ON COHOMOLOGY OF INVARIANT SUBMANIFOLDS OF
HAMILTONIAN ACTIONS

YILDIRAY OZAN

1. INTRODUCTION

In [5] the author proved that if there is a free algebraic circle action on
a nonsingular real algebraic variety \( X \) then the fundamental class is trivial
in any nonsingular projective complexification \( i : X \to X_\mathbb{C} \). The Kähler
forms on \( \mathbb{C}^N \) and \( \mathbb{C}P^N \) naturally induce symplectic structures on complex
algebraic affine or projective varieties and in case they are defined over reals,
their real parts, if not empty, are Lagrangian submanifolds.

The following result can be considered as a symplectic equivalent of au-
thor’s above result on real algebraic varieties.

**Theorem 1.1.** Assume that \( G \) is \( S^1 \) or \( S^3 \) acting on a compact symplectic
manifold \( (M, \omega) \) in a Hamiltonian fashion and \( L \) is an invariant closed
submanifold. If the \( G \)-action on \( L \) is locally free then the homomorphism
induced by the inclusion, \( i : L \to M, \)

\[
H_i(L, \mathbb{Q}) \to H_i(M, \mathbb{Q})
\]

is trivial for \( i \geq l - k + 1 \), where \( k = \dim(G) \). In particular, the fundamental
class \( [L] \) is trivial in \( H_1(M, \mathbb{Q}) \).

Moreover, if the corresponding sphere bundle \( S^k \to L \times EG \to LG \) has
non torsion Euler class then the homomorphism

\[
i_* : H_{l-k}(L, \mathbb{Q}) \to H_{l-k}(M, \mathbb{Q}),
\]

induced by the inclusion \( i : L \to M \), is also trivial (see Section 2 for the
definition of \( EG \) and \( LG \)).

Since any compact connected Lie group has a circle subgroup we deduce
the following immediate corollary.

**Corollary 1.2.** Let \( G \) be a compact connected Lie group acting on a compact
symplectic manifold \( (M, \omega) \) in a Hamiltonian fashion and \( L \) is an invariant
closed submanifold of dimension \( l \). If the \( G \)-action on \( L \) is locally free then
the fundamental class \( [L] \) is trivial in \( H_l(M, \mathbb{Q}) \).
Remark 1.3. 1) It is well known that the natural actions of $U(n)$ and $T^n$ on the complex projective space $\mathbb{C}P^{n-1}$ and hence on any smooth projective variety, regarded as a symplectic manifolds with their Fubini-Study forms, are Hamiltonian (cf. see p.163 in [4]).

2) Consider the 2-torus $T^2 = S^1 \times S^1$ with the symplectic (volume) form $d\theta_1 \wedge d\theta_2$. Then, the $S^1$ action on $T^2$, given by $z \cdot (w_1, w_2) = (zw_1, w_2)$ is clearly symplectic but not Hamiltonian (it has no fixed point). Let $L$ be the invariant submanifold $S^1 \times \{pt\}$, on which the circle action is free. Clearly, the homology class $[L]$ is not zero in $H_1(T^2, \mathbb{Q})$. Hence, the assumption in the above theorem that the action is Hamiltonian, is necessary.

3) Since $S^3 = SU(2)$ is semisimple any symplectic $SU(2)$-action is Hamiltonian (cf. see p.159 of [4]).

Example 1.4. Let $G$ be a compact Lie group acting linearly on a closed manifold $M$. Dovermann and Masuda proved that if the action is semifree or $G = S^1$ then there exists a nonsingular real algebraic variety $X$ with an algebraic $G$-action equivariantly diffeomorphic to $M$ (cf see [1]). If the linear action on $X$ extends to some nonsingular projective complexification $X_{\mathbb{C}}$, then by part (1) of the above remark the action will be Hamiltonian and thus the above results can be applied to the pair $X \subseteq X_{\mathbb{C}}$.

Symplectic reduction and the proofs of the above results give a somewhat stronger result.

Let $G = G_1 \times \cdots \times G_d$, where each $G_j$ is either $S^1$ or $S^3 = SU(2)$ and suppose that it acts in a Hamiltonian fashion on a closed symplectic manifold $M$, with moment map $\mu : M \to g^*$, where

$$\mu = (\mu_1, \ldots, \mu_d) : M \to (g_1^*, \ldots, g_d^*),$$

each $g_j^*$ being the dual of the Lie algebra of $G_j$. Assume that $L$ is an invariant submanifold contained in a level set $M^0 = \mu^{-1}(v_1, \ldots, v_d)$ of the moment map. Further assume that, we can form successive symplectic reductions first by $G_1$, for level set $\mu^{-1}(v_1)$, then by $G_2$, for the level set $(\mu^{-1}(v_1) \cap \mu_2^{-1}(v_2))/G_1$, and so on for all $G_j$ (note that we use the same notation for the moment map on the reduced spaces). These assumptions clearly imply that the $G$-action on $L$ is locally free.

