ASYMPTOTICS OF CERTAIN \( q \)-SERIES

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Abstract. In this work we study complete asymptotic expansions for the \( q \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^a} q^n \) and \( \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^b} q^n \) in the scale function \( (\log q)^n \) as \( q \to 1^- \), where \( a > 0 \), \( q \in (0, 1) \), \( b, \alpha \in \mathbb{C} \) and \( \sigma_\alpha(n) \) is the divisor function \( \sigma_\alpha(n) = \sum_{d|n} d^\alpha \).

1. Preliminaries

In this work we study complete asymptotic expansions for the \( q \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^a} q^n \) and \( \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^b} q^n \) in the scale function \( (\log q)^n \) as \( q \to 1^- \), where \( a > 0 \), \( q \in (0, 1) \), \( b, \alpha \in \mathbb{C} \) and \( \sigma_\alpha(n) \) is the divisor function \( \sigma_\alpha(n) = \sum_{d|n} d^\alpha \). Unlike methods used \[3, 4\], our method does not apply Fourier transform or the modular properties, it cannot give \( \sum_{n=1}^{\infty} \frac{1}{n^a} q^n \) a complete asymptotic expansion in exponential scales when \( a = 2 \) and \( b \) is an even integer. However, this shortcoming can be overcome by applying the functional equations for the corresponding zeta functions which are equivalent to the symmetry \( x \to 1/x \).

The Euler gamma function is defined by

\[
\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx, \quad \Re(z) > 0,
\]

and its analytic continuation is given by

\[
\Gamma(z) = \int_1^\infty e^{-x}x^{z-1}dx + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z(n+z)}, \quad z \in \mathbb{C}\setminus\mathbb{N}_0.
\]

Let \( a, b \in \mathbb{R} \) and \( a < b \), it is known that \(3, 5, 6\)

\[
\Gamma(\sigma + it) = O \left( e^{-\pi|t|/2} |t|^{|\sigma|-1/2} \right), \quad t \in \mathbb{R}
\]
as \( t \to \pm \infty \), uniformly with respect to \( \sigma \in [a, b] \). The digamma function is defined by

\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad z \in \mathbb{C}
\]

and the Euler’s constant is

\[
\gamma = -\psi(1) \approx 0.577216.
\]

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The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,$$

then its analytic continuation, which is also denoted as $\zeta(s)$, is an meromorphic function that has a simple pole at 1 with residue 1. The meromorphic function $\zeta(s)$ satisfies the functional equation [1, 2, 5, 6]

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{\pi s}{2} \right) \zeta(1-s).$$

For $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \sigma \leq \beta$, it is known that [6]

$$\zeta(\sigma + it) = O \left( |t|^{|\alpha|+1/2} \right)$$

as $t \to \pm \infty$, uniformly with respect to $\sigma$. The Stieltjes constants $\gamma_n$ are the coefficients in the Laurent expansion,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n,$$

where $\gamma_0 = \gamma$ and $\gamma_1 \approx -0.0728158$. Moreover, the Glaisher’s constant $A \approx 1.28243$ is defined as

$$\log A = \frac{1}{12} - \zeta'(-1).$$

The Bernoulli numbers $B_n$ are defined by

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi.$$

Then

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_{2n-1} = 0, \quad n \in \mathbb{N}.$$ 

By (1.7) we get

$$\zeta(-2n) = 0, \quad \zeta(1-2n) = -\frac{B_{2n}}{2n}, \quad n \in \mathbb{N}.$$

The function $\sigma_{\alpha}(n)$ for $\alpha \in \mathbb{C}$ is defined as the sum of the $\alpha$-th powers of the positive divisors of $n$, [2, 5]

$$\sigma_{\alpha}(n) = \sum_{d|n} d^\alpha,$$

where $d|n$ stands for "$d$ divides $n$". We also use the notations $d(n) = \sigma_0$ and $\sigma(n) = \sigma_1(n)$. It is known that [2, 5]

$$\sum_{n=1}^{\infty} \frac{\sigma_{\alpha}(n)}{n^s} = \zeta(s) \zeta(s-\alpha), \quad \Re(s) > \max \{ 1, \Re(\alpha) + 1 \}$$

and

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s) \quad \Re(s) > 1.$$
2. Main Results

**Theorem 1.** Given a positive integer $k$, let $a_j \in \mathbb{N}$, $b_j \in \mathbb{C}$ for all $j$ satisfying $1 \leq j \leq k$.

