Invariable generation of certain groups of piecewise linear homeomorphisms of the interval

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Abstract. Let $P$ be the group of all the orientation preserving piecewise linear homeomorphisms of the interval $[0, 1]$. Given any $a > 1$, let $P^a$ be the subgroup of $P$ consisting of all the elements with slopes in $a\mathbb{Z}$, and let $P^Q$ be the subgroup of $P$ consisting of all the elements with slopes and breaks in $\mathbb{Q}$. We show that the groups $P$, $P^a$, $P^Q$, as well as Thompson group $F$, are invariably generated.

1. Introduction

The concept of invariable generation for a group $G$ was introduced by J. Wiegold in [9].

Notation 1.1. For elements $g$ and $h$ of a group $G$, and a subgroup $H$ of $G$, we denote:

$h^g = ghg^{-1}$, $H^g = \{h^g \mid g \in G\}$, $H^g = \{h^g \mid h \in H\}$.

Definition 1.2. (1) A subgroup $H$ of $G$ is called classful if $H \cap g^G \neq \emptyset$ for any $g \in G$, or equivalently,

$$\bigcup_{g \in G} H^g = G.$$ (1.1)

(2) A group $G$ is said to be invariably generated if there are no classful subgroups other than $G$ itself.

Any finite group is invariably generated, as is shown by a counting argument on [9]. Much easier is the fact that any abelian group is invariably generated. In [9], it is shown that the invariable generation is extension closed. Therefore any virtually solvable group is invariably generated. It is also projection closed. Given a prime number $p > 10^{75}$, an infinite group whose arbitrary proper nontrivial subgroup is of order $p$ is constructed in [8]. Such groups are necessarily generated by arbitrary two elements not from the same proper subgroup, and is invariably

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1The convention $g^h$ is not the same as the customary one.
generated, provided there are more than one nontrivial conjugacy classes. The Grigorchuk group [5] is also invariably generated [7]. However the invariable generation is not subgroup closed: an example is given in [10]. It is also not direct union closed: the group of the permutations of \( N \) with finite support is not invariably generated, since the stabilizer of \( 1 \in N \) is classful. Infinite groups with one nontrivial conjugacy class, constructed in [6], are not invariably generated. Free groups of generators \( \geq 2 \) are not invariably generated [9]. More generally, nonelementary convergence groups are not invariably generated [3]. Acylindrically hyperbolic groups are not invariably generated [2]. Invariable generation of linear groups are discussed in [7].

The current paper is concerned with groups of piecewise linear (PL) homeomorphisms of the interval.

**Theorem 1.** Thompson group \( F \) is invariably generated.

Our method cannot give the finite invariance generation of \( F \) obtained in [4]. Denote by \( P \) the group formed by all the orientation preserving PL homeomorphisms of the unit interval \([0, 1]\), and by \( P^Q \) the subgroup of \( P \) formed by elements with slopes and breaks in \( Q \). Fix an arbitrary real number \( a > 1 \). Let \( P^a \) be the subgroup of \( P \) consisting of all the elements with slopes in \( aZ \).

**Theorem 2.** The group \( P^a \) is invariably generated.

**Theorem 3.** The group \( P \) is invariably generated.

**Theorem 4.** The group \( P^Q \) is invariably generated.

The proofs of the above theorems are quite similar. In Section 2, we summarize conditions for a subgroup \( G \) of \( P \) to be invariently generated. In later sections we show that \( F, P^a, P \) and \( P^Q \) satisfy these conditions independently.

### 2. Conditions for invariable generation

Let \( G \) be any subgroup of the group \( P \) of all the orientation preserving PL homeomorphisms of the interval \([0, 1]\). We shall raise three conditions for \( G \) to be invariably generated. Let \( X \) be a dense subset of \((0, 1)\) which is left invariant by \( G \), and let \( X^* = X \cup \{0, 1\} \). A closed interval \( I \subset [0, 1] \) is called an \( X \)-interval (resp. \( X^* \)-interval) if the endpoints of \( I \) are contained in \( X \) (resp. \( X^* \)).

**Definition 2.1.** For an \( X^* \)-interval \( I \), let us denote
\[
G(I) = \{ g|_I \mid g \in G, \text{ Supp}(g) \subset I \} \quad \text{and} \quad G|_I = \{ g|_I \mid g \in G, \ g(I) = I \}.
\]

The first condition is to fix the relation between \( G \) and \( X \).

