Representations of rational Cherednik algebras

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Abstract. This paper surveys the representation theory of rational Cherednik algebras. We also discuss the representations of the spherical subalgebras. We describe in particular the results on category $\mathcal{O}$. For type $A$, we explain relations with the Hilbert scheme of points on $\mathbb{C}^2$. We insist on the analogy with the representation theory of complex semi-simple Lie algebras.

Contents

1. Introduction 2
2. A motivation via Dunkl operators 3
2.1. Dimension 1 3
2.2. Dimension $n$ 4
3. Structure 5
3.1. The rational Cherednik algebra 5
3.2. The spherical subalgebra 7
4. Representation theory at $t \neq 0$ 8
4.1. Category $\mathcal{O}$ 8
4.2. Dunkl operators and KZ functor 10
4.3. Primitive ideals and supports 11
4.4. Harish-Chandra bimodules 12
5. Representation theory at $t = 0$ 13
5.1. General representations 13
5.2. 0-fiber 14
6. Type $A$ 14
6.1. Structure 14
6.2. Category $\mathcal{O}$ 15
6.3. Shift functors 16
6.4. Hilbert schemes 17
7. Type $A_1$ 20
7.1. Presentation 20
7.2. Category $\mathcal{O}$ and KZ 20
7.3. Spherical subalgebra 21
7.4. Double affine Hecke algebra 21
8. Generalizations 22
8.1. Complex and symplectic reflection groups 22
8.2. Characteristic $p > 0$ 23
9. Table of analogies 23
References 23
1. Introduction

Let $G$ be a complex reductive algebraic group, $T$ a maximal torus and $W = N_G(T)/T$ the Weyl group. Let $\mathfrak{t} = \text{Lie } T$.

There are several “Hecke” algebras associated to $G$ (or to $W$):

| finite Hecke algebra | affine Hecke algebra | double affine Hecke algebra |
|-----------------------|-----------------------|----------------------------|
| $H$                   | $C[T] \otimes H$      | $C[T] \otimes C[T^\vee] \otimes H$ |
| degenerate affine Hecke | $C[t] \otimes C[W]$ | degenerate (trigonometric) daha |
| | | $C[T] \otimes C[t^*] \otimes C[W]$ |
| | | doubly degenerate (rational) daha |
| | | $C[t] \otimes C[t^*] \otimes C[W]$ |

Here, “daha” stands for double affine Hecke algebra (also called Cherednik algebra) and the structure is given as a $C$-vector space. These algebras have various incarnations:

- The finite Hecke algebra is a quotient of the group algebra of the braid group, which is the fundamental group of $\mathfrak{t}_{\text{reg}}/W$. There is a similar description of the affine Hecke algebra (use the space $T_{\text{reg}}/W$) as well as of the double affine Hecke algebra [Che5].

- The finite Hecke algebra appears as a coset algebra for a group of the same type as $G$, over a finite field. There are similar realizations of the affine Hecke algebra (use a 1-dimensional local field) and of the double affine Hecke algebra (2-dimensional local field, cf [Kap]).

- There is a geometric realization of (a quotient of) the daha as the equivariant $K$-theory of the loop Steinberg variety [Vas] (cf also [GarGr, GiKapVas]), generalizing the realization of the affine Hecke algebra as the equivariant $K$-theory of the ordinary Steinberg variety. As pointed out in [BerEtGi2, §7.1], it is likely that there is an analogous description of the degenerate daha obtained by using homology instead of $K$-theory (generalizing the realization of the degenerate affine Hecke algebra as the equivariant homology of the Steinberg variety). There is no hint of existence of a geometric realization of the rational daha.

- When $W$ has type $A_{n-1}$, there are Schur-Weyl dualities between any of the six types of Hecke algebras and corresponding Lie algebras: $\mathfrak{sl}_n$ in the finite case, quantum $\mathfrak{sl}_n$ and Yangian $\mathfrak{sl}_n$ in the affine and degenerate affine case, toroidal quantum $\mathfrak{sl}_n$ [VarVas1] in the daha case, toroidal Yangian $\mathfrak{sl}_n$ in the degenerate daha case, and a subalgebra of the latter in the rational case [Gu2].

After suitable completions, the daha and the degenerate daha can be viewed as trivial deformations of the rational daha ([EtGi, p. 283], [Che6, p.65], [BerEtGi2, §7.1]). As a consequence, the categories of finite dimensional modules for ordinary, degenerate and rational daha’s are equivalent ([Che6], [BerEtGi2, Proposition 7.1], [VarVas2]). The categories $O$ for the ordinary and degenerate daha’s are related [VarVas2]. Finally, category $O$ for the rational daha can be realized as a full subcategory of category $O$ for the degenerate daha [Su].

Double affine Hecke algebras are related to combinatorics and they were introduced by Cherednik as a crucial instrument in the proof of the Macdonald’s constant term conjectures. The degenerations (trigonometric and rational) were obtained in a straightforward way.
In this survey, we are concerned with the representation theory of rational Cherednik algebras. Rational Cherednik algebras are connected to

- Finite Hecke algebras
- Resolutions and deformations of symplectic singularities
- Hilbert schemes of points on surfaces.

There are also the following connections, which we won’t discuss in this survey:

- Integrable systems: rings of quasi-invariants and quantum Calogero-Moser systems (cf [EtSt] for a survey)
- Analytic representation theory, Bessel functions, unitary representations (cf the book [Che7]).

Many of these aspects actually make sense in the framework of symplectic reflection algebras [EtGi].

The rational Cherednik algebra is a deformation of the algebra $C[t \times t^*] \rtimes W$ depending on parameters $t, c$.

The main idea in the study of representations of rational Cherednik algebras (at $t = 1$) is to handle them like universal enveloping algebras of semi-simple complex Lie algebras and study in particular a “category $O$”. We emphasize this in these notes, by expounding the analogies (cf also the table in §9). The (finite) Hecke algebra controls a large part of the representation theory, via the construction of Knizhnik-Zamolodchikov connections. Another important feature of category $O$ is that it generalizes, for any Weyl group (or even any complex reflection group), the construction of $q$-Schur algebras.

The usual interplay between representation theory and geometry is incomplete in general. In type $A$, there are more geometric objects at hand and the Hilbert scheme of points in $C^2$ plays the role of the cotangent bundle of the flag variety.

The representation theory is quite different for $t = 0$. It is related to (generalized) Calogero-Moser spaces.

Section §2 is independent from the rest of the text. It explains how the rational Cherednik algebras occur naturally in the study of a commuting family of operators deforming the partial derivatives.

Although the theory has been developed quite intensively over the last few years, many problems remain and we have listed a number of them.

I have tried to give detailed references for most results, I apologize in advance for possible omissions.

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2. A motivation via Dunkl operators

2.1. Dimension 1.

2.1.1. Fix $k \in \mathbb{R}$. Given $f : \mathbb{R} \to \mathbb{R}$ a function of class $C^1$, consider the function

$$T(f) : x \mapsto f'(x) + k \frac{f(x) - f(-x)}{x}.$$ 

The operator $T$ deforms the ordinary derivation, and presents new features for special values of $k$.

For example, one shows easily that there exists a non-constant polynomial killed by $T$ if and only if $k \in -\frac{1}{2} + \mathbb{Z}_{\leq 0}$.

2.1.2. Let us now study the spectrum of $T$. We consider the Banach algebra of functions $B = \{f = \sum_{n \geq 0} a_n X^n : |-1, 1[ \to \mathbb{R}, \sum_n |a_n| < \infty\}$.

We consider the Banach algebra of functions $B = \{f = \sum_{n \geq 0} a_n X^n : |-1, 1[ \to \mathbb{R}, \sum_n |a_n| < \infty\}$.

We want to solve the equation

(1) $T(f) = \lambda f$ and $f(0) = 1$

for some $\lambda \in \mathbb{R}$ and $f \in B$. 
Assume $k > 0$. Define
\[ \chi : B \to B, \ f \mapsto (x \mapsto \alpha \int_{-1}^{1} f(xt)(1-t)^{k-1}(1+t)^k dt) \]
where $\alpha = \left( \int_{-1}^{1}(1-t)^{k-1}(1+t)^k dt \right)^{-1}$ (so that $\chi(1) = 1$). Then, one shows that
\[ T \circ \chi = \chi \circ \frac{d}{dx} \]

So, $\chi(\exp(\lambda x))$ is the unique solution of (1) (it can be expressed in terms of Bessel functions).

More classical would be the study of the eigenfunctions of the operator $T^2$ acting on even functions: given $f$ with $f(-x) = f(x)$, then $T^2(f) = \frac{d^2 f}{dx^2} + 2k \frac{df}{dx}$.

We refer to [CheMa] and [Che7, §2] for a more detailed study, in particular of the analytic aspects (Hankel transform, truncated Bessel functions).

### 2.2. Dimension $n$.

#### 2.2.1. We are now going to generalize the previous construction to the case of $n \geq 2$ variables. We will focus on the algebraic aspects (polynomial functions) and work with complex coefficients. In particular, we take now $k \in \mathbb{C}$.

Let $V = \bigoplus_n \mathbb{C} \xi_i$ and $V^* = \bigoplus_n \mathbb{C} x_i$ with the dual basis. Let $\mathfrak{S}_n$ be the symmetric group on $\{1, \ldots, n\}$. It acts on $V$ by permutation of the coordinates, hence it acts on functions $V \to \mathbb{C}$. We denote by $\rho_{ij}$ the endomorphism of the space of functions $V \to \mathbb{C}$ given by the transposition $(ij)$.

Given $1 \leq i \leq n$ and $f : V \to \mathbb{C}$ smooth, we define
\[ T_i(f) = \frac{\partial f}{\partial \xi_i} + k \sum_{j \neq i} \frac{f - \rho_{ij}(f)}{X_i - X_j} \]

We have a family of operators (the Dunkl operators) deforming the ordinary partial derivatives. What makes this deformation interesting is Dunkl’s result:
\[ T_i \circ T_j = T_j \circ T_i \text{ for all } i, j. \]

Note also that $T_i$ sends a polynomial to a polynomial.

Let $\mathcal{E}$ be the set of values of $k$ for which there are non-constant polynomials killed by $T_1, \ldots, T_n$.

The case $n = 2$ here is related to the 1-dimensional case of §2.1 by restricting functions $\mathbb{C}^2 \to \mathbb{C}$ to the subspace $x_1 + x_2 = 0$.

Note that the study of functions, in a space similar to $B$ above, that are simultaneous eigenvectors for all $T_i’s$ can be done similarly, when $k \in \mathbb{R}_{>0}$. The endomorphism $\chi$ can be constructed first for polynomials, and then extended to $B$ by continuity. Nevertheless, there is no explicit integral form for $\chi$.

#### 2.2.2. We denote by $\mathbb{C}[V] = \mathbb{C}[X_1, \ldots, X_n]$ the algebra of polynomial functions on $V$. Let $H$ be the subalgebra of $\text{End}_\mathbb{C}(\mathbb{C}[V])$ generated by $T_1, \ldots, T_n, X_1, \ldots, X_n$ (acting by multiplication), and $\mathfrak{S}_n$ (acting by permutation on the $X_i$’s). This is the rational Cherednik algebra.

One shows that $k \in \mathcal{E}$ if and only if $\mathbb{C}[V]$ is not an irreducible representation of $H$. Let us now say more about the structure of $H$.

When $k = 0$, then $H = D(V) \rtimes \mathfrak{S}_n$, where $D(V)$ is the algebra of polynomial differential operators on $V$. In general, there is a vector space decomposition
\[ H = \mathbb{C}[T_1, \ldots, T_n] \otimes \mathbb{C}[\mathfrak{S}_n] \otimes \mathbb{C}[X_1, \ldots, X_n]. \]

This shows that $H$ (which depends on the parameter $k$) is a deformation of $D(V) \rtimes \mathfrak{S}_n$.

This is analogous to the decomposition $\mathfrak{g}l_n(\mathbb{C}) = n^+ \oplus \mathfrak{h} \oplus n^-$, where $n^+$ (resp. $n^-$) are strictly upper (resp. lower) triangular matrices and $\mathfrak{h}$ diagonal matrices, or rather analogous to the decomposition of the enveloping
algebra (Poincaré-Birkhoff-Witt Theorem)
\[ U(\mathfrak{g}l_n(\mathbb{C})) = U(n^+) \otimes U(h) \otimes U(n^-) \].

This analogy is a guide for the study of the representation theory of $H$.

First, one defines a category $\mathcal{O}$ of finitely generated $H$-modules on which the $T_i$’s act locally nilpotently. Given $E$ a complex irreducible representation of $\mathfrak{S}_n$, one gets $\Delta(E) = \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(E)$, an object of $\mathcal{O}$. It has a unique simple quotient $L(E)$, and one obtains this way all simple objects of $\mathcal{O}$. Note that $\Delta(C) = C[V]$ is the original faithful representation. Outside a countable set of values of $k$, then $\mathcal{O}$ is semi-simple.

