Two-Dimensional Yang-Mills Theory on Surfaces With Corners in Batalin-Vilkovisky Formalism

R. Iraso, a P. Mnev b

a International School of Advanced Studies (SISSA) via Bonomea 265, I-34136 Trieste, Italy
b University of Notre Dame Notre Dame, IN 46556

E-mail: riraso@sissa.it, pmnev@nd.edu

Abstract: In this paper we recover the non-perturbative partition function of 2D Yang-Mills theory from the perturbative path integral. To achieve this goal, we study the perturbative path integral quantization for 2D Yang-Mills theory on surfaces with boundaries and corners in the Batalin-Vilkovisky formalism (or, more precisely, in its adaptation to the setting with boundaries, compatible with gluing and cutting – the BV-BFV formalism). We prove that cutting a surface (e.g. a closed one) into simple enough pieces – building blocks – and choosing a convenient gauge-fixing on the pieces, and assembling back the partition function on the surface, one recovers the known non-perturbative answers for 2D Yang-Mills theory.

Contents

1 Introduction 3
1.1 Surfaces of non-negative Euler characteristic 4
1.2 General surfaces, surfaces with corners 6
1.3 Main results 9
1.4 Plan of the paper 10
1.5 Acknowledgements 10

2 Background: BV-BFV formalism 11
2.1 Classical BV-BFV 11
2.2 Quantum BV-BFV 12
2.3 Quantization 13
2.3.1 Perturbative expansion 14
2.3.2 Renormalization and globalization 15
2.4 BV-BFV formulation of 2D YM 16
2.4.1 Ω-cohomology in $A$-polarization on a circle 20
2.4.2 Hodge propagators and axial gauge 21
1 Introduction

In this paper we study the perturbative path integral quantization of 2D Yang-Mills theory, defined classically by the first-order action functional

\[ S_{\text{YM}}^{\text{cl}} = \int_{\Sigma} \langle B, dA + A \wedge A \rangle + \frac{1}{2} \mu (B, B), \]  

(1.1)

with fields \( A \), a 1-form on the surface \( \Sigma \) with coefficients in a semi-simple Lie algebra \( g = \text{Lie}(G) \), for \( G \) a compact simply-connected structure group, and \( B \), a 0-form valued in \( g^* \), and where \( \mu \) is a fixed background 2-form (the "area form") on \( \Sigma \). Here \( \langle \cdot , \cdot \rangle \) is the pairing between \( g \) and \( g^* \) and \( (\cdot , \cdot) \) is the inverse Killing form.\(^1\) We study 2D Yang-Mills theory in the Batalin-Vilkovisky formalism on oriented surfaces \( \Sigma \) with boundaries and corners allowed.\(^2\) The quantization is constructed in such a way that it is compatible with gluing and cutting of surfaces.

Our primary motivating goal is to construct explicit partition functions of 2D Yang-Mills theory on arbitrary surfaces via the perturbative path integral quantization, \( Z = \int e^{i \hbar S_{\text{YM}}} \), and to compare them with the known non-perturbative answers \([20, 24]\) formulated in terms of the representation-theoretic data of the structure group \( G \). There are two immediate problems to deal with:

**Gauge symmetry.** To define the Feynman diagrams giving the perturbative expansion of the path integral, one needs to deal with the gauge symmetry of the action creating the degeneracy of stationary phase points in the path integral. To do this, we employ the Batalin-Vilkovisky (BV) formalism. In the BV formalism, the classical fields \( A, B \) are promoted to non-homogeneous differential forms on the surface (whose homogeneous components are the original classical fields, the Faddeev-Popov ghost for the gauge symmetry and the anti-fields for those) and the gauge-fixing consists in the choice of a Lagrangian submanifold in the BV fields.

**Computability of the perturbative answers.** Generally, Feynman diagrams are given by certain integrals over configuration spaces of points on the surface, with the integrand given by a product of propagators which depend on the details of the gauge-fixing, and typically these integrals are very hard to compute. The remedy for this comes from two ideas:

1. Firstly, we employ the BV-BFV refinement of the Batalin-Vilkovisky formalism constructed in \([10, 13]\) – a refinement adapted to gauge theories on manifolds with boundaries allowed, compatible with gluing and cutting (thus, it is a version of BV quantization compatible with Atiyah-Segal functorial picture of QFT). In this formalism, we

---

\(^1\) The case \( \mu = 0 \) of the action functional (1.1) defines the so-called non-abelian BF theory, which is a topological field theory, i.e., is invariant under diffeomorphisms of surfaces.

\(^2\) We assume orientability for convenience of the formalism, but in fact one can define the theory on non-orientable surfaces as well, twisting the field \( B \) by the orientation line bundle and defining \( \mu \) to be a density on \( \Sigma \), rather than a 2-form. The integral (1.1) is then also understood as an integral of a density.
can recover the perturbative partition function on a surface from cutting the surface into pieces – the appropriate “building blocks of surfaces”, calculating the perturbative partition function on the pieces and then assembling back into the answer for the whole surface via the gluing formula.

2. Secondly, to compute the answers on our building blocks, we employ special gauge-fixings which allow for explicit computation of Feynman diagrams on the building block. E.g. we use the axial gauge for cylinders. This procedure is equivalent to imposing a very special gauge-fixing, involving the data of cutting into the building blocks, on the theory on the surface we started with.

For instance, the following Feynman graph for 2D Yang-Mills theory on a sphere

\[
\begin{array}{c}
\mu \\
\end{array}
\]

is given by a complicated integral in Lorenz gauge on the sphere (e.g. the one associated to the standard round metric), but in our approach it is explicitly computable, once we split the sphere into four pieces (figure 1).

![Figure 1](image1.png)

**Figure 1.** A two-loops Feynman diagram for 2D Yang-Mills on the sphere, computed by suitably cutting the surface. The darker regions are the ones where the 2-form $\mu$ is allowed to be nonzero.

**Remark 1.1.** Axial gauge, which we often use, corresponds to a singular propagator which can be obtained as a limit of metric propagators (given by smooth forms on the configuration space of two points) corresponding to the degeneration of geometry of the cylinder where the ratio of the circumference to the length tends either to zero or to infinity (corresponding to two versions of the axial gauge). Thus, our computable answers obtained in a convenient gauge arise as a limit (corresponding to a limiting point on a certain curve in the space of metrics on the surface $\Sigma$) of perturbative answers computed with smooth propagators.

### 1.1 Surfaces of non-negative Euler characteristic

Surfaces of non-negative Euler characteristic (possibly with boundary, but with no corners) can be decomposed into the following building blocks:
(I) Cylinder with polarizations $A, B$ fixed on the two boundary circles (i.e. with boundary conditions prescribing the pullback of $A$ to one boundary circle, and the pullback of $B$ to the other circle), with a nonzero area form $\mu$ allowed.

(II) Cylinder with polarization $B$ fixed on both boundaries, with the background 2-form $\mu = 0$.

(III) Cylinder with polarization $A$ fixed on both boundaries, with $\mu = 0$.

(IV) Disk with polarization $B$ on the boundary and with $\mu = 0$.

This is the premise of the BV-BFV formalism as in [13], where the connected components of the boundary are decorated with either $A$- or $B$-polarization (boundary condition), and one is allowed to glue an $A$-boundary circle of one surface to a $B$-boundary circle of another surface.

We manage to compute building blocks (I-III) explicitly using the axial gauge, whereas the building block (IV) can be computed in any gauge due to the vanishing of almost all Feynman diagrams by a degree counting argument. We use the fact that the partition function can only depend on the total area of a surface to concentrate the area form $\mu$ on building blocks (I).\footnote{More precisely, changing the area form $\mu$ by an exact 2-form $d\gamma$ amounts in the BV language to a canonical transformation of the action and the associated change of the partition function by a BV-exact term. Therefore, working modulo BV exact terms, one can concentrate the area term in an arbitrarily small region. We thank Alberto S. Cattaneo for this remark.}

Block (III) is the most complicated in this list. We only compute it modulo BV-exact terms: the latter ultimately become irrelevant once we pass from cochain-level answers to the reduced space of states and reduced partition functions (i.e., once we integrate out the bulk residual fields and pass to the cohomology of the boundary BFV differential $\Omega$).\footnote{In particular, we exploit the gauge-invariance of the answer, known a priori from the quantum master equation, to reduce to the case of constant connections on the boundary.}

The cohomology in degree 0 of the BFV differential $\Omega$ on (the BFV model for) the space of states on a circle $H_{S^1}^{\text{BFV}, A}$ in the $A$-polarization yields the standard (reduced) space of states of 2D Yang-Mills theory – the space of class functions on the group $G$, $H_{S^1}^{\text{red}} = L^2(G)^G$ (see Section 2.4.1). The following is the central result of this paper.

**Theorem I.** The BV-BFV partition function of 2D Yang-Mills theory for $\Sigma$ any surface with (possibly empty) boundary, with boundary circles decorated with $A$-polarization, induces, after integrating out the bulk residual fields and passing to the cohomology of the boundary
BFV operator \( \Omega \), the Migdal-Witten non-perturbative partition function of 2D Yang-Mills:

\[
\left[ \int_{\text{residual fields}} Z^{\text{BFV}}(\Sigma) \right] = \sum_{R} \left( (\dim R) \chi(\Sigma) e^{\frac{-i\hbar a}{2} C_2(R)} |R\rangle \otimes^n \right) \in (H_{S^1}^{\text{red}})^{\otimes n}.
\] (1.2)

Here on the left, \([\ldots]\) stands for passing to the class in zeroth \( \Omega \)-cohomology. On the right side, non-perturbative partition function is given as the sum over irreducible representations \( R \) of the structure group \( G \), \( \dim R \) is the dimension of the representation and \( C_2(R) \) is the value of the quadratic Casimir; \( |R\rangle \) is the class function on \( G \) corresponding to the character of the representation \( R \), mapping \( g \mapsto \text{tr} R g \); \( \chi(\Sigma) \) is the Euler characteristic; \( n \) is the number of boundary circles in \( \Sigma \); \( a = \int_{\Sigma} \mu \) is the total area of the surface.

We first prove the comparison (1.2) for the case of \( \Sigma \) a disk in Section 3.5, by presenting the disk as a gluing of building blocks (I), (III) and (IV) above. We prove the Theorem I for a general surface in Section 4.

The gluing property of the r.h.s. of (1.2) is

\[
Z(\Sigma_1 \cup_{S^1} \Sigma_2) = \langle Z(\Sigma_1), Z(\Sigma_2) \rangle_{H_{S^1}^{\text{red}}}.
\]

Here on the right side one has the pairing in the space of states for the circle over which the surfaces are being glued. In the BV-BFV framework it corresponds to gluing two \( A \)-boundary circles via an “infinitesimally short” \( B-B_c \) cylinder – our building block (II).

### 1.2 General surfaces, surfaces with corners

To extend the result (1.2) to general surfaces we have to consider gluing and cutting with corners. In this setting we continue to decorate the codimension 1 strata – circles and intervals – with a choice of polarization, \( A \) or \( B \), and we also decorate the codimension 2 corners with a choice of polarization, \( \alpha \) or \( \beta \) (corresponding to fixing the value of either field \( A \) or field \( B \) in the corner).\(^5\) For gluing, we require that if several domains are meeting at a corner, the respective corner polarizations are the same (unlike the situation with gluing over codimension 1 strata – those should have the opposite polarization coming from the two sides of

---

\(^5\) We think of a corner carrying a polarization as the result of a collapse of an interval carrying same polarization, see the discussion of the picture I and picture II for corners in section 4.
the stratum).\(^6\)

![Diagram](image)

In this setup, one can perform the following moves on codimension 1 strata:

(a) One can split an \(A\)-interval (or an \(A\)-circle) on the boundary of a surface into \(k \geq 2\) \(A\)-intervals separated by \(\alpha\)-corners. Then the partition function for a new surface is obtained by evaluating the partition function for the old surface evaluated on the concatenation of the fields \(A\) on the \(k\) intervals.

(b) One has the inverse of the move (a): one can merge \(k\) \(A\)-intervals separated by \(\alpha\)-corners into a single \(A\)-interval – this corresponds to evaluating the partition function on the field \(A\) restricted to the smaller sub-intervals and to the points separating them. One can do the same for the \(B/\beta\) polarizations.

(c) One can switch between the polarizations of the corner separating an \(A\)-interval and a \(B\)-interval.

(d) One can integrate out (the field corresponding to) the \(\beta\)-corner separating two \(A\)-intervals, merging them together. Likewise, one can integrate out an \(\alpha\)-corner separating two \(B\)-intervals.

The minimal set of building blocks, sufficient to construct all closed surfaces is the following:

\(^6\) Actually the BV-BFV formalism does not prescribe, in principle, a particular compatibility between polarizations for the gluing. What we describe here is a choice that simplifies the computations.
(i) A disk with boundary subdivided into $k$ intervals, all in $A$-polarization, and all corners taken in $\alpha$-polarization, with a possibly nonzero 2-form $\mu$.

This building block is computed via the $A$-disk which is expressed in terms of the building blocks (I,III,IV) above; then one applies the splitting move to the boundary. (Recall that in our convention the shaded regions are those that are allowed to carry a nonvanishing 2-form $\mu$.)

(ii) A “bean” – a disk with boundary subdivided into two $B$-intervals, with the two corners in $\alpha$-polarization.

This block is computed from considering an axial gauge on a square and collapsing two opposite sides to two points.

Then one can e.g. triangulate any surface, assign the building block (i) with $k = 3$ to each triangle and glue them using building blocks (ii), thickening the edges of the triangulation into “beans”.

This way one can construct the partition function for any surface with boundary and corners, as long as corners are all in $\alpha$-polarization (boundary intervals and circles can be in any polarization). To produce all decorations of both boundary components and corners, one needs the following additional building block:
(iii) Disk in $A$-polarization, with a single $\beta$-corner.

Then one can use the building block (iii), together with the moves on the boundary, to create any combination of polarizations of arcs and corners on the boundary of a surface.\footnote{One starts with a surface with the corners only in $\alpha$ polarization and creates the desired $\beta$-corners surrounded by two $A$-arcs by gluing in the block (iii) – see Figure 12 in Section 4.1 and formula (4.33). One creates the $\beta$-corners surrounded by two $B$-arcs by the splitting move and $\beta$-corners surrounded by an $A$-arc and a $B$-arc by the switch move.}

Using the building blocks (i), (ii), we immediately obtain the proof of Theorem I for a general surface $\Sigma$ (see Sections 4.6, 4.7).

\textbf{Remark 1.2.} Theorem I in fact applies also to non-orientable surfaces, assuming the theory in the non-orientable case is defined as in footnote 2.

\textbf{Remark 1.3.} In this paper we are constructing a “pragmatic” extension of the BV-BFV framework to codimension 2 corners in the case of 2D Yang-Mills theory, motivated by the problem of computing explicit partition functions on surfaces (e.g., closed ones) of arbitrary genus. The general theory of quantization with corners in the BV-BFV formalism is work in progress and will be expanded on in a separate publication.

1.3 Main results

• Construction, in terms of explicitly computed building blocks and the gluing rule, of the partition function of 2D Yang-Mills in BV-BFV formalism on any oriented surface with boundary and corners, with any combination of polarizations $\in \{A, B\}$ assigned to the codimension 1 strata and polarizations $\in \{\alpha, \beta\}$ assigned to codimension 2 strata.

• Theorem I above, providing the comparison between the perturbative BV-BFV result in the case of a surface with $A$-polarized boundary and the known non-perturbative result.

• In Section 4.2 we prove that:

  – The BV-BFV partition function on a surface $\Sigma$ with corners satisfies the modified quantum master equation $(\hbar^2 \Delta + \Omega_{\partial \Sigma})Z = 0$, with $\Delta$ the BV Laplacian on bulk residual fields (in the minimal realization, they are modelled on de Rham cohomology of the surface), and with $\Omega_{\partial \Sigma}$ the boundary BFV operator. We construct the operator $\Omega_{\partial \Sigma}$ explicitly. In particular, apart from the edge contributions it contains quite nontrivial corner contributions, expressed in terms of the generating function for Bernoulli numbers.

  – We prove that $\Omega$ squares to zero, and thus the space of states $\mathcal{H}$ for a stratified boundary is a cochain complex.

  – We show that the space of states for a stratified circle can be disassembled into contributions of edges and corners, as the tensor product of certain differential graded (dg) bimodules – spaces of states assigned to the intervals (depending on...
the polarization of the interval and of its endpoints) – over certain dg algebras – the spaces of states for the corners. In particular, an $\alpha$-corner gets assigned the supercommutative dg algebra $\wedge^* g^*$ with Chevalley-Eilenberg differential. A $\beta$-corner gets assigned the algebra $S^* g$ endowed with zero differential and a non-commutative star-product, written in terms of Baker-Campbell-Hausdorff formula.

This picture in particular establishes a link with Baez-Dolan-Lurie setup \cite{BAEZ, LURIE} of extended topological quantum field theory where (in one of the models) one maps strata of the spacetime manifold of codimension 2, 1, 0, respectively, to algebras, bimodules and bimodule morphisms.\footnote{A version of extension of Atiyah’s axioms accommodating the non-perturbative answers for 2D Yang-Mills with corners was previously suggested in \cite{23}. It can be obtained from our picture by fixing polarizations on all strata to $\lambda, \alpha$ and passing to the zeroth cohomology of the BFV differential $\Omega$.}

1.4 Plan of the paper

The paper is organized as follows. In Section 2 we will review, for the reader’s convenience, the basics of the BV-BFV formalism introduced in \cite{9-11, 13} emphasizing the constructions that we will use for the analysis in the rest of the paper.

Sections 3,4 contain the main original results of this work. We will first, in Section 3, compute the perturbative partition function for 2D Yang-Mills on disks and cylinders. Then, in the first part of Section 4, we will discuss the extension of the BV-BFV formalism to manifolds with corners for 2D YM. This extension will be used in the second part of Section 4 to compute the perturbative 2D YM partition function on surfaces of arbitrary genus, which induces in $\Omega$-cohomology the known non-perturbative answer \cite{20, 24}.

The reader who is well-acquainted with BV formalism and would like to take the shortest route to the proof of Theorem I, might want to read the sections in the following order. Sections 3.1, 3.3, 3.4 for the building blocks (I), (III), (IV), which are then assembled into the Yang-Mills on a disk in Sections 3.5.1, 3.5.2. Then in Section 4.1 the logic of extension to corners is explained and in Section 4.3 the “bean” is computed. Finally, in Sections 4.6, 4.7 polygons (obtained from the disk) are glued via beans into an arbitrary surface and thus the comparison theorem is proven.

In Appendix A we will discuss how to compute, in this setting, Wilson loop observables for both non-intersecting and intersecting loops, recovering in $\Omega$-cohomology the known non-perturbative result.

1.5 Acknowledgements

We would like to thank Alberto S. Cattaneo, Andrey S. Losev, Stephan Stolz, Konstantin Wernli for inspiring discussions. P. M. acknowledges partial support of RFBR Grant No. 17-01-00283a.
2 Background: BV-BFV formalism

We will start this section reviewing the basic constructions of the BV-BFV formalism and fixing the notation. We will then apply this construction to obtain the BV-BFV formulation of the non-abelian BF theory and Yang-Mills theory reviewing some of the known results. For a complete and detailed discussion of this topic we refer to [9–11, 13], where this formalism was first introduced.

2.1 Classical BV-BFV

Definition 2.1. A BFV manifold is given by the triple \((\mathcal{F}_\partial, \alpha_\partial, Q_\partial)\), where: the space of boundary fields \(\mathcal{F}_\partial\) is an exact graded symplectic manifold with 0-symplectic form \(\omega_\partial = d\alpha_\partial\) and \(Q_\partial\) is a homological symplectic vector field of degree 1.\(^9\)

In particular the condition \(L_{Q_\partial} d\alpha_\partial = 0\) for the vector field \(Q_\partial\), since \(|Q_\partial| + |\omega_\partial| \neq 0\), implies that it is also Hamiltonian: \(\iota_{Q_\partial} \omega_\partial = dS_\partial\). This defines the degree 1 Hamiltonian \(S_\partial\), which we will call the boundary BFV action.

Definition 2.2. A BV-BFV manifold, over a BFV manifold \((\mathcal{F}_\partial, \alpha_\partial, Q_\partial)\), is a quintuple \((\mathcal{F}, \omega, S, Q, \pi)\), where the space of bulk fields \((\mathcal{F}, \omega)\) is a \((-1)\)-symplectic manifold, the bulk action \(S\) is a function of the fields, the bulk BRST operator \(Q\) is a homological vector field of degree 1 and \(\pi: \mathcal{F} \to \mathcal{F}_\partial\) is a surjective submersion, satisfying the following two compatibility conditions:

i) the bulk homological vector field projects on the boundary homological vector field: \(d\pi Q = Q_\partial\);

ii) the modified Classical Master Equation (mCME) holds: \(\iota_{Q} \omega = dS + \pi^* \alpha_\partial\).

A classical BV-BFV theory is constructed for manifolds with boundaries of some fixed dimension \(n\). It consists of the association to each manifold with boundary \(\Sigma\) of a BV-BFV manifold \(\mathcal{F}_\Sigma\) over the BFV manifold \(\mathcal{F}_\partial\) associated to the boundary \(\partial \Sigma\). This association has to be compatible with disjoint union and “gluing” in the following sense:

i) a disjoint union maps to the direct product: \(\mathcal{F}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{F}_{\Sigma_1} \times \mathcal{F}_{\Sigma_2}\);

ii) a gluing of two manifolds maps to the fiber product over the space of fields associated to the gluing interface \(\gamma: \mathcal{F}_{\Sigma_1 \cup \gamma \Sigma_2} = \mathcal{F}_{\Sigma_1} \times_{\mathcal{F}_\gamma} \mathcal{F}_{\Sigma_2}\).

Remark 2.3. This can be interpreted as a covariant monoidal functor from the spacetime category, with \((n - 1)\) closed manifolds as objects and \(n\)-manifolds with boundary as morphisms with composition given by gluing,\(^10\) to the BFV category, where objects are BFV manifolds.

---

\(^9\) For simplicity we will consider, here and in the following, all the gradings to be \(\mathbb{Z}\) gradings. The parity, determining the commuting/anticommuting properties of coordinates, is given by the degree \(\text{mod } 2\).

\(^10\) Depending on the specific theory, the spacetime category could have additional structures: for examples manifolds could be oriented, Riemannian, etc. Also, depending on the specific theory, there may be subtleties to defining the spacetime category as an actual category. This discussion is beyond the scope (and is not relevant to) this paper.
manifolds, morphisms are BV-BFV manifolds over (products of in- and out-) BFV manifolds, and composition is given by fiber products. The monoidal structure on the spacetime category is given by the disjoint union, while on the BFV category side it is given by the direct product.

**Remark 2.4.** On closed manifolds this construction reduces to a classical BV theory, which gives a homological resolution of the space of classical states for Lagrangian gauge field theories and is the classical starting point for the BV quantization of such theories [3, 16].

### 2.2 Quantum BV-BFV

A quantum BV-BFV theory associates to an \((n−1)\) manifold \(\gamma\) a graded cochain complex \(H_\gamma\), the space of states, with a coboundary operator \(\Omega_\gamma\) called the quantum BFV charge. To \(n\)-manifolds with boundary \(\Sigma\) the quantum theory assigns a (finite-dimensional) \((-1)\)-symplectic manifold \((V_\Sigma, \omega_{V_\Sigma})\), the space of residual fields, and the partition function, which is an element \(Z_\Sigma \in H_{\partial \Sigma} \otimes \text{Dens}^2(V_\Sigma)\) in the boundary space of states tensored with the half-densities on the residual fields.\(^{11}\) The partition function has to satisfy the modified Quantum Master Equation (mQME):

\[
(\Omega_{\partial \Sigma} + \hbar^2 \Delta_{V_\Sigma}) Z_\Sigma = 0 ,
\]

where \(\Delta_{V_\Sigma}\) is the canonical BV Laplacian on the half-densities of the residual fields. The partition function is understood to be defined modulo \((\Omega_{\partial \Sigma} + \hbar^2 \Delta_{V_\Sigma})\)-exact terms. Also, the quantum theory satisfies compatibility conditions with respect to the disjoint union and the gluing of spacetime manifolds:

i) To disjoint unions, reflecting the quantum nature of the theory, the BV-BFV theory associates the tensor product of the spaces of states, \(H_{\gamma_1 \cup \gamma_2} = H_{\gamma_1} \otimes H_{\gamma_2}\), the direct product of residual fields \(V_{\Sigma_1 \cup \Sigma_2} = V_{\Sigma_1} \times V_{\Sigma_2}\) and the tensor product of partition functions, \(Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}\).

ii) To the gluing of two manifolds the theory associates the partition function obtained as the pairing, in the space of states of the gluing interface, of the partition functions of the constituent manifolds: \(Z_{\Sigma_1 \cup \gamma_\Sigma \Sigma_2} = \langle Z_{\Sigma_1}, Z_{\Sigma_2}\rangle_\gamma\).\(^{12}\)

Quantum observables are defined to be cohomology classes of the coboundary operator \(Z_\Sigma^{-1}(\Omega_{\partial \Sigma} + \hbar^2 \Delta_{V_\Sigma})(Z_\Sigma \cdot \cdot \cdot)\), with expectation value computed by a BV pushforward of a representative \(\mathcal{O}\) times the partition function, i.e. integrating their product over a Lagrangian \(L \subset V\):

\[
\langle \mathcal{O}\rangle_\Sigma := \int_L \mathcal{O} Z_\Sigma .
\]

\(^{11}\) The space of residual fields is not uniquely determined, but comes in a poset of different realizations. The partition function for a smaller realization can be reached with a BV-pushforward (see Subsection 2.3.2 for further discussion).

