THE GROTHENDIECK GROUP OF AN n-ANGULATED CATEGORY

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ABSTRACT. We define the Grothendieck group of an n-angulated category and show that for odd n its properties are as in the special case of n = 3, i.e. the triangulated case. In particular, its subgroups classify the dense and complete n-angulated subcategories via a bijective correspondence. For a tensor n-angulated category, the Grothendieck group becomes a ring, whose ideals classify the dense and complete n-angulated tensor ideals of the category.

1. INTRODUCTION

The Grothendieck group of a triangulated category is the free abelian group on the (set of) isomorphism classes of objects, modulo the Euler relations corresponding to the distinguished triangles. Thomason showed in [Tho] that the set of subgroups of the Grothendieck group classifies the dense triangulated subcategories. Namely, there is a bijective correspondence between the set of subgroups and the set of dense triangulated subcategories.

Recently, in [GKO], Geiss, Keller and Oppermann introduced “higher dimensional” analogues of triangulated categories, called n-angulated categories. An n-angulated category with n = 3 is nothing but a classical triangulated category, and they showed that certain cluster tilting subcategories of triangulated categories give rise to higher n-angulated categories.

In this paper, we define and study the Grothendieck group of an n-angulated category. As in the triangulated case, it is the free abelian group on the set of isomorphism classes of objects, modulo the Euler relations corresponding to the n-angles. Our main result shows that when n is odd, then the set of subgroups corresponds bijectively to the complete and dense n-angulated subcategories, thus providing a classification of these. When n = 3, that is, in the classical triangulated case, every triangulated subcategory is complete, hence we recover Thomason’s classification theorem. Our proof of the classification result only works when n is odd; we do not know whether a similar result holds when n is an even integer. The underlying reason for this is that in the odd case, the additive inverse of an element in the Grothendieck group is given by the suspension of the corresponding object. This is no longer true in the even case.

We also define tensor n-angulated categories, that is, n-angulated categories with a symmetric tensor structure (or symmetric monoidal structure) which is compatible with the n-angulation. For such a category, the Grothendieck group becomes a ring in a natural way, the Grothendieck ring. We show that the set of ideals in this ring classify the complete and dense n-angulated tensor ideals of the category.

This paper is organized as follows. In Section 2 we recall the definition of an n-angulated category, define its Grothendieck group, and prove some elementary properties. In Section 3 we specialize to the case of an n-angulated category arising from a cluster tilting subcategory of a triangulated category. We prove that there is a natural surjective homomorphism from the Grothendieck group of the n-angulated category to the Grothendieck group of the triangulated category. In Section 4 we prove our main result, the classification theorem which gives a bijective correspondence between subgroups and complete and dense n-angulated subcategories. Finally, in Section 5 we define tensor n-angulated categories, and prove the tensor version of the classification theorem.

2. THE GROTHENDIECK GROUP OF AN n-ANGULATED CATEGORY

Throughout this paper, every category will be assumed to be small, that is, the collection of isomorphism classes of objects forms a set.

We start by recalling the definition of an n-angulated category from [GKO]. Let C be an additive category with an automorphism Σ: C → C, and n an integer greater than or equal to three. An n-Σ-sequence in C is a sequence

\[ A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} \Sigma A_1 \]

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of objects and morphisms in \( \mathcal{C} \). We shall often denote such sequences by \( A_*, B_* \) etc. Its left and right rotations are the two \( n\Sigma \)-sequences

\[
A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \ldots \xrightarrow{a_n} \Sigma A_1 \xrightarrow{(-1)^n\Sigma a_1} \Sigma A_2
\]

and

\[
\Sigma^{-1} A_n \xrightarrow{(-1)^n\Sigma^{-1} a_n} A_1 \xrightarrow{a_1} \ldots \xrightarrow{a_{n-2}} A_{n-1} \xrightarrow{a_{n-1}} A_n
\]

respectively, and it is exact if the induced sequence

\[
\cdots \rightarrow \text{Hom}_{\mathcal{C}}(B, A_1) \xrightarrow{(a_1)^*} \text{Hom}_{\mathcal{C}}(B, A_2) \xrightarrow{(a_2)^*} \cdots \rightarrow \text{Hom}_{\mathcal{C}}(B, A_n) \xrightarrow{(a_n)^*} \text{Hom}_{\mathcal{C}}(B, \Sigma A_1) \rightarrow \cdots
\]

of abelian groups is exact for every object \( B \in \mathcal{C} \). A trivial \( n\Sigma \)-sequence is a sequence of the form

\[
A \xrightarrow{1} A \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma A
\]

or any of its rotations. A morphism \( A \xrightarrow{\varphi} B \) of \( n\Sigma \)-sequences is a sequence \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \) of morphisms in \( \mathcal{C} \) such that the diagram

\[
\begin{array}{c}
A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1 \\
\varphi_1 \downarrow \quad \varphi_2 \downarrow \quad \varphi_3 \downarrow \quad \ldots \quad \varphi_{n-1} \downarrow \quad \varphi_n \downarrow \\
B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots \xrightarrow{\beta_{n-1}} B_n \xrightarrow{\beta_n} \Sigma B_1
\end{array}
\]

commutes. It is an isomorphism if \( \varphi_1, \varphi_2, \ldots, \varphi_n \) are all isomorphisms in \( \mathcal{C} \), and a weak isomorphism if \( \varphi_i \) and \( \varphi_{i+1} \) are isomorphisms for some \( i \) (with \( \varphi_{n+1} := \Sigma \varphi_1 \)).

The category \( \mathcal{C} \) is pre-\( n \)-angulated if there exists a collection \( \mathcal{N} \) of \( n\Sigma \)-sequences satisfying the following three axioms:

(N1) (a) \( \mathcal{N} \) is closed under direct sums, direct summands and isomorphisms of \( n\Sigma \)-sequences.
(b) For all \( A \in \mathcal{C} \), the trivial \( n\Sigma \)-sequence

\[
A \xrightarrow{1} A \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma A
\]

belongs to \( \mathcal{N} \).
(c) For each morphism \( \alpha: A_1 \rightarrow A_2 \) in \( \mathcal{C} \), there exists an \( n\Sigma \)-sequence in \( \mathcal{N} \) whose first morphism is \( \alpha \).

(N2) An \( n\Sigma \)-sequence belongs to \( \mathcal{N} \) if and only if its left rotation belongs to \( \mathcal{N} \).

(N3) Each commutative diagram

\[
\begin{array}{c}
A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1 \\
\varphi_1 \downarrow \quad \varphi_2 \downarrow \quad \varphi_3 \downarrow \quad \ldots \quad \varphi_{n-1} \downarrow \quad \varphi_n \downarrow \\
B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots \xrightarrow{\beta_{n-1}} B_n \xrightarrow{\beta_n} \Sigma B_1
\end{array}
\]

with rows in \( \mathcal{N} \) can be completed to a morphism of \( n\Sigma \)-sequences.

In this case, the collection \( \mathcal{N} \) is a pre-\( n \)-angulation of the category \( \mathcal{C} \) (relative to the automorphism \( \Sigma \)), and the \( n\Sigma \)-sequences in \( \mathcal{N} \) are \( n \)-angles. If, in addition, the collection \( \mathcal{N} \) satisfies the following axiom, then it is an \( n \)-angulation of \( \mathcal{C} \), and the category is \( n \)-angulated:

(N4) In the situation of (N3), the morphisms \( \varphi_3, \varphi_4, \ldots, \varphi_n \) can be chosen such that the mapping cone

\[
\begin{array}{c}
A_2 \oplus B_1 \xrightarrow{\left[ \begin{array}{cc} -a_2 & 0 \\ \varphi_2 & \beta_1 \end{array} \right]} A_3 \oplus B_2 \xrightarrow{\left[ \begin{array}{cc} -a_3 & 0 \\ \varphi_3 & \beta_2 \end{array} \right]} \ldots \xrightarrow{\left[ \begin{array}{cc} -a_n & 0 \\ \varphi_n & \beta_{n-1} \end{array} \right]} \Sigma A_1 \oplus B_n \xrightarrow{\left[ \begin{array}{cc} -\Sigma a_1 & 0 \\ \Sigma \varphi_1 & \beta_1 \end{array} \right]} \Sigma A_2 \oplus \Sigma B_1
\end{array}
\]

belongs to \( \mathcal{N} \).

