VANISHING VISCOSITY LIMIT FOR GLOBAL ATTRACTORS FOR THE DAMPED NAVIER–STOKES SYSTEM WITH STRESS FREE BOUNDARY CONDITIONS

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To Edriss Titi on the occasion of his 60-th birthday with warmest regards

Abstract. We consider the damped and driven Navier–Stokes system with stress free boundary conditions and the damped Euler system in a bounded domain $\Omega \subset \mathbb{R}^2$. We show that the damped Euler system has a (strong) global attractor in $H^1(\Omega)$. We also show that in the vanishing viscosity limit the global attractors of the Navier–Stokes system converge in the non-symmetric Hausdorff distance in $H^1(\Omega)$ to the strong global attractor of the limiting damped Euler system (whose solutions are not necessarily unique).

1. Introduction

In this paper, we study from the point of view of global attractors the 2D damped and driven Navier–Stokes system

\[
\begin{aligned}
\partial_t u + (u, \nabla)u + \nabla p + ru &= \nu \Delta u + g(x), \\
\text{div } u &= 0, \quad u(0) = u^0,
\end{aligned}
\]  

and the corresponding limiting ($\nu = 0$) damped/driven Euler system

\[
\begin{aligned}
\partial_t u + (u, \nabla)u + \nabla p + ru &= g(x), \\
\text{div } u &= 0, \quad u(0) = u^0.
\end{aligned}
\]  

Both systems are considered in a bounded multiply connected smooth domain $\Omega \subset \mathbb{R}^2$ with standard non-penetration boundary condition

\[
u \cdot n|_{\partial \Omega} = 0, 
\]
while the system (1.1) is supplemented with the so-called stress free or slip boundary conditions
\[
\begin{align*}
\mathbf{u} \cdot 
abla \mathbf{n} & = 0, \\
\text{curl } 
abla \mathbf{u} & = 0.
\end{align*}
\]
(1.4)

The Laplace operator with (1.4) commutes the Leray projection. These boundary conditions guarantee the absence of the boundary layer and yield the conservation of enstrophy in the unforced and undamped case of (1.2). They are also convenient for studying the limit as \( \nu \to 0^+ \) of the individual solutions of the 2D Navier–Stokes system [4, 25].

Systems (1.1) and (1.2) are relevant in geophysical hydrodynamics and the damping term \(- ru\) describes the Rayleigh or Ekman friction and parameterizes the main dissipation occurring in the planetary boundary layer (see, for example, [27]). The viscous term \(- \nu \Delta \mathbf{u}\) in system (1.1) is responsible for the small scale dissipation. We also observe that in physically relevant cases we have \( \nu \ll r |\Omega| \).

The damped and driven 2D Euler and Navier-Stokes systems attracted considerable attention over the last years and were studied from different points of view. The regularity, uniqueness, and stability of the stationary solutions for (1.2) were studied in [5, 29, 33]. The vanishing viscosity limit for system (1.1) was studied for steady-state statistical solutions in [14].

In the presence of the damping term the weak attractor for the system (1.2) was constructed in [17] in the phase space \( H^1 \). In the trajectory phase space the weak attractor was constructed in [6, 7].

The dynamical effects of the damping term \(- ru\) in the case of the Navier–Stokes system (1.1) were studied in [21] on the torus, on the 2D sphere, and in bounded (simply connected) domain \( \Omega \) with boundary conditions (1.4). Specifically, it was shown that the fractal dimension of the global attractor \( \mathcal{A}_\nu \) satisfies the estimate
\[
\dim_f \mathcal{A}_\nu \leq \min \left( c_1(\Omega) \frac{\| \text{curl } g \| |\Omega|^{1/2}}{\nu r}, c_2(\Omega) \frac{\| \text{curl } g \| ^2}{\nu r^3} \right),
\]
(1.5)

where \( |\Omega| \) is the area of the spatial domain. This estimate is sharp in the limit \( \nu \to 0^+ \) and the lower bound is provided by the corresponding family of Kolmogorov flows. Furthermore, the constants \( c_1 \) and \( c_2 \) are given explicitly for the torus \( \Omega = \mathbb{T}^2 \) and for the sphere \( \Omega = S^2 \). The case of an elongated torus \( \mathbb{T}^2_\alpha \) with periods \( L \) and \( L/\alpha \), where \( \alpha \to 0^+ \) was studied in [24], where it was shown that (1.5) still holds for \( \mathbb{T}^2_\alpha \) and is sharp as both \( \alpha \to 0 \) and \( \nu \to 0 \).

The essential analytical tool used in the proof of (1.5), especially in finding explicit values of \( c_1 \) and \( c_2 \), is the Lieb–Thirring inequality. New
bounds for the Lieb–Thirring constants for the anisotropic torus were recently obtained in [20] with applications to the system (1.1) on $T^2_{\alpha}$.

One might expect that in the case of the damped Navier–Stokes system (1.1) in $\mathbb{R}^2$ in the space of finite energy solutions the attractor $\mathcal{A}_\nu$ exists and its fractal dimension is bounded by the second number on the right-hand side in (1.5). It was recently shown in [22] that it is indeed the case:

$$\dim_f \mathcal{A}_\nu \leq \frac{1}{16\sqrt{3}} \frac{\| \text{curl } g \|_2^2}{\nu^{\frac{3}{2}}}.$$  \hfill (1.6)

Moreover, due to convenient scaling available for $\mathbb{R}^2$ this estimate of the dimension is included in [22] in the family of estimates depending on the norm of $g$ in the scale of homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^2)$, $-1 \leq s \leq 1$; the case $s = 1$ being precisely (1.6).

Estimates for the degrees of freedom for the damped Navier–Stokes system (1.1) expressed in terms of various finite dimensional projections were obtained in [23]. They are also of the order (1.5).

We point out two important differences between the damped Navier–Stokes system (1.1) and the damped Euler system (1.2) which make the construction of the global attractor for (1.2) less straightforward. The first is the absence for (1.2) of the instantaneous smoothing property of solutions and explains why the existence of only a weak attractor was first established [17]. The second is that the uniqueness is only known for the solutions with bounded vorticity [34] and is not known in the natural Sobolev space $H^1$, which makes the trajectory attractors very convenient for (1.2), see [6, 7, 8, 9, 32]. The trajectory attractors for (1.2) in the weak topology of $H^1$ were constructed in [7] (see also [6]) and in [12] for the non-autonomous case. In addition, the upper semi-continuous dependence as $\nu \to 0+$ of the trajectory attractors of the system (1.1) on the torus was established in [7] in the weak topology of $H^1(T^2)$.

