EXT-QUIVERS OF HEARTS OF A-TYPE AND THE ORIENTATION OF ASSOCIAHEDRON

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Abstract. We classify the Ext-quivers of hearts in the bounded derived category $\mathcal{D}(A_n)$ and the finite-dimensional derived category $\mathcal{D}(\Gamma_N A_n)$ of the Calabi-Yau-N Ginzburg algebra $\Gamma_N A_n$. This provides the classification for Buan-Thomas' colored quiver for higher clusters of A-type. We also give explicit combinatorial constructions from a binary tree with $n + 2$ leaves to a torsion pair in $\text{mod } kA_n^*$ and a cluster tilting set in the corresponding cluster category, for the straight oriented A-type quiver $A_n^-$. As an application, we show that the orientation of the $n$-dimensional associahedron induced by poset structure of binary trees coincides with the orientation induced by poset structure of torsion pairs in $\text{mod } kA_n^*$ (under the correspondence above).

Key words: Ext-quiver, binary tree, torsion pair, cluster theory

Summary

Assem and Happel [1] gave a classification of iterated tilted algebras of A-type using tilting theory decants ago. In the first part of the paper (Section 1 and Section 2), we generalize their result to classify (Theorem 2.11) the Ext-quivers of hearts of A-type (i.e. in $\mathcal{D}(A_n)$), in terms of graded gentle trees. As an application, we describe (Corollary 2.12) the Ext-quivers of hearts in $\mathcal{D}(\Gamma_N A_n)$, the finite-dimensional derived category of the Calabi-Yau-N Ginzburg algebra $\Gamma_N A_n$, which correspond (cf. [9, Theorem 8.6]) to colored quivers for $(N - 1)$-clusters of A-type, in the sense of Buan-Thomas [3].

In the second part of the paper (Section 3), we give explicit combinatorial constructions (Proposition 3.2 and Proposition 3.3), from a binary trees with $n + 2$ leaves (for parenthesizing a word with $n + 2$ letters) to a torsion pair in $\text{mod } kA_n^*$ and a cluster tilting sets in the (normal) cluster category $\mathcal{C}(A_n)$, where $A_n^*$ is a straight oriented $A_n$ quiver. Thus, we obtain the bijections between these sets. As an application, we show (Theorem 3.5) that under the bijection above, the orientation of the $n$-dimensional associahedron induced by poset structure of binary trees (cf. [10]) coincides with the orientation induced by poset structure of torsion pairs (or hearts, in the sense of King-Qiu [9]).

Note that there are many potential orientations for the $n$-dimensional associahedron, arising from representation theory of quivers, cf. [9, Figure 4 and Theorem 9.6]). These orientations are also interested in physics (see [4]), as they are related to wall crossing formula, quantum dilogarithm identities and Bridgeland’s stability condition (cf. [7] and [11]).

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1. Preliminaries

1.1. Derived category and cluster category. Let $Q$ be a quiver of A-type with $n$ vertices and $k$ a fixed algebraic closed field. Let $kQ$ be the path algebra, $\mathcal{H}_Q = \text{mod}kQ$ its module category and let $\mathcal{D}(Q) = \mathcal{D}^b(\mathcal{H}_Q)$ be the bounded derived category. Note that $\mathcal{D}(Q)$ is independent of the orientation of $Q$ and we will write $A_n$ for $Q$ sometimes.

Denote by $\tau$ the AR-functor (cf. [2, Chapter IV]). Let $\mathcal{C}(A_n)$ be the cluster category of $\mathcal{D}(A_n)$, that is the orbit category of $\mathcal{D}(A_n)$ quotient by $[1] \circ \tau$. Denote by $\pi_n$ be the quotient map $\pi_n : \mathcal{D}(A_n) \to \mathcal{C}(A_n)$.

1.2. Calabi-Yau category. Denoted by $\Gamma_N Q$ the (degree $N$) Ginzburg’s differential graded algebra that associated to $Q$ which the dg algebra

$$k\langle e, x, x^*, e^* | e \in Q_0; x \in Q_1 \rangle$$

with degrees

$$\deg e = \deg x = 0, \quad \deg x^* = N - 2, \quad \deg e^* = N - 1$$

and differentials

$$d \sum_{e \in Q_0} e^* = \sum_{x \in Q_1} [x_k, x^*_k].$$

Let $\mathcal{D}(\Gamma_N Q)$ be the finite dimensional derived category of $\Gamma_N Q$ and $\mathcal{H}_\Gamma$ be its canonical heart. Notice that the derived categories are always triangulated. Again, since $\mathcal{D}(\Gamma_N Q)$ is independent of the orientation of $Q$, we will write $\Gamma_N A_n$ for $\Gamma_N Q$.

