Gravitational waves from instabilities in relativistic stars

Nils Andersson
Department of Mathematics, University of Southampton, Southampton SO17 1BJ, UK

Abstract. This article provides an overview of stellar instabilities as sources of gravitational waves. The aim is to put recent work on secular and dynamical instabilities in compact stars in context, and to summarize the current thinking about the detectability of gravitational waves from various scenarios. As a new generation of kilometer length interferometric detectors are now coming online this is a highly topical theme. The review is motivated by two key questions for future gravitational-wave astronomy: Are the gravitational waves from various instabilities detectable? If so, what can these gravitational-wave signals teach us about neutron star physics? Even though we may not have clear answers to these questions, recent studies of the dynamical bar-mode instability and the secular r-mode instability have provided new insights into many of the difficult issues involved in modelling unstable stars as gravitational-wave sources.

1. Introduction

Neutron stars may suffer a number of instabilities. These instabilities come in different flavours, but they have one general feature in common: They can be directly associated with unstable modes of oscillation. A study of the stability properties of a relativistic star is closely related to an investigation of the star’s various pulsation modes. Furthermore, non-axisymmetric stellar oscillations will inevitably lead to the production of gravitational radiation. Should these waves turn out to be detectable, they would provide a fingerprint that could be used to put constraints on the interior structure of the star [1]. This would be analogous to the recent success story of helioseismology, where the detailed spectrum of solar oscillation modes has been matched to theoretical models of the interior to provide insights into, for example, the sound speed at different depths in the Sun. In order for “gravitational-wave asteroseismology” to be a realistic proposition, one must find scenarios which lead to a star pulsating wildly. The most obvious situation where this may be the case is when a newly born neutron star settles down after the supernova collapse. Other promising possibilities are associated with instabilities. As an unstable pulsation mode grows it may reach a sufficiently large amplitude that the emerging gravitational waves can be detected.
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The aim of this article is to provide an overview of instabilities that may lead to detectable gravitational waves. This is a topical theme given that several large-scale interferometers (LIGO, GEO600, VIRGO and TAMA300) will be up and running in the near future. My intention is to put the current thinking about various stellar instabilities in context, and provide a foundation for future research in this area by summarizing what we currently know and, perhaps more importantly, what the key issues that need further attention are. This review is, however, by no means exhaustive. It does not provide a “complete” set of references to the literature or a detailed discussion of every possible instability that may occur in a relativistic star. I have simply focussed on the issues that seem (to me) to be the most interesting/important. The reader is encouraged to use the provided bibliography as an entry point to the vast literature on instabilities and other gravitational-wave sources [2, 3, 4, 5, 6], explore it for him/herself, and perhaps even draw conclusions that differ from mine.

Neutron stars are tremendously complicated objects. In essence, their modelling requires a detailed understanding of the very extremes of physics, including supranuclear physics, general relativity, superfluidity/superconductivity, strong magnetic fields, exotic particle physics etcetera [7]. To investigate the pulsation properties of any “realistic” neutron star model is therefore a serious challenge. While our understanding of the modes of a simple self-gravitating “ball of fluid” may be good, a significant effort is still required if we want to model the dynamics of astrophysical compact stars.

The instabilities that I will discuss fall into two main categories: They are either dynamical or secular. In the first case the unstable modes grow on a timescale similar to that of the oscillations, while in the second the growth takes place on a much longer timescale (eg. that associated with viscosity). A dynamical instability is likely to have dramatic effects on an equilibrium configuration, while a secular instability acts in a more subtle way. As we will see, this means that dynamical instabilities can be studied directly using fully nonlinear hydrodynamical simulations. The effects of a secular instability are difficult to explore in this way since the evolutionary timescale tends to be much longer than the dynamical timescale of the system. Hence, secular instability evolutions have so far mainly been discussed at a phenomenological level.

Most of the discussion in this article concerns rotating stars. This is natural since instabilities in spinning stars are thought to be particularly promising gravitational-wave sources. A question of key astrophysical importance concerns whether instabilities may limit the spin of a compact star to a rate significantly lower than the mass-shedding limit. This limit is reached when the equator rotates at the Kepler frequency of a particle in circular orbit around the star. It is well approximated by a rotation frequency

\[ \Omega_K \approx \frac{2}{3} \sqrt{\frac{\pi G}{\rho_0}} \]

where \( \rho_0 \) is the average density of the corresponding non-rotating star. This is known to be a good approximation for rigidly rotating Newtonian polytropes [8], and it remains reasonably accurate also for relativistic models [9]. However, we should keep in mind that the mass-shedding limit may be significantly different in the case of differentially
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rotating stars. In fact, the rotation frequency $\Omega$ is not a particularly useful parameter for differentially rotating configurations (see [10]). Instead, the ratio $\beta$ between the kinetic energy $T$ and the gravitational potential energy $W$ is often used in discussions of rotational instabilities. The values of $\beta$ are restricted (by the virial theorem) to the range $0 - 0.5$. For a uniformly rotating constant density star one can estimate that

$$ \beta = \frac{T}{|W|} \approx \frac{1}{9} \left( \frac{\Omega}{\Omega_K} \right)^2 $$

(2)

In other words, the mass shedding limit would correspond to $\beta_K \approx 0.11$. This simple approximation agrees quite well with detailed calculations for realistic supranuclear equations of state†, which typically indicate a maximum value of $\beta$ in the range 0.09-0.13. Depending on the rotation law, differentially rotating models may allow considerably larger values of $\beta$ (maybe as large as $\beta \approx 0.3$). Large maximum values of $\beta$ can also be reached for strange stars. Such self-bound objects may have $\beta$ significantly larger than 0.2 [11].

2. Gravitational-wave estimates

Among astrophysicists General Relativity is sometimes viewed as a “correction” (expected to alter results quantitatively at the 15-20% level). This attitude makes some sense since many astrophysical results remain essentially unaltered by a fully relativistic description. On the other hand, there are situations where General Relativity is a “leading order effect” which must be incorporated. The dynamics of compact stars provides an excellent example of this. It is well known that it is meaningless (or at least highly dubious) to use a realistic supranuclear equation of state in a Newtonian calculation of neutron star structure. The mass and radius of the resultant star may differ greatly from the results obtained from the Tolman-Oppenheimer-Volkoff equations (for a given central density), and in order to avoid confusion one should always use relativistic models for “realistic” neutron stars. In addition, gravitational waves are a purely relativistic phenomenon. So if we are interested in neutron stars as gravitational-wave sources it stands to reason that we should aim to build fully relativistic models. However, this is a far from simple task. Since much of the physics required for a realistic model of neutron star dynamics is poorly understood we are in practice often forced to work with Newtonian models, estimating gravitational waves via post-Newtonian formulas.

If we focus our attention on a single pulsation mode of a rotating Newtonian star then, assuming that the density and velocity perturbations, $\delta \rho$ and $\delta \vec{v}$, depend on time as $\exp(i \omega_r t)$, the associated gravitational-wave luminosity (measured in the rotating frame) can be estimated using [12]

$$ \frac{dE}{dt} = -\omega_r \sum_{l=2}^{\infty} N_l \omega_l^{2l+1} \left( |\delta D_{lm}|^2 + |\delta J_{lm}|^2 \right) , $$

(3)

† For a detailed review of various methods used to construct stationary rotating stellar models in General Relativity, see [9].
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where \( \omega_i \) (\( \omega_r \)) is the mode-frequency in the inertial (rotating) frame, and

\[
N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)(2l+1)!!} .
\]

(4)

The first term in the bracket of (3) represents radiation due to the mass multipoles. These are determined by

\[
\delta D_{lm} = \int \delta \rho r^l Y_{lm}^* dV .
\]

(5)

(where the asterisk represents complex conjugation). The second term in the bracket of (3) corresponds to the current multipoles, which follow from

\[
\delta J_{lm} = \frac{2}{c} \sqrt{\frac{l}{l+1}} \int r^l (\rho \delta \vec{v} + \delta \rho \vec{\omega}) \cdot \vec{Y}_{B\,lm}^* dV ,
\]

(6)

where

\[
\vec{Y}_{B\,lm} = \frac{1}{\sqrt{l(l+1)}} \hat{r} \times \nabla Y_l^m
\]

are the magnetic multipoles [12].

From the above formulas we can draw some general conclusions. First of all it is clear that any fluid motion that leads to significant density variations will radiate gravitationally predominantly through the mass multipoles. This follows from the fact that \( |\delta J|^2 \sim |\delta D|^2 / c^2 \) which means that the current multipole radiation is generally “one order higher” in the post-Newtonian approximation. However, there are situations where this standard consensus no longer holds and the current multipoles provide the main radiation mechanism. Most notably, this is the case for the unstable r-modes which are characterized by a large \( \delta \vec{v} \) and a small \( \delta \rho \).

In the relativistic case, the calculation of gravitational-wave emission from stellar pulsation is conceptually more straightforward. Because any non-axisymmetric fluid motion will lead to radiation, the various modes are distinguished by imposing outgoing-wave boundary conditions at infinity. This means that they are no longer normal modes of the system. Gravitational-wave dissipation leads to the mode-frequencies becoming complex (with the imaginary part representing the damping/growth due to the emitted radiation). As is well-known from studies of perturbed black holes the numerical determination of such “quasinormal modes” is not trivial. Several reliable methods for handling this difficulty for spherical stars have been developed [13], but a solution to the problem for rapidly rotating stars is still outstanding.

Once we have some idea of the character of a given gravitational-wave source it is relevant to try to estimate the detectability of the signal. The obtained estimates are often rough, but they provide useful guidelines for further work, both on the theoretical modelling side and the experimental side. If initial estimates make it seem plausible

\[\frac{\delta \rho}{c} \approx \frac{\delta \omega}{c} \approx \frac{1}{c^5} \]

we see that quadrupole perturbations \( l = 2 \) will lead to \( \dot{E} \sim 1/c^5 \) for the mass multipoles, while \( \dot{E} \sim 1/c^7 \) for the current multipole radiation. Using the standard way of counting orders, this means that the mass multipole radiation arises at 2.5 post-Newtonian order while the current multipole radiation is a 3.5 post-Newtonian effect.

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Table 1. Canonical parameter values used in various formulas in the article.

| Parameter | Value |
|-----------|-------|
| $M_{1.4}$ | $M/1.4M_\odot$ | mass |
| $R_{10}$  | $R/10\text{ km}$ | radius |
| $P_{-3}$  | $P/1\text{ ms}$ | rotation period |
| $T_9$     | $T/10^9\text{ K}$ | core temperature |
| $\rho_{15}$ | $\rho/10^{15}\text{ g/cm}^3$ | density |
| $f_{Hz}$  | $f/1\text{ Hz}$ | gravitational-wave frequency |
| $D_{15}$  | $D/15\text{ Mpc}$ | distance to source |

that the source can be detected then a detailed data analysis strategy needs to be developed. In the ideal case this strategy would be based on matched filtering, where an accurate theoretical template is used to search for a real signal in the noisy data stream. However, in many cases our understanding of the relevant physics is not at the level where we can expect to provide reliable theoretical models. This is certainly the case for neutron stars, where crucial pieces of physics like the equation of state are only known to within a factor of two or so. The detectability of a signal can nevertheless be improved considerably given only overall characteristics, like the duration and the frequency range. Any information we can extract from the theoretical models, even if they are rudimentary, could be valuable!

In order to estimate the strength of a signal, we can use the well-known flux formula which relates the luminosity to the gravitational-wave strain $h$:

$$\frac{c^3}{16\pi G} |\dot{h}|^2 = \frac{1}{4\pi D^2} \frac{dE}{dt}$$

(8)

where $D$ is the distance to the source. This relation is exact for the weak waves that bathe the Earth. To proceed we characterize the event by a timescale $\tau$ and assume that the signal is essentially monochromatic. Then we can use $\dot{h} \approx 2\pi f h$, and readily deduce that

$$h \approx 5 \times 10^{-22} \left( \frac{E}{10^{-3}M_\odot c^2} \right)^{1/2} \left( \frac{\tau}{1\text{ ms}} \right)^{-1/2} \frac{f_{Hz}^{-1} D_{15}^{-1}}{f_{Hz}}$$

(9)

Here we have taken the distance to be that of a source in the Virgo cluster. This is necessary to ensure a reasonable event rate for most astrophysical scenarios. At that distance one would expect to see many supernovae per year, which means that one can hope to see a few neutron stars being born during one year of observation.

We can proceed further and estimate an “effective amplitude” that reflects the fact that detailed knowledge of the signal can be used to dig deeper into the noise. A typical example is based on the use of matched filtering, for which the effective amplitude improves as the square root of the number of observed cycles $n$. Using $n \approx f\tau$ we arrive at

$$h_c \approx 5 \times 10^{-22} \left( \frac{E}{10^{-3}M_\odot c^2} \right)^{1/2} \left( \frac{f}{1\text{ kHz}} \right)^{-1/2} D_{15}^{-1}$$

(10)
From this relation we see that the “detector sensitivity” essentially depends only on the radiated energy and the characteristic frequency. Hence, an estimate of the total energy radiated and the frequency of the signal may be sufficient to assess the relevance of the event as a gravitational-wave source.

The above formulas can be applied to many interesting astrophysical scenarios, but there are obvious situations where they fail. One case, that will be discussed later, is when gravitational radiation reaction leads to a change in the character of the signal during the observation. This typically leads to a signal whose frequency varies with time. Provided that this variation is sufficiently slow one can use the method of stationary phase to show that

\[ h_c \approx h \sqrt{\int f^2 \frac{dt}{df}} \]  

(11)

For example, Owen et al [14] used this relation to assess the detectability of gravitational waves from unstable r-modes.

3. Stellar pulsation

In this Section I will describe the nature of those pulsation modes that are currently thought to be the most relevant from the gravitational-wave point of view. The main focus will be on the acoustic f-modes and the inertial r-modes, but I will also briefly mention the gravitational-wave w-modes. For more details, the reader is referred to standard textbooks [10, 15] and various review articles [16, 17, 18].

