From Dark Energy & Dark Matter to Dark Metric

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Abstract: It is nowadays clear that General Relativity cannot be the definitive theory of Gravitation due to several shortcomings that come out both from theoretical and experimental viewpoints. At large scales (astrophysical and cosmological) the attempts to match it with the latest observational data lead to invoke Dark Energy and Dark Matter as the bulk components of the cosmic fluid. Since no final evidence, at fundamental level, exists for such ingredients, it is clear that General Relativity presents shortcomings at infrared scales. On the other hand, the attempts to formulate more general theories than Einstein’s one give rise to mathematical difficulties that need workarounds that, in turn, generate problems from the interpretative viewpoint. We present here a completely new approach to the mathematical objects in terms of which a theory of Gravitation may be written in a first-order (à la Palatini) formalism, and introduce the concept of Dark Metric which could completely bypass the introduction of disturbing concepts as Dark Energy and Dark Matter.

1. Introduction

Einstein General Relativity (GR) is a self-consistent theory that dynamically describes space, time and matter under the same standard. The result is a deep and beautiful scheme that, starting from some first principles, is capable of explaining a huge number of gravitational phenomena, ranging from laboratory up to cosmological scales. Its predictions are well tested at Solar System scales and give rise to a comprehensive cosmological model that agrees with the Standard Model of particles, with the recession of galaxies, with the cosmic nucleosynthesis and so on.

Despite these good results, the recent advent of the so-called Precision Cosmology and several tests coming from the Solar System outskirts (\textit{e.g.} the Pioneer Anomaly) entail that the self-consistent scheme of GR seems to disagree with an increasingly high number of observational data, as \textit{e.g.} those coming from IA-type Supernovae, used as standard candles, large scale structure ranging from galaxies up to galaxy superclusters and so on. Furthermore, being not renormalizable, GR fails to be quantized in any “classical” way (see [1]). In other words, it seems then, from ultraviolet up to infrared scales, that GR is not and cannot be the definitive theory of Gravitation also if it successfully addresses a wide range of phenomena.

Many attempts have been therefore made both to recover the validity of GR at all scales, on the one hand, and to produce theories that suitably generalize Einstein’s one, on the other hand.

Besides, in order to interpret a large number of recent observational data inside the paradigm of GR,
the introduction of Dark Matter (DM) and Dark Energy (DE) has seemed to be necessary: the price of preserving the simplicity of the Hilbert Lagrangian has been, however, the introduction of rather odd-behaving physical entities which, up to now, have not been revealed by any experiment at fundamental scales. In other words, we are observing the large scale effects of missing matter (DM) and the accelerating behaviour of the Hubble flow (DE) but no final evidence of these ingredients exists, if we want to deal with them under the standard of quantum particles or quantum fields. In Section 3, we shall argue whether, after all, it is really preferable the use of the simplest Lagrangian.

An opposite approach resides in the so-called Non-Linear Theories of Gravitation (NLTGs, see later), that have been also investigated by many authors, also in connection with Scalar-Tensor Theories (STTs, see later). In this case, no ill-defined ingredients have to be required, at the price of big mathematical complications. None of the many efforts made up to now to solve this problem (see later) seems to be satisfactory from an interpretative viewpoint.

What we shortly present here is a completely new approach to the mathematical objects in terms of which a theory of Gravitation may be written, whereby Gravity is encoded from the very beginning in a (symmetric) linear connection in SpaceTime. At the end, we shall conclude that, although the gravitational field is a linear connection, the fundamental field of Gravity turns out a posteriori to be still a metric, but not the “obvious” one given from the very beginning (which we shall therefore call Apparent Metric). Rather we shall show the relevance of another metric, ensuing from gravitational dynamics, that we shall call Dark Metric since we claim it being a possible source of the apparently “Dark Side” of our Universe which reveals itself, at large scales, as missing matter (in clustered structures) and accelerating behaviour (in the Hubble fluid).

To complete our program, we need first to recall some facts regarding different (relativistic) theories of Gravitation. This will not be an historical compendium, but rather a collection of speculative hints useful to our aims.

