GROUPS HAVING A FAITHFUL IRREDUCIBLE REPRESENTATION

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Abstract. We address the problem of finding necessary and sufficient conditions for an arbitrary group, not necessarily finite, to admit a faithful irreducible representation over an arbitrary field.

1. Introduction

We are interested in the problem of finding necessary and sufficient conditions for a group to have a faithful irreducible linear representation. Various criteria have been found when the group in question is finite, as described in [2] so we will concentrate mainly on infinite groups.

Let $G$ be an arbitrary group. Recall that $\text{Soc}(G)$ is the subgroup of $G$ generated by all minimal normal subgroups of $G$. It is perfectly possible for $\text{Soc}(G)$ to be trivial. We denote by $\text{S}(G)$, $\text{T}(G)$ and $\text{F}(G)$ the subgroups of $\text{Soc}(G)$ generated by all minimal normal subgroups of $G$ that are non-abelian, torsion abelian and torsion-free abelian, respectively. For each prime $p$, let $\text{T}(G)_p$ be the $p$-part of $\text{T}(G)$. It is a vector space over $\mathbb{F}_p$. Let $\Pi(G)$ be the set of all primes $p$ such that $\text{T}(G)_p$ is non-trivial.

A normal subgroup $N$ of $G$ is said to be essential if every non-trivial normal subgroup of $G$ intersects $N$ non-trivially.

With this terminology, our main result can be stated as follows.

Theorem 1.1. Let $G$ be a group and let $K$ be a field. A necessary condition for $G$ to have a faithful irreducible representation over $K$ is that $\text{char}(K) \not\in \Pi(G)$ and $\text{T}(G)$ have subgroup $S$ such that $\text{T}(G)/S$ be locally cyclic and $S$ contain no non-trivial normal subgroup of $G$. This condition is sufficient provided $\text{Soc}(G)$ is essential and every non-abelian minimal normal subgroup of $G$ admits a non-trivial irreducible representation over $K$ (the latter holds automatically if $\text{char}(K) = 0$).

Although the requirement that $\text{Soc}(G)$ be essential plays a critical role for us, this is certainly not a necessary condition. We illustrate this by means of McLain’s group $M$ [M], which has no minimal normal subgroups. We use the special features of $M$ to show that it has a faithful irreducible representation over any field $K$, except when the underlying division ring has the same characteristic as $K$ and this is a prime $p$. In this case $M$ is a locally cyclic $p$-group, so its only irreducible representation in characteristic $p$ is trivial.

Our restriction on the non-abelian minimal normal subgroups of $G$ is fairly mild and is automatically satisfied in a wide range of cases, including when $\text{char}(K) = 0$.

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Indeed, it is easy to see (cf. Lemma 6.3) that if a non-trivial group $S$ has no non-trivial irreducible representation over a field $K$, then $K$ has prime characteristic $p$ and $S$ is a $p$-group. Now a minimal normal subgroup of $G$ is characteristically simple and a non-abelian characteristically simple group is perfect. A non-trivial perfect finitely generated group is known to admit non-trivial irreducible representations over any field (cf. [Pa2, Theorem 6.3]). On the other hand, McLain’s group shows that a characteristically simple $p$-group may lack non-trivial irreducible representations in prime characteristic $p$. This explains the reason behind our restriction. It seems to be an open question whether a non-abelian simple $p$-group exists lacking non-trivial irreducible representations in prime characteristic $p$.

The most general contributions to this problem have been made by Tushev. Theorem 1.1 extends [T2, Theorem 1], which imposed the additional assumption that every minimal normal subgroup of $G$ be finite. Under this assumption, $F(G)$ is trivial, every irreducible $ZG$-submodule of $T(G)$ is finite, and every non-abelian minimal normal subgroup $N$ of $G$ is a direct power of a finite non-abelian simple group. In particular, $N$ is not a $p$-group for any prime $p$, and it therefore has non-trivial irreducible representations over any field.

When $G$ itself is finite $\text{Soc}(G)$ is essential, so [T2, Theorem 1] and Theorem 1.1 yield a criterion originally obtained by Nakayama [N], and shortly afterwards by Kochendörfer [K], who were the first to study this problem over an arbitrary field.

In prior work, Tushev [T1] found necessary and sufficient conditions for a locally polycyclic, solvable group of finite Prüfer rank to have a faithful irreducible representation over an algebraic extension of a finite field. He recently [T3] extended this line of research to solvable groups of finite Prüfer rank over an arbitrary field.

Also recently, Bekka and de la Harpe [Bd] found a criterion for a countable group to have faithful irreducible unitary complex representation, although their sense of irreducibility differs from the purely algebraic meaning given in this paper.

2. A review of the finite case

The development of our problem for finite groups makes an interesting story which is often reported with a certain degree of inaccuracy in the literature. Moreover, all known criteria, obviously equivalent to each other, are easily obtained from one another by means of a straightforward argument that does not involve groups, their representations or whether they are faithful or not. This prompted us to review the history of this problem for finite groups in more detail than usual and to indicate how a direct translation between the various criteria can be carried out.

For the sake of our historical review we adopt the following conventions: $G$ stands for an arbitrary finite group and a representation of $G$ means a complex representation, unless otherwise stated.

At the dawn of the twentieth century, a well known and established necessary condition was that the center of $G$ be cyclic. A partial converse was shown by Fite [F] as early as 1906. He proved that if $G$ has prime power order or, more generally, if $G$ is the direct product of such groups (that is, if $G$ is nilpotent), then the above condition is also sufficient.

The first example of a finite centerless group that admits no faithful irreducible representation was given by Burnside [B, Note F] in 1911. It was the semidirect product

$$ (C_3 \times C_3) \rtimes C_2, $$




where $C_2$ acts on $C_3 \times C_3$ without non-trivial fixed points. (Another well-known example of a similar kind, due to Isaacs \[I, Exercise 2.19\] is

$$\text{(2.2) } (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3,$$

where, again, $C_3$ acts on $C_2 \times C_2 \times C_2 \times C_2$ without non-trivial fixed points.)

In \[B, Note F\] Burnside found a sufficient condition as well. He showed that if $G$ does not contain two distinct minimal normal subgroups whose orders are powers of the same prime, then $G$ has a faithful irreducible representation. This condition is not necessary, a fact recognized by Burnside. To illustrate this phenomenon, let

$$G = V \rtimes T,$$

where $V$ is a vector space of finite dimension $d > 1$ over a finite field $F_q$ with $q > 2$ elements, prime characteristic $p$, and $T$ is the diagonal subgroup of $\text{GL}(V)$ with respect to some basis of $V$. Clearly, $G$ is the direct product of $d$ copies of

$$F_q^+ \rtimes F_q^*,$$

which admits a faithful irreducible representation of dimension $q - 1$. When $q > 2$ this group has trivial center, so the corresponding tensor power representation of $G$ is not only irreducible but also faithful.

