AN ANALYSIS OF SYMMETRY GROUPS OF GENERALIZED 
\textit{m}-QUASI-EINSTEIN MANIFOLDS

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Abstract. In this paper emphasis is placed on how the behavior of the solutions of a PDE is affected by the geometry of the generalized \textit{m}-quasi-Einstein manifold, and vice versa. Considering a \textit{n}-dimensional generalized \textit{m}-quasi-Einstein manifold which is conformal to a pseudo-Euclidean space, we prove the most general symmetry group of maximal dimension. Moreover, we demonstrate that there is no different low dimensional symmetry group on a generalized \textit{m}-quasi-Einstein manifold. As an application, we use the invariant structure of the metric to provide a method to build explicit examples.

1. Introduction

An \textit{n}-dimensional Riemannian manifold \((M^n, g)\) is generalized \textit{m}-quasi-Einstein if there exist two smooth functions \(f\) and \(\lambda\) on \(M\) such that

\begin{equation}
Ric_g + Hess_g f - \frac{1}{m} df \otimes df = \lambda g, 
\end{equation}

where \(m \in (0, +\infty]\). When \(m \in (0, +\infty)\), we can make the change \(h = e^{-\frac{f}{m}}\) and get the equation

\begin{equation}
Ric_g - \frac{m}{h} Hess_g h = \lambda g. 
\end{equation}

In [5], Catino introduced the notion of generalized quasi-Einstein manifolds. He proved that a complete generalized quasi-Einstein manifold with harmonic Weyl tensor and vanishing radial Weyl curvature is locally a warped product with \((n-1)\)-dimensional Einstein fiber.

The importance of understanding and giving explicit solutions for generalized \textit{m}-quasi-Einstein manifolds arises from the fact that they are closely related to Einstein warped product manifolds (cf. [8]). Furthermore, it is well known that [14] generalizes the notion of gradient Ricci solitons, and Einstein manifolds. Also, [12] generalizes several important metrics, e.g., critical metrics and static metrics (cf. [7] and the references therein). Therefore, this problem has great importance in physics.

Throughout history, several methods of reduction (ansatz) of PDEs were used in differential geometry to provide examples or even full classification of metrics (cf. [1, 2, 8, 9, 14]). As examples of ansatz methods we refer to the Lie group theory and the method of characteristics (cf. [10, 11]). The reduction method used in this paper is based on those two (cf. Theorem 1 and Theorem 2).

Recently, Ribeiro Jr and Tenenblat [13] classified the \(n\)-dimensional \textit{m}-quasi-Einstein manifolds invariants under the action of a translation group, in which \(\lambda\) is constant. They also gave a complete classification when \(\lambda = 0, m \geq 1\) or \(m = 2 - n\).

In this work we study conformally flat generalized \(m\)-quasi-Einstein manifolds satisfying (1.2). However, unlike [13] we do not fix any kind of symmetry. In fact, we completely...
describe the most general ansatz capable of reducing the system of PDEs obtained from (1.2) to a system of ODEs (ordinary differential equations). Moreover, as far as we know there is no classification for conformally flat pseudo-Euclidean generalized $m$-quasi-Einstein manifolds.

Hereafter, we will establish the needed notations to announce our main results. Let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space with coordinates $(x_1, \ldots, x_n)$ and metric components $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$. We want to find smooth functions $\varphi$, $h$ and $\lambda$ defined on an open subset $\Omega \subset \mathbb{R}^n$ such that, for $\bar{g}$ given by

$$\bar{g} = \frac{g}{\varphi^2},$$

$(\Omega, \bar{g})$ is a generalized $m$-quasi-Einstein manifold with potential function $h$, i.e.,

$$Ric_{\bar{g}} - \frac{m}{n} Hess_{\bar{g}}(h) = \lambda \bar{g},$$

(1.3) where $m \in (0, +\infty)$, $Ric_{\bar{g}}$ and $Hess_{\bar{g}}(h)$ are, respectively, the Ricci tensor and the Hessian of the metric $\bar{g}$. Then, we will prove the most general form a smooth function $\xi : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, such that $h \circ \xi$, $\varphi \circ \xi$ and $\lambda \circ \xi$ satisfy (1.3), must have.

