NON-EXACT INTEGRAL FUNCTORS

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Abstract. We give a natural notion of (non-exact) integral functor $D_{\text{perf}}(X) \to D^b_c(Y)$ in the context of $k$-linear and graded categories. In this broader sense, we prove that every $k$-linear and graded functor is integral.

Introduction

Let $k$ be a field, $X$ and $Y$ two projective $k$-schemes and $K$ an object of $D^b_c(X \times_k Y)$. Let us denote $p: X \times Y \to Y$ and $q: X \times Y \to X$ the natural projections. One has a functor

$$\Phi_K: D_{\text{perf}}(X) \to D^b_c(Y),$$

$$M \mapsto p_* (K \otimes q^* M)$$

This functor is $k$-linear, graded and exact. We shall say that $\Phi_K$ is an exact integral functor of kernel $K$. We have then a functor

$$\Phi: D^b_c(X \times Y) \to \text{Hom}^\text{ex}_k(D_{\text{perf}}(X), D^b_c(Y))$$

$$K \mapsto \Phi_K$$

where $\text{Hom}^\text{ex}_k(D_{\text{perf}}(X), D^b_c(Y))$ denotes the category of $k$-linear, graded and exact functors (with $k$-linear and graded natural transformations). This functor is, in general, neither essentially injective or full (see [7]) or faithful (see [4]). However, one of the main open questions is whether it is essentially surjective. One has a positive answer for fully faithful functors $D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ due to Orlov and Lunts (see [9] and [11]). More generally, A. Canonaco and P. Stellari have shown in [5] (see also [6] for the result in the supported case) that any exact functor $F: D^b_c(X) \to D^b_c(Y)$ satisfying

$$\text{Hom}_{D^b_c(Y)}(F(A), F(B)[k]) = 0$$

for any sheaves $A$ and $B$ on $X$ and any integer $k < 0$, is integral. There are also generalizations of the fully faithful case to derived stacks (see [10]) and twisted categories (see [3]).

The best evidence for a positive answer in general is due to the results of Töen concerning dg-categories. Indeed, it is proved in [13] that all dg (quasi-)functors between the dg-categories of perfect complexes on smooth proper schemes are of Fourier-Mukai type. This result, together with the conjecture by Bondal, Larsen and Lunts in [2] that states that all exact functors between the bounded derived categories of coherent sheaves on smooth projective varieties should be liftable to dg (quasi-)functors between the corresponding dg-enhancements, would give a positive answer to the question.

In this paper we show that if we work in the context of $k$-linear and graded categories and $k$-linear and graded functors (i.e., we forget exactness) the answer is positive. Of
course we have to say what a \( k \)-linear and graded (may be non exact) integral functor means. The idea is very simple: since an object \( K \in D^b_c(X \times Y) \) may be thought of as an exact functor \( D_{\text{perf}}(X \times Y) \to D_{\text{perf}}(k) \), we shall instead consider, as a kernel, a \( k \)-linear and graded (may be non exact) functor \( \omega: D_{\text{perf}}(X \times Y) \to D(k) \). Let us be more precise:

For any \( k \)-scheme \( f: Z \to \text{Spec } k \), let us denote \( D_{\text{perf}}(Z)^* \) the category of \( k \)-linear and graded functors \( D_{\text{perf}}(Z) \to D(k) \). One has a natural functor, \( D^b_c(Z) \to D_{\text{perf}}(Z)^* \), \( K \mapsto \omega_K \), where \( \omega_K \) is the exact integral functor of kernel \( K \), i.e., \( \omega_K(M) = f_*(K \otimes M) \). This functor is fully faithful and its essential image is the subcategory \( D_{\text{perf}}(Z)^{\omega} \) of exact and perfect functors (perfect means that it takes values in \( D_{\text{perf}}(k) \)).

Now, let \( \omega \in D_{\text{perf}}(X \times Y)^* \). It induces, in the obvious way, a \( k \)-linear and graded functor \( \Phi_\omega: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^* \)
\[
M \mapsto \Phi_\omega(M)
\]

i.e., \( \Phi_\omega(M)(N) = \omega(q^*M \otimes p^*N) \) (see also \( \omega \) for an alternative description which is closer to the usual definition). We shall say that \( \Phi_\omega \) is an integral functor of kernel \( \omega \). If \( \omega \) is exact and perfect, then \( \omega \simeq \omega_K \) for an unique \( K \in D^b_c(X \times Y) \). Then \( \Phi_\omega \) takes values in \( D_{\text{perf}}(Y)^{\omega} \simeq D^b_c(Y) \) and \( \Phi_\omega \simeq \Phi_K \).

One has then a functor
\[
\Phi: D_{\text{perf}}(X \times Y)^* \to \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*)
\]
\[
\omega \mapsto \Phi_\omega
\]

extending the functor \( \Phi: D^b_c(X \times Y) \to \text{Hom}_k^c(D_{\text{perf}}(X), D^b_c(Y)) \).

The aim of this paper is to prove that \( \Phi: D_{\text{perf}}(X \times Y)^* \to \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \) is essentially surjective. Even more, we shall construct a right inverse of \( \Phi \), i.e., a functor
\[
\Psi: \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \to D_{\text{perf}}(X \times Y)^*
\]
such that \( \Phi \circ \Psi \) is isomorphic to the identity. In other words, for any \( k \)-linear and graded functor \( F: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^* \) there exist a kernel \( \omega_F \in D_{\text{perf}}(X \times Y)^* \) and an isomorphism \( F \simeq \Phi_{\omega_F} \) which are functorial on \( F \). This will be a consequence of an extension theorem (Theorem \( \text{2.8} \)) that states that if \( F: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^* \) is a \( k \)-linear and graded functor and \( S \) is any \( k \)-scheme, then \( F \) can be lifted to an \( S \)-linear functor \( F_S: D_{\text{fhd}/X,S}(X \times S) \to D_{\text{perf}}(Y \times S)^* \), where \( D_{\text{fhd}/X,S}(X \times S) \) is the category of objects in \( D^b_c(X \times S) \) of finite homological dimension over both \( X \) and \( S \) (see Definition \( \text{2.6} \)).

