A Remark On Field Theories On The Non-Commutative Torus

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Abstract:
We investigate field theories on the non-commutative torus upon varying $\theta$, the parameter of non-commutativity. We argue that one should think of Morita equivalence as a symmetry of algebras describing the same space rather than of theories living on different spaces (as is T-duality). Then we give arguments why physical observables depend on $\theta$ non-continuously.

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1. Introduction

Non-commutative geometry, a generalization of ordinary geometry to spaces with coordinate functions that generate non-commuting algebras, was pioneered by Connes and has been studied for a long time, see [C], for a review also [H]. There is the general belief that a theory of quantum gravity has to incorporate this generalization since Heisenberg's uncertainty relation together with the Schwarzschild relation between mass and radius of a black hole hint to a space-space uncertainty relation in a theory that incorporates both, quantum physics and general relativity.

More recently, it was realized by Douglas and coworkers [CDS] [DH] that string theory, the leading candidate for a theory of quantum gravity, indeed realizes non-commutative structures of the form introduced earlier in a pure mathematical context. Since that time and especially after the work of Seiberg and Witten [SW] on non-commutative geometry in string theory, a tremendous number of papers have been published that deal with that subject. For a review consult [DN].

In string theory, non-commutativity enters thru the vev of the Neveu-Schwarz two-form $B_{\mu\nu}$ that is the anti-symmetric cousin of the metric. In the simplest case in which the $B$ field is just constant, the pointwise product of functions is deformed to the $\ast$-product

$$(f \ast g)(x) = e^{\frac{i}{2} \theta_{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}} f(y)g(z) \bigg|_{x=y=z}$$

where $\theta$ is some function of $B$.

The nature of this background field entering as a deformation suggests that also physical observables are just continuously deformed from their values in the commutative theory. Here however we will meet a surprise: In the classical case a theorem by Gelfand and Naimark states that there is a one to one correspondence between Hausdorff spaces and commutative $C^\ast$-algebras via the spectrum, that is the algebra of functions on that space. This is no longer true in the non-commutative case: There, the mapping is one to many: There are several different (more precisely: unitary inequivalent) algebras that correspond to the same space. Thus, rather than considering single algebras as representatives of spaces one should group them into equivalence classes. This is the origin of Morita equivalence.

Our main example in this note will be the non-commutative two torus. Here, the $\ast$-product is given in terms of a single parameter $\theta = \theta^{12}$. It is well known that the Morita equivalence classes are orbits of the $SL(2, \mathbb{Z})$ action

$$\theta \mapsto \frac{a\theta + b}{c\theta + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1,$$

that is for rational $\theta = p/q$:

$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$
Thus in arbitrary small neighborhoods of two values of $\theta$ there are always values that are Morita equivalent and that therefore describe the same space.

Therefore it seems highly unlikely that properties of the non-commutative space and thus physics of field theories living on those spaces varies continously as one varies $\theta$. This dependence of physics on $\theta$ is the subject of this note. It has been studied from other perspectives in [GT] and [AGB].

The structure of this note is as follows: In the following section we discuss Morita equivalence more extensively. In section three we show how it acts on field theories on non-commutative spaces. The next section deals with solutions of a toy “equation of motion” for different values of $\theta$. We show that observables indeed vary discontinuously. The final section sets this in the context of string theory and discusses the relation to the large volume limit of the non-commutative plane.

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2. Morita equivalence of algebras describing a space

In many situations it is fruitful in order to study an object $X$ to adopt a dual perspective, that is to consider the algebra of morphism from $X$ to the complex numbers. This, for example is subject of non-commutative geometry (where $X$ is a topological Hausdorff space, perhaps endowed with additional structure like differentiable structure, metric etc.), of quantum groups (where $X$ is a Lie group) or quantum information theory (where $X$ is the classical state space of a physical system operating on information).

