COHOMOLOGY OF N-GRADED LIE ALGEBRAS OF MAXIMAL CLASS OVER $\mathbb{Z}_2$

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Abstract. We compute the cohomology with trivial coefficients of Lie algebras $m_0$ and $m_2$ of maximal class over the field $\mathbb{Z}_2$. In the infinite-dimensional case, we show that the cohomology rings $H^*(m_0)$ and $H^*(m_2)$ are isomorphic, in contrast with the case of the ground field of characteristic zero, and we obtain a complete description of them. In the finite-dimensional case, we find the first three Betti numbers of $m_0(n)$ and $m_2(n)$ over $\mathbb{Z}_2$.

1. Introduction

A Lie algebra $g$ is said to be $N$-graded, if it is the direct sum of subspaces $g_i$, $i \in \mathbb{N}$ (the homogeneous components), such that $[g_i, g_j] \subset g_{i+j}$. Obviously, finite-dimensional $N$-graded Lie algebras are necessarily nilpotent. A great deal of attention in the literature has been focused on $N$-graded Lie algebras for which the homogeneous components $g_i$ are “the smallest possible”, that is, all of dimension one or, in the finite-dimensional case, dim $g_i = 1$, for $i \leq n := \dim g$, and $g_i = 0$, for $i > n$. With the additional condition that $g$ is generated as an algebra by elements $e_1$ and $e_2$, spanning $g_1$ and $g_2$ respectively, one obtains that the subspaces $C_0 = g$, $C_k = \bigoplus_{i=k+2}^\infty g_i$, $k > 0$, are the terms of the central descending series. This defines the $N$-graded filiform Lie algebras in the finite-dimensional case [15] and the $N$-graded Lie algebras of maximal class [12] (also called narrow algebras). In characteristic zero, these algebras have been completely classified.

In the infinite-dimensional case, one gets just three algebras [7], and independently [12, Theorem 7.1]. We list them here with their presentations:

\begin{align*}
  m_0 & = \text{Span}(e_1, e_2, \ldots), & [e_1, e_i] = e_{i+1}, & i > 1, \\
  m_2 & = \text{Span}(e_1, e_2, \ldots), & [e_1, e_i] = e_{i+1}, & i > 1, \quad [e_2, e_j] = e_{j+2}, \quad j > 2, \\
  V & = \text{Span}(e_1, e_2, \ldots), & [e_i, e_j] = (j-i)e_{i+j}, & i, j \geq 1.
\end{align*}

In the finite-dimensional case in characteristic zero, the classification of finite-dimensional $N$-graded filiform Lie algebras was established in [11]; one obtains the “truncations” of the above three algebras, in particular,

\begin{align*}
  m_0(n) & = \text{Span}(e_1, \ldots, e_n), & [e_1, e_i] = e_{i+1}, & 1 < i < n, \\
  m_2(n) & = \text{Span}(e_1, \ldots, e_n), & [e_1, e_i] = e_{i+1}, & 1 < i < n, \quad [e_2, e_j] = e_{j+2}, \quad 2 < j < n - 1, \quad (5)
\end{align*}

and $V(n)$, plus another three infinite series, and five one-parameter families of low-dimensional algebras. The picture is more complicated in positive characteristic: by [5],

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there are uncountably many isomorphism classes of Lie algebras of maximal class; the
construction of all such algebras in odd characteristic is given in [8], and in characteristic
two, in [10], with \( m_0 \) and \( m_2 \) being the simplest possible cases.

The cohomology of \( \mathbb{N} \)-graded Lie algebras of maximal class has been studied exten-
sively over a field of characteristic zero \([7, 8, 15]\), and at present is well-understood. In
[8], Fialowski and Millionschikov gave a full description of the cohomology with trivial
coefficients of the algebras \( m_0 \) and \( m_2 \); the Betti numbers of \( V \) are found in [9]. In the
finite-dimensional case, the cohomology of \( m_0(n) \) were found in [8] (see also [2] and [8]).
However, already for \( m_2(n) \) over a field of characteristic zero, our present knowledge is
limited to the first two Betti numbers \([11, 15]\).

The study of the cohomology of Lie algebras of maximal class over fields of positive
characteristic is much less developed. The cohomology of the Heisenberg algebra is found
in \([5, 13]\). A recent result by Tsartsarflis \([14]\) states that over a field of characteristic
two, the algebras \( m_0(n) \) and \( m_2(n) \) have the same Betti numbers (in contrast with the
case of characteristic zero), and furthermore, every algebra of the so called Vergyne class
admits a dual, non-isomorphic algebra, with the same Betti numbers.

In this paper we study the cohomology with trivial coefficients of the Lie algebras \( m_0 \)
and \( m_2 \), and their finite dimensional truncations, \( m_0(n) \) and \( m_2(n) \), over the field \( \mathbb{Z}_2 \). Let
\( V = \text{Span}(e_1, e_2, \ldots) \) and let \( \{ e^i \} \) be the dual basis for \( V^* \). Define the operator \( D_1 \) on
\( V^* \) by \( D_1 e^1 = D_1 e^2 = 0, \) \( D_1 e^i = e^{i-1}, \) for \( i > 2, \) and extend it to \( \Lambda(V) \) as a derivation.
For \( \omega \in \Lambda(V) \) and \( e^i \in V^* \), define \( F(\omega, e^i) = \sum_{j=0}^{\infty} (D_1^j \omega) \wedge e^{i+j+1} \) (note that the sum on
the right-hand side is finite).

Our main result in the infinite-dimensional case is as follows.

**Theorem 1.** The cohomology rings \( H^*(m_0) \) and \( H^*(m_2) \) over the field \( \mathbb{Z}_2 \) are isomorphic.
The respective cohomology classes of the cocycles

\[
\begin{align*}
&e^1, e^2, F(e^1 \wedge e^{i_2} \wedge \cdots \wedge e^{i_q}, e^{i_q}), \\
\end{align*}
\]

where \( q \geq 1, \) \( 2 \leq i_1 < i_2 < \ldots < i_q, \) form a basis for \( H^*(m_0) \) and for \( H^*(m_2) \), respectively.

Note that \( H^*(m_0) \) over \( \mathbb{Z}_2 \) is “the same” as over a field of characteristic zero (compare
with \([8, \text{Theorem 3.4}]\)). In contrast, the fact that \( H^*(m_0) \) and \( H^*(m_2) \) over \( \mathbb{Z}_2 \) are isomorphic (note that \( m_0 \) and \( m_2 \) are not isomorphic over any ground field) is specific to the \( \mathbb{Z}_2 \) case: over a field of characteristic zero, \( H^*(m_2) \) is very different \([8, \text{Theorem 5.5}]\).

