End-point of the $rp$ process and periodic gravitational wave emission

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(Dated: June 27, 2018)

Publication of the first data from the GEO 600 and LIGO interferometer gravitational wave detectors is a notable event in the study of neutron stars. Upper limits for periodic emission were obtained, initially for PSR J1939+2134, and then for a further 27 isolated pulsars. The current rate of improvement in sensitivity makes it possible to detect gravitational wave emission from events at distances of several hundred kiloparsecs. The analysis of these data leads to the conclusion that the equation of state of neutron star matter is not uniform and that the mass quadrupole moment tensor components generated by it are not vanishing.

I. INTRODUCTION

The general end-point of the $rp$ process in rapidly accreting neutron stars is believed to be a surface deformation and gravitational wave emission. The equation of state is determined by the presence of nuclear formation enthalpy minima at the proton closed shells. At $\rho_{nd}$, a sequence of weak interactions with capture or emission of neutron pairs rapidly transform nuclei to the most accessible proton closed shell. Therefore, angular asymmetries in nuclear composition present in accreted matter at $\rho_{nd}$ are preserved during further compression toward densities $\sim 10^{14}$ g cm$^{-3}$ provided transition rates between closed shells are negligible. Although it has been confirmed that this condition is satisfied for predicted internal temperatures and for the formation enthalpy distribution used in this work, it would not be so if the true enthalpy differences between maxima and minima in the distribution were a factor of two smaller. For this reason, it does not appear possible to assert with any confidence that position-dependent surface composition can lead to significant angle-dependence of the equation of state and to potentially observable gravitational radiation. The effect of non-radial internal temperature gradients on angle-dependency of the equation of state is also not quantifiable.

PACS numbers: 97.60.Jd, 97.80.Jp, 95.55.Ym, 26.60.+c

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The inner part of the solid crust at $\rho > \rho_{nd}$ is more massive ($\sim 10^{-2} M_{\odot}$) and therefore the investigation of possible angle-dependency in composition and equation of state in this region is a very important problem. It has been considered recently by Uschomirsky, Cutler and Bildsten in calculations based on the work of Sato and of Ref. 12. The analysis of the evolution of nuclear charge at $\rho > \rho_{nd}$ given in the present paper is in substantial disagreement with Ref. 2 because these authors assume, following Sato and Ref. 12, that accreted nuclei at $\rho > \rho_{nd}$ follow an evolutionary path of decreasing $Z$, produced by successive electron captures, until transition rates for pycnonuclear fusion reactions become appreciable at $Z \approx 10$. Calculations of nuclear formation enthalpies at $\rho > \rho_{nd}$, though necessarily very approximate, show that local minima exist at the proton closed shells with $Z = 20, 28, 34, 40, 50$. Our analysis is that the presence of these minima determines the evolution of accreted matter during compression from the neutron-drip threshold until $\rho$ approaches the region of phase transitions to possible lower-dimensional structures, or the liquid core of the star. The internal temperature and the nature of the formation enthalpy distribution are crucial factors for whether or not the system reaches weak-interaction equilibrium (homogeneity in $Z$) in the $10^6 - 10^7$ yr interval of high accretion rates. Sec. II A contains a brief description of the $Z$-dependence of the nuclear formation enthalpy which is the basis of the present paper. A system in which more than one closed shell is populated must eventually become unstable, during compression to higher densities, against quantum-mechanical tunnelling of protons between adjacent nuclei, even if weak-interaction transition rates are negligibly small. We also consider the inward movement of the accreted matter and show in Sec. III A that the stress-response of the solid cannot, without qualification, be described as elastic. Calculation of the quadrupole moment produced by the angle-dependent equation of state is therefore not a simple problem and for this reason the solid crust is represented as a one-dimensional system in Sec. III B. A brief comparison is given with the one-dimensional treatment considered in Ref. 3.

Whether or not periodic signals are seen by interferometric detectors, it is important to have some understanding of the ways in which they might be generated. Magnetic stress is the more simple source, although questions such as the form of the field distribution and the nature of proton superconductivity present themselves. The main conclusion of this paper is that our present understanding of the relevant aspects of nuclear structure in the neutron-drip region is such that predictions of gravitational wave emission arising from compositional asymmetries cannot be made with confidence. But the significant factors are considered in Sec. IV.

