Exact solution of the one-dimensional super-symmetric $t$–$J$ model with unparallel boundary fields

Xin Zhang$^1$, Junpeng Cao$^{1,2}$, Wen-Li Yang$^{3,4}$, Kangjie Shi$^3$ and Yupeng Wang$^{1,2}$

$^1$ Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
$^2$ Collaborative Innovation Center of Quantum Matter, Beijing, People’s Republic of China
$^3$ Institute of Modern Physics, Northwest University, Xian 710069, People’s Republic of China
$^4$ Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing 100048, People’s Republic of China

E-mail: shezhangxin@163.com, junpengcao@iphy.ac.cn, wlyang@nwu.edu.cn, kjshi@nwu.edu.cn and yupeng@iphy.ac.cn

Received 16 January 2014
Accepted for publication 11 March 2014
Published 25 April 2014

Online at stacks.iop.org/JSTAT/2014/P04031
doi:10.1088/1742-5468/2014/04/P04031

Abstract. The exact solution of the one-dimensional super-symmetric $t$–$J$ model under generic integrable boundary conditions is obtained via the Bethe ansatz methods. With the coordinate Bethe ansatz, the corresponding $R$-matrix and $K$-matrices are derived for the second eigenvalue problem associated with spin degrees of freedom. It is found that the second eigenvalue problem can be transformed into that of the transfer matrix of the inhomogeneous XXX spin chain, which allows us to obtain the spectrum of the Hamiltonian and the associated Bethe ansatz equations by the off-diagonal Bethe ansatz method.

Keywords: integrable spin chains (vertex models), solvable lattice models

ArXiv ePrint: 1312.0376
1. Introduction

The $t$–$J$ model is one of the most important models for describing strongly correlated electronic systems, in particular for high-$T_c$ superconductivity [1]–[3]. The model is in fact a large $U$ limit of the well-known Hubbard model [4]–[8] and provides a non-phonon mechanism for high-$T_c$ superconductivity [9]–[11]. Interestingly, the model in one spatial dimension at the super-symmetric points $2t = \pm J$ [12]–[14], [27] is exactly solvable [15]–[18]. Based on this observation, some important physical properties such as the elementary excitations [19]–[21], the correlation functions [22] and the thermodynamics [23]–[26] were studied by many authors. The model with diagonal boundary fields in the spin sector (or in the charge sector with boundary chemical potentials) has been studied extensively by the nested Bethe ansatz method [27]–[37] or the off-shell Bethe ansatz [38, 39]. Even the most general integrable boundary condition (corresponding to the non-diagonal reflection matrix) was obtained in 1999 [40], however, an interesting issue of the exact solution for generic integrable boundary conditions is still left for this model.

In this paper, we study the exactly solvable $t$–$J$ model with generic integrable boundary fields. The Hamiltonian we shall consider is

$$
H = -t \sum_{\alpha,j=1}^{N-1} \mathcal{P}[C^\dagger_{j,\alpha}C_{j+1,\alpha} + C^\dagger_{j+1,\alpha}C_{j,\alpha}]\mathcal{P} + 2t \sum_{j=1}^{N-1} [\vec{S}_j \cdot \vec{S}_{j+1} - \frac{1}{4}n_jn_{j+1}] + \xi_1 n_1 + 2\vec{h}_1 \cdot \vec{S}_1 + \xi_N n_N + 2\vec{h}_N \cdot \vec{S}_N,
$$

(1.1)
doi:10.1088/1742-5468/2014/04/P04031
where $N$ is the total number of sites; $t$ is the hopping constant; $\mathcal{P}$ projects out double occupancies; $C_{j,\alpha}^\dagger$ and $C_{j,\alpha}$ are the creation and annihilation operators of electrons on the $j$th site with spin component $\alpha = \uparrow, \downarrow$; $\vec{S}_j = \frac{1}{2} \sum_{\alpha,\beta} C_{j,\alpha}^\dagger \vec{\sigma}_{\alpha,\beta} C_{j,\beta}$ are the spin operators and $\vec{\sigma}$ are the Pauli matrices; $n_{j,\alpha}$ are particle number operators; $\vec{h}_1 = (h_{1x}, h_{1y}, h_{1z})$ and $\vec{h}_N = (h_{Nx}, h_{Ny}, h_{Nz})$ are the boundary fields; $\xi_1$ and $\xi_N$ are the boundary chemical potentials.

In this paper we shall show that for the proper choices of the boundary chemical potentials the model is exactly solvable for arbitrary boundary magnetic fields, and give the exact solution of the model.

The paper is organized as follows. In section 2, we use the coordinate Bethe ansatz method to derive the corresponding two-body scattering matrix (or $R$-matrix) [41] and the reflection matrices (or $K$-matrices) due to the boundary interaction. In section 3, we transform the eigenvalue problem into that of the inhomogeneous XXX spin chain with boundary fields, which allows us to apply the recently proposed off-diagonal Bethe ansatz method [42]–[44] to solve it. The exact spectrum of the Hamiltonian and the Bethe ansatz equations are thus obtained. Section 4 is attributed to the reduction to the parallel or anti-parallel boundary case. As an application of our solution, the surface energy of the model with parallel fields is given in section 5. Concluding remarks are given in section 6.

2. Coordinate Bethe ansatz

Due to the fact that the total number of electrons of the super-symmetric $t-J$ model is conserved, we construct the eigenstate of the Hamiltonian (1.1) as follows:

$$\Psi = \sum_{j=1}^{M} \sum_{\alpha_j = \uparrow, \downarrow} \sum_{j=1}^{N} \Psi^{(\alpha)}(x_1, \ldots, x_M) C_{x_1,\alpha_1}^\dagger \cdots C_{x_M,\alpha_M}^\dagger |0\rangle,$$

(2.1)

where $\{\alpha\} = (\alpha_1, \ldots, \alpha_M)$ and $M$ is the total number of electrons. To exclude double occupancy, we need to impose the following condition on the wavefunction

$$\Psi^{(\alpha)}(\ldots, x_j, \ldots, x, \ldots) \equiv 0.$$

(2.2)

This eigenvalue equation can be rewritten as

$$-t \sum_{j=1}^{M} [(1 - \delta_{x_j,N}) \Psi^{(\alpha)}(\ldots, x_j + 1, \ldots) + (1 - \delta_{x_j,1}) \Psi^{(\alpha)}(\ldots, x_j - 1, \ldots)]
+ \sum_{j=1}^{M} \sum_{\beta_j = \uparrow, \downarrow} [\delta_{x_j,1} (\xi_1 + \vec{h}_1 \cdot \vec{\sigma}_{\alpha_j,\beta_j}) + \delta_{x_j,N} (\xi_N + \vec{h}_N \cdot \vec{\sigma}_{\alpha_j,\beta_j})]
\times \Psi^{(\alpha)}(x_1, \ldots, x_M) - t \sum_{j=1}^{M} \sum_{k \neq j} \sum_{l=1}^{M} \delta_{x_j,\beta_j} \delta_{x_k,\beta_k+1} \delta_{\alpha_1,-\alpha_k}
\times \Psi^{(\alpha)}(x_1, \ldots, x_k, \ldots) + \Psi^{(\alpha)}(\ldots, x_k, \ldots, x_l, \ldots)
= E \Psi^{(\alpha)}(x_1, \ldots, x_M),$$