The existence of the strong $H^1$ trajectory attractors for the dissipative Euler system (1.2) on the 2D torus was proved in [10] under the assumption that $\text{curl } g \in L^\infty$ which was used to prove the enstrophy equality. The strong attraction and compactness for the trajectory attractor were established using the energy method developed in [3, 16, 26, 28] for the equations in unbounded, non-smooth domains or for equations without uniqueness. This method is based on the corresponding energy balance for the solutions and leads to the asymptotic compactness of the solution semigroups or collections the trajectories.

Most closely related to the present work is the paper [13] where the strong global and trajectory $W^{1,p}$-attractors were constructed for the
system (1.2) in $\mathbb{R}^2$. The crucial equation of the enstrophy balance is proved there in the Sobolev spaces $W^{1,p}$, $2 \leq p < \infty$ without the assumption on $g$ that guarantees the uniqueness of solutions on the attractor. Instead the authors used the fact that in the 2D case the vorticity satisfies a scalar transport equation, and the required enstrophy equality directly follows from the results of [15].

In unbounded domains the damped Navier–Stokes and Euler systems can be studied from the point of view of uniformly local spaces (where the energy is infinite) and one of the main issues is the proof of the dissipative estimate, which is achieved by means of delicate weighted estimates. In the uniformly local spaces in the viscous case $\nu > 0$ the global attractors for (1.1) in the strong topology were constructed in [37], see also [35, 36] for similar results in channel-like domains. In the inviscid case the strong attractor for (1.2) in the uniformly local $H^1$ space was recently constructed in [11].

In the present paper we study the convergence of the global attractors $\mathcal{A}_\nu$ of the system (1.1), (1.4) in the vanishing viscosity limit $\nu \to 0^+$, and our main result is as follows. The system (1.2), (1.3) has a global attractor $\mathcal{A}_0 \subseteq H^1(\Omega)$. For every $\delta$-neighbourhood $\mathcal{O}_\delta$ of $\mathcal{A}_0$ in $H^1(\Omega)$ there exits $\nu(\delta) > 0$ such that

$$\mathcal{A}_\nu \subset \mathcal{O}_\delta(\mathcal{A}_0) \quad \text{for all} \quad \nu \leq \nu(\delta), \quad (1.7)$$

where $\mathcal{A}_\nu$ for $\nu > 0$ are the attractors of the damped Navier–Stokes system (1.1), (1.4).

We point out that despite the fact that the dimension of $\mathcal{A}_\nu$ can be of order $1/\nu$ as $\nu \to 0^+$ (at least in the periodic case and the special family Kolmogorov-type forcing terms) the limiting attractor $\mathcal{A}_0$ is, nonetheless, a compact set in $H^1(\Omega)$.

This paper has the following structure. In Section 2 we define the function spaces, paying attention the case when the domain $\Omega$ is multiply connected, and construct the global attractors $\mathcal{A}_\nu$ for (1.1), (1.4). In Section 3 we prove the existence of weak solutions of the damped Euler system (1.2). We adapt the theory of renormalized solutions from [15] to the vorticity equation in a bounded domain which gives us the crucial equation of the enstrophy balance for an arbitrary weak solution of (1.2). In Section 4 we consider the generalized solution semigroup for the system (1.2) and define weak and strong global attractors for the generalized semigroup. We first construct a weak global attractor $\mathcal{A}_0$ in $H^1$ for (1.2) and then we prove the asymptotic compactness of the generalized semigroup which gives that the weak global attractor $\mathcal{A}_0$ is, in fact, the $H^1$ strong global attractor. In Section 5 we prove (1.7).
2. Equations and function spaces

We shall be dealing with the damped and driven Navier–Stokes system (1.1) with boundary conditions (1.4) and the corresponding limiting (\( \nu = 0 \)) damped Euler system (1.2) with standard non-penetration condition (1.3).

Both systems are studied in a bounded domain \( \Omega \subset \mathbb{R}^2 \). We consider the general case when \( \Omega \) can be multiply connected with boundary \( \partial \Omega = \Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k \).

In other words, \( \Gamma_0 \) is the outer boundary, and the \( \Gamma_i \)'s are the boundaries of \( k \) islands inside \( \Gamma_0 \). We assume that \( \partial \Omega \) is smooth (\( C^2 \) will be enough) so that there exists a well-defined outward unit normal \( n \) and also an extension operator \( E \):

\[ E : H^2(\Omega) \to H^2(\mathbb{R}^2), \quad \| Eu \|_{H^2(\mathbb{R}^2)} \leq \text{const} \| u \|_{H^2(\Omega)}. \]

We now introduce the required function spaces and their orthogonal decompositions. We set

\[ H = \{ u \in L^2(\Omega), \ \text{div} u = 0, \ u \cdot n|_{\partial \Omega} = 0 \}. \]

The following orthogonal decomposition holds [31, Appendix 1]:

\[ H = H_0 \oplus H_c, \tag{2.1} \]

where

\[ H_0 = \{ u \in L^2(\Omega), \ \text{div} u = 0, \ u \cdot n|_{\partial \Omega} = 0, \ u = \nabla^\perp \varphi, \ \varphi \in H^1_0(\Omega) \}, \]

that is, the vector functions in \( H_0 \) have a unique single valued stream function \( \varphi \) vanishing at all components of the boundary \( \partial \Omega \). Here \( \varphi \) is a scalar function, and

\[ \nabla^\perp \varphi := \{ -\partial_2 \varphi, \partial_1 \varphi \} = -\text{curl} \varphi, \quad u^\perp := \{ -u^2, u^1 \}. \]

Accordingly, the orthogonal complement to \( H_0 \) in \( H \) is the \( k \)-dimensional space of harmonic (and hence infinitely smooth) vector functions:

\[ H_c = \{ u \in L^2(\Omega), \ \text{div} u = 0, \ u \cdot n|_{\partial \Omega} = 0 \}. \]

In the similar way, for smoothness of order one we have

\[ \mathcal{H}^1 := \{ u \in H^1(\Omega), \ \text{div} u = 0, \ u \cdot n|_{\partial \Omega} = 0 \} = H_1 \oplus H_c, \]

where \( H_c \) is as before and

\[ H_1 = \{ u = \nabla^\perp \varphi, \ \varphi \in H^2(\Omega) \cap H^1_0(\Omega) \}, \quad \| u \|_{H_1} = \| \Delta \varphi \|. \]