**Condition A:**
1. The breaks of any \( g \in G \) are contained in \( X \).
2. The group \( G \) acts on \( X \) transitively.
3. For any \( X \)-interval \( I \), \( G|_I = G(I) \).
4. For any \( X^* \)-interval \( I \), there is a PL homeomorphism \( \psi_I : [0, 1] \to I \) such that \( \psi_I(X^*) = X^* \cap I \) and \( G^{\psi_I} = G(I) \).

The other two conditions are concerned with an arbitrary classful subgroup \( H \) of \( G \).

**Condition B:** Any classful subgroup \( H \) acts on \( X \) transitively.
DEFINITION 2.2. For an $X^*$-interval $I$, let us denote $H(I) = \{ h|_I \in H, \text{Supp}(h) \subset I \}$ and $H|_I = \{ h|_I \in H, h(I) = I \}$.

**Condition C:** For any classful subgroup $H$, there is an $X$-interval $I_0$ such that $H|_{I_0} = G(I_0)$.

In this section, we show that if a subgroup $G$ of $P$ satisfies conditions A, B and C, then $G$ is invariably generated. Henceforth in this section, we assume $G$ satisfies conditions A, B and C. For $f \in G$, define $s(f) \in [0,1]$ by

\[ s(f) = \text{sup}(s | f|_{[0,s]} = \text{id}) \]

By condition A(1), $s(f)$ is contained in $X^*$. Let $H$ be an arbitrary classful subgroup of $G$.

**Lemma 2.3.** For any $X^*$-interval $I = [t,1] \subset [0,1]$, $H(I)$ is a classful subgroup of $G(I)$.

**Proof:** Given any $f|_I \in G(I)$ where $f \in G$ with Supp$(f) \subset I$, let us show that there is $g \in G$ such that Supp$(g) \subset I$ and $f^g \in H$. Since $H$ is classful, there is $g_1 \in G$ such that $f^{g_1} \in H$. Now $s(f^{g_1}) = g_1(s(f)) \in X$. Notice that $s(f) \geq t$ since Supp$(f) \subset I$. By condition B, there is $g_2 \in H$ such that $g_2(g_1(s(f))) = g(s(f))$. Then $f^{g_2g_1} = (f^{g_1})^{g_2}$ is an element in $H$ since $f^{g_1} \in H$ and $g_2 \in H$. Moreover $f^{g_2g_1}|_I$ belongs to $H(I)$ since it satisfies $s(f^{g_2g_1}) = s(f) \geq t$. Notice that $s(f)$ is a fixed point of $g_2g_1$. Now by condition A(3), there is an element $g \in G$ which is the identity on $[0,s(f)]$ and is equal to $g_2g_1$ on $[s(f),1]$. Then we have $f^g = f^{g_2g_1} \in H$, as is required.

For any $X^*$-interval $I = [t,1]$, choose a PL homeomorphism $\psi_I : [0,1] \rightarrow I$ such that $\psi_I(X) = X \cap (t,1)$ and $G^{\psi_I} = G(I)$ (condition A(4)). Since $H(I)$ is classful in $G(I)$, $H(I)^{\psi^{-1}}$ is classful in $G(I)^{\psi^{-1}} = G$. By condition B, $H(I)^{\psi^{-1}}$ acts transitively on $X$. Therefore $H(I)$ acts transitively on $X \cap (t,1)$. This way we get the following lemma.

**Lemma 2.4.** The classful subgroup $H$ acts doubly transitively on $X$. □

By the same argument as Lemma 2.3 applied to the inclusion of an $X$-interval $I = [t,t']$ into an $X^*$-interval $=[t,1]$, we get the following.

**Lemma 2.5.** For any $X$-interval $I$ of $(0,1)$, the group $H(I)$ is a classful subgroup of $G(I)$. □

We shall discuss consequences of condition C. Let $I_0$ be an $X$-interval such that $H|_{I_0} = G(I_0)$.