Note that the axioms given here are the original ones presented by Geiss, Keller and Oppermann in [GKO]. In [BeF], the authors gave a set of alternative axioms and showed that they are equivalent to the original ones. In particular, it was shown that axiom (N4) is equivalent to a “higher” version of Verdier’s original octahedral axiom.

The construction and properties of Grothendieck groups do not require axiom (N4). Therefore, the theory we present in this paper is valid for pre-\( n \)-angulated categories. However, we have chosen to state the definitions and results for \( n \)-angulated categories.

Having recalled the definition of an \( n \)-angulated category, we now define the Grothendieck group. As in the triangulated case, it is the free abelian group on the set of isomorphism classes of objects modulo
the Euler relations corresponding to the \( n \)-angles. It will be convenient to have a shorter notation for these Euler relations. Suppose therefore that \( \mathcal{C} \) is an \( n \)-angulated category, and let \( F(\mathcal{C}) \) be the free abelian group on the set of isomorphism classes \( (A) \) of objects \( A \) in \( \mathcal{C} \). Given an \( n \)-angle

\[ A_* : A_1 \to A_2 \to A_3 \to \cdots \to A_n \to \Sigma A_1 \]

in \( \mathcal{C} \), we denote the corresponding Euler relation in \( F(\mathcal{C}) \) by \( \chi(A_*) \), i.e.

\[ \chi(A_*) := \langle A_1 \rangle - \langle A_2 \rangle + \langle A_3 \rangle - \cdots + (-1)^{n+1} \langle A_n \rangle. \]

These generate the relations in the Grothendieck group when \( n \) is odd. When \( n \) is even, we also include the trivial relation \((0)\), in order for the residue class of \((0)\) to be the zero element; see Proposition 2.1.

**Definition 2.1.** Let \((\mathcal{C}, \Sigma)\) be an \( n \)-angulated category, and \( F(\mathcal{C}) \) the free abelian group on the set of isomorphism classes \((A)\) of objects \( A \) in \( \mathcal{C} \). Furthermore, let \( R(\mathcal{C}) \) be the subgroup of \( F(\mathcal{C}) \) generated by the following sets of elements

\[ \{ \chi(A_* ) | A_* \text{ \( n \)-angle in } \mathcal{C} \} \text{ if } n \text{ is odd} \]
\[ \{ (0) \} \cup \{ \chi(A_* ) | A_* \text{ \( n \)-angle in } \mathcal{C} \} \text{ if } n \text{ is even} \]

in \( \mathcal{C} \). The *Grothendieck group* \( K_0(\mathcal{C}) \) of \( \mathcal{C} \) is the quotient group \( F(\mathcal{C})/R(\mathcal{C}) \). Given an object \( A \in \mathcal{C} \), the residue class \( (A) + R(\mathcal{C}) \) in \( K_0(\mathcal{C}) \) is denoted by \([A]\).

Note that the definition of the Grothendieck group is the reason why we are only considering small categories: the collection of isomorphism classes in the category must form a set. Note also that it follows immediately from the definition that the Grothendieck group of \( \mathcal{C} \) is universal with respect to group homomorphisms from \( F(\mathcal{C}) \) to abelian groups satisfying the Euler relations. To be precise, let \( G \) be an abelian group and \( f : F(\mathcal{C}) \to G \) a homomorphism such that \( f(\chi(A_* )) = 0 \) for all \( n \)-angles \( A_* \) in \( \mathcal{C} \), and such that \( f((0)) = 0 \) when \( n \) is even. Then there exists a unique homomorphism \( \hat{f} : K_0(\mathcal{C}) \to G \) such that the diagram

\[ \begin{array}{ccc}
F(\mathcal{C}) & \xrightarrow{f} & G \\
\pi \downarrow & & \downarrow j \\
K_0(\mathcal{C}) & \xrightarrow{\hat{f}} & G
\end{array} \]

commutes, where \( \pi : F(\mathcal{C}) \to K_0(\mathcal{C}) \) is the natural projection.

We now prove some elementary properties of the Grothendieck group, properties which are well-known when \( n = 3 \), that is, when the category is triangulated. Note that for an arbitrary \( n \), the relation \([\Sigma A] = -[A]\) holds when \( n \) is odd, whereas \([\Sigma A] = [A]\) when \( n \) is even.

**Proposition 2.2.** Let \((\mathcal{C}, \Sigma)\) be an \( n \)-angulated category, and \( K_0(\mathcal{C}) \) its Grothendieck group.

1. The element \((0)\) is the zero element in \( K_0(\mathcal{C}) \).
2. If \( A \) and \( B \) are objects in \( \mathcal{C} \) then \([A \oplus B] = [A] + [B]\) and \([\Sigma A] = (-1)^n [A]\) in \( K_0(\mathcal{C}) \).
3. When \( n \) is odd, then every element in \( K_0(\mathcal{C}) \) is of the form \([A]\) for some object \( A \in \mathcal{C} \). When \( n \) is even, then every element in \( K_0(\mathcal{C}) \) is of the form \([A] - [B]\) for some objects \( A, B \in \mathcal{C} \).

**Proof.** (1) If \( n \) is even, then \((0)\) is zero in \( K_0(\mathcal{C}) \) by definition of \( R(\mathcal{C}) \). If \( n \) is odd, then the Euler relation corresponding to the trivial \( n \)-angle

\[ 0 \to 0 \to \cdots \to 0 \to \Sigma 0 \]

gives that \((0)\) is zero in \( K_0(\mathcal{C}) \).

(2) The two \( n \)-\( \Sigma \)-sequences

\[ A \xrightarrow{1} A \to 0 \to \cdots \to 0 \to \Sigma A \]
\[ 0 \to B \xrightarrow{1} B \to 0 \to \cdots \to 0 \to 0 \]

are \( n \)-angles, hence so is their direct sum \( S_* \). From (1), the Euler relation \( \chi(S_*) \) gives the equality \([A \oplus B] = [A] + [B]\) in \( K_0(\mathcal{C}) \). Moreover, the \( n \)-\( \Sigma \)-sequence

\[ T_* : A \to 0 \to \cdots \to 0 \to \Sigma A \xrightarrow{1^n} \Sigma A \]

is an \( n \)-angle. From (1), the Euler relation \( \chi(T_*) \) gives the equality \([A] + (-1)^{n+1} [\Sigma A] = 0\) in \( K_0(\mathcal{C}) \).

(3) Let \( x \) be an element in \( K_0(\mathcal{C}) \). If \( x = 0 \) then \( x = (0) \) by (1), and we are done. If \( x \) is nonzero, then there are non-negative integers \( a_1, \ldots, a_r, b_1, \ldots, b_l \) and objects \( A_1, \ldots, A_r, B_1, \ldots, B_l \) in \( \mathcal{C} \) with

\[ x = a_1[A_1] + \cdots + a_r[A_r] - b_1[B_1] - \cdots - b_l[B_l]. \]
Proof. (1) The relation is clearly reflexive and symmetric. Suppose that
\[ n \in \pi \in C \]
by definition, \( n \) is odd.

The next result provides an alternative interpretation of the Grothendieck group in the case when \( n \) is odd.

Proposition 2.3. Let \( n \geq 3 \) be an odd integer and \((\mathcal{C}, \Sigma)\) an \( n \)-angulated category. Consider the following relation on the set of objects of \( \mathcal{C} \): \( A \sim B \) if and only if there exist objects \( C_1, \ldots, C_n \) and two \( n \)-angles
\[
\begin{align*}
A \oplus C_1 & \xrightarrow{a_1} C_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} C_n \xrightarrow{a_n} \Sigma A \oplus \Sigma C_1 \\
B \oplus C_1 & \xrightarrow{b_1} C_2 \xrightarrow{b_2} \cdots \xrightarrow{b_{n-1}} C_n \xrightarrow{b_n} \Sigma B \oplus \Sigma C_1
\end{align*}
\]
in \( \mathcal{C} \). Then the following hold:

1. The relation is an equivalence relation.
2. The set \( \pi \) of equivalence classes \([A]\) of objects in \( \mathcal{C} \) forms an abelian group with addition \([A] + [B] = [A \oplus B] \). The inverse of an element \([A]\) is \([\Sigma A]\).
3. The groups \( \pi \) and \( K_0(\mathcal{C}) \) are isomorphic via the correspondence \([A] \mapsto [A]\).

