THE COHOMOLOGY RING OF THE COMPLEMENT OF A FINITE FAMILY OF LINEAR SUBSPACES IN A COMPLEX PROJECTIVE SPACE

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Abstract. The integral cohomology ring of the complement of an arrangement of linear subspaces of a finite dimensional complex projective space is determined by combinatorial data, i.e. the intersection poset and the dimension function.

1. Introduction

Let $V$ be an $(n+1)$–dimensional complex vector space and $\mathcal{A}$ a linear arrangement in $V$, i.e. a finite set of proper linear subspaces of $V$. We define $Q$ to be the partially ordered set (poset) of intersections of $\mathcal{A}$, that is the set $\{\bigcap C : C \subset \mathcal{A}\}$ ordered by inclusion. $Q$ is called the intersection poset of $\mathcal{A}$. Note that $Q$ has the maximum $V = \bigcap \emptyset$.

A recurrent theme in the study of arrangements are questions of the type if a certain property of an arrangement is determined by combinatorial data, and if so, how it can be described using only those. For example, De Concini and Procesi have shown that the intersection poset labelled with the dimensions of the intersections determines the rational cohomology ring $H^* (V \setminus \bigcup \mathcal{A}; \mathbb{Q})$ of the complement of the arrangement [DCP95]. Yuzvinsky derived an explicit formula from their work [Yuz02]. Deligne, Goresky, and MacPherson and independently Mark de Longueville and the current author extended this result to cohomology with integral coefficients and clarified the extent to which it holds for real arrangements [DGM00, dLS01].

We will generalize this result to the complement of the projective arrangement $\mathcal{PA} := \{PA : A \in \mathcal{A}\}$ in the $n$-dimensional complex projective space $PV$. This will be a generalization, because the complement of an affine arrangement is the same as the complement of a projective arrangement containing an additional hyperplane, details will be given in Remark 2.9 and Remark 9.2. It will not imply the results on real linear arrangements mentioned above, though.
1.1. **Notation.** For \( u \in Q \), we set
\[
d(u) := \dim_C(u) - 1 = \dim_C(Pu).
\]

We will find it useful to write
\[
Q_{[r,s)} := \{ u \in Q : r \leq d(u) < s \}
\]

etc. for certain subsets of \( Q \).

Our goal is thus to describe the cohomology ring \( H^*(PV \setminus \bigcup P\mathcal{A}) \) in terms of the intersection poset \( Q \) and the dimension function \( d \).

1.2. **Notation.** For a poset \( P \), \( \Delta P \) denotes the order complex of \( P \), i.e. the simplicial complex with \( P \) as the set of vertices and the chains in \( P \) as simplexes.

By Poincaré duality, \( H^i(PV \setminus \bigcup P\mathcal{A}) \cong H_{2n-i}(PV, \bigcup P\mathcal{A}) \), and the homology of \( (PV, \bigcup P\mathcal{A}) \) has been given by Goresky and MacPherson [GM88, Thm. D] as
\[
(1) \quad H_i(PV, \bigcup P\mathcal{A}) \cong \bigoplus_{k=0}^{|i/2|} H_{i-2k}(\Delta Q_{[k,n]}; \Delta Q_{[k,n]}) .
\]

Actually, they state the cohomology analogue of this formula.

The proof we give in Proposition 2.8 constructs the isomorphism using explicit homomorphisms
\[
(2) \quad h_k : H_{*} - 2k(\Delta Q_{[k,n]}; \Delta Q_{[k,n]}) \to H_{*}(PV, \bigcup P\mathcal{A})
\]

(see Definition 2.7) in the spirit of the analogous construction for linear arrangements employed by Ziegler and Živaljević [ZZ93].

Our goal now is to describe cup products in \( H^*(PV \setminus \bigcup P\mathcal{A}) \), respectively the corresponding intersection products in \( H_*(PV, \bigcup P\mathcal{A}) \). For the formulation of the corresponding result for linear arrangements in [dLS01], the following product on the homology of an order complex is central.

1.3. **Definition.** Noting that the map
\[
\wedge : Q \times Q \to Q
\]

\[
u \wedge v := u \cap v
\]

is a simplicial map \( \Delta Q \times \Delta Q \to \Delta Q \), we define
\[
\hat{\times} : C_*(\Delta Q) \otimes C_*(\Delta Q) \to C_*(\Delta Q)
\]

\[
c \hat{\times} d := \wedge_*(c \times d)
\]

and denote the corresponding operation on homology by the same symbol.
With this notation, our main result (Theorem 9.1) can be formulated: For $c \in H^*(\Delta Q_{[k,n]}); d \in H^*(\Delta Q_{[l,n]}), (3)
\begin{align*}
h_k(c) \cdot h_l(d) &= \begin{cases} h_{k+l-n}(c \times d), & k + l \geq n. \\ 0, & k + l < n. \end{cases}
\end{align*}
While our setting is very similar to that in [dLS01], the proof cannot be as direct as the one for linear arrangements presented there, as additional problems arise. In particular, in the case $k + l < n$ of (3) we will not be able to arrange things in such a way that the chains which will naturally represent $h_k(c)$ and $h_l(d)$ do not intersect geometrically, or only in $\bigcup PA$. This is in contrast to the corresponding case for linear arrangements (Prop. 4.3 in [dLS01]). One step towards the solution will be to not only consider the order complex $\Delta Q$, but also, as in [DGM00], to consider the the poset $Q$ itself to be a space, namely the space of strata, equipped with the quotient topology. The combination of both will prove useful in Proposition 6.6.

Finally, in Section 10 we specialize the result to the case of projective arrangements obtaining a presentation of the cohomology ring of the complement in terms of generators and relations in the spirit of Orlik and Solomon. This complements results on linear arrangements by Feichtner and Ziegler [FZ00] and proceeds in the same way as Yuzvinsky derivation of a presentation of the cohomology ring of a linear arrangement with geometric intersection poset [Yuz99].

This article is a presentation of the main result of the author’s thesis [Sch04]. The thesis describes the spectral sequence at the core of Section 2 and Section 3 in a less ad hoc manner, considers real arrangements in more detail, and inspects the relationship between affine and projective arrangements more closely.

I wish to thank my thesis advisor Elmar Vogt for the support during the work on the results presented here. They are a follow-up to joint work with Mark de Longueville. Working with him was a very pleasant experience, and I want to thank him for also always being open to discussions of the current work.

2. THE HOMOLOGY SPECTRAL SEQUENCE AND THE DIRECT SUM DECOMPOSITION OF THE HOMOLOGY OF THE ARRANGEMENT

In this section we construct the isomorphism (1)

2.1. Definition. We define a bigraded abelian group $X$ by
\[
X_{p,q} := \bigoplus_{u \in Q^{q+1}} S_p(Pu_0),
\]
where $S_*$ denotes the simplicial chain complex. Writing $c \otimes \langle u_0, \ldots, u_q \rangle$ for the element $c \in S_*(Pu_0)$ of the summand indexed by $(u_0, \ldots, u_q)$, we make this into a double chain complex by

$$\partial' : X_{p,q} \to X_{p-1,q}$$

$$c \otimes \langle u_0, \ldots, u_q \rangle \mapsto \partial c \otimes \langle u_0, \ldots, u_q \rangle$$

and

$$\partial'' : X_{p,q} \to X_{p,q-1}$$

$$c \otimes \langle u_0, \ldots, u_q \rangle \mapsto \sum_{i=0}^{q-1} (-1)^{p+i} c \otimes \langle u_0, \ldots, \hat{u}_i, \ldots, u_q \rangle,$$

satisfying $\partial'\partial'' + \partial''\partial' = 0$. Note that in the definition of $\partial''$ we regard $S_*(u_0)$ as a subcomplex of $S_*(u_1)$.

The rough idea of the following is that on one hand $H''(\partial H(X)) \cong 0$ for $q \neq 0$ and $H''(\partial H_0(X)) \cong H(PV, \bigcup PA)$, while on the other hand $H_2k(\partial H_0(X)) \cong H_2k(CP^k) \otimes H_q(\Delta Q_{[k,n]} \Delta Q_{[k,n]}), H_2k+1(\partial H_0(X)) \cong 0$, so that there exists a spectral sequence with $E^2$–term isomorphic to the latter that converges against $H(PV, \bigcup PA)$. Convergence alone will not suffice, though. Not only will the $E^2$–term equal the $E^\infty$–term, all extensions will be trivial.

2.2. Proposition. The map

$$\varepsilon : X_{p,q} \to S_p(PV, \bigcup PA)$$

$$c \otimes \langle u_0, \ldots, u_q \rangle \mapsto 0, \quad \text{if } q > 0,$$

$$c \otimes \langle V \rangle \mapsto c$$

is a chain map from the total complex of $X$ to the relative simplicial chain complex and induces an isomorphism $\varepsilon_* : H_*(X) \cong H_*(PV, \bigcup PA)$.\[\]

Proof. For a simplicial simplex $\sigma \in S_*(PV)$ the subset $\{u : \text{im } \sigma \subset Pu\}$ of $Q$ has a minimum that we call $u_\sigma$. We define

$$K : X_{p,q} \to X_{p,q+1}$$

$$\sigma \otimes \langle u_0, \ldots, u_q \rangle \mapsto \begin{cases} (-1)^p \sigma \otimes \langle u_\sigma, u_0, \ldots, u_q \rangle, & u_0 > u_\sigma, \\ 0, & u_0 = u_\sigma, \end{cases}$$

and (do not have to) calculate $(\partial'' K + K \partial')x = x$ for $x \in X_{p,q}$, $q > 0$, so that $H_q(X_{p,*}) = 0$ for $q > 0$. Furthermore im $\left( X_{p,1} \xrightarrow{\varepsilon'} X_{p,0} \right) = \left( \sum_{u \in Q_{[0,n]}} S_p(Pu) \right) \otimes \langle V \rangle$. The claim now follows from the homology version of [God58, Thm I.4.8.1] and the fact that the inclusion $\sum_{u \in Q_{[0,n]}} S_p(Pu) \to S_*(\bigcup PA)$ induces an isomorphism in homology. \[\]
We start the construction of the homomorphisms $h_k$ announced in [2].

2.3. Proposition. Let $k \in \mathbb{N}$. We consider all functions $x$ assigning to each $u \in Q_{[k,n]}$ a system $(x_j^u)_{0 \leq j \leq k}$ of $k+1$ vectors in $u$. We regard the set of all these functions as the complex affine space $\prod_{u \in Q_{[k,n]}} u^{k+1}$. Among those, the set of all functions such that for all $u_0, \ldots, u_r \in Q_{[k,n]}$, $\lambda_0, \ldots, \lambda_r \in \mathbb{C}$ with $u_0 < u_1 < \cdots < u_r$ and $(\lambda_0, \ldots, \lambda_r) \neq 0$, the system of vectors

$$\left(\sum_{i=0}^{r} \lambda_i x^u_j \right)_{0 \leq j \leq k}$$

is linearly independent form a non-empty, Zariski-open set.

Proof. Since a finite intersection of non-empty Zariski-open sets is again non-empty and Zariski-open, it suffices to consider a fixed chain $u_0 < u_1 < \cdots < u_r$, $u_i \in Q_{[k,n]}$. The set

$$\left\{ (x^0, \ldots, x^r) \in u_0^{k+1} \times \cdots \times u_r^{k+1} : \right. \left. \text{There exist } \lambda \in \mathbb{C}^{r+1}, \mu \in \mathbb{C}^{k+1}, \lambda, \mu \neq 0 \text{ with } \sum_{j=0}^{r} \sum_{i=0}^{k} \mu_j \lambda_i x^u_j = 0 \right\}$$

is algebraic by the main theorem of elimination theory [Sha91, I.5, thm 3], because the equation is homogenous in $\lambda$ and $\mu$. To see that the complement of this set is non-empty, we choose a basis $(e_l)_{l=0,\ldots,n}$ of $V$ such that $e_l \in u_i$ for $l \leq i + k$ and set $x^u_j := e_{j+i}$. Now if $\sum_j \sum_i \mu_j \lambda_i x^u_j = 0$ then $\sum_i \lambda_i \mu_{s-i} = 0$ for all $s$, and it follows that $\lambda = 0$, $\mu = 0$.