(2.3)

doi:10.1088/1742-5468/2014/04/P04031 3
Exact solution of the one-dimensional super-symmetric $t$–$J$ model with unparallel boundary fields

where $\{\alpha\}_j$ means $\alpha_j$ is replaced by $\beta_j$ in the set $\{\alpha\}$. Suppose the wavefunction is taking the following Bethe ansatz form [41]:

$$\Psi^{(\alpha)}(x_1, \ldots, x_M) = \sum_{P, Q, r} A^{(\alpha), r}_P(Q) \exp \left[ i \sum_{j=1}^M r_{P,j} k_{P,j} x_{Q,j} \right] \theta(x_{Q,1} < x_{Q,2} < \cdots < x_{Q,M}),$$

(2.4)

where $Q = (Q_1, \ldots, Q_M)$ and $P = (P_1, \ldots, P_M)$ are the permutations of $(1, \ldots, M)$; $r = (r_1, \ldots, r_M)$ with $r_j = \pm 1$; $\theta(x_1 < \cdots < x_M)$ is the generalized step function. For $x_j \neq 1, N$, $x_i \neq 1, N$, and $|x_i - x_j| > 1$, the corresponding eigenvalue is

$$E = -2t \sum_{j=1}^M \cos k_j.$$

(2.5)

When two electrons occupy two adjacent sites, namely $x_{Q,j} = x_{Q,j+1} - 1 = x$ and $x \neq 1, N$, the Schrödinger equation (2.3) becomes

$$[1 + e^{i r_{P,j} k_{P,j} + i r_{P,j+1} k_{P,j+1}} - (1 - P_{j,j+1}) e^{i r_{P,j} k_{P,j} + i r_{P,j+1} k_{P,j+1}}] A_P(Q)$$

$$+ [1 + e^{i r_{P,j} k_{P,j} + i r_{P,j+1} k_{P,j+1}} - (1 - P_{j,j+1}) e^{i r_{P,j} k_{P,j} + i r_{P,j+1} k_{P,j+1}}] A_P'(Q) = 0,$$

(2.6)

with $P' = (\ldots, P_{j+1}, P_j, \ldots)$, $Q' = (\ldots, Q_{j+1}, Q_j, \ldots)$ and $r' = (\ldots, r_{j+1}, r_j, \ldots)$. It is remarked that we have omitted the superscript $\{\alpha\}$ and treated $A^{(\alpha)}_P(Q)$ as a column vector in the spin space. For convenience, let us introduce the permutation operators $\tilde{P}_{i,j}$ and $P_{i,j}$ in the coordinate and spin sectors, respectively.

$$\tilde{P}_{j+1} A_P(Q) = A_P'(Q').$$

(2.7)

Since the wavefunction of fermions is completely antisymmetric, we have

$$-P_{j,j+1} A_P(Q) = A_{P'}(Q'),$$

(2.8)

and

$$A_{P'}(Q) = S(r_{P,j} \lambda_{P,j}, r_{P,j+1} \lambda_{P,j+1}) A_{P'}'(Q').$$

(2.9)

After introducing a new parametrization

$$e^{ik_j} = \frac{\lambda_j - i/2}{\lambda_j + i/2},$$

(2.10)

we obtain the following $S$-matrix

$$S(\lambda_j, \lambda_k) = S(\lambda_j - \lambda_k) = \frac{\lambda_j - \lambda_k + i P_{j,k}}{\lambda_j - \lambda_k + i}.$$  

(2.11)

The $S$-matrix possesses the following property:

$$S(\lambda)^{-1} = S(-\lambda).$$

doi:10.1088/1742-5468/2014/04/P04031
Now we consider the case of \(x_{Q_1} = 1, x_{Q_2} \neq 2\). In this case, the eigenvalue equation becomes
\[
-t \Psi^{(\alpha)}(2, \ldots) + \sum_{\beta_1} (\xi_1 + \vec{h}_1 \cdot \vec{\sigma}_{\alpha_1, \beta_1}) \Psi^{(\alpha)}(1, \ldots) = -2t \cos k_{P_1} \Psi^{(\alpha)}(1, \ldots).
\]
(2.12)

This induces
\[
\sum_{\beta_1} (\xi_1 + \vec{h}_1 \cdot \vec{\sigma}_{\alpha_1, \beta_1}) \Psi^{(\alpha)}(1, \ldots) = -t \Psi^{(\alpha)}(0, \ldots),
\]
(2.13)
or
\[
A^{(\ldots \cdot \cdot \cdot)}_P(Q) = -[t + (\xi_1 + \vec{h}_1 \cdot \vec{\sigma}_{P_1}) e^{ik_{P_1}}]^{-1} [t + (\xi_1 + \vec{h}_1 \cdot \vec{\sigma}_{P_1}) e^{-ik_{P_1}}] A^{(-\cdot \cdot \cdot)}_P(Q) = K^+_{P_1} (k_{P_1}) A^{(-\cdot \cdot \cdot)}_P(Q).
\]
(2.14)

With the identity
\[
(\vec{h}_1 \cdot \vec{\sigma})^2 = \vec{h}_1^2,
\]
(2.15)
one readily obtains
\[
K^+(k) = -\frac{t^2 + \xi_1^2 - \vec{h}_1^2 + 2\xi_1 t \cos k - 2it \sin k \vec{h}_1 \cdot \vec{\sigma}}{(t + \xi_1 e^{ik})^2 - \vec{h}_1^2 e^{2ik}},
\]
(2.16)
and in terms of the new parameters \(\lambda\) (2.10) the \(K\)-matrix is given by
\[
K^+(\lambda) = \frac{\lambda + i/2}{\lambda - i/2} \left[ (t + \xi_1) \lambda + \frac{i}{2} (t - \xi_1) \right]^2 - \left( \lambda - \frac{i}{2} \right)^2 \vec{h}_1^2 \right]^{-1} \times \left( (t^2 + \xi_1^2 - \vec{h}_1^2) (\lambda^2 + \frac{1}{4}) + 2\xi_1 t (\lambda^2 - \frac{1}{4}) + 2i\lambda t \vec{h}_1 \cdot \vec{\sigma} \right).
\]
(2.17)

For the case of \(x_{Q_{M}} = N, x_{Q_{M-1}} \neq N - 1\), the eigenvalue equation is
\[
-t \Psi^{(\alpha)}(\ldots, N - 1) + \sum_{\beta_M} (\xi_N + \vec{h}_N \cdot \vec{\sigma}_{\alpha_M, \beta_M}) \Psi^{(\alpha)}(\ldots, N) = -2t \cos k_{P_M} \Psi^{(\alpha)}(\ldots, N).
\]
(2.18)

This induces
\[
\sum_{\beta_M} (\xi_N + \vec{h}_N \cdot \vec{\sigma}_{\alpha_M, \beta_M}) \Psi^{(\alpha)}(\ldots, N) = -t \Psi^{(\alpha)}(\ldots, N + 1),
\]
(2.19)
or
\[
A^{(\ldots \cdot \cdot \cdot)}_P(Q) = -e^{2ik_{P_M} N} [t e^{-ik_{P_M}} + (\xi_N + \vec{h}_N \cdot \vec{\sigma}_{P_M})]^{-1} \times [t e^{ik_{P_M}} + (\xi_N + \vec{h}_N \cdot \vec{\sigma}_{P_M})] A^{(-\cdot \cdot \cdot)}_P(Q) = e^{2ik_{P_M} N} K^-_{P_M} (k_{P_M}) A^{(-\cdot \cdot \cdot)}_P(Q),
\]
(2.20)
doi:10.1088/1742-5468/2014/04/P04031

J. Stat. Mech. (2014) P04031
Exact solution of the one-dimensional super-symmetric $t-J$ model with unparallel boundary fields

