SIMPLE CLOSED GEODESICS IN DIMENSIONS $\geq 3$

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Abstract. We show that for a generic Riemannian or reversible Finsler metric on a compact differentiable manifold $M$ of dimension at least three all closed geodesics are simple and do not intersect each other. Using results by Contreras [8] [9] this shows that for a generic Riemannian metric on a compact and simply-connected manifold all closed geodesics are simple and the number $N(t)$ of geometrically distinct closed geodesics of length $\leq t$ grows exponentially.

1. Results

On a compact differentiable manifold $M$ endowed with a reversible Finsler metric $f$ the corresponding norm of a tangent vector $v$ is given by $\|v\| = f(v)$. In the particular case of a Riemannian metric $g$ we have $\|v\|^2 = g(v,v)$. The Finsler metric is called reversible if $f(-v) = f(v)$ for all tangent vectors $v$. With a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ on a differentiable manifold equipped with a Finsler metric $f$ the iterates $c^m$ defined by $c^m(t) = c(mt)$ are closed geodesics, too. We call a closed geodesic $c$ prime, if there is no $m > 1$ such that $c = c_1^m$ for a closed curve $c_1$. For a a reversible Finsler metric we call closed geodesics $c_1, c_2$ geometrically equivalent, if their traces $c_1(S^1) = c_2(S^1)$ coincide. Otherwise we call the closed geodesics $c_1, c_2$ geometrically distinct. The orthogonal group $O(2)$ acts canonically on the parameter space $S^1$ of loops, hence also on the free loop space $\Lambda M$ of the manifold $M$ by isometries leaving the energy functional $E : \Lambda M \to \mathbb{R}$; $2E(\sigma) = \int_0^1 \|\sigma'(t)\|^2 dt$ invariant, cf. [12, Sec.2.3] for the Riemannian case and [17, Sec.2] for the Finsler case. The closed geodesics are the critical points of the energy functional. For an arbitrary closed geodesic $c$ there is a prime closed geodesic $c_1$, such that all geometrically equivalent closed geodesics are of the form $O(2).c_1^m, m \geq 1$. A prime closed geodesic $c$ of a reversible Finsler metric can have at most finitely many self-intersections, i.e. there are parameter values $s, t \in S^1, s \neq t$ with $c(s) = c(t)$, cf. Lemma 2. In this case $c'(s), c'(t)$ are linearly independent. We say a prime closed geodesic is simple, if it does not have self-intersections, i.e. if the map $c : S^1 \to M$ is injective. We say that two closed geodesics $c_1, c_2$ do not intersect if their traces are disjoint, i.e. the intersection set $I(c_1, c_2) = c_1(S^1) \cap c_2(S^1) = \emptyset$ is empty.

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The main result of this paper is the following

**Theorem 1.** Let $M$ be a compact differentiable manifold of dimension $n \geq 3$. For a $C^r$-generic Riemannian metric with $r \geq 2$, resp. a $C^r$-generic reversible Finsler metric with $r \geq 4$, all prime closed geodesics are simple, and geometrically distinct closed geodesics do not intersect each other.

The proof uses the *bumpy metrics theorem* for Riemannian metrics due to Abraham [1] with a detailed proof given by Anosov [2]. The corresponding result for Finsler metrics has been obtained by Taimanov and the author, cf. [18, Thm.3]. We state in Theorem 3 that the bumpy metrics theorem also holds for reversible Finsler metrics.

And we prove the following perturbation results: Let $c : [0, a]/\{0, a\} = \mathbb{R}/(a\mathbb{Z}) \to M$ be a prime closed geodesic of a Riemannian metric $g$ with $p = c(0)$ parametrized by arc length and hence of length $a$. In Lemma 1 we show that for an arbitrary small $\eta > 0$ we can define a one-parameter family of Riemannian metrics $g(s), s \in [0, \delta]$ for some $\delta > 0$ with $g(0) = g$ such that the metrics $g(s)$ and $g$ only differ in an arbitrary small tubular neighborhood $U$ of the geodesic segments $c : [-2\eta, -\eta] \to M$ and $c : [\eta, 2\eta] \to M$ and such that the metric $g(s)$ for $s > 0$ has a closed geodesic $c_s : [0, a]/\{0, a\} \to M$ parametrized by arc length which coincides with $c$ on the interval $[2\eta, a - 2\eta]$. Furthermore the geodesic segment $c_s|[-\eta, \eta]$ has positive distance from the geodesic segment $c|[-\eta, \eta]$. This local perturbation argument can be used on manifolds of dimension $n \geq 3$ to perturb away self-intersection and intersection points of distinct closed geodesics, respectively, as we show in Lemma 3. And it is also stated in Lemma 3 that the analogous result holds for reversible Finsler metrics.

We can combine the genericity statement of Theorem 1 with other genericity statements. Let $N(t)$ be the number of geometrically distinct, closed geodesics of length $\leq t$. The author has shown in [15] that a $C^2$-generic Riemannian metric on a compact and simply-connected manifold is strongly bumpy and carries infinitely many geometrically distinct closed geodesics. This result has been used by Contreras [8], [9] to show that for an open and dense set of Riemannian metrics on a compact and simply-connected manifold with respect to the $C^2$-topology the geodesic flow contains a non-trivial basic hyperbolic set. In particular this implies that $N(t)$ grows exponentially. Hence we obtain from Theorem 1 and [8], [9]:

**Theorem 2.** Let $M$ be a compact and simply-connected manifold of dimension $n \geq 3$. For a $C^2$-generic Riemannian metric all prime closed geodesics are simple and do not intersect each other. Furthermore the number $N(t)$ of closed geodesics of length $\leq t$ grows exponentially, i.e.

$$\liminf_{t \to \infty} \frac{(\log N(t))}{t} > 0.$$
For surfaces results are quite different, cf. Remark 5. For example a generic Riemannian or reversible Finsler metric of positive curvature on a two-dimensional sphere $S^2$ has only finitely many simple closed geodesics but infinitely many geometrically distinct closed geodesics with self-intersections. Surveys for existence results for closed geodesics are [3], [14] and [20].

