THE PERIODIC $b$-EQUATION AND EULER EQUATIONS ON THE CIRCLE

JOACHIM ESCHER AND JÖRG SEILER

Abstract. In this note we show that the periodic $b$-equation can only be realized as an Euler equation on the Lie group $\text{Diff}^\infty(S^1)$ of all smooth and orientation preserving diffeomorphisms on the circle if $b = 2$, i.e. for the Camassa-Holm equation. In this case the inertia operator generating the metric on $\text{Diff}^\infty(S^1)$ is given by $A = 1 - \partial_x^2$. In contrast, the Degasperis-Procesi equation, for which $b = 3$, is not an Euler equation on $\text{Diff}^\infty(S^1)$ for any inertia operator. Our result generalizes a recent result of B. Kolev [24].

1. Introduction

In this note we are interested in the geometric interpretation of the so-called $b$-equation

\begin{equation}
mt = -(mxu + bmux), \quad t \in \mathbb{R}, \quad x \in S^1,
\end{equation}

with the momentum variable $m$ given by

\[m = u - u_{xx},\]

and where $b$ stands for a real parameter, cf. [13, 12, 19]. It was shown in [13, 19, 26, 20] that equation (1) is asymptotically integrable, a necessary condition for complete integrability, but only for the values $b = 2$ and $b = 3$. In case $b = 2$ we recover the Camassa-Holm equation (CH)

\begin{equation}
utt - u_{txx} + 3uux - 2u_xuxx - uu_{xxx} = 0,
\end{equation}

while for $b = 3$ we obtain the Degasperis-Procesi equation (DP)

\begin{equation}
utt - u_{txx} + 4uux - 3u_xuxx - uu_{xxx} = 0.
\end{equation}

Independent of (asymptotical) integrability, equation (1) possesses some hydrodynamic relevance, as described for instance in [22, 21, 11]. Each of these equations models the unidirectional irrotational free surface flow of a shallow layer of an inviscid fluid moving under the influence of gravity over a flat bed. In these models, $u(t, x)$ represents the wave’s height at the moment $t$ and at position $x$ above the flat bottom.

The periodic Camassa-Holm equation is known to correspond to the geodesic flow with respect to the metric induced by the inertia operator $1 - \partial_x^2$ on the diffeomorphism group of the circle, cf. [25]. Local existence of the geodesics and properties of the Riemannian exponential map were studied in [9, 10]. The whole family of $b$-equations and in particular (DP) can be
realized as (in general) non-metric Euler equations, i.e. as geodesic flows with respect to a linear connection which is not necessarily Riemannian in the sense that there may not exist a Riemannian metric which is preserved by this connection, cf. [15].

Besides various common properties of the individual members of the b-equation, there are also significant differences to report on. It is known that solutions of the CH-equation preserve the $H^1$-norm in time and that (CH) possesses global in time weak solutions, cf. [7, 5, 3]. In particular there are no shock waves for (CH), although finite time blow of classical solutions occurs, but in form of wave breaking: solutions to (CH) stay continuous and bounded but their slopes may blow up in finite time, cf. [6, 8]. Wave breaking is also observed for the (DP) but in a weaker form. It seems that the $H^1$-norm of solutions of (DP) cannot be uniformly bounded, but $L^\infty$-bounds for large classes of initial values are available [16, 17, 18]. Moreover shock waves, i.e. discontinuous global travelling wave solutions are known to exist. Indeed it was shown in [14] that

$$u_c(t, x) = \frac{\sinh (x - [x] - 1/2)}{t \cosh (1/2) + c \sinh (1/2)}, \quad x \in \mathbb{R}/\mathbb{Z},$$

is for any $c > 0$ a global weak solution to the (DP) equation.

In this note we disclose a further difference between the (CH) and the (DP) equation, by proving that in a fairly large class of Riemannian metrics on $\text{Diff}^\infty(\mathbb{S}^1)$ it is impossible to realize (DP) as a geodesic flow.

The note is organized as follows. In Section 2, we first introduce the concept geodesic flows and Euler equations on a general Lie group. Subsequently, in Section 3, the important special case of $\text{Diff}^\infty(\mathbb{S}^1)$ is discussed and Section 4 contains the proof of our main result.

2. THE EULER EQUATION ON A GENERAL LIE GROUP

In his famous article [1] Arnold established a deep geometrical connection between the Euler equations for a perfect fluid in two and three dimensions and the geodesic flow for right-invariant metrics on the Lie group of volume-preserving diffeomorphisms. After Arnold’s fundamental work a lot of effort was devoted to understand the geometric structure of other physical systems with a Lie group as configuration space.