1.3. Hearts of triangulated categories. A torsion pair in an abelian category $\mathcal{C}$ is a pair of full subcategories $\langle F, T \rangle$ of $\mathcal{C}$, such that $\text{Hom}(T, F) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \to E^T \to E \to E^F \to 0$ for some objects $E^T \in T$ and $E^F \in F$.

A t-structure on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$, satisfying $\mathcal{P}[1] \subset \mathcal{P}$ and being the torsion part of some torsion pair (with respect to triangles) $(\mathcal{P}, \mathcal{P}^\perp)$ in $\mathcal{D}$. A t-structure $\mathcal{P}$ is bounded if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j].$$

The heart of a t-structure $\mathcal{P}$ is the full subcategory

$$\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}$$

and any bounded t-structure is determined by its heart. In this paper, we only consider bounded t-structures and their hearts.

Recall that we can forward/backward tilts a heart $\mathcal{H}$ to get a new one, with respect to any torsion pair in $\mathcal{H}$ in the sense of Happel-Reiten-Smalø([5], see also [9, Proposition 3.2]). Further, all forward/backward tilts with respect to torsion pairs in $\mathcal{H}$, correspond one-one to all hearts between $\mathcal{H}$ and $\mathcal{H}[\pm 1]$ (in the sense of King-Qiu [9]).
In particular there is a special kind of tilting which is called simple tilting (cf.[11, Definition 3.6]). We denote by $H^S_♯$ and $H^S_♭$, respectively, the simple forward/backward tilts of a heart $H$, with respect to a simple $S$.

The exchange graph of a triangulated category $D$ to be the oriented graph, whose vertices are all hearts in $D$ and whose edges correspond to the simple forward tilting between them. Denote by $EG(A_n)$ the exchange graph of $D(A_n)$, and $EG^*(\Gamma N A_n)$ the principal component of the exchange graph of $D(\Gamma N A_n)$, that is, the connected component containing $H_\Gamma$.

2. Ext-quivers of A-type

2.1. Graded gentle tree. In [1], it gives the complete description of all iterated tilted algebra of type $A_n$, namely:

**Definition 2.1.** [2] Let $A$ be an quiver algebra with acyclic quiver $T_A$. The algebra $A \cong kT_A/I$ is called gentle if the bound quiver $(T_A, I)$ has the following properties:

1°. Each point of $T_A$ is the source and the target of at most two arrows.
2°. For each arrow $\alpha \in (T_A)_1$, there is at most one arrow $\beta$ and one arrow $\gamma$ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.
3°. For each arrow $\alpha \in (T_A)_1$, there is at most one arrow $\xi$ and one arrow $\zeta$ such that $\alpha\xi \in I$ and $\zeta\alpha \in I$.
4°. The ideal $I$ is generated by the paths in $3°$.

If $T_A$ is a tree, the gentle algebra $A \cong kT_A/I$ is called a gentle tree algebra.

**Theorem 2.2.** Let $A$ be a quiver algebra with bound quiver $(T_A, I)$. Then $A$ is (iterated) tilted algebras of type $A_n$ if and only if $(T_A, I)$ is a gentle trees algebra. (cf.[1], also [2])

Considering the special properties of $T_A$, we can color it into two colors, such that any two neighbor arrows $\alpha, \beta$ has the same color if and only if $\alpha\beta \in I$ or $\beta\alpha \in I$.

Alternatively, we can also color it into two colors, such that any two neighbor arrows $\alpha, \beta$ has the different color if and only if $\alpha\beta \in I$ or $\beta\alpha \in I$. By the properties above, either coloring is unique up to swapping colors. Hence we have another way to characterize gentle tree algebra as follows.

**Definition 2.3.** A gentle tree is a quiver $T$ with a 2-coloring, such that each vertex has at most one arrow of each color incoming or outgoing.

For a colored quiver $T$, there are two natural ideals

$I_T^+ :$ generated by all unicolor-paths of length two;

$I_T^- :$ generated by all alternating color paths of length two.