Proof. (1) The relation is clearly reflexive and symmetric. Suppose that \( A \sim B \) and \( B \sim D \). Then by definition, there exist four \( n \)-angles
\[
\begin{align*}
A \oplus C_1 & \xrightarrow{a_1} C_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} C_n \xrightarrow{a_n} \Sigma A \oplus \Sigma C_1 \\
B \oplus C_1 & \xrightarrow{b_1} C_2 \xrightarrow{b_2} \cdots \xrightarrow{b_{n-1}} C_n \xrightarrow{b_n} \Sigma B \oplus \Sigma C_1 \\
B \oplus C_1 & \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{n-1}} C_n \xrightarrow{\gamma_n} \Sigma B \oplus \Sigma C_1 \\
D \oplus C_1 & \xrightarrow{\delta_1} C_2 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} C_n \xrightarrow{\delta_n} \Sigma D \oplus \Sigma C_1
\end{align*}
\]
in \( \mathcal{C} \). Now form two new \( n \)-angles from these: the direct sum of the first and the second, and the direct sum of the second and the fourth. These two \( n \)-angles show that \( A \sim D \), hence the relation is also transitive. Consequently, the relation is an equivalence relation.

(2) The operation \([A] + [B] = [A \oplus B] \) in \( \pi \) is a well-defined associative binary operation, with \([0]\) as the identity element. Now for any object \( A \in \mathcal{C} \), consider the two \( n \)-angles
\[
\begin{align*}
A \oplus \Sigma A & \xrightarrow{0} \Sigma A \xrightarrow{0} \Sigma A \xrightarrow{0} \cdots \xrightarrow{0} \Sigma A \xrightarrow{\{0\}} \Sigma A \oplus \Sigma^2 A \\
0 \quad & \Sigma A \xrightarrow{1} \Sigma A \xrightarrow{0} \Sigma A \xrightarrow{1} \cdots \xrightarrow{1} \Sigma A \xrightarrow{0}
\end{align*}
\]
in \( \mathcal{C} \), each built from trivial \( n \)-angles. They show that \( A \oplus \Sigma A \sim 0 \), so that
\[
[A] + [\Sigma A] = [A \oplus \Sigma A] = [0]
\]
in \( \pi \). Hence \( \pi \) is an abelian group: the inverse of an element \([A]\) is \([\Sigma A]\).

(3) We first show that the Euler relations hold in the group \( \pi \). Suppose that
\[
A_*: \quad A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} \Sigma A_1
\]
is an \( n \)-angle in \( \mathcal{C} \). Take the direct sum of \( A_* \) and the following trivial \( n \)-angles:
\[
\begin{align*}
A_i & \xrightarrow{1} A_i \xrightarrow{0} \cdots \xrightarrow{0} \Sigma A_i \quad i = 3, 5, \ldots, n \\
0 & \xrightarrow{1} A_i \xrightarrow{0} \cdots \xrightarrow{0} \Sigma A_i \quad i = 4, 6, \ldots, n - 1 \\
0 & \xrightarrow{0} \cdots \xrightarrow{0} A_{n-1} \xrightarrow{1} A_{n-1} \xrightarrow{0} \cdots \xrightarrow{0} \Sigma A_i \\
0 & \xrightarrow{0} \cdots \xrightarrow{0} A_{n-1} \xrightarrow{1} A_n \xrightarrow{0} \cdots \xrightarrow{0}
\end{align*}
\]
and write the odd integer \( n \) as \( n = 2t + 1 \). The result is an \( n \)-angle of the form
\[
\begin{array}{c}
\bigoplus_{i=0}^{l} A_{2i+1} 
\overset{i}{\longrightarrow} \bigoplus_{i=2}^{n} A_{i} 
\overset{n}{\longrightarrow} \bigoplus_{i=3}^{n} A_{i} 
\cdots 
\longrightarrow A_{n-1} \oplus A_{n} 
\overset{A_{n}}{\longrightarrow} \bigoplus_{i=0}^{l} \Sigma A_{2i+1}
\end{array}
\]

in \( \mathcal{C} \). Now take the direct sum of the following trivial \( n \)-angles:

\[
\begin{align*}
A_{i} & \overset{1}{\longrightarrow} A_{i} \\
0 & \longrightarrow A_{i} \\
0 & \longrightarrow A_{i} \\
\cdots & \longrightarrow 0 \\
0 & \longrightarrow 0 \\
\cdots & \longrightarrow 0 \\
0 & \longrightarrow 0 \\
0 & \longrightarrow A_{n-1} \\
0 & \longrightarrow A_{n-1} \\
0 & \longrightarrow 0
\end{align*}
\]

The result is an \( n \)-angle of the form

\[
\begin{array}{c}
\bigoplus_{i=1}^{l} A_{2i} 
\overset{i}{\longrightarrow} \bigoplus_{i=2}^{n} A_{i} 
\overset{n}{\longrightarrow} \bigoplus_{i=3}^{n} A_{i} 
\cdots 
\longrightarrow A_{n-1} \oplus A_{n} 
\overset{A_{n}}{\longrightarrow} \bigoplus_{i=1}^{l} \Sigma A_{2i}
\end{array}
\]

in \( \mathcal{C} \). By definition of the equivalence relation, the two \( n \)-angles show that

\[ A_{1} \oplus A_{3} \oplus \cdots \oplus A_{n} \sim A_{2} \oplus A_{4} \oplus \cdots \oplus A_{n-1}, \]

so that the relation

\[ [A_{1}] + [A_{3}] + \cdots + [A_{n}] = [A_{2}] + [A_{4}] + \cdots + [A_{n-1}] \]

holds in the group \( \pi \). But this is precisely the Euler relation \( \sum_{i=1}^{n} (-1)^{i+1} [A_{i}] = 0 \) corresponding to the \( n \)-angle \( A \), we started with.

Since the Euler relations hold in \( \pi \), the map \( f : K_{0}(\mathcal{C}) \to \pi \) given by \( [A] \mapsto [A] \) is a well-defined surjective homomorphism of abelian groups. To show injectivity, suppose that \( f([A]) = 0 \). Then \( A \sim 0 \), and so there exist objects \( C_{1}, \ldots, C_{n} \) and two \( n \)-angles

\[
\begin{align*}
A \oplus C_{1} & \overset{a_{1}}{\longrightarrow} C_{2} \overset{a_{2}}{\longrightarrow} \cdots \overset{a_{n-1}}{\longrightarrow} C_{n} \overset{a_{n}}{\longrightarrow} \Sigma A \oplus \Sigma C_{1} \\
C_{1} & \overset{\beta_{1}}{\longrightarrow} C_{2} \overset{\beta_{2}}{\longrightarrow} \cdots \overset{\beta_{n-1}}{\longrightarrow} C_{n} \overset{\beta_{n}}{\longrightarrow} \Sigma C_{1}
\end{align*}
\]

in \( \mathcal{C} \). Combining the Euler relations corresponding to these two \( n \)-angles gives \( [A] = 0 \) in \( K_{0}(\mathcal{C}) \), hence the map \( f \) is an isomorphism.

We end this section with the following corollary to Proposition \[2.3\] it follows immediately from the definition of the isomorphism between the Grothendieck group and the group of equivalence classes. It provides a criterion for determining when two elements in the Grothendieck group are equal. For \( n = 3 \) this is \[Tho\] Lemma \[2.4\], which in turn was based on \[Lan\] 1.6, Corollary.

**Corollary 2.4.** Let \( n \geq 3 \) be an odd integer and \( (\mathcal{C}, \Sigma) \) an \( n \)-angulated category. Then for any objects \( A, B \in \mathcal{C} \) the following are equivalent:

1. \( [A] = [B] \) in \( K_{0}(\mathcal{C}) \).
2. There exist objects \( C_{1}, \ldots, C_{n} \) and two \( n \)-angles

\[
\begin{align*}
A \oplus C_{1} & \overset{a_{1}}{\longrightarrow} C_{2} \overset{a_{2}}{\longrightarrow} \cdots \overset{a_{n-1}}{\longrightarrow} C_{n} \overset{a_{n}}{\longrightarrow} \Sigma A \oplus \Sigma C_{1} \\
B \oplus C_{1} & \overset{\beta_{1}}{\longrightarrow} C_{2} \overset{\beta_{2}}{\longrightarrow} \cdots \overset{\beta_{n-1}}{\longrightarrow} C_{n} \overset{\beta_{n}}{\longrightarrow} \Sigma B \oplus \Sigma C_{1}
\end{align*}
\]

in \( \mathcal{C} \).
3. Example: Cluster Tilting Categories

The first class of examples of $n$-angulated categories appeared in [GKO], namely, those that arise from certain cluster tilting subcategories of triangulated categories. In this section, we show that there always exists a surjective homomorphism from the Grothendieck group of such an $n$-angulated category onto the Grothendieck group of the underlying triangulated category.

