Macroscopic approach to N-qudit systems

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Abstract
We develop a general scheme for an analysis of macroscopic N-qudit systems which includes: (a) a scheme to organize the information obtained from collective measurements in the form of distribution functions in a discrete low-dimensional space; (b) a set of collective operators appropriate for the characterization of N-qudit states; (c) two collective tomographic protocols both for general and fully symmetric N-qudit states. The example of N-qutrits is analyzed in detail and compared to the N-qubit case.

Keywords: qudits, macroscopic, phase-space

(Some figures may appear in colour only in the online journal)

1. Introduction

Basic problems arising in the analysis of macroscopic quantum systems include: (a) optimization of tomographic schemes in order to reduce the number of measurements [1–3]; (b) processing of available data so that meaningful information about the state of the system can be efficiently extracted [4]. Both of these tasks are rather complicated since the number of parameters (the number of measurement outcomes) required for a complete characterization of a multipartite quantum state grows exponentially with the number of particles. An intuitively appealing attempt to employ the statistical description of N-qudit (d-level) system, by mapping qudit states into distributions in a finite-dimensional grid [5–7], is inconvenient in the large N limit due to an overwhelming complexity of the resulting discrete functions. Even in the simplest case of an N-qubit system, the distributions corresponding to relatively simple quantum states have a rather complicated structure (usually represented in form of randomly located peaks [8]). In addition, the distributions in discrete phase-space do not have natural macroscopic limits, i.e. they do not acquire smooth shapes in the limit N → ∞, which makes it very difficult to study their analytical properties.

While ∼ d^2N parameters are needed for a full microscopic description of a generic N qudit state, the global properties can be captured by a significantly smaller number of collective observables. These observables are invariant under particle permutations and thus provide only
partial information about the system. On the other hand, frequently only these types of symmetric correlation functions can be efficiently assessed (e.g. due to particle indistinguishability) in macroscopic quantum systems [9].

A quantum state characterization is strongly connected to the specific set of the measurable collective variables. In N qubit systems, the moments of the collective spin operators

$$S_{x,y,z} = \sum_{i=1}^{N} \sigma_{i,x,y,z}^{(0)}$$

provide the global information, allowing access to all SU(2) invariant subspaces appearing in the tensor decomposition $\mathcal{H}^{\otimes N}$ (without distinguishing subspaces of the same dimension). The results of such collective measurements can be used for density matrix estimation [10].

From an analytical perspective, the macroscopic features of an N qubit system are completely described by the so-called projected $\tilde{Q}$-function, defined in a three-dimensional discrete space. This $Q$-function contains full and non-redundant information about the results of any collective measurement in an arbitrary (not necessarily symmetric) state [11, 12], and allows us to outline collective kinematic and dynamic properties in the limit of large number of qubits.

The situation is more complicated for macroscopic systems containing a large number of qudits. Although qudit systems seem to be useful in quantum information protocols [13], only a limited number of experimental reconstructions of one- and bi-partite qudit systems were reported [14–22]. Extension to multipartite qudit systems is quite challenging due to a rapid growth of the number of required experimental setups [23], and the deficiency of appropriate theoretical tools to deal with $N \gg 1$ qudit systems. In addition, in this case (in contrast to qubit systems) there are several ways to choose a basis set of symmetric (permutationally invariant) operators, while only some specific sets are convenient for studying macroscopic properties of large number of qudits.

In the present paper we develop a novel general framework for the analysis of global characteristics of multipartite qudit systems. In particular, we construct a $d^2 - 1$ dimensional discrete distribution function that contains full macroscopic information about an N-qudit system. In addition, we introduce a set of efficiently measurable collective observables that are compatible with the (discrete) phase-space description and have appealing algebraic properties. Finally, we propose two tomographic protocols for (partial) reconstruction of N-qudit density matrix in terms of $\sim N^{d^2 - 1}$ collective correlation functions. The example of N qutrits will be analyzed in detail and compared to N-qubit systems.

2. N qudit projected $\tilde{Q}$-function

Let us consider N-qudit Hilbert space $\mathcal{H}_{d^N} = \mathcal{H}_{d}^{\otimes N}$, where $d$ is a prime number, spanned by the computational basis $|\lambda\rangle = |l_1, \ldots, l_N\rangle$, $l_i \in \mathbb{Z}_d$. The standard unitary N-qudit operators are defined according to [6, 24]

$$Z_\alpha = \sum_\lambda \omega^{\alpha_\lambda} |\lambda\rangle \langle \lambda|, \quad X_\beta = \sum_\lambda |\lambda + \beta\rangle \langle \lambda|,$$

where

$$\alpha = (a_1, \ldots, a_N), \quad \beta = (b_1, \ldots, b_N), \quad a_i, b_i = 0, \ldots, d - 1, \quad (3)$$
are $d$—strings with elements from $\mathbb{Z}_d$, $\alpha \beta = a_1 b_1 + \cdots + a_N b_N$, $\omega = \exp \left( 2\pi i / d \right)$ sums and multiplications are taken by mod $d$. The operators (2) are factorized into tensor products,

$$Z_\alpha = \bigotimes_{i=1}^N Z_i^{a_i}, \quad X_\beta = \bigotimes_{i=1}^N X_i^{b_i},$$

of single qudit Pauli operators [25]

$$Z_i = \sum_{l=0}^{d-1} \omega^i |l\rangle \langle l|, \quad X_i = \sum_{l=0}^{d-1} |l+1\rangle \langle l|,$$

and satisfy the commutation relation

$$Z_\alpha X_\beta = \omega^{\alpha \beta} X_\beta Z_\alpha.$$  \hspace{1cm} (6)

Operators acting in $N$-qudit Hilbert space can be mapped into functions labeled by a pair of $N$-tuples $(\alpha, \beta)$. Two mutually dual maps, known as discrete $Q$-symbols and $P$-symbols [7, 8, 26], of an arbitrary operator $\hat{f}$, have the form

$$Q_f (\alpha, \beta) = \operatorname{Tr} \left( \hat{\Delta}^{-1} (\alpha, \beta) \hat{f} \right), \quad P_f (\alpha, \beta) = \operatorname{Tr} \left( \hat{\Delta}^1 (\alpha, \beta) \hat{f} \right),$$

see (A.2) for the explicit form of the kernels $\hat{\Delta}^{(\pm 1)}$. The first rank projectors

$$\hat{\Delta}^{-1} (\alpha, \beta) = |\alpha, \beta\rangle \langle \alpha, \beta|,$$

form an informationally complete (SIC) POVM condition [28]

$$|\langle a_i, b_i | a'_i, b'_i |^2 = \frac{1 + d \delta_{a_i a'_i} \delta_{b_i b'_i}}{1 + d}.$$  \hspace{1cm} (14)

An operator $\hat{f}$ can be decomposed on the basis of $\hat{\Delta}^{(\pm 1)} (\alpha, \beta)$ as

$$\hat{f} = \sum_{\alpha, \beta} Q_f (\alpha, \beta) \hat{\Delta}^{(\pm 1)} (\alpha, \beta) \hspace{1cm} (15)$$
\[= \sum_{\alpha, \beta} P_f (\alpha, \beta) \hat{\Delta}^{(-1)} (\alpha, \beta), \tag{16} \]

so that the average values are computed according to

\[
\langle \hat{f} \rangle = \sum_{\alpha, \beta} Q_f (\alpha, \beta) P_f (\alpha, \beta) = \sum_{\alpha, \beta} Q_f (\alpha, \beta) P_\rho (\alpha, \beta). \tag{17} \]

It follows from (9) and (12) that

\[P \hat{\Delta}^{(\pm 1)} (\alpha, \beta) P^\dagger = \hat{\Delta}^{(\pm 1)} (\pi \alpha, \pi \beta) \tag{18} \]

where \(P\) is the permutation operator and \(\pi \alpha\) is a permutation of the string \(\alpha\) corresponding to \(P\).

Then, \(P\) and \(Q\)-symbols of \textit{permutationally invariant} operators, \(\hat{s} = P \hat{s} P^\dagger\), are symmetric functions of their arguments, and thus depend only on the corresponding weights \(h (\alpha, \beta)\) [27] (see also appendix A)

\[P_s (\alpha, \beta) = \text{Tr} \left( \hat{\Delta}^{(1)} (\alpha, \beta) \hat{s} \right) \equiv P_s (\hat{h} (\alpha, \beta)). \tag{19} \]

The weights \(h (\alpha)\), being invariant under permutation characteristics of \(d\)-strings (3), \(h (\alpha) = h (\pi \alpha)\), are defined according to

\[h (\alpha) = \sum_{i=1}^{N} a_i = \sum_{k} k \sum_{i=1}^{N} \delta_{a_i k}, 0 \leq h(\alpha) \leq N, \tag{20} \]

and can be arranged in a vector

\[h (\alpha, \beta) = \{ h(k \alpha + l \beta); k, l \in \mathbb{Z}_d \}. \tag{21} \]

The \(d^2 - 1\) components of the vector \(h (\alpha, \beta)\) form a basis in the space of symmetric functions constructed on \((\alpha, \beta)\).

