Functional Inequalities for Convolution Probability Measures∗

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Abstract

Let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}^d$, where $\mu(dx) = e^{-V(x)}dx$ for some $V \in C^1(\mathbb{R}^d)$. Explicit sufficient conditions on $V$ and $\nu$ are presented such that $\mu * \nu$ satisfies the log-Sobolev, Poincaré and super Poincaré inequalities. In particular, the recent results on the log-Sobolev inequality derived in [18] for convolutions of the Gaussian measure and compactly supported probability measures are improved and extended.

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1 Introduction

Functional inequalities of Dirichlet forms are powerful tools in characterizing the properties of Markov semigroups and their generators, see e.g. [17] and references within. To establish functional inequalities for less explicit or less regular probability measures, one regards the measures as perturbations from better ones satisfying the underlying functional inequalities. For a probability measure $\mu$ on $\mathbb{R}^d$, the perturbation to $\mu$ can be

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made in the following two senses. The first type perturbation is in the sense of exponential potential: the perturbation of $\mu$ by a potential $W$ is given by $\mu_W(dx) := \frac{e^{W(x)}\mu(dx)}{\mu(e^W)}$, for which functional inequalities have been studied in many papers, see [2, 5, 10] and references within. Another kind of perturbation is in the sense of independent sum of random variables: the perturbation of $\mu$ by a probability measure $\nu$ on $\mathbb{R}^d$ is given by their convolution

$$(\mu * \nu)(A) := \int_{\mathbb{R}^d} 1_A(x + y)\mu(dx)\nu(dy).$$

Functional inequalities for the latter case is not yet well investigated, and the study is useful in characterizing distribution properties of random variables under independent perturbations, see e.g. [18, Section 3] for an application in the study of random matrices.

In general, let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}^d$. A straightforward result on functional inequalities of $\mu * \nu$ can be derived from the sub-additivity property; that is, if both $\mu$ and $\nu$ satisfy a type of functional inequality, $\mu * \nu$ will satisfy the same type inequality. In this paper, we will consider the Poincaré inequality and the super Poincaré inequality. We say that a probability measure $\mu$ satisfies the Poincaré inequality with constant $C > 0$, if

$$\mu(f^2) \leq C\mu(|\nabla f|^2) + \mu(f)^2, \quad f \in C^1_b(\mathbb{R}^d).$$

We say that $\mu$ satisfies the super Poincaré inequality with $\beta : (0, \infty) \to (0, \infty)$, if

$$\mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(|f|^2), \quad r > 0, f \in C^1_b(\mathbb{R}^d).$$

It is shown in [14, Corollary 3.3] or [15, Corollary 1.3] that the super Poincaré inequality holds with $\beta(r) = e^{c/r}$ for some constant $c > 0$ if and only if the following Gross log-Sobolev inequality (see [11]) holds for some constant $C > 0$:

$$\mu(f^2 \log f^2) \leq C\mu(|\nabla f|^2), \quad f \in C^1_b(\mathbb{R}^d), \mu(f^2) = 1.$$

**Proposition 1.1.** Let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}^d$.

1. If $\mu$ and $\nu$ satisfy the Poincaré (resp. log-Sobolev) inequality with constants $C_1$ and $C_2 > 0$ respectively, then $\mu * \nu$ satisfies the same inequality with constant $C = C_1 + C_2$.

2. If $\mu$ and $\nu$ satisfy the super Poincaré inequality with $\beta_1$ and $\beta_2$ respectively, then $\mu * \nu$ satisfies the super Poincaré inequality with

$$\beta(r) := \inf \left\{ \beta_1(r_1)\beta_2(r_2) : r_1, r_2 > 0, r_1 + r_2\beta_1(r_1) \leq r \right\}, \quad r > 0.$$

Since the proof of this result is almost trivial by using functional inequalities for product measures (cf. [9, Corollary 13]), we simply omit it. Due to Proposition 1.1 in this paper we will allow the perturbation $\nu$ does not satisfy the Poincaré inequality, it is in particular the case if the support of $\nu$ is disconnected.
Recently, when $\mu$ is the Gaussian measure with variance matrix $\delta I$ for some $\delta > 0$, it is proved in [18] that $\mu * \nu$ satisfies the log-Sobolev inequality if $\nu$ has a compact support and either $d = 1$ or $\delta > 2R^2d$, where $R$ is the radius of a ball containing the support of $\nu$, see [18, Theorem 2 and Theorem 17]. The first purpose of this paper is to extend this result to more general $\mu$ and to drop the restriction $\delta > 2R^2d$ for high dimensions. The main tool used in [18] is the Hardy type criterion for the log-Sobolev inequality due to [6], which is qualitatively sharp in dimension one. In this paper we will use a perturbation result of [2] and a Lyapunov type criterion introduced in [8] to derive more general and better results.

According to the above-mentioned results in [18], one may wish to prove that the log-Sobolev inequality is stable under convolution with compactly supported probability measures; i.e. if $\mu$ satisfies the log-Sobolev inequality, then so does $\mu * \nu$ for a probability measure $\nu$ having compact support. This is however not true, a simple counterexample is that $\mu = \delta_0$, the Dirac measure at point 0, which obviously satisfies the log-Sobolev inequality, but $\mu * \nu = \nu$ does not have to satisfy the log-Sobolev inequality even if $\nu$ is compactly supported. Thus, to ensure that $\mu * \nu$ satisfies the log-Sobolev inequality for any compactly supported probability measure $\nu$, one needs additional assumptions on $\mu$.

Throughout this paper, let $\mu(dx) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$ such that $V \in C^1(\mathbb{R}^d)$. For a probability measure $\nu$ on $\mathbb{R}^d$, we define

$$p_\nu(x) = \int_{\mathbb{R}^d} e^{-V(x-z)}\nu(dz), \quad V_\nu(x) = -\log p_\nu(x), \quad x \in \mathbb{R}^d.$$ 

Then

$$\mu * \nu(dx) = p_\nu(x)dx = e^{-V_\nu(x)}dx.$$ 

Moreover, let

$$\nu_x(dz) = \frac{1}{p_\nu(x)}e^{-V(x-z)}\nu(dz), \quad x \in \mathbb{R}^d.$$ 

In the following three sections we will consider the log-Sobolev inequality, Poincaré and super Poincaré inequalities for $\mu * \nu$ respectively.

## 2 Log-Sobolev Inequality

In this section we will use two different arguments to study the log-Sobolev inequality for $\mu * \nu$. One is the perturbation argument due to [1][2], and the other is the Lyapunov criterion presented in [8].

