SOFT RESTRICTIONS ON POSITIVELY CURVED RIEMANNIAN SUBMERSIONS

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Abstract. We bound the dimension of the fiber of a Riemannian submersion from a positively curved manifold in terms of the dimension of the base of the submersion and either its conjugate radius or the length of its shortest closed geodesic.

1. Introduction and statement of results

The main difficulty when studying positively curved manifolds is the small number of known examples. New ones appear in increasing periods of time, and at the present state of knowledge, Riemannian submersions are necessary in their construction: starting with the correct manifold with nonnegative sectional curvature as total space, one searches for some submersion that would guarantee a positively curved base thanks to the well-known O’Neill’s formula. However, this is not so easily done, pointing out to the possible presence of restrictions on the existence of such Riemannian submersions from an arbitrary non-negatively curved manifold.

In this note, we consider the case of positively curved domains. The following conjecture (attributed to F. Wilhelm) is of particular interest:

Conjecture: Let \( \pi : M^{n+k} \to B^n \) be a Riemannian submersion between compact positively curved Riemannian manifolds. Then \( k \leq n - 1 \).

The conjecture is similar to the Chern-Kuiper theorem ruling out isometric immersions from compact nonpositively curved manifolds of dimension \( n \) into nonpositively curved manifolds of dimension \( 2n - 1 \).

Partial progress towards the conjecture appears in the thesis of W. Jiménez [7] where he used results of Kim and Tondeur [8] to obtain...
that if $\sec_M \geq 1$ and $\sec_B \leq C$, then

\[(1) \quad k \leq \frac{1}{3} (C - 1) (n - 1).\]

It is worth noticing that O’Neill’s formula together with [13] guarantees that $C > 1$, and therefore the right hand side in (1) is positive.

For a different type of restrictions using rational homotopy theory methods, see [1].

In this paper, we examine the index of Lagrangian subspaces of Jacobi fields (see section 2.1 for the definitions) along horizontal geodesics to prove:

**Theorem A.** Let $M^{n+k}, B^n$ be a compact Riemannian manifolds with $\sec \geq 1$, and $\pi : M^{n+k} \to B^n$ a Riemannian submersion. Then

\[k \leq \left( \frac{\pi}{\conj(B)} - 1 \right) (n - 1),\]

where $\conj(B)$ is the conjugate radius of $B$.

Since the conjugate radius of a manifold with $1 \leq \sec \leq C$ is at least $\pi/\sqrt{C}$, Theorem A gives the following improvement of Jimenez’s result:

**Corollary 1.1.** Under the conditions of Theorem A, if $\sec_B \leq C$, then

\[k \leq (\sqrt{C} - 1) (n - 1)\]

This bound is better than Jimenez’s when $C > 4$.

The arguments in the proof of Theorem A extend to Riemannian foliations, giving the following bound.

**Corollary 1.2.** Let $\mathcal{F}$ be a metric foliation with leaves of dimension $k$ in an $n+k$-dimensional compact manifold $M$ with $\sec \geq 1$. Then

\[k \leq \left( \frac{\pi}{\foc(\mathcal{F})} - 1 \right) (n - 1),\]

where $\foc(\mathcal{F})$ is the focal radius of the foliation.

The definition of focal radius is included at the end of section 4.

It is also possible to give bounds on the fiber dimension related to the length of the shortest nontrivial closed geodesic in the base (that exists by a theorem of Fet and Lyusternik [14]).

**Theorem B.** Let $M^{n+k}, B^n$ be a compact Riemannian manifolds with $\sec \geq 1$, and $\pi : M^{n+k} \to B^n$ a Riemannian submersion. Denote by $l_0$ the length of the shortest closed geodesic in $B$. Then

\[k \leq \left( \frac{3\pi}{l_0} - 1 \right) (n - 1).\]
The paper is organized as follows: section 2 recalls some results in [9] about abstract Jacobi fields and reformulates some classical theorems in this setting; it ends with a brief summary of Wilking’s transverse equation. Section 3 bounds the index of Lagrangian subspaces of Jacobi fields in several situations needed for the proofs of Theorems A and B; these appear in sections 4 and 5.

