AN APPLICATION OF A $C^2$-ESTIMATE FOR A COMPLEX MONGE-AMPÈRE EQUATION

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Abstract. By studying a complex Monge-Ampère equation, we present an alternate proof to a recent result of Chu-Lee-Tam concerning the projectivity of a compact Kähler manifold $N^n$ with $\text{Ric}_k < 0$ for some integer $k$ with $1 < k < n$, and the ampleness of the canonical line bundle $K_N$.

Dedicated to the 110th anniversary of S. S. Chern with a great honor.

1. Introduction

In a recent preprint [4], the authors proved the following result.

Theorem 1.1 (Chu-Lee-Tam). Assume that $(N^n, \omega_q)$ is a compact Kähler manifold $(n = \dim_{\mathbb{C}}(N))$ with $\text{Ric}_k(X, \overline{X}) \leq -(k+1)\sigma |X|^2$ for some $\sigma \geq 0$. Then $K_N$ is nef and is ample if $\sigma > 0$.

The curvature notion $\text{Ric}_k$ is defined as the Ricci curvature of the $k$-dimensional holomorphic subspaces of the holomorphic tangent bundle $T'N$. Hence it coincides with the holomorphic sectional curvature $H(X)$ when $k = 1$, and with the Ricci curvature when $k = n = \dim_{\mathbb{C}}(N)$. The condition $\text{Ric}_k > 0$ is significantly different from its Riemannian analogue, i.e. the so-called $q$-Ricci of Bishop-Wu [2], since it exams only the holomorphic subspaces in $T'N$, thus unlike its Riemannian analogue, $\text{Ric}_k > 0$ ($< 0$) does not imply $\text{Ric}_{k+1} > 0$ ($< 0$). The study of the condition of $\text{Ric}_k < 0$ was initiated to generalize the hyperbolicity of Kobayashi to the $k$-hyperbolicity of a compact Kähler manifold by the second author [6]. It is closely related to the degeneracy of holomorphic mappings from $\mathbb{C}^k$ into concerned manifolds (cf. Theorem 1.3 of [6]). Moreover it was recently proved by the second author that $\text{Ric}_k > 0$ implies that $M$ is projective and rational connected. The above result of Chu-Lee-Tam answers a question raised by the second author in [6] regarding the projectivity of a compact Kähler manifold with $\text{Ric}_k < 0$ affirmatively. In view of the fact that $\text{Ric}_k < 0$ is the same as the holomorphic section curvature $H < 0$, the above result generalizes the earlier work of [13, 12].

The proof of [4] is via the study of a twisted Kähler-Ricci flow. In this note we prove the result via the Aubin-Yau solution [1, 16] to a complex Monge-Ampère equation (which is similar to the equation for the Kähler-Einstein metric in the negative first Chern class). This was the method utilized in [13]. Here a modification the Monge-Ampère equation (cf. (3.2)) is necessary to adapted the idea to the curvature condition $\text{Ric}_k < 0$ when $n > k > 1$.

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which then makes a more direct alternate proof possible. The method here also extends to
the more general setting considered in [4].

It was proved in [8] that any compact Kähler manifold with the second scalar curvature
$S_2 > 0$ (simply put $S_k$ is the average of $\text{Ric}_k$) must be projective. It remains an interesting
question if $S_2 < 0$, or $\text{Ric}_k^+ < 0$, also implies the projectivity. For more backgrounds and
references related to the theorem please refer to [7]. One can also find the definitions
and motivations of several other curvature notions, including $S_2$, $\text{Ric}_k^+$, and problems related
to them in [7].

2. Preliminaries

Here we collect some algebraic estimates as consequences of the assumption $\text{Ric}_k(\mathbf{X}, \overline{\mathbf{X}}) \leq
-(k + 1)\sigma|\mathbf{X}|^2$, $\forall (1,0)$-type tangent vector $\mathbf{X}$. They are useful for related estimate for a
Monge-Ampère equation (cf. [5,6] below) Since the result is known for $k = 1$, $n$ we always
assume that $1 < k < n$. The first is Lemma 2.1 of [4].

