Spatially inhomogeneous and irrotational geometries admitting Intrinsic Conformal Symmetries

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“Diagonal” spatially inhomogeneous (SI) models are introduced under the assumption of the existence of (proper) intrinsic symmetries and can be seen, in some sense, complementary to the Szekeres models. The structure of this class of spacetimes can be regarded as a generalization of the (twist-free) Locally Rotationally Symmetric (LRS) geometries without any global isometry containing, however, these models as special cases. We consider geometries where a six-dimensional algebra $\mathcal{Z}$ of Intrinsic Conformal Vector Fields (ICVFs) exists acting on a 2–dimensional (pseudo)-Riemannian manifold. Its members $X_\alpha$, constituted of 3 Intrinsic Killing Vector Fields (IKVFs) and 3 proper and gradient ICVFs, as well as the specific form of the gravitational field are given explicitly. An interesting consequence, in contrast with the Szekeres models, is the immediate existence of conserved quantities along null geodesics. We check computationally that the magnetic part $H_{ab}$ of the Weyl tensor vanishes whereas the shear $\sigma_{ab}$ and the electric part $E_{ab}$ share a common eigenframe irrespective of the fluid interpretation of the models. A side result is the fact that the spacetimes are foliated by a set of conformally flat 3-dimensional timelike slices when the anisotropy of the flux-free fluid is described only in terms of the 3 principal inhomogeneous “pressures” $p_a$, or equivalently when the Ricci tensor shares the same basis of eigenvectors with $\sigma_{ab}$ and $E_{ab}$. The conformal flatness also indicates that a 10-dimensional algebra of ICVFs $\Xi$ acting on the 3–dimensional timelike slices is highly possible to exist enriching in that way the set of conserved quantities admitted by the SI models found in the present paper.

I. INTRODUCTION

The inspection of the Einstein’s Field Equations (EFEs)

$$G^a_b \equiv R^a_b - \frac{1}{2} R \delta^a_b = T^a_b$$

reveals the rich and strong correlation between the geometry of spacetime and the dynamics. The latter is primarily encoded to a realistic Energy-Momentum (EM) tensor $T_{ab}$. However, even if we assume that the spacetime does not contain any dynamical fields, then $g_{ab}(x^c)$ becomes itself a dynamical variable showing the complexity that arises from this duality. It is thus evident that any intention to simplify $g_{ab}(x^c)$ with some kind of symmetry must take into account the fusion between the gravitational field and the spacetime geometry.

On the other hand observable quantities necessitate the existence of a unit timelike vector field $u^a$ representing an average velocity [2] and its kinematical quantities $\theta$ (volume expansion scalar), $\sigma_{ab}$ (anisotropic expansion trace-free tensor), $\omega_{ba}$ (congruence’s twist tensor), $\dot{u}^a$ (non-geodesic indication 1-form) describe the distortion of the integral curves of $u^a$ as measured in the rest space of a comoving observer

$$\theta \equiv u_{ab} h^{ab}, \quad \sigma_{ab} \equiv u_{(cd)} \left( h^c_a h^d_b - \frac{1}{3} h^{cd} h_{ab} \right)$$

$$\dot{u}_a \equiv u_{ab} u^b, \quad \omega_{ba} \equiv u_{[cd]} h^c_a h^d_b.$$  (2)

where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor normally to $u^a$. In the generic case there are no a priori reasons to impose special features on the timelike congruence and only the interplay of physics (plus observations) and geometry with the inclusion of appropriate boundary data (at spatial or null past/future infinity) should enforce the need of such characteristics.

The third constituent element in this “arena” is the presence of a matter fluid which is described in terms of the geometry and the kinematics as

$$T^a_b = \rho u^a u_b + p h^a_b + q^a u_b + u^a q_b + \pi^a_b$$  (3)

where $\rho, p$ are the energy density and the isotropic pressure respectively, $q^a$ is the direction of the momentum flow and $\pi_{ab}$ is the anisotropic and trace-free pressure tensor

$$\rho \equiv T_{ab} u^a u^b \quad p \equiv \frac{1}{3} T_{ab} h^{ab}$$

$$q_a \equiv -h^c_a T_{cd} u^d, \quad \pi_{ab} \equiv \left( h^c_a h^d_b - \frac{1}{3} h^{cd} h_{ab} \right) T_{cd}.$$  (4)

Each of the above dynamical components (must) has a phenomenologically sound meaning [2] that can be justified from observations in some acceptable cosmological
scale. It should be noticed that the choice of the observer is not unique and can be chosen either comoving \( u^a \) or non-comoving \( \tilde{u}^a (\neq u^a) \) in which case the interpretation for each one should be completely different leading to the notion of tilted models [3].

