Multiple attribute decision making based on Pythagorean fuzzy Aczel-Alsina average aggregation operators

Tapan Senapati1 · Guiyun Chen1 · Radko Mesiar2,3 · Abhijit Saha4

Received: 15 July 2021 / Accepted: 28 July 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
A useful expansion of the intuitionistic fuzzy set (IFS) for dealing with ambiguities in information is the Pythagorean fuzzy set (PFS), which is one of the most frequently used fuzzy sets in data science. Due to these circumstances, the Aczel-Alsina operations are used in this study to formulate several Pythagorean fuzzy (PF) Aczel-Alsina aggregation operators, which include the PF Aczel-Alsina weighted average (PFAAWA) operator, PF Aczel-Alsina order weighted average (PFAAOWA) operator, and PF Aczel-Alsina hybrid average (PFAAHA) operator. The distinguishing characteristics of these potential operators are studied in detail. The primary advantage of using an advanced operator is that it provides decision-makers with a more comprehensive understanding of the situation. If we compare the results of this study to those of prior strategies, we can see that the approach proposed in this study is more thorough, more precise, and more concrete. As a result, this technique makes a significant contribution to the solution of real-world problems. Eventually, the suggested operator is put into practice in order to overcome the issues related to multi-attribute decision-making under the PF data environment. A numerical example has been used to show that the suggested method is valid, useful, and effective.

Keywords
Aczel-Alsina operations · Pythagorean fuzzy elements · Pythagorean fuzzy Aczel-Alsina average aggregation operators · MADM

1 Introduction
Multi-attribute decision-making (MADM) issues are worth focusing on in many sorts of fields, for example, management, engineering, and economics. Conventionally, it becomes apparent that the data which gives access to the alternatives in the form of criteria and weight are stated in real numbers. But as a result of the unpredictability of the framework in our everyday lives, it is very hard for the decision-makers to settle on an ideal choice since the majority of the preferred value throughout the decision-making procedure is penetrated by uncertainty. To be able to handle the uncertainties, the IFS (Atanassov 1986) theory is among the fruitful augmentations of the fuzzy set theory (Zadeh 1965), which will be described by the membership and non-membership degrees. Over the last four decades, the IFS has gained increasing attention by presenting various types of aggregation operators and data quantified and used to address decision-making concerns. In contrast, the challenge of this research is to live up to the expectation that they are substantial just for those conditions whose degree sum is less than one. Be that as it may, in everyday
life, there are numerous circumstances where this condition is precluded. For example, if an individual assigns their inclination by means of membership and non-membership degrees concerning a specific object to 0.7 and 0.5, then it is clear that this case is not dealing with IFS. Yager (2013) considered the PFS by diminishing this sum condition to its square sum of less than one. By way of illustration, in accordance with the aforementioned example, we observe that \((0.7)^2 + (0.5)^2 = 0.74 < 1\), and thus PFS is an expansion of the prevailing IFS.

Yager and Abbasov (2013) presented the PF weighted averaging (PFWA) operator and the PF ordered weighted averaging (PFOWA) operator as a result of their pioneering efforts. Peng and Yang (2016) brought the Choquet integral into PF information aggregation. Zhang and Xu (2014) presented the thought of TOPSIS strategy utilizing PF numbers. Garg (2016b) offered a new accuracy function under interval-valued PF conditions in finding a solution to MADM issues. Wu and Wei (2017) developed several PF aggregation operators by utilizing Hamacher operations. Garg (2018) presented many ways for solving critical decision-making issues under the PF condition with immediate probabilities. Under PF conditions, Wei together with colleagues implemented various aggregation operators (Lu et al. 2017; Lu and Wei 2017) to handle MADM issues. Aydin et al. (2020) developed a harmonic aggregation operator for trapezoidal PF numbers. Liang and Xu (2017) presented a novel expansion of the TOPSIS technique for MCDM issues with hesitant PFS. Rahman et al. (2017) introduced PF weighted Einstein geometric aggregation operators. Liang et al. (2018) stretched out Bonferroni’s mean to PFS and utilized it to illuminate valuable application issues. Ren and his associates Ren et al. (2016) suggested the PF TODIM approach for MCDM issues. Zhang (2016) presented an MCDM strategy on the basis of the notion of the similarity measure. Gou et al. (2016) analyzed the features of continuous PFS. Li and Zeng (2018) suggested many different distance measures for PFS, which consist of the four parameters of PFS. Zeng et al. (2018) developed a PFIOWAWA operator for MADM issues.

Menger (1942) developed the concept of triangular norms (abbreviated as \(t\)-NMs) in his theory of probability-based topological spaces. It turns out that the \(t\)-NMs and their associated triangular conorms (in short, \(t\)-CNMs) are crucial operations in PFS, for instance, Einstein \(t\)-NM and \(t\)-CNM (Garg 2016a), Archimedean \(t\)-NM and \(t\)-CNM (Sarkar and Biswas 2019), Lukasiewicz \(t\)-NM and \(t\)-CNM (Venkatesan and Sriram 2019), Hamacher \(t\)-NM and \(t\)-CNM (Wei et al. 2018), etc. Klement and Mesiar (2000) conducted a detailed examination of the features and accompanying perspectives of \(t\)-NMs. Aczel and Alsina (1982) introduced novel operations called Aczel-Alsina \(t\)-NM (AA \(t\)-NM) and Aczel-Alsina \(t\)-CNM (AA \(t\)-CNM), which set high premium on parameter variation. Recently, Senapati and his associates have opened new horizons in decision-making theories using the AA \(t\)-NMs. They applied AA \(t\)-NMs to decision-making difficulties under IFS (Senapati et al. 2022a), interval-valued IFS (Senapati et al. 2022b), hesitant fuzzy (Senapati et al. 2022c), picture fuzzy set (Senapati 2022) environments.

Based on these examples and conversations, it is said that PFS has the successful unwavering quality to show the disputable and plausible data that rises in everyday life. The aforementioned decision-making issues in different fuzzy aggregation settings under \(t\)-NM and \(t\)-CNM motivated us to a great extent to produce this paper. The Aczel-Alsina operational rules are advanced mathematical operations that can be used to their benefit when familiarized with erroneous and ambiguous data. As a solution to the problems caused by PF MADM, we were motivated by these ideas to implement Aczel-Alsina operations on PF elements and to construct some PF Aczel-Alsina aggregation operators. The following are some of the ways in which the objectives of our method are conveyed:

1. Some new aggregation operators, namely PF Aczel-Alsina weighted average (PFAAWA) operator, PF Aczel-Alsina order weighted average (PFAAOWA) operator and PF Aczel-Alsina hybrid average (PFAAHA) operator, are proposed.
2. Study the characteristics and particular cases of these new operators.
3. Develop a technique to address MADM issues with PF data.
4. Propose a new MADM technique using the PFAAWA operator.
5. Prove the validity and superiority of the suggested technique.

The rest of this article is organised as follows: In Sect. 2, we introduce some basic concepts of PFS and Aczel-Alsina operators, including definitions, properties, and working rules. In Sect. 3, we propose Aczel-Alsina operations with respect to PFEs. In Sect. 4, we propose the PFAAHA operator, PFAAOWA operator, and PFAAHA operator. In the next Section, we contemplate a MADM technique based on the PFAAAWA operator. In Sect. 6, we explain the intended technique using a real-world case studies. In Sect. 7, we analyze the effect of a parameter on the decision-making results. In Sect. 8, we investigate the effect of criteria weights on ranking orders. Section 9 compares the proposed technique to other pertinent techniques in order to determine its suitability. Section 10 contains the conclusion.

## 2 Preliminaries

In this Section, we introduce a few important concepts connected with \(t\)-NM, \(t\)-CNM, Aczel-Alsina \(t\)-NM, and PFS.
2.1 t-NM, t-CNM, Aczel-Alsina t-NM

Generalizing the ideas of Menger (1942); Schweizer and Sklar (1960) proposed in 1960 the concept of t-NM. Note that their approach was done in the context of probabilistic metric spaces for generalizing the triangular inequality of metrics; however, within some years, they have been considered in several other branches, in particular in fuzzy set theory (there, t-NMs generate the fuzzy conjunctions, generalizing the original proposal of Zadeh (1965) considering the min operation when introducing the intersection of fuzzy sets). The dual operations to t-NMs, namely t-CNMs, were considered already in the framework of probabilistic metric spaces, but later also to cover the fuzzy disjunctions (Schweizer and Sklar 1961). Later, t-NMs and t-CNMs were considered in various generalizations of fuzzy set theory, such as interval-valued fuzzy set theory and fuzzy type-2 theory (Goguen 1967), IFS theory (Atanassov and Gargov 1989), IVIFS theory (Atanassov and Gargov 1989), and so on. For more details concerning t-NMs and t-CNMs, we recommend the monograph by Klement and Mesiar (2000).

