AN IMPROVEMENT ON THE RELATIVELY COMPACTNESS CRITERIA

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Abstract. This paper is devoted to the study of the relatively compact sets in Banach function spaces, providing an important improvement of the known results. As an application, we take the final step in establishing a relative compactness criteria for function spaces with any weight without any assumption.

1. Introduction

The characterization of relatively compactness in the classical $L^p$ Lebesgue spaces was discovered by Kolmogorov (see [13]) under some restrictive conditions. Then it was extended by Tamarkin [12] and Tulajkow [15]. At the same time, M. Riesz proved a similar result. More precisely, the complete version of classical Riesz-Kolmogorov theorem can be stated as follows:

Theorem A. (Classical Riesz-Kolmogorov theorem) Let $1 \leq p < \infty$. A subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$ is relatively compactness, if and only if the following three conditions hold:

(a) $\mathcal{F}$ is bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L^p(\mathbb{R}^n)} \lesssim 1$;

(b) $\mathcal{F}$ uniformly vanishes at infinity, that is,

$$\lim_{N \to \infty} \sup_{f \in \mathcal{F}} \|f\mathcal{I}_{B^c(0,N)}\|_{L^p(\mathbb{R}^n)} = 0;$$

(c) $\mathcal{F}$ is equicontinuous, that is,

$$\lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \|\tau_y f - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Here, $\tau_y$ denotes the translation operator: $\tau_y f(x) = f(x - y)$.

From then on, the compactness criteria were studied by many authors in various settings. Meanwhile, it has played an important role in the compactness results of certain bounded operators in the field of harmonic analysis. Among numerous of articles, we would like to mention some of them from the following two perspectives:

1. Extension to general settings. First, the Lebesgue metric measure space $(\mathbb{R}^n, |\cdot|, m)$ in the classical case, with Euclidean metric $|\cdot|$ and Lebesgue measure $m$, can be generalized to the metric measure space $(X, \rho, \mu)$ with metric $\rho$ and measure $\mu$. More precisely, one can study the relatively compactness property
on $L^p(X, \rho, \mu)$, see [6, 8, 11] for this direction. In this general case, if one also wants to establish an equivalent characterization theorem on $L^p(X, \rho, \mu)$ like Theorem A, the condition (c) in Theorem A should be replaced by the following condition:

$$(c^+) \quad \limsup_{r \to 0} \sup_{f \in F} \left\| \frac{1}{\mu(B(\cdot, r))} \int_{B(\cdot, r)} f \, d\mu - f \right\|_{L^p(X, \rho, \mu)} = 0,$$

where $B(x, r) = \{ y \in X : \rho(x, y) < r \}$. Recently, in a more general framework, the compactness criteria were studied by Górka–Rafeiro [7] in the setting of Banach function space. The main result [7, Theorem 3.1] is a new relatively compact criteria fitting more general cases, in which the equicontinuous condition is replaced by

$$(c^*) \quad \limsup_{r \to 0} \sup_{f \in F} \left\| \frac{1}{\mu(B(\cdot, r))} \int_{B(\cdot, r)} f \, d\mu - f \right\|_E = 0,$$

where $E$ is a Banach function space containing certain $\mu$-measurable functions.

We would like to point out that if $\mu = m$ is a Lebesgue measure on $\mathbb{R}^n$, and $\rho = | \cdot |$ is the Euclidean metric, $(c^*)$ can be deduced by $(c)$ since

$$\left\| \frac{1}{|B(\cdot, r)|} \int_{B(\cdot, r)} f(y) \, dy - f \right\|_E = \left\| \frac{1}{|B(0, r)|} \int_{B(0, r)} f(\cdot + y) \, dy - f \right\|_E \leq \sup_{y \in B(0, r)} \| \tau_y f - f \|_E,$$

for Banach function space $E$. While this inclusion relations between $(c^*)$ and $(c)$ is invalid for general $\mu$ and $\rho$. In [7], the authors also establish a necessity result [7, Theorem 3.2] under some reasonable assumptions of the Banach function spaces.