The general Euler equation was derived initially for the Levi-Civita connection of a one-sided invariant Riemannian metric on a Lie group $G$ (see [1] or [2]) but the theory is even valid in the more general setting of a one-sided invariant linear connection, see [15].

A right invariant metric on a Lie group $G$ is determined by its value at the unit element $e$ of the group, i.e. by an inner product on its Lie algebra $\mathfrak{g}$. This inner product can be expressed in terms of a symmetric linear operator $A : \mathfrak{g} \to \mathfrak{g}^*$, i.e.

$$\langle Au, v \rangle = \langle Av, u \rangle, \quad \text{for all } u, v \in \mathfrak{g},$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $\mathfrak{g}^* \times \mathfrak{g}$. Each symmetric isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$ is called an inertia operator on $G$. The corresponding metric on $G$ induced by $A$ is denoted by $\rho_A$. 
Let $\nabla$ denote the Levi-Civita connection on $G$ induced by $\rho_A$. Then

$$\nabla_{\xi_u} \xi_v = \frac{1}{2} [\xi_u, \xi_v] + B(\xi_u, \xi_v),$$

where $\xi_u$ is the right invariant vector field on $G$, generated by $u \in g$. Moreover, $[\cdot, \cdot]$ is the Lie bracket on Vect$(G)$, the smooth sections of the tangent bundle over $G$, and the bilinear operator $B$ is called Christoffel operator. It is defined by the following formula

$$B(u, v) = \frac{1}{2} \left( (\text{ad}_u)^* (v) + (\text{ad}_v)^* (u) \right),$$

where $(\text{ad}_u)^*$ is the adjoint with respect to $\rho_A$ of the natural action of $g$ on itself, given by

$$\text{ad}_u : g \to g, \quad v \mapsto [u, v].$$

A proof of the above statements as well as of the following proposition can be found in [15].

**Proposition 1.** A smooth curve $g(t)$ on a Lie group $G$ is a geodesic for a right invariant linear connection $\nabla$ defined by (3) iff its Eulerian velocity $u = g' \circ g^{-1}$ satisfies the first order equation

$$u_t = -B(u, u).$$

Equation (6) is known as the Euler equation.

### 3. The Euler equation on Diff$^\infty(S^1)$

Since the tangent bundle $T S^1 \cong S^1 \times \mathbb{R}$ is trivial, Vect$^\infty(S^1)$, the space of smooth vector fields on $S^1$, can be identified with $C^\infty(S^1)$, the space of real smooth functions on $S^1$. Furthermore, the group Diff$^\infty(S^1)$ is naturally equipped with a Fréchet manifold structure modeled over $C^\infty(S^1)$, cf. [15]. The Lie bracket on Vect$^\infty(S^1)$ is given by

$$[u, v] = u_x v - u v_x.$$

The topological dual space of Vect$^\infty(S^1)$ is C$^\infty(S^1)$ is given by the distributions Vect$'(S^1)$ on $S^1$. In order to get a convenient representation of the Christoffel operator $B$ we restrict ourselves to Vect$^*(S^1)$ iff there is a $g \in C^\infty(S^1)$ such that

$$\langle T, \varphi \rangle = \int_{S^1} g \varphi \, dx \quad \text{for all} \quad \varphi \in C^\infty(S^1).$$

By Riesz’ representation theorem we may identify Vect$^*(S^1)$ with C$^\infty(S^1)$. In the following we denote by $L^\text{sym}_{is}(C^\infty(S^1))$ the set of all continuous isomorphisms on C$^\infty(S^1)$, which are symmetric with respect to the $L_2(S^1)$ inner product.

**Definition 2.** Each $A \in L^\text{sym}_{is}(C^\infty(S^1))$ is called a regular inertia operator on Diff$^\infty(S^1)$.

---

1Notice that this bracket differs from the usual bracket of vector fields by a sign.
Proposition 3. Given \( A \in \mathcal{L}_{\text{is}}^{\text{sym}}(C^\infty(S^1)) \), we have that
\[
B(u, v) = \frac{1}{2} A^{-1} [2Au \cdot v_x + 2Av \cdot u_x + u \cdot (Av)_x + v \cdot (Au)_x]
\]
for all \( u, v \in C^\infty(S^1) \).

Proof. Let \( u, v, w \in C^\infty(S^1) \) be given. Recalling (5), integration by parts yields
\[
\rho_A((\text{ad}_u)^*v, w) = \rho_A(v, \text{ad}_u w) = \int_{S^1} Av \cdot (u_xw - uw_x) \, dx
\]
This shows that
\[
(\text{ad}_u)^*v = 2(Av)u_x + u(Av)_x.
\]
Symmetrization of this formula completes the proof, cf. (4). \( \square \)

Examples 4. It may be instructive to discuss two paradigmatic examples.

