Accelerated adiabatic dynamics in a triangular spin cluster

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Abstract. We propose a scheme of the fast forward of adiabatic spin dynamics in a triangular spin cluster. We settle the quasi-adiabatic spin dynamics (QASD) by adding the regularization terms to the original Hamiltonian and accelerate it with use of a large time-scaling factor which realizes QASD on shortened time scale. Assuming the candidate regularization Hamiltonian consisting of three-body interactions besides the pair-wise exchange interactions and magnetic field, we solved the regularization terms. These terms multiplied by the velocity function give rise to the state-dependent counter-diabatic terms (CDTs) for each of adiabatic states. Our fast forward Hamiltonian proves to generate the plural number of state-dependent CDTs due to the driving three-body interactions. Applying this scheme to a simple transverse Ising model, we find 2 CDTs which contains both two-body and three body interaction. The driving two and three-body interaction in the fast-forward scheme guarantees the complete fidelity of accelerated states.

1. Introduction
Current technological developments are concerned with manipulating and optimizing systems’ dynamics. On a microscopic scale, systems’ manipulation is performed on electrons or atomic particles and their wave functions. In particular, a shorter time in manufacturing products is becoming an important factor in nanotechnology. If we try to fabricate massive amount of nanoscale structures, we should accelerate the dynamics of each electron, atom or molecules to reach their desired target states in shorter time. In designing quantum computers, the coherence of systems is degraded by their interaction with the environment, and therefore the acceleration of adiabatic quantum dynamics is highly desirable. Masuda and Nakamura [1] proposed a theory of fast forward to accelerate quantum dynamics with use of additional phase and driving potential, and Khujakulov and Nakamura [2] showed that this theory is useful to enhance the quantum tunneling power. The theory of fast-forward guarantees to accelerate any given quantum evolution and to obtain the desired target state on shortened time scale, by fast forwarding the reference quantum dynamics. This theory was further developed to accelerate the quasi-static or adiabatic quantum dynamics [3, 4, 5], and constitutes one of the promising means to the shortcut to
adiabaticity (STA) [6, 7, 8, 9, 10, 11, 12, 13]. The relationship between the fast forward and STA is nowadays clear [5, 13, 14]. While the theory of fast forward has been limited to orbital dynamics of atoms, molecules or Bose-Einstein condensates, we recently proposed a scheme of fast forward of adiabatic spin dynamics of quantum entangled states [15]. Confining to coupled two spin systems, we settled the quasi-adiabatic dynamics by adding the regularization terms to the original (reference) Hamiltonian and then accelerated it with use of a large time-scaling factor. Assuming a candidate regularization Hamiltonian consisting of the pair-wise exchange interactions and magnetic field, we solved the regularization terms. These terms multiplied by the velocity function give rise to the plural number of state-dependent counter-diabatic terms for each of adiabatic states, which is distinct from the issue of the method of transitionless quantum driving by Demirplak-Rice-Berry (to be abbreviated below as DRB) [6, 7, 8].

As a starting point of many spin systems to be driven by multi-spin interactions, we can conceive triangular 3 spin cluster of Fe$^{2+}$ magnetic ions around each Rb$^+$ ion in a triangular RbFeCl$_3$ antiferromagnet, and also three trapped atomic $^{171}$Yb$^+$ ions, where the effective spin-1/2 system is represented by the $^2S_{1/2} F = 1, m_F = 0$ and $F = 0, m_F = 0$ hyperfine ‘clock’ states in each ion. The triangular 3 spin cluster was extensively studied in the context of quantum chaos [16], topological order parameter [17, 18] and quantum simulation of the frustration [19]. It is now worthwhile to investigate the triangular 3 spin cluster in the context of fast forward, even though the application of the fast forward scheme to $N(\geq 4)$-spin systems will remain as a future work.

