A novel perspective on the Sun-Jupiter-comet three-body system

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Abstract

Within the solar system, approximate realizations of the three-body problem occur when a comet approaches a planet which is at a certain distance from the Sun, and this configuration was investigated by Tisserand within the framework of Newtonian gravity. The exact relativistic treatment of the problem is not an easy task, but the present paper develops first an approximate calculational scheme which computes for the first time the tiny effective-gravity correction to the equation of the surface for all points of which it is equally advantageous to regard the heliocentric motion as being perturbed by the attraction of Jupiter, or the jovicentric motion as being perturbed by the attraction of the Sun. In the second part a fully relativistic treatment is instead developed, obtaining the relativistic modifications to the dynamics of comets and displaying their orbits. Eventually, the observational tests of corrections to Newtonian formulas for the Sun-Jupiter-comet system are discussed.

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I. INTRODUCTION

A complete understanding of potentialities, applications and limits of Newton’s and Einstein’s theories of gravity has required dedicated efforts along more than three centuries, by now. For example, at the end of nineteenth century the monumental treatise by Tisserand on celestial mechanics [1] presented in great detail the work of d’Alembert and Laplace on the motion of comets when they are approaching a planet. This analysis stimulated Fermi himself, when he wrote his Scuola Normale Superiore dissertation [2], devoted to an investigation of cometary orbits with the help of probability theory and of the classical theory of restricted three-body problems.

What has motivated our research has been therefore, on the one hand, the many (recent) investigations of three-body problems in general relativity [3–8] and effective-field-theory models of gravity [9–19], and on the other hand the consideration that the passage of comets provides in the solar system some very interesting realizations of three-body systems in celestial mechanics. Short-period comets [20] are thought to generate in the Kuiper belt and have predictable orbits with short periods, i.e., up to 200 years. Two major families of short-period comets are the Jupiter family with periods of less than 20 years and the Halley family with periods in between 20 and 200 years. Interestingly, even though their orbits can be predicted with some accuracy, some of these short-period comets might be gravitationally perturbed and become long-period objects. More precisely, gravitational effects of the outer planets can cause these bodies to alter their paths into highly elliptical orbits that take them close to the Sun. Long-period comets are instead thought to generate in the Oort cloud and have unpredictable orbits, with periods much longer than 200 years. Their detection is extremely difficult for mankind because they can return on their steps after thousands or even millions of year (or not at all).

Several processes deserve careful consideration, e.g., the capture of comets with parabolic orbit by Jupiter [21]. Another intriguing difficulty is the choice of the appropriate formalism for our analysis. In the last decade of the twentieth century, the outstanding work by Damour, Soffel and Xu [22–25] led indeed to a complete prescription for studying equations of motion of $N$ bodies in celestial mechanics, which involve a repeated application of the inverse of the wave operator and are therefore nonlocal. Since, for a three-body problem, we are still far from the level of accuracy reached for relativistic binary systems [26–28],

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we here resort first to a shortcut, i.e. an approximation method, a hybrid scheme, which is nevertheless of physical interest, to the same extent that ordinary quantum mechanics, despite not being a relativistic quantum theory, is of much help in evaluating bound states and transition probabilities. The milestones we rely upon are as follows.

(i) In the effective-field-theory approach to quantum gravity, one discovers that the Newtonian potential among two bodies of masses $M_A$ and $M_B$ should be replaced by the expression \cite{12,14,16,18}

$$V_E(r) = -\frac{GM_A M_B}{r_{AB}} \left[ 1 + \kappa_1 \frac{(L_A + L_B)}{r_{AB}} + \kappa_2 \left( \frac{L_P}{r_{AB}} \right)^2 + O(G^2) \right],$$

(1.1)

where $L_A$ and $L_B$ are their gravitational radii

$$L_A \equiv \frac{GM_A}{c^2}, \quad L_B \equiv \frac{GM_B}{c^2},$$

(1.2)

$L_P$ is the Planck length

$$L_P \equiv \sqrt{\frac{G\hbar}{c^3}},$$

(1.3)

the length $r_{AB}$ is the mutual distance among the bodies, while $\kappa_1$ and $\kappa_2$ are dimensionless parameters obtained either from a detailed application of Feynman diagrams’ technique \cite{14,16} or the modern on-shell unitary-based methods \cite{29,30}. It should be stressed that, although the numerical effect of $L_P$ is immaterial in the calculations devoted to Lagrangian points and their stability \cite{9–12}, the dependence of $\kappa_1$ on $\kappa_2$ implies that the weight factor $\kappa_1$ for the classical gravitational radii depends actually on the underlying quantum world.

(ii) The formula \cite{14,16} tells us that quantum corrections map the dimensionless ratio

$$U_A \equiv \frac{L_A}{r_{AB}},$$

(1.4)

into \cite{12}

$$V_A \sim \left[ 1 + \kappa_2 \left( \frac{L_P}{r_{AB}} \right)^2 \right] U_A + \kappa_1 (U_A)^2.$$  

(1.5)

Thus, if (1.5) is applied to replace $U_A$ and $U_B$ in the Newtonian formula of effective potential $W_{\text{eff}}$ for circular restricted three-body problem, one finds in $c = 1$ units \cite{12}, denoting by $\omega$ the angular frequency for rotation of the $\xi, \eta$ axes about the $\zeta$ axis \cite{4},

$$W_{\text{eff}} = \frac{\omega^2}{2} (\xi^2 + \eta^2) + \left[ U_A + U_B + \kappa_1 ((U_A)^2 + (U_B)^2) \right] + O(G^2).$$

(1.6)
Remarkably, the general relativity formula in \( c = 1 \) units is instead \[4, 12\]

\[
W_{\text{eff}} \sim \frac{\omega^2}{2}(\xi^2 + \eta^2) + \left[U_A + U_B - \frac{1}{2}((U_A)^2 + (U_B)^2)\right] + \text{remainder},
\]

(1.7)

and the first lines of (1.6) and (1.7) agree if \( \kappa_1 = -\frac{1}{2} \). The value \( \kappa_1 = -\frac{1}{2} \) is indeed allowed by the effective-field-theory approach to quantum gravity, and it corresponds to the so-called bound-state option for the underlying Feynman diagrammatics \[12, 14, 16, 18\]. This means that, upon application of (1.1) or (1.5) to the Newtonian formulas for three-body problems, one can obtain many of the relevant terms of the associated effective potential, so that a valuable recipe is available. According to it, one can insert (1.1) (or (1.5)) in all Newtonian formulas, and obtain valuable information on the otherwise (too) lengthy fully relativistic calculations. Bearing in mind also this feature, which was first pointed out in Ref. \[12\], the plan of our paper is as follows.

Section II describes the Newtonian formulation of perturbations of cometary motions when comets come very close to planets. Section III finds the leading relativistic corrections of this treatment with the help of the map (1.1), by focusing on the equation of the surface for all points of which it is equally advantageous to view the heliocentric motion as being perturbed by the attraction of Jupiter, or the jovicentric motion as being perturbed by the attraction of the Sun. The general relativistic analysis of the Sun-Jupiter system is instead developed in Secs. IV and V, arriving eventually at the equations that describe a comet step by step along its spacetime path towards the Sun. Observational tests are then discussed in Sec. VI and concluding remarks are made in Sec. VII.

II. PERTURBATIONS OF COMETARY MOTIONS IN NEWTONIAN GRAVITY

Following the monograph of Tisserand \[1\], we denote by \((x, y, z)\) the heliocentric rectangular coordinates of the comet, by \((x', y', z')\) the coordinates of the planet, here taken to be Jupiter, and by \((\xi, \eta, \zeta)\) the jovicentric coordinates of the comet, with respect to axes that are parallel to the fixed axes. Moreover, \(m_\odot = 2 \times 10^{30} \) Kg is the mass of the Sun, \(m'\) is the mass of Jupiter:

\[
\frac{m'}{m_\odot} = \frac{1}{1047},
\]

(2.1)
\( \rho \) is the Euclidean distance comet-Jupiter, \( r, r' \) the Euclidean distances comet-Sun and Jupiter-Sun, respectively. The Newtonian equations of motion turn out to be

\[
\begin{align*}
\frac{d^2}{dt^2} + \frac{Gm_\odot}{r^3} x &= Gm' \left( \frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right), \\
\frac{d^2}{dt^2} + \frac{Gm_\odot}{r^3} y &= Gm' \left( \frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right), \\
\frac{d^2}{dt^2} + \frac{Gm_\odot}{r^3} z &= Gm' \left( \frac{z' - z}{\rho^3} - \frac{z'}{r'^3} \right),
\end{align*}
\]

(2.2)

\[
\begin{align*}
\frac{d^2}{dt^2} + Gm'_\odot \rho^3 \xi &= Gm' \left( \frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right), \\
\frac{d^2}{dt^2} + Gm'_\odot \rho^3 \eta &= Gm' \left( \frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right), \\
\frac{d^2}{dt^2} + Gm'_\odot \rho^3 \zeta &= Gm' \left( \frac{z' - z}{\rho^3} - \frac{z'}{r'^3} \right),
\end{align*}
\]

(2.10)

where

\[
\Omega^2 = \frac{G(m_\odot + m')}{r'^3}.
\]

(2.8)

The unprimed, primed and Greek lower case coordinates are related by linear equations, i.e.

\[
x = x' + \xi, \quad y = y' + \eta, \quad z = z' + \zeta,
\]

(2.9)

and by virtue of (2.2)-(2.9) one obtains the system of second-order equations

\[
\begin{align*}
\frac{d^2}{dt^2} + \frac{Gm_\odot}{\rho^3} \xi &= Gm' \left( \frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right), \\
\frac{d^2}{dt^2} + \frac{Gm_\odot}{\rho^3} \eta &= Gm' \left( \frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right), \\
\frac{d^2}{dt^2} + \frac{Gm_\odot}{\rho^3} \zeta &= Gm' \left( \frac{z' - z}{\rho^3} - \frac{z'}{r'^3} \right),
\end{align*}
\]

(2.10)

Equations (2.2)-(2.4) pertain to the heliocentric motion of the comet, whereas (2.5)-(2.8) refer to the elliptical motion of Jupiter around the barycenter of the system, which for simplicity coincides exactly with the Sun position; let now \( R \) be the modulus of the force per unit mass resulting from the attraction of the Sun, and let \( F \) be the modulus of the perturbing force per unit mass. They are given by

\[
R = \frac{Gm_\odot}{r'^2},
\]

(2.13)
\[ F = Gm' \sqrt{\left( \frac{(x' - x)}{\rho^3} - \frac{x'}{r'^3} \right)^2 + \left( \frac{(y' - y)}{\rho^3} - \frac{y'}{r'^3} \right)^2 + \left( \frac{(z' - z)}{\rho^3} - \frac{z'}{r'^3} \right)^2}. \] (2.14)

Equations (2.10)-(2.12) pertain instead to the jovicentric motion produced by the attraction \( R' \) of Jupiter, and the perturbing force exerted by the Sun. The modulus of such forces per unit mass reads as

\[ R' = \frac{Gm'}{\rho^2}, \] (2.15)

and

\[ F' = Gm_\odot \sqrt{\left( \frac{x'}{r'^3} - \frac{x}{r^3} \right)^2 + \left( \frac{y'}{r'^3} - \frac{y}{r^3} \right)^2 + \left( \frac{z'}{r'^3} - \frac{z}{r^3} \right)^2}, \] (2.16)

respectively.

