THE KOBAYASHI METRIC FOR
NON-CONVEX COMPLEX ELLIPSOIDS

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Abstract. In the paper we give some necessary conditions for a mapping to be a \( \kappa \)-geodesic in non-convex complex ellipsoids. Using these results we calculate explicitly the Kobayashi metric in the ellipsoids \( \{|z_1|^2 + |z_2|^{2m} < 1\} \subset \mathbb{C}^2 \), where \( m < \frac{1}{2} \).

1. Introduction. By \( E \) we denote the unit disk in \( \mathbb{C} \). Let \( D \) be a domain in \( \mathbb{C}^n \). For \( (z,X) \in D \times \mathbb{C}^n \) we define

\[
\kappa_D(z;X) = \inf \{ \gamma_E(\lambda;\alpha) : \exists \varphi : E \rightarrow D \text{ holomorphic}, \varphi(\lambda) = z, \alpha \varphi'(\lambda) = X \},
\]

where \( \gamma_E(\lambda;\alpha) = \frac{|\alpha|}{|\lambda|} \). The function \( \kappa_D \) is called the Kobayashi pseudometric.

We say that a holomorphic mapping \( \varphi : E \rightarrow D \) is a \( \kappa_D \)-geodesic for \( (z,X) \) if \( \varphi \) is a function achieving the minimum in (1). Without loss of generality we may assume that \( \varphi(0) = z \). Similarly one has a notion of \( k \)-geodesic for a pair of points, where \( k \) denotes the Kobayashi pseudodistance (cf. [JP]). We know that if \( D \) is a taut domain then for any \( (z,X) \in D \times \mathbb{C}^n \) there is a \( \kappa_D \)-geodesic for \( (z,X) \) (see [JP]).

If \( D \) is a convex domain, then any \( \kappa_D \)-geodesic \( \varphi \) for \( (z,X) \) with \( X \neq 0 \) is a \( \kappa_D \)-geodesic for any \( (\varphi(\lambda),\varphi'(\lambda)) \), \( \lambda \in E \); moreover, the Kobayashi pseudometric and the Carathéodory pseudometric on \( D \) coincide (see [L]). In this case any \( \kappa_D \)-geodesic for some \( (z,X) \) with \( X \neq 0 \) is called a complex geodesic (see [Ves]).

Let us define

\[
\mathcal{E}(p) := \{|z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1\} \subset \mathbb{C}^n,
\]

where \( n > 1 \) and \( p = (p_1, \ldots, p_n) \) with \( p_j > 0 \). We call \( \mathcal{E}(p) \) a complex ellipsoid. It is well known that complex ellipsoids are taut domains and they are convex iff \( p_j \geq \frac{1}{2} \) for \( j = 1, \ldots, n \). Moreover, \( \partial \mathcal{E}(p) \) is \( C^\omega \) and strongly pseudoconvex at all points \( z \in (\partial \mathcal{E}(p)) \cap (\mathbb{C}^*)^n \).

Let us define the following mappings \( \varphi = (\varphi_1, \ldots, \varphi_n) : E \rightarrow \mathbb{C}^n \), which will be crucial for our considerations:

\[
\varphi_j(\lambda) = \left( \frac{\lambda - \alpha_j}{1 - \bar{\alpha}_j \lambda} \right)^{\gamma_j} \left( \frac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right)^{\gamma} \quad (j = 1, \ldots, n)
\]
fulfilling the following conditions:

\begin{align}
(3) & \quad a_j \in \mathbb{C}, \alpha_j \in \bar{E} \text{ for } j = 1, \ldots, n \text{ and } \alpha_0 \in E, \\
(4) & \quad r_j \in \{0, 1\} \text{ for } j = 1, \ldots, n \text{ and if } r_j = 1, \text{ then } \alpha_j \in E, \\
(5) & \quad \alpha_0 = \sum_{j=1}^{n} |a_j|^2 \alpha_j, \\
(6) & \quad 1 + |\alpha_0|^2 = \sum_{j=1}^{n} |a_j|^2 (1 + |\alpha_j|^2).
\end{align}

One can easily check that if \( \varphi \) is not a constant mapping, then \( \varphi(E) \subset \mathscr{E}(p) \).

The mappings given by the formulas above turn out to be the complex geodesics in convex complex ellipsoids (certainly under the condition that they are not constant). To simplify our notation we will speak in the following of \( \kappa \)-geodesics instead of \( \kappa_{\mathscr{E}(p)} \)-geodesics in the case of complex ellipsoids.

**Theorem 1** (see [JPZ]). A non-constant, bounded mapping \( \varphi = (\varphi_1, \ldots, \varphi_n) : E \to \mathbb{C}^n \), where \( \varphi_j \not\equiv 0 \) for \( j = 1, \ldots, n \), is a complex geodesic in the convex ellipsoid \( \mathscr{E}(p) \) iff \( \varphi \) is of the form as in (2) with (3)–(6).

Observe that the assumption \( \varphi_j \not\equiv 0 \) does not restrict the generality of this result, since mappings with 0-components can be thought as mappings into a lower dimensional situation.

The mappings of (2) are well-defined not only for convex ellipsoids but also for the non-convex ones. Therefore the natural question arises what these formulas represent in the general, not necessarily convex case. Below we shall deal with this problem. The results we get suggest that all \( \kappa \)-geodesics are necessarily of the form (2) with (3)–(6).