If

$$\prod_{j=1}^{k} \zeta(a_j s + b_j) = \sum_{n=1}^{\infty} \frac{f_k(n)}{n^s}, \quad \Re(s) > \max_{1 \leq j \leq k} \left\{ \frac{1 - \Re(b_j)}{a_j} \right\},$$

where $\zeta(s)$ is the Riemann zeta function, then for all $x, a > 0$, $b \in \mathbb{C}$ and $c > 0$ satisfying $c > \max_{1 \leq j \leq k} \left\{ \frac{1 - \Re(b_j + b_a a_j)}{aa_j} \right\}$ we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \prod_{j=1}^{k} \zeta(a_j s + b_j + b) \frac{ds}{x^s} = \sum_{n=1}^{\infty} \frac{f_k(n)}{n^b} e^{-n^s x}.$$ 

Furthermore,

$$\sum_{n=1}^{\infty} \frac{f_k(n)}{n^b} e^{-n^s x} = \sum_j \text{Residue} \left\{ g(s), s = \frac{1 - b - b_j}{aa_j} \right\} + \sum_n \text{Residue} \left\{ g(s), s = -n \right\}$$

as $x \to 0^+$, where the first sum is over all the distinct pairs $a_j b_j, j = 1, \ldots, k$ while the last sum is over all nonnegative integers $n$ such that

$$-n \neq \frac{1 - b - b_j}{aa_j}, \quad j = 1, \ldots, k.$$

**Proof.** For $\Re(s) > \max_{1 \leq j \leq k} \left\{ \frac{1 - \Re(b_j + b_a a_j)}{aa_j} \right\}$, since each factor of $\prod_{j=1}^{k} \zeta(a_j s + b_j + b)$ is an absolute convergent Dirichlet series, then the product itself is also an absolute convergent Dirichlet series. Let $s_0$ be any complex number satisfying

$$\sigma_0 = \Re(s_0) > \max_{1 \leq j \leq k} \left\{ \frac{1 - \Re(b_j + b_a a_j)}{aa_j} \right\},$$

then by the theory of Dirichlet series we know the partial sums \( \sum_{n \leq x} \frac{f_k(n)}{n^a s_0 + \Re(b)} \) are absolutely and uniformly bounded for all $x > 1$. Let $N$ be a large positive integer and

$$M = \sum_{n=1}^{\infty} \frac{|f_k(n)|}{n^a s_0 + \Re(b)}$$

in Lemma 2 of section 11.6 in [2] to get $|f_k(N)| \leq 4MN^\sigma_0$. Hence,

$$\sum_{n=1}^{\infty} \frac{|f_k(N)|}{n^a s_0 + \Re(b)} e^{-n^a x} < \infty, \quad a, x > 0, \; b \in \mathbb{C}.$$ 

By the inverse Mellin transform of $\Gamma(s)$ we get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{ds}{x^s} = e^{-x}$$

for all $x, c > 0$.

Let

$$g(s) = \Gamma(s) \prod_{j=1}^{k} \zeta(a_j s + b_j + b) x^{-s},$$
then for any positive \( c \) satisfying \( c > \max_{1 \leq j \leq k} \left\{ \frac{1 - \Re(b_j + b)}{a_{a_j}} \right\} \), by (2.6) we get

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{x^s} \left( \sum_{n=1}^{\infty} \frac{f_k(n)}{n^{s+1}} \right) \frac{ds}{x^s} \\
= \sum_{n=1}^{\infty} \frac{f_k(n)}{n^b} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{ds}{(n^a x)^s} = \sum_{n=1}^{\infty} \frac{f_k(n)}{n^b} e^{-n^a x},
\]

where we have applied (2.5) and the Fubini’s theorem to exchange the order of summation and integration.

