Independence of $\ell$ for the supports in the Decomposition Theorem

Shenghao Sun*

Abstract

In this note, we prove a result on the independence of $\ell$ for the supports of irreducible perverse sheaves occurring in the Decomposition Theorem, as well as for the family of local systems on each support. It generalizes Gabber’s result on the independence of $\ell$ of intersection cohomology to the relative case.

1 Introduction

Let $f : X \to Y$ be a proper morphism of algebraic varieties over a separably closed field $k$, and let $IC_X(\mathbb{Q}_\ell)$ be the $\ell$-adic intersection complex on $X$. By the decomposition theorem [1], $Rf_*IC_X(\mathbb{Q}_\ell)$ splits into a direct sum of shifted semisimple perverse sheaves. It is then natural to expect that the supports of the irreducible perverse sheaves occurring in the decomposition are independent of $\ell \neq p$. This is clear if $\text{char } k = 0$, by reducing to the case where $k = \mathbb{C}$ and comparing with $\mathbb{C}$-coefficient.

The question can be raised from another perspective, and made more general. One has the following conjecture on “independence of $\ell$” under pushforwards. Given an $\mathbb{F}_q$-morphism $f : X \to Y$ between $\mathbb{F}_q$-schemes of finite type, and given a compatible system $\{\mathcal{F}_i\}_I$ of irreducible lisse sheaves on $X$ (see Definition 2.1 (i) below for definition), one predicts that for each integer $n$, the family of constructible sheaves $\{R^n f_* \mathcal{F}_i\}_I$ are stratified by a common stratification of $Y$ and that on each stratum, their restrictions are compatible; similarly for the other operations. Note that the assumption on lissity of $\mathcal{F}_i$ is necessary, since a lisse sheaf (which may not have global sections) could be compatible with the direct sum of a punctual sheaf (which always have global sections) and another constructible sheaf; similarly, the assumption on irreducibility rules out for instance nontrivial extensions of constant sheaves. Some results in this direction are given by Illusie [7], who showed, in particular, that there is a stratification $\{Y_i\}_i$ of $Y$, such that all the sheaves $R^n f_* \mathbb{Z}_\ell$ (as well as $R^n f_* \mathbb{Z}_p$), as $n$ and $\ell \neq p$ vary, are lisse on each $Y_i$ (see loc. cit., Corollaire 2.7).

Our main result Theorem 3.2 could be regarded as a variant of this conjecture with respect to the perverse $t$-structure, namely, we prove that if $f$ is proper, then for each $n$, the family of semisimplified perverse sheaves $pR^n f_*(\mathcal{F}_i)^{ss}$ is “independent of $i$”: the supports of their irreducible constituents are the same, and the family of local systems on each support is compatible. As a corollary, we have

Theorem 1.1. When $k$ is the algebraic closure of a finite field, the set of supports occurring in the decomposition of $Rf_*IC_X(\mathbb{Q}_\ell)$ is independent of $\ell$. Moreover, as $\ell$ varies, the local systems occurring in the decomposition over each stratum form a compatible system (with respect to some model defined over a finite subfield of $k$).

*Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China; email: shsun@math.tsinghua.edu.cn. Partially supported by China NSF grant (11531007).
This can be regarded as the relative version of a result of Gabber [6].

1.2. We fix some notations and conventions.

1.2.1. In most part of this article, we work over a finite field \( \mathbb{F}_q \) of characteristic \( p \), although a few concepts and results apply to more general base fields \( k \) over which we have the formalism of triangulated \( \ell \)-adic derived categories \( D^b_c(X, \overline{\mathbb{Q}}_\ell) \) (see, for instance, [1, 2.2.14]). The reader may take \( k \) to be always \( \mathbb{F}_q \) if he wants. Here "\( \ell \)" as well as "\( \ell_i \)" in the sequel, denote prime numbers different from \( p \).

We always assume \( k \)-schemes to be of finite type. We only work with middle perversity [1 Section 4].

1.2.2. Let \( X \) be a \( k \)-scheme. We say that \( X \) is essentially smooth if \( (X_{\mathbb{F}_q})_{\text{red}} \) is smooth over \( \overline{k} \). When \( k \) is perfect (e.g. a finite field), this is equivalent to \( X_{\text{red}} \) being regular. An \( \ell \)-adic lisse sheaf is sometimes called an \( \ell \)-adic local system.

1.2.3. Let \( X \) be a \( k \)-scheme, and let \( Z \) be an essentially smooth irreducible locally closed subscheme of \( X \). Let \( j : Z \to \overline{Z} \) be the open immersion of \( Z \) into its closure in \( X \). Let \( L \) be an \( \ell \)-adic lisse sheaf on \( Z \). We will denote the intermediate extension \( j_*L[\dim Z] \) (see [1 p. 58]), or its extension-by-zero to \( X \), by \( IC_{\overline{Z}}(L) \), and call it the intersection complex defined by the local system \( L \).

1.2.4. Let \( X \) be an irreducible \( k \)-scheme, \( j : U \to X \) an essentially smooth open subset of \( X \), and let \( L \) be an \( \ell \)-adic local system on \( U \). We say that \( U \) is the maximal support of \( L \), if for any essentially smooth open subset \( V \) of \( X \), such that the local system \( L|_{U \cap V} \) can be extended to a local system on \( V \), we have \( V \subset U \). Given \( L \) on \( U \), its maximal support always exists: it is the union of all essentially smooth open set of \( X \) over which the sheaf \( j_*L \) is lisse.

1.2.5. Let \( X \) be a \( k \)-scheme and let \( P \) be an \( \ell \)-adic perverse sheaf on \( X \), with irreducible constituents \( IC_{\overline{X}}(L_{\alpha}) \), where for each \( \alpha \), \( L_{\alpha} \) is an irreducible local system with maximal support \( X_{\alpha} \subset \overline{X}_{\alpha} \). Then we say that \( X_{\alpha} \) (resp. \( \overline{X}_{\alpha} \)) is a support (resp. a closed support) occurring in \( P \).

