Determination of differential pencils with spectral parameter dependent boundary conditions from interior spectral data

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Second-order differential pencils \( L(p, q, h_0, h_1, H_0, H_1) \) on a finite interval with spectral parameter dependent boundary conditions are considered. We prove the following: (i) a set of values of eigenfunctions at the mid-point of the interval \( [0, \pi] \) and one full spectrum suffice to determine differential pencils \( L(p, q, h_0, h_1, H_0, H_1) \); and (ii) some information on eigenfunctions at some an internal point \( b \in (\frac{\pi}{2}, \pi) \) and parts of two spectra suffice to determine differential pencils \( L(p, q, h_0, h_1, H_0, H_1) \). Copyright © 2013 The Authors. Mathematical Methods in the Applied Sciences published by John Wiley & Sons, Ltd.

Keywords: differential pencils with spectral parameter dependent boundary conditions; interior spectral data; eigenvalue; inverse problem

1. Introduction

We consider the boundary value problem \( L \) of the form:

\[
\begin{align*}
\int_{0}^{x} y(x, \lambda) & \overset{\text{def}}{=} y''(x) + [\lambda^2 - 2\lambda p(x) - q(x)]y(x) = 0, \quad 0 < x < \pi \\
y'(0) + (i\lambda h_1 + h_0)y(0) & = 0 \\
y'(\pi) + (i\lambda H_1 + H_0)y(\pi) & = 0,
\end{align*}
\]

(1.1)

where \( i = \sqrt{-1} \), \( \lambda \) is a spectral parameter; \( p \in W^1_2[0, \pi], q \in W^2_2[0, \pi] \) are complex-valued functions; \( h_j, H_j \in \mathbb{C}, \ j = 0, 1, \ h_1 \neq \pm 1, \ H_1 \neq \pm 1 \). The latter conditions exclude from consideration Rendez-vous problems \[1\], which require separate investigation. Differential equations with a nonlinear dependence on the spectral parameter frequently appear in mathematics as well as in applications. Inverse spectral problems consist in recovering operators from given their spectral characteristics \[2–11\]. Some aspects of spectral problems for second-order differential pencils were studied in \[12–24\] and other papers.

Inverse problem for interior spectral data of the differential operator lies in reconstructing this operator by some eigenvalues and information on eigenfunctions at some an internal point in the interval considered. The similar problems for the Sturm–Liouville operators \[25, 26\] and differential pencils \[27\] were studied.

As far as I know, the inverse problem of interior spectral data for differential pencils \( L \) with spectral parameter dependent boundary conditions has not been considered before. The aim of this paper is to give two uniqueness theorems from some eigenvalues and information on eigenfunctions at some an internal point in the interval \( [0, \pi] \) provided that the spectrum is simple. The results obtained are new and a generalization of the well-known one for the classical Sturm–Liouville operator, which was studied in \[25\], for a special case that \( p(x) = 0, \ h_1 = H_1 = 0 \), and the parameters \( h \) and \( H \) are fixed. The results in this paper are also a generalization of theorems because of Yang and Guo \[27\], where authors consider a special case that \( h_1 = H_1 = 0 \) and assume either \( p(x) \) or \( q(x) \) is a prior known.

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In fact, the last restriction is unnecessary. The proof presented here follows Mochizuki and Trooshin’s proof in outline, but the results obtained are more general.

2. Main results

The eigenvalue asymptotics for pencils $L$ are studied in [15], and it is shown that eigenvalues are not necessarily real, and all but a finite number of the eigenvalues are algebraically simple. It is well-known that the eigenvalues of pencils $L$ consist of the sequence $\lambda_n, n \in A \doteq \{\pm 0, \pm 1, \pm 2, \cdots\}$, and large eigenvalue $\lambda_n$ satisfies the classical asymptotic form

$$\lambda_n = n + \omega + O\left(\frac{1}{n}\right),$$

(2.1)

where

$$\omega = \frac{1}{\pi} \int_0^{\pi} p(x)dx + \frac{1}{2\pi i} \ln \frac{(h_1 + 1)(h_1 - 1)}{(h_1 - 1)(h_1 + 1)}.$$  

To simplify the calculations later, we will assume that the eigenvalues $\lambda_n$ are all simple. Denote by $\lambda_n$ and $y_n(x), n \in A$, eigenvalues and corresponding eigenfunctions of the differential pencils $L$, respectively. We introduce the quantities (derivatives of the logarithm of $|y_n(x)|$)

$$\kappa_n(x) = \frac{y_n'(x)}{y_n(x)}.$$  

Consider a second differential pencils

$$\begin{cases}
\tilde{l}_\lambda y(x, \lambda) \overset{def}{=} y''(x) + \left[\lambda^2 - 2\lambda \tilde{p}(x) - \tilde{q}(x)\right] y(x) = 0, & 0 < x < \pi \\
y'(0) + (i\lambda \tilde{h}_1 + \tilde{h}_0) y(0) = 0 \\
y'(\pi) + (i\lambda \tilde{h}_1 + \tilde{h}_0) y(\pi) = 0,
\end{cases}$$

(2.2)

where $\tilde{h}_k, \tilde{h}_k, \tilde{q}(x)$ and $\tilde{p}(x)$ have the same properties of $h_k, h_k, q(x)$ and $p(x), k = 0, 1$. We agree that if a certain symbol $\delta$ denotes an object related to $L(p, q, h_0, h_1, H_0, H_1)$, then $\delta$ will denote an analogous object related to $L(\tilde{p}, \tilde{q}, \tilde{h}_0, \tilde{h}_1, \tilde{H}_0, \tilde{H}_1)$.

