Fragility and Robustness in Mean-payoff Adversarial Stackelberg Games

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Abstract

Two-player mean-payoff Stackelberg games are nonzero-sum infinite duration games played on a bi-weighted graph by Leader (Player 0) and Follower (Player 1). Such games are played sequentially: first, Leader announces her strategy, second, Follower chooses his best-response. If we cannot impose which best-response is chosen by Follower, we say that Follower, though strategic, is adversarial towards Leader. The maximal value that Leader can get in this nonzero-sum game is called the adversarial Stackelberg value (ASV) of the game.

We study the robustness of strategies for Leader in these games against two types of deviations: (i) Modeling imprecision - the weights on the edges of the game arena may not be exactly correct, they may be delta-away from the right one. (ii) Sub-optimal response - Follower may play epsilon-optimal best-responses instead of perfect best-responses. First, we show that if the game is zero-sum then robustness is guaranteed while in the nonzero-sum case, optimal strategies for ASV are fragile. Second, we provide a solution concept to obtain strategies for Leader that are robust to both modeling imprecision, and as well as to the epsilon-optimal responses of Follower, and study several properties and algorithmic problems related to this solution concept.

1 Introduction

Stackelberg games [13] were first introduced to model strategic interactions among rational agents in markets that consist of Leader and Follower(s). Leader in the market makes her strategy public and Follower(s) respond by playing an optimal response to this strategy. Here, we consider Stackelberg games as a framework for the synthesis of reactive programs [16] [2]. These programs maintain a continuous interaction with the environment in which they operate; they are deterministic functions that given a history of interactions so far choose an action. Our work is a contribution to rational synthesis [9] [15], a nonzero-sum game setting where both the program and the environment are considered as rational agents that have their own goals. While Boolean \( \omega \)-regular payoff functions have been studied in [9] [15], here we study the quantitative long-run average (mean-payoff) function.

We illustrate our setting with the example of a game graph as shown in Figure 1. The set \( V \) of vertices is partitioned into \( V_0 \) (represented by circles) and \( V_1 \) (represented by...
squares) that are owned by Leader (also called Player 0) and Follower (also called Player 1) respectively. In the tuple on the edges, the first element is the payoff of Leader, while the second one is the payoff of Follower (weights are omitted if they are both equal to 0). Each player’s objective is to maximize the long run average of the payoffs that she receives (a.k.a. mean-payoff). In the adversarial Stackelberg setting, Player 0 (Leader) first announces how she will play then Player 1 (Follower) chooses one of his best-responses to this strategy. Here, there are two choices for Player 0: L or R. As Player 1 is assumed to be rational, Player 0 deduces that she must play L. Indeed, the best response of Player 1 is then to play LL and the reward she obtains is 10. This is better than playing R, for which the best-response of Player 1 is RL, and the reward is 8 instead of 10. Note that if there are several possible best responses for Player 1, then we consider the worst-case: Player 0 has no control on the choice of best-responses by Player 1.

Quantitative models and robustness  The study of adversarial Stackelberg games with mean-payoff objectives has been started in [8] with the concept of adversarial Stackelberg value (ASV for short). ASV is the best value that Leader can obtain by fixing her strategy and facing any rational response by Follower. As this setting is quantitative, it naturally triggers questions about robustness that were left open in the above paper.

Robustness is a highly desirable property of quantitative models: small changes in the quantities appearing in a model \( M \) (e.g. rewards, probabilities, etc.) should have small impacts on the predictions made from \( M \), see e.g. [1]. Robustness is thus crucial because it accounts for modelling imprecision that are inherent in quantitative modelling and those imprecision may have important consequences. For instance, a reactive program synthesized from a model \( M \) should provide acceptable performances if it is executed in a real environment that differ slightly w.r.t. the quantities modeled in \( M \).

Some classes of models are robust. For instance, consider two-player zero-sum mean-payoff games where players have fully antagonistic objectives. The value of a two-player zero-sum mean-payoff \( G \) is the maximum mean-payoff that Player 0 can ensure against all strategies of Player 1. A strategy \( \sigma_0 \) that enforces the optimal value \( c \) in \( G \) is robust in the following sense. Let \( G^{\pm \delta} \) be the set of games obtained by increasing or decreasing the weights on the edges of \( G \) by at most \( \delta \). Then for all \( \delta > 0 \), and for all \( H \in G^{\pm \delta} \), the strategy \( \sigma_0 \) ensures in \( H \) a mean-payoff of at least \( c - \delta \) for Player 0 against any strategy of Player 1 (Theorem 1). So slight changes in the quantities appearing in the model have only a small impact on the worst-case value enforced by the strategy.

The situation is more complex and less satisfactory in nonzero-sum games. Strategies that enforce the ASV proposed in [8] may be fragile: slight differences in the weights of the game, or in the optimality of the response by Player 1, may lead to large differences in the value obtained by the strategy. We illustrate these difficulties on our running example. The strategy of Player 0 that chooses L in \( v_0 \) ensures her a payoff of 10 which is the ASV. Indeed, the unique best-response of Player 1 against L is to play LL from \( v_1 \). However, if the weights in \( G \) are changed by up to \( \pm \delta = \pm 0.6 \) then there is a game \( H \in G^{\pm \delta} \) in which the weight on the self-loop over vertex \( v_4 \) changes to e.g. 9.55, and the weight on the self-loop over \( v_3 \) changes to e.g. 9.45, and the action LR becomes better for Player 1. So the value of L in \( H \) against a rational adversary is now 0 instead of 10. Thus a slight change in the rewards for Player 1 (due to e.g. modelling imprecision) may have a dramatic effect on the value of the optimal strategy L computed on the model \( G \) when evaluated in \( H \).
Figure 1 A game in which the strategy of Leader that maximizes the adversarial Stackelberg is fragile while the strategy of Leader that maximizes the $\epsilon = 1$-adversarial Stackelberg value is robust.

Contributions As a remedy to this situation, we provide an alternative notion of value that is better-suited to synthesize strategies that are robust against perturbations. We consider two types of perturbations. First, the strategies computed for this value are robust against modeling imprecision: if a strategy has been synthesized from a weighted game graph with weights that are possibly slightly wrong, the value that this strategy delivers is guaranteed to be close to what the model predicts. Second, strategies computed for this value are robust against sub-optimal responses: small deviations from the best-response by the adversary have only limited effect on the value guaranteed by the strategy.

Our solution relies on relaxing the notion of best-responses of Player 1 in the original model $G$. More precisely, we define the $\epsilon$-adversarial Stackelberg value ($ASV_\epsilon$, for short) as the value that Leader can enforce against all $\epsilon$-best responses of Follower. Obviously, this directly accounts for the second type of perturbations. But we show that, additionally, this accounts for the first type of perturbations: if a strategy $\sigma_0$ enforces an $ASV_\epsilon$ equal to $c$ then for all games $H \in G^{\pm \epsilon}$, we have that $\sigma_0$ enforce a value larger than $c - \epsilon$ in $H$ (Theorem 6 and Theorem 5).

We illustrate this by considering again the example of Figure 1. Here, if we consider that the adversary can play $2\delta = 1.2$-best responses instead of best responses only, then the optimal strategy of Player 0 is now $R$ and it has a $ASV_\epsilon$ equal to 8. This value is guaranteed to be robust for all games $H \in G^{\pm \epsilon}$ as $R$ is guaranteed to enforce a payoff that is larger than $8 - \delta$ in all games in $G^{\pm \epsilon}$. Stated otherwise, we use the notion of $ASV_\epsilon$ in the original game to find a strategy for Player 0 that she uses in the perturbed model while playing against a rational adversary. Thus we show that in the event of modelling imprecision resulting in a perturbed model, the solution concept to be used is $ASV_\epsilon$ instead of $ASV$ since the former provides strategies that are robust to such perturbations.

| Adversarial best responses of Follower | Robustness | Threshold Problem | Computing ASV | Achievability |
|----------------------------------------|------------|-------------------|--------------|--------------|
| No [Prop 2, Prop 3]                    | NP [8]     | Finite Memory Strategy [Thm 18] | Theory of Reals [5] | No [5] |
|                                            |            | Memoryless Strategy [Thm 20] |              |              |
| Adversarial $\epsilon$-best responses of Follower | Yes [Thm 5, Thm 6] | NP [Thm 12] | Finite Memory Strategy [Thm 12] | Theory of Reals [Thm 25] | Yes [Thm 31] |
|                                            |            | Memoryless Strategy [Thm 20] | Solving LP in EXPTime [Thm 28] | (Requires Infinite Memory [Thm 34]) |

Table 1 Summary of our results

In addition to proving the fragility of the original concept introduced in [8] (Proposition 2) and the introduction of the new notion of value $ASV_\epsilon$ that is robust against modelling
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imprecision (Theorem 5), we provide algorithms to handle $\text{ASV}^\epsilon$. First, we show how to decide the threshold problem for $\text{ASV}^\epsilon$ in nondeterministic polynomial time and that finite memory strategies suffice (Theorem 12). Second, we provide an algorithm to compute $\text{ASV}^\epsilon$ when $\epsilon$ is fixed (Theorem 25). Third, we provide an algorithm that given a threshold value $c$, computes the largest possible $\epsilon$ such that $\text{ASV}^\epsilon > c$ (Corollary 30). These three results form the core technical contributions of this paper and they are presented in Section 4 and Section 5. Additionally, in Section 6, we show that $\text{ASV}^\epsilon$ is always achievable (Theorem 31), which is in contrast to the case in [8] where Follower only plays best-responses. Finally, we provide results that concern the memory needed for players to play optimally, and complexity results for subcases (for example when Players are assumed to play memoryless). Our contributions have been summarized in Table 1, where the results obtained in this work are in bold.

Related Works Stackelberg games on graphs have been first considered in [9], where the authors study rational synthesis for $\omega$-regular objectives with co-operative Follower(s). In [8], Stackelberg mean-payoff games in adversarial setting, and Stackelberg discounted sum games in both adversarial and co-operative setting have been considered. However, as pointed out earlier, the model of [8] is not robust to perturbations. In [10], mean-payoff Stackelberg games in the co-operative setting have been studied. In [13], the authors study the effects of limited memory on both Nash and Stackelberg (or leader) strategies in multi-player discounted sum games. Incentive equilibrium over bi-matrix games and over mean-payoff games in a co-operative setting have been studied in [11] and [12] respectively. In [15], adversarial rational synthesis for $\omega$-regular objectives have been studied. In [7], precise complexity results for various $\omega$-regular objectives have been established for both adversarial and co-operative settings. In [6, 4], secure Nash equilibrium has been studied, where each player first maximises her own payoff, and then minimises the payoff of the other player; Player 0 and Player 1 are symmetric there unlike in Stackelberg games. For discounted sum objectives, in [8], the gap problem has been studied. Given rationals $c$ and $\delta$, a solution to the gap problem can decide if $\text{ASV} > c + \delta$ or $\text{ASV} < c - \delta$. The threshold problem was left open in [8], and is technically challenging. We leave the case of analysing robustness for discounted sum objective for future work.

2 Preliminaries

We denote by $\mathbb{N}, \mathbb{N}^+, \mathbb{Q}$, and $\mathbb{R}$ the set of naturals, the set of naturals excluding 0, the set of rationals, and the set of reals respectively.

Arenas An (bi-weighted) arena $\mathcal{A} = (V, E, \langle V_0, V_1 \rangle, w_0, w_1)$ consists of a finite set $V$ of vertices, a set $E \subseteq V \times V$ of edges such that for all $v \in V$ there exists $v' \in V$ and $(v, v') \in E$, a partition $\langle V_0, V_1 \rangle$ of $V$, where $V_0$ (resp. $V_1$) is the set of vertices for Player 0 (resp. Player 1), and two edge weight functions $w_0 : E \to \mathbb{Z}$, $w_1 : E \to \mathbb{Z}$. In the sequel, we denote the maximum absolute value of a weight in $\mathcal{A}$ by $W$. A strongly connected component of a directed graph is a subgraph that is strongly connected. In the sequel, unless otherwise mentioned, we denote by $\text{SCC}$ a subgraph that is strongly connected, and which may or may not be maximal.

Plays and histories A play in $\mathcal{A}$ is an infinite sequence of vertices $\pi = \pi_0 \pi_1 \cdots \in V^\omega$ such that for all $k \in \mathbb{N}$, we have $(\pi_k, \pi_{k+1}) \in E$. A history in $\mathcal{A}$ is a (non-empty) prefix of a play.
in $\mathcal{A}$. Given $\pi = \pi_0 \pi_1 \cdots \in \text{Plays}_\mathcal{A}$ and $k \in \mathbb{N}$, the prefix $\pi_0 \pi_1 \ldots \pi_k$ of $\pi$ is denoted by $\pi_{\leq k}$. We denote by $\inf(\pi)$ the set of vertices $v$ that appear infinitely many times along $\pi$, i.e., $\inf(\pi) = \{ v \in V \mid \forall i \in \mathbb{N} \cdot \exists j \in \mathbb{N}, j \geq i : \pi(j) = v \}$. It is easy to see that $\inf(\pi)$ forms an SCC in the underlying graph of the arena $\mathcal{A}$. We denote by $\text{Plays}_\mathcal{A}$ and $\text{Hist}_\mathcal{A}$ the set of plays and the set of histories in $\mathcal{A}$ respectively; the symbol $A$ is omitted when clear from the context. Given $i \in \{0, 1\}$, the set $\text{Hist}_\iota$ denotes the set of histories such that their last vertex belongs to $V_i$. We denote the first vertex and the last vertex of a history $h$ by $\text{first}(h)$ and $\text{last}(h)$ respectively.

**Games** A mean-payoff game $\mathcal{G} = (\mathcal{A}, (\mathcal{MP}_0, \mathcal{MP}_1))$ consists of a bi-weighted arena $\mathcal{A}$, payoff functions $\mathcal{MP}_0 : \text{Plays}_\mathcal{A} \rightarrow \mathbb{R}$ and $\mathcal{MP}_1 : \text{Plays}_\mathcal{A} \rightarrow \mathbb{R}$ for Player 0 and Player 1 respectively which are defined as follows. Given a play $\pi \in \text{Plays}_\mathcal{A}$ and $i \in \{0, 1\}$, the payoff $\mathcal{MP}_i(\pi)$ is given by $\mathcal{MP}_i(\pi) = \liminf_{k \to \infty} \frac{1}{k} w_i(\pi_{\leq k})$, where the weight $w_i(h)$ of a history $h \in \text{Hist}$ is the sum of the weights assigned by $w_i$ to its edges. In our definition of the mean-payoff, we have used $\lim \inf$ as the limit of the successive average may not exist. We will also need the $\lim \sup$ case for technical reasons. Here is the formal definition together with its notation: $\mathcal{MP}_i(\pi) = \limsup_{k \to \infty} \frac{1}{k} w_i(\pi_{\leq k})$. The size of the game $\mathcal{G}$, denoted $|\mathcal{G}|$, is the sum of the number of vertices and edges appearing in the arena $\mathcal{A}$.

**Unfolding of a game** Let $V$ and $E$ be respectively the set of vertices and the set of edges of $\mathcal{G}$. The unfolding of the game $\mathcal{G}$ starting from a vertex $v \in V$ is a tree $T_v(\mathcal{G})$ of infinite depth with its root $v$ such that there is a one-to-one correspondence between the set of plays $\pi$ of $\mathcal{G}$ with $\text{first}(\pi) = v$ and the branches of $T_v(\mathcal{G})$. Every node $v_1$ of $T_v(\mathcal{G})$ belongs to $V$, and there is an edge from $v_1$ to $v_2$ in $T_v(\mathcal{G})$ iff $(v_1, v_2) \in E$. Every node $p \in V^+$ of $T_v(\mathcal{G})$ is a play $p = v_1 \ldots v_n$ in $\mathcal{G}$, where $v_1 = v$. There is an edge from $p = v_1 \ldots v_n$ to $p' = v_1 \ldots v'_n$ iff $(v_n, v'_n) \in E$.

**Strategies and payoffs** A strategy for Player $i \in \{0, 1\}$ in the game $\mathcal{G}$ is a function $\sigma : \text{Hist}_\iota \rightarrow V$ that maps histories ending in a vertex $v \in V_i$ to a successor of $v$. The set of all strategies of Player $i \in \{0, 1\}$ in the game $\mathcal{G}$ is denoted by $\Sigma_\iota(\mathcal{G})$, or $\Sigma_\iota$ when $\mathcal{G}$ is clear from the context. A strategy has memory $M$ if it can be realized as the output of a state machine with $M$ states. A memoryless strategy is a function that only depends on the last element of the history $h \in \text{Hist}$. We denote by $\Sigma_\iota^{\text{ML}}$ the set of memoryless strategies of Player $i$, and by $\Sigma_\iota^{\text{FM}}$ her set of finite memory strategies. A profile is a pair of strategies $\bar{\sigma} = (\sigma_0, \sigma_1)$, where $\sigma_0 \in \Sigma_0(\mathcal{G})$ and $\sigma_1 \in \Sigma_1(\mathcal{G})$. As we consider games with perfect information and deterministic transitions, any profile $\bar{\sigma}$ yields, from any history $h$, a unique play or outcome, denoted $\text{Out}_h(\mathcal{G}, \bar{\sigma})$. Formally, $\text{Out}_h(\mathcal{G}, \bar{\sigma})$ is the play $\pi$ such that $\pi_{\leq |h| - 1} = h$ and $\forall k \geq |h| - 1$ it holds that $\pi_{k+1} = \sigma_i(\pi_{\leq k})$ if $\pi_k \in V_i$. We write $h \preceq \pi$ whenever $h$ is a prefix of $\pi$. The set of outcomes compatible with a strategy $\sigma \in \Sigma_{\in\{0,1\}}(\mathcal{G})$ after a history $h$ is $\text{Out}_h(\mathcal{G}, \sigma) = \{ \pi | \exists \sigma' \in \Sigma_{\in\{0,1\}}(\mathcal{G}) \text{ such that } \pi = \text{Out}_h(\mathcal{G}, (\sigma, \sigma')) \}$. Each outcome $\pi \in \mathcal{G} = (\mathcal{A}, (\mathcal{MP}_0, \mathcal{MP}_1))$ yields a payoff $\mathcal{MP}(\pi) = (\mathcal{MP}_0(\pi), \mathcal{MP}_1(\pi))$.

