ON A GAUSS-KUZMIN TYPE PROBLEM FOR A FAMILY OF CONTINUED FRACTIONS

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Abstract

We study a family of continued fraction expansion of reals from the unit interval. The Perron-Frobenius operator of the transformation which generates this expansion under the invariant measure of this transformation is given. Using the ergodic behavior of homogeneous random system with complete connections associated with this expansion we solve a variant of Gauss-Kuzmin problem for this continued fraction expansion.

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1 Introduction

In this section we describe a family of continued fraction expansions different from the regular continued fraction expansion for a number \( x \) in the unit interval \( I = [0, 1] \) which has been actually considered in [3]. We give metric properties of this continued fraction expansion and we show formulae of probability about incomplete quotients.

Thus, Chan shows that any \( x \in I := [0, 1) \) can be written in the form

\[
x = \frac{m^{-a_1(x)}}{1 + \frac{(m-1)m^{-a_2(x)}}{1 + \frac{(m-1)m^{-a_3(x)}}{1 + \ldots}} := [[a_1(x), a_2(x), a_3(x), \ldots]] \quad (1.1)
\]

where \( m \in \mathbb{N}, m \geq 2 \) and \( a_n(x) \) are natural numbers, for any \( n \in \mathbb{N}_+ \), where \( \mathbb{N} := \{0, 1, 2, \ldots\} \) and \( \mathbb{N}_+ := \{1, 2, \ldots\} \).

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The naturals $a_n$ are called the incomplete quotients of $x$ and they are given by the relations

$$a_1(x) = \left\lfloor \frac{\log x^{-1}}{\log m} \right\rfloor, \quad x \neq 0, \quad a_1(0) = \infty,$$

(1.2)

$$a_n(x) = a_1\left(T_m^{n-1}(x)\right), \quad n \in \mathbb{N}_+, \quad n \geq 2.$$  

(1.3)

Let $\Omega$ be the set of all irrational numbers from $I$. In the case when $x \in I \setminus \Omega$, we have

$$a_n(x) = \infty, \forall n \geq k(x) \geq m, \quad \text{and} \quad a_n(x) \in \mathbb{N}, \forall n < k(x).$$

(1.4)

Therefore, in the rational case, the continued fraction expansion (1.1) is finite, unlike the irrational case, when we have an infinite number of natural incomplete quotients.

Next, for any $m \in \mathbb{N}$ with $m \geq 2$, we define on $I$ the shift transformation $T_m$ as follows:

**Definition 1.1** For $x \neq 0$,

$$T_m(x) := T_m([[a_1(x), a_2(x), \ldots]]) = [[a_2(x), a_3(x), \ldots]],$$

(1.5)

and for $x = 0$, $T_m(0) = 0$.

From (1.1) and (1.5), for $x \neq 0$ and $m \geq 2$, we have

$$x = \frac{m^{-a_1(x)}}{1 + (m - 1)T_m(x)}.$$  

(1.6)

Consequently, using (1.6) we can write the transformation $T_m$ of $I$ as

$$T_m(x) := \frac{\log x^{-1} - \left\lfloor \frac{\log x^{-1}}{\log m} \right\rfloor - 1}{m - 1}, \quad x \neq 0,$$

(1.7)

where $\lfloor \cdot \rfloor$ denotes the floor (entire) function.

It follows from the definitions of $T_m$ and $a_n$ that for any $\omega \in \Omega$ we have

$$T_m^{n-1}(\omega) = \frac{m^{-a_n(\omega)}}{1 + (m - 1)T_m^n(\omega)}, \quad n \in \mathbb{N}_+,$$

(1.8)

hence

$$\omega = \frac{m^{-a_1(\omega)}}{1 + (m - 1)m^{-a_2(\omega)}} \cdot \left(1 + \frac{(m - 1)m^{-a_3(\omega)}}{1 + T_m(\omega)}\right) \cdots$$

(1.9)

If we set

$$[[a_1]] = m^{-a_1}, \quad [[a_1, a_2]] = \frac{m^{-a_1}}{1 + (m - 1)m^{-a_2}},$$

(1.10)
Using (1.16) we obtain

\[ \frac{m^{-a_1}}{1 + (m-1)[a_2, a_3, \ldots, a_n]}, \quad n \geq 3, \tag{1.11} \]

then (1.9) can be written as

\[
\omega = \left[ a_1 + \frac{\log(1 + (m-1)T_m(\omega))}{\log m}, \right. \\
\omega = \left[ a_1, a_2 + \frac{\log(1 + (m-1)T^2_m(\omega))}{\log m}, \right. \\
\omega = \left[ a_1, a_2, \ldots, a_{n-1}, a_n + \frac{\log(1 + (m-1)T^n_m(\omega))}{\log m} \right]. \tag{1.12}
\]

**Definition 1.2** A finite truncation in (1.1), i.e.

\[
\omega_n(\omega) := \frac{p_n(\omega)}{q_n(\omega)} = [[a_1(\omega), a_2(\omega), \ldots, a_n(\omega)]], \quad n \in \mathbb{N}_+ \tag{1.13}
\]

is called the \(n\)-th convergent of \(\omega\).

