Some Remarks on Conformal Symmetries and Bartnik’s Splitting Conjecture

I. P. Costa e Silva, J. L. Flores and J. Herrera

Abstract. Inspired by the results in a recent paper by Galloway and Vega (Lett Math Phys 108(10):2285–2292, 2018), we investigate a number of geometric consequences of the existence of a timelike conformal Killing vector field in a globally hyperbolic space-time with compact Cauchy hypersurfaces, especially in connection with the so-called Bartnik’s splitting conjecture. In particular, we give a complementary result to the main theorem in [11].

Mathematics Subject Classification. 53C50, 83C75.

1. Introduction: Motivations and Statement of Main Results

In 1988, Bartnik [1] posed the following conjecture:

**Conjecture 1.1.** Let \((M^{n+1}, g)\) be a globally hyperbolic space-time with compact Cauchy hypersurfaces, satisfying the timelike convergence condition \((TCC) \, \text{Ric}(v, v) \geq 0\) for every \(v \in TM\) timelike. Then either \((M, g)\) is timelike geodesically incomplete or else it is globally isometric to a product space-time \((\mathbb{R} \times S, -dt^2 \oplus h)\), where \((S, h)\) is a compact Riemannian manifold.

Bartnik calls a globally hyperbolic space-time (of any dimension) possessing a compact Cauchy hypersurface and satisfying the TCC a *cosmological space-time*, because the “spatially closed” Robertson–Walker models in relativistic cosmology (with suitable matter content and cosmological constant) are obviously important examples of such space-times. The Bartnik’s conjecture then becomes the statement that *a cosmological space-time is either timelike geodesically incomplete or else splits isometrically as a product of a Lorentz line and a compact Riemannian manifold*. (For simplicity, we shall simply say then that the pertinent space-time *splits*; in this paper, this phrase will always refer to the specific kind of isometric splitting appearing in the Bartnik conjecture.)

The importance of Bartnik’s conjecture both for geometry and physics because it establishes a rigidity statement for the celebrated 1970 singularity theorem of Hawking and Penrose [16,24]. The latter theorem implies, in par-
ticular, that cosmological space-times which in addition satisfy the \textit{generic condition} have an incomplete timelike or null geodesic. Of course, it is well known (see, e.g., Chapters 2 and 14 of [2]) that the main technical effect of the generic condition is to induce pairs of conjugate points along complete nonspacelike geodesics; hence, it is natural to ask whether nonspacelike geodesic incompleteness still holds when one drops the generic condition. In this context, Bartnik’s conjecture (if true) implies that timelike geodesic completeness indeed holds only in exceptional, non-generic cases in the class of cosmological space-times.

Bartnik’s conjecture has been investigated by a number of researchers, especially Galloway and collaborators [1,7–11], who have proven alternative versions under a variety of additional assumptions. Two of the latter will especially concern us here. The first one is an early result by Galloway [8]: if a cosmological space-time \((M,g)\) is timelike geodesically complete and satisfies the \textit{no-observer-horizon (NOH) condition} \(M = I^\pm(\gamma)\) for any inextendible timelike curve \(\gamma\), then \((M,g)\) splits as in the Bartnik conjecture.

The NOH condition is relevant for the Bartnik’s conjecture because it implies that \((M,g)\) has \textit{no null geodesic lines}. To see this, let such a null geodesic line \(\gamma\) be given. In this case, we would have \(I^-(\gamma) \cap \text{Im}\gamma = \emptyset\) from the achronality of \(\gamma\). But \(I^-(\gamma)\) is an \textit{indecomposable terminal past set} (TIP) [12], and as such there would exist an inextendible \textit{timelike} curve \(\beta\) [12] such that \(I^-(\beta) = I^-(\gamma) \neq M\), violating the NOH condition.

Now, any globally hyperbolic space-time \((M,g)\) with compact Cauchy hypersurfaces admits either a null or a timelike geodesic line by standard arguments (see, e.g., Chapter 8 in [2]). Since the first situation is excluded if the NOH condition holds, we would then have a timelike line in \((M,g)\). But a timelike geodesically complete cosmological space-time with a timelike line necessarily splits by the Lorentzian splitting theorem [2,6,7].

The second germane partial result is much more recent [11], and assumes the presence of a \textit{timelike conformal Killing vector field} in a vacuum cosmological space-time:

\textbf{Theorem 1.2.} (Galloway and Vega [10]) \textit{Let }\(n \geq 2\)\textit{ and suppose }\((M^{n+1},g)\)\textit{ is a Ricci-flat, timelike geodesically complete cosmological space-time possessing a timelike conformal Killing vector field }\(X \in \mathfrak{X}(M)\).\textit{ Then }\(X\)\textit{ is in fact a Killing vector field and }\((M,g)\)\textit{ splits.}

Our goal in this paper is to clarify a little further the effects of the presence of a timelike conformal Killing vector field on the global geometry of space-times with compact Cauchy hypersurfaces. In view of Theorem 1.2 and our own results below, this investigation is directly relevant to Bartnik’s conjecture, and this is our main motivation; but a distinctive feature of our work here is that we also pursue results \textit{independently of any assumptions on the Ricci tensor}.

On the one hand, all Friedmann–Lemaître–Robertson–Walker geometries which are so highly useful in cosmology admit a timelike conformal
Killing vector field (usually not complete). This fact alone justifies pursuing a deeper understanding of the effects of the presence of that form of conformal symmetry in the geometric setting of Bartnik’s conjecture, not necessarily satisfying the TCC. On the other hand, we need not make any special assumptions about spatial isotropy/homogeneity in this paper.

Concretely, our first main result is to show that the main role of a complete timelike conformal Killing vector field is precisely to ensure that the NOH condition (and hence the absence of null geodesic lines) is satisfied:

**Theorem 1.3.** Suppose \((M, g)\) is a globally hyperbolic space-time with compact Cauchy hypersurfaces, and admits a complete timelike conformal Killing vector field \(X \in \mathfrak{X}(M)\). Then the no-observer-horizon condition (1) is satisfied in \((M, g)\).

We emphasize that this result in particular does not require the TCC, and indeed no condition on the Ricci tensor at all. Its proof is actually elementary after one recalls some well-known facts about stationary space-times; we include it here as a theorem both because it is important in the development of the ideas in this paper, but also because the result itself seems not to have been stated before explicitly in the literature. Its relevance in the Bartnik’s context, if the TCC is assumed, is given by the following.

**Corollary 1.4.** Let \((M, g)\) be a timelike geodesically complete cosmological space-time. Then the following are equivalent.

1. \((M, g)\) splits.
2. \((M, g)\) admits a timelike Killing vector field.
3. \((M, g)\) admits a complete timelike Killing vector field.
4. \((M, g)\) admits a complete timelike conformal Killing vector field.

Corollary 1.4 shows that the existence and completeness of some (conformal) Killing vector field are inescapable if the Bartnik’s conjecture is to hold. Indeed, in the proof of Theorem 1.2 as presented in [11], the crucial role of Ricci-flatness lies in showing that \(X\) is Killing. (More precisely, one needs that \(\mathcal{L}_X \text{Ric} \equiv 0\), where \(\mathcal{L}_X\) denotes the Lie derivative with respect to the conformal Killing field \(X\).) But once one does ensure that \(X\) is Killing, its completeness then follows from a result by Garfinkle and Harris [13], which is used both in [11] and to prove that (ii) implies (iii) in Corollary 1.4 (see Sect. 3 below).

The particular argument given in [11], although very elegant, seems hard to generalize (in a natural way) for more general Ricci tensors. This has in part motivated our investigations here.

Now, without the TCC it is perfectly possible for a space-time to be geodesically complete, globally hyperbolic with compact Cauchy hypersurfaces and yet to possess an incomplete conformal Killing vector field. A simple and very important example is given by de Sitter space-time, which is globally a warped product \((\mathbb{R} \times S^n, -dt^2 + a^2 \omega_n)\), where \((S^n, \omega_n)\) is the unit radius, round \(n\)-sphere, and the warping function is \(a : t \in \mathbb{R} \mapsto \cosh t \in \mathbb{R}\). In this case, \(X = a \partial_t\) is an incomplete conformal Killing vector field. Note,
however, that $\text{Ric}(X, X) < 0$, i.e. TCC is violated everywhere. Thus, de Sitter space-time is not cosmological in Bartnik’s technical sense, and hence it is still consistent both with Bartnik’s conjecture and with Corollary 1.4.