Since \( \zeta(s) \) has a simple pole at \( s = 1 \) and \( \Gamma(s) \) has simple poles at all non-positive integers, then all the possible poles of the meromorphic function \( g(s) \) are

\[
s = 1 - \frac{b - b_j}{a_{a_j}}, \quad j = 1, \ldots, k
\]

and all non-positive integers. Let \( N \in \mathbb{N} \) and \( M \in \mathbb{R} \) such that

\[
N > \max_{1 \leq j \leq k} \left\{ \frac{1 + |b + b_j|}{a_{a_j}} \right\} + 1, \quad M > \max_{1 \leq j \leq k} \left\{ \frac{1 + |b + b_j|}{a_{a_j}} \right\},
\]

we integrate \( g(s) \) over the rectangular contour \( \mathcal{R}(M, N) \) with vertices,

\[
c - iM, \quad c + iM, \quad -N - \frac{1}{2} + iM, \quad -N - \frac{1}{2} - iM.
\]

Then by Cauchy’s theorem we have

\[
\int_{\mathcal{R}(M, N)} g(s) ds = \sum_j \text{Residue} \left\{ g(s), s = \frac{1 - b - b_j}{a_{a_j}} \right\} + \sum_n \text{Residue} \left\{ g(s), s = -n \right\},
\]

where the first sum is over all the distinct pairs from \( a_{a_j}, b_j, j = 1, \ldots, k \) whereas the last sum is over all \( n \) satisfying \( 0 \leq n \leq N \) and (2.4).

On the other hand, we also have

\[
\int_{\mathcal{R}(M, N)} g(s) ds = \left\{ \int_{c-iM}^{c+iM} - \int_{-2N+1-iM}^{-2N+1+iM} \right\} g(s) ds + \left\{ \int_{-2N+1+iM}^{-2N+1-iM} - \int_{c-iM}^{c+iM} \right\} g(s) ds
\]

Fix \( N \) and \( x \), by (1.3) and (1.8), since the integrands of the last two integrals have the estimate

\[
g(s) = O \left( e^{-\epsilon/2} e^{M} \right), \quad M \to \infty,
\]

where \( \epsilon \) is an arbitrary positive number such that \( 0 < \epsilon < \frac{\pi}{2} \), then the last two integrals have limit 0 as \( M \to \infty \). Then by taking limit \( M \to \infty \) in (2.9) and (2.8) we get

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{x^s} \left( \sum_{n=1}^{\infty} \frac{f_k(n)}{n^{s+1}} \right) \frac{ds}{x^s} \\
+ \sum_j \text{Residue} \left\{ g(s), s = \frac{1 - b - b_j}{a_{a_j}} \right\} + \sum_n \text{Residue} \left\{ g(s), s = -n \right\},
\]

where the summations are the same as in (2.8).
Proof. When as \( x \to 0 \) then again by (1.3) and (1.8) we get

\[
\frac{1}{2\pi i} \int_{-\frac{2N+1}{2} - i\infty}^{\frac{2N+1}{2} + i\infty} g(s) ds = o\left(x^N\right)
\]

as \( x \to 0 \). Then by (2.10) and (2.11) we get

\[
\frac{1}{2\pi i} \int_{e^{2\pi i} - \frac{2N+1}{2} - i\infty}^{e^{2\pi i} + \frac{2N+1}{2} + i\infty} g(s) ds = \sum_j \text{Residue} \left\{ g(s), s = \frac{1 - b - b_j}{a a_j} \right\} + \sum_n \text{Residue} \left\{ g(s), s = -n \right\}
\]

as \( x \to 0 \), where the first sum is over all the distinct pairs from \( a_j b_j, j = 1, \ldots, k \) while the last sum is over all nonnegative integers \( n \) satisfying (2.4). Finally, (2.2) is obtained by combining (2.7) and (2.12).

