GEOMETRIC ASPECTS OF MIURA TRANSFORMATIONS

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ABSTRACT. The Miura transformation plays a crucial role in the study of integrable systems. There have been various extensions of the Miura transformation, which have been used to relate different kinds of integrable equations and to classify the bi-Hamiltonian structures. In this paper, we are mainly concerned with the geometric aspects of the Miura transformation. The generalized Miura transformations from the mKdV-type hierarchies to the KdV-type hierarchies are constructed under both algebraic and geometric settings. It is shown that the Miura transformations not only relate integrable curve flows in different geometries but also induce the transition between different moving frames. Other geometric formulations are also investigated.

Key words and phrases: Miura transformation; integrable system; KdV equation; modified KdV equation; integrable curve flow.

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1. Introduction

In [52], Miura introduced the remarkable transformation

\[ u = q^2 - q_x \]  

relating solutions of the KdV equation

\[ u_t = u_{xxx} - 6uu_x \]

with solutions of the modified KdV (mKdV) equation

\[ q_t = q_{xxx} - 6q^2q_x. \]

Nowadays, (1.1) is called the Miura transformation, which has been used to construct an infinite number of conservation laws of the KdV equation in that time [53]. And the generalized Miura transformation induces Bäcklund transformations for the Gelfand-Dickey hierarchy [1].

The Miura transformation adopts various extensions, and they have a number of applications in the study of integrable nonlinear dispersive equations. For examples, (1.1) is equivalent to the spectral problem of the Schrödinger operator for the KdV equation, it can be used to study the integrability of the PDE systems. The Miura transformation relates the Hamiltonian structures and conservation laws of the KdV equation with those of the mKdV equation. The Benjamin-Ono equation admits a two-parameter family of Miura transformations, which leads to its infinitely many

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family of conservation laws \[7\]. The Boussinesq equation is related to dispersive water wave system by the two-component Miura transformations \[43\]. Moreover, Miura-type transformations connect solutions of Sawada-Kotera (SK) \[60\] and Kaup-Kupershmidt (KK) equations \[41, 43\] to the solutions of Fordy-Gibbon-Jimbo-Miwa (FGJM) equation \[25\].

It is worth to point out that, the Miura transformation has been used to study the lower-regularity of the KdV equation by using the result of the mKdV equation \[15\]. The \(L^2\)-stability of solitons of the KdV equation was proved from the kink solution of the Gardner equation via the Miura transformation \[2, 51\]. Miura transformation admits various extensions, which have been applied to relate different kinds of integrable equations \[39, 46\]. Interestingly, the Miura-type transformations can be applied to classify dispersionless integrable systems \[19, 20\]. The Miura transformation relating \(L^G\)-opers to \(L^G\)-opers on the punctured disc gives an affine analogue of the Harish-Chandra homomorphism obtained by evaluating central elements on the Wakimoto modules \[27, 28\]. It was shown in \[21, 22, 28\] that the Miura transformations provides homomorphisms of different Poisson algebras. Recently, a direct correspondence between the operators obtained by the Miura transformation and those of the quantum toroidal algebra is found in \[36\].

Several ways have been developed to construct Miura transformations between integrable systems. In a series of papers by Fordy et.al. (c.f. \[3, 4, 6, 24\]), Miura transformations and its multi-component extensions can be constructed by the factorization of energy-dependent operators. Such construction can be utilized to obtain the Miura maps between super integrable systems. In \[67\], a direct scheme for constructing Miura transformations is presented, which also works for the discrete integrable systems. In the discrete systems, the bilinear transformation is a powerful method to construct Miura transformations \[38\]. The symmetry groups was developed to obtain Miura transformations (c.f. \[34, 35\]). Miura transformations between any two scalar evolution equations were classified in \[10\], which derive new Bäcklund transformations. For certain cases, Miura transformations also can be generated from the gauge transformation \[23\].

The mKdV equation has been highly involved in the study of differential geometry. In \[45\], Lamb used the mKdV equation to describe motion of curves with constant torsion. Chern and Tenenblat characterized the mKdV hierarchy as relations between local invariants of certain foliations on a surface of nonzero constant Gauss curvature in \[12\]. Doliwa and Santini \[17\] obtained the mKdV equation from nonstretching evolution of curves in \(S^2\). Moreover, the mKdV equation was the equation satisfied by the curvature of chiral shape arc-length preserving closed curves discussed by Goldstein and Petrich in \[31\].

The main goal of this paper is to explore the geometric aspect of Miura transformations. Our motivation in part comes from the following facts. First of all, it is found that Miura transformations connect with the planar curve flows in some imprimitive geometries in \(\mathbb{R}^2\) with the curve flows in the one-dimensional projective space \[13\]. Secondly, the Miura transformations connect the curvature in certain geometries with that in their sub-geometries. Furthermore, the Miura transformations enclose the rich algebraic structure of integrable systems. In this paper, we will set up a scheme to construct hierarchies of curve flows. Under certain parallel moving frame, the corresponding principle curvatures are solutions to the \(\hat{\mathcal{G}}^{(1)}\)-mKdV and
\( \hat{G}^{(2)} \)-mKdV hierarchies. This will give a geometric explanation of the mKdV-type hierarchies and generalized Miura transformations.

The organization of this paper is as follows. In Section 2, we give a brief discussion on geometric formations of Miura transformation based on the planar curve flows. A review on the construction of the \( \hat{G}^{(1)} \)-mKdV and \( \hat{G}^{(2)} \)-mKdV hierarchies is presented in Section 3. And the relation with pseudo-differential operators is discussed. In Section 4, a variety of curve flows are constructed and the relation to mKdV-type hierarchies is discussed. In Section 5, we give an explicit example of the Bousinessq equation to explain our scheme. And the last section is left for discussion and prospective projects.

2. Geometric formulations of Miura transformations

In this section, we give a brief discussion on the geometric aspect of the Miura transformation. According to the Erlangen program, for any Lie group \( G \) acting locally and effectively on an open set \( U \) in the plane so that its group action does not have a common fixed point, there is an associated Klein geometry, which is the theory of geometric invariants of the transformation groups. The Lie algebra \( \mathcal{G} \) of \( G \) consisting of all infinitesimal transformation acting on the plane has been classified up to local diffeomorphisms. They are divided into two classes: primitive and imprimitive cases. The corresponding real vector fields have also been classified (cf. [32], [55]). The group \( G \) is called imprimitive if there exists an invariant foliation in \( U \). For the transitive imprimitive Lie algebras, there are eleven types, among which there are six types of imprimitive Lie algebras of vector fields in the plane, including \( SL(2) \), \( SL'(2) \), \( GL(2) \), \( SL^2(2) \), \( SL(2,k) \) and \( GL(2,k)(k \in \mathbb{Z}^+) \). The invariant planar curve flows on those geometries can be projected to curve flows on \( \mathbb{R}P^1 \) [14].