Usually, we consider instances of games such that the players start playing at a fixed vertex $v_0$. Thus, we call an initialized game a pair $(\mathcal{G}, v_0)$, where $\mathcal{G}$ is a game and $v_0 \in V$ is the initial vertex. When $v_0$ is clear from context, we use $\mathcal{G}$, $\text{Out}(\mathcal{G}, \bar{\sigma})$, $\text{Out}(\mathcal{G}, (\sigma, \sigma'))$, $\mathcal{MP}(\bar{\sigma})$ instead of $\mathcal{G}_{v_0}$, $\text{Out}_{v_0}(\mathcal{G}, \bar{\sigma})$, $\text{Out}_{v_0}(\mathcal{G}, (\sigma, \sigma'))$, $\mathcal{MP}_{v_0}(\bar{\sigma})$. We sometimes omit $\mathcal{G}$ when it is clear from the context.
Best-responses, $\epsilon$-best-responses Let $G = (A, (MP_0, MP_1))$ be a two-dimensional mean-payoff game on the bi-weighted arena $A$. Given a strategy $\sigma_0$ for Player 0, we define

1. Player 1’s best responses to $\sigma_0$, denoted by $BR_1(\sigma_0)$, as:
   \[
   \{ \sigma_1 \in \Sigma_1 \mid \forall v \in V : \forall \sigma'_1 \in \Sigma_1 : MP_1(Out_v(\sigma_0, \sigma_1)) \geq MP_1(Out_v(\sigma_0, \sigma'_1)) \}
   \]
2. Player 1’s $\epsilon$-best-responses to $\sigma_0$, for $\epsilon > 0$, denoted by $BR^\epsilon_1(\sigma_0)$, as:
   \[
   \{ \sigma_1 \in \Sigma_1 \mid \forall v \in V : \forall \sigma'_1 \in \Sigma_1 : MP_1(Out_v(\sigma_0, \sigma_1)) > MP_1(Out_v(\sigma_0, \sigma'_1)) - \epsilon \}
   \]

We note here that the definitions of best-responses can also be defined if we consider limits instead of lim inf in the mean-payoff functions.

We also introduce the following notation for zero-sum games (that are needed as intermediary steps in our algorithms). Let $A$ be an arena, $v \in V$ one of its states, and $O \subseteq Plays_A$ be a set of plays (called objective), then we write $A, v \models \prec i \gg O$, if:

\[
\exists \sigma_i \in \Sigma_i \cdot \forall \sigma_{i-1} \in \Sigma_{i-1} : Out_v(A, (\sigma_i, \sigma_{i-1})) \in O, \text{ for } i \in \{0, 1\}
\]

All the zero-sum games we consider in this paper are determined meaning that for all $A$, for all objectives $O \subseteq Plays_A$ we have that $A, v \models \prec i \gg O \iff A, v \not\models \prec 1 - i \gg Plays_A \setminus O$. We sometimes omit $A$ when the arena being referenced is clear from the context.

Convex hull and $F_{\min}$ Given a finite dimension $d$, a finite set $X \subset \mathbb{Q}^d$ of rational vectors, we define the convex hull $CH(X) = \{ v \mid v = \sum_{x \in X} \alpha_x \cdot x \land \forall x \in X : \alpha_x \in [0, 1] \land \sum_{x \in X} \alpha_x = 1 \}$ as the set of all their convex combinations. Let $f_{\min}(X)$ be the vector $v = (v_1, v_2, \ldots, v_d)$ where $v_i = \min\{ c \mid \exists x \in X : x_i = c \}$ i.e. the vector $v$ is the pointwise minimum of the vectors in $X$. For $S \subseteq \mathbb{Q}^d$, we define $F_{\min}(S) = \{ f_{\min}(P) \mid P$ is a finite subset of $S \}$.

Mean-payoffs induced by simple cycles A cycle $c$ is a sequence of edges that starts and stops in a given vertex $v$, it is simple if it does not contain repetition of any other vertex. Given an SCC $S$, we write $C(S)$ for the set of simple cycles inside $S$. Given a simple cycle $c$, for $i \in \{0, 1\}$, let $MP_i(c) = w(c) / |c|$ be the mean of the weights in each dimension along the edges in the simple cycle $c$, and we call the pair $(MP_0(c), MP_1(c))$ the mean-payoff coordinate of the cycle $c$. We write $CH(C(S))$ for the convex-hull of the set of mean-payoff coordinates of simple cycles of $S$.

Adversarial Stackelberg Value for $MP$ Since the set of best-responses in mean-payoff games can be empty (See Lemma 3 of \cite{KynclL19}), we use the notion of $\epsilon$-best-responses for the definition of ASV which are guaranteed to always exist\footnote{Since we will use $\epsilon$ in ASV$^\epsilon$ to add robustness, we only consider the cases in which $\epsilon$ is strictly greater than 0.}. We define

\[
ASV(v) = \sup_{\sigma_0 \in \Sigma_0, \epsilon > 0} \inf_{\sigma_1 \in BR^\epsilon_1(\sigma_0)} MP_0(Out_v(\sigma_0, \sigma_1))\]

1 Since we will use $\epsilon$ in ASV$^\epsilon$ to add robustness, we only consider the cases in which $\epsilon$ is strictly greater than 0.
2 We do not use $MP_1$ since lim inf and lim sup are the same for a finite sequence of edges.
3 For a game $G$, we also use ASV$_G$ and ASV$_G^\epsilon$, and drop the subscript $G$ when it is clear from the context.
4 The definition of ASV, as it appears in \cite{KynclL19}, is syntactically different but the two definitions are equivalent, and the one presented here is simpler.
We also associate a (adversarial) value to a strategy $\sigma_0 \in \Sigma_0$ of Player 0, denoted
\[
ASV(\sigma_0)(v) = \sup_{\epsilon > 0} \inf_{\sigma_1 \in BR_{\epsilon}(\sigma_0)} MP_0(Out_v(\sigma_0, \sigma_1)).
\]
Clearly, we have that $ASV(v) = \sup_{\sigma_0 \in \Sigma_0} ASV(\sigma_0)(v)$.

In the sequel, unless otherwise mentioned, we refer to a two-dimensional nonzero-sum two-player mean-payoff game simply as a mean-payoff game.

**Zero-sum case** Zero-sum games are special cases of nonzero-sum games, where for all edges $e \in E$, we have that $w_0(e) = -w_1(e)$, i.e. the gain of one player is always equal to the opposite (the loss) of the other player. For zero-sum games, the classical concept is the notion of (worst-case) value. It is defined as
\[
Val_G(v) = \sup_{\sigma_0 \in \Sigma_0} \inf_{\sigma_1 \in \Sigma_1} MP_0(Out_v(\sigma_0, \sigma_1))
\]
Additionally, we define the value of a Player 0 strategy $\sigma_0$ from a vertex $v$ in a zero-sum mean-payoff game $G$ as $Val_G(\sigma_0)(v) = \inf_{\sigma_1 \in \Sigma_1} MP_0(Out_v(\sigma_0, \sigma_1))$.

### 3 Fragility and robustness in games

In this section, we study fragility and robustness properties in zero-sum and nonzero-sum games. Additionally, we provide a notion of value, for the nonzero-sum case, that is well-suited to synthesize strategies that are robust against two types of perturbations:

- **Modeling imprecision**: We want guarantees about the value that is obtained by a strategy in the Stackelberg game even if this strategy has been synthesized from a weighted game graph with weights that are possibly slightly wrong: small perturbations of the weight should have only limited effect on the value guaranteed by the strategy.
- **Sub-optimal responses**: We want guarantees about the value that is obtained by a strategy in the Stackelberg game even if the adversary responds with an $\epsilon$-best response instead of a perfectly optimal response (for some $\epsilon > 0$): small deviations from the best-response by the adversary should have only limited effect on the value guaranteed by the strategy.

**Formalizing deviations** To formalize modeling imprecision, we introduce the notion of a perturbed game graph. Given a game $G$ with arena $A_G = (V, E, \langle V_0, V_1 \rangle, w_0, w_1)$, and a value $\delta > 0$, we write $G^{\pm \delta}$ for the set $H$ of games with arena $A_H = (V, E, \langle V_0, V_1 \rangle, w'_0, w'_1)$ where edge weight functions respect the following constraints:
\[
\forall (v_1, v_2) \in E, \forall i \in \{0, 1\}, \quad w'_i(v_1, v_2) \in (w_i(v_1, v_2) + \delta, w_i(v_1, v_2) - \delta).
\]

We note that as the underlying game graph $(V, E)$ is not altered, for both players, the set of strategies in $G$ is identical to the set of strategies in $H$. Finally, to formalize sub-optimal responses, we naturally use the notion of $\epsilon$-best response introduced in the previous section.

**Robustness in zero-sum games** In zero-sum games, the worst-case value $Val_G(\sigma_0)$ is robust against both modeling imprecision and sub-optimal responses of Player 1.

**Proposition 1 (Robustness in zero-sum games).** For all zero-sum mean-payoff games $G$ with a set $V$ of vertices, for all Player 0 strategies $\sigma_0$, and for all vertices $v \in V$ we have that:
\[
\forall \delta, \epsilon > 0: \forall H \in G^{\pm \delta}, \inf_{\sigma_1 \in BR_{\epsilon}(\sigma_0)} MP_H(Out_v(\sigma_0, \sigma_1)) > Val_G(\sigma_0)(v) - \delta.
\]
Fragility in non-zero sum games

On the contrary, the adversarial Stackelberg value \( \text{ASV} \) is fragile against both modeling imprecision and sub-optimal responses.

**Proposition 2 (Fragility - modeling imprecision).** For all \( \mu > 0 \), we can construct a nonzero-sum mean-payoff game \( \mathcal{G} \) and a Player 0 strategy \( \sigma_0 \), such that there exist \( \delta > 0 \), a perturbed game \( \mathcal{H} \in \mathcal{G}^{\pm \delta} \), and a vertex \( v \) in \( \mathcal{G} \) with \( \text{ASV}_\mathcal{H}(\sigma_0)(v) < \text{ASV}_\mathcal{G}(\sigma_0)(v) - \mu \).

**Proof.** Consider the example in Figure 2 where we assume \( 0 < \epsilon/2 < \delta \). Note that for all values of \( \epsilon > 0 \), we have that \( \text{ASV}_\mathcal{G}(v_0) = 0 \), since Player 1 has no incentive to go to the left, thus the payoff of Player 0 is 0 corresponding to a best-response of Player 1. Thus \( \text{ASV}_\mathcal{G}(v_0) = 0 \).

Now we consider the perturbed game \( \mathcal{H} \in \mathcal{G}^{\pm \delta} \) in Figure 3. Taking the left edge and the right edge are equally good for Player 1, and so the value that Player 0 can ensure is at most \(-2\mu < 0 - \mu \). Hence \( \text{ASV}_\mathcal{H}(\sigma_0)(v_0) < \text{ASV}_\mathcal{G}(\sigma_0)(v_0) - \mu \).

**Proposition 3 (Fragility - sub-optimal responses).** For all \( \mu > 0 \), we can construct a nonzero-sum mean-payoff game \( \mathcal{G} \) and a Player 0 strategy \( \sigma_0 \), such that there exist \( \epsilon > 0 \) and a vertex \( v \) in \( \mathcal{G} \) with \( \inf_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_0^\mathcal{G}(\text{Out}_v(\sigma_0, \sigma_1)) < \text{ASV}_\mathcal{G}(\sigma_0)(v) - \mu \).

**Proof.** We again consider the example in Figure 2 where we now assume \( \epsilon > 0 \). Note that for all values of \( \mu > 0 \), we have that \( \text{ASV}_\mathcal{G}(\sigma_0)(v_0) = 0 \), since Player 1 has no incentive to go to the left, thus the payoff of Player 0 is 0 corresponding to a best-response of Player 1. Thus \( \text{ASV}_\mathcal{G}(\sigma_0)(v_0) = 0 \). We also note that for all \( \epsilon > \epsilon/2 \), we have that, in \( \mathcal{G} \), playing \( v_0 \to v_1 \) is an \( \epsilon \)-best response for Player 1. Thus, we have that \( \inf_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_0^\mathcal{G}(\text{Out}_{v_0}(\sigma_0, \sigma_1)) = -2\mu \).

Therefore, for all \( \mu > 0 \), we have that \( \inf_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_0^\mathcal{G}(\text{Out}_{v_0}(\sigma_0, \sigma_1)) < \text{ASV}_\mathcal{G}(\sigma_0)(v_0) - \mu \).
Relation between the two types of deviations

In nonzero-sum mean-payoff games, robustness against modeling imprecision does not imply robustness against sub-optimal responses.

Lemma 4. For all \( \mu, \delta, \epsilon > 0 \), we can construct a nonzero-sum mean-payoff game \( \mathcal{G} \) such that for all Player 0 strategies \( \sigma_0 \) and vertex \( v \) in \( \mathcal{G} \), we have that:

\[
\forall H \in \mathcal{G}^{\delta, \epsilon} : \text{ASV}_H(\sigma_0)(v) > \inf_{\sigma_1 \in \text{BR}^\epsilon_v} \text{MP}_H(\text{Out}_v(\sigma_0, \sigma_1)) + \mu.
\]

Proof. Consider the game \( \mathcal{G} \) shown in Figure 4. Here, since all the vertices are controlled by Player 1, the strategy of Player 0 is inconsequential. For every \( \delta > 0 \), we claim that the best strategy for Player 1 across all perturbed games \( H \in \mathcal{G}^{\delta, \epsilon} \) is to play \( v_1 \to v_1 \) forever. One such example of a perturbed game is shown in Figure 5. Here, for every \( 0 < \epsilon < \delta \), we have that \( v_1 \to v_1 \) is the only best-response for Player 1. Therefore, we have that

\[
\inf_{H \in \mathcal{G}^{\delta, \epsilon}} \text{ASV}_H(\sigma_0)(v_1) = \mu' - \delta,
\]

for all \( \delta > 0 \).

However, if we relax the assumption that Player 1 plays optimally and assume that he plays an \( \epsilon \)-best response in the game \( \mathcal{G} \), we note that Player 1 can play a strategy \( (v_1, v_2, v_2, v_2) \), for some \( k_1, k_2 \in \mathbb{N} \), such that \( \frac{2\delta}{k_1 + k_2 + 1} > 2\delta - \epsilon \), and Player 0 gets a payoff of \( \frac{k_1 \mu'}{k_1 + k_2 + 1} > \mu'(1 - \frac{\epsilon}{2\delta}) \). Thus, we have that

\[
\inf_{\sigma_1 \in \text{BR}^\epsilon_v} \text{MP}_H(\text{Out}_v(\sigma_0, \sigma_1)) = \mu'(1 - \frac{\epsilon}{2\delta} \frac{k_1}{k_1 + k_2 + 1} + \mu, \quad \text{i.e., we choose } \mu' \text{ to be large enough so that } \mu < \mu' \cdot \frac{k_1}{k_1 + k_2 + 1} - \delta.
\]

On the contrary, robustness against sub-optimal responses implies robustness against modeling imprecision.

Theorem 5 (Robust strategy in non-zero sum games). For all non-zero-sum mean-payoff games \( \mathcal{G} \) with a set \( V \) of vertices, for all \( \epsilon > 0 \), for all vertices \( v \in V \), for all strategies \( \sigma_0 \) of Player 0, we have that \( \forall H \in \mathcal{G}^{\epsilon, \epsilon} : \text{ASV}_H(\sigma_0)(v) > \inf_{\sigma_1 \in \text{BR}^\epsilon_v} \text{MP}_H(\text{Out}_v(\sigma_0, \sigma_1)) - \epsilon \).

Proof. Consider a nonzero-sum mean-payoff game \( \mathcal{G} \) and a vertex \( v \) in \( \mathcal{G} \) and a strategy \( \sigma_0 \) of Player 0. We let \( \inf_{\sigma_1 \in \text{BR}^\epsilon_v} \text{MP}_H(\text{Out}_v(\sigma_0, \sigma_1)) = c \), for some \( c \in \mathbb{Q} \). Let the supremum of the payoffs that Player 1 gets when Player 0 plays \( \sigma_0 \) be \( y \), where \( y \in \mathbb{Q} \), i.e., \( \sup_{\rho \in \text{Out}_v(\mathcal{G}, \sigma_0)}(\text{MP}_1(\rho) | \rho \in \text{Out}_v(\mathcal{G}, \sigma_0)) = y \). For all outcomes \( \rho \) which are in Player 1’s \( 2\epsilon \)-best response of \( \sigma_0 \), we have that \( \text{MP}_1(\rho) > y - 2\epsilon \) and \( \text{MP}_0(\rho) > c \).

Now, consider a game \( H \in \mathcal{G}^{\epsilon, \epsilon} \) and a Player 0 strategy \( \sigma_0 \) played in \( H \). We can see that the maximum payoff that Player 1 gets when Player 0 plays \( \sigma_0 \) is bounded by \( y + \epsilon \) and \( y - \epsilon \), i.e., \( y - \epsilon < \sup_{\rho \in \text{Out}_v(H, \sigma_0)}(\text{MP}_1(\rho) | \rho \in \text{Out}_v(H, \sigma_0)) < y + \epsilon \). We let this value be denoted by \( y_H \). We note that if \( \sup_{\rho \in \text{Out}_v(H, \sigma_0)}(\text{MP}_1(\rho) = y_H \), then for the corresponding play \( \rho_H \) in the game \( \mathcal{G} \), the mean-payoff of Player 1 in \( \rho_H \) is \( \text{MP}_1(\rho_H) > y - 2\epsilon \). Thus, in the game \( \mathcal{G} \),
we note that $\text{MP}_0(\rho_H) \geq c$ and for the corresponding play in $H$, we have $\text{MP}_0(\rho_H) > c - \epsilon$. Thus, we have $\text{ASV}_H(\sigma_0)(v) > c - \epsilon = \inf_{\sigma_1 \in \text{BR}_1^H} \text{MP}_0^H(\text{Out}_v(\sigma_0, \sigma_1)) - \epsilon$.