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2. Perturbing a Single Geodesic Segment

For the proof of our perturbation result we introduce at first geodesic coordinates in a tubular neighborhood of a geodesic segment. We first discuss in detail the case of a Riemannian metric $g$. Later we explain which changes are necessary in case of a reversible Finsler metric $f$.

On a compact Riemannian manifold $(M, g)$ there is a positive number $\tilde{\eta} < \text{inj}/3$, where inj is the injectivity radius, such that the following holds for any point $p \in M$, any unit tangent vector $v \in T_p^1 M$, any $0 < \eta \leq \tilde{\eta}$ and any sufficiently small $\epsilon > 0$ with $0 < 7\epsilon < \eta$ : Let $T_p^{1,v} M := \{ x \in T_p M ; \langle v, x \rangle = 0 \}$ be the orthogonal complement of the one-dimensional subspace generated by $v$. And for a linear subspace $W \subset T_p M$ define $B_\epsilon(W) := \{ x \in W ; \| x \| < \epsilon, \langle x, v \rangle = 0 \}$. Then the restriction

$$(1) \quad \exp_{\epsilon,v} : (t, x) \in [-2\eta, 2\eta] \times B_\epsilon(T_p^{1,v} M) \longrightarrow \exp(t \nu(\exp_{\epsilon}(x))) \in M$$

of the normal exponential map is a diffeomorphism onto the tubular neighborhood $T_{\nu}(\eta, \epsilon) = \exp_{\epsilon,v} \left( [-2\eta, 2\eta] \times B_\epsilon(T_p^{1,v} M) \right)$ of the geodesic $c_v : [-2\eta, 2\eta] \longrightarrow M$ with $c_v(0) = p, c_v'(0) = v$. Here $\nu : \Sigma_{p,v}(\epsilon) \longrightarrow T^1 M$ is the unit normal vector field defined on the local hypersurface $\Sigma_{p,v}(\epsilon) = \exp_{\epsilon}(B_\epsilon(T_p^{1,v} M))$ with $\nu(p) = v$.

For $q \in M, r > 0$ we denote by $B_q(r)$ the geodesic ball around $q$ of radius $r$, i.e. $B_q(r) = \{ x \in M ; d(q, x) \leq r \}$. Here $d$ is the distance induced by the Riemannian metric $g$. For statements about Riemannian geometry we refer to [12, ch.1]. We define the spherical shell $A_p(\eta, \epsilon) = B_p(\epsilon + 7\epsilon) - B_p(\eta)$ around $p$. And for $v \in T_p^1 M$ we have the tubular neighborhood $T_{\nu}(\eta, \epsilon)$ and the sets $U^-_v(\eta, \epsilon) = \exp_{\epsilon,v} \left( (-\eta - 6\epsilon, -\eta) \times B_\epsilon(T_p^{1,v} M) \right)$ and $U^+_v(\eta, \epsilon) = \exp_{\epsilon,v} \left( (\eta, \eta + 6\epsilon) \times B_\epsilon(T_p^{1,v} M) \right)$. Hence the union $U_v(\eta, \epsilon) = U^-_v(\eta, \epsilon) \cap U^+_v(\eta, \epsilon)$ is the disjoint union of the two connected sets $U^+_v(\eta, \epsilon)$ and $U^-_v(\eta, \epsilon) \subset A_p(\eta, \epsilon)$. If now $w \in T_p^1 M$ is a unit vector orthogonal to $v$ we can choose a local isometry

$$(2) \quad \zeta = \zeta_w : (B_\epsilon(T_p^{1,v} M), g_p) \longrightarrow D^{n-1}(\epsilon)$$
between $B_\epsilon(T^1_p M)$ with the Riemannian metric $g_p$, and an Euclidean disc $D^{n-1}(\epsilon)$ of radius $\epsilon$ and dimension $(n - 1)$ with $\zeta_w(w) = e_2$. Here $(e_1, e_2, \ldots, e_n)$ is an orthonormal basis for $\mathbb{R}^n$ with coordinates $x = \sum_{i=1}^n x_i e_i$ and $D^{n-1}(\epsilon) = \{(0, x_2, \ldots, x_n) \in \mathbb{R}^n; \sum_{j=2}^n x_j^2 < \epsilon\}$. We obtain the following diffeomorphism:

$$\xi = \xi_{v, w} : [-2\eta, 2\eta] \times D^{n-1}(\epsilon) \longrightarrow Tb_v(\eta, \epsilon); \ (t, y) \longmapsto \exp^v(t\nu(\exp_p(\zeta_w^{-1}(y)))) .$$

By definition for sufficiently small $\epsilon > 0$ the curves with $y = \text{const}$ are geodesics parametrized by arc length. The subset

$$P(v, w) = \{\exp^v(t\nu(\exp_p(sw))); t \in [-2\eta, 2\eta], -\epsilon < s < \epsilon\}$$

is a local surface defined in the tubular neighborhood $Tb_v(\eta, \epsilon)$ of the geodesic $c$.

**Remark 1** (Finsler case, orthogonal complement, exponential map). For facts about Finsler metrics we refer to the books [4] and [19]. If $f : TM \longrightarrow \mathbb{R}$ is a Finsler metric one obtains in each point $p \in M$ a whole family $g_v$ of Riemannian metrics parametrized by unit tangent vectors $v \in T^1_p M$. This Riemannian metric is defined for $x, y \in T_p M$ as follows:

$$g_v(x, y) = \langle x, y \rangle_v = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \bigg|_{s=t=0} f^2(v + sx + ty).$$

Then the Legendre transformation

$$L : TM \longrightarrow T^* M; \ v \longmapsto \langle ., v \rangle_v$$

is defined. The orthogonal complement of a unit vector $v \in T^1_p M$ can be defined as the kernel of the linear form $L(v)$, i.e. the orthogonal complement of $v$ with respect to the Riemannian metric $g_v$. The exponential mapping $\exp_p : B_\epsilon(T^1_p M) \longrightarrow B_p(\epsilon) \subset M$ restricted to tangent vectors of length $< \epsilon$, where $\epsilon$ is smaller than the injectivity radius, is a $C^1$-diffeomorphism, outside 0 the map is $C^\infty$.

