THERE IS NO MAXIMAL DECIDABLE EXPANSION OF THE \( \langle \mathbb{N}, \{<\} \rangle \) STRUCTURE

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Abstract. We are going to prove that if the theory of a structure \( \mathcal{M} = \langle \mathbb{N}, \Sigma \rangle \) is decidable and the standard order \(<\) on natural numbers \( \mathbb{N} \) is definable in \( \mathcal{M} \), then there is a nontrivial decidable expansion of \( \mathcal{M} \).

In the article [LRR] the authors stated the following

Problem. Does there exist any maximally decidable theory \( T \)?

The first order theory \( T \) of the structure \( \langle \mathbb{N}, \Sigma \rangle \) is called maximally decidable if \( T \) is decidable but the theory of \( \langle \mathbb{N}, \Sigma \cup \{P\} \rangle \) is undecidable for every predicate or function \( P \) on \( \mathbb{N} \) which is not already definable in \( \langle \mathbb{N}, \Sigma \rangle \).

There are few works [SOP1, BC1, BC2, BR1, BR2] with partial negative answers to this question.

Here we are going to show, that if the theory of a structure \( \mathcal{M} = \langle \mathbb{N}, \Sigma \rangle \) is definable in the signature \( \Sigma \) and standard order relation \(<\), then the theory of \( \langle \mathbb{N}, \Sigma \cup \{R\} \rangle \) is decidable as well.

In the next lemma we use definable orders on \( m^k \) and on subsets of \( m^k \) (we identify number \( m \) with the set \( \{0, \ldots, m-1\} \)). The order \(<\) on the set \( \mathbb{N}^k \) we define such that \( \max a > \max b \Rightarrow b < a \), \( \max a = \max b \Rightarrow b < a \Leftrightarrow \) (result of removing from the collection \( a \) one of its maximal item is "bigger" then similar collection, derived from \( b \)).

It is clear that with the such defined order the set \( m^k \) is an initial segment of \( (m+1)^k \). The order on subsets of set \( m^k \) will define such that subset \( A \) is less then \( B \) if \( \min (A \triangle B) \in A \). Note that

\( (*) \) if \( S \subset 2^{(m+1)^k} \) then \( m^k \cap \min S = \min \{m^k \cap s | s \in S\} \)

1. Lemma. There is such infinite decidable subset \( F \subset \mathbb{N} \) that for any formula \( Q(x, \bar{y}) \) in the signature \( \Sigma \) holds

(1) for any tuple \( \bar{a} \) the set \( \{x \in F | M \models Q(x, \bar{a})\} \) is finite or the set \( \{x \in F | M \models \neg Q(x, \bar{a})\} \) finite. Due to this property we call the set \( F \) filter.

(2) the relation \( P(\bar{y}) = \{x \in F | M \models Q(x, \bar{y})\} \) is finite) as well as the function \( f \) such, that \( P(\bar{y}) = \{x \in F | M \models Q(x, \bar{y})\} \subset [0, f(\bar{y})] \) is definable in \( \mathcal{M} \).

Proof. Construction of the set \( F \) is similar to construction of Morley sequence. Renumber all formulas in the \( \Sigma \) signature: \( Q_1(x, \bar{y}), \ldots, Q_n(x, \bar{y}), \ldots \) and construct the sequence of infinite definable subsets \( \mathbb{N} = F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots \) such that conditions (i, ii) holds when \( F = F_n, Q(x, \bar{y}) = Q_n(x, y_1, \ldots, y_k) \) Suppose that the set \( F_{n-1} \) is already constructed. For any \( a, m \in \mathbb{N} \) by \( S^m_a \) we denote the set \( \{v \in m^k | M \models Q_n(a, v)\} \) and by \( B_{a,m} = \{a' \in F_{n-1} | S_{a'}^m = S^m_a\} \). Let us take minimal \( S^m_a \), such that \( B_m = \{a' \in A_{n-1} | S_{a'}^m = S^m_a\} \) is infinite. It is easy to note,
that due the property (*) holds $B_{m+1} \subseteq B_m$. And finally let the definable set $\mathcal{F}_n = \{a_0, \ldots, a_m, \ldots\}$ be such that $a_i = \min\{a > i | a \in B_i\}$. Note that $\mathcal{F}_{n-1} \supseteq \mathcal{F}_n$.

Now as the set $\mathcal{F}$ we can choose any such decidable infinite sequence $a_0, \ldots, a_n, \ldots$, that $i > n \Rightarrow a_i \in \mathcal{F}_n$, and define the function $f$ such that $f(a) = \min\{a > i | a \in B_i\}$, where $i = \max \bar{a}$, and the relation $P(\bar{y}) = \bar{y} \notin B(\max \bar{y})$.

We define by induction on $k$ the standard notion of big set. A set $S \subseteq \mathbb{N}$ is big, if $S \cap \mathcal{F}$ is infinite. A set $S \subseteq \mathbb{N}^k$ is big, if $\{x_1 | \{x_2, \ldots, x_k\} \notin S\}$ is big.

