FedMM: Saddle Point Optimization for Federated Adversarial Domain Adaptation

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Abstract
Federated adversary domain adaptation is a unique distributed minimax training task due to the prevalence of label imbalance among clients, with each client only seeing a subset of the classes of labels required to train a global model. To tackle this problem, we propose a distributed minimax optimizer referred to as FedMM, designed specifically for the federated adversary domain adaptation problem. It works well even in the extreme case where each client has a different label classes and some clients only have unsupervised tasks. We prove that FedMM ensures convergence to a stationary point with domain-shifted unsupervised data. On a variety of benchmark datasets, extensive experiments show that FedMM consistently achieves either significant communication savings or significant accuracy improvements over federated optimizers based on the gradient descent ascent (GDA) algorithm. When training from scratch, for example, it outperforms other GDA based federated average methods by around 20% in accuracy over the same communication rounds; and it consistently outperforms when training from pre-trained models with an accuracy improvement from 5.4% to 9% for different networks.

1 Introduction
Federated Learning (FL) is gaining popularity because it enables multiple clients to train machine learning models iteratively and distributedly without directly sharing the potentially sensitive data with other clients (Kairouz et al., 2019; Li et al., 2020). The FL training pipeline involves exchanging local model parameters with a server to update the global model, and its communication overhead has been, in many cases, identified as the bottleneck (McMahan et al., 2017; Chen et al., 2020). Moreover, due to the heterogeneity, domain shift often exists between clients’ data (Quinonero-Candela et al., 2009), which is another characteristic feature of FL training, resulting from the data being sampled from different parts of the sample space on different clients. Because of the aforementioned two distinguishing features, FL training necessitates optimizers that converge on heterogeneous data among clients while requiring fewer communication rounds.

For data with distributional shifts, one of the most challenging settings is that each local client only has access to a subset of the label classes in order to train the global/common model. In this situation, the global model’s accuracy suffers considerably as a result of the gradient/model drift (McMahan et al., 2017). In the literature of domain adaptation, this problem is also known as label shift (Zhang et al., 2013; Tachet des Combes et al., 2020). Under the setting of FL, it is a natural occurrence due to the imbalance between clients’ label distributions, with the extreme case being individual clients with different domain labels, or clients without labels (unsupervised local model). Furthermore, recent techniques for domain adaptation with adversarial training (Ganin et al., 2016; Tzeng et al., 2017; Zhao et al., 2018) on minimax objectives complicates convergence even further.

One method is to use the gradient descent ascent (GDA) method (Lin et al., 2020a) directly as if the data are homogeneous and centralized globally where data are aggregated together to find saddle point solutions (Lin et al., 2020b). However, because of the domain shifts among clients in FL settings, a single client cannot access an unbiased sampling of the global objective (descent or ascent) gradient. A natural solution would be averaging on each client’s gradients, which exactly corresponds to the FedS-GDA approach in (Peng et al., 2019). Its training efficiency, on the other hand, is low due to the requirement of large communication rounds between the server and clients. Without considering the issue of domain shift, there are several works on communication-efficient FL algorithms, a large spectrum are variations of the FedAvg (McMahan et al., 2017). However, if the data are non-i.i.d among clients, especially in the case of imbalanced label distributions, the performance of FedAvg would be significantly lower than...
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For training a federated minimax objective, we show the typical pipeline of FedAvg with GDA, referred to as FedAvg-GDA with network, in Fig. 1. Specifically, in each client’s local oracle, only the source risk of the client’s local source data (if any) and the domain risk of the client’s source/target data are accessible. The federated domain adaptation algorithm optimizes the weighted sum of each client’s local loss functions in a collaborative minimax fashion. A detailed explanation is presented in the next section. However, the federated adversarial domain adaptation method is extremely sensitive to the unbalanced distributions of data labels, which has been analyzed theoretically in the literature (Zhao et al., 2019). We also empirically verified and confirmed this phenomenon, as shown in Fig. 2.

FedMM. We formulate this distributed saddle point optimization as a Federated MiniMax (FedMM) optimization on a sum of non-identical distributions. In particular, we use an augmented Lagrange function to enforce the global model consensus constraints. Furthermore, in each client’s local optimization oracle, FedMM deconstructs the global sum by solving the augmented Lagrange of each function individually. The collection of Lagrange dual variables locally compensates for client-to-client model divergence caused by data domain shift. We detail the algorithm in Section 4.

Contributions: Label imbalance is a natural and extremely challenging problem in federated domain adaptation. As demonstrated in Fig 2, FedAvg’s low performance is driven by the imbalance of domain label distributions across clients. Our paper aims to tackle these challenging issues. We summarize our key contributions as follows:

- We present, FedMM, a specifically designed distributed optimizer for federated minimax optimizations with non-separable minimization and maximization variables, as well as clients with uneven label class distributions. It works in the extreme case where each client has disjoint classes of labels and some clients even have unsupervised task.
- Under the generic federated saddle point optimization problem with a nonconvex-concave global objective function assumption, we prove that FedMM converges to a stationary point for the nonconvex-strongly-concave case. Based on our theoretical analysis, we show that FedMM converges to a stationary point even if the data distribution suffers from domain shifts.
- FedMM consistently achieves either significant communication savings or significant accuracy improvements over the federated gradient descent ascent (GDA) method on a variety of benchmark datasets with varying adversarial domain adaptation networks. For example, when training from scratch, it outperforms other GDA based federated average methods by around 20% in accuracy over the same communication rounds; and it consistently outperforms when training from pre-trained models with an accuracy improvement from 5.4% to 9% for different networks.

2 CENTRALIZED ADVERSARIAL DOMAIN ADAPTATION

Domain adaptation refers to the process of transferring knowledge from a labeled source domain to an unlabeled target domain (Ben-David et al., 2010; Zhao et al., 2019). Let $P$ and $Q$ be the source and target distributions, respectively. In a general formulation, the upper bound of the
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The target prediction error is given by [Ben-David et al., 2010]
\[ \text{err}_{\Omega}(\zeta) \leq \text{err}_{\Xi}(\zeta) + d_H(P, Q) + \min_{\zeta' \in \mathcal{F}} \{ \text{err}_{\Omega}(\zeta') + \text{err}_{\Xi}(\zeta') \}, \]
where \( \text{err}_{\Xi}(\zeta) \) denotes the population loss of \( \zeta \), i.e., \( \text{err}_{\Xi}(\zeta) \triangleq \mathbb{E}_{(x, y) \sim Q}[\ell(h_{\xi}(x), y)] \), and we use the parallel notation \( \text{err}_{\Omega}(\zeta) \) for the source domain. Besides, \( d_H(P, Q) \) is a discrepancy-based distance and \( \min_{\zeta' \in \mathcal{F}} \{ \text{err}_{\Omega}(\zeta') + \text{err}_{\Xi}(\zeta') \} \) is a lower bound on the sum of source and target domain’s population loss of \( \zeta \) in a hypothesis class \( \mathcal{F} \).

For the unsupervised domain adaptation problem, it has been proven that minimizing the upper bound, which is the r.h.s in (1), leads to an architecture consisting of a feature extractor parameterized by \( \omega \), i.e., \( \zeta_1^{\omega} \), a label predictor, parameterized also by \( \omega \), i.e., \( \zeta_2^{\omega} \triangleq \zeta_1^{\omega} \circ \zeta_0^{\omega} \), and a domain classifier parameterized by \( \psi \), i.e., \( h_{\psi} \), as shown in Fig 1.

The feature extractor generates the domain-independent feature representations, which are then fed into the domain classifier and label predictor. The domain classifier then tries to determine whether the extracted features belong to the source or target domain. Meanwhile, the label predictor predicts instance labels based on the extracted features of the labeled source-domain instances.

Minimizing the upper bound in (1) encourages the extracted feature to be both discriminative and invariant to changes between the source and target domains. The upper bound minimization corresponding to a saddle point over the parameter space of \( \omega \) and \( \psi \) has been demonstrated using \( \hat{\omega} \triangleq \arg \min_{\omega} L_1(\omega) - \nu L_2(\omega, \hat{\psi}) \) and \( \hat{\psi} \triangleq \arg \min_{\psi} L_2(\hat{\omega}, \psi) \) with an equivalent minimax compact form as
\[ \min_{\omega} \max_{\psi} F = \min_{\omega} \max_{\psi} L_1(\omega) - \nu L_2(\omega, \psi). \] (2)

In the majority of adversarial domain adaptation problems, \( L_1(\omega) \triangleq \mathbb{E}_{(x, y) \sim Q}[\ell(h_{\xi}(x), y)] \) is the supervised learning loss on \( \zeta \), \( L_2(\omega, \psi) \triangleq \mathbb{E}_{(x, y) \sim Q} D_{\psi}(h_{\omega}(\zeta_1(x))) - \mathbb{E}_{(x, y) \sim P} D_{\psi}(h_{\psi}(\zeta_0(x))) \) is the domain classification loss, and \( \nu \) is the trade-off coefficient between \( L_1(\omega) \) and \( L_2(\omega, \psi) \). With the commonly used cross-entropy loss for \( L_2 \), we have \( D_{\psi}(x) \triangleq 1 - \log(x) \) and \( D_{\psi}(x) \triangleq \log(1 - x) \). Besides, \( \zeta_0 \) is the feature and \( h_{\psi}(\cdot) : \mathbb{R}^D \rightarrow [0, 1] \) is the probabilistic prediction of the domain label. In general, \( \zeta_1 \) and \( h_{\psi}(\cdot) \) include, but is not limited to, the following cases:

- **Domain-Adversarial Neural Networks (DANN)** [Ganin & Lempitsky, 2015]: In DANN, the input of \( h_{\psi}(\cdot) \) is designed simply to be the domain invariant feature \( \zeta_1^{\omega}(x_i) \), i.e., \( h_{\psi}(\zeta_1^{\omega}(x_i)) \).

- **Margin Disparity Discrepancy (MDD)** [Zhang et al., 2019]: In MDD, the input of \( h_{\psi}(\cdot) \) is the concatenation of \( \zeta_0 \) and \( \arg \max_{\hat{c}} \zeta_{\omega}(x_i; c) \) with \( \hat{c} \) the class type i.e., \( h_{\psi}(\zeta_1^{\omega}(x_i), \arg \max_{\hat{c}} \zeta_{\omega}(x_i; c)) \).

- **Conditional Domain Adaptation Network (CDAN)** [Long et al., 2017]: In CDAN, the input of \( h_{\psi} \) is from the cross-product space of \( \zeta_1^{\omega}(x_i) \) and \( \zeta_{\omega}(x_i) \), i.e., \( h_{\psi}(\zeta_1^{\omega}(x_i) \otimes \zeta_{\omega}(x_i)) \).

Our FedMM is a generic federated adversarial domain adaptation framework in which each client is equipped with \( h_{\psi} \) and \( \eta_{\omega} \) depending on the availability of source data, target data, or both.

3 FEDERATED ADVERSARIAL DOMAIN ADAPTATION FORMULATION

Due to privacy concerns regarding sensitive data, the data cannot be shared among clients. As a result, federated adversarial domain adaption addresses the problem by training a transferred model among clients from a labeled source domain to an unlabeled target domain. A central server coordinates a loose federation of clients exchanging local models to solve the learning task.

To express the federated adversarial domain adaptation objective, we convert the joint learning objective in (2) into the form of a centralized average of all the clients’ objective functions, as given by
\[ \min_{\omega} \max_{\psi} f(\omega, \psi) \triangleq \min_{\omega} \max_{\psi} \frac{1}{N} \sum_{i=1}^{N} f_i(\omega, \psi), \] (3)
where \( N \) is the number of clients, and \( f_i(\omega, \psi) \) is the average loss function at the \( i \)-th client, which is computed by
\[ f_i(\omega, \psi) \triangleq \alpha_i \sum_{\xi_{ij}^{(i)} \in D_i} F_i(\omega, \psi; \xi_{ij}^{(i)}), \] (4)
where \( \alpha_i \) is the weight coefficient, and \( F_i(\omega, \psi; \xi_{ij}^{(i)}) \) is the loss function w.r.t the data point \( \xi_{ij}^{(i)} \triangleq \{ x_j, y_j \} \) in data set \( D_i \). The objective function at client \( i \) is specified based on whether the data is from the source domain or the target domain, i.e.,
\[ F_i(\omega, \psi; \xi_{ij}^{(i)}) \triangleq \begin{cases} \ell(\zeta_{\omega}(x_i), y_i) + \nu \log(1 - h_{\psi}(\zeta_1^{\omega}(x_i))), & \text{if} \ \xi_{ij}^{(i)} \in Q, \\ \nu \log(h_{\psi}(\zeta_0(x_i))), & \text{if} \ \xi_{ij}^{(i)} \in P. \end{cases} \]

This novel structure introduces additional challenges below in federated adversarial domain adaptation problems or the federated learning literature:

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2The parameters of \( \zeta_1 \) and \( \zeta_1 \) are not the same. In this case, we abuse the notation to simplify the expression.
Algorithm 1 FedSGDA Algorithm (Peng et al., 2019)

Require: $x^0, \eta_1, \eta_2, T$
1: for $t = 0, \ldots, T - 1$ do
2: for each client $i \in [N]$ in parallel do
3: $\omega^t_i = \omega^0_i$, $\psi^t_i = \psi^0_i$
4: # Local Update:
5: $\omega^{t+1}_i = \omega^t_i - \eta_1 \nabla_{\omega_i} f_i(\omega^t_i, \psi^t_i)$
6: $\psi^{t+1}_i = \psi^t_i + \eta_2 \nabla_{\psi_i} f_i(\omega^t_i, \psi^t_i)$
7: end for
8: # Global Update:
9: $\omega^{t+1}_0 = \frac{1}{N} \sum_{i=1}^N \omega^{t+1}_i$, $\psi^{t+1}_0 = \frac{1}{N} \sum_{i=1}^N \psi^{t+1}_i$
10: end for

- Clients are restricted to compute the minimax optimization in a distributed manner rather than the centralized minimax optimization.
- To train a common model, both the set of feature extractor variables $\omega$ and domain classifier variables $\psi$ are non-separable across clients.
- The marginal label distributions are class-imbalanced cross clients due to the uneven distribution of source domain data and target domain data. In extreme cases, each client may only access data from the target domain or the source domain; therefore, different data distributions and loss functions among clients degrade distributed learning performance.

3.1 Simple GDA based Algorithms

The majority of federated optimizers, such as FedSGD, FedAvg (McMahan et al., 2017), FedProx (Li et al., 2018), FedPD (Zhang et al., 2020), and others, optimize the local optimal minimum value. The federated adversarial domain adaptation, on the other hand, has a more difficult task of converging to a saddle point in a distributed manner.

Peng et al. (2019) propose FedSGDA algorithm by extending FedSGD with stochastic Gradient Descent Ascent (GDA) in the problem of federated domain adaptation. In order to make the paper self-contained, we summarize FedSGDA in Algorithm 1. However, due to its single descent/ascension step per communication round, SGDA has a massive communication overhead. Later in the experiments, we observe that FedSGDA requires more than 1000 rounds of communication.

FedSGDA inspires us to simply extend FedAvg, a more communication efficient scheme, by GDA, resulting in FedAvgGDA, as shown in Algorithm 2 where the server averages multi-step stochastic gradient descent w.r.t $\omega$ and stochastic gradient ascent w.r.t $\psi$ from all clients. Several works, including Reisizadeh et al. (2020) and Deng & Mahdavi (2021), use a similar or variant of FedAvgGDA for federated GAN training. Rasouli et al. (2020) use FedAvgGDA as well. However, due to the unique class-imbalance problem in federated adversarial domain adaptation, the inter-client drift of a local models from a multi-step stochastic gradient descent ascent using FedAvgGDA is no longer negligible. As illustrated in Fig. 2. We also extend Fedprox (Li et al., 2018) by GDA, which leads to FedProxGDA in Algorithm 2.

Motivated by the global consensus constraint in FedPD (Zhang et al., 2020), we address the problem of model drift from multiple steps of GDA by introducing a separate set of dual variables. The introduction of dual variables is intended to bridge the gradient gap between the distributed optimization and the centralized result.