Observe that while $G$ has $d > 1$ distinct minimal normal subgroups of order $q$, the normal subgroup they generate, namely $V$, is clearly a cyclic $F_p G$-module. This is precisely the condition that Burnside missed.

Since the center of a nilpotent group intersects every non-trivial normal subgroup non-trivially, Fite’s result follows from Burnside’s. The groups $F_q^+ \rtimes F_q^*$ used above show that the converse fails. Because of his work in \[B, Note F\] we will refer to the problem at hand as Burnside’s problem or Burnside’s question.

The relevance of $\text{Soc}(G)$ to his problem was already apparent to Burnside in 1911, although the nature of $\text{Soc}(G)$ was not sufficiently understood at the time for him to produce a solution. The structure of $\text{Soc}(G)$ was elucidated by Remak \[R\] in 1930, and his description has been implicitly or explicitly used by every author who eventually wrote about Burnside’s question.

The first paper addressing Burnside’s problem was written by Shoda \[S\] in 1930. An error in \[S\] was quickly pointed by Akizuki in a private letter to Shoda. This letter included the first correct solution to Burnside’s question. Akizuki’s criterion and a sketch of his proof appeared in a second paper by Shoda \[S2\] in 1931, which also included an independent proof of Akizuki’s criterion by Shoda.

Let us outline Akizuki’s criterion as it appears in \[S2\]. Let $T$ be one of the factors appearing in a decomposition of $\text{T}(G)$ as a direct product of abelian minimal normal subgroups of $G$, and let $s$ be the total number of these factors isomorphic to $T$ via an isomorphism that commutes with the inner automorphisms of $G$. The group $T$ is isomorphic to the direct product of, say $r$, copies of $\mathbb{Z}_p$ for some prime $p$. The total number of endomorphisms of $T$ that commute with all inner automorphisms of $G$ is of the form $p^g$ for some positive integer $g$. Akizuki’s criterion is that $G$ admits a faithful irreducible representation if and only if $sg \leq r$, and this holds for all factors $T$ indicated above.

The reader will likely be as baffled by this criterion as subsequent writers on this subject were. Akizuki’s condition seemed difficult to verify in practice, to say the least, and alternative criteria were sought.
A second criterion was produced by Weisner [W] in 1939. According to Weisner, $G$ has a faithful irreducible representation if and only if for every $p \in \Pi(G)$ there exists a maximal subgroup of $T(G)_p$ that contains no normal subgroup of $G$ other than the trivial subgroup.

Shortly after Weisner’s paper, Tazawa [T] extended Burnside’s problem and asked under what conditions $G$ would admit a faithful representation with $k$ irreducible constituents. His answer was given along the same lines as Akizuki’s criterion.

The next paper on the subject was written by Nakayama [N] in 1947. He was the first to consider Burnside’s problem over fields of prime characteristic, obtaining a full criterion which is a slight modification of Weisner’s criterion as stated over $\mathbb{C}$. Nakayama seems to have been unaware of Weisner’s prior work. A great deal of [N] is devoted to solve several generalizations of Burnside’s problem.

The following paper on Burnside’s question was written Kochendörffer [K] in 1948. He gave another proof of Akizuki’s criterion and, unaware of Weisner’s paper, stated and proved Weisner’s criterion. He wished to produce a condition that was easily verifiable in practice, and proved that if all Sylow subgroups of $G$ have cyclic center then $G$ has a faithful irreducible representation (cf. [I, Exercise 5.25]). This sufficient condition is an immediate consequence of the one proved by Burnside in 1911. In a second part of his paper Kochendörffer addressed and solved Burnside’s question for fields of prime characteristic, unaware of Nakayama’s prior solution.

The next solution to Burnside’s problem was given by Gaschütz [G] in 1954. According to Gaschütz, $G$ has a faithful irreducible representation if and only if $T(G)$ is a cyclic $\mathbb{Z}G$-module.

With the benefit of hindsight, let us try to explain directly why the criteria of Akizuki, Weisner and Gaschütz are equivalent to each other. For this purpose, let us slightly translate each of the above criteria into a more favorable language.

Since $T(G)$ is a completely reducible $\mathbb{Z}G$-module, it is readily seen that $T(G)$ is a cyclic $\mathbb{Z}G$-module if and only if $T(G)_p$ is a cyclic $\mathbb{F}_pG$-module for each $p \in \Pi(G)$.

Given $p \in \Pi(G)$, a maximal subgroup, say $M$, of $T(G)_p$ is the kernel a non-trivial linear functional $\lambda : T(G)_p \to \mathbb{F}_p$, and the largest normal subgroup of $G$ contained in $M$ is therefore

$$\bigcap_{g \in G} \ker(g\lambda) = \bigcap_{g \in G} \ker(\lambda g^{-1}) = \bigcap_{g \in G} g M g^{-1},$$

where $\lambda : T(G)_p \to \mathbb{F}_p$ is the linear character defined by

$$(\lambda v)(g) = \lambda(g^{-1}vg), \quad v \in T(G)_p.$$ 

Since a proper subspace of the dual space $T(G)_p^*$ annihilates a non-zero subspace of $T(G)_p$, we see that Weisner’s condition is that $T(G)_p^*$ be a cyclic $\mathbb{F}_pG$-module for every $p \in \Pi(G)$.

It is easily seen (cf. [B]) that if $K$ is a field, $A$ is a $K$-algebra with involution, and $V$ is a finite dimensional completely reducible $A$-module then $V$ is cyclic if and only if so is its dual $V^*$. This shows directly that the criteria of Weisner and Gaschütz are equivalent.

Let $T$ be an abelian minimal normal subgroup of $G$, that is, an irreducible $\mathbb{F}_pG$-module of $T(G)_p$, for some $p \in \Pi(G)$. The quantities $s, g$ and $r$ of Akizuki’s criterion are respectively equal to the multiplicity of $T$ in $T(G)_p$, the dimension of the field
$F = \text{End}_{F_pG}(T)$ over $F_p$, and the dimension of $T$ itself over $F_p$. Now $T$ is an $F$-vector space of dimension $r/g$, so Akizuki’s criterion is that the multiplicity of $T$ in $T(G)_p$ be less than or equal to $\dim_F(T)$. The above translation of Akizuki’s criterion was already known to Pálfy [P] in 1979, who gave another proof of Akizuki’s criterion, this time over a splitting field for $G$ of characteristic not dividing $T(G)$.

