Graviton-Dilaton Cosmology

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ABSTRACT. We show that the evolution of the universe is singularity free in a class of graviton-dilaton models.
We present a general analysis of the evolution of a homogeneous isotropic universe in graviton-dilaton theories and show that the evolution of the observed universe is singularity free in a class of models.

1. Action and Equations of Motion

In the absence of a dilaton potential, the most general graviton-dilaton action is given by

\[
S = -\frac{1}{2} \int d^4x \sqrt{-g} \left( \chi R + \frac{\omega(\chi)}{\chi} (\nabla \chi)^2 \right) + S_M(M, g_{\mu\nu}),
\]

where \( \chi \) is the dilaton with the range \( 0 \leq \chi \leq 1 \), \( \omega(\chi) \) is the arbitrary function that characterises the theory. In (1), the matter fields feel only the gravitational force and, hence, the physical metric is \( g_{\mu\nu} \) and the physical quantities are those directly obtained from \( g_{\mu\nu} \).

In the model derived in [1], the function \( \Omega(\chi) \equiv 2\omega(\chi) + 3 \) is required to satisfy

\[
\begin{align*}
\Omega(0) & = \Omega_0 \leq \frac{1}{3} \\
\frac{d^n\Omega}{d\chi^n} & = \text{finite} \quad \forall \ n \geq 1, \quad 0 \leq \chi < 1 \\
\Omega & \to \infty \quad \text{at} \ \chi = 1 \ \text{only} \\
\lim_{\chi \to 1} \Omega & = \Omega_1(1 - \chi)^{-2\alpha}, \quad \frac{1}{2} \leq \alpha < 1,
\end{align*}
\]

where \( \Omega_0 > 0 \) and \( \Omega_1 > 0 \) are constants. We further assume, for the sake of simplicity, that \( \Omega(\chi) \) is a strictly increasing function of \( \chi \). The function \( \Omega(\chi) \) is otherwise arbitrary.

In the following, we take “matter” to be a perfect fluid with density \( \rho \) and pressure \( p \), related by \( p = \gamma \rho \) where \( -1 \leq \gamma \leq 1 \) and take the line element to be given by

\[
ds^2 = -dt^2 + e^{2A(t)} \left( dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right).
\]

1The possibilities of such a model arising within string theory is discussed in [1, 2]. See also [3]. Other proposals for singularity free evolution of the universe in string theory can be found in [4].

2\( \chi(today) = 1 \) corresponding to a non zero value of the Newton’s constant.
The equations of motion from (1) are

\[ \dot{A} = -\frac{\dot{\chi}}{2\chi} + \epsilon \sqrt{\frac{\rho}{6\chi}} + \frac{\Omega\chi^2}{12\chi^2} \]

(4)

\[ \dot{\chi} + 3\dot{A}\dot{\chi} + \frac{\dot{\Omega}\dot{\chi}}{2\Omega} = \frac{(1 - 3\gamma)\rho}{2\Omega} \]

(5)

\[ \rho = \rho_0 e^{-(1+\gamma)A}, \]

(6)

where upper dots denote $t$-derivatives, $\epsilon = \pm 1$ and we have used $\Omega = 2\omega + 3$ and $p = \gamma \rho$. The square roots are to be taken with a positive sign always. Equations (4) and (5) can also be written as

\[ \dot{\chi}(t) = \frac{e^{-3A}}{\sqrt{\Omega}} (\sigma(t) + c) \]

(7)

\[ 2\chi \frac{dA}{d\chi} = -1 + \epsilon \text{ sign}(\dot{\chi}) \sqrt{K}, \]

(8)

where $c = \dot{\chi}\sqrt{\Omega}e^{3A}\big|_{t=t_i}$ is a constant, $t_i$ is an initial time, and

\[ \sigma(t) \equiv \frac{(1 - 3\gamma)\rho_0}{2} \int_{t_i}^{t} dt \frac{e^{-3\gamma A}}{\sqrt{\Omega}} \]

(9)

\[ K \equiv \frac{\Omega}{3} \left(1 + \frac{2\rho_0\chi e^{3(1-\gamma)A}}{(\sigma(t) + c)^2}\right). \]

(10)

Note that under time reversal $t \rightarrow -t$, $\epsilon \rightarrow -\epsilon$ in equation (4), whereas equation (5) is unchanged. Thus, evolution for $-\epsilon$ is same as that for $\epsilon$, but with the direction of time reversed.

Note also that, in general, there will appear in the solutions positive non zero arbitrary constant factors $e^{A_0}$, $\rho_0$, and $\chi_0$ in front of $e^A$, $\rho$, and $\chi$ respectively. However, it follows from (4) and (5) that they can all be set equal to 1 with no loss of generality if time $t$ is measured in units of $\sqrt{\frac{\chi_0 e^{3(1+\gamma)A_0}}{\rho_0}}$. Therefore, in the solutions below we often assume that $t$ is measured in appropriate units and, hence, set these constants equal to 1. Note, however, that the model dependent constants associated with $\Omega(\chi)$, such as $\Omega_0$ or $\Omega_1$ in (2), cannot be set equal to 1.

