General relativity principle and uniqueness in Einstein equations

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The issue of implementing the principle of general relativity in Einstein equations has been widely discussed, since Kretschmann’s well-known criticism stated that general covariance of the Einstein equations is not sufficient to express the principle of general relativity (the equivalence of all the coordinate systems). This failure is usually rooted in the fact that metric in Einstein equations is not univocally determined by the matter distribution. We show that the condition of univocal determination of the metric by the matter distribution is stronger than the requirement of equivalence of all coordinate systems. In order to separate the uniqueness problem in Einstein equations from the issue of the principle of general relativity, we define the “equivalence group” instead of the notion of covariance group which is empty of physical content. Moreover, we complement in a positive way Kretschmann’s objection by supplementing Einstein equations with a sufficient condition for the equivalence of all the coordinate systems.

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1. INTRODUCTION

Physics has a peculiarity that distinguishes it from any other science: its theories are expressed in a language that differs from the one in which they were developed. In the translation process some generality may be lost or the original idea may end up being contradicted. The latter is what happened to the principle of general relativity in its translation to Einstein equations. The aim of this work is to justify this assertion and indicate the way to a more faithful translation.

As it is well known, Einstein felt attracted by Mach’s thought around 1912. Mach’s criticism to Newton’s absolute space, namely his quoted comment regarding the rotating bucket, suggested to Einstein and other authors the formulation of a new mechanics where the inertial forces appearing in the so called non-inertial frames would not be ascribed to the acceleration relative to the absolute space but to the motion of the frame relative to the matter distribution in the universe. From this point of view, the force producing the parabolic shape of the surface of water in the frame fixed to the bucket should be ascribed to the rotating motion of the stars in that frame.

Initially, Einstein thought that Mach’s ideas could be expressed in a theory where the components of the metric and the affine connection in each frame were obtained from the distribution of matter of the universe. Certainly, the law linking the metric and the distribution of matter should not privilege any frame. Einstein convinced himself that this condition would be fulfilled if the set of equations for the metric were generally covariant.

As Einstein recognized after Kretschmann’s criticism (later clearly stated by Anderson and Friedman), the general covariance of the equations for the metric field does not imply a principle of general relativ-
ciple”. Bondi and Samuel mention ten different and non-equivalent statements of the Mach principle. The Ref. (2) can be consulted to have both a historical and an updated account of this subject.

We will briefly comment the References (2, 8, 9), which tackled the problem by proposing a relation metric-distribution of matter which results from using a “Green function” for the metric tensor, analogously to electromagnetism where the potential vector is expressed in terms of the current source. With some differences, these works try to implement the idea that the metric should be dictated by the distribution of matter. Only some subgroup of metrics solving the Einstein equations satisfies this requirement, which works as a selection rule to eliminate “non-Machian” solutions. Concerning the implementation of the principle of general relativity, we should point out that these works do not entirely reach their objective. First, given the non-linear character of the Einstein equations, the “Green function” contains information of the same metric. Therefore, the procedure is not a perfect translation of the requirement of univocal determination of the metric by means of the energy-momentum tensor. Second, the different formulations of Mach’s principle contain two intermingled questions: the determination of the geometry by the matter distribution, and the determination of the components of metric tensor by the form of the matter distribution in a certain frame. As this paper will show, the latter question, not the former, is the one involved in the principle of general relativity. Our main claim in this paper is that the condition of univocal determination of the metric by matter distribution is stronger than the requirement of equivalence of all the coordinate systems dictated by principle of general relativity.

Since our specific interest in this paper lies in the intersection of Mach principle and the principle of general relativity, we will focus exclusively on finding the requirements that a relation metric-matter distribution should fulfill to satisfy the principle of general relativity. In other words, we will try to find selection rules that are a faithful translation of the principle of general relativity.

The search for these rules has been guided by the following principle. As usual, finding a solution to Einstein equations is facilitated by the imposition of symmetries to the metric. However, in other theories where a field is determined by its source—for example, electrostatics—the imposition of symmetries of the source to the field solution is not merely a way to solve the equations but a consequence of the very laws of the field, as we will discuss in detail at the beginning of the next section.

This observation leads to the idea of elevating to the category of principle the association between symmetry of the source and symmetry of the solution in Einstein equations. In order to put this principle in practice, we consider that a supplementary equation should be added to Einstein equations. This extra condition on the solutions of General Relativity would cover part of the path towards the generalization of the principle of relativity.

In section 2 we analyze the electrostatic equations in order to draw a parallel with Einstein equations. In section 3 we trace the analogy between Einstein equations and the electrostatic ones. In section 4 we analyze the problem of the “determination of the metric” by “the matter distribution” by focusing on the meaning of both expressions. In section 5 we examine the root of the problem: the relation between the non-univocal determination of the metric in Einstein equations by the energy-momentum tensor and the general relativity principle. In section 6 we propose that a supplementary equation be added to Einstein equations to express the principle of general relativity. And we also show the results of applying the supplementary equation to certain solutions considered in the literature as “non-Machian”. Finally, we present a summary and comment on the problems left open for a future work.

2. THE ELECTROSTATIC ANALOGY

2.1 Covariance group and equivalence group

To begin with, we will consider the electrostatic equations as a toy model to study some aspects that should be also present in a theory of gravitation. The electrostatic field \( \vec{E}_C \) generated by a distribution of charge \( \rho \) is

\[
\vec{E}_C(\vec{r}) = \int \frac{\rho(\vec{r'}) (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3} \, d^3 r'
\]

(1)

The Coulombian field \( \vec{E}_C \) has relational character relative to the charge density; i.e., the differences between the components of the field in two different Euclidean frames are exclusively attributable to the different appearances of the charge distribution in each frame. This means that the theory given by (Coulombian) respects the isotropy and the homogeneity of the space or, in other words, the equivalence of all Euclidean frames. We will call GE, without distinction, both the set of Euclidean frames or the group of transformations connecting them.