**Proposition 2.4.** Let $A = kT/I$ be a bound quiver algebra. We have the following equivalent statement:

- $A$ is a gentle tree algebra.
- $T$ is some gentle tree with $I = I_T^+$ or $I = I_T^-$. 

**Proof.** By the one of two ways of coloring, the relations in the ideal and the coloring of the gentle tree can be determined uniquely by each other. □
Remark 2.5. In fact, there is an irrelevant but interesting result that for a gentle tree $T$, $kT/I_T^-$ and $kT/I_T^+$ are Koszul dual.

We are going to generalize Theorem 2.2 to describe all hearts in $D(A_n)$.

2.2. Ext-quivers of hearts. Recall that a heart $H$ is a finite, if the set of its simples, denoted by $\text{Sim} H$, is finite and generates $H$ by means of extensions,

Definition 2.6. Let $H$ be a finite heart in a triangulated category $D$ and $S = \bigoplus_{S \in \text{Sim} H} S$. The Ext-quiver $Q(H)$ is the (positively) graded quiver whose vertices are the simples of $H$ and whose graded edges correspond to a basis of $\text{End}^\bullet(S, S)$.

Note that, by [9], $H$ is finite, rigid and strongly monochromatic for any $H$ in $D(A_n)$. By [9, Lemma 3.3], we know that there are at most one arrow between any two vertices in $Q(H)$.

Definition 2.7. A graded gentle tree $G$ is a gentle tree with a positive grading for each arrow. The associated quiver $Q(G)$ of $G$, is a graded quiver with the same vertex set and an arrow $a : i \to j$ for each unicolored path $p : i \to j$ in $G$, with the grading of $p$.

Define a mutation $\mu$ on graded gentle tree as follow.

Definition 2.8. For a graded gentle tree $G$, let $V$ be a vertex with neighborhood

\[ \begin{array}{ccc}
R_1 & \gamma_1 & B_2 \\
& \delta_2 & \\
V & \gamma_2 & \\
& \delta_1 & B_1 \\
B_1 & \beta_1 & \end{array} \]

where $B_i, R_i$ are the sub trees and $\gamma_i, \delta_i$ are degrees, $i = 1, 2$. The straight line represent one color and the curly line represent the other color. Define the forward mutation $\mu_V$ at vertex $V$ (on $G$) as follows:

- if $\delta_1 \geq 1$, $\mu_V$ on the lower part of of the quiver is:

\[ \begin{array}{ccc}
R_2 & \delta_1 & B_1 \\
& \gamma_2 & \\
V & \gamma_1 & \\
& \delta_1 - 1 & R_2 \\
B_1 & \beta_1 & \end{array} \]

- if $\delta_1 = 1$, denote

\[ \begin{array}{ccc}
E_1 & \theta_1 & E_2 \\
& \beta_1 & \\
B_1 & \theta_2 & \end{array} \]

\[ \begin{array}{ccc}
\mathcal{L}_1 & \beta_1 & \mathcal{E}_2 \\
& \theta_2 & \\
W & \theta_1 & \end{array} \]
and $\mu_V$ on the lower part of the quiver is:

$$
\begin{array}{c}
\varepsilon_1 \\
\theta_1 \\
W \\
\beta \\
\varepsilon_2 \\
\theta_2 \\
R_2
\end{array}
\quad
\begin{array}{c}
\gamma_2 \\
\mu_i \\
V \\
1 \\
\varepsilon_1^x \\
\theta_2 \\
R_1
\end{array}
\quad
\begin{array}{c}
\beta \\
\varepsilon_2 \\
W \\
\gamma_2 \\
\varepsilon_1 \\
\theta_1 \\
R_2
\end{array}
\quad
\begin{array}{c}
\mu_i \\
V \\
1 \\
\gamma_2 \\
\varepsilon_1^x \\
\theta_2 \\
R_1
\end{array}
\quad
\begin{array}{c}
\varepsilon_1 \\
\theta_1 \\
W \\
\beta \\
\varepsilon_2 \\
\theta_2 \\
R_2
\end{array}

(2.1)

where $X^x$ is the operation of swapping colors on a graded gentle trees $X$.

$\bullet$ $\mu_V$ on the upper part follows the mirror rule of the lower part.

Dually, define the backward mutation $\mu_V^{-1}$ to be the reverse of $\mu_V$ (which follows a similar rule).

Clearly, the set of all graded gentle trees with $n$ vertexes is closed under such mutation. In fact, this set is also connected under (forward/backward) mutation.

