CARLESON MEASURE ESTIMATES AND $\varepsilon$-APPROXIMATION FOR BOUNDED HARMONIC FUNCTIONS, WITHOUT AHLFORS REGULARITY ASSUMPTIONS

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ABSTRACT. Let $\Omega$ be a domain in $\mathbb{R}^{d+1}$, $d \geq 1$. In [HMM2] and [GMT] it was proved that if $\Omega$ satisfies a corkscrew condition and if $\partial\Omega$ is $d$-Ahlfors regular, i.e. Hausdorff measure $H^d(B(x, r) \cap \partial\Omega) \sim r^d$ for all $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, then $\partial\Omega$ is uniformly rectifiable if and only if every bounded harmonic function on $\Omega$ satisfies either

(a) a square function Carleson measure estimate,

or

(b) an $\varepsilon$-approximation property for all $\varepsilon > 0$.

Here we explore (a) and (b) when $\partial\Omega$ is not required to be Ahlfors regular. We first prove that (a) and (b) hold for any domain $\Omega$ for which there exists a domain $\tilde{\Omega} \subset \Omega$ such that $\partial\Omega \subset \partial\tilde{\Omega}$ and $\partial\tilde{\Omega}$ is uniformly rectifiable. We next assume $\Omega$ satisfies a corkscrew condition and $\partial\Omega$ satisfies a capacity density condition. Under these assumptions we prove conversely that the existence of such $\tilde{\Omega}$ implies (a) and (b) hold on $\Omega$ and further that either (a) or (b) holds on a domain if and only if its harmonic measure satisfies a Carleson packing condition for diameters reminiscent of the corona decompositions for measures that characterize uniform rectifiability in the papers [GMT] and [HMM2].

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1. Introduction

Let $\Omega \subset \mathbb{R}^{d+1}$ be open. We say bounded harmonic functions on $\Omega$ satisfy a Carleson measure estimate if there is a constant $C > 0$ such that

$$
\frac{1}{r^d} \int_{B(x,r) \cap \Omega} |\nabla u(x)|^2 \text{dist}(x, \partial \Omega) \, dx \leq C \|u\|_{L^\infty(\Omega)}^2.
$$

whenever $x \in \partial \Omega$, $0 < r < \text{diam}(\Omega)$, and $u$ is a bounded harmonic function on $\Omega$. It is a famous theorem of C. Fefferman \cite{FS} that (1.1) holds for the half spaces $\mathbb{R}^{d+1}$.

If $u$ is a bounded harmonic function on $\Omega$ and if $\varepsilon > 0$, we say that $u$ is $\varepsilon$-approximable if there exists $g \in W^{1,1}_{\text{loc}}(\Omega)$ and $C > 0$ such that

$$
\|u - g\|_{L^\infty(\Omega)} < \varepsilon
$$

and for all $x \in \partial \Omega$ and all $r > 0$

$$
\frac{1}{r^d} \int_{B(x,r) \cap \Omega} |\nabla g(y)| \, dy \leq C.
$$

It is then clear by normal families that (1.2) and (1.3) hold for every bounded harmonic function on $\Omega$ with constant $C = C_\varepsilon$ depending only on $\varepsilon$ and $\Omega$. It is also clear that by local mollifications (1.2) and (1.3) will hold with $g \in C^\infty(\Omega)$; see \cite{Ga} page 347.

The notion of $\varepsilon$-approximation was introduced by Varopoulos in [Va1] and [Va2] in connection with corona theorems and $H^1 - \text{BMO}$ duality. Chapter VIII of [Ga] gave a proof for all $\varepsilon > 0$ on the upper half plane and Dahlberg \cite{Dal} extended the proof to Lipschitz domains using his work relating square functions to maximal functions. Later Kenig, Koch, Pipher and Toro \cite{KKoPT} applied $\varepsilon$-approximation to more general elliptic boundary value problems to prove that on any Lipschitz domain elliptic harmonic measure is $A_\infty$ equivalent to boundary surface measure. Further connections between $\varepsilon$-approximation, Carleson measure estimates, square functions, maximal functions, and $A_\infty$ conditions for elliptic measures have been obtained on Lipschitz domains by several authors, including [DKP], [JK], [HKMP], [KKiPT] and [Pi], and on domains with Ahlfors regular boundaries by [AGMT], [HLMN], [HM1], [HM2] and [HMM2], and more recently by [Az], [AHMMT], [BH], [H], [HMM3], [HMMTZ1] and [HMMTZ2].

The papers [HMM2] and [GMT] have connected $\varepsilon$-approximation and Carleson measures to rectifiability in domains with Ahlfors regular boundaries. To explain their work we give three definitions:

The open set $\Omega \subset \mathbb{R}^n$ satisfies a corkscrew condition if there exists a constant $\alpha$ such that if $x \in \partial \Omega$ and $0 < r < \text{diam}(\Omega)$ there exists ball $B(p, \alpha r)$ so that

$$
B(p, \alpha r) \subset \Omega \cap B(x, r).
$$

If $\Omega$ is a connected open set with the corkscrew condition we say $\Omega$ is a corkscrew domain.

For integers $n > d \geq 1$, a set $E \subset \mathbb{R}^n$ is called $d$-Ahlfors regular (or simply Ahlfors regular
if $d$ is clear from the context) if there exists a constant $c > 0$ such that for all $x \in E$ and $0 < r < \text{diam}(E)$,

$$\frac{1}{c} r^d \leq H^d(B(x, r) \cap E) \leq c r^d$$

(1.5)

where $H^d$ denotes $d$-dimensional Hausdorff measure. The set $E \subset \mathbb{R}^n$ is uniformly $d$-rectifiable if it is $d$-Ahlfors regular and there exist constants $c$ and $M > 0$ such that for all $x \in E$ and all $0 < r \leq \text{diam}(E)$ there is a Lipschitz mapping $g$ from the ball $B(0, r) \subset \mathbb{R}^d$ to $\mathbb{R}^n$ such that $\text{Lip}(g) \leq M$ and

$$H^d(E \cap B(x, r) \cap g(B_d(0, r))) \geq c r^d.$$  

(1.6)

Uniform rectifiability is a quantitative version of rectifiability. It was introduced in the pioneering works [DS1] and [DS2] and David and Semmes who proved it was a geometric condition under which all singular integrals with sufficiently smooth odd kernels are bounded in $L^2(\partial \Omega)$. Later [MMV] and [NT0V] proved conversely that the $L^2$ boundedness of the Cauchy integral or the Riesz transforms on an Ahlfors regular boundary $\partial \Omega$ implies $\partial \Omega$ is uniformly rectifiable.

The papers [HMM2] and [GMT] prove that if $\Omega \subset \mathbb{R}^{d+1}, d \geq 1$, is a corkscrew domain and $\partial \Omega$ is $d$-Ahlfors regular, then the following are equivalent:

(a) All bounded harmonic functions on $\Omega$ satisfy the Carleson measure estimate (1.1).

(b) Every bounded harmonic function on $\Omega$ is $\varepsilon$-approximable for all $\varepsilon > 0$.

(c) $\partial \Omega$ is uniformly $d$-rectifiable.

In fact, [HMM2] proved (c) implies (a) and (b) and [GMT] proved the converse statements. Here our goal is to understand the conditions (a) and (b) when $\partial \Omega$ is not necessarily Ahlfors regular. To state our results we need two more definitions. We will usually assume $\Omega$ satisfies a capacity density condition: there is $\beta > 0$ such that for all $x \in \partial \Omega$ and $r \leq \text{diam}(\Omega)$,

$$\text{Cap}(B(x, r) \setminus \Omega) \geq \begin{cases} \beta r & \text{if } d + 1 = 2, \\ \beta r^{d-1} & \text{if } d + 1 \geq 3. \end{cases}$$

(1.7)

where Cap is Newtonian capacity when $d + 1 \geq 3$ and logarithmic capacity when $d + 1 = 2$. If $\Omega$ satisfies (1.7) every point of $\partial \Omega$ is regular for the Dirichlet problem, so that for each $p \in \Omega$ there exists an unique Borel probability $\omega_p = \omega(p, \cdot, \Omega)$ on $\partial \Omega$ such that

$$u(p) = \int_{\partial \Omega} u(x) d\omega(p, x, \Omega)$$

(1.8)

if $u$ is continuous on $\partial \Omega$ and harmonic on $\Omega$. Moreover, if $u(x)$ is continuous on $\partial \Omega$, (1.8) defines a function harmonic on $\Omega$ which continuously extends $u$ from $\partial \Omega$ to $\overline{\Omega}$. Since $\Omega$ is connected it then follows from Harnack’s inequality that for all $p, q \in \Omega$ there is a constant $C_{p,q} = C_{p,q}(\Omega)$ such that $\omega_p \leq C_{p,q} \omega_q$. The measure $\omega_p$ is called harmonic measure for $p$. 

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We state four theorems. Theorem 1.1 gives a sufficient condition for (a) and (b) to hold for any domain, without assuming (1.4) or (1.7). It has a simple proof using [HMM2] and Whitney cubes.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^{d+1}, d \geq 1 \), be a domain. If there exists a domain \( \tilde{\Omega} \) such that

\[
\tilde{\Omega} \subset \Omega \quad \text{and} \quad \partial \Omega \subset \partial \tilde{\Omega}
\]

and \( \partial \tilde{\Omega} \) is uniformly rectifiable, then (a) and (b) holds for \( \Omega \).

The converse of Theorem 1.1 holds if \( \Omega \) satisfies (1.4) and (1.7).

**Theorem 1.2.** Conversely, if \( \Omega \) is a corkscrew domain with (1.7) and if (a) or (b) holds for \( \Omega \) then there exists a domain \( \tilde{\Omega} \) such that \( \partial \tilde{\Omega} \) uniformly rectifiable and (1.9) holds.

The proof of Theorem 1.2 will use Theorem 1.4 below and Proposition 5.1 from [GMT].