A. Heliocentric vs. jovicentric motion

A concept of crucial importance is expressed by the condition \[ \frac{F}{R} = \frac{F'}{R'}, \] (2.17)

which defines implicitly a surface for all points of which it is equally advantageous to regard the heliocentric motion as being perturbed by the attraction of Jupiter, or the jovicentric motion as being perturbed by the attraction of the Sun. On denoting simply by \( m' \) the ratio in (2.1) (this means that the Sun provides the unit of mass), condition (2.17) reads eventually as

\[ m'^2 r^2 \sqrt{\frac{1}{\rho^4} + \frac{1}{r'^4} + 2 \frac{x' \xi + y' \eta + z' \zeta}{\rho^4 r'^3}} = \rho^2 \sqrt{\frac{1}{r^4} + \frac{1}{r'^4} - 2 \frac{xx' + yy' + zz'}{r^3 r'^3}}, \] (2.18)

where we have exploited the identities

\[ r^2 = x^2 + y^2 + z^2, \quad r'^2 = x'^2 + y'^2 + z'^2, \quad \rho^2 = \xi^2 + \eta^2 + \zeta^2. \] (2.19)

At this stage, it is convenient to introduce the variables \( \theta \) and \( u \) by means of the definitions

\[ \cos(\theta) \equiv \frac{(x' \xi + y' \eta + z' \zeta)}{\rho r'}, \] (2.20)

\[ u \equiv \frac{\rho}{r'}. \] (2.21)
The dimensionless parameter $u$ approaches 0, since it is only at short distance from Jupiter that the transformation considered can be convenient. Moreover, by virtue of the linear relations, a term on the right-hand side of (2.18) reads as

\[ xx' + yy' + zz' = r'^2 (1 + u \cos(\theta)), \quad (2.22) \]

while the squared distance $r^2$ can be expressed in the form

\[ r^2 = (x' + \xi)^2 + (y' + \eta)^2 + (x' + \zeta)^2 = r'^2 (1 + 2u \cos(\theta) + u^2). \quad (2.23) \]

In light of (2.20)-(2.23), we can express the squared Jupiter mass in (2.18) by means of $u$ and $\theta$ only, after writing $\rho = ur'$ and then expressing $r'$ from (2.23). Hence one finds

\[ m' = \frac{u^4}{(1 + 2u \cos(\theta) + u^2)^2 \sqrt{1 + 2u^2 \cos(\theta) + u^4}} \sqrt{P(u, \theta)}. \quad (2.24) \]

The exact form of the function $P$ is

\[ P(u, \theta) \equiv 1 + (1 + 2u \cos(\theta) + u^2)^2 - 2\sqrt{1 + 2u \cos(\theta) + u^2} (1 + u \cos(\theta)). \quad (2.25) \]

Since $u$ approaches 0 in our physical model, we only need the small-$u$ expansion of $P(u, \theta)$. For this purpose, we first notice that, at fixed $\theta$,

\[ f(u) \equiv \sqrt{1 + 2u \cos(\theta) + u^2} = 1 + u \cos(\theta) + \frac{1}{2} \sin^2(\theta) u^2 - \frac{1}{2} \cos(\theta) \sin^2(\theta) u^3 + O(u^4), \quad (2.26) \]

and hence we find, after following patiently a number of exact or partial cancellations,

\[ P(u, \theta) = u^2[1 + 3 \cos^2(\theta)] + 4u^3 \cos(\theta) + O(u^4). \quad (2.27) \]

By virtue of (2.24) and (2.27), we obtain first the approximate formula

\[ m'^2 = u^4 \sqrt{u^2[1 + 3 \cos^2(\theta)] + 4u^3 \cos(\theta) + O(u^4)} \]

\[ = \frac{u^4(1 - 4u \cos(\theta) + O(u^2))\sqrt{1 + 3 \cos^2(\theta) + 4u \cos(\theta)}}{(1 + 4u \cos(\theta) + O(u^2))(1 + O(u^2))}, \quad (2.28) \]

and eventually we solve approximately for the dimensionless parameter $u$ in the form

\[ u = \left( \frac{m^2}{\sqrt{1 + 3 \cos^2(\theta)}} \right)^\frac{1}{8} \left( 1 + 2u \cos(\theta) \frac{[1 + 6 \cos^2(\theta)]}{[1 + 3 \cos^2(\theta)]} \right)^\frac{1}{8}, \]

\[ \approx \left( \frac{m^2}{\sqrt{1 + 3 \cos^2(\theta)}} \right)^\frac{1}{5} + \frac{2}{5} \cos(\theta) \left( \frac{m'^2}{\sqrt{1 + 3 \cos^2(\theta)}} \right)^\frac{2}{5} \frac{[1 + 6 \cos^2(\theta)]}{[1 + 3 \cos^2(\theta)]}. \quad (2.29) \]

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By virtue of (2.1), one has in all cases
\[
\left( \frac{m^2}{\sqrt{1 + 3 \cos^2(\theta)}} \right)^{\frac{1}{5}} < m^{\frac{2}{5}} = 0.062,
\]
and hence one can safely use the approximate formula
\[
\rho = r' \left( \frac{m^2}{\sqrt{1 + 3 \cos^2(\theta)}} \right)^{\frac{1}{5}}.
\]
This is the approximate equation of the desired surface in polar coordinates \( \rho \) and \( \theta \), the polar axis being obtained by extending the line \( SP \) joining the Sun with the planet, having set the origin at the center of the planet. This is a surface of revolution about the \( SP \) line, and does not differ much from a sphere, since \( \rho \) varies in between the two limits
\[
m^{\frac{2}{5}} r' \text{ and } \frac{m^{\frac{2}{5}}}{2^{\frac{1}{5}}} r',
\]
which correspond to \( \theta = \frac{\pi}{2} \) and \( \theta = 0 \), respectively, and whose ratio equals 1.15. One can therefore say, with little error, that the surface defined by the condition (2.17) is a sphere of radius \( m^{\frac{2}{5}} r' \), called by Laplace the sphere of influence of the planet.\(^1\) Outside of such a sphere, one has \( \frac{E}{R} < \frac{E'}{R'} \), and it is therefore advantageous to start from the heliocentric motion of the comet, and to evaluate the perturbations caused by the planet. Within the sphere of influence, one has instead \( \frac{E'}{R'} < \frac{E}{R} \), and it is more advantageous to consider the jovicentric motion, and to evaluate as a next step the perturbations resulting from the Sun.

III. PERTURBATIONS OF COMETARY MOTIONS IN EFFECTIVE FIELD THEORIES OF GRAVITY

The quantum effects considered in (1.1) affect the potential, whereas the Newtonian model outlined in Sec. II relies upon the evaluation of forces. Thus, we need to propose first a modified force formula in order to write down the effective-gravity counterpart of Sec. II. For this purpose, we assume that we can still express the force as minus the gradient of the effective potential, i.e.,
\[
F_k = (-\text{grad } V_E)_k, \quad k = 1, 2, 3, \quad x_1 = x, x_2 = y, x_3 = z,
\]
\(^1\) Strictly, Laplace considered the fifth root of half the square of \( m' \), as pointed out by Tisserand [I].
which implies that (since \( \frac{\partial r}{\partial x_k} = \frac{x_k}{r} \))

\[
F_k = -\frac{GM_AM_B}{r^3} x_k \left[ 1 + 2\kappa_1 \frac{(L_A + L_B)}{r} + 3\kappa_2 \left( \frac{L_P}{r} \right)^2 + O(G^2) \right].
\]

(3.2)

The Newtonian formulas (2.13) and (2.14) for the modulus of perturbing forces receive therefore an additional contribution from \( \kappa_1 \) according to the prescriptions (we use the subscript \( E \) to denote the influence of effective-gravity calculations, and we neglect the gravitational radius \( L_J \) of Jupiter with respect to the gravitational radius \( L_S \) of the Sun)

\[
R_E \sim \frac{G m_{\odot}}{r^2} \left[ 1 + 2\kappa_1 \frac{L_J}{r} + O(L_P^2) \right],
\]

(3.3)

\[
F_E \sim G m' \sqrt{P_1} \left[ \sum_{k=1}^{3} \left[ \frac{(x'_k - x_k)}{\rho^3} \right] \left( 1 + 2\kappa_1 \frac{L_J}{\rho} \right) - \frac{x'_k}{r^3} \left( 1 + 2\kappa_1 \frac{L_S}{r} \right) + O(L_P^2) \right]^2,
\]

(3.4)

where \( x'_1 = x', x'_2 = y', x'_3 = z' \). Now we point out that, up to \( O(L_P^2) \) terms,

\[
\left[ \frac{(x'_k - x_k)}{\rho^3} \right] \left( 1 + 2\kappa_1 \frac{L_J}{\rho} \right) - \frac{x'_k}{r^3} \left( 1 + 2\kappa_1 \frac{L_S}{r} \right)
\]

\[
= \left( \frac{(x'_k - x_k)}{\rho^3} - \frac{x'_k}{r^3} \right)^2 + 4\kappa_1 \left( \frac{(x'_k - x_k)}{\rho^3} - \frac{x'_k}{r^3} \right) \left( \frac{L_J}{\rho} \frac{(x'_k - x_k)}{\rho^3} - \frac{L_S}{r'} \frac{x'_k}{r^3} \right)
\]

\[
+ O(G^2),
\]

(3.5)

and hence we find

\[
F_E \sim G m' \sqrt{P_1},
\]

(3.6)

where

\[
P_1(u, \theta) = \frac{1}{\rho^4} + \frac{1}{r'^4} + 2 \frac{(x'\xi + y'\eta + z'\zeta)}{\rho^3 r'^3}
\]

\[
+ 4\kappa_1 \sum_{k=1}^{3} \left[ \left( \frac{(x'_k - x_k)}{\rho^3} - \frac{x'_k}{r^3} \right) \left( \frac{L_J}{\rho} \frac{(x'_k - x_k)}{\rho^3} - \frac{L_S}{r'} \frac{x'_k}{r^3} \right) \right] + O(G^2)
\]

\[
= \frac{1}{\rho^4} + \frac{1}{r'^4} + \frac{2}{(\rho r')^2} \cos(\theta) + 4\kappa_1 f_1 + O(G^2),
\]

(3.7)

having set (see (2.9))

\[
f_1 \equiv \sum_{k=1}^{3} \left[ \left( \frac{\xi_k}{\rho^3} + \frac{x'_k}{r^3} \right) \left( \frac{L_J}{\rho} \frac{\xi_k}{\rho^3} + \frac{L_S}{r'} \frac{x'_k}{r^3} \right) \right],
\]

(3.8)

with \( \xi_1 = \xi, \xi_2 = \eta, \xi_3 = \zeta \).
With analogous procedure, we propose to replace the Newtonian formulas (2.15) and (2.16) with their effective-gravity counterparts

\[ R_E' \sim \frac{Gm'}{\rho^2} \left( 1 + 2\kappa_1 \frac{L_J}{\rho} + O(L_P^2) \right), \quad (3.9) \]

\[ F_E' \sim Gm_\odot \sqrt{P_2} \sqrt{3} \sum_{k=1}^{3} \left[ \frac{x'_k}{r^3} - \frac{x_k}{r^3} + 2\kappa_1 \left( \frac{L_S x'_k}{r^4} - \frac{L_S x_k}{r^3} \right) \right] + O(L_P^2)^2. \quad (3.10) \]