It indicates that the Miura transformation arises from the relationship between integrable planar curve flows and one-dimensional projective space. Consider planar curve flows in the centro-affine geometry with imprimitive group \( GL(2) \). Given \( \gamma(p) \in \mathbb{R}^2 \setminus \{0\} \), such that \( \det(\gamma, \gamma_p) \neq 0 \). Then \( (\gamma, \gamma_p) \) is a natural moving frame along \( \gamma \). The curve flow \( \gamma(t,s) \) is governed by

\[
\gamma_t = U \gamma + W \gamma_s,
\]

where \( s \) is the arc-length parameter, defined by

\[
ds = \sqrt{\left| \frac{\det(\gamma_p, \gamma_{pp})}{\det(\gamma, \gamma_p)} \right| dp}.
\]

And the curvature \( \kappa \) of \( \gamma \) is given by

\[
\kappa = \frac{\det(\gamma, \gamma_{ss})}{\det(\gamma, \gamma_s)}.
\]

Then the structure equation for \( \gamma \) is

\[
(\gamma, \gamma_s)_s = (\gamma, \gamma_s) \left( \begin{array}{cc} 0 & 1 \\ 1 & \kappa \end{array} \right).
\]
Assume that the flow is arc-length preserving, then $W$ and $U$ satisfy

$$W_s - \frac{1}{2} \kappa U_s + \frac{1}{2} U_{ss} = 0,$$

while the curvature satisfies the following equation,

$$\kappa_t = \kappa_s W + 2U_s + \frac{1}{2} (\kappa_s U_s + \kappa^2 U_s - U_{sss}).$$

It is clear to see that the geometric flow

$$\gamma_t = -2 \kappa \gamma + (\kappa_s - \frac{1}{2} \kappa^2) \gamma_s$$

gives the mKdV equation

$$\kappa_t = \kappa_{sss} - \frac{3}{2} \kappa^2 \kappa_s - 4 \kappa_s.$$

One can verify by a straightforward computation that this mKdV equation is related to the KdV equation

$$u_t = u_{sss} - 6uu_s$$

by the Miura transformation

$$u = \frac{1}{2} \kappa_s + \frac{1}{4} \kappa^2 + \frac{2}{3}.$$

It is shown that such Miura transformation relates the mKdV-flow in $GL(2)$ to the KdV-flow in $\mathbb{R}P^1$ [14, 8].

Furthermore, it is noticed that the Lie algebra $GL(2)$ is a subalgebra of the Lie algebra $SL^2(2)$ generated by $\{\partial_x, x\partial_x, x^2\partial_x, \partial_u, u\partial_u, u^2\partial_u\}$. So the geometry $GL(2)$ is a subgeometry of the geometry $SL^2(2)$. Let $\kappa$ and $\phi$ be the curvatures of planar curves respectively in the geometries $SL^2(2)$ and $GL(2)$. Indeed, it is easy to check that their curvatures are related by the Miura transformation [14]

$$\kappa = 2\phi + \phi^2.$$

In the end of this section, we show that the Miura transformation connects the integrable curve flows with different moving frames. Consider the centro-equi-affine curve $\gamma(x) : \mathbb{R} \rightarrow \mathbb{R}^2 \setminus \{0\}$, with $x$ the centro-equi-affine arc-length parameter, i.e. $\det(\gamma, \gamma_x) = 1$. And $(\gamma, \gamma_x)$ is the centro-equi-affine moving frame along $\gamma$ with curvature $u$. It is known that (cf. [8], [59], [63]) if $\gamma$ is a solution of the following equation

$$\gamma_t = \frac{1}{4} u_x \gamma - \frac{1}{2} u \gamma_x,$$  \quad (2.1)$$

then $u$ satisfies the KdV equation

$$u_t = \frac{1}{4} (u_{xxx} - 6uu_x).$$

Let $\eta$ be a smooth vector field along $\gamma$ such that

$$\begin{cases}
\det(\gamma, \eta) = 1, \\
\det(\eta_x, \eta) = 0.
\end{cases}$$

Such $(\gamma, \eta)$ is called a centro-equi-affine parallel frame along $\gamma$.

Set $q = \det(\gamma, \eta_x)$, by $\det(\eta_x, \eta) = 0$, we get

$$u = q^2 - q_x,$$
which is the Miura transformation. And the transition matrix between the centro- 
equiaffine frame and parallel frame is 
\[(\gamma, \gamma_x) = (\gamma, \eta) \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}.\]

We call \(q\) a principle curvature of \(\gamma\) w.r.t the parallel frame \((\gamma, \eta)\). And (2.1) can be written in terms of the parallel frame 
\[\dot{\gamma}_t = \frac{1}{4} (2q^3 - q_{xx})\gamma - \frac{1}{2} (q^2 - 2q_x)\eta,\]
with \(q\) satisfying the mKdV equation 
\[q_t = \frac{1}{4}(q_{xxx} - 6q^2 q_x).\]

Note that if we consider the following third-ordered centro-equiaffine curve flow: 
\[\dot{\gamma}_t = \frac{1}{9} (uu_x - u_{xxx})\gamma + \frac{1}{9} (2u_{xx} - u^2)\gamma_x,\]
then the equal centro-equiaffine curvature \(u\) is a solution to the Sawada-Kotera (SK) equation [60]: 
\[u_t = \frac{1}{9} (u_{xxxx} - 5uu_{xxx} - 5u_x u_{xx} + 5u^2 u_x).\]

The same Miura transformation (1.1) takes the solution \(q\) of Fordy-Gibbons-Jimbo-Miwa (FGJM) equation (cf. [26], [37]): 
\[q_t = -\frac{1}{9} (qxxxx - 5q^2 q_{xx} - 5q_x q_{xx} - 5qq_x^2 + q^5)_x\]
to the solution of (2.2).

From the above discussion, we find that the Miura transformation used to obtain an infinite number of conservation laws has natural geometric formulations. Such formulations motivate us to investigate further applications of Miura transformations in other geometric settings.

3. The \(\hat{G}^{(1)}\)-mKdV and \(\hat{G}^{(2)}\)-mKdV hierarchies

In this section, we give a brief introduction to the construction of the mKdV-type hierarchies associated to affine Kac-Moody algebras introduced in [18] from Lie algebra splittings.

Let \(G\) be a non-compact, real simple Lie group, \(G\) its Lie algebra, and 
\[\hat{G}^{(1)} = L(G) = \left\{ \sum_{i \leq n_0} \xi_i \lambda^i \mid n_0 \text{ an integer, } \xi_i \in G \right\},\]
the set of smooth loops on \(G\).

Let 
\[\hat{G}^{(1)}_+ = \left\{ \sum_{i \geq 0} \xi_i \lambda^i \in L(G) \right\}, \quad \hat{G}^{(1)}_- = \left\{ \sum_{i < 0} \xi_i \lambda^i \in L(G) \right\}.
\]
Then \((\hat{G}^{(1)}_+, \hat{G}^{(1)}_-)\) is a splitting of \(\hat{G}^{(1)}\).
Let \( \{ \alpha_1, \ldots, \alpha_n \} \) be a simple root system of \( \mathcal{G} \), and \( C, B_+, B_-, N_+ \) the Cartan, Borel subalgebras of \( \mathcal{G} \) of non-negative roots, non-positive roots, and positive roots respectively. Let \( C, B_+, B_-, N_+ \) be connected subgroups of \( \mathcal{G} \) with Lie algebras \( C, B_+, B_-, N_+ \) respectively. Let

\[
J = \beta \lambda + b,
\]

where \( b = - \sum_{i=1}^{n} \alpha_i \) and \( \beta \) is the highest root.

**Theorem 3.1.** ([18], [61], [64]) Given \( q \in C^\infty(\mathbb{R}, B_+) \), then there exists a unique \( S(q, \lambda) = \sum_{i \leq 1} S_{1,i}(q) \lambda^i \in \hat{G}^{(1)} \) satisfying

\[
\begin{align*}
\{ [\partial_x + b + q, S(q, \lambda)] = 0, \\
m(S(q, \lambda)) = 0,
\end{align*}
\]

where \( m \) is the minimal polynomial of \( J \) defined by (3.1).

Assume that there is a sequence of increasing positive integers \( \{ n_j \mid j \geq 1 \} \) such that \( J^{n_j} \) lies in \( \hat{G}^{(1)}_+ \) for all \( j \geq 1 \). Then \( \{ J^{n_j} \in \hat{G}^{(1)}_+ \} \) is called a vacuum sequence. Write

\[
S^{n_j}(q, \lambda) = \sum_{i} S_{n_j,i}(q) \lambda^i
\]

Then the \( n_j \)-th flow in the \( \hat{G}^{(1)} \)-hierarchy for \( q : \mathbb{R}^2 \to B_+ \) is

\[
q_{n_j} = [\partial_x + b + q, S_{n_j,0}(q)].
\]

The \( \hat{G}^{1} \)-KdV hierarchy can be constructed from pushing down the \( \hat{G}^{1} \)-hierarchy to certain cross-section \( C^\infty(\mathbb{R}, V) \) along the orbit of the gauge action of \( C^\infty(\mathbb{R}, N_+) \) on \( C^\infty(\mathbb{R}, B_+) \) (cf. [18], [66]).

