Cob: a Multidimensional Byzantine Agreement Protocol for Asynchronous Incomplete Networks

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Abstract

In this paper we extend the Multidimensional Byzantine Agreement (MBA) Protocol, a leaderless Byzantine agreement for vectors of arbitrary values, into the Cob protocol, that works in Asynchronous Gossiping (AG) networks.

This generalization allows the consensus process to be run by an incomplete network of nodes provided with (non-synchronized) same-speed clocks. Not all nodes are active in every step, so the network size does not hamper the efficiency, as long as the gossiping broadcast delivers the messages to every node in reasonable time. These network assumptions model more closely real-life communication channels, so the Cob protocol may be applicable to a variety of practical problems, such as blockchain platforms implementing sharding.

The Cob protocol has the same Bernoulli-like distribution that upper bounds the number of steps required as the MBA protocol, and we prove its correctness and security assuming a supermajority of honest nodes in the network.

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Declarations

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1 Introduction

The MBA protocol, presented in [7], is a Byzantine agreement protocol [8] for synchronous and complete networks which allows the nodes to work in parallel on each component of a vector of

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arbitrary values in order to reach a democratic (i.e. leaderless) consensus on their values. The protocol is based on the extension to the multidimensional case of the protocols presented by Micali in [6, 10], and could find many applications in decentralized environments where it is required coordination between various entities that simultaneously modify the state of the system. In particular its leaderless approach (which distinguishes it from BFT protocols which use the primary-backup model [1, 11, 3, 12]) widens the agreement by taking account of multiple points of view, and thwarts attacks from malicious leaders.

Moreover its parallel approach enhances efficiency, however the network assumptions it requires are quite unrealistic, limiting practical applications. So, in this paper we tackle the same problem described in [7], namely reaching consensus on vector of arbitrary values, but we adapt the protocol in order to work in a more realistic network model, described in Section 2.1, and we drop the active participation of every node in every step of the protocol, enabling efficient executions in large networks. This makes the solution applicable to a variety of practical problems, for example blockchain platforms implementing sharding [9]. In fact, an extension that works in asynchronous incomplete networks would allow the nodes of the blockchain that are working on different shards to synchronize their operations, creating an extremely regulated environment that gives the right conditions for a practical reconciliation of the transactions recorded on the shards.

Outline  In Section 2.4 we establish the preliminaries necessary to present the Cob protocol. We define our network assumptions, we recall some useful notation, then we describe the sortition mechanism that selects which nodes are active in each step of the protocol, giving the necessary definitions. Finally we describe our assumptions on the honesty of the nodes.

In Section 3.2 we introduce the Cob protocol, presenting a reference list of all the parameters and then describing in detail every step of the protocol itself.

Then, in Section 4.2.2 we formally analyze the properties of the Cob protocol, arriving to prove that it indeed is a Byzantine agreement through a series of preparatory lemmas and propositions. The main theorem also gives a probabilistic upper bound on the steps that are necessary to halt the execution.

Finally in Section 5.1 we draw some conclusions and remarks, and outline future works to improve the applicability of the protocol.

2 Preliminaries

In this section we define some assumptions, preliminary concepts, and notations that will be used later on to describe the Cob protocol and prove its properties.

2.1 Network Assumptions

In complete networks the number of messages exchanged through the network grows exponentially with the number of network participants, so for practical applications it is more convenient to consider a different network model, namely the Asynchronous Gossiping Network (AG networks).

In this model messages are broadcast in the network in a gossiping fashion: a procedure characteristic of peer-to-peer communications where messages pass from one node to its neighbours and so on until they reach every node. In gossiping networks we rely on each member to pass messages along to its neighbours, therefore it is reasonable to envisage the network as an incomplete, connected and non directed graph. We assume that a message sent by an honest node reaches every
honest node within a time limit that depends on the size of the message itself. Since malicious
nodes can behave arbitrarily, this assumption means that malicious nodes cannot be cut vertices in
the network graph and the graph remains connected even without the edges connected to malicious
nodes. We will also require that the ratio of malicious or faulty nodes is less than $\frac{1}{3}$.