In the finite-dimensional case, which appears to be substantially harder than the
infinite-dimensional one, we compute the first three Betti numbers of \( m_0(n) \) and the
corresponding bases for \( H^i(m_0(n)) \), \( i = 1, 2, 3. \)

**Theorem 2.** The first three Betti numbers of the Lie algebra \( m_0(n) \) over \( \mathbb{Z}_2 \) are given by

\[
\begin{align*}
&\text{(a)} \quad b_1(m_0(n)) = 2, \\
&\text{(b)} \quad b_2(m_0(n)) = \left\lfloor \frac{1}{2} (n + 1) \right\rfloor, \text{ where } \lfloor . \rfloor \text{ denotes the integer part}, \\
&\text{(c)} \quad b_3(m_0(n)) = \left\lfloor \frac{1}{2} (2^p - 1)(2^{p-1} - 1) + \frac{1}{2} m(m - 1) + \frac{1}{2} (n-1) \right\rfloor, \text{ where } n = 2^p + m \text{ and} \\
&\quad \quad \quad 0 < m \leq 2^p.
\end{align*}
\]

An explicit form of the basis for \( H^3(m_0(n)) \) is given in Theorem 4 of Section 3. The-
orem 2 also gives us the first three Betti numbers of \( m_2(n) \) (Corollary 1 of Section 4),
which in characteristic two are simply the same as those for \( m_0(n) \), by \([13, \text{Theorem 1}] \).
The paper is organised as follows. We begin with some short preliminaries in Section 2. We treat the algebras \( m_0 \) and \( m_0(n) \) in Section 3. Parts (iii) and (iv) of Theorem 2 follow from Proposition 1. After some technical preparation similar to the arguments of [8], we prove Theorem 3, which is “the \( m_0 \) part” of Theorem 4. We then proceed to the proof of Theorem 4. This is the longest and most technically involved part of the paper. Finally, in Section 4 we use a construction similar to [14] to establish the isomorphism between \( H^*(m_0) \) and \( H^*(m_2) \), hence completing the proof of Theorem 4.

2. Preliminaries

Given a Lie algebra \( g \) over \( \mathbb{Z}_2 \) with a basis elements \( e_i \), we denote the dual basis elements \( e^i \). For convenience, we set \( e^0 = 0 \). For simplicity we write a monomial \( q \)-form \( e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_q} \in \Lambda^q(g) \) as \( e^{i_1} e^{i_2} \cdots e^{i_q} \). For a monomial \( e^{i_1} e^{i_2} \cdots e^{i_q} \), its degree is defined to be \( \sum_{j=1}^q i_j \). The homogeneous component \( \Lambda^k_q(g) \) of degree \( k \) and of rank \( q \) is the span of all the monomials of degree \( k \) and of rank \( q \). We set \( \Lambda_k(g) := \oplus_q \Lambda^k_q(g) \).

As usual, the differential \( d \) is defined by \( d\xi(X, Y) = \xi[X, Y] \) for one-forms \( \xi \), where \( X, Y \in g \), and then is extended to the exterior algebra \( \Lambda(g) \) as a derivation (so that \( d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \omega_2 + \omega_1 \wedge d(\omega_2) \)). Then \( d^2 = 0 \) and one define the \( q \)-th cohomology group \( H^q(g) \) (with trivial coefficients) by \( H^q(g) = \ker(d : \Lambda^q \to \Lambda^{q+1})/\text{Im}(d : \Lambda^{q-1} \to \Lambda^q) \). Then \( H^q(g) \) is a linear space over \( \mathbb{Z}_2 \); if its dimension is finite, it is called the \( q \)-th Betti number \( b_q(g) \). It is immediate from the definition that if \( \dim g = n \), then

\[
   b_q(g) = \dim \ker(d : \Lambda^q \to \Lambda^{q+1}) + \dim \ker(d : \Lambda^{q-1} \to \Lambda^q) - \binom{n}{q-1},
\]

so to compute the Betti numbers it suffices to know the dimensions of the kernels of \( d \) on the \( \Lambda^q \)'s. Also note that in the graded case (in particular, for the bases \{\( e_i \)\} from [1 – 5]), the operator \( d \) maps \( \Lambda^k_q(g) \) to \( \Lambda^{k+1}_q(g) \), and so \( H^q(g) \) is spanned by the classes of homogeneous elements; we get a decomposition (a bi-gradation) \( H^q(g) = \oplus_k H^q_k(g) \). The multiplicative structure in \( H(g) := \oplus_q H^q(g) \) is inherited from the wedge product.

3. Cohomology of \( m_0 \)

In this section, we compute the cohomology of the infinite-dimensional Lie algebra \( m_0 \) and also the first three Betti numbers of the finite-dimensional Lie algebras \( m_0(n) \) defined as follows [1, 4, 14]:

\[
   m_0 = \text{Span}(e_1, e_2, e_3, \ldots), \quad [e_1, e_i] = e_{i+1}, \quad \text{for } i \geq 2,
\]

\[
   m_0(n) = \text{Span}(e_1, e_2, e_3, \ldots, e_n), \quad [e_1, e_i] = e_{i+1}, \quad \text{for } 2 \leq i \leq n-1.
\]

In the first few paragraphs, we closely follow the approach and the results of [8, Section 3], adapting them to the case of the ground field \( \mathbb{Z}_2 \). In effect, the outcome is that in the infinite-dimensional case, for \( g = m_0 \), the cohomology is “the same” as that for a field of characteristic zero, while in the finite-dimensional case, for \( g = m_0(n) \), the situation is more delicate – not only the Betti numbers are different, but also the methods of [8, 2] and the very elegant approach of [3, Appendix B] do not work directly.
For a monomial \( e^{i_1i_2\ldots i_q} \in \Lambda^q(\mathfrak{g}) \), \( q \geq 1 \), \( i_1, i_2, \ldots, i_q \geq 1 \), (for both \( \mathfrak{g} = \mathfrak{m}_0 \) and \( \mathfrak{g} = \mathfrak{m}_0(n) \)) we have

\[
d(e^{i_1i_2\ldots i_q}) = e^{1(i_1-1)i_2\ldots i_q} + e^{1i_1(i_2-1)\ldots i_q} + \ldots + e^{1i_1i_2\ldots (i_q-1)} = e^1 \wedge (e^{i_1-1}e^{i_2} + e^{i_1i_2-1}e^1 + \ldots + e^{i_1i_2\ldots (i_q-1)}).
\] (8)

It follows from (8) that the subspaces \( \Lambda_k(\mathfrak{g}) \) are \( d \)-invariant.

Moreover, for any \( \omega \in \Lambda(\mathfrak{g}) \) we have \( d(\omega) = 0 \) and \( d(\omega) \in \Lambda(\mathfrak{g}) \). Set \( \mathfrak{h} := \text{Span}(e_2, e_3, \ldots) \) for \( \mathfrak{m}_0 \), and \( \mathfrak{h} := \text{Span}(e_2, e_3, \ldots, e_n) \) for \( \mathfrak{m}_0(n) \). Then \( \mathfrak{h} \) is abelian and from (8) it follows that there is a well-defined linear operator \( D \) on \( \Lambda(\mathfrak{h}) \) such that for \( \omega \in \Lambda(\mathfrak{h}) \), we have

\[
d\omega = e^1 \wedge (D\omega).
\] (9)

It is easy to see that

\[
D^2 e = 0, \quad De^i = e^{i-1} \quad \text{for } i > 2, \quad D(\xi \wedge \eta) = D(\xi) \wedge \eta + \xi \wedge D(\eta) \quad \text{for } \xi, \eta \in \Lambda(\mathfrak{h}),
\] (10)

so \( D \) is a derivation of \( \Lambda(\mathfrak{h}) \). Recall that the Lie derivative with respect to \( e_1 \) is defined by taking the operator \( (\text{ad}_{e_1})^* \) on \( \mathfrak{g}^* \) to be the dual to \( \text{ad}_{e_1} \) on \( \mathfrak{g} \), and then extending it as a derivation to \( \Lambda(\mathfrak{g}) \). Note that \( D \) is just the restriction of \( (\text{ad}_{e_1})^* \) to \( \Lambda(\mathfrak{h}) \). Furthermore, \( D(\Lambda_k^q(\mathfrak{h})) \subset \Lambda_{k-1}^{q+1}(\mathfrak{h}) \), so that \( D \) is “nilpotent”: for any \( \omega \in \Lambda(\mathfrak{h}) \) there exists \( N = N(\omega) \geq 0 \) such that \( D^N \omega = 0 \). For convenience, we define \( D^0 \) to be the identity map.

