Recurrence and Transience within Quantum Markov Chains

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Abstract. We introduce a new notion of recurrence and transience for the Quantum Markov Chains which are weaker than Accardi-Koroliuk’s recurrence and transience, respectively. We investigate some its properties.

1. Introduction
A Quantum Markov Chain (QMC) is a quantum generalization of a Classical Markov Chain where the state space is a Hilbert space, and the transition probability matrix of a Markov chain is replaced by a transition amplitude matrix, which describes the mathematical formalism of the discrete time evolution of open quantum systems, see [4]-[6],[10] for more details.

On the other hand, investigations of recurrence are motivated by a large number of papers extending it in different directions: see [7] for the notion of monitored recurrence for discrete-time quantum processes; see [11] for the recurrence of discrete time unitary evolutions, see [5] for the recurrence of the quantum Markov chains; see [9] for the recurrence of the quantum Markovian semigroups. In [5, 6] it was defined a notion of recurrence for QMC which was based on the transition expectation (see section 3 for definitions) and initial projection. When we look at QMC it depends on an initial state and a transition expectation, therefore, to define the recurrence within QMC scheme one needs to use the given state and the expectation. We point out that in the classical case, the recurrence is defined for every initial state (not for fixed one).

Due to this fact, in this paper, we introduce a new notion of recurrence and transience for QMC in terms on the state and transition expectation. We point out that this notion is more weaker than the one given in [5]. We stress that, in the classical setting, the introduced recurrence is also weaker than the usual one, since in our scheme, from the beginning we are fixing the initial state for the chain. Roughly speaking, we define the recurrence for a fixed Markov measure while in the classical setting the initial distributions are taken depending on the state. Note that applications of the introduced recurrences have been illustrated in [8], to QMC associated with open quantum random walks.

2. Quantum Markov Chains
In this section, we recall the definition of quantum Markov chains [1, 3, 4] and define a new notion called $\varphi$-recurrence.
For each $i \in \mathbb{Z}_+$, (here $\mathbb{Z}_+$ denotes the set of all non negative integers) let us associate identical copies of a separable Hilbert space $\mathcal{H}$ and $C^*$-subalgebra $M_0$ of $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$:

$$\mathcal{H}(i) = \mathcal{H},$$

$$\mathcal{A}(i) = M_0 \subset \mathcal{B}(\mathcal{H}) \text{ for each } i \in \mathbb{Z}_+ \tag{2.1}$$

We assume that any minimal projection in $M_0$ is one dimensional. For any bounded $A \subset \mathbb{Z}_+$, let

$$\mathcal{A}_A = \bigotimes_{i \in A} \mathcal{A}_i$$

$$\mathcal{A} = \left( \bigcup_{A \subset \mathbb{Z}_+, |A| < \infty} \mathcal{A}_A \right) =: \bigotimes_{i \in \mathbb{Z}_+} \mathcal{A}_i \tag{2.2}$$

where the bar denotes the norm closure.

For each $i \in \mathbb{Z}_+$, let $J_i$ be the canonical injection of $M_0$ to the $i$-th component of $\mathcal{A}$. For each $\Lambda \subset \mathbb{Z}_+$ we identity $\mathcal{A}_{\Lambda}$ as a subalgebra of $\mathcal{A}$.

The basic ingredients in the construction of a stationary generalized quantum Markov chain in the sense of Accardi and Frigerio [4] consist of a completely positive unital map (c.p.u. map) $\mathcal{E} : M_0 \otimes M_0 \rightarrow M_0$, called a transition expectation:

$$\mathcal{E}(1 \otimes 1) = 1, \tag{2.3}$$

and a state $\phi_0$ on $M_0$. In what follows, a pair $(\phi_0, \mathcal{E})$ is called a Markov pair.