II. THE STRUCTURE OF THE NEUTRON-DRIP SOLID

A. Shell effects above the neutron-drip threshold

The canonical view of the solid phase has been that strong and weak-interaction equilibria are maintained during cooling so that a Coulomb $bcc$ lattice of nuclei is formed on solidification, neutralized by a relativistic electron gas. The matter density at the neutron-drip threshold has some dependence on the details of the equation of state but is $\rho_{nd} \approx 4 \times 10^{11} \text{ g cm}^{-3}$. At $\rho > \rho_{nd}$, the neutron chemical potential exceeds its rest energy and the nuclei are in equilibrium with both the electrons and a continuous neutron medium, superfluid at temperatures below $T \sim 10^9$ K. The early and classic paper of Negele and Vautherin, involving microscopic calculations of the single-particle states for neutrons and protons inside a Wigner-Seitz cell, gave the structure of nuclei within the neutron medium. The energy differences per nucleon, at fixed baryon density, between lattices with different proton closed shells, for example, $Z = 40, 50$ were found to be very small, indicating that the standard assumption of a homogeneous $bcc$ lattice might be questionable. Much later, calculations of the formation enthalpies for monovacancies and for charge-impurity point-defects produced values which showed that the assumption could not be valid. In that work, a simple procedure was used to obtain the properties of homogeneous lattices as a basis from which the enthalpy changes produced by perturbations (the point-defects) could be calculated. Nuclei were described by the compressible liquid-drop model (CLDM) of Lattimer et al with their model parameter values. Table I contains the calculated lattice parameters used. The CLDM nuclear charge $Z$ is a continuous variable, and the neutron and proton chemical potentials are given with reference to the neutron rest energy. The formation enthalpy for a point-defect of charge $Z$ can be expressed as $H_{FZ} = C(Z - Z)^2$ relative to the homogeneous lattice. It is reasonable to expect that the order of magnitude of $H_{FZ}$ should be related to the monovacancy formation enthalpy $H_{Fv}$ through the expression $H_{FZ} \approx (ZZ^{-1} - 1)^2 H_{Fv}$, with $C \approx H_{Fv} Z^{-2}$. The values of $C$ contained in the final column are broadly consistent with this, given that $H_{Fv} \approx 15 \text{ MeV}$ in these $H_{FZ}$ values, determined solely by Coulomb lattice and bulk nuclear matter properties, were small and it became obvious that shell corrections should not be neglected. Their inclusion, based on the single-particle levels found by Negele and Vautherin and the Strutinski procedure (see Ref. 19), changes the $H_{FZ}$ to the formation enthalpies shown in Fig. 1. These are exclusive of integral multiples of the electron and neutron chemical potentials. The quadratic dependence on $Z - Z$ and the effect of proton closed shells are both obvious. We refer to Ref. 14 for further details of the methods of calculation of these $H_{FZ}$ and of the approximations and
assumptions on which they are based. They are necessarily subject to some uncertainty and it is also the case that, for laboratory nuclei far from stability, the dependence of the spin-orbit interaction on neutron excess remains a problem of current interest (see Schiffer et al. [20]). Although the formation enthalpies given in Fig. 1 are, as we have stressed, of uncertain quantitative value (which is why the simplification of approximating free energy by enthalpy has been made), it is fairly certain that shell corrections producing local minima in $H_{FZ}$ must exist, even if they are smaller than those of $\bar{H}_{FZ}$. An additional factor supporting this conclusion is the belief that the quadratic dependence of $H_{FZ}$ on $Z - Z$ has not been seriously underestimated. (The reason for this is that the monovacancy formation enthalpies obtained in [16] are large when expressed in units of the melting temperature, $H_{Fv} \approx 35k_BT_m$, whereas for alkali metals they are $\approx 10 - 15k_BT_m$.) Therefore, the qualitative features of Fig. 1, and specifically the existence of large potential barriers between closed-shell values of $Z$, are probably well-founded. The consequence is that weak-interaction equilibrium fails during cooling of primordial neutron-star matter [14] producing a $Z$-heterogeneous and amorphous solid. This is the primordial state of the solid in an isolated neutron star or in a binary system that has not yet evolved to a high rate of mass accretion.

**B. Elimination of heterogeneity by quantum-mechanical tunnelling**

The metastable state of matter described in Sec. IIA is not possible at matter densities approaching the phase transition to the core liquid. Lower-dimensional phases, if they exist at densities between those of the spherical nuclear phase and liquid core [21], are unable to support point-defects. But since such phases may not exist, it is necessary to investigate the stability of the $Z$-heterogeneous solid for values of $\rho$ larger than those of Table I. At finite temperatures, quantum-mechanical tunnelling of single protons and of pairs can occur between any nucleus and its nearest neighbours. Consider two nuclei with proton closed shells $Z_1, Z_2$. The tunnelling of a single proton forms particle and hole excitations, $Z_1 \rightarrow Z_1 \pm 1$ and $Z_2 \rightarrow Z_2 \mp 1$ with an associated enthalpy change $\delta H_F > 0$ which must be balanced by interaction with other degrees of freedom present in the system. (Pair tunnelling would allow formation of even $-Z$ nuclei but with much smaller transition rates.) The nature of the potential barrier in this unusual system is somewhat uncertain, but we have assumed that it is determined by the difference between that proton chemical potential $\mu^e_p$ which is defined by proceeding to the limit of zero proton number density within the neutron continuum, and the chemical potential $\mu_p$ in the nuclear volume. The values of $\mu^e_p$ given in Table I have been obtained from the Skyrme bulk nuclear matter pseudo-potential used by Lattimer et al. [18]. The order of magnitude of the barrier penetration factor, calculated for the homogeneous lattice of charge $Z$, in which $h^2\kappa^2 = 2m_p(\mu^e_p - \mu_p)$ and $x = 2(r WS - r N)$, is given in Table I.