Let us denote \( D_{\text{perf}}(X \times Y)^{\omega} \) the full subcategory of \( D_{\text{perf}}(X \times Y)^* \) whose objects are the \( \omega \in D_{\text{perf}}(X \times Y)^* \) which are exact and perfect on \( Y \), i.e. such that for any \( M \in D_{\text{perf}}(X) \), \( \Phi_\omega(M) \) belongs to \( D_{\text{perf}}(Y)^{\omega} \). Taking into account the equivalence \( D^b_c(Y) \to D_{\text{perf}}(Y)^{\omega} \), we obtain functors \( \Phi: D_{\text{perf}}(X \times Y)^{\omega} \to \text{Hom}_k(D_{\text{perf}}(X), D^b_c(Y)) \) and \( \Psi: \text{Hom}_k(D_{\text{perf}}(X), D^b_c(Y)) \to D_{\text{perf}}(X \times Y)^{\omega} \), such that \( \Phi \circ \Psi \) is isomorphic to the identity. Finally, if we denote \( D_{\text{perf}}(X \times Y)^{\omega} \) the full subcategory of \( D_{\text{perf}}(X \times Y)^* \) whose objects are the \( \omega \in D_{\text{perf}}(X \times Y)^* \) which are bi-exact (see Definition \( \text{2.17} \)), then we obtain functors \( \Phi: D_{\text{perf}}(X \times Y)^{\omega} \to \text{Hom}_k^c(D_{\text{perf}}(X), D^b_c(Y)) \) and \( \Psi: \text{Hom}_k^c(D_{\text{perf}}(X), D^b_c(Y)) \to D_{\text{perf}}(X \times Y)^{\omega} \), such that \( \Phi \circ \Psi \) is isomorphic to the identity.

We shall also give these results in the relative setting. That is, assume that \( X \) and \( Y \) are flat \( T \)-schemes and let \( p: X \times_T Y \to Y \), \( q: X \times_T Y \to X \) be the natural projections.
For each $M \in D^b_c(X \times_T Y)$ one has a $T$-linear functor $\Phi_K: D_{\text{perf}}(X) \to D^b_c(Y)$, $M \mapsto p_*(K \otimes q^* M)$; one has then a $T$-linear functor

$$\Phi: D^b_c(X \times_T Y) \to \text{Hom}^T(D_{\text{perf}}(X), D^b_c(Y))$$

More generally, for any $\omega \in D_{\text{perf}}(X \times_T Y)^*$ one has a $T$-linear functor $\Phi_\omega: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^*$; one has then a $T$-linear functor

$$\Phi: D_{\text{perf}}(X \times_T Y)^* \to \text{Hom}_T(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*)$$

As before, we shall construct a $T$-linear functor

$$\Psi: \text{Hom}_T(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \to D_{\text{perf}}(X \times_T Y)^*$$

such that $\Phi \circ \Psi$ is isomorphic to the identity.

1. Notations and basic results

Throughout the paper $k$ denotes a field. All the schemes are assumed to be proper $k$-schemes. If $f: X \to Y$ is a morphism of $T$-schemes, we shall still denote by $f$ the morphism $X \times_T T' \to Y \times_T T'$ induced by $f$ after a base change $T' \to T$.

For a scheme $X$, we denote by $D(X)$ the derived category of complexes of $O_X$-modules with quasi-coherent cohomology, $D^b_c(X)$ the full subcategory of complexes with bounded and coherent cohomology and $D_{\text{perf}}(X)$ the full subcategory of perfect complexes.

Since we shall deal with derived categories, we shall use the abbreviated notations $f_*, f^*, \otimes, \ldots$ for the derived functors $Rf_*, Lf^*, \otimes, \ldots$

We shall use extensively the following results of derived categories:

1. Projection formula: If $f: X \to Y$ is a morphism of schemes, then one has a natural isomorphism $f_*(M \otimes f^* L) \simeq (f_* M) \otimes L$, with $M \in D(X)$, $L \in D(Y)$.

2. Flat base change: Let us consider a cartesian diagram

$$\begin{array}{ccc}
X \times_T Y & \xrightarrow{p} & Y \\
q \downarrow & & \downarrow q \\
X & \xrightarrow{p} & T
\end{array}$$

with $p$ flat. For any $N \in D(Y)$ one has a natural isomorphism $p^* q_* N \xrightarrow{\sim} q_* p^* N$.

1.1. Non-exact integral functors.

Let $k$ be a field, $D(k)$ the derived category of complexes of $k$-vector spaces.

Definition 1.1. Let $p: Z \to \text{Spec} k$ be a $k$-scheme. A linear form on $Z$ is a $k$-linear and graded functor $\omega: D_{\text{perf}}(Z) \to D(k)$. We say that $\omega$ is perfect if it takes values in $D_{\text{perf}}(k)$. We say that $\omega$ is exact if it takes exact triangles into exact triangles.

A linear morphism $\omega \to \omega'$ between linear forms on $Z$ is just a morphism of $k$-linear and graded functors. We shall denote by $D_{\text{perf}}(Z)^*$ the category of $k$-linear forms on $Z$ and $k$-linear morphisms and by $D_{\text{perf}}(Z)^{\vee}$ the full subcategory of $D_{\text{perf}}(Z)^*$ whose objects are the exact and perfect linear forms on $Z$. Both $D_{\text{perf}}(Z)^*$ and $D_{\text{perf}}(Z)^{\vee}$ are $k$-linear and graded categories. $\triangle$

For any $K \in D^b_c(Z)$ one has a (perfect and exact) linear form $\omega_K$ on $Z$, defined by $\omega_K(M) = p_*(K \otimes M)$. Moreover one has the following
Proposition 1.2. Let $Z$ be a projective $k$-scheme. The functor $D_c^b(Z) \to D_{\text{perf}}(Z)^\vee$, $K \mapsto \omega_K$, is an equivalence (of $k$-linear and graded categories).

Proof. It is proved in [3] that any contravariant cohomological functor of finite type over $D_{\text{perf}}(Z)$ ($Z$ a projective scheme over $k$) is representable by a bounded complex with coherent homology. It follows that if $\omega: D_{\text{perf}}(Z) \to D_{\text{perf}}(k)$ is exact, then it has a right pseudo adjoint $\omega^*: D_{\text{perf}}(k) \to D_c^b(Z)$; that is, one has

$$\text{Hom}_{D(k)}(\omega(M), E) \simeq \text{Hom}_{D(Z)}(M, \omega^*(E))$$

for any $M \in D_{\text{perf}}(Z)$, $E \in D_{\text{perf}}(k)$. Since $\omega^*$ is $k$-linear and graded, one has $\omega^*(E) \simeq \omega^!(k) \otimes p^* E$, and then $\omega \simeq \omega_K$ with $K = R\text{Hom}(\omega^!(k), p^! k)$. Conclusion follows (see [12] for further details and a more general statement).

Definition 1.3. Tensor product, direct and inverse image.

1. $D_{\text{perf}}(Z)^*$ has a $D_{\text{perf}}(Z)$-module structure: for any $M \in D_{\text{perf}}(Z)$, $\omega \in D_{\text{perf}}(Z)^*$, we define $\omega \otimes M \in D_{\text{perf}}(Z)^*$ by the formula $(\omega \otimes M)(N) = \omega(M \otimes N)$.

