Nonlinear dynamics in problems of stability of complex media

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Abstract. The problem of stability of nonlinear viscoelastic bodies with respect to finite perturbations is considered in this article. The analysis of the basic process of deformation of a viscoelastic medium is reduced to solution of a nonlinear boundary value problem with variable coefficients. Solutions for perturbations of displacements are in the form of series in eigenfunctions. Using the principle of possible displacements the question on stability of the ground state of a variational nonlinear problem is reduced to investigation of stability of the zero solution of an infinite system of ordinary differential equations with constant coefficients. For the resulting system of equations we construct a function, which under certain restrictions on the initial perturbations is a Lyapunov function. The dimension of the strange attractor of the dynamical system which allows to limit the number of terms in a Bubnov-Galerkin set is found.

1. Introduction

The theory of the stability of deformable systems, the beginning of which was laid by Euler's work, has now turned into a highly ramified branch of mechanics that has numerous applications that have created their methods and approaches. In particular, there is no branch of industry and construction where the methods and results of the theory of stability of deformable media are not used. There are a lot of monographs and articles devoted to the theory of stability of elastic and inelastic systems [1], [2]. From the analysis of these works it follows that almost all the problems of stability of elastic and inelastic systems are fulfilled with two basic approaches: static and dynamic with the appropriate stability criteria and assumptions depending on the body model and, as a rule, reduce to the determination of a single critical parameter, in which together with the initial state, there can be a contiguous state of the system. As a rule, according to the statement of the problem, perturbations were taken as arbitrarily small, which made it possible to linearize the initial equations for the perturbations.

Further development of engineering activities related to the use and creation of new materials, the problems of precision in mechanical engineering, the problems of geomechanics, especially in the areas with increased seismicity, and the internal needs of the mechanics of deformable bodies have necessitated the development of a theory of stability under finite perturbations.

Solving the problem of stability of elements and objects with respect to finite disturbances is important not only for the purpose of reducing the weight of structures and objects, but also for ensuring the reliability by estimating the permissible boundary of the region with respect to finite...
perturbations for specified loading parameters and constructions. Thus, the development of the theory of elastic stability at finite perturbations makes it possible to trace the connection between this theory and the general theory of stability of motion on the one hand, as well as with the theory of bifurcations on the other hand, and evaluate the accuracy of the three-dimensional linearized stability theory for small and finite deformations simultaneously. Moreover, the constructed finite sequences of bifurcation points show that, in contrast to the classical stability theories, which determine a unique bifurcation point, there always exists a hierarchy of stable equilibrium states. These sequences make it possible to find the dimensions of a strange attractor for nonlinear systems, and, consequently, the number of terms in the Bubnov-Galerkin set, which allows to present the solutions to the equations in perturbations.

2. Materials and methods
We consider three states of the body [1]. The first state is a natural state when the stresses and deformations are absent in a continuous medium and the temperature $\theta_0$ is independent on the time and coordinates. The second state is obtained from the first state when the mechanical and thermal loads are constant over time after a time interval sufficient for the creep and relaxation processes to completely end in the body. This state corresponds to a displacement vector $u_0$, a strain tensor $E_0$, a temperature $\theta_0$, and these characteristics do not depend on the time $t$, i.e., they correspond to a certain equilibrium state. The third state of motion is achieved by thermomechanical perturbation of the second state. All the values of this state will be marked with a stroke and represented as a sum of quantities related to the second state and perturbations of the corresponding quantities. The perturbations will be considered as a finite and we will not mark them with any index [2], [3].

The third state is characterized by a displacement vector, a stress tensor, a temperature, etc., i.e.

$$u' = u + u_0, \quad E' = E + E_0, \quad S' = S + S_0, \quad \theta' = \theta + \theta_0, \quad \ldots$$

We represent the relations of thermomechanics in the Cartesian coordinates of the natural state $\xi = \{\xi_1, \xi_2, \xi_3\}$, which coincide with the material coordinates at the initial instant. The final deformation corresponding to the displacement $u = \{u_1, u_2, u_3\}$ is determined by the equations in [1], in which it is necessary to put the zero-marks over all quantities. The same values are satisfied by the quantities with primes. Taking this into account it is possible to obtain the relations for the perturbations assuming that the quantities with zero-marks are known [4].