4 FedMM Algorithm

Due to the distributed constraint in FL systems, the traditional centralized method introduced in Section 2 cannot perform the minimax optimization of (3). Simply decomposing (3) into local optimization and global average as in algorithms like FedSGDA, FedAvgGDA, and FedProxGDA results in a serve performance degradation because these distributed training algorithms diverge from the central optimizer in (3), as validated in Fig. 2. In this section, we look at how to reduce this divergence by reformulating the centralized problem in (3) into the federated saddle-point...
optimization problem with consensus constraints given by
\[
\min_{\omega_i, \psi_i, \omega, \psi} \max \ f(\omega, \psi) = \frac{1}{N} \sum_{i=1}^{N} f_i(\omega_i, \psi_i)
\]
\[
s.t. \quad \omega_i = \omega_0, \quad \psi_i = \psi_0, \quad \forall i \in [N].
\]
(5)
The corresponding augmented Lagrangian form for each client is defined as
\[
L_i(\omega_0, \omega_i, \lambda_i, \psi_0, \psi_i, \beta_i) = f_i(\omega_i, \psi_i) + \langle \lambda_i, \omega_i - \omega_0 \rangle + \frac{\mu_1}{2} \| \omega_i - \omega_0 \|^2 - \langle \beta_i, \psi_i - \psi_0 \rangle - \frac{\mu_2}{2} \| \psi_i - \psi_0 \|^2.
\]
(6)
The centralized optimization problem in (5) is then transformed into a saddle-point minimax optimization of augmented Lagrangian functions over all primal-dual pairs, i.e., \(\{\omega_i, \omega_0, \lambda_i, \psi_0, \psi_i, \beta_i\}\) for all clients \(i \in [N]\):
\[
\min_{\omega_i, \omega_0, \psi_i, \psi_0} \max_{\lambda_i, \beta_i} \mathcal{L} = \frac{1}{N} \sum_{i=1}^{N} L_i(\omega_0, \omega_i, \lambda_i, \psi_0, \psi_i, \beta_i).
\]
(7)
By fixing the global consensus variables \(\{\omega_0, \psi_0\}\), the above problem is separable w.r.t local pairs \(\{\omega_i, \psi_i, \lambda_i, \beta_i\}\) for all \(i \in [N]\). The decomposed task could be independently updated on local clients periodically without global communication. The only problem left is to align the update of global consensus \(\omega_0, \psi_0\) and local updates \(\omega_i, \psi_i\) for all \(i \in [N]\). Next, we demonstrate how to achieve distributed local updates and align local updates with global consensus.

By substituting (6) into (7), we obtain the augmented Lagrangian functions over all primal-dual parameters:
\[
\min_{\omega_i, \psi_i} \mathcal{L} = \frac{1}{N} \sum_{i=1}^{N} f_i(\omega_i, \psi_i) + \langle \lambda_i, \omega_i - \omega_0 \rangle + \frac{\mu_1}{2} \| \omega_i - \omega_0 \|^2 - \langle \beta_i, \psi_i - \psi_0 \rangle - \frac{\mu_2}{2} \| \psi_i - \psi_0 \|^2.
\]
(8)
The minimax optimization w.r.t the global consensus variable \(\omega_0\) and \(\psi_0\) is given by:
\[
\hat{\omega}_0 = \arg \min_{\omega_0} \frac{1}{N} \sum_{i=1}^{N} f_i(\omega_i, \psi_i) + \langle \lambda_i, \omega_i - \omega_0 \rangle + \frac{\mu_1}{2} \| \omega_i - \omega_0 \|^2 - \langle \beta_i, \psi_i - \psi_0 \rangle - \frac{\mu_2}{2} \| \psi_i - \psi_0 \|^2.
\]
\[
= \frac{1}{N} \sum_{i=1}^{N} \omega_i + \frac{1}{\mu_2} \lambda_i,
\]
(9)
where the closed-form solution is due to the quadratic optimization. Similarly, we obtain
\[
\hat{\psi}_0 = \frac{1}{N} \sum_{i=1}^{N} \psi_i + \frac{1}{\mu_2} \beta_i.
\]
(10)
Eqn. (2) and (10) provide guidance for local update alignment with global consensus. More specifically, in each round, we optimize each client’s individual \(\omega_i\) and \(\psi_i\), by fixing the global consensus constraints \(\omega_0\) and \(\psi_0\) and dual parameters \(\lambda_i, \beta_i\). Taking the \((t+1)\)-th round update as an example. Client \(i\) receives the global parameters \(\{\omega_0, \psi_0\}\) from the server and sets local parameters \(\hat{\omega}_i^t, \hat{\psi}_i^t = \psi_0^t\). Then, the local saddle-point optimization of (8) w.r.t \(\{\omega_i, \psi_i\}\) is updated by the local GDA:
\[
\hat{\omega}_i^{t+1} = \hat{\omega}_i^t - \eta_1 \nabla_{\omega_i} \mathcal{L}_i(\hat{\omega}_i^t, \hat{\psi}_i^t)
\]
\[
= \omega_i^t - \eta_1 \left[ \nabla_{\omega_i} f_i(\omega_i^t, \psi_i^t) + \lambda_i (\hat{\omega}_i^t - \omega_0) + \mu_1 (\hat{\omega}_i^t - \omega_0) \right],
\]
(11)
\[
\hat{\psi}_i^{t+1} = \hat{\psi}_i^t - \eta_2 \nabla_{\psi_i} \mathcal{L}_i(\hat{\omega}_i^t, \hat{\psi}_i^t)
\]
\[
= \psi_i^t - \eta_2 \left[ \nabla_{\psi_i} f_i(\omega_i^t, \psi_i^t) + \mu_2 (\hat{\psi}_i^t - \psi_0) - \beta_i \right].
\]
(12)
We denote \(\omega_i^{t+1} = \hat{\omega}_i^t\) and \(\psi_i^{t+1} = \hat{\psi}_i^t\) for the results of \(M_t\)-step local update. The dual parameters are then updated using GDA with
\[
\lambda_i^{t+1} = \lambda_i^t + \mu_1 (\omega_i^{t+1} - \omega_0),
\]
\[
\beta_i^{t+1} = \beta_i^t + \mu_2 (\psi_i^{t+1} - \psi_0).
\]
(13)
(14)
To align with the global consensus constraint obtained in (9) and (10), we set
\[
\omega_i^{t+1} = \omega_i^t + \frac{\eta_1}{\mu_1} \lambda_i^{t+1}, \quad \psi_i^{t+1} = \psi_i^t + \frac{\eta_2}{\mu_2} \beta_i^{t+1}.
\]
(15)
Therefore, the global consensus constraint is satisfied by the global update at the server with
\[
\omega_0^{t+1} = \frac{1}{N} \sum_{i=1}^{N} \omega_i^{t+1}, \quad \text{and} \quad \psi_0^{t+1} = \frac{1}{N} \sum_{i=1}^{N} \psi_i^{t+1}.
\]
(16)
It should be noted that we use an exponential decay factor \(\eta_i \leq 1\) in (15). We find that \(\eta_i\) helps the convergence even when the local training step \(M_t\) is insufficient.

We can now summarize one round of the FedMM algorithm, which consists of three major steps: (i) Parallel saddle-point optimization on all local augmented Lagrangian function \(\mathcal{L}_i\)'s. One optimization oracle example is based on stochastic GDA, as shown in (11) and (12). (ii) Local gradient
After that, the server averages the client’s parameters and uses a method such as FedMM that involves bounding client’s drift from local updates. In particular, Fig. 3 with FedMM algorithm summarizes in Algorithm 3.

\begin{algorithm}
\caption{FedMM Algorithm}
\begin{algorithmic}
    \Require{Initialize $\omega_0^i, \psi_0^i, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3, \{M_i\}_{i=0}^N, T$
    \For{$t = 0, \ldots, T-1$}
        \For{each client $i \in [N]$ in parallel}
            $\tilde{\omega}_i^t = \omega_i^t$, $\tilde{\psi}_i^t = \psi_0^i$
        \EndFor
        \State{\# Local Update:}
        \For{$m = 0, \ldots, M_i - 1$}
            \State{\# Gradient Descent:}
            $\omega_i^{m+1} = \omega_i^m - \eta_1 (\nabla_{\omega_i} f_i(\tilde{\omega}_i^m, \tilde{\psi}_i^m)) + \mu_1 (\omega_i^m - \omega_0^i) + \lambda_i^m$
            \State{\# Gradient Ascent:}
            $\psi_i^{m+1} = \psi_i^m + \eta_2 (\nabla_{\psi_i} f_i(\tilde{\omega}_i^m, \tilde{\psi}_i^m)) - \mu_2 (\psi_i^m - \psi_0^i) - \beta_i^m$
        \EndFor
        \State{$\omega_i^{t+1} = \omega_i^t$, $\psi_i^{t+1} = \tilde{\psi}_i^t$}
        \State{\# Dual Descent:}
        $\lambda_i^{t+1} = \lambda_i^t + \mu_1 (\omega_i^{t+1} - \omega_0^i)$
        \State{\# Dual Ascent:}
        $\beta_i^{t+1} = \beta_i^t + \mu_2 (\psi_i^{t+1} - \psi_0^i)$
        \State{$\omega_i^{t+1} = \omega_i^{t+1} + \frac{\beta_i^t}{\mu_1} \lambda_i^{t+1}$, $\psi_i^{t+1} = \psi_i^{t+1} + \frac{\beta_i^t}{\mu_2} \lambda_i^{t+1}$}
    \EndFor
    \State{$\omega_0^{t+1} = \frac{1}{N} \sum_{i=1}^N \omega_i^{t+1}$, $\psi_0^{t+1} = \frac{1}{N} \sum_{i=1}^N \psi_i^{t+1}$}
\end{algorithmic}
\end{algorithm}

After that, the server averages the client’s parameters and broadcasts them back to the client, completing one-round updates.

**5 Convergence Analysis**

Finding a global saddle point $\min_x \max_y f(x, y)$ in general is intractable [Lin et al., 2020b]. One approach is to equivalently reformulate the problem by $\min_x \{ \Phi(x) := \max_y f(x, y) \}$, and define an optimality notion for the local surrogate of global optimum of $\Phi$. A series of theoretical analyses on the stationary point convergence condition of $\Phi$ with first-order algorithm were carried out to extend the convex-concave assumption to assumptions of nonconvex-strongly-concave [Rafique et al., 2018], nonconvex-concave [Lin et al., 2020b, Nouriehed et al., 2019], and nonconvex-nonconcave [Jin et al., 2020]. Convergence analysis for a federated optimizer, such as FedMM that involves bounding client’s drift from

\[ f(x, \cdot) \] is not necessarily convex and, $f(\cdot, y)$ is strongly concave.

\[ f(x, \cdot) \] is not necessarily convex and, $f(\cdot, y)$ is not necessarily concave.

global parameter via primal-dual method, on the other hand, is more complicated. We establish our main convergence results in this section and show that FedMM converges to the stationary point for the nonconvex-strongly-concave case.

Let $\psi^*(\omega) = \arg \max_{\psi} f(\omega, \psi)$ be the optimal value of $\psi$ for the global objective function $f$ w.r.t $\omega$. Then (3) can be reformulated as $\min_{\omega} f(\omega, \psi) = \min_{\omega} \frac{1}{N} \sum_{i=1}^N \Phi_i(\omega)$ with

\[ \Phi_i(\omega) = f_i(\omega, \psi^*(\omega)), \quad \Phi(\omega) = \frac{1}{N} \sum_{i=1}^N \Phi_i(\omega). \] (17)

In this way, we equivalently reformulate the problem as $\min_{\omega} \{ \Phi(\omega) = \max_{\psi} f(\omega, \psi) \}$. To ease the presentation, we further define the augmented Lagrange of $\Phi$ by

\begin{align}
\mathcal{L}_a(\omega_i^t, \lambda_i^t) &= \Phi_i(\omega_i^t) + \langle \lambda_i^t, \omega_i^t - \omega_0^i \rangle + \frac{\mu_1}{2} \| \omega_i^t - \omega_0^i \|^2.
\end{align} (18)

For our theoretical analysis, we make the following standard assumptions that have been used in the literature [Lin et al., 2020a, Lin et al., 2020b, Luo et al., 2020, Lin et al., 2020b]:

**Assumption 1.** (Lipschitz continuous gradients) For all $i \in [N]$, there exists positive constants $L_{11}, L_{12}, L_{21},$ and $L_{22}$ such that for any $\omega, \omega' \in \mathbb{R}^d$, and $\psi, \psi' \in \mathbb{R}^d$, we have

\[ \| \nabla_{\omega} f_i(\omega, \psi) - \nabla_{\omega} f_i(\omega', \psi) \| \leq L_{11} \| \omega - \omega' \|, \]

\[ \| \nabla_{\psi} f_i(\omega, \psi) - \nabla_{\psi} f_i(\omega, \psi') \| \leq L_{12} \| \psi - \psi' \|, \]

\[ \| \nabla_{\psi} f_i(\omega, \psi) - \nabla_{\psi} f_i(\omega', \psi') \| \leq L_{21} \| \omega - \omega' \|, \]

\[ \| \nabla_{\psi} f_i(\omega, \psi) - \nabla_{\psi} f_i(\omega', \psi') \| \leq L_{22} \| \psi - \psi' \|. \]
**Theorem 1.** (Convergence on $\Phi(\omega)$) There exist positive constants $E_1, E_2, E_3, E_4$, and $\epsilon$ such that after $T$ rounds of global updates, the upper bound for the accumulate descent of $\Phi(\omega_0)$ is given by

$$
\Phi(\omega_0) - \Psi(\omega_0^T) \leq -E_1 \sum_{t=1}^{T} \|\nabla \Phi(\omega_0^t)\|^2 + E_4 T \epsilon 
+ E_2 \sum_{i=1}^{N} \|\psi_i^0 - \psi^*(\omega_i^0)\|^2 
+ E_3 \sum_{i=1}^{N} \sum_{j=1}^{N} \|\omega_i^0 - \omega_j^0\|^2.
$$

(25)

In particular, this implies $\limsup_{T \to \infty} \|\nabla \Phi(\omega_0^T)\| = O(\epsilon)$.

**Remark** Because the l.h.s. of (25) admits a lower bound, so is the r.h.s. As a result, $\limsup_{T \to \infty} \sum_{t=1}^{T} \|\nabla \Phi(\omega_0^t)\|^2$ must converge, which implies that $\Phi(\omega_0^T)$ converges to a $\epsilon$-stationary point. More specifically, Dividing both sides of (25) by $T$ and taking $\limsup_{T \to \infty}$, we obtain

$$
\limsup_{T \to \infty} \frac{\sum_{t=1}^{T} \|\nabla \Phi(\omega_0^t)\|^2}{T} \leq \frac{E_4 \epsilon}{E_1},
$$

which implies that $\sum_{t=1}^{T} \|\nabla \Phi(\omega_0^t)\|^2 = O(T \epsilon)$ and for sufficiently large $T$, $\|\nabla \Phi(\omega_0^T)\|^2 = O(\epsilon)$. In the special case of $\epsilon = 0$, i.e., strict optimality is obtained at each local client, this result shows that the limiting point is a stationary point. We provide the detailed proof in Appendix A.9.

**6 Related Work**

FedSGD (McMahan et al. 2017) suggests one-step local SGD update and then sends the gradients to the server for global update. It mimics the centralized SGD training. The high communication overhead, however, prevents it from being used in practice. FedAvg (McMahan et al. 2017) is a generalization of FedSGD, proposing multiple-step local SGD per communication round, with a good accuracy-to-communication trade-off. However, its accuracy suffers in non-i.i.d. scenarios. Several works have been developed to address non-optimal behavior on non-i.i.d. data, including FedProx (Li et al., 2018), FedPD (Zhang et al., 2020), SCAFFOLD (Karimireddy et al., 2020), FedNova (Wang et al., 2020), and FedDyn (Acar et al., 2021). These works aim to minimize a sum of non-identical functions, where...
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There are several works (Rasouli et al., 2020; Reisizadeh et al., 2020) that bring the communication efficiency to minimax optimization based on FedAvg, such as the federated GAN (Rasouli et al., 2020) that uses a binary classification function to distinguish between real and generated data. Because there is no label-imbalance problem across the training functions of local clients, it works well for distributed GAN learning. This type of FedAvgGDA is sensitive to data imbalance in the federated domain adaptation problem, as demonstrated later in the experiment. Furthermore, the federated robust optimization (Reisizadeh et al., 2020; Deng & Mahdavi, 2021) differs from the federated adversarial domain adaptation problem in that the set of maximization variables is separable across local client-side functions. These strategies, however, are unsuitable for federated domain adaptation due to structural differences. Note that the FLRA in Reisizadeh et al. (2020) corresponds to FedAvgGDA, and its convergence analysis cannot be directly borrowed to our case because each local client in our study is optimized on the augmented Lagrangian local function rather than the pure local function.

7 Experiments

On real-world data sets, FedMM is evaluated with three representative domain adaptation methods: DANN (Ganin & Lempitsky, 2015), MDD (Zhang et al., 2019), and CDAN (Long et al., 2017). Please refer to Section 3 for more information on these methods. Our experiments are primarily concerned with the federated saddle point optimization problems, such as the federated adversarial domain adaptation, which seeks a federated minimax optimization.

Datasets and Source/Target Data Distribution:
MNIST (Ganin et al., 2016) is a dataset that demonstrates domain adaptation by combining MNIST with randomly colored image patches from the BSD500 dataset (Arbelaez et al., 2010). 55,000 labeled images from the source domain and 55,000 unlabeled images from the target domain are used for training; and 55,000 images from the target domain are used for testing.

Office-31 (Saenko et al., 2010) is a typical domain adaptation dataset made up of three distinct domains with 31 categories in each domain. There are 4,652 images in total from 31 classes. We will focus on the worst-case scenario (as analyzed in Fig 2), where the source and target domain data are allocated to different clients for all datasets.

