Abstract

These lectures given in Montreal in Summer 1997 are mainly based on, and form a condensed survey of the book by N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser 1997. Various algebras arising naturally in Representation Theory such as the group algebra of a Weyl group, the universal enveloping algebra of a complex semisimple Lie algebra, a quantum group or the Iwahori-Hecke algebra of bi-invariant functions (under convolution) on a $p$-adic group, are considered. We give a uniform geometric construction of these algebras in terms of homology of an appropriate “Steinberg-type” variety $Z$ (or its modification, such as K-theory or elliptic cohomology of $Z$, or an equivariant version thereof). We then explain how to obtain a complete classification of finite dimensional irreducible representations of the algebras in question, using our geometric construction and perverse sheaves methods.

Similar techniques can be applied to other algebras, e.g. the double-affine Hecke algebras, elliptic algebras, quantum toroidal algebras.

Introduction

A new branch has emerged during the last decade within the part of mathematics dealing with Lie groups. That new branch may be called *Geometric Representation Theory*. As Beilinson-Bernstein put it in their seminal paper [BeBe], the discovery of $\mathcal{D}$-modules and of perverse sheaves has made Representation Theory, to a large extent, part of Algebraic Geometry. Among applications of perverse sheaf methods to Representation Theory that have already proved to be of primary importance we would like to mention here the proof of the Kazhdan-Lusztig conjecture by Beilinson-Bernstein [BeBe] and Brylinski-Kashiwara [BrKa], Lusztig’s construction of canonical bases in quantum groups [L1], and the work of Beilinson-Drinfeld on the Geometric Langlands conjecture, cf. [Gi3]. We refer the reader to [L2] for further applications.

In these notes we discuss another (not completely unrelated to the above) kind of applications of equivariant $K$-theory and perverse sheaves to representations of Hecke algebras
and quantum groups. We study various associative algebras that arise naturally in Representation Theory. These may be, for example, either the group algebra of a Weyl group, or the universal enveloping algebra of a complex semisimple Lie algebra, or a quantum group, or the Hecke algebra of bi-invariant functions (under convolution) on a $p$-adic group. Further examples, such as the double-affine Hecke algebra of Cherednik, elliptic algebras, quantum toroidal algebras, etc., fit into the same scheme, but will not be considered here; see [GaGr], [GKV1], [GKV2], [Na] for more details. In spite of the diversity of all the examples above, our strategy will always follow the same pattern that we now outline.

The first step consists of giving an “abstract-algebraic” presentation of our algebra $A$ in terms of a convenient set of generators and relations. The second step is to find a geometric construction of $A$. More specifically, we are looking for a complex manifold $M$ and a “correspondence” $Z \subset M \times M$ such that the algebra $A$ is isomorphic to the homology $H_\ast(Z, \mathbb{C})$ or its modification, such as $K$-theory or elliptic cohomology of $Z$, or an equivariant version thereof. Here the subvariety $Z$ that we are seeking should be thought of as the graph of a multivalued map $f : M \to M$, and this map $f$ should satisfy the idempotency equation: $f \circ f = f$. Such an equation rarely holds for genuine maps, but becomes not so rare for multivalued maps. We will see that the idempotency equation for $f$ gives rise to a multiplication-map on homology: $H_\ast(Z, \mathbb{C}) \times H_\ast(Z, \mathbb{C}) \to H_\ast(Z, \mathbb{C})$, called convolution. Such a convolution makes $H_\ast(Z, \mathbb{C})$ an associative, typically noncommutative, $\mathbb{C}$-algebra, and it is this algebra structure on homology (or $K$-theory or elliptic cohomology) of $Z$ that should be isomorphic to the one on $A$.

It should be mentioned that, in all examples above, the only known way of proving an isomorphism $A \simeq H_\ast(Z, \mathbb{C})$ is by showing that the algebra on the right hand side has the same set of generators and relations as were found for $A$ in Step 1. Of course, given an algebra $A$, there is no a-priori recipe helping to find a relevant geometric data $(M, Z)$; in each case this is a matter of good luck. Sometimes a partial indication towards finding (in a conceptual way) a geometric realization of our algebra $A$ as the convolution algebra $H_\ast(Z, \mathbb{C})$ comes from $\mathcal{D}$-modules; more precisely, from the notion of the characteristic cycle of a holonomic $\mathcal{D}$-module (this links our subject to that discussed in the first paragraph, see [Gi1]). It is fair to say, however, that it is still quite a mystery, why a geometric realization of the algebras $A$ that we are interested in is possible at all. But once a geometric realization is found, a complete classification of finite dimensional irreducible representations of $A$ can be obtained in a straightforward manner. This constitutes the last step of our approach, which we now outline and which is to be explained in more detail in Section 5.

The geometric realization of the algebra reduces the problem to the classification of finite dimensional irreducible representations of a convolution algebra, like $H_\ast(Z, \mathbb{C})$. This problem is solved as follows. First we show, using the techniques of sheaf theory (see Sections 4 and 5), that the convolution algebra is isomorphic to the Ext-algebra, $\text{Ext}^\ast(\mathcal{L}, \mathcal{L})$, equipped with the Yoneda product, where $\mathcal{L}$ is a certain constructible complex on an appropriate complex variety. The structure of $\mathcal{L}$ is then analyzed using the very deep Decomposition Theorem [BBD]. The theorem yields an explicit decomposition of the Ext-algebra as the sum of a nilpotent ideal and a direct sum of finitely many matrix algebras. Hence the nilpotent ideal is the radical, and each matrix algebra occurring in the direct sum gives an irreducible representation of the Ext-algebra. Therefore, the non-isomorphic irreducible representations are parametrized by the matrix algebras that occur in the decomposition above. Thus, the classification of finite dimensional irreducible representations of the original algebra $A$
can be read off from the decomposition of the constructible complex \( \mathcal{L} \) provided by the Decomposition Theorem.

These notes are mainly based on, and form a condensed survey of, the book [CG]. The reader is referred to the introduction to [CG] for more motivation and historical background. We have tried, however, to make our present exposition as complementary to [CG] as possible. For example, the geometric construction of Weyl groups and enveloping algebras given here is based on Fourier transform, whereas in [CG] another approach has been used. We also discuss here degenerate affine Hecke algebras and quantum affine algebras, which were not present in [CG].

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**1 Borel-Moore homology**

Borel-Moore homology will be the principal functor we use in these lectures for constructing representations of Weyl groups, enveloping algebras, and Hecke algebras. We review here the most essential properties of the Borel-Moore homology theory and refer the reader to the monographs [Bre] and [Iv] for a more detailed treatment of the subject. We have to say a few words about the kind of spaces we will be dealing with.

By a “space” (in the topological sense) we will mean a locally compact topological space \( X \) that has the homotopy type of a finite \( CW \)-complex; in particular, has finitely many connected components and finitely generated homotopy and homology groups (with \( \mathbb{Z} \)-coefficients). Furthermore, our space \( X \) is assumed to admit a closed embedding into a countable at infinity \( C^\infty \)-manifold \( M \) (in particular, \( X \) is paracompact). We assume also that there exists an open neighborhood \( U \supset X \) in \( M \) such that \( X \) is a homotopy retract of \( U \). It is known, cf. [GM], [RoSa], that any complex or real algebraic variety satisfies the above conditions. These are the spaces we will mainly use in applications.

Similarly, by a closed “subset” of a \( C^\infty \)-manifold we always mean a subset \( X \) which has an open neighborhood \( U \supset X \) such that \( X \) is a homotopy retract of \( U \). In that case one can also find a smaller closed neighborhood \( V \subset U \) such that \( X \) is a proper homotopy retract of \( V \) (recall that a continuous map \( f : X \to Y \) is called proper if the inverse image of any compact set is compact).

We now give a list of the various equivalent definitions of Borel-Moore homology of a
space $X$, see [BoMo], [Bre]. In what follows, all homology and cohomology is taken with complex coefficients, which may be replaced by any field of characteristic zero.

1. Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of $X$. Define $H_{BM}^*(X) = H_*(\hat{X}, \infty)$, where $H_*$ is ordinary relative homology of the pair $(\hat{X}, \infty)$.

2. Let $\overline{X}$ be an arbitrary compactification of $X$ such that $(\overline{X}, X \setminus X)$ is a $CW$-pair. Then, $H_{BM}^*(X) \simeq H_*(\overline{X}, \overline{X} \setminus X)$; see [Sp]. The fact that this definition agrees with (1) is proved in [Bre].

3. Let $C_{BM}^*(X)$ be the chain complex of infinite singular chains $\sum_{i=0}^\infty a_i \sigma_i$, where $\sigma_i$ is a singular simplex, $a_i \in \mathbb{C}$, and the sum is locally finite in the following sense: for any compact set $D \subset X$ there are only finitely many non-zero coefficients $a_i$ such that $D \cap \text{supp} \sigma_i \neq \emptyset$. The usual boundary map $\partial$ on singular chains is well defined on $C_{BM}^*(X)$ because taking boundaries cannot destroy the finiteness condition. We then have

$$H_{BM}^*(X) = H_*(C_{BM}^*(X), \partial).$$

4. Poincaré duality: let $M$ be a smooth, oriented manifold, and $\dim M = m$. Let $X$ be a closed subset of $M$ which has a closed neighborhood $U \subset M$ such that $X$ is a proper deformation retract of $U$. Then there is a canonical isomorphism ([Iv], [Bre]):

$$H_{BM}^i(X) \simeq H^{m-i}(M, M \setminus X),$$

where each side of the equality is understood to be with complex coefficients. In particular, setting $X = M$ we obtain, for any smooth not necessarily compact variety $M$, a canonical isomorphism (depending on the orientation of $M$)

$$H_{BM}^i(M) \simeq H^{m-i}(M).$$

We will often use the “Poincaré duality” definition (formula (1) above) to prove many of the basic theorems about Borel-Moore homology by appealing to the same theorems for singular cohomology. In these instances we will refer the reader to [Bre], [Sp] for the proofs in singular cohomology, despite the fact that Borel-Moore homology is not explicitly developed there.

**Notation** From now on $H_*$ will stand for $H_{BM}^*$ (since Borel-Moore homology is the main functor used in these notes).

We now study the functorial properties of Borel-Moore homology.

**Proper pushforward** Borel-Moore homology is a covariant functor with respect to proper maps. If $f : X \to Y$ is a proper map, then we may define the direct image (or proper push-forward) map

$$f_* : H_*(X) \to H_*(Y)$$

by extending $f$ to a map $\bar{f} : \hat{X} \to \hat{Y}$ where $\hat{X} = X \cup \{\infty\}$, resp. $\hat{Y} = Y \cup \{\infty\}$, and $f(\infty) = \infty$ (observe that $f$ being proper ensures that $\bar{f}$ is continuous).

**Long exact sequence of Borel-Moore homology** Given an open subset $U \subset X$ there is a natural restriction morphism $H_*(X) \to H_*(U)$ induced by the composition of maps:

$$H_*(X) = H_\text{ord}(\overline{X}, \overline{X} \setminus X) \to H_\text{ord}(\overline{X}, \overline{X} \setminus U) = H_*(U),$$
where $\overline{X}$ stands for a compactification of $X$, cf. definition (2) of Borel-Moore homology, and the map in the middle is induced by the natural morphism of pairs $(\overline{X}, \overline{X} \setminus X) \to (\overline{X}, \overline{X} \setminus U)$. For an alternative ad hoc definition of the restriction to an open subset see [IV].

Suppose that next $F$ is a closed subset of $X$. Write $i : F \hookrightarrow X$ for the (closed) embedding, set $U = X \setminus F$, and consider the diagram

$$F \xrightarrow{i} X \xleftarrow{j} U.$$  

Since $i$ is proper and $j$ is an open embedding, the functors $i_*$ and $j^*$ are defined. Then there is a natural long exact sequence in Borel-Moore homology (see [Bre], [Sp] for more details):

$$\cdots \to H_p(F) \to H_p(X) \to H_p(U) \to H_{p-1}(F) \to \cdots \tag{3}$$

To construct this long exact sequence, choose an embedding of $X$ as a closed subset of a smooth manifold $M$. Then the Poincaré duality isomorphism (1) gives:

$$H^{m-p}(M, M \setminus X) \simeq H_p(X) \quad \text{and} \quad H^{m-p}(M, M \setminus F) \simeq H_p(F).$$

Further, the set $U$ being locally closed in $M$, we may find an open subset $M' \subset M$ such that $U$ is a closed subset of $M'$. Then, the excision axiom, see [Sp], combined with Poincaré duality yields

$$H^{m-p}(M, M \setminus U) \simeq H^{m-p}(M', M' \setminus U) \simeq H_p(U).$$

Thus, we see that terms of the standard relative cohomology long exact sequence, cf. [Sp]:

$$\cdots \to H^k(M, M \setminus F) \to H^k(M, M \setminus X) \to H^k(M, M \setminus U) \to H^{k+1}(M, M \setminus F) \to \cdots \tag{4}$$

get identified via the above isomorphisms with the corresponding terms of (3). In this way we define (3) to be the exact sequence induced by the cohomology exact sequence (4).

**Fundamental class** The main reason we are using Borel-Moore homology is the existence of fundamental classes. Recall that any smooth oriented manifold $X$ has a well defined fundamental class in Borel-Moore homology:

$$[X] \in H_m(X), \quad m = \dim \mathbb{R} X.$$  

Note that there is no fundamental class in ordinary homology unless $X$ is compact.

The essential feature of Borel-Moore homology is the existence of a fundamental class, $[X]$, of any (not necessarily smooth or compact) complex algebraic variety $X$. If $X$ is irreducible of real dimension $m$, then $[X]$ is the unique class in $H_m(X)$ that restricts to the fundamental class of the non-singular part of $X$. More precisely, write $X^{\text{reg}}$ for the Zariski open dense subset consisting of the non-singular points of $X$. Being a smooth complex manifold, $X^{\text{reg}}$ has a canonical orientation coming from the complex structure, and hence a fundamental class $[X^{\text{reg}}] \in H_m(X^{\text{reg}})$. The inequality $\dim \mathbb{R}(X \setminus X^{\text{reg}}) \leq m - 2$ yields (say by definition (1) of Borel-Moore homology)

$$H_k(X \setminus X^{\text{reg}}) = 0 \quad \text{for any} \quad k > m - 2.$$
The long exact sequence of Borel-Moore homology \((\text{3})\) shows that the restriction \(H_\text{reg}(X) \to H_m(X)\) is an isomorphism. We define \([X]\) to be the preimage of \([X]_{\text{reg}}\) under this isomorphism. If \(X\) is an arbitrary complex algebraic variety with irreducible components \(X_1, X_2, \ldots, X_n\), then \([X]\) is set to be a non-homogeneous class equal to \(\sum [X_i]\).

The top Borel-Moore homology of a complex algebraic variety is particularly easy to understand in the light of the following proposition.

**Proposition 1.1**  
Let \(X\) be a complex algebraic variety of complex dimension \(n\) and let \(X_1, \ldots, X_m\) be the \(n\)-dimensional irreducible components of \(X\). Then the fundamental classes \([X_1], \ldots, [X_m]\) form a basis for the vector space \(H_{\text{top}}(X) = H_{2n}(X)\). \(\square\)

**Intersection Pairing** Let \(M\) be a smooth, oriented manifold and \(Z, \tilde{Z}\) two closed subsets (in the sense explained at the beginning of this section) in \(M\). We define a bilinear pairing

\[
\cap: H_i(Z) \times H_j(\tilde{Z}) \to H_{i+j-m}(Z \cap \tilde{Z}), \quad m = \dim \mathbb{R} M
\]

which refines the standard intersection of cycles in a smooth variety. The only new feature is that instead of regarding cycles as homology classes in the ambient manifold \(M\) we take their supports into account. So, given two singular chains with supports in the subsets \(Z, \tilde{Z}\), respectively, we would like to define their intersection to be a class in the homology of the set-theoretic intersection, \(Z \cap \tilde{Z}\). To that end we use the standard \(\cup\)-product in relative cohomology (cf. \[Sp\]):

\[
\cup: H^{m-i}(M, M \setminus Z) \times H^{m-j}(M, M \setminus \tilde{Z}) \to H^{2m-j-i}(M, (M \setminus Z) \cup (M \setminus \tilde{Z})).
\]

Applying Poincaré duality \((\text{1})\) to each term of this \(\cup\)-product we get the intersection pairing \((\text{5})\).

The intersection pairing introduced above has an especially clear geometric meaning in the case when \(M\) is a real analytic manifold and \(Z, \tilde{Z}\) are closed analytic subsets in \(M\). One can then use the definition of Borel-Moore homology as the homology of the complex formed by subanalytic chains, cf. e.g. \[GM2\] or \[KS\]. It is known further, see \[RoSa\], that the set \(Z \cap \tilde{Z}\) has an open neighborhood \(U\) in \(M\) such that \(Z \cap \tilde{Z}\) is a proper homotopy retract of \(U\), the closure of \(U\) (this is a general property of analytic sets). Now, given two subanalytic cycles \(c \in H_\ast(Z)\) and \(\tilde{c} \in H_\ast(\tilde{Z})\), one can give the following geometric construction of the class \(c \cap \tilde{c} \in H_\ast(Z \cap \tilde{Z})\).

First choose \(V\), an open neighborhood of \(Z\) in \(M\), such that \(Z\) is a proper homotopy retract of \(V\), and \(\overline{V} \cap \tilde{Z} \subset U\). Second, since \(V\) is smooth, one can find a subanalytic cycle \(c'\) in \(\overline{V}\) which is homologous to \(c\) in \(\overline{V}\) and such that the set-theoretic intersection of \(c'\) with \(\tilde{c}\) is contained in \(V\) and, moreover, \(c'\) intersects \(\tilde{c}\) transversely at smooth points of both \(c'\) and \(\tilde{c}\). Hence, the set-theoretic intersection \(c' \cap \tilde{c}\) gives a well-defined subanalytic cycle in \(H_\ast(\overline{V} \cap \tilde{Z})\), and therefore in \(H_\ast(U)\). Finally, one defines \(c \cap \tilde{c} \in H_\ast(Z \cap \tilde{Z})\) as the direct image of \(c' \cap \tilde{c}\) under a proper contraction \(U \to Z \cap \tilde{Z}\) which exists by assumption. It is fairly straightforward to check that this way one obtains the same class as the one defined in \((\text{5})\) via the \(\cup\)-product in cohomology. It follows in particular that the result of the geometric construction above does not depend on the choices involved in the construction.
2 Convolution in Borel-Moore homology

In this section we give a general construction of a convolution-type product in Borel-Moore homology. Though looking technically quite involved, the construction is essentially nothing but a “homology-valued” version of the standard definition of the composition of multi-valued maps.

Toy example
We begin with the trivial case of the convolution product. We write \( C(M) \) for the finite dimensional vector space of \( \mathbb{C} \)-valued functions on a finite set \( M \). Given finite sets \( M_1, M_2, M_3 \), define a convolution product:

\[
C(M_1 \times M_2) \otimes C(M_2 \times M_3) \to C(M_1 \times M_3)
\]

by the formula

\[
f_{12} \ast f_{23} : (m_1, m_3) \mapsto \sum_{m_2 \in M_2} f_{12}(m_1, m_2) \cdot f_{23}(m_2, m_3).
\]

Writing \( d_i \) for the cardinality of the finite set \( M_i \) we may naturally identify \( C(M_i \times M_j) \) with the vector space of \( d_i \times d_j \)-matrices with complex entries. Then, formula (6) turns into the standard formula for the matrix multiplication.

As a next step of our toy example we would like to find a similar convolution construction assuming that \( M_1, M_2, M_3 \) are smooth compact manifolds rather than finite sets (note that the compactness condition is a natural generalization of the finiteness condition. The latter was needed in order to make the sum in the RHS of (6) finite). As one knows from elementary analysis, it is usually the measures and not the functions that can be convoluted in a natural way. In differential geometry the role of measures is played by the differential forms. Thus, given a smooth manifold \( M \), we let \( \Omega^\bullet(M) \) denote the graded vector space of \( C^\infty \)-differential forms on \( M \). This is the right substitute for the vector space \( C(M) \) when a finite set is replaced by a manifold.

Let \( M_1, M_2, M_3 \) be smooth compact oriented manifolds, and \( p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j \) the projection to the \((i,j)\)-factor. Put \( d = \dim M_2 \). We now define a convolution product:

\[
\Omega^i(M_1 \times M_2) \otimes \Omega^j(M_2 \times M_3) \to \Omega^{i+j-d}(M_1 \times M_3)
\]

by the formula

\[
f_{12} \ast f_{23} = \int_{M_2} p_{12}^* f_{12} \wedge p_{23}^* f_{23}.
\]

Here \( \int_{M_2} \) stands for operation of integrating over the fibers of the projection \( p_{13} : M_1 \times M_2 \times M_3 \to M_1 \times M_3 \) (see [BtTu]).

The standard properties of differential calculus on manifolds show that the convolution (7) is compatible with the De Rham differential, i.e., we have

\[
d(f_{12} \ast f_{23}) = (df_{12}) \ast f_{23} + (-1)^j f_{12} \ast (df_{23}), \quad j = \deg f_{12}.
\]

It follows that the convolution product of differential forms induces a convolution product on the De Rham cohomology:

\[
H^i(M_1 \times M_2) \otimes H^j(M_2 \times M_3) \to H^{i+j-d}(M_1 \times M_3).
\]
The latter can be transported, via the Poincaré duality, to a similar convolution in homology.

In what follows, we are going to give an alternative “abstract” definition of the convolution product (8) in terms of algebraic topology. One advantage of such an “abstract” definition is that it works for any generalized homology theory, e.g., for K-theory. Such a K-theoretic convolution will be discussed below and applied to representation theory later. Another advantage of the “abstract” definition is that it enables us to make a refined convolution construction “with supports”.

**General case**

We proceed now to the “abstract” construction of the convolution product. Let $M_1, M_2, M_3$ be connected, oriented $C^\infty$-manifolds and let

$$Z_{12} \subset M_1 \times M_2, \quad Z_{23} \subset M_2 \times M_3$$

be closed subsets. Define the set-theoretic composition $Z_{12} \circ Z_{23}$ as follows

$$Z_{12} \circ Z_{23} = \{(m_1, m_3) \in M_1 \times M_3 \mid \text{there exists } m_2 \in M_2 \text{ such that } (m_1, m_2) \in Z_{12} \text{ and } (m_2, m_3) \in Z_{23}\}.$$  \hspace{1cm} (9)

If we think of $Z_{12}$ (resp. $Z_{23}$) as the graph of a multivalued map from $M_1$ to $M_2$ (resp. from $M_2$ to $M_3$), then $Z_{12} \circ Z_{23}$ may be viewed as the graph of the composition of $Z_{12}$ and $Z_{23}$.

**Example** Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be smooth maps. Then

$$\text{Graph}(f) \circ \text{Graph}(g) = \text{Graph}(g \circ f). \quad \Box$$

We will need another form of definition (8) in the future. Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection to the $(i, j)$-factor. From now on, we assume, in addition, that the map

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3 \text{ is proper.} \quad (10)$$

We observe that

$$p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) = (Z_{12} \times M_3) \cap (M_1 \times Z_{23}) = Z_{12} \times_{X_2} Z_{23}. $$

Therefore the set $Z_{12} \circ Z_{23}$ defined in (8) is equal to the image of the map (10). In particular, this set is a closed subset in $M_1 \times M_3$, since the map in (10) is proper.

Let $d = \dim R M_2$. We define a *convolution in Borel-Moore homology*, cf. also [FM],

$$H_i(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23}) \quad , \quad (c_{12}, c_{23}) \mapsto c_{12} \ast c_{23} \quad (11)$$

by translating the set theoretic composition into composition of cycles. Specifically put (compare with (8)):

$$c_{12} \ast c_{23} = (p_{13})_*(p_{12}^*c_{12} \cap p_{23}^*c_{23}) \in H_*(Z_{12} \circ Z_{23}),$$

where $p_{12}^*(c_{12}) := c_{12} \boxtimes [M_3]$, and $p_{23}^*(c_{23}) := [M_1] \boxtimes c_{23}$ are given by the Künneth formula, and the intersection pairing $\cap$ was defined in (8). Note that

$$((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})) \subset (Z_{12} \times M_3) \cap (M_1 \times Z_{23}),$$
so that the direct image is well defined due to the condition that the map $p_{13}$ in (10) is proper. The reader should be warned that although the ambient manifolds $M_i$ are not explicitly present in (10), the convolution map does depend on these ambient spaces in an essential way. Note that changing the orientation would change the sign of the fundamental classes $[M_i]$ in the formula, hence it would change the convolution product.

**Associativity of convolution** Given a fourth oriented manifold, $M_4$, and a closed subset $Z_{34} \subset M_3 \times M_4$, the following associativity equation holds in Borel-Moore homology.

$$(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34}),$$

where $c_{12} \in H_*(Z_{12})$, $c_{23} \in H_*(Z_{23})$, $c_{34} \in H_*(Z_{34})$. For the proof of the associativity equation see [CG].

**Remark** The same definition applies in the disconnected case as well, provided $[M_1]$, resp. $[M_3]$, is understood as the sum of the fundamental classes of connected components of $M_1$, resp. $M_3$.

**Variant: Convolution in equivariant K-theory**

A similar convolution construction works for any generalized homology theory that has pull-back morphisms for smooth maps, push forward morphisms for proper maps and an intersection pairing with supports. This is the case, e.g. for the topological K-homology theory used in [KL1] and also for the algebraic equivariant K-theory (though the latter is not a generalized homology theory).

Given a complex linear algebraic group $G$ and a complex algebraic $G$-variety $X$, let $\text{Coh}^G(X)$ denote the abelian category of $G$-equivariant coherent sheaves on $X$. Let $K^G(X)$ be the Grothendieck group of $\text{Coh}^G(X)$. Given $F \in \text{Coh}^G(X)$ let $[F]$ denote its class in $K^G(X)$. For any $X$, the $K$-group has a natural $R(G)$-module structure where $R(G) = K^G(pt)$ is the representation ring of $G$. We recall a few properties of equivariant K-theory (see [CG] for more details).

(a) For any proper map $f : X \to Y$ between two $G$-varieties $X$ and $Y$ there is a direct image $f_* : K^G(X) \to K^G(Y)$. The map $f_*$ is a group homomorphism.

