Abstract

It has recently been shown that the uniqueness theorem for stationary black holes cannot be extended to five dimensions. However, uniqueness is an important assumption of the string theory black hole entropy calculations. This paper partially justifies this assumption by proving two uniqueness theorems for supersymmetric black holes in five dimensions. Some remarks concerning general properties of non-supersymmetric higher dimensional black holes are made. It is conjectured that there exist new families of stationary higher dimensional black hole solutions with fewer symmetries than any known solution.

1 Introduction

One of the most impressive successes of string theory is a microscopic derivation of the entropy of certain supersymmetric black holes [1]. The idea is that a weakly coupled system of strings and branes wrapped around some compact dimensions turns into a black hole in the non-compact dimensions as the string coupling is increased. For fixed asymptotic charges (mass, angular momenta and gauge charges), the degeneracy of microstates can be calculated in the weakly coupled description. Provided sufficient supersymmetry is preserved, this is found to reproduce correctly the Bekenstein-Hawking entropy of the black hole (at least for black holes much larger than the string length).

These calculations were first performed for static supersymmetric black holes in five dimensions [1]. They were subsequently extended to static supersymmetric holes in four dimensions [2, 3], to rotating supersymmetric holes in five dimensions [4] and to nearly supersymmetric generalizations of all of these [5, 6, 7].

A key assumption made in this work is that the relevant black hole solutions are uniquely specified by their asymptotic charges. If this turned out to be untrue, i.e., if there existed distinct
supersymmetric black hole solutions with the same asymptotic charges, then there would be a problem with the conventional interpretation of the entropy calculations. The problem would be in identifying which sets of microstates should correspond to each black hole, as would be necessary in order to compute their respective entropies. The black holes would be distinguished macroscopically by their differing gravitational fields. However, there is no gravitational field present in the weakly coupled description used for the entropy calculations. Hence, at weak coupling, there would be no way of telling which microstates should correspond to each black hole. The distinction between different sets of microstates would only become apparent as the microscopic description became strongly coupled.

Given the importance of this assumption, one might ask how it was originally motivated. It seems that the only evidence in its favour is the existence of the black hole uniqueness theorems in four dimensions. These establish that stationary four dimensional black holes are indeed uniquely specified by their asymptotic charges, at least in Einstein-Maxwell theory. The uniqueness theorems assume a non-degenerate event horizon and therefore do not apply to supersymmetric black holes. Nevertheless, it would be very surprising if supersymmetric black holes turned out to be non-unique in four dimensions.\footnote{This paper will only discuss spacetimes containing a single black hole. Otherwise multi-black hole solutions \cite{8} would be an example of non-uniqueness.}

In five dimensions, the only evidence for the uniqueness assumption seems to have been that higher dimensional black holes appeared to have very similar properties to four dimensional ones. However, this is not really evidence at all because all known higher dimensional black hole solutions were derived using ans"atze based on simple generalizations of four dimensional black hole solutions, or were related by dualities to solutions based on such ans"atze. Therefore it is not very surprising that the known higher dimensional black holes had similar properties to four dimensional ones.

The situation has changed with the recent discovery of a class of five dimensional vacuum black holes that are completely unlike anything encountered in four dimensions – “black rings” \cite{9}. These are stationary black holes with event horizons of topology $S^1 \times S^2$. They can be regarded as rotating loops of black string, with the centrifugal force balancing the tendency of the ring to collapse under gravity. The existence of black rings implies that the uniqueness theorem for stationary black holes does not extend to five dimensions. This is because black rings can carry the same asymptotic charges as the black holes of spherical topology discovered by Myers and Perry \cite{10}.

In the first part of this paper, it will be suggested that there should be many more exotic black hole solutions in higher dimensions. Examining the steps that go into proving the uniqueness theorems in four dimensions suggests that a general stationary asymptotically flat black hole in higher dimensions should admit only two commuting Killing vector fields. However, all known higher dimensional black hole solutions have more symmetry. So there may exist large families of higher dimensional black hole solutions in addition to the known ones. This would imply that black hole uniqueness would always be badly violated in higher dimensions\footnote{It is tempting to conjecture that adding the requirement of stability would guarantee uniqueness \cite{11}, but there is no evidence for this since stability of higher dimensional black holes has never been studied.}, and emphasizes the importance of justifying the uniqueness assumption for supersymmetric black holes.

A uniqueness theorem has been proved for non-degenerate higher dimensional static black holes in Einstein-Maxwell \cite{12} and Einstein-Maxwell-dilaton theory \cite{13}, and for Einstein gravity
coupled to a $\sigma$-model [14]. The uniqueness assumption for static supersymmetric black holes in higher dimensions therefore seems plausible.

For rotating holes, it is not at all clear whether this assumption is correct. Rotating supersymmetric black holes seem to exist only in five dimensions – the first example was found by Breckenridge, Myers, Peet and Vafa (BMPV) [4]. It seems rather likely that charged black ring solutions should also exist, and if these had a regular supersymmetric limit then uniqueness of supersymmetric rotating black holes might be violated. Also, if there do exist higher dimensional black holes with fewer symmetries than any known solution then why not supersymmetric black holes with fewer symmetries than BMPV? It is clearly desirable to know whether this happens and, if not, whether a uniqueness theorem for supersymmetric black holes can be proved.

The main goal of this paper is to provide the first examples of uniqueness theorems for supersymmetric black holes, and thereby provide some justification for the uniqueness assumption made in the black hole entropy calculations. This is therefore a check on the consistency of the entropy calculations that can be performed at the level of classical supergravity.

The supergravity theory that will be considered is minimal $D = 5$ supergravity [15] because it is the simplest theory in which black hole uniqueness is known to be violated (the theory admits black ring solutions). Furthermore, the BMPV supersymmetric rotating black hole solution can be embedded in this theory [16]. In fact, this theory is sufficiently simple that it is possible to find all supersymmetric solutions [17]. Previously, the only theories for which this had been done were minimal $N = 2$, $D = 4$ supergravity [18] and some simple $D = 4$ generalizations [19].

The general supersymmetric solution obtained in [17] is sufficiently complicated that it is far from obvious which solutions correspond to black holes. In fact, the solution given in [17] is only valid away from any horizons that may be present in the spacetime. In this paper, it will be shown how a local analysis of the constraints imposed by supersymmetry in the neighbourhood of the horizon can be used to overcome this problem. The following theorem will be proved:

**Theorem 1.** Any supersymmetric solution with a (spatially) compact horizon has a near-horizon geometry that is locally isometric to one of the following maximally supersymmetric solutions: flat space, $AdS_3 \times S^2$, or the near-horizon geometry of the BMPV solution (of which $AdS_2 \times S^3$ is a special case). The geometry of the horizon in these three cases is the standard metric on $T^3$, $S^1 \times S^2$ or (a quotient of) a squashed $S^3$, respectively.

Obviously it is desirable to extend this near-horizon result to a global uniqueness theorem. We shall see that this can be done for the BMPV black hole subject to one additional assumption. Supersymmetry guarantees the existence of a globally defined Killing vector field which is everywhere timelike or null [20]. The assumption is that this Killing field has no null orbits outside the black hole horizon:

**Theorem 2.** The only asymptotically flat supersymmetric black hole solution with near-horizon geometry locally isometric to the near-horizon BMPV solution and with the supersymmetric Killing vector field timelike everywhere outside the horizon is the BMPV black hole.

These theorems constitute a uniqueness theorem for supersymmetric black holes whose near-horizon geometry is not flat space or $AdS_3 \times S^2$. Note that this is much stronger than simply proving uniqueness for given horizon topology – any supersymmetric black hole other than BMPV must have an event horizon with geometry $T^3$ or $S^1 \times S^2$. Of course, the latter possibility would describe a supersymmetric black ring so the above theorems do not exclude the existence
of such solutions. It will be interesting to see whether they exist, or whether the theorems can be strengthened to eliminate these exceptional cases.

This paper is organized as follows. Section 2 discusses general properties of higher dimensional black holes. Section 3 contains the uniqueness theorem. There is one appendix dealing with a special case that arises in the analysis.

## 2 Higher dimensional black holes

### 2.1 Black holes with fewer symmetries

All known stationary $D$-dimensional black hole solutions have at least $[(D + 1)/2]$ commuting isometries. The purpose of this section is to point out that this seems to be “too many”, i.e., in general one would expect fewer symmetries. Before explaining this, it is helpful to recall what happens for the analogous case of black string solutions.

Consider the uniform black string solution of the five dimensional vacuum Einstein equations. The metric is the product of the four dimensional Schwarzschild solution with a flat direction, so there are three commuting Killing vector fields, corresponding to time translations, rotations and spatial translations. If the string is compactified on a circle of asymptotic radius $L$ then one can define a dimensionless parameter

$$\eta = \frac{GM}{L^2},$$

(2.1)

where $M$ is the mass of the string. There is a particular value $\eta = \eta_c$ for which the uniform string solution admits a static zero-mode that breaks the translational symmetry [21]. This led to the conjecture [21, 22] that exact static black string solutions without translational symmetry should also exist. There is good perturbative [23] and numerical [24, 25] evidence that this is indeed the case, but the solutions are not known analytically.

These solutions have only two commuting Killing vector fields, which is one fewer than for the solutions that are known analytically.

To understand why there might also exist stationary black holes with fewer symmetries than any known solution, it is worth reviewing the steps that go into proving the uniqueness theorem for four dimensional black holes, and asking which steps can be generalized to higher dimensions. For simplicity, only vacuum black holes will be considered, although similar remarks should apply to non-degenerate charged black holes. It is probably also worth emphasizing that only asymptotically flat black holes will be considered in this paper.