**Lemma 2.6.** We have $H(I_0) = G(I_0)$.

**Proof:** Choose an arbitrary $f|_{I_0} \in G(I_0)$, where $f \in G$ with Supp$(f) \subset I_0$. Then since $H(I_0)$ is classful in $G(I_0)$ (Lemma 2.3), there is $g \in G$ such that Supp$(g) \subset I_0$ and $f^g \in H$. Since $g|_{I_0} \in G(I_0) = H|_{I_0}$, there is $h \in H$ such that $h(I_0) = I_0$ and $h|_{I_0} = g|_{I_0}$. Then since Supp$(f) \subset I_0$, we have $f^g = f^h$, and hence $f = (f^g)^{h^{-1}} \in H$. This, together with the assumption Supp$(f) \subset I_0$, implies that $f|_{I_0} \in H(I_0)$. □

**Corollary 2.7.** For any $X$-interval $I$ in $(0,1)$, we have $H(I) = G(I)$.
Proof. By double transitivity of the action of $H$ on $X$ (Lemma 2.4), there is $h \in H$ such that $h(I_0) = I$. Now
\[ H(I) = H(I_0)^h = G(I_0)^h = G(I), \]
as is required. \qed

Finally we shall prove that $H = G$. Let
\[ G_0 = \{ g \in G \mid g'(0) = g'(1) = 1 \}, \quad H_0 = \{ h \in H \mid h'(0) = h'(1) = 1 \}. \]
Let $\{J_n\}_{n \in \mathbb{N}}$ be an increasing sequence of $X$-intervals such that $\bigcup_n J_n = (0, 1)$.
We have
\[ H_0 = \bigcup_{n \in \mathbb{N}} H(J_n) \quad \text{and} \quad G_0 = \bigcup_{n \in \mathbb{N}} G(J_n). \]
Since by the previous lemma, $H(J_n) = G(J_n)$ for any $n \in \mathbb{N}$, we get $G_0 = H_0$.
Now for any $f \in G$, there is $g \in G$ such that $f^g \in H$. But then $f^g f^{-1} = [g, f] \in G_0 \subset H$, and therefore $f \in H$. This finishes the proof that if $G$ satisfies conditions A, B and C, then $G$ is invariably generated.

In the rest of the paper, we use the following terminology.

Definition 2.8. For $f \in P$, an interval $[0, \epsilon]$ or $[1 - \epsilon, 1]$ on which $f$ is linear is called an end linear zone of $f$.

3. The group $F$

Let us denote by $\mathbb{Z}[2^{-1}] \subset \mathbb{R}$ the set of dyadic rationals. Thompson group $F$ is the subgroup of $P$ consisting of all the elements with slopes in $2\mathbb{Z}$ and breaks in $\mathbb{Z}[2^{-1}]$. For $F$, we define $X$ in the previous section as $X = \mathbb{Z}[2^{-1}] \cap (0, 1)$. It is well known that $F$ satisfies condition A. For A(4), we can take $\psi_I$ to be any PL homeomorphism from $[0, 1]$ to $I$ with slopes in $2\mathbb{Z}$ and breaks in $\mathbb{Z}[2^{-1}]$. See [1] for the existence.

Definition 3.1. Define a homomorphism $\alpha : F \to \mathbb{Z}^2$ by
\[ \alpha(f) = (\log_2 f'(0), \log_2 f'(1)). \]
Notice that $f \in \text{Ker}(\alpha)$ if and only if $\text{Supp}(f) \subset (0, 1)$. It is well known [1] that $\text{Ker}(\alpha) = [F, F]$. Of course $\alpha$ is a class function: $\alpha(f) = \alpha(f^g)$. Let
\[ F_{1,-1} = \{ f \in F \mid f(x) > x, \forall x \in (0, 1), \alpha(f) = (1, -1) \}. \]
Given $f \in F_{1,-1}$, points $2^{-i}$ from an end linear zone of $f$ at 0 are contained in a single orbit of the $\langle f \rangle$-action. Their images by high iterates of $f$ which lie in an end linear zone at 1 are of the form $1 - k2^{-j}$ for some positive odd integer $k$.

Definition 3.2. Define a map $\beta : F_{1,-1} \to 2\mathbb{N} - 1$, by setting $\beta(f)$ to be the above odd integer $k$.

Lemma 3.3. The map $\beta$ is surjective.
Let \( k \in 2\mathbb{N} - 1 \) be given. Choose a large integer \( j \), and define \( g \in F \) by setting
\[
g(x) = 2x \quad \text{on} \quad [0, 2^{-j}],
g(x) = 2^{-1}(x - 1) + 1 \quad \text{on} \quad [1 - k2^{-j}.1],
\]
and \( g \) is a PL homeomorphism with slopes in \( 2\mathbb{Z} \) and breaks in \( \mathbb{Z}[2^{-1}] \) from the interval \([2^{-j}, 2^{-j+1}]\) to \([2^{-j+1}, 1 - k2^{-j}]\). Then we have \( \beta(g) = k \).