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2. The Jacobi equation

2.1. Jacobi fields in an abstract setting. This section collects a few facts on Jacobi fields from [9]. Let $E$ be a euclidean vector space of dimension $m$ with positive definite inner product $\langle \cdot, \cdot \rangle$. For a smooth one-parameter family of self adjoint linear maps $R : \mathbb{R} \to \text{Sym}(E)$, we consider the equation $J''(t) + R(t)J(t) = 0$ whose solutions we refer to as $R$-Jacobi fields (or just Jacobi fields if it is clear from the context to what $R$ we refer). We denote by $\text{Jac}^R$ the space of Jacobi fields, a vector space of dimension $2m$; $\text{Jac}^R$ is a symplectic vector space with form

$$\omega : \text{Jac}^R \times \text{Jac}^R \to \mathbb{R}, \quad \omega(X, Y) = \langle X, Y' \rangle - \langle X', Y \rangle$$

where the right hand side of $\omega$ is independent of the $t$ chosen. A subspace $W$ is called isotropic when $\omega$ vanishes in $W$; a maximal isotropic subspace is called a Lagrangian subspace, or simply, a Lagrangian. Since $\omega$ is nondegenerate, it is clear that Lagrangian subspaces are just isotropic subspaces of dimension $m$; in the literature, Lagrangian spaces have often been called maximal self-adjoint spaces for the Jacobi operator (see for instance [13] and [15]).

Since the inner product of $E$ is positive definite, zeros of Jacobi fields are isolated; we should mention that this is not true in the case of nonzero signature, as was noticed in [6] and further studied in [11]. Therefore, if $I \subset \mathbb{R}$ is an interval, we can define the index of an isotropic subspace $W \subset \text{Jac}^R$ in $I$ as the number of times (with multiplicity) that fields in $W$ vanish in $I$; a more precise definition appears in [9]. We will denote this index as $\text{ind}_{W}I$.

The indexes of different Lagrangians along the same interval are related by the following inequality in [9].

**Proposition 2.1.** Let $E, R, \text{Jac}^R$ be as previously described. Then for any Lagrangians $L_1, L_2 \subset \text{Jac}^R$ and any interval $I \subset \mathbb{R}$, we have

$$|\text{ind}_{L_1}I - \text{ind}_{L_2}I| \leq \dim E - \dim (L_1 \cap L_2)$$
2.2. The transverse Jacobi equation. Let \( W \) be an isotropic subspace of Jacobi fields of \((E, R)\) and \( t \in \mathbb{R} \), we define
\[
W(t) = \{ J(t) : J \in W \}, \quad W^t = \{ J \in W : J(t) = 0 \}.
\]
For each \( t \in \mathbb{R} \), the subspace
\[
\overline{W}(t) = W(t) \oplus \{ J'(t) : J \in W^t \}
\]
varies smoothly on \( t \) as was shown in [15]; denote by \( H(t) \) its orthogonal complement, and by \( e = e^h + e^v \) the splitting of a vector under the sum \( E = H(t) \oplus \overline{W}(t) \). We use \( \mathcal{H} \) to denote the vector bundle over \( \mathbb{R} \) formed by the \( H(t) \). There is a covariant derivative on \( \mathcal{H} \) induced from \( E \): if \( X : \mathbb{R} \to E \) is a section of \( \mathcal{H} \), we define
\[
\frac{D^h X}{dt}(t) = X'(t)^h.
\]
The covariant derivative \( D^h/dt \) defines parallel sections, and preserves the inner product induced on \( \mathcal{H} \) from \( E \). Let \( E_1 \) be an inner vector space of dimension the rank of \( \mathcal{H} \); using a parallel trivialization of \( \mathcal{H} \), we can identify sections of \( \mathcal{H} \) with maps \( X : \mathbb{R} \to E_1 \), and the covariant derivative \( D^h/dt \) with standard derivation.

Modulo these identifications, Wilking’s transverse equation reads as
\[
X''(t) + R^W(t)X(t) = 0, \quad R^W(t)X(t) = [R(t)X(t)]^h + 3A_tA^*_tX(t),
\]
where for each \( t \), the map \( A_t : W(t) \to H(t) \) is linear and its definition can be found in [15].

Thus we obtain a new Jacobi setting \((E_1, R^W)\) with \( R^W \) as the new curvature operator used to construct the Jacobi equation; Wilking proved that the projection of any \( R \)-Jacobi field onto \( \mathcal{H} \) is a solution of the transverse equation, i.e. an \( R^W \)-Jacobi field. Moreover, as Lytchak observed, any Lagrangian for \((E_1, R^W)\) is obtained projecting some Lagrangian that contains \( W \) and vice versa.

