Axially symmetric solution in Teleparallel Theory of Gravitation

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An exact solution has an axial symmetry is obtained in the teleparallel theory of gravitation. The associated metric has the structure function \( G(\xi) = 1 - \xi^2 - 2mA\xi^3 \). The cubic nature of the structure function can make calculations cumbersome. Using a coordinate transformation we get a tetrad that its associated metric has the structure function in a factorisable form. This new form has the advantage that its roots are now trivial to write down. The singularities of the obtained tetrad are studied. Using another coordinate transformation we get a tetrad that its associated metric gives the Schwarzschild spacetime. Calculate the energy content of this tetrad we get a meaningless result!


1. Introduction

The C-metric is well known to describe a pair of black holes undergoing uniform acceleration. It is usually written in a form first given by Kinnersley and Walker [1]

\[
ds^2 = \frac{1}{A^2(x-y)^2} \left[ G(y)dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + G(x)d\phi^2 \right],
\]

where the stricture function \( G \) is defined by

\[
G(\xi) = 1 - \xi^2 - 2mA\xi^3.
\]

Here \( m \) and \( A \) are positive parameters related to the mass and acceleration of the black hole, such that \( mA < 1/\sqrt{27} \). The fact that \( G \) is a cubic polynomial in \( \xi \) means that one can not in general write down simple expression for its roots. Since these roots play an important role in almost every analysis of the C-metric, most results have to be expressed implicitly in terms of them. Any calculation which requires their explicit forms would naturally be very tedious if not impossible to carry out [2, 3, 4].

Einstein’s general relativity (GR) is very successful in describing long distance (macroscopic) phenomena. This theory, however, encounters serious difficulties on microscopic distances. So far essential problems appear in all attempts to quantize the standard GR [5, 6]. Also, the Lagrangian structure of GR differs, in principle, from the ordinary microscopic gauge theories. In particular, a covariant conserved energy-momentum tensor for the gravitational field can not be constructed in the framework of GR. Consequently, the study of alternative models of gravity is justified from the physical as well as from the mathematical point of view. Even in the case when GR is unique true theory of gravity, consideration of close alternative models can shed light on the properties of GR itself.

Theories of gravity based on the geometry of distance parallelism [7]–[14] are commonly considered as the closest alternative to the general relativity theory. Teleparallel gravity models possess a number of attractive features both from the geometrical and physical viewpoints. Teleparallelism is naturally formulated by gauging external (spacetime) translation and underlain the Weitzenböck spacetime characterized by the metricity condition and by the vanishing of the curvature tensor. Translations are closely related to the group of general coordinate transformations which underlies general relativity. Therefore, the energy-momentum tensor represents the matter source in the field equations of tetradic theories of gravity like in general relativity.

It is the aim of the present work to derive an axially symmetric solution in teleparallel theory of gravitation. In section 2 we give brief review of the teleparallel theory of gravitation. A tetrad having four unknown functions is applied to the field equation of the teleparallel theory of gravity then, a solution of axial symmetry is obtained in section 3. A coordinate transformation is applied to the obtained tetrad in section 3, to put the structure function in a factorisable form. The advantage of this transformation is that it makes the roots of the original solution be factorisable. In section 4, the singularities of this tetrad are studied.
In section 5, another coordinate transformation is applied and a tetrad that its associated metric gives the Schwarzschild spacetime is obtained. The energy content of this tetrad is also calculated in section 5. Discussion and conclusion of the obtained results are given in the final section.