3.1. The equation of state

In order to close the system of equations that describe linear oscillations of a star it is necessary to provide an equation of state for matter. This problem is extremely difficult — at least if we expect to find the “correct” answer. The proposed equations of state for supranuclear matter come in many varieties and lead to equilibrium neutron stars whose bulk properties vary by up to factors of two. The available predictions also differ considerably on issues concerning the nature of matter at extreme densities. Many important questions remain to be resolved. Will a neutron star have an exotic core containing deconfined quarks and/or hyperons? Are there strange matter stars? At what density and temperature do various constituents become superfluid/superconducting?

As we will discuss later, the answers to these questions are crucial for an understanding of the stability properties of relativistic stars. At the time of writing, the permissible parameter space is enormous.

Detailed nuclear physics calculations typically provide a tabulated equation of state relating the pressure to the density for matter in beta equilibrium. This provides sufficient information for us to construct eg. rotating equilibria, and thus model unperturbed neutron stars. In these models matter is usually modelled as a perfect fluid. The temperature is typically (unless one considers newly born neutron stars) taken to
be zero, because even though neutron stars may seem hot on a “normal physics scale”, with interior temperatures in the range $10^6 - 10^8$ K, the thermal energy is considerably smaller than the Fermi energy of the fluid, $T_{\text{Fermi}} \sim 10^{12}$ K. As a consequence a one-parameter equation of state $p = f(\rho)$ is often sufficient to describe the matter. In this article stars constructed from such equations of state are referred to as “barotropic”. A simple yet reasonable class of equations of state are the polytropes

$$p = \kappa \rho^\Gamma, \quad \text{with} \quad \Gamma = 1 + \frac{1}{n}$$

(12)

where $n$ is the polytropic index. A comparison with proposed realistic equations of state suggests that $\Gamma \sim 2$, and hence model calculations are often carried out for $n = 1$ polytropes.

More detailed models account for the dependence on additional parameters, the most important of which may be the relative number density of protons ($x_p = \rho_p/\rho$) and exotic particles like hyperons and/or deconfined quarks. Stars described by such multi-parameter equations of state, $p = f(\rho, x_p, ...)$, are “non-barotropic” and have internal stratification. This stratification may affect the pulsation properties of the star significantly. For example, internal stratification associated with chemical composition gradients leads to the presence of distinct gravity g-modes.

Since we want to study oscillating stars we face a further challenge. The mode motion will inevitably force the fluid out of equilibrium, which means that the available tabulated equations of state only provide partial information. To address the pulsation problem we need to know the “exact” equation of state in order to relate the various (Eulerian) perturbations, eg.

$$\delta p = \frac{\partial p}{\partial \rho} \bigg|_{\rho,...} \delta \rho + \frac{\partial p}{\partial x_p} \bigg|_{\rho,...} \delta x_p + ...$$

(13)

This requires the knowledge of physics that is poorly understood. Hence it is customary to approach the problem in the following way: We simply allow the perturbations to be governed by an equation of state that differs from that which describes the background configuration. To do this we relate the Lagrangian pressure and density variations by

$$\Delta p = \frac{\Gamma_1 p}{\rho} \Delta \rho$$

(14)

where $\Gamma_1$ need not be equal to $\Gamma$. From this definition it immediately follows that the Eulerian variations are related by

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta \rho}{\rho} + \Gamma_1 \xi \cdot \left[ \nabla \log \rho - \frac{1}{\Gamma_1} \nabla \log p \right] = \Gamma_1 \frac{\delta \rho}{\rho} + \frac{\xi}{\rho} \cdot \nabla \rho (\Gamma_1 - \Gamma)$$

(15)

where $\xi$ is the Lagrangian displacement vector, i.e. $\delta \vec{v} = \partial_t \vec{\xi}$, and the last equality only holds for polytropes. Here it is customary to introduce the so-called Schwarzschild discriminant $A_s$. For a spherical stellar model where $p$ and $\rho$ depend only on the radial coordinate $r$ we have

$$A_s = \frac{1}{\rho} \frac{d\rho}{dr} - \frac{1}{\Gamma_1 p} \frac{dp}{dr} = \frac{\Gamma_1 - \Gamma}{\Gamma_1} \frac{1}{\rho} \frac{d\rho}{dr}$$

(16)
The key point of this approach is that one can easily account for the buoyancy due to internal stratification (whatever the physical reasons for its presence may be) by considering models with a non-zero Schwarzschild discriminant. This way we can parameterize our ignorance concerning the detailed physics involved in real stellar pulsations.

3.2. f- and r-modes

A realistic neutron star model has a large number of families of pulsation modes with more or less distinct character. In principle, one expects each physical “restoring force” that acts on a fluid element to lead to the presence of a new family of pulsation modes. An exhaustive description of the all these various modes goes far beyond the scope of the present article. Here we are mainly interested in modes that may become unstable and lead to a significant gravitational-wave signal. The modes that are currently thought to be the most important in this respect are the acoustic f-modes and the Coriolis restored r-modes.

In addition to being associated with different restoring forces, stellar pulsation modes can be classified by the way that the perturbations transform under parity. This is a useful classification since the overall parity of a perturbation is preserved when rotation is present. This means that one can follow the changes in a particular parity mode as $\Omega$ is varied, and meaningfully label the mode by its nature in the non-rotating limit (e.g. the spherical harmonic indices $l$ and $m$ used to describe the angular dependence and the number of radial nodes of the mode-eigenfunction).

A general velocity field can be represented by

$$r \delta \vec{v} = \sum_{lm} \{ W_l^m \vec{r} Y_l^m + V_l^m r \nabla Y_l^m - i U_l^m \vec{r} \times \nabla Y_l^m \}$$  \hspace{1cm} (17)

where $Y_l^m(\theta, \phi)$ are the standard spherical harmonics. The first two terms in the sum transform as $(-1)^l$ under parity, while the last term changes as $(-1)^{l+1}$. These two classes of perturbations are often (in relativity) referred to as polar and axial $\parallel$, respectively. This nomenclature was introduced by Chandrasekhar for perturbed black holes [20]. In a non-rotating star the various multipoles decouple, but in the rotating case they are coupled in such a way that the overall parity of the mode is preserved. That is, polar terms corresponding to even values of $l$ are coupled to axial terms for odd $l$, and vice versa.

The f-mode, which can be viewed as the fundamental pressure mode of the star, is present already in a non-rotating star. It corresponds to polar perturbations and, as was first shown by Kelvin, for a non-rotating uniform density star its frequency is given by

$$\omega^2 = \frac{2l(l - 1) GM}{2l + 1} \approx 1.5 \times 10^8 \frac{2l(l - 1)}{2l + 1} M_{1.4} R_{10}^3 \text{ s}^{-2}$$  \hspace{1cm} (18)

$\parallel$ In Newtonian stellar pulsation studies the two classes of modes are often called “spheroidal” and “toroidal”.
This is a reasonable approximation also for more realistic equations of state, and we can readily deduce that a typical neutron star has f-mode frequency \((f = \omega_r/2\pi)\) in the range 2–4 kHz (for \(l = 2\)). Mode frequencies and gravitational-wave damping rates for non-rotating stars constructed for many realistic equations of state are illustrated in Figure 1, cf. [21].

Because of the symmetry of the non-rotating problem, modes corresponding to different \(l\) and \(m\) decouple. In fact, it is sufficient to consider the \(m = 0\) case. Rotation complicates the problem considerably. First of all, it breaks the symmetry in such a way that the various \(-l \leq m \leq l\) modes become distinct. As a first approximation one finds that the f-mode frequencies change as

\[
\omega_i(\Omega) = \omega_r(\Omega = 0) + C_{lm}(\Omega) - m\Omega + O(\Omega^2)
\]

(19)

according to an inertial observer. Here, \(\omega_r(\Omega = 0)\) is the frequency of the mode in the non-rotating case, cf. [18], and \(C_{lm}\) is a function that depends on the mode-eigenfunction in a non-rotating star. Rotation also couples the various multipoles, which means that an increasing number of \(Y_{l^m}\)’s are needed to describe a mode as the rotation rate is increased. One must also account for coupling between the polar and axial vectors. The problem is further complicated by the rotationally induced change in shape of the star, which first contributes at \(O(\Omega^2)\) in the slow-rotation expansion.

As we will see later, it is often useful to consider the “pattern speed” of the mode. Every mode of an axially symmetric system can be assumed to be proportional to \(e^{i(m\phi + \omega t)}\), and surfaces of constant phase are therefore given by

\[
m\phi + \omega t = \text{constant}
\]

(20)
After differentiation this leads to
\[ \frac{d\phi}{dt} = -\frac{\omega}{m} = \sigma \] (21)
which defines the pattern speed \( \sigma \) of the mode. Having defined this quantity it is worth making two observations concerning the \( (l = m) \) f-modes. 1) From (18) we see that the frequency of these modes increases with \( m \) roughly as \( \omega_r \sim \sqrt{m} \). According to (21) this means that the pattern speed of the f-modes decreases as we increase \( m \). As a consequence one can always find an f-mode with arbitrarily small pattern speed (corresponding to a suitably large value of \( m \)) even though the high order f-modes have increasingly large frequencies. 2) We can also see that mode patterns corresponding to opposite signs of \( m \) tend to rotate around the star in different directions. Taking the positive direction to be that associated with \( \Omega \) we find that the \( l = \pm m \) modes are backwards and forwards moving (retro/prograde), respectively, in the limit of vanishing rotation. However, rotation may change the situation for the \( l = m \) f-modes. (The particular case of the \( m = \pm 2 \) modes of a Maclaurin spheroid is illustrated in Figure 5.) Combining (18) with (19) — where we neglect \( C_{lm} \) for simplicity — we can estimate that these modes becomes prograde for rotation rates above
\[ \Omega_s \approx \sqrt{\frac{3}{m}} \Omega_K \quad \text{or} \quad \beta_s \approx \frac{1}{3m} . \] (22)
In other words, all but the \( l = m = 2 \) f-modes are likely to change from backwards to forwards moving (according to an inertial observer) at realistic rates of rotation \( (\beta < \beta_K) \). This result is brought out clearly by more accurate calculations. Using their two-potential formalism, Ipser and Lindblom [8, 22] have calculated the \( l = m \) f-modes for rapidly rotating Newtonian polytropes. Their results are reproduced in Figure 2. Similar results have been obtained by Yoshida and Eriguchi [23]. The numerical calculations show that the \( m = 3 \) mode changes from retro- to prograde motion at \( \beta \approx 0.08 \) while the critical value for the \( m = 4 \) mode is \( \beta \approx 0.06 \).

A non-rotating fluid star has no non-trivial axial modes (in Newtonian theory). This situation is altered by rotation, which leads to the presence of inertial modes whose dynamics is governed by the Coriolis force (see for example [24] for a detailed discussion). A general inertial mode is such that \( \delta \vec{v} \) is composed of a mixture of polar and axial components to leading order, i.e. one can take \( [W_l, V_l, U_l] \sim \Omega \). With this ordering, all inertial modes are such that \( [\delta p, \delta \rho] \sim \Omega^2 \). The r-modes are a special subclass of inertial modes that (at least in Newtonian theory) are purely axial to leading order, i.e. they have \( U_l \sim \Omega \) while \( [W_l, V_l] \sim \Omega^2 \). The r-mode frequencies are easy to determine from the fact that the radial component of the vorticity is conserved by their fluid motion, and one finds that [25]
\[ \omega_r \approx \frac{2m\Omega}{l(l+1)} . \] (23)
A non-barotropic star has an infinite number of r-modes for each \( l \) and \( m \neq 0 \) [26]. In contrast, only a single r-mode (for each \( l = m \)) exists in a barotropic star. This
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mode is associated with the simple fact that the star is a sphere and not a cylinder (which is relevant since the Coriolis operator has cylindrical symmetry). As a result, this mode is only weakly dependent on internal stratification. In the general case the radial dependency of the r-mode eigenfunctions remains undetermined at order $\Omega$, and the calculation must be taken to order $\Omega^2$. The barotropic case is, again, different in that one can determine that $U_l \propto r^{l+1}$ already at leading order \[26\].

The pattern speed for a typical $l = m$ r-mode is

$$\sigma_r = -\frac{2\Omega}{l(l+1)}$$

(24)

according to an observer rotating with the star. Meanwhile, an inertial observer would find

$$\sigma_i = \Omega \frac{(l-1)(l+2)}{l(l+1)} .$$

(25)

That is, although the modes appear retrograde in the rotating system an inertial observer would view them as prograde at all rotation rates.

Given the eigenfunctions for a pulsation mode of a Newtonian star, we can use the multipole formulas in Section 2 to estimate the rate at which the oscillation is damped by gravitational radiation emission. As far as the f-modes are concerned, they are associated with significant density variations and one can easily show that the main contribution to the gravitational-wave damping comes from the mass multipoles. Detweiler [27] has shown that, for uniform density stars, the gravitational-wave damping timescale can be estimated as

$$t_{gw} \approx \frac{2}{3} \frac{(l-1)(2l+1)!!}{(l+1)(l+2)} \left[ \frac{2l+1}{2l(l-1)} \right]^l \left( \frac{c^2 R}{G M} \right)^{l+1} \frac{R}{c}$$

(26)
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or \( t_{gw} \approx 0.07 M_{1.4}^{-3} R_{10}^4 \) s for the quadrupole modes. From this we see that the typical damping rate for the quadrupole f-mode will be of the order of a tenth of a second. Detailed results for a collection of realistic equations of state are given in Figure 2 of [21].