**Corollary 2.** Let \( a, x > 0, b \in \mathbb{C} \). If \( b \neq 1 + an, \ n \in \mathbb{N} \cup \{0\} \), then

\[
\sum_{n=1}^{\infty} \frac{e^{-n^a x}}{n^b} = \frac{x^{b-1}}{a} \Gamma\left(\frac{1 - b}{a}\right) + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an)
\]

as \( x \to 0 \).

If there exists a \( n_0 \in \mathbb{N} \cup \{0\} \) such that \( b = 1 + an_0 \), then

\[
\sum_{n=1}^{\infty} \frac{e^{-n^a x}}{n^b} = \frac{(-x)^n}{an_0!} \left(\gamma a + \psi(n_0 + 1) - \log(x)\right) + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an)
\]

as \( x \to 0 \).

**Proof.** When \( b \neq 1 + an, \ n \in \mathbb{N} \cup \{0\} \), the integrand \( \frac{\Gamma(s) \zeta(as + b)}{x^s} \) is meromorphic and has the following simple poles

\[
s = \frac{1 - b}{a}, 0, -1, -2, \ldots
\]

with residues

\[
\text{Residue} \left\{ \Gamma(s) \zeta(as + b), s = \frac{1 - b}{a} \right\} = \frac{x^{(b-1)/a}}{a} \Gamma\left(\frac{1 - b}{a}\right)
\]

\[
\text{Residue} \left\{ \Gamma(s) \zeta(as + b), s = -n \right\} = \frac{(-x)^n}{n!} \zeta(b - an).
\]

Then (2.13) is obtained by applying Theorem 1

When \( b = 1 + an_0 \) for some nonnegative integer \( n_0 \), then

\[
\frac{\Gamma(s) \zeta(as + b)}{x^s} = \frac{\Gamma(s) \zeta(a(s + n_0) + 1)}{x^s}
\]

has a double pole at \(-n_0\) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s) \zeta(as + b)}{x^s}, s = -n_0 \right\} = \frac{(-1)^{n_0} x^{n_0} (\gamma a + \psi(n_0 + 1) - \log(x))}{an_0!},
\]
all the other nonpositive integers are simple poles with residues,

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)}{x^s} \right\}, s = -n \neq -n_0 = \frac{(-x)^n}{n!} \zeta(b - an).
\]

Then by Theorem 1 we have

\[
\sum_{n=1}^{\infty} e^{-n^a x} = \frac{(-x)^{n_0}(\gamma a + \psi(n_0 + 1) - \log(x))}{an_0!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an)
\]

\[
\text{as } x \to 0.
\]

\[\square\]

**Example 3.** When \(a = 2, b = 0\) we have

\[\sum_{n=1}^{\infty} e^{-n^2 x} = \frac{1}{2\sqrt{\pi x}} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(-2n) = \frac{1}{2\sqrt{\pi x}}\]

as \(x \to 0\), which means the error term is better than any \(x^n\). When \(a = 2, b = 1\),

\[
\sum_{n=1}^{\infty} e^{-n^2 x} = \frac{\gamma - \log(x)}{2} - \sum_{n=1}^{\infty} \frac{B_{2n} (-x)^n}{n!(2n)},
\]

as \(x \to 0\). When \(a = 2, b = -1\), then \(2n + 1 = -1\) has no nonnegative integer solutions. Thus,

\[
\sum_{n=1}^{\infty} ne^{-n^2 x} = \frac{1}{2x} + \frac{1}{2x} \sum_{n=1}^{\infty} \frac{B_{2n} (-x)^n}{n!},
\]

or

\[
\sum_{n=-\infty}^{\infty} |n|e^{-n^2 x} = \frac{1}{x} + \frac{1}{x} \sum_{n=1}^{\infty} \frac{B_{2n} (-x)^n}{n!}
\]

as \(x \to 0\).

