0,4}, X_{1,2}, X_{1,3}, X_{1,4}, X_{2,3}, X_{2,4}, X_{3,4}].$$

In Section 3, we first assume that $W$ is a complex Hadamard matrix. Then by Lemmas [1] and [2] $\alpha$ is a common zero of the polynomials

$$g(X_{i,j}, X_{i,k}, X_{j,k}) \quad \{i, j, k\} \in \binom{\{0, 1, 2, 3, 4\}}{3},$$

$$h(X_{i,j}, X_{i,k}, X_{i,l}, X_{j,k}, X_{j,l}, X_{k,l}) \quad \{i, j, k, l\} \in \binom{\{0, 1, 2, 3, 4\}}{4},$$

$$e_k \quad (k \in \{1, 2, 3, 4\}).$$

Let $\mathcal{I}$ be the ideal of $R$ generated by these polynomials. Calculating the ideal generated by $\mathcal{I}$ and $X_{0,4} - 2$, we find (i) and (ii) of Theorem [1].

Conversely, we assume that $w_4 = 1$ and $w_1, w_2, w_3$ are given in Theorem [1]. Then, to show that the matrix $W$ given in (2) is a complex Hadamard matrix, we check that $\alpha$ defined by (3), (17) is a zero of the polynomials (19), (20), (21). Moreover, we check that $-2 < a_{i_0, i_1} < 2$ holds for some $i_0, i_1$ with $0 \leq i_0 < i_1 \leq 4$.

All the computer calculations in this paper were performed with the help of Magma [2]. In order to facilitate computations covering all possible values of the integer $q$, we perform the computations in the polynomial ring with 12 variables $q, r = q^m$ and $X_{i,j}$ over the field of rational numbers, rather than the ring (18). The results valid for this generic setting are also valid for arbitrary integers $q$ and $m$. 

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5
3 Proof of Theorem 1

Recall $q \geq 4$ and $m \geq 2$, and $I$ is the ideal of the polynomial ring $R$ generated by the polynomials (19), (20), and (21). For the remainder of this section, we assume that $a_{0,4} = 2$, that is, $w_4 = 1$. Let $I_1$ denote the ideal generated by $I$ and $X_{0,4} - 2$. For Lemmas 3–5 we assume that $a$ defined in (17) is a common zero of the polynomials in $I_1$.

Lemma 3. We have

$$a_{1,2} = -2(q^{2m} - 2) / q^{2m}.$$  \hspace{1cm} (22)

Proof. We can verify that $I_1$ contains $X_{1,2} + 2(q^{2m} - 2)/q^{2m}$. Hence we have (22). \hfill \Box

Lemma 4. Assume $a_{0,1} = 2$. Then, $(w_1, w_2, w_3)$ is given in (i) of Theorem 1.

Proof. Let $I_2$ denote the ideal generated by $I_1$ and $X_{0,1} - 2$. Then we can verify that $I_2$ contains $(X_{0,3} - 2)^2$, that is, $a_{0,3} = 2$. Hence $w_1 = w_3 = 1$. Since $a_{1,2}$ is given in (22), the matrix $W$ given in (2) belongs to the Bose–Mesner algebra of the strongly regular graph defined by $R_2$. From [4] we have the condition of $w$ given in (i) of Theorem 1. \hfill \Box

Lemma 5. Assume that $a_{0,1}$ is given in (5). Then, $(w_1, w_2, w_3)$ is given in (ii) of Theorem 1.

Proof. Let $I_3$ denote the ideal generated by $I_1$ and $X_{0,1} - a_{0,1}$. Then we can verify that $I_3$ contains $q(q^{2m-1} + q - 2)X_{0,2} + 2(q^{2m} - q^2 + 2q - 2)$, that is,

$$a_{0,2} = -2(q^{2m} - q^2 + 2q - 2) / q(q^{2m-1} + q - 2).$$  \hspace{1cm} (23)