**Lemma 3.4.** The map \( \beta \) is class invariant, that is, \( \beta(g^f) = \beta(g) \) for any \( g \in F_{1,-1} \) and any \( f \in F \).

**Proof.** Assume \( \beta(g) = k \in 2\mathbb{N} - 1 \) for \( g \in F_{1,-1} \). Then there is an orbit \( O \) of \( g \) which contains \( 2^{-j} \) and \( 1 - k2^{-j} \) for any large \( j \). Choose an arbitrary element \( f \in F \) and assume that the slopes of \( f \) are \( 2^j \) near 0 and \( 2^j \) near 1. Then \( f \) maps \( O \) to an orbit of \( g^f \) which contains \( 2^{-j+jf} \) and \( 1 - k2^{-j+jf} \) for any large \( j \), showing that \( \beta(g^f) = k \).

Let \( H \) be an arbitrary classful subgroup of \( F \).

**Corollary 3.5.** The map \( \beta \) restricted to \( H \cap F_{1,-1} \) is surjective onto \( 2\mathbb{N} - 1 \).

The next lemma shows that condition B of Section 2 is satisfied by \( F \).

**Lemma 3.6.** The classful subgroup \( H \) acts transitively on \( X = \mathbb{Z}[2^{-1}] \cap (0, 1) \).

**Proof.** By Corollary 3.5 there is an element \( h_0 \in H \) such that \( \beta(h_0) = 1 \). Thus for any large \( j \), the points \( 2^{-j} \), as well as \( 1 - 2^{-j} \), are on one orbit of \( h_0 \). Again by Corollary 3.5 the \( H \) orbit of these points contains \( 1 - k2^{-j} \) for any \( k \in 2\mathbb{N} - 1 \) and any large \( j \). Applying negative iterates of \( h_0 \), we get that the \( H \) orbit contains all the points in \( X \).

We need more in order to establish condition C for \( F \). For \( n \) large, let \( I_n = [2^{-n-1}, 2^{-n}] \), \( J_n = [1 - 2^{-n}, 1 - 2^{-n-1}] \) and let \( \phi_n : [0, 1] \to I_n, \psi_n : [0, 1] \to J_n \) be the orientation preserving surjective linear map of slope \( 2^{-n-1} \). Let
\[
F_{1,-1,1} = \{g \in F_{1,-1} \mid \beta(g) = 1\}.
\]
Given any \( g \in F_{1,-1,1} \), if we choose \( n \) large enough, some iterate \( g^N \) maps \( I_n \) onto \( J_n \). The map \( \psi_n^{-1} \circ g^N \circ \phi_n \) is independent of the choice of \( n \). In fact, if \( k > 0 \), \( g^k \phi_{n+k} = \phi_n \) and \( g^k \psi_n = \psi_{n+k} \). Therefore we have
\[
\psi_{n+k}^{-1}g^{N+2k} \phi_{n+k} = (\psi_{n+k}^{-1}g^k)g^N(g^k \phi_{n+k}) = \psi_n^{-1}g^N \phi_n.
\]
Notice also that \( \psi_n^{-1}g^N \phi_n \) is an element of \( F \).

**Definition 3.7.** Define a map \( \gamma : F_{1,-1,1} \to F \) by \( \gamma(f) = \psi_n^{-1} \circ g^N \circ \phi_n \).

**Lemma 3.8.** The map \( \gamma \) is surjective.

We shall adopt a bit longer proof, which is applicable also to the group \( P^n \) in the next section.

**Proof.** Choose an arbitrary element \( g \in F_{1,-1,1} \) which is linear on \([0, 2^{-n}]\) and \([1 - 2^{-n}, 1]\). There is \( N > 0 \) such that \( g^N \) maps \( I_n \) onto \( J_n \). Let \( f_0 = \psi_1^{-1}g^N \phi_n \in F \). Any element of \( F \) can be written as \( f \circ f_0 \) for some \( f \in F \). The map \( \hat{f} = \psi_n \circ f \circ \psi_n^{-1} \) is a PL homeomorphism of the interval \( J_n \) with slopes in \( 2\mathbb{Z} \) and breaks in \( \mathbb{Z}[2^{-1}] \).
Define an element $g_1 \in F_{1,-1,1}$ to be equal to $\hat{f}g$ on $J_{n-1}$ and equal to $g$ elsewhere. Notice that $\hat{f}g$ is still linear on $[0, 2^{-n}]$ and $[1 - 2^{-n}, 1]$. We also have

$$\psi_n^{-1}g_1^N\phi_n = \psi_n^{-1}\hat{f}g^N\phi_n = (\psi_n^{-1}\hat{f}\psi_n)(\psi_n^{-1}g^N\phi_n) = ff_0.$$ 

