Equivariant differential characters and Chern-Simons bundles

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Abstract

We construct Chern-Simons bundles as $\text{Aut}^+ P$-equivariant $U(1)$-bundles with connection over the space of connections $\mathcal{A}_P$ on a principal $G$-bundle $P \to M$. We show that the Chern-Simons bundles are determined up to isomorphisms by their equivariant holonomy. The space of equivariant holonomies is shown to coincide with the space of equivariant differential characters of order 2. Furthermore, we prove that the Chern-Simons theory provides, in a natural way, an equivariant differential character that determines the Chern-Simons bundles. Our construction can be applied in the case in which $M$ is a compact manifold of even dimension and for arbitrary bundle $P$ and group $G$.

The results are also generalized to the case of the action of diffeomorphisms on the space of Riemannian metrics. In particular, in dimension 2 a Chern-Simons bundle over the Teichmüller space is obtained.

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1 Introduction

We introduce a geometric definition of Chern-Simons bundles valid for arbitrary Lie groups and principal bundles over even dimensional compact manifolds. Our definition is based on the concept of equivariant holonomy introduced in [11] and [13]. We show that the Chern-Simons bundles can be obtained from its equivariant holonomy, and that the equivariant holonomy is determined in a natural way from the Cheeger-Chern-Simons differential characters introduced in [6].
First we recall the classical construction of the Chern-Simons bundles. If $M$ is a closed 2-manifold, then the space of connections $\mathcal{A}_P$ on the trivial principal $SU(2)$-bundle $P = M \times SU(2) \to M$ is a symplectic manifold with the Atiyah-Bott symplectic structure. It is a classical construction in Chern-Simons theory (e.g. see [21]) that this symplectic manifold admits a $Gau(P)$-equivariant prequantization bundle $U_P \to \mathcal{A}_P$ with connection $\Theta_P$. By symplectic reduction, a prequantization of the Atiyah-Bott symplectic structure on the moduli space of flat connections is obtained. Furthermore, if $\tilde{M}$ is a compact 3-manifold with boundary $\partial \tilde{M} = M$, $\tilde{P} = \tilde{M} \times SU(2)$ and $\pi: \mathcal{A}_{\tilde{P}} \to \mathcal{A}_P$ is the restriction map, then the Chern-Simons action on $\tilde{M}$ can be considered as a section of the bundle $\pi^*U_{\tilde{P}}^{-1} \to \mathcal{A}_{\tilde{P}}$ (see also [14]). For this reason the bundle with connection $(U_P, \Theta_P)$ is called the Chern-Simons bundle of $M$. We also recall that it is possible (e.g. see [1]) to lift the action of the group of orientation preserving diffeomorphisms $D^+M$ to $U_P$ preserving the connection $\Theta_P$.

The construction of the Chern-Simons bundle in [21] can be easily extended to trivial bundles with arbitrary group $G$ (e.g. see Section 5.5). In this case the bundle is constructed by using the Chern-Simons action associated to a Weil polynomial of $G$. For connected and simply connected group $G$ any bundle over a 2 or 3-manifold is trivial, and hence the preceding construction can be applied. However, for nontrivial bundles and in higher dimensions this construction cannot be applied.

It is shown in [10] that it is possible to define the Chern-Simons bundle for an arbitrary principal $G$-bundle $P \to M$ with base a closed manifold $M$ of dimension $2k - 2$, $k \geq 2$. The Chern-Simons bundle is associated to a Weil polynomial $p \in I^k(G)$, a compatible characteristic class $\Upsilon \in H^{2k}(BG)$ and a background connection $A_0$. Moreover, the bundle is equivariant with respect to the action of the group $\text{Aut}^+P$ of automorphisms of $P$ (and not only under gauge transformations as in [21]). It is also proved in [10] that the bundles associated to different background connections $A_0$ are canonically isomorphic. The construction of the Chern-Simons bundle in [10] is rather technical and hard to interpret in geometrical terms. As commented above, in this paper we clarify the construction of Chern-Simons bundle by using the concept of equivariant holonomy that we recall (see [13] for details). Let a Lie group $G$ act on a manifold $N$. The ordinary holonomy is defined for closed curves. The equivariant analogue of a closed curve is a curve $\gamma$ such that $\gamma(1) = \phi_N(\gamma(0))$ for an element $\phi \in G$. Note that in this case, if $\pi: N \to N/G$ is the projection then $\pi \circ \gamma$ is a closed curve on $N/G$. Let $U \to N$ be a $G$-equivariant principal $U(1)$-bundle with a $G$-invariant connection $\Theta$. We define the $\phi$-equivariant logarithmic holonomy $\text{hol}^\Theta_\phi(\gamma) \in \mathbb{R}/\mathbb{Z}$ of $\gamma$ by the property $\gamma(1) = \phi_U(\gamma(0)) \cdot \exp(2\pi i \text{hol}^\Theta_\phi(\gamma))$ for any $\Theta$-horizontal lift $\tilde{\gamma}: [0, 1] \to U$ of $\gamma$. Moreover, the equivariant holonomy determines the $U(1)$-bundle with connection up to $G$-equivariant isomorphisms (see [13]).

In the case of the Chern-Simons bundle $(U_P, \Theta_P)$ over the space of connections of a trivial $SU(2)$-bundle it is possible to compute the equivariant holonomy of $\Theta_P$ (see Section 5.5) and the result is as follows. If $\phi \in \text{Aut} P$, then
a curve $\gamma$ on $A_P$ determines a connection $A^\gamma$ on $P \times I \to M \times I$ such that $A^\gamma|_{P \times \{t\}} = \gamma(t)$. We denote by $P_\phi = (P \times I)/ \sim_\phi$ the mapping torus of $P$, where $(y, 0) \sim_\phi (\phi P(y), 1)$ for any $y \in P$. The condition $\gamma(1) = \phi A_P(\gamma(0))$ implies that $A^\gamma$ projects onto a connection $A^\gamma_\phi$ on $P_\phi \to M_\phi$. We prove in Section 5.5 that if $\phi \in \text{Gau}P$, then we have
\[
\text{hol}^{\text{B}_P}(\gamma) = -\text{CS}(A^\gamma_\phi)
\] (1)

where $\text{CS}: A_{P_\phi} \to \mathbb{R}/\mathbb{Z}$ is the usual Chern-Simons action (CS is well defined because $P_\phi \to M \times S^1$ is a principal $SU(2)$-bundle over a 3-manifold, and hence trivializable).

As the equivariant holonomy determines the equivariant bundle up to an isomorphism, we can use equation (1) to define the Chern-Simons bundle for arbitrary bundles and dimensions. We recall (e.g. see [7]) that the Chern-Simons action can be extended to arbitrary bundles. We define a characteristic pair as a pair $\vec{p} = (p, \Upsilon)$, where $p \in I^r_2(G)$ is Weil polynomial and $\Upsilon \in H^{2r} (BG, \mathbb{Z})$ a compatible characteristic class. Given the characteristic pair $\vec{p}$, for any principal $G$-bundle $Q \to N$ over a closed manifold $N$ of dimension $2r-1$ the Chern-Simons action is defined $\text{CS}^\vec{p}: A_Q \to \mathbb{R}/\mathbb{Z}$ (see Section 5.1). Hence, if $M$ is a closed manifold of dimension $2r-2$ and $P \to M$ is a principal $G$-bundle, then for any $\phi \in \text{Gau}P$ we can define
\[
\Xi^\vec{p}_\phi(\phi, \gamma) = \text{CS}^\vec{p}(A^\gamma_\phi)
\] (2)

and we know that there is at most one (up to an isomorphism) $\text{Gau}P$-equivariant $U(1)$-bundle with connection whose $\text{Gau}P$-equivariant holonomy is given by $\Xi^\vec{p}_\phi$. Furthermore, we can compute (2) also for any $\phi \in \text{Aut}^+ P$.

Now the remaining question is if the Chern-Simons bundle exists, i.e., if there exists a $\text{Aut}^+ P$-equivariant $U(1)$-bundle with connection whose $\text{Gau}P$-equivariant holonomy is given by $\Xi^\vec{p}_\phi$. We prove that it exists by introducing the concept of equivariant differential character.

We recall that in the non-equivariant setting, the set of log-holonomies of connections on principal $U(1)$-bundles over $N$ is known to coincide with the space $\hat{H}^2(N)$ of Cheeger-Simons differential characters of degree 2 (see Section 4.2 for details). Furthermore, it is a classical result that if $\text{Bund}_{U(1)}^N(N)$ denotes the set of principal $U(1)$-bundles with connection over a manifold $N$ modulo isomorphisms, then the map that assigns to a connection its holonomy induces a bijection
\[
\text{Bund}_{U(1)}^N(N) \simeq \hat{H}^2(N).
\] (3)

In Section 5 we define the space of equivariant differential characters $\hat{H}^2_P(N)$ and we prove an equivariant version of the isomorphism (3) in the case in which $N$ is contractible (see Section 4.2 for the case of arbitrary $N$).

Finally, in the case of the Chern-Simons bundle, we prove in Section 5.4 that we have $\Xi^\vec{p}_\phi \in \hat{H}^2_{\text{Aut}^+ P}(A_P)$, where $\Xi^\vec{p}_\phi$ is defined by equation (2). We conclude that the Chern-Simons bundle exists as a $\text{Aut}^+ P$-equivariant bundle for any principal $G$-bundle $P \to M$. It is important to note that the equivariant
differential character $\Xi_{\vec{p}}^P$ is determined only by $\vec{p}$, but to obtain a concrete bundle and connection additional information is necessary. It is shown in Section 5.4 that if we fix a background connection $A_0 \in \mathcal{A}_P$, it is possible to obtain a concrete $\text{Aut}^+P$-equivariant bundle with connection. In this way we recover the result proved in [10], but our proofs are simpler and more conceptual.

1.1 Further applications

The equivariant differential character $\Xi_{\vec{p}}^P$ is the fundamental geometrical object from which other geometric constructions can be derived. For example, let $\mathcal{F}_P \subset \mathcal{A}_P$ be the space of flat connections and let $\mathcal{G}$ be a subgroup of $\text{Aut}^+P$ acting freely on $\mathcal{F}_P$. Then $\Xi_{\vec{p}}^P$ projects onto a differential character $\Xi_{\vec{p}, \mathcal{F}_P}^P \in \hat{H}_G^2(\mathcal{F}_P/\mathcal{G})$, and hence determines a $U(1)$-bundle $\mathcal{L}^\vec{p} \to \mathcal{F}_P/\mathcal{G}$ with connection $\Theta_{\vec{p}}^\mathcal{F}_P$ (defined up to an isomorphism). In the case in which $G = SU(2)$, $\vec{p}$ is the characteristic pair corresponding to the second Chern class, $M$ is a Riemann surface and $P = M \times SU(2)$ the bundle $(\mathcal{L}^\vec{p}, \Theta_{\vec{p}}^\mathcal{F}_P)$ is isomorphic to Quillen’s determinant line bundle. In Section 5.6 we study the restriction of $\Xi_{\vec{p}}^P$ to the action of the Gauge group, and it is shown how the classical constructions in Chern-Simons theory can be generalized to arbitrary bundles. In Section 5.8 we consider the action of the orientation preserving diffeomorphisms group $\mathcal{D}_M$ on the space of Riemannian metrics $\mathcal{M}_M$ on a compact oriented manifold $M$ of dimension $4k - 2$. Precisely, let $FM \to M$ be the frame bundle of $M$ and let $\vec{p}$ be the characteristic pair corresponding to the $k$-th Pontryagin class. Then we can pull-back the character $\Xi_{\vec{p}}^FM$ by the Levi-Civita map and we obtain an equivariant differential character $\Sigma_{\vec{p}, \mathcal{M}_M}^P \in \hat{H}_G^2(\mathcal{M}_M)$. In the case $k = 1$, if $M$ is a Riemann surface of genus $g > 1$, then $\Sigma_{\vec{p}, \mathcal{M}_M}^P$ projects onto an equivariant differential character $\Sigma_{\vec{p}, \mathcal{M}_M}^{\mathcal{T}_M} \in \hat{H}_G^2(\mathcal{T}_M)$, where $\mathcal{T}_M$ is the Teichmüller space of $M$ and $\Gamma_M$ is the mapping class group of $M$. Furthermore, the curvature of $\Sigma_{\vec{p}, \mathcal{M}_M}^{\mathcal{T}_M}$ is $\frac{1}{2\pi} \sigma_{WP}$, where $\sigma_{WP}$ is the symplectic form associated to the Weil-Petersson metric on $\mathcal{T}_M$. By the the equivariant version of isomorphisms [3] we obtain a $\Gamma_M$-equivariant prequantization bundle for $\frac{1}{2\pi} \sigma_{WP}$ (determined up to an isomorphism).