2. For any morphism of $k$-schemes $f: Z \to Z'$, we define $f_*: D_{\text{perf}}(Z)^* \to D_{\text{perf}}(Z')^*$ as the $k$-linear and graded functor induced by $f^*: D_{\text{perf}}(Z') \to D_{\text{perf}}(Z)$; that is, $(f_* \omega)(M') = \omega(f^* M')$, for $M' \in D_{\text{perf}}(Z')$, $\omega \in D_{\text{perf}}(Z)^*$.

3. If $f: Z \to Z'$ is flat, we define $f^*: D_{\text{perf}}(Z')^* \to D_{\text{perf}}(Z)^*$ as the $k$-linear and graded functor induced by $f_*: D_{\text{perf}}(Z) \to D_{\text{perf}}(Z')$; that is, $(f^* \omega)(M) = \omega(f_* M)$, for $M \in D_{\text{perf}}(Z)$, $\omega \in D_{\text{perf}}(Z')^*$.

Example 1.5. Assume that $X$ and $Y$ are projective $k$-schemes. If $\omega$ is exact and perfect, then $\omega \simeq \omega_K$ for a unique $K \in D_c^b(X \times Y)$ by Proposition [12]. Then $\Phi_{\omega}$ takes values in $D_{\text{perf}}(Y)^\vee \simeq D_c^b(Y)$ and $\Phi_{\omega}$ is isomorphic to the usual exact integral functor $\Phi_K$.

Let us denote $\text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*)$ the category of $k$-linear and graded functors from $D_{\text{perf}}(X)$ to $D_{\text{perf}}(Y)^*$ and $k$-linear and graded morphisms of functors. It is a $k$-linear and graded category in the obvious way. One has a $k$-linear and graded functor

$$\Phi: D_{\text{perf}}(X \times Y)^* \to \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*)$$

$$\omega \mapsto \Phi_{\omega}$$

and a commutative diagram

$$\begin{array}{ccc}
D_{\text{perf}}(X \times Y)^* & \xrightarrow{\Phi} & \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \\
\downarrow & & \downarrow \\
D_c^b(X \times Y) & \xrightarrow{\Phi} & \text{Hom}_k^c(D_{\text{perf}}(X), D_c^b(Y))
\end{array}$$

whose vertical maps are fully faithful.
2. Main results

The aim of this section is to construct a functor $\Psi: \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \to D_{\text{perf}}(X \times Y)^*$ such that $\Phi \circ \Psi$ is isomorphic to the identity. This will be a consequence of the following extension theorem: a $k$-linear and graded functor $F: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^*$ can be (functorially) extended to an $S$-linear functor $F_S: D_{\text{rhd}/X,S}(X \times S) \to D_{\text{perf}}(Y \times S)^*$, for any $k$-scheme $S$ (see Definition 2.6 for the meaning of $D_{\text{rhd}/X,S}(X \times S)$).

2.1. Cokernels of linear forms.

Let $E_* = \{E_1 \xrightarrow{d_0} E_0\}$ be two maps in $D(k)$. We define $\text{Coker}(E_*)$ as the cokernel of the morphisms of vector spaces $H(E_1) \xrightarrow{H(d_0)} H(E_0)$.

Now let $p: Z \to \text{Spec } k$ be a $k$-scheme and let $\omega_* = \{\omega_1 \xrightarrow{d_1} \omega_0\}$ be two morphisms in $D_{\text{perf}}(Z)^*$. For each $M \in D_{\text{perf}}(Z)$, let us denote $\text{Coker}(\omega_*) = \{\omega_1(M) \xrightarrow{d_1(M)} \omega_0(M)\}$. We define $\text{Coker}(\omega_*)$ as the object in $D_{\text{perf}}(Z)^*$ defined by

$$\text{Coker}(\omega_*)(M) = \text{Coker}(\omega_*(M)).$$

This is clearly functorial on $\omega_*$. The next proposition is immediate.

**Proposition 2.1.** Let $\omega_* = \{\omega_1 \xrightarrow{d_1} \omega_0\}$ be two morphisms in $D_{\text{perf}}(Z)^*$.

1. For any $M \in D_{\text{perf}}(Z)$ one has $\text{Coker}(\omega_* \otimes M) = \text{Coker}(\omega_*) \otimes M$.
2. For any morphism $f: Z \to Z'$, one has $f_* \text{Coker}(\omega_*) = \text{Coker}(f_* \omega_*)$.

2.2. The perfect-resolution.

Let $p: X \to \text{Spec } k$, $f: S \to \text{Spec } k$ be two $k$-schemes and $M \in D_c^b(X \times S)$ an object of finite homological dimension over $S$ (see Definition 2.5). We still denote by $p: X \times S \to S$ and $f: X \times S \to X$ the natural projections. For each $E \in D_{\text{perf}}(X)$ we shall denote $R_E(M) = f^*E \otimes p^*p_*(f^*E^* \otimes M)$, with $E^* = \mathcal{R}\text{Hom}_X^*(E, \mathcal{O}_X)$. One has a natural morphism $\rho_M: R_E(M) \to M$. We shall denote

$$R_0(M) = \bigoplus_{E \in D_{\text{perf}}(X)} R_E(M)$$

and $\rho_M: R_0(M) \to M$ the natural map. This is functorial on $M$.

Let $R_1(M) = R_0(R_0(M))$. One has two morphisms

$$R_1(M) \xrightarrow{d_0} R_0(M) \xrightarrow{d_1} R_0(M)$$

namely: $d_0 = \rho_{R_0(M)}$ and $d_1 = R_0(\rho_M)$. It is immediate to check that $\rho_M \circ d_0 = \rho_M \circ d_1$.

More explicitly, $R_1(M) = \bigoplus_{E_1, E_0 \in D_{\text{perf}}(X)} R_{E_1}R_{E_0}(M)$ and

$$R_{E_1}R_{E_0}(M) \simeq f^*E_1 \otimes p^*p_*(f^*E^*_1 \otimes E_0) \otimes p^*p_*(f^*E^*_0 \otimes M).$$

(2.1)
The differentials $d_0, d_1: R_1(M) \to R_0(M)$ are induced by the morphisms
\[
\rho_{R_0(M)}^{\varepsilon_1}: R_\varepsilon R_0(M) \to R_\varepsilon(M)
\]
\[
R_\varepsilon (\rho_M^{\varepsilon_0}): R_\varepsilon R_0(M) \to R_\varepsilon(M).
\]

**Proposition 2.2.** For any $L \in D_{\text{perf}}(X)$, $R_1(f^*L) \xrightarrow{d_0} R_0(f^*L) \xrightarrow{\rho_{f^*L}} f^*L$ is exact.