The first step is the proof that the event horizon of a stationary black hole must have $S^2$ topology [28, 29]. This relies on the Gauss-Bonnet theorem applied to the (two dimensional) horizon and therefore does not generalize to higher dimensions. An alternative proof in four dimensions is based on the notion of “topological censorship” [30]. Consider a spacelike slice $\Sigma$ that intersects the future event horizon and let $H$ denote the intersection. Topological censorship requires that $\Sigma$ be simply connected. Note that $\Sigma$ has two boundaries, namely $H$ and the sphere at spatial infinity. Hence topological censorship requires that $H$ be cobordant to a sphere via a simply connected cobordism. For a stationary black hole, this can be shown to imply that $H$ is a sphere [31].

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3 See [26, 27] for attempts to construct such solutions analytically.
Topological censorship is also valid for $D > 4$ but it is much less restrictive. First, if $D > 4$ and there exists a cobordism from $H$ to the sphere then there also exists a simply connected cobordism. Secondly, a cobordism from $H$ to the sphere exists if, and only if, $H$ has vanishing Pontrjagin and Stiefel-Whitney numbers. For $D = 5$, $H$ is an oriented 3-manifold and hence automatically has vanishing Pontrjagin and Stiefel-Whitney numbers so topological censorship does not restrict the topology of the event horizon for $D = 5$ black holes. For $D = 6$, $H$ is a 4-manifold and topological censorship excludes, for example, $H = \mathbb{C}P^2$ because it has non-vanishing Pontrjagin and Stiefel-Whitney numbers.

In summary, there are very few useful restrictions on the topology of the event horizon of a general stationary black hole in higher dimensions. However, black rings are the only known example of stationary black holes with non-spherical horizons.

The next step is the four dimensional uniqueness proof that a stationary black hole must either be static or have an ergoregion. This theorem is straightforward to extend to higher dimensions. In the static case, it can then be shown that the only solution is the Schwarzschild solution, and this theorem has recently been extended to higher dimensions. A simple corollary is that a static higher dimensional black hole must have a spherical horizon.

The possibility of an ergoregion disjoint from the event horizon was excluded for four dimensional black holes. This proof relies on a technical theorem concerning maximal hypersurfaces; it will be assumed here that it can be generalized to higher dimensions. This implies that the stationary Killing vector field of a stationary, non-static, higher dimensional black hole is spacelike on the event horizon.

In four dimensions, it can be argued that the tangent vector to the null geodesic generators of the event horizon can be extended to give a Killing vector field $\xi$ of the full spacetime, which commutes with the stationary Killing vector field. The latter cannot be equal to $\xi$ since it is spacelike on the horizon. One can therefore write (after appropriately scaling $\xi$)

$$\xi = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi},$$

with $\partial/\partial \phi$ spacelike and Killing. It seems likely that this theorem could be extended to higher dimensions although, since the topology of the horizon is not known, the geometrical interpretation of $\phi$ is not clear. Roughly speaking, this Killing vector should correspond to a symmetry in the direction of rotation.

In four dimensions, the existence of two commuting Killing vector fields implies that the metric has to take a fairly simple form, and it can then be argued that any such solution to the Einstein equations is uniquely determined by its mass and angular momentum and must belong to the Kerr family of solutions. In higher dimensions, two Killing vector fields is not enough symmetry to write the metric in a useful form, and the existence of black rings shows that uniqueness should not be expected even when more symmetry is present.

These general arguments suggest that all stationary higher dimensional black holes must have two commuting symmetries. However, no known higher dimensional black hole solution has only two commuting symmetries. This suggests that higher dimensional black holes may be similar to black strings in the sense that there may exist undiscovered stationary solutions with fewer symmetries than the presently known solutions. More precisely,
Conjecture. There exist stationary, asymptotically flat black hole solutions of the $D > 4$ dimensional vacuum Einstein equations that admit exactly two commuting Killing vector fields.

These solutions would have to be non-static (because of the uniqueness theorem for static black holes [12]). If such solutions do exist then it seems unlikely that the Schwarzschild solution would be recovered as a limit. This would imply that such solutions must have an angular momentum that is bounded below (in terms of their mass), just as occurs for black rings.

If the above conjecture is correct then higher dimensional black holes would exhibit similar behaviour to black strings. There would be known solutions with lots of symmetry and new solutions with less symmetry. It is tempting to push this analogy further. Consider the case of five dimensions with a single non-vanishing angular momentum. Define a dimensionless parameter $\eta$ by

$$\eta \equiv \frac{27 \pi J^2}{32GM^3},$$

(2.3)

where $J$ and $M$ are the angular momentum and mass of a black hole. The known solutions are the Myers-Perry solutions [10] (which exist for $\eta < 1$) and black rings [9] ($\eta > \eta_* \approx 0.84$). These solutions have three commuting Killing vector fields $\partial/\partial t, \partial/\partial \phi$ and $\partial/\partial \psi$ where $\phi$ is the direction of rotation. The above conjecture suggests looking for new solutions without symmetry in the $\psi$ direction. The analogy with black strings suggests that there might be some critical value $\eta = \eta_c$ for which the Myers-Perry solution (or black ring) admits a stationary zero-mode that breaks the symmetry in the $\psi$ direction. Finding such a mode would therefore be evidence in favour of the above conjecture. However, the absence of such a mode would not rule out the existence of new solutions. For example, the topology of the new solutions might differ from that of the Myers-Perry solutions and black rings, in which case they would not be seen in perturbation theory about the known solutions.

2.2 Magnetic rings

The existence of black rings implies that stationary black holes in five dimensions are not uniquely specified by their asymptotic charges. If the above conjecture is correct then there exist further black hole solutions, and therefore black hole uniqueness is more severely violated. It is clearly desirable to know how many stationary higher dimensional black hole solutions have a given set of asymptotic charges. Are there finitely many or infinitely many? The purpose of the present subsection is to suggest that there may be a continuous infinity of solutions with a given set of asymptotic charges.

The black ring solutions obtained in [9] are solutions of the vacuum Einstein equations in five dimensions. It is interesting to ask whether electromagnetic generalizations exist. Consider Einstein-Maxwell theory in five dimensions, possibly with a Chern-Simons term. This theory admits two types of static black string solution: electric and magnetic. The electric solution becomes nakedly singular in the extremal limit. The extremal solution is best viewed as a

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5Examining perturbations of Myers-Perry solutions would also be of interest in view of the conjecture [9] that a five dimensional Myers-Perry black hole with a single non-vanishing angular momentum is classically unstable for $\eta$ close to 1.
smeared distribution of black holes. The magnetic solution has a regular extremal limit: this is the supersymmetric black string of minimal $N = 1, D = 5$ supergravity.

Black rings can be regarded as rotating loops of black string. Consider a rotating loop of magnetic black string. If such a solution exists then it would have vanishing electric charge. The magnetic charge of a localized configuration must vanish in four spatial dimensions. Therefore the only asymptotic charges that would be carried by such a solution are its mass and angular momentum. However, the solution would presumably be characterized by a third parameter $\alpha$ measuring the strength of the magnetic field. Therefore, if magnetic black rings exist, then they would be an example of a continuous family (labelled by $\alpha$) of solutions with the same asymptotic charges.

3 Uniqueness theorems

3.1 Introduction

The above considerations highlight how little is known about general properties of higher dimensional stationary black holes, and suggest that such black holes are highly non-unique, if non-static. This casts doubt on the uniqueness assumption that underlies the entropy calculations for supersymmetric rotating black holes. A uniqueness theorem is required in order to justify this assumption. In this section, the theorems stated in the introduction will be proved.

Proving uniqueness theorems for supersymmetric black holes is much easier than, say, attempting to generalize the known black hole uniqueness theorems to include degenerate horizons. This is because the existence of a globally defined super-covariantly constant spinor highly constrains the form of the spacetime. In fact, for minimal $N = 2, D = 4$ supergravity, it fully determines the local form of the metric. For minimal $D = 5$ supergravity, a simple algorithm can be given for the construction of all supersymmetric solutions. This will be reviewed in subsection 3.2.

The method of yields the general supersymmetric solution in a coordinate system that does not cover any event horizons in the spacetime. Therefore, the first step in the uniqueness proof is to introduce a coordinate system valid in the neighbourhood of a Killing horizon, and to repeat some of the analysis of in these coordinates. It turns out that this fully determines the local form of the near-horizon geometry, thereby proving Theorem 1. Theorem 2 is proved by showing that the local form of the near-horizon geometry, together with asymptotic flatness, is sufficient to select a unique solution from the general solution of, which must therefore be the known BMPV solution.

