PHYSICAL MODELS OF SOLID MASS AND RELATED PROCESSES IN INTERACTION WITH FOUNDATIONS
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ARTICLE DETAILS
ABSTRACT

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In the article, interaction of foundations with foundation beds is described and represented on the basis of a general theory of mathematical modeling of large systems in performing three mathematical spaces: qualitative characteristics; discrete parameters (DP); continuous parameters (CP). This representation was made for the first time in comparison with the known works devoted to the theory of interaction of foundations with foundation beds. We have developed the theory of mathematical modeling of large systems, including models of their interaction, of physical processes in the interaction and optimization mathematical models based on the vector criterion in performing three mathematical spaces. They are widely used in the field of mining sciences. We have proved that the issue of interaction of foundations and foundation beds is multidimensional not only in terms of physical processes, described and systematized, but also due to the variety of foundations in the multidimensional space of their qualitative characteristics. This allowed us to cover all foundation beds and their properties. This article provides all well-known models of solid deformable body behavior presented in a systematized form: elastic deformable body, viscoelastic deformable body, models of plastic and flowing medium, models of fluid motion and gas flow, models of thermal conductivity and heat and mass exchange models. We have presented mathematical models of physical processes occurring in a rock mass in the interaction of foundations with foundation beds.

KEYWORDS
Mathematical models, large system, elastic body, physical processes, solid deformable body

1. INTRODUCTION

There are no comprehensive generalizing works on the interaction of foundation with foundation bed (rock formation) have been found that systematize and evaluate the study of this issue. In known sources, the necessity of considering this complex system from the position of mathematical modeling of large systems (LS) is not mentioned [1-3]. In terms of LS, parameters and characteristics are necessary to determine in performing three mathematical spaces [4, 5] for studied systems: artificial system – foundation ( ); natural environment – foundation bed (rock formation); . Namely: qualitative characteristics that do not have a reference scale; discrete parameters (DP); Continuous parameters (CP).

Introduction of these three mathematical spaces requires the use of two concepts: a point and trajectory of their behavior under or in performing these spaces. Disorder and intuitive consideration of characteristics has led researchers to explicit and implicit contradictions [6-8]. For example, this applies to mathematical and physical models for determining the interaction parameters of and systems.

This is especially typical for a random research on a huge variety of foundations, which characteristics have not been formalized in certain works [9-11]. Mutual interaction considered in a deterministic form is the second major drawback of the known works. Although, interaction of these two systems was proven to occur over the overlapping parameters, which are random variables and even functions that depend on time (t). Mathematical modeling of LS allows eliminating these two major drawbacks. Thus, the present solution will contribute to more accurate description and, consequently, to the creation of a real situation. This article substantiates the way of overcoming the first drawback. In subsequent works, the possibility of eliminating the second drawback will also be considered.

2. MATERIALS AND METHOD

There are only three methods for correct and effective study of any system. The first method is a direct experiment on the system in real time. It is not suitable for such LS as the interaction of foundations with foundation beds, since such LS has only one trajectory of behavior. Consequently, we cannot compare the results of such a method, even if they are theoretically possible. There is no possibility to rebuild the structure with other parameters if it was built once. The second method is a large-scale modeling of a system based on the well-known theory of model and system similarity. This method is not suitable for our LS – foundations and foundation bed, because any reproduction of rock formation in another scale does not make sense. At the same time, we should recognize that the separate LS subsystems foundation + foundation bed* can be studied on large-scale models of separate foundations designed to study the nature of their loading in order to select the most appropriate mathematical models of physical processes. The study of soil behavior – foundation under load on rock samples – is an important part of this method.

The only drawback here is that the studied samples are subjected to an experimental study when they are without load in comparison to their real state during the operation of buildings and structures. In this regard, only the third method is useful in studying the presented two systems (– foundations and – foundation bed) – mathematical modeling of LS. Obviously, we will be able to consider the entire variety of foundations interacting with any natural environment (rock formation) if we make an adequate description of physical processes and system operation (– foundations and – foundation bed) by mathematical language. Besides,
mathematical models of LS allow conducting any desired number of any experiments with a maximum approximation to their actual state. Experiments are available for any interval of LS "life" and for determining the sensitivity of all controlled parameters and characteristics of foundations and structures. Mathematical models of LS can be used in modeling the control and damping processes of ground vibrations in earthquakes and wind loads.

2.1 Parameters and characteristics of solid mass interacting with foundations

Any building technology T1, … , Tn interacts with natural or artificial rock mass and with fluids – water, gas or solutions. In construction, there are various physical processes in the solid mass that require studying in order to adapt building technologies to the environment. The most important physical processes for any considered building technologies are:

- state of the rock mass – soils, leading to a stressed state of rock formations;
- diffusion dissolution of any components in rock formation;
- filtration transfer of solutions or any fluids in porous medium.

Let us consider the well-known models of rock mass.

Models of solid deformable body

\( X_1 \) – elastic body;
\( X_2 \) – viscoelastic deformable body;
\( X_3 \) – elastic deformable body;
\( X_4 \) – flowing deformable medium;
\( X_5 \) – medium with a crack system;
\( X_6 \) – block medium with various organization systems of blocks.

Models of fluid motion and gas flow (filtration-diffusion models of medium)

\( X_7 \) – filtration (macroscopic pores);
\( X_8 \) – diffusion (Knudsen and Volmer);
\( X_9 \) – sorption microscopic pores;
\( X_{10} \) – non-deformable porous medium;
\( X_{11} \) – deformable porous medium.

