Stochastically perturbed flows: Delayed and interrupted evolution

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Abstract

We present analytical expressions for the time-dependent and stationary probability distributions corresponding to a stochastically perturbed one-dimensional flow with critical points, in two physically relevant situations: delayed evolution, in which the flow alternates with a quiescent state in which the variate remains frozen at its current value for random intervals of time; and interrupted evolution, in which the variate is also re-set in the quiescent state to a random value drawn from a fixed distribution. In the former case, the effect of the delay upon the first passage time statistics is analyzed. In the latter case, the conditions under which an extended stationary distribution can exist as a consequence of the competition between an attractor in the flow and the random re-setting are examined. We elucidate the role of

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the normalization condition in eliminating the singularities arising from the unstable critical points of the flow, and present a number of representative examples. A simple formula is obtained for the stationary distribution and interpreted physically. A similar interpretation is also given for the known formula for the stationary distribution in a full-fledged dichotomous flow.

**Keywords**: Randomly interrupted flow, critical points, time-dependent distribution, stationary distribution.
1 Introduction

Physical systems are invariably subject to stochastic perturbations of various kinds. The effects of noise upon the evolution of classical dynamical systems have been studied quite extensively for several decades now [1] - [4]. While much of the work in the area has been concerned with Gaussian noise (which offers several analytical advantages), other forms of noise are also relevant in specific instances. The inclusion of a finite correlation time for the noise (a closer approximation to physical reality) has also led to the uncovering of several interesting features that are absent in the idealized case of delta-correlated noise.

In many physical situations, the stochasticity affects the dynamics more strongly than as just a small perturbation of the deterministic evolution. An important case is that of random interruption of the deterministic flow of a system variable $x$; during each interruption (the ‘quiescent’ state), $x$ either remains frozen at its current value, or is re-set to a random value drawn from a prescribed distribution, and remains frozen there. Further evolution (the ‘active state’) is resumed after a random interval of time. We call these possibilities delayed evolution and interrupted evolution, respectively. A main purpose of this paper is to obtain exact expressions for both the time-dependent and the stationary probability distributions of the driven variable $x$ in both these situations. While problems of this general type have been studied right from the early days in a variety of specific contexts ranging
from chromatography \[5\] to the collision broadening of line shapes \[6\] and other relaxation phenomena \[7\], our focus here is on an altogether different aspect: the elucidation of the role of the stable and unstable critical points of the flow in determining these distributions.

The random interruptions are most simply modeled by assuming that they are switched on and off according to a Poisson pulse process of intensity $\lambda$ \[8\]. The mean duration $\lambda^{-1}$ of the active and quiescent states provides a time scale whose interplay with the characteristic time scale of the deterministic evolution will be of interest. (For simplicity we assume that the switching between the two states occurs at a common mean rate $\lambda$. All our results can be easily extended to the case of unequal switching rates in a straightforward manner.) We identify the active and quiescent states by the state labels $\xi = +1$ and $\xi = -1$ respectively. It is useful to note that the random variable $\xi(t)$ is in fact a symmetric, stationary dichotomous Markov process (DMP) with a correlation time $\tau_c = (2\lambda)^{-1}$. The conditional probability density $P(x, \xi, t|x_0, \xi_0)$ will occasionally be abbreviated to $P_{\pm}(x, t)$ when it is convenient to do so without causing confusion.

2 Delayed evolution
2.1 Solution for $P(x, \xi, t|x_0, \xi_0)$

Let the deterministic evolution (in the active or $\xi = +1$ state) be given by

$$\dot{x} = f_+(x). \tag{1}$$

The first step is to identify the critical points of the flow given by Eq. (1), i.e., the roots of $f_+(x) = 0$, and the direction in which the flow occurs in each interval between successive critical points. Clearly, for an initial condition $x(0) = x_0$ lying between two such points, $x(t)$ is restricted to this interval for all $t \geq 0$. We therefore expect $P_{\pm}(x, t)$ to involve step functions that reflect this fact.

The device we use to solve the problem of obtaining the exact time-dependent probability distributions for delayed evolution is to map the problem onto that of a particular dichotomous flow. Since $\dot{x}$ vanishes identically in the state $\xi = -1$, the evolution equation for $x$ at any time $t$ can be written as

$$\dot{x} = \left(\frac{1 + \xi(t)}{2}\right) f_+(x). \tag{2}$$

Equation (2) implies that \{x(t), \xi(t)\} constitute a stationary Markov process. Denoting the anti-derivative of $1/f_+(x)$ by $q(x)$, i.e., $q(x) = \int dx/f_+(x)$, we see that $\dot{q}$ is a DMP that switches between the values 1 and 0. Therefore the (non-stationary!) process $y = q - \frac{1}{2}t$ is given by the stochastic differential equation $\dot{y} = \zeta(t)$ where $\zeta(t)$ is a DMP switching between the values $\frac{1}{2}$ and $-\frac{1}{2}$: in other words, $y$ is a persistent diffusion process [11]. Its conditional
densities \( P_\pm(y, t) \) (retaining the symbol \( P \) for these, in a slight abuse of notation) therefore satisfy the telegrapher’s equation

\[
\left( \frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} - \frac{1}{4} \frac{\partial^2}{\partial y^2} \right) P_\pm(y, t) = 0 .
\]