**Lemma 2.1.** Under the assumption that $\text{Ric}_k(\mathbf{X}, \overline{\mathbf{X}}) \leq -(k + 1)\sigma|\mathbf{X}|^2$ the following estimate holds

$$(k - 1)|\mathbf{X}|^2 \text{Ric}(\mathbf{X}, \overline{\mathbf{X}}) + (n - k)R(\mathbf{X}, \overline{\mathbf{X}}, \mathbf{X}, \overline{\mathbf{X}}) \leq -(n - 1)(k + 1)\sigma|\mathbf{X}|^4. \quad (2.1)$$

The result follows by summing $\text{Ric}_k(\mathbf{X}, \overline{\mathbf{X}}) \leq -(k + 1)\sigma|\mathbf{X}|^2$ (the assumption) for a suitable
chosen unitary basis. By using a Royden’s trick [9] the following result was derived out of
Lemma 2.1 (cf. [4] Lemma 2.2).

**Lemma 2.2.** Let $(N, \omega)$ be a compact Kähler manifold with

$\text{Ric}_k(\mathbf{X}, \overline{\mathbf{X}}) \leq -(k + 1)\sigma|\mathbf{X}|^2 \quad (2.2)$

for some $\sigma \geq 0$. Let $\tilde{\omega} = \omega \tilde{g}$ be another Kähler metric on $N$. Set

$$G = \text{tr}_\omega \omega.$$

Then the following estimate holds

$$2\tilde{g}^{ij} \tilde{g}^{kl} R_{ijkl} \leq \frac{-n - 1}{n - k} G \frac{(G^2 + \gamma^2)}{G \cdot \text{tr}_{\tilde{g}} \text{Ric} - \frac{k - 1}{n - k} \langle \omega, \text{Ric} \rangle \tilde{g}}. \quad (2.3)$$

**Proof.** Here we provide a proof using the averaging technique (cf. Appendix of [6]) instead
of Royden’s trick since the argument is more transparent. Pick a normal frame $\{\frac{\partial}{\partial z^i}\}$ so
that $\tilde{g}_{ij} = \delta_{ij}$ and $g_{ij} = |\lambda_i|^2 \delta_{ij}$. Then $\frac{\partial}{\partial \overline{z}^i}$, with $g = \frac{\partial}{\partial \overline{z}^i}$, is a unitary frame for $\omega$. Lemma 2.1 implies that (Einstein convention applied)

$$2(n - k) R_{ijkl} |\lambda_i|^2 |\lambda_j|^2 + (k - 1)G \text{Ric}^\omega_s |\lambda_s|^2 + (k - 1) \text{Ric}^\omega_{ij} |\lambda_i|^4$$

$$= n(n + 1) \int_{\mathbb{S}^{2n-1}} (k - 1)|\mathbf{Y}|^2 \text{Ric}^\omega(\mathbf{Y}, \overline{\mathbf{Y}}) + (n - k)R^\omega(\mathbf{Y}, \overline{\mathbf{Y}}, \overline{\mathbf{Y}})$$

$$\leq -(n - 1)(k + 1)n(n + 1)\sigma \int_{\mathbb{S}^{2n-1}} |\mathbf{Y}|^4 = -(n - 1)(k + 1)\sigma (G^2 + |\lambda|_4^4).$$

Here $\mathbf{Y} = \lambda_i w^i \frac{\partial}{\partial \overline{w}^i}$ with respect to a normal frame $\{\frac{\partial}{\partial \overline{w}^i}\}$, $\text{Ric}^\omega$ and $R^\omega_{ijkl}$ are the Ricci and
curvature tensor expressed with respect to the metric $\omega$ (namely the unitary frame $\{\frac{\partial}{\partial \overline{w}^i}\}$). The result then follows by identifying the terms invariantly. $\square$
For the application it is useful to write (2.3) as

$$2\tilde{g}^{ij} \tilde{g}^{kl} R_{ijkl} \leq \frac{-(n-1)(k+1)\sigma}{n-k} \left( G^2 + |g|^2 \right) - \frac{k-1}{n-k} G \cdot \text{tr}_g \text{Ric}$$

$$+ \frac{k-1}{n-k} \left( \langle \text{Ric}, \bar{\omega} \rangle_g \langle \omega, \bar{\omega} \rangle_g - \langle \omega, \text{Ric} \rangle_g \right). \quad (2.4)$$