### 2.1 Perturbation Argument

**Theorem 2.1.** Assume that the log-Sobolev inequality (1.3) holds for $\mu$ with some constant $C > 0$. If $V \in C^1(\mathbb{R}^d)$ such that

$$\Phi_\nu(x) := \int_{\mathbb{R}^d} (\nabla e^{-V})(x-z)\nu(dz), \quad x \in \mathbb{R}^d$$

...
is well-defined and continuous, and there exists a constant $\delta > 1$ such that

\[
\int_{\mathbb{R}^d} \exp \left\{ \frac{\delta C}{4} \left( \int_{\mathbb{R}^d} |\nabla V(x) - \nabla V(x - z)|\nu_z(dz) \right)^2 \right\} \mu(dx) < \infty,
\]

then $\mu \ast \nu$ also satisfies the log-Sobolev inequality, i.e. for some constant $C' > 0$,

\[
(\mu \ast \nu)(f^2 \log f^2) \leq C'(\mu \ast \nu)(|\nabla f|^2), \quad f \in C^1_{\text{b}}(\mathbb{R}^d), (\mu \ast \nu)(f^2) = 1.
\]

Obviously, $\Phi_\nu \in C(\mathbb{R}^d, \mathbb{R}^d)$ holds if either $\nu$ has compact support or $\nabla e^{-V}$ is bounded. Moreover, (2.2) below holds for bounded Hess $V$ and compactly supported $\nu$. So, the following direct consequence of Theorem 2.1 improves the above-mentioned main results in [18].

**Corollary 2.2.** Assume that (1.3) holds and $\Phi_\nu$ is well defined and continuous. If $V \in C^2(\mathbb{R}^d)$ with bounded Hess $V$ such that

\[
\int_{\mathbb{R}^d} \exp \left\{ \frac{\delta C}{4} \|\nabla V\|^2_{\infty} \left( \int_{\mathbb{R}^d} |z|\nu_z(dz) \right)^2 \right\} \mu(dx) < \infty,
\]

holds for some constant $\delta > 1$, then $\mu \ast \nu$ satisfies the log-Sobolev inequality.

To prove Theorem 2.1, we introduce the following perturbation result due to [2, Lemma 3.1] and [1, Lemma 4.1].

**Lemma 2.3.** Assume that the probability measure $\mu(dx) = e^{-V(x)}dx$ satisfies the log-Sobolev inequality (1.3) with some constant $C > 0$. Let $\mu_V(dx) = e^{-V_0(x)}dx$ be a probability measure on $\mathbb{R}^d$. If $F := \frac{1}{2}(V - V_0) \in C^1(\mathbb{R}^d)$ such that

\[
\int_{\mathbb{R}^d} \exp(\delta C|\nabla F|^2)d\mu < \infty,
\]

holds for some constant $\delta > 1$, then the defective log-Sobolev inequality

\[
\mu_V(f^2 \log f^2) \leq C_1 \mu_V(|\nabla f|^2) + C_2, \quad f \in C^1_{\text{b}}(\mathbb{R}^d), \mu_V(f^2) = 1,
\]

holds for some constants $C_1, C_2 > 0$.

**Proof of Theorem 2.1.** Since by (1.4) we have $(\mu \ast \nu)(dx) = e^{-V_\nu(x)}dx$, to apply Lemma 2.3 we take $V_0 = V_\nu$, so that

\[
F(x) = \frac{1}{2}(V(x) - V_0(x)) = \frac{1}{2} \log \int_{\mathbb{R}^d} e^{V(x) - V_0(x)}\nu(dz).
\]

Since $\Phi_\nu$ is locally bounded, for any $x \in \mathbb{R}^d$ we have

\[
\lim_{y \to 0} (p_\nu(x + y) - p_\nu(x)) = \lim_{y \to 0} \int_0^1 \langle y, \Phi_\nu(x + sy) \rangle ds = 0.
\]
So, \( p_\nu \in C(\mathbb{R}^d) \). Then the continuity of \( \Phi_\nu \) implies that
\[
\Psi(x) := \int_{\mathbb{R}^d} (\nabla V)(x-z)\nu_x(dz) = -\frac{\Phi_\nu(x)}{p_\nu(x)}
\]
is continuous in \( x \) as well. Therefore, for any \( x, v \in \mathbb{R}^d \),
\[
\lim_{\varepsilon \downarrow 0} \frac{F(x+\varepsilon v) - F(x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^\varepsilon \langle v, \nabla V(x+sv) - \Psi(x+sv) \rangle ds
\]
= \( \frac{1}{2} \langle v, \nabla V(x) - \Psi(x) \rangle \).
Thus, by the continuity of \( \Psi \) and \( \nabla V \) we conclude that \( F \in C^1(\mathbb{R}^d) \) and
\[
|\nabla F(x)|^2 = \frac{1}{4} |\nabla V(x) - \Psi(x)|^2 \leq \frac{1}{4} \left( \int_{\mathbb{R}^d} |\nabla V(x) - \nabla V(x-z)|\nu_x(dz) \right)^2.
\]
Combining this with (2.1), we are able to apply Lemma 2.3 to derive the defective log-Sobolev inequality for \( \mu * \nu \). Moreover, the form
\[
\mathcal{E}(f,g) := \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle d(\mu * \nu), \quad f, g \in C_1^b(\mathbb{R}^d)
\]
is closable in \( L^2(\mu * \nu) \), and its closure is a symmetric, conservative, irreducible Dirichlet form. Thus, according to [16, Corollary 1.3] (see also [13, Theorem 1]), the defective log-Sobolev inequality implies the desired log-Sobolev inequality. Then the proof is finished.

To see that Corollary 2.2 has a broad range of application beyond [18, Theorem 2] and Proposition 1.1(1) for the log-Sobolev inequality, we present below an example where the support of \( \nu \) is unbounded and disconnected.

**Example 2.4.** Let \( d = 1, V(x) = \frac{1}{2} \log \pi + x^2 \) and
\[
\nu(dz) = \frac{1}{\gamma} \sum_{i \in \mathbb{Z}} e^{-\lambda i^2} \delta_i(dz), \quad \gamma := \sum_{i \in \mathbb{Z}} e^{-\lambda i^2},
\]
where \( \delta_i \) is the Dirac measure at point \( i \) and \( \lambda > 0 \). Then \( \mu * \nu \) satisfies the log-Sobolev inequality.