2. A critical excursus

The theory of Special Relativity (SR), published by A. Einstein in 1905 [2], was aimed to reconcile Mechanics with Electromagnetism, but leaved out matter and Gravitation (Minkowski SpaceTime is rigorously empty and flat). Then Einstein devoted more than ten years (1905–1915/1916) to develop a theory of Gravitation based on the following requirements (see [3]): principle of equivalence (Gravity and Inertia are indistinguishable; there exist observers in free fall, i.e. inertial motion under gravitational pull); principle of relativity (SR holds pointwise; the structure of SpaceTime is pointwise Minkowskian); principle of general covariance (“democracy” in Physics); principle of causality (all physical phenomena propagate respecting the light-cones). Einstein, who was also deeply influenced by Riemann’s teachings about the link between matter and curvature, decided then to describe Gravity by means of a (dynamic) SpaceTime $M$ endowed with a dynamic Lorentzian metric $g$. This appeared to be a good choice for a number of reasons: a metric is the right tool to define measurements (rods & clocks); the geodesics of a metric are good mathematical objects to describe the free fall; a Lorentzian manifold is pointwise Minkowskian, is suitable to be the domain of tensor fields, is compatible with a light-cones structure. And, after all, at that time, there was no other geometrical field Einstein could use to define the curvature of a differentiable manifold!

Following this way, Einstein deduced his famous equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1}$$

A linear concomitant of the Riemann curvature tensor of $g$, nowadays called the Einstein tensor $G_{\mu\nu}$, equals here the stress-energy tensor $T_{\mu\nu} \equiv \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ that reflects the properties of matter. Here $R_{\mu\nu}$ is the Ricci tensor of the metric $g$ and $R(g) \equiv g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature of the metric, while $L_{\text{mat}} \equiv L_{\text{mat}} ds$ is the matter Lagrangian and $G$ is the gravitational coupling constant.
In other words, the distribution of matter influences Gravity through 10 second-order field equations. Their structure, in a sense and *mutatis mutandis*, is the same as Newton second law of Dynamics: no forces mean geodesic motion, while the effects of sources are to produce curvature (just in motion in the Newtonian case, where the Space and Time are separately fixed and immutable; both in the structure of SpaceTime and in its motions in Einstein’s case).

GR has been a success: it admits an elegant and very simple Lagrangian formulation (the Lagrangian is $L_H := R(g) \sqrt{g}$ ds and it was first found by Hilbert in 1915) and most of its predictions have soon been experimentally verified and these have remained valid for many years after its introduction. So there was no reason for Einstein to be unhappy with his beautiful creation, at least for some time.

In GR, is $g$ the gravitational field? Einstein knew that it is not, since $g$ is a tensor, while the principle of equivalence holds true and implies that there exist frames in which the gravitational field can be inertially switched off, while a tensor cannot be set to vanish at a point in a frame, if it does not vanish at that point in all frames. Free fall in GR is in fact described by the geodesics of $(M, g)$

$$\ddot{x}^\lambda + \{^\lambda_{\mu\nu}\}_g \dot{x}^\mu \dot{x}^\nu = 0.$$  

Einstein himself argued that the right objects to represent the gravitational field have to be the Christoffel symbols $\{^\lambda_{\mu\nu}\}_g$: the metric $g$ is just the potential of the gravitational field, but being the Christoffel symbols algorithmically constructed out of $g$, the metric remains the fundamental variable: $g$ gives rise to the gravitational field, to causality, to the principle of equivalence as well as to rods & clocks.

In 1919, working on the theory of “parallelism” in manifolds, Tullio Levi-Civita understood that parallelism and curvature are non-metric features of space, but rather features of “affine” type, having to do with “congruences of privileged lines” (see [4]). Generalizing the case of the Christoffel symbols $\{^\lambda_{\mu\nu}\}_g$ of a metric $g$, Levi-Civita introduced the notion of **linear connection** as the most general object $\Gamma^\lambda_{\mu\nu}$, such that the equation of geodesics

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

is generally covariant. This revolutionary idea (that stands in fact at the heart of Non-Euclidean Geometries) has been immediately captured by Einstein who, unfortunately, did not further use it up to its real strength. We shall come back later on this topic, as this work is strongly based on it.