In this note, we will be concerned with non-commutative geometry which in its simplest form is about the Gelfand-Naimark isomorphism of the categories of topological Hausdorff spaces and abelian C* algebras via the map

$$X \in \{\text{Hausdorff spaces}\} \mapsto \mathcal{A}_X := \{f: X \to \mathbb{C}| f \text{ continuous, vanishing at } \infty\}$$

An important aspect of this correspondence is how to go in the opposite direction, that is given an abelian C* algebra $\mathcal{A}$, construct the topological space such that its algebra of continuous functions is given by $\mathcal{A}$. The first step in this direction is to recover what is the analog of a point $x \in X$ in the algebraic setting. It is well known (for a review, see for example [H]) one can associate two structures to $x$: The first is an irreducible representation of $\mathcal{A}$, as

$$\pi_x: \mathcal{A} \to \mathbb{C}, \quad \pi_x(f) := f(x)$$

is one and all of them are of this form. Furthermore, associated to $x$ there is a maximal ideal in $\mathcal{A}$, that is the vanishing ideal of $\pi_x$, namely

$$\{f \in \mathcal{A}| f(x) = 0\}.$$
Again, there is a one to one correspondence between maximal ideals, points and irreducible representations, at least as long as \( A \) is abelian.

If one now leaves the commutative realm and generalizes the above notions to non-commutative C* algebras in general, these notions do not coincide anymore and it is ambiguous what one should take as the non-commutative extension of the inverse Gelfand-Naimark map, that is what generalizes to the points of a non-commutative space that constitute the space.

Nevertheless, it is clear the structure of the space is — as in the commutative case — encoded in the representation theory of the algebra. But quite different from the commutative case where the representation theory is in one to one correspondence with topological Hausdorff spaces, there are inequivalent C* algebras with identical representation theory.

Thus, one should not take one algebra to be a description of a non-commutative space but rather an equivalence class of algebras. The appropriate equivalence is given by Morita equivalence. Two (unitary) inequivalent algebras that are nevertheless Morita equivalent should be thought of as two different coordinatizations of the same non-commutative space.

To precisely define Morita equivalence one needs the algebra \( K \) of compact operators on a Hilbert space, that is the norm closure of finite rank operators (in more physics parlance: All operators that can be obtained as limits of series of matrices with only a finite number of non-zero entries). Then, two algebras \( A \) and \( B \) are said to be Morita equivalent if \( A \otimes K \) is equivalent to \( B \otimes K \) in the ordinary (that is unitary) sense. (For an introduction see [1] or [8]). Note that, so far, we have been concerned only with the geometry of the space and have not mentioned field theories living on the space.

As an example, note that \( A \) is Morita equivalent to the algebra \( M(n \times n, A) = A \otimes M(n \times n, \mathbb{C}) \) of \( n \times n \) matrices with entries in \( A \).

The fact that Morita equivalent algebras also have the identical K-theory (which encodes the topology and possible stable branes in string theory) also supports the view that all algebras in an Morita equivalence class should be thought of as the same non-commutative space.

The next step is to consider field theories defined on these non-commutative spaces. If two algebras really describe the same space they should also support the same field theories. This is really the main motivation for this letter: One should think of Morita equivalence not as a property of field theories living on a non-commutative space but rather as a property of the algebras describing the space itself.

An alternative characterization of Morita equivalence is via the notion of Hilbert bi-modules. Take again two algebras \( A \) and \( B \), then a Hilbert bi-module is a left \( A \) and a right \( B \) module that fulfills certain natural algebraic relations. Morita equivalence of \( A \) and \( B \) is then equivalent to the existence of such a Hilbert bi-module.

To define a field theory on a space, one should first identify the appropriate description of the fields. In the commutative case, this is that of sections in bundles over the base space. The non-commutative analog of a bundle is a projective module (that is a module of finite rank that can be completed by a direct sum with an other module to a free module).
An important property of Hilbert bi-modules is that they can be used to turn projective \( B \) modules into projective \( A \) modules: Let \( P \) be a projective \( B \) module and \( H \) a Hilbert bi-module. Then,

\[
H \otimes_B P
\]
is a projective \( A \) module. Thus, every field on a space described by \( B \) can be translated to a field in the \( A \) description of the same space via the existence of a Hilbert bi-module.

We have seen that a bi-module that relates two algebraic descriptions of some space can also be used to translate the algebraic analogs of vector bundles, that is fields between the two descriptions. Under this mapping the “form” of the field will not be preserved generically. In the following section, we will give an example of a scalar field that is mapped to a matrix valued field and the change of the rank of the gauge group encountered in stringy realizations of gauge theories on non-commutative spaces is yet another manifestation of this phenomenon.