An order count based on the multipole formulas suggests that the r-modes are different. For \( l = m \) r-modes the dominant contribution to the gravitational radiation comes from the first term in the current multipole formula (5). That this is the case can be seen as follows. The \( l = m \) modes have axial displacement to leading order \( \sim \Omega \), while the density variation \( \delta \rho \) enters at order \( \Omega^2 \). Furthermore, we know that if the axial component corresponds to the \( l \)th multipole, then the polar components that arise from rotational coupling will correspond to \( l + 1 \). This means that we have \( \delta D_{l+1m} \sim \Omega^2 \) so \( \dot{E}_{\text{mass}} \sim \Omega^{2l+8} \), while \( \delta J_{lm} \sim \Omega \) leads to \( \dot{E}_{\text{current}} \sim \Omega^{2l+4} \). As a gravitational-wave source the r-modes are therefore quite unusual. The gravitational radiation that they emit comes primarily from the time-dependent mass currents, and is the gravitational analogue of magnetic multipole radiation.

3.3. Relativistic pulsations

Schematically, the relativistic problem consists of the perturbed Einstein equations

\[
\delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu}
\]

and the equations of motion

\[
\delta(T^\mu_{\nu,\mu}) = 0
\]

We need to solve (a subset of) these equations for the four-velocity associated with the mode and the perturbed spacetime metric \( \delta g_{\mu\nu} \). In this calculation, the fluid four velocity is decomposed in a way that is completely analogous to the Newtonian description, cf. (17), while the metric perturbations are represented by the corresponding tensor decomposition (see [12]). Just like in the Newtonian case, we can study axial and polar perturbations separately. The relativistic perturbation equations have been derived many times over, and explicit formulas are given in (for example) [28]. A nice gauge-invariant description can be found in [29]. The solutions that represent a pulsation mode satisfy the appropriate regularity conditions at the centre of the star and correspond to purely outgoing waves at infinity. As already mentioned, the imposition of this boundary condition presents a technical challenge [13]. Several reliable techniques have been developed to handle this difficulty, and the pulsations of non-rotating relativistic stars are by now well understood [21]. The spinning star problem still presents a challenge, however, and the problem of calculating oscillation modes of rapidly rotating neutron stars in General Relativity (including the imaginary parts of \( \omega \) associated with gravitational-wave damping) remains unsolved.

As we will see later, the neutral modes (representing the point where an originally retrograde mode becomes prograde) of a rotating star determine the onset of gravitational-wave driven instability. Consequently, one can focus attention on the
simplified problem of finding time-independent solutions to the perturbation equations. Such solutions would be marginally unstable. A numerical solution to this problem was obtained by Stergioulas and Friedman [30] for polytropic stars. Their results show that relativistic effects tend to destabilize a rotating star considerably. Interestingly, one finds that in General Relativity the \( m = 2 \) f-mode may have a neutral point for attainable rates of rotation. This result, which constrasts with the Newtonian case shown in Figure 3, has been shown to hold also for realistic equations of state [31]. An empirical fit to results for several realistic equations of state suggests that the \( m = 2 \) f-mode becomes secularly unstable at

\[
\beta_s \approx 0.115 - 0.048 \frac{M}{M_{\text{max}}(\Omega = 0)}
\]

where \( M_{\text{max}}(\Omega = 0) \) is the maximum allowed mass of a nonrotating star for the given equation of state. For a typical 1.4\( M_\odot \) star the \( m = 2 \) f-mode has a neutral point near \( \beta \approx 0.08 \) or \( \Omega \approx 0.85\Omega_K \). This result could be of considerable importance since mass quadrupole radiation is likely to lead to the fastest instability growth. One further step towards the calculation of f-modes of rotating relativistic stars was taken by Yoshida and Eriguchi [32, 33]. They solved the problem in the relativistic Cowling approximation, wherein one assumes that \( \delta g_{\mu\nu} = 0 \). The obtained results show that the Cowling approximation tends to overestimate the stability of the star, and therefore emphasize the conclusion that relativity has a destabilizing effect.

General Relativity also affects the r-modes in a significant way. This is perhaps not surprising since the relativistic framdragging is an order \( \Omega \) effect, which may affect inertial modes at “leading order”. Retaining only the leading order rotational effects, the metric of a stationary equilibrium can be written [34]

\[
ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2 - 2\omega(r)r^2\sin^2\theta dt d\varphi \quad (30)
\]

The frame-dragging is represented by the function \( \omega(r) \). (To avoid confusion with the mode-frequency \( \omega \) we will always retain the dependence on \( r \) in expressions involving the frame dragging.)

Although some details remain to be understood, significant progress has been made on the relativistic r-mode problem in the last couple of years. For barotropic relativistic stars one can prove that no purely axial inertial modes can exist [35]. All inertial modes of such stars are a hybrid mixture of axial and polar perturbations to leading order. For the particular modes that limit to the purely axial r-modes as \( M/R \to 0 \) one can show (for uniform density stars) that [35]

\[
\kappa = \frac{2}{(m+1)} \left[ 1 - \frac{4(m-1)(2m+11)}{5(2m+1)(2m+5)} \left( \frac{2M}{R} \right) + O \left( \frac{2M}{R} \right)^2 \right] \quad (31)
\]

where \( \kappa = (\omega + m\Omega)/\Omega \). Meanwhile, the velocity eigenfunctions take the form

\[
U_m(r) = \left( \frac{r}{R} \right)^{m+1} \left[ 1 + C \left( 1 - \frac{r^2}{R^2} \right) \left( \frac{2M}{R} \right) + O \left( \frac{2M}{R} \right)^2 \right] \quad (32)
\]
where $C$ is a constant, and

$$W_{m+1} \sim V_{m+1} \sim U_{m+2} \sim O\left(\frac{2M}{R}\right) \tag{33}$$

From this we see how the $l = m$ Newtonian r-modes of barotropic uniform density stars are affected by first order post-Newtonian corrections. All barotropic Newtonian r-modes with $m \geq 2$ pick up both axial and polar post-Newtonian terms. From Eq. (31) we also see that the r-mode frequency decreases because of the small relativistic correction. It is natural that General Relativity will have this effect. The gravitational redshift will tend to decrease the fluid oscillation frequency as measured by a distant inertial observer. Also, because these modes are rotationally restored they will be affected by the dragging of inertial frames induced by the star’s rotation. Specifically, since the Coriolis force is determined by the fluids angular velocity relative to that of the local inertial frame $[36]$, $\bar{\omega}(r) = \Omega - \omega(r)$ it decreases — and the modes oscillate less rapidly — as the dragging of inertial frames becomes more pronounced.

Just like in Newtonian theory, the non-barotropic problem is different. For non-barotropic stars one can still have purely axial inertial modes also in General Relativity. These modes are determined by a single ordinary differential equation for one of the perturbed metric components;

$$\left(\alpha - \bar{\omega}\right)\left\{e^{\nu - \chi}\frac{d}{dr}\left[e^{\nu - \chi}d\right] - \left[\frac{l(l+1)}{r^2} - \frac{4M}{r^3} + 8\pi(p + \rho)\right]h\right\}$$

$$+ 16\pi(p + \rho)\alpha h = 0, \tag{34}$$

where $h \propto \delta g_{\nu\phi}$, and we have used

$$\omega = -m\Omega \left[1 - \frac{2\alpha}{l(l+1)}\right], \tag{35}$$

as well as $\bar{\omega} = \bar{\omega}/\Omega$. This equation was first derived by Kojima $[37]$. The eigenvalues $\alpha$ and the corresponding eigenfunctions $h$ are not explicitly dependent on $m$, which means that if we find an acceptable mode-solution to (34) it will be relevant for all $m \neq 0$ for each given multipole $l$. This would be in accord with the non-barotropic Newtonian case where one finds a single r-mode for each combination of $l$ and $m$ at order $\Omega$. Given (34) one can prove two interesting results. First of all, non-trivial solutions may only exist provided that $\alpha - \bar{\omega}$ vanishes at some point in the interval $r \in [0, \infty]$ $[35]$. As first demonstrated by Lockitch et al $[35]$ for uniform density stars, one can find a single discrete mode solution to (34) with frequency in the required interval, cf. Figure 4. Secondly, the equation admits a continuous spectrum $[37, 38]$ in the range $\bar{\omega}(0) < \alpha < \bar{\omega}(R)$. The dynamical role of this continuous spectrum, or indeed if it remains present when higher order rotational corrections are included, is not clear at the present time. The presence of the continuous spectrum makes the non-barotropic r-mode problem difficult. When one considers softer equations of state one must typically determine a discrete mode lying inside the continuous spectrum. As is well known from studies of differentially rotating systems (where a continuous spectrum arises because of the presence of so-called co-rotation points) this is a notoriously difficult problem.
This technical difficulty has led to suggestions that the r-modes may not even “exist” for certain relativistic stars [41, 42]. However, such conclusions are likely premature: A failure to calculate the mode is more likely an indication of a breakdown in the approximations we have made to the physics. In the case of the r-modes it is clear already from (34) that the slow-rotation approximation is no longer consistent in regions where $\alpha - \tilde{\omega} \sim O(\Omega^2)$ or smaller. In principle, this means that the problem requires a “boundary layer” approach [43] where either $\Omega^2$ terms, viscosity or the coupling to polar perturbations are included in the analysis. Some very recent results [45, 46] support this view.

Figure 3. This figure shows the r-mode eigenfrequencies $\alpha$ for relativistic nonbarotropic uniform density stars ($n = 0$ polytrope) and a range of compactness ratios $M/R$. The Newtonian limit corresponds to $M/R \rightarrow 0$. Also shown are the corresponding values of the relativistic framedragging at the centre $\tilde{\omega}_c$ and surface $\tilde{\omega}_s$ of the star. The perturbation equations admit a continuous spectrum in the range $\tilde{\omega}_c < \alpha < \tilde{\omega}_s$. For such frequencies the eigenvalue problem is formally singular. As is clear from the data, the uniform density r-modes are always regular. However, it should be noted that most realistic equations of state lead to a singular problem. Also shown (as a dashed curve) are the eigenfrequencies for the axial-led inertial mode of a barotropic star that most resembles the Newtonian r-mode. Note that the hybrid/inertial mode problem is never singular.

Finally, it worth mentioning that there is a class of pulsation modes that arise only when the pulsation problem is considered in General Relativity. These are known as the w-modes [47], and they exist because the curvature of spacetime that is generated by the background density distribution can temporarily trap impinging gravitational waves. The w-modes typically have high frequencies (above 7 kHz) and they damp out in a fraction of a millisecond (see data provided in [21]). For ultracompact stars these modes may become very long lived [48]. Gravitational waves can be trapped inside the peak of the spacetime curvature barrier that is unveiled as $R < 3M$ (in the non-rotating case). Even though such extremely compact stars may not exist in the Universe (no proposed realistic equations of state permit stars more compact than $R \approx 3M$), they are still intriguing. In particular since, when rotating, they may admit the presence of an
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ergosphere, i.e. a region (located inside the star) in which all observers must be dragged along with the stars rotation. This could lead to [49] the w-modes becoming unstable due to the same mechanism that drives the CFS unstable fluid modes (see Section 7). This instability is unlikely to be astrophysically relevant [50, 51, 52], but since it is a uniquely relativistic effect it is of conceptual interest.

4. What do we learn from the ellipsoids?

Many aspects of the instabilities in spinning stars can be illustrated by classic results for rotating ellipsoids. This is, in fact, an age old problem that has attracted the attention of distinguished scientists for centuries¶. The relative mathematical simplicity makes a study of the equilibrium properties and stability of rotating self-gravitating fluid bodies with uniform density analytically tractable. In this Section we will discuss the key results and try to make contact with the pulsation properties of compressible stars.

The most studied (and therefore best understood) figures of rotating, self-gravitating, homogeneous and incompressible fluid bodies in equilibrium are the uniformly rotating Maclaurin spheroids, which are oblate in shape. In addition, there exist various triaxial equilibrium configurations. The Jacobi ellipsoids are rigidly rotating about the smallest axis, and have no vorticity when viewed from a rotating frame in which the figure appears stationary. For a given angular momentum, mass and volume the Jacobi ellipsoid has lower energy than the corresponding Maclaurin spheroid. The Dedekind ellipsoids have a stationary triaxial shape in the inertial frame. Hence, they are non-rotating, and their shape is entirely supported by internal motions of uniform vorticity. A Dedekind ellipsoid with the same mass and circulation as the corresponding Maclaurin configuration has lower angular momentum. These three families of equilibrium spheroids are subclasses of the so-called Riemann-S ellipsoids, which are distinguished by having rotation and vorticity vectors aligned with a symmetry axis of the figure. The Riemann sequences are characterized by having constant ratio \( \zeta/\Omega \), where \( \zeta \) is the vorticity in the rotating frame. The Jacobi and Dedekind families correspond to the special cases \( \zeta = 0 \) and \( \Omega = 0 \), respectively.

We begin by focussing our attention on the Maclaurin spheroids. Proceeding along the Maclaurin sequence towards more rapidly rotating configurations, e.g. increasing \( \beta \), one finds a bifurcation point at \( \beta_s \approx 0.14 \). At this point the Jacobi and Dedekind ellipsoids both branch off from the Maclaurin sequence. This is schematically illustrated in Figure 4. Given the existence of the alternative states with lower energy/angular momentum beyond the point of bifurcation it would be favourable for a perturbed Maclaurin spheroid to move towards either the Jacobi or the Dedekind sequence. However, this is not possible as long as the system conserves circulation and angular momentum. This means that the Maclaurin spheroid will remain stable unless we add dissipation to the dynamical equations. In other words, the bifurcation point at \( \beta_s \) indicates the onset of secular instabilities.