\(^{12}\) For oriented spacetime manifolds, this is the dual pairing between \(\mathcal{H}_\alpha\) and \(\mathcal{H}_\alpha^*\): since the two boundaries that are glued together must have opposite orientations, the associated vector spaces are dual to each other.
The Lagrangian submanifold $L$ has here the meaning of gauge-fixing for the integration over residual fields and the closedness of $\mathcal{O}Z_{\Sigma}$ with respect to $\Omega_{\partial\Sigma} + \hbar^2 \Delta V_{\Sigma}$ ensures that the $\Omega_{\partial\Sigma}$-cohomology class resulting from the integration does not depend on the particular choice of gauge fixing thanks to the following theorem [13, 22].

**Theorem 2.5.** Let $(M_1, \omega_2)$ and $(M_2, \omega_2)$ be two graded manifolds with odd symplectic forms $\omega_i$ and canonical Laplacians $\Delta_i$. Consider $\mathcal{M} = M_1 \times M_2$ with product symplectic form and canonical Laplacian $\Delta$ and let $L, L' \subset M_2$ be any two Lagrangian submanifolds which can be deformed into each other. For any half-density $f \in \text{Dens}^{\frac{1}{2}}(\mathcal{M})$ we have:

i) $\int_L \Delta f = \Delta_1 \int_{L'} f$

ii) $\int_L f - \int_{L'} f = \Delta_1 \xi$ for some $\xi \in \text{Dens}^{\frac{1}{2}}(M_1)$, if $\Delta f = 0$.

In particular, when $M_1$ is just a point, the r.h.s. of the two equations above vanishes.

### 2.3 Quantization

The quantization procedure is a way to get a quantum BV-BFV theory from the data of a classical BV-BFV theory. The first object to construct is the space of states $\mathcal{H}_\gamma$, which is obtained from the space of boundary fields $\mathfrak{F}_\gamma$ by choosing a Lagrangian foliation, or more generally a polarization $\mathcal{P}$. We will assume in the following that the 1-form $\alpha_\gamma$ vanishes along the fibers of $\mathcal{P}$; if this is not the case, we can use a gauge transformation:

$$\alpha_\gamma \mapsto \alpha_\gamma - df_\gamma, \quad \mathcal{S}_\Sigma \mapsto \mathcal{S}_\Sigma + \pi^* f_\gamma,$$

which uses an arbitrary function $f_\gamma$ of the boundary fields to shift $\alpha_\gamma$ and the bulk action in such a way that the mCME is preserved (cf. def. 2.2). The space of states of the quantum theory is defined as the space of complex-valued functions on the leaf space $\mathcal{B}_\gamma^\mathcal{P} = \mathfrak{F}_\gamma / \mathcal{P}$ (or more generally the space of polarized sections of the trivial “prequantum” $U(1)$-bundle over $\mathfrak{F}_\gamma$).

$$\mathcal{H}_\gamma := \text{Func}_C(\mathcal{B}_\gamma^\mathcal{P}).$$

In other words, the space of quantum states is obtained as the geometric quantization of the space of boundary fields [13].

The space of quantum states forms a cochain complex. The coboundary operator $\Omega_\gamma$ is constructed as the quantization of the boundary action $\mathcal{S}_\gamma$. Suppose we have Darboux coordinates $(q, p)$ on $\mathfrak{F}_\gamma$, where $q$ are also coordinates of $\mathcal{B}_\gamma^\mathcal{P}$. The operator $\Omega_\gamma$ is the standard-ordering quantization of the action:

$$\Omega_\gamma := \mathcal{S}_\gamma \left( q, -i\hbar \frac{\partial}{\partial q} \right),$$

13 Another possible model for states uses half-densities instead of functions. These two models are isomorphic, with the isomorphism given by multiplication by a fixed reference half-density.
where all the derivatives are positioned to the right. For the theories we will consider in this paper, with this definition $\Omega$, squares to zero; in general it could be needed to add quantum corrections to (2.5) for $\Omega$ to actually be a coboundary operator.\footnote{In general there might be cohomological obstructions to do that. Moreover, the partition function might be not compatible with the so constructed $\Omega$, causing the mQME to fail.}

Let us consider now the data associated to the bulk $n$-manifolds. The space of bulk fields has a fibration over $\mathcal{B}^P_{\partial \Sigma}$ defined by composing the projection to the boundary fields with the projection given by the polarization: $\mathcal{F}_\Sigma \xrightarrow{\pi} \mathcal{F}_{\partial \Sigma} \rightarrow \mathcal{B}^P_{\partial \Sigma}$. Suppose for simplicity that this is a trivial bundle: $\mathcal{F}_\Sigma = \tilde{\mathcal{B}}^P_{\partial \Sigma} \times \mathcal{Y}$, where $\tilde{\mathcal{B}}^P_{\partial \Sigma}$ is some bulk extension of $\mathcal{B}^P_{\partial \Sigma}$ and $\mathcal{Y}$ is some ($-1$)-symplectic manifold. This assumption will hold in all the theories considered in the following.

The space of residual fields can be taken to be any (finite-dimensional) symplectic subspace $\mathcal{V}$ of the space of fields,\footnote{In the framework of perturbation theory, the requirement that the integral (2.6) below is perturbatively well-defined, imposes restrictions on the possible choices of $\mathcal{V}$. E.g. $\mathcal{V}$ for perturbed BF theories has to be modelled on a deformation retract of the de Rham complex of the bulk manifold.} separating it as $\mathcal{Y} = \mathcal{V} \times \mathcal{Y}'$, where $\mathcal{Y}'$ is the space of fluctuations. The partition function is now defined as a BV pushforward of the exponentiated bulk action:

$$Z_\Sigma(\mathcal{P}; \mathcal{V}) := \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_\Sigma}, \quad (2.6)$$

where $\mathcal{L} \subset \mathcal{Y}'$ is a Lagrangian submanifold. If $\Delta_Y S_\Sigma = 0$, theorem 2.5 implies that the partition function is a solution of the mQME (2.1).\footnote{In the theories considered in the following this condition is verified. However, this is not always the case and there can be theories where the bulk action needs quantum corrections in order for the mQME to hold. This is connected in particular to the presence of quantum anomalies in the theory.}

Moreover, we have that $Z_\Sigma$ does not depend on (deformations of) the gauge-fixing Lagrangian $\mathcal{L}$ used in the BV pushforward, up to $(\Omega + \hbar^2 \Delta)$-exact terms.

The discussion, until now, assumes a finite-dimensional situation. This is usually not the setting of quantum field theories; for infinite-dimensional spaces one needs a more delicate analysis to prove the mQME and to prove that the dependence of the partition on the gauge-fixing is BV-exact. A way to make sense of infinite-dimensional integrals is through perturbation theory, as discussed in the following section.

### 2.3.1 Perturbative expansion

The space of fields $\mathcal{F}_\Sigma$ is typically infinite-dimensional, for example it can contain the de Rham complex of differential forms over $\Sigma$. As a consequence, the integral (2.6) defining the partition function is (almost always) ill-defined as a measure-theoretic integral and has to be understood as a perturbative series written in terms of the Feynman diagrams coming from the interactions in the bulk action expanded around a point $x_0 \in \mathcal{M}$ in the Euler-Lagrange moduli space – the space of solutions of classical equations of motion of $S_\Sigma$ (modulo gauge symmetries).
In order for the perturbative expansion to be well-defined, the *gauge-fixed action* – the restriction of $\mathcal{S}_\Sigma$ to the gauge-fixing Lagrangian $\mathcal{L}$ – needs to have isolated critical points. It is important to remark that this condition does not, in general, hold for every Lagrangian.

The existence of such a “good gauge-fixing” depends on the choice of residual fields. In particular the quadratic part of the bulk action can have *zero-modes* $\mathcal{V}_\Sigma^0$, i.e. bulk fields configurations that are annihilated by the kinetic operator.\(^{17}\) Zero-modes correspond to the tangent directions to the Euler-Lagrange moduli space (cf. [13], appendix F) and therefore their presence in the space of fluctuations indicates non-isolated critical points of the action and obstructs the perturbative expansion. Thus, the space of residual fields has to at least contain the space of zero-modes for a good gauge-fixing Lagrangian to exist: $\mathcal{V}_\Sigma^0 \subseteq \mathcal{V}_\Sigma$. When the residual fields coincide with zero modes we say that the perturbative partition function is in its *minimal realization*.

Another consequence of the infinite dimensions of $\mathfrak{F}_\Sigma$ is that also the Laplacian is ill-defined. The equations containing it are thus only formal (or require a regularization).\(^{18}\) In particular, theorem 2.5 is proved in a finite-dimensional setting. An important point is thus that even if the action is formally annihilated by the Laplacian, the mQME is only expected to hold and needs to be verified for each particular theory. For perturbed BF theories, including 2D Yang-Mills, the mQME has been proved in the infinite-dimensional perturbative setting in [13] and relies on the Stokes’ theorem for integrals over compactified configuration spaces of points.

### 2.3.2 Renormalization and globalization

A non-minimal realization of a theory is obtained when the zero-modes are a proper subset of the space of residual fields. Of course there are different, inequivalent, non-minimal realizations of any theory. Given a non-minimal realization, one can obtain a smaller one by a BV pushforward. If $\mathcal{V}_\Sigma' = \mathcal{V}_\Sigma'' \times \mathcal{Y}'$, with $\mathcal{V}_\Sigma^0 \subset \mathcal{V}_\Sigma''$, then:

$$Z_\Sigma(\mathcal{P}; \mathcal{V}'') = \int_{\mathcal{L}} Z_\Sigma(\mathcal{P}; \mathcal{V}')$$

for a Lagrangian submanifold $\mathcal{L} \subset \mathcal{Y}'$. The set of all possible realizations forms therefore a partially ordered set, with the final object given by the minimal realization. Passing from bigger to smaller realizations can be interpreted as following the *renormalization group flow*.

**Remark 2.6.** According to the gluing prescription (cf. section 2.2), the residual fields of the glued manifold are the direct product of the residual fields of the two manifolds being glued. In particular this means that, generally, if we glue together two partition functions in the minimal realization the result of the gluing will not be in the minimal realization. Let $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$; it typically happens that $\mathcal{V}_\Sigma^0 \subset \mathcal{V}_{\Sigma_1}^0 \times \mathcal{V}_{\Sigma_2}^0$. The minimal realization for the

---

\(^{17}\) See section 2.4.2 for the definition of zero modes in 2D YM.

\(^{18}\) The BV Laplacian becomes non-singular within the framework of renormalization theory on the level of residual fields. See also [14] for the discussion of how the RG flow regularizes the BV Laplacian.
glued manifold has then to be obtained via a BV pushforward:

$$Z_\Sigma(V_0) = \int L (Z_{\Sigma_1}(V_{\Sigma_1}^0), Z_{\Sigma_2}(V_{\Sigma_2}^0)) .$$

(2.8)

Because of its perturbative definition, the partition function depends on the point $x_0 \in \mathcal{M}$ around which we are expanding and carries only local information on field configurations infinitesimally close to $x_0$ (it is defined on a formal neighbourhood of $x_0$). Nevertheless, at least in the theories considered in this paper, its minimal realization is the Taylor expansion of a *global* half-density on the tangent bundle of the Euler-Lagrange moduli space (cf. [13], appendix F). Thus, under some assumptions, it can be integrated on the zero section of $T\mathcal{M}$. This corresponds to setting to zero all the zero-modes $\nu \in V_0^\Sigma$ and integrating the partition function on the Euler-Lagrange moduli space

$$Z_\Sigma(P) = \int_{\mathcal{M}} Z_\Sigma(P, V_0^\Sigma; x_0) |_{\nu = 0} \in \text{Dens}^\frac{1}{2}(B\mathcal{P}_{\partial \Sigma}) ,$$

(2.9)

obtaining a *globalized partition function* depending only on the boundary fields in $B\mathcal{P}_{\partial \Sigma}$.

Another way to obtain a partition function which does not depend on $V_0^\Sigma$ is to integrate its minimal realization over all the zero-modes, again using a BV pushforward. Notice that this cannot be done in a perturbative way – the propagator cannot be defined for zero modes – but since $V_0^\Sigma$ is a finite-dimensional space, the BV pushforward is well-defined as an ordinary integral on a supermanifold:

$$Z_\Sigma(P) = \int_{E \subset V_0^\Sigma} Z_\Sigma(P, V_0^\Sigma) .$$

(2.10)

This can be viewed as an alternative definition of a globalized partition function and in fact, when both this integral and the one in (2.9) can be computed explicitly, they coincide (cf. section 4.7). However, the precise relation between the two globalization procedures is to be understood better.

### 2.4 BV-BFV formulation of 2D YM

We will review in this section the BV-BFV construction for 2D Yang-Mills and non-abelian BF theories; for a deeper discussion and for some of the proofs we refer to [9, 13].

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $A$ be a connection 1-form on a principal $G$-bundle over a 2-dimensional surface $\Sigma$. In the first order formalism the classical YM action can be written in the following form:

$$S_\Sigma(A, B) = \int \langle B, F_A \rangle + \frac{1}{2} \int \langle B, B \rangle \mu ,$$

(2.11)

where the auxiliary field $B$ is a zero-form valued in $\mathfrak{g}^*$, $F_A$ is the curvature 2-form of $A$, $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, $(\cdot, \cdot)$ is an invariant non-degenerate pairing on $\mathfrak{g}^*$

---

19 One caveat is that one needs to take care to avoid possible overcounting when integrating over zero-modes, cf. Remark 3.12 below.
and $\mu$ is the volume 2-form associated to a metric on $\Sigma$. We see that 2D YM can be treated as a perturbation of 2D non-abelian BF theory, which can be obtained in the zero-area limit $\mu \to 0$. In the following sections we will generally find it useful to consider BF theory first, introducing the area term only afterwards.

On closed surfaces, the classical BV construction enhances the space of fields by adding differential forms of every degree, usually called ghosts and antifields for positive or negative internal degree respectively. The BV space of fields over $\Sigma$ is then

$$F_{\Sigma} = \Omega(\Sigma; g)[1] \oplus \Omega(\Sigma; g^*) \ni (A, B),$$

where $A$ and $B$ are the superfields associated to $A = A_{(1)}$ and $B = B_{(0)}$ which are their degree-zero components.\(^{20}\) The BV space of fields is a symplectic graded space, with $(-1)$-symplectic form given by:

$$\omega_{\Sigma} = \int_{\Sigma} \langle \delta B, \delta A \rangle.$$

The BV action on a closed manifold is

$$S_{\Sigma} = \int_{\Sigma} \langle B, dA + \frac{1}{2}[A, A] \rangle + \frac{1}{2} \int_{\Sigma} (B, B) \mu,$$

and the corresponding Hamiltonian vector field, the homological vector field $Q_{\Sigma}$, is:

$$Q_{\Sigma} = \int_{\Sigma} \left( \frac{dA + \frac{1}{2}[A, A]}{\delta A} \right) + \int_{\Sigma} \left( dB + ad_{A}B \cdot \frac{\delta}{\delta B} \right) + \int_{\Sigma} \left( B, \frac{\delta}{\delta B} \right) \mu,$$

In the BV-BFV construction the bulk fields, symplectic structure, action and homological vector field are again the ones described above. Notice that now, when $\Sigma$ has a non-empty boundary $\partial \Sigma$, the homological vector field (2.15) is not the Hamiltonian vector field of the action (2.14). Indeed, it is not even symplectic:

$$i_{Q_{\Sigma}} \omega_{\Sigma} = \int_{\partial \Sigma} \langle B, \delta A \rangle.$$

The boundary fields $\mathcal{F}_{\partial \Sigma}$ are defined analogously to the bulk:

$$\mathcal{F}_{\partial \Sigma} = \Omega(\partial \Sigma; g)[1] \oplus \Omega(\partial \Sigma; g^*) \ni (A, B).$$

We can thus define the projection $\pi: \mathcal{F}_{\Sigma} \to \mathcal{F}_{\partial \Sigma}$ to be just the restriction (pullback) to $\partial \Sigma$ of the bulk fields. This, taking into account the compatibility conditions of def. 2.2, fixes the remaining boundary data. From (2.16) we get

$$\alpha_{\partial \Sigma} = \int_{\partial \Sigma} \langle B, \delta A \rangle, \quad \omega_{\partial \Sigma} = \delta \alpha_{\partial \Sigma} = -\int_{\partial \Sigma} \langle \delta B, \delta A \rangle,$$

the boundary homological vector field is the projection of the bulk homological vector field

$$Q_{\partial \Sigma} = d\pi Q_{\Sigma} = \int_{\partial \Sigma} \left( \left\langle dA + \frac{1}{2}[A, A], \frac{\delta}{\delta A} \right\rangle + \left\langle dB + ad_{A}B, \frac{\delta}{\delta B} \right\rangle \right).$$

\(^{20}\) We will denote by $A_{(n)}$ the $n$-form component of a superfield $A$. 

and thus the boundary action is obtained as the Hamiltonian of $Q_{\partial \Sigma}$:

$$S_{\partial \Sigma} = \int_{\partial \Sigma} \langle B, dA + \frac{1}{2}[A, A] \rangle .$$

(2.20)

Notice that, for degree reasons, the area form $\mu$ does not appear in the boundary data. The boundary BFV manifold for 2D YM is thus exactly the same as in BF theory; actually, the only difference between the two theories is the area term in the bulk action and, consequently, the state (partition function) defined by the two quantum theories.

To quantize the theory, we need to choose a polarization of the space of boundary fields. From (2.18) we see that there are two simple choices of polarization: the $A$-polarization $P_A$, for which the leaf space is described by the $A$ fields

$$B_{\partial \Sigma}^{P_A} = \Omega(\partial \Sigma; g)[1]$$

(2.21)

and the $B$-polarization $P_B$, for which the leaf space is described by the $B$ fields

$$B_{\partial \Sigma}^{P_B} = \Omega(\partial \Sigma; g^*) .$$

(2.22)

We will, in the rest of this paper, always use these two transversal polarizations, arbitrarily splitting the boundary of a manifold into the disjoint union of two components, $\partial \Sigma := \partial_B \Sigma \sqcup \partial_A \Sigma$, and choosing the product polarization which assigns the $B$-polarization to the first and the $A$-polarization to the latter boundary component:

$$B_{\partial \Sigma}^P = \Omega(\partial_A \Sigma; g)[1] \oplus \Omega(\partial_B \Sigma; g^*) .$$

(2.23)

The boundary one-form $\alpha_{\partial \Sigma}$ does not vanish on the fibers of this polarization (cf. (2.18)) but it can be adapted to this choice using a gauge transformation (2.3):

$$\alpha_{\partial \Sigma}^P = \alpha_{\partial \Sigma} + \delta \int_{\partial_b \Sigma} \langle B, A \rangle = \int_{\partial_a \Sigma} \langle B, \delta A \rangle - \int_{\partial_b \Sigma} \langle \delta B, A \rangle ,$$

$$S_{\Sigma}^P = S_{\Sigma} - \int_{\partial_b \Sigma} \langle B, A \rangle .$$

(2.24)

We can now quantize, with the above polarization, the boundary action to obtain the coboundary operator $\Omega_{\partial \Sigma}^P$:

$$\Omega_{\partial \Sigma}^P = \int_{\partial_a \Sigma} i\hbar \left( dA^a + \frac{1}{2} f_{bc}^a A^b A^c \right) \frac{\delta}{\delta A^a} + \int_{\partial_b \Sigma} \left( i\hbar dB_a \frac{\delta}{\delta B_a} - \frac{\hbar^2}{2} f_{bc}^a B_b B_c \frac{\delta}{\delta B_b} \frac{\delta}{\delta B_c} \right) .$$

(2.25)

Here $f_{bc}^a$ are the structure constants of the Lie algebra $g$.

---

21 In the terminology of [13], this is “$A$-representation”, or “$\frac{\delta}{\delta A}$-polarization” (as those are the vector fields spanning the tangential Lagrangian distribution on the phase space).

22 However, when we start considering corners in section 4, the disjointness assumption will fail along codimension 2 strata.
To write the partition function we lift \( B^P = \mathcal{B}^P \times \mathcal{Y}^\Sigma \) by taking (discontinuous) bulk extensions \((\tilde{A}, \tilde{B})\) of the boundary fields (cf. the discussion in [13], Section 3.4). We can now split the bulk fields \( \mathcal{Y}^\Sigma \) into residual fields \((a, b) \in \mathcal{V}^\Sigma\) and fluctuations \((\alpha, \beta) \in \mathcal{Y}^\Sigma'\):

\[
A = \tilde{A} + a + \alpha, \quad B = \tilde{B} + b + \beta.
\]  

(2.26)

We can finally define the partition function as the (perturbative) path integral:

\[
Z^\Sigma[A, B; a, b] = \int_{\mathcal{L} \subset Y^\Sigma'} D[\alpha, \beta] e^{i/\hbar S^P_\Sigma(\tilde{A} + a + \alpha, \tilde{B} + b + \beta)}.
\]

(2.27)

**Remark 2.7.** To avoid the appearance of ill-defined derivatives of the discontinuous fields \((\tilde{A}, \tilde{B})\) in the bulk action \( S^P_\Sigma \), we integrate by parts rewriting it as:

\[
S^P_\Sigma(\tilde{A} + a + \alpha, \tilde{B} + b + \beta) = S^P_\Sigma(a + \alpha, b + \beta) + \frac{1}{2} \int_{\Sigma} (b + \beta, b + \beta) \mu \\
+ \int_{\partial_{\Sigma}} \langle b + \beta, \tilde{A} \rangle - \int_{\partial_{\Sigma}} \langle \tilde{B}, a + \alpha \rangle.
\]

(2.28)

The boundary fields thus act as currents in the perturbative expansion of the partition function.

We are now in the position of writing down the diagrammatic elements of the Feynman diagrams expansion of the theory:\(^{23}\)

\[
\text{propagator} \quad \text{B boundary source} \quad \text{A boundary source}
\]

\[
\eta \quad \text{...} \quad \text{...} \quad \text{...}
\]

\[
\text{BF interaction} \quad \text{YM interaction} \quad \text{b zero-modes} \quad \text{a zero-modes}
\]

\[
\mu \quad \text{b} \quad \text{a}
\]

(2.29)

With these vertices we can compose a large set of non-trivial Feynman diagrams (e.g. figure 2). The general strategy will be to cut the surface, and hence Feynman diagrams, in such a way that there is a simple choice of propagators on each component which allows us to compute the partition function for that surface. Then, using the gluing properties of BV-BFV theories, we can glue back all the pieces to recover the partition function on the original surface we started with. This procedure can be viewed as a method to construct a complicated propagator on the starting surface which, though, allows explicit computations.

\(^{23}\)Zero-modes are here understood as “loose” half-hedges.
2.4.1 $\Omega$-cohomology in $\Lambda$-polarization on a circle

In [13] it was proven that the partition function for 2D YM (and other perturbations of abelian BF) solves the mQME. In the following sections, to simplify some computations, we will exploit this fact by choosing a suitable representative for the cohomology class of the partition function. In particular it will be useful to know the cohomology of $\Omega$, in ghost degree zero, on the space of boundary fields in $\Lambda$ polarization. From equation (2.25) we see that $\Omega$ acts as a gauge transformation, thus $\Omega$-closed functionals of ghost degree zero are just gauge-invariant functionals of the connection, which are isomorphic to class functions on the simply connected Lie group $G$ integrating the Lie algebra $g$. An alternative approach is to split $\Omega$ into the “abelian” part, i.e. the de Rham differential $d$, plus a perturbation $\delta$ containing the structure constants of the Lie algebra:

$$\Omega = \int_{S^1} d\hat{A}_a(0) \frac{\delta}{\delta \hat{A}_a(1)} + \int_{S^1} \left( \frac{1}{2} f^a_{bc} \hat{A}_b(0) \hat{A}_c(0) \frac{\delta}{\delta \hat{A}_a(0)} + f^a_{bc} \hat{A}_b(0) \hat{A}_c(0) \frac{\delta}{\delta \hat{A}_a(0)} + \frac{1}{2} f^a_{bc} \hat{A}_b(1) \hat{A}_c(1) \frac{\delta}{\delta \hat{A}_a(1)} \right).$$

We can then compute the cohomology via the homological perturbation lemma [15]. The cohomology of $d$ is given by functions on the de Rham cohomology $H^*_{dR}(S^1; g)[1]$; choosing the coordinate $t$ on the circle, these can be represented as functions of the “constant fields” $\hat{\Lambda}_a(0)$ and $\hat{\Lambda}_a(1) dt$, where $\hat{\Lambda}_a(0) \in g[1]$ and $\hat{\Lambda}_a(1) \in g$. Now if we compute the cohomology of the induced differential

$$\delta = \frac{1}{2} f^a_{bc} \hat{A}_b(0) \hat{A}_c(0) \frac{\delta}{\delta \hat{\Lambda}_a(0)} + f^a_{bc} \hat{A}_b(0) \hat{A}_c(0) \frac{\delta}{\delta \hat{\Lambda}_a(0)} + \frac{1}{2} f^a_{bc} \hat{A}_b(1) \hat{A}_c(1) \frac{\delta}{\delta \hat{\Lambda}_a(1)},$$

on $H^*_{dR}(S^1; g)[1]$, we get that in ghost degree zero it is given by $G$-invariant functions on the Lie algebra $g$. Comparing with the previous answer, we see that the correct $\Omega$-cohomology corresponds to the subspace of $G$-invariant functions on $g$ coming as the pullback by the exponential map $\exp: g \rightarrow G$ of class functions on $G$. Such functions on $g$ are determined by their values on the fundamental domain $B_0$ of the exponential map (e.g. for $G = SU(2)$, $B_0$ is a ball in $g$ centered at the origin).\(^{24}\) The discrepancy between the correct cohomology of $\Omega$

---

\(^{24}\) Generally, $B_0$ is the connected component of the origin in $g - \phi^{-1}(0)$ where the function $\phi: g \rightarrow \mathbb{R}$,

$$\phi(x) := \det \frac{\sinh(a x_+ / 2)}{a x_+ / 2},$$

is the Jacobian of the exponential map. In other words, $B_0$ is the set of elements $x \in g$
The kinetic term in the YM action (2.14) is of the kind
\[ 2.4.2 \text{ Hodge propagators and axial gauge} \]

The useful remark coming from this discussion is that, modulo \( \Omega \)-exact terms, the partition function and the physical observables can be represented as a \((G\text{-invariant})\) function of \( \text{constant fields} \) valued in a neighbourhood of zero in \( g \). Moreover, in ghost degree zero, for \( \Omega \)-closed objects – depending only on \( \mathbb{A}_1 \) – the “reduced wavefunction” \( \Psi(\mathbb{A}_1) \) can be lifted to an \( \Omega \)-closed function in the non-reduced space of states by evaluating \( \Psi \) on the logarithm of the holonomy of \( \mathbb{A}_1 \).

