Complete Low-Energy Effective Action in $\mathcal{N}=4$ SYM:
a Direct $\mathcal{N} = 2$ Supergraph Calculation

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Abstract

Using the covariant $\mathcal{N} = 2$ harmonic supergraph techniques we calculate the one-loop low-energy effective action of $\mathcal{N} = 4$ super-Yang-Mills theory in Coulomb branch with
gauge group $SU(2)$ spontaneously broken down to $U(1)$. The full dependence of the low-
energy effective action on both the hypermultiplet and gauge fields is determined. The
direct quantum calculation confirms the correctness of the exact $\mathcal{N} = 4$ SYM low-energy
effective action derived in [hep-th/0111062] on the purely algebraic ground by invoking a
hidden $\mathcal{N} = 2$ supersymmetry which completes the manifest $\mathcal{N} = 2$ one to $\mathcal{N} = 4$. Our
results provide an exhaustive solution to the problem of finding out the exact completely
$\mathcal{N} = 4$ supersymmetric low-energy effective action for the theory under consideration.
1 Introduction

One of the remarkable features of $\mathcal{N} = 4$ supersymmetric quantum Yang-Mills theory is the opportunity to obtain exact results. At present, one can distinguish at least three trends in finding out exact solutions to some important quantities in this theory. These are, first, the study of low-energy effective action, second, computing the correlators of gauge invariant operators, and, third, computing the expectation values of Wilson loops (see [1]-[3] for a review and references).

In this paper we solve the open problem of calculating the exact low-energy effective action depending on all fields of $\mathcal{N} = 4$ gauge multiplet. Until recently, the well-established exact results for leading low-energy contributions to the effective action were obtained for $SU(2)$, $\mathcal{N} = 4$ SYM model in Coulomb branch [4, 5, 6] and only for the $\mathcal{N} = 2$ gauge multiplet sector of the effective action. These contributions are presented by a non-holomorphic effective potential of the form

$$\mathcal{H}(W, \bar{W}) = \frac{1}{(4\pi)^2} \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda}.$$  \hspace{1cm} (1.1)

Here $W$ and $\bar{W}$ are the $\mathcal{N} = 2$ $U(1)$ gauge superfield strengths and $\Lambda$ is an arbitrary scale. It was pointed out in [4, 8] that although the potential (1.1) was obtained in one-loop approximation, it receives neither perturbative nor non-perturbative corrections. Hence, the function $\mathcal{H}(W, \bar{W})$ (1.1) determines the exact low-energy effective action in the $\mathcal{N} = 2$ gauge field sector (to be more precise, the leading in external momenta part of the full effective action).

A generalization of the effective potential (1.1) to the Coulomb branch of $\mathcal{N} = 4$ SYM model with the gauge group $SU(N)$ broken to its Cartan subgroup $U(1)^{N-1}$ has been given in [8, 9, 10] (see also the review [1]). Despite the fact that the one-loop non-holomorphic potential in this case looks quite analogous to (1.1), it was argued in [8, 11] that the effective potential can contain, in principle, some extra non-logarithmic contributions (beginning at least with fifth loop [12]) which do not have the product form (1.1). However, in the actual computations such contributions have never been found.

From the standpoint of $\mathcal{N} = 2$ supersymmetry, the $\mathcal{N} = 4$ gauge multiplet is a sum of $\mathcal{N} = 2$ gauge multiplet and hypermultiplet. All the above mentioned results on the structure of non-holomorphic potential were obtained only for that part of the effective action which depends on the fields of $\mathcal{N} = 2$ gauge multiplet. The problem of constructing the complete effective action depending on both the $\mathcal{N} = 2$ gauge multiplet and hypermultiplet fields has been solved in a recent paper [13]. The construction of [13] is based upon the purely algebraic analysis exploring the existence of extra hidden on-shell $\mathcal{N} = 2$ supersymmetry in $\mathcal{N} = 4$ SYM theory formulated in $\mathcal{N} = 2$ harmonic superspace [14, 15]. The manifest off-shell $\mathcal{N} = 2$ supersymmetry and the hidden one constitute the full on-shell $\mathcal{N} = 4$ supersymmetry of $\mathcal{N} = 4$ SYM theory. It was shown that the potential (1.1), as well as its generalization to

\footnote{A possibility of non-holomorphic contributions having the form of product of two logarithms was earlier found in [7] for the effective action of a generic $\mathcal{N} = 2$ SYM model.}
SU(N) model, can be completed to $\mathcal{N} = 4$ supersymmetric form by adding the appropriate terms depending on hypermultiplet superfields. However, any non-logarithmic terms in the low-energy effective action do not admit such a completion and hence are ruled out by $\mathcal{N} = 4$ supersymmetry. An open problem to remain was to re-derive the effective action of \cite{13} by a direct supergraph computation in the quantum field theory framework. The present paper is devoted to solving this problem.

We specialize to the $SU(2) \, \mathcal{N} = 4$ SYM model formulated in harmonic superspace and consider its Coulomb branch. Since the effective potential (1.1) is generated solely by one-loop contributions, the corresponding hypermultiplet-dependent terms also have to be evaluated only in the one-loop approximation. We would like to point out that such a calculation is a very non-trivial problem. We have to calculate not just a contribution of single supergraph with a few external legs but the contributions of the supergraphs with arbitrary numbers of external legs on a background of non-zero $W$ and $\bar{W}$ and then to sum up all these contributions. We show that such calculations can be actually accomplished and the result yields just the effective action found in \cite{13}. A similar approach is used to give a new derivation of the one-loop non-holomorphic effective $W, \bar{W}$ potential. As the result we are aware of the unified approach allowing us to compute the low-energy effective action in both the gauge multiplet and hypermultiplet sectors. The calculations are carried out within the $\mathcal{N} = 2$ background field method \cite{17}, with making use of the quantum $\mathcal{N} = 2$ harmonic superfield techniques pioneered in \cite{14} and further advanced in \cite{12, 16–21}.

The paper is organized as follows. In section 2 we recapitulate a formulation of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ harmonic superspace and describe a general structure of low-energy effective action in this theory. Section 3 presents details of calculating the hypermultiplet-dependent contributions to effective action. In section 4, using a similar approach, we perform a manifestly $\mathcal{N} = 2$ supercovariant calculation of the non-holomorphic effective potential in the $\mathcal{N} = 2$ gauge fields sector. The results are summarized in section 5.

2 $\mathcal{N} = 4$ SYM theory in harmonic superspace and the problem of complete low-energy effective action

The ‘microscopic’ action of $\mathcal{N} = 4$ $SU(2)$ SYM theory written in $\mathcal{N} = 2$ harmonic superspace \cite{14, 15} reads

$$S = \frac{1}{2g^2} \text{tr} \int d^8z \, W^2 - \frac{1}{2} \text{tr} \int d\zeta^{(-4)} q^{+a}(D^{++} + iV^{++}) q_{a}^{+}.$$ (2.1)

Here $q_{a}^{+} = (q^{+}, \tilde{q}^{+})$, $q^{+a} = \epsilon^{ab} q_{b}^{+}$, is the hypermultiplet in the adjoint representation of the gauge group and $W$ is the covariant strength of $\mathcal{N} = 2$ analytic gauge superfield $V^{++}$. This action (i) is manifestly $\mathcal{N} = 2$ supersymmetric; (ii) possesses a second hidden $\mathcal{N} = 2$ supersymmetry completing the first one to $\mathcal{N} = 4$. The off-shell $\mathcal{N} = 4$ transformations for the full nonabelian
case can be found in [14, 13]. For fixing the one-loop effective action it suffices to know them for the abelian $U(1)$ components of $\mathcal{W}, q^{+a}$ only (with all off-diagonal superfields neglected) and on the free mass shell for these $U(1)$ superfields [13]:

$$\delta \mathcal{W} = \frac{1}{2}\epsilon^{\alpha a} \bar{D}^- q^+_a, \quad \delta \bar{\mathcal{W}} = \frac{1}{2}\epsilon^\alpha D^- q^+_a,$$

$$\delta q^+_a = \frac{1}{4}(\epsilon_a^\beta D^\beta \mathcal{W} + \epsilon^\beta_\alpha \bar{D}^\beta \bar{\mathcal{W}}), \quad \delta q^-_a = \frac{1}{4}(\epsilon_a^\beta D^\beta \mathcal{W} + \epsilon^\beta_\alpha \bar{D}^\beta \bar{\mathcal{W}}). \quad (2.2)$$

Here $q^{-a} \equiv D^{-} q^{+a}$, where $D^{-}$ is the second harmonic derivative.