**Proposition 2.** Let \( \varphi : E \to \mathscr{E}(p) \) be a \( \kappa \)-geodesic for \( (\varphi(0), \varphi'(0)) \) with \( \varphi'(0) \not\equiv 0 \), where \( \varphi_j \not\equiv 0 \) for \( j = 1, \ldots, n \). Then

\[
\varphi_j(\lambda) = B_j(\lambda) \left( a_j \frac{1 - \alpha_j \lambda}{1 - \alpha_0 \lambda} \right)^{\frac{1}{p_j}},
\]

where \( B_j \) is the Blaschke product and \( \alpha_j, \alpha_0, a_j \) fulfill the relations (3), (5) and (6). Moreover, if \( p_j \geq \frac{1}{2} \) for some \( j \), then we can choose either \( B_j \equiv 1 \) or \( B_j(\lambda) = \frac{\lambda - \alpha_j}{1 - \alpha_j \lambda} \) with \( |\alpha_j| < 1 \).

Additionally, if \( |\alpha_j| < 1 \) for all \( j = 1, \ldots, n \) then either \( B_j \equiv 1 \) or \( B_j(\lambda) = \frac{\lambda - \alpha_j}{1 - \alpha_j \lambda} \) for all \( j = 1, \ldots, n \).

Proposition 2 shows that \( \kappa \)-geodesics in \( \mathscr{E}(p) \) are, in general, of a similar form as in the convex case. Observe that we do not exclude the case that the Blaschke products appearing in the formulas for geodesics may have more than one zero. Even in the case they have only one, we do not claim that this one equals \( \alpha_j \). In particular, this is not clear for those \( j \)-th components with \( |\alpha_j| = 1 \) and \( p_j < \frac{1}{2} \).

Nevertheless, our experience so far has led to the conjecture that there are no \( \kappa \)-geodesics at all with some \( |\alpha_j| = 1 \), where \( p_j < \frac{1}{2} \).

As an easy consequence of Proposition 2 we get (cf. [Po]):
Corollary 3. Let \( \varphi : E \to \mathcal{E}(p) \) be a \( \kappa \)-geodesic for \( (\varphi(0), \varphi'(0)) \) with \( \varphi'(0) \neq 0 \). Then \( \varphi^*(\partial E) \subset \partial \mathcal{E}(p) \).

As usual, \( \varphi^* \) denotes the boundary values of \( \varphi \).

In the paper [BFKKMP] the authors delivered an effective formula for the Kobayashi metric in the convex ellipsoids of type \( \mathcal{E}(1, m) \) (i.e. for \( m \geq \frac{1}{2} \)). In our note we shall find the formulas for the Kobayashi metric of the ellipsoids \( \mathcal{E}(1, m) \) with \( m < \frac{1}{2} \). We obtain the formulas using Proposition 2, proceeding similarly as in [JP] (we even use similar notation), where the authors used Theorem 1 to get the formulas from [BFKKMP].

Let us make here one general remark: whenever, in the sequel, we shall consider \( \mathcal{E}(1, m) \), then we mean \( m < \frac{1}{2} \) (unless otherwise stated).

Remark that there are the following automorphisms

\[
\mathcal{E}(1, m) \ni (z_1, z_2) \mapsto \left( \frac{z_1 - a}{1 - \bar{a} z_1}, \frac{e^{i\theta}(1 - |a|^2)^{\frac{m}{2}} z_2}{(1 - \bar{a} z_1)^{\frac{m}{2}}} \right) \in \mathcal{E}(1, m),
\]

where \( a \in E, \theta \in \mathbb{R} \). Therefore, in order to find the formulas for the Kobayashi metric, it suffices to calculate the Kobayashi metric for \( ((0, b), (X, Y)) \), where \( b \geq 0 \).

The cases: \( b = 0, X = 0 \) or \( Y = 0 \) are as usual easily done.

In the case \( b > 0, XY \neq 0 \) we shall need some more work and much more calculations. To make the calculations simpler and the formulas for the Kobayashi metric clearer we introduce in this case some additional notation. Let us put

\[
v := \left( \frac{b|X|}{m|Y|} \right)^2.
\]

Without loss of generality we may assume that \( Y = 1 \). For \( v \leq \frac{1}{4m(1-m)} \) we define

\[
t := \frac{2m^2 v}{1 + 2m(m - 1)v + \sqrt{1 + 4m(m - 1)v}}.
\]

Observe that \( \frac{1}{4m(1-m)} > 1 \) and that \( t \) increases when \( v \) increases and, additionally, we have \( t \leq \frac{m}{1-m} < 1 \). Consequently, for any \( v \leq \frac{1}{4m(1-m)} \) there is exactly one solution (from the interval \((0, 1)\)) of the following equation:

\[
x^{2m} - tx^{2m-2} - (1-t)Y^{2m} = 0.
\]

Then the formulas for the Kobayashi metric (we often write \( \kappa(v) \) instead of writing \( \kappa_{\mathcal{E}(1, m)}((0, b); (X, Y)) \)) are given as follows:

**Theorem 4.** If \( m < \frac{1}{2} \) then the following formulas hold:

\[
\kappa_{\mathcal{E}(1, m)}((0, 0); (X, Y)) = h(X, Y),
\]

\[
\kappa_{\mathcal{E}(1, m)}((0, b); (0, Y)) = \frac{|Y|}{1 - b^2},
\]

\[
\kappa_{\mathcal{E}(1, m)}((0, b); (X, 0)) = \frac{|X|}{(1 - b^2m)^{\frac{m}{2}}},
\]
where $h$ is the Minkowski function for $E(1,m)$.