The general solution of Eq. (3) can be written down [12] (using, for instance, the method of characteristics), once the initial conditions \( P_\pm(y, 0) \) and \( \dot{P}_\pm(y, 0) \) are specified. To obtain the latter, we begin with the initial conditions on the original densities, namely,

\[
P(x, \pm, 0|x_0, \mp) = \delta(x - x_0) , \quad P(x, \mp, 0|x_0, \pm) = 0 .
\]

These translate into the conditions

\[
P(y, \pm, 0|y_0, \pm) = \delta(y - y_0) , \quad P(y, \mp, 0|y_0, \mp) = 0
\]

for the densities of the \( y \)-process, where \( y_0 \equiv q(x_0) \). The initial conditions on the time derivatives of the densities are found by recalling that \( P_+(y, t) \) and \( P_-(y, t) \) actually satisfy the coupled first order equations \((\partial_t \pm \frac{1}{2} \partial_y + \lambda)P_\pm(y, t) = \lambda P_\mp(y, t)\). Insertion of Eqs. (5) in the latter leads to the initial conditions

\[
\dot{P}(y, \pm, 0|y_0, \pm) = \mp \frac{1}{2} \delta'(y - y_0) - \lambda \delta(y - y_0) , \quad \dot{P}(y, \mp, 0|y_0, \mp) = \lambda \delta(y - y_0) .
\]

Using Eqs. (4) and (5) in the general solution of Eq. (3) and simplifying, we finally arrive at the solutions given in Eqs. (10)-(12) below for the set of densities \( P(x, \xi, t|x_0, \xi_0) \). Let

\[
T(x_0, x) \equiv \int_{x_0}^{x} dx' / f_+(x') = q(x) - q(x_0)
\]
denote the time taken to move from $x_0$ to a given point $x$ under the (autonomous) flow of Eq. (4). Correspondingly, the location of the point reached under this flow in a given time interval $t$, starting from $x_0$, is given by the expression

$$q^{-1}(q(x_0) + t) \equiv X(x_0, t).$$

(8)

In reverting from $y$ to the original variable $x$, use is made of the relationship

$$\delta(x - X(x_0, t)) = \frac{\delta(t - T(x_0, x))}{|f_+(x)|}.$$

(9)

Then, omitting the arguments in $X(x_0, t)$ and $T(x_0, x)$ for brevity, we find:

$$P(x, +, t|x_0, +) = e^{-\lambda t} \left[ \delta(x - X) + \lambda \frac{\lambda}{|f_+(x)|} \left( \frac{T}{t - T} \right)^{\frac{3}{2}} I_1 \left( 2\lambda \sqrt{T(t-T)} \right) \theta(x - x_0) \theta(X - x) \right],$$

(10)

$$P(x, +, t|x_0, -) = P(x, -, t|x_0, +)$$

$$= \lambda e^{-\lambda t} \left[ \delta(x - X) + \lambda \frac{\lambda}{|f_+(x)|} \left( \frac{T}{t - T} \right)^{\frac{3}{2}} I_1 \left( 2\lambda \sqrt{T(t-T)} \right) \theta(x - x_0) \theta(X - x) \right],$$

(11)

and

$$P(x, -, t|x_0, -) = e^{-\lambda t} \left[ \delta(x - x_0) + \lambda \frac{\lambda}{|f_+(x)|} \left( \frac{T}{t - T} \right)^{\frac{3}{2}} I_1 \left( 2\lambda \sqrt{T(t-T)} \right) \theta(x - x_0) \theta(X - x) \right].$$

(12)

where $I_n$ is the modified Bessel function of order $n$. The step functions in Eqs. (10)-(12) refer to the situation in which $f_+(x) > 0$ in the interval concerned. They originally appear as $\theta(T) \theta(t - T)$, so that when $f_+(x) < 0$ the product of step functions in these equations is replaced by $\theta(x_0 - x) \theta(x - X)$.
2.2 First passage time distributions

An interesting question that arises naturally in the case of evolution that is stochastically delayed as above is the following: what is the effect of the delay on the corresponding level crossing (or first passage time) statistics? As the flow does not reverse direction in any interval between two successive critical points, it is clear that \( x(t) \) is a non-decreasing [respectively, non-increasing] function of \( t \) for all \( t \geq 0 \) if \( f_+(x) > 0 \) [respectively, \( f_+(x) < 0 \)] in the interval concerned. It follows at once that \( P(x, +, t|x_0, \xi_0) \, dx \) is also equal to the probability \( Q(t, x|x_0, \xi_0) \, dt \) of reaching the point \( x \) for the first time at time \( t \). Thus the quantity \( |f_+(x)| \, P(x, +, t|x_0, \xi_0) \equiv Q(t, x|x_0, \xi_0) \), expressed in terms of the appropriate variable (namely, \( t \)), is the first passage time density for reaching the point \( x \), starting at \( t = 0 \) from the point \( x_0 \) and in the state \( \xi_0 \). With the help of the useful relationship in Eq. (9) we obtain, after a bit of simplification,

\[
Q(t, x|x_0, +) = e^{-\lambda t} \left[ \delta(t - T) + \frac{d}{dt} I_0 \left( \frac{2\lambda}{\sqrt{T}} \sqrt{t - T} \right) \theta(t - T) \right]
\] (13)

and

\[
Q(t, x|x_0, -) = \lambda e^{-\lambda t} I_0 \left( \frac{2\lambda}{\sqrt{T}} \sqrt{t - T} \right) \theta(t - T).
\] (14)

The corresponding generating functions (i.e., Laplace transforms) are given by relatively simple expressions:

\[
\tilde{Q}(s, x|x_0, +) = e^{-(s+\lambda)T} e^{\lambda^2 T/(s+\lambda)}
\] (15)
and

\[ \tilde{Q}(s, x|x_0, -) = \frac{\lambda}{(s + \lambda)} e^{-(s+\lambda)T} e^{\lambda^2 T / (s+\lambda)}. \] (16)

As these expressions reduce to unity when \( s = 0 \), it is evident that the first passage time distributions concerned are properly normalized.