1.2.6. For \( K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \), let \( [K] \) (resp. \( p^*[K] \)) be its image in the Grothendieck group of \( \ell \)-adic sheaves (resp. \( \ell \)-adic perverse sheaves), i.e.

\[
[K] = \sum_n (-1)^n [H^n_K] \quad \text{(resp. } p^*[K] = \sum_n (-1)^n p^*[H^n_K])\).

Given \( K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \), there exist two semisimple perverse sheaves \( P^+ \) and \( P^- \) without common irreducible factors, such that

\[
p^*[K] = p^*[P^+] - p^*[P^-],
\]

and they are unique up to isomorphism. We shall call it the canonical representative of \( p^*[K] \).

1.2.7. Let \( X \) be an \( \mathbb{F}_q \)-scheme, and \( x \in X(\mathbb{F}_q^*) \) (or \( x \in |X| \), namely a closed point). Let \( \mathcal{F} \) be a \( \mathbb{Q}_\ell \)-sheaf on \( X \). We denote by \( P^\mathcal{F}_x(t) \) the polynomial

\[
\det(1 - \text{Frob}_x t, \mathcal{F}) \in \overline{\mathbb{Q}}_\ell[t],
\]

and define the \( L \)-function \( L(\mathcal{F}, t) \) to be

\[
L(\mathcal{F}, t) = \prod_{x \in |X|} \frac{1}{P^\mathcal{F}_x(t^{\deg(x)})},
\]
where \( \deg(x) = [k(x) : \mathbb{F}_q] \) is the residue degree of \( x \). Both extend to \( \mathcal{F} \in D_c^b : \)

\[
P^\mathcal{F}_{x}(t) := \prod_{i \in \mathbb{Z}} P^\mathcal{F}_{x}^{(i)}(t)^{(-1)^{i}} \in \overline{\mathbb{Q}}(t),
\]

and depends only on the image \([\mathcal{F}]\) of \( \mathcal{F} \) in the Grothendieck group \( K_0(Sh(X)) \).

1.2.8. Given \( b \in \overline{\mathbb{Q}}(t) \), denote by \( \overline{\mathbb{Q}}(t)^{(b)} \) the Weil sheaf on \( \text{Spec} \mathbb{F}_q \) of rank 1 on which the geometric Frobenius acts as multiplication by \( b \). Given an \( \mathbb{F}_q \)-scheme \( X \) with structural map \( a : X \to \text{Spec} \mathbb{F}_q \), and an \( \ell \)-adic Weil sheaf \( \mathcal{F} \) on \( X \), let \( \mathcal{F}^{(b)} \) be \( \mathcal{F} \otimes a^{*} \overline{\mathbb{Q}}^{(b)} \), and call it a Tate twist deduced from \( \mathcal{F} \).

1.2.9. For the perverse \( t \)-structure on \( k \)-Artin stacks (always assumed of finite presentation), see [9]; for the \( L \)-functions of \( \mathbb{F}_q \)-Artin stacks, see [11, Definition 4.1]. The other notions then all generalize without much difficulty to Artin stacks.

**Acknowledgement.** I would like to thank Ofer Gabber, Martin Olsson and Weizhe Zheng for helpful discussions. In particular, Weizhe Zheng showed me how to generalize the result to algebraic stacks. Part of this work was done during the stay in Université Paris-Sud (UMR 8628), supported by ANR grant G-FIB, and in IHÉS. This work is partially supported by China NSF grant (11531007).

## 2 Perverse compatible systems

In this section, we review the notion of compatible systems, and define its variant for the perverse \( t \)-structure, called perverse compatible systems, and show their relations.

Let \( \overline{\mathbb{Q}} \) be the subfield of algebraic numbers in \( \mathbb{C} \); by a number field we understand a subfield \( E \subset \overline{\mathbb{Q}} \) of finite degree over \( \mathbb{Q} \), hence it comes with an embedding \( E \hookrightarrow \mathbb{C} \).

Let \( E \) be a number field, and let \( I \) be a set of pairs \( i = (\ell_i, \sigma_i) \), where \( \ell_i \) is a rational prime number not equal to \( p \), and \( \sigma_i : E \hookrightarrow \overline{\mathbb{Q}}_{\ell_i} \) is an embedding of fields. In order to talk about weights \( \mathbb{Z} \), we fix once and for all an embedding of fields \( \iota : \overline{\mathbb{Q}}_{\ell_i} \to \mathbb{C} \), such that the composite

\[
E \xrightarrow{\sigma_i} \overline{\mathbb{Q}}_{\ell_i} \xrightarrow{\iota} \mathbb{C}
\]

is the given embedding \( E \hookrightarrow \mathbb{C} \). Given any \( \sigma_i \), such an \( \iota \) certainly exists (assuming the axiom of choice).

By Lafforgue’s work on Langlands correspondence [8], one can show that, for any \( \mathbb{F}_q \)-scheme \( X \) (or even \( \mathbb{F}_q \)-Artin stack of finite presentation), for any \( \ell \)-adic sheaf \( \mathcal{F} \) on \( X \), and for any embedding \( \iota : \overline{\mathbb{Q}}_{\ell} \to \mathbb{C} \), we have that \( \mathcal{F} \) is \( \iota \)-mixed; see [11, Remark 2.8.1]. In particular, by [8, Théorème 3.4.1], any irreducible \( \ell \)-adic lisse sheaf \( \mathcal{F} \) on \( X \) is punctually \( \iota \)-pure of some weight \( w \in \mathbb{R} \) (certainly, \( w \) may depend on \( \iota \)).

Part (i) in the following definition, at least for \( \mathcal{K}_i \) sheaves, is well-known; see for instance [10, Chapter I, Section 2.3] for number fields.

**Definition 2.1.** Let \( X \) be an \( \mathbb{F}_q \)-scheme, and for each \( i \in I \), let \( \mathcal{K}_i \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_i}) \).

