GROTHENDIECK RINGS OF QUEER LIE SUPERALGEBRAS

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Abstract. We determine the Grothendieck rings of the category of finite-dimensional modules over Queer Lie superalgebras via their rings of characters. In particular, we show that the $\mathbb{Q}$-span of the ring of characters of the Queer Lie supergroup $Q(n)$ is isomorphic to the ring of Laurent polynomials in $x_1, \ldots, x_n$ for which the evaluation $x_{n-1} = -x_n = t$ is independent of $t$. We thus complete the description of Grothendieck rings for all classical Lie superalgebras.

1. Introduction

The Grothendieck group is an invariant of an abelian category defined to be the $\mathbb{Z}$-span of all objects, modulo the relation $[A] + [C] = [B]$ for every exact sequence $0 \to A \to B \to C \to 0$. When the category is a tensor category, the Grothendieck group inherits a ring structure via $[A] \cdot [B] = [A \otimes B]$. For finite-dimensional modules over semisimple Lie algebras, the Grothendieck ring is isomorphic to the ring of characters due to the highest weight description of simple modules, c.f. e.g. [SV, Prop. 4.2]. For Lie superalgebras, the ring of characters is the quotient of the Grothendieck ring by the relation identifying a module with its parity shift.

In this paper, we determine the ring of characters for queer Lie superalgebras. These algebras are one of the strange series in Kac’s classification of simple Lie superalgebras [K], and is the remaining classical series for which the Grothendieck ring is described. For the queer Lie supergroup $Q(n)$, the description yields the following theorem.

Theorem 1. The $\mathbb{Q}$-span of the ring of characters of $Q(n)$ is isomorphic to the ring

\[ \left\{ f \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} : f \mid_{x_1=-x_2=t} \text{ is independent of } t \right\}. \]

The description of the ring of characters over $\mathbb{Z}$, given in Proposition 3, is more subtle since the dimensions of weight spaces are certain powers of 2. Theorem 1 is a natural extension of the description of Grothendieck rings for the other classical Lie algebras and superalgebras: The ring of $S_n$-invariant Laurent polynomials is the ring of characters of the Lie algebra $\mathfrak{gl}(n)$. Sergeev and Veselov showed in [SV] that such an evaluation condition characterizes the ring of characters for basic-classical Lie superalgebras. In [IRS], Im, Serganova and the author determine the Grothendieck ring for periplectic Lie superalgebras.

For basic-classical Lie superalgebras, the ring of characters is isomorphic to the ring of supercharacters. For the queer Lie superalgebra this is not the case. In fact, Cheng [Che] showed that supercharacters of a finite-dimensional nontrivial simple $q(n)$-modules are zero. Another obstacle which arises is the fact that the dimensions of the highest weight spaces of a simple modules can be larger than one.

To prove that all characters satisfy invariance conditions, we restrict to low-rank subalgebras. To show the converse, namely that every invariant function is a linear combination of characters, we use a basis of the ring of characters which consists of Euler characteristics of...
certain cohomologies introduced by Penkov and Serganova [PS]. We first prove the result for
the queer Lie supergroup \(Q(n)\), and then extend it to the other queer Lie supergroups and
carry it over to Lie superalgebras. We also give a description of \(J_n\) in terms of invariance
under the Weyl groupoid.

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2. The queer Lie superalgebra and its finite-dimensional representations

We recall the structure and representation theory of the queer Lie superalgebras. One of
their main difficulty is that they do not posses an even invariant bilinear form, moreover
their Cartan subalgebras are not purely even.

2.1. Preliminaries. The family of queer Lie superalgebras consist of: the subalgebra \(q(n)\)
of \(gl(n|n)\) consisting of matrices of the form \[
\begin{bmatrix}
A & B \\
B & A \\
\end{bmatrix}
\]
where \(A, B\) are \(n \times n\) matrices,
\[
\text{sq}(n) := \left\{ \begin{bmatrix}
A & B \\
B & A \\
\end{bmatrix} \in q(n) : trB = 0 \right\},
\]
\[
\text{pq}(n) := q(n)/CI \text{ and } \text{psq}(n) := \text{sq}(n)/CI.
\]
The even parts of \(q(n)\) and \(sq(n)\) are isomorphic to \(gl(n)\), whereas the even parts of \(pq(n)\) and \(psq(n)\) are isomorphic to \(sl(n)\).