Models of thermodynamic processes in medium:

\( X_{12} \) – thermal conductivity model;
\( X_{13} \) – heat and mass transfer models.

Let us consider the state of the rock mass on the quasiordered graph \( G = (X, \Gamma) \).

Thus, in terms of \( X_1, … , X_6 \) we have: two conditions \( x_{j1} = 0; x_{j2} = 1; \) medium has to be strictly defined;
- in terms of \( X_5 \) and \( X_6 \), there might be different conditions:

\[
\begin{align*}
X_5 & = \{x_{5} \mid x_{51} = 0; \} \\
X_6 & = \{x_{6} \mid x_{61} = 0; \}
\end{align*}
\]

If \( k = 4; \ell = 4 \), we will have the following possible number of graph paths from \( X1 \) to \( X6 \) on the quasiordered graph:

\[
M(G) = 2^4 \cdot 4 \cdot 4 = 2^4 \cdot 2^2 = 2^6!
\]

Simple analysis shows that not all \( G \) paths are acceptable:

\[
\begin{align*}
\mu_1 & = [0, 0, …, 0] \text{ - no medium;} \\
\mu_2 & = [x_{1,2}, x_{2,2}, …] \text{ - conflict in } G.
\end{align*}
\]

For example, in terms of sub-graph \( G_1 \subset G \), we will get: \( G_1 = (X, \Gamma); \)

\[
X = \bigcup_{j=1}^{11} X_j \cup \bigcap_{j=1}^{11} X_j = \emptyset \text{ only 4 paths are acceptable:}
\]

\[
\begin{align*}
\mu_1 & = [x_{1} : 0, 0, 0]; \\
\mu_2 & = [x_{1} : 0, 0, 0]; \\
\mu_3 & = [0, x_{1} : 0, 0]; \\
\mu_4 & = [0, 0, x_{1} : 0, 0];
\end{align*}
\]

In terms of sub-graph \( G_2 = (X, \Gamma) \subset G, X = X_5 \cup X_6 \cup X_7 \cap X_8 = \emptyset \), the number of acceptable paths will be:

\[
M(G_2) = k + \ell.
\]

Although, there might be mediums having any block system with its own system of cracks. Then the number of acceptable paths will be:

\[
M(G_3) = k \cdot \ell.
\]

The total number of solid deformable rock formations will be:

\[
M = 4 + k \cdot \ell.
\]

Therefore, one has to find a model that corresponds to the real environment in each specific case of modeling foundation and foundation bed.

There are only 6 possible models to describe the processes of diffusion dissolution, filtration of fluids and sorption. They are represented on the sub-graph \( G = (X, \Gamma); \)

\[
X = \bigcup_{j=1}^{11} X_j \cup \bigcap_{j=1}^{11} X_j = \emptyset \text{ under } X_{j1} = 0; \quad j = 7, 11
\]

\[
\begin{align*}
\mu_1 & = [x_{7} : 0; 0; x_{10} : 0]; \\
\mu_2 & = [0; x_{8} : 0; x_{10} : 0]; \\
\mu_3 & = [0; 0; x_{8} : 0; x_{10} : 0]; \\
\mu_4 & = [x_{7} : 0; 0; x_{12}]; \\
\mu_5 & = [0; x_{8} : 0; x_{12}]; \\
\mu_6 & = [0; 0; x_{8} : 0; x_{12}];
\end{align*}
\]

There are 3 models to describe thermodynamic processes in foundation soil, represented on the sub-graph \( G_1 = (X, \Gamma) \subset G, X = X_{11} \cup X_{12} \) under \( x_{j1} = 0; \ j = 11, 12 \)

\[
\begin{align*}
\mu_1 & = [x_{1} : 0]; \\
\mu_2 & = [0; x_{12}]; \\
\mu_3 & = [0; 0; x_{12}];
\end{align*}
\]

Therefore, the total number of possible models designed to describe physical processes in rock mass with any foundation will be:

\[
M_g = 4 + k \cdot \ell + 6 + 3 = 13 + k \cdot \ell.
\]

If \( k = 4; \ell = 4 \), the total number of rock mass models is:

\[
M = 4 \cdot 4 + 13 = 29.
\]

Determining 6 specific conditions is a challenge, as well as determining which model more effectively reflects the rock mass with its various properties. Only separate models are usually considered to simplify the process, but not their joint combinations. Thus, the most commonly used models \( X_5 \) and \( X_6 \) are described in the book [12]. The models \( X_7 \), \( X_8 \), \( X_9 \), \( X_{10} \) and \( X_{11} \), thermal conductivity model – \( X_{12} \), and heat transfer model – \( X_{13} \) are considered in the same monograph [12]. A brief outline of these
models is provided below. M.V. Kurely and others have made a successful attempt to describe the model of technogenic geomechanical stress [6]. Here is a brief description of geomechanical models of rock deformation, according to an academy fellow M.V. Kurely and to the monograph [6, 12].

Continuum mechanics substantiates the following equilibrium equations at any point \((x, y, z)\) for time \(t\):

\[
\frac{\partial \tau_{x}}{\partial x} + \frac{\partial \tau_{y}}{\partial y} + \frac{\partial \tau_{z}}{\partial z} + \rho X = \frac{\partial^{2} u}{\partial t^{2}},
\]

\[
\frac{\partial \tau_{x}'}{\partial x} + \frac{\partial \tau_{y}'}{\partial y} + \frac{\partial \tau_{z}'}{\partial z} + \rho Y = \frac{\partial^{2} u'}{\partial t^{2}},
\]

\[
\frac{\partial \tau_{x}''}{\partial x} + \frac{\partial \tau_{y}''}{\partial y} + \frac{\partial \tau_{z}''}{\partial z} + \rho Z = \frac{\partial^{2} \omega}{\partial t^{2}}.
\]

(12)

Where: \(\sigma_x, \sigma_y, \sigma_z\) – normal components of stress tensor;

\(\tau_{x}, \tau_{y}, \tau_{z}\) – shear components of stress tensor;

\(u, v, \omega\) – projections of displacement vector on the axis \(x, y, z\);

\(X, Y, Z\) – projections of volume force vector on the axis \(x, y, z\);

\(\rho\) – rock density;

\(t\) – time.