### 2.4.2 Hodge propagators and axial gauge

The kinetic term in the YM action (2.14) is of the kind \( \int_Y (B, DA) \), where \( D \) is a differential on \( \mathcal{Y}_\Sigma^* \). Since the propagator is the integral kernel of the inverse of \( D \), we want to find where the differential can actually be inverted.

Let \((K, i, p)\) be a retraction of \((\mathcal{Y}_\Sigma^*, D)\) on its cohomology \((\mathcal{V}_\Sigma^*, 0)\), i.e. a triple where \( K: \mathcal{Y}_\Sigma^* \rightarrow \mathcal{Y}_\Sigma^{* - 1} \) is a chain homotopy, \( i: \mathcal{V}_\Sigma^* \hookrightarrow \mathcal{Y}_\Sigma^* \) a chain inclusion and \( p: \mathcal{Y}_\Sigma^* \rightarrow \mathcal{V}_\Sigma^* \) a chain projection satisfying:

\[ K^2 = p \circ K = K \circ i = 0, \quad i \circ p = \text{id}, \quad DK + KD = \text{id} - i \circ p. \quad (2.32) \]

Then the complex \( \mathcal{Y}_\Sigma^* \) has a weak Hodge decomposition:

\[ \mathcal{Y}_\Sigma^* \cong \Pi \mathcal{Y}_\Sigma^* \oplus K \mathcal{Y}_\Sigma^{* + 1} \oplus D \mathcal{Y}_\Sigma^{* - 1}, \]

where we have defined \( \Pi := i \circ p \). From eq. (2.32), the differential \( D \) is invertible as an operator from the image of the chain homotopy \( K \) to \( \text{D-exact cochains} \) \( D: \mathcal{Y}_\Sigma^{* + 1} \rightarrow \mathcal{Y}_\Sigma^{* - 1} \) and its inverse is precisely the chain homotopy itself: \( K = D^{-1} \).

The gauge can thus be fixed on the Lagrangian \( \mathcal{L} = K \mathcal{Y}_\Sigma \); the propagator \( \eta(x'; x) \), with this gauge-fixing, is defined as the integral kernel of the chain homotopy:

\[ K \omega(x) = \int_{\Sigma \ni x'} \eta(x; x') \wedge \omega(x'), \quad \omega \in \mathcal{Y}_\Sigma. \quad (2.34) \]

When spacetime is a product manifold, \( \Sigma = \Sigma_1 \times \Sigma_2 \), there is a particular class of propagators which can be induced on \( \Sigma \) from lower-dimensional propagators on the two factors [4]. Since the differential forms on a product manifold are the (closure) of the sum of products of the differential forms on the two factors, we have \( \mathcal{Y}_\Sigma = \mathcal{Y}_{\Sigma_1} \otimes \mathcal{Y}_{\Sigma_2} \). For each pair of contractions \((K_t, i_t, p_t)\) on the factors \( \mathcal{Y}_{\Sigma_t} \), we have an induced weak Hodge decomposition on \( \mathcal{Y}_\Sigma \):

\[ \mathcal{Y}_\Sigma = (\Pi_1 \mathcal{Y}_{\Sigma_1} \otimes \Pi_2 \mathcal{Y}_{\Sigma_2}) \oplus (\Pi_1 \mathcal{Y}_{\Sigma_1} \otimes K_2 \mathcal{Y}_{\Sigma_2}) \oplus (K_1 \mathcal{Y}_{\Sigma_1} \otimes \mathcal{Y}_{\Sigma_2}) \oplus (\Pi_1 \mathcal{Y}_{\Sigma_1} \otimes D_2 \mathcal{Y}_{\Sigma_2}) \oplus (D_1 \mathcal{Y}_{\Sigma_1} \otimes \mathcal{Y}_{\Sigma_2}) \quad (2.35) \]

such that all eigenvalues of \( \text{ad}_\Phi \) are contained in the interval \( (-2\pi i, 2\pi i) \subset \mathbb{R} \).

\( ^{25} \) This problem is a version of the Gribov ambiguity (Gribov copies) problem in 4d Yang-Mills theory – the problem of gauge-fixing “section” intersecting the gauge orbits more than once. For that reason, we will refer to \( B_0 \) as the “Gribov region”.

---

2 Background: BV-BFV formalism
The zero modes are the product of the zero modes on the two factors and the induced chain homotopy is $K = \Pi_1 \otimes K_2 \oplus K_1 \otimes \text{id}_{\Sigma_2}$. The associated gauge is called *axial gauge*. If we call $\pi_{\ell}$ the integral kernel of $\Pi_{\ell}$, the *axial gauge propagator* is:

$$
\eta(x_1, x_2; x'_1, x'_2) = \pi_1(x_1; x'_1) \wedge \eta_2(x_2; x'_2) + \eta_1(x_1; x'_1) \wedge \delta(x_2; x'_2). \quad (2.36)
$$

### 3 2D YM for surfaces of non-negative Euler characteristic

In this section we will consider 2D YM on manifolds with codimension 1 boundaries. With a good choice of propagators and exploiting the gluing properties of BV-BFV theories, we will be able in this setting to explicitly compute all Feynman diagrams and sum the perturbative series to find the complete partition function of this theory on disks and cylinders. The globalized realization of the partition function on a disk in the $A$ polarization will coincide with the well-known non-perturbative solution of 2D YM [20, 24].

We consider a set of generators, under gluing, for orientable surfaces of non-negative Euler characteristic: the disk and the cylinder. At the level of the field theory constructed on such surfaces, we have to also consider the data of the polarization associated to the boundaries. The building blocks for 2D YM can be thus chosen to be the disk in the $B$ polarization, the cylinder in $A - B$ polarization and the cylinder in the $B - B$ polarization. Moreover, using the invariance of the theory under area-preserving diffeomorphisms, we can concentrate the support of the volume form $\mu$ near the boundaries; this allows to use as generators the above surfaces in the limit of zero area, i.e. for BF theory, at the cost of introducing as fourth generator a YM cylinder in $A - B$ polarization with finite volume (figure 3).

**Figure 3.** Building blocks for 2D YM on surfaces with non-negative Euler characteristic.

#### 3.1 $A-B$ polarization on the cylinder

Let us start studying the BF theory on the cylinder, $\Sigma = S^1 \times I \ni (\tau, t), \; I = [0, 1]$ . We will firstly choose $B$ polarization on $S^1 \times \{0\} = \partial_B \Sigma$ and $A$ polarization on $S^1 \times \{1\} = \partial_A \Sigma$. The space of bulk fields, with this polarization, is $\mathcal{Y} = \Omega(\Sigma, \partial_B \Sigma; g)[1] \oplus \Omega(\Sigma, \partial_A \Sigma; g)$. Since

---

26 Although we can present, e.g., the sphere and the torus as assembled from building blocks considered in this section, globalization integrals for them are perturbatively obstructed, see Section 3.5.4. We obtain a non-singular globalized answer in these cases as a part of the general result of Section 4.
the relative cohomology \( H(\Sigma, \partial_i \Sigma) \) is trivial with the above choice of boundaries, we have no zero-modes. Thus the connected diagrams contributing to the effective action of the theory are trees with one root on \( \partial_B \Sigma \) and leafs on \( \partial_A \Sigma \) or 1-loop diagrams with trees rooted on a point of the loop and leafs on \( \partial_A \Sigma \) (figure 4).

![Figure 4. Connected diagrams for non-abelian BF on the cylinder in A-B polarization.](image)

To compute these diagrams we can use the axial-gauge, with propagator (cf. B.2):

\[
(t, \tau) \longrightarrow (t', \tau') = \eta(t, \tau; t', \tau') = -\Theta(t' - t)\delta(\tau - \tau')(dt' - d\tau). \tag{3.1}
\]

Looking at this propagator we immediately notice that it is a zero-form on the interval \( I \).

Since each bulk vertex carries an integration over \( S^1 \times I \), the differential form associated to a diagram has the right form components (that is, s.t. its integral over the configuration space doesn’t vanish) only if it doesn’t contain any bulk vertex. Thus there is only one non-vanishing diagram contributing to the effective action:

\[
S_{\text{BF}}^{\text{eff}}[B, A] = \int_{\partial_A \Sigma} \langle p^* B, A \rangle, \tag{3.2}
\]

where \( p: \Sigma \rightarrow \partial_B \Sigma \) is a projection to the \( B \)-boundary.

The Yang-Mills action can be rewritten as a perturbation of BF:

\[
S_{\text{YM}} = S_{\text{BF}} + \frac{1}{2} \int_\Sigma \mu \text{tr}(B^2). \tag{3.3}
\]

The additional bivalent interaction vertex is proportional to the volume form \( \mu \). For degree counting reasons analogous to the one described above, the only additional non-vanishing Feynman diagram is the one containing a single YM vertex:

\[
S_{\text{YM}}^{\text{eff}} = \int_{\partial_A \Sigma} \langle p^* B, A \rangle + \frac{1}{2} \int_{\partial_B \Sigma} p_\ast \mu \text{tr}B^2, \tag{3.4}
\]

where the last integral is the integral of a density, with \( p_\ast \mu \) the pushforward of \( \mu \), viewed as a density on the cylinder, to the \( B \)-circle. Thus we proved the following:

**Proposition 3.1 (YM on A-B cylinder).** The partition function for a YM cylinder in the A-B polarization is:

\[
Z[A, B] = \exp \left( \frac{i}{\hbar} \left( \int_{\partial_A \Sigma} \langle p^* B, A \rangle + \frac{1}{2} \int_{\partial_B \Sigma} p_\ast \mu \text{tr}B^2 \right) \right). \tag{3.5}
\]

A YM A-B cylinder can be glued to other YM surfaces with boundary to modify their volume. In particular in this way one can convert BF (\( \mu = 0 \)) to YM.
3.2 \( \mathbb{B}-\mathbb{B} \) polarization on the cylinder

Another possible choice is to take the \( \mathbb{B} \) polarization on both the boundary components of the cylinder. This time the bulk fields are \( \mathcal{V} = \Omega(M; \mathfrak{g})[1] \oplus \Omega(M, \partial M; \mathfrak{g}) \), with zero-modes \( \mathcal{V} = H(M; \mathfrak{g})[1] \oplus H(M, \partial M; \mathfrak{g}) \simeq H(S^1; \mathfrak{g})[1] \oplus H(S^1; \mathfrak{g})[-1] \). More explicitly, the zero-modes can be described expanding with respect to a basis \( [\chi^i] \) of \( H(S^1) \) and its dual \( [\chi^i] \):

\[ a = a^i \chi^i \in H(S^1; \mathfrak{g})[1] \,, \quad b = b^i \chi^i \wedge dt \in H(S^1; \mathfrak{g})[-1] \,. \tag{3.6} \]

We can again fix the gauge using the axial-gauge, obtaining the propagator (3.7):

\[ \eta(t, \tau; t', \tau') = (t' - \Theta(t' - t)) \delta(\tau' - \tau)(dt' - d\tau) + dt'(\Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2}) \,. \tag{3.7} \]

Now the effective action contains trees with root on one of the boundaries and leafs in the bulk or 1-loop diagrams with trees rooted on the loop and leafs in the bulk. Luckily, a lot of these diagrams vanish as it is shown by the following

**Lemma 3.2.** For BF theory on a cylinder with \( \mathbb{B}-\mathbb{B} \) polarization in the axial gauge, all the diagrams containing a bulk vertex with attached two \( a \) zero-modes vanish:

\[ \Gamma = 0 \,. \tag{3.8} \]

**Proof.** Consider any diagram of the kind depicted in formula (3.8). The associated differential form on the configuration space of the diagram will be of the kind

\[ \Gamma_c(t, \tau) \eta(t, \tau; t', \tau') f_{ab}^c a^a(\tau') b^b(\tau') \,.
\]

Since \( a \) has no form component along \( dt \), using the axial-gauge propagator (3.7) we have for the corresponding amplitude:

\[ \int \Gamma_c(t, \tau) f_{ab}^c a^a(\tau') b^b(\tau') \eta(t, \tau; t', \tau') = f_{ab}^c a^a a^b \int \Gamma_c(t, \tau) \int_{S^1} \chi^i(\tau') \chi_j(\tau') \left( \Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2} \right) = 0 \,. \tag{3.9} \]

In particular this means that contributions to the effective action only come from either one single zero-mode attached to one of the two boundaries or from 1-loop diagrams with \( n \geq 2 \) vertices, each attached to a single \( a \) zero-mode (figure 5). These diagrams can be explicitly evaluated and the perturbative series can be summed to obtain the effective action.

**Proposition 3.3.** The partition function for BF theory on the cylinder in \( \mathbb{B}-\mathbb{B} \) polarization is:

\[ Z[\mathbb{B}, \mathbb{B}, a, b] = \exp \left( \frac{i}{2\hbar} \int_{S^1} \langle b, [a, a] \rangle + \frac{i}{\hbar} \int_{S^1} \langle \mathbb{B} - \mathbb{B}, a \rangle + \sum_{n \geq 2} \frac{1}{n} \text{tr}(ad_{a_1})^n \frac{B_n}{n!} \right) \cdot \rho_V \]

\[ = e^{\frac{1}{2\hbar} I_{S^1} b [a, a] + \frac{i}{\hbar} \int_{S^1} \langle \mathbb{B} - \mathbb{B}, a \rangle \text{det} \left( \frac{\sinh(\text{ad}_{a_1}/2)}{\text{ad}_{a_1}/2} \right) \} \cdot \rho_V \,. \tag{3.10} \]

Here \( \rho_V = (-i\hbar)^{\text{dim} \mathfrak{g}} D_{a}^{1/2} D_{b}^{1/2} \) is the reference half-density on the space of zero-modes.
Proof. We refer for the proof to Appendix C. □

Remark 3.4 (Reference half-densities on residual fields). Generally, we choose the following reference half-density on the space of residual fields $\mathcal{V}$:

$$\rho_\mathcal{V} = \prod_{k=0}^{2} (\xi_k)^{d_k} \cdot D^{\frac{1}{2}}a \cdot D^{\frac{1}{2}}b .$$

Here $D^{\frac{1}{2}}a \cdot D^{\frac{1}{2}}b$ is the standard half-density on $\mathcal{V}$, inducing the standard Berezin-Lebesgue densities $da$, $db$ on the Lagrangians $b = 0$ and $a = 0$, respectively. Also, $d_k$ is the dimension of the subspace of $\mathcal{V}$ corresponding to $a$-fields of de Rham degree $k \in \{0, 1, 2\}$; in particular, for $\mathcal{V}$ the zero-modes, $d_k = \dim H^k(\Sigma, \partial \Sigma; g)$ are the Betti numbers of relative de Rham cohomology. Factors $\xi_k$ are as follows:

$$\xi_0 = -i\hbar, \quad \xi_1 = 1, \quad \xi_2 = \frac{1}{2\pi \hbar} .$$

The logic behind this normalization is that for $\mathcal{V} = W[1] \oplus W^*[−2]$ with $W$ a complex (concentrated in degrees 0, 1, 2) and $\mathcal{V}' = W'[1] \oplus W'^*[−2]$ with $W'$ a deformation retract of $W$, we would like the BV pushforward of the half-density $\rho_\mathcal{V} e^{i(b, da)}$ on $\mathcal{V}$ (corresponding to abstract abelian BF theory associated to $W$) to yield $\rho_{\mathcal{V}'} e^{i(b', da')} on \mathcal{V}'$. Thus, we recover the normalization of reference half-densities from the automorphicity with respect to BV pushforwards. Most general normalization satisfying this condition is:

$$\xi_0 = -i\phi \hbar, \quad \xi_1 = \phi^{-1}, \quad \xi_2 = \frac{\phi}{2\pi \hbar} ,$$

with $\phi \neq 0$ an arbitrary constant. Our choice is to set $\phi = 1$ which will ultimately lead to the number-valued partition function of 2D Yang-Mills with standard normalization. Choosing any other $\phi$ would induce a rescaling of partition functions by

$$Z_{\Sigma} \mapsto \phi^{\chi(\Sigma) \cdot \dim \mathfrak{g}} Z_{\Sigma} ,$$

which reflects an inherent ambiguity of the normalization of path integral measure. We refer the reader to [12] for details on the normalization of half-densities compatible with BV pushforwards.
3.3 $A$-$A$ polarization on the cylinder

The last polarization choice we will consider consists in taking $A$ polarization for both the boundaries of the cylinder. With this polarization the bulk fields are $\mathcal{V} = \Omega(M, \partial M; g)[1] \oplus \Omega(M; g)$. The zero-modes $\mathcal{V} = H(M, \partial M; g)[1] \oplus H(M; g) \simeq H(S^1; g) \oplus H(S^1; g)$ can be expanded as:

$$a = a_i \chi^i \wedge dt, \quad b = b^i \chi_i.$$ (3.15)

The axial-gauge propagator is now (B.9):

$$\eta(t, \tau; t', \tau') = \left( \Theta(t - t') - t \right) \delta(\tau' - \tau)(d\tau' - d\tau) - dt \left( \Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2} \right).$$ (3.16)

The effective action contains trees with the root in the bulk and leaves on one of the boundaries or 1-loop diagrams with trees rooted on the loop and leaves either on the boundary or decorated with the zero mode $a^1$. Also with this polarization an analogue of lemma 3.2 holds:

**Lemma 3.5.** For BF theory on a cylinder with $A$-$A$ polarization in the axial gauge, all the diagrams containing a bulk vertex with attached two $a$ zero-modes vanish:

$$\Gamma = 0.$$ (3.17)

**Proof.** The proof follows trivially from degree counting, since a zero-modes always have a component along $dt$. \qed

The diagrams contributing to the effective action can be restricted further by reducing to the case of constant fields $\bar{A}, \bar{\bar{A}}$. Indeed, the gauge-invariance of the partition function (expressed by the mQME) implies that it is sufficient to evaluate it on constant 1-form fields $\bar{A} = dt \bar{A}(1), \bar{\bar{A}} = dt \bar{\bar{A}}(1)$, with $\bar{A}(1), \bar{\bar{A}}(1) \in g$ two constants. Then the value of the partition function for generic fields $\bar{A}, \bar{\bar{A}}$ is recovered (modulo a BV-exact term, cf. (3.18) below) by evaluating the constant-field result on the logs of holonomies $\bar{A}(1) = \log U(\bar{A}), \bar{\bar{A}}(1) = \log U(\bar{\bar{A}})$. Here $U(\cdots)$ stands for the holonomy of a connection 1-form around a circle. In other words, using the language of homological perturbation theory, we have a quasi-isomorphism between the two models for the space of states for an $A$-circle:

(i) The full BFV model $\mathcal{H}^A = \text{Fun}_C(\Omega^*(S^1, g)[1])$ given by functions of a general differential form $A$ on the circle, with differential $\Omega$ defined by (2.30).

(ii) The constant-field model $\mathcal{H}^{A, \text{const}} = \text{Fun}_C(H^*(S^1, g)[1])$ – functions of a constant form $\bar{A}(0) + dt \bar{A}(1)$, with differential $\bar{\delta}$ defined by (2.31).

---

$^{27}$ For simplicity of notations we are omitting the subscript of the 1-form $A(1)$ when it appears in the holonomy $U(A)$.
We have two chain maps: first, the projection \( p: H^A \to H^A \), – evaluation of a wavefunction on constant forms or equivalently the pullback \( p = \iota^* \) by the inclusion of the cohomology by as constant forms \( H^*(S^1, g) \to \Omega^*(S^1, g) \). Second, the inclusion \( i: H^A, \text{const} \to H^A \) sending \( \Psi \mapsto \Psi(\mathcal{A}(0), \log U(\mathcal{A})) \) where \( p \) is the base point on the circle used to define the holonomy. Denoting \( K \) the chain homotopy for the retraction of chain complexes \( (H^A, \Omega) \to (H^A, \text{const}, \delta) \), we have the following (cf. the discussion of the reduced partition function in [12], section 7.4):

\[
i_H \circ p_H Z = (\text{id} - K_H \Omega - \Omega K_H)Z = Z + (\Omega + h^2 \Delta)(\cdots) ,
\]

where \( \cdots = -K_H Z \). The left hand side in (3.18) is exactly the partition function evaluated on constant 1-form fields having the same holonomy as the original non-constant ones.

**Lemma 3.6.** For BF theory on a cylinder with \( \Lambda - \Lambda \) polarization in the axial gauge, all the diagrams containing a bulk vertex with attached two boundary fields vanish, assuming that \( \Lambda \) and \( \tilde{\Lambda} \) are constant 1-forms.

\[
\Gamma \begin{array}{c}
 \Lambda, \tilde{\Lambda} \\
 \end{array} = \Lambda \begin{array}{c}
 \Gamma \\
 \end{array} = 0 .
\]

**Proof.** Using the assumption of constancy of boundary fields and the axial-gauge propagator (3.16), when we have two boundary fields connected to the same bulk vertex we find the amplitude:

\[
\int \Gamma_c(\tilde{t}, \tilde{\tau}) f_{ab}^c \Lambda^a \Lambda^b \eta(\tilde{t}, \tilde{\tau}; t, \tau) \eta(t, \tau; 0, \tau') \eta(t, \tau; 0, \tau'')
\]

\[
= \frac{1}{2} f_{ab}^c \Lambda^a \Lambda^b \int \Gamma_c(\tilde{t}, \tilde{\tau}) \eta(\tilde{t}, \tilde{\tau}; t, \tau) \int_{S^1} d\tau' \left( \Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2} \right) = 0 .
\]

Similar amplitudes are found also in the case one or both the boundary fields live on the boundary at \( t = 1 \).

**Figure 6.** Relevant loop diagrams for the globalized effective action of BF theory on a cylinder with \( \Lambda - \Lambda \) polarization

Like in the case of non-abelian BF theory on closed surfaces, the computation of the effective action greatly simplifies if we look at the globalized answer. In this case we can define the globalized partition function by integrating the perturbative partition function over
a Lagrangian submanifold of the space of residual fields. If $S_{\text{eff}}[A, B, a, b]$ is the perturbative effective action on the space of boundary fields and zero-modes and $\mathcal{L} \subset \mathcal{V}$ is a Lagrangian submanifold, the globalized partition function can be defined as:

$$Z = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_{\text{eff}}[A, B, a, b]} .$$

(3.21)

A possible choice for the Lagrangian is $\mathcal{L} := \{a = 0\}$, which in particular implies that all diagrams containing a zero-modes will not contribute to the globalized effective action. The effective action of BF theory is always linear in the $b$ zero-modes. Moreover, for the $A$-$A$ polarization on the cylinder in the axial gauge, lemma 3.6 implies that there are no tree diagrams with this $\mathcal{L}$. Thus:

$$Z = \int \frac{db}{(2\pi \hbar)^{\dim g}} e^{\frac{i}{\hbar} \int S_1 (b, A - \tilde{A}) + \frac{i}{\hbar} S_{\text{eff}}(A, \tilde{A}, a=0, b=0)} = \left(\frac{i}{\hbar}\right)^{\dim g} \delta(A, \tilde{A}) e^{\frac{i}{\hbar} S_{\text{eff}}(A, \tilde{A}, a=0, b=0)} .$$

(3.22)

The loop diagrams contributing to the globalized effective action are now only those where each loop vertex is connected to a boundary field with a single propagator and the fields on the two boundary components coincide (figure 6). The amplitude of such a diagram with $n$ boundary fields is:

$$\frac{-1}{n} \text{tr}(\text{ad}_A^n) \int_{(S^1)^n} d\tau_1 \cdots d\tau_n \eta_{S^1}(\tau_1; \tau_n) \eta_{S^1}(\tau_n; \tau_{n-1}) \cdots \eta_{S^1}(\tau_2; \tau_1) .$$

(3.23)

The integrals involved are exactly the same as the ones of the case of $B$-$B$ polarization (C.1). Thus we have:

**Proposition 3.7.** The globalized partition function for BF theory on the cylinder in the $\Lambda$-$\Lambda$ polarization is

$$Z[A, \Lambda] = \left(\frac{i}{\hbar}\right)^{\dim g} \delta(\log U(A), \log U(\Lambda)) \cdot \delta(A_p, \Lambda_p) \cdot \delta_{G}(U(A), U(\Lambda)) .$$

(3.24)

where $U(\cdots)$ is the holonomy of the connection around a circle and $A_p, \Lambda_p$ are the zero-form components of boundary fields $A, \Lambda$ evaluated at the base points $p, \tilde{p}$ on the two boundary circles.

**Remark 3.8.** Since $\det\left(\frac{\sinh(\text{ad}_x/2)}{\text{ad}_x/2}\right)$ is the determinant of the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, we can rewrite (3.24) in terms of the delta function on the Lie group:

$$Z[A, \Lambda] = \left(\frac{i}{\hbar}\right)^{\dim g} \delta(A_p, \Lambda_p) \cdot \delta_{G}(U(A), U(\Lambda)) .$$

(3.25)

### 3.4 $B$ polarization on the disk

Let us consider now non-abelian BF theory on the disk $D$. Using the $B$ polarization on the boundary, the bulk fields are $\mathcal{Y} = \Omega(D; \mathfrak{g})[1] \oplus \Omega(D, S^1; \mathfrak{g})$. The zero-modes are $\mathcal{V} =$
\[ \mathfrak{g}[1] \oplus \mathfrak{g}^*[−2] \] with generators the constant zero-form \([1 \cdot t_a]\) and an area 2-form \([\mu \cdot t^a]\), where \(t_a\) and \(t^a\) are dual basis of \(\mathfrak{g}\) and \(\mathfrak{g}^*\).