The Cartan subalgebra \(h = h_0 \oplus h_1\) of \(q(n)\) and \(sq(n)\) consists of matrices of the form
\[
\begin{bmatrix}
A & B \\
B & A \\
\end{bmatrix}
\]
where \(A\) and \(B\) are diagonal \(n \times n\) matrices. For \(i = 1, \ldots, n\), let \(H_i = E_{ii} +
E_{n+i,n+i}\). Let \(\varepsilon_1, \ldots, \varepsilon_n\) denote the basis of \(h_0^*\) dual to \(H_1, \ldots, H_n\). For \(pq(n)\) and \(psq(n)\), \(h_0^*\) and \(\varepsilon_1, \ldots, \varepsilon_n\) denote their images under the natural projection.

The set of roots is \(\Phi = \{\varepsilon_i - \varepsilon_j : i \neq j\}\) where for each root there is an odd root vector
and an even root vector. We choose the standard set of positive roots:
\[
\Phi_0^+ = \Phi_1^+ = \{\varepsilon_i - \varepsilon_j : i < j\}.
\]
The Weyl group \(W\) is isomorphic to \(S_n\). We denote the denominators by
\[
R_0 := \prod_{\alpha \in \Phi_0^+} (1 - e^{-\alpha}), \quad R_1 := \prod_{\alpha \in \Phi_1^+} (1 + e^{-\alpha}), \quad R = \frac{R_0}{R_1}.
\]
Let \(\rho_0 := (n - 1)\varepsilon_1 + (n - 2)\varepsilon_2 + \ldots + \varepsilon_{n-1}\). Note that \(e^{\rho_0} R_0\) is \(W\)-anti-invariant and \(e^{\rho_0} R_1\)
is \(W\)-invariant.

2.2. Finite-dimensional modules and their characters. We briefly recall the construction
of highest weight \(q(n)\)-modules, see [CW] Sec. 1.5.4 for more details. Given \(\lambda \in h_0^*\), one defines a symmetric bilinear form on \(h_1\) by \(\langle v, w \rangle_\lambda := \lambda ([v, w])\). Let \(h_1' \subseteq h_1\) be a maximal isotropic subspace with respect to \(\langle \cdot, \cdot \rangle_\lambda\) and define \(h' = h_0 \oplus h_1'\). The one dimensional \(h_0\)-
module \(\C v_\lambda\), defined by \(xv_\lambda = \lambda(x)v_\lambda\), extends to an \(h'\)-module by letting \(h_1'.v_\lambda = 0\). Define
the induced \(h\)-module
\[
W_\lambda := \text{Ind}^h_{h'} \C v_\lambda.
\]
Then \(\dim W_\lambda = 2^{h(\lambda)/2 + 1}\) where \(h(\lambda)\) is the number of nonzero \(\lambda_1, \ldots, \lambda_n\). The following
lemma is standard (see for example [CW] Lem. 1.42).
Lemma 2. The \( h \)-module \( W_\lambda \) is irreducible. Moreover, every finite-dimensional irreducible \( h \)-module is isomorphic to \( W_\lambda \) for some \( \lambda \in h_0 \).

A simple finite-dimensional \( q(n) \)-module \( L(\lambda) \) has \( W_\lambda \) as its highest weight space. Moreover, a highest-weight module \( L(\lambda) \) is finite-dimensional if and only if \( \lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \) where \( \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \) for every \( i = 1,\ldots,n-1 \), and \( \lambda_i = \lambda_{i+1} \) implies \( \lambda_i = 0 \) (see [Pe] Thm. 4 and [CW] Thm 2.18). We denote the set of these highest weights by \( \Lambda_n \). Any finite-dimensional simple \( q(n) \)-module is a highest weight module. A \( q(n) \)-module can be integrated to the Lie supergroup \( Q(n) \) if and only if \( \lambda \in \Lambda_n \cap \mathbb{Z}^n \). Let \( \Lambda_n := \Lambda_n \cap \mathbb{Z}^n \).