(b) If $f : X \to Y$ is flat (for instance an open embedding) or is a closed embedding of a smooth $G$-variety and $Y$ is smooth, there is an inverse image homomorphism (of groups) $f^* : K^G(Y) \to K^G(X)$.

Recall the general convolution setup. Let $M_1$, $M_2$ and $M_3$ be smooth $G$-varieties. Let

$$p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$$

be the projection along the factor not named. The $G$-action on each factor induces a natural $G$-action on the Cartesian product such that the projections $p_{ij}$ are $G$-equivariant. Let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be $G$-stable closed subvarieties such that (10) holds. Define a convolution map in K-theory

$$\ast : K^G(Z_{12}) \otimes K^G(Z_{23}) \to K^G(Z_{12} \circ Z_{23})$$

as follows. Let $F_{12}$, $F_{23}$ be two equivariant coherent sheaves on $Z_{12}$ and $Z_{23}$, respectively. Set

$$[F_{12}] \ast [F_{23}] = p_{13*} \left( p_{12*}^*[F_{12}] \otimes p_{23*}^*[F_{23}] \right).$$
In this formula, the upper star stands for the pullback morphism, well-defined on smooth maps, and $\otimes$ is defined by choosing a finite locally free $G$-equivariant resolution $F^*_i$ of $F_{i2}^*$ of $p_{i2}^*$ on the ambient smooth space $M_1 \times M_2 \times M_3$, and taking the simple complex associated with the double-complex $F^*_{i2} \otimes F^*_{23}$.

**Examples**

(i) Let $M_1 = M_2 = M_3 = M$ be smooth, and

$$Z_{12}, Z_{23} \subset M_{\Delta} \hookrightarrow M \times M,$$

where $M_{\Delta} \hookrightarrow M \times M$ is the diagonal embedding. If $Z_{12}$ and $Z_{23}$ are closed then $p_{13}$ in (10) is always proper, and moreover,

$$Z_{12} \circ Z_{23} = Z_{12} \cap Z_{23} \subset M_{\Delta} \subset M \times M.$$

In this case we see that the $\ast$-convolution product in homology reduces to the intersection $\cap$-product defined in (6) above, and $\ast$-convolution in K-theory reduces to the tensor product with supports (see [CG, Corollary 5.2.25]).

(ii) Let $M_1$ be a point and $f : M_2 \to M_3$ be a proper map of connected varieties. Set $Z_{12} = pt \times M_2 = M_2$, and $Z_{23} = \text{Graph}(f)$. Then $Z_{12} \circ Z_{23} = \text{Im}f \subset pt \times M_3 = M_3$. Let $c \in H_*(M_2) = H_*(Z_{12})$. Then we have $c \ast [\text{Graph}f] = f_*(c)$.

(iii) Let $M_3 = pt$ and $Z_{23} = M_2 \times pt$. Then the convolution

$$H_i(Z_{12}) \otimes H_j(M_2) \to H_{i+j-d}(M_1), \quad d = \dim M_2. \quad (13)$$

allows one to think of $H_*(Z_{12})$ as part of $\text{Hom}(H_*(M_2), H_*(M_1))$.

**The convolution algebra** [Gi1]

Let $M$ be a smooth complex manifold, let $N$ be a (possibly singular) variety, and let $\mu : M \to N$ be a proper map. Put $M_1 = M_2 = M_3 = M$ and $Z = Z_{12} = Z_{23} = M \times_M M$ in the general convolution setup. Explicitly, we have

$$Z = \{(m_1, m_2) \in M \times M \mid \mu(m_1) = \mu(m_2)\}.$$ 

It is obvious that $Z \circ Z = Z$. Therefore we have the convolution maps, cf. (11),

$$H_*(Z) \times H_*(Z) \to H_*(Z), \quad \text{resp. } K^G(Z) \times K^G(Z) \to K^G(Z)$$

in the $G$-equivariant setup.

The following corollary is an immediate consequence of (12).

**Corollary 2.1** (i) $H_*(Z)$ has a natural structure of an associative algebra with unit. Similarly, in the $G$-equivariant setup, $K^G(Z)$ has a natural structure of an associative $\mathbb{R}(G)$-algebra with unit.

(ii) The unit in $H_*(Z)$, resp. in $K^G(Z)$, is given by the fundamental class of $M_{\Delta} \subset Z$, resp. by the structure sheaf of $M_{\Delta}$. $\Box$

Choose $x \in N$ and set $M_x = \mu^{-1}(x)$. Apply the convolution construction for $M_1 = M_2 = M$ and $M_3$ a point. Let $Z = Z_{12} = M \times_N M$ and $Z_{23} = M_x \subset M \times \{pt\}$. We see immediately that $Z \circ M_x = M_x$. 

**Corollary 2.2** \( H_s(M_x) \) has a natural structure of a left \( H_s(Z) \)-module under the convolution map. \( \square \)

**Examples**

(i) Assume \( N = M \) and \( \mu : M \to N \) is the identity map. Then \( Z = M_\Delta \) is the diagonal in \( M \times M \). Then the convolution algebra \( H_s(Z) \) is isomorphic to the cohomology algebra \( H^*(M) \), this is easily derived from (2). In particular, the convolution algebra \( H_s(Z) \) is in this case a graded commutative local \( \mathbb{C} \)-algebra. Moreover, for any \( x \in N \), \( M_x = \mu^{-1}(x) = \{x\} \), so that \( H_s(M_x) \cong \mathbb{C} \) is the (only) simple module over this local algebra.

(ii) Assume \( M \) is smooth and compact, and \( N = pt \), so that \( \mu \) is a constant map. Then \( Z = M \times M \) and \( M_x = M, \{x\} = N \). Furthermore, the convolution action \( H_s(Z) \times H_s(M_x) \to H_s(M_z) \) can be seen to give an algebra isomorphism

\[
H_s(Z) = \text{End}_\mathbb{C} H_s(M).
\]

In particular, \( H_s(Z) \) is a simple (matrix) algebra.

In the general case of an arbitrary morphism \( \mu : M \to N \) the algebra \( H_s(Z) \) is, in a sense, a combination of the special cases (i) and (ii) considered above. In general, the variety \( Z \) is the union of the family \( \{Z_x = M_x \times M_x, x \in N\} \). The algebra \( H_s(Z) \) is neither simple nor local, and is, in a sense, “glued” from the “family” of simple algebras \( \{\text{End}_\mathbb{C} H_s(M_x), x \in N\} \). However, these simple algebras are “glued together” in a rather complicated way depending on how far the map \( \mu : M \to N \) is from a locally trivial fibration.

**The dimension property**

Let \( M_1, M_2, M_3 \) be smooth varieties of real dimensions \( m_1, m_2, m_3 \), respectively. Let \( Z_{12} \subset M_1 \times M_2 \) and \( Z_{23} \subset M_2 \times M_3 \), and let

\[
p = \frac{m_1 + m_2}{2}, \quad q = \frac{m_2 + m_3}{2}, \quad r = \frac{m_1 + m_3}{2}.
\]

Then it is obvious from (11) that convolution induces a map (assuming that \( p, q \) and \( r \) are integers)

\[
H_p(Z_{12}) \times H_q(Z_{23}) \to H_r(Z_{12} \circ Z_{23}).
\]

We say that this is the property that “the middle dimension part is always preserved.”

Therefore, in our convolution-algebra setup \( Z = M \times_N M \), and the dimension property yields:

**Corollary 2.3** [Gi1] \( H(Z) \) is a subalgebra of \( H_s(Z) \).

The last result is especially concrete in view of the following

**Lemma 2.4** Let \( \{Z_w\}_{w \in W} \) be the irreducible components of \( Z \) indexed by a finite index set \( W \). If all the components have the same dimension then the fundamental classes \( [Z_w] \) form a basis for the convolution algebra \( H(Z) \).

**Proof** This follows from Proposition 1.1. \( \square \)

In a similar way, one derives from formula (13):

**Corollary 2.5** The convolution action of the subalgebra \( H(Z) \subset H_s(Z) \) on \( H_s(M_x) \) is degree preserving, i.e., for any \( i \geq 0 \) we have \( H(Z) \ast H_j(M_x) \subset H_j(M_x) \). \( \square \)
3 Constructible complexes

This section contains definitions and theorems that will allow us later to interpret the Borel-Moore homology and the convolution product in sheaf-theoretic terms.

For any topological space $X$ (subject to conditions described at the beginning of Section 1), let $Sh(X)$ be the abelian category of sheaves of $\mathbb{C}$-vector spaces on $X$. Define the category $\text{Comp}^b(Sh(X))$ as the category whose objects are finite complexes of sheaves on $X$

$$A^\bullet = (0 \to A^{-m} \to A^{-m+1} \to \ldots \to A^{n-1} \to A^n \to 0), \quad m, n \gg 0,$$

and whose morphisms are morphisms of complexes $A^\bullet \to B^\bullet$ commuting with the differentials. Given a complex of sheaves $A^\bullet$ we let

$$\mathcal{H}^i(A^\bullet) = \text{Ker}(A^i \to A^{i+1})/\text{Im}(A^{i-1} \to A^i)$$

denote the $i$-th cohomology sheaf. A morphism of complexes is called a quasi-isomorphism provided it induces isomorphisms between cohomology sheaves.

The derived category, $D^b(Sh(X))$, is by definition the category with the same objects as $\text{Comp}^b(Sh(X))$ and with morphisms which are obtained from those in $\text{Comp}^b(Sh(X))$ by formally inverting all quasi-isomorphisms; thus quasi-isomorphisms become isomorphisms in the derived category. For example, we may (and will) identify $D^b(Sh(pt))$, the derived category on $X = pt$, with the derived category of bounded complexes of vector spaces. In general, the kernels and cokernels of morphisms are not well-defined in $D^b(Sh(X))$ so that this category is no longer abelian. It has instead the structure of a triangulated category. This structure involves, for each $n \in \mathbb{Z}$, a translation functor $[n] : A \mapsto A[n]$ such that $\mathcal{H}^i(A[n]) = \mathcal{H}^{i+n}(A)$, for all $i \in \mathbb{Z}$, and a class of distinguished triangles that come from all short exact sequences of complexes. The precise definition of the derived category is a bit more involved than this oversimplified exposition leads one to believe. For more on the derived category see [KS], [Iv], and [Ha] [Ver2].

The reason for introducing derived categories is that most of the natural functors on sheaves, like direct and inverse images, are not generally exact, i.e. do not take short exact sequences into short exact sequences. The exactness is preserved, however, provided the sheaves in the short exact sequences are injective. Now, the point is that any sheaf admits an injective resolution (possibly not unique) and, more generally, any complex of sheaves is quasi-isomorphic to a complex of injective sheaves. The notion of an “isomorphism” in $D^b(Sh(X))$ is defined so as to ensure that any object of $D^b(Sh(X))$ can be represented by a complex of injective sheaves. In this way, all the above-mentioned natural functors become exact, in a sense, when considered as functors on the derived category.

From now on we assume $X$ to be a complex algebraic variety. A sheaf $\mathcal{F}$ on $X$ is said to be constructible if there is a finite algebraic stratification $X = \bigsqcup X_\alpha$, such that for each $\alpha$, the stratum $X_\alpha$ is a locally closed smooth connected algebraic subvariety of $X$, and the restriction of $\mathcal{F}$ to the stratum $X_\alpha$ is a locally-constant sheaf of finite dimensional vector spaces (such locally-constant sheaves will be referred to as local systems in the future). An object $A \in D^b(Sh(X))$ is said to be a constructible complex if all the cohomology sheaves $\mathcal{H}^i(A)$ are constructible. Let $D^b(X)$ be the full subcategory of $D^b(Sh(X))$ formed by constructible complexes (full means that the morphisms remain the same as in $D^b(Sh(X))$). The category $D^b(X)$ is called the bounded derived category of constructible complexes on $X$ in spite of the fact that it is not the derived category of the category of constructible sheaves.
Our next objective is to give a definition of the dualizing complex and the Verdier duality functor on $D^b(X)$.

Let $i : X \hookrightarrow M$ be a closed embedding of a topological space $X$ into a smooth manifold $M$ (this always exists). We define a functor

$$i^!: \text{Sh}(M) \to \text{Sh}(X),$$

by taking germs of sections supported on $X$. Specifically, given a sheaf $F$ on $M$ and an open set $U \subset M$ set

$$\Gamma[\mathcal{X}](U,F) = \{f \in \Gamma(U,F) \mid \text{supp}(f) \subset X \cap U\}.$$

The stalk of the sheaf $i^! F$ at a point $x \in X$ is defined by the formula

$$(i^! F)_x = \lim_{\to} \Gamma[\mathcal{X}](U,F),$$

where the direct limit is taken over all open neighborhoods $U \supset x$. The functor $i^!$ is left exact, and we let $Ri^! : D^b(\text{Sh}(M)) \to D^b(\text{Sh}(X))$ denote the corresponding derived functor. If $X$ and $M$ are algebraic varieties one proves that $Ri^!$ sends $D^b(M)$ to $D^b(X)$.

Let $C_X \in D^b(X)$ be the constant sheaf, regarded as a complex concentrated in degree zero. Define the “dualizing complex” of $X$, denoted $\mathbb{D}_X$, to be

$$\mathbb{D}_X = Ri^!(C_M) \otimes \mathbb{L}^{2 \dim C_M},$$

where $i : X \hookrightarrow M$ as above.

The stalks of the cohomology sheaves of the dualizing complex are given by the formula

$$H^i_x(\mathbb{D}_X) = H^{i+2\dim C_M}(U, U \setminus (U \cap X)) = H^{BM}_{-2i}(U \cap X) \quad \text{for all } x \in X,$$

where $U \subset M$ is a small contractible open neighborhood of $x$ in $M$, and the last isomorphism is due to Poincaré duality (1).

**Proposition 3.1** (i) Let $i : N \hookrightarrow M$ be a closed embedding of a smooth complex variety $N$ into a smooth complex variety $M$. Then we have

$$Ri^!(C_M) = C_N[-2d], \quad \text{where } d = \dim C_M - \dim C_N.$$

(ii) The dualizing complex $\mathbb{D}_X$ does not depend on the choice of the embedding $i : X \hookrightarrow M$. Moreover, for a smooth variety $X$ we have

$$\mathbb{D}_X = C_X[2\dim C_X].$$

**Proof** See Lemma 8.3.3 and Proposition 8.3.4 in [CG]. ☐

From now on we will never make use of the functor $i^!$ itself and will only use the corresponding derived functor. Thus, to simplify notation we write $i^!$ for $Ri^!$, starting from this moment.

To any object $F \in D^b(X)$ and any integer $i \in \mathbb{Z}$ we assign the hyper-cohomology group $H^i(F) = H^i(X,F)$. This is, by definition, the $i$-th derived functor to the global sections functor $\Gamma : \text{Sh}(X) \to \{\text{complex vector spaces}\}$. Explicitly, to compute the derived functors
above, find a representative (up to quasi-isomorphism) of \( \mathcal{F} \in D^b(X) \) by a complex of injective sheaves \( \mathcal{I}^* \in \text{Comp}^b(Sh(X)) \). Then we have by definition of derived functors, see [Bo]:

\[
H^i(\mathcal{F}) := H^i(\Gamma(\mathcal{I}^*)) = H^i(\text{Hom}_{Sh(X)}(\mathbb{C}_X, \mathcal{I}^*)).
\]

We list the following basic isomorphisms, which we will use extensively:

\[
H^i(X) = H^i(X, \mathbb{C}_X) \quad , \quad H_i(X) = H^{-i}(X, \mathcal{D}_X). \tag{16}
\]

The second isomorphism is a global counterpart of (15). This can be seen as follows. The complex \( \mathcal{D}_X \) is obtained by applying the functor \( R\Gamma \) to the constant sheaf on an ambient smooth variety \( M \). The hyper-cohomology is the derived functor of the functor of global sections. Thus, \( H^\bullet(\mathcal{D}_X) \) is equal to the hyper-cohomology of \( R\Gamma_X \), the derived functor of the functor \( \Gamma_X \) of global sections supported on \( X \). But the hyper-cohomology of \( R\Gamma_X \), applied to the constant sheaf on \( M, M \setminus X \), and the isomorphism follows by Poincaré duality.

For any complexes \( A, B \in D^b(X) \), one defines Ext-groups in the derived category as shifted Hom’s, that is, \( \text{Ext}^k_{D^b(X)}(A, B) := \text{Hom}^k_{D^b(X)}(A, B[k]) \). There is also an internal Hom-complex, denoted \( \text{Hom}(A, B) \in D^b(X) \), such that the Ext-groups above can be expressed as

\[
\text{Ext}^k_{D^b(X)}(A, B) = H^k(X, \text{Hom}(A, B)). \tag{17}
\]

We now introduce the Verdier duality functor, \( A \mapsto A^\vee \), which is a contravariant functor on the category \( D^b(X) \) defined by the formula

\[
A^\vee = \text{Hom}(A, \mathcal{D}_X). \tag{18}
\]

Note that with this definition we have \( C_X^\vee = \mathcal{D}_X \). It is easy to show that for \( \mathcal{F} \in D^b(X) \),

\[
(\mathcal{F}[n])^\vee = (\mathcal{F}^\vee)[-n] \quad \text{and} \quad (\mathcal{F}^\vee)^\vee = \mathcal{F}. \tag{19}
\]

Given an arbitrary algebraic map \( f : X_1 \to X_2 \) we have the following four functors:

\[
f_*^!, f_1 : D^b(X_1) \to D^b(X_2) \quad , \quad f^!, f^1 : D^b(X_2) \to D^b(X_1). \tag{20}
\]

The functors \( f_*^!, f^1 \) are defined as the derived functors of sheaf-theoretic direct and inverse image functors, respectively. (We remark that sometimes what we call \( f_* \) is written \( Rf_* \) in this context, but as we will never use the sheaf theoretic pushforward we will not adopt the derived functor notation.) The other pair \( (f_1, f^1) \) is defined via Verdier duality:

\[
f_1 A_1 := (f_1(A_1^\vee))^\vee, \quad f^1 A_2 := (f^*(A_2^\vee))^\vee, \tag{21}
\]

for any \( A_1 \in D^b(X_1) \) and \( A_2 \in D^b(X_2) \). There is a direct image formula for hypercohomology:

\[
H^\bullet(X_2, f_* A_1) = H^\bullet(X_1, A_1) \tag{22}
\]

and two basic inverse image isomorphisms for “sheaves” (see [CG] for proofs):

\[
f^! \mathbb{C}_{X_2} = \mathbb{C}_{X_1}, \quad f^! \mathcal{D}_{X_2} = \mathcal{D}_{X_1}. \tag{23}
\]
It is further useful to remember that for a map \( f : X \to Y \) one has
\[ f_! = f_* \text{, if } f \text{ is proper;} \tag{22} \]
\[ f^! = f^*[2d] \text{, if } f \text{ is flat with smooth fibers of complex dimension } d. \]

One should mention that, for a closed embedding \( f : X_1 \hookrightarrow X_2 \), the functor \( f_! \) coincides with the derived functor of the “sections supported on \( X_1 \)” functor, which was used earlier in the definition of a dualizing complex.

The functors (19) are related by a base change formula, see [Ver2]. It says that, given a Cartesian square,
\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\tilde{f}} & Y \\
\downarrow \tilde{g} & & \downarrow g \\
X & \xrightarrow{f} & Z
\end{array}
\]
for any object \( A \in D^b(X) \), we have a canonical isomorphism:
\[ g^! f_* A = \tilde{f}_* \tilde{g}^! A \text{ holds in } D^b(Y). \tag{23} \]

Let \( i_\Delta : X \hookrightarrow X \times X \) be the diagonal embedding. We define two (derived) tensor product functors on \( D^b(X) \) by
\[ A \otimes B = i_\Delta^*(A \boxtimes B), \quad A \otimes^! B = i_\Delta^!(A \boxtimes B). \tag{24} \]

We will be using later the following canonical isomorphism in the derived category:
\[ \mathcal{H}om(A, B) = A^\vee \otimes B, \tag{25} \]
which is a sheaf-theoretic version of the well-known isomorphism \( \text{Hom}(V, W) \simeq V^* \otimes W \) for finite dimensional vector spaces.

Let \( N \) be a variety and \( A_1, A_2, A_3 \in D^b(N) \). For any \( p, q \in \mathbb{Z} \), the composition of morphisms in the category \( D^b(N) \) gives a bilinear product
\[ \text{Hom}_{D^b(N)}(A_1, A_2[p]) \times \text{Hom}_{D^b(N)}(A_2[p], A_3[p + q]) \to \text{Hom}_{D^b(N)}(A_1, A_3[p + q]). \]

Using that \( \text{Hom}_{D^b(N)}(A_2[p], A_3[p + q]) = \text{Hom}_{D^b(N)}(A_2, A_3) = \text{Ext}^q_{D^b(N)}(A_2, A_3) \), we can rewrite the composition above as a bilinear product of Ext-groups, called the Yoneda product,
\[ \text{Ext}^p_{D^b(N)}(A_1, A_2) \otimes \text{Ext}^q_{D^b(N)}(A_2, A_3) \to \text{Ext}^{p+q}_{D^b(N)}(A_1, A_3). \tag{26} \]

## 4 Perverse sheaves and the Decomposition Theorem

We will briefly recall some definitions and list a few basic results about the category of perverse sheaves on a complex algebraic variety. For a detailed treatment the reader is referred to [BBD].
A locally constant sheaf $\mathcal{L}$ will be referred to as a local system. Let $Y \subset X$ be a smooth locally closed subvariety of complex dimension $d$, and let $\mathcal{L}$ be a local system on $Y$. The intersection cohomology complex of Deligne-Goresky-MacPherson, $IC(Y, \mathcal{L})$, is an object of $D^b(X)$ supported on $\bar{Y}$, the closure of $Y$, that satisfies the following properties:

\begin{enumerate}[(a)]
  \item $\mathcal{H}^i IC(Y, \mathcal{L}) = 0$ if $i < -d$,
  \item $\mathcal{H}^{-d} IC(Y, \mathcal{L})|_Y = \mathcal{L}$,
  \item $\dim \text{supp} \mathcal{H}^i IC(Y, \mathcal{L}) < -i$, if $i > -d$,
  \item $\dim \text{supp} \mathcal{H}^i(\mathcal{L}(Y)^\vee) < -i$, if $i > -d$.
\end{enumerate}

An explicit construction of $IC(Y, \mathcal{L})$ given in [BBD] yields the following result:

**Proposition 4.1** Let $j : Y \hookrightarrow X$ be an embedding of a smooth connected locally closed subvariety of complex dimension $d > 0$ and $\bar{Y}$ the closure of the image. Then for any local system $\mathcal{L}$ on $Y$ there exists a unique object $IC(Y, \mathcal{L}) \in D^b(X)$ such that the above properties (a)-(d) hold. Moreover, one has:

\begin{enumerate}[(i)]
  \item The cohomology sheaves $\mathcal{H}^i IC(Y, \mathcal{L})$ vanish unless $-d \leq i < 0$;
  \item $\mathcal{H}^{-d} IC(Y, \mathcal{L}) = \mathcal{H}^0(j_* \mathcal{L})$;
  \item $IC(Y, \mathcal{L}^*) = IC(Y, \mathcal{L})^\vee$, where $\mathcal{L}^*$ denotes the local system dual to $\mathcal{L}$.
\end{enumerate}

If $X$ is a smooth connected variety, $Y = X$ and $\mathcal{L} = \mathcal{C}_X$, then we have $IC(X, \mathcal{C}_X) = \mathcal{C}_X[\dim \mathcal{C}_X X]$. This motivates the following definition. Given a smooth variety $X$ with irreducible components $X_i$ define a complex $\mathcal{C}_X$ on $X$ by the equality

$$\mathcal{C}_X|_{X_i} = \mathcal{C}_{X_i}[\dim \mathcal{C}_X X_i].$$

By Proposition 3.1, the complex $\mathcal{C}_X$ is self-dual: $\mathcal{C}_X^\vee = \mathcal{C}_X$. It will be referred to as the constant perverse sheaf on $X$, for it satisfies the conditions of the following definition.

**Definition 4.2** A complex $\mathcal{F} \in D^b(X)$ is called perverse sheaf if

\begin{enumerate}[(a)]
  \item $\dim \text{supp} \mathcal{H}^i \mathcal{F} \leq -i$,
  \item $\dim \text{supp} \mathcal{H}^i(\mathcal{F}^\vee) \leq -i$, for any $i$.
\end{enumerate}

Observe that the dimension estimates involved in the definition of the intersection complex $IC(Y, \mathcal{L})$ are similar to properties (c)-(d) in the definition of a perverse sheaf, except that the strict inequalities are relaxed to non-strict ones. Hence, any intersection complex is a perverse sheaf. If $\phi$ is a local system on an unspecified locally closed subvariety of $X$ we will sometimes write $IC_\phi$ for the corresponding intersection cohomology complex, i.e. if $\phi$ is a local system on $Y$, then by definition $IC_\phi = IC(Y, \phi)$.

**Exercise** Let $X = \mathbb{C}^2$ be the plane with coordinates $(x_1, x_2)$, and $Y = \{(x_1, x_2) \in \mathbb{C}^2 | x_1 \cdot x_2 = 0\}$ the “coordinate cross”. Check whether the complex $\mathcal{C}_Y[1]$, extended by 0 to $\mathbb{C}^2 \setminus Y$, is a perverse sheaf on $\mathbb{C}^2$.

**Theorem 4.3** [BBD] (i) The full subcategory of $D^b(X)$ whose objects are perverse sheaves on $X$ is an abelian category, $\text{Perv}(X)$.

(ii) The simple objects of $\text{Perv}(X)$ are the intersection complexes $IC(Y, \mathcal{L})$ as $\mathcal{L}$ runs through the irreducible locally constant sheaves on various smooth locally closed subvarieties $Y \subset X$. □

**Corollary 4.4**

(a) There are no negative degree global Ext-groups between perverse sheaves, in particular

$$\text{Ext}_{D^b(X)}^k (IC_\phi, IC_\psi) = 0 \text{ for all } k < 0.$$
(b) For any irreducible locally constant sheaves $\phi$ and $\psi$ we have

$$\Hom_{D^b(N)}(IC_{\phi},IC_{\psi}) = \Hom_{\text{Perv}(X)}(IC_{\phi},IC_{\psi}) = \mathbb{C} \cdot \delta_{\phi,\psi}.$$

Let $X^\circ$ be a smooth Zariski open subset in a (possibly singular) algebraic variety $X$.