Let $l(n), r(n)$ be sequences of natural numbers with properties

$$\begin{aligned}
l(n) = \frac{n}{\sigma_1} (1 + \epsilon_{1,n}), & \quad 0 < \sigma_1 \leq 1, \quad \epsilon_{1,n} \to 0, \\
r(n) = \frac{n}{\sigma_2} (1 + \epsilon_{2,n}), & \quad 0 < \sigma_2 \leq 1, \quad \epsilon_{2,n} \to 0;
\end{aligned}$$

(2.3)

and let $\mu_n$ be the eigenvalues of the pencils $L(p, q, h_0, h_1, H_0, H_1), H_1 \neq H_2 \in \mathbb{C}$.

Now, we state the main results of this work.

**Theorem 2.1**

If for any $n \in A$,

$$\lambda_n = \tilde{\lambda}_n, \quad \kappa_n\left(\frac{\pi}{2}\right) = \tilde{\kappa}_n\left(\frac{\pi}{2}\right),$$

then $L(p, q, h_0, h_1, H_0, H_1) = L(\tilde{p}, \tilde{q}, \tilde{h}_0, \tilde{h}_1, \tilde{H}_0, \tilde{H}_1)$.  

**Remark 2.2**

(1) Equation $L(p, q, h_0, h_1, H_0, H_1) = L(\tilde{p}, \tilde{q}, \tilde{h}_0, \tilde{h}_1, \tilde{H}_0, \tilde{H}_1)$ means that $p(x) = \tilde{p}(x)$ on $[0, \pi], q(x) = \tilde{q}(x)$ a.e. on $[0, \pi]$, and $h_k = \tilde{h}_k, h_k = \tilde{h}_k (k = 0, 1)$.

(2) The solution of inverse problem in Theorem 2.1 is not unique without condition $\kappa_n\left(\frac{\pi}{2}\right) = \tilde{\kappa}_n\left(\frac{\pi}{2}\right)$, because single spectrum cannot determine the pencils $L$. In particular, when $y_n(\frac{\pi}{2}) = 0$, equation $\kappa_n\left(\frac{\pi}{2}\right) = \tilde{\kappa}_n\left(\frac{\pi}{2}\right)$ is replaced by $y_n(\frac{\pi}{2}) \tilde{y}_n(\frac{\pi}{2}) = y_n'(\frac{\pi}{2}) \tilde{y}_n'(\frac{\pi}{2})$.

**Theorem 2.3**

Let $l(n), r(n)$ and $b \in (\frac{\pi}{2}, \pi)$ be such that $\sigma_1 > \frac{2b}{\pi} - 1, \sigma_2 > 2 - \frac{2b}{\pi}$. If for any $n \in A$,

$$\lambda_n = \tilde{\lambda}_n, \quad \mu_{l(n)} = \tilde{\mu}_{l(n)}, \quad \kappa_{r(n)}(b) = \tilde{\kappa}_{l(n)}(b),$$

then $L(p, q, h_0, h_1, H_0, H_1) = L(\tilde{p}, \tilde{q}, \tilde{h}_0, \tilde{h}_1, \tilde{H}_0, \tilde{H}_1)$ and $H_2 = \tilde{H}_2$.  

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3. Integral representation of products

In this section, we shall study two products of the eigenfunctions of two differential pencils, see (3.6). The result will be used to derive an integral equation for the functions \( p(x) - \bar{p}(x) \) and \( q(x) - \bar{q}(x) \), which does not involve the parameter \( \lambda \).

Let \( y_1(x, \lambda) \) be the solution of equation \( i_2 y_1(x, \lambda) = 0 \) satisfying the initial conditions \( y_1(0) = 1 \) and \( y'_1(0) = -h_0 \), then

\[
y_1(x, \lambda) = \cos[\lambda x - \alpha(x)] + \int_0^X A_1(x, t) \cos(\lambda t) dt + \int_0^X B_1(x, t) \sin(\lambda t) dt,
\]

where \( \alpha(x) = \int_0^x p(t) dt \), and kernels \( A_1(x, t) \) and \( B_1(x, t) \) are solutions of the following problem [19]:

\[
\begin{aligned}
\frac{\partial^2 A_1(x, t)}{\partial t^2} - 2p(x)\frac{\partial A_1(x, t)}{\partial t} - q(x)A_1(x, t) &= \frac{\partial^2 A_1(x, t)}{\partial x^2}, \\
\frac{\partial^2 B_1(x, t)}{\partial t^2} + 2p(x)\frac{\partial B_1(x, t)}{\partial t} - q(x)B_1(x, t) &= \frac{\partial^2 B_1(x, t)}{\partial x^2}, \\
A_1(0, 0) &= -h_0, \quad B_1(0, 0) = 0, \quad \frac{\partial A_1(x, t)}{\partial t} \mid_{t=0} = 0
\end{aligned}
\]

with

\[
2 \frac{d}{dx} [A_1(x, x) \cos \alpha(x) + B_1(x, x) \sin \alpha(x)] = q(x) + p^2(x).
\]