(i) We say that \( \{ \mathcal{K}_i \}_I \) is a weakly \((E, I)\)-compatible system, if for every integer \( v \geq 1 \) and for every point \( x \in X(\mathbb{F}_{q^v}) \), there exists a number \( t_x \in E \) such that

\[
\sigma_i(t_x) = \text{Tr(Frob}_x, \mathcal{K}_i)
\]

for all \( i \in I \).

(ii) We say that \( \{ \mathcal{K}_i \}_I \) is a strongly \((E, I)\)-compatible system, if for each \( n \in \mathbb{Z} \), the system of cohomology sheaves \( \{ \mathcal{H}^n \mathcal{K}_i \}_I \) is a weakly \((E, I)\)-compatible system.
(iii) Assume that $K_i = P_i$ are perverse sheaves. We say that $\{P_i\}_I$ is *perverse $(E, I)$-compatible*, if there exist a finite number of essentially smooth irreducible locally closed subschemes $X_\alpha \hookrightarrow X$, and for each $\alpha$ a weakly $(E, I)$-compatible system $\{L^i_\alpha\}_I$ of semisimple local systems on $X_\alpha$, having $X_\alpha$ as their maximal support (inside $\overline{X}_\alpha$), such that

$$P^\text{ss}_i \simeq \bigoplus_\alpha IC_{\overline{X}_\alpha}(L^i_\alpha)$$

for all $i$. Here $P^\text{ss}_i$ denotes the semi-simplification of $P_i$ in the abelian category of perverse sheaves.

(iv) Once again, let $K_i \in D^b_c(X, \mathbb{Q}_{\ell_i})$. We say that $\{K_i\}_I$ is a *weakly perverse $(E, I)$-compatible system* if, letting

$$p[K_i] = p[P^+_i] - p[P^-_i]$$

be the canonical representatives (1.2.6), both $\{P^+_i\}_I$ and $\{P^-_i\}_I$ are perverse $(E, I)$-compatible.

(v) We say that $\{K_i\}_I$ is a *strongly perverse $(E, I)$-compatible system*, if for each $n \in \mathbb{Z}$, the system of perverse cohomology sheaves $\{\mathcal{H}^nK_i\}_I$ is perverse $(E, I)$-compatible.

These notions apply to $\mathbb{F}_q$-Artin stacks $X$ as well.

**Remark 2.2.** (i) Basically a “weak” version of the notion depends only on the image in the Grothendieck group, so there could be cancellations between cohomologies of even indices and odd indices, and the weak notion could not see shifts of indices by even integers. A “strong” version is for the cohomology objects, so cancellations and shifts are not allowed.

(ii) For a system $\{\mathcal{F}_i\}_I$ of sheaves, the notions (i) and (ii) in Definition 2.1 are the same, and we sometimes just say that $\{\mathcal{F}_i\}_I$ is an $(E, I)$-compatible system of sheaves.

(iii) The condition in Definition 2.1 (i) is equivalent to that for each closed point $x \in X$, there exists a rational function $P_x(t) \in E(t)$ such that

$$\sigma_i(P_x(t)) = P^K_x(t), \ \forall i \in I,$$

which is the definition given in [6, 1.2]. This is because formally we have

$$-i \frac{d}{dt} \log P^K_x(t) = \sum_{r=1}^\infty \text{Tr}(\text{Frob}^r_x, K)t^r.$$

The advantage of our definition is that it applies to Artin stacks as well.

**Lemma 2.3.** Let $X$ be a connected $\mathbb{F}_q$-scheme and let $\{\mathcal{F}_i\}_I$ be an $(E, I)$-compatible system of punctually $\iota_i$-pure lisse sheaves (e.g. irreducible lisse sheaves) on $X$. Let $w_i \in \mathbb{R}$ be the $\iota_i$-weight of $\mathcal{F}_i$. Then the numbers $w_i$ ($i \in I$) are all equal.

**Proof.** Take any closed point $x \in X$; then by Remark 2.2 (iii) there is a polynomial $P_x(t) \in E[t]$ such that

$$\sigma_i(P_x(t)) = P^K_{\mathcal{F}_i}(t), \ \forall i \in I.$$

When we ask for the $\iota_i$-weight of $\mathcal{F}_i$, we consider the complex absolute value of the reciprocal roots of $\iota_i(P^K_{\mathcal{F}_i}(t))$, which is just $P_x(t)$ regarded as in $\mathbb{C}[t]$ via the given embedding $E \subset \mathbb{C}$. Therefore, these $\iota_i$-weights $w_i$ are independent of $i$. \hfill $\square$

In particular, [6, Theorem 3] has an $\iota_i$-variant, which is what we mean when we refer to this theorem in the sequel.

Following [4, 0.9], we give the following definition.
Definition 2.4. For a \( \mathcal{F}_q \)-sheaf \( \mathcal{F} \) on an \( \mathbb{F}_q \)-scheme \( X \), we say that \( \mathcal{F} \) is \textit{algebraic}, if \( \text{Tr}(\text{Frob}_x, \mathcal{F}) \) is an algebraic number in \( \overline{\mathbb{Q}}_q \) (i.e. algebraic over \( \mathbb{Q} \)), for any \( v \geq 1 \) and \( x \in X(\mathbb{F}_q^v) \).

If \( \{ \mathcal{F}_i \}_I \) is an \((E, I)\)-compatible system of sheaves on \( X \), then each \( \mathcal{F}_i \) is algebraic (recall that we require \( [E : \mathbb{Q}] < \infty \)).

Now we recall the following theorems of Deligne and of Drinfeld, which are key ingredients of this article.

Theorem 2.5. Let \( \mathcal{F} \) be an algebraic \( \mathcal{F}_q \)-sheaf on an \( \mathbb{F}_q \)-scheme \( X \). Then there exists a finite extension \( E \subset \overline{\mathbb{Q}}_q \) of \( \mathbb{Q} \), such that \( \text{Tr}(\text{Frob}_x, \mathcal{F}) \) belongs to \( E \), for any \( v \geq 1 \) and \( x \in X(\mathbb{F}_q^v) \).