From a more abstract point of view, Morita equivalence of two algebraic descriptions of one space maps the set of all field theories on that algebra to the set of field theories on the other algebra rather than the pairs one might naively expect: The scalar field on one algebra is not necessarily mapped to the scalar field but to a scalar field tensored with the bi-module which generically introduces matrix structure.

3. Relating field theories on Morita equivalent descriptions of a space

In the previous chapter we have mapped the fields, the constituents of a physical theory. It remains to translate the dynamics of the field theories between the two settings. This can, for example, be done by translating action functionals. Rather than continuing within the abstract setting, we will now exemplify this in the concrete example of the non-commutative torus with parameter \( \theta \) for which the Morita equivalence classes are given by the \( SL(2,\mathbb{Z}) \) orbits in the space of \( \theta \)'s. We will follow [GT], see also [AMNS].

The \( N \times N \) ’t Hooft clock and shift matrices

\[
Q = \begin{pmatrix}
1 & & \\
e^{\frac{2\pi i}{N}} & \ddots & \\
& \ddots & e^{\frac{2\pi i(N-1)}{N}} & \\
& & 1 & 
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & \ddots & \\
1 & & & 
\end{pmatrix}.
\]

obey the commutation relations

\[
PQ = QPe^{\frac{2\pi i}{N}}
\]
and generate the algebra of complex \( N \times N \) matrices as all \( P^nQ^m \) form a basis for \( n, m = 0, 1, \ldots, N - 1 \).

We are interested in fields \( \phi \) that take values in this algebra and live on a two-torus with both radii equal to \( R \). Let us demand that \( \phi \) obeys the twisted boundary conditions

\[
\phi(x_1 + 2\pi R, x_2) = P^{-m}\phi(x_1, x_2)P^m \\
\phi(x_1, x_1 + 2\pi R) = Q\phi(x_1, x_2)Q^{-1}
\]

(3.3)
for some given integer \( m \), later referred to as the magnetic flux. We assume that \( m \) and \( N \) are relatively prime and fulfill
\[
aN - cm = 1
\]
(or \( c \equiv m^{-1}(N) \)) for some integers \( a \) and \( c \). Note that \( \phi^\dagger \) fulfills the same boundary conditions.

We decompose \( \phi \) into its Fourier-modes as
\[
\phi(x_1, x_2) = \sum_{\vec{r}} \phi_{\vec{r}} Q^{-cr_1} P^{r_2} \exp \left( -i2\pi \frac{cr_1r_2}{N} \right) \exp \left( -i\frac{\vec{r} \cdot \vec{x}}{NR} \right) \quad (3.4).
\]

The first phase factor could be absorbed in the definition of \( \phi_{\vec{r}} \) but we put it here for later convenience.

Using these modes, we can now define a complex scalar field \( \hat{\phi} \) that lives on a larger torus with radii \( R' = NR \) as
\[
\hat{\phi}(x_1, x_2) = \sum_{\vec{r}} \phi_{\vec{r}} \exp \left( -i\frac{\vec{r} \cdot \vec{x}}{R'} \right).
\]

As can be checked by direct calculation, the operation “\(^\dagger\)” has the properties
\[
(i) \quad \hat{\partial}_i \hat{\phi} = \partial_i \hat{\phi} \\
(ii) \quad \frac{1}{N^2} \int_{T^2} d^2 x \hat{\phi} = \int_{T^2} d^2 x \text{tr}\phi \\
(iii) \quad \hat{\phi}_1 \hat{\phi}_2 = \hat{\phi}_1 \ast \hat{\phi}_2,
\]
where in the last line we introduced the Moyal product
\[
(\phi_1 \ast \phi_2)(\vec{x}) = e^{i\sum \frac{\partial \phi_{1}}{\partial x^i} \frac{\partial \phi_{2}}{\partial y^j} \phi_1(\vec{x})\phi_2(\vec{y})}|_{x=y}
\]
with non-commutativity parameter \( \vartheta_{ij} = \bar{\vartheta}_{ij} = -2\pi \frac{c}{N} R'^2 \epsilon_{ij} \).

