We present an analytic method, based on the Bohmian equations for quantum mechanics, for approaching the phase-retrieval problem in the following formulation: by knowing the probability density $|\psi(\vec{r}, t)|^2$ and the energy potential $V(\vec{r}, t)$ of a system, how can one determine the complex state $\psi(\vec{r}, t)$? We illustrate our method with three classic examples involving Gaussian states, suggesting applications to quantum state and Hamiltonian engineering.

1 Introduction

In this article, we should address an issue which emerged with the advent of the wave function concept: Is it possible to determine the wave function $\psi(\vec{r}, t)$, a complex entity, from the experimentally measured density of probability $P_r(\vec{r}, t) = |\psi(\vec{r}, t)|^2$, that is a real entity by definition? This is an aspect of the general mathematical/physical phase-retrieval problem [1–5] (or state determination problem [6]), i.e., the characterization of the phase of a general scalar field by its modulus [7,8], not necessarily of quantum nature. Indeed, we have early works of image signal/noise analysis on optical [1,9,10], electronic microscopy and holography [11,12] and, as mentioned in [13], control theory and crystallography. Already in the context of quantum wave function characterization, we have both theoretical and experimental approaches on optical microscopy [5], general quantum optics [3,14], cavity QED [15] and ion traps [16]. A wide general review can be found on Ref. [1,4,8,12].

The meaning of the solution of the wave equation proposed by Schrödinger [17,18],

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V(\vec{r}, t) \psi(\vec{r}, t), \quad (1)$$

(where we follow the common definitions $\hbar = \frac{\hbar}{2\pi}$, $\hbar$ is the Planck’s constant, $m$ the mass of the particle, $V(\vec{r}, t)$ the energy potential over the particle, $\vec{r}$ is the position vector and $t$ the time), was object of discussion by pioneers of quantum mechanics like Schrodinger [17–19] himself, Dirac [20] and Born [17,21,22] (the latter proposed the actually accepted

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probabilistic interpretation). The work described here is motivated by the structure of the Bohm’s equations [23,24], that are totally equivalent to the Schrödinger’s one, and allow us to expose a general and analytic method to solve the phase-retrieval problem.

