Some structural properties of vector valued $\varphi$-function sequence space

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\textbf{Abstract.} The sequence space $W(M)$, where $M$ is an Orlicz function was introduced by Parashar and Choudhary \cite{1} and Maddox \cite{2}. Let $f$ be $\varphi$-function and $X$ be a Banach space. In this work, we introduce vector valued sequence space defined by $f$, denoted by $W(X, f)$. We study some topological properties and inclusion relations of this space.

1. Introduction and Preliminaries

An \textit{Orlicz function} is a continuous, convex, non-decreasing function defined from $[0, \infty)$ to itself such that $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri \cite{3} introduced the sequence space $\ell^\infty(M)$ using Orlicz function $M$ as follows:

$$\ell^\infty(M) = \{ x = (x_k) : x_k \in \mathbb{R} \forall k \in \mathbb{N} \text{ and } \exists \rho > 0 \text{ such that } \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \}$$

The space $\ell^\infty(M)$ equipped with the Luxemburg norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an \textit{Orlicz sequence space}. The space $\ell^\infty(M)$ is closely related to the space $\ell_p$ with $1 \leq p < \infty$,

$$\ell_p = \left\{ x = (x_k) : x_k \in \mathbb{R} \forall k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}$$

which is an \textit{Orlicz sequence space} with $M(x) = x^p$. In the mathematical literature there exists various modifications of these definitions, where $\ell$ is replaced by another solid sequence space (see \cite{4–6}). A sequence space $X$ is said to be solid (or normal) if $(\lambda_k x_k) \in X$, whenever $(x_k) \in X$ and for all sequences $(\lambda_k)$ of scalars with $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$.

A norm $|| \cdot ||$ on a normal sequence space $X$ is said to be \textit{absolutely monotone norm} if $x = (x_k), y = (y_k) \in X$ and $|x_k| \leq |y_k|$ for all $k \in \mathbb{N}$ implies $||x_k|| \leq ||y_k||$. The norm

$$||x||_\infty = \sup |x_k|$$
over the classical sequence space $\ell_\infty$, $c$, $c_0$ and the norm

$$
\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}
$$

over $\ell_\infty$ for $p \geq 1$ are absolutely monotone.

A completed normed space $X$ is said to be a BK-space if the function $p_k : X \to \mathbb{R}$ where $p_k(x) = x_k$ is continuous in $X$ for every $x = (x_k) \in X$ and every $k \in \mathbb{N}$. An AK-space $X$ with the norm $\|\cdot\|$ is a BK-space and $\|x - x[n]\| \to 0$ as $n \to \infty$ for every $x \in X$, where $x[n]$ denotes the $n$-th section of $x$.

Let $X$ be a vector space. The collection of all vector valued sequences denoted by $\Omega(X)$. Any vector subspace of $\Omega(X)$ is called vector valued sequence space. The studies on vector valued sequence spaces are done by Rath and Srivastava [7], Das and Choudhary [8], Leonard [9], Srivastava and Srivastava [10] and many others.

A function $f : \mathbb{R} \to [0, \infty)$ which is continuous, vanishing at zero, non-decreasing on $[0, \infty)$ and even is called $\varphi$-function. A $\varphi$-function $f$ is said to satisfy $\Delta_2$-condition (written as $f \in \Delta_2$ for shortly), if there exists $K > 0$ such that $f(2x) \leq Kf(x)$ for every $x \geq 0$.

A functional $\rho : X \to [0, \infty)$ is called a convex modular if $\rho(x) = 0 \Rightarrow x = 0$, even, $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. In this case, we say that $X$ is a convex modular space.

Let $X = (X, \|\cdot\|_X)$ be a Banach space with an absolutely monotone norm $\|\cdot\|_X$ and $f$ is a $\varphi$-function. Using convex $\varphi$-function $f$, we introduce the following set, denoted by $W(X, f)$,

$$
W(X, f) = \left\{ x = (x(i)) \in \Omega(X) : \rho_f \left( \frac{x(i) - \ell}{\alpha} \right) \to 0 \text{ as } i \to \infty, \text{ for some } \alpha > 0 \text{ and } \ell \in X \right\}
$$

where

$$
\rho_f(x(i)) = \frac{1}{m} \sum_{i=1}^{m} f(\|x(i)\|_X), \text{ for every } x(i) \in X
$$

is a convex modular.

