AFFINE KAC-MOODY LIE ALGEBRAS AS TORSORS
OVER THE PUNCTURED LINE

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Abstract. We interpret and develop a theory of loop algebras as torsors (principal homogeneous spaces) over \( \text{Spec}(k[t, t^{-1}]) \). As an application, we recover Kac’s realization of affine Kac-Moody Lie algebras.

Introduction. There is a beautiful construction of Victor Kac’s, realizing affine Kac-Moody Lie algebras over the complex numbers as (twisted) loop algebras. The construction gives explicit generators for the algebras, which are then shown to satisfy the relations corresponding to the affine Cartan matrix at hand.

In this short note, we propose to look at loop algebras in a completely different way. The basic idea is to view loop algebras as algebras over a ring of Laurent polynomials \( R \), all of which become isomorphic after a flat covering \( R \rightarrow S \). Thus loop algebras become torsors over \( \text{Spec}(R) \) under the group of automorphisms of the algebra at hand. Since this point of view applies to arbitrary algebras and base fields, we are able to obtain some rather general new results about loop algebras in 8 and 10. As an application, we show how to recover from this Kac’s original result. This is done in 11.

We begin with a review of loop algebras in 1, and then recall Kac’s construction in 2. This is followed by some abstract results on algebraic groups (3 through 5) which are later needed. The description of loop algebras as torsors is given in 7.

0. Conventions and notation. Throughout this note \( k \) will denote a field. If \( G \) is a \( k \)-group and \( X = \text{Spec}(R) \) an affine \( k \)-scheme, the \( X \)-group \( G \times_{\text{Spec}(k)} X \) will be denoted by \( G_X \) or \( G_R \). For an \( X \)-group \( F \) the \( \text{C}^\text{\acute{e}ech} \) cohomology \( \hat{H}^1(X_{\text{\acute{e}t}}, F) \) on the \( \text{\acute{e}tale} \) site of \( X \) will be denoted simply by \( H^1(X, F) \). For terminology and results about schemes and principal homogeneous spaces (torsors), the reader is referred to [DG] Ch. 3.4, [Mln] Ch. 3, and [SGA1].

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We will throughout work with two copies of \( k \)-algebras of Laurent polynomials that for convenience we denote by

\[
R = k[t, t^{-1}] \text{ and } S = k[z, z^{-1}].
\]

For each positive integer \( m \) we view the ring \( S \) as an \( R \)-algebra via \( t \mapsto z^m \). This algebra will be denoted by \( S_m \). As usual in what follows \( \bar{\begin{array}{} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \end{array}} \) will denote the canonical map.

1. **Loop algebras.** Let \( A \) be an algebra over \( k \) of a certain “type” (eg. associative, Lie, Jordan, etc) which we assumed comes equipped with a \( \mathbb{Z}/m\mathbb{Z} \)-grading \( \Sigma \).

Thus \( A = \bigoplus_{0 \leq i < m} A_{\bar{i}} \) where the \( A_{\bar{i}} \) are subspaces of \( A \) such that \( A_{\bar{i}} A_{\bar{j}} \subset A_{\bar{i+j}} \). We then define the loop algebra of \( A \) with respect to the given grading \( \Sigma \) by

\[
L(A, \Sigma) := \bigoplus_{i \in \mathbb{Z}} A_{\bar{i}} \otimes z^i \subset A \otimes_k k[z, z^{-1}] := A \otimes_k S.
\]

Note that loop algebras are in a natural way \( k \)-algebras of the same type as \( A \) (eg. loop algebras of a Lie algebra are Lie algebras etc.), and are viewed as such in what follows. If the grading is trivial, namely if \( A_{\bar{0}} = A \), then \( L(A, \Sigma) \simeq A \otimes_k R \). Loop algebras isomorphic as \( k \)-algebras to \( A \otimes k R \) are said to be trivial.

