Replica Symmetry Breaking without Replicas

Simone Franchini

Abstract

We introduce a mathematical framework based on simple combinatorial arguments (Kernel Representation) that allows to deal successfully with spin glass problems, among others. Let $\Omega^N$ be the space of the configurations of an $N$-spins system, each spin having a finite set $\Omega$ of inner states, and let $\mu : \Omega^N \to [0, 1]$ be some probability measure. Here we give an argument to encode $\mu$ into a kernel function $M : [0, 1]^2 \to \Omega$, and use this notion to reinterpret the assumptions of the Replica Symmetry Breaking ansatz (RSB) of Parisi et Al. [1,2], without using replicas, nor averaging on the disorder.

1 Introduction

Originally introduced by Parisi [1] in his analysis of the Sherrington-Kirkpatrick model (SK) [1,2], the Replica Symmetry Breaking (RSB) ansatz proved to be an extremely valuable tool in explaining properties of disordered systems. Despite many technical advances, worth to cite is the proof of the free energy formula by Guerra and Talagrand [3,4], some of its fundamental features remain quite mysterious after forty years.

A central role is played by the elusive concept of pure state. Despite a precise definition is still lacking, it is widely acknowledged that they must satisfy some properties. For example, it is expected that the connected correlation functions associated to these states vanish in the thermodynamic limit [2],[12]. This imply that in some sense the measure conditioned to those states can be described by a mean field model of some kind (see Part III of [12], updated 2014 version, for a non-rigorous but detailed discussion of the finite volume pure states).

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1Goethe Mathematics Institute, 10 Robert Mayer Str, 60325 Frankfurt, Germany
2Sapienza Università di Roma, Piazzale Aldo Moro 1, 00185 Roma, Italy
1MSC2000: 82D30, 60F10, Keywords: Spin Glasses, Replica Symmetry Breaking, Pure States, Parisi Formula
Perhaps, the most striking and unconventional property is that the pure states have been predicted to have a hierarchical structure, such that the support of the overlaps is ultrametric [2]. A considerable amount of works have been published on this argument, that culminated in a proof of ultrametricity by Panchenko [14].

Anyway, whether the ultrametricity and other properties of the pure states hold in some general framework, including their representation as well defined mathematical objects, proved to be an extremely hard task and remains an open question.

Inspired by a remarkable series of papers by Coja-Oghlan and others, which introduce tools from Graph Theory to study Belief Propagation algorithms [16, 17, 18, 19], here we present an original framework that is based on elementary combinatorial arguments, and that allows to deal with many interesting physical systems, including spin glasses, without using replicas, nor averaging over the disorder.

The theory is presented both symbolically and by a graphical representation in terms of kernel functions of the kind

\[ M : [0, 1]^2 \to \{-, +\}. \]  

(1.1)

This object is intended to provide a simple visual encoding for probability measures, and it was central for us in understanding and developing the concepts we are going to explain.

We introduce the kernel representation in the Section 2 along with some notation, showing how to encode a probability measure into a kernel and bring it back, the commutation relations, the transposed measure, and other basic kernel features.

In the Section 2 we also introduce an analogue notion of pure states, that can be applied to any distribution, the space of these generalized states is charted by a simple partition of the spin space into disjoint subsets. This first section does not contain complex mathematical concepts, and it aims to carefully introduce probability in kernel language to connect with graph theoretic arguments, and highlight the bridges that exist between probability and graph theory.

Then in Section 3 we introduce more advanced kernel methods, for example it is possible to introduce a new type of convergence of random variables, ie convergence in “Cut Distance”, that is stronger than weak* convergence and allows to manage the kernels (then also Gibbs measures) directly in the thermodynamic limit. In section 3 we also give an alternative formulation that connects with the findings presented in [16, 17, 18, 19] and Graph Theory in general [31].

The readers mostly interested in the physics of the SK model may jump these two sections in first read, and go straight to the Section 4 where we apply some of the ideas
presented in Section 2 to the SK Hamiltonian

\[ H_{sk}(\sigma_V) = \frac{1}{\sqrt{N}} \sum_{i \in V} \sum_{j < i} \sigma_i J_{ij} \sigma_j \] (1.2)

and its kernel, and develop a scheme for the propagation of properties of the Gibbs distribution as it evolves along the cavity chain described in [27].

The main steps of our analysis will be as follows: the first is to define a sequence of SK models of increasing sizes, this is done by partitioning the vertex set \( V \) into \( L \) subsets \( V_\ell \), marked by the index \( \ell \), that we call layers. Then, we consider their union \( Q_\ell \) up to a certain \( \ell \), with \( Q_{\ell-1} \subseteq Q_\ell \) being a sequence of sets of size \( |Q_\ell| = q_\ell N \), such that \( V_\ell = Q_\ell \setminus Q_{\ell-1} \) and \( Q_L = V \), see Definition 4.

This construction converges to the actual system, and we interpret it as a layering scheme in which we grow the system layer by layer up to the original size. We find the Hamiltonian sequence of these layers in Lemma 9,

\[ H_{sk}(\sigma_V) = \sum_{\ell \leq L} H_\ell (\sigma_{Q_\ell}) \quad (1.3) \]

where \( H_\ell \) is the Hamiltonian describing the layer \( V_\ell \),

\[ H_\ell (\sigma_{Q_\ell}) = \sqrt{q_\ell - q_{\ell-1}} H_{sk}(\sigma_{V_\ell}) + \sqrt{q_{\ell-1}} \sum_{i \in V_\ell} \sigma_i h_i (\sigma_{Q_{\ell-1}}), \] (1.4)

the first term is the energy contribution coming from the interactions inside the layer itself, and is simply a smaller SK model, while \( h_i \) is a cavity field only depending on the spins of the previous layers, i.e. the spins of \( Q_{\ell-1} \). Then we show that, due to the mean field nature of the model, if the partition is fine enough the factor \( \sqrt{q_\ell - q_{\ell-1}} \) kills the contribution from the smaller SK, and the thermodynamics is dominated by the interface, i.e. the interaction term \( \sigma_{V_\ell} \cdot h_{V_\ell} \) between the newly added layer and the rest of the system up to that point of the sequence, see Lemma 10.

The interface is simply a collection of spins coupled to an external random field that comes from the previous layers, and can be solved exactly, for example by the same techniques developed in [7, 8, 9] for the Number Partitioning Problem. In Section 5 we show that the thermodynamics of the layer is equivalent to a Random Energy Model of the Derrida type [10] in the limit of very low temperature, that is Lemma 12.

Finally, in Section 6 we give an explicit application by combining the Cavity Method of [21, 22, 23, 27] with Lemma 4 to obtain a constructive derivation of the cavity variables, and compute the Parisi functional in a simple way.
2 Kernel representation

Before entering in the core of the discussion some preliminaries are mandatory in order to explain the notation and justify our later arguments. In particular, we describe how and why to encode a finite spin system into a kernel function. For this paper we indicate by \( I(A) \) the indicator function of the event \( A \), that is \( I(A) = 1 \) if \( A \) is verified and is zero otherwise. Also, given two ordered sets \( A \) and \( B \) we use the notation \( A \otimes B \) for the tensor product and just \( AB \) for the Cartesian product (ie same for number multiplication). The Hadamard product is denoted by the \( \circ \) symbol.

Consider a random spin system \( \sigma_V \) of \( N \) spins, distributed according to some law \( \mu(\sigma_V) \), and imagine to perform a sequence of independent measurements of such system. Formally, let \( V = \{1, 2, ..., N\} \) be a set of \( N \) vertices and put a spin \( \sigma_i \in \Omega \) of inner states \( \Omega = \{+,-\} \) on each vertex, we denote by

\[
\sigma_V = \{\sigma_i \in \Omega : i \in V\}
\] (2.1)

the generic magnetization state. Due to the finiteness of the spin number there is only a finite set of possible outcomes, in fact, each measurement will give as result some element of \( \Omega^V \) product space of the elementary spin spaces \( \Omega \).

The first important observation is that if the measurements are independent the order in which the states are observed cannot contain information of the underlying law, then we are free to regroup them to our convenience. Let order the states of \( \Omega^V \) by some index \( \alpha : \Omega^V \rightarrow S \), where \( S = \{1, 2, ..., 2^{|V|}\} \) is the span of the index. The set \( \Omega^V \) is then rewritten as follows

\[
\Omega^V = \{\tau^\alpha_V : \alpha \in S\}, \quad \tau^\alpha_V = \{\tau^\alpha_i \in \Omega : i \in V\}
\] (2.2)

with each state \( \tau^\alpha_V \) being uniquely identified by \( \alpha \), ie \( \tau^\alpha_V \neq \tau^\gamma_V \) if \( \alpha \neq \gamma \).

Since for finite \( V \) also \( \Omega^V \) has a finite number of states, for a large number of measurements the relative frequencies of the states \( \tau^\alpha_V \), that are rational numbers, approximate the probabilities \( \mu(\tau^\alpha_V) \in [0,1] \) that are associated to the occurrence of a given state \( \alpha \). Arranging them into vectors

\[
\mu = \{\mu(\tau^\alpha_V) \in [0,1] : \alpha \in S\},
\] (2.3)

we can also write a simple representation for the set of measures on \( \Omega^V \)

\[
\mathcal{P}(\Omega^V) = \{\mu \in [0,1]^V : \sum_{\alpha} \mu(\tau^\alpha_V) = 1\}.
\] (2.4)
Is easy to verify that the measure (probability mass function) can be reconstructed from the vector $\mu$. Explicitly, we can write the measure $\mu : \Omega^V \to [0,1]$ and its average applied to some test function $f : \Omega^V \to \mathbb{R}$ as follows

$$
\mu (\sigma_V) = \sum_{\alpha \in S} \mu (\tau^\alpha_V) \prod_{i \in V} \left( \frac{1 + \tau^\alpha_i \sigma_i}{2} \right), \quad \langle f (\sigma_V) \rangle_\mu = \sum_{\alpha \in S} \mu (\tau^\alpha_V) f (\tau^\alpha_V). \tag{2.5}
$$

This will be our preferential notation, and we will also use a dedicated symbol for the uniform measure $\nu_\alpha = 1/|S| = 1/2^N$ and call it support measure (see upper kernel of Figure 2.1)

$$
\nu (\sigma_V) = \frac{1}{2^N} \sum_{\alpha \in S} \prod_{i \in V} \left( \frac{1 + \tau^\alpha_i \sigma_i}{2} \right), \quad \langle f (\sigma_V) \rangle_\nu = \frac{1}{2^N} \sum_{\alpha \in S} f (\tau^\alpha_V). \tag{2.6}
$$

We can now introduce a powerful graphical tool to represent $(\mu, \Omega^V)$, that simply consists in rearranging the states into an array. In the following we show how to encode the probability pair $(\mu, \Omega^V)$ into a two dimensional function.

**Definition 1.** (Magnetization Kernel) Let $\mu \in \mathcal{P} (\Omega^V)$, then, the Magnetization Kernel $M : [0,1]^2 \to \Omega$ associated to $\mu$ is the step function

$$
M^\mu_{xy} = \sum_{\alpha \in S} \sum_{i \in V} \tau^\alpha_i I (x \in (x_{i-1}, x_i], y \in (y_{\alpha-1}, y_\alpha])
$$

with $I(A)$ indicator function of the event $A$. The sizes of the intervals are

$$
x_i - x_{i-1} = 1/N, \ y_\alpha - y_{\alpha-1} = \mu (\tau^\alpha_V). \tag{2.8}
$$

An explicit example is given in Figures 2.1 and 2.2 where the states are ordered according to the inverse binary map $\alpha (\sigma_V) = 1 + \sum_{i \leq N} 2^{N-i} \left( \frac{1 + \sigma_i}{2} \right)$.