Benchmarks: We compare FedMM with the FedSGDA in (Peng et al., 2019). Furthermore, most existing federated optimizers were designed to solve the loss function minimization, which is unsuitable for adversarial domain adaptation. To make a fair comparison, we extend FedAvg (McMahan et al., 2017) and FedProx (Li et al., 2018) with recently proposed minimax optimizer (Lin et al., 2020a) and refer to them as FedAvgGDA and FedProxGDA with details explained in Section 3.1 and summarized in Algorithm 2.

Networks: On MNIST, we use a three-layer convolutional network as the invariant feature extractor. On Office-31, we use the pre-trained MobileNetV2 (Sandler et al., 2018) on ImageNet (Russakovsky et al., 2015) as the feature extractor. Both the task classifier and the domain classifier are two-layer fully-connected neural networks.

Hyper-parameters: The dual variables, i.e., \( \{ \beta_i, \lambda_i \} \), are set to 0 at the start of training, \( \mu_1 \) and \( \mu_2 \) are set to 1.0 during all training settings. During local training, the learning rate \( \eta_1 \) is fixed to 0.01. In the experiment of training from scratch on MNIST, \( \eta_2 = 0.01 \), \( \mu_1 = 1 \) and \( \mu_2 = 1 \). In the experiment of training from pre-trained model on Office-31, we set customized layer-wise learning rate. In details, the learning rate of feature extractors is set as 0.0005, and \( \eta_2 = 0.1, 0.04 \) and 0.1 for MDD, DANN and CDAN methods, respectively. Besides, \( \nu = 0.1 \) for MDD, \( \nu = 0.25 \) for DANN and CDAN methods. The rest learning rate are all fixed at 0.01. For exponential decay parameter, we set \( \eta_3 = 1.0001^{-1} \) for MDD, \( \eta_3 = 1.0002^{-1} \) for CDAN and \( \eta_3 = 1.0005^{-1} \) for DANN.

Data Distribution: Fig 2 has already demonstrated that as the degree of inter-client domain shift (label imbalance) increases, federated learning performance degrades significantly. As a result, we will focus on the worst-case scenario, in which the source domain data and target domain data are allocated to different clients separately, i.e., \( p = 1.0 \), to verify the effectiveness of FedMM in the experiments.

Performance of Training from Scratch

We begin by examining the convergence property of our proposed FedMM algorithm when it is trained from scratch on MNIST.

Fig 3 compares the global communication rounds of our proposed FedMM to FedSGDA. We compare FedMM with \( M = 20 \), and \( M = 25 \). Thanks to the local multi-steps minimax optimization at each client, FedMM has a quick convergence rate saving more than 90% communication rounds compared to FedSGDA to achieve similar test accuracy. Furthermore, the FedMM convergence rate can be improved by increasing the local steps of primal and dual ascent descent.

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7The code is available at https://github.com/yshen22/fedmm
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Figure 4: Comparisons of convergence for the proposed FedMM with FedSGDA\cite{Peng2019} based on different adversarial domain networks, i.e., DANN, MDD, and CDAN.

Figure 5: Comparisons of convergence for the proposed FedMM with FedAvgGDA and FedProxGDA.
Table 1: Unsupervised target domain test accuracy (%) on Office-31.

| Source | FedAvgGDA | FedSGDA | FedMM |
|--------|-----------|---------|-------|
|        | DANN  | MDD   | CDAN  | DANN  | MDD   | CDAN  | DANN  | MDD   | CDAN  |
| A→W    | 60.1  | 73.2  | 62.9  | 60.3  | 76.4  | 55.3  | 65.5  | 79.7  | 64.7  |
| D→W    | 86.1  | 93.6  | 86.8  | 84.9  | 94.7  | 83.4  | 89.6  | 95.9  | 93.4  |
| W→D    | 93.6  | 97.8  | 94.2  | 93.7  | 98.3  | 94.0  | 96.7  | 98.5  | 94.0  |
| A→D    | 63.5  | 72.1  | 65.1  | 65.3  | 75.3  | 67.7  | 67.8  | 78.8  | 66.9  |
| D→A    | 33.7  | 47.9  | 40.3  | 36.9  | 49.2  | 47.1  | 44.3  | 60.3  | 51.4  |
| W→A    | 40.5  | 51.7  | 45.5  | 40.3  | 52.6  | 43.3  | 48.7  | 55.5  | 59.6  |
| Average| 62.9  | 72.7  | 65.8  | 65.8  | 74.4  | 65.1  | 68.7  | 78.1  | 71.7  |

Table 2: Communication rounds (×100) for training on Office-31.

| Source | FedAvgGDA | FedSGDA | FedMM |
|--------|-----------|---------|-------|
|        | DANN  | MDD   | CDAN  | DANN  | MDD   | CDAN  | DANN  | MDD   | CDAN  |
| A→W    | 10    | 31    | 17    | 59    | 255   | 78    | 13    | 23    | 29    |
| D→W    | 13    | 27    | 13    | 40    | 188   | 49    | 15    | 18    | 9     |
| W→D    | 8     | 11    | 7     | 29    | 92    | 16    | 14    | 19    | 10    |
| A→D    | 7     | 22    | 21    | 56    | 400   | 13    | 7.5   | 22    | 32    |
| D→A    | 24    | 31    | 34    | 48    | 300   | 95    | 39    | 19    | 17    |
| W→A    | 18    | 18    | 14    | 88    | 321   | 85    | 25    | 15    | 13    |
| Average| 13.3  | 23.3  | 17.7  | 53.3  | 259.3 | 65.1  | 18.9  | 19.3  | 18.3  |

In Fig. 5, the convergence property of our proposed FedMM is further compared with other representative federated training algorithms with multiple local descent and ascent updates, namely FedAvgGDA and FedProxGDA with $M_i = 20$ for different number of source/target clients settings. While both the FedAvgGDA and FedProxGDA algorithms converge, FedMM consistently outperforms them in terms of test accuracy for all three widely used domain adaptation methods. The results clearly show that FedMM has a superior test accuracy for training from scratch with more than 20% accuracy improvement.

This enormous improvement is understandable given that the FedMM is intended to bridge the gap between the distributed local model and the global model through distributed consensus in the minimax optimization context. Because of the unique structure of federated adversarial domain adaptation, when the source and target data are distributed across different clients, model drift becomes a severe problem (validated in Fig. 2), which did not occur in any previous federated learning problems in the literature.

**Performance of Training from Pre-trained Models**

We further examine how the proposed FedMM algorithm performs with the pre-trained MobileNetV2 as a feature extractor. In this part, all the experiments are conducted on Office-31. Test accuracy and training communication rounds using FedMM, FedAvgGDA, and FedSGDA for commonly used domain adaptation methods are included in Table 1 and Table 2, respectively. Note that FedMM's performance improvement is reduced when compared to the training from scratch case in Fig. 5. This is because feature extractor parameters in this pre-trained models have approached optimal values. Nevertheless, we take the best average results of FedAvgGDA and FedSGDA (averaged over all tasks) for DANN, MDD, and CDAN and compare them to FedMM. As highlighted in Table 1, FedMM improves by 8.2%, 5.4%, and 9.0% for DANN, MDD and CDAN, respectively. Besides, both FedAvgGDA and FedMM cost much less communication rounds than FedSGDA. However, FedMM does not have a significant communication advantage over FedAvgGDA due to the additional dual variables.

8 CONCLUSIONS

We propose FedMM for federated adversarial domain adaptation in this paper. FedMM is designed specifically for federated minimax optimizations with non-separable minimization and maximization variables, as well as clients with uneven label class distributions. We show that FedMM ensures convergence for clients by using both supervised source domain data and unsupervised target domain data. Experiments show that FedMM outperforms state-of-the-art algorithms in terms of communication rounds and test accuracy on various benchmark datasets. It outperforms other methods by around a 20% improvement in accuracy over the same communication rounds when training from scratch, and it also clearly outperforms other methods when training from pre-trained models.
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A Appendix: Convergence Analysis for FedMM in Algorithm

Because the proof is lengthy, we begin by demonstrating convergence to the stationary point by assuming sufficient local training is obtained to ensure local convergence (Section A.3, Section A.7). This assumption is further removed with the results being extended to the convergence proof with bounded local convergence error, as shown in Section A.9.

A.1 Notation

Let $\psi^*(\omega)$ be the optimal value of $\psi$ for the global objective function $f$ for $\omega$, which is given by

$$\psi^*(\omega) \triangleq \arg \max_{\psi} f(\omega, \psi).$$

Then (3) is reformulated as $\min_{\omega} f(\omega, \psi) = \frac{1}{N} \sum_{i=1}^{N} \Phi_i(\omega)$ with

$$\Phi_i(\omega) \triangleq f_i(\omega, \psi^*(\omega)), \quad \text{and} \quad \Phi(\omega) \triangleq \frac{1}{N} \sum_{i=1}^{N} \Phi_i(\omega).$$

In this way, we equivalently reformulate the problem as $\min_{\omega} \{ \Phi(\omega) = \max_{\phi} f(\omega, \phi) \}$. We further define the augmented Lagrange of $\Phi_i$ by

$$L^\psi_i(\omega_i^t, \omega_0^t, \lambda_i^t) = \Phi_i(\omega_i^t) + \langle \lambda_i^t, \omega_i^t - \omega_0^t \rangle + \frac{\mu_i}{2} \| \omega_i^t - \omega_0^t \|^2.$$  

In Table 3, some notations are further defined to represent some commonly used computations in the proof.

| Notation | Explanation |
|----------|-------------|
| $\tilde{\psi}_t = a_t = \sum_{i=1}^{N} \sum_{j \neq i} \| \psi^*_i - \psi^*_j \|^2$ | Average deviation among $\psi^*_i$'s. |
| $\bar{\psi}_t = b_t = \sum_{i=1}^{N} \| \psi_i^t - \psi_i^{t-1} \|^2$ | Average update increment for $\psi_i^t$. |
| $\epsilon_t = \sum_{i=1}^{N} \| \psi_i^t - \psi^*(\omega_i^t) \|^2$ | Average distance to optimum for $\psi_i^t$. |
| $\bar{\omega}_t = d_t = \sum_{i=1}^{N} \| \omega_i^t - \omega_i^{t-1} \|^2$ | Average update increment for $\omega_i^t$. |
| $\bar{\omega}_t = e_t = \sum_{i=1}^{N} \sum_{j \neq i} \| \omega_i^t - \omega_j^t \|^2$ | Average deviation among $\omega_i^t$. |

A.2 Assumptions

Assumption 5. (Lipschitz continuous gradients) For all $i \in [N]$, there exists positive constants $L_{11}$, $L_{12}$, $L_{21}$, and $L_{22}$ such that for any $\omega, \omega' \in \mathbb{R}^{d_1}$, and $\psi, \psi' \in \mathbb{R}^{d_2}$, we have

$$\| \nabla_{\omega} f_i(\omega, \psi) - \nabla_{\omega} f_i(\omega', \psi) \| \leq L_{11} \| \omega - \omega' \|, \quad \| \nabla_{\omega} f_i(\omega, \psi) - \nabla_{\omega} f_i(\omega, \psi') \| \leq L_{12} \| \psi - \psi' \|,$$

$$\| \nabla_{\psi} f_i(\omega, \psi) - \nabla_{\psi} f_i(\omega', \psi) \| \leq L_{21} \| \omega - \omega' \|, \quad \| \nabla_{\psi} f_i(\omega, \psi) - \nabla_{\psi} f_i(\omega, \psi') \| \leq L_{22} \| \psi - \psi' \|. $$

Assumption 6. (Strongly concave $f_i(\cdot, \psi)$) For all $i \in [N]$, $f_i(\omega, \psi)$ are strongly concave on $\psi$, i.e., there exists constant $B > 0$ such that for any $\omega \in \mathbb{R}^{d_1}$, and $\psi, \psi' \in \mathbb{R}^{d_2}$, we have

$$\langle \nabla_{\psi} f_i(\omega, \psi) - \nabla_{\psi} f_i(\omega, \psi'), \psi - \psi' \rangle \leq -B \| \psi - \psi' \|^2.$$  

Assumption 7. (Sufficient local training) For all $i \in [N]$, after $M_i$-step update, the gradients w.r.t. $\omega_i$ and $\psi_i$ are finite and denoted by

$$\| \nabla_{\omega} L_i(\omega_i^t, \psi_i^t) \| = e_{\omega_i^t}, \quad \| \nabla_{\psi} L_i(\omega_i^t, \psi_i^t) \| = e_{\psi_i^t}, \quad \forall t \in [T].$$

We set $\eta_3 = 1$ for the analysis without loss of generality.

Assumption 8. The $\kappa$-Lipschitz continuity of $\psi^*(\omega)$, i.e.,

$$\| \psi^*(\omega_i^{t-1}) - \psi^*(\omega_i^t) \| \leq \kappa \| \omega_i^{t-1} - \omega_i^t \|, \quad \forall t \in [T].$$
A.3 Basic Properties of FedMM

Proposition 1. In Algorithm 3, the following update of $\sum_{i=1}^{N} \psi_i^t$ is valid for all $t \in [T]$: 

$$\sum_{i=1}^{N} \psi_i^{t+1} = \sum_{i=1}^{N} \psi_i^t + \frac{1}{\mu_2} \sum_{i=1}^{N} \nabla_\psi f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right).$$  \hspace{1cm} (32) 

Proof. Applying Assumption 7 to (12) and replace $\omega_m$ as well as $\hat{\psi}^m$ with $\omega^{t+1}$ as well as $\hat{\psi}^{t+1}$ respectively, we obtain: 

$$\nabla_\psi L_i(\omega_i^{t+1}, \psi_i^{t+1}) = \nabla_\psi f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) - \mu_2 \left( \psi_i^{t+1} - \psi_0^t \right) - \beta_i^t = 0.$$  \hspace{1cm} (33) 

By further making a summation for all $i \in [N]$, we have 

$$\sum_{i=1}^{N} \nabla_\psi f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) - \mu_2 \sum_{i=1}^{N} \psi_i^{t+1} + \mu_2 N \psi_0^t = \sum_{i=1}^{N} \beta_i^t.$$  \hspace{1cm} (34) 

In addition, we get the following equation by substituting (15) into (16), which is given by 

$$\psi_0^t = \frac{1}{N} \sum_{i=1}^{N} \left( \psi_i^t + \frac{1}{\mu_2} \beta_i^t \right).$$  \hspace{1cm} (35) 

We finally prove (32) by substituting the above equation into (34). \hfill \Box

Proposition 2. In Algorithm 3, the following update of $\sum_{i=1}^{N} \omega_i^t$ is valid for all $t \in [T]$: 

$$\sum_{i=1}^{N} \omega_i^{t+1} = \sum_{i=1}^{N} \omega_i^t - \frac{1}{\mu_1} \sum_{i=1}^{N} \nabla_\omega f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right).$$  \hspace{1cm} (36) 

Proof. The proof procedure is similar to that for Proposition 1. The following are the specifics. Applying Assumption 7 to (11) and replace $\omega_m$ as well as $\hat{\psi}^m$ with $\omega^{t+1}$ as well as $\hat{\psi}^{t+1}$ respectively, we obtain: 

$$\nabla_\omega L_i(\omega_i^{t+1}, \psi_i^{t+1}) = \nabla_\omega f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) + \mu_1 \left( \psi_i^{t+1} - \psi_0^t \right) + \lambda_i^t = 0.$$  \hspace{1cm} (37) 

By further making a summation for all $i \in [N]$, we have 

$$\sum_{i=1}^{N} \nabla_\omega f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) + \mu_1 \sum_{i=1}^{N} \omega_i^{t+1} - \mu_1 N \omega_0^t = - \sum_{i=1}^{N} \lambda_i^t.$$  \hspace{1cm} (38) 

In addition, by substituting (15) into (16), the following holds: 

$$\omega_0^t = \frac{1}{N} \sum_{i=1}^{N} \left( \omega_i^t + \frac{1}{\mu_1} \lambda_i^t \right).$$  \hspace{1cm} (39) 

We finally prove (36) by substituting the above equation into (38). \hfill \Box

Proposition 3. In Algorithm 3, the update of $\hat{\psi}_i^t$ holds true for all $i \in [N]$ and $t \in [T]$: 

$$\mu_2 \left( \psi_i^{t+1} - \psi_0^t \right) = \nabla_\psi f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) - \nabla_\psi f_i \left( \omega_i^t, \psi_i^t \right).$$  \hspace{1cm} (40) 

Proof. Applying Assumption 7 to (12) and replace $\omega_m$ as well as $\hat{\psi}^m$ with $\omega^{t+1}$ as well as $\hat{\psi}^{t+1}$ respectively, we obtain: 

$$\nabla_\psi f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) - \mu_2 \left( \psi_i^{t+1} - \psi_0^t \right) - \beta_i^t = 0.$$  \hspace{1cm} (41) 

By substituting the $\beta_i^t$’s update equation in (14) into (41), we have 

$$\nabla_\psi f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) - \beta_i^{t+1} = 0.$$  \hspace{1cm} (42)
We prove Lemma 1 as follows. We repeat Lemma 1 in the following Lemma 4 to make the appendix self-contained.

By substituting \( \beta_{i}^{t+1} - \beta_{i}^{t} \) in (14) to the l.h.s of the above equation, we have proved (40).