Theorem 1.5. Assume the above setup. Then the induced homomorphism $i^* : H_i(L, \mathbb{Q}) \to H_i(M, \mathbb{Q})$ is trivial, for $i \geq l - k + 1$, where $k = \dim(G)$ and $i : L \to M$ is the inclusion map.

2. Proofs

A $G$-space $X$ is called equivariantly formal if its equivariant cohomology is isomorphic to its usual cohomology tensored by the cohomology of the classifying space. Equivariant formality implies that the equivariant cohomology of $X$ injects into the equivariant cohomology of the fixed point set, $H^*_G(X) \hookrightarrow H^*_G(X^G)$ (cf. see Theorem 11.4.5 in [2]). On the other hand,
Kirwan proved in [3] that a compact Hamiltonian space $M$ is equivariantly formal. Therefore, Theorem 1.1 is a consequence of Theorem 2.1 below and the Kirwan’s result.

Let $G$ act freely on $EG = S^\infty$, which is $\lim S^{2n-1}$ in case $G = S^1$ and $\lim S^{4n-1}$ in case $G = SU(2)$. Also let $BG = EG/G$, which is $\mathbb{C}P^\infty$ if $G = S^1$ and $\mathbb{H}P^\infty$ if $G = SU(2)$. For any $G$-space $X$, we will denote the twisted product $X \times_G EG$ by $X_G$, where $G$-action on $X \times EG$ is given by $g \cdot (x, h) = (g^{-1} \cdot x, h \cdot g)$, for any $g \in G$, $h \in EG$ and $x \in X$. Then for any coefficient ring $R$, the $G$-equivariant cohomology of $X$ is defined to be the ordinary cohomology of $X_G$:

$$H^*_G(X, R) = H^*(X_G, R).$$

Let $p : X \times EG \to X_G$ be the quotient map. Since $S^\infty$ is contractible the induced homomorphism by $p$ in cohomology can be regarded as a map $p^* : H^*_G(X, R) \to H^*(X, R)$.

**Theorem 2.1.** Let $M$ be an orientable closed manifold equipped with an equivariantly formal $G$-action, and $L^1$ an invariant closed submanifold. If the $G$-action on $L$ is locally free then the homomorphism induced by the inclusion, $i : L \to M$,

$$H_i(L, \mathbb{Q}) \to H_i(M, \mathbb{Q})$$

is trivial for $i \geq l - k + 1$, where $k = \dim(G)$. In particular, the fundamental class $[L]$ is trivial in $H_1(M, \mathbb{Q})$.

Moreover, if the corresponding sphere bundle $S^k \to L \times EG \to LG$ has non torsion Euler class then the homomorphism

$$i_* : H_{l-k}(L, \mathbb{Q}) \to H_{l-k}(M, \mathbb{Q}),$$

induced by the inclusion $i : L \to M$, is also trivial.

The above theorem is a consequence of a more general result which can be stated using equivariant cohomology.

**Theorem 2.2.** Let $M$ be an orientable closed manifold with an equivariantly formal $G$-action and $L^1$ an invariant closed submanifold. If $i : L \to M$ is the inclusion map then the image of $i^* : H^*(M, \mathbb{Q}) \to H^*(L, \mathbb{Q})$ lies in the image of $p^* : H^*_G(L, \mathbb{Q}) \to H^*(L, \mathbb{Q})$.

If the $G$-action on $X$ is free then $X_G \to X/G$ has contractible fibers and thus $X_G \to X/G$ is a homotopy equivalence. If the action is locally free then the fibers are finite cyclic quotients of contractible spaces and therefore the rational cohomology of $X_G$ and $X/G$ are still isomorphic. Hence, we obtain the following corollary.