Array encodings of the order parameters have been considered since the beginning in the context of the Spin Glasses theory (see the overlap matrix of [2]), but their use to represent probability distributions is recent enough. Before [16, 17], for example, the Aldous-Hoover theorem has been used in [15] to encode the replicated distribution of the SK model into a four dimensional spin tensor. We remark that $\mu (\sigma_V)$ is defined up to an arbitrary reshuffling of $\alpha$, if we apply the discrete invertible map $\alpha \to \theta (\alpha)$ still

$$
\sum_{\alpha \in S} \mu (\tau^\theta (\alpha)) \prod_{i \in V} \left( \frac{1 + \tau^\theta_i (\alpha) \sigma_i}{2} \right) = \sum_{\alpha \in S} \mu (\tau^\theta_V) \prod_{i \in V} \left( \frac{1 + \tau^\theta_i \sigma_i}{2} \right) \tag{2.9}
$$

because for probability measures the labeling of the support is a free parameter.
Figure 2.1: Kernel representation $M_\mu(x,y)$ of Eq. (2.7) (lower kernel B) and its support kernel $M_\nu(x,y)$ (upper kernel A) for a system of $N=4$ spins described by a trial distribution with $\mu(\tau^\alpha_\nu) > 0$ for $\alpha \in \{4,5,6,10,11\}$ and zero otherwise. Dark cells indicate spin down, bright cells spin up. Between the two kernels it is shown the action of $\mu$ on the support kernel $M_\nu$ to get the actual kernel $M_\mu$. The states are ordered following the $\alpha$–index of Definition 1, then $\tau^4_\nu = (+,+,−,−)$, $\tau^5_\nu = (−,−,+,$ $−)$, $\tau^6_\nu = (+,−,+,$ $−)$, $\tau^{10}_\nu = (+,−,−,−)$, $\tau^{11}_\nu = (−,+,−,−)$. 
Figure 2.2: Support kernel $M_\nu(x, y)$ associated to the support (uniform) measure $\nu$ for a system of $N = 12$ spins. As before, the spin up is in bright color and the spin down is in darker shade. The states have been disposed according to the $\alpha$–index of Definition 1 in increasing order from $\alpha = 1$, that is $(-, -, ..., -)$, to $\alpha = 2^N$, that is $(+, +, ... +)$. As one can appreciate from the figure, the index highlights a hierarchical structure that exist between the magnetization states.
Figure 2.3: Non compactified kernel of the support $\nu$ for a system of $N = 4$ spins, and its transposed version $\nu^\dagger$. The figure shows at bottom-left: $M$, a non compactified version of the kernel of $\nu$, like in Definition 2 but with $x_{i+1} - x_i = y_{\alpha+1} - y_{\alpha} = 1$. At top-right: $M^\dagger$, non-compactified kernel of the transposed support $\nu^\dagger$ of Eqs. (2.15) and (2.17). At top-left and bottom-right: $Q$ and $C$, the overlap matrix and the correlation matrix: color shades correspond to the possible values 1 (darkest shade), $1/2$, 0, $-1/2$, $-1$ (white).
Then, the kernel of $\mu$ is not unique, because there are a large number of possible choices for the map $\theta$ that points to the same $\mu$. But, in general, the order in which the states are observed gives informations about the pattern that minimizes the action. The physical meaning of the index $\alpha$ (and the $\theta$ maps in general) is best understood if the same experiment before is figured for spin systems that are not random, for example some Lagrangian system. In this setting two measurements are independent if taken at time intervals that are many times larger than the recurrence time of the system. By choosing a specific order for the states we are then fixing a time gauge, eventually adding some momentum to the Hamiltonian that describes the system.

Notice that the kernel function provides a powerful encoding of correlations and overlaps (and the event algebra in general). The following lemma express this important feature of the kernel representation. Clearly, we can write higher order correlation functions and overlaps using the same procedure:

**Definition 2. (Correlations and Overlaps)** Let $i, j \in V$ and select two rows of $M_\mu$ such that $z_i \in (x_{i-1}, x_i]$, $z_j \in (x_{j-1}, x_j]$, then the scalar product between the two rows $z_i$ and $z_j$ is the two point correlation function

$$\langle \sigma_i \sigma_j \rangle_\mu = \sum_{\tau_i, \tau_j} \mu(\tau_i) \mu(\tau_j) = \int_{y \in [0,1]} dy M_\mu^\tau z_i^\tau z_j^\tau. \quad (2.10)$$

Moreover, let $\sigma_V$ and $\tau_V$ be two magnetization states, let $t_\sigma \in S(\sigma_V)$ and $t_\tau \in S(\tau_V)$, then the scalar product between the columns $t_\sigma$ and $t_\tau$ of the kernel $M$ is the magnetization overlap between these states

$$q(\sigma_V, \tau_V) = \frac{1}{N} \sum_{i \in V} \sigma_i \tau_i = \int_{x \in [0,1]} dx M_\mu^{\sigma_i} M_\mu^{\tau_i}. \quad (2.11)$$

The property can be trivially verified by substituting the definition of $M$ into the above formulas.

Notice that this last statements admit an interesting operatorial description: if $M^\dagger$ is the transposed kernel, then $M^\dagger M = Q$ and $MM^\dagger = C$, where $Q$ and $C$ are the overlap and correlation matrices rescaled to the unitary square. In fact, according to the Definition 2 we can use kernels to represent creation and annihilation operators by the commutator

$$[M^\dagger, M] = M^\dagger M - MM^\dagger = Q - C, \quad (2.12)$$

where $Q$ and $C$ are the overlap and correlation matrices rescaled to the unitary square.
Two questions immediately arise, what happen if

\[
[M^\dagger, M] = 0 \quad (2.13)
\]

in some way, and the meaning of the measure \(\mu^\dagger\) associated to \(M^\dagger\).

If the commutator is zero the overlap matrix weakly converges to the correlation matrix, then the averages on \(V\) matches those on \(S\) after proper rescaling of the variable on which we take the average. Concerning the transposed measure, it describes a whole new spin system whose correlations and overlaps are exchanged in role respect to \(\mu\), we can give a simple definition as follows:

**Definition 3.** (Transposed measure \(\mu^\dagger\)) Let \(\mu \in \mathcal{P}(\Omega^V)\) be a probability measure describing a system of \(|V| = N\) spins, and let \(M\) be its kernel. Starting from \(\mu\) we can define a new sequence of probability measures \(\mu^\dagger_n \in \mathcal{P}(\Omega^R)\), each acting on a different spin space \(\Omega^R\) with different (eventually much larger) number of spins \(|R| = n\), such that for \(n \to \infty\) the sequence of the associated kernels weakly converges to \(M^\dagger\). We indicate this limit with the symbol \(\mu^\dagger\), and call it the transposed measure of \(\mu\).

Notice that the sequence \(\mu^\dagger_n\) is not unique, we can define many that converge to the same limit kernel. To precisely describe these concepts it would be in fact necessary to introduce the cut distance convergence and other graph theoretic arguments that are needed to work with limit kernels (an introduction can be found in Section 3), but there are already interesting cases in which \(\mu^\dagger\) can be defined also when \(n\) and \(N\) are finite. For example, consider the transposed support \(\nu^\dagger\). Let \(R = S\), and consider the spin vectors

\[
\sigma_S \in \Omega^S, \quad |\Omega^S| = |\Omega^{2N}| = 2^{2N},
\]

then we introduce \(\mathcal{S}\), that is a collection of \(N\) states of \(2^N\) spins

\[
\mathcal{S} := \{ \rho^S_i \in \Omega^S : i \in V \} \subset \Omega^S,
\]

these states are eventually the row vectors of \(\Omega^V\),

\[
\rho^S_i := \{ \rho^i_a \in \Omega : \rho^i_a = \tau^a_i, \alpha \in S \},
\]

and are obtained by the following construction; start from \(1_S\), a magnetization state with all positive spins, and flip half of them to get two groups: one with all positive spins and one with all negative. Then, apply this procedure iteratively inside each
group, until the state \( i = N \), where the spins oscillate in sign between each \( \alpha \) and \( \alpha + 1 \) (see Figure 2.3). Then, the transposed measure \( \nu^\dagger \) is

\[
\nu^\dagger (\sigma_S) = \frac{1}{N} \sum_{i \in V} \prod_{\alpha \in S} \left( 1 + \rho_i^\alpha \sigma_\alpha \right).
\]  

(2.17)

notice that due to their special construction, the states \( \rho_R^i \) are exactly orthogonal, ie this set has overlap exactly zero between any pair of states.

What allows to define \( \nu^\dagger \) with finite \( n \) is that the measure \( \nu \) is constant, then it can be reduced to a rational number apart from a global rescaling. For non uniform real measures we can only write an approximating sequence: let \( n > N \) and split \( R \) into \( 2^N + 1 \) disjoint regularized subsets \( R_\alpha \) with \( \alpha \in S \), plus one irregular \( R_0 \) that collects the real valued reminders:

\[
|R_\alpha| = |n \mu (\tau_\alpha^n)| / n \in \mathbb{N}, \quad |R_0| = 1 - \sum_\alpha |R_\alpha|.
\]  

(2.18)

When \( n \to \infty \) the reminder becomes irrelevant, and the sequence of measures

\[
\mu^\dagger_n (\sigma_R) := \frac{1}{N} \sum_{i \in V} \prod_{\alpha \in S, \sigma \in R_\alpha} \left( 1 + \rho_i^\alpha \sigma_\alpha \right)
\]  

(2.19)

converges to \( \mu^\dagger \). Luckily enough, the kernel we will deal with for the SK model (kernel of the eigenstates of magnetization) can be transposed for finite \( n \) like in the \( \nu \) case, and we can partially avoid the technicalities that one would need to manage with limit kernels.

The physical significance of these transposed kernels is in that we interpret them as those that actually describe the 1-RSB phase of the Parisi Ansatz, in fact, we interpret the Parisi full-RSB ansatz as a way to split the systems into sub-systems whose kernel weakly commute in the thermodynamic limit, so that the averages can be done with the transposed measure. We remark that the transposed measure is typically defined on an exponentially larger set of spins: this suggests a connection with the replicated system.

In the Section 4 of this paper, we will show that for SK a possible scheme to obtain such partition into commuting sub-systems is to simply split the spin group into small subgroups and apply the Bayes rule. The physical idea behind is in that any probability measure describing an actual physical spin system defined for variable number of spins must be coherent with the fact that such system has been constructed or created in some way. Then, it must always be possible to construct a sequence of systems of increasing size that eventually converge to the one we are looking at.
Consider, for example, an Ising Model in $d = 3$ in which $N$ spins are arranged into a cube: such system can be constructed starting with a single spin, then adding a layer of nearest neighbors, then add another and so on, until reaching the size $N$. Notice that this idea is not new at all, being the same that motivates the Cavity methods and the Grand Canonical ensemble.

The most convenient layering scheme depends on the system, but in general each layer has two kinds of energy contributions: those between the spin of the layer itself, that we call the core contribution, and those between the layer and the previous ones (just the previous in case of the finite dimensional Ising model before) that we call the interface.

In the case of fully connected models, we expect that if such partition is into very tiny layers, then the contributions from the core can be ignored respect to that of the interface, and that this will make the layer commute. In fact, notice that in case of the SK model (but the same holds for the Curie-Weiss model) there is no space structure, and any spin that is added form a layer itself, with a large interface. Following [27], we can formally define our analogue of the pure states of the RSB ansatz by partitioning the vertex set $V$ into $L$ subsets $V_\ell$.

\begin{equation}
\mathcal{V} = \{V_1, V_2, \ldots, V_L\},
\end{equation}

we label the parts by the ordered index $1 \leq \ell \leq L$ and also relabel the vertexes inside each $V_\ell$, for $i \in V_\ell$ we apply a map $i \to i_\ell$ such that $1 \leq i_\ell \leq |V_\ell|$. We will refer to

\begin{equation}
\sigma_{V_\ell} = \{\sigma_i \in \Omega : i \in V_\ell\}
\end{equation}

as layer magnetization states. Also, it will be convenient to express $\mathcal{V}$ in terms of the sequence $Q_\ell$, with $|Q_\ell| = q_\ell |V|$ and $0 \leq q_\ell \leq 1$. Starting from $Q_1 = V_1$ this sequence is defined recursively $Q_\ell = \bigcup_{t \leq \ell} V_t$ until the last step $\ell = L$, corresponding to the whole vertex set. The associated sequence of states is as follows:

\begin{equation}
\sigma_{Q_{\ell}} = \bigcup_{i \leq \ell} \sigma_{V_i} \in \Omega^{Q_{\ell}},
\end{equation}

they are composed by the first $\ell$ sub-states $\sigma_{V_i}$.

**Definition 4.** *(Filtration of $S$ induced by $\mathcal{V}$)* Let define the subsets

\begin{equation}
S(\sigma_{\mathcal{V}}) = \{\sigma_{\mathcal{V}}\}
\end{equation}
each composed by one element of $S$. Then, the filtration of $S$ induced by $\mathcal{V}$

$$\mathcal{S}(\mathcal{V}) = \{\mathcal{S}(\mathcal{V}) : 1 \leq \ell \leq L\}$$ (2.24)

is defined as the sequence of partitions

$$\mathcal{S}(\mathcal{V}) = \{S(\sigma_{\mathcal{Q}_\ell}) : \sigma_{\mathcal{Q}_\ell} \subseteq \mathcal{Q} \in \Omega^V\}$$, (2.25)

that is obtained by recursively joining the subsets according to the iteration

$$S(\sigma_{\mathcal{Q}_{\ell - 1}}) = \bigcup_{\sigma_{V} \in \Omega^V} S(\sigma_{\mathcal{Q}_\ell})$$, (2.26)

down to $\ell = 1$. Let $\mu \in \mathcal{P}(\Omega^V)$ and take some partition $\mathcal{V}$, we write

$$\mu(\sigma_{\mathcal{Q}_\ell}) = \sum_{\tau_{V} \in S(\sigma_{\mathcal{Q}_\ell})} \mu(\tau_{V})$$ (2.27)

for the probability mass of $S(\sigma_{\mathcal{Q}_\ell})$ under $\mu$. Let $f : \Omega^V \to \mathbb{R}$, then, the average value $\langle f(\sigma_{V}) \rangle_\mu$ according to $\mu$ is obtained starting from

$$f_L(\sigma_{\mathcal{Q}_L}) = f(\sigma_{\mathcal{Q}_L})$$, (2.28)

where $\mathcal{Q}_L = \mathcal{V}$, then we iterate the formula backward

$$f_{\ell - 1}(\sigma_{\mathcal{Q}_{\ell - 1}}) = \sum_{\sigma_{\mathcal{V}} \in \Omega^V} \xi_{\ell}(\sigma_{\mathcal{Q}_\ell}) f_{\ell}(\sigma_{\mathcal{Q}_\ell})$$, (2.29)

the average is taken according to the following distribution

$$\xi_{\ell}(\sigma_{\mathcal{Q}_\ell}) := \mu_{\ell}(\sigma_{\mathcal{Q}_\ell}) / \mu_{\ell - 1}(\sigma_{\mathcal{Q}_{\ell - 1}})$$, (2.30)

that is the distribution of the layer $\sigma_{\mathcal{V}_\ell}$ for a given $\sigma_{\mathcal{Q}_{\ell - 1}}$. [27]

This is enough to analyze the SK Hamiltonian, but before that, it will be useful to discuss the kernel $M$, shown in Figures [2.4, 2.5, 2.6, 2.7]. We identify our analogue of the RSB pure states in the sub-kernels associated to the partitions $\mathcal{S}(\mathcal{V})$ (see below).

