Gravitational wave defocussing in quadratic gravity

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We demonstrate that Huygens’ principle for gravitational waves fails in quadratic gravity models that exhibit conformal symmetry at high energies. This results in the blurring of gravitational wave signals over finite timescales related to the energy scale of new physics \( M_\star \). Furthermore, on very small scales the gravitational wave Green’s function reduces to that of the wave equation in two dimensions. In principle, \( M_\star \) could be constrained directly from gravitational wave observations.

I. INTRODUCTION

The recent direct detection of gravitational waves by LIGO [1] provides us with new ways of testing Einstein’s general relativity (GR) [2]. We are now able to directly measure the time-dependent, weak fluctuations of spacetime geometry first predicted over 100 years ago. A key prediction of GR is that, in many respects, these perturbations behave exactly like waves of electromagnetic radiation: Small amplitude gravitational waves propagate in sharply-defined spherical wavefronts traveling at a fixed speed. That is, gravitational waves obey the same Huygens’ principle that underpins our understanding of sound and light waves in three spatial dimensions.

A natural and interesting question is: does Huygens’ principle hold in theories of gravitation other than GR? Many such alternative theories [3–5] have been proposed, and they are often motivated by the failure of GR coupled to ordinary matter to explain observations of the universe at the largest scales. Others follow from the observation that naive attempts to quantize gravity are not renormalizable. Regardless of motivation, viable modified gravity theories tend to leave the predictions of GR unchanged on solar system scales, while significantly altering them over much smaller or larger distances. (See [4] for a recent review on observational tests of modified theories of gravity.)

One procedure for constructing alternate gravitational models involves adding functions of curvature scalars to the Einstein-Hilbert Lagrangian. These terms can be selected to accomplish a specific goals, as in many \( f(R) \) models [5], or they can be derived in an approximation to a more fundamental framework, as in string theory. In this work, we consider actions of the form

\[
S = \frac{M_\text{Pl}^2}{2} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{2}{M_\star^2} C^2 \right) + S_m. \tag{1}
\]

Here, \( C^2 = C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \) is the square of the Weyl tensor, \( M_\star \) is a fixed mass that sets the energy scale of new physics, \( \Lambda = 3H^2 \) is the cosmological constant (\( \Lambda > 0 \) for de Sitter space), \( S_m \) is the matter action, and \( M_\text{Pl}^2 = 1/8\pi G \) is the reduced Planck mass. This action belongs to the class of quadratic gravity theories [6–8]; i.e., models whose gravitational actions involving quadratic functions of curvature scalars. Any locally de Sitter solution of general relativity is a vacuum (\( S_m = 0 \)) solution of (1), including the family of Kerr black holes with cosmological constant.

We note that if we were to include an additional \( R^2 \) term in (1), we would have the most general form of quadratic gravity action. Such a term basically represents an \( f(R) \) contribution to the theory. For the linear perturbations that we consider below, such a contribution is fairly well studied [9]. We therefore omit this term for simplicity.

The spectrum of gravitational excitations about flat space in quadratic gravity is well understood [6]. In addition to the massless graviton of GR, the Weyl-squared term in the action induces a massive spin-2 field. When the action is expanded to second order in fluctuations, the sign of the massive graviton kinetic term is opposite to that of the massless graviton, indicating the presence of a quantum ghost instability. Though we will be exclusively concerned with classical physics in this work, this ghost mode implies that we should view (1) as an effective, as opposed to a fundamental, action. (A recent discussion of the ghost phenomenon in quadratic gravity can be found in [8], for example.) We do not speculate on the nature of the fundamental theory giving rise to (1), other than assuming that it is ghost-free and (1) is valid for scales \( \gtrsim 1/M_\star \), and may indeed be applicable over a range of shorter distances.

We can see that the \( C^2 \) term in (1) is negligible on

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1 If we included the \( R^2 \) term in the action, there would be an additional spin-0 degree of freedom in the theory [9]. Since modes of different helicity are decoupled in linear theory, metric perturbations from the \( R^2 \) term would just be added to the spin-2 fluctuations presented in this paper.

