The conjecture \( cr(C_m \times C_n) = (m - 2)n \) is true for all but finitely many \( n \), for each \( m \)

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Abstract. It has been long conjectured that the crossing number of \( C_m \times C_n \) is \( (m - 2)n \), for all \( m, n \) such that \( n \geq m \geq 3 \). In this paper it is proved that this conjecture holds for all but finitely many \( n \), for each \( m \). More specifically, it is shown that if \( n \geq (m/2)((m + 3)^2/2 + 1) \) and \( m \geq 3 \), then the crossing number of \( C_m \times C_n \) is exactly \( (m - 2)n \), as conjectured. The proof is largely based on the theory of arrangements, introduced by Adamsson and further developed by Adamsson and Richter.

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1. INTRODUCTION

In 1973, Harary, Kainen, and Schwenk proved that toroidal graphs can have arbitrarily large crossing numbers [7]. In the same paper, they put forward the following conjecture.

**Conjecture [HKS–Conjecture].** The crossing number $cr(C_m \times C_n)$ of the Cartesian product $C_m \times C_n$ is $(m - 2)n$, for all $m, n$ such that $n \geq m \geq 3$.

This has been proved for $m, n$ satisfying $n \geq m, m \leq 7$ [12, 5, 6, 11, 9, 3, 10, 4, 1]. Our aim in this paper is to show that the HKS–conjecture holds for all but finitely many $n$, for each fixed $m \geq 3$.

**Main Theorem.** Let $m, n$ be integers such that $n \geq (m/2)((m + 3)^2/2 + 1)$, $m \geq 3$. Then $cr(C_m \times C_n) = (m - 2)n$.

Although we do not use the notion of arrangement explicitly, the proof of the Main Theorem is largely based on the theory of arrangements, introduced by Adamsson [1], and further developed by Adamsson and Richter [2].

In the proof we make frequent use of the Jordan Curve Theorem. With this exception, the proof is self–contained. As we point out in the last section, the statement of the Main Theorem can be slightly improved using the general bound $cr(C_m \times C_n) \geq (m - 2)n/2$ [8].

The heart of the proof of the Main Theorem is the following.

**Theorem 1.** Let $m, n$ be integers such that $n \geq m \geq 3$. Then every robust drawing of $C_m \times C_n$ has at least $(m - 2)n$ crossings.

Roughly speaking (formal definitions are in Section 2), a drawing of $C_m \times C_n$ is robust if (i) for every three $m$–cycles $R, R', R''$, there is a component of $R^2 \setminus R$ that intersects both $R'$ and $R''$; and (ii) to every $m$–cycle $R$ we can assign two disjoint $m$–cycles $R', R''$, both disjoint from $R$, such that every cycle between (with respect to a circular relation defined below) $R'$ and $R$ is disjoint from $R''$.

The argument that shows that the Main Theorem follows from Theorem 1 can be outlined as follows. Let $m \geq 3$ be fixed, and let $n_0 = (m + 3)^2/2 + 1$. It is easy to check that the Main Theorem is a consequence of the following auxiliary statement: for all $n \geq n_0$, $cr(C_m \times C_n) \geq \min\{(m - 2)n, m(n - n_0)\}$. This statement is proved by induction on $n$. The base case is $n = n_0$, for which there is nothing to prove. Suppose the statement is true for $n = k - 1 \geq n_0$, and let $D$ be a drawing of $C_m \times C_k$. Since $k$ is large (enough) compared to $m$, then either $D$ is robust or some $m$–cycle has $m$ or more crossings. In the first case the statement follows directly (without using the inductive assumption) from Theorem 1. In the second case, the statement follows by applying the inductive assumption to the drawing obtained by removing $R$ from $D$.

The complete, formal proof that the Main Theorem is a consequence of Theorem 1 is given in Section 8. The rest of this paper is devoted to the proof of Theorem 1.
The strategy of the proof of Theorem 1 is to show that in every robust drawing of $C_m \times C_n$, we can associate to each of the $n$ $m$–cycles at least $m - 2$ crossings, in such a way that no crossing is associated to more than one $m$–cycle.

In Section 2 we introduce basic definitions, notation, and terminology. In Section 3 we analyze drawings of structures that consist of three closed curves plus a set of arcs that meet the curves in the same order. The major topological results needed in the proof of Theorem 1 (namely Corollaries 3 and 5) are established in this section; the rest of the proof consists mostly of combinatorial arguments. We chose to present these topological results at this early stage in order to prevent a disruption of the discussion in later sections.

In Section 4 we prove some basic facts on robust drawings. In Section 5 we specify the set of crossings associated to each $m$–cycle in a robust drawing. In Section 6 we show that no crossing is associated to more than one $m$–cycle, and in Section 7 we show that there are at least $m - 2$ crossings associated to each $m$–cycle. Finally, in Section 8 we prove Theorem 1 and the Main Theorem. Section 9 contains some final remarks.

2. BASIC DEFINITIONS, NOTATION, AND TERMINOLOGY

2.1 Addition, subtraction, and circular relation on $Z_k$

For each integer $k > 1$, we denote addition and subtraction on $Z_k$ by the symbols $\oplus_k$ and $\ominus_k$, respectively. We define the circular relation $\preceq$ on $Z_k$ by the rule $i \preceq j$ iff $j \ominus_k i \in \{0, \ldots, \lfloor k/2 \rfloor\}$. We write $i \prec j$ if $i \preceq j$ and $i \neq j$. This relation has the following properties: (i) if $i \not\preceq j$, then $j \prec i$ (we remark that if $k$ is even, then it is possible that $i \prec j$ and $j \prec i$); (ii) if $i \prec j \prec i \oplus_k l$ for some $l \neq 0$, then $j = i \oplus_k x$ for some $x \in \{1, \ldots, l \ominus_k 1\}$; and (iii) if $i, j < n/2$, then $i \preceq j$ iff $i \leq j$.

2.2 The Cartesian product $C_m \times C_n$

The Cartesian product $C_m \times C_n$ is a 4–regular graph with $mn$ vertices $v(i, j)$, where $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$. The vertices are labeled so that the vertices adjacent to $v(i, j)$ are $v(i \ominus_n 1, j), v(i \ominus_n 1, j), v(i, j \ominus_n 1), v(i, j \ominus_n 1)$, and $v(i, j \ominus_n 1)$.

The edge set of $C_m \times C_n$ is naturally partitioned into $m$ edge sets of $n$–cycles and $n$ edge sets of $m$–cycles. To help comprehension, we color the $n$–cycles blue and the $m$–cycles red. We label the blue cycles $(v(i, j), j \in Z_n)$, by $B(i), i \in Z_m$, and the red cycles $(v(i, j), i \in Z_m, j \in Z_n)$.

Let $i \in Z_m, j, k \in Z_n, j \neq k$. The blue edge that joins $v(i, j \ominus_n 1)$ to $v(i, j)$ is denoted $bl(i, j)$. The open blue path $P(i, j, k)$ is the sequence of edges and vertices $bl(i, j \ominus_n 1), v(i, j \ominus_n 1), bl(i, j \ominus_n 2), \ldots, v(i, k \ominus_n 1), bl(i, k)$. The closed blue path $\overline{P}(i, j, k)$ is obtained by adding $v(i, j)$ at the beginning and $v(i, k)$ at the end of $P(i, j, k)$.
2.3 Arcs, well-behaved collections of arcs, tangential intersections, crossings

An open arc $\gamma$ is the image of a local homeomorphism $f : (0, 1) \to \mathbb{R}^2$ (a building map for $\gamma$) with the property that the unique continuous extension $\overline{f}$ of $f$ to $[0, 1]$ is such that $\overline{f}(0) \neq \overline{f}(1)$ and $\overline{f}(0), \overline{f}(1) \notin \gamma$. Denote by $\overline{\gamma}$ the image of $\overline{f}$. The points $\overline{f}(0)$ and $\overline{f}(1)$ are the end points of both $\gamma$ and $\overline{\gamma}$. A closed arc $\gamma$ is the image of a local homeomorphism $f : S^1 \to \mathbb{R}^2$ (a building map for $\gamma$). If $\gamma$ is an (open or closed) arc that has some one-to-one building map, then $\gamma$ is simple.