2. Fast-forward of adiabatic spin dynamics.

For self-containedness, we shall sketch the scheme of fast forward of adiabatic spin dynamics [15]. Our strategy is as follows: (i) A given original Hamiltonian $H_0$ is assumed to change adiabatically and to generate a stationary state $\Psi_c$ which is an eigenstate of the time-independent Schrödinger equation with the instantaneous Hamiltonian. Then $H_0$ is regularized so that it should satisfy the time-dependent Schrödinger equation (TDSE); (ii) Taking the regularized state as a reference state, we shall change the time scaling with use of the scaling factor $a(t)$, where the mean value $\bar{a}$ of the infinitely-large time scaling factor $a(t)$ will be chosen to compensate the infinitesimally-small growth rate $\varepsilon$ of the quasi-adiabatic parameter and to satisfy $\bar{a} \times \varepsilon = \text{finite}$.

To be explicit, consider the Hamiltonian for spin systems to be characterized by a slowly time-changing parameter $R(t)$ such as the exchange interaction, magnetic field, etc. Then we can study the eigenvalue problem for the time-independent Schrödinger equation:

$$H_0(R)C(R) = E(R)C(R)$$

with

$$C(R) = \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix}$$

where

$$R \equiv R(t) = R_n + \varepsilon t$$

is the adiabatically-changing parameter with $\varepsilon \ll 1$. In Eq.(1), the quantum number $n$ for each eigenvalue and eigenstate is suppressed for simplicity. Let us assume

$$\Psi_0(R(t)) = C(R)e^{\frac{i}{\hbar} \int_0^t E(R(t'))dt'}e^{i\tilde{g}(R(t))},$$

with

$$E(R)C(R) = G(R)C(R),$$

and

$$G(R) = \begin{pmatrix} G_1(R) \\ \vdots \\ G_N(R) \end{pmatrix}.$$
to be a quasi-adiabatic state, i.e., adiabatically evolving state, where $\xi$ is the adiabatic phase:

$$\xi(R(t)) = i \int_0^t dt' C^\dagger \partial_t C = i \varepsilon \int_0^t dt' C^\dagger \partial_\varepsilon C. \quad (5)$$

To make $\Psi_0(R(t))$ in Eq.(4) to satisfy the TDSE, we must regularize the Hamiltonian as

$$H_0^\text{reg}(R(t)) = H_0(R(t)) + \varepsilon \hat{\mathcal{R}}_n(R(t)) \quad (6)$$

Then TDSE becomes

$$ih \frac{\partial}{\partial t} \Psi_0(R(t)) = (H_0 + \varepsilon \hat{\mathcal{R}}_n) \Psi_0(R(t)). \quad (7)$$

Here $\hat{\mathcal{R}}_n$ is the $n$-th state-dependent regularization term. Substituting $\Psi_0(R(t))$ in Eq.(4) into the above TDSE, we see Eq.(1) in order of $O(\varepsilon^0)$, and in order of $O(\varepsilon^1)$

$$\hat{\mathcal{R}}_n C(R) = i h \partial_R C(R) - i h (C^\dagger \partial_\varepsilon C) C(R), \quad (8)$$

which is the core equation of the present work.

The fast-forward state is defined by

$$\Psi_{FF}(t) \equiv C(R(\Lambda(t))) e^{-\frac{i}{\hbar} \int_0^t E(R(\Lambda(t))) dt'} e^{i \xi(R(\Lambda(t)))}, \quad (9)$$

where $\Lambda(t)$ is an advanced time defined by

$$\Lambda(t) = \int_0^t \alpha(t') dt', \quad (10)$$

with the standard time $t$. $\alpha(t)$ is a magnification time-scale factor given by $\alpha(0) = 1$, $\alpha(t) > 1$ ($0 < t < T_{FF}$) and $\alpha(t) = 1$ ($t \geq T_{FF}$). We consider the fast forward dynamics with a new time variable which reproduces the target state $\Psi_0(R(T))$ in a shorter final time $T_{FF}$ defined by

$$T = \int_0^{T_{FF}} \alpha(t) dt. \quad (11)$$

The explicit expression for $\alpha(t)$ in the fast-forward range ($0 \leq t \leq T_{FF}$) is typically given by [3] as:

$$\alpha(t) = \bar{\alpha} - (\bar{\alpha} - 1) \cos(\frac{2\pi}{T_{FF}} t), \quad (12)$$

where $\bar{\alpha}$ is the mean value of $\alpha(t)$ and is given by $\bar{\alpha} = T/T_{FF}$. Then by taking the time derivative of $\Psi_{FF}$ in Eq.(9) and using the equalities $\partial_\varepsilon C(R(\Lambda(t))) = \alpha \varepsilon \partial_R C$ and $\partial_\varepsilon \xi(R(\Lambda(t))) = i C^\dagger \partial_\varepsilon C = i \alpha \varepsilon C^\dagger \partial_\varepsilon C$, we have

$$ih \Psi_{FF} = \left[ i h \alpha \varepsilon (\partial_\varepsilon C - (C^\dagger \partial_\varepsilon C) C) + EC \right] e^{\frac{i}{\hbar} \int_0^t E(R(\Lambda(t))) dt'} e^{i \xi(R(\Lambda(t)))}, \quad (13)$$

The first and second terms in the angular bracket on the r.h.s are replaced by $\alpha \varepsilon \hat{\mathcal{R}}_n C(R(\Lambda(t)))$ and $H_0 C(R(\Lambda(t)))$, respectively, by using Eqs.(8) and (1). Using the definition of $\Psi_{FF}(t)$ and taking the asymptotic limit $\bar{\alpha} \to \infty$ and $\varepsilon \to 0$ under the constraint $\bar{\alpha} \cdot \varepsilon \equiv \bar{\alpha} \varepsilon = \text{finite}$, we obtain

$$ih \frac{\partial \Psi_{FF}}{\partial t} = (v(t) \hat{\mathcal{R}}_n(R(\Lambda(t))) + H_0(R(\Lambda(t)))) \Psi_{FF} \equiv H_{FF} \Psi_{FF} . \quad (14)$$

\[3\]
Here $v(t)$ is a velocity function available from $\alpha(t)$ in the asymptotic limit:

$$v(t) = \lim_{\varepsilon \to 0, R \to \infty} \varepsilon \alpha(t) = \tilde{v} \left( 1 - \cos \frac{2\pi T_{FF}}{\varepsilon} t \right).$$

(15)

Consequently, for $0 \leq t \leq T_{FF}$,

$$R(\Lambda(t)) = R_0 + \lim_{\varepsilon \to 0, R \to \infty} \varepsilon \Lambda(t) = R_0 + \int_0^t v(t') dt'$$

$$= R_0 + \tilde{v} \left[ t - \frac{T_{FF}}{2\pi} \sin \left( \frac{2\pi}{T_{FF}} t \right) \right].$$

(16)

$H_{FF}$ is the driving Hamiltonian and $\tilde{H}_n$ is the regularization term obtained from Eq.(8) to generate the fast-forward scheme in spin system.

As a simple example we consider the adiabatic dynamics of a single spin in the magnetic field which is rotated adiabatically from $z$ to $-z$ direction while its magnitude is kept constant. Let the magnetic field be written as

$$B = B \begin{pmatrix} \sin \theta(t) \cos \varphi \\ \sin \theta(t) \sin \varphi \\ \cos \theta(t) \end{pmatrix},$$

(17)

where

$$\theta(t) = R(t) = \varepsilon t,$$

(18)

and $\varphi = \text{const.}$ The corresponding spin Hamiltonian is given by

$$H(R(t)) = \frac{1}{2} \sigma \cdot B$$

$$= \frac{B}{2} \begin{pmatrix} \cos R(t) & \sin R(t) e^{-i\varphi} \\ \sin R(t) e^{i\varphi} & -\cos R(t) \end{pmatrix}.$$  

(19)

The eigenvalues are $\lambda_\pm = \pm \frac{B}{2}$, and the eigenstates are written as

$$\Psi_0^+ = \begin{pmatrix} c_1^+ \\ c_2^+ \end{pmatrix} = \begin{pmatrix} \cos \frac{R(t)}{2} \\ e^{i\varphi} \sin \frac{R(t)}{2} \end{pmatrix}.$$ 

