Optimal consumption with reference to past spending maximum

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Abstract
This paper studies the infinite-horizon optimal consumption problem with a path-dependent reference under exponential utility. The performance is measured by the difference between the nonnegative consumption rate and a fraction of the historical consumption maximum. The consumption running maximum process is chosen as an auxiliary state process, and hence the value function depends on two state variables. The Hamilton–Jacobi–Bellman (HJB) equation can be heuristically expressed in a piecewise manner across different regions to take into account all constraints. By employing the dual transform and smooth-fit principle, some thresholds of the wealth variable are derived such that a classical solution to the HJB equation and feedback optimal investment and consumption strategies can be obtained in closed form in each region. A complete proof of the verification theorem is provided, and numerical examples are presented to illustrate some financial implications.

Keywords Exponential utility · Consumption running maximum · Path-dependent reference · Piecewise feedback control · Verification theorem

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1 Introduction

The Merton problem, first studied in Merton [18, 19], has been one of the milestones in quantitative finance, which bridges investment decision making and some advanced mathematical tools such as PDE theory and stochastic analysis. By the dynamic programming principle, one can solve the stochastic control problem by looking for a solution of the associated HJB equation. Isoelastic utility and exponential utility have attracted dominant attention in academic research as they enjoy the merits of homogeneity and scaling property. In abundant work on terminal wealth optimisation, the value function can be conjectured in some separation forms or a change of variables can be applied. Consequently, a dimension reduction can be exercised to simplify the HJB equation. When intermediate consumption is taken into account, the study of exponential utility becomes relatively rare in the literature due to its unnatural allowance of negative consumption behaviour. To be precise, as the exponential utility is defined on the whole real line, the optimal consumption resulting from the first-order condition can be negative in general. For technical convenience, some existing literature such as Merton [18], Vayanos [23], Liu [16] and many subsequent works simply ignore the constraint or interpret negative consumption by different financial meanings so that a nonnegativity constraint on the controls can be avoided.

The case of exponential utility with nonnegative consumption has been examined before in Cox and Huang [5] by using the martingale method, in which the optimal consumption can be expressed in an integral form using the state price density process. As shown in [5], the value function and the optimal consumption differ substantially from the case when the constraint is neglected. Some technical endeavours are required to fulfill the nonnegativity constraint on the control process. In the present paper, we revisit this problem under exponential utility with a binding nonnegativity constraint on the consumption rate. In addition, our study goes beyond the conventional time-separable utilities, and we aim to investigate the consumption behaviour when an endogenous reference point is included inside the utility. Our proposed preference concerns how far the investor is away from the past consumption maximum level, and this intermediate gap is chosen as the metric to generate the utility of the investor in a dynamic way. Due to the consumption running maximum process in the utility, the martingale method developed in Cox and Huang [5] can no longer handle our path-dependent optimisation problem because it is difficult to conjecture the correct dual processes and the associated dual problem.

Our research is mainly motivated by the psychological viewpoint that the consumer’s satisfaction level and risk tolerance sometimes depend on recent changes instead of absolute rates. Some large expenditures, such as purchasing a car, a house or some luxury goods, not only spur some long-term continuing spending for maintenance and repair, but also lift up the investor’s standard of living gradually. A striking decline in future consumption plans may result in intolerable disappointment and discomfort. To depict the quantitative influence of a relative change towards the investor’s preference, it is reasonable to consider a utility that measures the distance between the consumption rate and a proportion of the past consumption peak. On the other hand, during some economic recession periods such as the recent global
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It is unrealistic to mandate that the investor needs to catch up with the past spending maximum all the time. To capture the possibility that the investor may strategically decrease the consumption budget to fall below the benchmark so that more wealth can be accumulated to meet future higher consumption plans, we choose to work with exponential utility that is defined on the positive real line. As a direct consequence, the investor can bear a negative gap between the current consumption and the reference level. The flexibility to compromise consumption plans below the reference point from time to time makes the model suitable to accommodate more versatile market environments.

Utility maximisation with a reference point has become an important topic in behavioural finance; see Tversky and Kahneman [22], He and Zhou [15], He and Yang [14] and He and Strub [13] on portfolio management with either a fixed or an adaptive reference level. Our paper differs from previous work as we do not distinguish the utility on gains and losses separately and our reference process is dynamically updated by the control itself. The impact of the path-dependent reference generated by the past consumption maximum becomes highly implicit in our model, which makes the mathematical problem appealing. Our formulation is also closely related to the consumption habit formation preference, which measures the deviation of the consumption from the standard of living conventionally defined as the weighted average of the consumption integral. Some previous work on addictive consumption habit formation includes Constantinides [4], Detemple and Zapatero [8], Schroder and Skiadas [21], Munk [20], Englezos and Karatzas [11], Yu [24, 25], and non-addictive consumption habit formation is studied in Detemple and Karatzas [7]. Recently, there has been some emerging research on the combination of a reference and habit formation, see Curatola [6] and Bilsen et al. [3], in which the reference level is generated by an endogenous habit formation process and different utility functions are used when consumption is above and below the habit, respectively. It will be interesting future work to consider such an S-shaped utility defined on the difference between the consumption and the consumption peak reference level and investigate the structure of the optimal consumption. Among the aforementioned work, it is worth noting that Detemple and Karatzas [7] consider a utility defined on the whole real line and also permit the admissible consumption to fall below the habit level from time to time. That is, the consumption habit is not addictive. Detemple and Karatzas [7] extend the martingale method in Cox and Huang [5] by using an adjusted state price density process, which produces a nice construction of the optimal consumption in a complete market model. However, the duality approach in [7] may not be applicable to our problem due to the presence of the running maximum process.

One main contribution of the present paper is to show that our path-dependent control problem can be solved under the umbrella of dynamic programming and the PDE approach. Comparing with the existing literature, the utility measures the difference between the control and its running maximum, and a nonnegativity constraint on consumption is imposed. The standard change of variables and the dimension reduction cannot be applied, and we confront a value function depending on two state variables, namely the wealth variable \( x \in \mathbb{R}_+ \) and the reference level variable \( h \in \mathbb{R}_+. \) By noting that the consumption control is restricted between 0 and the peak level, we first heuristically derive the HJB equation in different forms based on decomposing the domain \( \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ \} \) into disjoint regions of \( (x, h) \) such that
the feedback optimal consumption satisfies (i) \( c^*(x, h) = 0 \); (ii) \( 0 < c^*(x, h) < h \); (iii) \( c^*(x, h) = h \). To overcome the obstacle from nonlinearity, we apply the dual transformation only with respect to the state variable \( x \) and treat \( h \) as a parameter that is involved in some free boundary conditions. The linearised dual PDE can be handled as a piecewise ODE problem with parameter \( h \). By using the smooth-fit principle and some intrinsic boundary conditions, we obtain an explicit solution of the ODE that enables us to express the value function and the feedback optimal investment and consumption in terms of the primal variables after an inverse transform. We are able to find \( x_{zero}(h) \), \( x_{modr}(h) \), \( x_{aggr}(h) \) and \( x_{lavs}(h) \), indicating thresholds of zero, moderate, aggressive and lavish consumption for the wealth variable \( x \) as nonlinear function of the variable \( h \). The feedback optimal consumption can be characterised in the way that (i) \( c^*(x, h) = 0 \) when \( x \leq x_{zero}(h) \); (ii) \( 0 < c^*(x, h) < \lambda h \) when \( x_{zero}(h) < x < x_{modr}(h) \); (iii) \( \lambda h \leq c^*(x, h) < h \) when \( x_{modr}(h) \leq x < x_{aggr}(h) \); (iv) \( c^*(x, h) = h \) but the instant running maximum process \( (H^*_t) \) remains flat when \( x_{aggr}(h) \leq x < x_{lavs}(h) \); (v) \( c^*(x, h) = h \) and the instant \( c^*_t \) creates a new global maximum level when \( x = x_{lavs}(h) \). Moreover, due to the presence of the running maximum process inside the utility, the proof of the verification theorem involves many technical and non-standard arguments.

Building upon the closed-form value function and feedback optimal controls, some numerical examples are presented. The impacts of the variable \( h \) and the reference degree parameter \( \lambda \) on all boundary curves \( x_{zero}(h) \), \( x_{modr}(h) \), \( x_{aggr}(h) \) and \( x_{lavs}(h) \) can be numerically illustrated. We also perform a sensitivity analysis of the value function, the optimal consumption and portfolio on some model parameters, namely the reference degree parameter, the mean return and the volatility of the risky asset, and we discuss some quantitative properties and their financial implications.

The remainder of the paper is organised as follows. Section 2 introduces the market model and formulates the control problem under our utility with reference to the consumption peak. Section 3 presents the associated HJB equation for \( 0 < \lambda < 1 \) and some heuristic results to derive its explicit solution. Some numerical examples are presented in Sect. 4. Section 5 provides the proof of the verification theorem and other auxiliary results in the previous sections. Finally, the main result of the extreme case \( \lambda = 1 \) is given in Appendix A, and the lengthy proof of an auxiliary lemma is given in Appendix B.

## 2 Market model and problem formulation

Let \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be a filtered probability space, in which \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) satisfies the usual conditions. We consider a financial market consisting of one riskless asset and one risky asset. The riskless asset price satisfies \( dB_t = r B_t dt \), where \( r \geq 0 \) represents the constant interest rate. The risky asset price follows the dynamics

\[
\mathrm{d}S_t = S_t \mu dt + S_t \sigma dW_t,
\]

where \( W \) is an \( \mathbb{F} \)-Brownian motion and the drift \( \mu \) and volatility \( \sigma > 0 \) are given constants. The Sharpe ratio parameter is denoted by \( \kappa := \frac{\mu - r}{\sigma} \). It is worth noting
that our mathematical arguments and all conclusions can be readily generalised to a model with multiple risky assets as long as the market is complete. For the sake of simple presentation, we only focus on a model with a single risky asset. It is assumed that $\kappa > 0$ from this point onwards, i.e., $\mu > r$ so that the return of the risky asset is higher than the interest rate.

Let $(\pi_t)_{t \geq 0}$ represent the dynamic amount that the investor allocates in the risky asset and $(c_t)_{t \geq 0}$ denote the dynamic consumption rate of the investor. The resulting self-financing wealth process $(X_t)_{t \geq 0}$ satisfies

$$dX_t = r X_t dt + \pi_t (\mu - r) dt + \pi_t \sigma dW_t - c_t dt, \quad t \geq 0,$$

with the initial wealth $X_0 = x \geq 0$.

A consumption–portfolio pair $(c, \pi)$ is admissible, denoted by $(c, \pi) \in \mathcal{A}(x)$, if the consumption rate $c_t \geq 0$ a.s. for all $t \geq 0$, $c$ is $\mathbb{F}$-predictable, $\pi$ is $\mathbb{F}$-progressively measurable, and both satisfy the integrability condition $\int_0^\infty (c_t + \pi_t^2) dt < \infty$ a.s. Moreover, no bankruptcy is allowed in the sense that $X_t \geq 0$ a.s. for $t \geq 0$.

We focus on the exponential utility $U(x) = -\frac{1}{\beta} e^{-\beta x}$ with $\beta > 0$, $x \in \mathbb{R}$. We are interested in the infinite-horizon utility maximisation defined on the difference between the current consumption rate and its historical running maximum, i.e.,

$$u(x, h) = \sup_{(\pi, c) \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t - \lambda H_t) dt \right],$$

where

$$H_t = \max \left( h, \sup_{s \leq t} c_s \right), \quad H_0 = h \geq 0,$$

and the proportional constant $0 \leq \lambda \leq 1$ depicts the intensity towards the reference level that the investor adheres to the past spending pattern. Here, $H_0 = h \geq 0$ describes the reference level of consumption that the individual aims to surpass at the initial time.

One advantage of the exponential utility resides in the flexibility that the optimal consumption $c^*$ can fall below the reference level $\lambda H^*$, which matches better with the real-life situation that the investor can bear some unfulfilling consumption during economic recession periods. That is, to achieve the value function, it is not necessary for the optimal consumption to exceed the reference level at any time. However, the nonnegativity constraint $c_t \geq 0$ a.s. should be actively enforced for all $t \geq 0$. This control constraint spurs some new challenges when we handle the associated HJB equation using dynamic programming arguments in subsequent sections. We only focus on the more interesting case $0 < \lambda < 1$ in the main body of this paper. The extreme case $\lambda = 0$ is a standard Merton problem under exponential utility, which is omitted. Some main results in the other extreme case $\lambda = 1$ are reported in Appendix A.

## 3 Main results

For ease of presentation and technical convenience, we only consider the case that $\rho = r > 0$. The general cases (i) $\rho \neq r > 0$ and (ii) $r = 0$ and $\rho \geq 0$ can be handled...
similarly, leading to more complicated formulas. Additional parameter assumptions are then required in these general cases to support the optimality in the verification proof, which are beyond the scope of this paper. To embed the control problem into a Markovian framework and derive the HJB equation using dynamic programming arguments, we treat both \((X_t)\) and \((H_t)\) as controlled state processes given the control policy \((c, \pi)\). The value function \(u(x, h)\) depends on both variables \(x \geq 0\) and \(h \geq 0\), namely the initial wealth and the initial reference level. Let us consider

\[
\Gamma_t := e^{-rt} u(X_t, H_t) + \int_0^t e^{-rs} U(c_s - \lambda H_s) \, ds.
\]

Heuristically, by the martingale optimality principle, we have that \((\Gamma_t)_{t \geq 0}\) is a local supermartingale under all admissible controls and a local martingale under the optimal control (if it exists). If the function \(u(x, h)\) is smooth enough, by applying Itô’s formula to the process \((\Gamma_t)_{t \geq 0}\), we can derive that

\[
e^{rt} d\Gamma_t = \left( -ru + u_x \left( rX_t + \pi_t (\mu - r) - c_t \right) + \frac{1}{2} \sigma^2 \pi_t^2 u_{xx} + U(c_t - \lambda H_t) \right) dt
\]

\[
+ u_h dH_t + u_x \pi_t \sigma dW_t,
\]

which heuristically leads to the associated HJB variational inequality

\[
\begin{align*}
\sup_{c \in [0, h], \pi \in \mathbb{R}} \left( -ru + u_x \left( rX_t + \pi (\mu - r) - c \right) + \frac{1}{2} \sigma^2 \pi^2 u_{xx} - \frac{1}{\beta} e^{\beta(\lambda h - c)} \right) &= 0, \\
u_h(x, h) &= 0, \quad \text{on some set of } (x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ that will be characterised later.}
\end{align*}
\]

for \(x \geq 0\), \(h \geq 0\). The local martingale property of \(u(X^*, H^*)\) under the optimal control \((c^*, \pi^*)\) requires that \(u_h(X^*_t, H^*_t) = 0\) whenever the process \(H^*\) strictly increases, i.e., the current consumption rate \(c^*_t\) creates a new historical maximum level so that \(H^*_t = c^*_t\) and \(c^*_t > H^*_s\) for \(s < t\). This motivates us to mandate an important free boundary condition that \(u_h(x, h) = 0\) on some set of \((x, h)\) that will be determined explicitly later in (3.6) when we analyse the associated HJB equation.

In the present paper, we aim to find some deterministic functions \(\pi^*(x, h)\) and \(c^*(x, h)\) to provide a feedback form of the optimal portfolio and consumption strategy. To this end, if \(u(x, \cdot)\) is \(C^2\) with respect to the variable \(x\), the first-order condition gives the optimal portfolio in feedback form by \(\pi^*(x, h) = -\frac{\mu - r}{\sigma^2} \frac{u_x}{u_{xx}}\). The previous HJB variational inequality (3.1) can then first be written as

\[
\begin{align*}
\left( -\frac{1}{\beta} e^{\beta(\lambda h - c)} - cu_x \right) - ru + r Xu_x - \frac{\kappa^2}{2} \frac{u_x^2}{u_{xx}} &= 0, \\
u_h &= 0,
\end{align*}
\]

for \(x \geq 0\), \(h \geq 0\), together with the free boundary condition \(u_h = 0\) on some set of \((x, h) \in \mathbb{R}_+ \times \mathbb{R}_+\) that will be characterised later.
3.1 Heuristic solution to the HJB equation

Given that $0 \leq c_t \leq H_t$, we first decompose the domain $\{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+\}$ into three different regions such that the feedback optimal consumption strategy satisfies (1) $c^*(x, h) = 0$; (2) $0 < c^*(x, h) < h$; (3) $c^*(x, h) = h$. By applying the first order condition to the HJB equation (3.2), let us consider the auxiliary control $\hat{c}(x, h) := -\frac{1}{\beta} \ln u_x + \lambda h$, which facilitates the separation of the following regions.

Region I. On the set $\mathcal{R}_1 := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) \geq e^{\lambda h}\}$, we have $\hat{c}(x, h) \leq 0$ and the optimal consumption is therefore $c^*(x, h) = 0$. The HJB variational inequality becomes

$$-\frac{1}{\beta} e^{\lambda h} - ru + rxu_x - \frac{\kappa^2 u_x^2}{2u_{xx}} = 0, \quad u_h \leq 0. \quad (3.3)$$

Region II. On the set $\mathcal{R}_2 := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : e^{-(1-\lambda)\beta h} < u_x(x, h) < e^{\lambda h}\}$, we have $0 < \hat{c}(x, h) < h$ and the optimal consumption is $c^* = -\frac{1}{\beta} \ln u_x + \lambda h$. The HJB variational inequality becomes

$$-\frac{1}{\beta} u_x + u_x \left(\frac{1}{\beta} \ln u_x - \lambda h\right) - ru + rxu_x - \frac{\kappa^2 u_x^2}{2u_{xx}} = 0, \quad u_h \leq 0. \quad (3.4)$$

Remark 3.1 Based on $c^* = -\frac{1}{\beta} \ln u_x + \lambda h$ in Region II, we know that $c^* < \lambda H^*$ if and only if $(x, h)$ is in the subset $\{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : 1 < u_x(x, h) < e^{\lambda h}\}$. This subset can be further expressed later in Remark 3.6 as a threshold (depending on $h$) of the wealth level $x$.

Region III. On the set $\mathcal{R}_3 := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) \leq e^{-(1-\lambda)\beta h}\}$, we have $\hat{c}(x, h) \geq h$ and the optimal consumption is $c^*(x, h) = h$, which indicates that the instant consumption rate $c^*_t$ coincides with the running maximum $H^*_t$. However, two subtle cases may occur that motivate us to split this region further:

(i) In a certain region (to be determined), the historical maximum level is already attained at some previous time $s$ before time $t$, and the current optimal consumption rate is either to revisit this maximum level from below or to sit on the same maximum level. This is the case that the running maximum process $H$ keeps flat from time $s$ to time $t$, and the feedback consumption takes the form $c^*_t = H^*_s$ for some $s < t$.

(ii) In the complementary region, the optimal consumption rate creates a new record of the maximum level that is strictly larger than its past consumption, and the running maximum process $H$ is strictly increasing at time $t$. This corresponds to the case that $c^*_t = H^*_t$ is a singular control and $c^*_t > H^*_s$ for $s < t$ and we have to mandate the free boundary condition $u_h(x, h) = 0$ from the martingale optimality condition.

Restricted on the set $\{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) \leq e^{-(1-\lambda)\beta h}\}$, the case (ii) suggests to treat $H^*_t = c^*_t$ as a singular control instead of the state process. That is, the dimension of the problem can be reduced and we can first substitute $h = c$ in (3.2) and then apply the first order condition to $-\frac{1}{\beta} e^{\beta(\lambda c - c)} - cu_x$ with respect to $c$. We can define the auxiliary control $\hat{c}(x) := \frac{1}{\beta(\lambda - 1)} \ln u_x$. It is then convenient to see that
\(c_t^*\) can update \(H_t^*\) to a new level if and only if the feedback control \(c_t^* = \hat{c}(X_t^*) \geq H_t^*\)
so that \(H_t^*\) is instantly increasing. We therefore separate Region III into three subsets.