3. PROOF OF THEOREM 1.1

Assume that the canonical line bundle $K_N$ of $(N, \omega)$ is not nef. Then there exists $\epsilon_0 > 0$ such that $\epsilon_0 [\omega] - C_1(N)$ is nef but not Kähler. Thus, $\forall \epsilon > 0$, $(\epsilon + \epsilon_0)[\omega] - C_1(N)$ is Kähler. This means that there exists a smooth function $\phi_\epsilon$ such that

$$\omega_\epsilon := (\epsilon_0 + \epsilon)\omega - \text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} \phi_\epsilon > 0. \quad (3.1)$$

By Aubin-Yau’s existence theorem and a priori estimate for a complex Monge-Ampère equation, we first prove the theorem below.

**Theorem 3.1.** Let $(N, \omega)$ be a compact Kähler manifold which satisfies (2.2) for some $\sigma \geq 0$. Then $K_N$ is nef.

**Proof.** For any $\epsilon > 0$, we consider the complex Monge-Ampère equation for $\psi_\epsilon$,

$$\left( \left( \epsilon + \epsilon_0 \right) \omega - \text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} (\phi_\epsilon + \psi_\epsilon) \right)^n = e^{\phi_\epsilon + \psi_\epsilon + \frac{1}{2(n-k)} \langle \phi_\epsilon + \psi_\epsilon, \omega \rangle_g} \quad (3.2)$$

and

$$(\epsilon + \epsilon_0)\omega - \text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} (\phi_\epsilon + \psi_\epsilon) > 0. \quad (3.3)$$

By the Aubin-Yau theorem [11, 16], there is a unique solution $\psi_\epsilon$ of (3.2). For simplicity, we let

$$\tilde{\omega}_\epsilon := (\epsilon + \epsilon_0)\omega - \text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} (\phi_\epsilon + \psi_\epsilon) = \omega_\epsilon + \sqrt{-1} \partial \bar{\partial} \psi_\epsilon,$$

$$\sigma_\epsilon := \phi_\epsilon + \psi_\epsilon. \quad (3.4)$$

Then taking $\partial \bar{\partial} \log(\cdot)$ on both sides of (3.2), we see that (3.2) is equivalent to

$$\tilde{\text{Ric}}_\epsilon := \text{Ric}(\tilde{\omega}_\epsilon) = \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} \left( \sigma_\epsilon + \frac{k-1}{2(n-k)} \sigma_\epsilon \right)$$

$$= -\tilde{\omega}_\epsilon + (\epsilon + \epsilon_0)\omega - \sqrt{-1} \frac{k-1}{2(n-k)} \partial \bar{\partial} \sigma_\epsilon. \quad (3.5)$$

Let $G = G_\epsilon = \text{tr}_{\tilde{\omega}_\epsilon} \omega$. Then by the calculation in the proof of the Schwarz Lemma [15], and in particular (2.3) of [8] (also see computations in the earlier work of [9, 5]), as well as the $C^2$-estimate computation in [11, 10] (a slight different calculation was done in [16, 11]), we have that

$$\tilde{\Delta}_\epsilon (\log G) \geq \frac{1}{G} \left( \tilde{\text{Ric}}_{g_\epsilon} \tilde{g}^{ij} \tilde{g}^{kl} \tilde{R}_{ijkl} - \tilde{g}_\epsilon^{ij} \tilde{g}_\epsilon^{kl} \tilde{R}_{ijkl} \right). \quad (3.6)$$

Applying Lemma [2.2] (namely (2.1)) to $G = G_\epsilon$ we have the estimate

$$\frac{1}{G} \tilde{g}_\epsilon^{ij} \tilde{g}_\epsilon^{kl} \tilde{R}_{ijkl} \leq \frac{-(n-1)(k+1)\sigma}{2(n-k)} \left( G^2 + \frac{1}{2(n-k)} |g|^2 \right) - \frac{k-1}{n-k} \text{tr}_{\tilde{g}_\epsilon} \text{Ric}$$

$$+ \frac{k-1}{2(n-k)} \frac{1}{G} \left( G \cdot \text{tr}_{\tilde{g}_\epsilon} \text{Ric} - \langle \omega, \text{Ric} \rangle_{\tilde{g}_\epsilon} \right). \quad (3.7)$$
Choosing local coordinates such that \((\tilde{g}_e)_{ij} = \delta_{ij}\), \(g_{ij} = g_{ij}\delta_{ij}\), then we also have