Spatially Inhomogeneous (SI) models [4] provide a significant work field towards our understanding of the structure formation and the effect of local density and pressure fluctuations in the accelerated phase of the Universe. It is clear that they represent not an alternative of the linearized version of the perturbed Friedmann-Lemaître-Robertson-Walker (FLRW) models but exact perturbation solutions within a homogeneous and isotropic background. Although, up to date, a quite generic SI model without special characteristics (in the sense that will become transparent in the next sections) has not been found, the known exact SI solutions can be served, however, as toy models to various directions [3].

Szekeres solution [6] was the first SI model without flatness of the 3-dimensional slices \( t = \text{const.} \) which, geometrically, could be the reminiscence of the constant curvature of the 2-dimensional hypersurfaces \( t, r = \text{const.} \) and the subsequent existence of a 6-dimensional algebra of Intrinsic Conformal Vector Fields (ICVF) X satisfying [10]

\[
p^a p^b \mathcal{L}_X p_{cd} = 2 \phi(X) p_{ab} \tag{10}
\]

where \( p_{ab} = h_{ab} - x_a x_b \) is the projection tensor normal to the pair \( \{u^a, x^a\} \) and, given the structure of [10] or [17], represents the induced metric of the 2-dimensional manifold \( u \wedge x = 0 \).

The key feature of the family [11] or [12] is the conformal flatness of the 3-dimensional slices \( t = \text{const.} \) which, essentially equation [8] is true for the general diagonal metric \( (u^a = C^{-1} \delta^a_t) \)

\[
ds^2 = A^2 dt^2 + B^2 dz^2 - C^2 dt^2 + D^2 dy^2 = g_{ab} dx^a dx^b \tag{9}
\]

therefore it can be seen entirely as an “artifact” of the specific geometrical character of the tetrad \( \{u^a, x^a, y^a, z^a\} \) irrespective of further dynamical restrictions. Spacetimes that satisfy equation [8] are usually referred as purely “electrical” and a lot of work has been done regarding the dynamical structure and the existence of perfect fluid models (see e.g. [12] and references cited therein) with vanishing \( H_{ab} \). The analysis is focused mainly to perfect fluids with a barotropic equation of state \( p = p(\rho) \) or rotational dust (geodesic) models.

In addition the unit timelike vector field \( u^a \) is geodesic, consistent with a dust fluid content (7) provides a generalization of the Szekeres spacetime with \( p \neq 0 \) which results to the Szekeres family of quasi-symmetric models [9, 10].
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tion way and study their consequences in the kinematics and dynamics of the corresponding model. The fact that Szekeres models admit (proper) ICVFs acting on 2-dimensional (and possibly 3-dimensional) submanifolds shows that ICVFs could be more relevant and impose much less restrictions than the full CVFs-models which are very rare [4].

The purpose of the present paper is to extent the investigation of the existence of ICVFs to spacetimes with metric [19] thus providing a some kind of geometrical classification with respect to the intrinsic conformal algebra *without assuming* any matter content thus providing a much richer diversity of possible physically sound models than those that have been reported so far [13]. In particular, in Section II we assume that a 6-dimensional algebra of ICVFs exists, acting on the timelike distribution \( \mathbf{x} \times \mathbf{z} = 0 \) which implies that the latter has constant curvature and the resulting spacetimes can be referred to as quasi-symmetric. We give the explicit form of the ICVFs and the associated spacetime metrics and show computationally that the magnetic part \( H_{ab} \) of the Weyl tensor vanishes whereas the shear \( \sigma_{ab} \) and the electric part \( E_{ab} = C_{aabcd}e^a e^d \) share a common eigenframe irrespective of the fluid interpretation of the models. Furthermore non-tilted perfect fluids (where, in general, \( p \) and \( \rho \) do not satisfy a barotropic equation of state) cannot be excluded at once since the \( H \)-divergence constraint is trivially satisfied. Two interesting results then arise: in contrast with the Szekeres models, there exist *infinite conserved quantities* along null geodesics. Furthermore the hypersurfaces \( x = \text{const.} \) are *conformally flat* when the fluid is flux-free \( q^a = 0 \) and its anisotropy is described only in terms of the 3 principal inhomogeneous "pressures" \( p_a \) or, equivalently, when the Einstein tensor \( G^a_b \) is "diagonal". One should expect the existence of 10-dimensional algebra of ICVFs \( \Xi \) of the \( x \perp \) --distribution that satisfy