For smoothness of order two

\[ \mathcal{H}^2 := \{ u \in H^2(\Omega), \ \text{div} u = 0, \ u \cdot n|_{\partial \Omega} = 0 \} = H_2 \oplus H_c, \]
where
\[ H_2 = \{ u = \nabla^\perp \varphi, \varphi \in H^3(\Omega) \cap H^1_0(\Omega) \}, \quad \| u \|_{H_2} = \| \nabla \Delta \varphi \|. \]

Corresponding to the second boundary condition in (1.4) is the following closed subspace in \( H_2^2 \):
\[ H_0^2 = \{ u = \nabla^\perp \varphi, \varphi \in H^3(\Omega) \cap H^1_0(\Omega) \cap \{ \Delta \varphi|_{\partial \Omega} = 0 \} \}. \]

The space of all divergence free vector functions of class \( H^2(\Omega) \) satisfying the boundary conditions (1.4) is denoted by \( H^2_0 \):
\[ H^2_0 = H_2^0 \oplus H_c. \] (2.2)

The orthonormal basis in \( H_0 \) is made up of vector functions
\[ u_j = \lambda_j^{-1/2} \nabla^\perp \varphi_j, \]
where \( \lambda_j \) and \( \varphi_j \) are the eigenvalues and eigenfunctions of the scalar Dirichlet Laplacian \( -\Delta \varphi_j = \lambda_j \varphi_j, \varphi_j|_{\partial \Omega} = 0, \quad 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to +\infty. \)

In fact,
\[ \| u_j \|^2 = \lambda_j^{-1}(\nabla^\perp \varphi_j, \nabla^\perp \varphi_j) = \lambda_j^{-1}\| \nabla \varphi_j \|^2 = 1. \]

Furthermore, since on scalars
\[ \text{curl} \nabla^\perp = -\text{curl} \text{curl} = \Delta, \quad \text{curl} = -\nabla^\perp, \] (2.3)
the \( u_j \)'s satisfy (1.4), and the system \( \{ u_j \}_{j=1}^\infty \) is the complete orthonormal basis of eigen vector functions with eigenvalues \( \{ \lambda_j \}_{j=1}^\infty \) of the vector Laplacian
\[ \Delta = \nabla \text{div} - \text{curl} \text{curl} \] (2.4)
with boundary conditions (1.4):
\[ -\Delta u_j = \text{curl} \text{curl} u_j = \lambda_j u_j. \]

We can express the fact that a vector function \( u \) belongs to \( H_0, H_1, \) or \( H^2_0 \) in terms of its Fourier coefficients as follows. Let
\[ u = \sum_{j=1}^\infty c_j u_j, \quad c_j = (u, u_j) = \lambda_j^{-1/2}(u, \nabla^\perp \varphi_j), \] (2.5)
where (setting \( \omega := \text{curl} u \))
\[ (u, \nabla^\perp \varphi_j) = (u^\perp, \nabla \varphi_j) = -(\text{div} u^\perp, \varphi_j) = (\text{curl} u, \varphi_j) = (\omega, \varphi_j). \]
This gives that
\[ u \in H_0 \iff \sum_{j=1}^{\infty} c_j^2 = \|u\|^2 = \|\omega\|_{H^{-1}(\Omega)}^2 < \infty, \]
\[ u \in H_1 \iff \sum_{j=1}^{\infty} \lambda_j c_j^2 = \|\omega\|^2 < \infty, \]
\[ u \in H^0_2 \iff \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 = \|\text{curl} \text{ curl} u\|^2 = \|\omega\|_{H^1_0(\Omega)}^2 < \infty. \]

The basis in the $k$-dimensional space of harmonic vector functions $H_c$ is given in [31, Appendix 1, Lemma 1.2] in terms of the gradients of harmonic multi valued functions. In our 2D case it is more convenient to construct a basis in $H_c$ in terms of single valued stream functions.

**Lemma 2.1.** The system \( \{ \nabla \psi^j \}_{j=1}^{k} \) is a basis in $H_c$. Here $\psi^j$ is the solution in $\Omega$ of the equation $\Delta \psi^j = 0$, where $\psi^j = 0$ at all the components of the boundary $\Gamma$ except for $\Gamma_j$, where $\psi^j = 1$.

**Proof.** The vector functions $\nabla \psi^j \in H_c$ and are linearly independent. \( \square \)

Next, we consider the Leray projection $P$ from $L^2(\Omega)$ onto $H$. In accordance with (2.1) we have $P = P_0 \oplus P_c$. For the projection $P_0$ onto $H_0$ we have
\[ P_0 u = \nabla \perp (\Delta^D_\Omega)^{-1} \text{curl} u, \tag{2.6} \]
where $\Delta^D_\Omega$ is the (scalar) Dirichlet Laplacian, which is an isomorphism from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$.

**Lemma 2.2.** On $H^2_0$ the projection $P$ commutes with the Laplacian $\Delta$ with boundary conditions (1.4).

**Proof.** Since $P_c \Delta = \Delta P_c = 0$ on $H_c$, it suffices to consider $P_0$. Let $u \in H^2_0$, see (2.2), so that $P_0 u = u$. Then interpreting $\text{curl} u$ as a scalar and using (2.3) we obtain
\[ P_0 \Delta u = -\nabla \perp (\Delta^D_\Omega)^{-1} \text{curl} \text{ curl} u = \nabla \perp (\Delta^D_\Omega)^{-1} \text{curl} u = \nabla \perp \text{curl} u = -\text{curl} \text{ curl} u = \Delta u = \Delta P_0 u. \]

\( \square \)

This lemma makes the subsequent analysis very similar to the 2D periodic case or the case of a manifold without boundary.