**Proof.** For the present \( V \) it is well known from [11] that the log-Sobolev inequality [13] holds with \( C = 1 \). On the other hand, it is easy to see that for any \( i \in \mathbb{Z}, x \in \mathbb{R} \) and \( \lambda > 0 \), we have
\[
|x - i|^2 + \lambda i^2 = (1 + \lambda) \left( i - \frac{x}{\lambda + 1} \right)^2 + \frac{\lambda x^2}{1 + \lambda}.
\]
Let \( \tilde{p}(x) = \sum_{i \in \mathbb{Z}} e^{-(1+\lambda)(i-x)/(1+\lambda))^2} \). Then

\[
(2.6) \quad \nu(x) (dz) = \frac{1}{\gamma(x)} \sum_{i \in \mathbb{Z}} e^{-|x-i|^2-\lambda i^2} \delta_i (dz) = \frac{1}{\tilde{p}(x)} \sum_{i \in \mathbb{Z}} e^{-(1+\lambda)(i-x)/(1+\lambda))^2} \delta_i (dz),
\]

where \( \gamma(x) = \sum_{i \in \mathbb{Z}} e^{-|x-i|^2-\lambda i^2} \). So,

\[
\int_{\mathbb{R}^d} |z| \nu(x) (dz) = \frac{1}{\gamma(x)} \sum_{i \in \mathbb{Z}} |i| e^{-(1+\lambda)(i-x)/(1+\lambda))^2} \\
\leq \frac{|x|}{1+\lambda} + \frac{1}{\tilde{p}(x)} \sum_{i \in \mathbb{Z}} |i - x| e^{-(1+\lambda)(i-x)/(1+\lambda))^2} \\
\leq \frac{|x|}{1+\lambda} + c, \quad x \in \mathbb{R}
\]
holds for

\[
(2.7) \quad c := \sup_{x \in [0,1+\lambda]} \frac{1}{\tilde{p}(x)} \sum_{i \in \mathbb{Z}} |i - x| e^{-(1+\lambda)(i-x)/(1+\lambda))^2} < \infty
\]
since the underlying function is periodic with a period \([0,1+\lambda]\). Noting that \( C = 1 \) and \( \|\text{Hess} V\|^2 = 4 \), we conclude from this that condition (2.2) holds for \( \delta \in (1,1+\lambda) \). Then the proof is finished by Corollary 2.2. \( \square \)

Finally, the following example shows that Theorem 2.1 may also work for unbounded \( \text{Hess} V \).

**Example 2.5.** Let \( V(x) = c + |x|^p \) with \( p \in [2,4) \) for some constant \( c \) such that \( \mu(dx) := e^{-V(x)}dx \) is a probability measure on \( \mathbb{R}^d \). Let \( \nu \) be a probability measure on \( \mathbb{R}^d \) with compact support. Then \( \mu * \nu \) satisfies the log-Sobolev inequality.

**Proof.** Since \( p \geq 2 \), we have \( V \in C^2(\mathbb{R}^d) \) and \( \Phi_\nu \in C(\mathbb{R}^d, \mathbb{R}^d) \). Let \( R = \sup \{|z| : z \in \text{supp } \nu\} \). Then

\[
\int_{\mathbb{R}^d} |\nabla V(x) - \nabla V(x-z)| \nu(x) (dz) \leq R \sup_{z \in B(x,R)} |\text{Hess} V(z)| \leq C(R)(1 + |x|^{p-2})
\]
holds for some constant \( C(R) > 0 \) and all \( x \in \mathbb{R}^d \). Combining this with \( 2(p-2) < p \) implied by \( p < 4 \), we see that (2.1) holds. Then the proof is finished by Theorem 2.1. \( \square \)

We will see in Remark 4.1 below that the assertion in Example 2.5 remains true for \( p \geq 4 \). Indeed, when \( p > 2 \) the super Poincaré inequality presented in Example 4.4 below is stronger than the log-Sobolev inequality, see [14, Corollary 3.3].
2.2 Lyapunov Criterion

Theorem 2.6. Assume that $V \in C^2(\mathbb{R}^d)$ with bounded $\text{Hess}_V$ such that

\begin{equation}
\text{Hess}_V \geq K I \quad \text{outside a compact set}
\end{equation}

holds for some constant $K > 0$. Then $\mu \ast \nu$ satisfies the log-Sobolev inequality provided the following two conditions hold:

(C1) There exists a constant $c > 0$ such that

\[ \nu_x(f^2) - \nu_x(f)^2 \leq c\|\nabla f\|_\infty^2, \quad f \in C^1_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d. \]

(C2) $\limsup_{|x| \to \infty} \frac{\int_{\mathbb{R}^d} |\nabla V(-z)| \nu_x(dz)}{|x|} < K$.

We believe that Theorems 2.1 and 2.6 are incomparable, since (2.8) is neither necessary for (1.3) to hold, nor provides explicit upper bound of $C$ in (1.3) which is involved in condition (2.1) for Theorem 2.1. But it would be rather complicated to construct proper counterexamples confirming this observation.

The proof of Theorem 2.6 is based on the following Lyapunov type criterion due to [8, Theorem 1.2].

Lemma 2.7 ([8]). Let $\mu_0(dx) = e^{-V_0(x)}dx$ be a probability measure on $\mathbb{R}^d$ for some $V_0 \in C^2(\mathbb{R}^d)$. Then $\mu_0$ satisfies the log-Sobolev inequality provided the following two conditions hold:

(i) There exists a constant $K_0 \in \mathbb{R}$ such that $\text{Hess}_{V_0} \geq K_0 I$.

(ii) There exists $W \in C^2(\mathbb{R}^d)$ with $W \geq 1$ such that

\[ \Delta W(x) - \langle \nabla V_0, \nabla W \rangle(x) \leq (c_1 - c_2|x|^2)W(x), \quad x \in \mathbb{R}^d \]

holds for some constants $c_1, c_2 > 0$.

Proof of Theorem 2.6. By (1.4) and Lemma 2.7 it suffices to verify conditions (i) and (ii) for $V_0 = V_\nu := -\log p_\nu$.

(a) Proof of (i). By the boundedness of $\text{Hess}_V$ and the condition (2.8), it is to see that $p_\nu \in C^2(\mathbb{R}^d)$ and for any $X \in \mathbb{R}^d$ with $|X| = 1$, we have

\begin{equation}
\text{Hess}_{V_0}(X, X) = \frac{1}{p_\nu^2} \left( (\nabla_X p)^2 - p_\nu \text{Hess}_{p_\nu}(X, X) \right).
\end{equation}

Moreover,

\[ \nabla_X p_\nu(x) = -p_\nu(x) \int_{\mathbb{R}^d} (\nabla_X V(x - z)) \nu_x(dz). \]
Then, letting \( K_1 := \| \text{Hess}_\nu \| < \infty \), we obtain
\[
\text{Hess}_\nu(X, X)(x) = \int_{\mathbb{R}^d} \left( |\nabla V(x - z)|^2 - \text{Hess}_V(X, X)(x - z) \right) e^{-V(x - z)} \nu(dz)
\leq \nu(x) \int_{\mathbb{R}^d} |\nabla V(x - z)|^2 \nu(dz) + K_1 \nu(x).
\]
Combining these with (2.9) and (C1), we conclude that
\[
\text{Hess}_{10}(X, X)(x) \geq -K_1 - \int_{\mathbb{R}^d} (\nabla V(x - z))^2 \nu(dz) + \left( \int_{\mathbb{R}^d} \nabla V(x - z) \nu(dz) \right)^2
\geq -K_1 - cK^2_1.
\]
Thus, (i) holds for \( K_0 = -K_1 - cK^2_1 \).