**Corollary 4.** For all \(x, a > 0, b \in \mathbb{C}\) and \(c > \frac{1-\Re(b)}{a}\) we have

\[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta^2(as + b) \frac{ds}{x^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^b} e^{-n^a x}.
\]

Furthermore, if \(an \neq b - 1\) for all nonnegative integers \(n\), then

\[\sum_{n=1}^{\infty} \frac{d(n)}{n^b} e^{-n^a x} = \frac{x^{b-1} \Gamma \left( \frac{1-b}{a} \right) \left( \psi \left( \frac{1-b}{a} \right) + 2\gamma a - \log(x) \right)}{a^2} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta^2(b - an)
\]

as \(x \to 0^+\).
If \( am = b - 1 \) for certain nonnegative \( m \), then

\[
\sum_{n=1}^{\infty} \frac{d(n)}{n^b} e^{-n^ax} = \frac{(-x)^m}{a^2m!} \left\{ (a\gamma)^2 - 2a^2\gamma_1 + (2a\gamma - \log x) \psi(m + 1) - 2a\gamma \log x + \frac{\psi^2(m + 1) - \psi^{(1)}(m + 1) + \log^2(x)}{2} + \frac{\pi^2}{6} \right\} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta^2(b - an),
\]

as \( x \to 0^+ \).

**Proof.** When \( an \neq b - 1 \) for all nonnegative integers \( n \), then the meromorphic function \( \Gamma(s)\zeta^2(as + b)x^{-s} \) has a double pole at \( s = (1 - b)/a \) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta^2(as + b)}{x^s}, s = 1 - \frac{b}{a} \right\} = \frac{x^{\frac{b-1}{a}}\Gamma(\frac{1-b}{a}) (\psi(\frac{1-b}{a}) + 2\gamma a - \log(x))}{a^2}
\]

and simple poles at all nonpositive integers \( n \in \mathbb{N}_0 \) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta^2(as + b)}{x^s}, s = -n \right\} = \frac{(-x)^n}{n!} \zeta^2(b - an).
\]

Then by Theorem [1] we get

\[
\sum_{n=1}^{\infty} \frac{d(n)}{n^b} e^{-n^ax} = \frac{x^{\frac{b-1}{a}}\Gamma(\frac{1-b}{a}) (\psi(\frac{1-b}{a}) + 2\gamma a - \log(x))}{a^2} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta^2(b - an)
\]

as \( x \to 0^+ \).

When \( \alpha = 0, \ am = b - 1 \) for certain nonnegative integer \( m \), then the meromorphic function \( \Gamma(s)\zeta^2(as + b)x^{-s} \) has a triple pole at \( s = -m \) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta^2(as + b)}{x^s}, s = -m \right\} = \frac{(-x)^m}{a^2m!} \left\{ (a\gamma)^2 - 2a^2\gamma_1 + (2a\gamma - \log x) \psi(m + 1) - 2a\gamma \log x + \frac{\psi^2(m + 1) - \psi^{(1)}(m + 1) + \log^2(x)}{2} + \frac{\pi^2}{6} \right\}.
\]

It has simple poles at all other nonpositive integers with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta^2(as + b)}{x^s}, s = -n \right\} = \frac{(-x)^n}{n!} \zeta^2(b - an).
\]

Then by Theorem [1] we get

\[
\sum_{n=1}^{\infty} \frac{d(n)}{n^b} e^{-n^ax} = \frac{x^{\frac{b-1}{a}}\Gamma(\frac{1-b}{a}) (\psi(\frac{1-b}{a}) + 2\gamma a - \log(x))}{a^2} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta^2(b - an)
\]

as \( x \to 0^+ \).
Example 5. Let \( a = 2, \ b = 2 \), then by (2.20) to get
\[
\sum_{n=1}^{\infty} \frac{d(n)}{n^2} e^{-n^2 \pi^2} = \frac{\sqrt{\pi x} \left( \log x - \psi\left(\frac{1}{2}\right) - 4\gamma_1 \right)}{2} + \frac{\pi^4}{36}
\]
as \( x \to 0^+ \), the remainder here is better than any \( x^n \).