**Lemma 1.** Let $(M, g)$ be a compact Riemannian manifold, $p \in M$ and $v, w \in T^1_p M$ two unit vectors orthogonal to each other, i.e. $\langle v, w \rangle = 0$. Let $\eta > 0$ be sufficiently small, i.e. $\eta \leq \eta$, cf. the beginning of Section 2. Let $\epsilon > 0$ satisfy $0 < 6\epsilon < \eta$ and such that the map $\xi = \xi_{v, w}$ from Equation (3) is a diffeomorphism and $c(t) = \xi_{v, w}(t, 0) = \exp(tv), -2\eta \leq t \leq 2\eta$ is a geodesic parametrized by arc length.

Then there is a smooth one-parameter family of Riemannian metrics $g^{(s)}, s \in [0, \delta]$ for some sufficiently small $\delta > 0$ with $g = g^{(0)}$ such that the metric $g^{(s)}$ has a geodesic $c_s(t) = \xi(t, u_s(t), 0, \ldots, 0)$ with a scalar function $u_s : [-2\eta, 2\eta] \longrightarrow \mathbb{R}^{\geq 0}$ satisfying $u_s(t) = 0$ for all $t \in (-2\eta, -\eta-4\epsilon) \cup (\eta+4\epsilon, 2\eta)$ and $u_s(t) = s$ for all $t \in [-\eta-2\epsilon, \eta+2\epsilon]$ and sufficiently small $s > 0$, and $u'_s(t)t \leq 0$, i.e. $u_s$ is monotone increasing for $t \leq -\eta$ and monotone decreasing for $t \geq \eta$. In particular, for $s > 0$ the geodesic $c_s$ lies in the
local surface \( P(v,w) - \{p\} \) and has the same length \( 2\eta \) as \( c \). The metrics \( g^{(s)} \) and \( g \) coincide outside the set \( U_v(\eta, \epsilon) = U^-_v(\eta, \epsilon) \cup U^+_v(\eta, \epsilon) \).

**Proof.** Define for sufficiently small \( \delta > 0 \) the following vector field \( V(t,x_2,\ldots,x_n) \) for \((t,x) \in [-2\eta,2\eta] \times D^{n-1}(\epsilon) \) with \( x = (x_2,\ldots,x_n) :\)

\[
V(t,x_2,\ldots,x_n) = (0,\psi(t,x),0,\ldots,0) \quad \text{where} \quad (t,x) \in [-2\eta,2\eta] \times D^{n-1} \mapsto \psi(t,x) \in [0,1] \text{ is a smooth function, and} \quad \psi(t,x_2,0,\ldots,0) = 1 \text{ for } |t| \leq \eta + 2\epsilon, 0 \leq x_2 \leq \delta \text{ and } \psi(t,x) = 0 \text{ for } |t| \geq \eta + 4\epsilon \text{ or } |x| \geq 2\delta. \]

In addition we assume \( \psi(-t,x) = \psi(t,x) \) for all \( t, x \). This vector field determines an one-parameter group of diffeomorphisms \( \Psi^s : [-2\eta,2\eta] \times D^{n-1}(\epsilon) \longrightarrow [-2\eta,2\eta] \times D^{n-1}(\epsilon) \) with \( s \in \mathbb{R} \);

\[
\Psi^s(t,x) = (t,\tilde{\psi}(s,t,x)) \quad \text{for} \quad |t| \geq \eta + 4\epsilon \text{ or } \|x\| \geq \delta \text{ and } \tilde{\psi}(s,t,0,\ldots,0) = (t,s,0,\ldots,0) \text{ for } s \in [0,\delta] \text{ and } |t| \leq \eta + 2\epsilon. \]

In addition \( \tilde{\psi}^s(0) = (t,u_s(t),0,\ldots,0) \) with a smooth and even function \( t \mapsto u_s(t) \) which is monotone increasing for \( t < 0 \) and hence monotone decreasing for \( t > 0 \), and satisfies \( u_s(t) = s \) for \( 0 < s \leq \delta \) for all \( t \) with \( |t| \leq \eta + 2\epsilon \) and \( u_s(t) = 0 \) for all \( t \) with \( |t| \geq \eta + 4\epsilon \). With the help of the diffeomorphism \( \xi_{v,w} \) defined in Equation (3) we obtain a one-parameter group of diffeomorphisms \( \Psi^s : M \longrightarrow M \) with \( \Psi^s(y) = y \) for all \( y \in M - Tb_v(\eta, \epsilon) \) and \( s \geq 0 \), and

\[
(7) \quad \Psi^s(y) = \xi_{v,w} \left( \tilde{\psi}^s \left( \xi_{w,v}^{-1}(y) \right) \right)
\]

for all \( y \in Tb_v(\eta, \epsilon), s \in \mathbb{R} \). Hence for the geodesic \( c(t) = \xi_{v,w}(t,0), t \in [-2\eta,2\eta] \) we obtain

\[
(8) \quad c_s(t) = \Psi^s(c(t)) = \xi_{v,w}(t,u_s(t),0).
\]

And the curves \( c_s = c_s(t), c = c(t) \) coincide on \([-2\eta, -\eta - 4\epsilon] \cup [\eta + 4\epsilon, 2\eta]\).

For a positive number \( \theta > 0 \) choose smooth cut-off functions \( \beta_\theta, \beta_{\eta,\epsilon} : \mathbb{R} \longrightarrow [0,1] \) with \( \beta_\theta(-t) = \beta_\theta(t) \) and \( \beta_{\eta,\epsilon}(t) = \beta_{\eta,\epsilon}(-t) \) for all \( t \geq 0 \). The function \( \beta_\theta : [0,\infty] \longrightarrow \mathbb{R} \) is monotone decreasing and satisfies \( \beta_\theta(t) = 1 \) for all \( t \in [0,\theta] \), and \( \beta_\theta(t) = 0 \) for all \( t \geq 2\theta \). The function \( \beta_{\eta,\epsilon} : [0,\infty] \longrightarrow \mathbb{R} \) satisfies: \( \beta_{\eta,\epsilon}(t) = 1 \) for all \( t \) with \( |t| \in [\eta + \epsilon, \eta + 5\epsilon] \), and \( \beta_{\eta,\epsilon}(t) = 0 \) for all \( t \) with \( |t| \leq \eta \) or \( |t| \geq \eta + 6\epsilon \). And \( \beta_{\eta,\epsilon} \) is monotone increasing on \( [\eta, \eta + \epsilon] \) and monotone decreasing on \( [\eta + 5\epsilon, \eta + 6\epsilon] \). Then the smooth function \( \alpha = \alpha_{\eta,\epsilon,\delta} : M \longrightarrow [0,1] \) is defined as follows: \( \alpha(\xi_{v,w}(t,x)) = \beta_\delta(\|x\|)\beta_{\eta,\epsilon}(t) \) and \( \alpha(x) = 0 \) for \( x \not\in Tb_v(\eta, \epsilon) \). By definition of the function \( \alpha = \alpha_{\eta,\epsilon,\delta} \) it follows that \( \alpha \) vanishes outside the set \( U_v(\eta, \epsilon) \), i.e. \( \alpha(y) = 0 \) for \( y \not\in U_v(\eta, \epsilon) \). With this function we define the smooth perturbation \( g^{(s)}, s \in [0,s_0] \) of the metric \( g = g^{(0)} \) by the following convex combination of metrics:

\[
(9) \quad g^{(s)} = (1 - \alpha)g + \alpha(\Psi^{-s})^*(g).
\]

Here \( g_1^{(s)} = (\Psi^{-s})^*(g) \) is the pull-back of the Riemannian metric \( g \) via the diffeomorphism \( \Psi^{-s} \). By definition the mapping \( \Psi^s : (M,g) \longrightarrow (M,g_1^{(s)}) \) is an isometry between
these Riemannian manifolds. We conclude that the geodesic \( c : \mathbb{R} \to M \) with \( c(0) = p, c'(0) = v \) with respect to the metric \( g \) is mapped onto the geodesic \( c_s : \mathbb{R} \to M \) defined above of the metric \( g_1^{(s)} \). The restriction \( c_s : [-\eta - 2\epsilon, \eta + 2\epsilon] \to M \) is a geodesic parametrized by arc length of the metrics \( g \) and of the metric \( g_1^{(s)} = (\Psi^{-s})^*(g) \).

Hence \( c_s : [-\eta - 2\epsilon, \eta + 2\epsilon] \to M \) is also a geodesic of the convex combination \( g^{(s)} \) defined in Equation (9). This one can check easily since the energy \( E_s(\gamma) \) with respect to \( g^{(s)} \) of an arbitrary smooth curve \( \gamma(t) = \xi_{v,w}(t,x(t)) \), \( t \in [t_0, t_1] \subset [-\eta - 2\epsilon, \eta + 2\epsilon] \), with \( x(t_0) = x(t_1) \) satisfies \( E_s(\gamma) \geq (t_1 - t_0)/2 \) and the restriction \( c_s|_{[t_0, t_1]} \) satisfies \( E_s(c_s|_{[t_0, t_1]}) = (t_1 - t_0)/2 \). This is a consequence of the Gauß Lemma, cf. [12, sec.1.9] and the following Remark 2, since the curves \( t \in [-\eta - 2\epsilon, \eta + 2\epsilon] \to \xi_{v,w}(t,x) \in M \) for fixed \( x \in D^{n-1} \) are geodesics parametrized by arc length of the metric \( g \) and of the metric \( g_1^{(s)} \). Therefore the curve \( c_s : [-\eta - 2\epsilon, \eta + 2\epsilon] \to M \) is locally the shortest connection of its endpoints with respect to the metric \( g^{(s)} \), hence a geodesic. On the other hand the curve \( c_s : [-2\eta, 2\eta] \to M \) is a geodesic of \( g_1^{(s)} \), since \( \alpha(c_s(t)) = 1 \) for \( |t| \in [\eta + \epsilon, \eta + 5\epsilon] \) it also follows that \( c_s : [\eta + \epsilon, \eta + 5\epsilon] \to M \) as well as \( c_s : [-\eta - 5\epsilon, -\eta - \epsilon] \to M \) is also a geodesic of \( g^{(s)} \) since it is a geodesic of \( g_1^{(s)} \).

In the line following Equation (8) we already have seen that the curves \( c \) and \( c_s \) coincide on \([\eta + 4\epsilon, 2\eta]\) and \([-2\eta, -\eta - 4\epsilon]\). And since \( \Psi(\xi_w(t,x)) = \xi_{v,w}(t,x) \) for \( |t| \in [\eta + 4\epsilon, 2\eta] \) the metrics \( g, g_1^{(s)} \) and \( g^{(s)} \) coincide on the set \( \xi_{v,w}([-2\eta, -\eta - 4\epsilon] \cup [\eta + 4\epsilon, 2\eta]) \times D^{n-1} \).

Hence \( c_s : [-2\eta, 2\eta] \to M \) is a geodesic segment of the Riemannian metric \( g^{(s)} \).

We have shown that the smooth one-parameter family \( g^{(s)} \) of Riemannian metrics for sufficiently small \( s \) carries a geodesic \( c_s(t) \) with the following properties: \( c_s \) coincides outside \( Tb_v(\eta, \epsilon) \) with the geodesic \( c \) of the metric \( g \) and \( c_s(t) = \xi_{v,w}(t, u_s(t), 0, \ldots, 0) \) for \( t \in [-2\eta, 2\eta] \) and \( u_s(t) = s \) for \( t \in [-\eta - 2\epsilon, \eta + 2\epsilon] \). And \( u_s(t) = 0 \) for \( t \in [-2\eta, -\eta - 4\epsilon] \cup [\eta + 4\epsilon, 2\eta] \). In particular \( c_s([-2\eta, 2\eta]) \subset P(u,v) - \{p\} \), i.e. \( c_s \) lies in the local surface \( P(u,v) \) and avoids the point \( p \) for \( s > 0 \).