Let \( \pi_{bn} \) be the projection of \( \mathcal{G} \) onto \( B_- \) with respect to \( \mathcal{G} = B_- \oplus N_+ \). Then from a direct computation, the \( \hat{G}^{(1)} \)-hierarchy induces a flow on the space of \( C^\infty(\mathbb{R}^2, C) \), which is called the \( \hat{G}^{(1)}_+ \)-mKdV hierarchy.

The \( n_j \)-th \( \hat{G}^{(1)} \)-mKdV flow is

\[
q_{n_j} = [\partial_x + b + q, \pi_{bn}(S_{n_j,0}(q))].
\]

Let \( \sigma \) be a complex linear involution of \( \mathcal{G} \), and \( \mathcal{K}, \mathcal{P} \) the 1, -1 eigenspaces of \( \sigma \) respectively.

The \( \hat{G}^{(2)} \)-hierarchy is constructed from the splitting \((\hat{G}^{(2)}_+, \hat{G}^{(2)}_-)\) of \( \hat{G}^{(2)} \), where

\[
\hat{G}^{(2)}_+ = \{ \xi(\lambda) \in \hat{G}^{(1)} \mid \xi(\lambda) = \xi(\lambda), \sigma(\xi(-\lambda)) = \xi(\lambda) \}, \quad \hat{G}^{(2)}_- = \hat{G}^{(2)} \cap \hat{G}^{(1)}_+.
\]

If there is a simple root system of \( \mathcal{G} \) such that \( \beta \in \mathcal{P} \) and \( b \in \mathcal{K} \), then \( C^\infty(\mathbb{R}, \mathcal{K} \cap \mathcal{C}) \) is invariant under (3.2), which induce the \( \hat{G}^{(2)} \)-mKdV hierarchy. And the \( \hat{G}^{(2)} \)-KdV hierarchy can be constructed in a similar way from the \( \hat{G}^{(2)} \)-hierarchy as from the \( \hat{G}^{1} \)-hierarchy to the \( \hat{G}^{1} \)-KdV hierarchy.

**Definition 3.2.** ([18]) The generalized Miura transformation is the gauge transformation of \( C^\infty(\mathbb{R}, N_+) \) from the \( \hat{G}^{1} \)- (\( \hat{G}^{(2)}_- \), resp.) mKdV hierarchy to the \( \hat{G}^{(1)} \)- (\( \hat{G}^{(2)}_- \), resp.) KdV hierarchy.
Next we give some explicit examples of the mKdV-type hierarchies from the algorithm given by Theorem 3.1 and corresponding Miura-type transformations. Let
\[ B_n^+ = \{ y = (y_{ij}) \in \mathfrak{sl}(n, \mathbb{C}) \mid y_{ij} = 0, i > j \}, \]
\[ B_n^- = \{ y = (y_{ij}) \in \mathfrak{sl}(n, \mathbb{C}) \mid y_{ij} = 0, i < j \}, \]
\[ N_n^+ = \{ y = (y_{ij}) \in \mathfrak{sl}(n, \mathbb{C}) \mid y_{ij} = 0, i \geq j \}, \]
\[ T_n = \{ y \in gl(n, \mathbb{C}) \mid y_{ij} = 0, i \neq j \} \]
denote the subalgebras of upper triangular, lower triangular, strictly upper triangular matrices in \( \mathfrak{sl}(n, \mathbb{C}) \) and diagonal matrices in \( gl(n, \mathbb{C}) \) respectively. Let \( N_n^+ \) denote the corresponding Lie subgroup of \( N_n^+ \).

3.1. The \( \hat{A}_{n-1}^{(1)} \)-mKdV hierarchy.
In the case when \( \mathcal{G} = \mathfrak{sl}(n), b = \sum_{i=1}^{n-1} e_{i+1,i} \). The Catran subalgebra is \( T_n \). Let \( \pi_{bn} \) be the projection of \( \mathfrak{sl}(n) \) onto \( B_n^- \) with respect to \( \mathfrak{sl}(n) = B_n^- \oplus N_n^+ \). Let \( \pi_{bn} \) be the projection of \( \mathfrak{sl}(n) \) onto \( B_n^- \) with respect to \( \mathfrak{sl}(n) = B_n^- \oplus N_n^+ \). The \( j \)-th \( \hat{A}_{n-1}^{(1)} \)-mKdV flow is \( (3.2) \) for \( q = \text{diag}(q_1, \ldots, q_n) \in C^\infty(\mathbb{R}^2, T_n) \),
\[ q_t = [\partial_x + b + q, \pi_{bn}(S_j,0(q))]. \quad (3.3) \]
Note that the \( \hat{A}_{n-1}^{(1)} \)-KdV hierarchy is the Gelfand-Dickey hierarchy, and the \( \hat{A}_{n-1}^{(1)} \)-mKdV hierarchy is the Drinfeld-Sokolov \( n \times n \) mKdV hierarchy. In particular, the third \( \hat{A}_1^{(1)} \)-mKdV flow is the mKdV equation
\[ q_t = \frac{1}{4}(q_{xxx} - 6q^2q_x). \]

3.2. The \( \hat{A}_{2n}^{(2)} \)-mKdV hierarchy.
Let \( \mathbb{R}^{n+1,2} \) be the linear space \( \mathbb{R}^{2n+1} \) equipped with the non-degenerate bilinear form
\[ \langle X, Y \rangle = X^t \rho_n Y, \quad \rho = \sum_{i=1}^{2n+1} (-1)^{n+i-1} e_{i,2n+2-i}. \quad (3.4) \]
Let \( O_{\mathbb{C}}(n+1,n) \) be the group of linear isomorphisms of \( \mathbb{C}^{2n+1} \) that preserve \( \langle \ , \ \rangle \), i.e.
\[ O_{\mathbb{C}}(n+1,n) = \{ g \in SL(2n+1, \mathbb{C}) \mid g^t \rho_n g = \rho_n \}. \]
Its Lie algebra is \( o_{\mathbb{C}}(n+1,n) = \{ A \in \mathfrak{sl}(2n+1, \mathbb{C}) \mid A^t \rho_n + \rho_n A = 0 \} \). Note that \( A = (A_{ij}) \in o_{\mathbb{C}}(n+1,n) \) if and only if
(i) \( A_{ij} \)'s are symmetric (skew-symmetric resp.) with respect to the skew diagonal line \( i + j = 2n + 2 \) if \( i + j \) is odd (even resp.),
(ii) \( A_{ij} = 0 \) if \( i + j = 2n + 2 \).
Let
\[ J = e_{1,2n} \lambda + \sum_{i=1}^{2n} e_{i+1,i}. \quad (3.5) \]
The \((2j - 1)\)-th \(\hat{A}^{(2)}_{2n}\)-KdV flow [65] is an evolution equation for

\[
u = \sum_{i=1}^{n} u_i \beta_i, \quad \beta_i = e_{n+1-i,n+i} + e_{n+2-i,n+1+i}. \tag{3.6}
\]

In this case, \(\pi_{bn}\) is the projection of \(o(n + 1, n)\) onto \(B^{-}_{2n+1} \cap o(n + 1, n)\) with respect to

\[
o(n + 1, n) = (B^{-}_{2n+1} \cap o(n + 1, n)) \cap (N^+_{2n+1} \cap o(n + 1, n)).
\]

The \((2j - 1)\)-th \(\hat{A}^{(2)}_{2n}\)-mKdV flow is (3.2) for

\[
q = \sum_{i=1}^{n} q_i (e_{n+1-i,n+1-i} - e_{n+1+i,n+1+i}) \in C^\infty(\mathbb{R}^2, \mathcal{T}_{2n+1} \cap o(n + 1, n)).
\]

From a direct computation, we see that the fifth \(\hat{A}^{(2)}_{2n}\)-KdV flow for \(\text{diag}(q, 0, -q)\) is the Fordy-Gibbons-Jimbo-Miura (FGJM) equation:

\[
q_t = -\frac{1}{9}(q_{xxxx} - 5q^2q_{xx} - 5q_x^2 - 5qq_x^2 + q^5)_x. \tag{3.7}
\]

And the fifth \(\hat{A}^{(2)}_{2n}\)-KdV flow is the Kaup-Kupershmidt (KK) equation:

\[
u_t = -\frac{1}{9}(u_{xxxx} - 10uu_{xxx} - 25u_xu_{xx} + 2u^2u_x). \tag{3.8}
\]

And the Miura transformation \(u = q_x + \frac{1}{2}q^2\) take the solution of (3.7) to a solution of (3.8).