2. Analysis of the Evolution
Our main goal is to determine whether the evolution of the universe in the present model is singular or not. The task is trivial if one can solve the equations of motion (4) - (6) for arbitrary functions \( \Omega(\chi) \) (or \( \psi(\phi) \)). However, explicit solutions can be obtained only in special cases [5, 1] and the methods involved in obtaining such solutions are inapplicable in the general cases of interest here. Also, even when applicable, the details of the solution tend to obscure the general features of the evolution. Hence, a different approach is needed which is valid for any matter and for any arbitrary function \( \Omega(\chi) \) and which enables one to analyse the evolution for the general case and obtain its generic features even in the absence of explicit solutions. We will present such an approach below.

Note that for the singularities to be absent, all curvature invariants must remain finite. A sufficient condition, proved in [2], for all curvature invariants to be finite is that the quantities

\[ e^{-A}; \quad \frac{\rho}{\chi}, \quad \frac{\rho}{\chi\Omega}; \quad \frac{\dot{\chi}}{\chi}, \quad \frac{\Omega\chi^2}{\chi^2}; \quad \text{and} \quad \frac{\chi^n}{\Omega} \left( \frac{\dot{\chi}}{\chi} \right)^n, \quad \forall n \geq 1 \quad (11) \]

be all finite. To proceed, we also need initial values of \( \dot{A}, \dot{\chi}, \) and \( \chi \) or equivalently \( \Omega \) at initial time \( t_i \). However, their numerical values are not needed in our approach as long as they are generic [4] which we assume to be the case.

The value \( \Omega(t_i) \lesssim 3 \) will turn out to be special. Thus, there are 16 sets of possible initial conditions: 2 each for the signs of \( \dot{A}(t_i), \dot{\chi}(t_i), \) and \( \epsilon \), and 2 for whether \( \Omega(t_i) \lesssim 3 \) or not.

However, it is not necessary to analyse all of these 16 sets. The evolution for \(-\epsilon\) is same as that for \( \epsilon \), but with the direction of time reversed. Hence, only 8 sets need to be analysed. Now, let \( \epsilon = 1 \). If \( \dot{\chi} < 0 \) then \( \dot{A} \) cannot be negative for any value of \( \Omega \), see (4). Similarly, if \( \dot{\chi} > 0 \) and \( \Omega > 3 \) then also \( \dot{A} \) cannot be negative since

\[ \sqrt{\frac{\rho}{6\chi} + \frac{\Omega\chi^2}{12\chi^2}} \geq \sqrt{\frac{\Omega}{3} \frac{\dot{\chi}}{2\chi}} > \frac{\dot{\chi}}{2\chi} \]

in equation (4). Thus, there remain only 5 sets of possible initial conditions. We choose these sets so as to be of direct relevance to the evolution of observed universe and analyse each of them for \( t > t_i \). We then use these results in section 6 to describe the generic evolution of the universe. We also assume that \( \rho \neq 0 \) and \( \gamma \neq 1 \) identically, as is relevant to the observed universe.

\footnote{A non generic example: \( \dot{\chi}(t_i) = 0 \) and \( \chi(t_i) = 1 \), i.e. \( \Omega|_{t_i} = \infty \).}
Two remarks are now in order. First, the amount and the nature of the dominant “matter” in the universe are given by \( \rho_0 \) and \( \gamma \) where \( -1 \leq \gamma \leq 1 \). The later varies as the scale factor \( e^A \) of the universe evolves. If \( e^A \) is increasing, i.e. if \( \dot{A} > 0 \), then the universe is expanding, eventually becoming dominated by “matter” with \( \gamma < 1/3 \); when such “matter” is present. For the observed universe, which is known to contain dust for which \( \gamma = 0 \), one may take \( \gamma \leq 0 \). Thus, by choosing \( t_i \) suitably we may assume, with no loss of generality, that \( \gamma < 1/3 \) (\( \leq 0 \) for observed universe) if \( \dot{A} > 0 \). Similarly, if \( \dot{A} < 0 \) then we may assume, with no loss of generality, that \( \gamma \geq 1/3 \) when such “matter” is present. This is valid for the observed universe which is known to contain radiation for which \( \gamma = 1/3 \).

Second, consider \( \sigma(t) \) in (7). The integrand is positive and \( \propto 1/\sqrt{\Omega} \). (The dependence on \( \rho_0 \) can be absorbed into the unit of time, as explained in section 3.) Hence, the value of the integral is controlled by \( \Omega \). For example, in the limit \( \chi \to 1 \), it is controlled by the constant \( \Omega_1 \) in (2). We will use this fact in the following. Also, since \( t > t_i \), the sign of \( \sigma(t) \) is same as that of \( (1-3\gamma) \). Moreover, \( \sigma(t) = 0 \) if \( \rho_0 = 0 \) or \( \gamma = 1/3 \) corresponding to an universe containing, respectively, no matter or radiation only.

\[ 2 \text{ a. } \dot{A}(t_i) > 0, \quad \dot{\chi}(t_i) > 0, \quad \text{and } \Omega(t_i) > 3 \]

These conditions imply that \( \epsilon = +1 \) in (4) and the constant \( c > 0 \) in (7). For \( t > t_i \), the scale factor \( e^A \) increases and, eventually, \( \gamma \) can be taken to be \( < 1/3 \) when such “matter” is present. In fact, \( \gamma \leq 0 \) for the observed universe. Then, \( (1-3\gamma) > 0 \) and, hence, \( \sigma(t) \) increases. Thus, for \( t > t_i \), \( \chi > 0 \) and \( \chi \) and \( \Omega \) both increase. Since \( \Omega > 3 \), it follows from equation (4) that \( \dot{A} > 0 \) for \( t > t_i \) and, hence, \( e^A \) increases. Eventually, \( \chi \to 1 \) and \( \Omega \to \Omega_1 (1-\chi)^{-2\alpha} \).