Let $I_4$ denote the ideal generated by $I_3$ and $p_1(X_{0,2})$. Then we can verify that $I_4$ contains $X_{0,3} - 2$, that is, $w_3 = 1$. From (13) we obtain

$$w_2 = w_1^2 - 1 / a_{1,2}w_1 - a_{0,2}.$$  

Since $w_1^2 - a_{0,1}w_1 + 1 = 0$, we have (3) from (22), (23). \hfill \Box

Proof of Theorem 1. Suppose that the matrix (2) is a complex Hadamard matrix. For $i, j \in \{0, 1, 2, 3, 4\}$, define $a_{i,j}$ by (3). Let $a$ be given in (17). Then $a$ is a common zero of the polynomials in $I_1$ by Lemma 2. From Lemmas 3–5 we have (i) and (ii) of Theorem 1.

Conversely, assume that $w_1, w_2, w_3$, and $w_4$ are given in Theorem 1. Then, we show that the matrix given in (2) is a complex Hadamard matrix. To do this, we check that $a$ defined by (3) is a zero of the polynomials (19), (20), and (21), and $(w_1, w_2, w_3)$ are complex numbers of absolute value 1. The latter condition is satisfied if $-2 < a_{i_0,i_1} < 2$ holds for some $i_0, i_1$ with $0 \leq i_0 < i_1 \leq 4$.

Case (i) is done by [4].

Next consider Case (ii). From (3), (5), and (6) we have (22) and (23). Then we have

$$a = (a_{0,1}, a_{0,2}, 2, 2, a_{1,2}, a_{0,1}, a_{0,1}, a_{0,2}, a_{0,2}, 2).$$

This is a zero of the polynomials (19), (20), and (21). It is easy to check that $0 < a_{0,1} < 2$. \hfill \Box
4 Proof of Theorem 2

Since \( q^{2m} - 1 \) is a composite, there are uncountably many inequivalent complex Hadamard matrices of order \( q^{2m} - 1 \) by [6]. Indeed, such matrices can be constructed using generalized tensor products [8]. We show that none of our complex Hadamard matrices is equivalent to a nontrivial generalized tensor product. This is done by showing that the Nomura algebra of our complex Hadamard matrices has dimension 2. According to [8], the Nomura algebra of a nontrivial generalized tensor product of type-II matrices is imprimitive, and this is never the case when it has dimension 2.

Recall \( q \geq 4 \) and \( m \geq 2 \). The intersection matrices \( B_i = (p_{ij}^k) \) \( (i = 0, \ldots, 4) \) of \( X \) are given by the following:

\[
B_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
0 & 0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
0 & \frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0 & \frac{(q^2-1)q^{2m-2}}{4} \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & 0 \\
\frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4} & \frac{(q^2-1)q^{2m-2}}{4}
\end{bmatrix},
\]

\[
B_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
\frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} \\
\frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} \\
\frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} \\
\frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2} & \frac{q^2-1}{q^2}
\end{bmatrix}.
\]

**Lemma 6.** The algebra \( N(W) \) is symmetric.

**Proof.** Suppose that \( N(W) \) is not symmetric. Then by [8 Proposition 6(i)], there exists \((b, c) \in X^2\) with \( b \neq c \) such that

\[
\sum_{x \in X} x W_{x,b}^2 = 0.
\]

\[
W_{x,c}^2 = 0.
\]
This is equivalent to
\[ \sum_{j,k} p^i_{jk} \frac{w_j^2}{w_k^2} = 0 \]
for some \( i \in \{1, 2, 3, 4\} \). Using the notation (3), we have
\[
\sum_{j,k} p^i_{jk} \frac{w_j^2}{w_k^2} = \sum_{j<k} p^i_{jk} \left( \frac{w_j}{w_k} + \frac{w_k}{w_j} \right)^2 + \sum_{j=0}^{4} p^i_{jj}
\]
\[ = \sum_{j<k} p^i_{jk} \left( (w_j/w_k + w_k/w_j)^2 - 2 \right) + \sum_{j=0}^{4} p^i_{jj} \]  
\[ = \sum_{j<k} p^i_{jk} (a^2_{j,k} - 2) + \sum_{j=0}^{4} p^i_{jj}. \]  
(24)

It can be verified by computer that (24) is nonzero for each of the cases (i)–(ii) in Theorem 1.