However, it is well known (see, e.g. [15]) that de Sitter space-time (as well as other FLRW geometries) can be conformally embedded as an open set in a larger space-time (namely Einstein’s static universe $(\mathbb{R} \times S^n, -dt^2 \oplus \omega_n)$) in which $X$ can be extended as a complete unit timelike Killing vector field.

Our second main result is that this latter construction can be generalized as follows:

**Theorem 1.5.** Suppose $(M^{n+1}, g)$ is globally hyperbolic, and let $X$ be a timelike conformal Killing vector field in $(M, g)$. Then $(M, g)$ admits an open conformal embedding $\varphi : (M, g) \hookrightarrow (\hat{M}^{n+1}, \hat{g})$ with a complete, unit timelike Killing vector field $\hat{X} \in \mathcal{X}(\hat{M})$ extending the pushforwarded $\varphi_* X$ in $\hat{M}$. Moreover, if $S \subset M$ is a smooth, spacelike (hence acausal) compact Cauchy hypersurface for $(M, g)$, then $(\hat{M}, \hat{g})$ can be chosen so that $\varphi(S)$ is also a smooth, spacelike compact Cauchy hypersurface for $(\hat{M}, \hat{g})$.

Again, no assumptions on the Ricci tensor are made, and compactness of the Cauchy hypersurfaces is only required for the second part.

Our last main result extends Theorem 1.2 to a broader context insofar as we assume the TCC but not Ricci-flatness. However, in trying to prove the splitting result we have been unable to do without imposing an additional geometric condition to control the asymptotic behavior of the scalar curvature and the conformal Killing vector field. This control needs to be enforced only along one timelike geodesic.

**Theorem 1.6.** Assume that the following holds for the globally hyperbolic space-time $(M^{n+1}, g)$ with $n \geq 2$:

(i) the TCC is satisfied and $(M, g)$ has compact Cauchy hypersurfaces (so that $(M, g)$ is cosmological);

(ii) $(M, g)$ is timelike geodesically complete;

(iii) the timelike conformal Killing vector field $X$ satisfies at least one of the following conditions:

(iii.1) $X$ is complete, or

(iii.2) for some future-complete timelike geodesic $\gamma : [0, +\infty) \to M$ there exists a number $A > 0$ such that

$$\left(\sigma \circ \gamma\right)^2 - \frac{(\beta \circ \gamma)^2}{n(n-1)} R \circ \gamma \leq A \quad \text{along } \gamma, \quad (2)$$

where $R$ is the scalar curvature of $(M, g)$, $\beta := \sqrt{-g(X, X)}$, and $\sigma \in C^\infty(M)$ is as in Eq. (4) (see below), or

(iii.3) $\sigma$ is bounded.

Then $(M, g)$ splits as in the Bartnik conjecture.

**Remark 1.7.** The latter result is compatible with Theorem 1.2 in the following concrete sense. If $\text{Ric} = 0$, then we immediately have $R = 0$, and the results in Ref. [11] imply that $\sigma \equiv 0$. Therefore, the bounds in items (iii.2) and (iii.3) of Theorem 1.6 are automatically satisfied.
The rest of the paper is organized as follows. In Sect. 2, we give some basic technical preliminaries, mainly to establish the terminology and notation. In Sect. 3, the proofs of Theorem 1.3 and Corollary 1.4 are given, while in 4 we prove Theorem 1.5. In Sect. 5, however, not only Theorem 1.6 itself is proven, but we discuss in detail the geometric structure of the conformal embedding described in Theorem 1.5 when the conformal Killing vector field $X$ is incomplete. A number of ancillary results are proven therein which are of independent interest.

2. Preliminaries

Throughout this paper, we shall fix once and for all a space-time, i.e., a connected, Hausdorff, second-countable time-oriented $C^\infty$ Lorentzian manifold $(M^{n+1},g)$ with $\dim M = n + 1 \geq 2$. We shall assume that the reader is familiar with standard facts in Lorentz geometry and causal theory as given in the basic references [2,15,20].

In this section, we establish some preliminary technical facts which will be useful in the proof of our main theorems. We also recall a few standard definitions to settle the precise terminology and notation we shall use throughout the paper.

Let $Z \in \mathfrak{X}(M)$ be a timelike vector field (not necessarily conformal Killing). Its maximal integral curves define a one-dimensional foliation of $M$. We shall denote the set of leaves of this foliation by $Q = Q_Z$, and the standard projection which takes each $p \in M$ to the leaf through $p$ by $\pi = \pi_Z : M \to Q$. Then we have the following.

Proposition 2.1. Suppose $(M,g)$ is globally hyperbolic and let $Z \in \mathfrak{X}(M)$ be a timelike vector field. Then the following facts hold.

(i) There exists a unique topology and differentiable structure on $Q$ which makes it a smooth $n$-manifold for which $\pi : M \to Q$ is a smooth submersion. Moreover, $M$ is (non-canonically) diffeomorphic to $\mathbb{R} \times Q$. (In particular, $Q$ is homeomorphic to any given Cauchy hypersurface $S \subset M$.)

(ii) If, in addition, $Z$ is Killing, then there exists a unique Riemannian metric $h$ on $Q$ such that the canonical projection $\pi : M \to Q$ is a semi-Riemannian submersion, and the metric $g$ can then be written as

$$g = g(Z,Z)\theta_Z \otimes \theta_Z + \pi^*h,$$

where

$$\theta_Z := g\left(\frac{Z}{g(Z,Z)},\cdot\right).$$

Proof. (i)

Pick any positive $f \in C^\infty(M)$ for which $Y := f \cdot Z$ is complete. The maximal integral curves of $Y$ and $Z$ clearly define the same foliation, so in particular $Q_Y = Q_Z = Q$. Let $\phi : \mathbb{R} \times M \to M$ be the flow of $Y$. Since $Y$ is complete, $\phi$ is a smooth action of the abelian group $(\mathbb{R},+)$ on $M$. By Corollary 3.14
of \([5]\), the action \(\phi\) is free and proper. With respect of this action, \(\pi\) then defines on \(M\) the structure of a (necessarily trivial) principal \(\mathbb{R}\)-bundle over \(Q\). See, e.g., in \([19]\), p. 218, Theorem 9.16 of \([19]\), Sects. 12 and 13 of \([22]\), and Theorem 5.7, p. 58 of \([18]\) for more details.

(ii) Recall (see, e.g., Chapter 77, p. 212, Definition 44 of \([20]\)) that \(\pi: (M, g) \to (Q, h)\) being a semi-Riemannian submersion means that \(\forall x \in Q\), the fiber \(\pi^{-1}(x)\) is a semi-Riemannian submanifold of \(M\), which here is a trivial requirement, and \(d\pi\) preserves scalar products in vectors normal to fibers ('horizontal spaces'). The latter condition can be used to define \(h\): it is well defined because since \(Z\) is Killing, horizontal spaces are preserved along the integral curves thereof. The decomposition (3) then follows from a direct computation. \(\square\)

We shall say that \((M, g)\) is conformastationary if there exists some \(X \in \mathfrak{X}(M)\) which is everywhere a timelike conformal Killing vector field, i.e.,

\[
\mathcal{L}_X g = 2\sigma g
\]

for some function \(\sigma \in C^\infty(M)\), where here and hereafter \(\mathcal{L}_X\) denotes the Lie derivative with respect to \(X\). Of course, \(X\) is Killing if and only if \(\sigma \equiv 0\), in which case we say that \((M, g)\) is stationary (with respect to \(X\)). From (4), we easily deduce that

\[
\sigma := \frac{X(g(X, X))}{2g(X, X)}.
\]

In this paper, we do not take a given conformal Killing vector field \(X\) on \((M, g)\) to be complete, unless otherwise explicitly stated. On the other hand, we do require it to be timelike everywhere.\(^1\)

Example 2.2. (Standard (conforma)stationary space-times) Let \((M_0, g_0)\) be any smooth Riemannian \(n\)-manifold. On \(M_0\), pick a smooth, real-valued, strictly positive function \(\beta_0\), and a smooth 1-form \(\omega_0 \in \Omega^1(M_0)\). Fix also a strictly positive smooth function \(\Lambda_0 \in C^\infty(\mathbb{R} \times M_0)\). Then the standard conformastationary space-time associated with the data \((M_0, g_0, \beta_0, \omega_0, \Lambda_0)\) is \((M, g)\), where \(M := \mathbb{R} \times M_0\), and

\[
g = \Lambda_0^2(-\beta^2 d\pi_1 \otimes d\pi_1 + \omega \otimes d\pi_1 + d\pi_1 \otimes \omega + \pi_2^* g_0),
\]

where \(\beta := \beta_0 \circ \pi_2\), \(\omega := \pi_2^* \omega_0\), and \(\pi_1\) [resp. \(\pi_2\)] is the projection of \(M\) onto the \(\mathbb{R}\) [resp. \(M_0\)] factor. The time-orientation of \((M, g)\) is chosen such that \(\partial_t\), the lift to \(M\) of the standard vector field \(d/dt\) on \(\mathbb{R}\), is future-directed. The vector field \(\partial_t\) is then a timelike conformal Killing vector field and \((M, g)\) is indeed conformastationary. If \(\Lambda_0 \equiv 1\), then \((M, g)\) is said to be standard stationary (for the respective data), and if in addition \(\omega_0 \equiv 0\), then \((M, g)\) thus defined is said to be standard static.