The mean first passage time is then

\[ \langle t(x_0 \to x) \rangle_+ = 2T(x_0, x), \quad \langle t(x_0 \to x) \rangle_- = 2T(x_0, x) + \lambda^{-1}, \] (17)

where the subscripts on the angular brackets indicate the starting state. The extra \( \lambda^{-1} \) in \( \langle t \rangle_- \) is understood as follows. We expect \( \langle t \rangle_- \) to exceed \( \langle t \rangle_+ \) by just the mean forward recurrence time in the \( \xi = -1 \) state; and since the DMP is governed by an uncorrelated Poisson process (implying an exponential waiting time density), this is the same as the mean duration of this state, i.e., \( \lambda^{-1} \).

The mean squares are

\[ \langle t^2(x_0 \to x) \rangle_+ = 4T^2 + 2T\lambda^{-1}, \quad \langle t^2(x_0 \to x) \rangle_- = 4T^2 + 6T\lambda^{-1} + 2\lambda^{-2}. \] (18)

The stationary a priori probabilities of being in the active and quiescent states being equal, the net mean first passage time and its mean square are therefore

\[ \langle t(x_0 \to x) \rangle = \frac{1}{2}(\langle t \rangle_+ + \langle t \rangle_-) = 2T(x_0, x) + (2\lambda)^{-1} = 2T(x_0, x) + \tau_c \] (19)
and

$$\langle t^2(x_0 \rightarrow x) \rangle = \frac{1}{2} (\langle t^2 \rangle_+ + \langle t^2 \rangle_-) = (2 T(x_0, x) + \lambda^{-1})^2 = 4 (T(x_0, x) + \tau_c)^2 ,$$

(20)

where we recall that $\tau_c = (2\lambda)^{-1}$ is the correlation time of the DMP $\xi(t)$ controlling the switching between the evolving and quiescent states. The relative fluctuation in the first passage time (the ratio of its standard deviation to its mean) is therefore given by

$$\Delta t(x_0 \rightarrow x) = \sqrt{\tau_c (4T + 3\tau_c)} \frac{2T + \tau_c}{2T + \tau_c}. \quad (21)$$

We note that $\Delta t$ tends to a constant value ($= \sqrt{3}$) as $T(x_0, x)/\tau_c \rightarrow 0$, while it decays like $(T(x_0, x)/\tau_c)^{-\frac{1}{2}}$ for very large values of this ratio.

These results can easily be generalized to the case of unequal mean durations of the evolving and quiescent states.

3 Interrupted evolution

3.1 Solution for $P(x, \xi, t|x_0, \xi_0)$

We turn now to the case when $x$ is re-set instantaneously to a random value drawn from a prescribed normalized distribution $\phi_-(x)$ whenever a transition occurs from the active or evolving state to the quiescent state, and remains fixed at that value till the active state occurs again. It is evident that the re-setting can be taken to occur at any instant during the quiescent state,
without affecting the results. We note that \( x \) may be re-set randomly anywhere in its total range \((-\infty, \infty)\), and not just in the interval between the two successive zeroes of \( f_+(x) \) that contains its initial value \( x_0 \).

Once again, it is helpful to regard the switching between states as being triggered by the DMP \( \xi(t) \). Denoting the transition probabilities for this DMP by \( p(\xi, t|\xi_0) \), we have

\[
p(\pm, t|\pm) = e^{-\lambda t} \cosh \lambda t, \quad p(\pm, t|\mp) = e^{-\lambda t} \sinh \lambda t.
\]  

(22)

Each of these probabilities tends to \( \frac{1}{2} \) as \( t \to \infty \). We therefore have the limit

\[
\lim_{t \to \infty} P(x, -, t|x_0, \xi_0) = \frac{1}{2} \phi_-(x),
\]  

(23)

where

\[
\int_{-\infty}^{\infty} \phi_-(x) dx = 1.
\]  

(24)

Similarly, we may ask whether

\[
\lim_{t \to \infty} P(x, +, t|x_0, \xi_0) = \frac{1}{2} \phi_+(x),
\]  

(25)

where \( \phi_+(x) \) represents the stationary probability density of \( x \) in the active state. The conditions under which a normalizable, extended density \( \phi_+(x) \) exists will be examined in detail in the next subsection.

