Degree Formulae for Grassmann Bundles, II

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Abstract. Let $X$ be a non-singular quasi-projective variety over a field, and let $E$ be a vector bundle of rank $r$ over $X$. Let $G_X(d,E)$ be the Grassmann bundle of $E$ over $X$ parametrizing corank $d$ subbundles of $E$ with projection $\pi : G_X(d,E) \to X$, and let $Q \leftarrow \pi^*E$ be the universal quotient bundle of rank $d$ on $G_X(d,E)$. We denote by $\theta$ the first Chern class $c_1(\det Q) = c_1(Q)$ of $Q$, and call $\theta$ the Plücker class of $G_X(d,E)$: In fact, the determinant bundle $\det Q$ is isomorphic to the pull-back of the tautological line bundle $\mathcal{O}_{\mathbb{P}_X}(\wedge^d E)$ by the relative Plücker embedding over $X$.

The purpose of this article is to study the push-forward of powers of the Plücker class to $X$ by $\pi$, namely, $\pi_*(\theta^N)$, where $\pi_* : A^* + d(r-d)(G_X(d,E)) \to A^*(X)$ is the push-forward by $\pi$ between the Chow rings. The main result is a closed formula for the push-forward of $\text{ch}(\det Q) := \exp \theta = \sum_{N \geq 0} \frac{1}{N!} \theta^N$, the Chern character of $\det Q$ in terms of the Segre classes of $E$, as follows:

**Theorem 0.1.** We have

$$\pi_* \text{ch}(\det Q) = \sum_k \prod_{0 \leq i < j \leq d-1} (k_i - k_j - i + j) \prod_{0 \leq i \leq d-1} (r + k_i - i)! \prod_{0 \leq i \leq d-1} s_{k_i}(E)$$

in $A^*(X) \otimes \mathbb{Q}$, where $k = (k_0, \ldots, k_{d-1}) \in \mathbb{Z}_{\geq 0}^d$, and $s_i(E)$ is the $i$-th Segre class of $E$.

The Segre classes $s_i(E)$ here are the ones satisfying $s(E,t)c(E,-t) = 1$ as in [2], [7], [8], where $s(E,t)$ and $c(E,t)$ are respectively the Segre series and the Chern polynomial of $E$ in $t$. Note that our Segre class $s_i(E)$ differs by the sign $(-1)^{i}$ from the one in [3].

Theorem 0.1 yields

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Corollary 0.2 (Degree Formula for Grassmann Bundles). If $X$ is projective and $\wedge^d \mathcal{E}$ is very ample, then $\mathbb{G}(d, \mathcal{E})$ is embedded in the projective space $\mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$ by the tautological line bundle $\mathcal{O}_{\mathbb{G}(d, \mathcal{E})}(1)$, and its degree is given by

$$\deg \mathbb{G}(d, \mathcal{E}) = (d(r - d) + n)! \sum_{|k|=n} \prod_{0 \leq i < j \leq d-1} (k_i - k_j - i + j) \prod_{0 \leq i \leq d-1} (r + k_i - i)! \int_X \prod_{0 \leq i \leq d-1} s_{k_i}(\mathcal{E}),$$

where $|k| := \sum_i k_i$.

Here a vector bundle $\mathcal{F}$ over $X$ is said to be very ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ of $\mathbb{P}(\mathcal{F})$ is very ample.

We also give a proof for the following:

Theorem 0.3 ([3], [10]). We have

$$\pi_* \text{ch}(\det \mathcal{Q}) = \sum_{\lambda} \frac{1}{\lambda!} f^{\lambda + \epsilon} \Delta_{\lambda}(s(\mathcal{E}))$$

in $A^*(X) \otimes \mathbb{Q}$, where $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a partition with $|\lambda| := \sum_i \lambda_i$, $\epsilon := (r - d)^d = (r - d, \ldots, r - d)$, $f^{\lambda + \epsilon}$ is the number of standard Young tableaux with shape $\lambda + \epsilon$, and $\Delta_{\lambda}(s(\mathcal{E})) := \det[s_{\lambda_i + j-i}(\mathcal{E})]_{1 \leq i, j \leq d}$ is the Schur polynomial in the Segre classes of $\mathcal{E}$ corresponding to $\lambda$.

Note that our proofs for Theorem 0.3 as well as Theorem 0.1 do not use the push-forward formula of Józefiak-Lascoux-Pragacz [6], while the proofs given in [5], [10] do.

We establish instead a new push-forward formula, as follows: Let $\mathbb{F}_X^{d}(\mathcal{E})$ be the partial flag bundle of $\mathcal{E}$ on $X$, parametrizing flags of subbundles of corank 1 up to $d$ in $\mathcal{E}$, let $p : \mathbb{F}_X^{d}(\mathcal{E}) \to X$ be the projection, and denote by $p_* : A^{*+c}(\mathbb{F}_X^{d}(\mathcal{E})) \to A^*(X)$ the push-forward by $p$, where $c$ is the relative dimension of $\mathbb{F}_X^{d}(\mathcal{E})/X$. Let $\xi_0, \ldots, \xi_{d-1}$ be the set of Chern roots of $\mathcal{Q}$. It turns out (see §1) that one may consider $A^{*+c}(\mathbb{F}_X^{d}(\mathcal{E}))$ as an $A^*(X)$-algebra generated by the $\xi_i$. Then

Theorem 0.4 (Push-Forward Formula). For any polynomial $F \in A^*(X)[T_0, \ldots, T_{d-1}]$, we have

$$p_* F(\xi) = \text{const}_d \left( \Delta(\mathcal{U}) \prod_{i=0}^{d-1} t_i^{e-d} F(1/\mathcal{U}) \prod_{i=0}^{d-1} s(\mathcal{E}, t_i) \right),$$

in $A^*(X)$, where $\xi := (\xi_0, \ldots, \xi_{d-1})$, $\text{const}_d(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\mathcal{U} := (t_0, \ldots, t_{d-1})$, $\Delta(\mathcal{U}) := \prod_{0 \leq i < j \leq d-1} (t_i - t_j)$ and $F(1/\mathcal{U}) := F(1/t_0, \ldots, 1/t_{d-1})$.