\[
\hat{h}^a_c \hat{h}^b_d \mathcal{L} \hat{h}_{cd} = 2\phi(\Xi) \hat{h}_{ab}
\]

where \( \hat{h}_{ab} = g_{ab} - x_a x_b \) is regarded as the induced metric of \( \mathbf{x} \perp \). In section III, for completeness, we also give the 6-dimensional algebra of ICVFs acting on the \( \mathbf{x} \times \mathbf{y} = 0 \) spacelike distribution when \( a^a \neq 0 = x^a x^b \). As expected, the \( x \)-slices are also conformally flat provided that \( T^a_b = \text{diag}(\rho, p_1, p_2, p_3) \). Section IV includes our conclusions and further areas of research.

Throughout this paper, the following conventions have been used: the spacetime manifold is endowed with a Lorentzian metric of signature \( (- + + +) \), spacetime indices are denoted by lower case Latin letters \( a, b, \ldots = 0, 1, 2, 3 \), spatial frame indices are denoted by lower case Greek letters \( \alpha, \beta, \ldots = 1, 2, 3 \) and we have used geometrized units such that \( 8\pi G = 1 = c \).

**II. SPATIALLY INHOMOGENEOUS AND IRROTATIONAL MODELS OF TYPE II**

We consider a spacetime geometry where a unit timelike vector field \( u^a \) is twist-free \( \omega_{ab} = 0 \) but *non-geodesic* \( a^a \neq 0 \). We make the assumption that there exist 3 independent spacelike unit vector fields \( \{ \mathbf{x}, \mathbf{y}, \mathbf{z} \} \) normal to \( u^a \), and each of these has the property to be hypersurface orthogonal

\[
x_{[a x_{bc}]} = y_{[a y_{bc}]} = z_{[a z_{bc}]} = 0.
\]

The unit spacelike vector field \( x^a \) is taken to be geodesic i.e. \( (x_a)^* \equiv x_a x^b = 0 \) and the pairs \( \{ u^a, x^a \} \), \( \{ u^a, y^a \} \), \( \{ u^a, z^a \} \) are surface forming satisfying eq. [3].

Under these conditions the most general metric adapted to the geodesic coordinates of \( x^a \) has the following form

\[
ds^2 = g_{ab} dx^a dx^b = dx^2 + B^2 dz^2 - C^2 dt^2 + D^2 dy^2
\]

where the functions \( B(t, x, y, z) \), \( C(t, x, y, z) \) and \( D(t, x, y, z) \) depend on all four coordinates. It follows from \([13]\) that the magnetic part of the Weyl tensor w.r.t. \( u^a \) vanishes \( H_{ab} = 0 \) and, in general, the Petrov type is I that is \( E_{ab} = \text{diag}(0, E_1, E_2, E_3) \).

Essentially, the induced metric of the distribution \( \mathbf{x} \times \mathbf{z} = 0 \) is represented by the second order symmetric tensor \( p_{ab} \equiv g_{ab} - x_a x_b - z_a z_b \) where \( p_{ab}^k x^k = 0 = p^k_a z_k \). We assume that there exist a 6-dimensional algebra \( IC(\mathbf{X}_A) \)

\[
(A = 1, \ldots, 6) \text{ of ICVFs acting on } 2d \text{ pseudo-Riemannian manifold that obey}
\]

\[
p^k_a p^l_b \mathcal{L} \mathbf{x}_{ik} = p^k_a p^l_b \mathcal{L} \mathbf{x}_{lj} \equiv \nabla_v (\rho \mathbf{X}_a) = 2\phi(\mathbf{X}) p_{ab}
\]

where \( \phi(\mathbf{X}_A) \) are the conformal factors of the vectors \( \mathbf{X}_A \) that are lying and acting on the submanifold \( \mathbf{x} \times \mathbf{z} = 0 \) and \( \nabla_a \) represents a well defined covariant derivative

\[
\nabla_v p_{ab} = p^k_c p^l_d \nabla_k p_{lj} = 0
\]

for any tensorial quantity

\[
\nabla_v \Pi_{ab} = p^k_c p^l_d \Pi_{kj}.
\]