We consider a nonlinearly viscoelastic medium described in monographs [1], [5]. The problem of solution of such a system is very difficult, in this way some simplifications are necessary. The influence of connectivity of mechanical and thermal fields in the propagation of transient disturbances is important at relatively high rates of temperature increasing. Therefore, in order to simplify the problem we consider the dynamical unrelated problems of thermomechanics that are fall in to two distinct problems:
- the problem of thermal conductivity;
- the problem of thermoviscoelastics, in which the mechanical properties of the material are considered as dependent on the temperature and the initial deformation. Within the framework of these ideas the energy equation can be obtained from the disturbances in [4], [6] by neglecting the dissipation function and can be written as:

$$\rho_c \ddot{\theta} + h_{ij} \dot{u}_j = 0. \tag{1}$$

Here dots determine the time derivative of the corresponding quantity.
The equations of motion and boundary conditions have the same form. The corresponding relations [1], [5] can be written as:

\[ S = S^* + \int \left[ G^{(1)}(0, \xi - \xi'; E, \theta) \frac{\partial E(\tau')}{\partial \tau'} + G^{(2)}(0, \xi - \xi'; E, \theta) \frac{\partial \theta(\tau')}{\partial \tau'} \right] d\tau'; \]  

\[ \eta = \eta^* + \int G^{(2)}(0, \xi - \xi'; E, \theta) \frac{\partial \theta(\tau')}{\partial \tau'} d\tau'. \]  

For the heat flow we have:

\[ -h_k = M(E, \theta) g_k. \]  

Relations for the perturbation of stress tensor can be rewritten as:

\[ S^* = S^*(1) + S^*(2) + \cdots + S_{H}. \]  

The number of terms in (5) is specified by the form of elastic potential, and the ratio \( S_{H} \) takes the form:

\[ S_{H} = S^* + \int \left[ G^{(1)}(0, q - q'; E, \theta) \frac{\partial E(\tau')}{\partial \tau'} + G^{(2)}(0, q - q'; E, \theta) \frac{\partial \theta(\tau')}{\partial \tau'} \right] d\tau'. \]  

Let us suppose that the temperature distribution has the form:

\[ \theta = \theta(t, E, \theta, \xi). \]  

We construct the variational equations in the frame of the Bubnov-Galerkin method that corresponds to nonlinear boundary value problem. Assuming that the left sides of the equations of motion are taken with the opposite sign of some components of the volume forces, and the left sides of the equations of equilibrium contain some components of the surface forces we can construct the conditions that take into account the fact that the work of these forces on the possible displacements \( \delta u_m \) is zero.

\[ \int \left\{ \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \right] \right] \right\} T_{ijkl} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} + \rho_k X_i - \rho_k u_i \right] \delta u_i dV 

\[ + \int \left\{ \left[ \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \right] \right] T_{ijkl} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} \right\} n_{ij} - P_i \right] \delta u_i dS = 0. \]  

Without loss of generality, let us assume that there are no mass and surface forces in the body. Then, the variational equation can be written as:

\[ \int \left\{ \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \right] \right] \right\} T_{ijkl} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} \right] \delta u_i dV 

\[ + \int \left\{ \left[ \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \right] \right] T_{ijkl} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} + T_{ijkl} \frac{\partial u_i}{\partial \xi_k} \right\} n_{ij} \right] \delta u_i dS = 0. \]
We transform the first integral in (9) by the Gauss-Ostrogradsky theorem

\[-\frac{1}{2} \int \left[ \delta_{ik} \left( \frac{\partial u}{\partial \xi_k} + \frac{\partial u}{\partial \xi_k} \right) T^{0}_{ij} + \frac{\partial u}{\partial \xi_k} + \frac{\partial u}{\partial \xi_k} \right] \delta u_i \, dV = \int \left[ \left( \frac{\partial u}{\partial \xi_k} \right) T^{0}_{ij} + \frac{\partial u}{\partial \xi_k} \right] \delta u_i \, dS \]

\[+ \int \left[ \left( \frac{\partial u}{\partial \xi_k} \right) T^{0}_{ij} + \frac{\partial u}{\partial \xi_k} \right] \delta u_{ij} \, dV. \tag{10}\]

