Holographic viscoelastic hydrodynamics

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Relativistic hydrodynamics:

- ideal hydrodynamics,

\[ T^{\mu\nu} \equiv T^{\mu\nu}_{eq} = \epsilon \, u^\mu u^\nu + P(\epsilon) \, \Delta^{\mu\nu}, \quad u^\mu u_\mu = -1, \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \]

\{\epsilon, P\} — energy density and pressure of the fluid, \( u^\mu \) — local fluid 4-velocity;

- Navier—Stokes hydrodynamics,

\[ T^{\mu\nu} = T^{\mu\nu}_{eq} - \eta(\epsilon) \, \sigma^{\mu\nu} - \zeta(\epsilon) \, \Delta^{\mu\nu} \, (\nabla \cdot u) \]

\{\eta, \zeta\} — shear and bulk viscosities; \( \sigma^{\mu\nu} = \mathcal{O}(\nabla^\mu u^\nu) \)
all-orders,

$$T^{\mu\nu} \equiv T_{eq}^{\mu\nu} + \Pi^{\mu\nu} (\nabla u, \{(\nabla u)^2, \nabla^2 u\}, \cdots)$$

We will be interested in $n \to \infty$ order in the hydrodynamic expansion, i.e., focusing on terms $(\nabla u)^n$ or more generally

$$(\nabla^{k_1} u)^{p_1} (\nabla^{k_2} u)^{p_2} \cdots (\nabla^{k_m} u)^{p_m}$$

with $k_1p_1 + k_2p_2 + \cdots k_mp_m = n$

Too many indices, and too many different ways to describe flows....
We take the following steps to simplify index structure of the observables:

- we focus on the entropy density $s$ production rate,

$$\frac{d}{dt} \ln(s) = \frac{1}{T} S (\nabla u, \{(\nabla u)^2, \nabla^2 u\}, \cdots)$$

$$S = \left[ (\nabla \cdot u)^2 \frac{\zeta}{s} + \frac{2\eta}{s} \sigma_{\mu\nu}\sigma^{\mu\nu} \right] + \cdots$$

- and a specific flow, i.e., the homogeneous and isotropic expansion:

$$u^\mu = (1, 0, 0, 0), \quad \nabla_\mu u^\mu = 3\frac{\dot{a}}{a} = 3H = \text{const}$$

This flow can be alternatively though as a co-moving frame expansion of the fluid in de Sitter Universe

$$ds^2 = -dt^2 + a^2(t) \, d\mathbf{x}^2, \quad a(t) = e^{Ht}$$

Notice that for such a flow

$$\sigma^{\mu\nu} \equiv 0$$
The full co-moving entropy production is due to conformal symmetry breaking:

\[ \mathcal{L} = \mathcal{L}_{CFT} + \lambda_{4-\Delta} \mathcal{O}_\Delta \]

where \( \Delta \) is a dimension of the CFT breaking operator,

\[ \frac{d}{dt} \ln(a^3 s) \propto \frac{H^2}{T} \left( \frac{\lambda_{4-\Delta}}{T^{4-\Delta}} \right)^2 \Omega^2_\Delta \]

\[ \Omega_\Delta = \Omega_\Delta (\nabla u, \{(\nabla u)^2, \nabla^2 u\}, \cdots) = \Omega_\Delta (\frac{H}{T}) \]

for some models of holographic QGP fluids we can explicitly compute

\[ \Omega_\Delta = \sum_{n=0}^{\infty} c_n \left( \frac{H}{T} \right)^n \]

and find

\[ \frac{c_{n+1}}{c_n} \propto (n + 4 - \Delta) \quad \implies \quad c_n \propto \Gamma(n + 4 - \Delta) \sim n! \]
Thus:

- hydrodynamic expansion for fluids has zero radius of convergence
- the series in the derivative expansion can be Borel-resummed
- the poles in the Borel transform identify that the physical reason for the asymptotic character of the hydrodynamics are the non-hydrodynamic excitation in fluids (black brane QNMs in the dual holographic picture)

this is an old story [Michal Heller+Romuald Janik+, 2013]

Now, an even older story [Alex Buchel+Jim Sethna, 1996]:
Recall the Hooke’s Law:

\[ F = k \, x \]

where \( k \) is a spring constant.