In the limit as \( \chi \to 1 \), \( (\sigma(t) + c) > 0 \) is a constant to an excellent approximation. Then, equation (8) can be solved relating \( \chi \) and \( e^A \):

\[ e^A = (\text{constant}) (1-\chi)^{-\frac{2(1-\alpha)}{3(1-\gamma)}}. \quad (12) \]

Substituting this result in equation (3) then yields the unique solution in the limit \( \chi \to 1 \):

\[ e^A = e^{A_0} t^{\frac{1}{(1+\gamma)(1-\alpha)}}, \quad \chi = 1 - \chi_0 t^{-\frac{1}{(1+\gamma)(1-\alpha)}}, \quad (13) \]

where \( A_0 \) and \( \chi_0 > 0 \) are constants and \( t \) is measured in appropriate units. Since \( \chi \to 1 \), it follows from (13) that \( t \to \infty \). It can now be seen that
\((\sigma(t) + c) > 0\) is indeed a constant to an excellent approximation. Also, the quantities in (11) are all finite, implying that all the curvature invariants are finite. Thus, there is no singularity as \(\chi \to 1\) and, thus, for \(t > t_i\).

2 b. \(\dot{A}(t_i) > 0, \ \dot{\chi}(t_i) < 0, \ \Omega(t_i) > 3, \text{ and } \epsilon = 1\)

The constant \(c < 0\) in (7). For \(t > t_i\), \(\chi\) and \(\Omega\) decrease and the scale factor \(e^A\) increases. Equation (8) now becomes

\[
2\chi \frac{dA}{d\chi} = -1 - \sqrt{K},
\]

where \(K(t)\) is given in (8).

Consider first the case \(\rho = 0\) or \(\gamma = 1\), discussed in section 2a and which led to the present phase. Then, \(\rho_0(1 - 3\gamma) \leq 0\) and, therefore, \(\sigma(t) \leq 0\). Thus, \((\sigma(t) + c) \leq c < 0\) and, hence, \(\dot{\chi}(t)\) is negative and non zero, implying that \(\chi(t)\) decreases for \(t > t_i\). Also, \(K(t)\) remains finite and it follows from equation (8) that \(\frac{dA}{d\chi} < 0\). Thus, \(A > 0\) since \(\dot{\chi} < 0\) and, hence, \(e^A\) increases for \(t > t_i\).

Eventually, as \(t\) increases, \(\chi \to 0\) and \(e^A \to \infty\). Also, \(\frac{2\rho_0\chi e^{3(1-\gamma)A}}{(\sigma(t)+c)^2} \ll 1\) in (8) since \(\rho_0 = 0\) or \(\gamma = 1\). Equation (8) can then be solved relating \(\chi\) and \(e^A\):

\[
e^A = (\text{constant}) \chi^{\frac{3 + \sqrt{3\Omega_0}}{6}},
\]

where \(\chi \to 0\) and \(\Omega_0\) is given in (2). Substituting this result in equation (8) then yields the unique solution in the limit \(\chi \to 0\):

\[
e^A = e^{A_0}(t - t_0)^n, \quad \chi = \chi_0(t - t_0)^m,
\]

where \(A_0, \chi_0, \text{ and } t_0 > t_i\) are some constants and

\[
n = \frac{3 + \sqrt{3\Omega_0}}{3(1 + \sqrt{3\Omega_0})}, \quad m = \frac{-2}{1 + \sqrt{3\Omega_0}}.
\]

Note that \(m < 0\). Then, since \(\chi \to 0\), it follows from (15) that \(t \to \infty\).

Thus, as \(\chi \to 0, t \to \infty\) and \(e^A \to \infty\). It can also be seen that the quantities in (11) are all finite for \(t_i \leq t \leq \infty\), implying that all the curvature invariants are finite. Thus, there is no singularity for \(t_i \leq t \leq \infty\).
Consider now the case where $\rho$ and $\gamma$ are non zero and have generic values. We have $\dot{A}(t_i) > 0$ and $\dot{\chi}(t_i) < 0$. Hence, $e^A$ increases and $\chi$ decreases. As $e^A$ increases, $\gamma$ can eventually be taken to be $< \frac{1}{3}$, when such “matter” is present. In fact, $\gamma \leq 0$ for the observed universe. In such cases where $\gamma \leq 0$, $\sigma(t)$ grows faster than $t$ as can be seen from (9). Then, as $t$ increases, the factor $(\sigma(t) + c)$ becomes positive.

This implies that $\dot{\chi}$, initially negative, passes through zero and becomes positive. We then have $\dot{A} > 0$ and $\dot{\chi} > 0$. As $t$ increases further, $\chi$ and, hence, $\Omega(\chi)$ increase. If $\Omega > 3$ then further evolution proceeds as described in section 2a. If $\Omega < 3$ then, upon making a time reversal, the initial conditions become identical to the one described in section 2e. The evolution in the present case then proceeds as in section 2d for the case which leads to the initial conditions of section 2e.