This is \([4, \text{Théorème 3.1}]\).

Theorem 2.6. Let \( X \) be a smooth connected \( \mathbb{F}_q \)-scheme, and let \( \mathcal{F} \) be an irreducible lisse \( \ell \)-adic sheaf on \( X \), whose determinant has finite order. Then there exists a number field \( E \) with an embedding \( \sigma : E \rightarrow \overline{\mathbb{Q}}_q \), such that for each integer \( v \geq 1 \) and for any \( x \in X(\mathbb{F}_q^v) \), the polynomial \( P_{\mathcal{F}}^x(t) \in \overline{\mathbb{Q}}_q[t] \) has coefficients in \( \sigma(E) \), and that for every finite place \( \lambda \) of \( E \) not lying over \( p \), there exists a lisse \( E_\lambda \)-sheaf \( \mathcal{G} \) which is compatible with \( \mathcal{F} \).

This follows from \([5, \text{Proposition VII.7}]\) and \([5, \text{Theorem 1.1}]\). Note that such a \( \mathcal{G} \) is necessarily unique and irreducible: it is irreducible by looking at the order of pole of the \( L \)-function \( L(\mathcal{G} \otimes \mathcal{G}'^\vee, t) \) at \( t = q^{-\dim X} \), hence it is determined by its local characteristic polynomials \( P_{\mathcal{G}}^x(t) \). It is sometimes called a \textit{companion} of \( \mathcal{F} \).

Lemma 2.7. Let \( X \) be an irreducible \( \mathbb{F}_q \)-scheme, and let \( U_1 \) and \( U_2 \) be two essentially smooth open subsets in \( X \). Let \( L_i \) be an irreducible lisse \( \mathcal{F}_q \)-sheaf on \( U_i \) with \( \det(L_i) \) of finite order, having \( U_i \) as maximal support, and let \( \sigma_i : E \rightarrow \overline{\mathbb{Q}}_q \) be an embedding, for \( i = 1, 2 \). Let \( I = \{(\ell_i, \sigma_i) | i = 1, 2 \} \). If \( L_1 \) and \( L_2 \) are \((E, I)\)-compatible on some non-empty open set \( U \subset U_1 \cap U_2 \), then \( U_1 = U_2 \).

Proof. By Theorem 2.6 let \( M \) be the \( \ell_2 \)-companion of \( L_1 \) on \( U_1 \) with respect to \( \sigma_2 \). Then both \( M|U \) and \( L_2|U \) are \( \ell_2 \)-companions of \( L_1|U \), which remains irreducible on \( U \), so by uniqueness we have \( M|U \simeq L_2|U \). As \( \pi_1(U) \) is mapped onto both \( \pi_1(U_1) \) and \( \pi_1(U_2) \), we see that \( j_*L_2 \) is unramified over \( U_1 \cup U_2 \), where \( j : U_2 \rightarrow X \) is the open immersion. So \( U_1 \subset U_2 \), by the maximality of \( U_2 \). By symmetry we have \( U_2 \subset U_1 \) as well. \( \square \)

An \textit{extension} of the pair \((E, I)\) is another pair \((E', I')\) together with a bijection \( I \rightarrow I' \) of the form

\[ (\ell_i, \sigma_i) \rightarrow (\ell_i, \sigma'_i), \]

where \( E'/E \) is a finite subextension of \( \overline{\mathbb{Q}}/E \) and for each \( i \in I \), \( \sigma'_i \) is an extension of \( \sigma_i \) to \( E' \). Because of the bijectivity, the index set for \( \{ \ell_i \} \) is unchanged under such an extension.

Proposition 2.8. Let \( \{ P_i \}_I \) be a system of perverse sheaves on an \( \mathbb{F}_q \)-scheme \( X \). If they are weakly \((E, I)\)-compatible, then they are perverse \((E', I')\)-compatible for some extension \((E', I')\) of \((E, I)\).

Proof. Since \( [P_i] = [P_i^\text{ps}] \), we may assume that the \( P_i \)'s are semisimple perverse sheaves. We may assume that \( X \) is integral, or even geometrically integral if we replace \( \mathbb{F}_q \) by a finite extension.

To show that they have the same set of supports, we may assume that \( I = \{1, 2\} \). We may assume that either \( P_1 \) or \( P_2 \) has an open support; then so does the other. Let \( U \)
be the intersection of all open supports occurring in either \( P_1 \) or \( P_2 \), and let \( \mathcal{F}_i \) be the semisimple \( \mathcal{O}_{\ell_1} \)-local system \( \mathcal{H}^{-\dim X}(P_i|_{U}) \) on \( U \). Then \( U \) is regular and geometrically integral (hence satisfies the hypothesis at the beginning of [3 Section 1.3]), and \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are \((E, I)\)-compatible. Let

\[
\mathcal{F}_1 = L_1 \oplus \cdots \oplus L_n \quad \text{(resp. } \mathcal{F}_2 = M_1 \oplus \cdots \oplus M_r)\]

be the decomposition into irreducible lisse sheaves. We will show that \( n = r \), that \( L_j \) is \((E', I')\)-compatible with \( M_j \) after relabeling, for a certain extension \((E', I')\) of \((E, I)\), and that as a consequence, \( L_j \) and \( M_j \) have the same maximal support.