Furthermore, there are other important constructions in gauge theory that can be interpreted as equivariant differential characters. One important example is Witten global gravitational anomaly formula [23]. We recall that in [23], in order to study global gravitational anomalies, Witten studies the variation of the path integral $\int Z$ under the action of the diffeomorphisms group $\mathcal{D}_M$. He defines a number $w(\phi, \gamma) \in \mathbb{R}/\mathbb{Z}$ that measures the variation of $Z$ along a curve $\gamma: I \to \mathcal{M}_M$ such that $\gamma(1) = \phi(\gamma(0))$ for a diffeomorphism $\phi \in \mathcal{D}_M$. In more detail $w(\phi, \gamma) = \lim \eta(\delta_\phi)$, where $\eta$ denotes the Atiyah-Patodi-Singer $\eta$-invariant, $\delta_\phi$ is an elliptic operator on the mapping torus $M_\phi$ and $\lim$ denotes adiabatic limit.\footnote{1}$Z$ is defined as the regularized determinant of a $\mathcal{D}_M$-equivariant family of Dirac operators $\delta_\phi$ parametrized by Riemannian metrics $g \in \mathcal{M}_M$\footnote{2}Note the similarity between the definitions of $w$ and $\Xi_{\vec{p}}^P$.}
Later Witten’s formula was interpreted (e.g. see [15]) as a computation of the holonomy of the Bismut-Freed connection $\Theta^s$ on the quotient determinant line bundle $\det \delta / D_M \to \mathcal{M}_M / D_M$, or more precisely, as a computation of the $D_M$-equivariant holonomy on the equivariant determinant line bundle $\det \delta \to \mathcal{M}_M$ (see [12], [13]). In particular $w \in H^2_{D_M}(\mathcal{M}_M)$.

## 2 Equivariant cohomology in the Cartan model

First, we recall the definition of equivariant cohomology in the Cartan model (e.g. see [17]). Suppose that we have a left action of a connected Lie group $G$ on a manifold $M$. If $\phi \in G$ and $x \in M$, we denote by $\phi_M(x)$ or simply by $\phi \cdot x$ the action of $\phi$ on $x$. In a similar way, for $X \in \mathfrak{g}$ the fundamental vector field $X_M \in \mathfrak{X}(M)$ is defined by $X_M(x) = \frac{d}{dt} \big|_{t=0} \exp(-tX)_M(x)$.

We denote by $\Omega^k(M)^G$ the space of $G$-invariant $k$-forms on $M$. Let $\Omega^*_G(M) = \mathcal{P}^*(g, \Omega^*(M))^G$ be the space of $G$-invariant polynomials\(^3\) on $\mathfrak{g}$ with values in $\Omega^*(M)$, with the graduation $\deg(\alpha) = 2k + r$ if $\alpha$ is a polynomial of degree $k$ with values on the space $\Omega^r(M)$. Let $D: \Omega^2_G(M) \to \Omega^3_G(M)$ be the Cartan differential, $(D\alpha)(X) = d(\alpha(X)) - \iota_{X_M} \alpha(X)$ for $X \in \mathfrak{g}$. On $\Omega^*_G(M)$ we have $D^2 = 0$, and the equivariant cohomology (in the Cartan model) of $M$ with respect of the action of $G$ is defined as the cohomology of this complex. If $\varpi \in \Omega^2_G(M)$ is a $G$-equivariant $2$-form, then we have $\varpi = \omega + \mu$ where $\omega \in \Omega^2(M)$ is $G$-invariant and $\mu \in \text{Hom} (g, \Omega^0(M))^G$. We have $D\omega = 0$ if and only if $d\omega = 0$, and $\iota_{X_M} \omega = d(\mu_X)$ for every $X \in \mathfrak{g}$. Hence $\mu$ is a moment map for $\omega$.

Let a group $G$ act on a manifold $M$ and let $\rho: H \to G$ be a homomorphism. We denote by $d\rho: \mathfrak{h} \to \mathfrak{g}$ the induced map on Lie algebras. If $H$ acts on another manifold $N$ we say that $f: N \to M$ is $\rho$-equivariant if $f(\phi_N(x)) = \rho(\phi)_M(f(x))$ for any $x \in N$ and $\phi \in H$. In this case, we have a map $(f, \rho)^*: \Omega^r_G(M) \to \Omega^r_H(N)$ defined by $((f, \rho)^*\alpha)(X) = f^*(\alpha(\rho(\mathfrak{d}\rho(X))))$ for $X \in \mathfrak{h}$ and $\alpha \in \Omega^r_G(M)$.

We recall the definition of equivariant characteristic classes (see [3]). Let $H$ be a group that acts on a principal $G$-bundle $P \to M$ and let $A$ be a connection on $P$ invariant under the action of $H$. It can be proved that for every $X \in \mathfrak{h}$ the $\mathfrak{g}$-valued function $A(X_P)$ is of adjoint type and defines a section of the adjoint bundle $v_A(X) \in \Omega^0(M, \text{ad}P)$. We denote by $\Gamma^r(G)$ the space of Weil polynomials of degree $r$. For every $p \in \Gamma^r(G)$ the $H$-equivariant characteristic form $p^A_H \in \Omega^2_H(M)$ associated to $p$ and $A$, is defined by $p^A_H(X) = p(F_A - v_A(X))$ for every $X \in \mathfrak{h}$ and we have $Dp^A_H = 0$.

If $\alpha \in \Omega^k(M \times N)$ with $M$ compact and oriented we define $\int_M \alpha \in \Omega^{k-d}(N)$ by $(\int_M \alpha)_{y}(X_1, \ldots, X_{k-d}) = \int_M \iota_{X_{k-d}} \cdots \iota_{X_1} \alpha$ for $y \in N$, $X_1, \ldots, X_d \in T_y N$. If $k < d$ we define $\int_M \alpha = 0$. We have $\int_N \int_M \alpha = \int_{M \times N} \alpha$ and Stokes theorem $d \int_M \alpha = \int_M d\alpha - (-1)^{k-d} \int_M \alpha$. Furthermore, if a group $G$ acts on $M$ and $N$ then the integration map is extended to equivariant differential forms $\int_M: \Omega^k_G(M \times N) \to \Omega^{k-d}_G(N)$ by setting $(\int_M \alpha)(X) = \int_M (\alpha(X))$ for $X \in \mathfrak{g}$, and we have $D(\int_M \alpha) = \int_M D\alpha - (-1)^{k-d} \int_{D_M} \alpha$.

\(^3\)Continuous polynomials in the infinite dimensional case.
Proposition 1 If $U \to M$ is a $G$-equivariant principal $U(1)$-bundle, and $\Theta$ is a $G$-invariant connection on $U$, then for any $\phi, \phi' \in G$, $\gamma \in C^{\phi}(M)$ and $x \in M$ we have

- a) $\text{hol}_{\phi \cdot \gamma}^{\Theta} \cdot \phi' = \text{hol}_{\phi \cdot \gamma}^{\Theta}$.
- b) If $\gamma' \in C_{\gamma(1)}(M)$, then we have $\text{hol}_{\phi \cdot \phi'}^{\Theta} = \text{hol}_{\phi}^{\Theta} + \text{hol}_{\phi'}^{\Theta}$.  
- c) $\text{hol}_{\phi \cdot \phi'}^{\Theta} = \text{hol}_{\phi' \cdot \phi}^{\Theta}$.
- d) If $\gamma : I \to M$ is a curve on $M$ such that $\gamma(0) = (0)$ then $\text{hol}_{\phi}^{\Theta}(\gamma) = \text{hol}_{\phi}^{\Theta}(\gamma \cdot \phi)$.
- e) Let $U' \to M'$ be another $G$-equivariant $U(1)$-bundle with connection and $\Phi : U' \to U$ be a $G$-equivariant $U(1)$-bundle morphism that covers $\Phi : M' \to M$. The connection $\Theta' = \Phi^* \Theta$ is $G$-invariant and we have $\text{hol}_{\phi}^{\Theta'}(\gamma') = \text{hol}_{\phi}^{\Theta}(\Phi \circ \gamma')$ for any $\phi \in G$ and $\gamma' \in C^{\phi}(M')$.

If $U \to M, U' \to M$ are two $G$-equivariant $U(1)$-bundles then we write that $U' \simeq_G U$ if there exists a $G$-equivariant $U(1)$-bundle isomorphism $\Phi : U' \to U$.
covering the identity map of \( M \). We say that \( \mathcal{U} \) is a trivial \( G \)-equivariant \( U(1) \)-bundle if \( \mathcal{U} \cong_G \mathbb{M} \times U(1) \) for an action of \( G \) on \( M \) and where \( G \) acts trivially on \( U(1) \). A \( G \)-equivariant \( U(1) \)-bundle with connection is a pair \( (\mathcal{U}, \Theta) \), where \( \mathcal{U} \to M \) is a \( G \)-equivariant \( U(1) \)-bundle and \( \Theta \) is a \( G \)-invariant connection on \( \mathcal{U} \). We write that \( (\mathcal{U}, \Theta) \cong_G (\mathcal{U}', \Theta') \) if there exists a \( G \)-equivariant \( U(1) \)-bundle isomorphism \( \Phi : \mathcal{U}' \to \mathcal{U} \) covering the identity map of \( M \) such that \( \Phi^* \Theta = \Theta' \).

**Theorem 2** Let \( (\mathcal{U}, \Theta) \) and \( (\mathcal{U}', \Theta') \) be \( G \)-equivariant \( U(1) \)-bundles with connection over \( M \).

a) We have \( (\mathcal{U}, \Theta) \cong_G (\mathcal{U}', \Theta') \) if and only if \( \text{hol}_\phi^G (\gamma) = \text{hol}_\phi'^G (\gamma) \) for all \( \phi \in G \), and \( \gamma \in \mathcal{C}^0(M) \).

b) The bundle \( \mathcal{U} \to M \) is a trivial \( G \)-equivariant \( U(1) \)-bundle if and only if there exists a \( G \)-invariant 1-form \( \beta \in \Omega^1(M)^G \) such that \( \text{hol}_\phi^G (\gamma) = \int_\gamma \beta \mod \mathbb{Z} \) for any \( \phi \in G \) and \( \gamma \in \mathcal{C}^0(M) \).

### 3.1 Equivariant Curvature

If \( \Theta \) is a \( G \)-invariant connection on a principal \( U(1) \) bundle \( \mathcal{U} \to M \) then \( \tfrac{1}{2\pi} D(\Theta) \) is the pull-back of a closed \( G \)-equivariant 2-form \( \text{curv}_G(\Theta) \in \Omega^2_G(M) \) called the \( G \)-equivariant curvature of \( \Theta \). If \( X \in \mathfrak{g} \) then we have \( \text{curv}_G(\Theta)(X) = \text{curv}(\Theta) + \mu^\Theta_X \), where \( \mu^\Theta_X = -\tfrac{1}{2\pi} \Gamma(X_U) \) is called the momentum of \( \Theta \). As it is well known, for bundles with arbitrary group the curvature of \( \Theta \) measures the infinitesimal holonomy. For \( U(1) \)-bundles we have a more precise result that is a generalization of the classical Gauss-Bonnet Theorem for surfaces.

**Proposition 3** If \( \gamma \in \mathcal{C}^0(M) \) and \( \gamma = \partial \sigma \) for \( \sigma \in \mathcal{C}_2(M) \) then we have \( \text{hol}_\phi^G (\gamma) = \int_\sigma \text{curv}(\Theta) \mod \mathbb{Z} \).

In a similar way, the second term of the equivariant curvature, the moment \( \mu^\Theta \) measures the variation of \( \text{hol}_\phi^G (\gamma) \) with respect \( \phi \in G \). Precisely, we have the following result (see [13, Proposition 8])

**Proposition 4** For any \( X \in \mathfrak{g} \) and \( x \in M \) we have \( \text{hol}^\Theta_{\exp(X)}(\tau_{x,X}) = \mu^\Theta_x(x) \) where \( \tau_{x,X}(s) = \exp(sX)_M(x) \).

### 3.2 Contractible base

If \( M \) is a contractible manifold, then several aspects can be simplified. As in this paper we work with the spaces of connections and metrics, we study in detail this case. If \( M \) is contractible, then any principal \( U(1) \)-bundle is trivializable, and hence it is enough to study the case of the trivial bundle \( \mathcal{U} = M \times U(1) \to M \).