2 c. $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) > 3$

These conditions imply that $\epsilon = -1$ in (4) and the constant $c < 0$ in (7). For $t > t_i$, the scale factor $e^A$ decreases and, eventually, $\gamma$ can be taken to be $\geq \frac{1}{3}$ when such “matter” is present. In fact, this is the case for the observed universe. Then $(1 - 3\gamma) \leq 0$ and $\sigma(t)$ decreases or remains constant. Therefore, the factor $(\sigma(t) + c) \leq c < 0$ and, since $e^A$ decreases, we have that $\dot{\chi}(t) < \dot{\chi}(t_i) < 0$ for $t > t_i$. Thus $\chi$ and, hence, $\Omega$ decrease for $t > t_i$. Also, $\frac{dA}{d\chi} > 0$ since $\dot{\chi} < 0$ and $\dot{A} < 0$.

In (8), $K(t_i) > 1$ since $\Omega > 3$. From the behaviour of $e^A$, $\chi$, $\Omega(\chi)$, and $(\sigma(t) + c)$ described above, it follows that $K$ decreases monotonically. The lowest value of $K$ is $\frac{K_0}{3} \leq \frac{1}{3}$, achievable when $\chi$ vanishes. This, together with $K(t_i) > 1$ and the monotonic behaviour of $K(t)$, then implies that there exists a time, say $t = t_m > t_i$ where $K(t_m) = 1$ with $\chi(t_m) > 0$. Hence, $\frac{dA}{d\chi}(t_m) = 0$. Therefore, $A(t_m) = 0$ since $\dot{\chi}(t_m)$ is non zero. This is a critical point of $e^A$ and is a minimum. Also, equation (8) can be written as

$$A(t_i) - A(t_m) = \int_{\chi(t_m)}^{\chi(t_i)} \frac{d\chi}{2\chi} (-1 + \sqrt{K}) = \text{finite},$$

(17)

where the last equality follows because both the integrand and the interval of integration are finite. Therefore, $A(t_m)$ is finite and $> -\infty$ and, hence, $e^{A(t_m)}$ is finite and non vanishing.
Thus, for \( t > t_i \), the scale factor \( e^A \) continues to decrease and reaches a non-zero minimum at \( t = t_m \). The precise values of \( t_m \), \( A(t_m) \), \( \chi(t_m) \), and \( \Omega(t_m) \) are model dependent. Hence, nothing further can be said about them except, as follows from \( K(t_m) = 1 \), that \( \Omega(t_m) \lesssim 3 \) in general and \( = 3 \) when \( \rho_0 = 0 \).

However, the above information suffices for our purposes. It can now be seen that the quantities in (11) are all finite, implying that all the curvature invariants are finite. Thus, there is no singularity for \( t_i \leq t \leq t_m \).

For \( t > t_m \), one has \( \dot{A}(t) > 0 \), \( \dot{\chi}(t) < 0 \), and \( \Omega(\chi(t_m)) < \sim 3 \) by continuity. Further evolution is analysed below.

### 2 d. \( \dot{A}(t_i) > 0 \), \( \dot{\chi}(t_i) < 0 \), \( \Omega(t_i) \lesssim 3 \), and \( \epsilon = -1 \)

This implies that the constant \( c < 0 \) in (18) and that \( \frac{dA}{d\chi}(t_i) < 0 \) in (8). Hence, \( K(t_i) < 1 \). For \( t > t_i \), the scale factor \( e^A \) increases and, eventually, \( \gamma \) can be taken to be \( < \frac{1}{3} \) when such “matter” is present. In fact, \( \gamma \leq 0 \) for the observed universe. Then, \( (1 - 3\gamma) > 0 \) and, hence, \( \sigma(t) \) increases. Thus, \( e^A \) is increases, \( \chi \) decreases, and \( (\sigma(t) + c) \) increases. Now, depending on \( \rho_0, \gamma \), and the details of \( \Omega(\chi), K(t) \) in (8) may or may not remain \( < 1 \) for \( t > t_i \).

Consider first the case where \( K(t) \) remains \( < 1 \) for \( t > t_i \). It then follows that \( (\sigma(t) + c) \) cannot have a zero for \( t > t_i \) and, since \( c < 0 \), must remain negative and non infinitesimal. Note that \( \gamma \) must be \( > 0 \) for otherwise \( \sigma(t) \) grows as fast as or faster than \( t \), making \( (\sigma(t) + c) \) vanish at some finite time. Let

\[
\lim_{t \to \infty} (\sigma(t) + c) = c_e
\]

where \( c_e \) is a negative, non infinitesimal constant. We then have for \( t_i \leq t \leq \infty \), \( c \leq (\sigma(t) + c) \leq c_e < 0 \) and, hence, \( \dot{\chi}(t) < 0 \). Therefore, eventually \( \chi(t) \to 0 \). Since \( K(t) < 1 \) and \( \dot{\chi}(t) < 0 \), it also follows from (8) that \( \frac{dA}{d\chi} < 0 \) and, hence, \( \dot{A}(t) > 0 \). We will now consider the limit \( \chi \to 0 \).

\( K(t) < 1 \) implies that as \( \chi \to 0 \), \( K \to \frac{\Omega_e}{3} < 1 \) where

\[
\Omega_e \equiv \Omega_0 \left( 1 + \frac{2\rho_0}{c_e^2} \lim_{\chi \to 0} \chi e^{3(1-\gamma)A} \right)
\]

is a constant and \( \Omega_0 \) is given in (2). Now, as \( \chi \to 0 \), equation (8) can be solved relating \( \chi \) and \( e^A \):

\[
e^A = (\text{constant}) \chi^{\frac{2-\sqrt{3}}{3}}.
\]
Note that \(3 - \sqrt{3\Omega_e} > 0\) since \(\Omega_e < 3\). Hence, \(e^A \to \infty\) as \(\chi \to 0\).