Farther we will use results from works \cite{SOP1, SOP2}.

In the first one claimed

1. **Statement.** If a branching order is definable in the structure with decidable theory, then the structure is not maximal decidable. A partial order $\prec$ on a set $S$ is called branching, if holds the statement

   $$\forall a \in S \exists b, c \in \mathbb{N} \ (b < a \land c < a \land (\{x | x < b\} \cap \{x | x < c\} = \varnothing))$$

   In the second one discussed definable in the structure $\mathcal{M}$ trees. Without loss of generality we suppose that a tree $Tr$ on $\mathbb{N}$ is a family of finite subsets $\mathbb{N}$ such that if $s \in Tr$ then any initial segment of $s$ belongs to $Tr$ as well. There is the order $s \preceq s' \equiv s$ is initial segment of $s'$ on the tree. We say that a relation $Tr(x, y)$ defines the tree, if $\{s_i | s_i = \{x | Tr(x, i)\}\}$ is a tree.

   In particular is this work the notion of rank of a node is introduced:

   By induction we define a rank of nodes: a partial mapping $rk : Tr \rightarrow \mathbb{N}$. We set $rk(s) = 0$ if the subtree $\{s' | s \preceq s'\}$ is locally finite. Suppose that $rk(s) = k$ is defined for $k < n$. We say that a node $s$ is $n$-regular if

   $$(\exists a) (\forall s' \succ s)(s' \cap (\max(s), a) = \varnothing \rightarrow rk(s') < n)$$

To any $n$-regular node $s$ we assign the number $r(s) < n$:

   $$r(s) = \max\{k | (\forall a)(\exists s' > s)(s' \cap (\max(s), a) = \varnothing \land rk(s') = k)\}$$

   We set $r(s) = -1$ if $(\exists a)(\forall s' > s)(s' \cap (\max(s), a) = \varnothing)$. Now we set $rk(s) = n$ if (1) all nodes $s' \succeq s$ are $n$-regular and (2)max $\{r(s') | s' \succeq s\} = n - 1$.

   We say that a node $s$ of finite rank ($rk(s) < \infty$) if $rk(s)$ is defined, otherwise we say that $s$ of infinite rank ($rk(s) = \infty$). We say that a node $s$ is regular if it is $k$-regular for some $k$.

   The main lemmas about ranks are stated.

2. **Lemma.** Let a node $s$ has finite rank $rk(s) = n, n > 0$. Then

   (i) if $s_1 \succ s$, then $s_1$ has finite rank and $rk(s_1) \leq rk(s)$.

   (ii) there are infinitely many $s' \succ s$, such that $rk(s') = n - 1$.

   (iii) $rk(s) > r(s)$

3. **Lemma.** Consider a structure $\mathcal{M} = (\mathbb{N}, \{Tr, <\})$, where the relation $Tr(a, x, y)$ with a parameter $a$ defines a family of trees on $\mathbb{N}$ and $<$ is the usual order on $\mathbb{N}$. If the elementary theory of $\mathcal{M}$ is decidable, then there is such number $k$, that $rk(s) < k$ holds for all nodes $s$ of finite rank in all trees $Tr(a, x, y)$.
1. **Consequence.** Let a relation $Tr(y, x)$ defines a tree on $\mathbb{N}$, and elementary theory of the structure $\mathcal{M} = (\mathbb{N}, \{Tr, <\})$ is decidable. Then
   (i) the relation ”$s$ is a node of finite rank” is definable (in $\mathcal{M}$).
   (ii) if the set of nodes of infinite rank is not empty, then it contains a definable subtree isomorphic to $\mathbb{N}^\omega$.
   (iii) there is $k \in \mathbb{N}$ such that for any node $s$ of finite rank, $s = \{a_1 < a_2 < \cdots < a_n\}$, holds $k \geq |\{a_i \in s | a_{i+1} > \varphi(a_i)\}|$.

   From the definition of rank and Lemma 2 immediately follows

2. **Consequence.** For any definable tree $Tr(y, x)$ exists such definable mapping $\varphi: \mathbb{N} \to \mathbb{N}$, that for any subsegment $\{a_1 < a_2 < \cdots < a_n\}, a_{i+1} > \varphi(a_i)$ of node $s$ of finite rank holds $Tr, k \geq n$.

   According to items (i) and (ii) Consequence 1 if there is a definable tree with a node of infinite rank, then the branching order is definable in the structure. So hereinafter we consider any node in any tree as a node of finite rank. (So our reasonings are slightly nonconstructive: we claim existence of nontrivial decidable expansion, but can not construct it.)

   Now we start to construct undefinable subset $R \subset \mathbb{N}$ such that theory of the structure $\langle \mathbb{N}, \Sigma \cup \{R\} \rangle$ is decidable. Our construction based, of course, on forcing.

   A condition $p$ is a triple $\langle p^+, p^-, \varphi^p \rangle$, where $p^+$ – finite subset of $\mathbb{N}$, $p^-$ – small definable subset of $\mathbb{N}$, $p^+ \cap p^- = \emptyset$; $\varphi^p: \mathbb{N} \to \mathbb{N}$ – definable mapping.