Since \( N(W) \) is symmetric, the adjacency matrices of \( N(W) \) are the \((0, 1)\)-matrices representing the connected components of the Jones graph defined as follows (see [9] Sect. 3.3). The Jones graph of a type-II matrix \( W \in M_X(C) \) is the graph with vertex set \( X^2 \) such that two distinct vertices \((a, b) \) and \((c, d) \) are adjacent whenever \( \langle Y_{ab}, Y_{cd} \rangle \neq 0 \), where \( \langle , \rangle \) denotes the ordinary (not Hermitian) scalar product.

**Proof of Theorem 2** We claim that \((x, y)\) and \((x, z)\) belong to the same connected component in the Jones graph whenever \((x, y), (y, z), (z, x) \in R_4\). Indeed, if \((x, y)\) and \((x, z)\) belong to different connected components, then \((y, x)\) and \((z, x)\) belong to different connected components by Lemma 6. In particular,
\[ \langle Y_{xy}, Y_{xz} \rangle = \langle Y_{yx}, Y_{zx} \rangle = 0. \]
Let
\[ c_{i,j,k} = |\{ u \in X \mid (x, u) \in R_i, (y, u) \in R_j, (z, u) \in R_k \}|. \]
Since \( p^1_{1,3} = p^2_{2,3} = 0 \), we have
\[ c_{1,1,k} + c_{2,1,k} = c_{j,1,k} + c_{j,2,k} = c_{j,k,1} + c_{j,k,2} = p^4_{j,k} \]  
(25)
for \( j, k \in \{1, 2\} \). Then we have
\[ \sum_{i,j,k=0}^{4} c_{i,j,k} \frac{w_i^2}{w_j w_k} = \sum_{i,j,k=0}^{4} c_{i,j,k} \frac{w_j w_k}{w_i^2} = 0. \]  
(26)
Since the rank of the coefficient matrix in (25) is 7, we have one degree of freedom in (25). Combining (25) and (26), it can be verified by computer that these conditions give rise to a polynomial equation in \( q \) which has no solution in positive integers \( q \geq 4 \) for each of the cases (i) and (ii) in Theorem 1.

Therefore, we have proved the claim. This, together with Lemma 6, implies that, for each equivalence class \( C \) of the equivalence relation \( R_0 \cup R_4, (C \times C) \cap R_4 \) belongs to the same connected component in the Jones graph.
Let $C$ and $C'$ be two distinct equivalence classes of $R_0 \cup R_4$. We claim that, for any $(x, z) \in C \times C'$, there exist $y \in C$ such that $\langle Y_{xy}, Y_{xz} \rangle \neq 0$, and there exist $y' \in C'$ such that $\langle Y_{y'z}, Y_{xz} \rangle \neq 0$.

Suppose $(x, z) \in R_1$ and $\langle Y_{xy}, Y_{xz} \rangle = 0$ for all $y \in R_4(x)$. Then

$$0 = \sum_{y \in R_4(x)} \langle Y_{xy}, Y_{xz} \rangle$$

$$= \sum_{y \in R_4(x)} \sum_{u \in X} \langle Y_{xy}u, Y_{xz}u \rangle$$

$$= \sum_{y \in R_4(x)} \sum_{u \in X} \frac{W_{yu}^2 W_{zu}}{W_{yu} W_{zu}}$$

$$= \sum_{y \in R_4(x)} 4 \sum_{i,j=0}^4 \sum_{u \in R_i(x) \cap R_j(z)} \frac{W_{yu}^2}{W_{yu} W_{zu}}$$

$$= \sum_{i,j=0}^4 \sum_{u \in R_i(x) \cap R_j(z)} \sum_{k=0}^4 \frac{w_{ik}^2}{w_{ik} w_{kj}}$$

$$= \sum_{i,j=0}^4 \frac{p_{ij}^4 w_{ik}^2}{w_{ik} w_{kj}}.$$

It can be verified by computer that this leads to a polynomial equation in $q$ which has no solution in positive integers $q \geq 4$. Set $\ell \in \{2, 3\}$. Similarly, suppose $(x, z) \in R_\ell$ and $\langle Y_{xy}, Y_{xz} \rangle = 0$ for all $y \in C$. Then

