Classification of cosmology with arbitrary matter in the Hořava-Lifshitz model

Masato Minamitsuji

Center for Quantum Spacetime, Sogang University,
Shinsu-dong 1, Mapo-gu, 121-742 Seoul, South Korea

In this work, we discuss the cosmological evolutions in the nonrelativistic and possibly renormalizable gravitational theory, called the Hořava-Lifshitz (HL) theory. We consider the original HL model (type I), and the modified version obtained by an analytic continuation of parameters (type II). We classify the possible cosmological evolutions with arbitrary matter. We will find a variety of cosmology.

PACS numbers: 04.50.+h, 98.80.Cq
Keywords:

I. INTRODUCTION

Studies on ultra-violet (UV) completion of gravity have a long history. A lot of understanding has been obtained especially from string theory. Very recently, a new class of UV complete theory of gravity was proposed by Hořava [1], by generalizing the ideas discussed in Ref. [2]. The theory does not have the full diffeomorphism invariance as in Einstein’s general relativity, but only has a local Galilean invariance, and Einstein’s relativity may emerge at the infra-red (IR) fixed point. The model has been motivated by a scalar field model discussed by Lifshitz [3], to explain the quantum critical phenomena in condensed matter physics, in which the action has \( z = 2 \), where \( z \) represents the dynamical critical exponent given by

\[
t \to t^z, \quad x^i \to x^i.
\]

The model proposed in Ref. [1] has the scaling dimension \( z = 3 \) and is often called the Hořava-Lifshitz (HL) model.

There would be many implications of the HL model for cosmology, which were firstly discussed in Ref. [4]. In Ref. [5], it was suggested that the divergence of speed of light in the UV region may resolve the horizon problem. The model proposed in Ref. [1] has the scaling dimension \( z = 3 \) and is often called the Hořava-Lifshitz (HL) model. We briefly review the HL model. The metric which has the local Galilean invariance can be written in the form

\[
ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right).
\]

and \( N_i = g_{ij} N^j \), which is similar to the ADM decomposition in Einstein’s general relativity [23]. The scaling dimensions of various quantities in the momentum unit are given by \( [t] = -3 \), \( [x^i] = -1 \), \( [N] = [g_{ij}] = 0 \), and \( [N^i] = 2 \). The dynamical variables are \( N \), \( N_i \) and \( g_{ij} \), which is very similar to the ADM decomposition in Einstein’s general relativity [23].

We briefly review the HL model. The metric which has the local Galilean invariance can be written in the form

\[
ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right).
\]

and \( N_i = g_{ij} N^j \), which is similar to the ADM decomposition in Einstein’s general relativity [23]. The scaling dimensions of various quantities in the momentum unit are given by \( [t] = -3 \), \( [x^i] = -1 \), \( [N] = [g_{ij}] = 0 \), and \( [N^i] = 2 \). The dynamical variables are \( N \), \( N_i \) and \( g_{ij} \), which is very similar to the ADM decomposition in Einstein’s general relativity [23].

The total action is composed of the kinetic part and potential one: \( S = S_K + S_V \). The kinetic action is given by

\[
S_K = \frac{2}{\kappa^2} \int dtd^3x \sqrt{g} N \left( K_{ij} K^{ij} - \lambda K^2 \right),
\]

and \( K^{ij} = g^{ik} g^{jl} K_{kl} \), where the extrinsic curvature is defined by

\[
K_{ij} = \frac{1}{2N} \left( \frac{\partial}{\partial t} g_{ij} - \nabla_i N_j - \nabla_j N_i \right).
\]

\( \nabla_i \) denotes the covariant derivative with respect to \( g_{ij} \). Correspondingly, with the detailed-balance condition, the potential action is given as

\[
S_V = \int dtd^3x \sqrt{g} N \left[ -\frac{\kappa^2}{2\mu^2} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2\mu^2} \epsilon^{ijk} \nabla_i R_{jk} - \frac{\kappa^2 \mu^2}{8} R^{ij} R_{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( \frac{1 - 4\lambda}{4} R^2 + \lambda R - 3\lambda^2 \right) \right].
\]

II. HOŘAVA-LIFSHITZ MODEL AND COSMOLOGY

In this work, we discuss the cosmological evolutions in analogy with a point particle moving in a potential in classical mechanics. We classify the possible cosmological evolutions. In Sec. IV, we give a brief summary before closing the article.