The contents of this article are organized as follows: The general theories [8, §6], [11, §§0–1] on the structure of Chow ring of certain partial flag bundles are reviewed in §1. Then, Theorem 0.4 is proved in §2, by which it is shown that $\pi_* \text{ch}(\det \mathcal{Q})$ is given as the constant term of a certain Laurent series with coefficients in the Chow ring $A^*(X)$ of $X$, denoted by $P(\mathcal{U})$ (Proposition 2.3). To evaluate the constant term of $P(\mathcal{U})$, in §3, a linear form on the Laurent polynomial ring, denoted by $\Phi$, is introduced (Definition 3.1), and an evaluation formula is proved (Proposition 3.3): The evaluation formula is the key in the final step to prove Theorems 0.1 and 0.3. In §4, a generalization of Cauchy determinant formula is given (Proposition 4.1). This yields another proof of a push-forward formula for monomials of the $\xi_i$ (Lemma 2.2).
1. Set-up

Let $X$ be a non-singular quasi-projective variety of dimension $n$ defined over a field $k$, let $E$ be a vector bundle of rank $r$ on $X$, and let $\varpi : P(E) \to X$ be the projection. Denote by $\xi$ the first Chern class of the tautological line bundle $O_{P(E)}(1)$, and define a polynomial $P_\xi \in A^*(X)[T]$ associated to $E$ by setting

$$P_\xi(T) := T^r - c_1(E)T^{r-1} + \cdots + (-1)^r c_r(E),$$

where $A^*(X)$ is the Chow ring of $X$. Then, $P_\xi(\xi) = 0$ by definition of the Chern classes (Remark 3.2.4), and

$$A^*(P(E)) = \bigoplus_{0 \leq i \leq r-1} A^*(X)\xi^i \cong A^*(X)[T]/(P_\xi(T))$$

(Theorem 3.3 (b); Example 8.3.4]). Let $\varpi_* : A^{*+r-1}(P(E)) \to A^*(X)$ be the push-forward by $\varpi$. Then $\varpi_*\alpha$ is equal to the coefficient of $\alpha$ in $\xi^{r-1}$, denoted by $\text{coeff}_\xi(\alpha)$, with respect to the decomposition (1.1) for $\alpha \in A^{*+r-1}(P(E))$ (Proposition 3.1):

$$\varpi_*\alpha = \text{coeff}_\xi(\alpha)$$

Denote by $F^d_X(E)$ the partial flag bundle of $E$ on $X$, parametrizing flags of subbundles of corank 1 up to $d$ in $E$, and let $p : F^d_X(E) \to X$ be the projection. Set $E_0 := E$, and let $E_{i+1}$ be the kernel of the canonical surjection from the pull-back of $E_i$ to $P(E_i)$, to the tautological line bundle $O_{P(E_i)}(1)$, with $\text{rk} E_i = r - i$ ($i \geq 0$). Set $\xi_i := c_1(O_{P(E_i)}(1))$. We have an exact sequence on $P(E_i)$:

$$0 \to E_{i+1} \to E_i \to O_{F_i}(1) \to 0,$$

and an equation of Chern polynomials,

$$c(E_i, t) = c(E_{i+1}, t)(1 + \xi_i t),$$

where we omit the symbol of the pull-back by the projection $P_{F_i}(E_{i+1}) \to P(E_i)$. It is easily shown that the projection $p : F^d_X(E) \to X$ decomposes as a successive composition of projective space bundles, $P_{F_i}(E_{i+1}) \to P(E_i)$ ($i \geq 0$):

$$p : F^d_X(E) = P(E_{d-1}) \to P(E_{d-2}) \to \cdots \to P(E_1) \to P(E_0) \to X.$$ 

In fact, $P(E_i) \cong F^{i+1}_X(E)$ ($0 \leq i \leq d - 1$). Using (1.1) repeatedly, we see that the Chow ring of $F^d_X(E)$ is given as follows:

$$A^*(F^d_X(E)) = \bigoplus_{0 \leq i \leq r-1 \leq d-1} A^*(X)\xi_0^{i_0}\xi_1^{i_1} \cdots \xi_{d-1}^{i_{d-1}} = A^*(X)[T_0, T_1, \ldots, T_{d-1}] / \left(\{P_{E_i}(T_i) | 0 \leq i \leq d - 1\}\right).$$

Denote by $p_* : A^{*+c}(F^d_X(E)) \to A^*(X)$ the push-forward by $p$, where $c := \sum_{0 \leq i \leq d-1}(r - i - 1)$, the relative dimension of $F^d_X(E)/X$. Then, using (1.2) repeatedly, we see that

$$p_*\alpha = \text{coeff}_\xi(\alpha)$$

for $\alpha \in A^*(F^d_X(E))$, where $\text{coeff}_\xi(\alpha)$ denotes the coefficient of $\alpha$ in $\xi_0^{r-1}\xi_1^{r-2} \cdots \xi_{r-1}^{r-d}$ with respect to the decomposition (1.4).