Therefore, we can write

\[ F_E' \sim Gm_\odot \sqrt{P_2}, \quad (3.11) \]

where up to \( O(L_P^2) \) (see (2.22))

\[ P_2(u,\theta) = \frac{1}{r^4} + \frac{1}{r^4} - 2r^2(1 + u \cos(\theta)) + 4\kappa_1 L_S f_2 + O(G^2), \quad (3.12) \]

having set

\[ f_2 \equiv \sum_{k=1}^{3} \left[ \left( \frac{x'_k}{r^3} - \frac{x_k}{r^3} \right) \left( \frac{L_S x'_k}{r^4} - \frac{L_S x_k}{r^4} \right) \right]. \quad (3.13) \]

The effective-gravity counterpart of condition (2.17), i.e.,

\[ \frac{F_E}{R_E} = \frac{F_E'}{R_E'}, \quad (3.14) \]

leads therefore to the equation (cf. (2.18) and (2.24))

\[ m'^2 = \left( \frac{\rho}{r} \right)^2 \left( 1 + 2\kappa_1 \frac{L_S}{r} + O(L_P^2) \right) \left( 1 + 2\kappa_1 \frac{L_J}{\rho} + O(L_P^2) \right)^{-1} \sqrt{P_2(u,\theta) \over P_1(u,\theta)}. \quad (3.15) \]

At this stage, by exploiting Eqs. (2.9) and (2.19)-(2.23) we can rewrite Eqs. (3.8) and (3.13) as follows (the calculations below are both new and very instructive, hence they deserve our special care):

\[ f_1 = \frac{1}{r^7} \sum_{k=1}^{3} \left[ \left( \frac{x'_k + \xi_k}{w^3} \right) \left( \frac{L_S x'_k + L_J \xi_k}{w^4} \right) \right] \]

\[ = \frac{1}{\rho^5} \left[ L_J \rho^2 + (L_S u + L_J) \rho r' \cos(\theta) u^3 + L_S r'^2 u^7 \right] \]

\[ = \frac{1}{\rho^5} \left[ L_J + (u L_S + L_J) \cos(\theta) u^2 + L_S u^5 \right]. \quad (3.16) \]
\[ f_2 = \frac{1}{r_0^2} \sum_{k=1}^{3} \left\{ \left[ x'_k - \gamma^{-\frac{3}{2}} (x'_k + \xi_k) \right] \left[ x'_k - \gamma^{-2} (x'_k + \xi_k) \right] \right\} \]
\[ = \frac{1}{r_0^2} \left\{ r'^2 + \gamma^{-\frac{3}{2}} \left[ r^2 - (r'^2 + p r' \cos(\theta)) \left( \gamma^{-\frac{3}{2}} + \gamma^2 \right) \right] \right\} \]
\[ = \frac{u^5}{\rho^5} \left\{ 1 + \gamma^{-\frac{5}{2}} \left[ 1 - \left( \gamma^{\frac{1}{2}} + \gamma \right) (1 + u \cos(\theta)) \right] \right\}, \quad (3.17) \]

where
\[ \gamma \equiv 1 + 2u \cos(\theta) + u^2. \quad (3.18) \]

In view of the forthcoming calculations, it will be useful to write (3.17) as
\[ f_2 = \left( \frac{u}{\rho} \right)^5 B(\gamma), \quad (3.19) \]

with
\[ B(\gamma) \equiv 1 + \gamma^{-\frac{5}{2}} \left[ 1 - \left( \gamma^{\frac{1}{2}} + \gamma \right) (1 + u \cos(\theta)) \right], \quad (3.20) \]

and to take into account that (see (2.23))
\[ \rho \cdot r = u \left( \frac{r'}{r} \right) = u \left( \gamma^{-\frac{1}{2}} \right). \quad (3.21) \]

Bearing in mind the above calculations, we can write Eq. (3.15) as
\[ m'^2 = \frac{u^4 \left( 1 + 2\kappa_1 \frac{L_s}{r} + O(L_p^2) \right)}{\gamma^2 \left( 1 + 2\kappa_1 \frac{L_1}{\rho} + O(L_p^2) \right)} \sqrt{\frac{P_2(u, \theta)}{P_1(u, \theta)}} \quad (3.22) \]

Inspired by the classical calculations of Sec. II, we extract the factor \( 1/r^4 \) and \( 1/\rho^4 \) from \( P_2(u, \theta) \) and \( P_1(u, \theta) \), respectively, and hence on exploiting (3.21) the previous equation can be arranged as
\[ m'^2 = \frac{u^4 \left( 1 + 2\kappa_1 \frac{L_s}{r} + O(L_p^2) \right)}{\gamma^2 \left( 1 + 2\kappa_1 \frac{L_1}{\rho} + O(L_p^2) \right)} \sqrt{\frac{N(u, \theta)}{D(u, \theta)}} \quad (3.23) \]

where (cf. Eqs. (2.24) and (2.25))
\[ N(u, \theta) = 1 + \gamma^2 - 2\sqrt{\gamma} (1 + u \cos(\theta)) + 4\kappa_1 L_s (r^4 f_2) + O(G^2) = r^4 P_2(u, \theta), \quad (3.24) \]
\[ D(u, \theta) = 1 + 2u^2 \cos(\theta) + u^4 + 4\kappa_1 (\rho^4 f_1) + O(G^2) = \rho^4 P_1(u, \theta). \quad (3.25) \]

In order to consider the effective gravity version of the sphere of influence of the planet, we need to evaluate the small-\( u \) behaviour of Eq. (3.23). We begin with (3.24). The last term
of this equation represents a pure effective gravity effect and its expansion reads as (see Eqs. \(3.19\) - \(3.21\))

\[
4\kappa_1 L_S r^4 f_2 = 4\kappa_1 \left( \frac{L_S}{\rho} \right) [u \gamma^2 B(\gamma)] = 4\kappa_1 \left( \frac{L_S}{\rho} \right) \left[ (1 + 5 \cos^2(\theta)) u^3 + O(u^4) \right]. \tag{3.26}
\]

Therefore, the expansion of \(3.24\) leads to (cf. Eq. \(2.27\))

\[
N(u, \theta) = u^2 (1 + 3 \cos^2(\theta)) + 4u^3 \cos(\theta) + u^3 \left( 4\kappa_1 \frac{L_S}{\rho} \right) (1 + 5 \cos^2(\theta)) + O(u^4) + O(G^2). \tag{3.27}
\]

In order to obtain the expansion of \(3.25\), we first need to take into account that in our model we can safely write

\[
\frac{L_S}{r}, \frac{L_S}{\rho}, \frac{L_J}{\rho} \ll 1. \tag{3.28}
\]

This leads to (see Eq. \(3.16\))

\[
D(u, \theta) = 1 + 4\kappa_1 \left( \frac{L_J}{\rho} \right) + 2u^2 \cos(\theta) + 4\kappa_1 u^2 \left( \frac{L_J}{\rho} \right) \cos(\theta) + 4\kappa_1 u^3 \left( \frac{L_S}{\rho} \right) \cos(\theta) + O(u^4) + O(G^2). \tag{3.29}
\]

As a result of the last calculations, we can now evaluate the small-\(u\) behavior of \(3.23\), that is ruled by the following computation:

\[
\frac{u^4 \sqrt{N(u, \theta)}}{\gamma^2 \sqrt{D(u, \theta)}} = \frac{u^4 \sqrt{N(u, \theta)}}{\left( 1 + 4u \cos(\theta) + O(u^2) \right) \left( \sqrt{1 + \frac{4\kappa_1 L_J}{\rho}} + O(u^2) + O(G^2) \right)}
\]

\[
= \frac{u^4}{\sqrt{1 + \frac{4\kappa_1 L_J}{\rho}}} (1 - 4u \cos(\theta) + O(u^2) + O(G^2)) \sqrt{1 + 3 \cos^2(\theta)}
\]

\[
\times \left[ 1 + \frac{2u}{(1 + 3 \cos^2(\theta))} \left( \cos(\theta) + k_1 \frac{L_S}{\rho} + 5k_1 \frac{L_S}{\rho} \cos^2(\theta) \right) + O(u^2) + O(G^2) \right]
\]

\[
= \frac{u^5 \sqrt{1 + 3 \cos^2(\theta)}}{\sqrt{1 + 4\kappa_1 (L_J/\rho)}} \times \left[ 1 + \frac{-2u \cos(\theta) (1 + 6 \cos^2(\theta) - 5k_1 (L_S/\rho) \cos(\theta)) + 2u k_1 (L_S/\rho)}{(1 + 3 \cos^2(\theta))} + O(u^2) + O(G^2) \right]. \tag{3.30}
\]
Therefore, up to $O(u^2)$ and $O(G^2)$ terms, we have (cf. Eq. (2.28))

\[
m' = u^5 g(\kappa_1) \sqrt{1 + 3 \cos^2(\theta)} \times \left[ 1 + \frac{-2u \cos(\theta) (1 + 6 \cos^2(\theta) - 5\kappa_1 (L_S/\rho) \cos(\theta)) + 2u\kappa_1 (L_S/\rho)}{(1 + 3 \cos^2(\theta))} \right],
\]

where we have defined, up to $O(L_P^2)$,

\[
g(\kappa_1) \equiv \frac{1 + 2\kappa_1 (L_S/r)}{(1 + 2\kappa_1 (L_J/\rho)) \sqrt{1 + 4\kappa_1 (L_J/\rho)}}.
\]

At this stage, we can invert (3.31), finding

\[
u^5 = \frac{m^2}{g(\kappa_1) \sqrt{1 + 3 \cos^2(\theta)}} \times \left[ 1 + \frac{2u \cos(\theta) (1 + 6 \cos^2(\theta) - 5\kappa_1 (L_S/\rho) \cos(\theta)) - 2u\kappa_1 (L_S/\rho)}{(1 + 3 \cos^2(\theta))} \right],
\]

whose lowest order solution reads as

\[
u = \left( \frac{m^2}{g(\kappa_1) \sqrt{1 + 3 \cos^2(\theta)}} \right)^{1/5} \equiv u_0^{1/5}.
\]

Therefore, by solving approximately Eq. (3.33) we obtain

\[
u \approx u_0^{1/5} + \frac{2}{5} \cos(\theta) u_0^{2/5} \left[ 1 + 6 \cos^2(\theta) - 5\kappa_1 (L_S/\rho) \cos(\theta) \right] \left( 1 + 3 \cos^2(\theta) \right)
\]

\[
- \frac{2}{5} u_0^{2/5} \frac{\kappa_1 (L_S/\rho)}{(1 + 3 \cos^2(\theta))}.
\]

Since Eq. (3.28) implies that $g(\kappa_1) \approx 1$, we obtain a result similar to the Newtonian case

\[
u_0^{1/5} \approx \left( \frac{m'^2}{\sqrt{1 + 3 \cos^2(\theta)}} \right)^{1/5} < m'^{2/5},
\]

and hence from (3.35) we conclude that the (approximate) equation defining the sphere of influence of the planet within the effective field theories of gravity picture reads as

\[
\rho = r' \left( \frac{m'^2}{g(\kappa_1) \sqrt{1 + 3 \cos^2(\theta)}} \right)^{1/5}.
\]

From the above equation it is clear that the effective field theory framework produces a tiny variation of the radius of the sphere of influence by means of the factor $1/g(\kappa_1)$. 
A. Comet trajectories in effective and Newtonian gravity

In this section we show the trajectories followed by the comet both in Newtonian and in effective field theory of gravity. As we will see, the corrections introduced in the comet motion by the effective picture turn out to be too tiny to be testable.