Given a finite-dimensional \( q(n) \)-module \( M \) with weight space decomposition \( M = \bigoplus_{\lambda \in h_0} M_\lambda \) where \( M_\lambda = \{ v \in M \mid xv = \lambda(x)v \text{ for all } x \in h_0 \} \), we define the character of \( M \) to be

\[
\text{ch} M := \sum_{\lambda \in h_0} (\dim M_\lambda) e^\lambda.
\]

For a Lie superalgebra \( g \) (resp. Lie supergroup \( G \)), we denote by \( J(g) \) (resp. \( J(G) \)) the ring of all characters of finite-dimensional \( g \)-modules (resp. \( G \)-modules). We shall use the notation \( x_i := e^{\varepsilon_i}, i = 1,\ldots,n \).

An element in the ring of characters is also called a virtual character. Note that the ring of characters is isomorphic to the quotient of Grothendieck ring by the ideal generated by \( [M] - [\Pi M] \) where \( \Pi \) is the parity shift functor.

2.3. The Euler characteristic \( E(\lambda) \). Let \( \lambda \in \Lambda_n \). Note that \( W_\lambda \) is a module over the Cartan subgroup \( H \) that can be extended naturally to the Borel subgroup \( B \) corresponding to \( \Phi^+ \). We have the higher cohomology modules

\[
H^i(\lambda) := H^i(G/P, \mathcal{L}(W_\lambda)).
\]

Here \( P \) is the largest parabolic subgroup of \( G \) to which the \( B \)-module \( W_\lambda \) can be lifted, and \( \mathcal{L}(W_\lambda) \) denotes the \( G \)-equivariant \( \mathcal{O}_{G/P} \)-module induced by the \( P \)-module \( W_\lambda \). Since \( H^i(\lambda) \) is finite dimensional and is zero for large enough \( i \), the Euler characteristic

\[
E(\lambda) := \sum_{i \geq 0} (-1)^i [H^i(\lambda)]
\]

belongs to \( J(q(n)) \).

One has

\[
E(\lambda) = 2^{\frac{|\lambda|+1}{2}} R^{-1} \cdot \sum_{w \in W} (-1)^{l(w)} w \left( e^\lambda \prod_{\beta \in \Phi^+} \frac{1}{(1 + e^{-\beta})} \right).
\]

Recall the definition of Schur’s \( P \)-function \( p_\lambda \) for \( \lambda \in \Lambda_n \):

\[
p_{\lambda,n} = \sum_{w \in S_n/S_\lambda} w \left( x^\lambda \prod_{1 \leq i < j \leq n, \lambda_i > \lambda_j} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]

where \( S_n/S_\lambda \) denotes the set of minimal length coset representatives for the stabilizer of \( \lambda \) in \( S_n \). Then by [PS] Prop. 1, \( E(\lambda) = 2^{\frac{|\lambda|+1}{2}} \cdot p_\lambda \). By [B] Cor. 4.26, \( \{ E(\lambda) \mid \lambda \in \Lambda_n \} \) forms a basis to the character ring.
3. The ring of characters for queer Lie supergroups.

In this section we describe the ring of characters $Q(n)$ and deduce Theorem I as well as the description for the ring of character of $SQ(n)$.

**Proposition 3.** The ring of characters of $Q(n)$ is isomorphic to the ring

$$(3.1) \quad J_n = \left\{ f \in \bigoplus_{\lambda \in \mathbb{Z}^n} 2^{ \lfloor \frac{h(\lambda)+1}{2} \rfloor \mathbb{Z} x^\lambda} : f \big|_{x_1=-x_2=t} \text{ is independent in } t \right\}$$

where $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $h(\lambda)$ is the number of nonzero $\lambda_i$'s.

The following lemma shows that the ring of characters of $Q(n)$ and $SQ(n)$ are contained in $J_n$ (see (3.1)). The proof is similar to [SV, Prop. 4.3] except one replaces supercharacters by characters.

**Lemma 4.** Let $M$ be a module over $Q(n)$ or $SQ(n)$ then $chM$ is in $J_n$.

**Proof.** The character of $M$ is in $\bigoplus_{\lambda \in \mathbb{Z}^n} 2^{ \lfloor \frac{h(\lambda)+1}{2} \rfloor \mathbb{Z} x^\lambda}$ by Lemma 2 and the $S_n$-symmetry follows from restricting the representation of $f$ to the even part which is either $GL(n)$ or $SL(n)$.