**Corollary 2.3.** Assume that $M$ and $G$ are as in the above theorem and the $G$-action on $L$ is locally free. If $p : L \to B = L/G$ is the quotient map, then the image of $i^* : H^*(M, \mathbb{Q}) \to H^*(L, \mathbb{Q})$ lies in the image of $p^* : H^*(B, \mathbb{Q}) \to H^*(L, \mathbb{Q})$. 
Proof of Theorem 2.2 Consider the following commutative diagram of Gysin sequences corresponding to the sphere bundles $S^k = G \to M \times EG \to M_G$ and $S^k = G \to L \times EG \to L_G$ ($k = \dim(G)$):

\[
\cdots \to H^i(M, \mathbb{Q}) \xrightarrow{p^*} H^i(M, \mathbb{Q}) \xrightarrow{p^*} H^{i-k}(M, \mathbb{Q}) \xrightarrow{i_*} H^{i+1}(M, \mathbb{Q}) \to \cdots
\]

where $p^*$ is the connecting homomorphism (can be thought as integration along fiber) and $e \in H^{k+1}(M, \mathbb{Q})$ is the image of the Euler class of the sphere bundle of under the natural map $H^{k+1}(M_G, \mathbb{Z}) \to H^{k+1}(M, \mathbb{Q})$.

Note that to prove the theorem it suffices to show that the map $p^*$ in the top row is trivial. Indeed, we claim that the map $H^{i-k}(M, \mathbb{Q}) \xrightarrow{i_*} H^{i+1}(M, \mathbb{Q})$ is injective. To see this let $M_G$ denote the fixed point and consider the following commutative diagram:

\[
H^{i-k}_G(M, \mathbb{Q}) \xrightarrow{i_*} H^{i+1}_G(M, \mathbb{Q})
\]

By assumption the vertical arrows are injections and hence it is enough to show that the bottom row is injective. For the latter, note that the $G$-action on $M_G$ is trivial and hence the corresponding $G$-bundle for $M_G$ is

\[
G \to M_G \times S^\infty \xrightarrow{id \times h} M^G \times BG,
\]

where $BG = CP^\infty$ or $\mathbb{H}P^\infty$, depending on whether $G$ is $S^1$ or $SU(2) = S^3$. Moreover, the Euler class of the bundle is $e = (1, e_0) \in H^0(M_G, \mathbb{Q}) \times H^{k+1}(BG, \mathbb{Q})$, where $e_0$ is a generator of $H^{k+1}(BG, \mathbb{Q})$ and hence cup product with the Euler class is injective.

Proof of Theorem 2.1 By the Universal Coefficient Theorem it suffices to show that the map $i_* : H^i(M, \mathbb{Q}) \to H^i(L, \mathbb{Q})$ is trivial for $i \geq l - k + 1$. Therefore, by Corollary 2.3 it is enough to show that the map

\[
p^* : H^i(L, \mathbb{Q}) \to H^i(L, \mathbb{Q})
\]

is trivial. However, since the $G$-action on $L$ is locally free the rational (co)homology of $L_G$ is equal to that of the $l - k$-dimensional orbifold $B = L/G$. In particular, $H^i(L_G, \mathbb{Q}) = 0$ for $i \geq l - k + 1$. This finishes the proof of the first statement.

For the second statement consider the Gysin sequence corresponding to the $G$-bundle $p : L \times EG \to L_G$:

\[
\to H^{l-2k-1}(L_G, \mathbb{Q}) \xrightarrow{i_*} H^{l-k}(L_G, \mathbb{Q}) \xrightarrow{p^*} H^{l-k}(L, \mathbb{Q}) \xrightarrow{i_*} H^{l-k-1}(L_G, \mathbb{Q}) \to .
\]
Since $e \in H^{k+1}(B,\mathbb{Z})$ is not a torsion class, by Poincaré duality the map given by the cup product with the Euler class is onto. This implies that the map $p^*$ is trivial and hence the proof concludes as in the first statement.

Proof of Theorem 1.5 First we will prove that

$$\text{Im}(H^i(M,\mathbb{Q}) \rightarrow H^i(L,\mathbb{Q})) \subseteq \text{Im}(H^i(L/G,\mathbb{Q}) \rightarrow H^i(L,\mathbb{Q})).$$

Proof is by induction on $d$, the number of factors in the decomposition $G = G_1 \times \cdots \times G_d$. The case $d = 1$ is contained in Theorem 2.2. Suppose that the theorem holds for all integers $1, \ldots, d - 1$, where $d \geq 2$. Consider the action of $G_1$ on $M$, with the moment map $\mu : M \rightarrow \mathfrak{g}_1^*$, where

$$\mu = (\mu_1, \ldots, \mu_d) : M \rightarrow (\mathfrak{g}_1^*, \ldots, \mathfrak{g}_d^*).$$

Let $M_{\text{red}}$ denote the reduced space $M^1 = \mu_1^{-1}(v_1)/G_1$. Note that $G_{\text{red}} = G_2 \times \cdots \times G_d$ has an induced Hamiltonian action on the symplectic manifold $M_{\text{red}}$ and the moment map $\mu$ descends to a moment map

$$\mu_{\text{red}} = (\mu_2, \ldots, \mu_d) : M_{\text{red}} \rightarrow (\mathfrak{g}_2^*, \ldots, \mathfrak{g}_d^*),$$

satisfying the same hypothesis as $\mu$. Abusing the notation further, we will denote $L/G_1$ by $L_{\text{red}}$. Note that $L_{\text{red}}$ is an invariant submanifold of $M_{\text{red}}$.