To illustrate Theorem 1.1 and Theorem 1.2 we look at some Cantor sets. Let \( 0 < \lambda < \frac{1}{2} \) and in \( \mathbb{R}^2 \) consider the Cantor set \( K_\lambda = \bigcap_{n \geq 0} K_{\lambda,n} \) where \( K_{\lambda,0} = [0,1] \times [0,1] \), \( K_{\lambda,n+1} \subset K_{\lambda,n} \), and \( K_{\lambda,n+1} \) is the union of \( 4^{n+1} \) pairwise disjoint closed squares of side \( \lambda^{n+1} \), each containing one corner of a component square of \( K_{\lambda,n} \). Then (1.4) and (1.7) hold for \( \Omega_\lambda = \mathbb{R}^2 \setminus K_\lambda \). Theorem 1.1 and Theorem 1.2 imply that (a) or (b) holds for \( \Omega_\lambda \) if and only if \( \lambda < 1/4 \), but these facts have much easier proofs. If \( \lambda \geq 1/4 \), \( H^1 \) and harmonic measure for \( \mathbb{C} \setminus K_\lambda \) are mutually singular ([Ca2], [GM]) and then Lemma 3.3 and the proof of Proposition 4.2 below show (a) and (b) fail. The case \( \lambda < 1/4 \) is even easier because if \( u \) is harmonic on \( \Omega_\lambda \)

\[
\int_{B(x,R) \setminus K_\lambda} \left| \nabla u \right| dy \leq ||u||_{L^\infty} \int_{B(x,R) \setminus K_\lambda} \frac{dy}{\text{dist}(y,K_\lambda)} \leq CR||u||_{L^\infty}.
\]

When \( \lambda < 1/4 \) the domain \( \tilde{\Omega} \) can be obtained by removing from \( \Omega_\lambda \) a closed disc of radius \( c\lambda^n \) at the center of each \( K_{\lambda,n} \), and the proof of Theorem 1.2 amounts to constructing similar discs in the general case. There, it is helpful to note that for \( K_\lambda, \lambda < 1/4 \), the harmonic measures for \( \Omega \) and \( \tilde{\Omega} \) are mutually singular.

The third theorem, Theorem 1.4, gives a condition necessary for \( \Omega \) to have (a) or (b), assuming \( \Omega \) satisfies (1.4) and (1.7). Its proof applies a construction from [GMT] to a modified version of Christ-David cubes described in Theorem 1.3 below. Theorem 1.5 is a converse to Theorem 1.4. It is also proved using the modified cubes and ideas from [GMT].

To state Theorem 1.4 and Theorem 1.5 we first explain their setting, which will be discussed more fully in Section 5. Assuming \( \Omega \) satisfies (1.4), (1.7) and either (a) or (b), we will introduce a family of subsets of \( \partial \Omega \) which we call “cubes” by adapting an important construction of David [Da] and Christ [Ch] to situations where \( \partial \Omega \) may not be Ahlfors regular.

**Proposition 1.3.** Assume \( \Omega \) is a bounded corkscrew domain with (1.7) and assume (a) or (b) holds for \( \Omega \). Then there exists a positive integer \( N \) and a family.
\[ S = \bigcup_{j \geq 0} S_j \]

of Borel subsets of \( \partial \Omega \) which has properties (1.10), (1.11), (1.12), (1.13) and (1.14) listed below, and which also satisfies the “small boundary property” (1.15):

\[ \textup{diam } S \sim 2^{-Nj} \text{ if } S \in S_j; \]

\[ \partial \Omega = \bigcup_{S_j} S, \text{ for all } j; \]

\[ S \cap S' = \emptyset \text{ if } S, S' \in S_j \text{ and } S' \neq S; \]

\[ \text{if } j < k, S_j \in S_j \text{ and } S_k \in S_k, \text{ then } S_k \subset S_j \text{ or } S_k \cap S_j = \emptyset. \]

There exists constant \( c_0 \) such that for all \( S \in S \) there exists \( x_S \in S \) with

\[ B(x_S, c_0 \ell(S)) \cap \partial \Omega \subset S. \]

For \( 0 < \tau < 1 \) and \( S_j \in S_j \) define

\[ \Delta_\tau(S_j) = \left\{ y \in S_j : \text{dist}(y, \partial \Omega \setminus S_j) < \tau 2^{-Nj} \right\} \cup \left\{ y \in \partial \Omega \setminus S_j : \text{dist}(y, S_j) < \tau 2^{-Nj} \right\}, \]

let

\[ G(\tau 2^{-Nj}) = \left\{ K = \bigcap_{1 \leq i \leq d+1} \{ k_i \tau 2^{-Nj} \leq x_i \leq (k_i + 1) \tau 2^{-Nj} \}, k_i \in \mathbb{Z} \right\} \]

denote the set of closed dyadic cubes in \( \mathbb{R}^{d+1} \) of side \( 2^{-Nj} \), scaled down by \( \tau \), and define

\[ N_\tau(S_j) = \# \{ K \in G(\tau 2^{-Nj}) : K \cap \Delta_\tau(Q) \neq \emptyset \}. \]

Then there exists a constant \( C = C_{sb} \) so that

\[ N_\tau(S_j) \leq C \tau^{(1/C)-d} \]

for all \( \tau \) and all \( S \in S \).

When \( S \in S_j \) we write \( \ell(S) = 2^{-Nj} \). By (1.14), (1.10), and (1.4), to each \( S \in S \) there corresponds a “corkscrew ball” \( B(p, \alpha c_0 \ell(S)) \subset \Omega \) with \( \text{dist}(p, S) \leq c_0 \ell(S) \). Moreover, by (1.7) and Lemma 3.1 and Lemma 3.2 from Section 3, for any \( \varepsilon > 0 \) there exist constants

\[ 2^{-N-1} c_0 < c_3 < 4 c_3 < c_0 \]
depending on $\varepsilon$ and the constants in (1.4) and (1.7) but not on $N$, such that for every $S \in S$ there exists a ball $B_S = B(p_S, c_3 \ell(S))$ with the properties:

\[(1.17) \quad B_S = B(p_S, c_3 \ell(S)) \subset 4B_S = B(p_S, 4c_3 \ell(S)) \subset \Omega \cap B(x_S, \frac{c_0}{2} \ell(S)),\]

and

\[(1.18) \quad \inf_{p \in 2B_S} \left\{ \omega(p, S \cap B(x_S, c_0 \ell(S)), \Omega \cap B(x_S, c_0 \ell(S))) \right\} \geq 1 - \varepsilon.\]

We can also take $N$ so large that if $S \cap S' = \emptyset$

\[(1.19) \quad B_S \cap B_{S'} = \emptyset,\]

and if $\ell(S') > \ell(S)$

\[(1.20) \quad 2B_{S'} \cap B(x_S, c_0 \ell(S)) = \emptyset.\]

Indeed, when $\ell(S) \neq \ell(S')$ and $N$ is large, (1.19) follows from (1.10) and (1.13), and if $S \neq S'$, $\ell(S) = \ell(S')$ and $\varepsilon < \frac{1}{2}$, then (1.12) and (1.18) imply (1.19). If $\ell(S') > \ell(S)$ then (1.20) holds by (1.16).

For $S \in S$ and $\lambda > 1$ define

$$\lambda S = \{ x : \text{dist}(x, S) \leq (\lambda - 1)\ell(S) \}.$$ \[(0.21)\]

Let $0 < \delta \lesssim 1$ and $A \geq 1$ be fixed constants. For $S_0 \in S$ and $S \in S$ with $S \subset S_0$, we say $S \in \text{HD}(S_0)$ (for “high density”) if $S$ is a maximal cube for which

\[(1.21) \quad \inf_{p \in B_{S_0}} \omega(p, 2S) \geq A \left( \frac{\ell(S)}{\ell(S_0)} \right)^d,\]

and we say $S \in \text{LD}(S_0)$ (for “low density) if $S$ is maximal for

\[(1.22) \quad \sup_{p \in B_{S_0}} \omega(p, S) \leq \delta \left( \frac{\ell(S)}{\ell(S_0)} \right)^d.\]

By (1.17) and Harnack’s inequality

\[(1.23) \quad \sup_{p \in B_{S_0}} \omega(p, S) \leq c_5 \inf_{q \in B_{S_0}} \omega(q, 2S)\]

for some constant $c_5$ and we can assume $A > c_5 \delta$ so that $\text{HD}(S_0) \cap \text{LD}(S_0) = \emptyset$.

For each $S_0 \in S$ let

\[(1.24) \quad G_1(S_0) = \{ S \in \text{LD}(S_0) \cup \text{HD}(S_0) : S \text{ is maximal} \}.$$
We call $G_1(S_0)$ the first generation of descendants of $S_0$, and we define later generations inductively:

\begin{equation}
G_k(S_0) = \bigcup_{S \in G_{k-1}(S_0)} G_1(S).
\end{equation}

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^{d+1}$, $d \geq 1$, be a domain satisfying (1.4) and (1.7). Assume (a) or (b) holds for $\Omega$, and let $S$ be a family of subsets of $\partial \Omega$ satisfying Proposition 1.3. Then there exists $\varepsilon_0$ and $A_0$ such that if $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \frac{\varepsilon}{3}$, and $A > \text{Max}(A_0, c_5 \delta)$, there exists a constant $C = C(\varepsilon, \delta, d, A)$ such that for any $S_0 \in S$

\begin{equation}
\sum_{k=1}^{\infty} \sum_{G_k(S_0)} \ell(S)^d \leq C \ell(S_0)^d.
\end{equation}

**Theorem 1.5.** Conversely, assume $\Omega$ is a corkscrew domain with (1.7), assume there exists a family $S$ of subsets of $\partial \Omega$ satisfying Proposition 1.3 and (1.17), (1.18) and (1.19), and assume (1.21), (1.22), (1.24) and (1.25) hold for some $\varepsilon, \delta$ and $A$ with $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \frac{\varepsilon}{3}$, and $A > c_5 \delta$. Further assume

(i) $S$ satisfies (1.26)

(ii) there exists $C > 0$ such that for any ball $B$, if $\{S_j\} \subset S$, $\bigcup S_j \subset B$ and for $j \neq k$ $S_j \cap S_k = \emptyset$, then

\begin{equation}
\sum \ell(S_j)^d \leq C \text{diam}(B)^d.
\end{equation}

Then (a) and (b) hold for $\Omega$.

There are many results like Theorem 1.4, called “corona decompositions” in reference to Carleson’s proof of the $H^\infty$ corona theorem [Ca1]. Among them we only mention Proposition 3.1 and Proposition 5.1 from [GMT], which together give a corona decomposition equivalent to (a) and (b) in the case that $\partial \Omega$ is $d$-Ahlfors regular. We use these two propositions and ideas from their proofs to establish the converse results Theorem 1.2 and Theorem 1.5. The main difference between Theorem 1.4 and Theorem 1.5 and the propositions from [GMT] is that here “cubes” are defined by capacities while in [GMT] cubes are defined by Hausdorff measures. The “small boundary condition” (1.15) must be included in Proposition 1.3 because it is needed for Propositions 3.1 and 5.1 from [GMT].

Theorem 1.1 is proved in Section 2. using [HMM2] and Harnack’s inequality on Whitney cubes for $\Omega$. Section 3 recalls three lemmas from [An] and [GMT], and in Section 4 these lemmas and ideas from [GMT] are used to show that (a) or (b) implies the packing condition (1.27) and consequently that $\partial \Omega$ is upper Ahlfors regular, $\mathcal{H}^d(B(x, r) \cap \partial \Omega) \leq Cr^d$. The proofs of Theorem 1.2, Theorem 1.4 and Theorem 1.5 are convoluted: In Section 5 we construct the cube family $S$ and prove Proposition 1.3. In Section 6, $S$ and the construction
from [GMT] are used to prove Theorem 1.4 and to extend Proposition 3.1 of [GMT] to
domains with (1.4) and (1.7). Then in Section 7 we use the already proved Theorem 1.4
to construct a subdomain $\tilde{\Omega} \subset \Omega$ such that $\partial \Omega \subset \partial \tilde{\Omega}$ and $\partial \tilde{\Omega}$ is Ahlfors regular. In Section 8 the generation sums for $\tilde{\Omega}$ are controlled by the generation sums for $\Omega$, which with Lemma 7.2 and Proposition 5.1 of [GMT], implies that $\partial \tilde{\Omega}$ is uniformly rectifiable, thus proving Theorem 1.2. Finally, Theorem 1.5 then follows from Theorem 1.1 and the proof of Theorem 1.2.