The integer valued functions sequences \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) can be recursively defined by the formulae:

\[
p_n(\omega) = m^{a_n(\omega)}p_{n-1}(\omega) + (m-1)m^{a_{n-1}(\omega)}p_{n-2}(\omega), \quad n \geq 2, \tag{1.14}
\]

\[
q_n(\omega) = m^{a_n(\omega)}q_{n-1}(\omega) + (m-1)m^{a_{n-1}(\omega)}q_{n-2}(\omega), \quad n \geq 1, \tag{1.15}
\]

with \(p_0(\omega) = 0, q_0(\omega) = 1, p_1(\omega) = 1\) and \(a_0(\omega) = 0, q_1(\omega) = 0\).

By induction, it is easy to prove that, for any \(n \in \mathbb{N}_+\), we have

\[
p_n(\omega)q_{n-1}(\omega) - p_{n-1}(\omega)q_n(\omega) = (-1)^{n-1}(m-1)^{n-1}m^{a_1(\omega) + \ldots + a_{n-1}(\omega)}, \tag{1.16}
\]

and that

\[
\frac{m^{-a_1(\omega)}}{1 + \frac{(m-1)m^{-a_2(\omega)}}{1 + \frac{(m-1)m^{-a_3(\omega)}}{\ddots + \frac{(m-1)m^{-a_t(\omega)}}{1 + (m-1)t)}}} = \frac{p_n(\omega) + (m-1)tm^{a_n(\omega)}p_{n-1}(\omega)}{q_n(\omega) + (m-1)tm^{a_n(\omega)}q_{n-1}(\omega)}, \tag{1.17}
\]

with \(t \geq 0\). Now, (1.12) and (1.17) imply that

\[
\omega = \frac{p_n(\omega) + (m-1)T^n_m(\omega)m^{a_1(\omega) + \ldots + a_{n-1}(\omega)}}{q_n(\omega) + (m-1)T^n_m(\omega)m^{a_n(\omega)}q_{n-1}(\omega)}, \quad \omega \in \Omega, \quad n \in \mathbb{N}_+. \tag{1.18}
\]

Using (1.16) we obtain

\[
|\omega - \omega_n(\omega)| = \frac{(m-1)^nT^n_m(\omega)m^{a_1(\omega) + \ldots + a_n(\omega)}}{q_n(\omega)(q_n(\omega) + (m-1)T^n_m(\omega)m^{a_n(\omega)}q_{n-1}(\omega))}. \tag{1.19}
\]

At this moment, we are able to assert that this continued fraction expansion is convergent, i.e.

\[
\omega = \lim_{n \to \infty} [[a_1(\omega), a_2(\omega), \ldots, a_n(\omega)]], \quad \omega \in \Omega. \tag{1.20}
\]
**Definition 1.3** Let \( i^{(n)} = (i_1, \ldots, i_n), i_n \in \mathbb{N}, n \in \mathbb{N} \). We will say that

\[
I \left( i^{(n)} \right) = \{ \omega \in \Omega : a_k(\omega) = i_k, 1 \leq k \leq n \} \tag{1.21}
\]

is the fundamental interval of rank \( n \). We make the convention that \( I \left( i^{(0)} \right) = \Omega \).

For example, for any \( i \in \mathbb{N} \) we have

\[
I(i) = \{ \omega \in \Omega : a_1(\omega) = i \} = \Omega \cap \left( \frac{1}{m^{i+1}}, \frac{1}{mi} \right). \tag{1.22}
\]

We will write \( I(a_1, \ldots, a_n) = I(a^{(n)}) \), \( n \in \mathbb{N}_+ \). If \( n \geq 2 \) and \( i_n \in \mathbb{N} \), then we have

\[
I(a_1(\omega), \ldots, a_n(\omega)) = I \left( i^{(n)} \right). \tag{1.23}
\]

By (1.18) we have

\[
I(a^{(n)}) = \Omega \cap \left( u \left( a^{(n)} \right), v \left( a^{(n)} \right) \right) \tag{1.23}
\]

where

\[
u \left( a^{(n)} \right) = \begin{cases} \frac{p_n + (m-1)m^{a_n}p_{n-1}}{q_n + (m-1)m^{a_n}q_{n-1}}, & \text{if } n \text{ is odd} \\ \frac{p_n}{q_n}, & \text{if } n \text{ is even} \end{cases} \tag{1.24}
\]

and

\[
v \left( a^{(n)} \right) = \begin{cases} \frac{p_n}{q_n + (m-1)m^{a_n}q_{n-1}}, & \text{if } n \text{ is odd} \\ \frac{p_n + (m-1)m^{a_n}p_{n-1}}{q_n + (m-1)m^{a_n}q_{n-1}}, & \text{if } n \text{ is even} \end{cases} \tag{1.25}
\]