The Feynman graphs appearing in the effective action are trees, with root either in a boundary \(B\)-field or in a \(b\) zero-mode in the bulk, or 1-loop diagrams (figure 7).

![Figure 7. Connected diagrams for non-abelian BF on the disk in B polarization.](image)

**Proposition 3.9.** In the effective action for BF theory on the disk in \(B\)-polarization, all the diagrams containing at least one propagator are vanishing. In particular the partition function reads:

\[
Z[B, a, b] = \exp \frac{i}{\hbar} \left( -\int_{S^1} \langle B, a \rangle + \int_D \frac{1}{2} \langle b, [a, a] \rangle \right) \cdot \rho_V ,
\]

with \(\rho_V = (-i\hbar)^{\dim g} D^{\frac{3}{2}} a \ D^{\frac{1}{2}} b\) the reference half-density on residual fields.

**Proof.** The result follows from degree counting. Let us consider first a tree diagram rooted in the bulk. If \(n\) is the number of bulk vertices, then we have \(n-1\) propagators, \(n+1\) \(a\) zero-modes and one \(b\) zero-mode. Since propagators are 1-forms, \(a\) only has the zero-form component and \(b\) is a 2-form, then the differential form associated to the diagram is a \((n+1)\)-form. This has to be integrated on the configuration space of the diagram, which is of dimension \(2n\). Thus the only possibly non-vanishing diagram is for \(n = 1\). Its contribution is:

\[
\frac{1}{2} \int_D \langle b, [a, a] \rangle .
\]

Consider now a tree diagram rooted on the boundary with \(n\) bulk vertices. We have \(n\) propagators, \(n+1\) \(a\) zero-modes and one boundary field \(B\). Thus the differential form associated to the diagram is a \(n\)-form or a \((n+1)\)-form, depending on the form degree of \(B\), and has to be integrated again on a \((2n+1)\)-dimensional configuration space. In this case we only have a contribution with \(n = 0\):

\[
- \int_{S^1} \langle B, a \rangle .
\]

Last, for a 1-loop diagram with \(n \geq 1\) vertices in the loop and \(l\) vertices in the trees rooted on the \(n\) loop vertices, we have \(n + l\) propagators and \(n + l\) \(a\) zero-modes. Thus we have to integrate a differential form of degree \(n + l\) on a \(2(n + l)\)-dimensional configuration space and, since \(n \geq 1\), we have no non-vanishing contributions. \(\square\)
3.5 Gluing

We computed the YM partition function on the $A$-$B$ cylinder and the BF partition function on the $B$-disk and the $A$-$A$ cylinder. As we will show in this section, using the gluing property of BV-BFV theories, this is sufficient to prove a gluing formula between $A$-polarized boundaries and to find the YM state on any surface with non-negative Euler characteristic.

3.5.1 BF disk in $A$ polarization

The BF disk in $A$ polarization can be obtained changing polarization to the disk in $B$ polarization by gluing to it an $A$-$A$ BF cylinder (figure 8).

![Figure 8](image_url)

**Figure 8.** A disk as the gluing of a $B$ disk with an $A$-$A$ cylinder.

For the $A$-$A$ cylinder we only know the projection in $\Omega$ cohomology of the globalized answer; since both globalization and projection to cohomology commute with gluing, we are still able to compute the partition function for the disk. The glued partition function is:

$$Z[A] = \int d\tilde{a} d\tilde{\tilde{a}} e^{\frac{i}{\hbar} \int_{S^1} \langle \tilde{\tilde{a}}, a \rangle} \delta(\tilde{A}_p, \tilde{\bar{A}}_p)$$

$$\cdot \det \left( \frac{\sinh \left( \text{ad} \log U(\tilde{A})/2 \right)}{\text{ad} \log U(\tilde{A})/2} \right)^{-1} \delta(\log U(\tilde{A}), \log U(\tilde{A}))$$

$$= \det \left( \frac{\sinh \left( \text{ad} \log U(\tilde{A})/2 \right)}{\text{ad} \log U(\tilde{A})/2} \right)^{-1} \delta(\log U(\tilde{A}), 0) = \delta_G(e^{\log U(\tilde{A})}, I)$$

$$= \delta_G(U(\tilde{A}), I).$$

**Remark 3.10.** To have consistency with gluing, we assume that the integration measure over the boundary fields is normalized in such a way that

$$\int d\tilde{B} d\tilde{A} e^{\frac{i}{\hbar} \int_{S^1} \langle \tilde{\tilde{a}}, \tilde{a} \rangle} = 1.$$  \hspace{1cm} (3.30)

As a matter of convenience, we moreover distribute the normalization between $d\tilde{A}$ and $d\tilde{\tilde{A}}$ in such a way that

$$\int d\tilde{B} e^{\frac{i}{\hbar} \int_{S^1} \langle \tilde{\tilde{a}}, \tilde{a} \rangle} = \delta(\tilde{A}).$$  \hspace{1cm} (3.31)

3.5.2 YM disk in $A$ polarization

We can obtain the partition function for the YM disk in $A$ polarization gluing to the BF disk a YM cylinder in $A$-$B$ polarization (figure 9).

As the partition function for the BF disk, also the YM partition function coincides with the non-perturbative answer.
Proposition 3.11. The globalized partition function for 2D YM on the disk in $\mathcal{A}$-polarization is:

$$Z_{YM}[\mathcal{A}] = \sum_R (\dim R) \chi_R(U(\mathcal{A})) \ e^{-\frac{ia}{\hbar}C_2(R)},$$

(3.32)

where $a = \int \mu$ is the area of the disk, $\chi_R$ the character and $C_2(R)$ the quadratic Casimir of the representation $R$.

Proof. Gluing a BF disk to a YM cylinder in $\mathcal{A}-\mathcal{B}$ polarization we get:

$$Z_{YM}[\mathcal{A}] = \int d\tilde{\mathcal{B}} d\tilde{\mathcal{A}} e^{-\frac{i}{\hbar} \int_{S^1} \langle \tilde{\mathcal{B}}, (\tilde{\mathcal{A}} - \mathcal{A}) \rangle - \frac{i}{\hbar} \int_{S^1 \times I} \mu \ tr(\tilde{\mathcal{B}}^2) \delta(U(\mathcal{A}), I)}$$

$$= e^{\frac{ia}{\hbar} \left( \frac{\partial}{\partial \mathcal{A}}, \frac{\partial}{\partial \mathcal{A}} \right)} \delta(U(\mathcal{A}), I) = \langle I | e^{-\frac{i}{\hbar} H_{YM}} | U(\mathcal{A}) \rangle$$

$$= \sum_R (\dim R) \chi_R(U(\mathcal{A})) \ e^{-\frac{ia}{\hbar}C_2(R)}.$$  \hfill \square

3.5.3 Gluing circles in $\mathcal{A}$ polarization

Two boundaries in $\mathcal{A}$ polarization can be glued together using a BF cylinder in $\mathcal{B}-\mathcal{B}$ polarization. If $Z_{\Sigma_i}[\mathcal{A}_i]$ is the globalized partition function on a surface $\Sigma_i$, $i = 1, 2$, and $\Sigma$ is the gluing $\Sigma_1 \cup_{S^1} \Sigma_2$ along a common boundary in $\mathcal{A}$ polarization, we get:

$$Z_{\Sigma} = \int d\tilde{\mathcal{A}} d\mathcal{A}_1 d\mathcal{A}_2 d\mathcal{A}_1 e^{\frac{i}{\hbar} \int_{S^1} \langle \tilde{\mathcal{A}}, (\mathcal{A}_1 - \mathcal{A}_2) \rangle - \frac{i}{\hbar} \int_{S^1} \langle \mathcal{B}, (\mathcal{A}_1 - \mathcal{A}_2) \rangle + \frac{i}{\hbar} \langle \mathcal{B}, [\mathcal{A}_1^{(0)}, \mathcal{A}_2^{(0)}] \rangle}$$

$$\cdot \ det \left( \frac{\sinh \left( \frac{ad_{\mathcal{A}_1}}{2} \right)}{ad_{\mathcal{A}_1}/2} \right) Z_{\Sigma_1} [\mathcal{A}] Z_{\Sigma_2} [\tilde{\mathcal{A}}] \cdot \rho_\mathcal{V}$$

$$= \rho_\mathcal{V} \cdot e^{\frac{i}{\hbar} \langle \mathcal{B}^{(2)}, [\mathcal{A}_1^{(0)}, \mathcal{A}_2^{(0)}] \rangle} \int_G dU \ Z_{\Sigma_1} [U] Z_{\Sigma_2} [U],$$

(3.34)

which coincides with the gluing formula for YM known in literature [20, 24] up to a zero-mode dependent factor. Here, instead of integrating out the zero-modes on the $\mathcal{B}-\mathcal{B}$ cylinder completely (which would yield an ill-defined integral), we performed a partial integration (BV pushforward), retaining the zero-modes $a^{(0)}, b^{(2)}$. Here the index in brackets stands for the form degree of a zero-mode and $\rho_\mathcal{V} = (-i\hbar)^{\dim g} D^\frac{1}{2} a^{(0)} D^\frac{1}{2} b^{(2)}$ is the reference half-density on the remaining zero-modes.

Remark 3.12. In (3.34), the domain of integration over $\mathcal{A}_1$ is the “Gribov region” $B_0 \subset \mathfrak{g}$ (cf. subsection 2.4.1) – the preimage of an open dense subset of the group $G$ under the
exponential map \( \exp : g \to G \). On one hand, this is the domain corresponding to values of \( a \) for which the sum of Feynman diagrams converges. On the other hand, this corresponds to avoiding overcounting when performing the globalization via integrating over zero-modes as opposed to integrating over the moduli space of solutions of Euler-Lagrange equations.

### 3.5.4 Other surfaces of non-negative Euler characteristic

To obtain the YM cylinder in \( \mathbb{A} - \mathbb{A} \) polarization we can simply change polarization to the YM \( \mathbb{A} - \mathbb{B} \) cylinder by gluing a BF cylinder (figure 10):

\[
Z_{YM}[\mathbb{A}, \mathbb{A}'] = \left( \frac{i}{\hbar} \right)^{\dim g} \int \bar{d}\mathbb{B} d\mathbb{A} \, e^{-\frac{i}{\hbar} \int_{S^1} \langle \mathbb{B}, (\mathbb{A} - \mathbb{A}) \rangle - \frac{1}{2\pi} \int_{D_B} \mu \text{tr}(\mathbb{B}^2)} \delta_G(U(\mathbb{A}), U(\mathbb{A}')) \, \delta(\mathbb{A}'_p, \mathbb{A}_p) = \left( \frac{i}{\hbar} \right)^{\dim g} \delta(\mathbb{A}'_p, \mathbb{A}_p) \sum_R (\dim R) \chi_R(U^{-1}(\mathbb{A}) \cdot U(\mathbb{A})) e^{-\frac{\hbar}{2} C_2(R)}.
\]

(3.35)

**Figure 10.** YM \( \mathbb{A} - \mathbb{A} \) cylinder as the gluing of a BF \( \mathbb{A} - \mathbb{A} \) cylinder with a YM \( \mathbb{A} - \mathbb{B} \) cylinder.

**Remark 3.13.** The answer (3.35) does not coincide with the non-perturbative answer, which will be recovered perturbatively in Section 4 using manifolds with corners. This discrepancy is due to the presence of inequivalent gauge-fixings in the globalization process.

Let us now compute the YM partition function for a sphere \( S^2 \) obtained by the gluing of two disks: one with area \( a \) and in the \( \mathbb{A} \) polarization, the other with zero area and in \( \mathbb{B} \) polarization.

Using the *globalized* partition function (3.32) for the \( \mathbb{A} \)-disk and the *non-globalized* answer (3.26) for the \( \mathbb{B} \)-disk we get:

\[
Z_{YM}^{S^2}[a, b] = \rho \int d\mathbb{A} d\mathbb{B} \, e^{-\frac{i}{\hbar} \int_{S^1} \langle \mathbb{B}, (\mathbb{A} - \mathbb{A}) \rangle - \frac{1}{2\pi} \int_{D_B} \mu \text{tr}(\mathbb{B}^2)} \cdot Z_{YM}[\mathbb{A}] = \rho \cdot e^{\frac{1}{2\pi} \int_{D_B} \langle b, [a, b] \rangle} \sum_R (\dim R)^2 e^{-\frac{\hbar}{2} C_2(R)}.
\]

(3.36)
Here $\rho_V = (-i\hbar)^{\dim g} D_1^\gamma a D_2^\gamma b$ is the reference half-density on zero-modes. We immediately notice that this non-globalized answer consists of the product of a function of the zero-modes times the non-perturbative Migdal-Witten partition function for the sphere. Moreover, the partition function (3.36) does not produce well-defined global answers by integrating out the zero modes.

Similarly, trying to calculate the globalized partition function for the torus by gluing a YM cylinder in $A$-polarization (3.35) with a cylinder in $B$-$\bar{B}$ polarization (3.10), one obtains an ill-defined answer.

**Remark 3.14.** We remark that the form of the perturbative answer here – as the non-perturbative (number-valued) answer times the exponential of a cubic term in zero-modes – is similar to the form of the perturbative result for Chern-Simons theory in BV formalism on a rational homology 3-sphere [8]:

$$ Z_{CS} = \rho_V \cdot e^{\frac{i}{\hbar} \langle a^{(3)}, [a^{(0)}, a^{(0)}] \rangle} \cdot e^{\frac{i}{\hbar} \zeta(h)}. $$

Here $\zeta(h)$ is the sum of contributions of connected 3-valent graphs without leaves.

**Remark 3.15.** The gluing construction of Section 4 (gluing along edges rather than circles) produces a well-defined globalized answer for all surfaces – including the cylinder, the sphere and the torus – coinciding with the non-perturbative answer in case of surfaces with boundary in $A$-polarization (1.2). In particular, the gluing construction of Section 4 produces the answer for the sphere as in (3.36) but without the zero-mode factor. This discrepancy is due to inequivalence of gauge-fixings used in the two approaches.

### 4 2D Yang-Mills for general surfaces with boundaries and corners

To be able to compute the partition function of 2D YM for general surfaces we need to also consider corners, i.e. codimension 2 strata – marked points on the boundary. In topology surfaces can be described as collections of polygons modulo an equivalence relation which identifies pairs of edges. The idea is to transport this description to the level of field theory: if we can compute the partition function on polygons with arbitrary combinations of polarizations associated to the edges, then we can recover the partition function on surfaces with boundary by gluing pairs of edges with transversal polarizations.

In this section we will formulate a set of rules for corners dictated by the logic of the path integral and find a set of building blocks that generates under gluing 2D YM on all manifolds with boundaries and corners. We will then discuss the mQME in presence of corners and compute the partition function of the various building blocks. Finally, we will use the results of this analysis to prove a gluing formula in presence of corners and compute the 2D YM partition function on a generic surface with boundary, recovering the well known non-perturbative solution.
4.1 Corners and building blocks for 2D YM

The partition function is an element of the space of boundary states, which are defined by the data of a choice of polarization on the boundary; this choice reflects on the (fluctuations of the) bulk fields by imposing boundary conditions. In the presence of corners dividing two arcs with different polarizations, we have to consider mixed boundary conditions for the bulk fields. More generally we can associate a polarization also to corners, inducing boundary conditions for all adjacent bulk or boundary fields. In this case, corners can be considered as collapsed arcs, with associated polarization the same as the corner they represent, but carrying only some of the boundary fields, namely the ones pulled back from the corner (i.e. constant zero-forms).

Notice that the presence of a corner with the same polarization as one of the adjacent edges has no effect on the partition function (but could require modifications of $\Omega$: cf. section 4.2). For example taking a corner with the same polarization of both adjacent edges simply means that we are formally splitting the boundary field into two concatenating fields, but this doesn’t change the boundary conditions for the bulk fields:

$$Z(\begin{array}{c} A \\ \alpha \\ B \\ \beta \end{array}) \simeq Z(\begin{array}{c} B \\ \beta \\ B \\ \beta \end{array}) ;$$

$$Z(\begin{array}{c} A \\ \alpha \\ A \end{array}) \simeq Z(\begin{array}{c} B \\ \alpha \\ B \end{array}) ;$$

Moreover, by “freeing” the bulk fields from the boundary conditions imposed by a corner, i.e., upon integrating over all possible values of corner fields, the partition function of the surface without that corner is recovered:

$$\int D\beta Z(\begin{array}{c} B \\ \beta \\ B \end{array}) = Z(\begin{array}{c} B \\ \beta \\ B \end{array}) .$$

The gluing of two arcs is analogous to the case without corners, with the only additional requirement that the fields of the corners that will be identified by the gluing have to coincide:

$$\int D(\begin{array}{c} A \\ \alpha \\ B \end{array}) e^{-\frac{i}{\hbar} \int(\begin{array}{c} A \\ \alpha \\ B \end{array}) Z(\begin{array}{c} A \\ \alpha \\ A \end{array}) Z(\begin{array}{c} B \\ \beta \\ B \end{array}) = Z(\begin{array}{c} B \\ \beta \\ B \end{array}) .}$$

All these statements will be tested with explicit computations in the following sections.

**Remark 4.1.** Let the surface $\Sigma$ be the result of gluing of surfaces $\Sigma_1$ and $\Sigma_2$ along an interval $I$, as above. Assume that the partition functions for $\Sigma_1$, $\Sigma_2$ are computed perturbatively, using the propagators $\eta_1, \eta_2$. Then the gluing formula (4.3) above yields the partition function for the glued surface $\Sigma$ computed using the “glued propagator” $\eta = \eta_1 \ast \eta_2$ on $\Sigma$, constructed as follows:
• For $x, y \in \Sigma_1$, $\eta(x, y) = \eta_1(x, y)$.

• For $x, y \in \Sigma_2$, $\eta(x, y) = \eta_2(x, y)$.

• For $x \in \Sigma_2$, $y \in \Sigma_1$, $\eta(x, y) = 0$.

• For $x \in \Sigma_1$, $y \in \Sigma_2$, we have

$$\eta(x, y) = \int_{I \ni z} \eta_1(x, z) \eta_2(z, y) \quad (4.4)$$

This is precisely the gluing construction for propagators from [13], which turns out to work also in the setting with corners.

Assuming this set of rules for the corners, we have the following set of building blocks for 2D YM, as illustrated in figure 11. The disk in the $A$ polarization was already computed in section 3.5.2 and, using equation (4.1), it is equivalent to a polygon with an arbitrary number of edges where all the edges and the corners are in $A$-polarization. To change polarization of one of its edges, we can glue to it the BF disk with two corners in the $\alpha$ polarization and two edges in $B$ polarization. The last BF disk of figure 11, with only one $A$-edge and one corner in the opposite polarization, can be then used in combination with the other building blocks to change the polarization of one corner (figure 12). In this way we can obtain a polygon with any number of edges and with any combination of polarizations associated to edges and corners; thus we can also obtain the partition function for any given surface with boundary (and corners).

![Figure 11. Building blocks for 2D YM with corners.](image-url)
Figure 12. The polarization on a corner can be changed by gluing. In this picture it is illustrated how to convert a corner in $\alpha$ polarization to a corner in $\beta$ polarization using the building blocks of figure 11.

4.2 Corners, spaces of states and the modified quantum master equation

We have two pictures for surfaces with boundary and corners.

I. **(Non-polarized corners.)** Boundary circles are split into intervals by vertices (corners). Each interval carries a polarization $A$ or $B$, corresponding to imposing the boundary condition on the pullback to the interval of the bulk field $A$ or $B$. Corners do not carry a polarization.

II. **(Polarized corners.)** In addition to the intervals carrying a polarization $A$ or $B$, each corner is also equipped with a polarization $\alpha$ or $\beta$ corresponding to prescribing the pullback of either $A$ or $B$ field to the corner.

Picture II is our main framework in this paper. One can transition from picture I to picture II by **collapsing** every other arc on a circle (assuming that initially the number of arcs was even) into a vertex with the corresponding polarization, by the rule $A \rightarrow \alpha$, $B \rightarrow \beta$. One obtains the partition function $Z_{II}$ in the picture II by evaluating the partition function $Z_I$ of picture I on constant 0-form fields on the arcs that are being collapsed – pullbacks of the corner fields to the arc. E.g., for a disk with the boundary split into 4 arcs of alternating polarizations in picture I, collapsing the $B$-arcs into $\beta$-corners corresponds to the following:

$$Z_{II}(A_1, \beta_1, A_2, \beta_2; \text{zero-modes}) = Z_I(A_1, B_1 = \beta_1, A_2, B_2 = \beta_2; \text{zero-modes}).$$

(4.5)
4.2.1 Picture I: non-polarized corners. Modified quantum master equation

Consider a circle (thought of as a boundary component of a surface Σ) split by n points \( p_1, p_2, \ldots, p_{2m} = p_0 \) (“corners”) into intervals \( I_1, I_2, \ldots, I_{2m} \) with \( I_k = [p_{k-1}, p_k] \).

Assume that we fix the \( \mathbb{A} \)-polarization on the intervals \( I_k \) with \( k \) odd and the \( \mathbb{B} \)-polarization for \( k \) even. We understand that we can, by a tautological transformation, further subdivide each \( \mathbb{A} \)- or \( \mathbb{B} \)-interval into several intervals carrying the same polarization. No polarization data is assigned to the corners \( p_k \) (this is our “picture I” for corners).

The BFV space of states \( \mathcal{H} \), associated to the circle with such a stratification and a choice of polarizations, is the space of complex-valued functions of the fields on the intervals:

\[
\mathcal{H} = \left\{ \text{functions } \Psi(\mathbb{A}|_{I_1}, \mathbb{B}|_{I_2}, \ldots, \mathbb{B}|_{I_{2m}}) \right\}.
\]  

(4.6)

The space of states is equipped with the BFV operator (which with an appropriate refinement becomes a differential, see Remark 4.3 below)

\[
\Omega = \sum_{k \text{ odd}} \Omega^\mathbb{A}_{I_k} + \sum_{k \text{ even}} \Omega^\mathbb{B}_{I_k} + \sum_{k \text{ odd}} \Omega^{\mathbb{A}\mathbb{B}}_{p_k} + \sum_{k \text{ even}} \Omega^{\mathbb{B}\mathbb{A}}_{p_k}.
\]  

(4.7)

Here the edge contributions from the intervals, depending on the polarization, are:

\[
\Omega^\mathbb{A}_{I_k} = i\hbar \int \langle \mathbb{A}, \mathbb{A} \rangle + \frac{1}{2} \langle [\mathbb{A}, \mathbb{A}], \frac{\delta}{\delta \mathbb{A}} \rangle,
\]  

(4.8)

\[
\Omega^\mathbb{B}_{I_k} = \int i\hbar \langle \frac{\delta}{\delta \mathbb{B}}, \mathbb{B} \rangle + (i\hbar)^2 \langle \mathbb{B}, \frac{1}{2} \left[ \frac{\delta}{\delta \mathbb{B}}, \frac{\delta}{\delta \mathbb{B}} \right] \rangle.
\]  

(4.9)

The corner contributions from the vertices \( p_k \) are the multiplication operators by the product of the limiting values of the \( \mathbb{A} \)-field and the \( \mathbb{B} \)-field coming from the incident arcs, with a sign depending on the order of the arcs relative to the orientation:

\[
\Omega^{\mathbb{A}\mathbb{B}}_{p_k} = -\langle \mathbb{B}_p, \mathbb{A}_p \rangle, \quad \Omega^{\mathbb{B}\mathbb{A}}_{p_k} = \langle \mathbb{B}_p, \mathbb{A}_p \rangle.
\]  

(4.10)

These corner contributions to the boundary BFV operator \( \Omega \) and their necessity for the modified quantum master equation were observed by Alberto S. Cattaneo [5].

The following is a refinement of Lemma 4.11 in [13] for a surface with boundary, with non-polarized corners allowed, in the case of 2D Yang-Mills theory.
Proposition 4.2 (mQME in picture I). The BV-BFV partition function $Z$ of 2D Yang-Mills theory on a surface with boundary consisting of stratified circles decorated with a choice of $A, B$ polarizations on the codimension 1 strata (and no polarization data on codimension 2 strata) satisfies the mQME

$$ (\hbar^2 \Delta + \Omega) Z = 0, \quad (4.11) $$

where $\Omega$ is the sum of expressions (4.7) for the stratified boundary circles.

Sketch of proof. The proof follows the proof of Lemma 4.11 in [13] where we need to take care of collapses of point near a corner. Let $\Gamma$ be a Feynman graph for the partition function; its contribution to $Z$ is $\int_{C_{\Gamma}} \omega_{\Gamma}$: the integral over the configuration space $C_{\Gamma}$ – where vertices of $\Gamma$ are restricted to the respective strata of $\Sigma$ (bulk, boundary arcs or corners) – of $\omega_{\Gamma}$, the differential form on $C_{\Gamma}$, which is the product of propagators, boundary fields and zero-modes, as prescribed by the combinatorics of $\Gamma$.\(^{28}\) One considers the Stokes’ theorem for configuration space integrals:

$$ i\hbar \sum_{\Gamma} \int_{C_{\Gamma}} d\omega_{\Gamma} = i\hbar \sum_{\Gamma} \int_{\partial C_{\Gamma}} \omega_{\Gamma}. \quad (4.12) $$

On the left hand side, the terms with $d$ acting on the propagators assemble into $\hbar^2 \Delta Z$ and the terms with $d$ acting on $A, B$ fields assemble into $\Omega_0 Z$ where $\Omega_0 = i\hbar \int_{\partial_b \Sigma} \langle dA, \frac{\delta}{\delta A} \delta A \rangle + i\hbar \int_{\partial_B \Sigma} \langle dB, \frac{\delta}{\delta B} \delta B \rangle$. Here $\partial_b \Sigma$ and $\partial_B \Sigma$ are the parts of the boundary equipped with polarizations $A$ and $B$, respectively. Thus, the l.h.s. of (4.12) is $(\hbar^2 \Delta + \Omega_0) Z$. The r.h.s. contains several types of terms, corresponding to types of boundary strata of $C_{\Gamma}$:

(i) Collapses of 2 points in the bulk – cancel out when summed over graphs, due to the classical master equation satisfied by the BV action.

