Study on the Free Vibration of a Rubber Band Oscillator

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Abstract. Through theoretical analysis and experimental verification of the motion of a rubber band oscillator, this paper proves that the new model for describing the mechanical properties of rubber is effective. It is pointed out that there are both elastic force related to deformation and viscoelastic force related to deformation process in rubber. Within a certain range of deformation, elastic force satisfies Hooke's theorem, and viscoelastic force is proportional to fractional derivative of elongation with time. The mechanical properties of rubber materials can be described by static elastic modulus (elastic coefficient), viscoelastic degree and viscoelastic coefficient.

1. Introduction
Rubber is a widely used material in modern life and production. There are many kinds of rubbers and new rubber products are constantly produced. The measurement of mechanical, physical and chemical properties of rubber and the change of properties with temperature, pH and another external environment have been reported constantly. [1–5]

Due to the diversity of rubber materials, the measurement of various properties is also complex. There are nearly 100 national standards for mechanical property measurement alone in China. When it comes to the measurement of rubber mechanical properties, there are some complex cases such as static modulus and dynamic modulus, which increase the difficulty of application in engineering modeling. Rubber material is both hyperelastic and viscoelastic. The vibration of the system composed of rubber material and object is complex. The physical idea of vibration equation used in measuring the dynamic modulus of rubber material is not clear, so it is difficult to reveal the essential characteristics of material [6]. There is no experiment on rubber band in all kinds of basic mechanical experiments in college physics, which also shows the specificity of rubber material [7]. Through experiments and theoretical analysis, a new model of rubber mechanical properties is put forward in this paper: There are both elastic force due to deformation and viscoelastic force related to deformation process in rubber material; The mechanical properties of rubber material are determined by its static elastic modulus (elastic coefficient of rubber band) and the order of fractional derivative and coefficient of fractional derivative term.

2. The model of rubber band oscillator and its motion differential equation
If there is a light rubber band with one end fixed and the other end connected with a mass m, the mass can move freely on a smooth horizontal plane, we call such a system rubber band oscillator. In other words, the mass only affected by the light rubber band constitutes the rubber band oscillator.
In a certain temperature, environment and deformation range, the elastic force of the rubber band is approximately proportional to its elongation, and the proportion coefficient is the elastic coefficient $k$. Let the viscoelastic force of the rubber band be proportional to the fractional derivative of its elongation with time [8, 9]. The order of the fractional derivative is called viscoelastic degree, which is recorded as $\alpha$, while the coefficient of the fractional derivative term is called viscoelastic coefficient, which is recorded as $k_V$. The elastic coefficient $K$, the viscoelastic degree $\alpha$ and the viscoelastic coefficient $k_V$ are related to the properties of the rubber material, the thickness of the rubber band and the temperature.

**Figure 1.** Schematic diagram of rubber band oscillator

According to the above assumption, as shown in Figure 1, when the mass $m$ is pulled away from the equilibrium position at $x$, the resultant of elastic force and viscoelastic force it receives is

$$ F = -kx - k_V \frac{d^\alpha x}{dt^\alpha} $$

Where $1 \leq \alpha \leq 2$, the fractional derivative is defined by Caputo [10]. According to Newton's second law, the differential equation of particle motion is

$$ m \frac{d^2x}{dt^2} + k_V \frac{d^\alpha x}{dt^\alpha} + kx = 0 $$

If the rubber band vibrator is placed vertically, the equilibrium position of the mass is taken as the coordinate origin, and the coordinate system is established with the positive position downward, the motion equation of the mass can also be expressed by the above formula.

3. The characteristics of the motion of a rubber band oscillator

We can turn equation (2) into

$$ \frac{d^2x}{dt^2} + 2\beta \frac{d^\alpha x}{dt^\alpha} + \omega_0^2 x = 0 $$

Where, $2\beta = \frac{k_V}{m}$, $\omega_0^2 = \frac{k}{m}$, and the initial condition is set to be $x(0) = x_0$, $\dot{x}(0) = 0$.

When $\alpha = 1$, the equation (3) is known as the equation of motion of a harmonic oscillator with viscous damping. According to the magnitude of damping force, the oscillator can be in three vibration states: under damped, critical damped and over damped. The mass will tend to equilibrium position by attenuating vibration or unidirectional motion. When $0 < \alpha < 1$, the equation (3) is a second order ordinary differential equation with fractional derivative term, and its exact solution is difficult to find, so far there is no report. The numerical simulation shows that the order of fractional derivative and the coefficient of fractional derivative term affect not only the frequency of vibration, but also the attenuation of amplitude.