As it is well known (see for example [3, 21]), for the trivial bundle \( M \times U(1) \to M \) the action of \( G \) on \( M \times U(1) \) is determined by a map \( \alpha : G \times M \to \mathbb{R}/\mathbb{Z} \) characterized by the property

\[
\phi_U(x,u) = (\phi_M(x), u \cdot \exp(2\pi i \cdot \alpha_\phi(x))).
\]
It satisfies the cocycle condition $\alpha_{\phi', \phi}(x) = \alpha_{\phi}(x) + \alpha_{\phi'}(\phi(x))$. Conversely any cocycle determines an action of $G$ on $M \times U(1)$ by $U(1)$-bundle isomorphisms. In this case the equivariant holonomy can be studied in terms of the cocycle $\alpha_{\phi}(x)$ (e.g. see [11]). For the trivial bundle, a connection $\Theta$ is simply a form of the type $\Theta = \vartheta - 2\pi i\lambda$ for a form $\lambda \in \Omega^1(M)$ and where $\vartheta = z^{-1}dz$ is the Maurer-Cartan form on $U(1)$.

**Proposition 5** If $\Theta = \vartheta - 2\pi i\lambda$ is $G$-invariant, then for any $\phi \in G$ and $\gamma \in C^G_x(M)$ we have

$$\text{hol}^\Theta_{\phi}(\gamma) = \int_{\gamma} \lambda - \alpha_{\phi}(x).$$

### 4 Equivariant differential characters

In this section we define equivariant differential characters (of degree 2) as objects that satisfy properties similar to the equivariant log-holonomy. A similar definition is introduced in [20] (see Section 4.2 for details). Furthermore, a general definition of equivariant differential cohomology for arbitrary order in the context of Deligne Cohomology is introduced in [19]. Although our definition is valid for arbitrary manifolds, we study the case in which the manifold is contractible because this is the case that we need in our applications to gauge theory and the proofs are simpler because the equivariant $U(1)$-bundles can be studied in terms of group cocycles.

First we define differential characters of degree 2 as maps that satisfy the same properties than the holonomy of a connection (see Section 4.2 for another equivalent definition)

**Definition 6** A differential character of degree 2 is a map $\chi: C(M) \to \mathbb{R}/\mathbb{Z}$ such that there exists a closed 2-form $\text{curv}(\chi) \in \Omega^2(M)$ satisfying the following conditions

a) $\chi(\gamma' \star \gamma) = \chi(\gamma') + \chi(\gamma)$ for $\gamma, \gamma' \in C_x(M), x \in M$.

b) If $\gamma \in C(M)$ and $\gamma = \partial \sigma$ for $\sigma \in C^2(M)$ then $\chi(\gamma) = \int_{\sigma} \text{curv}(\chi) \mod \mathbb{Z}$.

The space of degree 2 differential characters on $M$ is denoted by $\check{H}^2(M)$, and the map that assigns to a connection its holonomy induces a bijection $\text{Bund}_{U(1)}(N) \simeq \check{H}^2(N)$ (e.g. see [13 Theorem 2.5.1]), where $\text{Bund}_{U(1)}(N)$ denotes the set of principal $U(1)$-bundles with connection over a manifold $N$ modulo isomorphisms (covering the identity on $M$).

In the equivariant case we can give a similar definition

**Definition 7** A $G$-equivariant differential character is a map $\chi: C^G(M) \to \mathbb{R}/\mathbb{Z}$ such that there exists a closed $G$-equivariant 2-form $\text{curv}_G(\chi) = \text{curv}(\chi) + \mu^\chi \in \Omega^2_G(M)$ satisfying the following conditions

i) $\chi(\phi' \cdot \phi, \gamma' \star \gamma') = \chi(\phi', \gamma') + \chi(\phi, \gamma)$ for $\gamma \in C^\Theta(M), \gamma' \in C^G_{\gamma(1)}(M)$.

For simplicity in the notation, we use the same notation for forms on $M$ and $U(1)$ and its pull-backs to $M \times U(1)$.
Remark 11

If \( \zeta \) is a curve on \( M \) such that \( \zeta(0) = \gamma(0) \), and \( \gamma \in C^0(M) \) then we have \( \chi(\phi, \zeta \ast \gamma \ast (\phi \cdot \zeta)) = \chi(\phi, \gamma) \).

ii) If \( \gamma \in C^0(M) \) and \( \gamma = \partial \sigma \) for a chain \( \sigma \in C_2(M) \) then \( \chi(e, \gamma) = \int_\sigma \text{curv}(\chi) \mod \mathbb{Z} \).

iii) If \( \gamma \in \text{Hom}(\chi, \nu \ast \nu \ast (\phi \cdot \chi)) \) then \( \chi(\phi, \chi) = \int \text{curv}(\chi) \mod \mathbb{Z} \).

iv) For any \( X \in \mathfrak{g} \) and \( x \in M \) we have \( \chi(\exp(X), \tau \cdot X) = \mu_X^\Theta(x) \) where \( \tau \cdot X(s) = \exp(sX)_M(x) \).

The space of \( G \)-equivariant differential characters on \( M \) is denoted by \( \hat{H}_G^2(M) \).

If the conditions i) and iii) are satisfied, then the condition

\[ \text{iv') For any } X \in \mathfrak{g} \text{ and } x \in M \text{ we have } \chi(\exp(X), \tau \cdot X) = \mu_X^\Theta(x) \text{ where } \tau \cdot X(s) = \exp(sX)_M(x). \]

We have a natural map \( \hat{H}_G^2(M) \to \hat{H}_G^2(M) \). If \( \Theta \) is a \( G \)-invariant connection on a \( U(1) \)-bundle, we denote by \( \text{hol}_{\Theta}^G \in \hat{H}_G^2(M) \) the equivariant differential character determined by \( \text{hol}_{\Theta}^G(\phi, \gamma) = \text{hol}_{\Theta}^G(\phi, \gamma) \) for \( (\phi, \gamma) \in \mathbb{C}^0(M) \).

Example 8 If \( \beta \in \Omega^1(M)^G \) then we can define \( \zeta(\beta) \in \hat{H}_G^2(M) \) by setting \( \zeta(\beta)(\phi, \gamma) = \int_\gamma \beta \mod \mathbb{Z} \text{ for } \gamma \in \mathbb{C}^0(M) \). We have \( \text{curv}_G(\zeta(\beta)) = D\beta \).

Example 9 If \( M/G \) is a manifold, \( \pi: M \to M/G \) is the projection and \( \chi \in H^2(M/G) \), then for any \( \gamma \in \mathbb{C}^0(M) \) the curve \( \pi \circ \gamma \) is a closed loop on \( M/G \). We define \( \pi_G^2 \chi(\phi, \gamma) = \chi(\pi \circ \gamma) \) and we have \( \pi_G^2 \chi \in \hat{H}_G^2(M) \) and \( \text{curv}_G(\pi_G^2 \chi) = \pi^* \text{curv}(\chi) \).

Example 10 If \( \xi \in \text{Hom}(G, \mathbb{R}/\mathbb{Z}) \) we define \( \chi(\phi, \gamma) = \xi(\phi) \) for \( \phi \in G \) and \( \gamma \in \mathbb{C}^0(M) \). We have \( \chi \in \hat{H}_G^2(M \) and \( \text{curv}_G(\chi) = d\xi \), where \( d\xi \in \text{Hom}(\mathfrak{g}, \mathbb{R}) \subset \text{Hom}(\mathfrak{g}, \Omega^0(M)) \) is the differential of \( \xi \).

Remark 11 It follows from the conditions i) and iii) that if \( \gamma \) and \( \gamma' \) differ by a reparametrization, then \( \chi(\phi, \gamma) = \chi(\phi, \gamma') \).

The condition iv) is equivalent to a weaker condition that it is easier to check in practice.

Proposition 12 If the conditions i) and iii) are satisfied, then the condition iv) is equivalent to the following condition

\[ \text{iv') For any } X \in \mathfrak{g} \text{ and } x \in M \text{ we have } \left. \frac{d}{dt} \right|_{t=0} \chi(\exp(tX), \nu_t \cdot X) = \mu_X^\Theta(x) \text{ where } \nu_t \cdot X(s) = \exp(stX)_M(x). \]

Proof. Clearly iv') follows from iv). We prove the converse. We define \( k(t) = \chi(\exp(tX), \nu_t \cdot X) \) and we have \( k(0) = 0 \) and \( k(1) = \chi(\exp(X), \tau \cdot X) \). As the curves \( \nu_t \cdot X \) and \( \nu_t \cdot X \) differ by a reparametrization, by the condition i) and Remark 11 we have \( k(t + t') = k(t) + k(t') \). Taking the derivative we obtain

\[ \frac{d}{dt} k(t) \bigg|_{t=0} = \lim_{h \to 0} \frac{k(t+h) - k(t)}{h} = \lim_{h \to 0} \frac{k(h)}{h} = \mu_X^\Theta(x) \text{, and by integration it follows that } k(t) = t \mu_X^\Theta(x) \text{ for any } t. \]

By taking \( t = 1 \) we obtain the condition iv). ■

Lemma 13 If \( M \) is connected, \( \chi \in \hat{H}_G^2(M) \) and \( \phi, \phi' \in G \), then \( \gamma, \gamma' \in \mathbb{C}^0(M) \) then we have

a) \( \chi(\phi^{-1}, \gamma) = -\chi(\phi, \gamma) \).

b) \( \chi(\phi \cdot \phi' \cdot (\phi')^{-1}, \phi' \cdot \gamma) = \chi(\phi, \gamma) \).
Theorem 15

The next construction will appear frequently in our applications to G-aug

Proof. a) By Properties i) and iii) we have \( \chi(\phi, \gamma) = \chi(e, \gamma * \gamma) = 0 \).

b) If \( v \) is a curve on \( M \) joining \( \gamma(0) \) and \( \phi'M(\gamma(0)) \) then by conditions i), ii) and property a) we have

\[ \chi(\phi, v) = \chi(\phi^*v * (\phi' \cdot \gamma)) = -\chi(\phi, \gamma) + \chi(\phi, v) + \chi(\phi' \cdot \phi \cdot (\phi')^{-1}, \phi' \cdot \gamma), \]

and hence \( \chi(\phi', \phi \cdot (\phi')^{-1}, \phi' \cdot \gamma) = \chi(\phi, \gamma). \blacksquare \)

The next construction will appear frequently in our applications to Gauge

Proposition 14 If \( f : N \to M \) is \( \rho \)-equivariant, then any differential character \( \chi \in H^2_G(M) \) defines a \( H \)-equivariant differential character \( (f, \rho)^*\chi \in H^2_H(N) \) by \( (f, \rho)^*\chi(\phi, \gamma) = \chi(\rho(\phi), f \circ \gamma). \) The \( H \)-equivariant curvature of \( (f, \rho)^*\chi \) is \( (f, \rho)^*(\text{curv}_G(\chi)). \)

It is shown in Section 14.2 that \( H^2_G(M) \) is isomorphic to the space \( G \)-equivariant \( U(1) \)-bundles with connection over \( M \) modulo isomorphisms (covering the identity on \( M \)). Next we show that in the particular case in which \( M \) is contractible, it is possible to determine a concrete bundle with connection that corresponds to a \( G \)-equivariant differential character \( \chi \in H^2_G(M) \). In Section 5.4 we apply this construction in order to define the Chern-Simons bundles.

Theorem 15 Let \( M \) be a contractible manifold, \( \chi \in H^2_G(M) \) and let \( \lambda \in \Omega^1(M) \) be a 1-form such that \( d\lambda = \text{curv}(\chi) \). Then there exists a unique lift of the action of \( G \) to \( M \times U(1) \) by \( U(1) \)-bundle automorphisms such that \( \Theta = \varphi - 2\pi i\lambda \) is \( G \)-invariant and \( \chi = \text{hol}_G(\varphi) \). Precisely, the action is defined by the cocycle \( \alpha_\varphi(x) = \int_\gamma \lambda - \chi(\varphi, \gamma) \) for any \( \gamma \in \mathcal{C}_G^2(M) \).

Proof. First we show that \( \alpha_\varphi(x) = \int_\gamma \lambda - \chi(\varphi, \gamma) \) does not depend on the curve \( \gamma \in \mathcal{C}_G^2(M) \). If \( \gamma, \gamma' \in \mathcal{C}_G^2(M) \) then \( \gamma * \gamma' \) is a closed loop on \( M \). If \( \Sigma \) is a submanifold of dimension 2 such that \( \partial \Sigma = \gamma * \gamma' \) (it exists by the contractibility of \( M \) then by Lemma 13 a) we have

\[ \chi(\phi, \gamma') - \chi(\phi, \gamma) = \chi(\phi, \gamma * \gamma') = \int_{\Sigma} \lambda = \int_{\gamma * \gamma'} \lambda = \int_\gamma \lambda - \int_\gamma \lambda, \]

and hence \( \int_\gamma \lambda - \chi(\phi, \gamma) = \int_\gamma \lambda - \chi(\phi, \gamma') \).