**Example 3.3** (The Miura transformation from \(\hat{A}^{(2)}_{4}\)-KdV to \(\hat{A}^{(2)}_{4}\)-mKdV).

In this case, the Miura transformation is induced by

\[
\Delta_n \in C^\infty(\mathbb{R}, N^+_{2n+1} \cap O(n + 1, n))
\]

such that

\[
\Delta_n \begin{pmatrix}
\partial_x + \\
0 & 0 & 0 & u_2 & \lambda \\
1 & 0 & u_1 & 0 & u_2 \\
0 & 1 & 0 & u_1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\Delta_n^{-1}
= \partial_x + \\
\begin{pmatrix}
q_2 & 0 & 0 & 0 & \lambda \\
1 & q_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -q_1 & 0 \\
0 & 0 & 0 & 1 & -q_2
\end{pmatrix}.
\]

The Miura transformation written in terms of \((q_1, q_2)\) is

\[
\begin{align*}
u_1 &= q_{1,x} + 2q_{2,x} + \frac{1}{2}(q_1^2 + q_2^2), \\
u_2 &= q_{2,xxx} + q_2(q_{2,xx} - q_{1,xx}) - q_{1,x}(2q_{2,x} + 2q_1q_2 + q_2^2) \\
&\quad - \frac{1}{2}q_{2,x}(q_{2,x}^2 + 2q_1^2) - \frac{1}{2}q_1^2q_2^2.
\end{align*}
\]
3.3. The $\hat{A}_n^j$-mKdV hierarchy.

Set

$$S_n = \sum_{i=1}^{2n} (-1)^{i+1} e_{i,2n+1-i}. \quad (3.9)$$

Let $\sigma$ be the order four automorphism of $sl(2n+1, \mathbb{C})$ defined by

$$\sigma(X) = -D_n X^t D_n^{-1}, \quad \text{where } D_n = \text{diag}(i, S_n),$$

and $S_n$ is given by (3.9). Let $G_j$ be the eigenspace of $\sigma$ with respect to the eigenvalue $i^j$ for $0 \leq j \leq 3$. It follows from a direct computation that we have

$$G_0 = \begin{pmatrix} 0 & 0 \\ 0 & \text{sp}(2n, \mathbb{C}) \end{pmatrix}, \quad G_1 = \begin{cases} \begin{pmatrix} 0 & \xi \\ S_n \xi^t & \eta \end{pmatrix} & | S_n \eta^t S_n^{-1} = -i \eta \end{cases},$$

$$G_2 = \begin{cases} \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \eta \end{pmatrix} & | S_n \eta^t S_n^{-1} = \eta \end{cases}, \quad G_3 = \begin{cases} \begin{pmatrix} 0 & \xi \\ -S_n \xi^t & \eta \end{pmatrix} & | S_n \eta^t S_n^{-1} = i \eta \end{cases}.$$

We will use the following notation

$$G_0 = \left\{ \hat{y} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \mid y \in \text{sp}(2n, \mathbb{C}) \right\}.$$

Let

$$\hat{A}_n^j = \left\{ A(\lambda) = \sum_{i \leq n_0} A_i \lambda^i \mid A_i \in \text{sl}(2n+1, \mathbb{C}), \sigma(A(-i \lambda)) = A(\lambda) \right\},$$

$$(\hat{A}_n^j)_+ = \left\{ \sum_{i \geq 0} A_i \lambda^i \in \hat{A}_n^j \right\}, \quad (\hat{A}_n^j)_- = \left\{ \sum_{i < 0} A_i \lambda^i \in \hat{A}_n^j \right\}.$$

Then $A(\lambda) \in \hat{A}_n^j$ if and only if $A_i \in G_j$, where $i \equiv j (\text{mod } 4)$, and $\hat{A}_n^j = (\hat{A}_n^j)_+ \oplus (\hat{A}_n^j)_-$ as a direct sum of linear subspaces.

Let

$$J_A = (e_{1,2n+1} + e_{2,1}) \lambda + \sum_{i=2}^{2n} e_{i+1,i} = (e_{1,2n+1} + e_{2,1}) \lambda + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Then $J_{A_n}^{j-1} \in \hat{A}_n^j$ for all $j \geq 1$. Let $\pi_{bn}$ be the projection of $G_0$ on to $G_0 \cap B_{2n+1}$ with respect to

$$G_0 = (G_0 \cap B_{2n+1}^+) \oplus (G_0 \cap N_{2n+1}^+).$$

It can be checked that $W_n = \oplus_{i=1}^n \mathbb{R} e_{i+2-i,n+1-i}$ is a cross-section of the gauge orbit, and the $\hat{A}_n^j$-KdV hierarchy is generated by

$$\partial_x + J_A + \sum_{i=1}^n u_i e_{n+2-i,n+1-i}.$$

The $(2j-1)$-th $\hat{A}_n^j$-mKdV flow is (3.3.2) for

$$q = \sum_{i=1}^n q_i (e_{n+2-i,n+2-i} - e_{n+1+i,n+i+i}) \in C^\infty(G_0 \cap T_{2n+1}).$$
Example 3.4.
The fifth $A_2^n$-KdV flow for $ue_{2t}$ is the Sawada-Kotera (SK) equation \((2.2)\).
The fifth $A_2^n$-mKdV flow for diag($0, -q, q$) is again the FGJM \((3.7)\). Moreover, the Miura transformation connecting the solution $q$ of the FGJM and a solution $u$ of the SK \((2.2)\) is $u = q^3 - q_x$.

Example 3.5.
The third $A_2^n$-KdV flow is a coupled fifth-order system for $u = u_1 e_{3,4} + u_2 e_{2,5}$:

$$
\begin{align*}
    u_{1,t} &= -2u_{1,xxx} + \frac{6}{5}u_{1,u,xx} + 3u_{2,x}, \\
    u_{2,t} &= -\frac{3}{5}u_{1,xxxx} + u_{2,xxx} + \frac{3}{5}(u_{1,u,xx} + u_{1,xx}u_{1,xx} + u_{1,xx}u_{2} - u_{1}u_{2,x}).
\end{align*}
$$

The Miura transformation from the $\hat{A}_2^n$-mKdV hierarchy to the $\hat{A}_2^n$-KdV hierarchy is

$$
\begin{align*}
    u_1 &= q_1^3 - q_2 - q_{1,x} - 3q_{2,x}, \\
    u_2 &= -2q_{2,xxx} + q_{2}q_{2,x} + (q_{1,x} - q_1^2)(q_2^2 - q_{2,x}) - q_2(q_{1,xx} - 2q_1q_{1,x}).
\end{align*}
$$

3.4. The $\hat{A}_{2n-1}^{(2)}$-mKdV hierarchy.
Let $\mathbb{R}^{2n}$ be the symplectic space with the symplectic form

$$
\omega(X, Y) = X^t S_n Y,
$$
where $S_n$ is as defined in \((3.3)\). $Sp(2n) = \{ g \in GL(2n, \mathbb{R}) \mid g^t S_n g = S_n \}$ the group of linear isomorphisms of $\mathbb{R}^{2n}$ that preserves $\omega$, and

$$
sp(2n) = \{ A \in sl(2n) \mid A^t S_n + S_n A = 0 \}
$$
is the corresponding Lie algebra of $Sp(2n)$.