2.2.3. We now relate $\mathcal{O}$ to the Hecke algebra $H$ of $\mathfrak{S}_n$, at $q = \exp(2i\pi k)$.

Let $V_{\text{reg}} = \{(z_1, \ldots, z_n) \in V | z_i \neq z_j \text{ for } i \neq j\}$. Note that $T_i \in D(V_{\text{reg}}) \rtimes \mathfrak{S}_n$ and one gets an embedding $H \subset D(V_{\text{reg}}) \rtimes \mathfrak{S}_n$. This induces an isomorphism of algebras
\[ H \otimes_{C[V]} C[V_{\text{reg}}] \cong D(V_{\text{reg}}) \rtimes \mathfrak{S}_n \]

After localization, the deformation becomes trivial!

Let $M \in \mathcal{O}$. Then, $M \otimes_{C[V]} C[V_{\text{reg}}]$ is an $\mathfrak{S}_n$-equivariant vector bundle on $V_{\text{reg}}$ with a flat connection, that is shown to have regular singularities (along the hyperplanes and at infinity). This provides us with a system of differential equations, and taking solutions we obtain an $\mathfrak{S}_n$-equivariant local system on $V_{\text{reg}}$, i.e., a local system on $V_{\text{reg}}/\mathfrak{S}_n$. This corresponds to a finite dimensional representation of the braid group $B_n = \pi_1(V_{\text{reg}}/\mathfrak{S}_n, (1, 2, \ldots, n))$. That representation is shown to come from a representation of $H$ and this defines a functor
\[ \text{KZ} : \mathcal{O} \rightarrow H\text{-mod} \]

(actually, a contravariant functor; one needs to dualize or equivalently use the de Rham functor instead of the solution functor in order to have a covariant functor). This functor has good homological properties and there is an equivalence $\mathcal{O} \cong \text{End}_H(P)$-mod, where $P$ is a certain $H$-module.

When $k \not\in \frac{1}{2} + \mathbb{Z}$, $P$ can be identified as the $q$- tensor space $L^{\otimes n}$, where $L$ is an $n$-dimensional vector space, and this identifies $\mathcal{O}$ with the category of modules over the $q$-Schur algebra, i.e., a full subcategory of the category of modules over the quantum general linear group $U_q(\mathfrak{g}l_n)$. The knowledge of character formulas for simple modules in that setting allows to deduce the multiplicities $[\Delta(E) : L(F)]$ for the algebra $H$. This is nicely described in terms of canonical basis for the Fock space, under the action of $\mathfrak{sl}_d$, where $d$ is the order of $k$ in $\mathbb{C}/\mathbb{Z}$.

Let us finally describe the set $E$:
\[ E = \left\{ -\frac{r}{s} | 2 \leq s \leq n, \ r \in \mathbb{Z}_{>0} \text{ and } (s, r) = 1 \right\}. \]

The space of polynomials killed by the $T_i$’s and its structure as a representation of $\mathfrak{S}_n$ can be determined (cf [DuDeJOp, Du4]).

The analytic aspects, which we are not considering here, are very interesting: multi-dimensional Bessel functions, Hankel transform [Op, DuDeJOp], and unitary representations of rational Cherenik algebras. It sheds new light on a number of classical results (cf [Du2], [Che7, Chapter 2]).

3. Structure

Let us start with the definition of the rational Cherednik algebras and some important subalgebras, and their main properties, following [EtGi, Part 1].

3.1. The rational Cherednik algebra.
3.1.1. Let $W$ be a finite reflection group on a finite dimensional real vector space $V_{\mathbb{R}}$ and let $V = C \otimes_{\mathbb{R}} V_{\mathbb{R}}$. Let $n = \dim C V$. Let $S$ be the set of reflections of $W$ and $\tilde{S} = S/W$. Given $s \in S$, let $v_s \in V$ (resp. $\alpha_s \in V^*$) be an $-1$ eigenvector for $s$ acting on $V$ (resp. $V^*$).

Let $c = \{c_s\}_{s \in \tilde{S}}$ be a family of variables and $A = C[c_s, t]$. Note that for types ADE, we have $|\tilde{S}| = 1$.

The rational Cherednik algebra $H$ associated to $(W, V)$ is the quotient of $A \otimes_{C} (T(V \oplus V^*) \times W)$ by the relations\footnote{In [GGOR], we use $\gamma_H = -2c_s s$ and $k_{i=1} = -c_s$, where $H$ is the reflecting hyperplane of $s$}

\[ [\xi, \eta] = 0 \text{ for } \xi, \eta \in V, \quad [x, y] = 0 \text{ for } x, y \in V^* \]
\[ [\xi, x] = t \langle \xi, x \rangle - 2 \sum_{s \in S} \frac{\langle \xi, \alpha_s \rangle}{\langle v_s, \alpha_s \rangle} c_s s \]

There is a filtration on $H$ given by

\[ F^0 H = A[W], \quad F^1 H = (V \oplus V^*) \otimes_{C} A[W] \otimes A[W], \quad \text{and} \quad F^i H = (F^1 H)^i \text{ for } i \geq 2. \]

Let $gr H = \bigoplus_{i \geq 0} F^i H/F^{i-1} H$. The canonical morphism of $A$-modules $(V \oplus V^*) \otimes_{C} A[W] \to gr H$ extends to a surjective morphism of $A$-algebras $A \otimes_{C} (S(V) \otimes S(V^*)) \rtimes W \to gr H$. The following result asserts it is actually an isomorphism. This gives a triangular decomposition of $H$ (Cherednik, [EtGi, Theorem 1.3]):

**Theorem 3.1.** We have a canonical isomorphism of $A$-modules $S(V) \otimes_{C} A[W] \otimes_{C} S(V^*) \sim H$. In particular, the canonical map of $A$-algebras

\[ A \otimes_{C} (S(V) \otimes S(V^*)) \rtimes W \sim gr H \]

is an isomorphism.

**About the Proof.** Note that it is enough to prove the isomorphism after applying $- \otimes_{A} C$ for any morphism $A \to C$, i.e., for specialized parameters. Then, one can use the faithful representation by Dunkl operators (cf §4.2.1 and 5.1.1).

Given $t \in C$ and $c = \{c_s\}_{s \in \tilde{S}} \in C^{\tilde{S}}$, we put $H_{t,c} = H \otimes_{A} C$, where the morphism $A \to C$ is given by $t \mapsto t$ and $c_s \mapsto c_s$. Theorem 3.1 shows that $H$ is a deformation of $H_{t,c}$.

**Analogy 1.** Let $G$ be a semi-simple complex algebraic group, $T$ a maximal torus, $B$ a Borel subgroup containing $T$, $U^+$ its unipotent radical and $U^-$ the opposite unipotent subgroup. Let $\mathfrak{g} = Lie G$, $\mathfrak{h} = Lie T$, $n^+ = Lie U^+$ and $n^- = Lie U^-$. We have (Poincaré-Birkhoff-Witt Theorem)

\[ U(\mathfrak{g}) = U(n^+) \otimes U(\mathfrak{h}) \otimes U(n^-). \]

We have a filtration of $U(\mathfrak{g})$ given by $F^0 U(\mathfrak{g}) = C$, $F^1 U(\mathfrak{g}) = \mathfrak{g} \oplus C$ and $F^i U(\mathfrak{g}) = (F^1 U(\mathfrak{g}))[i]$. There is a canonical isomorphism $S(\mathfrak{g}) \sim gr U(\mathfrak{g})$.

**Remark 3.2.** If $W = W_1 \times W_2$ and $V = V_1 \oplus V_2$ are compatible decompositions, then there are canonical isomorphisms $H_1 \otimes H_2 \sim H$, where $H_i$ is the rational Cherednik algebra of $(W_i, V_i)$.

Let $S'$ be a $W$-invariant subset of $S$ such that $s_0 = 0$ for $s \in S - S'$. Let $W'$ be the reflection subgroup of $W$ generated by $S'$. Then, there is an embedding $H' \subset H$ and $H$ is a twisted group algebra of $W/W'$ over $H'$. 

\[ \gamma_H = -2c_s s \text{ and } k_{i=1} = -c_s, \text{ where } H \text{ is the reflecting hyperplane of } s \]
3.1.2. Specializations. For $t \neq 0$, we have $H_{t,c} \cong H_{1,t-1,c}$. We put $H_c = H_{1,c}$.

Consider the case $c = 0$:

- We have $H_{0,0} = S(V \oplus V^*) \ltimes W$. So, $H_{0,c}$ is a deformation of $S(V \oplus V^*) \ltimes W$.
- We have $H_{1,0} = D(V) \ltimes W$, where $D(V)$ is the Weyl algebra of $V$ (algebra of algebraic differential operators on $V$). So, $H_{1,c}$ is a deformation of $D(V) \ltimes W$.

Analogy 2. The parameter space $C^S$ corresponds to $\mathfrak{h}^*/W$ and the analog of $H_{1,c}$ is $\bar{U}_\lambda(\mathfrak{g}) = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\lambda$, where $\lambda \in \mathfrak{h}^*/W$, and $\mathfrak{m}_\lambda$ is the maximal ideal of $Z(U(\mathfrak{g}))$ image of $\lambda$ by the canonical isomorphism $\mathfrak{h}^*/W \cong \text{Spec } Z(U(\mathfrak{g}))$.

Consider the induced filtration on $\bar{U}_\lambda(\mathfrak{g})$. Let $N$ be the nilpotent cone of $\mathfrak{g}$. Then, there is a canonical isomorphism $C[N] \cong \text{gr} \bar{U}_\lambda(\mathfrak{g})$.

3.1.3. Fourier transform. Cf [EtGi, §4,5].

Fix an isomorphism of $C[W]$-modules $F : V \cong V^*$. This extends to an automorphism $F$ of $H_c$ given by $V \ni \xi \mapsto F(\xi)$, $V^* \ni x \mapsto -F^{-1}(x)$ and $W \ni w \mapsto w$.

More generally, there is an action of $\text{SL}_2(\mathbb{A})$ on $H$. The action of \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} is given by $V \ni \xi \mapsto a_{22}\xi + a_{21}F(\xi)$, $V^* \ni x \mapsto a_{11}x + a_{12}F^{-1}(x)$ and $W \ni w \mapsto w$.

3.1.4. Twist by characters. Let $c \in C^S$ and $t \in C$. Let $\zeta : W \to \{\pm 1\}$ be a character. There is an isomorphism of $C$-algebras

$$H_{t,c} \cong H_{t,c\zeta}, \quad V \ni \xi \mapsto \xi, \quad V^* \ni x \mapsto x, \quad W \ni w \mapsto \zeta(w)w.$$

3.1.5. Deformed Euler vector field and canonical grading. We consider in the remaining part of §3.1 the algebra $H_c = H_{1,c} = H \otimes_{C[s]} C[t]/(t - 1)$.

Let $B$ be a basis of $V$ and $(b^\vee)_{b \in B}$ be the dual basis. Let $eu' = \sum_{b \in B} b^\vee b$ be the “deformed” Euler vector field, $z = \sum_{s \in S} c_s(s - 1)$ and $eu = eu' - z$. Let $h = \frac{1}{2} \sum_{b \in B} (bb^\vee + b^\vee b)$. Then, $h = eu + \frac{1}{2} \dim V - \sum_{s \in S} c_s$. An easy computation in $H_c$ shows that $[eu, \xi] = -\xi$, $[eu, x] = x$ and $[eu, w] = 0$ for $\xi \in V$, $x \in V^*$ and $w \in W$.

So, the eigenspace decomposition of $H_c$ under the action of $[eu, -]$ puts a grading on $H_c : W$ is in degree 0, $V^*$ in degree 1 and $V$ in degree $-1$ (note that this defines a grading on $H$ as well, before specializing $t$ to 1).

Let $M$ be an $H_c$-module. We denote by $M_\alpha$ the generalized $\alpha$-eigenspace for $eu$ acting on $M$. For certain modules, we have a decomposition $M = \bigoplus_{\alpha \in C} M_\alpha$, and this gives a canonical $C$-grading on $M$.

3.1.6. $\mathfrak{sl}_2$-triple. Cf [De, §4], [BerEtGi3, §3].

Let $p : V \times V \to C$ and $q : V^* \times V^* \to C$ be the $W$-equivariant perfect pairings induced by $F$. If $\mathcal{B}$ is orthonormal for $p$, then $p = \sum_b (b^\vee)^2$ and $q = \sum_b b^2$. An easy computation shows that $(\frac{1}{2}p, h, -\frac{1}{2}q)$ is an $\mathfrak{sl}_2$-triple in $H_c$.