If the right-hand sides of equations are not equal to zero, then (12) are dynamic equilibrium conditions; if the right-hand sides are equal to zero, we are dealing with a static equilibrium.

We have also found the following geometric relationships:

\[
E_x = \frac{\partial u}{\partial x}, \quad Y_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};
\]

\[
E_y = \frac{\partial v}{\partial y}, \quad Y_{yx} = \frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z};
\]

\[
E_z = \frac{\partial \omega}{\partial z}, \quad Y_{zx} = \frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial y}.
\]

(13)

In (13), \(E_x, E_y, E_z\) are linear components of strain tensor;

\(\gamma_{xy}, \gamma_{yx}, \gamma_{zx}\) – angular components of strain tensor at the point \((x, y, z)\) of the solid mass.

Strain tensor components in (13) must satisfy the following conditions for medium continuity in the absence of discontinuities:

\[
\frac{\partial^{2} e_x}{\partial x^{2}} + \frac{\partial^{2} e_y}{\partial x \partial y} = \frac{\partial^{2} \gamma_{xy}}{\partial y \partial x},
\]

\[
\frac{\partial^{2} e_y}{\partial y^{2}} + \frac{\partial^{2} e_x}{\partial y \partial x} = \frac{\partial^{2} \gamma_{yx}}{\partial x \partial y},
\]

\[
\frac{\partial^{2} e_z}{\partial z^{2}} + \frac{\partial^{2} e_x}{\partial z \partial x} + \frac{\partial^{2} e_y}{\partial z \partial y} = \frac{\partial^{2} \gamma_{zx}}{\partial x \partial z} + \frac{\partial^{2} \gamma_{zy}}{\partial y \partial z}.
\]

(14)

Equations (12), (13) and (14) are common for any continuum – rock formations, water, air, etc.

Physical equations for a linear deformable homogeneous rock mass relate stresses and deformations at any point:

\[
e_{x} = \frac{1}{E} \left[ \sigma_{x} - \nu (\sigma_{y} + \sigma_{z}) \right],
\]

\[
e_{y} = \frac{1}{E} \left[ \sigma_{y} - \nu (\sigma_{x} + \sigma_{z}) \right],
\]

\[
e_{z} = \frac{1}{E} \left[ \sigma_{z} - \nu (\sigma_{x} + \sigma_{y}) \right],
\]

(15)

Where: \(E\) – Young's modulus (elasticity modulus);

\(\nu\) – Poisson's ratio.

The listed equations (12)-(15) conceptually suffice to solve any geomechanical problem about the stress-strain state (SSS) of some continuous rock mass volume.

Equation analysis (12)-(15) shows that the rock mass is characterized here only by the following parameters:

\(\rho\) – rock density, ton/m³;

\(E\) – Young’s modulus, MPa;

\(\nu\) – Poisson’s ratio;

\(X, Y, Z\) – projections of volume force vector on the axis \(x, y, z\).

There are only three known force fields in a solid mass:

– force of gravity in terms of overlying rock mass and structures on the foundation, vertical force: \(Q_y = -gH\);

– tectonic force in a folded rock mass, which has a very complex nature and a wide manifestation, especially in mountain regions [13, 14], \(Q(x, y, z)\);

– seismic forces of two types – natural force during the earthquakes, and technogenic force during blasting operations, \(Q(x, y, z)\).

In terms of viscoelastic bodies (rock formations), necessary medium parameters are expanded by two rheological characteristics from the equation [14]:

\[
L(t, \tau) = \delta(t - \tau)^{a}
\]

(16)

Where: \(\delta\) and \(\alpha\) under \(0 < \alpha < 1\) are required to determine the SSS parameters of the solid mass.

Plastic and flowing medium are expanded by two parameters of rock formations:

\(C\) – soil adhesion;

\(\phi\) – angle of internal friction.

Two additional parameters are often used in SSS model of solid mass:

\(K\) – bulk modulus of elasticity;

\(G\) – shear modulus of elasticity.

These two parameters are related with already mentioned \(E\) and \(\nu\) parameters in the following simple dependencies:

\[
K = \frac{E}{3(1 - 2\nu)}.
\]

(17)
\[ G = \frac{E}{2(1 + \nu)}, \]  

(18)

Therefore, in dealing with theoretical works in the field of stress-strain state of broken rock mass, one has to know seven parameters:

\[ \{\rho; E; v; \bar{\sigma}; \alpha; C; \varphi\} \]  

(19)

and three vector fields of force distribution in soil mass:

\[ \{Q_i = -pM; Q_i(x, y, z); Q_i(x, y, z)\}. \]  

(20)

This paramount knowledge of medium will make it possible to solve any SSS problem of soil medium. However, this is possible only without taking into account the qualitative characteristics of the rock mass. We get an overall theoretical picture for describing geo-mechanical processes in the solid mass with due account for the possible qualitative characteristics displayed by acceptable sub-graph paths \( (G_1 \text{ and } G_2) \), when the total number of models for studying SSS of the rock mass is:

\[ M = 4 + k \cdot \ell. \]  

(21)

We should also emphasize that parameters (19) have their own values at every point of the space \( T_i = (x_i; y_i; z_i) \) – rock mass, which can be essentially different.