A state $\varphi$ defined on $\mathcal{A}$ associated with a Markov pair $(\phi_0, \mathcal{E})$, is called Quantum Markov Chain (QMC) if

$$\varphi(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = \phi_0(\mathcal{E}(x_0 \otimes \mathcal{E}(x_1 \otimes \ldots \otimes \mathcal{E}(x_n \otimes 1) \ldots)))). \tag{2.4}$$

In what follows, by $\mathcal{A}_{n}$ we denote the subalgebra of $\mathcal{A}$, generated by the first $(n+1)$ factors, i.e.

$$a_n = a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1_{[n+1]} = J_0(a_0)J_1(a_1) \cdots J_n(a_n),$$

with $a_0, a_1, \ldots, a_n \in M_0$. It is well known [2] that for each $n \in \mathbb{N}$ there exists a unique completely positive identity preserving mapping $E_{n} : \mathcal{A} \rightarrow \mathcal{A}_{n}$ such that

$$E_n(a_m) = a_0 \otimes \cdots \otimes a_{n-1} \otimes \mathcal{E}(a_n \otimes \mathcal{E}(a_{n+1} \otimes \cdots \otimes \mathcal{E}(a_m \otimes 1) \ldots)), \quad m > n \tag{2.5}$$

Remark 2.1. We notice that if the state $\phi_0$ satisfies the following condition:

$$\phi_0(\mathcal{E}(1 \otimes x)) = \phi_0(x), \quad x \in M_0 \tag{2.6}$$

then the Markov pair $(\phi_0, \mathcal{E})$ defines local states

$$\varphi_{[i,n]}(x_i \otimes x_{i+1} \otimes \ldots \otimes x_n) = \phi_0(\mathcal{E}(x_i \otimes \mathcal{E}(x_{i+1} \otimes \ldots \otimes \mathcal{E}(x_n \otimes 1) \ldots)).) \tag{2.7}$$

The family of local states $\{\varphi_{[i,n]}\}$, due to (2.3),(2.6), satisfies a compatibility condition, and therefore, the state $\varphi$ is well defined on $\mathcal{A}_{\mathbb{Z}} := \bigotimes_{i \in \mathbb{Z}} \mathcal{A}_i$. Moreover, $\varphi$ is translation invariant, i.e. it is invariant with respect to the shift $\alpha$, i.e. $\alpha(J_n(a)) = J_{n+1}(a)$. 


3. $\varphi$-Recurrence and $\varphi$-Transience

Following [5] we recall a definition of the stopping time associated with a projection $e \in M_0$, which is a sequence $\{\tau_k\}$ defined by

$$
\tau_0 = e \otimes I_1 = J_0(e),
\tau_1 = e^+ \otimes e \otimes I_2 = J_0(e^+)J_1(e),
\ldots,
\tau_k = e^+ \otimes \cdots \otimes e^+ \otimes e \otimes I_{k+1} = J_0(e^+) \cdots J_k(e^+)J_k(e),
\tau^n_{\infty} = e^+ \otimes \cdots \otimes e^+ \otimes I_{n+1} = J_0(e^+) \cdots J_n(e^+).
$$

Since the sequence $\{\tau^n_{\infty}\}$ is decreasing, therefore, its strong limit exists in $A''$, and it is denoted by

$$
\tau_{\infty} := \lim_{n \to \infty} \tau^n_{\infty}
$$

One can see that

$$
\sum_{k \geq 0} \tau_k = 1 - \tau_{\infty},
$$

where the sum is meant in the strong topology in $A''$.

In [5] it was defined the following

**Definition 3.1.** Let $(\phi_0, \mathcal{E})$ be a Markov pair. A projection $e$ is called

(i) **$\mathcal{E}$-completely accessible** if

$$
E_{0\|}(\tau_{\infty}) := \lim_{n \to \infty} E_{0\|}(\tau^n_{\infty}) = 0;
$$

(ii) **$\mathcal{E}$-recurrent** if $\text{Tr}(\mathcal{E}(e \otimes I)) < \infty$ and one has

$$
\frac{1}{\text{Tr}(\mathcal{E}(e \otimes I))} \text{Tr} \left( E_{0\|} \left( \sum_{n \geq 0} J_0(e) \otimes \tau_n \right) \right) = 1.
$$

**Definition 3.2.** Let $(\phi_0, \mathcal{E})$ be a Markov pair, and $e, f$ be two projections in $M_0$. A projection $f$ is called **$\mathcal{E}$-accessible** from $e$ if there is $n \in \mathbb{N}$ such that

$$
E_{0\|}(J_0(e) \otimes I_{n-1} \otimes J_n(f)) \neq 0.
$$

In this paper, we define weaker notions.