And in the remaining cases ($b>0, X \neq 0, Y=1$)

(i) if $v \leq 1,$ then
\[ \kappa(v) = \frac{m}{b} \frac{x^{2m-1}}{(1-m)x^{2m} + m x^{2m-2} - b^2 m^2} =: \kappa_1(v) ; \]

(ii) if $v \geq \frac{1}{4m(1-m)},$ then
\[ \kappa(v) = \frac{m}{b} \frac{\sqrt{(1-b^2m)v + b^2m}}{1-b^2m} =: \kappa_2(v) ; \]

(iii) if $1 < v < \frac{1}{4m(1-m)},$ then
\[ \kappa(v) = \min\{\kappa_1(v), \kappa_2(v)\}, \]

where $x$ is the only solution in $(0,1)$ of equation (8).

Moreover, in the formula (iii) the minimum is equal to $\kappa_1(v)$ for $v \leq v_0$ and equal to $\kappa_2(v)$ for $v > v_0$, where $v_0 := \frac{t_0}{(1-m)^2 + t_0^2}$, $t_0 := \frac{x_0^{2m} - b^2 m^2}{x_0^{2m} - b^2 m}$ and $x_0$ is the only solution in the interval $(0,1)$ of the equation
\[ x^{4m-2}(-1 - 2m + 2m^2 + b^2m) + x^{2m}(1 + (1-2m)b^2m) + x^{2m-2}(1 + (2m-1)b^2m) - (1-m)^2 x^{4m} - m^2 x^{4m} - b^2 m = 0. \]

The detailed discussion of this situation also leads to the following properties, which differ the non-convex case from the convex one (cf. [BFKKMP], [M], [JPZ]).

Corollary 5. In the ellipsoid $E(1,m)$ the following properties hold:

(i) There are non-constant analytic discs of the form (2) with (3)–(6), which are not $\kappa$-geodesics for $(\varphi(0), \varphi'(0))$.

(ii) There are non-constant analytic discs of the form (2) with (3)–(6), which are not $k$-geodesics for $(\varphi(\lambda_0), \varphi(\lambda_1))$ for some $\lambda_0, \lambda_1 \in E$ (see (i) and [Ven]).

(iii) For any $b>0$ there is exactly one $v \in (0,\infty)$ (equal to $v_0$ from Theorem 4) such that there are more than one (up to a Möbius transformation, exactly two) $\kappa$-geodesics with respect to the pair $(b,v)$.

(iv) For any $b>0$ the function $\kappa_E(1,m)(0,b); (\cdot, 1)$ is not differentiable at the point $X_0$ given by the equation $v_0 = \left(\frac{bX_0}{m}\right)^2$.

It seems that all the analytic discs of (2) with (3)–(6) describe all the local $\kappa$-extremals. In any case, our paper shows that there are a lot of local $\kappa$-extremal and stationary maps, which are not $\kappa$-extremals, even in the case of strongly pseudoconvex Reinhardt domains, which arise from this example by smoothing the corners (compare the examples of N. Sibony in [Pa]).

2. Proof of Proposition 2. In the sequel we shall make use of the following characterization of $\kappa$-geodesics in a smooth strongly pseudoconvex domain $\Omega$ (writing smooth we always mean $C^\infty$).

For a mapping $\varphi \in \mathcal{O}(E, \Omega) \cap C^1(\bar{E}, \bar{\Omega})$ with $\varphi(\partial E) \subset \partial \Omega$, $\varphi(0) \in \Omega$ one knows that
\[ \partial E \ni \lambda \longrightarrow \lambda \frac{\partial \rho}{\partial z} (\varphi(\lambda)) \cdot \varphi'(\lambda) \]
is a positive function (see [Pa]), where \( \Omega = \{ \rho < 0 \} \) and \( \rho \) is a defining function of \( \Omega \) plurisubharmonic in \( \mathbb{C}^n \). Here \( w \cdot z := w_1 z_1 + \ldots + w_n z_n \) for \( w = (w_1, \ldots, w_n) \), \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \).

We define

\[
p(\lambda)^{-1} := \lambda \frac{\partial \rho}{\partial \overline{z}} (\varphi(\lambda)) \cdot \varphi'(\lambda), \quad \lambda \in \partial E,
\]

\[
\hat{\varphi}(\lambda) := p(\lambda) \lambda^2 \frac{\partial \rho}{\partial z} (\varphi(\lambda)), \quad \lambda \in \partial E.
\]

**Definition 6** (see [Pa], [L]). A mapping \( \varphi : \bar{E} \to \bar{\Omega} \) is said to be stationary if \( \varphi \) is a \( C^1 \) mapping of \( \bar{E} \) into \( \bar{\Omega} \), holomorphic on \( E \) such that \( \varphi(\partial E) \subset \partial \Omega \), \( \varphi(0) \in \Omega \) and \( \hat{\varphi} \) extends to a continuous mapping on \( \bar{E} \), holomorphic on \( E \).

Observe that there are weaker notations of what "stationary" means (see [JP]).

Then in [Pa] the following result is shown:

**Theorem 7** (see [Pa]). Let \( \varphi : \bar{E} \to \bar{\Omega} \) be a \( C^1 \)-mapping with \( \varphi(\partial E) \subset \partial \Omega \), \( \varphi(0) \in E \), which is holomorphic on \( E \). If \( \varphi \) is a \( \kappa_\Omega \)-geodesic for \( (\varphi(0), \varphi'(0)) \in \Omega \times (\mathbb{C}^n \setminus \{0\}) \), then \( \varphi \) is stationary.

We point out that even more is shown in [Pa], namely, that any \( \varphi \) as in Theorem 7, which is a local extremal for \( (z, X) \) — observe that the term \( \kappa \)-geodesic means a global extremal w.r.t. \( (1) \) — is automatically stationary.

It may be possible that one can find the proof of Proposition 2 (or even a better description of \( \kappa \)-geodesics than the one given there) by modifying slightly the solution of the extremal problem in \( C^1 \) pseudoconvex domains given in [Po]. Because of Corollary 5(i) we are not interested in this here.