Definition 1 Schweizer and Sklar (1960); Klement and Mesiar (2000) Consider $F : [0, 1]^2 \rightarrow [0, 1]$ to be a commutative, associative and monotone function. Then, if $e = 1$ is its neutral element, $F(x, 1) = F(1, y) = x$ for all $x \in [0, 1]$, $F$ is said to be a t-NM. Similarly, if $e = 0$ is its neutral element, i.e., $F(x, 0) = F(0, x) = x$ for all $x \in [0, 1]$, then $F$ is said to be a t-CNM.

To have a clear distinction between t-NMs and t-CNMs in notation, we will consider the traditional notation $T$ for t-NMs and $S$ for t-CNMs. Note that these two classes are dual, i.e., for any t-NM $T$, the mapping $S : [0, 1]^2 \rightarrow [0, 1]$ presented by $S(x, y) = 1 - T(1 - x, 1 - y)$ is a t-CNM (also named a t-CNM dual to $T$), and for any t-CNM $S$, the mapping $T : [0, 1]^2 \rightarrow [0, 1]$ presented by $T(x, y) = 1 - S(1 - x, 1 - y)$ is a t-NM (t-NM dual to $S$).

It is self-evident that the strongest (greatest) t-NM is $T_M(x, y) = \min(x, y)$ following the notation from Klement and Mesiar (2000), while the smallest t-NM is the drastic product $T_D$, that is vanishing on $[0, 1]^2$ (clearly, if $\max(x, y) = 1$ then for any t-NM we have $T(x, y) = \min(x, y)$). Two prototypical t-NMs playing a vital role both in theory and applications are the product t-NM $T_P$ (standard product of reals), and the Lukasiewicz t-NM $T_L$ given by $T_L(x, y) = \max(0, x + y - 1)$. One of the most distinguished subclasses of the class of all t-NMs is formed by the continuous Archimedean t-NMs, i.e., t-NMs generated by a continuous additive generator. Their importance is clearly visible when $n$-ary extensions of t-NMs are considered. For deeper results and more details, see Klement and Mesiar (2000). In our paper, we will deal with some specially generated t-NMs, namely with strict t-NMs which are isomorphic to the product t-NM, and which are generated by decreasing bijective additive generators $t : [0, 1] \rightarrow [0, \infty]$. In this instance, $T(x, y) = t^{-1}(t(x) + t(y))$, and, considering the $n$-array extension (which is unique due to the associativity of t-NMs), $T(x_1, \ldots, x_n) = t^{-1}(\sum_{i=1}^{n} t(x_i))$. Keep in mind that both extremal t-NMs $T_M$ and $T_D$, as well as the product t-NM $T_P$ commute with the power functions, i.e., for any $\lambda > 0$, they satisfy the equality $T(x^\lambda, y^\lambda) = T(x, y)^\lambda$. (Aczel and Alsina 1982) have characterized all other t-NM solutions of the above functional equation, showing that these are just strict t-NMs generated by additive generators $t_\varphi$, $\varphi \in [0, \infty]$, given by $t_\varphi(x) = (-\log x)^\varphi$. The related t-NMs are denoted as $T_\varphi$ and called (strict) Aczel-Alsina t-NMs, and given by

$$T_\varphi(x, y) = \begin{cases} T_D(x, y), & \text{if } \varphi = 0 \\ \min(x, y), & \text{if } \varphi = \infty \\ e^{-(\log x)^{\varphi} + (\log y)^{\varphi})^{1/\varphi}}, & \text{otherwise.} \end{cases}$$

Observe that including the extremal t-NMs, we obtain their Aczel-Alsina family ($T_\varphi$), $\varphi \in [0, \infty]$ of t-NMs which is strictly increasing and continuous in parameter $\varphi$.

Due to the duality, similar notes and examples can be introduced for t-CNMs. There, the smallest t-CNM is $S_M = \max$ (dual to $T_M$), and the greatest t-CNM is the drastic product $S_D$ which is constant $1$ on $[0, 1]^2$. For any t-CNM $S$, if $\min(x, y) = 0$, then $S(x, y) = \max(x, y)$, Dual t-CNM $S_L$ to $T_L$ (Lukasiewicz t-CNM, called also a truncated sum) is given by $S_L(x, y) = \min(1, x + y)$, and the dual t-CNM $S_P$ to the product $T_P$ (called a probabilistic sum) is given by $S_P(x, y) = x + y - xy$. Continuous Archimedean t-CNMs are also generated by additive generators (which are increasing), and if $S$ is dual to a continuous Archimedean t-NM $T$ generated by an additive generator $t$, then $S$ is generated by an additive generator $s$ given by $s(x) = t(1 - x)$. In particular, dual t-CNMs $S_\varphi$ to strict Aczel-Alsina t-noms $T_\varphi$ are generated by additive generators $s_\varphi(x) = (-\log(1 - x))^{\varphi}$, and they are given by

$$S_\varphi(x, y) = \begin{cases} S_D(x, y), & \text{if } \varphi = 0 \\ \max(x, y), & \text{if } \varphi = \infty \\ 1 - e^{-(\log(1 - x))^{\varphi} + (\log(1 - y))^{\varphi})^{1/\varphi}}, & \text{otherwise.} \end{cases}$$

Observe that including the extremal t-CNMs, we obtain their Aczel-Alsina family ($S_\varphi$), $\varphi \in [0, \infty]$ of t-CNMs which is strictly decreasing and continuous in parameter $\varphi$. 
2.2 Pythagorean fuzzy sets

Some fundamental principles of PFS are concisely discussed in this section.

**Definition 2** Given that \( X \) is a standard set, then a PFS \( \delta \) in \( X \) is denoted by

\[
\delta = \{ (x, \gamma_\delta(x), \nu_\delta(x)) | x \in X \},
\]

where \( \gamma_\delta(x) \) and \( \nu_\delta(x) \) are functions from \( X \) to the closed interval \([0, 1]\), such that \( 0 \leq \gamma_\delta^2(x) + \nu_\delta^2(x) \leq 1 \), for all \( x \in X \) and they represent the membership degree and non-membership degree of \( x \) to set \( \delta \) respectively. The value \( \pi_\delta(x) = \sqrt{1 - \gamma_\delta^2(x) - \nu_\delta^2(x)} \) is generally called the indeterminacy degree of the member \( x \) to set \( \delta \). It is clear that \( 0 \leq \pi_\delta(x) \leq 1 \), for all \( x \in X \).

For the convenience of our work we call \( \delta = \{ (x, \gamma_\delta(x), \nu_\delta(x)) | x \in X \} \) as Pythagorean fuzzy element (PFE) and we write \( \delta = (\gamma_{\delta}, \nu_{\delta}) \) in this short form.

For comparing two PFEs, a score and an accuracy function is denoted as

**Definition 3** Zhang and Xu (2014) The score function \( \tilde{S}(\delta) \) and accuracy function \( \tilde{L}(\delta) \) of a PFE \( \delta = (\gamma_{\delta}, \nu_{\delta}) \) can be computed as:

\[
\tilde{S}(\delta) = \gamma_{\delta}^2 - \nu_{\delta}^2, \ \text{where} \ \tilde{S}(\delta) \in [-1, 1]
\]

\[
\tilde{L}(\delta) = \gamma_{\delta}^2 + \nu_{\delta}^2, \ \text{where} \ \tilde{L}(\delta) \in [0, 1].
\]

Based on these two functions, Zhang and Xu introduced a comparison method for ranking the PFEs in the following manner

**Definition 4** Zhang and Xu (2014) Assume that \( \delta_1 = (\gamma_{\delta_1}, \nu_{\delta_1}) \) and \( \delta_2 = (\gamma_{\delta_2}, \nu_{\delta_2}) \) are any two PFEs. Let \( \tilde{S}(\delta_1) \) and \( \tilde{S}(\delta_2) \) be the score functions and \( \tilde{L}(\delta_1), \tilde{L}(\delta_2) \) be the accuracy functions of \( \delta_1 \) and \( \delta_2 \). Then

1. if \( \tilde{S}(\delta_1) < \tilde{S}(\delta_2) \), then \( \delta_1 < \delta_2 \)
2. if \( \tilde{S}(\delta_1) > \tilde{S}(\delta_2) \), then \( \delta_1 > \delta_2 \)
3. if \( \tilde{S}(\delta_1) = \tilde{S}(\delta_2) \), then \( \delta_1 \sim \delta_2 \).