With

$$
\tilde{K}^-(k) = \frac{-\frac{2}{\lambda^2} - \frac{\xi_N^2}{\lambda^2} - \frac{1}{\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda}}{(e^{-ik} + e^{ik})^2 - \frac{\lambda^2}{\lambda^2} - \frac{\lambda^2}{\lambda^2}},
$$

and in terms of the new parameter $\lambda$ (2.10) it reads

$$
\tilde{K}^-(\lambda) = \frac{-\frac{1}{\lambda^2} + \frac{1}{\lambda} - \frac{1}{\lambda} + \frac{1}{\lambda} - \frac{1}{\lambda} - \frac{1}{\lambda}}{(e^{-ik} + e^{ik})^2 - \frac{\lambda^2}{\lambda^2} - \frac{\lambda^2}{\lambda^2}}.
$$

For the cases of $x_{Q_1} = 1$, $x_{Q_2} = 2$ and $x_{Q_{M-1}} = N - 1$, $x_{Q_M} = N$, one may check that (2.3) is self-consistent with the solutions of the $S$-matrix and $K$-matrices. The $S$-matrix and $K$-matrices allow us to construct the following relations:

$$
A^{(\cdots,+\cdots)} = S_{j-1,j}(k_{j-1}, k_j) \cdots S_{1,j}(k_1, k_j) A^{(+\cdots)},
$$

$$
A^{(+\cdots)} = K_j^+(k_j) A^{(-\cdots)},
$$

$$
A^{(-\cdots)} = S_{j,1}(-k_j, k_1) \cdots S_{j,j-1}(-k_j, k_{j-1}) S_{j,j+1}(-k_j, k_{j+1}) \cdots S_{j,M}(-k_j, k_M) A^{(\cdots,-\cdots)},
$$

$$
A^{(\cdots,+\cdots)} = e^{2ik_j N} K_j^-(k_j) A^{(\cdots,+\cdots)},
$$

$$
A^{(\cdots,+\cdots)} = S_{M,j}(k_M, k_j) \cdots S_{j+1,j}(k_{j+1}, k_j) A^{(\cdots,+\cdots)}.
$$

In terms of the parameter $\lambda$ (2.10), the above relations give rise to the following eigenvalue problem:

$$
\tilde{\tau}(\lambda_j) A^{(\cdots,+\cdots)} = \left(\frac{2\lambda_j - i}{2\lambda_j + i}\right)^{-2N} A^{(\cdots,+\cdots)},
$$

with the resulting operator

$$
\tilde{\tau}(u) = S_{j-1,j}(\lambda_{j-1}, u) \cdots S_{1,j}(\lambda_1, u) K_j^+(u)
$$

$$
\times S_{j,1}(-u, \lambda_1) \cdots S_{j,j-1}(-u, \lambda_{j-1})
$$

$$
\times S_{j,j+1}(-u, \lambda_{j+1}) \cdots S_{j,M}(-u, \lambda_M) K_j^-(u)
$$

$$
\times S_{M,j}(\lambda_M, u) \cdots S_{j+1,j}(\lambda_{j+1}, u)
$$

$$
= S_{j-1,j}(\lambda_{j-1} - u) \cdots S_{1,j}(\lambda_1 - u) K_j^+(u)
$$

$$
\times S_{j,1}(-u - \lambda_1) \cdots S_{j,j-1}(-u - \lambda_{j-1})
$$

$$
\times S_{j,j+1}(-u - \lambda_{j+1}) \cdots S_{j,M}(-u - \lambda_M) K_j^-(u)
$$

$$
\times S_{M,j}(\lambda_M - u) \cdots S_{j+1,j}(\lambda_{j+1} - u).
$$

To ensure the integrability of the model, namely to ensure that the resulting operators with different values of $u$ commute with each other, $[\tilde{\tau}(u), \tilde{\tau}(v)] = 0$, the corresponding

doi:10.1088/1742-5468/2014/04/P04031
$K$-matrices $\hat{K}^\pm(u)$ have to satisfy the following reflection equation [45]:

$$S_{1,2}(u_1 - u_2)\hat{K}^+_1(u_1)S_{1,2}(u_1 + u_2)\hat{K}^+_2(u_2) = \hat{K}^-_2(u_2)S_{1,2}(u_1 + u_2)\hat{K}^-_1(u_1)S_{1,2}(u_1 - u_2),$$

(2.25)

which induces the following integrable conditions of the model

$$(t + \xi_1)^2 = \tilde{h}_1^2, \quad (t + \xi_N)^2 = \tilde{h}_N^2.$$  

(2.26)

Under this restriction, the reflection matrices become

$$\tilde{K}^-(\lambda) = \frac{2\lambda - i}{2\lambda + i} \frac{\xi_N - 2i\lambda \tilde{h}_N \cdot \vec{\sigma}}{\xi_N + 2i\lambda(t + \xi_N)},$$

(2.27)

$$\tilde{K}^+(\lambda) = \frac{2\lambda + i}{2\lambda - i} \frac{\xi_1 - 2i\lambda \tilde{h}_1 \cdot \vec{\sigma}}{\xi_1 + 2i\lambda(t + \xi_1)},$$

(2.28)

Similarly to that of the Hubbard model with arbitrary boundary magnetic fields [46], in our case the eigenvalue problem (2.23) can be transformed into that of the transfer matrix of the inhomogeneous XXX spin chain model with arbitrary boundary fields and thus can be solved via the off-diagonal Bethe ansatz [42–44].

3. Off-diagonal Bethe ansatz

Before going further, let us introduce the following $R$-matrix and $K$-matrices:

$$R_{0,j}(u) = u + \eta P_{0,j},$$

(3.1)

$$K_0^-(u) = p + u\tilde{h}_N \cdot \vec{\sigma}_0,$$

(3.2)

$$K_0^+(u) = q - (u + \eta)\tilde{h}_1 \cdot \vec{\sigma}_0,$$

(3.3)

where

$$\eta = i, \quad p = \frac{\xi_N}{2i}, \quad q = -\frac{\xi_1}{2i}.$$  

We remark that the $K$-matrices (3.2) and (3.3) are the most general reflection matrices associated the XXX spin chain [47, 48]. The $R$-matrix has the following properties:

initial condition: $R_{1,2}(0) = \eta P_{1,2}$,  

(3.4)

unitarity relation: $R_{1,2}(u)R_{1,2}(-u) = -(u + \eta)(u - \eta) \text{id}$,  

(3.5)

crossing relation: $R_{12}(u) = V_1 R_{12}^\dagger(-u - \eta)V_1$, $V = -i\sigma^y$.  