**Region III-(i).** On the set

\[
\mathcal{D}_1 := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : (1 - \lambda) e^{-(1-\lambda)\beta h} < u_x(x, h) \leq e^{-(1-\lambda)\beta h}\},
\]

we have a contradiction that \(\hat{c}(x) < h\), and therefore \((c^*_t)\) is not a singular control. We should follow the previous feedback form \(c^*(x, h) = h\), in which \(h\) is a previously attained maximum level. The corresponding running maximum process remains flat at the instant time. In this region of \((x, h)\), we only know that \(u_h(x, h) \leq 0\) as we have \(dH_t = 0\). The HJB variational inequality is written as

\[
-\frac{1}{\beta} e^{\beta(\lambda h - h)} - hu_x - ru + rxu_x - \frac{\kappa^2 u_x^2}{2u_{xx}} = 0, \quad u_h \leq 0. \quad (3.5)
\]

**Region III-(ii).** On the set

\[
\mathcal{D}_2 := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) = (1 - \lambda) e^{-(1-\lambda)\beta h}\},
\]

we get \(\hat{c}(x) = h\) and the optimal consumption is \(c^*(x, h) = \frac{1}{\beta(\lambda - 1)} \ln \frac{u_x}{1-\lambda} = h\). This corresponds to the singular control \(c^*\) that creates a new peak for the whole path and \(H_t^* = c_t^* = \frac{1}{\beta(\lambda - 1)} \ln \frac{u_x(X_t^*, H_t^*)}{1-\lambda}\) is strictly increasing at the instant time so that \(H_t^* > H_s^*\) for \(s < t\) and we must require the free boundary condition that

\[
u_h(x, h) = 0 \quad \text{on} \quad \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) = (1 - \lambda) e^{-(1-\lambda)\beta h}\}. \quad (3.6)
\]

In this region, the HJB equation follows the same PDE (3.5), but together with the free boundary condition (3.6).

**Region III-(iii).** On the set

\[
\mathcal{D}_3 := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) < (1 - \lambda) e^{-(1-\lambda)\beta h}\},
\]

we get \(\hat{c}(x) > h\). This indicates that the initial reference level \(h\) is below the feedback control \(\hat{c}(x)\), and the optimal consumption is again \(c^*(x, h) = \frac{1}{\beta(\lambda - 1)} \ln \frac{u_x}{1-\lambda}\). As the running maximum \(H_t^*\) is updated immediately by \(c_t^*\), the optimal consumption pulls the associated \(H_t^*\) upward to the new value \(\frac{1}{\beta(\lambda - 1)} \ln \frac{u_x(X_t^*, H_t^*)}{1-\lambda}\) in the direction of \(h\) while \(X_t^*\) remains the same, in which \(u(x, h)\) is the solution of the HJB equation (3.5) on the set \(\mathcal{D}_2\). This suggests that for any given initial value \((x, h)\) in the set \(\mathcal{D}_3\), the feedback control \(c^*(x, h)\) pushes the value function jumping immediately to the point \((x, \hat{h})\) on the set \(\mathcal{D}_2\) where \(\hat{h} = \frac{1}{\beta(\lambda - 1)} \ln \frac{u_x(x, \hat{h})}{1-\lambda}\) for the given value of \(x\).

In summary, it is sufficient for us to only consider \((x, h)\) on the effective domain of the stochastic control problem defined by

\[
\mathcal{C} := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) \geq (1 - \lambda) e^{-(1-\lambda)\beta h}\}. \quad (3.7)
\]

Equivalently, we have \(\mathcal{C} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \subseteq \mathbb{R}^2_+\). Notice that the only possibility for \((x, h) \in \mathcal{D}_3 = \mathcal{C}^c\) occurs at the initial time \(t = 0\), and the value function is just

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equivalent to the value function of \((x, \hat{h})\) on the boundary \(D_2\) with the same \(x\). In other words, if the controlled process \((X^*, H^*)\) starts from \((x, h)\) in the region \(C\), then \((X^*, H^*)\) will always stay inside the region \(C\). On the other hand, if the process \((X^*, H^*)\) starts from the value \((x, h)\) inside the region \(D_3\), the optimal control enforces an instant jump (and the only jump) of the process \(H^*\) from \(H^*_0 = h\) to \(H^*_0 = \hat{h}\) on the set \(D_2\), and both processes \((X^*_{t})\) and \((H^*_{t})\) are continuous processes diffusing inside the effective domain \(C\) afterwards for \(t > 0\).

On the other hand, observe that as the wealth level \(x\) declines to zero, the consumption rate \(c\) will reach zero at some \(x^*\) (to be determined). If \(x\) continues to decrease to 0, the optimal investment \(\pi\) should also drop to 0. Otherwise, we confront the risk of bankruptcy by keeping trading with nearly zero wealth. Using the optimal portfolio \(\pi^*(x, h) = -\frac{\mu-r}{\sigma^2} \frac{ux}{uxx}\), the boundary condition can be described by

\[
\lim_{x \to 0} \frac{ux(x, h)}{uxx(x, h)} = 0. \tag{3.8}
\]

In addition, if we start with 0 initial wealth, the wealth level will never change as there is no trading according to the previous condition, and consumption should consequently stay at 0 forever. That is, we have another boundary condition that

\[
\lim_{x \to 0} u(x, h) = \int_0^\infty -\frac{1}{\beta} e^0 e^{-rt} dt = -\frac{1}{r\beta}. \tag{3.9}
\]

On the other hand, as wealth tends to infinity, one can consume as much as possible and a small variation in the wealth has a negligible effect on the change of the value function. It thus follows that

\[
\lim_{x \to \infty} u(x, h) = 0 \quad \text{and} \quad \lim_{x \to \infty} ux(x, h) = 0. \tag{3.10}
\]

To ensure global regularity of the solution, we also need to impose smooth-fit conditions along two free boundaries of \((x, h)\) via \(ux(x, h) = e^{\lambda \beta h}, ux(x, h) = e^{-(1-\lambda)\beta h}\), which separate the regions as discussed above.

We can then employ the dual transform approach to linearise the HJB equation. In particular, we apply the dual transform only with respect to the variable \(x\) and treat the variable \(h\) as a parameter. That is, for each fixed \(h \geq 0\), we consider \(x \geq 0\) such that \((x, h) \in C\) and define the dual function on the domain \(\{y \geq (1-\lambda)e^{-(1-\lambda)\beta h}\}\) by

\[
v(y, h) := \sup_{(x, h) \in C, x \geq 0} (u(x, h) - xy) \quad \text{for} \quad y \geq (1-\lambda)e^{-(1-\lambda)\beta h}.
\]

For the given \((x, h)\), if we define \(\hat{y}(x, h) := u_x(x, h)\) (short as \(\hat{y}\)), the dual representation implies \(u(x, h) = v(\hat{y}, h) + x\hat{y}\) as well as \(v_y(\hat{y}, h) = -x\). We then have

\[
u_h(x, h) = \frac{\partial}{\partial h} (v(\hat{y}, h) + x\hat{y}) = v_h(\hat{y}, h) + (v_y(\hat{y}, h) + x) \frac{d\hat{y}}{dh} = v_h(\hat{y}, h).
\]

In view of (3.6), we obtain the free boundary condition that

\[
v_h(y, h) = 0 \quad \text{on the set} \quad \{(y, h) \in (0, \infty) \times \mathbb{R}_+ : y = (1-\lambda)e^{(\lambda-1)\beta h}\}. \tag{3.11}
\]
To align with the nonlinear HJB variational inequality (3.3)–(3.5) in three different regions, the transformed dual variational inequality can be written as

\[
\frac{\kappa^2}{2} y^2 v_{yy} - rv = \begin{cases} 
\frac{1}{\beta} e^{\lambda \beta h}, & \text{if } y \geq e^{\lambda \beta h}, \\
\frac{1}{\beta} y - y \left( \frac{1}{\beta} \ln y - \lambda h \right), & \text{if } e^{(\lambda-1)\beta h} < y < e^{\lambda \beta h}, \\
\frac{1}{\beta} e^{(\lambda-1)\beta h} + hy, & \text{if } (1 - \lambda)e^{(\lambda-1)\beta h} \leq y \leq e^{(\lambda-1)\beta h},
\end{cases}
\]

(3.12)
together with the free boundary condition (3.11). As \( h \) is regarded as a parameter, we fix \( h \) and study the above equation as an ODE problem in the variable \( y \).

By virtue of the duality representation, the boundary conditions in (3.10) become

\[
\lim_{y \to 0} v_y(y, h) = -\infty \quad \text{and} \quad \lim_{y \to 0} \left( v(y, h) - y v_y(y, h) \right) = 0,
\]

(3.13)
and the boundary conditions (3.8) and (3.9) at \( x = 0 \) can be written as

\[
y v_{yy}(y, h) \rightarrow 0, \quad v(y, h) - y v_y(y, h) \rightarrow -\frac{1}{r \beta} e^{\lambda \beta h} \quad \text{as } v_y(y, h) \to 0.
\]

(3.14)
Based on these boundary conditions, we can solve the dual ODE (3.12) fully explicitly. The proof is given in Sect. 5.1.

**Proposition 3.2** Let \( h \geq 0 \) be a fixed parameter. Under the boundary conditions in (3.13) and (3.14) and the free boundary condition (3.11) as well as the smooth-fit conditions with respect to \( y \) at the boundary points \( y = e^{\lambda \beta h} \) and \( y = e^{(\lambda-1)\beta h} \), the ODE (3.12) in the domain \( \{ y \geq (1 - \lambda)e^{(\lambda-1)\beta h} \} \) admits the unique solution

\[
v(y, h) = \begin{cases} 
C_2(h) y^r - \frac{1}{r \beta} e^{\lambda \beta h}, & \text{if } y \geq e^{\lambda \beta h}, \\
C_3(h) y^r + C_4(h) y^r_2 - \frac{1}{r \beta} \ln y - \lambda \beta h + \frac{\kappa^2}{2r}, & \text{if } e^{(\lambda-1)\beta h} < y < e^{\lambda \beta h}, \\
C_5(h) y^r + C_6(h) y^r_2 - \frac{1}{r \beta} hy - \frac{1}{r \beta} e^{(\lambda-1)\beta h}, & \text{if } (1 - \lambda)e^{(\lambda-1)\beta h} \leq y \leq e^{(\lambda-1)\beta h},
\end{cases}
\]

where the functions \( C_2(h), C_3(h), C_4(h), C_5(h) \) and \( C_6(h) \) are given explicitly by

\[
C_2(h) := \frac{(1 - \lambda)^{r_1-r_2}}{(r_1 - r_2) \beta r} \left( \frac{1}{1 - r_2} e^{(\lambda-1)(1-r_2)\beta h} \right) - \frac{\lambda}{\lambda (1 - r_2) - (r_1 - r_2)} e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} + \frac{(1 - r_1) \kappa^2}{2(r_1 - r_2) \beta r^2} \left( e^{(\lambda-1)(1-r_2)\beta h} - e^{\lambda(1-r_2)\beta h} \right),
\]

(3.15)

\[
C_3(h) := \frac{(r_2 - 1) \kappa^2}{2(r_1 - r_2) \beta r^2} e^{\lambda(1-r_1)\beta h},
\]

(3.16)
Optimal consumption with reference to past spending maximum

\[ C_4(h) := \frac{(1 - \lambda)r_1 - r_2}{(r_1 - r_2)\beta r} \left( \frac{1}{1 - r_2} e^{(\lambda - 1)(1 - r_2)\beta h} \right. \]
\[ \quad - \left. \frac{\lambda}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(\lambda(1 - r_2) - (r_1 - r_2))\beta h} \right) \]
\[ + \frac{(1 - r_1)\kappa^2}{2(r_1 - r_2)\beta r^2} e^{(\lambda - 1)(1 - r_2)\beta h}, \quad (3.17) \]
\[ C_5(h) := \frac{(1 - r_2)\kappa^2}{2(r_1 - r_2)\beta r^2} (e^{(\lambda - 1)(1 - r_1)\beta h} - e^{(1 - r_1)\beta h}), \quad (3.18) \]
\[ C_6(h) := \frac{(1 - \lambda)r_1 - r_2}{(r_1 - r_2)\beta r} \left( \frac{1}{1 - r_2} e^{(\lambda - 1)(1 - r_2)\beta h} \right. \]
\[ \quad - \left. \frac{\lambda}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(\lambda(1 - r_2) - (r_1 - r_2))\beta h} \right). \quad (3.19) \]

Here, the constants \( r_1 > 1 \) and \( r_2 < 0 \) are the two roots of the quadratic equation \( z^2 - z - \frac{2r}{\kappa^2} = 0 \); they are given by

\[ r_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8r}{\kappa^2}} \right). \]

We can now present the main result of this paper, which provides the optimal investment and consumption strategies in piecewise feedback form using the variables \( y \) and \( h \). The complete proof is deferred to Sect. 5.2.

**Theorem 3.3** Let \((x, h) \in \mathcal{C}\), where \( \mathcal{C} \) is the effective domain (3.7). For \((y, h)\) in the set \((0, \infty) \times [0, \infty)\), let us define the feedback functions

\[ c^\dagger(y, h) = \begin{cases} 0, & \text{if } y \geq e^{\lambda \beta h}, \\ -\frac{1}{\beta} \ln y + \lambda h, & \text{if } e^{(\lambda - 1)\beta h} < y < e^{\lambda \beta h}, \\ h, & \text{if } (1 - \lambda) e^{(\lambda - 1)\beta h} < y \leq e^{(\lambda - 1)\beta h}, \end{cases} \]
\[ \pi^\dagger(y, h) = \begin{cases} \frac{2r}{\kappa^2} C_2(h) y^{r_2 - 1}, & \text{if } y \geq e^{\lambda \beta h}, \\ \frac{2r}{\kappa^2} C_3(h) y^{r_1 - 1} + \frac{1}{r_1^2}, & \text{if } e^{(\lambda - 1)\beta h} < y < e^{\lambda \beta h}, \\ \frac{2r}{\kappa^2} C_4(h) y^{r_1 - 1} + \frac{2r}{\kappa^2} C_6(h) y^{r_2 - 1}, & \text{if } (1 - \lambda) e^{(\lambda - 1)\beta h} \leq y \leq e^{(\lambda - 1)\beta h}. \end{cases} \quad (3.21) \]
We consider the process $Y_t(y) := ye^{rt}M_t$, where $M_t := e^{-(r+\frac{\kappa^2}{2})t-\kappa W_t}$ is the discounted state price density. Let the constant $y^* = y^*(x,h)$ be the unique solution to the budget constraint equation $\mathbb{E}\left[\int_0^\infty c^\dagger(Y_t(y), H_t^\dagger(y))M_t dt\right] = x$, where

$$H_t^\dagger(y) := h \vee \sup_{s \leq t} c^\dagger(Y_s(y), H_s^\dagger(y)) = h + \left(\frac{1}{\lambda - 1}\right) \ln \left(\frac{1}{1 - \lambda} \inf_{s \leq t} Y_s(y)\right)$$

is the optimal reference corresponding to any fixed $y > 0$. The value function $u(x,h)$ can be attained by employing the optimal consumption and portfolio strategies in the feedback form $c^*_t = c^\dagger(Y^*_t, H^*_t)$ and $\pi^*_t = \pi^\dagger(Y^*_t, H^*_t)$ for all $t \geq 0$, where $Y_t^* := Y_t(y^*)$ and $H_t^* = H_t^\dagger(y^*)$.

The process $H^*$ is strictly increasing at $t$ if and only if $Y_t^* = (1 - \lambda)e^{(\lambda - 1)\beta H_t^*}$. If we have $y^*(x,h) < (1 - \lambda)e^{(\lambda - 1)\beta h}$ at the initial time, the optimal consumption creates a new peak and brings $H_{t-}^* = h$ jumping immediately to a higher level $H_t^* = \frac{1}{(\lambda - 1)^\beta} \ln(\frac{1}{1 - \lambda} y^*(x,h))$ such that $t = 0$ becomes the only jump time of $H^*$.

Remark 3.4 Note that the feedback optimal consumption $c_t^* = c^\dagger(Y_t^*, H_t^*)$ coming from (3.20) is predictable. Indeed, if $Y_t^* > (1 - \lambda)e^{(\lambda - 1)\beta H_t^*}$, the optimal consumption at time $t$ is determined by the continuous process $Y^*$ and the past consumption maximum right before $t$, i.e., $H_{t-}^*$, which is predictable. In this case, the current consumption does not create the new maximum level. When $Y_t^* = (1 - \lambda)e^{(\lambda - 1)\beta H_t^*}$, the optimal consumption is determined directly by the continuous process $Y^*$, which is again predictable.

In Theorem 3.3, the feedback controls are given in terms of the dual value function and the dual variables. In what follows, we show that the inverse transformation can be exercised so that the primal value function $u(x,h)$ and the feedback controls can be expressed by $x$ and $h$. In the proof of Theorem 3.3, we take full advantage of the simplicity in the dual feedback controls and verify their optimality using the duality relationship and some estimations based on the dual process $Y^*$ under the optimal controls, we have to express the feedback controls in terms of $Y^*$ and $H^*$, and therefore the inverse dual transform becomes necessary, which will be carefully established as follows.

By using the dual relationship that $v(y,h) = \sup_{x>0}(u(x,h) - xy)$, we have that the optimal choice $x$ satisfying $u(x,h) = y$ admits the expression

$$x = g(y,h) := -v_y(y,h). \quad (3.22)$$

Defining $f(\cdot,h)$ as the inverse of $g(\cdot,h)$, we have that

$$u(x,h) = v(f(x,h),h) + xf(x,h). \quad (3.23)$$

Note that $v$ has different expressions in the regions $\{c = 0\}$, $\{0 < c < h\}$ and $\{c = h\}$; the function $f$ should also have a piecewise form across these regions. By the definition of $g$ in (3.22), the invertibility of the map $x \mapsto g(x,h)$ is guaranteed by the following important result. Its proof is deferred to Sect. 5.1.
Lemma 3.5 In all three regions, we have $v_{yy}(y, h) > 0$ for all $h > 0$, and the inverse Legendre transform $u(x, h) = \inf_{y \geq (1-\lambda)e^{-(1-\lambda)\beta h}}(v(y, h) + xy)$ is well defined. Moreover, this implies that the feedback optimal portfolio always satisfies $\pi^+(y, h) > 0$.

Using (3.22) and Proposition 3.2, the function $f$ is implicitly determined in different regions by the following equations:

(i) If $f(x, h) \geq e^{\lambda \beta h}$, $f(x, h) = f_1(x, h)$ can be determined by

$$x = -C_2(h) r_2 \left( f_1(x, h) \right)^{r_2-1}.$$  

(ii) If $e^{(1-\lambda)\beta h} < f(x, h) < e^{\lambda \beta h}$, Lemma 3.5 implies that $v_y(y, h)$ is strictly increasing in $y$ and $f(x, h) = f_2(x, h)$ is uniquely determined by

$$x = -C_3(h) r_1 \left( f_2(x, h) \right)^{r_1-1} - C_4(h) r_2 \left( f_2(x, h) \right)^{r_2-1}$$

$$- \frac{1}{r\beta} \left( \ln f_2(x, h) - \lambda \beta h + \frac{\kappa^2}{2r} \right) \quad (3.24).$$

(iii) If $(1-\lambda)e^{(1-\lambda)\beta h} \leq f(x, h) \leq e^{(1-\lambda)\beta h}$, Lemma 3.5 implies that $v_y(y, h)$ is strictly increasing in $y$ and $f(x, h) = f_3(x, h)$ is uniquely determined by

$$x = -C_5(h) r_1 \left( f_3(x, h) \right)^{r_1-1} - C_6(h) r_2 \left( f_3(x, h) \right)^{r_2-1} + \frac{h}{r} . \quad (3.25)$$

In the region $R_1$, we obtain $f_1(x, h) = \left( \frac{-x}{C_2(h) r_2} \right)^{r_2-1}$. In addition, $f_1(x, h) \geq e^{\lambda \beta h}$ if and only if $x \leq x_{\text{zero}}(h)$, where we define

$$x_{\text{zero}}(h) := -e^{\lambda \beta h (r_2-1)} C_2(h) r_2,$$  

which corresponds to the threshold such that the optimal consumption becomes zero whenever $x < x_{\text{zero}}(h)$.

In the region $R_2$, the function $f_2$ is uniquely (implicitly) determined by equation (3.24) when $x_{\text{zero}}(h) < x < x_{\text{aggr}}(h)$, where $x_{\text{aggr}}(h)$ is the solution of

$$f_2(x, h) = e^{(1-\lambda)\beta h}.$$  

In view of (3.24), we can obtain the boundary explicitly as

$$x_{\text{aggr}}(h) = -C_3(h) r_1 e^{(1-\lambda)(r_1-1)\beta h} - C_4(h) r_2 e^{(1-\lambda)(r_2-1)\beta h} + \frac{h}{r} - \frac{\kappa^2}{2r^2 \beta} . \quad (3.27)$$

which corresponds to the threshold such that the consumption stays below the historical maximum level whenever $x < x_{\text{aggr}}(h)$.