Next we prove that \( \alpha \) satisfies the cocycle condition. If \( \gamma \in \mathcal{C}_G^2(M) \) and \( \gamma' \in \mathcal{C}_G^2(M) \) we have

\[ \alpha_{\phi', \phi}(x) = \int_{(\phi', \phi) \gamma} \lambda - \chi(\phi', \phi, \phi' \cdot \gamma' \cdot \gamma) \]

\[ = \int_{(\phi', \phi) \gamma} \lambda + \int_\gamma \lambda - \chi(\phi', \phi, \phi' \cdot (\phi')^{-1}, \phi' \cdot \gamma') - \chi(\phi, \gamma) \]

\[ = \alpha_{\phi}(x) + \alpha_{\phi'}(\phi(x)) \]
If \(x, x' \in M\) and \(\zeta\) is a curve on \(M\) with \(\zeta(0) = x, \zeta(1) = x'\) and \(\gamma \in C^\bullet_\gamma(M)\) then \(\zeta * \gamma * (\phi \cdot \zeta) \in C^\bullet_\phi(M)\) and by property ii) we have

\[
\alpha_\phi(x') = \int_{\zeta*\gamma*\phi} \lambda - \chi(\phi, \zeta*\gamma*\phi) = \int_\phi \zeta*\lambda + \int_\gamma \zeta*\lambda - \chi(\phi, \zeta) = \int_\phi (\phi_M*\lambda - \lambda) + \alpha_\phi(x).
\]

It follows from this condition that \(\alpha_\phi(x)\) is differentiable with respect \(x\) and that

\[
d\alpha_\phi = \phi_M*\lambda - \lambda. \tag{4}
\]

The differentiability of \(\alpha\) with respect to \(\phi\) follows from condition iv) in the definition of equivariant differential character.

We define the connection form \(\Theta = \vartheta\) such that

\[
\phi_M*U(\Theta) = \phi_M*U(\vartheta - 2\pi i\lambda) = (\vartheta + 2\pi i d\alpha_\phi) - 2\pi i \phi_M*\lambda = \vartheta\).
\]

Hence \(\Theta\) is \(G\)-invariant and from Proposition 4 it follows that \(\text{hol}_G^\Theta = \chi\).

From the preceding theorem we conclude the following

**Corollary 16** If \(M\) is contractible, then for any \(G\)-equivariant differential character \(\chi \in \hat{H}_2^G(M)\) there exists a \(G\)-equivariant \(U(1)\)-bundle with connection \((U, \Theta)\) such that \(\chi(\phi, \gamma) = \text{hol}_G^\Theta(\gamma)\).

If we denote by \(\text{Bund}^G_{U(1)}(M)\) the space of \(G\)-equivariant \(U(1)\)-bundles with connection over \(M\) modulo isomorphisms (covering the identity on \(M\)), then from Corollary 16 and Theorem 2 a) we obtain

**Corollary 17** If \(M\) is contractible, then the map that assigns to a \(G\)-equivariant \(U(1)\)-bundle with connection its \(G\)-equivariant log-holonomy determines a bijection \(\text{Bund}^G_{U(1)}(M) \simeq \hat{H}_2^G(M)\).

This result is generalized to arbitrary manifolds in Section 4.2.

**Remark 18** We conclude that a \(G\)-equivariant differential character determines a \(G\)-equivariant \(U(1)\)-bundle with connection modulo an isomorphism. However, in order to determine a concrete bundle with connection, it is necessary to give additional information. In the case of contractible manifolds it is enough to give a form \(\lambda \in \Omega^1(M)\) such that \(d\lambda = \text{curv}(\chi)\).

Theorem 2 b) can be reinterpreted in terms of differential characters. Precisely we have the following

**Proposition 19** If \(M\) is contractible then the \(G\)-equivariant \(U(1)\)-bundle determined (up to an isomorphism) by \(\chi \in \hat{H}_2^G(M)\) is trivial if and only if there exists \(\beta \in \Omega^1(M)^G\) such that \(\chi = \varsigma(\beta)\).
4.1 Projectable Differential Characters

Suppose that $M/G$ is a manifold, the projection $\pi: M \to M/G$ is smooth. We say that a differential character $\chi \in \hat{H}^2(M)$ is $\pi$-projectable if there exists $\underline{\chi} \in \hat{H}^2(M/G)$ such that $\chi = \pi_G^* \underline{\chi}$. A necessary condition for $\chi$ to be $\pi$-projectable is $\mu^\chi = 0$. For free actions, this condition is also sufficient.

**Proposition 20** If $G$ acts freely on a contractible manifold $M$ and $\pi: M \to M/G$ is a principal $G$-bundle, then $\chi \in \hat{H}^2_G(M)$ is $\pi$-projectable if and only if $\mu^\chi = 0$.

**Proof.** Let $\lambda \in \Omega^1(M)$ be a 1-form such that $d\lambda = \text{curv}(\chi)$ and let $\Theta = \vartheta - 2\pi i\lambda$ be the corresponding connection. For any $X \in g$ we have $\iota_X \Theta = 2\pi i \mu^\chi_X = 0$, and as $\Theta$ is $G$-invariant, it projects onto a connection $\Theta$ on $(M \times U(1))/G \to M/G$. It is easily seen that $\chi = \pi_G^* \text{hol}^\Theta$.

We need also the following generalization of the preceding result, that can be proved in a similar way:

**Proposition 21** Let $H \subset G$ be a Lie subgroup of $G$ that acts freely on a contractible manifold $M$ and such that $\pi: M \to M/H$ is a principal $H$-bundle. Then $G$ acts on $M/H$, and if $\phi \in G$ and $\gamma \in C^0(M)$ then $\pi \circ \gamma \in C^0(M/H)$. If $\chi \in \hat{H}^2_G(M)$ then there exists $\underline{\chi} \in \hat{H}^2_G(M/H)$ such that $\chi(\phi, \gamma) = \underline{\chi}(\pi \circ \gamma)$ if and only if $\mu^\chi|_h = 0$.

**Remark 22** If $H \subset G$ is normal closed subgroup, we can also consider the action of the quotient group $G/H$ on $M/H$ and if $\mu^\chi|_h = 0$ we obtain an element of $\hat{H}^2_{G/H}(M/H)$.

4.2 Other definitions and arbitrary manifolds

The Definition of differential character is a direct generalization of the concept of holonomy, but it is not the usual definition. Usually the differential characters are defined on $Z_1(M)$ and not on $C(M)$ (e.g. see Section 5.1). In this Section we show that both definitions are equivalent, and we study their generalization to the equivariant case. First we need the following technical result:

**Definition 23** A chain $\nu \in C_2(M)$ is a thin chain if $\int_\nu \alpha = 0$ for any $\alpha \in \Omega^2(M)$.

**Lemma 24** If $\gamma_1, \gamma_2$ are curves on $M$ with $\gamma_2(0) = \gamma_1(1)$, then $\gamma_1 \ast \gamma_2 = \gamma_1 + \gamma_2 + \partial \nu$, with $\nu \in C_2(M)$ a thin chain.

The results of this section are not necessary for the study of the Chern-Simons bundles.
The explicit form of the chain $\nu$ can be found for example in [16, page 64].

If $\chi$ is a differential character as defined in Definition 6 then we can extend $\chi$ to $Z_1(M)$ in the following way: if $z \in Z_1(M)$ then we have $z = \gamma + \partial \sigma$, with $\gamma \in C(M)$ and $\sigma \in C_2(M)$.

That $\chi$ is well defined and that it is a group homomorphism $\chi: Z_1(M) \to \mathbb{R}/\mathbb{Z}$ easily follows from Lemma 24. We conclude that a definition equivalent to Definition 6 is the following

**Definition 25** A differential character of degree 2 is a group homomorphism $\chi: Z_1(M) \to \mathbb{R}/\mathbb{Z}$ such that there exists a closed 2-form $\text{curv}(\chi) \in \Omega^2(M)$ satisfying $\chi(\partial \sigma) = \int_\sigma \text{curv}(\chi) \mod \mathbb{Z}$ for any $\sigma \in C_2(M)$.

In [20] a definition of $G$-equivariant differential character similar to the preceding one is introduced. Let $C_{1,G}(M)$ be the free abelian group generated by pairs $(\phi, \gamma)$, with $\gamma$ a curve on $M$ and $\phi \in G$. We define $\partial_G : C_{1,G}(M) \to C_0(M)$ by setting $\partial_G(\phi, \gamma) = \phi(\gamma(1)) - \gamma(0)$ and $Z_{1,G}(M) = \ker \partial_G$. Note that $Z_{1,G}(M)$ is generated by chains of the form $(\phi_1, \gamma_1) + \ldots + (\phi_n, \gamma_n)$ that satisfy the following condition

$$
\phi_i(\gamma_i(1)) = \gamma_{i+1}(0), \quad i = 1, \ldots, n-1 \quad \text{and} \quad \phi_n(\gamma_n(1)) = \gamma_1(0) \quad (5)
$$

In particular, if $\gamma \in C^\phi(\gamma)$ then $(\phi^{-1}, \gamma) \in Z_{1,G}(M)$.

**Definition 26** A Lerman-Malkin $G$-equivariant differential character is a group homomorphism $\eta: Z_{1,G}(M) \to \mathbb{R}/\mathbb{Z}$ such that there exists a closed $G$-equivariant 2-form $\text{curv}_G(\eta) = \text{curv}(\eta) + \mu^\eta \in \Omega^2_G(M)$ satisfying the following conditions

a) $\eta((\phi, \xi) + (\phi^{-1}, \phi \cdot \xi)) = 0$ for any curve $\xi$ on $M$ and $\phi \in G$.

b) If $\sigma \in C_2(M)$ then $\eta(e, \partial \sigma) = \int_\sigma \text{curv}(\eta) \mod \mathbb{Z}$.

c) For any $X \in \mathfrak{g}$ and $x \in M$ we have $\eta(\exp(-X), \tau^{\mu_{X}}(s)) = \exp(sX)_{M}(x)$ where $\tau^{\mu_{X}}(s) = \exp(sX)_{M}(x)$

d) For any $x \in M$ the map $\phi \mapsto \eta(\phi, c_x)$ determines a group homomorphism $G_x \to \mathbb{R}/\mathbb{Z}$, where $G_x$ is the isotropy group of $x$, and $c_x$ the constant curve with value $x$.

Next we show that if $M$ is connected, then Definition 26 is equivalent to Definition 7. Let $\chi$ be an equivariant differential character and let $z = \sum_{i=1}^n (\phi_i, \gamma_i)$ be a cycle that satisfies the condition (5). We chose curves $\tau_i \in C^\phi_{\gamma_i}(M)$ on $M$ and we define $\eta(z) = \sum_{i=1}^n \chi(\phi_i, \tau_i) \star \gamma_i - \sum_{i=1}^n \chi(\phi_i, \tau_i)$. It is easily seen that this definition is independent of the curves $\tau_i$ chosen.

Clearly $\eta$ determines a group homomorphism $\eta: Z_{1,G}(M) \to \mathbb{R}/\mathbb{Z}$ that satisfies the conditions b), c) and d) in Definition 26. Next we prove that condition a) is basically equivalent to condition ii) in Definition 7. If $\xi$ is a curve on $M$ and $\phi \in G$, we choose $\tau_1 \in C^\phi_{\xi(1)}(M)$ and $\tau_2 \in C^\phi_{\xi(0)}(M)$. Then by the definition

---

6This fact follows from the surjectivity of the map $\pi_1(M) \to H_1(M)$.  

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of \( \eta \) and the conditions i) and ii) in Definition 7 we have
\[
\eta((\phi, \xi) + (\phi^{-1}, \phi \cdot \xi)) = \chi(e, \xi) \tau_1 \tau_2 - \chi(\phi, \tau_1) = \chi(\phi, \tau_2) = 0.
\]
Conversely, if \( \eta \) satisfies the conditions of Definition 26, then we can define \( \chi \): \( CG(M) \to R/Z \) by setting \( \chi(\phi, \gamma) = \eta(\phi^{-1}, \gamma) \) for \( \gamma \in C(\phi(M)) \). Clearly conditions iii) and iv) are satisfied. We prove condition ii). If \( \phi \in G \), we consider the mapping torus \( \pi: M \times I \to M_\phi \). For any curve \( \gamma \) on \( M \), we define \( \gamma = \pi \circ (\gamma \times \text{id}_I) \). The Lerman-Malkin \( G \)-equivariant differential character \( \eta \) projects onto a differential character \( \eta \in H^2(M_\phi) \) such that \( \eta(\gamma) = \eta(\phi^{-1}, \gamma) \) for any \( \gamma \in C(\phi(M)) \). Then we have
\[
\chi(\phi, \xi \gamma (\phi \cdot \xi)) = \eta(\phi^{-1}, \xi \gamma (\phi \cdot \xi)) = \eta(\xi \gamma (\phi \cdot \xi)) = \eta(\gamma) = \eta(\phi^{-1}, \gamma) = \chi(\phi, \gamma).
\]
The condition i) can be proved in a similar way, by replacing \( M_\phi \) with \( (M \times E)/H \), where \( H \) is the subgroup generated by \( \phi \) and \( \phi' \), \( E \) is a manifold in which \( H \) acts freely, and \( \text{id}_I \) is replaced by two curves \( \tau \in C(\phi(E)), \tau' \in C(\phi')(E) \).