Also, we have

\[
p_n + (m-1)m^{a_n}p_{n-1} \quad q_n + (m-1)m^{a_n}q_{n-1} = \begin{cases} \left[ a_1 + 1 \right], & \text{if } n = 1 \\ \left[ a_1, \ldots, a_{n-1}, a_n + 1 \right], & \text{if } n \geq 2 \end{cases} \tag{1.26}
\]

and

\[
\lambda \left( I \left( a^{(n)} \right) \right) = \frac{(m-1)^n m^{a_1 + \cdots + a_n}}{q_n (q_n + (m-1)m^{a_n}q_{n-1})}, \tag{1.27}
\]

where \( \lambda \) denotes the Lebesgue measure.

Now, we define the random variables \((s_n)_{n \in \mathbb{N}_+}\) recursively by

\[
s_n = m^{-a_n} \frac{q_n}{q_{n-1}} - 1, \quad s_1 = 0. \tag{1.28}
\]

Next, (1.15) implies that

\[
s_n = \frac{(m-1)m^{-a_n}}{1 + s_{n-1}}, \quad n \geq 2, \tag{1.29}
\]

hence

\[
s_n = \frac{(m-1)m^{-a_n}}{1 + \frac{(m-1)m^{-a_{n-1}}}{1 + \frac{(m-1)m^{-a_{n-2}}}{\cdots + \frac{(m-1)m^{-a_2(x)}}{1 + (m-1)m^{-a_2(x)}}}}} = (m-1)[[a_n, a_{n-1}, \ldots, a_2, \infty]], \tag{1.30}
\]
for \( n \geq 2 \).

The probabilistic structure of the sequence \((a_n)_{n \in \mathbb{N}_+}\) under \( \lambda \) is described by the equations

\[
\lambda(a_1 = i) = \frac{m - 1}{m^{i+1}}, i \in \mathbb{N}
\]

and

\[
\lambda(a_{n+1} = i|a_1, \ldots, a_n) = P_i(s_n), i \in \mathbb{N}, n \in \mathbb{N}_+,
\]

where

\[
P_i(x) = \frac{(m - 1)m^{-(i+1)}(x + 1)(x + m)}{(x + (m - 1)m^{i+1} + 1)(x + (m - 1)m^{-i+1} + 1)}, i \in \mathbb{N}.
\]

These relations result as consequences of a Brodén-Borel-Lévy formula type, namely

\[
\lambda(T^n_m < x|a_1, \ldots, a_n) = \frac{(s_n + m)x}{(s_n + (m - 1)x + 1)}, x \in I, n \in \mathbb{N}_+
\]

where \( s_n \) is defined by (1.29).

**Remark 1.4** The sequence \((s_n)_{n \in \mathbb{N}_+}\) with \( s_1 = 0 \) is an \( I \)-valued Markov chain [8] on \((I, \mathcal{B}_I, \lambda)\) with the following transition mechanism: from state \( s \in I \setminus \Omega \) the possible transitions are to any state \( m^{-i}/(1 + (m - 1)s) \) with corresponding transition probability \( P_i(s), i \in \mathbb{N} \).

## 2 Measure preserving transformation

Let \( \mathcal{B}_I \) denote the \( \sigma \)-algebra of Borel subsets of \( I \). The metric point of view in studying the sequences \((a_n)_{n \in \mathbb{N}_+}\) is to consider that the \( a_n, n \in \mathbb{N}_+ \), are non-negative integer-valued random variables which are defined almost surely on \((I, \mathcal{B}_I, \lambda)\) with respect to any probability measure on \( \mathcal{B}_I \) that assign probability 0 to the set \( I \setminus \Omega \) of rationals in \( I \). Such a measure is Lebesgue measure \( \lambda \).

Another measure on \( \mathcal{B}_I \) more important than Lebesgue measure, that assign probability 0 to the set of rationals in \( I \), is the invariant probability measure \( \gamma_m \) of the shift transformation \( T_m \) defined by:

\[
\gamma_m(A) = c_m \int_A \frac{dx}{(1 + (m - 1)x)(m + (m - 1)x)}, A \in \mathcal{B}_I,
\]

where \( c_m \) is the normalization constant chosen so that \( \gamma_m([0, 1]) = 1 \), i.e. \( c_m = \frac{(m-1)^2}{\log(m^2/(2m-1))} \). It can be easily checked that \( \gamma_m(A) = \gamma_m(T_m^{-1}(A)), A \in \mathcal{B}_I \).