For the evaluation property, restrict $M$ to $g_{12} := h_0 \oplus g_{i,\epsilon_1-\epsilon_2} \oplus g_{1,\epsilon_2-\epsilon_1}$. Then $M$ has composition factors of the form $e^\lambda + e^{\lambda-(\epsilon_1-\epsilon_2)}$ (typical case) or $e^\mu$ where $(\mu, \epsilon_1 + \epsilon_2) = 0$ (atypical case). The former is equal to $x_1^{\lambda_1} \cdots x_n^{\lambda_n} \left(1 + \frac{\epsilon_2}{x_1} \right)$ and the evaluation $x_1 = -x_2 = t$ equals zero. The latter is of the form $e^{\mu_1(\epsilon_1+\epsilon_2)+\mu_3\epsilon_3+\cdots+\mu_n\epsilon_n}$ and the evaluation gives equals to $-e^{\mu_3\epsilon_3+\cdots+\mu_n\epsilon_n}$. \hfill \Box

Note that since an element $f \in J_n$ is a $W$-invariant, the evaluation $f \big|_{x_{n-1}=-x_n=t}$ is also independent of $t$. Set $J_0 = \mathbb{Z}$.

**Definition 5.** The evaluation map $ev : J_n \to J_{n-2}$ is defined by $ev(f) = f \big|_{x_{n-1}=-x_n}$.

3.1. The kernel of the evaluation map. The following proposition is a generalization of the fact that the kernel of the evaluation map of $GL(m|n)$ is spanned by characters of Kac modules [HR, Thm. 17].

**Proposition 6.** The kernel of the evaluation map is in $J(Q(n))$.

**Proof.** Suppose that $ev(f) = 0$ for $f \in J_n$. Then $f$ is divisible by $x_{n-1} + x_n$ and by the $W$-invariance, also divisible by $\prod_{i<j} (x_i + x_j) = \prod_{i<j} (e^{\epsilon_i} + e^{\epsilon_j})$. So

$$f = \prod_{i<j} (e^{\epsilon_i} + e^{\epsilon_j}) \cdot g.$$ 

Since $f$ and $\prod_{i\neq j} (e^{\epsilon_i} + e^{\epsilon_j})$ are $W$-invariant, so is $g$ and by theory of symmetric functions, $g = \sum_{\text{finite}} a_{\lambda} s_{\lambda}$ where $a_{\lambda} \in \mathbb{Z}$ and $s_{\lambda}$ is the Schur Laurent polynomial

$$s_{\lambda} = e^{-\rho_0} R_0^{-1} \cdot \sum_{w \in W} (-1)^{l(w)} w \left( e^{\lambda+\rho_0} \right)$$
where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \lambda_1 \geq \ldots \geq \lambda_n \). Now
\[
f = \sum_{\lambda} a_{\lambda} \prod_{i \neq j} (e^{\epsilon_i} + e^{\epsilon_j}) e^{-\rho_0} R_0^{-1} \cdot \sum_{w \in W} (-1)^{l(w)} w (e^{\lambda + \rho_0})
\]
\[
= \sum_{\lambda} a_{\lambda} \prod_{i \neq j} (1 + e^{-(\epsilon_i - \epsilon_j)}) R_0^{-1} \cdot \sum_{w \in W} (-1)^{l(w)} w (e^{\lambda + \rho_0})
\]
\[
= \sum_{\lambda} a_{\lambda} p_{\lambda + \rho_0}.
\]
Note that the last equality uses the fact that \( \lambda + \rho_0 \) has a trivial stabilizer in \( W \), that is \( \lambda + \rho_0 \in \Lambda_n \). Now, since \( f \in J_n \), the maximal \( \lambda \) for which \( a_{\lambda} \neq 0 \) is divisible by \( 2^\left\lfloor \frac{h(\lambda)+1}{2} \right\rfloor \). Subtracting \( a_{\lambda} p_{\lambda + \rho_0} \) from \( f \), we can show that all \( a_{\alpha} \)'s are divisible by \( 2^\left\lfloor \frac{h(\lambda)+1}{2} \right\rfloor \). Thus \( f \) is in the \( \mathbb{Z} \)-span of the \( E(\lambda) \)'s and the assertion follows.

3.2. Surjectivity of the evaluation map.

**Proposition 7.** The evaluation map \( ev : J(Q(n)) \to J(Q(n-2)) \) is surjective.

To prove the proposition, we give for every \( p_{\lambda,n-2} \) a Schur function \( p_{\lambda,n} \) such that \( ev (p_{\lambda,n}) = p_{\lambda,n-2} \). For the convenience of the reader, we first present the argument on an example.