In an asynchronous gossiping network there does not exist a common clock, but we assume that
all network participants are provided with Same-Speed Clocks [5]. In other words, we assume that
each network participant has its own clock and that the clocks all have the same speed, even if they
are not synchronized in any other way.

We take as time frame reference the earliest clock in the network, and suppose that the protocol
execution starts at time 0, i.e. time starts when the first player begins the protocol execution.
Moreover we assume that the discrepancy between any two clocks is at most a constant $\lambda$, that also
upper bounds the time required to diffuse a “short” message of the protocol to the whole network
(see Section 3.1), so each player will start the execution of the protocol at a time comprised in
the interval $[0, \lambda]$. For example, this discrepancy could be observed in a scenario in which the first
player triggers the start of the protocol execution, broadcasting a signal and resetting its clock, and
then each player starts the execution (and resets its clock) when it receives this signal, with the
network delay causing the discrepancies. Afterwards the time discrepancies do not vary because of
the same-speed nature of the clocks.

2.2 A Cryptographic Sortition Mechanism

In the protocol described in Section 3.2 not every player in the network is always active (i.e.
authorized to broadcast messages), on the contrary at every step some players are selected to be
active, while the others have a passive role. In order to better clear up this distinction, from now
on in a specific step we will call players only the nodes selected to be active and broadcast their
message, while a generic node of the network will be referred as a user. We will denote with $P(s)$
the set of players of step $s$.

We want this selection to be random, and furthermore we would like it to be private and
performed without the aid of a trusted third party. With private we mean that each user should
be able to privately check if it will be selected to be active (i.e. a player) in a step, and then be
able to prove its selection to the other players. This concept is closely related to that of verifiable
random functions (VRF), i.e. pseudo-random functions which provide publicly verifiable proofs of
their outputs' correctness.

In our protocol the sortition is implemented through a cryptographic hash function $H$ (modeled
as a random oracle) and a digital signature scheme $(G, S, V)$ with the uniqueness property, which
is defined as follows.

Definition 2.1 (Digital Signature Scheme with Unique Signature). A digital signature scheme
with unique signature is a triple of algorithms $(G, S, V)$ such that:

- $G$ is the key generation algorithm that outputs a secret key $sk$ and a public key $pk$;
- $S$ is the signing algorithm, that given a message $m$ and a private key $sk$ outputs a signature
  $\sigma = S(sk, m)$;
- $V$ is the verification algorithm that given a message $m$, a signature $\sigma$ and a public key $pk$
  outputs either true or false, and such that:
2.3 Sortition Assumptions

- the scheme is correct, i.e. for every \((\sk, \pk)\) generated with \(G\) it holds
  \[
  V(\pk, S(\sk, m), m) = \text{true} \quad \forall m
  \] (1)
- the signature is unique, i.e. for any probabilistic polynomial time algorithm \(F\) that given a message outputs a public key \(\hat{\pk}\) and two distinct signatures \(\hat{\sigma} \neq \tilde{\sigma}\), we have that:
  \[
  \mathbb{P}(V(\hat{\pk}, \hat{\sigma}, m) = V(\hat{\pk}, \tilde{\sigma}, m) = \text{true}) < \varepsilon \quad \forall m
  \] (2)

where \(\varepsilon\) is negligible. Note that this property holds also for public keys whose relative private key is known, and even for values \(\pk\) that are not legitimately generated public keys.

Given this definition, we can now describe the sortition of the active players in each step of the protocol.