**Definition 3.3.** Let $\varphi$ be a QMC on $A$ associated with the pair $(\phi_0, \mathcal{E})$. A projection $e$ is called

(i) **$\varphi$-completely accessible** if $\varphi(\tau_{\infty}) = 0$;

(ii) **$\varphi$-recurrent** if $\varphi(J_0(e)) \neq 0$ and

$$
\frac{1}{\varphi(J_0(e))} \varphi \left( \sum_{n \geq 0} J_0(e) \otimes \tau_n \right) = 1
$$

(iii) **$\varphi$-transient** if $\varphi(J_0(e)) \neq 0$ and

$$
\frac{1}{\varphi(J_0(e))} \varphi \left( \sum_{n \geq 0} J_0(e) \otimes \tau_n \right) < 1.
$$
**Definition 3.4.** Let \( \varphi \) be a QMC on \( \mathcal{A} \) associated with the pair \((\phi_0, \mathcal{E})\) and \( e, f \) be two projections in \( M_0 \). A projection \( f \) is called \( \varphi\)-**accessible** from \( e \) (we denote it as \( e \to^\varphi f \)) if there is \( n \in \mathbb{N} \) such that

\[
\varphi(J_0(e) \otimes I_{n-1} \otimes J_n(f)) \neq 0.
\]

If \( e \to^\varphi f \) and \( f \to^\varphi e \), then \( e \) and \( f \) are called \( \varphi\)-**communicate** and one denotes \( e \leftrightarrow^\varphi f \).

**Remark 3.5.** From the definitions one can infer that, due to the Markov property of \( \varphi \), \( \mathcal{E}\)-accessibility and \( \mathcal{E}\)-recurrence imply \( \varphi\)-accessibility and \( \varphi\)-recurrence, respectively. The reverse is not true. Moreover, we stress that, in the classical setting, the introduced \( \varphi\)-recurrence is also weaker than the usual one, since in our scheme, from the beginning we are fixing the initial state for the chain. Roughly speaking, we define the recurrence for a fixed Markov measure while in the classical setting the initial distributions are taken depending on elements of the state space.

Now we are going to study several properties of \( \varphi\)-accessibility and \( \varphi\)-recurrence, respectively.

**Theorem 3.6.** Let \( \varphi \) be a QMC on \( \mathcal{A} \) associated with the pair \((\phi_0, \mathcal{E})\). The following statements hold:

1. \( \varphi(J_n(e)) = 0 \) for all \( n \in \mathbb{N} \) if and only if for every \( k \in \mathbb{N} \) one has \( \varphi(\beta^k(\tau_\infty)) = 1 \), where

\[
\beta(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = I \otimes a_0 \otimes a_1 \otimes \cdots a_n, \text{ for any } n \in \mathbb{N};
\]

2. \( e \) is \( \varphi\)-recurrent if and only if \( \varphi(J_0(e) \otimes \tau_\infty) = 0 \). In particular, if \( e \) is \( \varphi\)-completely accessible, then \( e \) is \( \varphi\)-recurrent;

3. \( e \) is \( \varphi\)-transient if and only if \( \varphi(J_0(e) \otimes \tau_\infty) > 0 \). In particularly, the \( \varphi\)-transience of \( e \) implies its non \( \varphi\)-completely accessibility.

4. if \( \varphi \) is faithful, then \( e \) is \( \varphi\)-completely accessible if and only if \( e \) is \( \varphi\)-recurrent;

5. if all projections in \( M_0 \) are \( \varphi\)-communicating and \( e \) is \( \varphi\)-recurrent, then \( e \) is \( \varphi\)-completely accessible.

**Proof.** (i) Let \( \varphi(J_n(e)) = 0 \) for every \( n \in \mathbb{N} \). For any \( k, m \in \mathbb{N} \) we have

\[
I_{m} \otimes \tau_k \leq I_{m+k-1} \otimes J_{m+k}(e) \otimes I_{m+k+1},
\]

therefore, one finds \( \varphi(\tau_k) \leq \varphi(J_{m+k}(e)) = 0 \). Hence, from (3.1) one gets \( \varphi(\beta^m(\tau_\infty)) = 1 \).

Now assume that \( \varphi(\beta^m(\tau_\infty)) = 1 \) any \( m \in \mathbb{N} \). Then again from (3.1) we obtain

\[
\varphi\left(\beta^m\left(\sum_{k \geq 0} \tau_k\right)\right) = 0,
\]

which implies \( \varphi(\beta^m(\tau_k)) = 0 \) for all \( k \in \mathbb{N} \). This means \( \varphi(\beta^m(\tau_0)) = \varphi(J_m(e)) = 0 \).

