On possible violation of the CHSH Bell inequality in a classical context

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It has been shown that there is a small possibility to experimentally violate the CHSH Bell inequality in a “classical” context. The probability of such a violation has been estimated in the framework of a classical probabilistic model in the language of a random-walk representation.

The Bell inequality (BI) is supposed to discriminate between “classical” world and “quantum” one. Any classical theory, i.e. causal theory governed by local (possibly, hidden) variables, should fulfill the BI. Furthermore, any possible violation of the BI is usually interpreted as a sign of quantumness. In other words, it is commonly believed that only quantum mechanics is allowed to violate the BI. Then, the important question emerges: is it possible to violate the BI in the framework of a classical theory? If so, the powerful role of the BI could be a little diminished. It appears that the answer can be in principle positive, but it depends on details. Actually, the first such case is related to the notion of the postselection [1], other one refers to the notion of the memory loophole [2], there is an analog of the detection loophole presented in the NMR context in [3], and finally in optics in [4]. One should emphasize that a lot of them can be observed in laboratories.

In this letter, we will show, and this is our main aim, that in finite number of measurement rounds it is possible to “classically” violate the BI, and we will estimate the probability of such a violation in the framework of a simple classical probabilistic model. To make things as simple as possible, we will confine ourselves to the BI in the version of Clauser, Horne, Shimony and Holt (CHSH) (see, [5] for a contemporary introduction), and a simple classical Bernoulli-like model.

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We would like to strongly stress that our considerations have nothing to do (though, perhaps, could be somehow linked to) with real errors inevitably being encountered in actual measurements of true experiments (see, our toy model as a simple demonstration of the point). Otherwise, our conclusions would be entirely trivial. In other words, our “measurements” are exact, but our model is statistical.

First of all, we would like to remove the seeming contradiction between the well-known proof of the CHSH inequality and our a bit controversially sounding statements. Namely, according to our interpretation, which plays an auxiliary role and is not crucial for our model, one should note that a delicate point in the CHSH argumentation consists in summing up (with one minus sign) and collecting the all four terms \( a_i(\lambda)b_j(\lambda), \ i, j = 1, 2 \), i.e.

\[
a_1(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda) + a_2(\lambda)b_1(\lambda) + a_2(\lambda)b_2(\lambda),
\]

under the common probability measure \( P(\lambda)d\lambda \), yielding

\[
\int P(\lambda) [a_1(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda) + a_2(\lambda)b_1(\lambda) + a_2(\lambda)b_2(\lambda)] \ d\lambda.
\]

Here, as usual, \( a_i(\lambda) = \pm 1, b_j(\lambda) = \pm 1 \) are values of the polarizations measured by the two (distant) observers \( A \) and \( B \), respectively. The indices \( i, j = 1, 2 \) label the two orthogonal directions of the polarizers \( A \) and \( B \), and \( \lambda \) is a collective hidden variable governed by the probability distribution \( P(\lambda) \) (see, \( [6] \) for a very short introduction).

According to our interpretation the loophole in the CHSH argumentation consists in the direct addition and integration in (1) and (2). One should take into account that, simply speaking, each of the four correlations in (1) is, in practice, being measured in different time instants. Since hidden variables are in principle allowed to evolve in time, we have in general different \( \lambda \)'s in each of the four terms, and (1) and (2) does not, in general, make sense. Seemingly, similar doubts, but formulated rather in mathematical than physical terms are presented in \( [7, 8, 9] \).

If somebody is still sceptical concerning our above interpretation with the time \( t \) in its central role we offer the following additional remarks: 1) Let \( \lambda \equiv \lambda(t) = t \), for example. Does Eq.((1)), Eq.((2)) or the standard normalization \( \int P(\lambda)d\lambda = 1 \) make sense? Rather not. 2) The role of time is to break correlations, which forbids making the valid sums of the type \( a_1 \pm a_2 \) being used in the proof of the CHSH inequality. Our mathematical colleagues have already noticed some inconsistencies of this type, referring it to the issue of different probabilistic spaces (see, earlier references). 3) Our toy model presents a kind of direct “experimental” explanation of this phenomenon. 4) Finally, one can skip this explanatory paragraph at all, if one dislikes it, and directly jump to our model and the results which are interpretation independent. Our interpretation plays only an auxiliary role in our paper.

Obviously, the very fact that the proof has a loophole does not automatically mean that the CHSH inequality can be actually classically violated unless we show it explicitly. That will be done in due course.

There is also another seeming contradiction in our proposal. Namely, nobody is reporting an experimental classical violation of the CHSH inequality due to finite statistics. The explanation is very simple. Actually, the probability of such a violation in a real
experiment is very small. In fact, it depends on the number of the measurement rounds. It will be shown that the greater the number of the measurement rounds the lower probability of the violation. The fact that the number of measurement rounds is finite is crucial.