The arguments can be carried over to the case of a reversible Finsler metric. In particular the Finsler metric \( f^{(s)} \) corresponds to the convex combination of Riemannian metrics given in Equation (9):

\[
    f^{(s)} = \sqrt{(1 - \alpha)f^2 + \alpha((\Psi^{-s})^*(f))^2}.
\]

This is a Finsler metric, for which \( c_s(t) = \xi_{v,w}(t,0) = \xi_{v,w}(t, u_s(t), 0, \ldots, 0) \) is a geodesic as in the Riemannian case. For the Gauß Lemma in Finsler geometry cf. [4, Sec.6.1].

\[ \square \]

**Remark 2 (Convex combination of Riemannian and Finsler metrics).** (a) In the proof we used the following local statement in a Riemannian manifold. Let \( (t, x) \in [-\eta - 2\epsilon, \eta + 2\epsilon] \times D^{n-1} \) be local coordinates and let \( g^{(0)}, g^{(1)} \) be Riemannian metrics for
which the $t$-lines are geodesics parametrized by arc length starting orthogonally from the hypersurface $t = 0$. These coordinates are also called *geodesic parallel coordinates* based on the hypersurface $t = 0$. Then the convex combination

$$g^* = \alpha g^{(0)} + (1 - \alpha) g^{(1)}$$

for a smooth function $(t,x) \in [-\eta + 2\epsilon, \eta + 2\epsilon] \times D^{n-1} \mapsto \alpha = \alpha(t,x) \in [0,1]$ is a Riemannian metric, for which also the $t$-coordinate lines are geodesics parametrized by arc length, i.e. the coordinates $(t,x)$ are also geodesic parallel coordinates for the Riemannian metric $g^*$ based on the hypersurface $t = 0$. As indicated in the previous proof the argument is that the curve $t \in [t_0, t_1] \mapsto (t,x) \in [-\eta - 2\epsilon, \eta + 2\epsilon] \times D^{n-1}$ for a fixed $x$ is locally the shortest curve joining $(t_0,x)$ and $(t_1,x)$ as a consequence of the Gauß Lemma.

The statement follows also if you write down the line elements of the metrics $g^{(0)}$, $g^{(1)}$ by

$$dt^2 + \sum_{i,j=1}^{n-1} g^{(l)}_{ij}(t,x) dx^i dx^j, \quad l = 0, 1$$

with respect to the coordinates $(t,x_1, \ldots, x_{n-1})$ for $t \in [-\eta - 2\epsilon, \eta + 2\epsilon]$. Then the line element of $g^*$ is of the form

$$dt^2 + \sum_{i,j=1}^{n-1} g^*_{ij}(t,x) dx^i dx^j,$$

with metric coefficients $g^*_{ij}(t,x) = \alpha(t,x)g^{(0)}_{ij}(t,x) + (1 - \alpha(t,x)) g^{(1)}_{ij}(t,x)$. Therefore the lines with $x = \text{const}$ are geodesics of the metric $g^*$.

(b) The analogous statement for reversible Finsler metrics is the following: Let $(t,x) \in [-\eta + 2\epsilon, \eta + 2\epsilon] \times D^{n-1}$ be local coordinates and $f^{(0)}$, $f^{(1)}$ be reversible Finsler metrics for which the $t$-lines are geodesics parametrized by arc length and starting orthogonally from the hypersurface $t = 0$. Then the convex combination

$$f^* = \sqrt{\alpha f^{(0)} + (1 - \alpha) f^{(1)}}$$

for a smooth function $(t,x) \in [-\eta + 2\epsilon, \eta + 2\epsilon] \times D^{n-1} \mapsto \alpha(t,x) \in [0,1]$ is a reversible Finsler metric for which also the $t$-lines are geodesics parametrized by arc length. This follows from the Gauß Lemma for Finsler metrics, cf. [4, Sec.6.1]

3. Intersection of geodesics

The following statement is well known for Riemannian metrics, cf. for example [5, Sec.2.3], the statements carry over to the case of reversible Finsler metrics, as we will show:
Lemma 2. Let \( g \) be a Riemannian metric resp. let \( f \) be a reversible Finsler metric and \( c,d : S^1 \to M \) be prime closed geodesics, which are geometrically distinct.

(a) The set of double points, resp. self-intersection points

\[
DP(c) := \# \{ t \in S^1 ; \# c^{-1}(c(t)) \geq 2 \}
\]

of the closed geodesic \( c \) is finite.

(b) The intersection \( I(c,d) := c(S^1) \cap d(S^1) \) is finite.

Proof. Since \( c \) is an immersion and since \( S^1 \) is compact the set

\[
c^{-1}(c(t)) = \{ s \in S^1 ; c(s) = c(t) \}
\]

is finite for all \( t \in S^1 \).

(a) If the set \( DP(c) \) is not finite then there exist sequences \( s_j,t_j \in S^1 \) converging to \( s^*,t^* \in S^1 \) with \( s_j \neq s^*; t_j \neq t^*; s_j \neq t_j \) for all \( j \) and \( c(t_j) = c(s_j) \) for all \( j \geq 1 \). Then we conclude \( p := c(t^*) = c(s^*) \). If \( c'(t^*) = \pm c'(s^*) \) then the closed geodesic is not prime. Hence for \( v = c'(s^*) \) we have \( v \neq \pm w \). Since \( \|c'\| \) is constant, we obtain \( \|v\| = \|w\| \). The exponential map \( \exp_p : B_\eta(T_pM) \to B_\eta(p) \) is injective, we conclude: Since \( c(t_j) = \exp_p((t_j-t^*)v) = c(s_j) = \exp_p((s_j-s^*)w) \) for sufficiently large \( j \) with \( |t_j - t^*|\|v\|, |s_j - s^*|\|w\| < \text{inj} \) we conclude that \( (t_j - t^*)v = (s_j - s^*)w \) holds, i.e. \( v = \pm w \), which is a contradiction.

(b) The argument is similar, if the set \( I(c,d) \) is infinite, then there are sequences \( s_j,t_j \in S^1 \) converging to \( s^*,t^* \in S^1 \) with \( s_j \neq s^*; t_j \neq t^*; s_j \neq t_j \) for all \( j \) and \( c(t_j) = d(s_j) \) for all \( j \geq 1 \). Then we conclude \( p := c(t^*) = d(s^*) \). If \( c'(t^*) = \pm d'(s^*) \) then the closed geodesics \( c,d \) are geometrically equivalent. Hence for \( v = c'(s^*) \) we have \( v \neq \pm w \). Since \( \|c'\| \) is constant, we obtain \( \|v\| = \|w\| \). Since \( c(t_j) = \exp_p((t_j-t^*)v) = d(s_j) = \exp_p((s_j-s^*)w) \) we conclude for sufficiently large \( j \) with \( |t_j - t^*|\|v\|, |s_j - s^*|\|w\| < \text{inj} \) that \( (t_j - t^*)v = (s_j - s^*)w \) holds, i.e. \( v/\|v\| = \pm w/\|w\| \), which is a contradiction. \( \square \)

Remark 3 (Self-intersection of closed geodesics for non-reversible Finsler metrics).