According to the effective field theory prescriptions (3.1)-(3.4) the heliocentric motion is described by the set of differential equations (hereafter we neglect terms $O(L^2)$)

$$
\frac{d^2}{dt^2} + \frac{Gm_\odot}{r^3} \left(1 + 2\kappa_1 \frac{L_S}{r}\right) x_k = Gm' \left[\left(\frac{x_k'}{r} - \frac{x_k}{\rho}\right) \left(1 + 2\kappa_1 \frac{L_J}{\rho}\right) - \frac{x_k'}{r^3} \left(1 + \frac{2\kappa_1 L_S}{r}\right)\right],
$$

whereas the motion of Jupiter is given in terms of (2.5)-(2.7) provided that (2.8) is subjected to the change

$$
\Omega^2 \rightarrow \tilde{\Omega}^2 = \frac{G(m_\odot + m')}{r_{\odot}^3} \left(1 + 2\kappa_1 \frac{L_S}{r}\right).
$$

Therefore, Eqs. (2.5)-(2.7) are replaced by

$$
\left(\frac{d^2}{dt^2} + \tilde{\Omega}^2\right) x_k' = 0, \quad (k = 1, 2, 3).
$$

The plots of the heliocentric trajectory of the comet in Newtonian gravity (see Eqs. (2.2)-(2.7)) and in effective field theory of gravity are displayed in Fig. 1a and 1b, respectively. The solution $\vec{r}(t) = (x(t), y(t), z(t))$ of (3.38)-(3.40) is drawn in Fig. 2.

Bearing in mind Eqs. (3.1), (3.2), (3.9), and (3.10), within the effective field theory domain the jovicentric motion of the comet is given by

$$
\frac{d^2}{dt^2} + \frac{Gm'}{\rho^3} \left(1 + 2\kappa_1 \frac{L_J}{\rho}\right) \xi_k = Gm_\odot \left[\left(\frac{x_k'}{r^3} - \frac{x_k}{\rho^3}\right) + 2\kappa_1 \left(\frac{L_S x_k'}{r^3} - \frac{L_S x_k}{r^3}\right)\right],
$$

along with Eqs. (3.39) and (3.40).

The jovicentric motion of the comet both in Newtonian (cf. (2.5)-(2.7) and (2.10)-(2.12)) and effective gravity is depicted in Fig. 3. The solution $\vec{\rho}(t) = (\xi(t), \eta(t), \zeta(t))$ can be read from Fig. 4.

In obtaining Figs. 1-4 the initial conditions regarding the motion of Jupiter reflect its average orbital speed and its average distance from the Sun. With these choices, the radius $\rho_0$ evaluated at the initial time $t_0 = 0$ of the (classical) sphere of influence of Jupiter reads
FIG. 1: Plots of the heliocentric comet motion. Equations (2.2)-(2.7) and their effective gravity counterpart (3.38)-(3.40) have been integrated by employing the following initial conditions at the time \( t_0 = 0 \):
\[
\vec{r}(t_0) = (7.2 \times 10^{11} \text{ m}, 3 \times 10^{10} \text{ m}, 3 \times 10^{10} \text{ m}), \quad \frac{d}{dt}\vec{r}(t_0) = (80 \text{ m/s}, 0, 0), \quad \vec{r}'(t_0) = (7.78 \times 10^{11} \text{ m}, 0, 0), \quad \frac{d}{dt}\vec{r}'(t_0) = (0, 1.3 \times 10^4 \text{ m/s}, 0). \]
Fig. 1a: Heliocentric comet motion in Newtonian gravity. Fig. 1b: Heliocentric comet motion in effective field theory of gravity.

FIG. 2: Plot of the components of the solution \( \vec{r}(t) \) of the effective field theory equations (3.38)-(3.40).
Fig. 2a: The function \( x(t) \). Fig. 2b: The function \( y(t) \). Fig. 2c: The function \( z(t) \).
FIG. 3: Plots of the jovicentric comet motion. Equations (2.5)-(2.7), (2.10)-(2.12) and their effective gravity counterpart (3.39)-(3.41) have been integrated by employing the following initial conditions at the time \( t = 0 \):
\[
\vec{\rho}(t_0) = (1.0 \times 10^7 \text{ m}, 1.0 \times 10^7 \text{ m}, 1.0 \times 10^8 \text{ m}), \quad \frac{d}{dt} \vec{\rho}(t_0) = (80 \text{ m/s}, 0, 0),
\]
\[
\vec{r}'(t_0) = (7.78 \times 10^{11} \text{ m}, 0, 0), \quad \frac{d}{dt} \vec{r}'(t_0) = (0, 1.3 \times 10^4 \text{ m/s}, 0).
\]
Fig. 3a: Jovicentric comet motion in Newtonian gravity. Fig. 3b: Jovicentric comet motion in effective field theory of gravity.

FIG. 4: Plot of the components of the solution \( \vec{\rho}(t) \) of the effective field theory equations (3.39)-(3.41). Fig. 4a: The function \( \xi(t) \). Fig. 4b: The function \( \eta(t) \). Fig. 4c: The function \( \zeta(t) \).
as (restoring the Sun mass $m_\odot$)

$$\rho_0 = \left(\frac{m'}{m_\odot}\right)^{2/5} r'(t_0) = 4.82 \times 10^{10} \text{ m.}$$

(3.42)

We have analysed several alternatives for the initial conditions of the comet motion. In all the cases investigated, the differences between the classical and the effective regime are unperceivable (see further comments in Sec. VII). The initial conditions featuring the heliocentric motion sketched in Figs. 1 and 2 are such that the initial distances comet-Sun $r_h(t_0)$ and comet-Jupiter $\rho_h(t_0)$ are, respectively,

$$r_h(t_0) = 7.21 \times 10^{11} \text{ m, } \rho_h(t_0) = 7.19 \times 10^{10} \text{ m,}$$

(3.43)

with the parameter $u_h(t_0)$ given by (cf. Eq. (2.21))

$$u_h(t_0) = \frac{\rho_h(t_0)}{r'(t_0)} = 0.092.$$  

(3.44)

On the other hand, the jovicentric motion (Figs. 3 and 4) is characterized by (the reader should be aware that, in Fig. 3, the closed orbit results from a particular choice of initial conditions, while for other choices an open orbit is instead obtained)

$$r_j(t_0) = 7.78 \times 10^{11} \text{ m, } \rho_j(t_0) = 1.01 \times 10^{8} \text{ m,}$$

(3.45)

so that

$$u_j(t_0) = \frac{\rho_j(t_0)}{r'(t_0)} = 1.30 \times 10^{-4}. $$

(3.46)

The analysis performed in this section clearly shows that the corrections to the comet motion resulting from the effective field theory approach are negligible. This agrees with the outcome of the previous section, where Eq. (3.37) describing the sphere of influence of the planet indicates a tiny departure from the classical case.

**IV. GENERAL RELATIVISTIC APPROACH FROM SCRATCH**

Since general relativity has been successfully verified in the Solar System so far, it is of course rather important to build the general relativistic description of the comet motion. As we will describe in detail, as a first useful approximation it suffices to consider a relativistic regime whose classical limit is represented by the Newtonian restricted three-body problem.
This implies a little departure from the classical picture employed in the previous sections, where no simplifying hypothesis on the motion of the Sun and Jupiter has been adopted.

Our investigation relies on the spacetime surrounding a spherical spinning mass. For this purpose, we introduce the parameters

\[ L_M = G \frac{M}{c^2} \equiv \mu, \quad j \equiv G \frac{J}{c^3}, \]

where \( M \) is the mass of the body and \( J \) is the modulus of its angular momentum.

The usual approximate form of the metric, using polar coordinates with the reference axis aligned with the angular momentum and the time variable \( \tau = ct \), is

\[
\begin{align*}
    ds^2 &= \left(1 - 2 \frac{\mu}{r}\right) d\tau^2 - \left(1 + 2 \frac{\mu}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 + 4 \frac{j}{r} \sin^2 \theta d\varphi d\tau.
\end{align*}
\]  

(4.1)

The last term is also known as being the origin of gravitomagnetic effects. The metric (4.1) may be deduced from the exact Kerr metric under the assumption that \( \frac{j}{\mu} \ll r \).

In the case of the Sun one finds \( \frac{j}{\mu} \simeq 3193 \) m, i.e., much less than the radius of the star (\( \sim 6.8 \times 10^8 \) m), hence the condition is satisfied: the biggest term neglected in the metric components at the surface of the Sun would be a fraction \( \sim 2.2 \times 10^{-11} \) of the smallest one kept. Consistently all terms proportional to \( G^2 \) have been neglected.

In the case of Jupiter \( \frac{j}{\mu} \simeq 363 \) m (the radius of the planet is \( \sim 7 \times 10^7 \) m). The neglected contribution at the surface of the planet would be \( \sim 2.7 \times 10^{-11} \) times the smallest kept term.

Let us tilt the axis of the reference frame by an angle \( \Theta \) from the axis around which the \( \varphi \) angles are measured. The only term of the line element which is affected is \( g_{0\varphi} \) which is split into two new terms, \( g_{0\theta} \) and \( g_{0\varphi}^* \). Both new terms are the original \( g_{0\varphi} \) multiplied by combinations of \( \sin \Theta \) and \( \cos \Theta \), such that when \( \Theta = 0 \) the first vanishes and the second recovers the initial form.

Considering the size of the tilt angle we see that in both cases the changes are either proportional to \( j \Theta \) or to \( j \left(1 - \frac{a^2}{2}\right) \). Since we have chosen the term containing \( j \) to be the lowest allowed approximation, we may safely drop the changes.

In the case of the Sun, the tilt angle is the one between the perpendicular to the ecliptic plane and the direction of the angular momentum of the star: \( \Theta_\odot = 1.304^\circ \). The relative expected correction to \( g_{0\varphi} \) would then be \( \Delta g_{0\varphi} = \frac{\delta g_{0\varphi}}{g_{0\varphi}} = -\frac{1}{2} \Theta^2 \sim -2.6 \times 10^{-4} \); the size of the new \( g_{0\theta} \) term would be of order one hundredth of \( g_{0\varphi} \). These numbers would have to be taken into account if the accuracy with which \( j \) is known were better.
For Jupiter the tilt is $\Theta_J = 3.131^\circ$, then the correction would be $\Delta_J \sim -1.5 \times 10^{-3}$ and the other mixed term would be some percent of $j_J$.

The general conclusion on the approximations to be used is that the approximate form of the metric both for the Sun and for Jupiter (in a nonrotating frame) may be assumed to be (4.1). Distances are of course from the center of each body and the $\varphi$ angle is measured from the same direction but with a different origin.