Let us consider now what can be called the natural extension \( \overline{T}_m \) of \( T_m \), namely, the transformation of \([0, 1] \times I \) defined by

\[
\overline{T}_m(x, y) = \left(T_m(x), \frac{m^{-a_1(x)}}{1+(m-1)y}\right), (x, y) \in [0, 1] \times I.
\]
This is a one-to-one transformation of $\Omega^2$ (the irrationals from $I^2$) with the inverse
\[
T_m^{-1}(\omega, \theta) = \left( \frac{m^{-a_1(\theta)}}{1 + (m-1)\omega}, T_m(\theta) \right), (\omega, \theta) \in \Omega^2. \tag{2.3}
\]
It is easy to check that for $n \geq 2$ we have
\[
T_m^n(\omega, \theta) = \left( T_m^m(\omega), \left[ a_n(\omega), a_{n-1}(\omega), \ldots, a_2(\omega), a_1(\omega) + \frac{\log(1 + (m-1)\omega)}{\log m} \right] \right), \tag{2.4}
\]
and
\[
T_m^{-n}(\omega, \theta) = \left( \left[ a_n(\theta), a_{n-1}(\theta), \ldots, a_2(\theta), a_1(\theta) + \frac{\log(1 + (m-1)\omega)}{\log m} \right], T_m^{-m}(\theta) \right). \tag{2.5}
\]
Now, define the extended measure $\gamma_m$ on $B_2^I$ as
\[
\gamma_m(B) = c_m \int_B dx dy \frac{dx dy}{(1 + (m-1)(x+y))^2}, B \in B_2^I. \tag{2.6}
\]
A simple calculus show us that
\[
\gamma_m(A \times I) = \gamma_m(I \times A) = \gamma_m(A), A \in B_2^I. \tag{2.7}
\]
The result below shows that $\gamma_m$ plays with respect to $T_m$ the part played by $\gamma_m$ with respect to $T_m$.

**Proposition 2.1** The extended measure $\gamma_m$ is preserved by $T_m$.

**Proof.** We should show that $\gamma_m(T_m^{-1}(B)) = \gamma_m(B)$ for any $B \in B_2^I$ or, equivalently, since $T_m$ is invertible on $\Omega^2$, that
\[
\gamma_m(T_m(B)) = \gamma_m(B) \text{ for any } B \in B_2^I. \tag{2.8}
\]
We start with $B = (a, b) \times (c, d)$, where
\[
a = m^{-(i-1)}, \quad b = m^{-i}, \quad i \in \mathbb{N}
\]
and $c$ and $d$ arbitrary numbers from $(0, 1)$. Then
\[
T_m(B) = \left( \left( T_m(x), \frac{m^{-a_1(x)}}{1 + (m-1)y} \right) | x \in (a, b), y \in (c, d) \right). \tag{2.9}
\]
Taking $x = m^{-(i+\theta)}$, $0 < \theta < 1$, we have
\[
T_m(x) = \frac{m^\theta - 1}{m - 1}, \quad a_1(x) = i
\]
such that
\[
T_m(B) = \left( (0, 1), \left( \frac{m^{-i}}{1 + (m-1)c}, \frac{m^{-i}}{1 + (m-1)c} \right) \right). \tag{2.10}
\]
A simple computation yields

$$\gamma_m(T_m(B)) = \int_0^1 dx \int_{\frac{m^{-i}}{1+(m-1)c}}^{\frac{m^{-i}}{1+1/m}} \frac{dy}{(1+(m-1)(x+y))^2}$$

$$= \int_{m^{-i}(c+1)}^{m^{-i}} dx \int_c^d \frac{dy}{(1+(m-1)(x+y))^2} = \gamma_m(B)$$

that is, (2.8) holds.

Next, we consider the case

$$a = \frac{m^{-i}}{1+(m-1)m^{-j}}, \quad b = \frac{m^{-i}}{1+(m-1)m^{-(j+1)}}, \quad i, j \in \mathbb{N}$$

and $c, d$ an arbitrary interval. Now, with

$$x = \frac{m^{-i}}{1+(m-1)m^{-j+\theta}},$$

we have

$$\left\{ \frac{x^{-1}}{\log m} \right\} = \left\{ i + \frac{\log (1+(m-1)m^{-(j+\theta)})}{\log m} \right\} = \frac{\log (1+(m-1)m^{-(j+\theta)})}{\log m}$$

and

$$a_1(x) = \left\lfloor \frac{x^{-1}}{\log m} \right\rfloor = i.$$

Thus,

$$(m-1)T_m(x) = m^{\log \left(1+(m-1)m^{-(j+\theta)}\right)} - 1 = (m-1)m^{-(j+\theta)}.$$

Hence,

$$T_m(B) = \left(\left(m^{-(j+1)}, m^{-j}\right), \left(\frac{m^{-i}}{1+1/m}, \frac{m^{-i}}{1+(m-1)c}\right)\right). \quad (2.11)$$

A straightforward calculation shows us that

$$\gamma_m\left(T_m(x)\right) = \int_{m^{-(j+1)}}^{m^{-j}} dx \int_{\frac{m^{-i}}{1+1/m}}^{\frac{m^{-i}}{1+(m-1)c}} \frac{dy}{(1+(m-1)(x+y))^2}$$

$$= \int_{m^{-(j+1)}}^{\frac{1+(m-1)m^{-(j+1)}}{1+(m-1)m^{-j}}} dx \int_c^d \frac{dy}{(1+(m-1)(x+y))^2} = \gamma_m(B)$$

that is, (2.8) holds. Since any arbitrary interval $(a, b)$ can be written as a reunion of fundamental intervals the proof is complete.
3 The Perron-Frobenius operator

In this section we derive the Perron-Frobenius operator of \( T_m \) under \( \gamma_m \). Also we restrict this operator to the space of functions of bounded variation.