Substituting (20) in equation (7) then yields the unique solution in the limit \(\chi \to 0\): For \(\Omega_e \neq \frac{1}{3}\),

\[e^A = e^{A_0} (t_0 - \text{sign}(m)t)^n, \quad \chi = \chi_0 (t_0 - \text{sign}(m)t)^m, \quad (21)\]

where \(t\) is measured in appropriate units, \(A_0, \chi_0\), and \(t_0 > t_i\) are some constants, and

\[n = \frac{3 - \sqrt{3\Omega_e}}{3(1 - \sqrt{3\Omega_e})}, \quad m = \frac{-2}{1 - \sqrt{3\Omega_e}}. \quad (22)\]

For \(\Omega_e = \frac{1}{3}\),

\[e^A = e^{A_0} e^{-c_0(t-t_0)}, \quad \chi = \chi_0 e^{c_0(t-t_0)}, \quad (23)\]

where \(c_0\) is defined in (18).

Let \(\Omega_e > \frac{1}{3}\). Hence, \(m > 0\). Also, \(n < 0\) because \(\Omega_e < 3\). Since \(\chi \to 0\), it follows from (21) that \(t \to t_0 > t_i\), which implies that \(\chi\) vanishes at a finite time \(t_0\). In this limit, the scale factor \(e^A \to \infty\) since \(n < 0\). The quantities in (11), for example \(\frac{\dot{\chi}}{\chi}\), also diverge, implying that the curvature invariants, including the Ricci scalar, diverge. Thus, for \(\Omega_e > \frac{1}{3}\), there is a singularity at a finite time \(t_0\).

Let \(\Omega_e \leq \frac{1}{3}\), which is the case in our model, see (2). When \(\Omega_e < \frac{1}{3}\), \(m < 0\). Also, \(n > 0\) because \(\Omega_e < 3\). Since \(\chi \to 0\), it follows from (21) that \(t \to \infty\). In this limit, the scale factor \(e^A \to \infty\) since \(n > 0\) now. The same result holds also for the solution in (23) with \(\Omega_e = \frac{1}{3}\).

For \(K(t)\) to be \(< 1\), \((\sigma(t) + c)\) must not be too small. Since \(c < 0\) and \(\sigma(t) > 0\), it necessarily implies that \(\sigma(t)\) must remain finite. It follows from the above solutions that \(\sigma(t)\) can remain finite only if \(\sqrt{3\Omega_e} > \frac{1-3\gamma}{1-\gamma}\), equivalently, \(\gamma > \frac{1-\sqrt{3\Omega_e}}{3-\sqrt{3\Omega_e}}\). Under this condition, it can be checked easily that \(\lim_{\chi \to 0} \chi e^{3(1-\gamma)A} = 0\). Hence, \(\Omega_e = \Omega_0\) as follows from (19).

It can now be seen, for \(\Omega_e = \Omega_0 \leq \frac{1}{3}\), that the quantities in (11) are all finite for \(t_i \leq t \leq \infty\), implying that all the curvature invariants are finite. Thus, there is no singularity for \(t_i \leq t \leq \infty\) when \(\Omega_0 \leq \frac{1}{3}\).

Consider the second case where \(K(t)\) in (8) may not remain \(< 1\) for all \(t > t_i\). This is the case for the observed universe where \(\gamma \leq 0\) and, hence, \(\sigma(t)\) grows as fast as or faster than \(t\). Then there is a time, say \(t = t_1\), when
the value of $K = 1$. It follows that $c < (\sigma(t_1) + c) < 0$ and, hence, $\dot{\chi}(t) < 0$ for $t_i \leq t \leq t_1$. Also, $\chi(t_1)$ is non vanishing since the case of $\chi(t_1) \to 0$ is same as the case described above. Therefore, we have that $\frac{dA}{d\chi}(t_1) = 0$ which implies, since $\dot{\chi}(t_1) \neq 0$, that $\dot{A}(t_1) = 0$. This is a critical point of $e^A$ and is a maximum.

For $t > t_1$, $K(t)$ becomes $> 1$ and, hence, $\dot{A}(t) < 0$. Also, $\dot{\chi} < 0$ and $\Omega(t) < 3$. Then $\epsilon = -1$ necessarily. Further evolution is analysed below.

2 e. $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) < 3$

The above conditions imply that $\epsilon = -1$, the constant $c < 0$ in (7), and $\frac{dA}{d\chi}(t_i) > 0$. Hence, $K(t_i) > 1$. For $t > t_i$, $(\sigma(t) + c)$ which is negative at $t_i$ may or may not vanish for non zero $\chi$.

Consider the case where $(\sigma(t) + c)$ does not vanish and remains negative for non zero $\chi$. Therefore, $\dot{\chi}(t) < 0$ and, hence, $\chi(t)$ decreases. For $t > t_i$, the scale factor $e^A$ decreases and, eventually, $\gamma$ can be taken to be $\geq \frac{1}{3}$ when such “matter” is present. In fact, this is the case for the observed universe. Then, $(1 - 3\gamma) < 0$ and $\sigma(t)$ decreases. Hence, $(\sigma(t) + c)$ also decreases.