From these properties it follows that any action \( S \) of a field theory in terms of \( \phi \)'s on \( T^2 \) is equivalent to a dual action \( \hat{S} \) in terms of the \( \hat{\phi} \)'s if all products are replaced by Moyal products. In one theory the non-commutativity lies in the matrix product in the other theory non-commutativity comes with the Moyal product.

The two dimensions of the tori need not be the only dimensions of space-time. All fields \( \phi \) and \( \hat{\phi} \) and therefore the modes \( \phi_{\vec{r}} \) could depend on any number of further uncompactified directions.

The classical observables of both theories can be translated to expressions of the modes \( \phi_{\vec{r}} \) and are therefore in a one-to-one correspondence. Of course, the localization of the same observable in the two theories is quite different, thus the question arises if it is possible to map observables between the two theories without knowing about the relation between the two mode expansions.
The puzzle is that, in the hatted theory, $\vartheta$ should be thought of as a real parameter, as a similar parameter arises as a background field in low-energy effective theories arising from string theory. On the other hand, the first model does not make sense for irrational values of

$$\theta = \frac{\vartheta}{2\pi R'^2}$$

and appears very different for close real values as $1/2$ and $1000000/2000001$.

It would be nice to identify properties (observables?) in the theories for similar values of $B$ that would show, that these theories are also similar in some sense. A similar question was addressed in [G1]. There, Wilson-loops were considered for gauge theories that are related in same fashion as above without a conclusive answer. Here, we would like to point out, that the same question arises in a much larger class of theories and should therefore have an answer beyond the consideration of Wilson loops.

Even more, this question should be easier to analyze in a theory without unphysical (gauge) degrees of freedom as this one where observables can be directly identified.

What is behind this discussion is of course Morita equivalence: There is a $SL(2, \mathbb{Z})$ action on $\theta$ as

$$\rho \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (B) = \frac{a\theta + b}{c\theta + d}$$

and the above relation is an explicitly applied Hilbert-bi-module. As the algebras

$$\mathcal{A} = \{ f: T^2 \to \mathbb{C}, \text{pointwise product} \}, \quad \mathcal{B} = \{ f: T^{2'} \to \mathbb{C}, *-\text{product} \}$$

are Morita equivalent. Thus they have the same representation theory and describe the same space from the perspective of non-commutative geometry. One would expect to find a one-to-one correspondence between field-theories on both spaces. This correspondence is spelled out in this note.

What we have seen is that a theory that appears to be a theory on a non-commutative space can be physically equivalent to theory of (possibly large) matrices on a commutative space with magnetic background flux. As in this realization the size of the matrices and the amount of flux depends non-continuously on $\theta$ it seems very unlikely that all physical theories on these spaces (keep in mind this transformation can be applied to any Lagrangian) share some “magic” property that nevertheless realizes physics continuously given the vastly different realizations on commutative spaces.

4. Solving equations on the non-commutative torus

We would like to investigate the dependence of physical quantities on $\theta$, the parameter of non-commutativity. As explained before, we are especially interested in whether physics is continuous in $\theta$. Ideally, one would like to calculate for example correlation functions of some quantum field theory and study their behavior as $\theta$ varies. Unfortunately, this is presently beyond our power. A more modest goal would be to solve classical field equations...
on a non-commutative torus and determine whether solutions depend continuously on $\theta$. Instead, here we even ask a simpler question: Can we solve “algebraic”* equations. The simplest example of this kind is: Find smooth functions $f: T_\theta \to \mathbb{C}$ that solve

$$f \ast f = f. \tag{4.1}$$

This equation for the non-commutative torus has been solved in terms of $\vartheta$-functions in [B] and was discussed in [MM], [BKMT], and [KS]. This equation is easily solved in the operator formalism that amounts to replacing functions $f$ subject to $\ast$-multiplication by operators $O(f)$ on a Hilbert space that implement the $\ast$-product via $O(f \ast g) = O(f)O(g)$. The inverse operation to this Moyal-Weil correspondence is given by

$$f(x, y) = \Tr \left( O(f) \int \frac{dk dl}{(2\pi)^2} e^{ikO(x) + ilO(y)} e^{-ikx - kly} \right).$$