This article is devoted to discuss the cosmological evolutions in HL model. We consider the original HL model (type I), and the modified model obtained by an analytic continuation of parameters (type II). We classify the evolutions of the universe with arbitrary kind of matter. The article is constructed as follows. In Sec. II, we briefly review the HL model and give equations of motion in the cosmological background. In Sec. III, we discuss the cosmological evolutions in analogy with a point particle...
where $R_{ij}$ is Ricci tensor associated with $g_{ij}$, and the Cotton tensor $C_{ijkl}$ is transverse, conserved and vanishing for all conformally flat spaces and has scaling dimension 3. The full action is given by

$$S_I = S_E + S_V$$

$$= \int dt d^3x \sqrt{g} N \left[ \frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2 w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu^2}{2 w^2} R_t \nabla_j R^j_k - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8 (1 - 3 \lambda)} \left( \frac{1 - 4 \lambda}{4} R^2 + \lambda R - 3 \Lambda^2 \right) \right],$$

where scaling dimensions of parameters are given by $[\kappa] = 0, [w] = 0, [\mu] = 1,$ and $[\Lambda] = 2$. $\lambda$ represents a dynamical coupling constant, susceptible to quantum corrections [1].

For convenience, one can rewrite the above action as

$$S_I = \int dt d^3x \sqrt{g} N \left[ \alpha \left( K_{ij} K^{ij} - \lambda K^2 \right) + \beta C_{ij} C^{ij} + \gamma \frac{\epsilon_{ijk}}{\sqrt{g}} R_t \nabla_j R^j_k + \zeta R_t R^{ij} + \eta R^2 + \xi R + \sigma \right],$$

where

$$\alpha = \frac{2}{\kappa^2}, \quad \beta = \frac{\kappa^2}{2 w^4}, \quad \gamma = \frac{\kappa^2 \mu^2}{8}, \quad \zeta = -\frac{\kappa^2 \mu^2}{8},$$

$$\eta = \frac{\kappa^2 \mu^2}{8 (1 - 3 \lambda)} - 4 \lambda, \quad \xi = \frac{\kappa^2 \mu^2}{8 (1 - 3 \lambda)} - 3 \Lambda^2.$$.

The action is invariant under the restricted class of diffeomorphisms (foliation-preserving diffeomorphism) $t' = h(t)$ and $(x')^i = h'(x^i)$. For $\lambda = 1/3$, the theory has classical, anisotropic conformal invariance. Comparing the action with that of general relativity in ADM formalism, one can read the emergent constants $c = \frac{\kappa^2 \mu}{4} \sqrt{\Lambda / (1 - 3 \lambda)}$, $G = \frac{\kappa^2}{32 \pi c}$, $\Lambda_E = \frac{3}{2} \Lambda$, appearing in IR. Einstein’s general relativity is recovered for $\lambda = 1$.