For calculating quantum corrections we use the background field method [17]. We start with carrying out background-quantum splitting by the rule $q^{+a} \to q^{+a} + Q^{+a}; \quad V^{++} \to V^{++} + g v^{++}$. Here $q^{+a}$ is a background hypermultiplet and $Q^{+a}$ a quantum one; $V^{++}$ is a background $\mathcal{N} = 2$ gauge superfield and $v^{++}$ a quantum one. For our purpose it suffices to retain in the action (2.1) only terms of the second order in quantum superfields:

$$S_2 = -\frac{1}{2}\operatorname{tr} \int d\zeta (-4) \left[ v^{++} \bar{\square} v^{++} + Q^{+a} (D^{++} + iV^{++}) Q^+_a + Q^{+a} (ig v^{++}) q^+_a + q^{+a} (ig v^{++}) Q^+_a \right]. \quad (2.3)$$

The operator $\bar{\square}$ includes background $\mathcal{N} = 2$ superfield strengths $\mathcal{W}, \bar{\mathcal{W}}$, is defined to act in the adjoint representation of the gauge group and has the form [16, 13]:

$$\bar{\square} = \square + \frac{i}{2}(D^{+a} \mathcal{W}) D^a - \frac{i}{2}(\bar{D}^a + \bar{\mathcal{W}} \bar{D}^a) + \frac{i}{8}[D^{+a}, \bar{D}^a] \mathcal{W}$$

$$- \frac{1}{4}(D^{+a} D^a) W D^{--} + \frac{1}{2}[\mathcal{W}, \bar{\mathcal{W}}]. \quad (2.4)$$

The $\mathcal{N} = 4$ SYM effective action in the Coulomb branch depends only on abelian background superfields $\mathcal{W}, \bar{\mathcal{W}}, q^{+a}$ which can be chosen to be on-shell, $[D^{+a}, D^-] \mathcal{W} = 0$, $(D^{+a} D^a) \mathcal{W} = 0$, $D^{++} q^{-a} = 0$ [13, 13]. Taking into account these conditions and the fact that all superfields are in the adjoint representation, we have

$$\bar{\square} v^{++} = \square v^{++} + \frac{i}{2}[D^{+a} \mathcal{W}, D^- v^{++}] + \frac{i}{2}[\bar{D}^{+a} \bar{\mathcal{W}}, \bar{D}^- v^{++}] + [\mathcal{W}, [\bar{\mathcal{W}}, v^{++}]],$$

$$Q^{+a} v^{++} Q^+_a = Q^{+a} [v^{++}, Q^+_a], \quad \text{etc}. \quad (2.5)$$

The quantum fields $v^{++}$ with values in $su(2)$ can be written as

$$v^{++} = v^{++}_i \tau_i, \quad (2.6)$$

where $\tau_i$ are generators of $su(2)$ related to Pauli matrices $\sigma_i$ as

$$\tau_i = \frac{1}{\sqrt{2}} \sigma_i. \quad (2.7)$$

2To avoid a possible confusion we note that, though these hidden supersymmetry transformations can be defined for off-shell superfields, they form, together with the manifest $\mathcal{N} = 2$ supersymmetry transformations, $\mathcal{N} = 4$ Poincaré supersymmetry only on shell.
They satisfy the relations
\[ [\tau_i, \tau_j] = i\sqrt{2}e_{ijk}\tau_k, \quad \text{tr}(\tau_i\tau_j) = \delta_{ij}. \quad (2.8) \]

The procedure of calculation of quantum correction for the gauge group $SU(N)$ broken to $U(1)$ was developed in [19]. The background $\mathcal{N} = 2$ strengths are expanded as $\mathcal{W} = \mathcal{W}_i\tau_i$. We calculate the effective action in the Coulomb branch, in which it depends only on the background fields taking values in the unbroken $u(1)$. The latter corresponds to the restriction $\mathcal{W} = \mathcal{W}_3\tau_3$, further we denote $\mathcal{W}_3$ as $\mathcal{W}$. Our first aim will be to derive the action (and hence, vertices and propagators) for fields $v^+_i$, $Q^+$. We also redefine $gq^a \to q^a$. For the Abelian background we have $\mathcal{D}^+\mathcal{W} = \mathcal{D}^+\mathcal{W}$, $\mathcal{D}^a\mathcal{W} = \mathcal{D}^a\mathcal{W}$, with $\mathcal{D}, \mathcal{D}$ being flat spinor derivatives, and we can make these replacements in (2.4).

Thus the quadratic part of the action can be expressed as
\[
S_2 = -\frac{1}{2} \int d\zeta^{(-4)} \left\{ v^{++}_i(\Box + 2\mathcal{W}\mathcal{W})v^{++}_i + v^{++}_i(\Box + 2\mathcal{W}\mathcal{W})v^{++}_j - \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j \right\}. \quad (2.9)
\]

Using (2.7), we can do all traces. After this the expression (2.9) is rewritten as
\[
S_2 = -\frac{1}{2} \int d\zeta^{(-4)} \left\{ v^{++}_i(\Box + 2\mathcal{W}\mathcal{W})v^{++}_i + v^{++}_i(\Box + 2\mathcal{W}\mathcal{W})v^{++}_j - \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j + \frac{1}{\sqrt{2}}v^{++}_i((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)v^{++}_j \right\}. \quad (2.10)
\]

From (2.10) we conclude that only the components carrying indices 1,2 have a non-trivial background-dependent propagator, while the quantum superfield components with index 3 do not interact with the background, totally decouple from the action and possess the free propagators. Then we can define new complex quantum fields
\[
\begin{align*}
\chi^{++} &= \frac{1}{\sqrt{2}}(v^{++}_1 - iv^{++}_i), \\
\bar{\chi}^{++} &= \frac{1}{\sqrt{2}}(v^{++}_1 + iv^{++}_i), \\
\eta^{+i} &= \frac{1}{\sqrt{2}}(Q^{+i}_1 + iQ^{+i}_2), \\
\bar{\eta}^{+i} &= \frac{1}{\sqrt{2}}(Q^{+i}_1 - iQ^{+i}_2).
\end{align*} \quad (2.11)
\]

As a result, the action (2.10) takes the form
\[
S_2 = -\int d\zeta^{(-4)} \left\{ \bar{\chi}^{++}(\Box + 2\mathcal{W}\mathcal{W})\chi^{++} - \frac{1}{\sqrt{2}}((D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^+ + (D^{+\bar{\alpha}}\mathcal{W})\bar{\mathcal{D}}_\bar{\alpha}^+)\chi^{++} + \bar{\eta}^{+i}(D^{++} - \sqrt{2}V^{++})\eta^{+i} + \frac{1}{2}\bar{v}^{++}_i\Box v^{++}_j + \frac{1}{2}Q^{+a}_3 D^{++}Q^{+a}_3 + i\chi^{++}\sqrt{2}q^a\bar{\eta}^{+a} + i\bar{\chi}^{++}\sqrt{2}q^a\eta^{+a} \right\}.
\]
It is convenient to rewrite the action (2.10) as

$$S_2 = S_0 + V + \ldots ,$$  \hspace{1cm} (2.12)

where

$$S_0 = -\int d\xi\{ \bar{\chi}^{++}(\Box + 2W\bar{W} - \frac{1}{\sqrt{2}}((D^{+\alpha}W)D^-_\alpha + (\bar{D}^{\alpha}\bar{W})\bar{D}^-_{\alpha})\chi^{++} +$$

$$+ \bar{\eta}^{+\alpha}(D^{++} - \sqrt{2}V^{++})\eta^+_\alpha \}$$ \hspace{1cm} (2.13)

is the part that is used for defining propagators, and

$$V = -i\sqrt{2}(\chi^{++}q^{+a}\bar{\eta}^+_a + \bar{\chi}^{++}q^{+a}\eta^+_a)$$ \hspace{1cm} (2.14)

is the interaction term. Dots in (2.12) stand for the quadratic action of the decoupling fields $Q^{++}_3, u_3^{++}$ which do not contribute to quantum corrections.

It is worth mentioning that for calculation of the effective action in the pure $\mathcal{N} = 2$ SYM sector one must also take into account the third ghosts \[17\]. Contributions of the third ghosts to effective action will be considered in Section 4. They do not contribute to the hypermultiplet-dependent quantum corrections computed in Section 3.

As the next step we introduce the operator

$$\bar{\Box}_w = \Box + 2W\bar{W} - \frac{1}{\sqrt{2}}((D^{+\alpha}W)D^-_\alpha + (\bar{D}^{\alpha}\bar{W})\bar{D}^-_{\alpha})$$ \hspace{1cm} (2.15)

which is obtained from $\bar{\Box}$ (2.4) by retaining in the latter only the superfields $W, \bar{W}$ associated with the unbroken $u(1)$ and putting them on shell. Note that the action (2.13) is manifestly analytic like (2.3). As we shall see, while calculating the hypermultiplet-dependent contributions it suffices to consider only that part of $\bar{\Box}_w$ which does not contain derivatives of the background superfields $W, \bar{W}$. In what follows we shall use the notation $\bar{\Box}_w = \bar{\Box}$, with hoping that this will not give rise to any confusion.