Since $ff_0$ is an arbitrary element of $F$, we are done. \hfill $\square$

**Lemma 3.9.** The map $\gamma$ is class invariant. Precisely, if $g \in F_{1,-1,1}$ and $f \in F$, then $\gamma(gf) = \gamma(g)$.

**Proof.** Choose $n$ large enough so that $g$ and $f$ are linear on the intervals $[0, 2^{-n}]$ and $[1 - 2^{-n}, 1]$. Since $g \in F_{1,-1,1}$, some iterate $g^N$ of $g$ maps $I_n$ to $J_n$. Put $k = \gamma(g) = \psi_n^{-1}g^N\phi_n$, and let us show that $\gamma(gf) = k$. We assume $f$ is of slope $2^{j_0}$ on $[0, 2^{-n}]$ and of slope $2^{j_1}$ on $[1 - 2^{-n}, 1]$. Then $g^f$ is linear (of slope 2) on $[0, 2^{-n+j_0}]$, and is linear (of slope $2^{-1}$) on $[1 - 2^{-n+j_1}, 1]$. The map $g^f$ maps $I_{n-j_0}$ onto $J_{n-j_1}$. Since $\phi_{n-j_0} = f\phi_n$ and $\psi_{n-j_1} = f\psi_n$, we have

$$\psi_{n-j_1}(g^f)^N\phi_{n-j_0} = \psi_n^{-1}f^{-1}(fgf^{-1})^Nf\phi_n = \psi_n^{-1}g^N\phi_n = k.$$ 

If $n$ is big enough compared with $j_0$ and $j_1$, we have $\phi_{n-j_0} = (g^f)^{j_0-j_1}\phi_{n-j_1}$. Therefore

$$k = \psi_{n-j_1}(g^f)^N\phi_{n-j_0} = \psi_n^{-1}(g^f)^{N+j_0-j_1}\phi_{n-j_1}.$$ 

This shows $\gamma(gf) = k$, as required. \hfill $\square$

**Corollary 3.10.** The map $\gamma$ restricted to $H_{1,-1,1} = H \cap F_{1,-1,1}$ is surjective onto $F$. \hfill $\square$

Fix once and for all an element $h_0 \in H_{1,-1,1}$ such that $\gamma(h_0) = \text{id}$. Thus there is $n > 0$ such that $h_0$ is linear on $[0, 2^{-n}]$ and $[1 - 2^{-n}, 1]$, that some iterate $h_0^n$ maps $I_n$ onto $J_n$ and that $\psi_n^{-1}h_0^n\phi_n = \text{id}$. The next lemma shows that the group $F$ satisfies condition C of Section 2.

**Lemma 3.11.** We have $H|I_n = F(I_n)$.

**Proof:** Choose an arbitrary element $\hat{f} \in F(I_n)$ and let $f = \phi_n^{-1}\hat{f}\phi_n \in F$. By Corollary 3.10 there is $h_1 \in H_{1,-1,1}$ such that $\gamma(h_1) = f$. More precisely, for some big $m > 0$, there is $N > 0$ such that $h_1^N(I_m) = J_m$ and that $\psi_n^{-1}h_1^N\phi_m = f$. One can choose $m$ to be greater than $n$ in the lemma. Then some iterate $h_0^N$ of $h_0$ maps $I_m$ onto $J_m$ and $\psi_m^{-1}h_0^N\phi_m$ is still the identity. Thus

$$\phi_m^{-1}h_0^{-N}h_1^N\phi_m = (\psi_m^{-1}h_0^{-N}\phi_m)^{-1}(\psi_m^{-1}h_1^N\phi_m) = \text{id}^{-1}f = f,$$

and since $h_0^{-m}\phi_n = \phi_m$,

$$\phi_n^{-1}h_0^{-m}(h_0^{-N}h_1^N)h_0^{-m}\phi_n = f.$$ 

But this means

$$h_0^{-m}(h_0^{-N}h_1^N)h_0^{-m}|I_n = \hat{f}.$$ 

Since $\hat{f} \in F(I_n)$ is arbitrary and the LHS is in $H|I_n$, we are done. \hfill $\square$
4. The group $P^a$

Let $a > 1$ be an arbitrary real number.