We anticipate that the following definition aims to generalize the concept of pure state to any spin distribution, and does not yet have all the properties of the construction that one finds in [20], which we refer to as SK pure states and discuss in Section 4.
Definition 5. (Analogue Pure States) Let $\mathcal{S}(\mathcal{V})$ be a filtration induced by the partition $\mathcal{V}$ as in Definition 4, let $\mathcal{S}_\ell(\mathcal{V})$ be the partition associated to the $\ell$-th level of refinement, and let $M$ the kernel associated to $\mu$. Then, we can identify a partition

$$M^{\ell}_x(\sigma_{Q_{\ell-1}}) = M^{\ell}_x \cdot I[y \in \tilde{S}(\sigma_{Q_{\ell-1}})].$$

(2.31)

where $\tilde{S}(\sigma_{Q_{\ell-1}})$ is the image of $S(\sigma_{Q_{\ell-1}})$ on $[0,1]$. Hereafter will refer to these sub-kernels as the Pure States of $M$ according to $\mathcal{S}_\ell(\mathcal{V})$.

The analogue pure states of the $\ell$-th level are identified with the partition that one gets after $\ell$ refinements of $S$, an example is in Figure 2.5, the sub-kernel associated to the first pure state of each level is highlighted in blue.

Notice that for any pair of $\sigma_V, \tau_V \in S(\sigma_{Q_{\ell}})$, holds that the overlap (scalar product) between the magnetization states $\sigma_V$ and $\tau_V$ satisfy the inequality $\sigma_V \cdot \tau_V \geq |Q_{\ell} - |V \setminus Q_{\ell}|$ because by definition $\sigma_i = \tau_i$ at least for any $i \in Q_{\ell}$. By [20], any overlap distribution inside an SK pure state is expected to concentrate on some nontrivial value for large systems, this is recovered under the additional assumption that $\sigma_i$ and $\tau_i$ behave independently for $i \in V \setminus Q_{\ell}$, giving $\sigma_V \cdot \tau_V = |Q_{\ell}| + o(N)$ almost surely.

Although the previous definition allows to connect with the usual objects of Spin Glass theory, this partition structure of $M$ is not the most natural that one can arrange. In the following we define a second version of the pure states, which we call Layer States. These are not directly related with the usual notion of Pure State that is found in Spin Glass (SG) literature, and we interpret them as the transposed version the 1RSB pure states.

Definition 6. (Layer states) Let $\mathcal{S}(\mathcal{V})$ be a filtration induced by the partition $\mathcal{V}$ as in Definition 4 and let $M$ the kernel associated to $\mu$. We can identify a partition of $M$ into the sub-kernels associated to the $\tilde{x}_l(\sigma_{Q_{\ell}})$ distributions

$$M^{\ell}_x(\sigma_{Q_{\ell-1}}) = M^{\ell}_x \cdot I[x \in \tilde{V}_\ell, y \in \tilde{S}(\sigma_{Q_{\ell-1}})].$$

(2.32)

where $\tilde{V}_\ell$ and $\tilde{S}(\sigma_{Q_{\ell-1}})$ are the images of $V_\ell$ and $S(\sigma_{Q_{\ell-1}})$ on the interval $[0,1]$, we will refer to these sub-kernels as the Layer States of $M$ according to $\mathcal{S}(\mathcal{V})$.

One can confront the kernel of Figure 2.6A with its partition according to the previous definition, Figure 2.7.

The next section contains an introduction to more advanced kernel methods, the reader mostly interested in the physics of the SK model can jump directly to Section 4 for the moment.
Figure 2.4: Kernel of a product measure $\mu_{V_1} \otimes \mu_{V_2} \otimes \mu_{V_3}$ (lower kernel B) and its support (upper kernel A). Here we show the special case of three replicas of the same measure $\mu$ of Figure 2.7 located at $V_1, V_2$ and $V_3$, i.e., we take $\mu_{V_1} = \mu_{V_2} = \mu_{V_3} = \mu$ (replicated kernel).
Figure 2.5: Kernel representation of the filtration process according to $\mathcal{F}(\tau)$ of Definition 4 for the same measure $\mu$ of Figure 2.4. The vertical lines highlight the pure states of each layer $M$, $M^{\alpha_1}$ and $M^{\alpha_1 \alpha_2}$ and the last kernel is $M_\mu$ itself, the first pure state of each level is highlighted in blue.
Figure 2.6: Binary index of Definition applied to the same kernel of Figure 2.4 (upper kernel A). In the lower kernel B we explicitly show the states classified according to the α—index.
Figure 2.7: Pure state layers of Definition 6 in kernel representation for the same $\mu$ of Figure 2.4 and 2.6. The upper kernel $A$ shows the locations of the Pure state layers $M_{V_1}^{\alpha_1 \ldots \alpha_{t-1}}$, while in the lower kernel $B$ we show the refinements $S_{\alpha_1 \ldots \alpha_{t-1}}$, highlighted by their weights $\xi^{\alpha_1 \ldots \alpha_{t-1}}$. 
3 Kernel filtration

We continue by presenting an alternative approximation scheme that is intended to give a formulation for the finite volume pure states of Marinari et al. in [12] that is also compatible with the kernel methods presented in [16, 17, 18], which allows to operate directly in the thermodynamic limit (TL). In principle, this representation is more general, as it is not based on any special filtration, and in fact contains the one we used for the computations of the SK model as special case. The connection between kernels and the pure states of the RSB ansatz has been first noticed in [16], where a kernel encoding of $\mu$ is introduced in order to prove the following

**Lemma.** (Bapst, Coja-Oghlan, 2016) For any measure $\mu \in \mathcal{P}(\Omega^V)$ it is possible to take some arbitrary small $\varepsilon > 0$ and a partition of $\Omega^V$ into a finite number $n \geq n(\varepsilon, |K|)$, not dependent from $N$, of disjoint subsets $S_\alpha$, $0 \leq \alpha \leq n$ such that $\mu(S_0) \leq \varepsilon$ and

$$\sum_{K \in \{1, \ldots, N\}^{|K|}} \| \mu^\alpha_K - \bigotimes_{i \in K} \mu_i^\alpha \|_{TV} \leq \varepsilon N^{|K|}, \forall \alpha, |K| \geq 1$$

(3.1)

if $N$ is chosen large enough (we denoted by $\| \cdot \|_{TV}$ the total variation distance). For example, in the case $|K| = 2$ we can write

$$\sum_{\{i,j\} \in \{1, \ldots, N\}^2} \left\| \mu^\alpha_{\{i,j\}} - \mu_i^\alpha \otimes \mu_j^\alpha \right\|_{TV} \leq \varepsilon N^2, \forall \alpha.$$  

(3.2)

**Proof.** It is essentially a measure theoretic version of the Szemerédi Regularity Lemma, see Chapter 9.2 and 9.3 of [31] for a detailed review. A proof of Eqs (3.1) and (3.2) can be found in the first part of [16], after the statements of Theorem 2.2 and Corollaries 2.3-2.5.

This result tells us that for any measure $\mu$ that describes a system of variables with finite set $\Omega$ of inner states we can decompose our sample space $\Omega^V$ into a finite number $n(\varepsilon, |K|)$ of regular disjoint subsets $S_\alpha$, $1 \leq \alpha \leq n(\varepsilon, |K|)$ plus one irregular $S_0$ with $\mu(S_0) \leq \varepsilon$ such that for any regular subset $S_\alpha$ the layers of $\mu^\alpha$ over a randomly chosen set $K$ can be approximated by a product measure in the sense of Eq. (3.1). Surprisingly, the number $n(\varepsilon, |K|)$ of such regular subsets only depends on $|K|$, $|\Omega|$ and the level of precision $\varepsilon$ we want to achieve for our approximation, and it does not depend on the size $N$ of the system. This and many other results can be obtained by noticing that both probability measures and graphs can be exactly encoded into kernel functions.

---

2Given two measures $\mu, \nu : S \to [0, 1]$ and some $A \subseteq S$ the total variation distance between $\mu$ and $\nu$ is given by the formula $\| \mu - \nu \|_{TV} = 2 \sup_A | \mu(A) - \nu(A) |.$
For example, in \[16, 17, 19\] a new distance on \(\mathcal{P}(\Omega^V)\) based on Graph Theory is introduced to characterize Gibbs Measures directly in the thermodynamic limit.

**Definition.** *(Cut Distance)* Let \(M, W\) be two kernels and let \(\theta = (\theta_1, \theta_2)\) be a pair of measure preserving maps. We call Cut Norm the positive quantity

\[
\|M\|\Box = \sup_{A,B \subseteq [0,1]} \left| \int_{x \in A} \int_{y \in B} dxdy M(x,y) \right| \tag{3.3}
\]

and define the Cut Distance as

\[
D_\Box (M,W) = \inf_\theta \|M - W^\theta\|\Box, \tag{3.4}
\]

where \(W^\theta\) stands for \(W(\theta_1(x), \theta_2(y))\) \[31\]. In the context of probability theory the cut distance between \(\mu, \nu : S \to [0,1]\) is the cut distance \(D_\Box (M_\mu, M_\nu)\) between the associated kernels \(M_\mu, M_\nu\) of Eq.(2.7) below.

It can be shown \[31\] that the kernel space is compact in the Cut distance, and that convergence in cut distance is stronger than the weak* convergence when dealing with intensive quantities, such as the free energy density associated to a Gibbs measure (see Chapter 8 of \[31\] and therein, or the first part of \[16, 17\], see Chapter 8.2 of \[31\], or \[17\] for the measure theoretic approach).

The above Lemma is in fact a probabilistic version of the Szemeredi Regularity Lemma (Chapter 9.2 and 9.3 of \[31\]). Since the arguments presented in the following do not require the use Szemeredi Partitions we won’t discuss this here, but we stress that these are useful mathematical concepts and we warmly advice the reader to look at \[29, 31\] for further reading on this important subject.

We can give an intuitive picture by considering two independent and equitable partitions of \(S\) and \(V\) into sub-sets \(S_a\) and \(V_\ell\), their number being \(n\) and \(L\) respectively. Then, define the magnetization averages inside the blocks

\[
m^\mu_\ell = \frac{1}{|V_\ell|} \sum_{i \in V_\ell} \langle \sigma_i \rangle_{\mu^e} = \frac{1}{|V_\ell||S_a|} \sum_{i \in V_\ell} \sum_{\alpha \in S_a} \mu^{\alpha} \tau^{\alpha}_i \tag{3.5}
\]

Then, let \(\eta^\alpha : \Omega^{V_\ell} \to [0,1]\) and \(\eta : \Omega^V \to [0,1]\) be defined as follows

\[
\eta^\alpha (\sigma_{V_\ell}) := \prod_{i \in V_\ell} \left( 1 + \frac{m^\mu_\ell \sigma_i}{2} \right), \quad \eta (\sigma_V) := \frac{1}{n} \sum_{\alpha \leq \alpha L} \prod_{\ell} \eta^\alpha (\sigma_{V_\ell}). \tag{3.6}
\]

Szemeredi lemma guarantees that, for any small \(\varepsilon\), if \(n\) and \(L\) are taken large enough
it is possible to find a kernel such that $D_{\square}(M_\mu,M_\eta) \leq \varepsilon$. This fact can already have some interesting applications because the number of parameters that controls the trial measure $\eta$ is finite, although could be very large if we require $\varepsilon$ to be very small. For example, let $H(\sigma_V)$ be some Hamiltonian and $Z$ the partition function. By definition

$$
Z := \sum_{\sigma_V \in \Omega^V} e^{-\beta H(\sigma_V)} = \langle \exp \left[ -\beta H(\sigma_V) - \log \mu(\sigma_V) \right] \rangle_\mu \leq \\
\leq \exp \left[ -\beta \langle H(\sigma_V) \rangle_\mu - \langle \log \mu(\sigma_V) \rangle_\mu \right] = \exp \left[ -\beta F_{\beta,H}(\mu) \right], \quad (3.7)
$$

where in the second row we applied Jensen inequality, and also introduced the Gibbs free energy functional

$$
\mathcal{F}_{\beta,H}(\mu) := \langle H(\sigma_V) \rangle_\mu + \beta^{-1} \langle \log \mu(\sigma_V) \rangle_\mu. \quad (3.8)
$$

We easily obtained a variational bound for the free energy that is optimized by the Gibbs measure $\mu^*$, that is

$$
F = -\beta^{-1} \log Z = \inf_{\mu \in \mathcal{P}(\Omega^V)} \mathcal{F}_{\beta,H}(\mu). \quad (3.9)
$$

Nonetheless, the function $\mathcal{F}_{\beta,H}(\mu)$ may be hard to handle, because the number of parameters that controls $\mu$ grows exponentially with the size of the system, it is much simpler is to minimize $\mathcal{F}_{\beta,H}(\eta)$ on the $nL$ parameters that control $\eta$,

$$
F' = \inf_{M} \mathcal{F}_{\beta,H}(\eta). \quad (3.10)
$$

It is possible to show that if the cut distance $D_{\square}(M_\mu,M_\eta) \to 0$, then also the free energy densities $F'/N \to F/N$ are weakly convergent.