Suppose now that \( k \) is algebraically closed of characteristic 0, and fix a compatible set \( \xi = (\xi_m) \) of primitive roots of unity in \( k \) (thus if \( n = cm \) then \( \xi_m^n = \xi_m \)). Let \( \sigma \) be an automorphism of \( A \) of period \( m \) (we use periods to be able to compare loop algebras coming from automorphisms of different order). Then \( A \) decomposes as the direct sum of eigenspaces \( A_{\bar{i}} \) where \( \sigma \) acts on \( A_{\bar{i}} \) as scalar multiplication by \( \xi_m^i \). We thus obtain a \( \mathbb{Z}/m\mathbb{Z} \)-grading \( \Sigma \) of \( A \) as above. Conversely, any such grading \( \Sigma \) comes from a period \( m \) automorphism \( \sigma \) of \( A \). The resulting loop algebra (which up to \( R \)-algebra isomorphism is independent of the choice of period \( m \) of \( \sigma \)) will be denoted by \( L(A, \sigma, \xi \xi \xi) \), or simply by \( L(\sigma) \) if \( A \) and \( \xi \) are fixed. Note that \( L(id_A) \) is trivial.

Here is the most remarkable application of loop algebras.

**Theorem 2.** Let \( g \) be a simple finite dimensional Lie algebra over \( \mathbb{C} \), and let \( - \) denote the canonical map from \( \text{Aut}(g) \) onto the group \( \text{Out}(g) \) of outer automorphisms of \( g \). Then.

(i) \( L(\sigma) \simeq L(\pi) \) as complex Lie algebras for all \( \sigma \in \text{Aut}(g) \).

(ii) Let \( l \) be an affine Kac-Moody Lie algebra, and let \( l \) be its derived algebra modulo its centre. There exists \( g \) and \( \pi \in \text{Out}(g) \) as above such that \( l \simeq L(\pi) \).

(iii) In (ii) above, \( g \) is unique up to isomorphism, and \( \pi \) unique up to conjugacy in \( \text{Out}(g) \).

**Proof.** Parts (i) and (ii) are due to Kac (see [Kac Theorem 8.3]). For (iii) one needs the conjugacy theorem of Peterson and Kac [PK].

\( \Box \)
Loop algebras are thus concrete realizations of the affine Kac-Moody Lie algebras. We propose now to give new insight into this theorem, as well as obtain new results about loop algebras in general, by interpreting such algebras as torsors over the punctured line. To this end, we begin with some results on the cohomology of algebraic groups.

The following result shows that the classical vanishing of $H^1$ theorems of Steinberg and of Borel and Springer, hold for certain semisimple group schemes over Dedekind domains.

**Proposition 3.** Let $D$ be a Dedekind domain and $K$ its field of quotients. Set $X = \text{Spec}(D)$. Let $G$ be a quasisplit semisimple connected $X$-group, $B$ a Borel subgroup of $G$, and $T$ a maximal torus of $B$.

(i) The natural map $H^1(X, T) \to H^1(X, G)$ induces a surjection $H^1(X, T) \to \text{Ker}(H^1(X, G) \to H^1(K, G_K))$. In particular, this kernel is trivial whenever $H^1(X, T)$ is trivial.

(ii) Assume $G$ is semisimple of either adjoint or simply connected type. If all connected étale coverings of $X$ have trivial Picard group, then $H^1(X, T)$ is trivial.

(iii) Assume $G$ and $X$ are as in (ii). If $K$ is of cohomological dimension 1, then $H^1(X, G)$ is trivial.

(iv) Assume that $X$ and $K$ are as in (iii). If $G$ is split semisimple then the canonical map $H^1(X, \text{Aut}(G)) \to H^1(X, \text{Out}(G))$ is bijective.

**Proof.** (i) This is the extent of Satz 3.2 in [Hrd]. Here is another proof based on an idea (used in [CTO] in the case of a base field) more in tune with the spirit of this note. A torsor $Y$ on the kernel in question is rationally trivial, i.e. it admits a section over a non-empty Zariski open of $X$. Consider the exact sequence

$$1 \to B \to G \to G/B \to 1$$

as well as the contracted product $Y \times^G G/B$. It is clear that the structure morphism $\kappa : Y \times^G G/B \to X$ admits a rational section. Since $\kappa$ is proper and $X$ is one dimensional and regular, this section extends to all of $X$ ([EGA] Ch. II Cor. 7.3.6). By [DG] Ch.III 4.4.6 the $G$-torsor $Y$ comes from $B$, thence from $T$ given that $H^1(X, \text{rad}^u(B)) = \{0\}$ ([SGA3] XXII Cor. 5.9.7). Note that this proof also works in the reductive case. That the elements of $H^1(X, T)$ are rationally trivial is proved in section 1.4 of [Hrd].