It follows from (8) that $K$, initially $> 1$, is now decreasing since $\chi$ and $e^A$ are decreasing and $(\sigma(t) + c)^2$ is increasing. Its lowest value $= \frac{\Omega_0}{3} \leq \frac{1}{9}$, achievable at $\chi = 0$. Therefore, there exists a time, say $t = t_M$, where $K = 1$ and, hence, $\frac{dA}{d\chi} = 0$. Clearly, $\chi(t_M)$ and $\dot{\chi}(t_M)$ are non zero for the same reasons as in section 2c, implying that $\dot{A}(t_M) = 0$. This is a critical point of $e^A$ and is a minimum.

The existence of this critical point of $e^A$ can be seen in another way also. As $(\sigma(t) + c) < 0$ grows in magnitude, it follows from equation (6) that $\Omega\dot{\chi}^2 \propto e^{-6A}$, whereas $\rho$ is given by (7). Note that $\dot{\chi}(t) < 0$ and, hence, $\chi(t)$ decreases and eventually $\to 0$. Then, in the limit $\chi \to 0$, the $\frac{\Omega\chi^2}{\dot{\chi}}$ term in equation (4) dominates $\frac{\dot{\chi}}{\chi}$ term. Hence, in this limit where $\Omega \to \Omega_0 \leq \frac{1}{3} < 3$, we have that $\dot{A} > 0$. Since $\dot{A} < 0$ initially, this implies the existence of a zero of $\dot{A}$. This is a critical point of $e^A$ and is a minimum.

Note that all quantities remain finite for $t_i \leq t \leq t_M$. In particular, the quantities in (11) are all finite, implying that all the curvature invariants are finite. Thus, there is no singularity for $t_i \leq t \leq t_M$.

For $t > t_M$, we have $\dot{A} > 0$, $\dot{\chi} < 0$, $\Omega < 3$, and $\epsilon = -1$. Further evolution then proceeds as described in section 2d. The evolution can thus become repetitive and oscillatory.
Consider now the case where \((\sigma(t) + c)\) vanishes, say at \(t = t_m\), for non zero \(\chi\). Hence, \(\dot{\chi}(t_m)\) vanishes and this is a minimum of \(\chi\). For \(t > t_m\), \((\sigma(t) + c)\) and \(\dot{\chi}\) are positive by continuity and, hence, \(\chi\) increases. Also, \(\dot{A} < 0\) and the scale factor \(e^A\) decreases. Eventually, \(\gamma\) can be taken to be \(\geq \frac{1}{3}\) when such “matter” is present. In fact, this is the case for the observed universe. Then, \((1 - 3\gamma) < 0\) and \(\sigma(t)\) decreases. Hence, \((\sigma(t) + c)\) also decreases.

Now, as \(\chi\) increases and \(\rightarrow 1\), \((\sigma(t) + c)\) may or may not vanish with \(\chi < 1\). If \((\sigma(t) + c)\) vanishes with \(\chi < 1\), then \(\dot{\chi}\) vanishes. This is a critical point of \(\chi\) and is a maximum, beyond which we have \(\dot{A} < 0\), \(\dot{\chi} < 0\), and \(\epsilon = -1\). Further evolution then proceeds as described in section 2c if \(\Omega > 3\), and as in section 2e if \(\Omega < 3\). The evolution can thus become repetitive and oscillatory.

If \((\sigma(t) + c)\) does not vanish and remain positive for \(\chi < 1\) then, eventually, \(\chi \rightarrow 1\). It follows from equation (4) that \(\Omega \chi^2 \propto e^{-6A}\), whereas \(\rho\) is given by (3). Note that \(\dot{A} < 0\) and \(e^A\) is decreasing. Then, in this limit, the \(\frac{\Omega \chi^2}{\dot{\chi}}\) term in equation (4) dominates or, if \(\gamma = 1\), is of the same order as the \(\frac{L}{\dot{\chi}}\) term.

Thus we have \(\dot{A} < 0\), \(\dot{\chi} > 0\), \(\chi \rightarrow 1\), equivalently \(\Omega \rightarrow \infty\), and \(\epsilon = -1\). Also, \(\frac{\Omega \chi^2}{\dot{\chi}}\) term in equation (4) dominates or, if \(\gamma = 1\), is of the same order as the \(\frac{L}{\dot{\chi}}\) term. But this is precisely the time reversed version of the evolution analysed in section 2b and in section 2a for the case which led to the initial conditions of section 2a, where the dynamics is clear in terms of \(\phi\) and \(\psi(\phi)\). Applying the results of sections 2a and 2b, it follows that \(\phi\) will cross the value 0 after which \(e^\psi\) and, hence, \(\chi\) begins to decrease. We then have \(\dot{A} < 0\), \(\dot{\chi} < 0\), and \(\Omega > 3\). Also, \(\epsilon = -1\) necessarily. Further evolution then proceeds as described in section 2c. The evolution can thus become repetitive and oscillatory.

3. Evolution of observed Universe

We now use the results of the analysis in section 2 to describe the generic evolution of the observed universe. Note that the observed universe certainly contains dust \((\gamma = 0)\) and radiation \((\gamma = \frac{1}{3})\). As follows from (3), “matter” with larger \(\gamma\) dominates the evolution for smaller value of \(e^A\) and vice versa. Therefore, when \(e^A\) is decreasing we take \(\gamma \geq \frac{1}{3}\) eventually, and when \(e^A\) is increasing we take \(\gamma \leq 0\) eventually. We start with an initial time \(t_i\), corresponding to a temperature, say \(\gtrsim 10^{16}\) GeV, such that GUT symmetry
breaking, inflation, and other (matter) model dependent phenomena may occur for \( t > t_i \) only. Our observed universe is expanding at \( t_i \) and, hence, \( \dot{A}(t_i) > 0 \). For the sake of definiteness, we take \( \dot{\chi}(t_i) > 0 \) and \( \Omega(t_i) > 3 \), equivalently \( \omega(t_i) > 0 \), as commonly assumed.