Note that \( L_1 \) is algebraic (Definition 2.4), since \( P_{x_1}^I(t) \) divides \( P_{x_1}^{P_i}(t) \), whose roots are algebraic numbers. By Theorem 2.5, there is a finite extension \( E' \) of \( E \), embedded in \( \mathcal{O}_{\ell_1} \), via some extension \( \sigma_1' \) of \( \sigma_1 \), containing all local traces of \( L_1 \). Let \( r \) be the rank of \( L_1 \), and for some \( x \in U(\mathbb{F}_q) \), let

\[
P_{x_1}^I(t) = 1 + a_1 t + \cdots + a_r t^r.
\]

Enlarging \( E' \) if necessary, we may assume that \((\sigma_1')^{-1}(1/a_r)\) has an \( r \)-th root in \( E' \), denoted by \( b \). Then the determinant of the \( \ell_1 \)-adic local system \( L_1^{(\sigma_1'(b))} \) deduced by Tate twist has finite order [3 1.3.6]. Therefore, by Theorem 2.6 for each extension \( \sigma_2' : E' \hookrightarrow \mathcal{O}_{\ell_2} \) of \( \sigma_2 \), there exists an \( \ell_2 \)-companion \( \mathcal{G} \) of \( L_1^{(\sigma_1'(b))} \), namely a lisse \( \mathcal{O}_{\ell_2} \)-sheaf on \( U \), such that for any closed point \( x \in U \), the coefficients of \( P_{x}^{\mathcal{G}}(t) \) are in \( \sigma_2'(E') \), and that

\[
\sigma_2' \cdot P_{x}^{\mathcal{G}}(t) = \sigma_1' \cdot P_{x}^{(\sigma_1'(b))}(t).
\]

Let \( L'_1 = \mathcal{G}^{(\sigma_1'(1/b))} \); so \( L'_1 \) is \((E', I')\)-compatible with \( L_1 \). Therefore, \( \mathcal{F}_2 \otimes L_1 \) is \((E', I')\)-compatible with \( \mathcal{F}_1 \otimes L'_1 \), which has the constant sheaf \( \mathcal{O}_{\ell_1, U} \) as a direct summand, thus \( \mathcal{F}_2 \otimes L'_1 \) also has \( \mathcal{O}_{\ell_2, U} \) as a direct summand, with the same multiplicity as that of \( \mathcal{O}_{\ell_1, U} \) in \( \mathcal{F}_1 \otimes L'_1 \), by looking at the order of pole of the \( L \)-function \( L(\mathcal{F}_2 \otimes L'_1, t) \) at \( t = q^{-\dim U} \).

Therefore, one of the \( M_j \)'s, say \( M_1 \), is isomorphic to \( L'_1 \). By induction we see that \( n = r \), and after relabelling we have that \( L_j \) and \( M_j \) are \((E', I')\)-compatible.

Now let \( U_1 \) (resp. \( U_2 \)) be the maximal support of \( L_1 \) (resp. \( M_1 \)); then it is also the maximal support of \( L_1^{(\sigma_1'(b))} \) (resp. \( M_1^{(\sigma_1'(b))} \)). By Lemma 2.7, we have \( U_1 = U_2 \). Also, \( L_1 \) and \( M_1 \), being \((E', I')\)-compatible on \( U \), are \((E', I')\)-compatible on their maximal support \( U_1 \), by [3 Theorem 3].

Again by loc. cit., \( IC_X(L_1) \) and \( IC_X(M_1) \) are weakly \((E', I')\)-compatible, so by the additivity of local traces in exact triangles, the quotient perverse sheaves \( P_1/IC_X(L_1) \) and \( P_2/IC_X(M_1) \) are weakly \((E', I')\)-compatible. By induction on the length of \( P_i \), we are done. \( \square \)

**Remark 2.9.** Note from the proof that, the field \( E' \) can be made explicit, once the information of some (any) \( P_1 \) is completely given, so there is no problem in going from the “\( |I| = 2 \)” case to the general case. Moreover, the local systems in \( P_{x}^{ss} \) on each closed support are \((E, I)\)-compatible. Precisely, let \( \{X_{\alpha}\} \) be the supports occurring in \( P_i \) (known to be independent of \( i \)), and for any closed support \( Z \) occurring in \( P_i \) (i.e. closure of some \( X_{\alpha} \)), let \( Z^0 \) be the intersection of all the \( X_{\alpha} \)’s such that \( \overline{X_{\alpha}} = Z \); then there exists an \((E, I)\)-compatible system \( \{L_{Z}^{x}\}_{I} \) of semisimple lisse sheaves on \( Z^0 \), such that

\[
P_{x}^{ss} \simeq \bigoplus_{Z} IC_{Z}(L_{Z}^{x})
\]

for all \( i \). Proof: We may assume that \( X \) is integral. Let \( Y \subset X \) be the union of all proper (i.e. \( \neq X \)) closed supports occurring in \( P_i \) (for any \( i \)), and let \( U \) be the intersection of \( X - Y \)
and $X^0$ (intersection of all the open supports $X_\alpha$ occurring in $P_i$). Then for all $i \in I$, the sheaves

$$L_X^i := \mathcal{H}^{-\dim X}(P_i|_U)$$

are lisse on $U$, and as $L_X^i = (P_i|_U)[-\dim X]$, they are $(E, I)$-compatible. So by [6] Theorem 3, the system $\{IC_X(L_X^i)\}_I$ is weakly $(E, I)$-compatible. Note that, since $L_X^i$ is semisimple, by [3] Théorème 3.4.1, it is a direct sum of $\ell$-punctually pure local systems, and the parts of a prescribed $\ell$-weight are also $(E, I)$-compatible (the contribution of this part to the local trace lives in some finite Galois extension $\overline{E}$ of $E$ and is fixed by the Galois group), hence [6] Theorem 3 is applicable. Therefore, the quotient perverse sheaves $P_i/IC_X(L_X^i)$, supported on $Y$, are again weakly $(E, I)$-compatible, and by noetherian induction we are done. We will not need this rationality in the sequel.

**Proposition 2.10.** If $\{K_i\}_I$, for $K_i \in D^b(X, \overline{\mathbb{Q}}_\ell)$, is weakly $(E, I)$-compatible, then it is weakly perverse $(E', I')$-compatible for some extension $(E', I')$ of $(E, I)$.