**Definition 5** Let \( \delta = (\gamma_{\delta}, \nu_{\delta}), \delta_1 = (\gamma_{\delta_1}, \nu_{\delta_1}) \) and \( \delta_2 = (\gamma_{\delta_2}, \nu_{\delta_2}) \) be three PFEs in which basic operations defined Yager and Abbasov (2013); Yager (2013) in a subsequent way:

1. \( \delta_1 \oplus \delta_2 = (\max\{\gamma_{\delta_1}, \gamma_{\delta_2}\}, \min\{\nu_{\delta_1}, \nu_{\delta_2}\}) \)
2. \( \delta_1 \odot \delta_2 = (\min\{\gamma_{\delta_1}, \gamma_{\delta_2}\}, \max\{\nu_{\delta_1}, \nu_{\delta_2}\}) \)

\( \Theta \) Springer

Note that, following Klement and Mesiar (2018), there is a trivial isomorphism \( \phi \) between the lattice \( L^* \) of all intuitionistic values, \( L^* = \{ (a, b) | a, b \in [0, 1], a + b \leq 1 \} \) and the lattice \( P^* \) of all Pythagorean values, \( P^* = \{ (a, v) | a, v \in [0, 1], a^2 + v^2 \leq 1 \} \). For any \( \phi(a, b) = (u, v) \), \( u = \sqrt{a} \) and \( v = \sqrt{b} \). Though all the basic operations recalled in Definition 5 were originally introduced by Yager (2014), they can be deduced from the related operations on \( L^* \) introduced by Atanassov (1986), considering the isomorphism \( \phi \). Similarly, the operations for Pythagorean values presented and discussed in Sections 3 and 4, including the study of their properties, could be deduced from similar operations for intuitionistic values, when these operations are based on the product \( t \)-NM. Note that in such a case, one should deal with two isomorphisms, namely with \( \phi \) linking \( L^* \) and \( P^* \), and an isomorphism \( \tau \) relating the product \( t \)-NM with a strict Aczel-Alsina \( t \)-NM. To ensure the self-contentedness and readability of our contribution, we state the discussed operations and their properties independently of the already known results.

3 Aczel-Alsina operations on Pythagorean fuzzy elements

In view of Aczel-Alsina \( t \)-NM furthermore Aczel-Alsina \( t \)-CNM, we described Aczel-Alsina operations with respect to PFEs.

**Definition 6** Let \( \delta = (\gamma_{\delta}, \nu_{\delta}), \delta_1 = (\gamma_{\delta_1}, \nu_{\delta_1}), \) and \( \delta_2 = (\gamma_{\delta_2}, \nu_{\delta_2}) \) be three PFEs, \( \phi_0 > 0 \) and \( \phi_0 > 0 \). Then, the Aczel-Alsina \( t \)-NM and \( t \)-CNM operations of PFEs are defined as:

1. \[ \delta_1 \oplus \delta_2 = \left( \sqrt{1 - e^{-e^{-log(1-t_{\delta_1}^2)\phi_0} + (-log(1-t_{\delta_2}^2)\phi_0)\phi_0} \phi_0}, \right. \]
2. \[ \left. \sqrt{e^{-e^{-log(1-t_{\delta_1}^2)\phi_0} + (-log(1-t_{\delta_2}^2)\phi_0)\phi_0} \phi_0} \right) \]
Multiple attribute decision making based on Pythagorean fuzzy Aczel-Alsina average aggregation operators

(ii) $\delta_1 \boxtimes \delta_2 = \left( \sqrt[3]{e^{-((-\log y_1^2 + (\log z_1^2)^2))))/3}}, \sqrt[3]{e^{-((\log (0.68)^2)+(-\log (0.49)^2)^2)))}} \right)$

(iii) $\phi\delta = \left( \sqrt[3]{1 - e^{-((-\log (0.34)^2)^2+(\log (0.49)^2)^2)))}}, \sqrt[3]{e^{-((-\log (0.68)^2)^2+(\log (0.49)^2)^2)))}} \right)$

(iv) $\delta_\varphi = \left( \sqrt[3]{e^{-((-\log (0.34)^2)^2+(\log (0.49)^2)^2)))}}, \sqrt[3]{1 - e^{-((\log (0.68)^2)^2+(\log (0.49)^2)^2)))}} \right)$

Example 1 Let $\delta = (0.75, 0.43)$, $\delta_1 = (0.34, 0.68)$ and $\delta_2 = (0.55, 0.49)$ be three PFEs, then using Aczel-Alsina operation on PFEs as defined in Definition 6 for $\varphi = 5$ and $\varphi = 3$, we get

(i) $\delta_1 \oplus \delta_2 = \left( \sqrt[3]{1 - e^{-((-\log (0.34)^2)^2+(\log (0.49)^2)^2)))}}, \sqrt[3]{e^{-((\log (0.68)^2)^2+(\log (0.49)^2)^2)))}} \right) = (0.550210184, 0.486839404)$

(ii) $\delta_1 \boxtimes \delta_2 = \left( \sqrt[3]{e^{-((-\log (0.34)^2)^2+(\log (0.49)^2)^2)))}}, \sqrt[3]{1 - e^{-((\log (0.68)^2)^2+(\log (0.49)^2)^2)))}} \right) = (0.336264471, 0.680824522)$

(iii) $3\delta = \left( \sqrt[3]{1 - e^{-((3\log (1-0.75)^2)^2)))}}, \sqrt[3]{e^{-((3\log (0.43)^2)^2)))}} \right) = (0.801828188, 0.349462287)$

(iv) $\delta^3 = \left( \sqrt[3]{e^{-((3\log (0.75)^2)^2)))}}, \sqrt[3]{1 - e^{-((3\log (1-0.43)^2)^2)))}} \right) = (0.69881135, 0.474170502)$

Theorem 1 Let $\delta = (y_\theta, o_\theta)$, $\delta_1 = (y_\theta, o_\theta)$ and $\delta_2 = (y_\theta, o_\theta)$ be three PFEs, then we have

(i) $\delta_1 \oplus \delta_2 = \delta_1 \oplus \delta_2$

(ii) $\delta_1 \boxtimes \delta_2 = \delta_2 \boxtimes \delta_1$

(iii) $\phi(\delta_1 \boxtimes \delta_2) = \phi\delta_1 \boxtimes \phi\delta_2$

(iv) $\phi(\delta_1 \oplus \delta_2) = \phi\delta_1 \oplus \phi\delta_2$

(v) $\phi(\delta_1 \boxtimes \delta_2 = \phi\delta_1 \boxtimes \phi\delta_2$

(vi) $\phi(\delta_1 \oplus \delta_2 = \phi\delta_1 \oplus \phi\delta_2$

The proof of Theorem 1 is provided in the Appendix.

4 PF Aczel-Alsina average aggregation operators

In this section, based on the new operational laws on PFEs, we present Aczel-Alsina average aggregation operators with PFEs, for instance, PFAAWA operator, PFAAOWA operator, and PFAAHA operator.

Definition 7 Let $\delta_\theta = (y_\theta, o_\theta)$ ($\theta = 1, 2, \ldots, \omega$) be several PFEs. Then PF Aczel-Alsina weighted average (PFAAWA) operator is a mapping $PFAWA : PFE^\omega \to PFE$ in such a way that

$PFAWA_\psi(\delta_1, \delta_2, \ldots, \delta_\omega) = \bigoplus_{\theta=1}^{\omega}(\psi_\theta \delta_\theta)$

$= \psi_1 \delta_1 \boxplus \psi_2 \delta_2 \boxplus \cdots \boxplus \psi_\omega \delta_\omega$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_\omega)^T$ is weight vector of $\delta_\theta$ ($\theta = 1, 2, \ldots, \omega$) with $\psi_\theta \in [0, 1]$ and $\sum_{\theta=1}^{\omega} \psi_\theta = 1$.

