On the use of the Reciprocity Gap Functional in inverse scattering with near-field data: an application to mammography

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Abstract. Microwave tomography is a non-invasive approach to the early diagnosis of breast cancer. However, the problem of visualizing tumors from diffracted microwaves is a difficult non-linear ill-posed inverse scattering problem. We propose a qualitative approach to the solution of such a problem, whereby the shape and location of cancerous tissues can be detected by means of a combination of the Reciprocity Gap Functional method and the Linear Sampling method. We validate this approach to synthetic near-fields produced by a finite element method for boundary integral equations, where the breast is mimicked by the axial view of two nested cylinders, the external one representing the skin and the internal one representing the fat tissue.

1. Introduction

Usual methods of mammography use X-rays. Nonetheless, since at these wavelengths (between 5 pm and 10 nm), the electromagnetic waves have an ionizing effect, they can be harmful for the human body and therefore cannot be used for screening plans.

Using microwaves instead of X-ray could be an interesting alternative. Owing to notable diffraction effects, microwave tomography is characterized by a limited spatial resolution; on the other hand, microwaves assure a low degree of invasivity and an optimal contrast and therefore, in principle, would be a valuable tool for screening procedures in breast cancer detection and monitoring. However, image reconstruction in microwave tomography is a very difficult mathematical problem. In fact, the imaging modality used to detect cancers using microwaves consists in illuminating a part of the human body (in our case the breast) with time-harmonic electromagnetic waves for sources at different positions and measuring the scattered field by this part of the body for each position of the sources. Then one has to process these measured data in order to achieve the reconstruction of defects in the screened part of the body. Due to its ill-posedness this inverse scattering problem is highly numerically unstable and, furthermore, at these wavelengths the non-linearity of the model cannot be reduced by means of some linearization based on physical assumptions. Non-linear optimization schemes for solving this kind of problems are typically very heavy from a computational viewpoint and need an accurate initialization to converge. In this paper we will describe a qualitative visualization...
method which can detect the presence and the shape of cancerous tissues in the breast within a very reduced computational time and without needing any a priori information. It is qualitative in the sense that, instead of typical optimization algorithms, this approach cannot provide quantitative information on the biological parameters (essentially, electrical permittivity and conductivity) of the tumor.

The method is based on the reciprocity relation for the Helmholtz equation. Briefly speaking, the measurements of two different waves scattered by the same object verify an integral relation. Considering this integral relation but for one wave scattered by a known object (this wave is usually numerically computed) and another wave scattered by the same object with a defect (this wave is the measured wave), the method consists in exploiting the default in the theoretical integral relation (Reciprocity Gap) in order to achieve the reconstruction of the defect. With respect to another qualitative approach like linear sampling, the main advantages of this method are that

- it utilizes cylindrical waves instead of plane waves, which are more difficult to generate in an actual experiment;
- it requires measurements of the near-field instead of the far-field;
- it deals with inhomogeneous backgrounds in a more natural way.

The plan of the paper is as follows: in Section 2 the method is described; Section 3 contains a numerical test; Section 4 introduces some open problems.

2. Description of the method
Locally we consider the female breast as two nested amagnetic circular cylinders, with an external skin surrounding a fat tissue. Then, using TM-polarized waves for the scattering problem, as shown in Figure 1 the model for breast cancer detection using electromagnetic probing can be described by the two dimensional stationary electromagnetic scattering problem in a radially symmetric three layered medium.

![Figure 1. Model for breast cancer detection](image-url)
The forward problem is described by the inhomogeneous Helmholtz equation with a Sommerfeld radiation condition. The wave number of the background medium without the tumor (vacuum and breast) is denoted by \( k(x) \), piecewise constant in \( \mathbb{R}^2 \). More precisely

- \( k(x) = k_0 \) in the vacuum where \( k_0 \in \mathbb{R}_+^* \)
- \( k(x) = k_2 \) in the skin where \( k_2^2 = k_0^2 n_2 \) with \( \Im(n_2) > 0 \)
- \( k(x) = k_1 \) in the fat where \( k_1^2 = k_0^2 n_1 \) with \( \Im(n_1) > 0 \)

**Remark 2.1** We did not explicitly write it but the refraction indices \( n_1 \) and \( n_2 \) depend on \( k_0 \).

The tumor is denoted by \( D \) where \( D \) is supposed to be an open bounded Lipschitz domain included in the fatty part of the breast.