(2) Applications to Harmonic analysis. In the field of harmonic analysis, in order to verify the compactness of a bounded operator, we usually apply the compactness criteria like Theorem A, for instance one can see [16, 3] for the unweighted case of compact commutator of singular integral, and see [1, 2, 4, 17] for the weighted case. Especially, in order to verify the compactness of a $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ bounded operator with some weight function $\omega$, the reasonable equicontinuous condition should be as follows:

$$\limsup_{r \to 0} \sup_{f \in F} \sup_{y \in B(0, r)} \| \tau_y f - f \|_{L^p(\mathbb{R}^n)}, \quad (1.1)$$

one can see [16, 3, 1, 2, 4, 17] for more details. Thus, the known results with condition $(c^+)$, in the setting of metric measure space is invalid here, since the above condition [1, 11] can not imply condition $(c^+)$ even we take $\rho = | \cdot |$ and $d\mu = \omega dx$. Even to this day, due to the incompleteness of weighted compactness criteria, the weighted version of Riesz-Kolmogorov theorem is still being improved, one can see a very recent article [18], in which the authors study the relatively compactness criteria for $L^p_\omega(\mathbb{R}^n)$. We remark that in [18], some additional assumptions are still needed for the weights (see [18, Theorem
4.2), although the assumptions permit the weights beyond the $A_p$ ($1 \leq p < \infty$) class.

Based on the above two directions of research, we have two natural considerations:

1. Can the weighted version of Riesz-Kolmogorov theorem be deduced by a more general theorem established in [7] on Banach function spaces?
2. Can the additional assumptions on weights be completely eliminated in the weighted version of Riesz-Kolmogorov theorem?

The main purpose of this article is to consider the two problems mentioned above. In fact, for general weights without additional assumption, the answer for the first problem is negative, one will see the detailed explanation in Section 2. In order to solve the second problem, we turn to establish a useful relatively compactness criteria in a suitable framework of Banach function spaces, which is not included in [7]. As an application, we take the final step in establishing a relative compactness criteria for function spaces with any weight.

The remainder of this paper is structured as follows. In Section 2, we give some required definitions and notations for the framework we are working on. And, we also explain why [7, Theorem 3.1] is invalid in our case. Section 3 is devoted to the proofs of our main results, including a relatively compactness criteria on Banach and Quasi-Banach function spaces. We also list some important applications on weighted function spaces.

We point out that in the setting of completed metric space, relatively compactness and totally boundedness are equivalent. Due to the technical convenience, we use totally boundedness in our theorems and their proofs.

2. Banach function spaces and weighted function spaces

First, we recall some basic definitions about Banach function spaces. In this paper, we only consider the class of Lebesgue measurable functions, denoted by $L^0(m)$, where $m$ denotes the Lebesgue measure on $\mathbb{R}^n$.

**Definition 2.1.** A normed space $(E, \| \cdot \|_E)$ with $E \subset L^0(m)$ is called a Banach function space if it satisfies the following conditions:

- $(B_0)$: if $\|f\|_E = 0 \iff f = 0$ a.e.;
- $(B_1)$: if $f \in E$, then $\|f\|_E = \|f\|_E$;
- $(B_2)$: if $0 \leq g \leq f$, then $\|g\|_E \leq \|f\|_E$;
- $(B_3)$: if $0 \leq f_n \uparrow f$, then $\|f_n\|_E \uparrow \|f\|_E$;
- $(B_4)$: if $A \subset \mathbb{R}^n$ with $m(A) < \infty$, then $\chi_A \in E$;
- $(B_5)$: if $f\chi_A \in E$ with $\|\chi_A\|_E \neq 0$, then there exists a point $x_0 \in A$ such that $f(x_0) < \infty$.

For a classical definition of Banach function space used in [7], one can see the book Edmunds-Evans [5], where the definition of $(B_5)$ is replaced by the following stronger one:
$(B^*_5)$: if $A \subset \mathbb{R}^n$ and $m(A) < \infty$, then there exists a constant $C(A)$ such that
\[
\int_A |f|dx \leq C(A)\|f\|_E.
\]

We also remark that even under our weaker conditions, one can verify that the Banach function space in Definition 2.1 is a Banach space (see Appendix A).