(3.6)

The following Yang–Baxter equation, the reflection equation and the dual reflection equation also hold:

$$R_{0,0'}(u - v)R_{0,1}(u)R_{0',1}(v) = R_{0',1}(v)R_{0,1}(u)R_{0,0'}(u - v),$$

(3.7)

$$R_{0,0'}(u - v)K_0^-(u)R_{0,0'}(u + v)K_0^-(v) = K_0^-(v)R_{0,0'}(u + v)K_0^-(u)R_{0,0'}(u - v),$$

(3.8)

doi:10.1088/1742-5468/2014/04/P04031
boundary fields and the transfer matrices

\[
R_{0,0'}(v-u)K_0^+(u)R_{0,0'}(-u-v-2\eta)K_0^+(v)
\]
\[
= K_0^+(v)R_{0,0'}(-u-v-2\eta)K_0^+(u)R_{0,0'}(v-u).
\]  (3.9)

We introduce the inhomogeneous double-row monodromy matrix

\[
T_0(u) = R_{0,1}(u - \lambda_1) \cdots R_{0,M}(u - \lambda_M)K_0^+(u)R_{M,0}(u + \lambda_M) \cdots R_{1,0}(u + \lambda_1),
\]  (3.10)

and the associated transfer matrix \(\tau(u)\) is given by [45]

\[
\tau(u) = \text{tr}_0\{K_0^+(u)T_0(u)\}. \tag{3.11}
\]

From (3.7), (3.8) and (3.9) one may derive

\[
R_{0,0'}(u-v)T_0(u)R_{0,0'}(u+v)T_0(v) = T_0(v)R_{0,0'}(u+v)T_0(u)R_{0,0'}(u-v),
\]  (3.12)

and thus the transfer matrices with different spectrum parameters commute with each other,

\[
[\tau(u), \tau(v)] = 0, \tag{3.13}
\]

which ensures the integrability of the associated spin chain. Let \(u = -\lambda_j\); using the initial condition (3.4) and the Yang–Baxter equation (3.7) we can express the transfer matrix at the special point in terms of the \(K\)-matrices and the \(R\)-matrix

\[
\tau(-\lambda_j) = R_{j-1,j}(\lambda_{j-1} - \lambda_j) \cdots R_{1,j}(\lambda_1 - \lambda_j)
\]
\[
\times \text{tr}_0\{K_0^+(\lambda_{j-1} - \lambda_j)R_{0,j}(\lambda_j - \lambda_1) \cdots R_{j-1,j}(\lambda_j - \lambda_{j-1})
\]
\[
\times R_{j,j+1}(\lambda_j - \lambda_{j+1}) \cdots R_{j,M}(\lambda_j - \lambda_M)
\]
\[
\times K_j^-(\lambda_j)R_{M,j}(\lambda_M - \lambda_j) \cdots R_{j+2,j}(\lambda_j + 2 - \lambda_j)R_{j+1,j}(\lambda_j + 1 - \lambda_j).
\]  (3.14)

Noticing that

\[
S_{j,i}(\lambda_j, \lambda_i) = \frac{R_{j,i}(\lambda_j - \lambda_i)}{\lambda_j - \lambda_i + \eta}, \tag{3.15}
\]
\[
S_{j,i}(-\lambda_j, \lambda_i) = \frac{R_{j,i}(\lambda_j - \lambda_i)}{-\lambda_j - \lambda_i + \eta}, \tag{3.16}
\]
\[
\bar{K}_j^-(\lambda_j) = \frac{2\lambda_j - \eta}{2\lambda_j + \eta} \frac{K_j^-(\lambda_j)}{p + \lambda_j(t + \xi_N)}, \tag{3.17}
\]
\[
\bar{K}_j^+(\lambda_j) = -\frac{\text{tr}_0\{K_0^+(\lambda_j)R_{0,j}(-2\lambda_j)R_{0,j}(0)\}}{2\eta(\lambda_j - \eta)[g - \lambda_j(t + \xi_1)]} \frac{2\lambda_j + \eta}{2\lambda_j - \eta}.
\]  (3.18)

we have the following important identification between the operators \(\{\tau(\lambda_j)\}\) given by (2.24) appeared in the eigenvalue problem of the super-symmetric \(t-J\) model with boundary fields and the transfer matrices \(\{\tau(\lambda_j)\}\) of the open XXX spin chain with

doi:10.1088/1742-5468/2014/04/P04031
boundary fields

\[
\tau(\lambda_j) = \prod_{l \neq j}^{M}(\lambda_j - \lambda_l - \eta)^{-1}(\lambda_j + \lambda_l - \eta)^{-1} \frac{1}{2\eta(\lambda_j - \eta)} \\
\times \frac{\tau(-\lambda_j)}{[p + \lambda_j(t + \xi_N)][-q + \lambda_j(t + \xi_1)]}.
\]  

(3.19)

The eigenvalue problem (2.23) is thus equivalent to that of diagonalizing the transfer matrix of the inhomogeneous open XXX chain model with boundary fields. Here we naturally have the ‘inhomogeneous’ parameters \(\lambda_j\), related to the quasi-momentum (2.10) of the electrons and the crossing parameter \(\eta = i\). Thanks to the works [42]–[44], the transfer matrix (3.11) of the open XXX chain with arbitrary boundary fields which is specified by the \(K\)-matrices \(K^{\pm}(u)\) (3.2) and (3.3) can be exactly diagonalized by the off-diagonal Bethe ansatz method. In the following, we shall use the method in [44] to solve the eigenvalue problem (2.23) of the super-symmetric \(t\)–\(J\) model with general boundary fields. Suppose \(|\Psi\rangle\) is an eigenstate of \(\tau(u)\) and the corresponding eigenvalue is \(\Lambda(u)\),

\[
\tau(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle.
\]  

(3.20)

Following the method in [44], we find that \(\Lambda(u)\) possesses the following properties:

\[
\Lambda(u) = \Lambda(-u - \eta),
\]  

(3.21)

\[
\Lambda(0) = 2pq \prod_{l=1}^{M} - (\lambda_l - \eta)(\lambda_l + \eta),
\]  

(3.22)

\[
\Lambda(u) \sim -2\vec{h}_1 \cdot \vec{h}_N u^{2M + 2} + \cdots, \quad u \to \pm \infty,
\]  

(3.23)

\[
\Lambda(\lambda_j)\Lambda(\lambda_j - \eta) = \frac{4(\lambda_j^2 - \eta^2)}{4\lambda_j^2 - \eta^2}(q^2 - \lambda_j^2\vec{h}_1^2)(p^2 - \lambda_j^2\vec{h}_N^2) \\
\times \prod_{l=1}^{M} [(\lambda_j + \lambda_l)^2 - \eta^2][(\lambda_j - \lambda_l)^2 - \eta^2], \quad j = 1, 2, 3 \ldots, M.
\]  

(3.24)

Moreover, the explicit expressions (3.1)–(3.3) of the \(R\)-matrix and \(K\)-matrices imply that \(\Lambda(u)\), as a function of \(u\), is a polynomial of degree \(2M + 2\), hence \(\Lambda(u)\) can be completely determined by (3.21)–(3.24).

For convenience, we introduce the following notations:

\[
A(u) = \prod_{l=1}^{M} (u - \lambda_l + \eta)(u + \lambda_l + \eta),
\]  

(3.25)

\[
a(u) = \frac{2u + 2\eta}{2u + \eta} [p + u \text{sgn}(\vec{h}_1 \cdot \vec{h}_N)]\vec{h}_N||q - u\vec{h}_1||A(u),
\]  

(3.26)

\[
d(u) = a(-u - \eta),
\]  

(3.27)

\[
c = 2[\text{sgn}(\vec{h}_1 \cdot \vec{h}_N)]\vec{h}_1||\vec{h}_N| - \vec{h}_1 \cdot \vec{h}_N].
\]  

(3.28)

doi:10.1088/1742-5468/2014/04/P04031
3.1. Even M case

As in [44], we make the following functional $T-Q$ ansatz for an even $M$:

$$\Lambda(u) = a(u)\frac{Q_1(u - \eta)}{Q_2(u)} + d(u)\frac{Q_2(u + \eta)}{Q_1(u)} + cu(u + \eta)A(u)A(-u - \eta)\frac{A(u)A(-u - \eta)}{Q_1(u)Q_2(u)},$$