2.4. Definition and Proposition. Let $k \in \mathbb{N}$ and $x$ be any function as in the preceding proposition, but linear independence of $\left(\sum_{i=0}^{r} \lambda_i x^u_j \right)$ only required for $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Writing points of $\langle u_0, \ldots, u_r \rangle$ as $\sum_{i=0}^{r} \lambda_i u_i$, we define a map

$$f_k: \mathbb{C}P^k \times \Delta Q_{[k,n]} \to PV$$

$$\left( [\mu_0 : \cdots : \mu_k], \sum_{i=0}^{r} \lambda_i u_i \right) \mapsto \left[ \sum_{j=0}^{k} \sum_{i=0}^{r} \mu_j \lambda_i x^u_j \right] .$$

The homotopy class of $f_k$ does not depend on the particular choice of $x$.

Proof. The non-dependence of the homotopy class of $f_k$ on the choice of $x$ follows from the space of possibly choices, which contains a Zariski-open set, being path connected.
2.5. Definition. We define another, albeit degenerated, double complex by

\[ Y_{p,q} := \begin{cases} 
C_q(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}), & p = 2k, \\
0, & p = 2k + 1,
\end{cases} \]

\[ \psi' = 0 \text{ and } \psi'' \text{ being the simplicial boundary operator.} \]

2.6. Proposition. Let \( o_k \in S_{2k}(\mathbb{C}P^k) \) denote a chain representing the orientation class. The map

\[ g : Y_{2k,q} \rightarrow X^\text{tot}_{2k+q} \]

\[ \langle u_0, \ldots, u_q \rangle \mapsto \sum_{i=0}^{q} f^k_*(o_k \times \langle u_0, \ldots, u_i \rangle) \otimes \langle u_i, \ldots, u_q \rangle \]

is a chain map between the total complexes and induces an isomorphism \( H_*(Y) \rightarrow H_*(X) \).

Proof. The fact that \( g \) is a chain map follows from the usual calculation used to verify this property for the Alexander-Whitney diagonal approximation.

\[ g[Y_{s,q}] \subset \bigoplus_{q' \leq q} X_{s,q'}, \text{ that is } g \text{ respects the filtrations by } q \text{ and therefore induces a homomorphism between the spectral sequences defined by those filtrations.} \]

The induced map between the \( E^1 \)-terms \( H_p(Y_{*,q}) = Y_{p,q} \text{ and } H_p(X_{*,q}) = \bigoplus_{u_0 < \ldots < u_q = v} H_p(Pu_0) \) is given, for \( p = 2k \), by

\[ \langle u_0, \ldots, u_q \rangle \mapsto [f^k_*(o_k \times \langle u_0 \rangle)] \otimes \langle u_0, \ldots, u_q \rangle. \]

\( f^k(\cdot, u_0) : \mathbb{C}P^k \rightarrow Pu_0 \text{ is a linear embedding and therefore } [f^k_*(o_k \times \langle u_0 \rangle)] \text{ a generator of } H_p(Pu_0). \text{ So } g \text{ induces an isomorphism between the } E^1 \text{–terms and by God58 Thm I.4.3.1} \text{ it follows that the induced map between the homology groups of the total complexes is also an isomorphism.} \]

\[ \square \]

2.7. Definition. We define a homomorphism

\[ h_k : H_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) \rightarrow H_{2k+r}(PV, \bigcup PA) \]

\[ c \mapsto f^k_*([\mathbb{C}P^k] \times c) \]

where \( [\mathbb{C}P^k] \) denotes the orientation class of \( \mathbb{C}P^k \).

We can now prove the main result of this section.

2.8. Proposition. The map

\[ \sum_{k=0}^{n} h_k : \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n]})[-2k] \rightarrow H_*(PV, \bigcup PA) \]
is an isomorphism of graded abelian groups, in particular

\[ H^{2n-r}(P V \setminus \bigcup PA) \cong H_r(P V, \bigcup PA) \]
\[ \cong \bigoplus_k H_{r-2k}(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) \]
\[ \cong \bigoplus_k \bar{H}_{r-2k-1}(\Delta Q_{[k,n]}). \]

**Proof.** The map \( \sum h_k \) is just the composition of the isomorphisms considered in [Proposition 2.6 and Proposition 2.2].

For the isomorphism \( H_s(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) \xrightarrow{\delta} \bar{H}_{s-1}(\Delta Q_{[k,n]}), \) note that \( \Delta Q_{[k,n]} \) is a cone and hence acyclic, because \( Q_{[k,n]} \) has a maximal element, \( V \).

2.9. **Remark.** If there is an \( A_0 \in A \) with \( d(A_0) = n - 1 \), i.e. if the arrangement contains a hyperplane, then the complement of \( A \) can be considered the complement of an affine arrangement in the affine space \( PV \setminus PA_0 \). Since every affine arrangement arises in this way by adding a hyperplane at infinity, the usual formula for the complement of an affine (or linear) arrangement can be derived from the formula for projective arrangements.

This is called an ‘interesting exercise’ in [GM88].

We sketch how to do this. Set \( Q' := Q \setminus \{ A_0 \land q : q \in Q \}. \) \( Q' \) is the poset of all intersections that are not contained in \( A_0 \) and is can be defined as the intersection poset of the affine arrangement in question, without reference to \( A_0 \). For \( 0 \leq k < n \), the simplicial complex \( \Delta(Q_{[k,n]} \setminus Q'_{[k]}) \) is acyclic, since it contains the cone \( \Delta \{ q : q \leq A_0, d(q) \geq k \} \) as a deformation retract. Therefore the first map in

\[ H(\Delta Q_{[k,n]}, \Delta Q'_{[k,n]}) \xrightarrow{\cong} H(\Delta Q_{[k,n]} \cup (Q_{[k,n]} \setminus Q'_{[k]})), \]

which is induced by inclusion, is an isomorphism. The second map is also induced by inclusion and is an isomorphism by excision. The last group splits as

\[ \bigoplus_{u \in Q'_{[k]}} H(\Delta [u, V], \Delta [u, V] \cup \Delta (u, V)) \xrightarrow{\cong} H(\Delta Q'_{[k,n]} \cup \Delta Q_{[k,n]}), \]
again by maps induced by inclusion. The isomorphisms also hold for the trivial case $k = n$. Putting all this together yields

$$
\bigoplus_{k=0}^{n} H(\Delta Q_{[k,n]}, \Delta Q_{[k,n]})[-2k] = \bigoplus_{u \in Q'_{[0,n]}} H(\Delta [u, V], \Delta [u, V] \cup \Delta(u, V))[-2d(u)]
$$

as a description of the cohomology of the complement of the affine arrangement in $PV \setminus PA_0$.

The following will be needed when deriving product information from the spectral sequence in Section 3.

2.10. Proposition. Let $F$ be the filtration on the total complex $X$ defined by $(F_sX)_t := \bigoplus_{q=0}^{s} X_{t-q,q}$. Then

$$\text{im} \left( H_i(F_sX) \xrightarrow{\varepsilon_s} H_i(PV \cup PA) \right) = \bigoplus_{2k \geq i-s} \text{im} \left( H_i-2k(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) \xrightarrow{h_k} H_i(PV \cup PA) \right).$$

Proof. Setting $(F_sY)_t := \bigoplus_{q=0}^{s} Y_{t-q,q}$, the proof of Proposition 2.6 shows that $g$ also induces isomorphisms $g_\ast : H_\ast(F_sY) \xrightarrow{\cong} H_\ast(F_sX)$. Therefore

$$\text{im} \left( H_i(F_sX) \xrightarrow{\varepsilon_s} H_i(PV \cup PA) \right) = \text{im} \left( H_i(F_sY) \xrightarrow{\varepsilon_s \circ g_\ast} H_i(PV \cup PA) \right)$$

as claimed. \qed

3. The graded ring structure

In this section we show the following graded version of (3).

3.1. Proposition. Let $c \in H_\ast(Q_{[k,n]}, Q_{[k,n]})$, $d \in H_\ast(Q_{[l,n]}, Q_{[l,n]})$, $k + l \geq n$. Then

$$h_k(c) \ast h_l(d) = h_{k+l-n}(c \ast d) + \sum_{i : k+l-n < i \leq n} h_i(r_i)$$

for classes $r_i \in H_\ast(Q_{[i,n]}, Q_{[i,n]})$. 

To prove this proposition we will make the spectral sequence of the preceding section into a multiplicative spectral sequence. We will use simplicial cohomology to achieve this.

Let $M$ be a triangulation of $PV$, which is a barycentric subdivision of a triangulation of which all $A \in PA$ are subcomplexes. In particular, all $Pu, u \in Q$, are full subcomplexes of $M$. We will denote the face poset of this triangulation by $FM$ and by $C(FM)$ the chain complex of ascending (from 0-simplices to $2n$-simplices) chains in $FM$.

3.2. Definition and Proposition. For a subcomplex $A$ of $M$, cap products

$$C^r(FM, FM \setminus FA) \otimes C_s(FM) \xrightarrow{\cap} C_{s-r}(FA)$$

$$C^r(FM \setminus FA) \otimes C_s(FM) \xrightarrow{\cap} C_{s-r}(FM, FA)$$

$$h \otimes \langle f_0, \ldots, f_s \rangle \mapsto (-1)^{r(s-r)} h(\langle f_{s-r}, \ldots, f_s \rangle) \langle f_0, \ldots, f_{s-r} \rangle$$

are defined, where

$$C^r(FM, FM \setminus FA) = \ker(\text{Hom}(C_r(FM), R) \to \text{Hom}(C_r(FM \setminus FA), R)),$$

$$C_{s-r}(FM, FA) = \coker(C_{s-r}(FA) \to C_{s-r}(FM)).$$

Proof. If $\langle f_0, \ldots, f_{s-r} \rangle$ is not in $C(FA)$, then $f_{s-r}$ is not in $A$ and therefore $\langle f_{s-r}, \ldots, f_s \rangle$ is in $C(FM \setminus FA)$, so that $h(\langle f_{s-r}, \ldots, f_s \rangle) = 0$ for the first kind of product or $h(\langle f_{s-r}, \ldots, f_s \rangle)$ is defined for the second kind of product. □

Since $\Delta(FM)$ is just the barycentric subdivision of $M$, we have $H(C(FM)) \cong H(M) = H(PV)$. Let $o \in C_{2n}(FM)$ represent the orientation class $[M] \in H_{2n}(M)$. Regarding $C(FA)$ as a subcomplex of the singular complex $S(A)$, this yields a map $C(FM, FM \setminus FA) \xrightarrow{\cap} S(A)$ which induces an isomorphism in homology, if $A$ is a full subcomplex. $\Delta(FM \setminus FA)$ is the subcomplex of the barycentric subdivision of $M$ that consists of all simplices which do not meet $A$. It is the complement of an open normal neighbourhood of $A$, if $A$ is a full subcomplex.