The formulations of phase-retrieval problem in quantum mechanics varied according to time and authors, as already emphasized by Weigert [25]—an early recapitulation on quantum mechanics was made on 40’s by Reichenbach [26]. Perhaps the first author [25,27] to approach this question was Pauli [28] who, after an analysis of the one-dimensional case for a free particle, questioned himself whether, in general, with the densities of probabilities \( P_r (\vec{r}, t) \) and \( P_p (\vec{p}, t) = |\tilde{\psi} (\vec{p}, t)|^2 \), it would be possible to uniquely determine a function \( \psi (\vec{r}, t) \) that satisfies both and is “physically compatible,” i.e., it is as a solution of the Schrödinger Eq. (1) for known \( V (\vec{r}, t) \). This formulation would become known as the Pauli problem [2,13,25,29–32]. Although Pauli posed the question for all time, several authors later limited the issue to a particular instant \( t_0 \). Reichenbach [26] talks about his own discussion with Bargmann about the ambiguity in the definition of \( \psi (\vec{r}, t_0) \) if we know only \( P_r (\vec{r}, t_0) \) and \( P_p (\vec{p}, t_0) \). Bargmann showed we can have sets of functions on position \( \{ \psi_1 (\vec{r}, t_0) , \psi_2 (\vec{r}, t_0) \} \) and momentum \( \{ \tilde{\psi}_1 (\vec{p}, t_0) , \tilde{\psi}_2 (\vec{p}, t_0) \} \) representation with the same probability distribution for these observables, i.e., \( P_r (\vec{r}, t_0) = |\psi_1 (\vec{r}, t_0)|^2 = |\psi_2 (\vec{r}, t_0)|^2 \) and \( P_p (\vec{p}, t_0) = |\tilde{\psi}_1 (\vec{p}, t_0)|^2 = |\tilde{\psi}_2 (\vec{p}, t_0)|^2 \), but with \( |\psi_1 (\vec{u}, t_0)|^2 \neq |\psi_2 (\vec{u}, t_0)|^2 \) and \( |\tilde{\psi}_1 (\vec{u}, t_0)|^2 \neq |\tilde{\psi}_2 (\vec{u}, t_0)|^2 \) where \( \vec{u} \) can be chosen in such a way that \( [\vec{u}, \vec{r}] \neq 0 \) and \( [\vec{u}, \vec{p}] \neq 0 \). Feenberg [26,33] presupposes knowledge in \( t_0 \) of the density of probability and its time differentiation as sufficient condition to the definition of \( \psi (\vec{r}, t_0) \). This hypothesis was later refuted by Gale et al. [29], who proposed that \( P_r (\vec{r}, t_0) \) and the probability current \( \vec{j} (\vec{r}, t_0) \) will be sufficient to determine \( \psi (\vec{r}, t_0) \) only for a spinless particle. Corbett and Hurst [13] showed that, for \( t_0 \) bound states are uniquely determined by \( P_r (\vec{r}, t_0) \) and \( P_p (\vec{p}, t_0) \), but scattering states are not. Orlowski and Paul [2] proposed an iterative-approximative method to determine the state in \( t_0 \) with the knowledge in this time of both densities of probabilities - they applied the method for finite superpositions of photon-number states. Richter [31] modifies the Orlowski and Paul algorithm, substituting the momentum distribution by the time differentiation of \( P_r (\vec{r}, t_0) \). Followed by Vaccaro and Barnett [14], Bialynicka-Birula[34] proposed a slight modification of Richter’s algorithm, considering a finite superposition of Fock states \( |\psi \rangle = \sum_n A_n \exp (i\alpha_n) |n\rangle \) projected in the phase-space vector \( |\phi \rangle \propto \sum_n \exp (-in\phi) |n\rangle \), resulting in \( \psi (\phi) \propto \sum_n A_n \exp (i\alpha_n) \exp (-in\phi) \): they require the knowledge of \( |\psi (\phi)|^2 \) and all the \( A_n^2 \).

Considering specifically a Stern-Gerlach apparatus [35], Weigert [25] proposed that the wave function characterization can be made with three measurements for the density of probability: one along the \( z \) direction, another on an infinitesimally rotated one, \( z' \), and, at last, a measurement of the spin component in a perpendicular orientation to the plane formed by \( z \) and \( z' \). More recently, Lundeen and collaborators [36] proposed a method for photon wave functions involving two measurements of two complementary variables, the first measurement being a weak one [37,38]. They did not consider the time explicitly and, to Weigert [25,39], the original Pauli problem was not solved yet. Other authors, instead of considering a single particular fixed time, presuppose measures referring to two or more instants. Kreinovitch [40] and Wiesbrock [41] proposed that position measurements carried out at different times (within a known \( V (\vec{r}, t) \)) are sufficient to reproduce the initial state of the ensemble in a unique form. By its turn, Johansen [42] supposed explicitly the knowledge of the potential \( V (\vec{r}, t) \) with the measurement of \( P_r (\vec{r}, t) \) for a discrete number of time values.
spaced by a short time interval. As we can see, the list of works in this field is very extensive [3, 5, 6, 16, 32, 43–46].

Our aim in this paper is to expose a general and analytic method, based on Bohm’s equations [23, 24], to solve a phase-retrieval problem. Here we determine the state phase as a function of both position and time and our method requires only the probability amplitude $|\psi(\vec{r}, t)|^2$ and $V(\vec{r}, t)$, in a formulation similar to that of Bardroff et al. [15] and Leonhardt and Raymer [47]. We will not concern ourselves here with how these quantities are measured. In the case of the probability amplitude, for example, we can experimentally measure enough data so that we can construct a grid in space and time with values of $P_r(\vec{r}, t)$. Then, using the numerical techniques involving smooth interpolation, like cubic splines for 3D grids [48], we can calculate the derivatives that are required in the equations we solve in our procedure.