We note that in the above theorem, we need to consider a strategy that is robust against $2\epsilon$-best-responses to ensure robustness against $\epsilon$ weight perturbations.

**$\epsilon$-Adversarial Stackelberg Value** The results above suggest that, in order to obtain some robustness guarantees in nonzero-sum mean-payoff games, we must consider a solution concept that accounts for $\epsilon$-best-responses of the adversary. This leads to the following definition: Given an $\epsilon > 0$, we define the adversarial value of Player 0 strategy $\sigma_0$ when Player 1 plays $\epsilon$-best-responses as

$$\text{ASV}^\epsilon(\sigma_0)(v) = \inf_{\sigma_1 \in \text{BR}_1^\epsilon(\sigma_0)} \text{MP}_0(\text{Out}_v(\sigma_0, \sigma_1))$$

and the $\epsilon$-Adversarial Stackelberg value at vertex $v$ is: $\text{ASV}^\epsilon(v) = \sup_{\sigma_0 \in \Sigma} \text{ASV}^\epsilon(\sigma_0)(v)$, and we note that $\text{ASV}(v) = \sup_{\epsilon > 0} \text{ASV}^\epsilon(v)$. We can now state a theorem about combined robustness of $\text{ASV}^\epsilon$.

**Theorem 6** (Combined robustness of $\text{ASV}^\epsilon$). For all nonzero-sum mean-payoff games $G$ with a set $V$ of vertices, for all $\epsilon > 0$, for all $\delta > 0$, for all $H \in G^{\pm \delta}$, for all vertices $v \in V$, and for all strategies $\sigma_0$, we have that $\text{ASV}^{2\epsilon+\delta}_G(\sigma_0)(v) > c$, then for all $H \in G^{\pm \delta}$, we have that $\inf_{\sigma_1 \in \text{BR}_1^\epsilon(\sigma_0)} \text{MP}_0^H(\text{Out}_v(\sigma_0, \sigma_1)) > c - \delta$.

**Proof.** The proof for Theorem 6 is very similar to the proof of Theorem 5 and involves looking at the set of $\epsilon$-best-responses in the game $H$ and showing that the corresponding plays lie in the set of $(2\delta + \epsilon)$-best-responses in the game $G$. This would imply that the corresponding Player 0 mean-payoffs for the $\epsilon$-best-responses of Player 1 in every perturbed game $H \in G^{\pm \delta}$ would always be greater than $c - \delta$. Therefore, we can extrapolate that $\text{ASV}^\epsilon_H(\sigma_0)(v) > c - \delta$.

In the rest of the paper we study properties of $\text{ASV}^\epsilon$ and solve the following two problems:

- **Threshold Problem of $\text{ASV}^\epsilon$:** Given $G$, $c \in \mathbb{Q}$, an $\epsilon > 0$, and a vertex $v$, we provide a nondeterministic polynomial time algorithm to decide if $\text{ASV}^\epsilon(v) > c$ (see Theorem 12).

- **Computation of $\text{ASV}^\epsilon$ and largest $c$:** Given $G$, an $\epsilon > 0$, and a vertex $v$, we provide an exponential time algorithm to compute $\text{ASV}^\epsilon(v)$ (see Theorem 25). We also establish that $\text{ASV}^\epsilon$ is achievable (see Theorem 31). Then we show, given a fixed threshold $c$, how to computation of largest $\epsilon$ such that $\text{ASV}^\epsilon(v) > c$. Formally, we compute $\sup\{\epsilon > 0 \mid \text{ASV}^\epsilon(v) > c\}$ (See Corollary 30).

**4 Threshold problem for the $\text{ASV}^\epsilon$**

In this section, given $c \in \mathbb{Q}$, and a vertex $v$ in game $G$, we study the threshold problem which is to determine if $\text{ASV}^\epsilon(v) > c$.

**Witnesses for $\text{ASV}^\epsilon$** For a game $G$ and $\epsilon > 0$, we associate with each vertex $v$ in $G$, sets of pairs or real numbers $(c, d)$ such that Player 1 has a strategy to ensure that the mean-payoffs of Player 0 and Player 1 are at most $c$ and greater than $d - \epsilon$ respectively. Formally, we have:

$$\Lambda^\epsilon(v) = \{(c, d) \in \mathbb{R}^2 \mid v \Vdash 1 \Rightarrow \text{MP}_0 \leq c \land \text{MP}_1 > d - \epsilon\}.$$
A vertex \( v \) is \((c, d)\)-bad if \((c, d) \in A^*(v)\). Let \( c' \in \mathbb{R} \). A play \( \pi \) in \( G \) is called a \((c', d)\)-witness of \( ASV(v) > c \) if \((\MP_{c'}(\pi), \MP_{d'}(\pi)) = (c', d) \) where \( c' > c \), and \( \pi \) does not contain any \((c, d)\)-bad vertex. A play \( \pi \) is called a witness for \( ASV(v) > c \) if it is a \((c', d)\)-witness for \( ASV(v) > c \) for some \( c', d \).

We first state the following technical lemma which states that if \( ASV'(v) > c \), then there exists a strategy \( \sigma_0 \) for Player 0 that enforces \( ASV'(\sigma_0)(v) > c \).

**Lemma 7.** For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), for all \( c > 0 \), and for all rationals \( c \), we have that \( ASV'(v) > c \) iff there exists a strategy \( \sigma_0 \in \Sigma_0 \) such that \( ASV'(\sigma_0)(v) > c \).

**Proof.** The right to left direction of the proof is trivial as \( \sigma_0 \) can play the role of witness for \( \sup_{\sigma \in \Sigma_0} \inf_{\chi_1, \chi_2 \in \BR(\sigma_0)} \MP_0(\Out_\chi_1(\sigma_0, \sigma_1)) > c \), i.e., if there exists a strategy \( \sigma_0 \) of Player 0 such that \( \inf_{\chi_1, \chi_2 \in \BR(\sigma_0)} \MP_0(\Out_\chi_1(\sigma_0, \sigma_1)) > c \), then \( \sup_{\sigma \in \Sigma_0} \inf_{\chi_1, \chi_2 \in \BR(\sigma_0)} \MP_0(\Out_\chi_1(\sigma_0, \sigma_1)) > c \). For the left to right direction of the proof, let \( \sup_{\sigma \in \Sigma_0} \inf_{\chi_1, \chi_2 \in \BR(\sigma_0)} \MP_0(\Out_\chi_1(\sigma_0, \sigma_1)) = c' \). By definition of \( c' \), there exists \( \sigma_0^\delta \) such that \( \inf_{\chi_1, \chi_2 \in \BR(\sigma_0^\delta)} \MP_0(\Out_\chi_1(\sigma_0^\delta, \sigma_1)) > c' - \delta \). Let us consider a \( \delta > 0 \) such that \( c' - \delta > c \). Such a \( \delta \) exists as \( c' > c \). Then we have that there exists \( \sigma_0 \) such that \( \inf_{\chi_1, \chi_2 \in \BR(\sigma_0)} \MP_0(\Out_\chi_1(\sigma_0, \sigma_1)) > c' - \delta > c \). □

The following theorem relates the existence of a witness and the threshold problem.

**Theorem 8.** For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), for all \( c > 0 \), and \( c \in \mathbb{Q} \), we have that \( ASV'(v) > c \) if and only if there exists a \((c', d)\)-witness of \( ASV'(v) > c \), where \( d \in \mathbb{Q} \).

Towards proving the existence of the witness, we first recall a result from [5] which states that for every pair of points \((x, y)\) present in \( F_{\min}(\CH(C(S))) \), we can construct a play \( \pi \) in the SCC \( S \) such that \((\MP_0(\pi), \MP_1(\pi)) = (x, y)\). Further, for every play \( \pi \) in SCC \( S \), we have that \((\MP_0(\pi), \MP_1(\pi)) \in \FR(\CH(C(S))) \).

**Lemma 9.** [5] Let \( S \) be an SCC in the arena \( A \) with a set \( V \) of vertices, and \( W \) be the maximum of the absolute values appearing on the edges in \( A \). We have that

1. for all \( \pi \in \P_{\MP_0}(A) \), if \( \inf(\pi) \subseteq S \), then \((\MP_0(\pi), \MP_1(\pi)) \in \FR(\CH(C(S))) \)
2. for all \((x, y) \in \FR(\CH(C(S))) \), there exists a play \( \pi \in \P_{\MP_0}(A) \) such that \( \inf(\pi) = S \) and \((\MP_0(\pi), \MP_1(\pi)) = (x, y) \)
3. The set \( \FR(\CH(C(S))) \) is effectively expressible in \( (\mathbb{R}, +, <) \) as a conjunction of \( O(m^2) \) linear inequalities, where \( m \) is the number of mean-payoff coordinates of simple cycles in \( S \), which is \( O(W \cdot |V|) \). Hence this set of inequalities can be pseudopolynomial in size.

Now we have the ingredients to prove Theorem 8

**Proof of Theorem 8** In [5] it has been shown that that \( ASV(v) > c \) if and only if there exists a witness for \( ASV(v) > c \). We begin by proving the right to left direction, that is, showing that the existence of a \((c', d)\)-witness implies that \( ASV'(v) > c \). The proof of this direction is similar to the proof in [5] for the case of \( ASV \). We are given a play \( \pi \) in \( G \) that starts from \( v \) and the play \( \pi \) is such that \((\MP_0(\pi), \MP_1(\pi)) = (c', d) \) for \( c' > c \) and does not cross a \((c, d)\)-bad vertex. We need to prove that \( ASV'(v) > c \). We do this by defining a strategy \( \sigma_0 \) for Player 0, such that \( ASV'(\sigma_0)(v) > c \):
1. \( \forall h \leq \pi, \) if \( \text{last}(h) \) is a Player 0 vertex, the strategy \( \sigma_0 \) is such that \( \sigma_0(h) \) follows \( \pi \).
2. \( \forall h \not\in \pi, \) where there has been a deviation from \( \pi \) by Player 1, we assume that Player 0 switches to a punishing strategy defined as follows: In the subgame after history \( h' \) where \( \text{last}(h') \) is the first vertex from which Player 1 deviates from \( \pi \), we know that Player 0 has a strategy to enforce the objective: \( \text{MP}_0 > c \lor \text{MP}_1 \leq d - \epsilon \). This is true because \( \pi \) does not cross any \((c,d)^\epsilon\)-bad vertex and since \( n \)-dimensional mean-payoff games are determined.

Let us now establish that the strategy \( \sigma_0 \) satisfies \( \text{ASV}'(\sigma_0)(v) > c \). First note that, since \( \text{MP}_1(\pi) = d \), we have that \( \sup_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_1(\text{Out}_v(\sigma_0, \sigma_1)) \geq d \). Now consider some strategy \( \sigma_1' \in \text{BR}_1(\sigma_0) \) and let \( \pi' = \text{Out}_v(\sigma_0, \sigma_1') \). Clearly, \( \pi' \) is such that \( \text{MP}_1(\pi') > \sup_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_1(\text{Out}_v(\sigma_0, \sigma_1)) - \epsilon \geq d - \epsilon \). If \( \pi' = \pi \), we know that \( \text{MP}_0(\pi') > c \). If \( \pi' \neq \pi \), then when \( \pi' \) deviates from \( \pi \) we know that Player 0 employs the punishing strategy, thus making sure that \( \text{MP}_0(\pi') > c \lor \text{MP}_1(\pi') \leq d - \epsilon \). Since \( \sigma_1' \in \text{BR}_1(\sigma_0) \), it must be true that \( \text{MP}_0(\pi') > c \). Thus, \( \forall \sigma_1' \in \text{BR}_1(\sigma_0) \), we have \( \text{MP}_0(\text{Out}_v(\sigma_0, \sigma_1')) > c \). Therefore, \( \text{ASV}'(\sigma_0)(v) > c \), which implies \( \text{ASV}'(v) > c \).

We now consider the left to right direction of the proof that requires new technical tools. We are given that \( \text{ASV}'(v) > c \). From Lemma 7 we have that \( \text{ASV}'(v) > c \) if there exists a strategy \( \sigma_0 \) of Player 0 such that \( \text{ASV}'(\sigma_0)(v) > c \). Thus, there exists a \( \delta > 0 \), such that

\[
\inf_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_1(\text{Out}_v(\sigma_0, \sigma_1)) = c' = c + \delta
\]

Let \( c = \sup_{\sigma_1 \in \text{BR}_1(\sigma_0)} \text{MP}_1(\text{Out}_v(\sigma_0, \sigma_1)) \). We first prove that for all \( \sigma_1 \in \text{BR}_1(\sigma_0) \), we have that \( \text{Out}_v(\sigma_0, \sigma_1) \) does not cross a \((c,d)^\epsilon\)-bad vertex. For every \( \sigma_1 \in \text{BR}_1(\sigma_0) \), we let \( \pi_{\sigma_1} = \text{Out}_v(\sigma_0, \sigma_1) \). We note that \( \text{MP}_1(\pi_{\sigma_1}) > d - \epsilon \) and \( \text{MP}_0(\pi_{\sigma_1}) > c \). For every \( \pi' \in \text{Out}_v(\sigma_0) \), we know that if \( \text{MP}_1(\pi') > d - \epsilon \) then there exists a strategy \( \sigma_1' \in \text{BR}_1(\sigma_0) \) such that \( \pi' = \text{Out}_v(\sigma_0, \sigma_1') \). This means that \( \text{MP}_0(\pi') > c \). Thus we can see that every deviation from \( \pi_{\sigma_1} \) either gives Player 1 a mean-payoff that is at most \( d - \epsilon \) or Player 0 a mean-payoff greater than \( c \). Therefore, we conclude that \( \pi_{\sigma_1} \) does not cross any \((c,d)^\epsilon\)-bad vertex.

Now consider a sequence \( (\sigma_i)_i \in \mathbb{N} \) of Player 1 strategies such that \( \sigma_i \in \text{BR}_1(\sigma_0) \) for all \( i \in \mathbb{N} \), and \( \lim_{i \to \infty} \text{MP}_1(\text{Out}_v(\sigma_0, \sigma_i)) = d \). Let \( \pi_i = \text{Out}_v(\sigma_0, \sigma_i) \). Let inf(\( \pi_i \)) be the set of vertices that occur infinitely often in \( \pi_i \), and let \( V_{\pi_i} \) be the set of vertices appearing along the play \( \pi_i \). Since there are finitely many SCCs, w.l.o.g., we can assume that for all \( i, j \in \mathbb{N} \), we have that inf(\( \pi_i \)) = inf(\( \pi_j \)), that is, all the plays end up in the same SCC, say \( S \), and also \( V_{\pi_i} = V_{\pi_j} = V_{\pi} \) (say). Note that \( S \subseteq V_{\pi} \).

Note that for every \( \epsilon \geq \delta > 0 \), there is a strategy \( \sigma_1^\epsilon \in \text{BR}_1(\sigma_0) \) of Player 1, and a corresponding play \( \pi' = \text{Out}_v(\sigma_0, \sigma_1^\epsilon) \) such that \( \text{MP}_1(\pi') > d - \delta \), and \( \text{MP}_0(\pi') \geq c' \). Also the set \( V_{\pi'} \) of vertices appearing in \( \pi' \) be such that \( V_{\pi'} \subseteq V_{\pi} \), and inf(\( \pi' \)) \( \subseteq S \).

Now since \( \text{F}_{\text{min}}(\text{CH}(\mathbb{C}(S))) \) is a closed set, we have that \( (\hat{c},d) \in \text{F}_{\text{min}}(\text{CH}(\mathbb{C}(S))) \) for some \( \hat{c} \geq c' > c \). We now use a result from [5] which states that for every pair of points \((x,y)\) in \( \text{F}_{\text{min}}(\text{CH}(\mathbb{C}(S))) \), we can construct a play \( \pi \) in the SCC \( S \) such that \((\text{MP}_0(\pi), \text{MP}_1(\pi)) = (x,y) \). Further, for every play \( \pi \) in SCC \( S \), we also have that \((\text{MP}_0(\pi), \text{MP}_1(\pi)) \in \text{F}_{\text{min}}(\text{CH}(\mathbb{C}(S))) \). This is formally stated as Lemma 3. Thus, we have that, there exists a play \( \pi^* \) such that \((\text{MP}_0(\pi^*), \text{MP}_1(\pi^*)) = (\hat{c},d) \). Also inf(\( \pi^* \)) \( \subseteq S \), and \( V_{\pi^*} \subseteq V_{\pi} \). The proof follows since for all vertices \( v \in V_{\pi} \), we have that \( v \) is not \((c,d)^\epsilon\)-bad. 

\[\blacksquare\]
Now, we establish a small witness property to show that the threshold problem is in \( \text{NP} \). We do this by demonstrating that the witness consists of two simple cycles \( l_1 \) and \( l_2 \) in a strongly connected component of the game graph that is reachable from \( v \) such that the convex combination of the payoffs of Player 0 in these cycles exceeds the threshold \( c \).

**Lemma 10.** For all mean-payoff games \( \mathcal{G} \), for all vertices \( v \) in \( \mathcal{G} \), for all \( c > 0 \), and for all rationals \( c \), we have that \( \text{ASV}'(v) > c \) if and only if there exist two simple cycles \( l_1, l_2 \), three simple paths \( \pi_1, \pi_2, \pi_3 \) from \( v \) to \( l_1 \), from \( l_1 \) to \( l_2 \), and from \( l_2 \) to \( l_3 \) respectively, and \( \alpha, \beta \in \mathbb{Q}^+ \), where \( \alpha + \beta = 1 \), such that

(i) \( \alpha \cdot \text{MP}_0(l_1) + \beta \cdot \text{MP}_0(l_2) = c' > c \), and

(ii) \( \alpha \cdot \text{MP}_1(l_1) + \beta \cdot \text{MP}_1(l_2) = d \), for some rational \( d \), and

(iii) there is no \((c,d)^{\epsilon}\)-bad vertex \( v' \) along \( \pi_1, \pi_2, \pi_3, l_1 \) and \( l_2 \).