Note that the last argument does not work for non-reversible Finsler metrics. If \( \gamma(t) \) is a geodesic with \( \gamma_v(0) = p, \gamma_v(0) = v, v \), then the equation \( \gamma_v(t) = \exp_p(tv) \) only holds for \( t \geq 0 \). In general \( \gamma_v(t) \neq \exp_p(-tv) = \gamma_{-v}(t) \). Note that two closed geodesics \( c_1, c_2 : S^1 \to M \) of a non-reversible Finsler metric are called geometrically equivalent only if their traces \( c_1(S^1) = c_2(S^1) \) and their orientations coincide. The Katok-examples as non-reversible perturbations of the standard Riemannian metric on a sphere yield metrics for which there are two geometrically distinct closed geodesics \( c_1, c_2 : S^1 \to S^n \) which have the same trace but different orientation and length. These metrics are explained in [21], [16, Sect.11], and [6]. Hence this is an example.
of a non-reversible Finsler metric for which there are geometrically distinct closed geodesics intersecting in an infinite number of points.

4. Perturbing intersecting geodesic segments

For a point \( p \in M \) and \( \eta, \epsilon > 0 \) with \( 6\epsilon < \eta \) we have defined the spherical shell \( A(\eta, \epsilon) = \{ x \in M : \eta < d(x, p) < \eta + 7\epsilon \} \) around \( p \). For a geodesic segment \( c_j : [-2\eta, 2\eta] \rightarrow M, j = 1, \ldots, N \) parametrized by arc length with \( p = c_j(0) \) and \( v_j := c'_j(0) \) we recall the definition of the sets \( U_j(\eta, \epsilon) = U_{v_j}(\eta, \epsilon) \) which are subsets of the of the spherical shell \( A(\eta, \epsilon) \) around \( p \), cf. Section 2 and Lemma 1.

**Lemma 3.** Let \( p \in M \) be a point on a compact manifold of dimension \( n \geq 3 \) with a Riemannian metric \( g \) or with a reversible Finsler metric \( f \). Assume that \( \eta > 0 \) satisfies \( \eta < \text{inj}/3 \), here \text{inj} is the injectivity radius. Let \( c_j : [-2\eta, 2\eta] \rightarrow M, j = 1, 2, \ldots, N \) be geodesic segments parametrized by arc length with \( c_j(0) = p \), for which the initial directions for \( j \neq k \) satisfy \( v_j = c'_j(0) \neq \pm c'_k(0) = \pm v_k \).

For sufficiently small \( \epsilon > 0 \) in any neighborhood of the metric \( g \) resp. \( f \) with respect to the strong \( C^r \)-topology with \( r \geq 2 \) resp. \( r \geq 4 \) there is a Riemannian metric \( \overline{g} \) resp. a reversible Finsler metric \( \overline{f} \) with geodesic segments \( \overline{c}_j : [-2\eta, 2\eta] \rightarrow M, j = 1, \ldots, N \) parametrized by arc length which coincide with \( c_j \) on the set \([-2\eta, -\eta - 4\epsilon] \cup [\eta + 4\epsilon, 2\eta]\]. These geodesic segments do not intersect each other, i.e. \( \overline{c}_j(S^1) \cap \overline{c}_k(S^1) = \emptyset \). The metrics \( g \) and \( \overline{g} \) resp. \( f \) and \( \overline{f} \) differ only on the union of the pairwise disjoint sets \( U_j(\eta, \epsilon), j = 1, \ldots, N \). This is a subset of the spherical shell \( A(\eta, \epsilon) \) around \( p \).

**Proof.** Since \( 3\eta < \text{inj} \) the geodesic segments are injective and the point \( p \) is the only point lying on distinct geodesic segments \( c_j \). Then one can choose \( \epsilon \in (0, \eta/6) \) sufficiently small as in Lemma 1 such that the maps \( \xi_{v_j,w_j} \) defined in Equation (3) are diffeomorphisms and the subsets \( U_j(\eta, \epsilon) \) of the spherical shell \( A(\eta, \epsilon) \) are pairwise disjoint.

We can choose \( \epsilon \in (0, \eta/6) \) sufficiently small such that the intersections \( T_{v_j}(\eta, \epsilon) \cap A(\eta, \epsilon), j = 1, 2, \ldots, N \) of the tubular neighborhoods \( T_{v_j}(\eta, \epsilon) \) of the geodesic \( c_j \) with the spherical shells \( A(\eta, \epsilon) \) are pairwise disjoint. Since the sets \( U_j(\eta, \epsilon) \) are subsets of the these intersections we conclude that also the sets \( U_j(\eta, \epsilon), j = 1, \ldots, N \) are pairwise disjoint.

If the dimension \( n \) is at least four, we can find unit vectors \( w_j, j = 1, \ldots, N \) which are pairwise distinct such that for sufficiently small \( \eta > 0, \epsilon \in (0, \eta/6) \) the local surfaces \( P_j = P(v_j, w_j), j = 1, 2, \ldots, N \) in a neighborhood of \( p \) defined by \( v_j, w_j \), cf. Equation 4, pairwise only meet in the point \( p \). I.e. \( (P_j - \{p\}) \cap (P_k - \{p\}) = \emptyset \) for all \( j, k, j \neq k \).
If the dimension $n = 3$ then the local surfaces $P_j, P_k$ for sufficiently small $\epsilon > 0$ for distinct $j \neq k$ intersect in a smooth curve

\begin{equation}
\gamma_{jk}(t) = \zeta_{v_j, w_j}(t_j, x_j(t), 0) = \zeta_{v_k, w_k}(t_k, x_k(t), 0)
\end{equation}

parametrized by arc length. Here the curve is defined on a sufficiently small interval $[u_1, u_2]$ with $u_1 < 0 < u_2$ such that $t_j, t_k, x_j, x_k : [u_1, u_2] \to \mathbb{R}$ are strictly monotone. This is possible since $\gamma'_{jk}(0)$ is neither a multiple of $c'_j(0)$ nor of $c'_k(0)$.