**Example 8.** Suppose \( \lambda = (3,1) \). Let \( \lambda = (3,1,0,0) \). Let us show that \( ev (p_{\lambda,4}) = p_{\lambda,2} \). If \( w \in S_4 / S_{\lambda} \) is such that the terms \( 1 + x_3^{-1} x_4 \) or \( 1 + x_4^{-1} x_3 \) appear in \( w \left( x^\lambda \prod_{i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right) \), then the evaluation of the term is zero. This means that for the evaluation to be nonzero \( w^{-1} (\{3,4\}) = \{3,4\} \), which in this example means that \( w \in S_2 : = \{1, (12)\} \). Then
\[
ev (p_{\lambda,4}) = \ev \sum_{w \in S_4 / S_{\lambda}} w \left( x^\lambda \prod_{i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]
\[
= \ev \sum_{w \in S_2} w \left( x^\lambda \prod_{\lambda_i > \lambda_j} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]
\[
= \sum_{w \in S_2} w \left( \ev x^\lambda \prod_{\lambda_i > \lambda_j} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]
\[
= \sum_{w \in S_2} \left( ev \left( x_1^2 x_2 + 1 + x_1^{-1} x_2 \frac{1 + x_2^{-1} t}{1 - x_1^{-1} t} \frac{1 - x_1^{-1} t}{1 + x_2^{-1} t} \frac{1 - x_2^{-1} t}{1 + x_2^{-1} t} \right) \right) = p_{\lambda,2}.
\]

**Proof of Proposition** For \( n = 2 \), the evaluation map is surjective since \( 1 \in J(q(2)) \). Suppose \( n \geq 3 \). Let \( \lambda \in \mathbb{Z}^{n-2} \) and extend it to \( \lambda \in \mathbb{Z}_+ \) by adding zeros, namely \( \{\lambda_1, \ldots, \lambda_{n-2}\} \cup \{0\} = \{\lambda_1, \ldots, \lambda_n\} \). We show that \( ev (E(\lambda)) = E(\lambda) \). Since \( E(\lambda) \), \( \lambda \in \Lambda_{n-2} \) span \( J(Q(n-2)) \) over \( \mathbb{Z} \), this implies surjectivity.
Let \( k \in \{1, \ldots, n-1\} \) be such that \( \lambda_k = \lambda_{k+1} = 0 \) (such \( k \) may not be unique). Denote \( \sigma_k = (k-n)(k+1)n \in S_n \) and let \( \text{ev}_k \) be a map on \( J_n \) such that \( \text{ev}_k(f) = f \mid_{x_k = x_{k+1}} \). Then by symmetry \( \text{ev} = \sigma_k \circ \text{ev}_k \). Now, if \( w \in S_n/S_\lambda \) is such that \( w^{-1}(\{k, k+1\}) \subseteq \{i, j \mid \lambda_i \neq \lambda_j \} \) then
\[
\text{ev}_k \left( w \left( x^\lambda \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right) \right) = 0.
\]

Denote by \( A \) the set of \( w \in S_n/S_\lambda \) for which \( w^{-1}(\{k, k+1\}) \subseteq \{i, j \mid \lambda_i = \lambda_j \} \). By the definition of \( \Lambda^n \), \( \lambda_i = \lambda_j \) only if \( \lambda_i = 0 \). Then, up to \( S_\lambda \), \( w^{-1}(\{k, k+1\}) = \{k, k+1\} \). Hence, \( w \) preserves \( \{k, k+1\} \) and consequently \( w \) preserves \( \{1, \ldots, n\} / \{k, k+1\} \). In particular, \( w \) commutes with \( \text{ev}_k \) and \( A \) is the permutation group of \( \{1, \ldots, n\} / \{k, k+1\} \). Thus
\[
\text{ev}(p_\lambda) = \sigma_k \circ \text{ev}_k(p_{\lambda,n})
\]
\[
= \sigma_k \sum_{w \in A} w \left( x^\lambda \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]
\[
= \sigma_k \sum_{w \in A} \text{ev}_k \left( x^\lambda \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]
\[
= \sigma_k \sum_{w \in A} \text{ev}_k \left( x^\lambda \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_{k+1}}{1 - x_i^{-1} x_{k+1}} \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_k}{1 - x_i^{-1} x_k} \prod_{\lambda_i \neq 0, \lambda_i \neq \lambda_j} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)
\]
\[
= \sum_{w \in S_{n-2}/S_\lambda} w \left( x^\lambda \prod_{1 \leq i < j \leq n-2} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right) = p_{\lambda,n-2}
\]
Since \( h(\lambda) = h(\lambda) \), we obtain the assertion. \( \square \)