\(^1\)The reader should be aware that such a completeness assumption is often included as part of the definition of 'conformastationary (resp. stationary) space-time' in the literature. Moreover, sometimes the timelike character is imposed only in some asymptotic sense, especially in the study of stationary black holes.
The following facts about the standard conformastationary metric (6) will be useful to us here:

1. The timelike conformal Killing vector field $\partial_t$ is complete.
2. $\pi_1$ is a smooth temporal function, i.e., it has timelike gradient [17], and hence the hypersurfaces $\{t\} \times M_0$ are acausal and spacelike for each $t \in \mathbb{R}$. In particular, the space-time is stably causal [17].

Next, consider the following well-known result, obtained in a broader context in Ref. [17], and with a much simplified proof in the particular case of globally hyperbolic space-times in [4]:

**Proposition 2.3.** Let $(M, g)$ be a globally hyperbolic space-time admitting a complete timelike conformal Killing vector field $X \in \mathfrak{X}(M)$. Then $(M, g)$ is isometric to a standard conformastationary space-time. Furthermore, if $(M, g)$ admits a compact Cauchy hypersurface, then, with the notation as in (6), $M_0$ can be also chosen to be a compact Cauchy hypersurface.

Recall that $P \subset M$ [resp. $F \subset M$] is a past set [resp. future set] if $P = I^-(P)$ [resp. $F = I^+(F)$]. Note that past and future sets are always open. A non-empty set $A \subset M$ is said to be an achronal boundary if $A = \partial P$ for some past set $P$. In this case, it is easy to check that $F := M \setminus \overline{P}$ is a future set and $A \equiv \partial F$, so we may alternatively define an achronal boundary as the—non-empty—boundary of a future set. Indeed, as discussed in detail in Ref. [10], for an achronal boundary $A$, there exists a unique disjoint decomposition $M = P \cup F \cup A$, where $P$ is a past set, $F$ is a future set, and $A = \partial P = \partial F$. In particular, any achronal boundary separates $M$, i.e., $M \setminus A$ is not connected.

As its name suggests, an achronal boundary $A$ is always an achronal edgeless set, but the converse is not true in general. The following result, however, exploits an exception which will be of key importance later on and has independent interest.

**Proposition 2.4.** Suppose $(M, g)$ is chronological (i.e., admits no closed timelike curves) and let $X \in \mathfrak{X}(M)$ be a complete timelike conformal Killing vector field with flow $\phi$. Then, any (non-empty) achronal edgeless set $A \subset M$ is an achronal boundary, and each orbit of $\phi$ intersects $A$ exactly once. Moreover,

$$\zeta = \phi|_{\mathbb{R} \times A} : (t, p) \in \mathbb{R} \times A \mapsto \phi_t(p) \in M$$

is a homeomorphism. In particular, $A$ is connected and separates $M$. If $(M, g)$ is globally hyperbolic, then $A$ is homeomorphic to any given Cauchy hypersurface $S \subset M$; and if in addition $A$ is compact then it is a Cauchy hypersurface.

**Proof.** The map $\zeta$ is continuous (smooth if $A$ is smooth), and one-to-one since $A$ is achronal. Since $A$ is a $C^0$ hypersurface, Invariance of Domain implies that $\zeta$ is then a homeomorphism onto a open subset $O \subset M$, and each orbit

---

2 A is then a closed $C^0$ (indeed Lipschitz) hypersurface in $M$, see, e.g., Corollary 26, Chapter 14 in [20].
of $\phi$ intersects $A$ at most once. Showing that each such orbit does indeed intersect $A$ and that $\zeta$ is a homeomorphism onto $M$ boil down, therefore, to showing that $\mathcal{O} \equiv M$. Since $\mathcal{O}$ is open and $M$ is connected, all we need to show is that $\mathcal{O}$ is closed. To this end, consider a sequence $(t_k, x_k)$ in $\mathbb{R} \times A$ and $p \in M$ such that $\phi_{t_k}(x_k) \to p$.

Assume first that $(t_k)$ is unbounded. We may assume, up to passing to a subsequence that $t_k \to +\infty$, the argument if $t_k \to -\infty$ being analogous. Let $\gamma$ be any maximal integral curve of the timelike conformal Killing vector field $X$. Since $X$ is assumed to be complete, we can apply the Corollary 3.2 of [14] to conclude that

$$M = I^\pm(\gamma).$$

In particular, taking $\gamma$ to be the orbit of $p$ by $\phi$ we have that $\phi_s(p) \in I^-(x_1)$ for some $s \in \mathbb{R}$. ($x_1$ being the first term in the sequence $(x_k)$!). But then

$$\phi_{t_k+s}(x_k) = \phi_s(\phi_{t_k}(x_k)) \to \phi_s(p),$$

so for large enough $k$ we have $t_k + s > 0$ and $\phi_{t_k+s}(x_k) \in I^-(x_1)$; hence,

$$x_k \ll \phi_{t_k+s}(x_k) \ll x_1,$$

which contradicts the achronality of $A$. Therefore, $(t_k)$ must be bounded. But in that case, up to passing to a subsequence we may assume that it converges, say, $t_k \to 0$. Let $x_0 := \phi_{-t_0}(p)$. Then

$$x_k = \phi(-t_k, \phi_{t_k}(x_k)) \to x_0,$$

and since $A$ is closed we conclude that $x_0 \in A$, and $\zeta(t_0, x_0) = p$, which shows that $\mathcal{O}$ is closed, as desired.

To see that $A$ is an achronal boundary, let $P := I^-(A)$. $P$ is clearly a past set, and the fact that $A$ is achronal implies that $A \subset \partial P$. Given any $p \in \partial P$, the previous results show that $\phi_t(p) \in A$ for some $t \in \mathbb{R}$. But $\partial P$ is achronal (it is an achronal boundary!), so we must have $t = 0$, which means that $p \in A$. We conclude that $A = \partial P$, so that $A$ is indeed an achronal boundary.

Now, assume that $(M, g)$ is globally hyperbolic. By Proposition 2.1(i), the leaf space $Q$ is a smooth $n$-manifold $Q$, and the standard projection $\pi : M \to Q$ is a smooth onto submersion. Thus, $\pi \circ \zeta \circ i$ is a homeomorphism between $A$ and $Q$, where $i : x \in A \hookrightarrow (0, x) \in \mathbb{R} \times A$. Since $Q$ is homeomorphic to any Cauchy hypersurface, then so is $A$.