Let $G := G_X(d, E)$ be the Grassmann bundle of corank $d$ subbundles of $E$ on $X$, and let $Q \leftarrow \pi^*E$ be the universal quotient bundle of rank $d$. Consider the flag bundle $F^d_{G^{-1}}(Q)$ of $Q$ on $G$, parametrizing flags of subbundles of corank 1 up to $d - 1$ in $Q$. Then, as in the case of $F^d_X(E)$, the projection $F^d_{G^{-1}}(Q) \to G$ decomposes as a successive composition of projective space bundles, $P_{F_i}(Q_{i+1}) \to P(Q_i)$ ($i \geq 0$):

$$q : F^d_{G^{-1}}(Q) = P(Q_{d-2}) \to P(Q_{d-2}) \to \cdots \to P(Q_1) \to P(Q_0) \to G,$$
where \( Q_0 := Q \), and \( Q_{i+1} \) is the kernel of the canonical surjection from the pull-back of \( Q \) to \( \mathbb{P}(Q_i) \), to the tautological line bundle \( \mathcal{O}_{\mathbb{P}(Q_i)}(1) \), with \( \text{rk } Q_i = d - i \) \((i \geq 0)\): In fact, \( \mathbb{P}(Q_i) \cong \mathbb{P}^{d-1}(Q) \) \((0 \leq i \leq d - 2)\) and \( \mathbb{P}(Q_{d-2})(Q_{d-1}) \cong \mathbb{P}(Q_{d-2}) \cong \mathbb{P}^{d-1}(Q) = \mathbb{P}^{d}(Q) \). It follows from the construction of the \( Q_i \), that the Plücker class \( \theta := c_1(\operatorname{det } Q) = c_1(Q) \) is equal to the sum of the first Chern classes \( c_1(\mathcal{O}_{\mathbb{P}(Q_i)}(1)) \) \((0 \leq i \leq d - 1)\) in \( A^*(\mathbb{P}^{d-1}(Q)) \),

where \( \mathcal{O}_{\mathbb{P}(Q_{d-1})}(1) = Q_{d-1} \) via \( \mathbb{P}(Q_{d-2})(Q_{d-1}) \cong \mathbb{P}(Q_{d-2}) \).

Thus we have

\[
q^*\theta = \xi_0 + \cdots + \xi_{d-1}
\]

in \( A^*(\mathbb{P}^d(E)) = A^*(\mathbb{P}^{d-1}(Q)) \). For details, we refer to [8 §6, [11 §§0–1].

2. Laurent series

We keep the same notation as in §1.

**Lemma 2.1.** For any non-negative integer \( p \),

\[
\operatorname{coeff} \xi (\xi^p) = \operatorname{const}_t (t^{-p+r+1}s(E, t)),
\]

where \( \operatorname{const}_t (\cdots) \) denotes the constant term in the Laurent expansion of \( \cdots \) in \( t \).

**Proof.** Set \( R_p(x_0, \ldots, x_{p-r}) := \sum_{i=0}^r (-1)^i c_i(E)x_{p-i} \), and consider a recurring relation, \( R_p(x_0, \ldots, x_{p-r}) = 0 \) \((p \geq r)\) for \( \{x_i\} \subseteq A^*(X) \). If \( a_p := \operatorname{coeff}\xi (\xi^p) \), then

\[
R_p(a_p, \ldots, a_{p-r}) = \operatorname{coeff}_t \left( \sum_{i=0}^r (-1)^i c_i(E)\xi^{p-i} \right) = 0
\]

by \( P_{\xi}(\xi) = 0 \). On the other hand, if \( b_p := \operatorname{const}_t (t^{-p+1+r}s(E, t)) \), then

\[
R_p(b_p, \ldots, b_{p-r}) = \operatorname{const}_t \left( \sum_{i=0}^r c_i(E)(-t)^i t^{-p+1+r}s(E, t) \right) = \operatorname{const}_t (t^{-p+1+r}) = 0
\]

by \( c(E, -t)s(E, t) = 1 \). Thus both of \( \{a_p\} \) and \( \{b_p\} \) satisfy the recurring relation \( R_p = 0 \), so that \( a_p = b_p \) for all \( p \): Indeed, \( \alpha_r = b_r = c_1(E) \), \( \alpha_{r-1} = b_{r-1} = 1 \) and \( a_p = b_p = 0 \) if \( 0 \leq p \leq r - 2 \). We here note that \( x_p \) is determined by \( x_{p-1}, \ldots, x_{p-r} \) if \( R_p(x_0, \ldots, x_{p-r}) = 0 \). \( \square \)

**Lemma 2.2.** For any non-negative integers \( p_0, \ldots, p_{d-1} \), we have

\[
\operatorname{coeff}_\xi (\xi^{p_0} \cdots \xi^{p_{d-1}}) = \operatorname{const}_t \left( \Delta(t) \prod_{i=0}^{d-1} t_i^{-p_i+r-d}s(E, t_i) \right),
\]

where \( \operatorname{const}_t (\cdots) \) denotes the constant term in the Laurent expansion of \( \cdots \) in \( t := (t_0, \ldots, t_{d-1}) \), and \( \Delta(t) := \prod_{0 \leq i < j \leq d-1} (t_i - t_j) \) is the Vandermonde polynomial of \( t \).