**Proof.** The morphism $\rho_{f^*L}^{\varepsilon_1}: R_{f^*L}(f^*L) \to f^*L$ has a natural section $h: f^*L \to R_{f^*L}(f^*L) = f^*L \otimes p^*p_*(f^*L^* \otimes f^*L)$ induced by the natural map $\mathcal{O}_S \to p_*(f^*L^* \otimes f^*L)$. Then we have a map $h_0: f^*L \to R_0(f^*L)$ which is $h$ in the $f^*L$-component and zero in the others. It is clear that $h_0$ is a section of $\rho_{f^*L}$. Now define $h_1: R_0(f^*L) \to R_1(f^*L)$ as $h_1 = -R_0(h)$. One can check that $(d_0 - d_1) \circ h_1 + h_0 \circ \rho_{f^*L} = \text{Id}$, hence the result. \(\square\)

We shall denote $R_{\bullet}^{\mathcal{X} \times S/S}(M) := \{R_1(M) \xrightarrow{d_0} R_0(M)\}$.

**Remark 2.3.** If $G: D_{\text{perf}}(X \times S) \to \mathcal{D}$ is an additive functor and $\mathcal{D}$ has infinite direct sums, we can define $G(R_1(M)) := \bigoplus_{\varepsilon_1 \varepsilon_0} G(R_{\varepsilon_1} R_{\varepsilon_0}(M))$ and $G(R_0(M)) := \bigoplus_{\varepsilon} G(R_{\varepsilon}(M))$.

One has differentials $G(d_0), G(d_1): G(R_1(M)) \to G(R_0(M))$. We shall denote
\[
G(R_{\bullet}(M)) = \{G(R_1(M)) \xrightarrow{G(d_0)} G(R_0(M))\}.
\]

For any $L \in D_{\text{perf}}(X)$, $G(R_1(f^*L)) \xrightarrow{G(d_0)} G(R_0(f^*L)) \xrightarrow{G(\rho_{f^*L})} G(f^*L)$ is exact. \(\triangle\)

In the next propositions we shall prove the $S$-linearity of $R_{\bullet}^{\mathcal{X} \times S/S}(M)$ and its compatibility with direct images.

**Proposition 2.4.** For any $V \in D_{\text{perf}}(S)$ one has a natural isomorphism $R_{\bullet}^{\mathcal{X} \times S/S}(M \otimes p^*V) \cong R_{\bullet}^{\mathcal{X} \times S/S}(M) \otimes p^*V$.

**Proof.** One has a natural isomorphism $R_{\varepsilon}(M \otimes p^*V) \cong R_{\varepsilon}(M) \otimes p^*V$. Indeed, by projection formula,
\[
R_{\varepsilon}(M \otimes p^*V) = f^*\varepsilon \otimes p^*p_*(f^*\varepsilon^* \otimes M \otimes p^*V) \cong f^*\varepsilon \otimes p^*p_*(f^*\varepsilon^* \otimes M) \otimes p^*V
\]
\[
= R_{\varepsilon}(M) \otimes p^*V.
\]

One checks that the diagram
\[
\begin{array}{ccc}
R_{\varepsilon}(M \otimes p^*V) & \xrightarrow{\rho_{M \otimes p^*V}} & M \otimes p^*V \\
\downarrow & & \downarrow \text{Id} \\
R_{\varepsilon}(M) \otimes p^*V & \xrightarrow{\rho_{p^*V \otimes 1}} & M \otimes p^*V
\end{array}
\]
is commutative. Hence one has an isomorphism $R_0(M \otimes p^*V) \cong R_0(M) \otimes p^*V$ and a commutative diagram
Conclusion follows. □

**Proposition 2.5.** One has a natural isomorphism $f_!\left[R^X_{\mathcal{E}}(M)\right] \cong R^X_{/k}(f_*M)$.

*Proof.* By projection formula and flat base change one has

$$f_!R^\mathcal{E}(M) = f_*\left[f^*\mathcal{E} \otimes p^*\rho_*(f^*\mathcal{E}^* \otimes M)\right] \cong \mathcal{E} \otimes f_*p^*\rho_*(f^*\mathcal{E}^* \otimes M) \cong \mathcal{E} \otimes p^*\rho_*(\mathcal{E}^* \otimes f_*M) = R^\mathcal{E}(f_*M)$$

Moreover, the diagram

$$\begin{array}{ccc}
f_!R^\mathcal{E}(M) & \xrightarrow{f_!\rho_{\mathcal{E},id}} & M \\
\downarrow \iota & & \downarrow \mathrm{Id} \\
R^\mathcal{E}(f_*M) \otimes p^*V & \xrightarrow{\rho_{f_*M}} & M
\end{array}$$

is commutative. One has then $f_*R_0(M) \cong R_0(f_*M)$ and a commutative diagram

$$\begin{array}{ccc}
f_*R_0(M) & \xrightarrow{f_*\rho_M} & M \\
\downarrow \iota & & \downarrow \mathrm{Id} \\
R_0(f_*M) \otimes p^*V & \xrightarrow{\rho_{f_*M}} & M
\end{array}$$

Conclusion follows. □

### 2.3. The extension theorem.

We need to introduce a relative notion of perfectness.

**Definition 2.6.** Let $f: Z \to T$ be a morphism of schemes. An object $M \in D^b_c(Z)$ is said to be of finite homological dimension over $T$ ($f\text{-hd}/T$ for short), if $M \otimes f^*N$ is bounded and coherent for any $N \in D^b_c(T)$. We shall denote by $D^{f\text{-hd}}_c(T)$ the faithful subcategory of the objects of finite homological dimension over $T$. △

The following properties of $f\text{-hd}$-objects are quite immediate (see [8, Section 1.2]).

**Proposition 2.7.**

1. If $f$ is flat, then $D^\text{perf}(Z) \subset D^{f\text{-hd}}_c(T)$.
2. If $M$ is $f\text{-hd}$ over $T$ and $f$ is proper, then $f_*M$ is perfect.
3. If $M$ is $f\text{-hd}$ over $T$ and $\mathcal{E} \in D^\text{perf}(Z)$, then $M \otimes \mathcal{E}$ is $f\text{-hd}$ over $T$.

Given two schemes $X$ and $S$, we shall denote by $D^{f\text{-hd}/X,S}(X \times S)$ the category of objects in $D^{f\text{-hd}}_c(X \times S)$ of finite homological dimension over both $X$ and $S$.

**Theorem 2.8.** Let $p: X \to \text{Spec } k$ and $q: Y \to \text{Spec } k$ be two proper $k$-schemes. Let $F: D^\text{perf}(X) \to D^\text{perf}(Y)^*$ be a $k$-linear and graded functor. For any proper $k$-scheme $f: S \to \text{Spec } k$ there exists a functor

$$F_S: D^{f\text{-hd}/X,S}(X \times S) \to D^\text{perf}(Y \times S)^*$$

such that:
1) $F_S$ is $S$-linear: one has a bi-functorial isomorphism $F_S(M \otimes p^*V) \simeq F_S(M) \otimes q^*V$ for any $M \in D_{\text{fnd}/X,S}(X \times S)$, $V \in D_{\text{perf}}(S)$.