A function $g : X \to \mathbb{R}$ is said to be a paranorm if $g(\theta) = 0$, $g(x) \geq 0$, $g(x+y) \leq g(x) + g(y)$, even and every scalar sequence $(\lambda_n)$ with $|\lambda_n - \lambda| \to 0$ and every sequence $(x_n)$ with $g(x_n - x) \to 0$ implies $g(\lambda_n x_n - \lambda x) \to 0$ for all $\lambda \in \mathbb{R}$ and $x \in X$, where $\theta$ is the zero in the linear space $X$. The notion of paranormed sequence space was introduced by Nakano [11] and Simons [12]. Later on it was further investigated by Rath and Tripathy [13], Tripathy and Sen [14].

In this work, we investigate some of topological properties of the set $W(X, f)$ equipped with a paranorm that we will define and study some inclusion relations of this set.

2. Main Results

In this section we examine some topological properties and inclusion relations of the set $W(X, f)$.

Lemma 1. If $x, y \in X$ such that $0 \leq x \leq y$, then $\rho_f(x) \leq \rho_f(y)$.

Theorem 2.1. If $\varphi$-function $f$ satisfies the $\Delta_2$-condition and convex, then $W(X, f)$ is a linear space.

Proof. Let $x = (x(i))$, $y = (y(i)) \in W(X, f)$, then there exist $\alpha_1, \alpha_2 > 0$ and $\ell_1, \ell_2 \in X$ such that

$$
\rho_f \left( \frac{x(i) - \ell_1}{\alpha_1} \right) \to 0 \text{ and } \rho_f \left( \frac{y(i) - \ell_2}{\alpha_2} \right) \to 0 \text{ as } i \to \infty
$$
Proof. Since $\rho_f$ is convex and $f \in \Delta_2$, there exists $K_1, K_2 > 0$ such that

$$\rho_f \left( \frac{(x(i) + y(i)) - \ell}{\alpha} \right) \leq K_1 \rho_f \left( \frac{x(i) - \ell_1}{\alpha_1} \right) + \frac{K_2}{2} \rho_f \left( \frac{y(i) - \ell_2}{\alpha_2} \right)$$

Consequently, $\rho_f \left( \frac{(x(i) + y(i)) - \ell}{\alpha} \right) \to 0$ as $i \to \infty$. Hence, $x + y \in W(X, f)$. Let $\beta \in \mathbb{R}$ and $x = (x(i)) \in W(X, f)$, then there exists $\alpha > 0$ and $\ell \in X$ such that

$$\rho_f \left( \frac{x(i) - \ell}{\alpha} \right) \to 0 \text{ as } i \to \infty$$

Choose $p = \beta \ell$. For $\beta = 0$, is clear that $\rho_f \left( \frac{\beta x(i) - p}{\alpha} \right) \to 0$ as $i \to \infty$. Now, assume that $\beta \neq 0$. Since $f \in \Delta_2$, then by using Archimedian there exists $n_0 \in \mathbb{N}$ and $K > 0$ such that

$$\rho_f \left( \frac{\beta x(i) - p}{\alpha} \right) \leq K^{n_0} \rho_f \left( \frac{x(i) - \ell}{\alpha} \right) \to 0 \text{ as } i \to \infty$$

Therefore, $\beta x \in W(X, f)$ and the proof is complete. \hfill \Box

**Theorem 2.2.** A function $g : W(X, f) \to \mathbb{R}$ with

$$g(x) = \inf \left\{ \alpha > 0 : \rho_f \left( \frac{x(i)}{\alpha} \right) \leq 1 \right\}$$

is a paranorm.

**Proof.** It is easy to show that $g(\theta) = 0$, $g(x) \geq 0$ and $g(-x) = g(x)$, for every $x \in W(X, f)$, where $\theta$ is the zero in the linear space $W(X, f)$. We shall now show the subadditivity of $g$. Let $x = (x(i)), y = (y(i)) \in W(X, f)$, then there exist $\alpha_1, \alpha_2 > 0$ such that

$$\rho_f \left( \frac{x(i)}{\alpha_1} \right) \leq 1 \text{ and } \rho_f \left( \frac{y(i)}{\alpha_2} \right) \leq 1$$