It is proved in [20, Theorem 4.3.1] that there exists a bijection between the space of Lerman-Malkin \( G \)-equivariant differential characters and \( \text{Bund}^{U(1)}_G(M) \). This result generalizes Corollary 17 to the case of an arbitrary manifold \( M \).

5 Integrated equivariant Cheeger-Chern-Simons differential characters

In this section we apply the preceding constructions to the case of the space of connections \( A_P \) on a principal bundle and the action of the group of automorphisms. We show that the Cheeger-Chern-Simons construction determines in a natural way an equivariant differential character \( \Xi_P \) on \( A_P \). In Section 5.5 we show that in the case of a trivial bundle \( \Xi_P \) coincides (up to a sign) with the equivariant holonomy of the Chern-Simons line bundle. For arbitrary bundles the Chern-Simons bundle can be defined by applying Theorem 15 to \( \Xi_P^\phi \). We also show that this bundle is isomorphic to the bundle defined in [10]. In the later sections we study how other constructions in Chern-Simons theory can be derived from the equivariant differential character \( \Xi_P^\phi \).

5.1 Cheeger-Chern-Simons differential characters and the Chern-Simons action

We recall the properties of the Cheeger-Chern-Simons differential characters introduced in [6]. If \( M \) is an oriented manifold then a differential character of degree \( k \) is a homomorphism \( \chi: Z_{k-1}(M) \to R/Z \) such that there exists...
ω ∈ Ωk(M) (called the curvature of χ) satisfying χ(∂u) = ∫ u ω mod Z for any cycle u ∈ Zk(M).

Let G be a Lie group with a finite number of connected components. A characteristic pair of degree r for the group G is a pair \( \tilde{p} = (p, \Upsilon) \), where \( p \in \Gamma^r(G) \) is a Weil polynomial of degree r, \( \Upsilon \in H^{2r}(BG, \mathbb{R}) \) a characteristic class, and they are compatible in the sense that they determine the same real characteristic class on \( H^{2r}(BG, \mathbb{R}) \). We denote by \( I^r_2(G) \) the subset of elements \( p \in \Gamma^r(G) \) that are compatible with a characteristic class \( \Upsilon \in H^{2r}(BG, \mathbb{R}) \).

For any principal G-bundle \( P → M \) with connection A, the pair \( \tilde{p} \) determines in a natural way a differential character \( \xi_A^p \in \hat{H}^{2r}(M) \) with curvature \( p(F_A) \in \Omega^{2r}(M) \). In particular, for any 2r-dimensional chain \( u ∈ C_{2r}(M) \) we have \( \xi_A^p(\partial u) = ∫_u p(F_A) \) mod Z. We recall that natural means that for any principal G-bundle \( P' \to N' \) and any G-bundle map \( F : P' → P \) we have

\[
χ_{F^∗A} = f^∗(χ_A),
\]

where \( f : N' → N \) is the map induced by \( F \). If \( A' \) is another connection on \( P \), then for any \( u ∈ Z_{2r−1}(M) \) we have

\[
χ_{A′}(u) = χ_A(u) + ∫_u Tp(A′, A),
\]

where \( Tp(A, A′) = r ∫_0^1 p(a, F_t, ([r]−1), F_t) dt \) is the Chern-Simons transgression form, with \( a = A − A′ ∈ Ω^1(M, \text{ad}P) \) and \( F_t \) the curvature of the connection \( A_t = tA + (1 − t)A′ \). Furthermore, we have the following result (see e.g. [6] Proposition 2.9)

**Lemma 27** If \( A_t \) is a smooth 1-parametric family of connections on \( P \) with \( A_0 = a ∈ Ω^1(M, \text{ad}P) \), then \( ∂\left| \frac{d}{dt} \right|_{t=0} χ_{A_t}(u) = r ∫_u p(a, F_0, ([r]−1), F_0) \) for every \( u ∈ Z_{2r−1}(M) \).

If \( M \) is compact and without boundary of dimension \( \dim M = 2r − 1 \) and \( P → M \) is a principal G-bundle, then the Chern-Simons action \( CS^p : A_P → \mathbb{R}/\mathbb{Z} \) is defined (e.g. see [7]) by setting \( CS^p(A) = \xi_A^p(M) \). It follows from equation (7) that if \( A, A′ \) are two connections on \( P \), then \( CS^p(A) = CS^p(A′) + ∫_M Tp(A, A′) \). Moreover, form the naturality condition (6) we conclude that if \( (P, A) \) is isomorphic to \( (P′, A′) \) then \( CS^p(A) = CS^p(A′) \). It also follows from (6) that if \( P \) is a trivializable bundle, and \( A_0 \) the connection associated to a trivialization then \( CS^p(A_0) = 0 \), and hence \( CS^p(A) = ∫_M Tp(A, A_0) \). In the particular case in which \( r = 2, G = SU(2) \), \( P \) is a trivializable bundle with a section \( S : M → P \) and \( p(X) = \frac{1}{8π^2} \text{tr}(X^2) \), then we have \( CS^p(A) = \frac{1}{8π^2} ∫_M \text{tr}(α ∧ dα + \frac{2}{3}α ∧ α ∧ α) \), where \( α = S^∗A \). Hence in this case \( CS^p \) which coincides with the classical Chern-Simons action (e.g. see [14]).

**Remark 28** For trivializable bundles the Chern-Simons action \( CS^p \) only depends on the polynomial \( p ∈ I^r_2(G) \) and it is independent of the characteristic class \( \Upsilon \). In this case we denote the Chern-Simons action by \( CS^p \).
5.2 Geometry of the space of connections

Let $P \to M$ be a principal $G$-bundle, and let $\mathcal{A}_P$ be the space of principal connections on this bundle, considered as an infinite dimensional Fréchet manifold. As $\mathcal{A}_P$ is an affine space modeled on $\Omega^1(M, adP)$, we have canonical isomorphisms $T_A \mathcal{A}_P \cong \Omega^1(M, adP)$ for any $A \in \mathcal{A}_P$. We denote by $\text{Aut}P$ the group of $G$-bundle automorphism of $P$, and by $\text{Gau}P$ the subgroup of bundle automorphism covering the identity on $M$. The Lie algebra of $\text{Aut}P$ is the space of $G$-invariant vector fields on $P$, $\text{aut}P \subset \mathfrak{x}(P)$, and the Lie algebra of $\text{Gau}P$ is the subspace $\text{gau}P$ of vertical $G$-invariant vector fields (see [2], [21] for more details on the Lie group structure of $\text{Aut}P$). The group $\text{Aut}P$ acts in a natural way on $\mathcal{A}_P$. If $M$ is oriented, we denote by $\text{Aut}^+P$ the group of $G$-bundle automorphism of $P$ preserving the orientation on $M$. We also recall that if $\text{Gau}^P$ is the subgroup of gauge transformations fixing a point of $P$, then $\text{Gau}^P$ acts freely on $\mathcal{A}_P$ and $\mathcal{A}_P \to \mathcal{A}_P/\text{Gau}^P$ is a principal $\text{Gau}^P$-bundle (e.g. see [8]).

The principal $G$-bundle $\mathbb{P} = P \times \mathcal{A}_P \to M \times \mathcal{A}_P$ has a tautological connection $\mathbb{A} \in \Omega^1(P \times \mathcal{A}_P, \mathfrak{g})$ defined by $\mathbb{A}_{(x,A)}(X, Y) = A_x(X)$ for $(x, A) \in P \times \mathcal{A}_P$. $X \in T_xP$, $Y \in T_A \mathcal{A}_P$. This connection is universal in the sense that for any $A \in \mathcal{A}_P$ we have $A = t_A^\ast(\mathbb{A})$, where $t_A: P \to \mathbb{P}$ is defined by $t_A(y) = (y, A)$ for any $y \in P$. We denote by $\mathbb{P}$ the curvature of $\mathbb{A}$. The group $\text{Aut}P$ acts on $\mathbb{P}$ by automorphisms and $\mathbb{A}$ is a $\text{Aut}P$-invariant connection.

Remark 29 As it is usual in Gauge theories (e.g. see [8], Section 5.1.1.), in place of working in $\mathcal{A}_P$ and with the group $\text{Aut}^+P$, it is possible to formulate our results in terms of families of connections. In our case, we need to consider a variation of this concept, that we call equivariant families. Precisely, let $P \to M$ be a principal $G$-bundle and let $\mathfrak{g}$ be a Lie group acting on a manifold $T$, and also on $P \to M$ by automorphisms preserving the orientation on $M$ (i.e., we have a Lie group homomorphism $\rho: \mathfrak{g} \to \text{Aut}^+P$). A $\mathfrak{g}$-equivariant family of connections parametrized by $T$ is a $\mathfrak{g}$-invariant connection $B$ on the product $P \times T \to M \times T$. It defines a $\rho$-equivariant map $b: T \to \mathcal{A}_P$, where $b(t) = B\big|_{P \times \{t\}} \to M \times \{t\}$. All of the following results and proofs are valid if we replace $(\text{Aut}^+P, \mathcal{A}_P, \mathbb{A})$ by $(\mathfrak{g}, T, B)$ (see also Remarks [37] and [37]).

As the connection $\mathbb{A}$ is $\text{Aut}P$-invariant, for any Weil polynomial $p \in \Gamma(G)$ we can define the $\text{Aut}P$-equivariant characteristic form $p^h_{\text{Aut}P} \in \Omega^r_{\text{Aut}P}(M \times \mathcal{A}_P)$ by $p^h_{\text{Aut}P}(X) = p(\mathbb{P} - v_\mathbb{A}(X))$ for $X \in \text{aut}P$. If $M$ is a compact oriented manifold of dimension $n$ without boundary the equivariant form $p^h_{\text{Aut}^+P} \in \Omega^r_{\text{Aut}^+P}(M \times \mathcal{A}_P)$ can be integrated over $M$ to obtain $\int_M p^h_{\text{Aut}^+P} \in \Omega^r_{\text{Aut}^+P}(\mathcal{A}_P)$. In particular, if $\dim M = 2r - 2$, we have $\omega^P_P = \int_M p^h_{\text{Aut}^+P} \in \Omega^2_{\text{Aut}^+P}(\mathcal{A}_P)$ that can be written $\omega^P_P = \omega^P_P + \mu^P_P$, with $\mu^P_P$ a comoment map for $\omega^P_P$. The explicit expressions of these forms are

\[
(\omega^P_P)_A(a, b) = r(r - 1)\int_M p(a, b, F_A, (r, -2), F_A),
(\mu^P_P)_A(X) = -r\int_M p(v_A(X), F_A, (r, -1), F_A)
\]

(8)

for $A \in \mathcal{A}_P$, $a, b \in T_A \mathcal{A}_P \cong \Omega^1(M, adP)$ and $X \in \text{aut}P$. 

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Let \( A_0 \) be a connection on \( P \to M \) and let \( \text{pr}_1 : P \times \mathcal{A}_P \to P \) denote the projection. Then \( \mathfrak{A} \) and \( \overline{\mathcal{A}_0} = \text{pr}_1^* A_0 \) are connections on the same bundle \( P \times \mathcal{A}_P \to M \times \mathcal{A}_P \), and hence we can define \( Tp(\mathfrak{A}, \overline{\mathcal{A}_0}) \in \Omega^{2r-1}(M \times \mathcal{A}_P) \). We have \( p(\mathfrak{A}) = dTp(\mathfrak{A}, \overline{\mathcal{A}_0}) + \text{pr}_1^* p(F_0) \). In particular, if \( 2r > n \) then \( p(\mathfrak{A}) = dTp(\mathfrak{A}, \overline{\mathcal{A}_0}) \) and hence \( \int_M p(\mathfrak{A}) = d\int_MTp(\mathfrak{A}, \overline{\mathcal{A}_0}) \).