This circumstance and a lack of mathematical description of vector fields (19) lead us to a suggestion that complete and adequate description of geomechanical processes that develop in real interaction of foundations with foundation bed is challenging.

Apparently, research on SSS has been based on direct measurements and on experiments with the widest spectrum of recorded parameters (19) and characteristics \( (x_i; x_j; x_k) \) of the rock mass for a long time.

2.2 Models of solid deformable body

Most mathematical models of physical processes associated with the stress-strain state of a rock mass are based on the methods of continuum mechanics. We will briefly present the necessary information – its basics – based on the papers [7]. The position of each point of the body is determined by its radius – vector with the components \( x_1 = x, x_2 = y, x_3 = z \) in a particular coordinate system. Any solid is deformed under external forces, namely – it changes its shape and volume.

The stress-strain state of a body at any of its points \( (x, y, z) \) is characterized by:

\[ \{\sigma_{xx}; \sigma_{xy}; \sigma_{xz}; \} \]  

(22)

Where: \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \) – normal stress; and \( \sigma_{xy}, \sigma_{xz}, \sigma_{yz} \) – shear stress at the sites perpendicular to coordinate axes \( x, y, z \);

b) symmetric strain tensor \( \varepsilon_{ii} \)

\[ \varepsilon_{ii} = \frac{1}{2} \left( \partial u_i / \partial x_i + \partial u_i / \partial x_i \right), \quad (i, k = 1, 2, 3), \]  

(23)

Where: \( \varepsilon_{ii} (i = 1, 2, 3) \) – strain components;

\[ u_{ii} (i, k = 1, 2, 3) \]  

shear components.

There are three mutually perpendicular sites with zero shear stress at each point of the medium. Directions of normal stress to these sites form the major axes of the tensor \( \sigma_\alpha \). Normal stresses on these sites are called the major ones. They are denoted by \( \sigma_1, \sigma_2, \sigma_3 \). It is usually determined that shear stress is in the cross-sections, bisecting the angles between the principal planes:

\[ \tau_1 = (\sigma_2 - \sigma_3)/2; \quad \tau_2 = (\sigma_3 - \sigma_1)/2; \quad \tau_3 = (\sigma_1 - \sigma_2)/2. \]  

(24)

Variable

\[ \sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3 \]  

(25)

called a mean stress. In the total number of components \( \sigma_\alpha \), only the diagonal components are nonzero by the major axes of the strain tensor \( \varepsilon_\alpha \):

\[ \varepsilon_1 = \varepsilon_1, \varepsilon_2 = \varepsilon_2, \varepsilon_3 = \varepsilon_3. \]  

(26)

These components are called principal extensions. The sum of diagonal components is equal to the relative change in volume:

\[ \varepsilon = \Delta V/V = (V' - V)/V = \text{div} \; \bar{u}, \]  

(27)

Where: \( \bar{u} = \bar{F} - \bar{F} \) – displacement vector.

The medium is subject to maximum (principal) shear on the sites located at the angle of 45° to the major axes of a strain tensor:

\[ \gamma_1 = \varepsilon_1 - \varepsilon_2; \quad \gamma_2 = \varepsilon_2 - \varepsilon_3; \quad \gamma_3 = \varepsilon_3 - \varepsilon_1, \]  

(28)

Where: \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) – principal extensions.

Two more characteristics of the stress-strain state have to be considered. Variable

\[ T = \sqrt{\left(\varepsilon_1 - \varepsilon_2\right)^2 + \left(\varepsilon_2 - \varepsilon_3\right)^2 + \left(\varepsilon_3 - \varepsilon_1\right)^2} / 6 \]  

(29)

called shear stress intensity. In the case of pure shear under

\[ \sigma_1 = -\sigma_2 = \tau, \quad \sigma_2 = 0, \quad T = |\tau|. \]  

(30)

another variable

\[ \Gamma = \sqrt{2/3} \sqrt{\left(\varepsilon_1 - \varepsilon_2\right)^2 + \left(\varepsilon_2 - \varepsilon_3\right)^2 + \left(\varepsilon_3 - \varepsilon_1\right)^2}, \]  

(31)

called shear strain intensity. In the case of pure shear \( \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0, \varepsilon_{12} = \gamma/2, \) the equation is \( \Gamma = |\gamma|. \)

2.3 Elastic deformable body

If the body subjected to deformation returns to the initial state after removing external loads, then deformations of this kind are called elastic. The Hooke’s law is an equation for the state of an elastic body, under which stresses, and strains are related by linear dependence. In expanded form, these equations are like:

\[ \{\varepsilon_{ii} = (\sigma_{ij} - \nu \sigma_{jj})]/E; \]  

\[ \varepsilon_{ij} = (1 + \nu)\sigma_{ij}/E, \]  

(32)

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Where: \( E \) — Young’s modulus and \( v \) — Poisson’s ratio. Bulk modulus of elasticity \( K \) and shear modulus of elasticity \( G \) are often used in solving problems related with the following formulas:

\[
K = \frac{E}{3(1 - 2v)}, \quad G = \frac{E}{2(1 + v)}.
\]

In the invariant form, Hooke’s law can be represented like:

\[
\mathbf{\sigma} = K \mathbf{\varepsilon}, \quad T = G \mathbf{\varepsilon},
\]