Without loss of generality we can assume that $v_j, w_j$ are orthogonal to each other, i.e. $\langle v_j, w_j \rangle = 0$ for all $j = 1, \ldots, N$.

It follows from Lemma 1 that for any $j = 1, \ldots, N$ there is a one-parameter family of Riemannian $g_j^{(s)}$, $s \in [0, s_0], s_0 > 0$ (resp. Finsler metrics) with $g_j^{(0)} = g$ which coincides with $g$ on the complement of $U_j^\pm(\eta, \epsilon)$ and satisfies the following: The metrics $g_j^{(s)}$ have injective geodesics $c_{j,s} : [-2\eta, 2\eta] \to M, j = 1, \ldots, N$ parametrized by arc length which coincide with $c_j$ on $[-2\eta, -\eta - 4\epsilon] \cup [\eta + 4\epsilon, 2\eta]$ and which are of the form $c_{j,s}(t) = \xi_{v_j, w_j}(t, u_{j,s}(t), 0, \ldots, 0)$ for $t \in [-2\eta, 2\eta]$ as described in Lemma 1. In particular the geodesic $c_{j,s}$ lies in the local surface $P_j$ and does not meet $p$ for $s > 0$, and for $t \in [-\eta, \eta]$ and sufficiently small positive $s$:

\begin{equation}
c_{j,s}(t) = \xi_{v_j, w_j}(t, s, 0, \ldots, 0).
\end{equation}

If $n \geq 4$ it follows that the geodesics $c_{j,s}$ do not intersect pairwise for sufficiently small and positive $s$, since the local surfaces $P_j, P_k$ for distinct $j \neq k$ do only intersect in the point $p$. If the dimension $n = 3$ the intersection of $P_j, P_k$ is described above, cf. Equation (10). Then we define for $s_1, \ldots, s_N > 0$ sufficiently small a metric $g^{(s_1, \ldots, s_N)}$ by $g_j^{(s_j)}$ on the set $U_j(\eta, \epsilon)$ and by $g$ outside the union $U_1(\eta, \epsilon) \cup \ldots \cup U_N(\eta, \epsilon)$.

From the form of the intersection of the local surfaces $P_j \cap P_k$ as described in Equation (10) and the form of the geodesics $c_{j,s}$ in Equation (11) we see that for a given $s > 0$ we can choose the parameters $s_1, s_2, \ldots, s_N \in (0, s)$ such that the geodesic segments $c_{j,s_j}(t) = \xi_{v_j, w_j}(t, s_j, 0), t \in [-\eta, \eta]$ for distinct $j, k$ do not intersect. This is possible since for distinct $j, k$ there are unique $s^*_{jk}, t^*_{jk}$ with $c_{j,s_j}(t) = c_{k,s_{jk}}(t^*_{jk})$, i.e. one has to choose $s_k \neq s^*_{jk}$, for all $k \neq j$, cf. the description of the intersection of the local surfaces $P_j \cap P_k$ given in Equation (10).

Remark 4. To understand the argument for dimension $n = 3$ it may be helpful to consider the following special case. Let $c_j : \mathbb{R} \to \mathbb{R}^3, j = 1, \ldots, N$ be $N$ pairwise distinct straight lines $c_j$ in Euclidean 3-space $\mathbb{R}^3$ intersecting in $p$. Then one can find $N$ pairwise distinct planes $E_j$ containing $c_j$ such that the intersection of the planes $E_j \cap E_k$ for distinct $j, k$ is a straight line $c_{j,k}$ which is different from any of the lines $c_j$. Then one can find for any $\epsilon > 0$ and any line $c_j, j = 1, \ldots, N$ a parallel line $c'_j$ lying in $E_j$ with distance $< \epsilon$ from $c_j$ such that the intersection $c'_j \cap c'_k$ for distinct $j, k$ is
empty. In the Proof the curves $c_j' = c_{j,s_j}$ correspond then to curves which near $p$ are the straight segments $c_j'$ and outside a spherical shell $A(\eta, \epsilon)$ coincide with the original straight line $c_j$. The case $N = 2$ is obvious, for $N \geq 3$ one has to choose the distances from $c_j$ and $c_j'$ to avoid intersections of the straight lines $c_j'$. Then the Lemma implies that we can perturb the Euclidean metric on $\mathbb{R}^3$ in a spherical shell around $p$ such that the curves $c_j''$ become geodesics.

5. Proof of Theorem 1

Proof. Let $\mathcal{G} = \mathcal{G}^r(M)$ be the space of Riemannian metrics with the strong $C^r$-topology with $r \geq 2$. A closed geodesic $c$ is non-degenerate if there is no periodic Jacobi field $Y = Y(t)$ which is orthogonal to the closed geodesic, i.e. $g(Y(t), c'(t)) = 0$ holds for all $t$. This also implies that 1 is not an eigenvalue of the linearized Poincaré map $P_c$. We conclude from the bumpy metrics theorem: For $a > 0$ the set $\mathcal{G}(a)$ of Riemannian metrics for which all closed geodesics with length $\leq a$ are non-degenerate, is an open and dense subset of $\mathcal{G} = \mathcal{G}(M)$.

It also follows that there are only finitely many geometrically distinct and prime closed geodesics $\tilde{c}_1, \ldots, \tilde{c}_r$ such that all closed geodesics of length $\leq a$ are geometrically equivalent to one of the closed geodesics $\tilde{c}_j, j = 1, \ldots, r$.

Let $\mathcal{G}^*(a) \subset \mathcal{G}(a)$ be the set of Riemannian metrics, such that all prime closed geodesics $\tilde{c}_1, \ldots, \tilde{c}_r$ of length $\leq a$ are simple and do not intersect each other. This is an open subset of the set $\mathcal{G}(a)$, since there are only finitely many geometrically distinct closed geodesics of length $\leq a$ in $\mathcal{G}^*(a)$, cf. [15, Lem.2.4], [2, §4]. It remains to prove that the set $\mathcal{G}^*(a) \subset \mathcal{G}(a)$ is dense.