We recall the construction of these $n$-angulated categories. Let $\mathcal{T}$ be a triangulated category with suspension $\Sigma$, and $\mathcal{C}$ a full subcategory. Recall that a morphism $C \xrightarrow{f} X$ in $\mathcal{T}$ is a right $\mathcal{C}$-approximation of the object $X$ if the following hold: the object $C$ belongs to $\mathcal{C}$, and for every morphism $C' \xrightarrow{g} X$ with $C' \in \mathcal{C}$ there exists a morphism $C' \xrightarrow{h} C$ such that $g = f \circ h$.

\[
\begin{array}{ccc}
C & \xrightarrow{h} & C' \\
\downarrow{f} & & \downarrow{g} \\
X & & X
\end{array}
\]

A left $\mathcal{C}$-approximation of $X$ is defined dually. The subcategory $\mathcal{C}$ is called contravariantly finite in $\mathcal{T}$ if every object in $\mathcal{T}$ admits a right $\mathcal{C}$-approximation, and covariantly finite in $\mathcal{T}$ if every object in $\mathcal{T}$ admits a left $\mathcal{C}$-approximation. If $\mathcal{C}$ is both contravariantly and covariantly finite in $\mathcal{T}$, then it is called functorially finite. Finally, if $t \geq 2$ is an integer, then the subcategory $\mathcal{C}$ is called a $t$-cluster tilting subcategory of $\mathcal{T}$ if the following hold:

1. $\mathcal{C}$ is functorially finite in $\mathcal{T}$.
2. $\mathcal{C}$ is the full subcategory of $\mathcal{T}$ given by

\[\mathcal{C} = \{ A \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(A, \Sigma^i C) = 0 \text{ for } 1 \leq i \leq t-1 \text{ and for all } C \in \mathcal{C}\}\]

\[= \{ B \in \mathcal{T} \mid \text{ Hom}_{\mathcal{T}}(\Sigma^i C, B) = 0 \text{ for } 1 \leq i \leq t-1 \text{ and for all } C \in \mathcal{C}\}\].

Suppose now that the subcategory $\mathcal{C}$ is $(n-2)$-cluster tilting for some integer $n$, and that $\mathcal{C}$ is closed under the automorphism $\Sigma^{n-2}$ of $\mathcal{T}$; we denote $\Sigma^{n-2}$ by $\hat{\Sigma}$. Let $\mathcal{N}$ be the collection of $n$-$\hat{\Sigma}$-sequences in $\mathcal{C}$ such that there exists a diagram

\[
\begin{array}{cccccccc}
A_1 & \xrightarrow{\delta_{n-2}} & X_1 & \xrightarrow{\delta_{n-3}} & X_2 & \cdots & \xrightarrow{\delta_{n-3}} & X_{n-3} & \xrightarrow{\delta_{n-2}} & A_n \\
\downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_{n-3}} & & \downarrow{f_{n-2}} & & \downarrow{f_1} \\
A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & \cdots & \xrightarrow{a_2} & A_{n-1} & \xrightarrow{a_{n-2}} & A_n
\end{array}
\]

in $\mathcal{T}$ with the following properties:

1. Each diagram triangle $\Delta$ is a triangle in $\mathcal{T}$, where a map $X \rightarrow Y$ denotes a map from $X$ to $\Sigma Y$.
2. The other diagram triangles commute.
3. The map $a_n$ equals the composition $\Sigma^{n-3} \delta_{n-2} \circ \Sigma^{n-4} \delta_{n-3} \circ \cdots \circ \delta_1$.

Then it is shown in [GKO], Section 3, Theorem 1 that $(\mathcal{C}, \hat{\Sigma})$ is an $n$-angulated category, with the collection $\mathcal{N}$ as $n$-angles.

The following result shows that, in the above situation, the “obvious” map from from the Grothendieck group of $\mathcal{C}$ to that of $\mathcal{T}$ is a surjective homomorphism of groups.

**Theorem 3.1.** Let $(\mathcal{T}, \Sigma)$ be a triangulated category with an $(n-2)$-cluster-tilting subcategory $\mathcal{C}$ which is closed under $\Sigma^{n-2}$. Furthermore, let $(\mathcal{C}, \hat{\Sigma})$ be the corresponding $n$-angulated category, where $\hat{\Sigma} = \Sigma^{n-2}$. Then the map

\[K_0(\mathcal{C}) \twoheadrightarrow K_0(\mathcal{T})
\]

\[|A| - |B| \mapsto |A| - |B|
\]

is a well-defined surjective homomorphism of groups.

**Remark 3.2.** (1) When $n$ is an odd integer, then we know from Proposition 2.2 that every element in $K_0(\mathcal{C})$ is just of the form $|A|$ for some object $A \in \mathcal{C}$. It is to be understood that in this case, the map sends $|A|$ in $K_0(\mathcal{C})$ to $|A|$ in $K_0(\mathcal{T})$.

(2) Using Corollary 3.4, it is possible to characterize the kernel of the surjective homomorphism $K_0(\mathcal{C}) \twoheadrightarrow K_0(\mathcal{T})$. However, this characterization is not very effective, since it uses triangles in $\mathcal{T}$. 


Proof. Let $F(\mathcal{C})$ be the free abelian group on the set of isomorphism classes $\langle A \rangle$ of objects $A \in \mathcal{C}$, and $f : F(\mathcal{C}) \to K_0(\mathcal{F})$ the group homomorphism given by

$$a_1 \langle A_1 \rangle + \cdots + a_i \langle A_i \rangle - b_1 \langle B_1 \rangle - \cdots - b_i \langle B_i \rangle \mapsto a_1 [A_1] + \cdots + a_i [A_i] - b_1 [B_1] - \cdots - b_i [B_i].$$

In order to show that $f$ extends to a homomorphism $\hat{f} : K_0(\mathcal{C}) \to K_0(\mathcal{F})$, we must prove that $f$ maps the subgroup $R(\mathcal{C})$ in Definition 2.1 to zero. In other words, we must show that for every $n$-angle in $(\mathcal{C}, \Sigma)$, the corresponding Euler relation holds in $K_0(\mathcal{F})$.

Let therefore $f : A_1 a_1 \to A_2 a_2, \ldots, a_{n-1} A_{n-1} a_{n-1} \Sigma A_1$ be an $n$-angle in $(\mathcal{C}, \Sigma)$. By definition, this $n$-angle is built from triangles in $(\mathcal{F}, \Sigma)$, that is, there exists a diagram

$$\begin{array}{cccccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} A_{n-1} & \xrightarrow{a_{n-1}} A_1 \\
\text{\Delta} & \xrightarrow{f_i} & \Delta & \xrightarrow{f_{i+1}} & \cdots & \xrightarrow{f_{n-2}} \Delta & \xrightarrow{f_{n-1}} \Delta \\
X_1 & \xrightarrow{\delta_{n-2}} & X_1 & \xrightarrow{\delta_{n-3}} & \cdots & \xrightarrow{\delta_1} A_1
\end{array}$$

in $\mathcal{F}$ with the three properties given prior to the theorem. In particular, the $n-2$ diagram triangles $\Delta$ are triangles in $(\mathcal{F}, \Sigma)$. The Euler relations in $K_0(\mathcal{F})$ therefore give

$$0 = [A_1] - [A_2] + [X_1]$$

$$= [A_1] - [A_2] + [A_3] - [X_2]$$

$$\vdots$$

$$= [A_1] - [A_2] + [A_3] - \cdots + (-1)^{n+1}[A_n],$$

so that $f$ maps the subgroup $R(\mathcal{C})$ of $F(\mathcal{C})$ to zero in $K_0(\mathcal{F})$. Consequently, the homomorphism $f$ extends to a homomorphism $\hat{f} : K_0(\mathcal{C}) \to K_0(\mathcal{F})$, and the latter is precisely the homomorphism given in the theorem.