**Proof.** Since \( s(E_{d-1}, t_{d-1}) = (1 - \xi_{d-2}t_{d-1})s(E_{d-2}, t_{d-1}) \) by (4.3), it follows from Lemma 2.1 that

\[
\operatorname{coeff}_\xi (\xi^{p_{d-1}}) = \operatorname{const}_t (t_{d-1}^{-p_{d-1}+r-d}(1 - \xi_{d-2}t_{d-1})s(E_{d-2}, t_{d-1})
\]

\[

\Delta(t) \prod_{i=0}^{d-1} t_i^{-p_i+r-d}s(E, t_i)
\]

\[

\prod_{i=0}^{d-1} t_i^{-p_i+r-d}s(E, t_i)
\]

\[

\prod_{i=0}^{d-1} t_i^{-p_i+r-d}s(E, t_i)
\]

\[

\prod_{i=0}^{d-1} t_i^{-p_i+r-d}s(E, t_i)
\]
in $A^*(\mathbb{P}(\mathcal{E}_{d-2}))$, where $\text{coeff}_{\xi_{d-1}}(\cdots)$ denotes the coefficient of $\cdots$ in $\xi_{d-1}^{r-d}$. Therefore, using Lemma 2.1 again, we have

$$\text{coeff}_{\xi_{d-2}, \xi_{d-1}}(\xi_{d-2}^{p_d-2} \xi_{d-1}^{p_d-1}) = \text{coeff}_{\xi_{d-2}}(\xi_{d-2}^{p_d-2}) \text{const}_{t_{d-1}}(t_{d-1}^{p_{d-1}+d} (1 - \xi_{d-2} t_{d-1}) s(\mathcal{E}_{d-2}, t_{d-1})) = \text{coeff}_{\xi_{d-2}}(\xi_{d-2}^{p_d-2}) \text{const}_{t_{d-1}}(t_{d-1}^{p_{d-1}+d} s(\mathcal{E}_{d-2}, t_{d-1}))) + \text{coeff}_{\xi_{d-2}}(\xi_{d-2}^{p_d-2}) \text{const}_{t_{d-1}}(t_{d-1}^{p_{d-1}+d} s(\mathcal{E}_{d-2}, t_{d-1}))) = \text{const}_{t_{d-2}, t_{d-1}}(t_{d-2}^{p_{d-1}+d} s(\mathcal{E}_{d-2}, t_{d-1})) \text{const}_{t_{d-1}}(t_{d-1}^{p_{d-1}+d} s(\mathcal{E}_{d-2}, t_{d-1}))) + \text{const}_{t_{d-2}, t_{d-1}}(t_{d-2}^{p_{d-1}+d} s(\mathcal{E}_{d-2}, t_{d-1})) \text{const}_{t_{d-1}}(t_{d-1}^{p_{d-1}+d} s(\mathcal{E}_{d-2}, t_{d-1}))) = \text{const}_{t_{d-2}, t_{d-1}}((t_{d-2} - t_{d-1}) \prod_{i=d-2}^{d-1} t_i^{p_{i}+d} s(\mathcal{E}_{d-2}, t_i))$$

in $A^*(\mathbb{P}(\mathcal{E}_{d-3}))$, where $\text{coeff}_{\xi_{d-2}, \xi_{d-1}}(\cdots)$ denotes the coefficient of $\cdots$ in $\xi_{d-2}^{r-d+1} \xi_{d-1}^{r-d}$, and $\text{coeff}_{\xi_{d-2}}(\cdots)$ the coefficient of $\cdots$ in $\xi_{d-2}^{r-d-1}$. Repeating this procedure, we obtain the conclusion. \hfill \Box

Remark 2.3. Expanding the determinant $\Delta(t)$ in the right-hand side in Lemma 2.2 using (1.5), we obtain a formula, $p_*(\xi_{0}^{p_{0}} \cdots \xi_{d-1}^{p_{d-1}}) = \text{det}[s_{p_{i+j-r+1}}(\mathcal{E})]_{0 \leq i, j \leq d-1}$ in terms of the Schur polynomials in Segre classes of $\mathcal{E}$, which is equivalent to the determinantal formula [7 8.1 Theorem] with $f_i(\xi_i) := \xi_i^{p_i}$ $(0 \leq i \leq d-1)$.

Proposition 2.4. For any polynomial $F \in A^*(X)[T_0, \ldots, T_{d-1}]$, we have

$$\text{coeff}_{\xi}(F(\xi)) = \text{const}_{\xi}(\Delta(t) \prod_{i=0}^{d-1} t_i^{r-d} F(1/t) \prod_{i=0}^{d-1} s(\mathcal{E}, t_i)),$$

where $\xi := (\xi_0, \ldots, \xi_{d-1})$, $\text{const}_{\xi}(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\xi := (t_0, \ldots, t_{d-1})$, $\Delta(t) := \prod_{0 \leq i < j \leq d-1} (t_i - t_j)$, and $F(1/t) := F(1/t_0, \ldots, 1/t_{d-1})$.