3.2. The spherical subalgebra.

3.2.1. Cf [EtGi, §2].

Let $e = \frac{1}{|W|} \sum_{w \in W} w$, an idempotent of $Z(C[W])$. Let $\mathcal{B} = eH_0$ be the spherical subalgebra of $H$. Theorem 3.1 gives a canonical isomorphism $A \otimes C(C[V^* \times V]/W) \cong \text{gr } \mathcal{B}$ (for the induced filtration on $\mathcal{B}$).

We have canonical isomorphisms $B_{1,0} \cong D(V)^W$ and $B_{0,0} \cong C[(V^* \times V)/W]$. So, $\mathcal{B}$ is a deformation of $D(V)^W$ and of $C[(V^* \times V)/W]$.

Analogy 3. The analogy between $\mathcal{B}$ and $U(\mathfrak{g})$ is more accurate than that with $H$: instead of the smooth orbifold $[(V^* \times V)/W]$, one gets the singular variety $(V^* \times V)/W$ as corresponding to the nilpotent cone. Also, for a Cherednik algebra of type $A_1$, then $B_{1,c}$ is isomorphic to an algebra $\tilde{U}_\lambda(\mathfrak{sl}_2)$, cf §7.3.3.
3.2.2. The center and the spherical subalgebra have different behaviours, depending on \( t \).
We have \( Z(H_{t,c}) = C \) if \( t \neq 0 \) [BrGo, Proposition 7.2].

The algebra \( B_{t,c} \) is commutative if and only if \( t = 0 \) [EtGi, Theorem 1.6]. In particular, we get a structure of Poisson algebra on \( B_{0,c} \).

We have [EtGi, Theorem 3.1]:

**Theorem 3.3 ("Satake isomorphism").** We have an isomorphism \( Z(H_{0,c}) \xrightarrow{\sim} B_{0,c}, \ z \mapsto ze \).

The Calogero-Moser space associated to \( W \) is \( CM_c = \text{Spec} \( Z(H_{0,c}) \)). It is a Gorenstein normal Poisson variety [EtGi, Theorem 1.5 and Lemma 3.5] and a symplectic variety when smooth [EtGi, Theorem 1.8].

There is an inclusion \( S(V)^W \otimes S(V^*)^W \subset Z(H_{0,c}) \) and \( Z(H_{0,c}) \) is a free \( (S(V)^W \otimes S(V^*)^W) \)-module of rank \( |W| \) [EtGi, Proposition 4.15]. This gives a finite surjective map \( \Upsilon : CM_c \rightarrow V^*/W \times V/W \).

3.2.3. There is a "double centralizer Theorem":

**Theorem 3.4 ([EtGi, Theorem 1.5]).** We have canonical isomorphisms \( B \xrightarrow{\sim} \text{End}_{H}(He) \) and \( H \xrightarrow{\sim} \text{End}_{B^+}(He) \).

The bimodule \( H_{t,c}e \) induces a Morita equivalence between \( H_{t,c} \) and \( B_{t,c} \) (i.e., \( H_{t,c}e \otimes_{B_{t,c}} - : B_{t,c} \)-mod \( \rightarrow H_{t,c} \)-mod is an equivalence) if and only if \( H_{t,c} = H_{t,c}e H_{t,c} \). Cf Theorems 4.14 and 6.6 for cases of Morita equivalence. Note that \( H_{0,0} \) is not Morita equivalent to \( B_{0,0} \) for \( W \neq 1 \) (the first algebra has finite global dimension while the second one doesn’t).

## 4. Representation theory at \( t \neq 0 \)

4.1. **Category \( \mathcal{O} \).** Cf [DuOp, §2], [Gu1], [GGOR, §2.3], [BerEtGi3, §2], [Gi, Corollary 6.7].

4.1.1. **Decomposition.** Fix a specialization \( \widetilde{H} = H_c \) (i.e., \( t = 1 \)). Let \( \mathcal{O}' \) be the category of finitely generated \( H \)-modules that are locally finite for \( S = S(V) \).

Let \( \lambda \in V^*/W \), i.e., \( \lambda \) is a morphism of algebras \( S^W \rightarrow C \). Let \( \mathcal{O}_\lambda \) be the subcategory of objects \( M \) in \( \mathcal{O}' \) such that for any \( m \in M \) and any \( \xi \in S^W \), then \( (\xi - \lambda(\xi))^m \cdot m = 0 \) for \( n \gg 0 \).

Then,

\[
\mathcal{O}' = \bigoplus_{\lambda \in V^*/W} \mathcal{O}_\lambda
\]

4.1.2. **Principal block.** We focus our study on \( \mathcal{O} = \mathcal{O}_0 \) (similar descriptions for arbitrary \( \lambda \) have been partially worked out). We write also \( \mathcal{O}_c \) for the category \( \mathcal{O} \) of \( H_c \).

Let \( \text{Irr}(W) \) be the set of isomorphism classes of irreducible complex representations of \( W \). Given \( E \in \text{Irr}(W) \), let \( N_E = \sum_{s \in S} \frac{\text{Tr}(|s|E)}{\dim E} c_s = \frac{\text{Tr}(z|E)}{\dim E} + \sum_{s \in S} c_s \). We define an order on \( \text{Irr}(W) \) by \( E < F \) if \( N_F - N_E \in \mathbb{Z}_{>0} \).

Given \( E \in \text{Irr}(W) \), let

\[
\Delta(E) = \text{Ind}_{S^W \times W}^H E
\]

where \( E \) is viewed as an \((S \times W)\)-module with \( V \) acting as 0.

Then, we have

**Theorem 4.1 ([GGOR, Theorem 2.19]).** \( \mathcal{O} \) is a highest weight category with standard objects the \( \Delta(E) \)'s.

**About the proof.** The approach is similar to the one for affine Lie algebras and makes crucial use of the canonical grading.

In particular, \( \Delta(E) \) has a unique simple quotient \( L(E) \) and \( \{L(E)\}_{E \in \text{Irr}(W)} \) is a complete set of representatives of isomorphism classes of simple objects of \( \mathcal{O} \).

It follows also that if no two distinct elements of \( \text{Irr}(W) \) are comparable, then \( \mathcal{O} \) is semi-simple.

**Corollary 4.2.** For generic values of \( c \), then \( \mathcal{O} \) is semi-simple and \( \Delta(E) = L(E) \) for all \( E \in \text{Irr}(W) \).
The costandard object \( \nabla(E) \) is the \( H \)-submodule of \( \text{Hom}_{S(V^*) \times W}(H, E) \) of elements that are locally nilpotent for \( S \) (where \( E \) is viewed as an \( (S(V^*) \times W) \)-module with \( V^* \) acting as \( 0 \)).

Simple objects don’t have self-extensions:

**Proposition 4.3** ([BerEtGi2, Proposition 1.12]). We have \( \text{Ext}_\mathcal{O}(L(E), L(E)) = 0 \) for any \( E \in \text{Irr}(W) \).

4.1.3. Dualities. Cf [GGOR, §4].

Let \( M \in \mathcal{O} \). Let \( M' \) be the \( k \)-submodule of \( S(V^*) \)-locally nilpotent elements of \( \text{Hom}_C(M, C) \). Consider the anti-involution on \( H_c \):

\[
\psi : H_c \xrightarrow{\sim} H_c^{opp}, \quad \xi \ni V \mapsto -F(\xi), \quad V^* \ni x \mapsto -F^{-1}(x), \quad W \ni w \mapsto w^{-1}.
\]

Then \( M^\vee = \psi_* M' \in \mathcal{O} \) and this defines a duality

\[
\mathcal{O} \xrightarrow{\sim} \mathcal{O}^{opp}, \quad M \mapsto M^\vee.
\]

We have \( L(E)^\vee \simeq L(E) \) and \( \Delta(E)^\vee \simeq \nabla(E) \).

The anti-involution

\[
H_c \xrightarrow{\sim} H_c^{opp}, \quad \xi \ni V \mapsto -\xi, \quad V^* \ni x \mapsto x, \quad W \ni w \mapsto w^{-1}
\]

provides \( H_c \) with a structure of \((H_c \otimes H_c)\)-bimodule and we obtain a duality

\[
\text{RHom}_{H_c}(-, H_c[n]) : D^b(H_c\text{-mod}) \xrightarrow{\sim} D^b(H_c\text{-mod})^{opp}.
\]

It restricts to a duality

\[
D : D^b(\mathcal{O}) \xrightarrow{\sim} D^b(\mathcal{O})^{opp}.
\]

We have \( D(\Delta(E))^\vee \simeq \nabla(E \otimes \det) \) and \( D(P(E))^\vee \simeq T(E \otimes \det) \) (a tilting module). As a consequence, \( \mathcal{O} \) is equivalent to its Ringel dual.

4.1.4. Dihedral groups. The structure of the standard modules for \( W = I_2(d) \) a dihedral group is given in [Chm]. Let us explain the results, in the simpler case \( d = 2m + 1 \) is odd. We denote by \( \tau_l \) the 2-dimensional irreducible representation whose first occurrence in \( S(V) \) is in degree \( l \) (where \( 1 \leq l \leq m \)). We assume \( c > 0 \) (cf §3.1.4 to deduce the case \( c < 0 \)).

- Assume first \( c = \frac{r}{2} \) for some \( r \in \mathbb{Z}_{>0}, d \mid r \). Let \( l \in \{1, \ldots, m\} \) such that \( r \equiv \pm l \pmod{d} \).
  - We have \( L(\rho) = \Delta(\rho) = P(\rho) \) if \( \rho \neq c, \det, \tau_l \) (they form simple blocks of category \( \mathcal{O} \)).
  - We give now the Loewy series of various modules in \( \mathcal{O} \) (socle and radical series coincide):
    \[
    \Delta(C) = P(C) = \frac{L(C)}{L(\tau_l)}, \quad \Delta(\det) = L(\det) = T(\det), \quad \Delta(\tau_l) = \frac{L(\tau_l)}{L(\det)}
    \]
    \[
    P(\det) = T(\tau_l) = \frac{L(\tau_l)}{L(\det)}, \quad P(\tau_l) = T(C) = \frac{L(\det) \oplus L(C)}{L(\tau_l)}.
    \]

The only simple finite dimensional module is \( L(C) \).

- Assume now \( c = \frac{r}{2} + \mathbb{Z}_{>0} \).
  - We have \( L(\rho) = \Delta(\rho) = P(\rho) \) if \( \rho \neq c, \det \) (they form simple blocks of category \( \mathcal{O} \)).
  - We have
    \[
    \Delta(C) = P(C) = \frac{L(C)}{L(\det)}, \quad \Delta(\det) = T(\det) = \frac{L(\det)}{L(\det)}, \quad P(\det) = T(C) = \frac{L(C)}{L(\det)}.
    \]

- If \( c \in \mathbb{Z}_{>0} \) or neither \( 2c \) nor \( dc \) are integers, then \( \mathcal{O} \) is semi-simple.
The structure is more complicated when $d$ is even, for special values of the parameter. In particular, finite dimensional modules need not be semi-simple (this occurs as well for $W$ of type $D_4$ with parameter $c = \frac{1}{2}$ [BerEtGi2, Example 6.4]).

4.1.5. In view of the analogy with complex semi-simple Lie algebras, finite-dimensional representations are particularly interesting. A particular class of finite dimensional quotients of $\Delta(C)$ has been studied (“perfect representations”): these are naturally commutative algebras, and they generalize the Verlinde algebras [Che6].

**Problem 1.**
- Find the multiplicities $[\Delta(E) : L(F)]$. They are known when $W$ is dihedral (cf §4.1.4) and when has type $A_n$ and $c \not\in \frac{1}{2} + \mathbb{Z}$ (cf Corollary 6.3).
- Describe the category of finite dimensional representations of $H$. For which values of $c$ does $H$ have non-zero finite dimensional representations? Cf [Che6, De, BerEtGi2, Go2, ChmEt, Vas] for studies of finite dimensional representations. These questions are solved in type $A_n$, cf Theorem 6.5.
- Is $\mathcal{O}$ Koszul? If so, is it its own Koszul dual (up to a change of parameters)?

**Analogy 4.** The category $\mathcal{O}'$ of finitely generated $U(g)$-modules that are diagonalizable for $U(b)$ and locally finite for $U(n^+)$ splits into a sum of subcategories corresponding to a fixed central character. The finite dimensional representations are semi-simple and the simple ones correspond to dominant weights. The principal block $\mathcal{O}$ is a highest weight category with standard objects being Verma modules. The parametrizing set is $W$, with the Bruhat order. The multiplicities of simple objects in standard objects are given by evaluation at 1 of the Kazhdan-Lusztig polynomials for $W$ (Kazhdan-Lusztig conjecture, proven by Beilinson-Bernstein and Brylinski-Kashiwara). The principal block $\mathcal{O}$ is Koszul and it is equivalent to its Koszul dual (Beilinson-Ginzburg-Söergel).