(Pálfy credited Kochendörffer for this result, unaware of Nakayama’s prior work.)

It is easily seen (cf. [§3]) that if $R$ is left artinian ring and $V$ is a completely reducible $R$-module then $V$ is cyclic if and only if for every irreducible submodule $W$ of $V$ the multiplicity of $W$ in $V$ does not exceed $\dim_D W$, where $D = \text{End}_R W$. This shows directly that the criteria of Akizuki, Weisner and Gaschütz are all equivalent to each other, with generic arguments that involve no groups at all.

Burnside’s question and its various solutions have also appeared in book form. See the books by Huppert [H], Doerk and Hawkes [DH], and Berkovich and Zhadn’ [BZ], for instance. Incidentally, Zhadn’ [Zm] found alternative necessary and sufficient conditions for $G$ to admit a faithful representation with exactly $k$ irreducible representations, a problem previously considered by Tazawa [T].

We close this section with an example. Given a finite field $F_q$ with $q$ elements and a vector space $V$ of finite dimension $d$ over $F_q$ we consider the subgroup

$G(d, q) = V \rtimes F_q^*$

of the affine group $V \rtimes \text{GL}(V)$ and set

$G(q) = F_q^+ \rtimes F_q^*.$

For any non-trivial linear character $\lambda : F_q^+ \rightarrow \mathbb{C}^*$ the induced character

$\chi = \text{ind}_{F_q^+}^G \lambda$

is readily seen to be a faithful and irreducible of degree $q - 1$. Any hyperplane $P$ of $V$ gives rise a group epimorphism

$G(d, q) \rightarrow G(q)$

with kernel $P$. This produces $(q^d - 1)/(q - 1)$ different irreducible characters of $G(d, q)$ of degree $q - 1$, each of them having a hyperplane of $V$ as kernel. On the other hand, $G(d, q)$ has $q - 1$ different linear characters with $V$ in the kernel. Since

$$\frac{q^d - 1}{q - 1} \times (q - 1)^2 + (q - 1) \times 1^2 = q^d(q - 1) = |G(d, q)|,$$

these are all the irreducible characters of $G(d, q)$. In particular, $G(d, q)$ has no faithful irreducible characters if $d > 1$. On the other hand, if $q > 2$ the center of $G(d, q)$ is trivial. Thus, to each finite field $F_q$, with $q > 2$, there corresponds an infinite family of finite centerless groups, namely $G(d, q)$ with $d > 1$, having no faithful irreducible characters for virtually tautological reasons. Moreover, every non-linear irreducible character of $G(d, q)$ is uniquely determined by its (non-trivial) kernel. Observe that the groups (2.1) and (2.2) are isomorphic to $G(2, 3)$ and $G(2, 4)$, respectively.

3. Cyclic modules and their duals

All modules appearing in this paper are assumed to be left modules.
Lemma 3.1. Let $R$ be a left artinian ring and let $V$ be a completely reducible $R$-module. Then $V$ is cyclic if and only if for each irreducible component $W$ of $V$ the multiplicity of $W$ in $V$ is less than or equal than $\dim_D(W)$, where $D = \text{End}_K(W)$.

Proof. Since there are finitely many isomorphisms types of irreducible $R$-modules, we may reduce to the case when all irreducible submodules of $V$ are isomorphic. Let $W$ be one of them. Replacing $R$ by its image in $\text{End}(V)$ we may also assume that $J(R) = 0$. Thus we may restrict to the case $R = M_n(D)$ and $W = D^n$. A cyclic $R$-module is nothing but a quotient of $R$ by a left ideal. This is just a left ideal of $R$, which is isomorphic to the direct sum of at most $n$ copies of $W$. □

Corollary 3.2. Let $K$ be a field and let $A$ be a $K$-algebra with involution. Suppose that $V$ is a completely reducible finite dimensional $A$-module. Then $V$ is cyclic if and only if so is $V^*$.

Proof. Since every submodule of $V$ is isomorphic to its double dual, we may reduce to the case when all irreducible submodules of $V$ are isomorphic. Let $W$ be one of them. Then the transpose map

$$\text{End}_K(W) \to \text{End}_K(W^*)$$

sends $D = \text{End}_A(W)$ onto $D^0 = \text{End}_A(W^*)$, the division algebra opposite to $D$. Since $\dim_K(W) = \dim_K W^*$, it follows that $\dim_D(W) = \dim_{D^0}(W^*)$. Now apply Lemma 3.1 □

Corollary 3.3. Let $K$ be a field and let $A$ be a $K$-algebra with involution. Suppose that $V$ is a finite dimensional $A$-module. Then $V$ is cyclic if and only if the irreducible components of $V/\text{Rad}(V)$ satisfy the conditions of Lemma 3.1 and $V^*$ is cyclic if and only if the irreducible components of $\text{Soc}(V)$ satisfy the conditions of Lemma 3.1

Proof. Clearly $V$ is cyclic if and only if so is $V/\text{Rad}(V)$. Moreover,

$$V^*/\text{Rad}(V^*) \cong \text{Soc}(V)^*,$$

and by Lemma 3.1 we know that $\text{Soc}(V)^*$ is cyclic if and only if so is $\text{Soc}(V)$. □

In particular, if $V/\text{Rad}(V)$ (resp. $\text{Soc}(V)$) is irreducible then $V$ (resp. $V^*$) is cyclic. In the extreme case when $V$ is uniserial the so is $V^*$ and, moreover, $V$ and $V^*$ are automatically cyclic.

4. Tensor products

Lemma 4.1. Let $K$ be a field. Let $A_1$ and $A_2$ be $K$-algebras with irreducible modules $V_1$ and $V_2$, respectively, such that $\text{End}_{A_2}(V_2) = K$, and set $C = A_1 \otimes_K A_2$. Then $V = V_1 \otimes_K V_2$ is an irreducible $C$-module and $\text{End}_C(V) = \text{End}_{A_1}(V_1) \otimes 1$.