**Proposition 4.** In Algorithm 3, the update of \( \omega_{i}^{t} \) holds true for all \( i \in [N] \) and \( t \in [T] \):

\[
\mu_{i}(\omega_{i}^{t+1} - \omega_{i}^{0}) = \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) - \nabla_{\omega} f_{i}(\omega_{i}^{t+1}, \psi_{i}^{t+1}) + \lambda_{i}^{t} = 0.
\]

**Proof.** The proof procedure is similar to that for Proposition 3. The following are the specifics. Applying Assumption 7 to \( \lambda \) in (11) and replace \( \omega_{m}^{t} \) as well as \( \tilde{\omega}_{m}^{t} \) with \( \omega_{i}^{t+1} \) as well as \( \tilde{\omega}_{i}^{t+1} \) respectively, we obtain:

\[
\nabla_{\omega} f_{i}(\omega_{i}^{t+1}, \psi_{i}^{t+1}) + \mu_{i}(\omega_{i}^{t+1} - \omega_{i}^{0}) + \lambda_{i}^{t} = 0.
\]

By substituting the \( \lambda_{i}^{t} \)’s the \( \beta_{i}^{t} \)’s update equation in (13) into (46), we have

\[
\nabla_{\omega} f_{i}(\omega_{i}^{t+1}, \psi_{i}^{t+1}) + \lambda_{i}^{t+1} = 0.
\]

By replacing \( t + 1 \) with \( t \) in the preceding equation, we get

\[
\nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) + \lambda_{i}^{t} = 0.
\]

By subtracting (47) from (48), we arrive at

\[
\lambda_{i}^{t+1} - \lambda_{i}^{t} = \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) - \nabla_{\omega} f_{i}(\omega_{i}^{t+1}, \psi_{i}^{t+1}) + \lambda_{i}^{t} = 0.
\]

By substituting \( \lambda_{i}^{t+1} - \lambda_{i}^{t} \) in (13) to l.h.s of the above equation, we have proved (45).

**A.4 Proof of Lemma 1**

We prove Lemma 1 as follows. We repeat Lemma 1 in the following Lemma 4 to make the appendix self-contained.

**Lemma 4.** After \( M_{i} \) step updates, the gradient of \( L_{i}^{\Phi}(\omega_{i}^{t}, \omega_{0}^{i}, \lambda_{i}^{t}) \) is bounded by

\[
\| \nabla_{\omega} L_{i}^{\Phi}(\omega_{i}^{t}, \omega_{0}^{i}, \lambda_{i}^{t}) \| \leq L_{12} \epsilon_{i}, \quad \forall i \in [N].
\]

**Proof.** After \( M_{i} \) step updates, we have

\[
\nabla_{\omega} L_{i}(\omega_{i}^{t}, \omega_{0}^{i}, \lambda_{i}^{t}) = 0,
\]

which implies that

\[
\nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) + \lambda_{i}^{t} + \mu_{i}(\omega_{i}^{t} - \omega_{0}^{i}) = 0.
\]

Since \( \Phi_{i} \) is differentiable with \( \nabla \Phi_{i}(\omega) = \nabla_{\omega} f_{i}(\omega, \psi^{*}(\omega)) \) and from Danskin’s theorem (Rockafellar 2015), we have

\[
\| \nabla_{\omega} L_{i}^{\Phi}(\omega_{i}^{t}, \omega_{0}^{i}, \lambda_{i}^{t}) \| = \| \nabla \Phi_{i}(\omega_{i}^{t}) + \lambda_{i}^{t} + \mu_{i}(\omega_{i}^{t} - \omega_{0}^{i}) \| = \| \nabla \Phi_{i}(\omega_{i}^{t}) - \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) \| = \| \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi^{*}(\omega_{i}^{t})) - \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) \|.
\]

From \( L_{12} \)-Lipschitz of \( \nabla_{\omega} f_{i}(\omega, \psi) \) on \( \psi \), we have

\[
\| \nabla_{\omega} L_{i}^{\Phi}(\omega_{i}^{t}, \omega_{0}^{i}, \lambda_{i}^{t}) \| = \| \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi^{*}(\omega_{i}^{t})) - \nabla_{\omega} f_{i}(\omega_{i}^{t}, \psi_{i}^{t}) \| \leq L_{12} \| \psi_{i}^{t} - \psi^{*}(\omega_{i}^{t}) \|.
\]
A.5 Proof of Lemma 2

Lemma 5. In Algorithm 3, the following inequality holds for $t = 1, \ldots, T$.

\[
\epsilon_t \leq \frac{8\mu_2}{NB} \epsilon_{t-1} + \frac{2\mu_2}{2\mu_2 + B} \epsilon_{t-1} + \frac{8\mu_2}{B} \kappa^2 \bar{\omega}_t + \frac{8(\mu_2 + L_2)^2}{N\mu_2 B} \bar{\psi}_t + \frac{8L_2^2}{N\mu_2 B} \bar{\omega}_t. \tag{55}
\]

Proof. Considering the fact that $\frac{1}{N} \sum_{i=1}^{N} \nabla_{\psi} f_i (\omega^t_i, \psi^t_i) = \nabla_{\psi} f (\omega^t_i, \psi^t_i)$, we reformulate (32) in Proposition 1 and after some tedious algebra manipulations, we obtain

\[
\psi^t_i - \psi^* (\omega^t_i) = \psi^{t-1}_i - \psi^* (\omega^{t-1}_i) + \frac{1}{\mu_2} \nabla_{\psi} f (\omega^t_i, \psi^t_i)
\]

\[
+ \frac{1}{N\mu_2} \sum_{j=1}^{N} \nabla_{\psi} f_j (\omega^t_j, \psi^t_j) - \nabla_{\psi} f_j (\omega^t_i, \psi^t_i)
\]

\[
+ \frac{1}{N\mu_2} \sum_{j=1}^{N} (-\mu_2 (\psi^{t-1}_i - \psi^{t-1}_j) + \mu_2 (\psi^t_i - \psi^t_j)). \tag{56}
\]

By taking norm on both sides of the above equation and considering the triangle inequality, we have

\[
\| \psi^t_i - \psi^* (\omega^t_i) - \frac{1}{\mu_2} \nabla_{\psi} f (\omega^t_i, \psi^t_i) \| \leq \| \psi^{t-1}_i - \psi^* (\omega^{t-1}_i) \|
\]

\[
+ \frac{1}{N\mu_2} \sum_{j=1}^{N} \| \nabla_{\psi} f_j (\omega^t_j, \psi^t_j) - \nabla_{\psi} f_j (\omega^t_i, \psi^t_i) \|
\]

\[
+ \frac{1}{N} \sum_{j=1}^{N} \| \psi^{t-1}_i - \psi^{t-1}_j \| + \frac{1}{N} \sum_{j=1}^{N} \| \psi^t_i - \psi^t_j \|. \tag{57}
\]

Besides, the $B$-strongly concavity of $f(\cdot, \psi)$ implies

\[
\sqrt{\frac{\mu_2 + B}{\mu_2}} \| \psi^t_i - \psi^* (\omega^t_i) \| \leq \| \psi^t_i - \psi^* (\omega^t_i) - \frac{1}{\mu_2} \nabla_{\psi} f (\omega^t_i, \psi^t_i) \|. \tag{58}
\]

Sum of the two preceding inequalities along the same sign direction leads to

\[
\sqrt{\frac{\mu_2 + B}{\mu_2}} \| \psi^t_i - \psi^* (\omega^t_i) \| \leq \| \psi^{t-1}_i - \psi^* (\omega^{t-1}_i) \|
\]

\[
+ \frac{1}{N\mu_2} \sum_{j \neq i} \| \nabla_{\psi} f_j (\omega^t_j, \psi^t_j) - \nabla_{\psi} f_j (\omega^t_i, \psi^t_i) \|
\]

\[
+ \frac{1}{N} \sum_{j=1}^{N} \| \psi^{t-1}_i - \psi^{t-1}_j \| + \frac{1}{N} \sum_{j=1}^{N} \| \psi^t_i - \psi^t_j \|. \tag{59}
\]
We further obtain
\[
\|\psi_i^t - \psi^* (\omega_i^t)\| \leq \sqrt{\frac{\mu_2}{\mu_2 + B}} \left( \|\psi_{i-1}^t - \psi^* (\omega_i^t)\| + \frac{1}{N} \sum_{j=1}^{N} \|\nabla_{\psi} f_j (\omega_j^t, \psi_j^t) - \nabla_{\psi} f_j (\omega_i^t, \psi_i^t)\| \right)
+ \frac{1}{N} \sum_{j=1}^{N} \|\psi_j^{t-1} - \psi_j^t\| + \frac{1}{N} \sum_{j=1}^{N} \|\psi_i^{t-1} - \psi_i^{t-1}\| + \frac{1}{N} \sum_{j=1}^{N} \|\psi_i^t - \psi_j^t\|. 
\] (60)

Furthermore, the Lipschitz continuity properties in Assumption 5 imply the following inequality:
\[
L_{21} \|\omega_j^t - \omega_i^t\| + L_{22} \|\psi_j^t - \psi_i^t\| \geq \|\nabla_{\psi} f_j (\omega_j^t, \psi_j^t) - \nabla_{\psi} f_j (\omega_i^t, \psi_i^t)\| 
+ \|\nabla_{\psi} f_j (\omega_j^t, \psi_j^t) - \nabla_{\psi} f_j (\omega_j^t, \psi_j^t)\|. 
\] (61)

Applying the triangle inequality in Euclidean geometry on the r.h.s of the above inequality, we further obtain
\[
L_{21} \|\omega_j^t - \omega_i^t\| + L_{22} \|\psi_j^t - \psi_i^t\| \geq \|\nabla_{\psi} f_j (\omega_j^t, \psi_j^t) - \nabla_{\psi} f_j (\omega_i^t, \psi_i^t)\|. 
\] (62)

By putting (58) and (62) back into the corresponding items in the r.h.s. of (57), we have
\[
\|\psi_i^t - \psi^* (\omega_i^t)\| \leq \sqrt{\frac{\mu_2}{\mu_2 + B}} \left( \|\psi_{i-1}^t - \psi^* (\omega_i^t)\| + \sqrt{\frac{\mu_2}{\mu_2 + B}} \frac{\mu_2 + L_{22}}{N\mu_2} \sum_{j=1, j \neq i}^{N} \|\psi_j^t - \psi_i^t\| 
+ \sqrt{\frac{\mu_2}{\mu_2 + B}} \frac{L_{21}}{N\mu_2} \sum_{j=1, j \neq i}^{N} \|\omega_j^t - \omega_i^t\| \right) \]
\[+ \sqrt{\frac{\mu_2}{\mu_2 + B}} N \sum_{j=1, j \neq i}^{N} \|\psi_j^t - \psi_i^t\| \]
\[+ \sqrt{\frac{\mu_2}{\mu_2 + B}} \frac{\mu_2 + L_{22}}{N\mu_2} \sum_{j=1, j \neq i}^{N} \|\psi_j^{t-1} - \psi_i^{t-1}\| \]
\[+ \sqrt{\frac{\mu_2}{\mu_2 + B}} N \sum_{j=1, j \neq i}^{N} \|\psi_i^{t-1} - \psi_i^{t-1}\|. 
\] (63)

According to the $\kappa$-Lipschitz continuity of $\psi^* (\omega)$, the r.h.s of (63) can be further amplified by applying the triangle inequality on the first item, and we obtain
\[
\|\psi_i^t - \psi^* (\omega_i^t)\| \leq \sqrt{\frac{\mu_2}{\mu_2 + B}} \left( \|\psi_{i-1}^t - \psi^* (\omega_i^{t-1})\| + \sqrt{\frac{\mu_2}{\mu_2 + B}} \kappa \|\omega_i^{t-1} - \omega_i^t\| 
+ \sqrt{\frac{\mu_2}{\mu_2 + B}} \frac{1}{N} \sum_{j=1, j \neq i}^{N} \|\psi_j^{t-1} - \psi_i^{t-1}\| \right) \]
\[+ \sqrt{\frac{\mu_2}{\mu_2 + B}} \frac{L_{21}}{N\mu_2} \sum_{j=1, j \neq i}^{N} \|\omega_j^t - \omega_i^t\| 
+ \sqrt{\frac{\mu_2}{\mu_2 + B}} \frac{\mu_2 + L_{22}}{N\mu_2} \sum_{j=1, j \neq i}^{N} \|\psi_j^t - \psi_i^t\| \]
\[+ \sqrt{\frac{\mu_2}{\mu_2 + B}} N \sum_{j=1, j \neq i}^{N} \|\psi_j^t - \psi_i^t\| \] \[+ \sqrt{\frac{\mu_2}{\mu_2 + B}} N \sum_{j=1, j \neq i}^{N} \|\psi_i^{t-1} - \psi_i^{t-1}\|. 
\] (64)

The following inequality is obtained by further applying Cauchy-Schwarz inequality on the above equation.
\[
\|\psi_i^t - \psi^* (\omega_i^t)\|^2 \leq \frac{2\mu_2}{2\mu_2 + B} \|\psi_{i-1}^t - \psi^* (\omega_i^{t-1})\|^2 + \frac{8\mu_2}{B} \kappa^2 \|\omega_i^{t-1} - \omega_i^t\|^2 
+ \frac{8\mu_2}{NB} \sum_{j=1, j \neq i}^{N} \|\psi_j^{t-1} - \psi_i^{t-1}\|^2 + \frac{8L_{21}^2}{N\mu_2 B} \sum_{j=1, j \neq i}^{N} \|\omega_j^t - \omega_i^t\|^2 
+ \frac{8(\mu_2 + L_{22})}{N\mu_2 B} \sum_{j=1, j \neq i}^{N} \|\psi_j^t - \psi_i^t\|^2. 
\] (65)

We have finally proved Lemma 5 by summing over $i$ on both sides of the above inequality. 

\[\square\]
Lemma 6. In Algorithm 3, the following inequality holds for all \( t \in [T] \).

\[
\frac{\mu_2^2}{2N} \tilde{\psi}_t \leq 4L_{22}^2 \tilde{\psi}_t + 4L_{21}^2 \tilde{\omega}_t.
\] (66)

Proof. By taking the absolute value at both sides of (40) in Proposition 3 we have

\[
\mu_2 \left\| \psi_t^i - \psi_0^{t-1} \right\| = \left\| \nabla \psi f_i(\omega_t^i, \psi_t^i) - \nabla \psi f_i(\omega_0^{t-1}, \psi_0^{t-1}) \right\|.
\] (67)

Since \( \nabla \psi f_i(\omega, \psi) \) is \( L_{21} \)-Lipschitz continuity on \( \omega \) and \( L_{22} \)-Lipschitz continuity on \( \psi \), the r.h.s. of the above equation can be further amplified, which leads to

\[
\mu_2 \left\| \psi_t^i - \psi_0^{t-1} \right\| \leq L_{22} \left\| \psi_t^i - \psi_0^{t-1} \right\| + L_{21} \left\| \omega_t^i - \omega_0^{t-1} \right\|.
\] (68)

Next, we focus on the l.h.s. of the above inequality. According to the the triangle inequality, it is evident that \( \left\| \psi_t^i - \psi_0^{t-1} \right\| \leq \left\| \psi_t^i - \psi_j^{t-1} \right\| + \left\| \psi_j^{t-1} - \psi_0^{t-1} \right\| \), \( \forall i, j \in [N] \). Then, we have

\[
\mu_2 \left\| \psi_t^i - \psi_j^{t-1} \right\| \leq L_{22} \left\| \psi_t^i - \psi_j^{t-1} \right\| + L_{21} \left\| \psi_j^{t-1} - \psi_0^{t-1} \right\| + L_{21} \left\| \psi_j^{t-1} - \psi_0^{t-1} \right\|.
\] (69)

According to the Cauchy-Schwarz inequality, the above inequality is equivalent to

\[
\mu_2^2 \left\| \psi_t^i - \psi_j^{t-1} \right\|^2 \leq 4L_{22}^2 \left\| \psi_t^i - \psi_j^{t-1} \right\|^2 + 4L_{21}^2 \left\| \psi_j^{t-1} - \psi_0^{t-1} \right\|^2.
\] (70)

Then by summing up on a set of \( \{i, j\} \) pairs with \( i, j \in [N] \) and \( i \neq j \), we have proved (66).

\[ \square \]

Lemma 7. In Algorithm 3, the following inequality holds for all \( t \in [T] \).

\[
\psi_t \leq \frac{4L_{22}^2}{\mu_2^2} \epsilon_t + \frac{4(\mu_2 + L_{22})^2}{N\mu_2^2} \psi_t + \frac{4L_{21}^2}{N\mu_2^2} \tilde{\omega}_t + \frac{4}{N} \psi_t^{t-1}.
\] (71)

Proof. From Proposition 1 we have

\[
\frac{1}{N} \sum_{i=1}^{N} \psi_t^{i+1} - \frac{1}{N} \sum_{i=1}^{N} \psi_t^i = \frac{1}{N\mu_2} \sum_{i=1}^{N} \nabla \psi f_i (\omega_t^{i+1}, \psi_t^{i+1}) ,
\] (72)

where the \( \psi_t^i \) is equivalently represented by

\[
\psi_t^i = \frac{1}{N} \sum_{i=1}^{N} \psi_t^i + \frac{1}{N} \sum_{j=1}^{N} (\psi_t^i - \psi_t^j) .
\] (73)