For the case of $0 < \beta < 1$, when $\beta$ can be regarded as a small parameter, the approximate analytical solution of the equation (3) can be obtained by means of the average method as follows

$$ x(t) = x_0 e^{-\beta \omega_0^{-1} \sin \frac{\pi \alpha t}{2}} \cos \left[ (\omega_0 + \beta \omega_0^{-1} \cos \frac{\pi \alpha}{2} \right] t] $$

The corresponding solution of equation (1) is

$$ x(t) = x_0 e^{-\frac{1}{\alpha} \left( \frac{\pi \alpha t}{2} \right)} \cos \left[ \frac{\pi \alpha t}{2} k \left( \frac{\pi \alpha}{2} \right) \right] $$

Use the following symbols
The equation (5) can be changed into (8)

\[ \Delta = \frac{1}{2m} k_v \left( \frac{k}{m} \right)^{a-1} \sin \frac{\alpha \pi}{2} \]  
\[ \omega_v = \frac{1}{2m} k_v \left( \frac{k}{m} \right)^{a-1} \cos \frac{\alpha \pi}{2} \]

The equation (5) can be changed into

\[ x(t) = x_0 e^{-\Delta t} \cos \left( \omega_0 + \omega_v \right) t \]

It can be seen from the above formula that when the viscoelastic coefficient \( k_v \) is not very large, the rubber band oscillator will vibrate with amplitude attenuation. The amplitude attenuation coefficient \( \Delta \) and the additional circular frequency \( \omega_v \) are closely related to the viscoelastic properties of the rubber band. If other conditions are not changed, the larger the viscoelastic degree \( \alpha \), the faster the amplitude attenuation of the rubber band oscillator, and the larger the vibration period. When other conditions are not changed, the larger the viscoelastic coefficient \( k_v \) is, the faster the amplitude of the rubber band oscillator decays, but the smaller the vibration period is.

4. Experimental verification

4.1. Experimental process and data processing

In order to test the correctness of the above model and the corresponding theory, the commercial industrial rubber band made in Vietnam was selected for the experiment. The length of rubber band is about 25cm, the cross section is about 1.2 \times 1.2mm^2 and the mass is \( m_0 = 1.14g \). The experiment was carried out on fd-gl-b-ii new type Jolly scale experiment instrument produced by Fudan Tianxin Science and Education Instrument Co., Ltd. One end of the rubber band is fixed on the iron frame platform, and the other end is connected with the hook code frame. The elastic coefficient of rubber band is measured by stretching method. The elastic coefficient \( k = 9.0968 \text{Nm}^{-1} \) of the rubber band is obtained by using the straight-line fitting of the experimental data of tensile increment and elongation. After selecting the mass of vibration object (hook code and additional magnetic steel) \( M = 79.37g \), pull the hook code down from the balance position, and release it from static state. The periodic vibration of the system with amplitude attenuation is observed, which is consistent with the expectation of rubber band oscillator theory. By using the method of controlling times of Hall switch triggered by magnetic steel and fd-ct-ii counter, the time of four cycles was measured many times, and it was calculated that each vibration cycle was \( t = 0.5533s \).

Because of the fast attenuation of the amplitude, it is difficult to measure the logarithmic decrement with the traditional method, so the vibration process of the rubber band vibration system is recorded video, and the amplitude after half cycle (or one cycle) is determined by slow motion playback. According to this method, the logarithmic reduction of the vibration is \( \Delta = 0.5651 \).

4.2. Mechanical parameters of rubber band under the new model

If the air resistance and the friction force at the suspension point are ignored, the converted mass comes from the vibrating object and the rubber band. At this time, the rubber band vibration system in the experiment can be regarded as the rubber band oscillator. That is to say, the above experimental system can be considered as a rubber band oscillator with elastic coefficient \( k = 9.0968 \text{Nm}^{-1} \) and the mass \( m = M + 1/3 m_0 = 79.75g \). According to the experimental data, it is obtained that as follows.

\[ \Delta = \frac{\Delta}{T} = \frac{0.5651}{0.5533} = 1.02135^{-1} \]
\[ \omega_v = \frac{2\pi}{T} - \sqrt{\frac{k}{m}} = 0.6756S^{-1} \]
So far, three mechanical parameters, i.e. elastic coefficient $k$, viscoelastic degree $\alpha$ and viscoelastic coefficient $k_r$, which are proposed above, are obtained to describe the mechanical properties of the experimental rubber band at room temperature of 20°C.

Although the new model proposed by the author needs three parameters as the traditional method in describing the mechanical properties of rubber band, it avoids the situation that the dynamic elastic coefficient has nothing to do with the static elastic coefficient, and the embarrassment that the observable circular frequency determined by the system property appears in the vibration equation when measuring the dynamic parameters of rubber band.

5. Conclusion

In this paper, we define the rubber band oscillator and proposes a fractional differential term to describe the viscoelastic force on the rubber band oscillator. The correctness of the approximate analytical solution of the rubber band oscillator is verified by experiments, which shows that it is effective to describe the mechanical properties of the rubber band with the three parameters of elastic coefficient, viscoelastic degree and viscoelastic coefficient proposed by the author. This paper provides a new perspective to examine rubber materials. It is proposed that the mechanical properties of rubber are reflected by both elasticity and viscoelasticity. The elasticity is the same in both static and dynamic state, while the viscoelasticity described by fractional derivative term is only reflected in dynamic state.

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