Since from (8), \( \ker d = e^1 \wedge \Lambda(\mathfrak{h}) \oplus \ker D \) to find the kernel of \( d \) we need to find the kernel of \( D \). This is given by the following lemma.

**Lemma 1.** (a) Let \( \mathfrak{g} = \mathfrak{m}_0 \). For any \( \omega \in \Lambda(\mathfrak{h}) \) and \( e^i \in \mathfrak{h} \) define

\[
F(\omega, e^i) = \sum_{l=0}^{\infty} D^l \omega \wedge e^{i+1+l} = \sum_{l=0}^{N(\omega)-1} D^l \omega \wedge e^{i+1+l}.
\] (11)

Then \( F(\omega, e^i) \in \ker D \) for \( \omega \wedge e^i = 0 \) and moreover, the elements

\[
F(e^{i_1i_2\ldots i_q}, e^j) = e^{i_1i_2\ldots i_qj+1} + D(e^{i_1i_2\ldots i_q} \wedge e^j) + \ldots \in \Lambda_k^{q+1}(\mathfrak{h}),
\]

where \( q \geq 1, 2 \leq i_1 < i_2 < \ldots < i_q, \ k = i_q + 1 + \sum_{j=1}^q i_j, \)

form a basis for the kernel of the restriction of \( D \) to \( \Lambda_k^{q+1}(\mathfrak{h}) \); the kernel of the restriction of \( D \) to \( \mathfrak{h}^* \) is spanned by \( e^2 \).

(b) Let \( \mathfrak{g} = \mathfrak{m}_0(n) \), viewed as the subspace of \( \mathfrak{m}_0 \) spanned by the first \( n \) vectors. Then \( \ker D \) is the intersection of \( \ker D \) constructed in (a) for the case \( \mathfrak{g} = \mathfrak{m}_0 \) with \( \mathfrak{m}_0(n) \).

Note that in the Introduction we used \( D_1 = (\text{ad}_{e_1})^* \) rather than \( D \) to define \( F \). This yields the same object, since in (8), \( D \) only acts on elements of \( \Lambda(\mathfrak{h}) \) and \( D \) is the restriction on \( D_1 \) to \( \Lambda(\mathfrak{h}) \). Notice however that Lemma 1 concerns \( \ker D \), which is different to \( \ker D_1 \).
Proof. (iii) The fact that $F(\omega, e^i) \in \ker D$ follows immediately, as from (i), for any $\omega \in \Lambda(\mathfrak{h})$ and $e^i \in \mathfrak{h}$ we have

$$DF(\omega, e^i) = D\left(\sum_{l=0}^{\infty} D^l \omega \wedge e^{i+l}\right)$$

$$= \sum_{l=0}^{\infty} D^{l+1} \omega \wedge e^{i+1+l} + \sum_{l=0}^{\infty} D^l \omega \wedge e^{i+l}$$

$$= \sum_{l=0}^{\infty} D^l \omega \wedge e^{i+l} + \sum_{l=0}^{\infty} D^l \omega \wedge e^{i+l}$$

$$= \omega \wedge e^i,$$

as we are working over $\mathbb{Z}_2$. Notice in passing that this also shows that $D$ is surjective.

The fact that the elements given by (ii) are linearly independent is also easy, as from among the monomials $e^{j_1 j_2 \cdots j_q h+1}$, $2 \leq j_1 < j_2 < \cdots < j_q < j_{q+1}$ which appear on the right-hand side of the expansion of $F(e^{j_{12} \cdots i_q}, e^{i_q})$, there is exactly one with the property that $j_{q+1} = j_q + 1$, namely the monomial $e^{j_{12} \cdots i_q h+1}$. The fact that they indeed span the kernel of the restriction of $D$ to $\Lambda_q^{q+1}(\mathfrak{h})$ follows from the same observation and from the dimension count. The elements $F(e^{j_{12} \cdots i_q}, e^{i_q}) \in \Lambda_k^{q+1}(\mathfrak{h})$ with $q \geq 1$, $2 \leq i_1 < i_2 < \cdots < i_q$, $i_q + 1 + \sum_{j=1}^{q-1} i_j = k$, are in one-to-one correspondence with the elements $e^{j_{12} \cdots j_q h+1} \in \Lambda_k^{q+1}(\mathfrak{h})$ with $2 \leq j_1 < j_2 < \cdots < j_q$. On the other hand, consider the linear operator $A : \Lambda_k^{q+1}(\mathfrak{h}) \rightarrow \Lambda_k^{q+1}(\mathfrak{h})$ defined on the monomials as follows: $A e^{j_{12} \cdots j_q h+1} = e^{j_{12} \cdots j_q h+1-1}$. Then $A$ is surjective and its kernel is spanned by the monomials $e^{j_{12} \cdots j_q h+1}$, so every surjective linear operator from $\Lambda_k^{q+1}(\mathfrak{h})$ to $\Lambda_k^{q+1}(\mathfrak{h})$ (in particular, $D$) has a kernel of the same dimension.

(ii) easily follows from the fact that for the operator $D$ defined for $\mathfrak{g} = \mathfrak{m}_0$, the subspace $\Lambda(\mathfrak{h})$ defined for $\mathfrak{m}_0(n)$ is $D$-invariant, and the restriction of $D$ to it is the operator $D$ defined for $\mathfrak{m}_0(n)$. \hfill \Box

With Lemma (i) we can easily finish the computation of the cohomology for $\mathfrak{g} = \mathfrak{m}_0$; we obtain the same answer as in [3, Theorem 3.4]:

**Theorem 3.** The cohomology classes of the cocycles

$$e^1, e^2, F(e^{j_{12} \cdots i_q}, e^{i_q}),$$

(13)

where $q \geq 1$, $2 \leq i_1 < i_2 < \cdots < i_q$, form a basis for $H^*(\mathfrak{m}_0)$ over the field $\mathbb{Z}_2$.

Furthermore, the dimensions of the homogeneous components of $H^*(\mathfrak{m}_0)$ over $\mathbb{Z}_2$ are the same as those over a field of characteristic zero, so in particular,

$$\dim H_k^{q+1} \pi(\mathfrak{m}_0) = P_q(k) - P_q(k-1),$$

where $P_q(k)$ is the number of partitions of a positive integer $k$ into $q$ parts. The products of the basis elements also have “the same” decomposition as in [3, Equation (8)], after reducing the coefficients modulo 2.

**Proof of Theorem 3.** From Lemma (i) we know $\ker D$, and so we know $\ker d = e^1 \wedge \Lambda(\mathfrak{h}) \oplus \ker D$. The image of $d$ is just $e^1 \wedge \Lambda(\mathfrak{h})$, by (ii) and from the surjectivity of $D$ (which has been established in the proof of Lemma (iii)). Putting these two facts together we get the claim. \hfill \Box
We now turn our attention to the case $g = \mathfrak{m}_0(n)$. We view $\mathfrak{m}_0(n)$ as a subspace of $\mathfrak{m}_0$ spanned by the first $n$ basis elements and for convenience, denote the operator $D$ defined for $\mathfrak{m}_0$ by $\mathcal{D}$. The following Proposition easily follows from Lemma 1.

**Proposition 1.** The space $H^1(\mathfrak{m}_0(n))$ is spanned by the classes of the elements $e^1, e^2$ and so $b_1(\mathfrak{m}_0(n)) = 2$. The space $H^2(\mathfrak{m}_0(n))$ is spanned by the classes of the elements $e^{2i}$, $F(e^i, e^i) = e^{i,i+1} + e^{i-1,i+3} + \cdots + e^{2,2i-1}$, $2 \leq i \leq \frac{1}{2}(n + 1)$, and so $b_2(\mathfrak{m}_0(n)) = \lceil \frac{1}{2}(n + 1) \rceil$.