Proof. This follows from Lemma 2.2 \hfill \Box

Proof of Theorem 0.4. The assertion follows from (1.5) and Proposition 2.4 \hfill \Box

Proposition 2.5. With the same notation as in §1, we have

$$\pi_* \text{ch}(\text{det} \mathcal{Q}) = \text{const}_{\xi}(P(\xi)),$$

where $\pi_* : A^{+d(r-d)}(G_X(d, \mathcal{E})) \otimes \mathcal{Q} \to A^*(X) \otimes \mathcal{Q}$ is the push-forward by $\pi$, $\text{ch}(\text{det} \mathcal{Q})$ is the Chern character of $\text{det} \mathcal{Q}$, $\text{const}_{\xi}(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\xi := (t_0, \ldots, t_{d-1})$, and

$$P(\xi) := \Delta(t) \prod_{i=0}^{d-1} t_i^{r-d-(d-1-i)} \exp \left( \sum_{i=0}^{d-1} \frac{1}{t_i} \right) \prod_{i=0}^{d-1} s(\mathcal{E}, t_i).$$

Note that, though $\exp \left( \sum_{i=0}^{d-1} \frac{1}{t_i} \right)$ is an element in $\mathbb{Q}[[\{\frac{1}{t_i}\}_{0 \leq i \leq d-1}]]$, $\text{const}_{\xi}(P(\xi))$ is well defined since the Segre series are polynomials in $\xi$.\hfill \Box
Proof. Since \( \mathbb{P}^{i+1}_G(Q) \rightarrow \mathbb{P}^d_G(Q) \) is a \( \mathbb{P}^{d-1-i} \)-bundle, using Proposition 3.1 repeatedly, for a non-negative integer \( N \), we have

\[
\theta^N = q_*(\xi_0^{d-1} \xi_1^{d-2} \cdots \xi_{d-2} q^* \theta^N),
\]

where \( q \) is the composition of the projections, \( \mathbb{P}^{d-1}_G(Q) \rightarrow \cdots \rightarrow \mathbb{P}^1_G(Q) \rightarrow G \). It follows from (1.6) and the commutativity of \( \Delta(\cdot) \) the polynomial defined by (1). The assertion is a direct consequence from the definition of \( \Phi \) and a property of \( \pi^* \theta^N \).

\[
\pi^* (\theta^N) = \pi_* q_*(\xi_0^{d-1} \xi_1^{d-2} \cdots \xi_{d-2} q^* \theta^N)
\]

\[
= \pi_* q_*(\prod_{i=0}^{d-1} \xi_i^{d-1-i} (\sum_{i=0}^{d-1} \xi_i)^N) = p_* (\prod_{i=0}^{d-1} \xi_i^{d-1-i} (\sum_{i=0}^{d-1} \xi_i)^N),
\]

where \( p \) is the composition of the projections, \( \mathbb{P}^d_X(E) \rightarrow \cdots \rightarrow \mathbb{P}^1_X(E) \rightarrow X \). Now, apply Theorem 0.4 with \( F := \prod_{i=0}^{d-1} T_i^{d-1-i} (\sum_{i=0}^{d-1} T_i) \). Then,

\[
p_* \left( \prod_{i=0}^{d-1} \xi_i^{d-1-i} (\sum_{i=0}^{d-1} \xi_i)^N \right) = \text{const}_\mathcal{L} \left( \Delta(\xi) \prod_{i=0}^{d-1} t_i^{r-d-(d-1-i)} (\sum_{i=0}^{d-1} t_i)^N \prod_{i=0}^{d-1} s(E, t_i) \right).
\]

Thus the conclusion follows with \( \text{ch}(\det Q) = \exp(\theta) \).

\[\square\]

3. A Linear Form on the Laurent Polynomial Ring

Definition 3.1. Let \( A \) be a \( \mathbb{Q} \)-algebra. We define a linear form \( \Phi : A[t_i, \frac{1}{t_i}]_{0 \leq i \leq d-1} \rightarrow A \) on the Laurent polynomial ring \( A[t_i, \frac{1}{t_i}]_{0 \leq i \leq d-1} \) by

\[
\Phi(f) := \text{const}_\mathcal{L} \left( \Delta(\xi) \exp \left( \sum_{i=0}^{d-1} \frac{1}{t_i} f(\xi^i) \right) \left( f \in A[t_i, \frac{1}{t_i}]_{0 \leq i \leq d-1} \right) \right),
\]

where \( \mathcal{L} := (t_0, \ldots, t_{d-1}) \).

Lemma 3.2. (1) Consider the natural action of the permutation group \( \mathfrak{S}_d \) on \( A[t_i, \frac{1}{t_i}]_{0 \leq i \leq d-1} \) with \( \sigma(t_i) := t_{\sigma(i)} \) (\( \sigma \in \mathfrak{S}_d \)). Then we have \( \Phi(\sigma(f)) = \text{sgn}(\sigma) \Phi(f) \).

As a consequence, we have

\[
\Phi \left( \prod_{i=0}^{d-1} t_i^{-(d-1-i)} f(\xi^i) \right) = (-1)^{d(d-1)/2} \Phi \left( \prod_{i=0}^{d-1} t_i^{-i} f(\xi^i) \right)
\]

for a symmetric function \( f(\xi^i) \).