3.3. Definition and Proposition. If $A, B$ are subcomplexes of $M$, a cup product

$$C^r(FM, FM \setminus FA) \otimes C^s(FM, FM \setminus FB) \to C^{r+s}(FM, FM \setminus F(A \cap B))$$

$$g \otimes h \mapsto g \cup h,$$

$$(g \cup h)(\langle f_0, \ldots, f_{r+s} \rangle) := (-1)^r g(\langle f_0, \ldots, f_r \rangle) h(\langle f_r, \ldots, f_{r+s} \rangle),$$

is defined.

Proof. If $\langle f_0, \ldots, f_{r+s} \rangle$ is in $C(FM \setminus F(A \cap B))$, then $f_0 \not\in F(A \cap B)$ and therefore either $f_0 \not\in FA$ and $\langle f_0, \ldots, f_r \rangle \in C(FM \setminus FA)$, or $f_0 \not\in FB$ and $\langle f_r, \ldots, f_{r+s} \rangle \in C(FM \setminus FB)$. In either case $(g \cup h)(\langle f_0, \ldots, f_{r+s} \rangle) = 0$. □
We now introduce the double complex that will serve as a dual of the double complex $X$.

3.4. Definition. We define a double complex $Z$ by

$$Z_{p,q} := \bigoplus_{u_0 < \cdots < u_q = V} C^{-p}(FM, FM \setminus F(Pu_0))$$

and, using notation as in Definition 2.1,

$$\delta': Z_{p,q} \rightarrow Z_{p-1,q}$$

$$f \otimes \langle u_0, \ldots, u_q \rangle \mapsto \delta f \otimes \langle u_0, \ldots, u_q \rangle$$

and

$$\delta'': Z_{p,q} \rightarrow Z_{p,q-1}$$

$$f \otimes \langle u_0, \ldots, u_q \rangle \mapsto \sum_{i=0}^{q-1} (-1)^{p+i} f \otimes \langle u_0, \ldots, \hat{u}_i, \ldots, u_q \rangle,$$

satisfying $\delta'\delta'' + \delta''\delta' = 0$.

3.5. Definition. We define products on $Z$ by

$$Z_{p,q} \otimes Z_{p',q'} \rightarrow Z_{p+p',q+q'}$$

$$(f \otimes c) \otimes (g \otimes d) \mapsto (-1)^{qp'} (f \lrcorner g) \otimes (c \times d)$$

and on $Y$ by

$$Y_{p,q} \otimes Y_{p',q'} \rightarrow Y_{p+p'-n,q+q'}$$

$$c \otimes d \mapsto c \times d.$$

3.6. Definition and Proposition. The map

$$g': Z_{p,q} \rightarrow X_{p+2n,q}$$

$$f \otimes c \mapsto (f \lrcorner o) \otimes c$$

leads to a commutative diagram

$$
\begin{array}{ccc}
H_*(Z) & \xrightarrow{\cong} & H_*(X) \\
\downarrow & & \downarrow \cong \\
H^*(PV \setminus \bigcup PA) & \xrightarrow{\sim [PV]} & H_*(PV, \bigcup PA),
\end{array}
$$

where the arrow on the left hand side is a ring homomorphism.
Proof. The diagram
\[
\begin{array}{ccc}
Z_{p,q} & \xrightarrow{g'} & X_{p+2n,q} \\
\downarrow{\varepsilon'} & & \downarrow{\varepsilon} \\
C^{-p,q}(FM \setminus F(\bigcup P\mathcal{A})) & \xrightarrow{\sim} & S_{2n+p+q}(PV, \bigcup P\mathcal{A}), \\
\downarrow{\sim} & & \\
S^{-p,q}(PV \setminus \bigcup P\mathcal{A}) & & 
\end{array}
\]
where
\[
\varepsilon': Z_{p,q} \to C^{-p,q}(FM \setminus F(\bigcup P\mathcal{A}))
\]
\[
f \otimes \langle u_0, \ldots, u_q \rangle \mapsto 0, \quad \text{if } q > 0,
\]
\[
f \otimes \langle V \rangle \mapsto f,
\]
is commutative. When checking that \( \varepsilon' \) is a chain map from the total complex, the interesting case is \( Z_{p,1} \). Here
\[
\varepsilon'(\partial(f \otimes \langle u, V \rangle)) = \varepsilon'(\delta f \otimes \langle u, V \rangle + (-1)^p f \otimes \langle V \rangle) = (-1)^p f,
\]
and since \( f \in C^{-p}(FM, FM \setminus F(Pu_0)) \), the restriction of \( f \) to the chains from \( C(FM \setminus F(\bigcup P\mathcal{A})) \) is zero. That \( \varepsilon' \) respects products is obvious.

The map \( g' \) induces an isomorphism \( \H^*(Z_{\bullet,q}) \to \H^*(X_{\bullet,q}) \) and therefore an isomorphism \( \H_*(Z) \to \H_*(X) \) between the total complexes. The map \( H^*(PV \setminus \bigcup P\mathcal{A}) \to H^*(C(FM \setminus F(\bigcup P\mathcal{A})) \) induced by the lower left arrow is also an isomorphism. \( \square \)

3.7. Proposition. The two isomorphisms
\[
E_1^1(Z) \xrightarrow{g'_*} E_1^1(X) \xleftarrow{g_*} E_1(Y)
\]
induce the same multiplication on \( E_1(X) \).

Proof. As noted in the proof of Proposition 2.6 we have
\[
E_{p,q}^1(X) \cong \H_p(X_{\bullet,q}) \cong \bigoplus_{u_0 < \ldots < u_q = V} H_p(Pu_0)
\]
and
\[
H_p(Pu_0) \cong \begin{cases} \Z, & p = 2k, 0 \leq k \leq d(u_0), \\ 0, & \text{otherwise.} \end{cases}
\]
For \( u \in Q_{[k,n]} \), there is a well defined generator \( e_k^u \in H_{2k}(Pu) \) with the property that \( e_k^u \) is the image of the canonical orientation class \([CP^k]\) under a complex embedding \( CP^k \to Pu \). In particular the map
\[
Y_{2k,q} = \H_{2k}(Y_{\bullet,q}) \cong E_{2k,q}^1(Y) \xrightarrow{g_1'} E_{2k,q}^1(X) \cong \bigoplus_{u_0 < \ldots < u_q = V} H_{2k}(Pu_0)
\]
takes \( \langle u_0, \ldots, u_q \rangle \) to \( e_k^{u_0} \otimes \langle u_0, \ldots, u_q \rangle \) and induces the multiplication

\[
(e_k^{u_0} \otimes \langle u_0, \ldots, u_q \rangle) \otimes (e_l^{v_0} \otimes \langle v_0, \ldots, v_{q'} \rangle)
\]

\[
\mapsto \begin{cases} 
  e_{k+l-n}^{u_0 \wedge v_0} \otimes (\langle u_0, \ldots, u_q \rangle \hat{\times} \langle v_0, \ldots, v_{q'} \rangle), & k + l \geq n, \\
  0, & k + l < n
\end{cases}
\]
on \( E^1(X) \).

On the other hand, the multiplication on \( E^1(X) \) induced via \( g' \) by that on \( Z \) is seen to be

\[
(a \otimes c) \otimes (b \otimes d) \mapsto (-1)^{|c|+|b|} (a \cdot b) \otimes (c \hat{\times} d),
\]

where the intersection product is defined by commutativity of

\[
H^*(PV, PV \setminus P u) \otimes H^*(PV, PV \setminus P v) \xrightarrow{\sim} \xrightarrow{\sim} H^*(PV, PV \setminus (P u \cap P v))
\]

\[
H_* (P u) \otimes H_* (P v) \xrightarrow{\bullet} H_* (P u \cap P v),
\]

and we have

\[
e_k^{u_0} \cdot e_l^{v_0} = \begin{cases} 
  e_{k+l-n}^{u_0 \wedge v_0}, & k + l \geq n, \\
  0, & k + l < n
\end{cases}
\]

so that the two products agree. \( \square \)

**Proof of Proposition 3.1.** By Proposition 3.6 the multiplication on \( H(X) \) induced by that on \( Z \) is carried by \( \varepsilon_* \) into the intersection product on \( H_* (P V, \bigcup P A) \). By Proposition 3.7 the complexes \( Z \) and \( Y \) have multiplicative spectral sequences which are isomorphic from the \( E^1 \)-terms on, and these induce a multiplicative structure on the spectral sequence of \( X \). The \( E^\infty \)-term of this spectral sequence (which degenerates at the \( E^2 \)-term) is the graded object corresponding to the filtration considered in Proposition 2.10. By Proposition 3.7 the \( E^\infty \)-term of \( X \) is isomorphic to that of \( Y \). \( \square \)

### 4. Intersecting with a Hyperplane

Let \( \Lambda : V \to \mathbb{C} \) be a linear functional that vanishes on no element of \( Q_{[0,n]} \) and \( H := \ker \Lambda \). \( \mathcal{A} \) induces an arrangement \( \mathcal{A}^H := \{ A \cap H : A \in \mathcal{A} \} \) in \( H \).

If we denote the intersection poset of \( \mathcal{A}^H \) by \( Q^H \),

\[
\eta : Q_{[0,n]} \to Q^H_{[0,n-1]}
\]

\[
q \mapsto q \cap H
\]

is an isomorphism lowering dimensions by one.
4.1. Proposition. Consider the inclusion map \( i: (PH, \bigcup PA^H) \to (PV, \bigcup PA) \). For \( c \in H_* (\Delta Q_{[k,n]} \setminus \Delta Q_{[k,n]}) \) we have

\[
\iota_i(h_k(c)) = \begin{cases} 
  h_{k-1}^H(\eta_*(c)), & k > 0, \\
  0, & k = 0.
\end{cases}
\]

In particular \( \ker \iota_i = \operatorname{im} h_0 \).

Proof. We first choose \((x_u^*)_{0 \leq j < k}\) with \(x_u^* \in u \cap H\) satisfying the conditions of [Definition 2.4] and therefore defining functions \(f^k\) and \(h_k^H\). Now for each \( u \in Q_{[k,n]} \) we choose \(x^u_k \in u\) with \(\Lambda(x^u_k) = 1\). \((x^u_j)_{0 \leq j \leq k}\) then also satisfies the conditions of [Definition 2.4] and can be used to define \(f^k\) and \(h_k\).

Indeed we calculate

\[
\Lambda \left( \sum_{j=0}^k \sum_{i=0}^r \mu_j \lambda_i x^u_j \right) = \sum_{i=0}^r \mu_k \lambda_i = \mu_k.
\]

First this implies that \(f^k(x, y) \in H\) iff \(x \in \mathbb{C}^{k-1} \subset \mathbb{C}^k\). In particular \(f^0\) misses \(H\), which proves that part of the proposition, and we now assume \(k > 0\). The equation also implies that for \(x = [x_0 : \cdots : x_{k-1}] \in \mathbb{C}^{k-1}\) and \(y \in \Delta(Q_{[k,n]})\) the map \(\mu \mapsto f^k([x_0 : \cdots : x_{k-1} : \mu], y)\) meets \(H\) transversally. Furthermore

\[
f^k \left( [\mu_0 : \cdots : \mu_{k-1} : 0], \sum_{i=0}^r \lambda_i u_i \right) = \left[ \sum_{j=0}^{k-1} \sum_{i=0}^r \mu_j \lambda_i x^u_j \right] = f_{H}^{k-1} \left( [\mu_0 : \cdots : \mu_{k-1}], \sum_{i=0}^r \lambda_i \eta(u_i) \right),
\]

which proves the proposition as we will now show in more detail.