(b) Proof of (ii). Let \( W(x) = e^{\varepsilon |x|^2} \) for some constant \( \varepsilon > 0 \). Then
\[
\Delta W - \langle \nabla V_0, \nabla W \rangle = 2d \varepsilon + 4\varepsilon^2 |x|^2 - \varepsilon \int_{\mathbb{R}^d} \langle x, \nabla V(x - z) \rangle \nu(dz).
\]
Since \( \text{Hess}_V \) is bounded and (2.8) holds, we know that \( \int_{\mathbb{R}^d} \langle x, \nabla V(x - z) \rangle \nu(dz) \) is well defined and locally bounded. By (2.8), there exists a constant \( r_0 > 0 \) such that \( \text{Hess}_V \geq KI \) holds on the set \( \{|z| \geq r_0\} \). So, for \( x \in \mathbb{R}^d \) with \( |x| > 2r_0 \),
\[
\langle \nabla V(x - z) - \nabla V(-z), x \rangle = |x| \int_0^{r|x|} \text{Hess}_V \left( \frac{x}{|x|}, \frac{r x}{|x|}, \frac{r x}{|x|} - z \right) dr
\geq K|x|^2 - K_1|x| \left\{ r \in [0, |x|] : \left| \frac{r x}{|x|} - z \right| \leq r_0 \right\}
\geq K|x|^2 - 2K_1 r_0 |x|.
\]
Combining this with (2.10) and (C2), and noting that
\[
\langle x, \nabla V(x - z) \rangle \leq \langle \nabla V(x - z) - \nabla V(-z), x \rangle + |x| \cdot |\nabla V(-z)|,
\]
we conclude that there exist constants \( C_1, C_2 > 0 \) such that
\[
\frac{\Delta W - \langle \nabla V_0, \nabla W \rangle}{W} \leq 2d \varepsilon + 4\varepsilon^2 |x|^2 - \varepsilon C_1 |x|^2 + \varepsilon C_2.
\]
Taking \( \varepsilon = \frac{C_1}{8} \), we prove (ii) for some constants \( c_1, c_2 > 0 \).

Since when \( \nu \) has compact support, we have
\[
\nu_x(f^2) - \nu_x(f)^2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(z) - f(y)|^2 \nu_x(dz) \nu_x(dy) \leq R^2 \| \nabla f \|^2_\infty,
\]
where \( R := \sup \{|z - y| : z, y \in \text{supp} \nu\} < \infty \), and
\[
\lim_{|x| \to \infty} \frac{\int_{\mathbb{R}^d} |\nabla V(-z)| \nu_x(dz)}{|x|} \leq \lim_{|x| \to \infty} \frac{\sup_{\text{supp} \nu} |\nabla V|}{|x|} = 0.
\]
The following direct consequence of Theorem 2.6 improves the above mentioned results in [18] as well.
Corollary 2.8. Assume that \( V \in C^2(\mathbb{R}^d) \) with bounded \( \text{Hess} V \) such that \((2.8)\) holds. Then \( \mu * \nu \) satisfies the log-Sobolev inequality for any compactly supported probability measure \( \nu \).

To show that Theorem 2.6 also has a range of application beyond Corollary 2.8 and Proposition 1.1(1) for the log-Sobolev inequality, we reprove Example 2.4 by using Theorem 2.6.

**Proof of Example 2.4 using Theorem 2.6.**

Obviously, \((2.8)\) holds for \( K = 2 \). Let

\[
\tilde{\nu}_x = \frac{1}{\tilde{\gamma}(x)} \sum_{i \in \mathbb{Z}} e^{-(1+\lambda)(i-x)^2} \delta_i, \quad \tilde{\gamma}(x) = \sum_{i \in \mathbb{Z}} e^{-(1+\lambda)(i-x)^2}.
\]

By \((2.5)\) we have \( \tilde{\nu}_x = \nu_{(1+\lambda)x} \). Thus, we only need to verify conditions \((C1)\) and \((C2)\) for \( \tilde{\nu}_x \) in place of \( \nu_x \).

(a) To prove condition \((C1)\), we make use of a Hardy type inequality for birth-death processes with Dirichlet boundary introduced in [12]. Let \( x \in \mathbb{R} \) be fixed. For any bounded function \( f \) on \( \mathbb{Z} \), let \( \tilde{f}(i) = f(i) - f(i_x) \), where \( i_x := \sup \{ i \in \mathbb{Z} : i \leq x \} \) is the integer part of \( x \). Then

\[
(2.11) \quad \tilde{\nu}_x(f^2) - \tilde{\nu}_x(f)^2 \leq \sum_{i=-\infty}^{i_x} \tilde{f}(i)^2 \tilde{\nu}_x(i) + \sum_{i=i_x}^{\infty} \tilde{f}(i)^2 \tilde{\nu}_x(i).
\]

It is easy to see that there exists a constant \( c > 0 \) independent of \( x \) such that for any \( m \geq i_x > x - 1 \),

\[
\sum_{i=i_x}^{m} e^{(1+\lambda)(i-x)^2} \leq ce^{(1+\lambda)(m-x)^2}, \quad \sum_{i=m+1}^{\infty} e^{-(1+\lambda)(i-x)^2} \leq ce^{-(1+\lambda)(m+1-x)^2}.
\]

Therefore,

\[
\sup_{m \geq i_x} \left( \sum_{i=i_x}^{m} e^{(1+\lambda)(i-x)^2} \right) \sum_{i=m+1}^{\infty} e^{-(1+\lambda)(i-x)^2} \leq c^2 e^{(1+\lambda)((m-x)^2-(m+1-x)^2)} = c^2 e^{(1+\lambda)(2(x-m)-1)} \leq c^2 e^{1+\lambda}.
\]

By this and the Hardy inequality (see [17, Theorem 1.3.9]), we have

\[
\sum_{i=i_x}^{\infty} \tilde{f}(i)^2 \tilde{\nu}_x(i) \leq 4c^2 e^{1+\lambda} \sum_{i=i_x}^{\infty} (f(i + 1) - f(i))^2 \tilde{\nu}_x(i).
\]

Similarly,

\[
\sum_{i=-\infty}^{i_x} \tilde{f}(i)^2 \tilde{\nu}_x(i) \leq 4c^2 e^{1+\lambda} \sum_{i=-\infty}^{i_x} (f(i - 1) - f(i))^2 \tilde{\nu}_x(i).
\]
Combining these with (2.11) we prove (C1) for $\tilde{\nu}_x$ and some constant $c > 0$ (independent of $x \in \mathbb{R}$).