Scattering of neutron superfluid quasiparticles by protons is the most important interaction with other degrees of freedom. A very approximate estimate of the transition rate for the tunnelling process $Z_1 \rightarrow Z_1 \pm 1$ and $Z_2 \rightarrow Z_2 \mp 1$ can be found from the Table I barrier penetration factor. It is of the order of

$$\gamma_q = \frac{m_n^2 k_{Fn}^2 k_B T |\bar{V}|^2}{2\pi^3 h^3} e^{-2\kappa x - \beta \delta H_F - \beta \Delta_n}$$

(1)

where $\beta^{-1} = k_B T$ and the parameters $k_{Fn}$ and $\Delta_n$ are the neutron Fermi wavenumber and energy gap within the nuclear volume. The order of magnitude of the strong-interaction scattering matrix element is $\bar{V} \sim 10^2$ MeV fm$^3$, leading to a transition rate

$$\gamma_q \sim 10^3 e^{-2\kappa x - \beta \delta H_F - \beta \Delta_n s^{-1}},$$
TABLE I: Homogeneous CLDM lattice properties are given for a set of matter densities $\rho$. These are the neutron continuum number density $n_n^c$, the nuclear charge $\tilde{Z}$ and radius $r_N$, the Wigner-Seitz radius $r_{WS}$, the neutron and proton chemical potentials $\mu_n, \mu_p$ and the pressure $P$. Also given is the CLDM proton chemical potential $\mu_p^c$ defined in the limit of zero proton number density within the neutron continuum. The final two columns give the proton barrier penetration factor $\exp(-\kappa x)$ and the parameter $C$.

| $\rho$     | $n_n^c$  | $\tilde{Z}$ | $r_N$  | $r_{WS}$ | $\mu_n$ | $\mu_p$ | $\mu_p^c$ | $P$       | $\exp(-\kappa x)$ | $C$   |
|------------|----------|-------------|--------|----------|---------|---------|-----------|-----------|-------------------|-------|
| (10$^{13}$ g cm$^{-3}$) | (10$^{-3}$ fm$^{-3}$) | (fm) | (fm) | (MeV) | (MeV) | (MeV) | (MeV fm$^{-3}$) | (MeV) | |
| 1.6        | 7.8      | 34.6        | 5.8    | 27.1    | 4.9     | -46.4   | -9.1      | 0.0176   | 2.0$\times$10$^{-25}$ | 0.0142 |
| 3.7        | 18.4     | 35.1        | 6.3    | 22.0    | 7.2     | -53.6   | -20.9     | 0.0526   | 7.2$\times$10$^{-18}$ | 0.0096 |
| 8.8        | 43.6     | 34.3        | 7.1    | 17.1    | 10.1    | -63.7   | -45.7     | 0.158    | 7.2$\times$10$^{-9}$  | 0.0051 |

at 10$^9$ K. For temperatures expected either during accretion or early in the life of an isolated neutron star, and for values $\delta H_F \sim 1$ MeV that are consistent with Fig. 1, the most significant exponential term in Eq. (1) is certainly the barrier penetration factor. Its rapid increase from negligibly small values is an unavoidable consequence of the variations of both $r_{WS}$ and $\mu_p^c - \mu_p$ with increasing $\rho$. At values of $\rho$ not much larger than the third row of Table I, the proton tunnelling transition rate must become so large that the metastable equilibria represented by the minima in Fig. 1 become short-lived and all nuclear charges approach that of the most favoured closed shell, probably $Z = 40$.

Our conclusion is that the state of matter at this density and beyond is independent of its past history and that, in consequence, any direct transition from the spherical nuclear phase to the liquid must occur at a unique pressure $P_c$ and density $\rho_c$. The same statement can be made of any intermediate transition to lower-dimensional solid phases which can support some anisotropic stress-tensor components. This has the consequence that, within the Cowling approximation, in which the perturbation to the gravitational potential caused by an angle-dependent solid crust composition is neglected, the static structure of the neutron star is very simple. The surface separating liquid and solid phases is necessarily isobaric and spherical with radius $R_c$, and these conditions are maintained by strong and electromagnetic interactions. This statement, of course, neglects rotation and the effect of thermal fluctuations which average to zero over macroscopic areas. The extent to which it would be true for the various modes of oscillation of a neutron star is not considered here.

III. EVOLUTION OF ACCRETED MATTER

A. The inward flow of matter

The predicted nuclear mass numbers produced in the rp process burning of accreted matter extend to $A \approx 110$ [5 8 4 10]. Widths and shapes of these mass number distributions depend on thermodynamic conditions which are determined principally by the local rate of accretion and its hydrogen-helium composition. Therefore, it is very likely that the mean mass number of nuclei entering the surface ocean of the star is a function of surface position specified by the angles $\theta, \phi$, with the possibility of significant differences between the magnetic polar and equatorial regions. The extent to which convection motions exist in the ocean is a complex problem but is known to depend on magnetic field strength and on the presence of a non-radial temperature gradient [23, 24]. The more conservative assumption is that, in general, mixing is incomplete over the whole surface area and that some composition angle-dependence remains as matter enters the solid phase of the crust. This inward movement of individual nuclei occurs at constant $A$, but with $Z$ decreasing owing to electron capture transitions. Given the width of the predicted distributions, the solid structure formed must be amorphous, with any order limited to very short lengths [13].