Furthermore, \( \alpha \), \( \beta \), and \( d \) can be chosen so that they can be represented with a polynomial number of bits.

**Proof.** This proof is similar to the proof of Lemma 8 in \cite{[3]}. For the right to left direction of the proof, we give finite acyclic plays \( \pi_1, \pi_2, \pi_3 \), simple cycles \( l_1 \) and \( l_2 \) and constants \( \alpha, \beta \), we consider the witness \( \pi = \pi_1 \alpha \pi_2 \beta \pi_3 \) where, for all \( i \in \mathbb{N} \), we let \( \rho_i = l_1^{(i)} \pi_1 l_2^{(i)} \beta \pi_3 \). We know that \( \text{MP}_0(\pi) = \alpha \cdot \text{MP}_0(l_1) + \beta \cdot \text{MP}_0(l_2) = d \) and \( \text{MP}_0(\pi) = \alpha \cdot \text{MP}_0(l_1) + \beta \cdot \text{MP}_0(l_2) > c \). For all vertices \( v \) in \( \pi_1, \pi_2, \pi_3, l_1 \) and \( l_2 \), it is given that \( v \) is not \((c,d)^{\epsilon}\)-bad. Therefore, \( \pi \) is a suitable witness thus proving from Theorem 8 that \( \text{ASV}'(v) > c \).

For the left to right direction of the proof, we give \( \text{ASV}'(v) > c \). Using Theorem 8, we can construct a play \( \pi \) such that \( \text{MP}_0(\pi) > c \) and \( \text{MP}_1(\pi) = d \), and \( \pi \) does not cross a \((c,d)^{\epsilon}\)-bad vertex, i.e., for all vertices \( v' \) appearing in \( \pi \), we have that \( v' \not\in (c,d)^{\epsilon}\)-bad vertices. Let \( \inf(\pi) = S \) be the set of vertices appearing infinitely often in \( \pi \). Note that \( S \) forms an SCC. By abuse of notation, we also denote this SCC by \( S \) here. By Lemma 3, we have that \((\text{MP}_0(\pi), \text{MP}_1(\pi)) \in F_{\text{min}}(\text{CH}(\mathbb{C}^{S})) \). From Proposition 1 of \cite{[3]}, for a bi-weighted arena, we have that \( \text{F}_{\text{min}}(\text{CH}(\mathbb{C}^{S})) = \text{CH}(\text{F}_{\text{min}}(\mathbb{C}^{S})) \). Since \( \text{CH}(F_{\text{min}}(\mathbb{C}^{S})) \) can be expressed using conjunctions of linear inequalities whose coefficients have polynomial number of bits, the same also follows for \( \text{F}_{\text{min}}(\text{CH}(\mathbb{C}^{S})) \) in a bi-weighted arena. In addition, it is proven in \cite{[3]} that the set \( \Lambda'(v') \) is definable by a disjunction of conjunctions of linear inequalities whose coefficients have polynomial number of bits in the descriptions of the game \( \mathcal{G} \) and of \( \epsilon \). Hence \( \Lambda'(v') \) is also definable by a disjunction of conjunctions of linear inequalities whose coefficients have polynomial number of bits in the descriptions of the game \( \mathcal{G} \) and of \( \epsilon \). As a consequence of Theorem 2 in \cite{[3]} which states that given a system of linear inequalities that is satisfiable, there exists a point with polynomial representation that satisfies the system, we have that \( d \) can be chosen such that \((c,d) \in \Lambda'(v') \) and \((\text{MP}_0(\pi), d) \in F_{\text{min}}(\text{CH}(\mathbb{C}^{S})) \), and hence \( d \) can be represented with a polynomial number of bits.

Second, by applying the Carathéodory barcenter theorem, we can find two simple cycles \( l_1, l_2 \) in the SCC \( S \) and acyclic finite plays \( \pi_1, \pi_2 \) and \( \pi_3 \) from \( v \), and two positive rational constants \( \alpha, \beta \in \mathbb{Q}^+ \), such that \( \text{first}(\pi_1) = v \), \( \text{first}(\pi_2) = \text{last}(\pi_1) \), \( \text{first}(\pi_3) = \text{last}(\pi_2) \), \( \text{first}(\pi_2) = \text{last}(\pi_3) \), and \( \text{first}(\pi_2) = \text{first}(\pi_1) \), and \( \text{first}(\pi_3) = \text{first}(\pi_2) \), and \( \alpha + \beta = 1 \), \( \alpha \cdot \text{MP}_0(l_1) + \beta \cdot \text{MP}_0(l_2) > c \) and \( \alpha \cdot \text{MP}_1(l_1) + \beta \cdot \text{MP}_1(l_2) = d \). Again, using Theorem 2 in \cite{[3]}. we can assume that \( \alpha \) and \( \beta \) are rational values that can be represented using a polynomial number of bits. We note that for all vertices \( v \) in \( \pi_1, \pi_2, \pi_3, l_1 \) and \( l_2 \), we have that \( v \) is not \((c,d)^{\epsilon}\)-bad.
A play \( \pi \) is called a regular-witness of \( A\Sigma V'(v) > c \) if it is a witness of \( A\Sigma V'(v) > c \) and can be expressed as \( \pi = u \cdot \nu^\omega \), where \( u \) is a prefix of a play, and where \( \nu \) is a finite sequence of edges. We prove in that following theorem there exists a regular witness for \( A\Sigma V'(v) > c \). The existence of a regular witness helps in the construction of a finite memory strategy for Player 0.

**Theorem 11.** For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), for all \( \epsilon > 0 \), and for all rationals \( c \), we have that \( A\Sigma V'(v) > c \) if and only if there exists a regular \( (c', d') \)-witness of \( A\Sigma V'(v) > c \), where \( d \) is some rational.

**Proof.** Consider the witness \( \pi \) in the proof of Lemma \([10]\). We construct a regular-witness \( \pi' \) for \( A\Sigma V'(v) > c \) where \( \pi' = \pi_1, (l_1^{|\alpha'|k}_1, \pi_2, l_2^{|\beta'|k}, \pi_3)^\omega \) and \( \alpha', \beta' \) are constants in \( \mathbb{Q} \) and \( k \) is some large integer. We construct \( \pi' \) by modifying \( \pi \) as follows. We need to consider the following cases.

**Case 1:** \( MP_0(l_1) > MP_0(l_2) \) and \( MP_1(l_1) < MP_1(l_2) \)

Here, one simple cycle, \( l_1 \), increases Player 0’s mean-payoff while the other simple cycle, \( l_2 \), increases Player 1’s mean-payoff. We can build a witness \( \pi' = \pi_1, (l_1^{|\alpha'|k}_1, \pi_2, l_2^{|\beta'|k}, \pi_3)^\omega \) for some very large \( k \in \mathbb{N} \) and for some small \( \tau > 0 \) such that \( MP_0(\pi') > c \) and \( MP_1(\pi') = d' \).

We note that \( k \) and \( \tau \) are polynomial in the size of \( G \), and the largest weight \( W \) appearing on the edges of \( G \).

**Case 2:** \( MP_0(l_1) < MP_0(l_2) \) and \( MP_1(l_1) > MP_1(l_2) \)

This is analogous to **case 1**, and proceeds as mentioned above.

**Case 3:** \( MP_0(l_1) > MP_0(l_2) \) and \( MP_1(l_1) > MP_1(l_2) \)

One cycle, \( l_1 \), increases both Player 0 and Player 1’s mean-payoffs, while the other, \( l_2 \), decreases it. In this case, we can just omit one of the cycles and consider the one that gives a larger mean-payoff, to get a finite memory strategy. Thus, \( \pi' = \pi_1, l_1^\tau \) and we get \( MP_0(\pi') > c \), \( MP_1(\pi') \geq d \). Suppose \( MP_1(\pi') = d' \geq d \). Since no vertex in \( \pi_1, \pi_2, \pi_3, l_1 \), and \( l_2 \) is \( (c, d') \)-bad, we also have that they are not \( (c, d') \)-bad, and thus \( \pi' \) is a witness for \( A\Sigma V'(v) > c \).

**Case 4:** \( MP_0(l_1) < MP_0(l_2) \) and \( MP_1(l_1) < MP_1(l_2) \)

This is analogous to **case 3**, and proceeds as mentioned above.

In each of these cases, we have that \( A\Sigma V'(v) > c \) \( MP_0(\pi') > c \) and \( MP_1(\pi') \geq d \). Since we know that \( \pi \) does not cross a \( (c, d') \)-bad vertex, and the vertices of the play \( \pi' \) are a subset of the vertices of the play \( \pi \), we have that \( \pi' \) is a witness for \( A\Sigma V'(v) > c \). \( \blacktriangleleft \)

The following statement can be obtained by exploiting the existence of finite regular witnesses of polynomial size proved above.

**Theorem 12.** For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), for all \( \epsilon > 0 \), and for all \( c \in \mathbb{Q} \), it can be decided in nondeterministic polynomial time if \( A\Sigma V'(v) > c \), and a pseudopolynomial memory strategy of Player 0 suffices for this threshold. Furthermore, this decision problem is at least as hard as solving zero-sum mean-payoff games.

For proving Theorem 12, we start by stating a property of multi-dimensional mean-payoff games proved in \([17]\) that we rephrase here for a two-dimensional mean-payoff game. This

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5 For more details, we refer the reader to the Appendix A.
property expresses a relation between mean-payoff lim sup and mean-payoff lim inf objectives. We recall that in [17], the objective of Player 1 is to maximize the payoff in each dimension, i.e., for two-dimensional setting, given two rational \( c \) and \( d \), Player 1 wins if he has a winning strategy for \( \text{MP}_0 \geq c \land \text{MP}_1 \geq d \); otherwise Player 0 wins due to determinacy of multi-dimensional mean-payoff games. We call the mean-payoff game setting in [17] 2D-max mean-payoff games to distinguish it from the mean-payoff games that we consider here. Later we will relate the two settings.

**Proposition 13.** (Lemma 14 in [17]) For all mean-payoff games \( \mathcal{G} \), for all vertices \( v \) in the game \( \mathcal{G} \), and for all rationals \( c,d \), we have

\[
v \models \langle 1 \rangle \gg \text{MP}_0 \geq c \land \text{MP}_1 \geq d
\]

if and only if

\[
v \models \langle 1 \rangle \gg \text{MP}_0 \geq c \land \text{MP}_1 \geq d
\]

We now recall another property of multi-dimensional mean-payoff games proved in [17] that we rephrase here for a 2D-max mean-payoff game. This property expresses a bound on the weight of every finite play \( \pi^f \in \text{Out}_v(\sigma_0) \) where \( \sigma_0 \) is a memoryless winning strategy for Player 0.

**Lemma 14.** (Lemma 10 in [17]) For all 2D-max mean-payoff games \( \mathcal{G} \), for all vertices \( v \) in \( \mathcal{G} \), and for all rationals \( c,d \), if Player 0 \( \mathcal{G} \) wins \( \text{MP}_0 < c \lor \text{MP}_1 < d \) from \( v \) then she has a memoryless winning strategy \( \sigma_0 \) to do so, and there exist three constants \( m_G, c_G, d_G \in \mathbb{R} \) such that: \( c_G < c, d_G < d \), and for all finite plays \( \pi^f \in \text{Out}_v(\sigma_0) \), i.e. starting from \( v \) and compatible with \( \sigma_0 \), we have that

\[
w_0(\pi^f) \leq m_G + c_G \cdot |\pi^f|
\]

or

\[
w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|
\]

We now relate the 2D-max mean-payoff game in [17] where the objective of Player 1 is to maximize the payoff in both dimensions to our setting where in a game \( \mathcal{G} \), Player 1 maximizes the payoff in the second dimension, and minimizes the payoff on the first dimension from his set of available responses to a strategy of Player 0. The objective of Player 0 then is to maximize the payoff in the first dimension and minimize the payoff in the second dimension, i.e. given two rationals \( c \) and \( d \), Player 0’s objective is to ensure \( v \models \langle 0 \rangle \gg \text{MP}_0 > c \lor \text{MP}_1 < d \). We now state a modification of Lemma 14 as follows:

**Lemma 15.** For all mean-payoff games \( \mathcal{G} \), for all vertices \( v \) in \( \mathcal{G} \), and for all rationals \( c,d \), if Player 0 wins \( \text{MP}_0 > c \lor \text{MP}_1 < d \) from \( v \) then she has a memoryless winning strategy \( \sigma_0 \) to do so, and there exist three constants \( m_G, c_G, d_G \in \mathbb{R} \) such that \( c_G > c, d_G < d \), and for all finite plays \( \pi^f \in \text{Out}_v(\sigma_0) \), i.e. starting in \( v \) and compatible with \( \sigma_0 \), we have that

\[
w_0(\pi^f) \geq -m_G + c_G \cdot |\pi^f|
\]

or

\[
w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|
\]

---

6 Player 0 is called Player 2 in [17].
Proof. We show this by a reduction to a 2D-max mean-payoff game where Player 0’s objective is to ensure \(v \not< 0 \Rightarrow \text{MP}_0 \not< c \lor \text{MP}_1 < d\) in a 2D-max mean-payoff game.

We prove this lemma in two parts. If Player 0 wins \(\text{MP}_0 > c \lor \text{MP}_1 < d\), we show (i) the existence of a memoryless strategy \(\sigma_0\) for Player 0, and (ii) that there exist three constants \(m_G, c_G, d_G \in \mathbb{R}\) such that \(c_G > c, d_G < d\), and for all finite plays \(\pi^f \in \text{Out}_v(\sigma_0)\), i.e. starting from \(v\) and compatible with \(\sigma_0\), we have that either \(w_0(\pi^f) \geq -c_G + c_G \cdot |\pi^f|\) or \(w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|\).

Assume that Player 0 has a winning strategy from vertex \(v\) in \(G\) for \(\text{MP}_0 > c \lor \text{MP}_1 < d\). To prove (i), we subtract \(2c\) from the weights on the first dimension of all the edges, followed by multiplying them with \(-1\). We call the resultant 2D-max mean-payoff game \(G'\), and we have that \(v \not< 0 \Rightarrow \text{MP}_0 > c \lor \text{MP}_1 < d\) in \(G\) if and only if \(v \not< 0 \Rightarrow \text{MP}_0 < c \lor \text{MP}_1 < d\) in \(G'\). Using Proposition 13 and determinacy of multi-dimensional mean-payoff games, it follows that \(v \not< 0 \Rightarrow \text{MP}_0 > c \lor \text{MP}_1 < d\) in \(G\) if and only if \(v \not< 0 \Rightarrow \text{MP}_0 < c \lor \text{MP}_1 < d\) in \(G'\).

Also from Proposition 13, we have that if Player 0 has a winning strategy in \(G'\) for \(\text{MP}_0 < c \lor \text{MP}_1 < d\), then she has a memoryless strategy \(\sigma_0\) for the same, and the proof of Lemma 14 in [17] shows that same memoryless strategy \(\sigma_0\) is also winning for \(\text{MP}_0 < c \lor \text{MP}_1 < d\), thus concluding that if Player 0 wins \(\text{MP}_0 > c \lor \text{MP}_1 < d\) in \(G\) from vertex \(v\), then she has a memoryless winning strategy.

We prove (ii) by contradiction. Assume that Player 0 wins \(\text{MP}_0 > c \lor \text{MP}_1 < d\) from vertex \(v\) in \(G\), and by part (i), she has a memoryless winning strategy \(\sigma_0\). Assume for contradiction, that there does not exist three constants \(m_G, c_G, d_G \in \mathbb{R}\) such that \(c_G > c, d_G < d\), that for all finite plays \(\pi^f \in \text{Out}_v(\sigma_0)\), i.e. starting in \(v\) and compatible with \(\sigma_0\), we have either \(w_0(\pi^f) \geq -m_G + c_G \cdot |\pi^f|\) or \(w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|\).

Consider the steps in the construction of \(G'\) as defined above. As we subtract \(2c\) from the weights on the first dimension of each edge, and multiply the resultant weights on the first dimension by \(-1\), we have that there does not exist three constants \(m_G, c_G, d_G \in \mathbb{R}\) such that \(c_G > c, d_G < d\), and for all finite plays \(\pi^f \in \text{Out}_v(\sigma_0)\) in \(G\), i.e. starting from \(v\) and compatible with \(\sigma_0\), we have either

\[
\begin{align*}
w_0(\pi^f) & \geq -m_G + c_G \cdot |\pi^f| \\
or \quad w_1(\pi^f) & \leq m_G + d_G \cdot |\pi^f|
\end{align*}
\]

if and only if there does not exist three constants \(m_G, c_G, d_G \in \mathbb{R}\) such that \(c_G > c, d_G < d\), and for all finite plays \(\pi^f \in \text{Out}_v(\sigma_0)\) in the 2D-max mean-payoff game \(G'\) for the objective \(\text{MP}_0 < c \lor \text{MP}_1 < d\), i.e. starting in \(v\) and compatible with \(\sigma_0\), we have either

\[
\begin{align*}
w_0(\pi^f) & \leq m_G + (2c - c_G) \cdot |\pi^f| \\
or \quad w_1(\pi^f) & \leq m_G + d_G \cdot |\pi^f|
\end{align*}
\]

Let \(2c - c_G = c_G'\), and we have that \(c_G' < c\). Now since \(\sigma_0\) is a winning for Player 0 for the objective \(\text{MP}_0 < c \lor \text{MP}_1 < d\) in \(G'\) from \(v\), we reach a contradiction by Lemma 14 and due to determinacy of multi-dimensional mean-payoff games. •

Using Lemma 15, we can now prove that Player 0 can ensure from a vertex \(v\) that \(v \not< 1 \Rightarrow \text{MP}_0 < c \land \text{MP}_1 > d\) if and only if she can also ensure that \(v \not< 1 \Rightarrow \text{MP}_0 < c \land \text{MP}_1 \geq d'\) for all \(d' > d\). This is established in the following lemma.
Lemma 16. For all mean-payoff games \( G \), for all vertices \( v \in G \), and for all rationals \( c, d \), we have that:

\[ v \models \ll 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 > d \]

if and only if there exists a \( d' \in \mathbb{R} \), where \( d' > d \) such that

\[ v \models \ll 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 \geq d' \]

Proof. For the right to left direction of the proof, it is trivial to see that if \( v \models \ll 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 \geq d' \) for some \( d' > d \), then we have that \( v \models \ll 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 > d \).