3. Bounds on the index

Recall that given an interval \( I \subset \mathbb{R} \) and a Lagrangian \( L \subset \text{Jac}^R \), we denote by \( \text{ind}_L I \) the index of \( L \) in \( I \) and by \( \text{ind}_L(t_0) \) the dimension of the vector subspace of \( L \) formed by those Jacobi fields in \( L \) that vanish at \( t_0 \).

Given \( a \in \mathbb{R} \), denote by \( L_a \) the Lagrangian subspace of \( \text{Jac}^R \) defined as
\[
L_a := \{ Y \in \text{Jac}^R : Y(a) = 0 \}.
\]
Recall that \( \dim E = m \).
3.1. **Upper bounds based on the conjugate radius.** All along this subsection we will assume that there is some positive number $c > 0$ such that for any $a \in \mathbb{R}$ and any Jacobi field with $Y(a) = 0$, $Y$ does not vanish again in $(a, a + c]$. Clearly

$$\text{ind}_{L_a}(a, a + c) = \text{ind}_{L_a}(a, a + c) = 0, \quad \text{ind}_{L_a}[a, a + c) = m.$$  

Inequality (2) shows that for an arbitrary Lagrangian $L$,

$$\text{ind}_L(a, a + c] \leq m.$$  

Our next aim is to improve this to larger intervals:

**Proposition 3.1.** For any Lagrangian $L$ and any positive integer $r$ we have

$$\text{ind}_L[a, a + rc] \leq (r + 1)m.$$  

**Proof.** Breaking the interval $[a, a + rc]$ into subintervals of length $c$ and using (4) repeatedly, we get that

$$\text{ind}_L[a, a + rc] = \text{ind}_L(0) + \sum_{i=0}^{r-1} \text{ind}_L[a + ic, a + (i + 1)c] \leq m + rm.$$  

\[\square\]

3.2. **Curvature-related lower bounds.** To get a lower bound on the index of a Lagrangian $L$, we need to establish the existence of conjugate points for the fields in $L$; Rauch’s theorem gives precisely that for a Lagrangian of the form $L_a$ as defined in (3). We will then use Proposition 2.1 to relate this to the index of an arbitrary Lagrangian.

We will say that the curvature $R$ satisfies $R(t) \geq \delta$ for all $t \in \mathbb{R}$ if $\langle R(t)v, v \rangle \geq \delta \|v\|^2$ for any vector $v \in E$. Our first result is a quantitative refinement of Corollary 10 in [15].

**Proposition 3.2.** Assume that there is some $\delta > 0$ such that the curvature $R$ satisfies $R(t) \geq \delta$ for all $t \in \mathbb{R}$. Then for any $a \in \mathbb{R}$, the set

$$\mathcal{A} = \left\{ Y \in L_a : Y(t) = 0 \text{ for some } t \in (a, a + \pi/\sqrt{\delta}] \right\}$$  

generates $L_a$.

**Proof.** The proof consists on using of Wilking’s transversal equation repeatedly. We describe how to proceed:

(1) We compare $R(t)$ to the constant curvature case $\bar{R}(t) = \delta I$; Rauch’s theorem gives us that there is some nonzero $Y_1$ in $\mathcal{A}$ vanishing for some $t_1 \in (a, a + \pi/\sqrt{\delta}]$. 

(2) Let $W_1 \subset L_a$ be the vector subspace generated by $Y_1$; we consider the transverse Jacobi equation induced by $W_1$ in $L_a$. In $L_a/W_1$ there is a Jacobi equation of the form

$$Y'' + R_1 Y = 0, \quad R_1(t) = R(t)^h + 3A_t A_t^*,$$

and therefore $\langle R_1(t)v, v \rangle \geq \langle R(t)v, v \rangle \geq \delta \|v\|^2$ for any $v \in W_1(t)^\perp \subset E$. Moreover, after taking the $W_1$-orthogonal component, the fields in $L_a$ give an $R_1$-Lagrangian $L_1$. It is clear that every vector field in $L_1$ vanishes at $t = a$.

(3) Once again, we compare $R_1(t)$ to $\delta I$ to obtain some nonzero $X_2 \in L_a$ such that $X_2$ vanishes at some time $t_2$ in $(a, a + \pi/\sqrt{\delta}]$; this merely means that $X_2(t_2) = \lambda Y_1(t_2)$ for some $\lambda \in \mathbb{R}$, and thus the field $Y_2 = X_2 - \lambda Y_1$ is linearly independent with respect to $Y_1$ and lies in $\mathcal{A}$.

(4) Clearly, the process can be iterated as needed until we obtain a basis of $L_a$.