Moreover, let $G_1$ and $G_2$ be groups and set $G = G_1 \times G_2$. Suppose $A_1 = KG_1$, $A_2 = KG_2$ (so that $C \cong KG$) and $\text{End}_{KG_1}(V_1) = K$, and let $\lambda_1 : Z(G_1) \to K^*$ and $\lambda_2 : Z(G_2) \to K^*$ be the linear characters of the centers of $G_1$ and $G_2$ corresponding to $V_1$ and $V_2$, respectively. Then $V$ is a faithful $G$-module if and only if $V_1$ and $V_2$ are faithful $G_1$ and $G_2$-modules, respectively, and $\lambda_1(g_1)\lambda_2(g_2) = 1$ implies $g_1 = 1 = g_2$. 
Proof. Let $v$ be an arbitrary non-zero element of $V$. Then $v = u_1 \otimes v_1 + \cdots + u_n \otimes v_n$, where $u_1, \ldots, u_n \in V_1$ are linearly independent and $w_1, \ldots, w_n \in V_2$ are non-zero. Since $\text{End}_{A_1}(V_1) = K$, the density theorem (cf. [He, Theorem 2.1.2]) ensures the existence of $r \in A_1$ such that $ru_1 = u_1$ and $ru_i = 0$ for $i > 1$. Thus
\[(r \otimes 1)v = u_1 \otimes w_1.\]

Let $u \in V_1$ and $w \in V_2$ be arbitrary. As $V_1$ and $V_2$ are irreducible, there are $s \in A_1$ and $t \in A_2$ such that $su_1 = u$ and $tw_1 = w$. Therefore
\[(s \otimes t)(u_1 \otimes w_1) = u \otimes w.\]

Since every element of $V$ is a sum of basic tensors, it follows that $Cv = V$. This proves that $V$ is irreducible.

The fact that $\text{End}_C(V) = \text{End}_{A_1}(V_1) \otimes 1$ follows from [SZ, Lemma 3.7].

Finally, it is clear that the stated conditions are necessary for $V$ to be faithful. Suppose, conversely, that they are satisfied. We wish to show that $V$ is faithful. Let $g_1 \in G_1$. If $g_1$ is not central then $g_1u$ must be linearly independent from $u$ for some $u \in V_1$. It is then clear that $g_1g_2$ is not in the kernel of the action of $G$ on $V$ for any $g_2 \in G_2$. The exact same argument is applicable to any element non-central of $G_2$. Thus, the kernel of the action of $G$ on $V$ is contained in the center of $G$. The stated conditions force this kernel to be trivial. \qed

Let $A$ be an algebra over a field $K$ and let $V$ be a $A$-module. Given a field extension $L$ of $K$, set $B = L \otimes_K A$ and consider the $B$-module $W = L \otimes_K V$. The $L$-linear imbedding
\[L \otimes_K \text{End}_A(V) \hookrightarrow \text{End}_B(W)\]

need not be surjective, in general, and it is certainly not surjective when $\dim_K V$ and $\dim_K L$ are both infinite and $A = K$. This explains the need for a proof in the following result.

Lemma 4.2. Let $A$ be an algebra over a field $K$ and let $V$ be an irreducible $A$-module such that $\text{End}_A(V) = K$. Let $L$ be a field extension of $K$, set $B = L \otimes_K A$ and consider the $B$-module $W = L \otimes_K V$. Then $W$ is irreducible and $\text{End}_B(W) = L$.

Proof. For simplicity of notation we identify $V$ with its copy $1 \otimes V$ inside $W$ and $A$ with its copy $1 \otimes A$ inside $B$.

We claim that $B$ acts 2-transitively on the $L$-vector space $W$, that is, given any $w_1, w_2 \in W$ linearly independent over $L$ and given any $w_3, w_4 \in W$ there exists $b \in B$ such that $bw_1 = w_3$ and $bw_2 = w_4$. The claims implies that $W$ is $B$-irreducible and $\text{End}_B(W) = L$ (see [He, Theorem 2.1.3]).

To verify the claim, let $w_1, w_2 \in W$. Then
\[w_1 = \ell_1 v_1 + \ell_2 v_2 + \cdots + \ell_n v_n, \quad w_2 = m_1 v_1 + m_2 v_2 + \cdots + m_n v_n,\]
where $v_1, \ldots, v_n \in V$ are linearly independent over $K$, and hence over $L$, and $\ell_i, m_i \in L$. Suppose $w_1, w_2$ are linearly independent over $L$. This means
\[M = \begin{pmatrix} \ell_i & m_i \\ \ell_j & m_j \end{pmatrix} \in \text{GL}_2(L)\]
for some $1 \leq i < j \leq n$. Re-labelling, if necessary, we may assume that $i = 1$ and $j = 2$. Let
\[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_2(L)\]
be the inverse of $M$. Since $V$ is $A$-irreducible, $\text{End}_A(V) = K$ and $v_1, v_2, \ldots, v_n$ are linearly independent over $K$, the density theorem implies the existence of $a_1, a_2, a_3, a_4 \in A$ satisfying $a_i v_j = 0$ for all $1 \leq i \leq 4$ and $2 < j \leq n$, and such that the actions of $a_1, a_2, a_3, a_4$ on the $K$-span of $v_1, v_2$, relative to this basis, are respectively represented by

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0
\end{pmatrix}.
$$

Set 

$$c = \alpha a_1 + \beta a_2 + \gamma a_3 + \delta a_4 \in B.$$ 

Then 

$$cw_1 = \alpha \ell_1 v_1 + \beta \ell_2 v_1 + \gamma \ell_1 v_2 + \delta \ell_2 v_2 = v_1$$

and 

$$cw_2 = \alpha m_1 v_1 + \beta m_2 v_1 + \gamma m_1 v_2 + \delta m_2 v_2 = v_2.$$ 

Now let $w_3, w_4 \in W$ be arbitrary. Then 

$$w_3 = r_1 u_1 + \cdots + r_k u_k, \quad w_4 = s_1 u_1 + \cdots + s_k u_k,$$

where $u_i \in V$ and $r_i, s_i \in L$. Since $V$ is $A$-irreducible, $\text{End}_A(V) = K$ and $v_1, v_2$ are linearly independent over $K$, the density theorem implies the existence of $e_1, \ldots, e_k, f_1, \ldots, f_k \in A$ such that 

$$e_i v_1 = u_i, \quad e_i v_2 = 0, \quad f_i v_2 = u_i, \quad f_i v_1 = 0.$$ 

Set 

$$d = r_1 e_1 + \cdots + r_k e_k + s_1 f_1 + \cdots + s_k f_k \in B.$$ 

Then 

$$d_1 v_1 = w_3, \quad d_2 v_2 = w_4.$$ 

Thus, letting $b = dc \in B$, we have 

$$bw_1 = w_3, \quad bw_2 = w_4,$$

as required. 

\qed

5. Modules lying over others

**Lemma 5.1.** Let $K$ be a field, let $G_1, \ldots, G_n$ be groups and set $G = G_1 \times \cdots \times G_n$. Let $J_1, \ldots, J_n$ be proper left ideals of $K G_1, \ldots, K G_n$, respectively. Then the left ideal of $G$ generated by $J_1, \ldots, J_n$ is proper.

