On calculating the mean values of quantum observables in the optical tomography representation

G. G. Amosov\textsuperscript{1}, Ya. A. Korennoy\textsuperscript{2}, V. I. Man’ko\textsuperscript{2}

\textsuperscript{1}Steklov Mathematical Institute
ul. Gubkina 8, Moscow 119991, Russia
\textsuperscript{2}P.N. Lebedev Physics Institute,
Leninsky prospect 53, 117924 Moscow, Russia

Abstract

Given a density operator $\hat{\rho}$ the optical tomography map defines a one-parameter set of probability distributions $w_\hat{\rho}(X, \phi)$, $\phi \in [0, 2\pi)$, on the real line allowing to reconstruct $\hat{\rho}$. We introduce a dual map from the special class $\mathcal{A}$ of quantum observables $\hat{a}$ to a special class of generalized functions $a(X, \phi)$ such that the mean value $<\hat{a}>_\hat{\rho} = \text{Tr}(\hat{\rho} \hat{a})$ is given by the formula $<\hat{a}>_\hat{\rho} = \frac{2\pi}{\int_{-\infty}^{+\infty}} \int_{0}^{2\pi} w_\hat{\rho}(X, \phi) a(X, \phi) dX d\phi$. The class $\mathcal{A}$ includes all the symmetrized polynomials of canonical variables $\hat{q}$ and $\hat{p}$.

1 Introduction

Given an observable (hermitian operator) $\hat{a}$ in a Hilbert space $H$ the spectral theorem reads

$$\hat{a} = \int_{\mathbb{R}} X d\hat{E}((\infty, X]),$$

where $\hat{E}$ in an orthogonal projection valued measure defined on all Borel sub-sets $\Omega \subset \mathbb{R}$ such that $\hat{E}(\Omega)$ is an orthogonal projection and the projections $\hat{E}(\Omega_1)$, $\hat{E}(\Omega_2)$ are orthogonal for all open $\Omega_1, \Omega_2 \subset \mathbb{R}$, $\Omega_1 \cap \Omega_2 = \emptyset$. Using the projection valued (spectral) measure $\hat{E}$ transforms the Hilbert space $H$ to the Hilbert space $H_\hat{a} = L^2(\mathbb{R})$ formed by wave functions $\psi_\hat{a}(\cdot)$ obtaining from $\psi \in H$ by the formula

$$\psi_\hat{a}(X) = \frac{d}{dX} \left( \hat{E}((\infty, X])(\psi) \right).$$

The Hilbert space $H_\hat{a}$ is said to be a space of representation associated with the observable $\hat{a}$.

Suppose that $\hat{\rho}$ is a density operator (positive unit-trace operator), then in any space of representation $H_\hat{a}$ it can be represented as an integral operator

$$(\hat{\rho}\psi_\hat{a})(X) = \int_{\mathbb{R}} \rho_\hat{a}(X, Y) \psi_\hat{a}(Y) dY,$$
ψ_a(⋅) ∈ H_a. In the case, the Hilbert-Schmidt kernel ρ̂_a(⋅, ⋅) is said to be a density matrix of ̂ρ in the space of representation H_a. Analogously, one can define the density matrix b(⋅, ⋅) (which can be a generalized function) associated with a observable b in the space of representation H_a.

In [1] the Wigner function W(q, p) associated with the density matrix ̂ρ(⋅, ⋅) in the space of representation associated with the position operator ̂q was introduced as

\[ W(q, p) = \frac{1}{2\pi} \int_R e^{-ipx} \rho \left( q + \frac{x}{2}, q - \frac{x}{2} \right) dx. \]

The Moyal representation of quantum mechanics [2] defines a map between quantum observables ̂a and functions a(q, p) on the phase space under which the mean value \( \langle ̂a \rangle_̂ρ = Tr( ̂ρ ̂a) \) is given by the formula

\[ \langle ̂a \rangle_̂ρ = \int \int W(q, p)a(q, p)dqdp. \]

Unfortunately, although the normalization rule \( \int \int W(q, p)dqdp = 1 \) holds, the Wigner function W(q, p) is not positive definite in general. In [3][4] the optical tomogram w(X, φ) which can be calculated under experimental measuring a generalized homodyne quadrature was introduced as the Radon transform of the Wigner function,

\[ w(X, \phi) = \int \int W(q, p)\delta(X - \cos(\phi)q - \sin(\phi)p)dqdp, \]

where ̂q and ̂p are the position and momentum operators. The one-parameter set \{ w(X, φ), φ ∈ [0, 2π] \} consists of probability distributions on the real line. The optical tomogram can be calculated from the density operator directly by means of the formula [5]

\[ w(X, \phi) = Tr( ̂ρ\delta(X - \cos(\phi) ̂q - \sin(\phi) ̂p)). \]

The inverse Radon transform [6] allows to reconstruct the Wigner function from the optical tomogram.