For each \( \lambda \in \mathbb{C}, \lambda \neq 0 \), let \( y_2(x, \lambda) \) be the solution of equation \( i_2 y_2(x, \lambda) = 0 \) satisfying the initial conditions \( y_2(0) = 0 \) and \( y'_2(0) = 1 \), then

\[
y_2(x, \lambda) = \frac{1}{\lambda} \left\{ \sin[\lambda x - \alpha(x)] + \int_0^X A_2(x, t) \cos(\lambda t) dt + \int_0^X B_2(x, t) \sin(\lambda t) dt \right\},
\]

where the kernels \( A_2(x, t) \) and \( B_2(x, t) \) are the solution of the problem

\[
\begin{aligned}
\frac{\partial^2 A_2(x, t)}{\partial t^2} - 2p(x)\frac{\partial A_2(x, t)}{\partial t} - q(x)A_2(x, t) &= \frac{\partial^2 A_2(x, t)}{\partial x^2}, \\
\frac{\partial^2 B_2(x, t)}{\partial t^2} + 2p(x)\frac{\partial B_2(x, t)}{\partial t} - q(x)B_2(x, t) &= \frac{\partial^2 B_2(x, t)}{\partial x^2}, \\
A_2(0, 0) &= p(0), \quad B_2(0, 0) = 0, \quad \frac{\partial A_2(x, t)}{\partial t} \mid_{t=0} = 0
\end{aligned}
\]

with \( \alpha(x) = \int_0^x p(t) dt \). Moreover, there holds

\[
2 \frac{d}{dx} [A_2(x, x) \sin \alpha(x) + B_2(x, x) \cos \alpha(x)] = q(x) + p^2(x).
\]

Let \( y(x, \lambda) \) be the solution to equation \( i_2 y(x, \lambda) = 0 \) with the initial conditions \( y(0) = 1 \) and \( y'(0) = -(i\lambda h_1 + h_0) \), then from (3.1) and (3.2), we have

\[
y(x, \lambda) = y_1(x, \lambda) - i\lambda h_1 y_2(x, \lambda).
\]

Moreover, we obtain

\[
y(x, \lambda) = \cos[\lambda x - \alpha(x)] - ih_1 \sin[\lambda x - \alpha(x)] + \int_0^X A(x, t) \cos(\lambda t) dt \\
+ \int_0^X B(x, t) \sin(\lambda t) dt,
\]

where

\[
A(x, t) = A_1(x, t) - ih_1 A_2(x, t), \quad B(x, t) = B_1(x, t) - ih_1 B_2(x, t).
\]

Simple calculations show that the characteristic equation of the pencils \( L \) can be reduced to \( \omega(\lambda) = 0 \), where

\[
\omega(\lambda) = y'(\sigma, \lambda) + (i\lambda h_1 + H_0) y(\sigma, \lambda) \\
= \lambda[(h_1 H_1 - 1) \sin \lambda x + i(h_1 - h_1) \cos \lambda x] + O(e^{\varepsilon x}) \\
= \lambda \sqrt{(1 - h_1^2)(1 - H_1^2)} \sin(\lambda - \omega) x + O(e^{\varepsilon x}), \quad t = |3\lambda|.
\]

Denote \( G_\delta = \{\lambda : |\lambda - k - \omega| \geq \delta, k = 0, \pm 1, \pm 2, \cdots \}, \delta > 0 \). Using the known method (see, e.g. [4]), one can prove the following estimates for sufficiently large \( |\lambda| \):

\[
|\omega(\lambda)| \geq C_\delta |\lambda|^{|\varepsilon\pi|}, \quad \lambda \in G_\delta.
\]
Similarly, for the solution $\tilde{y}(x, \lambda)$ of equation $\tilde{L}_x \tilde{y}(x, \lambda) = 0$ with the initial conditions $\tilde{y}(0) = 1$ and $\tilde{y}'(0) = -(i\lambda \tilde{h}_1 + \tilde{h}_0)$, there has the following analogous result:

$$\tilde{y}(x, \lambda) = \cos[\lambda x - \tilde{a}(x)] - ih_1 \sin[\lambda x - \tilde{a}(x)] + \int_0^x \tilde{a}(x, t) \cos(\lambda t) dt + \int_0^x \tilde{b}(x, t) \sin(\lambda t) dt,$$

(3.5)

where

$$\tilde{a}(x, t) = \tilde{A}_1(x, t) - ih_1 \tilde{A}_2(x, t), \quad \tilde{b}(x, t) = \tilde{B}_1(x, t) - ih_1 \tilde{B}_2(x, t).$$

Next, using (3.3) and (3.5), and by extending the range of $\tilde{A}(x, t), \tilde{b}(x, t)$ evenly with respect to the argument $t$ and $\tilde{b}(x, t)$ oddly with respect to the argument $t$ and some straightforward calculations, for brevity denoting $\theta_-(t) = \alpha(t) - \tilde{a}(t)$ and $\theta_+(t) = \alpha(t) + \tilde{a}(t)$, we can infer that

$$\tilde{y}\tilde{y}' = \cos[\lambda x - \alpha(x)] \cos[\lambda x - \tilde{a}(x)] - ih_1 \sin[\lambda x - \alpha(x)] \cos[\lambda x - \tilde{a}(x)]$$

$$- ih_1 \cos[\lambda x - \alpha(x)] \sin[\lambda x - \tilde{a}(x)] - h_1 \tilde{h}_1 \sin[\lambda x - \alpha(x)] \sin[\lambda x - \tilde{a}(x)]$$

$$+ \int_0^x H_c(x, t) \cos[2\lambda t - \theta_+(t)] dt + \int_0^x H_s(x, t) \sin[2\lambda t - \theta_+(t)] dt$$

$$= \frac{1 - h_1 \tilde{h}_1}{2} \cos \theta_-(x) + \frac{i}{2} (h_1 - \tilde{h}_1) \sin \theta_-(x)$$

(3.6)

$$+ \frac{1 + h_1 \tilde{h}_1}{2} \cos[2\lambda x - \theta_+(x)] - \frac{i}{2} (h_1 + \tilde{h}_1) \sin[2\lambda x - \theta_+(x)]$$

$$+ \int_0^x H_c(x, t) \cos[2\lambda t - \theta_+(t)] dt + \int_0^x H_s(x, t) \sin[2\lambda t - \theta_+(t)] dt,$$

where $H_c(x, t), H_s(x, t) \in W^2_2([0, \pi] \times [0, \pi])$.