**Definition 7. (Tree Index for S and V) Let $0 \leq t \leq T$, then, let introduce the following pair of tree indexes. The first is

$$
A_t := \alpha_1 \alpha_2 \ldots \alpha_t \in \prod_{k \leq t} \{1,2,\ldots,n_k\}, \quad (3.11)
$$

where each subindex $\alpha_t$ runs from 1 to some integer $n_t$. The second is

$$
I_t := i_1 i_2 \ldots i_t \in \prod_{k \leq t} \{1,2,\ldots,v_k\}. \quad (3.12)
$$

The integer parameters $n_t$ and $v_t$ are assumed the same for each level $t$. Let define the sets $S_A = \{A_T(\alpha)\}$ each composed by only one element of $S$ mapped onto the Tree
Index $A_T$ by the invertible map $A_T(\alpha)$. Then we call $\mathcal{S}$, filtration of $S$, the sequence of equitable refinements

$$\mathcal{S}_t = \{ S_{A_t} \subset S : A_t \in \prod_{k \leq t} \{1, 2, \ldots, n_k\} \},$$

(3.13)

obtained from joining the subsets $S_{A_t} \subseteq S$ from the last layer $S_{A_T}$ down to the root level $t = 0$, associated to $S$. Then, define $V_{I_t} = \{ I_t(i) \}$, containing only one element $i$ of $V$ mapped on the index $I_T$ by the invertible map $I_T(i)$. We call $\mathcal{V}$ filtration of $V$ the sequence of refinements of

$$\mathcal{V}_t = \{ V_{I_t} \subset V : I_t \in \prod_{k \leq t} \{1, 2, \ldots, v_k\} \},$$

(3.14)

obtained by joining the subsets $V_{I_t} \subseteq V$ from the last layer $V_{I_T}$ to the root level, associated with $V$ itself. We remark that the two filtrations above are defined independently, apart from the fact that must have the same number of levels $T$, the definition of pure state is the same given before in Definition 5, with the refinements $\mathcal{S}$ on behalf of $\mathcal{S}_t$ on behalf of $\mathcal{V}_t$.

Using both the filtrations $\mathcal{S}$ and $\mathcal{V}$ one can construct a sequence of kernels that approximate $M_\mu$ by progressively averaging (actually “de-averaging”) over the refinements, for example, like in the following nested approximation scheme for the magnetization kernel

**Definition 8. (Magnetization Averages)** Start from the last layer $t = T$, that is associated directly to the external nodes,

$$\Omega \ni m_{I_T}^A = \tau_{I_T}^{\alpha(A_T)} .$$

(3.15)

For all the other layers we define

$$[0, 1] \ni m_{I_t}^A = \frac{1}{|V_{I_t}||S_{A_t}|} \sum_{r \in V_{I_t}} \sum_{\alpha \in S_{A_t}} \mu^\alpha \sigma_*^\alpha,$$

(3.16)

down to the root level $t = 0$, for which we drop the tree index and use simply

$$m = \frac{1}{N|S|} \sum_{r \in V} \sum_{\alpha \in S} \mu^\alpha \sigma_*^\alpha = \frac{1}{N} \sum_{r \in V} \langle \sigma_i \rangle_\mu .$$

(3.17)

Starting from the Magnetization Averages we define the parameters

$$\delta m_{I_t}^A = m_{I_t}^A - m_{I_{t-1}}^A ,$$

(3.18)
that indicate the fluctuation of the magnetization respect to the average of the cell at the level \( t - 1 \). For the the root level \( t = 0 \) we simply write \( \delta m = m \). In this representation the measure \( \mu \) is obtained from the initial condition

\[
\eta^{A_r}_{t_r}(\sigma_{t_r}) = \frac{1}{2} \left( 1 + \sigma_{t_r} \sum_{i_{t_r}} \delta m_{i_{t_r}}^{A_r} \right) \tag{3.19}
\]

and then applying the recursive formula

\[
\eta^{A_{t-1}}_{t-1}(\sigma_{V_{t-1}}) = \frac{1}{n_t} \sum_{\alpha_{t-1}} \prod_{i_{t-1}} \eta^{A_t}_{i_{t-1}}(\sigma_{V_{t-1}}) \tag{3.20}
\]

to the last level, the measure \( \eta \) that we would like to adapt to \( \mu \),

\[
\eta(\sigma_V) = \frac{1}{n_1} \sum_{\alpha_1=1}^{n_1} \prod_{i_1=1}^{n_1} \eta^{\alpha_1}_{i_1}(\sigma_{V_{i_1}}). \tag{3.21}
\]

We remark that, by construction, the averages of any \( \delta m \) respect to the indexes \( \alpha_t \) and \( i_t \) is zero, and any averaged magnetization can be reconstructed from the increments by summing back them together. The scheme is shown using kernel representation in Figures 3.1 and 3.2.
Figure 3.1: Kernel representation of the filtration process in Definition 8 for a 3-RSB system\((L = 3)\), and partition parameters \(n_0 = 5, n_1 = 5\) and \(v_0 = 3, v_1 = 2\). The measure \(\mu\) is the same of Figure 2.4. The vertical lines highlight the pure states of each layer \(M\), \(M^{\alpha_1}\) and \(M^{\alpha_1\alpha_2}\) of the kernel \(M_\mu\). The last kernel is \(M_\mu\) itself. The filtration has been chosen to match that of Figure 2.5.
Figure 3.2: Detail of the kernel filtration shown in Figure (3.1), associated to the first effective level of the scheme, that is 1RSB. Each block of the kernel has constant magnetization $m_{i_1}^{\alpha_1}$ as given in Definition 8 with $\alpha_1 \in \{4, 5, 6, 10, 11\}$ and $1 \leq i_1 \leq 3$. 
4 The SK model

We can finally apply these concepts to the SK model. We start introducing the basic
quantities of the previous section in the case of Gibbs measures. Consider a system of
$N$ spins, governed by the Hamiltonian

$$ H : \Omega^V \to \mathbb{R}, $$

(4.1)

the associated Gibbs measure is

$$ \mu (\sigma_V) = \frac{1}{Z} e^{-\beta H(\sigma_V)}, $$

(4.2)

the normalization (partition function) is

$$ Z = \sum_{\sigma_V \in \Omega^V} e^{-\beta H(\sigma_V)} = \sum_{\sigma_V \in \Omega^V_1} \ldots \sum_{\sigma_V \in \Omega^V_L} e^{-\beta H(\sigma_{QL})}. $$

(4.3)

Now, the Hamiltonian of the Sherrington-Kirkpatrick (SK) model \[2, 23\] is

$$ H_{sk} (\sigma_V) := \frac{1}{\sqrt{N}} \sum_{i \in V} \sum_{j < i} \sigma_i J_{ij} \sigma_j, $$

(4.4)

with $J$ Gaussian (asymmetric) random matrix with normal independent entries of unitary variance. From now we will work with a random Hamiltonian instead of a single instance of it, so that we don’t have to add another index for the disorder when computing the Gaussian averages $E (\cdot)$ (for which we use this special notation). As before, we can define the partition function

$$ Z = \sum_{\sigma_V \in \Omega^V} e^{-\beta H_{sk}(\sigma_V)}, $$

(4.5)

that in this case is a $J$-dependent random quantity.

To simplify some of the coming manipulations we will consider the Asymmetric SK Hamiltonian (ASK),

$$ H (\sigma_V) := \frac{1}{\sqrt{N}} \sum_{i \in V} \sum_{j \in V} \sigma_i J_{ij} \sigma_j, $$

(4.6)

because, apart from vanishing finite size corrections, holds

$$ \sqrt{2} H_{sk} (\sigma_V) \overset{d}{=} H (\sigma_V) $$

(4.7)
in distribution. Using this definition the temperature is rescaled by a factor $\sqrt{2}$ respect to the usual Parisi functional. The functional for the original SK model is recovered from that of ASK by substituting $\beta$ with $\beta/\sqrt{2}$.

**Lemma 9.** *(Layer States of ASK)* Given some partition $\mathcal{V}$ the ASK Hamiltonian can be decomposed according to Definition 4 as follows

$$H(\sigma_{\mathcal{V}}) = \sum_{\ell} H_{\ell}(\sigma_{Q_{\ell}}), \quad (4.8)$$

where the $H_{\ell}$ are the layer Hamiltonians

$$H_{\ell}(\sigma_{Q_{\ell}}) = \frac{1}{\sqrt{N}} \sum_{(i,j) \in W_{\ell}} \sigma_{i}J_{ij}\sigma_{j}, \quad (4.9)$$

where we introduced the sequence of edges sets

$$W_{\ell} := Q_{\ell}^{2} \setminus Q_{\ell-1}^{2}. \quad (4.10)$$

In general, we can associate the (random) distributions

$$\xi_{\ell}(\sigma_{Q_{\ell}}) = \frac{1}{Z_{\ell}(\sigma_{Q_{\ell-1}})} e^{-\beta H_{\ell}(\sigma_{Q_{\ell}})}, \quad (4.11)$$

and the (random) partition functions

$$Z_{\ell}(\sigma_{Q_{\ell-1}}) = \sum_{\sigma_{\mathcal{V}} \in \Omega_{\mathcal{V}_{\ell}}} e^{-\beta H_{\ell}(\sigma_{Q_{\ell}})}. \quad (4.12)$$

**Proof.** The representation of Definition 4 for ASK is as follows. The partition of $V$ is into a number $L$ of subsets $\mathcal{V}_{\ell}$, each of macroscopic size $O(N)$. As before we write everything in terms of the sets $Q_{\ell}$ so that all is controlled by the parameters $|Q_{\ell}|/N = q_{\ell}$. The sizes of $\mathcal{V}_{\ell}$ are then $|\mathcal{V}_{\ell}|/N = q_{\ell} - q_{\ell-1}$. Let

$$W := \{(i,j) \in V^{2} : i,j \in V\} \quad (4.13)$$

be the edges set (for the SK this is a fully connected graph). It is easy to verify that the effect of $\mathcal{V}$ is to produce a corresponding partition of $W$ into subsets $W_{\ell}$ such that each $W_{\ell}$ contains all edges with both ends in $Q_{\ell}$ minus those with both ends in $Q_{\ell-1}$ (see
Figure 4.1: Partition of $W$ induced by $\mathcal{Y}$. Under $\mathcal{Y}$ the edges set $W$ is splitted into subsets $W_\ell$, containing all edges with both ends in $Q_\ell$ minus those with both ends in $Q_{\ell-1}$. As predicted in Definition 4, the contribution to the total energy given by $W_\ell$ is adapted to the spins of $Q_{\ell-1}$.

Figure 4.1). Then we can define a partition of $W$

$$\mathcal{W}(\mathcal{Y}) = \{W_1, W_2, \ldots, W_L\}$$

(4.14)

uniquely defined by the partition $\mathcal{Y}$, the sets are $W_\ell = Q_\ell^2 \setminus Q_{\ell-1}^2$. The contribution to the total energy given by $W_\ell$ is then

$$H_\ell(\sigma_Q) = \frac{1}{\sqrt{N}} \sum_{(i,j) \in W_\ell} \sigma_i J_{ij} \sigma_j =$$

$$= \frac{1}{\sqrt{N}} \sum_{(i,j) \in Q_\ell^2} \sigma_i J_{ij} \sigma_j - \frac{1}{\sqrt{N}} \sum_{(i,j) \in Q_{\ell-1}^2} \sigma_i J_{ij} \sigma_j. \quad (4.15)$$

\[\square\]

As said in the previous sections, this partition structure is inspired by the fact that if $H(\sigma_V)$ can be defined for arbitrary sizes $|V|$ then we should be able to represent it.
Figure 4.2: Kernel diagrams of the pair correlations and their $\mathcal{Y}$—partition. The smaller diagrams on top are the partitions of $S$ and $V^2$ described in the captions of the Figures 2.6, 2.7 and 4.1. The last shows the multi-kernel $[0, 1]^3 \rightarrow \Omega$ that encodes the tensor $V^2 S \rightarrow \Omega$ of the pair correlations $\tau_{\alpha \beta} \tau_{\alpha}$, all sub-kernels have been removed except the first sequence 11...1 (in gray and white) to highlight the structure of the pure states.
as the terminal point of the sequence $\sqrt{q_l}H (\sigma_{Q_l})$. Notice that the layer Hamiltonians $H_\ell$ of Eq.4.9 can in fact be expressed in terms of the difference between two ASK Hamiltonians, depending on $Q_\ell$ and $Q_{\ell-1}$ spins respectively, ie

$$H_\ell (\sigma_{Q_\ell}) = \sqrt{q_\ell}H (\sigma_{Q_\ell}) - \sqrt{q_{\ell-1}}H (\sigma_{Q_{\ell-1}}).$$

(4.16)

In this form the layer Hamiltonians allow a better reading of what we are actually doing, ie reconstructing the system growing it layer by layer toward a target size $|V| = N$. This can be seen also in the Figure 4.1 where a trial partition of the edges set $W$ is shown. Here the edges $(i, j)$ are represented as elements of the square $W = V^2$. Remember that the noise of $J_{ij}$ is independent from edge to edge, then also between different $W_\ell$. This means that we can average the noise independently for different $\ell$.