4.2. Dunkl operators and KZ functor.

4.2.1. Dunkl operators. Cf [Che7], [EtGi, §4], [DuOp, §2.2], [GGOR, §5.2].

Denote by $\mathcal{C}$ the trivial representation of $W$. Via the canonical isomorphism $\mathcal{C}[V] \xrightarrow{\sim} \Delta(\mathcal{C})$, we obtain an action of $H$ on $\mathcal{C}[V]$. The action of $W$ is the natural action, the action of $\mathcal{C}[V]$ is given by multiplication, and the action of $\xi \in V$ is given by the Dunkl operator (for type $A$, this is the same as §2)

$$T_{\xi} = \partial_{\xi} + \sum_{s \in S} \frac{\langle \xi, \alpha_s \rangle}{\alpha_s} c_s (s - 1).$$

This gives a morphism $\rho : H \to D(V_{\text{reg}}) \rtimes W$, where $V_{\text{reg}} = V - \bigcup_{s \in S} \ker \alpha_s$.

A fundamental property is the faithfulness of that representation (Cherednik and [EtGi, Proposition 4.5]):

**Theorem 4.4.** The morphism $\rho$ is injective and induces an isomorphism $H \otimes_{\mathcal{C}[V]} \mathcal{C}[V_{\text{reg}}] \xrightarrow{\sim} D(V_{\text{reg}}) \rtimes W$.

**About the proof.** One puts a filtration on $H$ with $W$ and $V^*$ in degree 0, and $V$ in degree 1. Then, $\rho$ is compatible with the filtration on $D(V_{\text{reg}})$ given by the order of differential operators and the associated graded map is injective. □

Note that $\rho(eu) = \sum_{b \in B} b^y b \in D(V) \rtimes W$ is the ordinary Euler vector field.

**Remark 4.5.** Via the canonical isomorphism $D(V_{\text{reg}})^W \xrightarrow{\sim} e D(V_{\text{reg}})^e$, $f \mapsto e f$, the restriction of $\rho$ to $B_{1,c}$ gives an injective morphism $B_{1,c} \to D(V_{\text{reg}})^W$.

4.2.2. Knizhnik-Zamolodchikov functor. Cf [GGOR, §5.3-5.4].

We are going to associate a vector bundle with a connection on $V_{\text{reg}}$ associated to an object of $\mathcal{O}$. In the case of a standard object $\Delta(E)$, this is essentially the Knizhnik-Zamolodchikov-Cherednik connection [Che2, Che4, Op] (cf the affine equation in the trigonometric setting in [Che4]). For generic values of the parameter, every object of $\mathcal{O}$ is a sum of $\Delta(E)$’s and this is used in the constructions below via deformation arguments.
Let $M \in \mathcal{O}$ and $M_{reg} = \rho_s(M \otimes_{\mathbb{C}[V]} \mathbb{C}[V_{reg}])$. This corresponds to a $W$-equivariant vector bundle on $V_{reg}$ with a flat connection. It is shown to have regular singularities, by treating first the case $M = \Delta(E)$.

Applying the de Rham functor $\text{Hom}_{\mathcal{O}(V_{reg})}(\mathcal{O}(V_{reg}), -)$ gives a $W$-equivariant locally constant sheaf on $V_{reg}$. This corresponds to a locally constant sheaf on $V_{reg}/W$, hence to a finite dimensional representation $F(M)$ of $B = \pi_1(V_{reg}/W)$ (relative to some base point). Fixing a base point in $(V_W)_{reg}$ provides a description of $W$ as a finite Coxeter group, with set of simple reflections $S_0$. It also provides an identification of $B$ as the corresponding braid group with set of generators $\{\sigma_s\}_{s \in S_0}$.

Let $\mathcal{H}$ be the Hecke algebra of $W$ with parameters $\{1, - \exp(2i\pi c_s)\}$, i.e., the quotient of $\mathbb{C}[B_W]$ by the relations $(\sigma_s - 1)(\sigma_s + \exp(2i\pi c_s)) = 0$.

Then, the representation $F(M)$ of $B$ factors through a representation $\text{KZ}(M)$ of $\mathcal{H}$. This is proven by first computing the eigenvalues of monodromy when $M$ is a standard module and for generic values of the parameter. A deformation argument shows the result in general.

Let $\mathcal{O}_{tor}$ be the full subcategory of $\mathcal{O}$ of objects $M$ such that $M_{reg} = 0$. The main properties of $\text{KZ}$ are given in the following Theorem [GGOR, Theorem 5.14, Theorem 5.16, and Proposition 5.9]:

**Theorem 4.6.** The functor $\text{KZ}$ is exact and it induces an equivalence $\mathcal{O}/\mathcal{O}_{tor} \sim \mathcal{H}\text{-mod}$.

Given $M, N \in \mathcal{O}$, the canonical map $\text{Hom}_{\mathcal{O}}(M, N) \rightarrow \text{Hom}_{\mathcal{H}}(\text{KZ}(M), \text{KZ}(N))$ is an isomorphism in the following cases:

- when $N$ is projective
- when $c_s \notin \frac{1}{2} + \mathbb{Z}$ for all $s \in S$ and $N$ is $\Delta$-filtered.

**Proof**. One shows that $\mathcal{O} \rightarrow \mathcal{O}/\mathcal{O}_{tor}$ is fully faithful on projective objects by using the duality $D$. The fully faithfulness of $\mathcal{O}/\mathcal{O}_{tor} \rightarrow \mathcal{H}\text{-mod}$ is a consequence of the Riemann-Hilbert correspondence (Deligne). The essential surjectivity follows from a deformation argument. The statement in the case where $N$ is $\Delta$-filtered is obtained by a computation of residues.

Let $P$ be a progenerator for $\mathcal{O}$. Then, $\mathcal{O} \simeq \text{End}_\mathcal{H}(\text{KZ}(P))\text{-mod}$. The algebra $\text{End}_\mathcal{H}(\text{KZ}(P))$ should be viewed as a “generalized $q$-Schur algebra” associated to $W$, cf Theorem 6.2.

**Corollary 4.7.** The category $\mathcal{O}$ is semi-simple if and only if $\mathcal{H}$ is semi-simple.

**Problem 2.** Provide an explicit construction of a progenerator.

- What is the image of a progenerator $P$ of $\mathcal{O}$? What is $\text{End}_\mathcal{H}(\text{KZ}(P))$? This is understood when $W$ has type $A_n$ and $c \notin \frac{1}{2} + \mathbb{Z}$, cf §6.2.1.

**Analogy 5.** Let $W$ be the Weyl group of $G$ and let $C = \mathbb{C}[h]/(\mathbb{C}[h]C[h]^+) \simeq H^*(G/B)$. Let $P = \mathcal{P}(v_0)$ be the “antidominant” projective of $\mathcal{O}$. There is an isomorphism $C \sim \text{End}_\mathcal{O}(P)$, the functor $\text{Hom}_\mathcal{O}(P, -) : \mathcal{O} \rightarrow C\text{-mod}$ is fully faithful when restricted to projectives, and the image of a suitable progenerator is

$$\bigoplus_{w \in W} \mathbb{C}[h] \otimes_{C[h]} \mathbb{C}[h] \otimes_{C[h]} \cdots \otimes_{C[h]} \mathbb{C}[h] \otimes C,$$

where $w = s_1 \cdots s_r$ is a reduced decomposition (Soergel).

**Remark 4.8.** When $s \mapsto c_s$ is constant (equal parameter case), the $\mathcal{H}$-modules $\text{KZ}(\Delta(E))$ are the “standard” modules occurring in Kazhdan-Lusztig theory [GGOR, Theorem 6.8] (cf §6.2.1 for type $A$).

**4.3. Primitive ideals and supports.** Cf [Gi, §6].
4.3.1. Given $M$ a finitely generated $H$-module, there is a structure of filtered $H$-module on $M$ such that $\text{gr}M$ is a finitely generated $\text{gr}H$-module (a “good filtration”). It is in particular a finitely generated $S(V \oplus V^*)$-module (cf §3.1.1). Let $\text{Supp}(M)$ be the support of that module, a $W$-stable closed subvariety of $V^* \times V$. It is independent of the choice of a good filtration of $M$.

If $M \in \mathcal{O}$, then $\text{Supp}(M) \subseteq \{0\} \times V$. When $c = 0$, Bernstein’s inequality asserts that $\dim \text{Supp}(M) \geq \dim V$. But there are values of $c$ and objects $M \in \mathcal{O}_c$ with $\dim \text{Supp}(M) < \dim V$, cf §7.2.

We have of course $\text{Supp}(\Delta(E)) = \{0\} \times V$, since the restriction of $\Delta(E)$ to $P$ is free.

4.3.2. Recall that an ideal of $H$ is primitive if it is the annihilator of a simple $H$-module.

**Theorem 4.9** ([Gi, Corollary 6.6]). Every primitive ideal of $H$ is the annihilator of a simple object of $\mathcal{O}$.

Let $I$ be an ideal of $H$. Give $I$ the filtration induced by the canonical filtration on $H$. Then, $\text{gr}I$ is an ideal of $\text{gr}H = C[V^* \times V] \rtimes W$, hence defines a $W$-invariant closed subvariety of $V^* \times V$, the associated variety of $I$.

A parabolic subgroup of $W$ is the pointwise stabilizer in $W$ of a subspace of $V$. We denote by $\text{Par}(W)$ the set of parabolic subgroups of $W$.

**Theorem 4.10** ([Gi, Proposition 6.4]). The associated variety of a primitive ideal of $H$ is of the form $W \cdot (V^* \times V)^{W'}$ for some $W' \in \text{Par}(W)$. In particular, its image in $(V^* \times V)/W$ is irreducible.

4.3.3.

**Theorem 4.11** ([Gi, Theorem 6.8]). Given $M$ simple in $\mathcal{O}$, there is $W' \in \text{Par}(W)$ such that $\text{Supp}(M) = \{0\} \times (W \cdot V^{W'})$.

**Problem 3.**

- Determine $\text{Supp} L(E)$. This generalizes the problem about finite dimensional $L(E)$’s (Problem 1), they correspond to the case $\text{Supp} L(E) = 0$.
- Study the order on $\text{Irr}(W)$ defined by $E \prec E'$ if $\text{Supp} L(E) \subseteq \text{Supp} L(E')$.

**Analogy 6.** The associated variety of a primitive ideal of $U(g)$ is the closure of a nilpotent class, hence it is irreducible (Borho-Brylinski, Joseph, Kashiwara-Tanisaki).

Every primitive ideal of $U(g)$ is the annihilator of some simple object of $\mathcal{O}'$ (Duflot).

The annihilator of $L(w)$ is contained in the annihilator of $L(w')$ if and only if $w$ is smaller than $w'$ for the “left cell order” (Joseph, Vogan).

4.4. Harish-Chandra bimodules.

4.4.1. The definitions and results of this section 4.4.1 follow [BerEtGi3, §3 and §8].

Let $c,c' \in \mathbb{C}^S$.

**Definition 4.12.** A $(H_c,H_{c'})$-bimodule is a Harish-Chandra bimodule if it is finitely generated and the action of $a \otimes 1 - 1 \otimes a$ is locally nilpotent for every $a \in S(V)^W \cup S(V^*)^W$.

Such a bimodule is finitely generated as a left $H_c$-module, as a right $H_{c'}$-module, as a $(S(V)^W, S(V^*)^W)$-bimodule and as a $(S(V^*)^W, S(V)^W)$-bimodule.

We denote by $\mathcal{HC}_{c,c'}$ the category of Harish-Chandra $(H_c,H_{c'})$-bimodules. The inclusion functor $\mathcal{HC}_{c,c'} \rightarrow (H_c \otimes H_{c'})^\text{opp}$-mod has a right adjoint $M \mapsto M_{\text{fin}}$.

Given $M \in \mathcal{HC}_{c,c'}$ and $N \in \mathcal{HC}_{c',c''}$, then $M \otimes_{H_{c'}} N \in \mathcal{HC}_{c,c''}$.

**Theorem 4.13.** Assume $c,c' \in \mathbb{Z}^S_{\geq 0}$. There is a parametrization $\{V_{c,c'}(E)\}_{E \in \text{Irr}(W)}$ of the set of isomorphism classes of simple objects of $\mathcal{HC}_{c,c'}$ such that given $E_1, E_2 \in \text{Irr}(W)$, then

$$\text{Hom}_{\mathcal{C}}(\Delta_{c'}(E_1) \otimes \Delta_{c'}(E_2))_{\text{fin}} \simeq \bigoplus_{E \in \text{Irr}(W)} \text{Hom}_{\mathcal{C}[W]}(E \otimes E_1, E_2) \otimes c V_{c,c'}(E).$$

Furthermore, $E \mapsto V_{c,c'}(E)$ extends to an equivalence of monoidal categories $\mathcal{C}[W] \rightarrow \mathcal{HC}_{c,c'}$. 
Problem 4. Describe the structure of the 2-category with set of objects $\mathcal{S}$, 1-arrows the objects of $\mathcal{HC}_{c,c'}$ and 2-arrows the morphisms of $\mathcal{HC}_{c,c'}$.