**Proof.** The claim for $H^1(\mathfrak{m}_0(n))$ is clear. For the second cohomology, by Lemma 1(a), the kernel of $\mathcal{D}$ is spanned by the elements $F(e^i, e^i) = e^{i,i+1} + e^{i-1,i+3} + \cdots + e^{2,2i-1}$. Since a sum of some number of the $F(e^i, e^i)$ belongs to $\mathfrak{m}_0(n)$ if and only if each of them does (no two monomials of the different $F(e^i, e^i)$ may possibly cancel), we get by Lemma 1(b):

$$\text{ker } D = \text{Span}(F(e^i, e^i) : 2 \leq i \leq \frac{1}{2}(n + 1)).$$

Then $\text{ker } d = e^1 \wedge \Lambda^1(\mathfrak{h}) \oplus \text{ker } D$ and so the second coboundary space is spanned by $e^{1i}, F(e^i, e^i), i = 2, \ldots, n - 1$. Then, as the image of $d$ on the space of one-forms is spanned by $e^1 \wedge e^i$, for $1 \leq i \leq n - 1$, the claim follows. \hfill $\Box$

Proposition 1 establishes parts (a) and (b) of Theorem 2. The first two Betti numbers of $\mathfrak{m}_0(n)$ over $\mathbb{Z}_2$ are the same as those over a field of characteristic zero \cite{2}, but $b_3$ is different, as Theorem 2(c) shows.

**Remark 1.** Explicitly, for small values of $n$, Theorem 2(c) gives:

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_3(\mathfrak{m}_0(n))$ | 1 | 2 | 3 | 4 | 7 | 10 | 11 | 12 | 15 | 18 | 23 | 28 | 35 | 42 | 43 | 44 | 47 | 50 |

The sequence $b_3(\mathfrak{m}_0(n))$ is the sequence A266540 in \cite{1}. To see that, we note that by the formula given in Theorem 2(c), $b_3(\mathfrak{m}_0(n)) = \frac{1}{2}(b_3(\mathfrak{m}_0(n - 1)) + b_3(\mathfrak{m}_0(n + 1)))$, for odd $n \geq 3$, and so it suffices to show that the even terms of the two sequences coincide, which is equivalent to the fact that the sequence $A_l := \frac{1}{2}b_3(\mathfrak{m}_0(2l)) = \frac{1}{3}(2^{2l-2} - 1) + s^2$, where $l = 2^{p-1} + s$, $0 < s \leq 2^{p-1}$, coincides with A256249. This is equivalent to the fact that $A_l$ is the $(l - 1)$st partial sum of the sequence A006257 given by $a_j = 2(j - 2^{\lfloor \log_2 j \rfloor}) + 1$. But the latter partial sum equals $l^2 - 1 - 2(2^{p-1}s + \sum_{i=0}^{p-2} 2^{2i})$, and the claim follows.

The proof of Theorem 2(c) is based on the following Proposition. For brevity, let us denote the vector space $\Lambda^3(e_2, \ldots, e_n)$ by $W$. Denote $\mathfrak{h} = \text{Span}(e_2, \ldots, e_n)$.

**Proposition 2.** For $m$ as defined in Theorem 2, there exists $\omega_k \in W$ for $2 \leq k \leq m$ such that

$$\text{ker } D_{\Lambda^3(\mathfrak{h})} = \text{ker } D_{|W} \oplus \text{Span}(e^n \wedge F(e^k, e^k) + \omega_k : 2 \leq k \leq m).$$

We first prove the theorem assuming the Proposition.

\footnote{The authors are thankful to Omar E. Pol for pointing this out.}
Proof of Theorem 2. For \( n = 3 \) the statement is easily verified: \( H^3(m_0(3)) \) is spanned by the class of the single element \( e^{123} \), so \( b_1(m_0(3)) = 1 \), as claimed.

Assume \( n \geq 4 \). Denote \( d_n \) the dimension of the kernel of the operator \( D \) constructed for the algebra \( m_0(n) \). Then from Proposition 2 we have \( d_n = d_{n-1} + m - 1 \). It follows that for \( n = 2^p + m \), \( 0 < m \leq 2^p \), we have \( d_n = d_{2^p} + \frac{1}{2}m(m - 1) \) and in particular,

\[
d_{2^p+1} = d_{2^p} + 2^{p-1}(2^p - 1).
\] (15)

We also have \( d_4 = 1 \), as for \( m_0(4) \) the space \( \ker D \) is spanned by \( e^{234} \). It follows from (15) that \( d_{2^p} = \frac{1}{3}(2^p - 1)(2^{p-1} - 1) \), and so \( d_n = \frac{1}{3}(2^p - 1)(2^{p-1} - 1) + \frac{1}{2}m(m - 1) \).

We have

\[
\dim(\ker(d : \Lambda^3(m_0(n)) \to \Lambda^4(m_0(n)))) = d_n + \dim(e^1 \wedge \Lambda^2(m_0(n))) = d_n + \frac{1}{2}(n - 1)(n - 2).
\]

On the other hand, from Proposition 1

\[
\dim(\ker(d : \Lambda^2(m_0(n)) \to \Lambda^3(m_0(n)))) = n - 2 + \left\lfloor \frac{n}{2} \right\rfloor - 1,
\]

and so the claim follows from (7).

Proof of Proposition 2. Any \( \omega \in \Lambda^3(h) \) can be uniquely represented as \( \omega = e^n \wedge \xi + \omega' \), with \( \xi \in \Lambda^2(e_2, \ldots, e_{n-1}) \), \( \omega' \in \Lambda^3(e_2, \ldots, e_{n-1}) = W \). For \( \omega \) to belong to \( \ker D \) it is necessary that \( D\xi = 0 \) (so that \( D\omega \) does not contain \( e^n \)). From the proof of Proposition 2 it follows that \( \xi \) must be a linear combination of \( F(e^k, e^k) \), \( k = 2, \ldots, [n/2] \). Extracting the homogeneous components we obtain that the proposition is equivalent to the following statement: for \( 2 \leq k \leq [n/2] \), there exists \( \omega_k \in W \) such that \( e^n \wedge F(e^k, e^k) + \omega_k \in \ker D \), if and only if \( k \leq m \).

The next step in the proof is the following lemma.

Lemma 2. For \( n \geq 4 \) and \( 2 \leq k \leq [n/2] \), define \( a = \left\lceil (n+2k+1)/3 \right\rceil \), \( b = [n/2] + k - 1 \). There exists \( \omega_k \in W \) such that \( e^n \wedge F(e^k, e^k) + \omega_k \in \ker D \) if and only if the linear system \( Ax = (1, 0, \ldots, 0)^t \in \mathbb{Z}_2^{b-a+1} \) has a solution \( x \in \mathbb{Z}_2^{b-a+1} \), where \( A \) is the \((k-1) \times (b-a+1)\)-matrix given by

\[
A_{ij} = \begin{cases} 
\frac{n - (a + j - 1) + 2(i - 1)}{(a + j - 1) + (i - 1) - k} & \text{mod } 2, \ 1 \leq i \leq k - 1, \ 1 \leq j \leq b - a + 1, \\
0 & \text{otherwise}.
\end{cases}
\] (16)

and as usual we set \( \binom{N}{t} = 0 \) if \( t < 0 \) or \( t > N \).

Proof. Suppose for some \( \omega_k \in W \), the three-form \( \omega = e^n \wedge F(e^k, e^k) + \omega_k \) belongs to \( \ker D \) (where \( 2 \leq k \leq [n/2] \)). Without loss of generality we can assume that \( \omega_k \) is homogeneous, of the same degree as \( e^n \wedge F(e^k, e^k) \), so that \( \omega \) is homogeneous of degree \( n + 2k + 1 \).