Finally, suppose $(M, g)$ is globally hyperbolic and $A$ is compact. Let $\alpha : (a, b) \subset \mathbb{R} \to M$ be any future-directed, inextendible timelike curve in $(M, g)$ $(-\infty \leq a < b \leq +\infty)$. Then $\zeta^{-1} \circ \alpha : \lambda \in (a, b) \subset \mathbb{R} \mapsto (t(\lambda), x(\lambda)) \in \mathbb{R} \times A$ is continuous, and the fact that $\alpha$ is future-directed implies that $t$ is an increasing function. Fix any $\lambda_0 \in (a, b)$, and suppose $\alpha$ does not intersect $A$. In this case, $t(\lambda)$ is never zero, and we may assume, say, that $t(\lambda) < 0$ for all $\lambda \in (a, b)$ since the case if it always $> 0$ is entirely analogous. But then (since the $t$-coordinate increases) the curve $\zeta^{-1} \circ \alpha|_{\lambda_0, b}$ stays imprisoned in the compact set $[t(\lambda_0), 0] \times A$, and hence the future-inextendible timelike curve $\alpha|_{\lambda_0, b}$ stays imprisoned in the compact set $\zeta([t(\lambda_0), 0] \times A) \subset M$, which contradicts the strong causality of $(M, g)$ (see for instance Lemma
13, Chapter 14 in [20]). We conclude that $\alpha$ intersects $A$. We know it does so exactly once by the achronality of $A$, so $A$ is indeed a Cauchy hypersurface. □

Another relevant fact, which is easy to check, is that if $X$ is a timelike conformal Killing vector field in $(M, g)$, then $X$ is timelike Killing for $(M, \tilde{g})$, where

$$\tilde{g} := (1/\beta^2)g, \quad \beta := \sqrt{-g(X, X)}. \quad (8)$$

The following characterization will be technically useful later.

**Proposition 2.5.** Assume that $(M, g)$ is globally hyperbolic with compact Cauchy hypersurfaces, and let $X \in \mathfrak{X}(M)$ be a timelike conformal Killing vector field in $(M, g)$. Then, using the notation in (8), we have that $X$ is complete if and only if $(M, \tilde{g})$ is geodesically complete.

**Proof.** We know that $X$ is a timelike Killing vector field for $(M, \tilde{g})$. If the latter is geodesically complete, then $X$ is complete by Proposition 30, Chapter 9, p. 254 of [20].

Conversely, assume that $X$ is complete. Then the flow $\phi : \mathbb{R} \times M \to M$ defines (cf. the proof and notation of Proposition 2.1 (i)) is a free proper $\mathbb{R}$-action on $M$ which gives $\pi : M \to Q$ the structure of a principal $\mathbb{R}$-bundle. Since $X$ is unit Killing for $\tilde{g}$, we have, by Proposition 2.1(ii), that it has the form

$$\tilde{g} = g = -\theta \otimes \theta + \pi^*h,$$

but now $\theta(= \theta_X)$ can clearly be viewed as a connection 1-form over the bundle $\pi : M \to Q$, and hence this space-time is precisely of the form considered in Example 2.4 in [21] (with $f \equiv 1$). Therefore, it is geodesically complete by Proposition 2.1 in that reference. □

**3. Proof of Theorem 1.3**

In view of Proposition 2.3, and up to a conformal factor (which does not affect the purely causal arguments needed here) we can suppose

$$M = \mathbb{R} \times M_0, \quad g = -dt^2 + 2\omega(\cdot)dt + g_0,$$

with $M_0$ compact. Let $\alpha$ be any inextendible timelike curve and let $P = I^- (\alpha)$. We wish to show that $P = M$. Pick an arbitrary point $(t_*, x_*) \in M$. There is no loss of generality in assuming that the domain of $\alpha$ is $[0, +\infty)$ and we do so for the rest of the argument; we wish to show that $(t_*, x_*) \ll \alpha(s_*) = (t(s_*), x(s_*))$ for a large enough $s_* \in [0, +\infty)$. Consider the curve $\gamma(s) = (\tau(s), y(s))$ defined on the interval $[0, 1]$, where $\tau(s) := (t(s_*), t_*) + s + t_*$, and $y(s)$ is chosen to be any minimizing $g_0$-geodesic with $y(0) = x_*, \quad y(1) = x(s_*)$. Clearly, $\gamma(0) = (t_*, x_*)$ and $\gamma(1) = (t(s_*), x(s_*)))$. It suffices to show that $\gamma$ is timelike for $s_*$ big enough. Since $M_0$ is compact, and $\omega, g_0$ are independent of the $t$-coordinate, we have, for large enough $s_*$,

$$\dot{t} = t(s_*) - t_* > \omega(\dot{y}(s)) + \sqrt{\omega(\dot{y}(s))^2 + g_0(\dot{y}(s), \dot{y}(s))} \quad \text{for all } s \in [0, 1].$$
Hence, \( g(\dot{\gamma}(s), \dot{\gamma}(s)) = -\ddot{\tau}(s)^2 + 2\omega(\dot{y}(s))\dot{\tau}(s) + g_0(\dot{y}(s), \dot{y}(s)) < 0 \) for all \( s \in [0, 1] \), as required.

Proof of Corollary 1.4. As mentioned in Sect. 1, implication (ii) \( \implies \) (iii) follows immediately from Lemma 1 of [13], so the chain of implications

\[(i) \implies (ii) \implies (iii) \implies (iv)\]

is clear. The final implication (iv) \( \implies \) (i) follows from Theorem 1.3 together with Galloway’s result [8] mentioned in Sect. 1. □

4. Proof of Theorem 1.5

To fix ideas, we may always assume that \( X \) is future directed. Let \( \phi : U \subset \mathbb{R} \times M \to M \) denote the global flow of \( X \), and fix a smooth spacelike Cauchy hypersuface \( S \) in \((M, g)\). We know that \( U \) contains \( \{0\} \times M \), is open in the product topology in \( \mathbb{R} \times M \), and it equals the latter set iff \( X \) is complete. Let \( U_S := (\mathbb{R} \times S) \cap U \) and consider the smooth map \( \phi_S := \phi|_{U_S} \). The achronality of \( S \) implies that \( \phi_S \) is one-to-one, and hence a smooth homeomorphism onto an open set of \( M \) by Invariance of Domain. Since \( X \) is in particular non-zero everywhere, \( \phi_S \) is actually a local diffeomorphism (by the Inverse Function Theorem) and hence a diffeomorphism onto its image. Finally, the fact that \( S \) is Cauchy implies that \( \phi_S \) is onto. We conclude that the map \( \phi_S : U_S \to M \) is a global diffeomorphism. In particular, \( U_S \) is an open set in \( \mathbb{R} \times S \) diffeomorphic to \( \mathbb{R} \times S \) itself.

We denote generic points of \( U_S \) by \((t, x)\), with \( t \) viewed as a time coordinate and \( x \in S \). We also denote by \( \partial_t \) the lift, by the projection \( \pi_1 : (t, x) \in \mathbb{R} \times S \mapsto t \in \mathbb{R} \), of the standard vector field \( d/dt \) on \( \mathbb{R} \), and \( dt := d\pi_1 \), so that

\[ dt(\partial_t) \equiv 1. \tag{9} \]

We also keep the notation \( \partial_t \) for the restriction of this vector field to \( U_S \) if there is no risk of confusion. Accordingly, on \( U_S \) we have by construction

\[ (\phi_S)_*(\partial_t) = X. \tag{10} \]

As mentioned in Sect. 2, \( X \) is a unit Killing vector field with respect to the metric \( \tilde{g} \) given in (8). We shall analyze the globally hyperbolic space-time \((M, \tilde{g})\) more closely.

Using the concepts and notation of Proposition 2.1, we have a semi-Riemannian submersion \( \pi : (M, \tilde{g}) \to (Q, h) \), for some (uniquely given) Riemannian metric \( h \) on the quotient space \( Q = Q_X \), and (cf. Eq. (3))

\[ \tilde{g} = -\theta \otimes \theta + \pi^* h, \]

where

\[ \theta = -\tilde{g}(X, \cdot) \equiv -g(X/\beta, \cdot), \]

with \( \beta = \sqrt{-g(X, X)} \). Consider the pullback metric

\[ g_S := \phi_S^* \tilde{g} \equiv -(\phi_S^* \theta) \otimes (\phi_S^* \theta) + (\pi \circ \phi_S)^*(h) \tag{11} \]
on $U_S$. Then $\phi_S : (U_S, g_S) \rightarrow (M, \hat{g})$ becomes an isometry by construction. In particular, note that (10) implies that $\partial_t$ is a unit Killing vector field in $(U_S, g_S)$. Moreover, since isometries take (spacelike) Cauchy hypersurfaces onto (spacelike) Cauchy hypersurfaces, $(U_S, g_S)$ is globally hyperbolic and the hypersurfaces of the form 

$$\{t\} \times S$$

which are contained in $U_S$ are then spacelike Cauchy hypersurfaces therein.