4.4.2.

Theorem 4.14 ([BerEtGi3, Theorem 3.1]). Assume $\mathcal{H}$ is semi-simple. Then, $H_c$ is a simple algebra and $He$ gives a Morita equivalence between $H_c$ and $B_c$.

About the proof. The key point is Theorem 4.9. It says in particular that $He$ gives a Morita equivalence if and only if $e$ kills no simple object of category $\mathcal{O}$.

Let $\varepsilon : W \to \{\pm 1\}$ be a one-dimensional representation of $W$. Let $e_{\varepsilon} = \frac{1}{|W|} \sum_{w \in W} \varepsilon(w)w$. Define $1_{\varepsilon} : S \to \mathbb{C}, s \mapsto \begin{cases} 1 & \text{if } \varepsilon(s) = -1 \\ 0 & \text{otherwise.} \end{cases}$

Proposition 4.15 ([BerEtGi3, Proposition 4.11]). Assume $\mathcal{H}$ is semi-simple. Then, the algebras $e_{\varepsilon}He_{\varepsilon}$ and $e_{\varepsilon}He_{\varepsilon}$ are isomorphic.

Theorem 4.16 ([BerEtGi3, Theorem 8.1]). Assume $\mathcal{H}$ is semi-simple. Let $m \in \mathbb{Z}^S$. Then, the algebras $H_c$ and $H_{c-m}$ are Morita equivalent.

Problem 5 ([BerEtGi3, Conjecture 8.12]). Assume $\mathcal{H}$ is semi-simple and let $c' \in \mathbb{C}^S$. If $H_c$ and $H_{c'}$ are Morita equivalent, show that there is $\zeta : W \to \{\pm 1\}$ a character such that $c\zeta - c' \in \mathbb{Z}^S$. Cf Theorem 6.8 for a partial answer in type $A$.

Analog 7. Two blocks of category $\mathcal{O}'$ associated to regular weights are equivalent via a translation functor.

5. Representation theory at $t = 0$

5.1. General representations.

5.1.1. Limit Dunkl operators. Following [EtGi, §4], the construction of §4.2.1 can be done for the algebra $H_{t,c}$, $t \neq 0$, and it then possible to pass to the limit $t = 0$. One obtains an injective algebra morphism $H_{0,c} \hookrightarrow \mathbb{C}[V^* \times V_{reg}] \rtimes W$, $x \mapsto x$, $\xi + \sum_{s \in S} c_s \frac{\langle \xi, \alpha_s \rangle}{\alpha_s} s$, $w \mapsto w$.

5.1.2. Since $H_{0,c}$ is a finitely generated module over its centre $Z(H_{0,c})$, it follows that all simple $H_{0,c}$-modules are finite dimensional. Furthermore, the category of finite dimensional $H_{0,c}$-modules decomposes into a sum of subcategories according to the central character (a point of $\mathcal{CM}_c$).

The smoothness of the Calogero-Moser space is related to representation theory of $H_{0,c}$:

Theorem 5.1 ([BrGo, Theorem 7.8], [EtGi, Theorems 1.7 and 3.7, and Proposition 3.8], [GoSm, Lemma 2.8]). Let $m \in \mathcal{CM}_c$. The following assertions are equivalent

- $m$ is a smooth point of $\mathcal{CM}_c$
- the Poisson bracket of $\mathcal{CM}_c$ is non-degenerate at $m$
- there is a unique simple $H_{0,c}$-module with central character $m$
- the simple $H_{0,c}$-modules with central character $m$ have dimension $\geq |W|$
- the simple $H_{0,c}$-modules with central character $m$ are isomorphic to the regular representation of $W$, as $\mathbb{C}[W]$-modules.

In particular, if $\mathcal{CM}_c$ is smooth, then its points parametrize the (isomorphism classes of) simple $H_{0,c}$-modules.
5.1.3. There is a stratification of $CM_c$ by symplectic leaves [BrGo, §3]. Given $I$ a Poisson prime ideal of $B_{0,c}$, the associated symplectic leaf is the set of points $m \in CM_c$ such that $I$ is a maximal Poisson ideal contained in $m$. There are only finitely many symplectic leaves [BrGo, Theorem 7.8].

The representation theory of $H_{0,c}$ doesn’t change inside a symplectic leaf:

**Theorem 5.2 ([BrGo, Theorem 4.2]).** Let $m, m' \in CM_c$ be two points in the same symplectic leaf. Then, $H_{0,c}/H_{0,c}m \simeq H_{0,c}/H_{0,c}m'$.

One has an irreducibility statement for associated varieties of Poisson ideals:

**Theorem 5.3 ([Ma, Corollary 3]).** The associated variety in $(V^* \times V)/W$ of a Poisson prime ideal of $B_{0,c}$ is irreducible.

5.2. 0-fiber.

5.2.1. Cf [Go1].

Let $I$ be the ideal of $H_{0,c}$ generated by $C[V]_+^W$ and $C[V^*]_+^W$ and let $\bar{H}_{0,c} = H_{0,c}/I$. The $\bar{H}_{0,c}$-modules are $H_{0,c}$-modules whose central character is in $\Upsilon^{-1}(0)$ and every simple $H_{0,c}$-module with such a central character is a $\bar{H}_{0,c}$-module. The blocks of $\bar{H}_{0,c}$ are given by the central character, i.e., are in bijection with $\Upsilon^{-1}(0)$:

$$\bar{H}_{0,c} = \bigoplus_{m \in \Upsilon^{-1}(0)} \bar{H}_{0,c}b_m,$$

where $b_m$ is the primitive central idempotent of $\bar{H}_{0,c}$ corresponding to $m$. Let $Z_m = \Gamma(\Upsilon^*(0)_m)$, where $\Upsilon^*(0)$ is the scheme theoretic fiber.

We have a vector space decomposition $\bar{H}_{0,c} = C \otimes C[W] \otimes C'$, where $C = C[V^*]/(C[V^*]C[V^*]^W)$ and $C' = C[V]/(C[V]C[V]^W)$ are the coinvariant algebras.

5.2.2. The Baby-Verma module associated to $E \in \text{Irr}(W)$ is $M(E) = \text{Ind}_{C^W \times W}^{C_{\Upsilon} \times W} E$. It has a unique simple quotient $L(E)$ and $\{L(E)\}_{E \in \text{Irr}(W)}$ is a complete set of representatives of isomorphism classes of simple $\bar{H}_{0,c}$-modules. Define a map $\text{Irr}(W) \to \Upsilon^{-1}(0)$, $E \mapsto m_E$, by the property that $L(E)$ is in the block $\bar{H}_{0,c}b_{m_E}$.

The blocks corresponding to smooth points can be described precisely:

**Theorem 5.4 ([Go1, Corollary 5.8]).** Let $m \in \Upsilon^{-1}(0)$ be a smooth point of $CM_c$. Then, $\bar{H}_{0,c}b_m \simeq \text{Mat}_{|W|}(Z_m)$.

If all points in $\Upsilon^{-1}(0)$ are smooth, then the canonical map $\text{Irr}(W) \to \Upsilon^{-1}(0)$ is bijective and $\text{dim } Z_{m_E} = (\text{dim } E)^2$.

**Problem 6.**

- Is $\bar{H}_{0,c}$ a symmetric algebra ?
- Find the (graded) multiplicities $[M(E) : L(F)]$. In particular, what is the block distribution of the $M(E)$’s ?

**Remark 5.5.** It is known ([EtGi, Proposition 16.4], [Go1, Proposition 7.3]) that $CM_c$ is singular for all values of $c$, when $W$ has type different from $A_n$, $B_n$ and $D_{2n+1}$. It is conjectured that it will always be singular in type $D_{2n+1}$ ($n \geq 2$).

Let us give an example where $E \mapsto m_E$ is not bijective [Go1, §7.4]. Let $W = G_2$ and fix a generic value of $c$. Given $E \neq F \in \text{Irr}(W)$, then $m_E = m_F$ if and only if $E$ and $F$ are the two distinct 2-dimensional representations.

6. Type $A$

6.1. Structure.
6.1.1. In this section, let $W = S_n$ be the symmetric group on $\{1, \ldots, n\}$ in its permutation representation on $V = \mathbb{C}^n$. We consider the canonical bases $(\xi_1, \ldots, \xi_n)$ of $V$ and $(x_1, \ldots, x_n)$ of $V^*$. Then, $H$ is the $\mathbb{C}[t, c]$-algebra with generators $S_n$, $x_1, \ldots, x_n$ and $\xi_1, \ldots, \xi_n$ and relations

$$[\xi_i, \xi_j] = [x_i, x_j] = 0 \quad \text{for all} \quad i, j \quad \text{and}$$

$$wxw^{-1} = x_{w(i)}, \quad w\xi w^{-1} = \xi_{w(i)}, \quad [\xi_i, x_j] = c \cdot (ij), \quad \text{if} \quad i \neq j \quad \text{and} \quad \xi_i, x_i = t - c \sum_{k \neq (ik)}.$$

One can also consider the action of $W$ on the hyperplane $V' = \ker(x_1 + \cdots + x_n)$. The rational Cherednik algebra of $(W, V')$ is canonically isomorphic to the subalgebra $H'$ of $H$ generated by $\xi_i - \xi_j$, $x_i - x_j$ and $W$, for $1 \leq i, j \leq n$ and we have a decomposition $H = H' \otimes H^1$, where $H^1$ is generated by $\xi_1 + \cdots + \xi_n$ and $x_1 + \cdots + x_n$, and is isomorphic to the first Weyl algebra.

The algebra $H'$ is interesting since it carries finite dimensional non-zero representations for certain values of $c$, while $H_c$ is the one that relates most directly to Hilbert schemes of points on the plane. Note that the categories $\mathcal{O}$ for $H_c$ and $H'_c$ are canonically equivalent.

6.1.2. The variety $CM_1$ is isomorphic to the “usual” Calogero-Moser space (a smooth symplectic variety)

$$\{(M, M') \in \text{Mat}_n(\mathbb{C}) \times \text{Mat}_n(\mathbb{C}) \mid \text{rank}([M, M'] + \text{Id}) = 1\} / \text{GL}_n(\mathbb{C}),$$

where $\text{GL}_n(\mathbb{C})$ acts diagonally by conjugation [EtGi, Theorem 11.16].

At the level of points, this isomorphism is constructed as follows [EtGi, Theorem 11.16]: let $L$ be a simple representation of $H_{0,1}$. Fix a basis of the $n$-dimensional space $L^{\otimes n-1}$. The actions of $\xi_n$ and $x_n$ on that space give matrices $M$ and $M'$ such that $\text{rank}([M, M'] + \text{Id}) = 1$.

The morphism $\Upsilon$ sends $(M, M')$ to the pair of roots of the characteristic polynomials of $M$ and $M'$.

Remark 6.1 (Etingof). Let $w \in W - \{1\}$. Then, there is $i \in \{1, \ldots, n\}$ such that $w(i) = j \neq i$. We have $[\xi_i, x_j(ij)] = [\xi_j, x_i(ij)] = w = w$, hence $w \in [H_{0,1}, H_{0,1}]$. It follows that the restriction to $W$ of a representation of $H_{0,1}$ is a multiple of the regular representation. This proves the smoothness of $CM_1$, via Theorem 5.1.

6.2. Category $\mathcal{O}$.

6.2.1. Let $q = \exp(2\pi i c)$. The Hecke algebra $H$ of $S_n$ with parameters $(1, -q)$ is the $\mathbb{C}$-algebra with generators $T_1, \ldots, T_{n-1}$ and relations

$$T_i T_j = T_j T_i \quad \text{if} \quad |i - j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{and} \quad (T_i - 1)(T_i + q) = 0.$$

We use the standard parametrization of $\text{Irr}(W)$ by partitions of $n$.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ be a partition of $n$. Let $H(\lambda)$ be the subalgebra of $H$ generated by $T_1, \ldots, T_{\lambda_1-1}, T_{\lambda_1+1}, \ldots, T_{\lambda_r+\lambda_{r-1}}$. It is isomorphic to the tensor product of the Hecke algebras of $S_{\lambda_1}, \ldots, S_{\lambda_r}$.

Given $\lambda$ a partition of $n$, let $d(\lambda)$ be the number of $r$-uples $(\beta_1, \ldots, \beta_r)$ whose associated multiset is that of $\lambda$.

Let $M(\lambda) = \text{Ind}^H_{H(\lambda)} C$, where $C$ is the one-dimensional representations of $H$ where the $T_i$’s act as 1 and let $M = \bigoplus \lambda M(\lambda)^{d(\lambda)}$.