For instance, in case of qubits the three-dimensional \(h\)-vector has the form

\[h (\alpha, \beta) = \{ h(\alpha), h(\beta), h(\alpha + \beta) \}, \tag{22} \]

while for qutrits the \(h\)-vector is eight-dimensional,

\[h (\alpha, \beta) = \{ h(\alpha), h(2 \alpha), h(\beta), h(2 \beta), h(\alpha + \beta), h(2 \alpha + 2 \beta), h(2 \alpha + \beta), h(\alpha + 2 \beta) \}. \tag{23} \]

The points of the two-dimensional grid \((\alpha, \beta)\) can be partially ordered according to the values of the weights (21) and an arbitrary function constructed on \((\alpha, \beta)\) satisfies the following summation rule

\[\sum_{\alpha, \beta} f (\alpha, \beta) = \sum_{m} \sum_{\alpha, \beta} \delta_{h(\alpha, \beta), m} f (\alpha, \beta), \tag{24} \]

where the components of the vector \(m\) take values from 0 to \((d - 1)N\).
Taking into account (19) and the summation rule (24) we observe that the average value of any permutationally invariant operator in an arbitrary (not necessarily symmetric) state is computed according to

$$\langle \hat{s} \rangle = \sum_{\alpha, \beta} P_s(\alpha, \beta) Q_s(\alpha, \beta) = \sum_{\mathbf{m}} \tilde{Q}_s(\mathbf{m}), \quad (25)$$

where

$$\tilde{Q}_s(\mathbf{m}) = \sum_{\alpha, \beta} Q_s(\alpha, \beta) \delta_{h(\alpha, \beta), \mathbf{m}} = \text{Tr}(\Delta^{(-1)}(\mathbf{m})\hat{\rho}). \quad (26)$$

$$\Delta^{(\pm 1)}(\mathbf{m}) = \sum_{\alpha, \beta} \Delta^{(\pm 1)}(\alpha, \beta) \delta_{h(\alpha, \beta), \mathbf{m}}. \quad (27)$$

It is convenient to establish the following correspondence between the vectors \( h (\alpha, \beta) \) and \( \mathbf{m} \),

$$\mathbf{m} = \{m_{kl} = h(k \alpha + l \beta), k, l = 0, \ldots, d - 1\}, \quad 0 \leq m_{kl} \leq (d - 1)N. \quad (28)$$

For instance, in the qubit case one has

$$\Delta^{(\pm 1)}(m_{10}, m_{01}, m_{11}) = \sum_{\alpha, \beta} \Delta^{(\pm 1)}(\alpha, \beta) \delta_{h(\alpha), m_{10}} \delta_{h(\beta), m_{01}} \delta_{h(\alpha + \beta), m_{11}}. \quad (29)$$

For symmetric states, \( \mathcal{P} \hat{\rho}_S \mathcal{P}^\dagger = \hat{\rho}_S \), where \( Q_{\hat{\rho}_S}(\alpha, \beta) \equiv Q_{\hat{\rho}_S}(\mathbf{h}(\alpha, \beta)) \), one has

$$\tilde{Q}_{\hat{\rho}_S}(\mathbf{m}) = Q_{\hat{\rho}_S}(\mathbf{m} = \mathbf{h}(\alpha, \beta)) R^{(d)}(\mathbf{m}), \quad (30)$$

$$R^{(d)}(\mathbf{m}) = \sum_{\alpha, \beta} \delta_{h(\alpha), \mathbf{m}} = \sum_{\alpha, \beta} \prod_{k,l=0 \atop \{k,l\} \neq \{0,0\}}^{d-1} \delta_{h(k \alpha + l \beta), m_{kl}}, \quad (31)$$

where \( R^{(d)}(\mathbf{m}) \) is the multiplicity of each particular set \( \mathbf{m} \), i.e. is the number of pairs \( (\alpha, \beta) \) of \( d \)-strings characterized by the same vector \( \mathbf{m} = \{m_{kl} = h(k \alpha + l \beta), k, l = 0, \ldots, d - 1\} \).

It follows from (25) that the \( \tilde{Q}_s(\mathbf{m}) \)-function contains complete and non-redundant information about all macroscopic properties of the state \( \hat{\rho} \) and can be considered as a discrete distribution in \( d^2 - 1 \) dimensional macroscopic measurement space \( \mathcal{M} \) spanned by the vectors (28). The total number of multiplets \( \{\mathbf{m} = \mathbf{h}(\alpha, \beta)\} \) (points in \( \mathcal{M} \)) is

$$\mathcal{N}_\mathcal{M} = \frac{(N + d^2 - 2)!}{(d^2 - 1)!N!}, \quad (32)$$

which is the amount of collective measurements fully determining the \( \tilde{Q}_s(\mathbf{m}) \)-function. In other words, the global variables ‘see’ an \( N \)-qudit state in the form of \( \tilde{Q}_s(\mathbf{m}) \) distribution in \( \mathcal{M} \). In the \( N \)-qubit case, the \( \tilde{Q} \)-function has been extensively studied in [11, 12] and applied to the analysis of pure states thermalization effect and quantum phase transitions [29].

In the macroscopic limit \( N \gg d \),

$$\mathcal{N}_\mathcal{M} \sim N^{d^2 - 1} / (d^2 - 1)!, \quad (33)$$

which is significantly smaller than the number of points in the full discrete phase-space \( \sim d^{2N} \). In practice, distributions corresponding to physically relevant macroscopic states tend
to smooth shapes located in certain areas of the measurement space $\mathcal{M}$, as was observed in the qubit case [11, 12], and shown in the following examples.

**Examples**

(a) For the fiducial state (12)–(14) one can easily obtain

$$
\tilde{Q}_\xi^{(d)}(\mathbf{m}) = (d + 1) \frac{2}{d(d-1)} \sum_{m_1,m_2} R_{m_1}^{(d)},
$$

which is a localized distribution (of size $\sim \sqrt{N}$) in the $\mathcal{M}$ - space, that tends to the Gaussian shape for $N \gg d$, (see appendix C). For instance, in the $N$-qubit case

$$
\tilde{Q}_\xi^{(2)}(\mathbf{m}) \sim \exp \left( -\frac{2}{N} \left( (m_{10} - \frac{3N}{8})^2 + (m_{01} - \frac{3N}{8})^2 + (m_{11} - \frac{3N}{8})^2 \right) \right),
$$

while in the $N$-qutrit case

$$
\tilde{Q}_\xi^{(3)}(\mathbf{m}) \sim \exp \left( -\sum_{k=0}^{k+1} \sum_{l=0}^{k+1} \left[ (m_{d1} - \frac{8N}{9})^2 + (m_{d2l} - \frac{8N}{9})^2 - \frac{8N}{9} (m_{d1} - \frac{8N}{9}) (m_{d2l} - \frac{8N}{9}) \right] \right).
$$

(b) For the GHZ-like $N$-qudit state

$$
|GHZ\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} |l \ldots l\rangle.
$$

one obtains by using (10) for the discrete $Q$-function (7) the following expression explicitly invariant under particle permutations

$$
Q(\alpha, \beta) = |\langle GHZ | \alpha, \beta \rangle|^2 = \frac{1}{d} \left| \sum_{l=0}^{d-1} \sum_{i} \omega^{\sum_{j=0}^{N-1} c_{i-l}^{(0)} c_{i-j}^{(0)}} \right|^2,
$$

where $a_i, b_i$ are components of the $d$–string (3) and $c_{i}^{(0)}$ are the expansion coefficients of the $i$th particle’s fiducial state in the computational basis,

$$
|\xi_i\rangle = \sum_{l=0}^{d-1} c_{i}^{(0)} |l\rangle,
$$

so that

$$
|\xi\rangle = \sum_{\lambda} c_{\lambda} |\lambda\rangle, \quad |\lambda\rangle = |l_1, \ldots, l_N\rangle, \quad c_{\lambda} = \prod_{i=1}^{N} c_{i}^{(0)},
$$

6
Thus, according to (30), we arrive at

$$\tilde{Q}_{\text{GHZ}}^{(d)}(m) = \frac{1}{d} \left| \sum_{i=0}^{d-1} \omega^{m_i} \prod_{j=1}^{N} \epsilon_{i-j}^{(d)} \right|^2 R^{(d)}_m,$$

where the product $\prod_{j=1}^{N} \epsilon_{i-j}^{(d)}$ is a function of $m_{ij}$, $r = 1, 2, \ldots , d - 1$ only.