Substituting (10) in to (9) we obtain:

\[\int \left[ \left( \delta_{ik} + u^0_{ik} \right) T^{0}_{ij} + \frac{\partial u}{\partial \xi_k} u_{ik} + T^{0}_{ij} u_{ik} \right] \delta u_{ij} \, dV + \rho_R \int \delta u_i \, dV = 0. \tag{11}\]

Let us search the solution of nonlinear boundary value problem (11) in the form of a series

\[u_i(\xi, t) = \sum_{n,m} f_{nm}(t) \cdot \phi_{nm}(\xi_i), \tag{12}\]

\[i = 1, 2, 3; n, m = 1, 2, \ldots \infty.\]

Here \(f_{nm}(t)\) are the undetermined coefficients depending on the time, \(\phi_{nm}(\xi_i)\) are the functions satisfying the geometric boundary conditions for \(u_i\). As a variation \(\delta u_i\) we consider the expressions

\[\delta u_i = \sum_{n,m} \phi_{nm}(\xi_i) \cdot \delta f_{nm}(t). \tag{13}\]

As a function \(\phi_{nm}\) we take the known forms of troughs that correspond to linearized problem.

Substituting (12) and (13) in to (11) we obtain

\[\int \left[ \left( \delta_{ik} + u^0_{ik} \right) T^{0}_{ij} + \frac{\partial u}{\partial \xi_k} u_{ik} + T^{0}_{ij} u_{ik} \right] \phi_{nm,i} \delta f_{nm} \, dV + \rho_R \int \phi_{nm} \delta f_{nm} \, dV = 0. \tag{14}\]

From (14) we obtain the equation (for each time moment, taking into account the arbitrariness of the variations \(\delta f_{nm}(t)\)):

\[\int \left[ \left( \delta_{ik} + u^0_{ik} \right) T^{0}_{ij} + \frac{\partial u}{\partial \xi_k} u_{ik} + T^{0}_{ij} u_{ik} \right] \phi_{nm,i} \delta f_{nm} \, dV + \rho_R \int \phi_{nm} \delta f_{nm} \, dV = 0. \tag{15}\]

It is obvious, that when specifying the perturbation potential the components of stress tensor are expressed in terms of the perturbation of the displacement, and the relation (5) can be represented in the form:

\[S^{-}_{ij}(1) = f_{nm} S^{-}_{(ijnm)}, \quad S^{-}_{ij}(2) = f_{nm} f_{n'm'} S^{-}_{(ijnmn'm')}, \ldots. \tag{16}\]

Here we assume that

\[T = \rho R S. \tag{17}\]
We introduce the notation:

\[ f_{nm} = f^{(1)}; \quad f_{kl} = f^{(2)}; \quad \ldots \quad S_{mn} = S^{(1)}; \quad S_{ijkl} = S^{(2)}; \quad \ldots \]  

(18)

Then, (16) can be rewritten as:

\[ S^{(n)} = f^{(1)} \cdot S^{(1)}; \quad S^{(2)} = f^{(1)} f^{(2)} \cdot S^{(2)}; \quad S^{(3)} = f^{(1)} f^{(2)} f^{(3)} \cdot S^{(3)}. \]  

(19)

In these notations the relation (12) can be written as:

\[ u_t (X_k, t) = f^{(1)} (t) \phi^{(1)} (\xi). \]  

(20)