We also recall the familiar formulas
\[ (\nabla \varphi, v) = -(\varphi, \text{div} v), \quad v \cdot n|_{\partial \Omega} = 0, \]
\[ (\text{curl} \varphi, v) = (\varphi, \text{curl} v), \quad \text{curl} v|_{\partial \Omega} = 0. \tag{2.7} \]
Lemma 2.3. [18] Let \( u \in H^0_2 \) (see (2.2)). Then
\[
((u, \nabla)u, \Delta u) = 0. \tag{2.8}
\]

Proof. We use the invariant expression for the convective term
\[
(u, \nabla)u = \text{curl} \, u \times u + \frac{1}{2} \nabla \cdot u^2.
\]

Let \( u = u_0 + u_c \), where \( u_0 \in H^0_2 \), \( u_c \in H_c \). Then, taking into account (2.4), for the second term in the above expression we have
\[
(\nabla u^2, \text{curl} \, u_0) = (\text{curl} \, \nabla u^2, \text{curl} \, u_0) = 0,
\]
since \( \text{curl} \, \nabla = 0 \) algebraically, and the first equality follows from (2.7) with boundary condition \( \text{curl} \, u_0 |_{\partial \Omega} = 0 \)

For the first term we have setting \( \omega = \text{curl} \, u_0 \) and using (2.3)
\[
(\text{curl} \, u_0 \times u, \text{curl} \, \text{curl} \, u_0) = -(\omega u^1, \nabla^1 \omega) =
-\frac{1}{2} (u, \nabla \omega^2) = \frac{1}{2} (\text{div} \, u, \omega^2) = 0,
\]
where we used \( u \cdot n |_{\partial \Omega} = 0 \) for the integration by parts. \( \square \)

We also recall the familiar orthogonality relation
\[
b(u, v, v) = 0, \tag{2.9}
\]
where the trilinear form \( b \)
\[
b(u, v, w) = \int_{\Omega} \sum_{i,j=1}^2 u^i \partial_i v^j w^j \, dx
\]
is continuous on \( H^1 \).

The space \( H_c \) of (infinitely smooth) harmonic vector functions is \( k \)-dimensional, and every Sobolev norm \( H^k(\Omega) \) is equivalent to the \( L^2(\Omega) \)-norm. Therefore the \( H^1(\Omega) \)-norm on \( H^1 \) for \( u = u_0 + u_c \in H^1_1 \oplus H_c \) can be given by
\[
\|u\|_1^2 := \|u\|^2 + \|\text{curl} \, u\|^2 = \|u\|^2 + \|\text{curl} \, u_0\|^2.
\]
Accordingly, the \( H^2(\Omega) \)-norm on \( H^2_0 \) is given by
\[
\|u\|_2^2 := \|u\|^2 + \|\text{curl} \, \text{curl} \, u\|^2 = \|u\|^2 + \|\text{curl} \, \text{curl} \, u_0\|^2.
\]

Theorem 2.4. Let the initial data \( u^0 \) and the right-hand side \( g \) in the damped Navier–Stokes system (1.1), (1.4) satisfy
\[
u^0 \in H^1, \quad g \in H^1.
\]
Then there exists a unique strong solution \( u \in C([0, T]; H^1) \cap L^2(0, T; H^2_0) \) of (1.1), (1.4). Thus, a semigroup of solution operators
\[
u(t) = S(t)u(0),
\]
corresponding to (1.1), (1.4) is well defined.

The solution satisfies the equation of balance of energy and enstrophy:

$$\frac{1}{2} \frac{d}{dt} \|u\|_1^2 + \nu \|\Delta u\|^2 + r \|u\|_1^2 = (g, u)_1,$$

(2.10)

where

$$(g, u)_1 := (g, u) + (\text{curl } g, \text{curl } u).$$

Proof. The proof is standard and uses the Galerkin method. We use the special basis (2.5) in $H_0^2 \subset H^1$ and supplement it with a $k$-dimensional basis in $H_c$, for example, with the one from Lemma 2.1 starting the enumeration from the basis in $H_c$.

Then for every approximate Galerkin solution

$$u = u^{(n)} = \sum_{k=1}^n c_k u_k \in H_0^2$$

we have the orthogonality relations (2.8), (2.9). We take the scalar product of (1.1) with $u$, and also with $\Delta u$, integrating by parts using (2.7), drop the $\nu$-terms and use Growwall’s inequality to obtain in the standard way the estimates

$$\|u(t)\|^2 \leq \|u(0)\| e^{-rt} + r^{-2}\|g\|^2,$$

$$\|\text{curl } u(t)\|^2 \leq \|\text{curl } u(0)\| e^{-rt} + r^{-2}\|\text{curl } g\|^2,$$

which gives

$$\|u(t)\|_1^2 \leq \|u(0)\|_1 e^{-rt} + r^{-2}\|g\|^2_1$$

(2.11)

for $u = u^{(n)}$, uniformly for $n$ and $\nu > 0$. The remaining assertions of the theorem are proved very similarly to the classical case of the 2D Navier–Stokes system with Dirichlet boundary conditions (even simpler, since we now have more regularity, see, for instance, [2], [31]).

We recall the following definition of the (strong) global attractor (see, for instance [2], [30]).

Definition 2.5. Let $S(t)$, $t \geq 0$, be a semigroup acting in a Banach space $B$. Then the set $\mathcal{A} \subset B$ is a global attractor of $S(t)$ if

1) $\mathcal{A}$ is compact in $B$: $\mathcal{A} \subseteq B$.

2) $\mathcal{A}$ is strictly invariant: $S(t)\mathcal{A} = \mathcal{A}$.

3) $\mathcal{A}$ is globally attracting, that is,

$$\lim_{t \to \infty} \text{dist}(S(t)B, \mathcal{A}) = 0,$$

for every bounded set $B \subset B$.

Theorem 2.6. The semigroup $S(t)$ corresponding to (1.1), (1.4) has a global attractor $\mathcal{A} \subset H^1$. 

Proof. It follows from (2.11) that the ball
\[ B_0 = \{ u \in H^1, \|u\|_1^2 \leq 2r^{-2}\|g\|_1^2 \} \] (2.12)
is the absorbing ball for \( S(t) \). The semigroup \( S(t) \) is continuous in \( H^1 \) and has the smoothing property (which can be shown similarly to the classical 2D Navier–Stokes system [2], [30]). Therefore the set
\[ B_1 = S(1)B_0 \]
is a compact absorbing set, which gives the existence of the attractor \( \mathcal{A} \subset H^1 \). We finally point out that for \( u(t) \in \mathcal{A} \) we have for all \( t \in \mathbb{R} \)
\[ \|u(t)\|_1 \leq r^{-1}\|g\|_1 \] (2.13)
uniformly with respect to \( \nu > 0 \).

3. Weak solutions for the Euler system and energy-enstrophy balance

We now turn to the damped and driven Euler system (1.2), (1.3).