A conclusive explanation of the both above mentioned seeming contradictions will be given in the form of the proposed model. Mathematically, the model is formulated in probabilistic terms. In other words, formally, speaking in the language of hidden variables, one could state that the hidden variables are purely random entities. From physical point of view, we could interpret the randomness of the hidden variables in various ways. They could be fundamentally random. Their randomness could follow from complicated internal classical statistical mechanics. Or possibly, their classical mechanical evolution could be fundamentally very complicated, e.g. chaotic. Anyway, we are only interested in this letter in a principal possibility of classical violation of the CHSH inequality independently of a possible physical mechanism, i.e. the nature of the hidden variables.

A notable precedent of our work is a multi-thread and polemical study undertaken by Gill in [10] and [11]. One of the relevant threads of the Gill’s papers is a polemic centered around the role of statistics in violation of the CHSH Bell inequality. Namely, some authors propose classical scenarios significantly, according to them, violating the CHSH Bell inequality. Gill argues, using advanced statistical calculus, that such large violations, suggested by them, are impossible.

To quickly illustrate the point, first, we will present a toy version of our model. To this end, let us confine ourselves (in this toy version) to only four measurement rounds (in the full version that number will be large). Let us perform the four measurements in four different time instants, denoted: 1, 2, 3 and 4. We could even identify time with the hidden variable governing the process, but it is optional. “Accidentally”, it appears that our experimental data are such as those given in Table 1. Obviously, nothing forbids to obtain such data. As it is clear from Table 1 we deal with maximal possible violation of the CHSH inequality. Namely, we obtain the (maximal) number 4 instead of the classical bound given by the number 2 or the quantum bound given by $2\sqrt{2}$. We can observe, enriching our earlier arguments, that the constraints used in the standard proof of the CHSH inequality do not work because the variables $a$ and $b$ in different time instants (or for different hidden variables) are totally unconstrained. Evidently, the data given in Table 1 constitute

| measurement No. (time instants) | a | b | $c \equiv ab$ | i | j | contributions to the CHSH BI |
|---------------------------------|---|---|-------------|---|---|-----------------------------|
| 1                               | +1| +1| 1           | 1 | 1 | 1                           |
| 2                               | +1| -1| -1          | 1 | 2 | 1                           |
| 3                               | +1| +1| 1           | 2 | 1 | 1                           |
| 4                               | +1| +1| 1           | 2 | 2 | 1                           |

**Table 1:** Data maximally violating the CHSH BI in a classically allowed experiment

a very particular configuration out of many others. How many? A simple calculus, reduced for simplicity to $c \equiv ab$ (which is equivalent to a more tedious calculus given in
terms of \( a \) and \( b \) separately) gives \( 2^4 = 16 \) all configurations. We have assumed that the \( i,j \) pairs are fixed. This is a simplification (also present in our full model)—in a more realistic situation only the total number of the measurement rounds should be fixed (to 4, in this example). Since there are 2 violating configurations (one for 4 and one for \(-4\)), the total probability \( p \) of the violation of the CHSH inequality in our toy model amounts to

\[
p = \frac{2}{2^4} = \frac{1}{8} = 0.125. \quad (3)
\]

Now, let us consider the full model. In other words, we will estimate the probability of the violation of the CHSH inequality, in the spirit of the toy model, for a larger number of measurement rounds. In terms of probability theory our model is described by a Bernoulli process \([12]\). But it is more convenient, from intuitive point of view as well as from calculational one, to recast the model into (four-dimensional) random walks. As we have just observed, we can think and work directly in terms of the outcome \( c \equiv ab \) instead of in terms of \( a \) and \( b \), separately. In our random-walk representation the value \( c = +1 \) corresponds to the step forward, and the value \( c = -1 \) corresponds to the step backward. The direction forward/backward (\( \pm 1 \)) is “decided” first. Next, the “particle” (we mean the abstract particle of the random-walk representation) should be “informed” which dimension out of 4 should be followed. There are 4 dimensions corresponding to the 4 possibilities (pairs) given by the orientations of the polarizers \( A \) and \( B \) (\( i = 1,2 \) and \( j = 1,2 \)). The total number of the steps in one of the four dimensions is denoted by \( n_i \) (\( i = 1,2,3,4 \)). Since each step assumes the value +1 or \(-1\), the actual \( i \)th coordinate of the position of the “particle” is the numerator of the \( i \)th correlation. In other words, each of the 4 correlations entering to the full CHSH correlation is of the form (“frequency definition”)

\[
\sum_{i=1}^{n_i} \frac{\pm 1}{n_i} \equiv \frac{m_i}{n_i}. \quad (4)
\]

Therefore, the experimentally read CHSH correlation is given by

\[
C = \frac{m_1}{n_1} - \frac{m_2}{n_2} + \frac{m_3}{n_3} + \frac{m_4}{n_4}. \quad (5)
\]

Now, the inequality corresponding to the (rare) violating cases we are interested in assumes the form

\[
|C| > 2. \quad (6)
\]

For a large number of the measurement rounds, i.e.