The proofs in this paper entail many constants. Constants $C$ or $C_j$ are large and may vary from use to use, but the constants $c_0, c_1, \ldots$ are small and sometimes interdependent. They are written so that $c_j$ can depend on $c_k$ only if $k < j$.

2. Proof of Theorem 1.1

We recall the Whitney decomposition of $\Omega$ into cubes $\Omega = \bigcup_\mathcal{W} Q$. Each $Q \in \mathcal{W} = \mathcal{W}(\Omega)$ is a closed dyadic cube,

$$\tag{2.1} Q = \bigcap_{1 \leq j \leq d+1} \{ k_j 2^{-n} \leq x_j \leq (k_j + 1) 2^{-n} \}$$

with $n$ and $k_j$ integers. If $Q_1, Q_2 \in \mathcal{W}$ then

$$\tag{2.2} Q_1 \subset Q_2, \ Q_2 \subset Q_1, \text{ or } Q_1^o \cap Q_2^o = \emptyset,$$

where $Q^o$ denotes the interior of $Q$. There are constants $1 < c_6 < c_7 < 3$ such that for all $Q \in \mathcal{W}$

$$\tag{2.3} c_6 Q \cap \partial \Omega = \emptyset \text{ but } c_7 Q \cap \partial \Omega \neq \emptyset,$$

where $\ell(Q)$ is the sidelength of $Q$ and $cQ$ is the concentric closed cube having sidelength $c\ell(Q)$.

Assume $\Omega$ and $\tilde{\Omega}$ satisfy condition (1.9) from Theorem 1.1, let $u$ be an harmonic function on $\Omega$ with $\sup_\Omega |u(y)| \leq 1$, and let $Q \in \mathcal{W}(\Omega)$. We fix a constant $1 < c_8 < c_6$ and consider two cases.

Case I: $c_8 Q \cap \partial \tilde{\Omega} = \emptyset$.

In this case there is $C_1 = C_1(d, c_7, c_8)$ such that $\text{dist}(y, \partial \Omega) \leq C_1 \text{dist}(y, \partial \tilde{\Omega})$ for all $y \in Q$, so that

$$\tag{2.4} \int_Q |\nabla u(y)|^2 \text{dist}(y, \partial \Omega) dy \leq C_1 \int_Q |\nabla u(y)|^2 \text{dist}(y, \partial \tilde{\Omega}) dy.$$

Case II: $c_8 Q \cap \partial \tilde{\Omega} \neq \emptyset$.

In this case Harnack’s inequality gives $\sup_Q |\nabla u(y)| \leq \frac{C_2}{\ell(Q)}$, for $C_2 = C_2(d, c_7)$, so that
\begin{align}
(2.5) \int_Q |\nabla u(y)|^2 \text{dist}(y, \partial \Omega) dy & \leq C_2^2 (1 + c_8) \frac{d+1}{d} \ell(Q)^d = C_3 \ell(Q)^d.
\end{align}

Now consider a ball $B = B(x, r)$, with $x \in \partial \Omega$, $r < \text{diam} \Omega$, and let
\[ \mathcal{W}_B = \{ Q \in \mathcal{W}(\Omega) : Q \cap B \neq \emptyset \}, \]
and for $J = I$ or $II$ let $\mathcal{W}_{B,J}$ be the set of Case J cubes in $\mathcal{W}_B$. Also note that by (2.3)
\begin{align}
(2.6) \quad \bigcup_{\mathcal{W}_B} c_6 Q & \subset B(x, C_4 r)
\end{align}
for a constant $C_4$ depending only $c_6$ and $c_7$. Since $H^{d+1}(\partial \tilde{\Omega} \setminus \partial \Omega) = 0$ (because $\partial \tilde{\Omega}$ is Ahlfors regular) we have
\[ \int_B |\nabla u(y)|^2 \text{dist}(y, \partial \Omega) dy \leq \sum_{\mathcal{W}_B} \sum_{\mathcal{W}_{B,J}} \int_Q |\nabla u(y)|^2 \text{dist}(y, \partial \Omega) dy = \sum_{\mathcal{W}_{B,I}} + \sum_{\mathcal{W}_{B,II}}. \]

To estimate $\sum_{\mathcal{W}_{B,I}}$ we use (2.4), (2.6), the uniform rectifiability of $\partial \tilde{\Omega}$, and the theorem of [HMM2] to get
\begin{align}
(2.7) \quad \sum_{\mathcal{W}_{B,I}} & \leq C_1 \int_{B(x, C_4 r)} |\nabla u(y)|^2 \text{dist}(y, \partial \tilde{\Omega}) dy \leq C (C_4 r)^d.
\end{align}

For estimating $\sum_{\mathcal{W}_{B,II}}$ the only available inequality is
\[ \sum_{\mathcal{W}_{B,II}} \leq C_3 \sum_{\mathcal{W}_{B,II}} \ell(Q)^d \]
from (2.5). But in Case II
\begin{align}
(2.8) \quad \ell(Q)^d & \leq C_5 H^d(c_6 Q \cap \partial \tilde{\Omega})
\end{align}
because $\partial \tilde{\Omega}$ is Ahlfors regular and by (2.2) and (2.3) no point lies in more than $N = N(c_6, c_7, d)$ cubes $c_6 Q$, $Q \in \mathcal{W}$. Therefore (2.5), (2.6), and the Ahlfors regularity of $\partial \tilde{\Omega}$ imply
\begin{align}
(2.9) \quad \sum_{\mathcal{W}_{B,II}} \leq C_5 \sum_{\mathcal{W}_{B,II}} \ell(Q)^d \leq C_5 N H^d(B(x, C_4 r)) \leq C_5 N (C_4 r)^d.
\end{align}

Thus by (2.7), (2.5) and (2.9), (a) holds for all bounded harmonic $u$.

To prove (b) let $u$ be an harmonic function on $\Omega$, let $\varepsilon > 0$ and consider the Case I and Case II cubes in $\mathcal{W}(\Omega)$. Write
\[ U = \bigcup_{\text{Case II}} Q, \quad V = \bigcup_{\text{Case I}} Q, \]
and
\[ \Gamma = \Omega \cap \partial V = \Omega \cap \partial U. \]

Let \( g \in W^{1,1}(\tilde{\Omega}) \) satisfy (1.2) and (1.3) for \( u \) on \( \tilde{\Omega} \) and define \( G = g\chi_U + u\chi_{V \cup \Gamma} \). Then \( ||u - G||_{L^\infty(\Omega)} < \epsilon \), and on \( \Omega \) the distribution
\[ \nabla G = \chi_U \nabla g + \chi_V \nabla u + \nu \]
where \( \nu \) is an \( \mathbb{R}^{d+1} \)-valued measure that accounts for the jump from \( g \) to \( u \) across \( \Gamma \) and has total variation \( ||\nu|| \leq \epsilon \chi_{\Gamma} \mathcal{H}^d \). Let \( x \in \partial \tilde{\Omega} \) and \( r > 0 \).

Then by the proof of (a)
\[ \int_{B(x,r) \cap (U \cup V)} |\nabla G| dy \leq Cr^d, \]
and because \( \partial \tilde{\Omega} \) is Ahlfors regular, (2.8) implies
\[ ||\nu||(B(x,r) \cap \Omega) \leq C\epsilon r^d. \]
Hence (1.3) holds for the vector measure \( \nabla G \).

To replace \( G \) by a \( W^{1,1}_{\text{loc}} \) function let \( \eta > 0 \) be small, write
\[ \psi_\eta(y) = \eta^{-(d+1)} \psi\left(\frac{y}{\eta}\right) \]
where \( \psi \in C^\infty(\mathbb{R}^{d+1}) \) is a non-negative radial function, compactly supported in \( B(0,1) \), and satisfying \( \int_{\mathbb{R}^{d+1}} \psi dy = 1 \), and for \( y \in \Omega \) define by convolution
\[ \tilde{G}(y) = G * \psi_{\eta \text{dist}(y, \partial \Omega)}(y). \]
Then \( \tilde{G} \in W^{1,1}_{\text{loc}}(\Omega) \) and (1.2) and (1.3) hold for \( \tilde{G} \) and \( u \).

3. Three Lemmas

Recall we assume (1.7) so that the harmonic measure \( \omega(p, E) = \omega(p, E, \Omega) \) exists for \( p \in \Omega \) and Borel \( E \subset \partial \Omega \). The first lemma is Lemma 3 from [An].

**Lemma 3.1.** \( \Omega \) satisfies (1.7) with constant \( \beta \) if and only if there exists \( \eta = \eta(\beta) < 1 \) such that for all \( x \in \partial \Omega \) and all \( r > 0 \)

\[ (3.1) \quad \sup_{B(x,r) \cap \Omega} \omega(p, \partial B(x, 2r) \setminus \Omega, \Omega \cap B(x, 2r)) \leq \eta. \]

The second lemma is a well-known consequence of Lemma 3.1 and induction.

**Lemma 3.2.** Assume \( \Omega \) satisfies (1.4) and (1.7) and let \( 0 < \varepsilon \leq \frac{1}{2} \). There are constants \( c_1 \) and \( c_2 \) depending only on \( \varepsilon \) and the constants \( \alpha \) and \( \beta \) in (1.4) and (1.7), such that whenever \( x \in \partial \Omega \) and \( r < \text{diam} \Omega \), there exists a ball \( B = B(p, c_1 r) \) such that

\[ (3.2) \quad 4B = B(p, 4 c_1 r) \subset \Omega \cap B(x, r), \]
\[(3.3) \quad \text{dist}(p, \partial \Omega) < c_2 r, \]

and

\[(3.4) \quad \inf_{q \in 2B} \omega(q, \partial \Omega \cap B(x, r), \Omega \cap B(x, r)) > 1 - \varepsilon. \]

**Proof.** By the maximum principle and induction (3.1) implies

\[(3.5) \quad \sup_{B(x, r) \cap \Omega} \omega(p, \partial B(x, 2^N r) \setminus \Omega, \Omega \cap B(x, 2^N r)) < \eta^N. \]

For \(\varepsilon > 0\) take \(N\) with \(\eta^N < \varepsilon\) and set \(C_1 = 1 + 2^N\). For any \(p \in \Omega\) take \(x \in \partial \Omega\) such that \(|x - p| = \text{dist}(p, \partial \Omega)\). Applying (3.5) with \(r = |x - p|\) yields

\[(3.6) \quad \omega(p, \partial \Omega \setminus B(p, C_1 \text{dist}(p, \partial \Omega)), \Omega) < \varepsilon. \]

By (1.4), \(\Omega \cap B(x, \frac{r}{1+C_1})\) contains a ball \(B = B(p, \frac{or}{1+C_1})\). Therefore (3.2) holds with

\[c_1 = \frac{\alpha}{4(1 + C_1)}\]

and (3.3) holds with

\[c_2 = \frac{1}{1 + C_1}. \]

If \(q \in 2B = B(p, \frac{or}{2(1+C_1)})\) then by (3.2) \(\text{dist}(q, \partial \Omega) \leq |q - x| \leq \frac{r}{1 + C_1}\). Therefore \(B(q, C_1 \text{dist}(q, \partial \Omega)) \subset B(x, r)\), so that (3.6) implies (3.4).