\[
G \cdot \mathrm{tr}_{\tilde{g}_e} \mathrm{Ric} - \langle \omega, \mathrm{Ric} \rangle_{\tilde{g}_e} = \sum_i \mathrm{Ric}_{\tilde{g}_e} \left( \sum_k g_{kk} - g_{ii} \right) = \sum_i \left( \mathrm{Ric}_{\tilde{g}_e} \left( \sum_{k \neq i} g_{kk} \right) \right) \\
\leq \sum_i ((\epsilon + \epsilon_0)g_{ii} + u_{e_{ii}}) \left( \sum_k g_{kk} - g_{ii} \right) \\
= (\epsilon + \epsilon_0)G^2 - (\epsilon + \epsilon_0)\langle \sqrt{-1}\partial\bar{\partial}u_e, \omega \rangle_{\tilde{g}_e}.
\]

Here we used (3.3) in the third line. Plugging this into (3.7), we have

\[
\frac{1}{G} \tilde{g}_e^{ij} g^i_{k} g^j_{l} R_{ijkl} \geq \frac{\sigma(n - 1)(k + 1) - (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} G \\
+ \frac{\sigma(n - 1)(k + 1) + (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} |g_{\tilde{g}_e}|^2 \\
- \frac{k - 1}{2(n - k)} \Delta_e u_e + \frac{k - 1}{2(n - k)} \frac{1}{G} \langle \sqrt{-1}\partial\bar{\partial}u_e, \omega \rangle_{\tilde{g}_e}.
\]

On the other hand, a direct calculation using (3.3) can express the first term in (3.6) as

\[
\frac{1}{G} \tilde{g}_e^{ij} g_{k}^{ij} \tilde{g}_e^i g_{l}^j = \frac{1}{G} \langle \mathrm{Ric}_{\tilde{g}_e}, \omega \rangle_{\tilde{g}_e} \\
= \frac{1}{G} \langle -\bar{w}_e + (\epsilon + \epsilon_0)\omega - \sqrt{-1}\partial\bar{\partial} \left( \frac{k - 1}{2(n - k)} u_e \right), \omega \rangle_{\tilde{g}_e} \\
= \frac{1}{G} \langle -\bar{w}_e + (\epsilon + \epsilon_0)\omega, \omega \rangle_{\tilde{g}_e} - \frac{1}{G} \frac{1}{2(n - k)} \langle \sqrt{-1}\partial\bar{\partial}u_e, \omega \rangle_{\tilde{g}_e}.
\]

Note that we used (3.5) in the third line above. Combining (3.6), (3.9) and (3.10), we have

\[
\tilde{\Delta}_e (\log G) \geq \frac{\sigma(n - 1)(k + 1) - (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} G \\
+ \frac{\sigma(n - 1)(k + 1) + (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} |g_{\tilde{g}_e}|^2 \\
+ \frac{(k - 1)}{(n - k)} \mathrm{tr}_{\tilde{g}_e} \mathrm{Ric} - \frac{(k - 1)}{2(n - k)} \tilde{\Delta}_e u_e \\
+ \frac{1}{G} \langle -\bar{w}_e + (\epsilon + \epsilon_0)\omega, \omega \rangle_{\tilde{g}_e}.
\]

Hence

\[
\tilde{\Delta}_e \left( \log G - \frac{k - 1}{2(n - k)} u_e \right) \geq \frac{\sigma(n - 1)(k + 1) - (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} G \\
+ \frac{\sigma(n - 1)(k + 1) + (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} |g_{\tilde{g}_e}|^2 \\
+ \frac{(k - 1)}{(n - k)} \left( \mathrm{tr}_{\tilde{g}_e} \mathrm{Ric} - \tilde{\Delta}_e u_e \right) \\
+ \frac{1}{G} \langle -\bar{w}_e + (\epsilon + \epsilon_0)\omega, \omega \rangle_{\tilde{g}_e}.
\]
Next, we observe that
\[ |g|_{\bar{g}_{\epsilon}}^2 \geq \frac{G^2}{n}, \]
\[ -\Delta_{\epsilon} u_{\epsilon} = -\bar{g}_{\epsilon ij} \left( \bar{\nabla}_{\epsilon} \bar{\nabla} - (\epsilon + \epsilon_0) g_{ij} \right) \]
\[ = -n - \bar{g}_{\epsilon} \text{Ric} + (\epsilon + \epsilon_0) G, \quad \text{and} \]
\[ \frac{1}{G}((\epsilon + \epsilon_0) \omega - \bar{\omega}_{\epsilon}, \omega)_{\bar{g}_{\epsilon}} = \frac{1}{G}(\epsilon + \epsilon_0)|g|^2_{\bar{g}_{\epsilon}} - \frac{1}{G}G \geq -1. \]