2 A classical theory that involves higher order curvature terms yet reduces to (1) on scales \( \gtrsim \alpha_m M_\star^{-1} \) (with \( \alpha_m \ll 1 \)) can be constructed as follows: Assume a gravitational Lagrangian \( \mathcal{L} = R - 2\Lambda - 2M_\star^{-2}C^2 + \alpha_3 M_\star^{-4} \mathcal{L}_3 + \alpha_4 M_\star^{-6} \mathcal{L}_4 \cdots \) where \( \mathcal{L}_3 \) is a term involving cubic curvature terms, \( \mathcal{L}_4 \) is a term involving quartic curvature terms, etc., and the \( \alpha_n \) are dimensionless parameters. Then, assume a hierarchy max\( \{ |\alpha_3|, |\alpha_4|, \ldots \} = \alpha_m \ll 1 \). It then follows that the dominant part of the action on scales \( \gtrsim \alpha_m M_\star^{-1} \) is exactly (1).

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large scales in the context of linear theory by a heuristic argument that will be confirmed more rigorously below: Linear metric fluctuations about symmetric backgrounds, such as Minkowski or de Sitter, may be expanded in terms of Fourier modes \( \propto e^{i k_\alpha x^\alpha} \). Since the Weyl term in (1) involves the square of second order derivatives of the metric, it will be negligible for long wavelength modes \( \max(|k_\alpha|, |k|) \ll M_* \) and the action will reduce to the familiar Einstein-Hilbert form. Conversely, for gravitational modes with short wavelengths \( \min(|k_\alpha|, |k|) \gg M_* \) we expect the \( C^2 \) term to dominate:

\[
S \approx -g^{\text{eff}} \int d^4x \sqrt{-g} C^2 + S_m, \quad g^{\text{eff}} = \frac{M_{\text{Pl}}}{M_*}, \tag{2}
\]

which yields the conformally invariant gravitational action originally written down by Weyl in 1918.

Such conformal theories are attractive because they involve no fixed length scales; that is, only angles are physically meaningful. Hence, all coupling constants are necessarily dimensionless, suggesting that these gravitational models are perturbatively renormalizable, unlike ordinary GR. However, if the universe really does possess an exact conformal symmetry, it must be spontaneously broken such that one recovers an effective Einstein-Hilbert action on astrophysical scales [10]. Equation (1) is a plausible ansatz for the action of theory with a broken conformal symmetry that behaves like GR in the infrared.

Now, since the Newtonian limit of GR has been verified in the laboratory on scales \( \gtrsim 10^{-8} \) m [11], this immediately suggests an upper bound on the Compton wavelength of the exotic physics scale: \( M_*^{-1} \lesssim 10^{-8} \) m. Enforcing this bound implies that the influence of the \( C^2 \) term on solar system tests or the phase evolution of binary inspirals will be negligibly small.

We can now attempt to anticipate the behaviour of classical gravitational waves governed by (1). As in GR, the details of wave propagation will be dictated by the Green’s function \( G(x,x’) \) associated with the relevant equation of motion. Due to local Lorentz invariance, the Green’s function can only be a function of the geodesic distance \( \ell(x,x’) \) between the field point \( x \) and the source point \( x’ \). When \( \ell \gg 1/M_* \), we should recover the Green’s function of ordinary GR with a cosmological constant. However, the behaviour of \( G \) for \( \ell \ll 1/M_* \) will be dictated by the conformally invariant part of (1); i.e., by the action (2). The crucial observation is as follows: The nullcone connecting \( x \) and \( x’ \) is defined by \( \ell(x,x’) = 0 \), which means that structure of the Green’s function for null and nearly-null separations is entirely determined by the Weyl-squared part of the action.

But Huygens’ principle is essentially the statement that the Green’s function has singular support on the future nullcone of the source point \( x’ \). Therefore, we have the strong suspicion that the principle will modified in the theory of gravity (1). In the following section, we demonstrate that this intuition is correct: Huygens’ principle does not hold for gravitational waves governed by (1) because their propagator is non-singular and has support for timelike separations of the field and source points. The \( \delta \)-function gravitational wave nullcone of GR is smeared out over a distance \( \approx 1/M_* \), implying an intrinsic blurriness of the signals detected by experiments such as LIGO.