$$\sum_{i,j,k=0}^4 p_{ij}^\ell p_{4k} \frac{w_{ik}^2}{w_{ik} w_{kj}} = 0,$$

and again this leads to a contradiction. Thus, there exists $y \in C$ such that $\langle Y_{xy}, Y_{xz} \rangle \neq 0$. Switching the role of $x$ and $z$, we see that there exists $y' \in C'$ such that $\langle Y_{y'z}, Y_{xz} \rangle \neq 0$. Therefore, we have proved the claim.

Since $C$ and $C'$ are arbitrary, the claim shows that, in the Jones graph, $R_4$ is contained in a single connected component, and that every element $(x, z) \in R_1 \cup R_2 \cup R_3$ is adjacent to an element of $R_4$. Thus, $R_1 \cup R_2 \cup R_3 \cup R_4$ is a connected component of the Jones graph. Therefore, $\dim N(W) = 2$.

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Proof of Theorem 1

\[ d := 4; \]
\[ d2s := \text{&cat}[[i,j]: j \text{ in } [i+1..d]]: i \text{ in } [0..d-1]]; \]
\[ d2 := [\text{Seqset}(s): s \text{ in } d2s]; \]
\[ R := \text{PolynomialRing}((\text{Rationals}()), \#d2+3); \]
\[ X := \text{func}<i,j|R.\text{Position}(d2,\{i,j\}); \]
\[ q := R.(\#d2+1); \]
\[ r := R.(\#d22+2); \]
\[ nz1 := R.(\#d2+3); \]
\[ NZ1 := nz1*(q-1)-1; \]
\[ qm := q*r; \]
\[ g := \text{func}<i,j,k|X(i,j)^2+X(i,k)^2+X(j,k)^2-X(i,j)*X(i,k)*X(j,k)-4>; \]
\[ h := \text{func}<i,j,k,l|(X(k,l)^2-4)*X(i,j) \]
\[ -X(k,1)*(X(k,i)*X(l,j)+X(k,j)*X(l,i)) \]
\[ +2*(X(k,i)*X(k,j)+X(l,i)*X(l,j)); \]
\[ \text{eigenP} := \text{Matrix}(R, 5, 5, [\]
\[ 1, 1/2*qm*r*(q-2), 1/2*qm^2, q*(r^2-1), q-2, \]
\[ 1, 1/2*r*(q-2), 1/2*qm, -(r+1)*(q-1), q-2, \]
\[ 1, -1/2*r*(q-2), -1/2*qm, (r-1)*(q-1), q-2, \]
\[ 1, 1/2*qm, -1/2*qm, 0, -1, \]
\[ 1, -1/2*qm, 1/2*qm, 0, -1 \]); \]
\[ P := \text{func}<i,j|\text{eigenP}[i+1,j+1]>; \]
\[ n := \text{&+(P(0,i): i \text{ in } [0..d])}; \]
\[ n \text{ eq } qm^2-1; \]
\[ e := \text{func}<i|-n+\text{&+(P(i,j)^2:j \text{ in } [0..d])}>; \]
\[ \text{eq7} := [\text{g}(i[1], i[2], i[3]): i \text{ in } \text{s3}] \text{ cat} \]
\[ [\text{h}(0^i, 1^i, 2^i, 3^i): i \text{ in } \text{Sym}([0..d])] \text{ cat} \]
\[ [\text{e}(i): i \text{ in } [1..d]]]; \]
\[ \text{I} := \text{ideal}<R|\text{eq7}>; \]

Proof of Lemma 3

\[ \text{I1} := \text{ideal}<R|X(0,4)-2>; \]
\[ \text{pa12} := qm^2*X(1,2)+2*(qm^2-2); \]
\[ \text{pa12 in I1}; //Lemma 3 \]