**Proof.** Arguing by induction, we are reduced to the case $n = 2$.

We may replace any $J_i$ by a proper left ideal $J'_i \supset J_i$ of $K G_i$ and $K$ by a field extension $L$, for if the result is true in the latter case it will also be true in the former ($L J'_i$ is still a proper left ideal of $L G_i$).

Let $M_1$ be a maximal left ideal of $K G_1$ containing $J_1$, set $V_1 = K G_1 / M_1$ and $D_1 = \text{End}_{K G_1}(V_1)$, and let $L_1$ be a maximal subfield of $D$ containing $K$ (the existence of $M_1$ and $L_1$ is ensured by Zorn’s lemma). Then $V_1$ is an irreducible $L_1 G_1$-module and $\text{End}_{L_1 G_1}(V_1) = L_1$. Set $v_1 = 1 + M_1 \in V_1$. We then have an epimorphism of $L_1 G_1$-modules $L_1 G_1 \to V_1$ given by $r \mapsto rv_1$. Since the action of $L_1 G_1$ on $V_1$ extends that of $K G_1$, we see that the kernel of this map, which is a maximal left ideal of $L_1 G_1$, say $N_1$, contains $M_1$.
Likewise, let $M_2$ be a maximal left ideal of $L_1G_2$ containing $J_2$ and set $V_2 = L_1G_2/M_2$ and $D_2 = \text{End}_{L_1G_2}(V_2)$. Let $L_2$ be a maximal subfield of $D_2$ containing $L_1$. Then $V_2$ is an irreducible $L_2G_2$-module and $\text{End}_{L_2G_2}(V_1) = L_2$. As above, $V_2 \cong L_2G_2/N_2$, where $N_2$ is the maximal left ideal of $L_2G_2$, containing $M_2$, that annihilates $v_2 = 1 + M_2 \in V_2$.

By Lemma 4.2, $W_1 = L_2 \otimes_{L_1} V_1$ is an irreducible $L_2G_1$-module. Using the generator $w_1 = 1 \otimes v_1$ of $W_1$ we see that $W_1 \cong L_2G_1/P_1$, where $P_1$ is a maximal left ideal of $L_2G_1$ containing $N_1$.

By Lemma 4.1, $W_1 \otimes_{L_2} V_2$ is an irreducible module for $G = G_1 \times G_2$ over $L_2$. Using the generator $w_1 \otimes v_2$ we see that $W_1 \otimes_{L_2} V_2 \cong L_2G/N$, where $N$ is a maximal left ideal of $L_2G \cong L_2G_1 \otimes_{L_2} L_2G_2$ containing $P_1$ and $N_2$, as required.

\[\square\]

Given a ring $R$, a subring $S$, an $R$-module $V$, and an $S$-module $U$, we say that $V$ lies over $U$ if $U$ is isomorphic to an $S$-submodule of $V$.

**Lemma 5.2.** Let $(G_i)_{i \in I}$ be a family of groups and let $G$ be the direct (resp. cartesian) product of them. Let $K$ be a field and for each $i \in I$ let $V_i$ be an irreducible $KG_i$-module. Then there is an irreducible $KG$-module $V$ lying over all $V_i$.

**Proof.** For each $i \in I$ we have $V_i \cong KG_i/J_i$, where $J_i$ is a maximal left ideal of $KG_i$. Suppose the result is false. Then

\[1 = x_1y_1 + \cdots + x_ny_n,\]

where $x_i \in KG$ and each $y_i$ is in some $J_k$. Let $I_0$ be the finite subset of $I$ formed by all $k$ arising in this way. Let $H_1$ be the product of all $G_k$ with $k \in I_0$ and for $I_1 = I \setminus I_0$ let $H_2$ be the direct (resp. cartesian) product of all $G_i$ with $i \in I_1$. By construction, $G = H_1 \times H_2$. Let $j: G \to H_1$ be the associated projection and let $\pi: KG \to KH_1$ be the corresponding epimorphism of $K$-algebras. Applying $\pi$ to (5.1) we see that the the left ideal of $KH_1$ generated by all $J_k$, $k \in I_0$, is $KH_1$, which contradicts Lemma 5.1.

\[\square\]

**Lemma 5.3.** Let $H$ be a subgroup of a group $G$ and let $K$ be a field. Let $V$ be an irreducible $KH$-module. Then there exists an irreducible $KG$-module lying over $V$.

**Proof.** As shown in [Pa, Lemma 6.1.2], a proper left ideal of $KH$ generates a proper left ideal of $KG$. The result is an immediate consequence of this observation.

6. Restricting scalars

**Lemma 6.1.** Let $B$ be a ring with an irreducible $B$-module $V$. Suppose there is a subring $A$ of $B$ and a family $(u_i)_{i \in I}$ of units of $B$ such that: $B$ is the sum of all subgroups $Au_i$; $u_iA = Au_i$ for all $i \in I$; given any $i, j \in I$ there is $k \in I$ such that $u_iu_jA = u_kA$. Then

(a) $V$ has maximal $A$-submodule if and only if $V$ has a minimal $A$-submodule, in which case $V$ is a completely reducible $A$-module (homogeneous if all $u_i$ commute elementwise with $A$).

(b) If $I$ is finite then $V$ is a completely reducible $A$-module of length $\leq |I|$.

**Proof.** (a) Suppose first that $V$ has maximal $A$-submodule $M$. Since $u_iA = Au_i$ for all $i$, it follows that every $u_iM$ is an $A$-submodule of $V$. But each $u_i$ is a unit, so $u_iM$ is a maximal $A$-submodule of $V$. Let $N$ be the intersection of all $u_iM$. Since $u_iu_jA$ is equal to some $u_kA$, we see that $N$ is invariant under all $u_i$. But $N$ is closed under addition and $B$ is the sum of all $Au_i$, so $N$ is $B$-invariant.
The irreducibility of $V$ implies that $N = 0$. Thus $V$ is isomorphic, as $A$-module, to a submodule of a completely reducible $A$-module, namely the direct sum of all $V/u_i M$. In particular, $V$ has an irreducible $A$-submodule.

Suppose next that $V$ has an irreducible $A$-submodule $W$. Then every $u_i W$ is also $A$-irreducible and $V$ is the sum of all of them. Thus $V$ is the direct sum of sum of them. Removing one summand produces a maximal $A$-submodule.

(b) In this case $V$ is a finitely generated $A$-module, so it has a maximal $A$-submodule by Zorn’s Lemma. By above, $V$ is an $A$-submodule of a completely reducible $A$-module of length $\leq |I|$.