Following Ref. [14], one can make an analytic continuation of parameters $\mu \rightarrow i \mu$ and $w^2 \rightarrow -iw^2$. Then, the above action can be rewritten as

$$S_{II} = \int dt d^3x \sqrt{g} N \left[ \alpha_2 \left( K_{ij} K^{ij} - \lambda K^2 \right) + \beta_2 C_{ij} C^{ij} + \gamma_2 \frac{\epsilon_{ijk}}{\sqrt{g}} R_t \nabla_j R^j_k + \zeta_2 R_t R^{ij} + \eta_2 R^2 + \xi_2 R + \sigma_2 \right],$$

where new parameters are defined by $\alpha_2 = \alpha$, $\beta_2 = -\beta$, $\gamma_2 = -\gamma$, $\zeta_2 = -\zeta$, $\eta_2 = -\eta$, $\xi_2 = -\xi$ and $\sigma_2 = -\sigma$. The emergent speed of light is given by $c = (\kappa^2 \mu/4) \sqrt{\Lambda / (3 \lambda - 1)}$. For convenience, in this article, we call the model Eq. (11) “type I” and the model Eq. (12) “type II”.

For the cosmological background, i.e., the homogeneous and isotropic background, $N = N(t)$, $N_t = 0$ and $g_{ij} = a^2(t) \gamma_{ij}$, where $\gamma_{ij}$ is maximally symmetric 3-space whose curvature is given by $R^i_{ij} = 2k\gamma_{ij}$ and $R^i = 6k$. In the cosmological spacetime, the Cotton tensor vanishes and the equations of motion get simplified.

In the type I theory, the equations of motion (Friedmann equations) are given by

$$H^2 + \frac{\kappa^4 \mu^2 (k - a^2 \Lambda)^2}{16(1 - 3 \lambda)^2 a^4} = \frac{\rho}{3 \alpha (3 \lambda - 1)},$$

$$2 \left( \dot{H} + \frac{2}{3} H^2 \right) - \frac{\kappa^4 \mu^2}{16 a^4 (1 - 3 \lambda)^2} (k + 3a^2 \Lambda) (k - a^2 \Lambda) = - \frac{\rho}{\alpha (3 \lambda - 1)},$$

(11)

where $H := \dot{a}/(aN)$ (in the later discussion, we simply set $N = 1$) and $\rho$ and $p$ are the energy density and pressure of the matter sector. In the type II theory, by changing $\mu^2 \rightarrow -\mu^2$, the equations of motion become

$$H^2 - \frac{\kappa^4 \mu^2 (k - a^2 \Lambda)^2}{16(1 - 3 \lambda)^2 a^4} = \frac{\rho}{3 \alpha (3 \lambda - 1)},$$

$$2 \left( \dot{H} + \frac{2}{3} H^2 \right) + \frac{\kappa^4 \mu^2}{16 a^4 (1 - 3 \lambda)^2} (k + 3a^2 \Lambda) (k - a^2 \Lambda) = - \frac{\rho}{\alpha (3 \lambda - 1)},$$

(12)

It is convenient to represent the Friedmann equation in analogy with a dynamics of a point particle in a potential in the classical mechanics, $a^2 + V_m(a) = 0$ ($m = I, II$), where

$$V_m(a) = \epsilon_m \frac{\kappa^4 \mu^2 (k - a^2 \Lambda)^2}{16(1 - 3 \lambda)^2 a^4} - \frac{\rho a^2}{3 \alpha (3 \lambda - 1)},$$

(13)

where $\epsilon_I = +1$ and $\epsilon_{II} = -1$, respectively. In the absence of the matter $\rho = 0$, the effective potential is not sensitive to the value of $\Lambda$ unless $\lambda = 1/3$. With matter, it depends on whether $\lambda$ is larger or smaller than $1/3$. In the regime $\lambda > 1/3$, since $\alpha > 0$, the sign of the energy density terms in Eq. (13) is positive. While, in the regime where $\lambda < 1/3$, the sign of the same term is flipped. In the case $\lambda = 1/3$, where the theory develops an anisotropic conformal invariance, the scale factor becomes non-dynamical as seen in Eq. (11) and (12). In the later discussions, we assume $\lambda \neq 1/3$. Note that the case of $\lambda < 1/3$ induces the repulsive gravitational force (see (13)), and also the perturbation about the flat background provides a ghost-like scalar mode [1], which is potentially dangerous if it is coupled to the matter. Thus, this branch may not be realistic. But for the mathematical completeness, in this article, we will include the case of $\lambda < 1/3$ into our analysis.