However, further structural rearrangement must occur at a very high rate during compression because the average nuclear cell radius decreases during the inward movement. Below the neutron-drip threshold, it is $\propto \rho^{-1/3}$ and at $\rho > \rho_{bd}$, the density-variation obtained from column 5 of Table I shows almost the same dependence. These variations are very much more rapid than those which accompany the inward radial movement of a lattice without change of form and necessarily imply that structural rearrangement occurs. Its effect can be seen by assuming that, at some instant of time, the strain components are $\varepsilon_{ij}$, defined relative to suitable fixed coordinates. Given the $p$-dependence of $r_{WS}$, it follows that nuclear rearrangements consequential to inward movement can, in principle, change the strain at a rate as large as $\varepsilon_{ij} \sim \tilde{v}/L_\rho$ where $L_\rho \sim 10^4$ cm is the scale-length for radial density variation and $\tilde{v}$ is the bulk inward radial velocity of matter at density $\rho$. Although we are unaware of any strictly analogous terrestrial system, the high rate of nuclear rearrangement has some similarity with viscoelasticity. The solid system considered here must have the same elastic stress-response as an amorphous solid for times several orders of magnitude shorter than $\varepsilon_{ij}L_\rho/\tilde{v}$, but over longer times, the strain is continuously changing.
owing to the inward movement of accreted matter. The important feature is that the solid, unlike a conventional visco-elastic structure, has no intrinsic relaxation time; the values of $\dot{\epsilon}_{ij}$ are determined by the accretion rate. Later in the evolution of the binary system, when the accretion rate has become negligible, the structure is again that of an amorphous solid.

Below $\rho_{nd}$, the equation of state of neutron star matter is polytropic, $P = K \rho^\Gamma$, with an adiabatic index $\Gamma$ very close to $4/3$ because the pressure $P$ is well approximated by the relativistic electron pressure $P_e$. In this region, an incremental change in nuclear composition from reference mean values $\bar{A}$ and $\bar{Z}$ produces a simple change in the constant $K$. At a fixed $P_e$, the matter density $\rho \propto \bar{A}/\bar{Z}$ to a high level of approximation. The adiabatic index is constant and therefore,

$$
\frac{\delta K}{K} = -\Gamma \frac{\delta(\bar{A}/\bar{Z})}{(\bar{A}/\bar{Z})}.
$$

(2)

No similar elementary result can be derived at $\rho > \rho_{nd}$ because the neutron continuum contributes the greater part of the pressure. But the CLDM equation of state underlying Table I is approximately polytropic, with adiabatic index $\Gamma = 1.29$, and we shall assume that composition changes in this region also give $K \to K + \delta K$ with no incremental change in $\Gamma$.

### B. Angle-dependent structure of the solid crust

Given the considerations of Sec. IIB and IIIA, the basic properties of the crust in an accreting neutron star can be found very simply using a polytropic equation of state and the Cowling approximation, in which perturbations of the gravitational potential are neglected. Suppose that the nuclear composition is angle-dependent, giving an equation of state with $K \to K + \delta K(\theta, \phi)$, but with the length-scale for non-radial variation of $\delta K$ at least of the same order of magnitude as the crust depth. A three-dimensional elastic-deformation treatment of the crust is appropriate for calculations of the adiabatic modes of oscillation such as those of McDermott, Van Horn and Hansen at zero or negligible accretion rate. Nevertheless, given the many uncertainties in the formulation of the present problem, a one-dimensional (radial) approximation for local values of $\theta, \phi$ has the merit of transparency and is adequate.

In the three-dimensional treatment, Eulerian deviations from the reference system ($\delta K = 0$) satisfy the full static equilibrium condition (or quasi-static in view of the accretion flow)

$$
\frac{\partial \delta \sigma_{ij}}{\partial x_j} + \delta \rho g_i = 0,
$$

(3)

in which the deviation in the stress tensor,

$$
\delta \sigma_{ij} = \delta \sigma_{ij}^h + \sigma_{ij}^M - \delta P \delta_{ij},
$$

(4)

is the sum of the deviation in the shear components of the elastic stress tensor, the Maxwell tensor, and the term $-\delta P$ which includes the deviations in all other isotropic stress tensor components of the system. Let us assume that $\delta K \neq 0$ within a pressure interval $0 < P < P_a$, where $P_a \leq P_c$. (It is possible that $P_a$ may be slightly composition and therefore angle-dependent, but this would have an effect one or more orders of magnitude smaller than those of first order in $\delta K$.) In the interval $P_a < P < P_c$ where matter has reached equilibrium, either through high rates of proton tunnelling or a transition to a lower-dimensional structure, the angle-dependence vanishes so that $\delta K = 0$. Eq. (3) satisfy the boundary conditions $\delta \sigma_{rr} = 0$ at pressures $P = 0, P_a$, though possibly with a small error if $P_a < P_c$. The density deviation, integrated from $R_a$ to the surface of the star is then

$$
\int_{R_a}^{R} \delta \rho dr = \frac{1}{g} \sum_j \int_{R_a}^{R} \frac{\partial \delta \sigma_{ij}}{\partial x_j} dr.
$$

(5)

In a true one-dimensional approximation, the integrated density deviation therefore vanishes. It is worth comparing Eq. (5) with Eq. (51) and (52) of Ref. 8 which satisfy identical boundary conditions. But these authors find, by reference to their three-dimensional solutions, that $| \delta \sigma_{rr} |$ is at least an order of magnitude larger than $| \delta \sigma_{rj} |$ throughout most of the interval of integration. They therefore conclude that a one-dimensional estimate of a quadrupole moment which they express, in Eq. (52) of their paper, as

$$
Q_{22} = R^4 \int \delta \rho dr
$$

(6)

is extremely poor owing to its independence from the major stress deviation $\delta \sigma_{rr}$. It is obvious that this is correct. (In the one-dimensional approximation which will be adopted here, the moment so defined would be $Q_{22} = 0$.) These authors observe that changes in the radial distribution of density produced by $\delta K \neq 0$ are a far larger contributor to quadrupole moments than the non-radial displacements which also arise. Although the latter can be large ($\sim 10^4$ cm), they are smaller than the typical scale length for non-radial variation of $\delta K$ and so are much less significant. For a thin crust, it must be the case that any calculation which gives the correct radial distribution of density deviations also gives a satisfactory approximation to the quadrupole moment.