The components of the strain tensor perturbations in (8) with (12) and (20) can be written as:

\[ E_{mn} (1) = \frac{1}{2} f_{ij} \left[ \phi_{mij,n} + \phi_{nij,m} + u_{k,m} \phi_{klm} + u_{k,m} \phi_{klm} \right]; \quad E_{mn} (2) = \frac{1}{2} f_{ij} f_{nj} \phi_{ij,n} \phi_{ij,n}. \]  

(21)

The time derivative of the strain tensor components will have the form:

\[ \dot{E}_{mn} (1) = \frac{1}{2} f_{ij} \left( \phi_{mij,n} + \phi_{nij,m} \right). \]  

(22)

The expression for the temperature perturbation (7) can also be presented in the form similar to (20):

\[ \theta_t (t, E, \theta, \xi_k) = \theta^{(1)} (t) \psi^{(1)} \left[ E, \theta, \xi_k \right]. \]  

(23)

We have:

\[ \dot{\theta} (t, E, \theta, \xi_k) = \dot{\theta}^{(1)} (t) \psi^{(1)} \left[ E, \theta, \xi_k \right]. \]  

(24)

Substituting (22) and (24) in to (6) we obtain

\[ S_H = \int \left[ G^{(1)} f^{(1)} E^{(1)} + G^{(2)} \theta^{(1)} \psi^{(1)} \right] d \tau', \]  

(25)

where we use the following notations:

\[ E_{ijmn}^{(0)} = \frac{1}{2} \left( \phi_{mij,n} + \phi_{nij,m} \right); \]
\[ E_{ijmn}^{(1)} = \frac{1}{2} \left( \phi_{mij,n} + \phi_{nij,m} + u_{k,m} \phi_{klm} + u_{k,m} \phi_{klm} \right); \]
\[ E_{ijkl}^{(2)} = \frac{1}{2} \phi_{ijkl} \phi_{ijkl}. \]  

(26)

Substituting (19), (25) and (16) in to (15) we obtain

\[ \rho_t \left[ \left( I + \frac{\partial}{\partial t} \right) \left( f^{(1)} S^{(1)} + f^{(2)} (S^{(2)} + \cdots + S_H^{(2)}) + \left( f^{(1)} S^{(1)} + f^{(1)} f^{(2)} S^{(12)} + \cdots + S_H^{(12)} + S_H^{(1)} \right) f^{(1)} H \right) \right] dV \]
\[ + \rho_t \int f^{(1)} \phi^{(1)} \dot{\phi}^{(1)} dV = 0. \]  

(27)
Finally, substituting the expression (25) in (27), we obtain the system of equations

\[
A f^{(1)} + B f^{(1)} + C f^{(1)} + D f^{(1)} + L f^{(1)} + K_1 f^{(1)} + K_2 f^{(1)} + K_3 f^{(1)} + \cdots = 0,
\]

where the coefficients are as follows:

\[
A = \rho A \int f^{(1)}(x) \phi^{(1)}(x) \, dV;
\]

\[
B f^{(1)} = \int f^{(1)}(x) \left[ \left( I + H \right) E^{(0)} H^{(1)} \right] \, dV G^{(2)}(x) \, dx;
\]

\[
D f^{(1)} = \int f^{(1)}(x) \left[ \left( I + H \right) \psi^{(0)} H^{(1)} \right] \, dV G^{(2)}(x) \, dx;
\]

\[
C f^{(1)} = \int f^{(1)}(x) \left[ \left( E^{(0)} H^{(1)} \right)^2 \right] \, dV G^{(2)}(x) \, dx;
\]

\[
L f^{(1)} = \int f^{(1)}(x) \left[ \left( I + H \right) S^{(1)} + S^{(1)} H^{(1)} \right] \, dV G^{(2)}(x) \, dx;
\]

\[
K_1 = \int \left( I + H \right) S^{(1)} + S^{(1)} H^{(1)} \, dV G^{(2)}(x) \, dx;
\]

\[
K_2 = \int \left( I + H \right) S^{(2)} + S^{(2)} H^{(1)} \, dV G^{(2)}(x) \, dx;
\]

\[
K_3 = \int \left( I + H \right) S^{(3)} + S^{(3)} H^{(1)} \, dV G^{(2)}(x) \, dx;
\]

\[
\text{etc.}
\]

The number of terms in the system of equations (28) and the kind of coefficients are specified by the form of elastic potential.