Remark 3.6 In addition, as in Remark 3.1, we know that the optimal consumption falls below the reference level if and only if $1 < f_2(x, h) < e^{\lambda \beta h}$. Using (3.24) again, we can determine the critical point $x_{\text{modr}}(h)$ by

$$x_{\text{modr}}(h) := -C_3(h) r_1 - C_4(h) r_2 + \frac{\lambda h}{r} - \frac{\kappa^2}{2r^2 \beta} .$$
It follows that the optimal consumption rate meets the moderate plan 0 < \( c_t^* < \lambda H_t^* \) if and only if the wealth level \( X_t^* \) satisfies 
\[
x_{\text{zero}}(H_t^*) < X_t^* < x_{\text{modr}}(H_t^*).
\]

In the region \( \mathcal{D}_1 \cup \mathcal{D}_2 \), the expression of \( f_3 \) is uniquely determined by the equation (3.25) when 
\[
x_{\text{aggr}}(h) \leq x \leq x_{\text{lavs}}(h),
\]
where \( x_3 \) is the solution of
\[
f_3(x, h) = (1 - \lambda)e^{(\lambda - 1)\beta h}.
\]

It follows from (3.25) that the boundary \( x_{\text{lavs}}(h) \) is explicitly given by
\[
x_{\text{lavs}}(h) := -C_5(h)r_1(1 - \lambda)r_1^{-1}e^{(\lambda - 1)(r_1 - 1)\beta h} - C_6(h)r_2(1 - \lambda)r_2^{-1}e^{(\lambda - 1)(r_2 - 1)\beta h} + \frac{h}{r},
\]
which corresponds to the threshold such that the optimal consumption is extremely lavish in that \( c_t^* \) creates the new maximum level whenever \( x = x_{\text{lavs}}(h) \).

Moreover, in view of the definitions of \( C_5(h) \) and \( C_6(h) \) in (3.18) and (3.19), one can check that \( x_{\text{lavs}}(h) \) is strictly increasing in \( h \), and hence we can define the inverse function
\[
\tilde{h}(x) := (x_3)^{-1}(x), \quad x \geq 0.
\]
Along the boundary \( x = x_{\text{lavs}}(h) \), the feedback form of the optimal consumption in (3.20) for \( y = (1 - \lambda)e^{(\lambda - 1)\beta h} \) is given by \( e^*(x) = \frac{1}{1 - \lambda} \ln(\frac{1}{1 - \lambda} f_3(x, \tilde{h}(x))) \), which only depends on the variable \( x \). That is, the optimal consumption can be determined by the current wealth \( X_t^* \), and the associated running maximum \( H_t^* \) is instantly increasing.

In Figure 1, we graph all boundary curves \( x_{\text{zero}}(h), x_{\text{modr}}(h), x_{\text{aggr}}(h) \) and \( x_{\text{lavs}}(h) \) as functions of \( h \geq 0 \) in the left panel and plot them in terms of the parameter \( \lambda \in [0.01, 0.98] \) in the right panel (recall that each \( C_i(h; \lambda) \) depends on \( \lambda \)). Although \( x_{\text{zero}}(h), x_{\text{modr}}(h), x_{\text{aggr}}(h) \) and \( x_{\text{lavs}}(h) \) are complicated nonlinear functions of \( h \), the left panel illustrates that all boundary curves are increasing in \( h \). This is consistent with the intuition that if the past reference level is higher, the investor would expect larger wealth thresholds to trigger a change of consumption patterns. Recall that we only consider the effective domain, i.e., the region below (and including) the boundary curve \( x_{\text{lavs}}(h) \). It is interesting to see from the right panel that \( x_{\text{zero}}(1; \lambda) \) and \( x_{\text{aggr}}(1; \lambda) \) are decreasing in \( \lambda \), while \( x_{\text{modr}}(1; \lambda) \) and \( x_{\text{lavs}}(1; \lambda) \) are instead increasing in \( \lambda \). That is, if the investor clings to a larger proportion of the past spending maximum, it is more likely that he will switch from zero consumption to positive consumption (for a low wealth level) and switch from a consumption \( c_t^* < H_t^* \) to the past maximum level \( H_t^* \) (for a high wealth level). On the other hand, with a higher proportion \( \lambda \), the investor foresees that any aggressive consumption may lead to a ratcheting high reference that will depress all future utilities. As a consequence, the investor will accumulate a larger wealth to change from the moderate consumption \( c_t^* < \lambda H_t^* \) to the pattern \( c_t^* \geq \lambda H_t^* \) or consume in a way creating a new maximum record that \( c_t^* > H_t^* \) for \( s < t \), which is consistent with the right panel.

In particular, the boundary curve \( x_{\text{modr}}(1; \lambda) \) in the right panel illustrates that the more the investor cares about the past consumption peak \( H_t^* \), the more conservative
he will become. This may partially explain the real-life situation that a constantly aggressive consumption behaviour may not result in long-term happiness. A high consumption plan also creates a high level of psychological competition and hence the aggressive consumption behaviour may not be sustainable for the lifetime. A wise investor who takes into account the past reference will strategically lower the consumption rate from time to time (triggered by a wealth threshold) below the dynamic reference such that the reference process can be maintained at a reasonable level and the overall performance can eventually become a win.

Plugging the different pieces of $f$ back into equation (3.23), we can readily get the next result, in which the value function $u$ and the optimal feedback controls are all given in terms of the primal variables $x$ and $h$, and the existence of a unique strong solution to the SDE (2.1) under optimal controls can be obtained.

**Corollary 3.7** For $(x, h) \in C$ and $0 < \lambda < 1$, let us define the piecewise function

$$f(x, h) = \begin{cases} (-x/C_2(h)r_2)^{1/r} \times, & \text{if } x \leq x_{\text{zero}}(h), \\ f_2(x, h), & \text{if } x_{\text{zero}}(h) < x < x_{\text{aggr}}(h), \\ f_3(x, h), & \text{if } x_{\text{aggr}}(h) \leq x \leq x_{\text{lavs}}(h), \end{cases}$$

where $f_2(x, h)$ and $f_3(x, h)$ are determined by (3.24) and (3.25). The value function $u(x, h)$ of the control problem in (2.2) is given by

$$u(x, h) = \begin{cases} C_2(h) f(x, h)^{r_2} - \frac{1}{r_\beta} e^{\lambda \beta h} + xf(x, h), & \text{if } x \leq x_{\text{zero}}(h), \\ C_3(h) (f(x, h))^{r_1} + C_4(h) (f(x, h))^{r_2} + \frac{f(x, h)}{r_\beta} (\ln f(x, h) - \lambda \beta h + \frac{x_r}{2} - 1 + x r \beta), & \text{if } x_{\text{zero}}(h) < x < x_{\text{aggr}}(h), \\ C_5(h) (f(x, h))^{r_1} + C_6(h) (f(x, h))^{r_2} - \frac{1}{r \beta} h f(x, h) - \frac{1}{r_\beta} e^{(\lambda - 1) \beta h} + xf(x, h), & \text{if } x_{\text{aggr}}(h) \leq x \leq x_{\text{lavs}}(h). \end{cases}$$
To distinguish the feedback functions $c^\dagger(y,h)$ and $\pi^\dagger(y,h)$ in \eqref{eq:3.20} and \eqref{eq:3.21} based on the dual variables, let us denote by $c^*(x,h)$ and $\pi^*(x,h)$ the feedback functions of the optimal consumption and portfolio using the primal variables $(x,h)$. We have that $c_t^* = c^*(X_t^*, H_t^*)$ and $\pi_t^* = \pi^*(X_t^*, H_t^*)$, where

$$c^*(x,h) = \begin{cases} 0, & \text{if } x \leq x_{\text{zero}}(h), \\ -\frac{1}{\beta} \ln f(x,h) + \lambda h, & \text{if } x_{\text{zero}}(h) < x < x_{\text{aggr}}(h), \\ h, & \text{if } x_{\text{aggr}}(h) \leq x < x_{\text{lavs}}(h), \\ \frac{1}{(x-1)^2} \ln \left( \frac{1}{1-x} f(x,\tilde{h}(x)) \right), & \text{if } x = x_{\text{lavs}}(h), \end{cases} \quad (3.30)$$

where $\tilde{h}(x)$ is given in \eqref{eq:3.29}, and

$$\pi^*(x,h) = \begin{cases} (1-r_2)x, & \text{if } x \leq x_{\text{zero}}(h), \\ \frac{2r}{k^2} C_3(h) f'^{-1}(x,h) + \frac{2r}{k^2} C_4(h) f^r(x,h) + \frac{1}{r^2}, & \text{if } x_{\text{zero}}(h) < x < x_{\text{aggr}}(h), \\ \frac{2r}{k^2} C_5(h) f'^{-1}(x,h) + \frac{2r}{k^2} C_6(h) f^r(x,h), & \text{if } x_{\text{aggr}}(h) \leq x \leq x_{\text{lavs}}(h). \end{cases} \quad (3.31)$$

We have that $0 < c_t^*(X_t^*, H_t^*) < \lambda H_t^*$ if and only if $x_{\text{zero}}(H_t^*) < X_t^* < x_{\text{modr}}(H_t^*)$.

Moreover, for any initial value $(X_0^*, H_0^*) = (x,h) \in \mathcal{C}$, the stochastic differential equation

$$dX_t^* = rX_t^* dt + \pi_t^*(\mu - r) dt + \pi_t^* \sigma dW_t - c_t^* dt \quad (3.32)$$

has a unique strong solution given the optimal feedback control $(c^*, \pi^*)$ as above.

**Proof** See Sect. 5.2, before Lemma 5.7. \hfill \square

Based on Corollary 3.7, we can readily obtain the next result on the asymptotic behaviour of the optimal consumption–wealth ratio $c_t^*/X_t^*$ and the investment amount $\pi_t^*$ when the wealth is sufficiently large. The proof is given in Sect. 5.1.

**Corollary 3.8** For $0 < \lambda < 1$, as $x \leq x_{\text{lavs}}(h)$, the asymptotic behaviour of large wealth $x \to \infty$ is equivalent to $\lim_{h \to \infty} x_{\text{lavs}}(h) = \infty$ thanks to the explicit expression of $x_{\text{lavs}}(h)$ in \eqref{eq:3.28}. We then have that

$$\lim_{x \to \infty} \frac{c^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)} = r, \quad \lim_{x \to \infty} \frac{\pi^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)} = \frac{(\mu - r)(1-\lambda)^{r^{-1}}}{r\beta\sigma^2}.$$ 

As the wealth level gets sufficiently large, the optimal consumption is asymptotically proportional to the wealth level in that $c_t^* \approx rX_t^*$, and the optimal investment converges to a constant level so that $\pi_t^* \approx (\mu - r) \frac{(1-\lambda)^{r^{-1}}}{r\beta\sigma^2}$. That is, the investor will only allocate a constant amount of wealth into the risky asset and save most of his wealth in the bank account.
Table 1 Comparison results on optimal consumption when the wealth level is extremely small and extremely large

|                      | when wealth is low                                                                 | when wealth is high                                                                 |
|----------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
| Arun [2]             | the problem is only well defined if the wealth level satisfies the constraint $X_t^* \geq \frac{a}{\lambda} H_t^*$; moreover, $c_t^* = a H_t^*$ for some $a < 1$, i.e., the optimal consumption is proportional to the optimal wealth | $c_t^* = b X_t^*$ for some $b > 0$, i.e., the optimal consumption is proportional to the optimal wealth |
| Guasoni et al. [12]  | when $X_t^* \leq a H_t^*$, $c_t^* = \frac{1}{a} X_t^*$ for some $a > 0$, i.e., the optimal consumption is proportional to the optimal wealth | when $X_t^* \geq b H_t^*$, $c_t^* = \frac{1}{b} X_t^*$ for some $b > 0$, i.e., the optimal consumption is proportional to the optimal wealth |
| the present paper    | when $X_t^* \leq x_{\text{zero}}(H_t^*)$, $c_t^* = 0$ even when $X_t^* > 0$ | when $X_t^* = x_{\text{max}}(H_t^*)$, $c_t^*$ is a nonlinear function of $X_t^*$, as $X_t^* \to \infty$, $c_t^* \approx r X_t^*$ is asymptotically proportional to the optimal wealth |

3.2 Comparison with some related works

Given the feedback optimal controls in (3.30) and (3.31), we briefly present here some comparison results with Arun [2] and Guasoni et al. [12] on how the optimal consumption is affected by the past spending maximum. We stress that [2] considers optimal consumption under a standard time separable power utility, i.e.,

$$\sup_{(c,\pi)} \mathbb{E}\left[ \int_0^\infty e^{-\rho t} c_t^p dt \right],$$

while the drawdown constraint $c_t \geq \lambda H_t^*$, $0 < \lambda < 1$, is only imposed in the set of admissible controls. See also Dybvig [9] with the ratcheting constraint for $\lambda = 1$. A similar optimal dividend control problem with a drawdown constraint is also formulated and studied in Angoshtari et al. [1]. On the other hand, the authors of [12] study the optimal consumption under a Cobb–Douglas utility, i.e.,

$$\sup_{(c,\pi)} \mathbb{E}\left[ \int_0^\infty \frac{(c_t / H_t^\alpha)^p}{p} dt \right],$$

where the utility is defined on the ratio of the consumption rate and the consumption running maximum. By virtue of the power utility on the consumption rate, the optimal consumption in [2] and [12] automatically satisfies $c_t^* \geq 0$. On the other hand, we are interested in exponential utility that measures the difference between the consumption rate and the consumption running maximum. The nonnegativity constraint $c_t \geq 0$ needs to be taken care of in solving the HJB equation. As our utility differs from [2] and [12], the feedback optimal consumption is naturally distinct from their results. But we can compare the optimal consumption behaviour when the wealth level becomes extremely low and extremely high. These major differences are summarised in Table 1.

In addition, in Arun [2] and Guasoni et al. [12] (see also Angoshtari et al. [1]), one can change variables and focus on the new state process $X/H$ to reduce the dimension. As a consequence, all thresholds for the wealth variable $x$ separating different regions for the piecewise optimal control in these works are simply linear functions of $h$. In contrary, our utility is defined on the difference $c_t - \lambda H_t$, and such a change of variables is no longer applicable. The HJB equation is genuinely two-dimensional.
which complicates the characterisation of all boundary curves separating different regions. We choose to apply the dual transform to $x$ and treat $h$ as a parameter in the whole analysis. The smooth-fit principle and inverse transform help us to identify these boundary curves explicitly. We finally can express the thresholds $x_{\text{zero}}(h)$, $x_{\text{aggr}}(h)$ and $x_{\text{lavs}}(h)$ in (3.26)–(3.28) as nonlinear functions of $h$ (see the left panel of Figure 1), which are much more complicated than their counterparts in [2] and [12].

4 Numerical examples and sensitivity analysis

We present here some numerical examples of a sensitivity analysis on the model parameters using the closed-form value function and feedback optimal controls in Corollary 3.7 and discuss some interesting financial implications.

Let us first examine the sensitivity with respect to the weight parameter $0 < \lambda < 1$ by plotting in Figure 2 some comparison graphs of the value function, the feedback optimal consumption and the feedback optimal portfolio. From the middle panel, we see again that $x_{\text{zero}}(1; \lambda)$ and $x_{\text{aggr}}(1; \lambda)$ are decreasing in $\lambda$, but $x_{\text{lavs}}(1; \lambda)$ is increasing in $\lambda$. More importantly, for each fixed $x$ with $x_{\text{zero}}(1; \lambda_{\text{max}}) < x < x_{\text{aggr}}(1; \lambda_{\text{min}})$, the optimal consumption $c^*(x, 1; \lambda)$ is increasing in the parameter $\lambda \in (\lambda_{\text{min}}, \lambda_{\text{max}})$, which matches with the intuition that a higher reference weight parameter induces a higher consumption. However, the middle panel also illustrates that this intuition is only partially correct as it only holds when the consumption does not surpass the historical maximum. When the wealth level gets higher, the investor can freely choose to consume in a lavish way. Then, for $x > x_{\text{lavs}}(1; \lambda_{\text{min}})$, we can see that a smaller $\lambda$ leads to an earlier lavish consumption $c^*(x, 1) > h = 1$ creating a new $h$. But when the wealth continues to increase, the consumption with a larger $\lambda$ will eventually dominate its counterpart with a smaller $\lambda$.

We can also observe from the right panel of Figure 2 that for fixed $x > 0$, $\pi^*(x, 1; \lambda)$ is decreasing in $\lambda$, which is consistent with the middle panel that the optimal consumption level is lifted up by a larger value of $\lambda$. When there is sufficient capital, the investor may strategically invest less in the market to save more cash to support a higher consumption plan induced by a larger $\lambda$. The left panel of Figure 2
Fig. 3 Parameters are $r = 0.05$, $\sigma = 0.25$, $\lambda = 0.5$, $\beta = 1$ and the variable $h = 1$. With changes of $\mu = 0.10, 0.12, 0.14, 0.16$ and $0.18$, we plot graphs of the value function $u(x, 1)$ (left), the optimal consumption $c^*(x, 1)$ (middle) and the optimal portfolio $\pi^*(x, 1)$ (right) for $x \in [0, 20]$ further shows that the value function $u(x, h; \lambda)$ is actually decreasing in $\lambda$. Note that our utility is measured via the difference of the consumption $c^*_t$ and the reference level $\lambda H^*_t$. When $\lambda$ increases, both $c^*_t$ and $\lambda H^*_t$ increase. We can see from the left panel that $\lambda H^*_t$ increases faster than the consumption $c^*_t$ during the life cycle, which leads to a drop of $c^*_t - \lambda H^*_t$ and a decline in the value function. It is also interesting to observe that when $\lambda$ is large, for $x_{\text{aggr}}(1; \lambda) < x < x_{\text{lavs}}(1; \lambda)$, the optimal portfolio $\pi^*(x, 1; \lambda)$ may decrease even when the wealth $x$ increases. But when $x$ continues to increase such that $x > x_{\text{lavs}}(1; \lambda)$, we can see that the optimal portfolio $\pi^*(x, 1; \lambda)$ starts to increase in $x$. That the optimal portfolio might be decreasing in $x$ differs from some existing works and is a consequence of our specific path-dependent preference. Indeed, when the wealth is sufficient to support the aggressive consumption $c^*_t > \lambda H^*_t$ but the reference level $\lambda H^*_t$ is not changed, the investor may strategically withdraw a portfolio amount from the financial market to support the consumption plan. This non-standard phenomenon is more likely to happen when the reference parameter $\lambda$ is large such that the resulting consumption $c^*_t > \lambda H^*_t$ is very high (see Figure 2), or when the risky asset performance is not good enough (low return as in Figure 3 or high volatility as in Figure 4). However, when $x$ gets abundant such that the investor starts to increase the reference level $\lambda H^*_t$, the large amount of consumption will quickly eat the capital, and the investor can no longer compromise the portfolio amount to support consumption. It turns out to be optimal for the investor to increase the portfolio and accumulate more wealth from the financial market to sustain the extremely high consumption decision. We stress that the non-standard phenomenon that $\pi^*(x, 1)$ may decrease in $x$ is a consequence of some complicated trade-offs of all model parameters. Under some appropriate model parameters, the optimal portfolio $\pi^*(x, 1)$ is always increasing in $x > 0$, which matches with the intuition that we invest more if we have more.