The background-dependent propagators for the superfields $\chi^{++}, \bar{\chi}^{++}, \bar{\eta}^{+\alpha}, \eta^+_a$ are then given by the expressions [17]

$$< \bar{\chi}^{++}(z_1, u_1) \chi^{++}(z_2, u_2) > = -\frac{i}{2} \frac{\Box(u_1)\Box(u_2)}{(D^+_1)^4(D^+_2)^4\{\delta^{12}(z_1 - z_2)(D^-_2)^2\delta^{(-2,2)}(u_1, u_2)\}},$$

$$< \bar{\eta}^{+\alpha}(z_1, u_1)\eta^+_a(z_2, u_2) > = i\delta^a_b \frac{1}{(u_1)^3(u_2)^3} \frac{(D^+_1)^4(D^+_2)^4}{\Box}\delta^{12}(z_1 - z_2).$$ \hspace{1cm} (2.16)

It is worth pointing out that the gauge superfield propagator $< \bar{\chi}^{++}\chi^{++} >$ can be written in two equivalent forms [22, 13]. The form (2.16) is advantageous in that it is analytic with respect to both superspace arguments.

The exact one-loop non-holomorphic effective $W, \bar{W}$ potential in the model under consideration was found in [4, 5, 6]. The hypermultiplet-dependent part of the effective action was
restored in the recent paper [13]. It was shown there by an algebraic analysis that the following expression

$$\int d^{12}z [\mathcal{H}(W, \bar{W}) + \mathcal{L}_q(W, \bar{W}, q^+)]$$

(2.17)

is invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations (2.2) (and hence under $\mathcal{N} = 4$ supersymmetry), if its hypermultiplet-dependent part has the form

$$\mathcal{L}_q(W, \bar{W}, q^+) = \frac{1}{(4\pi)^2} \sum_{n=1}^{\infty} c_n \left( \frac{q^+ q^-}{W \bar{W}} \right)^n,$$

(2.18)

with

$$c_n = \frac{(-2)^n}{n^2(n+1)}.$$

(2.19)

This sum converges to

$$\mathcal{L}_q(X) = \frac{1}{(4\pi)^2} \frac{1}{X} [X (\text{Li}_2(X) - 1) - (1 - X) \ln(1 - X)], \quad X \equiv -2 \frac{q^+ q^-}{W \bar{W}},$$

(2.20)

where $\text{Li}_2(X)$ is Euler dilogarithm [23]. Our main purpose will be to show that the expression (2.18), (2.20) can be directly derived from the quantum harmonic formalism by summing up the appropriate sequence of the one-loop harmonic supergraphs.

3 Hypermultiplet-dependent contributions to effective action

The one-loop hypermultiplet-dependent contributions to the effective action are presented by the following infinite sequence of supergraphs:

Here the external and internal straight lines denote, respectively, external background hypermultiplets and quantum hypermultiplet propagators, and the wavy lines stand for the gauge superfield propagators. The numbers 1,2,... mark harmonic arguments of the external hypermultiplets and vertices. We note that the whole dependence of the corresponding contributions on the external gauge superfield strengths $W, \bar{W}$ is already accounted for by the background-dependent propagators (2.16). This prevents one from considering ‘mixed’ supergraphs with both $\mathcal{N} = 2$ gauge and hypermultiplet external legs. It is enough to consider only the supergraphs shown above.
A generic supergraph with $2n$ external lines can be viewed as a ring consisting of $n$ links of the form $<\eta^+\eta^+><\chi^+\chi^+$ or $n$ links of the form $<\bar{\eta}^+\bar{\eta}^+><\bar{\chi}^+\bar{\chi}^+$. For external superfields slowly varying in space-time the total contribution of these two kinds of the $2n$-point supergraph is given by the following general expression

$$\Gamma_{2n} = \frac{4}{n} \int d^{12}z \int du_1\ldots du_{2n} \frac{1}{\Box (u_1) \Box^2 (u_2) \ldots \Box (u_{2n-1}) \Box^2 (u_{2n})} \times \frac{(\mathcal{D}^+(u_1))^4(\mathcal{D}^+(u_2))^4(\mathcal{D}_2^-)^2\delta^{(2,-2)}(u_2, u_3)(\mathcal{D}^+(u_3))^4(\mathcal{D}^+(u_4))^4(\mathcal{D}_4^-)^2\delta^{(2,-2)}(u_4, u_5)}{(u_1^+ u_2^+)^4(u_3^+ u_4^+)^3} \times \frac{(\mathcal{D}^+(u_5))^4(\mathcal{D}^+(u_{2n-1}))^4(\mathcal{D}^+(u_{2n}))^4(\mathcal{D}_2^-)^2\delta^{(2,-2)}(u_{2n}, u_1)}{(u_5^+ u_6^+)^3(u_7^+ u_8^+)^3\ldots(u_{2n-3}^+ u_{2n-2}^+)^3(u_{2n-1}^+ u_{2n}^+)^3} \times q_a^+(u_1)q^a(u_2)q^b_c(u_3)\ldots q^c_{c'}(u_{2n-1})q^{+c}(u_{2n}) .$$

(3.1)

Hereafter, the symbol $\Box$ stands for the operator defined by eq. (2.15). Since our aim consists in calculation of contribution depending only on $q^+, W, \bar{W}$ but not on their derivatives we can omit all the derivative-depending terms in $\Box$. The factor $4/n$ has the following origin. The contribution from the ring-type supergraph composed of $n$ repeating links $<\eta^+\eta^+><\chi^+\chi^+$ appears with the symmetry factor $2/n$. The same factor $2/n$ arises from the supergraph composed of $n$ repeating links $<\eta^+\bar{\eta}^+><\bar{\chi}^+\chi^+$. Further, each vertex brings the factor $-i$, every $<\eta^+\eta^+>$ and $<\bar{\chi}^+\chi^+>$ propagators contribute the factors $i$ and $i/2$, respectively (hence total of $n$ links contributes $2^{-n}$). Any vertex also carries the coefficient $\sqrt{2}$. This leads to the total factor $2^n$. Putting all these contributions together, we obtain just the coefficient $4/n$.

We calculate the expression (3.1) in the following way. It contains harmonic integrals of products of the expressions $(\mathcal{D}^+(u_{2k}))^4(\mathcal{D}_2^-)^2\delta^{(2,-2)}(u_{2k}, u_{2k+1})(\mathcal{D}^+(u_{2k+1}))^4$, $k = 1, 2\ldots n$ (with $u_{2n+1} \equiv u_1$). In every such term we use the identity $(\mathcal{D}_2^-)^2\delta^{(2,-2)}(u_{2k}, u_{2k+1}) = (\mathcal{D}_2^{-1})^2\delta^{(2,-2)}(u_{2k}, u_{2k+1})$. Then, integrating by parts, we throw $(\mathcal{D}_2^{-1})^2$ on $(\mathcal{D}^+(u_{2k+1}))^4$, integrate over $u_{2k+1}$ using the harmonic delta function and eventually obtain the block $(\mathcal{D}^+(u_{2k}))^4(\mathcal{D}_2^-)^2(\mathcal{D}^+(u_{2k}))^4$. Due to the identity [1]

$$\Box = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^-)^2$$

(3.2)

we can replace $(\mathcal{D}^+(u_{2k}))^4(\mathcal{D}_2^-)^2$ by $-2 \Box_{2k}$ with the operator $\Box$ given by (2.15). We carry out this replacement for all $k$ except for $k = n$ and, after some relabelling of harmonics ($u_{2k}$ is relabelled as $u_{k+1}$, $k = 1, 2\ldots n$), we obtain

$$\Gamma_{2n} = \frac{4(-1)^n2^{n-1}n!}{2n} \int d^{12}z \int du_1du_2\ldots du_n du_{n+1} \frac{1}{\Box (u_1) \Box (u_2) \ldots \Box (u_{n-1}) \Box^2 (u_n)} \times \frac{(\mathcal{D}^+(u_1))^4(\mathcal{D}^+(u_2))^4(\mathcal{D}^+(u_{n}))^4(\mathcal{D}^+(u_{n+1}))^4}{(u_1^+ u_2^+)^4(u_3^+ u_4^+)^3\ldots(u_n^+ u_{n+1}^+)^3} \times \delta^{12}(z-z')|_{z=z'} q^a_a(u_1)q^a(u_2)q^b_c(u_3)\ldots q^c_{c'}(u_{n+1}) .$$

(3.3)
This general relation is applicable for any $n$.