(ii) Collapses of $\geq 3$ points in the bulk – vanish by the standard vanishing arguments for hidden strata of the configuration spaces [17].

(iii) Collapses of one or more points at a point on a boundary arc. These contributions assemble into $-\Omega_1 Z$, where contributions to the differential operator $\Omega_1$ are given by the collapsed subgraphs.

(iv) Collapses of several points at a corner – they assemble into $-\Omega_2 Z$.

Thus, one obtains the modified quantum master equation (4.11) with $\Omega = \Omega_0 + \Omega_1 + \Omega_2$.

Analyzing the possible contributing collapses at an arc yields two graphs contributing to $\Omega_1$:

\[\begin{align*}
A & \quad \rightarrow i\hbar \int_{\partial_b \Sigma} \langle \frac{1}{2} [A, A], \frac{\delta}{\delta A} \delta A \rangle, \\
B & \quad \rightarrow (i\hbar)^2 \int_{\partial_B \Sigma} \langle B, \frac{1}{2} \left[ \frac{\delta}{\delta B}, \frac{\delta}{\delta B} \right] \rangle.
\end{align*}\]

\(\quad (4.13)\)

\(^{28}\) A tacit assumption in this proof is that the propagator is a smooth 1-form on the configuration space of two points. E.g., the “metric propagator” arising from Hodge theory satisfies this property. Singular propagators considered in this paper arise as limits of such smooth propagators.

38
For $\Omega_2$, the only contributing graphs are

\[
\begin{align*}
A & \rightarrow -\langle B, A \rangle, \\
B & \rightarrow \langle B, A \rangle.
\end{align*}
\] (4.14)

**Remark 4.3.** It was found out in [13] that, in order to have the property $\Omega^2 = 0$ for the BFV operator, generally one should consider a certain refinement of the space of states, allowing the states to depend on the so-called “composite fields” on the boundary, which correspond in Feynman diagrams to boundary vertices of valency $\geq 2$.\(^{29}\) We are not considering composite fields in this paper: below, in Section 4.2.2, we manage to construct $\Omega$ for the setting of polarized corners, which squares to zero on the nose, without having to introduce composite fields.

### 4.2.2 Picture II: polarized corners

Now consider a circle split by $n$ points $p_1, \ldots, p_n = p_0$ (“corners”) into intervals $I_1, \ldots, I_n$. Assume that for each $k$ we fix on the interval $I_k$ the polarization $P_k \in \{A, B\}$ – i.e. we prescribe either the pullback of $A_k$ or the pullback $B_k$ of the field $B$ on $I_k$ (by an abuse of notations, we denote the differential form $A_k$ or $B_k$ also by $P_k$). Likewise, we fix a polarization $\xi_k \in \{\alpha, \beta\}$ on the corners $p_k$.

\[
\begin{align*}
I_1 & \quad p_0 = p_n \\
p_1 & \quad \xi_1 & \quad P_1 \\
I_2 & \quad \xi_2 & \quad p_2 \\
p_2 & \quad \xi_{n-1} & \quad p_{n-1} \\
I_n & \quad P_n \\
\end{align*}
\] (4.15)

The BFV space of states $\mathcal{H}$, associated to the circle with such a stratification and a choice of polarizations, is the space of complex-valued functions of the fields on the intervals and the corners, subject to the natural corner value conditions:

\[
\mathcal{H} = \left\{ \text{functions } \Psi(P_1, \xi_1, P_2, \xi_2, \ldots, P_n, \xi_n) \middle| \begin{array}{l}
P_k|_{p_k} = \xi_k \text{ if polarizations } P_k \text{ and } \xi_k \text{ agree} \\
\xi_{k-1} \text{ if polarizations } P_k \text{ and } \xi_{k-1} \text{ agree}
\end{array} \right\}.
\] (4.16)

\(\text{In fact, the operator } \Omega \text{ constructed above (4.7) with edge contributions (4.8,4.9) and corner contributions (4.10) does not satisfy } \Omega^2 = 0 \text{ on the nose, whenever corners are present. In the setting of [13] this is remedied by adding corrections to } \Omega, \text{ depending on composite boundary fields. Then in addition to the diagrams (4.14) at a corner one should consider other diagrams, involving boundary vertices of valency } \geq 2.\)\(^{29}\)
Here we say that the polarization of an interval “agrees” with the polarization of the incident corner if this pair of polarizations is either \((\mathcal{A}, \alpha)\) or \((\mathcal{B}, \beta)\). The space of states is a cochain complex with the differential

\[
\Omega = \sum_k \omega^k_{I_k} + \sum_k \omega^k_{P_k} \xi_k P_{k+1},
\]

where the edge contributions are given by (4.8,4.9).

The corner contributions to \(\Omega\) depend on the polarization \(\xi_k\) at the corner and polarizations of the incident edges \(P_k, P_{k+1}\) and are assembled from the contribution of the corner itself and the contributions of the corner interacting with the incident edges:

\[
\omega^k_{P_k} \xi_k P_{k+1} = \omega^k_{P_k} \xi_k + \omega^k_{P_k} \xi_k P_{k+1} + \omega^k_{P_k} \xi_k P_{k+1}.
\]

Here the pure corner contributions are:

\[
\omega^\alpha_p = i\hbar \left\langle \frac{1}{2} [\alpha, \alpha], \frac{\partial}{\partial \alpha} \right\rangle, \quad \omega^\beta_p = 0.
\]

The corner-edge contributions \(\omega^\alpha_p, \omega^\beta_p\) vanish if the polarization \(\xi\) at the corner matches the polarization \(P\) of the incident edge. For mismatching corner-edge polarizations, we have nontrivial contributions to \(\Omega\):

\[
\begin{align*}
\mathcal{A} \quad \beta & \quad \rightarrow \quad \langle \beta, F_- \left( \text{ad}_{i\hbar \frac{\partial}{\partial \beta}} \right) \mathcal{A}_p \rangle, \\
\mathcal{B} \quad \alpha & \quad \rightarrow \quad \langle \beta, F_+ \left( \text{ad}_{i\hbar \frac{\partial}{\partial \beta}} \right) \mathcal{A}_p \rangle,
\end{align*}
\]

\[
\begin{align*}
\mathcal{B} \quad \alpha & \quad \rightarrow \quad \langle \mathcal{B}_p, F_- \left( \text{ad}_{i\hbar \frac{\partial}{\partial \beta}} \right) \alpha \rangle, \\
\mathcal{A} \quad \beta & \quad \rightarrow \quad \langle \mathcal{B}_p, F_+ \left( \text{ad}_{i\hbar \frac{\partial}{\partial \beta}} \right) \alpha \rangle.
\end{align*}
\]

Here we have introduced the following functions:

\[
F_+(x) = \frac{x}{1 - e^{-x}} = \sum_{j=0}^\infty (-1)^j \frac{B_j}{j!} x^j = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots,
\]

\[
F_-(x) = \frac{x}{1 - e^x} = -\sum_{j=0}^\infty \frac{B_j}{j!} x^j = -1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^4}{720} + \cdots,
\]

where \(B_j\) are the Bernoulli numbers \(B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots\).

In (4.20), functions \(F_\pm\) are evaluated on \(x = \text{ad}_{i\hbar \frac{\partial}{\partial \beta}}\), producing \(\text{End}(g)\)-valued derivations (of infinite order) of the space of functions of \(\beta\). Further in this section we will also need the following two functions, related to the generating functions for Bernoulli polynomials:

\[
G_+(t, x) = \frac{1 - e^{-tx}}{1 - e^{-x}}, \quad G_-(t, x) = \frac{1 - e^{(1-t)x}}{1 - e^x}.
\]
Note that, when acting on the partition function, the complicated operators \( \Omega_p^B, \Omega_p^{\alpha B} \) from (4.20) act simply as multiplication operators
\[
\Omega_p^B \sim \langle B_p, \alpha \rangle, \quad \Omega_p^{\alpha B} \sim -\langle B_p, \alpha \rangle,
\]
since the derivative in the corner value of the field \( B \) acts by zero. Thus, we have:
\[
\Omega_p^B = \langle B_p, \alpha \rangle + \cdots, \quad \Omega_p^{\alpha B} = -\langle B_p, \alpha \rangle + \cdots,
\]
where we have added the terms \( \cdots \) (irrelevant for the master equation) so as to have the property \( \Omega^2 = 0 \). To be precise, we impose the following mild restriction on the states.

**Assumption 4.4 (Admissible states).** We assume that the states do not depend explicitly on the limiting values of 1-form components of fields \( A, B \) at corners. I.e., for \( p \) a corner, the derivatives \( \frac{\partial}{\partial A_p^{(1)}}, \frac{\partial}{\partial B_p^{(1)}} \) act by zero on admissible states.

Then, by a direct computation, one verifies the following (we give the explicit proof in Appendix D).

**Proposition 4.5.** For a stratified circle, with any choice of polarizations on the strata, the operator \( \Omega \) as defined by (4.17,4.19,4.20) satisfies \( \Omega^2 = 0 \) on admissible states in the sense of Assumption 4.4.

Let us introduce the following terminology. For a product of intervals (or circles) \( I \times J \) with \( I \) parameterized by coordinate \( t \) and \( J \) parameterized by \( \tau \), we call the axial gauge propagator containing \( \delta(\tau - \tau') \parallel \) parallel to \( I \) (and \( \perpendicular \) perpendicular to \( J \)), since the intervals on which the \( \delta \)-term is supported are parallel to \( I \). Note that in all the computations of Section 3, the axial gauge was always chosen to be perpendicular to the boundary.

Consider a surface \( \Sigma \) with stratified boundary circles in picture \( \Pi \), as in (4.15), i.e., with arcs and corners carrying polarization data. We can view such a surface as a limit at \( s \to 0 \) of a family of surfaces \( \Sigma_s \), with corners of \( \Sigma \) expanded into arcs of corresponding polarization (thus, surfaces \( \Sigma_s \) for \( s > 0 \) are in picture I). Let \( \eta_s \) be a family of propagators (corresponding to a family of gauge-fixings) on the surfaces \( \Sigma_s \), converging to a propagator \( \eta \) on \( \Sigma \). We make the following assumption.

**Assumption 4.6 (Collapsible gauge condition).** The contraction of the propagator \( \eta_s \) with a 1-form \( B^{(1)} \) on the \( B \)-interval \( I_s \subset \partial \Sigma_s \) that is being collapsed into a \( \beta \)-corner \( p \) of \( \Sigma \), becomes supported at \( p \) in the limit \( s \to 0 \).

This assumption can be realized by considering an \( s \)-dependent family of metric gauge-fixings associated to equipping \( \Sigma_s \) with a metric \( g_s \) in which the \( B \)-arc undergoing the collapse is placed at the end of a long “tentacle”. Thus, at \( s \to 0 \), the \( \beta \)-corner is placed infinitely far from the rest of the surface.
Put another way, if both arcs adjacent to \( I_s \) are in \( A \)-polarization, the assumption requires that \( \eta_s \) asymptotically approaches the axial gauge propagator \( \eta(t, \tau; t', \tau') = (\Theta(t - t') - t) \cdot \delta(\tau - \tau') (d\tau' - d\tau) - dt \Theta'(\tau' - \tau) \) (the axial propagator parallel to \( I_s \)) near the collapsing interval \( I_s \), as \( s \to 0 \), with \( t \) the coordinate along \( I_s \) and \( \tau \) the coordinate along the “tentacle”.

**Proposition 4.7** (mQME in picture II). Under the assumption above, the partition function \( Z \) for the surface \( \Sigma \) with boundary and corners equipped with polarization data, satisfies the modified quantum master equation

\[
(h^2 \Delta + \Omega)Z = 0 ,
\]

where \( \Omega \) is given as the sum of expressions (4.17) over the boundary circles, with edge contributions given by (4.8,4.9) and corner contributions defined by (4.18,4.19,4.20).

Here \( Z \) is understood as the limit \( s \to 0 \) of the evaluation of partition function of picture I on \( \Sigma_s \) on the fields pulled back from edges and corners of \( \Sigma \) along the collapse map \( \Sigma_s \to \Sigma \). See Remark 4.8 below for an explicit example of the mQME with corners in the picture II. Also, in Remark 4.14 we will have a non-example showing that the mQME does indeed fail without the Assumption 4.6.

**Sketch of proof.** Proposition 4.7 arises as a corollary of Proposition 4.2, since \( Z \) is understood as a limit \( s \to 0 \), in the sense explained above, of partition functions \( Z_s \) on surfaces \( \Sigma_s \) in picture I, which do satisfy the modified quantum master equation by Proposition 4.2.

Contribution \( \frac{i}{\hbar} \langle [\alpha, \alpha], \frac{\partial}{\partial \alpha} \rangle \) to \( \Omega \) in picture II at a \( B - \alpha - B \) corner arises as an \( s \to 0 \) limit of the \( A \)-edge contribution (4.8) from the \( A \)-edge of \( \Sigma_s \) collapsing to the corner.

Next, consider an \( A - \beta - A \) corner where, in addition to Assumption 4.6, we assume for the moment that the 0-form component of the \( A \) field is continuous through the corner. In this case, the corner contribution to \( \Omega \) given by (4.18,4.20) simplifies to \( \Omega^{A \beta A} = i\hbar \langle [A_p, \beta], \frac{\partial}{\partial \beta} \rangle \) and it arises from the fact that \( Z_s \) depends on the 1-form field \( B^{(1)} \) at the edge \( I_s \) collapsing into the \( \beta \)-corner \( p \), and this dependence is important for the mQME in picture I. Using the Assumption 4.6 and the continuity of \( A^{(0)} \) through the corner, the dependence of \( Z_s \) on \( B^{(1)} \) for small \( s \) is: \( Z_s \sim e^{\frac{i}{\hbar} \int_{I_s} \langle [A_p, B^{(0)}], \frac{\partial}{\partial B^{(0)}} \rangle} Z_s \to i\hbar \langle [A_p, \beta], \frac{\partial}{\partial \beta} \rangle Z_s \).

Therefore, one can compensate for the loss of dependence on \( B^{(1)} \) during the collapse by inclusion of the term \( i\hbar \langle [A_p, \beta], \frac{\partial}{\partial \beta} \rangle \) in \( \Omega \).

Finally, consider the \( A - \beta - A \) corner without assuming the continuity of \( A^{(0)} \) through the corner. To analyze the dependence of \( Z_s \) on \( B^{(1)} \), we cut a rectangle \( \mathcal{R} \) out of \( \Sigma_s \) at the
Thus, we present the surface $\Sigma_s$ as $R \cup I_2 \tilde{\Sigma}$. Computing the partition function on the rectangle in the axial gauge,\footnote{We are using the axial gauge propagator parallel to $I_s$, i.e. $\eta(t, \tau; t', \tau') = (\Theta(t-t') - t) \delta(\tau - \tau') (d\tau' - d\tau) - dt \Theta(\tau - \tau)$ with $t, \tau$ the vertical and horizontal coordinate on the rectangle. This choice is the one consistent with the Assumption 4.6.} setting $A_1 = A_{p+0}$, $A_3 = A_{p-0}$ – constant zero-forms, the limiting values of $A^{(0)}$ to the right and left of the corner $p$ on $\Sigma$, and setting $A_2 = dt$ – a constant 1-form, we find the following:

$$Z_R = e^{\frac{i}{\hbar} \int_{I_s} dt \langle A^{(0)}(t), A \rangle + \langle A^{(1)}, G_{\pm}(t, \text{ad} A) A_{p+0} \rangle + \langle A^{(1)}, G_{\pm}(t, \text{ad} A) A_{p-0} \rangle},$$

(4.25)

with $G_{\pm}$ as in (4.22). Here the integral is over $t \in [0, 1]$, or equivalently over $I_s$ with reversed orientation. This implies

$$\left(\Omega_{I_s}^B + \langle \beta, A_{p+0} \rangle - \langle \beta, A_{p-0} \rangle\right) Z_R \bigg|_{B=\beta} = - \langle \beta, F_{\pm}(\text{ad} A) A_{p+0} + F_{\pm}(\text{ad} A) A_{p-0} \rangle e^{\frac{i}{\hbar} \langle \beta, A \rangle}.$$  

(4.26)

The operator acting on $Z_R$ on the left hand side is the part of $\Omega$ in picture I corresponding to the collapsing interval $I_s$ and its two endpoints. Thus, combining with the gluing formula for partition functions we have:

$$\left(\Omega_{I_s}^B + \langle \beta, A_{p+0} \rangle - \langle \beta, A_{p-0} \rangle\right) \int d\tilde{\Sigma} \frac{d\beta'}{Z_{\tilde{\Sigma}}} e^{-\frac{i}{\hbar} \langle \beta', A \rangle} \cdot Z_{\tilde{\Sigma}}(\beta', \cdots) =$$

$$= \langle \beta, F_{\pm}(\text{ad} \frac{\partial}{\partial \beta}) A_{p+0} + F_{\pm}(\text{ad} \frac{\partial}{\partial \beta}) A_{p-0} \rangle \cdot Z_{\tilde{\Sigma}}(\beta, \cdots).$$ (4.27)

Thus, the action on $Z_{\tilde{\Sigma}}$ of the part of $\Omega$ in picture I corresponding to the collapsing interval (with its endpoints) is compensated by the action on the partition function in picture II of the operator appearing on the right hand side – which is precisely our anticipated corner contribution in picture II, $\Omega^{A,\beta \Lambda} = \Omega^{A,\beta} + \Omega^{\beta \Lambda}$, see (4.20).

Remark 4.8. Another argument for the contribution to $\Omega$ from a $\beta$-corner is as follows. In Section 4.5 we will obtain the explicit partition function for an $A$-disk $D$ with a single $\beta$-corner

\begin{tikzpicture}
  % TikZ code for the diagram
\end{tikzpicture}

\textsuperscript{30}
4. 2D Yang-Mills for general surfaces with boundaries and corners

in the form

$$Z_D = e^{\frac{i}{\hbar} \langle \beta, \log U(\mathcal{A}) \rangle},$$

(4.29)

with $U(\mathcal{A})$ the holonomy of the 1-form field $\mathcal{A}^{(1)}$ along the boundary circle. From Baker-Campbell-Hausdorff formula, one finds

$$\Omega^A \log U(\mathcal{A}) = -i\hbar \left( F_+ (\text{ad}_{\log U(\mathcal{A})}) \mathcal{A}_{p-0} + F_- (\text{ad}_{\log U(\mathcal{A})}) \mathcal{A}_{p+0} \right),$$

(4.30)

which implies

$$\Omega^A Z_D = \langle \beta, F_+ (\text{ad}_{\log U(\mathcal{A})}) \mathcal{A}_{p-0} + F_- (\text{ad}_{\log U(\mathcal{A})}) \mathcal{A}_{p+0} \rangle \cdot Z_D.$$  

(4.31)

From this one immediately sees that

$$(\Omega^A + \Omega^\beta_\tau + \Omega^{A\beta}_\tau)Z_D = 0,$$

(4.32)

with the corner contributions as prescribed by (4.20). Thus, the mQME works by a direct computation. For a general surface $\Sigma$ containing a $\beta$-corner, surrounded by $\mathcal{A}$-edges, one can cut out a disk around the corner and the mQME will follow from the one we just checked for the disk and from the one for the remaining part of the surface (thus by induction one can reduce to the case of surfaces without $\mathcal{A} - \beta - \mathcal{A}$-corners).

Yet another approach to the proof of Proposition 4.7, explaining the corner contributions (4.20), is in the vein of the proof of Proposition 4.2, with $\Omega$ given by Feynman subgraphs collapsing at the boundary/corners. Consider e.g. a collapse at a $\mathcal{A} - \beta - \mathcal{A}$ corner. The following subgraphs are contributing:

One computes these contributions to $\Omega$ using the propagator $\eta = (\Theta(t-t') - t) \delta(\tau - \tau') (d\tau - d\tau') - dt \Theta(\tau' - \tau)$ in the rectangle that we see when zooming into the corner. In the zoomed-in
picture we are considering configurations of points modulo the horizontal rescalings $\tau \mapsto c \cdot \tau$. We fix a representative of the quotient by fixing the horizontal position of one marked vertex. Edges leaving the collapsing subgraph are assigned the expression $dt \cdot i\hbar \frac{\partial}{\partial \beta}$ (the factor $dt$ comes from the propagator associated to the external edge). The graphs in (4.34) are easily computed and yield, when summed over the number of external edges, $\Omega^{A,\beta} = \Omega^{A,\beta} + \Omega^{\beta A}$ with $\Omega^{A,\beta}, \Omega^{\beta A}$ given by the formulae (4.20).

4.2.3 Space of states for the stratified circle as assembled from spaces of states for edges and corners

One can regard the space of states (4.16) for the stratified circle as constructed from the spaces of states for individual edges. One assigns to an interval (with chosen polarization $P$ in the bulk and $\xi, \xi'$ on the endpoints) a space of states – a cochain complex – constructed as the space of functions on the $P$-field at the edge and fields at the corners (understood as independent fields if the corner and edge polarizations disagree; if the polarizations agree, the corner field is the limiting value of the edge field):

$$H_{I}^{\xi,\alpha,\xi'} = \text{Fun}_{C}\left(\left\{ g[1] \right\} \times_{g[1]} \Omega(I, g)[1] \times_{g[1]} \left\{ g[1] \right\} \right),$$

$$\Omega_{I}^{\xi,\alpha,\xi'} = \Omega_{\text{in}}^{\xi} + \Omega_{\text{in}}^{\alpha} + \Omega_{I}^{\alpha} + \Omega_{\text{out}}^{\alpha} + \Omega_{\text{out}}^{\xi'},$$

$$H_{I}^{\xi,\beta,\xi'} = \text{Fun}_{C}\left(\left\{ g[1] \right\} \times_{g[1]} \Omega(I, g^*)[1] \times_{g[1]} \left\{ g[1] \right\} \right),$$

$$\Omega_{I}^{\xi,\beta,\xi'} = \Omega_{\text{in}}^{\xi} + \Omega_{\text{in}}^{\beta} + \Omega_{I}^{\beta} + \Omega_{\text{out}}^{\beta} + \Omega_{\text{out}}^{\xi'}.$$  (4.35)

Here top/bottom choice for the fiber product factors on the left/right corresponds to $\alpha$ or $\beta$ polarization on the left/right endpoint. Note that the polarizations of the endpoints affect the BFV differential, which is given by the edge term defined by (4.8, 4.9) plus the two endpoint-edge terms defined by (4.20), plus two pure endpoint terms defined by (4.19).

One can also assign a space of states to a corner $p$ in $\alpha$- or $\beta$-polarization as follows:

$$H_{p}^{\alpha} = \text{Fun}_{C}(g[1]) = C \otimes \wedge^* g, \quad \Omega_{p}^{\alpha} = \frac{i\hbar}{2} \left\langle [\alpha, \alpha], \frac{\partial}{\partial \alpha} \right\rangle,$$

$$H_{p}^{\beta} = \text{Fun}_{C}(g^*) = C \otimes S^* g, \quad \Omega_{p}^{\beta} = 0.$$.  (4.36)

Note that, as a cochain complex, $H_{I}^{\alpha,\alpha}$ is quasi-isomorphic to $H_{p}^{\alpha}$ – the Chevalley-Eilenberg complex of the Lie algebra $g$. Geometrically, this corresponds to the collapse of an $A$-interval with endpoints in $\alpha$-polarization into a single $\alpha$-point. Likewise, the cochain complex $H_{I}^{\beta,\beta}$ is quasi-isomorphic to $H_{p}^{\beta}$:

$$H_{I}^{\alpha,\alpha} \sim H_{p}^{\alpha}, \quad H_{I}^{\beta,\beta} \sim H_{p}^{\beta}.$$  (4.37)

One can regard $H_{p}^{\alpha}$ and $H_{p}^{\beta}$ as differential graded algebras. The algebra structure on $H_{p}^{\alpha}$ is the standard supercommutative multiplication in the exterior algebra, while for $H_{p}^{\beta}$ we need
to deform the naïve commutative product in the symmetric algebra into a star-product $\ast_h$ —
the deformation quantization of the Kirillov-Kostant-Souriau Poisson structure on $\mathfrak{g}^*$, as we
explain below.

One can regard the space of states for the interval as a bimodule over the spaces of
states associated to the end-points. The action of the end-point algebra $\mathcal{H}_p^\beta$ on the space
of states $\mathcal{H}_I^{\beta,\mathbb{A},\mathbb{K}}$ for the edge is via multiplication in the algebra, e.g. $\psi(\alpha) \otimes \Psi(\alpha, \mathbb{P}, \mathbb{K}') \mapsto \psi(\alpha)\Psi(\alpha, \mathbb{P}, \mathbb{K}')$, $\psi(\beta) \otimes \Psi(\beta, \mathbb{P}, \mathbb{K}') \mapsto \psi(\beta) \ast_h \Psi(\beta, \mathbb{P}, \mathbb{K}')$. The reason we need to deform the product in $\mathcal{H}^\beta$ from the commutative one is that we want the edge to give a differential graded bimodule over the corner spaces. In particular, the module structure map $\mathcal{H}_p^\beta \otimes \mathcal{H}_I^{\beta,\mathbb{A},\mathbb{K}} \rightarrow \mathcal{H}_I^{\beta,\mathbb{A},\mathbb{K}}$ should be a chain map with respect to the differential $\Omega_p^{\beta,\mathbb{K}} + \Omega_I^{\beta,\mathbb{A},\mathbb{K}}$. This requirement is incompatible with the commutative product on $\mathcal{H}_p^\beta$ and forces the following associative non-commutative deformation $\ast_h : \mathcal{H}_p^\beta \otimes \mathcal{H}^\beta \rightarrow \mathcal{H}^\beta$.