Our first result tells us that a generalized $m$-quasi-Einstein manifold reduced by our ansatz method is invariant under the action of the pseudo-orthogonal group and the action of the translation group. Moreover, Theorem 2 demonstrates that there is no other group-invariant even in low dimensions.

It is worth pointing out that in [12] the authors used the Lie point symmetries to find metrics that would solve the Ricci curvature and the Einstein equations. They provided a large class of group-invariant solutions and examples of complete metrics defined globally in $\mathbb{R}^n$. Here, using a different approach, we will describe all the invariant groups for a generalized $m$-quasi-Einstein manifold conformal to a pseudo-Euclidean space, based upon our ansatz method. Furthermore, it is important to say that Theorem 1 and Theorem 2 also can be applied to $m$-quasi-Einstein manifolds.

Without further ado, we state our main results.

**Theorem 1.** Let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space, $n \geq 2$, with Cartesian coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\varepsilon_i$ and let $\Omega \subseteq \mathbb{R}^n$ be an open subset. Consider non-constant smooth functions $h$, $\varphi$, $\lambda : \Omega \rightarrow \mathbb{R}$. Then, there exists a smooth function $\xi : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\xi := \xi(x_1, \ldots, x_n),$$

such that

$$\left(\mathbb{R}^n, \bar{g} = \frac{1}{(\varphi \circ \xi)^2}g, h \circ \xi, \lambda \circ \xi\right)$$

is a generalized $m$-quasi-Einstein manifold satisfying (1.3) if, and only if,

$$\xi = P \left(\sum_{k=1}^{n} a_k x_k^2 + b_k x_k + c_k\right),$$

(1.4) where $a, b_k, c_k \in \mathbb{R}$ and $P$ is at least a $C^1$ function.

Now we show a result concerning the rigidity of generalized $m$-quasi-Einstein metrics. We prove that there is no other symmetry group of low dimension on a generalized $m$-quasi-Einstein manifold. This question was mentioned also in [8], where the authors proved all the maximal invariant groups for the gradient Ricci soliton by the same ansatz method used in this work. However, they did not provide the low dimensional symmetries for the gradient Ricci solitons.
The nonexistence of \( m \)-quasi-Einstein metrics is an important issue. As we have said before, it can indicate the impossibility to construct Einstein warped products metrics or even gradient Ricci solitons. On this subject, we recommend the reader to see [4].

**Theorem 2.** Let \((\mathbb{R}^n, g)\) be the standard pseudo-Euclidean space, \( n \geq 2 \), with Cartesian coordinates \( x = (x_1, \ldots, x_n) \), \( g_{ij} = \delta_{ij} \varepsilon_i \). Considering smooth functions \( h \circ \xi, \varphi \circ \xi \) and \( \lambda \circ \xi \) satisfying (1.3) where
\[
\xi = \xi(x_1, \ldots, x_{n-1}).
\]
Then,
\[
\xi = P \left( \sum_{k=1}^{n-1} a_k x_k + b_k \right),
\]
where \( a_k, b_k \in \mathbb{R} \) and \( P \) is at least a \( C^1 \) function. Moreover, if the invariant has a different form then the generalized \( m \)-quasi-Einstein manifold is trivial, i.e., either \( \varphi \) or \( h \) is a constant functions.

For instance, we can gather that \( \mathbb{R} \times S^{n-1} \) do not represent a group-invariant for a generalized \( m \)-quasi-Einstein manifold. Precisely, we obtain the next result.