For now, let us just reconsider GR in terms of the new concept introduced by Levi-Civita. In GR free-falls are described by equation (2). So, in GR, there is a connection $\Gamma$, but it is given from the very beginning as the Levi-Civita connection of the metric $g$: we write $\Gamma \equiv \Gamma_LC(g)$, i.e. locally $\Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\}_g$.

This has been in fact realized and used by Attilio Palatini in the same year to re-write the derivation of Einstein equations by using step-by-step only tensorial quantities depending on $g$ (see [5]). Let us notice that the relations $\Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\}_g$ are not “essential” equations: they express a founding assumption on SpaceTime structure. The connection $\Gamma_LC(g)$ has no independent dynamics. Only $g$ has dynamics and $\Gamma_LC(g)$ behaves accordingly. Thus, as we already said, the single object $g$ determines at the same time the causal structure (light cones), the measurements (rods & clocks) and the free fall of test particles (geodesics). SpaceTime is definitely a couple: $(M, g)$.

Even if it was clear to Einstein that Gravity induces “freely falling observers” and that the principle of equivalence selects, in fact, an object that cannot be a tensor, since it is capable of being “switched off” and set to vanish at least at a point, he was obliged to choose this object under the form of the linear connection $\Gamma_LC(g)$, fully determined by the metric structure itself. Einstein, for obvious reasons, was very satisfied of having reduced all SpaceTime structure and Gravity into a single geometrical object, namely the metric $g$.

Still, in 1919, Hermann Weyl tried (see [6]) to unify Gravity with Electromagnetism, using for the first time a linear connection defined over SpaceTime, assumed as a dynamical field non-trivially depending on a metric. Weyl’s idea, unfortunately, failed because of a wrong choice of the Lagrangian and few more issues, but it generated however a keypoint: connections may have a physically interesting dynamics.

Einstein soon showed a great interest in Weyl’s idea. He too began to play with connections, in the obsessed seek for the “geometrically” Unified Theory. But he never arrived to “dethronize” $g$ in the description of the gravitational field. Probably, at some moments, he was not so happy with the fact that
the gravitational field is not the fundamental object, but just a by-product of the metric; however, he never really changed his mind about the physical and mathematical role of \( g \).

In 1925 Einstein constructed a theory that depends on a metric \( g \) and a symmetric linear connection \( \Gamma \), to be varied independently (the so-called Palatini method, because of a misunderstanding with W. Pauli; see [5]); he defined a Lagrangian theory in which the gravitational Lagrangian is

\[
L_{PE} \equiv R(g, \Gamma) \sqrt{g} \, ds ,
\]

with

\[
R(g, \Gamma) \equiv g^{\mu\nu} R_{\mu\nu}(\Gamma, \partial \Gamma)
\]

the scalar curvature of both \( g \) and \( \Gamma \), being \( R_{\mu\nu}(\Gamma, \partial \Gamma) \) the Ricci tensor of \( \Gamma \).

There are now 10 + 40 independent variables and the field equations, in vacuum, are:

\[
\begin{align*}
R_{\mu\nu} - \frac{1}{2} R(g, \Gamma) g_{\mu\nu} &= 0 \\
\nabla_\alpha (\sqrt{g} g^{\mu\nu}) &= 0
\end{align*}
\]

(6)

where \( R_{(\mu\nu)} \) is the symmetric part of \( R_{\mu\nu}(\Gamma, \partial \Gamma) \) and \( \nabla \) denotes covariant derivative with respect to \( \Gamma \).

If the dimension \( m \) of SpaceTime is greater than 2, the second field equation (6)_2 constrains the connection \( \Gamma \), which is \textit{a priori} arbitrary, to coincide \textit{a posteriori} with the Levi-Civita connection of the metric \( g \) (Levi-Civita theorem). By substituting this information into the first field equation (6)_1, the vacuum Einstein equation for \( g \) is eventually obtained. In Palatini formalism, the metric \( g \) determines rods & clocks, while the connection \( \Gamma \) determines the free fall, but since \textit{a posteriori} the same result of GR is found, Einstein soon ceased to show a real interest in this formalism.