¶ The subject has been exhaustively summarized by Chandrasekhar [53].
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Figure 4. A schematic summary of the instability results for rotating ellipsoids ($a_2/a_1$ represents the axis ratio, i.e. the ellipticity of the configuration). For values of $\beta$ greater than 0.14 the Maclaurin spheroids are secularly unstable. Viscosity tends to drive the system towards a triaxial Jacobi ellipsoid, while gravitational radiation leads to an evolution towards a Dedekind configuration. Indicated in the figure is an evolution of this latter kind. Above $\beta \approx 0.27$ the Maclaurin spheroids are dynamically unstable, as there exists a Riemann-S ellipsoid with lower (free) energy. [For more details, see 54, 56]

Viscosity dissipates energy while preserving the angular momentum. The Maclaurin spheroids are therefore susceptible to a viscosity-driven instability once $\beta > \beta_s$, and the instability drives the system towards the Jacobi sequence [55]. Gravitational-wave dissipation, on the other hand, radiates angular momentum while conserving the internal circulation. Thus, the Maclaurin spheroids also suffer a gravitational-wave driven instability when $\beta > \beta_s$. The gravitational-wave instability tends to drive the system towards the Dedekind sequence (the members of which do not radiate gravitationally) [56].

These classic secular instabilities set in through the quadrupole f-modes of the ellipsoids. In Figure 5 we show the frequencies of the $l = |m| = 2$ Maclaurin spheroid f-modes. These modes are usually referred to as the “bar-modes”. The figure illustrates several general features of the pulsation problem for rotating stars. In particular we notice i) the rotational splitting of modes that are degenerate in the non-rotating limit, i.e. the $m = \pm 2$ modes become distinct in the rotating case, and ii) the symmetry with respect to $\omega = 0$, which reflects the fact that the governing equations are invariant under the change $[\omega, m] \rightarrow [-\omega, -m]$. In Figure 5 we also show the pattern speed for the two modes that have positive frequency in the non-rotating limit, cf. (18). From this figure we see that the $l = -m = 2$ mode, which is always prograde moving in the inertial frame, has zero pattern speed in the rotating frame at $\beta_s (\sigma_p = \Omega)$. At this point the mode becomes unstable to the viscosity driven instability. That the instability

+ Recent results concerning the stability of the Riemann-S ellipsoids complicates this picture considerably. These results, due to Lebovitz and Lifschitz [57], show that the Riemann-S ellipsoids suffer a “strain” instability in most of the parameter space. In particular, the Dedekind ellipsoids are always unstable due to this new instability.
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Figure 5. Results for the \( l = |m| = 2 \) f-modes of a Maclaurin spheroid. In the left frame we show the oscillation frequencies (solid lines) and imaginary parts (dashed lines) of the modes, while the right frame shows the mode pattern speed \( \sigma_i \) for the two modes that have positive frequency in the non-rotating limit (the pattern speeds for the modes which have negative frequency in the non-rotating limit are obtained by reversing the sign of \( m \)). All results are according to an observer in the inertial frame. The dashed curves in the right frame represent a vanishing pattern speed i) in the inertial frame (the horizontal line), and ii) in the rotating frame (the circular arc, which shows \( \Omega/\Omega_K \) as a function of \( \beta \)). The points where the Maclaurin ellipsoid becomes secularly (\( \beta_s \)) and dynamically (\( \beta_d \)) are indicated by vertical dotted lines.

should set in at this point is natural since the perturbed configuration is “Jacobi-like” when the mode is stationary in the rotating frame. Meanwhile, the gravitational-wave instability sets in through the originally retrograde moving \( l = m = 2 \) modes. At \( \beta_s \) these modes have zero pattern speed in the inertial frame (\( \sigma_p = 0 \)). At this point the perturbed configuration is “Dedekind-like” since the mode is stationary according to an inertial observer.

The evolution of the secular instabilities depends on the relative strength of the dissipation mechanisms. This tug-of-war is typical of these kinds of problems. Since the gravitational-wave driven mode involves differential rotation it is damped by viscosity, and since the viscosity-driven mode is triaxial it tends to be damped by gravitational-wave emission. A detailed understanding of the dissipation mechanisms is therefore crucial for any investigation into secular instabilities of spinning stars.

Given the competition between gravitational radiation and viscosity one would expect a “realistic” star to be stabilized beyond the point \( \beta_s \). Also, the secular instabilities are no longer realized in the extreme case of a perfect fluid which conserves both angular momentum and circulation\(^*\). Then the Maclaurin sequence remains stable up to the point \( \beta_d \approx 0.27 \). At this point there exists a bifurcation to the \( x = +1 \) Riemann-S sequence. These equilibria have lower “free energy” \(^{[56]}\) than the corresponding Maclaurin spheroid for the same angular momentum and circulation.

\(^*\) Note that in General Relativity all non-axisymmetric modes of oscillation radiate gravitational waves. Hence, this argument is only relevant in Newtonian gravity.
This means that a dynamical transition to a lower energy state may take place without violating any conservation laws. In other words, at $\beta_d$ the Maclaurin spheroids become dynamically unstable to $m = 2$ perturbations. This instability is usually referred to as the dynamical bar-mode instability.

In terms of the pulsation modes, the dynamical instability sets in at a point where two real-frequency modes merge, cf. Figure 5. At the bifurcation point $\beta_d$ the two modes have identical oscillation frequencies and their angular momenta will vanish. Given this, one of the degenerate modes can grow without violating the conservation of angular momentum. The physical conditions required for the dynamical instability are easily understood. The instability occurs when the originally backwards moving $f$-mode (which has $\delta J < 0$ for $\beta < \beta_d$) has been dragged forwards by rotation so much that it has “caught up” with the originally forwards moving mode (which has $\delta J > 0$ for $\beta > \beta_d$). In order for the modes to merge and become degenerate the perturbation must have vanishing angular momentum at $\beta_d$ ($\delta J = 0$).

5. Stability analysis

Stellar stability problems have traditionally been explored either via construction of suitable variational principles or direct mode-calculations. Each strategy has its merits and drawbacks. The variational-principle approach is appealing from a formal point of view, and it is well known from many branches of physics that variational principles are closely connected to stability criteria. Once one has defined a suitable “energy” for the perturbations, stability simply follows from its sign for any given perturbation. However, the obtained relations can be difficult to use in practice. It may also be difficult to conclusively rule out instabilities. The mode-calculation approach is less elegant, but has the advantage that one can test each individual mode for stability. On the other hand, a failure to find an unstable mode does not necessarily mean that one does not exist.

The main development of the theoretical framework for studying stellar stability in general relativity took place in the 1970s. Key early contributions were made by Chandrasekhar and Friedman [58, 59] and Schutz [60, 61]. Their work attempted to extend the conclusions from Newtonian studies — that the onset of non-axisymmetric instability is signalled by the appearance of a neutral (zero-frequency) mode. There are two main reasons why a relativistic analysis is significantly more complicated than the Newtonian one. First of all the problem is algebraically more complex because one must solve the Einstein field equations in addition to the fluid equations of motion. Secondly, one must account for the fact that a general perturbation will generate gravitational waves. This is a fundamental complication since a mode-based proof of stability would require some kind of completeness of the modes of the star. Since the relativistic modes are “quasinormal” (they have complex frequencies, with the imaginary part corresponding to the gravitational-wave damping) they are unlikely to be complete in any meaningful sense [13].
The work of Friedman and Schutz culminated in a series of impressive papers\[62, 63, 64\] in which the role that gravitational radiation plays in these problems was explained, and a foundation for subsequent research in this area was established. The main result was that gravitational radiation acts in the same way in the full theory as in Chandrasekhar’s post-Newtonian analysis of the Maclaurin spheroids \[65\]. If we consider a sequence of equilibrium models, then a mode becomes secularly unstable at the point where its frequency vanishes (in the inertial frame). Most importantly, the proof does not require the completeness of the modes of the system.

The Friedman-Schutz criterion for instability relies on the so-called canonical energy\[E_c\] being negative. The canonical energy is defined as

\[
E_c = \frac{1}{2} \int \left[ \rho |\partial_t \xi|^2 - \rho |\vec{u} \cdot \nabla \xi|^2 + \nabla p \cdot (\nabla \cdot \xi) + \xi^\ast \cdot \nabla p \nabla \cdot \xi + \chi^i \cdot \nabla \nabla \cdot \xi^j + \xi^i \xi^j \left( \nabla_i \nabla_j p + \rho \nabla_i \nabla_j \Phi \right) - \frac{1}{4\pi G} |\nabla \delta \Phi|^2 \right] dV
\]  

(36)

where \(\vec{u} = \Omega \times \hat{r}\) represents the background flow\[\#\]. One can also define a (conserved) canonical angular momentum

\[
J_c = -\text{Re} \int \rho \partial_x \xi^i (\partial_t \xi_i + \vec{u} \cdot \nabla \xi_i) dV
\]  

(37)

The canonical energy and angular momentum are conserved in absence of radiation and viscosity. This means that, in order to have a dynamical instability (unbounded growth of a linear mode of the inviscid problem) we must have \(E_c = J_c = 0\). If \(E_c\) is negative at the outset and the star is coupled to radiation in such a way that \(E_c\) must decrease with time, then the absolute value of \(E_c\) will increase and the associated mode will be unstable. Generally, an instability can be established in a mode-independent way by constructing (canonical) initial data \([\xi, \partial_t \xi]\) such that \(E_c\) is negative. To do this is, however, not a simple task. From the computational point of view it is easier to calculate a mode of a rotating star and then evaluate \(E_c\) to assess stability. It is sufficient to show that the displacement vector associated with the mode leads to \(E_c < 0\) to demonstrate the presence of an instability.

The simple intuitive instability criterion can be deduced from the relation

\[
E_c = -\frac{\omega_i}{m} J_c = \sigma_i J_c
\]  

(38)

which is a general property of linear waves. We see that \(E_c\) changes sign when the inertial frame pattern speed \(\sigma_i\) passes through zero. Beyond this point the mode moves forwards with respect to the inertial frame while it is still moving backwards in the rotating frame. Gravitational waves from such a mode carry positive angular momentum away from the star, but since the perturbed fluid actually rotates slower than it would in absence of the perturbation the angular momentum of the mode is negative. The emission of gravitational waves consequently makes the angular momentum of the mode increasingly negative and leads to the instability. This instability is often referred to as the Chandrasekhar-Friedman-Schutz (CFS) instability.

\[\#\] A fully relativistic expression for the canonical energy has been derived by Friedman \[65\].
It is interesting to contrast the secular radiation driven instability to that associated with viscosity. For uniformly rotating stars one can show that the combination
\[ \delta E - \Omega \delta J = E_c - \Omega J_c = -\frac{\omega_r}{m} J_c = \sigma_r J_c = E_{c,R} \] (39)
relating the first order changes in the kinetic energy and angular momentum to a mode-solution, is gauge-invariant. \( E_{c,R} \) can be viewed as the canonical energy in the rotating reference frame. Viscosity leads to \( E_{c,R} \) being a decreasing function of time. From (39) we can deduce that the onset of the viscosity driven instability is signalled by the vanishing of the mode pattern speed in the rotating frame (\( \sigma_r = 0 \)).

In order to investigate whether an instability is of astrophysical relevance, eg. whether it leads to a detectable gravitational-wave signal, one must address two main questions. First of all, one must understand under what circumstances the instability will be present and how likely it is that a star will evolve through the relevant part of parameter space. Secondly, one must establish that the unstable mode grows on a sufficiently short timescale. This is always the case for dynamical instabilities, but as we will see in the following sections the issues that decide whether a secular instability is relevant or not are much more delicate.

6. Results: Dynamical instabilities

6.1. Quasiradial modes

The most familiar stellar instability is probably that associated with the existence of a maximum mass configuration for any given equation of state. A spherical relativistic star will suffer a dynamical instability before the compactness reaches the Schwarzschild limit \[ R < 2.25M \] and no realistic equations of state permit stars more compact than \[ R \approx 3M \]. Once an accreting neutron star reaches the maximum mass limit it will become unstable and undergo gravitational collapse, most likely leading to the formation of a black hole.

The maximum mass limit provides one of very few handles that we currently have on the supranuclear equation of state. Several neutron star masses have been deduced from pulsar observations and, in order to be acceptable, a proposed equation of state must allow for masses at least as large as those observed. The data for the binary pulsar PSR1913+16 provides the constraint \( M_{\text{max}} \geq 1.44M_\odot \), which allows us to rule out extremely soft equations of state. A much more severe constraint on the theoretical models may be provided by the data for the Vela pulsar. Several studies have estimated the mass of Vela to be about \( 1.8M_\odot \). Many proposed equations of state would be in trouble if this result were to be confirmed. In particular, neutron star models with sizeable exotic cores composed of hyperons and/or deconfined quarks which tend to soften the equation of state \[ \] might then be ruled out.

The maximum mass instability is relatively easy to analyse in the case of non-rotating stars. It sets in through the radial \( (l = m = 0) \) f-modes. Since the equations that describe radial oscillations depend only on \( \omega^2 \) the mode frequencies come in pairs
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Calculations show that, as one increases the central density of the star (for a given equation of state) the absolute value of the stable (real valued) f-mode frequencies decreases. It passes through zero at the point at which the mass reaches an extremum. Beyond that “turning point”, the mode frequencies become a complex conjugate pair, and thus one of the modes is unstable.

The simplicity of the “turning point method” for locating the onset of instability along a sequence of equilibrium models is extremely appealing. It is, however, not immediately obvious that it will generalize to a more complicated setting — like rotating stars. The problem is, in fact, quite subtle. A dynamical stability analysis requires evaluation of a complicated energy functional, and then relies on non-degenerate perturbation theory to deduce stability. However, instabilities set in through zero-frequency modes, and the simple non-rotating barotropic models that one would typically consider in this context have a degenerate set of such neutral modes — the inertial modes. These would be distinct in a non-barotropic model (in which they become the g-modes), but in the barotropic case they all have zero frequency. A proper stability analysis must take the presence of these modes into account. In order to single out a “marginally unstable” mode from the sea of inertial modes one must use degenerate perturbation theory. This is not a simple task. Fortunately, Sorkin [69] has provided a detailed proof that the turning point method can be used to locate the onset of instability. The proof does not assume mode-completeness and hence neatly circumvents difficulties like that associated with the degenerate inertial modes. In the case of uniform rotation an extremum of the angular momentum $J$ along a sequence with constant baryon number limits the region of stable stars [70]. The (dynamically) stable region for a typical relativistic star (represented by the FPS equation of state) is illustrated in figure 6. As can be seen in the figure, rotation generally increases the maximum allowed mass by up to 20%. It is worth noting that there may exist a class of rotating stars that have no non-rotating counterpart. These “supramassive” stars must eventually collapse if they are spun down, eg. by magnetic dipole radiation [71].