(3.29)

where the functions $Q_1(u)$ and $Q_2(u)$ are parameterized by $M$ Bethe roots $\{\mu_j|j = 1, \ldots, M\}$ for a generic non-vanishing $c$ as follows:

$$Q_1(u) = \prod_{j=1}^{M}(u - \mu_j), \quad Q_2(u) = \prod_{j=1}^{M}(u + \mu_j + \eta) = Q_1(-u - \eta).$$

(3.30)

$\Lambda(u)$ becomes the eigenvalue of the transfer matrix $\tau(u)$ if the parameters $\{\mu_j|j = 1, \ldots, M\}$ satisfy the following Bethe ansatz equations:

$$c(\mu_j + \eta)(2\mu_j + \eta)$$

$$= -\prod_{l=1}^{M}(\mu_j + \mu_l + \eta)(\mu_j + \mu_l + 2\eta), \quad j = 1, \ldots, M.$$ 

(3.31)

With the identification (3.19), we get the other Bethe ansatz equations

$$\frac{[p - (\mu_j + \eta)\text{sgn}(\vec{h}_1 \cdot \vec{h}_N)] [q + (\mu_j + \eta)\vec{h}_1]}{[p + \lambda_j(t + \xi_N)] [q - \lambda_j(t + \xi_1)]} \left(\frac{2\lambda_j - \eta}{2\lambda_j + \eta}\right)^{2N}$$

$$= \prod_{i=1}^{M}\frac{\lambda_j - \mu_i - \eta}{\lambda_j + \mu_i + \eta}, \quad j = 1, \ldots, M.$$ 

(3.32)

Then from the solutions of the Bethe ansatz equations (3.31) and (3.32), one can reconstruct the exact wavefunctions (2.4) with even number of electrons for the supersymmetric $t-J$ model with unparallel boundary fields; the corresponding eigenvalues are given by (2.5).

3.2. Odd M case

For an odd $N$, we make the following functional $T-Q$ ansatz

$$\Lambda(u) = a(u)\frac{Q_1(u - \eta)}{Q_2(u)} + d(u)\frac{Q_2(u + \eta)}{Q_1(u)} + cu^2(u + \eta)^2A(u)A(-u - \eta)\frac{A(u)A(-u - \eta)}{Q_1(u)Q_2(u)},$$

(3.33)

where the functions $a(u)$, $d(u)$ and $A(u)$ and the parameter $c$ are given by (3.25)–(3.28) respectively. The functions $Q_1(u)$ and $Q_2(u)$ are parameterized by $M + 1$ Bethe roots $\{\mu_j|j = 1, \ldots, M + 1\}$ for a generic non-vanishing $c$ as follows:

$$Q_1(u) = \prod_{j=1}^{M+1}(u - \mu_j), \quad Q_2(u) = \prod_{j=1}^{M+1}(u + \mu_j + \eta) = Q_1(-u - \eta).$$

(3.34)
The $M$ quasi-momentum $\{k_j\}$ (or $\{\lambda_j\}$) and the $M+1$ parameters $\{\mu_j\}_{j=1,\ldots,M+1}$ need to satisfy the following Bethe ansatz equations:

$$\frac{[p - \lambda_j \text{sgn}(\vec{h}_1 \cdot \vec{h}_N)](q + \lambda_j|\vec{h}_1|)}{[p + \lambda_j(t + \xi_N)](q - \lambda_j(t + \xi_1))] \left(\frac{2\lambda_j - \eta}{2\lambda_j + \eta}\right)^{2N} = \prod_{l=1}^{M+1} \frac{\lambda_j - \mu_l - \eta}{\lambda_j + \mu_l + \eta}, \quad j = 1, \ldots, M,$$

$$2[p - (\mu_j + \eta) \text{sgn}(\vec{h}_1 \cdot \vec{h}_N)]|\vec{h}_N| |q + (\mu_j + \eta)|\vec{h}_1|] = \prod_{l=1}^{M+1} (\mu_j + \mu_l + 2\eta)/(\mu_j - \lambda_l + \eta)(\mu_j + \lambda_l + \eta), \quad j = 1, \ldots, M + 1. \quad (3.35)$$

Then from the solutions of the Bethe ansatz equations (3.35) and (3.36), one can reconstruct the exact wavefunctions (2.4) with odd number of electrons for the supersymmetric $t$–$J$ model with boundary fields; the corresponding eigenvalues are given by (2.5).

### 4. Reduction to the parallel boundary case

When the two boundary fields $\vec{h}_1$ and $\vec{h}_N$ are parallel or anti-parallel, the parameter $c$ is vanishing. The resulting $T$–$Q$ relation becomes the conventional one

$$\Lambda(u) = a(u) \frac{Q(u - \eta)}{Q(u)} + d(u) \frac{Q(u + \eta)}{Q(u)}, \quad (4.1)$$

with

$$Q(u) = \prod_{l=1}^{m} (u - \gamma_l)(u + \gamma_l + \eta) = Q(-u - \eta), \quad m = 0, 1, \ldots, M. \quad (4.2)$$

Parameter $\{\gamma_j\}$ and quasi-momentum $\{\lambda_j\}$ satisfy the following Bethe ansatz equations:

$$\frac{[p - \lambda_j \text{sgn}(\vec{h}_1 \cdot \vec{h}_N)]|\vec{h}_N| |q + \lambda_j|\vec{h}_1|]}{[p + \lambda_j(t + \xi_N)]|q - \lambda_j(t + \xi_1))] \left(\frac{2\lambda_j - \eta}{2\lambda_j + \eta}\right)^{2N} = \prod_{l=1}^{M} (\lambda_j - \gamma_l - \eta)(\lambda_j + \gamma_l + \eta)/(\lambda_j + \gamma_l + \eta)(\lambda_j - \gamma_l + \eta), \quad j = 1, \ldots, M,$$

$$\gamma_j[p - (\gamma_j + \eta) \text{sgn}(\vec{h}_1 \cdot \vec{h}_N)]|\vec{h}_N| |q + (\gamma_j + \eta)|\vec{h}_1|] \times \prod_{l=1}^{M} (\gamma_j + \lambda_l)(\gamma_j - \lambda_l)/(\gamma_j - \lambda_l + \eta)(\gamma_j + \lambda_l + \eta)$$

$$= \prod_{l=1}^{M} (\gamma_j - \gamma_l - \eta)(\gamma_j + \gamma_l)(\gamma_j + \gamma_l + 2\eta), \quad j = 1, \ldots, m. \quad (4.4)$$
Then from the solutions of the Bethe ansatz equations (4.3) and (4.4), one can reconstruct the exact wavefunctions (2.4) for the super-symmetric \( t-J \) model with parallel or anti-parallel boundary fields; the corresponding eigenvalues are given by (2.5).