Plugging these three inequalities/equation above into (3.12), we see that
\[ \hat{\Delta}_{\epsilon} (\log G - \frac{(k - 1)}{2(n - k)} u_{\epsilon}) \]
\[ \geq \left( \frac{n}{n} \cdot \frac{\sigma(n - 1)(k + 1) - (k - 1)(\epsilon + \epsilon_0)}{2(n - k)} \right) G + \left( \frac{(\epsilon + \epsilon_0)(k - 1)2n}{(n - k)2n} \right) G \]
\[ + \left( \frac{\sigma(n - 1)(k + 1) + (k - 1)(\epsilon + \epsilon_0)}{2(n - k)n} \right) G - 1 - n \frac{(k - 1)}{n - k} \]
\[ = \left( \frac{(n + 1)\sigma(n - 1)(k + 1)}{2(n - k)n} + \frac{(\epsilon + \epsilon_0)(k - 1)(n + 1)}{2(n - k)n} \right) G - 1 - n \frac{(k - 1)}{n - k} \]
\[ \geq \max \left\{ \left( \frac{(n + 1)\sigma(n - 1)(k + 1)}{2(n - k)n}, \frac{(\epsilon + \epsilon_0)(k - 1)(n + 1)}{2(n - k)n} \right) \right\} \cdot G - 1 - n \frac{(k - 1)}{n - k}. \]

Now we apply the maximum principle to get a lower estimate of $\tilde{\omega}_{\epsilon}$. At the maximum of $u_\epsilon$, say $x_0$, since $\sqrt{-1} \partial \bar{\partial} u_\epsilon \leq 0$ we have that $((\epsilon + \epsilon_0) \omega - \text{Ric}(\omega))(x_0) \geq \tilde{\omega}_{\epsilon} > 0$ and $e^{\frac{2(n - k)}{n}} \sup_N u_\epsilon = e^{\frac{2(n - k)}{n}} u_\epsilon(x_0) \leq \frac{((\epsilon + \epsilon_0) \omega - \text{Ric}(\omega))^n}{\tilde{\omega}_n} \leq C$, for some $C$ independent of $\epsilon$.

This proves a uniform upper bound for $u_\epsilon$, and hence that
\[ \sup_N \tilde{\omega}_n \leq C, \quad \text{equivalently } W_n \geq C^{-1}, \quad \text{with } W_n := \frac{\omega^n}{\tilde{\omega}_n}. \] (3.14)

Again we apply the maximum principle to log $G - \frac{(k - 1)}{2(n - k)} u_\epsilon$. By (3.13), at the point $x_0'$, where the maximum of log $G - \frac{(k - 1)}{2(n - k)} u_\epsilon$ is attained, we have that
\[ G(x_0') \leq C \text{ for some } C \text{ independent of } \epsilon. \] (3.15)

Since $GW_{\frac{k - 1}{n - 1}} = Ge^{-\frac{k - 1}{n - 1}u_\epsilon}$, we infer that $\sup_N \left( GW_{\frac{k - 1}{n - 1}} \right)$ is also attained at $x_0'$.

By GM-AM inequality $G \cdot W_{\frac{k - 1}{n - 1}} \leq \left( \frac{d}{n} \right)^{\frac{k - 1}{n - 1}} \leq C \left( \frac{d}{n} \right)^{\frac{k - 1}{n - 1}}$ at $x_0'$. This, together with (3.15), implies $\sup_N \left( GW_{\frac{k - 1}{n - 1}} \right) \leq C$ for some $C > 0$ independent of $\epsilon$. Combining this with (3.14) we have that
\[ G \leq C \text{ hence } \tilde{\omega}_\epsilon \geq A \omega, \] (3.16)

for a constant $A > 0$ independent of $\epsilon$. This is a contradiction to that $\epsilon_0 [\omega] - C_1(N)$ is not Kähler by taking $\epsilon \to 0$. This completes the proof of Theorem 3.1.