We next discuss the impact of the drift parameter $\mu$ based on the plots in Figure 3. Firstly, we can see from the left panel that both $x_{\text{zero}}(1; \mu)$ and $x_{\text{aggr}}(1; \mu)$ are decreasing in $\mu$. That is, the higher return the risky asset has, the less wealth the investor needs to start a positive consumption and initiate a lavish consumption to increase the reference level. Moreover, the feedback optimal consumption is also increasing in $\mu$. These observations are consistent with the real-life situation that a bull market helps the investor to accumulate more wealth so that he becomes more
optimistic to develop a more aggressive consumption pattern. Secondly, as one can expect, the right panel of Figure 3 illustrates that the optimal portfolio in the financial market increases as the return increases. In addition, the left panel shows that the primal value function is increasing in $\mu$. It illustrates that when the return $\mu$ increases, the increment in the optimal consumption rate $c^*_t$ dominates the increment in the reference level $\lambda H^*_t$ so that the value function is lifted up. Thirdly, combining Figures 2 and 3, for the same wealth level $x$, we see that the optimal portfolio $\pi^*(x, 1; \lambda, \mu)$ is decreasing in $\lambda$, but increasing in $\mu$. As a consequence, for those investors who are more addicted to the past reference level, the market premia need to be sufficiently high to attract them to invest in the risky asset. This may partially explain the observed equity premium puzzle (see Mehra and Prescott [17] and many subsequent works) from the perspective of our proposed path-dependent utility with past spending maximum.

Finally, we study the sensitivity with respect to the volatility $\sigma$ in Figure 4. From the middle panel of Figure 4, we observe that monotonicity of the threshold $x_{\text{zero}}(1; \sigma)$ and the optimal consumption $c^*(x, 1; \sigma)$ in the parameter $\sigma$ do not hold in general and become more subtle and complicated. Only when the wealth level is sufficiently large, the optimal consumption $c^*(x, 1; \sigma)$ is decreasing in $\sigma$. It is only clear that the threshold $x_{\text{aggr}}(1; \sigma)$ is increasing in $\sigma$. This can be explained by noting that when the wealth level is sufficiently high, the less volatile the risky asset is, the more optimistically the investor behaves in his wealth management and consumption plan. In other words, the investor consumes more when $\sigma$ is smaller and lowers the threshold to start some large expenditures such that the spending maximum is increased. However, when the wealth level is too low, the investor becomes more conservative towards the risky asset account and relies more on the interest rate to accumulate enough wealth to initiate a positive consumption. As a consequence, the threshold $x_{\text{zero}}(1; \sigma)$ is not necessarily monotonic in $\sigma$. The left and right panels of Figure 4 also show that both the value function and the optimal portfolio are decreasing in the parameter $\sigma$. These graphs are consistent with the real-life observation that if the risky asset has a higher volatility, the investor allocates less wealth in the risky asset and the life cycle value function also becomes lower.
5 Proofs of the main results

5.1 Proofs of some auxiliary results in Sect. 3

Proof of Proposition 3.2 We first obtain the special solution $v(y, h) = -\frac{1}{r^β} e^{λβh}$ for the first equation, then $v(y, h) = -\frac{y}{r^β} + \frac{y}{r^β} ( ln y - λβh + \frac{k^2}{2} )$ for the second equation, and finally $v(y, h) = -\frac{1}{r^β} hy - \frac{1}{r^β} e^{(λ−1)βh}$ for the third equation in (3.12). Therefore, we can summarise the general solution of the ODE (3.12) by

$$v(y, h) = \begin{cases} 
C_1(h) y^r_1 + C_2(h) y^r_2 - \frac{1}{r^β} e^{λβh}, & \text{if } y \geq e^{λβh}, \\
C_3(h) y^r_1 + C_4(h) y^r_2 - \frac{y}{r^β} \\
+ \frac{1}{r^β} ( ln y - λβh + \frac{k^2}{2} ), & \text{if } e^{(λ−1)βh} < y < e^{λβh}, \\
C_5(h) y^r_1 + C_6(h) y^r_2 \\
- \frac{1}{r^β} hy - \frac{1}{r^β} e^{(λ−1)βh}, & \text{if } (1 - λ)e^{(λ−1)βh} \leq y \leq e^{(λ−1)βh}, 
\end{cases} \quad (5.1)$$

in which $C_i(h), i = 1, \ldots, 6$, are functions of $h$ to be determined.

By the explicit form of $v(y, h)$ in (5.1) along the free boundary $y = (1 - λ)e^{(λ−1)βh}$, the condition $v_h(y, h) = 0$ in (3.11) implies that

$$C_5'(h)(1 - λ)r_1 e^{(λ−1)βhr_1} + C_6'(h)(1 - λ)r_2 e^{(λ−1)βhr_2} = 0 \quad (5.2)$$

Similarly to the case when $λ = 0$, the free boundary condition $v_y(y, h) \rightarrow 0$ in (3.14) implies that $y \rightarrow \infty$. Together with the free boundary conditions in (3.14) and the formula of $v(y, h)$ in the region $\{ y \geq e^{λβh} \}$, we deduce that $C_1(h) \equiv 0$. Moreover, it is easy to see that as $h \rightarrow \infty$, we get $y \rightarrow 0$ in the third region $\{ (1 - λ)e^{(λ−1)βh} \leq y \leq e^{(λ−1)βh} \}$, and therefore the boundary condition in (3.13) also implies the asymptotic condition that $C_6(h) \rightarrow 0$ as $h \rightarrow \infty$.

To determine the remaining parameters, we apply the smooth-fit conditions with respect to the variable $y$ at the two boundary points $y = e^{λβh}$ and $y = e^{(λ−1)βh}$. After simple manipulations, we can deduce the system of equations

$$\begin{align*}
C_2(h) e^{λβhr_2} &= C_3(h) e^{λβhr_1} + C_4(h) e^{λβhr_2} + \frac{1}{2r^2 β} e^{λβh} k^2, \\
C_2(h) r_2 e^{λβhr_2} &= C_3(h) r_1 e^{λβhr_1} + C_4(h) r_2 e^{λβhr_2} + \frac{1}{2r^2 β} e^{λβh} k^2, \\
C_3(h) e^{(λ−1)βhr_1} + C_4(h) e^{(λ−1)βhr_2} + \frac{1}{2r^2 β} e^{(λ−1)βh} k^2 &= C_5(h) e^{(λ−1)βhr_1} + C_6(h) e^{(λ−1)βhr_2}, \\
C_3(h) r_1 e^{(λ−1)βhr_1} + C_4(h) r_2 e^{(λ−1)βhr_2} + \frac{1}{2r^2 β} e^{(λ−1)βh} k^2 &= C_5(h) r_1 e^{(λ−1)βhr_1} + C_6(h) r_2 e^{(λ−1)βhr_2}.
\end{align*}$$
This system can be solved explicitly. To this end, the system can be regarded as linear equations in terms of the variables \( C_3(h) \), \( C_2(h) - C_4(h) \), \( C_4(h) - C_6(h) \) and \( C_3(h) - C_5(h) \). We can solve the first two equations and obtain \( C_3(h) \) explicitly as in (3.16) and \( C_2(h) - C_4(h) \). By solving the last two equations, we also get \( C_3(h) - C_5(h) \), which yields \( C_5(h) \) as in (3.18) by substituting the function \( C_3(h) \).

Plugging the derivative \( C_3'(h) \) back into the boundary condition (5.2), we obtain

\[
C_3'(h)(1 - \lambda)^2 e^{(\lambda - 1)\beta h r_2} = (1 - \lambda)^2 e^{(\lambda - 1)\beta h r_1} \frac{(r_2 - 1)\kappa^2}{2(r_1 - r_2)\beta r^2} \times \left((\lambda - 1)(1 - r_1)e^{(\lambda - 1)(1-r_1)\beta h} - \lambda(1 - r_1)e^{(1-r_1)\beta h}\right).
\]

By using the asymptotic condition that \( C_6(h) \rightarrow 0 \) when \( h \rightarrow \infty \) and the condition that \( \lambda(1 - r_2) - (r_1 - r_2) < 0 \), we can integrate the equation above on both sides and get \( C_6(h) \) explicitly as in (3.19). Substituting \( C_6(h) \) back into

\[
(r_1 - r_2)(C_6(h) - C_4(h))e^{(\lambda - 1)\beta h r_2} = (r_1 - 1)e^{(\lambda - 1)\beta h} \frac{2r^2 \beta - \kappa^2}{2r_2 \beta - \kappa^2},
\]

we can get \( C_4(h) \) as in (3.17). Substituting \( C_4(h) \) into the equation

\[
(r_1 - r_2)(C_2(h) - C_4(h))e^{\lambda \beta h r_2} = (r_1 - 1)e^{\lambda \beta h} \frac{2r^2 \beta - \kappa^2}{2r_2 \beta - \kappa^2},
\]

we finally obtain \( C_2(h) \) as in (3.15).

\[ \square \]

**Proof of Lemma 3.5** We analyse each region separately.

(i) In the region \( \{ y \geq e^{\lambda \beta h} \} \), we have \( v_{yy}(y,h) = r_2(r_2 - 1)C_2(h)y^{r_2 - 2} \) as \( r_2(r_2 - 1) = \frac{2r}{\kappa^2} > 0 \) and \( C_2(h) > 0 \) thanks to its expression (3.15). The conclusion holds trivially.

(ii) In the region \( \{(1 - \lambda)e^{(\lambda - 1)\beta h} \leq y \leq e^{(\lambda - 1)\beta h}\} \), we recall that

\[
v_{yy}(y,h) = r_1(r_1 - 1)C_5(h)y^{r_1 - 2} + r_2(r_2 - 1)C_6(h)y^{r_2 - 2}.
\]

The conclusion easily follows from the fact that \( C_5(h) > 0 \), \( C_6(h) > 0 \) and the identity \( r_1(r_1 - 1) = r_2(r_2 - 1) = \frac{2r}{\kappa^2} > 0 \).

(iii) In the region \( \{e^{(\lambda - 1)\beta h} < y < e^{\lambda \beta h}\} \), we proceed by the following two steps:

**Step 1.** To show \( v_{yy}(y,h) > 0 \), it is equivalent to check that \( yv_{yy}(y,h) > 0 \). Let us first show this at the two endpoints \( e^{(\lambda - 1)\beta h} \) and \( e^{\lambda \beta h} \). So at \( y = e^{(\lambda - 1)\beta h} \) and \( y = e^{\lambda \beta h} \), we need to prove the inequality

\[
\frac{2r}{\kappa^2}y^{r_2 - 1}(C_3(h)y^{r_1 - r_2} + C_4(h)) + \frac{1}{\beta r} > 0. \tag{5.3}
\]

By \( C_3(h) \) and \( C_4(h) \) in their explicit form, at the point \( e^{\lambda \beta h} \), (5.3) boils down to proving that

\[ \square \]
\[ e^{\lambda \beta h(r_2-1)} \frac{1}{(r_1 - r_2)\beta} \left( e^{\lambda \beta h(1-r_2)} \frac{r_2 - 1}{r} + e^{(\lambda-1)\beta h(1-r_2)} \frac{1 - r_1}{r} \right) \]

\[ + \frac{(r_2 - 1)(1 - \lambda)^{r_1-r_2}}{r} \times \left( \frac{1 - r_1}{1 - r_2} e^{(\lambda-1)(1-r_2)\beta h} \right) - \frac{\lambda(1 - r_1)}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} \right) \]

\[ + \frac{1}{r\beta} > 0. \]

Using the fact that \( e^{\lambda \beta h(1-r_2)} > e^{(\lambda-1)\beta h(1-r_2)} \), we can see that the left-hand side above is larger than

\[ e^{\lambda \beta h(r_2-1)} \frac{1}{(r_1 - r_2)\beta} \left( e^{\lambda \beta h(1-r_2)} \frac{r_2 - 1}{r} + e^{(\lambda-1)\beta h(1-r_2)} \frac{1 - r_1}{r} \right) \]

\[ + \frac{(r_2 - 1)(1 - \lambda)^{r_1-r_2}}{r} \times \left( \frac{1 - r_1}{1 - r_2} e^{(\lambda-1)(1-r_2)\beta h} \right) - \frac{\lambda(1 - r_1)}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} \right) \]

\[ = -e^{\lambda \beta h(r_2-1)} \frac{(1 - r_2)(1 - \lambda)^{r_1-r_2}}{(r_1 - r_2)\beta r} \times \left( \frac{1 - r_1}{1 - r_2} e^{(\lambda-1)(1-r_2)\beta h} \right) - \frac{\lambda(1 - r_1)}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} \right), \]

which is strictly positive. Hence we have \( yv_{yy}(y, h) > 0 \) at \( y = e^{\lambda \beta h} \).

To show (5.3) at the endpoint \( y = e^{(\lambda-1)\beta h} \), it is enough to show that

\[ e^{(\lambda-1)\beta h(r_2-1)} \frac{1}{(r_1 - r_2)\beta} \left( e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} \frac{r_2 - 1}{r} + e^{(\lambda-1)\beta h(1-r_2)} \frac{1 - r_1}{r} \right) \]

\[ + \frac{(r_2 - 1)(1 - \lambda)^{r_1-r_2}}{r} \times \left( \frac{1 - r_1}{1 - r_2} e^{(\lambda-1)(1-r_2)\beta h} \right) - \frac{\lambda(1 - r_1)}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} \right) \]

\[ + \frac{1}{r\beta} > 0. \]
By the fact that \( e^{(\lambda(1-r_2)-(r_1-r_2))\beta h} < e^{(\lambda-1)\beta h(1-r_2)} \) and similar calculations as for \( y = e^{\lambda \beta h} \), we can also show that the left-hand side above is strictly larger than
\[
-e^{(\lambda-1)\beta h(r_2-1)} \frac{(1-r_2)(1-\lambda)^{r_1-r_2}}{(r_1-r_2)\beta r} - e^{(\lambda-1)(1-r_2)\beta h} \frac{\lambda(1-r_1)}{\lambda(1-r_2)-(r_1-r_2)} e^{(\lambda(1-r_2)-(r_1-r_2))\beta h}
\]
and hence is strictly positive.

**Step 2.** In this step, we show that the function
\[
\gamma(y) := yv_y(y, h) = \frac{2r}{\kappa^2} C_3(h) y^{r_1-1} + \frac{2r}{\kappa^2} C_4(h) y^{r_2-1} + \frac{1}{r \beta}
\]
is either monotonic or first increasing and then decreasing. Combining this with Step 1, we obtain the statement of the lemma. Indeed, the extreme point \( \tilde{y} \) of \( \gamma(y) \) should satisfy the first-order condition \( \gamma'(\tilde{y}) = 0 \), i.e.,
\[
C_3(h)(r_1-1)(y^*)^{r_1-r_2} + C_4(h)(r_2-1) = 0.
\]
Note that \( C_3(h) < 0 \), while \( C_4(h) \) can be negative or positive. If \( C_4(h) \leq 0 \), there is no solution \( \tilde{y} \); hence \( \gamma(y) \) is monotonic. If \( C_4(h) > 0 \), there exists a unique real solution to the above equation, given by
\[
\tilde{y} = \left( \frac{C_4(h)(1-r_2)}{C_3(h)(r_1-1)} \right)^{\frac{1}{r_1-r_2}}
\]
which might fall into the interval \([e^{(\lambda-1)\beta h}, e^{\lambda \beta h}]\). As we have \( C_3(h) < 0 \) and
\[
\gamma'(y) = \frac{2r}{\kappa^2} y^{r_2-2} \left( C_3(h)(r_1-1)(y^{r_1-r_2} + C_4(h)(r_2-1)) \right),
\]
it follows that \( \gamma'(y) \geq 0 \) if and only if \( y \leq \tilde{y} \). Hence \( \gamma(y) \) is increasing in \( y \) before reaching \( \tilde{y} \) and is then decreasing in \( y \) after \( \tilde{y} \). \( \square \)

**Proof of Corollary 3.8** As we consider the asymptotic behaviour along the boundary \( x_{\text{lav}}(h) \), we first have
\[
\lim_{h \to \infty} \frac{c^*(x_{\text{lav}}(h), h)}{x_{\text{lav}}(h)} = \lim_{h \to \infty} \frac{h}{x_{\text{lav}}(h)}.
\]
Taking into account the explicit form of \( x_{\text{lav}}(h) \) in (3.28), we need to compute the two limits
\[
\lim_{h \to \infty} -\frac{C_5(h)r_1(1-\lambda)^{r_1-1} e^{(\lambda-1)(r_1-1)\beta h}}{h} = \lim_{h \to \infty} -\frac{-r_1(1-\lambda)^{r_1-1}(1-r_2)\kappa^2}{2(r_1-r_2)\beta r^2} (1-e^{(1-r_1)\beta h})
\]
\[
= \lim_{h \to \infty} -\frac{h}{h} = 0
\]
\( \square \)} Springer
Optimal consumption with reference to past spending maximum

and

\[
\lim_{h \to \infty} \frac{-C_6(h)r_2(1 - \lambda)r_2^{-1}e^{(\lambda-1)(r_2-1)\beta h}}{h} = \lim_{h \to \infty} \frac{-r_2(1 - \lambda)r_2^{-1}(r_2-1)\kappa^2}{2(r_1 - r_2)\beta r} \left( \frac{1 - r_1}{1 - r_2} - \frac{\lambda(1 - r_1)}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(1-r_1)\beta h} \right)
\]

Therefore, we obtain that

\[
\lim_{h \to \infty} \frac{c^*(x_{lavs}(h), h)}{x_{lavs}(h)} = r.
\]

Similarly, thanks to the explicit form of \( \pi^*(x, h) \) in (3.31), we need to compute the two limits

\[
\lim_{h \to \infty} \frac{2r}{\kappa^2} C_5(h)(1 - \lambda)r_1^{-1}e^{(\lambda-1)\beta h(r_1-1)} = \lim_{h \to \infty} \frac{(1 - \lambda)r_1^{-1}(1 - r_2)}{(r_1 - r_2)\beta r} (1 - e^{(1-r_1)\beta h}) = \frac{(1 - \lambda)r_1^{-1}(1 - r_2)}{(r_1 - r_2)\beta r}
\]

and

\[
\lim_{h \to \infty} \frac{2r}{\kappa^2} C_6(h)(1 - \lambda)r_2^{-1}e^{(\lambda-1)(r_2-1)\beta h} = \lim_{h \to \infty} \frac{(1 - \lambda)r_2^{-1}(r_2-1)}{(r_1 - r_2)\beta r} \left( \frac{1 - r_1}{1 - r_2} - \frac{\lambda(1 - r_1)}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(1-r_1)\beta h} \right)
\]

Therefore we conclude that

\[
\lim_{h \to \infty} \pi^*(x_{lavs}(h), h) = \frac{\mu - r}{\sigma^2} \left( \frac{(1 - \lambda)r_1^{-1}(1 - r_2)}{(r_1 - r_2)\beta r} + \frac{(1 - \lambda)r_2^{-1}(r_2-1)}{(r_1 - r_2)\beta r} \right)
\]

\[
= \frac{(\mu - r)(1 - \lambda)r_1^{-1}}{r\beta\sigma^2}.
\]

5.2 Proofs of Theorem 3.3 and Corollary 3.7

**Proof of Theorem 3.3** The proof of this verification theorem boils down to showing that the solution to the PDE indeed coincides with the value function. In other words, there exists \((\pi^*, c^*) \in \mathcal{A}(x)\) such that

\[
u(x, h) = \mathbb{E} \left[ \int_0^\infty e^{-rt} U(c^*_t - \lambda H^*_t) \, dt \right].
\]

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For any admissible strategy \((\pi, c) \in \mathcal{A}(x)\), similarly to the standard proof of Lemma 1 in Arun [2], we have the budget constraint
\[
\mathbb{E}\left[\int_0^\infty c_t M_t dt\right] \leq x.
\]
Regarding \((\lambda, h)\) as fixed parameters, we consider the dual transform of \(U\) with respect to \(c\) in the constrained domain, i.e.,
\[
V(y, h) := \sup_{0 \leq c \leq h} \left( U(c - \lambda h) - cy \right)
\]
\[
= \begin{cases} 
-\frac{1}{\beta} e^{\lambda \beta h}, & \text{if } y \geq e^{\lambda \beta h}, \\
-\frac{1}{\beta} y + y \left( \frac{1}{\beta} \ln y - \lambda h \right), & \text{if } e^{(\lambda - 1) \beta h} < y < e^{\lambda \beta h}, \\
-\frac{1}{\beta} e^{(\lambda - 1) \beta h} - hy, & \text{if } (1 - \lambda) e^{(\lambda - 1) \beta h} \leq y \leq e^{(\lambda - 1) \beta h}.
\end{cases}
\]
We remark that when \(\lambda = 0\), \(V(y, h)\) is independent of \(h\). Moreover, \(V(y, h)\) can be attained by the construction of the feedback function \(c^\dagger(y, h)\) given in (3.20).