**Proof.** The proof is similar to the previous one. Again we may assume that $X$ is geometrically integral. Let $p[K_i] = p[P_i^+] - p[P_i^-]$ be the canonical representatives.

To show, for instance, that the supports occurring in $P_i^+$ are independent of $i$, we may again assume that $I = \{1, 2\}$. If $P_1^+ \oplus P_1^-$ has an open support, so does $P_2^+ \oplus P_2^-$. Let $U$ be the intersection of all open supports occurring in any one of the $P_i^\pm$, and let $\mathcal{F}_i^\pm = \mathcal{H}^{-\dim X}(P_i^\pm|_U)$, which are semisimple lisse sheaves on $U$. Then $\{[\mathcal{F}_1^+ + [\mathcal{F}_1^-]\}_I$ is weakly $(E, I)$-compatible on $U$, namely for any $x \in U(F_q)$, we have

$$\sigma^{-1}(\Tr(Frob_x, \mathcal{F}_1^+) - \Tr(Frob_x, \mathcal{F}_1^-)) = \sigma^{-1}(\Tr(Frob_x, \mathcal{F}_2^+) - \Tr(Frob_x, \mathcal{F}_2^-)).$$

As in the proof of Proposition 2.8, $\mathcal{F}_-^+$ has a semisimple $\ell_2$-companion $\mathcal{F}_-^+$, and $\mathcal{F}_-^-$ has a semisimple $\ell_1$-companion $\mathcal{F}_-^-$, with respect to some extension $(E', I')$ of $(E, I)$. Therefore

$$\sigma^{-1}(\Tr(Frob_x, \mathcal{F}_1^+ \oplus \mathcal{F}_2^-)) = \sigma^{-1}(\Tr(Frob_x, \mathcal{F}_2^+ \oplus \mathcal{F}_1^-),$$

i.e. $\mathcal{F}_1^+ \oplus \mathcal{F}_2^-$ is $(E', I')$-compatible with $\mathcal{F}_2^+ \oplus \mathcal{F}_1^-$. Note that by definition, $\mathcal{F}_1^+$ and $\mathcal{F}_2^-$ have no irreducible factors in common. Let $L_1$ be an irreducible factor of $\mathcal{F}_1^+$, and let $L'_1$ be its $\ell_2$-companion (with respect to some further extension $(E'', I'')$, such that all local traces of $L_1$ live in $E''$, whose existence is guaranteed by Theorem 2.5). Since $L_1$ is not a factor of $\mathcal{F}_1^-$, we see (from $L$-functions) that $L'_1$ is not a factor of $\mathcal{F}_1^-$. As $L'_1$ is a factor of $\mathcal{F}_2^+ \oplus \mathcal{F}_1^-$, it then must be a factor of $\mathcal{F}_2^+$. Also $L_1$ and $L'_1$ have the same maximal support, by applying Lemma 2.1 upon their Tate twists, and they are $(E'', I'')$-compatible over this maximal support by [6] Theorem 3.

Finally we pass to $p[P_1^+/IC_X(L_1)] - p[P_1^-]$ and $p[P_2^+/IC_X(L'_1)] - p[P_2^-]$, and by induction we see that $\{P_i^\pm\}_I$ is perverse $(E', I')$-compatible for some $(E', I')$.

3 The main result

**Definition 3.1.** We say that a complex $\mathcal{F} \in D^b(X, \overline{\mathbb{Q}}_\ell)$ is perverse semisimple if it is isomorphic to a direct sum of shifted semisimple perverse sheaves, i.e. $\mathcal{F} \simeq \bigoplus_n p_\mathcal{H}^n(\mathcal{F})[-n]$ and each $p_\mathcal{H}^n(\mathcal{F})$ is a semisimple perverse sheaf on $X$.

**Theorem 3.2.** Let $f : X \to Y$ be a proper morphism of $\mathbb{F}_q$-schemes, and let $\{\mathcal{F}_i\}_I$ be a strongly perverse $(E, I)$-compatible system of perverse semisimple complexes on $X$. Then $\{Rf_\ast \mathcal{F}_i\}_I$ is strongly perverse $(E', I')$-compatible, for some extension $(E', I')$ of $(E, I)$. 

7
Proof. By noetherian induction, we may assume that \( \mathcal{F}_i = IC_{\mathcal{P}}(L_i) \), where \( \{L_i\}_I \) is an \( (E, I) \)-compatible system of semisimple local systems on a connected essentially smooth locally closed subset \( U \subset X \). Replacing \( X \) by the closure of \( U \), we may assume that \( U \) is open dense. Let

\[
L_i = \bigoplus_{j=1}^{n_i} L_{ij}
\]

be the decomposition into irreducible local systems. Then the numbers \( n_i \) are the same for all \( i \in I \), and after relabeling, for any fixed \( j \), the system \( \{L_{ij}\}_I \) is \( (E', I') \)-compatible for some extension \( (E', I') \) of \( (E, I) \), and the \( L_{ij} \)'s (with \( j \) fixed and \( i \) varying) have the same maximal support (Lemma 2.7). As

\[
^pR^n f_\ast IC_X(L_i) = \bigoplus_j ^pR^n f_\ast IC_X(L_{ij}),
\]

we may assume that \( L_i \) is irreducible (and consequently with \( (E, I) \) extended), hence punctually \( \iota_i \)-pure of some weight \( w \in \mathbb{R} \), for each \( i \), by Lemma 2.8.

By the purity theorems of Gabber [1, Corollaire 5.3.2] on middle extensions and of Deligne [3, Proposition 6.2.6] on proper pushforwards, \( K_i := Rf_\ast IC_X(L_i) \) is an \( \iota_i \)-pure complex on \( Y \), of weight \( w + \dim U \). Therefore by [1, Corollaire 5.4.4], \( ^p\mathcal{H}^n K_i \) is \( \iota_i \)-pure of weight \( w + \dim U + n \), for each \( n \in \mathbb{Z} \).