Following the same procedure, \( f (\psi_t^{i+1}, \omega_t^{i+1}) \) is equivalently denoted by

\[
f (\psi_t^{i+1}, \omega_t^{i+1}) = \frac{1}{N} \sum_{j=1}^{N} f_j (\psi_t^{i+1}, \omega_t^{i+1})
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} f_j (\psi_t^{i+1}, \omega_t^{i+1}) + \frac{1}{N} \sum_{j=1}^{N} (f_j (\psi_t^{i+1}, \omega_t^{i+1}) - f_j (\psi_t^{i+1}, \omega_t^{i+1})) .
\] (74)
By substituting the preceding two equations back to (72) and after some tedious algebra manipulations, we obtain

\[
\|\psi_i^{t+1} - \psi_t^i\| = \frac{1}{\mu_2} \nabla_\psi f_i (\omega_i^{t+1}, \psi_i^{t+1}) + \frac{1}{N \mu_2} \sum_{j=1}^N (\nabla_\psi f_j (\omega_j^{t+1}, \psi_j^{t+1}) - \nabla_\psi f_j (\omega_j^{t+1}, \psi_j^{t+1})) \\
- \nabla_\psi f_j (\omega_i^{t+1}, \psi_i^{t+1})) - \frac{1}{N} \sum_{j \neq i} (\psi_i^t - \psi_j^t) + \frac{1}{N} \sum_{j \neq i} (\psi_i^{t+1} - \psi_j^{t+1}) \\
\leq \frac{1}{\mu_2} \|\nabla_\psi f (\omega_i^{t+1}, \psi_i^{t+1})\| + \frac{1}{N \mu_2} \sum_{j=1}^N \|\nabla_\psi f_j (\omega_j^{t+1}, \psi_j^{t+1}) - \nabla_\psi f_j (\omega_j^{t+1}, \psi_j^{t+1})\| \\
+ \frac{1}{N} \sum_{j \neq i} \|\psi_i^t - \psi_j^t\| + \frac{1}{N} \sum_{j \neq i} \|\psi_i^{t+1} - \psi_j^{t+1}\|. \tag{75}
\]

According to the definition of \(\psi^* (\omega_i^{t+1})\) in (26), we have \(\nabla_\psi f(\omega_i^{t+1}, \psi^* (\omega_i^{t+1})) = 0\). By further taking into account that \(\nabla_\psi f_j(\omega, \psi)\) is \(L_{22}\)-Lipschitz continuity on \(\psi\), we have

\[
\|\nabla_\psi f (\omega_i^{t+1}, \psi_i^{t+1})\| = \|\nabla_\psi f(\omega_i^{t+1}, \psi_i^{t+1}) - \nabla_\psi f(\omega_i^{t+1}, \psi^* (\omega_i^{t+1}))\| \\
\leq L_{22} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\|. \tag{76}
\]

Besides, since \(\nabla_\psi f_j(\omega, \psi)\) is \(L_{21}\)-Lipschitz continuity on \(\omega\) and \(L_{22}\)-Lipschitz continuity on \(\psi\), we have

\[
\|\nabla_\psi f_j (\omega_j^{t+1}, \psi_j^{t+1}) - \nabla_\psi f_j (\omega_j^{t+1}, \psi_i^{t+1})\| \leq L_{21} \|\omega_j^{t+1} - \omega_i^{t+1}\| + L_{22} \|\psi_j^{t+1} - \psi_i^{t+1}\|. \tag{77}
\]

By substituting (76) and (77) back into the inequality in (75), we obtain:

\[
\|\psi_i^{t+1} - \psi_t^i\| \leq \frac{L_{22}}{\mu_2} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\| + \frac{\mu_2 + L_{22}}{N \mu_2} \sum_{j=1}^N \|\psi_j^{t+1} - \psi_i^{t+1}\| \\
+ \frac{L_{21}}{N \mu_2} \sum_{j \neq i} \|\omega_j^{t+1} - \omega_i^{t+1}\| + \frac{1}{N} \sum_{j \neq i} \|\psi_j^t - \psi_i^t\|. \tag{78}
\]

Finally, by applying Cauchy-Schwarz inequality, we have

\[
\|\psi_i^{t+1} - \psi_t^i\|^2 \leq \frac{4 L_{22}^2}{\mu_2^2} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\|^2 + \frac{4(\mu_2 + L_{22})^2}{N \mu_2^2} \sum_{j \neq i} \|\psi_j^{t+1} - \psi_i^{t+1}\|^2 \\
+ \frac{4 L_{21}^2}{N \mu_2^2} \sum_{j \neq i} \|\omega_j^{t+1} - \omega_i^{t+1}\|^2 + \frac{4}{N} \sum_{j \neq i} \|\psi_j^t - \psi_i^t\|^2. \tag{79}
\]

By further making a summation of the above inequality for all \(i \in [N]\), we have proved (71).

\[\square\]

**Lemma 8.** In Algorithm 3, the following inequality holds for all \(t \in [T]\).

\[
\frac{1}{2N} \omega_t \leq \frac{6L_{22}^2}{\mu_1^2} \omega_t + \frac{6L_{12}^2}{\mu_1^2} \epsilon_t + \frac{6L_{12}^2}{\mu_1} \epsilon_{t-1}. \tag{80}
\]

**Proof.** As stated in (45), we have

\[
\mu_1 (\omega_i^{t+1} - \omega_i^t) = \nabla_\omega f_i (\omega_i^{t+1}, \psi_i^{t+1}) - \nabla_\omega f_i (\omega_i^t, \psi_i^t), \tag{81}
\]
which is further simplified and results in

\begin{align}
\mu_1 \left( \omega_i^{t+1} - \omega_0^i \right) &= \nabla \Phi_i \left( \omega_i^{t+1} \right) - \nabla \Phi_i \left( \omega_i^t \right) + \nabla \omega f_i \left( \omega_i^{t+1}, \psi_i^{t+1} \right) \\
&\quad - \nabla \Phi_i \left( \omega_i^{t+1} \right) - \left( \nabla \omega f_i (\omega_i^t, \psi_i^t) - \nabla \Phi_i \left( \omega_i^t \right) \right). \quad (82)
\end{align}

By first taking norm on both sides of the above equation and then applying the triangle inequality on the r.h.s, we further obtain the following inequality:

\begin{align}
\mu_1 \| \omega_i^{t+1} - \omega_0^i \| &\leq \| \nabla \Phi_i (\omega_i^{t+1}) - \Phi_i (\omega_i^t) \| + \| \nabla \omega f_i (\omega_i^{t+1}, \psi_i^{t+1}) - \Phi_i (\omega_i^{t+1}) \| \\
&\quad + \| \nabla \omega f_i (\omega_i^t, \psi_i^t) - \Phi_i (\omega_i^t) \|. \quad (83)
\end{align}

According to the claim in Assumption 5 of the Lipschitz condition, the r.h.s of the above inequality is further amplified, which leads to the following inequality:

\begin{align}
\| \omega_i^{t+1} - \omega_0^i \| &\leq \frac{L_{\Phi}}{\mu_1} \| \omega_i^{t+1} - \omega_i^t \| + \frac{L_{12}}{\mu_1} \| \psi_i^{t+1} - \psi_i^t (\omega_i^{t+1}) \| + \frac{L_{12}}{\mu_1} \| \psi_i^t - \psi_i^* (\omega_i^t) \|. \quad (84)
\end{align}

By replacing \( \omega_i^{t+1} \) with \( \omega_j^{t+1} \) and following the similar analysis, we get the following inequality:

\begin{align}
\| \omega_j^{t+1} - \omega_0^j \| &\leq \frac{L_{\Phi}}{\mu_1} \| \omega_j^{t+1} - \omega_j^t \| + \frac{L_{12}}{\mu_1} \| \psi_j^{t+1} - \psi_j^t (\omega_j^{t+1}) \| + \frac{L_{12}}{\mu_1} \| \psi_j^t - \psi_j^* (\omega_j^t) \|. \quad (85)
\end{align}

Summing the two previous inequalities along the same sign direction and then applying the triangle inequality results in:

\begin{align}
\| \omega_i^{t+1} - \omega_j^{t+1} \| &\leq \| \omega_i^{t+1} - \omega_i^t \| + \| \omega_j^{t+1} - \omega_j^t \| \\
&\leq \frac{L_{\Phi}}{\mu_1} \| \omega_i^{t+1} - \omega_i^t \| + \frac{L_{\Phi}}{\mu_1} \| \omega_j^{t+1} - \omega_j^t \| + \frac{L_{12}}{\mu_1} \| \psi_i^t - \psi_i^* (\omega_i^t) \| \\
&\quad + \frac{L_{12}}{\mu_1} \| \psi_j^t - \psi_j^* (\omega_j^t) \| + \frac{L_{12}}{\mu_1} \| \psi_j^{t+1} - \psi_j^{t+1} (\omega_j^{t+1}) \|. \quad (86)
\end{align}

Finally, by applying Cauchy-Schwarz inequality on the r.h.s of the above equation, we have

\begin{align}
\| \omega_i^{t+1} - \omega_j^{t+1} \|^2 &\leq \frac{6L_{\Phi}^2}{\mu_1^2} \| \omega_i^{t+1} - \omega_i^t \|^2 + \frac{6L_{12}^2}{\mu_1^2} \| \omega_j^{t+1} - \omega_j^t \|^2 \\
&\quad + \frac{6L_{12}^2}{\mu_1^2} \| \psi_i^t - \psi_i^* (\omega_i^t) \|^2 + \frac{6L_{12}^2}{\mu_1^2} \| \psi_j^t - \psi_j^* (\omega_j^t) \|^2 \\
&\quad + \frac{6L_{12}^2}{\mu_1^2} \| \psi_j^{t+1} - \psi_j^{t+1} (\omega_j^{t+1}) \|^2. \quad (87)
\end{align}

By summing up on \( i, j \) and replacing \( t + 1 \) with \( t \) in the above inequality, we have finally shown \( 80 \).

**Lemma 9.** In Algorithm 3 the following inequality holds for all \( t \in [T] \).

\begin{align}
\bar{\psi}_t &\leq A_1 \epsilon_t + A_2 \bar{\omega}_t + A_3 \tilde{\omega}_t + A_4 \tilde{\psi}_{t-1}. \quad (88)
\end{align}

**Proof.** By substituting the result of Lemma 6 into that of Lemma 7 we obtain

\begin{align}
\frac{\mu_3^2}{2N} \sigma - 4L_{23}^2 \bar{\omega}_t &\leq \frac{16L_{12}^2}{\mu_2^2} \epsilon_t + \frac{16L_{22}^2}{N\mu_2^2} (\mu_2 + L_{22})^2 \sigma + \frac{16L_{23}^2L_{21}}{N\mu_2^2} \bar{\omega}_t + \frac{4L_{22}^2}{N} \tilde{\psi}_{t-1}, \quad (89)
\end{align}

which is further simplified and results in

\begin{align}
\left( \frac{\mu_3^2}{2N} - \frac{16L_{22}^2}{N\mu_2^2} (\mu_2 + L_{22})^2 \right) \bar{\psi}_t &\leq \frac{16L_{12}^4}{\mu_2^2} \epsilon_t + \frac{16L_{23}^2L_{21}^2}{N\mu_2^2} \bar{\omega}_t + \frac{16L_{22}^2}{N} \tilde{\psi}_{t-1} + 4L_{23}^2 \bar{\omega}_t. \quad (90)
\end{align}

Then there must exit positive constants \( A_1, A_2, A_3, \) and \( A_4 \) such that

\begin{align}
\bar{\psi}_t &\leq A_1 \epsilon_t + A_2 \bar{\omega}_t + A_3 \tilde{\omega}_t + A_4 \tilde{\psi}_{t-1}, \quad (91)
\end{align}
where we have
\[
A_1 = 16L_{22}^4 N / \left( \mu_2^4 - 16L_{22}^2 (\mu_2 + L_{22})^2 \right),
\]
\[
A_2 = 16L_{22}^2 L_{21}^2 / \left( \mu_2^4 - 16L_{22}^2 (\mu_2 + L_{22})^2 \right),
\]
\[
A_3 = 16L_{22}^2 \mu_2^2 / \left( \mu_2^4 - 16L_{22}^2 (\mu_2 + L_{22})^2 \right),
\]
\[
A_4 = 4L_{21}^2 N \mu_2^2 / \left( \mu_2^4 - 16L_{22}^2 (\mu_2 + L_{22})^2 \right).
\]

**Lemma 10.** In Algorithm 1, the following inequality holds for all \( t \in [T] \):
\[
\epsilon_t + B_1 \tilde{\psi}_t \leq B_2 (\epsilon_t + B_1 \psi_t) + B_3 \tilde{\omega}_t.
\]

**Proof.** By substituting the result of Lemma 8 back into the result of Lemma 5, we have
\[
\epsilon_t \leq \frac{8(\mu_2 + L_{22})^2}{NB \mu_2} \tilde{\psi}_t + \frac{8 \mu_2}{NB} \psi_{t-1} + \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} \epsilon_t + \left( \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} + \frac{2 \mu_2}{2 \mu_2 + B} \right) \epsilon_{t-1} + \left( \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} + \frac{8 \mu_2 \kappa_2^2}{B} \right) \tilde{\omega}_t.
\]

From the above inequality, there must exist \( \mu_1 \) and \( \mu_2 \) that construct the constants \( A_5, A_6, A_7, \) and \( A_8 \) such that
\[
\epsilon_t \leq A_5 \tilde{\psi}_t + A_6 \psi_{t-1} + A_7 \epsilon_{t-1} + A_8 \tilde{\omega}_t.
\]

After some tedious algebra manipulations from the formulas of \( A_1 - A_8 \), we obtain
\[
A_5 = \frac{8(\mu_2 + L_{22})^2}{NB \mu_2} / \left( 1 - \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} \right),
\]
\[
A_6 = \frac{8 \mu_2}{NB} / \left( 1 - \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} \right),
\]
\[
A_7 = \left( \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} + \frac{2 \mu_2}{2 \mu_2 + B} \right) / \left( 1 - \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} \right),
\]
\[
A_8 = \left( \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} + \frac{8 \mu_2 \kappa_2^2}{B} \right) / \left( 1 - \frac{96L_{21}^2 L_{12}^2}{\mu_2 \mu_1^2 B} \right).
\]

Then by substituting the result of Lemma 8 into that of Lemma 9, we have
\[
\tilde{\psi}_t \leq A_9 \epsilon_t + A_{10} \epsilon_{t-1} + A_4 \tilde{\psi}_{t-1} + A_{11} \tilde{\omega}_t.
\]

where \( A_9, A_{10}, \) and \( A_{11} \) are also positive constants, i.e.,
\[
A_9 = A_1 + A_3 \frac{12NL_{12}^2}{\mu_1^2},
\]
\[
A_{10} = A_3 \frac{12NL_{12}^2}{\mu_1^2},
\]
\[
A_{11} = A_2 + A_3 \frac{12NL_{12}^2}{\mu_1^2}.
\]

By taking tedious manipulation on the formulas of \( A_9, A_{10}, A_{11} \) from \( A_1 - A_8 \), we calculate \( A_9, A_{10}, A_{11} \) as the following forms
\[
A_9 = 16L_{22}^4 N / \left( \mu_2^4 - 16L_{22}^2 (\mu_2 + L_{22})^2 \right) + 192L_{22}^2 L_{12}^2 \mu_2^2 / \left( \mu_2^4 \mu_1^2 - 16L_{22}^2 \mu_1^2 (\mu_2 + L_{22})^2 \right),
\]
\[
A_{10} = 192L_{22}^2 L_{12}^2 \mu_2^2 / \left( \mu_2^4 \mu_1^2 - 16L_{22}^2 \mu_1^2 (\mu_2 + L_{22})^2 \right),
\]
\[
A_{11} = 16L_{22}^2 L_{21}^2 / \left( \mu_2^4 - 16L_{22}^2 (\mu_2 + L_{22})^2 \right) + 192L_{22}^2 L_{12}^2 \mu_2^2 / \left( \mu_2^4 \mu_1^2 - 16L_{22}^2 \mu_1^2 (\mu_2 + L_{22})^2 \right).
\]
By computing $\psi_t \leq (1 - \bar{B}_1)\epsilon_t + (\bar{B}_1 - A_5)\bar{\psi}_t \leq (\bar{B}_1 A_{10} + A_7)\epsilon_{t-1} + (\bar{B}_1 A_4 + A_6)\psi_{t-1} + (\bar{B}_1 A_{11} + A_8)\bar{\psi}_t$, (103)

And we further scale to the following form

$$\epsilon_t + B_1\bar{\psi}_t \leq B_2(\epsilon_t + B_1\bar{\psi}_t) + B_3\bar{\psi}_t, \tag{104}$$

where

$$B_1 = \frac{\bar{B}_1 - A_5}{1 - B_1 A_9}, \tag{105}$$

$$B_2 = \max\left\{\frac{\bar{B}_1 A_{10} + A_7}{1 - B_1 A_9}, \frac{\bar{B}_1 A_4 + A_6}{1 - A_5}\right\}, \tag{106}$$

$$B_3 = \frac{\bar{B}_1 A_{11} + A_8}{1 - B_1 A_9}. \tag{107}$$

We prove Lemma 2 as follows. We repeat Lemma 2 in the following Lemma 11 to make the appendix self-contained.