The next lemma is similar to Lemma 3.3 of [GMT].

**Lemma 3.3.** Assume \(\Omega\) satisfies (1.4) and (1.7). Then there exists \(\varepsilon_0 > 0\) and constants \(c_9\) and \(c_{10}\) depending only on \(d\) and the constants \(\alpha\) and \(\beta\) of (1.4) and (1.7) such that if \(0 < \varepsilon < \varepsilon_0\) and

(i) \(S \subset \partial \Omega\) is a Borel set, \(x \in S\), \(0 < r < \text{diam}(\Omega)\), and \(B(x, r) \cap \partial \Omega \subset S\),

(ii) the ball \(B_S = B(p_S, c_1 r)\) satisfies (3.2), (3.3) and (3.4) from Lemma 3.2,

(iii) \(E_S \subset S \cap B(x, r)\) is a compact set such that

\[(3.7) \quad \inf_{2B_S} \omega(q, E_S, \Omega) \geq 1 - \varepsilon, \]

then there exists a non-negative harmonic function \(u_S\) on \(\Omega\) and a Borel function \(f_S\) such that

\[0 \leq f_S \leq \chi_{E_S} \]

and for all \(p \in \Omega\),
\[ u_S(p) = \int_{E_S} f_S(y) d\omega(p, y, \Omega), \]  

(3.8)  

\[ \inf_{B_S} u_S(p) \geq c_9, \]  

(3.9)  

and there exists a unit vector \( \vec{e}_S \in \mathbb{R}^{d+1} \) such that  

(3.10)  

\[ \inf_{B_S} |\nabla u_S(p) \cdot \vec{e}_S| \geq \frac{c_{10}}{c_1 r}. \]  

The right side of (3.10) is so written to display the radius \( c_1 r \) of \( B_S \).  

Proof. Take \( q_S \in S \cap \partial \Omega \) with \( |q_S - p_S| < 2 \text{dist}(p_S, \partial \Omega) \). By (3.2) and (3.3) we have  

(3.11)  

\[ 4c_1 r < |p_S - q_S| < 2c_2 r. \]  

Case I. \( d \geq 2 \). By (1.7) and the definition of capacity there exists a positive measure \( \mu_S \) supported on \( \overline{B}(q_S, c_1 r) \cap \partial \Omega \) with \( \int d\mu_S > \beta(c_1 r)^{d-1} \) such that the potential  

\[ U_S(p) = \int |p - y|^{1-d} d\mu_S(y) \]  

is harmonic on \( \mathbb{R}^{d+1} \setminus \text{supp}\mu_S \supset \Omega \), and satisfies  

(3.12)  

\[ 0 < U_S(p) \leq 1 \]  

for all \( p \in \mathbb{R}^{d+1} \). By Egoroff’s theorem there is a compact set \( F_S \subset \overline{B}(q_0, c_1 r) \cap \partial \Omega \) such that \( \mu_S(F_S) \geq \beta(c_1 r)^{d-1} \) and  

\[ \int_{B(p, \eta)} |p - y|^{1-d} d\mu_S(y) \to 0 (\eta \to 0) \]  

uniformly on \( F_S \). Redefine \( U_S \) to be  

(3.13)  

\[ U_S(p) = \int_{F_S} |p - y|^{1-d} d\mu_S(y). \]  

Then \( U_S \) is continuous on \( \mathbb{R}^{d+1} \), harmonic on \( \mathbb{R}^{d+1} \setminus F_S \supset \Omega \), and satisfies (3.12).  

By (3.11) and (3.13),  

(3.14)  

\[ \inf_{2B_S} U_S(p) \geq \beta \left( \frac{c_1 r}{|p_S - q_S| + 3c_1 r} \right)^{d-1} = \beta r^{1-d} = c_9' \]  

Let \( \vec{e}_S = \frac{(q_S - p_S)}{|q_S - p_S|} \). Then by (3.11) we have
\[
\begin{align*}
(3.15) \quad \inf \left\{ \bar{e}_S \cdot \frac{(q-p)}{|q-p|} \mid q \in F_S, p \in B_S \right\} &= \frac{c_2}{c_1} = \frac{4}{\alpha}, \\
\text{Hence by (3.11), (3.13), (3.15), and the formula} \\
(3.16) \quad \nabla U_S(p) &= (1 - d) \int_{F_S} \frac{(p-y)}{|p-y|^{d+1}} d\mu_S(y),
\end{align*}
\]

we have on \( B_S \)

\[
(3.17) \quad |\nabla U_S(p) \cdot \bar{e}_S| \geq \frac{4}{\alpha} \left( \frac{d-1}{2c_1} \right)^{d-1} \frac{\alpha}{2c_1 r} = \frac{c'_{10}}{c_1 r},
\]

in which

\[
c'_{10} = \frac{d-1}{2c_1^{d-2}} \frac{\beta}{\alpha} \left( \frac{\alpha}{4 + \alpha} \right)^d
\]
depends only on \( d, \alpha \) and \( \beta \). Since \( U_S \) is continuous on \( \Omega \),

\[ U_S(p) = \int_{\partial \Omega} g_S(y) d\omega(p, y, \Omega) \]

with continuous \( g_S = U_S|_{\partial \Omega} \). Set \( f_S = \chi_{E_S} g_S \) and define \( u_S \) by (3.8). Finally take

\[ \varepsilon_0 < \min \left( \frac{c'_{10}}{2}, \frac{c'_{10}}{3} \right) \]
on \( f_Q \mid \leq 1 \),

(3.7) yields \( \sup_{2B_S} |U_S - u_S| \leq \varepsilon \). Hence (3.14) implies (4.4) for \( c_9 = \frac{c'_{10}}{2} \), and by (3.7) and Harnack’s inequality \( \sup_{B_S} |\nabla (U_S - u_S)| \leq \frac{2\varepsilon}{c_1 r} \). so that (3.7) implies (4.5) for \( c_{10} = \frac{c'_{10}}{3} \).

**Case II:** \( d = 1 \). Decreasing \( c_1 \) and \( c_2 \) if necessary, we have, again by Egoroff’s theorem, compact sets \( F_S^\pm \subset \overline{B(x, r)} \cap S \) such that \( \text{Cap}(F_S^\pm) \geq \frac{\beta c_1 r}{2} \equiv e^{-\gamma} \) and probability measures \( \mu_\pm \) supported on \( F_S^\pm \) so that the logarithmic potentials

\[ U_\pm(p) = \int_{F_S^\pm} \log \frac{1}{|p-y|} d\mu_\pm(y) \]

are continuous on \( \mathbb{R}^2 \) and harmonic on \( \mathbb{R}^2 \setminus F_S^\pm \) and satisfy \( U_\pm < \gamma \) on \( \mathbb{R}^2 \setminus F_S^\pm \) and for small \( \eta, \gamma - \eta \leq U_\pm \leq \gamma \) on \( F_S^\pm \). Because capacity is bounded by diameter, we can, by choices of \( c_1 \) and \( c_2 \), position \( F_S^\pm \) so that

\[ F_S^+ \subset B(p_S, 2c_2 r) \]

but

\[ F_S^- \subset \mathbb{R}^2 \setminus B(p_S, 4c_2 r). \]
Then on $\mathbb{R}^2 \setminus (F_Q^+ \cup F_S^-)$ the function $U^+ - U^-$ is harmonic and bounded, because the logarithmic singularities at $\infty$ cancel, and by the choices of $F_S^\pm$,

$$\sup_{F^+ \cup F^-} |U^+ - U^-| \leq \gamma - \log \left( \frac{1}{2c_2} \right) = \log \left( \frac{4c_2}{\beta c_1} \right),$$

$$\inf_{2B_S} (U^+ - U^-) \geq \log \left( \frac{1}{2c_2 r - 2c_1 r} \right) - \log \left( \frac{1}{4c_2 r + 2c_1 r} \right) = \log \left( \frac{2c_2 + c_1}{c_2 - c_1} \right),$$

and for some unit vector $\vec{e}_S$,

$$\inf_{B_S} \left| \nabla (U^+ - U^-) \cdot \vec{e}_S \right| \geq \frac{c_{10}}{r}.$$

Then (3.8), (3.9) and (3.10) hold for

$$f_S = \left( 2 \log \left( \frac{4c_2}{c_1} \right) \right)^{-1} \left( \log \left( \frac{4c_2}{c_1} \right) + U^+ - U^- \right) \chi_{E_S}.$$

\[ \square \]

4. AN UPPER BOUND FOR SUMS

Assume $\Omega \subset \mathbb{R}^{d+1}, d \geq 1$, is a domain satisfying (1.4), (1.7), and (a) or (b) and let $S$ be a family of subsets of $\partial \Omega$ satisfying the conclusions of Theorem 1.3 and hence also their consequences (1.17), (1.18), and (1.19).

**Lemma 4.1.** There is $C > 0$ such that if $B$ be any ball and if $\{S_j\} \subset S, \bigcup S_j \subset B$, and

$$S_j \cap S_k = \emptyset \text{ if } j \neq k,$$

then

$$\sum \ell(S_j)^d \leq C \text{ diam}(B)^d.$$

**Proof.** By (1.16), (1.17), (1.18) and Lemma 3.3, for each $S_j \subset B$ there exists $E_j \subset S_j$ and a Borel function $0 \leq f_j \leq \chi_{E_j}$, such that the harmonic function

$$u_j(p) = \int_{E_j} f_j(y) d\omega(p, y, \Omega),$$

satisfies

$$\inf_{B_{S_j}} u_j(p) \geq c_{11},$$

and there exists a unit vector $\vec{e}_j \in \mathbb{R}^{d+1}$ such that

$$\inf_{B_{S_j}} \left| \nabla u_j(p) \cdot \vec{e}_j \right| \geq \frac{c_{12}}{\ell(S_j)}.$$
Set \( u = \sum u_j \). Then by (1.17), (1.18) and (3.2), (3.4) and (3.8) we have \( \sup_{2B_{S_j}} |u - u_j| \leq \varepsilon \), so that by Harnack’s inequality \( \sup_{B_{S_j}} |\nabla (u - u_j)| \leq \frac{2\varepsilon}{\ell(S_j)} \). Therefore

\[
|\nabla u| > \frac{c_{11} - 3\varepsilon}{\ell(S_j)}
\]
on \( B_{S_j} \) and

\[
(4.6) \quad \int_{B_{S_j} \cap \Omega} |\nabla u(x)|^2 \text{dist}(x, \partial\Omega) \, dx \geq c_{12} \ell(S_j)^d.
\]

Assuming (a) holds on \( \Omega \) with constant \( C \) and summing, we obtain

\[
\sum_j \ell(S_j)^d \leq \frac{1}{c_{12}} \int_{\Omega} |\nabla u(x)|^2 \text{dist}(x, \partial\Omega) \leq C (\text{diam } B)^d,
\]

which yields (4.8) when (a) holds.