**Exercises**

(i) If $A \in \text{Perv}(X)$ then $A^\vee \in \text{Perv}(X)$.

(ii) If $\mathcal{L}$ is a local system on $X^\circ$ then $IC(X,\mathcal{L})$ has neither subobjects nor quotients in $\text{Perv}(X)$ supported on $X \setminus X^\circ$.

(iii) Deduce from (ii) the following

**Proposition 4.5 (Perverse Continuation Principle)** Any morphism $a : \mathcal{L}_1 \to \mathcal{L}_2$ of local systems on $X^\circ$ can be uniquely extended to a morphism $IC(a) : IC(X,\mathcal{L}_1) \to IC(X,\mathcal{L}_2)$, and the map $a \mapsto IC(a)$ gives an isomorphism

$$\Hom(\mathcal{L}_1,\mathcal{L}_2) \cong \Hom(IC(X,\mathcal{L}_1),IC(X,\mathcal{L}_2)). \quad \square$$

We will often be concerned with the homology or cohomology of the fibers $M_x = \mu^{-1}(x)$ of a proper algebraic morphism $\mu : M \to N$, where $M$ is a smooth and $N$ is an arbitrary complex algebraic variety. We first consider the simplest case where $\mu$ is a locally trivial (in the ordinary Hausdorff topology) topological fibration with connected base $N$. The (co)homology of the fibers then clearly form a local system on $N$. In the sheaf-theoretic language, one takes $\mu_* \mathbb{C}_M$, the derived direct image of the constant sheaf on $M$. Then the cohomology sheaf $H^j(\mu_* \mathbb{C}_M)$ is locally constant and its stalk at $x \in N$ equals $H^j(M_x)$. Replacing $\mathbb{C}_M$ by $\mathbb{D}_M$, the dualizing complex, one sees that the stalk at $x$ of the local system $H^{-j}(\mu_* \mathbb{D}_M)$ is isomorphic to $H_j(M_x)$.

Recall now that for any connected, locally simply connected topological space $N$, and a choice of base point $x \in N$, there is an equivalence of categories

$$\left\{ \text{local systems on } X \right\} \leftrightarrow \left\{ \text{representations of the fundamental group } \pi_1(N,x) \right\}$$

sending a local system to its fiber at $x$, which is naturally a $\pi_1(N,x)$-module via the monodromy action. In particular, given a locally trivial topological fibration $\mu : M \to N$ and a point $x \in N$, there is a natural $\pi_1(N,x)$-action on $H^\bullet(M_x)$ and on $H_\bullet(M_x)$, respectively. We will see below (as a very special, though not at all trivial, case of the Decomposition Theorem) that this action is completely reducible, that is, both $H^\bullet(M_x)$ and $H_\bullet(M_x)$ are direct sums of irreducible representations of the group $\pi_1(N,x)$. For an irreducible representation $\chi$ of $\pi_1(N,x)$, let $H_\bullet(M_x)_\chi = \text{Hom}_{\pi_1(N,x)}(\chi,H_\bullet(M_x,\mathbb{C}))$ be the $\chi$-isotypic component of the homology of the fiber with complex coefficients. (Up to now we could work with, say, rational homology. But since some irreducible representations of $\pi_1(N,x)$ may not be defined over $\mathbb{Q}$ we have to take $\mathbb{C}$ as the ground field from now on.) This way we get the direct sum decompositions into isotypic components with respect to the fundamental group

$$H^\bullet(M_x,\mathbb{C}) = \bigoplus_{\chi \in \pi_1(N,x)} \chi \otimes H^\bullet(M_x)_\chi, \quad H_\bullet(M_x,\mathbb{C}) = \bigoplus_{\chi \in \pi_1(N,x)} \chi \otimes H_\bullet(M_x)_\chi. \quad (28)$$
The first decomposition reflects the corresponding direct sum decomposition of local systems

\[ \mathcal{H}^\bullet(\mu_*C_M) = \bigoplus_{\chi \in \pi_1(N,x)} \chi \otimes \mathcal{H}^\bullet(M_x)_{\chi}, \]  

where now the LHS stands for the cohomology sheaves; \( \chi \) is viewed, by the correspondence (27), as an irreducible local system on \( N \), and the vector spaces \( \mathcal{H}^\bullet(M_x)_{\chi} \) play the role of multiplicities. Note that there is no need to write a second formula of this type, corresponding to homology (as opposed to cohomology), because on the smooth variety \( M \) one has \( D_M = \mathbb{C}M[2 \dim \mathbb{C}M] \), and the second decomposition is nothing but the one above shifted by \( [2 \dim \mathbb{C}M] \).

We recall that a morphism \( \mu : M \to N \) is called projective if it can be factored as a composition of a closed embedding \( M \hookrightarrow \mathbb{P}^n \times N \) and the projection \( \mathbb{P}^n \times N \to N \). Any proper algebraic map between quasi-projective varieties is known to be projective. In the case of a projective morphism our analysis will be based on the very deep “Decomposition Theorem”, which has no elementary proof and is deduced (see [BBD] and references therein) from the Weil conjectures by reduction to ground fields of finite characteristic.

**Decomposition Theorem 4.6 [BBD]** Let \( \mu : M \to N \) be a projective morphism and \( X \subset M \) a smooth locally closed subvariety. Then we have a finite direct sum decomposition in \( D^b(N) \)

\[ \mu_*IC(X,\mathcal{C}_X) = \bigoplus_{(i,Y,\chi)} L^Y,\chi (i) \otimes IC(Y,\chi)[i], \]

where \( Y \) runs over locally closed subvarieties of \( N \), \( \chi \) is an irreducible local system on \( Y \), \([i]\) stands for the shift in the derived category and \( L^Y,\chi (i) \) are certain finite dimensional vector spaces.

Now let \( M \) be a smooth complex algebraic variety, \( \mu : M \to N \) a projective morphism, and \( N = \bigsqcup N_\beta \) an algebraic stratification such that, for each \( \beta \), the restriction map \( \mu : \mu^{-1}(N_\beta) \to N_\beta \) is a locally trivial topological fibration (such a stratification always exists, see [Ver1]). Applying the Decomposition Theorem to \( \mu_*C_M \) we see that all the complexes on the RHS of the decomposition have locally constant cohomology sheaves along each stratum \( N_\beta \). Thus, the decomposition takes the form

\[ \mu_*C_M = \bigoplus_{k \in \mathbb{Z},(N_\beta,\chi_\beta)} L_{\phi}(k) \otimes IC_{\phi}[k], \]

where \( IC_{\phi} \) is the intersection cohomology complex associated with an irreducible local system \( \chi_\beta \) on a stratum \( N_\beta \).

**5 Sheaf-theoretic analysis of the convolution algebra**

Given a smooth complex variety \( M \) and a proper map \( \mu : M \to N \), where \( N \) is not necessarily smooth, following the setup of the end of section 2 we put \( Z = M \times_N M \). Then \( Z \circ Z = Z \) so that \( H_*(Z) \) has a natural associative algebra structure.

This construction can be “localized” with respect to the base \( N \) using sheaf-theoretic language as follows. Consider the constant perverse sheaf \( C_M \) and the complex vector space
Ext}_{db(N)}^\bullet (\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M). The latter space has a natural (non-commutative) graded \mathbb{C}\text{-algebra structure given by the Yoneda product of Ext-groups, see }[24]\text{. This Ext-algebra construction is “local” in the sense that one may replace the space }N\text{ here by any open subset }N' \subset N\text{ to obtain a similar Ext-algebra on }N'.

In the sequel we will often be dealing with linear maps between graded spaces that do not necessarily respect the gradings. It will be convenient to introduce the following.

**Notation** Given graded vector spaces \(V, W\), we write \(V \cong W\) for a linear isomorphism that does not necessarily preserve the gradings. We will also use the notation \(\cong\) to denote quasi-isomorphisms that only hold up to a shift in the derived category.

We are going to prove an algebra isomorphism \(H_\bullet(Z) \cong \text{Ext}_{db(N)}^\bullet (\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M)\). This important isomorphism will allow us to study the algebra structure of \(H_\bullet(Z)\) via the sheaf-theoretic decomposition of \(\mu_*\mathcal{C}_M\).

**Proposition 5.1** There exists a (not necessarily grading preserving) natural algebra isomorphism

\[H_\bullet(Z) \cong \text{Ext}_{db(N)}^\bullet (\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M).\]

**Proof** Since \(\mathcal{C}_M = \mathbb{C}_M[\dim M]\) we may replace \(\mathcal{C}_M\) by \(\mathbb{C}_M\) in the statement of the proposition without affecting the Ext-groups. Further, we have seen in \([16]\) that \(H_\bullet(Z) \cong H^\bullet(Z, \mathbb{D}_Z)\).

Now use the following Cartesian square:

\[
\begin{array}{ccc}
Z = M \times_N M & \xrightarrow{i} & M \times M \\
\mu \downarrow & & \mu \times \mu \\
N_\Delta & \xrightarrow{i} & N \times N
\end{array}
\]

to obtain (denoting \(\mu_*\mathcal{C}_M\) by \(\mathcal{L}\)):

\[
\begin{align*}
H^\bullet(Z, \mathbb{D}_Z) & \cong H^\bullet(Z, \mathbb{D}_M) \\
& \cong H^\bullet(N_\Delta, (\mu_*\mathcal{C}_M) \times \mathbb{D}_M) \\
& = H^\bullet(N_\Delta, \mathbb{D}_M) \\
& = H^\bullet(N_\Delta, \mathcal{L} \boxtimes \mathcal{L}) \\
& = H^\bullet(N_\Delta, \mathcal{L} \boxtimes \mathcal{L})
\end{align*}
\]

since \(M \times M\) is smooth

\[
\begin{align*}
& \quad \text{by }[21] \\
& \quad \text{by definition of }\mathcal{L} \\
& \quad \text{since }\mu\text{ is proper, and Verdier} \\
& \text{duality commutes with }\mu_*
\end{align*}
\]

\[
\begin{align*}
& \cong H^\bullet(N_\Delta, \mathcal{L} \boxtimes \mathcal{L}) \\
& \cong H^\bullet(N_\Delta, \mathcal{L} \boxtimes \mathcal{L}) \\
& = \text{Ext}_{db(N)}^\bullet (\mathcal{L}, \mathcal{L}) \quad \text{by definition of }\boxtimes
\end{align*}
\]

by \([23]\) and \([17]\).

This shows that the two spaces are isomorphic as vector spaces over \(\mathbb{C}\). The fact that this isomorphism agrees with the algebra structures is more complicated; it is proved in \([CG, \text{Theorem 8.6.7}]\). \(\square\)

Assume from now on that the morphism \(\mu : M \to N\) is projective and that \(N = \bigsqcup N_\beta\) is an algebraic stratification such that, for each \(\beta\), the restriction map \(\mu^{-1}(N_\beta) \to N_\beta\) is a locally trivial topological fibration. We will study the structure of the convolution algebra \(H_\bullet(Z)\) by combining Proposition 5.1 with the known structure of the complex \(\mu_*\mathcal{C}_M\), provided
by the Decomposition Theorem, see (30). In this way we will be able to find a complete

collection of simple $H_\bullet(Z)$-modules.

By Proposition 5.1 and (30) we have

$$H_\bullet(Z) = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^k_{\mathcal{D}_b(N)}(\mu_\ast \mathcal{C}_M, \mu_\ast \mathcal{C}_M)$$

$$= \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \text{Hom}_\mathcal{C}(L_\phi(i), L_\psi(j)) \otimes \text{Ext}^k_{\mathcal{D}_b(N)}(IC_\phi[i], IC_\psi[j])$$

$$= \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \text{Hom}_\mathcal{C}(L_\phi(i), L_\psi(j)) \otimes \text{Ext}^{k+j-i}_{\mathcal{D}_b(N)}(IC_\phi, IC_\psi).$$

Since the summation runs over all $i, j, k \in \mathbb{Z}$, the expression in the last line will not be affected

if $k + j - i$ is replaced by $k$. Thus, we obtain

$$H_\bullet(Z) = \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \text{Hom}_\mathcal{C}(L_\phi(i), L_\psi(j)) \otimes \text{Ext}^k_{\mathcal{D}_b(N)}(IC_\phi, IC_\psi).$$

Introduce the notation $L_\phi = \bigoplus_{i \in \mathbb{Z}} L_\phi(i)$. Using the vanishing of $\text{Ext}^k_{\mathcal{D}_b(N)}(IC_\phi, IC_\psi) = 0$ for

all $k < 0$ by Corollary 4.4, one finds

$$H_\bullet(Z) = \bigoplus_{k \geq 0, \phi, \psi} \text{Hom}_\mathcal{C}(L_\phi, L_\psi) \otimes \text{Ext}^k_{\mathcal{D}_b(N)}(IC_\phi, IC_\psi).$$

(31)

Observe that the RHS of this formula has an algebra structure, essentially via the Yoneda

product. Moreover, it is clear that decomposition with respect to $k$, the degree of the Ext-

group, puts a grading on this algebra, which is compatible with the product structure.

Recall further that $\text{Hom}(IC_\phi, IC_\psi) = 0$ unless $\phi = \psi$. This yields

$$H_\bullet(Z) = \bigoplus_\phi \text{End}_\mathcal{C}L_\phi \oplus \bigoplus_{\phi, \psi, k > 0} \text{Hom}_\mathcal{C}(L_\phi, L_\psi) \otimes \text{Ext}^k_{\mathcal{D}_b(N)}(IC_\phi, IC_\psi).$$

(32)

The first sum in this expression is a direct sum of the matrix algebras $\text{End} L_\phi$, hence is a

semisimple subalgebra (as any direct sum of matrix algebras). The second sum is concentra-

ted in degrees $k > 0$, hence is a nilpotent ideal $H_\bullet(Z)_+ \subset H_\bullet(Z)$. This nilpotent ideal is

the radical of our algebra, since

$$H_\bullet(Z)/H_\bullet(Z)_+ \simeq \bigoplus_\phi \text{End} (L_\phi)$$

is a semisimple algebra. Now, for each $\psi$, the composition

$$H_\bullet(Z) \to H_\bullet(Z)/H_\bullet(Z)_+ = \bigoplus_\phi \text{End} L_\phi \to \text{End} L_\psi$$

(33)

(where $\pi$ is projection to the $\psi$-summand) yields an irreducible representation of the algebra

$H_\bullet(Z)$ on the vector space $L_\psi$. Since $H_\bullet(Z)_+$ is the radical of our algebra, and all simple

modules of the semisimple algebra $\bigoplus_\phi \text{End} L_\phi$ are of the form $L_\psi$, one obtains in this way

the following result.
Theorem 5.2 The non-zero members of the collection \( \{ L_\phi \} \) (arising from (30)) form a complete set of the isomorphism classes of simple \( H_\bullet(Z) \)-modules.

Special case: Semi-small maps

In this subsection we fix a smooth complex algebraic variety \( M \) with connected components \( M_1, \ldots, M_r \) and assume that \( \mu : M \to N \) is projective. Given \( x \in N_\alpha \), we put \( M_x = \mu^{-1}(x) \) and \( M_{x,k} := M_x \cap M_k, \ k = 1, \ldots, r \).

Notation We introduce the following integers:

\[
m_k = \dim_\mathbb{C} M_k, \quad n_\alpha = \dim_\mathbb{C} N_\alpha, \quad d_{\alpha,k} = \dim_\mathbb{C} M_{x,k}, \quad \text{for } x \in N_\alpha
\]

If \( M \) is connected we simply write \( m = \dim_\mathbb{C} M \), and simplify \( d_{\alpha,k} \) to \( d_\alpha \). Given a stratum of \( N \) and a local system \( \chi \) on this stratum, we will write \( \phi = (N_\phi, \chi_\phi) \) for such a pair, and in this case we use the notation \( n_\phi \) and \( d_{\phi,k} \) for the corresponding dimensions.

The following notion is introduced in [GM], cf. also [BM].

Definition 5.3 The morphism \( \mu \) is called semi-small with respect to the stratification \( N = \bigsqcup N_\alpha \) if, for any component \( M_k \) we have \( n_\alpha + 2d_{\alpha,k} \leq m_k \) for all \( \alpha \) such that \( N_\alpha \subset \mu(M_k) \).

If we always have \( n_\alpha + 2d_{\alpha,k} = m_k \) we say that \( \mu \) is strictly semi-small; and if we have \( n_\alpha + 2d_{\alpha,k} < m_k \) for all \( N_\alpha \) that are not dense in an irreducible component of \( N \) we say that \( \mu \) is small.

The results below copy, to a large extent, the results we have already obtained before, but in the semi-small case all formulas become “cleaner”, since most shifts in the derived category disappear. The following theorem may be regarded as an especially nice version of the Decomposition Theorem and is one of the main reasons to single out the semi-small maps.

Denote \( M_i \times N M_j \) by \( Z_{ij} \). Set \( H(Z) = \bigoplus_{ij} H_{m_i + m_j}(Z_{ij}) \), where \( m_i = \dim_\mathbb{C} M_i \). Thus, \( H(Z) \) is the “middle-dimension” subalgebra of \( H_\bullet(Z) \).

Given \( x \in N_\alpha \), let \( M_x = \mu^{-1}(x) \). Put \( H(M_x) = \bigoplus_k H_{2d_{\alpha,k}}(M_{x,k}) \), the “top” homology of \( M_x \).

Theorem 5.4 (i) Let \( C_M \) be the constant perverse sheaf on \( M \). If \( \mu \) is semi-small then \( \mu_* C_M \) is perverse and we have a decomposition without shifts:

\[
\mu_* C_M = \bigoplus_{\phi = (N_\phi, \chi_\phi)} L_\phi \otimes IC_\phi. \tag{34}
\]

Furthermore, \( H(Z) \) is a subalgebra of \( H_\bullet(Z) \) and one has algebra isomorphisms:

\[
H(Z) = \text{Hom}(\mu_* C_M, \mu_* C_M) = \bigoplus_\phi \text{End}_\mathbb{C}(L_\phi).
\]

(ii) For any stratum \( N_\alpha \), the family of spaces \( \{ H(M_x), x \in N_\alpha \} \) forms a local system on \( N_\alpha \). If \( L_\phi \) is the multiplicity space in (30) such that \( N_\phi \supset x \) and \( \chi_\phi \) is the representation of \( \pi_1(N_\phi, x) \) associated with \( \phi \) then

\[
L_\phi = H(M_x)_\phi = \text{Hom}_{\pi_1(N_\phi,x)}(H(M_x), \chi_\phi)
\]

In other words, each multiplicity space \( L_\phi \) in (30) can be obtained by taking \( \chi_\phi \)-isotypic component of the local system on \( N_\phi \) formed by top degree Borel-Moore homology of the fibers.
(iii) If \( \mu \) is small and \( N \) is irreducible then \( \mu_*\mathcal{C}_M = IC(\mu_*\mathcal{C}|_{N_0}) \), where \( N_0 \) is the dense stratum (that is, the decomposition in (i) contains only summands coming from irreducible local systems on the open stratum \( N_0 \)).

**Proof**

(i) By the Decomposition Theorem it suffices to show that \( \mu_*\mathcal{C}_M \) is a perverse sheaf, and for this we may assume without loss of generality that \( M \) is connected of complex dimension \( m \) and \( N = \mu(M) \).

First check condition (a) in Definition 4.2. Fix any \( x \in N \) and write \( i_x : \{x\} \hookrightarrow N \) for the embedding. Then one has

\[
H^j i^*_x(\mu_*\mathcal{C}_M) = H^j i^*_x(\mu_*\mathcal{C}_M[m]) = H^{j+m}(\mathcal{M}_x)
\]

Hence if \( x \in N_\alpha \) then we have a chain of implications

\[
H^j i^*_x(\mu_*\mathcal{C}_M) \neq 0 \Rightarrow j + m \leq 2\dim C M_x \leq m - \dim C N_\alpha \text{ (by definition of semi-smallness)} \Rightarrow \dim C N_\alpha \leq -j
\]

and condition (a) of Definition 4.2 follows. Condition (b) follows automatically form (a) due to self-duality of \( \mu_*\mathcal{C}_M \).

To prove the second part, notice first that \( H(Z) \) is a subalgebra due to the dimension property of Section 2. We can repeat the proof of Proposition 5.1 (this time minding the superscripts) to get \( H_k(Z_{ij}) = \text{Ext}^{m_i+m_j-k}(\mu_*\mathcal{C}_M,\mu_*\mathcal{C}_M) \), and this proves \( H(Z) = \text{Hom}(\mu_*\mathcal{C}_M,\mu_*\mathcal{C}_M) \) which implies our assertion in view of Corollary 4.4 (b).

(ii) We can assume that \( M \) is connected of complex dimension \( m \) (since all the objects involved are direct sums over connected components of \( M \)). We use the obvious Cartesian square:

\[
\begin{array}{ccc}
M_x & \xrightarrow{i} & M \\
\mu \downarrow & & \downarrow \mu \\
\{x\} & \xrightarrow{i} & N
\end{array}
\]

Then

\[
H_k(M_x) = H^{-k}(M_x, \mathbb{D}M_x) = H^{-k}\left(\{x\}, \mu_*\mathbb{D}M_x\right) = H^{-k}\left(\{x\}, \mu_*\mathcal{C}_M[m]\right)
\]

by \([16] \) since \( M \) is smooth, hence \( \mathbb{D}M = \mathcal{C}_M[m] \) by base change.

Using this computation, definition of a semi-small map and \([34] \), we find:

\[
H(M_x) = H_{2d_\alpha}(M_x) = H^{m-2d_\alpha}\left(\{x\}, i^*_x\mu_*\mathcal{C}_M\right) = H^{n_\alpha}\left(\{x\}, i^*_x\mu_*\mathcal{C}_M\right)
\]

by \([6] \)

\[
= H^{n_\alpha}\left(\{x\}, i^*_x\mathcal{L}_\phi \otimes IC_\phi\right) = \bigoplus_{\phi} \mathcal{L}_\phi \otimes H^{n_\alpha}\left(\{x\}, i^*_xIC_\phi\right)
\]

If the closure of \( N_\phi \) does not contain \( N_\alpha \) then \( x \) is not contained in the support of \( IC_\phi \) and \( i^*_xIC_\phi = 0 \). If \( N_\alpha \subset N_\phi \) then we use \( i^*_xIC_\phi = (i^*_x(\mu_*\mathcal{C}))^\vee \). By Proposition 4.1 (i) we
obtain
\[
H^{n_{\alpha}}(\{x\}, i_x^* IC_{\phi}) = H^{n_{\alpha}}(\{x\}, (i_x^* IC(L^*_\phi))^\vee) = \left(H^{-n_{\alpha}}(\{x\}, i_x^* IC(L^*_\phi))\right)^* = \left(H^{-n_{\alpha}} IC(L^*_\phi)\right)^*.
\]

If \( N_\alpha \neq N_\phi \) then \( n_\alpha = \dim \mathbb{C} N_\alpha < \dim \mathbb{C} N_\phi \). Denote \( \dim \mathbb{C} N_\phi \) by \( d \) and apply property (c) of an IC-complex (cf. the beginning of Section 4) to \( i = -n_\alpha \). We find
\[
\dim(\text{supp } H^{-n_{\alpha}} IC(L^*_\phi)) < n_\alpha
\]
and since \( IC(L^*_\phi) \) is locally constant along \( N_\alpha \) this means that \( H_x^{-n_{\alpha}} IC(L^*_\phi) = 0 \) for all \( x \in N_\alpha \).

If \( N_\alpha = N_\phi \) then by property (b) of an IC-complex we have \( H_x^{-n_{\alpha}} IC(L^*_\phi) = (L^*_\phi)_x = (L^*_\phi)_x \), so \( H(M_x) = \bigoplus_{\phi} L^*_\phi \otimes (L^*_\phi)_x \), where the sum runs over all \( \phi \) satisfying \( N_\alpha = N_\phi \). Now claim (ii) follows from Schur’s Lemma.

(iii) Suppose the decomposition of (i) has a component \( L^*_\phi \otimes IC_{\phi} \) coming from a local system \( L^*_\phi \) on a stratum \( N_\alpha \) of dimension \( n_\alpha \) and \( \overline{N_\alpha} \neq N \). Then by property (b) of the intersection homology complex \( IC(L^*_\phi) \) (cf. the beginning of Section 4), for any point \( x \in N_\alpha \), we get
\[
H^{-n_{\alpha}} i_x^*(L^*_\phi \otimes IC_{\phi}) = (L^*_\phi \otimes L^*_\phi)_x \neq 0.
\]
As in the proof of (i) we have:
\[
H^{-n_{\alpha}} i_x^*(\mu_{\alpha} C_M) \neq 0 \Rightarrow -n_\alpha + m_k \leq 2d_{\alpha,k},
\]
while the definition of smallness requires “>” instead of the inequality “≤” that we just obtained. Contradiction. □

6 Representations of Weyl groups

Fix a complex semisimple connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \), often viewed as a \( G \)-module via the adjoint action. We introduce a few standard objects associated with a semisimple group (see [Bo3] for more details about the structure of algebraic groups). Let \( B \) be a Borel subgroup, i.e. a maximal solvable subgroup of \( G \), see [Bo2], and let \( T \) be a maximal torus contained in \( B \). Let \( U \) be the unipotent radical of \( B \) so that \( B = T \cdot U \); in particular \( B \) is connected. Let \( \mathfrak{b}, \mathfrak{t}, \mathfrak{n} \), denote the Lie algebra of \( B \), resp. \( T, U \), so that \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \). We also consider the normalizer \( N_G(T) \) of \( T \). The quotient \( W := N_G(T)/T \) is called the Weyl group of \( G \). It is known [Bo2], [Se] that \( W \) is a finite group generated by reflections, if viewed as a subgroup of \( GL(\mathfrak{h}) \). The main result of this section is a geometric description of the group algebra of \( W \) as well as a classification of all its irreducible representations.