By the Lefschetz trace formula, \( \{K_i\}_I \) is weakly \( (E, I) \)-compatible, hence is weakly perverse \( (E', I') \)-compatible for some extension \( (E', I') \), by Proposition 2.10. Let \( ^p[K_i] = ^p[P_i^+] - ^p[P_i^-] \) be the canonical representatives, and let

\[
P_i^+ \oplus P_i^- \simeq \bigoplus_{\alpha, \beta} IC_{Y_\alpha}(M_{i, \alpha, \beta}^i)
\]

be the decomposition into irreducible factors, where \( M_{i, \alpha, \beta}^i \) is an irreducible \( \ell_i \)-adic lisse sheaf on \( Y_\alpha \), having \( Y_\alpha \) as its maximal support. One sees from the proof of Proposition 2.10 that, for each \( \alpha \), the index set of \( \beta \) is independent of \( i \), and after relabeling the \( \beta \) index, we may assume that, for each \( \alpha \) and \( \beta \), the family \( \{M_{i, \alpha, \beta}^i\}_I \) is \( (E'', I'') \)-compatible, for some further extension \( (E'', I'') \).

There is no cancellation in

\[
^p[K_i] = \sum_n (-1)^n ^p[\mathcal{H}^n K_i],
\]

since the \( ^p\mathcal{H}^n K_i \)'s have different \( \iota_i \)-weights. So we have

\[
P_i^+ \simeq \bigoplus_{n \text{ even}} ^p\mathcal{H}^n(K_i)^{ss} \quad \text{and} \quad P_i^- \simeq \bigoplus_{n \text{ odd}} ^p\mathcal{H}^n(K_i)^{ss}.
\]

Thus for each \( n \), we have

\[
^p\mathcal{H}^n(K_i)^{ss} \simeq \bigoplus IC_{Y_\alpha}(M_{i, \alpha, \beta}^i),
\]

where the direct sum is taken over all intersection complexes such that \( M_{i, \alpha, \beta}^i \) is punctually \( \iota_i \)-pure of weight \( w + \dim U + n - \dim Y_\alpha \), a number that is independent of \( i \). If \( IC_{Y_\alpha}(M_{i, \alpha, \beta}^i) \) occurs in \( ^p\mathcal{H}^n(K_{i_0})^{ss} \) for some \( i_0 \in I \) and some indices \( \alpha \) and \( \beta \), then \( IC_{Y_\alpha}(M_{i, \alpha, \beta}^i) \) occurs in \( ^p\mathcal{H}^n(K_i)^{ss} \) for every \( i \in I \), because by Lemma 2.8 the \( \iota_{i_0} \)-weight of \( M_{i, \alpha, \beta}^i \) is the same as the \( \iota_i \)-weight of \( M_{i, \alpha, \beta}^i \). Therefore, \( \{^p\mathcal{H}^n K_i\}_I \) is perverse \( (E'', I'') \)-compatible for each \( n \), in other words, \( \{K_i\}_I \) is strongly perverse \( (E'', I'') \)-compatible. \( \square \)
The following is a “geometric” statement, in the sense that it is over the algebraic closure $\mathbb{F} := \overline{\mathbb{F}}_p$.

**Corollary 3.3.** Let $f : X \to Y$ be a proper morphism of schemes over $\mathbb{F}$. Then the supports occurring in the decomposition of $Rf_*IC_X(\mathbb{Q}_\ell)$ into shifted irreducible perverse sheaves, as well as the connected monodromy groups of the local systems on each support, are independent of $\ell$.

**Proof.** Let $f_0 : X_0 \to Y_0$ be a model of $f$ over a finite subfield $\mathbb{F}_q \subset \mathbb{F}$. By Theorem 3.2, $(Rf_0)_*IC_{X_0}(\mathbb{Q}_\ell))_{t \neq p}$ is strongly perverse $(E, I)$-compatible for certain $(E, I)$, so for each $n \in \mathbb{Z}$, we have a decomposition

$$pR^n f_0_* IC_{X_0}(\mathbb{Q}_\ell)^{ss} \simeq \bigoplus_{\alpha, \beta} IC_{Y_{\alpha,0}}(L^\ell_{\alpha,0,0}),$$

where $L^\ell_{\alpha,0,0}$ is an irreducible $\ell$-adic local system on $Y_{\alpha,0}$, having $Y_{\alpha,0}$ as its maximal support. Extending $(E, I)$ and relabeling the $\beta$ index if necessary, we may assume that $\{L^\ell_{\alpha,0,0}\}_{t \neq p}$ is $(E, I)$-compatible, for each pair of indices $(\alpha, \beta)$. Making a finite base extension $\mathbb{F}_q' / \mathbb{F}_q$ if necessary, we may assume that all supports $Y_{\alpha,0}$ are geometrically connected. The irreducible factors of $L^\ell_{\alpha,0}$, inverse image of $L^\ell_{\alpha,0,0}$ on $Y_\alpha = Y_{\alpha,0} \otimes \mathbb{F}$, are of geometric origin (cf. [1] 6.2.4], but with $\mathbb{C}$ replaced by $\mathbb{F}$), so they are defined over some finite subfield $\mathbb{F}_{q'^v}$ of $\mathbb{F}$. The degree $v_\ell$, $a$ priori dependant of $\ell$, is in fact independent of $\ell$. This is because the number of irreducible local systems on $Y_{\alpha,0}$ (i.e. the number of the $\beta$ indices, for a fixed $\alpha$) is independent of $\ell$, as we saw in the proof of Proposition 2.8 if all the local systems $L^\ell_{\alpha,0,0}$ occurring in the decomposition over $\mathbb{F}_{q'^v}$ are geometrically irreducible, then so are $L^\ell_{\alpha,0,0}$, for any other prime $\ell$, since if $L^\ell_{\alpha,0,0}$ were to split over a finite extension of $\mathbb{F}_{q'^v}$, then $L^\ell_{\alpha,0,0}$ would have to split too. Therefore, by making a further finite base extension, we may assume that the $L^\ell_{\alpha,0,0}$’s, for all $\ell, \alpha, \beta$, are geometrically irreducible. Then the supports occurring in $pR^n f_* IC_X(\mathbb{Q}_\ell)$ are the $Y_\alpha = Y_{\alpha,0} \otimes \mathbb{F}$, independent of $\ell$.