In the sequel, we distinguish the two reference processes \(H_t := h \vee \sup_{s \leq t} c_s\) and \(H^\dagger_t(y) := h \vee \sup_{s \leq t} c^\dagger(Y_s(y), H^\dagger_s(y))\) that correspond to the reference process under an arbitrary consumption process \(c\) and under the optimal consumption process \(c^\dagger\) with an arbitrary \(y > 0\). Note that the (global) optimal reference process will be defined later by \(H^*_t := H^\dagger_t(y^*)\) with \(y^* > 0\) to be determined. Let us further introduce
\[
\hat{H}_t(y) := h \vee \left( \frac{1}{(\lambda - 1) \beta} \ln \left( \frac{1}{1 - \lambda} \inf_{s \leq t} Y_s(y) \right) \right),
\]
where \(Y_t(y) = ye^{rt} M_t\), \(t \geq 0\), is the discounted martingale measure density process.

For any admissible \((\pi, c) \in \mathcal{A}(x)\) and all \(y > 0\), we have
\[
\mathbb{E}\left[\int_0^\infty e^{-rt} U(c_t - \lambda H_t) dt\right] = \mathbb{E}\left[\int_0^\infty e^{-rt} \left(U(c_t - \lambda H_t) - Y_t(y) c_t\right) dt\right] \\
+ y \mathbb{E}\left[\int_0^\infty c_t M_t dt\right] \\
\leq \mathbb{E}\left[\int_0^\infty e^{-rt} V(Y_t(y), H^\dagger_t(y)) dt\right] +yx \\
= \mathbb{E}\left[\int_0^\infty e^{-rt} V(Y_t(y), \hat{H}_t(y)) dt\right] +yx \\
= v(y, h) +yx,
\]
where the inequality follows from Lemma 5.4 below, the second equality comes from Lemma 5.3 below, and the last line from Lemma 5.2 below. In addition, by Lemma 5.4, the inequality becomes an equality with the choice of \(c^\dagger_t = c^\dagger(Y_t(y^*), H^\dagger_t(y^*))\), in which \(y^*\) uniquely solves \(\mathbb{E}[\int_0^\infty c^\dagger(Y_t(y), H^\dagger_t(y)) M_t dt] = x\) for the given \(x > 0\) and \(h \geq 0\).
In conclusion, we arrive at

$$u(x, h) = \sup_{(\pi, c) \in A(x)} \mathbb{E} \left[ \int_0^\infty e^{-rt} U(c_t - \lambda H_t) dt \right] = \inf_{y > 0} (v(y, h) + yx),$$

which completes the proof of the verification theorem. □

We now proceed to prove some auxiliary results that have been used in the previous proof of the main theorem. We use the following asymptotic results for the coefficients defined in Proposition 3.2.

Remark 5.1 Based on the explicit formulas in (3.15)–(3.17), we note that as $h \to \infty$, we have the asymptotics

$$C_2(h) = O(e^{(\lambda-1)(1-r_2)\beta h}) + O(e^{(\lambda(1-r_2)-(r_1-r_2))\beta h}) + O(e^{\lambda(1-r_2)\beta h}),$$

$$C_3(h) = O(e^{\beta h(1-r_1)}),$$

$$C_4(h) = O(e^{(\lambda-1)(1-r_2)\beta h}) + O(e^{(\lambda(1-r_2)-(r_1-r_2))\beta h}),$$

$$C_5(h) = O(e^{\beta h(1-r_1)}) + O(e^{(\lambda-1)(1-r_1)\beta h}),$$

$$C_6(h) = O(e^{(\lambda-1)(1-r_2)\beta h}) + O(e^{(\lambda(1-r_2)-(r_1-r_2))\beta h}).$$

Lemma 5.2

$$v(y, h) = \mathbb{E} \left[ \int_0^\infty e^{-rt} V(Y_t(y), \hat{H}_t(y)) dt \right].$$

Proof Note that the process $M$ satisfies the equation

$$dM_t = M_t (r dt - \kappa dW_t).$$

By (3.12), $v(y, h)$ satisfies the ODE

$$\frac{\kappa^2}{2} y^2 v_{yy} - rv + V(y, h) = 0.$$ 

By Itô’s formula, we have that

$$d \left( e^{-rt} v(Y_t(y), \hat{H}_t(y)) \right) = -e^{-rt} V(Y_t(y), \hat{H}_t(y)) dt$$

$$- \kappa e^{-rt} v_y(Y_t(y), \hat{H}_t(y)) Y_t(y) dW_t$$

$$+ e^{-rt} v_h(Y_t(y), \hat{H}_t(y)) d\hat{H}_t(y). \quad (5.6)$$

Let us define the stopping time

$$\tau_n := \inf \left\{ t \geq 0 : Y_t(y) \geq n, \hat{H}_t(y) \geq \frac{1}{(\lambda-1)\beta} \ln \frac{1}{(1-\lambda)n} \right\}.$$
Integrating (5.6) from 0 to \( T \land \tau_n \) and taking expectations on both sides gives

\[
v(y, h) = \mathbb{E}
\left[
\int_0^{T \land \tau_n} e^{-rt} V(Y_t(y), \hat{H}_t(y)) dt
\right]
+ \mathbb{E}
\left[
 e^{-r(T \land \tau_n)} v(Y_{T \land \tau_n}(y), \hat{H}_{T \land \tau_n}(y))
\right]. \tag{5.7}
\]

Note that the integral term with respect to \( d\hat{H}_t(y) \) vanishes as \( \hat{H}_t(y) \) increases only when \( c^*_t(y) = \hat{H}_t(y) \) and we have \( v_h(Y_t(y), \hat{H}_t(y)) = 0 \) by the free boundary condition. In addition, the expectation of the integral of \( dW_t \) vanishes as the local martingale

\[
\int_0^{\cdot \land \tau_n} \kappa v_Y(Y_t(y), \hat{H}_t(y)) y M_t dW_t
\]
becomes a true martingale thanks to the definition of \( \tau_n \) and the fact that \( v \) is of class \( C^2 \).

By passing to the limit as \( n \to \infty \), the first term in (5.7) tends to

\[
\mathbb{E}
\left[
\int_0^T e^{-rt} V(Y_t(y) \hat{H}_t(y)) dt
\right]
\]
by the monotone convergence theorem. This follows from two facts: first, we clearly have \( V < 0 \) by its definition; second, by some calculations similar to Guasoni et al. \([12, (A.25)]\), we can see that when \( n \to \infty \), we get \( \tau_n \geq T \) almost surely. Moreover, the second term in (5.7) can be written as

\[
\mathbb{E}
\left[
 e^{-rT} v(Y_T(y) \hat{H}_T(y))
\right]
= \mathbb{E}
\left[
 e^{-rT} v(Y_T(y) \hat{H}_T(y)) 1_{\{T < \tau_n\}}
\right]
+ \mathbb{E}
\left[
 e^{-r\tau_n} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) 1_{\{T \geq \tau_n\}}
\right]. \tag{5.8}
\]

As \( n \to \infty \), the first term in (5.8) clearly converges to \( \mathbb{E}[e^{-rT} v(Y_T(y), \hat{H}_T(y))] \) because \( \tau_n > T \) almost surely when \( n \to \infty \). We further show in Lemma 5.5 below that the transversality condition holds in the sense that \( \mathbb{E}[e^{-rT} v(Y_T(y), \hat{H}_T(y))] \) converges to 0 as \( T \to \infty \).

We claim that the second term in (5.8) also converges to 0 as \( n \to \infty \). To see this, note that the definition of \( \tau_n \) yields \( O(Y_{\tau_n}(y)^{r_1}) = O(n^{r_1}) \), \( O(Y_{\tau_n}(y)^{r_2}) = O(n^{r_2}) \) and \( O(e^{(\lambda - 1)\beta \hat{H}_{\tau_n}(y)}) = O(n) \). Using the expressions of \( v(y, h) \) in different regions, we can analyse the orders in terms of \( n \) of the expressions

\[
C_2(\hat{H}_{\tau_n}(y)) Y_{\tau_n}(y)^{r_2},
C_3(\hat{H}_{\tau_n}(y)) Y_{\tau_n}(y)^{r_1} + C_4(\hat{H}_{\tau_n}(y)) Y_{\tau_n}(y)^{r_2},
C_5(\hat{H}_{\tau_n}(y)) Y_{\tau_n}(y)^{r_1} + C_6(\hat{H}_{\tau_n}(y)) Y_{\tau_n}(y)^{r_2}.
\]

In view of Remark 5.1, we have for \( n \to \infty \) that
\[
O(e^{(\lambda - 1)(1-r_2)\hat{H}_t}(y)) = O(n^{r_2 - 1}), \\
O(e^{(\lambda (1-r_2)-(r_1-r_2))\hat{H}_t}(y)) = O(n^{\frac{(1-r_2)-(r_1-r_2)}{1-\lambda}}), \\
O(e^{\lambda (1-r_2)\hat{H}_t}(y)) = O(n^{\frac{\lambda}{1-\lambda}(1-r_2)}), \\
O(e^{\lambda (1-r_1)\hat{H}_t}(y)) = O(n^{\frac{\lambda}{1-\lambda}(1-r_1)}), \\
O(e^{(\lambda - 1)(1-r_1)\hat{H}_t}(y)) = O(n^{1-r_1}).
\]

Similarly to [12, proof of (A.25)], we can show that there exists some constant \(C\) such that

\[
\mathbb{E}[1_{\{\tau_n \leq T\}}] \leq n^{-2\phi} (1 + y^{2\phi}) e^{CT}
\]

for any \(\phi \geq 1\). In particular, if we choose

\[
\phi := \frac{1}{2} \max \left( r_2 + \frac{\lambda (1-r_2)}{1-\lambda}, \frac{\lambda}{1-\lambda}(r_1 - 1) + r_1 \right) + 1,
\]

we can deduce that the terms

\[
\mathbb{E}\left[C_2(\hat{H}_t(y))Y_{\tau_n}(y)r^21_{\{T \geq \tau_n\}}\right], \\
\mathbb{E}\left[C_3(\hat{H}_t(y))Y_{\tau_n}(y)r^1 + C_4(\hat{H}_t(y))Y_{\tau_n}(y)r^21_{\{T \geq \tau_n\}}\right], \\
\mathbb{E}\left[C_5(\hat{H}_t(y))Y_{\tau_n}(y)r^1 + C_6(\hat{H}_t(y))Y_{\tau_n}(y)r^21_{\{T \geq \tau_n\}}\right]
\]

tend to 0 as \(n \to \infty\). As a result, we obtain the desired claim that

\[
\lim_{n \to \infty} \mathbb{E}\left[e^{-rT} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y))1_{\{T > \tau_n\}}\right] = 0.
\]

\(\square\)

**Lemma 5.3** For all \(y > 0\), we have \(H_t^{\dagger}(y) = \hat{H}_t(y), t \geq 0\), and hence

\[
\mathbb{E}\left[\int_0^\infty e^{-rt} V(Y_t(y), H_t^{\dagger}(y))dt\right] = \mathbb{E}\left[\int_0^\infty e^{-rt} V(Y_t(y), \hat{H}_t(y))dt\right].
\]

**Proof** The proof is similar to [12, Lemma A.1], but for the sake of completeness, we present the argument here. Suppose that \(H_t^{\dagger}(y)\) is strictly increasing at \(t\). Then we have that \(H_t^{\dagger}(y) = c^\dagger(Y_t(y), H_t^{\dagger}(y))\), and the optimal consumption is given by

\[
c^\dagger(Y_t(y), H_t^{\dagger}(y)) = \frac{1}{(\lambda - 1)^2} \ln(\frac{1}{1-\lambda} Y_t(y)).
\]

Define

\[
\mathcal{I}_t := \{s \leq t : H_t^{\dagger}(y) \text{ is strictly increasing at } s\}.
\]

We can now derive that
\( H_t(y) = h \vee \sup_{s \in \mathcal{I}_t} c^\dagger (Y_s(y), H^\dagger_s(y)) = h \vee \sup_{s \in \mathcal{I}_t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) \)

\[= h \vee \sup_{s \in \mathcal{I}_t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) \vee \sup_{s \notin \mathcal{I}_t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) \]

\[= h \vee \sup_{s \leq t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) = \hat{H}_t(y). \]

In the second line above, we have used the following fact: for any \( s \notin \mathcal{I}_t \), from the condition that \( Y_s(y) > (1 - \lambda)e^{(\lambda - 1)\beta H^\dagger_s(y)} \), we obtain that

\[ \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) < H^\dagger_s(y) \leq H^\dagger_t(y) \]

\[= h \vee \sup_{s \in \mathcal{I}_t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) . \]

Hence

\[ \sup_{s \notin \mathcal{I}_t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) \leq h \vee \sup_{s \in \mathcal{I}_t} \frac{1}{\lambda - 1} \beta \ln \left( \frac{1}{1 - \lambda} Y_s(y) \right) . \]

Lemma 5.4 The inequality (5.5) holds, and it becomes an equality with the consumption control \( c_t^* = c^\dagger (Y_t(y^*), \hat{H}_t(y^*)) \), \( t \geq 0 \), with \( y^* = y^*(x, h) \) as the unique solution to

\[ \mathbb{E} \left[ \int_0^\infty c^\dagger (Y_t(y^*), \hat{H}_t(y^*)) M_t dt \right] = x. \]

Proof Using the definition of \( V \) gives \( U(c_t - \lambda H_t) - Y_t(y)c_t \leq V(Y_t(y), H_t) \) for all \((\pi, c) \in \mathcal{A}(x)\). Moreover, for any fixed \( y > 0 \), the inequality becomes an equality with the optimal feedback \( c^\dagger (Y_t(y), H^\dagger_t(y)) \). In other words, for any admissible \((c_t)_{0 \leq t \leq T}\), we have that for all \( t \in [0, T] \),

\[ U(c_t - \lambda H_t) - Y_t(y)c_t \leq U \left( c^\dagger (Y_t(y), H^\dagger_t(y)) - \lambda H^\dagger_t(y) \right) \]

\[= -Y_t(y)c^\dagger (Y_t(y), H^\dagger_t(y)) \]

\[= V(Y_t(y), H^\dagger_t(y)). \]

Multiplying both sides by \( e^{-rt} \) and integrating from 0 to \( T \), we have that

\[ \int_0^T e^{-rt} (U(c_t - \lambda H_t) - Y_t(y)c_t) dt \leq \int_0^T e^{-rt} V(Y_t(y), H^\dagger_t(y)) dt. \]

To turn (5.5) into an equality, the consumption should take the optimal feedback form \( c^\dagger \), and \( \mathbb{E} [\int_0^\infty c^\dagger (Y_t(y^*), \hat{H}_t(y^*)) M_t dt] = x \) should be valid with some \( y^* > 0 \). We now show the existence of such a \( y^* > 0 \). To this end, we introduce

\[ c^\dagger (Y_t(y), \hat{H}_t(y)) := \hat{H}_t(y) F_t(y, Y_t(y)), \]
where the function $F$ is defined as

$$F_t(y,z) := \begin{cases} 1 & (1-\lambda)e^{-(1-\lambda)\beta \hat{H}_t(y)} \leq z \leq e^{-(1-\lambda)\beta \hat{H}_t(y)} \\ \lambda - \frac{\ln z}{\beta \hat{H}_t(y)} & e^{-(1-\lambda)\beta \hat{H}_t(y)} \leq z \leq e^{\lambda \beta \hat{H}_t(y)} \end{cases}. $$

In view of the definition of $\hat{H}_t(y)$ in (5.4), one can obtain that (i) if $y \downarrow 0$, then $\hat{H}_t(y) \uparrow \infty$ and $F_t(y, Y_t(y)) > 0$, which yields $\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y)) \, dt] \uparrow \infty$; (ii) if $y \uparrow \infty$, then $\hat{H}_t(y) \downarrow h$ and $F_t(y, Y_t(y)) \downarrow 0$, which yields in turn that $\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y)) \, dt] \downarrow 0$.

The existence of $y^*$ satisfying $\mathbb{E}[\int_0^\infty c^\dagger(Y_t(y^*), \hat{H}_t(y^*)) \, M_t \, dt] = x$ is then a consequence of the asymptotic results (i) and (ii), given the fact that the mapping $y \mapsto \mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y)) \, dt]$ is continuous. □

We next prove the transversality condition, which is a key step in the proof of Lemma 5.2.

**Lemma 5.5** For all $y > 0$, we have the transversality condition

$$\lim_{T \to \infty} \mathbb{E}[e^{-rT} v(Y_T(y), \hat{H}_T(y))] = 0.$$ 

**Proof** Let us first recall that

$$\hat{H}_t(y) = h \lor \left(\frac{1}{(\lambda - 1)\beta} \ln \left(\frac{1}{1 - \lambda} \inf_{s \leq t} Y_s(y)\right)\right).$$

From Proposition 3.2, in the interval $\{e^{(\lambda - 1)\beta h} < y < e^{\lambda \beta h}\}$, which corresponds to the case $0 < c_t < H_t$, we have

$$v(y, h) = C_3(h) y r^1 + C_4(h) y r^2 - \frac{y}{r \beta} + \frac{y}{r \beta} \left(\ln y - \lambda \beta h + \frac{\kappa^2}{2r}\right).$$

In the interval $\{y \geq e^{\lambda \beta h}\}$, which corresponds to the case $c_t = 0$, we have

$$v(y, h) = C_2(h) y r^2 - \frac{1}{r \beta} e^{\lambda \beta h}.$$

In the interval $\{(1 - \lambda)e^{(\lambda - 1)\beta h} \leq y \leq e^{(\lambda - 1)\beta h}\}$, we have

$$v(y, h) = C_5(h) y r^1 + C_6(h) y r^2 - \frac{1}{r} h y - \frac{1}{r \beta} e^{(\lambda - 1)\beta h}.$$

a) We first deal with the case $0 < c_t < H_t$ and check the asymptotic behaviour of the expectation.
\[ \mathbb{E} \left[ e^{-rT} \left( C_3 (\hat{H}_T (y)) (Y_T (y)) r_1 + C_4 (\hat{H}_T (y)) (Y_T (y)) r_2 \right. \right. \]
\[ \left. - \frac{Y_T (y)}{r_\beta} + \frac{Y_T (y)}{r_\beta} \left( \ln Y_T (y) - \lambda_\beta \hat{H}_T (y) + \frac{\kappa^2}{2r} \right) \right] \].

We consider its asymptotic behaviour term by term.