Owing to the instantaneous re-setting of \( x \) in each occurrence of the quiescent state, it is no longer useful to map the problem to a stochastic differential equation. However, it is clear that it is only the last such re-setting that controls the location of the final point \( x \) at time \( t \), in each realization of
the random process. The exact expressions for \( P(x, \xi, t|x_0, \xi_0) \) are therefore easily found by enumerating the processes contributing to the probabilities and summing over the corresponding propagators. As \( \xi(t) \) is governed by a Poisson process of intensity \( \lambda \), the probability of a flip in \( \xi(t) \) occurring in an infinitesimal interval \( \delta t \) is \( \lambda \delta t \), while the probability of the persistence of either state of \( \xi \) for an interval \( t \) is \( \exp(-\lambda t) \). Using these facts, and starting with the simplest case, we find

\[
P(x, -, t|x_0, +) = \sum_{n=0}^{\infty} \int_0^t \frac{(\lambda t')^{2n}}{(2n)!} e^{-\lambda t'} e^{-\lambda(t-t')} \phi_-(x) = e^{-\lambda t} \sinh \lambda t \phi_-(x). \tag{26}\]

While this is of course identical to \( \phi_-(x) p(-, t|+) \) (and has no \( x_0 \)-dependence), as one would expect, it is interesting to note that \( P(x, -, t|x_0, -) \) is not simply equal to \( \phi_-(x) p(-, t|-) \). Owing to the zero-transition contribution we obtain, instead,

\[
P(x, -, t|x_0, -) = e^{-\lambda t} [\delta(x - x_0) + \phi_-(x) (\cosh \lambda t - 1)] . \tag{27}\]

The non-trivial cases (corresponding to a final state \( \xi(t) = +1 \)) can be analyzed similarly, to obtain the following results:

\[
P(x, +, t|x_0, +) = e^{-\lambda t} \left[ \delta(x - X(x_0, t)) + \lambda \int_{-\infty}^{\infty} dx' \int_0^t dt' \phi_-(x'(t - t')) \delta(x - X(x', t - t')) \sinh \lambda t' \right], \tag{28}\]

and

\[
P(x, +, t|x_0, -) = \lambda e^{-\lambda t} \int_{-\infty}^{\infty} dx' \int_0^t dt' \phi_-(x'(t - t')) \delta(x - X(x', t - t')) \cosh \lambda t'. \tag{29}\]
Here \( X(x', t - t') = q^{-1}(q(x') + t - t') \), in accordance with the definition in Eq. (8). The \( x_0 \)-dependence of \( P(x, +, t|x_0, -) \) is implicit in Eq. (29); it arises from the ‘one-transition’ contribution to \( P(x, +, t|x_0, -) \), and is given by \( \lambda t e^{-\lambda t} \phi_-(x_0) \). It is easily verified that the expressions in Eqs. (26)-(29) are correctly normalized, according to

\[
\int_{-\infty}^{\infty} P(x, \pm, t|x_0, \pm) \, dx = p(\pm, t|\pm),
\]

and

\[
\int_{-\infty}^{\infty} P(x, \pm, t|x_0, \mp) \, dx = p(\pm, t|\mp).
\]

The integration over \( x' \) in Eqs. (28) and (29) is obviously constrained by the \( \delta \)-function in each integrand: for a given \( x \), it is restricted to those points lying on the trajectory \( X(x', t - t') \) that are carried by the flow of Eq. (1), in a time ranging from 0 to \( t \), into the point \( x \) at precisely the instant \( t \). Using the \( \delta \)-functions to carry out the integration over \( t' \) rather than \( x' \) helps us express the results in a suggestive form. Equation (4) implies that

\[
\delta(x - X(x', t - t')) = \frac{\delta(t' - t + T(x', x))}{|f_+(x)|}.
\]

Using this in Eqs. (28) and (29), we finally get

\[
P(x, +, t|x_0, +) = e^{-\lambda t} \left[ \delta(x - X(x_0, t)) \right. + \frac{\lambda}{f_+(x)} \int_{X(x_0, -t)}^{x} dx' \phi_-(x') \sinh(\lambda(t - T(x', x))) \left. \right],
\]

and

\[
P(x, +, t|x_0, -) = \frac{\lambda e^{-\lambda t}}{f_+(x)} \int_{X(x_0, -t)}^{x} dx' \phi_-(x') \cosh(\lambda(t - T(x', x))).
\]

We note that it is \( f_+ \), rather than \( |f_+| \), that appears in these final expressions: The limits of integration as they stand here automatically ensure the
positivity of the probability densities concerned, because $X(x, -t)$ lies to the left [right] of $x$ if $f_+$ is positive [negative] in the relevant interval, for $t > 0$.

3.2 Stationary distribution in the active state

It is trivially seen from Eqs. (26) and (27) that $\lim_{t \to \infty} P(x, -t | x_0, \xi_0) = \frac{1}{2} \phi_-(x)$ as required. The interesting question is whether there exists a non-degenerate stationary distribution $\phi_+(x)$ in the active state, in accordance with Eq. (24). In other words, what is the eventual outcome of the competition between the attracting fixed point(s) towards which $x$ evolves each time it is in the active state, and the random re-setting it undergoes whenever it falls into the quiescent state?