**Corollary 1.** Let \((\mathbb{R}^n, g)\) be the standard pseudo-Euclidean space, \( n \geq 2 \), with Cartesian coordinates \( x = (x_1, \ldots, x_n) \), \( g_{ij} = \delta_{ij} \varepsilon_i \). Considering smooth functions \( h \circ \xi, \varphi \circ \xi \) and \( \lambda \circ \xi \) satisfying (1.3) where
\[
\xi = \sum_{k=1}^{n-1} \varepsilon_k x_k^2.
\]
Then,
\[
\left( \mathbb{R}^n, \bar{g} = \frac{1}{(\varphi \circ \xi)^2} g, h \circ \xi, \lambda \circ \xi \right)
\]
is a trivial generalized \( m \)-quasi-Einstein manifold, i.e., either \( \varphi \) or \( h \) is a constant functions.

In what follows, we provide the reduction of the PDE system (1.3) into a ODE, which is a consequence of Theorem 1.

**Theorem 3.** Under the same conditions of Theorem 1, for any function \( \varphi(\xi) \),
\[
\left( \mathbb{R}^n, \bar{g} = \frac{1}{\varphi^2} g \right)
\]
is a generalized \( m \)-quasi-Einstein manifold if, and only if, the function \( h \) is the solution of the ordinary differential equation
\[
(n - 2)h'' - m \varphi h'' - 2m \varphi'h' = 0,
\]
where the function \( \lambda \) is given by
\[
\lambda = 2a \varphi \left[ (n - 2)\varphi' - m \varphi \frac{h'}{h} \right]
\]
\[
+ \left[ \varphi \varphi'' - (n - 1)(\varphi')^2 + m \varphi \varphi' \frac{h'}{h} \right] (4a \xi + S) + 2ma \varphi \varphi'.
\]
Here \( S = \sum_{k=1}^{n} (\varepsilon_k b_k - 4a c_k) \) and \( a, b_k, c_k \in \mathbb{R} \) for all \( 1 \leq k \leq n \).

Now we explicitly provide families of solutions for generalized \( m \)-quasi-Einstein manifolds. The first one is invariant by rotations and the second example is invariant by translations. The last example is invariant by rotations and shows that, for a given choice of the conformal factor, the system has solution only in dimension 2.
Example 1. Let us consider \( r = \sum_{k=1}^{n} x_k^2 \) and the function \( \varphi(r) = e^{\alpha r + \beta} \), with \( \alpha, \beta \in \mathbb{R} \). From Theorem 3 we have that \( h \) is the solution of

\[
h'' + 2\alpha h' - \frac{(n-2)}{m} \alpha^2 h = 0.
\]

Then,

\[
h(r) = c_1 e^{r_1 r} + c_2 e^{r_2 r},
\]

where \( r_1 = -\alpha + |\alpha| \sqrt{\frac{n+m-2}{m}} \), \( r_2 = -\alpha - |\alpha| \sqrt{\frac{n+m-2}{m}} \) and \( c_1, c_2 \in \mathbb{R} \). Moreover,

\[
\lambda(r) = e^{2(\alpha r + \beta)} \left[ -4(n-2)\alpha^2 r + 4(n-1)\alpha + m(4\alpha r - 2) \frac{c_1 r_1 e^{r_1 r} + c_2 r_2 e^{r_2 r}}{c_1 e^{r_1 r} + c_2 e^{r_2 r}} \right].
\]

In this case, the solutions are globally defined. Taking \( \varphi(r) = e^{-r} \), we have that \( \varphi \) is bounded and the metric \( \bar{g} \) is complete.