A one-dimensional solution of the form given here in Eq. (6) is capable of this. But it is necessary to use the correct quadrupole moment definition given by Eq. (1) of Ref. 8 instead of Eq. (6).

Solutions of Eq. (3) are easily obtained in the one-dimensional approximation for the case in which $\delta K$ is independent of depth within the pressure interval $0 < P < P_a$. We replace $\delta \sigma_{rr}$ by $-\delta P$ and find that the incremental changes from the reference values of pressure
and density are,
\[
\delta P^{(1)} = \frac{P\delta K}{K(\Gamma - 1)} \left( \frac{\rho_a}{\rho} \right)^{\Gamma - 1} (\Gamma - 1),
\]
\[
\delta \rho^{(1)} = \frac{\rho\delta K}{K(\Gamma - 1)} \left( \frac{\rho_a}{\rho} \right)^{\Gamma - 1} - \Gamma.
\]
(7)

The radial displacement of a point fixed in Lagrangian coordinates is,
\[
\delta r^{(1)} = \frac{\delta K}{g_a(\Gamma - 1)} \left( \frac{\rho_a}{\rho} \right)^{\Gamma - 1} - \Gamma,
\]
where \(g_a\) is the magnitude of the gravitational acceleration \(g\) at pressure \(P_a\). There is a density discontinuity \(-\rho_a\delta K/K\Gamma\) at \(P_a\). The radius of the stellar surface \((P = 0)\) is angle-dependent and the case \(\delta K > 0\), a slightly more stiff equation of state, produces a local increase in radius.

Eq. (4) makes obvious the fact that the term \(\delta P\) produced by an angle-dependent \(\delta K \neq 0\) has an effect analogous with the isotropic components of the Maxwell tensor. The change of phase to the liquid occurs at a fixed pressure \(P_c\) and density \(\rho_c\). In the Cowling approximation, and neglecting the effect of rotation, the radius \(R_c\) is angle-independent, but its magnitude reduces very slowly with time for most equations of state during accretion as the total mass of the star increases. Solution of the three-dimensional Eq. (3) has not been attempted. It would not be simply a problem in elasticity because \(\epsilon_{ij}\) is time-dependent for non-zero accretion rates as we have described in Sec. IIIA. From Eq. (3) and (4), and with neglect of the Maxwell tensor, we can see that the non-radial components of the shear stress deviation \(\delta \sigma_{ij}^{sh}\) are of the same order of magnitude as \(\delta P\). Hence the maximum \(\delta P\) for which the solid will respond quasi-elastically is of the order of \(10^{-2}\bar{\mu}\) for an amorphous structure, where \(\bar{\mu}\) is here the shear modulus. For greater values of \(\delta P\), the behaviour of the system will approach that of a fluid. The very complex problem represented by this condition is not considered here. In the neutron-drip region, the pressure and shear modulus have almost identical \(\rho\)-dependences and so are approximately related by \(\bar{\mu} \approx 6 \times 10^{-3} P\). Therefore, the maximum \(\delta K\) for which the solid behaves quasi-elastically is given by \(\delta K/K \approx 10^{-2}\bar{\mu}/P \approx 6 \times 10^{-5}\). We can see that the inward movement of matter during accretion is basically radial but, given the non-radial gradient in \(\delta P\), it cannot be exactly so.

C. Nuclear transitions above the neutron-drip threshold

The neutron-drip region at \(\rho > \rho_{nd}\) contains almost all the mass of the crust. Some information about the distribution of \(A\) and \(Z\) for matter moving into this region has been provided by Haensel and Zdunik (see Fig. 1 of [13]). Successive electron captures in the outer crust, for the two mass number examples \(A = 56, 106\) considered by these authors, lead to the formation of nuclei with charges \(Z = 18, 32\), respectively, as the inward moving accreted matter is compressed to \(\rho_{nd}\). The assumption which appears to have been made by Sato [11] and by later authors [12] is that, with further compression inside the neutron-drip region, electron capture transitions continue at constant \(A\) until the transition rates for pycnonuclear fusion reactions become appreciable at \(Z \approx 10\). Comparison with calculations of the neutron-drip state, for example, those of Negele and Vautherin [13] shows that the charges found by Haensel and Zdunik \((Z = 18, 32)\) lie well below the equilibrium \(Z\). Therefore, unless some special case for the contrary can be made, the nuclear transitions which occur must follow a path of increasing \(Z\). They are, principally, the successive capture of neutron pairs followed by electron emission. For \(\mu_c > 0\), nuclei can be viewed as bound states of protons embedded in a neutron continuum. Thus there is no barrier to increasing nuclear \(A\) and \(Z\) values by these transitions. Pycnonuclear fusion rates are negligible except possibly at the low-\(Z\) end of the nuclear number density distribution.

