CLASSICAL ELECTRON THEORY
AND CONSERVATION LAWS

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ABSTRACT: It is shown that the traditional conservation laws for total charge, energy, linear and angular momentum, hold jointly in classical electron theory if and only if classical electron spin is included as dynamical degree of freedom.

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I. INTRODUCTION

‘Classical electron theory’ aims at a consistent description of the dynamics of electromagnetic fields and charged particles. While already equipped with some heuristic rules to the effect that charged particles occur in stable, ‘quantized’ atomic units, classical electron theory is dynamically a pre-quantum theory, built on Maxwell’s electromagnetism, Newton’s or Einstein’s classical mechanics, and the Lorentz force law [20]. It originated around the time of J.J. Thomson’s discovery of the corpuscular nature of cathode rays [37] with the work of H. A. Lorentz [21], just before the advent of relativity [23,8,28,25], and is most prominently associated with the names of Lorentz [21,22] and Abraham [1] (see also [2,24,33]). Since the simplest classical mathematical structure for an atomic charge, the point charge, produces infinite Lorentz self-forces and electromagnetic self-energies associated with the singularities of the Liénard-Wiechert electromagnetic field [19,40] of a moving point charge, Abraham and Lorentz were led to consider models with extended microscopic charges that experience a volume-averaged Lorentz force. While problems of how to bring classical electron theory in line with special relativity still persisted [28,24,11], quantum mechanics was invented, and classical electron theory dropped from the list of contenders for a fundamental theory of electromagnetic matter and fields. Nowadays classical electron theory has degenerated into a subject of mere historical value — so it would seem, especially in view of the impressive range of electromagnetic phenomena covered with spectacular precision by Quantum Electrodynamics (QED) [13,5]. However, classical electron theory has continually been revisited by physicists, an incomplete list being [11,7,30,26,31,32,41,12]. One reason, apparently, is that the perturbative series defining QED, while renormalizable in each order [9], is most likely to be merely asymptotic in character rather than convergent [10], such that precision results are achieved only when computations are terminated after a few iterations, with no rules as to what is meant by ‘few’. As R. Peierls put it, [27]: “because the use, as basic principle, of a semiconvergent series is unsatisfactory, there is still interest in theories that avoid singularities.”

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Recently, considerable advances have been made in controlling the type of dynamical equations that govern classical electron theory. A mechanical particle coupled to a scalar wave field has been treated in [14,15,18]. The equations of a classical electron coupled to the Maxwell fields are considered in [6,16,17,34]. All these papers deal with a semi-relativistic theory. This means that, while the wave field satisfies relativistic equations and the ‘material’ particle momentum is given by Einstein’s rather than Newton’s expression, with Abraham [2] one assumes that the particle rigidly retains its shape. A fully relativistic model, first devised in a monumental work by Nodvik [26] and most recently completed in [3], is considerably more involved but has begun to yield to a mathematical onslaught as well [3]. As a result, classical electron theory is about to become established as the first mathematically well posed, fully relativistic theory of electromagnetism that consistently describes the dynamics of discrete charges and the continuum electromagnetic field.

In view of the continued interest in classical electron theory it seems worthwhile to draw attention to a small observation regarding conservation laws which, to the author’s knowledge, has not been made before, and which seems to be sufficiently interesting in its own right to warrant publication in this separate note.

To be specific, [6,16,17,34,14,15,18] and most earlier works on classical electron theory only take translational degrees of freedom of the particles into account. Already Abraham [2] and Lorentz [23] insisted on the possibility of additional degrees of freedom of the extended charged particles associated with particle spin (not to be confused with the “rotation of the electrons” in Lorentz’ theory of the Zeeman effect [24], which refers to circular motion of the electron’s center of charge inside a Thomson atom), though it seems that only Abraham wrote down corresponding dynamical equations. However, neither of these authors pursued such a spinning particle motion any further. As a consequence, it seems to have gone completely unnoticed that omitting particle spin generally leads to a violation of the law of conservation of total classical angular momentum! The discrepancy term has the form of an internal torque on the particles. This strongly suggests to add classical spin to the degrees of freedom of the charge distribution.

In this note we will show that, if classical particle spin is included as degree of freedom in semi-relativistic classical electron theory, with Abraham’s spherical charge distribution, then all classical conservation laws are satisfied. We will also demonstrate that the arbitrary setting to zero of the internal angular velocities of the particles is incompatible with the classical law of angular momentum conservation. Interestingly, though, the classical expressions for total charge, energy, and linear momentum are conserved during the motion even if classical particle spin is omitted.