For $n > 1$ one can achieve a further simplification in (3.3). We make use of the identity (3.2) one time more, relabel for convenience the indices ($c$ for $a$, $a$ for $b$, etc.) and integrate over $u_{n+1}$, after which (3.3) for $n > 1$ is reduced to the expression

$$
\Gamma_{2n} = \frac{4(-1)^n 2^n}{n} \int d^{12}z \int du_1 du_2 \ldots du_n \frac{1}{\Box(u_1) \Box(u_2) \ldots \Box(u_{n-1}) \Box(u_n)} \\
\times \left( (D^+(u_1))^4(D^+(u_2))^4 \ldots (D^+(u_n))^4 \right) \\
\times \left( \frac{(u_1^+ u_2^+)^3(u_2^+ u_3^+)^3 \ldots (u_n^+ u_1^+)^3}{(u_1^+ u_2^+)^3(u_2^+ u_3^+)^3 \ldots (u_n^+ u_1^+)^3} \right) \delta^{12}(z-z')|_{z=z'} \\
\times q^+a(u_1)q^+_b(u_1)q^+b(u_2) \ldots q^+_a(u_n) .
$$

(3.4)

Notice that we could arrive at the same result by using the alternative expression for the propagator of $\mathcal{N} = 2$ gauge superfield:

$$
< v^+_{IA}(z_1, u_1) v^+_{JB}(z_2, u_2) > = \delta_{AB} \frac{i}{\Box_w(u_1)} (D^+_I)^4 \left\{ \delta^{12}(z_1 - z_2) \delta^{(-2,2)}(u_1, u_2) \right\} .
$$

(3.5)

This propagator, unlike (2.16) which is manifestly analytic in its both arguments, displays the manifest analyticity only with respect to $(z_1, u_1)$. The analyticity with respect to the second argument follows only with taking into account the harmonic delta function [15].

It is appropriate here to note that the expression (3.4) obtained with using this ‘short’ form of propagator is ill-defined for $n = 1$. So the $n = 1$ case needs a special treatment. Indeed, on the one hand, at $n = 1$ the numerator in (3.4) is proportional to $(D^+(u_1))^4 \delta^{12}(z-z')|_{z=z'}$ which is zero. On the other hand, the denominator of (3.4) for $n = 1$ is $(u_1^+ u_1^+)^3 = 0$. Therefore, we encounter a harmonic singularity of the form $\frac{0}{0}$. To obtain the correct result for the $n = 1$ contribution we should proceed from the expression (3.3) without reducing it to (3.4). This amounts to keeping the $\mathcal{N} = 2$ gauge superfield propagator in the manifestly analytic ‘long’ form (2.16). The necessity to use just the ‘long’ form of the propagator to avoid coincident harmonic singularities was emphasized in [22].

The expression (3.3) for $n = 1$ is

$$
\Gamma_2 = 4 \int d^{12}z \int du_1 du_2 \frac{1}{\Box_1 \Box_2} \left( (D^+(u_1))^4(D^+(u_2))^4 \right) \left( D^2_{-2} \right)^2 \delta^{(2,-2)}(u_2, u_1) \\
\times \delta^{12}(z-z')|_{z=z'} q^+a(u_1)q^+a(u_2) .
$$

(3.6)

Using the relation [17]

$$
(D^+(u))^4(D^+(u'))^4 \delta^8(\theta - \theta')|_{\theta = \theta'} = (u^+ u'^+)^4 ,
$$

(3.7)

one finds

$$
\Gamma_2 = 4 \int d^{12}z \int du_1 du_2(u_1^+ u_2^+) \left( D^2_{-2} \right)^2 \delta^{(2,-2)}(u_2, u_1) \\
\times \frac{1}{\Box_1 \Box_2} \delta^4(x-x')|_{x=x'} q^+a(u_1)q^+a(u_2) .
$$

(3.8)
Then one can take \((D^-)^2\) off a delta function. Since we are interested in contributions depending only on \(q,W,\bar{W}\) but not on their derivatives, at this step we can omit all the derivatives of the background strengths and replace \(\Box\) by \(\Box + 2WW\). As the result, the harmonic delta function becomes free of derivatives, and after integration over one set of harmonics (e.g. over \(u_2\)) one obtains

\[
\Gamma_2 = -2 \int d^2z \int du_1 \frac{1}{(\Box + 2WW)^4} \delta^4(x - x') |_{x = x'} q^{+\alpha}(u_1) q^{-\alpha}(u_1) . \tag{3.9}
\]

After performing Fourier transformation this correction can be written as

\[
\Gamma_2 = -2 \int d^2z \int du_1 q^{+\alpha}(u_1) q^{-\alpha}(u_1) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(-k^2 + 2WW)^3} . \tag{3.10}
\]

Doing the momenta integral, we finally cast it in the form

\[
\Gamma_2 = -2 \int d^2z \int du_1 q^{+\alpha}(u_1) q^{-\alpha}(u_1) \frac{1}{2(4\pi)^2WW} . \tag{3.11}
\]

This expression precisely matches with the general result (2.18) – (2.20) obtained by invoking ‘hidden’ \(\mathcal{N} = 2\) supersymmetry [13].

Let us turn to the generic case of \(n > 1\). Since we are interested in contributions which do not depend on space-time derivatives of background hypermultiplets, we can regard the latter to be space-time constants.

To simplify the expression (3.4), we proceed as follows. First, we represent \(q^+(u_1)\) in (3.4) as \(q^+(u_1) = D_1^{++} q^-(u_1)\) (this is possible since \(q^+\) sits on its free mass shell)

\[
\Gamma_{2n} = \frac{4(-1)^n}{2^{2n}n!} \int d^2z \int du_1 \ldots du_n \frac{1}{\Box (u_1) \Box (u_2) \ldots \Box (u_{n-1}) \Box (u_n)}
\times \frac{1}{(u_1^+ u_2^+)^4 (u_2^+ u_3^+)^4 \ldots (u_n^+ u_1^+)^4} \delta^{12}(z - z') |_{z = z'}
\times q^{+\alpha}(u_1) D_1^{++} q^{-\alpha}(u_1) q^{+\beta}(u_2) \ldots q^{-\alpha}(u_n) . \tag{3.12}
\]

Then, integrating by parts and using the mass-shell condition \(D_1^{++} q^{+\alpha}(u_1) = 0\), we throw the harmonic derivative \(D_1^{++}\) on the harmonic factor

\[
1 \quad \frac{1}{(u_1^+ u_2^+)^3 (u_2^+ u_3^+)^3 \ldots (u_n^+ u_1^+)^3} . \tag{3.13}
\]

We note that acting of \(D_1^{++}\) on \(\Box (u_1)\) would lead only to terms which are proportional to derivatives of the background superfield strengths and so are irrelevant for our purpose of calculating the leading contributions. The result of acting of \(D_1^{++}\) on (3.13) is

\[
\frac{1}{(u_1^+ u_2^+)^3 (u_2^+ u_3^+)^3 \ldots (u_n^+ u_1^+)^3} = -\frac{1}{2} [(D_2^-)^2 \delta^{(2,-2)}(u_1, u_2) \frac{1}{(u_1^+ u_2^+)^3} + (u_2 \leftrightarrow u_n)]
\times \frac{1}{(u_2^+ u_3^+)^3 \ldots (u_{n-1}^+ u_n^+)^3} . \tag{3.14}
\]
After substituting (3.14) in (3.12) we obtain

\[
\Gamma_{2n} = \frac{4(-1)^n2^n}{n} \int d^{12}z \int du_1 \ldots du_n \frac{1}{\mathcal{D}(u_1) \mathcal{D}(u_2) \ldots \mathcal{D}(u_{n-1}) \mathcal{D}(u_n)}
\]

\[
\times \frac{1}{2} \left[ (D_2^-)^2 \delta^{(2,-2)}(u_1, u_2) \frac{1}{(u_1^+ u_2^+)^3} + (u_2 \rightarrow u_n) \right] \frac{1}{(u_2^+ u_3^+)^3 \ldots (u_{n-1}^+ u_n^+)^3}
\]

\[
\times (D^+(u_1))^4(D^+(u_2))^4 \ldots (D^+(u_n))^4 \delta^{12}(z - z')_{z=z'}
\]

\[
\times q^{+a}(u_1)q^{-b}(u_1)q^{+c}(u_2) \ldots q^{+d}(u_n)q^{+a}(u_n).
\] (3.15)