Let \( \mu \) be a probability measure on \( B_I \) such that \( \mu (T^{-1}_m (A)) = 0 \) whenever \( \mu (A) = 0 \), \( A \in B_I \), where the transformation \( T_m \) is defined in \( 1.17 \). In particular, this condition is satisfied if \( T_m \) is \( \mu \)-preserving, that is, \( \mu T^{-1}_m = \mu \). It is known from \([9]\) that the Perron-Frobenius operator \( P_\mu \) of \( T_m \) under \( \mu \) is defined as the bounded linear operator on \( L^1_\mu = \{ f : I \to \mathbb{C} : \int_I |f| \, d\mu < \infty \} \) which takes \( f \in L^1_\mu \) into \( P_\mu f \in L^1_\mu \) with

\[
\int_A P_\mu f \, d\mu = \int_{T^{-1}_m (A)} f \, d\mu, \quad A \in B_I.
\] (3.1)

In particular, the Perron-Frobenius operator \( P_\lambda \) of \( T_m \) under the Lebesgue measure \( \lambda \) is

\[
P_\lambda (x) = \frac{d}{dx} \int_{T^{-1}_m ([0,x])} f \, d\lambda \quad \text{a.e. in } I.
\] (3.2)

**Proposition 3.1** The Perron-Frobenius operator \( P_{\gamma_m} := U_m \) of \( T_m \) under \( \gamma_m \) is given a.e. in \( I \) by the equation

\[
U_m f(x) = \sum_{i \in \mathbb{N}} P_i ((m-1)x) f(u_i(x)), \quad f \in L^1_{\gamma_m},
\] (3.3)

where \( P_i \) is defined in \( 1.33 \) and \( u_i(x) \) is given by the equation

\[
u_i(x) = \frac{m^{-i}}{1 + (m-1)x}, \quad x \in I.
\] (3.4)

**Proof.** Let \( T_{m,i} : I_i \to I \) denote the restriction of \( T_m \) to the interval \( I_i = \left( \frac{1}{m^i + 1}, \frac{1}{m^i} \right], \ i \in \mathbb{N} \), that is,

\[
T_{m,i}(x) = \frac{1}{m-1} \left( \frac{m^{-i}}{x} - 1 \right), \quad x \in I_i.
\] (3.5)

For any \( f \in L^1_{\gamma_m} \) and any \( A \in B_I \), we have

\[
\int_{T^{-1}_m (A)} f \, d\gamma_m = \sum_{i \in \mathbb{N}} \int_{T^{-1}_m (A \cap I_i)} f \, d\gamma_m = \sum_{i \in \mathbb{N}} \int_{T_{m,i}^{-1} (A)} f \, d\gamma_m.
\] (3.6)

For any \( i \in \mathbb{N} \), by the change of variable

\[
x = T_{m,i}^{-1} (y) = \frac{m^{-i}}{1 + (m-1)y},
\] (3.7)
we successively obtain

\[ \int_{T_m^{-1}(A)} f^{(n)} \gamma_{\epsilon_m} = c \int_{T_m^{-1}(A)} \left( 1 + (m-1)x \right) \left( m + (m-1)x \right) dx \]

\[ = c \int_{A} \left( 1 + (m-1)u_i(y) \right) \left( m + (m-1)u_i(y) \right) \frac{(m-1)^{m-1}}{(1 + (m-1)y)^{m-1}} dy \]

\[ = c \int_{A} f(u_i(y)) \left( (m-1)y \right)^{m-1} \left( (m-1)u_i(y) \right)^{m-1} dy \]

Now, (3.8) follows from (3.6) and (3.8).

**Proposition 3.2** Let \( \mu \) be a probability measure on \( B_I \). Assume that \( \mu \) is absolutely continuous with respect to \( \lambda \) (and denote \( \mu \ll \lambda \), i.e. if \( \mu(A) = 0 \) for every set \( A \) for which \( \lambda(A) = 0 \)) and let \( h = d\mu/d\lambda \). Then

\[ \mu \left( T_m^{-n}(A) \right) = \int_{A} U_m^n f(x) \left( 1 + (m-1)x \right) \left( m + (m-1)x \right) dx, \quad (3.9) \]

for any \( n \in \mathbb{N} \) and \( A \in B_I \), where \( f(x) = (1 + (m-1)x)\left( m + (m-1)x \right)h(x), \ x \in I \).