1. **Ranges**

Let us suppose that the accuracy of the presently available observations makes it possible to detect deviations from Minkowski space-time up to some (dimensionless) value $\varepsilon$. From (4.1) we deduce the range $\varrho$ within which relativistic gravitational effects may be revealed:

$$\varrho = 2 \frac{\mu}{\varepsilon}.$$  

Just as an example, choosing for instance (and arbitrarily) $\varepsilon \sim 10^{-15}$ and bearing in mind that for the Sun one has

$$L_S = \mu_\odot \simeq 1.48 \times 10^3 \text{ m}, \quad j_\odot \simeq 4.71 \times 10^6 \text{ m}^2,$$

we obtain

$$\varrho_\odot \simeq 2.95 \times 10^{18} \text{ m } \sim 312 \text{ light years}. \quad (4.2)$$

The distance at which it would be possible to perceive gravitomagnetic effects would be

$$\varrho_{gm} = 2 \sqrt{\frac{\mu}{\varepsilon}} \sin \theta,$$

which, in the case of the Sun, gives

$$\varrho_{gm\odot} \leq 1.37 \times 10^{11} \text{ m}. \quad (4.3)$$

The range of gravitomagnetic effects is much smaller than that of the gravitoelectric component: here the size of the interested volume is more or less 1 astronomical unit (AU). In general

$$\frac{\varrho_{gm}}{\varrho} \leq \sqrt{\frac{\varepsilon j}{\mu}}.$$

In the case of Jupiter one obtains

$$L_J = \mu_J = 1.41 \text{ m}, \quad j_J = 1024.3 \text{ m}^2.$$
Correspondingly, with the same hypothetical accuracy,

\[ \rho_J = 2 \frac{\mu_J}{\varepsilon} = \frac{\mu_J}{\mu_\odot} \rho_\odot \simeq 2.82 \times 10^{15} \text{ m}, \quad (4.4) \]

\[ \rho_{gmJ} \leq 2 \sqrt{\frac{j_J}{\varepsilon}} = \sqrt{\frac{j_J}{j_\odot}} \rho_{gm\odot} \simeq 2.02 \times 10^9 \text{ m}. \quad (4.5) \]

A. The spacetime of the pair

We are interested in the behavior of a test body (actually a comet) moving in the joint field of the Sun and of Jupiter. Thus, as a first step, we need to characterize, in the sense of general relativity, the space-time jointly surrounding the two bodies.

In the Newtonian approach we just superpose the fields of the two bodies and describe the total field as being due to the total mass of the two located, as a single body, at the barycenter of the pair. This cannot be done literally in the case of general relativity, since the pair is unstable, losing energy in the form of gravitational waves. As a consequence, the distance between the members of the pair is not strictly constant and the barycenter is an idealized concept. However, the orbital decay occurs at a rate of \( \sim 5 \times 10^{-20} \text{ m/s} \), hence we may assume that the orbits are stable over normal observation times. Unlike the previous sections (cf. Eqs. (2.5)-(2.8)), we employ the additional simplifying hypothesis according to which the orbits are circular around the barycenter defined in the classical way. The constant angular velocity of the orbital motion coincides with the average value of the actual instantaneous values over one revolution:

\[ \Omega = 1.683 \times 10^{-8} \text{ rad/s}. \]

In order to account for the orbital motion it is convenient to pass to a rotating frame, whose azimuth is \( \phi = \varphi - \frac{2}{c} \tau \). The change is not meant to affect the space coordinates only: from the viewpoint of the co-rotating observer, time is also affected. The rotation is accompanied by a Lorentz boost at the peripheral velocity of the orbiting observer (actually an infinite series of boosts along a direction tangent to the orbit): in this way proper times as well as tangential lengths are locally affected by a Lorentz factor

\[ \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - (\Omega r/c)^2}}. \quad (4.6) \]
Upon re-expressing the line element in the new reference frame and preserving the approximation level discussed in the first part of section IV we end up with

$$ds^2 \simeq \left(1 - 2\frac{\mu}{r} - r^2\frac{\Omega^2}{c^2}\sin^2\theta\right) d\tau^2 - \left(1 + 2\frac{\mu}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + 2\left(2\frac{j}{r^2} - r\frac{\Omega}{c} + 2\mu\frac{\Omega}{c}\right) r \sin^2\theta d\phi d\tau. \quad (4.7)$$

The term coupling the mass $\mu$ with the orbital velocity $\Omega$ is also known as expressing the de Sitter or geodesic effect. With the sensitivity level $\varepsilon$ we have assumed, the de Sitter contribution is not negligible for the Sun, but can be neglected for Jupiter.

The weakness of the deviation from flat spacetime allows to simply superpose the deviations due to the Sun and to Jupiter. We must however remember that the relevant distances are measured from different origins, and angles $\theta$ and $\phi$ also change according to the different origin. It is convenient to introduce one more change of coordinates, passing to a Cartesian space triad. Variable $x$ will be along the axis joining the Sun to Jupiter, oriented from the star to the planet; $y$ in the ecliptic plane; $z$ perpendicular to the other two axes. The conversion is quite simple:

$$r = \sqrt{x^2 + y^2 + z^2}; \quad dr = \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}; \quad d\theta = -\frac{dz}{\sqrt{(x^2 + y^2)}} + \frac{(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)} \frac{z}{\sqrt{(x^2 + y^2)}},$$

$$\tan \phi = \frac{y}{x}; \quad d\phi = \frac{xdy - ydx}{x^2 + y^2}.$$  

Another clarification is in order. The origin of the reference frame of the pair will be in the barycenter. The distance between the two bodies (assumed to be constant) is $R = a + b$ where $a$ and $b$ are the distances from the barycenter to Jupiter ($a$) and to the Sun ($b$). In practice when the metric is referred to the Sun (resp. Jupiter) we must replace $x$ with $x + b$.
(resp. \(x - a\)). Thus, the line element for the Sun only becomes

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\langle x, y, z \rangle &
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\begin{align*}
\langle x, y, z \rangle &
\end{align*}
\]

Thus, the line element for the Sun only becomes

\[
\begin{align*}
\langle x, y, z \rangle &
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\[
\begin{align*}
\langle x, y, z \rangle &
\end{align*}
\]
Thus, according to the classical definition of the center of mass, we can write

\[ a = \frac{\mu_\odot}{(\mu_\odot + \mu_J)} R \simeq 7.77 \times 10^{11} \text{ m}, \]
\[ b = \frac{\mu_J}{(\mu_\odot + \mu_J)} R \simeq 7.42 \times 10^{8} \text{ m}. \]

B. Volumes of influence

From (4.8) we see that the gravitational (gravitoelectric) perturbations induced by the two bodies equal each other when

\[ \frac{\mu_\odot}{\sqrt{\delta_b}} = \frac{\mu_J}{\sqrt{\delta_a}}. \]  

(4.10)

The corresponding surface has rotation symmetry about the Sun-Jupiter axis and its equation is

\[ \frac{\delta_a}{\delta_b} = \frac{\mu_J^2}{\mu_\odot^2}. \]  

(4.11)

By rearranging terms, we see that it describes a sphere centered at \( \left( \frac{a \mu_\odot^2 + b \mu_J^2}{(\mu_\odot^2 - \mu_J^2)}, 0, 0 \right) \), a bit farther than Jupiter, and having radius

\[ R_e \equiv \frac{\mu_J \mu_\odot}{(\mu_\odot^2 - \mu_J^2)} R \simeq 7.43 \times 10^8 \text{ m}. \]  

(4.12)

The explicit equation is

\[ \left( x - \frac{a \mu_\odot^2 + b \mu_J^2}{(\mu_\odot^2 - \mu_J^2)} \right)^2 + y^2 + z^2 = R_e^2. \]  

(4.13)

This suggests defining

\[ x_c \equiv \frac{a \mu_\odot^2 + b \mu_J^2}{(\mu_\odot^2 - \mu_J^2)} = \frac{\mu_J^2 - \mu_J \mu_\odot + \mu_\odot^2}{(\mu_\odot^2 - \mu_J^2)} R \simeq 7.77 \times 10^{11} \text{ m}, \]

and hence the distance from the position of Jupiter is

\[ \delta x = x_c - a = \frac{\mu_\odot^2}{(\mu_\odot^2 - \mu_J^2)} R \simeq 7.01 \times 10^5 \text{ m}. \]

What we have outlined here is the border between the prevalence of Jupiter and the prevalence of the Sun, but the result should be compared with what already written on the volumes out of which the gravitational influence of one or the other body becomes totally negligible (in our example the values were (4.2) and (4.4)).
1. Gravitomagnetic influence

The same procedure adopted in the previous section may be repeated for the effect of the angular momentum of both bodies. Here again a surface can be found where the two gravitomagnetic influences are equal; it is expressed by the equation

$$\frac{j_\odot \sqrt{(x + b)^2 + y^2}}{\delta_b^3} = \frac{j_J \sqrt{(x - a)^2 + y^2}}{\delta_a^3}.$$  

The first remark is that now the symmetry about the Sun-Jupiter axis is lost; only a mirror symmetry about the plane containing the axis and the two angular momenta and about the plane of the orbit survives.\(^2\) The distances from those planes are our \(y\) and \(z\) coordinates.

On considering the \(z = 0\) plane, the borderline is a circumference

$$(x + b - \frac{\dot{j}_\odot}{(\dot{j}_\odot - \dot{j}_J)} \mathcal{R})^2 + y^2 = \frac{\dot{j}_\odot \dot{j}_J}{(\dot{j}_J - \dot{j}_\odot)^2} \mathcal{R}^2,$$

whose center is at

$$x_{gmc} = \frac{\dot{j}_\odot}{(\dot{j}_\odot - \dot{j}_J)} \mathcal{R} - b \simeq 7.77 \times 10^{11} \text{ m}.$$  

In the plane \(y = 0\) the description is more complicated, but we may immediately remark that the maximal extension of the surface along the \(x\) axis is

$$x_{gmax} = \frac{aj_\odot + bj_J + \mathcal{R} \sqrt{jj_\odot}}{(\dot{j}_\odot - \dot{j}_J)} \simeq 7.88 \times 10^{11} \text{ m},$$

much longer than the extension of the volume within which the gravitomagnetic perturbations are not entirely negligible: it is enough to look at (4.3) and (4.5). The conclusion is that the two volumes where gravitomagnetism has to be taken into account do not intersect, hence it is completely useless to look for a balancing surface.

C. An important remark

From the considerations of the previous sections, it is clear that the classical limit of our relativistic framework is represented by the Newtonian restricted three-body problem. This is different from the pattern of Secs. II and III. A first important consequence of

\(^2\) The initial approximations we adopted assume that the two momenta are parallel to each other and perpendicular to the orbital plane.
this approach regards the definition of the sphere of influence of the planet (Sec. IV B), which simply singles out the points in space where the Sun and Jupiter exert the same gravitational attraction on the comet. Indeed, Eq. (4.10) involves the comparison between usual Newtonian gravitational forces, unlike (2.17) where the ratios between an attractive force and its perturbing counterpart appear. In other words, in Eqs. (4.10) and (4.11) the gravitational force experienced by the comet due to the presence of the Sun and Jupiter assumes always the same (Newtonian) functional form and no regime exists where one or the other body introduces a perturbing effect. Therefore, the difference between heliocentric and jovicentric motion becomes irrelevant in this approach.

Eventually, another important aspect of our investigation should be stressed. In our treatment we have considered an ideal comet since we have neglected nongravitational effects. The analysis of Secs. II and III is nevertheless original because we have taken into account the differences between heliocentric and jovicentric regime. As a first approach to the matter of comet motion our hypotheses can be justified.