(i) Let us start by considering the asymptotic behaviour of the third \( \mathbb{E} [-e^{-rT} \frac{Y_T (y)}{r_\beta}] \) and the fourth term \( \mathbb{E} \left[ e^{-rT} \frac{Y_T (y)}{r_\beta} \left( \ln Y_T (y) - \lambda \beta \hat{H}_T (y) + \frac{\kappa^2}{2r} \right) \right] \). For the third term, it is easy to see that
\[ \mathbb{E} \left[ ye^{-(r+\frac{\kappa^2}{2})T - \kappa W_T} \frac{1}{r_\beta} \right] = \frac{y}{r_\beta} e^{-(r+\frac{\kappa^2}{2})T} \mathbb{E} [e^{-\kappa W_T}] = \frac{y}{r_\beta} e^{-rT}, \quad (5.9) \]
which converges to 0 as \( T \to \infty \). For the fourth term, we have
\[ \frac{y M_T}{r_\beta} \left( \ln Y_T (y) - \lambda \beta \hat{H}_T (y) + \frac{\kappa^2}{2r} \right) \]
\[ = \frac{1}{r_\beta} \left( y M_T \left( rT + \ln y + \frac{\kappa^2}{2r} \right) + y M_T \ln M_T - y M_T \lambda \beta \hat{H}_T (y) \right). \]

Similarly to (5.9), we can show that \( \mathbb{E} [y M_T (rT + \ln y + \frac{\kappa^2}{2r})] \) converges to 0, and
\[ \mathbb{E} [y M_T \ln M_T] = -ye^{-(r+\frac{\kappa^2}{2})T} \left( \mathbb{E} [e^{-\kappa W_T}] + \left( r + \frac{\kappa^2}{2} \right) T \mathbb{E} [e^{-\kappa W_T}] \right) \]
\[ = -ye^{-rT} \left( r - \frac{\kappa^2}{2} \right) T, \]
which also converges to 0 as \( T \to \infty \). Furthermore, we can deduce that
\[ \mathbb{E} [y M_T \hat{H}_T (y)] \leq \mathbb{E} [y M_T h] + \mathbb{E} \left[ y M_T \frac{1}{(\lambda - 1)\beta} \ln \left( \frac{1}{1 - \lambda} \inf_{s \leq T} (e^{rs} M_s) \right) \right] \]
\[ = O(\mathbb{E} [y M_T]) + O \left( e^{-rT} \mathbb{E} \left[ e^{-\kappa W_T} - \frac{1}{2} \kappa^2 T \sup_{s \leq T} (\kappa W_s + \frac{1}{2} \kappa^2 s) \right] \right). \]
The first term \( O(\mathbb{E} [y M_T]) \) clearly vanishes as \( T \to \infty \) by repeating similar computations as above for showing that \( \mathbb{E} [y M_T \ln M_T] \to 0 \). For the second term, we first note that
\[ \mathbb{E} \left[ e^{-\kappa W_T} \sup_{s \leq T} W_s \right] \]
\[ = \sqrt{\frac{T}{2\pi}} - e^{\frac{1}{2} \kappa^2 T} \kappa T \Phi(-\kappa \sqrt{T}) + e^{\frac{1}{2} \kappa^2 T} \frac{1}{2\kappa} \left( \Phi(\kappa \sqrt{T}) - \Phi(-\kappa \sqrt{T}) \right) \].
Let us define on $\mathcal{F}_T$ the equivalent measure $\mathbb{Q}$ under which $W_t^{(\xi)} := W_t + \frac{\kappa}{2} t$, $0 \leq t \leq T$, is a Brownian motion, with the Radon–Nikodým derivative

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} := \exp\left( -\frac{1}{2} \kappa W_T - \frac{1}{8} \kappa^2 T \right).
$$

It follows by Girsanov’s theorem that

$$
e^{-rT} \mathbb{E}\left[ e^{-\kappa W_T - \frac{1}{2} \kappa^2 T} \sup_{s \leq T} \left( \kappa W_s + \frac{1}{2} \kappa^2 s \right) \right]
= \kappa e^{-rT} \mathbb{E}_{\mathbb{Q}}\left[ e^{-\kappa W_T^{(\xi)} - \frac{1}{2} \kappa^2 T} \sup_{s \leq T} W^{(\xi)}_s \exp\left( \frac{1}{2} \kappa W^{(\xi)}_s - \frac{1}{8} \kappa^2 t \right) \right]
= \kappa e^{-rT} \left( \sqrt{\frac{T}{2\pi}} e^{-\frac{1}{8} \kappa^2 T} - \frac{1}{2} \kappa T \Phi\left( -\frac{1}{2} \kappa \sqrt{T} \right) \right.
+ \left. \frac{1}{\kappa} \left( \Phi\left( \frac{1}{2} \kappa \sqrt{T} \right) - \Phi\left( -\frac{1}{2} \kappa \sqrt{T} \right) \right) \right),
$$

which clearly vanishes when $T \to \infty$.

(ii) Let us continue to consider the term with $C_3(h)$. In view of the constraint $Y_T(y) < e^{\lambda \beta \hat{H}_T}$, we have

$$
\lambda \beta \hat{H}_T (1 - r_1) < (1 - r_1) \ln Y_T(y)
$$

and it follows that

$$
\mathbb{E}\left[ e^{-rT} C_3(\hat{H}_T(y)) (Y_T(y))^{r_1} \right] = O\left( \mathbb{E}\left[ e^{\lambda \beta \hat{H}_T(y)(1-r_1)} e^{-rT} (Y_T(y))^{r_1} \right] \right)
\leq O\left( \mathbb{E}\left[ (Y_T(y))^{1-r_1} e^{-rT} (Y_T(y))^{r_1} \right] \right)
= O(\mathbb{E}[y M_T]) = O(e^{-rT}).
$$

It is thus verified that the term $\mathbb{E}[e^{-rT} C_3(\hat{H}_T(y))(Y_T(y))^{r_1}]$ converges to 0 as $T \to \infty$.

(iii) Now let us work with the term $C_4(\hat{H}_T(y)) e^{-rT} (Y_T(y))^{r_2}$. Remark 5.1 asserts that

$$
C_4(h) = O(e^{(\lambda-1)(1-r_2)\beta h}) + O(e^{(\lambda(1-r_2)-(r_1-r_2))\beta h}).
$$

Let us define the set

$$
A := \left\{ \frac{1}{\lambda - 1} \ln \left( \frac{1}{1 - \lambda} \inf_{s \leq T} (e^{r s} M_s) \right) \geq h \right\} = \left\{ \inf_{s \leq T} (e^{r s} M_s) \leq (1 - \lambda) e^{(\lambda-1)h} \frac{1}{y} \right\}
$$

and the two auxiliary random variables
\[ G^1 := \left( \frac{y}{1 - \lambda} \right)^{(1 - \tau_2)\beta} \inf_{s \leq T} (e^{r s} M_s)^{(1 - \tau_2)\beta}, \]

\[ G^2 := \left( \frac{y}{1 - \lambda} \right)^{\lambda(1 - \tau_2) - (\tau_1 - \tau_2)\beta} \inf_{s \leq T} (e^{r s} M_s)^{\lambda(1 - \tau_2) - (\tau_1 - \tau_2)\beta}. \]

Using the formula for \( \hat{H}_t \) in (5.4), we have

\[
\begin{align*}
\mathbb{E}[e^{-rT} C_4 (\hat{H}_T(y)) (Y_T(y)] & = O\left( \mathbb{E}[e^{(\lambda - 1)(1 - \tau_2)\beta} \hat{H}_T(y) e^{-rT} (Y_T(y)]^{\tau_2} \right) \\
& + O\left( \mathbb{E}[G^1 (e^{rT} M_T)^{\tau_2} e^{-rT} 1_A] \right) \\
& + O\left( \mathbb{E}[G^2 (e^{rT} M_T)^{\tau_2} e^{-rT} 1_{A^c}] \right) \\
& = \mathcal{O}(\gamma_1(T)) \lor \mathcal{O}(\gamma_2(T)) \lor \mathcal{O}\left( \mathbb{E}[e^{-rT} (Y_T(y)]^{\tau_2} 1_{A^c}] \right),
\end{align*}
\]

in which we define

\[
\gamma_1(T) := \mathbb{E} \left[ \left( \frac{y}{1 - \lambda} \right)^{(1 - \tau_2)\beta} \inf_{s \leq T} (e^{r s} M_s)^{(1 - \tau_2)\beta} (e^{rT} M_T)^{\tau_2} e^{-rT} 1_A \right],
\]

\[
\gamma_2(T) := \mathbb{E} \left[ \left( \frac{y}{1 - \lambda} \right)^{\lambda(1 - \tau_2) - (\tau_1 - \tau_2)\beta} \inf_{s \leq T} (e^{r s} M_s)^{\lambda(1 - \tau_2) - (\tau_1 - \tau_2)\beta} \right]
\times (e^{rT} M_T)^{\tau_2} e^{-rT} 1_A,
\]

and the third term \( O(\mathbb{E}[e^{-rT} (Y_T(y)]^{\tau_2} 1_{A^c}] \) comes from the level \( h \) in the definition of \( \hat{H}_t(y) \) with \( A^c \) being the complementary set of \( A \). We then proceed to show that all three terms \( \gamma_1(T), \gamma_2(T) \) and \( \mathbb{E}[e^{-rT} (Y_T(y)]^{\tau_2} 1_{A^c}] \) converge to 0 as \( T \to \infty \).

By setting

\[ a_1 = -\kappa r_2, \quad b_1 = -\kappa (1 - r_2)\beta, \quad b_2 = -\kappa \frac{\lambda(1 - r_2) - (r_1 - r_2)}{\lambda - 1} \beta, \quad \zeta = \frac{\kappa}{2}, \]

we can use [12, Corollary A.7] to obtain

\[
\begin{align*}
\lim_{T \to \infty} \frac{1}{T} \log \gamma_1(T) &= \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E} \left[ \left( \frac{y}{1 - \lambda} \right)^{(1 - \tau_2)\beta} \inf_{s \leq T} (e^{r s} M_s)^{(1 - \tau_2)\beta} (e^{rT} M_T)^{\tau_2} e^{-rT} 1_A \right] \\
& \leq \max \left( \frac{a_1(a_1 + 2\zeta)}{2} - r, \frac{(a_1 + b_1)(a_1 + b_1 + 2\zeta)}{2} - r, \frac{\zeta^2}{2} - r \right).
\end{align*}
\]
Similarly, we have that

\[
\lim_{T \to \infty} \frac{1}{T} \log \Upsilon_2(T)
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E}
\left[
\frac{Y}{1 - \lambda}
\right]
\left[
\left(\inf_{s \leq T} (e^{r_s M_s})^{\frac{\lambda(1-r_2)-(r_1-r_2)}{\lambda-1}}
\right)
\times
(e^{r^T M_T})^{r_2} e^{-r^T 1_A}
\right]
\leq \max \left(\frac{a_1(a_1+2\xi)}{2} - r, \frac{(a_1+b_2)(a_1+b_2+2\xi)}{2} - r, \frac{-\xi^2}{2} - r\right).
\]

(5.11)

We now show that the above bounds in (5.10) and (5.11) are either negative or not attainable. First, there is the same third bound \(-\frac{\xi^2}{2} - r < 0\) in both (5.10) and (5.11). For the same first bound \(\frac{a_1(a_1+2\xi)}{2} - r\) in (5.10) and (5.11), direct calculations lead to

\[
\frac{a_1(a_1+2\xi)}{2} - r = -\frac{1}{2} \kappa^2 r_1(1 - r_1) - r = 0.
\]

However, this zero bound can never be reached, as by [12, Lemma A.5 and Corollary A.7], the corresponding condition of attaining the bound \(\frac{a_1(a_1+2\xi)}{2} - r\) is \(a_1 + \xi < 0\), but we instead have that

\[
a_1 + \xi = -\kappa r_2 + \frac{\kappa}{2} = \frac{1}{\kappa} \sqrt{\left(-\frac{\kappa^2}{2}\right)^2 + 2r\kappa^2} > 0.
\]

We next claim that the second upper bound \(\frac{(a_1+b_1)(a_1+b_1+2\xi)}{2} - r\) in (5.10) is strictly negative. From [12, Corollary A.7], this bound is attained if and only if \(a_1 + b_1 + \xi > 0\) and \(2a_1 + b_1 + 2\xi > 0\). Noting that \(\kappa > 0\), we have the equivalence

\[
2a_1 + b_1 + 2\xi > 0 \iff -2\kappa r_2 - \kappa (1 - r_2) \beta + \kappa > 0
\]

\[
\iff \beta < \frac{1 - 2r_2}{1 - r_2}.
\]

Therefore, under the condition \(2a_1 + b_1 + 2\xi > 0\), i.e., \(\beta < \frac{1 - 2r_2}{1 - r_2}\), that upper bound \(\frac{(a_1+b_1)(a_1+b_1+2\xi)}{2} - r\) must be negative because

\[
(a_1 + b_1)(a_1 + b_1 + 2\xi) - 2r = \kappa^2 (r_2 + (1 - r_2) \beta)(r_2 + (1 - r_2) \beta - 1) - 2r
\]

\[
= \kappa^2 (r_2^2 + (1 - r_2)^2 \beta^2 + 2r_2(1 - r_2) \beta - r_2 - (1 - r_2) \beta) - 2r
\]

\[
= \kappa^2 ((1 - r_2)^2 \beta^2 + 2r_2(1 - r_2) \beta - (1 - r_2) \beta)
\]

\[
= \kappa^2 (1 - r_2) \beta ((1 - r_2) \beta + 2r_2 - 1) < 0.
\]
Finally, we claim that the second upper bound \( \frac{(a_1 + b_2)(a_1 + b_2 + 2\xi)}{2} - r \) in (5.11) is also strictly negative. Once again from [12, Corollary A.7], this upper bound is attained if and only if \( a_1 + b_2 + \xi > 0 \) and \( 2a_1 + b_2 + 2\xi > 0 \). As \( \kappa > 0 \), we obtain the equivalence

\[
2a_1 + b_2 + 2\xi > 0 \iff -2\kappa r_2 - \kappa \frac{\lambda(1 - r_2) - (r_1 - r_2)\beta + \kappa}{\lambda - 1} > 0
\]

\[
\iff \beta < \frac{1 - 2r_2}{\lambda^*},
\]

where we define \( \lambda^* := \frac{(r_1 - r_2) - \lambda(1 - r_2)}{1 - \lambda} \). Hence, when the bound \( \frac{(a_1 + b_2)(a_1 + b_2 + 2\xi)}{2} - r \) is attained, the condition \( 2a_1 + b_2 + 2\xi > 0 \), i.e., \( \beta < \frac{1 - 2r_2}{\lambda^*} \), guarantees that this bound must be negative because

\[
(a_1 + b_2)(a_1 + b_2 + 2\xi) - 2r = \kappa^2(r_2 + \lambda^* \beta)(r_2 + \lambda^* \beta - 1) - 2r
\]

\[
= \kappa^2(r_2^2 + \lambda^* 2\beta^2 + 2r_2 \lambda^* \beta - r_2 - \lambda^* \beta) - 2r
\]

\[
= \kappa^2(\lambda^* 2\beta^2 + 2r_2 \lambda^* \beta - \lambda^* \beta) = \kappa^2\lambda^* \beta(\lambda^* \beta + 2r_2 - 1) < 0.
\]

Putting everything together, we conclude that \( \Upsilon_1(T) \to 0 \) and \( \Upsilon_2(T) \to 0 \) as \( T \to \infty \).

The last term \( \mathbb{E}[e^{-rT}(Y_T(y))^2 I_A^c] \) converges to as \( T \to \infty \) by Lemma 5.6 with \( a = -\kappa r_2, b = 0, \eta = \frac{r}{\kappa r_2} + \frac{1}{2}\kappa \).

b) Let us now deal with the case \( c_t = 0 \). In this case, we need to calculate the order of \( e^{-rT}C_2(\hat{H}_T(y))(Y_T(y))^2 - e^{-rT} \frac{1}{r^\beta} e^{\lambda \beta \hat{H}_T(y)} \) when \( T \to \infty \). From Remark 5.1, we recall that as \( h \to \infty \), \( C_2(h) \) has the order

\[
O(e^{(\lambda - 1)(1 - \frac{r_2}{\beta} \beta h)} + O(e^{(\lambda(1 - \frac{r_2}{\beta} r_1 - \frac{r_2}{\beta} r_2) \beta h})) + O(e^{(\lambda(1 - \frac{r_2}{\beta} r_2) \beta h)}).
\]

As the first two terms \( O(e^{(\lambda - 1)(1 - \frac{r_2}{\beta} \beta h)}) \) and \( O(e^{(\lambda(1 - \frac{r_2}{\beta} r_1 - \frac{r_2}{\beta} r_2) \beta h}) \) are identical to the asymptotic expression of \( C_4(h) \) analysed in the previous case when \( 0 < c_t < H_t \), we only need to consider here \( O(e^{\lambda \beta H_t(y)}) \) and hence study the limit behaviour of \( e^{-rT}e^{\lambda \beta \hat{H}_T(y)}(1 - \frac{r_2}{\beta} (Y_T(y))^2 \) for \( T \to \infty \). Due to the condition \( e^{\lambda \beta \hat{H}_T(y)} < e^{T y M_T} \), we have

\[
e^{\lambda \beta \hat{H}_T(y)(1 - r_2)} < (e^{T y M_T})^{1 - r_2}.
\]

It follows that

\[
e^{-rT} e^{\lambda \beta \hat{H}_T(y)(1 - r_2)} (e^{T y M_T})^{r_2} < y M_T.
\]

The term \( e^{-rT} \frac{1}{r^\beta} e^{\lambda \beta \hat{H}_T(y)} \) is also bounded by \( \frac{1}{r^\beta} y M_T \), using that \( e^{\lambda \beta \hat{H}_T(y)} < e^{T y M_T} \). In the previous case \( 0 < c_t < H_t \), it has been shown that \( \mathbb{E}[e^{T y M_T}] \) converges to 0 as \( T \to \infty \), which verifies the claim in the present case.
c) We now turn to the proof of the case $c_t = h$. Similarly as before, we need to calculate the order of

$$e^{-rT} \left( C_5(\hat{H}_T(y))(Y_T(y))^{r_1} + C_6(\hat{H}_T(y))(Y_T(y))^{r_2} - \frac{1}{r} \hat{H}_T(y)Y_T(y) - \frac{1}{r\beta} e^{(\lambda - 1)\beta\hat{H}_T(y)} \right)$$

when $T \to \infty$. By Remark 5.1, when $h \to \infty$, we have

$$C_5(h) = O(e^{\lambda \beta h(1 - r_1)}) + O(e^{(\lambda - 1)(1 - r_1)\beta h}),$$

$$C_6(h) = O(e^{(\lambda - 1)(1 - r_2)\beta h}) + O(e^{(\lambda - 1)(1 - r_2)\beta h}).$$

Firstly, as $C_6(h)$ and $C_4(h)$ have the same asymptotic expressions, we can conclude that $\mathbb{E}[e^{-rT} C_6(\hat{H}_T(y))(Y_T(y))^{r_2}]$ converges to 0 as $T \to \infty$ thanks to the result $\lim_{T \to \infty} \mathbb{E}[e^{-rT} C_4(\hat{H}_T(y))(Y_T(y))^{r_2}] = 0$ in step 3 when $0 < c_t < H_t$. For the asymptotic form of $C_5(h)$, note that its first term $O(e^{\lambda \beta h(1 - r_1)})$ coincides with the asymptotic expression of $C_3(h)$ in Remark 5.1. We can see from step 2 in the case $0 < c_t < H_t$ that $\mathbb{E}[e^{-rT} e^{\lambda \beta \hat{H}_T(y)(1 - r_1)} Y_T(y))^{r_1}]$ converges to 0 as $T \to \infty$. For the second term $O(e^{(\lambda - 1)(1 - r_1)\beta h})$, thanks to the condition $e^{(\lambda - 1)\beta \hat{H}_T(y)} \leq Y_T(y)$, we hence have

$$e^{(\lambda - 1)(1 - r_1)\beta \hat{H}_T(y)} \leq (Y_T(y))^{1 - r_1}.$$

Following the same computations as in step 2 of the case $0 < c_t < H_t$, the desired result holds that $\mathbb{E}[e^{-rT} C_5(\hat{H}_T(y))(Y_T(y))^{r_1}]$ converges to 0 as $T \to \infty$. The term $e^{-rT} \frac{1}{r\beta} e^{(\lambda - 1)\beta \hat{H}_T(y)}$ is first bounded due to $e^{(\lambda - 1)\beta \hat{H}_T(y)} < e^{rT y M_T}$. Using similar arguments as for the case $c_t = 0$, we can obtain its convergence result that $\mathbb{E}[e^{-rT} \frac{1}{r\beta} e^{(\lambda - 1)\beta \hat{H}_T(y)}]$ converges to 0 as $T \to \infty$. The final term $\frac{1}{r} \hat{H}_T(y)Y_T(y)$ term has already been handled in the proof for the case $0 < c_t < h$, which eventually completes the whole proof.

The following result has been used in the previous proof; it is essentially the same as [12, Corollary A.7]. We present it here for completeness.

**Lemma 5.6** Let $B_t^{(c)} = B_t + \xi t$, where $B$ is a standard Brownian motion, and let $\hat{B}_t^{(\xi)}$ be the running maximum of $B_t^{(c)}$. Then for any constants $a, b, k$ with $2a + b + 2\xi \neq 0$, $k \geq 0$, we have
\[ \mathbb{E}\left[e^{aB_T^\xi + b\bar{B}_T^\xi} \mathbbm{1}_{\bar{B}_T^\xi \leq k}\right] = \frac{2(a + b + \zeta)}{2a + b + \zeta} \exp\left(\frac{(a + b)(a + b + 2\zeta)}{2} T\right) \times \left(\Phi\left((a + b + \zeta)\sqrt{T}\right) - \Phi\left((a + b + \zeta)\sqrt{T} - \frac{k}{\sqrt{T}}\right)\right) \]

\[ + \frac{2(a + \zeta)}{2a + b + 2\zeta} \left(\exp\left(\frac{a(a + 2\zeta)}{2} T\right) \Phi\left(-(a + \zeta)\sqrt{T}\right) - \exp\left((2a + b + 2\zeta)k + \frac{a(a + 2\zeta)}{2} T\right) \times \Phi\left(-(a + \zeta)\sqrt{T} - \frac{k}{\sqrt{T}}\right)\right). \]

In particular, we have

\[ \lim_{T \to \infty} \mathbb{E}\left[e^{aB_T^\xi + b\bar{B}_T^\xi} \mathbbm{1}_{\bar{B}_T^\xi \leq k}\right] = 0. \]

Finally, to prove Corollary 3.7, it is sufficient to prove the existence of a unique strong solution to the SDE (3.32) for \( X^* \) under the optimal controls. First, we need to establish some results concerning the regularity of the feedback functions \( e^*(x, h) \) and \( \pi^*(x, h) \). By the definition of \( g \) in (3.22) and the fact that \( f(\cdot, h) \) is the inverse of \( g(\cdot, h) \), we have the following results for the function \( f \).