We shall often make use of the decomposition theorem for a mapping \( f \in H^\infty(E) \), \( f \neq 0 \), which says that

\[
f(\lambda) = B(\lambda)A(\lambda) \quad \text{for} \quad \lambda \in E,
\]

where \( B \) is the Blaschke product of \( f \) and \( A \) is a nowhere vanishing function from \( H^\infty(E) \).

Now we start the proof of Proposition 2 proving a sequence of lemmas.

**Lemma 8.** Let \( \varphi : E \to \mathcal{E}(p) \) be a \( \kappa \)-geodesic for \( (\varphi(\lambda_0), \varphi'(\lambda_0)) \) with \( \varphi'(\lambda_0) \neq 0 \) for \( \lambda_0 \in E \), where \( \varphi_j \neq 0 \) for all \( j \). Assume that

\[
\varphi_j(\lambda) = B_j(\lambda)A_j(\lambda), \quad j = 1, \ldots, n,
\]

where \( B_j \) and \( A_j \) are, as above, the factors from the decomposition of \( \varphi_j \). Denote by \( Z_j \) the zeros of \( B_j \) (counted with multiplicity). Denote by \( \tilde{Z}_j \) a subset of \( Z_j \). Let us associate with \( \tilde{Z}_j \) the Blaschke product \( \tilde{B}_j \). Put

\[
\hat{\varphi}_j := \tilde{B}_j A_j \quad \text{and} \quad \hat{\varphi} := (\hat{\varphi}_1, \ldots, \hat{\varphi}_n).
\]

If \( \hat{\varphi} \) is non-constant, then \( \hat{\varphi} \) is a \( \kappa \)-geodesic for \( (\hat{\varphi}(\lambda_0), \hat{\varphi}'(\lambda_0)) \) and \( \hat{\varphi}'(\lambda_0) \neq 0 \).

**Proof.** At first we prove that \( \hat{\varphi}(E) \subset \mathcal{E}(p) \). To see this let us repeat inductively the following procedure. We divide the components \( \varphi_j \) by the Blaschke factor...
assigned to a zero from $\mathcal{Z}_j \setminus \tilde{\mathcal{Z}}_j$ (or if we have already exhausted $\mathcal{Z}_j \setminus \tilde{\mathcal{Z}}_j$, we leave the component without change). We proceed so till we have exhausted all the sets $\mathcal{Z}_j \setminus \tilde{\mathcal{Z}}_j$ ($j = 1, \ldots, n$), so, in other words, till we get $\tilde{\varphi}$. Let us put $\hat{h}(z) := |z_1|^{2p_1} + \ldots + |z_n|^{2p_n}$. Remark that in view of the maximum principle for subharmonic functions applied to the composition of the mappings obtained from $\varphi$ after a finite number of steps of the above-mentioned procedure with $\hat{h}$ we get that this composition is not larger than 1 on $E$. Consequently the limit function of the composition is not larger than 1. So we get $\hat{h} \circ \tilde{\varphi} \leq 1$ on $E$. The maximum principle implies that either $\hat{h} \circ \tilde{\varphi} \equiv 1$ or $\hat{h} \circ \tilde{\varphi} < 1$ on $E$. The first case gives, using local peak functions, that $\tilde{\varphi}$ is constant. This completes the proof of the inclusion above.

Suppose now that $\tilde{\varphi}$ is not a $\kappa$-geodesic for $(\tilde{\varphi}(\lambda_0), \tilde{\varphi}'(\lambda_0))$. Then there exists a map $\psi : E \to \mathcal{E}(p)$ such that

$$\tilde{\psi}(\lambda_0) = \tilde{\varphi}(\lambda_0), \quad \tilde{\psi}'(\lambda_0) = \tilde{\varphi}'(\lambda_0), \quad \tilde{\psi}(E) \subset \subset \mathcal{E}(p).$$

Let us define

$$\psi_j(\lambda) := \frac{B_j(\lambda)}{B_j(\lambda)} \psi_j(\lambda) \quad \text{for } \lambda \in E \quad \text{and } \psi := (\psi_1, \ldots, \psi_n).$$

We have

$$\psi(\lambda_0) = \varphi(\lambda_0), \quad \psi'(\lambda_0) = \varphi'(\lambda_0)$$

(recall that $\varphi_j(\lambda) = \frac{B_j(\lambda)}{B_j(\lambda)} \tilde{\varphi}(\lambda)$) and $\psi(E) \subset \subset \mathcal{E}(p)$. But this contradicts the fact that $\varphi$ is a $\kappa$-geodesic for $(\varphi(\lambda_0), \varphi'(\lambda_0))$.

To prove that $\tilde{\varphi}'(\lambda_0) \neq 0$ we proceed by contradiction. Thus if $\tilde{\varphi}'(\lambda_0) = 0$ then $\eta := (\eta_1, \ldots, \eta_n)$, $\eta_j(\lambda) := \frac{B_j(\lambda)}{B_j(\lambda)} \lambda_j(\lambda)$, is also a $\kappa$-geodesic for $(\varphi(\lambda_0), \varphi'(\lambda_0))$. But $\eta(E) \subset \subset \mathcal{E}(p)$ — a contradiction. □

**Lemma 9.** Let $\varphi : E \to \mathcal{E}(p)$ be a $\kappa$-geodesic for $(\varphi(\lambda_0), \varphi'(\lambda_0))$ with $\varphi'(\lambda_0) \neq 0$, where $\varphi_j \neq 0$, $j = 1, \ldots, n$. Then

$$\varphi_j(\lambda) = B_j(\lambda) \left( a_j \frac{1 - \lambda_j \lambda}{1 - \bar{\alpha}_j \lambda} \right)^{\frac{1}{\kappa_j}},$$

where $B_j$ is the Blaschke product and the coefficients $\alpha_j, \alpha_0, a_j$ for $j = 1, \ldots, n$ fulfill the relations (3), (5) and (6).