Hence, by the induction hypothesis

$$\text{Im}(H^i(M_{\text{red}},\mathbb{Q}) \rightarrow H^i(L_{\text{red}},\mathbb{Q}))$$

$$\subseteq \text{Im}(H^i(L_{\text{red}}/G_{\text{red}},\mathbb{Q}) \rightarrow H^i(L_{\text{red}},\mathbb{Q})). \quad (\ast)$$

Now, we will consider a ladder of exact sequences similar to the one used in the proof of Theorem 2.2

$$\cdots \rightarrow H^i(M_G,\mathbb{Q}) \xrightarrow{\rho^i} H^i(M,\mathbb{Q}) \xrightarrow{\rho^i} H^{i-k_1}(M_G,\mathbb{Q}) \xrightarrow{i^*} H^{i+1}(M_G,\mathbb{Q}) \rightarrow$$

$$\downarrow \kappa \quad \downarrow i^* \quad \downarrow \kappa \quad \downarrow i^*$$

$$\cdots \rightarrow H^i(M_{\text{red}},\mathbb{Q}) \xrightarrow{\rho^i} H^i(M^1,\mathbb{Q}) \xrightarrow{\rho^i} H^{i-k_1}(M_{\text{red}},\mathbb{Q}) \xrightarrow{i^*} H^{i+1}(M_{\text{red}},\mathbb{Q}) \rightarrow$$

$$\downarrow i^* \quad \downarrow i^* \quad \downarrow i^* \quad \downarrow i^*$$

$$\cdots \rightarrow H^i(L_{\text{red}},\mathbb{Q}) \xrightarrow{\rho^i} H^i(L,\mathbb{Q}) \xrightarrow{\rho^i} H^{i-k_1}(L_{\text{red}},\mathbb{Q}) \xrightarrow{i^*} H^{i+1}(L_{\text{red}},\mathbb{Q}) \rightarrow,$$

where $k_1 = \dim(G_1)$ and the maps from the top row to the middle one denoted by $\kappa$ and induced by inclusion maps also, are the Kirwan maps (3). As in the proof of Theorem 2.2, the map $\rho^i$ in the top row is trivial. Noting that $L_{\text{red}}/G_{\text{red}} = L/G$ it follows from $(\ast)$ and the above diagram that

$$\text{Im}(H^i(M,\mathbb{Q}) \rightarrow H^i(L,\mathbb{Q})) \subseteq \text{Im}(H^i(L/G,\mathbb{Q}) \rightarrow H^i(L,\mathbb{Q})).$$

Finally, the arguments in the first paragraph of the proof of Theorem 2.1 finishes the proof.
Remark 2.4. It is known that the Kirwan map $\kappa$ is surjective ([3]). Even though we don’t need this information for the above proof a diagram chase in the exact sequences implies the following corollary.

Corollary 2.5. Let $M, M_{\text{red}}, L$ and $L_{\text{red}}$ be as above. Then the map

$$p^* : \text{Im}(H^i(M_{\text{red}}, \mathbb{Q}) \to H^i(L_{\text{red}}, \mathbb{Q})) \to \text{Im}(H^i(M, \mathbb{Q}) \to H^i(L, \mathbb{Q}))$$

is onto for any $i$ and is an isomorphism for $i = 1$.

Acknowledgment. Some part of this research has been completed during a visit to Université de Rennes 1, Rennes France, in October 2002. I am grateful to Mathematics Department for the invitation and the warm hospitality. I am also grateful to Selman Akbulut, Heiner Dovermann and Yael Karshon for their comments on the earlier version of this note. I would like thank also Frances Kirwan for sending a copy of her book [3] to me to learn more about her results.

References

[1] K.H. Dovermann, *Equivariant algebraic realization of smooth manifolds and vector bundles*, Contemp. Math., vol. 182, Amer. Math. Soc., 1995, pp. 11-28.

[2] V. W. Guillemin, S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer-Verlag, Berlin, 1999.

[3] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes (31), Princeton University Press, Princeton, New Jersey, 1984.

[4] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford University Press, New York, 1997.

[5] Y. Ozan, *On homology of real algebraic varieties*, Proc. Amer. Math. Soc. 129 (2001), 3167-3175.