### 5.3 The bundle of connections

The preceding constructions have a finite dimensional analog in terms of the (finite dimensional) bundle of connections. We recall that given a principal \( G \)-bundle \( \pi : P \to M \), there exists a bundle \( q : C(P) \to M \) (called the bundle of connections) such that we have a natural identification \( \mathcal{A}_P \simeq \Gamma(M, C(P)) \). For example we can take \( C(P) = (J^1 P)/G \) where \( J^1 P \) is the first jet bundle of \( P \). We refer to \([5]\) for more details on the geometry of \( C(P) \). If \( A \in \mathcal{A}_P \), we denote by \( \sigma_A \) the corresponding section of \( C(P) \). The pull-back bundle \( P = q^* P \to C(P) \) admits a tautological connection defined by \( A_{(x,c)}(X, Y) = A_x(X) \) for \( (x, c) \in P \times C(P) \), \( X \in T_x P \), \( Y \in T_x C(P) \) and where \( A \) is any connection such that \( \sigma_A(x) = c \). This connection \( A \) has the following universal property: for any \( A \in \mathcal{A}_P \) we have

\[
\overline{\sigma}_A^* (A) = A
\]

where \( \bar{\sigma}_A : P \to q^* P \) is defined by \( \bar{\sigma}_A(y) = (y, \sigma_A(\pi(y))) \) for any \( y \in P \). The group \( \text{Aut} P \) acts on \( C(P) \) in a natural way and the connection \( A \) is \( \text{Aut} P \)-invariant. Furthermore, the evaluation map \( ev : M \times \mathcal{A}_P \to C(P) \) defined by \( ev(x, A) = s_A(x) \) is \( \text{Aut} P \)-invariant and we have \( ev^* A = \mathfrak{A} \).

### 5.4 Integrated equivariant Cheeger-Chern-Simons differential characters

Let \( G \) be a Lie group with a finite number of connected components, \( \tilde{\gamma} = (p, \Upsilon) \) a characteristic pair of degree \( r \) and let \( \phi : P \to M \) be a principal \( G \)-bundle over a compact oriented manifold without boundary of dimension \( n = 2r - 2 \).

If \( (\phi, \gamma) \in C^{\text{Aut}^+ P}(\mathcal{A}_P) \) then \( \gamma \) can be extended to a curve \( \tilde{\gamma} : \mathbb{R} \to \mathcal{A}_P \) by setting \( \tilde{\gamma}(t) = (\phi_{\mathcal{A}_P})^n(\gamma(s)) \) if \( t = n + s \) for \( n \in \mathbb{Z} \) and \( s \in [0, 1) \). We define an action of \( \mathbb{Z} \) on \( P \times \mathbb{R} \) by setting \( n \cdot (y, t) = (\phi^n(y), t + n) \) for \( n \in \mathbb{Z} \) and \( (y, t) \in P \times \mathbb{R} \), and a similar action on \( M \times \mathbb{R} \). A connection \( A^\gamma \in \Omega^1(P \times \mathbb{R}, g) \) is defined by \( A^\gamma(X, h) = \tilde{\gamma}(t)(X) \) for \( X \in TP \), \( h \in T_0 \mathbb{R} \), \( t \in \mathbb{R} \). Then \( A^\gamma \) is a \( \mathbb{Z} \)-invariant connection form on the principal \( G \)-bundle \( P \times \mathbb{R} \to M \times \mathbb{R} \). Hence \( A^\gamma \) projects onto a connection \( A^\gamma_\phi \) on the quotient bundle \( (P \times \mathbb{R})/\mathbb{Z} \to (M \times \mathbb{R})/\mathbb{Z} \), and this bundle coincides with the mapping torus bundle \( P_\phi \to M_\phi \). We define the integrated Cheeger-Chern-Simons equivariant differential character by setting

\[
\Xi^\phi_\phi(\phi, \gamma) = \text{CS}^\phi(A^\gamma_\phi) \in \mathbb{R}/\mathbb{Z}.
\]

**Remark 30** Strictly speaking, if \( \gamma \) is smooth then \( A^\gamma \) and \( A^\gamma_\phi \) are continuous but not differentiable in the \( t \) direction. This problem can be solved by considering a smooth non decreasing function \( v : I \to I \) such that \( v \) has constant value 0 in...
[0, ε] and 1 on [1 − ε, 1], and replacing γ with the reparametrization γ ◦ v. The bundles with connection \( (P_\phi, A_\phi^{\gamma \circ v}) \) corresponding to different v are isomorphic and hence \( \text{CS}^\phi(A_\phi^{\gamma \circ v}) \) does not depend on the function v chosen.

In a similar way, if γ is only piecewise smooth, we can consider a smooth reparametrization of γ in order to obtain a smooth connection on \( P_\phi \).

Next we prove that \( \Xi_P^\phi \) is an equivariant differential character with equivariant curvature \( \omega_P^\phi \). To do it, we need to consider a second equivalent definition of \( \Xi_P^\phi \).

**Remark 31** In the second definition we use the bundle of connections \( C(P) \) because it allows us to obtain the results by applying the Cheeger-Chern-Simons constructions only for finite dimensional bundles. It is also possible to obtain the same results by replacing the bundle \( P \to C(P) \) with the infinite dimensional bundle \( P \times A_P \to M \times A_P \). However, in this case, this requires the application of the generalization of the original Cheeger-Chern-Simons construction to infinite dimensional bundles. Furthermore, if we work with finite dimensional equivariant families of connections, then it is not necessary to use \( C(P) \). In this case we can replace \( P \to C(P) \) by the bundle \( P \times T \to M \times T \), the connection \( A \) by the connection \( B \) and the group \( \text{Aut}^+ P \) by \( \Phi \).

As commented above, \( \phi \in \text{Aut}^+ P \) defines an action of \( \mathbb{Z} \) on \( P \) and also on \( C(P) \) and \( P \). If \( \text{pr}: P \times \mathbb{R} \to P \) is the projection, the form \( \text{pr}^* A \) is \( \phi \)-invariant and projects onto a connection \( A_\phi \) on the quotient bundle \( (P \times E)/\mathcal{G} \). For any \( \gamma \in C_\phi^\gamma(A_P) \) we define \( f_\phi^\gamma: M_\phi \to (C(P))_\phi \) by \( f_\phi^\gamma([(x, s)],[\gamma]) = [(x, \sigma_\gamma(x), s)]_\phi \). It follows from equation (9) that \( (f_\phi^\gamma)^* A_\phi = A_\phi^\gamma \) and by the naturality of the Cheeger-Chern-Simons character (equation (6)) we obtain

\[
\Xi_P^\phi(\phi, \gamma) = \xi_{A_\phi}^\phi(f_\phi^\gamma).
\]  

(11)

More generally, let \( \mathcal{G} \) be a discrete group that acts on \( P \) by elements of \( \text{Aut}^+ P \) and let \( E \) be a connected manifold in which \( \mathcal{G} \) acts freely. If \( \text{pr}: P \times E \to P \) is the projection, the form \( \text{pr}^* A \) is \( \mathcal{G} \)-invariant and projects onto a connection \( A_\mathcal{G} \) on the quotient bundle \( (P \times E)/\mathcal{G} \). Given \( \phi \in \mathcal{G} \), we choose a point \( y \in E \) and a curve \( v \in C_\phi^\gamma(E) \). For any \( \gamma \in C_\phi^\gamma(A_P) \) we define \( F_\phi^{\gamma, y, v}: M_\phi \to (C(P) \times E)/\mathcal{G} \) by \( F_\phi^{\gamma, y, v}([(x, s)], \gamma) = [(x, \sigma_\gamma(x), s)]_\mathcal{G} \). Note that in the particular case in which \( \mathcal{G} = \mathbb{Z} \) and \( E = \mathbb{R} \), \( y = 0 \) and \( \rho \in C_0^\phi(\mathbb{R}) \) is the inclusion map \( \rho: I \to \mathbb{R} \), we have \( F_0^{\gamma, y, v} = f_\phi^\gamma \).

**Lemma 32** For any \( \phi \in \mathcal{G} \), \( \gamma \in C_\phi^\gamma(A_P) \), \( y \in E \) and \( v \in C_\phi^\gamma(E) \) we have \( \Xi_P^\phi(\phi, \gamma, y, v) = \xi_{A_\phi}^\phi(F_\phi^{\gamma, y, v}) \).

**Proof.** The element \( \phi \in \mathcal{G} \) induces an action of \( \mathbb{Z} \) on \( E \). We have natural maps

\[
(P \times E \times \mathbb{R})/\mathbb{Z} \overset{\eta_{\phi, y}}{\longrightarrow} (P \times E)/\mathcal{G}
\]

\[
(P \times \mathbb{R})/\mathbb{Z} \overset{\eta_{\phi}}{\longrightarrow} (P \times \mathbb{R})/\mathbb{Z}
\]

The natural maps

\[
\Xi_P^\phi(\phi, \gamma, y, v) = \xi_{A_\phi}^\phi(F_\phi^{\gamma, y, v})
\]

that preserve the action of \( \mathcal{G} \).
and \( \overline{q}_E \mathcal{A}^E_G = \mathcal{A}^E \times \mathbb{R} = \overline{q}_R \mathcal{A} \). Hence we have \( \overline{q}_E^* (\xi^{E}_{\mathcal{A}^E_G}) = \xi^{E}_{\mathcal{A}^E \times \mathbb{R}} = \overline{q}_R^* (\xi^{E}_{\mathcal{A}^E}) \). If \( \rho: I \to \mathbb{R} \) is the inclusion, then by applying equations (16) and (17) we obtain

\[
\xi^{E}_{\mathcal{A}^E_G} (F^\gamma_{\overline{\phi}} (y, y)) = \xi^{E}_{\mathcal{A}^E_P} (\overline{q}_E \circ F^\gamma_{\overline{\phi}} (y, y)(\nu, \rho))) = \xi^{E}_{\mathcal{A}^E} (F^\gamma_{\overline{\phi}} (y, y), \nu, \rho)
\]

Proposition 33 i) If \( \phi, \phi' \in \mathcal{G} \) and \( \gamma \in C^0(\mathcal{A}_P), \gamma' \in C^0_{\gamma(1)}(\mathcal{A}_P) \) then we have \( \mathcal{E}_{\phi}^P (\phi' \cdot \phi, \gamma' * \gamma) = \mathcal{E}_{\phi}^P (\phi', \gamma') + \mathcal{E}_{\phi}^P (\phi, \gamma) \).

ii) If \( \zeta \) is a curve on \( M \) such that \( \zeta(0) = \gamma(0) \), and \( \gamma \in C^0(M) \) then \( \overline{\zeta} * \gamma * (\phi \cdot \zeta) \in C^0(M) \) and \( \mathcal{E}_{\phi, \zeta}^P (\phi, \zeta * (\phi \cdot \zeta)) = \mathcal{E}_{\phi}^P (\phi, \gamma) \).

Proof. i) Let \( \mathcal{G} \) be the subgroup of \( \text{Aut}^+P \) generated by \( \phi \) and \( \phi' \) and \( E \) a connected manifold in which \( \mathcal{G} \) acts freely. We chose \( y \in E, \nu \in C^0_{\gamma(1)}(E), \nu' \in C^0_{\gamma'}(E) \) and we have \( F^{\gamma' * \gamma, \nu', \nu} = F^{\gamma', \nu} + F^{\nu', \nu} \) on \( Z_{2k-1}(P \times E) / \mathcal{G} \), and the result follows from Lemma 32.

ii) We denote by \( c_0 \) the constant curve with value \( 0 \in \mathbb{R} \) and we define \( \gamma_1 = \overline{\zeta} * \gamma * (\phi \cdot \zeta) \) and \( v_1 = c_0 * \nu * (\phi \cdot c_0) \in C^0_{\gamma}(E) \). Then \( F^{\gamma, \nu, \nu} = F^{\gamma, \nu} + F^{\nu, \nu} \) on \( Z_{2k-1}(P) \) and the result follows.

Next we compute the equivariant curvature of \( \overline{\mathcal{E}}_{\phi}^P \) and we show that the conditions iii) and iv) in the definition of equivariant differential character are satisfied.

Proposition 34 If \( \gamma \in C^0(\mathcal{A}_P) \) and \( \gamma = \partial \nu \) for \( \nu \in C_2(M) \) then \( \mathcal{E}_{\phi}^P (\phi, \gamma) = \int_M \mathcal{E}_{\phi}^P (\partial \nu) \).