Let \( \mathcal{B} \) be the set of all Borel subalgebras in \( \mathfrak{g} \). By definition, \( \mathcal{B} \) is the closed subvariety of the Grassmannian of \((\dim \mathfrak{b})\)-dimensional subspaces in \( \mathfrak{g} \) formed by all solvable Lie subalgebras. Hence, \( \mathcal{B} \) is a projective variety called flag variety. Recall that all Borel subalgebras are conjugate under the adjoint action of \( G \) and that \( G_{\mathfrak{b}} \), the isotropy subgroup of \( \mathfrak{b} \) in \( G \), is equal to \( B \) by (cf. [Bo2]). Thus, the assignment \( g \mapsto \text{Ad } g(\mathfrak{b}) \) gives a bijection
\[
G/B \simeq \mathcal{B}.
\]
Furthermore, the LHS has the natural structure of a smooth algebraic $G$-variety (cf. [Bo2]), and the above bijection becomes a $G$-equivariant isomorphism of algebraic varieties.

Recall that an element $x \in \mathfrak{g}$ is called \textit{nilpotent} if the operator $\text{ad} \, x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. This agrees with the usual notion of nilpotency when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Let $\mathcal{N}$ denote the set of all nilpotent elements of $\mathfrak{g}$. Clearly $\mathcal{N}$ is a closed $\text{Ad} \, G$-stable subvariety of $\mathfrak{g}$. The set $\mathcal{N}$ is also $\mathbb{C}^*$-stable with respect to dilations, i.e. $\mathcal{N}$ is a cone-variety.

Set $\tilde{\mathcal{N}} := \{(x, b) \in \mathcal{N} \times B \mid x \in b\}$. The fiber over a Borel subalgebra $b \in B$ of the second projection $\pi : \tilde{\mathcal{N}} \to B$ is formed by the nilpotent elements of $b$. It is clear that the operator $\text{ad} \, x \, , x \in b$, is nilpotent if and only if $x$ has no Cartan component in a decomposition $b = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is the nil-radical of $\mathfrak{b}$. Thus, an element of $\mathfrak{b}$ is nilpotent if and only if it belongs to $\mathfrak{n}$. It follows that the projection $\pi$ makes $\tilde{\mathcal{N}}$ a vector bundle over $B$ with fiber $\mathfrak{n}$. Furthermore, since any nilpotent element of $\mathfrak{g}$ is $G$-conjugate into $\mathfrak{n}$, we get a $G$-equivariant vector bundle isomorphism

$$
\tilde{\mathcal{N}} \simeq G \times_B \mathfrak{n} \xrightarrow{\pi} G/B = B,
$$

where $B$, the Borel subgroup of $G$ corresponding to a fixed Borel subalgebra $b$, acts on the second factor $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ by the adjoint action. In particular, $\mathcal{N}$ is a smooth variety, while $\tilde{\mathcal{N}}$ itself is always singular at the origin.

Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via the $G$-equivariant isomorphism given by an invariant bilinear form on $\mathfrak{g}$, e.g. the Killing form $(x, y) = \text{Tr}(\text{ad} \, x \cdot \text{ad} \, y)$, cf. [Hum], [Se].

\textbf{Lemma 6.1} (cf. e.g. [BoB]) \textit{There is a natural $G$-equivariant vector bundle isomorphism}

$$
\tilde{\mathcal{N}} \simeq T^* B \quad (= \text{cotangent bundle on } B).
$$

\textbf{Proof} \ Let $e = 1 \cdot B/B \in G/B$ be the base point. We have $T_e(G/B) = \mathfrak{g}/\mathfrak{b}$ and $T^*_e(G/B) = (\mathfrak{g}/\mathfrak{b})^* = \mathfrak{b}^\perp \subset \mathfrak{g}^*$. It follows that, for any $g \in G$,

$$
T^*_g e(G/B) = \text{Ad} \, g \left( \mathfrak{b}^\perp \right).
$$

This shows that the vector bundles $T^*(G/B)$ and $G \times_B \mathfrak{b}^\perp$ have the same fibers at each point of $G/B$, hence are equal as sets. To prove that they are isomorphic as manifolds, one can refine the argument as follows.

Consider the trivial bundle $g_{G/B} = \mathfrak{g} \times G/B$ on $G/B$ with fiber $\mathfrak{g}$. The infinitesimal $\mathfrak{g}$-action on $G/B$ gives rise to a vector bundle morphism $g_{G/B} \to T(G/B)$. It is clear that the kernel of this morphism is the subbundle $\mathfrak{b} \subset g_{G/B}$ whose fiber at a point $x \in G/B$ is the isotropy Lie algebra $b_x \subset \mathfrak{g}$ at $x$. This gives an isomorphism $T(G/B) \simeq g_{G/B}/\mathfrak{b}$. Hence, $T(G/B) \simeq G \times_B (\mathfrak{g}/\mathfrak{b})$. Taking the dual on each side we get $T^*(G/B) \simeq G \times_B (\mathfrak{g}/\mathfrak{b})^*$.

Note that $(\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{b}^\perp$ = annihilator in $\mathfrak{g}^*$ of the vector subspace $\mathfrak{b} \subset \mathfrak{g}$. Under the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$, the annihilator $\mathfrak{b}^\perp \subset \mathfrak{g}^*$ gets identified with the annihilator of $\mathfrak{b}$ in $\mathfrak{g}$ with respect to the invariant form. The latter is equal to $\mathfrak{n}$, the nil-radical of $\mathfrak{b}$. Thus, $T^* B = G \times_B \mathfrak{n} = \tilde{\mathcal{N}}$. \hfill $\square$

Define the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ to be the restriction to $\tilde{\mathcal{N}} \subset \mathcal{N} \times B$ of the first projection $\mathcal{N} \times B \to \mathcal{N}$.

\textbf{Theorem 6.2} \textit{The map } $\mu : T^* B = \tilde{\mathcal{N}} \to \mathcal{N}$ \textit{is proper and surjective. Moreover, }$\mathcal{N}$ \textit{is irreducible and }$\mu$ \textit{is a resolution of singularities for }$\mathcal{N}$. 

Proof First of all, $\mu$ is surjective since any nilpotent element of $\mathfrak{g}$ is known to be contained in the nil-radical of a Borel subalgebra (cf. [Hum]). The surjectivity of $\mu$ implies

(i) irreducibility of $\mathcal{N}$ (since $T^*\mathcal{B}$ is irreducible),
(ii) the dimension bound: $\dim \mathcal{N} \leq \dim T^*\mathcal{B} = 2\dim \mathcal{B} = 2\dim \mathfrak{n}$.

To prove $\dim \mathcal{N} \geq 2\dim \mathfrak{n}$, recall first that $\mathcal{N}$ can be also defined as a common zero set of all $G$-invariant polynomials on $\mathfrak{g}$ without constant term (cf. [CG, Proposition 3.2.5]). By the Chevalley Restriction Theorem we have $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{b}]^W$ (cf. for example [CG, Theorem 3.1.38]), hence one has exactly $\text{rk} \mathfrak{g}$ algebraically independent $G$-invariant polynomials and therefore $\dim \mathcal{N} \geq \dim \mathfrak{g} - \text{rk} \mathfrak{g} = 2\dim \mathfrak{n}$. Thus $T^*\mathcal{B}$ and $\mathcal{N}$ have the same dimension.

One can prove that the set of all regular nilpotent elements (that is, elements for which the dimension of their centralizer is equal to $\text{rk} \mathfrak{g}$) is a single conjugacy class which is Zariski-open and dense in $\mathcal{N}$ (cf. [CG, Proposition 3.2.10]). By $G$-equivariance and the dimension equality the preimage of any regular nilpotent element is a finite set. To prove that $\mu$ is generically one to one, by $G$-equivariance it is enough to show that some particular regular nilpotent element has just one point in its preimage. If $e_1, \ldots, e_l \in \mathfrak{n}$ are the root vectors corresponding to positive simple roots with respect to $\mathfrak{b}$, then one shows ([CG, 3.2]) that $n = e_1 + \ldots + e_l$ is a regular nilpotent which is contained in a unique Borel subalgebra, the one containing $e_1, \ldots, e_l$. This implies $\#(\mu^{-1}(n)) = 1$ and therefore a generic point of $\mathcal{N}$ has exactly one preimage in $T^*\mathcal{B}$, hence $\mu$ is a resolution of singularities. □

Definition 6.3 The map $\mu : \tilde{\mathcal{N}} = T^*\mathcal{B} \to \mathcal{N}$ is called the Springer resolution.

Remark The map $\mu$ is also the moment map with respect to the canonical Hamiltonian $G$-action on $T^*\mathcal{B}$.

Theorem 6.4 The Springer resolution is strictly semi-small (cf. Definition 5.3).

Proof See [CG, 3.3]. □

Now we can apply the machinery of Section 5 to the Springer resolution. Set $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$, and call it the Steinberg variety. Consider the convolution algebra $H(Z)$ associated with $Z$. If $x \in \mathcal{N}$ then it follows from the definitions that the set $M_x = \mu^{-1}(x)$ is formed by pairs $\{(x, b) | b \supseteq x\}$ where $b$ runs over the subset $\mathcal{B}_x \subset \mathcal{B}$ of $x$-invariant points of $\mathcal{B}$ (any element $x \in \mathfrak{g}$ induces a vector field $\xi_x$ on $\mathcal{B}$ and $\mathcal{B}_x$ is the subvariety of zeros of this vector field).

Denote by $G(x)$ the centralizer of $x \in \mathcal{N}$ in $G$ and by $G(x)^0$ its identity component. Since $\mu$ is $G$-equivariant, $G(x)$ acts on $\mathcal{B}_x$ and this induces a $G(x)$-action of $H_*(\mathcal{B}_x)$. Moreover, since the automorphism of $\mathcal{B}_x$ induced by any $g \in G(x)^0$ is homotopic to the identity, we conclude that the group $G(x)^0$ acts trivially on $H_*(\mathcal{B}_x)$. Hence we obtain an action of the finite group $A(x) := G(x)/G(X)^0$ on the homology groups $H_*(\mathcal{B}_x)$. Write $A(x)^\vee$ for the set of isomorphism classes of irreducible representations of $A(x)$ occurring in the top homology $H_{d(x)}(\mathcal{B}_x)$, $d(x) = \dim \mathcal{B}_x$, with non-zero multiplicity.

Our main result concerning representations of Weyl group is the following.

Theorem (Geometric Construction of $W$)

(i) There is an algebra isomorphism $H(Z) \simeq \mathbb{C}[W]$.

(ii) The collection $\{H(\mathcal{B}_x)_{\phi} \}$, where $(x, \phi)$ runs over $G$-conjugacy classes of pairs $x \in \mathcal{N}$, $\phi \in A(x)^\vee$, is a complete set of irreducible representations of $W$.

The proof of (i) will be given in Section 9. Then (ii) follows from part (i) and Theorem 5.4. Claim (ii) of the theorem is known as the “Springer construction of Weyl group representations”.

Geometric construction of $W$ and Chern-Mather classes

There is an interesting connection between the geometric construction of the Weyl group $W$ given above and a general construction of Chern classes for singular varieties, see [Sab], outlined below.

Let $X$ be a smooth complex variety of dimension $n$. A $C^i$-stable subvariety of $T^*X$ will be referred to as a cone-subvariety. Let $L_i(T^*X)$ be the group of algebraic cycles generated by isotropic cone-subvarieties in $T^*X$ of dimension $\leq i$. (Recall that $T^*X$ has a canonical symplectic 2-form $\omega$, and a subvariety $\Lambda \subset T^*X$ is called isotropic if the pull-back of $\omega$ to $\Lambda$ vanishes. An isotropic subvariety of pure dimension $n = \frac{1}{2} \dim T^*X$ is called “Lagrangian”.) Define the group of Lagrangian cone-cycles as $L(X) := L_n(T^*X)/L_{n-1}(T^*X)$.

**Example** If $Y \subset X$ is a smooth subvariety, then its conormal bundle, $T^*_Y X$, is a Lagrangian cone-subvariety in $T^*X$.

In general, given a closed (possibly singular) subvariety $Y \subset X$, write $Y^{\text{reg}}$ for the smooth locus of $Y$. Then $Y^{\text{reg}}$ is dense in $Y$, and $T^*_Y X$ is a locally closed Lagrangian cone-subvariety in $T^*X$. Let $\Lambda_Y := \overline{T^*_Y X}$ be its closure. Then $\Lambda_Y$ is a Lagrangian cone-cycle, hence an element of $L(X)$.

This example is in effect typical since one has:

**Lemma 6.5 (see [CG, Lemma 1.3.27])** Any irreducible closed Lagrangian cone-subvariety in $T^*X$ is of the form $\Lambda_Y$ for an appropriate locally closed smooth subvariety $Y \subset X$. 

**Corollary 6.6** The group $L(X)$ is spanned by classes of the form $\Lambda_Y$, $Y \subset X$. 

Let $Gr_k(TX)$ be the Grassmann bundle on $X$ formed by all $k$-dimensional subspaces in the tangent bundle $TX$. Given an irreducible $k$-dimensional (possibly singular) closed subvariety $Y \subset X$, the tangent spaces to $Y$ at the regular points of $Y$ give rise to a section $\tau : Y^{\text{reg}} \rightarrow Gr_k(TX)$. The closure $\overline{Y} := \overline{\tau(Y^{\text{reg}})} \subset Gr_k(TX)$ of the image of $\tau$ is called the Nash resolution of $Y$. The natural projection $Gr_k(TX) \rightarrow X$ restricts to a proper map $p : \overline{Y} \rightarrow Y$, which is an isomorphism over $Y^{\text{reg}}$. The variety $\overline{Y}$ is not necessarily smooth, but it carries a natural rank $k$ vector bundle $\overline{T_Y}$, the restriction of the tautological rank $k$ vector bundle on $Gr_k(TX)$. The bundle $\overline{T_Y}$ plays the role of the “tangent bundle of the singular variety $Y$” because we have $\overline{T_Y}|_{p^{-1}(Y^{\text{reg}})} = p^*(TY^{\text{reg}})$.

One defines the Chern-Mather class $c_M(Y) \in H_*(Y)$ of the singular variety $Y$ as follows. Write $c(\overline{T_Y}) \in H^*(\overline{Y})$ for the total Chern class of the vector bundle $\overline{T_Y}$. Let $c(\overline{T_Y}) \cap [\overline{Y}] \in H_*(\overline{Y})$ be the corresponding class in Borel-Moore homology. We set

$$c_M(Y) = p_*(c(\overline{T_Y}) \cap [\overline{Y}]).$$

The class in $H_*(Y)$ thus defined is independent of the choice of an ambient smooth variety $X$, see [Mac].

We use Chern-Mather classes and Lemma 6.5 to define a homomorphism $c_M : L(X) \rightarrow H_*(X)$ by the formula:

$$c_M : \Lambda_Y \mapsto i_* (c_M(Y)), \text{ where } i : Y \hookrightarrow X. \quad (35)$$

A totally different, but a posteriori equivalent construction of homomorphism [35], based on K-theory, is given in [Gi4].
One can extend the above construction to the bivariant framework [FM]. Thus, given two smooth varieties \(X_i, i = 1, 2\), one defines a homomorphism
\[
c_{biv} : L(X_1 \times X_2) \rightarrow H_*(X_1 \times X_2)
\]
by assigning to \(\Lambda_Y, Y \subset X_1 \times X_2\), the relative Chern-Mather class of the fibers of the projection \(Y \rightarrow X_2\). The main reason for introducing bivariant Chern-Mather classes is that they behave nicely with respect to the convolution. In the special case of a “push-forward”, see Example (ii) above formula (13); this has been shown by MacPherson [Mac]. In the general case, given smooth varieties \(X_i, i = 1, 2, 3\), one defines (under appropriate “properness” assumptions like in [Gi2]) a convolution map \(L(X_1 \times X_2) \times L(X_2 \times X_3) \rightarrow L(X_1 \times X_3)\). In [Gi2] we proved the following

**Theorem 6.7** The map \(c_{biv}\) commutes with convolution, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
L(X_1 \times X_2) \otimes L(X_2 \times X_3) & \xrightarrow{\text{convolution}} & L(X_1 \times X_3) \\
\downarrow_{c_{biv} \otimes c_{biv}} & & \downarrow_{c_{biv}} \\
H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow{\text{convolution}} & H_*(X_1 \times X_3)
\end{array}
\]

Now, let \(X_1 = X_2 = X_3 = B\) be the flag manifold for a semisimple group \(G\). Since the Steinberg variety \(Z \subset T^*(B \times B)\) is a Lagrangian cone-subvariety, see [CG, Corollary 3.3.4], we may view \(H(Z)\), the top homology of \(Z\), as a subgroup in \(L(B \times B)\). Recall that the convolution product makes both \(H(Z)\) and \(H_*(B \times B)\) an associative algebra. Thus Theorem 6.7 yields:

**Corollary 6.8** The map \(c_{biv} : H(Z) \rightarrow H_*(B \times B)\) is an algebra homomorphism.

Further, the theorem on the geometric construction of Weyl groups gives an algebra isomorphism \(\mathbb{C}[W] \simeq H(Z)\). Thus, the Chern-Mather homomorphism of Corollary 6.8 may be thought of as a homomorphism \(c_{biv} : \mathbb{C}[W] \rightarrow H_*(B \times B)\). We will describe the latter map quite explicitly as follows.

Let \(G\) act diagonally on \(B \times B\). Choose a basepoint \(e \in B\) fixed by the maximal torus \(T\). Assign to \(w \in W = N_G(T)/T\) the \(G\)-diagonal orbit through the point \((\bar{w} \cdot e, e)\), where \(\bar{w}\) is a representative of \(w\) in \(N_G(T)\). This assignment gives a canonical bijection between \(W\) and the set of \(G\)-diagonal orbits in \(B \times B\). We write \(O_w\) for the orbit corresponding to \(w \in W\), and let \(\overline{O_w}\) denote its closure, and \([O_w] \in H_*(B \times B)\) the fundamental class of \(\overline{O_w}\). In particular, for \(w = e = (\text{unit of } W)\), we have \(O_e = \Delta\), the diagonal in \(B \times B\). Recall further, that the Weyl group \(W\) acts naturally on \(H_*(B)\). Hence there is a \(W \times W\)-action on \(H_*(B \times B)\), and for any \(w \in W\) we may form the class \((e \boxtimes w)(\Delta)\).

**Proposition 6.9** For any simple reflection \(s \in W\) we have
\[
c_{biv}(s) = (e \boxtimes s)(\Delta) + [O_s]. \qed
\]

Recall (cf. [Se]) that the Weyl group is generated by simple reflections \(s_1, \ldots, s_l\), where \(l = rk g\), and that to each element \(w \in W\) we can associate its length \(l(w)\), equal to the number of factors in any minimal decomposition \(w = s_{i_1} \cdots s_{i_{l(w)}}\) into a product of simple reflections.
Put \( n = \dim_{\mathbb{C}} \mathcal{B} \). Since \( c_{\text{biv}} \) is an algebra map and the simple reflection generate \( W \), we deduce from the proposition:

**Corollary 6.10** For any \( w \in W \) we have

\[
c_{\text{biv}}(w) = (e \boxtimes w)(\Delta) + \sum_{i=1}^{l(w)} c_{i_{(w)}}^{\text{biv}}(w), \quad c_{i_{(w)}}^{\text{biv}}(w) \in H_{2(n+i)}(\mathcal{B} \times \mathcal{B}).
\]

Furthermore \( c_{i_{(w)}}^{\text{biv}}(w) = [O_w] \). \( \square \)

### 7 Springer theory for \( \mathcal{U}(\mathfrak{sl}_n) \)

We are going to demonstrate in this section that, as a special case of the general machinery developed above, one can construct representations of \( \mathfrak{sl}_n(\mathbb{C}) \) and, maybe, other semisimple algebras as well, cf. [Na]. Many of the objects we use here for studying the \( \mathfrak{sl}_n(\mathbb{C}) \)-case are analogous to the objects in the Weyl group case studied in the previous section.

We fix an integer \( n \geq 1 \) corresponding to \( \mathfrak{sl}_n(\mathbb{C}) \) whose representations we want to study. We also fix an integer \( d \geq 1 \) bearing no relation to \( n \).

An \( n \)-step partial flag \( F \) in the vector space \( \mathbb{C}^d \) is a sequence of subspaces \( 0 = F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^d \), where the inclusions are not necessarily proper. Write \( \mathcal{F} \) for the set of all \( n \)-step partial flags in \( \mathbb{C}^d \). In the current situation, \( \mathcal{F} \) will play the role that the flag variety \( \mathcal{B} \) played in representations of Weyl groups. The space \( \mathcal{F} \) is a smooth compact manifold with connected components parametrized by all partitions

\[
d = (d_1 + d_2 + \ldots + d_n = d), \quad d_i \in \mathbb{Z}_{\geq 0}.
\]

We emphasize that each \( d_i \) here may take any value \( 0 \leq d_i \leq d \), zero in particular. To the partition \( d = (d_1 + \ldots + d_n) \) we associate the connected component of \( \mathcal{F} \) consisting of flags

\[
\mathcal{F}_d = \{ F = (0 = F_0 \subset \ldots \subset F_n = \mathbb{C}^d) \mid \dim F_i/F_{i-1} = d_i \}. \quad (36)
\]

Next we introduce an analogue of the nilpotent variety in the current situation to be the set \( N = \{ x : \mathbb{C}^d \to \mathbb{C}^d \mid x \text{ is linear, } x^n = 0 \} \).

We are going to define an analogue of the Springer resolution. Write \( M \) for the set of pairs \( M = \{ (x, F) \in N \times \mathcal{F} \mid x(F_i) \subset F_{i-1}, i = 1, 2, \ldots, n \} \). Note that the requirement \( x \in N \) in this formula is superfluous because \( x(F_i) \subset F_{i-1} \), for all \( i \), necessarily implies that \( x^n = 0 \). The first and second projections give rise to a natural diagram:

\[
\begin{array}{ccc}
M & \longrightarrow & \mathcal{F} \\
\mu \downarrow & & \downarrow \mu \\
N & & \\
\end{array}
\]

The natural action of \( \text{GL}_d(\mathbb{C}) \) on \( \mathbb{C}^d \) gives rise to \( \text{GL}_d(\mathbb{C}) \)-actions on \( \mathcal{F}, N \) and \( M \) by conjugation. The projections clearly commute with the \( \text{GL}_d(\mathbb{C}) \)-action.

We have the following description of the cotangent bundle on \( \mathcal{F} \); its proof is entirely analogous to the proof in the case of the flag variety (see Lemma 6.1).
Proposition 7.1 There is a natural $GL_d(\mathbb{C})$-equivariant vector bundle isomorphism

$$M \cong T^*F$$

making the map $\pi$ above into the canonical projection $T^*F \to F$. □

The decomposition of $F$ into connected projection $T^*F \to F$. □

The decomposition of $F$ into connected components $F_d$ gives rise to a decomposition of $M$ into connected components according to $n$-step partitions of $d$:

$$M = \bigsqcup_d M_d, \quad M_d = T^*F_d.$$

The variety $N$ of nilpotent endomorphisms is naturally stratified by $GL_d(\mathbb{C})$-conjugacy classes, $N = \bigsqcup_a N_a$. For any point $x \in N$ the fiber $\mu^{-1}(x)$ consists of pairs $(x, F)$ satisfying $x(F_i) \subset F_{i-1}$, for all $i$, and may be identified with a subvariety of $F$ that we denote by $F_x$.

Lemma 7.2 [Spa] For any $x \in N$, and any $n$-step partition $d$, the set $F_x \cap M_d$ is a connected variety of pure dimension (that is, each irreducible component has the same dimension) and

$$\dim \mathcal{O}_x + 2 \cdot \dim(F_x \cap F_d) = 2 \cdot \dim F_d;$$

where $\mathcal{O}_x$ denotes the $GL_d(\mathbb{C})$ orbit of $x$.

This result was proved by Spaltenstein [Spa] via an explicit computation. The connectivity part of the lemma can be proved in a more conceptual way using Zariski’s Main Theorem (see [Mum]). This theorem works because $N$ (and, more generally, the closure of any nilpotent conjugacy class in $\mathfrak{sl}_d(\mathbb{C})$) is known to be a normal variety.

The second claim of the Lemma concerning dimension just says that the morphism $\mu : M \to N$ is strictly semi-small. The inequality between the dimensions $\text{LHS} \leq \text{RHS}$ (which amounts to saying that $\mu$ is semi-small) can be proved by showing that $Z = M \times_N M$ is a Lagrangian subvariety of $M \times M$. Proof of the strict equality as well as of the equidimensionality assertion given in [Spa] exploits some specific features of $SL_n$ in an essential way and will not be reproduced here. These assertions fail for simple groups of types other than $SL_n$. □

As before, we set $Z = M \times_N M$ and consider the convolution algebra $H_*(Z)$. By Lemma 7.2, the map $\mu$ is (strictly) semi-small and hence by Theorem 5.4 (i) the subspace $H(Z) \subset H_*(Z)$ spanned by the fundamental classes of the irreducible components of $Z$ is a semisimple subalgebra. As at the end of Section 2, the algebra $H_*(Z)$ acts on $H_*(F_x)$ by convolution. Using the dimension property of Section 2 we deduce that the subspace $H(F_x)$ spanned by the classes of the irreducible components of $F_x$ is stable under $H(Z)$-action.

Using the general prescription of Theorem 5.4 (iii) we should now decompose $H(F_x)$ into isotypic components with respect to the monodromy action and the multiplicity spaces will be the irreducible modules over $H(Z)$. In our particular case the decomposition simplifies due to the following lemma:

Lemma 7.3 The monodromy action on $H(F_x)$ is trivial for any $x \in N$. □

Corollary 7.4 (i) If $x, y \in N$ are $GL_d(\mathbb{C})$-conjugate then the $H(Z)$-modules $H(F_x)$ and $H(F_y)$ are isomorphic.

(ii) The spaces $\{H(F_x)\}$, where $x$ runs over the representatives of $GL_d(\mathbb{C})$-conjugacy classes in $N$, form a complete collection of irreducible $H(Z)$-modules. □
Now we identify the convolution algebra $H(Z)$. Consider the natural $\mathfrak{sl}_n$-action on $\mathbb{C}^n$ and the induced action of the universal enveloping algebra $U(\mathfrak{sl}_n)$ on $(\mathbb{C}^n)^{\otimes d}$, the $d$-th tensor power. Let
\[
I_d = \text{Ann}(\mathbb{C}^n)^{\otimes d} \subset U(\mathfrak{sl}_n)
\]
be the annihilator of $(\mathbb{C}^n)^{\otimes d}$, a two-sided ideal of finite codimension in $U(\mathfrak{sl}_n)$.