Proof of Lemma 4

\[ \text{I2} := \text{ideal}<R|X(0,1)-2>; \]
Proof of Lemma 5

\[ \text{pa01:=qm*r*(qm*r+q-2)*X(0,1)-2*(qm^2*r^2-qm^2-2*qm*r+2)}; \]
\[ \text{I3:=ideal<R|I1,pa01>;} \]
\[ \text{pa02:=q*(qm*r+q-2)*X(0,2)+2*qm^2-2*q^2+4*q-4;} \]
\[ \text{ff:=(qm^2-1)*(148*r^4-8103*r^2+8214*q^2-46102*q+42957)} \]
\[ *((q+1)*(2*q-1)*qm^4*r^6+(2*q+1)*(q^2-7*q+4)*qm^3*r^5 \]
\[ +(5*q^3+27*q^2-22*q-4)*qm^2*r^2-(q^3+11*q^2+46*q-56)*qm^2*r^2 \]
\[ +8*(q+6)*(q-1)*q*r^2-16*q+16); \]
\[ \text{pa02*ff in I3;} \]
\[ \text{I4:=ideal<R|I3,pa02>;} \]
\[ (r^2-1)*(X(0,3)-2)^2 \text{ in I4;} \]

//Total time: 88023.889 seconds, Total memory usage: 253.50MB

Proof of Theorem 1

\[ \text{Pqr:=PolynomialRing(Rationals(),2);} \]
\[ \text{Fqr<q,r>:=FieldOfFractions(Pqr);} \]
\[ \text{Rqr<z1>:=PolynomialRing(Fqr);} \]
\[ \text{qm:=q*r;} \]
\[ \text{a01:=2*(qm^2*r^2-(q+2)*qm*r+2)/(qm*r*(qm*r+q-2));} \]
\[ \text{a02:=-2*(qm^2-q^2+2*q-2)/(q*(qm*r+q-2));} \]
\[ \text{a12:=-2*(qm^2-2)/(qm^2);} \]
\[ \text{F<w1>:=FieldOfFractions(Rqr/ideal<Rqr|z1^2-a01*z1+1>);} \]
\[ \text{w1^2-a01*w1+1 eq 0;} \]
\[ \text{w2:=-((q-1)*qm*r*w1+qm*r+q-2)/((qm*r-1)*q);} \]
\[ \text{w1/w2+w2/w1 eq a12;} \]
\[ \text{w2+1/w2 eq a02;} \]
\[ \text{d:=4;} \]
\[ \text{d2s:=&cat[[[i,j]:j in [i+1..d]]:i in [0..d-1]];} \]
\[ \text{d2:=[Seqset(s):s in d2s];} \]
\[ \text{R:=PolynomialRing(F,#d2);} \]
\[ \text{X:=func<i,j|R.Position(d2,{i,j})>;} \]
\[ \text{g:=func<i,j,k|X(i,j)^2+X(i,k)^2+X(j,k)^2-X(i,j)*X(i,k)*X(j,k)-4>;} \]
\[ \text{h:=func<i,j,k,l|X(k,l)^2-4)*X(i,j) \]
\[ \text{-X(k,1)*X(k,i)*X(1,j)+X(k,j)*X(1,i)} \]
\[ \text{+2*(X(k,i)*X(k,j)+X(1,i)*X(1,j))>;} \]
\[ \text{eigenP:=Matrix(F,5,5,[} \]
\[ \text{1,1/2*qm*r*(q-2),1/2*qm^2,0-q* (r^2-1),q,-2,} \]
\[ \text{1,1/2*(q-2),1/2*qm,-(r+1)*(q-1),q,-2,} \]
\[ \text{1,-1/2*(q-2),-1/2*qm,(r-1)*(q-1),q,-2,} \]
\[ \text{1,1/2*qm,-1/2*qm,0,-1,} \]
\[1,-\frac{1}{2}\cdot qm,\frac{1}{2}\cdot qm,0,-1\]
\]

\[P := \text{func}\langle i, j \mid \text{eigenP}[i+1,j+1]\rangle;\]
\[n := \&+[P(0,i) : i \in [0..d]];\]
\[n \equiv qm^2-1;\]