We may deduce the result given as Theorem 2 regarding the new operational criteria for PFEs described in (i)-(iv) of Definition 6.

Theorem 2 Let $\delta_\theta = (y_\theta, o_\theta)$ ($\theta = 1, 2, \ldots, \omega$) be several PFE elements, then accumulated value by use of PFAWA operator is also a PFE, and

$PFAWA_\psi(\delta_1, \delta_2, \ldots, \delta_\omega) = \bigoplus_{\theta=1}^{\omega}(\psi_\theta \delta_\theta)$

$= \left(\sqrt[3]{1 - e^{-\left(\sum_{\theta=1}^{\omega} \psi_\theta (-\log (1-\varphi))\right)^{1/\varphi}}}, \sqrt[3]{e^{-(-\log (0.68)^2)^2+(\log (0.49)^2)^2)}\right})$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_\omega)$ is weight vector of $\delta_\theta$ ($\theta = 1, 2, \ldots, \omega$) in a manner that $\psi_\theta \in [0, 1]$ and $\sum_{\theta=1}^{\omega} \psi_\theta = 1$. Springer
**Proof** We may establish Theorem 2 in the prescribed sequence using the mathematical induction technic:

(i) When $\omega = 2$, based on Aczel-Alsina operations of PFEs, we obtain

\[
\psi_1 \delta_1 = \left\langle \sqrt{1 - e^{-\psi_1 (\log (1 - \gamma_1^2))^{1/p}}}, \sqrt{e^{-\psi_1 (\log \gamma_1^2)^{1/p}}} \right\rangle \quad \text{and} \quad \psi_2 \delta_2 = \left\langle \sqrt{1 - e^{-\psi_2 (\log (1 - \gamma_2^2))^{1/p}}}, \sqrt{e^{-\psi_2 (\log \gamma_2^2)^{1/p}}} \right\rangle.
\]

Based on Definition 6, we obtain

\[
PFAAWA_\psi(\delta_1, \delta_2) = \psi_1 \delta_1 \bigoplus \psi_2 \delta_2
\]

\[
= \left\langle \sqrt{1 - e^{-\left(\sum_{\theta=1}^k \psi_\theta (\log (1 - \gamma_\theta^2))^{1/p}\right)}}, \sqrt{\sum_{\theta=1}^k e^{-\left(\sum_{\theta=1}^k \psi_\theta (\log \gamma_\theta^2)^{1/p}\right)}} \right\rangle
\]

Thus, (1) is holds for $\omega = k + 1$.

As a result of (i) and (ii), we can deduce that (1) appears to be true for any $\omega$.

In the following, we will go through some of the characteristics of the PFAWA operator.

**Theorem 3** (Idempotency Property) If $\delta_\theta = (\gamma_\delta, \nu_\delta)$ for all $\theta = 1, 2, \ldots, \omega$ are several equal PFEs, i.e., $\delta_\theta = \delta$ for all $\theta$, then $PFAWA_\psi(\delta_1, \delta_2, \ldots, \delta_\omega) = \delta$.

**Proof** Since $\delta_\theta = (\gamma_\delta, \nu_\delta) = \delta$ for all $\theta = 1, 2, \ldots, \omega$, then we have by equation (1),

\[
PFAWA_\psi(\delta_1, \delta_2, \ldots, \delta_\omega) = \left\langle \sum_{\theta=1}^\omega e^{-\left(\sum_{\theta=1}^\omega \psi_\theta (\log (1 - \gamma_\theta^2))^{1/p}\right)} \right\rangle
\]

Now for $\omega = k + 1$, then

\[
PFAWA_\psi(\delta_1, \delta_2, \ldots, \delta_k) = \sum_{\theta=1}^k (\psi_\theta \delta_\theta)
\]

\[
= \left\langle 1 - e^{-\left(\sum_{\theta=1}^k \psi_\theta (\log (1 - \gamma_\theta^2))^{1/p}\right)}, \left(\sum_{\theta=1}^k e^{-\left(\sum_{\theta=1}^k \psi_\theta (\log \gamma_\theta^2)^{1/p}\right)} \right) \right\rangle
\]

Thus, $PFAWA_\psi(\delta_1, \delta_2, \ldots, \delta_\omega) = \delta$ holds.

**Theorem 4** (Boundedness Property) Let $\delta_\theta = (\gamma_\delta, \nu_\delta)$ for all $\theta = 1, 2, \ldots, \omega$ be a selection of PFEs. Let
Multiple attribute decision making based on Pythagorean fuzzy Aczel-Alsina average aggregation...

\( \delta^- = \min(\delta_1, \delta_2, \ldots, \delta_\omega) \) and \( \delta^+ = \max(\delta_1, \delta_2, \ldots, \delta_\omega) \). Then, \( \delta^- \leq \text{PFAAWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) \leq \delta^+ \).

**Proof** Let \( \delta_\theta = (y_{\delta_1}, v_{\delta_1}) (\theta = 1, 2, \ldots, \omega) \) be a number of PFEs. Let \( \delta^- = \min(\delta_1, \delta_2, \ldots, \delta_\omega) = (y^-_{\delta}, v^-_{\delta}) \) and \( \delta^+ = \max(\delta_1, \delta_2, \ldots, \delta_\omega) = (y^+_{\delta}, v^+_{\delta}) \). We have, \( y^-_{\delta} = \min(y_{\delta_\theta}, y^+_{\delta}) = \max(\delta_\theta) \), and \( v^-_{\delta} = \min(y_{\delta_\theta}, y^+_{\delta}) = \max(\delta_\theta) \). Hence, there have the subsequent inequalities,

\[
\sqrt{1 - e^{(\sum_{\delta} w_{\delta}(\log_2(1 - y^-_{\delta}^2))^{1/p})}} \leq \text{PFAAWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) \leq \sqrt{1 - e^{(\sum_{\delta} w_{\delta}(\log_2(1 - y^+_{\delta}^2))^{1/p})}}.
\]

Therefore, \( \delta^- \leq \text{PFAAWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) \leq \delta^+ \).

**Theorem 5** (Monotonicity Property) Let \( \delta_\theta \) and \( \delta'_\theta \) \((\theta = 1, 2, \ldots, \omega)\) be two sets of PFEs, if \( \delta_\theta \leq \delta'_\theta \) for all \( \theta \), then \( \text{PFAAWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) \leq \text{PFAAWA}_\Phi(\delta'_1, \delta'_2, \ldots, \delta'_\omega) \).

Furthermore, based on the PFAAOWA operator above, we shall develop the PF Aczel-Alsina ordered weighted averaging (PFAOWA) operator as follows:

**Definition 8** Assume that \( \delta_\theta = (y_{\delta_\theta}, v_{\delta_\theta}) (\theta = 1, 2, \ldots, \omega) \) are several PFEs. A \( \omega \)-dimensional PFAOWA operator is a function \( \text{PFAOWA}_\Phi : \text{PFE}^\omega \rightarrow \text{PFE} \) alongside relaying vector \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_\omega)^T \) in such a way as to allow \( \Phi_\theta \in [0, 1] \), and \( \sum_{\theta=1}^\omega \Phi_\theta = 1 \). Therefore,

\[
\text{PFAOWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) = \bigoplus_{\theta=1}^\omega (\Phi_\theta \delta_{\kappa(\theta)})
\]

where \((\kappa(1), \kappa(2), \ldots, \kappa(\omega))\) are permutation of \((\theta = 1, 2, \ldots, \omega)\), for which \( \delta_{\kappa(\theta-1)} \geq \delta_{\kappa(\theta)} \) for all \( \theta = 1, 2, \ldots, \omega \).

The succeeding theorem is developed on the basis of the Aczel-Alsina product operation on PFEs.

**Theorem 6** Let \( \delta_\theta = (y_{\delta_\theta}, v_{\delta_\theta}) (\theta = 1, 2, \ldots, \omega) \) be a number of PFEs. A \( \omega \)-dimensional PFAOWA operator is a function \( \text{PFAOWA}_\Phi : \text{PFE}^\omega \rightarrow \text{PFE} \) in such a way as to allow \( \Phi_\theta \in [0, 1] \), and \( \sum_{\theta=1}^\omega \Phi_\theta = 1 \). Then,

\[
\text{PFAOWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) = \bigoplus_{\theta=1}^\omega (\Phi_\theta \delta_{\kappa(\theta)})
\]

where \((\kappa(1), \kappa(2), \ldots, \kappa(\omega))\) are permutation of \((\theta = 1, 2, \ldots, \omega)\), in such a manner \( \delta_{\kappa(\theta-1)} \geq \delta_{\kappa(\theta)} \) for all \( \theta = 1, 2, \ldots, \omega \).