Let $\gamma$ be an open arc with building map $f : (0, 1) \to \mathbb{R}^2$. An arc $\delta$ is a subarc of $\gamma$ if there are $a, b, 0 \leq a < b \leq 1$, such that the map $g : (0, 1) \to \mathbb{R}^2$ defined by the rule $g(x) = f(x(b - a) + a)$ is a building map for $\delta$. If $0 < a < b < 1$, then $\delta$ is a totally proper subarc (or simply tp-subarc) of $\gamma$. Thus, if $\delta$ is a $\text{tp}$–subarc of $\gamma$, then no endpoint of $\gamma$ is an endpoint of $\delta$.

Let $C$ be a collection of closed arcs, and let $C, C', C'' \in C$. If no component of $\mathbb{R}^2 \setminus C$ intersects both $C'$ and $C''$, then $C$ separates $C'$ from $C''$. If no arc in $C$ separates two arcs in $C$ from each other, then $C$ is nonseparating.

Let $\gamma$ be an arc with building map $f : X \to \mathbb{R}^2$. For each $z \in \gamma$, the multiplicity of $z$ is $|\{y \in X \mid f(y) = z\}|$. If $z \in \gamma$ has multiplicity $\mu > 1$, then $z$ is a self-intersection of $\gamma$ of multiplicity $\mu$. It is easy to see that the multiplicity of a self-intersection of $\gamma$ is independent of the building map chosen for $\gamma$. Clearly, $\gamma$ is simple iff it has no self-intersections.

A collection $\mathcal{A}$ of arcs is well-behaved if, for every $\gamma, \delta$ in $\mathcal{A}$:

(i) every self-intersection of $\gamma$ has multiplicity 2, and has a neighborhood that contains no other self-intersections of $\gamma$;
(ii) $\gamma$ and $\delta$ intersect each other a finite number of times; and
(iii) every intersection between $\gamma$ and $\delta$ is a self-intersection of neither $\gamma$ nor $\delta$.

Let $\{\gamma, \delta\}$ be a well-behaved collection of arcs, and let $x$ be an intersection point between $\gamma$ and $\delta$ (a self-intersection if $\gamma = \delta$). Then there is a set $N \subset \mathbb{R}^2$ homeomorphic to a closed disc, whose boundary $\partial N$ is a simple closed arc, and whose interior $N^o$ contains $x$, such that $(\gamma \cup \delta) \cap N^o$ is the union of two open arcs $\alpha, \beta$, with the following properties:

(i) $\alpha$ and $\beta$ are simple $\text{tp}$–subarcs of $\gamma$ and $\delta$, respectively; (ii) $\overline{\alpha} \cap \overline{\beta} = \{x\}$; and (iii) $(\gamma \cup \delta) \cap N = \overline{\alpha} \cup \overline{\beta}$. If $x$ can be removed by an isotopy on $\overline{\alpha}$ totally contained in $N$, that leaves $\partial N$ fixed, then $x$ is tangential. Otherwise $x$ is a crossing.

Suppose that $\gamma = \delta$ (so $x$ is a self-intersection), and $x$ is a crossing. Let $a, a'$ and $b, b'$ be the end points of $\alpha$ and $\beta$, respectively. There are (uniquely determined) simple arcs $\phi, \psi$ contained in $\partial N$ such that (i) the end points of both $\phi$ and $\psi$ are in $\{a, a', b, b'\}$; (ii) $\overline{\phi} \cap \overline{\psi} = \emptyset$; and (iii) $\gamma' = (\gamma \setminus (\alpha \cup \beta)) \cup (\phi \cup \psi)$ is an arc. This arc $\gamma'$ has exactly one fewer self-intersection than $\gamma$. We say that $\gamma'$ is obtained from $\gamma$ by smoothing out the self-intersection $x$.

2.4 Drawings of graphs, definition of $\cap_D$
A drawing $\mathcal{D}$ of a simple graph $G$ is a representation of $G$ in the plane such that: (i) each vertex is represented by a point, and no two different vertices are represented by the same point; (ii) each edge $e$ is represented by an open arc, so that the end points of the representation of $e$ are precisely the points that represent the vertices incident with $e$; (iii) no representation of an edge contains a representation of a vertex.

Remark. For simplicity, if there is only one drawing under consideration, we often make no distinction between a substructure of the graph (such as a vertex, or a path, or a cycle) and the subset of $\mathbb{R}^2$ that represents it. Throughout this work, we have taken special care to ensure that no confusion arises from this practice.

Let $\mathcal{D}$ be a drawing of $C_m \times C_n$. Suppose that each of $H$ and $K$ is either an open path or a red cycle, and that no edge is in both $H$ and $K$. Denote by $H \cap_D K$ the set of pairwise intersections of edges in $\mathcal{D}$ that involve one edge in $H$ and one edge in $K$. If $\mathcal{D}$ is the only drawing under consideration, we omit the reference to $\mathcal{D}$ and simply write $\cap$.

A drawing of a simple graph is good if (i) no edge has a self–intersection; (ii) no two adjacent edges intersect; (iii) no two edges intersect each other more than once; and (iv) each intersection of edges is a crossing.

The crossing number $\text{cr}(G)$ of a graph $G$ is the minimum number of pairwise intersections of edges in a drawing of $G$ in the plane. An optimal drawing of $G$ is a drawing where the crossing number is attained. It is a routine exercise to show that every optimal drawing of a graph is a good drawing (hence the term crossing number, in view of (iv)).

2.5 Robust drawings of $C_m \times C_n$

Let $\mathcal{D}$ be a drawing of $C_m \times C_n$. Fix $j \in Z_n$. If $R(j) \cap_D R(k) = \emptyset$ for some red cycle $R(k)$, then let $b(\mathcal{D}, j) = \min\{b \in Z_n \mid R(j \ominus b) \cap_D R(j) = \emptyset\}$. If $b(\mathcal{D}, j)$ is defined, and there is a red cycle $R(l) \notin \{R(j \ominus b(\mathcal{D}, j)), R(j \ominus (b(\mathcal{D}, j) - 1)), \ldots, R(j)\}$ such that $R(j \ominus c) \cap_D R(l) = \emptyset$ for each $c$ such that $0 \leq c \leq b(\mathcal{D}, j)$, then define

$$a(\mathcal{D}, j) = \min\{a \in Z_n \mid \forall c, 0 \leq c \leq b(\mathcal{D}, j), R(j \ominus c) \cap_D R(j \ominus a) = \emptyset\}.$$

If there is only one drawing under consideration, we omit the reference to $\mathcal{D}$ and simply write $b(j)$ and $a(j)$.

A drawing $\mathcal{D}$ of $C_m \times C_n$ is red–nonseparating if $\{R(0), \ldots, R(j - 1)\}$ is nonseparating, and it is relaxed if

for every $j \in Z_n$, $a(\mathcal{D}, j), b(\mathcal{D}, j)$ are defined and $a(\mathcal{D}, j) + b(\mathcal{D}, j) < n/2$. If $\mathcal{D}$ is relaxed, then $B(\mathcal{D}) = \max\{a(\mathcal{D}, j), b(\mathcal{D}, j) \mid j \in Z_n\}$. Finally, $\mathcal{D}$ is robust if it is red–nonseparating and relaxed.

3. ANALYSIS OF CROSSINGS IN (3,s)–CONFIGURATIONS

Let $k \geq 0$ and $s \geq 1$ be integers. Let $\mathcal{C} = \{C_0, C_1, C_2\}$ be a collection of closed arcs, and let $\mathcal{A} = \{A_0, \ldots, A_{s-1}\}$ be a collection of open arcs. The pair $(\mathcal{C}, \mathcal{A})$ is a $k$–intersecting
(3, s)–configuration if the following are satisfied:

(i) \( \mathcal{C} \cup A \) is well–behaved;
(ii) \( \mathcal{C} \) is nonseparating;
(iii) \(|C_0 \cap C_1| = k\), and \( C_2 \) is disjoint from \( C_0 \cup C_1 \);
(iv) each \( A_i \) has one end point (the initial point \( t(i) \) of \( A_i \)) in \( C_0 \), and the other end point (the final point \( f(i) \) of \( A_i \)) in \( C_2 \);
(v) each \( A_i \) intersects \( C_1 \) in exactly one point (the middle point \( w(i) \) of \( A_i \)), and does so tangentially;
(vi) with the exception of the intersection points mentioned in (v), all the intersections in \( \mathcal{C} \cup A \) (including self–intersections) are crossings.