In a fully relativistic formulation [26,3] the kinematical effect of Thomas precession [35] enforces additional self rotation of the particle. However, it is the law of angular momentum conservation which compels us to introduce a ‘classical particle spin’ already at the level of a semi-relativistic formulation of classical electron theory, i.e. independently of Thomas precession.

The rest of the paper is organized as follows. We first present the dynamical equations (section II), next we prove that the traditional conservation laws are satisfied (section III), then we show (section IV) that angular momentum is not conserved if spin is omitted. In section V, we conclude with a brief historical musing.
II. THE EQUATIONS OF SEMI-RELATIVISTIC ELECTRON THEORY

We denote by \( \mathbf{E}(x, t) \in \mathbb{R}^3 \) the electric field, and by \( \mathbf{B}(x, t) \in \mathbb{R}^3 \) the magnetic induction, at the point \( x \in \mathbb{R}^3 \) at time \( t \in \mathbb{R} \). Let \( \mathcal{I} \) be a finite subset of the natural numbers \( \mathbb{N} \), of cardinality \( |\mathcal{I}| = N \). We consider \( N \) particles indexed by \( \mathcal{I} \). With Abraham, we assign to particle \( k \) a rigid shape given by a nonnegative form function \( f_k \in C_0^\infty(\mathbb{R}^3) \), having \( SO(3) \) symmetry and satisfying \( \int_{\mathbb{R}^3} f_k(x) d^3x = 1 \). The total charge \( q_k \) of particle \( k \) is distributed in space around the point \( y_k(t) \in \mathbb{R}^3 \) by \( \rho_k(x, t) = q_k f_k(x - y_k(t)) \), and rigidly rotating with angular velocity \( \mathbf{w}_k(t) \). We also assign to the particle a 'bare inertial mass' \( m_k \) ('material mass' in [24]) and a 'bare moment of inertia' \( I_k \). Moreover, \( \nabla \) denotes the gradient operator with respect to \( x \), and a dot on top of a quantity will signify derivative with respect to time, e.g. \( \dot{y}_k(t) \) is the linear velocity of particle \( k \).

In semi-relativistic classical electron theory, the fields \( \mathbf{E} \) and \( \mathbf{B} \) satisfy the Maxwell-Lorentz equations

\[
\begin{align*}
\frac{1}{c} \partial_t \mathbf{B}(x, t) + \nabla \times \mathbf{E}(x, t) &= 0, \\
-\frac{1}{c} \partial_t \mathbf{E}(x, t) + \nabla \times \mathbf{B}(x, t) &= 4\pi \frac{1}{c} \mathbf{j}(x, t) \\
\nabla \cdot \mathbf{B}(x, t) &= 0, \\
\nabla \cdot \mathbf{E}(x, t) &= 4\pi \rho(x, t)
\end{align*}
\]  

(2.1) (2.2) (2.3) (2.4)

where charge and current densities, \( \rho(x, t) \) and \( \mathbf{j}(x, t) \), are given by the Abraham-Lorentz expressions

\[
\begin{align*}
\rho(x, t) &= \sum_{k \in \mathcal{I}} q_k f_k(x - y_k(t)), \\
\mathbf{j}(x, t) &= \sum_{k \in \mathcal{I}} q_k f_k(x - y_k(t)) \mathbf{v}_k(x, t),
\end{align*}
\]  

(2.5) (2.6)

with

\[
\mathbf{v}_k(x, t) = \dot{y}_k(t) + \mathbf{w}_k(t) \times (x - y_k(t)).
\]  

(2.7)

The dynamical variables of particle \( k \), momentum \( \mathbf{p}_k(t) \in \mathbb{R}^3 \) and spin \( \mathbf{s}_k(t) \in \mathbb{R}^3 \), satisfy Newton’s, respectively Euler’s equation of motion, equipped with the Abraham-Lorentz expressions for the total force and torque acting on particle \( k \),

\[
\begin{align*}
\dot{\mathbf{p}}_k &= q_k \int_{\mathbb{R}^3} \left[ \mathbf{E}(x, t) + \frac{1}{c} \mathbf{v}_k(x, t) \times \mathbf{B}(x, t) \right] f_k(x - y_k(t)) d^3x, \\
\dot{\mathbf{s}}_k &= q_k \int_{\mathbb{R}^3} (x - y_k(t)) \times \left[ \mathbf{E}(x, t) + \frac{1}{c} \mathbf{v}_k(x, t) \times \mathbf{B}(x, t) \right] f_k(x - y_k(t)) d^3x.
\end{align*}
\]  