In harmonic analysis, a weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere (see [9]). For a weight function $\omega$, the $L^p_\omega(\mathbb{R}^n)$ function space with $p \in (0, \infty)$ is defined by
\[
L^p_\omega(\mathbb{R}^n) := \left\{ f \in L^0(m) : \|f\|_{L^p_\omega(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \right)^{1/p} < \infty \right\}.
\]

In order to find out whether the relative compactness criteria in [7] can be used in the case of weighted function spaces, let us first check that whether the condition $(B^*_5)$ holds in the case of weighted function spaces $E = L^p_\omega(\mathbb{R}^n)$ with $1 \leq p < \infty$. In fact in this case, the $(B^*_5)$ condition is just
\[
\int_A |f|dx \leq C(A) \left( \int_A |f(x)|^p \omega(x)dx \right)^{1/p} \quad \text{for } m(A) < \infty,
\]
which is equivalent to
\[
\int_A \omega(x)^{1-p'}dx < \infty \quad \text{for } m(A) < \infty,
\]
where the right term should be interpreted as $\|\omega^{-1} \chi_A\|_{L^\infty}$ for $p = 1$. It is well known that (2.2) can be deduced by the so-called $A_p$ condition or be as an independent assumption as in [18, Lemma 4.1]. We recall the definition of $A_p$ as follows.

**Definition 2.2** ([9]). For $1 < p < \infty$, the Muckenhoupt class $A_p$ is the set of locally integrable weights $\omega$ such that
\[
[\omega]_{A_p}^{1/p} := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x)dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-p'}dx \right)^{1/p'} < \infty,
\]
where $1/p + 1/p' = 1$.

**Definition 2.3** ([9]). A weight function $\omega$ is called an $A_1$ weight if
\[
\frac{1}{|Q|} \int_Q \omega(x)dx \leq [\omega]_{A_1, \text{ess.inf}}_{y \in Q} \omega(y).
\]

One can easily check that the condition (2.2) holds for $A_p$ weight. In fact, in [7] if we choose the Banach function space $E = L^p_\omega(\mathbb{R}^n)$ containing functions belong to $L^0(m)$, then the following result is a direct conclusion of [7, Theorem 3.1]

**Proposition 2.4.** Let $\omega$ be a weight satisfies (2.2). If the subset $\mathcal{F}$ of $L^p_\omega(\mathbb{R}^n)$ satisfies the following three conditions:

(a): $\mathcal{F}$ is bounded, i.e.,
\[
\sup_{f \in \mathcal{F}} \|f\|_{L^p_\omega(\mathbb{R}^n)} \lesssim 1;
\]
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(b): $\mathcal{F}$ uniformly vanishes at infinity, that is,
\[
\lim_{N \to \infty} \sup_{f \in \mathcal{F}} \| f \chi_{B^c(0,N)} \|_{L^p(\mathbb{R}^n)} = 0;
\]

(c\*): $\mathcal{F}$ is equicontinuous, that is,
\[
\lim_{r \to 0} \sup_{f \in \mathcal{F}} \left\| \frac{1}{B(\cdot, r)} \int_{B(\cdot, r)} f(y) dy - f \right\|_{L^p(\mathbb{R}^n)} = 0,
\]
then $\mathcal{F}$ is a totally bounded subset of $L^p_\omega(\mathbb{R}^n)$.

As mentioned in Section 1, the condition (c\*) in Proposition 2.4 can be replaced by the following stronger one:
\[
(c) \quad \lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0, r)} \| \tau_y f - f \|_{L^p(\mathbb{R}^n)} = 0. \tag{2.3}
\]

However, for more general weights, $(B_5^*)$ is too strong to apply. In order to explain this more precisely, we give a counterexample here. Let $A = B(0, 1)$, $\omega(x) = |x|^{n(p-1)+1}$ and $f_N = \chi_{B(0, 1/N)}$. A direct calculation yields that
\[
\sup_{f \in \mathcal{X}} \int_A |f(x)|^p \omega(x) dx < \infty \implies |f(x)|^p \omega(x) < \infty \quad a.e. \implies |f(x)| < \infty \quad a.e..
\]
This breaks the condition (2.1). Of course, in this counterexample, $\omega$ is chosen outside the $A_p$ class.