For a density operator ̂a one can define a function of complex variable z by the formula

\[ a(z, \phi) = -2\pi Tr( ̂a(z - \cos(\phi) ̂q - \sin(\phi) ̂p)^{-2}), \]

\[ z ∈ C, \ Im(z) \neq 0, \ φ ∈ [0, 2\pi]. \]

In the present paper we shall correct the mistake in [7]. Our goal is to prove the following statements.

**Theorem 1.** For any density operator ̂ρ the following identity holds,

\[ \lim_{\varepsilon \to +0} \int_0^{2\pi} \int_{-∞}^{+∞} w_̂ρ(X + iε, φ)a(X + iε, φ)dXdφ = Tr( ̂ρ ̂a). \]
Definition. We shall call the relation (1) a map dual to the optical tomogram map.

It should be noted that the notion of duality we introduce is different from the known concept of [8].

Denote \( D \) the convex set of density operators whose kernels in the coordinate representation belong to the Schwartz space \( S(\mathbb{R}^2) \). Then, optical tomograms corresponding to states from \( D \) belong to the space \( \Omega \) consisting of functions \( w(X, \phi) \) which are from the Schwartz space in \( x \) and infinitely differentiable in \( \phi \). Notice that \( A = D^* \) contains all bounded quantum observables at least.

Corollary 2. The dual map (1) can be extended to any \( \hat{a} \in A \). The extension \( a(X, \phi) \) belongs to the adjoint space \( \Omega^* \). Moreover, for any density operator \( \hat{\rho} \in D \) the equality

\[
\int_0^{2\pi} \int_{-\infty}^{+\infty} w_{\hat{\rho}}(X, \phi)a(X, \phi)dXd\phi = Tr(\hat{\rho} \hat{a})
\]

holds.

Let us define a symmetrized product of canonical quantum observables \( \hat{q}^m \hat{p}^n \) as

\[
\{\hat{q}^m \hat{p}^n\}_s = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \hat{q}^k \hat{p}^{n-k}.
\]

(2)

Below we use the trigonometric polynomials \( Q^m_n(\cos(\phi)) \) defined in Appendix.

Theorem 3. The action of the dual map (1) to the observables (2), gives rise to \( a_{mn}(X, \phi) \) of the form

\[
a_{mn}(X, \phi) = Q^m_{n+m}(\cos(\phi))X^{n+m}.
\]

2 The Parseval equality associated with the characteristic functions

Given a density operator \( \hat{\rho} \) the function \( F(\mu, \nu) = Tr(\hat{\rho} e^{i\mu \hat{q} + i\nu \hat{p}}) \) is said to be a characteristic function of the state \( \hat{\rho} \). The associated set of probability distributions is said to be a symplectic quantum tomogram [5]

\[
w(X, \mu, \nu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iXt} F(t\mu, t\nu)dt
\]

which is connected with the optical tomogram by the formula

\[
w(X, \phi) = w(X, \cos(\phi), \sin(\phi)).
\]
In this way,

\[ F(t \cos(\phi), t \sin(\phi)) = \int_{-\infty}^{+\infty} e^{itX} w(X, \phi) dX. \]  

The standard identity \( e^{i\mu \hat{q} + i\nu \hat{p}} = e^{i\frac{1}{2} \mu \nu} e^{i\mu \hat{q}} e^{i\nu \hat{p}} \) results in

\[ F(\mu, \nu) = \int_{-\infty}^{+\infty} e^{i\mu x} \rho \left( x + \frac{\nu}{2}, x - \frac{\nu}{2} \right) dx. \]  

It immediately follows from (4) that the following Parseval-type equality holds,

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\mu, \nu)|^2 d\mu d\nu = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(X, Y) dX dY = \frac{1}{2\pi} \text{Tr}(\hat{\rho}^2), \]

which is equivalent to

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu, \nu) F_{\hat{a}}(\mu, \nu) d\mu d\nu = \frac{1}{2\pi} \text{Tr}(\hat{\rho} \hat{a}) \]

for the characteristic functions of any two density operators \( \hat{\rho} \) and \( \hat{a} \).