Let us consider a local consequence of Eq. (1) by taking divergence in both members:

\[
\nabla \cdot \vec{E} = \rho
\]

(2)

We call the theory given by this sole equation TE1. Since the mapping \( \vec{E} \rightarrow \nabla \cdot \vec{E} \) is non-injective, TE1 also contains other fields different from \( \nabla \cdot \vec{E} \). So TE1 is a wider theory than the previous one. Does TE1 respect the equivalence of the elements of GE? Although this seems to be a well-posed question, depending on what we understand by “respects the equivalence of the elements of
GE" we will get opposite answers. If we are meaning that the fields \( \vec{E} \) and \( \rho \) satisfy the same equation in all the frames belonging to GE, the answer is yes. This is a consequence of the conjunction of two properties of the Eq. (2): its covariance under Euclidean transformations and the absence of any reference to a particular coordinate system.

As it was already said, the question admits another interpretation which concludes that TE1 does not respect the equivalence of the elements of GE: the theory TE1 contains solutions \( \vec{E} \) which are related with the source \( \rho \) in such a way that some particular coordinate system turn out to be privileged. In fact, the equation (2) admits both the solution \( E_o \) and the following non-relational solutions:

\[
E_o(\vec{r}) = \int \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3r' + E_o \, \hat{a}
\]

(3)

\[
E'_o(\vec{r}) = \int \frac{\rho(\vec{r})(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3r' + \int \frac{j(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3r' + \int \frac{j(\vec{r}) \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3r',
\]

(4)

where \( E_o \, \hat{a} \) is an uniform arbitrary vector, and \( j(\vec{r}) \) is an arbitrary axial vector field. In contrast to the relational field (4), the fields (3) and (4) are non-relational regarding the source \( \rho(\vec{r}) \). In fact, the differences between components of the fields in two different GE frames cannot be exclusively attributed to the appearance of the source \( \rho \) in each Euclidean frame, because they also depend on the appearance of \( E_o \, \hat{a} \) and \( j(\vec{r}) \) in each frame. In the laws (3) and (4) the only equivalent systems are those related by transformation that are symmetries of the vector \( \hat{a} \) and \( j \) respectively.

Summing up, the question of which coordinate systems are equivalent in the theory given by (2) admits two different readings. In the first case, we are asking about the group of transformations that leaves the form of Eq. (2) unchanged. This is the covariance group of Eq. (2). In the second case, we are asking about the group of transformations which does not modify any injective mapping \( \rho \rightarrow \vec{E} \) satisfying Eq. (2). This is the covariance group common to all equations defining injective mappings \( \rho \rightarrow \vec{E} \) that fulfills Eq. (2).

The ambiguity in the issue of the equivalence of frames can be eliminated by applying the notion of equivalence group (19). Let \( D_{\{y_i\}}(i = 1...N) \) be a set of equations with variables \( y_i \), and \( C_{D_{\{y_i\}}} \) their covariance group. We defined the equivalence group \( O_{D_{\{y_i\}}}^{(y_1,y_2,...,y_{r+1},...,y_N)} \) of \( D_{\{y_i\}} \) as the covariance group shared by all equations defining injective mappings \( (y_{r+1},...,y_N) \rightarrow (y_1,y_2,...,y_r) \) that satisfies \( D_{\{y_i\}} \). We call “dynamical” the variables \( y_1,...,y_r \), and “initial conditions” the variables \( y_{r+1},...,y_N \).

In this way, the issue of the equivalence group of a given theory leads to different answers according to these two possible roles of the variables entering an equation \( D \). We should underline the difference between the concepts of equivalence group and covariance group. As mentioned above, the latter is the set of transformations preserving the form of a given equation; its definition only depends on the form of the equation \( D \). Its relation with the equivalence group can be written in the following way:

\[
C_{D_{\{y_i\}}} = O_{D_{\{y_i\}}}^{(y_1,y_2,...,y_N)}
\]

(5)

where the right member displays all the variables in the role of initial conditions.

Returning to example of theory TE1, three different equivalence groups could be considered: \( O_{D_{\{E,\rho\}}}^{(\vec{E},\rho)} \), \( O_{D_{\{E,\rho\}}}^{(\vec{E},\rho)} \), \( O_{D_{\{E,\rho\}}}^{(\vec{E},\rho)} \). The different meanings of the question about the set of equivalent frames in TE1 arise from the different roles the variables \( \rho \) and \( \vec{E} \) can play. If we answer that all the frames of GE are equivalent, then we have asked about \( O_{D_{\{E,\rho\}}}^{(\vec{E},\rho)} \), where all the variables play the role of initial conditions. In this case, the group of equivalence agrees with the covariance group of the Eq. (2). If we answer that TE1 does not respect the equivalence of the frames of GE, then we have questioned on \( O_{D_{\{E,\rho\}}}^{(\vec{E},\rho)} \). In this case \( \vec{E} \) is a dynamical variable and \( \rho \) is an initial condition; then the equivalence group will be the covariance group shared by all the injective mappings of \( \rho \) in \( \vec{E} \) satisfying (2). \( O_{D_{\{E,\rho\}}}^{(\vec{E},\rho)} \) is empty. In fact, the solutions (3) and (4) belong to Eq. (2) and represent injective mappings of \( \rho \) in \( \vec{E} \). The covariance group of each mapping are the rotations around all axis parallel to \( \hat{a} \) and the symmetries of \( j \), respectively. Since \( \hat{a} \) and \( j \) are arbitrary, then the covariance group shared by all the mappings (3) and (4) is empty.