For the left to right direction of the proof, we prove the contrapositive, i.e., we assume that \( \forall d' > d \), we have \( v \nmodels \ll 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 > d' \). Now we prove that \( v \nmodels \ll 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 > d \).

Since \( \forall d' > d \), Player 1 loses \( \text{MP}_0 \leq c \land \text{MP}_1 > d' \) from a given vertex \( v \), due to determinacy of multi-dimensional mean-payoff games, Player 0 wins \( \text{MP}_0 > c \land \text{MP}_1 < d' \) from vertex \( v \). By Lemma 15, Player 0 has a memoryless strategy \( \sigma_0 \) to achieve the objective \( \text{MP}_0 > c \land \text{MP}_1 < d' \) from vertex \( v \). Note that Player 0 has only finitely many memoryless strategies. Therefore there exists a strategy \( \sigma_0^* \) that achieves the objective \( v \models \ll 0 \gg \text{MP}_0 > c \land \text{MP}_1 < d' \) for all \( d' > d \). Now from Lemma 14, for every \( d' > d \), there exists three constants \( m_G, c_G, d'_G \in \mathbb{R} \) such that \( c_G > c, d'_G < d' \), and for all finite plays \( \pi^f \in \text{Out}_v(\sigma_0^*) \), we have that

\[ w_0(\pi^f) \geq -m_G + c_G \cdot |\pi^f| \]

or

\[ w_1(\pi^f) \leq m_G + d'_G \cdot |\pi^f| \]

Note that since the above is true for every \( d' > d \), we can indeed consider a \( d_G \in \mathbb{R} \), where \( d_G \leq d \), such that for all \( d' > d \), and for all finite plays \( \pi^f \in \text{Out}_v(\sigma_0^*) \), we have that

\[ w_0(\pi^f) \geq -m_G + c_G \cdot |\pi^f| \]

or we have that

\[ w_1(\pi^f) \leq m_G + d'_G \cdot |\pi^f| \]

Hence, for every play \( \pi \in \text{Out}_v(\sigma_0^*) \), we have that

\[ \text{MP}_0(\pi) \geq c_G \land \text{MP}_1(\pi) \leq d_G \]

Thus, we get

\[ v \models \ll 0 \gg \text{MP}_0 \geq c_G \land \text{MP}_1 \leq d_G \]

\[ \iff v \models \ll 0 \gg \text{MP}_0 > c \land \text{MP}_1 < d' \] for every \( d' > d \), since \( c_G > c \) and \( d_G < d' \).

We now construct a 2D-max mean-payoff game \( G' \) from the given game \( G \) by multiplying the first dimension of the weights of all the edges by \(-1\). Thus, in the game \( G' \), we get

\[ v \models \ll 0 \gg \text{MP}_0 \leq -c \land \text{MP}_1 < d' \] for every \( d' > d \)

\[ \iff v \models \ll 0 \gg \text{MP}_0 < -c \land \text{MP}_1 < d' \] for every \( d' > d \) [from Proposition 13]
We are finally ready to prove Theorem 12.

Proof of Theorem 12. According to Lemma 15, we consider a nondeterministic Turing machine that establishes the membership to \( \text{NP} \) by guessing a reachable SCC \( S \), a finite play \( \pi_1 \) to reach \( S \) from \( v \), two simple cycles \( l_1, l_2 \), along with the finite play \( \pi_1 \) from \( v \) to \( l_1 \), and the two finite plays \( \pi_2 \) and \( \pi_3 \) that connects the two simple cycles, and parameters \( \alpha, \beta \in \mathbb{Q}^+ \). Additionally, for each vertex \( v' \) that appear along the plays \( \pi_1, \pi_2 \) and \( \pi_3 \), and on the simple cycles \( l_1 \) and \( l_2 \), the Turing machine guesses a memoryless strategy \( \sigma'_0 \) for Player 0 that establishes \( v' \not\equiv 1 \gg MP_0 > c \land MP_1 > d - \epsilon \), which implies by determinacy of multi-dimensional mean-payoff games, that \( v' \equiv \ll 0 \gg MP_0 > c \land MP_1 \leq d \).

Besides, from Theorem 11, we can obtain a regular witness \( \pi' \). Using \( \pi' \), we build a finite memory strategy \( \sigma'^{\text{FM}}_0 \) for Player 0 as stated below:

1. Player 0 follows \( \pi' \) if Player 1 does not deviate from \( \pi' \). The finite memory strategy stems from the finite \( k \) as required in the proof of Theorem 11.
2. For each vertex \( v' \in \pi' \), Player 0 employs the memoryless strategy \( \sigma'_0 \) that establishes \( v' \not\equiv 1 \gg MP_0 > c \land MP_1 > d - \epsilon \). The existence of such a memoryless strategy follows from the proof of Lemma 10.

It remains to show that all the guesses can be verified in polynomial time. The only difficult part concerns the memoryless strategies of Player 0 to punish deviations of Player 1 from the witness play \( \pi' \). These memoryless strategies are used to prove that the witness does not cross \( (c, d)^t \)-bad vertices. For vertex \( v' \in \pi' \), we consider a memoryless strategy \( \sigma'^t_{0,v} \), and we need to establish that it can enforce \( MP_0 > c \lor MP_1 \leq d - \epsilon \). Towards this, we adapt the proof of Lemma 10 in [17], which in turn is based on the polynomial time algorithm of Kosaraju and Sullivan [14] for detecting zero-cycles in multi-weighted directed graphs.
such that \( w(C) \geq 0 \) (i.e., in both the dimensions, the weight is non-negative). In [17], it has been shown that the problem of deciding if \( S \) has a non-negative multi-cycle can be solved in polynomial time by solving a set of linear inequalities. In our case, we are interested in multi-cycles such that \( w_0(C) \leq c \) and \( w_1(C) > d - \epsilon \). As in the proof of Lemma 10 in [17], this can be checked by defining the following set of linear constraints. Let \( V_S \) and \( E_S \) respectively denote the set of vertices and the set of edges in \( S \). For every edge \( e \in E_S \), we consider a variable \( \chi_e \).

(a) For \( v \in V_S \), let \( \text{In}(v) \) and \( \text{Out}(v) \) respectively denote the set of incoming edges to \( v \) and the set of outgoing edges from \( v \). For every \( v \in V_S \), we define the linear constraint
\[
\sum_{e \in \text{In}(v)} \chi_e = \sum_{e \in \text{Out}(v)} \chi_e \quad \text{which intuitively models flow constraints.}
\]
(b) For every \( e \in E_S \), we define the constraint \( \chi_e \geq 0 \).
(c) We also add the constraint \( \sum_{e \in E_S} \chi_e \cdot w_0(e) \leq c \) and \( \sum_{e \in E_S} \chi_e \cdot w_1(e) > d - \epsilon \).
(d) Finally, we define the constraint \( \sum_{e \in E_S} \chi_e \geq 1 \) that ensures that the multi-cycle is non-empty.

This set of linear constraints can be solved in polynomial time, and formally following the arguments from [17], it has a solution if and only if there exists a multi-cycle \( C \) such that \( w_0(C) \leq c \) and \( w_1(C) > d - \epsilon \). The NP-membership follows since we have linearly many maximal SCCs from each vertex \( v' \) in the bi-weighted graph that is obtained from \( G \) by fixing the choices of Player 0 according to the memoryless strategy \( \sigma_0' \), and there are linearly many vertices \( v' \) for which we need to check that \( v' \) is not \((c, d')\)-bad.

Now we show that the memory required by the strategy \( \sigma_0^\text{FM} \) as described above is pseudopolynomial in the input size. Recall from the proof of Theorem [11] that \( k \) and \( \tau \) are polynomial in the size of \( G \), and the largest weight \( W \) appearing on the edges of \( G \). Assuming that the weights are given in binary, the number of states in the finite state machine realizing this strategy is thus \( \text{poly}(|G|, W) \), and hence pseudopolynomial in the input size, assuming that the weights are given in binary.

Now we prove that the threshold problem is at least as hard as solving zero-sum mean-payoff games. We show the proof for \( \text{ASV}^\prime(v) > c \). The proof for the case of \( \text{ASV}(v) > c \) is exactly the same. Consider a zero-sum mean-payoff game \( \mathcal{G}_0 = (A, \text{MP}) \), where \( A = (V, E, (V_0, V_1), w) \). We construct a bi-weighted mean-payoff game \( \mathcal{G} = (A', \text{MP}_0, \text{MP}_1) \) from \( \mathcal{G}_0 \) simply by adding to the arena \( A \) a weight function \( w_1 \) that assigns a weight 0 to each edge. Stated formally, \( A' = (V, E, (V_0, V_1), w, w_1) \) such that for all \( e \in E \), we have that \( w_1(e) = 0 \).

Now consider that from a vertex \( v \in V \), Player 0 has a winning strategy \( \sigma_0 \) in \( \mathcal{G}_0 \) such that \( \text{MP}(\sigma_0, \sigma_1) > c \) for all Player 1 strategies \( \sigma_1 \), and where \( c \) is a rational. We show that by playing \( \sigma_0 \) from \( v \) in \( \mathcal{G} \), we have that \( \text{ASV}^\prime(v) > c \). For every play \( \pi \in \text{Out}_0(\sigma_0) \) in \( \mathcal{G} \), we have that \( \text{MP}_1(\pi) = 0 \), and Player 1 has a response \( \sigma_1 \) such that \( \text{Out}_1(\sigma_0, \sigma_1) = \pi \). In \( \mathcal{G} \), Player 1 thus chooses a strategy that minimizes the mean-payoff of Player 0, and since in \( \mathcal{G}_0 \), we have that \( \text{MP}(\sigma_0, \sigma_1) > c \) for all strategies \( \sigma_1 \) of Player 1, it follows that \( \text{ASV}^\prime(v) > c \).

Now in the other direction, consider that in \( \mathcal{G} \), we have \( \text{ASV}^\prime(v) > c \). Thus from Lemma [7] there exists a strategy \( \sigma_0 \) for Player 0 such that \( \text{ASV}^\prime(v)(\sigma_0) > c \). Using similar arguments as above, we see that \( \sigma_0 \) is also a winning strategy in \( \mathcal{G}_0 \) giving a mean-payoff greater than \( c \) to Player 0.

Finite memory strategies of Player 0 In [8], it has been shown that given a mean-payoff game \( \mathcal{G} \), a vertex \( v \) in \( \mathcal{G} \), and a rational \( c \), the problem of deciding if \( \text{ASV}(v) > c \) is in NP. The use of an infinite memory strategy \( \sigma_0 \) for Player 0 such that \( \text{ASV}(\sigma_0)(v) > c \) has been shown in [8]. Here we give an improvement to that result in [8] showing that if \( \text{ASV}(v) > c \),
then there exists a finite memory strategy $\sigma_0$ of Player 0 such that $\text{ASV}(\sigma_0)(v) > c$. The proof arguments are similar to that of Theorem 12 and hence omitted.

Towards this, we first define the notion of a witness for $\text{ASV}$ as it appears in [8].

**Witnesses for $\text{ASV}$** For a mean-payoff game $G$, we associate with each vertex $v$ in $G$, the following set of pairs of real numbers: $\Lambda(v) = \{(c,d) \in \mathbb{R}^2 \mid v \Vdash 1 \Rightarrow \text{MP}_0 \leq c \land \text{MP}_1 \geq d\}$. A vertex $v$ is said to be $(c,d)$-bad if $(c,d) \in \Lambda(v)$. Let $c' \in \mathbb{R}$, a play $\pi$ in $G$ is called a $(c',d)$-witness of $\text{ASV}(v) > c$ if $(\text{MP}_0(\pi), \text{MP}_1(\pi)) = (c',d)$ where $c'$ and $\pi$ does not contain any $(c,d)$-bad vertex. A play $\pi$ is called a witness of $\text{ASV}'(v) > c$ if it is a $(c',d)$-witness of $\text{ASV}(v) > c$ for some $c',d$.

We state the following theorem which is similar to Theorem 11 but in the context of $\text{ASV}$ instead of $\text{ASV}'$.

**Theorem 17.** For all mean-payoff games $G$, for all vertices $v$ in $G$, and for all rationals $c$, we have that $\text{ASV}(v) > c$ if and only if there exists a regular $(c',d)$-witness of $\text{ASV}(v) > c$.

The proof of this theorem is exactly the same as that of Theorem 11 and hence omitted.

Now using Lemma 8 in [8] (which is similar to Lemma 10 but in the context of $\text{ASV}$ instead of $\text{ASV}'$), and using Theorem 17 we obtain the following theorem.

**Theorem 18.** For all mean-payoff games $G$, for all vertices $v \in V$, and for all rationals $c$, if $\text{ASV}(v) > c$, then there exists a pseudopolynomial memory strategy $\sigma_0$ for Player 0 such that $\text{ASV}(\sigma_0)(v) > c$.

The proof follows since as in the proof of Theorem 11 the values of $k$ and $\tau$ are polynomial in the size of $G$, and the weights on the edges which are assumed to be given in binary. The proof arguments are similar to that of Theorem 12 and hence omitted.

We define the $c$-adversarial Stackelberg value for Player 0 when Player 0 is restricted to using finite memory strategies as :

$$\text{ASV}_{\text{FM}}(v) = \sup_{\sigma_0 \in \Sigma^\text{FM}_0} \inf_{\sigma_1 \in \text{BR}_0(\sigma_0)} \text{MP}_0(\text{Out}_v(\sigma_0,\sigma_1))$$

$$\text{ASV}'_{\text{FM}}(v) = \sup_{\sigma_0 \in \Sigma^\text{FM}_0} \inf_{\sigma_1 \in \text{BR}_0(\sigma_0)} \text{MP}_0(\text{Out}_v(\sigma_0,\sigma_1))$$

where $\Sigma^\text{FM}_0$ refers to the set of all finite memory strategies of Player 0. We note that for every finite memory strategy $\sigma_0$ of Player 0, a best-response of Player 1 to $\sigma_0$ always exists as noted in [8]. As a corollary of Theorem 12, we observe that in a mean-payoff game $G$, the $\text{ASV}'$ from every vertex $v$ does not change even if Player 0 is restricted to using only finite memory strategies. The result also holds for $\text{ASV}$ due to Theorem 13.

**Corollary 19.** For all games $G$, for all vertices $v$ in $G$, and for all $\epsilon > 0$, we have that $\text{ASV}'_{\text{FM}}(v) = \text{ASV}'(v)$ and $\text{ASV}_{\text{FM}}(v) = \text{ASV}(v)$.

Let $\text{ASV}'(v) = c$ ($\text{ASV}(v) = c$), for some $c \in \mathbb{Q}$. Note that for every $c' < c$, there exists a finite memory strategy $\sigma_0^{FM}$ for Player 0 such that $\text{ASV}'(\sigma_0^{FM})(v) > c'$ ($\text{ASV}(\sigma_0^{FM})(v) > c'$) leading to $\text{sup}_{\sigma_0 \in \Sigma^\text{FM}_0} \text{ASV}'(\sigma_0)(v) = \text{ASV}'(v) = c$ ($\text{sup}_{\sigma_0 \in \Sigma^\text{FM}_0} \text{ASV}(\sigma_0)(v) = \text{ASV}(v) = c$). This gives $\text{ASV}'_{\text{FM}}(v) = \text{ASV}'(v)$ ($\text{ASV}_{\text{FM}}(v) = \text{ASV}(v)$).

This corollary is important from a practical point of view as it implies that both the $\text{ASV}'$ and $\text{ASV}$ can be approximated to any precision with a finite memory strategy. Nevertheless, we show in Theorem 31 that infinite memory is necessary to achieve the exact $\text{ASV}'$. 
Figure 6 Reduction of the partition problem to the threshold problem where Player 0 is restricted to memoryless strategies.

Memoryless strategies of Player 0  We now establish that the threshold problem is NP-complete when Player 0 is restricted to play memoryless strategies. First we define

\[ \text{ASV}_{\text{ML}}(v) = \sup_{\sigma_0 \in \Sigma^0_{\text{ML}}} \inf_{\sigma_1 \in \text{BR}^1(\sigma_0)} \text{MP}_0(\text{Out}_e(\sigma_0, \sigma_1)) \]

and

\[ \text{ASV}_{\text{ML}}(v) = \sup_{\sigma_0 \in \Sigma^0_{\text{ML}}} \inf_{\sigma_1 \in \text{BR}^1(\sigma_0)} \text{MP}_0(\text{Out}_e(\sigma_0, \sigma_1)) \]

where \( \Sigma^0_{\text{ML}} \) is the set of all memoryless strategies of Player 0.

Theorem 20. For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), for all \( \epsilon > 0 \), and for all rationals \( c \), the problem of deciding if \( \text{ASV}'_{\text{ML}}(v) > c \) (\( \text{ASV}_{\text{ML}}(v) > c \)) is NP-Complete.