Proof. As \( \mathcal{A}_P \) is contractible, we can assume that \( \nu \) is a map \( \nu: D^2 \to M \), where \( D^2 \subset \mathbb{R}^2 \) is a disk. We recall that for \( \phi = e \) the mapping torus is simply \( M_e = M \times S^1 \), and we have a map \( f_{\mathcal{E}}: M \times S^1 \to C(P) \times S^1 \) such that \( \mathcal{E}_{\phi}^P (e, \gamma) = \xi_{\mathcal{A}_e}^E (f_{\mathcal{E}}) \). We define \( F^\nu: M \times D^2 \to C(P) \times D^2 \) by \( F^\nu (x, y) = (x, \sigma_{\nu(y)}(x), y) \) and we have \( D F^\nu = f_{\mathcal{E}} \). The connection \( \mathcal{A}_e \) has an obvious extension \( \overline{\mathcal{A}}_e \) to \( P \times D^2 \) and using equation (9) we have

\[
\mathcal{E}_{\phi}^P (\phi, \gamma) = \xi_{\mathcal{A}_e}^E (f_{\mathcal{E}}) = \xi_{\mathcal{A}_e}^E (\partial F^\nu) = \int_M \mathcal{E}_{\phi}^P (\partial F^\nu) = \int_M \mathcal{E}_{\phi}^P (\partial F^\nu) = \int_M \mathcal{E}_{\phi}^P (\partial F^\nu) = \int_M \mathcal{E}_{\phi}^P (\partial F^\nu).
\]

We conclude that \( \text{curv}(\overline{\mathcal{E}}_{\phi}^P) = \int_M p(\overline{F}) \) and that condition iii) is satisfied.

Finally we prove condition vi).

Proposition 35 Let \( X \in \text{aut}P \) and \( A \in \mathcal{A}_P \). If \( \nu^X_t (s) = \exp(tsX) \cdot A \) then we have \( \frac{d}{dt}|_{t=0} \mathcal{E}_{\phi}^P (\exp(tX), \nu^X_t) = \mu_{\phi}^P (A) \).

Proof. The map \( W_t^M: M \times \mathbb{R} \to M \times \mathbb{R}, \nu^M_t (x, s) = (\phi_t(x), s) \) satisfies \( W_t^M (e \cdot (x, s)) = \phi_t \cdot W_t^M (x, s) \) and hence it projects onto a diffeomorphism
$w^M_1: M \times S^1 \to M_{\phi_1}$, and we have similar maps for $P$ and $C(P)$. The composition $(w^C(P))^{-1} \circ f_\phi \circ w^M_1$

$M \times S^1 \xrightarrow{w^M_1} M_{\phi_1} \xrightarrow{f_\phi} (C(P))_{\phi_1} \xrightarrow{(w^C(P))^{-1}} C(P) \times S^1$

is the $t$-independent map $f^{\gamma_A}_t(x, s) = (x, \sigma_A(x), s)$. If $B_t = (w^P_t)^* A_{\phi_t}$, then $\Xi_p^B(\phi_t, \sigma_t) = \xi_p^B(f^{\gamma_A}_t)$ and by Lemma 27 we obtain

$$\frac{d}{dt}|_{t=0} \Xi_p^B(\phi_t, \sigma_t) = \frac{d}{dt}|_{t=0} \xi_p^B(f^{\gamma_A}_t) = r \int_{M \times S^1} (f^{\gamma_A}_t)^* p(B_0, F_1, (r^{-1}), F) = r \int_{M \times S^1} p((f^{\gamma_A}_t)^* B_0, F_A, (r^{-1}), F_A). \quad (12)$$

The connection $B_t$ is the projection of the connection $C_t = (W^P_t)^* p r^* A$ to $P \times S^1$. Hence $B_0$ is the projection of $C_0$. The vector field vector $Y \in \mathfrak{X}(P \times \mathbb{R})$ given by $Y(y, s) = (-sX_P(y), 0)$ has $W^P_t$ as its flow. If we define the vector $\overline{X}(y, s) = (X_P(y), 0)$ then by the Aut$^+ P$-invariance of $A$ we have $L_{\overline{X}}(pr^* A) = 0$ and

$$\hat{C}_0 = \frac{d}{dt}|_{t=0} (W^P_t)^* (pr^* A) = L_Y(pr^* A) = -sL_{\overline{X}}(pr^* A)) = -pr^*(v_A(X))ds$$

Using equations (9) and (12), we conclude that

$$\left. \frac{d}{dt} \right|_{t=0} \Xi_p^B(\phi_t, \sigma_t) = -r \int_{M \times S^1} p(v_A(X))ds, F_1, (r^{-1}), F)$$

$$= -r \int_{M} p(v_A(X), F_1, (r^{-1}), F) \cdot \int_{S^1} ds$$

$$= -r \int_{M} p(v_A(X), F_1, (r^{-1}), F) = \mu_p^B(X).$$

We conclude from the preceding results our main result:

**Theorem 36** $\Xi_p^B$ is a Aut$^+ P$-equivariant differential character on $\mathcal{A}_P$ with equivariant curvature $\omega_p^B$.

**Remark 37** If we work with $\mathfrak{G}$-equivariant families of connections as in Remark 29, then we obtain the following result: For any $\mathfrak{G}$-equivariant family $(\mathfrak{G}, T, B)$ of connections on $P$ the map $\Xi^\mathfrak{G}_p: \mathfrak{C}^\mathfrak{G}(T) \to \mathbb{R}/\mathbb{Z}$ defined by $\Xi^\mathfrak{G}_p, B(\phi, B) = CS^\mathfrak{G}(\mathcal{A}^B(\phi))$ (see Section 5.4 for the notation) is a $\mathfrak{G}$-equivariant differential character on $T$, i.e., $\Xi^\mathfrak{G}_p, B \in \mathcal{T}^2(G)$. In the particular case of the family $(\text{Aut}^+ P, \mathcal{A}_P, A_0)$ we obtain Theorem 36. Conversely, given $\Xi^\mathfrak{G}_p$, we have $\Xi^\mathfrak{G}_p, B = (b, \rho)^* \Xi^\mathfrak{G}_p$. Hence this result is an equivalent formulation of Theorem 36 that does not involve the infinite dimensional structures of $\mathcal{A}_P$ and Aut$^+ P$.
We can define the Chern-Simons bundle in terms of the equivariant differential character $\Xi_P^\gamma$, by choosing a background connection $A_0 \in \mathcal{A}_P$ and by applying Theorem \[15\] to the form $\lambda = \int_M Tp(\mathcal{A}, \mathcal{A}_0) \in \Omega^1(\mathcal{A}_P)$. Precisely, the Chern-Simons bundle is the $\text{Aut}^+P$-equivariant $U(1)$-bundle given by the trivial bundle $\mathcal{A}_P \times U(1) \to \mathcal{A}_P$ with the action defined by the cocycle $\alpha: \text{Aut}^+P \times \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ where $\alpha_\phi(A) = \int_\gamma \lambda - \Xi_P^\gamma(\gamma)$ for any $\gamma \in C^0_A(M)$. We show in the next section that in the case in which $P$ is a trivial bundle over a 2-manifold, this definition coincides with the usual definition of the Chern-Simons bundle.

5.5 The equivariant holonomy of the Chern-Simons bundle for trivial bundles

We recall the construction given in \[21\] of the Chern-Simons bundle for a trivial principal $G$-bundle $P = M \times G \to M$ over a compact oriented 2-manifold $M$ without boundary. In \[21\] it is considered the case of the group $G = SU(2)$ but the construction is valid for any trivial bundle. If $p \in I^2_\mathcal{P}(G)$, we define a cocycle $\alpha$ on $\mathcal{A}_P$ for the group $\mathcal{G} = \text{Gau}P \simeq C^\infty(M, G)$ in the following way. Let $\tilde{M}$ be a compact 3-manifold with boundary $\partial \tilde{M} = M$ and let $\tilde{P} = \tilde{M} \times G$. We denote by $A_0$ and $\tilde{A}_0$ the connections associated to the product structure on $M \times G$ and $\tilde{M} \times G$ respectively.

We define a cocycle $\alpha: \mathcal{G} \times \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ by setting for $A \in \mathcal{A}_P$ and $\phi \in \mathcal{G}$

$$\alpha_\phi(A) = \text{CS}^\phi(\tilde{\phi}(\tilde{A})) - \text{CS}^\phi(\tilde{A}) \mod \mathbb{Z},$$

where $\tilde{A} \in \mathcal{A}_P$ and $\tilde{\phi} \in \text{Gau}\tilde{P}$ are extensions of $A$ and $\phi$ to $\tilde{P}$. It is easily seen (see \[21\]) that the condition $p \in I^2_\mathcal{P}(G)$ implies that $\alpha_\phi(A) \mod \mathbb{Z}$ is independent of the extensions $\tilde{A}, \tilde{\phi}$ chosen, and that $\alpha$ satisfies the cocycle condition. Hence $\alpha$ defines a $\mathcal{G}$-equivariant $U(1)$-bundle $\mathcal{U}_P \to \mathcal{A}_P$. Furthermore, the form $\lambda = \int_M Tp(\mathcal{A}, \mathcal{A}_0) \in \Omega^1(\mathcal{A}_P)$ (see Section \[5.2\]) determines a $\mathcal{G}$-invariant connection $\Theta_P^\phi = \hat{\nu} + 2\pi i \lambda$ on $\mathcal{U}_P \to \mathcal{A}_P$. The $\mathcal{G}$-equivariant $U(1)$-bundle with connection $(\mathcal{U}_P, \Theta_P^\phi)$ is called the Chern-Simons bundle of $p$. Next we compute the $\mathcal{G}$-equivariant holonomy of $\Theta_P^\phi$ and we show that it coincides (up to a sign) with the $\mathcal{G}$-equivariant character $\Xi_P^\gamma$.

**Proposition 38** If $\phi \in \mathcal{G}$ and $\gamma \in C^0(\mathcal{A}_P)$ then we have $\text{hol}_{\Theta_P^\phi}^\phi(\gamma) = -\text{CS}^\phi(A_0^\gamma)$.

**Proof.** We denote by $\hat{\mathcal{A}}$ the tautological connection on $\tilde{M} \times \mathcal{A}_P$, by $\tilde{\mathcal{F}}$ its curvature, we set $\tilde{\mathcal{G}} = \text{Gau}\tilde{P}$ and we denote by $r: \mathcal{A}_P \to \mathcal{A}_P$ the restriction map. Given $\phi \in \mathcal{G}$ and $\gamma \in C^0_A(\mathcal{A}_P)$, we can find extensions $\tilde{\phi} \in \tilde{\mathcal{G}}, \tilde{A} \in \tilde{\mathcal{A}}_P$ of $\phi$ and $A$. We consider the mapping tori bundles $P_\phi = M_\phi, \tilde{P}_\tilde{\phi} \to \tilde{M}_\tilde{\phi}$ and we have $P_\phi = \partial \tilde{P}_\tilde{\phi}$. We choose an extension $A_P^\gamma$ of $A_0^\gamma$ to $\tilde{P}_\tilde{\phi}$, that corresponds to a curve $\tilde{\gamma} \in C^0_A(\tilde{P}_\tilde{\phi})$. If $\lambda = \int_M Tp(\mathcal{A}, \mathcal{A}_0) \in \Omega^1(\mathcal{A}_P)$ and $\beta = \int_{\tilde{M}} Tp(\mathcal{A}, \mathcal{A}_0) \in \Omega^0(\mathcal{A}_P)$ by Stokes Theorem we have $d\beta = \int_M p(\tilde{\mathcal{F}}) - \int_M Tp(\mathcal{A}, \mathcal{A}_0) = \int_{\tilde{M}} p(\tilde{\mathcal{F}}) - r^*\lambda$. By
applying Proposition\footnote{5} we obtain
\[
\text{hol}^\Theta_p(\gamma) = -\int_M \lambda - \alpha_\phi(x)
= -\int_M \lambda - \int_M T p(\bar{\gamma}(1), B_0) + \int_M T p(\bar{\gamma}(0), B_0)
= -\int_M r^*\lambda - \beta(\bar{\gamma}(1)) + \beta(\bar{\gamma}(0)) = -\int_M (r^*\lambda + d\beta)
= -\int_M \int_M T p(\bar{\nu}) = -\int_M \int_M T p(F_{\bar{A}}) = -\int_M \int_M T p(F_{\bar{A}})
= -\xi_{A^\gamma}(M_\phi) = -\text{CS}(A^\gamma).
\]

In the case of $G = SU(2)$ considered in \footnote{\cite{21}} any principal $SU(2)$-bundle over a manifold of dimension 2 or 3 is trivializable, and we can apply the preceding construction to define the Chern-Simons line bundle. For other groups (for example $G = U(1)$) there are nontrivial principal $G$-bundles and this construction cannot be applied. However, our construction in Section \footnote{\cite{5,4}} can be applied in this case. Furthermore, our construction is valid in any even dimension $m = 2k - 2$, for arbitrary group $G$ and $\Xi^\nu_p$ is equivariant with respect of the action of the group $\text{Aut}^+P$ (and not only for gauge transformations).