**Theorem (Geometric Construction of $U(\mathfrak{sl}_n)$)** There exists a natural algebra isomorphism
\[
U(\mathfrak{sl}_n)/I_d \simeq H(Z).
\]

This theorem will be proved in Section 10.

Recall next that the set of finite dimensional irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ is known to be in bijective correspondence with the set of all dominant weights of $\mathfrak{sl}_n(\mathbb{C})$. The latter can be viewed as $n$-tuples of integers $d_1 \geq \ldots \geq d_n$ modulo the $\mathbb{Z}$-action by simultaneous translation.

On the other hand, by the Geometric Construction of $U(\mathfrak{sl}_n)$, any simple $H(Z)$-module gives rise, via the projection $U(\mathfrak{sl}_n) \to H(Z)$, to an irreducible representation of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We wish to establish a relationship between $GL_d(\mathbb{C})$-conjugacy classes in $N$ parametrizing irreducible representations of $H(Z)$, and dominant weights parametrizing corresponding irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$.

Let $x \in N$ be a linear operator in $\mathbb{C}^d$ such that $x^n = 0$. Put formally $x^0 = Id$. Then there are two distinguished flags attached to $x$:
\[
F^{\max}(x) = (0 = \ker(x^0) \subset \ker(x) \subset \ker(x^2) \subset \ldots \subset \ker(x^n) = \mathbb{C}^d),
\]
\[
F^{\min}(x) = (0 = \im(x^n) \subset \im(x^{n-1}) \subset \ldots \subset \im(x) \subset \im(x^0) = \mathbb{C}^d).
\]
Observe that $F^{\max}(x), F^{\min}(x) \in \mathcal{F}_x$. We assign to each $x \in N$ the $n$-tuple
\[
d(x) = (d_1 + \ldots + d_n = d), \quad \text{where} \quad d_i = \dim \ker(x^i) - \dim \ker(x^{i-1}).
\]
This is the partition associated to the flag $F^{\max}(x)$.

**Lemma 7.5** The $n$-tuple $d(x)$ is a dominant weight.

**Proof** For any $i \geq 1$ we have $x(\ker(x^i)) \subset \ker(x^{i-1})$. Hence, the operator $x$ induces, for each $i \geq 1$, a linear map
\[
\frac{\ker(x^{i+1})}{\ker(x^i)} \hookrightarrow \frac{\ker(x^i)}{\ker(x^{i-1})}
\]
Observe that this map is injective. Whence $d_i \geq d_{i+1}$. The lemma follows. \(\square\)

**Remark** For any flag $F \in \mathcal{F}_x$ we have $F^{\min} \leq F \leq F^{\max}$ in the sense that $F^{\min}_i \subset F^{\max}_i \subset F^{\max}_i$, for each $i = 1, 2, \ldots, n$. To see that $F \leq F^{\max}$ note that, for any $F = (0 = F_0 \subset \ldots \subset F_n = \mathbb{C}^d) \in \mathcal{F}_x$ and any $i = 1, 2, \ldots, n$, one has $x^i(F_1) \subset x^{i-1}(F_{i-1}) \subset \ldots \subset x(F_1) = 0$. Hence, $F_i \subset \ker(x^i)$. The other inclusion is proved similarly. \(\square\)

Here is the main corollary of geometric construction of $U(\mathfrak{sl}_n)$. It provides a construction of all irreducible finite dimensional representations of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. 

Theorem 7.6 (Springer Theorem for $\mathfrak{u}(\mathfrak{sl}_n)$) For any $x \in N$, we have

(a) The simple $\mathfrak{sl}_n(\mathbb{C})$-module $H(F_x)$ has the highest weight

$$d(x) = (d_1 \geq d_2 \geq \ldots \geq d_n), \quad d_i = \dim \ker(x^i) - \dim \ker(x^{i-1})$$

In particular, every finite-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is of the form $H(F_x)$.

(b) The flags $F_{\text{max}}(x)$ and $F_{\text{min}}(x)$ are isolated points of the fiber $F_x$. The corresponding fundamental classes $[F_{\text{max}}(x)] \in H(F_x)$ and $[F_{\text{min}}(x)] \in H(F_x)$ are a highest weight and a lowest weight vector in $H(F_x)$, respectively.  

The fundamental classes of the irreducible components of the fiber $F_x$ form a distinguished basis in $H(F_x)$. This basis is a weight basis with respect to the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$. It is not known to the author whether or not this basis coincides with the canonical basis constructed by Lusztig and Kashiwara, see [KSa].

8 Fourier transform

In this section we will introduce the main tool used in the proofs of the geometric constructions of $W$ and $\mathcal{U}(\mathfrak{sl}_n)$, the Fourier transform. Given a complex manifold $X$ and a vector bundle $E \to X$, the Fourier transform sends certain complexes of sheaves on the total space of $E$ to complexes of sheaves on the total space of the dual bundle $E^*$. The first step is to define the appropriate categories.

Monodromic sheaves

Let $E \to X$ be a complex holomorphic vector bundle.

Definition 8.1 A sheaf of $\mathbb{C}$-vector spaces on the total space of the bundle $E$ is called monodromic if it is locally constant over the orbits of the natural $\mathbb{C}^*$-action on $E$. A complex of sheaves is called monodromic if all its cohomology sheaves are monodromic. Let $D_{\text{mon}}^b(E)$ be the derived category of the category of bounded complexes (of $\mathbb{C}$-vector spaces) with monodromic cohomology sheaves.

Definition 8.2 Denote by $\text{Perv}_{\text{mon}}(E)$ the full subcategory of the category of perverse sheaves, $\text{Perv}(E)$, on the total space of the bundle $E$ (see Definition 4.2) formed by the monodromic perverse sheaves.

Consider $E$ as a real vector bundle. Then the complex dual bundle $E^*$ can be identified with the real dual of $E$ via the pairing $x, \xi \mapsto \langle x, \xi \rangle = Re(\xi(x)) \in \mathbb{R}$. Let $I$ be a complex of injective sheaves on $E$ which is bounded below, and let $\tau : E \to X$, $\check{\tau} : E^* \to X$ be the projections. Given an open subset $U \subset E^*$, define $U^\circ \subset E$, the polar set for $U$, as the set of all $x \in E$ such that

(i) $\tau(x) \in \check{\tau}(U)$, and
(ii) $\langle x, \xi \rangle \geq 0$ for all $\xi \in U$ satisfying $\check{\tau}(\xi) = \tau(x)$.

Then $U^\circ$ is a closed subset of $\tilde{U} = \tau^{-1}\check{\tau}(U)$. The assignment, see [Br, p.63]:

$$U \mapsto \Gamma_{U^\circ}(\tilde{U}, I^*|_{\tilde{U}})$$

defines a complex of presheaves on $E^*$. 

Denote by \( \tilde{\mathcal{F}}(I^*) \) the sheafification of this complex. Using injective resolutions of monodromic complexes we can extend \( \tilde{\mathcal{F}} \) to a functor

\[
\tilde{\mathcal{F}} : D^b_{mon}(E) \to D^b_{mon}(E^*)
\]

cf. [Br, Proposition 6.11] for the proof of the fact that the image of a monodromic sheaf under \( \tilde{\mathcal{F}} \) is also monodromic. This definition is rather technical and will never be used here. What is only important are the properties of the Fourier transform listed in Proposition 8.3 below.

It is convenient to consider the shifted functor \( \mathcal{F} = \tilde{\mathcal{F}}[r] \), where \( r = rk(E) \).

**Remark** The above definition of Fourier transform looks mysterious; it is not even clear why \( \mathcal{F} \) should be called a Fourier transform functor. In fact, this functor comes from a functor on modules over the ring \( \mathcal{D}_E \) of algebraic linear differential operators on \( E \) (\( \mathcal{D}_E \)-modules for short). For any \( \mathcal{D}_E \)-module \( M \) the sheaf \( \tau_*(M) \) is a sheaf of modules over \( \tau_*(\mathcal{D}_E) \). In this way we can think of a \( \mathcal{D}_E \)-module as a sheaf on \( X \) having the structure of a (left) \( \tau_*(\mathcal{D}_E) \)-module.

Assuming \( E \) to be trivial (as it will be in applications), we can construct a natural isomorphism \( \tau_*(\mathcal{D}_E) \cong \tau_*(\mathcal{D}_E^*) \) which can be written as \( \mathcal{F}(x_i) = \partial/\partial x_i \), \( \mathcal{F}(\partial/\partial x_i) = -\xi_i \) in coordinates \( x_1, \ldots, x_r \) (resp. \( \xi_1, \ldots, \xi_r \)) along the fibers of \( E \) (resp. \( E^* \)). Recall that the classical Fourier transform also interchanges multiplication by a coordinate function with taking a partial derivative. This isomorphism establishes an equivalence between the category of \( \mathcal{D}_E \)-modules and \( \mathcal{D}_E^* \)-modules which we also denote by \( \mathcal{F} \). More generally, for a non-trivial bundle \( E \) one has \( \tau_*(\mathcal{D}_E) = (\Lambda^r E) \otimes_{\mathcal{O}_X} \tau_*(\mathcal{D}_E^*) \otimes_{\mathcal{O}_X} (\Lambda^r E)^{-1} \), where \( r \) is the rank of \( E \) and \( \mathcal{O}_X \) is the sheaf of regular functions on \( X \), cf. [BMV]. Hence if \( \mathcal{M} \) is a \( \tau_*(\mathcal{D}_E) \)-module, then \( (\Lambda^r E)^{-1} \otimes_{\mathcal{O}_X} \mathcal{M} \) is a \( \tau_*(\mathcal{D}_E^*) \)-module.

There is a natural De Rham functor from the category of \( \mathcal{D} \)-modules on a complex manifold \( X \) to the category of perverse sheaves on \( X \) which sends \( \mathcal{M} \) to \( DR(\mathcal{M}) := \Omega^*_X \otimes_{\mathcal{D}_X} \mathcal{M} \), where the sheaf \( \Omega^*_X \) of top degree holomorphic differential forms on \( X \) has a natural right \( \mathcal{D}_X \)-module structure.

One can show that the Fourier transform functor \( \mathcal{F} \) on perverse sheaves is obtained by applying \( DR(\cdot) \) to the Fourier transform of \( \mathcal{D} \)-modules, that is, the mysterious definition above was designed so that one has (cf. [Br, Corollary 7.22])

\[
\mathcal{F}_{Perv}(DR(\mathcal{M})) = DR(\mathcal{F}_{\mathcal{D}mod}(\mathcal{M})).
\]

One has other similarities between \( \mathcal{F} \) and the classical Fourier transform. For example, there is a \( * \)-convolution (not the convolution in Borel-Moore homology defined in Section 2) on \( D^b_{mon}(\mathbb{C}^n) \), similar to the classical convolution of functions on a group, defined by:

\[
\mathcal{F}^* \ast \mathcal{G}^* = s_!(\mathcal{F}^* \boxtimes \mathcal{G}^*),
\]

where \( s : E \times E \to E \) is the sum map: \( s(x, y) = x + y \).

One has an isomorphism of functors \( \mathcal{F}(\mathcal{F}^* \ast \mathcal{G}^*) = \mathcal{F}(\mathcal{F}^*) \otimes \mathcal{F}(\mathcal{G}^*) \) (cf. [Br, Corollary 6.3]) analogous to the corresponding classical result, saying that the Fourier transform takes the convolution of functions into the product of their Fourier transforms.

We summarize the properties of \( \mathcal{F} \) in the following

**Proposition 8.3**

(1) For a monodromic complex \( \mathcal{G}^* \) there exists a natural isomorphism \( \mathcal{F} \circ \mathcal{F}(\mathcal{G}^*) \cong a^* \mathcal{G}^* \)
where a is the automorphism of the total space of the vector bundle E given by multiplication by (-1).

(2) The image under F of a monodromic perverse sheaf is also a monodromic perverse sheaf.

(3) \( F \) sets up an equivalence of categories \( \text{Perv}_{\text{mon}}(E) \sim \text{Perv}_{\text{mon}}(E^*) \).

(4) Let \( i_V : V \hookrightarrow E \) be a subbundle and \( C_V = (i_V)_*(C) \). Then \( F(C_V) \cong C_{V^\perp} \), where \( i_{V^\perp} : V^\perp \hookrightarrow E^* \) is the embedding of the annihilator (in \( E^* \)) of the subbundle \( V \).

**Proof** (1) is proved in [Br, Proposition 6.11] and (2) follows from (1). (3) is obtained from a similar statement for monodromic \( D \)-modules (cf. [Br, Corollary 7.26] see also the remark above). To prove (4) notice first that if \( V = E \), then by definition \( F \) is the constant sheaf supported at zero section. The general case follows from this by functoriality of \( F \) with respect to \( i_V \) [Br, Theorem 6.1(2)]. \( \square \)

**Direct Image** Given a vector space \( E \) and a complex variety \( X \) we form a trivial vector bundle \( E_X = E \times X \). Since \( E \) can be viewed as a vector bundle over a point, we have two Fourier transforms defined on \( \text{Perv}_{\text{mon}}(E_X) \) and on \( \text{Perv}_{\text{mon}}(E) \), respectively.

**Claim 8.4** For a compact algebraic variety \( X \) the following diagram commutes:

\[
\begin{array}{ccc}
\text{Perv}_{\text{mon}}(E_X) & \xrightarrow{F_{E_X}} & \text{Perv}_{\text{mon}}(E^*_X) \\
(pr_E)_* & \downarrow & (pr_{E^*})_* \\
\text{Perv}_{\text{mon}}(E) & \xrightarrow{F_E} & \text{Perv}_{\text{mon}}(E^*)
\end{array}
\]  

(38)

**Proof** This follows from [Br, Proposition 6.8]. \( \square \)

9 Proof of the geometric construction of \( W \)

Recall that we want to construct an algebra isomorphism \( H(Z) \cong \mathbb{C}[W] \) where \( Z \), the Steinberg variety, arises from the Springer map \( \mu : \tilde{N} \rightarrow \mathcal{N} \) via the basic construction of §2.

**The Fourier transform reduction**

We have seen in Proposition 5.1 that \( H(Z) \subset \text{Ext}^\bullet(\mu_* C_{\tilde{N}}, \mu_* C_{\tilde{N}}) \). Since \( \mu \) is semi-small, \( \mu_* C_{\tilde{N}} \) is perverse, and therefore \( H(Z) = \text{End}_{\text{Perv}}(\mu_* C_{\tilde{N}}) \), by Theorem 5.4 (i).

Using the embedding \( \mathcal{N} \hookrightarrow g \) we can view perverse sheaves on \( \mathcal{N} \) as perverse sheaves on \( g \). In this way \( H(Z) \) is realized as an endomorphism algebra of the perverse sheaf \( \mu_* C_{\tilde{N}} \) on \( g \). The group \( \mathbb{C}^* \) acts both on \( \mathcal{N} \), since a multiple of a nilpotent element is also nilpotent, and on \( \tilde{N} \) (along the fibers of \( \tilde{N} \rightarrow B \)). Since \( \mu \) is \( \mathbb{C}^* \)-equivariant, all the direct image sheaves involved are \( \mathbb{C}^* \)-equivariant, in particular monodromic. Therefore we can apply the Fourier transform functor \( F \) to \( \mu_* C_{\tilde{N}} \). Then \( F(\mu_* C_{\tilde{N}}) \) is a perverse sheaf on \( g^* \) and since \( F \) is an equivalence of categories, \( H(Z) \cong \text{Hom}_{\text{Db}(g^*)}(F(\mu_* C_{\tilde{N}}), F(\mu_* C_{\tilde{N}})) \).

Recall the definition of the Springer variety:

\[
\tilde{N} = \{(x, b) \in g \times B | x \in n_b \}
\]
Proposition 9.1
The map \( \tilde{\mu} : \tilde{g} \to g \) is small. \( \square \)

Denote by \( g^{rs} \) the open subset of \( g \) of all regular semisimple elements. Then the restriction of \( \tilde{\mu} : \tilde{g} \to g^{rs} \) is a local system, \( \tilde{\mu}_* C_{\tilde{g}}|_{g^{rs}} \), and we write \( IC(\tilde{\mu}_* C_{\tilde{g}}|_{g^{rs}}) \) for the corresponding IC-complex on \( g \), as defined in Section 4.

Corollary 9.2
\( \tilde{\mu}_* C_{\tilde{g}} = IC(\tilde{\mu}_* C_{\tilde{g}}|_{g^{rs}}) \).

Proof
Follows from Proposition 9.1 and Theorem 5.4 (iii). \( \square \)

The basic idea of the Fourier transform reduction can be now summarized as follows:

The convolution algebra \( H(Z) \) is isomorphic to \( \text{End}(\mu_* C_{\tilde{N}}) \). View \( C_{\tilde{N}} \) as a perverse sheaf on \( g_b \) extended by zero from \( \tilde{N} \). Then

\[
H(Z) \simeq \text{End}(\mu_* C_{\tilde{N}}) \\
\simeq \text{End}(\text{F} (\mu_* C_{\tilde{N}})) \simeq \text{End} (\tilde{\mu}_* C_{\tilde{g}}) \\
\simeq \text{End}(IC(\tilde{\mu}_* C_{\tilde{g}}|_{g^{rs}})) \simeq \text{End}(\tilde{\mu}_* C_{\tilde{g}}|_{g^{rs}}),
\]

(39)

where \( pr_{g^{rs}} : g^{rs} \times B \to g^{rs} \) is the projection and \( \text{F}(C_{\tilde{N}}) \) is a perverse sheaf on \( g_{rs} \). The fourth isomorphism follows from Corollary 9.2 and the last one from the Perverse Continuation Principle 4.5.

Analysis of \( \tilde{\mu}|_{g^{rs}} \)

We denote \( \tilde{\mu}^{-1}(g^{rs}) \) by \( \tilde{g}^{rs} \). By definition of \( \tilde{\mu} \) the fiber of \( \tilde{\mu} : \tilde{g}^{rs} \to g^{rs} \) at a point \( x \in g^{rs} \) is the set of all Borel subalgebras that contain \( x \).

Recall that for a fixed maximal torus \( T \subset G \) the Weyl group \( W \) is defined as the quotient \( N_G(T)/T \).

Proposition 9.3

(i) There exists a free action of \( W \) on \( \tilde{g}^{rs} \) such that \( \tilde{g}^{rs}/W \) is isomorphic to \( g^{rs} \). Moreover, under this isomorphism \( \tilde{\mu} \) corresponds to the quotient map \( \tilde{g}^{rs} \to \tilde{g}^{rs}/W \).

(ii) The map \( \tilde{\mu} : \tilde{g}^{rs} \to g^{rs} \) is a regular (i.e. Galois) covering with automorphism group \( W \).
Proof Choose and fix a Cartan subalgebra $\mathfrak{h}$ (corresponding to a maximal torus $T$) and a Borel subalgebra $\mathfrak{b}$ (corresponding to a Borel subgroup $B$) containing $\mathfrak{h}$. Let $\mathfrak{h}^{rs} = \mathfrak{g}^{rs} \cap \mathfrak{h}$ be the set of regular elements in $\mathfrak{h}$. Define a map $\phi : G/T \times \mathfrak{h}^{rs} \to \mathfrak{g}^{rs}$ by $\phi(g, h) = (\text{Ad}(g)\mathfrak{b}, \text{Ad}(g)h)$. Since $T$ acts trivially on $\mathfrak{h}$ and maps $\mathfrak{b}$ into itself, $\phi(g, h)$ depends only on the image of $g$ in $G/T$.

We claim that $\phi$ is an isomorphism. To show that $\phi$ is injective assume that for $g_1, g_2 \in G$ we have $\text{Ad}(g_1)\mathfrak{b} = \text{Ad}(g_2)\mathfrak{b}$ and $\text{Ad}(g_1)h = \text{Ad}(g_2)h$. The first equality implies $g_2^{-1}g_1 \in N_G(B)$ and since $h$ is regular semisimple the second equality implies $g_2^{-1}g_1 \in N_G(T)$. By [Hum] $N_G(B) = B$ and $B \cap N_G(T) = T$, so $g_2^{-1}g_1 \in T$, hence $g_1$ and $g_2$ represent the same point in $G/T$.

To prove that $\phi$ is surjective assume that a point $(b', x') \in \mathfrak{g}^{rs}$ is given (i.e. $x' \in \mathfrak{b}'$ and $x'$ is regular semisimple). Since all Borel subalgebras are conjugate (cf. [Hum]) we have $\mathfrak{b}' = \text{Ad}(g')\mathfrak{b}$ for some $g' \in G$ and $B' = \text{Ad}(g')B$. Then $\text{Ad}(g')\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{b}$ hence by [Hum] there exists an element $u \in B' = \text{Ad}(g')B$ such that $\text{Ad}(u)x' \in \text{Ad}(g')\mathfrak{h}$. Denote $\text{Ad}((g')^{-1}u)x' \in \mathfrak{h}^{rs}$ by $x$ and $u^{-1}g'$ by $g$. Then $\text{Ad}(g)\mathfrak{b} = \text{Ad}(u^{-1})\text{Ad}(g')\mathfrak{b} = \text{Ad}(u^{-1})\mathfrak{b}' = \mathfrak{b}'$ and $\text{Ad}(g)x = \text{Ad}(u^{-1})\text{Ad}(g')(g'(g')^{-1}u)x' = x'$, so $\phi(g, x) = (b', x')$ and $\phi$ is surjective.

The Weyl group $W = N_G(T)/T$ acts on $G/T \times \mathfrak{h}^{rs}$ by $w \cdot (g, h) = (g\mu^{-1}, \text{Ad}(n)h)$ where $n \in N_G(T)$ is any representative of $w \in W$.

It follows from the definitions of $\phi$ and $\tilde{\mu}$ that $\tau = \tilde{\mu} \circ \phi : G/T \times \mathfrak{h}^{rs} \to \mathfrak{g}^{rs}$ is the quotient map for the action of $W$ on $G/T \times \mathfrak{h}^{rs}$.

To prove the second statement of the proposition one has to show (by general theory of coverings) that $\mathfrak{g}^{rs}$ is connected. Since $G/T$ is connected it suffices to show that $\mathfrak{h}^{rs}$ is connected. But $\mathfrak{h}^{rs}$ is a complement of finitely many complex hyperplanes in a complex vector space $\mathfrak{h}$ (cf. [Hum]), therefore it is connected. \(\square\)

Remark In fact, the action of $W$ on $\mathfrak{g}^{rs}$ does not depend on the choice of $T$ and $B$ if one views $W$ as an abstract Weyl group associated with the root system of $\mathfrak{g}$, cf. [CG, Chapter 3].

Next, for the covering $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$, one has a decomposition

$$
\tilde{\mu}_*\mathcal{C}_{\tilde{\mathfrak{g}}}|_{\mathfrak{g}^{rs}} = \bigoplus_{\psi} L_\psi \otimes \mathcal{L}_\psi
$$

of the local system $\tilde{\mu}_*\mathcal{C}_{\tilde{\mathfrak{g}}}|_{\mathfrak{g}^{rs}}$ into a direct sum of irreducible pairwise distinct local systems $\mathcal{L}_\psi$ (with multiplicity spaces $L_\psi$). Therefore we have by the Perverse Continuation principle

$$
\text{End}(\mu_*\mathcal{C}_{\mathfrak{g}}^{rs}) = \text{Hom}
\left(
\bigoplus_{\psi} L_\psi \otimes \mathcal{L}_\psi, \bigoplus_{\psi'} L_{\psi'} \otimes \mathcal{L}_{\psi'}
\right)
= \bigoplus_{\psi, \psi'} \text{Hom}(L_\psi, L_{\psi'}) \otimes \text{Hom}(\mathcal{L}_\psi, \mathcal{L}_{\psi'}) = \bigoplus_{\psi} \text{End}_C(L_\psi).
$$

Corollary 9.4 There exists a natural algebra isomorphism

$$
H(Z) \simeq \bigoplus_{\psi} \text{End}_C(L_\psi).
$$

Proof Follows from (38) and the computation above. \(\square\)
End of proof of the geometric construction of $W$. By Corollary 9.4 we just have to compute the multiplicity spaces $L_\psi$ above. By the correspondence between monodromy representations and local systems (see (27)) it suffices to decompose the monodromy representation of $\tilde{\mu}_*C_{\tilde{g}}$ into irreducible representations. The latter, by Proposition 9.3(ii), is nothing but the regular representation of $W$:

$$\mathbb{C}[W] = \bigoplus_{\chi \in W^\vee} L_\chi \otimes \chi,$$

where the summation is over the set $W^\vee$ of isomorphism classes of irreducible representations of $W$, and $L_\chi$ is isomorphic as a vector space to the dual of $\chi$. Moreover, the decomposition of vector spaces:

$$\mathbb{C}[W] = \bigoplus_{\chi \in W^\vee} \text{End}_C(L_\chi)$$

is an isomorphism of algebras, hence the theorem follows by Corollary 9.4. □

**Remark** Applying the inverse Fourier transform to the decomposition $\tilde{\mu}_*C_{\tilde{g}} = \bigoplus L_\psi \otimes IC_\psi$ we get $\mu_*C_{\bar{g}} = \bigoplus L_\psi \otimes IC_\psi$. Since $F$ is an equivalence of categories, $F^{-1}(IC_\psi)$ is a simple perverse sheaf, hence of the form $IC(L)$ for some irreducible local system $L$. Thus we have proved (!) the Decomposition Theorem in this case.

**Digression: the braid group $B_W$**

The proof of Proposition 9.3 allows us to analyze the fundamental group $\pi_1(\bar{g}^{rs})$. Firstly, one has an exact sequence

$$1 \to \pi_1(G/T \times \mathfrak{h}^{rs}) \to \pi_1(\bar{g}^{rs}) \to W \to 1.$$

The homogeneous space $G/T$ is naturally a fibration over $G/B$ with fibers isomorphic (non-canonically) to $B/T$. Since $B = G/B$ is simply-connected and $B/T$ is contractible, the exact sequence above turns into

$$1 \to \pi_1(\mathfrak{h}^{rs}) \to \pi_1(\mathfrak{g}^{rs}) \to W \to 1. \quad (40)$$

The map $\pi_1(\mathfrak{h}^{rs}) \to \pi_1(\mathfrak{g}^{rs})$ is induced by the restriction $\tau|_{\mathfrak{g} \times \mathfrak{h}^{rs}} : \mathfrak{h}^{rs} \to \mathfrak{g}^{rs}$ where $\tau : G/T \times \mathfrak{h}^{rs} \to \mathfrak{g}^{rs}$ is defined at the end of the proof of Proposition 9.3. This restriction coincides with the natural embedding $\mathfrak{h}^{rs} \hookrightarrow \mathfrak{g}^{rs}$.