(2) For a Schur polynomial \( s_\lambda(\mathcal{L}) \) and a symmetric function \( f(\xi^i) \), we have

\[
\Phi \left( \prod_{i=0}^{d-1} t_i^{-i} f(\xi^i) s_\lambda(\mathcal{L}) \right) = \Phi \left( \prod_{i=0}^{d-1} t_i^{-i+\lambda_{i+1}} f(\xi^i) s_\lambda(\mathcal{L}) \right).
\]

Here the Schur polynomial \( s_\lambda(\mathcal{L}) \) in \( \mathcal{L} = (t_0, \ldots, t_{d-1}) \) for a partition \( \lambda = (\lambda_1, \ldots, \lambda_d) \) is the polynomial defined by

\[
s_\lambda(\mathcal{L}) := \frac{\det[t_{ij}^{\lambda_i+j-d-i}]}{\det[t_{ij}^{d-i}]} = \frac{\det[t_{ij}^{\lambda_i+j-d-i}]}{\Delta(\mathcal{L})},
\]

where \( 1 \leq i \leq d, 0 \leq j \leq d-1 \) (see, e.g., [3] 14.5 and A.9, [2] Chapter I, §3).

Proof. (1). The assertion is a direct consequence from the definition of \( \Phi \) and a property of \( \Delta(\mathcal{L}) \).
(2). Using (1), we have
\[
\Phi \left( \prod_{i=0}^{d-1} t_i^{-i} f(t) s_\lambda(t) \right) = \frac{1}{d!} \Phi \left( \prod_{i=0}^{d-1} t_i^{-(d-1)} f(t) s_\lambda(t) \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \prod_{i=0}^{d-1} t_{\sigma(i)}^{-i} \right)
\]
\[
= \frac{1}{d!} \Phi \left( \prod_{i=0}^{d-1} t_i^{-(d-1)} f(t) s_\lambda(t) \Delta(t) \right)
\]
\[
= \frac{1}{d!} \Phi \left( \prod_{i=0}^{d-1} t_i^{-(d-1)} f(t) \det[t_j^{\lambda_j+d-l}]_{1 \leq t \leq d, 0 \leq j \leq d-1} \right)
\]
\[
= \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \Phi \left( \prod_{i=0}^{d-1} t_i^{i+\lambda_{i+1}} f(t) \right) = \Phi \left( \prod_{i=0}^{d-1} t_i^{i+\lambda_{i+1}} f(t) \right). \quad \square
\]

To simplify the notation, for a finite set of integers \{a_i\}_{0 \leq i \leq d-1}, set
\[
\{a_i\}! := \prod_{0 \leq i \leq d-1} a_i!, \quad \Delta(a_i) := \prod_{0 \leq i < j \leq d-1} (a_i - a_j).
\]

Setting \(m! := \Gamma(m + 1)\) for \(m \in \mathbb{Z}\), we have \(1/m! = 0\) if \(m < 0\).

**Proposition 3.3** (Evaluation Formula). For \(k = (k_0, \ldots, k_{d-1}) \in \mathbb{Z}_{\geq 0}^d\), we have
\[
\Phi \left( \prod_{i=0}^{d-1} t_i^{k_i} \right) = \frac{(-1)^{d(d-1)/2} \Delta(k_i)}{\{k_i + d - 1\}!}.
\]

**Proof.** We have
\[
\Phi \left( \prod_{i=0}^{d-1} t_i^{k_i} \right) = \text{const}_t \left( \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \prod_{i=0}^{d-1} \left( t_i^{k_i + d - 1 - \sigma(i)} \exp \left( \frac{1}{t_i} \right) \right) \right)
\]
\[
= \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \prod_{i=0}^{d-1} \text{const}_t \left( t_i^{k_i + d - 1 - \sigma(i)} \exp \left( \frac{1}{t_i} \right) \right)
\]
\[
= \sum_{\sigma \in \mathfrak{S}_d} \frac{\text{sgn}(\sigma)}{\{k_i + d - 1 - \sigma(i)\}!} = \text{det} \left[ \frac{1}{(k_i + d - 1 - j)!} \right]_{0 \leq i, j \leq d-1}
\]
\[
= \frac{(-1)^{d(d-1)/2} \Delta(k_i)}{\{k_i + d - 1\}!}.
\]

The last equality follows from the lemma below. \(\square\)

**Lemma 3.4** ([3, Example A.9.3]).
\[
\text{det} \left[ \frac{1}{(x_i + j)!} \right]_{0 \leq i, j \leq d-1} = \frac{\Delta(x_i)}{\{x_i + d - 1\}!}.
\]

**Proof of Theorem 0.1**. By Proposition 2.5 and Lemma 3.2, with \(A := A^*(X) \otimes \mathbb{Q}\), we have
\[
(3.1) \quad \pi_* \text{ch}(\det \mathcal{Q}) = \Phi \left( \prod_{i=0}^{d-1} t_i^{-(d-1)} \prod_{i=0}^{d-1} (t_i^{r-d} s(E, t_i)) \right)
\]
\[
= (-1)^{d(d-1)/2} \Phi \left( \prod_{i=0}^{d-1} t_i^{-i} \prod_{i=0}^{d-1} (t_i^{r-d} s(E, t_i)) \right).
\]
Since
\[ \prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \sum_{k \geq 0} \prod_{i=0}^{d-1} s_{k_i}(\mathcal{E}) t_i^{k_i}, \]
it follows from Proposition 3.3 that the most right-hand side of (3.1) is equal to
\[ (-1)^{d(d-1)/2} \sum_k \Phi \left( \prod_{i=0}^{d-1} t_i^{r_i-d-i} \prod_{i=0}^{d-1} s_{k_i}(\mathcal{E}) \right) = \sum_k \frac{\Delta(k_i-i)}{\{r+k_i-i-1\}} \prod_{i=0}^{d-1} s_{k_i}(\mathcal{E}), \]
where \( k = (k_0, \ldots, k_{d-1}) \in \mathbb{Z}_{\geq 0}^d \). Thus we obtain the conclusion. \( \square \)