(ii) By [SGA3] XXIV 3.13-3.15 and [Hrd] 1.4 there exists a finite family of connected étale coverings $X_i/X$ of $X$ such that

$$T = \prod_i R_{X_i/X}(\mathbb{G}_m),$$

where the $R_{X_i/X}$ are Weil restrictions. By Shapiro’s lemma one has

$$H^1(X, T) = \prod_i H^1(X_i, \mathbb{G}_m) = \prod_i \text{Pic}(X_i) = \{0\}.$$ 

(iii) By classical results of Steinberg and of Borel and Springer $H^1(K, G_K)$ is trivial. (See [BS] 8.2, [Stb], and [JPS1] Ch. 3.1 and 3.2 ). Now (iii) follows from (i) and (ii).
(iv) Recall from [SGA3] XXV the existence of a split exact sequence of $X$-groups

$$1 \to \text{Ad}(G) \to \text{Aut}(G) \to \text{Out}(G) \to 1$$

The group $\text{Out}(G)$ is a finite constant group, and admits a section $s : \text{Out}(G) \to \text{Aut}(G)$ whose image consists of elements of $\text{Aut}(G)$ that stabilize both $B$ and $T$ (We henceforth identify $\text{Out}(G)$ with a subgroup of $\text{Aut}(G)$ by means of this section). Passing to cohomology yields

$$H^1(X, \text{Ad}(G)) \to H^1(X, \text{Aut}(G)) \to H^1(X, \text{Out}(G)).$$

Let $Y$ be an $X$-torsor under $\text{Out}(G)$, and consider the groups $\gamma \text{Ad}(G)$ (resp. $\gamma \text{Ad}(B)$, $\gamma \text{Ad}(T)$) obtained from $\text{Ad}(G)$ (resp. $\text{Ad}(B)$, $\text{Ad}(T)$) by twisting by $Y$. Being a form of $\text{Ad}(G)$, the group $\gamma \text{Ad}(G)$ is semisimple and of adjoint type as well. It is also quasisplit by means of $(\gamma \text{Ad}(B), \gamma \text{Ad}(T))$. By (iii) then, $H^1(X, \gamma \text{Ad}(G))$ is trivial. From this it follows that the map in question is injective. The surjectivity is clear because the existence of the section $s$. \hfill \Box

**Note.** A priori $\gamma \text{Ad}(G)$ is a sheaf of groups on $X$. That it is an affine and smooth scheme over $X$ follows from descent. That its geometric fibers are reductive and connected follows from the analogous properties for $\text{Ad}(G)$. Thus $\gamma \text{Ad}(G)$ is a reductive group in the sense of [SGA3]. Along similar lines $\gamma \text{Ad}(B)$ is a Borel subgroup . . .

The following result shows that the assumption on étale coverings made in part (ii) of Proposition 3, holds in a crucial case.

**Proposition 4.** (P. Gille) Assume $k$ is of characteristic 0. Every connected finite étale covering of $\text{Spec}(k[t,t^{-1}])$ has trivial Picard group.

**Proof.** Let $Y \to X := \text{Spec}(k[t,t^{-1}])$ be one such covering. Fix an element $x \in X(k)$. From [SGA1] Exp. IX Th. 6.3.1 together with a Theorem of Grauert-Rehmert ([SGA4] Exp. XI Th. 4.3), as well as from [SGA1] Exp. XIII Cor. 2.12, it follows that the fundamental group $\Pi_1(X, x)$ is the semidirect product $(\text{inv lim } \mu_n(k_s)) \rtimes \text{Gal}(k_s/k)$, where the $\mu_n$ come from the Kummer coverings. There thus exists a positive integer $m$, a finite Galois field extension $L/k$ containing a primitive $m$-th root of unity, and a subgroup $\Gamma$ of $\mu_m(L) \rtimes \text{Gal}(L/k)$, such that $Y = Y_0/\Gamma$ where $Y_0$ is the $k$-variety defined by the morphism $Y_0 = X_L \to \text{Spec}(L) \to \text{Spec}(k)$. As the morphism $Y_0 \to Y$ is a Galois covering, the beginning of the Hochschild–Serre spectral sequence $E^{p,q}_2 = H^p(\Gamma, H^q(Y_0, \mathbb{G}_m)) \Rightarrow H^{p+q}(Y, \mathbb{G}_m)$ yields an exact sequence

$$0 \to H^1(\Gamma, H^0(Y_0, \mathbb{G}_m)) \to H^1(Y, \mathbb{G}_m) \to H^1(Y_0, \mathbb{G}_m)^{\Gamma}.$$