(20)

and

$$\Psi_0^- = \begin{pmatrix} c_1^- \\ c_2^- \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} \sin \frac{R(t)}{2} \\ -\cos \frac{R(t)}{2} \end{pmatrix}.$$ 

(21)

Now we choose one of the states $\Psi_0^+$ with $\lambda_+$, and consider the adiabatic dynamics. Noting $\xi = 0$, the adiabatically-evolving state is:

$$\Psi_0(t) = \begin{pmatrix} \cos \frac{R(t)}{2} \\ e^{i\varphi} \sin \frac{R(t)}{2} \end{pmatrix} e^{-iBt/2}. $$

(22)

Noting that $\tilde{H}_{ij}$ is traceless ($\tilde{H}_{11} = -\tilde{H}_{22}$) and Hermitian ($\tilde{H}_{11}^* = \tilde{H}_{11}$), Eq.(8) constitutes a rank = 2 linear algebraic equation for two unknowns ($\tilde{H}_{11}$ and $\tilde{H}_{12}$). We can solve Eq.(8) for $\tilde{H}$ as $\tilde{H}_{11} = 0$ and
The state-dependent counter-diabatic term and the fast-forward Hamiltonian are now written as

\[
\hat{H}_{12} = -\frac{\hbar}{2} (\sin \varphi + i \cos \varphi).
\]

The fast forward state is written as

\[
\hat{H} = v(t) \hat{H} = \begin{pmatrix} 0 & -v(t) \frac{\hbar}{2} (\sin \varphi + i \cos \varphi) \\ -v(t) \frac{\hbar}{2} (\sin \varphi - i \cos \varphi) & 0 \end{pmatrix}
\]

and

\[
H_{FF} = \begin{pmatrix} \frac{B}{2} \cos R(\Lambda(t)) & \frac{B}{2} \sin R(\Lambda(t)) e^{-i\varphi} - v(t) \frac{\hbar}{2} (\sin \varphi + i \cos \varphi) \\ \frac{B}{2} \sin R(\Lambda(t)) e^{i\varphi} - v(t) \frac{\hbar}{2} (\sin \varphi - i \cos \varphi) & -\frac{B}{2} \cos R(\Lambda(t)) \end{pmatrix},
\]

respectively.

Choosing another state \(\Psi_0^-\) with the eigenvalue \(\lambda_\perp\), we can obtain the same result in Eqs.(23) and (24). Therefore the counter-diabatic term here proves to be state-independent.

3. Triangular 3 spin cluster

As a model of three-spin systems, we consider the triangular 3 spin cluster shown in Fig.1. On the basis of this configuration, we shall investigate simple transverse Ising model. It should be noted: our interest lies in obtaining a variety of driving fields or counter-diabatic terms for the triangular spin cluster rather than finding the nontrivial ground state of many-spin systems.

Here, the number of independent equations in Eq.(8) is less than that of the unknown \(\{\hat{H}_{ij}\}\) \((1 \leq i, j \leq 3)\), noting the dimension of Hilbert space= \(2^3\) for the 3-spin system. Therefore some extra strategy should be introduced. We assume a form for the regularization term \(\hat{H}_{reg}\) in Eq.(8) so as to include the diagonal-exchange interaction \(\hat{J}_i = \hat{J}_i(\text{et})\), \(\hat{J}_j = \hat{J}_j(\text{et})\), \(\hat{J}_k = \hat{J}_k(\text{et})\), off-diagonal-exchange interaction \(\bar{W}_i = \bar{W}_i(\text{et})\), \(\bar{W}_j = \bar{W}_j(\text{et})\), \(\bar{W}_k = \bar{W}_k(\text{et})\), three-body interaction \(\bar{Q} = \bar{Q}(\text{et})\), and 3-component magnetic field \(\bar{B} = \bar{B}(\text{et})\). The candidate for regularization Hamiltonian \(\hat{H}\) takes the following form:

![Figure 1. Triangular 3 spin cluster](image)
The single-body \((\mathbf{B})\) and two-body \((\mathbf{J}, \mathbf{W})\) terms are just the extension of the corresponding terms in the two-spin system in [15]. The three-body interaction \((\mathbf{Q})\) is new and is introduced by being inspired by the scalar chirality [21, 22, 23]:

\[
\mathbf{Q} = \frac{1}{2} (\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3) \cdot \mathbf{B},
\]

which is invariant against the cyclic change of site indices.