**Definition 2.2** (Sortition Mechanism). Let \((G, S, V)\) be a digital signature scheme with unique signature, and suppose that every user \(1 \leq i \leq N\) is identified by a public key \(\pk_i\), let \(r\) be a random string independent from \(\sk\) for every \(1 \leq i \leq N\), and suppose that every user knows \(r\) and \(\{ \pk_i \}_{1 \leq i \leq N}\). Moreover let \(n \leq N\) be the desired number of players during each step \(s\) of the protocol, let \(H: \{0,1\}^* \rightarrow \{0,1\}\) be a hash function, and let \(\phi: \{0,1\}^d \rightarrow (0,1]\) be the standard decoding of a bit string into the unit interval \(\phi(h) = \frac{1 + \sum_{i=0}^{d-1} h_i 2^i}{2^d}\).

User \(i\) is selected to be a player during step \(s\) of the protocol, i.e. \(i \in \mathbb{P}^{(s)}\), if:

\[
\phi\left(H\left(\sigma_i^{(s)}\right)\right) \leq \frac{n}{N} \land V(\pk_i, \sigma_i^{(s)}, H(r||s)) = \text{true}.
\] (3)

The signature \(\sigma_i^{(s)}\) can then be used to prove that \(i \in \mathbb{P}^{(s)}\).

Note that, when \(H\) is modeled as a random oracle, \(\phi\left(H\left(\sigma_i^{(s)}\right)\right)\) is uniformly distributed, so the probability of a player to be selected is \(\frac{n}{N}\), and the expected number of active players is indeed \(n\).

Note that the same sortition mechanism can be implemented with a weaker notion of signature scheme that allows signing failures. That is, a scheme where the output of \(S\) is a special symbol \(\perp\) with fixed probability \(f\) (supposing the message to be uniform in \(\{0,1\}\)), with \(V(\pk, \perp, m) = \text{false}\) for every \(\pk\) and \(m\). The sole adjustment required is to increase the threshold \(\frac{n}{N}\) to account for the signing failure, using \(\frac{n}{N(1-f)}\) instead.

In some applications it might be desirable that players are selected with nonuniform probability. There is a simple trick to adjust the selection probability for each player: let \(p \in (0,1]\) be a fixed probability and \(t_i \geq 1\) be a publicly known threshold for player \(i\). The tweaked process selects player \(i\) if it can provide a pair signature-counter \((\sigma_i^{(s)}, c_i)\) such that \(\phi\left(H\left(\sigma_i^{(s)}\right)\right) \leq p,\)

\[
V(\pk_i, \sigma_i^{(s)}, H(r||s||c_i)) = \text{true}, \quad \text{and} \quad c_i \leq t_i.
\]

In other words the player \(i\) has \(t_i\) attempts to produce a winning signature, so its probability to be selected is \(1 - (1-p)^{t_i}\).

### 2.3 Sortition Assumptions

The MBA protocol requires a \(\frac{2}{3}\) supermajority of honest players to work properly, similarly the Cob protocol requires a \(\frac{1}{3}\) supermajority of honest players among the active ones at each step.
Since the active players are randomly selected, we require that at each step there are enough active honest players with high probability.

We now define a probabilistic concept which will be widely used throughout this paper.

**Definition 2.3.** We will write that an event \(E\) happens with **overwhelming probability** if \(P(E) \geq 1 - \epsilon\), where \(\epsilon \in (0, 1)\) is a parameter sufficiently close to 0.

A good choice for practical applications could be \(\epsilon = 10^{-12}\).

Let \(n\) be the expected number of active players in a step, we define the threshold \(t_H = \lfloor \frac{2n}{3} \rfloor + 1\). For every step \(s\) we choose the parameter \(n\) in a way that the following relationships between the number of honest players \(H_P(s)\) and the number of malicious players \(M_P(s)\) hold with overwhelming probability:

1. \(|H_P(s)| > t_H\)
2. \(|H_P(s)| + 2|M_P(s)| < 2t_H\)

Note that these two conditions imply that \(H_P > 2M_P\). In practice, they imply that with overwhelming probability:

- the protocol has, at each step, the required \(\frac{2}{3}\) honest majority of players.
- at every step there is a sufficient number of honest players who can certify a new vector or finalize a vector component;
- two distinct nodes can not finalize the same component with two distinct values.