Taking into account the Parseval-type equality (5) it is possible to extend the map \( \hat{\rho} \rightarrow F_{\hat{\rho}} \) to all operators of Hilbert-Schmidt class. Moreover, one can construct a tempered distribution \( F_{\hat{a}} \in S'(\mathbb{R}^2) \) associated with an observable \( \hat{a} \) such that

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu, \nu) F_{\hat{a}}(\mu, \nu) d\mu d\nu = \frac{1}{2\pi} \text{Tr}(\hat{\rho} \hat{a}) \]

for all density operators \( \hat{\rho} \in \mathcal{D} \). The following result is well-known (2) and we put it for the sake of completeness.

**Proposition 4.** The tempered distributions \( F_{\hat{a}} \equiv F_{mn} \) associated with the observables \( \hat{a} \) of the form (2) are given by the formula

\[ F_{mn}(\mu, \nu) = (-i)^{m+n} \delta^{(m)}(\mu) \delta^{(n)}(\nu). \]

Proof. Using the Parseval type identity (5) we get

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu, \nu) F_{00}(\mu, \nu) d\mu d\nu = \frac{1}{2\pi} \text{Tr}(\hat{\rho}) = \frac{1}{2\pi}. \]

Since \( F_{\hat{\rho}}(0,0) = 1 \) for all density operators \( \hat{\rho} \) it results in

\[ F_{00}(\mu, \nu) = \delta(\mu) \delta(\nu). \]
Notice that the statement holds if either \( m \) or \( n \) equals zero. Suppose that it is true for all integer numbers up to fixed \( m \) and \( n \), let us prove that it holds for \( m + 1 \) and \( n + 1 \). Using the equalities
\[
\hat{p}^k \hat{q}^m \hat{p}^{-k} = \hat{p}^k \hat{q}^{m+1} \hat{p}^{-k} - i(n-k)\hat{p}^k \hat{q}^m \hat{p}^{-k-1}
\]
and
\[
\nu \delta (\nu) = 0, \nu \delta^{(n)} (\nu) = -n \delta^{(n-1)} (\nu), \ n \geq 1,
\]
we get
\[
\frac{\partial}{\partial \mu} \left( \text{Tr} (\{ \hat{q}^m \hat{p}^n \}_s e^{i \mu \hat{q} + i \nu \hat{p}} ) \right) = \text{Tr} \left( \{ \hat{q}^m \hat{p}^n \}_s (i \hat{q} + \frac{i \nu}{2} e^{-i \mu \hat{q} e^{i \nu \hat{p}}} \right) =
\]
\[
i \text{Tr} \left( \{ \hat{q}^m \hat{p}^n \}_s \hat{q} e^{i \mu \hat{q} + i \nu \hat{p}} \right) - \frac{i}{2} n \delta^{(m)} (\mu) \delta^{(n-1)} (\nu) = i \text{Tr} \left( \{ \hat{q}^{m+1} \hat{p}^n \}_s e^{i \mu \hat{q} + i \nu \hat{p}} \right).
\]
On the other hand, the equality
\[
\hat{p} \frac{1}{2} \{ \hat{q}^m \hat{p}^n \}_s + \{ \hat{q}^m \hat{p}^n \}_s \frac{1}{2} \hat{p} = \{ \hat{q}^m \hat{p}^{n+1} \}_s
\]
results in
\[
\frac{\partial}{\partial \nu} \left( \text{Tr} (\{ \hat{q}^m \hat{p}^n \}_s e^{i \mu \hat{q} + i \nu \hat{p}} ) \right) = \text{Tr} \left( \{ \hat{q}^m \hat{p}^n \}_s e^{\frac{i \mu}{2} e^{i \nu \hat{p}}} (i \hat{p} + \frac{i \mu}{2} e^{i \nu \hat{p}}) \right) =
\]
\[
\text{Tr} \left( \frac{i \hat{p}}{2} \{ \hat{q}^m \hat{p}^n \}_s e^{\frac{i \mu}{2} e^{i \nu \hat{p}}} \right) + \text{Tr} \left( \{ \hat{q}^m \hat{p}^n \}_s e^{\frac{i \mu}{2} e^{i \nu \hat{p}}} \right) + \text{Tr} \left( \{ \hat{q}^m \hat{p}^n \}_s e^{\frac{i \mu}{2} e^{i \nu \hat{p}}} \right) + \frac{i m}{2} \delta^{(m-1)} (\mu) \delta^{(n)} (\nu) =
\]
i \text{Tr} \left( \{ \hat{q}^{m+1} \hat{p}^n \}_s e^{i \mu \hat{q} + i \nu \hat{p}} \right).
\]
\[\square\]

3 The dual map

To prove Theorem 1 and Corollary 2 we need the following result.