For each \( A_i \) in \( A \), let \( T_i \) denote the subarc of \( A_i \) whose end points are \( t(i) \) and \( w(i) \), and let \( F_i \) denote the subarc of \( A_i \) whose end points are \( w(i) \) and \( f(i) \). The subarcs \( T_i \) and \( F_i \) are the initial and final subarcs of \( A_i \), respectively. Thus, \( A_i = T_i \cup F_i \cup \{w(i)\} \).

A (3, s)–configuration is clean if for every \( A_i \in A \), (i) \( T_i \) does not intersect \( C_2 \), and (ii) \( F_i \) does not intersect \( C_0 \). An intersection between arcs in a (3, s)–configuration is good if it occurs either between a \( T_i \) and an \( F_j \), for some \( i \neq j \), or between two initial subarcs \( T_i, T_j \), for some \( i \neq j \). To emphasize the fact that every good intersection is necessarily a crossing, we also use the term good crossing to refer to a good intersection.

**Lemma 2.** Every clean 0–intersecting (3, s)–configuration has at least \( s – 2 \) good crossings.

Let \( (\mathcal{C}, A) = (\{C_0, C_1, C_2\}, \{A_0, \ldots, A_{s-1}\}) \) be a clean (3, s)–configuration. If \( s < 3 \), then there is nothing to prove. So we assume \( s \geq 3 \). Suppose the statement is true for \( s = 3 \) (that is, every clean (3, 3)–configuration has at least one good crossing). An elementary counting argument then shows that if \( s \geq 4 \), then every (3, s)–configuration has at least \( (\binom{s}{3})/(s – 2) \) \( s – 2 \) good crossings. Thus it suffices to show that every clean (3, 3)–configuration has at least one good crossing. Therefore we assume \( A = \{A_0, A_1, A_2\} \).

Each \( T_i \) has a unique subarc \( T'_i \) such that (i) one end point of \( T'_i \) is in \( C_0 \); (ii) the other end point of \( T'_i \) is \( w(i) \); and (iii) \( T'_i \) does not intersect \( C_0 \). Similarly, each \( F_i \) has a unique subarc \( F'_i \) such that (i) one end point of \( F'_i \) is in \( C_2 \); (ii) the other end point of \( F'_i \) is \( w(i) \); and (iii) \( F'_i \) does not intersect \( C_2 \). For each \( i \), let \( A'_i = T'_i \cup F'_i \cup \{w(i)\} \), and let \( \mathcal{A}' = \{A'_0, A'_1, A'_2\} \). It is easy to check that \( (\mathcal{C}, \mathcal{A}') \) is also a clean (3, 3)–configuration. Moreover, every good crossing of \( (\mathcal{C}, \mathcal{A}') \) is a good crossing of \( (\mathcal{C}, A) \). Thus it suffices to show that \( (\mathcal{C}, \mathcal{A}') \) has at least one good crossing.

By construction, no \( T'_i \) intersects \( C_0 \). Since \( (\mathcal{C}, \mathcal{A}') \) is clean, no \( T'_i \) intersects \( C_1 \cup C_2 \). Thus, no \( T'_i \) intersects \( C_0 \cup C_1 \cup C_2 \). Similarly, no \( F'_i \) intersects \( C_0 \cup C_1 \cup C_2 \).

We claim we can assume that \( C_0, C_1, \) and \( C_2 \) have no self–intersections. For suppose that \( x \) is a self–intersection of \( C_i \), for some \( i \in \{0, 1, 2\} \). Then \( x \) has multiplicity 2, and \( x \) is the end point of no arc in \( A \). Thus we can obtain a new closed arc \( C'_i \) by smoothing out \( x \) from \( C_i \), without modifying any arc in \( \mathcal{C} \cup A \) other than \( C_i \), so that \(((\mathcal{C} \setminus \{C_i\}) \cup \{C'_i\}, A)\)
is a clean \((3, 3)\)-configuration with the same number of good crossings as \((C, A)\).

Since an intersection involving \(T_i'\) and \(T_j'\) with \(i \neq j\) is good, we may assume \(T_i', T_j', T_2'\) are pairwise disjoint. Since \(C_2\) does not intersect \(C_0 \cup C_1 \cup T_i' \cup T_j' \cup T_2'\), it follows that \(C_2\) is contained in one of the five components of \(\mathbb{R}^2 \setminus (C_0 \cup C_1 \cup T_i' \cup T_j' \cup T_2')\).

The boundary of one of these five components (say \(U\)) is \(C_0\), and the boundary of another of these five components (say \(V\)) is \(C_1\). Since no \(F_k'\) intersects \(C_0\), it follows that \(C_2\) cannot be contained in \(U\). Since each \(w(i)\) is a tangential intersection, it follows that \(C_2\) cannot be contained in \(V\). Thus the boundary \(\partial W\) of the component \(W\) that contains \(C_2\) is the disjoint union of \(T_i', T_j'\) (for some \(i \neq j\)), one \(tp\)-subarc of \(C_0\), one \(tp\)-subarc of \(C_1\), and \(\{w(i), w(j), t(i), t(j)\}\). Let \(k\) be the integer in \(\{0, 1, 2\}\) different from \(i\) and \(j\). Since \(w(k)\) is not in \(\partial W\), it follows that \(F_k'\) must intersect (cross) \(\partial W\). Since \(F_k'\) does not intersect \(C_0 \cup C_1\), then \(F_k'\) must intersect \(T_i' \cup T_j'\). Since such an intersection is good, we are done.

**Corollary 3.** Let \((C, A) = (\{C_0, C_1, C_2\}, \{A_0, \ldots, A_{s−1}\})\) be a \(0\)-intersecting \((3, s)\)-configuration. Let \(x_1\) denote the number of good crossings of \((C, A)\), let \(x_2\) denote the number of initial arcs that cross \(C_2\), and let \(x_3\) denote the number of final arcs that cross \(C_0\). Then \(x_1 + x_2 + x_3 \geq s - 2\).

**Proof.** This follows from the definition of a clean configuration and Lemma 2.

**Lemma 4.** Let \((C, A) = (\{C_0, C_1, C_2\}, \{A_0, \ldots, A_{s−1}\})\) be a clean \(k\)-intersecting \((3, s)\)-configuration, with \(k > 0\). Then \((C, A)\) has at least \(s - k\) good crossings.

**Proof.** Using similar techniques as in the proof of Lemma 2, construct a set \(\mathcal{A}' = \{A_0', A_1', \ldots, A_{s−1}'\}\) of open arcs such that (i) \((C, \mathcal{A}')\) is a clean \(k\)-intersecting \((3, s)\)-configuration; (ii) the initial arc \(T_i'\) of each \(A_i'\) does not intersect \(C_0 \cup C_1 \cup C_2\); (iii) the final arc \(F_k'\) of each \(A_i'\) does not intersect \(C_0 \cup C_1 \cup C_2\); and (iv) every good crossing of \((C, \mathcal{A}')\) is a good crossing of \((C, A)\).

Thus, it suffices to show that \((C, \mathcal{A}')\) has at least \(s - k\) good crossings.

As in the proof of Lemma 2, we can assume that \(C_0, C_1,\) and \(C_2\) have no self-intersections. Let \(W\) denote the component of \(\mathbb{R}^2 \setminus (C_0 \cup C_1)\) that contains \(C_2\). Let \(\{D_1, D_2, \ldots, D_r\}\) denote the pairwise disjoint (necessarily \(tp\)-) subarcs of \(C_0\) on \(\partial W\) (clearly, there is at least one such segment). For each \(j \in \{1, \ldots, r\}\), let \(D_j'\) be a \(tp\)-subarc of \(D_j\) that contains all the \(t(i)\)'s in \(D_j\). Since \(C_0\) is a simple arc, each \(D_j'\) is also a simple arc.

For each \(j \in \{1, \ldots, r\}\), let \(\mathcal{A}_j'\) denote the set of arcs \(A_i'\) in \(\mathcal{A}'\) such that \(t(i) \in D_j'\). By assumption, (i) no \(T_i'\) and no \(F_j'\) intersect \(C_0 \cup C_1 \cup C_2\), and (ii) the only intersection between each \(A_i'\) and \(C_1\) is tangential. It is easy to check that it follows that each \(A_i'\) is contained in \(W\). In particular, each \(A_i'\) has its initial point \(t(i)\) in some \(D_j'\). Thus the collections \(\mathcal{A}_j'\) partition \(\mathcal{A}'\). Let \(s_j = |\mathcal{A}_j'|\).