Proof. The proof of hardness is a reduction from the partition problem while easiness is straightforwardly obtained by techniques used in the proof of Theorem 12. Note that Player 0 can guess a memoryless strategy \( \sigma_0 \) from a vertex \( v \) in NP, and we consider the bi-weighted graph obtained by fixing the choices of Player 0 according to the strategy \( \sigma_0 \). The set of maximal SCCs that are reachable from \( v \) in this bi-weighted graph can be computed in linear time. For each SCC \( S \), we need to check that Player 1 cannot achieve \( \text{MP}_0 \leq c \land \text{MP}_1 > d - \epsilon \) (\( \text{MP}_0 \leq c \land \text{MP}_1 \geq d \)) to ensure \( \text{ASV}'(\sigma_0) > c \) (\( \text{ASV}(\sigma_0) > c \)). This can be done in polynomial time as has been done in the proof of Theorem 12.

We prove the NP-hardness result by reducing an NP-complete problem, i.e., the partition problem, to solving the threshold problem in a two-player non-zero-sum mean-payoff game. The partition problem is described as follows: Given a set of natural numbers \( S = \{a_1, a_2, a_3, \ldots, a_n\} \), to decide if we can partition \( S \) into two sets \( R \) and \( M \) such that \( \sum_{a_i \in R} a_i = \sum_{a_j \in M} a_j \). W.l.o.g. assume that \( \sum_{a_i \in S} a_i = 2T \).

Given an instance of a partition problem, we construct a two-player non-zero-sum mean-payoff game as described in Figure 6 such that \( \text{ASV}'_{\text{ML}}(v_0) > \frac{T - 1}{n} \) for some \( \epsilon < \frac{1}{2n} \) if and only if there exists a solution to the partition problem.

Consider the case, where a solution to the partition problem exists. In the game in Figure 6 we can construct a Player 0 strategy \( \sigma_0 \) as follows: from each Player 0 vertex \( v_i \), for all \( i \in \{1, 2, \ldots, n\} \), Player 0 chooses to play \( (a_i, 0) \) (or the top-arrow) if \( a_i \in R \) or \( (0, a_i) \) (or the bottom-arrow) if \( a_i \in M \) and from \( v' \), she plays \( v' \rightarrow v' \). Thus, we have that from vertex \( v_0 \), if Player 1 chooses to play \( v_0 \rightarrow v_1 \), both players get a mean-payoff of \( \frac{T}{n} \), but if
he chooses to play \( v_0 \to v' \), Player 1 gets a mean-payoff of \( \frac{T-0.5}{n} \) which is not the \( \epsilon \)-best response for \( \epsilon < \frac{1}{2n} \). Thus, we can see that \( \mathsf{ASV_{ML}}(v_0) > \frac{T-1}{n} \).

Now consider the case, where the solution to the partition problem does not exist. We note that any memoryless strategy \( \sigma_0 \) for Player 0 would involve choosing to play from every vertex \( v_i \), where \( i \in \{1, 2, \ldots, n\} \), either \( (a_i, 0) \) (or the top-arrow) or \( (0, a_i) \) (or the bottom-arrow). This leads to a cycle from \( v_1 \to v_n \), which either gives Player 1 a mean-payoff of \( \leq \frac{T-1}{n} \) or Player 0 a mean-payoff of \( \leq \frac{T-1}{n} \). In the first case, the \( \epsilon \)-best response of Player 1 would be to play \( v_0 \to v' \), and thus \( \mathsf{ASV_{ML}}(v_0) = 0 \). For the second case, although Player 1’s \( \epsilon \)-best response is to play \( v_0 \to v_1 \), Player 0 would only get a mean-payoff which is \( \leq \frac{T-1}{n} \). Therefore in the two-player nonzero-sum mean-payoff game described in Figure 6, we have that \( \mathsf{ASV_{ML}}(v) > \frac{T-1}{n} \) if and only if there exists a solution to the partition problem for the set \( S \).

We can construct a similar argument for the case of \( \mathsf{ASV_{ML}}(v) \) by considering only best-responses (and not \( \epsilon \)-best responses) to the Player 0 strategy \( \sigma_0 \). Thus, in the two-player nonzero-sum mean-payoff game described in Figure 6, we have that \( \mathsf{ASV_{ML}}(v) > \frac{T-1}{n} \) if and only if there exists a solution to the partition problem for the set \( S \).

5 Computation of the \( \mathsf{ASV}^\epsilon \) and the largest \( \epsilon \) possible

Here, we express the \( \mathsf{ASV}^\epsilon \) as a formula in the theory of reals by adapting a method provided in [5] for \( \mathsf{ASV} \). We then provide a new \( \text{EXPTime} \) algorithm to compute the \( \mathsf{ASV}^\epsilon \) based on \( \text{LP} \) which in turn is applicable to \( \mathsf{ASV} \) as well.

**Extended mean-payoff game**  Given a mean-payoff game \( G = (A, \langle \mathsf{MP}_0, \mathsf{MP}_1 \rangle) \) with \( A = (V, E, \langle V_0, V_1 \rangle, w_0, w_1) \), we construct an extended mean-payoff game \( G^\text{ext} = (A^\text{ext}, \langle \mathsf{MP}_0, \mathsf{MP}_1 \rangle) \), and whose vertices and edges are defined as follows. The set of vertices is \( V^\text{ext} = V \times 2^V \). With every history \( h \) in \( G \), we associate a vertex in \( G^\text{ext} \) which is a pair \((v, P)\), where \( v = \text{last}(h) \) and \( P \) is the set of the vertices traversed along \( h \). Accordingly the set of edges and the weight functions are respectively defined as \( E^\text{ext} = \{(v, P), (v', P') \} | (v, v') \in E \land P' = P \cup \{v'\} \) and \( \mathsf{w}^\text{ext}(v, P, v', P') = w_i(v, v') \) for \( i \in \{0, 1\} \). We observe that there exists a bijection between the plays \( \pi \) in \( G \) and the plays \( \pi^\text{ext} \) in \( G^\text{ext} \) which start in vertices of the form \((v, \{i\})\), i.e. \( \pi^\text{ext} \) is mapped to the play \( \pi \) in \( G \) that is obtained by erasing the second dimension of its vertices. Note that the second component of the vertices of the play \( \pi^\text{ext} \) stabilises into a set of vertices of \( G \) which we denote by \( V^\epsilon(\pi^\text{ext}) \).

To relate the witnesses with the \( \mathsf{ASV}^\epsilon \) in the game \( G^\text{ext} \), we introduce the following proposition. This proposition is an adaptation of Proposition 10 in [5] that is used to compute \( \mathsf{ASV} \).

**Proposition 21.** For all mean-payoff games \( G \), the following holds:

- Let \( \pi^\text{ext} \) be an infinite play in the extended mean-payoff game and \( \pi \) be its projection on the original mean-payoff game \( G \) (over the first component of each vertex); the following properties hold:
  - For all \( i < j \), if \( \pi^\text{ext}(i) = (v_i, P_i) \) and \( \pi^\text{ext}(j) = (v_j, P_j) \), then \( P_i \subseteq P_j \)
  - \( \mathsf{MP}_i(\pi^\text{ext}) = \mathsf{MP}_i(\pi) \), for \( i \in \{0, 1\} \).
- The unfolding of \( G \) from \( v \) and the unfolding of \( G^\text{ext} \) from \((v, \{i\})\) are isomorphic and so \( \mathsf{ASV}^\epsilon(v) = \mathsf{ASV}^\epsilon(v, \{i\}) \).
By the first point of the above proposition and since the set of vertices of the mean-payoff game is finite, the second component of any play \( \pi^{\text{ext}} \), that keeps track of the set of vertices visited along \( \pi^{\text{ext}} \), stabilises into a set of vertices of \( G \) which we denote by \( V^*(\pi^{\text{ext}}) \).

We now characterize \( \Lambda^*(v) \) with the notion of witness introduced earlier and the decomposition of \( G^{\text{ext}} \) into SCCs. For a vertex \( v \) in \( V \), let \( \text{SCC}^{\text{ext}}(v) \) be the set of strongly-connected components in \( G^{\text{ext}} \) which are reachable from \( (v, \{v\}) \).

Lemma 22. For all mean-payoff games \( G \) and for all vertices \( v \) in \( G \), we have

\[
\Lambda^*(v) = \max_{S \in \text{SCC}^{\text{ext}}(v)} \sup \{ c \in \mathbb{R} \mid \exists \pi : \pi^{\text{ext}} \text{ is a witness for } \Lambda^*(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S \}
\]

Proof. First, we note the following sequence of inequalities:

\[
\Lambda^*(v) = \sup \{ c \in \mathbb{R} \mid \Lambda^*(v) \geq c \} = \sup \{ c \in \mathbb{R} \mid \Lambda^*(v) > c \} = \sup \{ c \in \mathbb{R} \mid \exists \pi : \pi^{\text{ext}} \text{ is a witness for } \Lambda^*(v) > c \} = \sup_{S \in \text{SCC}^{\text{ext}}(v)} \{ c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \Lambda^*(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S \}
\]

The first two equalities follow from the definition of the supremum and that \( \Lambda^*(v) \in \mathbb{R} \).

The third equality follows from Theorem 5 that guarantees the existence of witnesses for strict inequalities. The fourth equality is due to the second point in Proposition 21. The last equality is a consequence of first point in Proposition 21.

By definition of \( G^{\text{ext}} \), for every SCC \( S \) of \( G^{\text{ext}} \), there exists a set \( V^*(S) \) of vertices of \( G \) such that every vertex of \( S \) is of the form \( (v', V^*(S)) \), where \( v' \) is a vertex in \( G \).

Now, we define \( \Lambda_S^{\text{ext}} = \bigcup_{v \in V^*(S)} \Lambda^*(v) \) as the set of \((c, d)\) such that Player 1 can ensure \( v \models 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 \geq d - \epsilon \) from some vertex \( v \in S \). The set \( \Lambda_S^{\text{ext}} \) can be represented by a formula \( \Psi_e(x, y) \) in the first order theory of reals with addition, \( (\mathbb{R}, +, <) \), with two free variables. Before we begin to prove the above, we refer to the following lemma which has been established in \[8\].

Lemma 23. [Lemma 9 in \[8\]] For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), and for all rationals \( c, d \), we can effectively construct a formula \( \Psi_e(x, y) \) of \( (\mathbb{R}, +, <) \) with two free variables such that \((c, d) \in \Lambda(v)\) if and only if the formula \( \Psi_e(x, y)|[x/c, y/d] \) is true.

Using the above lemma, we can now compute an effective representation of the infinite set of pairs \( \Lambda^*(v) \) for each vertex \( v \) of the mean-payoff game. This is stated in the following lemma.

Lemma 24. For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), for all \( \epsilon > 0 \), and for all rationals \( c, d \), we can effectively construct a formula \( \Psi_e(x, y) \) of \( (\mathbb{R}, +, <) \) with two free variables such that \((c, d) \in \Lambda^*(v)\) if and only if the formula \( \Psi_e(x, y)|[x/c, y/d] \) is true.

Proof. From the definition of \( \Lambda(v) \) from \[3\], we know that a pair of real values \((c, d) \in \Lambda(v)\) if \( v \models 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 \geq d \). We now recall from the definition of \( \Lambda^*(v) \) that \((c, d) \in \Lambda^*(v)\) if \( v \models 1 \gg \text{MP}_0 \leq c \land \text{MP}_1 \geq d - \epsilon \). From this, we can see that

\[
\Psi_e(x, y) \equiv \exists c > 0 \cdot \Psi(x, y - \epsilon + c)
\]
We can now state the following theorem about the computability of $\text{ASV}'(v)$:

**Theorem 25.** For all mean-payoff games $G$, for all vertices $v$ in $G$ and for all $\epsilon > 0$, the $\text{ASV}'(v)$ can be effectively expressed by a formula in $\langle \mathbb{R}, +, \cdot, < \rangle$, and can be computed from this formula.

**Proof.** To prove this theorem, we build a formula in $\langle \mathbb{R}, +, < \rangle$ that is true iff $\text{ASV}'(v) = z$. Recall from Lemma 22 that

$$\text{ASV}'(v) = \max_{S \in \text{SCC}^*(v)} \sup \{ c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \text{ASV}'(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S \}$$

Since it is easy to express $\max_{S \in \text{SCC}^*(v)}$ in $\langle \mathbb{R}, +, < \rangle$, we concentrate on one SCC $S$ reachable from $(v, \{v\})$, and show how to express

$$\sup \{ c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \text{ASV}'(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S \}$$

in $\langle \mathbb{R}, +, < \rangle$.

Such a value of $c$ can be encoded by the following formula

$$\rho^S_v(c) \equiv \exists x, y : x > c \land \Phi_S(x, y) \land \neg \Psi^S_v(c, y)$$

where $\Phi_S(x, y)$ is the symbolic encoding of $F_{\min}(\text{CH}(\text{C}(S)))$ in $\langle \mathbb{R}, +, < \rangle$ as defined in Lemma 9. This states that the pair of values $(x, y)$ are the mean-payoff values realisable by some play in $S$. By Lemma 24, the formula $\neg \Psi^S_v(c, y)$ expresses that the play does not cross a $(c, y)$-bad vertex. So the conjunction $\exists x, y : x > c \land \Phi_S(x, y) \land \neg \Psi^S_v(c, y)$ establishes the existence of a witness with mean-payoff values $(x, y)$ for the threshold $c$, and hence satisfying this formula implies that $\text{ASV}'(v) > c$. Now we consider the formula

$$\rho^S_{\max,v}(z) \equiv \forall e > 0 : \rho^S_v(z - e) \land \forall c : \rho^S_v(c) \implies c < z$$

which is satisfied by a value that is the supremum over the set of values $c$ such that $c$ satisfies the formula $\rho^S_v$, and hence the formula $\rho^S_{\max,v}(z)$ expresses

$$\sup \{ c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \text{ASV}'(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S \}$$

From the formula $\rho^S_{\max,v}$, we can compute the $\text{ASV}'(v)$ by quantifier elimination in

$$\max_{S \in \text{SCC}^*(v)} \exists z : \rho^S_{\max,v}(z)$$

and obtain the unique value of $z$ that makes this formula true, and equals $\text{ASV}'(v)$.

**Example 26.** We illustrate the computation of $\text{ASV}'$ with an example. Consider the mean-payoff game $G$ depicted in Figure 7 and its extension $G^{\text{ext}}$ as shown in Figure 8.

Note that in $G^{\text{ext}}$ there exist three SCCs which are $S_1 = \{v'_{02}, v'_{1'}\}$, $S_2 = \{v'_{21}\}$, and $S_3 = \{v'_{02}\}$. The SCCs $S_2$ and $S_3$ are similar, and thus $\rho^S_{\max,v}(z)$ and $\rho^S_{\max,v}(z)$ would be equivalent. We start with SCC $S_1$ that contains two cycles $v'_1 \rightarrow v'_1$ and $v'_{02} \rightarrow v'_{1'}$, and SCC $S_2$ contains one cycle $v'_{02} \rightarrow v'_{21}$. Since $S_3$ is similar to $S_2$, we consider only $S_2$ in our example. Thus, the set $F_{\min}(\text{CH}(\text{C}(S_1)))$ is represented by the Cartesian points within the triangle represented by $(0, 2), (1, 1)$ and $(0, 1)$ and $F_{\min}(\text{CH}(\text{C}(S_2))) = \{(0, 1)\}$. Thus, we get that $\Phi_{S_1}(x, y) \equiv (x \geq 0 \land x \leq 1) \land (y \geq 1 \land y \leq 2) \land (x + y) \leq 2$ and $\Phi_{S_2}(x, y) \equiv x = 0 \land y = 1$. Now, we calculate $\Lambda'(v'_{02}), \Lambda'(v'_{21})$ and $\Lambda'(v'_{1'})$ for some value of $\epsilon$ less than 1. We note that the vertex $v'_{1'}$ is not $(0, 2 + \epsilon - \delta)$-bad, for all $0 < \delta < 1$, as Player 0 can always choose the edge
Note that the coordinate \((0,1)\) is obtained as the pointwise minimum over the two coordinates separately.