On the other hand, if the question refers to \( O_{D_{\{E,\rho\}}}^{(\rho,\vec{E})} \), the answer is GE, since the only injective mapping of \( \vec{E} \) (initial condition) in \( \rho \) (dynamical variable) satisfying the Eq. (2) is \( \rho = \vec{\nabla} \cdot \vec{E} \).

The theory TE1 is a simple example of Kretzschmann’s well known statement that -using the previously introduced language- establishes that the covariance group of an equation is generally smaller than the equivalence group of their solutions. Nevertheless, there is a case where the covariance group of a theory agrees with its equivalence group (besides the trivial case (3)). This happens when the equation \( D \) has a unique solution for the chosen initial conditions (uniqueness property).

2.2 Equations for a relational field \( \vec{E} \)

Although the equivalence group \( O_{D_{\{E,\rho\}}}^{(E,\rho)} \) corresponding to Eq. (2) is empty, it is possible to supplement the Eq. 
with other equations, so that the equivalence group of the extended system \( \mathcal{D}' \) is equal to some subgroup of \( \mathbf{GE} \). For instance, we could get an equivalence group \( \mathcal{C}'_{\mathbf{D}'}(\rho) \) equal to the covariance group \( \mathcal{C}_{\mathbf{D}} \) by adding equations leading to the uniqueness of the solution for each initial condition \( \rho \). In other words, certain conditions could be imposed in order to restrict the domain of the functional \( \vec{F} \rightarrow \nabla \cdot \vec{F} \) so that it admits the inverse mapping. Depending on the characteristics of these conditions, the emerging theory will have different equivalence groups which are subgroups of \( \mathbf{GE} \).

For instance, the theory that will be called \( \text{TE2} \), which results from adding to Eq. (2) the conditions

\[
\vec{\nabla} \times \vec{E} = 0 \quad \lim_{\vec{r} \rightarrow \infty} \vec{E}(\vec{r}) = E_o \hat{a}
\]

have the field (2) as the only solution associated to the source \( \rho \). It can be said that Eq. (2) has been included in a theory whose equivalence group is a subgroup of \( \mathbf{GE} \) containing the symmetries of the field \( E_o \hat{a} \).

If we want to increase the equivalence group so that it agrees with \( \mathbf{GE} \), we can add to (2) the following conditions:

\[
\vec{\nabla} \times \vec{E} = 0 \quad \lim_{\vec{r} \rightarrow \infty} \vec{E}(\vec{r}) = 0
\]

thus leading to the Coulombian Electrostatics for localized sources. Because Eqs. (2), (3,4) are covariant under \( \mathbf{GE} \) and determine univocally the field \( \vec{E} \) for a given \( \rho \), we can be sure that the equivalence group of this system will agree with \( \mathbf{GE} \). In fact, the relational field is the only solution to Eqs. (2), (3,4). So, the conditions (3,4) eliminate the non-relational solutions contained in Eq. (2).

The most general way to obtain that (2) be part of a theory with an equivalence group equal to \( \mathbf{GE} \) consists in adding an equation that imposes the relational character to any field solution. The relational character lies in the fact that the field \( \vec{E}(\vec{r}) \) must depend exclusively on the values of the charge distribution \( \rho \) and the position \( \vec{r} \) relative to each point where \( \rho \) is non-null.

Let \( \Lambda \) be a transformation belonging to the group of rotations, translations and reflections. \( \Lambda \) maps Cartesian frames into Cartesian frames. Then, the relational character of the field is expressed by the equation:

\[
\vec{E}^{(\rho)}(\mathbb{M}_{(\Lambda)}\vec{r}) = \mathbb{M}_{(\Lambda)}\vec{E}^{(\rho)}(\vec{r})
\]

\( \vec{E}^{(\rho)} \) being the field associated with the source \( \rho \), \( \rho^\Lambda \) the source transformed by \( \Lambda \) and \( \mathbb{M}_{(\Lambda)} \) the matrix representation of \( \Lambda \) in the space of polar vectors. The theory given by the systems (2)-(10) will be called \( \text{TER} \).

In order to prepare the treatment of the Einstein equations, we will rewrite the Eq. (10) in terms of the Lie derivative \( \mathcal{L} \). If \( \Lambda \) belongs to the subgroup of continuous transformations (translations and rotations) generated by a vectorial field \( \vec{\xi} \), i.e., if the infinitesimal coordinate transformation is \( \mathbb{M}_{(\Lambda)}\vec{r} = \vec{r} + \epsilon \vec{\xi} + O(\epsilon^2) \), then \( \rho^\Lambda \simeq \rho + \epsilon L_{\xi} \rho \), and the Eq. (11) is written as:

\[
\vec{E}^{\rho+\epsilon L_{\xi} \rho} - \vec{E}^\rho \simeq \epsilon L_{\xi} \vec{E}^\rho
\]

If the source admits some vector such that \( L_{\xi} \rho = 0 \), then the Eq. (11) adopts a particularly simple form. This happens when the source has a symmetry. By applying Eq. (11) together with (2) to the case of a spherically symmetrical charge \( \rho \), there results a Coulombian field (11) fulfilling Eqs. (3,4).

Note that, given an arbitrary charge distribution, the theory \( \text{TE2} \) does not univocally determine \( \vec{E} \) from \( \rho \). In other words, the Eq. (2) possesses relational solutions different to the field (11). For instance, the field:

\[
\vec{E}_{\text{non-linear}}(\vec{r}) = \vec{E}_C + \frac{k}{V_D} \int_D \vec{E}_C(\vec{r})d^3r,
\]

where \( D = \{ \vec{r}/\rho(\vec{r}) \neq 0 \} \) and \( k \) is an arbitrary constant, is a relational solution of Eq. (2). The second term in Eq. (12) is a constant field which does not privilege any particular direction of the space, since this term has been built exclusively from global properties of \( \rho \).