In the operator language, (4.1) just states that $f$ is a projector. Let us assume it has rank one, that is $O(f) = \ket{\psi}\bra{\psi}$. Let us translate this back into the language of $\ast$-products. For convenience, we use an $x$-space basis $\{ \ket{x} : x \in \mathbb{R} \}$ for the Hilbert space, that is $O(x)\ket{x} = x\ket{x}$ and $\bra{x}\psi = \psi(x)$:

$$f(x, y) = \Tr \left( \ket{\psi}\bra{\psi} \int \frac{dk dl}{(2\pi)^2} e^{ikO(x) + ilO(y)} e^{-ikx - kly} \right)$$

$$= \int dz \bra{z}\psi \int \frac{dk dl}{(2\pi)^2} \bra{\psi} e^{ikO(x)} e^{ilO(y)} e^{-\frac{1}{2}[ikO(x), ilO(y)]} e^{-ikx - kly}$$

$$= \int dz \int \frac{dk dl}{(2\pi)^2} \psi(z) \psi^*(z + l\theta) e^{ik(z + \frac{1}{2}\theta l - x)} e^{-ily}$$

$$= \int \frac{dl}{2\pi} \psi(x - \frac{1}{2}\theta l) \psi^*(x + \frac{1}{2}\theta l) e^{-ily}$$

Here we used the Baker-Campbel-Hausdorff formula and the fact that the operator $e^{ilO(y)}$ operates by shifting $x$ by $l\theta$.

So far, the calculation is equally valid on the non-commutative plane. Now, we have to make use of the fact that $f(x, y)$ is supposed to be a function that is defined on the torus. That is, it should be periodic in both $x$ and $y$ with unit period, say. This implies restrictions on the wave function $\psi(x)$. Periodicity in $x$ forces $|\psi|$ to have unit periodicity, too. On the other hand, $\psi(x - \frac{1}{2}\theta l) \psi^*(x + \frac{1}{2}\theta l)$ is the Fourier transform in $y$. In order to have unit periodicity in $y$, this Fourier transform must have support only for integer $l$.

We can make this even clearer if we decompose $\psi$ into its Fourier modes. Naively, one would take

$$\psi(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}. \tag{4.2}$$

* Of course, the $\ast$-product introduces derivatives and the equation is really a partial differential equation.
This gives

\[
f(x, y) = \int \frac{dl}{2\pi} \sum_{m,n} a_m a_n^* e^{i(m-n)x} e^{-il[(m+n)\theta+y]} = \sum_{m,n} a_m a_n^* e^{i(m-n)x} \delta((m+n)\theta+y).
\]

But we see that \(x\) appears only thru \(\exp(i(m-n)x)\). Thus, \(m\) and \(n\) do not need to be integer moded but only their difference has to be in order to ensure periodicity in \(x\). Therefore we should replace (4.2) by

\[
\psi(x) = \sum_{n} a_n e^{i(n+\phi)x}
\]

for some \(\phi\) and obtain

\[
f(x, y) = \sum_{m,n} a_m a_n^* e^{i(m-n)x} \delta((m+n+2\phi)\theta+y). \quad (4.3)
\]

Now, this expression has to be periodic in \(y\). That means, if it is non-zero for some \(y_0\) (meaning \((m+n+\phi)\theta = -y_0\) for some \(m\) and \(n\)) it also has to be for \(y_0 + 1\). This can only be if for some other \(m'\) and \(n'\) we have \((m+n-m'-n')\theta = 1\). Finally, we find \(\theta = 1/q\) for integer \(q\) especially, it cannot be irrational.

In order to find solutions for (4.1) if \(\theta = p/q\) one must generalize the above construction to the case of rank \(p\) projectors. In that case, one starts with \(p\) orthogonal wave functions

\[
\psi_i(x) = \sum_{n} a_{in} e^{i(n+\phi_i)x}
\]

with “shift angles” \(\phi_i = \phi_0 + i/2q\) that fill up the “missing” peaks for unit periodicity in \(y\). This fits well with the intuition from string theory: There, the non-commutative torus arises from compactifications on a torus in which a gauge field has a flux \(\theta\) thru the torus. This setup however is T-dual to a D1-brane that wraps \(p\) and \(q\) times the two basic holonomy cycles of the torus. Thus, above each \(x\) there are \(p\) instances of the brane. On the other hand, the rank of the projector has been argued to be related to the number of branes in the context of non-commutative instantons\[HKLM].