By Lemma 1 the form \( \omega \) viewed as a three-form on \( m_0 \), lies in the kernel of \( D \) and so is a linear combination of the forms \( F(e^{s-r}, e^r), 2 \leq s < r \), where by homogeneity we can assume that \( s + 2r + 1 = n + 2k + 1 \), from which it follows that \( s = n + 2k - 2r \). Then
$2 \leq s \leq r - 1$ gives $a \leq r \leq b$. Therefore for some $\mu_r \in \mathbb{Z}_2$, $r = a, \ldots, b$ we have

\[
\omega = F(e^k, e^k) \wedge e^n + \omega_k = \sum_{r=a}^{b} \mu_r F(e^{n+2k-2r}, e^r)
\]

\[
= \sum_{r=a}^{b} \mu_r \sum_{l=0}^{\infty} D^l(e^{n+2k-2r}, e) \wedge e^{l+r+1}
\]

\[
= \sum_{l=0}^{\infty} \sum_{r=a}^{b} \mu_r D^l(e^{n+2k-2r}, e) \wedge e^{l+r+1}.
\]

(17)

As $n + 2k - 2r = s < r \leq b$ and $b = \lceil n/2 \rceil + k - 1 \leq 2\lceil n/2 \rceil - 1 < n$, no terms $D^l(e^{n+2k-2r}, e)$ in the latter expression may possibly contain $e^N, N \geq n$. It follows that the only terms containing $e^N$ with $N \geq n$ in (17) are $\xi_N \wedge e^N$, where $\xi_N := \sum_{r=a}^{\min(b, N-1)} \mu_r D^{N-r-1}(e^{n+2k-2r}, e)$. In fact, since $\omega \in \Lambda^3(m_0(n))$, we have $\xi_N = 0$ for all $N > n$ and equating the terms containing $e^N$ we get $\xi_n = F(e^k, e^k)$. Conversely, if $\xi_n = F(e^k, e^k)$, then $\xi_N = 0$ for all $N > n$, as $\xi_{n+1} = D\xi_n = DF(e^k, e^k) = 0$, $\xi_{n+2} = D^2\xi_n = D^2F(e^k, e^k) = 0$, and so on. Thus a necessary and sufficient condition for the existence of $\omega_k \in W$ such that the three-form $\omega = e^n \wedge F(e^k, e^k) + \omega_k$ belongs to ker $D$ is the existence of $\mu_r \in \mathbb{Z}_2, r = a, \ldots, b$ such that

\[
F(e^k, e^k) = \xi_n = \sum_{r=a}^{b} \mu_r D^{n-r-1}(e^{n+2k-2r}, e).
\]

(18)

(the summation on the right-hand side is up to $b$ as $b \leq n - 1$). Note that both sides are homogeneous two-forms of degree $2k + 1$. Recall that $F(e^k, e^k) = e^{k,k+1} + e^{k-1,k+2} + \cdots + e^{2,2k-1}$, and observe that

\[
D^{n-r-1}(e^{n+2k-2r}, e) = \sum_{i=0}^{n-r-1} (n-r-1)_{i} e^{2k-r+i+1,-r-i}.
\]

So expanding and equating coefficients of the corresponding monomials we see that (18) is equivalent to the following system:

\[
\sum_{r=a}^{b} \mu_r \binom{n-r-1}{r-k} + \binom{n-r-1}{r-(k+1)} = 1 \mod 2,
\]

\[
\sum_{r=a}^{b} \mu_r \binom{n-r-1}{r-(k-1)} + \binom{n-r-1}{r-(k+2)} = 1 \mod 2,
\]

\[
\vdots
\]

\[
\sum_{r=a}^{b} \mu_r \binom{n-r-1}{r-2} + \binom{n-r-1}{r-(2k-1)} = 1 \mod 2.
\]

Now the linear combination of the first $s \leq k - 1$ of the above equations with the coefficients $\binom{2s-1}{s-1}, \binom{2s-1}{s-2}, \ldots, \binom{2s-1}{1}, \binom{2s-1}{0}$ respectively gives

\[
\sum_{r=a}^{b} \mu_r \left( \sum_{i=0}^{2s-1} \binom{2s-1}{i} \binom{n-r-1}{r-k-s+i} \right) = \sum_{r=a}^{b} \mu_r \binom{n-r+2s-2}{r-k+s-1}
\]

on the left-hand side (as $\sum_{i=0}^{l} \binom{l}{i} \binom{N}{l-i} = \sum_{i=0}^{l} \binom{N}{l-i} \binom{N+i}{l+i}$ by Vandermonde’s identity). On the right-hand side we obtain $\binom{2s-1}{s-1} + \binom{2s-1}{s-2} + \cdots + \binom{2s-1}{0} = \frac{1}{2} \times 2^{2s-1} = 2^{2s-2}$, which is odd when $s = 1$ and even otherwise. Thus the above system of equations is equivalent to the following one:

\[
\sum_{r=a}^{b} \mu_r \binom{n-r}{r-k} = 1 \mod 2, \quad \sum_{r=a}^{b} \mu_r \binom{n-r+2s-2}{r-k+s-1} = 0 \mod 2, \text{ for } 2 \leq s \leq k-1.
\]
This is equivalent to the claim of the lemma if we define \( x = (\mu_a, \mu_{a+1}, \ldots, \mu_n)^t \).

In order to use Lemma 2 to conclude the proof of the proposition, we need to show that the system \( Ax = (1, 0, \ldots, 0)^t \) has a solution if and only if \( k \leq m \). Even though we are working over \( \mathbb{Z}_2 \), let us say that vectors \( x, y \) are orthogonal if \( x^t y = 0 \).

To prove the **necessity** we show that, assuming \( k > m \), the first row of \( A \) belongs to the span of the next \( m - 1 \) rows, namely that

\[
\left( \binom{k-m-1}{0}, \binom{k-m-1}{1}, \ldots, \binom{k-m-1}{k-m-1}, 0, \ldots, 0 \right) A = 0 \mod 2.
\] (19)

Then any \( x \) orthogonal to all the rows of \( A \) starting from the second one, must also be orthogonal to the first row, and so the system \( Ax = (1, 0, \ldots, 0)^t \) has no solutions. To establish (19) we need to show that for every \( j = 1, \ldots, b - a + 1 \), we have

\[
\sum_{i=1}^{k-m} \binom{k-m-1}{i-1} \left( n - (a + j - 1) + 2(i - 1) \right) \left( a + j - 1 + (i - 1) - k \right) = 0 \mod 2.
\]

which is equivalent (by substitution \( r = a + j - 1, \ l = i - 1 \), \( N = k - m - 1 \), \( n = 2^p + m \)) to showing that for all \( r = a, \ldots, b \),

\[
\sum_{l=0}^{N} \binom{N}{l} \left( 2^p - 1 - (r - k + N - 2l) \right) = 0 \mod 2.
\] (20)

We require the following Lemma.

**Lemma 3.** Suppose \( p \geq 2 \) and let \( x, y \in \mathbb{Z} \).

(a) If \( 0 \leq x < y < 2^p \), then \( \binom{2^p + x}{y} = 0 \mod 2 \).

(b) If \( x, y \leq 2^p - 2 \) and \( y, x + y > 0 \), then \( \binom{2^p - 1 - x}{y} = \binom{y + x}{y} \mod 2 \).

**Proof.** By Kummer’s Theorem, a binomial coefficient \( \binom{t}{q} \) with \( 0 \leq t \) is odd if and only if there is a place in the binary representation where \( q \) has 0 and \( t \) has 1 and, when \( 0 \leq t \leq q \), if and only if there is a place in the binary representation where both \( q - t \) and \( t \) have 1.