Where: \( T \) and \( \varepsilon \) are determined by formulas (29), (31). The specific value for accumulated energy of elastic deformation of a solid body in general form is found from the expression:

\[
w = \frac{1}{2}K + T^2/2G.
\]

In solving practical problems in terms of the theory of elasticity, the Hooke’s law (32), (34) is used along with the equations of motion and equilibrium for an elastic body. Motion value of medium elements with density \( \rho \) and with applied force is determined by differential equations:

\[
\left[ \frac{\partial \sigma_{ik}}{\partial x_k} + (F_j - a_j) \right] \rho = 0, \quad (i, k = 1, 2, 3).
\]

In this case, summation is performed over the repeated index \( k \). In mathematical models of mining processes, the plane problem of the theory of elasticity is often used. Let’s assume that one of displacement vector components is equal to zero \((w_1)\), and all other variables depend only on \( u, v \). In this case, we deal with a plane deformation. Thus, the strain tensor components \( \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \) and the stress tensor components \( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \) vanish identically. The stress \( \alpha_{xy} \) is non-zero and is determined by:

\[
\sigma_{xy} = 3\varepsilon_{xy}.
\]

In this case, equilibrium equations are like:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0.
\]

2.4 Viscoelastic deformable body

In general cases, viscous bodies are medium models, in which mechanical processes are considered in time. Such models can also be called rheological models. The change of SSS of deformable bodies based on the rheological models is studied by the creep theory of heredity, flow theory, hardening theory and the theory of aging. The creep theory of heredity has received the most theoretical justification and practical application in studies of rock deformation processes. It was developed by L. Boltzmann and V. Volterra in the form of a mathematical description of linear creep phenomena. Subsequently, Yu. N. Rabotnov has proved that the theory of linear heredity can be formally considered as a problem in the theory of elasticity, in which time operators with a creep kernel are necessary to use instead of elastic constants. This position was called the Volterra principle [13]. Zh. S. Yevzhanov (1964) has given a systematic construction of the theory of rock creep. He has showed that the dependence of stress and strain on time is a linear phenomenon:

\[
L(t, \tau) = \delta(t - \tau) \tau,
\]

with Abel creep kernel

\[
E(t, t) = \frac{s}{E} \left[ 1 + \int_0^t \frac{L(t, \tau)}{\tau} d\tau \right] \quad \left( 0 < s < 1 \right)
\]

Typical creep curve for rock mass is shown in Figure 1.

![Figure 1: Creep curve for rock mass](image)

| Segment | Description |
|---------|-------------|
| (0-1)   | Initial instantaneous deformation \( a_0 \) |
| (1-2)   | To stage I of unsteady creep: const |
| (2-3)   | To stage II of steady-state creep: \( a_0 \) - const |
| (3-4)   | To stage III of progressive flow: destruction |

2.5 Models of plastic and flowing medium

In interaction of foundation with foundation bed, a significant part of deformed medium possesses dissipation properties. This includes plastic and flowing medium, which have to be considered separately.

2.6 Plastic deformable medium

If deformed body does not return to the initial state after removing the external load, but has permanent deformations, we are dealing with a plastic deformation. Plastic yield criterion is usually expressed mathematically by means of stress components.

2.6.1 Tresca-Saint-Venant plasticity condition:

\[
2\tau_{\max} = \sigma_1 = \sigma_3 \geq \sigma_2 \geq \sigma_0
\]

Where: \( \tau_{\max} \) — maximum shear stress; \( \sigma_0 \) — yield stress under uniaxial tension.

2.6.2 Huber-Mises-Hencky plasticity condition:

\[
T = \sigma_0 \sqrt{3} = \tau_0
\]

The \( \tau_0 \) is known not to differ much from \( \tau_{\max} \).

2.6.3 Mohr-Coulomb yield criterion is usually used to describe the plastic deformation of rock mass:

\[
\sigma_1 - \sigma_3 = 2C \cdot \cos \varphi + \left( \sigma_1 + \sigma_3 \right) \sin \varphi
\]

Where: \( C \) — soil adhesion;
$\varphi$ – angle of internal friction.

Mathematical theories of plasticity are based on the hypothetical understanding of rock formation properties. They include the so-called ideal rigid-plastic, elasto-plastic, hardening and softening solids with own functional relationship between stress and strain (Figure 2).

**Figure 2:** Linear strain models of solid bodies: a – rigid-plastic, b – elasto-plastic, c – elasto-plastic with softening, d – rigid-plastic with softening

Complete system of equations for plastic deformation contains also the equation of continuity

$$\partial \rho / \partial t + \partial \left( \rho \nabla \right) / \partial \xi = 0, \quad (47)$$

that reflects the law of conservation of mass.

**Viscoelastic body.** In reaching the ultimate shearing resistance, further deformation can be represented as a viscous fluid motion. Viscous liquid deformation is described by

$$\sigma_{ik}^* = \eta V_{ik}^* + \left( \eta/3 + \xi \right) V_{ik}^* \partial \nu_j / \partial \xi_j, \quad (48)$$

Where: $\eta$– viscosity factor;

$\sigma_{ik}^*$ – stress deviator;

$V_{ik}^*$ – strain-rate deviator;

$\xi$ – bulk viscosity (in hydrodynamics, it is usually $\xi = -\eta/3$).

Solutions for viscoplastic flow problems with Navier-Stokes equations are provided in [12-14]. It should be noted that there is a significant difficulty in determining viscosity factor of rock formations experimentally, especially since it is not a constant, but represents a particular function of stress and strain.

### 2.7 Flowing deformable medium

The study of soil behavior while foundation interacts with sand is associated with statics and dynamics of flowing medium.