   The order $\leq$ on conditions is defined such that $(q \leq p) \iff (p^- \subseteq q^-) \land (\varphi^q > \varphi^p) \land ((\forall x \in q^+) \land (x, \varphi^p(x)) \cap q^+ = \emptyset)$

   The notion of forcing $p \Vdash \Phi$ is defined in the standard way: by formula $\Phi$ in the signature $\Sigma \cup \{R\}$ induction: $p \Vdash R(a) \iff a \in p^+$, etc.

   Hereinafter exist arbitrary small $q$, that .. means that for any definable function $\varphi$ there is such condition $q, \varphi^q > \varphi$, that ... In the same way we define the notion for any small enough condition $q$, holds...

2. **Statement.** For any formula $\Phi$ and any condition $p$ exists definable families $A^+(\bar{a}, x), A^-(\bar{a}, x)$ and such condition $p^' \leq p$ that for any $q \leq p^'$ holds

   $q \Vdash \Phi \iff (\exists \bar{a})(\{x | A^+(\bar{a}, x) \subset q^+\} \land \{x | A^-(\bar{a}, x) \subset q^-\})$

   **Proof.** Proof by formula $\Phi$ induction. It is obvious for a quantifier-free formula as well as for cases $\forall, \exists$. The complicated case is certainly $\neg$ and this case study will take the rest of the text.

   We will need next

4. **Lemma.** For any condition $p$ and definable family $S$ of finite subsets there is such condition $p^' \leq p$ and subset $S' \subset \mathbb{N}$ such, that for $q \leq p^'$ holds

   $$(\exists s \in S)(s \subset q^+) \iff (q^+ \cap S' \neq \emptyset)$$

   **Proof.**

   Set $p^+ = p$. Consider the family $S' = \{s \setminus p^+ | s \in S\}$. Due to sequence 2 there are such number $n$ and definable function $\varphi$, that if $\varphi^q > \varphi, q \leq p^', s \subset q^+$ for some
\[ s \in S \text{ then } |q^+ \setminus p^+| < n. \] So we can count on each element of \( S' \) contains less than \( n \) items. If \( S' \) is the big set then choosing corresponding function \( \varphi^{p'} \) and small set \( p'^{+} \) we may suppose that if \( q \leq p' \) then
\[
(\exists s \in S)(s \subset q^+) \iff (\exists s' \in S')(s' \subset (q^+ \setminus p^+)) \iff (q^+ \cap \{x_1|\{x_2, \ldots x_n\}\}|x_1, \ldots x_n \in S' \})' \neq \emptyset
\]

So we consider the case \( p \models \neg P \) and for the formula \( P \) inductive hypothesis holds i.e. there are corresponding definable families \( A^+(\bar{a}, x), A^- (\bar{a}, x) \) of finite subsets of \( \mathbb{N} \). Adding, if necessary, one more item to each element of \( A^+(\bar{a}, x) \), we may suppose that for any \( \bar{a} \) holds \( \text{max } A^+(\bar{a}, x) > \text{ max } A^- (\bar{a}, x) \) We show that we can limit ourselves to the case the case when \( A^- (\bar{a}, x) \) is empty. Indeed according the lemma \( \Box \) we can count on elements of \( A^+(\bar{a}, x) \) are sigletons. Consider the following condition on triples \( c_1 < c_2 < c_3 \): there exists such \( \bar{a} \) that corresponding \( \{x|A^-(x, \bar{a})\} \subset (c_1, c_2); \{x|A^+(x, \bar{a})\} = \{c_3\} \). Temporary we will call such triple regular. Obviously, that if \( p' \leq p \) small enough and
\[
(*)q \models \Phi \iff (\exists \bar{a})(\{x|A^+(\bar{a}, x)\} \subset q^+) \wedge (\{x|A^- (\bar{a}, x)\} \subset q^-)
\]
for some \( q \leq p' \mid q^+ \setminus p^+ \mid > 3 \), then there is a regular triple in \( q^+ \) and conversely – if there is such triple , then (*) holds.

So we may suppose, that the family \( A^- \) is empty, and the family \( A^+ \) consists from triples or ( lemma \( \Box \) once more), singletons. Choose \( p' \) as in the lemma \( \Box \). Choose big enough \( \varphi^{p'} \) so for any \( q \leq p' \) or any triple from \( q^+ \setminus p^+ \) is regular or any triple from \( q^+ \setminus p^+ \) is unregular. Note, that if the set of regular triples is big, then for any \( q \leq p' \) there is \( q' \leq q \), \( q \models \Phi \), so \( q' \models \neg \Phi. \) If the set of regular triples is small then \( q' \leq q \), \( q \models \neg \Phi \) if all such triples (more exactly corresponding singletons) belongs to \( q^- \).

It completes the proof of the statement \( \Box \)

Now we can easily construct the decidable generic sequence \( p_0 \geq p_1 \geq \ldots p_n \geq \ldots \) such that \( R = \bigcup_i p_i^+ \) is undefinable.

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