Obviously, $(B_5)$ is weaker than $(B_5^*)$. Moreover, our new condition $(B_5)$ can be applied to a much wider range of weighted function spaces. In fact, $(B_5)$ is valid for $L^p_\omega(\mathbb{R}^n)$ with any weight $\omega$, since
\[
\int_A |f(x)|^p \omega(x) dx < \infty \implies |f(x)|^p \omega(x) < \infty \quad a.e. \implies |f(x)| < \infty \quad a.e..
\]
That is just the reason why the additional assumption on the weight can be completely eliminated in our framework.

3. RELATIVELY COMPACTNESS CRITERIA

3.1. On Banach function space. In this subsection, we establish the relatively compactness criteria in the framework of Banach function space. As mentioned above, we weaken the assumptions of Banach spaces, providing a more general framework fitting weighted function spaces with any weight. Meanwhile, in our following theorem, the equicontinuous condition is chosen to be the “(c) type” as in Theorem A. Although the ”(c\*) type” as in Proposition 2.4 is more weak, however, the relatively compactness criteria of “(c\*) type” relies heavily on the condition $(B_5^*)$, one can see the proof of [7, Theorem 3.1] for more details. On the other hand, the “(c) type” condition is more applicable in the field of harmonic analysis, see [16, 3, 1, 2, 4, 17]. Therefore, under our weaker assumptions on Banach function space, “(c) type” condition is reasonable and has strong applicability.

**Theorem 3.1.** Let $E$ be a Banach function space. If the family $\mathcal{F} \subset E$ satisfies the following conditions:
(a) $F$ is bounded, i.e.,
$$\sup_{f \in F} \|f\|_E \lesssim 1;$$
(b) $F$ uniformly vanishes at infinity, that is,
$$\lim_{N \to \infty} \sup_{f \in F} \|f \chi_{B^c(0,N)}\|_E = 0;$$
(c) $F$ is equicontinuous, that is,
$$\lim_{r \to 0} \sup_{f \in F} \sup_{y \in B(0,r)} \|\tau_y f - f\|_E = 0,$$
then the family $F$ is a totally bounded subset of $E$.

Proof. From condition (c), there exists a sufficiently small $\delta > 0$ such that
$$\int_{[-\delta, \delta]^n} \|\tau_y f - f\|_E dy \leq (2\delta)^n \sup_{y \in [-\delta, \delta]^n} \|\tau_y f - f\|_E < \infty. \quad (3.1)$$

In order to verify that $F$ is a totally bounded set, we only need to find the finite $\epsilon$-net of $F$ for each fixed $\epsilon$. Denote by $R_i := [-2^i, 2^i]^n$ for $i \in \mathbb{Z}$. By condition (b), there exists a sufficiently large positive integer $m$ such that
$$\|f - f \chi_{R_m}\|_E < \frac{\epsilon}{3}.$$ 
Thus, we only need to verify that the family of functions $\{f \chi_{R_m}\}_{f \in F}$ has a finite $\frac{2\epsilon}{3}$-net.