For qubits, $d = 2$, and the SIC POVM generating fiducial state

$$|\zeta\rangle_i = \frac{|0\rangle_i + \zeta|1\rangle_i}{\sqrt{1 + |\zeta|^2}}, \quad \zeta = \frac{\sqrt{3} - 1}{\sqrt{2}} e^{i\pi/4},$$

the expression (41) is reduced to

$$\tilde{Q}_{\text{GHZ}}^{(2)}(m) = \frac{1}{2 \left( 1 + |\zeta|^2 \right)^{N/2}} \left| \zeta^{m_0} + (-1)^{m_0} \zeta^{N - m_0} \right|^2 R^{(2)}_m,$$

where $R^{(2)}_m$ is given in (C.1). The distribution $\tilde{Q}_{\text{GHZ}}^{(2)}(m)$ has a form of two discrete clouds separated by a distance $\sim N$ along the axis $m_{01}$, each of size $\sim \sqrt{N}$ [11, 12] in three-dimensional measurement space. Each cloud acquires a Gaussian shape centered at $m_{01} = N(1 \pm 1/\sqrt{3})/2$ in the limit $N \gg 1$. In figure 1 we plot the projection of $\tilde{Q}_{\text{GHZ}}^{(2)}(m_{01}) = \sum_{m_{10}, m_{11}} \tilde{Q}_{\text{GHZ}}^{(2)}(m)$ on the axis $m_{01}$. $\tilde{Q}_{\text{GHZ}}^{(2)}(m_{01})$ has two maxima at $m_{01} = N(1 \pm 1/\sqrt{3})/2$

For qutrits, $d = 3$, one obtains

$$\tilde{Q}_{\text{GHZ}}^{(3)}(m) = \frac{1}{3^N} \left| \sum_{m_{10}, m_{11}} \delta_{m_{01}, 2m_{02}} + 2^N (\omega)^{m_{01}-m_{02}} \delta_{2m_{01}, m_{02}} + 2^N (\omega)^{m_{01}-m_{02}} \delta_{3m_{01}, m_{02}} \right|^2 R^{(3)}_m,$$
where the fiducial state is chosen as
\[
|\xi\rangle_i = \frac{1}{\sqrt{2}} (|0\rangle_i + e^{i\pi/3}|1\rangle_i),
\]
(45)
and \( R^{(3)}_m \) is given in (C.3). As it can be appreciated from the above expression, \( \overline{Q}^{(3)}_{\text{GHZ}}(m) \) is a superposition of three localized clusters, each tending to a Gaussian form centered at the plane \((m_{01}, m_{02})\) of eight-dimensional \( M \)-space at \((m_{01}, m_{02}) = (N, N/2), (N/2, N), (3N/2, 3N/2)\) see figure 2.

In the qubit case \( \overline{Q}_\rho(m) \) can be plotted as density distributions in a three-dimensional space. For higher dimensions \( \overline{Q}_\rho(m) \) can be represented only in form of projections into hyper-planes in the full \( M \)-space. Nevertheless, the analytical properties of the \( \overline{Q} \)-functions are very useful for analysis of the global properties of macroscopic systems, as will be shown below.

2.1. Collective operators

Let us introduce the following set of collective (invariant under particle permutations) Hermitian operators for an \( N \)-qudit system
\[
\hat{O}_{\alpha\beta} = N\hat{I} - \frac{2}{(d-1)d^N} \sum_{\alpha,\beta} h(k\alpha + l\beta) |\alpha, \beta\rangle \langle \alpha, \beta|, \quad k, l = 0, \ldots, d-1.
\]
(46)
Their \( P \)-symbols (7) are proportional to the corresponding weights:
\[
P_{\hat{O}_{\alpha\beta}} = \frac{1}{d^N} \left[ N - \frac{2}{d-1} h(k\alpha + l\beta) \right],
\]
(47)
which allows us to associate the axes (28) in the measurement space \( \mathcal{M} \) with the collective operators (46). This can be very helpful for determination of measurements required for a detection of quantum states. For instance, the main features of the qutrit GHZ-like state (38) can be recognized by measuring \( \hat{O}_{0,1} \) and \( \hat{O}_{0,2} \) operators and their moments, see figure 2.

It is proven in appendix B that the operators (46) are split into \( d + 1 \) disjoint sets of \( d \) commuting operators:

\[
\{ \hat{O}_{M,\lambda}, \lambda = 1, \ldots, d - 1 \}, \quad \{ \hat{O}_{0,1} \}, \quad \{ \hat{O}_{k,0} \}, \quad k, l = 1, \ldots, d - 1, \tag{48}
\]

where

\[
[\hat{O}_{M,\lambda}, \hat{O}_{L,\mu}] = 0, \quad [\hat{O}_{0,1}, \hat{O}_{0,1}^\dagger] = 0, \quad [\hat{O}_{k,0}, \hat{O}_{k',0}] = 0, \tag{49}
\]

and

\[
\text{tr} \left( \hat{O}_{k,0} \hat{O}_{k',0}^\dagger \right) = 0, \quad k' \neq k'. \tag{50}
\]

The collective observables (46) can be represented in form of a direct product of single-particle operators

\[
\hat{O}_{k,0} = \sum_{i=1}^{N} I \otimes \cdots \otimes \hat{O}_{i,k}^{(0)} \otimes \cdots \otimes I, \tag{51}
\]

\[
\hat{O}_{i,k}^{(0)} = \hat{I}^{(0)} - \frac{2}{d(d-1)} \sum_{a_i,b_i=0}^{d-1} \{ka_i + lb_i\} |a_i, b_i\rangle \langle a_i, b_i|, \tag{52}
\]

where \( |a_i, b_i\rangle \) are the states (13) and the operations \( \{ \cdot \} \) are taken mod \( d \). The operators (52) are normalized according to

\[
\text{Tr} \hat{O}_{k,0}^{(0)} = 0, \quad \text{Tr} \left( \left[ \hat{O}_{k,0}^{(0)} \right]^2 \right) = \frac{d}{3(d-1)}, \tag{53}
\]

and form a basis of the \( su(d) \) algebra similar to that introduced in [30]. The explicit form of the matrix elements of the operators \( \hat{O}_{k,0}^{(0)} \) in the computational basis is given in appendix B, equations (B.1) and (B.2).

Taking into account (13) one obtains that \( d - 1 \) operators \( \hat{O}_{0,1}^{(0)} \) are diagonal in the logical basis (see appendix B). The elements of the commuting sets containing non-diagonal operators \( \hat{O}_{k,0}^{(0)}, k \neq 0 \), are convenient to label as \( \hat{O}_{\lambda,\lambda m}^{(0)} \), which matrix elements in the computational basis are given in appendix B. The commuting sets \( \{ \hat{O}_{\lambda,\lambda m}, [\hat{O}_{\lambda,\lambda m}, \hat{O}_{\lambda,\lambda m'}] = 0, m, m' = 0, \ldots, d - 1 \} \) are obtained from the diagonal set \( \{ \hat{O}_{0,1} \} \) by \( SU(d) \) transformations, and thus can be efficiently measured.