5. Surface energy

As an application of our exact solution of the supersymmetry \( t-J \) model with boundary fields, here we study the surface energy of the supersymmetry \( t-J \) model with parallel boundary fields\(^5\). In the interesting paper [29], Essler calculated the surface energy for this particular case. Here we only list the main results and we refer the reader to [29] for more details. Let us introduce a new parameter

\[
\theta_j = \gamma_j + \frac{\eta}{2},
\]

(5.1)

5.1. The case of \( t + \xi_N = -|\vec{h}_N| \) and \( t + \xi_1 = -|\vec{h}_1| \)

In this case, the functions (4.3) and (4.4) become

\[
\left( \frac{\lambda_j - \eta/2}{\lambda_j + \eta/2} \right)^{2N} = \prod_{l=1}^{m} \frac{\lambda_j - \theta_l - \eta/2 \lambda_j + \theta_l - \eta/2}{\lambda_j - \theta_l + \eta/2 \lambda_j + \theta_l + \eta/2}, \quad j = 1, 2, \ldots, M,
\]

(5.2)

and

\[
\frac{\theta_j - \eta/2 \theta_j - c \eta \theta_j - d \eta \prod_{l=1}^{M} \theta_j + \lambda_l - \eta/2 \theta_j - \lambda_l - \eta/2}{\theta_j + \eta/2 \theta_j + c \eta \theta_j + d \eta \prod_{l=1}^{M} \theta_j + \lambda_l + \eta/2 \theta_j + \lambda_l + \eta/2} = - \prod_{l=1}^{M} \frac{\theta_j - \theta_l - \eta \theta_j + \theta_l - \eta}{\theta_j - \theta_l + \eta \theta_j + \theta_l + \eta}, \quad j = 1, 2, \ldots, m,
\]

(5.3)

where

\[
c = -\frac{\xi_N}{2|\vec{h}_N|} - \frac{1}{2}, \quad d = -\frac{\xi_1}{2|\vec{h}_1|} - \frac{1}{2}.
\]

(5.4)

The logarithms of equations (5.2) and (5.3) are

\[
2N \ln \frac{\lambda_j - \eta/2}{\lambda_j + \eta/2} = \sum_{l=1}^{m} \left\{ \ln \frac{\lambda_j - \theta_l - \eta/2}{\lambda_j - \theta_l + \eta/2} + \ln \frac{\lambda_j + \theta_l - \eta/2}{\lambda_j + \theta_l + \eta/2} \right\} + 2\pi \eta I_j, \quad j = 1, 2, \ldots, M,
\]

(5.5)

\(^5\) The generalization to the case of the generic non-diagonal boundary is nontrivial even for the XXX open spin chain case [49, 50].
Exact solution of the one-dimensional super-symmetric \( t-J \) model with unparallel boundary fields

and

\[
\ln \frac{\theta_j - \eta/2}{\theta_j + \eta/2} + \ln \frac{\theta_j - c\eta}{\theta_j + c\eta} + \ln \frac{\theta_j - d\eta}{\theta_j + d\eta} + \sum_{l=1}^{M} \left\{ \ln \frac{\theta_j + \lambda_l - \eta/2}{\theta_j + \lambda_l + \eta/2} + \ln \frac{\theta_j - \lambda_l - \eta/2}{\theta_j - \lambda_l + \eta/2} \right\} 
\]

\[
= \pi \eta + \sum_{l=1}^{m} \left\{ \ln \frac{\theta_j - \theta_l - \eta}{\theta_j - \theta_l + \eta} + \ln \frac{\theta_j + \theta_l - \eta}{\theta_j + \theta_l + \eta} \right\} 
\]

\[
+ 2\pi \eta I'_j, \quad j = 1, 2, \ldots, m, \quad (5.6)
\]

where \( I_j \) and \( I'_j \) are integers. In the thermodynamic limit \( \{\lambda_j\} \) and \( \{\theta_j\} \) distribute with densities \( \rho(\lambda) \) and \( \sigma(\theta) \) respectively. Due to the fact that \( \lambda_j = 0 \) and \( \theta_j = 0 \) are the solutions of (5.2) and (5.3) and they make the wavefunction vanishing, they should be excluded, namely, the densities corresponding to the ground state in the thermodynamic limit are [51, 52]

\[
\rho(\lambda) = \frac{1}{2N} \frac{dI}{d\lambda} - \frac{1}{2N} \delta(\lambda), \quad \sigma(\theta) = \frac{1}{2N} \frac{dI'}{d\theta} - \frac{1}{2N} \delta(\theta) \quad (5.7)
\]

and \( \rho(\lambda) = \rho(-\lambda), \sigma(\theta) = \sigma(-\theta) \).

Let us first consider the case \( t > 0 \), which is corresponding to \( c > 0 \) and \( d > 0 \). Taking the derivative of (5.5) and (5.6), we have

\[
a_1(\lambda) - \frac{1}{2N} \delta(\lambda) = \int_{-\infty}^{\infty} a_1(\lambda - x) \sigma(x) \, dx + \rho(\lambda), \quad (5.8)
\]

and

\[
\frac{1}{2N} [a_1(\theta) + a_2c(\theta) + a_2d(\theta)] + \int_{-\infty}^{\infty} a_1(\theta - y) \rho(y) \, dy 
\]

\[
= \int_{-\infty}^{\infty} a_2(\theta - y) \sigma(\theta) \, dy + \sigma(\theta) + \frac{1}{2N} \delta(\theta). \quad (5.9)
\]

Here the function \( a_n(z) \) is defined by

\[
a_n(z) = \frac{1}{2\pi} \frac{n}{z^2 + n^2/4}, \quad n > 0. \quad (5.10)
\]

Using the Fourier expansion

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) \, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{f}(\omega) \, d\omega, \quad (5.11)
\]

(5.8) and (5.9) become

\[
\tilde{a}_1(\omega) - \frac{1}{2N} = \tilde{a}_1(\omega) \tilde{\sigma}(\omega) + \tilde{\rho}(\omega), \quad (5.12)
\]

and

\[
\frac{1}{2N} [\tilde{a}_1(\omega) + \tilde{a}_2c(\omega) + \tilde{a}_2d(\omega) - 1] = [\tilde{a}_2(\omega) + 1] \tilde{\sigma}(\omega) - \tilde{a}_1(\omega) \tilde{\rho}(\omega), \quad (5.13)
\]

doi:10.1088/1742-5468/2014/04/P04031
where
\[ \tilde{a}_n(\omega) = e^{-n/2|\omega|}, \quad n > 0. \] (5.14)

Solving equations (5.12) and (5.13) yields that
\[ \tilde{\rho}(\omega) = 1 \] (5.15)
\[ C(\omega) = 1 \] (5.16)

For the periodic $t-J$ model, $C(\omega)$ is vanishing, the number of electrons in the ground state is
\[ M = N \int_{-\infty}^{\infty} \rho(\lambda) d\lambda = N \int_{-\infty}^{\infty} \tilde{\rho}(\omega) e^{-i\omega\lambda} d\omega d\lambda \]
\[ = N \tilde{\rho}(0) = \frac{3}{2} N \] (5.17)

and the ground state energy in the thermodynamic limit is
\[ E_0^g = -2t \sum_{j=1}^{M} \frac{\lambda_j^2 - 1/4}{\lambda_j^2 + 1/4} = -2Nt \int_{-\infty}^{\infty} \rho(\lambda) \frac{\lambda^2 - 1/4}{\lambda^2 + 1/4} d\lambda \]
\[ = -2Nt \int_{-\infty}^{\infty} \tilde{\rho}(\omega) \left[ \frac{1}{2} \mathbb{1}(\omega) \right] d\omega \]
\[ = -\frac{1}{3} Nt + \frac{\ln 3}{2} N t. \] (5.18)