By condition (c), we choose an integer $i_\epsilon$ such that $2^{i_\epsilon} < \delta$ and
$$\|\tau_y f - f\|_E < 2^{-n} \epsilon/3, \quad y \in R_{i_\epsilon}.$$
For $x \in R_m$, $Q_x$ means the dyadic cube of side length $2^{i_\epsilon}$ that contains $x$. Define
$$\Phi(f \chi_{R_m})(x) = \begin{cases} 
\frac{1}{|Q_x|} \int_{Q_x} f(y) dy, & x \in R_m, \\
0, & \text{otherwise}. 
\end{cases}$$
We claim that the map $\Phi$ is well-defined by
$$\int_{Q_x} |f(y)| dy < \infty, \quad \text{for } x \in R_m.$$
It follows by (3.1) that
$$\left\| \int_{R_{i_\epsilon}} |f(\cdot - y)| dy \right\|_E \leq \int_{R_{i_\epsilon}} \|f(\cdot - y)\|_E dy$$
$$\leq \int_{R_{i_\epsilon}} \|\tau_y f - f\|_E dy + \int_{R_{i_\epsilon}} \|f\|_E dy$$
$$\leq \int_{R_{i_\epsilon}} \|\tau_y f - f\|_E dy + |R_{i_\epsilon}| \|f\|_E < \infty.$$
For any fixed $Q_x$, since $\|\chi_{Q_x}\|_X \neq 0$, by $(B_5)$ there exists a point $x_0 \in Q_x$ such that
$$\int_{R_{i_\epsilon}} |f(x_0 - y)| dy < \infty.$$
Observing that $Q_x \subset x_0 - R_{i_\epsilon}$, we further have
\[ \int_{Q_x} |f(y)| dy \leq \int_{x_0 - R_i} |f(y)| dy = \int_{R_i} |f(x_0 - y)| dy < \infty. \]

Next, we turn to the estimate of \( \|f\chi_{R_m} - \Phi(f\chi_{R_m})\|_E \). A direct calculation yields that
\[
|(f - f_{Q_x})\chi_{Q_x}| = \left| \frac{1}{|Q_x|} \int_{Q_x} (f(x) - f(z)) dz \chi_{Q_x} \right| \\
\leq \frac{1}{|Q_x|} \int_{Q_x} |f(x) - f(z)| dz \chi_{Q_x} \\
\leq \frac{1}{|Q_x|} \int_{R_i} |f(x) - f(x - y)| dy \chi_{Q_x}. 
\]

It follows that
\[
\left| \sum_{Q_x \subset R_m} (f - f_{Q_x})\chi_{Q_x} \right| \leq \sum_{Q_x \subset R_m} |(f - f_{Q_x})\chi_{Q_x}| \\
\leq \sum_{Q_x \subset R_m} \frac{1}{|Q_x|} \int_{R_i} |f(x) - f(x - y)| dy \chi_{Q_x} \\
= 2^{-ni} \int_{R_i} |f(x) - f(x - y)| dy \chi_{R_m}(x) 
\]

Hence,
\[
\|f\chi_{R_m} - \Phi(f\chi_{R_m})\|_E = \left\| \sum_{Q_x \subset R_m} f\chi_{Q_x} - \sum_{Q_x \subset R_m} f_{Q_x}\chi_{Q_x} \right\|_E \\
= \left\| \sum_{Q_x \subset R_m} (f - f_{Q_x})\chi_{Q_x} \right\|_E \\
\leq 2^{-ni} \int_{R_i} |f(x) - f(x - y)| dy \chi_{R_m}(x) \right\|_E 
\]

Applying the Minkowski inequality, we have
\[
2^{-ni} \int_{R_i} |f(x) - f(x - y)| dy \chi_{R_m}(x) \right\|_E \\
\leq 2^{-ni} \int_{R_i} \|\tau_y f - f\|_E dy \leq 2^n \sup_{y \in R_i} \|\tau_y f - f\|_E < \epsilon/3. 
\]

The above two estimates imply that
\[ \|f\chi_{R_m} - \Phi(f\chi_{R_m})\|_E < \epsilon/3. \]

From this, to get our final conclusion, we only need to verify that the family of functions \( \{\Phi(f\chi_{R_m})\}_{f \in \mathcal{F}} \) has a finite \( \frac{\epsilon}{3} \)-net. This is true since this family is a bounded subset of
a finite dimensional Banach space. Let us check the boundedness by 
\[ \| \Phi(f \chi_{R_m}) \|_E \leq \| f \chi_{R_m} - \Phi(f \chi_{R_m}) \|_E + \| f \chi_{R_m} \|_E \leq \epsilon/3 + \| f \|_E \lesssim 1. \]
Now, we have completed this proof. □

Remark 3.2. Here, the proof is finished by a finite dimension argument. By adapting the arguments in \([10]\), the authors in \([4]\) also used a finite dimension argument to prove the relatively compactness criteria on \(L^p_\omega(\mathbb{R}^n)\) with \(\omega \in A_p(\mathbb{R}^n)\). Unfortunately, the method in \([4, 10]\) is invalid here, since it heavily depends on the \(A_p(\mathbb{R}^n)\) condition and the special structure of \(L^p_\omega(\mathbb{R}^n)\). By contrast, our new method is applicable for the case of Banach function spaces including weighted function spaces with any weight.