4. Completion of proofs

In this section, we shall complete the proofs of Theorems 2.1 and 2.3. The basic idea is to translate the integral equation (4.1) into an inhomogeneous integral equations that are independent of $\lambda$ and then show, step by step, that the inhomogeneous terms must vanish.

Now, we can give the proofs of theorems in this work.

**Proof of Theorem 2.1**

If we multiply the equation in (1.1) by $\tilde{y}(x)$ and the equation in (2.2) by $y(x)$ and subtract, after integrating on $[0, \frac{\pi}{2}]$, we obtain

$$(\tilde{y}y' - \tilde{y}'y) \frac{\pi}{2} + \int_0^{\pi/2} [(\bar{q} - q) + 2\lambda (\bar{p} - p)] \tilde{y}y dx = 0.$$  

(4.1)

Using the initial conditions at 0, then it yields

$$[\tilde{y} \left( \frac{\pi}{2}, \lambda \right) y' \left( \frac{\pi}{2}, \lambda \right) - y \left( \frac{\pi}{2}, \lambda \right) \tilde{y}' \left( \frac{\pi}{2}, \lambda \right)] + i\lambda (h_1 - \tilde{h}_1) + h_0 - \tilde{h}_0 + \int_0^{\pi/2} [(\bar{q} - q) + 2\lambda (\bar{p} - p)] \tilde{y}y dx = 0.$$  

(4.2)

Denote

$$Q(x) = \bar{q} - q, \quad P(x) = \bar{p} - p,$$

$$H(\lambda) = i\lambda (h_1 - \tilde{h}_1) + h_0 - \tilde{h}_0 + \int_0^{\pi/2} [Q(x) + 2\lambda P(x)] \tilde{y}y dx.$$  

(4.3)

Because $y(x, \lambda)$ and $\tilde{y}(x, \lambda)$ are entire functions in variable $\lambda$, $H(\lambda)$ is an entire function in variable $\lambda$. For $\lambda = \lambda_n$, by the given assumptions, it follows that

$$\tilde{y}_n \left( \frac{\pi}{2} \right) y'_n \left( \frac{\pi}{2} \right) - y_n \left( \frac{\pi}{2} \right) \tilde{y}'_n \left( \frac{\pi}{2} \right) = 0,$$

that is, the first term in (4.2) vanishes and hence

$$H(\lambda_n) = 0, \quad n \in A.$$
which implies that the eigenvalues $\lambda_n$ of pencils $L$ are contained in the set of zeros of $H(\lambda)$. The eigenvalues $\lambda_n$ with account of multiplicity coincide with the zeros of its characteristic function $\omega(\lambda)$. From (3.6) and (4.3), we find that for all complex number $\lambda$

$$|H(\lambda)| \leq (C_1 + C_2|\lambda|)|e^{i\pi}|$$

(4.4)

for some positive constants $C_1$ and $C_2$. Define

$$\Phi(\lambda) = \frac{H(\lambda)}{\omega(\lambda)}$$

(4.5)

which is an entire function from the earlier arguments and it follows from (3.4) and (4.4) that

$$\Phi(\lambda) = O(1)$$

for sufficiently large $|\lambda|, \lambda \in G_\delta$. Then by the maximum modulus principle for all complex number $\lambda$, we obtain that for all complex number $\lambda$

$$\Phi(\lambda) = C,$$

where $C$ is a constant.

Let us show that the constant $C = 0$. We can rewrite equation $H(\lambda) = C\omega(\lambda)$ in the form

$$i\lambda (h_1 - \tilde{h}_1) + h_0 - \tilde{h}_0 + \int_{0}^{\frac{\pi}{2}} [Q(x) + 2\lambda P(x)]\tilde{y}\tilde{dx} = C\lambda \sqrt{(1 - h_1^2)(1 - \tilde{h}_1^2)} \sin(\lambda - \omega)\pi + O\left(e^{x\pi}\right),$$

that is,

$$i(h_1 - \tilde{h}_1) + \frac{h_0 - \tilde{h}_0}{\lambda} + \int_{0}^{\frac{\pi}{2}} \left[Q(x)\lambda + 2P(x)\right]\tilde{y}\tilde{dx} = C\lambda \sqrt{(1 - h_1^2)(1 - \tilde{h}_1^2)} \sin(\lambda - \omega)\pi + O\left(e^{x\pi}/\lambda\right).$$

By use of the Riemann–Lebesgue Lemma, we see that the limit of the left-hand side of the earlier equality exists as $\lambda \to \infty, \lambda \in \mathbb{R}$. Thus, we obtain that the constant $C = 0$. So, we have proved

$$H(\lambda) = 0 \text{ for all complex number } \lambda.$$  

(4.6)