2. The tetrad theory of gravitation

In a spacetime with absolute parallelism the parallel vector fields $e^i_\mu$ define the nonsymmetric connection

$$\Gamma^\lambda_{\mu\nu} \equiv e^i_\lambda e^j_\mu \varepsilon^{ij}_{\mu\nu},$$

where $e^i_\mu = \partial_\mu e^j_\mu$. The curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$ is identically vanishing, however. The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{ij} b^i_\mu b^j_\nu,$$

with the Minkowski metric $\eta_{ij} = \text{diag}(+1, -1, -1, -1)$. We note that, associated with any tetrad field $e^i_\mu$ there is a metric field defined uniquely by (4), while a given metric $g^{\mu\nu}$ does not determine the tetrad field completely; for any local Lorentz transformation of the tetrads $e^i_\mu$ leads to a new set of tetrads which also satisfy (4). The gravitational Lagrangian $L_G$ has the form

$$L_G = \sqrt{-g} L_G = \sqrt{-g} \left( -\frac{1}{3\kappa} (t^{\mu\nu\lambda} t_{\mu\nu\lambda} - \Phi^\mu \Phi_\mu) + \xi a^\mu a_\mu \right),$$

where $t^{\mu\nu\lambda}$, $\Phi_\mu$ and $a_\mu$ are irreducible representation of the torsion tensor defined by

$$t^{\mu\nu\lambda} \equiv \frac{1}{2} (T^{\mu\nu\lambda} + T^{\nu\mu\lambda}) + \frac{1}{6} (g_{\mu\nu} \Phi^\lambda + g_{\lambda\nu} \Phi_\mu) - \frac{1}{3} g_{\mu\nu} \Phi_\lambda,$$

$$\Phi_\mu \equiv T^\lambda_{\mu\lambda}, \quad a_\mu \equiv \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma},$$

with $\epsilon_{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor normalized to

$$\epsilon_{0123} = -\sqrt{-g}, \quad \text{with} \quad g \equiv \det(g_{\mu\nu}),$$

and $T^{\lambda}_{\mu\lambda}$ is the torsion tensor defined by

$$T^\lambda_{\mu\lambda} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}.$$

$\kappa$ and $\xi$ are the Einstein gravitational constant and a free dimensionless parameter.$^\dagger$

The gravitational field equations for the system described by $L_G$ are the following:

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$^\dagger$Latin indices are raising and lowering with the aid of $\eta_{ij}$ and $\eta^{ij}$.

$^\dagger$Throughout this paper we use the relativistic units $c = G = 1$ and $\kappa = 8\pi$. 

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\[ G_{\mu\nu}(\{\}) + H_{\mu\nu} = -\kappa T_{\mu\nu}, \]  
\[ \partial_\mu (J^{ij\mu}) = 0, \]  
where the Einstein tensor \( G_{\mu\nu}(\{\}) \) is defined by
\[ G_{\mu\nu}(\{\}) \overset{\text{def.}}{=} R_{\mu\nu}(\{\}) - \frac{1}{2} g_{\mu\nu} R(\{\}), \]  
where \( R_{\mu\nu}(\{\}) \) is the Ricci tensor and \( R(\{\}) \) is the Ricci scalar. We assume that the energy-momentum tensor of matter fields is symmetric. The energy-momentum tensor of a source field with Lagrangian \( L_M \):
\[ \sqrt{-g} T^{\mu\nu} \overset{\text{def.}}{=} e_i^\mu \delta(-\sqrt{-g} L_M) \frac{\delta}{\delta e_i^\nu}. \]  
Here \( H_{\mu\nu} \) and \( J^{ij\mu} \) are given by
\[ H^{\mu\nu} \overset{\text{def.}}{=} \frac{k}{\lambda} \left[ \frac{1}{2} \{ e^{\mu\rho\sigma\lambda}(T^\rho_{\ \sigma} - T_{\rho\sigma}) + e^{\nu\rho\sigma\lambda}(T^\mu_{\ \rho\sigma} - T_{\rho\sigma}^\mu) \} a_\lambda - \frac{3}{2} a^\mu a^\nu - \frac{3}{4} g^{\mu\nu} a^\lambda a_\lambda \right], \]  
and
\[ J^{ij\mu} \overset{\text{def.}}{=} -\frac{1}{2} e^i_\rho e^j_\sigma e^{\rho\sigma\mu\nu} a_\nu, \]  
respectively, where
\[ \lambda \overset{\text{def.}}{=} \frac{4}{9} \xi + \frac{1}{3k}. \]  
Therefore, both \( H_{\mu\nu} \) and \( J_{ij\mu} \) vanish if the \( a_\mu \) is vanishing. In other words, when the \( a_\mu \) is found to vanish from the antisymmetric part of the field equations, (9), the symmetric part (8) coincides with the Einstein equation. When the dimensionless parameter \( \lambda = 0 \) then the theory reduce to Einstein teleparallel theory equivalent to general relativity.