Then we take the factor \((D_2^-)^2\) off the delta function. It is easy to see that this factor can give a non-vanishing result only when hitting \((D^+(u_2))^4\). Then we relabel \(u_2 \leftrightarrow u_n\) in the second term in the square bracket in (3.14), after which it becomes identical to the first one. Doing the same integral over \(u_1\), we arrive at the following intermediate expression for \(\Gamma_{2n}\)

\[
\Gamma_{2n} = -\frac{4(-1)^n2^n}{n} \int d^{12}z \int du_2 \ldots du_n \frac{1}{\mathcal{D}(u_2) \mathcal{D}(u_2) \ldots \mathcal{D}(u_{n-1}) \mathcal{D}(u_n)}
\]

\[
\times \frac{1}{(u_2^+ u_3^+)^3 \ldots (u_{n-1}^+ u_n^+)^3}
\]

\[
\times (D^+(u_2))^4[(D_2^-)^2(D^+(u_2))^4] \ldots (D^+(u_n))^4 \delta^{12}(z - z')_{z=z'}
\]

\[
\times q^{+a}(u_2)q^{-b}(u_2)q^{+c}(u_2) \ldots q^{+d}(u_n)q^{+a}(u_n).
\] (3.16)

At the next step we repeat above procedure for \(q^+(u_2)\) in (3.16), i.e. represent it in the form \(q^+_c(u_2) = D^+_2 q^-_c(u_2)\) and integrate by parts with respect to \(D^+_2\). Performing the same manipulations as in deriving (3.16), we bring \(\Gamma_{2n}\) into the form

\[
\Gamma_{2n} = (-1)^2 \frac{4(-1)^n2^n}{n} \int d^{12}z \int du_3 \ldots du_n \frac{1}{\mathcal{D}(u_2) \mathcal{D}(u_2) \ldots \mathcal{D}(u_{n-1}) \mathcal{D}(u_n)}
\]

\[
\times \frac{1}{(u_3^+ u_4^+)^3 \ldots (u_{n-1}^+ u_n^+)^3}
\]

\[
\times (D^+(u_3))^4[(D_3^-)^2(D^+(u_3))^4] \ldots (D^+(u_4))^4 \delta^{12}(z - z')_{z=z'}
\]

\[
\times q^{+a}(u_3)q^{-b}(u_3)q^{+c}(u_3) \ldots q^{+d}(u_n)q^{+a}(u_n).
\] (3.17)

Note that \(D^+_2\), when hitting \((D^+(u_2))^4[(D_2^-)^2(D^+(u_2))^4]\), gives rise to the structures like \((D^+(u))^4(D^+(u))^3 = 0\). After \(k\) analogous steps, we have

\[
\Gamma_{2n} = (-1)^k \frac{4(-1)^n2^n}{n} \int d^{12}z \int du_{n-k} \ldots du_n \frac{1}{\mathcal{D}^k(u_{k+1}) \ldots \mathcal{D}(u_{n-1}) \mathcal{D}(u_n)}
\]

\[
\times \frac{1}{(u_{k+1}^+ u_{k+2}^+)^3 \ldots (u_{n-1}^+ u_n^+)^3}
\]

\[
\times (D^+(u_k))^4[(D_k^-)^2(D^+(u_k))^4] \ldots (D^+(u_{k+1}))^4 \delta^{12}(z - z')_{z=z'}
\]

\[
\times q^{+a}(u_{n-k})q^{-b}(u_{n-k})q^{+c}(u_{n-k}) \ldots q^{+d}(u_{n-k})q^{+a}(u_{n-k}).
\] (3.18)
Repeating the same procedure further, we can reduce the harmonic integral to that over three sets of harmonics, \( u, u' \) and \( u'' \):

\[
\Gamma_{2n} = \frac{4(-1)^{2n-3}2^n}{n} \int d^{12}z \int dudu'' \frac{1}{\hat{\Box}_u^{n-2} \hat{\Box}_u' \hat{\Box}_u''} \times \left[(\mathcal{D}^+(u))^4(D^-)^2\right]^{n-3} (\mathcal{D}^+(u))^4(\mathcal{D}^+(u'))^4(\mathcal{D}^+(u''))^4\delta^{12}(z - z')|_{z = z'}
\]

\[
\times \left. \frac{1}{(u + u' + u'')^3} q^+ a(u)q^- b(u) \ldots q^+ e(u)q^- c(u)q^+ d(u)q^- e(u)q^+ a(u'') \right). 
\]

Then, using (3.2), we can reduce the number of spinor derivatives in the numerator by the relation

\[
[(\mathcal{D}^+(u))^4(D^-)^2]^{n-3} = (-2)^{n-3} \hat{\Box}_u^{n-3}. 
\]

After this we once again effect the previous procedure by writing \( q^+_c(u) = D^+_u q^- c(u) \) and throwing \( D^+ u'' \) on the harmonic factor. Repeating the same steps as above and using at the last step the relation (3.20), we can perform the \( u' \)-integration, thus arriving at the expression

\[
\Gamma_{2n} = -(-2)^n \int d^{12}z \int dudu'' \frac{1}{\hat{\Box}_u^{n-2} \hat{\Box}_u' \hat{\Box}_u''} \left[(\mathcal{D}^+(u))^4(\mathcal{D}^+(u''))^4\delta^{12}(z - z')|_{z = z'}
\times q^+ a(u)q^- b(u) \ldots q^+ e(u)q^- c(u)q^+ d(u)q^- e(u)q^+ a(u'') \right). 
\]

Now we suppress all the derivative-depending terms in \( \hat{\Box} \) by replacing \( \hat{\Box} \) with \( \Box + 2WW \). After this we use the identity (3.7). We find

\[
\Gamma_{2n} = -(-2)^n \int d^{12}z \int dudu'' \frac{1}{\Box + 2WW} \delta^4(x - x')|_{x = x'}
\times \frac{1}{(u + u' + u'')^2} q^+ a(u)q^- b(u) \ldots q^+ e(u)q^- c(u)q^+ d(u)q^- e(u)q^+ a(u'') \right). 
\]

The last step is to rewrite \( q^+_a(u'') = D^+_u q^- a(u'') \) and to throw \( D^+_u \) on \( 1/(u + u'')^2 \), which gives

\[
D^+_u q^- a(u'') = D^+_u \delta^{(2,-2)}(u'', u) = -D^+_u \delta^{(0,0)}(u'', u).
\]

Then we throw \( D^+_u \) on \( q^+_c(u) \), integrate over \( u'' \) and perform the Fourier transformation. The expression for the contribution of 2n-th order in hypermultiplets which we obtain at this step is as follows

\[
\Gamma_{2n} = (-2)^n \int d^{12}z \int du \int \frac{d^4k}{(2\pi)^4} \frac{1}{(-k^2 + 2WW)^{n+2}} \times q^+ a(u)q^- b(u) \ldots q^+ e(u)q^- a(u) \right).
\]

Since (in Minkowski space)

\[
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(-k^2 + 2WW)^{n+2}} = \frac{1}{(4\pi)^2 \Gamma(n+2)} \frac{\Gamma(n)}{(2WW)^n},
\]
we find the final expression in the form

\[ \Gamma_{2n} = \frac{(-2)^n}{n^2(n+1)(4\pi)^2} \int d^{12}z \frac{1}{(WW)^n} (q^{+a}(u)q_{a}^{-}(u))^n. \]  

(3.25)

This expression precisely coincides with the result \((2.18) - (2.20)\) obtained from the requirement of \(\mathcal{N} = 4\) supersymmetry. This expression can be represented as

\[ \Gamma_{2n} = \frac{1}{n^2(n+1)(4\pi)^2} \int d^{12}z X^n, \]  

(3.26)

where

\[ X = -2 \frac{q^{+a}q_{a}^{-}}{WW}. \]  

(3.27)

(note that in the central basis of \(\mathcal{N} = 2\) harmonic superspace the hypermultiplet superfields are expressed on shell as \(q^{\pm a} = q^{ia}(z)u_{\pm i}\), and so \(X\) and \(\Gamma_{2n}\) do not depend on harmonics \([13]\)). The result for \(n = 1\), eq. (3.11), is also incorporated by the general expression (3.26).

Thus the total one-loop effective action in the hypermultiplet sector explicitly computed using the harmonic supergraph techniques is given by the expression \((2.18) - (2.20)\) anticipated on the \(\mathcal{N} = 4\) supersymmetry ground in \([13]\). Combined with the non-holomorphic effective potential, it provides the exact \(\mathcal{N} = 4\) supersymmetric low-energy \(\mathcal{N} = 4\) SYM effective action \((2.17)\) \([13]\).

4 A manifestly \(\mathcal{N} = 2\) supersymmetric calculation of the non-holomorphic effective potential of \(W, \bar{W}\)

Here we demonstrate that the techniques developed in section 3 are equally applicable to calculating the non-holomorphic effective potential \((1.1)\), with taking into account some specific features of this problem. The calculation is carried out entirely within the \(\mathcal{N} = 2\) harmonic superfield formalism, with preserving manifest \(\mathcal{N} = 2\) supersymmetry\(^3\).