### 2.8 Perfect flowing medium

Let's consider a bulk medium under a certain volume $V$. If the medium consists of particles, which size $r$ satisfies the relation $r/\sqrt[3]{V} << 1$, and the interaction between them is characterized only by internal friction ($\varphi > 0$) and no soil adhesion ($C = 0$), then it is considered as a perfect flowing medium. According to V.V. Sokolovsky, limit equilibrium of flowing medium is reflected by the Mohr-Coulomb yield criterion [13].

$$\sigma_1 - \sigma_3 = 2C \cos \varphi + (\sigma_1 + \sigma_3) \sin \varphi \quad (49)$$

In terms of this equation, condition

$$\tau_n = C + \sigma_n \tan \varphi. \quad (50)$$

is at each point on the sites that make up acute angles $\gamma = \left( \frac{\pi}{4} - \frac{\varphi}{2} \right)$ with a direction of the major stress $\sigma_n$. These sites correspond to two families of flow lines. Solution of practical problems of bulk mechanics is connected with the assumption of its compressibility. In terms of a compressed flowing medium, its density at each point is a functional relation of a stress state at this point, $\rho = f(\sigma)$. In general case, internal friction angle $\varphi(\sigma)$ is also changing. It has been experimentally established that the angle of internal friction $\varphi = \varphi(\sigma)$ increases in terms of a nonlinear relationship while compressive stress $\sigma$ is also increasing.

### 2.9 Models of fluid motion and gas flow

A significant class of processes in solid mass is related to the mechanism of fluid motion and gas flow in a medium with variable porosity and permeability.

### 2.10 Filtration-diffusion models of medium

Currently, there are three groups of pores distinguished by physicochemical processes in porous medium: 1) filtration (macroscopic pores); 2) diffusion (Knudsen and Volmer); 3) sorption (microporous pores). This fact calls for a consideration of a three-phase model of porous structure.

The model of such a medium can be represented by a system of three types of balls placed into each other. Non-deformable balls with radius $r_1$ forming a sorption volume are placed in larger elastic deformable balls with radius $r_2$ corresponding to a diffusion volume of pores. Elastic deformable balls are placed in plastic deformable balls with radius $r_3$ corresponding to a filtration volume of pores. Parameters of the model with an average-static size of particles are determined according to differential porosity curve $f(t)$. Thus, this model reflects the actual porous medium more adequately. The clearance determines the surface porosity, the area ratio of filter channels and the entire filter. In terms of a model medium, clearance is defined by formula

$$n_j = 1 - N_j^2 + r_j^2, \quad j = 1, 2, 3. \quad (51)$$

Permeability of the medium is an ability to pass liquid and gas under a pressure gradient. The permeability coefficient is

$$k_3 = \frac{m_3^2}{12(1-m_1)}. \quad (52)$$

The properties of a deformable porous medium depend on the components of stress and strain tensors. In general cases, porosity is determined by

$$m(C) = 1 - (1 - m_1)/(1 + C), \quad (53)$$

Where: $C$ – bulk deformation of a porous medium.

Accordingly, the clearance in deformation plane is

$$n_{ik} = \frac{e_{ii} + e_{kk} + 2n_o}{2 + e_{ii} + e_{kk}}, \quad (i, k = 1, 2, 3). \quad (54)$$

Porosity and gas permeability vary exponentially depending on the mean stress:

$$m = m_oo \exp(-a \sigma), \quad k = k_oo \exp(-b \sigma). \quad (55)$$

Where: $m_oo$, $k_oo$ – porosity and gas permeability without mechanical stress ($\sigma = 0$).

If strain components are known for each direction, then the permeability coefficient $k_0$ with allowance for $(53), (54)$ will be

$$k_{jj} = \frac{9k_0}{4m^2 n_{ik}^2} (2 + e_{ii} + e_{kk}). \quad (56)$$
2.11 Equations for fluid motion and gas flow

Fluid or gas internal processes are described by thermodynamic equations of state:

\[ f = (P, \rho, T) = 0, \]  
(57)

Where: \( P, \rho \) – medium pressure and density; \( T \) – absolute temperature.

If the process is isothermal, then the temperature is constant at all points of the medium, and \( T \) is a parameter in equation (57), namely –

\[ P/\rho = RT, \]  
(58)

Where: \( R \) – gas constant.

If we exclude dissipative processes from true fluid, in particular – heat exchange, we will have a so-called adiabatic process, which equation is

\[ P/\rho^k = \text{const}, \]  
(59)

Where: \( k = c_p/c_v \) – specific heat ratio under constant pressure and constant volume.

Sorption processes are important in the mechanism of fluid motion and gas flow in the coal mass. They include surface and space absorption, as well as formation of a certain chemical compound sorbate/sorbent. Desorption is a process opposite to sorption – separation of previously absorbed substance. Sorption equilibrium is established as a result of opposite processes (sorption and desorption), which feature includes the equality of rates

\[ V_1 = V_2. \]  
(60)

In assuming that the adsorption rate is proportional to the magnitude \((1-\Theta)\) at any instant of time \( t \) and partial pressure \( P \), and the desorption rate is proportional only to a degree of surface filling by adsorbed substance \( \Theta \), we will get a Langmuir equation

\[ \Theta = \frac{k_1 P}{1 + k_2 P}, \]  
(61)

Where: \( k_1, k_2 \) – constants.

**Equations for mass transfer:** Filtration and diffusion are the major forms of gas flow and fluid motion in a porous medium. Laminar filtration is described by Darcy’s law

\[ \vec{v} = -k/\mu \text{ grad } P, \]  
(62)

Where: \( \mu \) – dynamic fluid viscosity.