Let us now make a digression. The situation does not change if matter is present through a matter Lagrangian \( L_{\text{mat}} \) (independent of \( \Gamma \) but just depending on \( g \) and other external matter fields), that generates an energy-momentum tensor \( T_{\mu\nu} \) as

\[
T_{\mu\nu} \equiv \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}.
\]

If the total Lagrangian is then assumed to be \( L_{\text{tot}} \equiv L_{PE} + L_{\text{mat}} \) field equation (6)_1 are replaced by

\[
R_{(\mu\nu)} - \frac{1}{2} R(g, \Gamma) g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

and again (6)_2 implies, \textit{a posteriori}, that (7) reduces eventually to Einstein equations (1).

Let us also emphasize that the dynamical coincidence between \( \Gamma \) and the Levi-Civita connection of \( g \) is entirely due to the particular Lagrangian considered by Einstein, which is the simplest, but not the only possible one! Furthermore, it seems to us that Einstein did not fully recognize that the Palatini method privileges the affine structure with respect to the metric structure. In fact, the Einstein-Palatini Lagrangian is of order zero in the metric, while it is first order in the connection; in other words, it contains (first order) derivatives of \( \Gamma \) but no derivatives of \( g \). The connection is the real dynamical field while the metric \( g \) has no dynamics, since it enters the Lagrangian just as a “Lagrange multiplier.” This time is the metric \( g \) to gain a dynamical meaning from that of \( \Gamma \), that plays the role of fundamental field.

Notice that, in this case (\textit{i.e.} in Palatini formalism), the relations

\[
\Gamma^\lambda_{\mu\nu} = \{ \lambda^{\lambda}_{\mu\nu} \}_g
\]

are field equations: the fact that \( \Gamma \) is the Levi-Civita connection of \( g \) is no longer an assumption \textit{a priori} but it is the outcome of field equations!

3. Among the different Theories of Gravitation, should we really prefer the simplest (in the sense of the one with the simplest Lagrangian)?
Let us shortly criticize the final points that we stressed at the end of Section 2. We are now in a situation that was already seen elsewhere in Mathematics and Physics, from which we should try to learn something. We refer to the onset of Non-Euclidean Geometry. Let us limit for simplicity ourselves to discuss plane (2-dimensional) Geometry. Notice first that Euclidean Geometry is based on two fundamental and a priori distinct structures: metricity and linearity (or, better, affinity). In modern language we say that the Euclidean plane carries the (a priori independent) structures of vector (or, better, affine) space and of metric space. One depends on the linear group (or its affine extension) while the other on the (subgroup) of orthogonal transformations.

In a plane, metricity selects the circles (i.e. the level sets of the distance function) while affinity the congruence of all straight lines and their parallelism properties. This corresponds to the well-known “compass & unmarked straightedge Geometry.”

One of the by-products of the aforementioned works of Levi-Civita, Weyl and Einstein is the demonstration that these two structures are in fact separated. One is not obtained directly from the other.

Galilean and Newtonian Physics are in fact strongly based on the assumption that Space carries a Euclidean structure. A famous wording of Galilei (“Il Saggiatore”) tells that: “to understand the Universe we have to know the language in which it is written and its characters. The language is Mathematics, while the characters are circles, triangles and other geometrical figures.” Galilei states that Space has two structures: the metric one (circles) and the linear one (triangles).

From a purely geometrical viewpoint, these two structures stand on an equal footing, but from a physical point of view the situation is however different: the principle of Inertia selects, in fact, the straight lines as the more fundamental structure, while circles limit their role to the definition of space distances! The whole Euclidean Geometry is a subtle mix-up of both structures, but the fundamental principle of Physics (the First Law, i.e. the principle of Inertia) privileges straight lines, i.e. (uniform) rectilinear motion, when forces are not present. Forces (and metricity) will enter the game only through the Second Law (i.e. Newton law \( \mathbf{F} = m \mathbf{a} \), where \( \mathbf{a} \) contains curvature, hence metric relations, and \( \mathbf{F} \) is the source that generates \( \mathbf{a} \) as a deviation from uniform and rectilinear motion).