Let \( n, \alpha, \beta \) be nonnegative integers, then by (2.21) to get
\[
\sum_{n=1}^{\infty} \frac{d(n)}{n} e^{-n^2 \pi^2} = \frac{6 \log^2 x - 45 \log x + 6 \gamma^2 + \pi^2 - 24 \gamma_1}{12} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{B_{2n}(-x)^n}{n(n+1)!}
\]
as \( x \to 0^+ \).

Corollary 6. Let \( \alpha \in \mathbb{C} \) and \( \alpha \neq 0 \), then for all \( x, a > 0, b \in \mathbb{C} \) and \( c > \max_{1 < k \leq b} \left( 1 - R(b, 1 - R(b, -\alpha)) \right) \)
we have
\[
\frac{1}{2\pi i} \int_{c+\infty}^{c-i\infty} \Gamma(s) \zeta(as + b) \zeta(as + b - \alpha) \frac{ds}{x^s} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^b} e^{-n^a x}.
\]
Furthermore, if \( an \neq b - 1 \) and \( an \neq b - 1 - \alpha \) for all nonnegative integers \( n \in \mathbb{N}_0 \), then
\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^b} e^{-n^a x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an) \zeta(b - an - \alpha) + \frac{x^{(b-1)/a}}{\alpha} \Gamma\left(1 - \frac{b}{a}\right) \zeta(1 - \alpha) + \frac{x^{(b-1-a)/a}}{\alpha} \Gamma\left(1 - \frac{b + \alpha}{a}\right) \zeta(1 + \alpha)
\]
as \( x \to 0^+ \).

If \( am = b - 1 \) for certain nonnegative integer \( m \) and \( an \neq b - 1 - \alpha \) for all nonnegative integers \( n \in \mathbb{N}_0 \), then
\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^b} e^{-n^a x} = \frac{(-x)^m (a\zeta'(1 - \alpha) + \zeta(1 - \alpha) (\gamma a + \psi(m + 1) - \log(x)))}{am!} + \frac{x^{m-a/a}}{\alpha} \Gamma\left(\frac{\alpha - ma}{a}\right) \zeta(1 + \alpha) + \sum_{n = 0}^{\infty} \frac{(-x)^n}{n!} \zeta(1 - a(n - m)) \zeta(1 - \alpha - a(n - m))
\]
as \( x \to 0^+ \).

If \( an \neq b - 1 \) for all nonnegative integers \( n \in \mathbb{N}_0 \) and \( am = b - 1 - \alpha \) for certain \( m \in \mathbb{N}_0 \), then
\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^b} e^{-n^a x} = \frac{(-x)^m (a\zeta'(1 + \alpha) + \zeta(1 + \alpha) (\gamma a + \psi(m + 1) - \log(x)))}{am!} + \sum_{n = 0}^{\infty} \frac{(-x)^n}{n!} \zeta(1 + \alpha - a(n - m)) \zeta(1 - a(n - m))
\]
as \( x \to 0^+ \).
If \( am = b - 1 \) and \( \alpha = a(m - \ell) \) for certain nonnegative integers \( m, \ell \in \mathbb{N}_0 \) with \( m \neq \ell \), then

\[
\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^b} e^{-n^a x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(1 - a(n - m)) \zeta(1 - a(n - \ell)) \\
+ \frac{(-x)^m}{am!} (a\zeta'(1 - a(m - \ell)) + \zeta(1 - a(m - \ell))(\gamma a + \psi(m + 1) - \log(x))) \\
+ \frac{(-x)^\ell}{a\ell!} (a\zeta'(1 + a(m - \ell)) + \zeta(1 + a(m - \ell))(\gamma a + \psi(\ell + 1) - \log(x)))
\]

as \( x \to 0^+ \).