1. First we choose \( A = \text{id} \). Then \( B(u, u) = -3uu_x \) and the corresponding Euler equation \( u_t + 3uu_x = 0 \) is known as the periodic inviscid Burgers equation, see e.g. [4, 23].

2. Next we choose \( A = \text{id} - \partial_x^2 \). Then the Euler equation reads as \( u_t = -(1 - \partial_x^2)^{-1} (3uu_x - 2u_xu_{xx} - uu_{xxx}) \), which equivalent to the periodic Camassa-Holm equation, cf. (2).

4. The family of \( b \)-equations

Each \( A \in \mathcal{L}_{\text{is}}^{\text{sym}}(C^\infty(S^1)) \) induces an Euler equation on \( \text{Diff}^\infty(S^1) \). Conversely, given \( b \in \mathbb{R} \), we may ask whether there exists a regular inertia operator such that the \( b \)-equation is the corresponding Euler equation on \( \text{Diff}^\infty(S^1) \). We know from Example 4.2 that the answer is positive if \( b = 2 \). The following result shows that the answer is negative if \( b \neq 2 \).

Theorem 5. Let \( b \in \mathbb{R} \) be given and suppose that there is a regular inertia operator \( A \in \mathcal{L}_{\text{is}}^{\text{sym}}(C^\infty(S^1)) \) such that the \( b \)-equation
\[
m_t = -(m_xu + bmux), \quad m = u - u_{xx}
\]
is the Euler equation on \( \text{Diff}^\infty(S^1) \) with respect to \( \rho_A \). Then \( b = 2 \) and \( A = \text{id} - \partial_x^2 \).

Corollary 6. The Degasperis-Procesi equation
\[
m_t = -(m_xu + 3mu_x), \quad m = u - u_{xx}
\]
cannot be realized as an Euler equation for any regular inertia operator \( A \in \mathcal{L}_{\text{is}}^{\text{sym}}(C^\infty(S^1)) \).

Proof of Theorem 5. Let \( b \in \mathbb{R} \) be given and assume that the \( b \)-equation is the Euler equation on \( \text{Diff}^\infty(S^1) \) with respect to \( \rho_A \). Letting \( L = 1 - \partial_x^2 \), we then get
\[
A^{-1} \left( 2(Au)u_x + u(Au)_x \right) = L^{-1} \left( b(Lu)u_x + u(Lu)_x \right)
\]
for all \( u \in C^\infty(S^1) \).
(a) Let $1$ denote the constant function with value 1. Choosing $u = 1$ in (7), we get $A^{-1}(1(A1)_x) = 0$ and hence $(A1)_x = 0$, i.e. $A1$ is constant. Scaling (7), we may assume that $A1 = 1$. Next we replace $u$ by $u + \lambda$ in (7). Then we find for the left-hand side that
\[
\frac{1}{\lambda} A^{-1} \left( 2(A(u + \lambda))(u + \lambda)_x + (u + \lambda)(A(u + \lambda))_x \right)
\]
\[
= \frac{1}{\lambda} A^{-1} \left( 2((Au) + \lambda)u_x + (u + \lambda)(Au)_x \right)
\]
\[
= A^{-1} \left( \frac{2(Au)_x + u(Au)_x}{\lambda} + 2u_x + (Au)_x \right)
\]
\[
\xrightarrow{\lambda \to \infty} A^{-1} (2u_x + (Au)_x),
\]
and similarly for the right-hand side:
\[
\frac{1}{\lambda} L^{-1} \left( b(L(u + \lambda))(u + \lambda)_x + (u + \lambda)(L(u + \lambda))_x \right)
\]
\[
\xrightarrow{\lambda \to \infty} L^{-1} (bu_x + (Lu)_x).
\]
Combining these limits, we conclude that
\[
(8) \quad A^{-1} (2u_x + (Au)_x) = L^{-1} (bu_x + (Lu)_x)
\]
for all $u \in C^\infty(S^1)$. Setting $u_n = e^{inx}$, we find that
\[
L^{-1} (b(u_n)_x + (Lu_n)_x) = i\alpha_n u_n,
\]
where $\alpha_n := n + \left( bn/(1 + n^2) \right)$. Applying $A$ to (8) with $u = u_n$ thus yields
\[
2inu_n + (Au_n)_x = i\alpha_n Au_n.
\]
Therefore $v_n := Au_n$ solves the ordinary differential equation
\[
(9) \quad v' - i\alpha_n v = -2inu_n.
\]
For $n \neq 0$, let us solve (9) explicitly. Assume first that $b = 0$. Then
\[
v(x) = (c - 2inx)u_n
\]
for some constant $c$. But this function is never $2\pi$-periodic. Thus we must have $b \neq 0$. However, in this case
\[
(10) \quad v_n = Au_n = \gamma_n e^{i\alpha_n x} + \beta_n u_n \quad (n \neq 0)
\]
with $\beta_n = \frac{2(1 + n^2)}{b}$ and suitable constants $\gamma_n$.