III. COSMOLOGICAL EVOLUTIONS

In this section, we discuss the classification of the cosmological evolutions with matter in the HL model. We
assume that the energy density of the matter is nonnegative \( \rho \geq 0 \), which can be parameterized as \( \rho = \rho_0 \alpha^n / a^n \), where \( n \) is arbitrary integer and \( \rho_0 \geq 0 \). One might think that it would be enough to consider the matter of \( n = 6 \) with the equation of state \( p = \rho \), because in the nonrelativistic theory with the dynamical critical exponent \( z \), the equation of state of the massless particles becomes \( p = (z/3) \rho \). But it seems to be too restrictive. The HL theory itself is purely a gravitational theory and does not specify the nature of the matter. In addition, there is no particular reason why the matter sector may not share the same dynamical critical exponent with the gravity sector, although it may be plausible. Therefore, for completeness of our analysis, it is appropriate to keep \( n \) to be an arbitrary integer.

It is important to note two important properties. In the type I model and in the presence of the matter \( \rho > 0 \) there is no solution for \( \lambda < 1/3 \), because the potential \( V_I(a) \) defined by (13) is always positive. In the type II model and in the presence of the matter \( \rho > 0 \) there are always monotonic solutions \(^1\) for \( \lambda > 1/3 \), because the potential \( V_{II}(a) \) defined by (13) is always negative. Thus, in the presence of the matter, we will basically focus on the case of \( \lambda > 1/3 \) for the type I model and on the case of \( \lambda < 1/3 \) for the type II model.

In the simplest case \( k = \Lambda = 0 \), both in type I and II theories, the solutions are given by

\[
a = \left( \frac{\rho_0 n^2 a_0^n}{12\alpha (3\lambda - 1)} \right)^{1/n} t^{2/n},
\]

which is only possible for \( \lambda > 1/3 \).

**A. The vacuum case**

Firstly, we consider the case of a vacuum solution \( \rho = 0 \).

a. **Type I model**: For the nonzero \( k \) and \( \Lambda \), there is only the static solution for \( k/\Lambda > 0 \), which is given by \( a_I = \sqrt{k/\Lambda} \), where the effective potential Eq. (13) is vanishing.

For \( k = 0 \) (but \( \Lambda \neq 0 \)) or for \( \Lambda = 0 \) (but \( k = 0 \)), the potential becomes positive everywhere and there is no cosmological solution.

b. **Type II model**: For \( k \neq 0 \) and \( k/\Lambda > 0 \), there is an exact solution given by

\[
a_{II}(t) = \sqrt{\frac{k}{\Lambda}} \sqrt{1 + c_1 e^{\pm 2H_0 t}}, \quad H_0 := \left| \frac{\kappa^2 \mu \Lambda}{4(3\lambda - 1)} \right|.
\]

where \( c_1 \) is an arbitrary constant, which can be positive or negative. There are four possibilities. The first one is

\(^1\) By "monotonic", we mean the universe which expands from \( a = 0 \) to \( a = \infty \) and contracts from \( a = \infty \) to \( a = 0 \), without any collapse or bounce.
where

\[ C = \frac{\kappa^4 \mu^2 k^2}{16(1-3\lambda)^2} + \frac{\rho_0 a_0^4}{3\alpha(1-3\lambda)}, \]
\[ H_0^2 := \frac{\kappa^4 \mu^2 \Lambda^2}{16(1-3\lambda)^2}, \quad K := \frac{\kappa^4 \mu^2 k\Lambda}{8(1-3\lambda)^2}. \] (21)

The solution only exists for \( K^2 > 4CH_0^2 \), which is consistent with \( \lambda > 1/3 \).