The $q$-Schur algebra of $S_n$ is $S(n) = \text{End}_H(M)$. Note that $S(n)$-mod is a highest category with parametrizing set the set of partitions of $n$.

The $q$-Schur algebra occurs also as a quotient of the quantum general linear group $U_q(\mathfrak{gl}_n)$ for $m \geq n$ (via its action on quantum tensor space $(\mathbb{C}^m)^{\otimes n}$) and when $q$ is a prime power, as a quotient of the group algebra of the finite group $GL_n(\mathbb{F}_q)$.

The category $\mathcal{O}$ is described as follows, as conjectured in [GGOR, Remark 5.17]:

**Theorem 6.2 ([Rou]).** Assume $c \not\in \frac{1}{2} + \mathbb{Z}$. Then, there is an equivalence $\mathcal{O} \sim \text{S(n)-mod}$ making the following diagram commutative

$$\begin{array}{ccc}
\mathcal{O} & \sim & S(n)-\text{mod} \\
\downarrow \text{KZ} & & \\
\mathcal{H}-\text{mod} & \downarrow M \otimes S(n)-\text{mod}
\end{array}$$
and sending $\Delta(\lambda)$ to the standard object of $S(n)$-mod associated to $\lambda$ if $c \leq 0$ and to the transposed partition of $\lambda$ if $c > 0$.

**About the proof.** The proof proceeds by deformation: the parameter ring becomes a discrete valuation ring and at the generic point the categories are semi-simple. Then, one shows that the image of the $\Delta$-filtered objects under the Schur and KZ-functors is a full subcategory closed under extensions.

In particular, there is a progenerator $P$ of $O$ such that $KZ(P) = M$. Furthermore, the modules $KZ(\Delta(\lambda))$ are the $q$-Specht modules.

Assume $c \in \mathbb{Q}$ and let $d$ be the order of $c$ in $\mathbb{Q}/\mathbb{Z}$.

Let $\text{Sym}$ be the space of symmetric functions. Given $\lambda$ a partition of $n$, let $s_\lambda$ be the corresponding Schur function.

The Fock space $\text{Sym}$ has a natural action of the affine Lie algebra $\hat{\mathfrak{sl}}_d$. There is a lower canonical basis $\{G^\lambda\}_\lambda$ a partition of $\text{Sym}$ [LeThi]. By [VarVas2], the multiplicity $[\Delta_{S(n)}(\lambda) : L_{S(n)}(\mu)]$ is the coefficient of $G^\mu$ in a decomposition of $s_\lambda$ in the lower canonical basis (a generalisation of the Lascoux-Leclerc-Thibon conjecture on Hecke algebras, proven by Ariki). So, we deduce the corresponding result for category $O$:

**Corollary 6.3.** Assume $c \not\in \frac{1}{2} + \mathbb{Z}$. Then, $[\Delta(\lambda) : L_\mu]$ is the coefficient of $G^\mu$ in a decomposition of $s_\lambda$ in the lower canonical basis.

**Remark 6.4.** There is a counterpart of Theorem 6.2 in the trigonometric case [VarVas2], which builds on an explicit computation of monodromy (this can’t be done in the rational case). It might be possible to deduce Theorem 6.2 from the trigonometric case by using [Su].

**6.2.2. Finite dimensional representations.** They are completely understood ([BerEtGi2, Theorem 1.2], cf also [Che6, §7.1]):

**Theorem 6.5.** The algebra $H_c$ has non-zero finite dimensional representations if and only if $c = \pm \frac{r}{n}$ for some $r \in \mathbb{Z}_{>0}$ with $(r, n) = 1$. When $c$ takes such a value, all finite dimensional representations are semi-simple and the only irreducible representation is $L(C)$ when $c > 0$ and $L(\det)$ when $c < 0$.

**6.3. Shift functors.**

**6.3.1.** We have $\rho(\delta^{-1}e_{\det}H_{c+1}e_{\det}\delta) = \rho(e_{H_c}c)$ [BerEtGi2, Proposition 4.1]. So, left and right multiplication make $Q^{c+1}_c = \rho(e_{H_{c+1}}e_{\det})$ into a $(B_{c+1}, B_c)$-bimodule. Let $S_c = Q^{c+1}_c \otimes_{B_c} - : B_c\text{-mod} \to B_{c+1}\text{-mod}$ ("Heckmann-Opdam shift functor").

The following result generalizes [BerEtGi2, Proposition 4.3].

**Theorem 6.6 ([GoSt1, Theorem 3.3 and Proposition 3.16]).** If $c \in \mathbb{R}_{>0}$ and $c \not\in \frac{1}{2} + \mathbb{Z}$, then

- $e_{H_c} \otimes_{H_c} - : H_c\text{-mod} \to B_c\text{-mod}$ is an equivalence
- $S_c : B_c\text{-mod} \to B_{c+1}\text{-mod}$ is an equivalence
- $M \mapsto H_{c+1}e_{\det}\delta \otimes_{B_c} eM : H_c\text{-mod} \to H_{c+1}\text{-mod}$ is an equivalence. It restricts to an equivalence $O_c \xrightarrow{\sim} O_{c+1}$ sending $\Delta_c(E)$ to $\Delta_{c+1}(E)$.

**About the proof.** The key point is to show that $H_{c+1}e_{\det}H_{c+1} = H_{c+1}$ and $H_c e_{H_c} = H_c$ for $c \geq 0$. Let us consider the first equality, the second one has a similar proof. If the equality fails, then $e_{\det}$ will kill some simple object of $O$ by Theorem 4.9. One uses the canonical $C$-grading on $\Delta(\lambda)$, $\lambda$ a partition. One shows that the lowest weight where the representation det of $W$ appears in $\Delta(\lambda)$ is strictly larger than the lowest weight where it appears in $\Delta(\mu)$ whenever $\lambda < \mu$ (this is where the assumption $c \geq 0$ enters). As a consequence, det occurs in $L(\mu)$, for any $\mu$, hence $e_{\det}L(\mu) \neq 0$ and we deduce the first equality. Note that the assumption $c \not\in \frac{1}{2} + \mathbb{Z}$ comes from the use of Theorem 4.6. To check that $\Delta_c(E)$ goes to $\Delta_{c+1}(E)$, one proves this after localizing to $V_{reg}$ and then show that the equivalence $O_c \xrightarrow{\sim} O_{c+1}$ must preserve the highest weight structure.
**Remark 6.7.** Note that, by Theorem 4.14 and §3.1.4, we deduce that $H_c$ and $B_c$ are Morita equivalent for every $c \in \mathbb{C}$ satisfying the conditions
\[ c \not\in \frac{1}{2} + \mathbb{Z} \quad \text{and} \quad c \not\in \left\{ \frac{m}{d} \mid m, d \in \mathbb{Z}, \ 2 \leq d \leq n, \ 0 < m < d \right\}. \]

### 6.3.2. Morita Equivalence Classifications

There is a Morita equivalence classification of the algebras $H_c$, when $c$ is not algebraic.

**Theorem 6.8 ([BerEtGi1, Theorem 2]).** Let $c \not\in \tilde{Q}$ and $c' \in \mathbb{C}$. The algebras $H_c$ and $H_{c'}$ are
- isomorphic if and only if $c' = \pm c$
- Morita equivalent if and only if $c \pm c' \in \mathbb{Z}$.

**About the Proof.** The criterion is obtained by computing the traces $K_0(H_c) \to HH_0(H_c)$.

### 6.4. Hilbert Schemes

The existence of a link between Hilbert schemes of points on $\mathbb{C}^2$ and rational Cherednik algebras of type $A$ was pointed out in [EtGi] and [BerEtGi2, §7.2]. We describe here some of the constructions and results of [GoSt1, GoSt2].

6.4.1. Quantumization. Cf [GoSt1, §4–6].

Let $\text{Hilb}^n \mathbb{C}^2$ be the Hilbert scheme of $n$ points in $\mathbb{C}^2$. Let $X_n$ be the reduced scheme of $\text{Hilb}^n \mathbb{C}^2 \times S^* \mathbb{C}^2 \mathbb{C}^{2n}$ (the isospectral Hilbert scheme). Following Haiman, we have a diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{f} & \mathbb{C}^{2n} = V^* \times V \\
\downarrow \text{flat} & & \downarrow \\
\text{Hilb}^n \mathbb{C}^2 & \xrightarrow{\tau} & S^n \mathbb{C}^2 = (V^* \times V)/W
\end{array}
\]

Denote by $Z_n = \tau^{-1}(0)$ the punctual Hilbert scheme.

**Fix** $c \in \mathbb{R}_{>0}$, $c \not\in \frac{1}{2} + \mathbb{Z}$.

**Given** $i > j \in \mathbb{Z}_{\geq 0}$, let $B^{ij} = B_{c+j}$ and $B^{ij} = Q_{c+i-j}^{c+i-j} \otimes B_{c+i-j-1}^{c+i-j-1} \otimes \cdots \otimes B_{c+j}^{c+j}$. Let $B = \bigoplus_{i,j \geq 0} B^{ij}$.

This is a non-unital algebra. We denote by $1_i$ the unit of $B^{ij}$. We denote by $B$-mod the category of finitely generated $B$-modules $M$ such that $M = \bigoplus_{i,j \geq 0} 1_i M$. We denote by $B$-qmod the abelian category quotient of $B$-mod by the Serre subcategory of objects $M$ such that $1_i M = 0$ for $i \gg 0$.

The filtration by the order of differential operators on $D(V_{reg}) \times W$ induces a filtration on $B$ and we denote by $\text{ogr} B = \bigoplus_{i,j \geq 0} \text{ogr} B^{ij}$ the associated graded (non-unital) algebra. We define a category (ogr $B$)-qmod as above.

**Theorem 6.9 ([GoSt1, Theorem 6.4]).** There are equivalences

$\text{ogr} B$-mod $\xrightarrow{\sim} B$-qmod

**Hilb}^n \mathbb{C}^2$-coh $\xrightarrow{\sim}$ (ogr $B$)-qmod.

**About the Proof.** The first equivalence is an immediate consequence of the fact that the $B^{ij}$’s induce Morita equivalences (Theorem 6.6).

Consider $A^1 = \mathbb{C}[\mathbb{C}^{2n}]^{\text{det}}$, the det-isotypic part for the action of $W$ on $\mathbb{C}[\mathbb{C}^{2n}]$ and let $A'_{c^1}$ be the ideal of $\mathbb{C}[\mathbb{C}^{2n}]$ generated by $A^1$. Let $A^d = (A^1)^d$ and $A'^d = (A'^1)^d$ (inside $\mathbb{C}[\mathbb{C}^{2n}]$) for $d \geq 1$ and let $A^0 = \mathbb{C}[\mathbb{C}^{2n}]^{S_n}$ and $A'^0 = \mathbb{C}[\mathbb{C}^{2n}]$. Let $A = \bigoplus_{d \geq 0} A^d$ and $A' = \bigoplus_{d \geq 0} A'^d$. 

Then, there are isomorphisms \( X_n \xrightarrow{\sim} \text{Proj} A' \) and \( \text{Hilb}^n \mathbb{C}^2 \xrightarrow{\sim} \text{Proj} A \) so that the diagram (2) above becomes the following diagram, with canonical maps (Haiman)

\[
\begin{array}{ccc}
\text{Proj} \bigoplus_{d \geq 0} A'^d & \longrightarrow & \text{Spec} A^0 \\
\downarrow & & \downarrow \\
\text{Proj} \bigoplus_{d \geq 0} A_d & \longrightarrow & \text{Spec} A_0
\end{array}
\]

Now, the Theorem follows from the equalities \( \text{ogr} B^{ij} = e \cdot A^{i-j} \delta^{i-j} e \) between subspaces of \( C[C^{2n}]^{\otimes n} \), whose delicate proof involves understanding the graded structure of these two subspaces. \( \square \)

6.4.2. Localization. Cf [GoSt2].

Consider the order filtration on \( H_e : \text{ord}^0 H_e = C[V] \times W, \text{ord}^1 H_e = V \cdot \text{ord}^0 H_e + \text{ord}^0 H_e \) and \( \text{ord}^i H_e = (\text{ord}^1 H_e)^i \) for \( i \geq 2 \). This induces a filtration on \( B_e \).

Theorem 6.9 gives a functor \( \Phi_c \) from the category \( B_c \)-filt of \( B_c \)-modules with a good filtration (for the order filtration) to the category \( \text{Hilb}^n \mathbb{C}^2 \)-coh of coherent sheaves over \( \text{Hilb}^n \mathbb{C}^2 \).