For each \(j \in \{1, \ldots, r\}\), let \(U_j\) denote the unique component of \(\mathbb{R}^2 \setminus (C_0 \cup C_1)\) different
from \( W \) that contains \( D'_j \) in its boundary. For each \( D'_j \), draw an open arc \( E_j \) very close to \( D'_j \), contained in \( U_j \), with the same endpoints as \( D'_j \). Let \( H_j \) denote the closed arc that consists of \( D'_j \cup E_j \) plus the (common) endpoints of \( D'_j \) and \( E_j \). For each \( j \) such that \( \mathcal{A}'_j \neq \emptyset \), \( \{H_j, C_1, C_2\}, \mathcal{A}'_j \) is a 0–intersecting \((3, s_j)\)–configuration. Each such \((3, s_j)\)–configuration has at least \( s_j - 2 \) good crossings, by Lemma 2. Therefore \((\mathcal{C}, \mathcal{A}')\) has at least \( \sum_{j=1}^{r}(s_j - 2) = -2r + \sum_{j=1}^{r} s_j \) good crossings.

It is readily checked that the total number \( k \) of crossings between \( C_0 \) and \( C_1 \) is at least \( 2r \), and so \( -2r \geq -k \). On the other hand, the collections \( \mathcal{A}'_j \) partition \( \mathcal{A}' \), and so \( \sum_{j=1}^{r} s_j = \sum_{j=1}^{r} |\mathcal{A}'_j| = |\mathcal{A}'| = s \). Thus \((\mathcal{C}, \mathcal{A}')\) has at least \( s - k \) good crossings.

**Corollary 5.** Let \((\mathcal{C}, \mathcal{A}) = \{\{C_0, C_1, C_2\}, \{A_0, \ldots, A_{s-1}\}\}\) be a \( k\)–intersecting \((3, s)\)–configuration, where \( k > 0 \). Let \( x_1 \) denote the number of good crossings of \((\mathcal{C}, \mathcal{A})\), let \( x_2 \) denote the number of initial arcs that cross \( C_2 \), and let \( x_3 \) denote the number of final arcs that cross \( C_0 \). Then \( x_1 + x_2 + x_3 + k \geq s \).

**Proof.** This follows from the definition of a clean configuration and Lemma 4. ■

### 4. PRELIMINARY RESULTS ON ROBUST DRAWINGS

Throughout this section, \( D \) is a fixed robust good drawing of \( C_m \times C_n \).

We have the following preliminary observations and conventions.

(1) As \( D \) is the only drawing under consideration, we shall omit the reference to \( D \) in the parameters \( a(D, j), b(D, j), \) and \( \mathcal{B}(D) \) (which are defined for all \( j \), since \( D \) is robust), and in the symbol \( \cap_D \), and simply write \( a(j), b(j), \mathcal{B}, \) and \( \cap \), respectively.

(2) Since \( D \) is good, all the intersections of edges are crossings. To emphasize this, we do not speak of intersections of edges but of crossings of edges.

(3) If \( H, K \) are (nonnecessarily different, nonnecessarily disjoint) subgraphs of \( C_m \times C_n \), then we say that \( H \) crosses \( K \) if some edge of \( H \) crosses some edge of \( K \).

An \((i, j)\)–crossing is a crossing between an edge in \( B(i) \) and an edge in \( R(j) \). If \( B(i) \) crosses \( R(j) \), then the first \((i, j)\)–crossing from \( v(i, j') \) (respectively last) is the first (respectively last) \((i, j)\)–crossing we find as we traverse \( B(i) \) starting at \( v(i, j') \) and finding the other vertices in \( B(i) \) in the order \( v(i, j' \oplus 1), v(i, j' \oplus 2), \ldots, v(i, j' \oplus n)\).

**Remark.** For each \( R(j) \), there is a unique component \( \Omega_j \) of \( \mathbb{R}^2 \setminus R(j) \) that intersects every red cycle different from \( R(j) \). To see this, first we note that, since \( D \) is robust, then there is an \( R(k) \) such that \( R(j) \cap R(k) = \emptyset \). Then \( \Omega_j \) is the component of \( \mathbb{R}^2 \setminus R(j) \) that contains \( R(k) \). Indeed, if \( R(l) \) does not intersect \( \Omega_j \) for some \( l \neq j \), then \( R(j) \) separates \( R(k) \) from \( R(l) \), contradicting the assumption that \( D \) is robust.

For each \( j \in \mathbb{Z}_n \), let \( \Phi_j = \mathbb{R}^2 \setminus (\Omega_j \cup R(j)) \).
Remark. If \( R(j) \cap R(k) = \emptyset \), then clearly \( R(k) \) is contained in \( \Omega_j \).

**Proposition 6.** Let \( j_1, j_2, \in Z_n \), and let \( a_1, b_1, a_2, b_2 \in Z_n \) satisfy \( 0 < b_1 \leq b(j_1) \), \( 0 < a_1 \leq a(j_1) \), \( 0 < b_2 \leq b(j_2) \), and \( 0 < a_2 \leq a(j_2) \). Then the following hold for every \( i \in Z_m \):

(i) If \( j_2 = j_1 \ominus b_1 \), then \( P(i, j_1, j_1 \ominus a_1) \) and \( P(i, j_2 \ominus_n B(D), j_2) \) have no common edges.

(ii) If \( j_2 = j_1 \ominus a_1 \), then \( P(i, j_1 \ominus_n b_1, j_1) \) and \( P(i, j_2 \ominus_n B(D)) \) have no common edges.

(iii) If \( j_1 \ominus_n b_1 = j_2 \ominus a_2 \), then \( P(i, j_2 \ominus_b b_2, j_2) \) and \( P(i, j_1 \ominus a_1) \) have no common edges.

(iv) If \( j_2 \leq j_1 \ominus b_1 \) and \( j_2 < j_1 \), then \( P(i, j_2 \ominus_b b_2, j_2) \) and \( P(i, j_1 \ominus a, b_1, j_1) \) have no common edges.

(v) Suppose that \( v(i, j_1) \in \Omega_{j_1 \ominus_n b_1} \), and \( R(j_1) \cap R(j_1 \ominus_n a_1) = R(j_1 \ominus_n b_1) \cap R(j_1 \ominus_n a_1) = \emptyset \). Then \( |P(i, j_1 \ominus_n b_1, j_1) \cap R(j_1 \ominus_n a_1)| \) is either zero or greater than one.

(vi) Suppose that \( R(j_1) \cap R(j_1 \ominus_n a_1) = R(j_1 \ominus_n b_1) \cap R(j_1 \ominus_n a_1) = \emptyset \). Then \( |P(i, j_1 \ominus_n b_1, j_1) \cap R(j_1 \ominus_n a_1)| \) is either zero or greater than one.

**Proof.** First we note that (v) and (vi) are straightforward consequences of the Jordan Curve Theorem. The arguments in the proofs of (i), (ii), (iii) and (iv) are quite similar to each other. For the sake of brevity, we only prove (iii) and (iv).

**Proof of (iii).** It clearly suffices to show that the closed paths \( \overline{P}(i, j_2 \ominus a_2, b_2, j_2), \overline{P}(i, j_1, j_1 \ominus_n a_1) \) have no vertex in common. Suppose that \( v(i, k) \) is in both closed paths. Then \( k = j_2 \ominus_n x = j_1 \ominus_n y \), where \( x \in \{0, \ldots, b_2\} \), \( y \in \{0, \ldots, a_1\} \). Thus \( 0 = j_1 \ominus_n j_2 \ominus_n x, y = b_1 \ominus_n a_2 \ominus_n x \ominus_n y \), and so \( b_1 + y + a_2 + x = kn \). Since \( b_1, a_2 \geq 1 \), then \( b_1 + y + a_2 + x \geq n \). But \( b_1 + y \leq b_1 + a_1 \leq b(j_1) + a(j_1) < n/2 \) and \( a_2 + x \leq a_2 + b_2 \leq a(j_2) + b(j_2) < n/2 \), since \( D \) is robust. Thus we obtain a contradiction.