**Proposition 5.** Given a density operator \( \hat{a} \) the relation between the dual map \[\text{[4]}\] and the characteristic function \( F_{\hat{a}} \) is given by
\[
\tau F_{\hat{a}} (t \cos (\phi), t \sin (\phi)) = \frac{1}{(2\pi)^2} \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} e^{itX a(X - i \varepsilon, \phi) dX, \ t > 0.}
\]

Proof. Let us consider the representation of \( \cos (\phi) \hat{p} + \sin (\phi) \hat{q} \) in the space \( H_\phi = L^2 (\mathbb{R}) \) such that
\[
((\cos (\phi) \hat{p} + \sin (\phi) \hat{q}) f)(x) = xf(x), \ f \in H_\phi.
\]

5
Then, given \( f, g \in H_\phi \)
\[
\int_{-\infty}^{+\infty} e^{itX} (g, (X - i\varepsilon - \cos(\phi)\hat{p} - \sin(\phi)\hat{q})^{-2}f) dX =
\]
\[
\int_{-\infty}^{+\infty} \overline{g(x)} f(x) \int_{-\infty}^{+\infty} e^{itX} \frac{1}{(X - x - i\varepsilon)^2} dX dx \equiv I
\]
Calculating the residue in \( z_0 = x + i\varepsilon \) we obtain
\[
I = 2\pi i \begin{cases} 
  it(g, e^{it(\cos(\phi)\hat{q} + \sin(\phi)\hat{p} + i\varepsilon)} f), & t > 0 \\
  0, & t < 0
\end{cases}
\]
□

Proof of Theorem 1.
Using the expression of \( \hat{w}_\rho \) through the characteristic function \( \hat{F}_\rho \) and the definition of \( a(z, \phi) \) we obtain
\[
\lim_{\varepsilon \to +0} \frac{2\pi}{2\pi} \int_{0}^{2\pi} \int_{-\infty}^{+\infty} w_\rho(X + i\varepsilon, \phi) a(X + i\varepsilon, \phi) dX d\phi =
\]
\[
- \lim_{\varepsilon \to +0} \int_{0}^{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-itX} F_\rho(t \cos(\phi), t \sin(\phi)) Tr(\hat{a}(X - i\varepsilon - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2}) dtdXd\phi =
\]
\[
- \lim_{\varepsilon \to +0} \int_{0}^{2\pi} \int_{-\infty}^{+\infty} F_\rho(t \cos(\phi), t \sin(\phi)) \left( \int_{-\infty}^{+\infty} Tr(\hat{a} e^{-itX} (X - i\varepsilon - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2}) dX \right) dtd\phi =
\]
\[
2\pi \lim_{\varepsilon \to 0} \int_{0}^{2\pi} \int_{-\infty}^{+\infty} F_\rho(t \cos(\phi), t \sin(\phi)) \left( -\frac{1}{2\pi} \int_{-\infty}^{+\infty} Tr(\hat{a} e^{itX} (X - i\varepsilon - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2}) dX \right) dtd\phi \equiv I
\]
Substituting the relation of Proposition 5 we get
\[
I = 2\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_\rho(\mu, \nu) \overline{F_\rho(\mu, \nu)} d\mu d\nu = Tr(\hat{a})
\]
□

Proof of Corollary 2.
If a density operator \( \hat{\rho} \in D \), i.e. the density matrix in the coordinate representation \( \rho(\cdot, \cdot) \in S(\mathbb{R}^2) \), then its characteristic function
\[
F_\rho(\mu, \nu) = \int_{-\infty}^{+\infty} e^{ix\mu} \rho \left( x + \frac{\nu}{2}, x - \frac{\nu}{2} \right) dx
\]
is also from the Schwartz space $S(\mathbb{R}^2)$. Thus, the corresponding optical tomo-
gram
\[
\omega(X, \phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itX} F_\rho(t \cos(\phi), t \sin(\phi)) dt
\]
belongs to $S(\mathbb{R})$ in $X$ and infinitely differentiable in $\phi$. Using the Parseval type
equation of Theorem 1
\[
\lim_{\varepsilon \to +0} \frac{2\pi}{0} \int_{-\infty}^{+\infty} w_\rho(X + i\varepsilon, \phi) a(X + i\varepsilon, \phi) dX d\phi = Tr(\hat{\rho} \hat{a})
\]
we can define the extension of dual tomographic map $\hat{\omega}_\rho \rightarrow a(X, \phi)$ such that
\[
< a, \omega_\rho > = Tr(\hat{\rho} \hat{a}).
\]
□