\[e := \text{func}\langle i \mid -n+\&+[P(i,j)^2 : j \in [0..d]] \]
\[+\&+[P(i,j[1])\cdot P(i,j[2])\cdot X(j[1],j[2]) : j \in d2s]\rangle;\]
\[s3 := [\text{Setseq}(x) : x \in \text{Subsets}([0..d],3)];\]
\[eq7 := [g(i[1],i[2],i[3]) : i \in s3] \text{ cat}\]
\[h(0^i,1^i,2^i,3^i) : i \in \text{Sym}([0..d])] \text{ cat}\]
\[e(i) : i \in [1..d]];\]

\[\text{subs1} := [a01,a02,2,2,a12,a01,a01,a02,2];\]
\[\text{and} [\text{Evaluate}(f, \text{subs1}) \equiv 0 : f \in \text{eq7}];\]
\[// \text{Total time: 0.440 seconds, Total memory usage: 32.09MB}\]

**Proof of Theorem 2**

Calculation of the intersection matrices \(\{B_i\}_{i=0}^4\):

\[P<\text{c111,c112,c121,c122,c211,c212,c221,c222,w1,w2,q,r}> := \text{PolynomialRing}(\text{Rationals}(),12);\]
\[F := \text{FieldOfFractions}(P);\]
\[qm := q\cdot r;\]
\[n := qm^2-1;\]

\[\text{eigenP} := \text{Matrix}(F,5,5,[\]
\[1,1/2\cdot qm\cdot r\cdot (q-2),1/2\cdot qm\cdot r\cdot (q-2),q\cdot (r^2-1),q-2,\]
\[1,1/2\cdot r\cdot (q-2),1/2\cdot qm\cdot (q-1),q-2,\]
\[1,-1/2\cdot r\cdot (q-2),-1/2\cdot qm\cdot (q-1),q-2,\]
\[1,1/2\cdot qm\cdot (q-1),1/2\cdot qm\cdot 0,-1,\]
\[1,-1/2\cdot qm\cdot 1/2\cdot qm\cdot 0,-1\]
\[]);\]

\[\text{intersectionMatrices} := \text{function}(P)\]
\[\text{d1} := \text{Nrows}(P);\]
\[n := \&+\text{Eltseq}(P[1]);\]
\[Q := n\cdot P^{(-1)};\]
\[\text{return } [\text{Matrix}(\text{Parent}(Q[1][1]),\text{d1},\text{d1},\]
\[[1/(n\cdot P[1,k])\ast+[ Q[1,1]\ast P[1,i]\ast P[1,j]\ast P[1,k] : l \in [1..\text{d1}] \]
\[ : k \in [1..\text{d1}] ] : j \in [1..\text{d1}] ]\]
\[): i \in [1..\text{d1}] ];\]
\[\text{end function};\]

\[B1 := \text{Matrix}(F,5,5,[0,1,0,0,0,\]
\[qm\cdot r\cdot (q-2)/2,r\cdot 2\cdot (q-2)^2/4,r\cdot 2\cdot (q-2)^2/4,r\cdot 2\cdot (q-2)^2/4,(q-4)\cdot qm\cdot r/4,\]
\[0,(q-2)\cdot qm\cdot r/4,(q-2)\cdot qm\cdot r/4,(q-2)\cdot qm\cdot r/4,qm^2/4,\]
\[13\]

0, (q-2)*(r^2-1)/2, (q-2)*(r^2-1)/2, (q-2)*r^2/2, 0,
0, 1/2*q-2, 1/2*q-1, 0, 0]);

B2:=Matrix(F,5,5,
[0, 0, 1, 0, 0,
0, (q-2)*qm*r/4, (q-2)*qm*r/4, (q-2)*qm*r/4, qm^2/4,
qm^2/2, qm^2/4, qm^2/4, qm^2/4, qm^2/4,
0, q*(r^2-1)/2, q*(r^2-1)/2, 1/2*qm*r, 0,
0, 1/2*q, 1/2*q-1, 0, 0]);

B3:=Matrix(F,5,5,
[0, 0, 0, 1, 0,
0, (q-2)*(r^2-1)/2, (q-2)*(r^2-1)/2, (q-2)*r^2/2, 0,
0, q*(r^2-1)/2, q*(r^2-1)/2, 1/2*qm*r, 0,
q*(r^2-1), r^2-1, r^2-1, r^2-2*q+1, q*(r^2-1),
0, 0, 0, q-2, 0]);