**Proof of (iv).** It clearly suffices to show that \( \overline{P}(i, j_2 \ominus_n b_2, j_2) \) and \( \overline{P}(i, j_1 \ominus_n b_1, j_1) \) have no vertex in common other than (possibly) \( v(i, j_2) \), which might be equal to \( v(i, j_1 \ominus_n b_1) \). Suppose that \( v(i, k) \) is in both closed paths, for some \( k \notin \{j_2, j_1 \ominus_n b_1\} \). Then \( k = j_1 \ominus_n x = j_2 \ominus_n y \), where \( x \in \{0, 1, \ldots, b_1 - 1\} \), \( y \in \{1, \ldots, b_2\} \). Let \( z = j_1 \ominus_n j_2 \). Since \( z \leq n/2 \) and \( y \leq b_2 \leq b(j_2) < n/2 \) (since \( D \) is robust), then \( x = z \ominus_n y = z + y \). Therefore \( 0 \leq z = x - y < b_1 \). On the other hand, \( j_1 \ominus_n b_1 \ominus_n j_2 = kn + z - b_1 \). Since \( -n/2 < z - b_1 < 0 \), then \( k = 1 \). Thus \( j_1 \ominus_n b_1 \ominus_n j_2 > n/2 \), contradicting the assumption that \( j_2 \leq j_1 \ominus_n b_1 \).

5. THE SET OF CROSSINGS ASSOCIATED TO A RED CYCLE

Throughout this section, \( D \) is a fixed robust good drawing of \( C_m \times C_n \). Thus, all the observations, conventions, and results from Section 4 apply.

The aim in this section is to define, for each red cycle \( R(j) \), a set \( \mathcal{I}(j) \) of crossings associated to \( R(j) \).

First we define, for each red cycle \( R(j) \), a partition \( \{C_j^+, C_j^-, T_j^+, T_j^-, T_j^0\} \) of \( Z_m \), according to the following rules:

(i) \( i \in C_j^+ \) iff for some neighborhood \( N \) of \( v(i, j) \), \( N \cap bl(i, j \ominus_n 1) \subset \Phi_j \).
(ii) \( i \in C_j^- \) iff for some neighborhood \( N \) of \( v(i, j) \), \( N \cap bl(i, j \oplus 1) \subseteq \Omega_j \) and \( N \cap bl(i, j) \subseteq \Phi_j \).

(iii) \( i \in T_j^+ \) iff for some neighborhood \( N \) of \( v(i, j) \), \( N \cap (P(i, j \ominus 1, j \ominus 1) \setminus \{v(i, j)\}) \subseteq \Omega_j \), and \( R(j) \cap P(i, j, j \ominus \mathcal{B}) \neq \emptyset \).

(iv) \( i \in T_j^- \) iff \( i \notin T_j^+ \) and, for some neighborhood \( N \) of \( v(i, j) \), \( N \cap (P(i, j \ominus 1, j \ominus 1) \setminus \{v(i, j)\}) \subseteq \Omega_j \), and \( R(j) \cap P(i, j \ominus \mathcal{B}, j \ominus \mathcal{B}) \neq \emptyset \).

(v) \( i \in T_j^0 \) iff for some neighborhood \( N \) of \( v(i, j) \), \( N \cap (P(i, j \ominus 1, j \ominus 1) \setminus \{v(i, j)\}) \subseteq \Omega_j \), and \( R(j) \cap P(i, j \ominus \mathcal{B}, j \ominus \mathcal{B}) = \emptyset \).

Each \( T_j^0 \) is, in turn, partitioned into subsets \( T(\beta, j) \). To define these sets we need some more notation. For each \( i \in Z_m, j \in Z_n \), let \( \mathcal{B}(i, j) = \min\{b \in Z_n \mid v(i, j) \in \Omega_{j \ominus b} \} \). Clearly, \( \mathcal{B}(i, j) \leq b(j) \) for every \( i \) and \( j \). For each \( j \in Z_n \), let \( S(j) = \{\beta, 0 < \beta \leq b(j) \mid \beta = \mathcal{B}(i, j) \text{ for some } i \in Z_m \} \). For each \( \beta \in S(j) \), define \( T(\beta, j) = \{i \in T_j^0 \mid \mathcal{B}(i, j) = \beta \} \).

**Proposition 7.** The following statements hold for each \( j \in Z_n \).

(i) If \( i \in C_j^+ \), then \( R(j) \cap P(i, j, j \oplus a(j)) \neq \emptyset \).

(ii) If \( i \in C_j^- \), then \( R(j) \cap P(i, j \ominus b(j), j) \neq \emptyset \).

(iii) If \( i \in T_j^+ \), then \( R(j) \cap P(i, j \ominus \mathcal{B}, j) \neq \emptyset \).

(iv) If \( i \in T_j^- \), then \( R(j) \cap P(i, j \ominus \mathcal{B}, j) \neq \emptyset \).

(v) If \( T(\beta, j) \cap T(\beta', j') \neq \emptyset \), and \( j < j' \), then \( j \preceq j' \ominus \beta' \).

**Proof.** Suppose \( i \in C_j^+ \). Then \( P(i, j, j \oplus a(j)) \ominus \Phi_j \neq \emptyset \). On the other hand, \( R(j \ominus a(j)) \subseteq \Omega_j \), and so \( P(i, j, j \oplus a(j)) \ominus \Omega_j \neq \emptyset \). Since \( P(i, j, j \oplus a(j)) \) contains no vertex in \( R(j) \), it follows that \( P(i, j, j \oplus a(j)) \) must cross \( R(j) \). Thus (i) follows. Statement (ii) is proved similarly, and (iii) and (iv) follow from the definitions of \( T_j^+ \) and \( T_j^- \), respectively.

Assume that \( i \in T(\beta, j) \cap T(\beta', j') \), and \( j < j' \). Suppose that \( j \not\preceq j' \ominus \beta' \). Then \( j' \ominus \beta' \prec j \), and so \( j = j' \ominus k \) for some \( k, 0 < k < \beta' \). By the definition of \( \mathcal{B}(i, j') \), \( v(i, j') \in \Phi_j \). But then \( P(i, j, j') \cap R(j) \neq \emptyset \), since \( i \in T_j^0 \). Each edge in \( P(i, j, j') \) is in \( P(i, j, j \ominus \mathcal{B}) \), since \( k < \beta' \leq b(j) \leq \mathcal{B} \). Thus \( P(i, j, j \ominus \mathcal{B}) \cap R(j) \neq \emptyset \), contradicting the assumption that \( i \in T(\beta, j) \subseteq T_j^0 \).

For each \( j \in Z_n \), and each \( \beta \in S(j) \), let \( X(\beta, j) \) denote the set of crossings of the following types:

(A) all the crossings between \( R(j \ominus \beta) \) and \( R(j) \);

(B) if \( i \in T(\beta, j) \) and \( R(j \ominus \beta) \cap P(i, j, j \ominus a(j)) \neq \emptyset \), the last \( (i, j \ominus \beta) \)-crossing from \( v(i, j \ominus a(j)) \);

(C) if \( i \in T(\beta, j) \) and \( P(i \ominus a(j)) \cap (i \ominus \beta, j) \neq \emptyset \), the first \( (i, j \ominus a(j)) \)-crossing from \( v(i, j \ominus \beta) \);

(D) if \( i, i' \in T(\beta, j), i \neq i' \), every crossing between \( P(i, j \ominus \beta, j) \) and \( P(i', j \ominus \beta, j) \), and every crossing between \( P(i, j \ominus \beta, j) \) and \( P(i', j \ominus \beta, j) \).
For each \( j \in \mathbb{Z}_n \), let \( \mathcal{Y}(j) \) denote the collection of crossings of the following types:

(I) for each \( i \in C_j^+ \cup T_j^+ \), the first \((i,j)\)-crossing from \( v(i,j) \); and

(II) for each \( i \in C_j^- \cup T_j^- \), the last \((i,j)\)-crossing from \( v(i,j) \).

We are now ready to define the set \( \mathcal{I}(j) \) of crossings associated to each red cycle \( R(j) \):

\[
\mathcal{I}(j) = \mathcal{Y}(j) \cup \left( \bigcup_{\beta \in \mathcal{S}(j)} \mathcal{X}(\beta,j) \right).
\]  

(1)

In the next section we show that if \( j \neq k \), then \( \mathcal{I}(j) \cap \mathcal{I}(k) = \emptyset \).

6. NO CROSSING IS ASSOCIATED TO MORE THAN ONE RED CYCLE

Throughout this section, \( \mathcal{D} \) is a fixed robust good drawing of \( C_m \times C_n \). Thus, all the observations, conventions, and results from Sections 4 and 5 apply.

The main result in this section is the following.

**Lemma 8.** If \( j \neq k \), then \( \mathcal{I}(j) \cap \mathcal{I}(k) = \emptyset \). That is, no crossing in \( \mathcal{D} \) is associated to more than one red cycle.