Note that the closer to 1 the ratio of honest users in the network is, the smaller the number of players for each step needs to be.

The parameter choice necessary to meet the requirements is done using variants of Chernoff bounds, as in Algorand [5] and the analysis of such bound can be found in [2].

### 2.4 Notation

We will typeset vectors in boldface and in general subscript will be used to denote the player who created the value and the index of vector components, while superscripts will refer to the protocol step in which the value has been produced. So \(v_i(s)\) will be a vector created by user \(i\) during step \(s\) of the protocol, while \(v_i^{(s)}(c)\) will denote the \(c\)-th component of said vector. As shorthand, \(1\) denotes a vector where each component is equal to 1, and similarly \(0\) denotes an all-zero vector.

As in [4] and [7], the notation \(\#_i^{(s)}(v, c)\), for \(1 \leq c \leq m\) represents the number of players from which player \(i\) has received during step \(s\) a valid message containing a vector \(v^{(i)} = (v_1, \ldots, v_m)\) such that \(v_c = v\) considering, possibly, also its own message. We recall that honest players consider at most one message from player \(j\) as valid (discarding all contrasting and not properly formatted messages, and counting identical messages as one), so only \emph{valid} messages are considered and counted, and \(\sum_c \#_i^{(s)}(v, c) \leq |P(s)|\) \forall i, s, c.

In the protocol the players try to reach agreement on a vector of arbitrary values, where each player \(j\) starts the protocol knowing an \(m\)-dimensional vector \(v_j = (v_{j,1}, \ldots, v_{j,m}) \in V = \prod_{c=1}^m V_c\). We say that the players have reached \emph{c-agreement}, where \(1 \leq c \leq m\) is a specific component, when
there exists \( v \in V_c \) such that for every honest player \( j \), \( v_{j,c} = v \). When \( c \)-agreement is reached on all the components of the vector, we have that for all honest players \( i, j \), \( v_i = v_j \), hence also agreement is reached.

3 Cob Protocol

We now present the Cob protocol that, as anticipated, allows a wide asynchronous gossiping network to reach agreement on a vector of arbitrary values. The properties of the protocol will be formally stated and proved in Section 4.2.2.

3.1 Protocol Parameters and Components

For the sake of clarity and easy reference, we now provide a list with the definition of the parameters and the notation that we will use to describe and analyze the Cob protocol:

- \( H \): a cryptographic hash function, modelled as a random oracle;
- \((G, S, V)\): a digital signature scheme with unique signature (see Definition 2.1);
- \( N \in \mathbb{Z}^+ \): the number of nodes in the network, i.e. the users of the protocol;
- \( \mathbf{PK} \): the set of public keys of the users, each user \( i \), with \( 1 \leq i \leq N \), is univocally identified by its public key \( \mathbf{pk}_i \in \mathbf{PK} \) and has a private key \( \mathbf{sk}_i \);
- \( \frac{2}{3} < h \leq 1 \): the ratio of the honest users in \( \mathbf{PK} \);
- \( r \): a reference string, i.e. a random string independent from every \( \mathbf{pk}_i \in \mathbf{PK} \) and known by every user;
- \( n \): the expected number of players active in each step of the protocol;
- \( t_H = \lfloor \frac{2n}{3} \rfloor + 1 \): a threshold used in the protocol, derived from the expected lower bound of the number of honest players in each step;
- \( m \): the number of components of the vector of arbitrary values upon which agreement has to be achieved;
- \( V = \prod_{c=1}^{m} V_c \): the set the vector to be agreed upon belongs to, each set \( V_c \) contains all the possible values of the \( c \)-th component of said vector;
- \( \bot \): a special value that represents a non meaningful value for any component, we require that \( \bot \in V_c \) for every \( 1 \leq c \leq m \);
- \( O_i \in V \): the vector built by player \( i \) at the start of the protocol, for every \( 1 \leq c \leq m \) we will say that \( c \) is an unambiguous component if \( O_{i,c} = O_{j,c} \) for every couple of honest users \( i, j \), otherwise, if there exist two honest users \( i \) and \( j \) such that \( O_{i,c} \neq O_{j,c} \), we will say that \( c \) is an ambiguous component;
- \( \Omega \): the amount of time spent by each player \( i \) at the start of the protocol to build its private vector \( O_i \) (e.g. by observing some events and reporting some relevant information about them), the protocol will then try to reconcile all these vectors into a shared one;
• $\Lambda$: the upper bound to the time needed to propagate the messages in each of the first two steps of the protocol;