After using the data to generate the smooth interpolated functions, the whole prescription here described can be applied without any modification. Anyway, this issue is addressed in several of the cited authors [2, 14–16, 26, 36, 46, 49–51] and here we will simply assume that both quantities are known as functions of both position and time.

The Bohm’s causal formulation of quantum mechanics, that emphasizes the concept of quantum trajectories and had as precedents the ideas of de Broglie [52] and Madelung [53], has a history of polemics, mainly related to the hidden variables question [19, 54]. However, such a philosophical (or ideological) controversy does not compromise the mathematical rigor of Bohm’s equations and our use of them will be merely mathematical and instrumental. As we will see below, Bohm’s equations emerge easily from Schrödinger’s one (1), leading to two equations widely known before the advent of quantum theory: the continuity and the Hamilton–Jacobi ones [55], the latter being added by what is known as the non-local quantum potential [56, 57]. The continuity equation is already used to support some approaches [6, 12] but, to our knowledge, before the present paper, only Ref. [58] used the Hamilton–Jacobi equation for this purpose. However, in their work the emphasis is on the numerical treatment of the experimental data to estimate the probability density function, and they just suggest the numerical solution of the Bohm’s equations, with no details, to obtain the phase. Here, on the other side, by the first time, we explain in detail how to approach the equations, with purely analytic examples.

We illustrate our method with three unidimensional pure state cases, described by Gaussians. In the first, we have an illustrative application of our method to the free particle case and it can be compared with the results on Ref. [6]. In the second case, we suppose a general oscillating Gaussian to illustrate possible extensions of our method for quantum state engineering [2, 46, 59, 60]. Finally, in the third, we suppose an unusual probability density to illustrate the application of our method to filter valid solutions for the Schrödinger’s equation for a given potential and to construct specific Hamiltonians with desired solutions [30, 59, 61, 62].

The states considered here have characteristics that enable a wide range of applications, since we can find numerous proposals have been devoted to the phase-retrieval of pure states [2, 5, 14–16, 25, 29, 31, 34, 36, 39, 41, 45, 46], considering unidimensional problems [2, 25, 32, 36, 41, 42], harmonic potential [2, 31, 34, 46], Gaussian states [30, 61] and time-dependent potentials [62].

The article is organized as follows: Sect. 2 provides a mathematical description of the method; Sect. 3 shows the examples; conclusions and further suggestions for applications are presented in Sect. 4.
2 The method

The Bohm’s equations [23,24] result from the solution of Eq. (1) in the polar form \(\psi (\vec{r}, t) = R(\vec{r}, t) \exp \left[ \frac{\mp i}{\hbar} S (\vec{r}, t) \right]\), where \(P(\vec{r}, t) = R^2(\vec{r}, t)\) (omitting the subscript \(r\) on \(P_r(\vec{r}, t)\) since the momentum distributions will not be used). If we substitute this polar form on (1) and separate the real and imaginary parts, we obtain

\[
\frac{\partial}{\partial t} S + \frac{(\vec{\nabla} S)^2}{2m} + V + U = 0, \tag{2}
\]

\[
\frac{\partial}{\partial t} P + \vec{\nabla} \cdot \left( \frac{\vec{p} \vec{\nabla} S}{m} \right) = 0. \tag{3}
\]

While Eq. (3) is the continuity equation, Eq. (2) is the Hamilton–Jacobi one [55], from which we can define the canonical momentum \(\vec{p} (\vec{r}, t) = \vec{\nabla} S (\vec{r}, t)\) and the non-local quantum potential

\[
U \equiv -\frac{\hbar^2}{4m} \left[ \nabla^2 P - \frac{(\vec{\nabla} P)^2}{2P^2} \right], \tag{4}
\]

which together with \(V(\vec{r}, t)\) gives the Bohm’s total potential \(V_B (\vec{r}, t) \equiv V (\vec{r}, t) + U (\vec{r}, t)\). An important fact of Eq. (2) is that the density of probability \(P(\vec{r}, t)\) is necessary and univocally related to the phase \(S (\vec{r}, t)\). Then, if we know \(V(\vec{r}, t)\) and \(P(\vec{r}, t)\), we can determine \(S (\vec{r}, t)\) and, then, the complete state \(\psi (\vec{r}, t)\). Our method involves the following steps:

(i) We first solve the continuity equation for \(\vec{p} (\vec{r}, t)\) by expanding the del operator, thus leading to a differential ordinary equation with only the spatial differentiation on \(\vec{p} (\vec{r}, t)\):

\[
\vec{\nabla} \cdot \vec{p} + \frac{\vec{\nabla} P}{P} \cdot \vec{p} = -\frac{m}{P} \frac{\partial}{\partial t} P. \tag{5}
\]

Solving it, we will obtain an expression for \(\vec{p} (\vec{r}, t)\) with an undetermined boundary condition \(\Theta (t)\), which is a function of time only.

(ii) Irrespectively to the first step, we then determine the Bohm’s potential \(V_B (\vec{r}, t)\).

(iii) We then solve the Hamilton–Jacobi Eq. (2) by differentiating it with respect to the spatial coordinates, to obtain

\[
\frac{\partial}{\partial t} \vec{p} + \frac{1}{m} \left( \vec{p} \cdot \vec{\nabla} \right) \vec{p} + \vec{\nabla} V_B = 0. \tag{6}
\]

By substituting \(\vec{p} (\vec{r}, t)\) found previously from the first step into Eq. (6), we can determine \(\Theta (t)\), but now with another time boundary condition \(f (t)\) for \(S (\vec{r}, t)\).

(iv) Finally, we substitute the function \(S (\vec{r}, t)\) obtained from the latter step on the Hamilton–Jacobi Eq. (2) to determine \(f (t)\), thus solving our problem apart from an irrelevant constant factor.
3 Illustrative examples

We illustrate our method considering, as anticipated above, two unidimensional examples: the free particle and the Harmonic oscillator.

3.1 Free particle

For a free particle on a Gaussian state, we have the probability density

\[ P(x, t) = \frac{1}{\sqrt{2\pi D(t)}} \exp \left[ -\left( \frac{x - \langle p \rangle t}{\sqrt{2D(t)}} \right)^2 \right]. \tag{7} \]

where we have defined the diffusion term

\[ D(t) = \frac{\hbar^2}{4 \langle \Delta p \rangle^2} + \frac{(\Delta p)^2}{m^2 t^2}, \]

with both the mean value \( \langle p \rangle \) and the variance \( \Delta p \) of the mechanical momentum being constants. As expected from our knowledge on the dynamics of a free quantum particle, the diffusion term depends on the variance of the mechanical momentum. Following the method exposed above, we start with the continuity equation for \( p(x, t) \), Eq. (5), whose solution is given by

\[ p(x, t) = \frac{1}{D(t)} \left( \frac{\hbar^2 \langle p \rangle}{4 \langle \Delta p \rangle^2} + \frac{(\Delta p)^2}{m^2 t^2} \right) + \Theta(t) \langle p \rangle \exp \left( \frac{x - 2 \langle p \rangle t}{2D(t)} \right), \tag{8} \]

where the boundary condition \( \Theta(t) \) remains to be determined. The Bohm’s potential for the free-particle becomes

\[ V_B(x, t) = \frac{\hbar^2}{4n D(t)} \left[ 1 - \left( \frac{x - \langle p \rangle t}{2D(t)} \right)^2 \right], \tag{9} \]

and by substituting Eqs. (7), (8) and (9) in the spatial derivative of the Hamilton–Jacobi Eq. (6), we are left with

\[ \frac{d\Theta}{dt} = \left( \frac{1}{D(t)} \frac{\hbar^2 \langle p \rangle^3}{4 \langle \Delta p \rangle^2} - (\Delta p)^2 \right) \frac{\Theta(t) t}{m^2 D(t)} \]

\[ - \frac{\langle p \rangle}{m} \left( x - \frac{\langle p \rangle t}{m} \right) D(t) \Theta(t) \exp \left[ -\frac{x}{2D(t)} \left( x - 2 \frac{\langle p \rangle}{m} t \right) \right]. \tag{10} \]