To better understand the physical meaning it will be convenient to introduce the cavity fields that makes the interface

$$h_{V_\ell} (\sigma_{Q_{\ell-1}}) := \left\{ h_i (\sigma_{Q_{\ell-1}}) \in \mathbb{R} : i \in V_\ell \right\},$$

(4.17)

with local components given by

$$h_i (\sigma_{Q_{\ell-1}}) := \frac{1}{\sqrt{|Q_{\ell-1}|}} \sum_{j \in Q_{\ell-1}} J_{ij} \sigma_j,$$

(4.18)

and same $J$ used for the Hamiltonian. Then, the layer can be rewritten as

$$H_\ell (\sigma_{Q_\ell}) = \sqrt{q_\ell - q_{\ell-1}}H (\sigma_{V_\ell}) + \sqrt{2q_{\ell-1}} \sigma_{V_\ell} h_{V_\ell} (\sigma_{Q_{\ell-1}}),$$

(4.19)

where the self-interaction is simply a smaller ASK Hamiltonian $H (\sigma_{V_\ell})$, while the contribution from the interface is mediated by the cavity fields

$$\tilde{H}_\ell [\sigma_{V_\ell}, h_{V_\ell} (\sigma_{Q_{\ell-1}})] := \sigma_{V_\ell} h_{V_\ell} (\sigma_{Q_{\ell-1}})$$

(4.20)

and match the Hamiltonian of an Asymmetric Bipartite SK model (ABSK, see [13]) at slightly shifted temperature, and with ratio between the group sizes that shrinks as $L$ increases. Introducing the auxiliary temperature variables

$$\tilde{\beta}_\ell := \sqrt{q_\ell - q_{\ell-1}}, \quad \beta_\ell := \sqrt{2q_{\ell-1}}$$

(4.21)
we arrive to the expression

\[ \beta H_\ell (\sigma_Q) = \beta^* H (\sigma_V) + \beta H_\ell \left[ \sigma_V, h_\ell (\sigma_{Q_{\ell-1}}) \right]. \]  

(4.22)

From this reformulation one can appreciate the structure of the interactions: the cavity fields \( h_\ell (\sigma_{Q_{\ell-1}}) \) act as random external fields that depend on the previous level, and towards which the system tries to align, while the thermal fluctuations and the Hamiltonian \( H (\sigma_V) \) act as perturbations that can introduce more directions for the eigenstates [27].

**Lemma 10.** *(IO model)* For large \( N \) and \( L \), the Gibbs measure associated to the layer \( \beta H_\ell \) converges in distribution to that of the interface \( \beta \bar{H}_\ell \).

**Proof.** First we notice that the term \( H (\sigma_V) \) is multiplied by \( \beta \sqrt{q - q_{\ell-1}} \) and its role becomes less important as \( L \) increases. Then, for any finite temperature \( \beta \) we can make \( N \) and \( L \) large enough to have a sequence for which \( \beta^* < \beta \) at any \( \ell \), and it is established since [5] and [6] by second moment methods that in the high temperature regime the annealed averages match the quenched ones.

The layers can be approximated in distribution by the (random) relative weights of the interface only (hereafter IO model),

\[ \bar{\xi}_\ell (\sigma_Q) := \frac{1}{Z_\ell (\sigma_{Q_{\ell-1}})} e^{-\beta H_\ell \left[ \sigma_V, h_\ell (\sigma_{Q_{\ell-1}}) \right]} \]  

(4.23)

with (random) partition function given by

\[ Z_\ell (\sigma_{Q_{\ell-1}}) := \sum_{\sigma_V \in \Omega_V} e^{-\beta H_\ell \left[ \sigma_V, h_\ell (\sigma_{Q_{\ell-1}}) \right]} = \sum_{\sigma_V \in \Omega_V} e^{-\beta H_\ell \left[ \sigma_V, h_\ell (\sigma_{Q_{\ell-1}}) \right]} = \prod_{i \in V_\ell} 2 \cosh \left[ \beta h_i (\sigma_{Q_{\ell-1}}) \right]. \]  

(4.24)

The interface is simply a group of independent spins coupled to a field, that is adapted to the previous layers but does not depend on the one on which acts. To make some stronger statement it will be convenient to introduce the (random) spin vector representing the direction of the external field \( h_\ell \),

\[ \omega_\ell (\sigma_{Q_{\ell-1}}) := \{ \omega_\ell (\sigma_{Q_{\ell-1}}) \in \Omega : i_\ell \in V_\ell \}. \]  

(4.25)
that we call master direction, its components are defined as follows:

\[ \omega_{i_\ell} (\sigma_{Q_{\ell-1}}) := h_{i_\ell} (\sigma_{Q_{\ell-1}}) / |h_{i_\ell} (\sigma_{Q_{\ell-1}})|. \] (4.26)

Notice that due to parity of the \( \cosh \) function the partition function \( \bar{Z}_\ell \) does not depend on \( \omega_{V_\ell} \) but only on the projections of \( \omega_{i_\ell} \) on the local fields \( h_{i_\ell} \). This vector has all positive entries \( |h_{i_\ell}| \) and can be represented by the Hadamard product between \( \omega_{V_\ell} \) and \( h_{V_\ell} \). Let also introduce the local overlap of the \( i_\ell \)-th spin with the direction of the external field

\[ m_{i_\ell} (\sigma_{Q_{\ell-1}}) := \langle \sigma_{i_\ell} \rangle \bar{\xi}_i (\sigma_{Q_{\ell-1}}) \omega_{i_\ell} (\sigma_{Q_{\ell-1}}) = \tanh [\beta_i h_{i_\ell} (\sigma_{Q_{\ell-1}})] \] (4.27)

this parameter is a measure of how much the spin \( \sigma_{i_\ell} \) is binded to the direction of the external field, and depends on the amplitude of \( h_{i_\ell} \).

The parameter \( m_{i_\ell} \) is an analogue of the local magnetization, is also related to the local overlap by the formula

\[ m_{i_\ell} (\sigma_{Q_{\ell-1}}) = \sqrt{\langle \sigma_{i_\ell} \rangle \bar{\xi}_i (\sigma_{Q_{\ell-1}}) \otimes \bar{\xi}_i (\sigma_{Q_{\ell-1}})}, \] (4.28)

and can be used as local order parameter. If the amplitude \( |h_{i_\ell}| \) (or \( \beta \)) is large, the spin will be forced to align with the field and \( m_{i_\ell} \to 1 \). On the contrary, when \( h_{i_\ell} \) is small, or \( \beta \) is small, then \( m_{i_\ell} \to 0 \) as the spin disentangles from the direction \( \omega_{i_\ell} \). The fluctuations of the interfaces can be characterized in detail by studying the kernel of \( \sigma^*_{i_\ell} := \sigma_{V_\ell} \circ \omega_{V_\ell} \) Hadamard product between \( \sigma_{V_\ell} \) and \( \omega_{V_\ell} \), the components

\[ \sigma^*_{i_\ell} (\sigma_{Q_{\ell-1}}) := \sigma_{i_\ell} \omega_{i_\ell} (\sigma_{Q_{\ell-1}}). \] (4.29)

Notice that the scalar product of spin states (overlap) equals the magnetization of their Hadamard product, also notice that the Hadamard product of a group of spin states by a common master spin state is a theta-map of the kind described in Section 2 and does not change the overlap between the states in the transformed group. Since we are now analyzing the behavior inside a given layer state, we can drop the dependence on previous layers. First notice that the layer Hamiltonian can be written in terms of the
the relative orientations of $\sigma^*_V$ and $\sigma_V$ are randomized by the multiplication with the random direction $\omega_i$, but are not independent, for example their overlap $\sigma^*_V \cdot \sigma_V$ is the total magnetization of the master direction $\omega_V$. Since $\sigma_V = \sigma^*_V \cdot \omega_V$, for a general non-random function $f$ it will be convenient to introduce the associated random function:

$$f^* (\sigma_V) := f (\sigma^*_V) = f (\sigma_V \cdot \omega_V), \quad (4.31)$$

then we can simplify the notation by rewriting $f (\sigma_V)$ as follows

$$f (\sigma_V) = f (\sigma^*_V \cdot \omega_V) = f^* (\sigma^*_V) \quad (4.32)$$

and recast the spin variables on which the average is applied:

$$\langle f (\sigma_{V_i}) \rangle_{\xi_i} = \frac{\sum_{\sigma_{V_i} \in \Omega^{V_i}} e^{-\beta_i \sum_{i \in V_i} \sigma_{V_i}^* |h_i|} f (\sigma_{V_i})}{\sum_{\sigma_{V_i} \in \Omega^{V_i}} e^{-\beta_i \sum_{i \in V_i} \sigma_{V_i}^* |h_i|}} = \frac{\sum_{\sigma_{V_i} \in \Omega^{V_i}} e^{-\beta_i \sum_{i \in V_i} \sigma_{V_i} |h_i|} f^* (\sigma_{V_i})}{\sum_{\sigma_{V_i} \in \Omega^{V_i}} e^{-\beta_i \sum_{i \in V_i} \sigma_{V_i} |h_i|}} \quad (4.33)$$

ie, we have changed the sum from $\sigma^*_V$ to $\sigma_V$, remembering that the two vectors are uniquely linked and both span the same space.

There is still a little technical difficulty in that $|h_i|$ are different from site to site. We can overcome this problem in various ways, for example by separating the average from the residual fluctuations of the field:

$$\psi_i := \frac{1}{|V_i|} \sum_{i \in V_i} |h_i|, \quad \varphi_i := |h_i| - \frac{1}{|V_i|} \sum_{i \in V_i} |h_i|. \quad (4.34)$$

Notice that for $N \to \infty$ the average amplitude $\psi_i$ converges to a deterministic constant that only depends on the input from the previous layers,

$$\sum_{i \in V_i} \sigma_i |h_i| = \psi_i M (\sigma_V) + \sigma_{V_i} \cdot \varphi_i, \quad (4.35)$$

where $M (\sigma_V)$ denotes the total magnetization of $\sigma_V$ (that should not be confused with
the kernel of the previous sections). As we shall see in short, this expression already allows to establish the connection with the Random Energy Model.

An alternative way is to realize that the layer Hamiltonian $H_\ell$ is essentially a Number Partitioning Problem (NPP) on the number sequence $|h_i|$: the NPR would actually be $|H_\ell|$ but the function $H_\ell^2$ is the same for both models. Then, following Borgs, Chayes and Pittel [7,8] (see also [9] for an informal discussion), we reorder the spins of $V_\ell$ such that the $|h_i|$ amplitudes form a non-decreasing sequence in $i$, ie such that $|h_{i+1}| \geq |h_i|$, this is possible because the fields $h_i$ are extracted independently for different index and there is no space structure to preserve. Then, we can further divide the layers into some large number $L'$ of sub-layers $V_{\ell'\ell}$, of equal volume, marked by $\ell'$: inside these sub-layers the fluctuations are bounded

$$
\psi_{\ell' \ell} \leq |h_i| \leq \psi_{\ell' \ell}^+, \quad \delta_{\ell' \ell} := |\psi_{\ell' \ell}^+ - \psi_{\ell' \ell}^-| \leq c_0/L'
$$

almost everywhere by some constant $c_0/L'$ with finite $c_0$, so that the fluctuations go to zero for large $L'$ and can be neglected. The sub-layers converge to

$$
\tilde{H}_{\ell' \ell} (\sigma_{\ell' \ell}, h_{V_{\ell' \ell}}) := \psi_{\ell' \ell} \mathcal{M} (\sigma_{\ell' \ell}),
$$

(4.37)

where $\psi_{\ell' \ell}$ is the average of $|h_i|$ inside $V_{\ell' \ell}$, then

$$
H_\ell \left[ \sigma_{\ell}, h_\ell (\sigma_{Q_{\ell-1}}) \right] \to \sum_{\ell' \leq \ell} \psi_{\ell' \ell} \left[ h_{V_{\ell}} (\sigma_{Q_{\ell-1}}) \right] \mathcal{M} (\sigma_{V_{\ell' \ell}}),
$$

(4.38)

notice that this last representation allows to study the sub-layer in terms of the magnetization eigenstates only, although this is done at the price of introducing a new level of partition.
5 Thermodynamics of the interface

To analyze the sub-layers it will be convenient to further simplify the notation and drop the $\ell\ell'$ index for this section. Let $\xi$ the sub-layer measure for some fixed pair $\ell$ and $\ell'$, then, the average of a function can be expressed as:

$$\langle f(\sigma_V) \rangle_\xi = \frac{\sum_{\sigma_V \in \Omega_V} e^{-\beta \psi M(\sigma_V) - \beta \sigma_V \cdot \phi} f^*(\sigma_V)}{\sum_{\sigma_V \in \Omega_V} e^{-\beta \psi M(\sigma_V) - \beta \sigma_V \cdot \phi}}.$$  

(5.1)

We first do the average according to $M$. Let introduce the set of magnetization eigenstates with given eigenvalue $M$ (see Figure 5.1):

$$\Omega(M, N) := \{ \sigma_V \in \Omega_V : M(\sigma_V) = M \},$$  

(5.2)

In what follows it will be convenient to also define a simplified notation $\Omega(m)$, to indicate set of the magnetization eigenstates with given eigenvalue $M = \lfloor mN \rfloor$, where $\lfloor mN \rfloor$ is the lower integer part of $mN$ and $m \in [-1, 1]$.