The more familiar approach to this question would be to start with the differential equation for $P(x, +, t | x_0, \xi_0)$, namely, $(\partial_t + \partial_x f_+ + \lambda) P_+(x, t) = \lambda P_-(x, t)$, in the limit $t \to \infty$: the stationary distribution $\phi_+(x)$ must satisfy the first order equation

$$\left( \frac{d}{dx} f_+(x) + \lambda \right) \phi_+(x) = \phi_-(x). \quad (34)$$

Moreover, it must be non-negative and normalizable. Equation (34) does make it clear that $\phi_-(x)$ acts as the ‘source’ for $\phi_+(x)$. However, the disadvantage of this approach is that a constant of integration is explicitly involved: if $\phi_+(x) = \bar{\phi}_+$ at $x = \bar{x}$, then the formal solution of Eq. (34) reads

$$\phi_+(x) = \bar{\phi}_+ e^{-\int_{\bar{x}}^x F(x') dx'} + \int_{\bar{x}}^x dx' \left( \frac{\phi_-(x')}{f_+(x')} \right) \left( e^{-\int_{x'}^x F(x'') dx''} \right) (35)$$
where \( F(x) = (f_+(x) + \lambda)/f_+(x) \). It is not immediately clear how to choose \( \bar{x} \) and determine \( \bar{\phi}_+ \), and how the explicit dependence on these quantities disappears in the final result for \( \phi_+(x) \), as it must. Presumably, the other requirements on \( \phi_+(x) \) such as its normalizability play a role here. As this is an interesting (and instructive) question in its own right, we shall return to it shortly. Before doing so, however, we point out that the exact time-dependent solution in Eq. (32) (or Eq. (33)) already provides us with the following complete answer, circumventing these problems.

Let \( \alpha, \beta \) denote two successive zeroes of \( f_+(x) \), and let the flow \( \dot{x} = f_+(x) \) be directed from \( \alpha \) to \( \beta \) in the interval \((\alpha, \beta)\). As \( \alpha \) is a repellor, we have \( X(x, -\infty) = \alpha \) for \( x \in (\alpha, \beta) \). Therefore, passing to the limit \( t \to \infty \) in Eq. (32) or (33), we obtain

\[
\phi_+(x) = \frac{\lambda}{f_+(x)} \int_{\alpha}^{x} \phi_-(x') e^{-\lambda T(x', x)} dx', \quad x \in (\alpha, \beta).
\]  

(36)

The same expression is valid in each such interval between successive critical points; the integration over \( x' \) runs from the repellor in that interval to the point \( x \). Equation (36) not only specifies the exact solution for the stationary distribution in the active state, but also helps determine when such a solution exists, whether it is normalizable or not, when a pile-up or a divergence of the density occurs at the attracting fixed point, and when there is a “leakage of probability” into the absorbing state. As all these aspects are best illustrated by considering a set of typical examples as case studies, we now turn to these.

In what follows, \( \gamma \) is a positive constant denoting a characteristic time scale.
of the deterministic dynamics given by $\dot{x} = f_+(x)$.

(i) **Stable drift:** Consider the simple case $f_+(x) = -\gamma x$, corresponding to a simple attractor at $x = 0$. For definiteness, let us confine our attention to the region $x \geq 0$. The repellor in this region is $x = +\infty$, so that the formula of Eq. (36) yields in this case the exact solution

$$\phi_+(x) = \frac{\lambda x^{-1+\lambda/\gamma}}{\gamma} \int_x^{\infty} \phi_-(x') \frac{1}{(x')^{\lambda/\gamma}} dx'. \quad (37)$$

On the other hand, the formal solution given by Eq. (35) yields

$$\phi_+(x) = x^{-1+\lambda/\gamma} \left( \frac{\bar{\phi}_+}{x^{-1+\lambda/\gamma}} - \frac{\lambda}{\gamma} \int_x^\infty \frac{\phi_-(x')}{(x')^{\lambda/\gamma}} dx' \right). \quad (38)$$

We require that $\phi_+(x)$ be normalizable (i.e., integrable). However, the factor $x^{-1+\lambda/\gamma}$ leads to a divergence at $+\infty$ (the repellor), unless the factor in parentheses in Eq. (38) vanishes as $x \to +\infty$. Therefore we must impose the condition

$$\frac{\bar{\phi}_+}{\bar{x}^{-1+\lambda/\gamma}} = \frac{\lambda}{\gamma} \int_x^{\infty} \frac{\phi_-(x')}{(x')^{\lambda/\gamma}} dx', \quad (39)$$

which not only removes the divergence, but also eliminates all dependence on $\bar{x}$ and $\bar{\phi}_+$ and leads to precisely the normalizable density given by Eq. (37).

(ii) **Unstable drift; power law tail:** Now let $f_+(x) = +\gamma x$, corresponding to an unstable critical point at the origin. We consider the region $x \geq 0$ as before. As $x = 0$ is now the repellor, Eq. (36) yields

$$\phi_+(x) = \frac{\lambda}{\gamma x^{1+\lambda/\gamma}} \int_0^x \phi_-(x') (x')^{\lambda/\gamma} dx'. \quad (40)$$
On the other hand, applying Eq. (35) we obtain

$$\phi_+(x) = \frac{1}{x^{1+\lambda/\gamma}} \left( \tilde{\phi}_+ \tilde{x}^{1+\lambda/\gamma} + \frac{\lambda}{\gamma} \int_x^\infty \phi_-(x') (x')^{\lambda/\gamma} \, dx' \right). \quad (41)$$

The problem of normalizability now arises from the factor $x^{-1-\lambda/\gamma}$, and again occurs at the repelling critical point (here, at $x = 0$). To eliminate it, the factor in parentheses in Eq. (41) must be required to vanish as $x \to 0$. The condition to be imposed is therefore

$$\tilde{\phi}_+ \tilde{x}^{1+\lambda/\gamma} = \frac{\lambda}{\gamma} \int_0^x \phi_-(x') (x')^{\lambda/\gamma} \, dx', \quad (42)$$

leading once again to the correct solution as given by Eq. (40).