\section{II. Gravitational Wave Green’s Function}

Variation of the action (1) with respect to the metric yields the equation of motion

\[
G_{\alpha\beta} + \Lambda g_{\alpha\beta} + 2M_*^{-2} B_{\alpha\beta} = M_{\text{Pl}}^{-2} T_{\alpha\beta}, \tag{3}
\]

where \( G_{\alpha\beta} \) and \( T_{\alpha\beta} \) are the Einstein and stress-energy tensors as usual, while \( B_{\alpha\beta} \) is the Bach tensor defined by:

\[
B_{\mu\nu} = -\nabla^\alpha \nabla_\alpha S_{\mu\nu} + \nabla^\alpha \nabla_\mu S_{\alpha\nu} + C_{\mu\alpha\nu\beta} S^{\alpha\beta}, \tag{4a}
\]

\[
S_{\alpha\beta} = \frac{1}{2} \left( R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \right). \tag{4b}
\]

Note that the Bach tensor involves fourth order derivatives of the metric, hence the action (1) represents a higher derivative theory of gravity.

The Bach tensor transforms trivially under conformal transformations, implying that \( B_{\alpha\beta} = 0 \) for conformally flat spacetimes. Assuming that \( \Lambda > 0 \), the vacuum solution of (3) with \( T_{\alpha\beta} = 0 \) is de Sitter space, which we take as the background geometry for perturbations. We work in the conformal time coordinate chart, given by

\[
ds^2 = (H\eta)^{-2} (-d\eta^2 + dx^2). \tag{5}
\]

An important quantity in de Sitter space for the calculation of Green’s functions is the geodesic distance \( \ell(x,x’) \) between spacetime points \( x = (\eta,x) \) and \( x’ = (\eta’,x’) \). This is given implicitly by the formula [12]:

\[
\sin^2 \left[ \frac{1}{2} H\ell(x,x’) \right] = \frac{1}{2} (\eta\eta’)^{-1} [-(\Delta \eta)^2 + r^2], \tag{6}
\]

where \( \Delta \eta = \eta - \eta’ \) and \( r = |x - x'| \). We note that in the limit that the geodesic distance is much less than the de Sitter curvature scale \( H|\ell| \ll 1 \), in which case we approach the flat spacetime limit, we have

\[
\ell^2(x,x’) \approx -(\Delta \eta)^2 + r^2, \quad H\eta \approx 1, \quad H\eta’ \approx 1. \tag{7}
\]

We consider gravitational wave fluctuations about such a de Sitter background. We will \textit{a priori} assume that the perturbations are transverse and traceless:

\[
\delta g_{\alpha\beta} = h_{\alpha\beta}, \quad g^{\alpha\beta} h_{\alpha\beta} = 0 = \nabla^\alpha h_{\alpha\beta}. \tag{8}
\]

Inserting (8) into the field equations and expanding to linear order in \( h_{\alpha\beta} \), one finds that the metric perturbation obeys a fourth order wave equation

\[
(2M_*^2)^{-1} (D^2 - 2H^2)(D^2 - 4H^2 - M_*^2) h_{\alpha\beta} = 0, \tag{9}
\]
where \( D^2 = \nabla_\alpha \nabla^\alpha \) is the covariant wave operator in de Sitter space. We further assume that \( h_{\alpha\beta} \) is purely spatial such that the perturbed line element is
\[
ds^2 = (H\eta)^{-2} \left[ -dT^2 + (\delta_{ij} + H\eta \psi_{ij})dx^i dx^j \right], \tag{10}
\]
where \( i, j = 1, 2, 3 \). It is straightforward to confirm that each component of the spatial tensor \( \psi_{ij} \) satisfies a fourth-order partial differential equation:
\[
\frac{H^3\eta}{2M_*^2} \left( \Box - \frac{M_*^2}{H^2\eta^2} \right) \eta^2 \left( \Box + \frac{2}{\eta^2} \right) \psi_{ij} = 0. \tag{11}
\]
Here, \( \Box = -\partial^2_t + \nabla^2 = -\partial^2_t + \partial^2_x + \partial^2_y + \partial^2_z \) is the ordinary d’Alembertian operator of flat space.

We note that our assumptions about the metric perturbations mean that we are considering gravitational waves with the same polarizations as found in GR, which is certainly not the most general perturbative ansatz. Of course, it would be interesting to consider other gravitational wave polarizations that may be expected due to the presence of an effective massive graviton [6]; however, for simplicity, we defer that discussion for future work.