B4:=Matrix(F,5,5,
[0, 0, 0, 0, 1,
0, 1/2*q-2, 1/2*q-1, 0, 0,
0, 1/2*q, 1/2*q-1, 0, 0,
0, 0, 0, q-2, 0,
q-2, 0, 0, q-3]);

BB:=[ScalarMatrix(5,F!1), B1, B2, B3, B4];
BB eq intersectionMatrices(eigenP);
pijk:=func<i,j,k|P!BB[i+1][j+1,k+1];

Proof of Lemma 6

isSymNbas:=function(ajk)
  aijs:=[[ajk[1], ajk[2], ajk[3], ajk[4]],
         [1, ajk[5], ajk[6], ajk[7]],
         [1, ajk[8], ajk[9]],
         [1, 1, 1, ajk[10]]];
  aij:=[func<i,j|aijs[i+1,j+1]>;
  ff:=F|&+[pijk(j,k,i)*(aij(j,k)^2-2):j,k in [0..4]|j lt k]
     +&+[pijk(j,j,i):j in [0..4]]:i in [1..4]]:
  return [Numerator(ff[i]):i in [1..4]]
end function;

x02:=-(2*q^2*r^2-4)/(q^2*r^2);
aa:=[2, x02, 2, 2, x02, 2, 2, x02, x02, 2];
isSymNbas(aa) eq [ (qm^2-1)*(qm^2-4) :i in [1..4]];

a01:=2*(qm^2*r^2-(q+2)*qm*r+2)/(qm*r*(qm*r+q-2));
a02:=-2*(qm^2-2)*q-2)/(q*(qm*r+q-2));
a12:=-2*(qm^2-2)/(qm^2);
aa:=[a01, a02, 2, 2, a12, a01, a01, a02, a02, 2];
pp:=qm^5*r+2*(q^2-10*q+14)*qm^3*r+q*(q-2)*(q-3)*q-2*q-2*q+16)*r^2-4*(q-2)*(q-2*q+4);
isSymNbas(aa) eq [ (qm^2-1)*pp : i in [1..3] ] cat [ (qm^2-1)*(qm^2-4) ];

Proof of Theorem 2

The first claim:

\[
\text{varname:=[[i,j,k]:i,j,k in [1,2]]};
\]
\[
c:=\text{func}<i,j,k|R.\text{Position(varname,[i,j,k])}>;
\]
\[
w0:=1;
\]
\[
cijk:=\text{function}(i,j,k)
\]
\[
\text{if 0 in }\{i,j,k\} \text{ then}
\]
\[
\text{if }\{i,j,k\} \text{ in }\{[0,4,4],[4,0,4],[4,4,0]\} \text{ then return 1;}
\]
\[
\text{else return 0;}
\]
\[
\text{end if;}
\]
\[
\text{else}
\]
\[
\text{if 3 in }\{i,j,k\} \text{ then}
\]
\[
\text{if }\{3\} \text{ eq }\{i,j,k\} \text{ then return pijk(3,3,4);}
\]
\[
\text{else return 0;}
\]
\[
\text{end if;}
\]
\[
\text{else}
\]
\[
\text{if 4 in }\{i,j,k\} \text{ then}
\]
\[
\text{if }\{4\} \text{ eq }\{i,j,k\} \text{ then return pijk(4,4,4)-1;}
\]
\[
\text{else return 0;}
\]
\[
\text{end if;}
\]
\[
\text{else return c(i,j,k);}
\]
\[
\text{end if;}
\]
\[
\text{end if;}
\]
\[
\text{end function;}
\]
\[
\text{fx:=[cijk(1,j,k)+cijk(2,j,k)-pijk(j,k,4):j,k in [1,2]]};
\]
\[
\text{fy:=[cijk(j,1,k)+cijk(j,2,k)-pijk(j,k,4):j,k in [1,2]]};
\]
\[
\text{fz:=[cijk(j,k,1)+cijk(j,k,2)-pijk(j,k,4):j,k in [1,2]]};
\]
\[
\text{fxyz:=fx cat fy cat fz;}
\]
\[
a02N1:=-(2*q^2*r^2-4);
\]
\[
a02D1:=q^2*r^2;
\]
\[
a021:=a02N1/a02D1;
\]
\[
a021:=a02D1*w2^2-a02N1*w2+1;
\[ \text{ww1:=}[w0,1,w2,1,1]; \]
\[
\alpha_1:=\text{func}<i|\text{ww1}[i+1]>; \\
f f_1:=\&+[cijk(i,j,k)*\alpha_1(i)^2/(\alpha_1(j)*\alpha_1(k)) \\
:i,j,k \in [0..4]]; \\
g g_1:=\&+[cijk(i,j,k)*\alpha_1(j)*\alpha_1(k)/\alpha_1(i)^2 \\
:i,j,k \in [0..4]]; \\
I_1:=\text{ideal}<\mathbb{R}|[\text{fa21},\text{Numerator}(ff_1),\text{Numerator}(gg_1)] \text{ cat fxyz}>; \\
(qm^2-1)*(5*qm^6-90*qm^4+313*qm^2-128) \text{ in } I_1; \\
\text{Irreducible}(5*qm^6-90*qm^4+313*qm^2-128); \\
a_{01N2}:=2*(qm^2*r^2-(q+2)*qm*r+2); \\
a_{01D2}:=(qm*r+q-2)*qm*r; \\
a_{01}:=a_{01N2}/a_{01D2}; \\
a_{022}:=-2*(qm^2-q^2+2*q-2)/(q*(qm*r+q-2)); \\
a_{12}:=a_{01D2}/\text{w1}^2-a_{01N2}/\text{w1}+a_{01D2}; \\
w_2:=(\text{w1}^2-1)/(a_{12}*\text{w1}-a_{02}); \\
w_2:=[w0,w1,w2,1,1]; \\
\alpha_2:=\text{func}<i|\text{ww2}[i+1]>; \\
f f_2:=\&+[cijk(i,j,k)*\alpha_2(i)^2/(\alpha_2(j)*\alpha_2(k)) \\
:i,j,k \in [0..4]]; \\
g g_2:=\&+[cijk(i,j,k)*\alpha_2(j)*\alpha_2(k)/\alpha_2(i)^2 \\
:i,j,k \in [0..4]]; \\
I_2:=\text{ideal}<\mathbb{R}|[\text{fa1},\text{Numerator}(ff_2),\text{Numerator}(gg_2)] \text{ cat fxyz}>; \\
\text{Basis}(\text{EliminationIdeal}(I_2,\{q,r\})) \\
\text{eq } [qm^10*(qm^2-1)^3*(qm*r+q-2)^4*(qm*r-1)^5]; \\
\]

The second claim:

\[ tl:=\text{function}(l) \\
\text{return } \&+[\text{pijk}(i,j,l)*\text{pijk}(4,k,i)*\alpha_1(i)^2/(\alpha_1(j)*\alpha_1(k)) \\
:i,j,k \in [0..4]]; \]
end function; \\
\text{and}[ (q-2)*(qm^2-1)*(5*qm^6-90*qm^4+313*qm^2-128) \text{ in } \\
\text{ideal}<\mathbb{R}|[\text{fa21},\text{Numerator}(tl(l))]> : l \text{ in } [1..3]; \\
\]

\[ sl:=\text{function}(l) \\
\text{return } \&+[\text{pijk}(i,j,l)*\text{pijk}(4,k,i)*\alpha_2(i)^2/(\alpha_2(j)*\alpha_2(k)) \\
:i,j,k \in [0..4]]; \]
end function; \\
\text{and}[ qm^7*r*(q-2)*(qm^2-1)^3*(qm*r-1)^5 \text{ in } \\
\text{ideal}<\mathbb{R}|[\text{fa1},\text{Numerator}(sl(l))]> : l \text{ in } [1..3]; \\
//Total time: 34.890 seconds, Total memory usage: 82.78MB}