Sorkin’s theorem states that for barotropic stars a turning point in $J$ marks the location of a zero-frequency axisymmetric mode and the onset of dynamical instability. For non-barotropic stars the situation is a bit more subtle. In that case the turning point method locates a point where a secular instability sets in. Consequently, a realistic star is likely to undergo a secular instability phase before it reaches the dynamical instability point and collapses.

From an intuitive point of view one might expect gravitational collapse to lead to a very strong gravitational-wave signal. However, it is also conceivable that the level of radiation may be low. The outcome depends entirely on the asymmetry of the collapse process. A purely spherical collapse will obviously not radiate gravitationally at all, while the collapse of a strongly deformed body could release a copious amount of gravitational waves. The main reason why it is very difficult to make “reliable” estimates for the energy released is that the answer depends entirely on the route that the system follows towards the final configuration. This is immediately clear from
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Figure 6. Left panel: The permissible region of stable stars described by the FPS equation of state. Rotation increases the maximum mass by roughly 20% and also leads to the presence of a family of stars that have no non-rotating counterpart. These configurations are located between the mass-shedding curve and the dashed curve, which represents the most massive rotating star that has a stable non-rotating counterpart. A particular sequence (with constant baryon mass) of such supramassive stars in indicated by a thin solid line. An isolated star that spins down due to magnetic dipole radiation would evolve along this sequence until it reaches the state represented by the filled circle. At this point it will become unstable and undergo gravitational collapse. Right panel: The instability of a supramassive star sequence (with constant baryon mass) sets in at the point where the angular momentum \( J \) has an extremum as a function of the central density.

the post-Newtonian formulas (3), which show that the gravitational-wave luminosity depends on high time-derivatives of the various multipoles.

It was originally thought that supernova core-collapse would lead to strong gravitational-wave signals. This expectation has not been confirmed by detailed simulations. In fact, the available numerical simulations paint a somewhat pessimistic picture. Typical results suggest that an energy equivalent to \( 8 \times 10^{-8} M_\odot c^2 \) may be radiated [72]. The signal from a collapse that leads to the formation of a black hole is likely to be dominated by the slowest damped quadrupole quasinormal mode of a black hole, i.e. have frequency

\[
f_{Hz} \approx 1200 \left( \frac{10 M_\odot}{M} \right)^{10} \quad (40)
\]

If we assume that a typical timescale for the event is of the order of a millisecond we find, cf. (10), that the gravitational-wave amplitude may be of the order of \( h_c \sim 10^{-22} \) for a source in the Virgo cluster. This estimate (which accords reasonably well with full numerical simulations) suggests that such sources are unlikely to be observable beyond the local group of galaxies.

Until very recently studies of core collapse focussed mainly on the non-rotating case (the classic work by Stark and Piran [73] being a notable exception). It is perhaps natural to expect that studies of the fully three-dimensional collapse problem may lead
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To an enhanced radiated energy and maybe even a prediction of clearly detectable waves. To solve this problem is, of course, a far from trivial task. Nevertheless, there has been interesting recent progress in this direction [74, 75]. Numerical relativity is rapidly maturing and it may not be too optimistic to expect that this problem will be manageable in the not too distant future. Having said that, it is clear that there are conceptual issues that may be very hard to resolve, in particular regarding the initial data. Numerical relativists have so far almost exclusively considered initial data obtained using York’s conformally flat prescription. This approach is taken because of its mathematical convenience. It is reasonable to worry about this since there is absolutely no physical reason why “realistic” initial data should be conformally flat. This is a key issue that needs to be addressed in the future.

In absence of detailed numerical results, it is interesting to speculate about the outcome of the gravitational collapse of a rapidly rotating star. It stands to reason that this would be a promising source for gravitational radiation, since rotation will couple the various multipoles in such a way that even quasi-radial modes will radiate. Furthermore, it is well known that rapidly rotating stars have a multipolar structure that differs significantly from that of a Kerr black hole (cf. the results in [76]). In order to form a black hole the collapsing star must in some way shed the “difference” in the various multipole moments. Presumably, this will be done mainly through the emission gravitational waves. Of course, a sceptic would argue that the collapse event may proceed in such a way that the level of radiation is minimal. Future multidimensional simulations will have to provide the real answer.

A collapse related scenario that has been discussed in the literature concerns internal phase-transitions. As a neutron star is spinning down, eg. due to magnetic dipole radiation, the central density will increase. Various theoretical models suggest that the equation of state may soften significantly once the central density increases beyond a critical value (several times nuclear density). This could be due to the formation of pion/kaon condensates, the creation of a significant hyperon core or quark deconfinement. Should this happen it is likely to result in a “mini-collapse” during which some gravitational potential energy may be released as radiation. Such phase transitions have been suggested as sources for both detectable gravitational waves [77] and gamma-ray bursts. However, most of the available estimates seem somewhat optimistic. The reason for this is very simple. It is typically assumed that the entire change in potential energy incurred during the contraction is radiated away. This is at variance with detailed studies which show that the radiated energy is at best only a few percent of this [78]. Most of the “lost” potential energy is transferred into internal energy (i.e. it heats the star up). Using results for a uniform density sphere, one can estimate that the change in potential energy \( \delta E \) associated with a change in radius \( \delta R \) is

\[
\delta E \approx \frac{3}{5} \frac{GM^2}{R^2} \delta R
\]

Suppose that the contraction associated with a phase-transition in the core of a neutron star leads to \( \delta R \approx 10 \text{ m} \), and that 1% of \( \delta E \) is radiated as gravitational waves (which
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does not seem too unreasonable). Then we find that \( h_c \approx 10^{-23} \) (assuming \( f_{Hz} = 10^3 \)) for a source at the distance of the Virgo cluster. This is probably too weak to be detected (at least in the next few years). Still, there are several reasons why these scenarios should not be ignored. First of all, the event may be more violent than I have assumed here. Secondly, a unique event from within our galaxy could be detectable. Given the estimated event rate (unlikely larger than \( 10^{-5}/\text{yr}/\text{galaxy} \)) we would obviously be very lucky to see such an event, but the information that an observation would provide about physics beyond supranuclear density would be extremely valuable.

6.2. The bar-mode instability

Even though realistic neutron star equations of state do not allow values of \( \beta \) much larger than 0.1 in uniformly rotating neutron stars, several scenarios may lead to a compact star becoming dynamically unstable to the bar-mode. For example, since \( \beta \sim 1/R \) one might expect a collapsing star to suffer a triaxial instability at some point during its evolution. A key parameter that determines the outcome is the degree of differential rotation. In fact, the maximum attainable \( \beta \) changes dramatically if the star is differentially rotating. This is not surprising since the presence of differential rotation may lead to an increase of the mass-shedding limit by allowing the equator to rotate slower than the central parts of the star.

The critical value at which the bar-mode becomes dynamically unstable remains close to the result for Maclaurin spheroids, \( \beta_d \approx 0.27 \), for models with varying compressibility. Moreover, detailed studies suggest that the onset of instability is in general weakly dependent also on the chosen differential rotation law \cite{79}. Having said this, there are extreme angular momentum distributions for which \( \beta_d \) becomes very small \cite{80}, and for which “spiral instabilities” tend to dominate. This is interesting as it indicates that dynamical instabilities could play a role also for relatively slowly rotating stars. This possibility is illustrated by recent numerical work by Centrella et al \cite{81}. Their simulations of differentially rotating \( \Gamma = 4/3 \) polytropes indicate the presence of a dynamical instability for \( \beta \approx 0.14 \) (i.e. similar to the point where secular instabilities are expected to set in). The simulations show that an \( m = 1 \) mode plays a dominant role in determining the evolution of the system. The nature of this instability, i.e. to what extent it is a generic feature, is not yet well understood. Very recent work by Shibata et al \cite{82} supports the notion that dynamical instabilities may operate at very low values of \( \beta \). In fact, they find that a dynamical instability may set in at values as low as \( \beta \approx 0.03 \)! These intriguing results show that we still have much to learn about the oscillations and instabilities of differentially rotating stars.

In order to study the nonlinear bar-mode evolution one must resort to large scale numerical simulations. Such work, carried out over the last two decades, shows that the nature of the bar-mode instability depends on the magnitude of \( \beta \) compared to the critical value. For large values, \( \beta >> \beta_d \), the initial exponential growth of the unstable mode (on the dynamical timescale) is followed by the formation of spiral arms.
Gravitational torques on the spiral arms lead to the shedding of a mass and angular momentum. Through this process the unstable mode saturates and the star reaches a dynamically stable state. In this scenario gravitational waves are emitted in a relatively short burst \[83\]. For some time it was thought that this was the generic behaviour, but recent work \[84\] indicates that when $\beta$ is only slightly larger than $\beta_d$ a long-lived ellipsoidal structure may be formed. If this is the case, the bar-mode may decay slowly (on the viscosity/gravitational-wave timescale) until the star reaches the point where it is secularly stable \[84\]. This could lead to a relatively long lasting gravitational-wave signal. Snapshots from a typical bar-mode evolution are shown in Figure 8.

![Figure 7](Image)

**Figure 7.** A few snapshots from a nonlinear evolution showing the development of typical bar-mode. The initial data for the evolution is a model with uniform vortensity and $\beta = 0.282$, i.e. slightly above $\beta_d$. The frames show density contours along with vectors representing the momenta in the equatorial plane at different times. The timescale indicated in the three frames is in units of the dynamical time $t_{dyn} = 1/\sqrt{\pi G\rho_0}$. [Reproduced from \[85\] by kind permission from the authors.]

It is straightforward to estimate the strength of the gravitational waves emitted by a sizeable bar-mode. Let us assume that the mode saturates at an amplitude $\eta$ represented by the axis ratio of the ellipsoidal structure. Typical values \[86\] may lie in the range $\eta \approx 0.2 - 0.4$. From the standard results for a rotating solid body with a given ellipticity we have

$$\frac{dE}{dt} \approx -\eta^2 \frac{GM^2R^4\Omega^6}{c^5}$$

which leads to

$$h \approx 4 \times 10^{-23} \left( \frac{\eta}{0.2} \right) \left( \frac{f}{2 \text{ kHz}} \right)^2 M_{1.4}R_{10}^{2}D_{15}^{-1}$$

where we have used the fact that the gravitational-wave frequency $f$ is twice the rotation frequency. This estimate compares reasonably well with the more detailed results available in the literature (see, for example, Table 7 in \[83\]). A signal with this strength may be detectable for sources in local galaxy group. Of course, the detectability of the signal is significantly improved if the instability leads to the formation of a persistent bar-like structure. Should a long-lived bar form and last for hundreds of rotation periods,
one can easily gain a factor of ten in the signal-to-noise ratio. Since such factors could
be crucial it is important that the long-term evolution of the bar-mode instability is
studied further and understood in detail.

To perform this kind of simulations within numerical relativity (with a dynamical
spacetime) has only recently become feasible. Shibata et al. [86] have performed the first
fully relativistic studies of the bar-mode problem. They consider models with varying
degrees of differential rotation, and draw conclusions that agree well with those of the
Newtonian work. In addition, they show that general relativistic effects enhance the
dynamical instability only very slightly and hardly change the critical value $\beta_d$ at all.
Although the limited size of their numerical grid means that the gravitational waveforms
cannot be directly calculated, the results of Shibata et al. [86] seem to be in agreement
with the Newtonian estimates of the strength of the gravitational-wave signal.

7. Results: Secular instabilities

7.1. The CFS instability

As was first proved by Friedman and Schutz [63], the radiation driven instability is
generic in rotating stars. That is, for any given $\Omega$ one can always find an unstable mode
(in an inviscid star). This is easy to see from (19): Regardless of the value of $\Omega$ there will
always exist an $m$ that is large enough that the associated mode satisfies the instability
criterion, i.e. is retrograde in the rotating frame but prograde according to an inertial
observer. However, this does not mean that the large $m$ modes lead to the strongest
instability. As one can readily deduce from (3) the higher order modes tend to radiate
less efficiently, and thus they will grow slower than the low order modes. In fact, Comins
[87, 88] has shown that the growth time ($t_{gw}$) increases exponentially with $m$ for modes
of the Maclaurin spheroids. We can obtain a rough estimate of the growth time for
the unstable f-modes from formulas used in [22]. Neglecting all rotational corrections
(taking $\alpha = \beta = \gamma = 1$ in the relevant equations in [22]), one finds that

$$t_{gw}(\Omega) \approx t_{gw}(\Omega = 0) \left(1 - \sqrt{\frac{m}{3} \frac{\Omega}{\Omega_K}}\right)^{-2m-1}$$

(44)

for the $l = m$ modes. This estimate captures the overall features of the full numerical
results and hence provides a useful illustration, cf. Figure 8. More detailed calculations
show that only f-modes with $m \leq 5$ are expected to grow fast enough to lead to an
astrophysically relevant instability. On the other hand, the low order modes only become
unstable at extremely high rotation rates (and the quadrupole mode may not be unstable
at all). Taking also this into consideration, one finds that the $l = m = 4$ f-mode is the
most strongly unstable mode in a Newtonian star [89].