Diffusion is determined according to the Fick's First Law

\[ \vec{g} = -D \text{ grad } C, \]  
(63)

Where: \( D \) – diffusion coefficient;

\( C \) – Concentration of diffusible substance (gas or fluid).

Equation of continuity for non-equilibrium (with allowance for sorption processes) gas flow is

\[ \partial P/\partial t + \text{div}(\rho \vec{v}) = \beta (\rho - \rho_o), \]  
(64)

Where: \( \beta \) – kinetic desorption coefficient.

The parameter \( \beta \) is often assumed to be \( \beta = 0 \). In this case, the system of equations (58), (61) + (64) reduces to one differential equation:

\[ \frac{\partial}{\partial x} \left( k \frac{\partial P}{\partial x} \right) = \alpha \frac{\partial P}{\partial t}, \quad (j = 1, 2, 3), \]  
(65)

Where:

\[ \alpha = 2 \mu \left[ m + \frac{a_o b_o RT}{1 + a_o \sqrt{P_o}} \right], \]  
(66)

\[ P = p^2 \text{ – the square of rock pressure (index 0 refers to original pressure)}; \]

\[ a_o, b_o \text{ – Langmuir adsorption constants}. \]

If there is one-dimensional distribution of gas pressure near a moving exposed surface, then equation (64) will be

\[ a_1 \frac{\partial}{\partial x} \left( k \frac{\partial P}{\partial x} \right) + v \frac{\partial P}{\partial x} = \frac{\partial P}{\partial t}, \]  
(67)

Where: \( a_1 = 2 \sqrt{P_o}/\alpha \text{ – equation parameter}; \)

\[ v \text{ – advanced ate of a face}. \]

Equation (67) for stationary case was solved by an academy fellow S.A. Khristianovich.

Transformation by a generalized L.S. Lebenson’s functional relation, the so-called potential of mass filtration rate is an effective method for solving equations (65) and (66):

\[ F(P) = \frac{1}{\mu} \int \left( k(P) \rho(P) \right) dP. \]  
(68)

In this case, we get an equation of gas flow in the form of thermal conductivity equation:

\[ a^2 \Delta F = \partial F/\partial t. \]  
(69)

The fluid motion rate is finite \( W = \sqrt{a^2/\tau_*, \tau_*} \text{ for filtration in a porous medium (for example, while watering)} \text{ – relaxation period of a pressure gradient. Thus, equation (67) is transformed to a hyperbolic form:} \]

\[ a^2 \Delta F = \partial F/\partial t + \tau_\alpha \partial^2 F/\partial t^2. \]  
(70)

It should be noted that permeability coefficient is assumed to be variable in each direction while solving practical problems of fluid and gas filtration in a porous deformable medium. Such problems are usually solved numerically by computer-based calculations under complex boundary conditions.
2.12 Thermodynamic models

Processes associated with heat and mass transfer are studied due to a necessity in maintaining prescribed temperature conditions, as well as due to a need to combat underground fires and other thermal phenomena. Thermodynamic models are a mathematical basis of these processes [13].

2.13 Thermal conductivity models

The law of conservation and transformation of energy in thermodynamics. Let's consider a physical system interacting with the environment. In this case, there is an energy exchange \( U \) between the system and the environment in the form of heat and in the form of activity \( A \)

\[
dU = dQ - da. \tag{71}
\]

Equation (71) is a differential representation of the first law of thermodynamics. In thermodynamics, there are considered those important characteristics of substance energy, which are the thermodynamic functions of its state: internal energy \( U \), free energy \( E \), enthalpy \( J \), thermodynamic potential \( \Phi \) and entropy \( S \). Free energy, enthalpy and thermodynamic potential are currently determined as:

\[
E = U - TS; \quad J = U - PV; \quad \Phi = J - TS,
\tag{72}
\]

Where: \( P, V, T \) - pressure, volume and absolute temperature.

Relations (72) can be used to find the thermodynamic parameters of the system if they are represented by corresponding variables. The increase in the internal energy \( dU \) due to heat absorption is equal to

\[
\partial U = c_u dT, \tag{73}
\]

Where: \( c_u \) - specific heat under constant volume. The ability of a substance to transmit heat under temperature gradient is called thermal conductivity. Temperature distribution of various body parts and a corresponding change in their internal energy are possible because of thermal conductivity.

Equation of thermal conductivity. Let's say that the temperature of each point of the body \( V T (r, t) \) at the time \( t \) is determined by its radius vector.

The amount of heat passing in the \( n \) direction per a time unit is determined by the Fourier's law

\[
dQ = - \lambda dT / \partial n, \tag{74}
\]

Where: \( \lambda \) - thermal conductivity coefficient.

Let us allocate a certain bounded closed surface \( S \). If the heat sources inside the body are with dense \( f (r, t) \), then the energy conservation law will be written as follows:

\[
\oint_v f (r, t) dv = \int_v c_u \lambda (r) dV - \int_s \lambda (r) \frac{dT}{\partial n} ds, \tag{75}
\]

Where: \( S \) - heat flux surface.