The work of Ref. [13] shows that the inward moving accreted matter may have a fairly wide distribution of \(Z\) at \(\rho_{nd}\), but with an rms value \(Z_\alpha\) which is probably much smaller than either the equilibrium values calculated by Negele and Vautherin [13] or those given in Table I. In order to see how the accreted matter changes as it is further compressed to \(\rho > \rho_{nd}\) it is convenient to use, with some modifications, the CLDM procedure which is the basis for Table I. Strong and electromagnetic-equilibrium constraints are retained, but the condition \(\delta \mu_i = 0\), where
\[
\delta \mu_i = \mu_n - \mu_{pi} - \mu_e - (m_p - m_e)\epsilon^2 - \frac{\partial f_c}{\partial Z}
\]
and \(f_c\) is the Coulomb energy of the Wigner-Seitz cell ([13]), is removed so that the nuclear charge \(Z\) can be treated as a constant parameter. For the modified CLDM procedure, the distribution of \(Z\)-values is approximated by a binary system \((i = 1, 2)\). The proper-frame neutron chemical potential \(\rho_{n}\) is related with the time-like component of the metric and so, to a good approximation, can be assumed a fixed quantity at any point. However, the electron chemical potential \(\mu_e\), defined here as the Fermi energy, necessarily has a common value for the binary system which has to be determined locally. This has been achieved by modifying the constraint [13] relating the nuclear surface thermodynamic potential density, defined here as \(\sigma_i\), with the Coulomb energy of the Wigner-Seitz cell. (In the binary system and as a matter of convenience, we refer to the electrically neutral sphere, with nucleus \(Z_i\) at the origin, as the Wigner-Seitz cell.) Clearly this cannot be satisfied simultaneously for the binary mixture \(i = 1, 2\) and so we have adopted the form
\[
\sum_i a_i (2\pi r_N^2\sigma_i - f_{ci}) = 0,
\]
(10)
for this constraint, where \( r_{N_i} \) is the nuclear radius and \( a_i \) the number density fraction. These are the two modifications to the CLDM procedure that have been made.

The two remaining constraints in the CLDM procedure, those of neutron chemical potential and nuclear pressure equilibria are satisfied by both nuclei of the binary system.

Although the modification given by Eq. (10) is a very elementary approximation for the binary system, the most important strong and electromagnetic-interaction constraints have been retained. Calculations of the equilibrium, using this CLDM procedure, confirm that the basic properties of accreted matter change in an intuitively obvious way as it is compressed to \( \rho > \rho_{nd} \). For the case in which the binary system mean square nuclear charge \( \sum a_i Z_i^2 = \bar{Z}^2 \), where \( \bar{Z} \) is the equilibrium charge given in Table I (for the lowest matter density, and with \( Z_1 = 28 \) and \( Z_2 = 40 \)), we find that the computed values of \( \delta \mu \), are very small and that there are negligible changes in \( P \), \( \rho \) and chemical potential per baryon. This is consistent with the discussion of formation enthalpy differences given in Sec. II A. But the mean square nuclear charge \( Z_i^2 \) of accreted matter at \( \rho_{nd} \) is smaller than \( \bar{Z}^2 \). Therefore, we have computed binary model cases in which \( Z_1 = \bar{Z} \) and \( Z_2 = 20 \) and find that the chemical potential imbalance defined by Eq. (9) is an approximately linear function of \( a_2 \), and is given by \( \delta \mu_2 \approx 0.6 + 27 a_2 \) Mev. The implication is that weak-interaction transitions rapidly reduce the value of \( a_2 \) at \( \rho > \rho_{nd} \) and change the system toward the equilibrium state of Table I. Ultimately, for small values of \( \bar{Z} - Z_1 \) and therefore of \( a_2 \), the nuclei of charge \( Z_2 \) can be regarded as impurities in the \( \bar{Z} \) system so that the formation enthalpy distribution of Fig. 1 becomes valid and the speed and efficiency of this process are determined by the temperature and by the formation enthalpy barriers between the closed shells at \( Z = 20, 28, 34 \) described in Sec. II A.

Nuclear reactions caused by compression of the accreted matter are a volume source of heat. Ref. [2] gives temperature distributions for an accretion rate of \( 10^{-8} M_\odot \) yr\(^{-1} \) calculated using various assumptions about neutron superfluidity. They are fairly slowly varying functions of \( \rho \) with maxima of \( 7 - 8 \times 10^8 \) K at \( \rho_{nd} \). (These high values show that a classical neutron gas must be present in a finite region with \( \mu_n < m_n c^2 \).) For this accretion rate, the inward radial velocity is \( 6 \times 10^{-8} \) cm s\(^{-1} \), at the threshold \( \rho_{nd} = 6 \times 10^{11} \) g cm\(^{-3} \). Initially, for large imbalances \( \delta \mu_1 > 0 \), weak-interaction transitions have no formation enthalpy barrier and are rapid. But as the value of \( Z_a \) increases toward \( \bar{Z} \) and the formation enthalpy distribution of Fig. 1 becomes valid, the question arises of the extent to which thermal excitation of the weak transitions allows movement between proton closed-shells. Low enough formation enthalpy barriers would allow all nuclei to reach a unique proton closed-shell well within the transit time of accreted matter in the crust.