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6Hence it could not saturate the Bogomolnyi bound appropriate to an asymptotically flat spacetime and therefore would not be supersymmetric.
3.2 Minimal five dimensional supergravity

Minimal $N = 1$, $D = 5$ supergravity was constructed in [15]. The bosonic sector has action

$$S = \frac{1}{4\pi G} \int \left( \frac{1}{4} R \ast 1 - \frac{1}{2} F \wedge \ast F - \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right).$$

(3.1)

All purely bosonic supersymmetric solutions of this theory were obtained in [17] as follows. Starting from a commuting super-covariantly constant (Dirac) spinor $\epsilon$, one can construct a real scalar field $f$, a real vector field $V$ and three real two-form fields $X^{(i)}$:

$$f \sim i\bar{\epsilon}\epsilon, \quad V^\alpha \sim \bar{\epsilon}\gamma^\alpha \epsilon, \quad \left(X^{(1)} + i X^{(2)}\right)_{\alpha\beta} \sim \epsilon^T C\gamma_{\alpha\beta} \epsilon, \quad X^{(3)}_{\alpha\beta} \sim \bar{\epsilon}\gamma_{\alpha\beta} \epsilon.$$  

(3.2)

Fierz identities imply various algebraic identities between these quantities, for example

$$f^2 = -V^2, \quad i_V X^{(i)} = 0, \quad i_V \ast X^{(i)} = -f X^{(i)},$$

(3.3)  

(3.4)  

(3.5)

$$X^{(i)}_{\gamma\alpha} X^{(j)}_{\gamma\beta} = \delta_{ij} \left(f^2 \eta_{\alpha\beta} + V_{\alpha} V_{\beta}\right) - f \epsilon_{ijk} X^{(k)}_{\alpha\beta},$$

(3.6)

where $\epsilon_{123} = +1$ and, for a $p$-form $A$ and a vector $Y$, $i_Y A$ denotes the $(p - 1)$-form obtained by contracting $Y$ with the first index of $A$. Equation (3.3) implies that the vector field $V$ is timelike, null or zero. Since $V_0 \sim \epsilon^\dagger \epsilon$, the latter possibility occurs if, and only if, $\epsilon$ vanishes. Since $\epsilon$ is super-covariantly constant, the above quantities must also satisfy certain differential constraints [17]:

$$df = -\frac{2}{\sqrt{3}} i_V F,$$

(3.7)

$$D (\alpha V_{\beta}) = 0,$$

(3.8)

$$dV = -\frac{4}{\sqrt{3}} f F - \frac{2}{\sqrt{3}} \ast (F \wedge V),$$

(3.9)

and

$$D_\alpha X^{(i)}_{\beta\gamma} = \frac{1}{\sqrt{3}} \left[ 2 F^{\delta}_{\alpha} \left( \ast X^{(i)} \right)_{\delta\beta\gamma} - 2 F_{\beta\gamma}^{\delta} \left( \ast X^{(i)} \right)_{\gamma\alpha\delta} + \eta_{\alpha[\beta} F^\delta_{\gamma] \delta} \left( \ast X^{(i)} \right)_{\gamma]\delta\epsilon} \right],$$

(3.10)

which implies

$$dX^{(i)} = 0.$$  

(3.11)

These equations imply that $V$ is a Killing vector field that preserves the field strength (i.e. $\mathcal{L}_V F = 0$ where $\mathcal{L}$ denotes the Lie derivative), i.e., $V$ generates a symmetry of the full solution.

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7 Conventions: the metric has positive signature, curvature is defined so that de Sitter space has positive Ricci scalar. Curved indices are denoted by $\mu, \nu, \ldots$ and tangent space indices by $\alpha, \beta, \ldots$.

8 The precise definition of these objects is given in [17] in terms of symplectic-Majorana spinors. Converting to Dirac spinors may introduce numerical factors, which have not been calculated here.
If \( p \) is a point at which \( V \) vanishes then consider a timelike geodesic through \( p \). Let \( U \) denote the tangent vector to this geodesic. \( V \) is a Killing vector field so \( V \cdot U \) is conserved along the geodesic, and must therefore vanish because it vanishes at \( p \). Therefore \( U \) and \( V \) are orthogonal along the geodesic. However, \( U \) is timelike and \( V \) is non-spacelike so this implies that \( V \) must vanish everywhere along the geodesic, and therefore so must \( \epsilon \). This applies to all timelike geodesics through \( p \). Hence \( \epsilon \) vanishes in open regions to the future and past of \( p \). By analyticity, \( \epsilon \) must then vanish everywhere, which contradicts the assumption that the spacetime admits a super-covariantly constant spinor. Hence there cannot exist any point in the spacetime at which \( V \) or \( \epsilon \) vanishes.

Either \( f \) vanishes throughout the spacetime or there is some point \( p \) at which \( f \neq 0 \). These will be referred to as the “null case” and “timelike case” respectively. In the null case, \( V \) is a globally defined null Killing vector field \( V \). In fact the general solution in this case is a plane-fronted wave \([17]\). Special cases of this general solution include the magnetic black string solution \([41]\) and its near horizon geometry, \( AdS_3 \times S^2 \). The existence of a globally defined null Killing vector field implies that these solutions cannot describe black holes.

In the timelike case, by continuity, there is some topologically trivial neighbourhood \( U \) of \( p \) in which \( f \neq 0 \). Therefore \( V \) is a timelike Killing vector field in \( U \). It will be assumed that \( f > 0 \) without loss of generality \([17]\). Coordinates can be introduced so that the metric in \( U \) can be written \([17]\)

\[
ds^2 = -f^2 (dt + \omega) + f^{-1} ds_4^2,
\]

where \( V = \partial / \partial t \) and \( ds_4^2 \) is the metric on a four dimensional Riemannian “base space” orthogonal to the orbits of \( V \). Note that all metric components must be independent of \( t \). \( \omega \) is a 1-form that is defined by the equations

\[
i_V \omega = 0, \quad d\omega = -d \left( f^{-2} V \right).
\]

This determines \( \omega \) up to a gradient, which reflects the freedom to choose the \( t = 0 \) hypersurface. Supersymmetry requires that the base space be hyper-Kähler, with \( X^{(i)} \) the three complex structures and a volume form \( \eta_4 \) chosen so that these are anti-self-dual. This volume form is related to the volume form \( \eta \) on the five dimensional spacetime by

\[
\eta_4 = f i_V \eta.
\]

\( d\omega \) can be regarded as a 2-form on the base space and can therefore be decomposed into self-dual and anti-self-dual parts with respect to the base space:

\[
f d\omega = G^+ + G^-.
\]

It is then possible to solve for the field strength \([17]\):

\[
F = -\sqrt{3} d \left[ f^{-1} V \right] - \frac{1}{\sqrt{3}} G^+.
\]

The Bianchi identity for \( F \) yields

\[
dG^+ = 0,
\]
and the equation of motion for $F$ gives

$$
\Delta f^{-1} = \frac{4}{9}(G^+)^2,
$$

(3.18)

where $\Delta$ is the Laplacian associated with the base space metric and

$$(G^+)^2 \equiv \frac{1}{2}(G^+)^{mn}(G^+)^{mn},$$

(3.19)

where $m, n$ are indices on the base space, raised with the base space metric. The above equations guarantee that equations 3.12 and 3.16 yield a supersymmetric solution of the supergravity theory [17].

Any supersymmetric black hole solution must belong to the timelike class. Therefore the full black hole spacetime is determined by analytic continuation of a solution of the above form. The only known supersymmetric black hole solution of this theory is the BMPV black hole [4, 16], which has base space $R^4$, with metric

$$
d s_4^2 = d \rho^2 + \frac{\rho^2}{4} \left[ (\sigma^1_R)^2 + (\sigma^2_R)^2 + (\sigma^3_R)^2 \right],
$$

(3.20)

where $\sigma^i_R$ are left invariant 1-forms on $SU(2)$ – see [17] for details. The solution has

$$
\omega = \frac{j}{2\rho^2} \sigma^3_R,
$$

(3.21)

which implies $G^+ = 0$. The solution for $f$ is

$$
f^{-1} = 1 + \frac{\mu}{\rho^2}.
$$

(3.22)

The global properties of this solutions were investigated in detail in [16, 42]. The solution describes a black hole provided $j^2 < \mu^3$. If this bound is violated then it instead describes a regular spacetime with naked closed causal curves [42] and the microscopic description becomes non-unitary [43]. There exists evidence [42, 44] that it is physically impossible to add angular momentum to the black hole and violate the above bound.

### 3.3 Introduction of coordinates

The coordinate system introduced above is only valid locally, and does not cover regions in which $f$ vanishes, for example the event horizon of a black hole. In this section, a new set of coordinate will be introduced that do cover such a horizon. However, before doing this, it is necessary to argue that, for a supersymmetric black hole solution, the Killing vector field $V$ has the usual properties associated with the stationary Killing vector field of an equilibrium black hole spacetime.

First, consider the possibility that $V$ becomes null at some point $p$ outside the black hole. Equation 3.7 implies $V \cdot \partial f = 0$, so $f$ is constant along the orbits of $V$. Since $f$ vanishes at $p$, it must vanish along the orbit through $p$. Hence $V$ is null on this orbit. However, one would not expect a spacetime describing the rest frame of a single black hole to admit a Killing vector field...
with a null orbit outside the black hole. This is because such an orbit would correspond to an observer moving at the speed of light for whom the gravitational field would appear unchanging. We would expect that this is only possible on the event horizon. Therefore, we assume that $V$ is timelike everywhere outside the black hole, i.e. it will be assumed that $f > 0$ everywhere outside the black hole.

Now consider the behaviour of $V$ at infinity. If $V$ were to vanish at some point $p$ on $\mathcal{I}^+$ then consider an affinely parametrized outgoing null geodesic with an endpoint at $p$. Let $k$ denote the tangent vector to this geodesic. $V \cdot k$ is constant along the geodesic and vanishes at infinity. Hence $V \cdot k$ vanishes everywhere on the geodesic. Therefore $V$ must be proportional to $k$, and therefore null, along this geodesic. But the argument above excludes the possibility of $V$ being null outside the black hole. Hence $V$ cannot vanish on $\mathcal{I}^+$. Similarly, $V$ cannot vanish on $\mathcal{I}^-$.

If $V$ were to become null at some point $p$ on $\mathcal{I}^+$ then it is easy to see that $V$ must be everywhere tangent to the null geodesic generator of $\mathcal{I}^+$ through $p$. Once again, such a null symmetry would not be expected of a spacetime describing a black hole at rest so $V$ cannot be null anywhere on $\mathcal{I}^+$. Similarly, $V$ cannot be null anywhere on $\mathcal{I}^-$. These considerations establish that $V$ must be timelike everywhere outside the black hole and also on $\mathcal{I}^\pm$. If $f$ were to diverge anywhere on $\mathcal{I}^\pm$ then $V$ would be behaving as a boost symmetry, which is not expected for a black hole in its rest frame. Hence $f$ must be non-zero and bounded on $\mathcal{I}^\pm$.