Example 2. Considering \( \xi = \sum_{k=1}^{n} b_k x_k \) and the function \( \varphi(\xi) = e^{\alpha \xi + b} \), with \( a, b \in \mathbb{R} \). By Theorem 3 we have that \( h \) is given by

\[
h(\xi) = c_1 e^{r_1 \xi} + c_2 e^{r_2 \xi},
\]

where \( r_1 = -a + |a| \sqrt{\frac{n+m-2}{m}} \), \( r_2 = -a - |a| \sqrt{\frac{n+m-2}{m}} \) and \( c_1, c_2 \in \mathbb{R} \). Therefore,

\[
\lambda(\xi) = e^{2(a \xi + b)} \left[ m \frac{c_1 r_1 e^{r_1 \xi} + c_2 r_2 e^{r_2 \xi}}{c_1 e^{r_1 \xi} + c_2 e^{r_2 \xi}} - (n-2)a \right],
\]

where \( \varepsilon_{i0} = \sum_k \varepsilon_{k0} b_k^2 \). In this case, the solutions are globally defined. We remark that if \( b = \sum_k b_k \frac{\partial}{\partial x_k} \) is a null vector (lightlike) we have \( \varepsilon_{i0} = 0 \), and therefore \( \lambda = 0 \), i.e., a steady quasi-Einstein manifold.

Example 3. Consider \( r = \sum_{k=1}^{n} x_k^2 \), and \( \varphi = \sqrt{r} \), such that \( r > 0 \), in Theorem 3. By (1.5), \( h \) is a solution of the Euler equation

\[
r^2 h'' + rh' + \frac{(n-2)}{4m} h = 0,
\]

which implies that \( n = 2 \) and

\[
h(x, y) = c_1 + c_2 \log(x^2 + y^2),
\]

with \( c_i \in \mathbb{R} \). From (1.6) we have \( \lambda = 0 \), and therefore

\[
\left( \mathbb{R}^2, \bar{g} = \frac{dx^2 + dy^2}{x^2 + y^2}, h \right)
\]

is a complete steady \( m \)-quasi-Einstein manifold satisfying (1.3), with null curvature (cf. [6]).
2. Background

We denote \( \varphi, i \) and \( h, i \) the first order derivatives, and \( \varphi, ij \) and \( h, ij \) as the second order derivatives of the functions \( \varphi \) and \( h \) with respect to \( x_i \) and \( x_i x_j \), respectively.

Let \( (\mathbb{R}^n, g) \) is the pseudo-Euclidean space with coordinates \( x = (x_1, ..., x_n) \), \( g_{ij} = \delta_{ij} \varepsilon_i \).

From the conformal structure (see [3]), if \( \bar{g} \) we obtain that

\[
(2.1) \quad \text{Ric} - \text{Ric}_g = \frac{1}{\varphi^2} \left\{ (n - 2) \varphi \text{Hess}_g \varphi + \left[ \varphi \Delta_g \varphi - (n - 1) |\nabla \varphi|^2 \right] g \right\}.
\]

Thus, for a tangent base \( X_1, ..., X_n \) of \( \mathbb{R}^n \) we get

\[
(2.2) \quad \left\{ \begin{array}{l}
(\text{Hess}_g(h))_{ij} = h_{ij} + \frac{\varphi_{ij}}{\varphi} h_{ii} + \frac{\varphi_{ii}}{\varphi} h_{ij}, \quad i \neq j \\
(\text{Hess}_g(h))_{ii} = h_{ii} + 2 \frac{\varphi_i}{\varphi} h_{ii} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\varphi_{ik}}{\varphi} h_{ik}, \quad i = j,
\end{array} \right.
\]

where \( \text{Hess}_g(h)(X_i, X_j) = (\text{Hess}_g(h))_{ij} \).

Remember that

\[
(2.3) \quad \text{Ric} - \frac{m}{h} \text{Hess}_g h = \lambda \bar{g}, \quad \lambda \in C^\infty(\mathbb{R}^n).
\]