**Lemma 11.** In Algorithm 3, the following inequality holds.

$$\sum_{t=1}^{T} \sum_{i=0}^{N} ||\psi_i^t - \psi^*(\omega_i^{t-1})||^2 \leq C_1 \sum_{i=1}^{T} ||\omega_i^{t-1} - \omega_i^t||^2 + C_2 \sum_{i=1}^{N} ||\psi_i^0 - \psi^*(\omega_i^0)||^2$$

$$+ C_3 \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} ||\omega_i^0 - \omega_j^0||^2. \tag{108}$$

**Proof.** By the recursive update (96) in Lemma 10, we obtain

$$\epsilon_t + B_1\bar{\psi}_t \leq B_2(\epsilon_0 + B_1\bar{\psi}_0) + \sum_{t'=0}^{t} B_3 B_2^{t-t'}\bar{\psi}_t. \tag{109}$$

By summing up on $t$, we get

$$\sum_{t=1}^{T} (\epsilon_t + B_1\bar{\psi}_t) \leq \sum_{t=1}^{T} B_2^t(\epsilon_0 + B_1\bar{\psi}_0) + \sum_{t=1}^{T} \sum_{t'=1}^{T} B_3 B_2^{t-t'}\bar{\psi}_t, \tag{110}$$

where the r.h.s. can be further amplified by the sum of finite exponential series, which result in

$$\sum_{t=1}^{T} (\epsilon_t + B_1\bar{\psi}_t) \leq \frac{1}{1 - B_2} (\epsilon_0 + B_1\bar{\psi}_0) + \frac{B_3}{1 - B_2} \sum_{t=1}^{T} \omega_t. \tag{111}$$

Finally, we reach

$$\sum_{t=1}^{T} \epsilon_t \leq C_1 \sum_{t=1}^{T} \omega_t + C_2\epsilon_0 + C_3\bar{\psi}_0, \tag{112}$$

where

$$C_1 = \frac{B_3}{1 - B_2}, \tag{113}$$

$$C_2 = \frac{1}{1 - B_2} \tag{114}$$

$$C_3 = \frac{B_1}{1 - B_2}. \tag{115}$$
A.6 Proof of Lemma 3

Lemma 12. (One Global Round Descent on $L_{\Phi}$) In Algorithm 3 the following inequality holds.

\[
\frac{1}{N} \sum_{i=1}^{N} \left( L_{\Phi}^i (\omega_{t+1}^i, \omega_{0}^i, \lambda_{t}^i) - L_{\Phi}^i (\omega_{0}^i, \lambda_{t}^i) \right) \leq \frac{1}{N} \sum_{i=1}^{N} \left( - \frac{\mu_1}{2} (\omega_{t+1}^i - \omega_{t}^i)^2 + \frac{1}{\mu_1} \|\lambda_{t+1}^i - \lambda_{t}^i\|^2 + \frac{L_{\Phi}^2}{2} \|\psi_{t+1}^i - \psi^* (\omega_{t+1}^i)\|^2 - \frac{\mu_1}{2} (\omega_{t+1} - \omega_{t})^2 \right). \tag{116}
\]

Proof. From the $L_{\Phi}$-Lipschitz on $\Phi_i(\omega)$, we have

\[
-\Phi_i(\omega_{t}^i) \leq -\Phi_i(\omega_{t+1}^i) + \langle -\nabla \Phi_i (\omega_{t+1}^i), \omega_{t}^i - \omega_{t+1}^i \rangle + \frac{L_{\Phi}}{2} \|\omega_{t}^i - \omega_{t+1}^i\|^2. \tag{117}
\]

By considering the definition of $L_{\Phi}^i (\omega_{t}^i, \omega_{0}^i, \lambda_{t}^i)$ in [28], we obtain the following inequalities to bound one global round descent, i.e.,

\[
L_{\Phi}^i (\omega_{t+1}^i, \omega_{0}^i, \lambda_{t}^i) - L_{\Phi}^i (\omega_{0}^i, \lambda_{t}^i) \leq \langle \nabla \Phi_i (\omega_{t+1}^i, \omega_{0}^i - \omega_{t}^i), \phi_{t+1}^i \rangle + \frac{L_{\Phi}}{2} \|\omega_{t+1}^i - \omega_{t}^i\|^2 + \langle \lambda_{t}^i, \omega_{t+1}^i - \omega_{t}^i \rangle + \frac{\mu_1}{2} \|\omega_{t+1}^i - \omega_{0}^i\|^2 - \frac{\mu_1}{2} \|\omega_{t}^i - \omega_{0}^i\|^2.
\]

(118)

where the second last inequality is due to the inequality of arithmetic and geometric means (AM-GM). Next, from the iteration on $\lambda_{t}^i$ in Algorithm 3 we bound on the ascent from the iteration $\lambda_{t}^i$, i.e.,

\[
L_{\Phi}^i (\omega_{t+1}^i, \omega_{0}^i, \lambda_{t}^i) - L_{\Phi}^i (\omega_{0}^i, \lambda_{t}^i) = \langle \lambda_{t+1}^i - \lambda_{t}^i, \omega_{t+1}^i - \omega_{0}^i \rangle = \frac{1}{\mu_1} \|\lambda_{t+1}^i - \lambda_{t}^i\|^2. \tag{119}
\]

Finally, we bound on the descent from the iteration on $\omega_{0}^i$ in Algorithm 3 we have

\[
\sum_{i=1}^{N} \left( L_{\Phi}^i (\omega_{t+1}^i, \omega_{0}^i, \lambda_{t}^i) - L_{\Phi}^i (\omega_{t}^i, \omega_{0}^i, \lambda_{t}^i) \right) = \frac{\mu_1}{2} N \|\omega_{t}^i - \omega_{0}^i\|^2 + \frac{1}{\mu_1} N \sum_{i=1}^{N} \|\omega_{t}^i + \frac{1}{\mu_1} \lambda_{t}^i\|^2. \tag{120}
\]

According to the definition of $\omega_{0}^i$, i.e., $\omega_{0}^i = \frac{1}{N} \sum_{i=1}^{N} (\omega_{t}^i + \frac{1}{\mu_1} \lambda_{t}^i)$, we have

\[
\sum_{i=1}^{N} \left( L_{\Phi}^i (\omega_{t+1}^i, \omega_{0}^i, \lambda_{t}^i) - L_{\Phi}^i (\omega_{t}^i, \omega_{0}^i, \lambda_{t}^i) \right) = \frac{\mu_1}{2} N \|\omega_{t}^i - \omega_{0}^i + \frac{1}{\mu_1} \lambda_{t}^i\|^2. \tag{121}
\]

Making a summation of (118), (119), (120), and (121) along the same sign direction of the inequalities, and after some algebraic manipulations, we prove (185).
Lemma 13. (Bound on $\lambda_i$’s Iteration) In Algorithm 3, the following inequality holds for all $i \in [N]$ and $t \in [T]$.

$$
\|\lambda_i^t - \lambda_i^{t+1}\| \leq 2L_F \|\omega_i^t - \omega_i^{t+1}\|^2 + 4L_{12} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\|^2 + 4L_{12} \|\psi_i^t - \psi^* (\omega_i^t)\|^2.
$$

(122)

Proof. By simply replacing $t + 1$ with $t$ in the above inequality, we further obtain

$$
\lambda_i^t - \lambda_i^{t+1} = \nabla \omega f_i (\omega_i^{t+1}, \psi_i^{t+1}) - \nabla \omega f_i (\omega_i^t, \psi_i^t).
$$

(123)

Taking the norm on both sides, we have

$$
\|\lambda_i^t - \lambda_i^{t+1}\| = \| - \nabla \Phi_i (\omega_i^{t+1}) + \nabla \Phi_i (\omega_i^t) - \nabla \omega f_i (\omega_i^{t+1}, \psi_i^{t+1}) + \nabla \Phi_i (\omega_i^t) \|
$$

$$
\leq \| - \nabla \Phi_i (\omega_i^{t+1}) - \nabla \Phi_i (\omega_i^t) \| + \| \nabla \omega f_i (\omega_i^{t+1}, \psi_i^{t+1}) - \nabla \Phi_i (\omega_i^{t+1}) \|
$$

$$
\leq L_F \|\omega_i^t - \omega_i^{t+1}\| + L_{12} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\| + L_{12} \|\psi_i^t - \psi^* (\omega_i^t)\|,
$$

(124)

where the first inequality is due to triangle inequality, and the second is due to the Lipschitz continuous property. By applying Cauchy-Schwarz inequality on the last inequality of (123), we finally prove (126).

Lemma 14. (Descent on $L_\Phi$) In Algorithm 3, the following inequality holds.

$$
\sum_{t=1}^{T} \sum_{i=1}^{N} \left( L_i^\Phi (\omega_i^T, \omega_i^0, \lambda_i^t) - L_i^\Phi (\omega_i^0, \omega_i^0, \lambda_i^t) \right)
$$

$$
\leq - \mu_i^2 - 2L_\Phi \mu_i - 4L^2 \frac{N \mu_i}{2} \sum_{t=1}^{T} \|\omega_i^{t+1} - \omega_i^t\|^2 - \sum_{t=1}^{T} \sum_{i=1}^{N} \mu_i \|\omega_i^0 - \omega_i^{t+1}\|^2
$$

(125)

+ 2(\mu_1 + 8L_\Phi) \frac{L_{12}^2}{L_\Phi \mu_1} \sum_{i=1}^{N} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\|^2.

Proof. By summing all the inequalities of (185) in Lemma 12 along the same sign direction for all $t = [T]$, we obtain

$$
\sum_{i=1}^{N} \left( L_i^\Phi (\omega_i^{t+1}, \omega_i^0, \lambda_i^{t+1}, \lambda_i^t) \right)
$$

$$
\leq \sum_{i=1}^{N} \sum_{t=1}^{T} - \mu_i^2 - 2L_\Phi \mu_i \|\omega_i^{t+1} - \omega_i^t\|^2 + \frac{1}{\mu_i} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\lambda_i^{t+1} - \lambda_i^t\|^2
$$

$$
+ L_{12}^2 \sum_{i=1}^{N} \sum_{t=1}^{T} \|\psi_i^{t+1} - \psi^* (\omega_i^{t+1})\|^2 - \frac{N \mu_i}{2} \|\omega_i^0 - \omega_i^{t+1}\|^2.
$$

(126)

By substituting (186) in Lemma 13 into the above inequality, we have proved (125).

Lemma 15. (Lower Bound on $L_i^\Phi$) In Algorithm 3, the following inequality holds for all $t \in [T]$ and $i \in [N]$.

$$
L_i^\Phi (\omega_i^t, \omega_i^0, \lambda_i^t) \geq \Phi_i (\omega_i^0) - \frac{L_{12}^2}{L_\Phi} \| \psi_i^t - \psi^* (\omega_i^t) \|^2 + \frac{\mu_i}{2} - \frac{2L_\Phi}{2} \|\omega_i^0 - \omega_i^t\|^2.
$$

(127)

Proof. From the $L_\Phi$-Lipschitz on $\nabla \Phi_i (\omega)$, we have

$$
\Phi_i (\omega_i^0) - \Phi_i (\omega_i^t) \leq \langle \nabla \Phi_i (\omega_i^t), \omega_i^0 - \omega_i^t \rangle + \frac{L_\Phi}{2} \|\omega_i^0 - \omega_i^t\|^2.
$$

(128)
Then we have
\[ L^\Phi_i(\omega^t_i, \omega^0_i, \lambda^0_i) - \Phi_i(\omega^0_i) \geq \langle \nabla \Phi_i(\omega^t_i) + \lambda^0_i, \omega^0_i - \omega^t_i \rangle + \frac{\mu_1 - L\Phi}{2} \| \omega^0_i - \omega^t_i \|^2 \]
\[ \geq \langle \nabla \Phi_i(\omega^t_i) - \nabla \omega f_i(\omega^t_i, \psi^t_i), \omega^0_i - \omega^t_i \rangle + \frac{\mu_1 - L\Phi}{2} \| \omega^0_i - \omega^t_i \|^2. \]  
(129)

Besides, by applying the AM-GM inequality, we obtain
\[ \langle \nabla \Phi_i(\omega^t_i) - \nabla \omega f_i(\omega^t_i, \psi^t_i), \omega^0_i - \omega^t_i \rangle \geq - \frac{1}{2L\Phi} \| \nabla \Phi_i(\omega^t_i) - \nabla \omega f_i(\omega^t_i, \psi^t_i) \|^2 \]
\[ \geq - \frac{L^2}{2L\Phi} \| \psi^t_i - \psi^* (\omega^t_i) \|^2. \]  
(130)

Substituting the above result back to (128), we have proved (127).

We prove Lemma 3 as follows. Note that we repeat Lemma 3 in the following Lemma 16 to make the appendix self-contained.

**Lemma 16. (Descent on \( \Phi \)) In Algorithm 3, the following inequality holds.**

\[ \Phi(\omega^T_0) - \Phi(\omega^0_0) \leq - \frac{\mu_1^2 - 2L\Phi \mu_1 - 4L^2}{2\mu_1 N} \sum_{i=1}^{N} \sum_{t=1}^{T} \| \omega^t_{i+1} - \omega^t_i \|^2 - \frac{\mu_1}{4} \sum_{i=1}^{T} \| \omega^0_i - \omega^0_0 \|^2 \]
\[ + \frac{3 \mu_1 + 16L\Phi}{\mu_1 NL\Phi} \sum_{i=1}^{N} \sum_{t=1}^{T} \| \psi^t_1 - \psi^* (\omega^t_1) \|^2. \]  
(131)

**Proof.** From the initial condition of \( \lambda^0_i = 0 \) and \( \omega^0_i = \omega^0_0 \), we have
\[ \Phi_i(\omega^0_0) = \frac{1}{N} \sum_{i=1}^{N} L^\Phi_i(\omega^0_i, \omega^0_0, \lambda^0_i). \]  
(132)

Substituting (127) in Lemma 3 into (125) in Lemma 14 results in
\[ \frac{1}{N} \sum_{i=1}^{N} L^\Phi_i(\omega^t_i, \omega^0_i, \lambda^0_i) \geq \Phi(\omega^0_0) - \frac{L^2}{NL\Phi} \sum_{i=1}^{N} \| \psi^t_i - \psi^* (\omega^t_i) \|^2 + \sum_{i=1}^{N} \frac{\mu_1 - 2L\Phi}{2N} \| \omega^0_i - \omega^0_0 \|^2 \]
\[ \geq \Phi(\omega^0_0) - \frac{L^2}{NL\Phi} \sum_{i=1}^{N} \| \psi^t_i - \psi^* (\omega^t_i) \|^2. \]  
(133)

By substituting the above result into (125) in Lemma 14 we have proved (131).