**Proposition 4.9.** The associative product structure $\ast_h$ on $\mathcal{H}^\beta$ is fixed uniquely by the two properties:

(i) The module structure map $m : \mathcal{H}_p^\beta \otimes \mathcal{H}_I^{\beta,\mathbb{A},\mathbb{K}} \rightarrow \mathcal{H}_I^{\beta,\mathbb{A},\mathbb{K}}$, obtained by extending $\ast_h$ by
linearity in the second factor, is a chain map.

(ii) $\ast_h$ is unital with $\psi(\beta) = 1$ the unit.

The product $\ast_h$ is explicitly described as follows:

$$e^{-\frac{i}{\hbar}\langle \beta, x \rangle} \ast_h e^{-\frac{i}{\hbar}\langle \beta, y \rangle} = e^{-\frac{i}{\hbar}\langle \beta, \text{BCH}(x,y) \rangle} \tag{4.38}$$

Here $x, y \in \mathfrak{g}$ are arbitrary parameters in the Lie algebra and $\text{BCH}(x,y) = \log(e^x e^y)$ is the Baker-Campbell-Hausdorff group law.

**Proof.** Let us check that the star-product (4.38) does indeed make the module structure map
a chain map. Note that, for $\psi(\beta) = e^{-\frac{i}{\hbar}\langle \beta, x \rangle}$, the action of $\Omega_p^{\beta,\mathbb{K}}$ on $\psi$ can be written as

$$\Omega_p^{\beta,\mathbb{K}} \psi = \left( \langle \beta, F_+(\text{ad}_x) \mathbb{A}_p \rangle \psi \right) = \frac{\hbar}{i} \frac{d}{d\epsilon} \bigg| _{\epsilon = 0} e^{\frac{i}{\hbar} \langle \beta, \text{BCH}(x,\epsilon \mathbb{A}_p) \rangle} = \frac{\hbar}{i} \frac{d}{d\epsilon} \bigg| _{\epsilon = 0} \psi \ast_h e^{-\frac{i}{\hbar}\langle \beta, \epsilon \mathbb{A}_p \rangle} \tag{4.39}$$

with $\epsilon$ an odd, ghost degree $-1$ parameter and $\ast_h$ defined by (4.38). Here we have used the identity $\text{BCH}(x,y) = x + F_+(\text{ad}_x)y + \mathcal{O}(y^2)$. This implies that for any $\Psi \in \mathcal{H}_I^{\beta,\mathbb{A},\mathbb{K}}$ we have:

$$\Omega_p^{\beta,\mathbb{K}} \Psi = \frac{\hbar}{i} \frac{d}{d\epsilon} \bigg| _{\epsilon = 0} \Psi \ast_h e^{-\frac{i}{\hbar}\langle \beta, \epsilon \mathbb{A}_p \rangle} \tag{4.40}.$$  

Therefore, for any $\tilde{\psi} \in \mathcal{H}_p^\beta$ we have:

$$m \circ (\text{id} \otimes \Omega^{\beta,\mathbb{K}})(\tilde{\psi} \otimes \Psi) = \frac{\hbar}{i} \frac{d}{d\epsilon} \bigg| _{\epsilon = 0} \left( \tilde{\psi} \ast_h \left( \Psi \ast_h e^{-\frac{i}{\hbar}\langle \beta, \epsilon \mathbb{A}_p \rangle} \right) \right) \tag{4.41}$$

$$\quad = \frac{\hbar}{i} \frac{d}{d\epsilon} \bigg| _{\epsilon = 0} \left( \tilde{\psi} \ast_h \Psi \ast_h e^{-\frac{i}{\hbar}\langle \beta, \epsilon \mathbb{A}_p \rangle} \right) = \Omega_p^{\beta,\mathbb{K}} \circ m(\tilde{\psi} \otimes \Psi).$$

46
Here we used the associativity of the star-product (4.38). Note that the other pieces of the differential, $\Omega^A_{\mathcal{I}}$ and $\Omega_{\mathcal{I}'p}$, clearly commute with the module structure map $m$. Thus we have proven that $m$, defined by (4.38) and extended by Fun($\mathcal{A},\xi'$)-linearity in the second factor, is indeed a chain map.

Moreover, assume that $\bullet$ is some unital associative product on $H^\beta$ with $\psi(\beta) = 1$ the unit. Then the argument above shows that the module structure map $m$ defined using $\bullet$ is a chain map if and only if $\psi_1 \bullet (\psi_2 \ast_h \psi_3) = (\psi_1 \bullet \psi_2) \ast_h \psi_3$ for any $\psi_{1,2,3} \in H^\beta$. Choosing $\psi_2 = 1$, we obtain $\psi_1 \bullet \psi_3 = \psi_1 \ast_h \psi_3$. This proves uniqueness of the star-product (4.38).

Gluing two intervals over a point corresponds to taking the tensor product of the spaces of states for the intervals over the algebra associated to the point:

$$\mathcal{H}_{I_1}^{\xi_1} \otimes \mathcal{H}_{I_2}^{\xi_2}.$$  (4.42)

The space of states (4.16) for the stratified circle can then be written, in terms of the spaces of states for intervals and corners introduced above, as:

$$\mathcal{H} = \left( \mathcal{H}_{I_1}^{\xi_1,p_1} \otimes \mathcal{H}_{I_2}^{\xi_2,p_2} \cdots \otimes \mathcal{H}_{I_{n-1}}^{\xi_{n-1},p_{n-1}} \right) \otimes \mathcal{H}_{p_n}^\ast \otimes \left( \mathcal{H}_{I_n}^{\xi_n} \right)^{op} \mathcal{H}_{p_n}^\ast.  \quad (4.43)$$

Here the superscript op stands for the opposite algebra.

**Remark 4.10.** Denote $\mathcal{I} = \bigcup_{k=1}^l I_k$ be the union of $l$ consecutive intervals on the stratified circle (4.15) and $\mathcal{J} = \bigcup_{k=l+1}^{n} I_k$ the union of the remaining intervals, and let $p = p_0$, $q = p_l$ be the points separating $\mathcal{I}$ and $\mathcal{J}$. The globalized partition function $Z$ for the disk $D$ filling the stratified circle is, by the mQME, an $\Omega$-closed element of the space of states

$$\mathcal{H}_{S^1} = \mathcal{H}_{\mathcal{I}} \otimes_{\mathcal{H}_{q}} \mathcal{H}_{\mathcal{J}} \otimes_{\mathcal{H}_{p}} \mathcal{H}_{p} \cong \text{Hom}(\mathcal{H}_{q},\mathcal{H}_{p})_{\text{bimod}}(\mathcal{H}_{\mathcal{I}},\mathcal{H}_{\mathcal{J}}). \quad (4.44)$$

Here on the right hand side we have the space of morphisms of dg bimodules over $\mathcal{H}_q$ on the left and $\mathcal{H}_p$ on the right; bar on $\mathcal{I}$ stands for orientation reversal. Thus, the partition function for a disk can be seen as a bimodule morphism between two bimodules associated to the two arcs constituting the boundary.

Note that the picture for 2D Yang-Mills we just described, mapping points to algebras, intervals to bimodules and disks to morphisms of bimodules, is in agreement with Baez-Dolan-Lurie setting of extended topological quantum field theory [1, 19], with the correction

$\overset{31}{\text{Note that, in dg setting, when taking the tensor product } M_1 \otimes_A M_2 \text{ of a right } A\text{-module } M_1 \text{ and a left } A\text{-module } M_2 \text{ over a dg algebra } A, \text{ the total differential is the sum of the differentials on } M_1, M_2 \text{ minus the differential on } A.}$
that our spaces of states depend on the choice of polarization and partition functions depend
on the area of the surface (and pre-globalization partition functions additionally depend on
residual fields).

Towards quantization of codimension 2 corners in more general BV-BFV theories

The algebra $H^\beta \star \hbar$ is isomorphic to $U_\hbar(g)$ – the enveloping algebra of $g$ with the normalized
Lie bracket $i\hbar [-, -]$. This algebra arises as Kontsevich’s deformation quantization [6, 17] of
the algebra of functions on $g^*$ equipped with the Kirillov-Kostant-Souriau linear Poisson
structure. This observation fits well into the following expected picture of quantization of
corners of codimension 2.

In a general gauge theory, a codimension 2 stratum $\gamma$ is classically associated a BFV “cor-
ner phase space” [10] $\Phi_\gamma$ equipped with a degree +1 symplectic form $\omega_\gamma$ and a BFV charge $S_\gamma$
of degree +2. On the level of quantization, we impose a polarization $\Phi_\gamma \simeq T^*\mathcal{B}_\gamma$. The
BFV charge $S_\gamma$ generates a $P_\infty$ (Poisson up-to-homotopy) algebra structure on $C^\infty(\mathcal{B}_\gamma)$,
coming from interpreting $S_\gamma$ as a self-commuting polyvector $\Pi$ on $\mathcal{B}_\gamma$. Then the quantum
space of states $H_\gamma$ is expected to be the $A_\infty$ algebra obtained as Kontsevich’s deformation
quantization of the $P_\infty$ algebra $C^\infty(\mathcal{B}_\gamma)$. In particular, the $A_\infty$ structure maps arise from
Feynman diagrams on a thickening of $\gamma$ to $\gamma \times D$, with $D$ a 2-disk, for a field theory coming
from the AKSZ construction on the mapping space $\text{Map}(T[1]D, \Phi_\gamma)$. We plan to revisit this
construction in more detail in a future paper on corners on BV-BFV formalism.

Note that, in the case of 2D Yang-Mills theory, the corner phase space is $\Phi_\rho = g[1] \oplus g^*$, with
$\omega_\rho = \langle d\beta, d\alpha \rangle$, $S_\rho = \frac{1}{2}\langle \beta, [\alpha, \alpha] \rangle$. Deformation quantization of $C^\infty(g^*)$ with Poisson
bivector $\Pi = \frac{1}{2}\langle \beta, [\frac{\partial}{\partial \beta}, \frac{\partial}{\partial \beta}] \rangle$ yields the algebra $H^\beta$. Taking the opposite polarization, one
gets the deformation quantization of $C^\infty(g[1]) = \wedge g^*$ with 1-vector $\Pi = \frac{1}{2}\langle [\alpha, \alpha], \frac{\partial}{\partial \alpha} \rangle$, which
is the dg algebra $H^\alpha$.

In general, one expects all the structure maps on (and between) the spaces of states
associated to various strata to come from Feynman diagrams.

A related picture was obtained in [7] in the context of Poisson sigma model on a disk
with intervals on the boundary decorated with coisotropic submanifolds $C_i$ of the Poisson
target $M$. In this setting the quantization yields algebras assigned to intervals (deformation
quantization of the rings of functions on $C_i$) and bimodules assigned to the corners separating
the intervals. In particular, the algebra $H^\beta$ arises in this context as a quantization of the
space-filling coisotropic in $M = g^*$. This picture can be thought of as Poincaré dual to our
picture on the boundary of a disk.

---

32 Equivalently, the $P_\infty$ structure arises from $S_\gamma$ via the derived bracket construction, $\{\psi_1, \ldots, \psi_n\}_\Pi :=$ $(\cdots (S_\gamma, \psi_1), \ldots, \psi_n)$ with $\psi_1, \ldots, \psi_n \in C^\infty(\mathcal{B}_\gamma)$. The brackets $(-, -)$ on the r.h.s. are the Poisson brackets
on functions on the phase space $\Phi_\gamma$ defined by $\omega_\gamma$. 48
4.2.4 Gluing regions along an interval and the Fourier transform property of BFV differentials

Recall that the BFV differentials for an $A$-circle and a $B$-circle are related by Fourier transform. This property in particular implies that mQME is compatible with gluing: if $\Sigma = \Sigma_1 \cup_s \Sigma_2$ a union of surfaces over a circle and if partition functions $Z_{\Sigma_1}, Z_{\Sigma_2}$ are known to satisfy mQME, then the glued partition function $Z_\Sigma = \langle Z_{\Sigma_1}, Z_{\Sigma_2} \rangle_{H_s}$ automatically satisfies mQME on the glued surface.

One has an analogous property in the setting with corners. Consider e.g. an $A$-interval $I$ parameterized by $t \in [0, 1]$ with endpoints in polarizations $\xi, \xi'$ and consider a $B$-interval $\tilde{I}$ parameterized by $\tilde{t} \in [0, 1]$, with endpoints in $\xi', \xi$. Let $r: I \to \tilde{I}$ be an orientation-reversing diffeomorphism $t \mapsto \tilde{t} = 1 - t$. Gluing along $r$ corresponds to the following pairing of states on $I$ and $\tilde{I}$:

\[
\langle -, - \rangle_I: H_{\xi, A, \xi'}^I \otimes H_{\xi', B, \xi}^\tilde{I} \to \mathbb{C}
\]

\[
\psi_1 \otimes \psi_2 \mapsto \int DA DB \psi_1(\xi, A, \xi') \cdot e^{-\frac{i}{\hbar} \int_I \langle r^* B, A \rangle} \cdot \psi_2(\xi', B, \xi)
\]

One easily verifies the following:

\[
\langle (\Omega_{A, i}^{\xi} + \Omega_{A, o}^{\xi'}) \psi_1, \psi_2 \rangle_I = -\langle \psi_1, (\Omega_{A, o}^{\xi} + \Omega_{A, i}^{\xi'}) \psi_2 \rangle_I.
\]

Here we are making the Assumption 4.4 on states $\psi_1, \psi_2$. In other words, the operators $\Omega_{A, i}^{\xi} + \Omega_{A, o}^{\xi'}$ and $\Omega_{A, o}^{\xi} + \Omega_{A, i}^{\xi'}$ are, up to sign, the Fourier transform of each other (when acting on admissible states).

This immediately implies the following. Assume that $\Sigma$ is a result of gluing of surfaces $\Sigma_1$ and $\Sigma_2$ via attaching an interval $I \subset \partial \Sigma_1$ to $\tilde{I} \subset \partial \Sigma_2$ along the diffeomorphism $r$. Then for $\Psi_1 \in H_{\partial \Sigma_1}, \Psi_2 \in H_{\partial \Sigma_2}$ any two states on the boundary of $\Sigma_1, \Sigma_2$, we have

\[
\Omega_{\partial \Sigma} \langle \Psi_1, \Psi_2 \rangle_I = \langle \Omega_{\partial \Sigma_1} \Psi_1, \Psi_2 \rangle_I + \langle \Psi_1, \Omega_{\partial \Sigma_2} \Psi_2 \rangle_I,
\]

where $\langle \Psi_1, \Psi_2 \rangle_I \in H_{\partial \Sigma}$ is understood as the “gluing” of states $\Psi_1, \Psi_2$ along $I$.

In particular, if the partition functions on $\Sigma_1, \Sigma_2$ are known to satisfy the mQME, the glued partition function $Z_\Sigma = \langle Z_{\Sigma_1}, Z_{\Sigma_2} \rangle_I$ automatically satisfies the mQME on $\Sigma$. 
4.2.5 Small model for states on an $A$-interval

In preparation for the calculations of section 4.5, we want to present a “small model” for the space of states on an $A$-interval, corresponding to the passage to a constant 1-form field $A^{(1)}$ on the interval. This is an extension of the discussion of section 3.3 (and in particular, formula (3.18)), and of section 2.4.1.

Consider a single interval in $A$-polarization:

$$\begin{array}{c}
\mathcal{A}_0 \quad \mathcal{A} \quad \mathcal{A}_1 \\
p_0 \quad I \quad p_1
\end{array}$$

We view its endpoints as corners in picture I (non-polarized), with $\mathcal{A}_0, \mathcal{A}_1$ the limiting values of the 0-form field $\mathcal{A}$ at the endpoints $p_0, p_1$. Equivalently, we can treat the endpoint in picture II, putting $\alpha$-polarization on them, with corner fields $\alpha_0, \alpha_1$ identified with $\mathcal{A}_0, \mathcal{A}_1$.

The space of states for the interval (4.48) is a cochain complex

$$H = \text{Fun}_C(\Omega^*(I, g)[1]) \equiv \{ \Psi(\mathcal{A}) \}$$

with differential

$$\Omega = i\hbar \left( \int \left\langle d\mathcal{A}^{(0)} + [\mathcal{A}^{(0)}, \mathcal{A}^{(1)}], \frac{\delta}{\delta \mathcal{A}^{(1)}} \right\rangle + \int \left\langle \frac{1}{2} [\mathcal{A}^{(0)}, \mathcal{A}^{(0)}], \frac{\delta}{\delta \mathcal{A}^{(0)}} \right\rangle \right).$$

One has the following “small” quasi-isomorphic model for the space of states – the cochain complex

$$H^{\text{small}} = \text{Fun}_C(C^*(I, g)[1]) \equiv \{ \Psi(\mathcal{A}_0, \mathcal{A}, \mathcal{A}_1) \}.$$ 

Here $C^*(I, g) = g \oplus g[-1] \oplus g$ is the complex of $g$-valued cellular cochains on the interval $I$ endowed with the standard CW complex structure, with two 0-cells $p_0, p_1$ and a single 1-cell $I$. Variables $\mathcal{A}_0, \mathcal{A}_1 \in g[1]$ are the values of the cochain on the 0-cells of the cochain on the 0-cells $p_0$ and $p_1$ (endpoints), respectively, and $\mathcal{A} \in g$ is the value of the cochain on the 1-cell $I$ itself. The differential on $H^{\text{small}}$ is given by:

$$\Omega^{\text{small}} = i\hbar \left( \left\langle \frac{1}{2} [\mathcal{A}_0, \mathcal{A}_0], \frac{\partial}{\partial \mathcal{A}_0} \right\rangle + \left\langle \frac{1}{2} [\mathcal{A}_1, \mathcal{A}_1], \frac{\partial}{\partial \mathcal{A}_1} \right\rangle + \right.$$

$$\left. - \left\langle F_- (\text{ad}_\mathcal{A}) \circ \mathcal{A}_0 + F_+ (\text{ad}_\mathcal{A}) \circ \mathcal{A}_1, \frac{\partial}{\partial \mathcal{A}} \right\rangle \right).$$

The chain projection $p_H: H \to H^{\text{small}}$ is the following map:

$$\Psi \mapsto \left( \Psi: \{ \mathcal{A}_0, \mathcal{A}, \mathcal{A}_1 \} \mapsto \Psi \left( \mathcal{A}^{(0)} = G_- (t, \text{ad}_\mathcal{A}) \circ \mathcal{A}_0 + G_+ (t, \text{ad}_\mathcal{A}) \circ \mathcal{A}_1, \mathcal{A}^{(1)} = dt \cdot \mathcal{A} \right) \right).$$

Here we parameterize the interval by the coordinate $t \in [0, 1]$ and $G_\pm$ are the generating functions for Bernoulli polynomials (4.22).

The chain inclusion $i_H: H^{\text{small}} \to H$ is given as follows:

$$i_H: \Psi \mapsto \left( \Psi: \mathcal{A} \mapsto \Psi \left( \mathcal{A}_0 = \mathcal{A}_0, \mathcal{A}_1 = \mathcal{A}_1, \mathcal{A} = \log U(\mathcal{A}) \right) \right).$$
Here the group element $U(\cdots) \in G$ is the holonomy of the connection 1-form along the interval $I$.

**Remark 4.11.** The space of states $\mathcal{H}, \Omega$ is the Chevalley-Eilenberg complex (or the dual of the bar complex) of the differential graded Lie algebra of $g$-valued differential forms on the interval, $\Omega^*(I, g), d, [-, -]$. Likewise, $\mathcal{H}^{\text{small}}, \Omega^{\text{small}}$ is the Chevalley-Eilenberg complex for the $L_\infty$ algebra structure on $g$-valued cellular cochains on an interval, constructed in [21, 22, see also [18, 25]]. This $L_\infty$ algebra arises as the homotopy transfer of the “big” algebra $\Omega^*(I, g)$ onto the deformation retract $C^*(I, g)$ – cochains, realized as Whitney forms on the interval. Chain map (4.53) corresponds to the $L_\infty$ morphism from $C^*(I, g)$ to $\Omega^*(I, g)$ constructed explicitly in [21, 22, – Statement 14]; it is a non-abelian deformation of the inclusion of cochains as Whitney forms. The map (4.54), constructed via holonomies, corresponds to the $L_\infty$ morphism from forms to cochains - the non-abelian version of the integration-over-cells map, cf. [2]. We give a proof of the chain map property of (4.54) in Appendix E.

One has similar small models for the space of states on the $A$-interval with endpoints in any combination of polarizations $\xi_0, \xi_1$. E.g. for both endpoints in $\beta$-polarization, we have the small model (4.51,4.52) and the maps (4.53,4.54), where we adjoin the corner variables $\beta_0, \beta_1$ on which the wavefunctions $\Psi, \bar{\Psi}$ are allowed to depend, and we add corner-edge terms $\Omega^{\beta \beta}, \Omega^{\beta \beta}$ (4.20) to the respective differentials $\Omega$ and $\Omega^{\text{small}}$.

Finally, consider a surface $\Sigma$ with stratified boundary circles decorated with an arbitrary combination of polarizations of arcs and corners. By the discussion above, we have a small quasi-isomorphic model $\mathcal{H}'$ for the space of states $\mathcal{H}$ corresponding to replacing the states on some (or all) $A$-arcs with respective small models for $A$-arcs in the formula (4.43), and we have chain maps $p_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}'$, $i_{\mathcal{H}} : \mathcal{H}' \to \mathcal{H}$. They correspond to a quasi-isomorphism of complexes and thus there exists a chain homotopy $K_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ between the identity and the projection $i_{\mathcal{H}} \circ p_{\mathcal{H}}$. Therefore, we can apply the argument (3.18) to the partition function $Z$ on $\Sigma$:

$$i_{\mathcal{H}} \circ p_{\mathcal{H}} Z = Z + (\Omega + h^2 \Delta)(\cdots). \tag{4.55}$$

In particular, one can recover $Z$, modulo BV exact terms, by evaluating it on constant 1-forms $dt : \mathbb{A}_k$ on the boundary arcs, provided that their holonomy coincides with the holonomy of the original $A^{(1)}$ field along the respective intervals, i.e. $\mathbb{A}_k = \log U_{I_k}(A)$.  

51
4.3 BF $\mathbb{B}$-disk with two $\alpha$ corners

Let us consider now the case of a BF disk with the boundary split into two arcs $\gamma_i: [0, 1] \to S^1$, $i = 1, 2$, with $\gamma_i(0) = v_0$ and $\gamma_i(1) = v_1$, both in $\mathbb{B}$ polarization. On both vertices of the arcs we fix the value $\alpha_i$ for the restriction of the bulk $A$ fields. Expanding the vertices $v_i$ into two edges in $\tilde{A}$ polarization we can think of this disk as a square (figure 13).

The square can be viewed as the product of two intervals, with $A$ or $\tilde{B}$ polarization on both endpoints respectively. The zero-modes now contain 1-form components for the $a$ and $b$ fields:

$$a = a_1 d\tau, \quad b = b_1 dt.$$  

A possible choice for axial-gauge propagator is (cf. appendix (B.1)):

$$\eta(t, \tau; t', \tau') = (\Theta(\tau - \tau') - \tau')dt' - (\Theta(t' - t) - t')\delta(\tau' - \tau)(d\tau' - d\tau). \quad (4.56)$$

The contributing Feynman diagrams to the effective action are wheels with $n$ $a$ zero-modes and trees, rooted either on the $\mathbb{B}(1)$ boundary field or on the $b$ zero-mode and ending on one $\tilde{A}(0)$ boundary field, with no bifurcations and the insertion of $n$ leaves decorated with $a$ zero-modes (Figure 14).

**Figure 13.** $\mathbb{B}$ disk with the boundary split into two arcs separated by points in $A$-polarization.

**Figure 14.** Examples of the tree and 1-loop Feynman diagrams contributing to the effective action for the BF disk in $\mathbb{B}$ polarization with two $\alpha$ corners.

**Proposition 4.12.** The partition function for the BF disk in $\mathbb{B}$ polarization with two $\alpha$ cor-
The effective action of the BF model on a disk with one \( \alpha \) corner is:

\[
Z = \exp \left( \frac{i}{\hbar} \int_\gamma \langle \mathbb{B}, a + G_+(\tau, \text{ad}_a)\alpha + G_-(\tau, \text{ad}_a)\bar{\alpha} \rangle + \frac{i}{\hbar} \int_{\tilde{\gamma}} \langle \tilde{\mathbb{B}}, a + G_+(\tau, \text{ad}_a)\alpha + G_-(\tau, \text{ad}_a)\bar{\alpha} \rangle + \frac{i}{\hbar} \langle \mathbf{b}, F_+(\text{ad}_a)\alpha + F_-(\text{ad}_a)\bar{\alpha} \rangle \right) \det \left( \frac{\sinh (\text{ad}_a / 2)}{\text{ad}_a / 2} \right) \cdot \rho_V ,
\]

where \( \rho_V = D^1 a \) is the reference half-density on zero-modes.

**Proof.** See Appendix C.

### 4.4 BF \( \mathbb{B} \)-disk with one \( \alpha \) corner

Let us consider a disk in \( \mathbb{B} \)-polarization with a single corner in \( \alpha \)-polarization. We will denote by \( \alpha \) the value of the zero-form component of the \( A \)-fields on the corner. Notice that the space of zero-modes is now empty, in contrast to the \( \mathbb{B} \)-disk without corners or with two \( \alpha \) corners.