Let \(\vartheta \in H^2(PV, PV \setminus PH)\) be the Thom class of (the normal bundle of) \(PH\) in \(PV\), i.e. the class satisfying \(\vartheta \sim [PV] = [PH]\). By the above calculations \((f^k)^* (\vartheta) \in H^2(\mathbb{C}^k \times \Delta Q_{[k,n]} \setminus (\mathbb{C}^k \times \mathbb{C}^{k-1}) \times \Delta Q_{[k,n]}\)) is the Thom class of \(\mathbb{C}^{k-1} \times \Delta Q_{[k,n]}\) in \(\mathbb{C}^k \times \Delta Q_{[k,n]}\) which equals the class \(\alpha \times 1\) where \(\alpha \in H^2(\mathbb{C}^k, \mathbb{C}^k \setminus \mathbb{C}^{k-1})\) which is again a Thom class and
maps to the canonical generator of $H^*(\mathbb{C}P^k)$. We finally calculate

$$i_!(h_k(c)) = \vartheta \cdot h_k(c)$$

$$= \vartheta \cdot f_!^k\left([\mathbb{C}P^k] \times c\right)$$

$$= f_!^k\left((f^k)_!(\vartheta) \cdot ([\mathbb{C}P^k] \times c)\right)$$

$$= f_!^k\left((\alpha \times 1) \cdot ([\mathbb{C}P^k] \times c)\right)$$

$$= f_!^k\left((\alpha \cdot [\mathbb{C}P^k]) \times (1 \cdot c)\right)$$

$$= f_!^k\left([\mathbb{C}P^{k-1}] \times c\right)$$

$$= h_{k-1}(\eta_*(c))$$

as claimed. 

This proposition will enable us to prove statements about arrangements by induction on the dimension of $V$. An example is the derivation of Theorem 9.1 from Propositions 3.1 and 8.1 in Section 9.

5. Real projective arrangements and an example

We will investigate, without giving full proofs, what of the above remains true for real projective arrangements, see how the analogue of the product formula we are trying to prove fails for real projective arrangements, and sketch the difference between real and complex arrangements that will allow us to prove the formula for complex arrangements.

For a real projective arrangement, functions $f^k$ as in Definition 2.4 exist but are not uniquely defined up to homotopy. A direct sum decomposition as in Proposition 2.8 will not be achieved for integral coefficients, but working over the ring $\mathbb{Z}_2$, we obtain an isomorphism

$$\sum_{k=0}^{n} h_k : \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}; \mathbb{Z}_2)[-k] \xrightarrow{\cong} H_*(PV, \bigcup PA; \mathbb{Z}_2).$$

The proof of Proposition 3.1 will require no additional changes.

Let $k, l \geq 0$, $n := k + l + 1$. We consider the following subspaces of $\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^2$.

$$u := \mathbb{R}^k \times \{0\} \times (\mathbb{R} \cdot (0, 1)), \quad \tilde{u} := \{0\} \times \mathbb{R}^l \times (\mathbb{R} \cdot (1, 1)),$$

$$v := \mathbb{R}^k \times \{0\} \times (\mathbb{R} \cdot (4, 1)), \quad \tilde{v} := \{0\} \times \mathbb{R}^l \times (\mathbb{R} \cdot (5, 1)).$$

In the arrangement $\tilde{A} := \{u, v, \tilde{u}, \tilde{v}\}$ we will find classes $c \in H(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}), \ d \in H(\Delta Q_{[l,n]}, \Delta Q_{[l,n]})$ with $h_k(c) \cdot h_l(d) \neq 0$, although $k + l < n$. 

\[\]
The combinatorial data of $\hat{A}$ are given by the intersection poset

$$Q_{[0,n]} = \begin{array}{c}
\hat{Q} & \xrightarrow{V} \\
\hat{u} & \xleftarrow{\hat{v}} \\
& \hat{u} \cap \hat{v}
\end{array}$$

with $u \cap v$ and $\hat{u} \cap \hat{v}$ only present for $k > 0$ and $l > 0$ respectively, and the dimensions $d(u) = d(v) = k$, $d(u \cap v) = k - 1$, $d(\hat{u}) = d(\hat{v}) = l$, $d(\hat{u} \cap \hat{v}) = l - 1$, $d(V) = n$. In case of $k = l$, we can, if we want to, avoid the intersections $u \cap v$ and $\hat{u} \cap \hat{v}$ by a small change of $u$ and $\hat{u}$ without substantially affecting the calculations below. This shows that, in contrast to the case of affine arrangements in [DGM00] and [dLS01], a simple condition on the occurring codimensions will not be enough for the product formula to extend from complex to real arrangements.

To simplify the pictures below and to have the notation parallel that of Section 8, we consider $\hat{A}$ to be the union of the two arrangements $A := \{u, v\}$ and $\hat{A} := \{\hat{u}, \hat{v}\}$. The arrangement $\hat{A}$ is in general position with respect to $A$ as in Definition 7.4.

Denoting the intersection posets of $A$ and $\hat{A}$ by $Q$ and $\hat{Q}$ respectively, we have $H_1(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $c := [\langle u, V \rangle + \langle v, V \rangle]$, and $H_1(\Delta \hat{Q}_{[l,n]}, \Delta \hat{Q}_{[l,n]}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $d := [\langle \hat{u}, V \rangle + \langle \hat{v}, V \rangle]$. For the definition of

$$f^k: \mathbb{R}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) \to (V, \bigcup PA),$$

$$f^l: \mathbb{R}P^l \times (\Delta \hat{Q}_{[l,n]}, \Delta \hat{Q}_{[l,n]}) \to (V, \bigcup P\hat{A}),$$

and hence of

$$h_k: H_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}; \mathbb{Z}_2) \to H_{r+k}(V, \bigcup PA; \mathbb{Z}_2),$$

$$\hat{h}_l: H_r(\Delta \hat{Q}_{[l,n]}, \Delta \hat{Q}_{[l,n]}; \mathbb{Z}_2) \to H_{r+l}(V, \bigcup P\hat{A}; \mathbb{Z}_2)$$

we set, with $(e_0, \ldots, e_n)$ the standard basis of $V = \mathbb{R}^{n+1}$,

$$x^u_j := \begin{cases} e_j, & j < k, \\
e_{k+l+1}, & j = k, \end{cases}$$

$$\tilde{x}^{\hat{u}}_j := \begin{cases} e_{k+j}, & j < l, \\
e_{k+l} + e_{k+l+1}, & j = l, \end{cases}$$

$$x^v_j := \begin{cases} e_j, & j < k, \\
4e_{k+l} + e_{k+l+1}, & j = k, \end{cases}$$

$$\tilde{x}^{\hat{v}}_j := \begin{cases} e_{k+j}, & j < l, \\
5e_{k+l} + e_{k+l+1}, & j = l, \end{cases}$$

$$x^\hat{v}_j := \begin{cases} e_j, & j < k, \\
2e_{k+l} + e_{k+l+1}, & j = k, \end{cases}$$

$$\tilde{x}^{\hat{\hat{v}}}_j := \begin{cases} e_{k+j}, & j < l, \\
3e_{k+l} + e_{k+l+1}, & j = l. \end{cases}$$

To determine $h_k(c) \bullet h_l(d)$, we first have a look at the geometric intersection $S := f^k[\mathbb{R}P^k \times \Delta Q_{[k,n]}] \cap f^l[\mathbb{R}P^l \times \Delta \hat{Q}_{[l,n]}]$. For $x \in \Delta(Q_{[k,n]})$, $y \in \Delta(\hat{Q}_{[l,n]})$, the intersection $f[\mathbb{R}P^k \times \{x\}] \cap f[\mathbb{R}P^l \times \{y\}]$ is either empty or
consists of a single point. The left of the following two pictures shows the
two dimensional simplicial complex $\Delta Q_{[k,n]} \times \Delta \tilde{Q}_{[l,n]} = \Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]})$.

![Diagram](image)

The dotted line depicts the set $\tilde{S}$ of those points $(x,y)$ for which the intersection is nonempty. $S$ is a connected 1-dimensional manifold with boundary, and we can see from the picture that one boundary point lies in $u \cap V = u$ and the other one in $V \cap \tilde{v} = \tilde{v}$. A closer look at $S$, which is the intersection of two manifolds that meet transversely, shows that indeed $h_k(c) \cdot \tilde{h}_l(d) = \tilde{h}_0([\langle u, V \rangle + \langle \tilde{v}, V \rangle])$. $\langle u, V \rangle + \langle \tilde{v}, V \rangle$ is a generator of $H_1(\Delta \tilde{Q}_{[0,n]} \times \Delta \tilde{Q}_{[0,n]} ; \mathbb{Z}_2)$, therefore $h_k(c) \cdot \tilde{h}_l(d) \neq 0$.

We equip $Q \times \tilde{Q}$ with a dimension function $d(p,q) := d(p) + d(q)$. The map $Q \times \tilde{Q} \to \tilde{Q}$, $(p,q) \mapsto p \cap q$, sends $((Q \times \tilde{Q})_{[n,2n]} , (Q \times \tilde{Q})_{[2n,2n]})$ to $(\tilde{Q}_{[0,n]} , \tilde{Q}_{[0,n]})$. In the picture, the border of the square is $\Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} ) \times \tilde{Q}_{[0,n]}$ and the four vertices at the corners are $\Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[0,n]}$. Under the composition of maps

$$
(\Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} ) \setminus \Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[0,n]} , \Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[0,n]} \setminus \Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[0,n]} ) \\
(\Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[n,2n]} , \Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[n,2n]} ) \\
(\Delta \tilde{Q}_{[0,n]} , \Delta \tilde{Q}_{[0,n]} )
$$

the first being a deformation retraction and the second given by inclusion, in our example $(\tilde{S}, \partial \tilde{S})$ is mapped to the dotted line in the picture on the right. This set carries the relative cycle $\langle u, V \rangle + \langle \tilde{v}, V \rangle$ representing $h_k(c) \cdot \tilde{h}_l(d)$. We will see in Section 8 that this is not just a coincidence.

When considering complex arrangements we will see that in the above situation we gain one dimension compared to real arrangements, and $\tilde{S}$ will miss the cone with top the vertex $(V, V)$ and base $\Delta (Q_{[k,n]} \times \tilde{Q}_{[l,n]} )_{[0,n]}$. The map of $(\tilde{S}, \partial \tilde{S})$ to $(\Delta \tilde{Q}_{[0,n]} , \Delta \tilde{Q}_{[0,n]} )$ will therefore miss the vertex $V$ and be homotopic to a map with image in $\Delta \tilde{Q}_{[0,n]}$. This is the idea behind the proof of Proposition 8.2 although it will be technically a bit different.
6. Recovering the direct sum decomposition

When discussing the real example, it seemed plausible that a certain subset of the order complex of the intersection poset should carry the inverse image of the considered intersection product under the isomorphism $\sum_k h_k$. We now develop tools that allow to actually prove this kind of proposition.

More generally, given a class in $H_*(PV, \bigcup P_A)$ we want to determine the corresponding element of $\bigoplus_k H_*(\Delta Q_{[0,n]}, \Delta Q_{[k,n]})$. Because of Proposition 4.1 it will suffice to identify the part in the summand $H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n]})$. The key to this will be to not only consider the map $f_0: \Delta Q_{[0,n]} \rightarrow PV$, but also a map $PV \rightarrow Q_{[0,n]}$, where the poset $Q$ is topologized in an appropriate way yielding the space of strata. While we have up to this point used only the former map, in [DGM00] a description of the cohomology ring of the complement of an affine arrangement is obtained using exclusively the latter map. Here the interplay of both maps will be important.

6.1. Definition. Let $P$ be a poset. We make $P$ into a topological space by calling a set $O \subset P$ open, iff $x \in O$ implies $y \in O$ for all $y \geq x$.