This is an important question because the modified CLDM procedure described above shows that a nuclear charge imbalance, of the form \( Z_2 < \bar{Z} \), has a significant effect on the equation of state. This should not be unexpected because both the nuclear pressure equilibrium constraint and Eq. (10) depend quadratically on \( Z \). For the binary system charges \( Z_1 = \bar{Z} \) and \( Z_2 = 20 \), at the lowest density of Table I, the incremental changes \( \delta P \) and \( \delta \rho \) with respect to the system with mean charge \( \bar{Z} \) give an incremental change in the equation of state,

\[
\frac{\delta K}{K} = \frac{\delta P}{P} - \frac{\Gamma \delta \rho}{\rho} = 0.15 a_2,
\]

for an assumed constant \( \Gamma = 4/3 \). In terms of quadrupole-moment generation, this would be a potentially enormous effect if its presence were angle-dependent. The problem becomes one of estimating the interval of \( \rho \) over which \( a_2 \) is non-negligible.

Assuming the formation enthalpies given in Fig. 1 (lowest density), the rates for \( Z = 23 \rightarrow 24 \) and \( Z = 25 \rightarrow 26 \) have been calculated as functions of temperature. The approximations made in calculating weak-interaction transition rates are precisely those described in previous papers [14, 17]. In the temperature region of interest, the time-constant for the depletion of a small value of \( a_2 \), with \( Z_2 = 20 \), is \( \tau \approx 3.3 \exp(51/K T) \) s, where \( T_9 \) is the temperature in units of \( 10^9 \) K. The times, for the temperatures predicted at \( \rho_{nd} \) in Ref. [2], are so long that there is no question that significant non-zero \( a_2 \) can survive the \( \sim 10^{14} \) s interval of rapid accretion. But we have previously stressed the many sources of uncertainty in the details of Fig. 1. The existence of these problems is unfortunate because halving the formation enthalpy barrier (roughly equivalent to halving the exponent in the time-constant expression) reduces the depletion time constant to values within one or two orders of magnitude of the accretion time-interval.

In summary, the evolution of accreted matter in the neutron-drip region is as follows. A nucleus of charge \( Z \) at \( \rho_{nd} \) undergoes rapid weak-interaction transitions, with capture or emission of neutron pairs, toward the most accessible proton closed shell configuration. The calculations of burning [2, 8, 12] and of evolution below the neutron-drip threshold [15] indicate \( Z \)-distributions which would evolve by transitions toward those at \( Z = 20, 28, 34 \) and possibly \( 40 \). Their relative populations reflect the width and form of the distribution of \( A \) formed in the atmosphere. But the mean square charge \( Z_a^2 \) at \( \rho_{nd} \) is appreciably smaller than \( \bar{Z}^2 \) so that, as our CLDM binary system calculations confirm, there will be further rapid transitions toward one or more closed shells near \( \bar{Z} \) until the populations of the other closed shells become small, of the order of \( a_i \sim 10^{-2} \). At this stage, the system may be metastable or, depending on temperature and on formation enthalpy barrier height, there may be further transitions to a unique closed-shell \( Z \). As we have emphasized in the previous paragraph, it is unfortunately the case that we cannot decide between these two possibilities with any confidence. The sequence of forma-
tion enthalpy distributions shown in Fig. 1 indicate that compression to $\rho > \rho_{nd}$ may itself change the closed-shell populations. Thus the $Z = 28$ minimum becomes less pronounced in the vicinity of $3.7 \times 10^{14}$ g cm$^{-3}$ and it is possible that its population is transferred firstly to $Z = 34$ and then, near $8.8 \times 10^{13}$ g cm$^{-3}$, to $Z = 40$. But it must be re-emphasized that the specific values of $H_{FZ}$ in Fig. 1 should not be taken too seriously and that although they show the general way in which the system evolves, any details that have been presented for illustrative purposes are not necessarily reliable. These changes under compression continue until the density reaches $\rho_a$, where the transition rates for quantum-mechanical proton tunnelling become so large that any metastability disappears.

Non-radial temperature gradients, if present in the solid crust, are an independent source of composition angle-dependency on any isobaric surface at $\rho < \rho_{nd}$. This arises, in principle, because the weak-interaction transition rates which change $Z$ as $\rho$ increases are temperature-dependent. Above $\rho_{nd}$, the initial progression of a nucleus toward a particular proton closed shell is rapid because there is no formation enthalpy barrier. Thus temperature is unlikely to be a significant factor. But sequences of transitions between adjacent closed shells are, as we have described, extremely temperature-dependent and it is not possible to predict these rates with any confidence.

**IV. CONCLUSIONS**

Unfortunately, the main conclusion reached in this paper is that it is not possible to calculate the angle-dependence of the equation of state (the incremental function $\delta K(\theta, \phi)$) arising from either an angle-dependent atmospheric composition or non-radial temperature gradients in an accreting neutron star. We have shown that such a calculation depends on nuclear structure properties that are very unlikely to be established with any degree of certainty in the immediate future. But this conclusion, though unfortunately negative, is not without value in relation to future developments in gravitational wave detection.