It will be assumed that the future event horizon $\mathcal{H}^+$ has a single connected component. Since $V$ is an isometry, it must leave this horizon invariant and must therefore be null on $\mathcal{H}^+$. Let $\Sigma$ be a Cauchy surface for the exterior region of the black hole such that $\Sigma$ has a boundary $H$ on the future event horizon. A null Gaussian coordinate system can be set up in a neighbourhood of $H$ as follows (see [45] for more details). Introduce local coordinates $x^A$ ($A = 1, 2, 3$) on $H$. Let $p$ be a point on $H$ with coordinates $x^A$. Consider the future directed null geodesic generator of $\mathcal{H}^+$ that passes through $p$, with tangent vector $V$. The coordinates of a point affine parameter distance $u$ from $p$ along this generator will be defined to be $(u, x^A)$. This defines coordinates on a neighbourhood $\mathcal{U}$ of $H$ in $\mathcal{H}^+$ with $V = \partial/\partial u$. Now let $n$ be the unique past directed null vector field defined on $\mathcal{U}$ by $V \cdot n = 1$ and $n \cdot X = 0$ for all $X$ tangent to surfaces of constant $u$. Finally, consider the null geodesic from a point $p \in \mathcal{U}$ with tangent $n$. Let the coordinates of a point affine parameter distance $r$ along this geodesic be $(u, r, x^A)$ where $(u, x^A)$ are the coordinate of $p$.

It is easy to check that $\mathcal{L}_V n = 0$ on $\mathcal{H}^+$. Moreover, $V$ is a Killing vector field and hence geodesics are mapped to geodesics under the flow of $V$. Putting these facts together, under the flow of $V$ through a parameter distance $\delta$, the point with coordinates $(u, r, x^A)$ is mapped to the point with coordinates $(u + \delta, r, x^A)$. Hence

$$V = \partial/\partial u$$

(3.23)

everywhere, not just on the horizon.

Since $f$ vanishes at $r = 0$, differentiability implies

$$f = r \Delta(r, x^A),$$

(3.24)

for some function $\Delta$ independent of $u$ (as $V \cdot \partial f = 0$). The exterior of the black hole is $r > 0$ so $\Delta$ must be positive for $r > 0$. $\partial/\partial x^A$ is tangent to surfaces of constant $u$ in $\mathcal{H}^+$ and hence
orthogonal to $V$ at $r = 0$. Therefore $g_{uA} = r h_A(r, x^B)$ for some functions $h_A$ independent of $u$ (as $V$ is Killing). The full metric must take the form \[ds^2 = -r^2 \Delta^2 du^2 + 2dudr + 2rh_Audx^A + \gamma_{AB}dx^A dx^B,\] where $\gamma_{AB}$ is a function of $r$ and $x^A$. It was argued above that black holes must belong to the timelike family of solutions, for which $\Delta > 0$ for $r > 0$. However, the above line element is clearly also valid in the neighbourhood of a Killing horizon of $V$ in the null family, for which $\Delta \equiv 0$. Also note that the form of this metric guarantees the existence of a regular near horizon geometry, defined by the limit $r = \epsilon \tilde{r}$, $u = \tilde{u}/\epsilon$ and $\epsilon \to 0$.

3.4 Supersymmetry near the horizon

The next step in the proof is to examine the constraints imposed by supersymmetry in the above coordinate system.

Using the above form for the metric, equations [3.4] and [3.5] imply that the two forms can be written

$$X^{(i)} = dr \wedge Z^{(i)} + r \left( h \wedge Z^{(i)} - \Delta \ast_3 Z^{(i)} \right),$$

where $Z^{(i)} \equiv Z_A^{(i)}dx^A$, $h \equiv h_A dx^A$ and $\ast_3$ denotes the Hodge dual with respect to $\gamma_{AB}$. $X^{(i)}$ is globally defined so $Z^{(i)}$ are well-defined in a neighbourhood of $H$. The algebraic relations satisfied by $X^{(i)}$ imply

$$<Z^{(i)}, Z^{(j)}> = \delta^{ij}, \quad Z^{(i)} \wedge Z^{(j)} = \epsilon_{ijk} \ast_3 Z^{(k)},$$

where $<,>$ denotes the inner product defined by $\gamma_{AB}$. Closure of $X^{(i)}$ (equation 3.11) yields

$$\hat{d}Z^{(i)} = -\frac{1}{2} \partial_r (r \Delta) \epsilon_{ijk} Z^{(i)} \wedge Z^{(j)} + \partial_r (rh) \wedge Z^{(i)} - r \Delta \epsilon_{ijk} \partial_r Z^{(j)} \wedge Z^{(k)} + rh \wedge \partial_r Z^{(i)},$$

and

$$\ast_3 \hat{d}h - \hat{d}\Delta - \Delta h + r \partial_r \Delta h - 2r \Delta \partial_r h - r \ast_3 (h \wedge \partial_r h) - r \Delta^2 \epsilon_{ijk} Z^{(i)} < Z^{(j)}, \partial_r Z^{(k)} >= 0,$$

where, for a $p$-form $Y$ with only $A, B, C$ indices,

$$(\hat{d}Y)_{ABC...} \equiv (p+1) \partial_{[A} Y_{BC...]}. \quad (3.30)$$

For $r > 0$, $\omega$ can be defined as in equation 3.13, giving

$$\omega = -\frac{dr}{r^2 \Delta^2} - \frac{1}{r \Delta^2} h, \quad (3.31)$$

where an arbitrary gradient can be absorbed by shifting the $u = 0$ surface. The definition of $G^+$ can be rewritten as

$$G^+ = \frac{1}{2} (fd\omega + i_V * d\omega). \quad (3.32)$$

Computing $G^+$ then gives

$$G^+ = dr \wedge \mathcal{G} + r (h \wedge \mathcal{G} + \Delta \ast_3 \mathcal{G}), \quad (3.33)$$
where
\[
G = -\frac{3}{2r\Delta^2}\Delta + \frac{3}{2\Delta^2}\partial_r\Delta h - \frac{3}{2\Delta}\partial_r h - \frac{1}{2}\epsilon_{ijk}Z^{(i)} < Z^{(j)}, \partial_r Z^{(k)} > .
\] (3.34)

Equation 3.29 was used to eliminate \( \hat{d}h \) from this expression. Note the similarity between equations 3.26 and 3.33; this structure is a consequence of (anti)-self-duality on the base space.

Equation 3.16 now gives
\[
F = \frac{\sqrt{3}}{2} \left[-\partial_r (r\Delta)du \wedge dr - rdu \wedge \Delta h + \frac{1}{3}\epsilon_{ijk}dr \wedge Z^{(i)} < Z^{(j)}, \partial_r Z^{(k)} > - *_3 h
- r *_3 \partial_r h + \frac{r}{3}\epsilon_{ijk} \left(-2\Delta *_3 Z^{(i)} + h \wedge Z^{(i)} < Z^{(j)}, \partial_r Z^{(k)} > \right) \right].
\] (3.35)

Note that this is well-defined at \( r = 0 \), and has a well-defined near-horizon limit, even though \( G^+ \) need not be regular at \( r = 0 \).

The \( ABC \) component of the Bianchi identity for \( F \) (or equivalently, equation 3.17) now yields
\[
\hat{d} *_3 h = \mathcal{O}(r).
\] (3.36)

Now equations 3.29 and 3.36 give
\[
\hat{d} \Delta \wedge *_3 \hat{d} \Delta = \hat{d} \left[ \Delta \hat{d} h - \frac{1}{2}\Delta^2 *_3 h \right] + \mathcal{O}(r).
\] (3.37)

Integrating this equation over the compact 3-manifold \( H \) (which is at \( r = 0 \)) then implies
\[
\hat{d} \Delta = 0 \quad \text{on } H,
\] (3.38)

hence \( \Delta \) is constant on the event horizon. Equations 3.29 and 3.36 now yield
\[
\hat{d} h = \Delta *_3 h, \quad \hat{d} *_3 h = 0 \quad \text{on } H.
\] (3.39)

The \( ArB \) component of equation 3.10 gives
\[
\nabla_A Z^{(i)}_B = -\frac{1}{2}\Delta \left(*_3 Z^{(i)}\right)_A^B + \gamma_{AB} < h, Z^{(i)} > - Z^{(i)}_A h_B + \mathcal{O}(r),
\] (3.40)

where \( \nabla \) is the connection associated with \( \gamma_{AB} \). Taking another derivative and antisymmetrizing then yields an expression for the Riemann tensor of \( H \). From this, the Ricci tensor of \( H \) is (using 3.39)
\[
R_{AB} = \left(\frac{\Delta^2}{2} + h^2\right) \gamma_{AB} - \nabla_{(A} h_{B)} - h_A h_B,
\] (3.41)

where \( h^2 = h_A h^A \), raising indices with \( \gamma^{AB} \) on \( H \). Equations 3.39 imply
\[
\nabla^2 h_A = R_{AB} h^B - \Delta^2 h_A \quad \text{on } H.
\] (3.42)

Now consider
\[
I = \int_H \nabla_{(A} h_B) \nabla^{(A} h^{B)}.
\] (3.43)
Integrating by parts and using 3.42 and $\nabla_A h^A = 0$ (from 3.39) gives

$$I = \int_H \left( \Delta^2 h^2 - 2R_{AB} h^A h^B \right). \quad (3.44)$$

Finally, substituting in equation 3.41 and integrating by parts yields $I = 0$. Hence

$$\nabla_A h_B = 0 \quad \text{on } H. \quad (3.45)$$

Therefore, if non-zero, then $h$ is a Killing vector field on $H$. Substituting into 3.41 gives the Ricci tensor of $H$

$$R_{AB} = \left( \frac{\Delta^2}{2} + h^2 \right) \gamma_{AB} - h_A h_B. \quad (3.46)$$