**Lemma 11.** Let $\gamma$ be the Gibbs measure associated to the Hamiltonian $M(\sigma_V)$:

$$\gamma(\sigma_V) := \frac{1}{Z} e^{-\beta \psi M(\sigma_V)},$$  

(5.3)

the partition function is simply $\log Z = N \log 2 \cosh (\beta \psi)$, for large systems the average of some test function $f : \Omega_V \rightarrow \mathbb{R}$ respect to $\gamma$ converges to

$$\langle f(\sigma_V) \rangle_\gamma = \langle f(\sigma_V) \rangle_{\Omega(m)},$$  

(5.4)

the order parameter $m := \tanh (\psi \beta)$ is the limit magnetization at which the test function $f$ is sampled.

**Proof.** The set $\Omega(m)$ can be studied using Large Deviations Theory (LDT), even at the sample-path LDT level. For example, following the methods presented in the proof sections of [11], ie by some simple applications of the Varadhan Integral Lemma and other standard LDT theory tools, one can compute $|\Omega(m)|$ and find that is proportional to $\exp [N \phi (m)]$ with $\phi$ convex function of $m$. Given that these methods are well known we only give the essential features.
Figure 5.1: Example of a partition of $\Omega^V$, with $N = 4$, into the magnetization eigenstates $\Omega(M, N)$ of Eq. (5.2). The spin states are organized by decreasing total magnetization. The sub-kernels associated to groups of spins with given magnetization $M$ are highlighted in various shades. Let $X$ be the number of spin up in a given spin state, then, the kernels with fixed $M$ are equivalent to self-avoiding lattice gases of $X = N/2 - M/2$ particles on a lattice of size $N$. 
The average according to $\gamma$ can be expressed in terms of the magnetizations eigenstates of as follows:

$$\langle f(\sigma_V) \rangle_{\gamma} = \sum_M |\Omega(M,N)| e^{-\beta M} \langle f(\sigma_V) \rangle_{\Omega(M,N)},$$  \hspace{1cm} (5.5)$$

In the last formula we introduced a braket notation for the uniform average on the eigenstates of magnetization: for integer $M$ we write

$$\langle f(\sigma_V) \rangle_{\Omega(M,N)} := \frac{1}{|\Omega(M,N)|} \sum_{\sigma_V \in \Omega(M,N)} f(\sigma_V),$$  \hspace{1cm} (5.6)$$

that simplifies to $\langle f(\sigma_V) \rangle_{\Omega(m)}$ in case $M = [mN]$. In this case is also possible to represent the average in integral form: for any test function $f$

$$\sum_{\sigma_V \in \Omega'} e^{-\beta M(\sigma_V)} f(\sigma_V) \propto \int_{m \in [-1,1]} dm e^{-N p(\beta, m)} \langle f(\sigma_V) \rangle_{\Omega(m)}. \hspace{1cm} (5.7)$$

The value at which $m$ concentrates can be the computed using a saddle point method applied to the pressure functional $p(\beta, m) := \beta m - \phi(m)$ with entropy functional $\phi(m)$ given by the formula

$$\phi(m) = \frac{1 + m}{2} \log \frac{1 + m}{2} + \frac{1 - m}{2} \log \frac{1 - m}{2} = - \log 2 + \frac{1}{2} \log (1 - m^2) + \frac{m}{2} \log \frac{1 + m}{1 - m}. \hspace{1cm} (5.8)$$

Follows $\partial_m \phi(m) = \tanh^{-1}(m)$, and $\partial^2_m \phi(m) = 1/(1 - m^2)$. Then we compute the average magnetization $m(\beta)$ by putting the derivative of the pressure to zero, $\partial_m p(\beta, m) = 0$, that is equivalent to impose $\partial_m \phi(m) = \beta$. In the end one obtains that $m(\beta) = \tanh(\beta)$ and finds

$$\sum_{\sigma_V \in \Omega'} e^{-\beta M(\sigma_V)} f(\sigma_V) \propto \langle f(\sigma_V) \rangle_{\Omega(m(\beta))}, \hspace{1cm} (5.9)$$

we remark once again that the average of $\sigma_V$ on the states $\Omega(m)$ is taken with equal weights for each magnetization eigenstate.

Then, in the limit of large $N$ the sub-layer average is

$$\langle f(\sigma_V) \rangle_{\xi} = \frac{\sum_{\sigma_V \in \Omega(m)} e^{-\beta \sigma_V \varphi_V} f^*(\sigma_V)}{\sum_{\sigma_V \in \Omega(m)} e^{-\beta \sigma_V \varphi_V}}, \hspace{1cm} (5.10)$$

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the next lemma we show that at very low temperature the sub-layer fluctuations converge in distribution to the Random Energy Model of Derrida (REM, see also [10, 14, 23] for reviews).

Lemma 12. (Random Energy Model) Let $\varphi_V \in \mathbb{R}^V$ be some random vector with $N$ independent entries of variance $\delta$ as defined in Eq. (4.34), and let $\Omega(m)$ be the set of magnetization eigenstates of eigenvalue $\lfloor mN \rfloor$.

$$m := \tanh (\psi \beta).$$  \hfill (5.11)

Let $\xi : \Omega^V \to [0, 1]$ be the measure of the generic sub-layer

$$\xi (\sigma_V) = \frac{e^{-\beta \sigma_V \varphi_V}}{\sum_{\sigma_V \in \Omega(m)} e^{-\beta \sigma_V \varphi_V}} I(\sigma_V \in \Omega(m)), \hfill (5.12)$$

at low temperatures, the fluctuations of the sub-layer $\xi$ converge in distribution to a Random Energy Model.

Proof. For this proof we assume that the support of $\sigma_V$ is the set of magnetization eigenstates for some given magnetization parameter. Start from

$$\sum_{i \in V} \sigma_i \varphi_i = \sum_{i \in V} \varphi_i - \sum_{i \in V} (1 - \sigma_i) \varphi_i \hfill (5.13)$$

and define the following subset of $V$:

$$X (\sigma_V) := \{ j \in V : \sigma_j = -1 \} \hfill (5.14)$$

that contains only those sites of $V$ such that $\sigma_j$ is flipped respect to the master direction $\omega_j$. Notice that the size of the set $X$ is fixed,

$$\frac{1}{N} |X (\sigma_V)| = \frac{1 - m}{2} = : \varepsilon. \hfill (5.15)$$

Further noticing that for the term on the right in Eq. (5.13) holds

$$\sum_{i \in V} (1 - \sigma_i) \varphi_i = \sum_{i \in X(\sigma_V)} 2 \varphi_i \hfill (5.16)$$

we can define two new cavity variables, a constant offset

$$\varphi_1 := \frac{1}{\sqrt{N}} \sum_{i \in V} \varphi_i \hfill (5.17)$$
and the actual fluctuating term, that is given by
\[ \hat{\phi}(\sigma_V) := \frac{1}{\sqrt{\varepsilon N}} \sum_{i \in X(\sigma_V)} \varphi_i, \]  
(5.18)
then, the fluctuations can be rewritten as
\[ \sum_{i \in V} \sigma_i \varphi_i = \sqrt{N} \varphi(1V) - 2 \sqrt{\varepsilon N} \hat{\phi}(\sigma_V) \]  
(5.19)
the constant offset \( \varphi(1V) \) cancels out in the average formula, and it is possible to rewrite the average in terms of the \( \hat{\phi}(\sigma_V) \) variables only: we can concentrate on the actually fluctuating component. Notice that by the Central Limit Theorem \( \hat{\phi}(\sigma_V) \) can be approximated in distribution by a sum of Gaussian variables with variance \( \delta \), ie
\[ \hat{J}(\sigma_V) := \hat{\phi}(\sigma_V) / \sqrt{\delta}, \]  
(5.20)
is approximately a Gaussian variable of unitary variance. Here is the final step: the overlap between the flipped sets \( X(\sigma_V) \) and \( X(\tau_V) \) concentrates on \( \varepsilon^2 N \) in the TL,
\[ \lim_{N \to \infty} \frac{1}{N} \langle \left| X(\sigma_V) \cap X(\tau_V) \right| \rangle_{\Omega(m) \otimes \Omega(m)} = \varepsilon^2, \]  
(5.21)
and vanishes faster than the size of \( X(\sigma_V) \) as \( \beta \to \infty \), then, the fluctuations \( \hat{J} \) become asymptotically independent for each input, like in the REM.

**Lemma 13.** Let \( f_0 \) be a positive function of the fluctuations of the cavity fields around the master direction (ground state):
\[ \beta \Delta^* (\sigma_V) := \frac{\beta}{\sqrt{\varepsilon N}} \sum_{i \in V} J^*_i (1 - \sigma_i), \]  
(5.22)
defined with an independent noise vector \( J^*_V \). Then, the average of \( f_0 \) according to \( \xi \) can be approximated by a Poisson Point Process of rate \( \lambda \):
\[ \langle f_0 [\beta \Delta^* (\sigma_V)] \rangle^{\frac{\lambda}{\xi}} = K_0 \langle f_0 \hat{\beta} \Delta^* (\sigma_V) \rangle^{\lambda}, \]  
(5.23)
where \( K_0 \) is a constant, and \( \lambda \) and \( \hat{\beta} \) are deterministic parameters that does not depend on the spins \( \sigma_V \) of the considered layer (although may still depend from those of the previous).
Proof. Form the proof of Lemma 12 before, the average formula is
\[
\langle f_0 | \beta \Delta^* (\sigma_V) || \xi \rangle = \frac{\sum_{\sigma_V \in \Omega(m)} e^{-2\beta \sqrt{N} \beta \sigma_V \sqrt{\hat{J}}(\sigma_V) \sqrt{\hat{N}}} f_0 | \beta \Delta^* (\sigma_V) \rangle}{\sum_{\sigma_V \in \Omega(m)} e^{-2\beta \sqrt{N} \beta \sigma_V \sqrt{\hat{J}}(\sigma_V) \sqrt{\hat{N}}}}.
\] (5.24)
and in the low temperature limit we can use the properties of the REM to study it. It is known that for a REM of (random) Gibbs distribution
\[
\eta(\sigma_V) = \frac{e^{-\beta J(\sigma_V) \sqrt{\hat{N}}}}{\sum_{\tau_V \in \Omega^V} e^{-\beta J(\tau_V) \sqrt{\hat{N}}}},
\] (5.25)
with \(J(\sigma_V)\) independent and normally distributed, holds that for any positive test function \(f: \Omega^V \rightarrow \mathbb{R}^+\) the (random) average is equal to
\[
\sum_{\sigma_V \in \Omega^V} \eta(\sigma_V) f(\sigma_V) \overset{d}{=} K_0 \langle f(\sigma) \lambda \rangle^{1/\lambda} (5.26)
\] with rate parameter \(\lambda = \sqrt{\log 2 / \beta}\) for \(\beta > \sqrt{2 \log 2}\), and \(\lambda = 1\) otherwise.

This result is well known: at low temperatures the weights are proportional in distribution to a Poisson Point Process (PPP) of rate \(\lambda\), due to concentration of the measure on the states with lowest energy, and by applying the fundamental averaging property of PPP [14, 23] (see also Little Theorem of [25]) Eq.(5.26) follows. Above the threshold the sampling of the test function is dense and the average is unaffected. We can adapt the formula of Eq. (5.26) to our case using the scaling properties of REM: first rescale the number of spins to take into account the size of \(\Omega(m)\): we define \(K := N/N_c\) with \(N_c := \log 2 / \phi(m)\), then rescale the fluctuations of the cavity fields from \(N\) to \(K\) spins
\[
\Delta^*(\sigma_V) := \frac{1}{\sqrt{\hat{N}}} \sum_{i \in V} J_i^* - \frac{1}{\sqrt{\hat{N}}} \sum_{i \in V} J_i^* \sigma_i =
\]
\[
= \frac{2}{\sqrt{\hat{N}}} \sum_{i \in X(\sigma_V)} J_i^* = 2 \sqrt{\hat{\epsilon} \hat{J}^*(\sigma_V)} \overset{d}{=} 2 \sqrt{\epsilon N_c \hat{J}^*(\sigma_K)},
\] (5.27)
introducing the modified temperature
\[
\beta^* := 2 \beta \sqrt{\epsilon N_c} = \sqrt{\frac{1}{\phi(m)}} \frac{1 - m}{\phi(m)}
\] (5.28)
and substituting in the formula before we arrive to the expression

$$
\langle f_0 [\beta \Delta^* (\sigma_V)] \rangle_{\xi} = \frac{d}{\sum_{\alpha \leq 2K} e^{\beta \sqrt{\phi'(\tau^\alpha_{V})}} f_0 [\beta \Delta^* (\sigma^\alpha_{V})]}} \sum_{\alpha \leq 2K} e^{\beta \sqrt{\phi'(\tau^\alpha_{V})}} f_0 [\beta \Delta^* (\sigma^\alpha_{V})] \rangle_{\xi},
$$

(5.29)

now the REM average formula of Eq. (5.26) can be applied in straight fashion, and we see that, at least in the limit of zero temperature

$$
\langle f_0 [\beta \Delta^* (\sigma_V)] \rangle_{\xi} = K \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} f_0 (\hat{\beta} x)^\lambda,
$$

(5.30)

in this limit the rate parameter $\lambda$ is given by

$$
\lambda = \frac{1}{\beta \sqrt{2\delta}} \sqrt{\frac{\phi(m)}{1 - m}},
$$

(5.31)

expanding $\phi(m)$ for $\beta \to 1$ we find that near zero temperature

$$
\hat{\beta} \approx \sqrt{2 \log 2} \sqrt{\frac{\beta}{\psi}}, \quad \lambda \approx \sqrt{\frac{\psi}{2\beta \delta}}.
$$

(5.32)

It is an interesting fact that the ratio $N_c \approx e^{2\beta \psi}$ between the original number of spins and the number $K$ at which the REM average is computed is exponentially diverging in $\beta$: this suggests a relation with the transposed measure of Section 2, that can be related to the replicated system, and we interpret as the measure that is actually used in the 1RSB ansatz with replicas.