V. FREE FALL TRAJECTORIES

In order to deduce the free fall trajectories, i.e., the geodesics of the space-time surrounding Jupiter and the Sun, we consider first a Lagrangian approach. The Lagrangian of a freely falling test body will be obtained from the line element (4.8) (see Appendix A):

\[
L = \frac{1}{2} \left( \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2 \right) + \frac{\Omega c^2}{\sqrt{\delta_b}} \left( \frac{y}{\frac{dx}{ds}} - \frac{x}{\frac{dy}{ds}} \right) \frac{d\tau}{ds} - \left( \frac{\mu_\odot}{\sqrt{\delta_b}} + \frac{\mu_J}{\sqrt{\delta_a}} \right) \left( \frac{x}{\frac{dx}{ds}} + \frac{y}{\frac{dy}{ds}} + \frac{z}{\frac{dz}{ds}} \right)^2 \frac{dx}{ds} \frac{dy}{ds} \frac{d\tau}{ds}
- \frac{j_\odot}{\delta_b^{\frac{3}{2}}} \left( \frac{dx}{ds} \right)^2 + \frac{j_J}{\delta_a^{\frac{3}{2}}} \left( \frac{dx}{ds} \right)^2 \frac{y}{\sqrt{x^2 + y^2}} + \frac{j_\odot}{\delta_b^{\frac{3}{2}}} \left( \frac{dx}{ds} \right)^2 + \frac{j_J}{\delta_a^{\frac{3}{2}}} \left( \frac{dx}{ds} \right)^2 \frac{x}{\sqrt{x^2 + y^2}} \right) \frac{d\tau}{ds}.
\]

The Lagrangian does not depend on time, hence Noether’s theorem provides a constant
of motion $E$, which reads as

$$
\left( 1 - \left( x^2 + y^2 \right) \frac{\Omega^2}{c^2} - 2 \frac{\mu_{\odot}}{\sqrt{\delta_b}} - 2 \frac{\mu_J}{\sqrt{\delta_a}} \right) \frac{d\tau}{ds} - 2 \left( \frac{j_{\odot} \sqrt{(x+b)^2 + y^2}}{\delta_b^3} + \frac{j_J \sqrt{(x-a)^2 + y^2}}{\delta_a^3} \right) - \frac{\Omega}{2c} \left( 1 - 2 \frac{\mu_{\odot}}{\sqrt{\delta_b}} \right) \sqrt{x^2 + y^2} \frac{y}{\sqrt{x^2 + y^2}} ds
$$

$$
+ 2 \left( \frac{j_{\odot} \sqrt{(x+b)^2 + y^2}}{\delta_b^3} + \frac{j_J \sqrt{(x-a)^2 + y^2}}{\delta_a^3} \right) - \frac{\Omega}{2c} \left( 1 - 2 \frac{\mu_{\odot}}{\sqrt{\delta_b}} \right) \sqrt{x^2 + y^2} \frac{x}{\sqrt{x^2 + y^2}} ds
$$

$$
= E. \tag{5.2}
$$

The presence of this constant of motion has represented a precious resource for the integration process of the Lagrange equations that we have performed in Sec. V C.

Besides this analysis, it is crucial to outline also the details of a perturbative strategy, which turns out to be useful when gravitomagnetic effects can be neglected. The following sections are devoted to this task.

A. Following a comet step by step

Let us consider the motion of a body (e.g., a comet, but not necessarily) falling towards the Sun from the Oort cloud. Initially (i.e., far away) we are outside the region of relevance of Jupiter (distance bigger than $\varrho_J$) and even more out of the volumes where the gravitomagnetic fields of the two main bodies are not negligible. We are however in the range of the influence of the Sun (distance less than $\varrho_{\odot}$). Under the usual sphericity and time independence hypotheses the relevant metric has the typical Schwarzschild form (using the corresponding coordinates in a non-rotating frame)

$$
ds^2 = \left( 1 - 2 \frac{\mu_{\odot}}{r} \right) d\tau^2 - \frac{dr^2}{\left( 1 - 2 \frac{\mu_{\odot}}{r} \right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.
$$

Motion takes place in a plane, ruled by the equations

$$
\theta = \theta_0; \quad \frac{d\theta}{ds} = \frac{d^2\theta}{ds^2} = 0,
$$

and there are two constants of motion:

$$
\left( 1 - 2 \frac{\mu_{\odot}}{r} \right) \frac{d\tau}{ds} = E, \quad r^2 \sin^2 \theta_0 \frac{d\varphi}{ds} = \Lambda.
$$
One ($E$: the energy per unit mass) will remain a constant throughout the whole treatment, provided the condition of time independence of the metric will be maintained. The second ($\Lambda$: the angular momentum per unit mass) will not really be a constant when perturbations from Jupiter and the gravitomagnetic fields will no longer be negligible.

Let us limit the consideration to a finite orbit. As long as the other influences do not appear, the properties of the trajectory are well known and the difference from the Newtonian treatment shows up close to the Sun (precession of the perihelion) and over a number of orbits (which are not closed).

Upon looking for the radial acceleration of a freely falling body in the situation treated in this section, we may resort to Lagrangian equations (see Eq. (A4)) or (which is equivalent) to the equation of geodesics:

$$
\dddot{x}^\nu = \frac{d}{ds} \left( \sum_{\rho=0}^{3} u^\rho u^\nu_{\rho} \right) + \sum_{\alpha,\beta=0}^{3} \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta = 0.
$$

Either way leads to the result

$$
\frac{d^2 r}{d\tau^2} = -\frac{1}{r^2 \left( 1 - 2\frac{\mu_{\odot}}{r} \right)} + \frac{2}{r^3 \left( 1 - 2\frac{\mu_{\odot}}{r} \right)} \left( 1 - 3\frac{\mu_{\odot}}{r} \right),
$$

and the general relativity corrections are of order $\mu_{\odot}^2 / r^2$ and $\Lambda^2 \mu_{\odot} / r^3$.

1. Angular momentum of the Sun

If the trajectory stays out of the domain where Jupiter must be accounted for, but gets close enough to the Sun to perceive its gravitomagnetic field, the reference metric becomes (4.1). We have again two constants of motion

$$
\left( 1 - 2\frac{\mu_{\odot}}{r} \right) \left( \frac{d\tau}{ds} \right) + \frac{\dot{j}_{\odot}}{r} \sin^2 \theta \frac{d\varphi}{ds} = E,
$$

$$
-r^2 \sin^2 \theta \left( \frac{d\varphi}{ds} \right) + \frac{\dot{j}_{\odot}}{r} \sin^2 \theta \frac{d\tau}{ds} = -\Lambda,
$$

wherefrom

$$
\frac{d\tau}{ds} = \frac{r^4 E - 2\dot{j}_{\odot} r \Lambda}{(4\dot{j}_{\odot}^2 \sin^2 \theta - 2\mu_{\odot} r^3 + r^4)} \sim \left( 1 + 2\frac{\mu_{\odot}}{r} \right) E - 2\frac{\dot{j}_{\odot}}{r^3} \Lambda,
$$

$$
\frac{d\varphi}{ds} = \frac{r^2 \Lambda - 2\mu_{\odot} r \Lambda + 2\dot{j}_{\odot} r E \sin^2 \theta}{(4\dot{j}_{\odot} \dot{j}_{\odot}^2 \sin^4 \theta + r^4 \sin^2 \theta - 2\mu_{\odot} r^3 \sin^2 \theta)} \sim \frac{1}{r^2 \sin^2 \theta} \frac{\Lambda}{\dot{j}_{\odot}} + \frac{2\dot{j}_{\odot}}{r^3} E.
$$

\footnote{Dots denote derivatives with respect to $s$.}
Of course, the terms containing $j_⊙$ are additional perturbations with respect to the Newtonian trajectory and contribute an additional term to the radial acceleration of the freely falling body. Furthermore, now the only plane trajectories are the ones contained in the ecliptic plane (or, better to say, perpendicular to the angular momentum of the Sun).

B. Inclusion of Jupiter

What we have written in the previous sections holds as far as the comet is at a distance from Jupiter bigger than $\varrho_J$. When this condition is violated (and still assuming to be outside the volumes where gravitomagnetism is relevant) the metric to be considered is obtained from (4.8) in a co-rotating frame (polar coordinates):

$$ds^2 \simeq \left(1 - \frac{\Omega^2}{c^2} r^2 \sin^2 \theta - 2 \frac{\mu_⊙}{\sqrt{\chi}} - 2 \frac{\mu_J}{\sqrt{\psi}}\right) d\tau^2$$

$$- \left(1 + 2 \frac{\mu_⊙}{\sqrt{\chi}} + 2 \frac{\mu_J}{\sqrt{\psi}}\right) dr^2$$

$$- (r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + 2 \frac{\Omega}{c} \left(1 - 2 \frac{\mu_⊙}{\sqrt{\chi}}\right) r^2 \sin^2 \theta d\phi d\tau,$$

(5.4)

where (cf. Eq. (4.9))

$$\chi \equiv b^2 + 2 br \sin \theta \cos \phi + r^2, \quad \psi \equiv a^2 - 2 ar \sin \theta \cos \phi + r^2. \quad (5.5)$$

A constant of motion is

$$\left(1 - \frac{\Omega^2}{c^2} r^2 \sin^2 \theta - 2 \frac{\mu_⊙}{\sqrt{\chi}} - 2 \frac{\mu_J}{\sqrt{\psi}}\right) \frac{d\tau}{ds} + \frac{\Omega}{c} \left(1 - 2 \frac{\mu_⊙}{\sqrt{\chi}}\right) r^2 \sin^2 \theta \frac{d\phi}{ds} = E.$$

We have already seen that the influences of the two main bodies equate each other on a sphere of radius $R_e$ as in (4.12), but as pointed out in Sec. IV C this information is not really relevant in this approach since the influence of both main bodies is accounted for without caring which one is bigger and where. Now Cartesian coordinates may be a bit simpler to handle, and the resulting form of the metric reads as

$$ds^2 \simeq \left(1 - \frac{\Omega^2}{c^2} (x^2 + y^2) - 2 \frac{\mu_⊙}{\sqrt{\delta_b}} - 2 \frac{\mu_J}{\sqrt{\delta_a}}\right) d\tau^2$$

$$- 2 \left(\frac{\mu_⊙}{\sqrt{\delta_b}} + \frac{\mu_J}{\sqrt{\delta_a}}\right) \frac{(xdx + ydy + zdz)^2}{(x^2 + y^2 + z^2)}$$

$$- (dx^2 + dy^2 + dz^2) + 2 \frac{\Omega}{c} \left(1 - 2 \frac{\mu_⊙}{\sqrt{\delta_b}}\right) (xdy - ydx) d\tau.$$

(5.6)
The constant of motion in these coordinates reads as

\[
\left(1 - \frac{\Omega^2}{c^2} (x^2 + y^2) - 2\frac{\mu_\odot}{\sqrt{\delta_b}} - 2\frac{\mu_J}{\sqrt{\psi}}\right) \frac{d\tau}{ds} + \frac{\Omega}{c} \left(1 - 2\frac{\mu_\odot}{\sqrt{\delta_b}}\right) \left(x \frac{dy}{ds} - y \frac{dx}{ds}\right) = E. \tag{5.7}
\]

1. Geodesics

An alternative to the direct application of Lagrange equations is to resort to the geodesics of the spacetime described by (5.4). The general form of the equations uses Christoffel symbols, i.e.,

\[
\frac{d^2 x^\nu}{ds^2} = - \sum_{\alpha,\beta=0}^3 \Gamma^\nu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}. \tag{5.8}
\]