The theory \( \text{TE2} \) contains all the solutions of the extended system that result from supplementing the Eq. (2) with the equations:

\[
\vec{\nabla} \times \vec{E} = \vec{J}(\rho) \quad \lim_{\vec{r} \rightarrow \infty} \vec{E}(\vec{r}) = \vec{E}(\rho)
\]

for any vectors \( \vec{E} \) and \( \vec{J} \) built exclusively from the charge distribution.

Nevertheless, the system (2)+(11), whose equivalence group is \( \mathbf{GE} \), acquires the uniqueness property by additionally demanding that the field \( \vec{E}^\rho \) be a linear functional of its source, i.e., that \( \vec{E}^{\rho_1+\rho_2} = \vec{E}^{\rho_1} + \vec{E}^{\rho_2} \) (this requirement is feasible due to the linearity of Eq. (2)). In this way, the linear theory included in \( \text{TE2} \) is equivalent to \( \text{TEC} \). In fact, we already said that the Coulombian field is the solution of Eqs. (2)+(11) for a spherically symmetric charge distribution. This field satisfies the system (3)+(4). The linearity implies that these same conditions will be satisfied for all distribution \( \rho \), since all charge distribution can be expressed as a superposition of spherically symmetric distributions. We have therefore proved that the system (11)-(2) supplemented with the linearity requirement is equivalent to the system (2)-(3)-(4).

To sum up, the equivalence group of \( \text{TEC} \) is \( \mathbf{GE} \). We have defined the theory \( \text{TE1} \) which is given by the sole equation (2). The field (11) belongs to \( \text{TE1} \), but this theory contains other solutions because the map \( \vec{E} \rightarrow \nabla \cdot \vec{E} \)
is non-injective and, therefore, it does not have inverse mapping. The new theory has an equivalence group \( O(\tilde{E}(\rho)) \) empty, although its covariance is the same as in the theory TEC. This reduction of the equivalence group happens because the space of solutions of Eq. (2) contains both relational and non-relational fields.

The Eq. (2) can appear in a theory whose equivalence group is a subgroup of \( GE \) (for instance the theory TE2 which involves fields that privilege a particular direction of the space). To render Eq. (2) a part of the theory \( TE2 \) whose equivalence group includes all Euclidean transformations- Eq. (2) was supplemented with the condition (10) that imposes the relational character of the field.

Although the equivalence group of \( TER \) is the set of all Euclidean transformations, the field \( \tilde{E} \) is not univocally determined by the source \( \rho \) in this theory. In other words, the existence of the theory \( TER \) proves that the solution of Eq. (2) cannot be univocally fixed by demanding that the equivalence group coincide with \( GE \). Besides we must demand linearity, a requirement that has nothing to do with the issue of the equivalence group here considered. In this way we have shown that the requirement of univocal determination of the field by its source is stronger than the requirement of equivalence of \( GE \).

3. THE EINSTEIN EQUATIONS

3.1 The equivalence group and other definitions

All that has been said about Eq. (2) can be immediately extended to Einstein equations:

\[
G(g) = kT
\]

where \( k \) is the Einstein constant, and \( G(g) \) is the Einstein tensor associated with the Lorentzian metric tensor \( g \). In fact, one should replace “Euclidean transformations” with “general coordinates transformations”, “charge distributions” with “matter-energy distribution” and “electric field” with “metrics”. Therefore, although the form of Einstein equations is generally covariant, it is not guaranteed that any relation metric-distribution of matter satisfying Einstein equations will respect the equivalence of all coordinate systems. General covariance of Einstein equations implies only that the equivalence group \( O(D_{(g,T)}) \) (here \( D \) are the Einstein equations) is the group of general coordinate transformations, which will be called \( GG \). However, this is not the desired property, as explained in the introduction. We are interested in the equivalence group \( O(D_{(g)}) \). Like Eq. (2), Einstein equations could be part of an enlarged system of equations with an arbitrary equivalence group. The equivalence group depends on the supplementary conditions added for univocally determining the metric. So, we must study the way of reducing the domain of the functional \( G : g_{ij} \rightarrow G_{ij}(g) \) so that the non-relational character be eliminated.

Although the electrostatic analogy can help us to tackle a more involved problem, we should stress a fundamental difference between Eq. (2) and Einstein equations. Eq. (2) was derived from the relation field-source (4), which respects the equivalence of the Euclidean systems. Therefore, Eq. (2) does contain the relational field, and only needs proper conditions to select the relational field as its sole solution. Moreover, the supplementary equations (8, 9) were deduced from the knowledge of the relational field. However, Einstein equations have not been deduced from any relation metric-matter distribution respecting the equivalence of all coordinate systems. Therefore, we do not have the guidance of such a solution for finding supplementary conditions for Einstein equations. Furthermore, we do not even know whether such solution exists.

In spite of it, we have shown that the relational field (11) can be obtained by adding to Eq. (2) the condition (11), which directly expresses the relational character of the field (i.e. the equivalence of the Euclidean systems in the relation field-source) plus the requirement of linearity. Omitting this last condition, the system (2), (11) has only relational solutions although it lacks the uniqueness property. This will be our strategy to find supplementary conditions for Einstein equations that express the relational character of the metric.

Before moving on, we will explicitly separate the problem we have posed from the Cauchy problem. The Cauchy problem is concerned with the initial conditions which are needed to univocally determine a solution of the equations. Once given the components of the energy-momentum tensor on the manifold, we select a solution of Einstein equations by prescribing the components of the metric tensor and their normal derivatives (subject to the constraint equations) on a certain space-like hypersurface. Although this procedure does lead to a unique set of components of the metric tensor, the solution obtained does not have, in general, relational character. This is so because the Einstein equations tell us the dynamics of the metric tensor, but say nothing about the values of its components. This situation is comparable to Newtonian dynamics, whose Galilean covariant laws cannot regulate the non-invariant velocities of the bodies. However, we intend to get a theory where the metric, not just its dynamics, is dictated by the distribution of matter. This plan retains Einstein’s original ideas motivated by Mach.