To compute the one-loop low-energy effective action of \(W, \bar{W}\) in the \(\mathcal{N} = 2\) harmonic background field method, we again make the splitting \(V^{++} \rightarrow V^{++} + g v^{++}\) in (2.1), with \(V^{++}\) being a background field and \(v^{++}\) a quantum one. This part of the full effective action of the theory under consideration is determined by the contributions from the pure \(\mathcal{N} = 2\) SYM sector, hypermultiplet sector, Faddeev-Popov (FP) ghosts and third ghosts. However, the contributions from the hypermultiplet and FP ghost sectors in the one-loop approximation are known to exactly cancel each other \([10]\) (see also \([3]\)). The reason for this cancellation is as follows. It was shown in \([17]\) that the action of FP ghosts (with choosing the background-gauge covariant version of Fermi gauge for the quantum superfields \(v^{++}\)) has the same form as

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\(^3\)A different method of manifestly \(\mathcal{N} = 2\) supersymmetric calculation of the potential \((1.1)\) was developed in \([20]\) using proper-time techniques.
the action of massless hypermultiplet in the adjoint representation. The $\mathcal{N} = 4$ SYM theory is $\mathcal{N} = 2$ SYM plus a hypermultiplet in adjoint representation. So the contribution of FP ghosts to $\mathcal{N} = 4$ SYM one-loop effective action is equal to that of hypermultiplets taken with the opposite sign (hypermultiplets are bosons, while FP ghosts are fermions). Therefore, that part of the full one-loop effective action which depends only on $\mathcal{N} = 2$ gauge superfields is constituted by contributions from the pure $\mathcal{N} = 2$ SYM sector and the third ghosts sector.

The general form of this part of the one-loop effective action was found in [5, 6, 10]:

$$\Gamma^{(1)} = \frac{i}{2} \text{Tr}(2,2) \ln \Box - \frac{i}{2} \text{Tr}(4,0) \ln \Box .$$  \hspace{1cm} (4.1)$$

Here the symbol $\text{Tr}$ means both the functional trace and the one with respect to matrix indices. The operator $\Box$ is given by the on-shell expression (2.15), with $W, \bar{W}$ being matrices which take values in the Cartan subalgebra of $su(2)$. The exact matrix structure of this operator will be explored further. For our calculation in this Section it will be important that $\Box$ includes both the $W\bar{W}$ and derivative-depending terms. First contribution in (4.1) originates from the $\mathcal{N} = 2$ SYM sector, and the second one is determined by the third ghosts $\rho^{(+4)}, \sigma$ [6]. The propagator of these ghosts reads [6]

$$G^{(4,0)}(1, 2)_{\rho^{(+4)}(1)}^{\sigma(2)} = (D^{++})^2 G^{(0,0)}(1, 2) ,$$ \hspace{1cm} (4.2)$$

where $G^{(0,0)}$ is the propagator of an uncharged analytic superfield $\omega$ [6]. The precise form of $G^{(0,0)}$ will be of no need for our further purposes.

Let us consider a theory of uncharged analytic superfield $\sigma$ with the action

$$-\frac{1}{2} \text{tr} \int d\zeta (-4) \sigma (D^{++})^2 \Box \sigma .$$ \hspace{1cm} (4.3)$$

This theory leads to the same Green function (4.2) and hence to the same contribution to the effective action (4.1). Therefore the action (4.3) can be treated as an alternative form of the action of third ghosts. It will prove to be more convenient for our purposes. Hereafter, the symbol ‘tr’ denotes trace over matrix indices only.

Thus the quadratic action of quantum superfields is given by

$$-\frac{1}{2} \text{tr} \int d\zeta (-4) v^{++} \Box v^{++} - \frac{1}{2} \text{tr} \int d\zeta (-4) \sigma (D^{++})^2 \Box \sigma .$$ \hspace{1cm} (4.4)$$

As the next important step, we separate $v^{++}$ in the two orthogonal projections

$$v^{++} = v_T^{++} + D^{++} \xi ,$$  \hspace{1cm} (4.5)$$

$$D^{++} v_T^{++} = 0 .$$  \hspace{1cm} (4.6)$$

The corresponding projecting operator is given in [20] and it is a covariantization of the analogous operator introduced in [14, 15]. The covariantized projecting operator $\Pi_T^{(2,2)}(1, 2)$ is defined by the expression

$$v_T^{++}(1) = \int d\zeta_2 (-4) \Pi_T^{(2,2)}(1, 2) v^{++}(2) ,$$ \hspace{1cm} (4.7)$$

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and preserves the analyticity and the harmonic constraint in (4.6). Its explicit form can be found in [20].

After splitting $v^{++}$ as in (4.5), the action (4.4) can be rewritten as

$$-\frac{1}{2} \text{tr} \int d\zeta (-4) v^{++}_T \Box v^{++}_T + \frac{1}{2} \text{tr} \int d\zeta (-4) \xi (D^{++})^2 \Box \xi - \frac{1}{2} \text{tr} \int d\zeta (-4) \sigma (D^{++})^2 \Box \sigma .$$  (4.8)

We see that $\xi$ is a boson, while $\sigma$ is a fermion and they have the same actions. Clearly, their contributions to the one-loop effective action are equal up to the sign and hence cancel each other. Therefore it is sufficient to consider only the following part of the quadratic action:

$$S_2 = -\frac{1}{2} \text{tr} \int d\zeta (-4) v^{++}_T \Box v^{++}_T .$$  (4.9)

Further, quantum fields $v^{++}_T$ with values in $su(2)$ can be written in the form (2.6) where $\tau_i$ are generators of $su(2)$ related to Pauli matrices $\sigma_i$ by (2.7). They satisfy the relations (2.8). The background $N = 2$ superfields are expanded as $W = W_i \tau_i$. We calculate the effective action with the background superfields taking values in the unbroken $u(1)$. This corresponds to the restriction $W = W_3 \tau_3$. As before, we denote $W_3$ by $W$.

The quadratic action of quantum superfields for the given background can be read off from (2.13), with the background hypermultiplets being put equal to zero:

$$S_2 = -\int d\zeta (-4) \left\{ \chi^{++}_T (\Box + 2W \bar{W}) \chi^{++}_T - \frac{1}{\sqrt{2}} \chi^{++}_T ((D^{+\alpha} W) D^-_{\alpha} + (D^{+\bar{\alpha}} W) \bar{D}^\alpha) \chi^{++}_T \right\} .$$  (4.10)

The one-loop effective action corresponding to (4.10) is [20]

$$\Gamma^{(1)} = i Tr^T_{(2,2)} \ln \Box ,$$  (4.11)

where the symbol $Tr^T$ means that the trace is taken over subspace of the analytic superfields $v^{++}_T$ satisfying the constraint $D^{++} v^{++}_T = 0$. Hereafter the operator $\Box$ acting on $\chi^{++}_T$ is given by the expression (2.13). Eq. (4.11) leads to the following expression for the one-loop effective action [20]:

$$\Gamma^{(1)} = i \int d\zeta (-4) (\ln \Box) \Pi^{(2,2)}_T (1, 2) |_{\zeta_1 = \zeta_2, u_1 = u_2} .$$  (4.12)

The projector $\Pi^{(2,2)}_T$ (1.7) at the coinciding harmonics is given by the expression [20]

$$\Pi^{(2,2)}_T (1, 2) |_{u_1 = u_2} = (D^+_1)^4 \delta^{12} (z_1 - z_2) .$$  (4.13)

Therefore the one-loop effective action is

$$\Gamma^{(1)} = i \int d\zeta (-4) (D^+_1)^4 (\ln \Box) \delta^{12} (z_2 - z_1) |_{z_1 = z_2} .$$  (4.14)

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4I.L.B. is very thankful to S.M. Kuzenko for clarifying the technical details of calculations in Ref. [20], especially those related to the structure and properties of the projection operator $\Pi^{(2,2)}_T$.  