Since, by hypothesis, \( \Theta \) must be a function of time only, the spatial differentiation of Eq. (10) must be null and the only admissible condition is \( \Theta(t) \equiv 0 \). Consequently, from the canonical momentum definition, it follows the action

\[ S(x, t) = \frac{1}{2D(t)} \left( \frac{\hbar^2 \langle p \rangle}{2 \langle \Delta p \rangle^2} x + \frac{(\Delta p)^2}{m} x^2 \right) + f(t). \tag{11} \]
In order to determine \( f(t) \), we substitute Eqs. (11) and (9) in the original Hamilton–Jacobi equation to find

\[
f(t) = f(0) - \frac{\hbar^2}{8m} \langle p \rangle^2 \frac{t}{D(t)} - \frac{\hbar}{2} \arctan \left( \frac{2 \langle \Delta p \rangle^2}{\hbar m} t \right),
\]

(12)
apart from an irrelevant overall factor \( f(0) \).

### 3.2 Harmonic oscillator

We now turn to the harmonic oscillator case whose potential is given by \( V(x) = \frac{m\omega^2}{2} x^2 \), where \( \omega \) is the natural oscillation frequency. Here, instead of assuming a completely definite probability density as we did in the case of a free particle, we assume a general Gaussian form

\[
P(x, t) = \frac{|a|}{\sqrt{\pi}} e^{-a^2 [x + b \cos(\omega t)]^2},
\]

(13)
whose shape remains constant throughout its time evolution. The parameters \( a \) and \( b \) are thus left to be determined under this coherence assumption. Starting with the continuity Eq. (5) for \( p(x, t) \), we thus derive the solution

\[
p(x, t) = \frac{m\omega}{a} \left( ab \sin(\omega t) + \Theta(t) e^{a^2 [x^2 + 2bx \cos(\omega t)]} \right),
\]

(14)
again with the boundary condition \( \Theta(t) \). Going to the Bohm’s potential, from Eqs. (4) and (13) we obtain

\[
V_B(x, t) = U(x, t) + \frac{1}{2}m\omega^2 x^2,
\]

where

\[
U(x, t) = \frac{(\hbar a)^2}{2m} \left[ 1 - a^2 [x + b \cos(\omega t)]^2 \right].
\]

(15)

By substituting Eqs. (13) and (14) on the expression resulting from the spatial differentiation of the Hamilton–Jacobi equation, we obtain

\[
\frac{d\Theta}{dt} = m\omega \left( \frac{\hbar a}{m\omega} \right)^2 a^2 - 1 \left[ x + b \cos(\omega t) \right] e^{-a^2 [x^2 + 2bx \cos(\omega t)]}
- \omega a \Theta(t) \left[ ab^2 \sin(2\omega t) + 2\Theta(t) [x + b \cos(\omega t)] e^{a^2 [x^2 + 2bx \cos(\omega t)]} \right],
\]

(16)
Once again, by hypothesis \( \Theta \) must be only a function of \( t \), such that the spatial derivative of Eq. (16) becomes null only if \( \Theta(t) \equiv 0 \), and Eq. (14) turns out to be

\[
p(x, t) = m\omega b \sin(\omega t).
\]

(17)
From the definition of the canonical momentum, it follows that

\[
S(x, t) = m\omega b \sin(\omega t) x + f(t),
\]

(18)
where \( a, b, \) and \( f(t) \) remain to be determined. By substituting Eqs. (17) and (18) on the Hamilton–Jacobi equation, we obtain

\[
\frac{df}{dt} + \frac{(\hbar a)^2}{2m} + \frac{b^2}{2} \left[ m\omega^2 \sin^2(\omega t) - \frac{(\hbar a)^2}{m} a^2 \cos^2(\omega t) \right]
+ \left( m\omega^2 - \frac{(\hbar a)^2}{m} a^2 \right) \left[ b \cos(\omega t) x + \frac{1}{2}x^2 \right] = 0
\]