**Proof.** We know (from the decomposition theorem) that $\varphi_j = B_j A_j$, where $B_j$ is the Blaschke product and $A_j$ has no zero in $E$.

If $A_j$ is constant for $j = 1, \ldots, n$, then we are done with $\alpha_0 = \ldots = \alpha_n = 0$ and $a_j = A_j^{\bar{a}_j}$ because $|B_j| = 1$ a.e. on $\partial E$ implies $\sum_{j=1}^{n} |A_j|^{2p_j} = 1$ (otherwise, if the sum is smaller than 1, then $\varphi(E) \subset \subset \mathcal{E}(p)$, hence $\varphi$ is not a $\kappa$-geodesic). So we may assume that some $A_j$ is not constant. For $j = 1, \ldots, n$ let us put $\psi_j := A_j$.

Then in view of Lemma 8 the mapping

$$\psi := (\psi_1, \ldots, \psi_n)$$

is a $\kappa$-geodesic for $(\psi(\lambda_0), \psi'(\lambda_0))$ with $\psi'(\lambda_0) \neq 0$. 

(9)
Let us take \( k \in \mathbb{N} \) such that \( q_j := p_j k \geq \frac{1}{2} \) for \( j = 1, \ldots, n \). Put \( \tilde{\psi}_j := \psi_j^+ \), \( q := (q_1, \ldots, q_n) \), \( \psi := (\tilde{\psi}_1, \ldots, \tilde{\psi}_n) \). Remark that \( \tilde{\psi}(E) \subset \mathcal{E}(q) \) and that \( \mathcal{E}(q) \) is a convex ellipsoid.

Now we will prove that \( \tilde{\psi} \) is a \( \kappa \)-geodesic for \((\tilde{\psi}(\lambda_0), \tilde{\psi}'(\lambda_0))\) in \( \mathcal{E}(q) \) (with \( \tilde{\psi}'(\lambda_0) \neq 0 \)), so; consequently, it is a complex geodesic.

To see this, remark that otherwise there would be \( \eta : E \rightarrow \mathcal{E}(q) \) such that \( \eta(\lambda_0) = \tilde{\psi}(\lambda_0) \), \( \eta'(\lambda_0) = \tilde{\psi}'(\lambda_0) \) and \( \eta(E) \subset \subset \mathcal{E}(q) \). Hence \( \eta^k := (\eta_1^k, \ldots, \eta_n^k) \) maps \( E \) into \( \mathcal{E}(p) \) with \( \eta^k(\lambda_0) = \psi(\lambda_0) \), \( (\eta^k)'(\lambda_0) = \psi'(\lambda_0) \) and \( \eta^k(E) \subset \subset \mathcal{E}(p) \), which contradicts (9).

Observe that \( \tilde{\psi}_j \) is without zeros in \( E \), so in view of Theorem 1 we know that

\[
\tilde{\psi}_j(\lambda) = \left( a_j \frac{1 - \bar{a}_j \lambda}{1 - \bar{\alpha}_j \lambda} \right)^{r_j / k},
\]

where \( a_j, \alpha_j, \alpha_0 \) fulfil the relations (3), (5) and (6). This completes the proof. \( \square \)

**Lemma 10.** Let \( \varphi : E \rightarrow \mathcal{E}(p) \) be a \( \kappa \)-geodesic for \((\varphi(\lambda_0), \varphi'(\lambda_0))\) with \( \varphi'(\lambda_0) \neq 0 \), where \( \varphi_j \neq 0 \) for \( j = 1, \ldots, n \). Assume that \( p_1, \ldots, p_k \geq \frac{1}{2} \) for some \( 0 \leq k \leq n \).

Then \( B_j, j = 1, \ldots, k \), from Lemma 9 can be chosen as \( \left( \frac{\lambda - \alpha_j}{1 - \alpha_j \lambda} \right)^{r_j} \) with \( r_j = 0 \) or 1 (observe that the \( a_j \)'s have to be modified by rotations); the \( \alpha_j \)'s are those from Lemma 9.

**Proof.** From Lemma 9 we get

\[
\varphi_j(\lambda) = B_j(\lambda) \left( a_j \frac{1 - \bar{a}_j \lambda}{1 - \bar{\alpha}_j \lambda} \right)^{r_j / k},
\]

for \( j = 1, \ldots, n \), where \( B_j \) is the Blaschke product and the coefficients \( \alpha_j, \alpha_0, a_j \) fulfil the relations (3), (5) and (6).

Put

\[
\psi_j := \varphi_j, j = 1, \ldots, k, \quad \psi_j(\lambda) := \left( a_j \frac{1 - \bar{a}_j \lambda}{1 - \bar{\alpha}_j \lambda} \right)^{r_j / k}, j = k + 1, \ldots, n,
\]

where \( \ell \in \mathbb{N} \) is such that \( \ell p_j \geq \frac{1}{2}, j = k + 1, \ldots, n \). If \( \psi := (\psi_1, \ldots, \psi_n) \) is constant then we are done. Assume now that \( \psi \) is not constant. Then exactly as in the proof of Lemma 9 and in view of Lemma 8 we get that \( \psi \) is a complex geodesic for \((\psi(\lambda_0), \psi'(\lambda_0))\) with \( \psi'(\lambda_0) \neq 0 \) in \( \mathcal{E}(p_1, \ldots, p_k, \ell p_{k+1}, \ldots, \ell p_n) \) — a convex ellipsoid — which in view of Theorem 1 completes the proof. \( \square \)

**Proof of Proposition 2.** If we combine Lemmas 9 and 10, then we only have to prove the last part of the proposition.

In view of Lemma 8 it suffices to discuss the case when the Blaschke products of all components \( \varphi_j \) of \( \varphi \) have at most a finite number of zeros.