For the connected monodromy groups, i.e. identity components of the Zariski closures of the images of the representations

$$\rho^\ell_{\alpha,\beta} : \pi_1(Y_\alpha) \to GL(V^\ell_{\alpha,\beta})$$

corresponding to various irreducible local systems $L^\ell_{\alpha,0}$ on $Y_\alpha$ occurring in the decomposition of $pR^n f_* IC_X(\mathbb{Q}_\ell)$, we may replace $Y_\alpha$ by $(Y_{\alpha,0})_{red}$ to assume it regular. Also, the irreducible local systems $\{L^\ell_{\alpha,0,0}\}_{t \neq p}$ are pure of some integer weight $w_{\alpha,\beta}$, independent of $\ell$. Then the last assertion follows from [2] Theorem 1.6], generalized to higher dimensions (see the remark after loc. cit.).

**Remark 3.4.** (i) Weizhe Zheng suggested to me the following variant of Theorem 3.2.

Following [13] Definition 3.2.1, we say that a complex $K \in D^b_c(X, \mathbb{Q}_\ell)$ is a split complex if it is isomorphic to a direct sum of shifted perverse sheaves.

**Theorem 3.4.1.** Let $f : X \to Y$ be a proper morphism of $\mathbb{F}_q$-schemes, $w$ be an integer, and let $\{\mathcal{F}_i\}_I$ be a weakly $(E, I)$-compatible system of split complexes on $X$, each of which is pure of weight $w$. Then $\{Rf_* \mathcal{F}_i\}_I$ is strongly perverse $(E', I')$-compatible for some extension $(E', I')$ of $(E, I)$.

**Proof.** By [6] Theorem 2 and [13] Corollary 3.2.5, the system $\{Rf_* \mathcal{F}_i\}_I$ is a weakly $(E, I)$-compatible system of split pure complexes of weight $w$. By [1] Corollaire 5.4.4, each $pR^n f_* \mathcal{F}_i$ is pure of weight $n + w$, so by [14] Proposition 2.7, they are weakly $(E, I)$-compatible, hence perverse $(E', I')$-compatible for some $(E', I')$, by Proposition 2.8.

[9]
(ii) Weizhe Zheng has generalized Theorems 2.5 and 2.6 to Artin stacks with affine stabilizers [15], as well as [6, Theorem 3] to Artin stacks (see the end of the article [14] for the claim; as for the proof, combine [14, Proposition 5.7] and [9, Lemma 6.2] to reduce to the case of schemes), and the author has generalized Deligne’s purity theorem for proper push-forwards to such stacks [12, Proposition 3.9 (iii)]. Therefore, one can generalize Theorem 3.2 to such stacks as well, and the proof is verbatim.

**Theorem 3.4.2.** Let \( f : X \to Y \) be a proper morphism of finite diagonal between \( \mathbb{F}_q \)-Artin stacks with affine stabilizers. Let \( \{ \mathcal{F}_i \}_I \) be a strongly perverse \((E, I)\)-compatible system of perverse semisimple complexes on \( X \). Then \( \{ Rf_* \mathcal{F}_i \}_I \) is strongly perverse \((E', I')\)-compatible, for some extension \((E', I')\) of \((E, I)\).

**References**

[1] Alexander A. Beilinson, Joseph Bernstein, Pierre Deligne, *Faisceaux pervers*, Astérisque, 100, 5-171, Soc. Math. France, Paris (1982).

[2] Cheewhye Chin, *Independence of \( \ell \) of monodromy groups*, Journal of the AMS, 17:3 (2004), 723-747.

[3] Pierre Deligne, *La conjecture de Weil: II*, Publications Mathématiques de l’I.H.É.S., 52 (1980), 137-252.

[4] Pierre Deligne, *Finitude de l’extension de \( \mathbb{Q} \) engendrée par des traces de Frobenius, en caractéristique finie*, Moscow Math. J., 12:3 (2012), 497-514.

[5] Vladimir Drinfeld, *On a conjecture of Deligne*, Moscow Math. J., 12:3 (2012), 515-542.

[6] Kazuhiro Fujiwara, *Independence of \( \ell \) for Intersection Cohomology* (after Gabber), Advanced Studies in Pure Mathematics 36 (2002), Algebraic Geometry 2000, Azumino, 145-151.

[7] Luc Illusie, *Constructibilité générique et Uniformité en \( \ell \)*, preprint, available on his webpage.

[8] Laurent Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Inv. Math. 147 (2002), 1-241.

[9] Yves Laszlo, Martin Olsson, *Perverse sheaves on Artin stacks*, Math. Zeit. 261, (2009), 737-748.

[10] Jean-Pierre Serre, *Abelian \( l \)-Adic Representations and Elliptic Curves*, Addison-Wesley Publ. Company (1968).

[11] Shenghao Sun, *L-series of Artin stacks over finite fields*, Algebra and Number Theory, 6:1 (2012), 47-122.

[12] Shenghao Sun, *Decomposition theorem for perverse sheaves on Artin stacks over finite fields*, Duke Math. J., 161:12 (2012), 2297-2310.

[13] Shenghao Sun, Weizhe Zheng, *Parity and symmetry in intersection and ordinary cohomology*, available at [arXiv:1402.1292](http://arXiv.org/1402.1292), submitted.
[14] Weizhe Zheng, *Sur l’indépendance de l en cohomologie l-adique sur les corps locaux*, Annales Scientifiques de l’École Normale Supérieure, quatrième série - tome 42, fascicule 2, mars-avril 2009.

[15] Weizhe Zheng, *Companions on Artin stacks*, preprint.