It remains to show that $\hat{f}$ is surjective. Let therefore $[X]$ be an element in $K_0(\mathcal{F})$. By [KeR, 5.5, Proposition], since $\mathcal{C}$ is an $(n-2)$-cluster-tilting subcategory of $\mathcal{F}$, every object in $\mathcal{F}$ admits a $\mathcal{C}$-resolution of length $n-3$. Consequently, there exist $n-3$ triangles

$$\begin{array}{ccc}
X_1 & \xrightarrow{A_0} & X \xrightarrow{\Sigma X_1} \\
X_2 & \xrightarrow{A_1} & X_1 \xrightarrow{\Sigma X_2} \\
\vdots & \vdots & \vdots \\
X_{n-4} & \xrightarrow{A_{n-5}} & X_{n-5} \xrightarrow{\Sigma X_{n-4}} \\
A_{n-3} & \xrightarrow{A_{n-4}} & X_{n-4} \xrightarrow{\Sigma A_{n-3}}
\end{array}$$

in $\mathcal{F}$, with $A_i \in \mathcal{C}$ for all $i$. The Euler relations in $K_0(\mathcal{C})$ now gives

$$[X] = [A_0] - [X_1]$$

$$= [A_0] - [A_1] + [X_2]$$

$$\vdots$$

$$= [A_0] - [A_1] + [A_2] - \cdots + (-1)^{n-3}[A_{n-3}],$$

hence $[X] = \hat{f} (\sum_{i=0}^{n-3} (-1)^i [A_i])$. This shows that the homomorphism $\hat{f}$ is surjective. \hfill \Box

4. Classifying subcategories

In this section we prove a generalized version of Thomason’s classification theorem [Tho, Theorem 2.1]. Thomason’s theorem states that the subgroups of the Grothendieck group of a triangulated category correspond bijectively to the so-called dense triangulated subcategories. We generalize this to $n$-angulated categories, for $n$ odd. The reason why our proof does not work when $n$ is even is very simple: in this case the relation $[\Sigma C] = -[C]$ does not hold in the Grothendieck group (as it does when $n$ is odd), but $[\Sigma C] = [C]$ instead.
We start by defining $n$-angulated subcategories of an $n$-angulated category $(\mathcal{C}, \Sigma)$. In general, if $(\mathcal{C}', \Sigma')$ is another $n$-angulated category, then a functor $L: \mathcal{C} \to \mathcal{C}'$ is $n$-\textit{angulated} if it has the following three properties:

1. $L$ is additive.
2. There exists a natural isomorphism $\eta: L \circ \Sigma \cong \Sigma' \circ L$.
3. $L$ preserves $n$-angles: if $A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} \Sigma A_1$ is an $n$-angle in $\mathcal{C}$, then
   $$L(A_1) \xrightarrow{L(a_1)} L(A_2) \xrightarrow{L(a_2)} \cdots \xrightarrow{L(a_{n-1})} L(A_n) \xrightarrow{\eta L(a_n)} \Sigma'L(A_1)$$
is an $n$-angle in $\mathcal{C}'$.

As in the triangulated case, the key requirement of an $n$-angulated subcategory is that the inclusion functor is $n$-angulated.

**Definition 4.1.** Let $(\mathcal{C}, \Sigma)$ be an $n$-angulated category.

1. An $n$-\textit{angulated subcategory} of $\mathcal{C}$ is a full subcategory $\mathcal{A}$ such that $(\mathcal{A}, \Sigma)$ is $n$-angulated, closed under isomorphisms and the inclusion functor $\iota: \mathcal{A} \to \mathcal{C}$ is an $n$-angulated functor.
2. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is dense in $\mathcal{C}$ if the following holds: every object in $\mathcal{C}$ is a direct summand of an object in $\mathcal{A}$.
3. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is complete if the following holds: given an $n$-angle in $\mathcal{C}$ in which $n-1$ of the vertices are in $\mathcal{A}$, then the last vertex is an object in $\mathcal{A}$.

Note that in the triangulated case, i.e. when $n = 3$, then every triangulated subcategory of a triangulated category is complete. This is the well-known "two out of three" property for triangulated subcategories, and this was how triangulated subcategories were originally defined (cf. [Tho] 1.1 and [Ver] §1, no. 1, 2-3]). The reason is that, up to isomorphism, there is only one way to complete a given map to a triangle (using axiom (N1)(c)), a consequence of the Five Lemma. Thus, when $\mathcal{C}$ is a triangulated category and $\mathcal{A}$ is a triangulated subcategory, then the triangles in $\mathcal{A}$ are characterized as follows: a $3$-$\Sigma$-sequence in $\mathcal{A}$ (or equivalently, a $3$-$\Sigma$-sequence in $\mathcal{C}$ with all its objects in $\mathcal{A}$) is a triangle in $\mathcal{A}$ if and only if it is a triangle in $\mathcal{C}$. As the following lemma shows, this holds for general $n$-angulated categories.

**Lemma 4.2.** Let $(\mathcal{C}, \Sigma)$ be an $n$-angulated category, and $(\mathcal{A}, \Sigma)$ an $n$-angulated subcategory. Then an $n$-$\Sigma$-sequence in $\mathcal{A}$ is an $n$-angle in $\mathcal{A}$ if and only if it is an $n$-angle in $\mathcal{C}$.

**Proof.** Consider an $n$-$\Sigma$-sequence

$$A_\ast : \quad A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} \Sigma A_1$$
in $\mathcal{A}$. If $A_\ast$ is an $n$-angle in $\mathcal{A}$, then it is also an $n$-angle in $\mathcal{C}$, since the inclusion functor $\iota: \mathcal{A} \to \mathcal{C}$ is $n$-angulated. Conversely, suppose that $A_\ast$ is an $n$-angle in $\mathcal{C}$. Using axiom (N1)(c), we can complete the map $a_1: A_1 \to A_2$ to an $n$-angle

$$B_\ast : \quad B_1 \xrightarrow{a_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots \xrightarrow{\beta_{n-1}} B_n \xrightarrow{\beta_n} \Sigma A_1$$
in $\mathcal{A}$. As above, this $n$-angle is also an $n$-angle in $\mathcal{C}$. Now use axiom (N3) to obtain a morphism

$$\varphi : \quad A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} \Sigma A_1$$
of $n$-angles in $\mathcal{C}$. Since $\mathcal{A}$ is a full subcategory of $\mathcal{C}$, the maps $\varphi_1, \ldots, \varphi_n$ belong to $\mathcal{A}$. For the same reason, the $n$-$\Sigma$-sequence $A_\ast$ is exact in $\mathcal{A}$, since it is exact in $\mathcal{C}$ by [GKO] Proposition 1.5. Consequently, the morphism $\varphi: B_\ast \to A_\ast$ is a weak isomorphism from an $n$-angle in $\mathcal{A}$ to an exact $n$-$\Sigma$-sequence in $\mathcal{A}$. By [GKO] Lemma 1.4, the $n$-$\Sigma$-sequence $A_\ast$ is an $n$-angle in $\mathcal{A}$. □

Our proof of the classification theorem is modeled on that provided by Thomason. We start with the following technical lemma.

**Lemma 4.3.** Let $(\mathcal{C}, \Sigma)$ be an $n$-angulated category, $(\mathcal{A}, \Sigma)$ a complete $n$-angulated subcategory, and $C$ an object in $\mathcal{C}$. If there exists and object $A \in \mathcal{A}$ such that $C \oplus A \in \mathcal{A}$, then $C \in \mathcal{A}$.
The $n$-$\Sigma$-sequence
\[
\begin{array}{cccccccc}
0 & \to & C & \to & A & \to & \cdots & \to & 0
\end{array}
\]
is an $n$-angle in $\mathcal{C}$, since it is the direct sum of trivial $n$-angles. The $n-1$ last vertices are all objects in $\mathcal{A}$, hence, since $\mathcal{A}$ is complete, so is $C$. \qed

From now on we need to restrict to the case when $n$ is odd, as explained in the beginning of this section. The next result is an $n$-angulated version of [H] Lemma 2.2. Note first that if $(\mathcal{C}, \Sigma)$ is an $n$-angulated category and $(\mathcal{A}, \Sigma)$ an $n$-angulated subcategory, then the inclusion functor $i \colon \mathcal{A} \to \mathcal{C}$ induces a homomorphism
\[
K_0(\mathcal{A}) \to K_0(\mathcal{C})
\]
of Grothendieck groups, since the functor maps $n$-angles in $\mathcal{A}$ to $n$-angles in $\mathcal{C}$. We shall denote the image of this homomorphism by $K_0(\mathcal{A})$; this is a subgroup of $K_0(\mathcal{C})$.

**Lemma 4.4.** Let $n \geq 3$ be an odd integer, let $(\mathcal{C}, \Sigma)$ be an $n$-angulated category and $(\mathcal{A}, \Sigma)$ a complete and dense $n$-angulated subcategory of $(\mathcal{C}, \Sigma)$. Then for any object $C \in \mathcal{C}$ the following holds: $C$ belongs to $\mathcal{A}$ if and only if $[C] = 0$ in $K_0(\mathcal{C})/\text{Im} K_0(\mathcal{A})$.