**Definition 9.5** The fundamental group $\pi_1(\mathfrak{g}^{rs})$ is called the braid group of $\mathfrak{g}$.

Recall (cf. [Se]) that the Weyl group is generated by simple reflections $s_1, \ldots, s_l$ where $l = \text{rk} \mathfrak{g}$ and $\{s_i\}_{i=1,\ldots,l}$ satisfy the following relations:

(i) $s_i^2 = e$,

(ii) For any $i \neq j$ one has $s_is_js_i\ldots = s_js_i\ldots s_j$ with $m_{ij}$ terms on each side, where $m_{ij}$ is defined in the standard way from the corresponding Dynkin diagram (cf. [Se]).
The relations (ii) are called Coxeter relations.

There are two different (equivalent) definitions of the braid group associated to $W$:

1st description: the braid group is an abstract group, $B_W$, generated by the elements $T_1, \ldots, T_l$, subject to the Coxeter relations (ii) associated with $W$ (but with relations (i) omitted).

2nd description: the braid group is the group generated by the elements $T_w, w \in W$, satisfying $T_wT_{w'} = T_{ww'}$ whenever $l(w) + l(w') = l(ww')$.

In particular, $W$ is embedded as a set (not as a subgroup) into $B_W$ via $w \mapsto T_w$. One also has a surjective group homomorphism $B_W \twoheadrightarrow W$. Its kernel is called the “colored braid group”.

The second description is “better” in some categorical sense explained in [D].

10 Proof of the geometric construction of $\mathcal{U}(\mathfrak{sl}_n)$

The proof\footnote{The idea of using the Fourier transform to prove the geometric construction of $\mathcal{U}(\mathfrak{sl}_n)$ is due to A.Braverman and D. Gaitsgory.} of the geometric construction of $\mathcal{U}(\mathfrak{sl}_n)$ follows the same pattern as the proof of geometric construction of $W$. First notice that we have a diagram

\[
\begin{array}{ccc}
M & \longrightarrow & \mathfrak{gl}_d \times \mathcal{F} \\
\mu \downarrow & & \mu \downarrow \\
N & \longrightarrow & \mathfrak{gl}_d
\end{array}
\]

The convolution algebra $H(Z)$ is isomorphic to $\text{Hom}(\mu_* \mathbb{C}_M, \mu_* \mathbb{C}_M)$ (by Lemma 7.2 and Theorem 5.4).

We can view $\mathbb{C}_M$ as a sheaf on the total space of the bundle $\mathfrak{gl}_d \times \mathcal{F} \rightarrow \mathcal{F}$ supported on the subbundle $M \hookrightarrow \mathfrak{gl}_d \times \mathcal{F}$ and $\mu_* \mathbb{C}_M$ as a sheaf on $\mathfrak{gl}_d$. Applying the Fourier transform as in the proof of the geometric construction of $W$, we obtain by Proposition 8.3:

$H(Z) \simeq \text{Hom}(pr_* \mathbb{C}_{M^\perp}, pr_* \mathbb{C}_{M^\perp})$,

where $pr$ is the projection $(\mathfrak{gl}_d)^* \times \mathcal{F} \rightarrow (\mathfrak{gl}_d)^*$ and $M^\perp$ is the subbundle of the trivial bundle $(\mathfrak{gl}_d)^* \times \mathcal{F} \rightarrow \mathcal{F}$ annihilating $M$.

Let $\tilde{\mu} : M^\perp \rightarrow \mathfrak{gl}_d$ be the restriction of $pr : (\mathfrak{gl}_d)^* \times \mathcal{F} \rightarrow (\mathfrak{gl}_d)^*$ to $M^\perp$. If we identify $(\mathfrak{gl}_d)^*$ with $\mathfrak{gl}_d$ via the form $\langle A, B \rangle = Tr(AB)$ then, for any flag $F \in \mathcal{F}$, the fiber of the vector bundle $M^\perp \hookrightarrow \mathcal{F}$ over $F$ is the subspace $\{x \in \mathfrak{gl}_d \mid x(F_i) \subset F_i \quad \forall i = 1, \ldots, n\}$. Thus for $x \in \mathfrak{gl}_d$ the fiber $\tilde{\mu}^{-1}(x)$ over $x$ is equal to the set of all flags $F = (0 = F_0 \subset \ldots \subset F_n = \mathbb{C}^d)$ such that $x(F_i) \subset F_i$, for all $i$. We see that

$\tilde{\mu}^{-1}(x) = \{n$-step partial flags in $\mathbb{C}^d$ fixed by $x\}$.

Notice that this fiber is non-empty for any $x \in \mathfrak{gl}_d$ as is clear from the Jordan normal form of $x$. Again as in Section 9 one proves:

**Proposition 10.1** The map $\tilde{\mu} : M^\perp \rightarrow \mathfrak{gl}_d$ is small (cf. Definition 5.3).\textvisiblespace$\Box$
First we analyze the fibers of $\tilde{\mu}$ over a regular semisimple element $x \in \mathfrak{gl}_d^r$, that is, a diagonalizable $d \times d$ matrix with pairwise distinct eigenvalues. Choose the basis $(e_1, \ldots, e_d)$ of $\mathbb{C}^d$ in which $x$ is diagonal. Then any subspace $F_i$ satisfying $x(F_i) \subset F_i$ is spanned by a subset of our basis. Hence we can describe $\tilde{\mu}^{-1}(x)$ as the set of maps $\phi : \{1 \ldots d\} \to \{1 \ldots n\}$, where $\phi(i)$ is defined for each $i = 1, \ldots, d$ as the minimal number such that $e_i \in F_{\phi(i)}$.

Therefore the stalk $(\tilde{\mu}_s C_{M^\perp})_x$ of the local system $\tilde{\mu}_s C_{M^\perp}|_{\mathfrak{gl}_d^r}$ is isomorphic to the complex vector space with base $\{\phi \mid \phi \in \text{Maps} \{1, \ldots, d\}, \{1, \ldots, n\}\}$. This space can be identified with $(\mathbb{C}^n)^{\otimes d}$ by choosing a basis $f_1, \ldots, f_n$ of $\mathbb{C}^n$ and mapping $\phi$ to $f_{\phi(1)} \otimes \ldots \otimes f_{\phi(d)} \in (\mathbb{C}^n)^{\otimes d}$.

**Lemma 10.2** For $x \in \mathfrak{gl}_d^r$, the monodromy action of $\pi_1(\mathfrak{gl}_d^r, x)$ on $(\mathbb{C}^n)^{\otimes d}$ factors through the natural representation of the symmetric group $S_d$ on $(\mathbb{C}^n)^{\otimes d}$.

**Proof** First notice that Proposition 9.3 is in fact valid for any reductive Lie algebra $\mathfrak{g}$ (not necessarily semisimple). In particular we can take $\mathfrak{g} = \mathfrak{gl}_d$. By (41) the monodromy action factors through $W = S_d$ if and only if the restriction of the local system on $\mathfrak{h}^r$ is trivial. But in our case, $\mathfrak{h}^r$ is the space of diagonal $(d \times d)$-matrices with pairwise distinct eigenvalues. Hence the fibers of $\tilde{\mu}$ are canonically identified with each other, and the local system $\tilde{\mu}_s \mathbb{C}$ is constant on $\mathfrak{h}^r$.

To show that the action of $S_d$ on $(\mathbb{C}^n)^{\otimes d}$ coincides with the natural one, notice that a lift of an element $w \in W$ to a loop in $\mathfrak{g}^r$ is the image (under $\tau : G/T \times \mathfrak{h}^r \to \mathfrak{g}^r$ defined in the proof of Proposition 9.3) of some path in $\psi : [0, 1] \to (G/T) \times \mathfrak{h}^r$ connecting $(n^{-1}, w \cdot h)$ with $(e, h)$ (where $n \in N_G(T)$ is any preimage of $w \in W$). Let $\psi(t) = (g(t), h(t))$. Then the fiber of $\tilde{\mu}$ over $\tau(g(t), h(t))$ as a subset of $\mathcal{F}$ is isomorphic of $g(t) \cdot \tilde{\mu}^{-1}(h)$. Since $g^{-1}(w \cdot h)$ is canonically identified with $\tilde{\mu}^{-1}(h)$ and $g(1) = n^{-1}$, our assertion follows.

As in Corollary 9.4 we have a decomposition of local systems on $\mathfrak{gl}_d^r$,

$$
\tilde{\mu}_s C_{M^\perp} = \bigoplus_{\psi} L_\psi \otimes \mathcal{L}_\psi,
$$

into a direct sum of irreducible local systems with multiplicities (the vector spaces $L_\psi$). Using the connection between monodromy and local systems (27) and Lemma 10.2, we can reformulate the conclusion of Corollary 9.4 as an algebra isomorphism

$$
H(Z) \simeq \text{End}_{S_d}((\mathbb{C}^n)^{\otimes d}).
$$

Recall that $I_d = \text{Ann}(\mathbb{C}^n)^{\otimes d} \subset \mathcal{U}(\mathfrak{sl}_n)$, see §7. The following Lemma is a classical result of H.Weyl which is at the origin of Schur-Weyl duality.

**Lemma 10.3** The image of natural homomorphism $\mathcal{U}(\mathfrak{sl}_n) \to \text{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes d})$ commutes with the $S_d$-action and induces an algebra isomorphism

$$
\text{End}_{S_d}((\mathbb{C}^n)^{\otimes d}) \simeq \mathcal{U}(\mathfrak{sl}_n)/I_d. \quad \square
$$

Lemma 10.3 completes the proof of the geometric construction of $\mathcal{U}(\mathfrak{sl}_n)$. \(\square\)
11 q-Deformations: Hecke algebras and a quantum group

In this section we will state generalizations of the geometric constructions of $W$ and $U(\mathfrak{sl}_n)$ that will give geometric interpretations to “quantized” versions of these algebras.

First, we want to recall the notations of Section 6 and introduce some more. Let $G$ be a complex semisimple simply-connected Lie group. Choose and fix a Borel subgroup $B \subset G$ and let $T$ be a maximal torus contained in $B$. The Lie algebra $\mathfrak{h}$ of $T$ acts on the Lie algebra $\mathfrak{g}$ of $G$ via the adjoint representation. Denote by $\Delta \subset \mathfrak{h}^*$ the set of all roots of $G$. For any root $\alpha \in \Delta$ one has a reflection $s_\alpha \in W$. Write $Q \subset \mathfrak{h}^*$ and $Q^I \subset \mathfrak{h}$ for the root and coroot lattices, respectively, and let $X^*(T) = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$ and $X_*(T) = \text{Hom}_{\text{alg}}(\mathbb{C}^*, T)$ be the corresponding weight and coweight lattices (where $\text{Hom}_{\text{alg}}$ stands for ‘algebraic group homomorphisms’). We have

\[ Q \subset X^*(T) \quad \text{and} \quad Q^I \subset X_*(T). \]

The group $\Pi = X^*(T)/Q$ is finite and is known to be isomorphic to the fundamental group of $G$. We also denote by $\theta^I$ the maximal coroot of $G$.

**Definition 11.1**

(a) The **affine Weyl group** $W^a$ of $G$ is defined as the group of affine transformations of $\mathfrak{h}^*$ generated by $W$ and $s_0$, an additional reflection with respect to the affine hyperplane \( \{ h \in \mathfrak{h}^* : (\theta^I, h) + 1 = 0 \} \). Thus $W^a$ is a Coxeter group with generators $s_i, i = 0, ..., l$ (see Definition 9.6). It is known that $W^a$ is a semidirect product of $W$ and the co-root lattice $Q$.

(b) The **extended affine Weyl group** $\tilde{W}$ of $G$ is defined as a semidirect product of $W$ and $X^*(T)$ and is not, in general, a Coxeter group. It is clear that $W^a \subset \tilde{W}$ is a normal subgroup and we have $\tilde{W}/W^a = \Pi$. For $\pi \in \Pi$ and $i \in \{0, ..., l\}$, the transformation $\pi s_i \pi^{-1} : \mathfrak{h}^* \to \mathfrak{h}^*$ is a simple reflection again, which we denote by $s_{\pi(i)}$. Thus, $\tilde{W} \cong \Pi \ltimes W^a$ with the commutation relations $\pi s_i \pi^{-1} = s_{\pi(i)}$.

(c) For $w \in W^a$ we define the length function $l(w)$ as we did above Corollary 6.10, and we extend it to $\tilde{W}$ by requiring $l(\pi \cdot w) = l(w)$ for all $\pi \in \Pi, w \in W^a$.

The **affine Hecke algebra** $H$ of $G$ can be defined in two different (but equivalent) ways. They are analogous to the two different definitions of the braid group.

**First definition 11.2** [KL2] The algebra $H$ is the free $\mathbb{Z}[q, q^{-1}]$-algebra with basis $T_w, w \in \tilde{W}$ and multiplication given by the rules:

\[
(T_w + 1)(T_w - q) = 0, \quad w \in \{ s_0, ..., s_l \}, \quad (41) \\
T_w T_y = T_{wy}, \quad \text{if} \quad l(wy) = l(w) + l(y). \quad (42)
\]

Denote by $H^a \subset H$ the subspace spanned by the $T_w, w \in W^a$ only. This is clearly a subalgebra, and $H \cong H^a[\Pi]$ is the twisted group algebra for the $\Pi$-action on $H^a$. In other words, $H$ is generated by the sets $\{ T_w, w \in W^a \}$ and $\{ T_\pi, \pi \in \Pi \}$ with the relations (\[\Pi\]), (\[\Pi\]) for the $T_w$’s, and the relations

\[ T_\pi T_\pi' = T_{\pi + \pi'}; \quad T_\pi T_{s_i} T_\pi = T_{s_{\pi(i)}}, \quad i = 0, ..., l. \]

**Second definition 11.3** The algebra $H$ is the $\mathbb{Z}[q, q^{-1}]$-algebra with generators $\{ T_w, w \in W \}$ and $\{ Y_\lambda, \lambda \in X^*(T) \}$ subject to the relations:
(i) The $T_w, w \in W$, satisfy (41) and (42).

(ii) $Y_\lambda Y_\mu = Y_{\lambda + \mu}$.

(iii) $T_s Y_\lambda - Y_\lambda T_s = (1 - \frac{q}{q-1}) \frac{Y_{\lambda(s^{-1})} Y_{\lambda^{-1}}}{Y_{\lambda^{-1}} Y_{\lambda^{-1}} s^{-1}}, \quad i = 1, \ldots, n.$

It is known that the elements $T_w Y_\lambda, w \in W, \lambda \in X^*(T)$, form a $C$-basis of $H$.

**Remarks**

1. The elements $\{Y_\lambda, \lambda \in X^*(T)\}$ span a commutative subalgebra in $H$ isomorphic to $\mathbb{Z}[q,q^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[X^*(T)]$, the group algebra of the lattice $X^*(T)$ over $\mathbb{Z}[q,q^{-1}]$. The latter is also isomorphic to $R(T)[q,q^{-1}]$.

2. We have $\tilde{W} \simeq W \times X^*(T)$. Accordingly, one has a presentation of $H$ as $H_W \otimes_{\mathbb{Z}} R(T)[q,q^{-1}]$, where $H_W$ is the subalgebra generated by the $T_w$ (and the $\otimes$ above is only tensor product as $\mathbb{Z}$-modules, not algebras).

3. Given a Coxeter group $C$, like $W$ or $W^a$, or its close cousin like $\tilde{W}$, write $H_C$ for the corresponding Hecke algebra. Thus $H = H_{\tilde{W}}$, and $H_W$ is the “finite” Hecke algebra. Recall further that associated with $C$ is the corresponding Braid group $B_C$, see Definition 9.5. It is clear from (12) that the Hecke algebra $H_C$ is the quotient of the group algebra $\mathbb{Z}[q,q^{-1}] [B_C]$ modulo quadratic relations of type (11).

4. Hecke algebras arise naturally in Lie theory in at least three different contexts. First of all, the effect of taking the quotient of $\mathbb{Z}[q,q^{-1}] [B_C]$ modulo (11) is to get an algebra of the same “size” as the group algebra of the group $C$. More formally, the Hecke algebra $H_C$ is flat over $\mathbb{Z}[q,q^{-1}]$ and its specialization at $q = 1$ is the group algebra $\mathbb{C}[C]$. Thus, $H_C$ may be thought of as a “$q$-deformation” of the group algebra.

Recall further that the Braid group $B_W$ has a topological interpretation as the fundamental group $\pi_1(\mathfrak{g}^{rs})$. Thus, the Hecke algebra $H_W$ may be viewed as a quotient of $\mathbb{Z}[q,q^{-1}] [\pi_1(\mathfrak{g}^{rs})]$ modulo certain quadratic relations. Replacing here the Lie algebra $\mathfrak{g}$ by the group $G$ we get a similar interpretation of the *affine* Hecke algebra. Specifically, write $\mathfrak{W} := W \rtimes X_*(T)$ for the semidirect product of $W$ with the coweight lattice, and let $H_{\mathfrak{W}}$ be the corresponding Hecke algebra (note that the group $\tilde{W}$ was defined as the semidirect product of $W$ with $X^*(T)$, the dual lattice. Thus, replacing $\tilde{W}$ by $\mathfrak{W}$ amounts to replacing the root system by its dual, or equivalently, replacing the group $G$ by its Langlands dual $L(G)$). Then one proves, repeating the argument in Proposition 9.3, that

$$\pi_1(G^{rs}) \simeq B_{\mathfrak{W}}.$$ 

The group $B_{\mathfrak{W}}$ appears on the RHS instead of $B_W$ because, in the group case, the Cartan subalgebra $\mathfrak{h}$ gets replaced by the corresponding maximal torus $T$ which has non-trivial fundamental group $\pi_1(T) \simeq X_*(T)$. This fundamental group is responsible for the extra generators in $B_{\mathfrak{W}}$ as compared to $B_W$. Thus we may view $H_{\mathfrak{W}}$ as a quotient of the group algebra of $\pi_1(G^{rs})$ modulo quadratic relations.

Most fundamentally, Hecke algebras arise as convolution algebras. Specifically, let $G$ be a split simply-connected reductive group over $\mathbb{Z}$ with Borel subgroup $B$. Fix a finite field $\mathbb{F}_q$. Write $G(\mathbb{F}_q)$, $B(\mathbb{F}_q)$ for the corresponding finite groups of $\mathbb{F}_q$-rational points. Then $\mathbb{C}[B(\mathbb{F}_q) \setminus G(\mathbb{F}_q) / B(\mathbb{F}_q)]$, the algebra (under convolution) of $\mathbb{C}$-valued $B(\mathbb{F}_q)$-biinvariant functions on $G(\mathbb{F}_q)$ is known to be isomorphic to the Hecke algebra $H_W$ specialized at $q = q$, i.e.

$$\mathbb{C}[B(\mathbb{F}_q) \setminus G(\mathbb{F}_q) / B(\mathbb{F}_q)] \simeq H_W|_{q=q}. \quad (43)$$
Note that the convolution algebra on the left is the algebra of intertwiners of the induced module $\text{Ind}_{B(F_q)}^G 1$. Thus, decomposition of the induced module into irreducible $G(F_q)$-modules is governed by representation theory of the Hecke algebra $H_{W|q=q}$.

One has a similar interpretation of affine Hecke algebras in terms of $p$-adic groups. Specifically, let $\mathbb{Q}_p$ be a $p$-adic field with the ring of integers $\mathbb{Z}_p$ and the residue class field $\mathbb{F}_p = \mathbb{Z}_p / p \mathbb{Z}_p$. Then the ring maps on the left (below) induce the following group homomorphisms on the right (below):

$$\mathbb{F}_p \xrightarrow{\pi} \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p, \quad G(\mathbb{F}_p) \xrightarrow{\pi} G(\mathbb{Z}_p) \hookrightarrow G(\mathbb{Q}_p).$$

The preimage of $B(\mathbb{F}_p)$ under the projection $\pi$ is a compact subgroup $I \subset G(\mathbb{Z}_p)$, called an Iwahori subgroup. We may consider the algebra $\mathbb{C}[I \backslash G(\mathbb{Q}_p) / I]$ of $\mathbb{C}$-valued $I$-biinvariant functions on $G(\mathbb{Q}_p)$ with compact support. Similarly to (43) one establishes an algebra isomorphism

$$\mathbb{C}[I \backslash G(\mathbb{F}_q) / I] \simeq H_{W|q=p}.\tag{44}$$

Now let $\rho : G(\mathbb{Q}_p) \to \text{End}(V)$ be an admissible representation of $G(\mathbb{Q}_p)$. For any $I$-biinvariant compactly supported function $f$ on $G(\mathbb{Q}_p)$, the formula:

$$\rho(f) : v \mapsto \int_{G(\mathbb{Q}_p)} f(g) \cdot \rho(g)v \, dg, \quad v \in V^I$$

defines a $\mathbb{C}[I \backslash G(\mathbb{Q}_p) / I]$-module structure on the vector space $V^I$ of $I$-fixed vectors. Moreover, the space $V^I$ turns out to be finite-dimensional, and the assignment $V \mapsto V^I$ is known to provide an equivalence between the category of admissible $G(\mathbb{Q}_p)$-modules generated by $I$-fixed vectors and the category of finite dimensional $H_{W|q=p}$-modules.

The interpretations of Hecke algebras given above show the importance of having a classification of their finite dimensional irreducible representations. However, none of the above interpretations helps in finding such a classification. For this, one needs a totally different geometric interpretation that we are now going to explain.

Let $\mu : \widetilde{N} = T^*B \to N$ be the Springer resolution and $Z = \widetilde{N} \times_{\widetilde{N}} \widetilde{N}$ the Steinberg variety. Since $\mu : T^*B \to N$ is $G$-equivariant, $Z$ is a $G$-variety and since $Z \circ Z = Z$, the $K$-group $K^G(Z)$ has the structure of an associative convolution algebra.

Let $Z_\Delta \subset T^*B \times T^*B$ be the diagonal copy of $T^*B$. The variety $Z_\Delta$ gets identified with $T^*_\mathcal{B}_\Delta$, the conormal bundle to the diagonal $\mathcal{B}_\Delta \subset B \times B$. This yields the following canonical isomorphisms of $R(G)$-algebras

$$K^G(Z_\Delta) = K^G(T^*_\mathcal{B}_\Delta) \simeq K^G(\mathcal{B}_\Delta) \simeq K^G(G/B) \simeq R(T),\tag{44}$$

where the second isomorphism is the Thom isomorphism (cf. [CG, Lemma 5.4.9]), and the last one is the induction isomorphism (cf. [CG, Lemma 6.1.6]).

The following result is a $G$-equivariant extension of the geometric construction of $\mathcal{Z}[W]$ given in Section 6.
Theorem 11.4 (see [CG, Theorem 7.2.2]) There is a natural algebra isomorphism $K^G(Z) \simeq Z[\tilde{W}]$ making the following diagram commute

$$
\begin{align*}
K^G(Z_{\Delta}) & \leftrightarrow K^G(Z) \\
\downarrow \cong & \downarrow \cong \\
R(T) & \leftrightarrow Z[\tilde{W}].
\end{align*}
$$

\[ \square \]

Affine Hecke algebras

Notice that our picture has an extra symmetry: the group $\mathbb{C}^*$ acts on $\mathcal{N}$, and also on $T^*(\mathcal{B} \times \mathcal{B})$, along the fibers, by the formula $\mathbb{C}^* \ni z : x \mapsto z^{-1}.x$. Then, the Steinberg variety $Z = T^*\mathcal{B} \times \mathcal{N} T^*\mathcal{B}$ is a $G \times \mathbb{C}^*$ stable subvariety of $T^*(\mathcal{B} \times \mathcal{B})$ and we can consider $K^{G \times \mathbb{C}^*}$-theory of $Z$.

Note that any irreducible representation of $\mathbb{C}^*$ has the form $z \mapsto z^m$ for some $m \in \mathbb{Z}$. Therefore we have the natural ring isomorphism

$$
R(T) \simeq Z[q, q^{-1}],
$$

where $q$ is the tautological representation $q : \mathbb{C}^* \to \mathbb{C}^*$ given by the identity map. One can prove the following “$G \times \mathbb{C}^*$-counterpart” of (44)

$$
K^{G \times \mathbb{C}^*}(Z_{\Delta}) \simeq R(T \times \mathbb{C}^*) \simeq R(T)[q, q^{-1}].
$$

Theorem 11.5 (see [CG, Theorem 7.2.5]) There is a natural algebra isomorphism $K^{G \times \mathbb{C}^*}(Z) \simeq H$ making the following diagram commute

$$
\begin{align*}
K^{G \times \mathbb{C}^*}(Z_{\Delta}) & \leftrightarrow K^{G \times \mathbb{C}^*}(Z) \\
\downarrow \cong & \downarrow \cong \\
R(T)[q, q^{-1}] & \leftrightarrow H. 
\end{align*}
$$

\[ \square \]

Remarks

(1) For any $G \times \mathbb{C}^*$-variety $M$, the group $K^{G \times \mathbb{C}^*}(M)$ is a module over $K^{G \times \mathbb{C}^*}(pt) = R(G \times \mathbb{C}^*)$, the representation ring of $G \times \mathbb{C}^*$. Restriction to $T \times \mathbb{C}^*$ gives a ring isomorphism $R(G \times \mathbb{C}^*) \simeq R(T)^W[q, q^{-1}]$ where the RHS stands for $W$-invariants in $R(T)[q, q^{-1}]$ (here $q$ comes from the representation ring of $\mathbb{C}^*$). This ring $R(T)^W[q, q^{-1}]$ gets identified, via the second definition of $H$, with a subalgebra of $\tilde{H}$. One can prove [CG, Proposition 7.1.14] that it coincides with $Z(\tilde{H})$, the center of $\tilde{H}$.