3.2 The distribution of matter

The matter distribution is characterized by the energy-momentum tensor \( T \) in the right member of Eq. (13). We
are not interested here in the structure of the energy-
momentum tensor but only in its value at each point of
space-time. Since $T$ is in the image of the functional $G$
mapping $g_{ij}$ in $G_{ij}$, then the divergence of the energy-
momentum tensor is identically null. This divergence
involves the non-predetermined metric affine connection
associated with the pre-image of $T$.

3.3 The distinguishability of coordinate systems

In the context of Electrostatics, different Euclidean
frames can be distinguished by the appearance of the
charge in each one of them. If a relational electrostatic
field looks different in two different Euclidean frames,
then the charge looks different in each frame. Analog-
ously, a non-null energy-momentum tensor $T$ provides
a physical way of distinguishing different coordinate sys-
tems. Although different coordinate systems can be dis-
tinguished by means of the distribution of energy-matter,
this does not mean that they are non-equivalent.

3.4 The metric tensor

The points—the events—in space-time can be identified
by means of different coordinate systems. Thus, a given
geometry $g$ can be represented by different values of its
components at each point $x$: $g_{ij}(x) \equiv< g, e_i \otimes e_j > (x)$, where $\{e_i(x)\}$ is a coordinate basis at $x$. For instance,
we could introduce a flat geometry by giving any of the
equivalent matrices $g_{ij}(x)$ corresponding to a flat geo-
metry. In this case, each choice for $g_{ij}(x)$ amounts to the
choice of the transformation connecting the basis $\{e_i(x)\}$
with the Euclidean basis. From a physical point of view,
these different ways of introducing a geometry in space-
time are indistinguishable, unless the different coordinate
systems are physically distinguishable. The energy-
momentum tensor $T$ allows to distinguish coordinate sys-
tems, as it was already stated.

In principle, any space-time could be provided with
a metric not subject to fulfilling the Einstein equations.
Even so, the procedure to introduce the metric involves
two steps. First, a geometry is chosen and written as
one of the equivalent matrices $g_{ij}$, and second, the ba-
sis $\{e_i(x)\}$ where the metric has the components $g_{ij}$
is physically identified by giving the components of $T$ in
that basis or the transformation linking $\{e_i(x)\}$ with some
privileged basis selected by $T$.

4. THE DETERMINATION OF METRIC IN
EINSTEIN EQUATIONS

The Einstein equations do not determine the metric $g$
exclusively from the energy-momentum tensor $T$. In fact,
the functional $G$ mapping $g_{ij}$ in $G_{ij}$ is non-injective, i.e.,
there exists $g_{ij} \neq g'_{ij}$ so that $G(g_{ij}) = G(g'_{ij})$. Therefore,
the functional $G$ lacks of inverse mapping, unless its do-
main is properly restricted. However, our goal is to study
a more weaker condition: the reduction on the domain
in order to eliminate the non-relational metric.

In order to study the reduction of the domain, the
above mentioned aspects of the specification of the metric
should be reexamined due to the relation between metric
and energy-momentum tensor established by the Einstein
equations. Three sets are involved in this issue:

- $A_{(G_{ij})}$: matrices $g_{ij}$ that are physically identified
  by giving the components of the energy-momentum tensor
- $B_{(G_{ij})}$: geometries $g$ that respect all the axioms of the
  functional $G_{ij}$
- $C_{(g,G_{ij})}$: matrices $g_{ij} \in A_{(G_{ij})}$ associated with
  a given geometry $g \in B_{(G_{ij})}$

We notice that the set $C_{(g,G_{ij})}$ does not contain all the
equivalent forms of geometry $g$ but only those satisfying
the Einstein equations for a certain expression $G_{ij}$ of the
tensor $G$.

Our aim is to replace the set $A_{(G_{ij})}$ by a subset
$A'_{(G_{ij})} \subset A_{(G_{ij})}$ containing only one representation of
each geometry. In other words, $\#C'_{(g,G_{ij})} = 1$. This pro-
cess can be accomplished in two steps: i) an element is
chosen from $B_{(G_{ij})}$; ii) an element is chosen from $C_{(g,G_{ij})}$.
The first step only concerns geometric aspects; then, it
respects the equivalence of all coordinate systems. For in-
stance, we consider the Minkowski metric and the exact
plane wave $\Phi$. They are two different geometries asso-
ciated with the null energy-momentum tensor. A choice
of some of them in the set $B_{(G_{ij}=0)}$ does not privilege any
coordinate system.

The choice in step (ii) is the one that produces a con-
ict with the principle of general relativity. Since it is
involved with the coordinate system, that choice must
be dictated by the only quantity associated with a given
coordinate system: the components $G_{ij}$ of $G$. A clear
example of this conflict is the case where $G_{ij}$ is null in a
finite region of space-time. In fact, since general coor-
dinate changes are local, we can build two coordinate bases
$\{e(x)\}$ and $\{e'(x)\}$ such that they agree at all point of
space-time, except in the empty region $\mathcal{V}$. Therefore,
the components of the energy-momentum tensor agree in
the whole space-time:

$$< T, e_i \otimes e_j > (x) =< T, e'_i \otimes e'_j > (x), \quad \forall x$$  \hspace{1cm} (16)

In spite of this, the components of the metric tensor will
differ in the region where the coordinate bases do not agree:

$$< g, e_i \otimes e_j > (x) \neq< g, e'_i \otimes e'_j > (x), \quad x \in \mathcal{V}$$  \hspace{1cm} (17)