This completely determines the curvature of $H$ because $H$ is a 3-manifold. Combining equations 3.39 and 3.45 gives

$$\nabla_A h_B = \frac{1}{2} \Delta \eta_{ABC} h^C, \quad (3.47)$$

where $\eta_{ABC}$ is the volume form of $H$. Note that this implies that $h^2$ is constant on $H$. Furthermore, combining equations 3.40 and 3.47 gives

$$[h, Z^{(i)}] = 0 \quad \text{on } H. \quad (3.48)$$

### 3.5 A special case

This subsection will consider the case in which $\Delta$ vanishes on $H$. If this happens then, on $H$,

$$\hat{d}Z^{(i)} = h \wedge Z^{(i)}, \quad (3.49)$$

which implies that the 1-forms $Z^{(i)}$ are hypersurface orthogonal, i.e., there exist functions $z^i$ and $K^{(i)}$ defined on $H$ so that $Z^{(i)} = K^{(i)} d z^i$ (no summation on $i$). Equation 3.49 then requires that the functions $K^{(i)}$ be proportional. The constants of proportionality can be absorbed into $z^i$, so $K^{(i)} = K$ for $i = 1, 2, 3$, i.e.,

$$\hat{d}Z^{(i)} = K d z^i. \quad (3.50)$$

Equation 3.49 also implies

$$h = \hat{d} \log K. \quad (3.51)$$

The functions $z^i$ can be used as local coordinates on $H$, i.e., $\{x^A\} = \{z^i\}$. Orthonormality of $Z^{(i)}$ implies that the metric on $H$ is conformally flat:

$$\gamma_{AB} dx^A dx^B = K^2 d z^i d z^i. \quad (3.52)$$

Equation 3.47 says that $h$ is covariantly constant. In these coordinates, this gives

$$K^{-1} \partial_i \partial_j K - 3 K^{-2} \partial_i K \partial_j K + K^{-2} \partial_k K \partial_k K \delta_{ij} = 0. \quad (3.53)$$

This equation was encountered in [17]. By shifting the origin and rescaling the coordinates $z^i$, the solutions can be written

$$K = 1, \quad \text{or} \quad K = \frac{L}{R}, \quad (3.54)$$
where \( R = \sqrt{z^iz^i} \). In the first case, this implies that the metric near \( H \) can be written
\[
 ds^2 = 2drdu + (\delta_{ij} + \mathcal{O}(r))dz^idz^j + \mathcal{O}(r^4)du^2 + \mathcal{O}(r^2)dudz^i. 
\] (3.55)

The near horizon limit of this solution is locally isometric to flat space with vanishing gauge field. Globally it must differ by some discrete identifications because \( H \) is assumed compact. The metric on \( H \) is flat so \( H \) must be some quotient of \( R^3 \) with its flat metric \([46] \), i.e., \( R^3 \) identified with respect to some subgroup of its isometry group. However, these identifications have to preserve the 1-forms \( Z^{(i)} \), which implies that they must be translations. So \( H \) is a compact manifold obtained by identifying \( R^3 \) with respect to certain translations, i.e., \( H \) must be a 3-torus \( T^3 \).

In the second case, it is convenient to use spherical polar coordinates \( \{x^A\} = (R, \theta, \phi) \). The solution can be written
\[
 ds^2 = 2drdu - 2\frac{r}{R}dudR + L^2\left(\frac{dR^2}{R^2} + d\theta^2 + \sin^2\theta d\phi^2\right) + \mathcal{O}(r^4)du^2 + \mathcal{O}(r^2)dudx^A + \mathcal{O}(r)dx^A dx^B. 
\] (3.56)

The near horizon limit of such a solution is locally isometric to the (maximally supersymmetric) \( AdS_3 \times S^2 \) solution (to see this, let \( r = vR/L \) for some new coordinate \( v \)). Globally, it must differ because \( H \) is compact. The metric on \( H \) can be written
\[
 ds_3^2 = L^2\left(dZ^2 + d\theta^2 + \sin^2\theta d\phi^2\right), 
\] (3.57)

where \( Z = \log R \) (note that \( R \) must be bounded away from zero because the 1-forms \( Z^{(i)} \) are well-defined on \( H \)). Hence \( H \) is locally isometric to the standard metric on \( R \times S^2 \), which presumably implies that \( H \) can be obtained as a quotient of \( R \times S^2 \). However, the only elements of the isometry group of \( R \times S^2 \) which preserve the 1-forms \( Z^{(i)} \) are translations \( Z \rightarrow Z + \text{constant} \). Hence \( H \) must be globally isometric to the standard metric on \( S^1 \times S^2 \) with \( Z \sim Z + l \) for some \( l \).

The \( AdS_3 \times S^2 \) solution arises as the near-horizon geometry of a momentum-carrying black string wrapped around a compact Kaluza-Klein direction \([10] \). To fully understand uniqueness of supersymmetric black holes it is necessary to know whether \( AdS_3 \times S^2 \) can also arise as the near-horizon geometry of a black hole. Such a solution would have horizon topology \( S^1 \times S^2 \), i.e., it would be a supersymmetric black ring. Some restrictions on the geometry of the base space of such solutions (if they exist) are obtained in the Appendix.

### 3.6 Near horizon geometry

Now consider the case in which \( \Delta > 0 \) on \( H \). It will be shown that the near horizon geometry must be locally isometric to that of the BMPV black hole. First define a set of 1-forms on \( H \) by \([10] \)
\[
 \sigma^i_L = \frac{\Delta^2 + h^2}{\Delta} Z^{(i)} + \frac{1}{\Delta} d\left(h^A Z^{(i)}_A \right). 
\] (3.58)

\(^9\) If the string does not carry momentum then it cannot be identified to yield a regular compact event horizon \([11] \). \(^{10}\) All equations in this subsection are evaluated on \( H \), so hats will not be included on exterior derivatives.
Using equations 3.40 and 3.47, it can be shown that these 1-forms obey
\[ d\sigma^i_L = -\frac{1}{2}\epsilon_{ijk}\sigma^j_L \wedge \sigma^k_L, \] (3.59)

and
\[ [h, \sigma^i_L] = 0. \] (3.60)

The vector fields dual to these 1-forms are
\[ \xi^i_L = \frac{\Delta}{\Delta^2 + h^2}Z^{(i)} - \frac{1}{\Delta^2 + h^2} h^3 \left( h \wedge Z^{(i)} \right). \] (3.61)

Equations 3.40 and 3.47 imply that these are Killing vector fields:
\[ \nabla_A \left( \xi^i_L \right)_B = 0, \] (3.62)

and satisfy the commutation relations of $SU(2)$:
\[ [\xi^i_L, \xi^j_L] = \epsilon_{ijk}\xi^k_L. \] (3.63)

Furthermore, these Killing vector fields commute with $h$:
\[ [h, \xi^i_L] = 0. \] (3.64)

These considerations show that, if $h \neq 0$ then $H$ admits four globally defined Killing vector fields satisfying the commutation relations of $SU(2) \times U(1)$.

If $h \neq 0$ then local coordinates can now be introduced as follows. Let $\hat{h} = h/\sqrt{h^2}$, and
\[ \hat{x}^i = \hat{h}^A Z_A^{(i)} \quad \text{so} \quad \hat{x}^i \hat{x}^i = 1. \] (3.65)

Equations 3.47 and 3.40 imply
\[ \left( d\hat{x}^i d\hat{x}^i \right)_{AB} = \left( \Delta^2 + h^2 \right) \left( \gamma_{AB} - \hat{h}_A \hat{h}_B \right). \] (3.66)

Note that $\mathcal{L}_h \hat{x}^i = 0$ so $\hat{x}^i$ is constant along integral curves of $h$. In some open set it is therefore possible to use $\hat{x}^i$ and the parameter along these curves as coordinates. It is convenient to define
\[ \mu = \frac{4}{\Delta^2 + h^2}, \quad j = \pm \frac{8\sqrt{h^2}}{\left( \Delta^2 + h^2 \right)^2}, \] (3.67)

where the sign of $j$ will be left arbitrary. Note that $j^2 < \mu^3$. To bring the metric to standard form, introduce polar coordinates
\[ \hat{x}^1 = -\cos \phi \sin \theta, \quad \hat{x}^2 = \sin \phi \sin \theta, \quad \hat{x}^3 = \cos \theta, \] (3.68)

and let $\psi$ be the parameter along the integral curves of $h$, normalized so that
\[ h = -4j\mu^{-5/2} \left( 1 - \frac{j^2}{\mu^3} \right)^{-1/2} \frac{\partial}{\partial \psi}. \] (3.69)
The metric must take the form

\[ ds_3^2 = \gamma_{AB} dx^A dx^B = \frac{\mu}{4} \left[ \left( 1 - \frac{j^2}{\mu^3} \right) (d\psi + A)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right], \]

for some locally defined 1-form \( A \) on \( H \). Equation 3.47 determines \( A \) up to a gradient, which is just the freedom to choose the \( \psi = 0 \) surface. A convenient choice is

\[ A = \cos \theta d\phi. \]

which also fixes the orientation of \( H \) so that \( d\theta \wedge d\psi \wedge d\phi \) is positively oriented. The metric on \( H \) now takes the form

\[ ds_3^2 = \frac{\mu}{4} \left[ \left( 1 - \frac{j^2}{\mu^3} \right) (d\psi + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right], \]

which is the standard form of the metric on a squashed \( S^3 \). However, this is a local result – globally, \( H \) could differ from a squashed \( S^3 \) by discrete identifications.

Writing \( h \) as a 1-form gives

\[ h = -j \mu^{-3/2} \left( 1 - \frac{j^2}{\mu^3} \right)^{1/2} (d\psi + \cos \theta d\phi). \]

The case \( h = 0 \) (i.e. \( j = 0 \)) is much simpler. Equation 3.46 establishes that \( H \) is a three dimensional compact Einstein space of positive curvature and hence locally isometric to \( S^3 \) with its round metric. Therefore the above local coordinates can also be introduced in this case, and the metric is as above with \( j = 0 \).