We wish to show now that $g_S$ will have the general form (6) of a standard stationary metric. To do that, we define on $U_S$ a 1-form $\omega \in \Omega^1(U_S)$ given by

$$\omega := dt - \phi_S^* \theta.$$  

Now, using (10) and the definition of $\theta$, we conclude

$$(\phi_S^* \theta)(\partial_t) = \theta((\phi_S)_*(\partial_t)) = \theta(X) \equiv 1.$$  

Therefore,

$$\omega(\partial_t) \equiv 0,$$

and clearly, $\mathcal{L}_X \theta = 0$, so that

$$\mathcal{L}_{\partial_t} \omega = 0.$$  

We conclude that there exists a unique 1-form $\omega_0 \in \Omega^1(S)$ for which

$$\omega = \pi_2^* \omega_0,$$  

(12)

where $\pi_2 : \mathbb{R} \times S \rightarrow S$ is the canonical projection onto the second factor.

To proceed, consider the mapping $m : S \rightarrow Q$ given by

$$m : x \in S \mapsto \pi \circ \phi_S(0, x) \in Q.$$  

It not hard to check $m$ is a smooth diffeomorphism. Use it to define a Riemannian metric $h_0$ on $S$ by the pullback:

$$h_0 := m^* h.$$  

Finally, note that

$$m \circ \pi_2(t, x) = \pi(\phi_S(0, x)) \equiv \pi(\phi_S(t, x)), \forall (t, x) \in U_S,$$

i.e. $m \circ \pi_2|_{U_S} \equiv \pi \circ \phi_S$, whence we conclude that

$$(\pi \circ \phi_S)^* h = \pi_2^* h_0.$$  

(13)

Substituting (12) and (13) in (11) and rearranging, we get

$$g_S = -dt \otimes dt - \pi_2^* \omega_0 \otimes dt - dt \otimes \pi_2^* \omega_0 + \pi_2^* g_0,$$  

(14)

where we have defined

$$g_0 := h_0 - \omega_0 \otimes \omega_0.$$  

(15)

Note that the smooth $(0,2)$-tensor $g_0$ is actually the induced metric on each $t = \text{const.}$ hypersurface in $U_S$, and hence positive-definite. Equation (14) is the desired standard stationary form.

The next step is now clear: we define $(\hat{M}, \hat{g})$ as the standard stationary space-time (cf. Example 2.2) associated with the data $(S, g_0, \beta_0 \equiv 1, \omega_0)$,
which has precisely the form (14), and a complete unit Killing vector field \( \hat{X} \equiv \partial_t \) (cf. item (1) in Example 2.2). Clearly, this space-time is an isometric (trivial) extension of \((U_S, g_S)\), and the map

\[ \varphi := \phi_S^{-1} \]

gives the desired conformal embedding.

To complete the proof, assume that \( S \) is compact. We wish to show that \( \varphi(S) \equiv \{0\} \times S \) is a Cauchy hypersurface for \((\hat{M}, \hat{g})\). Let \( \alpha : \lambda \in (a, b) \subset \mathbb{R} \mapsto (t(\lambda), x(\lambda)) \in \mathbb{R} \times S \) be any future-directed causal inextendible curve in \((\hat{M}, \hat{g})\) \((-\infty \leq a < b \leq +\infty)\). In particular, the fact that this curve is future-directed implies that we have \( \dot{t}(\lambda) > 0 \), where the dot indicates derivative with respect to the curve parameter. Fix any \( \lambda_0 \in (a, b) \), and suppose \( \alpha \) does not intersect \( \varphi(S) \). In this case, we may assume, say, that \( t(\lambda) < 0 \) for all \( \lambda \in I \), since the case if it always \( > 0 \) is entirely analogous. But then (since the \( t \)-coordinate increases) the future-inextendible curve \( \alpha|_{[\lambda_0, b)} \) stays imprisoned in the compact set \([t(\lambda_0), 0] \times S\), which contradicts the strong causality of \((\hat{M}, \hat{g})\) (cf. (2) in Example 2.2). We conclude that \( \alpha \) does intersect \( \varphi(S) \), and since \( \alpha \) is arbitrary, and \( \varphi(S) \) is acausal (again by (2) in Example 2.2), it is indeed a Cauchy hypersurface as claimed. \( \square \)

5. Proof of Theorem 1.6

Our goal in this section is to understand in a more detailed fashion what can be said about the structure of the space-time in the context of the Bartnik conjecture when one has an incomplete conformal Killing vector field defined thereon.

We again assume, throughout this section, that \((M, g)\) is globally hyperbolic with compact Cauchy hypersurfaces and admits a timelike conformal Killing vector field \( X \in \mathfrak{X}(M) \). No a priori assumption on the Ricci tensor is made. We shall make extensive use of the conformally extended space-time \((\hat{M}, \hat{g})\) wherein \( X \) extends to a complete unit timelike Killing vector field \( \hat{X} \), as described in Theorem 1.5. We denote the boundary of \( M \) when viewed as an open subset of \( \hat{M} \) by \( \partial_{\hat{M}} M \equiv \partial M \).

The proof of Theorem 1.5 makes it clear that a point \( p \in \hat{M} \) is on \( \partial M \) if and only if it is an endpoint in \( \hat{M} \) of an incomplete maximal integral curve of \( X \). In particular, \( \partial M = \emptyset \) if and only if \( X \) is itself complete in \( M \). We start our discussion by examining more closely some facts about \( M \) and \( \partial M \).

**Lemma 5.1.** \( M \subset \hat{M} \) is causally convex, i.e., any causal curve segment in \((\hat{M}, \hat{g})\) with endpoints in \( M \) is contained in \( M \).

**Proof.** Let \( \alpha : [0, 1] \rightarrow \hat{M} \) be a future-directed causal curve segment in \((\hat{M}, \hat{g})\) with \( \alpha(0), \alpha(1) \in M \), and \( \alpha(t_0) \notin M \) for some \( 0 < t_0 < 1 \). Let \( S \) be a smooth spacelike (hence acausal) Cauchy hypersurface of \((M, g)\) with \( \alpha(1) \in S \). Since \( S \) is also an acausal Cauchy hypersurface of \((\hat{M}, \hat{g})\), \( \alpha(t) \notin S \) for any \( t \in [0, 1) \).
In particular, $\alpha(0) \in I^+(S, M) \cup I^-(S, M)$. However, if $\alpha(0) \in I^+(S, M) \subset I^+(\hat{M}, \hat{g})$, then for some $x \in S$, we would have
\[ x \ll_{(\hat{M}, \hat{g})} \alpha(0) \leq_{(\hat{M}, \hat{g})} \alpha(1) \Rightarrow x \ll_{(\hat{M}, \hat{g})} \alpha(1), \]
which violates the achronality of $S$ in $(\hat{M}, \hat{g})$. We conclude that $\alpha(0) \notin I^+(S, M)$.

Now, let
\[ s_0 := \sup\{t \in [0, 1] : \alpha([0, t]) \subset M\}. \]
Then, $s_0 \leq t_0 < 1$ and hence $\alpha|_{[0, s_0]}$ is a future-inextendible causal curve in $(M, g)$ starting at $\alpha(0)$ which does not intersect $S$, an absurd since $S$ is Cauchy.

**Lemma 5.2.** If $S, S' \subset M$ are two smooth spacelike Cauchy hypersurfaces in $(M, g)$, then
\[ I^\pm(S, \hat{M}) \cap \partial M = I^\pm(S', \hat{M}) \cap \partial M. \tag{16} \]

**Proof.** We only need to show that
\[ I^+(S, \hat{M}) \cap \partial M \subset I^+(S', \hat{M}) \cap \partial M \]
since then the opposite inclusion is immediate by exchanging the roles of $S$ and $S'$, and the past case proceeds by time-dual arguments.

But note that $S'$ being also Cauchy in $(\hat{M}, \hat{g})$ means that any putative $p \in I^+(S, \hat{M}) \cap \partial M \backslash I^+(S', \hat{M})$ would be in $J^-(S', \hat{M})$, and hence we might juxtapose a future-directed timelike curve from $S$ to $p$ with a future-directed causal curve from $p$ to $S'$, both in $(\hat{M}, \hat{g})$. However, since the resulting composition of these two curves could be viewed as a causal curve in $(\hat{M}, \hat{g})$ with endpoints in $M$ which leaves $M$, this would contradict 5.1, and hence the desired inclusion must hold.