The PFAOWA operator has some properties similar to those of the PFAAWA operator.

**Theorem 7** (Idempotency Property) If \( \delta_\theta = (y_{\delta_\theta}, v_{\delta_\theta}) (\theta = 1, 2, \ldots, \omega) \) are several PFEs, which are all equal, i.e. \( \delta_\theta = \delta \) for all \( \theta \), then \( \text{PFAOWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) = \delta \).

**Theorem 8** (Boundness Property) Let \( \delta_\theta = (y_{\delta_\theta}, v_{\delta_\theta}) (\theta = 1, 2, \ldots, \omega) \) be a selection of PFEs. Let \( \delta^- = \min \delta_\theta \) and \( \delta^+ = \max \delta_\theta \). Then

\( \delta^- \leq \text{PFAOWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) \leq \delta^+ \).

**Theorem 9** (Monotonicity Property) Let \( \delta_\theta = (y_{\delta_\theta}, v_{\delta_\theta}) \) and \( \delta'_\theta = (y'_{\delta_\theta}, v'_{\delta_\theta}) (\theta = 1, 2, \ldots, \omega) \) be two sets of PFEs, if \( \delta_\theta \leq \delta'_\theta \) for all \( \theta \), then \( \text{PFAOWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) \leq \text{PFAOWA}_\Phi(\delta'_1, \delta'_2, \ldots, \delta'_\omega) \).

**Theorem 10** (Commutativity Property) Let \( \delta_\theta = (y_{\delta_\theta}, v_{\delta_\theta}) \) and \( \delta'_\theta = (y'_{\delta_\theta}, v'_{\delta_\theta}) (\theta = 1, 2, \ldots, \omega) \) be two sets of PFEs, then \( \text{PFAOWA}_\Phi(\delta_1, \delta_2, \ldots, \delta_\omega) = \text{PFAOWA}_\Phi(\delta'_1, \delta'_2, \ldots, \delta'_\omega) \) where \( \delta'_\theta ((\theta = 1, 2, \ldots, \omega)) \) is any permutation of \( \delta_\theta \) \((\theta = 1, 2, \ldots, \omega)\).

In Definition 7, we realize that the PFAOWA operator weights only the PFEs, and in Definition 8, we realize that the PFAOWA operator weights only the ordered positions of the PFEs. As such, the above-mentioned two average operators indicate the weights in two ways. But the weight can be shown by one operator without being directed in two ways by two operators. To overcome this difficulty, we put forward the PF Aczel-Alsina hybrid averaging (PFAHA) operator, which weights both the given PFE and its ordered position.
Definition 9 Let \( \delta_\theta (\theta = 1, 2, \ldots, \omega) \) be a number of PFEs. An \( \omega \)-dimensional PF Aczel-Alsina hybrid average (PFAAHA) operator is a function \( \text{PFAAHA} : \text{PFE}^\omega \to \text{PFE} \), in such a way as to allow
\[
\text{PFAAHA}_{\psi, \phi} (\delta_1, \delta_2, \ldots, \delta_\omega) = \bigoplus_{\theta=1}^\omega (\Phi_\theta \delta_{\kappa(\theta)})
\]
where \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_\omega)^T \) is weighting vector closely related to the PFAAHA operator, with \( \Phi_\theta \in [0, 1] \) \((\theta = 1, 2, \ldots, \omega)\) and \( \sum_{\theta=1}^\omega \Phi_\theta = 1 \); \( \delta_\theta = \omega \psi_\theta \delta_\theta, \theta = 1, 2, \ldots, \omega \), \((\delta_{\kappa(1)}, \delta_{\kappa(2)}, \ldots, \delta_{\kappa(\omega)})\) is any permutation of weighted PFEs \((\delta_1, \delta_2, \ldots, \delta_\omega)\), so as to allow \( \delta_{\kappa(\theta-1)} \geq \delta_{\kappa(\theta)} (\theta = 1, 2, \ldots, \omega) \); \( \psi = (\psi_1, \psi_2, \ldots, \psi_\omega)^T \) is weight vector of \( \delta_\theta (\theta = 1, 2, \ldots, \omega) \), with \( \psi_\theta \in [0, 1] (\theta = 1, 2, \ldots, \omega) \) and \( \sum_{\theta=1}^\omega \psi_\theta = 1 \), and \( \omega \) is the balancing coefficient, which is involved in maintaining equilibrium.

The underlying theorem can be deduced using Aczel-Alsina operations with PFEs information.

Theorem 11 Assume that \( \delta_\theta (\theta = 1, 2, \ldots, \omega) \) is a family of PFEs. Then the value acquired by the PFAAHA operator remains a PFE, and
\[
\text{PFAAHA}_{\psi, \phi} (\delta_1, \delta_2, \ldots, \delta_\omega) = \bigoplus_{\theta=1}^\omega (\Phi_\theta \delta_{\kappa(\theta)})
\]

Proof Just like Theorem 2, we can conventionally get Theorem 11.

Theorem 12 The PFAWA and PFAAOWA operators are particular instances of the PFAAHA operator.
Proof (1) Assume \( \Phi = (1/\omega, 1/\omega, \ldots, 1/\omega)^T \). Then
\[
\text{PFAAHA}_{\psi, \phi} (\delta_1, \delta_2, \ldots, \delta_\omega)
\]

\[
= \Phi_1 \delta_{\kappa(1)} \bigoplus \Phi_2 \delta_{\kappa(2)} \bigoplus \cdots \bigoplus \Phi_\omega \delta_{\kappa(\omega)}
\]

\[
= \left( \frac{1}{\omega} \sum_{\theta=1}^\omega (\Phi_\theta \delta_{\kappa(\theta)}) \right)
\]

\[
= \psi_1 \delta_1 \bigoplus \psi_2 \delta_2 \bigoplus \cdots \bigoplus \psi_\omega \delta_\omega
\]

\[
= \text{PFAWA}_{\psi, \phi} (\delta_1, \delta_2, \ldots, \delta_\omega),
\]

(2) Assume \( \psi = (1/\omega, 1/\omega, \ldots, 1/\omega)^T \). Then \( \hat{\delta}_\theta = \delta_\theta (\theta = 1, 2, \ldots, \omega) \) and
\[
\text{PFAAHA}_{\psi, \phi} (\delta_1, \delta_2, \ldots, \delta_\omega)
\]

\[
= \Phi_1 \hat{\delta}_{\kappa(1)} \bigoplus \Phi_2 \hat{\delta}_{\kappa(2)} \bigoplus \cdots \bigoplus \Phi_\omega \hat{\delta}_{\kappa(\omega)}
\]

\[
= \Phi_1 \delta_{\kappa(1)} \bigoplus \Phi_2 \delta_{\kappa(2)} \bigoplus \cdots \bigoplus \Phi_\omega \delta_{\kappa(\omega)}
\]

\[
= \text{PFAAOWA}_{\phi} (\delta_1, \delta_2, \ldots, \delta_\omega),
\]

which completes the proof.

5 Model for MADM using PF data

In this section, we will use the suggested operators to remedy a MADM problem in a PF context.

5.1 Formal illustration of MADM using PFEs

MADM approaches provide decision alternatives by examining the tradeoffs of alternative exhibits across several attributes (Hwang 1981). A MADM technique requires attribute values or performance measures (individual assessments of alternatives against each attribute), attribute weights (considering the importance of each attribute to the entire decision problem), and a system to integrate these data into an accumulated value or evaluation for each alternative. Simply put, the goal of a MADM problem is to find a good compromise solution among all the possible options that can be evaluated on a number of different criteria.