Now assume (b) holds for \( \Omega \) and \( \varepsilon < \frac{c_{11}}{3} \). If \( g \in W^{1,1}_{\text{loc}}(\Omega) \) satisfies (1.2) for \( u \) and \( \varepsilon < \frac{c_{11}}{3} \), then, using (4.4) and (4.5) for \( u_{S_j} \), we obtain

\[
\int_{B_{S_j}} |\nabla g(x)| \, dx \geq c_1^{13} \ell(S_j)^d.
\]

Thus from (a) or (b) we conclude that (4.2) holds. \( \square \)

As an aside we note that the proof of Lemma 4.1 yields two necessary conditions for (a) and (b).

**Proposition 4.2.** Let \( \Omega \subset \mathbb{R}^{d+1} \) be a domain for which (1.4), (1.7), and (a) or (b) holds. For \( 0 < \varepsilon < 1/2 \) there is \( C > 0 \), depending only on \( \varepsilon \) and the constants in (a) or (b) such that if \( x \in \partial\Omega, R > 0, \) and \( \{Q_j\} \) is a family of Whitney cubes for \( \Omega \) such that \( Q_j \subset B(x, R) \cap \Omega \) and there exists \( q_j \in Q_j \) and Borel \( E_j \subset \partial\Omega \) with \( \omega(q_j, E_j, \Omega) > 1 - \varepsilon \) and \( E_j \cap E_k = \emptyset, j \neq k \), then

\[
\sum_j \ell(Q_j)^d \leq CR^d.
\]

The proof is the same as the proof of Lemma 4.1.

**Proposition 4.3.** Let \( \Omega \subset \mathbb{R}^{d+1} \) be a corkscrew domain for which (1.7) holds. If (a) or (b) holds for \( \Omega \) then there a constant \( C > 0 \) such that for all \( x \in \partial\Omega \) and all \( r > 0 \),

\[
(4.7) \quad \mathcal{H}^d(B(x, r) \cap \partial\Omega) \leq Cr^d.
\]

**Proof.** Fix \( q \in \partial\Omega \) and \( R > 0 \). For large \( n \) let \( \mathcal{E}_n \) be the set of dyadic cubes \( Q \) of side \( 2^{-n} < 2 \) such that \( Q \cap \partial\Omega \cap B(q, R) \neq \emptyset \). It is enough to prove

\[
(4.8) \quad \sup_n 2^{-nd} \# \mathcal{E}_n \leq C 2^d
\]
for some constant $C$. Let $\mathcal{F}_n$ be a maximal subset of $\mathcal{E}_n$ such that $\frac{5}{4}Q \cap \frac{5}{4}Q' = \emptyset$ whenever $Q, Q' \in \mathcal{F}_n$. Then $\# \mathcal{E}_n \leq 3^{d+1} \# \mathcal{F}_n$. Let $\varepsilon$ be small, say $\varepsilon < \frac{c_4}{3}$. For each $Q \in \mathcal{F}_n$ there exists by (1.7) and (1.4) a function $u_Q$, a ball $B_Q$, and a set $E_Q \subset \frac{5}{4}Q$ satisfying the conclusions of Lemma 3.2 and Lemma 3.3.

Now set

$$ u = \sum_{Q \in \mathcal{F}_n} u_Q. $$

Then $u$ is harmonic on $\Omega$ and $0 < u < 1$ by (3.7) and the maximum principle since $E_Q \cap E_{Q'} = \emptyset$ for $Q \neq Q' \in \mathcal{F}_n$. For any $Q \in \mathcal{F}_n$, $|\nabla u_Q| \geq \frac{c_{14}}{\ell(Q)}$ by (4.5).

On the other hand, by (3.4), (3.8), and (3.2), $\sup_{2B_Q} |u - u_Q| \leq \varepsilon$, so that by Harnack’s inequality $\sup_{B_Q} |\nabla (u - u_Q)| \leq \frac{2\varepsilon}{\ell(Q)}$. Therefore

$$ |\nabla u| > \frac{c_{14} - 3\varepsilon}{\ell(Q)} $$
on $B_Q$ and

(4.9) $$ \int_{B_Q \cap \Omega} |\nabla u(x)|^2 \text{dist}(x, \partial \Omega) dx \geq c_{15} 2^{-nd}. $$

Assuming (a) holds on $\Omega$ with constant $C$ and summing over $Q \in \mathcal{F}_n$, we obtain

$$ \# \mathcal{E}_n \leq 3^{d+1} \# \mathcal{F}_n \leq \frac{1}{c_{15}} 2^{nd} \int_{B(0,2R) \cap \Omega} |\nabla u(x)|^2 \text{dist}(x, \partial \Omega) dx \leq 2^{nd} C 2^{d, d}. $$

which yields (4.8) when (a) holds.

Now assume (b) holds for $\Omega$ and $\varepsilon < \frac{c_{14}}{3}$. If $g \in W^{1,1}_\text{loc}(\Omega)$ satisfies (1.2) for $u$ and $\varepsilon < \frac{c_{14}}{3}$ then, using (4.4) and (4.5) for $u_{S_j}$, we obtain

$$ \int_{B_{S_j}} |\nabla g(x)| dx \geq c'_{16}(\ell(S_j))^d. $$

Thus from (a) or (b)

$$ 2^{-nd} \# \mathcal{E}_n \leq \text{Max} \left( \frac{1}{c_{15}}, \frac{1}{c'_{16}} \right) R^d. $$

Merged with the results of [HMM2] and [GMT] Proposition 4.3 yields:

**Corollary 4.4.** If $\Omega \subset \mathbb{R}^{d+1}$ is a corkscrew domain for which there exists constant $c > 0$ such that for all $x \in \partial \Omega$ and all $0 < R < \text{diam}(\partial \Omega)$,

(4.10) $$ \mathcal{H}^d (B(x, r) \cap \partial \Omega) \geq cr^d, $$

then (a) or (b) holds for $\Omega$ if and only if $\partial \Omega$ is uniformly rectifiable.
5. Modified Christ-David Cubes

To prove Theorem 1.3, we follow the construction in [Da] very closely, although the arguments from [Ch] and [DM] would also work. To start we use (a) or (b) to get a grip on the small boundary condition (1.15).

Lemma 5.1. Let $0 < \eta < 1$ and let $N$ be a positive integer. Assume $\Omega$ is a bounded corkscrew domain with (1.7) and assume (a) or (b) holds for $\Omega$. Then for any $x \in \partial \Omega$ and any $j \in \mathbb{N}$ there exists an open ball $B_j(x) = B_j(x, r)$ having center $x$ and radius

$$r \in (2^{-Nj}, (1 + \eta)2^{-Nj})$$

such that if

$$\Delta_j(x) = B_j(x) \cap \partial \Omega,$$

$$E_j(x) = \{ y \in \Delta_j(x) : \text{dist}(y, \partial \Omega \setminus \Delta_j(x)) < \eta^22^{-Nj} \}$$

$$\cup \{ y \in \partial \Omega \setminus \Delta_j(x) : \text{dist}(y, \Delta_j(x)) < \eta^22^{-Nj} \}$$

and $m_j(x)$ is the minimum number of closed balls $\overline{B(p, \eta^22^{-Nj})}$ needed to cover $E_j(x)$, then

$$m_j(x) \leq C_d\eta^{1-2d},$$

in which the constant $C_d$ depends only on $d$ and the constants in (a) and (b).

Proof. Partition the closed shell

$$\overline{B(x, (1 + \eta)2^{-Nj}) \setminus B(x, 2^{-Nj})}$$

into a family $\mathcal{E}$ of at most $1 + \lfloor \frac{1}{\eta} \rfloor$ closed shells of width $\eta^22^{-Nj}$. Fix $2^{-n} \sim \eta^22^{-Nj}$, let $\mathcal{E}'$ be the set of closed dyadic cubes $Q$ of side $2^{-n}$ such that $Q \cap \partial \Omega \neq \emptyset$ and let $M = \# \mathcal{E}'$. Choose a maximal subset $\mathcal{E}_0 \subset \mathcal{E}$ of pairwise disjoint closed cubes. Then $\mathcal{E}_0$ has cardinality $\# \mathcal{E}_0 \geq c_{16}3^{-d-1}M$ and the enlarged cubes $\frac{5}{4}Q, Q \in \mathcal{E}_0$ are pairwise disjoint. For each $Q \in \mathcal{E}_0$ there exists by (1.7) a compact set $E_Q \subset \frac{5}{4}Q \cap \partial \Omega$ and a ball $B(p_Q, \alpha\eta^22^{-j}) \subset \frac{5}{4}Q \cap \Omega$ satisfying the conclusions of Lemma 3.2 and Lemma 3.3. Now we can follow the proof of Lemma 4.1 or Proposition 4.3 to build functions $u_Q, Q \in \mathcal{E}_0$ and use assumption (a) or (b) with $u = \sum_{\mathcal{E}_0} u_Q$ to conclude that $\# \mathcal{E}_0(\eta^22^{-Nj}) \leq C'2^{-Njd}$. Hence $M \leq C_d\eta^{-2d}$ and there exists a pair of adjacent closed subrings in $\mathcal{E}$ whose union meets at most $c_{17}C_d\eta^{1-2d}$ dyadic cubes from $\mathcal{E}$. That implies (5.1).