Finally we relate the character $\Xi^\nu_p$ with the bundles defined in \footnote{\cite{10}}. If $pr: P \times A_P \times \mathbb{R} \to P \times A_P$ is the projection, for any $\phi \in \text{Aut}^+P$ the form $pr^*\lambda_\phi$ is $\phi$-invariant and projects onto a connection $A_\phi$ on $(P \times A_P)_\phi \to (M \times A_P)_\phi$. The differential character $\xi^\nu_{A_\phi} \in \check{H}^2((M \times A_P)_\phi)$ can be integrated over $M$ and we obtain a differential character $\int_M \xi^\nu_{A_\phi} \in \check{H}^2((A_P)_\phi)$. If $\gamma \in C^\nu_\phi(A_P)$ then we can define a curve $\gamma_\phi$ on $(A_P)_\phi$ by setting $\gamma_\phi(t) = [\gamma(t), t]_{-\phi}$. We define $\Lambda^\nu_p(\phi, \gamma) = \left(\int_M \xi^\nu_{A_\phi}\right)(\gamma_\phi)$. It is shown in \footnote{\cite{10}} that if $A_0$ is a connection on $P \to M$, then the form $\lambda = \int_M T p(\bar{\lambda}, \bar{\lambda}_0) \in \Omega^1(A_P)$ satisfies $d\lambda = \omega^\nu_p$, the map $\beta_\phi(A) = \int_M \lambda - \Lambda^\nu_p(\phi, \gamma)$ for $\phi \in \text{Aut}^+P, \gamma \in C^\nu_\phi(A_P)$ satisfies the cocycle condition and the connection $\Theta^\nu = \vartheta - 2\pi i\lambda$ is invariant under the action of $\text{Aut}^+P$ on $A_P \times U(1)$ induced by the cocycle $\lambda$. It follows from Proposition\footnote{5} that $\Lambda^\nu_p = \text{hol}^{\text{Aut}^+P}$. It can be proved using the definition of the fiber integral of differential characters and equation \footnote{\cite{10}} that $\Xi^\nu_p = \Lambda^\nu_p = \text{hol}^{\text{Aut}^+P}$. This result provides an alternative proof of the fact that $\Xi^\nu_p \in \check{H}^2_{\text{Aut}^+P}(A_P)$, but it needs to use fiber integration of differential characters and also the Cheeger-Chern-Simons construction applied to infinite dimensional bundles.

In the rest of the paper we show how the results of \footnote{\cite{10}} can be obtained form the equivariant differential character $\Xi^\nu_p$.

**Example 39** Let $M$ be a Riemann surface, $P = M \times SU(2)$ the trivial principal $SU(2)$-bundle and $\tilde{\nu}$ the characteristic pair corresponding to the second Chern class. In this case $\Xi^\nu_p$, coincide with the Atiyah-Bott symplectic structure $\omega \in \Omega^2(A_P)$ and moment map given by $\omega^\nu_p(a, b) = -\frac{1}{4\pi^2} \int_M \text{tr}(a \wedge b)$ and $(\mu^\nu_P)_X(A) = \frac{1}{4\pi^2} \int_M \text{tr}(v_A(X) \wedge F)$, for $A \in A_P, a, b \in \Omega^2(\Sigma, \text{ad}P) \simeq T_A(A_P)$
and $X \in \text{aut} P$. We have a $\text{Aut}^+ P$-equivariant differential character $\Xi^\nabla_{\partial P} \in \check{H}^2_{\text{Aut}^+ P}(A_P)$ that by Corollary 17 determines (up to an isomorphism) a $\text{Aut}^+ P$-equivariant $U(1)$-bundle $\check{U}^\nabla \to A_P$ with connection $\check{\nabla}$.

If $\check{\mathcal{F}}_P \subset A_P$ is the space of flat connections, then we have $\check{\mathcal{F}}_P \subset (\rho P_p)^{-1}(0)$, and by Proposition 21 $\Xi^\nabla_{\partial P}$ projects onto a $\text{Aut}^+ P$-equivariant differential character $\Xi^\nabla_{\partial P} \in \check{H}^2_{\text{Aut}^+ P}(\check{\mathcal{F}}_P/\text{Gau}^* P)$ on the moduli space of flat connections. The connection $\check{\nabla}$ projects to the quotient $\check{U}^\nabla/\text{Gau}^* P \to \check{\mathcal{F}}_P/\text{Gau}^* P$ and this bundle is isomorphic to Quillen’s determinant line bundle (see [10]).

### 5.6 Action by Gauge transformations

Now we consider that $M$ is an arbitrary oriented manifold, and $P \to M$ a principal $G$-bundle. Let $C$ be a compact oriented manifold of dimension $2r - 2$ and let $c: C \to M$ be a smooth map (for example $C$ can be a submanifold of $M$). Then we have a group homomorphism $\rho: \text{Gau} P \to \text{Aut}^+ (c^* P)$ and a $\rho$-equivariant map $f: A_P \to A_{c^* P}$. By Proposition 14 we have an equivariant differential character $(f, \rho)^* \Xi^\nabla_{\partial P} \in \check{H}^2_{\text{Gau} P}(A_P)$.

### 5.7 Manifolds with boundary

Let $M$ be an oriented manifold with compact boundary $\partial M$. We denote by $v: \partial M \to M$ the inclusion map. If $P \to M$ is a principal $G$-bundle, then we denote by $\partial P$ the bundle $v^* P \to \partial M$. We have a group homomorphism $\rho: \text{Aut}^+ P \to \text{Aut}^+ \partial P$ and a $\rho$-equivariant map $f: A_P \to A_{\partial P}$.

If $\check{\vec{p}} = (p, \Theta)$ is a characteristic pair of degree $r$ and $\dim M = 2r - 1$, then we have the Chern-Simons character $\Xi^\nabla_{\partial P} \in \check{H}^2(\partial P)$. By Proposition 13 these data determine a differential character $(f, \rho)^* \Xi^\nabla_{\partial P} \in \check{H}^2_{\text{Aut}^+ P}(A_P)$. As commented in the Introduction, the equivariant bundles associated to the character $\Xi^\nabla_{\partial P}$ are called the Chern-Simons bundles because the Chern-Simons action on $M$ determines a $\text{Aut}^+ P$-equivariant section of $(f, \rho)^* \Xi^\nabla_{\partial P}$ (see [10]). This fact depends on the background connection $A_0$ chosen in the definition of the bundle and of the Chern-Simons action. We present an intrinsic version of this result.

If $q \in I'(M)$ is a Weil polynomial of degree $r$ and $M$ is a manifold with compact boundary of dimension $2r - 1$, then $\int_M q(\check{\mathcal{F}}) \in \Omega^1(A_P)^{\text{Aut}^+ P}$.

**Proposition 40** If $q \in I'(M)$ is a Weil polynomial of degree $r$ and $M$ is a compact oriented manifold of dimension $2r - 1$, then $\int_M q(\check{\mathcal{F}}) \in \Omega^1(A_P)^{\text{Aut}^+ P}$ and $\varsigma(\int_M q(\check{\mathcal{F}}))(\phi, \gamma) = \int_M q(F_{A_0})$.

**Proof.** For any $(\phi, \gamma) \in \mathcal{C}^{\text{Aut}^+ P}(A_P)$ we have $\varsigma(\int_M q(\check{\mathcal{F}}))(\phi, \gamma) = \int_M q(\check{\mathcal{F}}) = \int_{M \times A} q(\check{\mathcal{F}}) = \int_M q(\check{\mathcal{F}}_{\partial A}) = \int_{M_0} q(\check{\mathcal{F}}_{\partial A_0})$. ■

**Proposition 41** We have $(f, \rho)^* \Xi^\nabla_{\partial P} = \varsigma(\int_M p(\check{\mathcal{F}}))$.  

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Proof. Let us fix \((\phi, \gamma) \in C^{\operatorname{Aut}+P}(A_P)\) and define \(\chi = (f, \rho)^* \Xi_{\phi}^\beta\). The curve \(\gamma \in C^0(A_P)\) induces a curve \(\partial \gamma \in C^0(A_{\partial P})\). If \(P_\phi \to M_\phi\) is the mapping torus bundle of \(P\), then we have \(\partial (P_\phi) = (\partial P)_\phi\). Furthermore, by the properties of the Cheeger-Chern-Simons characters and the preceding proposition we have \(\chi(\phi, \gamma) = \xi_{A_\phi}^\beta((\partial M)_\phi) = \xi_{A_\phi}^\beta(\partial (M_\phi)) = \int_{M_\phi} p(F_{\phi}) = \varsigma(\int_{M_\phi} p(F))(\phi, \gamma)\). \(\blacksquare\)

In particular, it follows from Proposition \([19]\) that the \(\operatorname{Aut}^+ P\)-equivariant \(U(1)\)-bundle associated to \((f, \rho)^* \Xi_{\phi}^\beta\) is trivial and hence it admits a \(\operatorname{Aut}^+ P\)-invariant section. We hope that our approach using equivariant differential characters could be used to study Chern-Simons theory for arbitrary bundles and groups.

5.8 Riemannian metrics and diffeomorphisms

Let \(\vec{\rho} = (\rho, \Upsilon)\) a characteristic pair of degree \(2k\) for the group \(G\ell(4k - 2, \mathbb{R})\). For example we can consider the pair corresponding to the \(k\)-th Pontryagin class. Let \(M\) be a compact oriented manifold without boundary of dimension \(n = 4k - 2\). We denote by \(FM \to M\) the frame bundle of \(M\), by \(\mathcal{M}_M\) the space of Riemannian metrics on \(M\) and by \(D_M^+\) the group of orientation preserving diffeomorphism of \(M\). As \(D_M^+\) acts in a natural way on \(F(M)\) by automorphisms, we have a natural homomorphism \(\rho: D_M^+ \to \operatorname{Aut}^+ F(M)\). The Levi-Civita map \(LC: \mathcal{M}_M \to A_{FM}\) is \(\rho\)-equivariant and hence by Proposition \([14]\) we have an equivariant differential character \(\Sigma_{\mathcal{M}_M}^\beta = (LC, \rho)^* \Xi_{\phi}^\beta \in H_{D_M^+}^2(\mathcal{M}_M)\) with curvature \(\sigma_{\mathcal{M}_M}^\beta = (LC, \rho)^* \sigma_{FM}^\beta\), that can be written \(\sigma_{\mathcal{M}_M}^\beta = \omega_{\mathcal{M}_M}^\beta + \mu_{\mathcal{M}_M}^\beta\).

We consider in more detail the case \(k = 1\). If \(M\) is a Riemann surface of genus \(g > 1\), and \(\mathcal{M}_M^0\) is the space of metrics of constant curvature \(-1\), then we have \(\mathcal{M}_M^0 \subset \mu_{\mathcal{M}_M}^\beta\) (see \([9]\)). If \(D_M^0\) denotes the connected component with the identity on \(D_M^0\), then \(D_M^0\) acts freely on \(\mathcal{M}_M^0\) and the Teichmüller space of \(M\) is defined by \(\mathcal{T}_M = \mathcal{M}_M^0 / D_M^0\). As it is well known (e.g. see \([22]\)), \(\mathcal{T}(M)\) is a contractible manifold of real dimension \(6g - 6\). Furthermore, it is proved in \([9]\) that the form obtained on \(\mathcal{T}_M\) from \(\omega_{\mathcal{M}_M}^\beta\) by symplectic reduction is \(\frac{1}{2\pi} \sigma_{WP}\), where \(\sigma_{WP}\) is the symplectic form of the Weil-Petersson metric on \(\mathcal{T}_M\).

By Remark \([22]\) we obtain an equivariant differential character \(\Sigma_{\mathcal{M}_M}^\beta \in H_{D_M^+}^2(\mathcal{T}_M)\) with curvature \(\frac{1}{2\pi} \sigma_{WP}\), where \(\Gamma_M = D_M^+ / D_M^0\) is the mapping class group of \(M\). By Corollary \([10]\) \(\Sigma_{\mathcal{M}_M}^\beta\) determines (up to an isomorphism) a \(\Gamma_M\)-equivariant \(U(1)\)-bundle with connection over \(\mathcal{T}_M\).

5.8.1 Manifolds with boundary

Let \(M\) be an oriented manifold of dimension \(n = 4k - 1\) with compact boundary \(\partial M\). We have a homomorphism \(\rho: D_M^+ \to \operatorname{Aut}^+ F(\partial M)\) and a \(\rho\)-equivariant map \(f: \mathcal{M}_M \to A_{F(\partial M)}\). By Proposition \([14]\) we obtain an equivariant differential character \((f, \rho)^* \Xi_{F(\partial M)}^\beta \in H_{D_M^+}^2(\mathcal{M}_M)\), and by Corollary \([16]\) a \(D_M^+\)-equivariant \(U(1)\)-bundle with connection over \(\mathcal{M}_M\). We also have a result analogous to Proposition \([11]\)
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