**Examples**

(a) Qubits, \( d = 2 \). The fiducial state (42) leads to the natural representation

\[
\hat{O}_{0,1}^{(0)} = \frac{1}{\sqrt{3}} \sigma_x^{(0)}, \quad \hat{O}_{1,0}^{(0)} = \frac{1}{\sqrt{3}} \sigma_x^{(0)}, \quad \hat{O}_{1,1}^{(0)} = \frac{1}{\sqrt{3}} \sigma_y^{(0)}, \tag{54}
\]

so that the operators (46) coincide up to a constant factor with the spin collective operators (1).
(b) Qutrits, $d = 3$. Taking the fiducial state (45) we obtain the following four commutative sets of cyclic (up to a constant) operators, $\hat{O}_{i,j}^{(l)} = 4 \left( \hat{O}_{i,j}^{(l)} \right)^{1}$, (compare with [30])

\[
\hat{O}_{0,1}^{(0)} = \frac{i}{2} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \hat{O}_{0,2}^{(0)} = \frac{1}{2} \begin{bmatrix} 0 & e^{-i\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & 0 & e^{-i\pi/6} \\ e^{-i\pi/6} & e^{i\pi/6} & 0 \end{bmatrix},
\]

(55)

\[
\hat{O}_{1,0}^{(0)} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 0 & -i & e^{-i5\pi/6} \\ i & 0 & e^{i\pi/6} \\ e^{i5\pi/6} & e^{-i\pi/6} & 0 \end{bmatrix}, \quad \hat{O}_{2,0}^{(0)} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 0 & e^{-i\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & 0 & e^{-i\pi/6} \\ e^{-i\pi/6} & e^{i\pi/6} & 0 \end{bmatrix},
\]

(56)

\[
\hat{O}_{1,1}^{(0)} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 0 & -i & e^{-i5\pi/6} \\ e^{i\pi/6} & 0 & e^{-i\pi/6} \\ e^{i5\pi/6} & e^{-i\pi/6} & 0 \end{bmatrix}, \quad \hat{O}_{2,1}^{(0)} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 0 & e^{-i\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & 0 & e^{-i\pi/6} \\ e^{-i\pi/6} & e^{i\pi/6} & 0 \end{bmatrix},
\]

(57)

\[
\hat{O}_{2,2}^{(0)} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 0 & -i & e^{-i5\pi/6} \\ e^{i\pi/6} & 0 & e^{-i\pi/6} \\ e^{i5\pi/6} & e^{-i\pi/6} & 0 \end{bmatrix}, \quad \hat{O}_{1,2}^{(0)} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 0 & e^{-i\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & 0 & e^{-i\pi/6} \\ e^{-i\pi/6} & e^{i\pi/6} & 0 \end{bmatrix}.
\]

(58)

The sets (56)–(58) can be obtained form the set (55) by $SU(3)$ rotations.

3. Collective tomography

By a direct substitution one can prove that the symmetrized operators (27) form a bi-orthogonal set

\[
\text{Tr} \left[ \hat{\Delta}^{(1)} (m) \hat{\Delta}^{(-1)} (m') \right] = R_{m}^{(d)} \delta_{m,m'}.
\]

(59)

This suggests the approximation of the density matrix by ‘inverting’ equation (26) in the form similar to equation (15)

\[
\hat{\rho} \approx \hat{\rho}_{nc} = \sum_{m} \hat{Q}_{\rho} (m) \left( R_{m}^{(d)} \right)^{-1} \hat{\Delta}^{(1)} (m).
\]

(60)

The above equation is a formal expression for the state reconstruction by using the results of all possible collective measurements stored in $Q_{\rho} (m)$. The ‘tomographic representation’ (60) is incomplete since the map (26) is not faithful. In other words, equation (60) should be considered as a form of arranging the information obtained from $\sim N^{d^{2}-1}$ collective measurements (corresponding to the total number of multiplets $\{ m \}$) in a $d^{N} \times d^{N}$ matrix. Obviously, measuring collective operators we can access only to $SU(N)$ invariant subspaces, that appear in the
decomposition of $N$-qudit density matrix, with no possibility to distinguish subspaces of the same dimensions.

Equation (60) can be explicitly rewritten in terms of the average values of symmetrized monomials (2) as follows

$$\hat{\rho}_{\text{rec}} = d^{-N} \sum_{m} (R_{m}^d)^{-1} \langle \hat{D}_m \rangle \hat{D}_m^\dag,$$  

$$\hat{D}_m = \sum_{\alpha, \beta} \delta_{h(\alpha, \beta), m} Z_{\alpha}, \langle \hat{D}_m \rangle = \text{Tr}(\hat{\rho} \hat{D}_m),$$

which is a generalization of results obtained in [31]. The operators (62) can be always expressed in terms of polynomials of the collective operators (46)

$$\hat{D}_m = \sum_{p} c^m(p) \prod_{k, l, m_{kl}} O_{p k, l} \leq \sum_{k, l} m_{kl}.$$

(see appendix D), and thus, accessed from Von Neumann measurements.

Since any state obtained from $\hat{\rho}$ by particle permutations, $\mathcal{P} \hat{\rho} \mathcal{P}^\dagger$, leads to the same $\hat{\rho}_{\text{rec}}$, the density matrix reconstructed according to equation (61) is related to the original one by the full symmetrization,

$$\hat{\rho}_{\text{rec}} = \frac{1}{N!} \sum_{\mathcal{P}} \mathcal{P} \hat{\rho} \mathcal{P}^\dagger$$

Thus, the reconstruction (61) is exact for permutationally invariant (symmetric) states. It is straightforward to obtain a closed form expression for the reconstruction fidelity of a pure state $\hat{\rho} = |\psi\rangle \langle \psi|$

$$F = \text{Tr}(\hat{\rho} \hat{\rho}_{\text{rec}}) = \frac{1}{N!} \sum_{\mathcal{P}} |\langle \psi| \mathcal{P}| \psi\rangle|^2,$$

which reaches its minimum value

$$F_{\text{min}} = \frac{1}{N!},$$

for states that become orthogonal under any particle permutation, e.g. for $N$-qudit, $d \geq N$, elements of the computational basis $|\lambda\rangle = |l_1, \ldots, l_N\rangle$, with $l_i \neq l_j$.

For mixed states the minimum value of the fidelity can be substantially smaller than given in equation (66). For instance, for 2 and 3 qubits one obtains $F_{\text{min}} = 1/2$ and $F_{\text{min}} = 1/8$ correspondingly; for 2 and 3 qutrits $F_{\text{min}} = 1/9$ and $F_{\text{min}} = 1/27$.

### 3.1. Symmetric space reconstruction

In the particular case of fully symmetric states, the reconstruction expressions (60) and (61) become exact and can be rewritten in an explicit form by projecting the expansion (15) into the symmetric subspace.

Let us consider a symmetric density matrix, i.e. $\hat{\rho}_s = \hat{\Pi}_s \hat{\rho}_s \hat{\Pi}_s$, where $\hat{\Pi}_s$ is the projection operator on the symmetric subspace $\mathcal{H}_{\text{sym}}$ of $N$ qudits

$$\hat{\Pi}_s = \sum_{p} |p; N\rangle \langle p; N|,$$
and permutationally invariant states
\[ |p; N\rangle = |p_1, \ldots, p_{d-1}; N\rangle, \tag{68} \]
\[ p_j = 0, \ldots, N, \quad \sum_{j=1}^{d-1} p_j \leq N, \tag{69} \]
are elements of an orthonormal basis, \[ \langle p', N | p, N \rangle = \delta_{p', p}, \] in \( d_{\text{sym}} \)-dimensional Hilbert space \( H_{\text{sym}} \).

\[ d_{\text{sym}} = \frac{(N + d - 1)!}{(d - 1)!N!}. \tag{70} \]

The states (68) are expanded in the computational basis according to

\[ |p, N\rangle = N_p \sum_{\lambda} \prod_{i=1}^{d-1} \delta_{\eta_i^{(\lambda)}} |\lambda\rangle, \tag{71} \]

where

\[ \eta_i^{(\lambda)} = \frac{(-1)^{i+1}}{(d - 1)!} \sum_{j=1}^{N} \prod_{k=0}^{d-1} (k - l_j), \tag{72} \]
denotes the number of coefficients \( l_j \) equal to \( i = 1, \ldots, d - 1 \) in the state \( |\lambda\rangle = |l_1, \ldots, l_n\rangle \), and

\[ N_p = \sqrt{p_1! \cdots p_{d-1}!(N - p_1 - \cdots - p_{d-1})!}, \tag{73} \]
is the normalization constant. It is easy to see that \( \eta_i^{(\lambda)} \) are symmetric functions of \( \lambda \), \( \eta_i^{(\lambda)} = \eta_i(h(\lambda)) \).

A density matrix \( \hat{\rho}_s \) can be represented according to equation (15) as follows

\[ \hat{\rho}_s = \sum_{\alpha, \beta} \text{tr} \left( \hat{\rho}_s \hat{\Delta}_{s}^{-1}(\alpha, \beta) \right) \hat{\Delta}_{s}^{(1)}(\alpha, \beta), \tag{74} \]

where \( \hat{\Delta}_{s}^{(1)}(\alpha, \beta) \) are the kernels (A.2) projected into the symmetric subspace,

\[ \hat{\Delta}_{s}^{(1)}(\alpha, \beta) = \hat{\Pi}_s \hat{\Delta}^{(1)}(\alpha, \beta) \hat{\Pi}_s \equiv \hat{\Delta}_{s}^{(1)}(h(\alpha, \beta)), \tag{75} \]
acting in \( H_{\text{sym}} \).