**A.7 Proof of Theorem 1**

**Lemma 17.** In Algorithm 1, the following inequality holds for all \( t \in [T] \) and \( i \in [N] \).
\[ \| \omega^t_i - \omega^0_i \| \leq \| \omega^{t-1}_0 - \omega^0_i \| + \frac{L\Phi}{\mu_1} \| \omega^{t-1}_i - \omega^t_i \| + \frac{L^2}{\mu_1} \| \psi^t_i - \psi^* (\omega^t_i) \| + \frac{L^2}{\mu_1} \| \psi^{t-1} + \psi^* (\omega^{t-1}_i) \| \]  
(134)

**Proof.** Using the triangle inequality, we have
\[ \| \omega^{t+1}_i - \omega^0_i \| \leq \| \omega^{t+1}_0 - \omega^0_i \| + \| \omega^{t+1}_i - \omega^t_i \|. \]  
(135)

Substituting (84) into the above inequality results in
\[ \| \omega^{t+1}_i - \omega^0_i \| \leq \| \omega^{t+1}_0 - \omega^0_i \| + \frac{L\Phi}{\mu_1} \| \omega^{t+1}_i - \omega^t_i \| + \frac{L^2}{\mu_1} \| \psi^t_i - \psi^* (\omega^t_i) \| + \frac{L^2}{\mu_1} \| \psi^{t+1}_i + \psi^* (\omega^{t+1}_i) \|. \]  
(136)

By replacing \( t + 1 \) with \( t \), we have proved (189).
Lemma 18. (Bounded $||\nabla \Psi (\omega_0^t)||^2$) In Algorithm 4, the following inequality holds for $t \in [T]$.

$$
\begin{align*}
||\nabla \Phi (\omega_0^t)|| &\leq L_{\Phi} ||\omega_0^t - \omega_0^{t-1}|| + \frac{\mu_1^2 + L_2^2}{N\mu_1} \sum_{i=1}^{N} ||\omega_i^{t-1} - \omega_i^t|| + \frac{L_{12}(L_{\Phi} + \mu_1)}{N\mu_1} \sum_{i=1}^{N} ||\psi_i^{t-1} - \psi^*(\omega_i^{t-1})|| \\
&\quad + \frac{L_{12}L_{\Phi}}{N\mu_1} \sum_{i=1}^{N} ||\psi_i^t - \psi^*(\omega_i^t)|| .
\end{align*}
$$

(137)

Proof. From Lemma 36, we have

$$
\sum_{i=1}^{N} \omega_i^t = \sum_{i=1}^{N} \omega_i^{t-1} - \frac{1}{\mu_1} \sum_{i=1}^{N} \nabla \omega f_i (\omega_i^t, \psi_i^t),
$$

(138)

which can be equivalently represented by

$$
\begin{align*}
\sum_{i=1}^{N} \omega_i^t &= \sum_{i=1}^{N} \omega_i^{t-1} - \frac{1}{\mu_1} \sum_{i=1}^{N} (\nabla \Phi_i (\omega_i^t) + (\nabla \Phi_i (\omega_i^t) - \nabla \Phi_i (\omega_0^t))) \\
&\quad + (\nabla \omega f_i (\omega_i^t, \psi_i^t) - \nabla \Phi_i (\omega_i^t)) .
\end{align*}
$$

(139)

Next, we get

$$
\nabla \Phi (\omega_0^t) = \frac{\mu_1}{N} \sum_{i=1}^{N} \omega_i^{t-1} - \frac{\mu_1}{N} \sum_{i=1}^{N} \nabla \Phi_i (\omega_i^t) + \frac{1}{N} (\nabla \omega f_i (\omega_i^t, \psi_i^t) - \nabla \Phi_i (\omega_i^t)) .
$$

(140)

According to the triangle inequality, we have

$$
||\nabla \Phi (\omega_0^t)|| \leq \frac{\mu_1}{N} \sum_{i=1}^{N} ||\omega_i^t - \omega_i^{t-1}|| + \frac{\mu_1}{N} ||\nabla \Phi_i (\omega_i^t) - \nabla \Phi_i (\omega_0^t)|| + \frac{1}{N} ||\nabla \omega f_i (\omega_i^t, \psi_i^t) - \nabla \Phi_i (\omega_i^t)|| .
$$

(141)

By sing the Lipschitz condition, we further obtain

$$
||\nabla \Phi (\omega_0^t)|| \leq \frac{L_{\Phi}}{N} \sum_{i=1}^{N} ||\omega_i^t - \omega_i^{t-1}|| + \frac{\mu_1}{N} \sum_{i=1}^{N} ||\omega_i^{t-1} - \omega_i^t|| + \frac{L_{12}}{N} \sum_{i=1}^{N} ||\psi_i^{t-1} - \psi^*(\omega_i^{t-1})|| .
$$

(142)

By substituting (139) in Lemma 17 into the above inequality, we have proved (190). □

Lemma 19. In Algorithm 4, the following inequality holds.

$$
\begin{align*}
\sum_{t=1}^{T} ||\nabla \Phi (\omega_0^t)||^2 &\leq \frac{4L_2^2\mu_1^2 N + 4C_1 L_{12}^2 ((L_{\Phi} + \mu_1)^2 + L_{\Phi}^2)}{N\mu_1^2} \sum_{t=1}^{T} ||\omega_0^t - \omega_0^{t-1}||^2 \\
&\quad + \frac{4(\mu_1^2 + L_2^2)^2}{N\mu_1^2} \sum_{i=0}^{T} \sum_{t=1}^{T} ||\omega_i^{t-1} - \omega_i^t||^2 \\
&\quad + \frac{4C_2 L_{12}^2 ((L_{\Phi} + \mu_1)^2 + L_{\Phi}^2)}{N\mu_1^2} \sum_{i=1}^{N} ||\psi_i^0 - \psi^*(\omega_i^0)||^2 \\
&\quad + \frac{4C_3 L_{12}^2 ((L_{\Phi} + \mu_1)^2 + L_{\Phi}^2)}{N\mu_1^2} \sum_{i=1}^{N} \sum_{j \neq i} ||\omega_i^0 - \omega_j^0||^2 .
\end{align*}
$$

(143)
Proof. According to the Cauchy-Schwarz inequality, we have

\[
\| \nabla \Phi(\omega_0^t) \|^2 \leq 4 L_\Phi^2 \| \omega_0^t - \omega_0^{t-1} \|^2 + \frac{4(\mu_1^2 + L_\Phi^2)^2}{N \mu_1^2} \sum_{i=1}^{N} \| \omega_{i}^{t-1} - \omega_i^t \|^2 \\
+ \frac{4 L^2 T (L_\Phi + \mu_1)^2}{N \mu_1^2} \sum_{i=1}^{N} \| \psi_i^{t-1} - \psi^*(\omega_i^{t-1}) \|^2 \\
+ \frac{4 L^2 T^2}{N \mu_1^2} \sum_{i=1}^{N} \| \psi_i^t - \psi^*(\omega_i^t) \|^2.
\]

(144)

By making a summation of all the inequalities for \( t \in [T] \), we further obtain

\[
\sum_{t=1}^{T} \| \nabla \Phi(\omega_0^t) \|^2 \leq 4 L_\Phi^2 \sum_{t} \| \omega_0^t - \omega_0^{t-1} \|^2 + \frac{4(\mu_1^2 + L_\Phi^2)^2}{N \mu_1^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \| \omega_{i}^{t-1} - \omega_i^t \|^2 \\
+ \frac{4 L^2 T (L_\Phi + \mu_1)^2 + L_\Phi^2}{N \mu_1^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \| \psi_i^{t-1} - \psi^*(\omega_i^{t-1}) \|^2 \\
+ \frac{4 L^2 T^2}{N \mu_1^2} \sum_{i=1}^{N} \| \psi_i^t - \psi^*(\omega_i^t) \|^2.
\]

(145)

By substituting (108) in Lemma 11 into (190) in Lemma 18 and further substituting the result into the r.h.s. of (145), we prove (143).

We are now ready to prove Theorem 1, which is replicated in the following Theorem 2 to make the appendix self-contained.

**Theorem 2.** In Algorithm 3, the following inequality holds.

\[
\Phi(\omega_0^0) - \Psi(\omega_0^T) \leq -E_1 \sum_{t=1}^{T} \| \nabla \Phi(\omega_0^t) \|^2 + E_2 \sum_{i=1}^{N} \| \psi_i^0 - \psi^*(\omega_0^0) \|^2 + E_3 \sum_{i=1}^{N} \sum_{j \neq i} \| \omega_i^0 - \omega_j^0 \|^2.
\]

(146)

**Proof.** By substituting the result of Lemma 11 into (131) in Lemma 16 and after some algebraic manipulations, we obtain

\[
\Phi(\omega_0^0) - \Phi(\omega_0^T) \leq \left( -\frac{\mu_1^2 - 2 L_\Phi \mu_1 - 4 L_\Phi^2}{2 \mu_1 N} \right) \sum_{i=1}^{N} \| \omega_{i}^{t+1} - \omega_i^t \|^2 \\
- \frac{\mu_1}{4} \sum_{t=1}^{T} \| \omega_0^t - \omega_0^{t+1} \|^2 + \frac{C_1 (3 \mu_1 + 16 L_\Phi) L_{12}^2}{2 \mu_1} \sum_{i=1}^{N} \| \psi_i^0 - \psi^*(\omega_i^0) \|^2 \\
+ \frac{C_2 (3 \mu_1 + 16 L_\Phi) L_{12}^2}{2 \mu_1} \sum_{i=1}^{N} \| \omega_i^0 - \omega_j^0 \|^2.
\]

(147)

By substituting (143) in Lemma 19 into the r.h.s. of the above equation, we have

\[
\Phi(\omega_0^0) - \Phi(\omega_0^T) \leq -E_1 \sum_{t=1}^{T} \| \nabla \Phi(\omega_0^t) \|^2 + E_2 \sum_{i=0}^{N} \| \psi_i^0 - \psi^*(\omega_i^0) \|^2 + E_3 \sum_{i=0}^{N} \sum_{j \neq i} \| \omega_i^0 - \omega_j^0 \|^2.
\]

(148)
where

\[ E_1 = \min \left\{ \frac{D_3}{D_1}, \frac{D_4}{D_2} \right\}, \]  

(149)

\[ E_2 = \frac{4E_1 C_2 ((L_\Phi + \mu_1)^2 + L_\Phi^2) L_{12}^2}{N \mu_1^3} + \frac{C_2 (3\mu_1 + 16L_\Phi) L_{12}^2}{NL_\Phi \mu_1}, \]  

(150)

\[ E_3 = \frac{4E_1 C_3 ((L_\Phi + \mu_1)^2 + L_\Phi^2) L_{12}^2}{N \mu_1^3} + \frac{C_3 (3\mu_1 + 16L_\Phi) L_{12}^2}{NL_\Phi \mu_1}, \]  

(151)

\[ D_1 = \frac{4L_\Phi^2 \mu_1^3 N + 4C_1 L_{12}^2 (L_\Phi + \mu_1)^2}{N \mu_1^3}, \]  

(152)

\[ D_2 = \frac{4C_4 L_{12}^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1^2}, \]  

(153)

\[ D_3 = \frac{\mu_1^2 - 2L_\Phi \mu_1 - 4L_\Phi^2}{2\mu_1 N} - \frac{C^2_1 (3\mu_1 + 16L_\Phi) L_{12}^2}{NL_\Phi \mu_1}, \]  

(154)

\[ D_4 = \frac{\mu_1}{4}. \]  

(155)

In particular, taking \( \lim \inf \) on both sides of the inequality w.r.t. \( T \), we have:

\[ \Phi(\omega_0^0) - \Psi(\omega_0^T) \leq \lim \inf_{T \to \infty} -E_1 \sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2 + E_2 \sum_{i=1}^{N} \left\| \psi_i^0 - \psi^* (\omega_i^0) \right\|^2 + E_3 \sum_{i=1}^{N} \sum_{j \neq i} \left\| \omega_i^0 - \omega_j^0 \right\|^2. \]  

(156)

Note that in the inequality (156), all the other terms are independent of \( T \) except for \( \lim \inf_{T \to \infty} -E_1 \sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2 \). Rearranging the terms independent of \( T \) to the other side, yielding:

\[ \lim \inf_{T \to \infty} -E_1 \sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2 \geq C, \]

where \( C \) is a constant independent of \( T \). In particular, this implies that

\[ \lim \sup_{T \to \infty} \sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2 \leq -C/E_1. \]

Given that the sequence \( \{ \left\| \nabla \Phi (\omega_t^0) \right\|^2 \} \) is nonnegative, we must have

\[ \lim_{t \to \infty} \left\| \nabla \Phi (\omega_t^0) \right\|^2 = 0, \]

which completes the proof. 

A.8 Discussion on the Range of \( \mu_1 \) and \( \mu_2 \) for Theoretically Convergence Guarantee

Let we chose a large enough \( l \) such that

\[ l \geq \max \left\{ \frac{L_{22}}{B}, \frac{L_{12}}{B} \right\}, \]  

(157)

By choosing \( \mu_1 \) and \( \mu_2 \) in the following range,

\[ \max \{64L_{22}l, 64L_{12}l\} \leq \mu_2 \leq 64l^2 B \]  

(158)

\[ \mu_1 \geq \max \left\{ \frac{24576}{B\mu_2}, \frac{512N}{1600}, \frac{512N}{4L_\Phi}, \frac{7 \cdot 164092^2l^2 \kappa^4 L_{12}^2}{L_\Phi} \right\} \]  

(159)

**Lemma 20.** (Conditions on \( B_2 < 1 \)) Let we chose a large enough \( l \) such that

\[ l \geq \max \left\{ \frac{L_{22}}{B}, \frac{L_{12}}{B} \right\}. \]  

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With the following assumptions holds,

\[
\frac{L_{22}}{\mu_2} \leq \frac{1}{64l},
\]
\[
\frac{L_{12}}{\mu_2} \leq \frac{1}{64l},
\]
\[
B \geq \frac{1}{64l^2},
\]
\[
\frac{192L_{21}^2L_{12}^2}{\mu_2^2B} \leq \frac{1}{128l^2},
\]
\[
\frac{L_{12}^2N}{1600\mu_1^2l^2} \leq \frac{1}{512l^2}.
\]

By properly choosing \(\bar{B}_1\) in Lemma 10, we have \(B_2 \leq 1\).

Proof. Following (92), we have

\[
A_1 = 16L_{22}^3N/(\mu_2^4 - 16L_{22}^2(\mu_2 + L_{22})^2) \leq \frac{N}{614400l^4},
\]
\[
A_3 = 16L_{22}^2\mu_2^2/(\mu_2^4 - 16L_{22}^2(\mu_2 + L_{22})^2) \leq \frac{1}{1600l^2}.
\]

So we have

\[
A_{10} = A_3 \frac{12NL_{12}^2}{\mu_1^2} \leq \frac{L_{12}^2}{1600\mu_1^2l^2},
\]
\[
A_7 = \left(\frac{96L_{21}^2L_{12}^2}{\mu_2^2}\right)/\left(1 - \frac{96L_{21}^2L_{12}^2}{\mu_2^2}\right) \leq \left(1 - \frac{B}{2\mu_2 + B} + \frac{192L_{21}^2L_{12}^2}{\mu_2^2}\right) \leq 1 - \frac{1}{128l^2},
\]
\[
A_9 = A_1 + A_3 \frac{12NL_{12}^2}{\mu_2^2} \leq \frac{N}{614400l^4} + \frac{L_{12}^2N}{1600\mu_1^2l^2},
\]
\[
A_6 = \frac{8\mu_2}{NB} \left(1 - \frac{96L_{21}^2L_{12}^2}{\mu_2^2}\right) \leq \frac{516}{Nl^2},
\]
\[
A_4 = 4L_{21}^2N\mu_2^2/(\mu_2^4 - 16L_{22}^2(\mu_2 + L_{22})^2) \leq \frac{N}{6400l^2},
\]
\[
A_5 = \frac{8(\mu_2 + L_{22})^2}{NB\mu_2} \left(1 - \frac{96L_{21}^2L_{12}^2}{\mu_2^2}\right) \leq \frac{573}{Nl^2}.
\]

By choosing \(\bar{B}_1 = \frac{1200}{Nl^2}\), we have

\[
\frac{\bar{B}_1A_{10} + A_7}{1 - \bar{B}_1A_9} \leq 1 - \frac{1}{512l^2},
\]
\[
\frac{\bar{B}_1A_4 + A_6}{B_1 - A_5} \leq \frac{6}{7}.
\]

Thus, we bound the decay factor \(B_2\) by

\[
B_2 = \max\left\{\frac{\bar{B}_1A_{10} + A_7}{1 - \bar{B}_1A_9}, \frac{\bar{B}_1A_4 + A_6}{B_1 - A_5}\right\} \leq 1 - \frac{1}{512l^2}.
\]
PROOF OF POSITIVE OF $E_1$. As we have

$$B_1 = \frac{\bar{B}_1A_{11} + A_8}{1 - \bar{B}_1A_9} \leq 516\kappa^2,$$

$$B_3 = \frac{\bar{B}_1 - A_5}{1 - \bar{B}_1A_9} \leq \frac{700}{N\ell^2}.$$

So we have

$$C_1 = \frac{B_3}{1 - B_2} \leq 164092\ell^2\kappa^2,$$

$$C_2 = \frac{1}{1 - B_2} \leq 512\ell^2,$$

$$C_3 = \frac{B_1}{1 - B_2} \leq \frac{358400}{N}.$$

As from Lemma [11] we have

$$C_1 \leq 164092\ell^2\kappa^2. \tag{163}$$

So the $D_3$ is lower bounded by

$$D_3 \geq \frac{\mu_1^2 - 2L\Phi\mu_1 - 4L^2\kappa^2}{2\mu_1 N} - 164092\ell^4\kappa^4(3\mu_1 + 16L\Phi)L^2_{12}. \tag{164}$$

By setting

$$\mu_1 \geq \max \left( 4L\Phi, \frac{7 \cdot 164092\ell^4\kappa^4L^2_{12}}{L\Phi} \right),$$

we finally prove that

$$D_3 \geq \frac{\mu_1}{8N} - \frac{7 \cdot 164092\ell^4\kappa^4L^2_{12}}{NL\Phi} \geq 0. \tag{165}$$

It is then evident that $E_1$, $E_2$, and $E_3$ are positive.