From the inspection of equations [14] it follows that \( C = D \) and the general solution shows that \( \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \) are *Intrinsic Killing Vector Fields* (IKVFs) and \( \mathbf{X}_4, \mathbf{X}_5, \mathbf{X}_6 \) are *proper and gradient* ICVFs i.e. their associated bivectors vanish identically \( \nabla_v (\rho \mathbf{X}_a) = 0 \)

\[
\mathbf{X}_1 = M_{gt} = (y - Y) \partial_t + (t - T) \partial_y
\]

\[
\mathbf{X}_2 = \left\{ \frac{k}{4} \left[ (y - Y)^2 + (t - T)^2 \right] - 1 \right\} \partial_t +
\]

\[
+ \frac{k}{2} (y - Y) (t - T) \partial_y
\]

(17)
\[ X_3 = \frac{k}{2} (y - Y)(t - T) \partial_t + \]
\[ \left\{ 1 + \frac{k}{4} \left[ (y - Y)^2 + (t - T)^2 \right] \right\} \partial_y \]
\[ X_4 = H = (t - T) \partial_t + (y - Y) \partial_y \]
\[ X_5 = \left\{ \frac{k}{4} \left[ (t - T)^2 + (y - Y)^2 \right] + 1 \right\} \partial_t + \]
\[ \left\{ \frac{k}{4} \left[ (y - Y)^2 + (t - T)^2 \right] - 1 \right\} \partial_y. \]

with associated conformal factors
\[ \phi(X_1) = \phi(X_2) = \phi(X_3) = 0 \]
\[ \phi(X_4) = \left\{ 1 - \frac{k}{4} \left[ (y - Y)^2 - (t - T)^2 \right] \right\} N \]
\[ \phi(X_5) = kN(t - T), \quad \phi(X_6) = kN(y - Y). \]

Consequently the 2d manifold \( x \wedge z = 0 \) has (locally) constant curvature and the metric \((13)\) takes the form
\[ ds^2 = dx^2 + B^2 dz^2 + \frac{S^2}{V^2} \left\{ 1 + \frac{\epsilon}{4V^2} \left[ (y - Y)^2 - (t - T)^2 \right] \right\}^2 \]

where \( S(x, z), Y(z), T(z), V(z) \) are arbitrary functions of their arguments and \( \epsilon = \pm 1 \) \((\neq 0)\) corresponds to the constant curvature of the hypersurfaces \( x, z = \text{const.} \)

Defining the function \( E(t, y, z) \) according to \((k = \epsilon/V^2)\)
\[ E(t, y, z) = V \left\{ 1 + \frac{k}{4} \left[ (y - Y)^2 - (t - T)^2 \right] \right\} \]
then
\[ N(t, y, z) = \frac{1}{E(t, y, z)} \]
and the metric becomes
\[ ds^2 = dx^2 + B^2 dz^2 + \frac{S^2}{E^2} (-dt^2 + dy^2). \]

The case where the distribution \( x \wedge z = 0 \) has zero curvature is treated similarly. The ICVF's are
\[ X_1 = M_{yy} = (y - Y) \partial_t + (t - T) \partial_y \]
\[ X_2 = \left[ (y - Y)^2 + (t - T)^2 \right] \partial_t + \]
\[ + 2 (y - Y) (t - T) \partial_y \]
\[ X_3 = 2 (y - Y) (t - T) \partial_t + \]
\[ + \left[ (y - Y)^2 + (t - T)^2 \right] \partial_y \]
\[ X_4 = H = (t - T) \partial_t + (y - Y) \partial_y \]
\[ X_5 = \partial_t, \quad X_6 = \partial_y \]

with conformal factors
\[ \phi(X_1) = \phi(X_2) = \phi(X_3) = 0 \]
\[ \phi(X_4) = -1 \]
\[ \phi(X_5) = \frac{2(t - T)}{(y - Y)^2 - (t - T)^2} \]
\[ \phi(X_6) = \frac{2(y - Y)}{(t - T)^2 - (y - Y)^2} \]

and the metric function \( E(t, y, z) \) is given by
\[ E(t, y, z) = \frac{1}{N(t, y, z)} = \frac{1}{4} \left[ (y - Y)^2 - (t - T)^2 \right]. \]

A potential application of the IC algebra \( \mathcal{ZC}(X_A) \) found in the present section could be the existence of conserved currents and quantities. For example consider a null geodesic vector field \( l^a \) lying in the 2d manifold \( x \wedge z = 0 \) and the quantities \( Q_A = l^a X_A(\alpha) \). It is easy to see that \( Q_A \) are conserved along the null geodesics since
\[ [Q_A]_{,a} l^a = l^a_{,a} X_A(\alpha) + l^a l^b X_A(\alpha)_{,b} = 0. \]

For the metric \((28)\) a null geodesic vector field is \( l^a = f(u^a + y^a) = fn^a \) where \( f(x^a) \) satisfies \( f(t)_{,b} = -f n^a_{,b} n^k \) (we note that \( n^a = u^a + y^a \) is not geodesic for a generic form of \((28)\)).