(b) Assume that all $\gamma_n$ vanish, i.e. $Au_n = \beta_n u_n$ for all $n \neq 0$ and $A1 = 1$. In particular, $A$ is a Fourier multiplication operator and thus commutes with $L$. Therefore we can write (7) as
\[
L \left( 2(Au)_x + u(Au)_x \right) = A \left( b(Lu)_x + u(Lu)_x \right).
\]
Inserting $u = u_n$ a direct computation yields
\[
3(1 + 4n^2)\beta_n = (1 + b)(1 + n^2)\beta_{2n}.
\]
Using that $\beta_n = 2(1 + n^2)/b$, this is equivalent to $b = 2$. Then $\beta_n = 1 + n^2$ and therefore $A$ coincides with $L$. 
(c) Let us assume there is a \( p \in \mathbb{Z} \setminus \{0\} \) such that \( \gamma_p \neq 0 \). We shall derive a contradiction. Since \( v_p = Au_p \) must be \( 2\pi \)-periodic, \( \alpha_p \) is an integer. This implies that \( b = k(1 + p^2)/p \) for some non-zero integer \( k \). We set
\[
m := \alpha_p.
\]
Observe that \( m \neq p \), since \( b \neq 0 \). Thus \( (u_p | u_m)_L_2 = 0 \) and (10) implies that
\[
(Au_p | u_m)_L_2 = (\gamma_p e^{i\alpha_m x} | u_m)_L_2 = \gamma_p.
\]
By the symmetry of \( A \) we also find that
\[
\gamma_p = (Au_p | u_m)_L_2 = (u_p | Au_m)_L_2 = \gamma m (u_p | e^{i\alpha_m x})_L_2.
\]
Since \( \gamma_p \neq 0 \) we must have \( \gamma_m \neq 0 \). Again by periodicity we conclude that \( \alpha_m \in \mathbb{Z} \). But then \( \alpha_m = p \), since otherwise we would have \( (u_p | e^{i\alpha_m x})_L_2 = 0 \) and thus again \( \gamma_p = 0 \). We know already that \( b = k(1 + p^2)/p \). Thus \( m = \alpha_p = p + k \) by (11) and the definition \( \alpha_p \). Now we calculate
\[
p = \alpha_m = \alpha_{p+k} = p + k + b \frac{p + k}{1 + (p + k)^2}
\]
\[
= p + k + \frac{k(1 + p^2)}{p} \frac{p + k}{1 + (p + k)^2},
\]
and we find
\[
p (1 + (p + k)^2) k + k(1 + p^2)(p + k) = 0.
\]
Observing that \( k \neq 0 \), an elementary calculation yields
\[
2p^3 + 3p^2 k + pk + 2p + k = 0.
\]
From this we conclude that there is an \( l \in \mathbb{Z} \) such that \( k = pl \). With this we infer from (12) that
\[
(l + 2) ((l + 1)p^2 + 1) = 0.
\]
The only integer solution of this equation is \( l = -2 \). In fact, the solution \( l = \frac{1}{p^2} - 1 \) is only possible if \( p^2 = 1 \) and thus again \( l = -2 \), since for \( p^2 \neq 1 \) we have \( l \notin \mathbb{Z} \). Therefore \( b = -2(1 + p^2) \) and thus \( \alpha_p = -p \).

Moreover, we can conclude that \( \gamma_n = 0 \) whenever \( n \neq 0 \) does not coincide with \( p \) or \( -p \), since otherwise the same calculation as before would show \( b = -2(1 + n^2) \) contradicting \( b = -2(1 + p^2) \).