Proof of Corollary 0.2. By the assumption \( \mathbb{G}_X(d, \mathcal{E}) \) is projective and the tautological line bundle \( \mathcal{O}_{\mathbb{F}_X(\wedge^d \mathcal{E})}(1) \) defines an embedding \( \mathbb{P}_X(\wedge^d \mathcal{E}) \hookrightarrow \mathbb{P}(H^0(X, \wedge^d \mathcal{E})) \). Therefore \( \mathbb{G}_X(d, \mathcal{E}) \) is considered to be a projective variety in \( \mathbb{P}(H^0(X, \wedge^d \mathcal{E})) \) via the relative Plücker embedding \( \mathbb{G}_X(d, \mathcal{E}) \hookrightarrow \mathbb{P}_X(\wedge^d \mathcal{E}) \) over \( X \) defined by the quotient \( \wedge^d \pi^* \mathcal{E} \to \wedge^d \mathcal{Q} = \det \mathcal{Q} \). Since the hyperplane section class of \( \mathbb{G}_X(d, \mathcal{E}) \) is equal to the Plücker class \( \theta \), we obtain the conclusion, taking the degree of the equality in Theorem 0.1. \( \square \)

Proof of Theorem 0.3. By Lemmas 3.5 below, 3.2 (2) and Proposition 3.3 the most right-hand side of (3.1) is equal to
\[ (-1)^{d(d-1)/2} \sum_{\lambda} \Phi \left( \prod_{i=0}^{d-1} t_i^{r_i-d-i} s_{\lambda}(t) \right) \Delta(\mathcal{E}) \]
\[ = (-1)^{d(d-1)/2} \sum_{\lambda} \frac{\Delta(r-d-i+\lambda_{i+1})}{\{r-d-i+\lambda_{i+1}+(d-1)!\}} \Delta(\mathcal{E}) \]
\[ = \sum_{\lambda} \frac{\Delta(\lambda_{i+1}-(i+1))}{\lambda_{i+1}+r-(i+1)!} \Delta(s(\mathcal{E})) = \sum_{\lambda} \frac{f_{\lambda+i}}{\lambda+i!} \Delta(\mathcal{E}). \]
\( \square \)

Lemma 3.5.
\[ \prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \sum_{\lambda} \Delta(\mathcal{E}) s_{\lambda}(t). \]

Proof. Using Cauchy identity [9, Chapter I, (4.3)] and Jacobi-Trudi identity [3, Lemma A.9.3], we have
\[ \prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \prod_{i=0}^{d-1} c(\mathcal{E}, -t_i) = \prod_{i=0}^{d-1} \prod_{j=1}^{r} \frac{1}{1-\alpha_j t_i} = \sum_{\lambda} s(\alpha) s_{\lambda}(t) = \sum_{\lambda} \Delta(\mathcal{E}) s_{\lambda}(t), \]
where \( \alpha = \{\alpha_1, \ldots, \alpha_r\} \) are the Chern roots of the vector bundle \( \mathcal{E} \). \( \square \)

4. Appendix: A generalization of Cauchy Determinant Formula

Consider a polynomial ring \( R_1 := A[\xi_0, \ldots, \xi_{r-1}] \) with \( r \) variables over a \( \mathbb{Q} \)-algebra \( A \). Denote by \( c_{ij}^r \) the \( i \)-th elementary symmetric polynomial in \( \xi_d, \ldots, \xi_{r-1} \), and by \( c_i \) the \( i \)-th elementary symmetric polynomial in \( \xi_0, \ldots, \xi_{r-1} \). We define the Segre series \( s(t) \) by
\[ s(t) := \frac{1}{\prod_{i=0}^{r-1}(1-\xi_i t)}. \]
Set \( R_2 := A[\xi_0, \ldots, \xi_{d-1}, \xi''_l, \ldots, \xi''_{r-d}] \), and \( R_3 := A[c_1, \ldots, c_r] \). Then, \( R_1 \supset R_2 \supset R_3 \), and \( R_1 \) (resp. \( R_2 \)) is a free \( R_3 \)-module generated by \( \{\xi''_0, \ldots, \xi''_{r-1}\} \) (resp. \( \{\xi''_0, \ldots, \xi''_{d-1}\} \)), where \( 0 \leq i_l \leq r - l - 1 \) (see, e.g., [11 Chapitre 4, §6], [11 §§2–3]). In particular, we have a decomposition,

\[
R_2 = \bigoplus_{0 \leq i_l \leq r - l - 1 \atop 0 \leq i \leq d - 1} R_3 \cdot \xi''_0 \xi''_1 \cdots \xi''_{d-1}.
\]

For \( \alpha \in R_2 \), we denote by \( \text{coeff}_\xi(\alpha) \) the coefficient of \( \alpha \) in \( \xi''_0 \cdots \xi''_{d-1} \) with respect to the decomposition (4.1).

Let \( \mathcal{A} \) (resp. \( \mathcal{A}', \mathcal{A}'' \)) be the anti-symmetrizer for variables \( \{\xi_0, \ldots, \xi_{r-1}\} \) (resp. \( \{\xi_0, \ldots, \xi_{d-1}\}; \{\xi_{d}, \ldots, \xi_{r-1}\} \)), that is, \( \mathcal{A}(\alpha) := \sum_{\sigma \in S_r} \text{sgn}(\sigma) \sigma(\alpha) \) (\( \alpha \in R_1 \)), for instance.