In the search for fluid solutions we usually start by analyzing the structure of the constraints of the EFESs \((11)\). The "temporal" constraint \( G^0_{,\alpha} = 0 \) for the metric \((28)\) reduce to
\[ SB_{tx} - B_{,t} S_{,x} = 0 \]

where \( B \) is the \( 1 \) form.
\[ B_{,y}E_{,t} + B_{,t}E_{,y} + EB_{,ty} = 0 \quad (41) \]
\[ BS(EE_{,zt} - E_{,t}E_{,z}) + EB_{,t}(E_{,S}z - SE_{,z}) = 0 \quad (42) \]
whereas the “spatial” constraints \( G^{a}_{\beta} = 0 \) have the form
\[ B_{,y}S_{,x} - SB_{,yx} = 0 \quad (43) \]
\[ B(ES_{,zx} - S_{,z}E_{,x}) + B_{,x}(SE_{,z} - ES_{,z}) = 0 \quad (44) \]
\[ BS(EE_{,yz} - E_{,y}E_{,z}) + EB_{,y}(E_{,S}z - SE_{,z}) = 0 \quad (45) \]
where a “\( \partial \)” denotes partial differentiation w.r.t. the corresponding coordinate.

The general solution of the above set of coupled differential equations is
\[ B = \frac{S[\ln (S/E)]_{,z}}{\sqrt{\varepsilon + F(z)}} \quad (46) \]
where \( F(z) \) is an arbitrary function and \( E(t, y, z) \) is given in \[20\] or \[38\].

It should be emphasized that the existence of the \( J\mathcal{C}(X_{A}) \) intrinsic conformal algebra is a direct consequence of the general solution \[10\], \[20\] or \[38\] therefore in order to determine the exact form of \( X_{A} \) we could simply apply the methodology of \[10\] avoiding in that way eqs. \[14\]. Furthermore we can verify that the Petrov type is D i.e. the eigenvalues of the electric part of the Weyl tensor \( E_{1} = E_{2} \) (in contrast with the Szekeres models where \( E_{2} = E_{3} \)).

The EFEs \[11\] then become
\[ G^{a}_{b} = T^{a}_{b} = \text{diag}(p, p_{1}, p_{2}, p_{3}) \quad (47) \]
i.e. the Ricci tensor \( R^{a}_{b} \) shares the same basis of eigenvectors with \( \sigma_{ab} \) and \( E_{ab} \).

The directional and inhomogeneous “pressures” \( p_{a} \) are not necessarily equal and the fluid is, in general, anisotropic for the comoving observers \( u^{a} = (E/S)\partial^{a}_{t} \). In order to show if a specific perfect fluid solution exists (i.e. \( p_{1} = p_{2} = p_{3} \)) one must monitor the integrability conditions i.e. the consistent evolution of the non-trivial constraints. We can prove, however, that the div\( - H \) constraint is trivially satisfied. We observe computationally that the three mutually orthogonal and unit spacelike vector fields \( \{x^{a}, y^{a}, z^{a}\} \) are eigenvectors of \( E_{ab} = \text{diag}(0, E_{1}, E_{2}, E_{1}) \) and \( \sigma_{ab} = \text{diag}(0, \sigma_{3}, \sigma_{2}, \sigma_{3}) \).

Because \( H_{ab} \) vanishes identically for the metric \[28\] the further requirement \( p_{1} = p_{2} = p_{3} \) gives \( \pi_{ab} = 0 \) and the \( H \)−divergence equation \[11\]
\[ \epsilon^{\alpha\beta\gamma}\sigma_{\beta\delta}E_{\delta}^{\gamma} = 0 \]
implies that the shear \( \sigma_{ab} \) and the electric part \( E_{ab} \) tensors commute i.e. they must share a common eigenframe as actually do.

An important consequence of the solution \[20\] or \[38\] and \[46\] is that the Cotton-York tensor \[19\], \[20\]
\[ C_{abc} = 2(R_{a[b} - \frac{1}{4}Rg_{[b|c]} \quad (48) \]
vanishes i.e. the hypersurfaces \( x = \text{const.} \) are conformally flat. Therefore in complete analogy with the 2d case, one should expect the existence of 10-dimensional algebra \( \mathcal{E} \) of the \( x_{1} \)−distribution that satisfy
\[ \hat{h}_{a}^{b}[\hat{h}_{d}^{c}L_{\Xi}^{e}h_{cd} = 2\phi(\Xi)\hat{h}_{ab} \quad (49) \]
where \( \hat{h}_{ab} = g_{ab} - x_{a}x_{b} \) is regarded as the induced metric of \( x_{1} \).