(b) Let $\tilde{\rho}(x) = \sum_{i \in \mathbb{Z}} e^{-(1+\lambda)(i-x/(1+\lambda))^2}$. Noting that $\nabla V(z) = 2z$, by (2.6) we obtain

$$\int_{\mathbb{R}^d} |\nabla V(-z)| \nu_x(dz) = \frac{2}{\tilde{\rho}(x)} \sum_{i \in \mathbb{Z}} |i| e^{-(1+\lambda)(i-x/(1+\lambda))^2}$$

$$\leq \frac{2|x|}{1+\lambda} + \frac{2}{\tilde{\rho}(x)} \sum_{i \in \mathbb{Z}} \left| i - \frac{x}{1+\lambda} \right| e^{-(1+\lambda)(i-x/(1+\lambda))^2}$$

$$\leq c + \frac{2|x|}{1+\lambda}$$

for $c > 0$ in (2.7). Therefore,

$$\limsup_{|x| \to \infty} \frac{\int_{\mathbb{R}^d} |\nabla V(-z)| \nu_x(dz)}{|x|} \leq \frac{2}{1+\lambda} < 2 = K.$$

Thus, condition (C2) holds.

\[\square\]

## 3 Poincaré inequality

In the spirit of the proof of Theorem 2.6, in this section we study the Poincaré inequality for convolution measures using the Lyapunov conditions presented in [4, 3]. One may also wish to use the following easy to check perturbation result on the Poincaré inequality corresponding to Lemma 2.3.

If the probability measure $\mu_V(dx) = e^{-V(x)}dx$ satisfies the Poincaré inequality (1.1) with some constant $C > 0$, then for any $V_0 \in C^1(\mathbb{R}^d)$ such that $\int e^{-V_0(x)}dx = 1$ and $C\|\nabla(V-V_0)\|_\infty < 2$, the probability measure $\nu_{V_0}(dx) = e^{-V_0(x)}dx$ satisfies the Poincaré inequality (1.1) (with a different constant) as well.

Since the boundedness condition on $\nabla(V-V_0)$ is rather strong (for instance, it excludes Example 3.3(1) below for $p > 2$), here, and in the next section for the super Poincaré inequality, we will only use the Lyapunov criteria rather than this perturbation result.

**Theorem 3.1.** Let $\mu(dx) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$ and let $\nu$ be a probability measure on $\mathbb{R}^d$. Assume that $\Phi_{\nu}$ in Theorem 2.1 is well-defined and continuous. Then $\mu \ast \nu$ satisfies the Poincaré inequality (1.1), if at least one of the following conditions holds:

1. $V \in C^1(\mathbb{R}^d)$ such that $\liminf_{|x| \to \infty} \frac{\int_{\mathbb{R}^d} (x, \nabla V(x-z)) \nu_x(dz)}{|x|} > 0$.

2. $V \in C^2(\mathbb{R}^d)$ such that $\Phi_{\nu_x}(x) := \int_{\mathbb{R}^d} (\nabla^2 V)(x-z) \nu_x(dz)$ is well-defined and continuous in $x$, and there is a constant $\delta \in (0, 1)$ such that

$$\liminf_{|x| \to \infty} \int_{\mathbb{R}^d} \left( \delta |\nabla V(x-z)|^2 - \Delta V(x-z) \right) \nu_x(dz) > 0.$$
Proof. Let $L_\nu = \Delta - \nabla V_\nu$. According to [4, Theorem 3.5] or [3, Theorem 1.4], $(\mu \ast \nu)(dx) := e^{-V_\nu(x)}dx$ satisfies the Poincaré inequality if there exist a $C^2$-function $W \geq 1$ and some positive constants $\theta, b, R$ such that for all $x \in \mathbb{R}^d$,

\begin{equation}
L_\nu W(x) \leq -\theta W(x) + b1_{B(0,R)}(x).
\end{equation}

In particular, by [3, Corollary 1.6], if either

\begin{equation}
\liminf_{|x| \to \infty} \frac{\langle \nabla V_\nu(x), x \rangle}{|x|} > 0,
\end{equation}

or there is a constant $\delta \in (0, 1)$ such that

\begin{equation}
\liminf_{|x| \to \infty} \left( \delta |\nabla V_\nu(x)|^2 - \Delta V_\nu(x) \right) > 0,
\end{equation}

then the inequality (3.1) fulfills.

Now, as shown in the proof of Theorem 2.1 that the continuity of $\Phi_\nu$ implies that $V_\nu \in C^1(\mathbb{R}^d)$ and

\[ \langle \nabla V_\nu(x), x \rangle = \int_{\mathbb{R}^d} \langle \nabla V(x - z), x \rangle \nu_z(dz). \]

Then condition (1) in Theorem 3.1 implies (3.2), and hence the Poincaré inequality for $\mu \ast \nu$.

On the other hand, repeating the argument leading to $F \in C^1(\mathbb{R}^d)$ in the proof of Theorem 2.1, we conclude that the continuity of $\Phi_\nu$ and $\tilde{\Phi}_\nu$ implies $V_\nu \in C^2(\mathbb{R}^d)$ and

\[ |\nabla V_\nu(x)|^2 = \left( \int_{\mathbb{R}^d} \nabla V(x - z) \nu_x(dz) \right)^2, \]

\[ \Delta V_\nu(x) = |\nabla V_\nu(x)|^2 + \int_{\mathbb{R}^d} \{\Delta V(x - z) - |\nabla V(x - z)|^2\} \nu_z(dz). \]

Then for any $\delta \in (0, 1)$,

\[ \delta |\nabla V_\nu(x)|^2 - \Delta V_\nu(x) = \int_{\mathbb{R}^d} \left( |\nabla V(x - z)|^2 - \Delta V(x - z) \right) \nu_x(dz) - (1 - \delta) |\nabla V_\nu(x)|^2 \]

\[ \geq \int_{\mathbb{R}^d} \left( \delta |\nabla V(x - z)|^2 - \Delta V(x - z) \right) \nu_x(dz). \]

Combining this with condition (2) in Theorem 3.1, we prove (3.3), and hence the Poincaré inequality for $\mu \ast \nu$. \qed

When the measure $\nu$ is compactly supported, we have the following consequence of Theorem 3.1.
**Corollary 3.2.** Let $\nu$ be a probability measure on $\mathbb{R}^d$ with compact support such that $R := \sup \{|z| : z \in \text{supp } \nu| < \infty$. If either $V \in C^1(\mathbb{R}^d)$ with

$$\liminf_{|x| \to \infty} \frac{\langle \nabla V(x), x \rangle - R|\nabla V(x)|}{|x|} > 0,$$

or $V \in C^2(\mathbb{R}^d)$ and there is a constant $\delta \in (0, 1)$ such that

$$\liminf_{|x| \to \infty} (\delta|\nabla V(x)|^2 - \Delta V(x)) > 0,$$

then $\mu \ast \nu$ satisfies the Poincaré inequality.