Definition 3.1. Let \( u(0), g \in H^1 \). A vector function \( u = u(t, x) \) is called a weak solution of (1.2), (1.3) if \( u \in L^\infty(0, T; H^1) \) and satisfies the integral identity
\[
-\int_0^T (u, v\eta'(t))dt + \int_0^T b(u, u, v\eta(t))dt +
\]
\[
+ r \int_0^T (u, v\eta(t))dt = \int_0^T (g, v\eta(t))dt
\]
(3.1)
for all \( \eta \in C^\infty_0(0, T) \) and all \( v \in H^1 \).

Theorem 3.2. There exists at least one solution of the damped Euler system (1.2), (1.3). Moreover, every weak solution in the sense of Definition 3.1 is of class \( C([0, T]; H^1) \) and satisfies the equation of balance of energy
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + r\|u(t)\|^2 = (g, u(t)).
\]
(3.2)

Proof. As before we use the special basis and see that approximate Galerkin solutions \( u^n \) satisfy (2.11) and therefore we obtain that uniformly with respect to \( n \)
\[ u^n \in L^\infty(0, T; H^1) \).

Next, we see from equation (1.2) that \( \partial_t u^n \) is bounded in \( L^2(0, T; H^{-1}) \). Therefore we can extract a subsequence (still denoted by \( u^n \)) such that
\[ u^n \to u \text{-weakly in } L^\infty(0, T; H^1) \text{ and strongly in } L^2(0, T; H). \]
This is enough to pass to the limit in the non-linear term in (3.1) and therefore to verify that \( u \) satisfies (3.1). Since \( \partial_t u \in L^2(0, T; \mathcal{H}^{-1}) \), it follows that we can take the scalar product of (1.2) with the solution \( u \) to obtain (3.2), see [31]. □

We now derive the scalar equation for \( \omega = \text{curl} \ u \). We set in (3.1) \( v = \text{curl} \varphi, \ \varphi \in C_0^\infty(\Omega) \) and integrate by parts the linear terms in (3.1) by using the second formula in (2.7). For the non-linear term we have

\[
\begin{align*}
    b(u, u, v) &= \int_\Omega (u, \nabla) u \cdot \text{curl} \varphi \, dx = \int_\Omega (\text{curl} u \times u) \cdot \text{curl} \varphi \, dx = \int_\Omega (\omega u^\perp) \cdot \text{curl} \varphi \, dx = \int_\Omega \text{curl}(\omega u^\perp) \varphi \, dx = \int_\Omega u \nabla \omega \varphi \, dx, \\
    \text{since algebraically} \quad \text{curl}(\omega u^\perp) &= \omega \text{div} u + u \nabla \omega.
\end{align*}
\]

Thus, we have shown that \( \omega = \text{curl} \ u \) satisfies in \( \Omega \) the following equation (in the sense of distributions)

\[
\begin{align*}
    \partial_t \omega + u \nabla \omega + r \omega &= G, \\
    \omega(0) &= \omega^0.
\end{align*}
\]  

(3.4)

where \( G = \text{curl} \, g, \ \omega^0 = \text{curl} \, u(0) \).

We observe that we can integrate by parts the last term in (3.3) another time using the boundary condition for \( u \) only: \( u \cdot n|_{\partial \Omega} = 0 \). Namely, for every \( \varphi \in C^\infty(\bar{\Omega}) \) it holds

\[
\int_\Omega u \nabla \omega \varphi \, dx = -\int_\Omega \omega \text{div}(u \varphi) \, dx,
\]

where \( \varphi \) does not necessarily vanish at \( \partial \Omega \), we use \( u \cdot n|_{\partial \Omega} = 0 \) instead.

The above argument shows that if \( u \) is a weak solution of the Euler system (1.2), then \( \omega = \text{curl} \ u \) satisfies the following integral identity:

\[
\begin{align*}
    -\int_0^T \int_\Omega \omega \varphi \eta'(t) \, dxdt - \int_0^T \int_\Omega \omega \text{div}(u \varphi) \eta(t) \, dxdt + \\
    + r \int_0^T \int_\Omega \omega \varphi \eta(t) \, dxdt &= \int_0^T \int_\Omega G \varphi \eta(t) \, dxdt,
\end{align*}
\]

(3.5)

holding for all \( \varphi \in C^\infty(\bar{\Omega}) \).

We now extend \( \omega \) by zero outside \( \Omega \) setting for all \( t \)

\[
\tilde{\omega} = \begin{cases} 
    \omega, & \text{in } \Omega; \\
    0, & \text{in } \Omega^c = \mathbb{R}^2 \setminus \Omega.
\end{cases}
\]
In the similar way by define $\tilde{G}$. The vector function $u$ is extended to a $\tilde{u} \in H^1(\mathbb{R}^2)$ in a certain way that will be specified later. Since $\varphi$ in $u \in H^1(\mathbb{R}^2)$ is an arbitrary smooth function in $C^\infty(\Omega)$, it follows that the following integral identity holds in the whole $\mathbb{R}^2$

$$-\int_0^T \int_{\mathbb{R}^2} \tilde{\omega} \varphi'(t)dxdt - \int_0^T \int_{\mathbb{R}^2} \tilde{\omega} \text{div}(\tilde{u}\varphi)\eta(t)dxdt +$$

$$+ r \int_0^T \int_{\mathbb{R}^2} \tilde{\omega} \varphi\eta(t)dxdt = \int_0^T \int_{\mathbb{R}^2} \tilde{G}\varphi\eta(t)dxdt,$$

(3.6)

holding for all $\varphi \in C_0^\infty(\mathbb{R}^2)$, $\eta \in C_0^\infty(0,T)$.

In other words, we have shown that $\tilde{\omega}$ is a weak solution in the whole $\mathbb{R}^2$ of the equation

$$\partial_t \tilde{\omega} + \tilde{u}\nabla \tilde{\omega} + r\tilde{\omega} = \tilde{G},$$

$$\tilde{\omega}(0) = \tilde{\omega}^0.$$ (3.7)

We shall now specify the construction of $\tilde{u}$. Recall that $u = u_0 \oplus u_c$, $u_0 \in H_1$, $u_c \in H_c$, where $u \in L^\infty(0,T;H^1)$, and where $u_0$ has a single valued stream function $\psi_0$: $u_0 = \nabla^\perp \psi_0$, $\psi_0 \in H^2(\Omega)$ (we do not use the additional information that $\psi_0 = 0$ at $\Gamma$). In view of Lemma 2.1 so does $u_c$: $u_c = \nabla^\perp \psi_c$, where $\psi_c \in H^2(\Omega)$ (at least). We set $\tilde{\psi} = \psi_0 + \psi_c$ and apply the extension operator $E$: $\tilde{\psi} = E\psi$, $\tilde{\psi} \in H^2(\mathbb{R}^2)$, $||\tilde{\psi}||_{H^2(\mathbb{R}^2)} \leq c(\Omega)||\psi||_{H^2(\Omega)}$.