3.3. The ring of characters. We can now prove the main theorem.

Proof of Theorem. We prove the theorem by induction on \( n \). For \( n = 0 \), \( J_n = \mathbb{Z} \) by definition. For \( n = 1 \), \( Q(1) \) admits a 1-dimensional representation whose character is a scalar multiple of \( e^{m_1 x_1} \) for every \( m \in \mathbb{Z} \). Then \( J(Q(1)) = \mathbb{Z}[x_1^{\pm 1}] = J_1 \).

Suppose that \( J(Q(n-2)) = J_{n-2} \). Recall the evaluation map \( \text{ev} : J_n \to J_{n-2} \) and that \( J(Q(n)) \subseteq J_n \). By Proposition \( \text{ev}(J(Q(n))) = J(Q(n-2)) = J_{n-2} \). So, any element in \( J_n \) is a sum of an element from \( J(Q(n)) \) and an element in the kernel of \( \text{ev} \). By Proposition \( \text{the kernel is also contained in } J(Q(n)) \) and the assertion is obtained. \( \square \)

Corollary 9. The ring of characters of \( SQ(n) \) is also isomorphic to \( J_n \). Indeed, since \( SQ(n) \) is a subgroup of \( Q(n) \), every \( Q(n) \)-module restricts to \( SQ(n) \) and so \( J(SQ(n)) \supseteq J(Q(n)) = \ldots \)
The ring of characters of the category of finite-dimensional half-integer weights is isomorphic to the ring of characters of the category of finite-dimensional integer (non-integer) weights is divisible by $\mathbb{Z}$.

Indeed, a module over $Q(n)$ (resp. $SQ(n)$) is lifts to a module over $PQ(n)$ (resp. $PSQ(n)$) if and only if the identity matrix acts by zero. This holds if and only if all the weights $\lambda$ of the module satisfy that $\sum_{i=1}^{n} \lambda_i = 0$. When the module is simple, this is true if and only if one weight $\lambda$ of the module satisfies that $\sum_{i=1}^{n} \lambda_i = 0$.

**Remark** 11. The category of finite-dimensional $Q(n)$-modules admits a subcategory of polynomial representations, namely those which appear as compositions factors in the tensor algebra of the natural representations. The Grothendieck ring of this subcategory is the set of all polynomials in $J_n$. This ring is isomorphic to the center of $U(q(n))$ and was described in [Pr, Thm. 2.11] (see also [CW, A.3.4]).

## 4. The Ring of Characters for Queer Lie Superalgebras

We first describe the ring of characters for the category of finite-dimensional modules whose weights are half integers. The simple modules in this category were studied in [CK].

**Proposition 12.** Let $J_n$ be as in (3.1) and

$$J_{\frac{1}{2}, n} = \left( \text{span}_x \frac{1}{2} \left[ \frac{x_{i+1}^1}{x_{i}^{k_1}} \cdots x_{n}^{k_n} \right] \mid k_1, \ldots, k_n \in \frac{1}{2} + \mathbb{Z} \right) S_n.$$

The ring of characters of the category of finite-dimensional half-integer weights is isomorphic to

$$J_n \oplus \prod_{1 \leq i < j \leq n} (x_i + x_j) J_{\frac{1}{2}, n}.$$

**Proof.** The elements in $\prod_{1 \leq i < j \leq n} (x_i + x_j) J_{\frac{1}{2}, n}$ are in the kernel of the evaluation map and so belong to the ring of characters for the same reasoning as in Proposition 3.