Arranging the bases as \(\uparrow\uparrow\uparrow, \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow, \downarrow\downarrow\uparrow, \downarrow\downarrow\downarrow, \) and \(\uparrow\downarrow\downarrow\), we obtain

\[
\begin{pmatrix}
E & A & A & A & A + B & A + B & A + B & 0 \\
A' & E_1 & E_2 & E_2 & C & C & 0 & B + D \\
A' & E_2 & E_1 & E_2 & C & 0 & C & B + D \\
A' & E_2 & E_2 & E_1 & 0 & C & C & B + D \\
A' & E_2 & E_2 & E_2 & E_1 & E_1 & E_1 & F \\
A' & E_2 & E_2 & E_1 & E_2 & E_1 & E_1 & F \\
A' & E_2 & E_2 & E_1 & E_2 & E_1 & E_1 & F \\
A' & E_2 & E_2 & E_1 & E_2 & E_1 & E_1 & F \\
\end{pmatrix}
\]

where \(A = -i2\mathbf{W}_2 + 2\mathbf{W}_2 + C\), \(B = \mathbf{J}_1 + \mathbf{J}_2 + 2i\mathbf{W}_1\), \(C = \frac{1}{2}(\mathbf{B}_x - i\mathbf{B}_y)\), \(D = -2i\mathbf{Q}\), \(E = 3\mathbf{J}_3 + \frac{3\mathbf{E}}{2}\), \(E_1 = -\mathbf{J}_3 + \frac{\mathbf{E}}{2}\), \(E_2 = \mathbf{J}_1 + \mathbf{J}_2\), \(E_3 = 3\mathbf{J}_3 - \frac{3\mathbf{E}}{2}\), and \(F = i2\mathbf{W}_2 - 2\mathbf{W}_3 + C\). The explicit expression for \(\mathbf{R}_t\) in Eq. (27) will help us to solve Eq. (8).

### 3.1 Simple transverse Ising model

For the transverse Ising model [20], Hamiltonian for 3 spin systems is written as

\[
H_0 = J(R(t))(\sigma_1^x\sigma_2^x + \sigma_2^x\sigma_3^x + \sigma_3^x\sigma_1^x) - \frac{1}{2} (\sigma_1^x + \sigma_2^x + \sigma_3^x)B_x(R(t))
\]

By using the bases : \(\uparrow\uparrow\uparrow, \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow, \downarrow\downarrow\uparrow, \downarrow\downarrow\downarrow\) and \(\uparrow\downarrow\downarrow\), we obtain
The eigenvalues are:
\[
\begin{align*}
\lambda_1 &= \lambda_2 = -\frac{B_x}{2} - J, \\
\lambda_3 &= \lambda_4 = \frac{B_x}{2} - J, \\
\lambda_5 &= -\sqrt{\frac{B_x^2}{2} + 2B_xJ + 4J^2} - \frac{B_x}{2} + J, \\
\lambda_6 &= -\sqrt{\frac{B_x^2}{2} - 2B_xJ + 4J^2} + \frac{B_x}{2} + J, \\
\lambda_7 &= \sqrt{\frac{B_x^2}{2} + 2B_xJ + 4J^2} - \frac{B_x}{2} + J, \quad \text{and} \\
\lambda_8 &= \sqrt{\frac{B_x^2}{2} - 2B_xJ + 4J^2} + \frac{B_x}{2} + J. 
\end{align*}
\]

The dependence of these values are shown in Fig. 2(a).