The explicit approximate form of the Christoffels has been worked out in the Appendix B in polar coordinates. The resulting equations of motion are

\[
\frac{d^2 \tau}{ds^2} = - \frac{\Omega \mu_\odot b}{c} r \sin \theta \sin \phi \left(\frac{d\tau}{ds}\right)^2 - \frac{\Omega \mu_\odot b}{c} r \sin \theta \sin \phi \left(\frac{dr}{ds}\right)^2
- 2 \left(\frac{\mu_\odot}{\chi^2} + \mu_J \frac{r - a \sin \theta \cos \phi}{\psi^2}\right) \frac{d\tau}{ds} \frac{dr}{ds}
- 2 \left(\frac{\mu_\odot b}{\chi^2} - \frac{\mu_J a}{\psi^2}\right) r \cos \theta \cos \phi \frac{d\tau}{ds} \frac{d\theta}{ds} + 2 \left(\frac{\mu_\odot b}{\chi^2} - \frac{\mu_J a}{\psi^2}\right) r \sin \theta \sin \phi \frac{d\tau}{ds} \frac{d\phi}{ds}
- 4 \frac{\Omega \mu_\odot b}{\sqrt{\chi}} r \sin^2 \theta \frac{dr}{ds} \frac{d\phi}{ds} - 2 \frac{\Omega \mu_\odot b}{c} \cos \phi \frac{d\tau}{ds} \frac{d\theta}{ds} - 2 \frac{\Omega \mu_\odot b}{c} \cos \phi \frac{d\tau}{ds} \frac{d\phi}{ds}, \tag{5.9}
\]

\[
\frac{d^2 r}{ds^2} = \left(\frac{\sin^2 \theta}{c^2} r \Omega^2 - \mu_\odot \frac{r + b \sin \theta \cos \phi}{\chi^2} - \mu_J \frac{r - a \sin \theta \cos \phi}{\psi^2}\right) \left(\frac{d\tau}{ds}\right)^2
+ \left(\frac{r + b \sin \theta \cos \phi}{\chi^2} + \mu_J \frac{r - a \sin \theta \cos \phi}{\psi^2}\right) \left(\frac{dr}{ds}\right)^2
+ \left(1 - 2\frac{\mu_J}{\sqrt{\psi}} - \frac{2\mu_\odot}{\sqrt{\chi}}\right) \frac{d\theta}{ds}^2 + \left(1 - 2\frac{\mu_J}{\sqrt{\psi}} - \frac{2\mu_\odot}{\sqrt{\chi}}\right) r \sin^2 \theta \left(\frac{d\phi}{ds}\right)^2
+ 2 \left(\frac{\mu_\odot b}{\chi^2} - \frac{\mu_J a}{\psi^2}\right) r \cos \theta \cos \phi \frac{d\theta}{ds} \frac{dr}{ds}
- 2 \left(1 - \mu_\odot 4b^2 + 7br \cos \phi \sin \theta + 3r^2\right) \frac{\Omega}{c} r \sin^2 \theta \frac{d\tau}{ds} \frac{d\phi}{ds}
- 2 \left(\frac{\mu_\odot b}{\chi^2} - \frac{\mu_J a}{\psi^2}\right) r \sin \theta \sin \phi \frac{d\tau}{ds} \frac{d\phi}{ds}, \tag{5.10}
\]
\[
\frac{d^2 \theta}{ds^2} = \frac{1}{r^2} \left( \frac{r^2 \Omega^2}{c^2} \cos \theta \sin \theta - \mu_\odot \frac{br \cos \theta \cos \phi}{\chi^2} + \mu_J \frac{a r \cos \theta \cos \phi}{\psi^2} \right) \left( \frac{d \tau}{ds} \right)^2 \\
- \frac{1}{r} \left( \mu_\odot \frac{b}{\chi^2} - \frac{\mu_J a}{\psi^2} \right) \cos \theta \cos \phi \left( \frac{dr}{ds} \right)^2 - \frac{2 dr}{r} \frac{d \theta}{ds}
\]
\[
- \frac{2}{\Omega} \left( 1 - \mu_\odot \frac{2 (b^2 + r^2) + 3br \sin \theta \cos \phi}{\chi^2} \right) \cos \theta \sin \theta \frac{d \tau}{ds} \frac{d \phi}{ds}
\]
\[+ \cos \theta \sin \theta \left( \frac{d \phi}{ds} \right)^2, \quad (5.11)\]

\[
\frac{d^2 \phi}{ds^2} = \frac{\sin \phi}{r \sin \theta} \left( \mu_\odot \frac{b}{\chi^2} - \frac{\mu_J a}{\psi^2} \right) \left( \frac{d \tau}{ds} \right)^2 + \frac{\sin \phi}{r \sin \theta} \left( \mu_\odot \frac{b}{\chi^2} - \frac{\mu_J a}{\psi^2} \right) \left( \frac{dr}{ds} \right)^2 \\
+ \frac{2 \Omega}{r c} \left( 1 - \frac{4 \mu_\odot}{\sqrt{\chi}} \right) \frac{d \tau}{ds} \frac{dr}{ds} + \frac{2 \Omega \cos \theta}{r c \sin \theta} \left( 1 - \frac{4 \mu_\odot}{\sqrt{\chi}} \right) \frac{d \tau}{ds} \frac{d \theta}{ds}
\]
\[+ \frac{2 \Omega b \mu_\odot}{c \chi^2} r \sin \theta \sin \phi \frac{d \tau}{ds} \frac{d \phi}{ds} - \frac{2}{r} \left( 1 - \frac{2 \mu_\odot}{\sqrt{\chi}} - \frac{2 \mu_J}{\sqrt{\psi}} \right) \frac{d \tau}{ds} \frac{d \phi}{ds} \\
- \frac{2}{\sin \theta} \cos \theta \left( 1 - \frac{2 \mu_\odot}{\sqrt{\chi}} - \frac{2 \mu_J}{\sqrt{\psi}} \right) \frac{d \theta}{ds} \frac{d \phi}{ds}, \quad (5.12)\]

C. Comet trajectories in general relativity and Newtonian gravity

We have integrated by means of numerical tools both the Lagrangian equations resulting from (5.1) and Eqs. (5.9)-(5.12) for several choices of the initial conditions. For the comet trajectories depicted in Figs. 5 and 6, the Euclidean distances Sun-comet and Jupiter-comet are, respectively,

\[r_\odot(0) = 1.82 \times 10^9 \text{ m, } r_J(0) = 7.77 \times 10^{11} \text{ m.}\]

Therefore, being \(r_\odot(0) < a_{gm\odot}\) (cf. Eq. (4.13)), for this specific case we have determined the comet dynamics starting from the Lagrangian (5.1)\(^4\). During the integration process, we have exploited the existence of the constant energy (5.2) to lower the number of dynamical equations to be solved. The value of this constant can be determined from the initial conditions jointly with Eq. (A2). For the case considered in Figs. 5 and 6 we have

\[E = 1.01.\]

\(^4\) The initial conditions of Figs. 5 and 6 correspond to a motion starting outside the sphere of influence of the planet (see Eq. (4.13)). However, as pointed out before, this information is of no interest within this approach.
The Newtonian equations describing the comet dynamics coincide with those describing the motion of the planetoid in the restricted three-body problem. In a co-rotating reference frame having the origin in the barycenter of the Sun and Jupiter they can be derived from the Lagrangian

\[ \mathcal{L}_N = GU(x, y, z) + \frac{1}{2} \left[ \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2 \right] + \frac{\Omega}{c} \left( x \frac{dy}{ds} - y \frac{dx}{ds} \right), \]  

(5.13)

where

\[ GU(x, y, z) \equiv \frac{1}{2} \frac{\Omega^2}{c^2} (x^2 + y^2) + \frac{\mu_{\odot}}{\sqrt{\delta_b}} + \frac{\mu_J}{\sqrt{\delta_a}}. \]  

(5.14)

FIG. 5: Plots of the three-dimensional comet motion in Newtonian gravity and in general relativity with \( 0 < s < 1.0 \times 10^{18} \) m. Lagrangian equations derived from (5.13) and (5.1) have been integrated by employing the following initial conditions at \( s_0 = 0 \): \( x(s_0) = y(s_0) = z(s_0) = R_e, \frac{dx}{ds}(s_0) = 0.1, \frac{dy}{ds}(s_0) = \frac{dz}{ds}(s_0) = 0.01 \). Fig. 5a: Three-dimensional comet motion in Newtonian gravity. Fig. 5b: Three-dimensional general relativistic comet motion.

As is clear from Figs. 5 and 6, the discrepancies between Newtonian and relativistic frameworks are imperceptible. This result matches with the analysis performed in Sec.
FIG. 6: Plots of the two-dimensional comet motion in Newtonian gravity and in general relativity with $0 < s < 1.0 \times 10^{17}$ m. Lagrangian equations derived from (5.13) and (5.1) have been integrated by employing the following initial conditions at $s = 0$: $x(s_0) = y(s_0) = z(s_0) = R_e$, $\frac{d}{ds} x(s_0) = 0.1$, $\frac{d}{ds} y(s_0) = \frac{d}{ds} z(s_0) = 0.01$. Fig. 6a: Two-dimensional comet motion in Newtonian gravity. Fig. 6b: Two-dimensional general relativistic comet motion.

III A where no clear trace of dissimilarity was found between the comet trajectories foretold by Newtonian and effective field theory regimes.

VI. OBSERVATIONAL TESTS

In order to experimentally check the different effects of Newtonian versus general relativity treatments in our three-body problem, we need a precise orbit determination for comets. If the orbit is reconstructed just by means of observation from the ground or from space telescopes, the accuracy will hardly be of order of (hundred, maybe thousand) kilometers. The corrections we are interested in are much stricter than that, which means that much better positioning strategies must be deployed. An appropriate technique is based on RPS (Relativistic Positioning System) [31] [32]: the idea is to consider space-time as a four-dimensional curved map on which the coordinated lines of the topographic grid are drawn by electromagnetic signals. Positioning on the “map” is attained by identifying the “cell” in which the object to be followed is located and then using a simple algorithm to work out the detailed coordinates in the cell.

The hard structure underlying such an idea are two (if the expected trajectory of the
target is in the ecliptic plane), three (if the object is out of that plane) or four (if we also want to independently determine a time coordinate) stations emitting regular electromagnetic signals. Furthermore the target must be equipped with a receiver and a clock: the position with respect to the emitters is obtained locally on the base of the sequences of arrival times of the signals emitted by the “fixed” beacons. All this implies a mission, like Rosetta, aimed at laying down a lander on the surface of the comet to be studied. The most appropriate locations for the emitters would be the Lagrange points of the pair Sun/planet, which occupy a stable position with respect to the two main bodies and move around “rigidly” with them. The method has already been proposed, with reference to the Sun/earth pair [33]; here the most appropriate set would be the L-points of the Sun/Jupiter pair, though not easy to reach.

In principle the accuracy of the purely electromagnetic segment would depend on the stability of the emission of the signals from the beacons and on the accuracy of the clock of the device deposited on the nucleus of the comet: the corresponding positioning would be within centimeters’ range. Beyond that, the limiting factors would be the real knowledge of the instantaneous positions of the emitters: the spacecraft carrying them would move about the nominal position along Lissajous or halo orbits.

VII. CONCLUDING REMARKS

Despite the extreme difficulty of observational tests, the comparison between effective-gravity and general relativity descriptions on the one hand, and the dedicated efforts in the literature on the problem of covariant equations of motion [34–39], provide a strong motivation for continuing the investigation of celestial mechanics from all points of view. In this paper, we have performed a detailed and novel analysis of the motion of a comet in effective field theories, Einstein and Newtonian gravity.

For the first time in the literature, the dynamics of a comet has been investigated within a framework where the dimensionless weight factors for gravitational radii are obtained from quantum field theory. This topic comprises the first part of the manuscript, Secs. II and III. We have proposed that the effective-gravity regime should rely on the set of original prescriptions (3.1)-(3.4) and we have found that our model predicts tiny departures from the Newtonian picture. This is witnessed both by Eq. (3.37) defining the quantum corrected
sphere of influence of Jupiter and by the compared comet trajectories of Figs. 1 and 3.