We denote by \( k_D(x) \) the wave number in the tumor \( D \) where \( k_D \in L^\infty(D) \) with \( k_D^2(x) = k_0^2 n_D(x) \) and \( \Im(n_D(x)) > c_D > 0 \) (as in the remark 2.1, \( n_D \) also depends on \( k_0 \)).

We will denote by \( \tilde{k} \) the function defined in \( \mathbb{R}^2 \) by

\[
\tilde{k}(x) = k(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus \bar{D} \quad \text{and} \quad \tilde{k}(x) = k_D(x) \quad \text{for} \quad x \in D
\]

Then we can formulate the direct scattering problem as that of finding a solution \( u \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{x_0\}) \) of

\[
\begin{align*}
\Delta u + \tilde{k}^2(x)u &= 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{x_0\} \\
u &= u^s + u^i \\
\lim_{r \to \infty} \sqrt{r}(\partial_r u^s - ik_0 u^s) &= 0
\end{align*}
\]

(1)

where

- \( x_0 \) is the location of a point source
- \( u(\cdot, x_0) = u^i(\cdot, x_0) + u^s(\cdot, x_0), u^i(\cdot, x_0) \) being the incident field equal to the Green’s function of the background medium without the tumor and \( u^s(\cdot, x_0) \) is the scattered wave verifying the Sommerfeld radiation condition

**Theorem 2.1** There exists a unique solution to problem 1

**Remark 2.2** For \( x_0 \in \mathbb{R}^2 \) we will also denote the function \( u^i(\cdot, x_0) \) by \( G(\cdot, x_0) \) or \( G_{x_0} \).

Our aim is to solve the inverse scattering problem of finding the location and the shape of a tumor \( D \) using measurements of the field \( u(\cdot, x_0) \) for different positions of the sources \( x_0 \). In [1] a combination of the Reciprocity Gap Functional Method (RGFM) and the Linear Sampling Method (LSM) is performed. Nonetheless the advantage provided by this approach, which is to avoid the need to compute Green’s function for the background medium, implies some restrictions which are not compatible with the structure of our problem. So giving up this advantage, with few modifications it is possible to adapt the original method to the scheme for breast cancer detection represented in the previous figure. The final approach is very similar to LSM but can be applied in the case of near-field data.

We consider a circle \( C \) larger than the breast which will be the location of the sources and we consider another circle \( \Gamma \) larger than the breast and smaller than \( C \) (cf Figure 1) which will be the location of the measurements. We denote by \( \Omega \) the disc of boundary \( \Gamma \). For \( u, v \) sufficiently regular we can define the Reciprocity Gap Functional \( \mathcal{R} \) by

\[
\mathcal{R}(u, v) = \int_\Gamma (u \partial_\nu v - v \partial_\nu u) ds
\]

(2)

Using this functional, for \( z \) in the breast we look at the solutions \( g_z \) of the following equation

\[
\mathcal{R}(u(\cdot, x_0), g_z) = \mathcal{R}(u(\cdot, x_0), u^i(\cdot, z)) \quad \text{for all} \quad x_0 \in C
\]

(3)
where \( s_{g_z} \) is the simple layer potential of density \( g_z \) on \( C \). This equation has no solution in general, nonetheless we can find regularized solutions. Then, as for the classical Linear Sampling Method [3], we can show that when \( z \) is taken inside the tumor the norm \( ||g_z|| \) of the approximate solutions remains bounded whereas when \( z \) is taken outside the tumor the norm \( ||g_z|| \) of the regularized solutions can become arbitrarily large. Thus studying the norm of the approximate solutions of equation (3) we can find the location of the tumor. More precisely, we can prove