Let $\kappa$ be the involution of $sl(2n, \mathbb{C})$ defined by

$$
\kappa(X) = -S_n X^t S_n^{-1},
$$
where $S_n$ is as in \((3.3)\).

Let

$$
\hat{A}_{2n-1}^{(2)} = \left\{ A(\lambda) = \sum_{i \leq m_0} A_i \lambda^i \mid A_i \in sl(2n, \mathbb{R}), \kappa(A(-\lambda)) = A(\lambda) \right\}
$$
and

$$
(\hat{A}_{2n-1}^{(2)})_+ = \left\{ \sum_{i \geq 0} A_i \lambda^i \in \hat{A}_{2n-1}^{(2)} \right\}, \quad (\hat{A}_{2n-1}^{(2)})_- = \left\{ \sum_{i < 0} A_i \lambda^i \in \hat{A}_{2n-1}^{(2)} \right\}.
$$

Then $((\hat{A}_{2n-1}^{(2)})_+, (\hat{A}_{2n-1}^{(2)})_-)$ is a splitting of $\hat{A}_{2n-1}^{(2)}$.

In this case, $J_n = \frac{1}{2}(e_{1,2n-1} + e_{1,2n})\lambda + \sum_{i=1}^{2n-1} e_{i+1,i}$, and $\pi_{2n}$ be the projection of $sp(2n, \mathbb{C})$ on to $sp(2n, \mathbb{C}) \cap B_{2n}^-$ with resect to

$$
sp(2n, \mathbb{C}) = (sp(2n, \mathbb{C}) \cap B_{2n}^-) \oplus (sp(2n, \mathbb{C}) \cap N_{2n}^+).
$$
The $(2j-1)$-th $\hat{A}^{(2)}_{2n-1}$-mKdV flow is \( \{3.2\} \) for

$$q = \sum_{i=1}^{n} q_i(e_{n+1-i,n+1-i} - e_{n+i,n+i}) \in C^\infty(\text{sp}(2n, \mathbb{C}) \cap \mathcal{T}_{2n}).$$

**Example 3.6.**

The third $\hat{A}^3_3$-mKdV flow for $q = q_2(e_{11} - e_{44}) + q_1(e_{22} - e_{33})$ is

$$\begin{cases}
q_{2,t} = -4q_{2,xxx} + 4(q_{1,x}q_2 + q_1^2q_2)_x, \\
q_{1,t} = -\frac{2}{3}(6q_{2,x}q_2 + 2q_1q_{2,x} - 4q_1^2q_2 - 8q_1q_2^2)_x.
\end{cases}$$

The third $\hat{A}^3_3$-KdV flow for $u = u_1e_{23} + u_2e_{14}$ is

$$\begin{cases}
q_{1,t} = 3u_{2,x}, \\
q_{2,t} = u_{2,xxx} - (u_1u_2)_x.
\end{cases}$$

And the explicit formula of the Miura transformation is

$$\begin{cases}
u_1 = 3q_{2,x} + q_{1,x} + q_1^2 + q_2^2, \\
u_2 = q_{2,xxx} + q_2q_{2,xx} - q_{1,xxx}q_2 - q_{1,x}(q_{2,x} + q_2^2 + 2q_1q_2) - q_1^2(q_{2,x} + q_2^2).
\end{cases}$$

### 3.5. The $\hat{B}^{(1)}_n$-mKdV hierarchy.

Let $\hat{B}^{(1)}_n$ be the Lie algebra of formal power series $\xi(\lambda) = \sum_{i \geq n_0} \xi_i\lambda^i$ with some integer $n_0$ that satisfy

$$\rho_n\xi(\lambda) + \xi(\lambda)^t\rho_n = 0, \quad \xi(\lambda) = \xi(\lambda),$$

where $\rho$ is as defined in \( \{3.4\} \).

Let $(\hat{B}^{(1)}_n)^+$ and $(\hat{B}^{(1)}_n)^-$ be the sub-algebras of $\hat{B}^{(1)}_n$ defined by

$$(\hat{B}^{(1)}_n)^+ = \{\xi(\lambda) = \sum_{i \geq 0} \xi_i\lambda^i \in \hat{B}^{(1)}_n\},$$

$$(\hat{B}^{(1)}_n)^- = \{\xi(\lambda) = \sum_{i < 0} \xi_i\lambda^i \in \hat{B}^{(1)}_n\}.$$

Let

$$J_B(\lambda) = \beta\lambda + b,$$

where

$$\beta = \frac{1}{2}(e_{1,2n} + e_{2,2n+1}), \quad b = \sum_{i=1}^{2n} e_{i+1,i}.$$

Note that $J_B^{2j} \notin \hat{B}^{(1)}_n$, $J_B^{2j-1}(j \geq 1) \in (\hat{B}^{(1)}_n)^+$, and

$$J_B^{2n+1}(\lambda) = \lambda J_B(\lambda).$$

Let $\pi_{bn}$ is the projection of $o(n+1,n)$ onto $B_{2n+1}^- \cap o(n+1,n)$ with respect to $o(n+1,n) = (B_{2n+1}^- \cap o(n+1,n)) \cap (\mathcal{N}_{2n+1}^+ \cap o(n+1,n))$. 

The $(2j-1)$-th $\hat{B}_n^{(1)}$-mKdV flow is (3.2) for
\[ q = \sum_{i=1}^{n} q_i(e_{n+1-i,n+1}-e_{n+1+i,n+1+i}) \in C^\infty(\mathbb{R}^2, T_{2n+1} \cap o(n+1,n)). \]

**Example 3.7.**

The $\hat{B}_1^{(1)}$-mKdV hierarchy is the same as the mKdV hierarchy, and the third $\hat{B}_1^{(1)}$-mKdV flow for diag$(-q,0,q)$ is
\[ q_t = q_{xxx} - q^2 q_x. \]

The Miura transformation from diag$(-q,0,q)$ to $u(e_{12} + e_{23})$ is
\[ u = -q_x + \frac{1}{2} q^2. \]

The Miura transformation in the case of $\hat{B}_2^{(1)}$-mKdV to the $\hat{B}_2^{(1)}$-KdV is
\[
\begin{align*}
    u_1 &= 2q_{2,x} + q_{1,x} + \frac{1}{2}(q_1^2 + q_2^2), \\
    u_2 &= q_{2,xxx} - q_{1,xx}q_2 + q_2q_{2,xx} - 2q_{1,x}q_2 - \frac{1}{2} q_2^2 \\
        &- q_1^2 q_{2,x} - q_1q_{2}q_2 - q_1q_1q_2 - \frac{1}{2} q_1^2 q_2^2,
\end{align*}
\]

where the phase space of the $\hat{B}_2^{(1)}$-KdV is of the form $u = u_1(e_{23} + e_{34}) + u_2(e_{14} + e_{25})$.