The first-rank projectors

\[ \hat{\Delta}_{s}^{(-1)}(h(\alpha, \beta)) = |\phi_{h(\alpha, \beta)}\rangle \langle \phi_{h(\alpha, \beta)}|, \tag{76} \]

where

\[ |\phi_{h(\alpha, \beta)}\rangle = \hat{\Pi}_s |\alpha, \beta\rangle = \sum_p N_p \Upsilon_p(h(\alpha, \beta)) |p, N\rangle, \tag{77} \]

\[ \Upsilon_p(h(\alpha, \beta)) = p \sum_{\lambda} \omega^{\alpha \lambda} c_{\lambda - p} \prod_{i=1}^{d-1} \delta_{\eta_i^{(\lambda)}}, \tag{78} \]
being $c_\mu$, the fiducial state expansion coefficients \eqref{eq:40}, define the measurement sets and satisfy the (POVM) completeness condition

$$\sum_m \hat{E}_m^{(d)} = \hat{1}, \quad \hat{E}_m^{(d)} = d^{-N} R_m^{(d)} \Delta_{s}^{-1}(m = \hbar(\alpha, \beta)), \quad (79)$$

where $R_m^{(d)}$ is defined in \eqref{eq:31} and $\hat{1}$ acts as an identity operator in $\mathcal{H}_{\text{sym}}$. After some straightforward algebra one can show that the $\hat{\Delta}_s^{(1)}$ kernel \eqref{eq:75} in the basis \eqref{eq:68} acquires the form

$$\langle t, N \mid \hat{\Delta}_s^{(1)}(m) \mid t', N \rangle = d^{-2N} \sum_{t, t'} N_t N_{t'} \left[ \sum_q \left( R_q^{(d)} \right)^{-2} g^{(d)}(q, m) f^{(d)}(q) C_{t', t}(q) \right], \quad (80)$$

where

$$g^{(d)}(q, m) = \sum_{\alpha, \beta} \omega^{(\alpha - \beta)t} \sum_{\eta} \delta_{h(\alpha, \beta), m} \delta_{h(\gamma, \beta), q}, \quad (81)$$

$$C_{t', t}(q) = \sum_{\gamma, \delta, \lambda, \eta} \omega^{\gamma \lambda} \prod_{i=1}^{d-1} \delta_{h(\gamma, \delta), i} \prod_{j=1}^{d-1} \delta_{h(\gamma, \delta), j} \delta_{h(\gamma, \beta), q}, \quad (82)$$

are the special discrete functions (special functions of discrete variables) and the coefficient

$$f^{(d)}(q) = \sum_{\gamma, \delta} \left( \langle \zeta \mid Z_{\gamma} X_{\delta} \zeta \rangle \right)^{-1} \delta_{h(\gamma, \beta), q} = R_q^{(d)} \left( \langle \zeta \mid Z_{\gamma} X_{\delta} \zeta \rangle \right)^{-1} |_{h(\gamma, \beta) = q}, \quad (83)$$

can be always evaluated analytically, see \eqref{eq:A.5}.

Finally, the explicit reconstruction expression in the symmetric subspace $\mathcal{H}_{\text{sym}}$ acquires the form

$$\hat{\rho}_s = d^N \sum_m \sigma_m \left( R_m^{(d)} \right)^{-1} \hat{\Delta}_s^{(1)}(m), \quad (84)$$

where

$$\sigma_m = \text{Tr} \left( \hat{E}_m^{(d)} \rho_s \right), \quad \sum_m \sigma_m = 1, \quad (85)$$

are measured probabilities. The total number of projections required in this protocol is given by equation \eqref{eq:32}, while the density matrix of a fully symmetric state contains at most $d_{\text{sym}}^2 - 1$ independent parameters. Such a redundancy occurs because the probabilities \eqref{eq:85} are not linearly independent and satisfy the following self-consistency conditions

$$\sigma_q = \sum_m \sigma_m \left( R_m^{(d)} \right)^{-1} R_q^{(d)} \langle \phi_q \mid \hat{\Delta}_s^{(1)}(m) \mid \phi_q \rangle, \quad (86)$$

In order to estimate the accuracy of the reconstruction scheme \eqref{eq:84} we numerically studied the minimum square error (MSE) of the quadratic Hilbert–Schmidt distance between a real state $\rho_s$ and its estimate $\hat{\rho}_s$ according to the Crâmer–Rao lower bound \cite{33} for qubit and qutrit states, see appendix \textit{E}.
(a) In the $N$-qubit case, $d = 2$, a single index (72) defines symmetric states,

$$\eta_i^{(\lambda)} = \sum_{i=1}^{N} l_i = h(\lambda).$$

(87)

The elements of the basis in the symmetric subspaces are the well known Dicke states [32]

$$|p; N\rangle = \mathcal{N}_{p_1} \sum_{\lambda} \delta_{h(\lambda), p_1} |\lambda\rangle,$$

(88)

$$\mathcal{N}_{p_1} = \sqrt{\frac{p_1!(N - p_1)!}{N!}}.$$ (89)

The symmetrized discrete coherent states (77) are expanded in the Dicke basis as follows

$$|\phi_{h}(\alpha, \beta)\rangle = \sum_{p_1} \mathcal{N}_{p_1} T_{p_1} (h(\alpha, \beta)) |p_1; N\rangle,$$

(90)

$$T_{p_1} (h(\alpha, \beta)) = \left(1 + |\zeta|^2\right)^{-N/2} \sum_{\lambda} (-1)^{\zeta h(\lambda + \beta)} \delta_{h(\lambda), p_1},$$

(91)

where $\zeta$ is defined in equation (42). The functions (81) and (83) have the form

$$g^{(2)}(\mathbf{q}, \mathbf{m}) = \sum_{\alpha, \beta} (-1)^{q_{10} + q_{01}} \delta_{h(\alpha, \beta), m} \delta_{h(\gamma, \delta), q},$$

$$f^{(2)}(\mathbf{q}) = 3 \int_{t_0}^{t_1} \mathbf{q}^{(2)} \, dt,$$

where $h(\gamma) = q_{10}, h(\delta) = q_{01}, h(\gamma + \delta) = q_{11}$ and

$$C_{\gamma, \delta, \lambda} = \sum_{\gamma, \lambda, \delta} (-1)^{\delta_{h(\lambda), \lambda}} \delta_{h(\lambda), \lambda} \delta_{h(\gamma), q_{10}} \delta_{h(\delta), q_{01}} \delta_{h(\gamma + \delta), q_{11}}$$

(92)

can be expressed in terms of \(_4F_3\)-functions.

We have numerically found that the MSE \(\sqrt{\langle \langle E_{\text{min}}^2 \rangle \rangle}\), averaged over 200 pure and mixed states, is inversely proportional to the square root of the number of trials $M$,

$$\sqrt{\langle \langle E_{\text{min}}^2 \rangle \rangle} \approx \frac{\lambda}{\sqrt{M}}.$$ (93)

In figure 3 we plot the proportionality constant $\lambda$ for pure and mixed states for $N = 1, \ldots, 6$ qubits and compare with the values corresponding to the SIC POVM tomographic protocol [28].

(b) In $N$-qutrit case, $d = 3$, there are two indices (72) that define qutrit symmetric space,

$$\eta_i^{(\lambda)} = \sum_{i=1}^{N} l_i (2 - l_i) = \frac{2}{3} \left[h(2\lambda) - \frac{1}{2} h(\lambda)\right] = p_1,$$

(94)

$$\eta_i^{(\lambda)} = \sum_{i=1}^{N} l_i (l_i - 1) = \frac{2}{3} \left[h(\lambda) - \frac{1}{2} h(2\lambda)\right] = p_2.$$ (95)
Figure 3. First two columns (blue and purple): the proportionality constant \( \lambda \) appearing in the minimum square error (93) for pure (left column) and mixed (right column) states \( d_{\text{sym}} = N + 1 \) for \( N = 1, \ldots, 6 \) qubits corresponding to the reconstruction skill (84); second two columns (gray and black): the corresponding values of \( \lambda \) for SIC tomographic protocol.

thus, the basis in the symmetric subspace has the form

\[
|p_1, p_2; N\rangle = N_{p_1, p_2} \sum_\lambda \delta_{h(\lambda), p_1} + 2p_2 \delta_{h(2\lambda), 2p_1 + p_2} |\lambda\rangle,
\]

(96)

\[
N_{p_1, p_2} = \sqrt{p_1! p_2! (N - p_1 - p_2)! N!}.
\]

(97)