With Lemma 5.2 in mind, we may define, for some smooth spacelike Cauchy hypersurface $S \subset M$,
\[ \partial_{\pm} M := I^\pm(S, \hat{M}) \cap \partial M, \]
and it is thus guaranteed that this definition does not depend on the particular choice of $S$. Yet, the achronality of $S$ implies that $\partial_{+} M \cap \partial_{-} M = \emptyset$, while the fact that $S$ is Cauchy for $(\hat{M}, \hat{g})$ ensures that $\partial M = \partial_{+} M \cup \partial_{-} M$.

The following result summarizes the structural properties of the partial boundaries $\partial_{+} M$ and $\partial_{-} M$.

**Proposition 5.3.** The sets $\partial_{+} M$ and $\partial_{-} M$ are edgeless achronal sets in $(\hat{M}, \hat{g})$. Indeed, if $\partial_{\pm} M$ is non-empty, it is a Cauchy hypersurface in $(\hat{M}, \hat{g})$, and hence homeomorphic to any given Cauchy hypersurface of $(M, g)$. Moreover, if both $\partial_{+} M$ and $\partial_{-} M$ are both non-empty, then
\[ M = I^-(\partial_{+} M, \hat{M}) \cap I^+(\partial_{-} M, \hat{M}). \tag{17} \]
In particular, $\overline{M}$ is compact in this case.
Proof. To show that $\partial_{\pm} M$ is edgeless and achronal in $(\hat{M}, \hat{g})$, we again need to argue only for, say, $\partial_{+} M$, the other case following from time duality.

Suppose, by way of contradiction, that there exist $p, q \in \partial_{+} M$ such that $p \ll (\hat{M}, \hat{g}) q$. From the definition of $\partial_{+} M$ together with the openness of the chronological relation, we can pick $p', q' \in \hat{M}$ and $S \subset M$ smooth spacelike Cauchy hypersurface such that (i) $p' \in I^{+}(S, \hat{M}) \setminus M$ and (ii) $q' \in I^{+}(p', \hat{M}) \cap M$. Again by suitably juxtaposing causal curves we can obtain a causal curve in $(\hat{M}, \hat{g})$ with endpoints in $M$ but leaving $M$, in contradiction with Lemma 5.1. We conclude that $\partial_{+} M$ is indeed achronal in $(\hat{M}, \hat{g})$.

Now, we show that $\text{edge}(\partial_{+} M) = \emptyset$. Suppose, again by way of contradiction, that $p \in \text{edge}(\partial_{+} M)$. Fix some smooth spacelike Cauchy hypersurface $S \subset M$ for which $p \in I^{+}(S, \hat{M}) = \cup U$. Since $U$ is open in $\hat{M}$ and $p$ is an edge point, there exists a timelike curve segment $\beta : [0, 1] \to U$ starting at $I^{-}(p, U)$ and ending at $I^{+}(p, U)$ which does not cross $\partial_{+} M$. Note, however, that such a curve is entirely contained in $U$. We claim that $\beta(0) \in M$ and $\beta(1) \notin M$. Note that in this case $\beta$ would intersect $\partial M \cap U = \partial_{+} M$, yielding the desired contradiction.

But if $\beta(0) \notin M$, since $q \ll (\hat{M}, \hat{g}) \beta(0) \ll (\hat{M}, \hat{g}) p$ for some $q \in M$ and $p \in \partial M$, then $q \ll (\hat{M}, \hat{g}) \beta(0) \ll (\hat{M}, \hat{g}) p'$ for some $p' \in M$, which violates Lemma 5.1. Thus, $\beta(0) \in M$. An entirely analogous reasoning establishes that $\beta(1) \notin M$. This finishes the proof that $\partial_{+} M$ is edgeless.

Therefore, if $\partial_{\pm} M \neq \emptyset$, Proposition 2.4 establishes it is a Cauchy hypersurface in $(\hat{M}, \hat{g})$ since the Killing vector field $\hat{X}$ is complete.

Finally, suppose $\partial_{\pm} M$ are both non-empty. It has been established that these are Cauchy hypersurfaces in $(\hat{M}, \hat{g})$, and the arguments in the previous paragraphs actually serve to prove that points to the future of $\partial_{+} M$ and to the past of $\partial_{-} M$, respectively, cannot be in $M$. Therefore,

$$M \subset I^{-}(\partial_{+} M, \hat{M}) \cap I^{+}(\partial_{-} M, \hat{M}).$$

Now, let $p \in I^{-}(\partial_{+} M, \hat{M}) \cap I^{+}(\partial_{-} M, \hat{M})$, and consider a future-directed timelike curve segment $\gamma$ of $(\hat{M}, \hat{g})$ with endpoints on $\partial_{+} M$ and $\partial_{-} M$ and passing through $p$. If $p \notin M$, then “perturbing” the endpoints of $\gamma$ a little, one might get another timelike curve with endpoints in $M$ passing through $p$, or in other words leaving $M$, again contradicting Lemma 5.1. So $p \in M$, and the equality (17) is established. Moreover, the global hyperbolicity of $(\hat{M}, \hat{g})$ implies that the right-hand side of (17) is precompact since $\partial_{+} M$ and $\partial_{-} M$ are compact, thus showing that $\overline{M}$ is compact in $\hat{M}$. □

Proposition 5.4. Any future-directed null geodesic line in $(M, g)$ has a future endpoint on $\partial_{+} M$ and a past endpoint on $\partial_{-} M$. In particular, if such a null line exists, then $\partial_{+} M$ and $\partial_{-} M$ are both non-empty, $\overline{M}$ is compact in $\hat{M}$, and every integral curve of $\hat{X}$ is incomplete both to the past and to the future.

Proof. Let $\eta : (a, b) \to M$ be a future-directed null geodesic line $(-\infty \leq a < b \leq +\infty)$, and pick any $t_0 \in (a, b)$. Suppose that $\eta_{0} := \eta|_{(t_0, b)}$ is future-inextendible in $(\hat{M}, \hat{g})$. Since $\hat{X}$ is a complete timelike Killing vector field, we
may use 1.3 to conclude that the NOH holds in \((\hat{M}, \hat{g})\). Therefore, \(I^- (\eta_0) \equiv \hat{M}\). But then there exists \(s_0 \in (t_0, b)\) for which \(\eta(t_0) \not\ll_{(\hat{M}, \hat{g})} \eta(s_0)\), and in view of Lemma 5.1 this implies that \(\eta(t_0) \not\ll_{(M, g)} \eta(s_0)\), contradicting the maximality of the null segment \(\eta_{[t_0,s_0]}\).

We, therefore, conclude that \(\eta_0\), and hence \(\eta\) cannot be future-inextendible in \((\hat{M}, \hat{g})\), and thus must have an endpoint therein, which in addition is clearly on \(\partial M\), say \(p \in \partial M\). Given an smooth spacelike Cauchy hypersurface \(S \subset M\), since \(S\) is acausal in \((\hat{M}, \hat{g})\), \(\eta_0\) must enter in \(I^+ (S, \hat{M})\) and remains there, so \(p \in I^+ (S, M) \cap \hat{M} \equiv \partial_+ M\) as desired. An analogous, time-dual argument establishes that \(\eta\) has a past endpoint on \(\partial_- M\).

To complete the proof, note that Proposition 5.3 shows that \(\partial_+ M\) and \(\partial_- M\) are Cauchy hypersurfaces in \((\hat{M}, \hat{g})\), and hence every integral curve of \(\hat{X}\) will intersect both exactly once at finite value of its parameters, which yields the incompleteness of each integral curve of \(X\).

\[\square\]

**Corollary 5.5.** Assume that the following holds for the globally hyperbolic space-time \((M, g)\):

(i) the TCC is satisfied and \((M, g)\) has compact Cauchy hypersurfaces (so that \((M, g)\) is cosmological);

(ii) \((M, g)\) is timelike geodesically complete.

Then either \((M, g)\) splits as in the Bartnik conjecture, or else

1. any future-directed integral curve of \(X\) has a future endpoint on \(\partial_+ M\) and a past endpoint on \(\partial_- M\). In particular, \(\partial_+ M\) and \(\partial_- M\) are both non-empty, and

2. \(\overline{M}\) is compact in \(\hat{M}\), and every integral curve of \(X\) is incomplete both to the past and to the future.