**Lemma 5.7** The function \( f \) is \( C^1 \) within each of the three subsets of \( \mathbb{R}_+^2 \) given by \( \{x \leq x_{\text{zero}}(h)\} \), \( \{x_{\text{zero}}(h) < x < x_{\text{aggr}}(h)\} \) and \( \{x_{\text{aggr}}(h) \leq x \leq x_{\text{lavs}}(h)\} \), and it is continuous at the points \( x = x_{\text{aggr}}(h) \) and \( x = x_{\text{lavs}}(h) \). Moreover, we have

\[ f_x(x, h) \]

\[ = \frac{1}{g_y(f, h)} \]

\[ = \begin{cases} 
(-C_2(h)r_2(r_2 - 1)(f_1(x, h))^{r_2-2})^{-1}, & \text{if } x \leq x_{\text{zero}}(h), \\
(-C_3(h)r_1(r_1 - 1)(f_2(x, h))^{r_1-2} - C_4(h)r_2(r_2 - 1)(f_2(x, h))^{r_2-2} - \frac{1}{r_1f_2(x, h)})^{-1}, & \text{if } x_{\text{zero}}(h) < x < x_{\text{aggr}}(h), \\
(-C_5(h)r_1(r_1 - 1)(f_3(x, h))^{r_1-2} - C_6(h)r_2(r_2 - 1)(f_3(x, h))^{r_2-2} + \frac{1}{r}h)^{-1}, & \text{if } x_{\text{aggr}}(h) \leq x \leq x_{\text{lavs}}(h),
\end{cases} \tag{5.12} \]

and

\[ f_h(x, h) = -g_h(f(x, h), h) f_x(x, h). \tag{5.13} \]
Proof This is similar to Elie and Touzi [10, proof of Lemma 6.1]. As the inverse of \( g \), the function \( f \) satisfies
\[
g\left(f(x, h), h\right) = x \quad \text{for} \ (x, h) \in \mathbb{R}^2_+.
\]
From the definition of \( g \) in (3.22), we know that for fixed \( h \), the map \( x \mapsto g(x, h) \) is \( C^1 \) and decreasing. By the inverse function theorem, the map \( x \mapsto f(x, h) \) is also \( C^1 \) and decreasing, for any \( h > 0 \).

Using the expression of \( v \) in Proposition 3.2 and the definition of \( g \) in (3.22), one can directly calculate the partial derivative of \( g \) with respect to its first argument and then get (5.12). Similarly, we can calculate the partial derivative \( gh \) explicitly. As \( gh \) is clearly a continuous function in each of the closed intervals \( \{0 \leq x \leq x_{zero}(h)\}, \{x_{zero}(h) \leq x \leq x_{aggr}(h)\} \), \( \{x_{aggr}(h) \leq x \leq x_{lavs}(h)\} \), it is bounded, i.e., there is a constant \( \alpha > 0 \) such that \( gh(x, h) \leq \alpha \) for all \( (x, h) \in \mathbb{R}^2_+ \).

Now in order to prove that \( f \) is \( C^1 \) within all three intervals \( \{0 \leq x \leq x_{zero}(h)\}, \{x_{zero}(h) < x < x_{aggr}(h)\} \) and \( \{x_{aggr}(h) \leq x \leq x_{lavs}(h)\} \), we can verify that \( f \) is differentiable in each variable with continuous partial derivative.

First, let us prove that \( f \) belongs to \( C^0 \) in the three regions of \((x, h) \in \mathbb{R}^2_+ \) given by \( \{0 \leq x \leq x_{zero}(h)\}, \{x_{zero}(h) \leq x \leq x_{aggr}(h)\} \) and \( \{x_{aggr}(h) \leq x \leq x_{lavs}(h)\} \), which implies that \( f_x \in C^0 \) in each of the three regions (as \( f_x \) is a differentiable function of \( f(x, h) \)). Indeed, for a pair \((x, h)\) belonging to one of the intervals and an \( \ell_2 \) small enough, we have
\[
g\left(f(x, h + \ell_2), h\right) - x = g\left(f(x, h + \ell_2), h\right) - g\left(f(x, h + \ell_2), h + \ell_2\right) \\
\leq \alpha \ell_2 \longrightarrow 0 \quad \text{as} \ \ell_2 \to 0.
\]
Now using the continuity of \( f(\cdot, h) \), we obtain
\[
f(x, h + \ell_2) - f(x, h) = f\left(g\left(f(x, h + \ell_2), h\right), h\right) - f(x, h) \longrightarrow 0 \quad \text{as} \ \ell_2 \to 0.
\]
Finally, for sufficiently small \( \ell_1 \), we have
\[
f(x + \ell_1, h + \ell_2) - f(x, h) = f_x(x, h + \ell_2) \ell_1 + f(x, h + \ell_2) - f(x, h),
\]
which will tend to 0 when \( \ell_1, \ell_2 \) tend to 0, and this shows that \( f \) is continuous at an arbitrary point \((x, h)\).

Secondly, let us show that \( f \) is differentiable with respect to \( h \) with continuous partial derivatives. Take a pair \((x, h)\) in a certain interval and \( \ell \) small enough such that \((x, h + \ell)\) is in the same interval. We have
\[
\frac{1}{\ell}\left(f(x, h + \ell) - f(x, h)\right) = \frac{1}{\ell}\left(f(x, h + \ell) - f\left(g\left(f(x, h), h + \ell\right), h + \ell\right)\right) \\
= f_x(x, h + \ell)\frac{1}{\ell}\left(g\left(f(x, h), h\right) - g\left(f(x, h), h + \ell\right)\right).
\]
for some $x_\ell \in [x, x + g(f(x, h), h + \ell)]$. As $f_x \in C^0$ and $g_h(f(x, h), \cdot)$ is continuous, we obtain

$$\lim_{\ell \to 0} \frac{1}{\ell} \left( f(x, h + \ell) - f(x, \ell) \right) = -f_x(x, h)g_h(f(x, h), h),$$

which gives (5.13). Then the continuity of $f_h$ follows from (5.13) and the continuity of $f$. \hfill $\square$

The next result guarantees the existence of a strong solution in Proposition 5.9 below. Its proof is standard and lengthy and will be reported in Appendix B.

**Lemma 5.8** The function $c^*$ is locally Lipschitz on $C$, and the function $\pi^*$ is Lipschitz on $C$.

We are now ready to verify that the SDE (3.32) has a unique strong solution.

**Proposition 5.9** The SDE (3.32) has a unique strong solution $(X^*, H^*)$ for any initial condition $(x, h) \in C$.

**Proof** First, we show that the stochastic differential equation

$$d \tilde{X}_t = r \tilde{X}_t dt + \pi^*(\tilde{X}_t, \tilde{H}_t)(\mu - r)dt + \pi^*(\tilde{X}_t, \tilde{H}_t)\sigma dW_t \quad (5.14)$$

has a unique strong solution. To see this, we introduce the functionals

$$G_1(t, x(t), h(t)) := rx(t) + \pi^*(x(t), h(t))(\mu - r),$$
$$G_2(t, x(t), h(t)) := \pi^*(x(t), h(t)).$$

By Lemma 5.8, we obtain that both $G_1$ and $G_2$ are Lipschitz functions. This justifies the existence of a strong solution for the SDE (5.14).

Now consider the equation (3.32). Since $c^*$ is locally Lipschitz on $C$, a similar argument as above implies the local existence and uniqueness of a solution to the stochastic differential equation (3.32). Using the fact that $c^* > 0$, it follows that $0 \leq X_t^* \leq \tilde{X}_t$; hence there is no explosion of the local solution. \hfill $\square$

**Appendix A: The extreme case when $\lambda = 1$**

We present here some computational results when $\lambda = 1$. Solving the HJB equation essentially follows the same arguments as when $0 < \lambda < 1$. However, the effective domain $\mathcal{C}$ defined in (3.7) needs to be modified to

$$\mathcal{C} := \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x, h) \geq 0\}.$$  

Equivalently, $\mathcal{C} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 = \mathbb{R}_+^2$, where $\mathcal{R}_1$, $\mathcal{R}_2$ are defined in the same way as in Sect. 3.1 for $0 < \lambda < 1$.  

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In particular, we now consider $R_3 = \{(x, h) \in \mathbb{R}_+ \times \mathbb{R}_+: 0 \leq u_x(x, h) \leq 1\}$ and note that the previous auxiliary singular control

$$\hat{c}(x) = \frac{1}{\beta(\lambda - 1)} \ln \frac{u_x}{1 - \lambda}$$

for the case $0 < \lambda < 1$ is no longer well defined if we have $\lambda = 1$.

In fact, in the extreme case $\lambda = 1$, there is no need to consider a singular optimal consumption that exceeds the previous maximum level $h$. In the whole region $R_3$, the optimal consumption is no longer unique, but one feedback optimal consumption is to constantly consume the initial level $H^*_0 = h$such that $c^*(x, h) \leq h$ can be guaranteed for any $x \geq 0$. That $H^*$ will never increase for $\lambda = 1$ results from the formulation $U(c_t - H_t)$ where the utility is defined on the difference. For the case $0 < \lambda < 1$, the utility $U(c_t - \lambda H_t)$ allows the investor to gain a positive outperformance $c_t - \lambda H_t > 0$ if he chooses a large $c_t$ to increase $H_t$. However, for the case $\lambda = 1$, the investor can only obtain $0 = c_t - H_t$ by choosing to consume more than the past maximum. But the investor can also easily achieve the same goal of zero difference $c_t - H_t$ by following the previously attained maximum level without creating any new record. Therefore, to achieve the largest gap $c_t - H_t = 0$, one optimal way is to sit on the previous consumption peak, and then the investor has no incentive to increase the reference process $H$ at any time. Consequently, we only adopt the feedback control $c^*(x, h) = h$ in the region $R_3$.

Based on the observations above, if the wealth $x$ is larger than or equal to the subsistence level $x^* := \frac{h}{r}$, the investor can always choose to invest the zero amount $\pi^*_t = 0$ in the risky asset and save the initial wealth $x^*$ in the bank account such that the interest rate can support the constant consumption at the initial reference level $c^*_t = H_0 = h$, $t \geq 0$. That is, we have $c^*_t - H^*_t = 0$ for $t \geq 0$. As a consequence, the value function in (2.2) attains its maximum value $u(x, h) = -\frac{1}{r\beta}$ for $x \geq \frac{h}{r}$. The primal value function $u(x, h)$ for $\lambda = 1$ is no longer strictly concave and $u(x, h)$ remains constant (and $u_x(x, h) = 0$) for $x \geq \frac{h}{r}$, which differs substantially from the case $0 < \lambda < 1$. Therefore, we have the asymptotic conditions that

$$\lim_{x \to \frac{h}{r}} u_x(x, h) = 0 \quad \text{and} \quad \lim_{x \to \frac{h}{r}} u(x, h) = -\frac{1}{r\beta}. \quad (A.1)$$

For each $h \geq 0$, we expect that the value function $x \mapsto u(x, h)$ is strictly concave for $0 \leq x < \frac{h}{r}$ and the dual transform method in the previous sections can still be applied on this interval $[0, \frac{h}{r})$. In view of the set $C$ when $\lambda = 1$, we now consider $y > 0$ for the dual problem and define

$$v(y, h) := \sup_{0 \leq x < \frac{h}{r}} \left( u(x, h) - xy \right) \quad \text{for } y > 0.$$ 

As a consequence of (A.1), we have the asymptotic conditions that

$$\lim_{y \to 0} v_y(y, h) = -\frac{h}{r} \quad \text{and} \quad \lim_{y \to 0} \left( v(y, h) - yv_y(y, h) \right) = -\frac{1}{r\beta}. \quad (A.2)$$

which are different from (3.13) for $0 < \lambda < 1$. 
Based on the same analysis as in the case $0 < \lambda < 1$, we can write down the linear dual ODE for the case $\lambda = 1$ as

$$
\frac{\kappa^2}{2} y^2 v_{yy} - rv = \begin{cases} 
\frac{1}{\beta} e^{\beta h}, & \text{if } y \geq e^{\beta h}, \\
\frac{1}{\beta} y - y(\frac{1}{\beta} \ln y - h), & \text{if } 1 < y < e^{\beta h}, \\
\frac{1}{\beta} + hy, & \text{if } 0 < y \leq 1.
\end{cases}
$$

(A.3)

By following the same arguments as in the proof of Proposition 3.2, and replacing the free boundary condition (3.11) by the new boundary condition (A.2) as $y \to 0$ in the third region, we can establish the next result.

**Proposition A.1** For a given parameter $h \geq 0$, the ODE (A.3) admits the unique explicit solution

$$
v(y, h) = \begin{cases}
C_2(h) y^{r_2} - \frac{1}{r\beta} e^{\beta h}, & \text{if } y \geq e^{\beta h}, \\
C_3(h) y^{r_1} + C_4(h) y^{r_2} - \frac{\kappa}{r\beta} + \frac{y}{r\beta} (\ln y - \beta h + \frac{\kappa^2}{2r}), & \text{if } 1 < y < e^{\beta h}, \\
C_5(h) y^{r_1} - \frac{1}{r} hy - \frac{1}{r\beta}, & \text{if } 0 < y \leq 1,
\end{cases}
$$

where $C_i(h)$, $i = 2, 3, 4, 5$, are defined in (3.15)–(3.18) in Proposition 3.2 by setting $\lambda = 1$.

By using the dual value function $v(y, h)$ and applying the inverse transform

$$
u(x, h) = \inf_{y > 0} \left( v(y, h) + xy \right) \text{ for } 0 \leq x < \frac{h}{r}
$$

and $u(x, h) = -\frac{1}{r\beta}$ for $x \geq \frac{h}{r}$, we can readily get the next result.

**Corollary A.2** For $(x, h) \in \mathbb{R}_+^2$ and $\lambda = 1$, let us define the boundaries

$$
\bar{x}_{\text{zero}}(h) := -e^{\beta h (r_2 - 1)} C_2(h) r_2,
$$

$$
\bar{x}_{\text{aggr}}(h) := -C_3(h) r_1 - C_4(h) r_2 + \frac{h}{r} - \frac{\kappa^2}{2r^2 \beta},
$$

and the piecewise function

$$
f(x, h) = \begin{cases}
\left( -x/C_2(h) r_2 \right)^{\frac{1}{r_2 - 1}}, & \text{if } x \leq \bar{x}_{\text{zero}}(h), \\
\bar{f}_2(x, h), & \text{if } \bar{x}_{\text{zero}}(h) < x < \bar{x}_{\text{aggr}}(h), \\
\left( h - xr/C_5(h) r_1 r \right)^{\frac{1}{r_1 - 1}}, & \text{if } \bar{x}_{\text{aggr}}(h) \leq x < \frac{h}{r},
\end{cases}
$$

where $\bar{f}_2(x, h)$ is uniquely determined by
\[
x = -C_3(h)r_1(\bar{f}_2(x, h))^{r_1-1} - C_4(h)r_2(\bar{f}_2(x, h))^{r_2-1} - \frac{1}{r\beta} \left( \ln \bar{f}_2(x, h) - \beta h + \frac{\kappa^2}{2r} \right).
\]

The value function \(u(x, h)\) of the control problem in (2.2) can be explicitly expressed as

\[
u(x, h) = \begin{cases} 
C_2(h)(\frac{-x}{C_2(h)r_2})^{r_2-1} - \frac{1}{r\beta} e^{\beta h} + x(\frac{-x}{C_2(h)r_2})^{\frac{1}{r_2-1}}, & \text{if } x \leq \bar{\text{zero}}(h), \\
C_3(h)(f(x, h))^{r_1} + C_4(h)(f(x, h))^{r_2} + \frac{f(x, h)}{r\beta} (\ln f(x, h) - \beta h + \frac{\kappa^2}{2r} - 1 + xr\beta), & \text{if } \bar{\text{zero}}(h) < x < \bar{\text{aggr}}(h), \\
C_5(h)(\frac{h-x}{C_5(h)r_1})^{r_1-1} - \frac{1}{r\beta} h(\frac{h-x}{C_5(h)r_1})^{\frac{1}{r_1-1}} - \frac{1}{r\beta} + x(\frac{h-x}{C_5(h)r_1})^{\frac{1}{r_1-1}}, & \text{if } \bar{\text{aggr}}(h) \leq x < \frac{h}{r}, \\
-1 \beta, & \text{if } h \leq x.
\end{cases}
\]

The feedback functions of the optimal consumption and portfolio are

\[
c^*(x, h) = \begin{cases} 
0, & \text{if } x \leq \bar{\text{zero}}(h), \\
-\frac{1}{\beta} \ln f(x, h) + h, & \text{if } \bar{\text{zero}}(h) < x < \bar{\text{aggr}}(h), \\
h, & \text{if } \bar{\text{aggr}}(h) \leq x,
\end{cases}
\]

and

\[
\pi^*(x, h) = \frac{\mu - r}{\sigma^2} \begin{cases} 
(1 - r_2)x, & \text{if } x \leq \bar{\text{zero}}(h), \\
\frac{2r}{\kappa^2} C_3(h) f^{r_1-1}(x, h) + \frac{2\kappa}{\kappa^2} C_4(h) f^{r_2-1}(x, h) + \frac{1}{r\beta}, & \text{if } \bar{\text{zero}}(h) < x < \bar{\text{aggr}}(h), \\
\frac{2r}{\kappa^2 r_1} \left( \frac{h}{r} - x \right), & \text{if } \bar{\text{aggr}}(h) \leq x < \frac{h}{r}, \\
0, & \text{if } h \leq x,
\end{cases}
\]

and the resulting consumption running maximum process is constant, \(H^*_t = H^*_0 = h\) for \(t \geq 0\).

**Appendix B: Proof of Lemma 5.8**

By (3.20), (3.21) and the inverse transform, we can express \(c^*\) and \(\pi^*\) in terms of the primal variables as in (3.30) and (3.31). Combining the expressions of \(c^*\) and \(\pi^*\) with Proposition 3.2 (implying that the coefficient functions \((C_i(h))_{2 \leq i \leq 5}\) are \(C^1\),

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Lemma 5.7 (implying the $C^1$-regularity of $f$) and the continuity of $f$ at the boundary between the three regions, we can draw the conclusion that $(x, h) \mapsto e^*(x, h)$ and $(x, h) \mapsto \pi^*(x, h)$ are locally Lipschitz on $\mathcal{C}$.