**Proof.** Since the support of $\nu$ is compact, the continuity of $\Phi_\nu$ when $V \in C^1(\mathbb{R}^d)$ and that of $\tilde{\Phi}_\nu$ when $V \in C^2(\mathbb{R}^d)$ are obvious. Below we prove conditions (1) and (2) in Theorem 3.1 using (3.4) and (3.5) respectively.

(a) By (3.4) we obtain

$$\int_{\mathbb{R}^d} \langle x, \nabla V(x-z) \rangle \nu_x(dz) = \int_{\mathbb{R}^d} \left( \langle x-z, \nabla V(x-z) \rangle + \langle z, \nabla V(x-z) \rangle \right) \nu_x(dz)$$

$$\geq \int_{\mathbb{R}^d} \left( \langle x-z, \nabla V(x-z) \rangle - R|\nabla V(x-z)| \right) \nu_x(dz)$$

$$\geq \int_{\mathbb{R}^d} (c_1|x-z| - c_2) \nu_x(dz)$$

$$\geq c_1(|x| - R)^+ - c_2$$

for some constants $c_1, c_2 > 0$. Then condition (1) in Theorem 3.1 holds.

(b) According to (3.5), there are constants $r_1, c_3$ and $c_4 > 0$ such that for all $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \left( \delta|\nabla V(x-z)|^2 - \Delta V(x-z) \right) \nu_x(dz)$$

$$\geq c_3 \int_{\{|x-z| > r_1\}} \nu_x(dz) - c_4 \int_{\{|x-z| \leq r_1\}} \nu_x(dz).$$

Since for $x \in \mathbb{R}^d$ with $|x| > R + r_1$ we have

$$\int_{\{|x-z| > r_1\}} \nu_x(dz) \geq \int_{\{|z| \leq R\}} \nu_x(dz) = 1$$

and

$$\int_{\{|x-z| \leq r_1\}} \nu_x(dz) \leq \int_{\{|z| > R\}} \nu_x(dz) = 0,$$

then (3.6) implies condition (2) in Theorem 3.1.

Finally, we present the following examples to illustrate Theorem 3.1 and Corollary 3.2.
Example 3.3. (1) Let $V(x) = c + |x|^p$ for some $p \geq 1$ and constant $c$ such that $\mu(dx) := e^{-V(x)}dx$ is a probability measure on $\mathbb{R}^d$. Then $\mu \ast \nu$ satisfies the Poincaré inequality for every compactly supported probability measure $\nu$ on $\mathbb{R}^d$.

(2) Let $d = 1$, $V(x) = c + \sqrt{1 + x^2}$ and

$$\nu(dz) = \frac{1}{\gamma} \sum_{i \in \mathbb{Z}} e^{-|i|} \delta_i(dz), \quad \gamma := \sum_{i \in \mathbb{Z}} e^{-|i|},$$

where $c = \log \int_{\mathbb{R}} e^{-\sqrt{1 + x^2}} dx$ and $\delta_i$ is the Dirac measure at point $i$. Then $\mu \ast \nu$ satisfies the Poincaré inequality.

Proof. Since when $p < 2$ the function $V(x) = c + |x|^p$ is not in $C^2$ at point 0, we take $\tilde{V} \in C^2(\mathbb{R}^d)$ such that $\tilde{V}(x) = V(x)$ for $|x| \geq 1$. Let $\tilde{\mu}(dx) = C e^{-\tilde{V}(x)}dx$, where $C > 0$ is a constant such that $\tilde{\mu}$ is a probability measure. By the stability of Poincaré inequality under the bounded perturbations (e.g. see [9, Proposition 17]), it suffices to prove that $\tilde{\mu} \ast \nu$ satisfies the Poincaré inequality.

In case (1) the assertion is a direct consequence of Corollary 3.2. So, we only have to verify condition (1) in Theorem 3.1 for case (2). For simplicity, we only verify for $x \to \infty$, i.e.

$$\lim_{x \to \infty} \frac{\int_{\mathbb{R}} xV'(x-z)\nu_x(dz)}{|x|} > 0.$$  

Let $i_x$ be the integer part of $x$, and $h_x = x - i_x$. Note that for any $x > 0$,

$$\frac{\int_{\mathbb{R}} xV'(x-z)\nu_x(dz)}{|x|} = \int_{\mathbb{R}} V'(x-z)\nu_x(dz) = \sum_{i \in \mathbb{Z}} \frac{x-i}{\sqrt{1+(x-i)^2}} e^{-\sqrt{1+(x-i)^2}-|i|}$$

$$= \sum_{i \in \mathbb{Z}} e^{-\sqrt{1+(x-i)^2}-|i|}$$

$$= \sum_{k \in \mathbb{Z}} \frac{h_x+k}{\sqrt{1+(h_x+k)^2}} e^{-\sqrt{1+(h_x+k)^2}-|i_x-k|}$$

$$= : 1 - p_x(x)^{-1} \sum_{k \in \mathbb{Z}} (a_k b_k)(x),$$

where

$$a_k(x) := \frac{\sqrt{1+(h_x+k)^2} - (h_x+k)}{\sqrt{1+(h_x+k)^2}},$$

$$b_k(x) := e^{-\sqrt{1+(h_x+k)^2}-|i_x-k|}, \quad p_x(x) = \sum_{k \in \mathbb{Z}} b_k(x).$$

It is easy to see that

$$0 \leq a_k(x) \leq \begin{cases} (1+k^2)^{-1/2}, & k \geq 0, \\ 2, & k < 0. \end{cases}$$
Then for any $n \geq 1$,
\[
\sum_{k \in \mathbb{Z}} (a_k b_k)(x) = \sum_{k \leq 0}(a_k b_k)(x) + \sum_{k=1}^{n}(a_k b_k)(x) + \sum_{k=n+1}^{\infty} a_k b_k(x)
\leq 2 \sum_{k \leq 0} b_k(x) + \sum_{k=1}^{n} b_k(x) + \frac{1}{n+1} \sum_{k=n+1}^{\infty} b_k(x).
\]