Proof of Theorem 3.
Given an optical tomogram $\omega_\rho(X, \phi)$ of a density operator $\hat{\rho}$ we get
\[
\int_{-\infty}^{+\infty} \int_{0}^{2\pi} X^{n+m} Q_{n+m}^m(\cos(\phi)) \omega_\rho(X, \phi) dX d\phi =
\]
\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} X^{n+m} Q_{n+m}^m(\cos(\phi)) \int_{-\infty}^{+\infty} e^{-itX} F_\rho(t \cos(\phi), t \sin(\phi)) dt dX d\phi =
\]
\[
i^n \int_{-\infty}^{+\infty} \int_{0}^{2\pi} \delta^{(n+m)}(t) F_\rho(t \cos(\phi), t \sin(\phi)))Q_{n+m}^m(\cos(\phi)) dt d\phi =
\]
\[
(-i)^n \sum_{k=0}^{n+m} C_{n+m}^k \frac{\partial^{n+m} F_\rho}{\partial \mu^k \partial \nu^{n+m-k}}(0, 0) \cos^k(\phi) \sin^{m+k}(\phi) Q_{n+m}^m(\cos(\phi)) d\phi =
\]
\[
(-i)^n \frac{\partial^{n+m} F_\rho}{\partial \mu^m \partial \nu^n}(0, 0).
\]
Now the result follows from Proposition 4. □

Appendix

Let us consider the trigonometric system $\{\sin^k(\phi) \cos^{n-k}(\phi), 0 \leq k \leq n\}$. Taking
derivatives of $\sin^k(\phi) \cos^{n-k}(\phi)$ give rise to linear combinations of these elements. It follows that $\sin^k(\phi) \cos^{n-k}(\phi)$ satisfy to the linear differential equation of $n + 1$th order. Notice that
\[
(sin^k(\phi) \cos^{n-k}(\phi))^{(s)} = 0, \ 0 \leq s < k, \ (\sin^k(\phi) \cos^{n-k}(\phi))^{(k)} = k!, \ if \ \phi = 0.
\]
Hence the Wronskian \( w(0) = \prod_{k=0}^{n} k! \neq 0 \) and the elements of this system are linear independent on the segment \([0, 2\pi]\). Thus, there exists the biorthogonal system \( \tilde{Q}_n^m(\cos(\phi)) \) consisting of polynomials in \( \sin^k(\phi) \cos^{n-k}(\phi) \) such that

\[
\int_0^{2\pi} \sin^k(\phi) \cos^{n-k}(\phi) \tilde{Q}_n^m(\cos(\phi)) d\phi = \delta_{km}.
\]

Put \( Q_n^m(\cos(\phi)) = \frac{1}{n!} \tilde{Q}_n^m(\cos(\phi)) \). The first several polynomials are

\[
Q_0^0(\cos(\phi)) = \frac{1}{2\pi}, \quad Q_1^0(\cos(\phi)) = \frac{1}{\pi} \cos(\phi), \quad Q_1^1(\cos(\phi)) = \frac{1}{\pi} \sin(\phi),
\]

\[
Q_2^0(\cos(\phi)) = -\frac{1}{2\pi} \cos^2(\phi) + \frac{3}{2\pi} \sin^2(\phi),
\]

\[
Q_2^1(\cos(\phi)) = \frac{2}{\pi} \sin(\phi) \cos(\phi),
\]

\[
Q_2^2(\cos(\phi)) = \frac{3}{2\pi} \cos^2(\phi) - \frac{1}{2\pi} \sin^2(\phi).
\]

**Acknowledgments**

The work of GGA and VIM is partially supported by RFBR, grant 09-02-00142, 10-02-00312, 11-02-00456.

**References**

[1] Wigner E. Phys. Rev. 40, 749 (1932).

[2] Moyal J.E. Proc. Cambr. Phil. Soc. 45, 91 (1949).

[3] Bertrand J. and Bertrand P. Found. Phys. 17, 397 (1987).

[4] Vogel K., Risken H. Phys. Rev. A 40, 2847 (1989).

[5] Mancini S., Man’ko V.I., Tombesi P. Quantum Semiclass. Opt., 7, 615 (1995).

[6] d’Ariano G.M., Leonhardt U., Paul H. Phys. Rev. A 52, 1801 (1995).

[7] Amosov G.G., Man’ko V.I. J. Russ. Las. Res., 30, 435 (2009).

[8] Man’ko O., Man’ko V.I. J. Russ. Las. Res., 18, 407 (1997).