• $\lambda$: the upper bound to the time needed to propagate the messages of the third and following steps of the protocol. The difference between $\lambda$ and $\Lambda$ depends on the size of the elements of $V$. We assume that $\Lambda = O(\lambda)$;

• $s \in \mathbb{Z}^+$: the current step of the protocol;

• $P(s)$: the active players that partake in step $s$ of the protocol;

• $\sigma^{(s)}_i = S(sk_i, H(r||s))$: the credential of user $i$ for step $s$, used to check if $i \in P(s)$;

• $p \in (0,1)$: for each time-slot $s$, each user in $\mathbb{PK}$ is chosen to be in $P(s)$ with probability $p = \frac{n}{N}$;

• $MP^{(s)}$ and $HP^{(s)}$: they are respectively the set of malicious and honest players in step $s$, note that $MP^{(s)} \cup HP^{(s)} = P(s)$ and $MP^{(s)} \cap HP^{(s)} = \emptyset$;

• $m^{(s)}_i$: the message broadcast by player $i$ during step $s$;

• $v^{(s)}_i$: the vector of information contained in the message $m^{(s)}_i$, we will see that $v^{(s)}_i \in V$ if $s \leq 2$, $v^{(s)}_i \in \{0,1\}^m$ if $s \geq 3$;

• $\operatorname{sig}^{(s)}_i(x) = (x, S(sk_i, s||x))$: the value $x$ broadcast by player $i$ during step $s$ certified by its signature, it is included in $m^{(s)}_i$;

• $C_i$: the certificate built by player $i$ which attests that the final vector has network agreement, each user $i$ continues running the protocol until it can build a certificate $C_i$;

• $\alpha_i \in [0, \lambda]$: the time at which user $i$ starts the execution of each step of the protocol;

• $\beta^{(s)}_i$: the time at which user $i$ ends the execution of step $s$ of the protocol;

• $t^{(s)}$: the amount of time that players of step $s$ have to wait in order to harvest all the information required to compute the message to broadcast, if $i \in P(s)$ then $t^{(s)} = \beta^{(s)}_i - \alpha_i$;

• $T_c$: the time at which the first honest user finalizes component $c$;

• $T$: the time at which the first honest user produces a certificate;

• $L$: a random variable representing the number of Bernoulli trials needed to see the output 1, when each trial outputs 1 with probability $\frac{1}{2}$;

• $\chi_{lh}$: a random variable representing the number of steps required to end the probability game described in [7] with parameters $l$ and $h$. 

\section{Protocol Parameters and Components}

3.1


3.2 Cob Protocol description

We now describe in detail how the Cob protocol works. The honest users will be the ones who follow the protocol described below, and, even if not elected as players on any step, they are supposed to stay online to support message propagation during the whole protocol execution.

The protocol is a variant of the MBA, where the first three steps are essentially the Multidimensional Graded Consensus presented in [7], then from the step 4 onward it is a three-step loop that corresponds to the Multidimensional Binary Byzantine Algorithm, also presented in [7].