![Figure 15. \( \mathbb{B} \) disk with one \( \alpha \) corner as the “collapse” of three edges in a square.](image)

The corner can be expanded to an \( \mathbb{A} \)-polarized edge with \( \mathbb{A} = \alpha \), which can be then split in three consecutive edges. We then get a square, which is the product of an \( \mathbb{A}-\mathbb{A} \) interval times an \( \mathbb{A}-\mathbb{B} \) interval (figure 15). We can thus choose the axial gauge propagator to compute the effective action. If we denote with \( t \) the coordinate on the \( \mathbb{A}-\mathbb{A} \) interval and with \( \tau \) the coordinate on the \( \mathbb{A}-\mathbb{B} \) interval we have:

\[
\eta(t, \tau; t', \tau') = -\Theta(\tau' - \tau)\delta(t' - t)(dt' - dt) .
\]

Since there are no zero-modes and the boundary \( \mathbb{A} \)-field has only the zero-form component \( \alpha \), from degree counting we get that the only non-vanishing diagrams contributing to the partition function are the ones containing no interaction vertices:

\[
Z[\mathbb{B}, \alpha] = e^{-\frac{i}{\hbar} \int_{\gamma} \mathcal{L}[\mathbb{B}, \alpha]} .
\]

**Remark 4.13.** If we compare this effective action with the one of the \( \mathbb{B} \)-disk without corners (3.26) we notice that the corner field \( \alpha \) plays here the role of the \( a \) zero-mode (the other term for action (3.26), containing only the zero-modes, is vanishing when restricted to the
globalizing Lagrangian $\mathcal{L} = \{b = 0\}$). Thus, integrating over the fields on the corner reproduces the globalized effective action for the $\mathbb{B}$-disk without corners.

We can also compare (4.59) with the partition function of the $\mathbb{B}$-disk with two corners computed in (4.57). We recover the partition function for the disk with one corner globalizing (4.57) over $\mathcal{L} = \{a = 0\}$ and then integrating out one corner field $\alpha$.

4.5 BF $\mathbb{A}$-disk with one $\beta$-corner

In order to calculate the one remaining building block of the theory, the partition function for an $\mathbb{A}$-disk with a single $\beta$-corner, we do the following. We first consider a disk $D$ with boundary split into two intervals in $\mathbb{A}$ and $\mathbb{B}$-polarization with the two corners not decorated by polarization data (i.e. in the setting of the “picture I” for corners, cf. subsection 4.2).\[33\]

\[
\begin{array}{c}
\xymatrix{ \mathbb{B} & \mathbb{A} \\
p' \ar[rr]_{r} & & p \\
} \\
\end{array}
\]

The partition function is easily computed by expanding the corners into two intervals (with arbitrary polarization) and putting the axial gauge on the square.\[34\] This yields the answer

\[
Z(\mathbb{A}, \mathbb{B}) = e^{\frac{i}{\hbar} \int_{\partial A} (r^\dagger)^{\mathbb{B}, \mathbb{A}}},
\]

where $r$ is an orientation-reversing involution on the boundary of the disk, mapping the $\mathbb{A}$-arc diffeomorphically onto the $\mathbb{B}$-arc, having the two corners as fixed points (in terms of the square, $r$ is the involution $(t, 0) \leftrightarrow (t, 1)$). This partition function satisfies the mQME, $\Omega Z = 0$, with $\Omega = \Omega_{\partial A}^{\mathbb{A}} + \Omega_{\partial B}^{\mathbb{B}} + \langle \mathbb{B}_p, \mathbb{A}_p \rangle - \langle \mathbb{B}_{p'}, \mathbb{A}_{p'} \rangle$, as per Proposition 4.2, and as one can easily check explicitly.

Remark 4.14. One can consider collapsing the $\mathbb{A}$- or $\mathbb{B}$-arc on the boundary of the disk (4.60):

\[33\] In fact, we can decorate the two corners with an arbitrary choice of polarizations $\xi, \xi'$. The partition function does not depend on this choice.

\[34\] Here we use the axial gauge with the propagator $\eta(t, \tau; t', \tau') = \delta(t' - t)(dt' - dt) \Theta(\tau' - \tau)$.
• Collapsing the $A$-arc into an $\alpha$-corner, we obtain a $B$-disk with a single $\alpha$-corner (in the picture II). Moreover, evaluating the partition function (4.61) on $B = \beta$, we obtain the partition function $Z(B, \alpha) = e^{\frac{i}{\hbar} \int_{\partial D} \langle B, \alpha \rangle}$, which agrees with our result (4.59) from Section 4.4 and, indeed, satisfies the mQME with $\Omega = \Omega_{\partial D}^B - \langle B \rvert^{+0}_{-0}, \alpha \rangle + i\hbar \langle \frac{1}{2} [\alpha, \alpha], \frac{\partial}{\partial \alpha} \rangle$. Here $B \rvert^{+0}_{-0}$ is the jump of the field $B$ when passing through the $\alpha$-corner in positive direction.

• Collapsing the $B$-arc of the disk (4.60) into a $\beta$-corner, we obtain a $B$-disk with a single $\beta$-corner. However, evaluating the partition function (4.61) on $B = \beta$ yields $e^{\frac{i}{\hbar} \int_{\partial D} \langle \beta, A \rangle}$ which does not satisfy the mQME! The reason for this is that the gauge-fixing on the disk (4.60) which was used to compute the partition function (4.61), which in turn came from the axial gauge on a square, is not “collapsible”, i.e. fails Assumption 4.6, and therefore Proposition 4.7 does not apply and we obtained a nonsensical answer after the collapse of the $B$-arc.

Using the construction of Section 4.2.5, we can consider the projection $p_H$ to the “small model” for the states on the $A$-arc followed by respective inclusion $i_H$, cf. (4.53,4.54). Thus we obtain a version of the partition function, factored through the small model for $A$-states:

$$Z(A, B) = i_H \circ p_H Z$$
$$= e^{\frac{i}{\hbar} \int_{\partial D} \langle r^* B(0), \text{d} t \log U(A) \rvert^{+0}_{-0}, \alpha \rangle + \langle r^* B(1), G_-(t, \text{ad}_{\log U(A)}) \rangle + G_+(t, \text{ad}_{\log U(A)}) \rangle \cdot}$$

(4.62)

Note that, by (4.55), $\tilde{Z} = Z + \Omega(\cdots)$ – a modification of the answer (4.61) by an $\Omega$-exact term; this deformation can be interpreted as corresponding to a computation in a different gauge.\(^{35}\) Also, observe that in (4.62), the field $B(1)$ only interacts with the corner values of $A(0)$, and thus the gauge corresponding to the answer (4.62) is “collapsible”, i.e., satisfies the Assumption 4.6. Therefore, we can collapse the $B$-arc into a $\beta$-corner, as in Section 4.2.2, by setting $B(0) = \beta$, $B(1) = 0$ in (4.62). Thus we finally arrive to the following result.

**Proposition 4.15.** The partition function for an $A$-disk with a single $\beta$-corner is:

$$Z(A, \beta) = e^{\frac{i}{\hbar} \langle \beta, \log U(A) \rangle}.$$  

(4.63)

Note that this answer has a rigidity property: it cannot be changed by a BV-exact term $\Omega(\cdots)$ for a degree reason – there no boundary/corner fields of negative degree needed to construct a degree $-1$ primitive. The answer (4.63) does indeed satisfy the mQME, i.e. is $\Omega$-closed, as we have verified explicitly in Remark 4.8 above.

### 4.6 Gluing arcs in $A$ polarization

We want now to recover the YM gluing law of two arcs in $A$ polarization. To compute this gluing law we can use an intermediate BF disk with the boundary split in two arcs with $B$
polarization, separated by points in $\alpha$ polarization (figure 16). Thus, gluing together two $A$-arcs with endpoints in $\alpha$-polarization, via the “bean” (4.57), for the partition function of the glued surface we obtain the following:

$$Z_\Sigma = \int dA \tilde{B} d\tilde{A} dA_1 e^{\frac{i}{\hbar} \int_\gamma B_\alpha a_\beta - B_\alpha (\tau, ad_{a_\alpha}) \alpha_0 + G_+ (\tau, ad_{a_\alpha}) \alpha_1} d\frac{\sinh (ad_{a_\alpha}/2)}{ad_{a_\alpha}/2} Z_{\Sigma_1} [A] Z_{\Sigma_2} [\tilde{A}]$$

$$= \int_{a_1 \in B_0} da_1 \det \left( \frac{\sinh (ad_{a_\alpha}/2)}{ad_{a_\alpha}/2} \right) Z_{\Sigma_1} [A = a + G_- (\tau, ad_{a_\alpha}) \alpha_0 + G_+ (\tau, ad_{a_\alpha}) \alpha_1] \cdot Z_{\Sigma_2} [\tilde{A} = a + G_- (\tau, ad_{a_\alpha}) \alpha_0 + G_+ (\tau, ad_{a_\alpha}) \alpha_1] .$$

(4.64)

Here the integration domain for the zero-mode $a_1$ is the “Gribov region” $B_0 \subset g$. Notice that in this gluing formula the states on the $A$-arcs factor through the “small model” for the space of states introduced in Section 4.2.5.

If we assume also that the all boundary strata of $\Sigma_1$, $\Sigma_2$ are in $A$-polarization and that partition functions $Z_{\Sigma_1}$, $Z_{\Sigma_2}$ are globalized, then the partition functions of $\Sigma_1$, $\Sigma_2$ does not depend on the ghost fields $A_0(0), \tilde{A}_0(0)$ and so the gluing formula reduces to

$$Z_\Sigma = \int_G dU(A) Z_{\Sigma_1} [U(A)] Z_{\Sigma_2} [U(A)] ,$$

(4.65)

which coincides with the gluing formula for YM known in literature [20, 24].

Figure 16. Two $A$-polarized boundaries glued together using an intermediate $B$-$B$ $BF$ disk.

---

Independence on the ghosts can be seen by assembling the surface with $A$-boundary by gluing $A$-polygons using beans as above. Partition functions for polygons do not depend on the ghosts and the gluing formula (4.64) does not generate ghost dependence. A curious point is that the answer for $A$-$A$ cylinder in Section 3.3 did contain ghost dependence which seems to contradict what we are saying here. In fact, there is no contradiction, rather there are inequivalent gauge-fixings: one can obtain an $A$-$A$ cylinder from an $A$-square, gluing two opposite sides using the bean (4.57). Choosing the gauge-fixing for the globalization on the bean as in (4.64) – integrating over $a_1$ – we get the answer for the cylinder without the ghost delta-function. If instead we use the opposite globalization on the bean – integrating over $b^1$ – we obtain the answer of Section 3.3, involving the ghost delta-function.
4.7 2D YM partition function on surfaces with boundaries

We can now compute the partition function on a general surface with boundaries. Indeed, any surface with boundary can be obtained by gluing edges of some polygon (or a collection of polygons – any triangulation or a cellular decomposition gives a presentation of the surface of this kind). Thus using the gluing properties of BV-BFV theories we can compute the YM partition function on a general surface with boundary starting from the partition function on the disk with the boundary split in several arcs \( \gamma_i \):

\[
Z_{D^2}[A_1, \ldots, A_n] = \sum \left( \text{dim } R \right) \chi_R(U_{\gamma_1}(A_1) \cdots U_{\gamma_n}(A_n)) e^{-\frac{i\hbar}{2} C_2(R)}. \tag{4.66}
\]

Each time we glue together two arcs, using the property (4.65), we have an integral of the kind:

\[
\begin{align*}
\int_G dU \chi_R(VUWU^\dagger) &= \frac{\chi_R(V)\chi_R(W)}{\text{dim } R}, \\
\int_G dU \chi_R(VU)\chi_R(WU^\dagger) &= \frac{\chi_R(VW)}{\text{dim } R}.
\end{align*} \tag{4.67}
\]

This way we get the following result.

**Theorem 4.16.** The globalized YM partition function on a surface with genus \( g \) and \( b \) boundaries in the \( A \) polarization is:

\[
Z_{\Sigma_{g,b}}[A_1, \ldots, A_b] = \sum \left( \text{dim } R \right)^{2g-b-e^{-\frac{i\hbar}{2} C_2(R)} \prod_{i=1}^b \chi_R(U_{b_i}(A_i))}. \tag{4.68}
\]

## A Wilson loop observables

Let us consider now observables in 2D YM. These are operators on the Hilbert space \( H_\Gamma \) associated to some boundary \( \Gamma \) which, in \( A \) polarization, is the space of functions of the holonomy \( U_\Gamma(A) \).

Let us consider for example the multiplication operator for the factor \( \chi_R(U_\Gamma(A)) \) for some representation \( R \). We can compute the matrix element of this operator between two states defined by the partition functions on two surfaces \( \Sigma_1 \) and \( \Sigma_2 \) with the same boundary \( \Gamma = S^1 \). Using the gluing rule (3.34) for boundaries in \( A \) polarization we get:

\[
\begin{align*}
\langle Z_{\Sigma_1} | \chi_R(U_\Gamma(A)) | Z_{\Sigma_2} \rangle &= \int_G dU \ Z_{\Sigma_1}(U^\dagger)\chi_R(U)Z_{\Sigma_2}(U) \\
&= \sum_{R_1, R_2} \left( \text{dim } R_1 \right)^{1-2g_1} \left( \text{dim } R_2 \right)^{1-2g_2} e^{-\frac{i\hbar}{2} C_2(R_1)-\frac{i\hbar}{2} C_2(R_2)} \int_G dU \ \chi_{R_1}(U^\dagger)\chi_R(U)\chi_{R_2}(U) \\
&= \sum_{R_1, R_2} \left( \text{dim } R_1 \right)^{1-2g_1} \left( \text{dim } R_2 \right)^{1-2g_2} e^{-\frac{i\hbar}{2} C_2(R_1)-\frac{i\hbar}{2} C_2(R_2)} N_{R_1, R_2}. \tag{A.1}
\end{align*}
\]
where we used the expression (4.68) for the partition functions of the surfaces $\Sigma_i$, with genus $g_i$, and where $N^{R_1}_{R,R_2}$ are the fusion numbers defined by the decomposition of the product of irreducible representations: $R \otimes R_2 = \oplus N^{R_1}_{R,R_2} R_1$. This quite obviously corresponds to the computation of the expectation value of a non self-intersecting Wilson loop $W_R(\Gamma)$ on the surface $\Sigma_1 \cup \Gamma \Sigma_2$.

More generally, we can consider operators going from some space of “inbound” states to some “outbound” states: $\mathcal{O}: \mathcal{H}_{in} \rightarrow \mathcal{H}_{out}$. Such an operator can be represented by a surface (possibly with corners) with the appropriate boundary components, i.e. such that the boundary Hilbert space is $\mathcal{H}^*_\text{in} \otimes \mathcal{H}_{\text{out}}$, and a particular state corresponding to $\mathcal{O}$. The operator now acts on the inbound states by gluing. For example to the (non self-intersecting) Wilson loop we computed above we can associate a cylinder in A-A polarization and the state $\chi_R(U(\mathcal{A}))(\mathcal{U}_{\mathcal{A}})$.

![Figure 17. Two intersecting Wilson loops $\Gamma$ and $\Gamma'$ on the 2-sphere.](image)

Consider now the case of two Wilson loops $W_R(\Gamma)W_{R'}(\Gamma')$ intersecting in 2 points: $\Gamma \cap \Gamma' = v_1 \cup v_2$. We can view them as 4 separate arcs $\gamma_i, \gamma'_i, i = 1, 2$, joining the two intersection points. These intersecting Wilson loops can be thought as a multiplication operator on the space of states of the 4 arcs – multiplication by the factor $\chi_R(U(\mathcal{A}))\delta(U(\mathcal{A}), U(\mathcal{A}'))$.

We can thus compute matrix elements between states defined by surfaces with opportune boundary components. Let us consider for example four disks, each with the boundary circle separated into two arcs, glued into a sphere with two intersecting Wilson loops as in figure 17:

$$\langle W_R(\Gamma)W_{R'}(\Gamma') \rangle_{S^2} = \sum_{R_1,R_2,R_3,R_4} e^{-\frac{i\pi}{2} \sum_i \alpha_i C_2(R_i)} \prod_i (\dim R_i) \int_G dU_1 dU_2 dU_3 dU_4 \chi_R(U_1U_3)$$

$$\cdot \chi_{R'}(U_2U_4)\chi_{R_1}(U_4U_1)\chi_{R_2}(U_1U_2)\chi_{R_3}(U_2U_3)\chi_{R_4}(U_3U_4).$$

This integral can be evaluated using the Peter-Weyl theorem (part 3) which implies:

$$\int_G dU \ R_1(U)_{ij}^{(R_1)} R_2(U)_{ij}^{(R_2)} R_3(U)_{ij}^{(R_3)} = \frac{1}{\dim R_3} \sum_{\mu} C_\mu(R_1, R_2; R_3)_{i}^{j} C^*_{\mu}(R_1, R_2; R_3)_{k}^{j'},$$

(A.3)
where \( C_\mu(R_1, R_2; R_3)_{ij}^{kl} \) are Clebsch-Gordan coefficients.\(^{37}\) We get:

\[
\langle W_T(R)W_{T'}(R') \rangle_{S^2} = \sum_{R_1, R_2, R_3, R_4} e^{-\frac{i \theta}{2} \sum_i a_i C_2(R_i)} \frac{\dim R_1}{\dim R_3} \sum_{\mu_1, \mu_2} C_{\mu \mu_1} (R, R_1; R_2)_{ij}^{kl} \cdot C_{\mu_2} (R_1', R_2', R_3')_{ij'}^{kl'} \cdot C_{\mu_3} (R, R_4; R_3)_{ij}^{kl} \cdot C_{\mu_4} (R_1, R_4; R_3)_{ij}^{kl} \cdot C_{\mu_1} (R, R_1; R_2)_{ij}^{kl} \cdot C_{\mu_2} (R_1', R_2', R_3')_{ij'}^{kl'} \cdot C_{\mu_3} (R, R_4; R_3)_{ij}^{kl} \cdot C_{\mu_4} (R_1, R_4; R_3)_{ij}^{kl} .
\]

where \( \{ R R' R_3 R_4 \}_{\mu_1 \mu_2} \) are Wigner 6-j symbols.\(^{38}\)

We can generalize this to compute the value of any number of (possibly intersecting) Wilson loops over any surface with boundary. Given a set of Wilson loops we can consider separately the various Wilson lines connecting intersection points.\(^{39}\) Each line carries a group variable and and contributes with the integral \((A.3)\), where \( R_1 \) is the representation of the Wilson loop containing that line and \( R_2, R_3 \) are the representations carried by the two regions adjacent to that line. The main observation is that this integral factorises into the product of two Clebsch-Gordan coefficients, each depending only on indices living on one of the two edges of the line.\(^{40}\) Thus when 4 lines meet at an intersection point, the factors associated to that intersection combine to give a 6-j symbol, as in the previous example:

\[
\begin{align*}
\begin{array}{c}
R_1 \quad R_2 \quad R_3 \quad R_4 \\
\mu_1 \quad \mu_2 \\
R' \quad \mu_3 \quad \mu_4
\end{array}
\end{align*}
\Rightarrow \begin{align*}
\{ R R_1 R_2 \}_{\mu_1 \mu_2}^{\mu_1 \mu_2} \\
\{ R' \mu_4 R_3 R_4 \}_{\mu_3 \mu_4}^{\mu_3 \mu_4}
\end{align*} = G(R, R'; \ldots).
\]

\(^{37}\) If we have representations \( R_1, R_2 \) we can decompose their product into the sum of irreducible representations. Let \( \{ e_1^a \} \) and \( \{ e_2^a \} \) be two basis of the representation spaces of \( R_1 \) and \( R_2 \) respectively, and let \( \{ e_\mu^a \} \) be a basis of their tensor product such that the product representation is in the block-diagonal form, where \( a \) denotes the irreps and \( \mu_a \) labels the various copies of the representation \( R_a \) appearing in the product \( R_1 \otimes R_2 \). The Clebsch-Gordan coefficients are defined as the basis changing coefficients:

\[
e_1^a \otimes e_2^b = \sum_{\mu_a} C_{\mu \mu_1} (1, 2; a)_{ij}^{kl} e_\nu^k .
\]

\(^{38}\) 6-j symbols are defined by:

\[
\begin{align*}
\{ R_1 R_2 R_3 \}_{\mu_1 \mu_2}^{\mu_1 \mu_2} \\
\{ R_4 R_5 R_6 \}_{\mu_3 \mu_4}^{\mu_3 \mu_4}
\end{align*} = \prod_{ij} C_{\mu_1} (R_1, R_2; R_3)_{ij}^{kl} C_{\mu_2} (R_4, R_5; R_6)_{ij}^{kl} C_{\mu_3} (R_1, R_6; R_3)_{ij}^{kl} C_{\mu_4} (R_1, R_4; R_3)_{ij}^{kl} .
\]

\(^{39}\) If a loop has no intersections, then its contribution will be \( N_{R_2 R_3}^{R_1} \) as in equation \((A.1)\).

\(^{40}\) Each oriented boundary, or Wilson loop, carries the character of the holonomy of \( \mathbb{k} \). If we split the circle into various arcs, then it will carry the character of the products of the holonomies over different arcs, multiplied according to the orientation of the loop. This defines inbound and outbound indices for the holonomy over each oriented arc.
Finally, the expectation value of a set of Wilson loops \( \{ \Gamma_i \} \) on a surface \( \Sigma \) is given by the following formula:

\[
\langle \prod_i W_{\Gamma_i}(R_i) \rangle_{\Sigma} = \sum_{R_{\lambda}} \prod_{\lambda} e^{-\frac{b_\lambda}{\alpha_s} C_2(R_{\lambda}) (\dim R_{\lambda})^2} \prod_{\lambda} \chi_{R_{\lambda}}(U_{b_\lambda}) \cdot \prod_{v} \sum_{\mu_v} G_v(R_{\mu_v}; R_{\lambda}) \prod_{l_0} N(R_{l_0}; R_{\lambda}) .
\]

(A.6)

where the index \( \lambda \) runs over connected components \( \Sigma_{\lambda} \) of the surface obtained by cutting \( \Sigma \) along the Wilson loops, \( b_\lambda \) labels the boundaries of \( \Sigma \) contained in \( \Sigma_{\lambda} \), \( v \) labels the intersections between loops, \( G_v \) indicates the 6-j symbol at the vertex \( v \) evaluated on the surrounding representations according to (A.5) and \( N(R_{l_0}; R_{\lambda}) \) denotes the fusion numbers for the non-intersecting Wilson loops labelled by \( l_0 \).

\section{B Propagators}

We collect in this appendix the computations of the propagators used in this paper. We will firstly consider one-dimensional BF propagator on the circle and on the interval with the various possible polarizations on the two end-points. Then we will use these to compute the axial-gauge propagator on 2D surfaces, in particular on the cylinder \( S^1 \times I \) and on the square \( I \times I \).

\subsection*{B.1 One-dimensional propagators}

\subsection*{B.1.1 Propagator on the circle}

Let us consider non-abelian BF theory on the circle \( S^1 \). We are looking for the propagator when we expand the action with respect to the trivial connection. In this case the kinetic term is \( \int_{S^1} \langle B, dA \rangle \). The space of zero-modes is thus given by the de Rham cohomology: \( \mathcal{V} = H_{\text{dR}}(S^1; \mathfrak{g})[1] \oplus H_{\text{dR}}(S^1; \mathfrak{g}^*) \). If \( \tau \in [0, 1] \) is the coordinate of the circle, we have the corresponding basis \([1], [d\tau]\) for the cohomology of the circle and the following coordinate expression for the zero modes:

\[
a = a_0 + a_1 d\tau , \quad b = b^0 + b^1 d\tau ,
\]

(B.1)

where \( a_1(i) \in \mathfrak{g} \) and \( b_0(i) \in \mathfrak{g}^* \). A Hodge decomposition for the de Rham complex of the circle is given by the following induction data:

\[
\Pi \omega(\tau) = \int_{S^1} (d\tau' - d\tau)\omega(\tau') , \\
K \omega(\tau) = \int_{S^1} \left( \Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2} \right)\omega(\tau') .
\]

(B.2)

The extension to the space of fields –Lie-algebra valued differential forms– is immediate and the resulting propagator is: \( 41 \eta_{S^1} \)

\[
b, \tau \to \eta_{S^1} a, \tau' = \eta_{S^1}(\tau, b; \tau', a) = \left( \Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2} \right)\delta_b^a .
\]

(B.3)

\( 41 \) The Lie-algebra part of the propagator in this paper is always the identity \( \delta_b^a \) and will be often omitted.
B.1.2 Propagators on the interval

Interval in $A$-$B$ polarization

Let us consider now BF theory on the unit interval $I = [0, 1]$ with $B$ polarization at $\{0\}$ and $A$ polarization at $\{1\}$. The space of bulk fields is now given by differential forms with Dirichlet boundary conditions on one of the two endpoints. The cohomology of the differential $d$ on this space of differential forms is vanishing, thus the space of zero-modes is empty $\mathcal{V} = 0$.

The chain homotopy $K$ is now

$$K\omega(t) = \int_{t}^{1} \omega(t')$$

and we have the corresponding propagator

$$\eta(t, t') = \Theta(t' - t) .$$

Notice that propagation can only occur if $t < t'$, i.e. moving away from the $B$ endpoint and toward the $A$ endpoint of the interval.