Proposition 3.2 allows us to obtain good lower bounds for the index of a Lagrangian over long intervals. They can also be obtained using the Morse-Schoenberg lemma [12] and Proposition 2.1.

**Proposition 3.3.** Let $a \in \mathbb{R}$; when $R \geq \delta$, the index of any Lagrangian subspace $L$ of Jacobi fields satisfies

$$\text{ind}_L \left[ a, a + r\pi/\sqrt{\delta} \right] \geq rm + \text{ind}_L(a)$$

for any positive integer $r$.

**Proof.** Without loss of generality we can assume that $a = 0$ and write the proof for this case. Consider the closed intervals

$$I_j = \left[ j\pi/\sqrt{\delta}, (j + 1)\pi/\sqrt{\delta} \right].$$

Proposition 3.2 says that

$$\text{ind}_{L_{j\pi/\sqrt{\delta}}} I_j \geq 2m;$$

while Proposition 2.1 gives us

$$\text{ind}_L I_j \geq \text{ind}_{L_{j\pi/\sqrt{\delta}}} I_j - m + \dim(L \cap L_{j\pi/\sqrt{\delta}}) \geq m + \text{ind}_L(j\pi/\sqrt{\delta}).$$
Breaking the interval \([0, r\pi/\sqrt{\delta}]\) into the \(I_j\)'s, we conclude that

\[
\text{ind}_L \left[0, r\pi/\sqrt{\delta}\right] = \sum_{j=0}^{r-1} \text{ind}_L I_j - \sum_{j=1}^{r-1} \text{ind}_L \left(j\pi/\sqrt{\delta}\right) \geq \sum_{j=0}^{r-1} \left(m + \text{ind}_L \left(j\pi/\sqrt{\delta}\right)\right) - \sum_{j=1}^{r-1} \text{ind}_L \left(j\pi/\sqrt{\delta}\right) = rm + \text{ind}_L(0).
\]

\(\square\)

A consequence of the last results is the following extension of Proposition 3.2 to arbitrary Lagrangians:

**Proposition 3.4.** Let \(a \in \mathbb{R}\); when \(R \geq \delta\), for every Lagrangian subspace \(L\) of \(\text{Jac}^R\) the set

\[
\left\{ Y \in L : Y(t) = 0 \text{ for some } t \in (a, a + \pi/\sqrt{\delta}] \right\}
\]

spans \(L\).

**Proof.** Proposition 3.3 for \(r = 1\) gives

\[
\text{ind}_L[a, a + \pi/\sqrt{\delta}] \geq m + \text{ind}_L(a).
\]

Therefore there exists a \(Y_1 \in L\) such that \(Y_1(t_1) = 0\) for some \(t_1 \in (a, a + \pi/\sqrt{\delta}]\). The proof is then identical to that of Proposition 3.2 \(\square\)

This has the following geometric application:

**Theorem 3.5.** Let \(M\) be an \(n\)-dimensional manifold with \(\sec \geq 1\) and \(\alpha : \mathbb{R} \to M\) a geodesic orthogonal to a submanifold \(N\) at \(\alpha(0)\). Then there are at least \(n - 1\) focal points of \(N\) along \(\alpha\) in the interval \((0, \pi]\).

**Proof.** This follows from Proposition 3.4 after choosing the Lagrangian of \(N\)-Jacobi fields along \(\alpha\) defined as

\[
L^N = \left\{ J \in \text{Jac}^R : J(0) \in T_{\alpha(0)}N, J'(0) + S_{\alpha'(0)}J(0) \perp T_{\alpha(0)}N \right\}
\]

\(\square\)

3.3. Index bounds for periodic Jacobi fields. In this section we examine the index of Lagrangians when the solutions of the Jacobi equation are periodic with common period. We will show that such index is always bounded above by some linear function related to multiples of the period.
Proposition 3.6. Suppose there is some $l > 0$ such that for every Jacobi field $J$, the field $t \to J(t + l)$ is also a Jacobi field. Then for any Lagrangian $L$ in $\text{Jac}^R$ we have

$$\text{ind}_L[a, a + rl] \leq r(m + \text{ind}_L[a, a + l]) + m$$

for any positive integer $r$.