For the other inclusion, we need to show that the character of of any module $M$ a half-integer (non-integer) weights is divisible by $(x_i + x_j)$ for every $i \neq j$. Note that $\alpha = \varepsilon_i - \varepsilon_j$ is also an even root. Restrict $M$ to the corresponding $\mathfrak{s}(2)$-triple $\mathfrak{s}_{\alpha}$. Since the weights are not integers, zero is not a weight and the restriction of $\mathfrak{s}_{\alpha}$ is a direct sum of $\mathfrak{s}(2)$-strings of even length. In particular, as an $\mathfrak{h}_0$-module, $M = \bigoplus_{\lambda \in B} N_\lambda$ where $B$ is a finite subset of $\mathfrak{h}_0$ and $\text{ch} N_\lambda = e^\lambda + e^{-\lambda} = x^\lambda + x^{-\lambda} x_i^{-1} x_j = (x_i + x_j) x^\lambda x_i^{-1}$.

Similarly to $J_{\frac{1}{2}, n}$, we can define the vector space

$$J_{a, n} := \left( \text{span}_x \left\{ \frac{1}{2} \left[ \frac{x_{i+1}^1}{x_{i}^{k_1}} \cdots x_{n}^{k_n} \right] \mid k_1, \ldots, k_n \in a + \mathbb{Z} \right\} \right) S_n$$

for any $a \in \mathbb{C}/\mathbb{Z}$.

**Proposition 13.** The ring of characters of finite-dimensional $q(n)$-modules is isomorphic to

$$J_n \oplus \bigoplus_{a \in \mathbb{C}/\mathbb{Z}} \prod_{1 \leq i < j \leq n} (x_i + x_j) J_{a, n}.$$
Proof. Similarly to the previous proposition the elements in \( \prod_{1 \leq i < j \leq n} (x_i + x_j) J_{a,n} \) belong to the ring of characters for the same reasoning as they are in the kernel of the evaluation map.

For the other inclusion, let \( L(\lambda) \) be a simple highest weight module for which \( \lambda \in a + \mathbb{Z}^n \). Suppose \( a \neq \frac{1}{2} ) \), then, \( \lambda_i \neq -\lambda_j \) for every \( i \neq j \) and \( L(\lambda) \) is a typical module. By [Pe, Thm. 2] (see also [CW, Thm. 2.5.2.]),

\[
\text{ch} L(\lambda) = 2^{\left\lfloor \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{2} \right\rfloor} R \cdot \sum_{w \in W} (-1)^{|w|} e^{w(\lambda)}.
\]

In particular \( \text{ch} L(\lambda) \) is divisible by \( e^{a0} \prod_{i<j} (1 + e^{-\varepsilon_i + \varepsilon_j}) = \prod_{i<j} (x_i + x_j) \). For \( a = \frac{1}{2} \), the characters are divisible by \( \prod_{i<j} (x_i + x_j) \) by the previous proposition and the assertion follows. \( \square \)

Remark 14. Note that \( q(n) \) and \( spq(n) \) admit the same rings of characters by a similar argument as in Corollary 9.

Corollary 15. Similarly to Corollary 14,

\[
J(pq(n)) = J(q(n)) \cap \text{span}_\mathbb{Z}\left\{ x^\lambda \mid \lambda \in \mathbb{Z}^n, \sum_{i=1}^n \lambda_i = 0 \right\}
\]

\[
J(spq(n)) = J(sq(n)) \cap \text{span}_\mathbb{Z}\left\{ x^\lambda \mid \lambda \in \mathbb{Z}^n, \sum_{i=1}^n \lambda_i = 0 \right\}.
\]

5. The Weyl Groupoid for \( q(n) \).

We describe the polynomial invariants of certain affine action of the super Weyl groupoid \( \mathcal{W} \) of \( q(n) \). We follow [SV, Sec. 9] (for the periplectic case, see [IRS, Sec. 5.4]).

We identify \( \mathfrak{h}_0^* \) with its dual using the scalar product in which \( (\varepsilon_i, \varepsilon_j) = \delta_{ij} \). Let \( \Sigma \) be a groupoid with the set of objects \( \{ [\varepsilon_i + \varepsilon_j], [-\varepsilon_i - \varepsilon_j] : i < j \} \) and the set of morphisms \( \text{Mor}_\Sigma ([\alpha], [\beta]) = \emptyset \) if \( \alpha \neq \pm \beta \), and \( \text{Mor}_\Sigma ([\alpha], [\pm \alpha]) \) is of cardinality 1. For every object \([\alpha] \), we denote by \( r_{\alpha} \) the unique morphism in \( \text{Mor}_\Sigma ([\alpha], [-\alpha]) \), and impose the relation \( r_{\alpha} r_{-\alpha} = \text{id}_{[-\alpha]} \).