Here, we may suggest a MADM process for regulating PF Aczel-Alsina aggregation operators in which attribute values are represented by PFEs and attribute weights are represented by real numbers. Assume \( \sigma = \{ \sigma_1, \sigma_2, \ldots, \sigma_m \} \) are discretely arranged alternatives, \( \chi = \{ \chi_1, \chi_2, \ldots, \chi_m \} \) are discretely arranged attributes, and \( \psi = (\psi_1, \psi_2, \ldots, \psi_\omega) \) is the weight vector of the attribute \( \chi_\theta (\theta = 1, 2, \ldots, \omega) \) so that \( \psi_\theta \in [0, 1] \) and \( \sum_{\theta=1}^\omega \psi_\theta = 1 \). Consider

\[
R = (\gamma_{\delta, \alpha})_{m \times \omega} = (\gamma_{\delta, \alpha}, v_{\delta, \alpha})_{m \times \omega}
\]

to be the PF decision matrix, as shown in Fig. 1, where \( \gamma_{\delta, \alpha} \) represent membership degree by which alternative \( \alpha \) fulfills the attribute \( \chi_\delta \), and \( v_{\delta, \alpha} \) represent non-membership degree by which alternative \( \alpha \) does not fulfill the attribute \( \chi_\delta \) where \( \gamma_{\delta, \alpha} \subseteq [0, 1] \), and \( v_{\delta, \alpha} \subseteq [0, 1] \) allowing \( 0 \leq \gamma_{\delta, \alpha} = v_{\delta, \alpha} \leq 1 \), \((s = 1, 2, \ldots, m)\) and \((\theta = 1, 2, \ldots, \omega)\).
Multiple attribute decision making based on Pythagorean fuzzy Aczel-Alsina average aggregation…

$$R = \left( \delta_{s\theta} \right)_{m \times n}$$

$$= \begin{pmatrix}
\chi_1 \\
(\gamma_{s11},v_{s11}) \\
(\gamma_{s12},v_{s12}) \\
(\gamma_{s1m},v_{s1m}) \\
\vdots \\
(\gamma_{s21},v_{s21}) \\
(\gamma_{s22},v_{s22}) \\
(\gamma_{s2m},v_{s2m}) \\
\vdots \\
(\gamma_{sm1},v_{sm1}) \\
(\gamma_{sm2},v_{sm2}) \\
(\gamma_{smm},v_{smm})
\end{pmatrix}$$

Fig. 1 Pythagorean fuzzy decision matrix

5.2 Algorithms of the proposed method

In the following algorithm, we develop a method for identifying the best alternative(s) in light of the PFAAWA operators for the MADM problem, which involves the following steps:

Step I: For a MADM issue involving PFEs, we construct the PF decision matrix $R = \left( \delta_{s\theta} \right)_{m \times n}$ where the components $\delta_{s\theta}(s = 1, 2, \ldots, m; \theta = 1, 2, \ldots, \omega)$ are the appraisals of the alternative $O_s$ regarding the criterion $X_{\theta}$.

Step II: In the event that there are several kinds of criteria, for example, benefit (B) and cost (C), at that point we convert the PF decision matrix $R = \left( \delta_{s\theta} \right)_{m \times n}$ into the normalized one $\tilde{R} = \left( \sigma_{s\theta} \right)_{m \times n}$ by utilizing the accompanying equation

$$\sigma_{s\theta} = \begin{cases}
\delta_{s\theta}, & \theta \in B \\
\delta_{s\theta}^c, & \theta \in C,
\end{cases}$$

where $\delta_{s\theta}^c$ is the complement of $\delta_{s\theta}$.

Step III: On the basis of the decision matrix $\tilde{R}$, as derived from Step II, the total accumulated value of the alternative $O_s (s = 1, 2, \ldots, m)$ under the different criteria $X_{\theta}$ is acquired by utilizing PFAAWA operator and hence get the general decision values $O_s (s = 1, 2, \ldots, m)$ for each alternative $O_s$, i.e.,

$$O_s = \text{PFAAWA}(\sigma_{s1}, \sigma_{s2}, \ldots, \sigma_{sn}) = \sum_{\theta=1}^{\omega} (\psi_{\theta} \sigma_{s\theta})$$

where $\psi_{\theta}$ are the weights of the criteria.

Step IV: We calculate the score values $S(\sigma_s)$ ($s = 1, 2, \ldots, m$) based on the general PF data $O_s$ ($s = 1, 2, \ldots, m$) with the purpose of ranking all the alternative $O_s (s = 1, 2, \ldots, m)$ and selecting flawless choice $O_s$. If we find equal score functions $S(\sigma_s)$ and $S(\sigma_\theta)$, we proceed to calculate accuracy degrees of $\hat{L}(\sigma_s)$ and $\hat{L}(\sigma_\theta)$ based on the general PF information of $O_s$ and $O_\theta$, and then order the choice $O_s$ according to the accuracy degrees of $\hat{L}(\sigma_s)$ and $\hat{L}(\sigma_\theta)$.

Step V: We rank all the alternative $O_s (s = 1, 2, \ldots, m)$ in accordance with the descending value of the score values and accordingly, select the most desirable alternative.

Step VI: End.

6 Numerical example

In this section, we will present a MADM issue to demonstrate the execution and flexibility of the proposed technique.

Consider the following scenario: a multinational company in China is developing a plan for its monetary system for the upcoming year in accordance with a group strategy target. As a result of their preliminary evaluation, the following five alternatives are identified as follows: ($O_1$) investing in the “Central Asian markets”; ($O_2$) investing in the “East Asian markets”; ($O_3$) investing in the “South Asian markets”; ($O_4$) investing in the “Southeast Asian markets”; and ($O_5$) investing in the “West Asian markets.” This evaluation continues from the four components, which are as $X_1$: “the growth analysis”; $X_2$: “the social-political impact analysis” $X_3$: “the risk analysis”; and $X_4$: “the environmental impact analysis”. The decision makers evaluate the five possibilities $O_\theta(\theta = 1, 2, \ldots, 5)$ in the context of PF information under four attributes, whose attributes weight is $\psi = (0.2, 0.1, 0.3, 0.4)^T$, where decision matrix $\tilde{R} = \left( \sigma_{s\theta} \right)_{5 \times 4}$ which is illustrated in Table 1, where $O_s$ are expressed in terms of PFEs.

Considering that attributes $X_2$ and $X_3$ are the cost attributes and all others are benefit attributes, we considered that using Eq. (2), transformed the PF decision matrix $\tilde{R}$ into the subsequently normalized matrix $\tilde{R}$, indicated in Table 2.

Table 1 PF decision matrix

| $O_1$ | $O_2$ | $O_3$ | $O_4$ | $O_5$ |
|-------|-------|-------|-------|-------|
| 0.80,0.30 | 0.55,0.49 | 0.15,0.50 | 0.78,0.31 | 0.69,0.39 |
| 0.51,0.55 | 0.57,0.39 | 0.26,0.74 | 0.76,0.23 | 0.54,0.25 |
| 0.23,0.76 | 0.43,0.46 | 0.25,0.77 | 0.26,0.74 | 0.21,0.84 |
| 0.35,0.74 | 0.68,0.31 | 0.86,0.23 | 0.66,0.34 | 0.61,0.42 |

Table 2 Normalized PF decision matrix

| $O_1$ | $O_2$ | $O_3$ | $O_4$ | $O_5$ |
|-------|-------|-------|-------|-------|
| 0.80,0.30 | 0.55,0.49 | 0.15,0.50 | 0.78,0.31 | 0.69,0.39 |
| 0.55,0.51 | 0.39,0.57 | 0.74,0.26 | 0.23,0.76 | 0.25,0.54 |
| 0.76,0.23 | 0.46,0.43 | 0.77,0.25 | 0.74,0.26 | 0.84,0.21 |
| 0.35,0.74 | 0.68,0.31 | 0.86,0.23 | 0.66,0.34 | 0.61,0.42 |
7 Analysis of the influence of parameter $\phi$ on decision-making consequences

To illustrate the effect of the operating parameters $\phi$ on MADM findings, we shall employ several $\phi$ estimations to evaluate the alternatives. The outcomes of the score function and order of priority of the alternatives $\sigma_s (\theta = 1, 2, \ldots, 5)$ in the range of $1 \leq \phi \leq 100$ on the basis of the PFAAWA operator are exhibited in Table 3, and expressed graphically in Fig. 2.

As seen in Table 3, if the magnitude of $\phi$ for the PFAAWA operator is altered, the order of preferences changes, but the best alternatives remain the same. When, $1 \leq \phi \leq 3$, the order of preference is $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$. When $4 \leq \phi \leq 100$, the corresponding ranking is $\sigma_3 > \sigma_5 > \sigma_1 > \sigma_4 > \sigma_2$, but the best one is $\sigma_3$.