These two (a priori distinct) structures may nevertheless be intertwined by the simplest variational principle: “straight lines are the shortest path between any two points.” This variational principle has two advantages and one disadvantage. Advantages: (1) it connects the two fundamental (a priori distinct) structures of Euclidean Geometry and of Newtonian Physics; (2) it is the simplest. Disadvantage: our world is not Euclidean! (Physics clearly shows that).

In Geometry, we are therefore forced by Nature to consider variational principles different from the simplest and to introduce metric geometries that differ from the simple Euclidean one. This is exactly the way in which Non-Euclidean Geometry arises (Gauss, 1836; Riemann, 1856). The metric defines a distance function: circles are replaced by the level sets of it; straight-lines are replaced by “geodesics,” i.e. lines of minimal length. The Euclidean plane is just an example where geodesics are straight lines and circles.

Why should be so strange that the same happens in Gravity?

Moreover, in the light of Levi-Civita’s work, we should also notice that this variational assumption can be replaced by a different one. In a space endowed with a (symmetric) linear connection \( \Gamma \), geodesics are no longer defined by a variational prescription, but are defined as self-parallel lines (i.e., their tangent vector is parallel-transported by the connection). In Euclidean Space, where a metric is given first, we could envisage the following situation, that summarizes the content of the already evoked Levi-Civita theorem: suppose a symmetric connection is given in it: claim that its self-parallel curves are the straight lines, i.e. those lines that minimize distance; then the connection is forced, a posteriori, to be the Levi-Civita connection of the Euclidean metric. The same will happen in any Riemannian manifold \( (M,g) \): if we give a symmetric \( \Gamma \) on it and pretend that the self-parallel curves of \( \Gamma \) minimize the Riemannian length, then \( \Gamma \) is forced to be the metric connection \( LCO(g) \). In a sense, there is a variational prescription that relates the apparently unrelated structures implied by \( g \) and \( \Gamma \) independently. In a sense, this variational prescription is the “Palatini prescription” based on the Lagrangian \( R(g, \Gamma) := g^{\mu \nu} R_{\mu \nu}(\Gamma, \partial \Gamma) \).
4. So then?

All that said, we believe we should first seriously reconsider NL TGs, without being unsensitive with respect to the appeal of simplicity, in the spirit of Occam Razor. This is why we begin to restrict ourselves to the first level of generalization of GR, the so-called $f(R)$-theories of metric type (see e.g. [7] for a review of the results concerning these theories). Here $f$ denotes any “reasonable” function of one-real variable. The Lagrangian is assumed to be

$$L_{NL}(g) := f(R(g)) \, \sqrt{g} \, ds$$

(9)

Of course, from $f(R)$-theories, we know that GR is retrieved in, and only in, the particular case $f(R) \equiv R$, i.e. if and only if the Lagrangian is linear in $R$.

Let us recall here just a few keypoints on metric $f(R)$-theories.

When treated in the purely metric formalism, these theories are mathematically much more complicated than GR. These theories do in fact produce field equations that are of the fourth order in the metric:

$$f'(R(g)) \, R_{\mu\nu}(g) - \frac{1}{2} \, f(R(g)) \, g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}) \, f'(R(g)) = 8\pi \, G \, T_{\mu\nu}$$

(10)

where $f'$ denotes the derivative of $f$ with respect to its real argument. This is something that cannot be accepted if one believes that physical laws should be governed by second order equations. In (10) we see a second order part that resembles Einstein tensor (and reduces identically to it if and only if $f(R) \equiv R$, i.e. if and only if $f'(R) \equiv 1$) and a fourth order “curvature term” (that again reduces to zero if and only if $f(R) \equiv R$).

A first workaround that was suggested long ago to this problem is to push the 4th order part $(\nabla_\mu \nabla_\nu - g_{\mu\nu}) \, f'(R(g))$ to the r.h.s. This lets us to interpret it as an “extra gravitational stress” $T^{\text{curv}}_{\mu\nu}$ due to higher-order curvature effects, much in the spirit of Riemann. In any case, however, the fourth order character of these equations makes them unsuitable under several aspects, so that they were eventually abandoned for long time and only recently they have regained interest (see [7] and references therein).