**An \(\text{EXPTime}\) algorithm for computing \(\text{ASV}^v\)** Now we provide a new linear programming based method that extends the previous approach to compute the \(\text{ASV}^v\), and show that the value can be computed in \(\text{EXPTime}\). We first illustrate our approach with the help of the following example by constructing the \(\text{LP}\) formulation for \(\rho^S(c)\) for each SCC \(S\) thereby building the system of LPs for computing \(\text{ASV}^v(v_0)\).
Example 27. We previously showed that the ASV\(^r\)(v\(_0\)) can be computed by quantifier elimination of a formula in the theory of reals with addition. Now, we compute the ASV\(^r\)(v\(_0\)) by solving a set of linear programs for every SCC in \(G^{\text{ext}}\). We recall that there are three SCCs \(S_1, S_2\) and \(S_3\) in \(G^{\text{ext}}\). From Lemma 2, we have that \(F_{\text{min}}(\text{CH}(C(S_i)))\) for \(i \in \{1, 2, 3\}\) can be defined using a set of linear inequalities. Now recall that \(F_{\text{min}}(\text{CH}(C(S_2))) = F_{\text{min}}(\text{CH}(C(S_3))) = \{(0, 1)\}\), and \(F_{\text{min}}(\text{CH}(C(S_1)))\) is represented by the set of points enclosed by the triangle formed by connecting the points \((0, 1), (1, 1)\) and \((0, 2)\) as shown in Figure 7, and \(\Lambda^r(v'_{02}) = \Lambda^r(v'_{21}) = \Lambda^r(v'_{22}) = \Lambda^r(v'_1) = \{(c, y) | c \geq 0 \land y < 1 + \epsilon\}\). Now, we consider the SCC \(S_1\), and the formula \(\neg \Psi^r_{S_1}\). We start this by finding the complement of \(\Lambda^r(v'_{21})\) and \(\Lambda^r(v'_1)\), that is, \(\overline{\Lambda}(v'_{21}) = \overline{\Lambda}(v'_1) = \{c < 0 \lor y \geq 1 + \epsilon\}\). Similarly for the SCC \(S_2\), and \(\neg \Psi^r_{S_2}\), we can fix a value of \(\rho\) in \(\{0\}\) and \(\neg \Psi^r_{S_2} = \overline{\Lambda}(v'_{22}) = \{c < 0 \lor y \geq 1 + \epsilon\}\) and \(\neg \Psi^r_{S_2} = \overline{\Lambda}(v'_{22}) = \{c < 0 \lor y \geq 1 + \epsilon\}\) and \(\neg \Psi^r_{S_3} = \overline{\Lambda}(v'_{23}) = \{c < 0 \lor y \geq 1 + \epsilon\}\). Note that the formulas \(\Phi_{S_1}(x, y)\) and \(\Phi_{S_2}(x, y)\) are represented by the set of linear inequalities \(x = 0 \land y = 1\) and the formula \(\Phi_{S_2}(x, y)\) is represented by the set of linear inequalities \(y \geq 1 \land y \leq 2 \land x \leq 1 \land (x + y) \leq 2 \land c < 0 \lor y \geq 1 + \epsilon\). Now the formula \(\rho_{S_1}^r(c)\) can be expressed using a set of linear equations and inequalities as follows: \(x > c \land y \geq 1 \land y \leq 2 \land x \leq 1 \land (x + y) \leq 2 \land c < 0 \lor y \geq 1 + \epsilon\) and the formula \(\rho_{S_2}^r(c)\) can be expressed using a set of linear equations and inequalities as follows: \(x > c \land y = 0 \land y \geq 1 \land c < 0 \lor y \geq 1 + \epsilon\). We maximise the value of \(c\) in the formula \(\rho_{S_1}^r(c)\) to get the following two linear programs: \text{maximise } c \text{ in } (x > c \land y \geq 1 \land y \leq 2 \land x \leq 1 \land (x + y) \leq 2 \land c < 0) \text{ which gives a solution } \{0\}\) and \text{maximise } c \text{ in } (x > c \land y \geq 1 \land y \leq 2 \land x \leq 1 \land (x + y) \leq 2 \land y \geq (1 + \epsilon)) \text{ which gives us a solution } \{(1 - \epsilon)\}\). Similarly, maximising \(c\) in the formulas \(\rho_{S_2}^r(c)\) and \(\rho_{S_3}^r(c)\) would give us the following two linear programs: \text{maximise } c \text{ in } (x > c \land x = 0 \land y = 1 \land c < 0) \text{ which gives a solution } \{0\}\) and \text{maximise } c \text{ in } (x > c \land x = 0 \land y = 1 \land y \geq (1 + \epsilon)) \text{ which gives us a solution } \{0\}. Thus, we conclude that ASV\(^r\)(v\(_0\)) = 1 - \epsilon which is the maximum value amongst all the SCCs. Note that in an LP, the strict inequalities are replaced with non-strict inequalities, and computing the supremum in the objective function is replaced by maximizing the objective function.

Again, for every SCC \(S\) and for every LP corresponding to \(S\), we can fix a value of \(c\) and change the objective function to maximise \(\epsilon\) from maximise \(c\) in order to obtain the maximum value of \(c\) that allows ASV\(^r\)(v\(_0\)) > \(c\). For example, consider the LP \((x > c \land y \geq 1 \land y \leq 2 \land x \leq 1 \land (x + y) \leq 2 \land y \geq (1 + \epsilon))\) in SCC \(S_1\) and fix a value of \(c\), and then maximize the value of \(c\). Doing this over all linear programs in an SCC, and over all SCCs, reachable from \(v_0\) for a fixed \(c\) gives us the supremum value of \(c\) such that we have ASV\(^r\)(v\(_0\)) > \(c\).
of linear inequalities forms an equivalence class, also called *cells* [3]. Let \( V_G \) denote the set of mean-payoff coordinates of simple cycles in \( G \), and we have that \( |V_G| = \mathcal{O}(V^{poly}(d)) \). Let \( B(V_G) \) denote the set of geometric centres where each geometric centre is a centre of at most \( d+1 \) points from \( V_G \). Thus \( |B(V_G)| = \mathcal{O}(|V_G|^{d+1}) \). From Lemma 6 of [3] that uses Carathéodory’s baricenter theorem in turn, we have that \( \Lambda'(v) \) can be represented as a union of all cells that contain a point from \( B(V_G) \) which is in \( \Lambda'(v) \). Each cell is a polyhedron that can be represented by \( \mathcal{O}(V) \) extremal points, or equivalently, by Theorem 3 of [3], by \( \mathcal{O}(V_G) \cdot 2^d \) inequalities. It follows that \( \Lambda'(v) \) can also be represented as a union of \( \mathcal{O}(|V_G|^{d+1}) \) polyhedra. Hence \( \bigcap_{v \in V(G)} \Lambda'(v) \) can also be represented as a union of \( \mathcal{O}(|V_G|^{d+1}) \) polyhedra. Thus we have exponentially many linear programs corresponding to \( \bigcap_{v \in V(G)} \Lambda'(v) \), since the weights on the edges are given in binary. In our case, we have \( d = 2 \). Further, from Lemma 29 for bi-weighted arena, we have that \( \Lambda'(v) \) can be expressed as a set of linear programs in the following manner. For each linear program, we have two variables \( x, y \) that correspond to \( \Lambda'(v) \), and hence \( \bigwedge \Psi_S^v \) in the formula \( \rho_S^v(c) \) corresponds to the set \( \bigcap_{v \in V(G)} \Lambda'(v) \). We show in Lemma 29 that this set can be represented as a union of exponentially many systems of strict and non-strict inequalities.

For each SCC \( S \) in the mean-payoff game \( G^\text{ext} \), we see that the value satisfying the formula \( \rho_S^v \) can be expressed as a set of linear programs in the following manner. For each linear program, we have two variables \( x \) and \( y \) that represent the mean-payoff values of Player 0 and Player 1 respectively, corresponding to the formula \( \Phi_S(x,y) \), and equivalently for plays in \( F_{\text{min}}(\text{CH}(S)) \), and a third variable \( c \) represents the \( c \) in the formula \( \rho_S^v(c) \). The formula \( \rho_S^v(c) \) can be expressed as a disjunction of a set of linear equations and inequalities, i.e., \( \bigvee_{\text{Cst}_i} \), where each \( \text{Cst}_i \) is a conjunction of linear inequalities. We use the variables \( (c,d) \) in the linear inequalities which represent the set \( \bigcap_{v \in V(G)} \Lambda'(v) \), and the variables \( (x,y) \) in the linear inequalities representing the set \( F_{\text{min}}(\text{CH}(S)) \) as mentioned above. We also include the inequation \( x > c \). Further, the variable \( d \) should assume the value of \( y \) that corresponds to the mean-payoff of Player 1. In the LP formulation, each strict inequation is replaced by a non-strict inequation, and supremum in the objective function is replaced with maximizing the objective. The set of linear programs we solve is \( \text{maximise } c \) under the linear constraints \( \text{Cst}_i \) for each \( \text{Cst}_i \in \bigvee_{\text{Cst}_i} \).

We solve the above sets of LPs for each SCC \( S \) present in \( G \) to get a set of values satisfying the formula \( \rho_S^v \). We choose the maximum of this set of values which is the \( \text{ASV}^v \). The algorithm runs in \( \text{EXPTime} \) as there can be exponentially many SCCs.

On the other hand, as illustrated in Example 27 we note that if we fix a value of \( c \) in every linear program corresponding to an SCC \( S \), and replace the objective function \( \text{maximise } c \) with the function \( \text{maximise } c \), then we can find the supremum over \( c \) which allows \( \text{ASV}^v > c \) in \( S \). Again, taking the maximum over all SCCs reachable from \( (v, [v]) \), we get the largest \( c \) possible so that we have \( \text{ASV}^v > c \). Thus we get the following corollary.
Corollary 30. For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), and for all \( c \in Q \), we can compute in EXPTime the maximum possible value of \( \epsilon \) such that ASV\( ^\epsilon (v) > c \).

6 Additional Properties of ASV\( ^\epsilon \)

In this section, we first show that the ASV\( ^\epsilon \) is achievable, i.e., there exists a Player 0 strategy that achieves ASV\( ^\epsilon \). Then we study the memory requirement in strategies of Player 0 for achieving the ASV\( ^\epsilon \), as well as the memory requirement by Player 1 for playing the the \( \epsilon \)-best-responses.

Achievability of the ASV\( ^\epsilon \) We formally define achievability as follows. Given \( \epsilon > 0 \), we say that ASV\( ^\epsilon (v) = c \) is achievable from a vertex \( v \), if there exists a strategy \( \sigma_0 \) for Player 0 such that \( \forall \sigma_1 \in BR_1^\epsilon (\sigma_0) : MP_0^\epsilon (\text{Out}_v(\sigma_0, \sigma_1)) \geq c \). We note that this result is in contrast to the case for ASV as shown in [8].

Theorem 31. For all mean-payoff games \( G \), for all vertices \( v \) in \( G \), and for all \( \epsilon > 0 \), we have that the ASV\( ^\epsilon (v) \) is achievable.

The rest of this section is devoted to proving Theorem 31. We start by defining the notion of a witness for ASV\( ^\epsilon (\sigma_0)(v) \) for a strategy \( \sigma_0 \) of Player 0.

Witness for ASV\( ^\epsilon (\sigma_0)(v) \) Given a mean-payoff game \( G \), a vertex \( v \) in \( G \), and an \( \epsilon > 0 \), we say that a play \( \pi \) is a witness for ASV\( ^\epsilon (\sigma_0)(v) > c \) for a strategy \( \sigma_0 \) of Player 0 if (i) \( \pi \in \text{Out}_v(\sigma_0) \), and (ii) \( \pi \) is a witness for ASV\( ^\epsilon (v) > c \) when Player 0 uses strategy \( \sigma_0 \) where the strategy \( \sigma_0 \) is defined as follows:

1. \( \sigma_0 \) follows \( \pi \) if Player 1 does not deviate from \( \pi \).
2. If Player 1 deviates \( \pi \), then for each vertex \( v \in \pi \), we have that \( \sigma_0 \) consists of a memoryless strategy that establishes \( v \not\approx 1 \gg MP_0 \leq c \land MP_1 > d - \epsilon \), where \( d = MP_1(\pi) \). The existence of such a memoryless strategy of Player 0 has been established in Section 3.

Assume that the ASV\( ^\epsilon (v) \) cannot be achieved by a finite memory strategy. We show that for such cases, it can indeed be achieved by an infinite memory strategy.

Let ASV\( ^\epsilon (v) = c \). For every \( c' < c \), from Theorem 12, there exists a finite memory strategy \( \sigma_0 \) such that ASV\( ^\epsilon (\sigma_0)(v) > c' \), and recall from Theorem 11 that there exists a corresponding regular witness. First we state the following proposition.

Proposition 32. There exists a sequence of increasing real numbers, \( c_1 < c_2 < c_3 < \ldots < c \), such that the sequence converges to \( c \), and a set of finite memory strategies \( \sigma_0^1, \sigma_0^2, \sigma_0^3, \ldots \) of Player 0 such that for each \( c_i \), we have ASV\( ^\epsilon (\sigma_0^i)(v) > c_i \), and there exists a play \( \pi^i \) that is a witness for ASV\( ^\epsilon (\sigma_0^i)(v) > c_i \), where \( \pi^i = \pi_1(l_1^{\alpha k_i} \cdot \pi_2 \cdot l_2^{\beta k_i} \cdot \pi_3)^\omega \), and \( \pi_1, \pi_2, \pi_3 \) are simple finite plays, and \( l_1, l_2 \) are simple cycles in the arena of the game \( G \).

Proof. Consider the play \( \pi^i = \pi_1(l_1^{\alpha k_i} \cdot \pi_2 \cdot l_2^{\beta k_i} \cdot \pi_3)^\omega \) which is a witness for ASV\( ^\epsilon (\sigma_0^i)(v) > c_i \) for the strategy \( \sigma_0^i \). Let MP\( _0(\pi^i) = c_i' > c_i \).

We have that MP\( _0(\pi^i) \) increases proportionally with \( i \) as \( \alpha \cdot k_i \) and \( \beta \cdot k_i \) increase with increasing \( k_i \). This follows because we disregard the cases where ASV\( ^\epsilon (v) \) is achievable with some finite memory strategy of Player 0, i.e., we only consider the case where ASV\( ^\epsilon (v) \) is not achievable by a finite memory strategy of Player 0.

Note that there are finitely many possible simple plays and simple cycles. Thus w.l.o.g. we can assume that in the sequence \( (\pi^i)_{i \in N^+} \), the finite plays \( \pi_1, \pi_2, \pi_3 \), and the simple
cycles $l_i, l_{i+1}$ are the same for different values of $i$. Thus, $\MP_0(\pi_{2i}(\alpha_k, \beta_k), \pi_{3i+1}(\alpha_k, \beta_k)) = c_i > c_{i+1}$, $\MP_0(\pi_{2i+1}(\alpha_k, \beta_k), \pi_{3i+2}(\alpha_k, \beta_k)) = c_{i+1} > c_{i+2}$, and so on, and the only difference in the strategies $\sigma_i^0$ as $i$ changes is the value of $k_i$, i.e., we increase the value of $\alpha \cdot k_i$ and $\beta \cdot k_i$ with increasing $k_i$ such that the effect of $\pi_2$ and $\pi_3$ on the mean-payoff is minimised. Thus, at the limit, as $i \to \infty$, the sequence $(c_i)_{i \in \mathbb{N}^+}$ converges to $\alpha \cdot \MP_0(l_1) + \beta \cdot \MP_0(l_2) = c$. ▲

These witnesses or plays in the sequence are regular, and they differ from each other only in the value of $k_i$ that they use.

To show that $\lim_{i \to \infty} \ASV^\pi(\sigma_i^0)(v) = c$, we construct a play $\pi^*$ that starts from $v$, follows $\pi^1$ until the mean-payoff of Player 0 over the prefix becomes greater than $c_1$. Then for $i \in \{2, 3, \ldots\}$, starting from first($l_i$), it follows $\pi^i$, excluding the initial simple finite play $\pi_1$, until the mean-payoff of the prefix of $\pi^i$ becomes greater than $c_i$. Then the play $\pi^*$ follows the prefix of the play $\pi^i$, excluding the initial finite play $\pi_1$, and so on. Clearly, we have that $\MP_1(\pi^*) = c$. We let $\MP_0(\pi^*) = d = \alpha \cdot \MP_1(l_1) + \beta \cdot \MP_1(l_2)$.

For the sequence of plays $(\pi^i)_{i \in \mathbb{N}^+}$ which are witnesses for $(\ASV^\pi(\sigma_i^0)(v) > c_i)_{i \in \mathbb{N}^+}$ for the strategies $(\sigma_i^0)_{i \in \mathbb{N}^+}$, we let $\MP_0(\pi_i) = d_i$. We state the following proposition.

**Proposition 33.** The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonic, and it converges to $d$ in the limit.

**Proof.** Recall that $\MP_0(\pi^i)$ increases monotonically with increasing $i$. Since the effect of the finite simple plays $\pi_2$ and $\pi_3$ decreases with increasing $\alpha \cdot k_i$ and $\beta \cdot k_i$, the mean-payoff on the second dimension also changes monotonically. If $\alpha \cdot \MP_1(l_1) + \beta \cdot \MP_1(l_2) \geq \frac{w_1(\pi_2) + w_1(\pi_3)}{|\pi_2| + |\pi_3|}$, then the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically non-decreasing. Otherwise, the sequence is monotonically decreasing.

The fact that this sequence converges to $d$ in the limit can be seen from the construction of $\pi^*$ as described above. ▲

The above two propositions establish the existence of an infinite sequence of regular witnesses $\ASV^\pi(\sigma_i^0)(v) > c_i$ for a sequence of increasing numbers $c_1 < c_2 < \ldots < c$, such that the mean-payoffs of the witnesses are monotonic and at the limit, the mean-payoffs of the witnesses converge to $c$ and $d$ for Player 0 and Player 1 respectively. These observations show the existence of a witness $\pi^*$ which gives Player 0 a mean-payoff value at least $c$ and Player 1 a mean-payoff value equal to $d$. Assuming that Player 0 has a corresponding strategy $\sigma_0$, we show that Player 1 does not have an $\epsilon$-best response to $\sigma_0$ that gives Player 0 a payoff less than $c$. Now, we have the ingredients to prove Theorem 31.

**Proof of Theorem 31.** We consider a sequence of increasing numbers $c_1 < c_2 < c_3 < \ldots < c$ such that for every $i \in \mathbb{N}^+$, by Theorem 12, we consider a finite memory strategy $\sigma_i^0$ of Player 0 that ensures $\ASV^\pi(\sigma_i^0)(v) > c_i$.

If the $\ASV^\pi$ is not achievable, then there exists a strategy of Player 1 to enforce some play $\pi'$ such that $\MP_0(\pi') = c' < c$ and $\MP_1(\pi') = d' > d - \epsilon$. Now, we use the monotonicity of the sequence $(d_i)_{i \in \mathbb{N}^+}$ established in Proposition 33 to show a contradiction. Since the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonic, there can be two cases:

1. The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically non-decreasing.
2. The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically decreasing.

We start with the first case where the sequence $(d_i)_{i \in \mathbb{N}^+}$ is non-decreasing. Assume for contradiction that $\ASV^\pi(\pi')$ is not achievable, i.e., Player 1 deviates from $\pi^*$ to enforce the play $\pi'$ such that $\MP_0(\pi') = c' < c$ and $\MP_1(\pi') = d'$. The proof of Theorem 31 follows from this assumption.
Since \((d_i)_{i \in \mathbb{N}^+}\) is non-decreasing, and thus \(d \geq d_i\) for all \(i \in \mathbb{N}^+\), and since Player 1 can let his payoff to be reduced by an amount that is less than \(\epsilon\) in order to reduce the payoff of Player 0, for all \(i \in \mathbb{N}^+\) we have that \(d' > d_i - \epsilon\). We know that the sequence \((c_i)_{i \in \mathbb{N}^+}\) is increasing. Thus, there exists a \(j \in \mathbb{N}\) such that \(c' < c_j\). Note that if \(\pi' = \pi^r\) for some index \(r\), we consider some \(j\) which is also greater than \(r\).