**Proof.** If $C$ belongs to $\mathcal{A}$, then in $K_0(\mathcal{C})$ the element $[C]$ obviously belongs to the subgroup $\text{Im} K_0(\mathcal{A})$. Thus $[C] = 0$ in $K_0(\mathcal{C})/\text{Im} K_0(\mathcal{A})$.

Conversely, suppose that $[C] = 0$ in $K_0(\mathcal{C})/\text{Im} K_0(\mathcal{A})$. Then, as above, in $K_0(\mathcal{C})$ the element $[C]$ belongs to the subgroup $\text{Im} K_0(\mathcal{A})$, hence there is an object $A \in \mathcal{A}$ such that $[C] = [A]$ in $K_0(\mathcal{C})$. By Corollary 2.4 there exist objects $C_1, \ldots, C_n$ and two $n$-angles
\[
C \oplus C_1 \xrightarrow{\alpha_1} C_2 \oplus C_2 \oplus \cdots \oplus C_n \xrightarrow{\alpha_n} \Sigma C \oplus \Sigma C_1
\]
\[
A \oplus C_1 \xrightarrow{\beta_1} C_2 \oplus C_2 \oplus \cdots \oplus C_n \xrightarrow{\beta_n} \Sigma A \oplus \Sigma C_1
\]
in $\mathcal{C}$. For each $i$, choose an object $C_i'$ such that $C_i \oplus C_i'$ belongs to $\mathcal{A}$; there exists such an object since $\mathcal{A}$ is dense. Furthermore, define the object $C$ by
\[
C = C_1 \oplus C_2 \oplus C_3 \oplus \cdots \oplus C_n \oplus C_n
\]
By adding trivial $n$-angles involving the objects $C_i$ and $C_i'$ to the two $n$-angles above, we obtain two new $n$-angles
\[
\begin{array}{cccccccc}
& C \oplus C & \xrightarrow{n \ (C_i \oplus C_i')} & \oplus \ (C_i \oplus C_i') & \cdots & \oplus \ (C_i \oplus C_i') & \rightarrow & \Sigma C \oplus \Sigma C
\end{array}
\]
\[
\begin{array}{cccccccc}
A \oplus C & \xrightarrow{n \ (C_i \oplus C_i')} & \oplus \ (C_i \oplus C_i') & \cdots & \oplus \ (C_i \oplus C_i') & \rightarrow & \Sigma A \oplus \Sigma C
\end{array}
\]
in $\mathcal{C}$. In both of these, the last $n-1$ vertices are objects in $\mathcal{A}$, hence, since $\mathcal{A}$ is dense, so are the objects $C \oplus C$ and $A \oplus C$. Now, all the three objects $A$, $A \oplus C$ and $C \oplus C$ belong to $\mathcal{A}$. Using Lemma 4.7 twice, we see that $C$ and $C$ also belong to $\mathcal{A}$. \qed

We need one more lemma before we can prove the classification theorem.

**Lemma 4.5.** Let $n \geq 3$ be an odd integer and $(\mathcal{C}, \Sigma)$ an $n$-angulated category. For a subgroup $H$ of $K_0(\mathcal{C})$, denote by $\mathcal{A}_H$ the full subcategory of $\mathcal{C}$ consisting of those objects $A \in \mathcal{C}$ such that $[A] \in H \subseteq K_0(\mathcal{C})$. Then $(\mathcal{A}_H, \Sigma)$ is a complete and dense $n$-angulated subcategory of $(\mathcal{C}, \Sigma)$, by declaring the $n$-angles in $\mathcal{C}$ with all objects in $\mathcal{A}_H$ to be the $n$-angles in $\mathcal{A}_H$.

**Proof.** We first show that $\mathcal{A}_H$ is dense and complete. If $C$ is any object in $\mathcal{C}$, then
\[
[C \oplus \Sigma C] = [C] + [\Sigma C] = [C] - [C] = 0 \in H
\]
in $K_0(\mathcal{C})$, hence $C \oplus \Sigma C$ belongs to $\mathcal{A}_H$ by definition. This shows that $\mathcal{A}_H$ is dense in $\mathcal{C}$. To prove completeness, suppose that
\[
C_1 \to C_2 \to \cdots \to C_n \to \Sigma C_1
\]
is an $n$-angle in $\mathcal{C}$ in which all but possibly one of the vertices, say $C_t$, are objects in $\mathcal{A}_H$. The Euler relation in $K_0(\mathcal{C})$ gives $\Sigma C_t = 0 \in H$, and since all the $[C_t]$ with $i \neq t$ are elements in $H$, we obtain $[C_t] \in H$. But then $C_t \in \mathcal{A}_H$ by definition, hence $\mathcal{A}_H$ is complete.

To show that $(\mathcal{A}_H, \Sigma)$ is an $n$-angulated category, we must prove that $\mathcal{A}_H$ is closed under the automorphism $\Sigma$, and verify that the collection of declared $n$-angles in $\mathcal{A}_H$ satisfies the four axioms (N1)–(N4).
The first part is easy. For if \( A \in \mathcal{A}_H \), then by definition \([A] \in H \) in \( K_0(\mathcal{C}) \), and then \(-[A] \in H \) since \( H \) is a subgroup of \( K_0(\mathcal{C}) \). But \([\Sigma A] = -[A] = [-\Sigma^{-1} A] \), so both \( \Sigma A \) and \( \Sigma^{-1} A \) belong to \( \mathcal{A}_H \). Hence \( \mathcal{A}_H \) is closed under the automorphism \( \Sigma \).

Next, we verify that the collection \( \mathcal{N}_{\mathcal{A}_H} \) of declared \( n \)-angles in \( \mathcal{A}_H \) satisfies the four \( n \)-angle axioms. Note first that \( \mathcal{A}_H \) is closed under isomorphisms. For if \( A \in \mathcal{A}_H \) and \( C \in \mathcal{C} \) are isomorphic objects, then \([C] = [A] \in H \) in \( K_0(\mathcal{C}) \), so that \( C \in \mathcal{A}_H \). Now combine this fact with the fact that \( \mathcal{A}_H \) is a full subcategory of \( \mathcal{C} \), and that the collection \( \mathcal{N}_{\mathcal{A}_H} \) consists precisely of those \( n \)-angles in \( \mathcal{C} \) with all objects in \( \mathcal{A}_H \). It follows that the collection \( \mathcal{N}_{\mathcal{A}_H} \) satisfies all the axioms (N1)–(N4), except possibly axiom (N1)(c), which we must verify directly. Let therefore \( \alpha : A_1 \to A_2 \) be a morphism in \( \mathcal{A}_H \). By applying axiom (N1)(c) in \( \mathcal{C} \), we can complete this morphism to an \( n \)-angle

\[
A_1 \xrightarrow{\alpha} A_2 \to C_3 \to \cdots \to C_n \to \Sigma A_1
\]

in \( \mathcal{C} \). Since \( \mathcal{A}_H \) is dense in \( \mathcal{C} \), there exist objects \( C_3', \ldots, C_n' \) such that \( C_i \oplus C_i' \in \mathcal{A}_H \) for \( 3 \leq i \leq n \). Now take the above \( n \)-angle and add trivial \( n \)-angles involving the objects \( C_i \) and \( C_i' \). The result is an \( n \)-angle

\[
A_1 \xrightarrow{\alpha} A_2 \to C_3 \oplus C_3' \xrightarrow{4} \bigoplus_{i=3}^{i=4} (C_i \oplus C_i') \to \cdots \to \bigoplus_{i=3}^{i=n-1} (C_i \oplus C_i') \xrightarrow{\oplus} C \to \Sigma A_1
\]

in \( \mathcal{C} \), with \( C = C_n \oplus C_{n-1} \oplus C_{n-2} \oplus \cdots \oplus C_3 \oplus C_3' \). In this \( n \)-angle, all the \( n-1 \) first vertices are objects in \( \mathcal{A}_H \), hence so is \( C \) since \( \mathcal{A}_H \) is complete. Consequently, this \( n \)-angle belongs to the collection \( \mathcal{N}_{\mathcal{A}_H} \), and we have shown that the collection also satisfies axiom (N1)(c). Therefore \( (\mathcal{A}_H, \Sigma) \) is an \( n \)-angulated category.