**Corollary 6.2.** Let $G$ be a group and let $L/K$ be a finite radical field extension. Let $V$ be an irreducible $LG$-module. Then $V$ is a completely reducible homogeneous $KG$-module. In particular, $V$ has an irreducible $KG$-submodule.

**Proof.** Arguing by induction, we may assume that $L = K[x]$, where $x^n \in K$ for some $n$, so Lemma 6.1 applies with $B = LG, A = KG$ and $u_i = x^i$ for $1 \leq i \leq n$. □

**Lemma 6.3.** Let $G$ be a non-trivial group and let $K$ be a field. If $K$ has prime characteristic $p$ we assume that $G$ is not a $p$-group. Then $G$ has a non-trivial irreducible representation over $K$.

**Proof.** Our hypothesis ensures the existence of $x \in G$ and a non-trivial linear character $\lambda : \langle x \rangle \to K[\zeta]$, where $\zeta$ is a root of unity. By Lemma 6.3 there is an irreducible $G$-module $V$ over $K[\zeta]$ lying over $\lambda$. By Corollary 6.2 there is an irreducible $KG$-submodule $U$ of $V$. Since $V = K[\zeta]U$ is not trivial, neither is $U$. □

**Note 6.4.** Lemma 6.1 is a generalization of [Pa] Theorem 7.2.16], which states that an irreducible $KG$-module $V$ has an irreducible $KN$-submodule, for a normal $N$ subgroup of $G$, provided $[G : N]$ is finite. Our statement of Lemma 6.1 was designed to accommodate Corollary 6.2 as well.

Observe that Corollary 6.2 fails if $L/K$ is an arbitrary field extension, even if $\dim_L(V) = 1$. Indeed, let $K$ be an arbitrary field, $L = K((t)), G = U(K[[t]])$ and $V = L$. Then $V$ is an irreducible $LG$-module of dimension one. However, when viewed as a $KG$-module, the submodules of $V$ form the following doubly descending/ascending infinite chain, where $R = K[[t]]$:

$$V \supset \cdots \supset Rt^{-2} \supset Rt^{-1} \supset R \supset Rt \supset Rt^2 \supset \cdots \supset 0.$$

Given a group $G$, a representation of a subgroup of $G$ is said to be $G$-faithful if its kernel contains no non-trivial normal subgroups of $G$.

**Lemma 6.5.** Let $G$ be a group and let $K$ be a field. Let $p$ be a prime different from $\text{char}(K)$. Suppose $N$ is a normal subgroup of $G$ contained in $T(G)_p$ with a maximal subgroup containing no non-trivial normal subgroup of $G$. Then $N$ has a $G$-faithful irreducible representation over $K$.

**Proof.** By assumption there exists a one-dimensional $G$-faithful module $V$ of $N$ over a $K[\zeta]$, where $\zeta^p = 1$. Since $[K[\zeta] : K] < \infty$, there is an irreducible $KN$-submodule $W$ of $V$. As $K[\zeta]W = V$, it is clear that $W$ is $G$-faithful. □

7. THE TORSION-FREE ABELIAN PART OF SOc($G$)

Recall that a torsion-free abelian characteristically simple group is divisible and hence a vector space over $\mathbb{Q}$.  

Lemma 7.1. Let $A$ be a non-trivial torsion-free divisible abelian group, let $K$ be a field and let $p$ be a prime different from $\text{char}(K)$. Then there exists an irreducible representation $R : A \to \text{GL}(V)$ over $K$ such that $R(A) \cong \mathbb{Z}_{p^\infty}$.

Proof. There is a subgroup $S$ of $A$ such that $B = A/S \cong \mathbb{Z}_{p^\infty}$. Let $L$ be a splitting field over $K$ for the family of polynomials $t^{p^i} - 1$ for all $i \geq 1$. Then $G = L^*$ acts $K$-linearly on the $K$-vector space $V = L$ by multiplication. We have an injective linear character $\lambda : B \to G$ whose image $\lambda(B) = C$ consists of all roots of $t^{p^i} - 1$, $i \geq 1$, in $L$. Since $L/K$ is algebraic, $K[C] = K(C) = L$. Thus, the representation $A \to B \to G \to \text{GL}(V)$ satisfies our requirements.

Lemma 7.2. Let $G$ be a group and let $K$ be a field. Suppose $A = \prod_{i \in I} A_i$, where each $A_i$ is an irreducible $\mathbb{Z}G$-submodule of $\text{F}(G)$ and $|I| \leq 8_0$. Then $A$ has a $G$-faithful irreducible module over $K$.

Proof. There is an injection $i \mapsto p_i$, where each $p_i$ is a prime different from $\text{char}(K)$. Lemma 7.1 ensures the existence, for each $i \in I$, of an irreducible representation $R_i : A_i \to \text{GL}(V_i)$ over $K$ such that $R(A_i) \cong \mathbb{Z}_{p_i^\infty}$. By Lemma 7.2, there is an irreducible representation $R : A \to \text{GL}(V)$ lying over all $R_i$. For a fixed $i \in I$, we have $V = \sum_{a \in A} aV_i = \oplus aV_i$ for some subset $J$ of $A$, so $R(A_i) \cong \mathbb{Z}_{p_i^\infty}$.

Suppose, if possible, that $A$ contains a non-trivial normal $N$ subgroup of $G$ which acts trivially on $U$, and let $1 \neq x \in N$. Then $x = x_{i_1} \cdots x_{i_n}$, where $1 \neq x_{i_k} \in A_i$. Since $A_i$ is an irreducible $\mathbb{Z}G$-module, there is $r \in \mathbb{Z}G$ such that $r \cdot x_{i_k}$ does not act trivially on $V_{i_k}$. Let $y_{i_k} = r \cdot x_{i_k} \in A_{i_k}$ for $1 \leq k \leq n$. Then $r \cdot x = y_{i_1} \cdots y_{i_n} \in N$ and $T(y_{i_1}) \cdots T(y_{i_n}) = 1_V$, which is impossible since these factors commute pairwise, the order of $T(y_{i_k})$ is a positive power of $p_{i_k}$, and the order of every $T(y_{i_k})$, $k > 1$, is a power of $p_{i_k}$.

Lemma 7.3. Let $G$ be a group and let $K$ be a field. Suppose $A = \prod_{i \in I} A_i$, where each $A_i$ is an irreducible $\mathbb{Z}G$-submodule of $\text{F}(G)$ and $|I|$ is infinite. Then $A$ has a $G$-faithful irreducible module over $K$.