3.2. on weighted function space. Obviously, Theorem 3.1 can be applied to \(L^p_\omega(\mathbb{R}^n)\) spaces with any weights. Here, we would like to show a more general case in which the "weight function \(v\)" can even disappear on a positive measurable set.

**Theorem 3.3.** Let \(1 \leq p < \infty, v \in L^0(m)\) be a nonnegative function. Define
\[ \| f \|_{L^p_v(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}. \]
If the family \(\mathcal{F} \subset L^p_v(\mathbb{R}^n)\) satisfies the following conditions:
(a) \(\mathcal{F}\) is bounded, i.e.,
\[ \sup_{f \in \mathcal{F}} \| f \|_{L^p_v(\mathbb{R}^n)} \lesssim 1; \]
(b) \(\mathcal{F}\) uniformly vanishes at infinity, that is,
\[ \lim_{N \to \infty} \sup_{f \in \mathcal{F}} \| f \chi_{B^c(0,N)} \|_{L^p_v(\mathbb{R}^n)} = 0; \]
(c) \(\mathcal{F}\) is equicontinuous, that is,
\[ \limsup_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \| \tau_y f - f \|_{L^p_v(\mathbb{R}^n)} = 0, \]
then the family \(\mathcal{F}\) is a totally bounded subset of \(L^p_v(\mathbb{R}^n)\).

**Proof.** As in the proof of Theorem 3.1, we only need to verify that the family of functions \(\{f \chi_{R_m}\}_{f \in \mathcal{F}}\) has a finite \(\frac{2^m}{3}\)-net. Take \(i_\epsilon\) as in the proof of Theorem 3.1. For \(x \in R_m, Q_x\) means the dyadic cube of side length \(2^m\) that contains \(x\). Define
\[ \Phi(f \chi_{R_m}) = \begin{cases} f_{Q_x}, & x \in R_m, \| \chi_{Q_x} \|_{L^p_v(\mathbb{R}^n)} \neq 0, \\ 0, & \text{otherwise.} \end{cases} \]
We claim that the map \(\Phi\) is well-defined by
\[ \int_{Q_x} |f(y)| dy < \infty, \quad \text{for } x \in R_m, \| \chi_{Q_x} \|_{L^p_v(\mathbb{R}^n)} \neq 0. \]
In fact, by the same estimate as in the proof of Theorem 3.1 we have
\[ \left\| \int_{R_{i_\epsilon}} |f(\cdot - y)| dy \right\|_{L^p_v(\mathbb{R}^n)} < \infty. \]
From this, we have

\[ \left| \int_{R_i} |f(x - y)| dy \right|^p v(x) < \infty \quad \text{a.e. } x \in \mathbb{R}^n. \]

If \( \| \chi_{Q_x} \|_{L^p_\omega(\mathbb{R}^n)} \neq 0 \), then \( v(x) \neq 0 \) on a positive measurable subset of \( Q_x \). Then, there exists a point \( x_0 \in Q_x \) such that

\[ \int_{R_i} |f(x_0 - y)| dy < \infty. \]

Observing that \( Q_x \subset x_0 - R_i \), the reasonable definition of \( \Phi \) is assured by

\[ \int_{Q_x} |f(y)| dy \leq \int_{x_0 - R_i} |f(y)| dy = \int_{R_i} |f(x_0 - y)| dy < \infty. \]

The remaining part of this proof is the same as the proof of Theorem 3.1. □

3.3. on Quasi-Banach space.