A.9 Convergence Analysis with Bounded Local Gradient Error

In the previous subsections, we assume that each client fully optimizes their local augmented Lagrangian function and assume $\|\nabla L_i(\omega_t^i, \psi_t^i)\| = 0$. In this subsection, we remove this assumption by assuming that there exists a local residue gradient error, i.e.,

$$\|\nabla L_i(\omega_t^i, \psi_t^i)\|^2 \leq \epsilon. \tag{166}$$

More specifically, we define the residue of gradient as

$$\nabla_{\omega} L_i(\omega_t^i, \psi_t^i) = e_{\omega,i}, \quad \nabla_{\psi} L_i(\omega_t^i, \psi_t^i) = e_{\psi,i}. \tag{167}$$

Following the above assumptions, the generalization of previous results is straightforward but tedious; thus, we provide the key results in the following directly. Proposition [11] can be simply generalized to the form that

$$\sum_{i=1}^{N} \psi_t^{i+1} = \sum_{i=1}^{N} \psi_t^i + \frac{1}{\mu_2} \sum_{i=1}^{N} \nabla_{\omega} f_i (\omega_t^{i+1}, \psi_t^{i+1}) - \frac{1}{\mu_1} \sum_{i=1}^{N} \omega_t^{i+1}. \tag{168}$$

Similarly, Proposition [2] becomes

$$\sum_{i=1}^{N} \omega_t^{i+1} = \sum_{i=1}^{N} \omega_t^i - \frac{1}{\mu_1} \sum_{i=1}^{N} \nabla_{\psi} f_i (\omega_t^{i+1}, \psi_t^{i+1}) - \frac{1}{\mu_1} \sum_{i=1}^{N} \omega_t^{i+1}. \tag{169}$$
We get a similar result as Lemma 5 by using the Cauchy-Schwarz inequality to the above equation, which is given by

$$\mu_2 \left( \psi_{t+1}^i - \psi_0^i \right) = \nabla \psi f_i \left( \omega_{t+1}, \psi_{t+1}^i \right) - \nabla \psi f_i \left( \omega_t, \psi_t^i \right) + e_{\psi,i}^t - e_{\psi,i}^{t+1}. \quad (170)$$

Finally, Proposition 4 can be simply generalized to

$$\mu_1 \left( \omega_{t+1}^i - \omega_0^i \right) = \nabla \omega f_i \left( \omega_t, \psi_t^i \right) - \nabla \omega f_i \left( \omega_{t+1}, \psi_{t+1}^i \right) - e_{\omega,i}^t + e_{\omega,i}^{t+1}. \quad (171)$$

Then by algebraic manipulations, Eqn. (64) is further generalized to

$$\left\| \psi_t^i - \psi^* \left( \omega_t^i \right) \right\| \leq \sqrt{\frac{\mu_2}{\mu_2 + B}} \left\| \psi_{t-1}^i - \psi^* \left( \omega_{t-1}^i \right) \right\| + \sqrt{\mu_2 \frac{1}{\mu_2 + B} \sum_{j=1}^{N} \left( \left\| \psi_{t-1}^j - \psi^* \left( \omega_{t-1}^j \right) \right\| + \sqrt{10 L_{21} N \mu_2} \sum_{j=1}^{N} \left\| \omega_{t-1}^j - \omega_{t-1}^j \right\| \right)} \quad (172)$$

We get a similar result as Lemma 5 by using the Cauchy-Schwarz inequality to the above equation, which is given by

$$\left\| \psi_t^i - \psi^* \left( \omega_t^i \right) \right\|^2 \leq \frac{2 \mu_2}{2 \mu_2 + B} \left\| \psi_{t-1}^i - \psi^* \left( \omega_{t-1}^i \right) \right\|^2 + \frac{10 \mu_2}{B} \left\| \psi_{t-1}^i - \psi_{t-1}^i \right\|^2 + \frac{10 \mu_2}{B} \sum_{j=1}^{N} \left( \left\| \psi_{t-1}^j - \psi_{t-1}^j \right\|^2 + \frac{10 L_{21}^2 N \mu_2}{B} \sum_{j=1}^{N} \left\| \omega_{t-1}^j - \omega_{t-1}^j \right\|^2 \right) \quad (173)$$

Similarly, Lemma 8 can be simply generalized to the following expression

$$\frac{1}{2N} \bar{\omega}_t \leq \frac{8 L_{\phi}^2 \mu_1}{\mu_1^2} \bar{\omega}_t + \frac{8 L_{\phi}^2 \mu_1}{\mu_1^2} \bar{\omega}_{t-1} + \frac{8}{\mu_1} \epsilon. \quad (174)$$

Then we reach a similar result as that of Lemma 9 given by

$$\bar{\psi}_t \leq A_1 \epsilon_t + A_2 \bar{\omega}_t + A_3 \bar{\omega}_{t-1} + A_4 \bar{\psi}_{t-1} + A_5 \epsilon. \quad (175)$$

Note that the value of $A_1, A_2, A_3, A_4$ is different with that of Lemma 9 because the coefficient is different in the previous equations. But the deriving steps are similar.

We get the same result as Lemma 10 by using similar algebraic manipulations:

$$\epsilon_t + B_1 \bar{\psi}_t \leq B_2 (\epsilon_t + B_3 \bar{\psi}_0) + B_3 \bar{\omega}_t + B_4 \epsilon. \quad (176)$$

Finding a $B_2 < 1$ would be achieved by following by the similar steps in section A.8 with $\mu_1$ and $\mu_2$ in given convergence range.

By recursive update and summing up on Eqn (176) from $t = 1$ to $T$, we have

$$\sum_{t=1}^{T} (\epsilon_t + B_1 \bar{\psi}_t) \leq \sum_{t=1}^{T} B_2^t (\epsilon_0 + B_1 \bar{\psi}_0) + \sum_{t=1}^{T} \sum_{t \leq t} B_3 B_2^{t-t'} \bar{\omega}_t + \sum_{t=1}^{T} \sum_{t \leq t} B_4 B_2^{t-t'} \epsilon. \quad (177)$$

By following the similar steps of amplifying by the sum of finite exponential series, we would reach

$$\sum_{t=1}^{T} (\epsilon_t + B_1 \bar{\psi}_t) \leq \frac{1}{1 - B_2} (\epsilon_0 + B_1 \bar{\psi}_0) + \frac{B_3}{1 - B_2} \sum_{t=1}^{T} \bar{\omega}_t + \frac{B_4}{1 - B_2} \sum_{t=1}^{T} \epsilon. \quad (178)$$
Finally, we reach
\[ \sum_{t=1}^{T} c_t \leq C_1 \sum_{t=1}^{T} \bar{\omega}_t + C_2 \epsilon_0 + C_3 \bar{\psi}_0 + C_4 \epsilon T, \]  
(179)
where
\[ C_1 = \frac{B_3}{1 - B_2}, \]  
(180)
\[ C_2 = \frac{1}{1 - B_2}, \]  
(181)
\[ C_3 = \frac{B_3}{1 - B_2}, \]  
(182)
\[ C_4 = \frac{B_3}{1 - B_2}. \]  
(183)

Note that the value of\( C_1, C_2, C_3 \) and \( C_4 \) is different from that value of \( \delta \mathcal{L}_i(\omega_i^t, \psi_i^t) = 0 \). Because the convergence range of \( \mu_1 \) and \( \mu_2 \) and \( B_1, B_2, B_3 \) is different in our analysis). Thus, we reach a result similar to Lemma[12] which is given by
\[ \sum_{t=1}^{T} \sum_{i=0}^{N} \|\psi_i^t - \psi^*(\omega_i^{t-1})\|^2 \leq C_1 \sum_{t=1}^{T} \|\omega_i^{t-1} - \omega_i^t\|^2 + C_2 \sum_{i=1}^{N} \|\psi_i^0 - \psi^*(\omega_i^0)\|^2 \\
+ C_3 \sum_{i=1}^{N} \sum_{j=1}^{N} \|\omega_i^0 - \omega_j^0\|^2 + TC_4 \epsilon. \]  
(184)

Next, following the similar step in Lemma[12] we obtain
\[ \frac{1}{N} \sum_{i=1}^{N} \left( \mathcal{L}_i^\Phi(\omega_i^{t+1}, \omega_i^0, \lambda_i^1) - \mathcal{L}_i^\Phi(\omega_i^0, \lambda_i^1) \right) \leq \frac{1}{N} \sum_{i=1}^{N} \left( - \frac{\mu_1 - 2L\Phi}{2} \|\omega_i^{t+1} - \omega_i^t\|^2 + \frac{1}{\mu_1} \|\lambda_i^{t+1} - \lambda_i^t\|^2 \right) \\
+ 2L_2^2 \|\psi_i^{t+1} - \psi^*(\omega_i^{t+1})\|^2 + 2\epsilon - \frac{\mu_1}{2} \|\omega_i^{t+1} - \omega_i^0\|^2. \]  
(185)

Then following the similar steps in the proof of Lemma[13] we have
\[ \|\lambda_i^t - \lambda_i^{t+1}\|^2 \leq 2L_2^2 \|\omega_i^t - \omega_i^{t+1}\|^2 + 8L_{12}^2 \|\psi_i^{t+1} - \psi^*(\omega_i^{t+1})\|^2 + 8L_{12}^2 \|\psi_i^t - \psi^*(\omega_i^t)\|^2 + 8\epsilon. \]  
(186)

Then we arrive at a similar conclusion with Lemma[14] that
\[ \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathcal{L}_i^\Phi(\omega_i^t, \omega_i^0, \lambda_i^t) - \mathcal{L}_i^\Phi(\omega_i^0, \lambda_i^t) \right) \leq - \frac{\mu_1 - 2L\Phi}{2\mu_1} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\omega_i^{t+1} - \omega_i^t\|^2 - \frac{N\mu_1}{4} \|\omega_i^0 - \omega_i^{t+1}\|^2 \\
+ \frac{4(\mu_1 + 8L\Phi)L_{12}^2}{L\Phi\mu_1} \sum_{i=1}^{N} \|\psi_i^{t+1} - \psi^*(\omega_i^{t+1})\|^2 + (2 + \frac{8}{\mu_1})T\epsilon, \]  
(187)
as well as a similar result as Lemma[3] which is given by
\[ \Phi(\omega_i^0) - \Phi(\omega_i^0) \leq - \frac{\mu_1 - 2L\Phi}{2\mu_1 N} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\omega_i^{t+1} - \omega_i^t\|^2 - \frac{N\mu_1}{4} \sum_{t=1}^{T} \|\omega_i^0 - \omega_i^{t+1}\|^2 \\
+ \frac{(5\mu_1 + 32L\Phi)N_{12}^2}{\mu_1 NL\Phi} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\psi_i^{t+1} - \psi^*(\omega_i^{t+1})\|^2 + \left(2 + \frac{8}{\mu_1}\right)T\epsilon. \]  
(188)

Next, we have the similar result as Lemma[7]
\[ \|\omega_i^0 - \omega_i^t\| \leq \|\omega_i^{t-1} - \omega_i^t\| + \frac{L_2^2}{\mu_1} \|\omega_i^{t-1} - \omega_i^t\| + \frac{L_{12}}{\mu_1} \|\psi_i^t - \psi^*(\omega_i^t)\| + \frac{L_{12}^2}{\mu_1} \|\psi_i^{t-1} - \psi^*(\omega_i^{t-1})\| + \frac{1}{\mu_1} \|\epsilon_{i,0}\| + \frac{1}{\mu_1} \|\epsilon_{i,0}^{t-1}\|. \]  
(189)
Then we bound $\|\nabla \Psi (\omega_t^o)\|^2$ with the following result similar to Lemma 18:

\[
\|\nabla \Phi (\omega_t^o)\| \leq L_\Phi \|\omega_t^0 - \omega_{t-1}^0\| + \frac{\mu_1^2 + L_\Phi^2}{N \mu_1} \sum_{i=1}^{N} \|\omega_i^{t-1} - \omega_i^t\| + \frac{L_12(L_\Phi + \mu_1)}{N \mu_1} \sum_{i=1}^{N} \|\psi_i^{t-1} - \psi^*(\omega_i^{t-1})\| \\
+ \frac{L_12 L_\Phi}{N \mu_1} \sum_{i=1}^{N} \|\psi_i^t - \psi^*(\omega_i^t)\| + \frac{\mu_1 + L_\Phi}{N \mu_1} \sum_{i=1}^{N} \|e_{\omega,i}^t\| + \frac{L_\Phi}{N \mu_1} \sum_{i=1}^{N} \|e_{\omega,i}^{t-1}\|.
\]

(190)

By applying the Cauchy-Schwarz inequality and summing up $t$ from 1 to $T$, we have a similar result as Eqn. (145), with the particular form as

\[
\sum_{t=1}^{T} \|\nabla \Phi (\omega_t^0)\|^2 \leq 4L_\Phi^2 \sum_{t=1}^{T} \|\omega_t^0 - \omega_{t-1}^0\|^2 + \frac{4(\mu_1^2 + L_\Phi^2)^2}{N \mu_1^2} \sum_{t=1}^{T} \sum_{i=1}^{N} \|\omega_i^{t-1} - \omega_i^t\|^2 \\
+ \frac{8L_12^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\psi_i^t - \psi^*(\omega_i^t)\|^2 + 8(\mu_1 + 2L_\Phi)^2 T \epsilon.
\]

(191)

Then by substituting Eqn. (184) into the previous equation, we have the following result

\[
\sum_{t=1}^{T} \|\nabla \Phi (\omega_t^0)\|^2 \leq \frac{4L_\Phi^2 \mu_1^2 N + 8C_1 L_12^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1^2} \sum_{t=1}^{T} \|\omega_t^0 - \omega_{t-1}^0\|^2 \\
+ \frac{4(\mu_1^2 + L_\Phi^2)^2}{N \mu_1^2} \sum_{t=0}^{T} \sum_{i=1}^{N} \|\omega_i^{t-1} - \omega_i^t\|^2 \\
+ \frac{8C_2 L_12^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\psi_i^0 - \psi^*(\omega_i^0)\|^2 \\
+ \frac{8C_3 L_12^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1^2} \sum_{i=1}^{N} \sum_{j \neq i} \|\omega_i^0 - \omega_j^0\|^2 \\
+ \left(\frac{8(\mu_1 + 2L_\Phi)^2}{\mu_1^2} + \frac{8C_4 L_12^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1^2}\right) T \epsilon.
\]

(192)

Finally, we reach the convergence result, which is the counterpart of Theorem 2 by following similar manipulations in our final proof of Theorem 2 substituting the above Eqn into Eqn (188), which is given by

\[
\Phi(\omega_0^o) - \Psi(\omega_T^o) \leq -E_1 \sum_{t=1}^{T} \|\nabla \Phi (\omega_t^0)\|^2 + E_2 \sum_{i=1}^{N} \|\psi_i^0 - \psi^*(\omega_i^0)\|^2 + E_3 \sum_{i=1}^{N} \sum_{j \neq i} \|\omega_i^0 - \omega_j^0\|^2 + E_4 T \epsilon.
\]

(193)
where

\[ E_1 = \min \left\{ \frac{D_1}{D_2}, \frac{D_4}{D_2} \right\}, \]  
\[ E_2 = \frac{8E_1C_2 ((L_\Phi + \mu_1)^2 + L_\Phi^2) L_{12}^2}{N \mu_1} + \frac{C_2(5\mu_1 + 32L_\Phi)L_{12}^2}{NL_\Phi \mu_1}, \]  
\[ E_3 = \frac{8E_1C_3 ((L_\Phi + \mu_1)^2 + L_\Phi^2) L_{12}^2}{N \mu_1} + \frac{C_3(5\mu_1 + 32L_\Phi)L_{12}^2}{NL_\Phi \mu_1}, \]  
\[ E_4 = \frac{8NE_1(\mu_1 + 2L_\Phi)^2 + 8E_1C_4L_{12}^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1} + C_4(2 + \frac{8}{\mu_1}), \]  
\[ D_1 = \frac{4L_\Phi^2 \mu_1^2 N + 4C_1 L_{12}^2 (L_\Phi + \mu_1)^2}{N \mu_1}, \]  
\[ D_2 = \frac{8C_4 L_{12}^2 ((L_\Phi + \mu_1)^2 + L_\Phi^2)}{N \mu_1}, \]  
\[ D_3 = \frac{\mu_1^2 - 2L_\Phi \mu_1 - 4L_\Phi^2}{2 \mu_1 N} - \frac{C_2^2(5\mu_1 + 32L_\Phi)L_{12}^2}{NL_\Phi \mu_1}, \]  
\[ D_4 = \frac{\mu_1}{4}. \]  

Similarly, since \( \Phi(\omega_0) - \Psi(\omega_0^T) \) is lower-bounded by a constant \( C \). Rearranging terms on the RHS of (193), we have

\[
\sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2 \leq \frac{E_2 \sum_{i=1}^{N} \left\| \psi_i^0 - \psi^* (\omega_i^0) \right\|^2 + E_3 \sum_{i=1}^{N} \sum_{j \neq i} \left\| \omega_i^0 - \omega_j^0 \right\|^2 - C}{E_1} + \frac{E_4 \epsilon}{E_1}.
\]

Dividing both sides by \( T \) and taking \( \limsup_{T \to \infty} \), we obtain

\[
\limsup_{T \to \infty} \frac{\sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2}{T} \leq \frac{E_2 \sum_{i=1}^{N} \left\| \psi_i^0 - \psi^* (\omega_i^0) \right\|^2 + E_3 \sum_{i=1}^{N} \sum_{j \neq i} \left\| \omega_i^0 - \omega_j^0 \right\|^2 - C}{TE_1} + \frac{E_4 \epsilon}{E_1} \]

which implies that \( \sum_{t=1}^{T} \left\| \nabla \Phi (\omega_t^0) \right\|^2 = O(T\epsilon) \), and for sufficiently large \( t \), \( \left\| \nabla \Phi (\omega_t^0) \right\|^2 = O(\epsilon) \), completing the proof.