It only remains to show that \( \mathcal{A}_H \) is an \( n \)-angulated subcategory of \( \mathcal{C} \), but this is easy. The subcategory \( \mathcal{A}_H \) is full by definition, and it is \( n \)-angulated and closed under isomorphism by the above. Finally, the inclusion functor \( i : \mathcal{A}_H \to \mathcal{C} \) is, by definition of the collection \( \mathcal{N}_{\mathcal{A}_H} \), an \( n \)-angulated functor.

We are now ready to prove the main result. It shows that the subgroups of the Grothendieck group classify the complete and dense \( n \)-angulated subcategories via a bijective correspondence. As mentioned after Definition 3.1 in the triangulated case every triangulated subcategory is complete, hence we recover Thomason’s classification theorem [Tho, Theorem 2.1].

**Theorem 4.6.** Let \( n \geq 3 \) be an odd integer and \( (\mathcal{C}, \Sigma) \) an \( n \)-angulated category. Denote the set of complete \( n \)-angulated subcategories of \( \mathcal{C} \) by \( \text{Comp}(\mathcal{C}) \), the set of dense \( n \)-angulated subcategories of \( \mathcal{C} \) by \( \text{Dense}(\mathcal{C}) \), and the set of subgroups of \( K_0(\mathcal{C}) \) by \( \text{Sub}(K_0(\mathcal{C})) \). Then there is a one-to-one correspondence

\[
\text{Comp}(\mathcal{C}) \cap \text{Dense}(\mathcal{C}) \longleftrightarrow \text{Sub}(K_0(\mathcal{C}))
\]

\[
\mathcal{A} \longrightarrow \text{Im} K_0(\mathcal{A})
\]

\[
\mathcal{A}_H \longrightarrow H
\]

where \( \mathcal{A}_H \) is the subcategory of \( \mathcal{C} \) consisting of those objects \( A \) in \( \mathcal{C} \) such that \([A] \in H \leq K_0(\mathcal{C}) \).

**Proof.** By Lemma 4.5 for every subgroup \( H \) of \( K_0(\mathcal{C}) \), the category \( (\mathcal{A}_H, \Sigma) \) is a complete and dense \( n \)-angulated subcategory of \( \mathcal{C} \). Therefore, the two maps

\[
\text{Comp}(\mathcal{C}) \cap \text{Dense}(\mathcal{C}) \xrightarrow{\Phi} \text{Sub}(K_0(\mathcal{C}))
\]

\[
\text{Sub}(K_0(\mathcal{C})) \xrightarrow{\Psi} \text{Comp}(\mathcal{C}) \cap \text{Dense}(\mathcal{C})
\]

given by \( \Phi(\mathcal{A}) = \text{Im} K_0(\mathcal{A}) \) and \( \Psi(H) = \mathcal{A}_H \) are well-defined. It follows from Lemma 4.2 that the composition \( \Psi \circ \Phi \) is the identity on \( \text{Comp}(\mathcal{C}) \cap \text{Dense}(\mathcal{C}) \). For a subgroup \( H \) of \( K_0(\mathcal{C}) \), it is easy to see that \( \Phi \circ \Psi(H) \) is a subgroup of \( H \), i.e. \( \text{Im} K_0(\mathcal{A}_H) \leq H \). But every element of \( H \) is of the form \([C] \) for some \( C \in \mathcal{C} \), hence \( C \in \mathcal{A}_H \) and then \([C] \in \text{Im} K_0(\mathcal{A}_H) \). This gives \( H \leq \text{Im} K_0(\mathcal{A}_H) \), and so the composition \( \Phi \circ \Psi \) is the identity on \( \text{Sub}(K_0(\mathcal{C})) \). \( \square \)

5. Tensor \( n \)-angulated categories

In this final section we briefly discuss the classification theorem in the context of \( n \)-angulated categories that admit a symmetric tensor (or monoidal) structure compatible with the \( n \)-angulated structure. As in the previous section, we need to restrict to the case when \( n \) is an odd integer. The Grothendieck group of such a category becomes a commutative ring, whose ideals classify the complete and dense \( n \)-angulated tensor ideals.

Recall that an additive category \( \mathcal{C} \) is a symmetric tensor category (or symmetric monoidal category) if there is an additive bifunctor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and a unit object \( I \in \mathcal{C} \) satisfying the following axioms:
(1) (Associativity Axiom) There is a natural isomorphism $\alpha : (- \otimes (- \otimes -)) \to ((- \otimes -) \otimes -)$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(2) (Unit Axiom) There are natural isomorphisms $\lambda : (I \otimes -) \to (-)$ and $\rho : (- \otimes I) \to (-)$ of functors $\mathcal{C} \to \mathcal{C}$.

(3) (Pentagon Axiom) The diagram

\[
\begin{array}{ccc}
(A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & (A \otimes (B \otimes C)) \otimes D \\
& & \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha} & A \otimes (B \otimes (C \otimes D))
\end{array}
\]

commutes for all objects $A, B, C, D \in \mathcal{C}$.

(4) (Triangle Axiom) The diagram

\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{\alpha} & (A \otimes I) \otimes B \\
& & \\
A \otimes (I \otimes B) & \xrightarrow{\alpha} & A \otimes (I \otimes B)
\end{array}
\]

commutes for all objects $A, B \in \mathcal{C}$.

(5) (Symmetry Axiom) For all objects $A, B \in \mathcal{C}$ there is an isomorphism $\gamma : A \otimes B \to B \otimes A$, natural in both $A$ and $B$. The three diagrams

\[
\begin{array}{ccc}
B \otimes A & \xrightarrow{\gamma} & A \otimes B \\
& & \\
A \otimes B & \xrightarrow{1} & A \otimes B
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{ccc}
A \otimes I & \xrightarrow{\gamma} & C \otimes (A \otimes B) \\
& & \\
A \otimes (C \otimes B) & \xrightarrow{1} & (A \otimes C) \otimes B
\end{array}
\]

commute for all objects $A, B, C \in \mathcal{C}$.

For further details, we refer to [Mac, VII.1 and VII.7]. Strictly speaking, one should refer to the symmetric tensor category $\mathcal{C}$ as the tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$. However, we shall keep the notation at a minimum and only refer to “the symmetric tensor category $\mathcal{C}$.”

Now let $(\mathcal{C}, \Sigma)$ be an $n$-angulated category. Then $\mathcal{C}$ is tensor $n$-angulated if it admits a symmetric tensor structure which is compatible with the $n$-angulated structure. Specifically, this means that $\mathcal{C}$ is also a symmetric tensor category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ satisfying the following axioms:

(1) There are natural isomorphisms $l : (\Sigma(-) \otimes -) \to \Sigma(- \otimes -)$ and $r : (\Sigma(-) \otimes -) \to \Sigma(- \otimes -)$ of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(2) For every object $A \in \mathcal{C}$, the endofunctors $(A \otimes -)$ and $(- \otimes A)$ on $\mathcal{C}$ are $n$-angulated functors, together with the natural isomorphisms $l$ and $r$, respectively.

(3) The two diagrams

\[
\begin{array}{ccc}
I \otimes \Sigma A & \xrightarrow{\lambda} & \Sigma A \\
& \xrightarrow{l} & \\
\Sigma (I \otimes A) & \xrightarrow{\Sigma \lambda} & \Sigma A
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{ccc}
\Sigma A \otimes I & \xrightarrow{\rho} & \Sigma A \\
\Sigma (A \otimes I) & \xrightarrow{\Sigma r} & \Sigma A
\end{array}
\]

commute for every object $A \in \mathcal{C}$.

(4) The diagram

\[
\begin{array}{ccc}
\end{array}
\]

\[
\begin{array}{ccc}
\end{array}
\]
anti-commutes for all objects $A, B \in \mathcal{C}$.

Note that axiom (2) can be reformulated as follows: for every object $A$ and $n$-angle

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} \Sigma A_1$$

in $\mathcal{C}$, the two $n$-$\Sigma$-sequences

$$A \otimes A_1 \xrightarrow{1 \otimes a_1} A \otimes A_2 \xrightarrow{1 \otimes a_2} \cdots \xrightarrow{1 \otimes a_{n-1}} A \otimes A_n \xrightarrow{1 \otimes (a_n \Sigma)} \Sigma (A \otimes A)$$

$$A_1 \otimes A \xrightarrow{a_1 \otimes 1} A_2 \otimes A \xrightarrow{a_2 \otimes 1} \cdots \xrightarrow{a_{n-1} \otimes 1} A_n \otimes A \xrightarrow{a_n \otimes (\Sigma A_1)} \Sigma (A_1 \otimes A)$$

are also $n$-angles in $\mathcal{C}$. Note also that in the triangulated case, i.e. when $n = 3$, then some authors include further axioms for a tensor triangulated category, cf. [Bal] Remark 4 and [HPS][KeN][May]. However, we will not need $n$-angulated versions of these axioms.