We note that relaxing the flux-free restrictions (equations \[11\], \[12\]), exact perfect fluid models could be exist for non-comoving (tilted) observers \( \tilde{u}^{a} \) similar to the case of Spatially Homogeneous (SH) tilted perfect models (e.g. \[21\], \[25\]) which necessitates the presence of non-zero vorticity \[24\]. This could be also possible for the Szekeres geometries i.e. when \( \tilde{u}^{a} \) are comoving with the (perfect) fluid in which case the \( u^{a} \)−observers will interpret it as imperfect\(^2\). Again, if such a solution exists, it must be proved that evolves consistently along \( u^{a} \) that “see” an anisotropic and non-zero flux matter fluid.

### III. Spatially Inhomogeneous and Irrotational Models of Type III

We are interested to the case where the induced metric of the distribution \( x \wedge u = 0 \), represented by the second order symmetric tensor \( p_{ab} \equiv g_{ab} - x_{a}x_{b} + u_{a}u_{b} \) where \( p_{a}^{k}x_{k} = 0 = p_{a}^{k}u_{k} \), admits the 6-dimensional algebra \( J\mathcal{C}(X_{A}) \quad (A = 1, ..., 6) \) of ICVF's \( p_{a}^{c}p_{d}^{e}L_{x}p_{cd} = 2\phi(X)p_{ab} \quad (50) \)

\[ x_{1} = M_{yz} \quad (51) \]
\[ x_{2} = \left\{ 1 + \frac{k}{4} \left[ (y - Y)^{2} - (z - Z)^{2} \right] \right\} \partial_{y} + \frac{k}{2} (y - Y)(z - Z) \partial_{z} \quad (52) \]
\[ x_{3} = \frac{k}{2} (y - Y)(z - Z) \partial_{y} + \left\{ 1 + \frac{k}{4} \left[ (z - Z)^{2} - (y - Y)^{2} \right] \right\} \partial_{z} \quad (53) \]

\(^2\) In \[23\] the “environment” is completely different since the comoving interpretation remains that of a perfect fluid (i.e. the exact Szekeres model) and the tilted observers are derived from a Lorentz boost of \( u^{a} \).
\[ X_4 = \mathbf{H} = (y - Y) \partial_y + (z - Z) \partial_z \] (54)
\[ X_5 = \left\{ \frac{k}{4} \left[ (y - Y)^2 - (z - Z)^2 \right] - 1 \right\} \partial_y + \frac{k}{2} (y - Y) (z - Z) \partial_z \] (55)
\[ X_6 = \frac{k}{2} (y - Y) (z - Z) \partial_y + \left\{ \frac{k}{4} \left[ (z - Z)^2 - (y - Y)^2 \right] - 1 \right\} \partial_z. \] (56)

The \( X_4, X_5, X_6 \) are proper and gradient ICVFs and the conformal factors are given by
\[ \phi(X_1) = \phi(X_2) = \phi(X_3) = \phi(X_4) = 0 \] (57)
\[ \phi(X_5) = kN(y - Y), \quad \phi(X_6) = kN(z - Z). \] (59)

The 2d manifold \( \mathbf{x} \wedge \mathbf{u} = 0 \) is of constant curvature and the metric (13) is
\[ ds^2 = dx^2 - C^2 dt^2 + \frac{S^2}{V^2} \left\{ \frac{dy^2 + dz^2}{\left[ (y - Y)^2 + (z - Z)^2 \right]^2} \right\} \] (60)

where \( S(t, x) \) and \( Y(t), Z(t), V(t) \) are now arbitrary functions of \( t \) and \( \epsilon = \pm 1 \) (\( \neq 0 \)) corresponds to the constant curvature of the hypersurfaces \( x, t = \text{const.} \)

Similarly with type II we define the function \( E(t, y, z) \) according to \( (k = \epsilon / V^2) \)

\[ E(t, y, z) = V \left\{ 1 + \frac{k}{4} \left[ (y - Y)^2 + (z - Z)^2 \right] \right\} \] (61)

with
\[ N(t, y, z) = \frac{1}{E(t, y, z)} \] (62)

and the metric becomes
\[ ds^2 = dx^2 - C^2 dt^2 + \frac{S^2}{E^2} (dy^2 + dz^2). \] (63)