Proof of Proposition 1.3. For $j \geq 0$ let $V_j$ be a maximal subset of $\partial \Omega$ such that when $x, x' \in V_j$ $|x - x'| \geq 2^{-jN}$ and for $x \in V_j$ let $B_j(x)$ be the ball given by Lemma 5.1 and set $\Delta_j(x) = \partial \Omega \cap B_j(x)$. Put a total order, written $x < y$, on the finite set $V_j$ and define
\[ \Delta_j^*(x) = \Delta_j(x) \setminus \bigcup_{y < x} \Delta_j(y). \]

Then for each \( j \), (1.10), (1.11), and (1.12) hold for the family \( \{\Delta_j^*(x)\} \) and because the balls \( B(x, (1 - \eta)2^{-Nj}) \), \( x \in V_j \) are disjoint we have

\[ B(x, (1 - \eta)2^{-Nj}) \subset \Delta_j^*(x) \]

for every \( x \in V_j \). Because \( \partial \Omega \subset \mathbb{R}^{d+1} \) there is constant \( M_d \) independent of \( j \) such that

\[ \# \{ y \in V_j : y < x \text{ and } B_j(y) \cap B_j(x) \neq \emptyset \} \leq M_d. \]

Therefore by (5.1) the minimum number \( m_j^* \) of closed balls \( B(p, \eta^22^{-N_j}) \) needed to cover \( E_j^*(x) = \{ y \in \Delta_j^*(x) : \text{dist}(y, \partial \Omega) < \eta^22^{-N_j} \} \cup \{ y \in \partial \Omega \setminus \Delta_j^*(x) : \text{dist}(y, \Delta_j^*(x)) < \eta^22^{-N_j} \} \) has the upper bound

\[ m_j^*(x) \leq C_d M_d \eta^{1-2d}. \]

However the families \( \{\Delta_j^*\}_{j \geq 0} \) may not satisfy the nesting condition (1.13) or the small boundary condition (1.15). For those reasons we further refine each set \( \Delta_j^* \), still following [Da]. If \( x \in V_j, j \geq 1 \), there exists by (1.11) and (1.12) a unique \( \varphi(x) \in V_{j-1} \) such that \( x \in \Delta_j^*_{j-1}(\varphi(x)) \). For any \( j \) and \( x \in V_j \) define \( D_{j,0}(x) = \Delta_j^*(x) \) and for \( n \in \mathbb{N} \),

\[ D_{j,n}(x) = \bigcup \{ \Delta_j^*_{j+n}(y) : \varphi^n(y) = x \}. \]

Then for any \( j, n \)

\[ \bigcup \{ D_{j,n}(x) : x \in V_j \} = \partial \Omega \]

and by induction

\[ D_{j,n}(x) = \bigcup \{ D_{j,n-k}(y) : \varphi^k(y) = x \}. \]

for \( 0 \leq k \leq n \).

Write \( \text{dist}_H(A, B) \) for the Hausdorff distance between subsets \( A, B \) of \( \mathbb{R}^{d+1} \). Since \( \text{diam}(\Delta_j^*) \leq (1 + \eta)2^{-Nj} \) we have

\[ \text{dist}_H(D_{j,1}(x), \Delta_j^*(x)) \leq (1 + \eta)2^{-N(j+1)}, \]

so that by (5.6) and induction

\[ \text{dist}_H(D_{j,n}(x), D_{j,n+1}(x)) \leq (1 + \eta)2^{-N(j+n)}. \]
Hence for each \( j \) and \( x \in V_j \) the sequence of \( \{D_{j,n}(x)\} \) of compact sets converges in Hausdorff metric to a compact set \( R_j(x) \). It is clear from (5.5) that for any fixed \( j \)

\[
(5.8) \quad \bigcup \{ R_j(x) : x \in V_j \} = \partial \Omega
\]
because if \( y \in \partial \Omega \) then \( y \in D_{j,n}(x^{(n)}) \) for some \( x^{(n)} \in V(j) \) and because \( V(j) \) is finite there is \( x \in V(j) \) with \( y \in D_{j,n}(x) \) for infinitely many \( n \).

Since we took closures (1.12) may not hold for the sets \( \{ R_j(x) \} \), and like [Da] we must alter them one final time. By induction we can choose the ordering on the finite set \( V_j, j \geq 1 \) so that \( x < y \) if \( \varphi(x) < \varphi(y) \). Then define, for all \( j \) and \( x \in V(j) \)

\[
(5.9) \quad S_j(x) = R_j(x) \setminus \bigcup_{V(j) \ni y < x} R_j(y).
\]

Then it is clear from (5.8) that (1.12) and (1.13) hold for the family \( S = \bigcup_j \{ S_j \} \), and since by (5.7)

\[
(5.10) \quad \text{diam}(S_j(x)) \leq \text{diam}(R_j(x)) \leq \sum_{k=j}^{\infty} 2(1 + \eta)2^{-Nk} \leq 4(1 + \eta)2^{-Nj}.
\]

To obtain the lower bound in (1.10) and also (1.13), (1.14) and (1.15) we need \( 2^{-N} \) to be small compared to \( \eta \). Assume

\[
(5.11) \quad 2^{-N} \sim \eta^2 < \frac{1}{9}.
\]

Then by (5.2) and (5.7) we have for \( x \in V_j \),

\[
\text{dist}(x, \partial \Omega \setminus D_{j,n}) \geq (1 - \eta)2^{-Nj} - \sum_{k>j} 2(1 + \eta)2^{-Nk} \\
\geq 2^{-Nj} \left( 1 - \eta - 2(1 + \eta) \frac{2^{-N}}{1 - 2^{-N}} \right) \geq \frac{2^{-Nj}}{3}.
\]

This implies (1.14) and with (5.10) it also implies (1.10).

To show (1.13) suppose \( u \in \Delta_j(x) \cap \Delta_{j+1}(y) \). Then by (5.7) \( u = \lim x_n \) where \( x_n \in V_n, x_{n+1} \in \Delta_n^*(x_n) \) and \( x_j = x \), and \( u = \lim y_n \) where \( y_n \in V_n, y_{n+1} \in \Delta_n^*(y_n) \) and \( y_{j+1} = y \). Hence \( u \in \bigcap_{n \geq j} R_n(x_n) \cap \bigcap_{n \geq j+1} R_n(y_n) \) so that by the definition (5.9) \( y_n = x_n \) for all \( n \geq j + 1 \) and \( S_{j+1}(y) \subset S_j(x) \).

To verify the small boundary condition (1.15) we can by (5.2) assume \( \tau = 2^{-Nk}, k \geq 1 \). Let \( x \in V_j \) and write \( S = S_j(x) \). Then by (5.7) and (5.10) \( N_\tau(S) \) is comparable to

\[
\# \{ y \in V_{j+k} : S_{j+k}^*(y) \cap \Delta_\tau(S) \neq \emptyset \},
\]

and by (5.4) and (5.11) this number is bounded by \( (C_dM_d\eta^{1-2d})^k \sim (C_dM_d)^k \tau^{1/2} \), which, for \( C' > 2 \) and \( \tau \) small, is bounded by \( C\tau^{1/C-d} \).
6. A CORONA DECOMPOSITION AND THE PROOF OF THEOREM 1.4

We prove Theorem 1.4. Assume $\Omega \subset \mathbb{R}^{d+1}, d \geq 1$, is a domain satisfying (1.4), (1.7), and either (a) or (b) and let $S$ be a family of subsets of $\partial \Omega$ satisfying the conclusions of Proposition 1.3. We shall prove there exist constants $\varepsilon_0, A_0$ and $C$ such that (1.24) holds with constant $C$ whenever $0 < \delta < \frac{\varepsilon_0}{3} < \frac{2\delta}{3}$ and $A > A_0, S_0 \in S$, and $G_k(S_0)$ are its generations defined for $\delta$ and $A$. Recall that by Proposition 1.3 the family $S$ has the properties (1.17), (1.18), and (1.19).

**Lemma 6.1.** Let $S \in S$ and let $\{S_j\} \subset S$ be a family of cubes $S_j \subset S$ satisfying (4.1). If $S_j \in \text{HD}(S)$ for all $j$, then

(6.1) \[ \sum \ell(S_j)^d \leq \frac{C_1}{A} \ell(S)^d, \]

while if $S_j \in \text{LD}(S)$ for all $j$, then

(6.2) \[ \sup_{B_S} \sum_{S_j} \omega(p, S_j) \leq C_2, \]

where $C_1$ and $C_2$ depend only on $d$, $\delta$ and the constants $\alpha$ and $\beta$ in (a) and (b).

**Proof.** Assertion (6.2) follows from (1.22) and Lemma 4.1, with constant $C_2$ depending only on $\delta$ and $\alpha$ or $\beta$ from (a) or (b).

Since the definition of HD entails $\omega(p_S, 2S_j, \Omega)$ and not $\omega(p_S, S_j, \Omega)$, the proof of (6.1) requires more work. Note that if $2S_k \cap 2S_j \neq \emptyset$ and $\ell(S_k) \leq \ell(S_j)$ then by (1.10)

\[ S_k \subset B(x_{S_j}, C\ell(S_j)), \]

in which the constant $C$ depends only on the upper bound in (1.10) and thus only on $\alpha$, $\beta$ and $d$. Hence by Lemma 4.1

\[ \sum \left\{ \ell(S_k)^d : 2S_k \cap 2S_j \neq \emptyset, \ell(S_k) \leq \ell(S_j) \right\} \leq C_1 \ell(S_k)^d, \]

and by a Vitali argument there exists $\{S'_j\} \subset \{S_j\}$ with $2S'_j \cap 2S'_{j'} = \emptyset$ and

\[ \sum \ell(S_j)^d \leq C_1 \sum \ell(S'_j)^d \leq C_1 \frac{A}{A} \sum \omega(p_S, 2S'_j, \Omega) \ell(S)^d \leq C_1 \ell(S)^d. \]

Now assume $A > 2C_1$. To prove (1.28) we separate high and low density cubes. For $S \in S$ let $GH_1(S)$ be a family of high density cubes $S' \in G_1(S)$ and by induction

(6.3) \[ GH_k(S) = \bigcup_{S' \in GH_{k-1}(S)} GH_1(S'). \]

Thus if $S_k \in GH_k(S)$, then

(6.4) \[ S_k \subset S_{k-1} \subset \ldots \subset S_1 \subset S_0 = S \]
in which for \( j > 0 \)

\[ S_{j+1} \in \text{HD}(S_j), \]

so that all ancestors of \( S_k \) except possibly the first are HD cubes. Write

\[ GH(S) = \bigcup_{k \geq 1} GH_k(S). \]

Then by (6.1)

\[ \sum_{GH(S)} \ell(S')^d = \sum_{k=1}^{\infty} \sum_{GH_k(S)} \ell(S')^d \leq \frac{C_1}{A-C_1} \ell(S)^d. \]

Similarly, let \( GL_1(S) \) be a family of low density cubes \( S_j \in G_1(S) \) and by induction

\[ GL_k(S) = \bigcup_{S' \in GL_{k-1}(S)} GL_1(S'). \]

Thus if \( S_k \in GL_k(S) \), then

\[ S_k \subset S_{k-1} \subset \ldots S_1 \subset S_0 = S \]

and \( S_{j+1} \in \text{LD}(S_j) \) for \( j > 0 \). Write

\[ GL(S) = \bigcup_{k \geq 1} GL_k(S). \]

**Lemma 6.2.** Assume \( \varepsilon \) in (1.2) is small and \( \delta \leq \varepsilon \). Then there exists constant \( C_2 \) such that for any \( S_0 \in S \)

\[ \sum_{GL(S_0)} \ell(S)^d = \sum_{k=1}^{\infty} \sum_{GL_k(S_0)} \ell(S)^d \leq C_2 \ell(S_0)^d. \]

**Proof.** The proof is like the proof of Lemma 4.1 or Lemma 3.7 of [GMT]. For any \( S \in GL(S) \) define

\[ E_S = S \setminus \bigcup_{S' \in GL_1(S)} S'. \]

Then by (6.2) and (1.22) there exists an harmonic function \( u_S \) on \( \Omega \) such that

\[ 0 \leq u_S \leq 1 \]

\[ u_S(x) = \int_{E_S} f_S(y)d\omega(x,dy), 0 \leq f_S \leq \chi_{E_S}, \]

and
\[
\inf_{B_S} |\nabla u_S| \geq \frac{c(\epsilon)}{\ell(S)}.
\]
Consequently
\[
u = \sum_{GL(S_0)} u_S
\]
is harmonic on \(\Omega\), satisfies \(0 \leq u \leq 1\), and has large oscillation on \(B_S\) for all \(S \in GL(S_0)\) so that by the proof of Lemma 4.1 inequality (6.5) holds with \(C_2\) depending only on the constants in (1.1) and (1.3).