Interval in $A$-$A$ polarization

If we take the $A$ polarization on both endpoints of the interval, the $A$ fields will have Dirichlet boundary conditions at the endpoints while the $B$ fields will have free boundary conditions: $\mathcal{V} = \Omega(I, \partial I; g)[1] \oplus \Omega(I; g^*)$. The cohomology is concentrated in form-degree 1 for the $A$ fields and in form-degree 0 for the $B$ fields

$$\mathcal{V} = H(I, \partial I; g)[1] \oplus H(I; g^*) \simeq g[1] \oplus g^* ,$$

so that the form-degree expansion of the zero modes is $a = a_1 dt, \ b = b^0$. The chain retraction is given by the following data:

$$\eta(t, t') = \Theta(t' - t) - t , \quad \pi(t, t') = -dt .$$

Interval in $B$-$B$ polarization

The interval with $B$ polarization on both endpoints has the role of $A$ and $B$ fields reversed with respect to the previous case. The space of bulk fields is $\mathcal{V} = \Omega(I; g)[1] \oplus \Omega(I, \partial I; g^*)$ and the zero-modes are: $a = a_0, \ b = b^1 dt$. The propagator and the projection to cohomology are:

$$\eta(t, t') = -\Theta(t' - t) + t' , \quad \pi(t, t') = dt' .$$

B.2 Axial gauge propagators on the cylinder

Consider now a cylinder $S^1 \times I$ and let $t$ denote the coordinate of the interval, $\tau$ the coordinate along the circle, $\chi_i$ a basis for the cohomology of $S^1$ and $\chi^i$ its dual basis. Using the 1-dimensional propagators of appendix B.1, from the axial-gauge formula (2.36) we get the
following propagators on the cylinder.

| Polarization | zero-modes | $\tau, t \to \eta(\tau, t; \tau', t') \to \tau', t'$ |
|--------------|------------|----------------------------------|
| $A - B$      | 0          | $-\Theta(t' - t)\delta(\tau - \tau')(d\tau' - d\tau)$ |
| $A - A$      | $a = a_i \, dt \wedge \chi^i$ | $(\Theta(t - t') - t)\delta(\tau' - \tau')(d\tau' - d\tau) - dt(\Theta(\tau - \tau') - \tau - \tau' - \frac{1}{2})$ |
| $B - B$      | $a = a_i \, \chi^i$ | $(t' - \Theta(t' - t))\delta(\tau' - \tau')(d\tau' - d\tau) + dt'(\Theta(\tau - \tau') - \tau - \tau' - \frac{1}{2})$ |

Reversing the role of the circle and the interval in formula (2.36) we would obtain different expressions for the propagator, called for the cylinder horizontal gauge, but we don’t need this choice in this paper.

C  Computations of some Feynman diagrams

We present here the proofs of Propositions 3.3, 4.12, consisting in the evaluation of tree and loop diagrams in the axial gauge. These computations are variations of the ones contained in [22], Lemma 3 and 4, obtained in the 1-dimensional setting.

Proof of Proposition 3.3. We have to evaluate the 1-loop diagrams of figure 5. The amplitude for a diagram with $n \geq 2$ vertices is:

$$\frac{1}{n} \text{tr}(\text{ad}_{a_1}^n) \int_{(S^1)^n} d\tau_1 \cdots d\tau_n \eta_{S^1}(\tau_1; \tau_2) \cdots \eta_{S^1}(\tau_{n-1}; \tau_n) \eta_{S^1}(\tau_1; \tau_1)$$

$$= \frac{1}{n} \text{tr}(\text{ad}_{a_1}^n)^n \text{tr}(K(\chi_1 \wedge \bullet))^n.$$  \hspace{1cm} (C.1)

where we chose the basis $\chi_0 = 1, \chi_1 = d\tau$ for $H^\bullet(S^1)$ and $K$ is the chain homotopy with integral kernel $\eta_{S^1}(\tau; \tau') = \Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2}$. We will compute $\text{tr}(K(\chi_1 \wedge \bullet))^n$ in the monomial basis $1, \tau, \tau^2, \ldots$. Let us define the generating function:

$$f_m(x, \tau) = \sum_{n=0}^{\infty} x^n(K(\chi_1 \wedge \bullet))^n \tau^m.$$  \hspace{1cm} (C.2)

Applying $xK(\chi_1 \wedge \bullet)$ on both sides we get

$$xK(\chi_1 f_m)(x, \tau) = f_m(x, \tau) - \tau^m$$  \hspace{1cm} (C.3)

and, differentiating w.r.t. $\tau$, we obtain the differential equation:

$$\frac{\partial}{\partial \tau} f_m = xf_m + m \tau^{m-1} - x \int_0^1 d\tau \ f_m.$$  \hspace{1cm} (C.4)
Solutions to the above equation are of the form
\[ f_m(x, \tau) = A(x)e^{x\tau} + B(x) + e^{x\tau} \int_0^\tau \, d\tilde{\tau} \, m\tilde{\tau}^{m-1}e^{-x\tilde{\tau}}, \]
(C.5)
where \( A(x) = \frac{1}{e^x - 1}(1 - e^x \int_0^1 \, d\tau \, m\tau^{m-1}e^{-x\tau}) \) and \( B(x) \) is to be determined from the boundary conditions. Since \( K(\chi_1 f_m)(x, 0) = K(\chi_1 f_m)(x, 1) \), from (C.3) we have:
\[ f_m(x, 1) - 1 = f_m(x, 0) = xK(\chi_1 f_m)(x, 0) \]
\[ = x \int_0^1 \, d\tau \, (A(x)e^{x\tau} + B(x) + e^{x\tau} \int_0^\tau \, d\tilde{\tau} \, m\tilde{\tau}^{m-1}e^{-x\tilde{\tau}})(\tau - \frac{1}{2}) \]
(C.6)
\[ = A(x)g(x) + C(x), \]
where \( g(x) = x \int_0^1 \, d\tilde{\tau} \, (\tilde{\tau} - \frac{1}{2})e^{x\tau} \) and \( C(x) = x \int_0^1 \, d\tau \, e^{x\tau}(\tau - \frac{1}{2}) \int_0^\tau \, d\tilde{\tau} \, m\tilde{\tau}^{m-1}e^{-x\tilde{\tau}} + \int_0^\tau \, d\tilde{\tau} \, m\tilde{\tau}^{m-1}e^{-x\tilde{\tau}} + f_m(x, 0). \)
(C.7)
We can now extract the trace of powers of \( \mathcal{M} := K(\chi_1 \wedge \bullet) \) from the series of the coefficients of \( \tau^m \) in the expansion of \( f_m \):
\[ f_{mm}(x) := \sum_{n=0}^\infty \langle \tau^m | x^n \mathcal{M}^n | \tau^m \rangle, \]
(C.8)
\[ \Rightarrow \sum_{m=1}^\infty (f_{mm}(x) - 1) = \sum_{n=1}^\infty x^n \text{tr} \mathcal{M}^n. \]

The coefficients \( f_{mm}(x) \) can be read from (C.7):
\[ f_m(x, \tau) = \frac{e^{x\tau} - 1}{e^x - 1} \sum_{k=0}^{m-1} \frac{m!}{(m-k)!} x^{-k} - \sum_{k=0}^{m-1} \frac{m!}{(m-k)!} \tau^{m-k}x^{-k} + f_m(x, 0), \]
\[ \Rightarrow f_{mm}(x) = 1 - \frac{1}{e^x - 1} \sum_{k=m+1}^\infty \frac{x^k}{k!}. \]
(C.9)
Thus we get:
\[ \sum_{n=1}^\infty x^n \text{tr} \mathcal{M}^n = -\frac{1}{e^x - 1} \sum_{m=1}^\infty \sum_{k=m+1}^\infty \frac{x^k}{k!} = -\frac{1}{e^x - 1} \sum_{k=2}^\infty \frac{k-1}{k!}x^k \]
\[ = 1 - x - \frac{x}{e^x - 1} = -\frac{1}{2}x - \sum_{n=2}^\infty \frac{B_n}{n!}x^n, \]
\[ \Rightarrow \text{tr} \mathcal{M}^n = -\frac{B_n}{n!} \quad \text{for } n \geq 2, \]
(C.10)
where \( B_n \) are the Bernoulli numbers.
Proof of Proposition 4.12. We have to evaluate the diagrams of the kind depicted in figure 14. The amplitude \( I_n \) for a tree rooted on a \( \mathbb{B} \) boundary field and ending on \( \alpha \) is:

\[
I_n = (-1)^{n+1} \int_{I_{\times(n+1)}} \langle \mathbb{B}(1) \rangle (\tau_0), \eta_I(\tau_0, \tau_1) \cdots \eta_I(\tau_{n-1}, \tau_n) \eta_I(\tau_n, 1) \text{ad}_{\alpha_1}^n \alpha \rangle d\tau_0 \cdots d\tau_n , \tag{C.11}
\]

where \( \eta_I(\tau, \tau') = \Theta(\tau - \tau') - \tau \). The result of this integral can be expressed in terms of the Bernoulli polynomials:

\[
I_n = (-1)^n \int \langle \mathbb{B}(1) \rangle (\tau), \frac{B_{n+1}(\tau) - B_{n+1}}{(n+1)!} \text{ad}_\alpha^n \alpha \rangle . \tag{C.12}
\]

To prove (C.12), let’s define the operator \( Kg(\tau) := \int_I \eta_I(\tau, \tau') g(\tau') d\tau' \) and the generating function

\[
f(x; \tau) := \sum_{n=0}^{\infty} x^n K^n(t) . \tag{C.13}
\]

This function satisfies the differential equation:

\[
\frac{\partial}{\partial \tau} f(x; \tau) = xf(x; \tau) + 1 - \int_I f(x; \tau') d\tau' = xf(x; \tau) + C(x) , \tag{C.14}
\]

where \( C(x) \) does not depend on \( \tau \). Since only the term \( n = 0 \) contributes to \( f \) evaluated on the endpoints \( \tau = 0, 1, f \) satisfies \( f(x; 0) = 0, f(x; 1) = 1 \). Solving the differential equation with this boundary conditions we get:

\[
f(x; \tau) = \frac{1 - e^{xt}}{1 - e^x} = \frac{1}{x} \left( \frac{x}{1 - e^x} - \frac{xe^{xt}}{1 - e^x} \right) = \sum_{n=0}^{\infty} \frac{B_{n+1}(\tau) - B_{n+1}}{(n+1)!} x^n . \tag{C.15}
\]

Since \( K(1) = 0 \), we have \( K^n(\eta_I(\tau; 1)) = -K^n(\eta_I(\tau; 0)) \). Thus, similar contributions to (C.12) come from trees ending on \( \tilde{\alpha} \) (the main difference being in the term for \( n = 0 \)) or rooted on \( \tilde{\mathbb{B}} \) or on \( b \). By summing over \( n \) we get, for the tree part of the effective action:

\[
S_{\text{tree}}^{\text{eff.}} = \int_0^1 \langle \mathbb{B}(1) \rangle (\tau) - \langle \mathbb{B}(1) \rangle (\tau), G_+(\tau, \text{ad}_{\alpha_1}) \alpha + G_-(\tau, \text{ad}_{\alpha_1}) \tilde{\alpha} \rangle d\tau + \langle b^1, F_+(\text{ad}_{\alpha_1}) \alpha + F_-(\text{ad}_{\alpha_1}) \tilde{\alpha} \rangle . \tag{C.16}
\]

The amplitude for a wheel diagram is:

\[
-\frac{ih}{n} \text{tr}(\text{ad}_{\alpha_1}^n) \int_I d\tau_1 \cdots d\tau_n \eta_I(\tau_1; \tau_2) \cdots \eta_I(\tau_{n-1}; \tau_n) \eta_I(\tau_n; \tau_1) . \tag{C.17}
\]

This integral is the same as the one appearing in [22] and can be computed with the technique used to prove equation C.1. The result for the loop contribution to the effective action is thus:

\[
S_{\text{loop}}^{\text{eff.}} = -ih \sum_{n \geq 2} \frac{1}{n!} \text{tr}(\text{ad}_{\alpha_1}^n) \frac{B_n}{n} = -ih \text{ tr} \left( \log \left( \frac{\sinh(\text{ad}_{\alpha_1}/2)}{\text{ad}_{\alpha_1}/2} \right) \right) . \tag{C.18}
\]
D Proof of Proposition 4.5

Here we present a direct computational proof that $\Omega^2 = 0$ for a stratified circle, with any choice of polarizations on the edges and corners.

First note that edge contributions $\Omega^A_I$, $\Omega^B_I$ and pure corner contributions $\Omega^\alpha_p$, $\Omega^\beta_p$ all square to zero. Also, edge contributions and pure corner contributions commute. In particular, we have

$$\Omega^2 = \sum_k (\Omega^{{\epsilon}_k}_{P_k})^2 + [\Omega^{{\epsilon}_k}_{P_k}, \Omega^{{\epsilon}_k}_{I_k} + \Omega^{{\epsilon}_{k+1}}_{I_{k+1}}]. \quad (D.1)$$

Denote $\text{BCH}(x, y) = \log(e^x e^y)$ for $x, y \in g$ – the Baker-Campbell-Hausdorff group law. We will need the following identities

$$\text{BCH}(x, y) = x + F_+(\text{ad}_x)y + \mathcal{O}(y^2), \quad \text{BCH}(x, y) = y - F_-(\text{ad}_y)x + \mathcal{O}(x^2) \quad (D.2)$$

which are the cases of the BCH formula when either first or second argument is infinitesimal.

Let us study e.g. a $\beta$-corner $p$ surrounded by a $B$-edge $I'$ on the left and an $A$-edge $I$ on the right. We have $\Omega^{{\beta}A}_p \beta A = \Omega^{{\beta}A}_p$ given by (4.20). Applying this operator to a wavefunction of form $\psi(\beta) = e^{-\frac{i}{\hbar}\langle \beta, \text{BCH}(x, \epsilon A_p) \rangle}$, with $x \in g$ a parameter, yields

$$\Omega^{{\beta}A}_p \psi = \langle \beta, F_+(\text{ad}_x)A_p \rangle \psi = i\hbar \frac{d}{d\epsilon} e^{-\frac{i}{\hbar}\langle \beta, \text{BCH}(x, \epsilon A_p) \rangle}, \quad (D.3)$$

with $\epsilon$ an odd parameter. Here we have used the first identity in (D.2). Note that similarly one can write $\Omega^{{\beta}A}_p \psi = -i\hbar \frac{d}{d\epsilon} e^{-\frac{i}{\hbar}\langle \beta, \text{BCH}(\epsilon A_p, x) \rangle}$, using the second identity in (D.2). This implies

$$\left(\Omega^{{\beta}A}_p\right)^2 \psi = (i\hbar)^2 \frac{d}{d\epsilon_2} \frac{d}{d\epsilon_1} e^{-\frac{i}{\hbar}\langle \beta, \text{BCH}(x, \epsilon_1 A_p, \epsilon_2 A_p) \rangle} \quad (D.4)$$

Note that the main trick of this computation is the use of associativity of the BCH formula. Operators $(\Omega^{{\beta}A}_p)^2$, $[\Omega^A_I, \Omega^{{\beta}A}_p]$ are multiplication operators in the variable $A_p$, thus the computation above, for $\psi$ independent of $A_p$ is sufficient to ascertain that $(\Omega^{{\beta}A}_p)^2 + [\Omega^{{\beta}A}_p, \Omega^A_I] = 0$ as operators. Further, note that $[\Omega^{{\beta}A}_p, \Omega^B_I]$ contains derivatives in $B^{(1)}_p$ and therefore vanishes on admissible states, in the sense of Assumption 4.4. This proves that the contribution of a $B\beta A$ corner to $\Omega^2$ (cf. the right hand side of (D.1)) vanishes. The case $A_1\beta B$ is an orientation reversal of the case we just studied; it is treated analogously and also yields a zero contribution to the r.h.s. of (D.1).
E A check of the chain map property of the inclusion of the small model for $\mathcal{A}$-states on an interval into the full model

Case of an $\mathcal{A}\beta\mathcal{A}$ corner is treated similarly. Here $\Omega_{\mathcal{P}}^{\mathcal{A}\beta\mathcal{A}} = \Omega_{\mathcal{P}}^{\mathcal{A}\beta} + \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{A}}$. We have $(\Omega_{\mathcal{P}}^{\mathcal{A}\beta})^2 + [\Omega_{\mathcal{P}}^{\mathcal{A}\beta}, \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{A}}] = 0$ as above, and similarly $(\Omega_{\mathcal{P}}^{\mathcal{A}\beta})^2 + [\Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{B}}, \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{B}}] = 0$. We also need to understand the term $[\Omega_{\mathcal{P}}^{\mathcal{A}\beta}, \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{A}}]$, which is done similarly to (D.4):

$$\Omega_{\mathcal{P}}^{\mathcal{A}\beta} \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{A}} \psi = -(ih)^2 \frac{d}{de_2\,de_1} e^{-\frac{i}{\hbar} \langle \beta, \text{BCH}(\epsilon_2 A_{p_0}, \text{BCH}(\epsilon_1 A_{p+0})) \rangle}$$

$$= -(ih)^2 \frac{d}{de_2\,de_1} e^{-\frac{i}{\hbar} \langle \beta, \text{BCH}(\epsilon_2 A_{p-0}, \epsilon_1 A_{p+0}) \rangle} = -\Omega_{\mathcal{P}}^{\mathcal{A}\beta} \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{A}} \psi.$$ (D.5)

Hence, $[\Omega_{\mathcal{P}}^{\mathcal{A}\beta}, \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{A}}] = 0$ and the contribution of an $\mathcal{A}\beta\mathcal{A}$ corner to the r.h.s. of (D.1) vanishes.

In the case of an $\mathcal{A}\alpha\mathcal{B}$ corner, we have $\Omega_{\mathcal{P}}^{\mathcal{A}\alpha\mathcal{B}} = \Omega_{\mathcal{P}}^{\mathcal{A}\alpha} + \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{B}}$. By a computation similar to (D.4), one shows that $((\Omega_{\mathcal{P}}^{\mathcal{A}\alpha})^2 + [\Omega_{\mathcal{P}}^{\mathcal{A}\alpha}, \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{B}}])\psi = 0$ for $\psi = e^{-\frac{i}{\hbar} \langle \beta, x \rangle}$. Together with $(\Omega_{\mathcal{P}}^{\mathcal{A}\alpha})^2 = 0$, this shows that $(\Omega_{\mathcal{P}}^{\mathcal{A}\alpha\mathcal{B}})^2 = 0$. Furthermore, $[\Omega_{\mathcal{P}}^{\mathcal{A}\alpha\mathcal{B}}, \Omega_{\mathcal{P}}^{\mathcal{A}\mathcal{B}}] = 0$ and $[\Omega_{\mathcal{P}}^{\mathcal{A}\alpha\mathcal{B}}, \Omega_{\mathcal{P}}^{\mathcal{A}\alpha\mathcal{B}}] = 0$ on admissible states. Thus, the contribution an $\mathcal{A}\alpha\mathcal{B}$ to the r.h.s. of (D.1) also vanishes. Orientation-reversed case $\mathcal{B}\alpha\mathcal{A}$ is similar.

Case of a $\mathcal{B}\alpha\mathcal{B}$ corner is similar to the above: we have $\Omega_{\mathcal{P}}^{\mathcal{B}\alpha\mathcal{B}} = \Omega_{\mathcal{P}}^{\mathcal{B}\alpha} + \Omega_{\mathcal{P}}^{\mathcal{B}\mathcal{B}}$. As above, we have $(\Omega_{\mathcal{P}}^{\mathcal{B}\alpha})^2 + [\Omega_{\mathcal{P}}^{\mathcal{B}\alpha}, \Omega_{\mathcal{P}}^{\mathcal{B}\mathcal{B}}] = 0$ and similarly $(\Omega_{\mathcal{P}}^{\mathcal{B}\alpha})^2 + [\Omega_{\mathcal{P}}^{\mathcal{B}\mathcal{B}}, \Omega_{\mathcal{P}}^{\mathcal{B}\mathcal{B}}] = 0$. One also trivially has $[\Omega_{\mathcal{P}}^{\mathcal{B}\alpha}, \Omega_{\mathcal{P}}^{\mathcal{B}\mathcal{B}}] = 0$. Thus, $(\Omega_{\mathcal{P}}^{\mathcal{B}\alpha\mathcal{B}})^2 = 0$. Also, the corner contribution to $\Omega$ commutes with the edge terms on admissible states. This proves that the contribution of a $\mathcal{B}\alpha\mathcal{B}$ corner to the r.h.s. of (D.1) vanishes, too.

Cases of $\mathcal{A}\alpha\mathcal{A}$ and $\mathcal{B}\beta\mathcal{B}$ corners are trivial. This finishes the proof that all terms in the sum (D.1) over the corners vanish, and thus $\Omega^2 = 0$ for an arbitrarily stratified and polarized circle.

E A check of the chain map property of the inclusion of the small model for $\mathcal{A}$-states on an interval into the full model

One can check directly that (4.54) is indeed a chain map. First, it is clearly an algebra morphism (w.r.t. the standard supercommutative pointwise product on $\text{Fun}(\cdots)$), so it is enough to check the chain map property on a set of generators of $\mathcal{H}^\text{small}$. Assume for simplicity that $g \subset \text{Mat}_N$ is a matrix Lie algebra and choose as generators

$$f_{k,\rho} := \text{tr} \rho \mathbb{A}_k, \quad g_\rho := \text{tr} \rho e^\mathbb{A},$$ (E.1)

with $\rho \in \text{Mat}_N$ arbitrary parameter and $k = 0, 1$. From (4.52) and (4.54), we immediately obtain that $i_\mathcal{H} \circ \Omega^\text{small} = \Omega \circ i_\mathcal{H}$ when applied to the generators $f_{k,\rho}$. For $g_\rho$, from (4.52,4.54) and from the rule for the deformation of holonomy under an infinitesimal gauge transformation, we obtain:

$$\left(i_\mathcal{H} \circ \Omega^\text{small}\right)g_\rho = (\Omega \circ i_\mathcal{H})g_\rho = -ih \text{tr} \rho \left(U(\mathbb{A}) \cdot \mathbb{A}_1 - \mathbb{A}_0 \cdot U(\mathbb{A})\right).$$ (E.2)
Here we used the observation that $\Omega^{\text{small}} g_\rho = -i \hbar \text{tr } \rho(e^X c_1 - c_0 e^X)$ with shorthand notation $c_0 = A_0$, $c_1 = A_1$, $X = A$. Indeed, we have

$$\frac{i}{\hbar} \Omega^{\text{small}} g_\rho = \text{tr } \rho \sum_{p,q \geq 0} \frac{1}{(p+q+1)!} X^p (F_+(\text{ad}_X)c_1 + F_-(\text{ad}_X)c_0) X^q$$

$$= \text{tr } \rho \sum_{p,q,j,l \geq 0} \frac{1}{(p+q+1)!} \left( (-1)^j B^+_{j+l} X^{j+p} c_1 X^{l+q} - (-1)^j B^-_{j+l} X^{j+p} c_0 X^{l+q} \right) , \quad (E.3)$$

where $B^+_i$ and $B^-_i$ are the Taylor coefficients of $F_+(x)$ and $-F_-(x)$, respectively. Note that, for $x, y$ scalars, we have

$$\sum_{p,q,j,l \geq 0} \frac{1}{(p+q+1)!} (-1)^j B^+_{j+l} x^{j+p} y^{l+q} = \left( \sum_{p,q \geq 0} \frac{x^p y^q}{(p+q+1)!} \right) \left( \sum_{j,l \geq 0} \frac{(-1)^j}{j!!} x^j y^l \right)$$

$$= e^x - e^y \cdot \frac{x - y}{x - y} = e^x$$

and, similarly, $\sum_{p,q,j,l \geq 0} \frac{1}{(p+q+1)!} (-1)^j B^-_{j+l} x^{j+p} y^{l+q} = e^y$. Thus:

$$\frac{i}{\hbar} \Omega^{\text{small}} g_\rho = \text{tr } \rho(e^X c_1 - c_0 e^X) \quad (E.5)$$
as claimed.

References

[1] J. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36, 1998.

[2] R. Bandiera and F. Schaeetz. How to discretize the differential forms on the interval. *arXiv:1607.03654 [math.QA]*, 2016.

[3] I. A. Batalin and G. A. Vilkovisky. Gauge algebra and quantization. *Physics Letters B102*, pages 27–31, 1981.

[4] F. Bonechi, A. S. Cattaneo, and P. Mnev. The Poisson sigma model on closed surfaces. *Journal of High Energy Physics, Volume 2012*, 2012.

[5] A. S. Cattaneo. *Private communications*, 2017.

[6] A. S. Cattaneo and G. Felder. A path integral approach to the Kontsevich quantization formula. *Commun.Math.Phys. 212*, pages 591–611, 2000.

[7] A. S. Cattaneo and G. Felder. Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model. *Letters in Mathematical Physics*, 69, 2003.

[8] A. S. Cattaneo and P. Mnev. Remarks on Chern–Simons invariants. *Communications in Mathematical Physics*, 293, 2008.

[9] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. Classical and quantum lagrangian field theories with boundary. *Proceedings of Science*, 2012.
[10] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. Classical BV theories on manifolds with boundary. *Communications in Mathematical Physics*, 332(2):535–603, 2014.

[11] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. Perturbative BV theories with Segal-like gluing. *arXiv:1602.00741 [math-ph]*, 2016.

[12] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. A cellular topological field theory. *arXiv:1701.05874 [math.AT]*, 2017.

[13] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. Perturbative quantum gauge theories on manifolds with boundary. *Communications in Mathematical Physics*, 357(2):631–730, 2018.

[14] K. J. Costello. Renormalisation and the Batalin-Vilkovisky formalism. *arXiv:0706.1533 [math.QA]*, 2007.

[15] V. K. A. M. Gugenheim and L. A. Lambe. Perturbation theory in differential homological algebra I. *Illinois J. Math.* 33, pages 566–582, 1989.

[16] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, 1994.

[17] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[18] R. Lawrence and D. Sullivan. A formula for topology/deformations and its significance. *Fundamenta Mathematicae*, 225, 2014.

[19] J. Lurie. On the classification of topological field theories. *Current Developments in Mathematics*, 2009.

[20] A. A. Migdal. Recursion Equations in Gauge Theories. *Sov. Phys. JETP*, 42:413, 1975.

[21] P. Mnev. Notes on simplicial BF theory. *Moscow Math. J.*, 9, 2006.

[22] P. Mnev. Discrete BF theory. *arXiv:0809.1160 [hep-th]*, 2008.

[23] R. Oeckl. Two-dimensional quantum Yang–Mills theory with corners. *Journal of Physics A: Mathematical and Theoretical*, 41(13):135401, 2008.

[24] E. Witten. On quantum gauge theories in two dimensions. *Commun. Math. Phys.* 141, pages 153–209, 1991.

[25] Xue Zhi Cheng and E. Getzler. Transferring homotopy commutative algebraic structures. *Journal of Pure and Applied Algebra*, 212, 2006.