The hypothesis that the limiting rotation frequencies of neutron stars in low-mass X-ray binary (LMXB) systems are determined by gravitational radiation specifies a fairly compact range of values for the mass-quadrupole tensor component. Integration of either Eq. (5) or (7) shows that an increment $\delta K \neq 0$ produces no change in the total mass within the interval $0 < P < P_a$ though there is a density discontinuity $\delta \rho_{pa}$ at $P_a$. Therefore, the resultant mass-quadrupole tensor component, as defined by Eq. (1) of Ref. 15 with $\delta \rho_{pm}$ replaced by $\delta \rho^{(1)}$, arises solely from the mass rearrangement given by Eq. (7) and is

$$Q_{22} \approx \frac{3}{20} h^2 \rho_a R_a^3 \frac{\delta K}{K},$$

where $h$ is the depth of the $\delta K \neq 0$ layers. If these are confined to $\rho < \rho_{nd}$, for layers such that $h^2 \rho_a \approx 10^{19}$ g cm$^{-1}$, the predicted mass-quadrupole tensor is several orders of magnitude smaller than the specified value, $Q_{22} = 3.5 \times 10^{37}$ g cm$^2$. In the neutron-drip region, values of $h^2 \rho_a \approx 10^{24}$ g cm$^{-1}$ are possible, for which the specified $Q_{22}$ would be given by $\delta K \approx 2 \times 10^{-4} K$. It is also interesting that these $\delta K$ are not much different from the approximate value corresponding with the quasi-elastic limit noted at the end of Sec. IIIB. This is not inconsistent with the conclusions of Ref. 15. With reference to Eq. (11), we can see that only a very small change in charge imbalance, if suitably angle-dependent, is needed to produce these $\delta K$ at a given density. But, as we have emphasized, it is not possible to predict whether or not the metastable population of proton closed shells can be sufficiently long-lived to maintain this.

The above conclusions are for angle-dependent composition asymmetries formed in the atmosphere of the star and make no reference to composition asymmetries produced by non-radial internal temperature gradients which may be present. Those formed at $\rho < \rho_{nd}$ have too small an effect to be of significance, but at $\rho > \rho_{nd}$ there is a possibility that the conclusions of Sec. IIIC may require modification. The value of $Z_a$ at $\rho_{nd}$ may have a small temperature-dependence, but this will not affect the population of proton closed shells initially formed. The major effect of temperature asymmetry is in changing the transition rates from these closed shells to the closed-shell $Z$ of complete weak-interaction equilibrium. As emphasized in Sec. IIIC, these rates are exponentially-dependent on formation enthalpy differences that are not well-known. Although they are probably many orders of magnitude too small or large for conceivable temperature gradients to have any effect, there is a non-quantifiable though small probability that transition rate orders of magnitude may allow the formation of angle-dependent proton closed-shell populations within an appreciable interval of $\rho$.

It is worth considering briefly the extent to which intrinsic temperature-dependence of the equation of state can lead to asymmetry in the presence of non-radial temperature gradients. There appears to be no published work on this problem for the case of the solid phase of neutron star matter. Calculations at high temperatures assume a normal neutron continuum, also the presence of classical (translational) degrees of freedom, and so cannot be extrapolated to $T < T_m$. In this region, the most important source is likely to be the temperature dependence of the electron partial pressure $P_e$. For the order of magnitude of temperature variation assumed in Ref. 16 ($\delta T \sim 0.05T$ at $T \approx 4 \times 10^8$ K), the resultant increment of $\delta K \sim 4 \times 10^{-8} K$ at a typical chemical potential $\mu_e = 60$ MeV is too small to be of significance.

The origin of the differences between this paper and Ref. 15 have been described in Sec. III. The evolutionary path of accreted matter at $\rho > \rho_{nd}$ is the major area of disagreement. Our analysis is that it is determined by the
proton closed-shell structure of nuclei. The disagreement concerning the treatment of the solid crust is much less important and in Sec. IIIB we have shown how it arises.

A three-dimensional calculation of the structure for an angle-dependent equation of state forms a large part of Ref. [5]. These authors assume an elastic response and the analysis given draws on that of McDermott, Van Horn and Hansen [25]. But we have shown in Sec. IIIA that the elastic-response assumption is not valid during high accretion-rate intervals which necessarily produce nuclear rearrangement within the solid structure. In Sec. IIB and IIIB, we have shown that the basic boundary condition in the accretion problem is that the pressure \( P_c \) at the crust-core boundary is a constant. In the Cowling approximation, and with neglect of the effects of rotation and thermal fluctuations, the boundary surface is a sphere whose radius \( R_c \), for most equations of state, decreases slowly as the mass of the star increases. There is no impediment to the maintenance of this condition by the rapid transfer of matter between the solid and liquid phases through electromagnetic and strong interactions, and there is no immediate local dependence on weak-interactions transition rates. This condition leads to the very simple one-dimensional solutions Eq. (7) and (8) of Sec. IIIB which appear hydrostatic in nature. Nonetheless, the stability of the system obviously depends on the shear modulus of the solid.

As a source of gravitational radiation, it appears from our analysis that the deformation produced by angle-dependence of the \( rp \) process end-point is not quantifiable and may be no more important than that derived from the Maxwell tensor in the solid crust. The two sources are not distinguishable unless there is some \textit{a priori} information about the internal field. The uncertainties exposed by our analysis seem unlikely to be removed in the near future and indicate that it will not be possible to decide with any confidence whether angle-dependent composition or the magnetic structure and superconductivity of the core is the more probable origin of any periodic signals seen in future gravitational wave experiments [1].

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