For irrational \(\theta\), one could superimpose an infinite number of D-branes densely. This yields a projector of infinite rank that is not well described in the formalism we employ here. Those projectors have been constructed in [B] in terms of \(\vartheta\)-functions, see also [MM] for a discussion.

So far in this section, we have concentrated on the distinction between rational and irrational \(\theta\). But of course we should also come back to the question of the preceding section of whether physical quantities are similar for similar values of \(\theta\) like \(1/2\) and \(1000001/2000000\). As explained above, we intend to model a classical field equation with (4.1). In fact, such projectors have been used heavily to construct soliton like solutions in
non-commutative field theories in the limit of vanishing derivatives (usually called limit of large non-commutativity, but as we show here, it does not really make sense to talk about limits of $\theta$ on compact spaces) as pioneered by [GMS]. For classical field theories without gauge invariances the observables the values of the fields and their derivatives at points in space-time.

Once again, form the structure of (4.3), it is clear that possible functions $f(x, y)$ differ significantly since the denominator of $\theta$ already determines the periodicity. From this we can conclude that there is no notion of continuity in $\theta$ on the torus and consequently limits with respect to $\theta$ cannot be defined as already claimed several times.

One might wonder whether this behavior is a peculiarity of the classical theory and it goes away once the theory is quantized. To us, thus sounds highly unlikely although we do not have a proof for this (mainly due to the lack of reliable information non-perturbative quantum observables of such theories). At least we can comment on the “$\theta$ enters only as a parameter in the vertices” argument. First of all, already this should also apply to the classical theory if in the calculation, one includes tree-level diagrams only. The conclusion of this note is to show that this cannot be true. So, what is the fallacy?

By using this argument one forgets that it should be understood in a path-integral context. In the non-commutative setting considered here, one should be careful in choosing the correct set of functions to integrate over. Namely, one should only integrate over functions that are compatible with the periodicity of the non-commutative torus. In fact, this is what stands behind our reasoning: The non-locality of the $\ast$-product makes the issue of periodicity subtle and non-continuous in $\theta$ and thus the set of allowed functions depends non-trivially on $\theta$. This resolves the apparent puzzle that naively $\theta$ enters Feynman rules only as a parameter in the vertices.

Let us end this section by giving an example of a system that behaves discontinuously with respect to a parameter just as the non-commutative torus. It consists of a field $\phi$ obeying the two dimensional wave-equation $(\partial_x^2 - \partial_t^2)\phi = 0$. But instead of giving initial values for $\phi$ and $\dot{\phi}$, we put the field in a rectangular, $a \times b$ sized, space-time box and impose Dirichlet boundary conditions. For example we require the field to vanish for $x = 0$ or $x = a$ or $t = 0$ or $t = b$. See fig. 1.

![Fig. 1: The field $\phi$ in a space-time box](image)

We solve the field equation in terms of left- and right-movers as $\phi(t, x) = u(x+t) + v(x-t)$ in the bulk and impose the boundary conditions. From $\phi(0, x) = 0$ we find $u = -v$, form $\phi(t, 0) = 0$ we find $u(t) = -u(-t)$. Hence, the solution is determined in terms of one
antisymmetric function. Furthermore $\phi(b, x) = 0$ implies $u(x + b) = u(x - b)$, thus $u$ is 2$b$-periodic. On the other hand, from $\phi(t, a) = 0$ follows $u(t + a) = u(t - a)$, so $u$ is also 2$a$-periodic. Whether this double periodicity can be fulfilled for other functions than $\phi(t, x) = 0$ depends on the dimensionless ratio $\eta = a/b$: If $\eta$ is irrational, the trivial one is the only smooth solution (as an aside we note that if we allow for discontinuous solutions, the remaining freedom of $u$ is precisely to pick a function on the irrational foliations of a torus. But this space of foliations is described by a non-commutative torus with irrational $\theta$, see [C]).

If the ratio is rational $\eta = \frac{p}{q}$ the periodicity depends on nominator and denominator separately; it is given by $\frac{a}{p} = \frac{b}{q}$. The solution is a linear combination of functions

$$u(x + t) = -v(x - t) = \sum b_n \sin \left( \frac{\pi n p x}{a} \right).$$

Just as for the projectors on the non-commutative torus we have a moduli space of solutions that depends very critically on the dimensionless ratio $\eta = a/b$. The connection between the two problems is that also in the case of the non-commutative torus the $*$-product implies relations among the values of the solution at points that are separated by $\theta$ and the periodicity in $x$ implies relations for points at unit distance.