Take $\alpha = \max\{2\alpha_1, 2\alpha_2\}$. Considering Lemma 1 and using the convexity of $\rho_f$, we have

$$\rho_f \left( \frac{x(i) + y(i)}{\alpha} \right) \leq \frac{1}{2} \rho_f \left( \frac{x(i)}{\alpha_1} \right) + \frac{1}{2} \rho_f \left( \frac{y(i)}{\alpha_2} \right)$$

$$\leq \rho_f \left( \frac{x(i)}{\alpha_1} \right) + \rho_f \left( \frac{y(i)}{\alpha_2} \right)$$

Therefore, $g(x + y) \leq g(x) + g(y)$ for every $x, y \in W(X, f)$. Finally, we show that scalar multiplication is continuous. Let $(\lambda_n)$ be any scalar sequence and $(x_n(i)) \subset W(X, f)$, with $|\lambda_n - \lambda| \to 0$ and $g(x_n(i) - x(i)) \to 0$ as $n \to \infty$. Considering Lemma 1 and using the the convexity of $\rho_f$, we have

$$\rho_f \left( \frac{\lambda_n x_n(i) - \lambda x(i)}{\alpha} \right) \leq \rho_f \left( \frac{|(\lambda_n - \lambda)x_n(i)|}{\alpha} \right) + \frac{\lambda |x_n(i) - x(i)|}{\alpha}$$

$$\leq \frac{1}{2} \rho_f \left( 2 |\lambda_n - \lambda| \frac{x_n(i)}{\alpha} \right) + \frac{1}{2} \rho_f \left( 2 |\lambda| \frac{x_n(i) - x(i)}{\alpha} \right)$$

$$\leq \rho_f \left( 2 |\lambda_n - \lambda| \frac{x_n(i)}{\alpha} \right) + \rho_f \left( 2 |\lambda| \frac{x_n(i) - x(i)}{\alpha} \right)$$
Therefore,

\[ g(\lambda_n x_n(i) - \lambda x(i)) = \inf \left\{ \alpha > 0 : \rho_f \left( \frac{\lambda_n x_n(i) - \lambda x(i)}{\alpha} \right) \leq 1 \right\} \]

\[ \leq 2|\lambda_n - \lambda| \inf \left\{ \alpha^* = \left( \frac{\alpha}{2|\lambda_n - \lambda|} \right) > 0 : \rho_f \left( \frac{x_n(i)}{\alpha^*} \right) \leq 1 \right\} \]

\[ + 2|\lambda| \inf \left\{ \alpha^{**} = \left( \frac{\alpha}{2|\lambda|} \right) > 0 : \rho_f \left( \frac{x_n(i) - x(i)}{\alpha^{**}} \right) \leq 1 \right\} \]

\[ = 2|\lambda_n - \lambda| g(x_n(i)) + 2|\lambda| g(x_n(i) - x(i)) \to 0 \]

Hence, \( g(\lambda_n x_n(i) - \lambda x(i)) \to 0 \). This completes the proof of the theorem. \( \square \)

**Theorem 2.3.** The linear space \( W(X, f) \) is a complete paranormed sequence space.

**Proof.** Let \( (x_n) \) be any Cauchy sequence in \( W(X, f) \) where \( (x_n) = (x_n(i)) = (x_n(1), x_n(2), ...) \). This implies for any \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for every \( m \geq n \geq n_0 \), we get \( g(x_m - x_n) < \epsilon \). Consequently, \( \rho_f \left( \frac{x_m(i) - x_n(i)}{\epsilon} \right) \leq 1 \). Since \( \rho_f \) is convex, we have \( \rho_f(x_m(i) - x_n(i)) \leq \epsilon \).

Using the continuity of \( f \), it follows that \( \|x_m(i) - x_n(i)\|_X < \epsilon \) for every \( \epsilon > 0 \). Hence, for every fixed \( i \), the sequence \( (x_n(i)) \) is a Cauchy sequence in \( X \). It converges since \( X \) is complete. Say, \( x_n(i) \to x(i) \) as \( n \to \infty \). Using these limits, we define \( x = (x(i)) \) and show that \( x \in W(X, f) \) and \( g(x_n - x) \to 0 \). Since \( X = (X, \| \cdot \|_X) \) is a Banach space, we get

\[ \|x_m(i) - x(i)\|_X = \|x_m(i) - \lim_{n \to \infty} x_n(i)\|_X = \lim_{n \to \infty} \|x_m(i) - x_n(i)\|_X < \epsilon^2 \]