Here we chose \( J = 1 \), but we have numerically confirmed that the order of 8 eigenvalue remains unchanged in the range \( 0 \leq J \leq 10 \). The components of the eigenvector for the ground state (\( \lambda_6 \)) are:
\[
C_1 = V_1 \zeta, \quad C_2 = V_2 \zeta, \quad C_3 = V_3 \zeta, \quad C_4 = V_4 \zeta, \quad C_5 = V_5 \zeta, \quad C_6 = V_6 \zeta, \quad C_7 = V_7 \zeta, \quad C_8 = V_8 \zeta,
\]
where \( V_1 = V_8 = 1, \quad V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = \frac{2\left(\frac{B_x^2}{2} + 2B_xJ + 4J^2 + \frac{B_x}{2} - J\right)}{3B_x} \) and \( \zeta = \frac{1}{\sqrt{2 + 6V_2}} \).

Here we see the symmetry: \( C_1 = C_8, \quad C_2 = C_3 = C_4 = C_5 = C_6 = C_7 \). From \( R \)-derivative of the normalization \( \sum_{j=1}^{8} C_j^2 = 2C_1^2 + 6C_2^2 = 1 \), we see
\[
C_1 \frac{\partial C_1}{\partial R} + 3C_2 \frac{\partial C_2}{\partial R} = 0, \quad \text{(30)}
\]
and then the adiabatic phase \( \xi = 0 \). Due to the symmetry of \( \{C_j\} \) and noting the real nature of \( \{\tilde{J}, \tilde{W}, \tilde{Q}, \tilde{B}\} \), Eq.(8) for regularization terms \( \tilde{H} \) reduces to
\[
\begin{align*}
\text{i}h \frac{\partial C_1}{\partial R} &= \tilde{A}_1 C_1 + \tilde{A}_2 C_2, \\
\text{i}h \frac{\partial C_2}{\partial R} &= \tilde{A}_3 C_1 + \tilde{A}_4 C_2, \quad \text{(31)}
\end{align*}
\]
where \( \tilde{A}_1 = \tilde{H}_{11} + \tilde{H}_{18} = 3J \), \( \tilde{A}_2 = \tilde{H}_{12} + \tilde{H}_{13} + \tilde{H}_{14} + \tilde{H}_{15} + \tilde{H}_{16} + \tilde{H}_{17} = -6\tilde{W}_2 - i6\tilde{Q} + 3(\tilde{J}_1 - \tilde{J}_2) + 3\tilde{B}_x \), \( \tilde{A}_3 = \tilde{H}_{21} + \tilde{H}_{28} = 2\text{i} \tilde{W}_2 + 2\text{i} \tilde{Q} + \tilde{J}_1 - \tilde{J}_2 + \frac{8\tilde{B}_x}{2} \), \( \tilde{A}_4 = \tilde{H}_{22} + \tilde{H}_{23} + \tilde{H}_{24} + \tilde{H}_{25} + \tilde{H}_{26} + \tilde{H}_{27} = -\tilde{J}_2 + 2(\tilde{J}_1 + \tilde{J}_2) + \tilde{B}_x \). To solve the two-component simultaneous linear equations in Eq.(31), we should choose two independent real variables out of 6 real variables \( \{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{Q}, \tilde{W}_2, \tilde{B}_x\} \) appearing in \( \{\tilde{A}_1\} \). Among 6 choices, however, we should pick up only the cases where \( 2 \times 2 \) coefficient matrix for the unknown \( \{\tilde{J}, \tilde{W}, \tilde{Q}, \tilde{B}\} \) is regular and each of two solution is real. For example there is a choice where \( \tilde{J}_3 \) and \( \tilde{W}_2 \) are independent variables with others.
zero, such that Eq. (31) can be rewritten as

$$\begin{align*}
\hbar \frac{\partial c_1}{\partial t} &= 3f_3 c_1 - i6\tilde{W}_2 c_2, \\
\hbar \frac{\partial c_2}{\partial t} &= 2i\tilde{W}_2 c_1 - f_3 c_2.
\end{align*}$$

(32)