The following point is noteworthy. An inspection of Eq. (40) shows clearly that, in this case, the density $\phi_+(x)$ always has a power law tail. (When $\phi_-(x)$ is an even function of $x$, replacing $x$ with $|x|$ in Eq. (40) yields $\phi_+(x)$ in the entire range $-\infty < x < \infty$.) The random re-setting of $x$ in the quiescent state does compensate for the unstable drift towards infinity in the evolving state, but only to the extent of producing a normalizable stationary distribution in the latter state. It is insufficient to prevent a slow (power-law) fall-off of the latter, no matter how rapidly the fixed density $\phi_-(x)$ falls off for large $|x|$, or even if its support is compact. Even then, the variance of $x$ diverges unless $\lambda/\gamma > 2$, i.e., the switching rate is greater than twice the drift coefficient.

This example also represents a very direct way in which simple dynamics can generate the whole family of stable (Levy) distributions. As the (cumulative) distribution function of $x$ has a tail $\sim x^{-\lambda/\gamma}$, a (suitably re-scaled and
shifted) sum of such variates, independently distributed, will have a stable
distribution with exponent $\lambda/\gamma$, going over into a normal distribution for
$\lambda/\gamma \geq 2$.

(iii) *Higher order critical point; ‘leakage’ of probability*: Next, consider the
case $f_+(x) = \gamma x^2$. The flow is towards $x = 0$ for all negative $x$, while it is
directed away from this point and towards $+\infty$ for positive $x$. As before,
the exact solution for $\phi_+(x)$ may be found by using Eq. (36), or by using
Eq. (35) and imposing the conditions required to eliminate divergences and
ensure normalizability (at $-\infty$ for negative $x$, and at $x = 0$ for positive $x$).
The result is, for $x > 0$,

$$
\phi_+(x) = \frac{\lambda e^{\lambda/(\gamma x)}}{\gamma x^2} \int_0^x \phi_-(x') e^{-\lambda/(\gamma x')} \, dx' .
$$

(43)

For $x < 0$, the lower limit of integration is $-\infty$ instead of 0.

As in the preceding case, $\phi_+(x)$ has a power law tail. But the degenerate
critical point at the origin has an even stronger effect. Any positive initial
value $x_0$ reaches $+\infty$ in a *finite* time $(\gamma x_0)^{-1}$ under the flow in the evolving
state. We may therefore expect some sort of “absorption” at $+\infty$: in terms
of probability distributions, this would show up as a “leakage of probability”
leading to a deficit in the total probability. This is indeed borne out: we find
that

$$
\int_{-\infty}^0 \phi_+(x) \, dx = \int_{-\infty}^0 \phi_-(x) \, dx ,
$$

(44)

but

$$
\int_0^\infty \phi_+(x) \, dx = \int_0^\infty \phi_-(x) \left(1 - e^{-\lambda/(\gamma x)}\right) \, dx .
$$

(45)
This loss in probability measure can also be given a simple interpretation. The probability of being injected into $(x, x + dx)$ under the random re-setting is $\phi_-(x)dx$. The time taken to reach $+\infty$ from any $x > 0$ under the flow is $(\gamma x)^{-1}$. The probability of remaining in the evolving state for this duration is $\exp(-\lambda/\gamma x)$. Therefore the total loss in measure is $\int_0^\infty \phi_-(x) \exp(-\lambda/\gamma x) dx$.

In general, such a loss of measure occurs whenever the absorbing state at the attracting critical point is reached in a finite time under the deterministic flow, as will become clear from the examples to follow.

It is also an instructive exercise to “unfold” the degenerate critical point at $x = 0$ by starting, for instance, with $f_+(x) = \gamma x(x - \epsilon)$, and then examining how various quantities behave as $\epsilon \to 0$.

(iv) Periodic boundary conditions; single critical point : In many physical applications, $x$ is an angular variable, so that its range is compact, and periodic boundary conditions apply. Consider the case $\dot{x} = \gamma |\sin x|$, with a fundamental period equal to $\pi$. There is only one critical point, and the flow is directed from 0 towards $\pi$. Given that $\phi_-(x) = \phi_-(x + \pi)$, we seek a solution $\phi_+(x)$ that has the same periodicity property. Using Eq. (36), we find

$$\phi_+(x) = \frac{\lambda}{\gamma \sin x} \int_0^x \left( \frac{\tan x'/2}{\tan x/2} \right)^{\lambda/\gamma} \phi_-(x') dx' , \quad x \in [0, \pi) , \quad (46)$$

together with $\phi_+(x) = \phi_+(x + \pi)$. The time taken to reach $\pi$ from any $x$ in $0 < x < \pi$ is infinite (although the range is finite, the velocity $f_+(x)$ vanishes as $x \to \pi$). Therefore there is no loss of measure in this case. However, a
pile-up of the density can occur at $x = \pi$, depending on the value of the ratio $\lambda/\gamma$: for a uniform $\phi_-(x)$, for instance, we can show from Eq. (46) that $\phi_+(x)$ is finite at $x = \pi$ only if the switching rate $\lambda > \gamma$. As $x \uparrow \pi$, we find $\phi_+(x) \sim -\ln (\pi - x)$ for $\lambda = \gamma$, and $\phi_+(x) \sim (\pi - x)^{-1+\lambda/\gamma}$ for $\lambda < \gamma$. Thus $\phi_+(x)$ is divergent (though integrable) at the attractor, unless the switching rate exceeds the drift coefficient.