Thus, for any $x > 0$ and $1 \leq n \leq i_x$,
\[
\sum_{k \leq 0} b_k(x) \leq e^{-x} + \sum_{k=-\infty}^{-1} e^{-(k-h_x)-(i_x-k)} \leq (2e^2 + 1)e^{-x},
\]
\[
\sum_{k=1}^{n} b_k(x) \leq ne^{-x}, \quad p_{\nu}(x) \geq \sum_{k=1}^{i_x} b_k(x) \geq i_x e^{-x-1}.
\]

Then for any $n \geq 1$,
\[
\limsup_{x \to \infty} \frac{1}{p_{\nu}(x)} \sum_{k \in \mathbb{Z}} (a_k b_k)(x) \leq \lim_{x \to \infty} \left\{ \frac{e^{x+1}(2e^2 + 1 + n)e^{-x}}{i_x} + \frac{1}{n+1} \right\} = \frac{1}{n+1}.
\]

Letting $n \to \infty$ we obtain $\lim_{x \to \infty} p_{\nu}(x)^{-1} \sum_{k \in \mathbb{Z}} (a_k b_k)(x) = 0$. Combining this with (3.8) we prove (3.7).

\section{Super Poincaré Inequality}

In this section we extend the results in Section 3 for the super Poincaré inequality.

\begin{theorem}
Let $\mu(dx) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$ and let $\nu$ be a probability measure on $\mathbb{R}^d$. Define
\[
\alpha(r, s) = (1 + s^{-d/2}) \left( \frac{\sup_{|x| \leq r} e^{-V(x)}}{\inf_{|x| \leq r} e^{-V(x)}} \right)^{d/2+1}, \quad s, r > 0.
\]

(1) If $V \in C^1(\mathbb{R}^d)$ such that
\begin{equation}
\liminf_{|x| \to \infty} \frac{\int_{\mathbb{R}^d} (x, \nabla V(x-z)) \nu_z(dz)}{|x|} = \infty,
\end{equation}

then $\mu \ast \nu$ satisfies the super Poincaré inequality (1.2) with
\[
\beta(r) = c_1 \left( 1 + \alpha(\psi(2/r), r/2) \right),
\]
Theorem 4.1. Let $\mu(dx) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$ and let $\nu$ be a probability measure on $\mathbb{R}^d$. Define
\[
\alpha(r, s) = (1 + s^{-d/2}) \left( \frac{\sup_{|x| \leq r} e^{-V(x)}}{\inf_{|x| \leq r} e^{-V(x)}} \right)^{d/2+1}, \quad s, r > 0.
\]

(1) If $V \in C^1(\mathbb{R}^d)$ such that
\begin{equation}
\liminf_{|x| \to \infty} \frac{\int_{\mathbb{R}^d} (x, \nabla V(x-z)) \nu_z(dz)}{|x|} = \infty,
\end{equation}

then $\mu \ast \nu$ satisfies the super Poincaré inequality (1.2) with
\[
\beta(r) = c_1 \left( 1 + \alpha(\psi(2/r), r/2) \right),
\]

where
\[
\psi(r) := \inf \left\{ s > 0 : \inf_{|x| \geq s} \frac{\int_{\mathbb{R}^d} (x, \nabla V(x-z)) \nu_z(dz)}{|x|} \geq r \right\} < \infty, \quad r > 0.
\]
Lemma 4.2. Let $V \in C^2(\mathbb{R}^d)$ and there is a constant $\delta \in (0,1)$ such that

$$\liminf_{|x| \to \infty} \int_{\mathbb{R}^d} \left( \delta |\nabla V(x-z)|^2 - \Delta V(x-z) \right) \nu_x(dx) = \infty.$$ \hfill (4.2)

Then, $\mu * \nu$ satisfies the super Poincaré inequality [1,2] with

$$\beta(r) = c_2 \left( 1 + \alpha(\tilde{\psi}(2/r), r/2) \right),$$

where

$$\tilde{\psi}(r) := \inf \left\{ s > 0 : \inf_{|x| \geq s} \int_{\mathbb{R}^d} \left( \delta |\nabla V(x-z)|^2 - \Delta V(x-z) \right) \nu_x(dx) \geq r \right\} < \infty, \ r > 0.$$

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.2. Let $\mu(dx) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^d$. Assume that there are functions $W \geq 1$, $\phi > 0$ with $\liminf_{|x| \to \infty} \phi(x) = \infty$ and constants $b, r_0 > 0$ such that

$$\frac{\Delta W - \langle \nabla V, \nabla \rangle}{W} \leq -\phi + b1_{B(0,r_0)}.$$

Then, the following super Poincaré inequality holds

$$\mu_V(f^2) \leq r \mu_V(|\nabla f|^2) + \beta(r) \mu_V(|f|)^2,$$

with

$$\beta(r) = c_0 \left( 1 + \alpha(\psi_\phi(2/r), r/2) \right), \ r > 0$$

for some constant $c_0 > 0$ and

$$\psi_\phi(r) := \inf \left\{ s > 0 : \inf_{|x| \geq s} \phi(x) \geq r \right\}.$$

Proof. It is well known that (e.g. see [7, Proposition 3.1]) there exists a constant $C > 0$ such that for any $t, s > 0$ and $f \in C^1(\mathbb{R}^d)$,

$$\int_{B(0,t)} f^2(x)dx \leq s \int_{B(0,t)} |\nabla f(x)|^2dx + C(1 + s^{-d/2}) \left( \int_{B(0,t)} |f|(x)dx \right)^2.$$
Taking $s = r \inf \{ |x| \leq t e^{-V(x)} \}$ in the inequality above, we arrive at that for any $t, r > 0$ and $f \in C^1(\mathbb{R}^d)$,
\[ \int_{B(0,t)} f^2(x) \mu_V(dx) \leq r \mu_V(|\nabla f|^2) + C \alpha(t, r) \mu_V(|f|^2). \]

Thus, the proof is finished by [7, Theorem 2.10] and the fact that the function $\alpha(r, s)$ is increasing with respect to $r$ and decreasing with respect to $s$.

\[ \square \]

\textbf{Proof of Theorem 4.1.} As the same to the proof of Theorem 3.1, let $L_\nu = \Delta - \nabla V_\nu$.