By solving Eq. (32), we obtain

$$\begin{align*}
\tilde{J}_3 &= \frac{a c_1 + 3b c_2}{3(c_1^2 - c_2^2)} = 0, \\
\tilde{W}_2 &= -\frac{i(a c_1 + 3b c_2)}{6(c_1^2 - c_2^2)},
\end{align*}$$

(33)

where $a = \hbar \frac{\partial c_1}{\partial t}$, and $b = \hbar \frac{\partial c_2}{\partial t}$. Noting Eq. (30), we find $\tilde{J}_3 = 0$. The regularization terms and the fast forward Hamiltonian are written as

$$\tilde{H} = \begin{pmatrix}
0 & -i2\tilde{W}_2 & -i\tilde{W}_2 & -i2\tilde{W}_2 & 0 & 0 & 0 & 0 \\
i2\tilde{W}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i2\tilde{W}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i2\tilde{W}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i2\tilde{W}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i2\tilde{W}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i2\tilde{W}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i2\tilde{W}_2 & 0
\end{pmatrix},$$

(34)

$$H_{xx} = H_0(R(\Lambda(t))) + v(t)\tilde{H}_n(R(\Lambda(t)))$$

with $H_0 = I(R(\Lambda(t)))(\sigma_1^x \sigma_2^x + \sigma_3^x \sigma_3^x + \sigma_4^x \sigma_4^x) - \frac{1}{2}(\sigma_1^x + \sigma_2^x + \sigma_3^x)B_n(R(\Lambda(t))).$ and $v\tilde{H}_n = v(t)\tilde{W}_2(R(\Lambda(t)))(\sigma_1^y \sigma_2^y + \sigma_3^y \sigma_3^y + \sigma_4^y \sigma_4^y) + (\sigma_2^y \sigma_3^y + \sigma_3^y \sigma_2^y) + (\sigma_3^y \sigma_4^y + \sigma_4^y \sigma_3^y).$

We find that each solution consists of two real variables among which one comes from 2 candidates ($\tilde{W}_2$ and $\tilde{Q}$) responsible to the imaginary part of $\tilde{H}$ in Eq. (27) and the other comes from its real part. To conclude we have two solutions. The other solution is $(\tilde{J}_3 = 0, \tilde{Q} = -\frac{i(a c_1 + 3b c_2)}{6(c_1^2 - c_2^2)}).$ The fast-forward Hamiltonian for this has the state-dependent counter-diabatic part besides the common original part $H_0$ as follows:

$$v\tilde{H}_n = v(t)\tilde{Q}(R(\Lambda(t)))(\sigma_1^x \sigma_2^x + \sigma_3^x \sigma_4^x) + \sigma_1^y (\sigma_2^y \sigma_3^y + \sigma_3^y \sigma_2^y) + \sigma_2^y (\sigma_3^y \sigma_4^x + \sigma_4^x \sigma_3^y).$$

(36)

The interaction strength of counter-diabatic terms, $\tilde{W}_2$ and $\tilde{Q}$, has the degenerate expression and its time dependence is shown in the solid line of Fig. 2(b), where $v(t)$ in Eq. (15) and $R(\Lambda(t))$ in Eq. (16) are used. It should be emphasized: Among 2 solutions for the counter-diabatic terms, one of them does not contain the three-body interaction, i.e., $\tilde{Q} = 0.$
Figure 2: (a) $B_y$ dependence of eigenvalues ($\lambda_j$) in the antiferromagnetic case ($J > 0$) (b) The time dependence of 2 degenerate solutions of counter-diabatic terms: $W_z = \nu(t)\tilde{W}_z$ and $Q = \nu(t)\tilde{Q}$. Solid and dashed lines are for the ground states of the antiferromagnetic ($J = R(\Lambda(t))$) and ferromagnetic ($J = -R(\Lambda(t))$) cases, respectively. $B_y = B_0 - R(\Lambda(t))$ with $\tilde{\nu} = 100$, $T_{FF} = 0.1$, $R_0 = 0$, and $B_0 = 10$.

Figure 3. The time dependence of $|C_2^{FF}|$ (solid line) and $|C_2^{FF}|$ (dashed line) in the antiferromagnetic case ($J = R(\Lambda(t))$). Other parameter values are the same as in Fig.2.