**Type II model (\( \lambda < 1/3 \))**: If \( \frac{\kappa^4 \mu^2 k^2}{16(1-3\lambda)^2} < \frac{\rho_0 a_0^4}{3\alpha} \) (22)

there are bouncing solutions. If \( \frac{\kappa^4 \mu^2 k^2}{16(1-3\lambda)^2} > \frac{\rho_0 a_0^4}{3\alpha} \) (23)

there are monotonic solutions.

For \( k = 0 \) (but \( \Lambda \neq 0 \)), which is just the special case of Eq. (22), there are bouncing solutions. For \( \Lambda = 0 \) (but \( k \neq 0 \)), there is no solution for Eq. (22) and there are monotonic solutions for Eq. (23).

The exact solution is given by

\[ a_{II}(t) = \frac{1}{\sqrt{2H_0}} \sqrt{K \pm (K^2 - 4CH_0^2)^{1/2} \cosh(2H(t-t_0))} \] (24)

for \( K^2 > 4CH_0^2 \), and

\[ a_{II}(t) = \frac{1}{\sqrt{2H_0}} \sqrt{K \pm (4CH_0^2 - K^2)^{1/2} \sinh(2H(t-t_0))} \] (25)

for \( K^2 < 4CH_0^2 \), where

\[ C = \frac{\kappa^4 \mu^2 k^2}{16(1-3\lambda)^2} + \frac{\rho_0 a_0^4}{3\alpha(1-3\lambda-1)}, \]
\[ H_0^2 := \frac{\kappa^4 \mu^2 \Lambda^2}{16(1-3\lambda)^2}, \quad K := \frac{\kappa^4 \mu^2 k\Lambda}{8(1-3\lambda)^2}. \] (26)

The first and second solution corresponds to \( \lambda < 1/3 \) (bouncing solutions) and \( \lambda > 1/3 \) (monotonic solutions).

**D. The case \( n = 3 \)**

The case \( n = 3 \) corresponds to that of massive particles: \( \rho \sim mn \propto 1/a^3 \), where \( n \) is the number density of particles with mass \( m \).

**Type I model (\( \lambda > 1/3 \))**: In the type I model, there is no cosmological solution unless \( \lambda > 1/3 \). For \( \lambda < 1/3 \) In this case, there is no solution or a cyclic universe is available. For \( k = 0 \) (but \( \Lambda \neq 0 \)), big crunch solutions are obtained, while for \( \Lambda = 0 \) (but \( k \neq 0 \)) bouncing universes are.

**Type II model**: For \( \lambda < 1/3 \), there are monotonic, or big crunch, or bouncing solutions. For \( k = 0 \) (but \( \Lambda \neq 0 \)), which just the special case of Eq. (22), there are bouncing solutions. For the case of \( \Lambda = 0 \) (but \( k \neq 0 \)), there are big crunch solutions.

For \( \lambda > 1/3 \) and \( k = 0 \) (but \( \Lambda \neq 0 \)), there is the exact solution

\[ a_{II}(t) = \left\{ \frac{4}{\kappa^2 \mu \lambda} \left( \frac{\kappa^2(3\lambda-1)\rho_0 a_0^3}{6} \times \sinh \left( \frac{3\kappa^2 \mu \lambda}{8(3\lambda-1)(t-t_0)} \right) \right)^{2/3} \right\} \] (27)

**E. The case \( n = 1 \) and \( n = 2 \)**

**Type I model (\( \lambda > 1/3 \))**: For \( \lambda > 1/3 \), in this case, there is no solution, or there are cyclic solutions. For \( k = 0 \) (but \( \Lambda \neq 0 \)), there are big crunch solutions. For \( \Lambda = 0 \) (but \( k \neq 0 \)), there are bouncing solutions.

**Type II model (\( \lambda < 1/3 \))**: There are monotonic, or big crunch, or bouncing solutions. For \( k = 0 \) (but \( \Lambda \neq 0 \)), there are bouncing solutions. For \( \Lambda = 0 \) (but \( k \neq 0 \)), there are big crunch solutions.