Now, consider the strategy \(\sigma'_0\) of Player 0 which follows the play \(\pi_j\). We know that \(\text{MP}_0(\pi_j) > c_j\) and \(\text{MP}_i(\pi_j) = d_j\). We also know from Lemma 10 that the play \(\pi_j\) does not cross a \((c_j, d_j)\)-bad vertex. Since the construction of \(\pi^r\), and by Proposition 32, the set of vertices appearing in the play \(\pi^r\) is the same as the set of vertices appearing in the play \(\pi_j\), for every vertex \(v\) in \(\pi^r\), we have that Player 1 does not have a strategy such that \(v \in \ll 1 \gg \text{MP}_0 \leq c_j \land \text{MP}_1 > d_j - \epsilon\). Since \(c' < c_j\) and \(d' > d_j - \epsilon\), it also follows that Player 1 does not have a strategy such that \(v \in \ll 1 \gg \text{MP}_0 \leq c' \land \text{MP}_1 > d'\). Stated otherwise, from the determinacy of multi-player mean-payoff games, we have that Player 0 has a strategy to ensure \(v \not\in \ll 1 \gg \text{MP}_0 \leq c' \land \text{MP}_1 > d'\) for every vertex \(v\) appearing in \(\pi^r\). In fact, Player 0 can ensure \(v \not\in \ll 1 \gg \text{MP}_0 \leq c' \land \text{MP}_1 > d'\) by choosing the strategy \(\sigma_{j'}\) for some \(j' \geq j\). Since \(\text{ASV}^r(\sigma_0)(v) = \sup_{\sigma_0 \in \Sigma_0} \text{ASV}^r(\sigma_0)(v)\), and the sequence \((c_i)_{i \in \mathbb{N}^+}\) is increasing, and we have that \(\text{ASV}^r(\sigma_i)(v) > c_i\) for all \(i \in \mathbb{N}^+\), it follows that the existence of \(\pi'\) is a contradiction.

Now, we consider the case where the sequence \((d_i)_{i \in \mathbb{N}^+}\) is monotonically decreasing. Again, assume for contradiction that \(\text{ASV}^r(\sigma_0)(v)\) is not achievable, i.e., Player 1 deviates from \(\sigma'_0\) to enforce the play \(\pi'\) such that \(\text{MP}_0(\pi') = c' < c\) and \(\text{MP}_i(\pi') = d'\). Since the sequence \((d_i)_{i \in \mathbb{N}^+}\) is monotonically decreasing, we know that there must exist a \(j \in \mathbb{N}\) such that \(i) d' > d_j - \epsilon\), and \(i) c_j > c',\) which follows since \((c_i)_{i \in \mathbb{N}^+}\) is a strictly increasing sequence. Thus for every vertex \(v\) in \(\pi^r\), Player 1 does not have a strategy such that there exists a play \(\pi\) in \(\text{Out}_i(\sigma_j)\), and \(v \in \ll 1 \gg \text{MP}_0 \leq c_j \land \text{MP}_1 > d_j - \epsilon\).

Finally, using the fact that \(c' < c_j\) and \(d' > d_j - \epsilon\), the contradiction follows exactly as above where the sequence \((d_i)_{i \in \mathbb{N}^+}\) is monotonically non-decreasing\(\footnote{We note that in the definition of \(\epsilon\)-best response in Section 2 if we allow \(\text{MP}_0(\text{Out}_v(\sigma_0, \sigma_1)) \geq \text{MP}_0(\text{Out}_v(\sigma_0, \sigma'_1)) - \epsilon\), then the existence of a \(j\) for the case when the sequence \((d_i)_{i \in \mathbb{N}^+}\) is monotonically decreasing, does not necessarily hold true.}.

**Figure 12** Finite memory strategy of Player 0 may not achieve \(\text{ASV}^r(v_0)\). Also, no finite memory \(\epsilon\)-best response exists for Player 1 for the strategy \(\sigma_0\) of Player 0.

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**Memory requirements of the players’ strategies** First we show that there exists a mean-payoff game \(\mathcal{G}\) in which Player 0 needs an infinite memory strategy to achieve the \(\text{ASV}^r\).

**Theorem 34.** There exist a mean-payoff game \(\mathcal{G}\), a vertex \(v\) in \(\mathcal{G}\), and an \(\epsilon > 0\) such that Player 0 needs an infinite memory strategy to achieve the \(\text{ASV}^r(v)\).
Proof. Consider the example in Figure 12. We show that in this example the $\text{ASV}^\epsilon(v_0) = 1$, and that this value can only be achieved using an infinite memory strategy. Assume a strategy $\sigma_0$ for Player 0 such that the game is played in rounds. In round $k$: (i) if Player 1 plays $v_0 \rightarrow v_0$ repeatedly at least $k$ times before playing $v_0 \rightarrow v_1$, then from $v_1$, play $v_1 \rightarrow v_1$ repeatedly $k$ times and then play $v_1 \rightarrow v_0$ and move to round $k+1$; (ii) else, if Player 1 plays $v_0 \rightarrow v_0$ less than $k$ times before playing $v_0 \rightarrow v_1$, then from $v_1$, play $v_1 \rightarrow v_0$. Note that $\sigma_0$ is an infinite memory strategy.

The best-response for Player 1 to strategy $\sigma_0$ would be to choose $k$ sequentially as $k = 1, 2, 3, \ldots$, to get a play $\pi = ((v_0)^k(v_1)^k)_{\epsilon \in \mathbb{N}}$. We have that $\text{MP}_0(\pi) = 1 + \epsilon$ and $\text{MP}_0(\sigma_0) = 1$. Player 1 can only sacrifice an amount that is less than $\epsilon$ to minimize the mean-payoff of Player 0, and thus he would not play $v_0 \rightarrow v_2$.

We now show that $\text{ASV}^\epsilon(\sigma_0)(v_0) = \text{ASV}^\epsilon(v_0)$, and that no finite memory strategy of Player 0 can achieve an $\text{ASV}^\epsilon(v_0)$ of 1. First we observe in Figure 12 that a strategy $\sigma_1$ of Player 1 that prescribes playing the edge $v_0 \rightarrow v_2$ some time yields a mean-payoff of 1 for Player 1, and hence we conclude that $\sigma_1 \notin \text{BR}_1^1(\sigma_0)$. Player 1 cannot play any other strategy without increasing the mean-payoff of Player 0 and/or decreasing his own payoff. We can see that Player 1 does not have a finite memory best-response strategy. Thus, the $\text{ASV}^\epsilon(\sigma_0)(v_0) = 1$.

We now prove the claim $\text{ASV}^\epsilon(\sigma_0)(v_0) = \text{ASV}^\epsilon(v_0)$. For every strategy $\sigma_1$ of Player 1 such that $\sigma_1 \in \text{BR}_1^1(\sigma_0)$, we note that the higher the payoff Player 1 has, the lower is the payoff for Player 0. For every other strategy $\sigma_0'$ of Player 0, if best-response of Player 1 to $\sigma_0'$ gives a mean-payoff less than $1 + \epsilon$, then Player 1 will switch to $v_2$, thus giving Player 0 a payoff of 0. If best-response of Player 1 to $\sigma_0'$ gives him a mean-payoff greater than $1 + \epsilon$, then Player 0 will have a lower $\text{ASV}^\epsilon(\sigma_0')$.

Now we show that a finite memory strategy of Player 0 cannot achieve an $\text{ASV}^\epsilon(v_0)$ of 1. In order to achieve $\text{ASV}^\epsilon(v_0)$ of 1, the edge $v_0 \rightarrow v_1$ and the edge $v_1 \rightarrow v_0$ need to be taken infinitely many times. If this does not happen, we note that either Player 1 would play $v_0 \rightarrow v_1$ and Player 0 would loop forever on $v_1$ or Player 1 would play $v_0 \rightarrow v_2$ and Player 0 would loop forever on $v_2$, both of which yield a mean-payoff of 0 to Player 0. Additionally, we note that the two edges $v_0 \rightarrow v_1$ and $v_1 \rightarrow v_0$ have edge weights $(0, 0)$, i.e., it gives a payoff of 0 to both players. Therefore, we need a strategy which suppresses the effect of the edges on the mean-payoff of a play. This can be achieved by choosing these edges less and less frequently. We now show that this cannot be done with any finite memory strategy, and we indeed need an infinite memory strategy as described by the Player 0 strategy $\sigma_0$. Consider a finite memory strategy of Player 0 and an infinite memory $\epsilon$-best-response for Player 1 for which no finite memory strategy exists. Given that he owns only one vertex which is $v_0$, such an infinite memory strategy can only lead to looping over $v_0$ more and more, thus giving him a payoff which is eventually 0. Thus consider a finite memory response of Player 1 to the finite memory strategy of Player 0. Note that Player 0 would choose a finite memory strategy such that the best-response of Player 1 gives him a value of at least $1 + \epsilon$. Also since both players have finite memory strategies, the resultant outcome is a regular play over vertices $v_0$ and $v_1$. In such a regular play, the effect of the edge from $v_0$ to $v_1$ and the edge from $v_1$ to $v_0$ is non-negligible, and hence if the payoff of Player 1 is at least $1 + \epsilon$, the payoff of Player 0 will be less than 1. Thus no finite memory strategy can achieve an $\text{ASV}^\epsilon$ that is equal to 1.

We also show that exist mean-payoff games in which a finite memory (but not memoryless) strategy for Player 0 can achieve the $\text{ASV}^\epsilon$. □
Theorem 35. There exists a mean-payoff game $G$, a vertex $v$ in $G$, and an $\epsilon > 0$ such that a finite memory strategy of Player 0 that can not be represented by any memoryless strategy suffices to achieve $\text{ASV}^\epsilon(v)$.

Proof. To show the existence of mean-payoff games in which player 0 can achieve the adversarial value with finite memory (but not memoryless) strategies, we consider the example in Figure 13. We show that $\text{ASV}^\epsilon(v_0) = 1 - \epsilon$. Assume a strategy $\sigma_0$ for Player 0 defined as: repeat forever, from $v_1$ play $j$ times $v_1 \rightarrow v_1$, and then repeat playing $v_1 \rightarrow v_0$ for $k$ times, with $j$ and $k$ chosen such that mean-payoff for Player 0 is equal to $1 - \epsilon$. For every rational $\epsilon$, such a $k$ always exists. In this example, we have that $k = \frac{1 - \epsilon}{2\epsilon} j$. The $\epsilon$-best-response of Player 1 to $\sigma_0$ is to always play $v_0 \rightarrow v_1$ as by playing this edge forever, Player 1 gets a mean-payoff equal to $1 + \epsilon$, whereas if Player 1 plays $v_0 \rightarrow v_2$, then Player 1 receives a payoff of 1. Since $1 \not\geq (1 + \epsilon - \epsilon)$, a strategy of Player 1 that chooses $v_0 \rightarrow v_2$ is not an $\epsilon$-best-response for Player 1, thus forcing Player 1 to play $v_0 \rightarrow v_1$. Thus $\text{ASV}^\epsilon(v_0)$ is achieved with a finite memory strategy of size $k$ for Player 0. Note that this size $k$ is a function of $\epsilon$.

Further, we show that there exist games such that for a strategy $\sigma_0$ of Player 0, and an $\epsilon > 0$, there does not exist any finite memory best-response of Player 1 to the strategy $\sigma_0$.

Theorem 36. There exist a mean-payoff game $G$, an $\epsilon > 0$, and a Player 0 strategy $\sigma_0$ in $G$ such that every Player 1 strategy $\sigma_1 \in \text{BR}_1^\epsilon(\sigma_0)$ is an infinite memory strategy.

Proof. Consider the example in Figure 12 and the strategy $\sigma_0$ of Player 0 where the game is played in rounds as described in Theorem 34. In round $k$: (i) if Player 1 plays $v_0 \rightarrow v_0$ repeatedly at least $k$ times before playing $v_0 \rightarrow v_1$, then from $v_1$, play $v_1 \rightarrow v_1$ repeatedly $k$ times and then play $v_1 \rightarrow v_0$ and move to round $k + 1$; (ii) else, if Player 1 plays $v_0 \rightarrow v_0$ less than $k$ times before playing $v_0 \rightarrow v_1$, then from $v_1$, play $v_1 \rightarrow v_0$. We can see that for all finite memory strategies, Player 1 gets at most 1, and thus no finite memory strategy is an $\epsilon$-best-response to $\sigma_0$. For each round $k$, Player 1’s $\epsilon$-best-response is to play $v_0 \rightarrow v_0$ repeatedly at least $k$ times before playing $v_0 \rightarrow v_1$, which is an infinite memory strategy for Player 1.

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A Additional details in the proof of Theorem 11

Below we compute the expressions for $k$ and $\tau$ for case 1 in the proof of Theorem 11. We know that for play $\pi = \pi_1, \rho_1, \rho_2, \ldots$, where $\rho_i = i^{[a_i]}, \pi_2, \pi_3$, constants $\alpha, \beta \in \mathbb{R}^+$ are chosen such that:

$$\alpha \cdot \text{MP}_0(l_1) + \beta \cdot \text{MP}_0(l_2) = c'' \text{ where } c'' > c$$
$$\alpha \cdot \text{MP}_1(l_1) + \beta \cdot \text{MP}_1(l_2) = d$$
$$\alpha + \beta = 1$$

We assume here that $\text{MP}_0(l_1) > \text{MP}_0(l_2)$ and $\text{MP}_1(l_1) < \text{MP}_1(l_2)$. This implies that one simple cycle, $l_1$, increases Player 0’s mean-payoff while the other simple cycle, $l_2$, increases Player 1’s mean-payoff. We build a play $\pi' = \pi_1 \cdot (l_1^{[a]} \cdot \pi_2 \cdot l_2^{[b]} \cdot \pi_3)\omega$ where we choose constant $k \in \mathbb{N}$ and constant $\tau > 0$ such that $\text{MP}_0(\pi') = c'$ and $\text{MP}_1(\pi') = d$ for some $c' > c$.

We try to express the conditions for $k$ and $\tau$ below:

$$\text{MP}_0(\pi') = \frac{k \cdot \alpha \cdot w_0(l_1) + (k + \tau) \cdot \beta \cdot w_0(l_2) + w_0(\pi_2) + w_0(\pi_3)}{k \cdot \alpha \cdot \lvert l_1 \rvert + (k + \tau) \cdot \beta \cdot \lvert l_2 \rvert + \lvert \pi_2 \rvert + \lvert \pi_3 \rvert}$$

$$\text{MP}_1(\pi') = \frac{k \cdot (\alpha \cdot w_1(l_1) + \beta \cdot w_1(l_2)) + \tau \cdot \beta \cdot w_1(l_2) + w_1(\pi_2) + w_1(\pi_3)}{k \cdot (\alpha \cdot \lvert l_1 \rvert + \beta \cdot \lvert l_2 \rvert) + \tau \cdot \beta + \lvert \pi_2 \rvert + \lvert \pi_3 \rvert}$$

Let $\lvert \pi_2 \rvert + \lvert \pi_3 \rvert = \nu$, $\alpha \cdot w_0(l_1) + \beta \cdot w_0(l_2) = x_0$, $\alpha \cdot w_1(l_1) + \beta \cdot w_1(l_2) = x_1$, $\alpha \cdot \lvert l_1 \rvert + \beta \cdot \lvert l_2 \rvert = y$, $w_0(\pi_2) + w_0(\pi_3) = z_0$ and $w_1(\pi_2) + w_1(\pi_3) = z_1$. We simplify the inequalities above to get:

$$\text{MP}_0(\pi') = \frac{k \cdot x_0 + \tau \cdot \beta \cdot w_0(l_2) + z_0}{k \cdot y + \tau \cdot \beta + v}$$

$$\text{MP}_1(\pi') = \frac{k \cdot x_1 + \tau \cdot \beta \cdot w_1(l_2) + z_1}{k \cdot y + \tau \cdot \beta + v}$$

We know that $\text{MP}_0(\pi') = c'$ and $\text{MP}_1(\pi') = d$. Thus,

$$\frac{k \cdot x_0 + \tau \cdot \beta \cdot w_0(l_2) + z_0}{k \cdot y + \tau \cdot \beta + v} = c'$$
$$\frac{k \cdot x_1 + \tau \cdot \beta \cdot w_1(l_2) + z_1}{k \cdot y + \tau \cdot \beta + v} = d$$

Simplifying the above inequalities we get:

$$k \cdot (x_0 - c' \cdot y) = c' \cdot v + \tau \cdot \beta \cdot (c' - w_0(l_2)) - z_0$$
$$\tau \cdot \beta \cdot (w_1(l_2) - d) = v \cdot d + k \cdot (d \cdot y - x_1) - z_1$$
Finally, after substitution of \( \tau \) in the first inequality expression and further simplification of both expressions, we finally get:

\[
\begin{align*}
k &= \frac{(c' \cdot v - z_0)(w_1(l_2) - d) + (c' - w_0(l_2))(v \cdot d - z_1)}{(x_0 - c' \cdot y)(w_1(l_2) - d) - (d \cdot y - x_1)} \\
\tau &= \frac{v \cdot d - z_1}{\beta(w_1(l_2) - d) + d \cdot y - x_1 \cdot k} \\
\beta &= \frac{d \cdot y - x_1}{w_1(l_2) - d} \
\end{align*}
\]

The above two inequalities specify the range from which we can choose a suitable \( k \) and \( \tau \), such that the requirements \( MP_0(\pi') = c' \) and \( MP_1(\pi') = d \) are met. We note that \( k \) and \( \tau \) are polynomial in size of the game and the weights on the edges, i.e., \( O(k) = |W|^2 \cdot |V|^3 \) and \( O(\tau) = |W|^3 \cdot |V|^5 \).