**Theorem 2.2** Let us suppose that \( k_0 \) is not a transmission eigenvalue of the following problem: Find \( u, v \in H^1_0(D) \) such that \( u - v \in H^1(D) \) and

\[
\begin{align*}
\Delta u + k_0^2 n_1(k_0)u &= 0 & \text{in } D \\
\Delta v + k_0^2 n_D(k_0)(x)v &= 0 & \text{in } D \\
(u - v) &= 0 & \text{on } \partial D \\
\partial_n(u - v) &= 0 & \text{on } \partial D
\end{align*}
\]

(4)

where

\[
H^1_0(D) = \{ u \in H^1(D) : \Delta u \in L^2(D) \} \\
H^1(D) = \{ u \in L^2(D) : \Delta u \in L^2(D) \}
\]

Then considering \( z \in \Omega \) we have :

a) If \( z \in D \) then there exists a sequence \( (g_p) \) such that

\[
\lim_{p \to \infty} R(u(\cdot, x_0), s_{g_p}) = R(u(\cdot, x_0), u^i(\cdot, z)) \quad \text{for all } x_0 \in C
\]

Moreover \( (s_{g_p}) \) converges in \( L^2(D) \).

b) If \( z \in \Omega \setminus D \) then there exists a sequence \( (g_p) \) such that

\[
\lim_{p \to \infty} R(u(\cdot, x_0), s_{g_p}) = R(u(\cdot, x_0), u^i(\cdot, z)) \quad \text{for all } x_0 \in C
\]

and for all sequence \( (g_p) \) verifying the previous equality we have

\[
\lim_{p \to \infty} ||s_{g_p}||_{L^2(D)} = \infty
\]

The first step in implementing the reciprocity gap functional method consists in discretizing equation (3) which is an infinite set of equations parameterized by \( x_0 \in C \) and \( z \in \Omega \). Discretizing the set of parametrization \( C \) with \( N \) points, which physically corresponds to a number \( N \) of positions of the sources on \( C \) and discretizing the integrals with \( N \) points for each equation, which physically corresponds to a number \( N \) of receivers on \( \Gamma \), equation (3) is reduced to a set of \( N \times N \) systems parameterized by \( z \in \Omega \). Denoting by \( R_C \) the radius of the circle \( C \), if the discretization points are chosen equidistant then these systems are given for each \( z \in \Omega \) by

\[
Ag(z) = b(z)
\]

(5)

where

\[
A = \frac{2\pi R_C}{N}(U^T L - V^T G),
\]

(6)

and

\[
b(z) = U^T l(z) - V^T h(z)
\]

(7)
Then Theorem 2.2 inspires the following visualization algorithm: plot the map $z \mapsto \Psi(z)$; then the boundary of the tumor will be given by the set of points where $\Psi(z)$ becomes large.

Let us define $B$ as a disc contained in $\Omega$. Then (5) can be considered as a functional equation in $[L^2(B)]^N$

$$\mathcal{A} g(\cdot) = b(\cdot),$$

where $\mathcal{A} : [L^2(B)]^N \rightarrow [L^2(B)]^N$ is defined by

$$[\mathcal{A} g(\cdot)](\cdot) := \left\{ \sum_{p=0}^{N-1} A_{ij} g_j(\cdot) \right\}_{i=0}^{N-1}. \quad (10)$$

It is easy to prove that the Tikhonov regularized solution of (10) is

$$g_\alpha(\cdot) = \sum_{p=0}^{N-1} \frac{\sigma_p}{(\sigma_p)^2 + \alpha} \langle b(\cdot), v_p \rangle_{CN} u_p, \quad (11)$$

where $\{\sigma_p, u_p, v_p\}_{p=0}^{N-1}$ is the singular system of $A$ and for any $f(\cdot) \in [L^2(B)]^N$ and $w \in \mathbb{C}^N$, $\langle f(\cdot), w \rangle_{CN}$ is the element of $L^2(B)$ defined as

$$\langle f(\cdot), w \rangle_{CN} : B \rightarrow \mathbb{C} \quad z \mapsto \langle f(z), w \rangle_{CN} \quad \text{f. a. a.} \quad z \in B. \quad (12)$$

$(\cdot, \cdot)_{CN}$ being is the canonic inner product on $\mathbb{C}^N$.