2) It is compatible with $F$: for any $M \in D_{\text{fnd}/X,S}(X \times S)$ one has a natural isomorphism $f_*F_S(M) \simeq F(f_*M)$.

3) $F_S$ is functorial on $F$.

**Proof.** Let $M \in D_{\text{fnd}/X,S}(X \times S)$. Let us take the “perfect” resolution of $M$, $R^X_{\bullet \times S/S}(M)$, constructed in section 2.2. Recall that $R^\xi(M) = f^*E \otimes p^*\rho_\xi(f^*\mathbb{E}^{*} \otimes M)$. Since $M$ is fnd over $S$, $R^\xi(M)$ belongs to $p^*D_{\text{perf}}(S) \otimes f^*D_{\text{perf}}(X)$. In particular, $R^\xi(M)$ is fnd over $S$ and then $R^\xi_1R^\xi_0(M)$ belongs also to $p^*D_{\text{perf}}(S) \otimes f^*D_{\text{perf}}(X)$.

Taking in mind the explicit expression of $R^\xi(M)$ and $R^\xi_1R^\xi_0(M)$ (see (2.1)), let us put

$$\tilde{F}(R^\xi(M)) := f^*F(E) \otimes q^*\rho_\xi(f^*\mathbb{E}^{*} \otimes M)$$

$$\tilde{F}(R^\xi_1R^\xi_0(M)) := f^*F(E_1) \otimes q^*\rho_\xi f^*(\mathbb{E}_1^{*} \otimes \mathbb{E}_0) \otimes q^*\rho_\xi(f^*\mathbb{E}_0^{*} \otimes M)$$

and

$$\tilde{F}(R^0(M)) := \bigoplus_{\mathbb{E} \in D_{\text{perf}}(X)} \tilde{F}(R^\xi(M))$$

$$\tilde{F}(R^1(M)) := \bigoplus_{\mathbb{E}_1,\mathbb{E}_0 \in D_{\text{perf}}(X)} \tilde{F}(R^\xi_1R^\xi_0(M))$$

By Lemma 2.9 (see below) the morphisms $d_0,d_1: R^0_1(M) \to R^0(M)$ induce morphisms $\tilde{d}_0, \tilde{d}_1: \tilde{F}(R^0_1(M)) \to \tilde{F}(R^0(M))$ in $D_{\text{perf}}(Y \times S)^*$ which are functorial on $M$. Finally, we define $F_S(M) := \text{Coker}(\tilde{F}(R^\xi_0(M)))$

It is clear that $F_S(M)$ is functorial on $M$, hence we obtain a functor $F_S: D_{\text{perf}}(X \times S) \to D_{\text{perf}}(Y \times S)^*$. By construction, $F_S$ satisfies 3).

**Lemma 2.9.** Let $V,V' \in D_{\text{perf}}(S)$ and $\mathbb{E},\mathbb{E}' \in D_{\text{perf}}(X)$. One has a natural map

$$\text{Hom}_{D_{\text{perf}}(X \times S)}(p^*V \otimes f^*\mathbb{E},p^*V' \otimes f^*\mathbb{E}') \to \text{Hom}_{D_{\text{perf}}(Y \times S)^*}(q^*V \otimes f^*F(\mathbb{E}),q^*V' \otimes f^*F(\mathbb{E}'))$$

This map is compatible with composition; moreover, it is $S$-linear and extends $F$ (the precise meaning of these will be given in the proof).

**Proof.** A morphism $h: p^*V \otimes f^*\mathbb{E} \to p^*V' \otimes f^*\mathbb{E}'$ corresponds with a morphism $\tilde{h}: \mathbb{E} \to \mathbb{E}' \otimes_k \text{RHom}^*(V,V')$. Since $F$ is $k$-linear and graded, it induces a morphism $F(\tilde{h}): F(\mathbb{E}) \to F(\mathbb{E}') \otimes_k \text{RHom}^*(V,V')$. Hence, for any $N \in D_{\text{perf}}(Y \times S)$ one has morphisms

$$(q^*V \otimes f^*F(\mathbb{E}))(N) \to F(\mathbb{E})(f_*q^*V \otimes N)) \xrightarrow{F(\tilde{h})} [F(\mathbb{E}') \otimes_k \text{RHom}^*(V,V')](f_*q^*V \otimes N))$$

$$= F(\mathbb{E}')(f_*q^*V \otimes N) \otimes_k \text{RHom}^*(V,V')) \xrightarrow{(\ast)} F(\mathbb{E}')(f_*q^*V \otimes N))$$

where $(\ast)$ is the morphism induced by the natural evaluation map $f_*q^*V \otimes N) \otimes_k \text{RHom}^*(V,V') \to f_*q^*V' \otimes N)$. That is, one obtains a morphism $\tilde{h}: q^*V \otimes f^*F(\mathbb{E}) \to q^*V' \otimes f^*F(\mathbb{E}')$.

One can check from the construction that $(f \circ g) = \tilde{f} \circ \tilde{g}$, for any $f: p^*V' \otimes f^*\mathbb{E}' \to p^*V'' \otimes f^*\mathbb{E}''$ and $g: p^*V \otimes f^*\mathbb{E} \to p^*V' \otimes f^*\mathbb{E}'$. Moreover, if $f = p^*(f_1) \otimes f_2$ for some $f_1: V \to V'$ and $f_2: f^*\mathbb{E} \to f^*\mathbb{E}'$, then $\tilde{f} = q^*(f_1) \otimes \tilde{f}_2$. Finally, if $V = V'$ and $f = \text{Id} \otimes f^*(f_2)$ for some $f_2: \mathbb{E} \to \mathbb{E}'$, then $\tilde{f} = \text{Id} \otimes f^*F(f_2)$.

$\square$
Proposition 2.10. One has natural isomorphisms:

a) \( \tilde{F}(R_\bullet(M)) \otimes q^*N \simeq \tilde{F}(R_\bullet(M \otimes p^*N)) \)

b) \( f_\ast \tilde{F}(R_\bullet(M)) \simeq F(f_\ast R_\bullet(M)) \) (see Remark 2.3 for the definition of \( F(f_\ast R_\bullet(M)) \)).