The situation is slightly different for the r-modes. As discussed previously, the r-
modes are always retrograde in the rotating frame and prograde in the inertial frame.
This means that they satisfy the CFS instability criterion at all rates of rotation [90, 91].
In other words, the r-modes are generically unstable in rotating perfect fluid stars. For the $l = m = 2$ r-mode one can show that the growth time is

$$t_{gw} \approx -47 M_{1.4}^{-1} R_{10}^{-4} P_{6}^{-3} \text{s}$$

for $n = 1$ polytropes [12, 93]. It is interesting to compare this result to the (presumably) most important f-mode. Consider a particular $n = 1$ polytropic stellar model with mass $1.5M_{\odot}$ and radius $12.533$ km [22], for which the Kepler limit would correspond to a period of $0.8$ ms. For a star spinning at this rate the r-mode would grow on a timescale of roughly $4$ s, while one finds $t_{gw} \approx 5 \times 10^5$ s for the $m = 4$ f-mode [22]. A similar comparison for other rotation rates is provided in Figure 8. These estimates indicate that the r-mode instability is significantly stronger than that of the f-mode. Having said this, one must be somewhat careful before drawing definite conclusions since differential rotation could change the picture considerably. In particular, an unstable $m = 2$ f-mode could become competitive with the r-mode. For example, Lai and Shapiro [54] estimate that the quadrupole f-mode grows on a timescale $t_{gw} \approx 1$ s for $\beta \approx 0.24$. One must also keep in mind that f-mode instability is strengthened by relativistic effects, and that the $m = 2$ f-mode may become unstable at reasonable rates of rotation, cf. Section 3.3. An important challenge for future work in this area concerns the calculation of growth/damping timescales of the f-modes of rapidly rotating fully relativistic stars.

![Figure 8.](image)

**Figure 8.** The growth/damping timescales of the most relevant unstable modes of a uniformly rotating $n = 1$ polytrope with $M = 1.5M_{\odot}$ and $R = 12.533$ km. The $l = m = 4$ f-mode (dashed curve) is compared to the $l = m = 2$ r-mode (solid line). We also show the rough estimate [44] for the f-mode (dotted curve). This figure captures the qualitative features of the problem (eg. that the f-mode becomes unstable above a critical rotation rate $\Omega_c \approx 0.85\Omega_K$), but comes with several disclaimers. Most importantly, these results will be affected by differential rotation and general relativistic effects. [The f-mode data was provided by Lee Lindblom.]

To date, there have been three studies of the growth timescale for the unstable relativistic r-modes. Two of these, [44] and [94], concern the non-barotropic problem while the third was for inertial modes of barotropic stars [95]. (Interestingly, the three methods used to extract the gravitational-wave dissipation rates are different.) The
studies all agree that the post-Newtonian estimates of the instability growth time are rather good. As can be seen in Figure 9, the growth time $t_{gw}$ approaches the post-Newtonian result as $M/R$ decreases. The fully relativistic timescales begin to deviate from the post-Newtonian ones as the star reaches the compactness of a typical neutron star, $M/R \approx 0.15$. This weakening of the instability is presumably due to the increased influence of the spacetime curvature as the star becomes more compact, and reflects the increased backscattering of gravitational waves.

![Figure 9](image_url)

**Figure 9.** The growth times for the inertial modes of a canonical $1.4M_\odot$ relativistic barotropic (uniform density) star. The data is for the modes that correspond to the first few $l = m$ r-modes in the Newtonian limit ($M/R \to 0$) \[95\].

The growth time estimates indicate that various unstable modes may grow fast enough to be of significance for rapidly spinning neutron stars. However, in order to assess the true relevance of these instabilities we must also consider possible damping effects. In particular, an unstable mode must grow fast enough that it is not completely damped out by viscosity in order to be relevant. To assess the strength of viscous damping one typically considers the effects of bulk and shear viscosity. These are due to rather different physical mechanisms. At relatively low temperatures (below a few times $10^9$ K) the main viscous dissipation mechanism in a fluid star arises from momentum transport due to particle scattering. In the standard approach these scattering events are modelled in terms of a macroscopic shear viscosity. In a normal fluid star neutron-neutron scattering provides the most important contribution. The effect of the corresponding shear viscosity is usually estimated using the viscosity coefficient

$$\eta = 2 \times 10^{18} \rho_1^{9/4} T_9^{-2} \text{g/cms}. \quad (46)$$

For the r-modes, this leads to a dissipation time-scale

$$t_{sv} \approx 6.7 \times 10^7 M_\odot^{-5/4} R_{10}^{23/4} T_9^2 \text{s} \quad (47)$$

At high temperatures (above a few times $10^9$ K) bulk viscosity is the dominant dissipation mechanism. Bulk viscosity arises as the mode oscillation drives the fluid
away from beta equilibrium. It corresponds to an estimate of the extent to which energy is dissipated from the fluid motion as weak interactions try to re-establish equilibrium. The mode energy lost through bulk viscosity is carried away by neutrinos. An estimate of the bulk viscosity damping rate for the r-modes is complicated by the fact that one must determine the Lagrangian density perturbation \[93\]. For r-modes this quantity vanishes at leading order so the calculation must be carried at least to order \(\Omega^2\) in the slow-rotation expansion. In the standard case, where \(\beta\)-equilibrium is regulated by the modified URCA reactions, the relevant bulk viscosity coefficient is

\[
\zeta = 6 \times 10^{25} \rho_{15}^2 T_6^6 \left(\frac{\omega_r}{1\,\text{Hz}}\right)^{-2} \text{g/cm} \cdot \text{s} .
\]

(48)

This leads to an estimated bulk viscosity timescale for \(n = 1\) polytropes \[10\]

\[
t_{\text{BV}} \approx 2.7 \times 10^{11} M_{1.4} P_{-10}^2 T_9^{-6} \text{ s} \quad (49)
\]

This result, which was obtained within the Cowling approximation agrees reasonably well with a calculation including also the perturbed gravitational potential \[96\].

![Figure 10](image-url)

**Figure 10.** Schematic illustration of the CFS instability window: At low temperatures (region I) dissipation due to shear viscosity counteracts the instability. At temperatures of the order of \(10^{10} \text{ K}\) bulk viscosity suppresses the instability (region III). At very high temperatures (region IV) the nuclear reactions that lead to the bulk viscosity are suppressed and an unstable mode can, in principle, grow. However, this region may only be relevant for the first few tens of seconds following the birth of a neutron star. The main instability window is expected at temperatures near \(T_c \approx 10^9 \text{ K}\) (region II). Provided that gravitational radiation drives the unstable mode strongly enough the instability may govern the spin-evolution of a hot young neutron star. For the \(m = 4\) f-mode one finds \(\Omega_c \approx 0.95\Omega_K\), while the \(l = m = 2\) r-mode leads to \(\Omega_c \approx 0.04\Omega_K\). The instability may change considerably if we add more detailed pieces of physics, like superfluidity and the presence of hyperons, to the model.

From the above estimates we can deduce that a gravitational-wave driven instability will only be active in a certain temperature range. To have an instability we need \(t_{\text{gw}}\) to be smaller in magnitude than both \(t_{\text{sv}}\) and \(t_{\text{BV}}\). From the estimates above we see that shear viscosity will completely suppress the r-mode instability at core temperatures...
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below $10^5$ K. This corresponds to region I in Figure 10. Similarly, bulk viscosity will prevent the mode from growing in a star that is hotter than a few times $10^{10}$ K. This is the case in region III of Figure 11. However, if the core becomes very hot (as in region IV in Figure 10), then the star is no longer transparent to neutrinos and the bulk viscosity is strongly suppressed. This region is likely not very relevant in an astrophysical context as neutron stars will not remain at such extreme temperatures for long enough that an unstable mode can grow to a large amplitude. Finally, in the intermediate region II in figure 10, there is a temperature window where the growth time due to gravitational radiation is short enough to overcome the viscous damping and drive the mode unstable. This general picture holds both for the unstable f-modes and the r-modes. We find the relevant critical rotation rate, above which the mode is unstable, by solving

$$\frac{1}{2E} \frac{dE}{dt} = \frac{1}{t_{gw}} + \frac{1}{t_{bv}} + \frac{1}{t_{sv}} = 0 \quad (50)$$

for a range of temperatures. Detailed calculations show that viscosity stabilizes all f-modes below $\Omega_c \approx 0.95\Omega_K$ [22], while the r-modes are stable below $\Omega_c \approx 0.04\Omega_K$ [92, 93]. For a typical neutron star, the latter would imply that the r-modes are unstable at rotation periods shorter than 25 ms. The fact that this estimate is close to the initial spin period of the Crab pulsar inferred from observational data, $P_o \approx 19$ ms, led to the suggestion that the r-mode instability may play a significant role in the spin-evolution of nascent neutron stars. This possibility caused some excitement and spawned a multitude of studies into the r-modes and the instability mechanism. Much of this work is reviewed in [16]. The detailed studies illustrate that this is an incredibly difficult problem, and that we need to understand many extremes of physics before we can draw any reliable conclusions regarding the relevance of the unstable r-modes. At the time of writing, the most important/interesting problems that need to be approached concern:

**Differential rotation:** Neutron stars are likely to be born differentially rotating. This means that any serious attempt to model their subsequent spin-evolution, eg. driven by a gravitational-wave instability, must be based on the oscillations of a differentially rotating star. This is a very difficult problem, but some progress has nevertheless been made. Imamura et al [97] have considered the onset of the secular f-mode instability for a wide range of angular momentum distributions. They find that differential rotation can have a significant effect. In their sample of parameter space, neutral $m = 2$ f-modes occur in the range $0.093 < \beta_s < 0.14$. This means that the critical point where the f-mode goes secularly unstable may be lowered by as much as 30% by differential rotation. The smallest values of $\beta_s$ are found for angular momentum distributions that are strongly peaked at the equator. There are as yet only one investigation into the effect that differential rotation has on the r-modes [98]. This study shows that the presence of corotation points complicates the problem considerably (see [10]).

**Superfluid stars:** As a neutron star cools below a few times $10^9$ K the extreme density in the core is expected to lead to the formation of various superfluids. The superfluid constituents play a crucial role in determining the dynamical properties of a rotating neutron star. In particular, the interplay between the lattice nuclei and the
superfluid in the inner crust is a key agent in the standard model for pulsar glitches. Intuitively, one would expect the presence of a superfluid to have a considerable effect also on the various modes of pulsation.

Studies of oscillations in superfluid stars have so far been based on models allowing for two distinct, dynamically coupled, fluids. These represent the superfluid neutrons and the “protons”. The latter can be viewed as a conglomerate of all charged components in the star. These are assumed to be electromagnetically coupled and hence become comoving on a very short timescale. Epstein [99] (and later Mendell [100]) argued, using a simple counting of the fluid degrees of freedom, that there ought to exist a new class of modes in a superfluid star. These “superfluid” modes have the protons moving oppositely to the neutrons, unlike an “ordinary” fluid mode that has the neutrons and protons moving more or less together (see [101, 102] for detailed discussions). An important feature of the superfluid problem is that a momentum induced in one of the constituents will cause some of the mass of the other to be carried along. Because of this “entrainment”, the flow of neutrons around the neutron fluid vortices (recall that a superfluid mimics large scale rotation by forming a large number of vortices) will induce a flow in a fraction of the protons, leading to magnetic fields being formed around the vortices. But since the electrons are coupled to the protons on very short timescales, some electrons will track the entrained protons. Mutual friction is the dissipative scattering of these electrons off of the magnetic fields associated with the vortices [103].

Superfluid mutual friction has been shown to suppress the instability of the f-mode in a rotating (Newtonian) star [104]. It was originally thought to be the dominant damping agent also on the r-modes, but a calculation by Lindblom and Mendell [105] suggests that a typical result for r-mode dissipation due to mutual friction is

\[ t_{mf} \approx 2 \times 10^5 P_5^{\frac{5}{3}} \text{ s} \]  

(51)

This would mean that the r-mode instability window is essentially unaffected by the inclusion of mutual friction in the model. However, \( t_{mf} \) is sensitive to changes in the entrainment parameter [105], and there seems to be critical values for which the mutual friction timescale becomes very short. These “resonances” are as yet not well understood [106] and it is clear that much more detailed studies into the pulsation properties of superfluid neutron stars and the associated dissipation mechanisms are needed [11].

Ekman layers: In addition to the core superfluid, we need to consider effects due to the presence of a solid crust in a mature neutron star. The melting temperature of the crust is usually estimated to be of the order of \( 10^{10} \) K (for a non-accreting star), so the crust may form shortly after the neutron star is born. The presence of a solid crust will have a crucial effect on oscillations of the core fluid. If the crust is assumed to be rigid the fluid motion must essentially fall to zero at the base of the crust. One can estimate the relevance of the crust using viscous boundary layer theory [108]. The

\[ \text{This point is further emphasized by the very recent demonstration that a so-called two-stream instability can operate in superfluid systems [107].} \]
region immediately beneath the crust then corresponds to a so-called Ekman layer. The thickness of the boundary layer ($\delta$) can be deduced by balancing the Coriolis force and shear viscosity:

$$\delta \sim \left( \frac{\eta}{\rho \Omega} \right)^{1/2}$$

(52)

where $\eta$ is the shear viscosity coefficient. After putting numbers into this relation we see that $\delta$ will typically be a few centimetres for a rapidly rotating neutron star. For a core r-mode the dissipation timescale due to the presence of the Ekman layer has been estimated as

$$t_{Ek} \approx 830 T_9 P_{1/2}^{-3} \text{ s}.$$  

(53)

From this estimate one can see that a solid crust would have a greater influence than many other dissipation mechanisms. For example, one finds that all neutron stars with a rigid crust are stable at rotation periods longer than roughly 5 ms, or $\Omega_c \approx 0.4$.