Of course, if can not be a full story:

\[ F = k \, x + k_2 \, x^2 + k_3 \, x^3 + \cdots \]

where \( k_i \) are non-linear elastic coefficients.

We argued that in brittle materials (those that can develop cracks under the stress), the Hooke’s Law is the first term in otherwise asymptotic series, \( i.e., \)

Elastic theory has zero radius of convergence.
Specifically,

- consider the fully non-linear in external pressure $P$ expression for the bulk modulus $K$ of a solid:

$$\frac{1}{K(P)} = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = c_0 + c_1 \, P + c_2 \, P^2 + \cdots$$

- $c_0$ represents the Hooke’s Law and $c_i, i \geq 1$ are higher-order coefficients

- as $n \to \infty$, for 2D elastic materials at temperature $T$, the crack surface tension $\alpha$, Yong’s modulus $Y$ and the Poisson’s ratio $\sigma$,

$$\frac{c_{n+1}}{c_n} \rightarrow -n^{1/2} \left( \frac{\pi T (1 - \sigma^2)}{8 Y \alpha^2} \right)^{1/2}$$

or

$$c_n \propto \Gamma\left(\frac{n+1}{2}\right) \sim \left(\frac{n}{2}\right)!$$
Elastic theory and hydrodynamics are similar:

- both have a well-defined effective description, akin to derivative expansion in EFT;
- both expansions are asymptotic series (gradient expansion in fluids, powers of strain expansion in solids);
- both have 'non-perturbative’ effects responsible for zero radius of convergence of effective description.

Elastic theory and hydrodynamics are different:

- non-perturbative effects in hydrodynamics: non-hydro modes in plasma
- non-perturbative effects in theory of elasticity: cracks
BUT solids and fluids are rather different:

- there is no shear in fluids; as a result the transverse long-wave length fluctuations are non-propagating, i.e., purely dissipative:
  \[ \omega = -iDq^2 \]
  where \( D \) is the diffusive constant, \( TD = \frac{n}{s} \)

- on the contrary, in solids we have transverse sound waves:
  \[ \omega = c_\perp q, \quad c_\perp^2 = \frac{\mu}{\epsilon + P} \]
  where \( \mu \) is the shear elastic modulus
In this talk

solids + fluids = viscoelastic materials

- Embed viscoelastic materials in holography
- Have a control parameter that interpolates from more solid like—to—more fluid like
- study all-derivative viscoelastic hydrodynamics
- signature of holographic cracks?
The holographic model (think in microcanonical ensemble — we are interested in dynamics)

- start with the holographic superconductor

\[
S = \frac{1}{16\pi G_N} \int_{\mathcal{M}_5} d^5x \sqrt{-g} \left[ R + 12 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} F^2 + \frac{\Delta(\Delta - 4)}{2} \phi^2 \right]
\]

as usual, for a fixed charge density \( Q \), below some critical energy density \( \epsilon \) below which \( \phi \) condenses

- add a 'lattice' (J. Gauntlett + others)

\[
\left[ \ldots - \frac{1}{2} \phi^2 \sum_{i=1}^{3} \left\{ \lambda_1 (\partial \psi_i)^2 + \lambda_2 ((\partial \psi_i)^2)^2 \right\} \right]
\]

where \( \lambda_i > 0 \) are coupling constants; we will be turning on the non-normalizable component for \( \psi_i \) as

\[
\psi_i = k \delta^i_j x_j , \quad \text{where} \quad \{i, j\} = 1 \ldots 3 \quad \text{and} \quad k = \text{const}
\]
where is the lattice?

- for simplicity, set $\lambda_1 = 1$ and $\lambda_2 = 0$;

\[
\{\phi, \psi_i\} \implies \text{field redefinition} \implies \Phi_i \equiv \frac{\phi}{\sqrt{2}} e^{i\sqrt{2}\psi_i}
\]

results in a standard kinetic term for 3 complex fields $\Phi_i$:

\[
- \delta^{ij} \partial_i \Phi_j \partial_j \Phi^*_j
\]

and identifies $\psi_i$ as axions:

\[
\psi_i \sim \psi_i + \pi \sqrt{2}
\]