Proof. a) Completely analogous arguments to that of Proposition 2.4 yield isomorphisms \( \tilde{F}(R_\bullet(M \otimes p^*V)) \simeq \tilde{F}(R_{\bullet\ast}R_{\bullet 0}(M \otimes p^*V)) \simeq \tilde{F}(R_{\bullet\ast}R_{\bullet 0}(M)) \otimes q^*V \). One checks that these isomorphisms are compatible with the differentials.

b) Completely analogous arguments to that of Proposition 2.5 yield isomorphisms \( f_\ast \tilde{F}(R_{\bullet\ast}(M)) \simeq F(f_\ast R_{\bullet\ast}(M)) \) and \( f_\ast \tilde{F}(R_{\bullet\ast}R_{\bullet 0}(M)) \simeq F(f_\ast R_{\bullet\ast}R_{\bullet 0}(M)) \). Again, one checks that these isomorphisms are compatible with the differentials. □

It follows immediately that \( F_S \) satisfies 1). For 2), one has

\[
\begin{align*}
  f_\ast F_S(M) &= f_\ast \text{Coker}(\tilde{F}(R_\bullet(M))) = \text{Coker}(f_\ast \tilde{F}(R_\bullet(M))) \\
  &= \text{Coker}(f_\ast R_\bullet(M)) \\
  &\overset{2.10}{=} \text{Coker}(f_\ast R_\bullet(M)) \\
  &\overset{2.3}{=} \text{Coker}(f_\ast R_\bullet(M))
\end{align*}
\]

Finally, \( \text{Coker}(F(R_\bullet(f_\ast M)) \simeq F(f_\ast M) \) by Proposition 2.2 and Remark 2.3.

This concludes the proof of Theorem 2.8. □

Remark 2.11. The lifting \( F_S \) of \( F \) is functorial but it is not unique. Let us show an alternative lifting \( F'_S \). Instead of considering the “resolution” \( R_1(M) \to R_0(M) \), let us consider the complex of objects in \( D(X \times S) \)

\[
R_\bullet(M) := \{ \cdots \to R_n(M) \to R_{n-1}(M) \to \cdots \to R_1(M) \to R_0(M) \}
\]

where \( R_n(M) = R_0(R_{n-1}(M)) \) and the differential \( R_n(M) \to R_{n-1}(M) \) is the alternate sum of the \( n + 1 \) natural maps from \( R_n(M) \) to \( R_{n-1}(M) \). As in the proof of the theorem, we can define \( \tilde{F}(R_\bullet(M)) \), which is a complex of objects in \( D_{\text{perf}}(Y \times S)^* \).

Then one defines \( F'_S(M) \) as the “simple complex” associated to \( \tilde{F}(R_\bullet(M)) \), i.e., for any \( N \in D_{\text{perf}}(Y \times S) \) we define \( F'_S(M)(N) \) as the simple complex associated to the complex of vector spaces

\[
\cdots \to H(\tilde{F}(R_n(M))(N)) \to H(\tilde{F}(R_{n-1}(M))(N)) \to \cdots \to H(\tilde{F}(R_0(M))(N))
\]

This functor \( F'_S \) also satisfies properties 1), 2) and 3). Moreover, it has an extra “exact” property: first notice that \( F'_S(M) \) is in fact a functor from \( D_{\text{perf}}(Y \times S) \) to the category of complexes of vector spaces (i.e, if \( h: N \to N' \) is a morphism in \( D_{\text{perf}}(Y \times S) \), then \( F'_S(M)(h) \) is a morphism of complexes); the exact property is the following: if \( N_1 \to N_2 \to N_3 \) is an exact triangle in \( D_{\text{perf}}(Y \times S) \), then

\[
F'_S(M)(N_1) \to F'_S(M)(N_2) \to F'_S(M)(N_3)
\]

is an exact sequence of complexes (but may be not an exact triangle). △

Let us see now how the extension theorem yields the integrality theorem.

Theorem 2.12. Let \( X \) and \( Y \) be two proper \( k \)-schemes and \( F: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^* \) a \( k \)-linear graded functor. Then there exists an object \( \omega \) in \( D_{\text{perf}}(X \times Y)^* \) such that \( F \simeq \Phi_\omega \).
Proof. Let \(F_S: D_{\text{fhd}/X,S}(X \times S) \to D_{\text{perf}}(Y \times S)^*\) be the \(S\)-linear functor given by Theorem 2.8. Take \(S = X, f = p, \delta: X \to X \times S\) the diagonal map and \(\mathcal{O}_\Delta = \delta_*\mathcal{O}_X\). Notice that \(\mathcal{O}_\Delta\) is fhd over both \(X\) and \(S\). Then, by properties 1) and 2) of \(F_S\),

\[
F(M) \simeq F(f_* (\mathcal{O}_\Delta \otimes p^* M)) \simeq f_* F_S(\mathcal{O}_\Delta \otimes p^* M) \simeq f_* (F_S(\mathcal{O}_\Delta) \otimes q^* M)
\]

So it is enough to take \(\omega = F_S(\mathcal{O}_\Delta)\).

Since \(F_X\) is functorial on \(F\) we obtain:

**Corollary 2.13.** One has a functor

\[
\Psi: \text{Hom}_k(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \to D_{\text{perf}}(X \times Y)^*
\]

\[
F \mapsto F_X(\mathcal{O}_\Delta)
\]

and the composition \(\Phi \circ \Psi\) is isomorphic to the identity.

2.4. Exactness.

**Definition 2.14.** A linear form, \(\omega: D_{\text{perf}}(X \times Y) \to D(k)\), on \(X \times Y\) is said to be exact and perfect on \(Y\) if for any \(M \in D_{\text{perf}}(X)\), the functor \(\Phi_\omega(M): D_{\text{perf}}(Y) \to D(k)\) is exact and perfect, i.e., \(\Phi_\omega(M) \in D(Y)^\vee\).

We shall denote \(D(X \times Y)^{Y - \vee}\) the full subcategory of \(D(X \times Y)^*\) whose objects are the linear forms on \(X \times Y\) which are exact and perfect on \(Y\).

Taking into account that \(D_c^b(Y) \to D_{\text{perf}}(Y)^\vee\) is an equivalence (\(Y\) projective), we obtain

**Corollary 2.15.** If \(Y\) is projective, one has functors

\[
\Phi: D_{\text{perf}}(X \times Y)^{Y - \vee} \to \text{Hom}_k(D_{\text{perf}}(X), D_c^b(Y))
\]

\[
\omega \mapsto \Phi_\omega
\]

and

\[
\Psi: \text{Hom}_k(D_{\text{perf}}(X), D_c^b(Y)) \to D_{\text{perf}}(X \times Y)^{Y - \vee}
\]

\[
F \mapsto F_X(\mathcal{O}_\Delta)
\]

and the composition \(\Psi \circ \Phi\) is isomorphic to the identity.

A linear form \(\omega\) on \(X \times Y\) also defines an integral functor in the opposite direction (i.e., from \(Y\) to \(X\)), which we shall denote by \(\overline{\Phi}_\omega: D_{\text{perf}}(Y) \to D_{\text{perf}}(X)^*\).

**Proposition 2.16.** Assume that \(X\) and \(Y\) are projective. Let \(\omega\) be a linear form on \(X \times Y\). The following conditions are equivalent:

1. \(\omega\) is exact and perfect on \(Y\) and the functor \(\Phi_\omega: D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^\vee \simeq D_c^b(Y)\) is exact.