It is useful to re-write (11) in terms of differential operators:
\[
\mathcal{L}_x \psi_{ij}(x) = 0, \quad \mathcal{L}_x = D_{2,x} \mathcal{E}_{2,x} \mathcal{D}_{1,x} \mathcal{L}_{1,x}, \tag{12}
\]
where
\[
\mathcal{L}_{1,x} = \Box + 2/\eta^2, \quad \mathcal{D}_{1,x} = \eta^2, \quad \mathcal{L}_{2,x} = \Box - \zeta/\eta^2, \quad \mathcal{D}_{2,x} = H\eta/2\zeta, \tag{13}
\]
and \( \zeta = M_*^2/H^2 \). The subscript \( x \) notation on the operators is meant to convey that they act with respect to the \( x = (\eta, x) \) spacetime point.

Now, we note that the inverses of \( \mathcal{D}_{1,x} \) and \( \mathcal{D}_{2,x} \) are trivial:
\[
\mathcal{D}_{1,x}^{-1} = 1/\eta^2, \quad \mathcal{D}_{2,x}^{-1} = 2\zeta/H\eta, \tag{14}
\]
and we have the following relationships between operators
\[
\mathcal{L}_{1,x} - \mathcal{L}_{2,x} = (2 + \zeta)\mathcal{D}_{1,x}^{-1}, \tag{15a}
\]
\[
\mathcal{L}_{1,x} \mathcal{D}_{1,x} \mathcal{L}_{2,x} = \mathcal{L}_{2,x} \mathcal{D}_{1,x} \mathcal{L}_{1,x}. \tag{15b}
\]
We now show how the identities (15) allow us to calculate the Green’s function \( G \) of \( \mathcal{L} \) given knowledge of the Green’s functions \( G_1 \) and \( G_2 \) of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), respectively. The respective Green’s functions satisfy
\[
\mathcal{L}_x G(x, x') = \delta^{(4)}(x - x'), \tag{16a}
\]
\[
\mathcal{L}_{1,x} G_1(x, x') = \delta^{(4)}(x - x'), \tag{16b}
\]
\[
\mathcal{L}_{2,x} G_2(x, x') = \delta^{(4)}(x - x'). \tag{16c}
\]
We claim that
\[
G(x, x') = \frac{2M_*^2[G_2(x, x') - G_1(x, x')]}{H\eta^2 (M_*^2 + 2H^2)}, \tag{17}
\]
or, re-written in a more compact form:
\[
G_{xx'} = (2 + \zeta)^{-1}(G_{xx'x} - G_{xx'}) \mathcal{D}_{2,x}^{-1}, \tag{18}
\]
where \( G_{xx'} = G(x, x') \), etc. To establish that (17) is indeed a solution of (16a), we operate \( \mathcal{L} \) on (18):
\[
\mathcal{L}_x G_{xx'} = \mathcal{L}_x(2 + \zeta)^{-1}(G_{xx'x} - G_{xx'}) \mathcal{D}_{2,x}^{-1}
\]
\[
= (2 + \zeta)^{-1}\mathcal{D}_{2,x} \mathcal{D}_{2,x}^{-1}(\mathcal{L}_{1,x} \mathcal{D}_{1,x} \mathcal{L}_{2,x} G_{xx'})
\]
\[
- \mathcal{L}_{2,x} \mathcal{D}_{1,x} \mathcal{L}_{1,x} G_{xx'}
\]
\[
= (2 + \zeta)^{-1}\mathcal{D}_{2,x} \mathcal{D}_{2,x}^{-1}(\mathcal{L}_{1,x} - \mathcal{L}_{2,x}) \mathcal{D}_{1,x} \delta^{(4)}
\]
\[
= \delta^{(4)}, \tag{19}
\]
where we have used (15b) to go from the first to the second line, (15a) to go from the third to the fourth line, and the shorthand notation \( \delta^{(4)}(x - x') \). Hence, our claim (17) is confirmed.