(2.8) (2.9)

Here,

\[
\mathbf{s}_k = I_k \mathbf{w}_k
\]  

(2.10)
is the classical particle spin associated with the bare moment of inertia, and

\[ p_k = \begin{cases} 
    \frac{m_k \dot{y}_k}{m_k} & \text{(Newtonian)} \\
    \frac{m_k \dot{y}_k}{\sqrt{1 - \left| \frac{\dot{y}_k}{c} \right|^2}} & \text{(Einsteinian)}
\end{cases} \tag{2.11}
\]

is the particle momentum associated with the bare mass. Defining the translational kinetic energy associated with the bare mass,

\[ T_k(p_k) = \begin{cases} 
    \frac{1}{2} \frac{|p_k|^2}{m_k} & \text{(Newtonian)} \\
    m_k c^2 \sqrt{1 + \frac{|p_k|^2}{m_k^2 c^2}} & \text{(Einsteinian)},
\end{cases} \tag{2.12}
\]

we notice that velocity \( \dot{y}_k \) and momentum \( p_k \) are, in either case, related by

\[ \dot{y}_k = \frac{\partial T_k}{\partial p_k}. \tag{2.13} \]

Both Newtonian [2,24] and Einsteinian [24,6,16,17,34,14,15,18] momenta have been used in semi-relativistic variants of classical electron theory. We therefore discuss both cases of (2.11), but only the nonrelativistic Euler form (2.10) for spin.

Naturally, we want to treat these equations as a Cauchy problem, with initial data posed at time \( t = t_0 \). For the mechanical variables of the particles, the data are \( y_k(t_0), \dot{y}_k(t_0), \) and \( w_k(t_0) \); and for the fields, \( B(x,t_0) \) satisfying (2.3), and \( E(x,t_0) \) satisfying (2.4) at \( t = t_0 \). Actually, one should also think of (2.3) and (2.4) rather as initial conditions, to be imposed only at \( t = t_0 \), on the initial data \( B(x,t_0) \) and \( E(x,t_0) \), for the above set of equations is slightly redundant. In fact, (2.3) and (2.4) are automatically satisfied for all \( t \) if they are satisfied at \( t = t_0 \). For (2.3) this is seen by taking the divergence of (2.1). For (2.4) this is seen by taking the divergence of (2.2) and the time-derivative of (2.4), then using the continuity equation for the charge, which is proven below to hold as consequence of (2.5), (2.6), and (2.7) alone.

Finally, a few remarks are in order regarding the bare inertias, \( m_k \) and \( I_k \). By following up on Thomson’s discovery [36] that the electromagnetic field of a particle contributes to its inertia, Abraham [2] in particular, but also Lorentz [24,23], suggested that inertia is entirely due to electromagnetic effects, and consequently proposed to set \( m_k = 0 = I_k \) in (2.11) and (2.10). However, setting \( m_k = 0 \) and/or \( I_k = 0 \) is in serious conflict with the mathematical structure of a Cauchy problem, see [3] for more on this. In another vein, ever since Dirac’s work [7] there has been quite some interest in letting \( m_k \rightarrow -\infty \) associated with the mass-renormalized point particle limit \( f_k(x - y_k) \rightarrow \delta(x - y_k) \), see [4,29,34]. However, stability problems emerge when \( m_k < 0 \) and/or \( I_k < 0 \), cf. [6] for \( m_k < 0 \) when only translational motions are considered. All these problems together suggest that one should choose the bare inertias strictly positive, i.e. \( m_k > 0 \) and \( I_k > 0 \). Formally though, as can be seen upon inspection of our proofs below, the conservation laws hold for all regular solutions of (2.1)–(2.11), with any values of \( m_k \in \mathbb{R} \) and \( I_k \in \mathbb{R} \).
III. CONSERVATION LAWS

We assume that the initial conditions correspond to finite charge, total energy, linear and angular momentum. Then, because of the finite propagation speed for the electromagnetic fields and the non-singular shape function, it is reasonable to expect (but not proven here) that the particle speeds remain bounded and the motions and fields regular, so that all surface integrals over the fields at infinity vanish at all times. We now prove that, as a consequence of these hypotheses, the traditional expressions of total charge, total energy, total linear and total angular momentum, are conserved quantities for the dynamical equations (2.1)-(2.11).

IIIa. Charge conservation

The total charge

$$Q = \int_{\mathbb{R}^3} \rho(x, t) \, d^3x \quad (3.1)$$

is conserved.