Since $H^1(Y_0, \mathbb{G}_m) = \text{Pic}(Y_0) = \{0\}$, we get an isomorphism $H^1(\Gamma, H^0(Y_0, \mathbb{G}_m)) \simeq \text{Pic}(Y)$. One has an exact sequence of $\Gamma$–modules (and of $\mu_m(L) \rtimes \text{Gal}(L/k)$–modules)

$$0 \to L^\times \to H^0(Y_0, \mathbb{G}_m) \to \mathbb{Z} \to 0,$$
where $\mathbb{Z}$ has trivial $\Gamma$-action. By Hilbert’s theorem 90 one has $H^1(\Gamma, L^\times) = \{0\}$, and as $\Gamma$ is finite, one also has $H^1(\Gamma, \mathbb{Z}) = \{0\}$. Thus $H^1(\Gamma, H^0(Y_0, G_m)) = \{0\}$ and therefore Pic($Y$) = $\{0\}$ as desired.

Part (i) of the next result is an easy, but useful generalization of a result of [CTO].

**Proposition 5.** Let $k$ be an infinite perfect field and let $G$ be a smooth connected linear algebraic $k$-group. Let $X$ be a nonempty open subscheme of Spec$(k[t]) = \mathbb{A}^1$. Then.

(i) The canonical map $H^1(X, G_X) \to H^1(k(t), G_{k(t)})$ has trivial kernel.

(ii) $H^1(X, G_X)$ is trivial in the following three cases.

(a) If $H^1(k(t), G_{k(t)})$ is trivial.

(b) If $k$ is algebraically closed and of characteristic 0.

(c) If $k$ is algebraically closed and $G$ is reductive.

**Proof.** Let us begin by showing that every Zariski $G$-torsor over $X$ (i.e. a torsor that can be trivialized by a Zariski covering of $X$) extends to $\mathbb{A}^1$ (this much holds for any group scheme $G$). Indeed. The underlying space of $X$ is obtained by removing a finite set $F$ of points from the affine line. Let now $Y$ be a Zariski $G$-torsor over $X$. Let $U$ be the intersection of a finite number of non empty open subschemes of $X$ that cover $X$, and over which $Y$ is trivial. Finally, let $Z$ be the trivial $G$-torsor over the open subscheme of $\mathbb{A}^1$ corresponding to $U \cup F$. Then by gluing $Y$ with $Z$ along $U$ we obtain a torsor over $\mathbb{A}^1$ as desired.

(i) By reasoning as in Théorème 2.1 of [CTO] we may assume that $G$ is reductive and connected. A Theorem of Nisnevich [Nsn] then shows that the kernel of the map in question is comprised precisely of Zariski $G$-torsors over $X$. But we have seen that any such torsor extends to the full affine line. Now (i) follows from [CTO] Corollaire 2.3 which asserts that (i) does hold for $\mathbb{A}^1$.

(ii) By the theorems of Steinberg and Borel-Springer mentioned above, $H^1(k(t), G_{k(t)})$ is trivial under the assumptions of either (b) or (c). Thus (b) and (c) reduce to (a), and this last holds by (i).

**Remark 6.** Assume that $k$ contains a primitive $m$-th root of unity. Then the finite covering Spec$(S_m) \to$ Spec$(R)$ is étale and in fact Galois, with Galois group $\Gamma \simeq \mathbb{Z}/m\mathbb{Z}$ (see 7 infra). Assume now that we are under one of the cases of Proposition 5 (ii). Then the usual non-abelian cohomology $H^1(\Gamma, G(S_m))$ vanishes. If $k = \mathbb{C}$ and $G$ is semisimple, we recover the observation made in [Kac] §8.9.

**7. Loop algebras as torsors.** We return now to our general setup of 1 and consider an arbitrary $k$-algebra $A$, together with $\mathbb{Z}/m\mathbb{Z}$-grading $\Sigma$. For convenience we henceforth denote Spec$(R) :=$ Spec$(k[t, t^{-1}])$ by $X$ and Spec$(S) :=$ Spec$(k[z, z^{-1}])$ by $Y$.