**F. The case \( n = 0 \)**

**Type I model (\( \lambda > 1/3 \))**: For \( \Lambda \neq 0 \) and \( k \neq 0 \), if

\[ \frac{\kappa^4 \mu^2 \Lambda^2}{16(3\lambda-1)} > \frac{\rho_0}{3\alpha} \] (28)

there is no solution, or there are cyclic solutions. If

\[ \frac{\kappa^4 \mu^2 \Lambda^2}{16(3\lambda-1)} < \frac{\rho_0}{3\alpha} \] (29)

there are bouncing solutions.

For \( k = 0 \) (but \( \Lambda \neq 0 \)), if Eq. (28) there is no solution, and if Eq. (29) there are monotonic solutions. For \( \Lambda = 0 \) (but \( k \neq 0 \)), there are bouncing solutions.

The exact solution is given by Eq. (20) where

\[ C = \frac{\kappa^4 \mu^2 k^2}{16(1-3\lambda)^2}, \quad K := \frac{\kappa^4 \mu^2 k\Lambda}{8(1-3\lambda)^2}, \]
\[ H_0^2 := \frac{\kappa^4 \mu^2 \Lambda^2}{16(1-3\lambda)^2} + \frac{3\alpha(1-3\lambda)}{3\alpha}. \] (30)

The solution only exists for \( K^2 > 4CH_0^2 \), which is consistent with \( \lambda > 1/3 \).

**Type II model (\( \lambda < 1/3 \))**: For \( \Lambda \neq 0 \) and \( k \neq 0 \), if

\[ \frac{\kappa^4 \mu^2 \Lambda^2}{16(1-3\lambda)^2} > \frac{\rho_0}{3\alpha} \] (31)

there are monotonic, or big crunch or bouncing solutions. If

\[ \frac{\kappa^4 \mu^2 \Lambda^2}{16(1-3\lambda)^2} < \frac{\rho_0}{3\alpha} \] (32)
there are big crunch solutions.

For $k = 0$ (but $\Lambda \neq 0$), if Eq. $31$ there are monotonic solutions, and if Eq. $29$ there is no solution. For $\Lambda = 0$ (but $k \neq 0$), there are big crunch solutions.

The exact solution is given by Eq. $24$ for $K^2 > 4CH_0^2$, and Eq. $25$ for $K^2 < 4CH_0^2$, where
\[
C = \frac{k^4\mu^2k^2}{16(1-3\lambda)^2}, \quad H_0^2 := \frac{k^4\mu^2\Lambda}{16(1-3\lambda)^2} + \frac{\rho_0}{3\alpha(1-3\lambda)}, \quad K := \frac{k^4\mu^2k\Lambda}{8(1-3\lambda)^2}.
\]

The first and second solution corresponds to $\lambda < 1/3$ and $\lambda > 1/3$.

G. The case $n \leq -1$

Type I model ($\lambda > 1/3$): For all the choices of $\Lambda$ and $k$, there are bouncing solutions.

Type II model ($\lambda < 1/3$): For all the choices of $\Lambda$ and $k$, there are big crunch solutions.

IV. SUMMARY

We investigated evolutions of homogeneous and isotropic universe in the recently proposed nonrelativistic, renormalizable gravitational theory with an anisotropic scaling $z = 3$ (called the Horava-Lifshitz (HL) theory), where $z$ is the dynamical critical exponent defined in Eq. $11$. We considered the original theory (type I) and the modified model obtained by an analytic continuations of parameters in the original theory (type II). The dimensionless, dynamical coupling parameter $\lambda$ is contained into the theory, which is sensible to the quantum corrections. The theory has IR fixed point and for $\lambda = 1$ the theory coincides with general relativity in IR regime. For $\lambda = 1/3$, the theory has an anisotropic conformal invariance and scale factor becomes non-dynamical. In this article, we did not restrict the range of $\lambda$. We also have assumed that the matter energy density is positive $\rho > 0$.