Replacing (2.1) and (2.2) in (2.3), provided that \( \Delta_g \varphi = \sum_{k=1}^n \varepsilon_k \varphi, kk \) and \( |\nabla \varphi|^2 = \sum_{k=1}^n \varepsilon_k \varphi, k^2 \), we get

\[
(2.4) \quad \left\{ \begin{array}{l}
(n - 2) h \varphi, ij - m \left( \varphi h_{ij} + \varphi, ij h_{ii} + \varphi, i h_{ij} \right) = 0; \quad \forall i \neq j, \\
(n - 2) h \varphi, ii + \varepsilon_i h \sum_{k=1}^n \varepsilon_k \left[ \varphi \varphi, kk - (n - 1) \varphi, k^2 \right] \\
- m \left[ \varphi^2 h_{ii} + 2 \varphi \varphi, ii h_{ii} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \varphi \varphi, k h_{ik} \right] = \varepsilon_i \lambda h; \quad \text{for} \quad i = j.
\end{array} \right.
\]

3. Proof of the Main Results

\textbf{Proof of Theorem} \[1\] Consider that (2.4) admits non-trivial solutions such that \( f \circ \xi, h \circ \xi, \varphi \circ \xi \) and \( \lambda \circ \xi \), where \( \xi : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function. Then, from the first equation of (2.4) we obtain that

\[
(3.1) \quad [(n - 2) h \varphi'' - m \varphi h'' - 2 m \varphi' h'] \xi_{ij} + [(n - 2) h \varphi' - m \varphi h'] \xi_{ij} = 0.
\]

Note that \( (n - 2) h \varphi' - m \varphi h' \neq 0 \). In fact, if \( (n - 2) h \varphi' - m \varphi h' = 0 \) then

\[
(n - 2) \frac{\varphi'}{\varphi} = m \frac{h'}{h}.
\]

From the above identity we get

\[
(3.2) \quad \frac{\varphi''}{\varphi} = \left( \frac{\varphi'}{\varphi} \right)' + \left( \frac{\varphi'}{\varphi} \right)^2,
\]
from (3.1) we can infer that either $\varphi$ or $h$ is constant, which is a contradiction. Thus, from (3.1) we get

$$\frac{\xi_{i,j}}{\xi_{i,j}} = \frac{-(n-2)h\varphi'' - m\varphi h'' - 2m\varphi' h'}{(n-2)h\varphi' - m\varphi h'} = F(\xi).$$

So,

$$\log(\xi_{i,i}) = \int F(\xi) d\xi + F_i(\hat{x}_j),$$

where the symbol $\hat{x}_j$ denotes that the function does not depend on $x_j$. That is,

$$\xi_{i,i} = e^{\int F(\xi) d\xi} e^{F_i(\hat{x}_j)}.$$ 

Since this is true for all $i \neq j$, denote by $G(\xi) = e^{\int F(\xi) d\xi}$ and $G_i(x_i) = e^{F_i(\hat{x}_j)}$, therefore,

$$\xi_{i,i} = G(\xi)G_i(x_i).$$

Now, contracting (1.2) we obtain

$$((n-1)\sum_{k=1}^{n} \varepsilon_k (2\varphi_kkk - n\varphi_k^2) - \frac{n}{h} \sum_{k=1}^{n} \varepsilon_k [\varphi^2 h_{kk} - (n-2)\varphi_k h_{k,k}] = n\lambda.$$ 

Multiplying this equation by $h$ and the second equation of (2.4) by $\varepsilon_i n$, we can conclude that

$$\left([(n-2)h\varphi'' - m\varphi h'' - 2m\varphi' h'] \left(\varepsilon_i n\xi_{i,i}^2 - \sum_{k=1}^{n} \varepsilon_k \xi_{i,k}^2\right)\right) + [(n-2)h\varphi' - m\varphi h'] \left(\varepsilon_i n\xi_{i,i} - \sum_{k=1}^{n} \varepsilon_k \xi_{i,k}\right) = 0.$$ 

Thus, from (3.3) we have

$$\varepsilon_i n[\xi_{i,i} - F(\xi)\xi_{i,i}^2] = \sum_{k=1}^{n} \varepsilon_k [\xi_{i,k} - F(\xi)\xi_{i,k}^2].$$