For completeness we give the corresponding expressions for the ICVFs and the metric for the case where the curvature of \( \mathbf{x} \wedge \mathbf{u} = 0 \) vanishes
\[ X_1 = \mathbf{M}_{yz} = (z - Z)\partial_y - (y - Y)\partial_z \] (64)
\[ X_2 = \left[ (y - Y)^2 - (z - Z)^2 \right] \partial_y + 2 (y - Y) (z - Z) \partial_z \] (65)
\[ X_3 = 2 (y - Y) (z - Z) \partial_y + \left[ (z - Z)^2 - (y - Y)^2 \right] \partial_z \] (66)
\[ X_4 = \mathbf{H} = (y - Y) \partial_y + (z - Z) \partial_z \] (67)
\[ X_5 = \partial_y, \quad X_6 = \partial_z. \] (68)

The conformal factors are
\[ \phi(X_1) = \phi(X_2) = \phi(X_3) = 0 \] (69)
\[ \phi(X_4) = -1 \] (70)
\[ \phi(X_5) = - \frac{2(y - Y)}{(y - Y)^2 + (z - Z)^2} \] (71)
\[ \phi(X_6) = - \frac{2(z - Z)}{(y - Y)^2 + (z - Z)^2} \] (72)

and the metric function \( E(t, y, z) \) assumes the form
\[ E(t, y, z) = \frac{1}{N(t, y, z)} = \frac{1}{4} \left[ (y - Y)^2 + (z - Z)^2 \right]. \] (73)

In contrast with the previous case, the spacetime (25) does not allow the existence of conserved currents and quantities constructed from null vector fields “living” in \( \mathbf{x} \wedge \mathbf{u} = 0 \) due to the positive-definite character of the quasi-symmetric 2d metric \( p_{ab} \). However we can check for a flux-free solution which implies the “temporal” constraints \( G^0_{\alpha} = 0 \)

\[ C (E S_{tx} - S_x E_t) + C_z (SE_{t} - ES_t) = 0 \] (74)
\[ CS (E E_{ty} - E_t E_{ty}) + EC_y (E S_t - SE_t) = 0. \] (75)
\[ CS (E E_{zt} - E_z E_t) + EC_z (E S_t - SE_t) = 0 \] (76)

and the associated “spatial” constraints \( G^\alpha_\beta = 0 \)

\[ SC_{yz} - C_y S_{xz} = 0 \] (77)
\[ SC_{zx} - C_z S_{xy} = 0 \] (78)
\[ C_y E_{z} + C_z E_{y} + EC_{yz} = 0. \] (79)
We can verify that the general solution of (74)-(79) is
\[ C = \frac{S [\ln (S/E)]_t}{\sqrt{t + F(t)}} \]  
(80)
with \( F(t) \) an arbitrary function and \( E(t, y, z) \) is given in (61) or (73). It becomes evident that also in this type, the existence of the \( IC(X_A) \) intrinsic conformal algebra is a direct consequence of the general solution (60), (61) or (73).

Using the same arguments, the directional “pressures” \( p_a \) are, in general, not equal and the fluid is anisotropic. This, however, does not exclude a priori a perfect fluid (not tilted) solution once the consistency of the integrability conditions is established. In addition, the existence of the general solution of (74)-(79) is equivalent with the fact that the \( x \)-slices are conformally flat and timelike which indicates a 10-dimensional algebra of ICVFs \( \mathcal{A} \) which will give rise to conserved quantities along null geodesics originated from the ICVFs admitted by the 2d submanifold (in type II) or the \( x_1 \)-submanifold (in types II and III). It could be therefore enlightening the determination of the 10-dimensional algebra of ICVFs due to the emerged conformal flatness of the hypersurfaces \( x = \text{const} \). when the fluid is flux-free \( q^a = 0 \) and its anisotropy is described only in terms of the 3 principal inhomogeneous “pressures” \( p_a \) (or equivalently when the Einstein tensor \( G^a_b \) is diagonal).

As we have seen, a perfect fluid solution was not excluded a priori. In this direction it would be interesting to allow the inclusion of a cosmological constant \( \Lambda \) similar to the case of the Szekeres models (27) or the Petrov type I silent universes (28) where exact solutions has been shown to exist. Furthermore the models of type II (28), (46) with (40) and (35) or type III (33), (50) with (41) and (73) could be also relevant of studying the effect of small anisotropic and inhomogeneous “pressures” to the expansion dynamics either as the relic of various physical sources (29) or as the result of backreaction terms of the density fluctuations (30, 31) provided that the use of purely phenomenological laws governing the appearance of “pressures” is consistent with the kinetic theory approach of the fluid thermodynamics.