**Lemma 2.9.** Any graded gentle tree with $n$ vertices can be iterally mutated from another graded gentle tree with $n$ vertices.

**Proof.** Use induction, starting from the trivial case when $n = 1$. Suppose that the lemma follows for $n = m$ and consider the case for $n = m + 1$. We only need to show that any graded gentle tree $G$ with $m + 1$ vertices can be iterally mutated from an unicolor graded gentle tree with all degrees equal zero. Let $V$ be a sink in $G$ and the subtree of $G$ by deleting $V$ is $G'$ while the connecting arrow from $G'$ to $V$ has degree $d$. By backward mutating on $V$, we can increase $d$ as large as possible without changing $G'$. Then the mutation at a vertex other than $V$ on $G'$ restricted to $G'$ will be the same as mutating at that vertex on $G'$. Thus, by the induction assumption, we can mutate $G$ such that $G'$ becomes unicolor with all degrees equal zero. Then, repeatedly forward mutating many times on $V$ will turn $G$ into unicolor with all degrees equal zero. \(\square\)

Using Lemma A.4, a direct calculation gives the following proposition.

**Proposition 2.10.** Let $G$ be a graded gentle tree and $H$ be a heart in $D(A_n)$. If $Q(G) = Q(H)$ with vertex $V$ in $G$ corresponding to the simple $S$ in $H$, then

$$
Q(H_S^{'}) = Q(\mu_V G), \quad Q(H_S^{'}) = Q(\mu_V^{-1} G).
$$

(2.2)

Now we can describe all Ext-quiver of hearts of $A$-type.

**Theorem 2.11.** The Ext-quivers of hearts in $D(A_n)$ are precisely the associated quivers of graded gentle trees with $n$ vertices.

**Proof.** Note that any heart in $D(A_n)$ can be iterated tilted from the standard heart $H_Q$. By [8]. Without lose of generality, let $Q$ has straight orientation. Then $Q(H_Q)$ certainly is the associated quiver for the graded gentle tree $G_Q$ with the same orientation and alternating colored arrow. Then, inducting from $H_Q$ and using (2.2), we deduce that the Ext-quiver of any heart in $D(A_n)$ is the associated quivers of some graded gentle tree with $n$ vertices. On the other hand, the set of graded gentle trees with $n$ vertices is
Then, also by induction, we deduce that the associated quiver of any graded gentle tree with \( n \) vertices is the Ext-quiver of some heart, because (2.2) and the fact that we can forward/backward tilt any simples in any heart in \( D(A_n) \) ([9, Theorem 5.7]). \( \square \)

Recall that we can CY-N double a graded quiver in the sense of [9, Definition 6.2]. Then we have the following corollary.

**Corollary 2.12.** The Ext-quivers of hearts in \( \text{EG}^0(\Gamma_N A_n) \) are precisely the CY-N double of the associated quivers of graded gentle trees with \( n \) vertices.

**Proof.** By [9, Corollary 8.3], any heart \( \mathcal{H} \) in \( \text{EG}^0(\Gamma_N A_n) \) is induced from some heart \( \mathcal{H}' \) in \( D(A_n) \), while Ext-quiver \( Q(\mathcal{H}) \) is the CY-N double of \( Q(\mathcal{H}') \) by [9, Proposition 7.5]. Thus the corollary follows from Theorem 2.11. \( \square \)

By [9, Proposition 8.6], the augmented graded quivers of colored quivers for \((N-1)\)-clusters (cf. [9, Definition 6.1] and [3]) of type \( A_n \) are also precisely the CY-N double of the associated quivers of graded gentle trees.

### 3. Associahedron

#### 3.1. Binary trees

Let \( \text{BT}_m \) be the set of binary trees with \( m + 1 \) leaves (and hence with \( m \) internal vertices), which can be used to parenthesize a word with \( m + 1 \) letters (see Figure 1 and cf. [10]). Let \( G_m \) be the full subgraph of the grid \( \mathbb{Z}^2 \) inducing by

\[
G_m = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq m\} \subset \mathbb{Z}^2.
\]

It is well-known that a binary tree with \( m + 1 \) leaves has a normal form as a subgraph of \( G_m \), such that the leaves are \( \{(x, m-x)\}_{x=0}^m \), and we will identify the binary tree with such normal form (see Figure 1).