It is worth summarizing what has been shown. Local coordinates have been introduced in a neighbourhood of the horizon \(( r = 0 \) and explicit expressions for the local behaviour of \( \Delta \), \( h_A \) and \( \gamma_{AB} \) at \( r = 0 \) has been obtained. Using the expressions for the metric and field strength (equations 3.25 and 3.35), it is now clear that the above analysis has fully determined the local form of the near-horizon solution. Since this local form is unique, it must agree with that of the BMPV solution. So the above analysis proves that the near horizon geometry of any supersymmetric black hole solution with \( \Delta > 0 \) on \( H \) must be locally isometric to that of the BMPV solution.

It has been proved that \( H \) is locally isometric to a squashed \( S^3 \) when \( j \neq 0 \) and a round \( S^3 \) when \( j = 0 \). In the latter case, \( H \) must be globally isometric to a discrete quotient of a round \( S^3 \) (since \( H \) is a positive Einstein 3-manifold) and in the former case, \( H \) is presumably globally isometric to a discrete quotient of a squashed \( S^3 \). The question of which quotients are consistent with supersymmetry can be deduced from the existence of the vector fields \( \xi^L \) and \( h \). Whatever quotient is taken must preserve these vector fields.

If \( j = 0 \) then \( \xi^L \) generate the \( SU(2)_L \) subgroup of the \( SU(2)_L \times SU(2)_R \) isometry group of \( S^3 \). The allowed quotients must therefore be subgroups of \( SU(2)_R \). So \( H \) is of the form \( S^3/\Gamma \), where \( \Gamma \) is a discrete subgroup of \( SU(2)_R \).

If \( j \neq 0 \) then \( \xi^L \) generate the \( SU(2) \) subgroup, and \( h \) the \( U(1) \) subgroup of the \( SU(2) \times U(1) \) isometry group of a squashed \( S^3 \). The allowed quotients must be subgroups of \( U(1) \), and therefore cyclic groups. Hence \( H \) must be a quotient of a squashed \( S^3 \) by a cyclic group, i.e., \( H \) is a squashed lens space.
3.7 Global constraints

The purpose of this subsection is to prove Theorem 2, i.e., to show that a supersymmetric black hole whose near-horizon geometry is locally isometric to that of the BMPV solution must actually be globally isometric to the BMPV solution. This is where detailed knowledge of the general supersymmetric solution [17] of minimal $D = 5$ supergravity is required.

The first step is to write the near-horizon solution in the form of equation 3.12. Doing so, the metric on the base space is

$$ds_4^2 = r\Delta \left( \gamma_{AB} + \frac{1}{\Delta^2} h_A h_B \right) dx^A dx^B + \frac{dr^2}{r\Delta} + \frac{2}{\Delta} dr h_A dx^A. \quad (3.74)$$

A priori, there is no reason why this metric should be regular at $r = 0$ because the form 3.12 is only valid for $r > 0$. It has just been demonstrated that coordinates $x^A = (\psi, \theta, \phi)$ can be introduced so that the metric on $H$ is locally isometric to that of a squashed $S^3$. In these coordinates,

$$ds_4^2 = \frac{r^{1/2}}{2} \left( 1 - \frac{j^2}{\mu^3} \right)^{1/2} \left[ (d\psi + \cos \theta d\phi - j r^{-1} \mu^{-3/2} \left( 1 - \frac{j^2}{\mu^3} \right)^{-1/2} dr)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right]$$

$$+ \frac{dr^2}{2r^{1/2}} \left( 1 - \frac{j^2}{\mu^3} \right)^{1/2} + \mathcal{O}(r^0) dr^2 + \mathcal{O}(r) dr A + \mathcal{O}(r^2) dx^A dx^B. \quad (3.75)$$

Now let

$$R = 2^{1/2} r^{1/2} \mu^{1/4} \left( 1 - \frac{j^2}{\mu^3} \right)^{1/4}, \quad (3.76)$$

and

$$\psi' = \psi - 2j \mu^{-3/2} \left( 1 - \frac{j^2}{\mu^3} \right)^{-1/2} \log R, \quad (3.77)$$

so

$$ds_4^2 = dR^2 + \frac{R^2}{4} \left[ (d\psi' + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right]$$

$$+ \mathcal{O}(R^2) dR^2 + \mathcal{O}(R^3) dR A + \mathcal{O}(R^4) dx^A dx^B, \quad (3.78)$$

where $x^A = (\psi', \theta, \phi)$. It should be emphasized that this is only valid locally – no assumptions have been made about the ranges of the coordinates $(\psi, \theta, \phi)$. However, it can be seen that the base space metric is locally flat near $R = 0$. Near $R = 0$, the surfaces of constant $R$ are positive Einstein spaces and must therefore be globally isometric to $S^3$ with its round metric, identified under some subgroup $\Gamma$ of its $SU(2) \times SU(2)$ isometry group. Generally, this implies that there will be a conical singularity at $R = 0$. The metric has to be hyper-Kähler for $R > 0$, so the holonomy has to be a subgroup of $SU(2)$. Therefore the singularity at $R = 0$ has to be an $A - D - E$ orbifold singularity (see [14] for a review).\footnote{\textsuperscript{11} This result also follows from the properties of $H$ deduced in the last subsection.}

Now, $f$ approaches a constant at infinity (since $V$ is the stationary Killing vector field), and in equation 3.12, $\omega$ must vanish fast enough at infinity for the ADM angular momentum to
be well-defined. Asymptotic flatness then implies that the base space has to be asymptotically Euclidean. Near $R = 0$, the metric is well-approximated by flat space with an $A - D - E$ singularity at the origin, and it is known [47] that such singularities can be resolved by “blowing up” the singularity. This produces a complete asymptotically Euclidean hyper-Kähler space. However, the only such space is $\mathbb{R}^4$ with its standard metric [48]. In other words, there can’t have been a singularity at $R = 0$ after all!

It follows that the metric 3.78 describes a portion of global flat space, i.e., the coordinates $(\theta, \phi, \psi)$ must have their standard ranges $0 \leq \theta \leq \pi$, $\phi \sim \phi + 2\pi$ and $\psi' \sim \psi' + 4\pi$ (which implies $\psi \sim \psi + 4\pi$). Hence $H$ has $S^3$ topology. The base space is globally flat, with metric

$$ds^2 = d\rho^2 + \frac{\rho^2}{4} \left[ (\sigma_R^1)^2 + (\sigma_R^2)^2 + (\sigma_R^3)^2 \right],$$

(3.79)

where

$$R = \rho + O(\rho^3),$$

(3.80)

and

$$\sigma_R^1 = -\sin \psi' d\theta + \cos \psi' \sin \theta d\phi$$
$$\sigma_R^2 = \cos \psi' d\theta + \sin \psi' \sin \theta d\phi$$
$$\sigma_R^3 = d\psi' + \cos \theta d\phi.$$  

(3.81)

Having established that the base space is flat, the next step is to show that the solution must have $G^+ = 0$. Consider equation 3.33. $\Delta$ is constant and non-zero at $r = 0$, which implies that $G^+$ is well-defined in a neighbourhood of $H$. (In fact, $G^+$ vanishes when restricted to $H$.) It is easy to see from the behaviour of $G^+$ near $r = 0$ that $G^+$ is regular at the origin of the base space. Hence $G^+$ is globally defined on $\mathbb{R}^4$. $G^+$ must vanish at infinity in $\mathbb{R}^4$ (because $\omega$ has to decay fast enough for the solution to be asymptotically flat). Furthermore $G^+$ is closed and must therefore belong to $H^2_{\text{cpt}}(\mathbb{R}^4, R)$, the second compactly supported cohomology group on $\mathbb{R}^4$. However, this group is trivial, so there exists a 1-form $\Gamma$ globally defined on $\mathbb{R}^4$ such that $G^+ = d\Gamma$ with $\Gamma$ vanishing at infinity. Now consider

$$0 = \int_{S^3} \Gamma \wedge G^+ = \int_{\mathbb{R}^4} G^+ \wedge G^+ = \int_{\mathbb{R}^4} d^4x \left( G^+ \right)^2,$$

(3.82)

where the first integral is taken over the three sphere at infinity, and the final equality follows from the self-duality of $G^+$. Hence $G^+$ must vanish everywhere on the base space and therefore everywhere in the spacetime.

The vanishing of $G^+$ implies (equation 3.18) that $f^{-1}$ is harmonic on the base space. Near the origin,

$$f^{-1} = \frac{1}{r\Delta} = \frac{\mu}{\rho^2} + g,$$

(3.83)

where $g$ is $O(\rho^0)$ near $\rho = 0$. As mentioned above, for a single black hole, it can be assumed that $f$ is positive everywhere outside the black hole, and hence $f^{-1}$ must be finite for $\rho > 0$.