(2) Recall that the restriction to the Steinberg variety $Z \subset T^*\mathcal{B} \times T^*\mathcal{B}$ of either of the two projections $T^*\mathcal{B} \times T^*\mathcal{B} \to T^*\mathcal{B}$ is proper. Thus, the convolution product yields a $K^{G \times \mathbb{C}^*}(Z)$-module structure on $K^{G \times \mathbb{C}^*}(T^*\mathcal{B})$. Recall that $K^{G \times \mathbb{C}^*}(T^*\mathcal{B}) \simeq R(T)[q, q^{-1}]$. Hence, for any $s_\alpha \in W$ convolution action of the class in $K^{G \times \mathbb{C}^*}(Z)$, corresponding to the element $T_{s_\alpha}$ via Theorem 11.5, gives an operator $\hat{T}_{s_\alpha} \in \text{End}_{Z[q, q^{-1}]} R(T)[q, q^{-1}]$ (cf. Definition 11.2). One can find the following explicit formula for the action of $\hat{T}_{s_\alpha}$

$$
\hat{T}_{s_\alpha} : Y_\lambda \mapsto \frac{Y_\lambda - Y_{s_\alpha(\lambda)}}{Y_\lambda - 1} - q \frac{Y_\lambda - Y_{s_\alpha(\lambda) + \alpha}}{Y_\lambda - 1}.
$$

(45)
This formula, discovered by Lusztig, was a starting point of the K-theoretic approach to Hecke algebras.

(3) Theorem 11.5 implies Deligne-Langlands-Lusztig conjecture for \( H \), see [CG], [KL1].

(4) A Fourier transform argument does not work for Theorems 11.2 and 11.5.

One can give a description of finite-dimensional irreducible complex representations of \( H \) similar to that of the Springer Theorem for \( W \). Firstly, by Remark (1) above, we have a canonical algebra isomorphism \( Z(H) \simeq R(G \times \mathbb{C}^*) \). On any irreducible representation the center \( Z(H) \) of \( H \) acts via an algebra homomorphism \( Z(H) \to \mathbb{C} \), due to Schur’s lemma. Any such homomorphism may be identified [CG, 8.1] with the evaluation homomorphism sending a character \( z \in R(G \times \mathbb{C}^*) \) to \( z(a) \), the value of \( z \) at a semisimple element \( a = (s, q) \in G \times \mathbb{C}^* \).

In particular, the indeterminate \( q \) specializes to a complex number \( q \in \mathbb{C}^* \).

Given a semisimple element \( a = (s, q) \in G \times \mathbb{C}^* \), let \( C_a \) be the 1-dimensional complex vector space \( \mathbb{C} \) viewed as a \( Z(H) \), equivalently \( R(G \times \mathbb{C}^*) \)-module, via the action

\[
R(G \times \mathbb{C}^*) \times \mathbb{C} \to \mathbb{C}, \quad (z, x) \mapsto z(a) \cdot x,
\]

where \( z \mapsto z(a) \) is the corresponding evaluation homomorphism at \( a \).

**Definition 11.6** The tensor product \( H_a := C_a \otimes_{Z(H)} H \) is called the Hecke algebra **specialized** at \( a \).

Thus, for any simple \( H \)-module \( M \) there exists a semisimple element \( a = (s, t) \in G \times \mathbb{C}^* \), such that the action of \( H \) factors through an action of the specialized Hecke algebra \( H_a \).

From now on we will fix \( a \) and consider representations of \( H_a \).

Recall [CG, 6.2] that \( a \) acts on \( N \) by the formula \((s, q) : x \mapsto q^{-1} \cdot sx \cdot s^{-1} \) (note the **inverse** power of \( q \)) and this action agrees with the action of \( a \) on \( N = T^*B \), given by a similar formula. Denote by \( \hat{N}^a \), \( N^a \) and \( Z^a \) the corresponding \( a \)-fixed point subvarieties. The variety \( \hat{N}^a \) is smooth due to [CG, Lemma 5.11.1], since \( \hat{N} \) is smooth. Observe further that we have \( Z^a = \hat{N}^a \times_{\hat{N}} \hat{N}^a \). Therefore \( Z^a \) may be viewed as a subvariety in \( \hat{N}^a \times \hat{N}^a \) such that

\[ Z^a \circ Z^a = Z^a. \]

Our general construction of Section 2 (Corollary 2.1) makes the Borel-Moore homology \( H_*(Z^a) \) an associative algebra via convolution.

**Proposition 11.7** [CG, Proposition 8.1.5] Let \( a = (s, q) \in G \times \mathbb{C}^* \) be a semisimple element. Then there is a natural algebra isomorphism

\[ H_a \simeq H_*(Z^a, \mathbb{C}). \]

**Remark on the proof of Proposition 11.7** Let \( A \) be the closed subgroup of \( G \times \mathbb{C}^* \) generated by \( a \), that is, the algebraic closure in \( G \times \mathbb{C}^* \) of the cyclic group \( \{a^n, n \in \mathbb{Z}\} \). Clearly \( A \) is an abelian reductive subgroup of \( G \times \mathbb{C}^* \), and we have \( Z^a = Z^A \). The isomorphism of Proposition 11.7 is constructed as a composite of the following chain of algebra isomorphisms (cf. [CG, 8.1.6])

\[
C_a \otimes_{Z(H)} H \xrightarrow{\sim} C_a \otimes_{R(G \times \mathbb{C}^*)} K^{G \times \mathbb{C}^*}(Z) \simeq C_a \otimes_{R(A)} K^A(Z) \\
\xrightarrow{r_a} C_a \otimes K^A(Z^A) \overset{ev}{\simeq} K_C(Z^A) \xrightarrow{RR} H_*(Z^A, \mathbb{C}) = H_*(Z^a, \mathbb{C}).
\]
The first isomorphism here is given by Theorem 11.5 and the first remark after it, the second is given by the restriction property for \( A \subset G \times \mathbb{C}^* \) (cf. [CG, section 4.2(6)]). The third map is given by the algebra homomorphism \( r_a \) of localization (cf. [CG, Theorem 5.11.10]). The fourth map \( \text{ev} : \mathbb{C}_a \otimes_{R(A)} K^A(Z^A) \cong \mathbb{C}_a \otimes_{R(A)} (R(A) \otimes K(Z^A)) \rightarrow K_C(Z^A) \) is the evaluation map sending \( 1 \otimes f \otimes |F| \) to \( f(a) \otimes |F| \) where \( f \in R(A) \) is viewed as a character function on \( A \) and \( |F| \in K^A(Z^A) \). The last isomorphism is the map \( RR \) given by the bivariant Riemann-Roch theorem (cf. [CG, Theorem 5.11.11]). \( \square \)

We will construct, for each semisimple \( a = (s,q) \in G \times \mathbb{C}^* \), a complete collection of simple \( H_a \)-modules, which will yield a complete collection of simple \( H \)-modules as \( a \) runs over all semisimple conjugacy classes in \( G \times \mathbb{C}^* \).

To that end, consider the map \( \mu : \widetilde{\mathcal{N}}^a \rightarrow \mathcal{N}^a \), the restriction of the Springer resolution to the fixed point varieties. Explicitly, we have

\[
\mathcal{N}^a = \{ x \in \mathcal{N} | sx \cdot s^{-1} = q \cdot x \}, \quad \widetilde{\mathcal{N}}^a = \{ (x,b) \in \mathcal{N}^a \times B^a | x \in b \}.
\]

Let \( x \in \mathcal{N}^a \). The fiber \( \mu^{-1}(x) \subset \widetilde{\mathcal{N}}^a \) may be identified via the projection \( \mathcal{N}^a \rightarrow B \), \( (x,b) \mapsto b \), with the subvariety \( B_x^a \subset B \) of all Borel subalgebras simultaneously fixed by \( s \) and \( x \).

**Remark** The variety \( B_x^a \) is non-empty.

**Proof** Recall that \( a = (s,q) \) and the relation \( sx \cdot s^{-1} = q \cdot x \) holds. Let \( u = \exp(z \cdot x) \in G \), \( z \in \mathbb{C} \). Then \( u \) is a unipotent element of \( G \) and clearly \( sus^{-1} = \exp(z \cdot q \cdot x) \). We see that the elements \( s \) and \( \exp(z \cdot x) \), \( z \in \mathbb{C} \), generate a solvable subgroup of \( G \). Hence there exists a Borel subgroup \( B \) containing this solvable subgroup. It follows that its Lie algebra is in \( B_x^a \). \( \square \)

By our general construction in the end of Section 2, the Borel-Moore homology of the fibers of the map \( \mu : \widetilde{\mathcal{N}}^a \rightarrow \mathcal{N}^a \) have a natural \( H_a(Z^a) \)-module structure via convolution. Hence, for any \( x \in \mathcal{N}^a \), we get an \( H_a(Z^a) \)-action on \( H_*(B_x^a) \). Further, let \( G(s,x) \) be the simultaneous centralizer in \( G \) of \( s \) and \( x \), and let \( G(s,x)^\circ \) be the connected component of the identity. As at the end of Section 6, we define \( A(s,x) = G(s,x)/G(s,x)^\circ \) and notice that the action of \( G(s,x) \) on \( B_x^a \) induces an action of \( A(s,x) \) on \( H_*(B_x^a) \). We write \( A(s,x)^\vee \) for the set of all irreducible representations of \( A(s,x) \) that occur with non-zero multiplicity in the homology of \( B_x^a \).

**Remark** Notice that in Section 6 we used only the top homology to define \( A^\vee \) and to state the analogue of Proposition 11.7. However, here we use all homology groups since Proposition 11.7 arises from the K-theoretic Theorem 11.5, and the K-groups are not graded by dimension.

The variety \( \mathcal{N}^a \) is stable under the adjoint action of the group \( G(s) \), the centralizer of \( s \) in \( G \). Moreover, \( \mathcal{N}^a \) is known [CG, Proposition 8.1.17] to be a finite union of \( G(s) \)-orbits.

Now we can apply the direct image decomposition (46) to the morphism \( \mu : \widetilde{\mathcal{N}}^a \rightarrow \mathcal{N}^a \), to get

\[
\mu_* \mathcal{C}_{\widetilde{\mathcal{N}}^a} = \bigoplus_{\phi} L_{\phi}(k) \otimes IC_{\phi}[k],
\]

where \( \phi \) runs over the set of pairs: \( (G(s) \text{-orbit in } \mathcal{N}^a, G(s) \text{-equivariant irreducible local system on this orbit}) \). Choosing a base point, we can write each \( G(s) \)-orbit in the form \( G(s) \cdot x \), \( x \in \mathcal{N}^a \); then giving an equivariant irreducible local system on \( G(s) \cdot x \) amounts to giving an
irreducible representation \( \chi \) of the group \( A(s, x) \). Thus, Theorem 5.2 says that the multiplicity spaces \( L_\phi = \bigoplus_{k \in \mathbb{Z}} L_\phi(k) \), where \( \phi = (G(s) \cdot x, \chi) \), form a complete collection of simple \( H_*(Z^a, \mathbb{C}) \)-modules, to be denoted \( L_{a,x,\chi} := L_\phi \). We will now specify more precisely the range of the parameters \( (a, x, \chi) \) labelling the isomorphism classes of non-zero modules \( L_{a,x,\chi} \) occurring in this parametrization.

We say that two pairs \( (x, \chi) \) and \( (x', \chi') \) are \( G(s) \)-conjugate if there is a \( g \in G(s) \) such that \( x' = g x g^{-1} \) and conjugation by \( g \) intertwines the \( A(s, x) \)-module \( \chi \) with the \( A(s, x') \)-module \( \chi' \). Write \( M \) for the set of \( G \)-conjugacy classes of the triple data

\[
M = \{ a = (s, q) \in G \times \mathbb{C}^*, x \in N^a, \chi \in A(s, x)^\vee \mid s \text{ is semisimple} \}/\text{Ad}G
\]

The main result on representations of \( H \) is now deduced from Theorem 5.2 and reads as follows:

**Theorem 11.8 (cf. [CG, 8.1.13-16] and [KL1])** Assume that \( q \in \mathbb{C}^* \) is not a root of unity. Then

(i) Two modules \( L_{a,x,\chi} \) and \( L_{a,x',\chi'} \) are isomorphic if and only if the pairs \( (x, \chi) \) and \( (x', \chi') \) are \( G(s) \)-conjugate to each other.

(ii) \( L_{a,x,\chi} \) is non-zero simple \( H_*(Z^a) \)-module, for any \( (a, x, \chi) \in M \).

(iii) The collection \( \{ L_{a,x,\chi} \}_{(a,x,\chi) \in M} \) is a complete set of irreducible \( H \)-modules such that \( q \) acts by \( q \). \( \square \)

**Remark**

(i) Theorem 11.8 fails if \( q \) is a root of unity. In fact, in this case some of the modules \( L_{a,x,\chi} \) may be zero. The classification of Theorem 11.8(iii) was first obtained in [KL1] in a different way, see [CG, Introduction] for more historical remarks.

(ii) Note that the morphism \( \mu : \bar{N}^a \to N^a \) is not semi-small and, as a result (cf. Proposition 5.1), the algebra \( H_a \) is not semisimple.

**Quantized loop algebra of \( \mathfrak{gl}_n \)**

Recall briefly the setup of Section 7. We have a variety \( N \) of endomorphisms \( x : \mathbb{C}^d \to \mathbb{C}^d \) that satisfy \( x^n = 0 \), a smooth variety \( \mathcal{F} \) of \( n \)-step partial flags in \( \mathbb{C}^d \), and a (semi-small) map \( \mu : M = T^*\mathcal{F} \to N \). Denote as usual \( Z = M \times_N M \).

The group \( GL_d \times \mathbb{C}^* \) acts on \( M \) and \( N \). Since the map \( \mu : M \to N \) is \( GL_d \times \mathbb{C}^*-\)equivariant, the variety \( Z \) is a \( GL_d \times \mathbb{C}^*-\)stable subvariety of \( M \times M \) and we can consider \( K^{GL_d \times \mathbb{C}^*}(Z) \).

To describe an algebraic object arising from equivariant K-theory of \( Z \), consider the loop Lie algebra \( \mathfrak{L}_q = \mathfrak{gl}_n \otimes_{\mathbb{C}} \mathbb{C}[u,u^{-1}] \) which may be thought of as the space of polynomial maps \( \mathbb{C}^* \to \mathfrak{gl}_n \). One can define (cf. [Dr1], [CP]) a deformation \( \mathcal{U}_q(\mathfrak{L}_q) \) of the universal enveloping algebra \( \mathcal{U}(\mathfrak{L}_q) \). We will call this deformation the quantized loop algebra of \( \mathfrak{gl}_n \).

We assume, throughout, that \( q \) is not a root of unity. The following theorem is announced in [GV] and proved in [V]:

**Theorem 11.9** There exists a surjective algebra homomorphism

\[
\mathcal{U}_q(\mathfrak{L}_q) \to \mathbb{C} \otimes_{\mathbb{Z}} K^{GL_d \times \mathbb{C}^*}(Z). \quad \square
\]

Recall that the algebra \( \mathcal{U}_q(\mathfrak{L}_q) \) has generators \( E_{i,r}, F_{i,r}, K_i, i = 1, \ldots, n, r \in \mathbb{Z} \) and \( H_{i,r}, i = 1, \ldots, n, r \in \mathbb{Z} \setminus \{0\} \) subject to a certain explicit set of relations (cf. [CP]). Recall further
that there is a tensor product decomposition as a vector space

$$U_q(Lg) = U^+ \otimes_{\mathbb{C}[q; q^{-1}]} U^0 \otimes_{\mathbb{C}[q; q^{-1}]} U^-,$$

where $U^+, U^0, U^-$ are the subalgebras generated by $\{E_{i,r}\}$, $\{K_i, H_{i,r}\}$ and $\{F_{i,r}\}$, respectively.

$U^0$ is a large commutative subalgebra of $U_q(Lg)$. A finite-dimensional irreducible representation $V$ of $U_q(Lg)$ is said to be of type 1 if $K_1, \ldots, K_n$ act semisimply on $V$ with eigenvalues which are half-integer powers of $q$, the specialization of $q$-action in $V$.

We say that $v \in V$ is a pseudo-highest weight vector if it is annihilated by $U^+$ and is a weight vector for $U^0$. We write $k_i(V), h_{i,r}(V), i = 1, \ldots, n, r \in \mathbb{Z} \setminus \{0\}$, for the corresponding eigenvalues of the elements $K_i, H_{i,r}$, and call the collection $\{k_i(V), h_{i,r}(V)\}$ quasi-highest weight. The following result about irreducible finite-dimensional representations of $U_q(Lg)$ is essentially due to Drinfel'd [Dr2], see [CP, Theorems 12.2.3 and 12.2.6].

**Proposition 11.10**

(i) Any finite-dimensional simple $U_q(Lg)$-module $V$ of type 1 has a unique quasi-highest weight vector.

(ii) Two simple modules of type 1 are isomorphic iff their quasi-highest weights are the same.

(iii) A collection $\{k_i, h_{i,r}\}$ is the quasi-highest weight of a finite-dimensional irreducible representation $V$ iff there exist unique monic polynomials $P_{i,V} \in \mathbb{C}[u], i = 1, \ldots, n$, all with non-zero constant term, such that, setting $d_i = \deg(P_{i,V})$, we have:

$$k_i \exp((q^{1/2} - q^{-1/2}) \sum_{s=1}^{\infty} h_{i,s} u^s) = q^{\frac{d_i}{2}} \frac{P_{i,V}(q^{-1/2} u)}{P_{i,V}(u)} = k_i^{-1} \exp((q^{-1/2} - q^{1/2}) \sum_{s=1}^{\infty} h_{i,-s} u^{-s})$$

in the sense that the left- and right-hand sides are the Laurent expansions of the middle term about 0 and $\infty$, respectively.

Moreover, the polynomials $P_{i,V}$ define the representation $V$ uniquely, and every $n$-tuple $(P_i)_{i=1,\ldots,n}$ of monic polynomials with non-zero constant term arises from a finite-dimensional irreducible $U_q(Lg)$-module of type 1 in this way. $\square$

The collection $\{P_{i,V}, i = 1, \ldots, n\}$ is called the Drinfeld polynomials associated to the irreducible representation $V$.

For any pair $a = (s, q), s \in GL_d(\mathbb{C}), q \in \mathbb{C}^*$, we consider the fixed point variety $Z^a$ and construct as in Proposition 11.6 a surjection $\mathbb{C} \otimes K^{GL_d(\mathbb{C}) \times \mathbb{C}^*}(Z) \to H_s(Z^a)$. As in the case of affine Hecke algebras we obtain from Theorem 5.2 a complete set of irreducible finite dimensional $H_s(Z^a)$-modules $L_{a,x}$ labeled by pairs $(a, x)$, where $a = (s, q)$ is a semisimple element of $GL_d \times \mathbb{C}^*$ and $x \in N$ (as in Lemma 7.3, the monodromy action is trivial, hence $L_{a,x}$ does not depend on the third parameter $\chi$).

Observe that every simple $K^{GL_d(\mathbb{C}) \times \mathbb{C}^*}(Z)$-module can be pulled back via the surjection $U_q(Lg) \to \mathbb{C} \otimes Z K^{GL_d \times \mathbb{C}^*}(Z)$ of Theorem 11.8 to give a simple $U_q(Lg)$-module. Now we are ready to identify the modules $L_{a,x}$ as representations of $U_q(Lg)$.

Fix $(a, x)$ such that $a = (s, q) \in GL_d \times \mathbb{C}^*$, where $q$ is not a root of unity, $x \in N$, $s$ is semisimple and $sxs^{-1} = qx$. Recall that we have defined before Lemma 7.5 two $n$-step flags $F^{\text{min}}(x)$ and $F^{\text{max}}(x)$ in $\mathbb{C}^d$. Since $sxs^{-1} = qx$, the flags $F^{\text{min}}(x)$ and $F^{\text{max}}(x)$ are both preserved by $s$ and we can consider, for each $i = 1, \ldots, n$, the action of $s$ on $F^{\text{max}}(x)/F^{\text{min}}(x)$. 

Let $L_{a,x}$ be the $K^{GL_d \times C^*}(Z)$-module viewed, due to the Theorem 11.8, as a $U_q(L_g)$-module. The theorem below is a quantized version of the Springer theorem for $U(sl_n)$ given in section 7.

**Theorem 11.11** The $i$-th Drinfeld polynomial, $P_{i,L_{a,x}}$, is equal to the characteristic polynomial of the $s$-action on $F_{i,\max}(x)/F_{i,\min}(x)$, i.e.:

$$P_{i,L_{a,x}}(u) = \det(u \cdot Id - s; F_{i,\max}(x)/F_{i,\min}(x))$$

for all $i = 1, 2, \ldots, n$.

In particular, every irreducible finite-dimensional $U_q(L_g)$-module of type 1 is of the form $L_{a,x}$. □

### 12 Equivariant cohomology and degenerate versions

In this last section we will study degenerate versions of Hecke algebras and quantized enveloping algebras as the deformation parameter $q \to 1$. Of course, in the limit $q = 1$, the algebras in question reduce to their classical counterparts: the Hecke algebra specializes to the group algebra of the corresponding Weyl group and the quantized enveloping algebra specializes to the corresponding classical enveloping algebra. We will consider however another, more interesting limit, which corresponds in a sense to taking the “first derivative” with respect to the deformation parameter at $q = 1$, rather than the value at $q = 1$ itself. We will see that taking “first derivative at $q = 1$” amounts geometrically to replacing equivariant K-theory by equivariant cohomology.

**The degenerate Hecke algebra**

Let $\epsilon$ be an indeterminate. Write $C[h, \epsilon]$ for polynomials in $\epsilon$ with coefficients in the ring $C[h]$ of polynomial functions on $h$.

**Definition 12.1** The degenerate affine Hecke algebra $H_{deg}$ of $G$ is the unital associative free $C[\epsilon]$-algebra defined by the following properties:

(i) $H_{deg} \simeq C[W] \otimes C[h, \epsilon]$ as a $C$-vector space.

(ii) the subspaces $C[W]$ and $C[h, \epsilon]$ are subalgebras of $H_{deg}$.

(iii) the following relations hold in $H_{deg}$:

$$s_i \lambda - s_i(\lambda)s_i = -\epsilon \cdot \langle \alpha_i^\vee, \lambda \rangle, \quad i = 1, \ldots, l, \quad \lambda \in h^* \subset C[h].$$

The algebra $H_{deg}$ has a natural grading defined by $\text{deg}(s_i) = 0$, $\text{deg}(\alpha) = 1$, $\text{deg}(\epsilon) = 1$.

**Remarks**

(1) Sometimes in the definition of the degenerate affine Hecke algebra one takes a quotient modulo relation $\epsilon = 1$. In fact, all algebras with $\epsilon$ specialized to a non-zero complex number are isomorphic. We prefer the homogeneous version above since we want to relate it to the equivariant cohomology with its natural grading.

(2) The degenerate affine Hecke algebra $H_{deg}$ can be obtained from the affine Hecke algebra $H$ by the following procedure: we formally make substitution $q \to \exp(2\epsilon)$, $Y_\lambda \to \exp(\epsilon \lambda)$, $T_{\alpha} \to s_\alpha$ in the relations defining the affine Hecke algebra (cf. Definition 11.2), and then take the homogeneous components of the minimal degree with respect to the grading above. This last step is sometimes expressed as “taking $\lim_{\epsilon \to 0}$".
Comparing the defining relations in $H^\wedge$

where the RHS denotes the associated graded algebra of $H$

Proposition 12.3

Recall that algebraic group homomorphisms $\lambda \in X^*(T)$ clearly form a $\mathbb{Z}$-basis of the representation ring $R(T)$, that is, $R(T)$ may be identified with the group algebra of the lattice $X^*(T)$. The Weyl group $W$ acts naturally on $R(T)$ and on $\mathbb{C}[h]$; we write $P \mapsto w(P)$ for the action of $w \in W$.

To each simple reflection $s_\alpha \in W$ we have associated in $[\mathcal{H}]$ a $\mathbb{Z}[q, q^{-1}]$-linear map $\tilde{T}_\alpha : R(T)[q, q^{-1}] \to R(T)[q, q^{-1}]$ given by the Demazure-Lusztig formula.

Similarly, we define a $\mathbb{C}[\varepsilon]$-linear map $S_\alpha : \mathbb{C}[h, \varepsilon] \to \mathbb{C}[h, \varepsilon]$ by the formula, due to [BGG]:

$$S_\alpha : P \mapsto s_\alpha(P) + \varepsilon \frac{s_\alpha(P) - P}{\alpha}, \quad P \in \mathbb{C}[h].$$

Proposition 12.2 (see [Dr2], [Lu3], [CG, Theorem 7.2.16])

(i) The $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\text{End}_{\mathbb{Z}[q, q^{-1}]} R(T)[q, q^{-1}]$ generated by the operators $\{\tilde{T}_\alpha, \alpha \text{ simple root}\}$ and by all the multiplication operators $P \mapsto f \cdot P$, $f \in R(T)$, is isomorphic to the affine Hecke algebra $H$.

(ii) The $\mathbb{C}[\varepsilon]$-subalgebra of $\text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[h, \varepsilon]$ generated by the operators $\{S_\alpha, \alpha \text{ simple root}\}$ and by all the multiplication operators $P \mapsto f \cdot P$, $f \in \mathbb{C}[h]$, is isomorphic to the degenerate affine Hecke algebra $H_{\text{deg}}$. $\square$

Using this proposition we will frequently identify $H, H_{\text{deg}}$ with the corresponding algebras of operators.

Let $C[T] = C \otimes_{\mathbb{Z}} R(T)$ be the algebra of regular functions on $T$. Pull-back via the exponential map $\exp : h \to T$ gives an embedding $\exp^* : C[T] \hookrightarrow C[h]$. Similarly, one gets an embedding $\exp^* : C[T \times C^*] = C[T][q, q^{-1}] \hookrightarrow C[h, \varepsilon]$ (here $\varepsilon$ is viewed as a base vector in $\text{Lie } C^*$ and the map $\exp^*$ takes $q$ to $\exp(\varepsilon) = \sum \frac{\varepsilon^k}{k!}$). Using the latter embedding and Proposition 12.2 we may (and will) view $H$ as a subalgebra of $\text{End}_{\mathbb{C}[\varepsilon]} C[[h, \varepsilon]]$ and form the $\mathbb{C}[\varepsilon]$-algebra $\widehat{H} := C[[\varepsilon]] \otimes_{\mathbb{Z}[q, q^{-1}]} H$.