**Definition 4.1.** Given two compact intervals $I$ and $J$, we denote by $PL^a(I, J)$ the space of the PL homeomorphisms from $I$ to $J$ with slopes in $\mathbb{A}^Z$. Such a map is called a $PL^a$ homeomorphism.

**Lemma 4.2.** For any compact interval $I$ and $J$, the space $PL^a(I, J)$ is nonempty.

**Proof.** Let $I = [p, q]$ and $J = [r, s]$. Consider a line $L \subset \mathbb{R}^2$ of slope $a^n$, $n > 1$, passing through the point $(p, r)$, and another line $L'$ of slope $a^{-m}$, $m > 1$, passing through $(q, s)$. If $n$ and $m$ are sufficiently large, $L$ and $L'$ intersect at a point in the open rectangle $(p, q) \times (r, s)$, yielding the graph of a desired map in $PL^a(I, J)$. \hfill $\square$

**Definition 4.3.** Define a group $P^a$ by $P^a = PL^a([0, 1], [0, 1])$.

We choose $X = (0, 1)$ in condition A. Then the group $P^a$ satisfies $A(1), A(2)$ and $A(3)$ by virtue of Lemma 4.2. For $A(4)$, we just take $\psi : [0, 1] \to I$ to be the orientation preserving linear homeomorphism. Therefore in this section, $X^*$-intervals are just closed intervals. In the rest we shall establish conditions B and C for $P^a$ by almost the same method as in Section 3.

Define a homomorphism $\alpha : P^a \to \mathbb{Z}^2$ by $\alpha(f) = (\log_a f'(0), \log_a f'(1))$.

Clearly $\alpha$ is a surjective class function. Let $P^a_{1,-1} = \{g \in P^a \mid g(x) > x, \forall x \in (0, 1), \alpha(g) = (1, -1)\}$.

Given $g \in P^a_{1,-1}$, points $a^{-i}$ for $i$ large are contained in a single orbit of $g$. Consider their images by high iterates of $g$ which are near 1. They are of the form $1 - \xi a^{-j}$ for some number $\xi \in (a^{-1}, 1]$.

**Definition 4.4.** Define a map $\beta : P^a_{1,-1} \to (a^{-1}, 1]$, by setting $\beta(g)$ to be the above number $\xi$.

Then one can show that the map $\beta$ is a surjective class function just as Lemmas 3.3 and 3.4 in Section 3. In particular, the map $\beta$ restricted to $H \cap P^a_{1,-1}$ is surjective onto $(a^{-1}, 1]$, where $H$ is an arbitrary classful subgroup of $P^a$. Then by the same method as Lemma 3.5 we get the following lemma, which establishes condition B.

**Lemma 4.5.** Any classful subgroup $H$ acts transitively on $(0, 1)$.

For a positive integer $n$, let $I_n = [a^{-n-1}, a^{-n}]$, $J_n = [1 - a^{-n}, 1 - a^{-n-1}]$ and let $\phi_n : [0, 1] \to I_n$, $\psi_n : [0, 1] \to J_n$ be the orientation preserving linear homeomorphism of the same slope $a^{-n}(1 - a^{-1})$. Let $P^a_{1,-1,1} = \{g \in P^a_{1,-1} \mid \beta(g) = 1\}$.

Given any $g \in P^a_{1,-1,1}$, if we choose $n$ large enough, then $g$ is linear on the intervals $[0, a^{-n}]$ and $[1 - a^{-n}, 1]$. By the definition of $P^a_{1,-1,1}$, some iterate $g^N$ of $g$ sends $I_n$ to $J_n$, and the map $\psi_n^{-1} \circ g^N \circ \phi_n$ is independent of the choice of $n$. Notice also that $\psi_n^{-1} \circ g^N \circ \phi_n$ is an element of $P^a$, since $\phi_n$ and $\psi_n$ are linear homeomorphisms of the same slope.