(a) For \( \binom{2^p + x}{y} = 1 \mod 2 \), the binary representation of \( 2^p + x \) must have a 1 at all the places where the binary representation of \( y \) does. But as \( y < 2^p \), this implies that the binary representation of \( x \) has 1 at all the places where the binary representation of \( y \) does, which contradicts the fact that \( y > x \).

(b) First suppose \( x \geq 0 \). Then \( \binom{2^p - 1 - x}{y} \) is even if and only if there is a place in the binary representation where \( 2^p - 1 - x \) has 0 and \( y \) has 1 if and only if there is a place in the binary representation where \( x \) has 1 and \( y \) has 1 if and only if \( \binom{y + x}{y} \) is even.

Now let \( x < 0 \). So \( \binom{y + x}{y} = 0 \). Denote \( z = -x - 1 \geq 0 \). Then \( \binom{2^p - 1 - x}{y} = \binom{2^p + z}{y} \) and \( 0 \leq z < y \leq 2^p - 2 \) by our assumption. By part (a), \( \binom{z}{y} = \binom{y}{y} \mod 2 \), and \( \binom{y}{y} = 0 \) as \( z < y \). So \( \binom{y + x}{y} = \binom{2^p + z}{y} \mod 2 \). □

To apply Lemma 3 to the binomial coefficients \( \binom{2^p - 1 - (r - k + N - 2l)}{r - k + l} \) from (20) we need to check few inequalities. We have \( r - k \geq a - k = \left[ \frac{1}{2}(n - k + 1) \right] \geq \left[ \frac{1}{2}(n - \left[ \frac{1}{2}n \right] + 1) \right] = \left[ \frac{1}{2}(\left[ \frac{1}{2}n \right] + 1) \right] \geq 1 \) and so \( r - k + l \geq 1 \) and \( (r - k + l) + (r - k + N - 2l) \geq 1 \). Furthermore, \( r - k + l, r - k + N - 2l \leq r - k + N \leq b - k + N = \left[ \frac{1}{2}n \right] + N - 1 \), and \( \left[ \frac{1}{2}n \right] + N - 1 = \left[ \frac{1}{2}n \right] + k - m - 2 \leq 2\left[ \frac{1}{2}n \right] - m - 2 = 2[2^{p-1} + \frac{1}{2}m] - m - 2 \leq 2^{p-2} \).
So the hypotheses of Lemma 3(b) are satisfied with \( x = r - k + N - 2l, y = r - k + l \). So Lemma 3(b) gives \( (2^{r-1} - (r-k-N-2l)) \mod 2 \), for every \( l = 0, \ldots, N \). Vandermonde’s identity gives \( (2^{r-k} + N-1)_{r-k+l} \mod 2 \), and hence the left-hand side of (20) is congruent modulo 2 to

\[
\sum_{l=0}^{N} \sum_{i=0}^{N-l} \left( \begin{array}{c} N-l \\ i \end{array} \right) \left( \begin{array}{c} 2(r-k) \\ r-k+l-i \end{array} \right) = \sum_{i,l \geq 0, i+l \leq N} \left( \begin{array}{c} N \\ i,l,N-l-i \end{array} \right) \left( \begin{array}{c} 2(r-k) \\ r-k+l-i \end{array} \right) + \sum_{i,l \geq 0, 2i \leq N} \left( \begin{array}{c} N \\ i,i,N-2i \end{array} \right) \left( \begin{array}{c} 2(r-k) \\ r-k \end{array} \right)
\]

as \( (2^{r-k})_{r-k} = (2^{r-k})_{r-k} \relax \) and \( (2^{r-k})_{r-k} = 2^{(2^{r-k}-1)} \). This completes the proof of necessity.

To prove the sufficiency we explicitly produce, for any \( 2 \leq k \leq m \), a vector \( x \in \mathbb{Z}_2^{b-a+1} \) such that \( Ax = (1,0,\ldots,0)^T \in \mathbb{Z}_2^{b-a+1} \).

\[
x_j = \sum_{s=0}^{p-1} \left( \frac{m-k}{n-(a+j-1)-2s} \right), \quad j = 1, \ldots, b-a+1. \tag{21}
\]

By Lemma 2 we need to show that for all \( i = 1, \ldots, k - 1 \),

\[
\sum_{j=1}^{b-a+1} \left( \frac{n-(a+j-1)+2(i-1)}{(a+j-1)+i-1-k} \right) \sum_{s=0}^{p-1} \left( \frac{m-k}{n-(a+j-1)-2s} \right) \mod 2 = \delta_{1i}. \tag{22}
\]

We first show that the expression on the left-hand side of (22) can be rewritten as

\[
\sum_{s=0}^{p-1} \sum_{j \in \mathbb{Z}} \left( \frac{n-(a+j-1)+2(i-1)}{(a+j-1)+(i-1)-k} \right) \left( \frac{m-k}{n-(a+j-1)-2s} \right) \mod 2,
\]

so that there is no contribution from the values \( j \leq 0 \) and \( j \geq b-a+1 \). The latter is easy: for the first binomial coefficient to be nonzero we need to have \( n-(a+j-1)+2(i-1) \geq (a+j-1)+(i-1)-k \) which gives \( 2j \leq n+k+i+1-2a \leq n+2k-2a \), as \( i \leq k-1 \), so \( j \leq \lfloor n/2 \rfloor + k - a = b - a + 1 \). To prove the former, we first look at the second binomial coefficient, from which we get \( m-k \geq n-(a+j-1)-2s \), so \( j \geq n-a+1+k-m-2s \geq n-\frac{1}{3}(n+2k+1)-\frac{2}{3}+1+k-m-2s = \frac{1}{3}(2^{p+1}+k-m-3 \cdot 2s) \). Now if \( s < p-1 \) the expression on the right-hand side is positive, as \( m \leq 2^p \), and we are done. Suppose \( s = p-1 \). Then we have \( j \geq \frac{1}{3}(2^{p+1}+k-m) \), which still implies \( j > 0 \) unless \( m = 2^{p-1}+k+l \), \( l \geq 0 \), in which case we have \( j \geq -\frac{1}{3}l \). Then

\[
a = \left\lfloor \frac{2^p + m + 2k + 1}{3} \right\rfloor = \left\lfloor \frac{2^p + 2^{p-1} + 3k + l + 1}{3} \right\rfloor = 2^{p-1} + k + \left\lfloor \frac{l + 1}{3} \right\rfloor
\]
and the first binomial coefficient has the form \( \binom{2^p + x}{y} \), where

\[
x = n - (a + j - 1) + 2(i - 1) - 2^p = m - (a + j - 1) + 2(i - 1)
\]

\[
y = (a + j - 1) + (i - 1) - k = 2^{p-1} + \frac{(l + 1)/3}{2} + j + i - 2.
\]

Note that as \( i \geq 1 \), we have \( x \geq 0 \) if \( j \leq 0 \). Also if \( j \leq 0 \), then as \( i \leq k - 1 \), we have \( y \leq 2^{p-1} + \frac{(l + 1)/3}{2} + k - 3 \leq 2^{p-1} + l + k - 2 = m - 2 < 2^p \). Moreover, if \( j \leq 0 \), then as \( i \leq k - 1 \), we have \( y - x = (2^{p-1} + \frac{(l + 1)/3}{2} + j + i - 2) - (l + 1 - \frac{(l + 1)/3}{2} + 2(i - 1) - j) = 2^{p-1} + 2 \frac{(l + 1)/3}{2} - (l + 1) + 2j - i \geq 2^{p-1} + 2 - (l + 1) - k = 2^p - m + 1 > 0 \). So the hypotheses of Lemma 3.8 are satisfied, and hence the binomial coefficient \( \binom{2^p + x}{y} \) is even. So it remains to establish that

\[
\sum_{s=0}^{p-1} \sum_{j \in \mathbb{Z}} \binom{n - (a + j - 1) + 2(i - 1)}{(a + j - 1) + (i - 1) - k} \binom{m - k}{n - (a + j - 1) - 2^s} \mod 2 = \delta_{ii}, \tag{23}
\]

for all \( i = 1, \ldots, k - 1 \).