Proof. Let $L$ be an extension of $K$ that is algebraically closed and satisfies $|L| = |I|$. Since $L^*$ is a divisible group, we have $L^* = T \times R$, where $T$ is the torsion subgroup of $L^*$ and $R$ and is torsion-free. Moreover, $R = \prod_{i \in I} R_i$, where $R_i \cong \mathbb{Q}$. Relabelling the factors of $A$, we have $A = B \times C$, where $B = \prod_{n \geq 1} B_n$ and $C = \prod_{i \in I} C_i$. Let $p_1, p_2, \ldots$ be the list of all primes different from $\text{char}(K)$. For each $p_k$, let $T_k$ be the $p_k$-part of $T$.

There is a family of linear characters $\lambda_i : C_i \to L^*$ such that $\lambda_i(C_i) = R_i$ for every $i \in I$ and a family of linear characters $\mu_k : B_k \to L^*$ such that $\mu_k(B_k) = T_k$. Let $\alpha : A \to L^*$ be the linear character determined by the $\lambda_i$ and $\mu_k$, so that $\alpha(A) = L^*$.

Every $\lambda_i$ and $\mu_k$ is non-trivial, each $A_i$ is an irreducible $\mathbb{Z}G$-module, and the $R_i$ and $T_k$ are independent. We infer it $\alpha$ is $G$-faithful. The group $G = L^*$ acts via $K$-linear automorphisms on the $K$-vector space $V = L$ by multiplication. The composition $A \to L^* \to \text{GL}(V)$ is a linear representation of $A$ on $V$ over $K$. As such, it is irreducible since $\alpha(A) = L^*$, and it is $G$-faithful as so is $\alpha$. □
8. Extending scalars

Lemma 8.1. Let $A$ be an associative algebra over $K$ and suppose that $V$ is an irreducible $A$-module. Let $L$ be a finite field extension of $K$ and consider the $L$-algebra $B = L \otimes_K A$. Then $W = L \otimes_K V$ has an irreducible $B$-submodule.

Proof. Clearly $W$ is the direct sum of $[L : K]$ copies of $V$ as an $A$-module. Since $V$ is irreducible, the length of any descending chain of non-zero $A$-submodules of $W$ is bounded by $[L : K]$, a bound that remains valid for $B$-submodules of $W$. □

Lemma 8.2. Suppose that $G$ has a faithful irreducible module $V$ over $K$ and let $L$ be a finite field extension of $K$. Then $G$ has a faithful irreducible module over $L$.

Proof. We know from Lemma 8.1 that $L \otimes_K V$ has an irreducible $LG$-submodule, say $U$. Let $N$ be the kernel of the action of $G$ on $U$ and let $S$ be the trivial $KN$-module. Then $L \otimes_K S$ is the trivial $LN$-module and hence an $LN$-submodule of $U$. Using again that $L/K$ is finite, we see that

$$L \otimes_K \text{Hom}_{KN}(S, V) \cong \text{Hom}_{LN}(L \otimes_F S, L \otimes_F V) \neq 0.$$ 

Thus $N$ has a fixed point on $V$. Since $N$ is normal and $V$ is irreducible, it follows that $N$ acts trivially on $V$, whence $N$ is the trivial group. □

9. Groups with a faithful irreducible representation

Theorem 9.1. Let $G$ be a group and let $K$ be a field. Suppose $G$ has a normal abelian subgroup $N$ of prime exponent $p$. Then

(a) If $\text{char}(K) = p$ then $G$ has no faithful irreducible representation over $K$.

(b) If $\text{char}(K) \neq p$ and every maximal subgroup of $N$ contains a non-trivial normal subgroup of $G$ then $G$ has no faithful irreducible representation over $K$.

Proof. Let $V$ be an irreducible $KG$-module.

Case 1. $\text{char}(K) \neq p$. By Lemma 8.2 we may assume that $K$ contains a primitive $p$th root of unity.

Given $x \in N$ consider the subgroup $S = \langle x \rangle$ of $G$, which has order 1 or $p$. For any non-zero $v \in V$ the $KS$-submodule $Ksv$ of $V$ is finite dimensional, so $Ksv$ is the sum of 1-dimensional $KS$-modules. Therefore, $x$ acts diagonalizably on $V$.

The well-ordering theorem, which is an equivalent version of the axiom of choice, ensures that $N$ can be well-ordered. For $x \in N$, set $N_x = \{y \in N \mid y < x\}$. Suppose $x \in N$ is not the first element of $N$ and there is a basis of $V$ that simultaneously diagonalizes all $R(y)$ with $y \in N_x$. Therefore

$$V = \bigoplus_{\alpha \in M_x} V(\alpha),$$

where $M_x$ is a set of functions $\alpha : N_x \to K^*$ and for each $\alpha \in M_x$

$$V(\alpha) = \{v \in V \mid yv = \alpha(y)v \text{ for all } y \in N_x\} \neq 0.$$ 

Since $N$ is abelian, each $V(\alpha)$ is $R(x)$-stable. By above, $R(x)$ is diagonalizable and therefore so are its restrictions to each $V(\alpha)$. We infer the existence of a basis of $V$ that simultaneously diagonalizes all $R(y)$, $y \leq x$. It follows that all $R(x)$, $x \in N$, are simultaneously diagonalizable. In particular, there is a common eigenvector for the action of $N$ on $V$, upon which $N$ acts via a linear character, say $\mu : N \to K^*$.

By assumption there is a non-trivial normal subgroup $M$ of $G$ contained in $\ker(\mu)$. 

Thus $M$ has a fixed point in $V$. Since $M$ is normal, the fixed points of $M$ are $G$-invariant. But $V$ is irreducible, so $M$ acts trivially on $V$.

**Case 2.** $\text{char}(K) = p$. In this case,

$$(R(x) - 1_N)^p = 0, \quad x \in N.$$ 

Well-order the abelian group $N$ and argue as above, mutatis mutandis, to see that $N$ has a fixed point, whence $N$ acts trivially on the irreducible $KG$-module $V$. □

**Example 9.2.** Here we extend the example given at the end of §2. Let $F$ be a field of prime characteristic $p$, let $V$ be an $F$-vector space of dimension $> 1$ (not necessarily finite), and set $G = V \rtimes F^*$. We claim that, provided $F$ is finite, $G$ has no faithful irreducible representation over any field. By Theorem 9.1 it suffices to show that every $F_p$-hyperplane of $V$ contains a non-zero $F$-subspace of $V$. The verification can be reduced to the case $\dim_F(V) = 2$, in which case a routine counting argument yields the desired result (which fails, in general, for $F$ infinite).