**Proof.** We will use mathematical induction. For \( n = 0 \), the equation (3.9) is reduced to

\[ \mu(A) = \int_{A} h(x) dx, \ A \in B_I, \]

which is obviously true. Assume that (3.9) holds for some \( n \in \mathbb{N} \). Then

\[ \mu \left( T_m^{-(n+1)}(A) \right) = \mu \left( T_m^{-n}(T_m^{-1}(A)) \right) \]

\[ = \int_{T_m^{-1}(A)} U_m^n f(x) \left( 1 + (m-1)x \right) \left( m + (m-1)x \right) dx \]

\[ = \frac{1}{c_m} \int_{T_m^{-1}(A)} U_m^n f(x) d \gamma_{\epsilon_m}(x). \]

By the very definition of the Perron-Frobenius operator \( U_m = P_{\gamma_{\epsilon_m}} \) we have

\[ \int_{T_m^{-1}(A)} U_m^n f d \gamma_{\epsilon_m} = \int_{A} U_m^{n+1} f d \gamma_{\epsilon_m}. \]

Therefore,

\[ \mu \left( T_m^{-(n+1)}(A) \right) = \frac{1}{c_m} \int_{A} U_m^{n+1} f d \gamma_{\epsilon_m} \]

\[ = \int_{A} \frac{U_m^{n+1} f(x)}{(1 + (m-1)x)\left( m + (m-1)x \right)} dx. \]
which ends the proof.

In the sequel we shall restrict the Perron-Frobenius operator $U$ to $BV(I)$ the linear space of all complex-valued functions of bounded variation. Let $B(I)$ denote the Banach space of bounded measurable complex-valued functions $f$ on $I$ under the supremum norm

$$|f| = \sup_{x \in I} |f(x)|. \quad (3.10)$$

The variation $\var_A f$ over $A \subset I$ of a function $f \in B(I)$ is defined as

$$\sup_{k-1} \sum_{i=1}^k |f(t_i) - f(t_{i-1})| \quad (3.11)$$

the supremum being taken over all points $t_1 < \ldots < t_k$ in $A$, for $k \geq 2$. We write simply $\var f$ for $\var_B f$, and if $\var f < \infty$, then $f$ is called a function of bounded variation. Note that under the norm $\|f\|_V = |f| + \var f$, $f \in BV(I)$, the linear space $BV(I)$ is a commutative Banach algebra with unit.

**Proposition 3.3** If $f \in BV(I)$ is a real-valued function, then

$$\var U_m f \leq K_m \var f, \quad (3.12)$$

where $K_m = \frac{(m-1)(3m^2-3m+1)}{(2m-1)(m^2+m-1)}$. The constant cannot be lowered.

*Proof.* For $x, y \in I$ we have

$$U_m f(x) - U_m f(y) = \sum_{i \in \mathbb{N}} (P_i((m-1)x)f(u_i(x)) - P_i((m-1)y)f(u_i(y)))
\begin{align*}
&= \sum_{i \in \mathbb{N}} (P_i((m-1)x) - P_i((m-1)y))(f(u_i(x)) - f(u_0(x))) \\
&\quad + \sum_{i \in \mathbb{N}} P_i((m-1)y)(f(u_i(x)) - f(u_i(y))) \\
&= \sum_{i \in \mathbb{N}^+} (P_i((m-1)x) - P_i((m-1)y))(f(u_i(x)) - f(u_0(x))) \\
&\quad + \sum_{i \in \mathbb{N}} P_i((m-1)y)(f(u_i(x)) - f(u_i(y))).
\end{align*}$$

Note that the function $P_0$ is increasing, while the functions $P_i$, $i \in \mathbb{N}^+$, are all decreasing. Let $x < y$, with $x, y \in I$. It follows from the above equation that

$$|U_m f(x) - U_m f(y)| \leq \left( \sum_{i \in \mathbb{N}^+} (P_i((m-1)x) - P_i((m-1)y)) \right) \var f
\begin{align*}
&\quad + \sup_{y \in I, i \in \mathbb{N}} P_i((m-1)y) \sum_{i \in \mathbb{N}} \var_{[x,y]} f \circ u_i(x) \\
&= (1 - P_0((m-1)x) - 1 + P_0((m-1)y)) \var f
\begin{align*}
&\quad + P_0(m-1) \sum_{i \in \mathbb{N}} \var_{[x,y]} f \circ u_i(x).
\end{align*}$$
Hence
\[
\text{var } U_m f \leq (2P_0(m - 1) - P_0(0))\text{var } f = \left( \frac{2m(m - 1)}{m^2 + m - 1} - \frac{m - 1}{2m - 1} \right)\text{var } f
\]
\[
= \left( \frac{(m - 1)(3m^2 - 3m + 1)}{(2m - 1)(m^2 + m - 1)} \right)\text{var } f.
\]

For \( f \) defined by \( f(x) = 0, \ 0 \leq x \leq \frac{1}{m} \), and \( f(x) = 1, \ \frac{1}{m} < x \leq 1 \), we have \( U_m f(x) = P_0(x), \ 0 \leq x < 1 \) and \( U_m f(1) = 0 \). Since \( \text{var } U_m f = (m - 1)(3m^2 - 3m + 1) \) \((2m - 1)(m^2 + m - 1)\) and \( \text{var } f = 1 \), it follows that the constant \( K_m \) cannot be lowered.