At first glance, (17) may be surprising since it states that the Green’s function of the fourth order differential operator \( \mathcal{L} \) is given by a superposition of the Green’s functions of two second order operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). However, we note that \( G_1 \) and \( G_2 \) are the Green’s functions for massless and massive gravitons in de Sitter space, respectively, which are both expected fields in the spectrum of quadratic gravity. Furthermore, the relative sign between \( G_1 \) and \( G_2 \) is reminiscent of the sign difference in the kinetic terms of these modes that leads to the ghost mode in the quantum version of the theory. We will soon see that this relative sign means the singular parts of \( G_1 \) and \( G_2 \) cancel out in (17), making \( G \) non-singular.

The Green’s function is not completely specified by (17) until we select boundary conditions. Causality dictates that the relevant Green’s function for gravitational wave propagation satisfies retarded boundary conditions; that is, \( G(x, x') \) vanishes if \( x \) is not in the causal future of \( x' \). We can satisfy this by selecting \( G_1 \) and \( G_2 \) to be the retarded Green’s functions of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), respectively. Fortunately, it is relatively easy to find \( G_1 \) and \( G_2 \) by either inverting the second order operators using standard techniques [13] or looking up the answer in the literature [14]. Either approach yields that
\[
G(x, x') = \frac{2M_*^2}{H\eta^2 (M_*^2 + 2H^2)} \left[ \frac{\delta(\Delta\eta - r)}{4\pi r} + \frac{\theta(\Delta\eta - r)}{4\pi\eta^2} + Q_\nu(x, x') \right], \tag{20}
\]
where \( \theta \) is the Heaviside function, \( \nu^2 = 1/4 - M_*^2/H^2 \),
\[
Q_\nu(x, x') = \theta(\Delta\eta) \frac{\sqrt{\eta^2}}{4\pi r} \int_0^\infty dk k \sin kr \times [J_\nu(-k\eta)Y_\nu(-k\eta') - J_\nu(-k\eta')Y_\nu(-k\eta)], \tag{21}
\]
and \( J_\nu \) and \( Y_\nu \) are Bessel functions. In Appendix A, we show how it is possible, but not overly instructive, to evaluate the integral in terms of an associated Legendre
function. A more useful expression is obtained by using the asymptotic expansion of the Bessel functions for large argument and integrating term-by-term:

\[
Q_\nu(x, x') = -\frac{\delta(\Delta \eta - r)}{4\pi r} - \frac{\theta(\Delta \eta - r)}{4\pi \eta r'} - \frac{(4\nu^2 - 9\theta(\Delta \eta - r))}{32\pi \eta r'} \sum_{n=0}^{\infty} c_n \phi^{2n},
\]

where \(\phi = M\ell(x, x')\). The \(c_n\) coefficients are explicitly calculable; we find that the first few are given by

\[
c_0 = 1, \quad c_1 = \frac{1}{8}, \quad c_2 = \frac{-4\nu^2 - 17}{768}, \quad c_4 = \frac{80\nu^4 - 840\nu^2 + 2381}{737280}.
\]

Inserting (22) into (20) yields

\[
G(x, x') = \frac{M_x^2 \theta(\Delta \eta - r)}{4\pi (H \eta)(H \eta')^2} \sum_{n=0}^{\infty} c_n \phi^{2n}.
\]

We see that the Green’s function contains no \(\delta\)-functions; that is, \(G(x, x')\) is finite everywhere. As mentioned in the Introduction, we expect \(M^{-1} \ll 10^{-8}\) m while current observations imply \(H^{-1}\) is of order the size of the observable universe; i.e., we can take \(M \gg H\). This implies that there are three regimes we can consider:

I. \(H\ell \gg 1\) (the spacetime separation between \(x\) and \(x'\) is greater than the cosmological horizon);

II. \(H\ell \ll 1 \ll M_x\ell\) (the spacetime separation between \(x\) and \(x'\) is smaller than the cosmic horizon but larger than the exotic physics scale); and

III. \(M_x\ell \ll 1\) (the spacetime separation between \(x\) and \(x'\) is smaller than the exotic physics scale).

We call (III) the “near nullcone regime” since it involves the smallest values of \(|\ell|\). In the near nullcone region we have

\[
G(x, x') = \frac{M_x^2 \theta(\Delta \eta - r)}{4\pi}.
\]

This is precisely the retarded Green’s function for gravitational waves satisfying the wave equation obtained from the linearization of (3) if \(G_{\alpha\beta}\) and \(\Lambda\) are ignored; i.e., \((2M_x)^{-2} \Box^2 \psi_{ij} = 0\) [15]. This confirms our heuristic argument that the behaviour of the Green’s function near the nullcone will be dominated by the Weyl-squared term in (1), and hence be radically different from GR.