(19)
However, as \( f \) must depend only of \( t \), the last term on the left-hand side of Eq. (19) must be null, what follows from the complete determination of the parameter \( a = \pm \sqrt{m \omega / \hbar} \), leaving us with the simplified differential equation

\[
\frac{df}{dt} = \frac{1}{2} \hbar \omega (ab)^2 \cos (2\omega t) - \frac{1}{2} \hbar \omega,
\]

whose solution is

\[
f (t) = f (0) + \frac{1}{4} \hbar (ab)^2 \sin (2\omega t) - \frac{1}{2} \hbar \omega t.
\]

Whereas we have no constraint for the parameter \( b \), which may assume any real value, the initial value \( f (0) \) makes no difference on the final result. Then, we have two solutions for \( \psi (x, t) \), one coming from \( a = \sqrt{m \omega / \hbar} \):

\[
\begin{align*}
R (x, t) &= \left( \frac{m \omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m \omega}{2\hbar} |x+b \cos (\omega t)|^2}, \\
S (x, t) &= m \omega b \left[ x \sin (\omega t) + \frac{1}{4} \hbar a^2 b \sin (2\omega t) \right] - \frac{1}{2} \hbar \omega t,
\end{align*}
\]

and the other from \( a = -\sqrt{m \omega / \hbar} \):

\[
\begin{align*}
R (x, t) &= \left( \frac{m \omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m \omega}{2\hbar} |x-b \cos (\omega t)|^2}, \\
S (x, t) &= -m \omega b \left[ x \sin (\omega t) - \frac{1}{4} \hbar a^2 b \sin (2\omega t) \right] - \frac{1}{2} \hbar \omega t,
\end{align*}
\]

which correspond, as expected [35], to both coherent states

\[
\psi_{\pm} (x, t) = \left( \frac{m \omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m \omega}{2\hbar} \left[ x^2 \pm b \cos (\omega t) + 2x |e^{-i\omega t} + i \frac{b}{m \omega} \right]}
\]

Therefore, starting only from the general shape of the probability density and assuming a behavior consistent with the features of the harmonic potential, we were able to determine from the Bohmian equations the required parameters \( a \) and \( b \), the first assuming only two possible values and the latter any real value. Although we have considered a Gaussian probability density (13) that led to the coherent state (23), the method can be applied to more general densities \( P (x, t, \{a_i\}) \), where \( \{a_i\} \) is a set of parameters to be determined considering the potential \( V (x, t) \) (in the previous example, we have \( a_1 \leftrightarrow a \) and \( a_2 \leftrightarrow b \)), enabling the quantum state engineering [2,46,59,60]. However, it is not always possible to associate a probability density with a given \( a \) \textit{priori} potential, and we will illustrate this with the following example. Still considering the harmonic potential, we will try to construct a Gaussian state whose center remains stopped at the same point in space, with the oscillatory character of the harmonic potential manifesting itself in a periodic variation of its width:

\[
P (x, t) = \frac{1}{\sqrt{\pi} |b|} \left\{ 1 + \varepsilon \sin (\omega t) \right\}^2
\]

Here, \( a, b, \varepsilon \in \mathbb{R} \) and \( |\varepsilon| < 1 \) and all these parameters are left to be determined. Following the method, we start with the continuity equation for \( p (x, t) \), Eq. (5), whose solution is given by
\[ p(x, t) = m \omega e^{-\frac{\cos(\omega t)}{1 + \epsilon \sin(\omega t)}} (x - a) + m \omega \Theta(t) \ e^{-\frac{x(2a - x)}{b^2[1 + \epsilon \sin(\omega t)]^2}} \]  

(25)

where the boundary condition \( \Theta(t) \) remains to be determined. The Bohm’s potential for Eq. (24) becomes

\[ V_B(x, t) = \frac{1}{2} \frac{\hbar^2}{m b^2} \left\{ \frac{(x - a)^2}{b^2[1 + \epsilon \sin(\omega t)]^4} - \frac{1}{[1 + \epsilon \sin(\omega t)]^2} \right\} \]  