**Definition 4.6.** Define a map $\gamma : P^a_{1,-1,1} \to P^a$ by $\gamma(f) = \psi_n^{-1} \circ g^N \circ \phi_n$. 

One can show that the map $\gamma$ is a surjective class function just as in Lemmas \ref{lem:5.5} and \ref{lem:5.9}. Fix once and for all an element $h_0 \in H \cap P_{1,-1,1}$ such that $\gamma(h_0) = \text{id}$. Thus there is $n > 0$ such that $h_0$ is linear on $[0, a^{-n}]$ and $[1 - a^{-n}, 1]$, that some iterate $h_0^N$ maps $I_n$ onto $J_n$ and that $\psi_n^{-1}h_0^N\phi_n = \text{id}$. Just as in Lemma \ref{lem:5.11} we get the following lemma which establishes condition $C$.

**Lemma 4.7.** We have $H|_{I_n} = P^a(I_n)$.

## 5. The groups $P$ and $P^Q$

In this section, we mainly deal with the group $P$ of all the orientation preserving PL homeomorphisms of $[0,1]$. In the last part, we remark one word for necessary modifications with the group $P^Q$. For $P$, put $X = (0,1)$ as in Section 4. Then condition $A$ is trivially fulfilled. Let $H$ be an arbitrary classful subgroup of $P$. First we shall establish conditions $B$.

**Lemma 5.1.** The group $H$ acts transitively on $(0,1)$.

**Proof.** There is an element $h_0 \in H$ such that $h_0(0) = 2$ and that $h(x) > x$ for any $x \in (0,1)$. Assume $h_0(x) = 2x$ on the interval $[0,2^{-n}]$ for some $n > 0$. The interval $[2^{-n-1}, 2^{-n}]$ is a fundamental domain of the action of the group $(h_0)$. Thus it suffices to show that for any $\xi \in (2^{-1},1]$, there is an element of $H$ which maps $2^{-n}$ to $\xi 2^{-n}$. Choose an element $h_1 \in H$ such that $h_1'(0) = \xi$. Assume $h_1$ is linear on an interval $[0, 2^{-m}]$ for some $m > n$. Then $h_1(2^{-m}) = \xi 2^{-m}$, and hence $h_0^{m-n}h_1h_0^{-m}(2^{-n}) = \xi 2^{-n}$, as is required. \hfill $\Box$

In the rest of this section, we shall establish condition $C$ by the following lemma.

**Lemma 5.2.** For some closed interval $I_0 \subset (0,1)$, we have $H|_{I_0} = P(I_0)$.

For any closed interval $I \subset [0,1]$, denote by $\phi_I : [0,1] \to I$ the orientation preserving bijective linear map. Define

$$P_{1,-1} = \{ g \in P \mid g(x) > x, \forall x \in (0,1), \ g'(0) = 2, \ g'(1) = 2^{-1} \}.$$ 

Let $I$ (resp. $J$) be a fundamental domain of $g \in P_{1,-1}$ contained in an end linear zone of $g$ (Definition \ref{def:2.8}) at $0$ (resp. at $1$). Thus $I = [a, 2a]$ for some $a > 0$ and $J = [1 - 2b, 1 - b]$ for some $b > 0$. If there is $N > 0$ such that $g^N(I) = J$, we say that the pair $I$ and $J$ are monitoring intervals for $g$. The map $f = \phi_I^{-1}g^N\phi_I \in P$ is called the information of $g$ monitored by $I$ and $J$. We also say that $I$ and $J$ monitor the information $f$.

**Definition 5.3.** For any $g \in P_{1,-1}$, denote by $\mathcal{I}(g) \subset P$ the set of all the monitored informations of $g$.

**Lemma 5.4.** For any $f \in P$, there is $g \in P_{1,-1}$ such that $f \in \mathcal{I}(g)$.

**Proof.** The proof is almost the same as Lemma \ref{lem:5.3}. \hfill $\Box$

**Lemma 5.5.** Given $g \in P_{1,-1}$ and $f \in \mathcal{I}(g)$, the intervals which monitor the information $f$ can be chosen arbitrarily near $0$ and $1$.

**Proof.** If the intervals $I$ and $J$ monitor the information $f$, and if $n > 0$, then clearly the intervals $g^{-n}(I)$ and $g^n(J)$ monitor the same information $f$. \hfill $\Box$

**Lemma 5.6.** If $g \in P_{1,-1}$ and $g_1 \in P$, then $\mathcal{I}(g^{g_1}) = \mathcal{I}(g)$.
PROOF. Let $g$ and $g_1$ be as in the lemma, and let $f \in \mathcal{I}(g)$. It suffices to show that $f \in \mathcal{I}(g^{\phi_1})$. By Lemma 5.5, one can choose the monitoring intervals $I$, $J$ of $g$ which monitor the information $f$ in the end linear zones of $g_1$. Then $g_1(I)$ and $g_1(J)$ are monitoring intervals of $g^{\phi_1}$, with information $f$ since $\phi_{g_1(I)} = g_1\phi_I$ and $\phi_{g_1(J)} = g_1\phi_J$.