6.2. Lemma. Let $X$ be a space, $P$ a poset, $A \subset X$, $R \subset P$. If $f, g: (X, A) \rightarrow (P, R)$ are continuous maps with $f(x) \geq g(x)$ for all $x \in X$, then $f \simeq g$.

Proof. The desired homotopy is given by

$$H: (X, A) \times I \rightarrow (P, R)$$

$$(x, t) \mapsto \begin{cases} f(x), & t < 1, \\ g(x), & t = 1. \end{cases}$$

This map is continuous, since $g^{-1}[O] \subset f^{-1}[O]$ for open $O \subset X$, and therefore $H^{-1}[O] = f^{-1}[O] \times [0,1) \cup g^{-1}[O] \times \{1\} = f^{-1}[O] \times [0,1) \cup g^{-1}[O] \times I$. □

6.3. Lemma. If $P$ has a minimum or a maximum, then $P$ is contractible.

Proof. By the preceding lemma, the constant map to the minimum respectively the maximum is homotopic to the identity. □

6.4. Definition. Let $X$ be a space equipped with a covering $\mathcal{C}$ by closed sets and let $P$ be the poset $P := \{\bigcap M: \emptyset \neq M \subset \mathcal{C}, \bigcap M \neq \emptyset\}$, ordered by inclusion. We define a continuous map $s: X \rightarrow P$

$$x \mapsto \min\{p \in P: x \in P\} = \bigcap\{C \in \mathcal{C}: x \in C\}.$$ 

In particular we consider the following two kinds of maps. For our arrangement $A$ we consider the map $s^A: PV \rightarrow Q_{[0,n]}$ corresponding to the covering
For a poset $P$ which has unique minima in the sense that for $M \subset P$, $M \neq \emptyset$, the set $\{p \in P : p \leq q \text{ for all } q \in M\}$ is either empty or of the form $\{p : p \leq q\}$ for a $q \in P$, we consider the map $s^P : \Delta P \to P$ arising from the covering of $\Delta P$ by the subspaces $\Delta(\{p : p' \leq p\})$, $p \in P$.

6.5. **Lemma.** For a finite poset $P$ and $R \subset P$, both satisfying the condition regarding minima of the preceding definition, the map $s^P : H_*(\Delta P, \Delta R) \cong H_*(P, R)$ is an isomorphism.

**Proof.** We may assume $R = \emptyset$, because the general case will follow by an application of the five lemma.

If $P = \emptyset$, $H_*(\Delta P) \cong 0 \cong H_*(P)$.

If $P \neq \emptyset$, $P$ has a minimal element $m$. We set $M := \{p : p \geq m\}$. $P = M \cup (P \setminus \{m\})$ is an open covering, and $\Delta P = \Delta M \cup \Delta (P \setminus \{m\})$ a covering by subcomplexes, and $\Delta M \cap \Delta (P \setminus \{m\}) = \Delta (M \setminus \{m\})$. By induction on the number of elements, the maps $H_*(\Delta (P \setminus \{m\})) \to H_*(P \setminus \{m\})$ and $H_*(\Delta (M \setminus \{m\})) \to H_*(M \setminus \{m\})$ are isomorphisms. $H_*(\Delta M) \to H_*(M)$ is an isomorphism, because both spaces are contractible. It follows by the Mayer-Vietoris theorem and the five-lemma, that $H_*(\Delta P) \to H_*(P)$ is also an isomorphism.

6.6. **Proposition.** The composition

$$H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) \xrightarrow{h_k} H_*(PV, \bigcup PA) \xrightarrow{s_A^*} H_*(Q_{[0,n]}, Q_{[0,n]})$$

is an isomorphism for $k = 0$ and zero for $k > 0$.

**Proof.** Consider the diagram

$$\begin{array}{ccc}
\mathbb{C}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n]}) & \xrightarrow{f^k} & (PV, \bigcup PA) \\
\downarrow \pi & & \downarrow s_A^*
\end{array}$$

where $i$ is the inclusion map and $\pi$ the projection onto the second factor. By construction of $f^k$, $f^k \{x\} \times \{u_0, \ldots, u_q\} \subset Pu_q$, that is $s_A^*(f^k(x, y)) \leq s_Q^*(y)$, and by **Lemma 6.2** this implies the homotopy commutativity of the diagram.

For $k = 0$, $\pi$ is a homeomorphism, $i$ the identity, and $h_0$ equals $f_0^*$. By an isomorphism. Therefore $s_A^* \circ h_0$ is an isomorphism, because $s_A^* Q_{[0,n]}$ is an isomorphism by **Lemma 6.5**.

For $k > 0$, $s_A^*(h_k(c)) = s_A^*(f_k^*([\mathbb{C}P^k] \times c)) = (i \circ s_Q^*[k,n])_*(\pi_*([\mathbb{C}P^k] \times c)) = (i \circ s_Q^*[k,n])_*(0) = 0$. 

\[\square\]
The intersection of two sets can be identified with the intersection of their cartesian product with a diagonal. Similarly the intersection product of two homology classes equals the image of their cross product under the image of the transfer map associated with the diagonal map. We therefore study the products of two arrangements.

Already in the real example we have discussed, it was useful to assume the homology classes of which the product was to be determined to be carried by different arrangements. So we now assume to be given a second arrangement \( \tilde{A} \) in \( V \) with intersection poset \( \tilde{Q} \). For the first part of this section the arrangement could be in a vector space different from \( V \), but we will have no use for this generality later on.

We equip the poset \( Q \times \tilde{Q} \) with a dimension function \( d \) by \( d(u,v) := d(u) + d(v) \). The counterpart of \( h \) for \( \tilde{A} \) will be denoted by \( \tilde{h} \) and so on.

As noted above, we will be interested in cross products.

7.1. Definition and Proposition. Any choice of \( (y_i^{u,v})_{i=0,\ldots,k}, (z_i^{u,v})_{i=0,\ldots,l} \) for \( (u,v) \in Q[k,n] \times \tilde{Q}[l,n] \) with \( y_i^{u,v} \in u \), \( z_i^{u,v} \in v \) and such that for all \( (u_0,v_0) < \cdots < (u_m,v_m) \) and \( \lambda \in \Delta^m \) the system \( \sum_j \lambda_j y_i^{u_j,v_j} \) as well as the system \( \sum_j \lambda_j z_i^{u_j,v_j} \) is linearly independent, yields a map

\[
\begin{align*}
g : \mathbb{C}^k \times \mathbb{C}^l & \to \mathbb{C}^{k \times l} \\
([u_0;\ldots;u_k],[v_0;\ldots;v_l], \sum_j \lambda_j (u_j,v_j)) & \mapsto \left( \sum_i \lambda_i u_i^{u_0,\ldots,u_k} z_i^{v_0,\ldots,v_l} \right),
\end{align*}
\]

and as in Definition 2.4 any two such maps are homotopic. \( \square \)

7.2. Proposition. For \( c \in H_*(\Delta Q[k,n], \Delta \tilde{Q}[l,n]), d \in H_*(\Delta \tilde{Q}[l,n], \Delta \tilde{Q}[l,n]) \), we have \( h_k(c) \times \tilde{h}_l(d) = g_c \left( [\mathbb{C}^k] \times [\mathbb{C}^l] \times (c \times d) \right) \).

Proof. For the choice \( y_i^{u,v} = x_i^u, z_i^{u,v} = \tilde{x}_i^v \), we just get the map \( f^k \times \tilde{f}^l \) up to identification of \( \mathbb{C}^k \times \mathbb{C}^l \times \left( \Delta (Q[k,n] \times \tilde{Q}[l,n]), \Delta (Q[k,n] \times \tilde{Q}[l,n]) \right) \) with \( \mathbb{C}^k \times (\Delta Q[k,n], \Delta \tilde{Q}[l,n]) \times \mathbb{C}^l \times (\Delta \tilde{Q}[l,n], \Delta \tilde{Q}[l,n]) \). \( \square \)

As noted after discussing the real example, it will be important to control the codimension of a set corresponding to the dotted line in [4]. We will now work towards this and start with an algebraic lemma.

7.3. Lemma. Let \( u, v \) be subspaces of \( V \) in general position with respect to each other, \( \dim u = r \geq k+1, \dim v = s \geq l+1, \dim V = n+1, k+l < n \). Let \( O \) be the open subspace of the affine space \( u^{k+1} \times v^{l+1} \) defined by

\[
O := \{(y_0, \ldots, y_k, z_0, \ldots, z_l) : \dim(\text{span}\{y_i\}) = k+1, \dim(\text{span}\{z_i\}) = l+1\}
\]
and algebraic subsets $\cdots \subset S_1 \subset S_0 \subset O$ defined by

$S_m := \{(y_0, \ldots, y_k, z_0, \ldots, z_l) : \dim(\text{span} \{y_i \cup \{z_j\}) < k + l + 2 - m \}$

Then $S_m \setminus S_{m+1}$ is a complex submanifold of codimension $(1 + m)(n - k - l + m)$.

**Proof.** We consider $(y_0, \ldots, y_k, z_0, \ldots, z_l) \in S_m \setminus S_{m+1}$. This implies $u + v = V$. We set $Y := \text{span} \{y_i\}$, $t := n - k - s + \dim(Y \cap v)$, and choose a basis $(e_0, \ldots, e_n)$ of $V$ such that $\text{span} \{e_0, \ldots, e_{r-1}\} = u$, $\text{span} \{e_{n-s+1}, \ldots, e_n\} = v$, $\text{span} \{e_1, \ldots, e_{k+1}\} = Y$. Let $A$ be the $(n+1) \times (k+l+2)$ matrix with columns $(y_0, \ldots, y_k, z_0, \ldots, z_l)$ expressed using this basis. Elements of $O$ are represented by matrices $A' = (a'_{ij})$ with $a'_{ij} = 0$ for $r \leq i \leq n$, $0 \leq j \leq k$ and for $0 \leq i \leq n - s$, $k + 1 \leq j \leq k + l + 1$ such that the first $k+1$ and the last $l+1$ columns are linearly independent, and $A = (a_{ij})$ has the additional property that the first $t$ rows are zero.

There are sets $I$ and $J$ with $\{t, \ldots, k + t\} \subset I \subset \{t, \ldots, n\}$, $\{0, \ldots, k\} \subset J \subset \{0, \ldots, k + l + 1\}$ and $|I| = |J| = k + l + 1 - m$ such that the matrix $B := (a_{ij})_{i \in I, j \in J}$ is regular. Similarly, there exist $I'$, $J'$ with $I' \subset \{t, \ldots, n\} \setminus M$, $\{k+1, \ldots, k + l + 1\} \subset J' \subset \{0, \ldots, k + l + 1\}$ and $|I'| = |J'| = k + l + 1 - m$ such that the matrix $C := (a'_{ij})_{i \in I', j \in J'}$ is regular.

Let $U \subset O$ be a neighbourhood of $A$ such that for every $A' = (a'_{ij}) \in U$ the matrices $(a'_{ij})_{i \in I, j \in J}$ and $(a'_{ij})_{i \in I', j \in J'}$ are regular. Then an $A' \in U$ is in $S_m$ if and only if the equations

$f_{i_0j_0}(A') := \det(a'_{ij})_{i \in I(U=i_0), j \in J(U=j_0)} = 0$ for all $i_0 \in I_0 := \{n + 1 - s, \ldots, n\} \setminus I$,

\[ j_0 \in J_0 := \{0, \ldots, k + l + 1\} \setminus J \]

and

$g_{i_0j_0}(A') := \det(a'_{ij})_{i \in I'(U=i_0), j \in J'(U=j_0)} = 0$ for all $i_0 \in \{0, \ldots, t - 1\}$,

\[ j_0 \in J'_{0} := \{0, \ldots, k + l + 1\} \setminus J' \]

hold. To see this, assume $A' \notin S_m$, i.e. $\text{rk} A' > k + l + 1 - m$. If the rank of the matrix $A'$ with the first $t$ rows deleted is greater than $k + l + 1 - m$, one of the functions $f_{i_0j_0}$ becomes non-zero, otherwise one of the functions $g_{i_0j_0}$.