Then $\tilde{u} := \nabla^\perp \tilde{\psi}$ is the required extension of the vector function $u$ with $||\tilde{u}||_{H^1(\mathbb{R}^2)} \leq c(\Omega)||u||_{H^1}$, $\text{div} \tilde{u} = 0$ in the whole $\mathbb{R}^2$.

We are now in a position to apply the theory developed in [13]. In particular, it follows from [13] Theorem II.3 that the weak solution $\tilde{\omega}$ of (3.7) in the sense (3.6) is a renormalized solution, that is, satisfies

$$\partial_t \beta(\tilde{\omega}) + \tilde{u} \nabla \beta(\tilde{\omega}) + r\tilde{\omega} \beta'(\tilde{\omega}) = \beta'(\tilde{\omega})\tilde{G}$$

for all $\beta \in C^1_b(\mathbb{R})$ with $\beta(0) = 0$. This gives that

$$\frac{d}{dt} \int_{\mathbb{R}^2} \beta(\tilde{\omega})dx + r \int_{\mathbb{R}^2} \tilde{\omega} \beta'(\tilde{\omega})dx = \int_{\mathbb{R}^2} \tilde{G}\beta'(\tilde{\omega})dx.$$ 

Since $\beta(0) = 0$ and $\tilde{\omega} = 0$ outside $\Omega$, the last equation goes over to

$$\frac{d}{dt} \int_{\Omega} \beta(\omega)dx + r \int_{\Omega} \omega \beta'(\omega)dx = \int_{\Omega} G\beta'(\omega)dx.$$
Choosing now for $\beta$ appropriate approximations of the function $s \rightarrow s^2$ we finally obtain
\[
\frac{1}{2} \frac{d}{dt} \|\omega(t)\|^2 + r\|\omega(t)\|^2 = (\omega(t), G).
\] (3.8)

Thus, we have proved the following result.

**Theorem 3.3.** Every weak solution of the damped and driven Euler equation is of class $C([0,T]; H^1)$ and satisfies the equation of balance of energy and enstrophy
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_1 + r\|u\|^2_1 = (u, g)_1.
\] (3.9)

**Proof.** The equation of balance (3.9) follows from (3.2) and (3.8). The continuity in $H^1$ follows from the continuity in $H^1$ and, hence, weak continuity in $H^1$ and the continuity of the norm $t \rightarrow \|\omega(t)\|^2$, which follows from (3.8), see [31]. □

4. **Global attractor for the damped Euler system**

For every solution of the damped Euler system we obtain from (3.9) that
\[
\frac{d}{dt}\|u\|^2_1 + 2r\|u\|^2_1 = 2(g, u) \leq 2\|g\|_1\|u\|_1 \leq r\|u\|^2_1 + r^{-1}\|g\|^2_1,
\]
so that by the Gronwall inequality
\[
\|u\|^2_1 \leq \|u(0)\|^2_1 + r^{-2}\|g\|^2_1(1 - e^{-rt})
\]
the ball (2.12) is also the absorbing ball for the generalized semigroup of solution operators
\[
S(t)u^0 = \{u(t)\}
\]
for the damped Euler system, where $\{u(t)\}$ is the section at time $t$ of all weak solutions with $u(0) = u^0$.

Our goal is to show that the generalized semigroup $S(t)$ has a weak $(H^1, H^1_w)$ attractor in the sense of the following definition (see [1], [2]).

**Definition 4.1.** A set $\mathcal{A} \subset H^1$ is called an $(H^1, H^1_w)$ attractor of the generalized semigroup $S(t)$ if
1) $\mathcal{A}$ is compact in the weak topology $H^1_w$.
2) $\mathcal{A}$ is strictly invariant: $S(t)\mathcal{A} = \mathcal{A}$.
3) $\mathcal{A}$ attracts in the weak topology $H^1_w$ bounded sets in $H^1$.

We first show that $S(t)$ a semigroup in the generalized sense.
Lemma 4.2. The family $S(t)$ has the semigroup property

$$S(t + \tau)u^0 = S(t)S(\tau)u^0$$

in the sense of the equality of sets.

Proof. The inclusion $S(t + \tau)u^0 \subset S(t)S(\tau)u^0$ holds since every solution in the sense of Definition 3.1 on the interval $[0, T]$ is also a solution on every smaller interval $[\tau, T]$. Let us prove the converse inclusion:

$$S(t)S(\tau)u^0 \subset S(t + \tau)u^0$$

Any solution $u(t)$ satisfies on $[0, T]$ the integral identity

$$-\int_0^T (u, v\eta'(t))dt + \int_0^T b(u, u, v\eta(t))dt + r \int_0^T (u, v\eta(t))dt -$$

$$-\int_0^T (g, v\eta(t))dt = (u(0), v\eta(0)) - (u(T), v\eta(T))$$

for every $v \in H^1$ and $\eta \in C^\infty[0, T]$. If this identity holds on the intervals $[0, \tau]$ and $[\tau, t + \tau]$, then adding them we see that it holds on $[0, t + \tau]$ for every $\eta \in C^\infty[0, t + \tau]$. This proves (4.2). □

The generalized semigroup is not known to be continuous (the uniqueness is not proved), however, the following two properties of it are, in a sense, a substitution for the continuity and make it possible to construct a weak attractor [1], [2].

Lemma 4.3. The generalized semigroup $S(t)$ satisfies the following:

1) $[S(t)X]_w \subset S(t)[X]_w$ for any $X \subset B_0$.
2) for every $y \in H^1$ the set $S(t)^{-1}y \cap B_0$ is compact in $H^1_w$.

Here $B_0$ is the absorbing ball (2.12), and $[\ ]_w$ is the closure in $H^1_w$.