A second way to tackle the problem has been proposed in 1987 (see [8], based on earlier work by the same authors [9], together with the references quoted therein). Notice that these are the first papers where the Legendre transformation that introduces an extra scalar field has been ever considered in literature (it has been later “re-discovered” by other authors), so that its priority should always be appropriately quoted when dealing with “metric” $f(R)$-theories. This is a method à la Hamilton, in which, whenever one has a non-linear gravitational metric Lagrangian of the most general type $L_{GNL}(g) := f(g, \text{Ric}(g))$, one defines a second metric $p$ as

$$p_{\mu\nu} := \frac{\partial L_{\text{grav}}}{\partial R_{\mu\nu}}.$$  

(11)

1 This idea corresponds to an Einstein’s attempt, dating back to 1925, to construct a “purely affine” theory (see [10]), i.e. a theory in which the only dynamical field is a linear connection. In this theory no metric is given from the beginning, but since it is obviously necessary to have a metric, the problem arises of how to construct it out of a connection. Einstein firstly tried to define the metric as the symmetric part of the Ricci tensor constructed out of the connection. But this idea could not work (unless for quadratic Lagrangians). A. Eddington then proposed a recipe analogous to (11). In this way Einstein and Eddington obtained a theory that reproduces GR, without introducing anything new. That is why Einstein eventually abandoned it too. About these purely affine theories see also [11] where J. Kijowski correctly pointed out that in the purely affine framework the prescription (11) of Einstein and Eddington is nothing but the assumption that the metric can be considered as a momentum canonically conjugated to the connection.
In this way the second metric \( p \), a canonically conjugated momentum for \( g \), is a function of \( g \) together with its first and second derivatives, since it is a function of \( g \) and \( \text{Ric}(g) \), the Ricci tensor of \( g \). Notice that this leads to two equations of the second order in \( g \) and \( p \), as Hamilton method always halves the order of the equations by doubling the variables. Following this method in the simpler \( f(R) \) case one gets that the “auxiliary” metric \( p \) is related to the original one \( g \) just by a conformal transformation:

\[
p \equiv \phi \, g , \quad \phi \equiv f'(R(g)) .
\]

The Lagrangian equations (10) are then rewritable as a Hamiltonian system:

\[
\begin{align*}
\{ \text{Ein}(p) = T_{\text{mat}} + T_{\text{KGl}} \\
\text{KGl}(\phi) = 0
\end{align*}
\]

where KGl means non-linear Klein-Gordon (because of a potential depending on \( f \); see [7], [8], [9] for details). Rewritten in this form, the theory has now two variables: the “auxiliary” metric \( p \) (or, equivalently, the original one \( g \)) and the scalar field \( \phi \). This is why these theories belong to a wide sector of theories that are called Scalar-Tensor Theories. For more details, and in particular for their application in Cosmology and Extragalactic Astrophysics see, e.g., [7] and the references quoted therein. See also [12], where equivalence properties between NLGTs and STTs are discussed in detail.

Notice that [8], [9] and all subsequent literature left in fact open a few fundamental problems: Who really are the second metric \( p \) and the scalar field \( \phi \)? How to interpret them (the scalar field \( \phi \) survives even in vacuum)? And... what about the original metric \( g \)?

Fortunately there is a third method to solve the problem.

5. Palatini formalism revisited and the Dark Metric

The third method anticipated at the end of the previous Section is the Palatini method applied to the case of \( f(R) \)-theories. Now SpaceTime is no longer a couple \((M, g)\) but rather a triple \((M, g, \Gamma)\), with \( \Gamma \) symmetric. The Lagrangian is assumed to be the non-linear Palatini-Einstein Lagrangian

\[
L_{\text{NLPE}}(g, \Gamma) := f(R(g, \Gamma)) \sqrt{g} \, \text{ds}
\]

with \( R(g, \Gamma) := g^\mu\nu \, R_{\mu\nu}(\Gamma, \partial \Gamma) \) and \( f \) “reasonable.” Field equations (6) are now replaced by the following:

\[
\begin{align*}
\left\{ f'(R(g, \Gamma)) \, R(\mu\nu) - \frac{1}{2} f(R(g, \Gamma)) \, g_{\mu\nu} &= 8 \pi \, G \, T_{\mu\nu} \\
\nabla_\alpha (f'(R(g, \Gamma)) \sqrt{g} \, g^{\mu\nu}) &= 0
\end{align*}
\]

that take into account a possible Lagrangian of the type \( L_{\text{mat}} \equiv L_{\text{mat}}(g, \psi) \), with \( \psi \) arbitrary fields coupled to \( g \) alone (and not to \( \Gamma \)). Notice that (15)_1 reduces to (7) if and only if \( f(R) \equiv R \). Notice also that the trace of equation (15)_1 gives

\[
R(g, \Gamma) \, f'(R(g, \Gamma)) - \frac{m}{2} f(R(g, \Gamma)) = 8 \pi \, G \, \tau
\]

being \( \tau \equiv g^{\mu\nu} \, T_{\mu\nu} \) the “trace” of the energy-momentum tensor. This equation was called the master equation in [13] and was there at the basis of a subtle discussion about “universality” of Einstein equations in non-linear special cases (i.e., when \( \tau \equiv 0 \)). Notice also the analogy of (16) with the trace of (10), i.e.

\[
f'(R(g)) \, R(g) - \frac{m}{2} f(R(g)) + (m - 1) \Box f'(R(g)) = 8 \pi \, G \, \tau
\]
and notice that only in the peculiar case $f(R) \equiv R$ they reduce to the same equation, namely (7). In all other cases (17) entails that non-linearity ($f' \neq 1$) produces, in the metric formalism, effects due to the scalar factor $f'(R)$, i.e. depending eventually on a scalar field tuned up by curvature.

Approaching $f(R)$-theories à la Palatini, we may now follow [13] step-by-step and make a number of considerations (well summarized also in the recent critical review [14]). At the end of these considerations we may conclude that:

(1) When (and only when) $f(R(g, \Gamma)) \equiv R(g, \Gamma)$ we “fully” recover GR for the given metric $g$.

(2) For a generic $f(R(g, \Gamma))$ and in presence of matter such that $\tau \equiv g'^{\mu\nu} T_{\mu\nu} \equiv 0$ (and thence, in particular, in vacuum), the theory is still equivalent to GR for the given metric $g$ with a “quantized” cosmological constant $\Lambda$ and a modified coupling constant. In this case, in fact, the master equation (16) implies that the scalar curvature $R(g, \Gamma)$ has to be a suitable constant $\alpha$, possibly and usually not unique but always chosen in a set that depends on $f$, so that (15)$^2$ still implies (8) with an additional cosmological term $\Lambda(\alpha) g_{\mu\nu}$.

(3) For a generic $f(R(g, \Gamma))$ but in presence of matter such that $\tau \equiv g'^{\mu\nu} T_{\mu\nu} \neq 0$ one can (implicitly) solve the master equation (16), for $m \neq 2$, and obtain $R(g, \Gamma)$ as a function $R(\tau)$ of the given trace $\tau$. Then, knowing $f$, one gets implicitly $f(R(g, \Gamma)) = f(\tau)$ and $f'(R(g, \Gamma)) = f'(\tau)$, so that equation (15)$^2$ tells us that $\Gamma$ is forced to be the Levi-Civita connection $I_{LC}(h)$ of a new metric $h$, conformally related to the original one $g$ by the relation

$$h_{\mu\nu} \equiv f'(\tau) g_{\mu\nu} \equiv f'(R(g, \Gamma)) g_{\mu\nu} .$$

Then, using again equation (15)$^1$, we see that the theory could be still rewritable as in (12) in a purely metric setting, but with far less interpretative problems, as we can immediately show. But, from a viewpoint “à la Palatini” in a genuine sense the method has in fact generated a completely new perspective. The remaining field equations (15)$^1$, in fact, are still equivalent to Einstein equations with matter (and cosmological constant) provided one changes the metric from $g$ to $h$!

What conclusions may be drawn?