In the type I model, we found that

- For the vacuum universe, there is a static solution, $a = \sqrt{k/\Lambda}$, for $\Lambda \neq 0$ and $k/\Lambda > 0$.
- With arbitrary kind of matter, there is no cosmological solution for $\lambda < 1/3$
- For $\Lambda = 0$ (but $k \neq 0$) and with matter ($\lambda > 1/3$), there are bouncing solutions for $n \leq 3$, and there are big crunch solutions for $n \geq 5$. The case $n = 4$ is marginal: If Eq. $14$ there is no solution, but if Eq. $13$, there are monotonic solutions.
- For $k = 0$ (but $\Lambda \neq 0$) and with matter ($\lambda > 1/3$), there are bouncing solutions for $n \leq -1$, and there are big crunch solutions for $n \geq 1$. The case $n = 0$ is marginal: If Eq. $25$ there is no solution, but if Eq. $24$, there are monotonic solutions.

In the type II model, we found that

- For $\Lambda \neq 0$ (and $k/\Lambda > 0$) and in the absence of matter, there are exact solutions given by Eq. $11$.
- With arbitrary kind of matter ($\lambda > 1/3$), there are monotonic solutions for any $n$.
- For $\Lambda = 0$ (but $k \neq 0$) and with matter ($\lambda < 1/3$), there are big crunch solutions for $n \leq 3$, and there are bouncing solutions for $n \geq 5$. The case $n = 4$ is marginal: If Eq. $22$ there is no solution, but if Eq. $21$, there are monotonic solutions.
- For $k = 0$ (but $\Lambda \neq 0$) and with matter ($\lambda < 1/3$), there are big crunch solutions for $n \leq -1$, and there are bouncing solutions for $n \geq 1$. The case $n = 0$ is marginal: If Eq. $32$ there is no solution, but if Eq. $31$, there are monotonic solutions.

It may be helpful to clarify the characteristic differences of the cosmological evolutions in the HL theory from those in general relativity.

- In the case of the type I theory: supposing that $\rho > 0$ and $\lambda > 1/3$ (the gravity is attractive), in the case of $k = 0$ and $\Lambda = 0$, there is no essential difference from the case of general relativity.
- In the case of nonzero $k$, the particular difference from the case of general relativity is the existence of the bouncing evolutions in the absence of matter which breaks the weak energy condition. Note that in general relativity, it is impossible to realize the bouncing Universe under the weak energy condition. In the HL theory, the corrections due to the higher spatial curvature terms could effectively break it. Therefore, this is the novel effect in the HL gravity. On the other hand, even in the flat ($k = 0$) or open ($k < 0$) Universe, with matter of $n \geq 0$, the presence of non-zero $\Lambda$ induces the maximal size of the Universe, i.e., the big crunch solutions.

- In the case of the type II theory: supposing that $\rho > 0$ and $\lambda > 1/3$ (the gravity is attractive), in the case of $k = 0$ and $\Lambda = 0$, there is no essential difference from the case of general relativity. However, in all the cases, the effective potential Eq. $13$ is always non-positive. Thus, even in the case of the closed ($k > 0$) Universe, there is no recollapsing Universe, which is the crucial difference from the case of general relativity.

In this article, for simplicity, the detailed balance condition was assumed, but it may not be essential. The
absence of the detailed balance condition introduces an additional $a^{-4}$ contribution to the effective potential Eq. (12) (See e.g., [23]), and leads to a richer variety of the cosmological evolutions. In addition, in the type I theory with the detailed balance condition, to obtain the attractive gravitational force, one has to choose $\lambda > 1/3$ (see Eq. (13)). Then in order to have the real emergent speed of light, from Eq. (9) the condition of $\Lambda < 0$ has to be imposed. Namely there is only the anti-de Sitter vacuum ($\Lambda_E < 0$). Although this is not the case for the type II theory, the detailed balance may not be suitable to obtain the realistic cosmology. For the detailed classifications in the case without the detailed balance condition, see [26].