Substituting (3.6) into (4.3), we obtain

$$H(\lambda) = i\lambda (h_1 - \tilde{h}_1) + h_0 - \tilde{h}_0 + \frac{1 - h_1\tilde{h}_1}{2} \int_{0}^{\frac{\pi}{2}} Q(x) \cos\theta_-(x)dx$$

$$+ \frac{i}{2} (h_1 - \tilde{h}_1) \int_{0}^{\frac{\pi}{2}} Q(x) \sin\theta_-(x)dx$$

$$+ \frac{1 + h_1\tilde{h}_1}{2} \int_{0}^{\frac{\pi}{2}} Q(x) \cos[2\lambda x - \theta_+(x)]dx$$

$$- \frac{i}{2} (h_1 + \tilde{h}_1) \int_{0}^{\frac{\pi}{2}} Q(x) \sin[2\lambda x - \theta_+(x)]dx$$

$$+ \int_{0}^{\frac{\pi}{2}} Q(x) \int_{0}^{x} H_c(x, t) \cos[2\lambda t - \theta_+(t)]dt dx$$

$$+ \int_{0}^{\frac{\pi}{2}} Q(x) \int_{0}^{x} H_s(x, t) \sin[2\lambda t - \theta_+(t)]dt dx$$

$$+ \lambda (1 - h_1\tilde{h}_1) \int_{0}^{\frac{\pi}{2}} P(x) \cos\theta_-(x)dx$$

$$+ i\lambda (h_1 - \tilde{h}_1) \int_{0}^{\frac{\pi}{2}} P(x) \sin\theta_-(x)dx$$

$$+ \lambda (1 + h_1\tilde{h}_1) \int_{0}^{\frac{\pi}{2}} P(x) \cos[2\lambda x - \theta_+(x)]dx$$

$$- i\lambda (h_1 + \tilde{h}_1) \int_{0}^{\frac{\pi}{2}} P(x) \sin[2\lambda x - \theta_+(x)]dx$$

$$+ 2\lambda \int_{0}^{\frac{\pi}{2}} P(x) \int_{0}^{x} H_c(x, t) \cos[2\lambda t - \theta_+(t)]dt dx$$

$$+ 2\lambda \int_{0}^{\frac{\pi}{2}} P(x) \int_{0}^{x} H_s(x, t) \sin[2\lambda t - \theta_+(t)]dt dx.$$
Moreover, from $H(\lambda) = 0$ for all complex number $\lambda$ and by use of the Riemann–Lebesgue Lemma as $\lambda \to \infty, \lambda \in \mathbb{R}$, we obtain that
\[
i(h_1 - \tilde{h}_1) + (1 - h_1 \tilde{h}_1) \int_0^\pi q(x) \cos \theta_-(x)dx + i(h_1 - \tilde{h}_1) \int_0^\pi q(x) \sin \theta_-(x)dx = 0. \tag{4.8}
\]

Thus, we have
\[
H(\lambda) = h_0 - \tilde{h}_0 + \frac{1 - h_1 \tilde{h}_1}{2} \int_0^\pi q(x) \cos \theta_-(x)dx \\
+ \frac{i}{2} (h_1 - \tilde{h}_1) \int_0^\pi q(x) \sin \theta_-(x)dx \\
+ \frac{1 + h_1 \tilde{h}_1}{2} \int_0^\pi q(x) \cos [2\lambda x - \theta_+(x)]dx \\
- \frac{i}{2} (h_1 + \tilde{h}_1) \int_0^\pi q(x) \sin [2\lambda x - \theta_+(x)]dx \\
+ \int_0^\pi q(x) \int_0^x H_c(x, t) \cos [2\lambda t - \theta_+(t)]dt dx \\
+ \int_0^\pi q(x) \int_0^x H_s(x, t) \sin [2\lambda t - \theta_+(t)]dt dx \\
+ \lambda (1 + h_1 \tilde{h}_1) \int_0^\pi p(x) \cos [2\lambda x - \theta_+(x)]dx \\
- \frac{i\lambda}{2} (h_1 + \tilde{h}_1) \int_0^\pi p(x) \sin [2\lambda x - \theta_+(x)]dx \\
+ 2\lambda \int_0^\pi p(x) \int_0^x H_c(x, t) \cos [2\lambda t - \theta_+(t)]dt dx \\
+ 2\lambda \int_0^\pi p(x) \int_0^x H_s(x, t) \sin [2\lambda t - \theta_+(t)]dt dx.
\tag{4.9}
\]

Introduce
\[
Q_1(t) = (1 + h_1 \tilde{h}_1) Q(t) + \int_1^t q(x) H_c(x, t)dx, \\
Q_2(t) = -i (h_1 + \tilde{h}_1) Q(t) + \int_1^t q(x) H_s(x, t)dx, \\
P_1(t) = (1 + h_1 \tilde{h}_1) P(t) + \int_1^t 2p(x) H_s(x, t)dx, \\
P_2(t) = -i (h_1 + \tilde{h}_1) P(t) + \int_1^t 2p(x) H_c(x, t)dx.
\tag{4.10}
\]