Since \( (x_n(i)) \in W(X, f) \), there exists \( \alpha > 0 \) and \( \ell \in X \) such that

\[ \rho_f \left( \frac{x_n(i) - \ell}{\alpha} \right) \to 0 \text{ as } i \to \infty \]

Using the continuity of \( f \), we obtain

\[ \rho_f \left( \frac{x(i) - \ell}{\alpha} \right) = \rho_f \left( \frac{\lim_{n \to \infty} x_n(i) - \ell}{\alpha} \right) = \lim_{n \to \infty} \rho_f \left( \frac{x_n(i) - \ell}{\alpha} \right) \to 0 \text{ as } i \to \infty \]

It follows that \( x \in W(X, f) \). We will show that \( g(x_n - x) \to 0 \). Since \( f \) is continuous, then

\[ \rho_f \left( \frac{x_n(i) - x(i)}{\alpha} \right) = \rho_f \left( \frac{x_n(i) - \lim_{m \to \infty} x_m(i)}{\alpha} \right) \leq 1 \]

Therefore, \( g(x_n - x) = \inf \left\{ \alpha > 0 : \rho_f \left( \frac{x_n(i) - x(i)}{\alpha} \right) \leq 1 \right\} \). Hence, there exists sequence \( \left( \frac{x_n(i)}{\alpha} \right), n \geq 1 \), for a real number \( c \) with \( g(x_n - x) < \frac{c}{\alpha} \), for every \( n \geq 1 \). Therefore, we get \( g(x_n - x) \to 0 \). We can conclude that \( W(X, f) \) is a complete paranormed space. \( \square \)

**Theorem 2.4.** The linear space \( W(X, f) \) is an AK space.

**Proof.** Let \( x = (x(i)) \in W(X, f) \), then there exists \( \alpha > 0 \) and \( \ell \in X \) such that

\[ \frac{1}{m} \sum_{i=1}^{m} f \left( \left\| \frac{x(i) - \ell}{\alpha} \right\|_X \right) \to 0 \text{ as } i \to \infty \]
It follows that for every $i = 1, \ldots, m$, we have $\|x(i) - \ell\|_X \to 0$, as $i \to \infty$. Consequently, 

$$\frac{1}{m} \sum_{i=1}^{m} f \left( \left\| \frac{x(i) - \ell}{\alpha} \right\|_X \right) \to 0 \text{ as } i \to \infty$$

Hence, $\rho_f \left( \frac{x-x[n]}{\alpha} \right) \to 0$ as $n \to \infty$, where $x[n]$ denotes the $n$-th section of $x$. Therefore, for $\epsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we obtain $\rho_f \left( \frac{x-x[n]}{\alpha} \right) \leq 1$. It follows that $g(x-x[n]) \to 0$ as $n \to \infty$. This completes the proof.

**Theorem 2.5.** Let $f$ and $g$ be two $\varphi -$ functions, then

(i) $W(X, f) \subseteq W(X, g \circ f)$

(ii) $W(X, f) \cap W(X, g) \subseteq W(X, f + g)$

**Proof.** (i) Let $x = (x(i)) \in W(X, f)$, then there exists $\alpha > 0$ and $\ell \in X$ such that

$$\frac{1}{m} \sum_{i=1}^{m} f \left( \left\| \frac{x(i) - \ell}{\alpha} \right\|_X \right) \to 0 \text{ as } i \to \infty$$

Hence, for every $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, we have

$$\frac{1}{m} \sum_{i=1}^{m} f \left( \left\| \frac{x(i) - \ell}{\alpha} \right\|_X \right) < \epsilon$$

It follows that for every $i = 1, \ldots, m$, we have $f \left( \left\| \frac{x(i) - \ell}{\alpha} \right\|_X \right) \to 0$ as $i \to \infty$. Since $g$ is a $\varphi -$ function, we have $g \left( f \left( \left\| \frac{x(i) - \ell}{\alpha} \right\|_X \right) \right) \to 0$ as $i \to \infty$. Hence, $\rho_{gof} \left( \frac{x(i) - \ell}{\alpha} \right) \to 0$ as $i \to \infty$. This implies $x \in W(X, g \circ f)$. This concludes the proof.

(ii) The result of this point is obvious. 

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