Finally we compute for $(i_1, j_1) \in I_0 \times J_0$

$$\frac{\partial f_{i_0j_0}(A)}{\partial a_{i_1j_1}} = \begin{cases} \det B, & (i_0, j_0) = (i_1, j_1), \\ 0, & (i_0, j_0) \neq (i_1, j_1), \end{cases}$$

$$\frac{\partial g_{i_0j_0}(A)}{\partial a_{i_1j_1}} = 0,$$

$(i_0, j_0) \in \{0, \ldots, t - 1\} \times J'_0$.
and for \((i_1, j_1) \in \{0, \ldots, t-1\} \times J'_0\)

\[
\frac{\partial f_{i_0 j_0}}{\partial a_{i_1 j_1}}(A) = 0, \quad (i_0, j_0) \in I_0 \times J_0,
\]

\[
\left| \frac{\partial g_{i_0 j_0}}{\partial a_{i_1 j_1}}(A) \right| = \begin{cases} \det C, & (i_0, j_0) = (i_1, j_1), \\ 0, & (i_0, j_0) \neq (i_1, j_1), \end{cases} \quad (i_0, j_0) \in \{0, \ldots, t-1\} \times J'_0
\]

and \(|I_0 \times J_0 \cup \{0, \ldots, t-1\} \times J'_0| = (n + 1 - |I|)(m + 1) = (n - k - l + m) \cdot (m + 1)\).

7.4. Definition. We say that the arrangement \(\tilde{A}\) is in general position with respect to the arrangement \(A\), if for all \(u \in Q\) and \(v \in Q\), we have \(u \cap v = 0\) whenever \(d(u) + d(v) < n\) and \(d(u \cap v) = d(u) + d(v) - n\) otherwise.

7.5. Proposition. Let \(k + l < n\), \(D \subset PV \times PV\) be the diagonal and \(S \subset \Delta(Q_{[k,n]} \times Q_{[l,n]})\) be defined as the set of all points \(x\) such that \(g\left[\mathbb{C}P^k \times \mathbb{C}P^l \times \{x\}\right] \cap D \neq \emptyset\). For a generic choice of the points \(y_{i,v}^u\) and \(z_{i,v}^u\) defining \(g\), the set \(S\) intersects every open simplex of \(\Delta(Q_{[k,n]} \times Q_{[l,n]})\) in an algebraic set of real codimension \(2(n - k - l)\).

Proof. In regard of Lemma 7.3 all that is required is that for each chain \((u_0, v_0) < \cdots < (u_t, v_t)\) the affine plane in \(u_t^{k+1} \times v_t^{k+1}\) spanned by the \(t + 1\) points \((y_0^{u_0, v_0}, z_0^{u_0, v_0}), \ldots, (y_t^{u_t, v_t}, z_t^{u_t, v_t})\) meets the algebraic set \(S_0\) transversely. Assuming that the affine plane spanned by the first \(t\) of these points already meets \(S_0\) transversely, this will be fulfilled for a generic choice of \((y_t^{u_t, v_t}, z_t^{u_t, v_t}) \in u_t^{k+1} \times v_t^{k+1}\).

8. THE VANISHING OF THE INTERSECTION PRODUCT OF CLASSES OF DEGREES NOT ADDING UP TO \(n\).

In this section we will prove the following proposition. It is one half of 3 and the other half will be derived from it using an inductive argument relying also on Proposition 4.1 and Proposition 3.1.

8.1. Proposition. Let \(c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}), \ d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n]}), \ k + l < n\). Then

\[
h_k(c) \cdot h_l(d) = 0.
\]

We will prove the proposition in three steps.

We would like to have the classes \(h_k(c)\) and \(h_l(d)\) represented by chains as much as possible in general position with respect to each other. To this end we consider an arrangement that is the union of two arrangements \(A\) and \(\tilde{A}\) with intersection posets \(Q\) and \(\tilde{Q}\) such that \(\tilde{A}\) is in general position with respect to \(A\) (see Definition 7.4).
We will denote the intersection poset of the arrangement \( \tilde{A} := A \cup \tilde{A} \) by \( \tilde{Q} \) and so on. The map
\[
\sigma: (Q \times \tilde{Q})_{[n,2n]} \to \tilde{Q}_{[0,n]}
(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cap \mathbf{v}
\] (6)
is an isomorphism.

### 8.2. Proposition
In the above situation, let \( c \in H_*(\Delta \tilde{Q}_{[k,n]}, \Delta Q_{[k,n]}) \), \( d \in H_*(\Delta \tilde{Q}_{[l,n]}, \Delta Q_{[l,n]}) \), \( k + l < n \). Then
\[
h_k(c) \cdot \tilde{h}_l(d) = \sum_{i > 0} \tilde{h}_i(r_i)
\]
for classes \( r_i \in H_*(\Delta \tilde{Q}_{[i,n]}, \Delta Q_{[i,n]}) \).

**Proof.** By Proposition 6.6 we will have to show \( s^A_*(h_k(c) \cdot \tilde{h}_l(d)) = 0 \). It will be in doing so that we employ the ideas laid out in the discussion of the real example.

We set \( (X, A) := CP^k \times CP^l \times \left( \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n]} \right) \), \( D := \{(x, x) \in PV \times PV, CD := (PV \times PV) \setminus D \) and use the map \( g \) from Definition 7.1. We denote the diagonal map \( PV \to PV \times PV \) by \( \Delta \) and define \( \tilde{g}: (g^{-1}[D], g^{-1}[D] \cap A) \to (PV, \bigcup P A \cup \bigcup P \tilde{A}) \) by \( \Delta \circ \tilde{g} = g \). Note that in the real example the projection of \( g^{-1}[D] \) to the order complex is the set represented by a dotted line in (4). We will first show \( h_k(c) \cdot \tilde{h}_l(d) \in \text{im} \tilde{g}_* \) and then \( s^A_* \circ \tilde{g}_* = 0 \).

There is a commutative diagram
\[
\begin{array}{ccc}
H^*(PV \times PV, CD) \otimes H_*(X, A) & \xrightarrow{id \otimes \tilde{g}_*} & H_*(PV \times PV, CD) \otimes (PV, \bigcup P A) \\
| \downarrow \quad g^* \otimes id & & \downarrow \quad \Delta_* \\
H^*(X, X \setminus g^{-1}[D]) \otimes H_*(X, A) & & H_*(PV, \bigcup P A \cup \bigcup P \tilde{A}) \\
| & \xrightarrow{g_*} & \cong \Delta_* \\
H_*(g^{-1}[D], g^{-1}[D] \cap A) & \xrightarrow{g_*} & H_*(D, D \cap (\bigcup P A \times PV \cup PV \times \bigcup P \tilde{A}))
\end{array}
\]
Regarding the existence of the cap products in this diagram and commutativity, note that we are entirely dealing with algebraic sets and polynomial
maps. Now, if $\vartheta \in H^*(PV \times PV, CD)$ is the Thom class determined by $\vartheta \sim [PV \times PV] = \Delta_s([PV])$, then

$$h_k(c) \cdot \tilde{h}_l(d) = \Delta_t(h_k(c) \times \tilde{h}_l(d))$$

$$= \Delta^{-1}_t(\vartheta \sim (h_k(c) \times \tilde{h}_l(d)))$$

$$= \Delta^{-1}_s(\vartheta \sim g_*([CP]^k \times [CP]^l \times (c \times d)))$$

$$= \Delta^{-1}_s(g_*^{\vartheta} \sim ([CP]^k \times [CP]^l \times (c \times d)))$$

$$= g_*^{\vartheta} \sim ([CP]^k \times [CP]^l \times (c \times d)).$$

By construction of $g, \tilde{g}(x, y, \sum_{j=0}^r \lambda_j(u_j, v_j)) \in u_r \cap u_r$. Firstly this implies that $g^{-1}[D]$ misses $CP^k \times CP^l \times \Delta(Q \times \tilde{Q}[0, n])$, and secondly from the reformulation $s^A(g(x, y, z)) \leq \sigma(s^{Q \times \tilde{Q}(z)})$, where $\sigma$ is the isomorphism from [6] and Lemma 6.2 it can be seen that the diagram

$$\xymatrix{ (Q[n, n \times \tilde{Q}[l, n]) \ar[rr]^{s^A \circ \tilde{g}} \ar[d]_{\pi} & & (Q[n, n \times \tilde{Q}[l, n]) \ar[d]^{\sigma} \ar@{^{(}->}[d] \ar@{^{(}->}[r]^{s^{Q \times \tilde{Q}(z)}} & (Q \times \tilde{Q}[n, 2n] \times (Q \times \tilde{Q}[n, 2n]) \ar@{^{(}->}[d] \ar@{^{(}->}[r] & (Q \times \tilde{Q}[n, 2n] \times (Q \times \tilde{Q}[n, 2n])}$$

where $\pi$ denotes projection onto the third factor, is homotopy commutative. The two arrows on the right hand side of the diagram should be compared to [5].

By Proposition 7.5 and because the subcomplex $\Delta((Q[n, n \times \tilde{Q}[l, n]) \cap (Q \times \tilde{Q}[0, n]))$ has dimension at most $n - 1 - k - l$, we may assume that $\pi[g^{-1}[D]]$ will not miss this subcomplex, but any cone over it. Therefore $\pi$ factorizes over the pair

$$\Delta(Q[k, n] \times \tilde{Q}[l, n]) \setminus \Delta((Q \times \tilde{Q})[0, n] \cup \{(V, V)\}),$$

$$\Delta(Q[k, n] \times \tilde{Q}[l, n]) \setminus \Delta(Q[k, n] \times \tilde{Q}[l, n]) \setminus \Delta(Q \times \tilde{Q}[0, n]).$$

This pair is homeomorphic to

$$\Delta((Q[k, n] \times \tilde{Q}[l, n]) \setminus \Delta(Q \times \tilde{Q}[0, n]) \times ([0, 1], \{0\})$$

and has trivial homology. So $(s^A \circ \tilde{g})_* = s^{Q \times \tilde{Q}} \circ \pi_* = s_* \circ \pi_* = 0 = 0$.