The crust-core interface has been the focus of several recent studies. These studies add further dimensions to the problem. The interplay between the r-modes in the fluid core and modes in the solid crust is particularly interesting. Calculations by Yoshida and Lee \[109\] show that as the spin of the star increases the r-modes will undergo a series of so-called avoided crossings with modes that are due to the elasticity of the crust. Furthermore, Levin and Ushomirsky \[110\] have shown that the assumption of a rigid crust, which was made in most estimates of the Ekman layer dissipation rate, is not warranted. They show that the r-mode typically extends into the crust, and as a consequence the true dissipation may well be a factor of perhaps a hundred weaker than (53). Finally, Lindblom et al \[111\] have discussed whether r-mode heating at the crust-core interface may melt the crust. They show that this is likely to be the case, and argue that the outcome may lead to the formation of partly frozen, partly melted crust regions analogous to ice chunks in the Arctic.

Despite some recent advances in our understanding of the effects that a solid crust may have on the r-modes, several crucial issues remain to be investigated in detail. For example, the inner crust of a neutron star (out to the neutron drip density) will likely be permeated by superfluid neutrons. It is not at all clear at the moment whether one should expect these neutrons to be strongly pinned to the crust nuclei or not \[112, 113\]. But if the superfluid is at all free to move relative to the crust it will likely lead to a weaker Ekman layer dissipation rate on the r-modes. Insights into this problem could also help shed light on the dynamics of pulsar glitches.

Magnetic fields: Given that a strong magnetic field is a key element of neutron star physics it is somewhat surprising that there have been few discussions of the effect that these fields may have on radiation instabilities. Particularly since the CFS instability is not unique to gravitational radiation: Any radiative mechanism will do, and it would not be surprising if a detailed investigation were to unveil interesting instabilities driven by electromagnetic radiation. But even if this does not turn out to be the case, the interplay between a large amplitude pulsation mode and the magnetic field is interesting.
Especially since the consequences may be observable electromagnetically. Of course, it is very difficult to incorporate a realistic magnetic field in a study of neutron star dynamics.

There have been some discussions of magnetic fields in connection with the gravitational-wave instability of the r-modes. In particular concerning the relation between an unstable mode and differential rotation. Spruit [114] has suggested that gravitational radiation reaction induces strong differential rotation in the star. This leads to a winding up of the interior magnetic field until a point is reached where the field becomes unstable due to buoyancy. This scenario was proposed as a model for gamma-ray bursts. The question whether an unstable r-mode leads to differential rotation, and whether the mode can be prevented from growing by the magnetic field was discussed by Rezzolla, Lamb and Shapiro [115]. They studied the so-called Stokes drift (due to which fluid elements undergo a secular drift when a wave is present in the system), the magnitude of which depends on the latitude of the fluid element and the r-mode amplitude. Key questions concern how this differential drift affects the magnetic field of the star, and what the backreaction on the mode may be. It was estimated that the instability could operate in young neutron stars \( B \approx 10^{12} \) G and recycled ones \( B \approx 10^{8} \) G provided that they spin fast enough. Just like in Spruit’s model, the differential drift due to the r-mode twists the magnetic field which affects its strength and nature. Rezzolla et al predict that the r-mode instability generates strong azimuthal magnetic fields in young pulsars.

Several other issues regarding the possible role of the magnetic field were discussed by Ho and Lai [116]. In particular, they attempt to quantify the extent to which electromagnetic radiation from the r-modes affects the growth rate of the instability. They find that the electromagnetic driving of the mode would become competitive with gravitational radiation for \( B \approx 10^{15} \) G. Ho and Lai also point out that the mode-oscillations will generate Alfvén waves in the magnetosphere. These would strengthen the instability somewhat. Finally, Mendell [117] has extended the crust/core boundary layer approach to the case of a magnetized core. His results indicate a significant damping of the r-modes for magnetic fields of the order of \( 10^{12} \) G and stronger. These various results indicate that magnetic field effects must be included in any detailed r-mode model, and also point to some potentially interesting repercussions for the instability.

Exotic bulk viscosity: The presence of exotic particles in the core of a neutron star may lead to significantly stronger viscous damping than assumed in the “standard” instability analysis [22, 92, 93]. Of particular relevance may be the presence of hyperons. Jones has estimated that the r-mode instability is almost completely suppressed in a star with a sizeable hyperon core [118]. A more detailed analysis of the role of hyperons has been carried out by Lindblom and Owen [119]. Their results show that the dissipation due to hyperons is, indeed, overwhelming. Most importantly, the hyperon bulk viscosity has the same temperature dependence as the shear viscosity, and thus it limits an instability at low temperatures. The available estimates suggest that the instability is unlikely to be significant for neutron stars with significant hyperon fractions. However,
it is important to emphasize the many uncertainties concerning exotic neutron star cores. For example, hyperons would act as a very efficient refrigerant. In fact, a neutron star with a hyperon core should cool rapidly to temperatures much lower than those suggested by observational data. This discrepancy can be avoided if the hyperons are superfluid and the relevant nuclear reactions are suppressed. But if this is the case then the bulk viscosity is also suppressed, and in addition one must discuss the role of the additional degree(s) of freedom that follows with having a star with superfluid components.

Interestingly, a strong viscous damping could work in favour of the r-modes as a gravitational-wave source. This is illustrated by the case of strange stars. The observational evidence for the existence of strange stars is tenuous, but they may exist if strange matter is indeed the most stable form of matter at high density. As was first pointed out by Madsen [120], the r-mode instability is affected by the fact that the bulk viscosity of strange matter is many orders of magnitude stronger than its neutron star counterpart. This means that the main instability window is located at comparatively low temperatures in a strange star. In Figure 10 we would have $T_c \approx 10^7$ K and $\Omega_c \approx 0.25\Omega_K$. The fact that the instability window is shifted to lower temperatures means that an accreting strange star may become a persistent gravitational-wave source once it reaches the spin-rate where the r-modes become unstable. This possibility will be discussed further in the next section.

Nonlinear saturation mechanisms: To model the gravitational waves associated with a secular instability we need to understand how an unstable mode evolves and what the “backreaction” on the bulk of the star is. This is a very difficult problem. The early growth phase of an unstable mode can obviously be described by linear theory, but we need to model what happens when the mode enters the nonlinear regime. Intuitively, one would expect the growth of the mode to be halted at some finite amplitude. It seems plausible that the excess angular momentum will be radiated away as the mode saturates, and that the star will spin down as a consequence. This general picture is supported by studies of instabilities in rotating ellipsoids [121, 122, 123]. In particular, Lai and Shapiro have discussed gravitational-wave signals associated with unstable ellipsoids in great detail. Their evolutions are driven by an unstable f-mode, and they argue that the characteristic gravitational-wave amplitude is

$$h_c \approx 1.2 \times 10^{-22} M_{1.4}^{3/4} R_{10}^{1/4} f_{Hz}^{1/2} D_{15}^{-1}$$

As the star spins down, the frequency of the signal varies in such a way that $f_{Hz} \approx 10^3 \rightarrow 0$, i.e. it moves through the region where ground-based interferometers will be the most sensitive. However, equation (54) is likely overly optimistic because it assumes that the spin-down is governed by an $m = 2$ mode. As discussed above, this mode leads to the fastest instability growth time, but it is not clear that the $m = 2$ f-mode will be unstable in a realistic (compressible) model. Having said that, one should not rule out the possibility since both General Relativity and differential rotation tend to destabilize the f-mode.
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As far as the unstable r-modes are concerned, most models to date have been phenomenological. The first such model, developed by Owen et al. [14], was based on expressing the conservation of energy and angular momentum as a system of evolution equations for the bulk rotation rate $\Omega$ and the mode-amplitude $\alpha$. The latter was defined by

$$\delta \vec{v} \approx \alpha \Omega R \left( \frac{r}{R} \right)^l Y_{ll}^{B} e^{i\omega_{r} t}$$

(55)

In order to account for nonlinear mode-saturation it was simply assumed that $\alpha$ stopped growing at a suitably large value. Qualitatively, such evolutions would be in accordance with the results for the analogous problem for ellipsoids [121, 122, 123].

A key question concerns what the maximum value of $\alpha$ is. In the first studies, it was assumed that $\alpha$ could reach values of order unity. This would lead to a newly born neutron star spinning down significantly on a timescale of a few weeks-months. The resultant gravitational-wave signal would be detectable by LIGOII for sources in the Virgo cluster [14]. The mode-saturation has recently been investigated in two different ways.

The first studies of this problem were based on fully nonlinear hydrodynamics. Stergioulas and Font [124] carried out a numerical evolution of the exact relativistic Euler equations on a background spacetime using a large-amplitude r-mode as initial data. They found that there was no apparent energy transfer into other modes until $\alpha$ was substantially larger than unity. This result was confirmed by evolutions of the full Newtonian hydro equations by Lindblom, Tohline and Vallisneri [125, 126]. Because the actual growth time due to radiation reaction is impractically long, they studied the maximum amplitude for a perturbation driven by an enhanced radiation reaction force [127]. They again found saturation at an amplitude large compared to unity. The limit appeared to be set by a shock wave, a dramatic wave-breaking on the star’s surface. A clear implication of these numerical evolutions was that coupling to other low-order modes did not set a stringent limit on the r-mode amplitude. At least not within the timescale of the evolutions (about ten rotational periods). To what extent these simulations represent the true physical behaviour was not clear, given the limitations in resolution and evolution time. Ideally, one would want to study the onset of instability and follow the mode through to saturation and see what effect the instability has on the spin of the star. Computationally, this means that one would need a numerical evolution that resolves the mode-oscillation (on a millisecond timescale) and tracks the star through hundreds of seconds. This is not possible given current technology.

A complementary way to address questions concerning nonlinear mode-coupling proceeds via second-order perturbation theory. By omitting higher-order couplings one can reduce the nonlinear evolution to a set of coupled ordinary differential equations for the amplitudes of the various modes of the system. The development of this perturbation theory for rotating stars was a major undertaking, which was recently completed by Arras et al [128]. They find that a strong resonant coupling to short wavelength inertial
modes would lead to r-mode saturation at an amplitude
\[ \alpha \approx 8 \times 10^{-3} P^{-5/2} \]
for a canonical star. This value is two orders of magnitude smaller (for rotation rates near the Kepler limit) than that used in the early studies of the r-modes [14]. This means that the effect of the instability would be much less dramatic.

At first sight it may seem as if the outcome of the perturbation calculation contradicts the nonlinear evolution results. However, the work of Arras et al [128] is consistent with the nonlinear evolutions in the sense that the coupling they find to low-order modes would lead to a saturation amplitude of order unity. It is the coupling to many short-wavelength modes, with frequencies comparable to that of the r-mode, that saturates the mode at a small amplitude. Given the coarse resolution of the numerical grids one would not expect these short wavelength oscillations to show up in the nonlinear evolutions. This interpretation is supported by the most recent fully nonlinear numerical evolution, due to Gressman et al. [129], which shows that increasing the resolution yields rapid nonlinear decay of the mode at decreased amplitude. This work points the way to future studies of the problem. The challenge is to test the saturation amplitude predicted by perturbation theory, perhaps via high resolution numerical evolutions. At the present time, the r-mode saturation remains an issue that requires further detailed consideration.

7.2. The viscosity driven instability

As discussed in Section 4, viscosity can drive an instability in prograde moving oscillation modes of a spinning star [55]. This instability sets in at the point where the rotational corrections to the mode frequency are such that the mode has zero frequency in the rotating frame, cf. (39). A viscosity unstable mode lowers the kinetic energy of the star by converting it into heat. In contrast to the radiation driven instability, this instability is not generic. The reason for this is that the quadrupole f-mode is the first to go unstable. Higher multipoles require faster rotation to reach their respective instability points.

By an interesting “coincidence” the critical rotation rates for the viscous instability and the radiation driven CFS instability are identical in a rotating Newtonian ellipsoid. Both instabilities become active at \( \beta_s \approx 0.14 \). This result no longer holds for compressible stars. In fact, one finds that the viscous instability will not be present unless the equation of state is unexpectedly stiff. For example, in Newtonian polytropes it is present only if \( n < 0.808 \) [10]. The viscosity driven instability is also not favoured by General Relativity. Relativistic effects tend to stabilize the modes that are susceptible to the viscosity instability, and one can show that the instability may operate in 1.4\( M_\odot \) relativistic stars only if \( n \leq 0.67 \) [130]. The neutron star equation of state is expected to be significantly softer than this. Nevertheless, the first relativistic study of the problem [131] indicates that a subset of the proposed equations of state allow the instability to
operate near the mass-shedding limit. One should also not rule out the possibility that rapidly spinning strange stars may undergo the viscous instability [11].

There have only been two studies of the possible gravitational-wave signal from a star undergoing the viscous instability. Almost twenty years ago, Ipser and Managan [132] estimated that an energy of $10^{-4} M_\odot c^2$ would be radiated through a signal with $f_{Hz} \approx 1250$. The signal would have very narrow bandwidth (a few Hz) since the star would not migrate far from the bifurcation point (cf. figure [4]). This is a reasonable assumption since the viscosity driven instability is likely to exist (if at all) only near the mass-shedding limit. These conclusions were confirmed, and somewhat refined, by Lai and Shapiro [54] who considered unstable ellipsoids.

However, it seems likely that damping due to gravitational-wave emission will suppress the viscous instability in all astrophysical neutron stars. The argument for this is quite simple. Just like in the case of the CFS instability the mechanism that drives the unstable mode must overcome all damping agents. In this case the viscous timescale associated with the unstable mode must be shorter than the gravitational-wave damping timescale. For the modes that are the most likely to suffer the viscosity driven instability (the prograde moving quadrupole f-modes) the various timescales will be weakly affected by the stellar rotation. This means that we can use the non-rotating values to estimate when the viscous timescale is shorter than that of gravitational-wave damping. For the shear viscosity we have [133]

$$t_{sv} \approx \frac{1}{5} \frac{\rho R^2}{\eta} \quad (57)$$

while the gravitational-wave damping should be adequately approximated by [23]. Combining these estimates we find that we must have $T < 2 \times 10^4$ K in order to have $t_{sv} < t_{gw}$. This shows that the viscosity driven instability may only operate in extremely cold neutron stars. However, neutron stars as cold as this are unlikely to exist in the Universe as accretion from the interstellar medium may prevent cooling significantly below $10^5$ K [134].