We need to show that $\dot{Q} = 0$. For this it suffices to prove that the continuity equation

$$\partial_t \rho(x, t) + \nabla \cdot j(x, t) = 0 \quad (3.2)$$

is satisfied.

We take the partial derivative of (2.5) with respect to time, finding

$$\partial_t \rho(x, t) = -\sum_{k \in \mathcal{I}} q_k \dot{y}_k(t) \cdot \nabla f_k(x - y_k(t)), \quad (3.3)$$

where we used that $\partial_y f_k(x - y) = -\partial_x f_k(x - y)$, with the identification $\partial_x f_k(x - y) = \nabla f_k(x - y)$. Next we take the divergence of (2.6) and obtain

$$\nabla \cdot j(x, t) = \sum_{k \in \mathcal{I}} q_k \left( v_k(x, t) \cdot \nabla f_k(x - y_k(t)) + f_k(x - y_k(t)) \nabla \cdot v_k(x, t) \right). \quad (3.4)$$

Noting that

$$w_k(t) \times (x - y_k(t)) = -\nabla \times \left( \frac{1}{2} w_k(t) |x - y_k(t)|^2 \right), \quad (3.5)$$

it follows that

$$\nabla \cdot v_k(x, t) = \nabla \cdot \left( \dot{y}_k(t) + w_k(t) \times (x - y_k(t)) \right) = 0, \quad (3.6)$$

whence,

$$\nabla \cdot j(x, t) = \sum_{k \in \mathcal{I}} q_k \dot{y}_k(t) \cdot \nabla f_k(x - y_k(t)). \quad (3.7)$$

In view of (3.7) and (3.3), (3.2) holds. Thus, conservation of charge (3.1) is proved.
IIIb. Energy conservation

The total energy

\[ W = \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( |E|^2 + |B|^2 \right) d^3x + \sum_{k \in I} \left( T_k + \frac{1}{2I_k} |s_k|^2 \right) \]  

is conserved.

We need to show that

\[ \dot{W} = \frac{1}{4\pi} \int_{\mathbb{R}^3} (E \cdot \partial_t E + B \cdot \partial_t B) d^3x + \sum_{k \in I} (\dot{y}_k \cdot \dot{p}_k + w_k \cdot \dot{s}_k) = 0. \]  

In the field integral in (3.9) we use (2.1) to express \( \partial_t B \) in terms of \( \nabla \times E \), and (2.2) to express \( \partial_t E \) in terms of \( \nabla \times B \) and \( j \), then use the standard identity (e.g. [12])

\[ B \cdot \nabla \times E - E \cdot \nabla \times B = \nabla \cdot (E \times B), \]  

apply Gauss’ theorem, notice that the surface integral at infinity vanishes, and get

\[ \int_{\mathbb{R}^3} (E \cdot \partial_t E + B \cdot \partial_t B) d^3x = -\int_{\mathbb{R}^3} (c \nabla \cdot (E \times B) + 4\pi E \cdot j) d^3x \]

\[ = -4\pi \int_{\mathbb{R}^3} E \cdot j d^3x. \]  

As for the sum over particles, we insert the right-hand side of (2.8) for \( \dot{p}_k \), and the right-hand side of (2.9) for \( \dot{s}_k \). We notice that (2.8) contains a term \( \perp \dot{y}_k \), which vanishes under the dot product with \( \dot{y}_k \) in (3.9). Similarly, inserting \( X = x - y_k, \ Y = w_k \) and \( Z = B \) in the vector identity \( X \times (X \times Y) \times Z = (X \times Y)(X \cdot Z) \), we see that (2.9) contains a term \( \perp w_k \), which vanishes under the dot product with \( w_k \) in (3.9). This, and a little vector algebra, gives us

\[ \sum_{k \in I} (\dot{y}_k \cdot \dot{p}_k + w_k \cdot \dot{s}_k) \]

\[ = \sum_{k \in I} q_k \int_{\mathbb{R}^3} \left( \dot{y}_k \cdot \left[ E(x, t) + \frac{1}{c} \left( w_k(t) \times (x - y_k(t)) \right) \right] \times B(x, t) \right. \]

\[ + \left. w_k \cdot \left( (x - y_k(t)) \times \left[ E(x, t) + \frac{1}{c} \dot{y}_k(t) \times B(x, t) \right] \right) \right) f_k(x - y_k(t)) d^3x \]

\[ = \sum_{k \in I} q_k \int_{\mathbb{R}^3} f_k(x - y_k(t)) \left( \dot{y}_k + w_k(t) \times (x - y_k(t)) \right) \cdot E(x, t) d^3x \]

\[ = \int_{\mathbb{R}^3} \sum_{k \in I} q_k f_k(x - y_k(t)) v_k(x, t) \cdot E(x, t) d^3x. \]  