We have

\[ \Phi_c (B_c) \simeq \mathcal{O}_{\text{Hilb}^n \mathbb{C}^2} \]

The image of \( eH_e \) with the order filtration is the Procesi bundle [GoSt2, Theorem 4.5]:

\[ \Phi_c (eH_e) \simeq p_* \mathcal{O}_{X_n}. \]

We have [GoSt2, Proposition 5.4] (this relates to [BerEtGi2, Conjectures 7.2 and 7.3]):

\[ \Phi_{1/n} (L_{1/n}(C)) \xrightarrow{\sim} \mathcal{O}_{Z_n}. \]

Given \( M \in B_c \)-filt, there is an induced tensor product filtration on \( S_c(M) = Q_{c+1} \otimes_{B_c} M \). Let \( L \) be the determinant of the universal rank \( n \) vector bundle on \( \text{Hilb}^n \mathbb{C}^2 \) (an ample line bundle). The geometric importance of \( S_c \) is provided by the next result, which explains the constructions of Gordon-Stafford.

**Theorem 6.10 ([GoSt2, Lemma 4.4]).** There is an isomorphism of functors \( B_c \)-filt \( \rightarrow \) \( \text{Hilb}^n \mathbb{C}^2 \)-coh:

\[ \Phi_{c+1} \circ S_c (-) \xrightarrow{\sim} L \otimes \Phi_c (-). \]

**Problem 7 ([GoSt2, Question 1]).** Let \( M \) be an \( H_c \)-module with a good filtration. Then, \( \text{gr} M \) is a finitely generated \( (C[V^* \times V] \times W) \)-module. Let \( \tilde{\Phi}(M) \in D^b(\text{Hilb}^n \mathbb{C}^2 \text{-coh}) \) be its image under the equivalence of derived categories (Bridgeland-King-Reid, Haiman):

\[ (p_* Lf^*(-))W : D^b((C[V^* \times V] \times W) \text{-mod}) \xrightarrow{\sim} D^b(\text{Hilb}^n \mathbb{C}^2 \text{-coh}). \]

Is there an isomorphism \( \tilde{\Phi}(M) \xrightarrow{\sim} \Phi(eM) \)? Gordon and Stafford construct a morphism which is surjective on \( H^0 \). A related question is to understand which \( (C[V^* \times V] \times W) \)-modules can be quantized to \( H_c \)-modules for some value of \( c \in \mathbb{R}_{\geq 0} \).

**Remark 6.11.** Gordon and Stafford actually work with \( H'_c \) and they consider the \( W \)-Hilbert scheme \( \text{Hilb}(n) \) of \( (V')^* \times V' \). There is an isomorphism \( \text{Hilb}^n \mathbb{C}^2 \xrightarrow{\sim} \text{Hilb}(n) \times \mathbb{C}^2 \), so the geometric properties of \( \text{Hilb}(n) \) and \( \text{Hilb}^n \mathbb{C}^2 \) are easily related [GoSt1, §4.9].
6.4.3. Characteristic cycles. CF [GoSt2, §6].
Let $Z = Z(n) = \tau^{-1}(\{0\} \times V/W)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ be a partition of $n$ and $S^\lambda \mathbb{C}^2$ be the subvariety of $S^\lambda \mathbb{C}^2$ of 0-cycles of $\mathbb{C}^2$ of the form $\sum_i \lambda_i x_i$, where $x_1, \ldots, x_r \in \mathbb{C}^2$ are distinct.

Let $Z_\lambda$ be the closure of $Z \cap \tau^{-1}(S^\lambda \mathbb{C}^2)$. This is a Lagrangian subvariety of $\text{Hilb}^n \mathbb{C}^2$ and the $Z_\lambda$’s, where $\lambda$ runs over the partitions of $n$, are the irreducible components of $Z$ (Grojnowski, Nakajima).

Given $\lambda$ a partition of $n$, let $m_\lambda \in \text{Sym}$ be the corresponding monomial symmetric function. There is an isomorphism (Grojnowski, Nakajima)

$$\xi : \bigoplus_{n \geq 0} H_n(Z(n)) \xrightarrow{\sim} \text{Sym}, \quad [Z_\lambda] \mapsto m_\lambda$$

where $H_n$ is the Borel-Moore homology with complex coefficients.

Recall that the cycle support of a coherent sheaf $F$ is the cycle $\sum_i n_i [Z_i]$, where $Z_i$ runs over irreducible components of the support of $F$ and $n_i$ is the dimension of $F$ at the generic point of $Z_i$.

Let $M \in \mathcal{O}_x$. Fix a good filtration of $eM$ and consider the part of the cycle support of $\Phi(eM)$ involving only subvarieties of dimension $n$. This is independent of the choice of the good filtration and this gives an isomorphism [GoSt2, Corollary 6.10]

$$\gamma : K(\mathcal{O}_x) \otimes \mathbb{C} \xrightarrow{\sim} H_n(Z)$$

The following Theorem describes the characteristic cycle of $\Delta(\mu)$.

**Theorem 6.12** ([GoSt2, Theorem 6.7]). Let $\mu$ be a partition of $n$. The support of $\Phi(e\Delta(\mu))$ is a union of $Z_\lambda$’s. We have

$$\xi \mu([\Delta(\mu)]) = s_\mu.$$

**Problem 8** ([GoSt2, Question 4.9 and §6.8]). From Corollary 6.3, one deduces that $\xi \mu([L(\mu)])$ is the lower canonical basis element of the Fock space corresponding to $\mu$. Are the irreducible components of the support of $\Phi(eL(\mu))$ all of dimension $n$? If so, this would completely describe the characteristic cycle of $L(\mu)$.

**Problem 9** ([GoSt2, Problem 7.7]). Gordon and Stafford show [GoSt2, Lemma 7.7] that the top Borel-Moore homology of $\text{Hilb}^n \mathbb{C}^2 \times_{S^\lambda \mathbb{C}^2} \text{Hilb}^n \mathbb{C}^2$ with the convolution product is isomorphic to the representation ring of $W$. The determination of the $(\mathbb{C}^\times)^2$-equivariant $K$-theory ring under convolution remains to be done.

**Remark 6.13.** There is a different geometric approach started in [GanGi], which also leads to the construction of characteristic cycles on the Hilbert scheme.

6.4.4. Let us summarize

$$\begin{array}{ccc}
B & \xrightarrow{\sim} & B_c \\
\downarrow\text{quantization} & & \downarrow\text{quantization} \\
\text{Hilb}^n \mathbb{C}^2 & \xrightarrow{\text{resolution}} & (V^* \times V)/W
\end{array}$$

The algebra $H_c$ is a “quantization” of the orbifold $[(V^* \times V)/W]$. Furthermore, $X_n$, viewed as a scheme over $\text{Hilb}^n \mathbb{C}^2$ and over $[(V^* \times V)/W]$, becomes after “quantization” the $(B_c, H_c)$-bimodule $eH_c$.

The transform with kernel $\mathcal{O}_{X_n}$ gives an equivalence of triangulated categories (“McKay correspondence”) $D^b(\text{Hilb}^n \mathbb{C}^2\text{-coh}) \xrightarrow{\sim} D^b([(V^* \times V)/W]\text{-coh})$. This is simpler in the non-commutative case, where $eH_c$ gives an equivalence of abelian categories $B_c\text{-mod} \xrightarrow{\sim} U_c\text{-mod}$ (for suitable $c$’s).

**Analogy 8.** Let $B = G/B$ be the flag variety of $G$. Given $w \in W$, we put $B_w = BwB/B$. We have the Springer resolution of singularities $T^*B \to N$. We have a canonical isomorphism $\hat{U}_0(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{B}, \mathcal{D}_B)$ and we
have $\Gamma(\mathcal{B}, \mathcal{D}_\mathcal{B})\text{-mod} \simeq \mathcal{D}_\mathcal{B}\text{-coh}$ (Beilinson-Bernstein). We have $\text{gr}\mathcal{D}_\mathcal{B} \xrightarrow{\sim} \mathcal{O}_{T^*\mathcal{B}}$.

Given an object $M$ of $\mathcal{O}'$, we can consider the characteristic cycle of the corresponding $\mathcal{D}_\mathcal{B}$-module, an element of $\bigoplus_{w \in W} \mathbb{Z}[T_{B_w}^\mathcal{B}]$.

There is a canonical isomorphism of algebras between the top homology of the Steinberg variety $Z = T^*\mathcal{B} \times_{\mathcal{X}} T^*\mathcal{B}$ and $\mathcal{C}W$ (Kazhdan-Lusztig). The images of the components of $Z$ give a basis $(b_w)_{w \in W}$ of $\mathcal{C}W$. Define $\langle \beta_{w',w} \rangle_{w,w'}$ by $w = \sum_w \beta_{w,w'} b_w'$. Then, the characteristic cycle of $\Delta(w)$ is $\sum_{w'} \beta_{w,w'} [T_{B_w'}^\mathcal{B}]$ (Kashiwara-Tanisaki).

The $(G \times \mathbb{C}^*)$-equivariant $K$-theory of $Z$ is isomorphic to the affine Hecke algebra of type $W$ (Kazhdan-Lusztig, Chriss-Ginzburg).

**Problem 10.** Can one deform the category of $W$-equivariant mixed Hodge modules on $V$?

**7. Type $A_1$**

**7.1. Presentation.** Take $W = A_1 = \langle s \rangle$ acting on $V = \mathbb{C}\xi$ and put $V^* = \mathbb{C}x$ where $\langle \xi, x \rangle = 1$. Then, $H$ is the $\mathbb{C}[t, c]$-algebra with generators $s, x, \xi$ and relations

$$s^2 = 1, \; sxs = -s, \; s\xi s = -\xi, \; [\xi, x] = t - 2cs.$$  

We have $c = \frac{1}{2}(1 + s)$.

Fix $t, c \in \mathbb{C}$. Recall that $H_{t,c} \simeq H_{1,t^{-1}c}$ if $t \neq 0$ and $H_{0,c} \simeq H_{0,1}$ if $c \neq 0$. So, there are three types of algebras in the family : $H_{1,c}, H_{0,1}$ and $H_{0,0}$.

**7.2. Category $\mathcal{O}$ and KZ.**

**7.2.1.** We take $t = 1$.

We identify $\Delta(\mathbb{C})$ with $\mathbb{C}[x]$. The action of the generators is given by

$$x : x^i \mapsto x^{i+1}$$
$$s : x^i \mapsto (-1)^i x^i$$
$$\xi : x^i \mapsto \begin{cases} ix^{i-1} & \text{if } i \text{ is even} \\ (i - 2c)x^{i-1} & \text{if } i \text{ is odd} \end{cases}$$

In particular, $\Delta(\mathbb{C}) = L(\mathbb{C})$ if and only if $c \notin \frac{1}{2} + \mathbb{Z}_{\geq 0}$. If $c = \frac{1}{2} + n$ with $n \geq 0$, then $x^{2n+1}\mathbb{C}[x]$ is the radical of $\Delta(\mathbb{C})$ and $L(\mathbb{C}) = \mathbb{C}[x]/(x^{2n+1})$.

The Dunkl operator is $T_\xi = \frac{\partial}{\partial x} + \xi (s - 1)$. The connection on the trivial line bundle over $V_{\text{reg}}$ is $\frac{d}{dx}$. The solutions are constant functions, the monodromy operator is trivial.

**7.2.2.** We identify $\Delta(\text{det})$ with $\mathbb{C}[x]$. The action of the generators is given by

$$x : x^i \mapsto x^{i+1}$$
$$s : x^i \mapsto (-1)^{i+1} x^i$$
$$\xi : x^i \mapsto \begin{cases} ix^{i-1} & \text{if } i \text{ is even} \\ (i + 2c)x^{i-1} & \text{if } i \text{ is odd} \end{cases}$$

In particular, $\Delta(\text{det}) = L(\text{det})$ if and only if $c \notin -(\frac{1}{2} + \mathbb{Z}_{\geq 0})$. If $c = -(\frac{1}{2} + n)$ with $n \geq 0$, then $x^{2n+1}\mathbb{C}[x]$ is the radical of $\Delta(\text{det})$ and $L(\text{det}) = \mathbb{C}[x]/(x^{2n+1})$. 
The connection on the trivial line bundle over $V$ is $\frac{dx}{x} + \frac{2c}{x}$. In a neighborhood of $x = 1$, we have the solution $f = x^{-2c}$ with $f(1) = 1$. Analytical continuation in a tubular neighborhood of the path $t \in [0, 1] \mapsto \exp(it\pi t)$ gives a function with value $\exp(-2it\pi c)$ at $-1$. The action of the monodromy operator is $-\exp(-2it\pi c)$. The KZ-construction involves the de Rham functor while here we considered the solution functors (=horizontal rational dA).

It is generated as a $\mathfrak{sl}_2$-algebra generated by $h$, $e$, $f$ with the relations
\[ [h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \]

7.2.3. When $c \notin \frac{1}{2} + \mathbb{Z}$, then $O$ is semi-simple.

When $c \in (-\left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)$, then we have an exact sequence $0 \to \Delta(C) \to \Delta(\det) \to L(\det) \to 0$. We have $P(\det) = \Delta(\det)$ and $P(C) = H \otimes \mathbb{C}[\xi] \otimes W S(\xi)/(\xi)^2$.