**Proposition 4.1** (Generalization of Cauchy Determinant Formula). We have an equality

\[
\mathcal{A}\left( \frac{\Delta(\xi_0, \ldots, \xi_{d-1}) \Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i, j \leq d-1} (\tau_j - \xi_i)} \right) = \frac{\Delta(\xi_0, \ldots, \xi_{r-1})}{\prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1} (\tau_j - \xi_i)}. \tag{4.1}
\]

By setting \( \tau_i := \frac{1}{t_i} \), we have

\[
\mathcal{A}\left( \frac{\Delta(\xi_0, \ldots, \xi_{d-1}) \Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i, j \leq d-1} (1 - \xi_i t_j)} \right) = \frac{\Delta(\xi_0, \ldots, \xi_{r-1}) \prod_{i=0}^{d-1} t_i^{r-d} \prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1} (1 - \xi_i t_j)}{\prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1} (1 - \xi_i \tau_j)}. \tag{4.2}
\]

Proof. The fractional expression,

\[
\mathcal{A}\left( \frac{\Delta(\xi_0, \ldots, \xi_{d-1}) \Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i, j \leq d-1} (\tau_j - \xi_i)} \right) \prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1} (\tau_j - \xi_i)
\]

is actually a homogeneous polynomial in the variables, \( \xi_0, \ldots, \xi_{r-1}; \tau_0, \ldots, \tau_{r-1} \), with degree \( d(d-1)/2 + (r-d)(r-d-1)/2 - d^2 + rd = r(r-1)/2 \), and anti-symmetric with respect to the \( \xi_i \). Therefore it is a multiple of \( \Delta(\xi_0, \ldots, \xi_{r-1}) \). By comparing the coefficient of \( \xi''_0 \cdots \xi''_{r-1} \), we see that those polynomials are equal to each other, and we obtain the first equality. The second equality follows from the first one. \( \square \)

**Another Proof of Lemma 2.2.** Let \( G(t) \) be the generating function of \( \text{coeff}_\xi(\xi''_0 \cdots \xi''_{d-1}) \), that is,

\[
G(t) := \sum_{p_0, \ldots, p_{d-1} \geq 0} \text{coeff}_\xi(\xi''_0 \cdots \xi''_{d-1}) t_0^{p_0} \cdots t_{d-1}^{p_{d-1}}.
\]

For \( 0 \leq i_l \leq r - l - 1 \), we have

\[
\mathcal{A}(\xi''_0 \cdots \xi''_{r-1}) = \begin{cases} 
\Delta(\xi_0, \ldots, \xi_{r-1}), & (i_0, \ldots, i_{r-1}) = (r-1, \ldots, 0), \\
0, & (i_0, \ldots, i_{r-1}) \neq (r-1, \ldots, 0).
\end{cases}
\]

Since \( \mathcal{A} \) is \( R_3 \)-linear, we have an equality

\[
\mathcal{A}(\alpha \cdot \xi''_{d-r} \cdots \xi''_0) = \text{coeff}_\xi(\alpha) \Delta(\xi_0, \ldots, \xi_{r-1})
\]
4.1, we see that
\[ \Delta(\xi_0, \ldots, \xi_{r-1})G(t) = \sum_{p_0, \ldots, p_{d-1} \geq 0} \mathcal{A}(\xi_0^{p_0}, \ldots, \xi_{d-1}^{p_{d-1}}, \xi_r^{r-d-1}, \ldots, \xi_{r-1}^{0}) t_0^{p_0} \cdots t_{d-1}^{p_{d-1}} \]
\[ = \mathcal{A}\left(\frac{\xi_r^{r-d-1} \cdot \xi_{r-1}^0}{(1 - \xi_0 t_0) \cdots (1 - \xi_{d-1} t_{d-1})}\right) \]
\[ = \mathcal{A}\left(\mathcal{A}'(1 - \xi_0 t_0) \cdots (1 - \xi_{d-1} t_{d-1})\right) \mathcal{A}''(\xi_r^{r-d-1} \cdot \xi_{r-1}^0) \]
\[ = \mathcal{A}\left(\frac{\Delta(t_0, \ldots, t_{d-1}) \Delta(\xi_0, \ldots, \xi_{d-1}) \Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i, j \leq d-1} (1 - \xi_i t_j)}\right) \]
\[ = \mathcal{A}\left(\frac{\Delta(\xi_0, \ldots, \xi_{d-1}) \Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i, j \leq d-1} (1 - \xi_i t_j)}\right) \Delta(t_0, \ldots, t_{d-1}). \]

Here we used the equality,
\[ \mathcal{A}(f(\xi_0, \ldots, \xi_{d-1}) g(\xi_d, \ldots, \xi_{r-1})) = \mathcal{A}(\mathcal{A}'(f(\xi_0, \ldots, \xi_{d-1})) \mathcal{A}''(g(\xi_d, \ldots, \xi_{r-1}))) \]
and Cauchy determinant formula ([9 p.67, I.4, Example 6]). Finally, using Proposition 4.1 we see that
\[ G(t) = \frac{\Delta(t_0, \ldots, t_{d-1}) \prod_{i=0}^{d-1} t_i^{r-d}}{\prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1} (1 - \xi_i t_j)} \]
and this proves Lemma 2.2 with \( R_1 := A^*(X) \) and \( R_2 := A^*(\mathbb{F}_Y^d \chi(E)) = A^*(\mathbb{F}_G^d \chi(Q)). \)

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