8.3. Proposition. In the above situation let $c \in H_s(\Delta Q[k, n] \times \Delta Q[l, n]), d \in H_s(\Delta Q[k, n] \times \Delta Q[l, n]), k + l < n$. Then

$$h_k(c) \cdot \tilde{h}_l(d) = 0.$$
Proof. We choose a hyperplane $H$ in $V$ in general position with respect to the arrangement $\tilde{A} = A \cup \tilde{A}$ and use notation as in Section 4. By Proposition 4.1 and induction on the dimension of $V$

$$i_!(h_k(c) \cdot \tilde{h}_l(d)) = i_!(\hat{h}_k(c) \cdot i_!(\tilde{h}_l(d))) = h_{k-1}^H(\eta_*(c)) \cdot \hat{h}_{k-1}^H(\eta_*(d)) = 0,$$

since the arrangement $A^H$ is again in general position with respect to $\tilde{A}^H$ and $(k - 1) + (l - 1) < n - 1$. This implies $h_k(c) \cdot \tilde{h}_l(d) \in \ker \hat{i}_! = \im \hat{h}_0$ by Proposition 4.1. But $h_k(c) \cdot \tilde{h}_l(d) \in \bigoplus_{j > 0} \im \hat{h}_j$ by Proposition 8.2 and hence $h_k(c) \cdot \tilde{h}_l(d) = 0$. \hfill $\square$

Proof of Proposition 8.1. We choose a neighbourhood $U$ of $\bigcup P \tilde{A}$ such that the inclusion $(PV, \bigcup P \tilde{A}) \to (PV, U)$ is a homotopy equivalence. We then choose a copy $\tilde{A}$ of $A$ also contained in $U$, in general position with respect to $A$ and such that the diagram

$$\begin{array}{ccc}
\mathbb{CP}^n \times (\Delta Q_{[k,n]}, \Delta Q_{[l,n]}) & \xrightarrow{f^i} & (PV, \bigcup P \tilde{A}) \\
\downarrow f^i & & \downarrow \text{incl.} \\
(PV, \bigcup P \tilde{A}) & \xrightarrow{\text{incl.}} & (PV, U)
\end{array}$$

commutes up to homotopy. Because of the commutativity of

$$\begin{array}{cccc}
H_*(PV, \bigcup P \tilde{A}) \otimes H_*(PV, \bigcup P \tilde{A}) & \xrightarrow{\bullet} & H_*(PV, \bigcup P \tilde{A}) \\
\downarrow h_k \otimes \tilde{h}_l & & \downarrow \text{incl.} & \cong \\
H_*(\Delta Q_{[k,n]}, \Delta Q_{[l,n]}) \otimes H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n]}) & \xrightarrow{h_k \otimes \tilde{h}_l} & H_*(PV, U) \\
\downarrow h_k \otimes \tilde{h}_l & & \downarrow \text{incl.} & \cong \\
H_*(PV, \bigcup P \tilde{A}) \otimes H_*(PV, \bigcup P \tilde{A}) & \xrightarrow{\bullet} & H_*(PV, \bigcup P \tilde{A} \cup \bigcup P \tilde{A})
\end{array}$$

the result follows from Proposition 8.3. \hfill $\square$

9. The main theorem

9.1. Theorem. Let $c \in H_*(Q_{[k,n]}, Q_{[k,n]})$, $d \in H_*(Q_{[l,n]}, Q_{[l,n]})$. Then

$$h_k(c) \cdot h_l(d) = h_{k+l-n}(c \hat{\times} d)$$

with the convention that $h_i = 0$ for $i < 0$.

Proof. We proceed by induction on the dimension of $V$.

For $k + l < n$ the conjecture is covered by Proposition 8.1. For $k + l \geq n$ and $\dim V = 1$ it is covered by Proposition 3.1 (and trivial anyway). For $k + l \geq n$ and $\dim V > 1$ we choose a hyperplane in general position with respect
to the arrangement and adopt the notation of Section 4. By induction and Proposition 4.1 (for \( k + l > n \)) or Proposition 8.1 (for \( k + l = n \))

\[
i_t(h_k(c) \cdot h_l(d)) = i_t(h_k(c)) \cdot i_t(h_l(d)) = h_l^H(i_t(h_k(c))) \cdot h_{l-1}^H(i_t(h_0(c))) =
\]

\[
= h_{k+l-n-1}^H(i_t(h_k(c) \times i_t(h_0(c)))) = h_{k+l-n-1}^H(i_t(h_k(c) \times d)).
\]

Again by Proposition 4.1, this implies \( h_k(c) \cdot h_l(d) - h_{k+l-n}(c \times d) \in \ker i_t = \operatorname{im} h_0 \). But \( h_k(c) \cdot h_l(d) - h_{k+l-n}(c \times d) \in \bigoplus_{i > k + l - n} \operatorname{im} h_i \) by Proposition 3.1. Therefore \( h_k(c) \cdot h_l(d) - h_{k+l-n}(c \times d) = 0 \). \( \square \)

9.2. Remark. Continuing Remark 2.9, this yields a description of the cohomology ring of an affine complex arrangement with intersection poset \( Q' \). All of the isomorphisms in that remark are induced by inclusions and therefore respect the product \( \times \). Denoting, for \( u \in Q'[0,n] \), the map \( H(\Delta[u,V], \Delta[u,V] \cup \Delta(u,V)) \to H(PV, \cup PA) \) arising from these isomorphisms and \( h_{d(u)} \) by \( h_u \) we obtain

\[
h_u(c) \cdot h_v(d) = \begin{cases} 
  h_{u \wedge v}(c \times d), & \text{if } u \wedge v \in Q'[0,n] \\
  0, & \text{otherwise},
\end{cases}
\]

for \( c \in H_s(\Delta[u,V], \Delta[u,V] \cup \Delta(u,V)) \), \( d \in H_s(\Delta[v,V], \Delta[v,V] \cup \Delta(v,V)) \).

10. Projective c-arrangements

For special classes of arrangements, one can hope to derive a simpler description of the cohomology ring of the complement from Theorem 9.1. The case that is probably easiest to handle is that of \( c \)-arrangements.

10.1. Definition. For a positive integer \( c \), we call \( A \) a \( c \)-arrangement, if every \( A \in \mathcal{A} \) is a subspace of codimension \( c \) and \( d(q) \) is an integral multiple of \( c \) for every \( q \in Q \).

A presentation of the cohomology ring.

10.2. Definition. We call a subset \( M \) of \( \mathcal{A} \) independent, if \( n - d(\bigcap M) = \sum_{A \in M} (n - d(A)) \), dependent, if it is not independent, and minimally dependent, if it is dependent but all of its proper subsets are independent.

In this section we will prove:

10.3. Theorem. Let \( \mathcal{A} \) be a \( c \)-arrangement, \( |\mathcal{A}| - 1 =: t \geq 0 \), \( \mathcal{A} = \{ A_0, \ldots, A_t \} \). Let \( R \) be the free graded commutative (in the graded sense) ring over the set of generators \( \{ x \} \cup \{ y_i : 1 \leq i \leq t \} \) with \( |x| = 2 \), \( |y_i| = \)}
2c − 1. Let \( I \) be the ideal generated by
\[
\left\{ \sum_{j=0}^{r} (-1)^j y_{i_0} \cdots y_{i_j} : i_0 < \cdots < i_r, \{A_{i_j}\} \text{ is minimally dependent.} \right\}
\]
\[\cup \left\{ y_{i_1} \cdots y_{i_r} : i_1 < \cdots < i_r, \{A_0\} \cup \{A_{i_j}\} \text{ is minimally dependent.} \right\}
\]
\[\cup \{ x^c \}. \]

The map
\[
\pi: R \to H^* \left( PV \setminus \bigcup P \mathcal{A} \right),
\]
\[
x \mapsto P(h_{n-1}([V])),
\]
\[
y_i \mapsto P(h_{n-c}([\langle A_i, V \rangle - \langle A_0, V \rangle])),
\]
where \( P: H_*(PV, \bigcup P \mathcal{A}) \overset{\cong}{\to} H^*(PV \setminus \bigcup P \mathcal{A}) \) denotes Poincaré duality, is an epimorphism and \( \ker \pi = I \).

We now fix the arrangement \( \mathcal{A} = \{A_0, \ldots, A_t\} \).

10.4. **Remark.** For \( c = 1 \) the complement \( PV \setminus \bigcup P \mathcal{A} \) can, as in Remark 2.9, be regarded as the complement in the affine space \( PV \setminus PA_0 \) of the linear hyperplane arrangement \( \mathcal{A}' := \{PA_i \setminus PA_0 : 1 \leq i \leq t\} \). In this case, the generator \( x \) and the corresponding relation can be omitted.

If \( A_0 \) is in general position with respect to \( \mathcal{A} \setminus \{A_0\} \), the second kind of generators does not occur. This is in particular the case if the arrangement \( \mathcal{A}' \) is central, i.e. if \( \bigcap \mathcal{A}' \neq \emptyset \). In this case the theorem reduces to the description of the cohomology ring of the complement of \( \mathcal{A}' \) given by Orlik and Solomon [OSS80].

**The atomic complex.** We now turn to the proof of the theorem. When using simplicial chain complexes, we will always use the complex of non-degenerate simplices and view it as the complex of all simplices modulo degenerate simplices if necessary.

10.5. **Definition.** For an integer \( k \) with \( 0 \leq k \leq n \), we define \( S_k \) to be the simplicial complex which has the vertex set \( \{0, \ldots, t\} \) and as simplices the sets \( I \subset \{0, \ldots, t\} \) with \( d(\bigcap_{i \in I} A_i) \geq k \). This is the **atomic complex** of \( Q_{[k,n]} \). We also define \( D_k \) to be the reduced ordered (using the natural order of \( \{0, \ldots, t\} \)) simplicial chain complex of \( S_k \) shifted by one, i.e. \( D_k = \tilde{C}_{r-1}(S_k) \) and in particular \( D_0 \cong \mathbb{Z} \) generated by the empty simplex.

As is well known, the atomic complex and the order complex, of \( Q_{[k,n]} \) in this case, are homotopy equivalent. We describe a homotopy equivalence to fix a concrete isomorphism between their homology groups. Before doing this, we state a useful lemma.
10.6. Lemma. Let $P_0$, $P_1$ be posets, $P'_i \subset P_i$. If $f, g: (P_0, P'_0) \to (P_1, P'_1)$ are order preserving functions such that $f(p) \leq g(p)$ for all $p \in P_0$, then the maps $f, g: (\Delta P_0, \Delta P'_0) \to (\Delta P_1, \Delta P'_1)$ are homotopic.

Proof. The map $H: \{0,1\} \times P_0 \to P_1$ defined by $H(0, x) := f(x)$, $H(1, x) := g(x)$ is order preserving and hence yields the desired homotopy

$I \times (\Delta P_0, \Delta P'_0) \cong (\Delta(\{0,1\} \times P_0), \Delta(\{0,1\} \times P'_0)) \xrightarrow{H} (\Delta P_1, \Delta P'_1)$,

where we view $\{0,1\}$ as a poset. □

10.7. Remark. This lemma is a special case of [Seg68, Prop. 2.1] which is proved in the same way.

10.8. Definition and Proposition. We denote the face poset of $S_k$ by $FS_k$, but order it by $M \leq M'$ if $M'$ is a face of $M$, that is if $M' \subset M$. We also set $\hat{FS}_k := FS_k \cup \{0\}$. The map

$s: (\hat{FS}_k, FS_k) \to (Q_{[k,n]}, Q_{[k,n]})$

$M \mapsto \bigcap \{A_i : i \in M\}$

is then order preserving and moreover satisfies $s(M \cap M') = s(M \cup M') = s(M) \cap s(M') = s(M) \cap s(M')$, if one side, and therefore the other, exists.

With these definitions, the map $s: (\Delta \hat{FS}_k, \Delta FS_k) \to (\Delta Q_{[k,n]}, \Delta Q_{[k,n]})$ is a homotopy equivalence.

10.9. Remark. $\Delta FS_k$ is the barycentric subdivision of $S_k$, and $\Delta \hat{FS}_k$ is a cone over $\Delta FS_k$.

Proof. We define an order preserving map

$r: (Q_{[k,n]}, Q_{[k,n]}) \to (\hat{FS}_k, FS_k),$

$q \mapsto \{i : A_i \supset q\}$.

We have $s(r(q)) \geq q$ for $q \in Q_{[k,n]}$ and $r(s(i)) \leq i$ for $i \in \hat{FS}_k$. Hence, by the preceding lemma $r$ is a homotopy inverse to $s$, when both maps are regarded as simplicial maps between order complexes. □

10.10. Definition and Proposition. We define chain maps

$f^k: D^k \to C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n]}))$

$\langle i_1, \ldots, i_r \rangle \mapsto \langle A_{i_1}, V \rangle \hat{\cdots} \hat{\langle A_{i_r}, V \rangle}$.