A clear advantage of (23) is that it “takes care of itself” — we do not have to worry about the limits. Changing the summation variable in (23) to \( h = n - (a + j - 1) - 2^s \) we obtain that (23) is equivalent to

\[
\sum_{s=0}^{p-1} \sum_{h \in \mathbb{Z}} \binom{2^s + 2(i - 1) + h}{n - 2^s + (i - 1) - k - h} \binom{m - k}{h} \mod 2 = \delta_{ii}. \tag{24}
\]

Now for a polynomial \( P \in \mathbb{Z}_2[t] \) and \( l \in \mathbb{Z} \) we denote \( \{P\}_l \) the coefficient of \( t^l \) in \( P \). Consider the polynomial \( P_{x,y}(t) = (t^2 + t)^x(t^2 + t + 1)^y \). We have

\[
P_{x,y}(t) = \sum_{h \in \mathbb{Z}} \binom{y}{h} \binom{x + h}{s} t^{x + h + s} = \sum_{l \in \mathbb{Z}} \sum_{h \in \mathbb{Z}} \binom{x + h}{l - x - h} \binom{y}{h} t^l,
\]

so the left-hand side of (24) equals

\[
\sum_{s=0}^{p-1} \{P_{2^s + 2(i-1), m-k}\}_{n+3(i-1)-k} = \left\{ \sum_{s=0}^{p-1} (t^2 + t)^{2^s + 2(i-1)}(t^2 + t + 1)^{m-k} \right\}_{n+3(i-1)-k}
\]

modulo 2 (since \( (t^2 + t)^{2^s} = t^{2^s+1} + t^{2^s} \) in \( \mathbb{Z}_2[t] \) and so \( \sum_{s=0}^{p-1} (t^2 + t)^{2^s} = t^{2^p+1} + t \mod 2 \)). Now, if in the expansion of the latter polynomial we take \( t \) from the first parentheses, then the maximal degree of \( t \) in the resulting terms will be \( 1 + 4(i-1) + 2(m-k) \leq 2m - 1 + 3(i-1) - k < n + 3(i-1) - k \), as \( i \leq k - 1 \) and \( n = 2^p + m \), \( m \leq 2^p \). It follows
that
\[
\sum_{s=0}^{p-1} \left\{ P_{2^s+2(i-1),m-k} \right\}_{n+3(i-1)-k} = \left\{ t^{2^s}(t^2 + t)^{2(i-1)}(t^2 + t + 1)^{m-k} \right\}_{n+3(i-1)-k}
\]
\[
= \left\{ (t + 1)^{2(i-1)}(t^2 + t + 1)^{m-k} \right\}_{m+(i-1)-k}
\]
\[
= \sum_{l \in \mathbb{Z}} \left\{ (t + 1)^{2(i-1)} \right\}_{i-1+l} \left\{ (t^2 + t + 1)^{m-k} \right\}_{m-k-l}
\]
\[
= \left\{ (t + 1)^{2(i-1)} \right\}_{i-1} \left\{ (t^2 + t + 1)^{m-k} \right\}_{m-k} \mod 2,
\]
where the last equality follows from the symmetry: for the polynomial \( f(t) = (t+1)^{2(i-1)} \) we have \( f(t) = t^{2(i-1)} f(t^{-1}) \), so \( \left\{ (t + 1)^{2(i-1)} \right\}_{i-1+l} = \left\{ (t + 1)^{2(i-1)} \right\}_{i-1-l} \), and similarly \( \left\{ (t^2 + t + 1)^{m-k} \right\}_{m-k-l} = \left\{ (t^2 + t + 1)^{m-k} \right\}_{m-k+l} \).

Now if \( i > 1 \) we obtain \( \left\{ (t + 1)^{2(i-1)} \right\}_{i-1} = (2^{i-1}) = 0 \mod 2 \), as required. If \( i = 1 \) we get \( \left\{ (t + 1)^{2(i-1)} \right\}_{m-k} = \sum \left\{ \binom{m-k}{l}(t^2 + t)^l \right\}_{m-k} = \sum_{l} \left( \binom{m-k}{l} \right)_{m-k-l} = \sum_{s} \binom{m-k}{s} \binom{m-k-s}{l} \), where \( s = m - k - l \). The terms with \( s < 0 \) vanish, and the term with \( s = 0 \) is 1. For \( s > 0 \), consider the first place, counting from the right, where the binary expansion of \( s \) has a 1. Then by Kummer’s Theorem, for \( \binom{m-k}{s} \) to be nonzero, the binary expansion of \( m - k \) must have a 1 at the same place, so the binary expansion of \( m - k - s \) will have zero at that place, thus \( \binom{m-k-s}{l} = 0 \). Hence \( \left\{ (t + 1)^{2(i-1)} \right\}_{m-k} = 1 \mod 2 \), as required. This concludes the proof of Proposition 2 and hence of Theorem 2.\( \square \)

Note that one can extract from the above proof an explicit basis for the space of three-cocycles of \( m_0(n) \) (and hence for \( H^3(m_0(n)) \)). We have the following theorem.

**Theorem 4.** For \( n \geq 4 \), \( n = 2^p + m \), \( 0 < m \leq 2^p \) and for \( 2 \leq k \leq m \), define the numbers \( a = \lceil (n+2k+1)/3 \rceil \), \( b = \lfloor n/2 \rfloor + k - 1 \). Let \( B_n \) be the set of elements of \( m_0(n) \) of the form
\[
\sum_{r=a}^{b} \sum_{s=0}^{p-1} \binom{m-k}{n - r - 2s} F(e^{n+2k-2r}e^r) = \sum_{r=a}^{b} \sum_{s=0}^{p-1} \binom{m-k}{n - r - 2s} D^l(e^{n+2k-2r}e^r) \wedge e^{r+l+1},
\]
for \( 2 \leq k \leq m \), where \( D \) is the linear operator defined by \( 1 \) and the binomial coefficients are taken modulo 2. Then classes of the elements of the set
\[
\{ e^{1,i-1,j}, \quad 2 + \lfloor n/2 \rfloor \leq i \leq n \} \cup \bigcup_{4 \leq t \leq n} B_t.
\]
is a basis for the cohomology space \( H^3(m_0(n)) \), \( n \geq 4 \), over the field \( \mathbb{Z}_2 \).

**Proof.** We start with the elements \( e^{1,i-1,j}, \) \( 2 + \lfloor n/2 \rfloor \leq i \leq n \). They are linearly independent cocycles and the space spanned by them has the correct dimension, which is the codimension of the space of coboundaries in the space spanned by \( e^{1,j} \), \( 1 < i < j \leq n \), by Proposition 1. It suffices to show that neither of them is a coboundary. But if it were so, then by homogeneity we would have had that \( e^{1,i-1,j} \) is the coboundary of a linear combination of the elements \( e^{kl}, \) \( 2 \leq k < l \leq n \), \( k + l = 2i \), that is, of the elements \( e^{i-k,i+k}, \) \( k = 1, \ldots, n - i \) (note that as \( i \geq 2 + \lfloor n/2 \rfloor \), we have
2i − n − 1 ≥ 2). But the coboundary of any such element is the sum of exactly two monomials, \( e^{1,i−k−1,i+k} + e^{i,i−k,i+k−1} \), so the coboundary of any linear combination of them is a sum of an even number of monomials, hence cannot be equal to \( e^{1,i,i} \).