2. \(\omega\) is exact and perfect on \(X\) and the functor \(\overline{\Phi}_\omega: D_{\text{perf}}(Y) \to D_{\text{perf}}(X)^\vee \simeq D_c^b(X)\) is exact.

**Proof.** For any \(M \in D_{\text{perf}}(X), N \in D_{\text{perf}}(Y),\) one has \(\Phi_\omega(M)(N) = \overline{\Phi}_\omega(N)(M)\). Let us see that \((1) \Rightarrow (2)\).

From the equality \(\Phi_\omega(M)(N) = \overline{\Phi}_\omega(N)(M)\) and \((1)\) it follows immediately that \(\omega\) is exact and perfect on \(X\). It remains to prove that \(\overline{\Phi}_\omega: D_{\text{perf}}(Y) \to D_{\text{perf}}(X)^\vee \simeq D_c^b(X)\) is exact.

For each object \(E \in D_{\text{perf}}(Y)\), let us denote \(E^\#: = R\mathcal{H}om(Y^*, (E, q^* k))\). The functor \(H: D_{\text{perf}}(X) \to \text{Vect}(k)\) defined by \(H(M) = \text{Hom}_{D(Y)}(\Phi_\omega(M), E^\#)\) is a contravariant
cohomological functor of finite type, hence it is representable by an object \( \Phi^\#(M) \in D^b_c(X) \). Hence we obtain a pseudo right adjoint of \( \Phi \):
\[
\Phi^\#: D^\#_{\text{perf}}(Y) \to D^b_c(X),
\]
where \( D^\#_{\text{perf}}(Y) \) is the full subcategory of \( D^b_c(Y) \) whose objects are of the form \( E^\# \), with \( E \in D^\#_{\text{perf}}(Y) \). That is, one has
\[
\text{Hom}_{D(Y)}(\Phi(M), E^\#) = \text{Hom}_{D(X)}(M, \Phi^\#(E^\#))
\]
for any \( M \in D^\#_{\text{perf}}(X) \), \( E \in D^\#_{\text{perf}}(Y) \). Now, since \( \Phi \) is exact, \( \Phi^\# \) is also exact (one can copy the same proof than [1, Lemma 4.11]). Finally, it is easy to see that the equality \( \Phi(M)(N) = \Phi^\#(N)(M) \) implies that \( \Phi(M) = [\Phi^\#(N)]^\# \). Hence \( \Phi \) is exact. \( \square \)

**Definition 2.17.** A linear form \( \omega : D^\#_{\text{perf}}(X \times Y) \to D(k) \) is called bi-exact if it satisfies any of the equivalent conditions of Proposition 2.16. \( \triangle \)

We denote by \( D^\#_{\text{perf}}(X \times Y)^{\bigvee} \) the full subcategory \( D^\#_{\text{perf}}(X \times Y)^* \) whose objects are the bi-exact linear forms on \( X \times Y \). We have then embeddings
\[
D^b_c(X \times Y) \cong D^\#_{\text{perf}}(X \times Y)^{\bigvee} \hookrightarrow D^\#_{\text{perf}}(X \times Y)^{\bigvee} \hookrightarrow D^\#_{\text{perf}}(X \times Y)^{\bigvee} \hookrightarrow D^\#_{\text{perf}}(X \times Y)^*
\]

Finally, for bi-exact linear forms we have:

**Corollary 2.18.** Assume that \( X \) and \( Y \) are projective. One has functors
\[
\Phi : D^\#_{\text{perf}}(X \times Y)^{\bigvee} \to \text{Hom}^\text{ex}(D^\#_{\text{perf}}(X), D^b_c(Y))
\]
\[
\omega \mapsto \Phi_\omega
\]
and
\[
\Psi : \text{Hom}^\text{ex}(D^\#_{\text{perf}}(X), D^b_c(Y)) \to D^\#_{\text{perf}}(X \times Y)^{\bigvee}
\]
\[
F \mapsto F_X(\mathcal{O}_{\Delta})
\]
and the composition \( \Psi \circ \Phi \) is isomorphic to the identity.

### 3. Relative Integral Functors

In this section we shall reproduce the main results of the previous section for relative schemes. Let \( p : X \to T \) and \( q : Y \to T \) be two proper \( T \)-schemes. Let us still denote by \( p : X \times_T Y \to Y \) and \( q : X \times_T Y \to Y \) the natural morphisms. For each object \( K \in D^b_c(X \times_T Y) \) one has the (relative) exact integral functor
\[
\Phi_K : D^\#_{\text{perf}}(X) \to D^b_c(Y)
\]
\[
M \mapsto p_\ast(K \otimes q^*M)
\]
This functor is \( T \)-linear: for any \( M \in D^\#_{\text{perf}}(X) \), \( E \in D^\#_{\text{perf}}(T) \) one has a natural isomorphism \( \Phi_K(M \otimes p^*E) \cong \Phi_K(M) \otimes q^*E \).

If we replace \( K \) by an object \( \omega \in D^\#_{\text{perf}}(X \times_T Y)^* \), then we have a functor
\[
\Phi_\omega : D^\#_{\text{perf}}(X) \to D^\#_{\text{perf}}(Y)^*
\]
\[
M \mapsto p_\ast(\omega \otimes q^*M)
\]
which is also \( T \)-linear (in a natural sense, see below). This will be called a (relative non-exact) integral functor. Our aim is to show that (under flatness hypothesis of \( p \) and \( q \)) any \( T \)-linear functor \( D^\#_{\text{perf}}(X) \to D^\#_{\text{perf}}(Y)^* \) is integral, i.e., it is isomorphic to \( \Phi_\omega \) for some \( \omega \in D^\#_{\text{perf}}(X \times_T Y)^* \).

We shall first give some natural definitions about \( T \)-linear categories and functors.
Definition 3.1. Let $T$ be a scheme. A $T$-linear structure on an additive graded category $\mathcal{D}$ is a biadditive and bigraded functor

\[ D_{\text{perf}}(T) \times \mathcal{D} \to \mathcal{D} \]

\[(\mathcal{E}, P) \mapsto \mathcal{E} \otimes P \]

satisfying functorial isomorphisms:

1. $\phi_P: \mathcal{O}_T \otimes P \simeq P$.
2. $\psi_{\mathcal{E}_1, \mathcal{E}_2, P}: \mathcal{E}_1 \otimes (\mathcal{E}_2 \otimes P) \simeq (\mathcal{E}_1 \otimes \mathcal{E}_2) \otimes P$.

\[ \triangle \]

Definition 3.2. A $T$-linear category is a graded category endowed with a $T$-linear structure. A $T$-linear functor $F: \mathcal{D} \to \mathcal{D}'$ between $T$-linear categories is a functor endowed with a bi-additive and bi-graded bi-functorial isomorphism $\theta_{F}: F(\mathcal{E} \otimes P) \simeq \mathcal{E} \otimes F(P)$, $F \in D_{\text{perf}}(T)$, $P \in \mathcal{D}$, which is compatible with $\phi_P$ and $\psi_{\mathcal{E}_1, \mathcal{E}_2, P}$ in the obvious sense. That is, a $T$-linear functor is a pair $(F, \theta_F)$, though we shall usually denote it by $F$.