The sole acceptable changes of a relational metric tensor,
when evaluated in different coordinate systems, should
be those arising from changes of the components of the
energy-momentum tensor in some region of space-time.
5. OBSTACLES TO IMPLEMENTING A PRINCIPLE OF GENERAL RELATIVITY

In the former section we showed the existence of non-relational metrics in Einstein-equation: those whose Einstein tensor $G(g)$ is null in a finite region of space-time. The non-relational character of this kind of metric is clear: the values of the components $g_{ij} = <g, e_i \otimes e_j>$ of a metric $g$ in two coordinate systems differing only inside the region where $T$ is null, cannot be attributed to differences in the appearances of matter distribution because the components of $T$ are identical in both systems. As a corollary, a distribution of matter compatible with the principle of general relativity must “fill” all the space-time. Evidently, the requirement of an everywhere non-null energy-momentum tensor is produced by the interest in having a theory with a general equivalence group which includes local transformations. This requirement would not be needed in a theory with a global equivalence group (in the sense that its transformations would not depend on $x$, as it happens with the Cartesian transformations in Electrostatics). In fact, in that case it would be impossible to perform a coordinate change in the empty region $\mathcal{V}$ without simultaneously modifying the components of the energy-momentum tensor in the region where they are non-null.

Actually, the Einstein equations are far from being a pure expression of the principle of general relativity in two ways. On the one hand, the mere covariance of their form is not enough to express the principle. On the other hand, the Einstein equations govern the geometry, which is a property that has nothing to do with coordinates. Therefore, both steps mentioned in Section 4 are entangled in Einstein equations. This fact hinders the analysis of the problem we are interested in, which is the one corresponding to step (ii). A theory straightforwardly expressing the equivalence of all coordinate systems should only establish that the choice of $g_{ij}$ in step (ii) depend exclusively on the matter distribution given by the components of the energy-momentum tensor in the considered coordinate system. This would express Mach’s idea that the inertial forces in each frame are exclusively determined by the motion of the frame relative to the matter of the universe. But Einstein not only wanted to formulate the principle of general relativity but also geometrize the gravitational field, which led to the entanglement of the issues considered in steps (i-ii).

6. A SUPPLEMENTARY CONDITION IN A THEORY WITH AN GENERAL EQUIVALENCE GROUP

In the light of the analysis of Electrostatics made in Section 2, it is natural to consider supplementing the Einstein equations with a condition analogous to Eq. (11). The supplementary condition should avoid those solutions privileging coordinate systems beyond the natural privilege that can be induced by the source. Then, the supplementary condition should reduce the domain of the functional $G$ by imposing a dependency between the components of $g$ and the components of $T$. As in Electrostatics, this procedure would not lead to a unique relation metric-source. Let us denote with $g_{ij}^{(T_s)}$ the components of a metric fulfilling the supplementary condition. We will call this metric relational. Following the electrostatic example, the relational character will be expressed by an equation similar to Eq. (11). Since we are going to implement a general equivalence group, we must replace the vectors $\xi$ generating Euclidean transformations by others generating arbitrary coordinate changes. Therefore we propose supplementing the Einstein equations with the condition

$$g_{ij}^{(T_s(\xi))} = g_{ij}^{(T_s)} \equiv \varepsilon (G \xi g^{(T_s)})_{ij} \quad (18)$$

where $\equiv$ means that the equality is true in first order in $\varepsilon$, and $T_s(\xi) \equiv T_{rs} + \varepsilon (G \xi T)_{rs}$. While $\varepsilon (G \xi g^{(T_s)})_{ij}$ is the change of components of the metric under the infinitesimal coordinate change generated by the vector $\xi$, the left member of Eq. (18) compares two solutions of Einstein equations associated, respectively, with the sources $T_{rs}$ and $T_{rs} + \varepsilon (L \xi T)_{rs}$. However, $T_{rs}(\xi)$ are nothing but the components of $T$ in the transformed coordinate system.

We cannot state that the Eqs. (15)-18 univocally determine the metric from the distribution of matter. But this point has nothing to do with the principle of general relativity. The Eqs. (15)-18 could contain different relations metric-source, even those corresponding to different geometries. This feature is also present in the Eqs. (2), which admit solutions such as (12). A different solution to (2) for a same source $\rho$ differs in properties which are invariant under Euclidean transformations. Instead, we claim that, once the geometry has been fixed, Eq. (18) carries out the additional desired work: it turns injective the relation components of $T$ - components of $g$. In fact, Eq. (18) establishes that different components of the metric are only ascribable to different components of the energy-momentum tensor $T$. Thus, Eq. (18) reduces the domain of $G$ to accomplish the principle of general relativity. We compare Electrostatics and Gravitation in Table 1.

Although the Eq. (18) is equivalent to Eq. (11), we cannot directly pass to the analogous of Eq. (15)-18 because the Einstein equations are not linear. This fact hinders the detailed analysis of the content of Eq. (18). However, two corollaries of Eq. (18) are evident:

1) The Eq. (18) excludes those distributions of matter that are null in some space-time regions. In fact, if $T$ was null in a region of space-time, we could arbitrarily choose vectors $\xi$ being different from zero only in that region. In this case, $T_{rs}(\xi)$ would be equal to $T$ in the
whole space-time. Thus, Eq. (18) would imply a null metric.

2) If the vector field $\zeta$ is associated with a symmetry of $T$, i.e., $\mathcal{L}_\zeta(T) = 0$, then $\mathcal{L}_\zeta(q) = 0$. In this way the Eq. (18) raises to the rank of principle the conditions implicitly contained in the catalogue of symmetric solutions for Einstein equations. Moreover, Eq. (18) regulates not only symmetric solutions but even those solutions without symmetries. We will now examine the role played by Eq. (18) in some known solutions of General Relativity.