The Weyl group \( W = S_n \) acts on \( \Delta(\mathfrak{g}_0) \) and we have the naturally defined homomorphism \( \Phi \) from \( W \) to the group of autoequivalences of \( \Sigma \). This yields the semidirect product groupoid \( \hat{\Sigma} = W \ltimes \Sigma \). The objects of \( \hat{\Sigma} \) are the same as the objects of \( \Sigma \). The morphisms in \( \hat{\Sigma} \) are generated by \( r_{\alpha} \) and \( w_{\alpha}^{w_0} \in \text{Mor}_\Sigma ([\alpha], [wa]) \), where \( w \in W \) and the following diagrams commute:

\[
\begin{array}{c}
[\alpha] \quad \xrightarrow{(w)_{\alpha}^{w_0}} \quad [uwa] \\
\downarrow w_{\alpha}^{w_0} \quad \searrow u_{\alpha}^{w_0} \quad \downarrow w_{\alpha}^{w_0} \\
[w_0a] \quad \xrightarrow{u_{\alpha}^{w_0}} \quad [w_0a] \\
\end{array}
\quad \quad \begin{array}{c}
[\alpha] \quad \xrightarrow{(w)_{\alpha}^{w_0}} \quad [wa] \\
\downarrow \quad \downarrow \quad r_{-\alpha} \quad \downarrow \quad r_{-\alpha} \\
[-\alpha] \quad \xrightarrow{w_{-\alpha}^{w_0}} \quad [-wa] \\
\end{array}
\]

The Weyl groupoid is defined as

\[
\mathcal{W} := W \ltimes \hat{\Sigma},
\]

where \( W \) is considered as a groupoid with a single point base \([W]\).
Example 16. For \( q(2) \), let \( s \) be the nontrivial element of \( S_2 \). Then \( \mathcal{W} \) takes the following form

\[
\begin{array}{c}
\rho_{s - \varepsilon_1, \varepsilon_2} \\
\rho_{\varepsilon_2 - \varepsilon_1, \varepsilon_2}
\end{array}
\]

\[
\begin{bmatrix}
\varepsilon_1 - \varepsilon_2 \\
\varepsilon_2 - \varepsilon_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
s \\
s
\end{bmatrix}
\]

\[
\begin{bmatrix}
r_{\varepsilon_2 - \varepsilon_1, \varepsilon_2} \\
r_{\varepsilon_2 - \varepsilon_1, \varepsilon_2}
\end{bmatrix}
\]

Let \( \mathcal{H} \) denote the category of all affine subspaces of \( \mathfrak{h} \) with morphisms given by affine transformations. For any \( \alpha = \pm (\varepsilon_i - \varepsilon_j) \), let

\[
\Pi_\alpha = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : x_i = -x_j \}.
\]

Note that \( \Pi_\alpha = \Pi_{-\alpha} \). Define \( \tau_\alpha \in \text{Mor}_\mathcal{H}(\Pi_\alpha, \Pi_\alpha) \) by the formula

\[
\tau_\alpha(x_1, \ldots, x_n) = (x_1, \ldots, x_i + 1, \ldots, x_j - 1, \ldots, x_n).
\]

Define the functor \( F : \mathcal{W} \to \mathcal{H} \) by setting

\[
F([\alpha]) = \Pi_\alpha, \quad F([W]) = \mathbb{C}^n, \quad F(r_\alpha) = \tau_\alpha, \quad F(w) = w, \quad F(w_{\alpha, w}) = \text{Res}_{\Pi_\alpha} w.
\]

A function \( f \) on \( \mathfrak{h} \) is called \( \mathcal{W} \)-invariant if for any \( \varphi \in \text{Mor}_\mathcal{W}(A, B) \)

\[
F(\varphi)\ast \text{Res}_{FB} f = \text{Res}_{FA} f.
\]

The description of \( J(Q(n)) \) can be formulated as follows:

**Theorem 17.** The \( \mathbb{Q} \)-span of the ring of characters \( J(Q(n)) \) of finite-dimensional representations of the supergroup \( Q(n) \) is isomorphic to the ring \( (\mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])^{\mathcal{W}} \) of invariants under the Weyl groupoid \( \mathcal{W} \) as defined above.

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