(3.12)
Recalling (2.5) and (2.6), we finally get

$$\sum_{k \in I} (\dot{y}_k \cdot \dot{p}_k + w_k \cdot \dot{s}_k) = \int_{\mathbb{R}^3} E \cdot j \, d^3 x. \quad (3.13)$$

With (3.13) and (3.11) we see that the integral and sum in (3.9) cancel in a manifest way, yielding $\dot{W} = 0$. Energy conservation is proved.

**IIIc. Momentum conservation**

*The total linear momentum*

$$P = \frac{1}{4\pi c} \int_{\mathbb{R}^3} E \times B \, d^3 x + \sum_{k \in I} p_k \quad (3.14)$$

is conserved.

We need to prove that

$$\dot{P} = \frac{1}{4\pi c} \int_{\mathbb{R}^3} (E \times \partial_t B - B \times \partial_t E) \, d^3 x + \sum_{k \in I} \dot{p}_k = 0. \quad (3.15)$$

In the field integral in (3.15) we use (2.1) to express $\partial_t B$ in terms of $\nabla \times E$ and (2.2) to express $\partial_t E$ in terms of $\nabla \times B$ and $\mathbf{j}$, then integrate the standard vanishing identity

$$0 = B \nabla \cdot B + E \nabla \cdot E - 4\pi \rho E \quad (3.16)$$

over $\mathbb{R}^3$, divide by $4\pi$, and add the result to our integral. We next recall that

$$E \nabla \cdot E + B \nabla \cdot B - E \times \nabla \times E - B \times \nabla \times B = 4\pi \nabla \cdot \mathcal{M}, \quad (3.17)$$

where

$$\mathcal{M} = \frac{1}{4\pi} \left( E \otimes E + B \otimes B - \frac{1}{2} \left(|E|^2 + |B|^2\right) I \right) \quad (3.18)$$

is Maxwell’s symmetric stress tensor, with $I$ the identity $3 \times 3$ tensor. But

$$\int_{\mathbb{R}^3} \nabla \cdot \mathcal{M} \, d^3 x = 0, \quad (3.19)$$

since, by Gauss’ theorem, the integral on the left can be transformed into a surface integral at $\infty$, where it vanishes, by our hypotheses. Thus, we have

$$\frac{1}{4\pi c} \int_{\mathbb{R}^3} (E \times \partial_t B - B \times \partial_t E) \, d^3 x = -\int_{\mathbb{R}^3} \left(\rho E + \frac{1}{c} j \times B\right) \, d^3 x, \quad (3.20)$$

with $\rho$ given by (2.5), and $j$ by (2.6).
As for the sum over particles, we insert the right-hand side of (2.8) for $\dot{p}_k$, then exchange summation and integration, recall (2.5) and (2.6), and obtain,

$$
\sum_{k \in \mathcal{I}} \dot{p}_k = \int_{\mathbb{R}^3} \sum_{k \in \mathcal{I}} q_k f_k(x - y_k(t)) \left[ E(x, t) + \frac{1}{c} v_k(x, t) \times B(x, t) \right] d^3x \\
= \int_{\mathbb{R}^3} \left( \rho E + \frac{1}{c} j \times B \right) d^3x. \quad (3.21)
$$

With (3.20) and (3.21) inserted into (3.15), we obtain $\dot{P} = 0$. Linear momentum conservation is proved.

**IIIId. Angular momentum conservation**

*The total angular momentum*

$$
J = \frac{1}{4\pi c} \int_{\mathbb{R}^3} x \times (E \times B) d^3x + \sum_{k \in \mathcal{I}} (y_k \times p_k + s_k) \quad (3.22)
$$

is conserved.