### 3. Axially symmetric solution

In this section we will assume the parallel vector fields to have the form
\[ (b^i_\mu) = \begin{pmatrix} A(x, y) & 0 & 0 & 0 \\ 0 & B(x, y) & 0 & 0 \\ 0 & 0 & C(x, y) & 0 \\ 0 & 0 & 0 & D(x, y) \end{pmatrix}, \]  
\[ (b^j_\mu) = \begin{pmatrix} A(x, y) & 0 & 0 & 0 \\ 0 & B(x, y) & 0 & 0 \\ 0 & 0 & C(x, y) & 0 \\ 0 & 0 & 0 & D(x, y) \end{pmatrix}. \]
where \( A(x, y), B(x, y), C(x, y), D(x, y) \) are unknown functions. Applying (15) to the field equations (8) and (9) one can obtains the unknown functions in the form

\[
A(x, y) = \frac{\sqrt{G(y)}}{A(x + y)}, \quad B(x, y) = \frac{1}{A(x + y) \sqrt{G(x)}},
\]

\[
C(x, y) = \frac{1}{A(x + y) \sqrt{G(y)}}, \quad D(x, y) = \frac{\sqrt{G(x)}}{A(x + y)},
\]

(16)

where \( G(\xi) = 1 - \xi^2 - 2mA\xi^3 \), \( A \) and \( m \) are positive parameters related to the mass and acceleration of the black hole and satisfying \( Am < \frac{1}{\sqrt{27}} \) [15]. The associated metric of solution (16) has the form (1) which is the C-metric. As it is stated in the introduction that in general one can not easily write down simple expression of the roots of \( G \). Therefore, one must find some coordinate transformation which makes the roots of \( G \) written explicitly and this would in turn simplify certain analysis of the C-metric. This coordinate transformation has the form [15]

\[
x = \sqrt{\frac{1 + 6\bar{m}\bar{A}c_1}{1 + c_1^2 + 4\bar{m}\bar{A}c_1^3}} (\bar{x} - c_1), \quad y = \sqrt{\frac{1 + 6\bar{m}\bar{A}c_1}{1 + c_1^2 + 4\bar{m}\bar{A}c_1^3}} (\bar{y} - c_1),
\]

\[
\phi = \sqrt{(1 + 6\bar{m}\bar{A}c_1)(1 + c_1^2 + 4\bar{m}\bar{A}c_1^3)} \bar{\phi}, \quad t = \sqrt{(1 + 6\bar{m}\bar{A}c_1)(1 + c_1^2 + 4\bar{m}\bar{A}c_1^3)} \bar{t},
\]

(17)

where \( \bar{x}, \bar{y}, \bar{\phi}, \bar{t} \) are the new coordinate and

\[
m = \frac{\bar{m}}{(1 + 6\bar{m}\bar{A}c_1)^{3/2}}, \quad A = \sqrt{(1 + c_1^2 + 4\bar{m}\bar{A}c_1^3)} \bar{A}.
\]

(18)

Applying the coordinate transformation (17) to the tetrad (15) with solution (16) we obtain

\[
(b'_{\mu}) = \begin{pmatrix}
\frac{G(\bar{y})}{A(\bar{x} - \bar{y})} & 0 & 0 & 0 \\
0 & \frac{1}{A(\bar{x} - \bar{y})G(\bar{x})} & 0 & 0 \\
0 & 0 & \frac{1}{A(\bar{x} - \bar{y})G(\bar{y})} & 0 \\
0 & 0 & 0 & \frac{G(\bar{x})}{A(\bar{x} - \bar{y})}
\end{pmatrix},
\]