Let $G$ be a group. Two normal subgroups $N_1$ and $N_2$ of $G$ are said to be $G$-isomorphic if there is a group isomorphism between them that commutes with all inner automorphisms of $G$. When $N_1$ and $N_2$ are abelian, this means that $N_1$ and $N_2$ are isomorphic as $ZG$-modules. The $G$-homogeneous component of a minimal normal subgroup $N$ of $G$ is the subgroup of $\text{Soc}(G)$ generated by all normal subgroups of $G$ that are $G$-isomorphic to $N$.

**Lemma 9.3.** Let $G$ be a group and let $N$ be a subgroup of $G$ generated by a family $(N_i)_{i \in I}$ of minimal normal subgroups of $G$. Then every normal subgroup $M$ of $G$ contained in $N$ is equal to the direct product of the intersection of $M$ with each $G$-homogeneous component of $N$. Moreover, if each $N_i$ is non-abelian, then $M$ is equal to the direct product of all $M \cap N_i$.

**Proof.** See [10, §3.3]. □

**Theorem 9.4.** Let $G$ be a group and let $K$ be a field. A necessary condition for $G$ to have a faithful irreducible representation over $K$ is that $\text{char}(K) \not\in \Pi(G)$ and $T(G)$ have subgroup $S$ such that $T(G)/S$ be locally cyclic and $S$ contain no non-trivial normal subgroup of $G$. This condition is sufficient provided $\text{Soc}(G)$ is essential and every non-abelian minimal normal subgroup of $G$ admits a non-trivial irreducible representation over $K$ (the latter holds automatically if $\text{char}(K) = 0$).

**Proof.** Necessity follows from Theorem 9.1 and Lemma 9.3. As for sufficiency, in view of Lemmas 5.2, 5.3 and 9.3, it is enough to show that every $G$-homogeneous component of $S(G)$, $F(G)$ and $T(G)$, respectively, has a $G$-faithful irreducible module over $K$.

For $S(G)$ this follows from another application of Lemmas 5.2 and 9.3 for $F(G)$ we use Lemmas 7.2 and 7.3 and for $T(G)$ we appeal to Lemma 6.5. □

**Example 9.5.** Let $M$ be the McLain group $M$ defined over a division ring $D$ and let $K$ be any field. If $\text{char}(D) = \text{char}(K) = p$ is prime the only irreducible representation of $M$ over $K$ is the trivial one. In every other case, $M$ has a faithful irreducible representation over $K$.

The first assertion follows from the fact that when $\text{char}(D) = p$ is prime, $M$ is a locally finite $p$-group. For each $\alpha < \beta \in \mathbb{Q}$ consider the subgroup $N_{\alpha, \beta} = \{1 + ae_{\alpha, \beta} | a \in D\}$ of $M$. Let $I$ be the set of all pairs $(\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q}$ such that
α < 0 and 1 < β and let N be the subgroup of M generated by all \(N_{\alpha,\beta}\) with \((\alpha, \beta) \in I\). Then N is an essential normal abelian subgroup of M, equal to the direct product of all \(N_{\alpha,\beta}\) with \((\alpha, \beta) \in I\).

Suppose first D has prime characteristic \(p \neq \text{char}(K)\). For each \((\alpha, \beta) \in I\) let \(\lambda_{\alpha,\beta} : N_{\alpha,\beta} \to K[\zeta]^{*}\), \(\zeta^p = 1\), be a non-trivial linear character and let \(\lambda : N \to K[\zeta]^{*}\) be the unique extension of the \(\lambda_{\alpha,\beta}\) to N. Then \(\lambda\) is \(M\)-faithful, for every non-trivial normal subgroup of \(M\) contains at least one (in fact, infinitely many) \(N_{\alpha,\beta}\) with \((\alpha, \beta) \in I\). It follows from Lemma 5.5 that \(M\) has a faithful irreducible module \(U\) over \(K[\zeta]\). By Lemma 6.2, there is an irreducible \(KM\)-submodule \(V\) of \(U\). Since \(K[\zeta]V = U\), it is clear that \(V\) is faithful.

Suppose next that \(\text{char}(D) = 0\). Choose any \(0 \neq x \in D^{+}\) and any prime \(p \neq \text{char}(K)\). Then there is a non-trivial linear character \(\mu : \langle x \rangle \to K[\zeta]^{*}\), where \(\zeta^p = 1\). By Lemma 5.5, there is an irreducible \(D^{+}\)-module \(U\) over \(K[\zeta]\) lying over \(\mu\). For each \((\alpha, \beta) \in I\) let \(U_{\alpha,\beta}\) be the irreducible \(N_{\alpha,\beta}\)-module over \(K[\zeta]\) obtained from \(U\) via the isomorphisms \(D^{+} \to N_{\alpha,\beta}\) given by \(a \mapsto 1 + ae_{\alpha,\beta}\). By Lemma 5.2, there is an irreducible \(N\)-module \(V\) over \(K[\zeta]\) lying over all \(U_{\alpha,\beta}\). This implies, as before, that \(M\) has a faithful irreducible module over \(K\).

10. NECESSARY AND SUFFICIENT CONDITIONS FOR NILPOTENT GROUPS

**Proposition 10.1.** Let \(G\) be nilpotent group. Let \(Z\) be the center of \(G\) and let \(T\) be the torsion subgroup of \(Z\). Then the following conditions are equivalent:

(a) \(G\) admits faithful irreducible representation.

(b) \(Z\) is isomorphic to a subgroup of the multiplicative group of a field.

(c) \(Z\) admits faithful irreducible representation.

(d) \(T\) is locally cyclic.

(e) \(T\) is a subgroup of \(\mathbb{Q}/\mathbb{Z}\).

(f) For each prime \(p\), the \(p\)-part of \(T\) is a subgroup of \(\mathbb{Z}_{p\infty}\).

**Proof.** Suppose first that \(V\) is a faithful irreducible \(G\)-module over a field \(K\) and let \(D = \text{End}_{KG}(V)\). Let \(L\) be a maximal subfield \(L\) of \(D\) containing \(K\). Then \(L = \text{End}_{LG}(V)\), which yields an injective group homomorphism \(Z \to L^{*}\). This shows that (a) implies (b), which obviously implies (c).

Suppose next that \(V\) is a faithful irreducible \(Z\)-module over a field \(K\). By Lemma 5.3, there is an irreducible \(KG\)-module \(U\) lying over \(V\). Suppose, if possible, that \(N\) is a non-trivial normal subgroup of \(G\) acting trivially on \(V\). Since \(G\) is nilpotent, \(N \cap Z(G)\) is a non-trivial normal subgroup of \(G\) acting trivially on \(U\) and hence on \(V\), a contradiction. Thus (c) implies (a).

Clearly (b) implies (d). The converse was proven by Cohn [C]. It is well-known that (d),(e) and (f) are equivalent.

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