Proof. As usual, we will write the proof for $a = 0$. We start by choosing some basis of $L$, given by $X_1, \ldots, X_m$; for any positive integer $r$, consider the Jacobi fields defined as

$$X_i^r(t) = X_i(t + rl), \quad i = 1, \ldots, m.$$

Let $L^r$ the subspace generated by $X_1^r, \ldots, X_m^r$; it is Lagrangian, with $L^0 = L$. Clearly

$$\text{ind}_L[jl, (j + 1)l] = \text{ind}_{L^r}[0, l].$$

Using (2), we have that

$$\text{ind}_{L^r}[0, rl] = \sum_{j=0}^{r-1} \text{ind}_{L^r}[0, l] + \text{ind}_{L^r}(0) \leq \sum_{j=0}^{r-1} (\text{ind}_{L^r}[0, l] + m) + m,$$

as claimed.

\[\square\]

4. Proof of Theorem A

We recall some basic facts about Riemannian submersions before proving Theorem A. Let $\pi : M \to B$ be such a submersion where $M$ and $B$ have dimensions $n + k$ and $n$ respectively. We will usually overline the notation for objects in the base, to distinguish them from those in $M$. To facilitate the reading, we recall briefly some of the main facts about projectable Jacobi fields; the reader can find more information about them in [10] and [5, section 1.6]; we will use, in particular, the notation from this latter reference.

Definition 4.1. Choose some unit speed geodesic $\alpha : I \to M$ horizontal for the submersion, and denote by $\bar{\alpha}$ its image $\pi \circ \alpha$ in $B$. A Jacobi field $Y$ along $\alpha$ is projectable if it satisfies

$$Y^r = -S_{\alpha^r}Y^r - A_{\alpha^r}Y^h$$

where $S_{\alpha^r}$ and $A_{\alpha^r}$ are the second fundamental forms of the fibers and the O’Neill tensor of the submersion respectively.
The interest of projectable Jacobi fields is that they arise from variations by horizontal geodesics. As such, if $Y$ is a projectable Jacobi field, $\pi_*Y$ is a Jacobi field along $\bar{\alpha}$ in the base. Conversely, we have the following

**Lemma 4.2.** Let $\bar{Y}$ be a Jacobi field of $B$ along $\bar{\alpha}$, and $v$ a vertical vector at $\alpha(0)$; then there is a unique projectable Jacobi field $Y$ along $\alpha$ such that $\pi_*Y = \bar{Y}$ and $Y(0)^v = v$.

A particular case of projectable Jacobi fields arises from taking geodesic variations obtained from lifting a geodesic in the base; such fields are called holonomy Jacobi fields, and they satisfy the stronger condition

$$J' = -A^\alpha_*J - S^\alpha_*J$$

It is clear that they agree with those projectable Jacobi fields mapping to the zero field under $\pi_*$.

**Proof of Theorem A.** Let $\alpha : \mathbb{R} \to M$ be a horizontal geodesic and $\bar{\alpha} = \pi \circ \alpha$ its projection by the submersion. The Lagrangian subspace $\bar{L}_0$ can be lifted to $\alpha$ by considering the subspace spanned by projectable Jacobi fields $Y$ that vanish at $t = 0$ (and will therefore have horizontal $Y'(0)$), and by holonomy Jacobi fields along $\alpha$. We use $L$ to denote such Lagrangian, and $W$ to denote the subspace generated by the holonomy fields. It is interesting to observe that $L$ agrees with the $L^N$ from the proof of Theorem 3.5 when $N$ is the fiber through $\alpha(0)$. By Lemma 3.1 in [9],

$$\text{ind}_{L/W} + \text{ind}_W = \text{ind}_L$$

along any interval, where in $L/W$ we are using the transverse Jacobi equation induced by $W \subset L$. Observe that since holonomy Jacobi fields never vanish, $\text{ind}_W = 0$ over any interval. We claim that $\text{ind}_{L/W} = \text{ind}_{L_0}$. To prove it, we use that, as stated in [9] section 3.2, the transverse Jacobi equation corresponding to $W$ along $\alpha$ agrees with the usual Jacobi equation along $\bar{\alpha}$. Since Lagrangians for the Jacobi equation project to Lagrangians for the transverse Jacobi equation, and every field $Y$ in $L$ satisfies $Y(0) \in W(0)$, we have the mentioned equivalence of indices. Thus we have $\text{ind}_{L_0} = \text{ind}_L$.

We will estimate this common value over the intervals $[0, r\pi]$ using some of the previous inequalities on the index; choose an arbitrary $c < c_0$ where $c_0$ is the conjugate radius of $B$:

$$\text{ind}_{L_0}[0, r\pi] = \text{ind}_{L_0}\left[0, \frac{r\pi}{c}\right] = \left(\left[\frac{r\pi}{c}\right] + 1\right)(n - 1)$$

$$\text{ind}_L[0, r\pi] \geq r(n - 1 + k) + \text{ind}_L(0) = r(n - 1 + k) + (n - 1)$$
by propositions 3.1 and 3.3 respectively.