Relaxing the flux-free restrictions (equations (40)-(42) or (74)-(76)) opens the possibility that exact tilted perfect fluid solutions could be found for the spacetimes presented in this paper. Unlike the symmetric Lemaître-Tolman-Bondi (LTB) subclass (32) or the Lorentzian plane/hyperbolic analogues where a tilted (twisted) perfect fluid solution cannot exist (28) (due to the locally rotational symmetry), it is far from obvious that the intrinsic locally rotational symmetry induced from the ICVF could be strong enough to forbid a non-moving perfect fluid interpretation.

We emphasize that every attempt to assign a dynamical (vacuum or non-vacuum) interpretation to the spacetimes presented in this paper must take into account the induced (non-symmetry) integrability conditions. This can be done by examining whether a suitable set of initial data evolves consistently which is equivalent to demand that the constraints (spatial divergence and curl equations encoded in the set of the initial data), are consistent with the evolution equations hence, they are preserved identically along the timelike congruence \( u^a \) without imposing new geometrical, kinematical or dynamical restrictions (11-12). Therefore it is necessary to formulate covariantly the necessary and sufficient conditions, coming from the existence of the symmetry, and study their consequences in the dynamics. All the above we believe that are physically sound and require further investigation.

\[ \text{IV. CONCLUSIONS} \]

It should be noticed that the existence of the 6-dimensional algebra of ICVFs acting on 2d manifolds is independent from the geodesic assumption of the unit spacelike vector field \( x^a \) and the form of the metrics (28), (43) is altered only by an arbitrary function in the \( g_{xx} \)--component with a subsequent change in the dynamics. As such, the structure of the class of spacetimes presented in this paper can be regarded as a generalization of the (irrotational) Locally Rotationally Symmetric (LRS) geometries without any global isometry containing, however, these models as special cases (26).

An interesting aspect of the analysis of the preceding sections is the existence of infinite conserved quantities along null geodesics originated from the ICVFs admitted by the 2d submanifold (in type II) or the \( x_1 \)-submanifold (in types II and III). It could be therefore enlightening the determination of the 10-dimensional algebra of ICVFs due to the emerged conformal flatness of the hypersurfaces \( x = \text{const} \). when the fluid is flux-free \( q^a = 0 \) and its anisotropy is described only in terms of the 3 principal inhomogeneous “pressures” \( p_a \) (or equivalently when the Einstein tensor \( G^a_b \) is diagonal).

As we have seen, a perfect fluid solution was not excluded a priori. In this direction it would be interesting to allow the inclusion of a cosmological constant \( \Lambda \) similar to the case of the Szekeres models (27) or the Petrov type I silent universes (28) where exact solutions has been shown to exist. Furthermore the models of type II (28), (46) with (40) and (35) or type III (33), (50) with (41) and (73) could be also relevant of studying the effect of small anisotropic and inhomogeneous “pressures” to the expansion dynamics either as the relic of various physical sources (29) or as the result of backreaction terms of the density fluctuations (30, 31) provided that the use of purely phenomenological laws governing the appearance of “pressures” is consistent with the kinetic theory approach of the fluid thermodynamics.

Relaxing the flux-free restrictions (equations (40)-(42) or (74)-(76)) opens the possibility that exact tilted perfect fluid solutions could be found for the spacetimes presented in this paper. Unlike the symmetric Lemaître-Tolman-Bondi (LTB) subclass (32) or the Lorentzian plane/hyperbolic analogues where a tilted (twisted) perfect fluid solution cannot exist (28) (due to the locally rotational symmetry), it is far from obvious that the intrinsic locally rotational symmetry induced from the ICVF could be strong enough to forbid a non-moving perfect fluid interpretation.

We emphasize that every attempt to assign a dynamical (vacuum or non-vacuum) interpretation to the spacetimes presented in this paper must take into account the induced (non-symmetry) integrability conditions. This can be done by examining whether a suitable set of initial data evolves consistently which is equivalent to demand that the constraints (spatial divergence and curl equations encoded in the set of the initial data), are consistent with the evolution equations hence, they are preserved identically along the timelike congruence \( u^a \) without imposing new geometrical, kinematical or dynamical restrictions (11-12). Therefore it is necessary to formulate covariantly the necessary and sufficient conditions, coming from the existence of the symmetry, and study their consequences in the dynamics. All the above we believe that are physically sound and require further investigation.

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