Multiplying the system of equations (28) to the \( f \) (where \( f = \{ f^{(1)}, f^{(2)}, \ldots, f^{(N)} \} \)) we obtain:

\[
A f f + B f f + C f f + D f f + L f f + K_1 f f + K_2 f f + K_3 f f + \cdots = 0.
\]

The obtained relation can be rewritten in the form:

\[
\frac{d}{dt} \left[ \frac{1}{2} A f f + \frac{1}{2} K_1 f f + \frac{1}{3} K_2 f f + \frac{1}{4} K_3 f f + \cdots + \int B f f \, f \, d\tau \right]
\]

\[
+ \int C f f \, f \, d\tau + \int D f f \, f \, d\tau + \int L f f \, f \, d\tau = 0.
\]

We introduce the function
\[
P = \frac{1}{2} A \ddot{f} \dot{f} + \frac{1}{2} K_{1,ff} + \frac{1}{3} K_{2,fff} + \frac{1}{4} K_{3,ffff} + \cdots + \int_{-\infty}^{\infty} B \left( \dot{f} \right) \dot{f} \, d\tau 
+ \int_{-\infty}^{\infty} C \left( \dot{f} \right) \dot{f} \, d\tau + \int_{-\infty}^{\infty} D \left( \dot{\theta} \right) \dot{f} \, d\tau + \int_{-\infty}^{\infty} L \left( \dot{\theta} \right) \dot{f} \, d\tau. \tag{32}
\]

If the function (32) is positive defined for the values of the initial coordinates and velocities that do not exceed the values found from the relations

\[
\left( \frac{\partial P}{\partial f} \right)_0 = 0; \quad \left( \frac{\partial P}{\partial \dot{f}} \right)_0 = 0; \quad \left( \frac{\partial P}{\partial \dot{\theta}} \right)_0 = 0, \tag{33}
\]

then, according to the first Lyapunov stability theorem the homogeneous solution of the system (28) will be stable in this area, because the derivative of the function \( P \) by the time is non-positive.

Let us consider the structure of the function (32). It can be divided into several terms

\[
P = P_1 + P_2 + P_3 + P_4 + P_5, \tag{34}
\]

where

\[
P_1 = \frac{1}{2} A \ddot{f} \dot{f} + \frac{1}{2} K_{1,ff} + \frac{1}{3} K_{2,fff} + \frac{1}{4} K_{3,ffff} + \cdots;
\]

\[
P_2 = \int_{-\infty}^{\infty} B \left( \dot{f} \right) \dot{f} \, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \left( I + H \right) E^{(1)}H^{(1)} \right] \psi \left( \tau_1 \right) f \left( \tau_1 \right) \, d\tau_1 \, d\tau; \tag{36}
\]

\[
P_3 = \int_{-\infty}^{\infty} C \left( \dot{f} \right) \dot{f} \, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \left( I + H \right) E^{(1)}H^{(1)} \right] \psi \left( \tau_1 \right) f \left( \tau_1 \right) \, d\tau_1 \, d\tau; \tag{37}
\]

\[
P_4 = \int_{-\infty}^{\infty} D \left( \dot{\theta} \right) \dot{f} \, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \left( I + H \right) \psi^{(1)}H^{(1)} \right] \psi \left( \tau_1 \right) \dot{\theta} \left( \tau_1 \right) \, d\tau_1 \, d\tau; \tag{38}
\]

\[
P_5 = \int_{-\infty}^{\infty} L \left( \dot{\theta} \right) \dot{f} \, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \psi^{(1)}H^{(1)}H^{(1)} \right] \psi \left( \tau_1 \right) \dot{\theta} \left( \tau_1 \right) f \left( \tau_1 \right) \, d\tau_1 \, d\tau. \tag{39}
\]