We first describe the evolution for \( t > t_i \) taking, as initial conditions \( \dot{A}(t_i) > 0 \), \( \dot{\chi}(t_i) > 0 \), and \( \Omega(t_i) > 3 \). Then, \( \epsilon = 1 \) necessarily. To describe the evolution for \( t < t_i \), we reverse the direction of time and take, as initial conditions, \( \dot{A}(t_i) < 0 \), \( \dot{\chi}(t_i) < 0 \), and \( \Omega(t_i) > 3 \). Then, \( \epsilon = -1 \) necessarily. The required evolution is that for \( t > t_i \) in terms of the reversed time variable, which is also denoted as \( t \). The results of section 2 can then be applied directly.

\[
\dot{A}(t) > 0, \quad \dot{\chi}(t) > 0, \quad \Omega(t) > 3
\]

Initially, we have \( \dot{A}(t_i) > 0 \), \( \dot{\chi}(t_i) > 0 \), and \( \Omega(t_i) > 3 \). Then, \( \epsilon = 1 \) necessarily. For \( t > t_i \), the evolution proceeds as described in section 2a. Both \( e^A \) and \( \chi \) increase. Hence, \( \Omega \) increases. Eventually, as \( e^A \) increases, we can take \( \gamma \leq 0 \). Then, as \( t \to \infty \), \( e^A \) and \( \chi \) evolve as given in (13).

In particular, it can be seen that the quantities in (14) are all finite for \( t_i \leq t \leq \infty \), implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

Thus, in the present day universe in our model, \( \chi(\text{today}) \to 1 \) and \( \Omega(\text{today}) \to \infty \). Also, as follows from (2),

\[
\frac{1}{\Omega^2} \frac{d\Omega}{d\chi}(\text{today}) = -\frac{2a}{\Omega^2} (1 - \chi)^{4a-1} \to 0
\]

Therefore, our model satisfies the observational constraints imposed by solar system experiments, namely \( \Omega(\text{today}) > 2000 \) and \( \frac{1}{\Omega^2} \frac{d\Omega}{d\chi}(\text{today}) < 0.0002 \).

\[
t < t_i, \text{ equivalently } (-t) > (-t_i)
\]

To describe the evolution for \( t < t_i \), we reverse the direction of time. Then, in terms of the reversed time variable, also denoted as \( t \), we have \( \dot{A}(t_i) < 0 \), \( \dot{\chi}(t_i) < 0 \), and \( \Omega(t_i) > 3 \). Then, \( \epsilon = -1 \) necessarily. For \( t > t_i \), the evolution proceeds as described in section 2c. Both \( e^A \) and \( \chi \) decrease. Hence, \( \Omega \) decreases. Then, as shown in section 2c, there exists a time, say \( t = t_m > t_i \) where \( K(t_m) = 1 \) in equation (8), \( e^{A(t_m)} > 0 \), \( \chi(t_m) > 0 \), and \( \dot{\chi}(t_m) < 0 \). Note that \( \Omega(t_m) < 3 \).

As shown in section 2c, \( K(t_m) = 1 \) implies that \( \dot{A}(t_m) = 0 \). Hence, the scale factor \( e^A \) reaches a minimum. For \( t > t_m \), we then have \( \dot{A} > 0 \), \( \dot{\chi} < 0 \),
$\Omega < 3$, and $\epsilon = -1$. The evolution proceeds as described in section 2d. Now, however, the evolution for $t > t_m$ is complicated since the universe is known to contain “matter” with $\gamma = 0$. Nevertheless, all qualitative features of its evolution can be obtained using the results of section 2.

For $t \geq t_m$, $\dot{A} > 0$, $\dot{\chi} < 0$, and $(\sigma(t) + c) < 0$. Hence, $e^A$ increases and $\chi$ decreases and, eventually, “matter” with $\gamma \leq 0$ dominates the evolution. Then $(1 - 3\gamma) > 0$ and $\sigma(t)$ begins to increase. Hence $(\sigma(t) + c)$, which is negative, also begins to increase. For $\gamma \leq 0$, $\sigma(t)$ grows at least as fast as $t$, as follows from (5). Then eventually, as described in section 2d, $e^A$ reaches a maximum eventually at time, say $t = t_1$. Also, $(\sigma(t_1) + c) < 0$ and, hence, $\dot{\chi}(t_1) < 0$. Note that this critical point of $e^A$ exists independent of the details of the model, as long as “matter” with $\gamma \leq 0$ is present. Further evolution then proceeds as described in section 2e.

For $t > t_1$, we then have $\dot{A} < 0$, $\dot{\chi} < 0$, and $\Omega < 3$. Also, $(\sigma(t) + c) < 0$ but $\sigma(t)$ remains increasing. Hence, depending now on the details of the model, $(\sigma(t) + c)$ may remain negative for all $t > t_1$, or it may reach a zero and become positive.