For the $t-J$ model with open boundary condition, we find that there are no boundary strings and that $C(\omega)$ given by (5.16) does not vanish. Then the ground state energy in the thermodynamic limit is
\[ E_g = -2t \sum_{j=1}^{M} \frac{\lambda_j^2 - 1/4}{\lambda_j^2 + 1/4} = -2Nt \int_{-\infty}^{\infty} \rho(\lambda) \frac{\lambda^2 - 1/4}{\lambda^2 + 1/4} d\lambda \]
\[ = -2Nt \int_{-\infty}^{\infty} \tilde{\rho}(\omega) \left[ \frac{1}{2} \mathbb{1}(\omega) \right] d\omega \]
\[ = E_0^g + 2Nt \int_{-\infty}^{\infty} \frac{C(\omega)}{2\tilde{a}_2(\omega) + 1} \left[ \frac{1}{2} \mathbb{1}(\omega) \right] d\omega \]
\[ = E_0^g + \frac{\ln 3}{2} t - \frac{2}{3} t - B_c t - B_d t, \] (5.19)

with
\[ B_{p>1} = \int_0^1 \frac{x^p}{2x+1} dx = \int_0^{1/2} \frac{x^p}{2x+1} dx + \int_{1/2}^1 \frac{1}{2^{p+1}} \frac{x^{p-1}}{1+2x} dx \]
\[ = \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n 2^n x^{n+p} dx + \int_{1/2}^1 \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{n+1}} x^{p-1-n} dx \]
Exact solution of the one-dimensional super-symmetric $t$–$J$ model with unparallel boundary fields

$$
\begin{align*}
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2p+1} \frac{1}{p+n+1} + \sum_{n=0}^{\infty} (1 - \delta_{p,n})(-1)^n \left( \frac{1}{2n+1} - \frac{1}{2p+1} \right) \frac{1}{p-n} \\
&\quad + \delta_{p,n} (-1)^p \frac{1}{2p+1} \ln 2.
\end{align*}
$$

(5.20)

For the case of $t < 0$, which is corresponding to $c < 0$ and $d < 0$, (5.19) is given by

$$
E_g = E_g^0 + \frac{\ln 3}{2} t - 2t + B_c|t| + B_d|t|.
$$

(5.21)

The surface energy is

$$
E_{\text{surf}} = E_g - E_g^0 = \frac{\ln 3}{2} t + \frac{1}{3} [\text{sgn}(c) + \text{sgn}(d) - 4] t - \text{sgn}(c) B_c|t| - \text{sgn}(d) B_d|t|.
$$

(5.22)

5.2. The case of $t + \xi_N = |\vec{h}_N|$ and $t + \xi_1 = -|\vec{h}_1|$ 

In this case, the Bethe ansatz equation (5.2) will become

$$
\begin{align*}
\left( \frac{\lambda_j - \eta/2}{\lambda_j + \eta/2} \right)^{2N} = -\frac{\lambda_j - g\eta}{\lambda_j + g\eta} \prod_{l=1}^{m} \frac{\lambda_j - \theta_l - \eta/2 \lambda_j + \theta_l - \eta/2}{\lambda_j - \theta_l + \eta/2 \lambda_j + \theta_l + \eta/2}, \quad j = 1, 2, \ldots, M,
\end{align*}
$$

(5.23)

with

$$
g = \frac{\xi_N}{2|\vec{h}_N|}.
$$

(5.24)

We find that now there does exist a boundary string located at $\lambda_0 = g\eta$ in the thermodynamics limit when $\xi_N > 0$. Suppose that the densities of $\lambda$ and $\theta$ in the state with boundary string are $\tilde{\rho}(\lambda)$ and $\tilde{\sigma}(\theta)$ and the densities of $\lambda$ and $\theta$ in the state without boundary string are $\rho(\lambda)$ and $\sigma(\theta)$, thus we have

$$
a_1(\lambda) - \frac{1}{2N} \delta(\lambda) = \int_{-\infty}^{\infty} a_1(\lambda - x) \sigma(x) \, dx + \tilde{\rho}(\lambda) + \frac{1}{2N} a_2g(\lambda),
$$

(5.25)

and

$$
\begin{align*}
\frac{1}{2N} [a_1(\theta) + a_2c(\theta) + a_2d(\theta)] + \int_{-\infty}^{\infty} a_1(\theta - y) \tilde{\rho}(y) \, dy + \frac{1}{2N} [a_1(\theta + \lambda_0) + a_1(\theta - \lambda_0)]
&= \frac{1}{2N} \delta(\theta) + \tilde{\sigma}(\theta) + \int_{-\infty}^{\infty} a_2(\theta - y) \tilde{\sigma}(\theta) \, dy.
\end{align*}
$$

(5.26)
Then we have
\[
\int_{-\infty}^{\infty} a_1(\lambda - x)\delta\sigma(x)\,dx + \delta\rho(\lambda) = 0,
\]
\[
\int_{-\infty}^{\infty} a_2(\theta - y)\delta\sigma(y)\,dy + \delta\sigma(\theta) - \int_{-\infty}^{\infty} a_1(\theta - y)\delta\rho(y)\,dy
= \frac{1}{2N}[a_1(\theta + \lambda_0) + a_1(\theta - \lambda_0)],
\]
with
\[
\delta\rho(\lambda) = \tilde{\rho}(\lambda) - \rho(\lambda), \quad \delta\sigma(\theta) = \tilde{\sigma}(\theta) - \sigma(\theta).
\]
Using the Fourier expansion, we have
\[
\tilde{a}_1(\omega)\delta\tilde{\sigma}(\omega) + \delta\tilde{\rho}(\omega) = 0,
\]
\[
[\tilde{a}_2(\omega) + 1]\delta\tilde{\sigma}(\omega) - \tilde{a}_1(\omega)\delta\tilde{\rho}(\omega) = \frac{1}{2N}A(\omega),
\]
with \(A(\omega) = \tilde{a}_{2g+1}(\omega) - \tilde{a}_{2g-1}(\omega)\) if \(g > \frac{1}{2}\) and \(A(\omega) = \tilde{a}_{1-2g}(\omega) + \tilde{a}_{2g+1}(\omega)\) if \(0 < g < \frac{1}{2}\).
Solving equations (5.29) and (5.30) gives rise to
\[
\delta\tilde{\rho}(\omega) = -\frac{1}{2N\tilde{a}_2(\omega) + 1} \tilde{a}_1(\omega)A(\omega),
\]
\[
\Delta E_1 = -2Nt \int_{-\infty}^{\infty} \delta\tilde{\rho}(\omega) \left[\delta(\omega) - \frac{1}{2}\tilde{a}_1(\omega)\right] d\omega - 2t\frac{\lambda_0^2 - 1/4}{\lambda_0^3 + 1/4}
= -2t - \frac{t}{g^2 - 1/4} + tB_{g-1/2} - tB_{g+1/2} > 0.
\]
When \(g > \frac{1}{2}\), which is corresponding to \(t < 0\), the difference between the energy of the state with the string and \(E_g\) is
\[
\Delta E_2 = -2Nt \int_{-\infty}^{\infty} \delta\tilde{\rho}(\omega) \left[\delta(\omega) - \frac{1}{2}\tilde{a}_1(\omega)\right] d\omega - 2t\frac{\lambda_0^2 - 1/4}{\lambda_0^3 + 1/4}
= -\frac{4}{3}t + \frac{t}{1/4 - g^2} - tB_{1/2-g} - tB_{1/2+g} > 0.
\]
When \(0 < g < \frac{1}{2}\), which is corresponding to \(t > 0\), the difference between the energy of the state with the string and \(E_g\) is
\[
\Delta E_2 = -2Nt \int_{-\infty}^{\infty} \delta\tilde{\rho}(\omega) \left[\delta(\omega) - \frac{1}{2}\tilde{a}_1(\omega)\right] d\omega - 2t\frac{\lambda_0^2 - 1/4}{\lambda_0^3 + 1/4}
= -\frac{4}{3}t + \frac{t}{1/4 - g^2} - tB_{1/2-g} - tB_{1/2+g} > 0.
\]
We can obtain a similar result when \(t + \xi_N = -|\vec{h}_N|\) and \(t + \xi_1 = |\vec{h}_1|\), which shows that the correct ground state contains only real roots when the two boundary fields \(\vec{h}_1\) and \(\vec{h}_N\) are parallel. The surface energy in this case is given by
\[
E_{\text{surf}} = \frac{\ln 3}{2}t + \frac{1}{3}[2\text{sgn}(g) + \text{sgn}(c) + \text{sgn}(d) - 4]t - [\text{sgn}(c)B_{|c|} - \text{sgn}(d)B_{|d|}
+ \text{sgn}(g)B_{|g|-1/2} + \text{sgn}(g)B_{|g|+1/2}t].
\]
\[\text{doi:10.1088/1742-5468/2014/04/P04031}\]
6. Conclusion