Proof. When \( \alpha \neq 0 \), \( an \neq b - 1 \) and \( an \neq b - 1 - \alpha \) for all nonnegative integers \( n \in \mathbb{N}_0 \), the meromorphic function \( \Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)x^{-s} \) has simple poles at

\[
\frac{1 - b}{a}, \frac{1 - b + \alpha}{a}, 0, -1, -2, \ldots
\]

with residues

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = 1 - \frac{b}{a} \right\} = \frac{x^{(b-1)/a}}{a} \Gamma \left( \frac{1 - b}{a} \right) \zeta(1 - \alpha),
\]

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = 1 - \frac{b + \alpha}{a} \right\} = \frac{x^{(b-1)/a}}{a} \Gamma \left( \frac{1 - b + \alpha}{a} \right) \zeta(1 + \alpha)
\]

and

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -n \right\} = \frac{(-x)^n}{n!} \zeta(b - an) \zeta(b - an - \alpha)
\]

for \( n \in \mathbb{N}_0 \). Then by Theorem II we get

\[
\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^b} e^{-n^a x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an) \zeta(b - an - \alpha) \\
+ \frac{x^{(b-1)/a}}{a} \Gamma \left( \frac{1 - b}{a} \right) \zeta(1 - \alpha) + \frac{x^{(b-1)/a}}{a} \Gamma \left( \frac{1 - b + \alpha}{a} \right) \zeta(1 + \alpha)
\]

as \( x \to 0^+ \).

When \( am = b - 1 \) for certain nonnegative integer \( m \) and \( an \neq b - 1 - \alpha \) for all nonnegative integers \( n \in \mathbb{N}_0 \), then the meromorphic function \( \Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)x^{-s} \) has a double pole at \( s = -m \) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -m \right\} = \frac{(-x)^m}{am!} (a\zeta'(1 - \alpha) + \zeta(1 - \alpha)(\gamma a + \psi(m + 1) - \log(x)))
\]

and a simple pole at \( s = -m + \frac{\alpha}{a} \) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -m + \frac{\alpha}{a} \right\} = \frac{2^{m-\alpha/a}}{a} \Gamma \left( \frac{\alpha - ma}{a} \right) \zeta(1 + \alpha)
\]
and simple poles at all nonpositive integers other than \(-m\) with residues

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -n \right\} = \frac{(-x)^n}{n!} \zeta(b - an)\zeta(b - \alpha - an).
\]

Hence,

\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^b} e^{-nx} = \frac{(-x)^m (a\zeta'(1 + \alpha) + \zeta(1 + \alpha)(\gamma a + \psi(m + 1) - \log(x)))}{am!} + \frac{x^{m-a/a}}{a} \Gamma \left( \frac{\alpha - ma}{a} \right) \zeta \left( 1 + \alpha \right) + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an)\zeta(b - \alpha - an)
\]
as \(x \to 0^+\).

When \(am \neq b - 1\) for all nonnegative integers \(n \in \mathbb{N}_0\) and \(am = b - 1 - \alpha\) for certain \(m \in \mathbb{N}_0\), then the meromorphic function \(\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)x^{-s}\) has a double simple pole \(s = -m\) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -m \right\} = \frac{(-x)^m (a\zeta'(1 + \alpha) + \zeta(1 + \alpha)(\gamma a + \psi(m + 1) - \log(x)))}{am!}
\]

and a simple pole \(s = -m - \frac{a}{\alpha}\) with residue

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -m - \frac{\alpha}{a} \right\} = \frac{x^{m+a/a}}{a} \Gamma \left( \frac{-am + \alpha}{a} \right) \zeta \left( 1 - \alpha \right)
\]

and simple poles at all nonpositive integers other than \(-m\) with residues

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -n \right\} = \frac{(-x)^n}{n!} \zeta(b - an)\zeta(b - \alpha - an).
\]

Thus,

\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^b} e^{-nx} = \frac{(-x)^m (a\zeta'(1 + \alpha) + \zeta(1 + \alpha)(\gamma a + \psi(m + 1) - \log(x)))}{am!} + \frac{x^{m+a/a}}{a} \Gamma \left( \frac{-am + \alpha}{a} \right) \zeta \left( 1 - \alpha \right) + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta(b - an)\zeta(b - \alpha - an)
\]
as \(x \to 0^+\).