In the first case, $(\sigma(t) + c)$ remains negative for all $t > t_1$. Hence, $\dot{\chi} < 0$ and $\chi$ continues to decrease. The scale factor $e^A$, as shown in section 2e, reaches a minimum at time, say $t = t_M > t_1$, with $\chi(t_M) > 0$ non vanishing. For $t > t_M$, $e^A$ increases, and we have $\dot{A} > 0$, $\dot{\chi} < 0$, and $\Omega < 3$. Further evolution then proceeds as described in section 2d.

Note that this means that the scale factor increases and reaches a maximum, then decreases and reaches a minimum, then increases and so on. The existence of maxima of $e^A$ is model independent as long as “matter” with $\gamma \leq 0$ is present, which is the case for the observed universe. The minima of $e^A$ are all non zero for the reasons described in section 2c. Their existence depends on whether $(\sigma(t) + c)$ remains negative or not, but is otherwise model independent as long as “matter” with $\gamma \geq \frac{1}{3}$ is present, which is the case for the observed universe. The evolution can thus become repetitive and oscillatory.

In the second case, $(\sigma(t) + c)$ reaches a zero at time, say $t = t_{m'} > t_1$ and becomes positive. It then follows that $\dot{\chi}(t_{m'}) = 0$ and that this is a minimum of $\chi$. For $t > t_{m'}$, we have $\dot{A} < 0$, $\dot{\chi} > 0$, and $\Omega < 3$. Hence, $e^A$ decreases and $\chi$ increases. Eventually, we can take $\gamma \geq \frac{1}{3}$.

Now, $(1 - 3\gamma) \leq 0$ and $\sigma(t)$ begins to decrease or remains constant. Hence $(\sigma(t) + c)$, which is positive, also begins to decrease or remains constant.
Thus, depending on the details of $\rho_0$ and $\gamma$, $(\sigma(t) + c)$ may or may not vanish with $\chi < 1$. If $(\sigma(t) + c)$ vanishes with $\chi < 1$, then $\dot{\chi}$ vanishes. This critical point is a maximum of $\chi$, beyond which we have $\dot{A} < 0$, $\dot{\chi} < 0$, and $\epsilon = -1$. Further evolution then proceeds as described in section 2c if $\Omega > 3$ and as in section 2e if $\Omega < 3$. The evolution can thus become repetitive and oscillatory.

If $(\sigma(t) + c)$ does not vanish and remain positive for $\chi < 1$ then, eventually, $\chi \to 1$. It follows from equation (7) that $\Omega \dot{\chi}^2 \propto e^{-6A}$, whereas $\rho$ is given by (11). Note that $\dot{A} < 0$ and $e^A$ is decreasing. Then, in this limit, the $\frac{\Omega\dot{\chi}^2}{\chi^2}$ term in equation (4) dominates or, if $\gamma = 1$, is of the order of the $\frac{\dot{\chi}}{\chi}$ term.

Thus we have $\dot{A} < 0$, $\dot{\chi} > 0$, $\chi \to 1$, equivalently $\Omega \to \infty$, and $\epsilon = -1$. Also, $\frac{\Omega\dot{\chi}^2}{\chi^2}$ term in equation (4) dominates or is of the order of the $\frac{\dot{\chi}}{\chi}$ term. But this is precisely the time reversed version of the evolution analysed in section 2b, where the dynamics is clear in terms of $\phi$ and $\psi(\phi)$. Applying the results of section 2b, it then follows that $\phi$ will cross the value 0 after which $e^\phi$ and, hence, $\chi$ will begin to decrease. We then have $\dot{A} < 0$, $\dot{\chi} < 0$, $\Omega > 3$. The evolution then proceeds as described in section 2c. The evolution can thus become repetitive and oscillatory.

Thus it is clear that depending on the details of the model, the universe undergoes oscillations, perhaps infinitely many. As follows from the above description, the oscillations can stop, if at all, only in the limit $\chi \to 0$. The solutions then are given by (21) - (23). In particular, however, the quantities in (11) all remain finite during the oscillations, implying that all the curvature invariants remain finite. They also remain finite in the solutions (21) - (23) if $\Omega_0 \leq \frac{1}{3}$ in (3).

Thus, it follows that if $\Omega_0 \leq \frac{1}{3}$ then the quantities in (11) are all finite for $t_i \leq t \leq \infty$, implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

In summary, the evolution of a realistic universe, such as our observed one, with the initial conditions $\dot{A}(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$ proceeds in the present model as follows. For $t > t_i$, the scale factor $e^A$ increases continuously to $\infty$. The field $\chi$ increases continuously to 1. Correspondingly, $\Omega$ increases continuously to $\infty$.

For $t < t_i$, the scale factor $e^A$ decreases and reaches a non zero minimum. It then increases and reaches a maximum, then decreases and reaches a minimum, then increases and so on, perhaps ad infinitum. The oscillations can stop, if at all, only in the limit $\chi \to 0$. The solutions then are given by
The field $\chi$ may, depending on the details of model, continuously decrease to 0, or undergo oscillations, its maxima always being $\leq 1$.

Also, the curvature invariants all remain finite for $-\infty \leq t \leq \infty$. Hence, the evolution, although complicated and model dependent in details, is completely singularity free if $\Omega(\chi)$ satisfies (2). Thus we have that a homogeneous isotropic universe, such as our observed one, evolves with no big bang or any other singularity in a class of models where $\Omega(\chi)$ satisfies (2). The time continues indefinitely into the past and the future, without encountering any singularity.

When the initial conditions are different, the evolution can again be analysed along similar lines. However, the main result that the evolution is singularity free remains unchanged.

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