It follows from Lemma 2 that the union $DIP(a) = DIP(\tilde{c}_1) \cup \ldots \cup DIP(\tilde{c}_r) \cup \bigcup_{j \neq k} I(\tilde{c}_j, \tilde{c}_k)$ of the double points $DIP(\tilde{c}_j), j = 1, \ldots, r$ of the prime closed geodesics of length $\leq a$ and the intersection points $I(\tilde{c}_j, \tilde{c}_k)$ for distinct $j, k$ is a finite set. Then we can find a sufficiently small $\eta > 0$ such that the geodesic balls $B_p(2\eta)$ for $p \in DIP(a)$ are disjoint and such that the following holds:

For $p \in DIP(a)$ the intersection $B_p(2\eta) \cap \{\tilde{c}_1(S^1) \cup \ldots \cup \tilde{c}_r(S^1)\}$ of the geodesic ball $B_p(2\eta)$ of radius $2\eta$ around $p$ and the traces of the prime closed geodesics $\tilde{c}_1, \ldots, \tilde{c}_r$ consists only of $N$ geodesic segments of length $4\eta$ with midpoint $p$. These are geodesic segments of one of the closed geodesics $\tilde{c}_1, \ldots, \tilde{c}_r$ if $p$ is a double point of this closed geodesic. If a geodesic $\tilde{c}_j$ enters the geodesic ball $B_p(2\eta)$, i.e. if $c_j(s) \in B_p(2\eta)$ for some $s \in S^1$ then there is a parameter $s_1$ with $|s - s_1| < 2\eta$ such that $p = \tilde{c}_j(s_1)$.

By a linear change of the parametrization these $N$ geodesic segments are geodesic segments $c_1, \ldots, c_N : [-2\eta, 2\eta] \to M$ parametrized by arc length with $c_j(0) = p$ and with $v_j = c_j'(0), j = 1, \ldots, N$, such that $v_j \neq \pm v_k$ for distinct $j \neq k$. 


These geodesic segments satisfy the assumptions of Lemma 3. Since the geodesic balls of radius $2\eta$ around the points $p \in DIP(a)$ are disjoint we can apply Lemma 3 for every point $p \in DIP(a)$ separately and change the metric in the geodesic ball $B_p(2\eta)$.

Hence we obtain in any neighborhood of $g \in \mathcal{G}(a)$ a metric $\overline{g}$ with prime closed geodesics $\tau_1, \ldots, \tau_r$. The length $L(\tau_j)$ equals the length $L(c_j)$ of $c_j$. And for every $j = 1, \ldots, N$ the closed geodesic $\tilde{c}_j$ coincides with $c_j$ outside the union

$$\bigcup_{p \in DIP(a)} B_p(2\eta)$$

of the geodesic balls of radius $2\eta$ around the finitely many points $p \in DIP(a)$.

In any sufficiently small neighborhood of a metric $g \in \mathcal{G}(a)$ the number of closed geodesics of length $\leq a$ cannot increase, since the closed geodesics seen as periodic orbits of the geodesic flow are non-degenerate, cf. [2, §4, i]). Therefore the geodesics $\tau_1, \ldots, \tau_r$ are the prime closed geodesics of length $\leq a$ of the metric $\overline{g}$ up to geometric equivalence. Since the geodesics $\tau_j$, $j = 1, \ldots, N$ are simple and do not intersect each other we have shown that $\overline{g} \in \mathcal{G}^*(a)$. Therefore we have shown that in any neighborhood of the metric $g$ there is a metric $\overline{g} \in \mathcal{G}^*(a)$.

Then the intersection

$$\mathcal{G}^* = \bigcap_{k \in \mathbb{N}} \mathcal{G}^*(k)$$

is a residual subset. The set $\mathcal{G}^*(k)$ is the set of Riemannian metrics for which the finitely many geometrically distinct prime closed geodesics of length $\leq k$ are simple, do not intersect each other and are non-degenerate. Therefore $\mathcal{G}^*$ is the set of Riemannian metrics for which all closed geodesics are non-degenerate and all prime closed geodesics are simple. In addition distinct closed geodesics do not intersect.

The argument in the Finsler case is the same, we only have to use the following bumpy metrics theorem for reversible Finsler metrics:

\begin{proof}
We consider the space $\mathcal{F}^{\text{rev}}_{C^r}(M)$ of reversible Finsler metrics on the compact manifold $M$ with respect to the strong $C^r$-topology. The only necessary modification in the proof of [18, Thm.3] is that we have to choose the function $\phi : \mathbb{R}^n \to \mathbb{R}$ in addition to be even, i.e. $\phi(-y) = \phi(y)$ for all $y \in \mathbb{R}^n$.
\end{proof}

\begin{remark}[Surfaces]
The case of dimension 2, i.e. surfaces, is quite different. One can show that for any Riemannian metric and any reversible Finsler metric on a closed surface there exists a simple closed geodesic. If the surface is not simply-connected one can see easily that the shortest non-contractible closed geodesic is simple. If the
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surface is simply-connected it is the famous result by Lusternik and Schnirelman [13] that there exist three simple closed geodesics. For a detailed recent proof, which also works for reversible Finsler metrics, see [10]. Calabi and Cao have shown that the shortest closed geodesic of a convex surface is simple, cf. [7]. For a convex surface with a Riemannian metric with sectional curvature $K \geq \delta > 0$ a simple closed geodesic has length $\leq 2\pi/\sqrt{\delta}$, this result is due to Toponogov, cf. [12, 3.4.10]. Hence a $C^r$-generic Riemannian metric on $S^2$ of positive curvature with $r \geq 2$ has only finitely many geometrically distinct, simple closed geodesics. On any convex surface two closed geodesics intersect, this statement holds for Riemannian metrics as well as for reversible Finsler metrics, cf. [6]. On the other hand there are non-reversible Finsler metric of positive flag curvature with two simple closed geodesics which do not intersect, cf. [17] and [6].

In case of negative curvature on a surface of genus $g$ the number $N(t)$ of closed geodesics of length $\leq t$ grows exponentially, whereas the number $N_1(t)$ of simple closed geodesics of length $\leq t$ grows polynomially of order $6g - 6$. Mirzakhani has been able to compute the asymptotic growth rate for $N_1(t)$, cf. [11].

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