The function \( P_1 \) is similar to the corresponding function obtained in [3], [4]. The functions \( P_i \) \((i = 2, 3, 4, 5)\) are the kernel multilinear forms, and the question of their positive definiteness is solved within the positive definiteness of the kernels of these forms. The specific form of the kernels in the (36)–(39) is determined from the experiments. The expressions (36) and (38) are the kernel bilinear forms. If we take the linear dependence relative to perturbations, the relations \( P_3 \) and \( P_5 \) in (37) and (39) can be considered equal to zero.

The equations (33) with respect to the initial perturbation amplitudes are the following:

\[
\left( \frac{\partial P}{\partial f} \right)_0 = A \dot{f} \left( 0 \right) + B \left( \dot{f} \left( 0 \right) \right) \dot{f} \left( 0 \right) + C \left( f \left( 0 \right) \dot{f} \left( 0 \right) \right) + D \left( \dot{\theta} \left( 0 \right) \right) f \left( 0 \right) \dot{f} \left( 0 \right) + \cdots = 0; \tag{40}
\]
\[
\left( \frac{\partial P}{\partial f} \right)_0 = K_1 f(0) + K_2 f(0) f(0) + K_3 f(0) f(0) f(0) + \cdots = 0. \quad (41)
\]

These relations represent a system of algebraic equations for the values of the initial perturbations and their velocities. The solution of (40), (41) gives the area of initial perturbations and velocities, wherein the homogeneous solution of the system (28) is stable.

3. Results and discussion

Thus, we see that the initial perturbation region, for which the zero solution of the system (28) is stable, consistent with the same output for a nonlinear elastic body [3]. For nonlinear elastic body the obtained velocities of initial perturbation equal to zero. For nonlinear viscoelastic body from the equation (40) it can be obtained that there is a region of initial velocities of perturbation, wherein the homogeneous solution is stable (except the limiting case). The system of equations (40), (41) can be solved numerically by setting the initial deformation and fixed strain rate. The difference between the presented approach and the linearized theory is that for given values of initial deformations and fixed strain rates (load history) a nonlinear viscoelastic body can always lose stability if the amplitudes of initial perturbations exceed a certain value. Outside this region the time derivative of the function \( P \) is negative definite and according to Lyapunov’s first theorem on instability, the zero solution of system (28) will be unstable.

It is known that in the phase space of nonlinear dynamical systems the strange attractors arise. The strange attractor is an attracting region characterized by the regime of steady-state non-periodic oscillations [7]. The conditions for the appearance of a strange attractor are a combination of a global compression with a local instability. The regime of the strange attractor is realized only in the dissipative systems and is characterized by the presence of positive indicators in the spectrum of Liapunov’s characteristic exponents. In this case, the attractor is in some region of the phase space and includes the Cantor’s set of hypersurfaces. The limit set of trajectories that correspond to a strange attractor is not a manifold. The set of trajectories corresponding to a strange attractor is characterized by Lyapunov instability, but Poisson stability.

The fractal dimension \( D_e \) of an arbitrary limit set \( G \) in \( N \)-dimensional space can be calculated by the expression [7]:

\[
D_e = \lim_{\varepsilon \to 0} \left\{ \ln M(\varepsilon) \cdot \left[ \ln \left( \varepsilon^{-1} \right) \right]^{-1} \right\}, \quad (42)
\]

where \( M(\varepsilon) \) is the minimal number of \( N \)-dimensional cubes with a side \( \varepsilon \) needed to cover all the elements of the set.