We need to show that

$$
\dot{J} = \frac{1}{4\pi c} \int_{\mathbb{R}^3} x \times (E \times \partial_t B - B \times \partial_t E) d^3x + \sum_{k \in \mathcal{I}} (y_k \times \dot{p}_k + \dot{s}_k) = 0. \quad (3.23)
$$

Turning first to the field integral in (3.23), we use (2.1) to express $\partial_t B$ in terms of $\nabla \times E$ and (2.2) to express $\partial_t E$ in terms of $\nabla \times B$ and $j$. Next we take the cross product of (3.16) with $x$, obtaining the vanishing identity

$$
0 = x \times (B \nabla \cdot B + E \nabla \cdot E - 4\pi \rho E). \quad (3.24)
$$

We integrate (3.24) over $\mathbb{R}^3$, divide by $4\pi$, and add the result to our integral in (3.23). With the help of (3.17), this gives us

$$
\frac{1}{4\pi c} \int_{\mathbb{R}^3} x \times (E \times \partial_t B - B \times \partial_t E) d^3x = \int_{\mathbb{R}^3} x \times \left( \nabla \cdot \mathcal{M} - \rho E - \frac{1}{c} j \times B \right) d^3x, \quad (3.25)
$$

with $\rho$ given by (2.5) and $j$ by (2.6). Finally, recalling the identity [12]

$$
\int_{\mathbb{R}^3} x \times \nabla \cdot \mathcal{M} d^3x = \lim_{R \to \infty} \oint_{S_R^2} (\mathcal{M} \cdot x) \times n d\sigma = 0, \quad (3.26)
$$

(where we also used that $\nabla \times x = 0$), we finally have

$$
\frac{1}{4\pi c} \int_{\mathbb{R}^3} x \times (E \times \partial_t B - B \times \partial_t E) d^3x = -\int_{\mathbb{R}^3} x \times \left( \rho E + \frac{1}{c} j \times B \right) d^3x. \quad (3.27)
$$
Coming to the sum over particles, we insert the right-hand side of (2.8) for $\dot{p}_k$, the right-hand side of (2.9) for $\dot{s}_k$, notice some obvious cancelations, and obtain

$$\sum_{k \in I} (y_k \times \dot{p}_k + \dot{s}_k)$$

$$= \sum_{k \in I} q_k \left( y_k \times \int_{\mathbb{R}^3} \left[ E(x, t) + \frac{1}{c} v_k(x, t) \times B(x, t) \right] f_k(x - y_k(t)) \, d^3x \right.$$

$$+ \left. \int_{\mathbb{R}^3} (x - y_k(t)) \times \left[ E(x, t) + \frac{1}{c} v_k(x, t) \times B(x, t) \right] f_k(x - y_k(t)) \, d^3x \right)$$

$$= \sum_{k \in I} q_k \int_{\mathbb{R}^3} x \times \left[ E(x, t) + \frac{1}{c} v_k(x, t) \times B(x, t) \right] f_k(x - y_k(t)) \, d^3x. \quad (3.28)$$

In the last expression we can exchange summation and integration. Recalling (2.5) and (2.6), we find

$$\sum_{k \in I} (y_k \times \dot{p}_k + \dot{s}_k) = \int_{\mathbb{R}^3} x \times \left( \rho E + \frac{1}{c} \dot{J} \times B \right) \, d^3x. \quad (3.29)$$

With (3.29) and (3.27) inserted in (3.23), we have $\dot{J} = 0$. Conservation of total angular momentum is proved.

**IV. NON-CONSERVATION OF ANGULAR MOMENTUM WHEN $w \equiv 0$**

It is now easily seen that, upon setting $w_k \equiv 0$ everywhere in the equations of motion, the traditional expressions for charge, energy and linear momentum are still conserved, but the one for angular momentum is not. Indeed, if in the equations of motion we set $w_k \equiv 0$ for all $k$, and then follow through the computations of section III step by step, with $w_k \equiv 0$ in place everywhere, we easily verify that the conclusions of subsections III.a, III.b, and III.c still hold. However, if we go through the steps of subsection III.d, with $w_k \equiv 0$ in place everywhere, we obtain

$$\dot{J} = -\sum_{k \in I} q_k \int_{\mathbb{R}^3} (x - y_k(t)) \times \left[ E(x, t) + \frac{1}{c} \dot{y}_k(t) \times B(x, t) \right] f_k(x - y_k(t)) \, d^3x. \quad (4.1)$$

The right side in (4.1) is, in general, an uncompensated sum of torques. Hence, except for some special highly symmetric situations, there will be a non-vanishing rate of change of total angular momentum.

**V. CLOSING REMARK**

Our observation could have been made at the beginning of the 20th century, by Abraham, Lorentz or Poincaré, but apparently it wasn’t. So it was left to Uhlenbeck and Goudsmit [39] to re-invent particle spin for the interpretation of spectral data. It is amusing to contemplate that the story of spin [38] could have been a different one.

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