When \(am = b - 1\) and \(\alpha = a(m - \ell)\) for certain nonnegative integers \(m, \ell \in \mathbb{N}_0\) with \(m \neq \ell\), then the meromorphic function \(\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)x^{-s}\) has two double poles at \(s = -m\) and \(s = -\ell\) with residues

\[
\text{Residue} \left\{ \frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -m \right\} = \frac{(-x)^m (a\zeta'(1 - a(m - \ell)) + \zeta(1 - a(m - \ell))(\gamma a + \psi(m + 1) - \log(x)))}{am!}
\]
and

\[
\text{Residue}\left\{\frac{\Gamma(s)\zeta(as + b)\zeta(as + b - \alpha)}{x^s}, s = -\ell\right\} = \frac{(-x)^\ell}{\ell!} \left(\alpha\zeta'(1 + a(m - \ell)) + \zeta(1 + a(m - \ell))(\gamma a + \psi(\ell + 1) - \log(x))\right)
\]

giving residues at all other nonpositive integers other than \(-m, -\ell\) with residues

\[
-\frac{(-x)\zeta(1 - a(n - m))\zeta(1 - a(n - \ell))}{n!}.
\]

Then,

\[
\sum_{n=1}^\infty \frac{\sigma_n(n)}{n^b} e^{-nx} = \sum_{n=0}^\infty \frac{(-x)^n}{n!}\zeta(1 - a(n - m))\zeta(1 - a(n - \ell)) + \frac{(-x)^m}{m!} \frac{\alpha\zeta'(1 - a(m - \ell)) + \zeta(1 - a(m - \ell))(\gamma a + \psi(m + 1) - \log(x))}{a\ell!} + \frac{(-x)^\ell}{\ell!} \left(\alpha\zeta'(1 + a(m - \ell)) + \zeta(1 + a(m - \ell))(\gamma a + \psi(\ell + 1) - \log(x))\right)
\]

as \(x \to 0^+\).

**Example 7.** When \(a = 2, \alpha = 1, b = \frac{1}{2}\), by (2.28) we get

(2.29)

\[
\sum_{n=1}^\infty \frac{\sigma(n)}{n^{b/2}} e^{-n^{1/2}x} = \frac{\pi^2}{9} \frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} x^{-\frac{1}{4}} + \sum_{n=0}^\infty \frac{(-x)^n}{n!} \zeta\left(\frac{1 - 4n}{2}\right) \zeta\left(\frac{-1 - 4n}{2}\right)
\]

as \(x \to 0^+\). When \(a = 2, \alpha = 1, b = 1\), then \(m = 0\) in (2.26). Then

\[
\sum_{n=1}^\infty \frac{\sigma(n)}{n} e^{-n^2x} = \frac{\pi^{5/2}}{12\sqrt{x}} + \log x - \log 2\pi - \gamma - \frac{\gamma}{4},
\]

as \(x \to 0^+\), it implies that the difference between two sides of the above formula is smaller than any \(x^n\). Let \(a = 2, b = 1, \alpha = -2\) in (2.28), then \(m = 0, \ell = 1\). Then,

\[
\sum_{n=1}^\infty \frac{\sigma_{-2}(n)}{n} e^{-n^2x} = -\frac{\zeta(3)}{2} \log x + \frac{2\zeta'(3) + \zeta(3)\gamma}{2} - x \log x + 24\log A + \gamma + 1 + \frac{24}{n} \sum_{n=2}^\infty \frac{B_2(n-1)B_{2n} (-x)^n}{(n-1)n n!}
\]

as \(x \to 0^+\).

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