(26)

and by substituting Eqs. (24), (25) and (26) in the spatial derivative of the Hamilton–Jacobi Eq. (6), we note that the only admissible condition is \( \Theta(t) \equiv 0 \). Consequently, from the canonical momentum definition, it follows the action

\[ S(x, t) = \frac{m \omega e^{-\frac{\cos(\omega t)}{1 + \epsilon \sin(\omega t)}} x (x - 2a) + f(t)}{2} \]  

(27)

The analysis of the Hamilton–Jacobi Eqs. (2) and (6), we will find that it is not possible a valid solution for both \( b \) and \( \epsilon \) time-independent, therefore we conclude that Eq. (24) is not a possible probability density to obtain for the harmonic potential. However, it is known in the scientific literature [30,61] the problem of investigating which Hamiltonians possess Gaussians as their solutions and, then, starting with (24) or other convenient Gaussian behavior and maintaining \( V(x, t) \) initially undefined, we can follow the method to find an adequate \( V(x, t) \) and define the entire Hamiltonian—a possible application of the time-dependent potential generated is on quantum computing by qubit arrays [62].

4 Conclusions

In this paper, we presented a method, based on the Bohmian equations, to reconstruct a wave-function \( \psi(\vec{r}, t) \) only from the probability density \( |\psi(\vec{r}, t)|^2 \) and the potential energy of the system \( V(\vec{r}, t) \), an approach that resembles some prior proposals [15,47]. The determined final state \( \psi(\vec{r}, t) \) has no phase ambiguities, except by a constant factor \( e^{if(0)} \), \( f(0) \in \mathbb{R} \). The strength of our analytical method is that it is entirely based on the Schrödinger Eq. (1) in its totally equivalent form given by Bohm [23] where the magnitude and phase of the wave function are separated from one another by construction. We have here an associated innovative character since, to the continuity equation already used in previous approaches [6,12], we now add the Hamilton–Jacobi equation. Since the phase-retrieval problem dealt here is based on different assumptions from those widely discussed [2,13,25–29,31–33], our method circumvents the limitations pointed out by the cited authors.

In addition to practical applications in general [1,7–12] and quantum physics [3,5,14–16], we can raise issues from the fundamental point of view. Pauli emphasized [28] that the wave functions themselves are not directly observable and only constitute a mathematical tool to establish \( P_r(\vec{r}, t) \) and its relation with \( P_p(\vec{p}, t) = |\vec{p}|(\vec{p}, t)|^2 \). However, there were recent attempts to measure the wave-function itself [16,36] and, with the method presented here, we can have a \( \psi \)-meter, as defined in Ref. [16], if we measure \( |\psi(\vec{r}, t)|^2 \).

Our examples are all unidimensional, but the structure of the Bohm’s equations allows for generalization to two or three dimensions, as suggested by Holland [56]. Our next step is to test the methodology we have developed with more challenging problems like Non-Gaussian probability densities and multidimensional spaces. The form of the Hamilton–Jacobi resulting equation depends on the potential \( V_B(\vec{r}, t) \) form, and it can be difficult to give a general analytic method to solve it [55], but there are advanced techniques [63–65] that
can be used when necessary, even through numerical implementations [65]. There are also different approaches to the continuity equation [55,56,63,65,66], mainly in fluid dynamics [67,68]. Even the pure-states presented here are not a limitation, because there are studies to expand the Bohm’s equation to mixed states [69,70], and even to open quantum systems [71,72].

It is worth stressing that our method can be used for state engineering [2,59,60] under some known potential $V(\vec{r}, t)$, or for Hamiltonian engineering [30,59,61,62] starting from a general form for the desired probability density $P_r(\vec{r}, t, \{a_i\})$, with applications on quantum computing [62] and quantum optics [59]. The fact that our method is analytical in its foundations and offers a clear answer to a fundamental question of quantum theory, involving the determination of its most basic entity, gives it a broad appeal.

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