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12 If one relaxes the condition of asymptotic flatness then an $A - D - E$ singularity can occur at $R = 0$. Such behaviour can by obtained by taking appropriate quotients of the BMPV solution.
Furthermore, $f$ must approach a constant at infinity. This implies that $g$ is regular and bounded on $R^4$. But $g$ must be harmonic (as $f^{-1}$ is) and hence constant. By rescaling the coordinates, $g$ can be set to 1, so

$$f^{-1} = 1 + \frac{\mu}{\rho^2}. \quad (3.84)$$

The final step is to prove that $\omega$ is uniquely determined. Topologically, it is clear that $\omega$ can be globally defined defined on $R^4 - \{0\}$ by equation 3.13. Now

$$-f^{-2} V = du - \frac{dr}{r^2 \Delta^2} - \frac{h}{r \Delta^2} \quad \Rightarrow \quad \omega = \frac{j}{4r} \mu^{-1/2} \left(1 - \frac{j^2}{\mu^2}\right)^{-1/2} \sigma_R^3 + \mathcal{O}(r^{-1}) \, dr + \mathcal{O}(r^0) \, dx^A, \quad (3.85)$$

where

$$t = u + \frac{\mu}{4r} \quad (3.86)$$

is a time coordinate defined for $r > 0$. Hence, up to a gradient (which can be absorbed into $t$), $\omega$ is given by

$$\omega = \frac{j}{4r} \mu^{-1/2} \left(1 - \frac{j^2}{\mu^2}\right)^{-1/2} \sigma_R^3 + \mathcal{O}(r^{-1}) \, dr + \mathcal{O}(r^0) \, dx^A, \quad (3.87)$$

For $\rho > 0$, $\omega$ has to satisfy the following criteria. First, the vanishing of $G^+$ implies that $d\omega$ has to be anti-self-dual with respect to the metric on the base space. Secondly, asymptotic flatness requires $\omega = \mathcal{O}(\rho^{-3})$ as $\rho \to \infty$. Finally, as $\rho \to 0$, $\omega$ must be given by equation 3.87. To prove uniqueness of $\omega$, assume that there were two solutions $\omega_1$ and $\omega_2$ satisfying these criteria. Let $\tilde{\omega} = \omega_1 - \omega_2$. Hence $\tilde{\omega}$ is $\mathcal{O}(\rho^{-3})$ as $\rho \to \infty$, and, near $\rho = 0$,

$$\tilde{\omega} = \mathcal{O}(\rho^{-1}) d\rho + \mathcal{O}(\rho^0) dx^A. \quad (3.88)$$

Therefore $\tilde{\omega}$ can be written, on $R^4 - \{0\}$, as

$$\tilde{\omega} = \frac{\alpha}{\rho} \, d\rho + \nu, \quad (3.89)$$

where $\nu \equiv \nu_A dx^A$ is a ($\rho$-dependent) 1-form defined on $S^3$. The quantities $\alpha$ and $\nu$ are well-behaved as $\rho \to 0$. Anti-self duality of $d\tilde{\omega}$ reduces to

$$\hat{d} \alpha = *_3 \hat{d} \nu + \rho \partial_\rho \nu, \quad (3.90)$$

where $\hat{d}$ and $*_3$ are now defined on the round unit $S^3$. This equation implies that $\alpha$ becomes a harmonic function on $S^3$ as $\rho \to 0$. However, the only harmonic functions on $S^3$ are constant, so $\alpha$ must become constant as $\rho \to 0$, i.e., $d\alpha \to 0$ as $\rho \to 0$. Therefore $d\nu \to 0$ as $\rho \to 0$. However this implies

$$0 = \lim_{\epsilon \to 0} \int_{\rho = \epsilon} \tilde{\omega} \wedge d\tilde{\omega} = \int_{R^4 - \{0\}} d\tilde{\omega} \wedge d\tilde{\omega} = -\int_{R^4 - \{0\}} d^4 x (d\tilde{\omega})^2, \quad (3.91)$$
where the surface term at infinity vanishes because of the boundary conditions on \( \tilde{\omega} \), and the final equality is a consequence of anti-self-duality. It follows then, that \( \tilde{\omega} \) is closed and hence \( \tilde{\omega} = d\lambda \) for some function \( \lambda \) defined for \( \rho > 0 \). Therefore, \( \omega_1 \) and \( \omega_2 \) differ at most by a gradient. Hence the general solution for \( \omega \) must agree with the BMPV solution up to a gradient:

\[
\omega = \frac{j}{2\rho^2} \sigma^3_R + d\lambda.
\] (3.92)

Finally, this gradient can be absorbed into the time coordinate \( t \), and then

\[
V = -f^2 \left( dt + \frac{j}{2\rho^2} \sigma^3_R \right)
\] (3.93)

for \( r > 0 \). Since the base space is flat, and \( f \) is given by equation [3.84], the solution is identical to the BMPV solution for \( r > 0 \).

### 3.8 Discussion

The above results constitute the first examples of uniqueness theorems for supersymmetric black holes. Theorem 1 was proved by a local analysis of the constraints imposed by supersymmetry in a neighbourhood of the horizon. It was shown that the near-horizon geometry is completely determined. The proof of Theorem 2 was based on the general form for all supersymmetric solutions determined in [17]. Asymptotic flatness and the boundary conditions obtained from the near-horizon geometry select a unique solution from this class.

Note that Theorem 1 determines the near-horizon geometry of any (i.e. not necessarily asymptotically flat) supersymmetric solution that admits a compact Killing horizon preserved by \( V \). The near-horizon geometry of such a solution has to be locally isometric to flat space, to \( AdS_3 \times S^2 \), or the near-horizon geometry of the BMPV solution (of which \( AdS_2 \times S^3 \) is a special case). Furthermore, in each case, the allowed possibilities for the spatial geometry of the event horizon (i.e. of \( H \)) have been determined. In the flat case, \( H \) must be \( T^3 \) with its flat metric and in the \( AdS_3 \times S^2 \) case, \( H \) must be \( S^1 \times S^2 \) with the usual metric. In the case of a BMPV near-horizon geometry there are more possibilities. If \( j \neq 0 \) then \( H \) must be a squashed lens space. If \( j = 0 \) then the near-horizon BMPV geometry reduces to \( AdS_2 \times S^3 \), and in this case \( H \) must be a quotient of a round \( S^3 \) by a discrete subgroup of \( SU(2) \).

Even if a spacetime has a non-compact Killing horizon, it is often possible to make identifications to render the horizon compact. For example, \( AdS_3 \times S^2 \) arises as the near-horizon geometry of a magnetic black string wrapped around a compact Kaluza-Klein direction, with momentum around this direction. A supersymmetric spacetimes admitting a Killing horizon for which the analysis of this paper does not determine the near-horizon geometry would have to satisfy one of two criteria. Either the horizon would not be preserved by \( V \), or the horizon would be non-compact and could not be rendered compact by identifications without breaking supersymmetry. The former case is not of much physical interest since one is usually interested in event horizons, which must be preserved by all Killing vector fields.

It would be interesting to see whether the above method could be extended to prove uniqueness theorems for other supergravity theories. For example, proving uniqueness of supersymmetric black holes in minimal \( N = 2, D = 4 \) supergravity amounts to proving the long-standing
conjecture [8] that the only black holes in the Israel-Wilson-Perjes (IWP) class of solutions are the Mujumdar-Papapetrou multi-black hole solutions. This is because all supersymmetric solutions of this theory are known [18], and fall into a timelike and a null class, as for the minimal $D = 5$ theory. The IWP solutions constitute the timelike class. It seems very likely that this conjecture could be proved easily using the methods of this paper, i.e., first constructing bosonic objects from the super-covariantly constant spinor, using these to determine the form of the near horizon geometry (presumably either flat space or $AdS_2 \times S^2$), and then showing that this information together with asymptotic flatness determines a unique member of the IWP family.

Of more physical interest would be the extension of the above results to more complicated supergravity theories, for example the maximally supersymmetric theories in $D = 4, 5$. It seems rather unlikely that the general supersymmetric solution of these theories could be obtained using the methods of [18, 17], so a complete uniqueness proof is probably not possible using the methods of this paper. However, it might be possible to determine all possible near-horizon geometries of solutions with compact Killing horizons (one would start with the metric 3.25 and assume that $\Delta$, $h_A$ and $\gamma_{AB}$ are independent of $r$, since this is what happens in the near-horizon limit). This information might lead to an understanding (in classical supergravity) of why supersymmetric rotating black hole solutions only seem to exist in $D = 5$. Finally, it might be possible to use the methods of the present paper to classify possible near-horizon geometries of supersymmetric solutions of $D = 10, 11$ supergravity theories.

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Appendix

The purpose of this appendix is to examine the possibility of supersymmetric black holes with near-horizon geometry $AdS_3 \times S^2$. Some constraints on the form of the base space of such a solution will be obtained. It is best read after subsection 3.7 because it relies on results developed there.

Since $\Delta$ vanishes at $r = 0$, but is not identically zero (this would correspond to the null class) then by analyticity it must be possible to write

$$\Delta = r^p \tilde{\Delta} ,$$

where $p$ is a positive integer, $\tilde{\Delta}$ is not identically zero on $H$, and $\tilde{\Delta} > 0$ for $r > 0$. The Maxwell equation 3.18 then gives

$$\nabla^2 \tilde{\Delta} - (2p - 1) h^A \nabla_A \tilde{\Delta} = -p(p - 1) \tilde{\Delta} h^2 + O(r).$$

Integrating this equation over $H$ (using $\nabla_A h^A = 0$) implies

$$p(p - 1) \tilde{\Delta} h^2 = 0 \quad \text{on } H$$

in the first version of this paper. This work was supported by PPARC.
Since $h^2 \neq 0$ on $H$ for the solution $3.36$, it follows that this solution must have $p = 1$. Substituting this back into equation $A.2$ multiplying by $\tilde{\Delta}$ and integrating over $H$ gives

$$\tilde{\Delta} = \tilde{\Delta}_0 \quad \text{on} \ H,$$

where $\tilde{\Delta}_0$ is a positive constant. Next, from subsection $3.5$ $h$ can be written as

$$h = \frac{dR}{R} + rh_1 + r^2h_2,$$

where $h_1$ is independent of $r$ and $h_2$ is smooth at $r = 0$. Equation $3.29$ implies

$$\hat{d}(Rh_1) = 0.$$

Hence there is some function $\lambda$ defined locally on $H$ such that

$$h_1 = \frac{L}{R}d\lambda.$$

The base space metric can be calculated from equation $3.74$

$$ds_4^2 = \tilde{\Delta}^{-1} \left( \frac{r}{R} \right)^2 \left[ d \left( L\lambda - \frac{R}{r} \right) + Rh_2 \right]^2 + r^2\tilde{\Delta}\gamma_{AB}dx^Adx^B.$$