**Lemma 5.1.** Let $(\mathcal{C}, \Sigma)$ be a tensor $n$-angulated category, with $n$ an odd integer. Then the Grothendieck group $K_0(\mathcal{C})$ is a commutative ring with multiplication given by $[A] [B] = [A \otimes B]$ for objects $A, B \in \mathcal{C}$.

**Proof.** Suppose that $A, B$, and $B'$ are objects in $\mathcal{C}$ with $[B] = [B']$ in $K_0(\mathcal{C})$. We first have to show that the multiplication is well-defined, i.e. that $[A] [B] = [A] [B']$. By Corollary 2.4, there exist objects $C_1, \ldots, C_n$ and two $n$-angles

$$B \otimes C_1 \xrightarrow{a_1} C_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} C_n \xrightarrow{a_n} \Sigma A \otimes \Sigma C_1$$

$$B' \otimes C_1 \xrightarrow{b_1} C_2 \xrightarrow{b_2} \cdots \xrightarrow{b_{n-1}} C_n \xrightarrow{b_n} \Sigma B \otimes \Sigma C_1$$

in $\mathcal{C}$. Tensoring these with the object $A$ yields two new $n$-angles, and together with Corollary 2.4 again these show that

$$[A] [B] = [A \otimes B] = [A \otimes B'] = [A] [B']$$

in $K_0(\mathcal{C})$. The multiplication is therefore well-defined.

The associativity axiom and symmetry axiom for a symmetric tensor category guarantees that the multiplication in $K_0(\mathcal{C})$ is associative and commutative. Furthermore, by the unit axiom, the image $[I]$ in $K_0(\mathcal{C})$ of the tensor unit object $I \in \mathcal{C}$ is the multiplicative identity. Finally, the equalities

$$[A] ([B] + [C]) = [A] [B \otimes C] = [(A \otimes B) \otimes (A \otimes C)] = [A \otimes (B + A \otimes C)] = [A \otimes B] + [A \otimes C] = [A] [B] + [A] [C]$$

show that the distributive law holds.

We may therefore speak of the Grothendieck ring $K_0(\mathcal{C})$ of a tensor $n$-angulated category $(\mathcal{C}, \Sigma)$. We end with a tensor version of Theorem 4.4 showing that the set of ideals in this ring classifies the “ideals” in $\mathcal{C}$. Namely, let $\mathcal{A}$ be a subcategory of $\mathcal{C}$. Then $\mathcal{A}$ is an $n$-angulated tensor ideal of $\mathcal{C}$ if it is an $n$-angulated subcategory with the following additional property: for objects $A \in \mathcal{A}$ and $C \in \mathcal{C}$ the tensor product $A \otimes C$ belongs to $\mathcal{A}$. Furthermore, we say that $\mathcal{A}$ is an $n$-angulated tensor prime ideal of $\mathcal{C}$ if it is an $n$-angulated tensor ideal with the following property: if $C \otimes C'$ belongs to $\mathcal{A}$ for objects $C, C' \in \mathcal{C}$, then either $C$ or $C'$ belongs to $\mathcal{A}$.

**Theorem 5.2.** Let $n \geq 3$ be an odd integer and $(\mathcal{C}, \Sigma)$ a tensor $n$-angulated category. Denote the set of complete $n$-angulated tensor ideals of $\mathcal{C}$ by $\text{Comp}^n(\mathcal{C})$, the set of dense $n$-angulated tensor ideals of $\mathcal{C}$ by $\text{Dense}^n(\mathcal{C})$, and the set of ideals of $K_0(\mathcal{C})$ by $\text{Ideal}(K_0(\mathcal{C}))$. Furthermore, denote the set of complete $n$-angulated tensor prime ideals of $\mathcal{C}$ by $\text{Comp}_p^n(\mathcal{C})$, the set of dense $n$-angulated tensor prime ideals of $\mathcal{C}$ by $\text{Dense}_p^n(\mathcal{C})$, and the set of prime ideals of $K_0(\mathcal{C})$ by $\text{Prime}(K_0(\mathcal{C}))$. Then there are one-to-one correspondences

$$\text{Comp}^n(\mathcal{C}) \cap \text{Dense}^n(\mathcal{C}) \longleftrightarrow \text{Ideal}(K_0(\mathcal{C}))$$

$$\text{Comp}_p^n(\mathcal{C}) \cap \text{Dense}_p^n(\mathcal{C}) \longleftrightarrow \text{Prime}(K_0(\mathcal{C}))$$

$$\mathcal{A} \longleftrightarrow \text{Im} K_0(\mathcal{A})$$

$$\mathcal{A}_H \longleftrightarrow H$$

where $\mathcal{A}_H$ is the subcategory of $\mathcal{C}$ consisting of those objects $A$ in $\mathcal{C}$ such that $[A] \in H \leq K_0(\mathcal{C})$. 


Proof. In view of Theorem 4.6 it suffices to show that the given correspondences map complete and dense $n$-angulated tensor (prime) ideals of $\mathcal{C}$ to (prime) ideals of the Grothendieck ring, and vice versa.

Let $H$ be an ideal in $K_0(\mathcal{C})$, and consider the subcategory $\mathcal{A}_H$ of $\mathcal{C}$. We know from Theorem 4.6 that $\mathcal{A}_H$ is a complete and dense $n$-angulated subcategory of $\mathcal{C}$. If $A \in \mathcal{A}_H$ and $C \in \mathcal{C}$, then by definition $[A]$ belongs to $H$ in $K_0(\mathcal{C})$. Since $H$ is an ideal, the element $[C][A]$ also belongs to $H$, i.e. $[C \otimes A] \in H$. But then $C \otimes A$ belongs to $\mathcal{A}_H$, hence $\mathcal{A}_H$ is an $n$-angulated tensor ideal of $\mathcal{C}$. Suppose now that $H$ is a prime ideal, and let $C, C' \in \mathcal{C}$ be objects with $C \otimes C' \in \mathcal{A}_H$. Then $[C \otimes C']$ belongs to $H$ in $K_0(\mathcal{C})$, i.e. $[C][C'] \in H$. Since $H$ is a prime ideal, either $[C]$ or $[C']$ belongs to $H$, hence either $C$ or $C'$ belongs to $\mathcal{A}_H$. This shows that $\mathcal{A}_H$ is an $n$-angulated tensor prime ideal of $\mathcal{C}$.

Conversely, let $\mathcal{A}$ be a complete and dense $n$-angulated tensor ideal of $\mathcal{C}$, and consider the subgroup $\text{Im} K_0(\mathcal{A})$ of $K_0(\mathcal{C})$. Let $[C]$ and $[A]$ be elements in $K_0(\mathcal{C})$ with $[A] \in \text{Im} K_0(\mathcal{A})$. Using Theorem 4.6 again, we see that the object $A$ then belongs to $\mathcal{A}$. Since $\mathcal{A}$ is an $n$-angulated tensor ideal of $\mathcal{C}$, the object $C \otimes A$ also belongs to $\mathcal{A}$, hence $[C \otimes A] \in \text{Im} K_0(\mathcal{A})$ in $K_0(\mathcal{C})$. But then $[C][A] \in \text{Im} K_0(\mathcal{A})$, hence $\text{Im} K_0(\mathcal{A})$ is an ideal of $K_0(\mathcal{C})$. Suppose now that $\mathcal{A}$ is an $n$-angulated tensor prime ideal of $\mathcal{C}$, and let $[C], [C'] \in K_0(\mathcal{C})$ be elements with $[C][C'] \in \text{Im} K_0(\mathcal{A})$. Then $[C \otimes C']$ belongs to $\text{Im} K_0(\mathcal{A})$, and as above this implies that the object $C \otimes C'$ belongs to $\mathcal{A}$. Since $\mathcal{A}$ is an $n$-angulated tensor prime ideal of $\mathcal{C}$, either $C$ or $C'$ belongs to $\mathcal{A}$, hence either $[C]$ or $[C']$ belongs to $\text{Im} K_0(\mathcal{A})$ in $K_0(\mathcal{C})$. This shows that $\text{Im} K_0(\mathcal{A})$ is a prime ideal of $K_0(\mathcal{C})$.

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