8. Astrophysical scenarios

It is not difficult to argue that neutron stars should be born rapidly spinning. By simply assuming conservation of angular momentum during the collapse of a solar-mass core from a radius of more than 1000 km to perhaps ten km, one finds that neutron stars ought to be born rotating as fast as they possibly can, i.e. near the break-up limit. However, this may be oversimplifying the problem greatly. In fact, Spruit and Phinney [135] have argued that magnetic locking between the core and the envelope of the progenitor star may prevent the collapsing core from spinning rapidly. In their model the main stellar rotation is due to the “kick” mechanism that may also cause a large linear momentum. Consequently, one would expect only a small subset of neutron stars to be born with spin periods shorter than (say) 10 ms.
Observational data regarding the initial spin rate of young pulsars is not reliable enough to piece together a coherent picture [130]. The best studied case is the Crab pulsar, whose initial period is estimated (assuming the standard magnetic braking model) to have been about 19 ms. The recently discovered young 16 ms X-ray pulsar in the supernova remnant N157B should have been born spinning faster, but it still probably had an initial period no shorter than a few ms [137]. These estimates should be compared to the shortest known period of a recycled pulsar of 1.56 ms, and with the theoretical lower limit on the period of about 0.5 to 2 ms [9], depending on the equation of state.

One possible explanation for the absence of young pulsars spinning near the Kepler limit could be that rotational instabilities play a significant role in the spin-evolution of nascent neutron stars, leading to a loss of angular momentum during or immediately after the initial collapse. In this respect the first estimates for the r-mode instability looked promising [92, 93]. Phenomenological spin-evolution models suggested that the unstable r-modes would be able to spin a newly born neutron star down to a rotation rate of roughly 20 ms, in good agreement with the observations of the Crab pulsar. It was also pointed out [93] that the r-mode instability might have the consequence that young neutron stars can only reach rotation periods shorter than (say) 3-5 ms if they are recycled by accretion in a binary system. The alternative model, that these stars are formed by accretion-induced collapse of a white dwarf, was not consistent with the r-mode results. The collapse would simply form a star hot enough that it would be expected to spin down because of the instability. Given more detailed studies of the relevant dissipation and mode-saturation mechanisms [16] it is not clear to what extent the original r-mode scenario is still viable. The emerging picture is complex, and the outcome depends crucially on physics that is not well understood.

There have not been many fully nonlinear studies aimed at establishing whether rotational instabilities play a role in gravitational collapse scenarios. The reason for this is that three dimensional studies of rotating core collapse are still prohibitively difficult (given current computational technology). The issue was, however, discussed by Rampp et al [138] who obtained evidence that instabilities are unlikely to be relevant during the collapse event itself. They found that there were events in the Zwerger-Müller collapse catalogue [72] that led to $\beta > \beta_d$, and therefore in principle the presence of a dynamically unstable bar-mode, at the time of core bounce. But since the core tended to reexpand before the unstable mode had time to grow to a large amplitude no enhancement in the gravitational-wave signal was observed. This may well be a generic feature, and we should perhaps not expect instabilities to emerge from collapse events involving a strong bounce (at least not until the remnant settles down). The strongest gravitational-wave signal estimated from the Zwerger-Müller data would correspond to a radiated energy of $E \approx 8 \times 10^{-8} M_{\odot} c^2$ with a characteristic frequency of a few hundred Hz. Such a signal is unlikely to be detected from sources beyond the local group of galaxies. The event rate for such sources is unfortunately very low, with only a few supernovae expected in 100 years in this volume of space.
Other studies have provided more promising (from the gravitational-wave point of view) results. Houser, Centrella and Smith [139] studied the development of unstable bars which grow while the collapse is stalled by the centrifugal force. Their results suggest that as much as $E \approx 10^{-3} M_\odot c^2$ could be radiated as gravitational waves. However, since the typical frequency of the waves is in the kHz range, this is probably not enough to make the signal detectable from events in the Virgo cluster. Furthermore, the study comes with a significant disclaimer since the entire collapse is not followed. The initial configuration chosen for the simulation may be one that is never actually reached in a “realistic” core collapse. This is an issue that needs to be investigated in more detail in the future: We need to understand what the “physical” region of the parameter space is.

It has been suggested that gravitational collapse may under some circumstances be halted by the centrifugal force and/or thermal pressure between white dwarf and neutron star densities. In this region there are no stable equilibrium configurations. In the original scenario, the formed object was assumed to be dynamically but not secularly stable. This would lead to a phase during which angular momentum is lost due to gravitational-wave driven instabilities. The system slowly spins down and contracts as angular momentum is lost until a neutron star is formed. Since this corresponds to a failed supernova this non-explosive avenue for neutron star formation became known as a “fizzler”. However, Shapiro and Lightman [140] showed that the scenario is unlikely to work for a star governed by a standard “cold” equation of state. The fizzler state will only exist for a part of the evolution — the system will become dynamically unstable before it reaches neutron star densities. Recent work by Hayashi et al [141] revives the idea by showing that the fizzler scenario could work for hot stars. A rotating configuration that is described by a hot equation of state can be dynamically stable because of the thermal pressure associated with a large lepton fraction. A preliminary study [142] indicates that relativistic effects tend to make the formation of fizzlers more difficult, but they cannot be ruled out.

In an interesting recent paper, Imamura and Durisen [143] have argued that the fizzler evolution it likely to be dominated by dynamical, rather than secular, instabilities. This result is based on the notion that deleptonization and cooling leads to constraction of the spinning configuration. As a consequence, the system approaches the dynamical instability point. Imamura and Durisen estimate the energy radiated as gravitational waves during the fizzler phase to be

$$\frac{\Delta E}{M c^2} \sim 10^{-3} \left( \frac{M}{M_\odot} \right) \left( \frac{P}{10 \text{ ms}} \right)^{-1}$$

where $P$ is the rotation period of the marginally stable state.

Fryer et al [144] have recently argued that the formation of medium size black holes from first generation stars could lead to fizzler-type gravitational-wave signals. The key idea is that angular momentum delays collapse of a $\sim 50M_\odot$ core at a radius $R \sim 1000$ km. If a secular instability develops (as in the original fizzler scenario) then it could lead to a significant amount of energy being released as gravitational waves.
This may lead to a signal strain of \( h \sim 10^{-21}/d \) Gpc, where the typical distance to the source will be greater than 5 Gpc. The frequency of the waves will be \( f_{Hz} \sim 10^{-2} \) and thus the signal could be of relevance for LISA. A similar scenario was proposed by New and Shapiro \[145\]. They consider the collapse of supermassive stars, with \( M \approx 10^6 M_{\odot} \). Should the collapse of such stars be stalled at a radius of \( 10^{17} \) m, a secular instability could lead to a gravitational-wave signal of \( h \approx 10^{-15} \) for a source at 1 Gpc. The corresponding wave-frequency would again be \( f_{Hz} \approx 10^{-2} \).

All the fizzler-type scenarios are interesting, and could well lead to relevant gravitational-wave signals. But it is clear that much more thought needs to go into the relevant stability criteria and the various viscous dissipation mechanisms that may affect the evolution of the proposed secular/dynamical instabilities. We also need to establish whether rotating core collapse can lead to the formation of objects in the range \( \beta_s < \beta < \beta_d \). So far we may only have scratched the surface of this challenging problem.

A secular gravitational-wave driven instability may also play a role in accreting systems. The possibility that gravitational waves from CFS unstable f-modes could balance the accretion torque, and hence halt the associated spin-up, was first discussed by Papaloizou and Pringle \[146\] and Wagoner \[147\]. The analogous scenario for the r-modes was analysed in detail by Andersson, Kokkotas and Stergioulas \[148\] (see also Bildsten \[149\]). It was originally thought that an accreting star in which an unstable mode could be excited to a significant level would reach a spin-equilibrium where gravitational-wave emission balances the torque. Should this happen, the neutron stars in Low-Mass X-ray Binaries (LMXBs) would be promising sources for detectable gravitational waves. For example, if gravitational-wave emission provides a limit on the spin of Sco X1, which is the strongest X-ray source in the sky, and the average accretion rate onto the neutron star is \( \dot{M} \approx 3 \times 10^{-9} M_{\odot}/yr \) then we can deduce that \( h \approx 3.5 \times 10^{-26} \rightarrow h_c \sim 10^{-21} \) (for \( P \approx 4 \) ms and \( f_{Hz} \approx 330 \)) after two weeks worth of signal has been accumulated \[148\]. In other words, this kind of source ought the be detectable by large-scale interferometers (perhaps requiring a narrowbanded advanced detector).

However, the original scenario may not work for neutron stars \[114, 150\]. In addition to generating gravitational waves that dissipate angular momentum from the system, the unstable mode will heat the star up (via the shear viscosity). If the viscosity gets weaker as the temperature increases (as is likely to be the case for an accreting neutron star since the core temperature is expected to be \( 10^8 \) K, which is smaller than \( T_c \approx 10^9 \) K, cf. Figure \[11\]), the mode-heating triggers a thermal runaway and in a few weeks/months the star may spin down to a comparatively low rotation rate. This could rule out the r-modes in galactic LMXBs as a source of detectable gravitational waves, since they will only radiate for a tiny fraction of the systems lifetime (although see \[151\]). Interestingly, one can show that the instability operates in a different way in strange quark stars \[120\]. Because of the significantly stronger bulk viscosity the r-mode instability window shifts towards lower temperatures. This means that, once the onset of instability is reached, an
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accreting strange star will reach a quasi-equilibrium state on a timescale of about a year. Hence, the r-modes in an accreting strange star may emit a persistent gravitational-wave signal that would provide unique evidence for the existence of such stars [152]. This is an exciting possibility since a proof of the existence of strange stars in the Universe would put useful constraints on the parameters of QCD. Very recent results [153] seem to indicate that accreting neutron stars with a superfluid hyperon core may evolve in a similar way, further strengthening the case for the r-mode instability to be relevant in LMXBs.

9. Concluding remarks

The aim of writing this article was to provide an overview of what we currently know about dynamical and secular instabilities in relativistic stars as gravitational-wave sources. There are many exciting possibilities (the various estimates are summarized in Figure 11) and it would be unwise to rule out any of the proposed scenarios given our current (lack of) knowledge of the detailed physics. For example, in the case of the r-modes the most recent evidence suggests that i) the gravitational-wave driven instability may be completely suppressed in a star with a sizeable hyperon core, and ii) mode-coupling to very short wavelength inertial modes could lead to saturation at low amplitudes. To what extent these results represent the “final say” on these issues is difficult to assess. First of all, the reliability of the relativistic mean-field equations of state used in the hyperon model is doubted by some nuclear physicists [154], and (if we want to take an extreme point of view) it is not clear that hyperons will be present at all in a neutron star. Secondly, the mode-coupling estimates were based on a simple barotropic neutron star model. At some level this assumption will be an oversimplification. For example, what happens to the mode-coupling if the neutron star core is superfluid? It is also important to realize that, even though the main focus of attention has been on the r-modes in the last few years, the secular instability of the f-modes is by no means dead and buried. The current Newtonian evidence may suggest that this instability is less relevant for uniformly rotating stars, but we know that this will change when we account for differential rotation and relativistic effects. Most importantly, if the $m = 2$ f-mode becomes unstable it may grow on a timescale comparable to that of the r-mode. There are also uncertainties concerning the dynamical bar-mode instability. Key questions concern whether configurations with $\beta > 0.27$ will ever occur in a realistic collapse scenarios, and whether the recently discovered instabilities that operate at much lower values of $\beta$ are of astrophysical significance.

If we want to have an accurate description of the emerging gravitational-wave signals from these scenarios we need to improve our theoretical models considerably. These are wonderful problems that require an understanding of many extremes of physics for their solution. The question is if it is realistic to expect us to be able to actually “solve them”. In many ways it seems likely that we will need observational data to put constraints on our theoretical models. As the new interferometers may open the gravitational-wave
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Figure 11. This figure provides a (quite optimistic) summary of the gravitational-wave sources discussed in the article. The effective (dimensionless) gravitational-wave strain $h_c$ is compared to the expected sensitivity of the new generation of interferometers (from top to bottom at the right end of the diagram: GEO600, LIGO I, VIRGO, LIGO II). The various source estimates correspond to: r-mode spindown of a nascent neutron star (assuming the mode can grow to a large enough amplitude and that it survives saturation etcetera), the analogous f-mode spindown (assuming that the $m = 2$ mode is unstable), gravitational collapse radiating only an energy equivalent to $8 \times 10^{-8} M_\odot c^2$, and black hole formation (quadrupole QNM ringing assuming an energy of $10^{-3} M_\odot c^2$). The bar-mode instability is represented by the grey region, where the upper limit corresponds to a source at 50 kpc and the lower limit to 15 Mpc (the Virgo cluster). Finally, the square indicates an accreting star which is prevented from spinning up by an unstable r-mode (assuming parameters relevant for Sco X1).

window to the Universe in the next few months, it is appropriate to ask to what extent the various theoretical models are falsifiable by observations. For example, given the current ideas about the r-mode instability one might predict that the mechanism may not operate in an accreting neutron star, but that it could lead to an accreting strange star emitting a more or less persistent gravitational-wave signal. The detection of such a signal from a galactic LMXB might therefore indicate the presence of a star governed by an “exotic” equation of state. Similarly, a newly born neutron star may not undergo a significant r-mode instability phase if it is massive enough to have a large hyperon core. But if hyperons are not present, or the relevant nuclear reactions are suppressed by the hyperons being superfluid, then the instability may play a role. There are many such comparisons that one can make, and it is clear that they could provide highly relevant insights into physics at extreme densities. Such results would be very exciting and would truly herald in the new era of “gravitational-wave astronomy”.

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