**Example 3.1.** Let

\[
G_m^+ = G_m \cap \{(x, y) \mid xy > 0\}, \quad G_m^* = G_m - \{(0, 0)\}.
\]
Consider the $A_n$-quiver $\overrightarrow{A_n} : n \to \cdots \to 1$ and let $\mathcal{H}_n = \text{mod } k\overrightarrow{A_n}$ with corresponding simples $S_1, \ldots, S_n$. Then, there are canonical bijections (cf. Figure 2)

$$
\xi_n : G_{n+1}^+ \to \text{Ind}(\mathcal{H}_n),
\varsigma_n : G_n^* \to \text{Ind}(\mathcal{H}_n) \cup \text{Proj}(\mathcal{H}_n[1])
$$

satisfying $\xi_n(i, j) = \varsigma_n(i - 1, j) = M_{i,j}$, where $M_{i,j} \in \text{Ind}(\mathcal{H}_n)$ is determined by

$$[M_{i,j}] = \sum_{i}^{n+1-j} [S_k]. \quad (3.1)$$

Let $\zeta_n = \pi_n \circ \varsigma_n : G_n^* \to \text{Ind}(\mathcal{C}(A_n))$. 

It is known that the following sets (see [6] for more possible sets) can parameterize the vertex set of an $n$-dimensional associahedron:

1°. the set $BT_{n+1}$ of binary trees with with $n + 2$ leaves;
2°. the set of triangulations of regular $(n + 3)$-gon.
3°. the set $\text{CEG}(A_n)$ of (2-)cluster tilting sets in $\mathcal{C}(A_n)$;
4°. the set $\text{TP}(\overrightarrow{A_n})$ of torsion pairs in $\mathcal{H}_n$ (cf. [9] and [5]),
5°. the set $\text{EG}(\mathcal{H}_n, \mathcal{H}_n[1])$ of hearts in $\mathcal{D}(A_n)$ between $\mathcal{H}_n$ and $\mathcal{H}_n[1]$ (in the sense of King-Qiu, [9]).

Therefore, there are bijections between these sets.

Furthermore, by [9, Section 9], the poset structure of torsion pairs (hearts) gives an orientation $O_t$ of the $n$-dimensional associahedron, i.e. the orientation of $\text{EG}(\mathcal{H}_n, \mathcal{H}_n[1])$ (considered as a subgraph of $\text{EG}(A_n)$).

On the other hand, there is a poset structure of binary trees, inducing by locally flipping a binary tree (as shown in Figure 3), or equivalently, changing the corresponding parenthesizing of words from $(A \cdot B) \cdot C$ to $A \cdot (B \cdot C)$ (see [10] for details). This poset structure also gives an orientation $O_p$ for the associahedron. We aim to prove $O_t = O_p$ this section.

3.2. Combinatorial constructions. First, we give explicit construction of torsion pairs from binary trees. For any $p \in \mathbb{Z}^2$ with coordinate $(x_p, y_p)$, let $L(p)$ be the edge
connecting \((x_p - 1, y_p)\) and \(p\) and \(R(p)\) be the edge connecting \((x_p, y_p - 1)\) and \(p\). Define
\[
T(b) = \langle \xi_n(p) \mid p \in G_{n+1}^+, L(p) \in b \rangle, \quad F(b) = \langle \xi_n(p) \mid p \in G_{n+1}^+, R(p) \in b \rangle,
\]
where \(\langle \rangle\) means generating by extension.

**Proposition 3.2.** There is a bijection \(\Theta_n : BT_{n+1} \to TP(\tilde{A}_n)\), sending \(b \in BT_{n+1} \) to \(\langle T(b), F(b) \rangle\).

**Proof.** We only need to show that \(\Theta_n : b \mapsto \langle T(b), F(b) \rangle\) is well-defined (and obviously injective) and hence bijective since both sets have \(n\) elements.