When $c \in \left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)$, then we have an exact sequence $0 \to \Delta(\det) \to \Delta(C) \to L(C) \to 0$. We have $P(C) = \Delta(C)$ and $P(\det) = H \otimes \mathbb{C}[\xi] \otimes W (S(\xi)/(\xi)^2 \otimes \det)$.

So, when $c \in \frac{1}{2} + \mathbb{Z}$, then $O$ is equivalent to the principal block of category $O$ for $\mathfrak{sl}_2(\mathbb{C})$ (this is no miracle, cf §7.3.3).

7.2.4. We have $ex + xe = 2x$, $\epsilon \xi + \xi e = 2\xi$, so $\bigoplus_{i,j} Cx^i\xi^j \subset \mathcal{H}(\mathcal{E}).$ Since $[\xi, x] = t - 2cs$, we have $t - 2cs \in H \mathcal{E}$. So, $2c + t \in H \mathcal{E}$. It follows that $H \mathcal{E} = H$ if $2c + t \neq 0$.

Assume now $2c = -t$. We put a structure of $H$-module on $L = \mathbb{C}$ by letting $x$ and $\xi$ act by $0$ and $s$ by $-1$. Then, $e$ acts by $0$, so $H \mathcal{E}$ annihilates $L$, hence $H \mathcal{E} \neq H$. So, we have $H \mathcal{E} = H$ if and only if $2c \neq -t$. Note that when $c = \frac{-1}{2}$ and $t = 1$, via the identification $\Delta(\det) = \mathcal{C}[x]$, then $L \simeq L(\det) = \mathcal{C}[x]/x\mathcal{C}[x]$. Note also that the algebra $\mathcal{B}_{-\frac{1}{2}}$ is simple.

Finally, the following assertions are equivalent:

- $H_{t,c} \cdot c H_{t,c} = H_{t,c}$
- $H_{t,c}$ and $B_{t,c}$ are Morita equivalent
- $c \neq -\frac{1}{2} t$.

7.3. Spherical subalgebra.

7.3.1. One has $ex^i\xi^j = \begin{cases} ex^i\xi^j, & \text{if } i + j \text{ is even} \\ 0, & \text{otherwise} \end{cases}$ and $ex^i\xi^j e = \pm ex^i s\xi^j e$. So, $B$ has a basis $(ex^i\xi^j)_{i,j \geq 0, i+j \text{ even}}$.

It is generated as a $\mathbb{C}[t, c]$-algebra by $u = \frac{1}{2} x^2 e$, $v = -\frac{1}{2} \epsilon x^2 e$ and $w = ex\xi e$. We obtain an isomorphism between $B$ and the $\mathbb{C}[t, c]$-algebra generated by $u$, $v$ and $w$ with the relations $w(w - t - 2c) = -4uv$ and $[u, v] = tw + t(\frac{1}{2}t - c)$.

7.3.2. We have $\mathcal{C}[u, v, w]/(w(w - 2c) + 4uv) \simeq B_{0,c}$. The variety $\text{Spec} B_{0,c}$ is smooth if and only if $c \neq 0$.

7.3.3. For $t = 1$, we have $[u, v] = w + \frac{1}{2} - c$, $[v, w] = 2v$ and $[u, w] = -2u$. Let $e_+$, $e_-$ and $h$ be the standard generators of $\mathfrak{sl}_2(\mathbb{C})$ and let $C = e_+ e_- + e_- e_+ + \frac{1}{2} h^2$ be the Casimir element. We have an isomorphism $\mathcal{E}$, Proposition 8.2]

\[ U(\mathfrak{sl}_2)/(C - \frac{1}{2}(c - 1)(c + \frac{3}{2})) \simeq B_{1,c}, \quad e_+ \mapsto u, \quad e_- \mapsto v. \]

We have $B_{1,c}$ Morita equivalent to $B_{1,c+1}$ if and only if $c \neq -\frac{3}{2}, -\frac{1}{2}$.

7.3.4. The representation theory of $U(\mathfrak{sl}_2)$ is related to $T^* \mathcal{P}_1$, since the flag variety of $\mathfrak{sl}_2$ is a projective line. Note that $\text{Hilb}^W(V^* \times X) \simeq T^* \mathcal{P}_1$, the minimal resolution of singularities of the cone $\text{Spec} B_{0,0}$.

7.4. Double affine Hecke algebra. We finish with some words on the dA and its relation with the rational dA.

7.4.1. The double affine Hecke algebra $H^{dA}$ is the $\mathbb{C}[\tau^\pm, q^\pm]$-algebra with generators $X^\pm, Y^\pm, T$ and relations
\[ (T - \tau)(T + \tau^{-1}) = 0, \quad TXY = X^{-1}, \quad TY^{-1}T = Y \quad \text{and} \quad Y^{-1}X^{-1}YXT^2 = q. \]

There is a triangular decomposition $H^{dA} = \mathbb{C}[X^\pm] \otimes \mathcal{H} \otimes \mathbb{C}[Y^\pm]$ [Che7, §1.4.2]. Let us consider the $H^{dA}$-module $\text{Ind}_{\mathcal{H}}(\mathbb{C}[Y^\pm] \otimes \mathcal{H} \mathcal{C})$, where $Y$ and $T$ act on $\mathcal{C}$ by multiplication by $\tau$ [Che7, Proof of Lemma 1.4.5].
We identify this module with $\mathbb{C}[X^\pm 1]$. This gives a faithful representation of $H^{\text{ell}}$. The action of $H^{\text{ell}}$ is given by
\[
X : X^i \mapsto X^{i+1} \\
T : X^i \mapsto X^{-i}T + (\tau^{-1} - \tau)(X^{i-2} + X^{i-4} + \cdots + X^{-i}) \\
YT^{-1} : X^i \mapsto q^{-i}X^{-i}
\]
So, $T$ acts by $\tau s + \frac{1}{\tau} (s - 1)$ and $Y$ acts by $spT$, where $p(f)(X) = f(q^{-1}X)$.

7.4.2. We follow [EtGi, Proposition 4.10].

Fix $c \in \mathbb{C}$. We consider the ring $\mathbb{C}[[h]]$ with its $h$-adic topology. Let $\hat{H}^{\text{ell}}$ be the $\mathbb{C}[[h]]$-algebra topologically generated by three elements $s$, $x$ and $y$ with relations (3) for $X = e^{hx}$, $Y = e^{hy}$, $T = se^{hx}x$, $q = e^{ht}$ and $\tau = e^{h^2c}$.

The first relation, taken at order $0$, gives $(s - 1)(s + 1) = 0$. The second and third relations, taken at order $1$, give $sxs = -x$ and $sys = -y$. Finally, the last relation, taken at order $2$, gives $yx - xy = 1 - 2cs$. This gives rise to an isomorphism of $\mathbb{C}$-algebras $H \sim \hat{H}^{\text{ell}} \otimes_{\mathbb{C}[[h]]} \mathbb{C}[[h]]/(h)$. Note finally that $\hat{H}^{\text{ell}}$ is actually a trivial deformation of $H$, i.e., there is an isomorphism of topological $\mathbb{C}[[h]]$-algebras $H \otimes \mathbb{C}[[h]] \sim \hat{H}^{\text{ell}}$ [Che6, p.65].

8. Generalizations

8.1. Complex and symplectic reflection groups.

8.1.1. The definition of the rational Cherednik algebra generalizes to the case where $W$ is a complex reflection group on $V$ and more generally, when $W$ is a symplectic reflection group on a space $L$ (which is $V \oplus V^*$ in the complex reflection case). Theorem 3.1 remains valid in that setting. The construction and the main results on category $O$ generalize to complex reflection groups.

Such deformations have been introduced and studied before in [CrBoHo] in the case of a symplectic reflection group acting on a symplectic space of dimension 2.

These constructions of symplectic reflection algebras are special cases of a more general construction [Dr]. An even more general construction is given in [EtGanGi], where finite groups are replaced by reductive groups.

Another direction of generalization is globalization, where $V$ is replaced by an algebraic variety acted on by a finite group [Et].

8.1.2. Many results of §6 should generalize to the complex reflection groups $B_n(d) \simeq \mathbb{Z}/d! \rtimes S_n$ (cf [Mu] for a generalization of some of the constructions of §6.4). Some geometric aspects (should) generalize even to $\Gamma \times S_n$, where $\Gamma$ is a finite subgroup of $\text{SL}_2(\mathbb{C})$. For example, the Hilbert scheme to consider is $\text{Hilb}^n(\text{Hilb}^n \mathbb{C}^2)$. The description of multiplicities for $B_n(d)$ should generalize Corollary 6.3 using suitable canonical bases of higher level Fock spaces [Rou].

Some finite dimensional representations have been constructed in the case $\Gamma \times S_n$. Those where the $x_i$’s and $\xi_i$’s act by zero have been classified in [Mo1]. More general finite dimensional simple representations have been shown to exist by cohomological methods [EtMo, Mo2]. These representations come from ones where the $x_i$’s and the $\xi_i$’s act by zero via reflection functors and every simple finite dimensional representation for parameters “close to 0” is of this form [Gan].

8.1.3. Assume $W = \Gamma \rtimes S_n$ acting on $\mathbb{C}^{2n}$, for some finite subgroup $\Gamma$ of $\text{SL}_2(\mathbb{C})$.

There is a crepant (=symplectic) resolution $X_c \to \mathcal{CM}_c$ and an equivalence [GoSm, Theorem 1.2]
\[
D^b(H_{0,c}\text{-mod}) \sim D^b(X_c\text{-coh})
\]
The variety $X_c$ is constructed as a moduli space of representations of $H_c$ and the kernel of the equivalence is the universal bundle. The case $c = 0$ is the McKay correspondence.

The algebra $H_{0,c}$ is actually a non-commutative crepant resolution of $\mathcal{CM}_c$, in the sense of Van den Bergh [GoSm, Lemma 3.10].
Note that $CM_c$ is smooth for generic values of $c$ [EtGi, Proposition 11.11] and it has then a description generalizing that of §6.1.2 [EtGi, Theorem 11.16].

8.2. Characteristic $p > 0$. The rational Cherednik algebra can be defined over $\mathbb{Z}$, and in particular over an algebraic closure $\overline{\mathbb{F}}_p$ of the field with $p$ elements. The representation theory in characteristic $p > 0$ is quite different from that in characteristic 0, due to the presence of a big center: we have $gr\mathbb{Z}(H_c) \cong (C[V^* \times V]^p)^W$ when $p > n$ (Etingof, cf [BezFiGi, Theorem 10.1.1]). In particular, $H_c$ is a finite dimensional module over its center, hence all its simple representations are finite-dimensional.

There is a localization result in characteristic $p$.

**Theorem 8.1** ([BezFiGi, Theorem 7.3.2]). Assume $c \leq 0$ and $c \not\in \frac{1}{2} + \mathbb{Z}$. Assume $p$ is large enough.

Then, there is a sheaf $\mathcal{F}_c$ of Azumaya algebras over the Frobenius twist $\text{Hilb}^{(1)}$ of $\text{Hilb}^n\mathbb{A}^2$ and an equivalence $D^b(\mathcal{F}_c\text{-mod}) \sim D^b(H_c\text{-mod})$.

This comes from an isomorphism $H^0(\mathcal{F}_c) \cong H_c$ and from the vanishing $H^{>0}(\mathcal{F}_c, \mathcal{F}_c) = 0$.

Cf [La] for the determination of irreducible representations of Cherednik algebras associated to the rank 1 groups $W = \mathbb{Z}/d$, over a field of characteristic $p | d$, making explicit results of [CrBoHo].

9. Table of analogies

| $H$ or $B$ | $H_{i,c}$ or $B_{i,c}$ | $H^i(W)$ | $\mathcal{H}$ | $\text{Hilb}^n\mathbb{C}^2$ (type $A_{n-1}$) |
|-----------|-----------------|---------|---------|------------------|
| $S(V) \otimes C[W] \otimes S(V^*)$ | $U(g)$ | $U(g) = U(n^+) \otimes U(h) \otimes U(n^-)$ | $\mathfrak{h}^*/\mathfrak{w}$ | $U(g)/U(g)\mathfrak{m}_\lambda$ |
| $\mathcal{S}$ | $h^*/\mathfrak{w}$ | $\mathcal{N}$ | $\mathcal{W}$ | $\mathcal{N}$ |
| parabolic subgroups of $W$ | $\mathfrak{h}^*/\mathfrak{w}$ | nilpotent classes | $W$ | $\mathcal{W}$ |
| $\mathcal{H}$ | $S(\mathfrak{h}^+)/S(\mathfrak{h}^+)^W \simeq H^*(G/B)$ | dominant weights | $T^*G/B$ | |

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