\[
n_i \gg 1, \quad (7)
\]

we can use a continuous approximation. The probability distribution of the one-dimensional discrete random walk \([12]\) is summarized by:

\[
P_n(m) = \begin{cases} 
\left(\frac{1}{2}\right)^n \frac{n_1}{n_1 + n_2}, & \text{for } m \equiv n (\text{mod } 2) \\
0, & \text{for } m \not\equiv n (\text{mod } 2),
\end{cases} \quad (8)
\]
which in turn can be approximated, by virtue of the Stirling formula, by

\[ P_n(x) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}}. \]  

(9)

Though our (auxiliary) random walk is four-dimensional, the probability distribution for

\[ \log_2 N = n_1 + n_2 + n_3 + n_4, \]

in a doubly-logarithmic scale. 3 variants—solid line: \( n_1 = n_2 = n_3 = n_4 \); dashed line: \( 10n_1 = n_2 = n_3 = n_4 \); dotted line: \( 100n_1 = n_2 = n_3 = n_4 \). The five vertical intervals correspond to the exact (discrete) calculations easily performed for small \( N \) and equal \( n_i \). The lowest points of the intervals correspond to \( C > 2 \), whereas the highest ones correspond to \( C \geq 2 \).

Each coordinate is independent. Therefore, the full four-dimensional probability measure is given by

\[ dP_{n_1 n_2 n_3 n_4}(x_1, x_2, x_3, x_4) = \frac{1}{\sqrt{2\pi n_1}} \frac{1}{\sqrt{2\pi n_2}} \frac{1}{\sqrt{2\pi n_3}} \frac{1}{\sqrt{2\pi n_4}} e^{-\frac{x_1^2}{2n_1} - \frac{x_2^2}{2n_2} - \frac{x_3^2}{2n_3} - \frac{x_4^2}{2n_4}} d^4 x. \]  

(10)
From mathematical point of view the task is to calculate the probability of finding the walking particle in the four-dimensional space outside the layer $L$ bounded by the two hyperplanes $C_2$ and $C_{-2}$ (see, Eq. (5) and (6)), where

$$C_{\pm 2} : \frac{x_1}{n_1} - \frac{x_2}{n_2} + \frac{x_3}{n_3} + \frac{x_4}{n_4} = \pm 2. \quad (11)$$

For technical reasons we will “isotropize” the coordinates of the probability measure by the following change of variables:

$$x_i = \sqrt{2n_i} z_i, \quad i = 1, 2, 3, 4. \quad (12)$$

Now, the (“isotropic”) probability measure (10) is of the form

$$dP_{n_1n_2n_3n_4}(z_1, z_2, z_3, z_4) = \frac{1}{\pi^2} e^{-z_1^2 - z_2^2 - z_3^2 - z_4^2} d^4 z, \quad (13)$$

whereas the equations of the bounding hyperplanes (11) are

$$C_{\pm 2} : \sqrt{\frac{2}{n_1}} z_1 - \sqrt{\frac{2}{n_2}} z_2 + \sqrt{\frac{2}{n_3}} z_3 + \sqrt{\frac{2}{n_4}} z_4 = \pm 2. \quad (14)$$

The distance $d$ between the hyperplane given by the equation $\sum_i a_i x_i = b$ and the beginning of the coordinate system is expressed by the formula:

$$d = \frac{|b|}{\sqrt{\sum_i a_i^2}}. \quad (15)$$

Making use of the isotropy of the probability measure we can rotate the hyperplanes so that they are orthogonal to the coordinate $z_1$, say. Therefore, the probability $p$ we are interested in is given by the integral

$$p(n_1, n_2, n_3, n_4) = \frac{2}{\pi^2} \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} d^3 z \ e^{-z_1^2 - z_2^2 - z_3^2 - z_4^2} \equiv \text{erfc}(d)$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \ e^{-t^2} \equiv \text{erfc}(d) \quad (16)$$

where in the last line we have made use of (14) and (15). Since $p$ is very quickly damped by large $n_i$, the probability of the violation of the CHSH inequality for $n_i \gg 1$ is very small.

A visual summary of our results is given in Fig. 1. Violation of the CHSH BI, as well as its scale, is now evident. For simplicity, in our model, we have assumed fixed values of $n_i$. In a more realistic model only the sum $N = n_1 + n_2 + n_3 + n_4$ should be fixed,
which would roughly correspond to a mean of \((16)\) with respect to \(n_i\). But, obviously, our technical simplification, qualitatively, does not change our final conclusion.

The continuous approximation is only a simplifying, technical trick because discrete calculations, as combinatorial ones, are straightforward only for small \(N\). Nevertheless, we observe that the continuous approach does work quite good also for small \(N\).

Not to cause any misunderstandings in the field we would like to stress that in the limit of infinite number of measurement rounds the CHSH Bell inequality is restored in the classical domain. In other words, assuming the “infinity” condition as a premise in the CHSH Bell theorem removes the classical violation but evidently the “infinity” condition is experimentally unrealistic.

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