Now insert \( u = u_p \) in (7). The left-hand side then equals
\[
A^{-1} \left( ip\gamma_p 1 - 3ipu_{2p} \right) = ip\gamma_p 1 - \frac{3ip}{\beta_{2p}} u_{2p};
\]
for the latter identity note that \( 2p \) does not coincide with \( p \) or \( -p \), so that \( \gamma_{2p} = 0 \), and hence \( A^{-1} u_{2p} = u_{2p}/\beta_{2p} \). Note also that \( \beta_p = -1 \). For the right-hand side we get
\[
i(1 + b) p(1 + p^2) \frac{1}{1 + 4p^2} u_{2p}.
\]
Comparing both expressions we conclude that \( p\gamma_p = 0 \) which is a contradiction to \( p \neq 0 \) and \( \gamma_p \neq 0 \). \( \square \)
THE PERIODIC b-EQUATION AND EULER EQUATIONS ON THE CIRCLE

References

1. V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble) 16 (1966), no. fasc. 1, 319–361. MR 0202082
2. V. I. Arnold and B. Khesin, Topological methods in hydrodynamics, Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York, 1998. MR 1612569
3. A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, Arch. Rational Mech. Anal. 183 (2007), 215–239. MR 2278406
4. J. M. Burgers, A mathematical model illustrating the theory of turbulence. Advances in Applied Mechanics, Academic Press Inc., New York, N. Y., 1948, edited by Richard von Mises and Theodore von Kármán., pp. 171–199. MR 0027195
5. A. Constantin and J. Escher, Global weak solutions for a shallow water equation, Indiana Univ. Math. J. 47 (1998), 1527–1546. MR 1687106
6. ———, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (1998), no. 2, 229–243. MR 1668586
7. ———, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, Math. Z. 233 (2000), no. 1, 75–91. MR 1738352
8. A. Constantin and B. Kolev, On the geometric approach to the motion of inertial mechanical systems, J. Phys. A 35 (2002), no. 32, R51–R79. MR 1930889
9. ———, Geodesic flow on the diffeomorphism group of the circle, Comment. Math. Helv. 78 (2003), no. 4, 787–804. MR 2016696
10. A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, Arch. Ration. Mech. Anal. 192 (2009), no. 1, 165–186. MR 2481064
11. A. Degasperis, D. D. Holm, and A. N. I. Hone, A new integrable equation with peakon solutions, Teoret. Mat. Fiz. 133 (2002), no. 2, 170–183. MR 2001531
12. A. Degasperis and M. Procesi, Asymptotic integrability, Symmetry and perturbation theory (Rome, 1998), World Sci. Publ., River Edge, NJ, 1999, pp. 23–37. MR 1844104
13. J. Escher, Wave breaking and shock waves for a periodic shallow water equation, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 365 (2007), no. 1858, 2281–2289. MR 2329148
14. J. Escher and Kolev, The Degasperis-Procesi equation as a non-metric Euler equation, preprint, arXiv:0908.0508, 2010.
15. J. Escher, Y. Liu, and Z. Yin, Global weak solutions and blow-up structure for the Degasperis-Procesi equation, J. Funct. Anal. 241 (2006), no. 2, 457–485. MR 2271927
16. ———, Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation, Indiana Univ. Math. J. 56 (2007), no. 1, 87–117. MR 2309391
17. J. Escher and Z. Yin, Well-posedness, blow-up phenomena, and global solutions for the b-equation, J. Reine Angew. Math. 624 (2008), 51–80. MR 2456624
18. A. N. W. Hone and J. P. Wang, Prolongation algebras and Hamiltonian operators for peakon equations, Inverse Problems 19 (2003), no. 1, 129–145. MR 1964254
19. R. I. Ivanov, On the integrability of a class of nonlinear dispersive waves equations, J. Nonlinear Math. Phys. 12 (2005), no. 4, 462–468. MR 2171998
20. ———, Water waves and integrability, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 365 (2007), no. 1858, 2267–2280. MR 2329147
21. R. S. Johnson, The classical problem of water waves: a reservoir of integrable and nearly-integrable equations, J. Nonlinear Math. Phys. 10 (2003), no. suppl. 1, 72–92. MR 2063546
22. T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Ration. Mech Anal. 58 (1975), 181–205.
23. B. Kolev, Some geometric investigations on the Degasperis-Procesi shallow water equation, Wave Motion 46 (2009), 412–419.
25. S. Kouranbaeva, *The Camassa-Holm equation as a geodesic flow on the diffeomorphism group*, J. Math. Phys. 40 (1999), no. 2, 857–868. MR 1674267

26. A. V. Mikhailov and V. S. Novikov, *Perturbative symmetry approach*, J. Phys. A 35 (2002), no. 22, 4775–4790. MR 1908645

Institute for Applied Mathematics, University of Hannover, D-30167 Hannover, Germany

E-mail address: escher@ifam.uni-hannover.de

Department of Mathematical Sciences, Loughborough University, Leicestershire LE11 3TU, United Kingdom

E-mail address: J.Seiler@lboro.ac.uk