Proof. 1) Let $u = u^T \in [S(T)X]_w$. Then there exists a sequence $x_n \in X$ such that $S(T)x_n \rightarrow u^T$ weakly in $H^1_w$. The sequence $\{x_n\}$ is bounded in $H^1$ and contains a subsequence weakly converging to $x_0 \in [X]_w$.

The set of all solutions $u_n(t) = S(t)x_n$ is bounded in $C([0, T]; H^1)$, where the set $\partial_t u_n$ is bounded in $L^\infty(0, T; L^{2-\varepsilon}(\Omega))$. Therefore we can extract a subsequence $u_n$ such that

$$u_n \rightarrow u \ast\text{-weakly in } L^\infty(0, T; H^1) \text{ and strongly in } L^2(0, T; H).$$

Each $u_n$ satisfies (4.3):

$$-\int_0^T (u_n, v\eta'(t))dt + \int_0^T b(u_n, u_n, v\eta(t))dt + r \int_0^T (u_n, v\eta(t))dt -$$

$$-\int_0^T (g, v\eta(t))dt = (x_n, v\eta(0)) - (S(T)x_n, v\eta(T)).$$
The convergence (4.4) makes it possible to pass to the limit in the integral terms, while by hypotheses we have
\[(S(T)x_n, v\eta(T)) \to (u^T, v\varphi(T)), \quad (x_n, v\eta(T)) \to (u^T, v\eta(T)).\]
This proves 1), since \(u\) is a solution with \(u(0) = x_0\) and \(u(T) = u^T\), where \(x_0 \in [X]_w\).
2) The second property is proved similarly. Let
\[u_n(0) = x_n, \quad u_n(t) = y, \quad x_n \in B_0, \quad x_n \to x \in B_0 \text{ weakly in } H^1.\]
Passing to the limit as in in part 1) we obtain that the limiting function \(u\) is a solution with \(u(0) = x, \ u(t) = y, \ x \in B_0\).

This lemma shows that the hypotheses of [1, Theorem 6.1] or [2, Theorem II.1.1] are satisfied for the generalized semigroup \(S(t)\). As a result we have proved the existence of the weak attractor.

**Theorem 4.4.** The generalized semigroup \(S(t)\) corresponding to the damped Euler system has a weak \((H^1, H^1_w)-\)attractor \(\mathcal{A}\).

Our next goal is to show that the attractor \(\mathcal{A}\) is in fact a (strong) global attractor in the sense of Definition 2.5, the only difference being that the semigroup \(S(t)\) now is a generalized (multi-valued) semigroup. The key role below is played by the equation of balance of energy and enstrophy (3.9).

**Theorem 4.5.** The attractor \(\mathcal{A}\) is the (strong) global attractor.

**Proof.** We have to prove the asymptotic compactness of \(S(t)\), that is, for every sequence \(\{u_n^0\}\) bounded in \(H^1\) and every sequence \(t_n \to +\infty\) the sequence (of sets) \(S(t_n)u_n^0\) is precompact in \(H^1\).

Let \(u_n(t), \ t \geq -t_n\) be a sequence of solutions of the damped Euler system:

\[
\begin{cases}
\partial_t u_n + (u_n, \nabla)u_n + \nabla p_n + ru_n = g(x), \\
\text{div} \ u_n = 0, \quad u_n|_{t=-t_n} = u_n^0.
\end{cases}
\]

Then \(u_n(0) \in S(t_n)u_n^0\) and we have to verify that \(\{u_n(0)\}_{n=0}^{\infty}\) is precompact in \(H^1\).

The solutions \(u_n(t), \ t \geq -t_n\), are bounded in \(C_b([-T, \infty), H^1]\) for \(T \leq t_n\) and we can extract a subsequence

\[u_n(0) \to \bar{u} \in H^1 \text{ weakly in } H^1.\]

Along a further subsequence we have
\[u_n \to u \ast\text{-weakly in } L^\infty(-T, T; H^1) \text{ and strongly in } L^2(-T, T; H).\]
This is enough to pass to the limit in the integral identities satisfied by $u_n$ to obtain that the following integral identity holds for $u$:

$$
- \int_{\mathbb{R}} (u, v\eta'(t)) dt + \int_{\mathbb{R}} b(u, u, v\eta(t)) dt + \int_{\mathbb{R}} r (u, v\eta(t)) dt - \int_{\mathbb{R}} (g, v\eta(t)) dt = 0, \quad \eta \in C^\infty_0(\mathbb{R}),
$$

which gives that $u$ is a solution of the damped Euler system bounded on $t \in \mathbb{R}$. Next, we have

$$u(0) = \bar{u}. \quad (4.5)$$

This is standard [31]. On one hand, for $\eta(0) \neq 0$ we have

$$
- \int_{0}^{0} (u, v\eta'(t)) dt + \int_{-\infty}^{0} b(u, u, v\eta(t)) dt + \int_{-\infty}^{0} r (u, v\eta(t)) dt - \int_{-\infty}^{0} (g, v\eta(t)) dt = -(\bar{u}, v) \eta(0), \quad (4.6)
$$

On the other hand, multiplying the equation

$$\frac{d}{dt} (u, v) + b(u, u, v) + r(u, v) = (g, v)$$

by the same $\eta$ and integrating from $-\infty$ to 0 we obtain equality (4.6) with the right-hand side equal to $-(u(0), v)\eta(0)$. This gives (4.5).

Thus, we have that $u_n(0) \to u(0)$ weakly in $H^1$, we now show that $u_n(0) \to u(0)$ strongly in $H^1$. We multiply the balance equation (3.9) for $u_n$ by $e^{2rt}$ and integrate from $-t_n$ to 0. We obtain

$$
\|u_n(0)\|_{H^1}^2 = \|u_n(-t_n)\|_{H^1}^2 e^{-2rt_n} + 2 \int_{-t_n}^{0} (u_n(t), g) e^{2rt} dt.
$$

Since $u_n(-t_n)$ are uniformly bounded in $H^1$ and

$$u_n \to u \quad \ast\text{-weakly in} \quad L^\infty_{loc}(\mathbb{R}; H^1)$$

we can pass to the limit as $n \to \infty$ to obtain

$$
\lim_{n \to \infty} \|u_n(0)\|_{H^1}^2 = 2 \int_{-\infty}^{0} (u(t), g) e^{2rt} dt.
$$