**Theorem 3.4.** Let \( E \) be a quasi-Banach function space satisfies the conditions \((B_1) - (B_5)\) with quasi-norm \( \| \cdot \|_E \). If there exists a positive integer \( N \) such that \( \| \cdot \|^N \|_E \) is a norm with

\[ \| \prod_{j=1}^N f_j \|_E \leq \prod_{j=1}^N \| f_j \|^N \|_E, \]

then Theorem 3.1 is also valid.

**Proof.** We only need to consider the case of nonnegative function. Denote by \( \| \cdot \|_Y := \| \cdot \|^N \|_E \). Since the family \( \mathcal{F} \) is uniformly bounded on \( E \), for \( f, g \in \mathcal{F} \) we have

\[ \| f - g \|_E = \left\| (f^{1/N} - g^{1/N}) \sum_{i+j=N-1} f_i^{1/N} g_j^{1/N} \right\|_E \]

\[ \leq \| f^{1/N} - g^{1/N} \|_Y \sum_{i+j=N-1} \| f_i^{1/N} \|_E \| g_j^{1/N} \|_E \leq \| f^{1/N} - g^{1/N} \|_Y. \]

From this, we only need to verify that the family \( \mathcal{F}_N := \{ f^{1/N} : f \in \mathcal{F} \} \) is totally bounded in \( Y \). This can be proved by Theorem 3.1 since \( \mathcal{F}_N \) satisfies all the conditions \((a) - (c)\) in Theorem 3.1 with norm \( \| \cdot \|_Y \). We only check the condition \((c)\) by

\[ \| \tau_y f^{1/N} - f^{1/N} \|_Y \leq \| \tau_y f - f \|^{1/N} = \| \tau_y f - f \|^{1/N}_E. \]

□

The proof of Theorem 3.1, based on the Minkowski inequality, is invalid for the Quasi-Banach case here. In order to deal with the Quasi-Banach case, we use a transfer method to reduce the proof to the Banach case. Although our method is more clear, we remark that the original idea of reduction to the Banach case should be due to Tsuji [14]. As a useful conclusion, we show the final version of relatively compactness criteria for \( L^p_\omega(\mathbb{R}^n) \) as follows.
Corollary 3.5 (Relatively compactness criteria for $L^p_w(\mathbb{R}^n)$ with any weight). Let $p \in (0, \infty)$, $\omega$ be a weight. A subset $\mathcal{F}$ of $L^p_w(\mathbb{R}^n)$ is relatively compact (or totally bounded) if the following statements are valid:

(a) $\mathcal{F}$ is bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L^p(\omega)} \lesssim 1$;

(b) $\mathcal{F}$ uniformly vanishes at infinity, that is,

$$\lim_{N \to \infty} \sup_{f \in \mathcal{F}} \|f \chi_{B^c(0,N)}\|_{L^p(\mathbb{R}^n)} = 0;$$

(c) $\mathcal{F}$ is equicontinuous, that is,

$$\lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \|\tau_y f - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Proof. The case $p \geq 1$ follows by Theorem 3.3. When $p \in (0, 1)$, this conclusion follows by Theorem 3.4 with $N = [1/p]+1$. □

Remark 3.6. Although in this paper, we only give the specific application for the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$, our main result Theorem 3.1 can be applied to weighted function spaces with variable exponent, weighted Morrey spaces, function spaces with one sided weight, and many other weighted (Quasi-)Banach function spaces.

Appendix A.

Here, we give a proof for that the Banach function spaces in Definition 2.1 is a Banach (completed normed) space.

For a Cauchy sequence $\{f_n\}$ in $E$, take $\{n_j\}_{j \in \mathbb{N}}$ with $n_j < n_{j+1}$ such that

$$\|f_{n_j} - f_{n_{j+1}}\|_E \leq 1/2^j.$$

Formally set

$$f(x) = f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x)) \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (A.1)$$