In the second part of the paper, Secs. IV and V, we have considered the comet dynamics within Einstein’s theory. We have provided for the details of two approaches depending on whether gravitomagnetic effects are significant (Lagrangian equations resulting from (5.1)) or not (geodesic equations (5.9)-(5.12)). The Newtonian limit of our relativistic approach is different from the one adopted in the first part, being represented by the restricted three-body problem. This implies no difference between the jovicentric and the heliocentric comet motion. However, the mismatch with the classical theory is again found to be impalpable (see Figs. 5 and 6). Although we are aware that one of the main obstacles in modern cometary orbit-determination is the search of the proper model describing nongravitational perturbations due to the rocket-like thrusting of the outgassing cometary nucleus [40], we claim that our paper contains, to the best of our knowledge, a substantial improvement in the fully general relativistic treatment of comet motion in the presence of Sun and Jupiter. Indeed, we have superseded the analysis of Refs. [41, 42], where an investigation is carried out of the relativistic modifications to the dynamics of comets (and asteroids such as 1566 Icarus) by employing (the slow-motion, weak-field approximation of) Schwarzschild solution. On the other hand, in this paper we have adopted a more realistic setting by using Kerr geometry and, unlike Refs. [41, 42], we have also explicitly exhibited the orbits of the comet.

This paper opens us some interesting issues to be addressed in future works. First of all, a fascinating task would consist in devising a relativistic setup having as its classical counterpart Tisserand’s theory of perturbations to comet motion. This would integrate the original effective field theory pattern described in the first part of this paper. It would also be interesting to analyze the secular relativistic effects generated by the Sun on the argument of the perihelion and the mean anomaly of an orbit described recently in the literature [43, 44] within the quantum corrected regime ruled by effective field theories of gravity. On the numerical side, the difference between classical and effective-gravity regime (see comments after Eq. (3.42)) might become clearer by studying how much the orbits differ at some specific points to be chosen.

Moreover, one should bear in mind that three-body systems display chaotic behavior over sufficiently long times [45]. This implies that there might exist critical combinations of some parameters, and in the neighborhood of such critical values, even a very small perturbation could give rise to orbits that differ a lot from each other. From this point of view, even the
detailed calculations of Sec. III are not just of academical interest. Furthermore, a numerical or analytic investigation of nongravitational effects such as evaporation of the comet’s head, and radiation pressure, might prove useful. Last, but not least, on the gravitational side it remains to be seen whether gravitomagnetic effects can drive the capture of a comet by Jupiter.

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Appendix A: Lagrangian equations and Killing vectors

In this appendix we recall the tensor form of the laws and principles from which the equations of motion and the related constants are deduced in any reference frame. Let us start with the components of the four-velocity of a test particle:

\[ u^\nu = \frac{dx^\nu}{ds}. \]  

(A1)

By construction, the four-velocity is normalised to 1:

\[ \langle u, u \rangle = g(u, u) = \sum_{\alpha,\beta=0}^{3} g^{\alpha\beta} u^\alpha u^\beta = 1. \]  

(A2)

However, the normalization condition may temporarily be released in order to define the Lagrange function of the test particle freely falling in an external gravitational field, identified by the metric tensor, i.e.

\[ \mathcal{L} = \frac{1}{2} \langle u, u \rangle = \frac{1}{2} \sum_{\alpha,\beta=0}^{3} g_{\alpha\beta} u^\alpha u^\beta. \]  

(A3)

By relying upon these definitions, the equations of motion are derived from a variational principle:

\[ \frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial u^\nu} \right) - \frac{\partial \mathcal{L}}{\partial x^\nu} = 0. \]  

(A4)
Noether’s theorem, when the Lagrangian does not explicitly depend on a coordinate, leads to a corresponding constant of motion. Another way to say the same thing is to refer to the Killing vectors $K$ of the system. The Lie derivative of the metric $g$ along $K$ vanishes, and in tensorial notation one writes therefore that the symmetrized covariant derivative of $K$ vanishes:

$$\nabla_{(\alpha} K_{\beta)} = 0. \quad (A5)$$

In all situations analyzed in this paper we have assumed that the metric tensor does not depend on time. This leads to a translational invariance (a symmetry) along the time axis; the corresponding Killing vector is simply

$$K_\tau = \frac{\partial}{\partial \tau} \equiv \partial_\tau. \quad (A6)$$

As an example, an axial symmetry, where the angle $\phi$ is about the symmetry axis, corresponds to the Killing vector $K_\phi = \partial_\phi$.

Once the Killing vectors are known, the constants of motion are obtained from equations like:

$$\langle K, u \rangle = g(K, u) = \sum_{\alpha, \beta = 0}^{3} g_{\alpha\beta} K^\alpha u^\beta = \text{constant}. \quad (A7)$$

**Appendix B: Connection coefficients**

For the reader’s convenience we list here the explicit formulas for the Christoffel symbols of the metric in Eq. (5.4). The approximation level corresponds to the conventional threshold $\varepsilon = 10^{-15}$: dimensionless terms smaller than that are neglected. Under these conditions the contravariant components of the metric are

$$(g^{-1})^{00} = 1 + \frac{2\mu J}{\sqrt{\psi}} + \frac{2\mu \odot}{\sqrt{\chi}},$$

$$(g^{-1})^{0\phi} = \frac{\Omega}{c},$$

$$(g^{-1})^{rr} = -1 + \frac{2\mu J}{\sqrt{\psi}} + \frac{2\mu \odot}{\sqrt{\chi}},$$

$$(g^{-1})^{\theta\theta} = -\frac{1}{r^2},$$

$$(g^{-1})^{\phi\phi} = -\frac{1}{r^2 \sin^2 \theta} \left(1 - \frac{\Omega^2}{c^2} r^2 \sin^2 \theta - \frac{2\mu \odot}{\sqrt{\chi}} - \frac{2\mu J}{\sqrt{\psi}} \right).$$
The general formula for the Christoffel symbols is

$$
\Gamma^\alpha_{\beta\gamma} = \sum_{\nu=0}^{3} \frac{1}{2} (g^{-1})^{\alpha\nu} \left( \frac{\partial g_{\beta\nu}}{\partial x^{\gamma}} + \frac{\partial g_{\nu\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\nu}} \right).
$$

The approximate explicit formulas for the nonvanishing connection coefficients are

$$
\begin{align*}
\Gamma^0_{00} &= \frac{\Omega}{c} \frac{\mu_{\psi \phi}}{\chi^2} \sin \theta \sin \phi \\
\Gamma^0_{0r} &= \mu_{\psi} r \frac{b r \sin \theta \cos \phi}{\chi^2} + \mu_{J} r \frac{a \sin \theta \cos \phi}{\psi^2} \\
\Gamma^0_{rr} &= r \frac{\Omega}{c} \frac{\mu_{\psi \phi}}{\chi^2} \sin \theta \sin \phi \\
\Gamma^0_{\theta \phi} &= \mu_{\psi} r \frac{b r \sin \theta \sin \phi}{\chi^2} + \mu_{J} r \frac{a \sin \theta \sin \phi}{\psi^2} \\
\Gamma^0_{r \phi} &= 2 r \frac{\Omega}{c} \frac{\mu_{\psi \phi}}{\psi^2} \sin^2 \theta \\
\Gamma^0_{\theta \theta} &= r^3 \frac{\Omega}{c} \frac{\mu_{\psi \phi} \cos \phi}{\chi^2} \cos \theta \sin^2 \theta, \\
\end{align*}
$$

$$
\begin{align*}
\Gamma^r_{00} &= -\left( \frac{\sin^2 \theta}{c^2} r \frac{\Omega^2}{c^2} - \mu_{\psi} \frac{r + b r \sin \theta \cos \phi}{\chi^2} + \mu_{J} \frac{a \sin \theta \cos \phi}{\psi^2} \right) \\
\Gamma^r_{rr} &= -\left( \mu_{\psi} \frac{r + b r \sin \theta \cos \phi}{\chi^2} + \mu_{J} \frac{a \sin \theta \cos \phi}{\psi^2} \right) \\
\Gamma^r_{\theta \theta} &= \left( -1 + \frac{2 \mu_{J}}{\sqrt{\psi}} + \frac{2 \mu_{\psi \phi}}{\sqrt{\chi}} \right) r \\
\Gamma^r_{\phi \phi} &= r \Omega \frac{\sin^2 \theta}{c^2} \left( 1 - \mu_{\psi} \frac{4 b^2 \psi^2 + 7 (\cos \theta \sin \phi) b r - 3 \psi^2}{\chi^2} \right) \\
\Gamma^r_{\phi \theta} &= \left( \mu_{\psi} \frac{b \psi}{\chi^2} - \mu_{J} \frac{\alpha}{\psi^2} \right) r \sin \theta \sin \phi \\
\Gamma^r_{\phi \phi} &= \left( -1 + \frac{2 \mu_{J}}{\sqrt{\psi}} + \frac{2 \mu_{\psi \phi}}{\sqrt{\chi}} \right) r \sin^2 \theta, \\
\end{align*}
$$

$$
\begin{align*}
\Gamma^0_{00} &= -\frac{1}{r^2} \left( \frac{\sin^2 \theta}{c^2} r \frac{\Omega^2}{c^2} \cos \theta \sin \phi - \mu_{\psi} \frac{b r \cos \theta \cos \phi}{\chi^2} + \mu_{J} \frac{a \cos \theta \cos \phi}{\psi^2} \right) \\
\Gamma^0_{rr} &= \frac{1}{r} \left( \mu_{\psi} \frac{b \psi}{\chi^2} - \mu_{J} \frac{\alpha}{\psi^2} \right) \cos \theta \cos \phi \\
\Gamma^0_{\theta \theta} &= \frac{1}{r} \left( \mu_{\psi} \frac{\psi}{\chi^2} - \mu_{J} \frac{\alpha}{\psi^2} \right) \cos \theta \cos \phi \\
\Gamma^0_{\phi \phi} &= \frac{\Omega}{c} \cos \theta \sin \phi \left( 1 - \mu_{\psi} \frac{(\psi^2 + r^2) + 3 b \sin \theta \cos \phi}{\psi^2} \right) \\
\Gamma^0_{\phi \theta} &= -\cos \theta \sin \phi, \\
\end{align*}
$$

38
\[
\begin{align*}
\Gamma_{00}^\phi &= -\frac{\sin \phi}{r \sin \theta} \left( \frac{b_{\mu \phi}}{\chi^2} - \frac{a_{\mu \phi}}{\psi^2} \right), \\
\Gamma_{0r}^\phi &= -\frac{\Omega}{c} \frac{1}{r} \left( 1 - \frac{4 \mu}{\sqrt{\chi}} \right), \\
\Gamma_{rr}^\phi &= -\frac{\sin \phi}{r \sin \theta} \left( \frac{\mu_{\chi b}}{\chi^2} - \frac{\mu \psi}{2 \psi^2} \right), \\
\Gamma_{0\theta}^\phi &= -\frac{\Omega \cos \theta}{c \sin \theta} \left( 1 - \frac{4 \mu}{\sqrt{\chi}} \right), \\
\Gamma_{0\phi}^\phi &= -\frac{\Omega \sin \theta}{c \sin \theta} \left( 1 - \frac{2 \mu}{\sqrt{\chi}} - \frac{2 \mu \psi}{\sqrt{\psi}} \right), \\
\Gamma_{r\phi}^\phi &= \frac{1}{r} \left( 1 - \frac{2 \mu}{\sqrt{\chi}} - \frac{2 \mu \psi}{\sqrt{\psi}} \right), \\
\Gamma_{\theta\phi}^\phi &= \frac{\cos \theta}{\sin \theta} \frac{1}{r} \left( 1 - \frac{2 \mu}{\sqrt{\chi}} - \frac{2 \mu \psi}{\sqrt{\psi}} \right).
\end{align*}
\]

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