4 The ergodic behaviour of the RSCC

**Proposition 4.1** The function \( P(x, i) = P_i(x) \) from (1.33) defines a transition probability function from \((I, B_I)\) to \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\).

**Proof.** We have to verify that
\[
\sum_{i \in \mathbb{N}} P(x, i) = 1, \ \text{for all } x \in I.
\]

Since
\[
P(x, i) = \frac{(x + 1)(x + m)}{m - 1} \left( \frac{1}{x + (m - 1)m^{-(i + 1)} + 1} - \frac{1}{x + (m - 1)m^{-i} + 1} \right),
\]
then it is an easy task to show that \( \sum_{i \geq 0} P(x, i) = 1 \).

Proposition 4.1 allows us to consider the random system with complete connections (RSCC) [8]
\[
((I, \mathcal{B}_I), (\mathbb{N}_+, \mathcal{P}(\mathbb{N}_+)), u, P),
\]
where \( P \) is \( P_i \) defined in (1.33), while \( u : I \times \mathbb{N} \to I \) is \( u_i \) given by
\[
u_i(x) = \frac{m^{-i}}{1 + (m - 1)x}, \ x \in I.
\]

We denote by \( U_m \) the associated Markov operator of the RSCC (4.2) with the transition probability function \( Q_m \).

In this section we study the ergodic behaviour of RSCC (4.2). The ergodic behaviour of the RSCC (4.2) allows us to find the limiting distribution function \( F = F_\infty \) and the invariant measure induced by \( F \). To study the ergodicity of the RSCC (4.2), we consider the norm \( \|\cdot\|_L \) defined on \( L(I) \) (the space of Lipschitz real-functions defined on \( I \)) by
\[
\|f\|_L = \sup_{x \in I} |f(x)| + \sup_{x' \neq x''} \frac{|f(x') - f(x'')|}{|x' - x''|}, \ f \in L(I).
\]
Proposition 4.2 The RSCC [12] is an RSCC with contraction and its associated Markov operator $U_m$ is regular with respect to $L(I)$.

**Proof.** We have
\[
\frac{d}{dx} u(x, i) = -\frac{(m-1)m^{-i}}{(1+(m-1)x)^2},
\]
and
\[
\frac{d}{dx} P(x, i) = \frac{2(m-1)x + m + 1}{(m-1)x + (m-1)m^{-(i+1)} + 1} - \frac{2(m-1)x + m + 1}{(m-1)x + (m-1)m^{-i} + 1}
\]
for any $x \in I$ and $i \in \mathbb{N}$. Therefore
\[
\sup_{x \in I} \left| \frac{d}{dx} u(x, i) \right| = (m-1)m^{-i}, \quad \sup_{x \in I} \left| \frac{d}{dx} P(x, i) \right| < \infty.
\]
It follows that $R_1 < \infty$ and $r_1 < \infty$, that is, the requirements of definition of an RSCC with contraction are fulfilled (Definition 3.1.15 in [8]). To prove the regularity of $U_m$ with respect to $L(I)$, define the recurrence relation $x_{n+1} = \frac{1}{x_n + 1}, \ n \in \mathbb{N}$, with $x_0 = x$. A criterion of regularity is expressed in Theorem 3.2.13 in [8], in terms of supports $\sigma_n(x)$ of the $n$-step transition probability functions $Q^n(x, \cdot)$, $n \in \mathbb{N}_+$.) Clearly $x_{n+1} \in \sigma_n(x)$ and therefore Lemma 3.2.14 in [8] and an induction argument lead us to the conclusion that $x_n \in \sigma_n(x)$, $n \in \mathbb{N}_+$. But, $\lim_{n \to \infty} x_n = \sqrt{2} - 1$ for any $x \in I$. Hence
\[
d \left( \sigma_n(x), \sqrt{2} - 1 \right) \leq \left| x_n - \sqrt{2} - 1 \right| \to 0, \ n \to \infty,
\]
where $d(x, y) = |x - y|$, for any $x, y \in I$. The regularity of $U_m$ with respect to $L(I)$ follows from Theorem 3.2.13 in [8].