### 3.6. The $\hat{C}_n^{(1)}$-mKdV hierarchy.

Let
\[
\hat{C}_n^{(1)} := \left\{ A = \sum_i A_i \lambda^i \mid A_i \in sp(2n) \right\},
\]

\[
(\hat{C}_n^{(1)})_+ = \left\{ \sum_{i \geq 0} A_i \lambda^i \in \hat{C}_n^{(1)} \right\}, \quad (\hat{C}_n^{(1)})_- = \left\{ \sum_{i < 0} A_i \lambda^i \in \hat{C}_n^{(1)} \right\},
\]

\[ J_C = e_{1,2n} + \sum_{i=1}^{2n-1} e_{i+1,i}. \]

Let $\pi_{bn}$ be the projection of $sp(2n, \mathbb{C})$ on to $sp(2n, \mathbb{C}) \cap B_{2n}^{-}$ with respect to
\[ sp(2n, \mathbb{C}) = (sp(2n, \mathbb{C}) \cap B_{2n}^{-}) \cap (sp(2n, \mathbb{C}) \cap N_{2n}^{+}). \]

The $(2j-1)$-th $\hat{C}_n^{(1)}$-mKdV hierarchy is (3.2) for
\[ q = \sum_{i=1}^{n} q_i(e_{n+1-i,n+1-i} - e_{n+i,n+i}) \in sp(2n, \mathbb{C}) \cap T_{2n}. \]

Note that the $\hat{C}_1^{(1)}$-mKdV hierarchy is the $2 \times 2$-mKdV hierarchy.
3.7. Pseudo-differential operator correspondence.

Drinfeld and Sokolov [18] have shown that there are various KdV-type hierarchies and one mKdV-type associated to each affine Kac-Moody algebra. For example, the \( \hat{A}^{(1)}_{n-1} \)-KdV hierarchy with phase space \( u = \sum_{i=1}^{n-1} u_i e_{i,n} \) is equivalent to the Gelfand-Dickey hierarchy [16] generated by the pseudo-differential operator

\[
L_1 = \partial^n - u_{n-1} \partial^{n-2} - \ldots - u_2 \partial - u_1.
\]

Moreover, the \( \hat{A}^{(1)}_{n-1} \)-mKdV hierarchy with phase space \( q = \text{diag}(q_1, \ldots, q_n) \) is equivalent to the generalized mKdV hierarchy generated by

\[
L_2 = (\partial - q_n) \cdots (\partial - q_1).
\]

And the transformation of written \( L_1 \) into the form of \( L_2 \) introduces the generalized Miura transformation.

Following a similar argument as in [18], we can get a series of results concerning the pseudo-differential operator correspondence for certain \( \hat{G}^{(1)} \)- and \( \hat{G}^{(2)} \)-mKdV hierarchies.

**Proposition 3.8.**

1. The \( \hat{A}^{(1)}_{2n} \)-mKdV hierarchy is equivalent to the KP-type hierarchy generated by

\[
L = (\partial + q_n) \cdots (\partial + q_1)(\partial - q_1) \cdots (\partial - q_n)\partial.
\]

2. The \( \hat{A}^{(2)}_{2n} \)-mKdV hierarchy is equivalent to the KP-type hierarchy generated by

\[
L = (\partial + q_n) \cdots (\partial + q_1)\partial(\partial - q_1) \cdots (\partial - q_n)\partial.
\]

3. The \( \hat{C}^{(1)}_n \)-mKdV hierarchy is equivalent to the reduced KP hierarchy generated by

\[
L = (\partial + q_n) \cdots (\partial + q_1)(\partial - q_1) \cdots (\partial - q_n)\partial.
\]

**Corollary 3.9.** The Miura transformation from the \( \hat{C}^{(1)}_n \)-mKdV hierarchy to the \( \hat{C}^{(1)}_n \)-KdV hierarchy is the same as the one from the \( \hat{A}^{(1)}_{2n} \)-mKdV hierarchy to the \( \hat{A}^{(1)}_{2n} \)-KdV hierarchy.

4. Geometric Miura transformations

It is known that there are natural curve flows explanation for KdV-type hierarchies in the background of different group actions (cf. [13], [14], [47]-[49]). In the geometry for curves, the moving frames along curves are induced by the group actions and give the corresponding local differential invariants. If we are able to establish a connection between the group actions and the algebraic structure of the integrable hierarchies, there will be a correspondence between the set of geometric invariants and the phase space of integrable equations. For example, associated to affine Kac-Moody algebra of type A, B, and C, there are centro-eqiuiaffine, isotropic and Lagrangian curves w.r.t. the group action of \( SL(n) \), \( O(n+1,n) \) and \( Sp(n) \) respectively (cf. [64]-[66]). In this section, we show that the generalized Miura transformations between the mKdV-type and KdV-type hierarchies discussed in the previous sections can be induced by the transition between moving frames along curve flows.
4.1. **Centro-equiaffine mKdV curve flow.**

Consider the following submanifold of centro-equiaffine curves in $\mathbb{R}^n \setminus \{0\}$,

$$\mathcal{M}_n(I) = \{ \gamma : I \to \mathbb{R}^n \setminus \{0\} \mid \det(\gamma, \gamma_\times, \ldots, \gamma_\times^{(n-1)}) = 1 \}, \quad I = S^1 \text{ or } \mathbb{R}. \quad (4.1)$$

Let $\gamma \in \mathcal{M}_n(I)$, $g = (\gamma, \gamma_\times, \ldots, \gamma_\times^{(n-1)})$ and $u = \sum_{i=1}^{n-1} u_i e_i$, the central-equiaffine moving frame and centro-equiaffine curvature along $\gamma$. There is a natural connection between the centro-equiaffine curve flows and the $\hat{A}_{n-1}$-KdV hierarchy \cite{91}.

**Definition 4.1.** A frame $g \in SL(n, \mathbb{R})$ is called a parallel frame along $\gamma$ if there exists smooth functions $q_1, \ldots, q_{n-1}$ along $\gamma$ such that $g e_1 = \gamma$ and

$$g_\times = g(b + \text{diag}(q_1, \ldots, q_{n-1}, - \sum_{i=1}^{n-1} q_i)).$$

We call these $\{q_1, \ldots, q_{n-1}\}$ a set of central-equiaffine principle curvatures along $\gamma$.

Write $g = (\gamma, \eta_2, \ldots, \eta_{n+1})$. If $g = (\gamma, \tilde{\eta}_2, \ldots, \tilde{\eta}_{n+1})$ is another parallel frame along $\gamma$ with parallel curvature $\tilde{q}_1, \ldots, \tilde{q}_{n-1}$. Then there exists $C \in C^\infty(\mathbb{R}, N_n^+)\setminus\{0\}$ such that $\tilde{g} = g C$, and

$$\tilde{q} = C^{-1} q C + C^{-1} C_\times$$

gives one type of the Bäcklund transformations for the $\hat{A}_{n-1}$-mKdV hierarchy.

**Proposition 4.2.** Let $\gamma \in \mathcal{M}_n(R)$, and $g$ be a parallel frame along $\gamma$. If its central-equiaffine principle curvatures $q$ satisfies the $j$-th $\hat{A}_{n-1}$-mKdV flow (3.3), then

$$\xi(\gamma) = g S_{j,0}(q) e_1 \in T_\gamma \mathcal{M}_n(\mathbb{R}).$$

**Proof.** Let $V_n = \sum_{i=1}^{n-1} u_i e_{i,n}$. Since $q$ is a solution of the $j$-th $\hat{A}_{n-1}$-mKdV flow, there exists $M \in C^\infty(\mathbb{R}^2, N_n^+)$, such that

$$M(\partial_x + J + q)M^{-1} = \partial_x + J + u, u \in C^\infty(\mathbb{R}^2, V_n).$$

Then $u$ is the solution $j$-th $\hat{A}_{n-1}$-KdV flow. Moreover, let $\tilde{g}$ be the central-equiaffine moving frame of $\gamma$. Then $\tilde{g} = g M^{-1}$, and $u$ is the central-equiaffine curvature of $\gamma$.