5. The non-commutative torus and the non-commutative plane

In what we have said above about the non-commutative torus it was crucial that the torus is compact. In general, the background field $\vartheta^{ij}(x)$ that describes the non-commutativity of the space via

$$[x^i, x^j] = i\vartheta^{ij}(x)$$

has dimension of length squared. For a dimensionful quantity, of course, there is no notion of rational or irrational as it would depend on the units used.

Therefore, in infinite space we would not expect any effects as explored in the previous two sections. Using string theory to realize field theories on the non-commutative plane, $\vartheta$ is really just a background field on which physics depends continuously. In perturbative calculations it only enters as a parameter in the vertices and as such can be thought of as a coupling constant.

In contrast, on compact spaces, there is the volume of the space that can be used to form a dimensionless number $\theta$ from $\vartheta$ and it is this dimensionless number that we have used in our discussion as we fixed our units by requiring the torus to have unit periodicity. More precisely, from a stringy perspective, $\theta$ is related to the flux of the $B$-field or a gauge field thru the torus. For such a flux a quantization does not come unexpected. In fact, multiplied by the correct electric charge, a rational flux is integer in the correct normalization[BGKL].

This can also be viewed from a T-dual perspective: The space on which the field theory lives is really a stack of D2-branes wrapping the torus. Now, we can apply T-duality in
one of the directions. This turns the D2-branes into D1-branes wrapping the other torus
direction. The non-commutativity $\theta$ can be attributed to a gauge field flux $F$ before the
duality map is applied.

Under T-duality, this flux is mapped to the slope of the D1, the angle being given by
cot $\alpha = F$. If $F$ (and thus $\theta$) is irrational the D1 never hits itself again and wraps the torus
densely. There is an infinity of arbitrary light open strings that span between different but
close wrappings that will dominate the dynamics. If $F$ is rational ($p/q$ say) the D1-brane
will hit itself after wrapping the torus $q$ times and the distance (and thus the mass of the
strings stretching) of the different wrappings stays finite.

Thus again we find totally different behavior in the two cases which once again leads us
to the conclusion that physics on the torus cannot be continuous with respect to $\theta$.

Instead of the gauge field $F$ we could have also considered the NS two-form $B$ since they
only appear in the gauge invariant combination $F = B - F$. The geometric picture in that
case differs as $B$ does not appear as the slope of the D1-brane but rather as the tilt of the
T-dual torus: Together with the volume it forms the Kähler modulus $\rho = B + iV$ of the
torus that gets exchanged with the complex structure modulus $\tau$ upon T-duality.

Indeed, one can see that the winding strings introduce a non-locality that connects points
on the D1-brane that are apart by integer multiples of $\theta$ just as in (4.3). The resulting
theory in the $V \to 0$ limit on $\mathbb{R}/\mathbb{Z} \times \theta \mathbb{Z}$ is once again the theory on the non-commutative
torus as described by Douglas and Hull [DH].
6. Conclusions

In this note we have studied field theories on non-commutative tori. We paid special
attention to their behavior under variations of $\theta$ the parameter of non-commutative
ativity.

We explained that Morita equivalence of spaces induces relations between field theories
living on these spaces that imply relations between theories for different $\theta$ and varying
magnetic flux.

Especially we constructed solutions to the projector equation and found non-trivial depen-
dence on $\theta$. We argued that it is likely that this is the generic behavior of field equations
on compact non-commutative spaces. Therefore, taking limits with respect to $\theta$ seems
ill-defined. Likewise, the IR-limit that uncompactifies the torus should be taken with care
as the compact and non-compact cases are qualitatively different. It would also be interest-
ing to study this in the light of the UV-IR-correspondence of [MvRS] that transforms UV
singularities of a theory into IR singularities on the non-commutative plane. As Morita
equivalence can be used to transform non-commutative tori to commutative tori with flux, this
relation will be more involved on compact spaces. We hope to come back to this
question in the future.

7. References

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