We comment that (25) bears a striking resemblance to the Green’s function of \(\Box\) in (1 + 1)-dimensions. In other words, gravitational waves governed by (1) propagate as if they were in two dimensions over short distances. This is congruent with similar reductions in spacetime dimensionality observed in [16].

It is also interesting to examine (24) in regime (II) mentioned above. Physically, this is the Green’s function governing the propagation of gravitational waves over sub-cosmological distances larger than the exotic physics scale. Under these circumstances, the series in (24) can be re-summed into a Bessel function \(J_1\) of the first kind:

\[
G(x, x') = \frac{\theta(\Delta \eta)}{2\pi} \delta_{M_x}(\sigma(x, x')),
\]

\[
\delta_{M_x}(\sigma) = \frac{\theta(-\sigma) M_x J_1(M_x \sqrt{-2\sigma})}{\sqrt{-2\sigma}}.
\]

where \(\sigma(x, x') = \frac{1}{2}\ell^2(x, x')\) is Synge’s world function. It is easy to confirm that \(\delta_{M_x}\) is peaked about \(\sigma = 0\) with width \(\sim 1/M_x\) and that \(\int_{-\infty}^{\infty} \delta_{M_x}(\sigma) d\sigma = 1\). Hence, \(\delta_{M_x}\) is a delta function sequence, and we have

\[
\lim_{M_x \to \infty} G(x, x') = \frac{\theta(\Delta \eta) \delta(\sigma)}{2\pi} = \frac{\delta(\Delta \eta - r)}{2\pi r}.
\]

This is the familiar Green’s function for the wave equation \(-\Box \psi_{ij} = 0\); i.e., for gravitational waves in GR. Therefore, when \(M_x\) is finite but much larger than \(H\), the Green’s function over sub-horizon distances is a non-singular smoothed version of a \(\delta\)-distribution centred on the nullcone. Huygens’ principle only holds in an approximate form: instead of sharp propagation along null geodesics, gravitational waves will be defocussed.

It is perhaps useful to pause and discuss why this defocussing effect occurs. It certainly seems reasonable to assume that it rooted in the presence of the massive spin-2 mode in this type of quadratic gravity. However, the mere existence of such a mode is not sufficient to ensure a regular Green’s function on the null cone; if it were, we would expect the retarded propagator of a single massive field in \(3 + 1\) dimensions to be non-singular, which it is not. Actually, the essential feature is the existence two effective spin-2 degrees of freedom in the action arranged such that the total Green’s function (17) is the difference of the individual propagators. This particular property of quadratic gravity is not expected to be true; i.e., there ought to be other modified gravity models that exhibit similar behaviour. Furthermore, we can conclude the spin-2 fields need not be massive to generate a finite Green’s function. This is most easily seen by the counterexample of \(\Box^2 \psi_{ij} = 0\), which has a nonsingular propagator and no mass parameter. Finally, we observe that it does not seem to be possible to capture this effect from a dispersion relation only. For example, plane wave solutions \(\psi_{ij} \propto e^{-i k_0 t + i k \cdot x}\) of both \(\Box \psi_{ij} = 0\) and \(\Box^2 \psi_{ij} = 0\) propagate with dispersion relation \(k_\sigma^2 = |k|^2\), yet the Green’s function is singular in the first case and non-singular in the second.