By changing the order of integration, (4.9) can be rewritten as
\[
H(\lambda) = h_0 - \tilde{h}_0 + \frac{1 - h_1 \tilde{h}_1}{2} \int_0^\pi q(x) \cos \theta_0(x)dx \\
+ \frac{i}{2} (h_1 - \tilde{h}_1) \int_0^\pi q(x) \sin \theta_0(x)dx \\
+ \frac{1}{2} \int_0^\pi R_1(t) e^{2i\lambda t} dt + \frac{1}{2} \int_0^\pi R_2(t) e^{-2i\lambda t} dt \\
+ \lambda \left( \int_0^\pi T_1(t) e^{2i\lambda t} dt + \int_0^\pi T_2(t) e^{-2i\lambda t} dt \right),
\tag{4.11}
\]

where
\[
R_1(t) = \frac{Q_1(t) - iQ_2(t)}{2} e^{-i\theta_0(t)}, \quad R_2(t) = \frac{Q_1(t) + iQ_2(t)}{2} e^{i\theta_0(t)} \\
T_1(t) = \frac{P_1(t) - iP_2(t)}{2} e^{-i\theta_0(t)}, \quad T_2(t) = \frac{P_1(t) + iP_2(t)}{2} e^{i\theta_0(t)}.
\]
By integration by parts in (4.11), we have

\[ H(\lambda) = h_0 - \tilde{h}_0 + \frac{1 - h_1 \tilde{h}_1}{2} \int_0^{\frac{\pi}{2}} Q(x) \cos \theta_-(x) dx \\
+ \frac{i}{2} \left( h_1 - \tilde{h}_1 \right) \int_0^{\frac{\pi}{2}} Q(x) \sin \theta_-(x) dx \\
+ \frac{1}{2} \int_0^{\frac{\pi}{2}} R_1(t) e^{i2\lambda t} dt + \frac{1}{2} \int_0^{\frac{\pi}{2}} R_2(t) e^{-2i\lambda t} dt \\
+ \frac{1}{2} \left( \frac{\pi}{2} \right) \sin [\lambda \pi - \theta_+ (\pi/2)] - \frac{1}{2} P_2(0) \\
+ \frac{i}{2} \int_0^{\frac{\pi}{2}} T'_1(t) e^{i2\lambda t} dt - \frac{i}{2} \int_0^{\frac{\pi}{2}} T'_2(t) e^{-2i\lambda t} dt. \]  

(4.12)

Moreover, from \( H(\lambda) = 0 \) for all complex number \( \lambda \) and by use of the Riemann–Lebesgue Lemma as \( \lambda \to \infty, \lambda \in \mathbb{R} \), we obtain that

\[ P \left( \frac{\pi}{2} \right) = 0, \]  

(4.13)

\[ h_0 - \tilde{h}_0 + \frac{1 - h_1 \tilde{h}_1}{2} \int_0^{\frac{\pi}{2}} Q(x) \cos \theta_-(x) dx + \frac{i}{2} \left( h_1 - \tilde{h}_1 \right) \int_0^{\frac{\pi}{2}} Q(x) \sin \theta_-(x) dx - \frac{1}{2} P_2(0) = 0, \]  

(4.14)

and

\[ \int_0^{\frac{\pi}{2}} \left[ R_1(t) + iT'_1(t) \right] e^{2i\lambda t} dt + \int_0^{\frac{\pi}{2}} \left[ R_2(t) - iT'_2(t) \right] e^{-2i\lambda t} dt = 0. \]  

(4.15)

Because the exponential system \( \left\{ (e^{2i\lambda t}, e^{-2i\lambda t})^T : \lambda \in \mathbb{R} \right\} \) is complete in \((L^2(0, \pi/2))^2\), consequently,

\[ R_1(t) + iT'_1(t) = 0 = R_2(t) - iT'_2(t) \text{ on } (0, \pi/2). \]

From the definitions of \( R_1(t), R_2(t), T_1(t), T_2(t) \) by (4.11), we can infer

\[ Q_1(t) + P_1(t) \theta'_+(t) + P'_2(t) = 0 = Q_2(t) + P_2(t) \theta'_+(t) - P'_1(t). \]  

(4.16)

Substituting (4.10) into (4.16), together with \( P(\pi/2) = 0 \), it follows that

\[
\begin{pmatrix}
1 + h_1 \tilde{h}_1 & \left( 1 + h_1 \tilde{h}_1 \right) \theta''_+(t) - 2H_c(t, t) \\
-i \left( h_1 + \tilde{h}_1 \right) P'(t) - \int_{\frac{\pi}{2}}^t H_c(x, t) Q(x) dx \\
+ \int_{\frac{\pi}{2}}^t \left[ 2\theta'_+(t) H_c(x, t) + 2 \frac{\beta}{m} H_c(x, t) \right] P(x) dx = 0,
\end{pmatrix}
\]

\[
\begin{pmatrix}
-i \left( h_1 + \tilde{h}_1 \right) Q(t) + \left[ -i \left( h_1 + \tilde{h}_1 \right) \theta''_+(t) + 2H_c(t, t) \right] P(t) \\
- \left( 1 + h_1 \tilde{h}_1 \right) P'(t) + \int_{\frac{\pi}{2}}^t H_c(x, t) Q(x) dx \\
+ \int_{\frac{\pi}{2}}^t \left[ 2\theta'_+(t) H_c(x, t) - 2 \frac{\beta}{m} H_c(x, t) \right] P(x) dx = 0,
\end{pmatrix}
\]

(4.17)

\[ P(t) + \int_{\frac{\pi}{2}}^t P'(x) dx = 0. \]