Hence,

$$\varepsilon_i n \left(\xi_{i,i} e^{-\int F d\xi}\right)_{i,i} = \sum_{k=1}^{n} \varepsilon_k \left(\xi_{i,k} e^{-\int F d\xi}\right)_{i,k}. $$

Then, by (3.4) we can infer that

$$\varepsilon_i n G_i' = \sum_{k=1}^{n} \varepsilon_k G_k'.$$

Since the left-hand side depends only on $x_i$, we have

$$\varepsilon_i G_i' = \varepsilon_j G_j', \forall i \neq j.$$ 

Thus $G_i(x_i) = 2a\varepsilon_i x_i + b_i$, with $a, b_i \in \mathbb{R}$. Therefore, $\frac{\xi_{i,i}}{G_i} = G$ implies that

$$\frac{\xi_{i,i}}{2a\varepsilon_i x_i + b_i} = \frac{\xi_{i,j}}{2a\varepsilon_j x_j + b_j},$$

where we conclude that $\xi$ is of the form (1.4).

The reverse statement is a straightforward computation. 

\[\square\]
Proof of Theorem 3: Now, considering \( \xi = \xi(x_1, \ldots, x_{n-1}) \), for all \( 1 \leq i, j \leq n-1 \) we have

\[
\varphi_{i} = \xi_{i} \varphi' \quad \varphi_{ij} = \xi_{i} \xi_{j} \varphi'' + \xi_{ij} \varphi' \quad \varphi_{ii} = \xi_{i}^{2} \varphi'' + \xi_{ii} \varphi'
\]

\[
h_{i} = \xi_{i} \ h' \quad h_{ij} = \xi_{i} \xi_{j} \ h'' + \xi_{ij} \ h' \quad h_{ii} = \xi_{i}^{2} \ h'' + \xi_{ii} \ h'
\]

and

\[
\varphi_{n} = \varphi_{n} = h_{n} = h_{n} = h_{nn} = 0, \quad \text{for all} \quad i = 1, \ldots, n.
\]

Then, from the first equation of (2.4), for \( i \neq j \neq n \), we obtain

\[
[(n - 2)h_{i} \varphi'' - m_{i} \varphi'' - 2m_{i} \varphi']\xi_{i} + [(n - 2)h_{i} \varphi' - m_{i} \varphi']\xi_{i, j} = 0.
\]

Considering \( i = n \) or \( j = n \), the first equation of (2.4) is trivially satisfied.

Now, from the second equation of (2.4), for \( i \neq n \) we obtain

\[
\varphi'[(n - 2)h_{i} \varphi'' - m_{i} \varphi'' - 2m_{i} \varphi']\xi_{i, n} + [\varphi[(n - 2)h_{i} \varphi' - m_{i} \varphi']\xi_{i, i} + \sum_{j=1}^{n-1} \varepsilon_{i j} \xi_{i} + \varepsilon_{i n} \xi_{i, n} \xi_{i, n} = \varepsilon_{i} \lambda h.
\]

On the other hand, for \( i = n \) we get

\[
\sum_{j=1}^{n-1} \varepsilon_{i j} \xi_{i} + \varepsilon_{i n} \xi_{i, n} \xi_{i, n} = \lambda h.
\]

Using the above identity in (3.6) leads us to

\[
[(n - 2)h_{i} \varphi'' - m_{i} \varphi'' - 2m_{i} \varphi']\xi_{i}^{2} + [\varphi[(n - 2)h_{i} \varphi' - m_{i} \varphi']\xi_{i, i} + \sum_{j=1}^{n-1} \varepsilon_{i j} \xi_{i} + \varepsilon_{i n} \xi_{i, n} \xi_{i, n} = 0.
\]

From (3.5) and (3.7) follows that if \( \varphi[(n - 2)h_{i} \varphi' - m_{i} \varphi'] \neq 0 \),

\[
\frac{\xi_{i, n}}{\xi_{i} \xi_{n}} = -\frac{[\varphi[(n - 2)h_{i} \varphi' - m_{i} \varphi']}{(n - 2)h_{i} \varphi' - m_{i} \varphi'},
\]