The loop algebra $L(\Sigma) = L(A, \Sigma)$ is naturally an $R$-algebra, and it is not hard to show that $L(\Sigma) \otimes_R S_m \simeq A \otimes_k S$ as $S$-algebras (see [ABP]). In other words, $L(\Sigma)$ is an $S_m/R$-form of $A \otimes_k R$. Since $S_m/R$ is faithfully flat and finitely presented (fppf), $L(\Sigma)$ is an $\text{Aut}(A_X)$-torsor over $X$. In this context, $\text{Aut}(A_X)$ stands for the sheaf of groups on the flat site of $X$ that attaches to $X'/X$, the group of $\mathcal{O}(X')$-algebra automorphism of $A \otimes_k \mathcal{O}(X')$ of the same type as $A$.  

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The isomorphism class of the $R$-algebra $L(\Sigma)$ is thus an element of $H^1(X_{\text{fppf}}, \text{Aut}(A_X))$ that we denote by $L^1(\Sigma)$. To say that $L^1(\Sigma_1) = L^1(\Sigma_2)$ is to say that $L(\Sigma_1)$ and $L(\Sigma_2)$ are isomorphic as algebras over $R$. It is clear then that $L(\Sigma_1)$ and $L(\Sigma_2)$ are a fortiori isomorphic as algebras over $k$. The converse does not hold in general, but as we shall see in 10 and 11, it does hold in some very interesting cases.

Next we assume that the base field $k$ is algebraically closed and of characteristic zero, and go on to describe explicitly how to construct $L^1(\sigma)$ from $\sigma$. The finite covering $Y \to X$ is étale and in fact Galois. Its Galois group $\Gamma$ can be identified with $X_m$.

Example 9. If $A$ is the associative unital algebra of $n$ by $n$ matrices with entries in an algebraically closed field $k$, then all its covering algebras are trivial. Indeed, $\text{Aut}(A) \simeq \text{PGL}_n$.

Proposition 10. Let $k$ be an algebraically closed field of characteristic 0. Assume that the $k$-algebra $A$ is finite dimensional, and that the linear algebraic $k$-group $\text{Aut}(A)$ is smooth and connected. If either of the conditions of Proposition 5 (ii) hold, then all loop algebras of $A$ are trivial.

Proof. By 7, it suffices to show that $H^1(X_{\text{fppf}}, \text{Aut}(A_X))$ is trivial. Since $\text{Aut}(A_X)$ is smooth, we may replace the fppf by the étale topology. Now apply Proposition 5 (ii).}$\square$

Example 9. If $A$ is the associative unital algebra of $n$ by $n$ matrices with entries in an algebraically closed field $k$, then all its covering algebras are trivial. Indeed, $\text{Aut}(A) \simeq \text{PGL}_n$.

Proposition 10. Let $k$ be an algebraically closed field of characteristic 0. Assume that the $k$-algebra $A$ is finite dimensional, and that the linear algebraic $k$-group $\text{Aut}(A)$ coincides with the group of automorphisms $\text{Aut}(G)$ of some semisimple algebraic $k$-group $G$ of either adjoint or simply connected type. Set $X = \text{Spec} \, k[t, t^{-1}]$. Then there exists a canonical bijection between the following four sets:

1. $H^1(X, \text{Aut}(G_X))$
2. $H^1(X, \text{Out}(G_X))$
3. Conjugacy classes of the (abstract) finite group $\text{Out}(G) := \text{Out}(G)(k)$.
4. Isomorphism classes in $R$-alg of loop algebras of $A$.

In particular, all these sets are finite. If in addition in the group $\text{Out}(G)$ all elements are conjugate to their inverses, and if the centroid of all loop algebras of $A$ coincides with $R$, then (4) above is equivalent to

5. Isomorphism classes in $k$-alg of loop algebras of $A$.

Proof. (1) $\simeq$ (2). This was established in Theorem 3 (iv).
(2) $\simeq$ (3). Let $E$ be the set of continuous group homomorphisms from the fundamental group $\Pi(X, x)$ into $\text{Out}(G)$. Clearly $\text{Out}(G)$ acts on $E$ by conjugation. Since $\text{Out}(G_X)$
is finite and constant, $H^1(X, \text{Out}(G_X))$ can be computed as the quotient set $E/\text{Out}(G)$ [SGA1]. Now from Proposition 4 we know that $\Pi(X, x) \simeq \varprojlim \mathbb{Z}/n\mathbb{Z}$. It follows that the elements of $E$ can be identified in a natural way with elements of $\text{Out}(G)$, and that then two elements of $E$ are equivalent under the action of $\text{Out}(G)$ if and only if the corresponding elements of $\text{Out}(G)$ are conjugate.