A direct computation implies $S^j(u) = M^{-1} S^j(q) M$. Therefore, the $j$-th $\hat{A}_{n-1}$-KdV flow can be written as

$$u_{t_j} = [\partial_x + b + u, M^{-1} S_{j,0}(q) M - \zeta_j(u)].$$

In other words, $[\partial_x + b + u, M^{-1} S_{j,0}(q) M - \zeta_j(u)] \in V_n$. Then from the result in \cite{64},

$$\xi(\gamma) = \tilde{g} M^{-1} S_{j,0}(q) Me_1 = g S_{j,0}(q) Me_1 \in T_\gamma \mathcal{M}_n(\mathbb{R}).$$

Since $M \in N_n^+, g S_{j,0}(q) Me_1 = g S_{j,0}(q)e_1$. This proves the Proposition. $\square$

Next we give some examples of the central-equiaffine mKdV flow:

**Example 4.3.**

1. Let $g = (\gamma, \eta)$ be the parallel frame for $\gamma \in \mathcal{M}_2$ and $q$ its parallel curvature. The third flow on $\mathcal{M}_2$ is

$$\gamma_t = \left(\frac{1}{2} q^3 - \frac{1}{4} q_{xx}\right) \gamma - \frac{1}{2} (q^2 - q_x) \eta.$$
And the parallel curvature $q$ satisfies the mKdV equation:

$$q_t = \frac{1}{4}(q_{xxx} - 6q^2q_x).$$

(2) For general $n \geq 3$, let $g = (\gamma, \eta_2, \eta_3, \ldots, \eta_{2n+1})$ and

$$q = \text{diag} \left( q_1, \ldots, q_{n-1}, - \sum_{i=1}^{n-1} q_i \right)$$

be the parallel frame and curvature respectively. The second mKdV central-equiaffine curve flow is

$$\gamma_t = -\frac{1}{n}((2n-1)q_{1,x} + (n-1)q_1^2 + \sum_{i=2}^{n-1} ((n-i)q_{i,x} - q_i(\sum_{j=1}^{i} q_j)))\gamma$$

$$+ (q_1 + q_2)\eta_2 + \eta_3.$$

These $q_i$’s satisfy the second $\hat{A}_{n-1}^{(1)}$-mKdV flow.

4.2. Isotropic mKdV flow.

Let $\gamma$ be an isotropic curve in $\mathbb{R}^{n+1,n}$, and $u_1, \ldots, u_n$ its isotropic curvatures. That is,

1. $\{\gamma, \gamma_x, \gamma_{(n-1)}^x\}$ forms a maximal isotropic subspace of $\mathbb{R}^{n+1,n}$.
2. $\langle \gamma_{(n)}^x, \gamma_{(n)}^x \rangle = 1$.
3. There exists unique $\tilde{g} = (\gamma, \gamma_x, \ldots, \gamma_{(n)}^x, p_{n+2}, \ldots, p_{2n+1})$ such that

$$\tilde{g}_x = \tilde{g}(\sum_{i=1}^{2n} e_{i+1,i} + \sum_{i=1}^{n} u_i \beta_i) = \tilde{g}(b + u),$$

where $\beta_i$ is as define in (3.6).

It is known that there are two types of isotropic curve flows associated to the $\hat{A}_{2n}^{(2)}$-KdV and $\hat{C}_n^{(1)}$-KdV hierarchies respectively (cf. [65]).

A parallel frame $g$ along $\gamma$ is for $g = (\gamma, \eta_2, \ldots, \eta_{2n+1}) \in O(n + 1,n)$ such that

$$g_x = g(b + \text{diag}(q_n, \ldots, q_1, 0, -q_1, \ldots, -q_n))$$

for some smooth functions $q_1, \ldots, q_n$ along $\gamma$. These $q_i$’s are called isotropic principle curvatures along $\gamma$. Let $\tilde{g}$ be the isotropic moving frame along $\gamma$, and $u_1, \ldots, u_n$ its isotropic curvatures. That is

$$\tilde{g} = (\gamma, \gamma_x, \ldots, \gamma_{(n)}^x, p_{n+2}, \ldots, p_{2n+1}),$$

$$\tilde{g}_x = \tilde{g}(\sum_{i=1}^{2n} e_{i+1,i} + \sum_{i=1}^{n} u_i \beta_i) = \tilde{g}(b + u),$$

where $\beta_i$ is as define in (3.6).

Then there exist $M \in \mathbb{N}^{2n+1}$ such that $\tilde{g} = gM$, and

$$M^{-1}(b + q)M + M^{-1}M_x = b + u.$$

This also gives the Miura transformation from the $\hat{A}_{2n}^{(2)}$-mKdV hierarchy to the $\hat{A}_{2n}$-KdV hierarchy.
Example 4.4 (Isotropic mKdV curve flow of A-type).

(1) Let $g = (\gamma, \eta, \eta_x)$ be a parallel frame along isotropic curve $\gamma \in \mathbb{R}^{2,1}$, with principle curvatures $q$. The fifth isotropic mKdV flow is

$$\gamma_t = \frac{1}{9}(-q_{xxxx} + 5q_xq_{xx} + 5q^2_q - q^5_q)\gamma + \frac{1}{3}(q_{xxx} - 3q_x^2 + q_{xx} - 4q^2_q - 4q^4)\eta.$$ 

The principle curvature $q$ satisfies the FGJM equation (3.7).

(2) Let $g = (\gamma, \eta_2, \eta_3, \eta_4, \eta_5)$ be a parallel frame along isotropic curve $\gamma \in \mathbb{R}^{3,2}$, and $q_1, q_2$ its principle curvatures. The third isotropic mKdV flow is

$$\gamma_t = -\frac{1}{2}(q_{2,xx} + 3q_{1,xx} + 12(q_1^2 + q_2^2)x + 6q_{1,x}q_2 + 3q_1^3q_2 + 3q_1^3 - 2q_1^2)\gamma - \frac{1}{5}(2q_{2,x} + q_{1,x} - 2(q_1^2 + q_2^2) + 5q_1q_2)\eta_2 - (q_1 + q_2)\eta_3 + \eta_{3,x}.$$ 

Example 4.5 (Isotropic mKdV curve flow of B-type).

(1) Let $g = (\gamma, \eta, \eta_x)$ be a parallel frame along isotropic curve $\gamma \in \mathbb{R}^{2,1}$, and $q$ the principle curvature along $\gamma$. The third isotropic mKdV curve flow of B-type is

$$\gamma_t = \frac{1}{3}(q^3 - q_{xx})\gamma + (q_x - \frac{1}{2}q^2)\eta.$$ 

(2) Let $g = (\gamma, \eta_2, \eta_3, \eta_4, \eta_5)$ be a parallel frame along isotropic curve $\gamma \in \mathbb{R}^{3,2}$, and $q_1, q_2$ its principle curvatures. The third isotropic mKdV flow of B-type is

$$\gamma_t = (q_{2,xx} - 3q_{1,x}q_2 - \frac{3}{2}q_1^3q_2 + \frac{3}{4}q_1^2)\gamma - \frac{1}{4}(4q_{2,x} + 2q_{1,x} - 3q_1q_2 - 2q_2^2 - q_1^2)\eta_2 + (q_1 + q_2)\eta_3 + \eta_{3,x}.$$ 

4.3. Lagrangian mKdV curve flow.

Definition 4.6. ([66])

(1) A linear subspace $V$ of $\mathbb{R}^{2n}$ is isotropic if $\omega(x,y) = 0$ for all $x,y \in V$. A maximal isotropic subspace has dimension $n$, and is called Lagrangian.

(2) A smooth map $\gamma : \mathbb{R} \to \mathbb{R}^{2n}$ is a Lagrangian curve if

(a) $\gamma(s), \gamma_s(s), \ldots, \gamma_s^{(2n-1)}(s)$ are linearly independent for all $s \in \mathbb{R}$,

(b) the span of $\gamma(s), \ldots, \gamma_s^{(n-1)}(s)$ is a Lagrangian subspace of $\mathbb{R}^{2n}$ for all $s \in \mathbb{R}$.