Let $I$ denote the augmentation ideal in $C[[h, \varepsilon]]$. The powers $I^n, n = 1, 2, \ldots$, form the decreasing $I$-adic filtration on $C[[h, \varepsilon]]$. We define a decreasing $\mathbb{Z}$-filtration $F^\bullet$ on $\text{End}_{\mathbb{C}[\varepsilon]} C[[h, \varepsilon]]$ as follows

$$F^j = \{u \mid u(I^n) \subset I^{n+j}, \text{ for all } n > -j\}, \quad j \in \mathbb{Z}.$$

We will write $F^\bullet \widehat{H}$ for the induced filtration on the subalgebra $\widehat{H} \subset \text{End}_{\mathbb{C}[\varepsilon]} C[[h, \varepsilon]]$. Comparing the defining relations in $H$ and $H_{\text{deg}}$, one deduces from Proposition 12.2 the following

Proposition 12.3 There is a graded algebra isomorphism

$$H_{\text{deg}} \simeq \text{gr}_F \widehat{H},$$

where the RHS denotes the associated graded algebra of $\widehat{H}$. $\square$

Note that although filtration terms $F^j \text{End}_{\mathbb{C}[\varepsilon]} C[[h, \varepsilon]]$ are non-zero for both positive and negative values of $j$, formula (47) shows that the induced filtration $F^j \widehat{H}$ has non-zero terms only for $j \geq 0$. 
Nil-Hecke algebra

Let \( \text{Nil} \) be the \( \mathbb{C} \)-algebra with base \( \{ r_w, w \in W \} \) and the following multiplication rules:

(i) \( r_w \cdot r_{w'} = r_{ww'} \) whenever \( l(w) + l(w') = l(ww') \),

(ii) \( r_w \cdot r_{w'} = 0 \) if \( l(w) + l(w') > l(ww') \).

Note that \( r_e \) is the unit of the algebra \( \text{Nil} \).

**Definition 12.4** The *nil-Hecke algebra* \( H_{nil} \) is the unital associative algebra defined by the following properties:

1. \( H_{nil} = \text{Nil} \otimes \mathbb{C}[h] \) as a \( \mathbb{C} \)-vector space;
2. The natural embeddings \( \text{Nil} \hookrightarrow H_{nil} \) and \( \mathbb{C}[h] \hookrightarrow H_{nil} \) are algebra homomorphisms;
3. For any simple root \( \alpha \in \mathfrak{h}^* \) and any linear function \( \lambda \in \mathfrak{h}^* \subset \mathbb{C}[h] \) the following relations hold in \( H_{nil} \):

\[
 r_{s_{\alpha}} \cdot \lambda - s_{\alpha}(\lambda) \cdot r_{s_{\alpha}} = -⟨\lambda, \alpha^\vee⟩. 
\]

To formulate an analogue of Proposition 12.2, consider for each simple root \( \alpha \) the linear operator \( R_{\alpha} : \mathbb{C}[h] \rightarrow \mathbb{C}[h] \) given by:

\[
 R_{\alpha} : P \mapsto \frac{s_{\alpha}(P) - P}{\alpha}, \quad P \in \mathbb{C}[h].
\]

**Proposition 12.5 (cf. [KK])** The subalgebra of \( \mathbb{C} \)-linear endomorphisms of \( \mathbb{C}[h] \) generated by \( \{ R_{\alpha}, \alpha \text{ a simple root} \} \) and operators of multiplication by the elements of \( \mathbb{C}[h] \), is isomorphic to the nil-Hecke algebra \( H_{nil} \). \( \square \)

**Remarks**

1. Comparison of formulas (15) and (18) shows that the algebra \( H_{nil} \) may be thought of as the limit of the affine Hecke algebra \( H \) at \( q \rightarrow 0 \). Note that since the variable \( q \) is invertible in \( H \) one cannot simply set \( q \) equal to 0.

2. The algebra \( H_{nil} \) has a natural \( \mathbb{Z} \)-grading defined on generators by

\[
 \text{deg } r_w = -l(w), \quad \text{deg } \lambda = 1 \quad \text{for all } \lambda \in \mathfrak{h}^* \subset \mathbb{C}[h].
\]

There is also an interpretation of the nil-Hecke algebra as an associated graded algebra, analogous to Proposition 12.3. To explain it, fix a complex number \( q \in \mathbb{C}^*, q \neq 1 \). Specializing in the Demazure-Lusztig formula (15) and in Proposition 12.2, the variable \( q \) to \( q \) we obtain an action of \( H_q \), the specialized affine Hecke algebra, on the \( \mathbb{C} \)-vector space \( \mathbb{C}[T] \). Using the embedding \( \mathbb{C}[T] \hookrightarrow \mathbb{C}[[h]] \) induced by the exponential map \( : \mathfrak{h} \rightarrow T \) we view \( H_q \) as a subalgebra of the \( \mathbb{C} \)-algebra \( \text{End}_\mathbb{C}\mathbb{C}[[h]] \) (a “specialized” analogue of Proposition 12.3 with \( q = q \) and no “\( e \)” at all). Following the same pattern as before Proposition 12.3, we endow \( \mathbb{C}[[h]] \) with \( J \)-adic filtration, where \( J \) is the augmentation ideal in \( \mathbb{C}[[h]] \). We further introduce a decreasing filtration \( F^j, j \in \mathbb{Z} \), on \( \text{End}_\mathbb{C}\mathbb{C}[[h]] \) by the formula

\[
 F^j \text{End}_\mathbb{C}\mathbb{C}[[h]] = \{ u \mid u(J^k) \subset J^{k+j}, \text{ for all } k > j \}.
\]

and let \( F^j H_q := H_q \cap F^j \text{End}_\mathbb{C}\mathbb{C}[[h]] \) be the induced filtration on \( H_q \). Note that unlike the filtration \( F^* H \) considered before Proposition 12.3, the filtration \( F^j H_q \) is non-trivial for both negative and positive values of \( j \).

Comparison of formulas (15), (18) and Proposition 12.5 yield the following result.
**Proposition 12.6** For any $q \neq 1$ there is a graded algebra isomorphism

$$H_{nil} \simeq \text{gr}_\mathcal{F} H_q.$$  

We now turn to the geometric interpretation.

**Equivariant cohomology**

Let $M$ be a manifold equipped with smooth action of a Lie group $G$. If the action of $G$ is free, then $M/G$ is a manifold, but in general it is not. We define a space $M_G$ (the *homotopy quotient*) by replacing $M$ by another space with the same homotopy type on which $G$ acts freely. This is done by introducing a contractible space $EG$ on which $G$ acts freely and defining

$$M_G = (EG \times M)/G.$$  

$M_G$ is well defined up to homotopy equivalence.

The first and the second projections $EG \leftarrow M \times EG \to M$ induce, by passing to $G$-orbits, a double fibration

$$M_G \xrightarrow{p} M/G \xleftarrow{\pi} BG$$

where the *classifying space* $BG$ is defined as $BG = EG/G$. Here we consider $M/G$ a space of orbits which may have non-separated topology if the action of $G$ is not free (if $G$ is compact then $M/G$ is always separated, but may have singularities even for smooth $M$).

The projection $\pi : M_G \to BG$ behaves much better. It is a locally trivial fibration with fiber $M$.

We define the *equivariant cohomology* of $M$ by $H^*_G(M) = H^*(M_G)$. Clearly $H^*_G(M)$ is a module over $H^*(BG)$ via $J_1$.

If $G$ acts freely on $M$ then the map $p : M_G \to M/G$ becomes a fibration with contractible fiber $EG$, and hence

$$H^*_G(M) = H^*(M/G).$$

**Example** The action of $\mathbb{C}^*$ on $\mathbb{C}^n \setminus \{0\}$ is free. Although $\mathbb{C}^n$ is not contractible, we may form $\mathbb{C}^\infty = \{(x_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{C}, x_i = 0 \text{ for almost all } i\}$ and one shows that $\mathbb{C}^\infty \setminus \{0\}$ is a contractible space on which $\mathbb{C}^*$ acts freely. Thus, $\mathbb{C}^\infty \setminus \{0\}$ is a model for $E\mathbb{C}^*$, and $B\mathbb{C}^*$ is homotopy equivalent to $\mathbb{C}^{\mathbb{P}^\infty} = (\mathbb{C}^\infty \setminus \{0\})/\mathbb{C}^*$.

**Remarks**

1. In our applications $G$ will be a complex reductive Lie group, and for such a group the spaces $EG$ and $BG$ are necessarily infinite-dimensional. The reader who doesn’t feel comfortable with leaving the category of algebraic varieties may choose the approach of [EG] where $EG$ and $BG$ are approximated by open subspaces of complex vector spaces with linear $G$-action.

2. If $K \subset G$, $G$ reductive, is a maximal compact subgroup, then $K$ acts freely on $EG$, so we can consider $EG$ as a model for $EK$. Hence we have a map $BK = EG/K \to BG = EG/G$ which is a homotopy equivalence. Therefore, for any $G$-variety $M$, the space $M_G$ is homotopy equivalent to $M_K$, hence $H^*_K(M) \simeq H^*_G(M)$.
Let \( M \) be a smooth complex algebraic variety with an algebraic action of a linear reductive group \( G \) and let \( F \) be a \( G \)-equivariant sheaf on \( M \). We may construct a sheaf \( F_G \) on \( M_G \) as follows. First, pullback \( F \) to \( M \times EG \) via the first projection, and regard the pull-back \( pr_1^*F \) as a \( G \)-equivariant sheaf on \( M \times EG \) relative to the diagonal \( G \)-action. The action being free, the equivariant descent property [CG, 5.2.15] (cf. also the first remark above) ensures that there exists a uniquely defined sheaf \( F_G \) on \( M_G \) such that \( pr_1^*F = \pi^*F_G \).

If \( F \) is locally free we define equivariant characteristic classes of \( F \) to be the corresponding classes of \( F_G \) (viewed as a vector bundle on \( M_G \)) in \( H^*(M_G) = H^*_G(M) \). In the general case, since \( M \) is smooth, we can choose a finite \( G \)-equivariant resolution of \( F \) by locally free sheaves and define equivariant characteristic classes via the usual formulas involving the classes of these locally free sheaves.

For any finite-dimensional representation \( V \) of \( G \) we may form the associated vector bundle \( V_G \) on \( BG \) corresponding to the principal \( G \)-bundle \( EG \to BG \). The assignment \( V \mapsto V_G \) gives a ring homomorphism \( R(G) \to K(BG) \). It is known that the homomorphism is injective and \( K(BG) = \hat{R}(G) \) is the completion of the image of \( R(G) \) at the augmentation ideal. Thus if \( G \) is reductive we have

\[
H^*(BG, \mathbb{C}) \xrightarrow{\text{Chern charact.}} \mathbb{C} \otimes K(BG) = \mathbb{C} \otimes \hat{R}(G) = \mathbb{C}[G]^G \cong \mathbb{C}[\mathfrak{g}]^G,
\]

where the last isomorphism is given by pulling back \( \text{Ad} \)-invariant functions from \( G \) to \( \mathfrak{g} = \text{Lie} G \) via the exponential map \( \exp : \mathfrak{g} = \text{Lie} G \to G \).

Similarly, for most of the \( G \)-spaces we consider, e.g. for cellular fibrations ([CG, 5.5]), the Chern character gives an isomorphism \( \hat{H}_G(M) = \hat{K}_G(M) \), where \( \hat{H}_G(M) \) stands for the completion of \( H^*_G(M) \) at the augmentation ideal of the ground ring \( H^*(BG) \), i.e.

\[
\hat{H}_G(M) = \prod_{i=0}^\infty H^*_G(M).
\]

Given a smooth \( G \)-manifold \( \bar{M} \) and a closed \( G \)-stable subset \( Z \subset \bar{M} \), we define the \( G \)-equivariant cohomology of \( \bar{M} \) with support in \( Z \) by the formula

\[
H^*(Z|\bar{M}; G) := H^*_G(\bar{M}, \bar{M} \setminus Z) = H^*(\bar{M}_G; (\bar{M} \setminus Z)_G)
\]

(compare this definition with (3)).

Recall now the setup of Section 6: we have the Steinberg variety \( Z \), embedded naturally as a \( G \)-invariant subvariety in \( \bar{M} = T^*B \times T^*B \). Since \( Z \circ Z = Z \) we can repeat all our constructions with convolutions in the “relative situation over \( BG \)” (i.e. we view \( Z_G \) as a fibration over \( BG \) which has an embedding into \( \bar{M}_G \), an infinite-dimensional smooth manifold, commuting with the projection to \( BG \), etc.) to define the structure of a convolution algebra on \( H^*(Z|\bar{M}; G) \).

We are going to interpret the degenerate Hecke algebra geometrically using equivariant cohomology, in the same way as the affine Hecke algebra \( \mathbf{H} \) was described via equivariant K-theory. Identify \( H^2_{\text{geom}}(\text{pt}) = H^2(\mathbb{C}P^\infty) \) with the ring \( \mathbb{C}[\epsilon] \), so that a generator \( \epsilon \in H^2(\mathbb{C}P^\infty) \) is viewed as a degree 1 element in the polynomial ring \( \mathbb{C}[\epsilon] \).
**Theorem 12.7** Let $Z$ be the Steinberg variety of $G$ with the natural action of $G \times \mathbb{C}^*$. Then the convolution algebra $H^\bullet(Z|\mathbb{M}; G \times \mathbb{C}^*)$ is naturally isomorphic to the degenerate affine Hecke algebra $H_{\text{deg}}$.

**Remarks**

1. We write $H^\bullet(Z|\mathbb{M}; G \times \mathbb{C}^*)$ since there are no equivariant odd cohomology groups with support in $Z$, and the isomorphism of Theorem 12.7 is doubling degrees, i.e. the degree of an element in the equivariant cohomology is twice the degree of the corresponding element in $H_{\text{deg}}$.

2. Theorems 12.7, 11.5 and 11.4 allow us to express all the results on convolution algebras of the Steinberg variety $Z$ presented in these lectures, in the following diagram:

\[
\begin{array}{cccccc}
H & \xrightarrow{K^{G\times\mathbb{C}^*}(Z)} & \xrightarrow{\text{ev}^*_{G\times\mathbb{C}^*}} & \xrightarrow{H(Z|\mathbb{M}; G \times \mathbb{C}^*)} & \xrightarrow{\hat{H}(Z|\mathbb{M}; G \times \mathbb{C}^*)} & \xrightarrow{H_{\text{deg}}} \\
\downarrow_{\text{q=1}} & \downarrow_{\text{forgetting \ C^*\text{-action}}} & \downarrow_{\text{forgetting \ C^*\text{-action}}} & \downarrow_{\text{forgetting \ G\text{-action}}} & \downarrow_{\text{forgetting \ G\text{-action}}} & \downarrow_{\text{proj}} \\
\mathbb{Z}[\hat{W}] & \xrightarrow{K^G(Z)} & \xrightarrow{\text{ev}^*} & \xrightarrow{H(Z|\mathbb{M}; G \times \mathbb{C}^*)} & \xrightarrow{H(Z, \mathbb{C})} & \xrightarrow{A[W]} \\
\downarrow_{\text{proj}} & \downarrow_{\text{forgetting \ G\text{-action}}} & \downarrow_{\text{forgetting \ G\text{-action}}} & \downarrow_{\text{forgetting \ G\text{-action}}} & \downarrow_{\text{proj}} & \\
\mathbb{A}[W] & \xrightarrow{K_{\mathbb{C}}(Z)} & \xrightarrow{\text{ev}^*} & \xrightarrow{H(Z, \mathbb{C})} & \xrightarrow{A[W]} & \\
\downarrow_{\text{ev}} & \downarrow_{\text{support \ cycle}} & \downarrow_{\text{top \ homology}} & & & \\
\mathbb{C}[W] & \xrightarrow{H(Z)} & \xrightarrow{H(Z)} & \xrightarrow{\mathbb{C}[W]} & &
\end{array}
\]

where going from top to bottom leads to forgetting some amount of structure and where the following notation has been used:

- $\mathbb{A} = \mathbb{C}[T]/I_T \overset{\text{exp}}{\sim} \mathbb{C}[h]/I_\mathfrak{h}$, where $I_T = \text{ideal generated by } W\text{-invariant functions on } T \text{ vanishing at } 1$,
- $I_\mathfrak{h} = \text{ideal generated by } W\text{-invariant power series on } \mathfrak{h} \text{ vanishing at } 0$,
- $\text{ev} : \mathbb{A}[W] \rightarrow \mathbb{C}[W]$ is taking value at $1 \in T$, resp. $0 \in \mathfrak{h}$,
- $\text{exp}^* : \mathbb{C}[T]/I_T \rightarrow \mathbb{C}[h]/I_\mathfrak{h}$ is the pullback via $\text{exp} : \mathfrak{h} \rightarrow T$,
- $\text{proj} : \mathbb{R}(T) \rightarrow \mathbb{A}$, resp. $\text{proj} : \mathbb{C}[h]/I_\mathfrak{h} \rightarrow \mathbb{A}$ is a natural projection,
- $\hat{H}_{\text{deg}} = \text{completed version of } H_{\text{deg}}$, with $S^\bullet(h^*)[\epsilon]$ replaced by $\mathbb{C}[h, \epsilon]$.

**Sketch of proof of Theorem 12.7** Recall that $\mu : T^*B \rightarrow \mathcal{N}$ denotes the Springer resolution. Since $B = \mu^{-1}(0)$, we have a natural $H^\bullet(Z|\mathbb{M}; G \times \mathbb{C}^*)$-action on $H_{G\times\mathbb{C}^*}^\bullet(B) = H_{G\times\mathbb{C}^*}^\bullet(G/B) = H_{B\times\mathbb{C}^*}^\bullet(pt) = H_{T\times\mathbb{C}^*}^\bullet(pt) = \mathbb{C}[h, \epsilon]$.

This action gives an algebra homomorphism $\rho : H^\bullet(Z|\mathbb{M}; G \times \mathbb{C}^*) \rightarrow \text{End}_{\mathbb{C}[h]} \mathbb{C}[h, \epsilon]$.

One shows first, by the same argument as has been used in [CG, Claim 7.6.7] to prove a similar result in equivariant K-theory, that the map $\rho$ is injective. The isomorphism of the theorem will then follow from Proposition 12.5 provided we show that the image of $\rho$ coincides with the subalgebra of $\text{End}_{\mathbb{C}[h]} \mathbb{C}[h, \epsilon]$ described in Proposition 12.5. To prove the latter we argue as follows.
View $H^*(Z[M; G \times \mathbb{C}^*])$ as a subalgebra of $\text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[\mathfrak{h}, \varepsilon]$ via $\rho$ and recall the filtration $F^* \text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[\mathfrak{h}, \varepsilon]$ introduced after Proposition 12.2. The convolution product on $H^*(Z[M; G \times \mathbb{C}^*])$ is continuous in the topology defined by the filtration $F^*$, hence extends to the completion. This way one makes $\widehat{H}(Z[M; G \times \mathbb{C}^*]) := \prod_{i=0}^{\infty} H^i(Z[M; G \times \mathbb{C}^*])$ a filtered algebra under convolution and defines an action of this algebra on

$$\widehat{H}_{G \times \mathbb{C}^*}(B) = \prod_{i=0}^{\infty} H^i_{G \times \mathbb{C}^*}(B) = \mathbb{C}[\mathfrak{h}, \varepsilon].$$

Thus, the homomorphism $\rho : H^*(Z[M; G \times \mathbb{C}^*]) \to \text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[\mathfrak{h}, \varepsilon]$, given by the action, can be extended to a continuous algebra homomorphism

$$\widehat{\rho} : \widehat{H}(Z[M; G \times \mathbb{C}^*]) \to \text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]].$$

Further, we use an equivariant version (i.e. relative version for fibrations over $BG$) of the Bivariant Riemann-Roch Theorem [CG, Theorem 5.11.11] to construct an algebra homomorphism

$$RR : K^{G \times \mathbb{C}^*}(Z) \to \widehat{H}(Z[M; G \times \mathbb{C}^*]).$$

We now apply the isomorphism $H \simeq K^{G \times \mathbb{C}^*}(Z)$ of Theorem 11.5 so that composing the maps $RR$ and $\widehat{\rho}$ yields an algebra homomorphism

$$\widehat{\rho} \circ RR : H \simeq K^{G \times \mathbb{C}^*}(Z) \to \text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]].$$

By our construction, we see by means of Remark (2) after Theorem 11.5 that the map $\widehat{\rho} \circ RR$ above, arising from the geometric convolution action, is in effect nothing but the embedding $H \hookrightarrow \text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]]$ used in Proposition 12.3 for the definition of the filtration $F^*$ on $\widehat{H}$. In particular, the map

$$\overline{\rho} \circ RR : \widehat{H} = \mathbb{C}[[\varepsilon]] \otimes_{\mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]} H \to \text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]]$$

induced by $\rho \circ RR$ is compatible with the filtration $F^*$ on both sides. Furthermore, the isomorphism $\text{gr}_F \widehat{H} \simeq H_{\text{deg}}$ of Proposition 12.3 implies that the image of the associated graded map

$$\text{gr}_F(\rho \circ RR) : \text{gr}_F(\widehat{H}) \to \text{gr}_F(\text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]])$$

can be identified with $H_{\text{deg}}$ if we identify $\text{gr}_F(\text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]])$ with $\text{End}_{\mathbb{C}[\varepsilon]} \mathbb{C}[[\mathfrak{h}, \varepsilon]]$ in a natural way (the associated graded of $\mathbb{C}[[\mathfrak{h}, \varepsilon]]$ with respect to the $I$-adic filtration is equal to $\mathbb{C}[[\mathfrak{h}, \varepsilon]]$). Also, one clearly has $\text{gr}_F \widehat{H}(Z[M; G \times \mathbb{C}^*]) \simeq H^*(Z[M; G \times \mathbb{C}^*]).$

Thus we conclude:

$$\text{Image}(\rho) = \text{Image} \left( \text{gr}_F(\rho) \right) \overset{\alpha}{\hookrightarrow} \text{Image} \left( \text{gr}_F(\rho \circ RR) \right) = H_{\text{deg}}.$$

To complete the proof of the theorem it suffices to show that the inclusion $\alpha$ above is in fact an equality. This follows from the fact that the map $RR$ has dense image, which is proved by the same argument as was used in [CG, Theorem 6.2.4] to show that the non-equivariant Riemann-Roch map $\mathbb{C} \otimes_{\mathbb{Z}} K(Z) \to H_*(Z)$ is a bijection. $\Box$
Geometric realization of the Nil-Hecke algebra

The algebra construction of Corollary 2.1 in the special case of \( M = B, N = pt \) (see Example (ii) after Corollary 2.2) makes \( H_*(B \times B) \) an associative algebra with convolution product, and makes \( H_*(B) \) a simple \( H_*(B \times B) \)-module. Since \( B \) is smooth we have by Poincaré duality \( H_*(B) \cong H^*(B) \). Applying further the Borel isomorphism we may identify \( H_*\left( B \right) \) with \( \mathbb{C}[h]/I_h \), where \( I_h \) is the ideal generated by \( W \)-invariant polynomials without constant term.

Recall that \( G \)-diagonal orbits on \( B \times B \) are in canonical bijection with the Weyl group, \( W \); we write \( [O_w] \in H_*(B \times B) \) for the fundamental class of the closure, \( \overline{O_w} \), of the orbit corresponding to \( w \in W \). The following result is essentially due to [BGG].

**Proposition 12.8**

(i) The classes \( [O_w], w \in W, \) span a \( \#W \)-dimensional subalgebra in the convolution algebra \( H_*(B \times B) \). Moreover, the map \( r_w \mapsto [O_w], w \in W, \) gives an isomorphism of this subalgebra with the algebra \( \text{Nil} \).

(ii) For any simple reflection \( s_\alpha \), the action of the class \( [O_{s_\alpha}] \) in the \( H_*(B \times B) \)-module \( H_*(B) \cong \mathbb{C}[h]/I_h \) is given by the operator \( R_\alpha \) defined in (48). ✷

**Remarks**

(1) The operators \( R_\alpha \) in (48) commute with the multiplication operator by a \( W \)-invariant polynomial, hence descend to well-defined operators on \( \mathbb{C}[h]/I_h \).

(2) One can get an analogue of Proposition 12.8 with the whole nil-Hecke algebra \( H_{nil} \) instead of \( \text{Nil} \), and with the operators \( R_\alpha \) acting on \( \mathbb{C}[h] \) instead of \( \mathbb{C}[h]/I_h \), by replacing the ordinary homology by \( G \)-equivariant (co-)homology; see [Ar].

Further, it follows from the formula of Corollary 6.10 that the map \( c^{biv} \) gives an embedding \( c^{biv} : \mathbb{C}[W] \hookrightarrow H_*(B \times B) \). We can therefore define a filtration on \( \mathbb{C}[W] \) by

\[
E_j(\mathbb{C}[W]) = \{ u \mid c^{biv}(u) \in \bigoplus_{i \leq j} H_{2(n+i)}(B \times B) \},
\]

which is analogous to the filtration used in Proposition 12.3. From Proposition 6.9 and Corollary 6.10 we obtain the following result:

**Corollary 12.9**

(i) For any simple reflection \( s_\alpha \), the convolution action of \( c^{biv}(s_\alpha) \in H_*(B \times B) \) on \( H_*(B) = \mathbb{C}[h]/I_h \) is given by the operator \( S_\alpha \), see (17), specialized at \( \epsilon = 1 \).

(ii) The assignment \( w \mapsto r_w, w \in W, \) gives a graded algebra isomorphism

\[
\text{gr}^F \mathbb{C}[W] \cong \text{Nil}.
\]

**Final remark** Most of the constructions of this section can be extended to quantized enveloping algebra instead of Hecke algebra. The analogue of the degenerate Hecke algebra \( H_{deg} \) is known, in the context of quantized enveloping algebras, as the Yangian, see [CP]. There is an analogue of Theorem 12.7 for Yangians.
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