The most enlightening case is that of $f(R)$ with generic matter and $\tau \neq 0$. Here, in fact, the universality property (see again [13]) does not hold in its strict form, but in an interesting wider interpretation: the dynamics of the connection $\Gamma$ still forces $\Gamma$ itself to be the Levi-Civita connection of a metric, but not of the “original” metric $g$, which we shall prefer to call the apparent metric for a reason we clarify in a moment. Instead, the dynamics of $\Gamma$ identifies a new metric $h$, conformally related to the apparent one $g$, which we call the Dark Metric. The Dark Metric $h$, we claim, is the true origin of the “Dark Side of the Universe”!

The apparent metric $g$ is in fact the one by means of which we perform measurements in our local laboratories. In other words, the metric $g$ is the one we have to use every day to construct and read instruments (rods & clocks). This is why we like to call it the “apparent” metric. But we claim that the right metric we have to use as the fundamental object to describe Gravity is, by obvious reasons, the Dark Metric, since it is the one responsible for gravitational free fall through the identification $\Gamma \equiv I_{LC}(h)$. Notice, incidentally, that photon world-lines and causality are not changed, since the light-cones structure of $g$ and $h$ are the same by conformal invariance.

In other words, in our laboratories we have to use the apparent metric $g$, but in our Gravity theories the dark one $h$. The translation from one “language” to the other is nothing but the conformal factor $f'(R)$, which manifestly depends on the theory and on the content in ordinary matter. Let us also notice explicitly that this in particular implies that if a certain metric $h$ is expected to be a solution of a problem, from a theoretical point of view, it is rather important to look for $h$ in experiments. Testing our theories with $g$, in a sense, is wrong, since it is the conformally related metric $h$ to be searched for instead!

6. Subtle is the Lord…and malicious He is too!

Quoting [15] we can say that “subtle is the Lord,” since the apparent metric $g$ is not the right one in terms of which we can describe the gravitational field. But differently from [15] we eventually conclude that
“malicious He is too,” because the Dark Metric is... hidden, until one considers theories more complicated than GR, namely \( f(R) \)-theories. The linear Hilbert Lagrangian, in (fact), hides completely the Dark Metric (thence the name we gave it), since only in this specific and rather peculiar case (both with and without matter) the Dark Metric \( h \) coincides with the apparent one \( g \). In all other cases \( h \neq g \) and the (unknown) conformal factor \( \phi \equiv f'(R,g,\Gamma) \equiv f'(\tau) \) has to be phenomenologically tested against observational data in order to find which is the (class of) Lagrangian(s) \( f \) that, given \( \tau \) (i.e., given the “visible matter”), allow one to interpret the “supposed Dark Matter” (and Energy) as a curvature effect \([16]\) due to the curvature of \( \Gamma \) (namely, Gravity) which, in turn, pointwise changes rods & clocks according to the conformal curvature-dependent (and matter-dependent) rescaling of Eq. (18). In forthcoming studies, we will give hints on how DE (accelerated cosmic behaviour) and DM (clustered structures) could be interpreted as Dark Metric effects. We shall also address the intriguing problem of the relations that certainly exist between our geometrical approach to the conformal factor \( \phi = f'(R) \) and the so-called AWE hypotheses of Alimi and coworkers (see, e.g., \([17]\) and references quoted therein).

As a final remark, we would also like to stress that changing the metric from a given one, needed to define measurements in the laboratory, to a different one which seems to be more fundamental for the description of Physics, as it might be suggested by the theoretical structure or by phenomenology, is not at all new in Physics. Recall in fact that in Newtonian Physics, the metric of Space is rigidly assumed to be the Euclidean one \( e \); nevertheless, when dealing with (holonomically) constrained systems of particles, this metric and the kinetic energy \( T \) induce together another metric \( g \) in the appropriate configuration space, whereby dynamics is eventually governed by a metric Lagrangian of the form \( T - U \). Analogously when dealing with continua (with or without internal structure), where the Euclidean metric induces the deformation metric of the body. In all these cases Physical properties of the system are governed by another metric, which encompasses at the same time the Euclidean one (i.e., the one that is used to perform measurements in the laboratory) and the realm of physical phenomena. We claim that here something analogous happens, since the dark metric contains both the given local metric \( g \) as well as the gravitational pull of curvature as induced by the conformal factor that, in turn, is reminiscent of the linear connection.

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