We have found that, at low temperatures, the layer converges asymptotically to a REM, and can be computed using the properties of PPP. This is not new, in fact, a result similar to Lemma 12 has been obtained for the NPP in [7, 8, 9].

We remark that Eq. (5.30) can be extended to any temperature by noticing that if holds at zero and infinite temperature, then must hold at any intermediate temperature, although with different coefficients. In fact, it is possible to show that for $\beta \geq \beta'$ holds

$$
E \langle f_0 [\beta \Delta^* (\sigma_V)] \rangle_{\xi(\beta)} \leq E \langle f_0 [\beta' \Delta^* (\sigma_V)] \rangle_{\xi(\beta')}
$$

(5.33)

due to the fact that the variance of $\beta \Delta^* (\sigma_V)$ decreases in $m(\beta)$. Also, when $\beta \geq \beta'$ holds $\lambda(\hat{\beta}) \leq \lambda(\hat{\beta}')$, then by Jensen inequality

$$
E \langle f_0 [\beta \Delta^* (\sigma_V)] \rangle_{\lambda(\hat{\beta})}^{1/\lambda(\hat{\beta})} \leq E \langle f_0 [\beta' \Delta^* (\sigma_V)] \rangle_{\lambda(\hat{\beta}')}^{1/\lambda(\hat{\beta}')},
$$

(5.34)
Then, notice that in the high temperature limit obviously holds

$$\lim_{\beta \to 0} E \langle f_0[\beta \Delta^*(\sigma_V)] \rangle_{\xi(\beta)} = \lim_{\tilde{\beta} \to 0} E \langle f_0[\tilde{\beta} \Delta^*(\sigma_V)]^{1/\lambda(\tilde{\beta})} \rangle_{\nu^{1/\lambda(\tilde{\beta})}}, \quad (5.35)$$

while from Lemma 13 we have found that

$$\lim_{\beta \to \infty} E \langle f_0[\beta \Delta^*(\sigma_V)] \rangle_{\xi(\beta)} = \lim_{\tilde{\beta} \to \infty} E \langle f_0[\tilde{\beta} \Delta^*(\sigma_V)]^{1/\lambda(\tilde{\beta})} \rangle_{\nu^{1/\lambda(\tilde{\beta})}}. \quad (5.36)$$

Then, in the TL there must be a mapping between $\beta$ and $\tilde{\beta}$ for any $\beta$, although may be different from that in the proof of Lemma 13 above, that only holds in the low temperature limit.

Lemma 12 completes the mandatory tools to obtain the lower bound of the Parisi formula, using the Cavity Method in the version of [22], but notice that in our reasoning we still did not addressed the distribution of the master direction $\omega_V$ itself, that is in fact not strictly necessary to compute the free energy by Cavity Method. We propose a conjecture for the full kernel in Figures 5.2 and 5.3 but a detailed argument will be given elsewhere.
Figure 5.2: Sub-kernel from of the first layer $V_1$ at a low temperature and for some fixed realization of the noise. The support has been relabeled according to a new index (dependent from the measure) that orders the states by their probability mass and where some states have been been removed (call \textit{purified} index). For some positive $\varepsilon$ we select $S_P \subset \Omega(\left\lfloor m_1 N_1 \right\rfloor, N_1)$ as the subset of mass $|\tilde{S}_P| \geq 1 - \varepsilon$ with the smallest cardinality, then, the purified index run from $p = 1$, the state with largest weight, to $p = |S_P|$, the last before the truncation. We call $\Omega(\left\lfloor m_1 N_1 \right\rfloor, N_1) \setminus S_P$ the irregular set. The ansatz predicts that the states $S_P$ on which $\mu$ concentrates most are randomly selected from $\Omega(\left\lfloor m_1 N_1 \right\rfloor, N_1)$, which means that the overlap between two states $p$ and $t$ is concentrated on a deterministic value $q(m_1)$ for all pairs $p \neq t$. We expect that in the low temperature phase only few states, eventually only one, will carry most of the probability mass. Notice that for low temperature the the states overlap almost perfectly with the master direction, except for some small subset of spins that get flipped. The localization of such subset is different for each state.
Figure 5.3: Kernel representation of the RSB ansatz with $L = 3$ for fixed noise. Also in this case we call $S_P$ the set of states that carries most of the probability mass for large $N$, $L$ and $\beta$, and the states have been relabeled according to a purified index $p_1...p_L$ where the states of $S \setminus S_P$ have been removed. The ansatz predicts that the layer states are independently extracted from $\Omega(\lfloor m_\ell N_\ell \rfloor, N_\ell)$ for any $p_1...p_{\ell-1} \neq t_1...t_{\ell-1}$. It is easy to realize that this implies an ultrametric organization of the overlaps. In fact, consider the states $p_1...p_{\ell+1}...t_L$ and $p_1...p_{\ell+1}...r_L$ of $S_P$ with $t_{\ell+1}...t_L \neq r_{\ell+1}...r_L$, the spins of $Q_\ell$ will overlap perfectly, while those restricted to $Q \setminus Q_\ell$ will concentrate on some deterministic overlap that is smaller than one.
6 ROSt variables and Parisi functional

In this end section we will apply our previous findings and methods to the cavity representation of the SK incremental pressure (see for example [22, 23, 27]) and show that by Lemma 12 it can be rewritten to match the functional of the Parisi formula for the SK model (Parisi functional). First, we need to represent the the (random) pressure

\[ p := \lim_{N \to \infty} \frac{1}{N} \log Z \quad (6.1) \]

in terms of some (tractable) functional of \( \mu \), this can be done by cavity method. Following the Random Overlap Structure (ROSt) oriented derivation of [22, 23] we define the ROSt cavity variables, ie the cavity field and the correction term respectively

\[ \tilde{x}(\sigma_V) = \frac{1}{\sqrt{N}} \sum_i \tilde{J}_{ii} \sigma_i, \quad (6.2) \]

\[ \tilde{y}(\sigma_V) = \frac{1}{N} \sum_{i < j} \sigma_i \tilde{J}_{ij} \sigma_j = \frac{1}{\sqrt{N}} \tilde{H}_{sk}(\sigma_V) = \frac{1}{\sqrt{N}} \tilde{H}_{sk}(\sigma_V), \quad (6.3) \]

the last proportional to the Hamiltonian in distribution. Notice that the above variables are obtained from a noise matrix \( \tilde{J} \) that is independent from \( J \). We keep a tilde on those variables that depends on the new noise.

Apart from vanishing finite size corrections the Cavity representation of the incremental pressure in the version of Aizenmann et al. is [14, 21, 22, 23]

**Lemma 14.** *(Incremental pressure)* The pressure of the Sherrington-Kirkpatrick model is equal in distribution to the limit

\[ p \overset{d}{=} \lim_{N \to \infty} A(\tilde{x}, \tilde{y}, \mu), \quad (6.4) \]

where \( \mu \) is the SK Gibbs measure, \( \tilde{x} \) and \( \tilde{y} \) are defined in Eq.s (6.2), (6.3) and the functional is

\[ A(\tilde{x}, \tilde{y}, \mu) = \log 2 + \log \sum_{\sigma_V \in \Omega^V} \mu(\sigma_V) \cosh[\beta \tilde{x}(\sigma_V)] + \]

\[ -\log \sum_{\sigma_V \in \Omega^V} \mu(\sigma_V) \exp[\beta \tilde{y}(\sigma_V)]. \quad (6.5) \]

**Proof.** The result is well known [2] and there are multiple routes, here we follow the derivation that can be found in the last two pages of [21], originally due to Aizenmann et al. [22], see also [14, 23]. The idea is to relate the partition function of an \( N \)-spin...
system with that of a larger \((N+1)\)–system, then compute the difference between the logarithms of the partition functions. The key ingredient is the Gaussian sum rule

\[
J_{ij} \sqrt{a+b} \equiv J_{ij} \sqrt{a} + J_{ij} \sqrt{b},
\]

(6.6)

where \(\tilde{J}\) is a new noise matrix independent from the \(J\). Applying this to the Hamiltonian gives the following relation that holds in distribution

\[
H_{sk}(\sigma_V) = \frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} \sigma_i J_{ij} \sigma_j \equiv \\
= \frac{d}{\sqrt{N+1}} \sum_{1 \leq i < j \leq N} \sigma_i \tilde{J}_{ij} \sigma_j + \frac{1}{\sqrt{N(N+1)}} \sum_{1 \leq i < j \leq N} \sigma_i \tilde{J}_{ij} \sigma_j,
\]

(6.7)

We applied the Gaussian trick before to isolate the correction term, the partition function can be written as

\[
Z \equiv \sum_{\sigma_V \in \Omega^d} \exp \left[ \beta \sqrt{\frac{N}{N+1}} \tilde{x}(\sigma_V) \right] e^{-\beta \sqrt{\frac{N}{N+1}} H_{sk}(\sigma_V)},
\]

(6.8)

Now consider the system of \(N+1\) spins, isolating the last spin gives

\[
H_{sk}(\sigma_V \cup \{N+1\}) = \frac{1}{\sqrt{N+1}} \sum_{1 \leq i < j \leq N+1} \sigma_i J_{ij} \sigma_j = \\
= \frac{1}{\sqrt{N+1}} \sum_{1 \leq i < j \leq N} \sigma_i J_{ij} \sigma_j + \frac{1}{\sqrt{N+1}} \sigma_{N+1} \sum_{1 \leq i \leq N} J_{i,N+1} \sigma_i,
\]

(6.9)

Since the sequence \(J_{i,N+1}\) is independent from the other \(J\) entries, we can write a more pleasant formula by using the diagonal terms of \(\tilde{J}\) on behalf, ie we take \(J_{i,N+1} = \tilde{J}_{ii}\) so that the noise relative to the vertex \(N+1\) is all expressed in terms of the \(\tilde{J}\) matrix. The associated partition function is

\[
Z^+ = \sum_{\sigma_V \in \Omega^d} 2 \cosh \left[ \beta \sqrt{\frac{N}{N+1}} \tilde{x}(\sigma_V) \right] e^{-\beta \sqrt{\frac{N}{N+1}} H_{sk}(\sigma_V)},
\]

(6.10)

we have written everything in terms of averages respect to a \(N\)–system at slightly shifted temperature. Calling its partition function with

\[
Z^* = \sum_{\sigma_V \in \Omega^d} e^{-\beta \sqrt{\frac{N}{N+1}} H_{sk}(\sigma_V)}
\]

(6.11)
and dividing by this quantity both $Z^+$ and $Z$ we arrive to the expression given in the statement by taking $A(\tilde{x}, \tilde{y}, \mu) = \log(Z^+/Z^*) - \log(Z/Z^*)$, apart from a rescaling of the temperature that becomes negligible in the TL. Notice that since this expression is a representation for the incremental free energy and not the actual free energy, then the proper relation with $p$ would be rather

$$p \geq \liminf_{N \to \infty} A(\tilde{x}, \tilde{y}, \mu), \quad (6.12)$$

but for the SK model it can be shown that the bound is tight $[14,23]$. \hfill \square

The cavity formula is then written using Lemma 4 as

$$A(\tilde{x}, \tilde{y}, \xi) = \log 2 + \log \langle \cdots \langle \cosh [\beta \tilde{x}(\sigma_Q)] \rangle_{\xi_L} \cdots \rangle_{\xi_1} + \log \langle \cdots \langle \exp [\beta \tilde{y}(\sigma_Q)] \rangle_{\xi_L} \cdots \rangle_{\xi_1}, \quad (6.13)$$

Here is an important step. Let rewrite the formula once again according to the partition $\mathcal{V}$ by introducing the variable

$$\tilde{z}_\ell(\sigma_V) \sqrt{|V_\ell|} := \sum_{i \in V_\ell} \tilde{J}_{ii} \sigma_i, \quad (6.14)$$

that is the $V_\ell$ component of the cavity field normalized by the square root of the number of spins $|V_\ell|$, and the variable

$$\tilde{g}_\ell(\sigma_Q) \sqrt{|W_\ell|} := \sum_{(i,j) \in W_\ell} \sigma_i \tilde{J}_{ij} \sigma_j, \quad (6.15)$$

that is the $W_\ell$ component of the correction term, this normalized with the square root of the number of terms $|W_\ell|$ contributing to $\tilde{H}_\ell(\sigma_Q)$ of Lemma 9.