(19)

with the structure function defined by

\[
G(\xi) \overset{\text{def.}}{=} (1 - \xi^2)(1 + 2\bar{m}\bar{A}\xi)
\]

(20)

and the associated metric has the form (1) with the structure function has the form (20). As is clear from (20) that one can gets the roots easily which has the form

\[
\xi_{1,2} = \pm 1, \quad \xi_3 = -\frac{1}{2\bar{m}\bar{A}}, \quad \text{which obey} \quad \xi_3 < \xi_2 < \xi_1.
\]

(21)
Now we are going to study the physics of tetrad (19) by studying the singularities.

4. Singularities

In teleparallel theories we mean by singularity of spacetime [16] the singularity of the scalar concomitants of the curvature and torsion tensors.

Using the definitions of the Riemann Christoffel, Ricci tensors, Ricci scalar, torsion tensor, basic vector, traceless part and the axial vector part [17] we obtain for the solution (19)

\[ R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = 48 \bar{A}^6 (\bar{x} - \bar{y})^6 \bar{m}^2, \quad R^{\mu\nu} R_{\mu\nu} = 0, \quad R = 0, \]

\[ T^{\mu\nu\lambda} T_{\mu\nu\lambda} = \frac{F_1(\bar{x}, \bar{y})}{(1 - \bar{x}^2)(1 + 2\bar{x}\bar{m}A)(1 - \bar{y}^2)(1 + 2\bar{y}\bar{m}A)}, \]

\[ \Phi^\mu \Phi_\mu = \frac{F_2(\bar{x}, \bar{y})}{(1 - \bar{x}^2)(1 + 2\bar{x}\bar{m}A)(1 - \bar{y}^2)(1 + 2\bar{y}\bar{m}A)}, \]

\[ t^{\mu\nu\lambda} t_{\mu\nu\lambda} = \frac{F_2(\bar{x}, \bar{y})}{(1 - \bar{x}^2)(1 + 2\bar{x}\bar{m}A)(1 - \bar{y}^2)(1 + 2\bar{y}\bar{m}A)}, \]

\[ a^\mu a_\mu = 0. \]  \hspace{1cm} (22)

As is clear from (22) that the scalars of torsion, basic vector and traceless part have the same singularities, let us discuss these singularities.

1) When \( \bar{x} = \bar{y} = \xi_2 \) then all the scalars of (22) have a singularities which is called asymptotic infinity [15].

2) When \( \bar{y} = \xi_3 \), there is a singularity which is called black hole event horizon [15].

3) When \( \bar{y} = \xi_2 \) there is also a singularity which is acceleration horizon.

4) When \( \bar{x} = \xi_1 \) there is a singularity which makes symmetry axis between event and acceleration horizons.

5) When \( \bar{x} = \xi_2 \) there is a singularity which makes a symmetry axis joining between event horizon with asymptotic horizon.

6) When \( \bar{x} = \xi_2 \) and \( \bar{y} = \xi_1 \) there will be a conical singularity [15].

5. Energy content

Now we are going to use the following coordinate transformation [15]

\[ \bar{x} = \cos(\theta), \quad \bar{y} = -\frac{1}{A r}, \quad \bar{t} = \bar{A} t_1, \]  \hspace{1cm} (23)
where \( r, \theta, t_1 \) are the new coordinate. Applying transformation (23) to tetrad (19) we get

\[
\left( b^i_{\mu} \right) = \left( \begin{array}{cccc}
-\sqrt{\frac{-2m\sqrt{A^2} - 1}{r(Ar \cos \theta + 1)}} & 0 & 0 & 0 \\
0 & 0 & \frac{r}{\sqrt{1+2mA cos \theta(\bar{A}r \cos \theta + 1)}} & 0 \\
0 & -\sqrt{\frac{-2m\sqrt{A^2} - 1(\bar{A}r \cos \theta + 1)}} & 0 & 0 \\
0 & 0 & 0 & \sin \theta \sqrt{\frac{1+2mA cos \theta}{\bar{A}r \cos \theta + 1}} \\
\end{array} \right).
\]

Taking the limit \( \bar{A} \to 0 \) in (24), the associate metric will have the Schwarzschild form. Now we are going to study some physics of this solution by calculate its energy content.