That proves (1.24) and Theorem 1.4.

7. A Domain \(\tilde{\Omega}\)

Assume \(\Omega\) is a corkscrew domain satisfying (1.7) and \(\mathcal{S}\) is a family of subsets of \(\partial \Omega\) having properties (1.10) - (1.15) of Theorem 1.3, and their consequences (1.17), (1.18) and (1.19). Fix constants \(\epsilon, \delta, N, A\) and \(C\) with \(0 < \delta < \frac{\epsilon}{3}\) and \(A\) so large that (1.24) holds for any \(S_0 \in \mathcal{S}\) when the generations \(G_k(S_0)\) are define by (1.21) and (1.22). Also assume \(\mathcal{S}\) satisfies the conclusion of Lemma 4.1 or, equivalently, hypothesis (ii) of Theorem 1.5. Under those assumptions we construct a domain \(\tilde{\Omega} \subset \Omega\) with \(\partial \tilde{\Omega} \subset \partial \Omega\) and a \(d\)-Ahlfors
regular measure $\sigma$ supported on $\partial\tilde{\Omega}$ and boundedly mutually absolutely continuous with $\chi_{\partial\tilde{\Omega}}\mathcal{H}^d$.

For any $S \in \mathcal{S}$ let

$$\Gamma_S \subset 2B_S \setminus B_S$$

be a finite union of separated closed spherical caps such that

$$\mathcal{H}^d(\Gamma_S) = c_{18}\ell(S)^d.$$  \hspace{1cm} (7.1)

Since $B_S$ has diameter $2c_1\ell(S)$ we can (and do) require $\Gamma_S$ to be uniformly rectifiable with constants depending only on $c_0,\ldots,c_{18}$ but not on $S$. Note that (taking $c_{18}$ carefully) we have

$$\omega(p_S,\Gamma_S,\Omega^*) \sim 1/2,$$  \hspace{1cm} (7.2)

for any domain $\Omega^*$ such that

$$(\Omega \setminus \Gamma_S) \cap B(x_S,c_0\ell(S)) \subset \Omega^* \subset \Omega,$$

and by (3.4)

$$\omega(p_S,S \cup \Gamma_S,\Omega^*) > 1 - \varepsilon$$  \hspace{1cm} (7.3)

for all such $\Omega^*$. Define $\Omega_0 = \Omega$ and assume $\text{diam}(\partial\Omega) \sim 1$ so that $\partial\Omega = S_0 \in \mathcal{S}$. Fix $\lambda > 1$ so that

$$\lambda - 1 < \text{dist}(S,4B_S)$$  \hspace{1cm} (7.4)

and define

$$\tilde{\mathcal{H}}(S_0) = \left\{ S_1 \in \mathcal{S}, S_1 \subset S_0 : \omega(p_{S_0},\lambda(S),\Omega_0) \geq A\left(\frac{\ell(S_1)}{\ell(S_0)}\right)^d, S_1 \text{ maximal} \right\},$$

$$\tilde{\mathcal{L}}(S_0) = \left\{ S_1 \in \mathcal{S}, S_1 \subset S_0 : \omega(p_{S_0},S_1,\Omega_n) \leq \delta\left(\frac{\ell(S_1)}{\ell(S_0)}\right)^d, S_1 \text{ maximal} \right\},$$

$$\tilde{G}_1 = \tilde{G}_1(S) = \left\{ S' \in \tilde{\mathcal{H}}(S) \cup \tilde{\mathcal{L}}(S), S' \text{ maximal} \right\},$$

$$K_1 = S_0 \setminus \bigcup_{\tilde{G}_1(S_0)} S,$$

$$\text{Tree}(S_0) = \left\{ S \in \mathcal{S} : S \not\subset S' \text{ for all } S' \in \tilde{G}_1(S_0) \right\},$$

$$\Omega_1 = \Omega \setminus \bigcup_{\tilde{G}_1(S_0)} \Gamma_S,$$

$$\mu_1(\cdot) = \ell(S_0)^d \chi_{K_1} \omega(p_{S_0},\cdot,\Omega_0),$$
\[ \nu_1 = \sum_{\tilde{G}_1(S_0)} \chi_{\Gamma_S} \mathcal{H}^d, \]

and

\[ \sigma_1 = \mu_1 + \nu_1. \]

Then \( \sigma_1 \) is a finite measure on \( \partial \Omega_1 \).

For \( S \in \mathcal{S} \) define

\[ S^1 = S \cup \bigcup \{ \Gamma_{S'} : S' \in \tilde{G}_1, S' \subset S \} \]

and declare \( \ell(S^1) = \ell(S) \).

**Lemma 7.1.** There are constants \( c_{20} \) and \( c_{21} \) such that if \( S \in \text{Tree}(S_0) \),

\[ c_{20} \ell(S) \leq \sigma_1(S^1) \leq c_{21} \ell(S). \]

**Proof.** For the upper bound we have

\[ \mu_1(S^1) \leq A \frac{\ell(S)^d}{\ell(S_0)^d}, \]

since \( S \in \text{Tree}(S_0) \), and

\[ \nu_1(S^1) \leq C_1 \ell(S)^d \]

by Lemma 4.1.

For the lower bound note that

\[ \sigma_1(S^1) = \ell(S_0)^d \omega(p_{S_0}, S, \Omega_0) - \ell(S_0)^d \sum_{\tilde{G}_1(S_0) \ni S' \subset S} \omega(p_{S_0}, S', \Omega_0) + \sum_{\tilde{G}_1(S_0) \ni S' \subset S} \mathcal{H}^d(\Gamma_{S'}), \]

in which

\[ \ell(S_0)^d \omega(p_{S_0}, S, \Omega_0) \geq \delta \ell(S)^d \]

while by the definition of \( G_1(S_0) \)

\[ \ell(S_0)^d \sum_{\tilde{G}_1(S_0) \ni S' \subset S} \omega(p_{S_0}, S', \Omega) \leq C_1 2^{2Nd} A \sum_{\tilde{G}_1(S_0) \ni S' \subset S} \ell(S')^d. \]

Thus if

\[ \sum_{\tilde{G}_1(S_0) \ni S' \subset S} \ell(S)^d \leq \frac{\delta}{C_1 2^{2Nd+1} A} \ell(S)^d \]

(7.6) the lower bound holds with \( c_{20} = \frac{\delta}{2} \). On the other hand, if (7.6) fails, then \( \mu_1(S^1) \geq 0 \) and

\[ \nu_1(S^1) \geq \frac{c_{18}}{C_1 2^{2Nd+1} A} \delta. \]
Now continue by induction. For \( n \geq 1 \) assume we have defined \( \tilde{G}_n = \tilde{G}_n(S_0), \Omega_n \), and \( S^n \) for all \( S \in \mathcal{S} \). Then for each \( S \in \tilde{G}_n(S_0) \) define
\[
\tilde{H}(S) = \left\{ S_1 \in \mathcal{S}, S_1 \subset S : \omega(p_S, \lambda(S_1)^n, \Omega_n) \geq A\left( \frac{\ell(S_1)}{\ell(S)} \right)^{d}, S_1 \text{ maximal} \right\},
\]
\[
\tilde{L}(S) = \left\{ S_1 \in \mathcal{S}, S_1 \subset S : \omega(p_S, (S_1)^n, \Omega_n) \leq \delta\left( \frac{\ell(S_1)}{\ell(S)} \right)^{d}, S_1 \text{ maximal} \right\},
\]
\[
\tilde{G}_1(S) = \left\{ S' \in \tilde{H}(S) \cup \tilde{L}(S), S' \text{ maximal} \right\}
\]
\[
\text{Tree}(S) = \left\{ S' \in \mathcal{S} : S' \subset S, S' \nsubseteq S_1 \text{ for all } S_1 \in \tilde{G}_1(S) \right\},
\]
\[
\tilde{G}_{n+1}(S_0) = \bigcup_{\tilde{G}_n(S_0)} \tilde{G}_1(S),
\]
\[
K_{n+1} = \bigcup_{\tilde{G}_n(S_0)} (S \setminus \bigcup_{\tilde{G}_1(S)} S_1),
\]
\[
\Omega_{n+1} = \Omega_n \setminus \bigcup_{\tilde{G}_{n+1}(S_0)} \Gamma_S,
\]
\[
\mu_{n+1}(.) = \sum_{S \in \tilde{G}_n} \ell(S)^d \chi_{S \cap K_{n+1}} \omega(p_S, \Omega_n),
\]
\[
\nu_{n+1} = \sum_{\tilde{G}_{n+1}(S_0)} \chi_{\Gamma_S} \mathcal{H}^d,
\]
and define
\[
\sigma_{n+1} = \mu_{n+1} + \nu_{n+1}.
\]
Then \( \sigma_{n+1} \) is a finite measure on \( \partial \Omega_{n+1} \).