Remark that in this case the mapping \( \varphi \) is holomorphic in a neighborhood of \( E \) and moreover (since the zeros of the \( \varphi_j \)'s are lying in \( E \) and their number is finite) it is 'far' from the points, where the boundary of the ellipsoid is not strongly pseudoconvex. Therefore it is a \( \kappa \)-geodesic in some smooth strongly pseudoconvex subdomain of \( \mathcal{E}(p) \), whose boundary coincides with the boundary of \( \mathcal{E}(p) \) everywhere.
except for a small neighborhood of that part of \( \partial E(p) \), where the strong pseudoconvexity breaks down. In particular, we may assume that the defining function of the new domain coincides with the defining function of \( E(p) \) (= \( |z_1|^{p_1} + \ldots + |z_n|^{p_n} - 1 \)) near \( \varphi(\partial E) \). Applying Theorem 7 we get that \( \varphi \) is stationary. Therefore

\[
\frac{1}{\lambda} \varphi_j(\lambda) \varphi_j(\lambda) = p(\lambda)p_j \left| \frac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right|^2 |a_j|^2 > 0 \quad \text{on} \quad \partial E, \quad j = 1, \ldots, n.
\]

In view of Gentili’s result (see [G]) we obtain that

\[
\varphi_j(\lambda) \varphi_j(\lambda) = r_j(\lambda - \gamma_j)(1 - \bar{\gamma}_j \lambda) \quad \text{for} \quad \lambda \in \bar{E}, \quad r_j > 0, \quad \gamma_j \in \bar{E}.
\]

This implies that \( \varphi_j \) has at most one zero in \( E \).

From the previous formulas we have:

\[
p(\lambda)p_j |a_j|^2 \left| \frac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right|^2 = |\varphi_j(\lambda)\varphi_j(\lambda)| = r_j|1 - \bar{\gamma}_j \lambda|^2
\]

for \( \lambda \in \partial E \). So

\[
\frac{r_j|1 - \bar{\gamma}_j \lambda|^2}{p_j |a_j|^2 |1 - \bar{\alpha}_j \lambda|^2} = \frac{p(\lambda)}{|1 - \bar{\alpha}_0 \lambda|^2}, \quad \text{if} \quad \lambda \in \partial E.
\]

Consequently either

\[
\gamma_1 = \cdots = \gamma_n, \quad \alpha_1 = \cdots = \alpha_n
\]

or

\[
\gamma_j = \alpha_j, \quad \text{for} \quad j = 1, \ldots, n.
\]

In the second case we are done. If we assume the first one, then (from the formulas (5) and (6))

\[
\alpha_0 = \left( \sum_{j=1}^{n} |a_j|^2 \right) \alpha_1,
\]

\[
1 + |\alpha_0|^2 = \left( \sum_{j=1}^{n} |a_j|^2 \right) (1 + |\alpha_1|^2).
\]

It follows that

\[
\alpha_1 (1 + |\alpha_0|^2) = \alpha_0 (1 + |\alpha_1|^2), \quad \text{so}
\]

\[
\alpha_1 - \alpha_0 = \alpha_0 \alpha_1 (\bar{\alpha}_1 - \bar{\alpha}_0).
\]

If \( \alpha_1 \neq \alpha_0 \), then \( |\alpha_0 \alpha_1| = 1 \) — a contradiction. If \( \alpha_1 = \alpha_0 \), then we have \( \gamma_1 = \cdots = \gamma_n \in E \). We may define \( \alpha_j = \alpha_0 := \gamma_1 \), which does not spoil earlier relations and then we are done. □

3. **Proof of Theorem 4.** To get the formulas for the Kobayashi metric in the 'extremal' cases \( b = 0, X = 0 \) or \( Y = 0 \) we need the following lemma.
Lemma 11. Put
\[ \varphi : E \ni \lambda \mapsto (\lambda^{r_1}z_1^0, \ldots, \lambda^{r_n}z_n^0) \in \mathcal{E}(p), \]
where \( z^0 = (z_1^0, \ldots, z_n^0) \in \partial \mathcal{E}(p), \) \( r_j \in \{0, 1\}, j = 1, \ldots, n \) and \( \# \{ j : r_j = 1 \} \geq 1. \) Then \( \varphi \) is a \( \kappa \)-geodesic for \( (\varphi(0), \varphi'(0)) \).

Proof. Without loss of generality we assume that
\[ \varphi(\lambda) = \left( \lambda z_0^1, \ldots, \lambda z_0^k, z_{k+1}^0, \ldots, z_n^0 \right), \]
where \( z_0^1 \cdots z_0^k \neq 0, k+1 \geq 2, z^0 \in \partial \mathcal{E}(p). \)

Let us take a holomorphic mapping \( \psi : E \rightarrow \mathcal{E}(p) \) with \( \psi(0) = (0, 1, \ldots, 0, z_{k+1}^0, \ldots, z_n^0) \) and
\[ \psi'(0) = t(z_1^0, \ldots, z_k^0, 0, \ldots, 0) = t\varphi'(0), t > 0. \]

Without loss of generality we may assume that \( \psi \) is continuous on \( \bar{E} \). Let \( \tilde{h}(z) := \sum_{j=1}^n |z_j|^{2p_j}, \tilde{h} \in \text{PSH}. \) In view of (11) we may write
\[ \psi(\lambda) = (\lambda A_1(\lambda), \ldots, \lambda A_k(\lambda), \psi_{k+1}(\lambda), \ldots, \psi_n(\lambda)). \]