A $T$-linear morphism $\phi: F \to F'$ between $T$-linear functors is a morphism of functors which is compatible with the $\theta$’s, i.e., such that the diagram

\[
\begin{array}{ccc}
F(\mathcal{E} \otimes P) & \xrightarrow{\sim} & \mathcal{E} \otimes F(P) \\
\phi(\mathcal{E} \otimes P) \downarrow & & \downarrow_{1 \otimes \phi(P)} \\
F'(\mathcal{E} \otimes P) & \xrightarrow{\sim} & E \otimes F'(P)
\end{array}
\]

is commutative.

\[ \triangle \]

Is $S = \text{Spec } k$, the above notion of $k$-linear category coincides with the usual notion of $k$-linear graded category.

We shall denote by $\text{Hom}_T(\mathcal{D}, \mathcal{D}')$ the category of $T$-linear functors from $\mathcal{D}$ to $\mathcal{D}'$ and $T$-linear morphisms. It has a natural $T$-linear structure. If $\mathcal{D}$ and $\mathcal{D}'$ are triangulated categories, we shall denote by $\text{Hom}^T_{\text{ex}}(\mathcal{D}, \mathcal{D}')$ the full subcategory of exact $T$-linear functors.

Example 3.3. If $p: X \to T$ is a $T$-scheme, then $D_{\text{perf}}(X)$ (or $D(X)$, $D^b_c(X)$) has a natural $T$-linear structure, namely: $\mathcal{E} \otimes P := p^* \mathcal{E} \otimes_{\mathcal{O}_X} P$. If $q: Y \to T$ is another $T$-scheme and $K \in D^b_c(X \times_T Y)$, then the integral functor $\Phi_K : D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ is a $T$-linear functor (with the $\theta$ induced by the projection formula).

For any $k$-linear category $\mathcal{D}$, we shall denote $\mathcal{D}^* = \text{Hom}_k(\mathcal{D}, D(k))$. If $\mathcal{D}$ is a triangulated category, we shall denote $\mathcal{D}^\vee = \text{Hom}^\text{ex}_k(\mathcal{D}, D_{\text{perf}}(k))$. If $\mathcal{D}$ is a $T$-linear category, then $\mathcal{D}^\vee$ has a natural $T$-linear structure, defining $(\mathcal{E} \otimes \omega)(M) = \omega(\mathcal{E} \otimes M)$. Moreover, if $\mathcal{D}$ is triangulated and $\mathcal{E} \otimes (-): \mathcal{D} \to \mathcal{D}$ is exact for any $\mathcal{E} \in D_{\text{perf}}(T)$, then $\mathcal{D}^\vee$ is also a $T$-linear category.

For any $T$-scheme $p: Z \to T$, the equivalence $D^b_c(Z) \sim D_{\text{perf}}(Z)^*$ of Proposition 1.2 is $T$-linear (Z is a projective $k$-scheme).

Let $X$ and $Y$ be two $T$-schemes and $\omega : D_{\text{perf}}(X \times_T Y) \to D(k)$ a $k$-linear form on $X \times_T Y$. Let us denote $p: X \times_T Y \to X$ and $q: X \times_T Y \to Y$ the natural projections. For each $M \in D_{\text{perf}}(X)$ we have a $k$-linear functor

$\Phi_\omega(M): D_{\text{perf}}(Y) \to D(k)$
defined by $\Phi_\omega(M)(N) = \omega(p^* M \otimes q^* N)$. We have then a $T$-linear functor

$$\Phi_\omega : D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^*.$$  

**Definition 3.4.** We say that $\Phi_\omega : D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^*$ is a relative integral functor of kernel $\omega$. \hfill \triangle$

**Example 3.5.** If $\omega$ is exact and perfect, then $\omega \simeq \omega_K$ for a unique $K \in D^b_c(X \times T Y)$, $\Phi_\omega$ takes values in $D(Y)^\vee \simeq D^b_c(Y)$ and $\Phi_\omega \simeq \Phi_K$. \hfill \triangle

The extension theorem has now the following form:

**Theorem 3.6.** Let $p : X \to T$ and $q : Y \to T$ be two proper and flat $T$-schemes. Let $F : D_{\text{perf}}(X) \to D_{\text{perf}}(Y)^*$ be a $T$-linear functor. For any proper and flat $T$-scheme $f : S \to T$ there exists a functor

$$F_S : D_{\text{fhd}/X,S}(X \times_T S) \to D_{\text{perf}}(Y \times_T S)^*$$

such that:

1) $F_S$ is $S$-linear: one has a bi-functorial isomorphism $F_S(M \otimes p^* N) \simeq F_S(M) \otimes q^* N$, for any $M \in D_{\text{fhd}/X,S}(X \times_T S)$, $N \in D_{\text{perf}}(S)$.

2) It is compatible with $F$: for any $M \in D_{\text{fhd}/X,S}(X \times_T S)$ one has a natural isomorphism $f_* F_S(M) \simeq F(f_* M)$.

3) $F_S$ is functorial on $F$.

**Proof.** The proof is completely analogous to that of Theorem 2.8. One constructs the relative version of the “perfect resolution” of section 2.2 just replacing $k$ by $T$. The $T$-linearity of $F$ is necessary to reproduce Lemma 2.9 in the relative setting. The flatness hypothesis is necessary for the use of flat base change and 1) of Proposition 2.7. \hfill \square

As in the absolute case, we obtain corollaries:

**Corollary 3.7.** One has a $T$-linear functor

$$\Psi : \text{Hom}_T(D_{\text{perf}}(X), D_{\text{perf}}(Y)^*) \to D_{\text{perf}}(X \times_T Y)^*$$

$$F \mapsto F_X(\mathcal{O}_\Delta)$$

and the composition $\Phi \circ \Psi$ is isomorphic to the identity.

**Corollary 3.8.** Assume that $X$, $Y$ and $T$ are projective $k$-schemes. One has $T$-linear functors

$$\Phi : D_{\text{perf}}(X \times_T Y)^{\text{bi-}} \to \text{Hom}_T^\text{ex}(D_{\text{perf}}(X), D^b_c(Y))$$

$$\omega \mapsto \Phi_\omega$$

and

$$\Psi : \text{Hom}_T^\text{ex}(D_{\text{perf}}(X), D^b_c(Y)) \to D_{\text{perf}}(X \times_T Y)^{\text{bi-}}$$

$$F \mapsto F_X(\mathcal{O}_\Delta)$$

and the composition $\Psi \circ \Phi$ is isomorphic to the identity.

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