In case (1), we consider a smooth function such that $W(x) = e^{2|x|}$ for $|x| \geq 1$ and $W(x) \geq 1$ for all $x \in \mathbb{R}^d$. We have
\[ \frac{L_\nu W(x)}{W(x)} \leq - \frac{\langle x, \nabla V_\nu(x) \rangle}{|x|} 1_{\{|x| \geq 1\}} + b 1_{\{|x| \leq 1\}} \]
for some constant $b > 0$. Then, the required assertion follows from Lemma 4.2 and the proof of Theorem 3.1(1).

In case (2), we consider a smooth function such that $W(x) = e^{(1-\delta)V(x)}$ for $|x| \geq 1$ and $W(x) \geq 1$ for all $x \in \mathbb{R}^d$. Then,
\[ \frac{L_\nu W(x)}{W(x)} \leq - (1 - \delta)(\Delta V(x) - \delta |\nabla V(x)|^2) + b 1_{\{|x| \leq 1\}} \]
for some constant $b > 0$. This along with Lemma 4.2 and the proof of Theorem 3.1(2) also yields the desired assertion.

\[ \square \]

According to the proof of Corollary 3.2, when the measure $\nu$ has the compact support, we can obtain the following statement from Theorem 4.1.

\textbf{Corollary 4.3.} Let $\nu$ be a probability measure on $\mathbb{R}^d$ with compact support such that $R := \sup \{|z| : z \in \supp \nu\} < \infty$.

(1) If
\[ \liminf_{|x| \to \infty} \frac{\langle \nabla V(x), x \rangle - R|\nabla V(x)|}{|x|} = \infty, \]
then $\mu * \nu$ satisfies the super Poincaré inequality (1.2) with
\[ \beta(r) = c_3 \left( 1 + \alpha(\psi(2/r), r/2) \right), \]
where
\[ \psi(r) := \inf \left\{ s > 0 : \inf_{|x| \geq 2s} \frac{\langle \nabla V(x), x \rangle - R|\nabla V(x)|}{|x|} \geq s \right\}. \]

(2) If there is a constant $\delta \in (0, 1)$ such that
\[ \liminf_{|x| \to \infty} \left( \delta |\nabla V(x)|^2 - \Delta V(x) \right) = \infty, \]

(4.4)
then $\mu \ast \nu$ satisfies the super Poincaré inequality (1.2) with
\[
\beta(r) = c_4 \left(1 + \alpha(\bar{\psi}(2/r), r/2)\right),
\]
where
\[
\bar{\psi}(r) := \inf \left\{ s > 0 : \inf_{|x| \geq 2s} (\delta|\nabla V(x)|^2 - \Delta V(x)) \geq r \right\}.
\]

The proof of Corollary 4.3 is similar to that of Corollary 3.2, and so we omit it here. Finally, we consider the following example to illustrate Corollary 4.3.

**Example 4.4.** Let $V(x) = c + |x|^p$ for some $p > 1$ and $c \in \mathbb{R}$ such that $\mu(dx) := e^{-V(x)}dx$ is a probability measure on $\mathbb{R}^d$. Then for any compactly supported probability measure $\nu$, $\mu \ast \nu$ satisfies the super Poincaré inequality (1.2) with

\[
\beta(r) = \exp \left( c_5 (1 + r^{-\frac{p}{2(p-1)}}) \right).
\]

**Proof.** As explained in the proof of Example 3.3 up to a bounded perturbation, we may simply assume that $V \in C^2(\mathbb{R}^d)$. For any $\delta \in (0, 1)$ and any $x \in \mathbb{R}^d$ with $|x|$ large enough,
\[
\delta|\nabla V(x)|^2 - \Delta V(x) \geq \eta(V(x)),
\]
where $\eta$ is a non-decreasing function such that $\eta(r) = \delta r^{2(p-2)/p}$ for $r \geq 1$. This along with Corollary 4.3(2) yields the desired assertion.

**Remark 4.1** According to [14, Corollary 3.3], when $p \geq 2$ the super Poincaré inequality with $\beta$ in (4.3) implies the defective log-Sobolev inequality, which is equivalent to the log-Sobolev inequality due to [16, Corollary 1.3] or [13, Theorem 1].

**References**

[1] S. Aida, Uniform positivity improving property, Sobolev inequalities, and spectral gaps, J. Funct. Anal. 158(1998), 152–185.

[2] S. Aida, I. Shigekawa, Logarithmic Sobolev inequalities and spectral gaps: Perturbation theory, J. Funct. Anal. 126(1994), 448–475.

[3] D. Bakry, F. Barthe, P. Cattiaux, A. Guillin, A simple proof of the Poincaré inequality for a large class of measures including the logconcave case, Electron. Comm. Probab. 13(2008), 60–66.

[4] D. Bakry, P. Cattiaux, A. Guillin, Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré, J. Funct. Anal. 254(2008), 727–759.

[5] D. Bakry, M. Ledoux, F.-Y. Wang, Perturbations of functional inequalities using growth conditions, J. Math. Pures Appl. 87(2007), 394–407.
[6] S. G. Bobkov, F. Götze, *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal. 163(1999), 1–28.

[7] P. Cattiaux, A. Guillin, F.-Y. Wang, L. Wu, *Lyapunov conditions for Super Poincaré inequalities*, J. Funct. Anal. 256(2009), 1821–1841.

[8] P. Cattiaux, A. Guillin, L. Wu, *A note on Talagrand’s transportation inequality and logarithmic Sobolev inequality*, Probab. Theory Relat. Fields 148(2010), 285–304.

[9] D. Chafai, *Entropies, convexity, and functional inequalities*, J. Math. Kyoto Uni. 44(2010), 325–363.

[10] X. Chen, F.-Y. Wang, J. Wang, *Perturbations of functional inequalities for Lévy type Dirichlet forms*, arXiv:1303.7349.

[11] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97(1975), 1061–1083.

[12] L. Miclo, *An example of application of discrete Hardy’s inequalities*, Markov Proc. Relat. Fields 2(1996), 263–284.

[13] L. Miclo, *On hyperboundedness and spectrum of Markov operators*, http://hal.archives-ouvertes.fr/hal-00777146v2.

[14] F.-Y. Wang, *Functional inequalities for empty essential spectrum*, J. Funct. Anal. 170(2000), 219–245.

[15] F.-Y. Wang, *Functional inequalities, semigroup properties and spectrum estimates*, Infin. Dimens. Anal. Quant. Probab. Relat. Topics 3(2000), 263–295.

[16] F.-Y. Wang, *Criteria of spectral gap for Markov operators*, arXiv:1305.4460.

[17] F.-Y. Wang, *Functional Inequalities, Markov Processes and Spectral Theory*, Science Press, Beijing, 2005.

[18] D. Zimmermann, *Logarithmic Sobolev inequalities for mollified compactly supported measures*, J. Funct. Anal. 265(2013), 1064–1083.