(3) $\mathcal{M}_{2n} = \{ \gamma \in \mathbb{R}^{2n} | \gamma \text{ is Lagrangian}, \omega(\gamma_s^{(n)}, \gamma_s^{(n-1)}) = (-1)^n \}.$

A parallel frame $g$ for a Lagrangian curve $\gamma \in \mathcal{M}_{2n}$ on symplectic space $(\mathbb{R}^{2n}, \omega)$ is $g = (\gamma, \eta_2, \ldots, \eta_{2n}) \in Sp(2n)$ such that

$$g_x = g(b + \text{diag}(q_n, \ldots, q_1, -q_1, \ldots, -q_n)),$$

and $q_1, \ldots, q_n$ are called the Lagrangian principle curvatures along $\gamma$. 
Example 4.7 (Lagrangian mKdV curve flow of A-type).
The third Lagrangian mKdV curve flow of A-type on $\mathbb{R}^{2,2}$ is
\[
\gamma_t = 4(q_2^2 q_3 + q_1 q_3 - q_2 q_3)\gamma + 4(q_3 - q_1 q_2)\eta_2 - 4q_2 \eta_3 + \eta_4.
\]

Example 4.8 (Lagrangian mKdV curve flow of C-type).
The third Lagrangian mKdV curve flow of C-type on $\mathbb{R}^{2,2}$ is
\[
\begin{align*}
\gamma_t & = -\frac{1}{8}(q_{2,xx} + 3q_{1,x}q_2 + 6q_{1,x}q_3 + 2q_2(q_1^2 - q_2^2))\gamma \\
& \quad + \frac{1}{4}(q_{1,x} - q_{2,x} + q_1^2 + q_2^2 + 4q_1 q_2)\eta_2 + q_2 \eta_3 + \eta_4.
\end{align*}
\]

5. The Bousinnessq hierarchy

In this section, we give an explicit example of the Bousinessq (or the $\hat{A}_2^{(1)}$-KdV) hierarchy. The second flow of the $\hat{A}_2^{(1)}$-KdV hierarchy is the following system:
\[
\begin{cases}
u_{1,t} = \nu_{1,xx} - \frac{2}{3} u_{2,xxx} + \frac{2}{3} u_{2} u_{2,x} , \\
u_{2,t} = -u_{2,xx} + 2u_{1,x}.
\end{cases}
\tag{5.1}
\]

It gives rise to solutions of the Boussinesq equation:
\[
u_{2,tt} = -\frac{1}{3} u_{2,xxxx} + \frac{4}{3} u_{2,xx} + \frac{4}{3} u_{2} u_{2,xx}.
\]

I. Let $\gamma$ be a centro-equiaffine curve on $\mathbb{R}^3 \setminus \{0\}$, and $\Gamma = (\gamma, \gamma_x, \gamma_{xx})$ the centro-equiaffine moving frame along $\gamma$ with $u_1, u_2$ the centro-equiaffine curvature. That is
\[
(\gamma, \gamma_x, \gamma_{xx})_x = (\gamma, \gamma_x, \gamma_{xx}) \begin{pmatrix} 0 & 0 & u_1 \\ 1 & 0 & u_2 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Note that if $\gamma$ satisfies the equation:
\[
\gamma_t = -\frac{2}{3} u_2 \gamma + \gamma_{xx}.
\]
Then $(u_1, u_2)$ is a solution of (5.1) (cf. [9]).

Consider the following centro-equiaffine parallel frame $g = (\gamma, \eta_2, \eta_3)$ with principle curvature $q_1, q_2$:
\[
(\gamma, \eta_2, \eta_3)_x = (\gamma, \eta_2, \eta_3) \begin{pmatrix} q_1 & 0 & 0 \\ 1 & q_2 & 0 \\ 0 & 1 & -q_1 - q_2 \end{pmatrix}.
\]

Note that
\[
\Gamma = g \begin{pmatrix} 1 & q_1 & q_1, x + q_1^2 \\ 0 & 1 & q_1 + q_2 \\ 0 & 0 & 1 \end{pmatrix}.
\]

This induces the Miura transformation from the principle curvatures $q_1, q_2$ to the central-equiaffine curvature $u_1$ and $u_2$:
\[
\begin{cases}
u_1 = q_{1,xx} + q_{1,x} q_1 - q_1 q_2, - (q_1 + q_2) q_1 q_2, \\
u_2 = 2q_{1,xx} + q_{2,xx} + q_1^2 + q_1 q_2 + q_2^2.
\end{cases}
\]
Consider the following bi-linear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$:

$$\langle X, Y \rangle = X^t \rho_1 Y, \quad \rho_1 = e_{22} - e_{13} - e_{31}.$$ 

If $\langle \gamma, \gamma \rangle = 1$, and $\langle \gamma_x, \gamma_x \rangle = 1$. Then there exist unique $p_3 \in \mathbb{R}^3$ satisfying

$$\langle \gamma, \gamma_x, p_3 \rangle \in O(2, 1).$$

This is the isotropic moving frame of $\gamma$, and the structure equation is

$$\langle \gamma, \gamma_x, p_3 \rangle_x = \langle \gamma, \gamma_x, p_3 \rangle \begin{pmatrix} 0 & u & 0 \\ 1 & 0 & u \\ 0 & 1 & 0 \end{pmatrix}.$$ 

If $\gamma$ is solution of

$$\gamma_t = -\frac{1}{9}(u_{xxx} - 8uu_x)\gamma + \frac{1}{9}(u_{xx} - 4u^2)\gamma_x,$$

then $u$ satisfies the KK equation (3.8).

On the other hand, let $(\gamma, \hat{\eta}_2, \hat{\eta}_3)$ be an isotropic parallel frame for $\gamma$, which means

$$\langle \gamma, \hat{\eta}_2, \hat{\eta}_3 \rangle_x = \langle \gamma, \hat{\eta}_2, \hat{\eta}_3 \rangle \begin{pmatrix} q & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -q \end{pmatrix}.$$ 

Then the isotropic moving frame and the isotropic parallel frame are related by the following gauge transformation:

$$\langle \gamma, \gamma_x, p_3 \rangle = \langle \gamma, \hat{\eta}_2, \hat{\eta}_3 \rangle \begin{pmatrix} 1 & q & \frac{1}{2}q^2 \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}.$$ 

So far, we can obtain the Miura transformations among the isotropic curvature $(u)$, isotropic principle curvature $(q)$, and the centro-equiaffine curvatures $(u_1, u_2)$ by the following formula:

$$u = \frac{1}{2}u_2 = q_x + \frac{1}{2}q^2.$$ 

And the isotropic condition induces the following reduction on central-equiaffine curvatures and principle curvatures:

$$u_1 = \frac{1}{2}u_{2,x}, \quad q_2 = 0.$$ 

6. Concluding remarks

In this paper, we present a scheme to study the geometric aspect of the Miura transformations among integrable hierarchies. In general, a transitive group action induces a natural frame along the space of curves that are invariant under the group. From differential geometry, there exists a set of parallel frames along such curves. The transformations from certain parallel frame to the frame induced by the group action naturally generate the generalized Miura transformation from the $G(1)(G(2)-mKdV$ hierarchy to the $G(1)(G(2))-KdV$ hierarchy associated. It also turns out that the Miura transformations set up the correspondence between invariant geometric flows in different geometries.
From algebraic structure of the mKdV-type hierarchies present in this paper, it is promising to set up corresponding factorization theory and construct Darboux transformations. Via the Miura transformation, it will give the Darboux transformations for the KdV-type hierarchies. Also being benefited from the algebraic structure, we are able to write down the explicit form of the differential operators which generated the mKdV-type hierarchies as invariant submanifolds of the Kadomtsev-Petviashvili (KP) hierarchy.

In an intriguing paper due to Olver and Rosenau [57], the tri-Hamiltonian duality approach was used to generate several kinds of Camassa-Holm-type equations or systems. It has been shown that the Liouville transformations can set up the correspondences between the hierarchies of the classical integrable equations and those of Camassa-Holm-type equations [39, 40, 50]. Using the Miura transformations between classical integrable and the Liouville correspondences, one is able to obtain the generalized Miura transformations for Camassa-Holm type equations. However, the geometric formulation of the generalized Miura transformations is not clear, and should be investigated further.

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