Introduce

\[ F(t) = (Q(t), P(t), P'(t))^T, \]

\[ K_1(t) = \begin{pmatrix}
1 + h_1 \tilde{h}_1 & \left( 1 + h_1 \tilde{h}_1 \right) \theta''_+(t) - 2H_c(t, t) & -i \left( h_1 + \tilde{h}_1 \right) \\
-i \left( h_1 + \tilde{h}_1 \right) & -i \left( h_1 + \tilde{h}_1 \right) \theta''_+(t) + 2H_c(t, t) & -i \left( h_1 + \tilde{h}_1 \right) \\
0 & 1 & 0
\end{pmatrix}, \]

and

\[ K_2(x, t) = \begin{pmatrix}
H_c(x, t) - 2\theta'_+(t) H_c(x, t) + 2 \frac{\beta}{m} H_c(x, t) & 0 \\
H_c(x, t) - 2\theta'_+(t) H_c(x, t) - 2 \frac{\beta}{m} H_c(x, t) & 0 \\
0 & 0 & 1
\end{pmatrix}. \]
Equation (4.17) can readily be reduced to a vector form

\[ K_1(t)F(t) + \int_t^{\pi/2} K_2(x, t)F(x)\,dx = 0 \quad \text{for} \quad 0 < t < \frac{\pi}{2}, \tag{4.18} \]

Because \( \det K_1(t) = -(1 - h_1^2)(1 - \tilde{h}_1^2) \neq 0 \) under the assumption that \( h_1 \neq \pm 1 \) and \( \tilde{h}_1 \neq \pm 1 \), (4.18) can be rewritten as

\[ F(t) + \int_t^{\pi/2} K_1^{-1}(t)K_2(x, t)F(x)\,dx = 0 \quad \text{for} \quad 0 < t < \frac{\pi}{2}. \tag{4.19} \]

But this is a homogeneous Volterra integral equation, and its solution is identically zero a.e. Thus, we have obtained

\[ F(t) = 0 \quad \text{a.e. on} \quad [0, \pi/2], \]

which yields that

\[ Q(t) = P(t) = 0 \quad \text{a.e. on} \quad [0, \pi/2]. \]

Therefore, we have proven

\[ p(x) = \tilde{p}(x) \quad \text{on} \quad [0, \pi/2] \quad \text{and} \quad q(x) = \tilde{q}(x) \quad \text{a.e. on} \quad [0, \pi/2]. \]

Moreover, from (4.8) and (4.14), it is obvious that \( h_k = \tilde{h}_k \) \( (k = 0, 1) \).

To prove that

\[ p(x) = \tilde{p}(x) \quad \text{on} \quad [\pi/2, \pi] \quad \text{and} \quad q(x) = \tilde{q}(x) \quad \text{a.e. on} \quad [\pi/2, \pi] \tag{4.20} \]

and \( H_k = \tilde{H}_k \) \( (k = 0, 1) \), we should repeat the earlier argument for the supplementary problem

\[
\begin{align*}
\lambda''(x) + \left[ \lambda^2 - 2\lambda \rho_1(x) - q_1(x) \right] y(x) &= 0, \quad x \in [0, \pi], \\
y'(0) + (-i\lambda h_1 - h_0) y(0) &= 0, \\
y'(\pi) + (-i\lambda \tilde{h}_1 - h_0) y(\pi) &= 0,
\end{align*}
\]

where \( q_1(x) = q(\pi - x) \) and \( \rho_1(x) = \rho(\pi - x) \). Then, we obtain \( P(\pi - t) = 0 = Q(\pi - t) \) a.e. on \( [0, \pi/2] \), that is, (4.20) holds and \( H_k = \tilde{H}_k \) \( (k = 0, 1) \). The proof of theorem is finished. \( \square \)

To prove Theorem 2.3 in this paper, we first give a Lemma.

Let \( m(n) \) be a sequence of natural numbers such that

\[ m(n) = \frac{n}{\sigma}(1 + \epsilon_n), \quad 0 < \sigma \leq 1, \quad \epsilon_n \to 0. \tag{4.21} \]

**Lemma 4.1**

1. Let \( m(n) \) and \( b \in (0, \frac{\pi}{2}) \) be such that \( \sigma > \frac{2b}{\pi} \). If for any \( n \in A \),

\[ \lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \quad \kappa_{m(n)}(b) = \tilde{\kappa}_{m(n)}(b), \]

then

\[ p(x) = \tilde{p}(x) \quad \text{on} \quad [0, b] \quad \text{and} \quad q(x) = \tilde{q}(x) \quad \text{a.e. on} \quad [0, b], \]

and \( h_k = \tilde{h}_k \) \( (k = 0, 1) \).

2. Let \( m(n) \) and \( b \in \left( \frac{\pi}{2}, \pi \right) \) be such that \( \sigma > 2 - \frac{2b}{\pi} \). If for any \( n \in A \),

\[ \lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \quad \kappa_{m(n)}(b) = \tilde{\kappa}_{m(n)}(b), \]

then

\[ p(x) = \tilde{p}(x) \quad \text{on} \quad [b, \pi] \quad \text{and} \quad q(x) = \tilde{q}(x) \quad \text{a.e. on} \quad [b, \pi], \]

and \( H_k = \tilde{H}_k \) \( (k = 0, 1) \).
Proof

(1) Let \( y(x, \lambda) \) be the solution to
\[
- y''(x, \lambda) + [q(x) + \frac{2\lambda}{x^2} p(x)] y(x, \lambda) = \lambda^2 y(x, \lambda)
\]
with the initial conditions \( y(0) = 1 \) and \( y'(0) = -i(\lambda h_1 + h_0) \). Similarly, let \( \tilde{y}(x, \lambda) \) be the solution of
\[
- \tilde{y}''(x, \lambda) + [\tilde{q}(x) + \frac{2\lambda}{x^2} \tilde{p}(x)] \tilde{y}(x, \lambda) = \lambda^2 \tilde{y}(x, \lambda)
\]
with the initial conditions \( \tilde{y}(0) = 1 \) and \( \tilde{y}'(0) = -i(\lambda \tilde{h}_1 + \tilde{h}_0) \).