A remark is appropriate to compare the above proof with that of [13]. The idea of using an Aubin-Yau solution is the same. The difference lies in the details. First we came up with a modified Monge-Ampère equation to accommodate the new curvature condition. Secondly Wu-Yau’s proof [13] of the $C^2$-estimate can be obtained by a direct application
of Royden’s version of Yau’s Schwarz lemma (precisely, Theorem 1, p554 of [9]). Namely no additional proof is necessary for bounding $G$ under the assumption of [13] (namely the holomorphic sectional curvature $H < 0$), in view of an obvious lower bound on $\text{Ric}(\tilde{\omega})$ from (3.3). By comparison, some nontrivial manipulations are needed above (at the least to the best knowledge of the authors) to get the $C^2$-estimate since one can not infer any useful information from (3.6) directly under $\text{Ric}_k < 0$ for some $k > 1$.

Once the nefness of $K_N$ is established, the ampleness of $K_N$ follows as Theorem 7 of [13] provided that $\sigma > 0$. In this case we take $\epsilon_0 = 0$. By considering the Monge-Ampère equation (3.2), repeating the argument above, since $\sigma > 0$ is assumed now, we still can have the uniform estimates (3.14), (3.16) and the upper bound of $u_\epsilon$ from the key estimate (3.13) independent of $\epsilon$.

Moreover the elementary inequality $\text{tr}_{\omega} \tilde{\omega}_\epsilon \leq \frac{1}{(n-1)!} (\text{tr}_{\omega_0} \omega)^{n-1} \frac{\omega^n}{\omega_{\epsilon}}$ implies that $\text{tr}_{\omega} \tilde{\omega}_\epsilon \leq C$. Hence we have that $\tilde{\omega}_\epsilon$ and $\omega$ are equivalent. Namely for some $C > 0$ independent of $\epsilon$

$$C^{-1} \omega \leq \tilde{\omega}_\epsilon \leq C \omega. \tag{3.17}$$

This also gives the $C^0$-estimate (namely the lower bound of $u_\epsilon$) by the equation (3.24). The $C^3$-estimate of Calabi [1, 16, 11] also applies here (cf. [12] for an adapted calculation to a settings similar to (3.2)). Alternatively one can also use the $C^{2, \alpha}$-estimate of Evans as in [10]. Uniform estimates for up to the third order derivatives of $u_\epsilon$ allow one to apply the Arzela-Ascoli compactness to get a convergent subsequence out of $u_\epsilon$ as $\epsilon \to 0$.

Taking $\epsilon \to 0$, and letting $u_\infty := \lim_{\epsilon \to 0} u_\epsilon$ and $\omega_\infty := -\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u_\infty > 0$, then it is easy to see that (3.2) becomes

$$(-\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u_\infty)^n = e^{u_\infty + \frac{k-1}{2(n-k)} u_\infty} \omega^n.$$

Taking $\partial \bar{\partial} \log(\cdot)$ on both sides of the above equation we have that

$$\text{Ric}(\omega_\infty) = -\omega_\infty - \frac{k-1}{2(n-k)} \sqrt{-1} \partial \bar{\partial} u_\infty.$$

This implies that $K_N$ is ample. The existence of a Kähler-Einstein metric is known by Aubin-Yau’s theorem.

We also remark that the argument can be easily modified to prove the same result under the assumption:

$$\alpha |X|^2 \text{Ric}(X, \overline{X}) + \beta R(X, \overline{X}, X, \overline{X}) \leq -\sigma |X|^4, \forall X \text{ of } (1,0)-\text{type},$$

for some positive constants $\alpha, \beta$ and $\sigma > 0$. The existing literatures (e.g. [14]) is enough to extend Theorem 1.1 to the case that $\text{Ric}_k$ is quasi-negative (as well as $\sigma$ above is quasi-positive) proving that $K_N$ is big. We leave the details to interested readers.

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