To finish the proof, divide both inequalities by $r$ and make it tend to infinity to conclude that

$$k \leq \left( \frac{\pi}{c} - 1 \right) (n - 1).$$

Letting $c$ tend to $c_0$ gives us Theorem A.

The above proof can be easily extended to metric foliations. In order to do this, we define the focal radius of a metric foliation $\mathcal{F}$, $\text{foc}(\mathcal{F})$, as the infimum over all the leaves of $\mathcal{F}$ of the focal radius of each leaf, i.e., the minimum distance to $N$ at which its first focal point appears.

**Proof of Corollary 1.2.** Let $F$ be a leaf of $\mathcal{F}$ and $\alpha : \mathbb{R} \to M$ a geodesic orthogonal to $F$ with $\alpha(0) \in F$. Denote by $W$ the set of holonomy Jacobi fields along $\alpha$, and by $L$ the Lagrangian spanned by $W$ and those Jacobi fields along $\alpha$ with $J(0) = 0$, $J'(0) \perp T_{\alpha(0)}F$. Since $\text{ind}_W I = 0$, equation 6 gives

$$\text{ind}_L I = \text{ind}_{L/W} I$$

for any interval $I$. Observe that $L/W$ corresponds to the Lagrangian $\bar{L}_0 = \{ J : J(0) = 0 \}$ of Jacobi fields for Wilking’s transverse equation for the isotropic $W$.

From the definition of the focal radius of $\mathcal{F}$ it follows that for every $c < \text{foc}(\mathcal{F})$,

$$\text{ind}_{\bar{L}_0} (0, c] = 0,$$

and therefore we are in the situation of Proposition 3.4, thus

$$r(n - 1 + k) \leq \text{ind}_L [0, r\pi] = \text{ind}_{\bar{L}_0} [0, r\pi] \leq \left( \left\lceil \frac{r\pi}{c} \right\rceil + 1 \right) (n - 1)$$

for any integer $r > 0$. As before, divide both sides by $r$ and let it tend to zero to obtain the inequality claimed in the corollary.

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5. **Proof of Theorem B**

Let $m$ be the smallest positive integer with $\pi_i B = 0$ when $i = 1, \ldots, m - 1$ and $\pi_m B \neq 0$. Hurewicz’s theorem implies that $m \leq n$. If $A B$ denotes the free loop space of $B$, then $\tilde{\pi}_{m-1} A B = \pi_m B$, and Lyusternik-Schnirelmann theory implies that there is a closed geodesic $\bar{\alpha} : [0, \ell] \to B$ such that the number of conjugate points to $\bar{\alpha}(0)$ along $\bar{\alpha}$ in the interval $(0, \ell)$ does not exceed $m - 1$ (see [2 Theorem 1.3]).

Denote by $\alpha : \mathbb{R} \to M$ some horizontal lift of $\bar{\alpha}$ to $M$.

Choose along $\alpha$ the Lagrangian $L$ of Jacobi fields spanned by the vertical holonomy Jacobi fields and projectable Jacobi fields that vanish at $t = 0$. As in the proof of Theorem A we have
We are going to use this equality in intervals of the form \([0, r\pi]\) for \(r\) a positive integer; the left hand side in (9) can be bound with the help of Proposition 3.3 giving
\[
\sum_{k=0}^{r-1} a_k \leq n - 1 + \text{ind}_L(0) \leq \text{ind}_L[0, r\pi];
\]
on the other hand the right hand side can be bound with Proposition 3.6 to get
\[
\text{ind}_{\bar{L}_0}[0, r\pi] \leq \left(\left\lfloor \frac{\pi}{\ell} \right\rfloor + 1\right) (n - 1 + \text{ind}_{\bar{L}_0}[0, \ell]).
\]
Dividing by \(r\) and letting it tend to infinity gives
\[
n - 1 + k \leq \frac{\pi}{\ell} (n - 1 + \text{ind}_{\bar{L}_0}[0, \ell]).
\]
But from the bound on the number of conjugate points of \(\bar{\alpha}\) in \([0, \ell]\) we get that
\[
n - 1 + k \leq \frac{3\pi}{\ell_0} (n - 1) \leq \frac{3\pi}{\ell_0}(n - 1),
\]
which proves Theorem B. \(\square\)

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