The symmetrized discrete coherent states (77) are

\[
|\phi_{h(\alpha, \beta)}\rangle = \sum_{p_1, p_2} N_{p_1, p_2} \Upsilon_{p_1, p_2} (h(\alpha, \beta)) |p_1, p_2; N\rangle,
\]

(98)

\[
\Upsilon_{p_1, p_2} (h(\alpha, \beta)) = \sum_\lambda \omega^{\lambda x} c_{\lambda, \beta} \delta_{h(\lambda), p_1} + 2p_2 \delta_{h(2\lambda), 2p_1 + p_2},
\]

(99)

where \( \omega = \exp(2\pi i/3) \) and the \( c_{\lambda, \beta} \) correspond to the fiducial state (45). The matrix element in equation (83) is

\[
\left( \langle \xi | Z_\gamma X_\delta | \xi \rangle \right)^{-1} |_{h(\gamma, \delta) = q} = 2^\gamma \sum_{\xi, \eta} N_{\xi, \eta} e^{\frac{\pi}{6} \sum_{q_1, q_2} N_{q_1, q_2} g_{q_1, q_2}},
\]

(100)

where \( q_{li} = h(k_l + l\delta) \) and

\[
C_{q_1, q_2} (q) = \sum_{\gamma, \lambda, l, l'} \omega^{\lambda x} \delta_{h(\gamma, \delta), q} \delta_{h(\lambda), l} + 2r_2 \delta_{h(2\lambda), 2l_1 + l_2} \delta_{h(-\lambda, -\delta), l_1' + l_2'} + 2r_2' \delta_{h(2\lambda, -2\delta), 2l_1' + l_2'},
\]

(101)

We have numerically found that for qutrits the minimum error \( \sqrt{\langle \xi_i^2 \rangle_{\text{min}}} \) also behaves according to equation (93). In figure 4 we plot the constant \( \lambda \) for pure and mixed qutrit states, \( N = 1, 2, 3 \).
3.2. Conclusions

We have developed a novel general framework for the analysis of $N$-qudit systems in the macroscopic limit, which includes:

(a) A scheme to organize the information obtained from collective measurements in the form of distribution functions in a discrete low-dimensional space spanned by the symmetric functions (21). The analysis of these projected $\tilde{Q}_\rho$-functions provides a useful insight into the global properties of multipartite quantum states. It allows, for instance, the identification of the relevant set of collective measurements for a given state, which are not always obvious from the state decomposition in the computational basis. In addition, $\tilde{Q}_\rho$-functions have well defined continuous limit when $N \to \infty$ and being plotted as projections into three dimensional hyper-planes, see e.g. figures 1 and 2, provide an intuitive graphical representation of $N$-qudit states (in contrast to the full discrete $d^N \times d^N$ phase-space [7, 8]).

(b) A set of collective operators (46) appropriate for the characterization of $N$-qudit states and the representation in the $M$-space, equations (47) and (B.11). The appealing algebraic properties (49), (50), (63) of these operators allow their efficient measurements with further applications to the collective tomography protocol (61):

(c) Explicit expressions for the state reconstruction from collective measurements, both in the whole $d^N$-dimensional space (60), (61) and in the symmetric subspace (84) along with the corresponding fidelity (65) and the minimum square error (93) estimations. The symmetric tomographic protocol based on projective measurements performed in POVM (76) is characterized by a larger MSE than the optimal SIC reconstruction scheme for $N$-qubit
states. Nevertheless, for larger $d$-dimensions the relative difference between both reconstruction methods becomes smaller. The advantage of the proposed method is the possibility to generate the first rank collective POVM (76) for $N$ qudit systems in a systematic way for an arbitrary number of particles.

A deeper analysis of the projected $N$-qudit $Q$-functions and their applications is in progress and will be published elsewhere.

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Appendix A

We outline the general method for computing $P$-symbols of collective operators. Consider a generic (non-Hermitian) collective operator

$$\sum_{i=1}^{N} Z_i^m X_i^n = \hat{s}_{mn}, \quad m, n \in Z_d. \tag{A.1}$$

Any other collective operator can be constructed from (A.1). Taking into account the explicit form of the kernels (7)

$$\hat{\Delta}^{(\pm 1)}(\alpha, \beta) = \frac{1}{d^{(\pm 1)N}} \sum_{\gamma, \delta} \omega^{\alpha \gamma - \beta \delta - \frac{1}{2}(\pm 1)\gamma \delta} (\langle \xi | Z_{\gamma} X_{\delta} | \xi \rangle) ^{\pm 1} Z_{\gamma} X_{\delta}, \tag{A.2}$$

$$\text{Tr} \hat{\Delta}^{(\pm 1)}(\alpha, \beta) = d^{\frac{1}{2}(\pm 1)N}, \quad \sum_{\alpha, \beta} \hat{\Delta}^{(\pm 1)}(\alpha, \beta) = d^\frac{1}{2}(\pm 1)^N \hat{I}, \tag{A.3}$$

the $P$-function of (A.1) is represented as follows according to (7)

$$P_{mn} = \frac{\omega^{mn}}{d^{N}} (\langle \xi | Z^{-m} X^{-n} | \xi \rangle)^{-1} \sum_{i=1}^{N} \omega^{mb_i - na_i}. \tag{A.4}$$

Observe that for the fiducial state of the form (12)–(14) the matrix element $\langle \xi | Z_{\gamma} X_{\delta} | \xi \rangle$ is a symmetric function of its argument,

$$\langle \xi | Z_{\gamma} X_{\delta} | \xi \rangle = \langle \xi | Z_{\gamma} X_{\delta} | \xi \rangle _{b(\gamma, \delta) = p}. \tag{A.5}$$

The sum $\sum \omega^{mb_i - na_i}$ that appears in the above equation can be rewritten as

$$\sum_{i=1}^{N} \omega^{mb_i - na_i} = \sum_{i=1}^{N} \left[ \frac{1}{(d-1)! b_i} + \frac{\omega^m}{(d-2)! (1 - b_i)} + \frac{\omega^{2m}}{-2(d-3)! (2 - b_i)} + \ldots \right]$$

$$\left[ \frac{1}{(d-1)! a_i} + \frac{\omega^{-n}}{(d-2)! (1 - a_i)} + \frac{\omega^{-2n}}{-2(d-3)! (2 - a_i)} + \ldots \right] \prod_{k=0}^{d-1} (k - b_i) (k - a_i), \tag{A.6}$$

The sum $\sum \omega^{mb_i - na_i}$ that appears in the above equation can be rewritten as

$$\sum_{i=1}^{N} \omega^{mb_i - na_i} = \sum_{i=1}^{N} \left[ \frac{1}{(d-1)! b_i} + \frac{\omega^m}{(d-2)! (1 - b_i)} + \frac{\omega^{2m}}{-2(d-3)! (2 - b_i)} + \ldots \right]$$

$$\left[ \frac{1}{(d-1)! a_i} + \frac{\omega^{-n}}{(d-2)! (1 - a_i)} + \frac{\omega^{-2n}}{-2(d-3)! (2 - a_i)} + \ldots \right] \prod_{k=0}^{d-1} (k - b_i) (k - a_i), \tag{A.6}$$
which is always reduced to the form
\[ \sum_{i=1}^{N} d_i^b j_i = \sum_{k,j=0}^{d-1} B_{kj}^{(p,q)} h (k\alpha + l\beta), \]  
(A.7)

where \( B_{kj}^{(p,q)} \) are some coefficients. Thus, the \( P \)-function (A.4) is a function of the weights (21).