If the transition probability function of the associated Markov operator $Q_m$ is given by
\[
Q_m(x, A) = \sum_{i \in \mathbb{N}, A(x) \in A} P_i(x), \quad x \in I, \ A \in \mathcal{B}_I. \quad (4.5)
\]
then the $n$-step transition probability function $Q^n_m(\cdot, \cdot)$ converges uniformly to a probability measure $Q^\infty_m$ and that there exist two positive constants $q < 1$ and $k$ such that
\[
\|U_m^n f - U_m^\infty f\|_L \leq kq^n \|f\|_L, \quad n \in \mathbb{N}_+, \ f \in L(I), \quad (4.6)
\]
where
\[
U_m^n f(\cdot) = \int f(y) Q^n_m (\cdot, dy), \quad U_m^\infty f = \int f(y) Q^\infty_m (dy). \quad (4.7)
\]

**Proposition 4.3** The invariant probability measure $Q^\infty_m$ of the transformation $T_m$ has the density $\rho_m(x) = \frac{1}{(1+(m-1)x)(m+(m-1)x)^2}$, $x \in I$, with the normalizing factor $c_m = \frac{1}{\log((m^2)(2m-1))}$. 

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Proof. On account of the uniqueness of $Q^\infty_m$ we have to show that
\[ \int_0^1 Q_m(x, A)Q^\infty_m(dx) = Q^\infty_m(A), \quad A \in \mathcal{B}_I. \] (4.8)
Since the intervals $[0, u) \subset [0, 1)$ generate $\mathcal{B}_I$, it is sufficient to check the equation (4.8) just for $A = [0, u)$, $0 < u \leq 1$. Let
\[ E(x, m) = \left\lfloor \frac{\log u((m-1)x+1)}{\log m-1} \right\rfloor + 1. \] (4.9)
Now, from (4.5), we have
\[ Q_m(x, [0, u)) = \sum_{i \in \mathbb{N} : 0 \leq u_i(x) < u} P_i(x) = \sum_{i \geq E(x, m)} P_i(x) \]
\[ = \frac{(x+1)(x+m)}{m-1} \left( \frac{1}{x+1} - \frac{1}{x+m-E(x, m)+1} \right), \]
so that,
\[ c_m \int_0^1 Q_m(x, [0, u)) \rho_m(x)dx = \frac{c_m}{(m-1)^2} \log \frac{m((m-1)u+1)}{(m-1)u+m} = Q^\infty_m([0, u)). \]

5 The Gauss-Kuzmin type theorem

Our aim is to prove a Gauss-Kuzmin type theorem for this new expansion. The ergodic behaviour of the RSCC introduced in §4 allows us to obtain a convergence rate result.

The Gauss-Kuzmin type theorem shows the limiting distribution function
\[ F(x) = F^\infty(x) = \lim_{n \to \infty} \mu(T^n_m < x). \] (5.1)

**Theorem 5.1** (Gauss-Kuzmin type theorem) If the density $F'_0$ of $\mu$ is a Riemann integrable function, then
\[ \lim_{n \to \infty} \mu(T^n_m < x) = \frac{c_m}{(m-1)^2} \log \frac{m((m-1)x+1)}{(m-1)x+m}, \quad x \in I. \] (5.2)
If the density $F'_0$ of $\mu$ is a Lipschitz function, then there exist two positive constants $q < 1$ and $k$ such that for all $x \in I$ and $n \in \mathbb{N}_+$
\[ \mu(T^n_m < x) = \frac{c_m}{(m-1)^2} (1 + \theta q^n) \log \frac{m((m-1)x+1)}{(m-1)x+m}, \] (5.3)
where $\theta = \theta(\mu, n, x)$, with $|\theta| \leq k$.

Proof. Let $F'_0 \in L(I)$. Then $f_0 \in L(I)$ and by the virtue of (4.7) we have
\[ U_m f_0 = \int f_0(y)Q^\infty_m(dy) = \int_0^1 F'_0(x)dx = c_m. \] (5.4)
According to (4.6) there exist two constants $q < 1$ and $k$ such that
\[ U^n_m f_0 = U^\infty_m f_0 + T^n_m f_0, \quad n \in \mathbb{N}_+ \] (5.5)
with $\|T^n_m f_0\|_L \leq k q^n$.

Further, consider $C(I)$ the metric space of real continuous functions defined on $I$ with the supremum norm $\|f\| = \sup_{x \in I} |f(x)|$. Since $L(I)$ is a dense subset of $C(I)$ we have
\[ \lim_{n \to \infty} \|T^n_m f_0\| = 0, \] (5.6)
for $f_0 \in C(I)$. Therefore, (5.6) is valid for a measurable function $f_0$ which is $Q^\infty_m$-almost surely continuous, that is, for a Riemann-integrable function $f_0$.

Thus, we have
\[
\lim_{n \to \infty} \mu (T^n_m < x) = \lim_{n \to \infty} \int_0^x U^n_m f_0(u) \rho(u) du \\
= \int_0^x U^\infty_m f_0(u) \rho(u) du \\
= c_m \int_0^x \rho(u) du \\
= \frac{c_m}{(m-1)^2} \log \left( \frac{(m-1)u + 1}{(m-1)u + m} \right) \\
= \frac{c_m}{(m-1)^2} \log \left( \frac{m((m-1)x + 1)}{(m-1)x + m} \right).
\]

Hence the statement is proved.

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