Put \( \tilde{\psi} := (A_1, \ldots, A_k, \psi_{k+1}, \ldots, \psi_n). \)

Since \( (\tilde{h} \circ \tilde{\psi})(\lambda) \leq 1 \) on \( \partial E \), we get that \( \tilde{h} \circ \psi \leq 1 \) on \( E \), so consequently \( \tilde{h} \circ \tilde{\psi}(0) \leq 1 \) or
\[ \sum_{j=1}^k |A_j(0)|^{2p_j} + \sum_{j=k+1}^n |\psi_j(0)|^{2p_j} \leq 1. \]

In view of (10), (11) and (12) we see that
\[ \sum_{j=1}^k t^{2p_j} |z_j^0|^{2p_j} \leq 1 - \sum_{j=k+1}^n |z_j^0|^{2p_j} = \sum_{j=1}^k |z_j^0|^{2p_j}. \]

Hence we obtain
\[ \sum_{j=1}^k |z_j^0|^{2p_j}(t^{2p_j} - 1) \leq 0. \]

But \( p_j > 0 \), so \( t \leq 1 \), which completes the proof of the lemma. \( \Box \)

Proof of Theorem 4. In view of Lemma 11 we get the formulas in the ‘extremal’ cases. Therefore we may assume that \( b > 0, X \neq 0 \) and \( Y = 1. \)

Below we consider only the mappings of the following forms
\[ \varphi(\lambda) = \left( \frac{a_1\lambda}{1 - \alpha_0\lambda}, \left( \frac{a_2}{1 - \alpha_0\lambda} \right)^{-\frac{1}{p}} \right), \]
\[ \varphi(\lambda) = \left( \frac{a_1\lambda}{1 - \alpha_0\lambda}, \frac{\lambda - \alpha_2}{1 - \alpha_2\lambda} \left( \frac{a_2}{1 - \alpha_0\lambda} \right)^{-\frac{1}{p}} \right). \]

such that

\[(15) \quad \varphi(0) = (0, b), \, \tau \varphi'(0) = (X, Y)\]

and \(\alpha_j, \alpha_0, a_j\) fulfil (3), (5) and (6).

In the sequel we shall find a mapping of form (13) or (14) such that \(|\tau|\) from (15) is the smallest and, moreover, we shall see that

\[(16) \quad \text{the smallest value of } |\tau| \text{ in (15) is never achieved by a mapping of type (13) with } |\alpha_2| = 1.\]

We claim that this smallest \(|\tau|\) will be the value of the Kobayashi metric for \(((0, b), (X, Y))\) (or equivalently for suitable \(v\)). Why is it so? Theoretically the smallest \(|\tau|\) may be provided by a mapping \(\varphi\) with a second component such that \(|\alpha_2| = 1\) whose Blaschke product has some zeros (see Proposition 2 or Lemma 10). But if this were the case then also the mapping with the 'deleted' Blaschke product of the second component would be a \(\kappa\)-geodesic for some other \(v\) (see Lemma 7). So the new mapping would deliver the minimum of the values of \(|\tau|\) between the mappings of the forms (13) and (14) with (15) (for some other data connected with our new \(v\)), which however contradicts our earlier remark (see (16)).

At this point we repeat, at least partially, the reasoning from [JP], where the authors calculate the Kobayashi metric for convex ellipsoids \(E(1, m)\).

Remark that there is a mapping \(\varphi\) of form (13) iff \(v \geq 1\) and in this case \(|\tau|\) is given by the formula (see (8.4.25) from [JP])

\[(17) \quad |\tau| = \frac{m}{b} \sqrt{\frac{(1 - b^{2m})v + b^{2m}}{1 - b^{2m}}}.\]

Moreover, there is a mapping \(\varphi\) as in (14) fulfilling (15) iff there is \(0 < x < 1\) (and then \(\alpha_1 = x\)) such that \(x^{2m-1} > b^{2m}\) (but this holds always since \(2m - 1 < 0\) !) and

\[(18) \quad v((m - 1)x^{2m} - mx^{2m-2} + b^{2m})^2 - (x^{4m-2} - b^{2m}x^{2m} - b^{2m}x^{2m-2} + b^{4m}) = 0.\]

And then our \(|\tau|\) is given by the formula (one can easily check that the denominator in the formula below is positive).

\[|\tau| = \frac{mx^{2m-1}}{b((1 - m)x^{2m} + mx^{2m-2} - b^{2m})} \quad (\text{see [JP] (8.4.32))}.\]

Recall that the formula for \(t\) (see (7)) has sense if \(v \leq \frac{1}{4m(1-m)}\). We also remark that \(v\) calculated from (18) fulfils the inequality \(v \leq \frac{1}{4m(1-m)}\) (this implies that there may exist a mapping of the form (14) fulfilling (15) only for \(v \leq \frac{1}{4m(1-m)}\)); moreover, we shall see that for any \(v \leq \frac{1}{4m(1-m)}\) there exists \(x \in (0, 1)\) fulfilling (18) — this will imply that in this case there is always a mapping of the form (14) with (15).
The equality (18) is equivalent to

$$x^{2m} - tx^{2m-2} - (1-t)b^{2m} = 0$$

or

$$(m-1)^2vx^{2m} - \frac{m^2v}{t}x^{2m-2} + \frac{1-v}{1-t}b^{2m} = 0.$$  

Remark that the equation (19) has exactly one solution $x \in (0, 1)$ for $0 < v \leq \frac{1}{4m(1-m)}$ and that (20) has no solution in the interval $(0, 1)$ for $0 < v < 1$ but has exactly one solution in $(0, 1)$ if $1 < v \leq \frac{1}{4m(1-m)}$ (for $v = 1$ we have $x = 1$ and this gives a mapping from (13) as above).