We claim that (A.1) defines a function $f \in E$ and then $f_n \to f$ in $E$. For $m = 1, 2, \ldots$, set

$$g_m(x) = \sum_{j=1}^{m} |f_{n_{j+1}}(x) - f_{n_j}(x)|.$$

Since $g_m \leq g_{m+1}$, we write

$$g(x) = \lim_{m \to \infty} g_m(x).$$

By condition $(B_3)$ we have

$$\|g\|_E = \lim_{m \to \infty} \|g_m\|_E \leq \lim_{m \to \infty} \sum_{j=1}^{m} \frac{1}{2^j} \leq 1.$$
Thus $g \in E$. Observe that conditions $(B_1)$ and $(B_5)$ imply that $g(x) < \infty$ a.e. $x \in \mathbb{R}^n$. It follows that the limit
\[
\lim_{j \to \infty} (f_{n+1}(x) - f_n(x))
\]
exists for a.e. $x \in \mathbb{R}^n$. From this and $|f(x)| \leq |f_{n_1}(x)| + |g(x)|$, we have $f \in E$. Finally, for every integer $k$,
\[
\|f - f_{n_k}\|_E \leq \sum_{j=k}^{\infty} \|f_{n_{j+1}} - f_{n_j}\|_E \leq \sum_{j=k}^{\infty} 1/2^j = 1/2^{k-1} \to 0.
\]
Thus, $f_{n_k}$ tends to $f$ in $E$. By a standard argument for Cauchy sequence, on can verify that $f_k$ tends to $f$ in $E$.

**References**

[1] A. Bényi, W. Damián, K. Moen, and R. H. Torres. Compact bilinear commutators: the weighted case. *Michigan Math. J.*, 64(1):39–51, 2015.

[2] L. Chaffee, P. Chen, Y. Han, R. H. Torres, and L. A. Ward. Characterization of compactness of commutators of bilinear singular integral operators. *Proc. Amer. Math. Soc.*, 146(9):3943–3953, 2018.

[3] Y. Chen, Y. Ding, and X. Wang. Compactness of commutators for singular integrals on Morrey spaces. *Canad. J. Math.*, 64(2):257–281, 2012.

[4] A. Clop and V. Cruz. Weighted estimates for Beltrami equations. *Ann. Acad. Sci. Fenn. Math.*, 38(1):91–113, 2013.

[5] D. E. Edmunds and W. D. Evans. *Hardy operators, function spaces and embeddings*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[6] P. a. Górka and A. Macíos. The Riesz-Kolmogorov theorem on metric spaces. *Miskolc Math. Notes*, 15(2):459–465, 2014.

[7] P. a. Górka and H. Rafeiro. From Arzelà-Ascoli to Riesz-Kolmogorov. *Nonlinear Anal.*, 144:23–31, 2016.

[8] P. a. Górka and H. Rafeiro. Light side of compactness in Lebesgue spaces: Sudakov theorem. *Ann. Acad. Sci. Fenn. Math.*, 42(1):135–139, 2017.

[9] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[10] H. Hanche-Olsen and H. Holden. The Kolmogorov-Riesz compactness theorem. *Expo. Math.*, 28(4):385–394, 2010.

[11] V. G. Krotov. Compactness criteria in the spaces $L^p$, $p \geq 0$. *Mat. Sb.*, 203(7):129–148, 2012.

[12] J. D. Tamarkin. On the compactness of the space $L^p$. *Bull. Amer. Math. Soc.*, 38(2):79–84, 1932.

[13] V. Tikhomirov. On the compactness of sets of functions in the case of convergence in mean. In *Selected Works of AN Kolmogorov*, pages 147–150. Springer, 1991.

[14] M. Tsuji. On the compactness of space $l^p(p > 0)$ and its application to integral operators. *Kodai Math. Sem. Rep.*, 3101:33–36, 1951.

[15] A. Tulajkov. Zur kompaktheit im raum l für $p = 1$. (in german). *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl. I.*, 39:167–170, 1933.

[16] A. Uchiyama. On the compactness of operators of Hankel type. *Tohoku Math. J. (2)*, 30(1):163–171, 1978.

[17] Q. Xue, K. Yabuta, and J. Yan. Weighted Fréchet-Kolmogorov theorem and compactness of vector-valued multilinear operators. *arXiv preprint arXiv:1806.06656*, 2018.
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