(v) Periodic boundary conditions; degenerate critical point: As the periodic analog of Case (iii) above, consider $f_+(x) = \gamma \sin^2 x$ with the fundamental interval $[0, \pi)$. We find the periodic solution

$$\phi_+(x) = \frac{\lambda}{\gamma \sin^2 x} e^{(\lambda/\gamma) \cot x} \int_0^x \phi_-(x') e^{-(\lambda/\gamma) \cot x'} \, dx', \quad x \in [0, \pi). \quad (47)$$

Again, as the time taken to reach $\pi$ from any $x$ in $0 < x < \pi$ is infinite, there is no loss of measure owing to absorption at $\pi$. Moreover, now there is no divergence of $\phi_+(x)$ at $x = \pi$, either, because the velocity $f_+(x)$ vanishes more rapidly than linearly as $x \to \pi$.

A flow like $\dot{x} = \gamma x$ with periodic boundary conditions ($[0, 1]$ being the fundamental interval, for instance), will produce a leakage of probability into the (finite) absorbing point at $x = 1$: now the velocity does not vanish as $x \uparrow 1$. It is easily shown that $\int_0^1 \phi_+(x) \, dx = \int_0^1 \phi_-(x) (1 - x^{\lambda/\gamma}) \, dx$ in this case.

(vi) Periodic boundary conditions; two critical points: Finally, let us consider the case $\dot{x} = \gamma \sin x$, with a period equal to $2\pi$. The velocity $f_+(x)$ changes sign in the fundamental interval, and there is a repellor (at 0) as well as
an attractor (at \( \pi \)) in it. In the range \( 0 \leq x < \pi \), the solution for \( \phi_+(x) \) is exactly as in Eq. (46); for \( \pi \leq x < 2\pi \), the lower limit of integration is replaced by \( 2\pi \). The divergent behavior found as \( x \uparrow \pi \) when \( \lambda \leq \gamma \) in Case (iv) above now occurs symmetrically on both sides of \( x = \pi \), as one would expect.

4 Stationary distribution in dichotomous flow: some remarks

The form of the stationary distribution in Eq. (36) and the discussion following it suggest a direct physical interpretation (amounting, in fact, to a simple heuristic derivation) of the well-known formula [13], [14] for the stationary distribution that obtains (under certain conditions) in the case of the full-fledged dichotomous flow given by the stochastic differential equation

\[ \dot{x} = f(x) + g(x)\xi(t) . \]  

(48)

Such a flow corresponds to the random alternation of two deterministic flows, given by \( \dot{x} = f_+(x) \) and \( \dot{x} = f_-(x) \), respectively, where \( f_\pm(x) \equiv f(x) \pm g(x) \).

It serves as a model for numerous physical phenomena (see, e.g., Refs. [14], [12]).

The time-dependent probability densities \( P(x, \xi, t|x_0, \xi_0) \) (denoted by \( P_\pm(x, t) \) for short) now satisfy the coupled first order equations (\( \partial_t + \partial_x f_\pm + \lambda \)) \( P_\pm(x, t) = \lambda P_\mp(x, t) \). In general, it is not possible to obtain a partial differential equa-
tion of finite order for the total probability density \( P(x, t) \equiv P_+(x, t) + P_-(x, t) \): this possibility is restricted to certain special cases [10]-[18]. However, if the deterministic flow is “dynamically stable”, the stationary density \( P^{st}(x) = \lim_{t \to \infty} P(x, t) \) is found to satisfy the first order ordinary differential equation

\[
\frac{d}{dx} \left( (f + g)(1 - \frac{f}{g})P^{st} \right) - 2\lambda \frac{f}{g} P^{st} = 0.
\] (49)

Here, the term “dynamic stability” denotes precisely the situation of interest in the present context: the existence of two different stable critical points in the flows \( \dot{x} = f_+(x) \) and \( \dot{x} = f_-(x) \) respectively, with no other critical point of either flow in between them. What is sought is the stationary distribution to which \( x \) settles down under the competition between the two attractors. (Thus the second attractor takes on the role played in the preceding section by the random re-setting of \( x \).) A typical case that helps visualize the situation is as follows: a repellor at \( x = \beta \) and an attractor at \( x = b \) in the flow \( f_+(x) \); and an attractor at \( x = a \) and a repellor at \( x = \alpha \) in the flow \( f_-(x) \). If \( \beta \leq a < b \leq \alpha \), the interval \([a, b]\) acts as a trapping region supporting a non-trivial \( P^{st}(x) \). The latter is given [13], [14] by the solution of Eq. (49), namely,

\[
P^{st}(x) = N \frac{g}{g^2 - f^2} \exp \left( 2\lambda \int \frac{f dx}{g^2 - f^2} \right),
\] (50)

where \( N \) is the normalization constant. We give a simple interpretation of this formula along the lines of that presented for Eq. (36) above, which suggests how the expression in Eq. (50) can virtually be written down by
inspection.