In applying the Ostrogradsky formula with due account for the randomness of \( V \), we get the equation of thermal conductivity:

\[
c_u \lambda \frac{\partial T}{\partial t} = \text{div}(\lambda \text{grad} T) + f. \tag{76}
\]

In general cases, parameters \( \lambda, c, \rho \) are functions of the temperature \( T \), the radius vector \( r \) and the time \( t \). For example, , the one-dimensional equation (76) is written in Cartesian coordinate system as follows:

\[
c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + f(x, t). \tag{77}
\]

The equation (76) with constant coefficient \( (\lambda, c, \rho) \) is:

\[
\frac{\partial T}{\partial t} = a^2 \Delta T, \tag{78}
\]

Where: \( a^2 = \lambda / c_p \) - thermal diffusivity;

\( \Delta \) - Laplace operator.

Equation (78) characterizes the non-steady temperature field in three-dimensional space without heat sources. Equation (76) shows that the steady temperature field will be described by expression:

\[
\text{div} (\lambda \text{grad} T) = -f. \tag{79}
\]

In this case, \( \lambda, f \) and \( T \) depend on \( t \). Moreover, if \( \lambda \) is const, we end with the Poisson's equation:

\[
\Delta T = - f / \lambda, \tag{80}
\]

and equation (78) is normalized by the Laplace's equation:

\[
\Delta T = 0. \tag{81}
\]

These expressions form a basis for one-dimensional and two-dimensional equations of thermal conductivity.

2.14 Heat and mass transfer models

Regular thermal conditions usually occur when the bodies are heated and cooled in a medium with constant temperature and with a constant heat-exchange coefficient. Let's say that environment temperature is \( T_s \) and the bulk body temperature - \( T_b \). In this case, regular thermal conditions are described by equation

\[
T_v = m(T_s < T_v), \tag{82}
\]

Where: \( m \) reflects the rate of temperature logarithm change and the dot denotes differentiation with respect to time.

\[
d(\ln T)/dt = m. \tag{83}
\]

Equation (82) reflects the heating process, if \( T_s > T_v \), and the cooling, if \( T_s < T_v \).

Regular heat transfer in a potential flow. Heat exchange between a fluid or a gas, as well as between solid bodies in a moving medium, is described by equations of hydrodynamics, of continuity and of convective heat transfer. The last equation describes the temperature in a moving medium. It is called the Fourier-Kirchhoff equation:

\[
c \rho \vec{v} = \text{div}(\lambda \text{grad} T) - \rho \vec{g}(\text{grad} T, \vec{v}). \tag{84}
\]

Where: \( \vec{v} \) - flow velocity vector.

In integrating the system of equations, boundary and initial conditions that arise in connection with specific problems are required to be taken into account. The major problems of non-steady heat exchange are formed as co-stressed ones. This leads to the solution of equations with partial differential coefficients of various types. The general solution of the Fourier-Kirchhoff equation with due account for the convection in a moving medium is very difficult. Therefore, we are considering specific equations that accept solutions in the form of different thermal potentials. Heat transfer in a two-dimensional flow is an important case, because the Kirchhoff-Fourier equation can be simplified here by conformal transformations. Let's say that the medium is moving and the temperature \( T \) satisfies the equation

\[
\partial^2 T / \partial x_i^2 = k \left( \nabla \cdot \frac{\partial T}{\partial t} + \nabla T / \partial t \right), \tag{85}
\]

Where: \( k = c \rho / \lambda \) - constant for a fluid;

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\( v_i \) – flow velocity components \((i = 1, 2)\) on the axes \( x, y \).

In equation (85), summation is carried out over the index \( i \). Let’s transform the complex plane \( z = x + iy \) into a plane \( \zeta = \xi + i\eta \) by means of a function \( \zeta = f(z) \). We will denote the velocity potential and the flow function by \( \phi(x, y) \) and \( \psi(x, y) \), and will introduce new variables

\[
\zeta = \psi(x, y), \quad \eta = \phi(x, y).
\]  

(86)

In this case, equation (83) will be:

\[
\frac{\partial^2 T}{\partial \zeta^2} + \frac{\partial^2 T}{\partial \eta^2} = k \frac{\partial^2 T}{\partial \xi^2} + k \frac{\partial^2 T}{\partial \eta^2} + v^2 \frac{\partial^2 T}{\partial \zeta^2}.
\]  

(87)

Where: \( v^2 = v_x^2 + v_y^2 \).

Equation (87) represents the Fourier–Kirchhoff equation for transformed flow. In the case of a stationary problem \((dt/dt = 0)\), it does not depend on \( v \). This fact obviously simplifies its solution.

**RESULTS AND DISCUSSION**

We have considered the theory of interaction of any foundation with any foundation bed following the mathematical modeling of large systems in performing three mathematical spaces:

- \( H^m \) – qualitative characteristics;
- \( H^m \) – discrete parameters (DP);
- \( H^m \) – continuous parameters (CP).

This allows us to take into account all possible and conceivable variety of foundation soils (rock formations). Well-known mathematical models of deformable body behavior have been systematized for the first time. This includes elastic deformable body, viscoelastic deformable body; models of plastic and flowing medium; models of fluid motion and gas flow; models of thermal conductivity and of heat and mass transfer. Developed basis for mathematical modeling of interaction of foundations with foundation bed allows bringing the knowledge about these processes to real situation in the most adequate way.

Mathematical model presented in the paper makes allowing any experiment along with the representation of real systems \( S_1 \) – foundation and \( S_2 \) – foundation bed in any medium and for any time interval. The final conclusion of the research on this issue gives grounds to suggest that alternative solutions do not exist. In popular foreign works, such approach to the issue is not found [15, 16]. The theory of mathematical modeling of LS is a linear phenomenon, according to the way of foundations interacting foundation beds. In the Republic of Kazakhstan, it is widely used by authors in the field of mining sciences. In our opinion, science will be developing if the theoretical bases of mathematical modeling of LS presented in the article are applied.

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