**Proof.** This is an immediate consequence of Proposition 9 below.

**Proposition 9.** Let \( j,k \in \mathbb{Z}_n \), \( \beta \in \mathcal{S}(j) \), and \( \beta' \in \mathcal{S}(k) \). Then:

(a) If \( j \neq k \), then \( \mathcal{Y}(j) \cap \mathcal{Y}(k) = \emptyset \).

(b) \( \mathcal{Y}(j) \cap \mathcal{X}(\beta',k) = \emptyset \).

(c) If \( j \neq k \) or \( \beta \neq \beta' \), then \( \mathcal{X}(\beta,j) \cap \mathcal{X}(\beta',k) = \emptyset \).

**Proof of (a).** Suppose \( j \neq k \). Each crossing in \( \mathcal{Y}(j) \) (respectively \( \mathcal{Y}(k) \)) is a bichromatic crossing whose red edge involved is in \( R(j) \) (respectively \( R(k) \)). Since \( j \neq k \), (a) follows.

**Proof of (b).** Seeking a contradiction, suppose that for some \( j,k \in \mathbb{Z}_n \), \( \beta' \in \mathcal{S}(k) \), some (necessarily bichromatic, by the definition of \( \mathcal{Y}(j) \)) crossing \( x \) belongs to both \( \mathcal{Y}(j) \) and \( \mathcal{X}(\beta',k) \). Let \( \overline{\ell}(x) \) denote the blue edge involved in \( x \). Since \( x \) is in \( \mathcal{Y}(j) \), the red edge involved in \( x \) is in \( R(j) \). On the other hand, a bichromatic crossing in \( \mathcal{X}(\beta',k) \) involves an edge in \( R(j) \) only if \( j \) is either \( k \ominus \beta' \) or \( k \oplus_a(k) \). Thus, either \( j = k \ominus \beta' \) or \( j = k \oplus_a(k) \).

We analyze these cases separately.

**Case 1.** \( j = k \ominus \beta' \). By the definition of \( \mathcal{X}(\beta',k) \), \( \overline{\ell}(x) \) is in \( P(i,k,k \ominus_a(k)) \). Moreover, \( x \) is the last \((i,k \ominus_a(k))\)-crossing (that is, \((i,j)\)-crossing) from \( v(i,k \ominus_a(k)) \). Since \( x \) is in \( \mathcal{Y}(j) \), \( i \) is either in \( C_j^- \cup T_j^- \) or in \( C_j^+ \cup T_j^+ \).
Suppose that \( i \in C^{-}_j \cup T^{-}_j \). Then, \( x \) occurs between \( R(j) \) and \( P(i,j \ominus a) \). Thus \( \overline{l}(x) \) is in both \( P(i,j \ominus a) \) and \( P(i,k \oplus a(k)) \). But this is impossible, since these open blue paths have no edges in common, by (i) in Proposition 6.

Suppose now that \( i \in C^{+}_j \cup T^{+}_j \). Thus \( x \) is the first \((i,j)\)-crossing from \( v(i,j) \). On the other hand, \( x \) is the last \((i,j)\)-crossing from \( v(i,k \oplus a(k)) \), and \( \overline{l}(x) \) is in \( P(i,k \oplus a(k)) \). It follows from the definitions of \( T(\beta', k) \) and \( a(k) \), and (v) in Proposition 6, that \( P(i,k \oplus a(k)) \) crosses \( R(j) \) at least twice (since they cross at least once). Thus we obtain a contradiction: \( x \) cannot be at the same time the first \((i,j)\)-crossing from \( v(i,j) \) and the last \((i,j)\)-crossing from \( v(i,k \oplus a(k)) \), since these crossings are different.

**Case 2.** \( j = k \ominus a(k) \). By the definition of \( X(\beta', k) \), \( \overline{l}(x) \) is in \( P(i,k \ominus \beta', k) \). Moreover, \( x \) is the first \((i,k \ominus a(k))\)-crossing (that is, \((i,j)\)-crossing) from \( v(i,k \ominus \beta') \). Since \( x \) is in \( Y(j) \), \( i \) is either in \( C^{-}_j \cup T^{-}_j \) or in \( C^{+}_j \cup T^{+}_j \).

Suppose that \( i \in C^{-}_j \cup T^{-}_j \). Then \( x \) is the last \((i,j)\)-crossing from \( v(i,j) \). On the other hand, \( x \) is the first \((i,j)\)-crossing from \( v(i,k \ominus \beta') \), and \( \overline{l}(x) \) is in \( P(i,k \ominus \beta', k) \). It follows from the definitions of \( T(\beta', k) \) and \( a(k) \), and (vi) in Proposition 6, that \( P(i,k \ominus \beta', k) \) crosses \( R(j) \) at least twice (since they cross at least once). Thus we obtain a contradiction: \( x \) cannot be at the same time the last \((i,j)\)-crossing from \( v(i,j) \) and the first \((i,j)\)-crossing from \( v(i,k \ominus \beta') \), since these crossings are different.

Suppose that \( i \in C^{+}_j \cup T^{+}_j \). Then \( x \) occurs between \( R(j) \) and \( P(i,j,j \ominus a) \). Thus \( \overline{l}(x) \) is in both \( P(i,j,j \ominus a) \) and \( P(i,k \ominus \beta', k) \). But this is impossible, since these open blue paths have no edges in common, by (ii) in Proposition 6.

**Proof of (c).** We derive a contradiction from the assumption that the following hold: (i) either \( j \neq k \) or \( \beta \neq \beta' \); and (ii) there is a crossing \( x \) in both \( X(\beta,j) \) and \( X(\beta', k) \).

It follows from the definitions of \( X(\beta,j) \) and \( X(\beta', k) \) that if \( j = k \) and \( \beta \neq \beta' \), then no crossing can belong to both \( X(\beta,j) \) and \( X(\beta', k) \). Thus we can assume without loss of generality that \( j \prec k \). If \( x \) involves only red edges, then \( x \in R(j \ominus \beta) \cap R(j) \) and \( x \in R(k \ominus \beta') \cap R(k) \). This clearly cannot happen, since there are at least three different cycles in \( \{R(j \ominus \beta), R(j), R(k \ominus \beta'), R(k)\} \). Thus \( x \) involves at least an edge from a blue cycle \( B(i) \), such that \( i \in T(\beta,j) \) and \( i \in T(\beta', k) \). By Statement (v) in Proposition 7, \( j \preceq k \ominus \beta' \). The crossing \( x \) involves either one blue edge and one red edge or two blue edges. We analyze these cases separately.

**Case 1.** \( x \) involves one blue edge \( \overline{l}(x) \) and one red edge \( \overline{r}(x) \). Each bichromatic crossing in \( X(\beta,j) \) (respectively \( X(\beta', k) \)) involves a red edge in either \( R(j \ominus \beta) \) or \( R(j \ominus a(j)) \) (respectively \( R(k \ominus \beta') \) or \( R(k \ominus a(k)) \)). Therefore, since by assumption \( x \) is in both \( X(\beta,j) \) and \( X(\beta', k) \), it follows that either (a) \( j \ominus a(j) = k \ominus a(k) \); or (b) \( j \ominus \beta = k \ominus a(k) \); or (c) \( j \ominus \beta = k \ominus \beta' \); or (d) \( j \ominus a(j) = k \ominus a(k) \). Since \( j \preceq k \ominus \beta' \) and \( \beta < n/2 \), it follows that \( j \ominus a(j) = k \ominus a(k) \). Therefore (a) cannot hold. Thus we analyze (b), (c), and (d).

Suppose that \( j \ominus a(j) = k \ominus a(k) \). It follows from the definitions of \( X(\beta,j) \) and \( X(\beta', k) \) that \( \overline{l}(x) \) is in both \( P(i,j,j \ominus a(j)) \) and \( P(i,k \ominus \beta', k) \). This contradicts (iii) in Proposition 6, and so (b) cannot hold. A similar argument shows that (c) cannot hold either.
Finally, suppose that \( j \oplus a(j) = k \oplus a(k) \). It follows from the definitions of \( \mathcal{X}(\beta, j) \) and \( \mathcal{X}(\beta', k) \) that \( \overline{7}(x) \) is in both \( P(i, j \ominus \beta, j) \) and \( P(i, k \ominus \beta', k) \). This contradicts (iv) in Proposition 6. Thus (d) cannot hold.