Consider, first, what happens if the system alternates regularly between the two flows, with a constant duration \( \tau \) in each state. Let us denote the solution trajectories in the two flows by \( X_+(x, t) \) and \( X_-(x, t) \), respectively. Thus, if we start in the + state from the point \( x_0 \), we have \( x(\tau) = X_+(x_0, \tau) \), \( x(2\tau) = X_-(x(\tau), \tau) \), and so on. Setting \( x(2n\tau) \equiv \eta_n \) and \( x((2n+1)\tau) \equiv \zeta_n \) (where \( n = 0, 1, \ldots \)), the dynamics is given by the two-dimensional map

\[
\zeta_n = X_+(\eta_n), \quad \eta_n = X_-(\zeta_n).
\] (51)

In the “dynamically stable” situation referred to above, there exist stable fixed points \( \eta^* \) and \( \zeta^* \) such that \( \eta^* = X_-(X_+(\eta^*)) \), \( \zeta^* = X_+(X_-(\zeta^*)) \), implying a stable period-two cycle into which the original variable \( x(n\tau) \) falls asymptotically as \( n \to \infty \). Correspondingly, in continuous time the trajectory \( x(t) \) zig-zags between the values \( \eta^* \) and \( \zeta^* \), alternately following the two evolution rules. An invariant density for \( x \) can therefore be defined in each of the two distinct alternating segments. Calling these \( \phi_+(x) \) and \( \phi_-(x) \), respectively, it is evident that \( \phi_\pm(x)dx |_{\pm} = \tau^{-1}dt \), or \( \phi_\pm(x) = (\tau |f_\pm(x)|)^{-1} \), i.e., essentially just the reciprocal of the corresponding Jacobian. As the + and − states are equally probable, the invariant density of \( x \) becomes

\[
P^{\text{st}}(x) = \frac{1}{2\tau} \left( \frac{1}{|f_+(x)|} + \frac{1}{|f_-(x)|} \right).
\] (52)

Taking into account the fact that the flows are in opposite directions, i.e., that \( f_+(x) \) and \( f_-(x) \) have opposite signs, the quantity in parentheses in Eq.
is proportional to \( f_+^{-1} - f_-^{-1} = g(g^2 - f^2)^{-1} \). This explains the origin of the factor preceding the exponential in Eq. \((50)\).

It remains to understand the exponential factor in Eq. \((50)\). Returning to the case of random switching between the two flows, it is evident that the weight factor \( \tau^{-1} \) in Eq. \((52)\), applicable when the switching occurs at regular intervals, must be replaced by a product of two probability factors, one for each of the two flows. This factor is the same as that already discussed in relation with Eq. \((36)\) in the case of interrupted evolution, namely, an exponential of the time required to reach the point \( x \) from the corresponding repellor - or rather, from the attractor in the other flow, recalling that the trapping region (the support of \( P_{st}(x) \)) is bounded by these points. Thus, in the example mentioned above (in which \( \beta \leq a < b \leq \alpha \)), we have the factor \( \exp[-\lambda T_+(a, x)] \exp[-\lambda T_-(b, x)] \), where we have used subscripts on \( T \) to indicate the corresponding flow. Since \( T_\pm(u, x) = \int_a^u dx'/f_\pm(x') \) and \( f_\pm = f \pm g \), we find

\[
T_+(a, x) + T_-(b, x) = \int_a^x \frac{dx'}{g(x') + f(x')} + \int_x^b \frac{dx'}{g(x') - f(x')} = -2 \int_a^x \frac{f \, dx'}{g^2 - f^2} + \text{const.} \tag{53}
\]

This yields precisely the exponential factor occurring in the solution given by Eq. \((50)\). The “mechanism” underlying this formula is therefore made more manifest by re-writing it in the form

\[
P_{st}(x) = \frac{\mathcal{N}}{2} \left( |f_+(x)|^{-1} + |f_-(x)|^{-1} \right) e^{-\lambda T_+(a, x)} e^{-\lambda T_-(b, x)} \tag{54}
\]
for the configuration of attractors under consideration, with obvious minor modifications for the other possible configurations.

A simple example is provided by the exactly solvable case \cite{17} of linear dichotomous flow, \( \dot{x} = -\gamma x + c \xi(t) \), where \( \gamma \) and \( c \) are positive constants. Here \( \beta = -\infty \), \( a = -c/\gamma \), \( b = c/\gamma \), \( \alpha = \infty \). The stationary density of \( x \) has support in the interval \([-c/\gamma, c/\gamma]\), and is proportional to \( (c^2 - \gamma^2 x^2)^{-1+\lambda/\gamma} \).

We can now readily understand its interesting features as follows. Since \( T_+(x_0, c/\gamma) \) and \( T_-(x_0, -c/\gamma) \) diverge, there is no loss of probability measure owing to the absorption at the attractors in this instance. However, as we may now anticipate (based on the remarks made in Case (iv) above), \( P^{st}(x) \) is indeed divergent at the (finite) endpoints \( \pm c/\gamma \) when the switching rate \( \lambda \) is smaller than the drift coefficient \( \gamma \).

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[8] This is the assumption generally made in modeling the effective Liouville operator corresponding to stochastically interrupted deterministic evolution (or unitary evolution, in the quantum mechanical case) in most physical applications (see, e.g., Ref. [⁶]). It represents a kind of Markovian assumption (e.g., $x$ then turns out to be a component of a Markov process). Non-Markovian generalizations based on interruptions
governed by an arbitrary renewal process rather than a Poisson process are possible [10]. However, as stated earlier, our present emphasis is on a different aspect of the problem.

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