Examples

(a) Qubits, \( d = 2 \)

The sum (A.6) takes the form, \( \omega = -1 \)
\[ N \sum_{i=1}^{N} \omega^{n_b - n_q} = N - 2\delta_{m,0} \delta_{n,1} h (\alpha) - 4\delta_{m,1} \delta_{n,0} h (\beta) - 4\delta_{m,1} \delta_{n,1} h (\alpha + \beta), \]  
(A.8)

so that, for the fiducial state (42)
\[ P_{mm} (\alpha, \beta) = \frac{\sqrt{3}}{2N} \left[ N - 2 \left( h (\alpha) \delta_{m,0} \delta_{n,1} + h (\beta) \delta_{m,1} \delta_{n,0} + h (\alpha + \beta) \delta_{m,1} \delta_{n,1} \right) \right]. \]  
(A.9)

(b) Qutrits, \( d = 3 \)

The sum (A.6) is significantly more complicated, \( \omega = \exp(2\pi i/3), \)
\[ N \sum_{i=1}^{N} \omega^{n_b - n_q} = N - \left[ \delta_{m,0} \delta_{n,1} e^{-i\pi/3} + \delta_{m,0} \delta_{n,2} e^{i\pi/3} \right] h (\alpha) \]
\[ - \left[ \delta_{m,1} \delta_{n,0} e^{i\pi/3} + \delta_{m,2} \delta_{n,0} e^{-i\pi/3} \right] h (2\alpha) \]
\[ - \left[ \delta_{m,1} \delta_{n,0} e^{-i\pi/3} + \delta_{m,2} \delta_{n,0} e^{i\pi/3} \right] h (\beta) \]
\[ - \left[ \delta_{m,1} \delta_{n,1} e^{-i\pi/3} + \delta_{m,2} \delta_{n,1} e^{i\pi/3} \right] h (\alpha + \beta) \]
\[ - \left[ \delta_{m,1} \delta_{n,2} e^{-i\pi/3} + \delta_{m,2} \delta_{n,1} e^{i\pi/3} \right] h (2\alpha + 2\beta) \]
\[ - \left[ -\delta_{m,1} \delta_{n,0} e^{-i\pi/3} + \delta_{m,2} \delta_{n,2} e^{i\pi/3} \right] h (2\alpha + \beta) \]
\[ - \left[ \delta_{m,1} \delta_{n,1} e^{-i\pi/3} + \delta_{m,2} \delta_{n,0} e^{i\pi/3} \right] h (\alpha + 2\beta). \]

Appendix B

The matrix elements of the operators (46) have the form
\[ \langle p| \hat{O}^{(p)}_{ij} |q \rangle = \delta_{p,q} \left[ 1 - \frac{2}{(d-1)} \sum_{r=0}^{d-1} \{ |r \} \langle \hat{O}^{(r)}_{ij} \rangle \right]^2, \]  
(B.1)
\[ \langle p| \hat{O}^{(p)}_{\lambda,lm} |q \rangle = \delta_{p,q} \sum_{r=0}^{d-1} \omega^{-m(r-q-p)} \langle \hat{O}^{(r)}_{p-r} \rangle \langle \hat{O}^{(r)}_{q-r} \rangle, \]  
(B.2)
where \( e_p^i \) are the expansion coefficients (39), and \( \tau_{q,\lambda} = \sum_{r=0}^{d-1} r \omega^{\mu \lambda - 1} \), \( \lambda = 1, \ldots, d - 1 \) \( m = 0, \ldots, d - 1 \). The trace of two operators (46) has the form

\[
\text{tr} \left( \hat{O}_{k,l}\hat{O}_{k',l'} \right) = N^2 d^N - \frac{2N}{(d - 1)d^N} \left( \sum_{\alpha,\beta} h(k\alpha + l\beta) + \sum_{\alpha,\beta} h(k'\alpha + l'\beta) \right) + \frac{1}{(d - 1)^2 d^{2N} d^N + 1} \left( \sum_{\alpha,\beta,\alpha',\beta'} h(k\alpha + l\beta) \right. \\
- \left. \sum_{\alpha,\beta} h(k\alpha + l\beta) \right) \left( \sum_{\alpha,\beta} h(k'\alpha + l'\beta) \right) + \frac{4}{(d - 1)^2 d^{2N} d^N} \sum_{\alpha,\beta} h(k\alpha + l\beta) h(k'\alpha + l'\beta),
\]

where we have used the SIC POVM condition (14).

Taking into account that

\[
\sum_{\mu} h(\mu) = \frac{N(d - 1)d^N}{2}, \quad \sum_{\alpha,\beta} h(k\alpha + l\beta) = \frac{N(d - 1)d^{2N}}{2},
\]

we obtain

\[
\text{tr} \left( \hat{O}_{k,l}\hat{O}_{k',l'} \right) = -\frac{N^2 d^N}{d^N + 1} + \frac{4}{d^N(d - 1)^2 d^{2N} + 1} \sum_{\alpha,\beta} h(k\alpha + l\beta) h(k'\alpha + l'\beta). \quad (B.5)
\]

In the case when \( k \neq \lambda k' \) and \( l \neq \lambda l' \), \( \lambda = 1, \ldots, d - 1 \) we can sum over the independent variables \( \mu = k\alpha + l\beta, \nu = \lambda(k'\alpha + l'\beta) \), so that the last sum in (B.5) is just a square of (B.3) and thus

\[
\text{tr} \left( \hat{O}_{k,l}\hat{O}_{k',l'} \right) = 0; \quad (B.6)
\]

if \( k = \lambda k' \) and \( l = \lambda l' \), then

\[
\text{tr} \left( \hat{O}_{k,l}\hat{O}_{k',l'} \right) = -\frac{N^2 d^N}{d^N + 1} + \frac{4}{(d - 1)^2 d^{2N} + 1} \sum_{\mu} h(\mu) h(\lambda\mu). \quad (B.7)
\]

Taking into account the representation (16) and the form of the \( P \)-function (47)

\[
\hat{O}_{ij} = \sum_{\alpha,\beta} \frac{1}{d^N} \left[ N - \frac{2}{d - 1} h(k\alpha + l\beta) \right] \Delta^{(-1)}(\alpha,\beta) = N\hat{1} - \frac{2}{d^N(d - 1)} \sum_{\gamma,\delta} \omega^{\gamma \delta} \left[ \sum_{\alpha,\beta} h(k\alpha + l\beta) \omega^{\alpha \beta - \gamma \delta} \right] \langle \xi | Z_{-\gamma} X_{-\delta} | \xi \rangle Z_{\gamma} X_{\delta}
\]

\[19\]
we immediately observe that
\[ [\hat{O}_{k,0}, \hat{O}_{\lambda,k}] = 0. \] 
(B.8)

If \( k \neq 0, l \neq 0 \) for \( d > 2 \) then after a change of variables
\[ \alpha = (\mu + \nu)(2k)^{-1}, \]
\[ \beta = (\mu - \nu)(2l)^{-1}, \]
where \((2k)^{-1}\) is understood as the inverse element in \( \mathbb{Z}_d \), we represent \( \hat{O}_{k,l} \) as
\[ \hat{O}_{k,l} = N! \sum_{\gamma} \omega^{2\gamma^2 - 1} \left( \sum_{\mu} h(\mu) \omega^{2\mu^2 - 1} \right) \langle \xi | Z_{\gamma} X_{k-1}^{\gamma} | \xi \rangle Z_{\gamma} X_{k-1}^{\gamma}, \] (B.9)
so that
\[ [\hat{O}_{k,l}, \hat{O}_{\lambda,k}] = 0. \] (B.10)

It is worth noting that the average values of the collective operators (46) in the states (77) are proportional to the corresponding \( P \)-symbols (47)
\[ \langle \phi_{h(\alpha,\beta)=m} | \hat{O}_{k,l} | \phi_{h(\alpha,\beta)=m} \rangle = \frac{d^N}{d+1} P_{k,l}. \] (B.11)

### Appendix C

Here we present explicit expressions for the \( R_{q,d} \)-functions for qubits and qutrits:
\[ R_{q,2} = \binom{N}{q_{10} + q_{01} + q_{11}} \binom{N}{q_{10} + q_{01} + q_{11}} \binom{N}{q_{10} + q_{01} + q_{11}} \binom{N}{q_{10} + q_{01} + q_{11}} \binom{N}{q_{10} + q_{01} + q_{11}}, \] (C.1)
\[ 0 \leq q_{kl} = h(k\gamma + l\delta) \leq N, \] (C.2)
and
\[ (R_{q,3})^{-1} = \frac{1}{N!} \left( N - \frac{q_{10} + q_{20} + q_{01} + q_{11} + q_{21} + q_{12}}{9} \right) \times \]
\[ \left( \frac{2q_{10} - q_{20} + 2q_{01} - q_{11} + 2q_{22} - q_{21} - q_{12}}{9} \right) \times \]
\[ \left( \frac{2q_{10} - q_{20} - q_{11} + 2q_{22} - q_{21} - q_{12}}{9} \right) \times \]
\[ \left( \frac{2q_{10} - q_{20} - q_{01} + 2q_{11} - q_{22} + 2q_{21} - q_{12}}{9} \right) \times \]
\[ \left( \frac{2q_{10} - q_{20} - q_{11} + 2q_{22} - q_{21} + 2q_{12}}{9} \right) \times \]
\[ \left( \frac{-q_{10} + 2q_{20} + 2q_{01} - q_{11} - q_{22} + q_{21} + 2q_{12}}{9} \right) \times \] (C.3)
In the limit $N \gg 1$ the function $R^{(h)}_q$ tends to a Gaussian form localized in the vicinity of the point $q_0 = \frac{d-1}{2} (N, \ldots, N)$, for instance,

$$R^{(2)}_{q(0)q(1)} \sim \exp(-2(q - q_0)^2/N), \quad (C.5)$$

$$R^{(3)}_q \sim \exp \left( -\sum_{k=0}^{k+1} \sum_{l=0}^{l+1} \frac{[q_{kl} - N]^2 + (q_{2k2l} - N)^2 - (q_{kl} - N)(q_{2k2l} - N)}{N} \right), \quad (C.6)$$