Now in order to prove the Lipschitz property of $\pi^*$, we show separately that the partial derivatives $\frac{\partial \pi^*}{\partial x}$ and $\frac{\partial \pi^*}{\partial h}$ are bounded. To obtain more compact expressions, we introduce the intervals

\[ J_1 := [0, x_{\text{zero}}(h)], \]
\[ J_2 := (x_{\text{zero}}(h), x_{\text{aggr}}(h)), \]
\[ J_3 := [x_{\text{aggr}}(h), x_{\text{lavs}}(h)]. \]

**Step 1: Boundedness of $\frac{\partial \pi^*}{\partial x}$.** First, using $\pi^*$ in (3.31), we have

\[
\frac{\partial \pi^*}{\partial x} (x, h) = \frac{\mu - r \sigma^2}{\sigma^2} \begin{cases} 
(1 - r_2), & \text{if } x \in J_1, \\
+ \frac{2r}{\kappa^2} C_3(h)(r_1 - 1) f_2^{r_1 - 2}(x, h) \frac{\partial f_2}{\partial x} + \frac{2r}{\kappa^2} C_4(h)(r_2 - 1) f_2^{r_2 - 2}(x, h) \frac{\partial f_2}{\partial x}, & \text{if } x \in J_2, \\
+ \frac{2r}{\kappa^2} C_5(h)(r_1 - 1) f_3^{r_1 - 2}(x, h) \frac{\partial f_3}{\partial x} + \frac{2r}{\kappa^2} C_6(h)(r_2 - 1) f_3^{r_2 - 2}(x, h) \frac{\partial f_3}{\partial x}, & \text{if } x \in J_3. 
\end{cases}
\]

(B.1)

Note that the first case expression is constant and hence bounded. For the second, by differentiating (3.24) and using $r_1(r_1 - 1) = r_2(r_2 - 1) = \frac{2r}{\kappa^2}$, we have

\[
1 = -2r \kappa^2 C_3(h) f_2^{r_1 - 2}(x, h) \frac{\partial f_2}{\partial x} - 2r \kappa^2 C_4(h) f_2^{r_2 - 2}(x, h) \frac{\partial f_2}{\partial x} - \frac{1}{r \beta} \frac{1}{f_2} \frac{\partial f_2}{\partial x},
\]

Plugging this back into $\frac{\partial \pi^*}{\partial x}$, we obtain

\[
\frac{\partial \pi^*}{\partial x} (x, h) = \frac{\mu - r}{\sigma^2} \left( 2r \kappa^2 C_3(h) f_2^{r_1 - 1}(x, h) \frac{\partial f_2}{\partial x}(r_1 - r_2) + (1 - r_2) \left( 1 - r_2 \right) \frac{1}{r \beta} \frac{1}{f_2} \frac{\partial f_2}{\partial x} \right)
\]

\[ = \frac{\mu - r}{\sigma^2} \left( 2r \kappa^2 C_3(h) f_2^{r_1 - 1}(x, h) (r_1 - r_2) + (1 - r_2) \frac{1}{r \beta} \frac{1}{f_2} \frac{\partial f_2}{\partial x} + (1 - r_2) \right). \]

Combining the above with (5.7), we obtain

\[
\frac{\partial \pi^*}{\partial x} (x, h) = \frac{\mu - r}{\sigma^2} \left( \frac{A(x, h)}{B(x, h)} + (1 - r_2) \right),
\]

(B.2)

where
\[ A(x, h) := \frac{2r}{\kappa^2} C_3(h) f_2^{r_1-1}(x, h)(r_1 - r_2) + \frac{1 - r_2}{r^\beta}, \]
\[ B(x, h) := \frac{2r}{\kappa^2} \left( - C_3(h) f_2^{r_1-1}(x, h) - C_4(h) f_2^{r_2-1}(x, h) \right) - \frac{1}{r^\beta}. \]  
(B.3)

In what follows, we show that there exist two constants \( A_0 > 0, B_0 < 0 \) independent from \( h \) such that \( 0 \leq A(x, h) \leq A_0 \) and \( B(x, h) \leq B_0 \). Combining with (B.2), this shows that

\[ \frac{\mu - r}{\sigma^2} \left( \frac{A_0}{B_0} + (1 - r_2) \right) \leq \frac{\partial \pi^*}{\partial x}(x, h) \leq \frac{\mu - r}{\sigma^2}(1 - r_2) \]

so that \( \frac{\partial \pi^*}{\partial x} \) is bounded.

To find \( A_0 \) and \( B_0 \), we first note that because \( C_3 < 0 \) and \( r_1 > 1 > r_2 \), the map 
\[ y \mapsto \frac{2r}{\kappa^2} C_3(h) y^{r_1-1} (r_1 - r_2) + (1 - r_2) \frac{1}{r^\beta} \]

is decreasing. Plugging the lower and upper bounds \( e^{(\lambda - 1)\beta h} \) and \( e^{\lambda \beta h} \) into \( f_2 \), we have

\[ A(x, h) \geq \frac{2r}{\kappa^2} C_3(h) e^{(\lambda - 1)\beta h} (r_1 - r_2) + (1 - r_2) \frac{1}{r^\beta} = 0 \]

and

\[ A(x, h) \leq \frac{2r}{\kappa^2} C_3(h) e^{\lambda \beta h} (r_1 - r_2) + (1 - r_2) \frac{1}{r^\beta} \]

\[ = (1 - r_2) \frac{1}{r^\beta}(1 - e^{-(r_1 - 1)\beta h}) \leq (1 - r_2) \frac{1}{r^\beta} =: A_0. \]

For the function \( B \), we know from the proof of Lemma 3.5 that in the closed interval
\( [e^{(\lambda - 1)\beta h}, e^{\lambda \beta h}] \), the map \( z : y \mapsto \frac{2r}{\kappa^2} \left( - C_3(h) y^{r_1-1} - C_4(h) y^{r_2-1} \right) - \frac{1}{r^\beta} \)

is strictly negative. Moreover, it is either monotonic or first decreasing and then increasing, as shown in Step 2 of the proof of Lemma 3.5. Therefore, the upper bound of \( z \) is attained at either \( y = e^{\lambda \beta h} \) or \( y = e^{(\lambda - 1)\beta h} \).

At the boundary point \( y = e^{\lambda \beta h} \), we have

\[ z(e^{\lambda \beta h}) \]

\[ = - \frac{(1 - \lambda)^{r_1 - r_2}}{(r_1 - r_2) \beta r} \left( \frac{1}{1 - r_2} e^{-(1 - r_2)\beta h} - \frac{\lambda}{\lambda (1 - r_2) - (r_1 - r_2)} e^{-(r_1 - r_2)\beta h} \right) \]

\[ - \frac{1}{(r_1 - r_2) \beta r} (r_1 - 1) (1 - e^{-(1 - r_2)\beta h}) =: L_1(h). \]

This expression is strictly negative for \( h \in [H_0, \infty) \). Indeed, when \( h \to \infty \), we have
\[ \lim_{h \to \infty} L_1(h) = - \frac{r_1 - 1}{r_1 - r_2} \beta r < 0. \]

Moreover, we have
\[ L_1(0) = - \frac{(1 - \lambda)^{r_1 - r_2}}{(r_1 - r_2) \beta r} \left( \frac{1}{1 - r_2} - \frac{\lambda}{\lambda (1 - r_2) - (r_1 - r_2)} \right) < 0. \]
We now analyse the monotonicity of each term in the function \( h \mapsto \mathcal{L}_1(h) \). The first term is strictly increasing on \([0, \infty)\) and takes values in the interval \([\mathcal{L}_1(0), 0)\). The second term is strictly decreasing on \([0, \infty)\) with values in the interval \((\lim_{h \to \infty} \mathcal{L}_1(h), 0]\). Hence we can define \( B'_0 := \sup_{0 \leq h < \infty} \mathcal{L}_1(h) < 0 \). On the other hand, at the boundary point \( y = e^{(\lambda-1)\beta h} \), we have

\[
z(e^{(\lambda-1)\beta h}) = -\frac{(1 - \lambda)^{r_1-r_2}}{(r_1 - r_2)\beta r} \frac{2r}{\kappa^2} \left( \frac{1}{1 - r_2} - \frac{\lambda}{\lambda(1 - r_2) - (r_1 - r_2)} e^{-(r_1-1)\beta h} \right) - \frac{1}{(r_1 - r_2)\beta r} (1 - r_2)(1 - e^{-(r_1-1)\beta h}) =: \mathcal{L}_2(h).
\]

Similar arguments as for the point \( e^{\lambda \beta h} \) lead to \( B''_0 := \sup_{0 \leq h < \infty} \mathcal{L}_2(h) < 0 \). We can finally set \( B_0 := \max(B'_0, B''_0) \).

For the third case expression of (B.1), similar calculations as for the second lead to

\[
\frac{\partial \pi^*}{\partial x}(x, h) = \frac{\mu - r}{\sigma^2} \left( \frac{2r}{\kappa^2} C_5(h) f_3^{r_1-1}(x, h)(r_1 - r_2) \right) \frac{1}{f_3} \frac{\partial f_3}{\partial x} + (1 - r_2)
\]

\[
= \frac{\mu - r}{\sigma^2} \left( \frac{\tilde{A}(x, h)}{\tilde{B}(x, h)} + (1 - r_2) \right),
\]

where

\[
\tilde{A}(x, h) := \frac{2r}{\kappa^2} C_5(h) f_3^{r_1-1}(x, h)(r_1 - r_2),
\]

\[
\tilde{B}(x, h) := \frac{2r}{\kappa^2} \left( - C_5(h) (f_3(x, h))^{r_1-1} - C_6(h) (f_3(x, h))^{r_2-1} \right).
\]  \((B.4)\)

We claim that there exist two constants \( A_1 > 0, B_1 < 0 \) which are independent from \( h \) and satisfy that \( 0 \leq \tilde{A}(x, h) \leq A_1 \) and \( \tilde{B}(x, h) \leq B_1 \). This implies that

\[
\frac{\mu - r}{\sigma^2} \left( \frac{A_1}{B_1} + (1 - r_2) \right) \leq \frac{\partial \pi^*}{\partial x}(x, h) \leq \frac{\mu - r}{\sigma^2} (1 - r_2)
\]

so that \( \frac{\partial \pi^*}{\partial x} \) is bounded in the third region \( J_3 \) as well.

Now we show how to find two constants \( A_1 \) and \( B_1 \). As \( C_5 > 0 \) and \( r_1 > 1 > r_2 \), we know that the map \( y \mapsto \frac{2r}{\kappa^2} C_5(h) y^{r_1-1}(r_1 - r_2) \) is increasing and clearly \( \tilde{A} \geq 0 \).

Using the condition that \( (1 - \lambda) e^{(\lambda-1)\beta h} \leq f_3 \leq e^{(\lambda-1)\beta h} \), we have

\[
\tilde{A}(x, h) \leq \frac{2r}{\kappa^2} C_5(h) (e^{(\lambda-1)\beta h})^{r_1-1}(r_1 - r_2)
\]

\[
= \frac{1 - r_2}{(r_1 - r_2)\beta r} (1 - e^{(1-r_1)\beta h}) \leq \frac{1 - r_2}{(r_1 - r_2)\beta r} =: A_1 > 0.
\]

As \( C_5 > 0, C_6 > 0 \) and \( (1 - \lambda) e^{(\lambda-1)\beta h} \leq f_3 \leq e^{(\lambda-1)\beta h} \), we have
\[ \tilde{B}(x, h) \leq -\frac{2r}{\kappa^2} C_5(h) ((1 - \lambda) e^{(\lambda - 1)\beta h})^{r_1-1} - \frac{2r}{\kappa^2} C_6(h) (e^{(\lambda - 1)\beta h})^{r_2-1} \]
\[ = -(1 - \lambda)^{r_1-1} \frac{1 - r_2}{(r_1 - r_2) \beta r} (1 - e^{(1-r_1)\beta h}) \]
\[ - \frac{2r}{\kappa^2} (1 - \lambda)^{r_1-r_2} \frac{1}{(1 - r_2) \beta r} \left( \frac{\lambda}{\lambda(1 - r_2) - (r_1 - r_2)} e^{(1-r_1)\beta h} \right) \]
\[ =: L_3(h). \]

As \( h \mapsto L_3(h) \) is strictly decreasing, we have
\[ \tilde{B}(x, h) \leq L_3(0) = -\frac{2r}{\kappa^2} (1 - \lambda)^{r_1-r_2} \frac{1}{(1 - r_2) \beta r} \left( \frac{\lambda}{\lambda(1 - r_2) - (r_1 - r_2)} \right) \]
\[ =: B_1 < 0. \]

**Step 2: Boundedness of \( \frac{\partial \pi^*}{\partial h} \).** First, using (5.13) and the definition of \( g \) in (3.22), we have
\[ f_h(x, h) = -g_h(f, h) f_x(x, h) \]
\[ = \begin{cases} 
- C_2'(h) r_2 f_1(x, h)^{r_2-1} - C_2(h) \frac{\nu}{\kappa^2} f_1(x, h)^{r_2-2}, & \text{if } x \in J_1, \\
(C_3'(h) r_1 f_2(x, h)^{r_1-1} + C_4'(h) r_2 f_2(x, h)^{r_2-1} - \frac{\lambda \beta}{r \beta}) \times \frac{1}{-C_3(h) \frac{\nu}{\kappa^2} f_2(x, h)^{r_2-2} - C_4(h) \frac{\nu}{\kappa^2} f_2(x, h)^{r_2-2} - \frac{\lambda \beta}{r \beta}}, & \text{if } x \in J_2, \\
(C_5'(h) r_1 f_3(x, h)^{r_1-1} + C_6'(h) r_2 f_3(x, h)^{r_2-1} - \frac{1}{r}) \times \frac{1}{-C_5(h) \frac{\nu}{\kappa^2} f_3(x, h)^{r_2-2} - C_6(h) \frac{\nu}{\kappa^2} f_3(x, h)^{r_2-2}}, & \text{if } x \in J_3. 
\end{cases} \]

We analyse the derivative \( \frac{\partial \pi^*}{\partial h} \) in different regions separately. In the region \( J_1 \), we have \( \frac{\partial \pi^*}{\partial h} = 0 \) so that it is bounded. In the region \( J_2 \), we have
\[ \frac{\partial \pi^*}{\partial h} = \frac{\mu - r}{\sigma^2} \left( \frac{2r}{\kappa^2} C_3'(h) f_2(x, h)^{r_1-1} + \frac{2r}{\kappa^2} C_3(h) (r_1 - 1) f_2(x, h)^{r_1-2} \frac{\partial f_2}{\partial h} \right. \]
\[ + \frac{2r}{\kappa^2} C_4'(h) f_2(x, h)^{r_2-1} + \frac{2r}{\kappa^2} C_4(h) (r_2 - 1) f_2(x, h)^{r_2-2} \frac{\partial f_2}{\partial h} \bigg) . \]

By differentiating (3.24) and using \( r_1(r_1 - 1) = r_2(r_2 - 1) = \frac{2r}{\kappa^2} \), we get
\[ C_4'(h) \frac{2r}{\kappa^2} f_2(x, h)^{r_2-1} + C_4(h) \frac{2r}{\kappa^2} f_2(x, h)^{r_2-2} \frac{\partial f_2}{\partial h} \]
\[ = -C_3'(h) r_1 (r_2 - 1) f_2(x, h)^{r_1-1} - C_3(h) \frac{2r}{\kappa^2} (r_2 - 1) \frac{\partial f_2}{\partial h} f_2(x, h)^{r_1-2} \]
\[ - \frac{1}{r \beta} (r_2 - 1) \left( \frac{1}{f_2} \frac{\partial f_2(x, h)}{\partial h} - \lambda \beta \right). \]
Replacing this back into the previous expression of \( \frac{\partial \pi^*}{\partial h} \), we obtain

\[
\frac{\partial \pi^*}{\partial h} = \frac{\mu - r}{\sigma^2} \left( \left( \frac{2r}{\kappa^2} (r_1 - r_2) C_3(h) f_2(x, h)^{r_1-1} + \frac{1}{r\beta} (1 - r_2) \right) \frac{1}{f_2} \frac{\partial f_2}{\partial h} 
\right.
\]

\[
+ r_1 (r_1 - r_2) C'_3(h) f_2(x, h)^{r_1-1} - \frac{\lambda \beta}{r \beta} (1 - r_2) 
\right)
\]

\[
= \frac{\mu - r}{\sigma^2} \left( A(x, h) \frac{1}{f_2} \frac{\partial f_2}{\partial h} + r_1 (r_1 - r_2) C'_3(h) f_2(x, h)^{r_1-1} - \frac{\lambda \beta}{r \beta} (1 - r_2) \right), \quad (B.5)
\]

where \( A(x, h) \) is defined in (B.3). In (B.5), the third term is a constant. For the second term, we can calculate that \( C'_3(h) \) and \( C'_4(h) \) are given respectively by

\[
C'_3(h) = \frac{1}{(r_1 - r_2) \beta \lambda r e^{\lambda (1-r_1) \beta h}} > 0,
\]

\[
C'_4(h) = \frac{(1 - \lambda)^{r_1-r_2}}{(r_1 - r_2) \beta r} \left( (\lambda - 1) \beta e^{\lambda (1-r_1) \beta h} - \lambda \beta e^{(\lambda - 1 - (r_1-r_2) \beta h)} \right)
\]

\[
- \frac{1}{(r_1 - r_2) \beta r} (\lambda - 1) \beta e^{(\lambda - 1) (1-r_2) \beta h}.
\]

Using the fact that \( e^{(\lambda - 1) \beta h} < f_2(x, h) < e^{\lambda \beta h} \) and \( r_1 > 1 \), we have

\[
0 \leq \frac{\lambda r_1}{r} e^{-\beta h (r_1-1)} \leq r_1 (r_1 - r_2) C'_3(h) f_2(x, h)^{r_1-1} \leq \frac{\lambda r_1}{r},
\]

so that the second term in (B.5) is also bounded. Now we consider the first term in the bracket of (B.5). From Step 1, we know that \( 0 \leq A(x, h) \leq A_0 \); hence it is enough to show that \( \frac{1}{f_2} \frac{\partial f_2}{\partial h} \) is bounded. Indeed, we have

\[
\frac{1}{f_2} \frac{\partial f_2}{\partial h} = \left( C'_3(h) r_1 f_2(x, h)^{r_1-1} + C'_4(h) r_2 f_2(x, h)^{r_2-1} - \frac{\lambda \beta}{r \beta} \right) \frac{1}{B(x, h)},
\]

where \( B(x, h) \) is defined in (B.3). From the expression of \( C'_4(h) \), we can derive that

\[
C'_4(h) r_2 f_2(x, h)^{r_2-1} \geq r_2 \frac{1 - \lambda}{(r_1 - r_2) r} (1 - (1 - \lambda)^{r_1-r_2})
\]

\[
- r_2 \frac{(1 - \lambda)^{r_1-r_2}}{(r_1 - r_2) r} \lambda e^{-(r_1-r_2) \beta h}
\]

\[
\geq r_2 \frac{1 - \lambda}{(r_1 - r_2) r} (1 - (1 - \lambda)^{r_1-r_2})
\]

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and
\[ C_4'(h)r_2 f_2(x, h)r_2^{-1} \leq r_2 \frac{1 - \lambda}{(r_1 - r_2)r} \left( 1 - (1 - \lambda)^{r_1-r_2} \right) e^{-(1-r_2)\beta h} - r_2 \frac{1 - \lambda}{(r_1 - r_2)r} \lambda e^{-(r_1-1)\beta h} \]
\[ \leq -r_2 \frac{(1 - \lambda)^{r_1-r_2}}{(r_1 - r_2)r} \lambda. \]

Using the fact that \( B(x, h) \leq B_0 < 0 \), we conclude that \( \frac{1}{f_2^2} \frac{\partial f_2}{\partial h} \) is bounded.

Finally, in the region \( J_3 \), similar calculations lead to
\[
\frac{\partial \pi^*}{\partial h} = \mu - r \left( \frac{\tilde{A}(x, h)}{\tilde{B}(x, h)} \left( C_5'(h)r_1 f_3(x, h)r_1^{-1} + C_6'(h)r_2 f_3(x, h)r_2^{-1} - \frac{1}{r} \right) + r_1(r_1 - r_2)C_5'(h)f_3(x, h)r_1^{-1} - \frac{1}{r}(1 - r_2) \right),
\]
where \( \tilde{A}(x, h) \) and \( \tilde{B}(x, h) \) are defined in (B.4). Now combining the fact that \( C_5'(h) \) has terms of \( e^{(\lambda-1)(1-r_1)\beta h} \) and \( e^{(\lambda-1)\beta h} \), \( C_6'(h) \) has terms of \( e^{(\lambda-1)(1-r_2)\beta h} \) and \( e^{(\lambda-(1-r_2)-(r_1-r_2)\beta h} \), that \( 0 \leq \tilde{A}(x, h) \leq A_1 \) and \( \tilde{B}(x, h) \leq B_1 < 0 \), and finally that \( (1 - \lambda)e^{(\lambda-1)\beta h} \leq f_3 \leq e^{(\lambda-1)\beta h} \), we obtain the desired boundedness result. The calculations are similar as before and we omit the details here. Putting all the pieces together, we obtain the desired boundedness of \( \frac{\partial \pi^*}{\partial h} \). \( \square \)

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