In order to choose a mapping of type (13) or (14) with (15) such that $|\tau|$ is minimal we have no problems for $v < \frac{1}{4m(1-m)}$ (respectively $v > \frac{1}{4m(1-m)}$); these are the mappings of type (14) with $\alpha_1 := x$, where $x$ is a solution of (19) (respectively of type (13)). The problem is with $1 \leq v \leq \frac{1}{4m(1-m)}$ (or equivalently $\frac{m}{1-m} \leq t \leq \frac{m}{1-m} < 1$). In this case we have to choose one of three (at most) mappings:

$$\tau_1(v) = \frac{m}{b} \frac{x_1^{2m-1}}{(1-m)x_1^{2m} + mx_1^{2m-2} - b^{2m}};$$

$$\tau_2(v) = \frac{m}{b} \frac{\sqrt{(1-b^{2m})v + b^{2m}}}{1-b^{2m}};$$

$$\tau_3(v) = \frac{m}{b} \frac{x_2^{2m-1}}{(1-m)x_2^{2m} + mx_2^{2m-2} - b^{2m}},$$

where $x_1$ is the solution in $(0, 1)$ of (19) and $x_2$ is the solution in $(0, 1)$ of (20) (if $v = 1$, then $x_2 = 1$).

One can easily check that

$$(m-1)^2vt^2 - (1+2m(m-1)v)t + m^2v = 0,$$

so

$$v = \frac{t}{(t(1-m) + m)^2}.$$  

Therefore we get from (20) (the equation w.r.t. $x_2$)

$$(m-1)^2tx_2^{2m} - b^{2m} = m^2x_2^{2m-2} - b^{2m}. $$

Substituting (19) into the formula (21) we may write (we consider now $\tau_j$’s as the functions of $t$)

$$\tau_1(t) = \frac{m}{b} \frac{x_1^{2m-1}}{(x_1^{2m-2} - b^{2m})(t(1-m) + m)}.$$  

Substituting (24) into (22) we get

$$\tau_2(t) = \frac{m}{b} \frac{\sqrt{(1-b^{2m})t + (t(1-m) + m)^2b^{2m}}}{(1-b^{2m})(t(1-m) + m)}.$$
Substituting (25) into (23) we get

\[ \tau_3(t) = \frac{m}{b} \frac{x_2^{2m-1}}{(x_2^{2m-2} - b^{2m})} \frac{(1-m)t}{m(m + (1-m)t)}. \]

To obtain the formulas for the Kobayashi metric as in the theorem it is enough to prove that

\[ \tau_1(t) < \tau_3(t) \text{ for } t \in \left( \left( \frac{m}{1-m} \right)^2, \frac{m}{1-m} \right) \]

(for \( t = \frac{m}{1-m} \) we have \( \tau_1(\frac{m}{1-m}) = \tau_2(\frac{m}{1-m}) \)). In particular, the inequality for \( t = \left( \frac{m}{1-m} \right)^2 \) (in other words for \( v = 1 \)) will prove that the mapping of type (13) with \( |\alpha_2| = 1 \) is never a \( \kappa \)-geodesic (see (16)).

Therefore, to prove (26), we calculate (using the formulas for \( \tau_1 \) and \( \tau_3 \) and obtaining \( x_1'(t) \) and \( x_2'(t) \) from (19) and (25))

\[ \tau_1'(t) - \tau_3'(t) = \frac{m}{2b} \frac{(m + t(m - 1))(tx_2(1-m) - mx_1)}{(m + (1-m)t)^2 (mx_1^2 + t(1-m)(1-m)x_2^2 + m)} \]

One sees that \( m + t(m - 1) > 0 \) and, after some calculations, that \( tx_2(1-m) - mx_1 < 0 \) for \( t \in \left( \left( \frac{m}{1-m} \right)^2, \frac{m}{1-m} \right) \). This implies that \( \tau_1'(t) - \tau_3'(t) \) is positive for these \( t \). Hence (26) holds and this completes the proof of the formula for the Kobayashi metric.

To prove the last part of the theorem observe that it is enough to show (use the continuity of the Kobayashi metric of \( \mathcal{E}(1, m) \), the continuity of \( \tau_1 \) and \( \tau_2 \) and the fact that the Kobayashi metric equals \( \tau_1(v) \) for \( v \leq 1 \) and \( \tau_2(v) \) for \( v \geq \frac{1}{4m(1-m)} \) that there is only one \( t_0 \in \left( \left( \frac{m}{1-m} \right)^2, \frac{m}{1-m} \right) \) such that

\[ \frac{x_1^{2m-1}}{x_1^{2m-2} - b^{2m}} = \frac{\tau_1(t_0) = \tau_2(t_0)}{\sqrt{(1-b^{2m})t_0 + (t_0(1-m) + m)^2b^{2m}}} \]

where

\[ x_1^{2m} - t_0x_1^{2m-2} - (1-t_0)b^{2m} = 0. \]

Substituting the second equality into the first one we end up with the following equality

\[ x_1^{4m-2}(-1 - 2m + 2m^2 + b^{2m}) + x_1^{2m}(1 + (1-2m)b^{2m}) + x_1^{2m-2}(1 + (2m-1)b^{2m}) - (1-m)^2x_1^{4m} - m^2x_1^{4m-4} - b^{2m} = 0, \]

which turns out to have exactly one solution \( x_1 \in (0, 1) \). This completes the proof. \( \square \)
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