On the other hand, dimensionful constants like \(M_x\) appearing in dispersion relations will indeed dictate the timescale over which gravitational waves signal are blurred, as we now demonstrate: Let us fix the source point to be at the origin of 4-dimensional Minkowski coordinates \(x' = (0, 0)\), and assume the observer follows an inertial trajectory that intersects the future lightcone of \(x'\) at a point \(x'' = r(1, \hat{r})\). Here, \(\hat{r}\) is a spatial unit vector
and \( r \) is the observer’s spatial distance from the source. The observer’s worldline is parametrized by

\[
x(\tau) = x'' + \tau \sqrt{1 - v^2}, \quad v^2 = \mathbf{v} \cdot \mathbf{v},
\]

where \( \tau \) is the observer’s proper time and \( \mathbf{v} \) is the observer’s 3-velocity in these coordinates. Note that \( \tau = 0 \) corresponds to the instant when the observer crosses the light cone of \( x' \). Evaluating the Green’s function on the observer’s worldline yields:

\[
G(x(\tau), x') = \theta(\tau) \frac{M_* J_1(M_* \sqrt{\tau^2 + 2\tau r})}{\sqrt{\tau^2 + 2\tau r}}.
\]

where \( \kappa \) is the relativistic Doppler factor

\[
\kappa = \frac{1 - v \cos \theta}{\sqrt{1 - v^2}}, \quad v \cos \theta = \mathbf{v} \cdot \hat{n};
\]

i.e., \( \theta \) is the angle between the observer’s 3-velocity and the line of sight from the source to \( x' \). In most astrophysical situations, one would expect \( \kappa = \mathcal{O}(1) \). Assuming the amount of time that the observer measures the gravitational wave signal is much less than the distance to the source (i.e. \( 0 < \tau \ll \kappa r \)) we have

\[
G(x(\tau), x') = \theta(\tau) \frac{M_* J_1(M_* \sqrt{2\tau r})}{\sqrt{2\tau r}}.
\]

This implies that the characteristic decay time \( \tau_d \) of the gravitational wave signal in the observer’s rest frame is

\[
\tau_d = \frac{1}{2\kappa M_*^2 r}.
\]

If one imposes \( M_* \lesssim 10^{-8} \) m (as suggested by laboratory tests [11]) and \( r \) is an astrophysically relevant distance, then \( \tau_d \) is an extremely short timescale. Conversely, if one has a particularly small Doppler factor \( \kappa \rightarrow 0 \), then it is easy to see that the characteristic decay time will be

\[
\tau_d = \frac{1}{M_*},
\]

which will typically be much larger than in the \( \kappa \sim 1 \) case. This leads us to conclude that the best case scenario for observing the temporal broadening of the gravitational wave Green’s function is when the source is highly redshifted.

It is possible get a crude estimate of \( \tau_d \) for events similar to the first gravitational wave event GW150914 observed by LIGO [1], assuming that the centre of mass for such events is at rest in its home galaxy. Consider an event at cosmological redshift \( z \) at a comoving distance of \( r \). The redshift is related to the Doppler factor by \( \kappa = (1+z)^{-1} \), which yields

\[
\frac{\tau_d}{t_{Pl}} = \left( 1 + z \right) \left( \frac{r}{l_{Pl}} \right)^{-1} \left( \frac{M_*}{l_{Pl}} \right)^{-2},
\]

where \( l_{Pl} \) and \( t_{Pl} \) are the reduced Planck length and time, respectively. This can be rewritten as

\[
\frac{\tau_d}{10^{-6} \text{sec}} = \left( \frac{1 + z}{2} \right) \left( \frac{r}{450 \text{Mpc}} \right)^{-1} \left( \frac{M_*}{350 \text{AU}} \right)^{2},
\]

GW150914 occurred at a moderate redshift of \( z \approx 0.093 \) and distance of \( r \approx 440 \text{Mpc} \). If LIGO were to be able rule out any gravitational wave blurring with characteristic timescale \( \gtrsim 10^{-6} \) sec for this event, then we could infer that \( M_*^{-1} \lesssim 350 \text{AU} \). While such a constraint is not competitive with laboratory tests of Newton’s law, it does have the virtue of being obtained from an independent observation. Also, we note that the LIGO detector noise curve is (relatively) high for frequencies \( \gtrsim 10^4 \) Hz, so it might be challenging to resolve time differences of order \( \tau_d \sim 10^{-6} \) sec. Finally, obtaining more stringent constraints on \( M_* \) might be possible for events that are much closer (i.e., within the Milky Way), have a high centre of mass velocity relative to the detector, or both.