**Proof.** Indeed, we know that \((M, g)\) admits either a null or a timelike line. But under assumptions (i)–(ii), if the latter occurs then \((M, g)\) splits by the Lorentzian splitting theorem and we are done. We can, therefore, assume that \((M, g)\) admits a null line, in which case the conclusions follow from Proposition 5.4.

\[\square\]

**Corollary 5.6.** Assume that conditions (i)–(ii) of Corollary 5.5 hold for the globally hyperbolic space-time \((M, g)\). If \((M, g)\) does not split, then the norm \(\beta := \sqrt{-g(X,X)}\) of the conformal Killing vector field \(X\) is unbounded along every future- or past-inextendible timelike geodesic.

**Proof.** Since \((\hat{M}, \hat{g})\) is in particular stably causal, we may choose a temporal function, i.e., a smooth function \(f : \hat{M} \to \mathbb{R}\) with past-directed timelike gradient \(\hat{\nabla} f\). For the sake of clarity, we adopt for the rest of this proof the following notation. Given \(v \in TM\) its norm with respect to \(g\) and \(\hat{g}\) will be denoted by \(|v|\) and \(\|v\|\), respectively. In particular, if \(v \in T_p M\), then

\[\|v\| \equiv |v|/\beta(p). \tag{18}\]

As long as we are assuming here that \((M, g)\) does not split, Corollary 5.5 implies in particular that \(\overline{M}\) is compact in \(\hat{M}\), so \(f\) is bounded in \(M\), and we can also pick a constant \(c > 0\) for which
Consider a future-inextendible (and thus future-complete) timelike $g$-geodesic $\gamma : [0, +\infty) \to M$. If $\beta \circ \gamma$ were bounded, then
\[
\frac{1}{\beta \circ \gamma} \geq A \text{ for some positive number } A.
\] (20)
But then, we have, for each $t \in [0, +\infty)$,
\[
f(\gamma(t)) - f(\gamma(0)) = \int_0^t (f \circ \gamma)'(s)ds = \int_0^t \hat{g}(\nabla f(\gamma(s)), \gamma'(s))ds
\] (21)
\[
\geq \int_0^t \|\hat{\nabla} f(\gamma(s))\|\|\gamma'(s)\|ds \geq cA \int_0^t |\gamma'(s)|ds,
\] (22)
where we have used the reverse Cauchy inequality for timelike vectors for the first inequality on the right-hand side and the bounds in (19) and (20), as well as the relationship in (18), for the second inequality. However, since the last term on the last equation diverges as $t$ goes to infinity since $\gamma$ is future complete, we would conclude that $f$ is unbounded in $M$, a contradiction. □

We finally have

Proof of Theorem 1.6. Assume, on the contrary, that $(M,g)$ does not split. Then, due to Corollary 1.4, we may assume that $X$ is incomplete. Thus, under our assumptions, either alternative (iii.2) or alternative (iii.3) in the statement of Theorem 1.6 hold.

Let $\gamma : [0, +\infty) \to M$ be a future-complete unit timelike $g$-geodesic. Then its $\hat{g}$-arc length reparametrization will be denoted by $\hat{\gamma} : [0, \hat{\ell}) \to M$.

The first key point to note is that, due to Corollary 5.5, $\hat{M}$ is compact in $\hat{M}$, and hence $\hat{\gamma}$ cannot be future-inextendible in $(\hat{M}, \hat{g})$; consequently, $\hat{\ell} < \infty$.

Now, if (iii.3) holds, then $\sigma \circ \hat{\gamma}$ is bounded. However, since $\sigma \equiv X(\log \beta)$ (cf. Eq. (5)), this would imply that so is $\beta \circ \hat{\gamma}$, which contradicts Corollary 5.6.

Therefore, we shall assume, for the rest of the proof, that alternative (iii.2) holds.

The next ingredient is the well-known formula relating the Ricci tensor of conformally rescaled metrics (see, e.g., p. 59 of [3])
\[
\text{Ric} = \hat{\text{Ric}} + (1 - n) \left[ \hat{\text{Hess}}_{\log \beta} - d \log \beta \otimes d \log \beta \right] - \left[ \hat{\Delta} \log \beta + (n - 1)\hat{g}(\hat{\nabla} \log \beta, \hat{\nabla} \log \beta) \right] \hat{g}.
\] (23)

After a straightforward calculation with formula (23), we get, on the one hand,
\[
\text{Ric}(\hat{\gamma}', \hat{\gamma}') = \hat{\text{Ric}}(\hat{\gamma}', \hat{\gamma}') - (n - 1)(\log \beta \circ \hat{\gamma})'' + (\hat{\Delta} \log \beta) \circ \hat{\gamma},
\] (24)
and on the other hand, tracing (23) we get, after a few rearrangements,
\[
\hat{\Delta} \log \beta = \frac{\hat{R} - \beta^2 R}{2n} - \frac{(n - 1)}{2} \hat{g}(\nabla \log \beta, \nabla \log \beta).
\] (25)
An easy computation also reveals that
\[
\hat{\nabla} \log \beta = \beta^2 \nabla \log \beta = -\beta \sigma U + \beta^2 \nabla_U U,
\]
where \(U := X/\beta\). Gathering together Eqs. (24), (25) and (26), we end up with
\[
\text{Ric}(\hat{\gamma}', \hat{\gamma}') = \hat{\text{Ric}}(\hat{\gamma}', \hat{\gamma}') - \frac{(n-1)(\log \beta \circ \hat{\gamma})''}{2n} \frac{\beta \circ U}{\beta} + \frac{(n-1)}{2} \left[ \sigma^2 - \frac{\beta^2 R}{n(n-1)} \right] \circ \hat{\gamma}.
\]

(27)

Finally, about Eq. (27), we note the following points.
1. \(\hat{R} \circ \hat{\gamma}\) is bounded, as per the first key point above;
2. \(g(\nabla_U U, \nabla_U U) \geq 0\), since \(U\) is a \(g\)-unit timelike field, and hence \(g(\nabla_U U, U) \equiv 0\);
3. \((\log \beta \circ \hat{\gamma})''\) is not bounded above. Indeed, \(\log \beta \circ \hat{\gamma}\) and \((\log \beta \circ \hat{\gamma})'\) are defined over a finite interval (recall that \(\ell < \infty\)), and hence it is unbounded due to Corollary 5.6.

We now argue that the first term on the right-hand side of Eq. (24), namely \(\hat{\text{Ric}}(\hat{\gamma}', \hat{\gamma}')\), is also bounded in \(\hat{M}\). The upper bound (2) on the last term of Eq. (27) and the TCC condition taken together imply that the following inequality holds:
\[
(n-1)(\log \beta \circ \hat{\gamma})'' \leq \hat{\text{Ric}}(\hat{\gamma}', \hat{\gamma}') + D \quad \text{for some } D > 0.
\]

(28)

On the other hand, given a Cauchy surface \(S\), the space-time \((\hat{M}, \hat{g})\) can be seen as the standard stationary space-time (cf. Example 2.2) associated with the data \((S, g_0, \beta_0 = 1, \omega_0)\) (recall the proof of Theorem 1.5). So, taking into account that \(\hat{\gamma} \equiv (\hat{t}, \hat{x})\) is timelike, we deduce the existence of some \(\mu > 0\) such that
\[
\hat{t}'^2 \geq \mu g_0(\hat{x}', \hat{x}') = \mu \|\hat{x}'\|^2_0.
\]

So, if we define the Riemannian metric \(g_R\) on \(\hat{M}\) as follows:
\[
g_R := dt^2 + g_0,
\]
then
\[
\|\hat{\gamma}'\|^2_R = \hat{t}'^2 + \|\hat{x}'\|^2_0 \leq \left(\frac{1}{\mu} + 1\right) \hat{t}'^2.
\]

(29)

On the other hand, the compactness of \(\overline{M}\) in \(\hat{M}\) gives
\[
\hat{\text{Ric}}(\hat{\gamma}', \hat{\gamma}') \leq C \|\hat{\gamma}'\|^2_R \quad \text{for some } C > 0.
\]

(30)