6.1 The exterior and interior Schwarzschild metric

The exterior Schwarzschild metric can be regarded as the external solution for a spherically symmetric matter distribution subject to the asymptotically flat condition. For this reason it is mentioned as an example of non-Machian metric, since it is determined partly by the source and partly by the boundary condition at the infinite. In fact, according to some definitions of Mach principle, the geometrical properties of the metric must be dictated exclusively by the matter distribution.

Nevertheless, concerning the principle of general relativity, this aspect of the selection of the metric does not cause any trouble because the asymptotically flat boundary condition is a geometrical property (it does not privilege any particular coordinate system).

The true conflict between the exterior Schwarzschild metric and the principle of general relativity is in the fact that this metric corresponds to an energy-momentum tensor which is null in the outer finite region $r > R$ (where $R$ is Schwarzschild’s radius, and $r$ is the standard radial coordinate). Due to this feature, this solution is ruled out by (18). In other words, the coordinate system where the metric asymptotically adopts the standard Minkowskian form should be determined by the appearance of the matter distribution in it. However, such desired state cannot be reached because the energy-momentum tensor is null in the exterior region.

Instead, the symmetry of the interior metric is not implied by Einstein equations but imposed by Eq. (18). If the spherically symmetric source fills space-time, then the interior solution satisfies the two necessary conditions involved in Eq. (18). The form of this solution is not completely fixed by the source; we can apply the Birkhoff theorem here. But this non-uniqueness aspect of the metric does not cause any a priori conflict with the principle of general relativity.

6.2 The Reissner-Nordstrom metric

The Reissner-Nordstrom metric is the outer solution for a spherically symmetric charged object. As such, it is a good example of a metric that satisfies both necessary conditions involved in Eq. (18) the energy-momentum is non-null in all space-time, and the metric shares the symmetry of the source. Unlike the exterior Schwarzschild metric, the asymptotically flat boundary condition does not exhibit, a priori, a conflict with the principle of general relativity. The difference resides in the fact that there is a non-null matter distribution which distinguishes the coordinate systems in the asymptotic region.

6.3 The metric associated to a rotating mass shell

Brill and Cohen have studied the approximate solution to Einstein equations corresponding to a rotating mass shell [12, 13, 14], and Pfister and Braun extended the approximation to a higher order [15, 16]. Actually, Eq. (18) rules out this kind of solutions, because the space-time is empty outside the shell. Nevertheless, we could consider a different solution in which a distribution of matter is added in the empty region, being sufficiently weak and asymptotically null, so that the metric is slightly changed. We could even consider the solution found in Ref. [17], where rotating concentric shells with variable densities fill space-time. For our purpose, it will be enough to think in the simpler approximate Brill & Cohen solution, which consists in a perturbation of the (non rotating) Schwarzschild solution. They consider a rotating dust shell localized at $r = R$, whose four-velocity field is: $u^\theta = 0 = u^r$, $u^\varphi = u^0 \omega(t)$, where $u^0$ does not depend on $\theta$ and $\varphi$. Brill & Cohen propose the outer solution

$$ds^2 = -(\frac{2r-m}{2r+m})^2 dt^2 + (1 + \frac{m}{2r})^4 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta (d\varphi - \Omega_n(t)dt)^2)]$$

which can be regarded as a perturbation of Schwarzschild’s solution in isotropic coordinates. This metric is subject to the following requirements:

1. It must fulfill Einstein equations in first order in $\Omega$ and $\omega$. 

2. It can be matched with the flat interior metric, regarding the junction conditions associated with the presence of the shell at $r = R$.

3. The metric must be asymptotically Minkowskian. Then, $\lim_{r \to \infty} \Omega_n(r,t) = 0$.

The inner metric is

$$ds^2 = -(\frac{2r-m}{2r+m})^2 dt^2 + (1 + \frac{m}{2r})^4 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta (d\varphi - \Omega_n(t)dt)^2)]$$

where $\Omega_n(t)$ is determined by junction conditions. In Eq. (20), the form $d\varphi - \Omega_n(t)dt$ is exact ($d\varphi - \Omega_n(t)dt = d\varphi - \int \Omega_n(t)dt = d\varphi$). So the inner metric can adopt the Minkowskian form by means of a coordinate change. It is evident that the distribution of matter admits a coordinate system where it looks spherically symmetric.
However, the metric in this coordinate system could not reflect the symmetry of the source because \(d\varphi - \Omega(t, r)\ dt\) is not an exact 1-form. In other words, the only outer spherically symmetric solution of Einstein equations is Schwarzschild’s solution, but the metric \([18]\) cannot be transformed to the Schwarzschild form. Therefore, Brill & Cohen’s solution is not compatible with the supplementary Eq. \([18]\).

We should add that the former problem becomes more complex when contributions of higher order in \(\Omega\) are considered. In that case, the rotation of the shell cannot be rigid but the speed \(\omega\) must depend on \(\theta\). \([13, 14]\) Although the spherical symmetry of the source cannot be retained for higher orders in \(\Omega\), the supplementary Eq. \([18]\) would be enough to rule out this kind of solution in the lowest order. Since the supplementary requirement \([18]\) could be applied to each order, then we can state that this kind of solution is not relational in the sense above defined.

7. SUMMARY

We have exploited the electrostatic simile to study the way of implementing a principle of general relativity. This strategy begins by analyzing the additional conditions that we should add to the Eq. \([2]\) in order to rise its covariance group–the set of Euclidean transformations–to the rank of equivalence group. Since \([11]\) demands that solutions to Eq. \([2]\) must be relational with its source, the equivalence group of the extended system \([2, 11]\) is the Euclidean transformations group. Nevertheless, if we do not impose the condition of field linearity of the field with respect to its source, the system \([2, 11]\) does not contain a unique relation field-source. Thus, this discussion allowed us to distinguish between the requirement of univocal determination by the source and the requirement of equivalence of all Euclidean systems in the relation field-source. After imposing the requirement of linearity with respect to the source, these two requirements agree. In such case, the system \([2, 11]\) turns out to be equivalent to the system \([2, 3, 9]\), whose sole solution is the Coulombian field \(\text{I}1\).