### Table 3 The ranking order of the alternatives with different parameter $\phi$ by PFAAWA operator

| $\phi$ | $\hat{S}(\sigma_1)$ | $\hat{S}(\sigma_2)$ | $\hat{S}(\sigma_3)$ | $\hat{S}(\sigma_4)$ | $\hat{S}(\sigma_5)$ | Order of preferences |
|--------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 1      | 0.248018             | 0.176464             | 0.512284             | 0.371592             | 0.378986             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 2      | 0.356500             | 0.209747             | 0.564775             | 0.402436             | 0.437581             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 3      | 0.414427             | 0.236557             | 0.589449             | 0.419831             | 0.48135              | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 4      | 0.447909             | 0.257328             | 0.605307             | 0.432182             | 0.514435             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 5      | 0.469421             | 0.273331             | 0.616799             | 0.441966             | 0.539259             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 6      | 0.484481             | 0.285783             | 0.625600             | 0.450123             | 0.557952             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 7      | 0.495723             | 0.295618             | 0.632552             | 0.457102             | 0.572216             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 8      | 0.504523             | 0.303513             | 0.638160             | 0.463168             | 0.583297             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 9      | 0.511661             | 0.309947             | 0.642759             | 0.468495             | 0.59207              | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 10     | 0.517610             | 0.315266             | 0.646583             | 0.473212             | 0.599146             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 50     | 0.57143              | 0.356031             | 0.677496             | 0.524526             | 0.649410             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |
| 100    | 0.579286             | 0.361173             | 0.682051             | 0.532679             | 0.655482             | $\sigma_3 > \sigma_5 > \sigma_4 > \sigma_1 > \sigma_2$ |

**Fig. 2** Score values of the alternatives for different values $\phi$ by PFAAWA operator
Table 4 Various weight sets of criteria

| Weight sets | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|-------------|---------|---------|---------|---------|
| S1          | 0.2     | 0.1     | 0.3     | 0.4     |
| S2          | 0.2     | 0.1     | 0.4     | 0.3     |
| S3          | 0.2     | 0.4     | 0.3     | 0.1     |
| S4          | 0.2     | 0.4     | 0.1     | 0.3     |
| S5          | 0.2     | 0.3     | 0.4     | 0.1     |
| S6          | 0.2     | 0.3     | 0.1     | 0.4     |
| S7          | 0.1     | 0.2     | 0.3     | 0.4     |
| S8          | 0.1     | 0.2     | 0.4     | 0.3     |

Table 5 Priority order of alternatives for diverse weight sets

| Ranking order | Ranking order | Ranking order |
|---------------|---------------|---------------|
| S1 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S9 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S17 $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4$ |
| S2 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S10 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S18 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |
| S3 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S11 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S19 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |
| S4 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S12 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S20 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |
| S5 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S13 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S21 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |
| S6 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S14 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S22 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |
| S7 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S15 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S23 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |
| S8 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S16 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ | S24 $\sigma_3 > \sigma_4 > \sigma_1 > \sigma_2$ |

Fig. 3 Final utility values of alternatives for various criteria weight sets

8 Sensitivity analysis (SA) of criteria weights

We propose a sensitivity analysis to explore the impact of criterion weights on the ranking order. This is accomplished by employing 24 different weight sets, namely S1, S2, ..., S24 (Table 4), which are produced by examining all possible combinations of the criteria weights $\psi_1 = 0.2$, $\psi_2 = 0.1$, $\psi_3 = 0.3$, $\psi_4 = 0.4$. This is notably essential to attain a broader range of criterion weights when assessing the influence of the constructed model. Figure 3 displays the cumulative scores of the alternatives, while Table 5 provides their relative ranking orders. Upon examining the ranking order of alternatives, it is seen that $\sigma_1$ holds the first rank in 87.5% of the scenarios when the PFAAWA operator (taking $\gamma = 2$) is applied. Hence, the priority of alternatives acquired by utilizing our developed method is credible.

9 Comparative analysis

In this section, we compare our suggested methods with other existing techniques, namely the PF weighted averaging (PFWA) operator (Zhang 2016), the PF Einstein weighted averaging (PFEWA) operator (Garg 2016a), the PF Hamacher weighted averaging (PFHWA) operator (Wu and Wei 2017), the PF Dombi weighted averaging (PFDWA) operator (Jana et al. 2019) and PF Einstein weighted geometric (PFWG*) operator (Rahman et al. 2017). The comparative results are listed in Table 6, and outlined graphically in Fig. 4. Such findings demonstrate the efficacy of the recommended operators and methodology. Furthermore, when compared to other authors’ operators and techniques, our operators and methodology have significant capabilities:

1. The PFWA operator is based on algebraic $t$-norm and $t$-conorm throughout the research (Zhang 2016), but the PFAAWA operator is based on AA $t$-norm and $t$-conorm in this paper. The PFWA operator developed in the literature (Zhang 2016) is a special instance of our suggested PFAAWA operator, according to
Tables 3 and 6, occurring when $\wp = 1$. As a result, the operators and procedures proposed in this study are still more broad and adaptable than any of those previously published (Zhang 2016).

(2) As shown in Table 3, the parameter $\wp$ indicates the decision makers’ inclinations, and the decision makers can select the appropriate value for $\wp$ based on their tendencies. By varying the value of the parameter $\wp$, we can construct distinct scoring functions and thus distinct ranks for the alternative. Thus, when used with parameters, the established aggregation operators focus on providing us with more options and versatility than the current aggregation operators (Zhang 2016; Garg 2016a; Wu and Wei 2017; Rahman et al. 2017), since they enable us to have positive variations for the parameter focusing on various real scenarios, which would be an intriguing topic and one that merits additional research.

10 Conclusions

Several new operating laws for PFEs depending on Aczel-Alsina $t$-NM and Aczel-Alsina $t$-CNM are proposed in this work. On the basis of these operating laws, new aggregation operators such as PFAAWA operator, PF-AAOWA operator, and PFAAHA operator are defined. Several characteristics of the suggested operators are discussed. The PFAAWA and PFAAOWA operators are special cases of the PFAAHA operator. In addition to evaluating these operators, a method has been developed to address MADM issues in the PF context. A mathematical example has been used to demonstrate the methodology’s viability and its reliable performance. From the results obtained by applying the aggregation operators, it is evident that we can utilise our method to solve the MADM problem in a very elegant manner. By comparing our method to other

| Techniques          | $\hat{S}(\wp_1)$ | $\hat{S}(\wp_2)$ | $\hat{S}(\wp_3)$ | $\hat{S}(\wp_4)$ | $\hat{S}(\wp_5)$ | Preference order |
|---------------------|------------------|------------------|------------------|------------------|------------------|-----------------|
| Zhang (2016)        | 0.248018254      | 0.176464107      | 0.512284086      | 0.37159183       | 0.37898593       | $\wp_4 > \wp_3 > \wp_2 > \wp_1 > \wp_5$ |
| Garg (2016a)        | 0.218347045      | 0.169569595      | 0.493425812      | 0.362376466      | 0.364774609      | $\wp_4 > \wp_3 > \wp_2 > \wp_1 > \wp_5$ |
| Wu and Wei (2017)   | 0.202065469      | 0.165659894      | 0.481081797      | 0.356747673      | 0.357003775      | $\wp_4 > \wp_3 > \wp_2 > \wp_1 > \wp_5$ |
| Jana et al. (2019)  | 0.545362494      | 0.335373981      | 0.66464365       | 0.496885228      | 0.630524594      | $\wp_4 > \wp_3 > \wp_2 > \wp_1 > \wp_5$ |
| Rahman et al. (2017)| 0.015472954      | 0.129345579      | 0.277830162      | 0.267524867      | 0.268023903      | $\wp_4 > \wp_3 > \wp_2 > \wp_1 > \wp_5$ |
| Proposed method     | 0.579285835      | 0.36117282       | 0.682050575      | 0.53269371       | 0.655481568      | $\wp_4 > \wp_3 > \wp_2 > \wp_1 > \wp_5$ |

Fig. 4 Comparison analysis with a few prevailing techniques
existing MADM approaches, we have demonstrated its benefits. The suggested operators offer a good new direction for the theory of quantitative studies and an easier way to deal with uncertainty during the decision-making process. We intend to apply the proposed framework to certain other domains in the future, including smart e-tourism applications, combating COVID-19, bridge construction techniques, software engineering pupil ability, selection of residential places, resolution of diversified and large data sets generated by patients with multiple chronic diseases, personalized individual semantics-based social networking, IoT-based real-time wearable health data sensors, biomass feedstock selection, and so on.