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Let us consider the general structure of the quantum corrections. We substitute the expression \( \bar{\partial} = \Box + 2W\bar{W} + R \) in (4.14) and obtain

\[
\Gamma^{(1)} = i \int d\zeta^{(-4)} (\mathcal{D}_1^+)^4 \ln(\Box + 2W\bar{W} + R) \delta^{12}(z_2 - z_1)|_{z_2 = z_1},
\]

where

\[
R \equiv -\frac{1}{\sqrt{2}}((\mathcal{D}^+\alpha W)\mathcal{D}_\alpha^- + (\mathcal{D}^+\bar{\alpha}\bar{W})\mathcal{D}_\bar{\alpha}^-).
\]

Note that the expression (4.15) is free of the harmonic singularities since it does not contain harmonic delta functions which could lead, in principle, to appearance of such singularities in harmonic supergraphs. To find the effective action, we expand (4.15) in power series with respect to \( R \), i.e. spinor derivatives of \( W, \bar{W} \)

\[
\ln(\Box + 2W\bar{W} + R) = \ln(\Box + 2W\bar{W}) + \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right)^{n-1} \left( \frac{R}{\Box + 2W\bar{W}} \right)^n.
\]

We obtain

\[
\Gamma^{(1)} = i \text{tr} \int d\zeta^{(-4)} (\mathcal{D}_1^+)^4 \left\{ \ln(\Box + 2W\bar{W}) + \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right)^{n-1} \left( \frac{1}{\Box + 2W\bar{W}} \right)^n R^n \right\} \delta^{12}(z_2 - z_1)|_{z_2 = z_1}.
\]

We observe that the first term in (4.18), i.e. the one containing \( \ln(\Box + 2W\bar{W}) \) carries only four spinor derivatives coming from \((\mathcal{D}^+)^4\). On the other hand, shrinking any \( \theta \) loop into a point in \( \theta \)-space by the rule (3.7) requires the presence of at least eight spinor derivatives. Hence, the first term in (4.18) cannot give contribution to the non-holomorphic effective potential.

The expansion of \( \Gamma^{(1)} \) given by (4.18) with the first term omitted can be represented by a sequence of harmonic supergraphs. Each supergraph is a ring with \( n \) vertices and \( n \) internal lines. The internal lines are represented by the propagators \((\Box + 2W\bar{W})^{-1} \delta^{12}(z_1 - z_2)\) defined in full \( \mathcal{N} = 2 \) superspace. There is one common factor \((\mathcal{D}^+)^4\), and at each vertex an external line \( \mathcal{D}^+W \) or \( \mathcal{D}^+\bar{W} \) is attached, with the factors \( \mathcal{D}^- \) or \( \mathcal{D}^- \), respectively. The analyticity of the contribution of any supergraph is guaranteed by the factor \((\mathcal{D}^+)^4\). Thus the contribution of a generic supergraph contains \((\mathcal{D}_1^+)^4\) from (4.18) and \( n \) \( \mathcal{D}^- \) factors, each being associated with each of \( n \) vertices.

Our aim is to calculate the contribution to the nonholomorphic effective potential which has the form \( \int d^{12}zduF(W,\bar{W}) \), with \( F(W,\bar{W}) \) being some function of the background strengths \( W,\bar{W} \), but not of their derivatives. We can rewrite this contribution as an integral over the analytic harmonic subspace by the rule

\[
\int d^{12}zdu F(W,\bar{W}) = \int d\zeta^{(-4)} (\mathcal{D}^+)^4 F(W,\bar{W}).
\]

This representation shows that the contribution to the nonholomorphic effective potential rewritten as an integral over the analytic subspace contains exactly two chiral derivatives and
two antichiral ones acting on the background $W, \bar{W}$. To extract such corrections we must represent all vertices but one as integrals over $d^{12}z$, and keep one vertex in the form of integral over $d\zeta$. From (4.18) we can derive that the contribution of an arbitrary supergraph with $n = a + b$ vertices, $a$ vertices containing $\mathcal{D}W$ and $b$ vertices containing $\bar{\mathcal{D}}\bar{W}$, is proportional to the expression

$$
\Gamma_n \propto \int d\zeta (-4) (\mathcal{D}W)^a (\bar{\mathcal{D}}\bar{W})^b [\mathcal{D}^a \bar{\mathcal{D}}^b (\mathcal{D}^+)^4 \delta^{12}(z - z')|_{z = z'}].
$$

Since for obtaining the contribution to nonholomorphic effective potential we must keep only the terms containing four supercovariant derivatives on $W, \bar{W}$ (two chiral and two antichiral ones), we are led to specialize to the case of $a = b = 2$. For $a, b > 2$ we would obtain contributions which remain to be derivative-dependent after representing them as integrals over $d^{12}z$. For $a, b < 2$ we would have not enough $\mathcal{D}$-factors to shrink a $\theta$-loop into a point. Therefore the only terms in the expansion of $\ln \bar{\Box}$ which can be relevant for our purpose are those of the second order in both $\mathcal{D}W$ and $\bar{\mathcal{D}}\bar{W}$. There is only one such term in the expansion of $\ln \bar{\Box}$. It is the $n = 4$ term the explicit form of which is

$$
- \left( \frac{1}{\Box + 2W\bar{W}} \right)^4 \frac{1}{16} \left[ (\mathcal{D}^{+a}W)\mathcal{D}_a^- + (\bar{\mathcal{D}}^{+\dot{a}}\bar{W})\bar{\mathcal{D}}_{\dot{a}}^- \right]^4
$$

$$
= - \frac{6}{16} \left( \frac{1}{\Box + 2W\bar{W}} \right)^4 \left[ (\mathcal{D}^{+a}W)\mathcal{D}_a^- \right]^2 \left[ (\bar{\mathcal{D}}^{+\dot{a}}\bar{W})\bar{\mathcal{D}}_{\dot{a}}^- \right]^2 + \ldots . \tag{4.21}
$$

The coefficient $-1/16$ arose due to $-1/4$ from the expansion of logarithm and the factor $(1/\sqrt{2})^4$. The coefficient 6 appeared from the binomial expansion $\left[ i(\mathcal{D}^{+a}W)\mathcal{D}_a^- + i(\bar{\mathcal{D}}^{+\dot{a}}\bar{W})\bar{\mathcal{D}}_{\dot{a}}^- \right]^4 = 6((\mathcal{D}^{+a}W)\mathcal{D}_a^-)^2((\bar{\mathcal{D}}^{+\dot{a}}\bar{W})\bar{\mathcal{D}}_{\dot{a}}^-)^2 + \ldots$. Here and in (4.21) dots stand for terms giving zero trace.

After keeping only the $n = 4$ term (4.21) in $\Gamma^{(1)}$, eq. (4.18), the latter can be represented by a four-point supergraph of the form

![Diagram](attachment:diagram.png)

Here external legs represent derivatives of superfield strengths, while bold internal lines stand for the “free” propagators $1/(\Box + 2W\bar{W}) \delta^{12}(z_1 - z_2)$. The full contribution of this graph is given by the following expression:

$$
\Gamma_4 = - \frac{6}{16} i \int d\zeta (-4) d^{12}z_2 d^{12}z_3 d^{12}z_4 (\mathcal{D}^{+a}(u_1)W(x_1, \theta_1))\mathcal{D}_a^-(u_1)(\bar{\mathcal{D}}^{+\dot{a}}(u_1)\bar{W}(x_2, \theta_2))\bar{\mathcal{D}}_{\dot{a}}^-(u_1)
$$
We observe that this expression contains only one integral over harmonics $u_1$ (recall that the harmonic integral is included into $d\zeta_1^{(-4)}$) and does not involve harmonic delta functions due to the structure of (4.15), (4.18). Therefore it is automatically free of harmonic singularities. Then we anticommutate $D^{-\alpha}(u)$ with $D_\alpha (u)$, which produces factor $-1$. As the result we obtain

$$
\Gamma_4 = \frac{6}{16} i \int d\zeta_1^{(-4)} d^4 x d^4 y d^4 z d^4 \bar{z} (D^{\alpha} (u_1) \bar{W}(x_1, \theta_1))(D^{+\alpha} (u_1) \bar{W}(x_2, \theta_2))
$$

$$
\times (D^{+\beta} (u_1) W(x_3, \theta_3))(D^{\beta} (u_1) W(x_4, \theta_4)) (D_{\alpha}^{+} (u_1)) (D_{\alpha}^{-} (u_1)) (D^{\beta} (u_1)) (D^{\alpha} (u_1))^4 \delta^8 (\theta_1 - \theta_2)
$$

Further we integrate over $\theta_3, \theta_4$ with the help of delta functions and shrink a loop into a point by the rule

$$
\delta^8 (\theta_1 - \theta_2) (D^{\alpha} (u_1)) (D^{\beta} (u_1)) (D_{\alpha}^{+} (u_1)) (D_{\alpha}^{-} (u_1)) (D^{\beta} (u_1)) (D^{\alpha} (u_1))^4 \delta^8 (\theta_1 - \theta_2) = 4 \epsilon_{\alpha \beta} \epsilon^{\alpha \beta} \delta^8 (\theta_1 - \theta_2).
$$

The (non-holomorphic) effective potential by definition is an effective Lagrangian with the background strengths slowly varying in space-time. Therefore we can put

$$
(D^{+\alpha} (u_1) W(x_1, \theta))(D^{\alpha} (u_1) W(x_2, \theta))(D^{+\beta} (u_1) \bar{W}(x_3, \theta))(D^{\beta} (u_1) \bar{W}(x_4, \theta))
$$

$$
\simeq (D^{+\alpha} (u_1) W(x_1, \theta))(D^{\alpha} (u_1) W(x_1, \theta))(D^{+\beta} (u_1) \bar{W}(x_1, \theta))(D^{\beta} (u_1) \bar{W}(x_1, \theta)).
$$