To do so, we first show that any object \(M \in H_n\) admits a short exact sequence
\[
0 \to T \to M \to F \to 0 \quad (3.3)
\]
for some \(T \in T(b)\) and \(F \in F(b)\). Let \(m = \xi_n^{-1}(M) \in G_{n+1}^+\). If \(m \in b\) then \(M \in T(b) \cup F(b)\) and we have a trivial short exact sequence\((3.3)\). If \(m \notin b\), let \(t\) be the vertex in \(b \cap \{ (x_m, j) \mid j \geq y_m \}\) with minimal \(y\)-coordinate and \(f\) be the vertex in \(b \cap \{ (i, y_m) \mid i \geq x_m \}\) with minimal \(x\)-coordinate; let \(a\) and \(b\) be the vertices with coordinates \((n + 1 - y_t, y_t)\) and \((x_f, n + 1 - x_f)\), see Figure 4. By construction and the property of the binary tree, we know that

- edges in the line segments, from \(m\) to \(t\) and from \(m\) to \(f\), are not in \(b\);
edges in the line segments, from \(a\) to \(t\) and from \(b\) to \(f\), are in \(b\);
\[(x_a, y_a) + (1, -1) = (x_b, y_b), \text{i.e.} \ a, b \text{ are neighbors in the line } x + y = n + 1;\]
\(L(t)\) and \(R(f)\) are in \(b\).

Thus \(T = \xi_n(t) \in \mathcal{T}(b)\) and \(F = \xi_n(f) \in \mathcal{F}(b)\). By (3.1), a direct calculation shows that \([M] = [T] + [F]\), which implies we have (3.3), by Lemma A.1, as required.

To finish, we need to show that \(\text{Hom}(\mathcal{T}(b), \mathcal{F}(b)) = 0\). Let \(F = \xi_n(f) \in \mathcal{F}(b)\). As above, edges in the line segments from \(b\) to \(f\) are in \(b\). By the property of binary tree, the horizontal edges (i.e. parallelling to x-axis) in the shadow area in Figure 4 are not in \(b\), which implies, by Lemma A.2, that the modules in \(H_n\) that has nonzero maps to \(F\) are not in \(\mathcal{T}(b)\), as required.

Next, we identify cluster tilting sets from binary trees via \(\zeta_n\). For any \(b \in \text{BT}_{n+1}\), let \(iv(b)\) be set of the internal vertices expect \((0, 0)\) so that \(#iv(b) = n\). Denote by \(\text{Proj} H\) a complete set of indecomposable projectives of a heart \(H\). Recall ([9, Section 2]) that 
\[P \in \text{Proj} H \iff P \in \text{Ind}(P \cap \tau^{-1}P^\perp),\]
where \(\mathcal{P}\) is the t-structure corresponding to \(H\).

**Proposition 3.3.** Let \(b \in \text{BT}_{n+1}\) and \(H(b)\) be the heart corresponding to the torsion pair \(\Theta_n(b)\) in \(H_n\). Then we have \(\text{Proj} H(b) = \zeta_n(iv(b))\) and there is a bijection \(\zeta_n \circ iv : \text{BT}_{n+1} \to \text{CEG}(A_n)\).

**Proof.** By [9, Corollary 5.12], we know that \(\pi_n \text{Proj} H(b) \in \text{CEG}(A_n)\) and hence the second claim follows immediately from the first one.

Let \(p \in iv(b)\), which is the intersection of the edges \(L(r)\) and \(R(q)\), where \(q, r\) be the points with coordinates \((x_p, y_p + 1)\) and \((x_p + 1, y_p)\) (see Figure 5). Note that \(p\) is not in the line \(x_p + y_p \leq n\) and thus \(q, r \in G_{n+1}\). Let \(\mathcal{P}(b)\) be the t-structure corresponding to \(H(b)\). Note that 
\[\mathcal{P}(b) = \mathcal{T}(b) \cup \bigcup_{j > 0} H_n[j], \quad \mathcal{P}(b)^\perp = \mathcal{F}(b) \cup \bigcup_{j < 0} H_n[j].\]

If \(r \in G_{n+1}^+\), then \(P = \zeta_n(p) = \xi_n(r)\) is in \(\mathcal{T}(b)\); otherwise, \(y_p = 0\) and then \(P \in H_n[1]\). Either way, \(P \in \mathcal{P}(b)\). Similarly, if \(q \in G_{n+1}^+\), then \(\tau P = \xi_n(q)\) is in \(\mathcal{F}(b)\); otherwise,
$x_p = 0$ and then $\tau P \in \mathcal{H}_{n}[−1]$. Either way, $\tau P \in \mathcal{P}(b)^\perp$. Therefore $P \in \text{Proj} \mathcal{H}(b)$ by (3.4). Thus $\text{Proj} \mathcal{H}(b)$ contains, and hence equals $\varsigma_n(\text{iv}(b))$ as required, noticing that $\# \text{Proj} \mathcal{H}(b) = n = \# \text{iv}(b)$. □

Example 3.4. Keep the notation in Example 3.1. Then the binary tree in Figure 1 corresponds to the torsion pair $\mathcal{T} = (M_{2,2}, M_{1,2}), \mathcal{F} = (M_{1,3}, M_{3,1})$ and the cluster tilting set $\{M_{1,2}, M_{2,2}, M_{1,1}[1]\}$.