**Appendix**

**Proof of Theorem 1** For the three PFEs \( \delta, \delta_1 \) and \( \delta_2 \), and \( \varphi, \varphi_1, \varphi_2 > 0 \), as provided in Definition 6, we may obtain

(i) \[
\varphi_1 \delta_1 \ominus \delta_2 = \varphi \sqrt{1 - e^{-\varphi_1 \varphi \left( -\log(1 - \delta_1^2) + \left(-\log(1 - \delta_2^2)^p\right)^{1/p} \right)}}
\]

(ii) It is straightforward.

(iii) Let \( t = \sqrt{1 - e^{-\left(-\log(1 - \delta_1^2) + \left(-\log(1 - \delta_2^2)^p\right)^{1/p} \right)}} \).

Then \( \log(1 - t) = \left(-\log(1 - \delta_1^2) + \left(-\log(1 - \delta_2^2)^p\right)^{1/p} \right) \).

Using this, we get

\[
\varphi(\delta_1 \oplus \delta_2) = \varphi \sqrt{1 - e^{-\left(-\log(1 - \delta_1^2) + \left(-\log(1 - \delta_2^2)^p\right)^{1/p} \right)}}
\]

\[
\varphi(\delta_2 \oplus \delta_1) = \varphi \sqrt{1 - e^{-\left(-\log(1 - \delta_2^2) + \left(-\log(1 - \delta_1^2)^p\right)^{1/p} \right)}}
\]

\[
\varphi(\delta_1 \ominus \delta_2) = \varphi \sqrt{1 - e^{-\left(-\log(1 - \delta_1^2) + \left(-\log(1 - \delta_2^2)^p\right)^{1/p} \right)}}
\]

\[
\varphi(\delta_2 \ominus \delta_1) = \varphi \sqrt{1 - e^{-\left(-\log(1 - \delta_2^2) + \left(-\log(1 - \delta_1^2)^p\right)^{1/p} \right)}}
\]
(v) \[ (\delta_1 \otimes \delta_2)^p = \left\{ e^{-((\log r_{11}^{(p)})^p + (\log r_{22}^{(p)})^p)/p}, \sqrt{1 - e^{-((\log(1-r_{11}^{(p)}))^p + (\log(1-r_{22}^{(p)}))^p)/p}} \right\}^p \]

\[ = \left\{ e^{-((\log r_{11}^{(p)})^p + (\log r_{22}^{(p)})^p)/p}, \sqrt{1 - e^{-((\log(1-r_{11}^{(p)}))^p + (\log(1-r_{22}^{(p)}))^p)/p}} \right\}^p \]

\[ = \delta_1^p \otimes \delta_2^p \]

References

Aczel J, Alsina C (1982) Characterization of some classes of quasi-linear functions with applications to triangular norms and to synthesizing judgements. Aequationes Math 25(1):313–315

Alsina C, Frank MJ (2006) Schweizer B (2006) Associative Functions-Triangular Norms and Copulas. World Scientific Publishing, Danvers, MA

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

Atanassov KT, Gargov G (1989) Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31(3):343–349

Aydin S, Kahraman C, Kabak M (2020) Development of harmonic aggregation operator with trapezoidal Pythagorean fuzzy numbers. Soft Comput 24:11791–11803

Garg H (2016) A new generalized pythagorean fuzzy information aggregation using einstein operations and its application to decision making. Int J Intell Syst 31(9):886–920

Garg H (2016) A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-criteria decision making problem. J Intell Fuzzy Syst 31(1):529–540

Garg H (2018) Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment. J Intell Fuzzy Syst 31(4):687–712

Goguen JA (1967) L-fuzzy sets. J Math Anal Appl 14:145–174

Gou X, Xu Z, Ren P (2016) The properties of continuous Pythagorean fuzzy information. Int J Intell Syst 31:401–424

Hwang C (1981) Yoon K (1981) Multiple Attribute Decision Making: Methods and Applications. Springer, Berlin, Heidelberg

Jana C, Senapati T, Pal M (2019) Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision making. Int J Intell Syst 34:2019–2038

Klement EP, Mesiar R (2000) Pap E (2000) Triangular Norms. Kluwer Academic Publishers, Dordrecht

Klement EP, Mesiar R (2018) L-Fuzzy Sets and Isomorphic Lattices: Are All the “New” Results Really New? Mathematics 6:146; https://doi.org/10.3390/math6090146

Li D, Zeng W (2018) Distance measure of pythagorean fuzzy sets. Int J Intell Syst 33(2):348–361

Lu M, Wei G, Alsaaadi FE, Hayat T, Alsaaadi A (2017) Hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making. J Intell Fuzzy Syst 33(2):1105–1117

Lu M, Wei G (2017) Pythagorean uncertain linguistic aggregation operators for multiple attribute decision making. Int J Knowledge-Based Intell Eng Syst 21(3):165–179

Menger K (1942) Statistical metrics. Proceedings of the National Academy of Sciences USA 8:535–537

Peng X, Yang Y (2016) Pythagorean fuzzy Choquet integral based MABAC method for multiple attribute group decision making. Int J Intell Syst 31(10):989–1020

Rahman K, Abdullah S, Ahmed R, Ullah M (2017) Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making. J Intell Fuzzy Syst 33(3):635–647

Ren P, Xu Z, Gou X (2016) Pythagorean fuzzy TODIM approach to multi-criteria decision making. Appl Soft Comput 42:246–259

Sarkar A, Biswas A (2019) Multicriteria decision-making using Archimedean aggregation operators in Pythagorean hesitant fuzzy environment. Int J Intell Syst 34(7):1361–1386

Schweizer B, Sklar A (1960) Statistical metric spaces. Pacific J Math 10:313–334

Schweizer B, Sklar A (1961) Associative functions and statistical triangle inequalities. Publ Math Debrecen 8:169–186

Senapati T (2022) Approaches to multi-attribute decision making based on picture fuzzy Aczel-Alsina average aggregation

Funding information This work was supported by the National Natural Science Foundation of China (Grant No-12071376), the Slovak Research and Development Agency (Grant No. APVV-18-0052) and the IGA project of the Faculty of Science Palacky University Olomouc (Grant No-PrF2019015).
Multiple attribute decision making based on Pythagorean fuzzy Aczel-Alsina average aggregation operators. Comput Appl Math 41(40):1–28. https://doi.org/10.1007/s40314-021-01742-w

Senapati T, Chen G, Yager RR (2022a) Aczel-Alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making. Int J Intell Syst 37(2):1529–1551

Senapati T, Chen G, Mesiar R, Yager RR (2022b) Novel Aczel-Alsina operations-based interval-valued intuitionistic fuzzy aggregation operators and its applications in multiple attribute decision-making process. Int J Intell Syst 37(8):5059–5081

Senapati T, Chen G, Mesiar R, Yager RR, Saha A (2022c) Novel Aczel-Alsina operations-based hesitant fuzzy aggregation operators and their applications in cyclone disaster assessment. Int J Gen Syst 51(5):511–546

Venkatesan D, Sriram S (2019) On Lukasiewicz disjunction and conjunction of Pythagorean fuzzy matrices. Int J Comput Sci Eng 7(6):861–865

Wei G, Lu M, Tang X, Wei Y (2018) Pythagorean hesitant fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Int J Intell Syst 33(6):1197–1233

Wu S, Wei G (2017) Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Int J Knowl-Based Intell Eng Syst 21:189–201

Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers and decision making. Int J Intell Syst 28:436–452

Yager RR (2013) Pythagorean fuzzy subsets. Proc. Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, pp 57–61

Yager RR (2014) Pythagorean membership grades in multicriteria decision making. IEEE Trans Fuzzy Syst 22:958–965

Zadeh LA (1965) Fuzzy sets. Inform. Control 8:338–353

Zeng S, Mu Z, Balezentis T (2018) A novel aggregation method for Pythagorean fuzzy multiple attribute group decision making. J Intell Fuzzy Syst 33(3):573–585

Zhang X, Xu Z (2014) Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 29:1061–1078

Zhang X (2016) A novel approach based on similarity measure for pythagorean fuzzy multiple criteria group decision making. Int J Intell Syst 31:593–611

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.