Before closing this article, we shall note some properties of the model, which would be important for future studies. We mainly focused on the possible cosmological evolutions in the UV regime and ignored the process of the recovery of the cosmology in Einstein’s relativity. The UV cosmology should be smoothly matched to that of IR. But, there still seem to be some unclear points about this connection. First, we assumed that $\lambda$ is a constant but actually $\lambda$ would be sensitive to the perturbative corrections. $\lambda$ would eventually approach some IR fixed point and variation of $\lambda$ would affect the cosmology. Second, in Ref. [1, 2, 9, 11, 16], it has been pointed out that in the HL theory there is another (scalar) degree of freedom other than those appearing in general relativity, which corresponds to a scalar mode in the linearized theory. The analysis of perturbations about a flat spacetime indicates that the kinetic term of this scalar mode becomes a ghost for $\lambda < 1/3$ and $\lambda \geq 1$, and may be important for stability of the solution, if this mode is coupled to the matter perturbations [1, 2, 16]. It also has been pointed out that in the cosmological background this mode may be useful to produce the scale invariant spectrum without inflaton in the UV regime, if the detailed balance condition is broken [17]. The decoupling of scalar mode in approaching IR would also be essential to recover the relativity.

Acknowledgements

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant No. R11-2005-021, founded by the Korean Government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University.

[1] P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].
[2] P. Horava, JHEP 0903, 020 (2009) [arXiv:0812.4287 [hep-th]].
[3] E.M. Lifshitz, On the Theory of Second-Order Phase Transitions I and II, Zh. Eksp. Teor. Fiz. 11, 255 and 269 (1941).
[4] G. Calcagni, arXiv:0901.0829 [hep-th].
[5] E. Kiritsis and G. Kofinas, arXiv:0904.1334 [hep-th].
[6] S. Mukohyama, arXiv:0904.2190 [hep-th].
[7] R. Brandenberger, arXiv:0904.2835 [hep-th].
[8] Y. S. Piao, arXiv:0904.4117 [hep-th].
[9] B. Chen, S. Pi and J. Z. Tang, arXiv:0905.2300 [hep-th].
[10] X. Gao, arXiv:0904.4187 [hep-th].
[11] R. G. Cai, B. Hu and H. B. Zhang, arXiv:0905.0255 [hep-th].
[12] T. Takahashi and J. Soda, arXiv:0904.0554 [hep-th].
[13] S. Mukohyama, K. Nakayama, F. Takahashi and S. Yokoyama, arXiv:0905.0055 [hep-th].
[14] H. Lu, J. Mei and C. N. Pope, arXiv:0904.1595 [hep-th].
[15] Y. S. Myung and Y. W. Kim, arXiv:0905.0179 [hep-th].
[16] A. Kehagias and K. Sfetsos, arXiv:0905.0477 [hep-th].
[17] R. G. Cai, L. M. Cao and N. Ohta, arXiv:0905.0751 [hep-th].
[18] Y. S. Myung, arXiv:0905.0957 [hep-th].
[19] R. A. Konoplya, arXiv:0905.1523 [hep-th].
[20] J. Kluson, arXiv:0904.1343 [hep-th].
[21] H. Nastase, arXiv:0904.3604 [hep-th].
[22] R. G. Cai, Y. Liu and Y. W. Sun, arXiv:0904.4104 [hep-th].
[23] T. Sotiriou, M. Visser and S. Weinhardt, arXiv:0904.4464 [hep-th].
[24] D. Orlando and S. Reffert, arXiv:0905.0301 [hep-th].
[25] R. L. Arnowitt, S. Deser and C. W. Misner, arXiv:gr-qc/0405109.
[26] M. Minamitsuji, in preparation.