The complete trajectory $u(t)$ also satisfies the balance equation, and acting similarly we obtain

$$
\|u(0)\|_{H^1}^2 = 2 \int_{-\infty}^{0} (u(t), g) e^{2rt} dt.
$$
Thus, we have shown that
\[
\lim_{n \to \infty} \|u_n(0)\|_1^2 = \|u(0)\|_1^2,
\]
which along with the established weak convergence gives that
\[
u_n(0) \to u(0) \text{ strongly in } H^1,
\]
and completes the proof. \(\square\)

5. Upper semi-continuity of the attractors in the limit of vanishing viscosity

In this concluding section we study the dependence of the attractors \(A_\nu\) of the damped Navier–Stokes system on the viscosity coefficient \(\nu\) as \(\nu \to 0^+\). In the previous section we have shown that the damped Euler system (with \(\nu = 0\)) has the global attractor
\[
A_{\nu=0} = A_0.
\]
Furthermore, uniformly for \(\nu \geq 0\) the following estimate holds:
\[
\sup_{u \in A_\nu} \|u\|_1 \leq \|g\|_1. \tag{5.1}
\]

\textbf{Theorem 5.1.} The attractors \(A_\nu\) depend upper semi-continuously on \(\nu\) as \(\nu \to 0^+\). In other words
\[
\lim_{\nu \to 0^+} \text{dist}_{H^1}(A_\nu, A_0) = 0, \tag{5.1}
\]
where
\[
\text{dist}_{H^1}(X, Y) := \sup_{x \in X} \inf_{y \in Y} \|x - y\|_{H^1}. \tag{5.2}
\]

\textbf{Proof.} We take an arbitrary sequence \(\nu_n \to 0^+\), and for every \(\nu_n\) choose a point on the attractor \(A_{\nu_n}\) of equation (1.1) with \(\nu = \nu_n\). Specifically, we choose the point on \(A_{\nu_n}\), whose distance from \(A_0\) is equal to the distance from \(A_{\nu_n}\) to \(A_0\). In view of the compactness of \(A_{\nu_n}\) and \(A_0\) such a point exists. These points lie on \(A_{\nu_n}\) and therefore there are complete trajectories passing through them, and we can denote these points by \(u_n(0)\), so that
\[
u_n(0) \in A_{\nu_n}, \quad u_n \in C_b(\mathbb{R}, H^1), \quad \|u_n\|_{C_b(\mathbb{R}, H^1)} \leq r^{-1}\|g\|_1, \tag{5.3}
\]
and in view of our choice
\[
\text{dist}_{H^1}(u_n(0), A_0) = \text{dist}_{H^1}(A_{\nu_n}, A_0). \tag{5.4}
\]
In view of (5.3) we can extract a subsequence \(u_{\nu_n}\) for which for a \(\bar{u} \in H^1\)
\[
u_{\nu_n}(0) \to \bar{u} \text{ weakly in } H^1 \text{ as } n \to \infty,
\]
and along a further subsequence we have

$$u_{\nu_n}(0) \to u_0$$ *-weakly in $L^\infty_{\text{loc}}(\mathbb{R}, \mathcal{H}^1)$ and strongly in $L^2_{\text{loc}}(\mathbb{R}, \mathcal{H})$.

The solutions $u_{\nu_n}$, by definition, satisfy the integral identity

$$- \int_\mathbb{R} (u_{\nu_n}, v\eta'(t)) \, dt + \int_\mathbb{R} b(u_{\nu_n}, u_{\nu_n}, v\eta(t)) \, dt + \int_\mathbb{R} \nu_n (\text{curl } u_{\nu_n}, \text{curl } v\eta(t)) \, dt + r \int_\mathbb{R} (u_{\nu_n}, v\eta(t)) \, dt - \int_\mathbb{R} (g, v\eta(t)) \, dt = 0.$$

We now pass to the limit in this identity taking into account that

$$\nu_n (\text{curl } u_{\nu_n}, \text{curl } v) \to 0 \text{ as } \nu_n \to 0,$$

and obtain that $u_0$ is a solution (a complete trajectory) of the damped Euler system and therefore satisfies the balance equation (3.9). In addition, as in Theorem 4.5, we can show that $u(0) = \bar{u}$, so that

$$u_{\nu_n}(0) \to u(0) \text{ weakly in } \mathcal{H}^1.$$

The complete trajectories $u_n = u_n(t)$ of the damped Navier–Stokes system (1.1) satisfy the balance equation (2.10). We drop there the second (non-negative) term multiply the resulting inequality by $e^{2rt}$ and integrate from $-t_n$ to 0, where $t_n \to +\infty$. We obtain

$$\|u_{\nu_n}(0)\|_1^2 \leq \|u_{\nu_n}(-t_n)\|_1^2 e^{-2rt_n} + 2 \int_{-t_n}^0 (u_{\nu_n}(t), g) e^{2rt} \, dt.$$

In the limit as $n \to \infty$ this gives that

$$\limsup_{n \to \infty} \|u_{\nu_n}(0)\|_1^2 \leq 2 \int_{-\infty}^0 (u_0(t), g) e^{2rt} \, dt.$$

For the solution $u_0$ as in Theorem 4.5 we have

$$\|u_0(0)\|_1^2 = 2 \int_{-\infty}^0 (u_0(t), g) e^{2rt} \, dt,$$

and together with the previous inequality this gives that

$$\limsup_{n \to \infty} \|u_{\nu_n}(0)\|_1^2 \leq \|u_0(0)\|_1^2.$$

Since by the weak convergence we always have

$$\|u_0(0)\|_1 \leq \liminf_{n \to \infty} \|u_n(0)\|_1,$$

it follows from (5.5) that

$$\lim_{n \to \infty} \|u_{\nu_n}(0)\|_1 = \|u_0(0)\|_1,$$

and, finally, that

$$\lim_{n \to \infty} \|u_{\nu_n}(0) - u_0(0)\|_1 = 0.$$
Taking into account (5.4) we obtain that
\[
\lim_{n \to \infty} \text{dist}_{H^1}(A_{\nu_n}, A_0) = 0.
\] (5.6)

Since in the course of the proof we have been several times passing to subsequences we have actually shown that
\[
\liminf_{\nu_n \to 0^+} \text{dist}_{H^1}(A_{\nu_n}, A_0) = 0
\] (5.7)
for any sequence \(\nu_n \to 0^+\). This obviously implies (5.1). The proof is complete. \(\square\)

**Remark 5.2.** A similar result in \(\mathbb{R}^2\) was recently obtained in [19].

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