### Appendix D

Any product of collective operators (46) can be directly expanded in the monomial basis (4):

$$\prod_{k, l} O_{k, l}^{p} = \sum_{\gamma, \delta} b_{(\gamma, \delta), q} Z_{\gamma} X_{\delta}, \quad (D.1)$$

The required representation of the symmetrized monomials in terms of measurables operators (46)

$$\sum_{\mu, \nu} Z_{\mu} X_{\nu} \delta_{b(\mu, \nu), m} = \sum_{p} c_{m}^{p} \prod_{k, l} O_{k, l}^{p}, \quad \sum_{k, l} p_{k, l} \leq \sum_{m} m_{kl}, \quad (D.2)$$

is thus obtained by substituting (D.1) into (D.2) and comparing the coefficients of the same basis elements. This leads to the following set of equations for the coefficients $C_{m}^{p}$

$$\delta_{b(\mu, \nu), m} = \sum_{p} c_{m}^{p} b_{(\mu, \nu), p}. \quad (D.3)$$

In the case of qubits, (D.1) acquires the following form for diagonal operators

$$\sum_{\mu} Z_{\mu} \delta_{b(\mu), k} = \sum_{p=0}^{k} C_{p}^{k} S_{\mu}. \quad (D.4)$$

Taking into account that the collective spin operator in the computational basis has the form

$$S_{\mu} = \sum_{\nu} (N - h(\nu)) |\nu\rangle \langle \nu|, \quad (D.5)$$
we arrive at the following expression for the coefficient \( b \)

\[
b_{h(\mu),p} = \frac{1}{2^{N}} \sum_{\nu} (N - h(\nu))^{p} \chi(\mu \nu) = \frac{1}{2^{N}} \sum_{m=0}^{N} \binom{N}{m} (N - m)^{p} g^{(2)}(m),
\]

where

\[
g^{(2)}(m) = \sum_{\nu} (-1)^{\mu \nu} \delta_{h(\nu),m},
\]

is a discrete special function, which can be expressed in terms of \( P^{(r)}_{n}(z) \) Jacobi polynomials.

The system (D.3) takes the form

\[
\delta h(\mu),p = \sum_{p \leq m} C^{(p)}_{m} b_{h(\mu),p},
\]

leading in particular to

\[
\sum_{\mu} Z_{\mu} \delta h(\mu),1 = \hat{S}_{z},
\]

\[
\sum_{\mu} Z_{\mu} \delta h(\mu),2 = \hat{S}_{z}^{2} - \hat{N} \hat{I}.
\]

Similar calculations can be carried out for qudits:

\[
\sum_{\mu} Z_{\mu} \delta h(\mu),m_{1} \delta h(2\mu),m_{2} = \sum_{p_{1} + p_{2} \leq m_{1} + m_{2}} C^{(m_{1},m_{2})}_{p_{1},p_{2}} \hat{O}^{p_{1}}_{0,1} \hat{O}^{p_{2}}_{0,2},
\]

where

\[
\hat{O}^{p}_{0,l} = \sum_{\nu} \left( N - \sum_{\varepsilon} h(\varepsilon) |c_{\nu - \varepsilon}|^{2} \right)^{p} |\nu\rangle \langle \nu|,
\]

c\_\_ being expansion coefficients (40) of the fiducial state in the computational basis.

Thus the \( b_{h(\mu),p} \) coefficients (D.3) have the form

\[
b_{h(\mu),p_{1},p_{2}} = \frac{1}{3^{p_{1} + p_{2}}} \sum_{m_{1},m_{2}} (m_{2} - m_{1})^{p_{1}} (N - m_{1})^{p_{2}} g^{(3)}(m_{1}, m_{2}),
\]

\[
g^{(3)}(m_{1}, m_{2}) = \sum_{\nu} \delta h(\nu),m_{1} \delta h(2\nu),m_{2} \omega^{\mu \nu}, \quad \omega = \exp(2\pi i/3),
\]

where we used the explicit expansion (45). The set of equation (D.3) to be inverted is now

\[
\delta b_{h(\mu),m} = \sum_{p_{1} + p_{2} \leq m_{1} + m_{2}} C^{(m_{1},m_{2})}_{p_{1},p_{2}} b_{h(\mu),p_{1},p_{2}}.
\]

In particular, we find,

\[
\sum_{\mu} Z_{\mu} \delta h(\mu),1 = 2 \omega \hat{O}_{0,1} + 2 \hat{O}_{0,2},
\]
\[
\sum_{\mu} Z_{\mu} \delta h(\mu),2 = 4 \left( \omega \hat{O}_{0,1} + \hat{O}_{0,2} \right)^{2} + 2 \left( \omega^{2} \hat{O}_{0,1} + \hat{O}_{0,2} \right).
\]
Appendix E

The performance of a reconstruction scheme can be measured by the statistical average of the Hilbert–Schmidt distance between the real $\rho$ and estimated $\tilde{\rho}$ states,

$$\langle E^2 \rangle = \langle \text{Tr}[(\rho - \tilde{\rho})^2] \rangle,$$

(E.1)

when only a finite number of copies ($M$) are involved in the measurement process [34].

The multinomial measurement statistics associated to the protocol (84)

$$\mathcal{P} \propto \prod_p \tilde{\sigma}_p^{n_p} \cdot \sum_p n_p = M,$$

(E.2)

allows us to relate the estimated probabilities $\tilde{\sigma}_p$ with the corresponding frequencies $n_p/M$, according to

$$\langle n_p \rangle = M \tilde{\sigma}_p, \quad \langle n_p^2 \rangle = M \tilde{\sigma}_p (1 + (M - 1) \tilde{\sigma}_p), \quad \langle n_p n_q \rangle = M (M - 1) \tilde{\sigma}_p \tilde{\sigma}_q.$$

(E.3)

Using the explicit reconstruction form (84) we obtain the square error in terms of the deviation between probabilities $\sigma_p$ and their estimates $\tilde{\sigma}_p$,

$$\langle E^2 \rangle = \sum_{p,q} A_{p,q} \Delta \sigma_p \Delta \sigma_q,$$

(E.4)

where $\Delta \sigma_p = \sigma_p - \tilde{\sigma}_p$ and

$$A_{p,q} = d^{2N} \left( \tilde{\rho}_p^{(d)} \tilde{\rho}_q^{(d)} \right)^{-1} \text{Tr} \left( \hat{\Delta}_s^{(1)}(p) \hat{\Delta}_s^{(1)}(q) \right).$$

(E.5)

Taking into account the redundancy (86) and normalization (85) conditions we obtain the square error in terms of the deviation between the independent probabilities $\sigma_{p'}$ and their estimates $\tilde{\sigma}_{p'}$,

$$\langle E^2 \rangle = \sum_{p',q'} A'_{p',q'} \Delta \sigma_{p'} \Delta \sigma_{q'},$$

(E.6)

where the sum is taken only on the indices $p', q'$ labeling the independent probabilities; the coefficients $A'_{p',q'}$ are not explicitly given here due to of their cumbersome form.

The minimum square error (MSE) is given by the Cramér–Rao bound [33],

$$\langle E^2 \rangle \geq \text{Tr}(\mathcal{A}' \mathcal{F}^{-1}),$$

(E.7)

where $\mathcal{A}'$ is the $(d_{sym}^2 - 1) \times (d_{sym}^2 - 1)$ matrix with the elements $\mathcal{A}'_{p',q'}$, while $\mathcal{F}$ is the Fisher matrix for the independent probabilities,

$$\mathcal{F}_{p',q'} = \left\langle \frac{\partial}{\partial \sigma'_{p'}} \text{Tr} \ln \mathcal{P} \frac{\partial}{\partial \sigma'_{q'}} \text{Tr} \ln \mathcal{P} \right\rangle.$$

(E.8)

To estimate the average minimum square error we numerically compute the statistical mean value over the symmetric states,

$$\sqrt{\langle \langle E^2_{\text{min}} \rangle \rangle} = \sqrt{\text{Tr}(\mathcal{A}' \mathcal{F}^{-1})}.$$

(E.9)
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