The superpotential is given by

\[
U_{\mu}^{\nu \lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi \rho \sigma}^{\tau \nu \lambda} \left[ \Phi^\rho g^{\sigma \chi} g_{\mu \tau} - \lambda g_{\tau \mu} \gamma^{\chi \rho \sigma} - (1 - 2\lambda) g_{\tau \mu} \gamma^{\sigma \rho \chi} \right], \tag{25}
\]

where \( P_{\chi \rho \sigma}^{\tau \nu \lambda} \) is

\[
P_{\chi \rho \sigma}^{\tau \nu \lambda} \overset{\text{def}}{=} \delta_\chi^\tau g_{\rho \sigma}^{\nu \lambda} + \delta_\rho^\tau g_{\sigma \chi}^{\nu \lambda} - \delta_\sigma^\tau g_{\chi \rho}^{\nu \lambda} \tag{26}
\]

with \( g_{\rho \sigma}^{\nu \lambda} \) being a tensor defined by

\[
g_{\rho \sigma}^{\nu \lambda} \overset{\text{def}}{=} \delta_\rho^\nu \delta_\sigma^\lambda - \delta_\sigma^\nu \delta_\rho^\lambda. \tag{27}
\]

The energy is expressed by the surface integral [18]

\[
E = \lim_{r \to \infty} \int_{r=\text{constant}} U_0^{a \alpha} n_\alpha dS, \tag{28}
\]

where \( n_\alpha \) is the unit 3-vector normal to the surface element \( dS \).

Now we are in a position to calculate the energy associated with solution (24) using the superpotential (25). As is clear from (28), the only components which contributes to the energy is \( U_0^{0a} \). Thus substituting from solution (24) into (25) we obtain the following non-vanishing value

\[
U_0^{0a} = -\frac{x^a}{\kappa r^3(x^2 + y^2)} \left( r^3 + (r - 4M)(x^2 + y^2) \right), \quad a = 1, 2, \quad U_0^{03} = -\frac{z}{\kappa r^3} (r - 4M). \tag{29}
\]

Substituting from (29) into (28) we get

\[
E(r) = 2M - r! \tag{30}
\]
6. Main results and Discussion

We begin with tetrad (15) which is the square root of metric (1). We then, preform the coordinate transformation (17) to tetrad (15) with solution (16). The associated metric of tetrad (19) has the structure function (20), that is offers advantage over the traditional form (1). With a factorisable structure function, its roots can be explicitly read off from it, leading to simplifications in the analysis of the C-metric. Besides the simplifying known results, the new form of the C-metric opens up the possibility of performing calculations that were previously impractical if not possible. We also study the singularities of tetrad (19) as we see from (22) that we easily find the singularities which are the roots of the structure function (20).

Perform the coordinate transformation (23) to the tetrad (19), we obtain the tetrad (24) whose associated metric is the Schwarzschild spacetime. Calculating the energy content of this solution we obtain (30) which is a very strange results! In fact if we take the limit of (30) when $r \to \infty$ we will get a meaningless result. We can offer the following reasons for such strange result

i) Is the tetrad (15) that we begin with it its structure is inconsistence? If this is true this means that the square root of metric that will constitute the tetrad is not a physical one.

ii) Another possibility for this strange result is that the transformation (17) and (23) makes the tetrad (15) in its final form (24) unphysics!

In the present time it is important to find a consistence tetrad which can reproduce the structure function (1), then preforming the coordinate transformation (17) and (23). Calculations of the energy content of the tetrad that reproduce the Schwarzschild spacetime must be coincide with inertial mass. Work in this direction is in progress.
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