Denoting by

\[
F(\xi) = -\frac{[\varphi[(n - 2)h_{i} \varphi' - m_{i} \varphi']}{(n - 2)h_{i} \varphi' - m_{i} \varphi'},
\]

from the first equality of (3.8) we have

\[
\log \xi_{i} = \int F(\xi) d\xi + F(\xi_{j}), \forall i \neq j.
\]

Hence,

\[
\xi_{i} = e^{\int F(\xi) d\xi} F(\xi_{j}) = G(\xi)G(\xi_{j}).
\]

Similarly, by the second equality of (3.8),

\[
\log \xi_{i} = \int F(\xi) d\xi + L(\xi_{j}), \forall i.
\]

Then,

\[
\xi_{i} = e^{\int F(\xi) d\xi} L(\xi_{j}) = G(\xi)K(\xi_{j}.
\]

From (3.9) and (3.10) we concluded that \( G(\xi_{i}) = K(\xi_{i}) = a_{i}, a_{i} \) constant.

Thus, we get

\[
\frac{\xi_{i}}{a_{i}} = \frac{\xi_{i, j}}{a_{j}}
\]

Then, for all \( i \neq j \neq n \), the characteristic for the above equation implies that

\[
\xi = P \left( \sum_{k=1}^{n-1} a_{k} x_{k} + b_{k} \right),
\]

where \( a_{k}, b_{k} \in \mathbb{R} \) and \( P \) is at least a \( C^{1} \) function.
Now, if \((n - 2)h\varphi' - m\varphi h' = 0\), then
\[
(n - 2)\frac{\varphi'}{\varphi} = \frac{h'}{h}.
\]
Using (3.2), from (3.5) or (3.7) we conclude that either \(\varphi\) or \(h\) is constant.

**Proof of Theorem 3** From Theorem 1 we can suppose that
\[
\xi = \sum_{k=1}^{n} U_k(x_k).
\]
Therefore,
\[
\varphi, i = \varphi' U_i', \quad \varphi, ij = \varphi'' U_i' U_j', \quad \varphi, ii = \varphi''' (U_i')^2 + \varphi' U_i'';
\]
\[
h, i = h' U_i', \quad h, ij = h'' U_i' U_j', \quad h, ii = h''' (U_i')^2 + h' U_i''.
\]
Thus, from the first equation of (2.4) we obtain
\[
[(n - 2)h\varphi'' - m(\varphi h'' + 2\varphi' h')] U_i' U_j' = 0,
\]
from which we conclude that
\[
(n - 2)h\varphi'' - m(\varphi h'' + 2\varphi' h') = 0. \quad (3.11)
\]
From the second equation of (2.4) we have
\[
\varphi [(n - 2)h\varphi'' - m(\varphi h'' + 2\varphi' h')] (U_i')^2 + \varphi [(n - 2)h\varphi' - m\varphi h'] U_i''
\]
\[
+ \epsilon_i \sum_{k=1}^{n} \epsilon_k \{[h\varphi'' - n(\varphi')^2] (U_k')^2 + m\varphi\varphi' h' (U_k')^2 + h\varphi U_k''\} = \epsilon_i \lambda h.
\]
Note that \(\sum_{k=1}^{n} \epsilon_k (U_k')^2 = 4a\xi + S\) and \(\sum_{k=1}^{n} \epsilon_k U_k'' = 2na\), where \(S = \sum_{k=1}^{n} (\epsilon_k b_k - 4a \epsilon_k)\).

Hence, using the equation (3.11) we obtain
\[
2a \varphi [(n - 2)h\varphi' - m\varphi h']
\]
\[
+ [h\varphi'' - (n - 1)h(\varphi')^2 + m\varphi\varphi' h' (4a \xi + S) + 2nah\varphi\varphi'] = \lambda h. \quad (3.12)
\]
Finally, from (3.11) and (3.12) the result follows.

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