For $r = 0$ this is to be understood as $f^k(\langle \rangle) = \langle V \rangle$.

10.11. Notation. To simplify the following calculations, we set

$\alpha_i := \langle A_i, V \rangle \in C_1(\Delta Q_{[n-c,n]})$

and sometimes write the multiplication $\hat{\cdots}$ as juxtaposition.
Proof. To see that $f^k$ is well-defined, we have to check that the right hand side is in $C_\ast(\Delta Q_{[k,n]})$. But $d(A_{i_1} \wedge \cdots \wedge A_{i_r}) \geq k$ by definition of $S_k$ and hence $D^k_r$.

To see that $f^k$ is a chain map, we calculate

$$\mathcal{D}(f^k((i_1, \ldots, i_r))) = \sum_{j=1}^r (-1)^{j+1} \langle A_{i_1}, V \rangle \wedge \cdots \wedge \langle A_{i_j}, V \rangle \wedge \cdots \wedge \langle A_{i_r}, V \rangle$$

$$= \sum_{j=1}^r (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \alpha_{i_{j+1}} \cdots \alpha_{i_r}$$

$$- \sum_{j=1}^r (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \alpha_{i_1} \cdots \alpha_{i_{j+1}} \cdots \alpha_{i_r}$$

$$= \sum_{j=1}^r (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \alpha_{i_j} \alpha_{i_{j+1}} \cdots \alpha_{i_r}$$

$$- \sum_{j=1}^r (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \alpha_{i_1} \cdots \alpha_{i_{j+1}} \cdots \alpha_{i_r}.$$
Proof of the presentation. The chain maps \( f^k \) would be more useful in a situation in which the chains \( \langle A_i, V \rangle \) are cycles. For example, think of affine arrangements, where, according to Remark 2.9, \( (\Delta Q[k,n], \Delta Q[k,n] \cup \Delta Q[k,n]) \) takes the place of \( (\Delta Q[k,n], \Delta Q[k,n]) \). In our situation they are not. The chains \( \langle A_i, V \rangle - \langle A_j, V \rangle \) however are cycles, we will therefore replace the maps \( f^k \) by the following maps.

10.13. Definition and Proposition. For a \( c \)-arrangement \( A \), we define chain maps

\[
g^k : D^k \rightarrow C_0(\Delta Q[k,n], \Delta Q[k,n])
\]

\[
\langle i_1, \ldots, i_r \rangle \mapsto \begin{cases} (\langle A_i, V \rangle - \langle A_0, V \rangle) \hat{\times} \cdots \hat{\times} (\langle A_i, V \rangle - \langle A_0, V \rangle), & r = a, \\
0, & r \neq a,
\end{cases}
\]

where \( a \) is defined by \( n - (a + 1)c < k \leq n - ac \).

Proof. We check that \( g^k \) is a well-defined chain map. For \( r > a \) we have \( C_r(\Delta Q[k,n], \Delta Q[k,n]) \cong 0 \), since \( n - k < (a + 1)c \leq rc \). So we just have to show that \( g^k(\langle i_1, \ldots, i_a \rangle) \) is a cycle in \( C_0(\Delta Q[k,n], \Delta Q[k,n]) \). This is true, because each \( \langle A_i, V \rangle - \langle A_0, V \rangle \) is a cycle in \( C_1(\Delta Q[n-c,n], \Delta Q[n-c,n]) \) and \( n - ac \geq k \).

10.14. Proposition. The maps \( f^k \) and \( g^k \) are chain homotopic.

Proof. We define

\[
K : D^k \rightarrow C_{r+1}(\Delta Q[k,n], \Delta Q[k,n])
\]

\[
\langle i_1, \ldots, i_r \rangle \mapsto \begin{cases} f^k(\langle 0, i_1, \ldots, i_r \rangle), & r < a, \\
0, & r \geq a.
\end{cases}
\]

The right hand side is well defined, because for \( r < a \) we have

\[
d(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_r}) \geq n - (r + 1)c \geq n - ac \geq k.
\]

We calculate \( K \mathfrak{d} + \mathfrak{d}K \).

For \( r < a \):

\[
(K \mathfrak{d} + \mathfrak{d}K)\langle i_1, \ldots, i_r \rangle
\]

\[
= f^k \left( \sum_{j=1}^{r} (-1)^j (\langle 0, i_1, \ldots, \hat{i}_j, \ldots, i_r \rangle) \right) + \mathfrak{d}f^k(\langle 0, i_1, \ldots, i_r \rangle)
\]

\[
= f^k \left( \sum_{j=1}^{r} (-1)^j (\langle 0, i_1, \ldots, \hat{i}_j, \ldots, i_r \rangle) + \mathfrak{d}(0, i_1, \ldots, i_r) \right)
\]

\[
= f^k(\langle i_1, \ldots, i_r \rangle) = (f^k - g^k)\langle i_1, \ldots, i_r \rangle.
\]
For $r = a$: We first calculate
\[
g^k(\langle i_1, \ldots, i_a \rangle) = (\alpha_{i_1} - \alpha_0) \cdots (\alpha_{i_a} - \alpha_0)
= \alpha_{i_1} \cdots \alpha_{i_a} - \sum_{j=1}^a \alpha_{i_1} \cdots \alpha_{i_{j-1}} \alpha_0 \alpha_{i_{j+1}} \cdots \alpha_{i_a}
= \alpha_{i_1} \cdots \alpha_{i_a} + \sum_{j=1}^a (-1)^j \alpha_0 \alpha_{i_1} \cdots \hat{\alpha}_{i_j} \cdots \alpha_{i_a}
\]
and with this
\[
(K \delta + \delta K)\langle i_1, \ldots, i_a \rangle = f^k \left( \sum_{j=1}^a (-1)^{j+1} \langle 0, i_1, \ldots, \hat{i}_j, \ldots, i_a \rangle \right)
= \sum_{j=1}^a (-1)^{j+1} \alpha_0 \alpha_{i_1} \cdots \hat{\alpha}_{i_j} \cdots \alpha_{i_a}
= (f^k - g^k)\langle i_1, \ldots, i_a \rangle.
\]

For $r > a$ we have $(K \delta + \delta K)\langle i_1, \ldots, i_r \rangle = 0 = (f^k - g^k)\langle i_1, \ldots, i_r \rangle$, since $C_r(\Delta Q[k,n], \Delta Q[k,n]) \cong 0$ as noted before.

10.15. Proposition. The map $\pi$ is surjective.

Proof. By Proposition 10.12, Proposition 10.14, and of course Proposition 2.8, $H^*(PV \setminus \bigcup PA)$ is additively generated by the elements $P(h_k ([g^k(i_1, \ldots, i_r)])$ with $k \leq n - rc$. By Theorem 9.1
\[
P \left( h_k \left( [g^k(i_1, \ldots, i_r)] \right) \right)
= P(h_k([\alpha_{i_1} - \alpha_0] \times \cdots \times [\alpha_{i_r} - \alpha_0]))
= P(h_n-1([V]))^{n-k-rc} P(h_{n-c}([\alpha_{i_1} - \alpha_0])) \cdots P(h_{n-c}([\alpha_{i_r} - \alpha_0]))
= \pi(x)^{n-k-rc} \pi(y_{i_1}) \cdots \pi(y_{i_r}).
\]
This shows that $\pi$ is surjective. □

10.16. Proposition. $I \subset \ker \pi$.

Proof. First of all
\[
\pi(x^c) = P(h_{n-1}([V]))^c
= P(h_{n-c}([V]))
= P(h_{n-c}([\delta(A_0,V)])) = P(h_{n-c}(0)) = 0.
\]
If \( \{A_{i_0}, \ldots, A_{i_r}\} \) is minimally dependent, then \( d \left( \bigcap_j A_{i_j} \right) = n - rc \) and
\[
0 = P(h_{n-rc}(g^{n-rc}_n(\mathfrak{d}(i_0, \ldots, i_r))))
\]
\[
= (P \circ h_{n-rc} \circ g^{n-rc}_n) \left( \sum_{j=0}^{n} (-1)^j \langle i_0, \ldots, \hat{i}_j, \ldots, i_r \rangle \right)
\]
\[
= \pi \left( \sum_{j=0}^{r} (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \right)
\]
and similarly if \( \{A_0, A_{i_1}, \ldots, A_{i_r}\} \) is minimally dependent, then
\[
0 = P(h_{n-rc}(g^{n-rc}_n(\mathfrak{d}(0, i_1, \ldots, i_r))))
\]
\[
= (P \circ h_{n-rc} \circ g^{n-rc}_n) \left( \langle i_1, \ldots, i_r \rangle + \sum_{j=1}^{r} (-1)^j \langle 0, i_1, \ldots, \hat{i}_j, \ldots, i_r \rangle \right)
\]
\[
= \pi (y_{i_1} \cdots y_{i_r})
\]
as claimed. \( \square \)

10.17. Lemma. If \( i_0 < \cdots < i_r \) and \( \{A_{i_j}\} \) is dependent, then \( y_{i_0} \cdots y_{i_r} \in I \) and \( \sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \in I \).

Proof. Let \( \{A_{i_0}, \ldots, A_{i_r}\} \) be dependent. To show \( y_{i_0} \cdots y_{i_r} \in I \) we may assume that the set is minimally dependent. Then \( \sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \in I \) and \( y_{i_0} \left( \sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \right) = y_{i_0} \cdots y_{i_r} \) since \( y_0^2 = 0 \).

For the second part of the lemma we may assume that \( \{A_{i_j} : j \leq s\} \) is minimally dependent. Then
\[
\sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} =
\]
\[
\left( \sum_{j=0}^{s} (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_s} \right) y_{i_{s+1}} \cdots y_{i_r}
\]
\[
+ \underbrace{y_{i_0} \cdots y_{i_s}}_{\in I} \sum_{j=s+1}^{r} (-1)^j y_{i_{s+1}} \cdots \hat{y}_{i_j} \cdots y_{i_r} \in I
\]
as claimed. \( \square \)

10.18. Proposition. \( \ker \pi \subset I \).

Proof. Let \( z \in \ker \pi \). We want to show \( z \in I \). We may assume that \( z \) is a linear combination of elements \( x^s y_{i_1} \cdots y_{i_r} \) with \( 0 \leq s < c, i_1 < \cdots < i_r \).
and \( \{A_i, \ldots, A_r\} \) independent. Since \( \pi(x^s y_{i_1} \cdots y_{i_r}) \in \text{im}(P \circ h_{n-cr-s}) \) and \( r \) and \( s \) are determined by \( cr+s \), we may assume \( z \) to be homogenous in \( r \) and \( s \), i.e. \( z = x^s \sum_{i_1 < \cdots < i_r} \lambda_i y_{i_1} \cdots y_{i_r} \). We set \( k := n - cr - s \). The chain

\[
z' := \sum_{i_1 < \cdots < i_r} \lambda_i \langle i_1, \ldots, i_r \rangle + \sum_{i_1 < \cdots < i_r} \lambda_i \sum_{j=1}^r (-1)^j \langle 0, i_1, \ldots, \hat{i_j}, \ldots, i_r \rangle
\]

is a cycle in \( D^k_r \) (the second summand is a cone over the boundary of the first summand), \( 0 = \pi(z) = (P \circ h_k)(g_k ((z'))) \), and therefore \( [z'] = 0 \) by Proposition 10.12 and Proposition 10.14. Comparing coefficients and sorting the simplices by whether the first vertex is 0 yields

\[
z = \sum_i \mu_i y_{i_1} \cdots y_{i_r} + \sum_i \nu_i \sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \in I
\]
as claimed.  

This completes the proof of Theorem 10.3. 

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