As to the element from the sets \( B_i \), no linear combination of them is a coboundary (as any coboundary is a multiple of \( e^1 \)). Moreover, from Proposition 2 (both the statement and the proof) it follows that they form a basis for the kernel of \( D \), where the form of the elements given in the statement follows from Lemma 2 and Equation (21).

**Example 1.** For \( n = 4, \ldots, 12 \), the space of 3-cocycles of \( \mathfrak{m}_0(n) \) is spanned by the three-forms \( e^{1,j} \), \( 1 < i < j \leq n \), and the three-forms from the following table in the rows labelled by the numbers less than or equal to \( n \).

| 4  | \( e^{234} \) |
|----|------------|
| 5  |            |
| 6  | \( e^{245} + e^{236} \) |
| 7  | \( e^{345} + e^{246} + e^{237} + e^{356} + e^{257} + e^{347} \) |
| 8  | \( e^{256} + e^{247} + e^{238} + e^{456} + e^{357} + e^{258} + e^{348} + e^{467} + e^{278} + e^{368} + e^{458} \) |
| 9  |            |
| 10 | \( e^{267} + e^{258} + e^{249} + e^{23(10)} \) |
| 11 | \( e^{367} + e^{268} + e^{358} + e^{349} + e^{24(10)} + e^{23(11)} \) |
| 12 | \( e^{467} + e^{368} + e^{458} + e^{269} + e^{25(10)} + e^{24(11)} + e^{23(12)} \) |

### 4. Cohomology of \( \mathfrak{m}_2 \)

In this section, we compute the cohomology of the infinite-dimensional Lie algebra \( \mathfrak{m}_2 \) given by (2):

\[
\mathfrak{m}_2 = \text{Span}(e_1, e_2, \ldots), \quad [e_1, e_i] = e_{i+1}, \quad i > 1, \quad [e_2, e_j] = e_{j+2}, \quad j > 2,
\]

hence completing the proof of Theorem 1. First we state the following result for the truncation \( \mathfrak{m}_2(n) \).

**Corollary 1.** The first three Betti numbers of the Lie algebra \( \mathfrak{m}_2(n) \), \( n \geq 5 \), over \( \mathbb{Z}_2 \) are given by \( b_1(\mathfrak{m}_2(n)) = 2 \), \( b_2(\mathfrak{m}_2(n)) = [\frac{1}{2}(n + 1)] \), and

\[
b_3(\mathfrak{m}_2(n)) = \frac{1}{3}(2^n - 1)(2^{n-1} - 1) + \frac{1}{2}m(m - 1) + [\frac{1}{2}(n - 1)],
\]

where \( n = 2^p + m, \quad 0 < m \leq 2^p \).

**Proof.** By Theorem 1, the Betti numbers of \( \mathfrak{m}_2(n) \) and of \( \mathfrak{m}_0(n) \) over \( \mathbb{Z}_2 \) are the same. The claim then follows from Theorem 2.

**Remark 2.** It is easy to see that \( H^1(\mathfrak{m}_2(n)) \) is spanned by the cohomology classes of \( e^1 \) and \( e^2 \) and that \( H^2(\mathfrak{m}_2(n)) \) is spanned by the cohomology classes of the elements \( e^{1n} + e^{2,n-1}, \quad e^{i,i+1} + e^{i-1,i+3} + \cdots + e^{2i-1} \), where \( 2 \leq i \leq \frac{1}{2}(n + 1) \). A basis for \( H^3(\mathfrak{m}_2(n)) \)
can be found by applying the map $f$ from \[14\] Definition 3] (see below) to the elements of the basis for $H^2(\mathfrak{m}_0(n))$ constructed in Theorem 4; the resulting basis is the same.

In the infinite-dimensional case, we follow the construction of \[14\]. As in the Introduction, let $V = \text{Span}(e_1, e_2, \ldots)$, and define the operator $D_1$ on $V^*$ by $D_1 e^1 = D_1 e^2 = 0$, $D_1 e^i = e^{i-1}$, for $i > 2$, and then extend it to $\Lambda(V)$ as a derivation. Note that any $\omega \in \Lambda^q(V)$, $q \geq 2$, has a unique presentation in the form $\omega = e^1 \wedge \xi + e^2 \wedge \eta + \zeta$, where $\xi \in \Lambda^{q-1}(e_2, e_3, \ldots)$, $\eta \in \Lambda^{q-1}(e_3, e_4, \ldots)$ and $\zeta \in \Lambda^q(e_3, e_4, \ldots)$. Note that $\xi, \eta$ and $\zeta$ linearly depend on $\omega$.

Define the linear map $f$ on $\Lambda(V)$ by setting $f(e^1 \wedge \xi + e^2 \wedge \eta + \zeta) = e^1 \wedge \xi + e^2 \wedge (\eta + D_1 \xi) + \zeta$ on the forms of rank at least two, and taking it to be the identity on $V^*$. The following properties of $f$ are easy to check:

- $f$ is an involution, hence a bijection, and $f^{-1} = f$,
- the restriction of $f$ to $\Lambda(e_2, e_3, \ldots)$ is the identity,
- $f$ preserves the homogeneous components: $f(\Lambda^q_k(V)) = \Lambda^q_k(V)$.

The main feature of $f$ is the fact that it interweaves the differentials of $\mathfrak{m}_0$ and $\mathfrak{m}_2$. More precisely, consider $\mathfrak{m}_0$ and $\mathfrak{m}_2$ to have the same underlying linear space $V$, but to be defined by the brackets (1) and (2) respectively relative to the same basis $\{e_1, e_2, \ldots\}$ for $V$. Then for all $\omega \in \Lambda(\omega)$, we have

$$fd_0\omega = d_2 f\omega, \quad fd_2\omega = d_0 f\omega,$$

where $d_0$ and $d_2$ are the differentials on $\mathfrak{m}_0$ and $\mathfrak{m}_2$ respectively. The first equation is easily verified for $\omega = e^i$, and the proof for $\omega \in \Lambda^q(V)$, $q \geq 2$, is identical to the proof of \[14\] Proposition 1. The second one follows, as $f$ is an involution.

**Proof of Theorem 1** By (25), $f$ bijectively maps cocycles and coboundaries of $\mathfrak{m}_0$ to cocycles and coboundaries of $\mathfrak{m}_2$ respectively. It follows that $H^*(\mathfrak{m}_2)$ is spanned by the classes of the images under $f$ of the elements \[13\]. As $f$ acts on all those elements as the identity, we obtain that the basis for $H^*(\mathfrak{m}_2)$ is the set of the classes of the same cocycles.

The fact that the multiplicative structure is preserved follows from the fact that the restriction of $f$ to $\Lambda(e_2, e_3, \ldots)$ is the identity and that multiplication by $e^1$ is trivial in both $H^*(\mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2)$. Multiplication by $e^1$ is trivial in $H^*(\mathfrak{m}_0)$ because $e^1 \wedge \omega$ is a $d_0$-coboundary, for any $\omega$ (see the proof of Theorem 3). To see that multiplication by $e^1$ is trivial in $H^*(\mathfrak{m}_2)$, notice that for any $\omega$ in the list \[13\], one has $D_2 \omega = 0$ (which is essentially assertion $(a)$ of Lemma 1), and so $f(e^1 \wedge \omega) = e^1 \wedge \omega$, which is then a $d_2$-coboundary, as $f$ maps coboundaries to coboundaries.

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