(3) $\simeq$ (4). First note that since we are in characteristic 0 all loop algebras are of the form $L(\sigma)$ for some $\sigma \in \text{Aut}(A) = \text{Aut}(A)(k) = \text{Aut}(G)(k)$. The correspondence assigns to the $R$-isomorphism class of $L(\sigma)$ the conjugacy class of $\sigma$. That such map exists and is bijective follows from the interpretation of the $L(\sigma)$’s as torsors, the explicit construction of $L^1(\sigma)$ given in 7, and Proposition 3(iv).

Assume now that $A$ above is such that in the group $\text{Out}(G)$ all elements are conjugate to their inverses, and also such that the centroid $\mathcal{Z}$ of all the $L(\sigma)$’s (viewed as $k$-algebras) coincide with $R$ (namely $\mathcal{Z} = k[z^m, z^{-m}] \simeq R$ where $m$ is a chosen period of $\sigma$). We must show that if two loop algebras $L(\sigma_1)$ and $L(\sigma_2)$ are isomorphic as $k$-algebras then they are isomorphic as $R$-algebras. Since all the isomorphism classes in question do not depend on the choice of period, we may assume that both $\sigma_1$ and $\sigma_2$ have period $m$. We now reason as in [ABP]. Fix a $k$-algebra isomorphism $\psi$ between $L(\sigma_1)$ and $L(\sigma_2)$. Then $\psi$ induces an automorphism $\psi_2$ of the centroid $\mathcal{Z}$ of $L(\sigma_1)$. There are two “types” of automorphisms of $\mathcal{Z} \simeq R$: those that do not interchange $kt$ and $kt^{-1}$, and those which do. The upshot of this is that $L(\sigma_1)$ is isomorphic as an $R$-algebra to either $L(\sigma_2)$ or $L(\sigma_2^{-1})$ depending on the type of $\psi_2$.

Having establish this, we appeal to (3) $\simeq$ (4) to conclude that $\sigma_1$ and $\sigma_2$ are conjugate (because in $\text{Out}(G)$ elements are conjugate to their inverses), and hence that $L(\sigma_1)$ and $L(\sigma_2)$ are isomorphic as $R$-algebras (again by (3) $\simeq$ (4)).

We can now recover the parts of Theorem 2 concerning loop algebras.

**Corollary 11.** Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over an algebraically closed field $k$ of characteristic 0. Let $\sigma_1$ and $\sigma_2$ be two automorphisms of $\mathfrak{g}$ of finite order. For $L(\sigma_1)$ to be isomorphic to $L(\sigma_2)$ as algebras over $k$, it is necessary and sufficient that $\sigma_1$ and $\sigma_2$ be conjugate in $\text{Out}(\mathfrak{g})$.

**Proof.** Let $G$ be the simply connected Chevalley-Demazure group corresponding to $\mathfrak{g}$. Then $\text{Aut}(G_X) \simeq \text{Aut}(\mathfrak{g}_X)$ with $\text{Out}(G)$ corresponding to $\text{Out}(\mathfrak{g})$ ([SGA3] XXV). Furthermore $\text{Out}(G)$ is the group of automorphisms of the corresponding Dynkin diagram, and one knows that in these groups all elements are conjugate to their inverses. That the centroids of the loop algebras of $\mathfrak{g}$ coincide with $R$ is easy to check (see [ABP] for details). \(\square\)

**Remark 12** It is natural to ask if Proposition 10 holds for symmetrizable Kac-Moody Lie algebras. We look at this problem in [ABP], and make good progress by using the Gantmacher-like decomposition of automorphisms described in [KW].

It would be interesting to know if the answer to this question can be had by purely cohomological methods (as the finite dimensional case here as well as [Pzl1] and [Pzl2] seem to suggest is possible). The abstract construction of section 7 applies, but the real difficulty of course comes when one tries to recreate Proposition 3 for the various group schemes attached to Kac-Moody algebras (See [Tts]). This appears to be a very difficult question, but in the affine case at least, progress should be possible.
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