- since we are turning on $\psi_i = k \delta^j_i x_j$, the (boundary) spatial coordinates $x_j$ must be periodically identified:

\[
x_j \sim x_j + \frac{\pi \sqrt{2}}{k}
\]

since we have a lattice, it will not be a surprise that we have nonzero elastic modulus;
- turns out, elastic modulus in the model exists robustly for any set of \( \{\lambda_1, \lambda_2\} \);
- elastic modulus exists independently whether or not the non-normalizable component of \( \phi \) is turned on:
  - in the former case transverse phonons are gapped
  - in the latter case transverse phonons are gapless, with expected dispersion relation dictated by the shear elastic modulus
- enhance the 'lattice' effects in the model

\[
-\frac{1}{4}F^2 \quad \Rightarrow \quad -\frac{1}{4}(1 + \gamma \phi^2)F^2, \quad \gamma > 0
\]

\[\Rightarrow\] I will now highlight the computational results in the model introduced
Thermodynamics (energy density $\epsilon$, charge density $Q$, entropy density $s$):

- **red**: RN black hole;
- **orange**: broken phase at $k = 0$;
- **green**: broken phase at $\frac{k}{\epsilon^{1/4}} = 1$;
- **purple**: broken phase at $\frac{k}{\epsilon^{1/4}} = 10$
Elastic shear modulus $G \propto k^4 \tilde{G}$ and the shear viscosity $4\pi \eta / S = 1 + \tilde{\eta}$ in the model:

The reduced shear elastic modulus $\tilde{G} = 16\pi G_N G / k^4$ (left panel) and the reduced shear viscosity $\tilde{\eta} = (4\pi \eta / S - 1)$ (right panel) as functions of $k/T$ for select values of $T / \mu = \{1/12, 1/6\}$, {red,green} curves, at the criticality.
To study large-order hydrodynamics of our holographic viscoelastic model we focus on a divergent series for $\Omega_\Delta$:

$$
\Omega_\Delta = \sum_{n=0}^{\infty} c_n g^n
$$

- construct a Borel transform

$$
\Omega^{(B)}_\Delta(\xi) = \sum_{n=0}^{\infty} \frac{c_n \xi^n}{n!}
$$

- Borel resummation is performed as

$$
\Omega^{(R)}_\Delta = \int_C d\xi \, e^{-\xi} \, \Omega^{(B)}_\Delta(\xi, g) \equiv \frac{1}{g} \int_C d\xi \, e^{-\xi/g} \, \Omega^{(B)}_\Delta(\xi)
$$

where the contour $C$ connects 0 and $\infty$.

- Ambiguities in $\Omega^{(R)}_\Delta$ come from the poles in $\Omega^{(B)}_\Delta(\xi)$:

$$
\delta \Omega^{(R)}_\Delta \sim e^{-\xi_0/g}, \quad \text{once} \quad \frac{1}{\Omega^{(B)}_\Delta(\xi_0)} = 0
$$

For small $g$, poles in $\Omega^{(B)}_\Delta(\xi)$ generate essential singularity in $\Omega^{(R)}_\Delta$, responsible for the asymptotic character of $\Omega_\Delta$. 
$\rightarrow k = 0$ case (fluid)

- blue filled circles: poles of the (Pade approximation of the) Borel transform of $\Omega_{\Delta=2}$
- green crosses: Starinets-Nunez QNMs
$\Rightarrow \ k \ T = 100 \ case \ (viscoelastic)$

- red crosses: QNMs in the model at $\ k \ T = 100$
- orange lines: spectral flows of QNMs from $\ k \ T = 0$ to $\ k \ T = 100$
\[ \frac{k}{T} = 1000 \text{ case (solid)} \]
I did not have time to discuss:

- $G$ with explicit symmetry breaking
- elastic bulk modulus $\mathcal{K}$
- gapped-vs.-gapless phonons
- general $\Delta$ results
- how large orders of the hydrodynamics know about spontaneous symmetry breaking
- how and why $G$ depends on the charge density
- critical exponents of $G$ and $\mathcal{K}$ for spontaneous symmetry breaking

Open questions:

- what are limitations of Pade approximation of Borel transform?
- where are 'cracks' in the model?
- or it is not a brittle solid?
- is there a physics in the wall-of-Borel-poles?
- can we study boost-invariant expansion of the viscoelastic model?