One might conceive that, by including in Eq.(27) antisymmetric terms like $\sigma^x_1\sigma^x_2 - \sigma^y_1\sigma^z_2$ and $\sigma^z_1(\sigma^z_2\sigma^z_3 - \sigma^x_2\sigma^x_3)$, etc, more solutions would be available. We have confirmed, however, that there is no solution that includes such terms. Therefore the ansatz in Eq.(27) is justified.

With use of each of 2 solutions for the counter-diabatic term, we numerically solve TDSE in Eq.(14). The result is shown in Fig.3. Here the parameter values are the same as in Fig.2. The initial state ($C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = 0.3535$) is a linear combination of $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ and $\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow$ states. As $J$ is increased and $B_y$ is decreased, the system rapidly changes to the final state, a linear combination of $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$, $\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow$, and $\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow$. Figure 3 is common to both of 2 counter-diabatic terms, and exactly agrees with the time dependence of the ground state (with eigenvalue $\lambda_2$). The time-dependent fidelity of the wavefunction solution $\Psi_{FF}(t)$ of TDSE in Eq.(14) to the eigenfunction $\Psi_0(R(\Lambda(t)))$ in Eq.(4) is defined by $\langle\Psi_{FF}(t)|\Psi_0(R(\Lambda(t)))\rangle = |\Sigma_j |C_{j,F}(t)C_j(R(\Lambda(t)))|$. We numerically confirmed the perfect fidelity, i.e., fidelity = 1 during the fast-forward time range $0 \leq t \leq T_{FF}$. 


Choosing another eigenstate, we can obtain the similar 2 solutions as above. However, they are expressed in terms of components of the different eigenstate. Therefore the corresponding counter-diabatic terms have the time dependence which differs from those of the ground state.

The above analysis was concerned with the antiferromagnetic (frustrated) case, noting \( J = R(\Lambda(t)) > 0 \). We also explored the case of ferromagnetic exchange interaction, by choosing \( J = -R(\Lambda(t)) \). The quasi-adiabatic dynamics of the corresponding ground state can be fast forwarded. We find 2 solutions for the regularization terms, which have the same expressions as in the antiferromagnetic case. The solutions \( \tilde{Q} \) and \( \tilde{W}_2 \) are again degenerate and now characterized by the ground-state eigenvector of the ferromagnetic case. Figure 2(b) includes a profile for these degenerate solutions multiplied by the velocity function. We find that the sign of counter-diabatic terms are opposite to that in the antiferromagnetic case.

To conclude, in the simple transverse Ising model for the triangular spin cluster, we obtained 2 state-dependent counter-diabatic terms for each of eigenstates: one of them is the driving 3-body interactions which are symmetric with respect to exchange of spin variables as seen in \( a_1^x (a_2^x a_3^x + a_2^y a_3^y) \), and the other one contains the two-body interactions only.

4. Conclusion
We presented a scheme of the fast forward of adiabatic spin dynamics in a triangular spin cluster. We settled the quasi-adiabatic dynamics by adding the regularization terms to the original Hamiltonian and then accelerated it with use of a large time-scaling factor. The candidate regularization Hamiltonian can include three-body interactions which are absent in the original Hamiltonian consisting of the Zeeman energy and the pair-wise exchange interactions. The driving 3-body interactions are symmetric with respect to exchange of the sites of spin variables like \( a_1^x (a_2^x a_3^x + a_2^y a_3^y) \). The present scheme requires no necessity of truncating the counter-diabatic terms and guarantees the complete fidelity of accelerated adiabatic states.

In case of a simple transverse Ising model, we find 2 counter-diabatic terms (CDTs) for each of adiabatic states, one of which includes the 3-body interaction. This fast-forward scheme also indicates a way of obtaining a driving short-range few-body (e.g., three-body) interaction without truncating the longer-range multiple interaction of the exact counter-diabatic terms determined by the Demirplak-Rice-Berry’s transitionless quantum driving.

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