Define a new coordinate $\rho$ by

$$r = \rho X,$$

where

$$X = 1 + \frac{L\lambda\rho}{R}.$$

The base space metric is then

$$ds_4^2 = X^{-2} \left\{ \tilde{\Delta}^{-1} \left[ \frac{d\rho}{\rho} - \frac{dR}{R} + \frac{\rho^2h_2}{X} \right]^2 + L^2\rho^2\tilde{\Delta} \left[ \frac{dR^2}{R^2} + d\Omega^2 + \mathcal{O}(\rho)dx^Adx^B \right] \right\},$$

where the metric on $H$ deduced in subsection $3.5$ has been used and $x^A = (R, \theta, \phi)$. Completing the square on $dR/R$ gives

$$ds_4^2 = X^{-2} \left\{ \tilde{\Delta}^{-1}Y \left[ \frac{d\rho}{\rho} + \frac{\rho^2h_2}{X} \right]^2 + \tilde{\Delta}Y^{-1}L^2\rho^2 \left[ \frac{d\rho}{\rho} + \frac{\rho^2h_2}{X} \right]^2 ight. + \left. L^2\rho^2\tilde{\Delta}d\Omega^2 + \mathcal{O}(\rho^3)dx^Adx^B \right\},$$

where

$$Y \equiv 1 + L^2\rho^2\tilde{\Delta}^2.$$

Hence

$$ds_4^2 = (1 + \mathcal{O}(\rho)) \left\{ \tilde{\Delta}_0^{-1} \left[ \frac{d\rho}{\rho} + \mathcal{O}(\rho)d\rho + \mathcal{O}(\rho^2)dx^A \right]^2 + L^2\tilde{\Delta}_0 \left( \rho^2 + \rho^2d\Omega^2 \right) + \mathcal{O}(\rho)d\rho^2 + \mathcal{O}(\rho^3)d\rho dx^A + \mathcal{O}(\rho^3)dx^Adx^B \right\}.$$
Now define
\[ x^4 = \tilde{\Delta}_0^{-1/2} \log \frac{R}{\rho}, \quad \tilde{\rho} = L \tilde{\Delta}_0^{1/2} \rho. \tag{A.15} \]

Then
\[
\begin{align*}
ds_4^2 &= d\tilde{\rho}^2 + \tilde{\rho}^2 d\Omega^2 + (dx^4)^2 + \mathcal{O}(\tilde{\rho})d\tilde{\rho} dx^4 + \mathcal{O}(\tilde{\rho}^3)d\tilde{\rho} dx'^A \\
&\quad + \mathcal{O}(\tilde{\rho})(dx^4)^2 + \mathcal{O}(\tilde{\rho}^2)dx^4 dx'^A + \mathcal{O}(\tilde{\rho}^3)dx'^A dx'^B,
\end{align*}
\tag{A.16}
\]
where \(x'^A = (x^4, \theta, \phi)\). The identifications inherited from \(H\) imply that \(\theta\) and \(\phi\) parametrize a two-sphere and \(x^4\) is periodically identified. Superficially, the base space is therefore regular at \(\tilde{\rho} = 0\). However, more careful inspection reveals that not all of the correction terms above are necessarily smooth at \(\tilde{\rho} = 0\) so this conclusion may be incorrect. Asymptotic flatness requires the base space to be asymptotically Euclidean. Hence if the base space is regular then it must be global \(\mathbb{R}^4\) with its flat metric.

It is not clear whether supersymmetric black holes with near-horizon geometry \(AdS_3 \times S^2\) actually exist. Of course \(AdS_3 \times S^2\) does arise as the near horizon geometry of a black \textit{string}. Such strings exist in both the null class and the timelike class. To get such a string from the timelike class, one can take the base space to be \(\mathbb{R}^4\) parametrized as in (A.16) (neglecting the corrections) and then follow the method of section 3.7 of [17] (with \(H = 1, \chi_i = \omega_i = 0\, \text{and point sources for the harmonic functions})

References

[1] A. Strominger and C. Vafa, Phys. Lett. B379, 99 (1996). [arXiv:hep-th/9601029]
[2] J.M. Maldacena and A. Strominger, Phys. Rev. Lett. 77, 428 (1996). [arXiv:hep-th/9603060]
[3] C.V. Johnson, R.R. Khuri and R.C. Myers, Phys. Lett. B378, 78 (1996). [arXiv:hep-th/9603061]
[4] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, Phys. Lett. B391, 93 (1997). [arXiv:hep-th/9602065]
[5] G.T. Horowitz, J.M. Maldacena and A. Strominger, Phys. Lett. B383, 151 (1996). [arXiv:hep-th/9603109]
[6] J. C. Brekenridge, D.A. Lowe, R. C. Myers, A. W. Peet, A. Strominger and C. Vafa, Phys. Lett. B381, 423 (1996). [arXiv:hep-th/9603078]
[7] G.T. Horowitz, D.A. Lowe and J.M. Maldacena, Phys. Rev. Lett. 77, 430 (1996). [arXiv:hep-th/9603195]
[8] J.B. Hartle and S.W. Hawking, Commun. Math. Phys. 26, 87 (1972).
[9] R. Emparan and H. S. Reall, Phys. Rev. Lett. 88, 101101 (2002). [arXiv:hep-th/0110260].
[10] R. C. Myers and M. J. Perry, Ann. Phys. 172, 304 (1986).

[11] B. Kol. arXiv:hep-th/0208056.

[12] G.W. Gibbons, D. Ida and T. Shiromizu, Phys. Rev. Lett. 89, 041101 (2002). arXiv:hep-th/0206049.

[13] G.W. Gibbons, D. Ida and T. Shiromizu, Phys. Rev. D66, 044010 (2002). arXiv:hep-th/0206136

[14] M. Rogatko, Class. Quant. Grav. 19, L151 (2002). arXiv:hep-th/0207187

[15] E. Cremmer in Superspace and supergravity, edited by S.W Hawking and M. Rocek, CUP 1981.

[16] J. P. Gauntlett, R. C. Myers and P. K. Townsend, Class. Quant. Grav. 16, 1 (1999). arXiv:hep-th/9810204

[17] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, arXiv:hep-th/0209114.

[18] K. P. Tod, Phys. Lett. B121, 241 (1983).

[19] K.P. Tod, Class. Quant. Grav. 12, 1801 (1995).

[20] G. W. Gibbons, D. Kastor, L. A. London, P. K. Townsend and J. Traschen, Nucl. Phys. B416, 850 (1994). arXiv:hep-th/9310118.

[21] R. Gregory and R. Laflamme, Phys. Rev. D37, 305 (1988).

[22] G. T. Horowitz and K. Maeda, Phys. Rev. Lett. 87, 131301 (2001). arXiv:hep-th/0105111

[23] S. S. Gubser, Class. Quant. Grav. 19, 4825 (2002). arXiv:hep-th/0110193

[24] T. Wiseman, arXiv:hep-th/0209051

[25] T. Wiseman, arXiv:hep-th/0211028

[26] T. Harmark and N. A. Obers, JHEP 0205, 032 (2002). arXiv:hep-th/0204047.

[27] P. J. De Smet, Class. Quant.Grav. 19, 4877 (2002). arXiv:hep-th/0206106.

[28] S.W. Hawking, Commun. Math. Phys. 25, 152 (1972).

[29] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press (1973).

[30] J.L. Friedman, K. Schleich and D.M. Witt, Phys. Rev. Lett. 71, 1486 (1993), Erratum-ibid.75, 1872 (1995). arXiv:gr-qc/9305017.

[31] P.T. Chrusciel and R.M. Wald, Class. Quant. Grav. 11, L147 (1994). arXiv:gr-qc/9410004.

[32] S.A. Hartnoll, arXiv:hep-th/0302072
[33] G.J. Galloway, K. Schleich, D.M. Witt and E. Woolgar, Phys. Lett. B505, 255 (2001). [arXiv:hep-th/9912119].

[34] W. Israel, Phys. Rev. 164, 1776 (1967).

[35] G.L. Bunting and A.K.M Massood-ul-Alam, Gen. Rel. Grav. 19, 147 (1987).

[36] D. Sudarsky and R.M. Wald, Phys. Rev. D46, 1453 (1992).

[37] P.T. Chrusciel and R.M. Wald, Commun. Math. Phys. 163, 561 (1994). [arXiv:gr-qc/9304009].

[38] B. Carter, Phys. Rev. Lett. 26, 331 (1971).

[39] D.C. Robinson, Phys. Rev. Lett. 34, 905 (1975).

[40] F. Dowker, J. P. Gauntlett, G. W. Gibbons and G. T. Horowitz, Phys. Rev. D53, 7115 (1996). [arXiv:hep-th/9512154].

[41] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Class. Quant. Grav. 12, 297 (1995). [arXiv:hep-th/9410073].

[42] G. W. Gibbons and C. A. Herdeiro, Class. Quant. Grav. 16, 3619 (1999). [arXiv:hep-th/9906098].

[43] C. A. Herdeiro, Nucl. Phys. B582, 363 (2000). [arXiv:hep-th/0003063].

[44] L. Järv and C.V. Johnson, [arXiv:hep-th/0211097].

[45] H. Friedrich, I. Rácz and R.M. Wald, Commun. Math. Phys. 204, 691 (1999). [arXiv:gr-qc/9811021].

[46] D. Giulini, Int. J. Theor. Phys. 33, 913 (1994). [arXiv:gr-qc/9308008].

[47] P.S. Aspinwall, [arXiv:hep-th/9611137].

[48] G.W. Gibbons and C.N. Pope, Comm. Math. Phys. 66, 267 (1979).