**Case 2.** \( x \) involves two blue edges. By the definition of \( \mathcal{X}(\beta, j) \), \( x \) occurs between edges in different blue cycles \( B(i) \) and \( B(i') \). We can assume without loss of generality that \( x \) involves an edge in \( P(i, j \ominus \beta, j) \), and an edge in either \( P(i', j \ominus \beta, j) \) or \( P(i', j \ominus a(j)) \). On the other hand, since \( x \) is in \( \mathcal{X}(\beta', k) \), \( x \) involves either (i) an edge in \( P(i, k \ominus \beta', k) \) and an edge in \( P(i', k \ominus \beta', k) \); or (ii) an edge in \( P(i, k \ominus \beta', k) \) and an edge in \( P(i', k \ominus a(k)) \); or (iii) an edge in \( P(i', k \ominus \beta', k) \) and an edge in \( P(i, k \ominus a(k)) \). Now (i) and (ii) cannot hold, since by (iv) in Proposition 6 \( P(i, j \ominus \beta, j) \) has no edge in common with \( P(i, k \ominus \beta', k) \).

On the other hand, it is easily checked that (iii) holds only if (I) \( P(i', k \ominus \beta', k) \) and \( P(i', j \ominus a(j)) \) have a common edge and (II) \( P(i, k \ominus a(k)) \) and \( P(i, j \ominus \beta, j) \) have a common edge. But (I) and (II) hold simultaneously only if \( \beta + \beta' + a(j) + a(k) > n \). But this is impossible, since the assumption that \( \mathcal{D} \) is robust implies that \( \beta + a(j) \leq b(j) + a(j) < n/2 \) and \( \beta' + a(k) \leq b(k) + a(k) < n/2 \).

**Corollary 10.** Each of the unions on the right hand side of Eq. (1) is a disjoint union.

**7. AT LEAST \( m - 2 \) CROSSINGS ARE ASSOCIATED TO EACH RED CYCLE**

Throughout this section, \( \mathcal{D} \) is a fixed robust good drawing of \( C_m \times C_n \). Thus, all the observations, conventions, and results from Sections 4, 5, and 6 apply.

The purpose of this section is to prove the following.

**Lemma 11.** For each \( j \in Z_n \), \( |\mathcal{I}(j)| \geq m - 2 \). In other words, there are at least \( m - 2 \) crossings associated to each red cycle.

**Proof.** Suppose that \( b(j) \in \mathcal{S}(j) \). Then, by Corollary 10, \( |\mathcal{I}(j)| = |\mathcal{Y}(j)| + (\sum_{\beta \in \mathcal{S}(j), \beta \neq b(j)} |\mathcal{X}(\beta, j)|) + |\mathcal{X}(b(j), j)| \). Applying (i), (ii), and (iii) in Proposition 12 below, \( |\mathcal{I}(j)| \geq |\mathcal{C}_j^+| + |\mathcal{C}_j^-| + |\mathcal{T}_j^+| + |\mathcal{T}_j^-| + (\sum_{\beta \in \mathcal{S}(j), \beta \neq b(j)} |\mathcal{T}(\beta, j)|) + |\mathcal{T}(b(j), j)| - 2 \). Since \( Z_m \) is the disjoint union of \( \mathcal{C}_j^+, \mathcal{C}_j^-, \mathcal{T}_j^+, \mathcal{T}_j^- \), and the sets \( \mathcal{T}(\beta, j) \) (for all \( \beta \in \mathcal{S}(j) \)), it follows that \( |\mathcal{I}(j)| \geq m - 2 \), as required.

Now suppose \( b(j) \notin \mathcal{S}(j) \). Then, by Corollary 10, \( |\mathcal{I}(j)| = |\mathcal{Y}(j)| + (\sum_{\beta \in \mathcal{S}(j), \beta \neq b(j)} |\mathcal{X}(\beta, j)|) \). Applying (i) and (ii) in Proposition 12 ((iii) does not apply, since \( b(j) \notin \mathcal{S}(j) \)), \( |\mathcal{I}(j)| \geq |\mathcal{C}_j^+| + |\mathcal{C}_j^-| + |\mathcal{T}_j^+| + |\mathcal{T}_j^-| + (\sum_{\beta \in \mathcal{S}(j), \beta \neq b(j)} |\mathcal{T}(\beta, j)|) \). Since \( Z_m \) is the disjoint union of \( \mathcal{C}_j^+, \mathcal{C}_j^-, \mathcal{T}_j^+, \mathcal{T}_j^- \), and the sets \( \mathcal{T}(\beta, j) \) (for all \( \beta \in \mathcal{S}(j) \)), we obtain \( |\mathcal{I}(j)| \geq m \).

**Proposition 12.** For each \( j \in Z_n \), the following statements hold.
(i) \(|\mathcal{Y}(j)| \geq |C_j^+| + |C_j^-| + |T_j^+| + |T_j^-|\).
(ii) For each \(\beta \in S(j), \beta \neq b(j)\), \(|\mathcal{X}(\beta, j)| \geq |T(\beta, j)|\).
(iii) If \(b(j) \in S(j)\), then \(|\mathcal{X}(b(j), j)| \geq |T(b(j), j)| - 2\).

Proof of (i). If \(i \in C_j^+\), then by Proposition 7 \(P(i, j \ominus_n a(j)) \cap R(j) \neq \emptyset\), and by the definition of \(\mathcal{Y}(j)\), one of these crossings is in \(\mathcal{Y}(j)\). If \(i \in C_j^-\), then by Proposition 7 \(P(i, j \ominus b(j), j) \cap R(j) \neq \emptyset\), and by the definition of \(\mathcal{Y}(j)\), one of these crossings is in \(\mathcal{Y}(j)\). If \(i \in T_j^+\), then by Proposition 7 \(P(i, j \ominus B, j) \cap R(j) \neq \emptyset\), and by the definition of \(\mathcal{Y}(j)\), one of these crossings is in \(\mathcal{Y}(j)\). Since \(C_j^+, \mathcal{C}_j^-, T_j^+, \) and \(T_j^-\) are pairwise disjoint, (i) follows.

Proof of (ii). Let \(\beta \in S(j), \beta \neq b(j)\). Let \(A = \{P(i, j \ominus, j \ominus a(j) \mid i \in T(\beta, j))\). Since \(\beta < b(j)\), \(R(j \ominus, \beta) \cap R(j) = k > 0\). Thus it follows from the definition of \(T(\beta, j)\) that \((\{R(j \ominus, \beta), R(j), R(j \ominus a(j))\}, A)\) is a \(k\)-intersecting \(\{|T(\beta, j)|\}\)-configuration. Note that the crossings in \((\{R(j \ominus, \beta), R(j), R(j \ominus a(j))\}, A)\) that are in \(\mathcal{X}(\beta, j)\) are (a) the good crossings; (b) one crossing for each initial arc that crosses \(R(j \ominus a(j))\); (c) one crossing for each final arc that crosses \(R(j \ominus, \beta)\); and (d) the \(k\) crossings between \(R(j \ominus, \beta)\) and \(R(j)\). By Corollary 5, there are at least \(|A| = |T(\beta, j)|\) such crossings.

Proof of (iii). Suppose that \(b(j) \in S(j)\). Let \(A = \{P(i, j \ominus b(j), j \ominus a(j) \mid i \in T(b(j), j))\). By the definition of \(b(j)\), \(R(j \ominus b(j)) \cap R(j) = 0\). Thus it follows from the definition of \(T(b(j), j)\) that \((\{R(j \ominus b(j)), R(j), R(j \ominus a(j))\}, A)\) is a 0-intersecting \(\{|T(b(j), j)|\}\)-configuration. Note that the crossings in \((\{R(j \ominus b(j)), R(j), R(j \ominus a(j))\}, A)\), that are in \(\mathcal{X}(b(j), j)\) are (a) the good crossings; (b) one crossing for each initial arc that crosses \(R(j \ominus a(j))\); and (c) one crossing for each final arc that crosses \(R(j \ominus b(j))\). By Corollary 3, there are at least \(|A| = |T(b(j), j)| - 2\) such crossings. □

8. PROOFS OF THEOREM 1 AND THE MAIN THEOREM

First we show that if \(n\) is sufficiently large compared to \(m\), then every drawing of \(C_m \times C_n\) either is robust or has a red cycle with at least \(m\) crossings.

Proposition 13. Let \(m, n\) be such that \(n \geq (m + 3)^2/2 + 1, m \geq 3\). Let \(D\) be a drawing of \(C_m \times C_n\). Then either \(D\) is robust or there is a red cycle with at least \(m\) crossings in \(D\).