Thus, taking into account (29) and (30) in (28), we deduce
\[
(n-1)(\log \beta \circ \hat{\gamma})'' \leq C \left(\frac{1}{\mu} + 1\right) \hat{t}'^2 + D.
\]

(31)

Next, we integrate both sides of (31) along \([0, \hat{s}]\), for any \(0 \leq \hat{s} < \hat{\ell}\),
\[
(\log \beta \circ \hat{\gamma})' |_{0}^{\hat{s}} \leq C' \int_{0}^{\hat{s}} \hat{t}'^2 + D'.
\]

(32)
If the integral in the previous inequality remains bounded for all \( \hat{s} \in [0, \hat{l}) \), then we have a contradiction with point (3) above. Hence, we can assume that it is not bounded above, and so, since \( \hat{\gamma} \) is future directed, \( \hat{t}' \) is (positive and) unbounded from above. On the other hand, note that

\[
\int_{\hat{s}}^{\hat{l}} \hat{t}' \, d\hat{s} = (\hat{t}(\hat{s})\hat{t}'(\hat{s}) - \hat{t}(0)\hat{t}'(0)) - \int_{0}^{\hat{s}} \hat{t}'' \, d\hat{s}.
\]  
(33)

We need to show that

\[
\int_{\hat{s}}^{\hat{l}} \hat{t}'' \, d\hat{s} > 0.
\]  
(34)

As \( \hat{t} \) is continuous, it is bounded on \( M \), and it is strictly increasing, so there is no loss of generality in assuming it is positive on \([0, \hat{l})\). Let

\[
J_+ := \{s \in (0, \hat{l}) : \hat{t}''(s) > 0\}
\]

and

\[
J_- := \{s \in (0, \hat{l}) : \hat{t}''(s) < 0\}.
\]

Proving (34) is equivalent to showing that

\[
\int_{J_+} \hat{t}(s)|\hat{t}''(s)| \, ds > \int_{J_-} \hat{t}(s)|\hat{t}''(s)| \, ds.
\]  
(35)

But because \( \hat{t}' \) diverges to infinity on \([0, \hat{l})\), it is not difficult to see that for any interval \( I \subset [0, \hat{l}) \) with \( \hat{t}'' < 0 \) on \( I \) and \( \sup(I) \leq \hat{t}_0 \) for some \( \hat{t}_0 \in [0, \hat{l}) \), there exists some subset \( J_I \subset [0, \hat{l}) \) with \( \hat{t}_0 \leq \inf(J_I) < \sup(J_I) < \hat{l} \) such that \( \int_{J_I} \hat{t}'' > -\int_{I} \hat{t}'' \); for example, take \( \hat{t}_0 < \hat{s}_1 < \hat{s}_2 < \hat{l} \) such that \( \hat{t}'(s_2) - \hat{t}'(s_1) > -\Delta \hat{t}' \). Define then \( J_I := \{s \in [\hat{s}_1, \hat{s}_2] : \hat{t}''(\hat{s}) > 0\} \). Since \( \inf(J_I) > \sup(I) \),

\[
\int_{J_I} \hat{t}'' > \int_{J_I} \hat{t}' \hat{t}'' > -\int_{I} \hat{t}' \hat{t}'' > -\int_{J_I} \hat{t}''.
\]

Hence, (35), and hence (34), holds.

Next, we replace (33) in (32), and take into account (34) and the boundedness of \( \hat{t} \) on \([0, \hat{l})\), and deduce

\[
(\log \beta \circ \hat{\gamma})(\hat{s}) \leq C'' \hat{t}'(\hat{s}) + D'' \quad \text{for } \hat{s} \text{ near } \hat{l}.
\]  
(36)

Thus, if we integrate both sides of (36), and use again that \( \hat{t}(\hat{s}) \) remains bounded on \([0, \hat{l})\), we arrive at a contradiction with point (3) above. \( \Box \)

**Acknowledgements**

We wish to thank Marcelo M. Cavalcanti, Alberto Enciso and Rafael Ortega for useful discussions. The authors are partially supported by the Spanish Grant MTM2016-78807-C2-2-P (MINECO and FEDER funds). The second author also wishes to acknowledge the Department of Mathematics, Universidade Federal de Santa Catarina (Brazil), for the kind hospitality while part of this research was being carried out.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Bartnik, R.: Remarks on cosmological spacetimes and constant mean curvature surfaces. Comm. Math. Phys. 117(4), 615–624 (1988)
[2] Beem, J.K., Ehrlich, P.E., Easley, K.L.: Global Lorentzian Geometry, 2nd ed. Marcel Dekker, New York (1996)
[3] Besse, A.L.: Einstein manifolds. Springer, Berlin Heidelberg (1987)
[4] Candela, A.M., Flores, J.L., Sánchez, M.: Global hyperbolicity and Palais–Smale condition for action functionals in stationary spacetimes. Adv. Math. 218, 515–536 (2008)
[5] Costa e Silva, I.P., Flores, J.L.: On the splitting problem for Lorentzian manifolds with an $\mathbb{R}$-action with causal orbits. Ann. Henri Poincaré 18(5), 1635–1670 (2017)
[6] Eschenburg, J.-H.: The splitting theorem for space-times with strong energy condition. J. Differ. Geom. 27(3), 477–491 (1988)
[7] Eschenburg, J.-H., Galloway, G.J.: Lines in space-times. Comm. Math. Phys. 148(1), 209–216 (1992)
[8] Galloway, G.J.: Splitting theorems for spatially closed space-times. Comm. Math. Phys. 96(4), 423–429 (1984)
[9] Galloway, G.J.: Some rigidity results for spatially closed spacetimes, Mathematics of gravitation, Part I (Warsaw, 1996), Banach Center Publ., vol. 41, Polish Acad. Sci., Warsaw, pp. 21–34 (1997)
[10] Galloway, G.J., Vega, C.: Hausdorff closed limits and rigidity in Lorentzian geometry. Ann. Henri Poincaré 18(10), 3399–3426 (2017)
[11] Galloway, G.J., Vega, C.: Rigidity in vacuum under conformal symmetry. Lett. Math. Phys. 108(10), 2285–2292 (2018)
[12] Geroch, R.P., Kronheimer, E.H., Penrose, R.: Ideal points in spacetime. Proc. R. Soc. Lond. A 327, 545–567 (1972)
[13] Garfinkle, D., Harris, S.G.: Ricci fall-off in static and stationary, globally hyperbolic, non-singular spacetimes. Class. Quantum Grav. 14(1), 139–151 (1997)
[14] Harris, S.G., Low, R.J.: Causal monotonicity, omniscient foliations and the shape of space. Class. Quantum Grav. 18, 27–43 (2001)
[15] Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-time. Cambridge University Press, Cambridge (1973)
[16] Hawking, S.W., Penrose, R.: The singularities of gravitational collapse and cosmology. Proc. R. Soc. Lond. A 314, 529–548 (1970)
[17] Javaloyes, M.A., Sánchez, M.: A note on the existence of standard splittings for conformally stationary spacetimes. Class. Quantum Grav. 25, 168001 (2008)
[18] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. 1. Wiley, New York (1962)
[19] Lee, J.M.: Introduction to Smooth Manifolds. Springer, New York (2003)
[20] O’Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)
[21] Romero, A., Sánchez, M.: On completeness of certain families of semi-Riemannian manifolds. Geom. Dedicata 53(1), 103–117 (1994)
[22] van den Ban, E.P.: Lecture notes on Lie groups. http://www.staff.science.uu.nl/~ban00101/lecnotes/lie2010.pdf
[23] Wald, R.M.: General Relativity. University of Chicago Press, Chicago (1984)
[24] Yau, S.-T.: Problem section, Annals of Math. Studies, No. 102, Princeton University Press, Princeton, N. J., pp. 669–706 (1982)

I. P. Costa e Silva
Department of Mathematics
Universidade Federal de Santa Catarina
Florianópolis
SC 88.040-900
Brazil

J. L. Flores
Departamento de Álgebra, Geometría y Topología,
Facultad de Ciencias
Universidad de Málaga
Campus Teatinos
29071 Málaga
Spain

J. Herrera
Departamento de Matemáticas
Edificio Albert Einstein,
Universidad de Córdoba
Campus de Rabanales
14071 Córdoba
Spain
e-mail: jherrera@uco.es

Received: January 21, 2019.
Revised: June 8, 2019.
Accepted: December 2, 2019.