If the spectrum of the Lyapunov exponents \( \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n \) is known, then the concept of Lyapunov dimension can be introduced, namely:

\[
D_e = j + \sum_{i=1}^{n} \lambda_i \left| \frac{\lambda_i}{\lambda_{j1}} \right|,
\]

where \( j \) is the largest number satisfying the condition

\[
L = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_j > 0. \quad (43)
\]

As it follows from the works of Takens [8], the dimension of a strange attractor allows us to quantify the number of phase variables involved in the motion. If this dimension is finite and relatively small, then, in the distributed systems the simulation of the processes is possible within a finite number of ordinary differential equations the number of which corresponds to the dimension of the phase space the strange attractor is embedded in to.
If the sequence of experimental values for the variables is known, then on the basis of this sequence we construct the spaces of dimensions \( m = 2, 3, \ldots \), for which the correlation dimension can be calculated [7], [9], [10]:

\[
\gamma_m = d^m = \lim_{\varepsilon \to 0} \frac{\ln C_m(\varepsilon)}{\ln \varepsilon},
\]

where

\[
C_m(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\varepsilon - |\xi(t_i) - \xi(t_j)|),
\]

where \( \Theta \) is the Heaviside step-function.

In the case when \( \varepsilon \to 0, \ m \to \infty \) we have:

\[
C_m(\varepsilon) \approx \varepsilon^{d_m} \exp(-m\tau L).
\]

The dimension of a strange attractor is defined as follows:

\[
L = \frac{1}{\tau} \ln \frac{C_m}{C_{m+1}},
\]

where \( \tau \) is the time interval between the measurements of neighbor bifurcation points on the trajectory of the dynamical system.

The process of calculation of the correlation dimension continues until the value of \( d^m \) stabilizes for some \( m = m^* \).

According to [8], the dynamic stochasticity can develop after a finite sequence of bifurcations that ensure the achievement of a chaotic regime. For a viscoelastic body many bifurcation points can be obtained, which can be taken as traces of the solution of a strange attractor that has fallen into the region of attraction. Therefore, instead of the sequence of experimental values of \( T \) it is proposed to take a sequence of bifurcation values of \( \{ f_m \} \), for which it is possible to carry out the calculation of the correlation dimension of a strange attractor. This will calculate the dimension of the phase space of the dynamical system, which simulates the processes occurring in the original system. Knowing the dimension of the strange attractor we can confine ourselves to the series of the expansion of displacements by coordinate functions by the number of terms equal to the dimension of the space the strange attractor is embedded in to.

From the expression (46) it is possible to obtain the value of the predictability interval, namely:

\[
T = \frac{1}{L}.
\]

This interval corresponds to the time necessary for the solution of the system of differential equations describing the process of loss of stability of nonlinear elastic and nonlinear viscoelastic media through a bounded sequence of bifurcations to be entered into the regime of a strange attractor.

For the obtained dynamical system using the approach presented in [6], [8] we can obtain the dimension of the strange attractor – \( \gamma_m \), which allows to limit the number of terms in the Bubnov-Galerkin series (12), in which the perturbations of displacements can be decomposed.

As a result, we can obtain the graphical representations of dependencies between the disturbance module and load parameters.

4. Conclusions
Using the approach described above, the stability problems for the concrete structures [4], [6] have been solved. Namely, we obtained the following facts:

- a decrease in the ratio of the dimensions of the structures reduces the stability region both with respect to the initial perturbations and relative to the elongations; the replacement of the compressible material by an incompressible material leads to the same results;

- for all considered cases, the dimension of the strange attractor decreases with increasing of the load parameter.

We also emphasize that the dimension of the strange attractor, found for all the considered problems makes it possible to recommend the number of terms in the Bubnov-Galerkin set in the case when the structure operates in different ranges of variation of the initial stresses and strains.

The use of the stability criterion with respect to finite perturbations allows to obtain a limited sequence of admissible initial perturbations (for a specific value of the load parameter) in which the basic process of deformation will be stable and a hierarchy of stable equilibrium states is observed.

Setting the maximum allowable value of the initial perturbation leads to finding the region of the load parameter variation in which the main process of deformation will be stable.

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