Corollary 5.7. For any $f \in P$, there is $h \in H \cap P_{1,-1}$ such that $f \in \mathcal{I}(h)$.

Choose an element $h_0 \in H \cap P_{1,-1}$ so that $id \in \mathcal{I}(h_0)$, and let $I_0$ and $J_0$ be monitoring intervals of $h_0$ with information $id$. That is, there is $N_0 > 0$ such that $h_0^{-N_0}(I_0) = J_0$ and $\phi_{h_0}^{-1}h_0^{N_0}\phi_{I_0} = id$. Let $\hat{f}$ be an arbitrary element of $P(I_0)$, and let $f = \phi_{I_0}^{-1}\hat{f}\phi_{I_0} \in P$. By Corollary 5.7, there is $h_1 \in H \cap P_{1,-1}$ such that $f \in \mathcal{I}(h_1)$. Let $I_1$, $J_1$ be the corresponding monitoring intervals: we assume $h_1^{-N_1}(I_1) = J_1$ for some $N_1 > 0$ and $\phi_{h_1}^{-1}h_1^{N_1}\phi_{I_1} = f$. Put $I_0 = [a, 2a]$, $J_0 = [1 - 2b, 1 - b]$, $I_1 = [c, 2c]$ and $J_1 = [1 - 2d, 1 - d]$ for some $a, b, c, d > 0$. Choose an element $h_2 \in H$ such that $h_2(0) = c/a$ and $h_2(1) = d/b$. Choose a big $n > 0$ so that both intervals $h_0^{-n}(I_0)$ and $h_0^{n}(J_0)$ are in the end linear zones of $h_2$. Direct computation shows that $h_2(h_0^{-n}(I_0)) = h_1^{-n}(I_1)$ and $h_2(h_0^{n}(J_0)) = h_1^{n}(J_1)$. See the figure.

The equality $\phi_{h_0}^{-1}h_0^{N_0}\phi_{I_0} = id$ implies (cf. the proof of Lemma 5.5)

$$\phi_{h_0}^{-1}(J_0)h_0^{N_0+2n}\phi_{h_0^{-n}(I_0)} = id.$$ 

Likewise we have

$$\phi_{h_1}^{-1}(J_1)h_1^{N_1+2n}\phi_{h_1^{-n}(I_1)} = f.$$ 

These equalities show

$$\phi_{h_0}^{-1}(J_0)h_0^{-N_0-2n}\phi_{h_0^{n}(J_0)}\phi_{h_1}^{-1}(J_1)h_1^{N_1+2n}\phi_{h_1^{-n}(I_1)} = f.$$ 

Since $h_2$ is linear on the intervals $h_0^{-n}(I_0)$ and $h_0^{n}(J_0)$, we have $h_2\phi_{h_0^{n}(J_0)} = \phi_{h_1^{-n}(I_1)}$ and $h_2\phi_{h_0^{-n}(I_0)} = \phi_{h_1^{n}(J_1)}$. Therefore

$$\phi_{h_0}^{-1}(J_0)h_0^{-N_0-2n}h_2^{-1}h_1^{N_1+2n}h_2\phi_{h_0^{-n}(I_0)} = f.$$ 

Finally since $h_0^{-N_0}\phi_{h_0^{-n}(I_0)} = \phi_{I_0}$, we get

$$\phi_{I_0}^{-1}h_0^{-N_0-n}h_2^{-1}h_1^{N_1+2n}h_2h_0^{-n}\phi_{I_0} = f.$$ 

This implies

$$h_0^{-N_0-n}h_2^{-1}h_1^{N_1+2n}h_2h_0^{-n}|_{I_0} = \hat{f}.$$
Since $f \in P(I_0)$ is arbitrary, and the map on the LHS is from $H|_{I_0}$, the proof of Lemma 5.2 is now complete.

For the subgroup $P^Q$ of $P$ consisting of all the elements with slopes and breaks in $\mathbb{Q}$, we define $X = \mathbb{Q} \cap (0,1)$. The argument for $P^Q$ is the same for $P$ under necessary modifications.

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