If we multiply (4.22) by \( \tilde{y}(x, \lambda) \) and (4.23) by \( y(x, \lambda) \) and subtract, after integrating on \([0, b]\), we obtain
\[
G(\lambda) \overset{\text{def}}{=} \int_0^b \left[ \tilde{q}(x) - q(x) \right] y(x, \lambda) \tilde{y}(x, \lambda) dx
+ 2\lambda \int_0^b \left[ \tilde{p}(x) - p(x) \right] y(x, \lambda) \tilde{y}(x, \lambda) dx
+ i\lambda (h_1 - \tilde{h}_1) + h_0 - \tilde{h}_0
= \left[ \tilde{y}'(x, \lambda) y(x, \lambda) - y(x, \lambda) \tilde{y}'(x, \lambda) \right]_{x=b}.
\]

By the assumption \( \kappa_{m(n)}(b) = \tilde{\kappa}_{m(n)}(b) \), it follows that
\[
G(\lambda_{m(n)}) = 0, \quad n \in A.
\]

Using the same method in [27] (pp. 291–292), we can show that \( G(\lambda) = 0 \) on the whole complex plane. As we already mentioned, if \( G(\lambda) = 0 \), then the conclusion of Lemma is true.

(2) Note that the interval \([b, \pi]\) can be converted to an interval \([0, \pi - b]\) by a transformation of variable \( x \mapsto \pi - x \).

To prove (2), we should repeat arguments in part (1) for the supplementary problem \( \tilde{L} \):
\[
\begin{cases}
\tilde{L} y(x) \overset{\text{def}}{=} y''(x) + [\lambda^2 - 2\lambda \rho_1(x) - q_1(x)] y(x) = 0, \quad x \in [0, \pi], \\
y'(0) + (-i\lambda H_1 - H_0) y(0) = 0, \\
y'(\pi) + (-i\lambda \tilde{h}_1 - \tilde{h}_0) y(\pi) = 0,
\end{cases}
\]
where \( q_1(x) = q(\pi - x) \) and \( \rho_1(x) = \rho(\pi - x) \). A direct calculation implies that \( \tilde{y}_n(x) = y_n(\pi - x) \) is a solution to the supplementary problem \( \tilde{L} \) and \( \tilde{y}_n(\pi - b) = y_n(b) \). Note that \( \pi - b \in (0, \frac{\pi}{2}) \). Thus, for the supplementary problem \( \tilde{L} \), the assumption in the case (1) is satisfied still. If we repeat the earlier arguments, we can obtain the proof of Lemma.

Proof of Theorem 2.3

Because
\[
\lambda_{r(n)} = \tilde{\lambda}_{r(n)}, \quad \kappa_{r(n)}(b) = \tilde{\kappa}_{r(n)}(b),
\]
where \( r(n) \) satisfies (2.3) and \( \sigma_2 > 2 - \frac{2b}{\pi} \), by Lemma 4.1, we obtain that
\[
p(x) = \tilde{p}(x) \text{ on } [b, \pi] \text{ and } q(x) = \tilde{q}(x) \text{ a.e. on } [b, \pi],
\]
and \( H_k = \tilde{H}_k \) \((k = 0, 1)\). Thus, we only need to prove that
\[
p(x) = \tilde{p}(x) \text{ on } [0, b] \text{ and } q(x) = \tilde{q}(x) \text{ a.e. on } [0, b]
\]
and \( h_k = \tilde{h}_k \) \((k = 0, 1), H_2 = \tilde{H}_2\).

Similar to (4.24), in the case \( b \in (\frac{\pi}{2}, \pi) \), we have
\[
G(\lambda) \overset{\text{def}}{=} \int_0^b \left[ \tilde{q}(x) - q(x) \right] y(x, \lambda) \tilde{y}(x, \lambda) dx
+ 2\lambda \int_0^b \left[ \tilde{p}(x) - p(x) \right] y(x, \lambda) \tilde{y}(x, \lambda) dx
+ i\lambda (h_1 - \tilde{h}_1) + h_0 - \tilde{h}_0
= \left[ \tilde{y}'(x, \lambda) y(x, \lambda) - y(x, \lambda) \tilde{y}'(x, \lambda) \right]_{x=b}.
\]

Using the same method in [27] (pp. 292–293), we can prove that \( G(\lambda) = 0 \) on the whole \( \lambda \)-plane. This implies that
\[
p(x) = \tilde{p}(x) \text{ on } [0, b] \text{ and } q(x) = \tilde{q}(x) \text{ a.e. on } [0, b],
\]
and \( h_k = \tilde{h}_k \) \((k = 0, 1)\).

Because \( \mu_n \) are eigenvalues of differential pencils \( L(p, q, h_0, h_1, H_0, H_2) \) and \( \tilde{\mu}_n \) are eigenvalues of differential pencils \( L \left( \tilde{p}, \tilde{q}, \tilde{h}_0, \tilde{h}_1, \tilde{H}_0, \tilde{H}_2 \right) \), by the relation \( \left( p, h_1, \mu_{m(n)} \right) = \left( \tilde{p}, \tilde{h}_1, \tilde{\mu}_{m(n)} \right) \) and the asymptotic expression (2.1), we obtain \( H_2 = \tilde{H}_2 \). The proof is complete. \( \square \)
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