For \( S \in \mathcal{S} \) define
\[
S^{n+1} = S^n \cup \bigcup \{ \Gamma_{S'} : S' \in \tilde{G}_{n+1}, S' \subset S \}
\]
and define
\[
\ell(S^{n+1}) = \ell(S).
\]
Note that by the proof of Lemma 7.1,
\[
(7.7) \quad c_{17} \ell(S)^d \leq \sigma_{n+1}(S^{n+1}) \leq (c_{18} + 1) \ell(S)^d
\]
for all \( S \in \text{Tree}(S'), S' \in \tilde{G}_n(S_0) \).
Define

\[ \tilde{\Omega} = \bigcap_n \Omega_n, \]

which, as we will see, is a connected open set, and

\[ \mu = \sum_{n \geq 1} \mu_n, \]

\[ \nu = \sum_{n \geq 1} \nu_n, \]

\[ \sigma = \mu + \nu, \]

and, for \( S \in \mathcal{S} \),

\[ S^\infty = \bigcup S^n. \]

**Lemma 7.2.** Let \( S \in \tilde{G}_n \). Then

\[ \sum_{H(S)} \left( \ell(S_1) / \ell(S) \right)^d \leq C \frac{1}{A}, \]

and

\[ \sum_{L(S)} \omega(p_S, S_1, \Omega) \leq C \delta + \varepsilon, \]

where

\[ \inf_{T \in S} \inf_{p \in \Gamma} \omega(p, T, \Omega) \geq 1 - \varepsilon. \]

**Proof.** The proof of (7.8) is the same as the proof of (6.6) because by Lemma 4.1 (or hypothesis (ii) of Theorem 1.5) the Vitali argument used there will still apply. To prove (7.9) let \( S \in \tilde{G}_n \) and for \( 1 \leq k \leq (n - 1) \), let \( T_k(S) \) be that unique \( T \in \tilde{G}_k \) such that \( S \subset T_k \). Let \( S_1 \in L(S) \). Then \( S_1 \subset \partial \Omega \subset \partial \Omega_n \) and

\[ \omega(p_S, S_1, \Omega) = \omega(p_S, S_1, \Omega_n) + \sum_{k=1}^n \sum_{T \in \tilde{G}_k \setminus \{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega) d\omega(p_S, p, \Omega_n). \]

By definition and Lemma 4.1

\[ \sum_{S_1 \in L(S)} \omega(p_S, S_1, \Omega_n) \leq \delta \sum_{L(S)} \left( \frac{\ell(S_1)}{\ell(S)} \right)^d \leq C \delta, \]

while
\[\sum_{S_1 \in \text{LD}(S)} \sum_{k=1}^{n} \sum_{T \in \tilde{G}_k \setminus \{S_1\}} \int_{\Gamma_T} \omega(p, S_1', 1, \Omega) d\omega(p_S, p, \Omega_n)\]

\[= \int_{\Gamma_S} \sum_{S_1 \in \text{LD}(S)} \omega(p, S_1, \Omega) d\omega(p_S, p, \Omega_n)\]

\[+ \sum_{k=1}^{n} \sum_{T \in \tilde{G}_k} \int_{\Gamma_T} \sum_{S_1 \in \text{LD}(S)} \omega(p, S_1, \Omega) d\omega(p_S, p, \Omega_n)\]

\[+ \sum_{k=1}^{n-1} \int_{\Gamma_{T_k}} \sum_{S_1 \in \text{LD}(S)} \omega(p, S_1, \Omega) d\omega(p_S, p, \Omega_n)\]

\[= I + II + III.\]

By (7.2) and Harnack’s inequality we have

\[I \leq \frac{2}{3} \sum_{S_1 \in \text{LD}(S)} \omega(p_S, S_1, \Omega),\]

and we can move term \(I\) to the left side of (7.9).

For \(II\), note that

\[(S \cup \Gamma_S) \cap \bigcup_{1 \leq k \leq n} \bigcup_{\{T \in \tilde{G}_k \cap S = \emptyset\}} \Gamma_T = \emptyset\]

so that by (7.3) we have \(II \leq \varepsilon\).

For \(III\) recall that \(\text{dist}(p_{T_k}, S) \geq c_2 2^{N(n-k)\ell(S)}\). Therefore

\[B(x_S, c_0 \ell(S)) \cap \bigcup_{1 \leq k \leq n-1} \Gamma_{T_k} = \emptyset,\]

so that by (1.20) \(III < C\varepsilon\).

That established (7.9) and Lemma 7.2.

\[\square\]

If \(C\delta + \varepsilon\) is small, Lemma 7.2 and the proof of Lemma 6.2 yield

\[(7.10) \quad \sum_{k=1}^{\infty} \sum_{\tilde{G}_k} \left(\frac{\ell(S_1)}{\ell(S)}\right)^d \leq C_3\]

for any \(S \in \mathcal{S}\).

By (7.1) and (7.10) \(\tilde{\Omega} = \bigcup \tilde{\Omega}_n\) is a connected open set and

\[\partial \tilde{\Omega} = \partial \Omega \cup \bigcup_{n=1}^{\infty} \bigcup_{S \in \tilde{G}_n} \Gamma_S.\]
By (7.7) \( \sigma \) is a finite measure on \( \partial \tilde{\Omega} \) such that for all \( S \in \mathcal{S} \)

\[
\beta_1 \ell(S)^d \leq \sigma(S) \leq \gamma_1 \ell(S)^d
\]

and by Lemma 7.1 and the definition of \( \nu_{n+1} \)

\[
\sigma(E) = \mathcal{H}^d(E)
\]

for all Borel \( E \subset \bigcup \Gamma_S \). In view of properties (1.10) and (1.14) of \( S \), these imply that \( \sigma \) is a \( d \)-Ahlfors regular measure with closed support \( \partial \tilde{\Omega} \) and hence that \( \partial \tilde{\Omega} \) is \( d \)-Ahlfors regular.

Moreover, the family

\[
S^\infty = \bigcup_{S \in \mathcal{S}} S^\infty \cup \bigcup_{S \in \cup_n \mathcal{G}_n} \mathcal{F}_S,
\]

where \( \mathcal{F}_S \) is the dyadic decomposition of \( \Gamma_S \) in spherical coordinates, is a family of Christ-David cubes for \( \partial \tilde{\Omega} \).

8. PROOFS OF THEOREM 1.2 AND THEOREM 1.5

To prove Theorem 1.2 we assume \( \Omega \) is a corkscrew domain satisfying (1.4) and either (a) or (b) and we let \( \tilde{\Omega} \) be the domain constructed from \( \Omega \) in Section 7. Recall that \( \partial \tilde{\Omega} \) is \( d \)-Ahlfors regular. We will prove \( \partial \tilde{\Omega} \) is uniformly rectifiable by repeating the proof of Lemma 6.2 and applying Proposition 5.1 of [GMT]. Define

\[
\text{HD}(S^\infty) = \left\{ S_1^\infty \in S^\infty : S_1^\infty \subset S^\infty, \omega(p_S, \lambda S_1^\infty, \tilde{\Omega}) \geq A \left( \frac{\ell(S_1)}{\ell(S)} \right)^d, S_1^\infty \text{ maximal} \right\},
\]

\[
\text{LD}(S^\infty) = \left\{ S_1^\infty \in S^\infty : S_1^\infty \subset S^\infty, \omega(p_S, S_1^\infty, \tilde{\Omega}) \leq \delta \left( \frac{\ell(S_1)}{\ell(S)} \right)^d, S_1^\infty \text{ maximal} \right\},
\]

\[
G_1^1(S^\infty) = \left\{ S_1^\infty \in \text{HD}(S^\infty) \cup \text{LD}(S^\infty), S_1^\infty \text{ maximal} \right\},
\]

\[
\text{Tree}(S^\infty) = \left\{ S_1^\infty \in S : S_1^\infty \subset S^\infty, S_1^\infty \not\subset S^\infty \text{ for all } S_2^\infty \in G_1^1(S^\infty) \right\},
\]

and

\[
G_{n+1}^* = \bigcup_{S^\infty \in G_n^*} G_1^1(S^\infty).
\]

Lemma 8.1. Let \( S^\infty \in G_n^* \). Then

\[
\sum_{S_1^\infty \in \text{HD}(S^\infty)} \left( \frac{\ell(S_1)}{\ell(S)} \right)^d \leq \frac{C_1}{A},
\]

and

\[
\sum_{S_1^\infty \in \text{LD}(S^\infty)} \omega(p_S, S_1, \tilde{\Omega}) \leq C \delta + \varepsilon,
\]
where

\[ \inf_{T \in S} \inf_{p \in \Gamma_T} \omega(p, T, \Omega) \geq 1 - \varepsilon. \]

**Proof.** The proof of (8.1) is the same as the proof of (6.8). To prove (8.2) we follow the proof of (6.9) and (7.9). Let \( S_1^\infty \in \text{LD}(S^\infty) \). Then

\[ \omega(p_S, S_1, \Omega) \leq \omega(p_S, S_1^\infty, \bar{\Omega}) + \sum_{k \geq 1} \sum_{G_k^* \setminus \{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega) d\omega(p_S, p, \bar{\Omega}). \]

By definition and Lemma 4.1,

\[ \sum_{S_1^\infty \in \text{LD}(S^\infty)} \omega(p_S, S_1^\infty, \bar{\Omega}) \leq \delta \sum_{\text{LD}(S^\infty)} \left( \frac{\ell(S_1^\infty)}{\ell(S^\infty)} \right)^d \leq C\delta, \]

and

\[
\begin{align*}
\sum_{S_1^\infty \in \text{LD}(S^\infty)} \sum_{k=1}^\infty \sum_{T \in G_k^* \setminus \{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega) d\omega(p_S, dp, \bar{\Omega}) \\
= \int_{\Gamma_S} \sum_{S_1^\infty \in \text{LD}(S^\infty)} \omega(p, S_1, \Omega) d\omega(p_S, dp, \bar{\Omega}) \\
+ \sum_{k=1}^\infty \sum_{T \in G_k^* \setminus \{S_1\}} \int_{\Gamma_T} \sum_{S_1^\infty \in \text{LD}(S^\infty)} \omega(p, S_1, \Omega) d\omega(p_S, p, \bar{\Omega}) \\
+ \sum_{k=1}^{n-1} \int_{\Gamma_T} \sum_{S_1^\infty \in \text{LD}(S^\infty)} \omega(p, S_1, \Omega) d\omega(p_S, p, \bar{\Omega}) \\
+ \sum_{S_1 \in G_1(S)} \sum_{T \in \bigcup_k G_k(S_1)} \int_{\Gamma_T} \omega(p, S_1, \Omega) d\omega(p_S, p, \bar{\Omega}) \\
= I' + II' + III' + IV'.
\end{align*}
\]

Here \( I', II' \) and \( III' \) can be handled the same way as \( I, II, \) and \( III \) were, while \( IV' \leq C\varepsilon \) by (8.3).

Thus if \( C\delta + \varepsilon \) is small, Lemma 7.2 and Lemma 6.3 yield

\[ \sum_{k=1}^\infty \sum_{G_k^* \setminus \{S_1\}} \left( \frac{\ell(S_1)}{\ell(S)} \right)^d \leq C_3 \]

for any \( S^\infty \in S^\infty \) and any \( S_1^\infty \in \text{Tree}(S^\infty) \),

\[ \delta \frac{\ell(S_1)}{\ell(S)} \leq \omega(p_S, \lambda S_1^\infty, \bar{\Omega}) \leq A \left( \frac{\ell(S_1)}{\ell(S)} \right)^d. \]
By (8.5) and Proposition 5.1 of [GMT] this proves $\partial \tilde{\Omega}$ is uniformly rectifiable, and that established Theorem 1.2.

To prove Theorem 1.5 note that the arguments in Section 7 and Section 8 show that when $\Omega$ satisfies the hypotheses of Theorem 1.5, the constructed domain $\tilde{\Omega}$ has uniformly rectifiable boundary. Therefore by Theorem 1.1, (a) and (b) hold for $\Omega$.

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