### III. SUMMARY

We have derived the retarded Green’s function for source-free gravitational waves propagating in the de Sitter vacuum of a quadratic gravity model described by (1). When the length scale of exotic physics is much smaller than the de Sitter radius, the singular Green’s function of GR is effectively smoothed-out about the nullcone. Since the gravitational wave propagator is non-singular and has support inside the nullcone, we conclude that Huygens’ principle does not hold in this version of quadratic gravity.

The retarded Green’s function can be used to write down an explicit solution of the gravitational wave initial value problem—usually called Kirchhoff’s formula—in both GR [13] and the current model [17]. In GR, one finds that gravitational waves at a given point \( x \) depend only on initial data on the point’s past nullcone. Conversely, for quadratic gravity the smoothed Green’s function (24) implies that the radiation at \( x \) depends on initial data inside the past nullcone of \( x \).

To grasp the significance of this fact, it is useful to imagine the same effect applied to more familiar physics: If the Green’s function of electromagnetism were regulated as in (24), it would be impossible to see a moving object clearly; at any given time the image detected by our eyes would be a superposition of the state of the object at multiple times.

The implication for gravitational wave observatories is clear: gravitational wave signals predicted by nonlinear simulations will be blurred as they travel from sources to detectors. This effect will be most pronounced for highly redshifted sources where the defocussing timescale is \( M_*^{-1} \). Depending on the noise characteristics of the detector and the theoretical uncertainties in the gravitational waveforms, it might be possible to directly con-
strain \( M_* \) independently of solar system and laboratory tests.

We note that gravitational wave sources in the early universe can plausibly have size of order \( M_*^{-1} \) or smaller, and are also intrinsically highly redshifted. Hence, the investigation of gravitational waves governed by (1) and produced during such epochs could be very interesting.

**Appendix A**

In this Appendix, we demonstrate how (21) can be evaluated in terms of an associated Legendre function. We first rewrite \( Q_\nu(x, x') \) as

\[
Q_\nu(x, x') = -\frac{\Theta(\Delta \eta)}{4\pi r^2} M_\nu \left( -\frac{\eta}{r}, -\frac{\eta'}{r} \right),
\]

where,

\[
M_\nu(\alpha, \beta) = \sqrt{\alpha \beta} \text{Im} \left[ \int_0^\infty dz \sin z \times \right. \\
\left. H_\nu^{(1)}(\alpha z) H_\nu^{(1)}(\beta z) \right], \quad \beta \geq \alpha > 0.
\]

This can be rewritten in terms of modified Bessel functions of the second kind:

\[
M_\nu(\alpha, \beta) = \frac{4\sqrt{\alpha \beta}}{\pi^2} \text{Im} \left[ \int_0^\infty dz \sin z \times \\
K_\nu(-i\alpha z) K_\nu(i\beta z) \right].
\]

The integral can be evaluated if we displace \( \alpha \) and \( \beta \) off the real axis

\[
\alpha \mapsto \alpha + i\epsilon, \quad \beta \mapsto \beta - i\epsilon, \quad 0 < \epsilon \ll 1.
\]

Then we get \([18, \text{eq. 6.692.2}]\)

\[
M_\nu(\alpha, \beta) = -\frac{\text{sec}(\pi \nu)}{\alpha \beta} \text{Im} \left[ \frac{P_{\nu-1/2}^{-1}(u + i\delta)}{\sqrt{(u + i\delta)^2 - 1}} \right],
\]

where \( P_{\nu-1/2}^{-1}(u) \) is an associated Legendre function and

\[
u = -1 + \frac{1 - (\alpha - \beta)^2}{2\alpha \beta}, \quad \delta = \frac{(\beta - \alpha)(1 - u)}{\alpha \beta} \epsilon.
\]

Making use of (6), we see that

\[
u = -\cos H \ell(x, x').
\]

It follows that \( 0 < \delta \ll 1 \), so we obtain

\[
Q_\nu(x, x') = \frac{\Theta(\Delta \eta) \text{sec}(\pi \nu)}{4\pi \eta'} \text{Im} \left[ \frac{P_{\nu-1/2}^{-1}(u + i0^+)}{\sqrt{(u + i0^+)^2 - 1}} \right].
\]

Note that in order to actually use this expression, one needs to select the branch of \( P_{\nu-1/2}^{-1} \) carefully. For this reason, it is more straightforward to use the series expansion employed in Section II to obtain an explicit form of the Green’s function.

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