3.3. The orientation. Now we apply the constructions above to show that $O_t = O_p$.

Theorem 3.5. Under the bijection $\Theta_n$ in Proposition 3.2, the orientations $O_t$ and $O_p$ of the $n$-dimensional associahedron coincide.

Proof. Consider an edge $e : b_1 \rightarrow b_2$ in $\text{BT}_{n+1}$, which corresponds to a local flip as in Figure 3. Let $\mathcal{H}(b_i)$ forward tilt of $\mathcal{H}_{n}$ with respect to $\Theta_n(b_i)$. We only need to show that $\mathcal{H}(b_2)$ is a simple forward tilt of $\mathcal{H}(b_1)$.

By Proposition 3.3, we know that $\text{Proj} \mathcal{H}(b_i) = \varsigma_n(\text{iv}(b_i))$, for $i = 1, 2$, defer by one object. Denote by $P_i \in \text{Proj} \mathcal{H}(b_i)$ the different objects. Thus, $\pi_n \text{Proj} \mathcal{H}(b_i) \in \text{CEG}(A_n)$ are related by one mutation, which implies $\mathcal{H}(b_i)$ are related by a single simple tilting, by [9, Corollary 5.12], and $P_1, P_2$ are related by some triangle $P_j \rightarrow M \rightarrow P_k \rightarrow P_j[1]$ in $\mathcal{D}(A_n)$ for some ordering $\{j, k\} = \{1, 2\}$. By Lemma A.3, $P_j$ is a predecessor of $P_k$. But, from the flip we know that $P_1$ is the predecessor of $P_2$, which implies $j = 1$ and $k = 2$. Thus the forward simple tiling is from $\mathcal{H}(b_1)$ to $\mathcal{H}(b_2)$ as required. □

Example 3.6. Figure 6 is the orientation of the 2-dimensional associahedron, induced by poset structure of binary trees, which is the oriented pentagon in [9, Figure 3] and [11, (3.5)], cf. also [7, Figure 5].

![Figure 6. The orientation of the 2-dimensional associahedron](image-url)
APPENDIX A. MAPS AND TRIANGLES IN D(An)

In this appendix, we collect several facts about the maps and triangles in D(A_n). See [2, Chapter IX] for the proofs of the first three lemmas.

Recall there are notions of sectional paths and predecessors in D(A_n) cf. [11, Section 2.2].

Lemma A.1. Let M, A, B ∈ Ind D(A_n) such that A ∈ Ps^{-1}(M) and B ∈ Ps(M) − Ps(A). Then there is a short exact sequence 0 → A → M → B → 0 if and only if [M] = [A] + [B].

Lemma A.2. Let M, L ∈ Ind D(A_n). Then Hom(M, L) ≠ 0 if and only if L ∈ [Ps(M), Ps^{-1}(τ(M[1]))], M ∈ [Ps(τ^{-1}(L[-1])), Ps^{-1}(L)].

Lemma A.3. If Hom(L, M[1]) ≠ 0 for some M and L in Ind D(A_n), then M is a predecessor of L. Any two non-isomorphic indecomposables in D(A_n) can not be predecessors of each other.

Lemma A.4. Let H be a heart in D(A_n). If there are the following full sub-quivers

in the Ext-quiver Q(H) for some S, T, A, B, C, D ∈ Sim H and positive integer a, b, c, d, then there are following full sub-quivers

in the Ext-quiver Q(H^1_S), where R is the nontrivial extension of T on top of S.

Proof. We only prove the first case while the other cases are similar. By [9, Theorem 5.7], we know that the simples in H^1_S corresponding to S, T and A are S[1], R and A. By [11, Lemma 3.3], we have an isomorphism Hom^1(T, S) ⊗ Hom^a(S, A) → Hom^{a+1}(T, A). Thus, applying Hom(−, A) to the triangle S → R → T → S[1] gives Hom^*(R, A) = 0. Similarly, a direct calculation of other Hom^• between S[1], R, A shows the new sub-quiver is as required. □

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