The one-dimensional super-symmetric $t$–$J$ model with unparallel boundary magnetic fields described by the Hamiltonian (1.1) is studied by combining the coordinate Bethe ansatz and off-diagonal Bethe ansatz methods. With the coordinate Bethe ansatz, eigenfunctions of the Hamiltonian of the model are given in terms of some quasi-momentum $\{k_j\}$ as (2.4). The constraints (2.23) on this quasi-momentum are transformed into the eigenvalue problem of the resulting transfer matrix of the associated open XXX spin chain with arbitrary boundary fields. The second eigenvalue problem is then solved via the off-diagonal Bethe ansatz method. We remark that further study on the correlation functions would be an interesting issue.

Acknowledgments

The financial support from the National Natural Science Foundation of China (Grant Nos 11174335, 11031005, 11375141, 11374334), the National Program for Basic Research of MOST (973 project under grant No. 2011CB921700), the State Education Ministry of China (Grant No. 20116101110017) and BCMIIS are gratefully acknowledged.

References

[1] Zhang F C and Rice T M, 1988 Phys. Rev. B 37 3759
[2] Anderson P W, 1987 Science 235 1196
[3] Essler F H L, Korepin V E and Schoutens K, 1992 Phys. Revs. Lett. 68 2960
[4] Hu Z-N and Pu F-C, 1999 Nucl. Phys. B 546 691
[5] Eskes H and Sawatzky G A, 1988 Phys. Rev. Lett. 61 1415
[6] McMahan A K, Martin R M and Satpathy S, 1989 Phys. Rev. B 38 6650
[7] Hybertsen M S, Schlüter M and Christensen N E, 1989 Phys. Rev. B 39 9028
[8] Hybertsen M S, Stechel E B, Schlüter M and Jennison D R, 1990 Phys. Rev. B 41 11068
[9] Anderson P W, 1990 Phys. Rev. Lett. 65 2306
[10] Ogata M, Luchini M U, Sorella S and Assaad F F, 1991 Phys. Rev. Lett. 66 2388
[11] Putikka W O, Luchini M U and Rice T M, 1992 Phys. Rev. Lett. 68 538
[12] Wiegmann P B, 1988 Phys. Rev. Lett. 60 821
[13] Förster D, 1989 Phys. Rev. Lett. 63 2140
[14] Foerster A and Karowski M, 1992 Phys. Rev. B 46 9234
[15] Foerster A and Karowski M, 1993 Nucl. Phys. B 396 611
[16] Sutherland B, 1975 Phys. Rev. B 12 3795
[17] Schloettman P, 1987 Phys. Rev. B 36 5177
[18] Schulz H J, 1990 Phys. Rev. Lett. 64 2831
[19] Essler F H L and Korepin V E, 1992 Phys. Rev. B 46 9147
[20] Bares P-A, Blatter G and Ogata M, 1991 Phys. Rev. B 44 130
[21] Mischenko A S and Nagaosa N, 2004 Phys. Rev. Lett. 93 036402
[22] Martinez G and Horsch P, 1991 Phys. Rev. B 44 317
[23] Kawakami N and Yang S-K, 1990 Phys. Rev. Lett. 65 2309
[24] Williams E D, 1993 arXiv:cond-mat/9304009v1
[25] Jüttner G, Klümpfer A and Suzuki J, 1997 Nucl. Phys. B 487 650
[26] Giamarchi T and Hluillier C, 1991 Phys. Rev. B 43 12
[27] Emery V J, Kivelson S A and Lin H Q, 1991 Phys. Rev. Lett. 64 475
[28] Foerster A and Karowski M, 1993 Nucl. Phys. B 408 512
[29] Gonzalez-Ruiz A, 1994 Nucl. Phys. B 24 166
[30] Essler F H L, 1996 J. Phys. A: Math. Gen. 29 6183

doi:10.1088/1742-5468/2014/04/P04031
Exact solution of the one-dimensional super-symmetric $t$--$J$ model with unparallel boundary fields

[30] Wang Y, Dai J, Hu Z and Pu F-C, 1997 Phys. Rev. Lett. 79 1901
[31] Fan H, Hou B and Shi K, 1999 Nucl. Phys. B 541 483
[32] Zhou Y-K and Batchelor M T, 1997 Nucl. Phys. B 490 576
[33] Fan H and Wadati M, 2001 Nucl. Phys. B 599 561
[34] Fan H, Wadati M and Yue R, 2000 J. Phys. A: Math. Gen. 33 6187
[35] Bedu¨ rftig G and Frahm H, 1999 J. Phys. A: Math. Gen. 32 4585
[36] Hu Z-N, Pu F-C and Wang Y, 1998 J. Phys. A: Math. Gen. 31 5241
[37] Galleas W, 2007 Nucl. Phys. B 777 352
[38] Babujian H M and Flume R, 1994 Mod. Phys. Lett. A 9 2029
[39] Babujian H M, Foerster A and Karowski M, 2008 J. Phys. A: Math. Theor. 41 275202
Babujian H M, Foerster A and Karowski M, 2012 J. Phys. A: Math. Theor. 45 055207
[40] Lima-Santos A, 1999 Nucl. Phys. B 558 637
[41] Yang C N, 1967 Phys. Rev. Lett. 19 1312
[42] Cao J, Yang W-L, Shi K and Wang Y, 2013 Phys. Rev. Lett. 111 137201
[43] Cao J, Yang W-L, Shi K and Wang Y, 2013 Nucl. Phys. B 875 152
[44] Cao J, Yang W-L, Shi K and Wang Y, 2013 Nucl. Phys. B 877 152
[45] Sklyanin E K, 1988 J. Phys. A: Math. Gen. 21 2375
[46] Li Y-Y, Cao J, Yang W-L, Shi K and Wang Y, 2014 Nucl. Phys. B 879 98
[47] de Vega H J and González-Ruiz A, 1993 J. Phys. A: Math. Gen. 26 L519
[48] Ghoshal S and Zamolodchikov A B, 1994 Int. J. Mod. Phys. A 9 3841
[49] Jiang Y, Cui S, Cao J, Yang W-L and Wang Y, Completeness and Bethe root distribution of the spin-1/2 Heisenberg chain with arbitrary boundary fields, 2013 arXiv:1309.6456
[50] Nepomechie R I and Wang C, 2014 J. Phys. A: Math. Theor. 47 032001
[51] Alcaraz F, Barber M, Batchelor M, Baxter R and Quispel G, 1987 J. Phys. A: Math. Gen. 20 6397
[52] Kapustin A and Skorik S, 1995 J. Phys. A: Math. Gen. 29 1629

doi:10.1088/1742-5468/2014/04/P04031