After this we perform Fourier transformation which leads to

$$
\Gamma_4 = -\frac{6}{4} i \int d\zeta_1^{(-4)} (D^{+\alpha} (u_1) W)^2 (x, \theta)(D^{\alpha} (u_1) \bar{W})^2 (x, \theta) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - 2W)^4}.
$$
After doing the loop integration by the rule

$$
\int \frac{d^4k}{(2\pi)^4 (k^2 - 2WW)^4} = \frac{1}{6} i \frac{1}{(4\pi)^2} \frac{1}{(2WW)^2}
$$

we have

$$
\Gamma_4 = \frac{1}{16} \int d\zeta^{(-4)} (D^{+\alpha}(u_1)W)^2 (D_{\dot{\alpha}}^{+}(u_1)\bar{W})^2 \frac{1}{(WW)^2} \frac{1}{(4\pi)^2} \frac{1}{(W\bar{W})^2}.
$$

Then we note that

$$
(D^{+\alpha}(u_1)W)^2 (D_{\dot{\alpha}}^{+}(u_1)\bar{W})^2 \frac{1}{(WW)^2} = 16 (D^{+}(u_1))^4 \left( \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda} \right),
$$

whence

$$
\Gamma_4 = \frac{1}{(4\pi)^2} \int d\zeta^{(-4)} (D^{+}(u_1))^4 \left( \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda} \right).
$$

Finally, we pass to the integration over the full harmonic superspace, \( \int d\zeta^{(-4)} (D^{+}(u_1))^4 = \int d^{12}zu_1 \), and take into account that the \( u_1 \) integral in (4.31) can be taken away because the integrand does not depend on harmonics. Hence we finally obtain the complete one-loop non-holomorphic effective potential just in the form (1.1) given in [4, 3, 6].

We conclude that the non-holomorphic potential of \( W, \bar{W} \), like the hypermultiplet-dependent part of the full effective action, can be self-consistently derived in the framework of the quantum harmonic superspace approach.

## 5 Summary

To summarize, in this paper we addressed the problem of calculating \( \mathcal{N} = 4 \) supersymmetric low-energy effective action of \( \mathcal{N} = 4 \) \( SU(2) \) SYM theory in the Coulomb branch formulated in harmonic superspace as \( \mathcal{N} = 2 \) SYM theory coupled to the hypermultiplet in adjoint representation of gauge group. We have developed a universal procedure of computing both the hypermultiplet-dependent and purely \( \mathcal{N} = 2 \) SYM parts of the effective action, based on the covariant harmonic supergraph techniques ensuring a manifest \( \mathcal{N} = 2 \) supersymmetry at every stage of calculation. The directly computed hypermultiplet part of the effective action was proved to coincide with the expression found earlier in [13] by invoking the requirement of hidden \( \mathcal{N} = 2 \) supersymmetry which completes the manifest \( \mathcal{N} = 2 \) one to the full \( \mathcal{N} = 4 \) supersymmetry. Thus the direct quantum computation reproduces the effective Lagrangian found in [13]. Also, we demonstrated that the same \( \mathcal{N} = 2 \) covariant harmonic supergraph techniques allow one to derive the non-holomorphic \( W, \bar{W} \) potential. We conclude that our approach sets up a generic manifestly \( \mathcal{N} = 2 \) supersymmetric framework for analysing the dependence of the full low-energy effective action of \( \mathcal{N} = 4 \) SYM theory on all fields of \( \mathcal{N} = 4 \) gauge multiplet. The results of [13] and this paper can be regarded as providing the ultimate
solution to the problem of constructing the manifestly $\mathcal{N} = 2$ supersymmetric exact low-energy effective action in $\mathcal{N} = 4$ SYM theory. Whereas we considered here the simplest case of the gauge group $SU(2)$, it is straightforward to extend our study to the general case of gauge group $SU(N)$ broken down to $U(1)^{N-1}$.

It would be interesting to find, using both the quantum and algebraic methods, the full $\mathcal{N} = 4$ supersymmetric form of some subleading $W, \bar{W}$ terms in the effective action of $\mathcal{N} = 4$ SYM theory, e.g. of those studied in [18, 19]. This could provide further checks of the supergravity/super Yang-Mills correspondence and be of direct relevance to the closely related problem of constructing $\mathcal{N} = 4$ superconformally invariant extension of the Dirac-Born-Infeld theory in $\mathcal{N} = 2$ superfield formulation [24].

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References

[1] E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, Physics of Particles and Nuclei, 32 (2001) 641.

[2] E. D’Hoker, D.Z. Freedman, ‘Supersymmetric Gauge Theories and the ADS/CFT Correspondence’, Lectures given at Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions, Boulder, Colorado, 3-29 June 2001; hep-th/0201253.

[3] J.K. Erickson, G.W. Semenoff, K. Zarembo, Nucl. Phys. B582 (2000) 155; hep-th/0003055; N. Drukker, D.J. Gross, J. Math. Phys. 42 (2001) 2896; hep-th/0010274; M. Bianchi, M.B. Green, S. Kovacs, JHEP 0204 (2002) 040; hep-th/0202003.

[4] M. Dine, N. Seiberg, Phys. Lett. B409 (1997) 239; hep-th/9705054.

[5] V. Periwal, R. von Unge, Phys. Lett. B430 (1998) 71; hep-th/9801121; F. Gonzalez-Rey, M. Rocek, Phys. Lett. B434 (1998) 303; hep-th/9804010.
[6] I.L. Buchbinder, S.M. Kuzenko, Mod. Phys. Lett. A13 (1998) 1629; hep-th/9804168.

[7] B. de Wit, M.T. Grisaru, M. Roček, Phys. Lett. B374 (1996) 297; hep-th/9601113; U. Lindstrom, F. Gonzalez-Rey, M. Roček, R. von Unge, Phys. Lett. B339 (1996) 581; hep-th/9607089.

[8] D.A. Lowe, R. von Unge, JHEP 9811 (1998) 014; hep-th/9811017.

[9] I. Chepelev, A.A. Tseytlin, Nucl. Phys. B511 (1998) 629; hep-th/9705120; F. Gonzalez-Rey, B. Kulik, I.J. Park, M. Roček, Nucl. Phys. B544 (1999) 218; hep-th/9810152.

[10] E.I. Buchbinder, I.L. Buchbinder, S.M. Kuzenko, Phys. Lett. B446 (1999) 216; hep-th/9810239.

[11] M. Dine, J. Gray, Phys. Lett. B481 (2000) 427; hep-th/9909020.

[12] I.L. Buchbinder, A.Yu. Petrov, Phys. Lett. B482 (2000) 429; hep-th/0003265.

[13] I.L. Buchbinder, E.A. Ivanov, Phys. Lett. B524 (2002) 208; hep-th/0111062.

[14] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 1 (1984) 469; A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 2 (1985) 601; Class. Quant. Grav. 2 (1985) 617.

[15] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, ‘Harmonic Superspace’, Cambridge, UK: Univ. Press (2001) 306 p.

[16] I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B412 (1997) 309; hep-th/9703147; E.A. Ivanov, S.V. Ketov, B.M. Zupnik, Nucl. Phys. B509 (1997) 52; hep-th/9706078; E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S. M. Kuzenko, Mod. Phys. Lett. A13 (1998) 1071; hep-th/9803176; S. Eremin, E. Ivanov, Mod. Phys. Lett. A15 (2000) 1859; hep-th/9908054; I.L. Buchbinder, I.B. Samsonov, Mod. Phys. Lett. A14 (1999) 2537; hep-th/9909183.

[17] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B417 (1998) 61; hep-th/9704214; I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B433 (1998) 335; hep-th/9710142.

[18] I.L. Buchbinder, S.M. Kuzenko, A.A. Tseytlin, Phys.Rev. D62 (2000) 045001; hep-th/9911221.

[19] I.L. Buchbinder, A.Yu. Petrov, A.A. Tseytlin, Nucl.Phys. B621 (2002) 179; hep-th/0110173.

[20] S.M. Kuzenko, I.N. McArthur, Phys. Lett. B506 (2001) 140; hep-th/0101127.

[21] S.M. Kuzenko, I.N. McArthur, Phys. Lett. B513 (2001) 213; hep-th/0105121.
[22] A. Galperin, Nguen Anh Ky, E. Sokatchev, Mod. Phys. Lett. A2 (1987) 33.

[23] H. Bateman, A. Erdelyi, ‘Higher Transcendental Functions’, vol.1, Mc Grow-Hill, 1953.

[24] E. Ivanov, ‘Towards Higher N Superextensions of Born-Infeld Theory’, hep-th/0202201.