(ii) Let \( e \) be \( \varphi\)-recurrent. Then from the definition and (3.1) one finds

\[
\varphi(J_0(e)) = \varphi\left(J_0(e) \otimes \sum_{k \geq 0} \tau_k\right) = \varphi(J_0(e)) - \varphi(J_0(e) \otimes \tau_\infty),
\]

which means \( \varphi(J_0(e) \otimes \tau_\infty) = 0 \). The reverse implication is obvious.

The proof of the statement (iii) immediately follows from the definition of the \( \varphi\)-transience and (3.2).

(iv) if \( \varphi \) is faithful, then \( \varphi\)-completely accessibility of \( e \) is equivalent to \( \tau_\infty = 0 \), then from (ii) we have that \( e \) is \( \varphi\)-recurrent. Conversely, if \( e \) is \( \varphi\)-recurrent, then due to the faithfulness of \( \varphi \) with (ii) one gets \( J_0(e) \otimes \tau_\infty = 0 \), so \( \tau_\infty = 0 \) which means \( \varphi\)-completely accessibility of \( e \).
(v) Assume that $e$ is not $\varphi$-completely accessible, this means $\varphi(\tau_\infty) > 0$. Due to $\varphi$-recurrence one has $\varphi(J_0(e) \otimes \tau_\infty) = 0$, which implies that

$$\lim_{n \to \infty} \varphi(J_0(e) \otimes \tau_\infty^n) = 0.$$ 

The last equality yields that

$$\lim_{n \to \infty} \varphi(J_0(e) \otimes J_1(e) \otimes \tau_\infty^n) = 0, \quad \lim_{n \to \infty} \varphi(J_0(e) \otimes J_1(e^+) \otimes \tau_\infty^n) = 0,$$

so

$$\lim_{n \to \infty} \varphi(J_0(e) \otimes \mathbf{1} \otimes \tau_\infty^n) = 0.$$

Hence, iterating the last equality, for every $k \in \mathbb{N}$ one finds

$$\varphi(J_0(e) \otimes \mathbf{1}_{k-1} \otimes \tau_\infty^k) = 0. \quad (3.3)$$

Since $\varphi(\tau_\infty) > 0$, then one can find a projection $p \in M_0$ ($p \neq 0$) such that $\tau_\infty \geq \lambda p$ for some positive number $\lambda$. Then from (3.3) we infer that

$$\varphi(J_0(e) \otimes \mathbf{1}_{k-1} \otimes p) \leq \frac{1}{\lambda} \varphi(J_0(e) \otimes \mathbf{1}_{k-1} \otimes \tau_\infty^k) = 0$$

this implies that $e$ and $p$ are not $\varphi$-communicate. This is a contradiction. This completes the proof. \hfill \Box

**Corollary 3.7.** Let $\varphi$ be a QMC on $\mathcal{A}_\mathbb{Z}$ associated with the pair $(\phi_0, \mathcal{E})$. The following statements hold:

(i) $\varphi(e) = 0$ if and only if $\varphi(\tau_\infty) = 1$;

(ii) $e$ is $\varphi$-recurrent if and only if $e$ is $\varphi$-completely accessible;

(iii) if $\varphi$ is faithful, then $e$ is $\varphi$-completely accessible if and only if $\mathcal{E}$-completely accessible.

**References**

[1] Accardi L 1975 *Funct. Anal. Appl.* **8** 1–8.
[2] Accardi L 1979 *J. Math. Anal. Appl.* **72** 34–69.
[3] Accardi L., Fidaleo F and Mukhamedov F 2007 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **10** 165–183
[4] Accardi L and Frigerio A 1983 *Proc. Royal Irish Acad.* **83A** 251-263.
[5] Accardi L and Koroliuk D 1992 *J. Theor. Probab.* **5** 521-535
[6] Accardi L and Koroliuk D 1991 In book: *Quantum Prob. and Related Topics VII*, 63–73
[7] Bourgain J, Grünbaum F A, Velázquez L and Wilkening J 2014 *Commun. Math. Phys.* **329** 1031–1067.
[8] Dhahri A and Mukhamedov F arXiv:1608.01065
[9] Fagnola F and Rebolledo R 2003 *Probab. Theory Relat. Fields* **126** 289-306.
[10] Fidaleo F and Mukhamedov F 2004 *Probab. Math. Stat.* **24** 401–418
[11] Grünbaum F A, Velázquez L, Werner A H and Werner R F. 2013 *Commun. Math. Phys.* **320** 543-569