Proof. Let \(m, n\) satisfy the inequalities in the statement of the proposition. Let \(D\) be a drawing of \(C_m \times C_n\), and suppose that no red cycle has \(m\) or more crossings in \(D\). We will show that then \(D\) is robust. Let \(R\) denote the set of all red cycles.

Since two red cycles that cross do so in at least two points, it follows that each red cycle crosses at most \((m - 1)/2\) other red cycles in \(D\). Since \(n > (m - 1)/2 + 1\), for each \(R(j)\) there is a cycle in \(R \setminus \{R(j)\}\) that does not cross \(R(j)\). This shows that \(b(D, j)\) is defined for every \(j \in Z_n\). Moreover, \(b(D, j) \leq (m - 1)/2 + 1 = (m + 1)/2\) for every \(j \in Z_n\).
For each \( j \in \mathbb{Z}_n \), let \( \mathcal{R}(j) = \{ R(j \subset_b (\mathcal{D}, j)), R(j \subset_a (\mathcal{D}, j) - 1), \ldots, R(j) \} \). Thus, \( |\mathcal{R}(j)| \leq (m + 3)/2 \). We now show that \( a(\mathcal{D}, j) \) exists and is at most \((m + 1)(m + 3)/4 + 1\) for every \( j \in \mathbb{Z}_n \). Let \( j \in \mathbb{Z}_n \) be fixed. Since every red cycle crosses at most \((m - 1)/2\) other red cycles, then the collection of red cycles that either are in \( \mathcal{R}(j) \) or cross a cycle in \( \mathcal{R}(j) \) has size at most \(((m + 3)/2) + ((m + 3)/2)((m - 1)/2) = (m + 1)(m + 3)/4 \). Since \( n > (m + 1)(m + 3)/4 + 1 \), it follows that there is a red cycle not in \( \mathcal{R}(j) \) that crosses no cycle in \( \mathcal{R}(j) \). Moreover, it follows that \( a(\mathcal{D}, j) \leq (m + 1)(m + 3)/4 + 1 \).

Finally, we note that \( b(\mathcal{D}, j) + a(\mathcal{D}, j) \leq (m + 1)/2 + (m + 1)(m + 3)/4 + 1 = (m + 3)^2/4 < n/2 \). Thus \( \mathcal{D} \) is robust.

\textbf{Proof of Theorem 1.} It follows immediately from the definition of \( \mathcal{I}(j) \), Lemma 8, and Lemma 11.

\textbf{Proof of Main Theorem.} It is easy to exhibit drawings of \( C_m \times C_n \) with exactly \((m - 2)n\) crossings, for all \( m, n \) such that \( n \geq m \geq 3 \). Thus we need to show that \( \text{cr}(C_m \times C_n) \geq (m - 2)n \), for all \( m, n \) such that \( n \geq (m/2)((m + 3)^2/2 + 1), m \geq 3 \).

Let \( m \geq 3 \) be fixed. Let \( n_0 = (m + 3)^2/2 + 1 \). For every \( n \geq (m/2)((m + 3)^2/2 + 1) \), \( \min\{(m - 2)n, m(n - n_0)\} = (m - 2)n \). Therefore it suffices to show that if \( n \geq n_0 \), then \( \text{cr}(C_m \times C_n) \geq \min\{(m - 2)n, m(n - n_0)\} \). We prove this by induction on \( n \).

The base case is \( n = n_0 \), for which there is nothing to prove. Suppose that the statement holds for \( n = k - 1 \geq n_0 \), and consider a drawing \( \mathcal{D} \) of \( C_m \times C_k \). If \( \mathcal{D} \) is robust, then we are done, since by Theorem 1 \( \mathcal{D} \) has at least \((m - 2)k\) crossings. Thus we assume \( \mathcal{D} \) is not robust. Since \( k > n_0 \), it follows from Proposition 13 that there is a red cycle \( R \) with \( m \) or more crossings. The drawing \( \mathcal{D}' \) that results by removing \( R \) from \( \mathcal{D} \) has, by the induction hypothesis, at least \( \min\{(m - 2)(k - 1), m(k - 1 - n_0)\} \) crossings, and so \( \mathcal{D} \) has at least \( \min\{(m - 2)(k - 1) + m, m(k - 1 - n_0) + m\} = \min\{(mk - 2(k - 1), m(k - n_0)\} \) crossings. Since \( mk - 2(k - 1) > (m - 2)k \), then \( \mathcal{D} \) has at least \( \min\{(m - 2)k, m(k - n_0)\} \) crossings, as required.

\textbf{9. CONCLUDING REMARKS}

As we mentioned in Section 1, although we do not use the notion of an arrangement explicitly, our proof of the Main Theorem is largely based on the theory of arrangements introduced by Adamsson [1] and further developed by Adamsson and Richter [2].

An \((m, n)\)-circular arrangement consists of two collections \( \mathcal{B}, \mathcal{R} \) of \((\text{blue and red, respectively})\) closed curves. The red curves are cyclically ordered, and each blue curve intersects the red curves in the given cyclic order. Clearly, each drawing of \( C_m \times C_n \) yields an \((m, n)\)-circular arrangement, if we regard each vertex as an intersection between a red curve and a blue curve. Moreover, lower bounds on the number of intersections of \((m, n)\)-circular arrangements imply lower bounds for \( \text{cr}(C_m \times C_n) \). Using this approach, Adamsson proved that large classes of drawings of \( C_m \times C_n \) have at least \((m - 2)n\) crossings. He also used this approach to show that \( \text{cr}(C_7 \times C_n) = 5n \), as conjectured.
A nice aspect of the Adamsson and Richter approach to the HKS–conjecture is that, although it draws from and generalizes ideas introduced in previous work, it is virtually self–contained. A similar observation holds for this paper, where no results from previous work on the HKS–conjecture are used to prove the Main Theorem.

In the proof of the Main Theorem we used as a base case of the induction the (obviously true) inequality $\text{cr}(C_m \times C_n) \geq 0$. It is natural to ask whether the statement of the Main Theorem is substantially improved if instead we use a nontrivial bound for $\text{cr}(C_m \times C_n)$. The best general lower bound known for the crossing number of $C_m \times C_n$ (for $n \geq m \geq 3$) is $\text{cr}(C_m \times C_n) \geq (1/2)(m - 2)n$ [8]. Using this bound, we obtain the following slightly improved version of the Main Theorem.

Main Theorem [Improved version]. Let $m, n$ be integers such that $n \geq (m/4 + 1/2)((m + 3)^2/2 + 1)$, $m \geq 3$. Then $\text{cr}(C_m \times C_n) = (m - 2)n$.

As we mentioned above, $(m, n)$–circular arrangements are more general structures than drawings of $C_m \times C_n$. Thus, Adamsson’s results actually imply lower bounds for the crossing numbers of families of graphs more general than $C_m \times C_n$. A similar observation holds for the work in this paper. Let us say that a 4–regular graph $G$ is an $(m, n)$–graph if $G$ consists of $n$ pairwise disjoint, cyclically ordered (red) $m$–cycles $\{R(0), R(1), \ldots, R(n-1)\}$, plus $mn$ (blue) edges, such that for each vertex $v$ in $R(j)$, one blue edge incident with $v$ is incident with $R(j - 1)$, and the other blue edge incident with $v$ is incident with $R(j + 1)$. It can be checked that the techniques developed above yield the following more general version of the Main Theorem.

Theorem. Let $m, n$ be integers such that $n \geq (m/2)((m + 3)^2/2 + 1)$, $m \geq 3$. Let $G$ be an $(m, n)$–graph. Then $\text{cr}(G) \geq (m - 2)n$.

While the Main Theorem settles the HKS–Conjecture for all but finitely many values of $n$, for each $m$, the HKS–Conjecture remains open for $n < (m/2)((m + 3)^2/2 + 1), m \geq 8$. For values of $n$ sufficiently close to $m$ (more precisely, for $m, n$ such that $m \geq 8$, $m \leq n \leq 5(m - 1)/4$), it is known that $\text{cr}(C_m \times C_n) \geq (5/7)mn$ [13]. For $n$ between $5(m - 1)/4$ and $(m/4 + 1/2)((m + 3)^2/2 + 1)$, the best general lower bound known is $\text{cr}(C_m \times C_n) \geq (m - 2)n/2$ [8].

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