Since (9) is a single functional equation independent of $x$, also the regularization parameter $\alpha$ is independent of $z$. It can be fixed to an optimal value $\alpha^*$ by using the generalized discrepancy principle [5]. We introduce the indicator function

$$\Psi(z) := \|g_{\alpha^*}(z)\|^2_{CN} = \sum_{p=0}^{N-1} \frac{\sigma_p^2}{(\sigma_p^2 + \alpha^*)^2} \|\langle b(z), v_p \rangle_{CN}\|^2. \quad (13)$$

Then Theorem 2.2 inspires the following visualization algorithm: plot the map $z \mapsto \Psi(z)$; then the boundary of the tumor will be given by the set of points where $\Psi(z)$ becomes large.
3. Numerical results

To generate the data used to test the inverse code we chose to implement a finite elements method for boundary integral equations on the tumor for the forward code. The reason is that for to inverse code we have to compute the Green’s function of the background medium without defect. Moreover the radial symmetry of the background medium allows us to compute it rather simply. Then instead of using a 2D finite elements method or a boundary elements method on the interface of each tissue (vacuum–skin, skin–fat, fat–tumor) it seems to be quite natural to generate the data using this Green’s function that we have to compute for the inverse code. Since the code for the scattering by a penetrable obstacle in the breast is not available for the moment, for the numerical results we will suppose that the considered tumors have a high conductivity so that they can be considered as perfect conductors. Here are the geometric characteristics of the considered tissues, the breast is supposed to be radially symmetric and the tumors are supposed to be discs (in the screening plane). Then the geometric characteristics of the tissues will be denoted by the position of the center and the radius of the tissue, except for the skin which is a corona and then denoted by two radii.

Table 1. Geometric characteristics

| Tissue        | Center Position (cm) | Radius (cm) |
|---------------|----------------------|-------------|
| Skin          | (0,0)                | 4.2–4       |
| Fat           | (0,0)                | 4           |
| First Tumor   | (-1,-1)              | 0.25        |
| Second Tumor  | (1,1)                | 0.25        |

For the imaging we chose a frequency of 2.5GHz, then the dielectric properties of the different tissues are given in the following table.

Table 2. Dielectric properties

| Tissue | Conductivity (S/m) | Relative permittivity | Wavelength (cm) |
|--------|--------------------|-----------------------|-----------------|
| Vacuum | –                  | 1                     | 11.992          |
| Skin   | 1.4876             | 37.951                | 1.9278          |
| Fat    | 0.14068            | 5.137                 | 5.2656          |

The emitters are taken on the circle of radius 8cm centered on O and the receivers are taken on the circle of radius 5cm centered on O. The number of emitters/receivers is 16. A gaussian noise of 10% is added to the data computed by the forward code.

The plotting of the indicator function is done for a regular circular grid with 16 radial points and 64 angular points in the fatty part of the breast, ie with 961 points.
Here is the scheme of the numerical experiment ($\lambda_1$ is the wavelength in the fat).

*Figure 2. Scheme of the experiment*

We obtain the following plotting of the indicator function $\Psi$ for the reconstruction.

*Figure 3. Plotting of the indicator function $\Psi$*
Concerning the computational time, the forward code takes around 1s whereas the inverse code takes around 0.5s.

4. Open issues
We presented a fast method for visualizing breast tumors using microwaves. The method is tested in the two-dimensional case, using synthetic data corresponding to a simple breast model and small conductors mimicking the presence of tumors. Next validations of the method will be concerned with more realistic models for both the breast and the tumors, accounting for the presence of glandules and veins in the fat tissue and more realistic values of the electrical parameters for the tumor. Furthermore the no-sampling implementation of the method suggests that an extension to a three-dimensional framework is computationally feasible.

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