Notice that for fixed $\sigma_Q$, both variables are normally distributed. We rewrite the cavity variables in terms of the previous, the first is as follows

$$\tilde{x}(\sigma_V) \sqrt{N} = \sum_{\ell \leq L} \tilde{z}_\ell(\sigma_V) \sqrt{|V_\ell|} \quad (6.16)$$

while for the correction term we can follow the same decomposition presented in Lemma 9 and find

$$\tilde{y}(\sigma_V) \sqrt{2N} = \sum_{\ell \leq L} \tilde{g}_\ell(\sigma_Q) \sqrt{|W_\ell|} \quad (6.17)$$

where $1/\sqrt{2}$ (and not $1/2$) comes from removing the $i < j$ constraint under the as-
sumption that $\tilde{J}$ is asymmetric almost surely. Recall that by definition the sizes of the sets are
\begin{equation}
|V_\ell| = |Q_\ell| - |Q_{\ell-1}| = (q_\ell - q_{\ell-1})N, \tag{6.18}
\end{equation}
\begin{equation}
|W_\ell| = |Q_\ell|^2 - |Q_{\ell-1}|^2 = (q_\ell^2 - q_{\ell-1}^2)N^2. \tag{6.19}
\end{equation}

We can already recognize two familiar coefficients, in particular, these relations allow to identify the sizes of the sets $|Q_\ell|$ with the overlap parameters $q_\ell$ as usually intended in the RSB theory. Then, substituting these expressions into the cavity formula before we arrive to

\begin{equation}
A(q, \tilde{z}, \tilde{g}, \xi) = \log \left( \cdots \left( 2 \cosh \left[ \beta \sum_\ell \tilde{z}_\ell \left( \sigma_{V_\ell} \right) \sqrt{q_\ell - q_{\ell-1}} \right] \cdots \right) \tilde{z}_1 \right) + \nonumber
- \log \left( \cdots \left( \exp \left[ \beta \sum_\ell \tilde{g}_\ell \left( \sigma_{Q_\ell} \right) \sqrt{q_\ell^2 - q_{\ell-1}^2} \right] \cdots \right) \tilde{z}_1 \right). \tag{6.20}
\end{equation}

As one can easily see, the Definition 4 provides a natural description of the ROSt probability space and its variables. Notice that up to this point all the things that we did on $A$ depend on the partition of $V$ and hold in general, nonetheless, both the cavity field and the correction term (called fugacity variable in [22]) are now expressed using a common kernel base. In the following theorem, we show how to obtain the functional that appear in the celebrated variational formula by Parisi. Concerning the version of the functional, we refer to the one given in reference [23] for a comparison.

**Theorem 15.** (Parisi functional) Lemma 12 applied to the cavity representation of Eq. (6.20) gives the Parisi functional

\begin{equation}
A_P(q, \lambda) = \log 2 + \log Y_0 - \frac{\beta^2}{4} \sum_\ell \lambda_\ell \left( q_\ell^2 - q_{\ell-1}^2 \right), \tag{6.21}
\end{equation}

where $Y_0$ given by the recursion $Y_{\ell+1}^{\lambda_\ell} = E\gamma Y_\ell^{\lambda_\ell}$ applied to the initial condition

\begin{equation}
Y_{L+1} = \cosh \left( \beta \sum_\ell \tilde{z}_\ell \sqrt{q_\ell - q_{\ell-1}} \right), \tag{6.22}
\end{equation}

with $\tilde{z}_\ell$ i.i.d. normally distributed and $E_\ell$ normal average acting on $\tilde{z}_\ell$.

**Proof.** From Lemma 12 applied to Eq. (6.20) we find

\begin{equation}
\sum_{\sigma_{V_\ell}, \sigma_{Q_\ell}} \xi_\ell \left( \sigma_{Q_\ell} \right) f \left[ \tilde{z}_\ell \left( \sigma_{V_\ell} \right) \right] \overset{\text{d}}{=} K_\ell \left( f \left[ \tilde{z}_\ell \left( \sigma_{V_\ell} \right) \right] \right)^{1/\lambda_\ell} \tag{6.23}
\end{equation}
in distribution for some constant $K_\ell$, and the same can be done for $\tilde{g}_\ell$, since in the Eq.

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If the uniform average $\nu$ is applied, we can safely take

$$\tilde{z}_\ell (\sigma_{V_\ell}) \overset{d}{=} \tilde{z}_\ell, \tilde{g}_\ell (\sigma_{Q_\ell}) \overset{d}{=} \tilde{g}_\ell \quad (6.24)$$

with $\tilde{z}_\ell$ and $\tilde{g}_\ell$ independent and normally distributed for all $\ell$, then

$$\langle f [\tilde{z}_\ell (\sigma_{V_\ell})]^{1/\lambda_\ell} \rangle_\xi \overset{d}{=} \langle [E_\ell f (\tilde{z}_\ell)^{1/\lambda_\ell}] \rangle_\xi, \quad (6.25)$$

and do the same for $\tilde{g} (\sigma_{Q_\ell})$. Now, start from the initial condition $Y_{L+1}$ and apply the averages down to $\ell = 0$. We arrive at

$$\langle \cdots \langle \cosh \left[ \beta \sum_\ell \tilde{z}_\ell (\sigma_{V_\ell}) \sqrt{q_\ell - q_{\ell-1}} \right] \rangle_{\tilde{z}_\ell} \cdots \rangle_{\xi_1} \overset{d}{=} Y_0 \exp \left( \sum_\ell \log K_\ell \right) \quad (6.26)$$

Then we can compute the correction term in the same way, finding

$$\langle \cdots \langle \exp \left[ \frac{\beta}{\sqrt{2}} \sum_\ell \tilde{g}_\ell (\sigma_{Q_\ell}) \sqrt{q_\ell^2 - q_{\ell-1}^2} \right] \rangle_{\tilde{z}_\ell} \cdots \rangle_{\xi_1} \overset{d}{=} \exp \left[ \frac{\beta^2}{4} \sum_\ell \lambda_\ell \left( q_\ell^2 - q_{\ell-1}^2 \right) + \sum_\ell \log K_\ell \right]. \quad (6.27)$$

Putting together, the contributions depending from $K_\ell$ cancel out,

$$A (q, \tilde{z}, \tilde{g}, \xi) \overset{d}{=} A_\nu (q, \lambda), \quad (6.28)$$

and we obtained the Parisi functional as is presented in [23].

At this point we can easily understand also the origin of the functional parameters appearing in the Parisi formula. The variables $q_\ell$ control the energy contributions due to the new spin (actually is the absence of it) and are determined by the relative sizes of the sets in the partition $\mathcal{V}$, while the lambda parameters $\lambda_\ell$ control the cascade of Point Processes.

A fundamental aspect of the Parisi ansatz, which we do not address here, is the special direction of the variational principle to obtain the pressure (one takes the inferior limit of the functional instead of the superior as in Boltzmann theory). Concerning our previous computations, up to now we assumed $q_\ell$ and $\lambda_\ell$ fixed to the correct SK value, but one can immediately write the lower bound by varying them

$$E (p) \geq \inf_{q, \lambda} A_\nu (q, \lambda). \quad (6.29)$$

Clearly, the hard part is to prove that this inequality is tight. At least for the SK model.
this has been obtained by proving a matching upper bound via Gaussian interpolation techniques (Guerra-Toninelli interpolation [3]).

Then, the physical meaning of the Parisi variational formula would be in some equilibration condition between the original system and the contributions from the new layer, here composed by just one spin (but one can add more and find the same result). Moreover, it has been proven in [26] that the Parisi functional $A_P(q, \lambda)$ has a unique minimizer, indicating that such equilibration process has only one result. Since the Parisi principle prescribes to maximize the incremental free energy functional the exact mechanism behind is still not evident, by the way, assuming that the thermodynamic limit exists, then starting from Eq.(6.20) and applying Jensen inequality

$$E(p) \leq \log E[\exp A(q, \tilde{z}, \tilde{g}, \tilde{\xi})], \quad (6.30)$$

and since by Eq.(6.28) the functional is distributed like $A_P(q, \lambda)$, that is non-random, one finds that $A_P(q, \lambda)$ is also an upper bound for the expectation of the incremental free energy for any value of $q$ and $\lambda$. 
7 Conclusions and outlooks

We have described a mathematical formalism that provides a rigorous framework to handle spin glass problems at finite and infinite volume and in a constructive way. In particular, the analogue pure states of Definition 4 allow a constructive approach to the cavity method with ROSt (Sections 4 and 6), and eventually provide a scheme to go from 1RSB to L-RSB once the 1RSB approximation is known. In Section 4 we give a detailed analysis of the martingale representation of [27] and its relation to the Random Energy Model. These manipulations show that a constructive approach to RSB is at least possible without averages and non-standard algebraic tricks.

Concerning the extensibility of the method beyond the SK model, we remark that the arguments of the first three sections and their mathematical formalism are very general, and allow for much in-depth analysis and generalizations. For example, it can be used to deal with any Hamiltonian

\[ H(\sigma_v) = \frac{1}{g(H)} \sum_{i \in V} \sum_{j \in V} \sigma_i H_{ij} \sigma_j, \quad (7.1) \]

with \( H_{ij} \) being any interaction matrix with fancy underlying topology, and generalized to bipartite and multi-spin interactions by considering higher dimensional kernels (multi-kernels). Also, the kernel formalism allowed to introduce the transposed measure, an interesting probabilistic object that we interpret in the proof of Lemma 13 as physically significant to the replica space.

The arguments of Sections 4, 5 and 6 are also very general, and they can be directly applied for noise that is not Gaussian, and also to the multipartite models. For example, it is possible to repeat the same analysis for the Little model [28], a bipartite system defined by the Hamiltonian

\[ H_{\text{Little}}(\sigma_V, \tau_V) := \frac{1}{\sqrt{N}} \sum_{i \in V} \sum_{j \in V} \sigma_i J_{ij} \tau_j. \quad (7.2) \]

We can write the same layer decomposition of Lemma 9 obtaining

\[
H_\ell \left[ \sigma_V, h_V \left( \sigma_{Q_{\ell-1}} \right), \tau_V, h_V \left( \tau_{Q_{\ell-1}} \right) \right] = \\
\sqrt{q_\ell - q_{\ell-1}} H \left( \sigma_V, \tau_V \right) + \sqrt{q_{\ell-1}} \sigma_V \cdot h_V \left( \tau_{Q_{\ell-1}} \right) + \\
+ \sqrt{q_{\ell-1}} \tau_V \cdot h_V \left( \sigma_{Q_{\ell-1}} \right),
\]  

\[(7.3)\]

then we apply Lemma 10 that establish the irrelevance of the energy contribution from
the core $H\left(\sigma_{V_l}, \tau_{V_l}\right)$, and then proceed as in the SK case.

There is a limitation in that Lemma 10 is expected to hold for fully connected models only. In fact, in the case of sparsely connected models the contribution from the layer core to the energy could be still relevant. By the way, detailed computations should be possible for the Bethe Lattice and other sparse but mean-field structured models, in this respect it would be interesting to confront with the loop expansion method of Chertkov, Chernyak, Xiao and Zhou [32].

Concerning finite dimensional lattice models, consider for example the Ising Spin Glass on a $d-$dimensional lattice, the Hamiltonian is

$$H_d(\sigma_V) := \frac{1}{g(\Lambda)} \sum_{i \in V} \sum_{i < j} \sigma_i J_{ij} \Lambda_{ij} \sigma_j,$$

(7.4)

where $\Lambda_{ij}$ is the adjacency matrix of the considered lattice, and

$$h_i(\sigma_{V_{l-1}}) := \frac{1}{g(\Lambda)} \sum_{j \in V_l} J_{ij} \Lambda_{ij} \sigma_j$$

(7.5)

are cavity fields that describe the energy contributions from the interface between the spins of $V_l$ and $V_{l-1}$. Then we can apply the layer decomposition to obtain

$$H_l\left[\sigma_{V_l}, h_{V_l}(\sigma_{V_{l-1}})\right] = H_{d-1}(\sigma_{V_l}) + \sum_{j \in V_l} \sigma_i \cdot h_i(\sigma_{V_{l-1}}),$$

(7.6)

where $H_{d-1}$ is the Hamiltonian of the layer’s core, that is a $d-1$ dimensional model. In this case the Lemma 10 cannot be applied, because the core and the interface contributions to the layer energy have comparable sizes, but it should be possible to obtain results with a proper choice of the $Q_l$ sequence and the lattice geometry, a short discussion of this can be found in [27].

Although we stop here for this paper, we are persuaded that the kernel framework could provide the ground to systematize many of the known relations between the spin glass problems and other important fields of physics and mathematics, including graph theory [31] (the Section 5 provides the connection with [16] [17] [18] [19] and [31]), neural networks [33] [42] (notice that the structure proposed in Section 4 already resembles a layer neural network), polymers (the layer states of Definition 4 have been used already to compute an urn model approximation of the Range Problem on finite dimensional lattices in [34]) soft granular matter [35], Nelson mechanics [36] [37] (through the commutation properties expressed by Lemma 2), analytic number theory [7] [8] [9] [38] [39] [40] and algorithmic optimization (see [41] for a review). Further in-
vestigations will establish where this could eventually bring.

Apart from the spin glass theory and RSB, the kernel representation provides a formal ground to confront datasets from real experiments with kernels from spin glass problems. We remark that datasets of this kind are already available, for example, from neural activity measurements in mammals, most interesting are those from single neuron spike detections (see the kernels obtained in the remarkable experiments of Clawson et al. in [42]).
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