In Einstein equations, the univocal determination of the metric by the matter distribution is a condition stronger than the requirement of equivalence of all coordinate systems dictated by the principle of general relativity. We have shown in section 3.4 that there are two steps in the specification of a given metric. The principle of general relativity implies that the choice in the step (ii) should be dictated by the matter distribution, the choice in the step (i) being arbitrary. For this purpose, we have supplemented Einstein equations with the condition \([18]\), which says that the metric solution to Eq. \([2]\) should be relational. In this way, the equivalence group of the extended system \([16, 18]\) is the set of all coordinate transformations.

This so extended system of equations contains two straightforward general results:

I) The allowed energy-momentum tensor cannot be null in any region of the space-time. Therefore Mach’s criticism to the Newton bucket experiment should be reformulated by demanding that the parabolic form of the water surface is due to the motion relative to the electromagnetic radiation filling the universe.

II) The symmetries of the matter distribution are also symmetries of the metric. Although it is well known that symmetric solutions of Einstein equations do have this property, this is not a mandatory consequence of Einstein equations. The Eq. \([18]\) shows that this association is a consequence of applying the principle of general relativity.

Then, we have given a positive complement to Kretschmann’s objection: the Eq. \([18]\) replaces the requirement of general covariance which is empty of physical content. In spite of the effort required for a more exhaustive analysis of the content of Eq. \([18]\), we can state that Eq. \([18]\), together with Einstein equations, implements the principle of general relativity in the so called Theory of General Relativity.

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[1] E. Kretschman, Ann. Phys. 53, 576 (1917).
[2] J.L. Anderson, Gen. Rel. Grav. 2, 161 (1971).
[3] J.L. Anderson, Principles of relativity physics, Academic Press (N.Y.) (1967).
[4] M. Friedman, Foundations of Space-Time Theories, Princeton University Press (1986).
[5] J.B. Barbour and H. Pfister (eds.)Mach’s Principle: From Newton’s Bucket to Quantum Gravity, Einstein Studies, Vol. 6, Birhauser (1995).
[6] H. Bondi and J. Samuel, The Lense-Thirring Effect and Mach’s Principle, gr-qc/9607009.
[7] C.G. Gilman, Phys. Rev. D 2 1300 (1970).
[8] D.W. Sciama, P.C. Waylen and R.C. Gilman, Phys. Rev. 187, 1762 (1969).
[9] D. Lyden-Bell, Mon. Not. R. Astron. Soc. 135, 413 (1967).
[10] H. Bondi, F.A.E. Pirani and I. Robinson, Proc. Roy. Soc. London A 251, 519 (1959).
[11] D.J. Raine, Mon. Not. R. Astron. Soc. 171, 507 (1975).
[12] D.R. Brill and J.M. Cohen, Phys. Rev. 143, 1011 (1966).
[13] J.M. Cohen, Phys. Rev. 175, 1258 (1967).
[14] L. Lindblom and D.R. Brill, Phys. Rev. 10, 3151 (1974).
[15] H. Pfister and K.H. Braun, Class. Quant. Grav. 2, 909 (1985).
[16] H. Pfister and K.H. Braun, Class. Quant. Grav. 3, 335 (1985).
Remarkably, the mere covariance of the Eq. 2 does not imply the equivalence of the elements of $GE$. In fact the covariance group of Eq. 2 could be increased by using the covariant derivative associated with a flat connection $\Gamma$. In this case, the Eq. 2 would acquire a covariant form under general coordinate changes. However, the cost of increasing the covariance is the need to introduce the connection $\Gamma$ as an additional object. In other words, the equation 2 is identical in all the coordinate systems if written with $\vec{E}, \rho, \Gamma$; but it is not if written with $\vec{E}, \rho$. Unless some additional structure is introduced, the covariance group of Eq. 2 is $GE$.

For a formal definition of related terms connected with the covariance of a space-time theory, see Ref. [4].

### Table I: Comparison between Electrostatics and Gravitation

|                        | Electrostatics                                      | Gravitation                      |
|------------------------|-----------------------------------------------------|----------------------------------|
| I.- Differential equation | $\vec{\nabla} \cdot \vec{E} = \rho$                | $G(\mathfrak{g}) = kT$           |
| Covariance group       | Euclidean transformations                           | general coordinate changes       |
| Examples of non-relational solutions | $\vec{E}_0(\vec{r}) = \int \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^2} \ d^3r' + \vec{E}_0 \ \hat{a}$ | any solution containing an empty region |
| II.- Supplementary equation imposing an equivalence group equal to the covariance group of the equation (I) | $\vec{E}^{\rho} + \epsilon \mathcal{L}_\xi \rho - \vec{E}^{\rho} \simeq \epsilon \mathcal{L}_\xi \vec{E}^{\rho}$ | $g^{\rho}_{\alpha \beta} - g^{\rho}_{\beta \alpha} \simeq \epsilon (\mathcal{L}_\xi g^{\rho}_{\alpha \beta})_{ij}$ |
| Examples of injective mappings fulfilling (I) + (II) | $\vec{E}_C(\vec{r}) = \int \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^2} \ d^3r'$ | not found yet |

(II) is equivalent to:

$$\vec{\nabla} \times \vec{E} = \vec{J}(\rho)$$

$$\lim_{r \to \infty} \vec{E}(\vec{r}) = \vec{E}(\rho)$$

for all vectors $\vec{E}$ and $\vec{J}$ built exclusively from the charge distribution. There are many vectors $\vec{E}$ and $\vec{J}$ for a given distribution $\rho$.

existence of different geometries associated to the same tensor $G$