As in the protocol MBA, each user $i$ privately saves a vector $f_i$ initialized to $0$ that keeps track of the finalization of the components. A component is finalized when the network is in agreement on it, and from that moment on the protocol will not change it anymore. Once every component has been finalized, the protocol enables the creation of certificates that attest that the vector is indeed shared by the network, and then terminates. That is, the three-step loop is repeated until the ending condition is met, which corresponds to the creation of a certificate for the agreed-upon final vector.

Every honest user $i$ in the system starts the protocol execution when its own private clock signs $0$. Note that, right from the start, each user $i$ can build its credentials $\sigma_i^{(s)}$ and check for which $s$ it will be $\phi(H(\sigma_i^{(s)})) \leq p$ and therefore $i \in P(s)$.

We now describe the Cob protocol, followed by the Ending Condition to be performed in each step $s \geq 4$ to determine whether agreement has been achieved.

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**STEP 1** *(first step of $m$-dimensional GC)*

- Each user $i$ computes its credential $\sigma_i^{(1)}$ and checks if $i \in P^{(1)}$;
- if $i \not\in P^{(1)}$ then $i$ ends its step 1 right away;
- if $i \in P^{(1)}$, $i$ spends $t^{(1)} = \Omega$ time building its own private vector $O_i \in V$, then:
  - sets $v_i^{(1)} = O_i$;
  - broadcasts the message $m_i^{(1)} = \left( 1, \sigma_i^{(1)}, \text{sig}_i^{(1)}(v_i^{(1)}) \right)$.

**STEP 2** *(second step of $m$-dimensional GC)*

- Each user $i$ computes its credential $\sigma_i^{(2)}$, if $i \not\in P^{(2)}$ then $i$ ends its step 2;
- if $i \in P^{(2)}$, after waiting an amount of time $t^{(2)} = t^{(1)} + \Lambda + \lambda$, player $i$ does the following:
  - sets $v_i^{(2)}$, where $v_i^{(2)} = v_c \neq \perp$ if and only if $\#_i (v_c, c) \geq t_R$, and $v_i^{(2)} = \perp$ otherwise;
  - broadcasts the message $m_i^{(2)} = \left( 2, \sigma_i^{(2)}, \text{sig}_i^{(2)}(v_i^{(2)}) \right)$. 

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**STEP 3** (output determination of m-dimensional GC and starting broadcast in m-dimensional BBA)

- Each user \( i \) collects and locally saves the messages received from the players of step 2;
- after waiting an amount of time \( t^{(3)} = t^{(2)} + \lambda + \Lambda \), \( i \) updates its private vector \( O_i \) and computes the vector \( g_i \) where, for each component \( c \), \( O_{i,c} \) and \( g_{i,c} \) are computed as follows:
  - if, for some \( x \neq \perp \), \( \#_{\overline{2}}^{(2)}(x, c) \geq t_R \), then \( (O_{i,c}, g_{i,c}) = (x, 2) \);
  - else, if, for some \( x \neq \perp \), \( \#_{\overline{2}}^{(2)}(x, c) \geq \frac{t}{2} \), then \( (O_{i,c}, g_{i,c}) = (x, 1) \);
  - otherwise, \( (O_{i,c}, g_{i,c}) = (\perp, 0) \);
- if \( i \notin P^{(3)} \), then \( i \) ends the execution of step 3;
- if \( i \in P^{(3)} \), then player \( i \) does the following:
  - builds the vector \( v^{(3)}_i \in \{0, 1\}^m \) such that \( v^{(3)}_{i,c} = 0 \) if \( g_{i,c} = 2 \), \( v^{(3)}_{i,c} = 1 \) otherwise;
  - computes the vector \( \Theta^{(3)}_i \) such that \( \Theta^{(3)}_{i,c} = \perp \) if \( v^{(3)}_{i,c} = 1 \), \( \Theta^{(3)}_{i,c} = O_{i,c} \) when \( v^{(3)}_{i,c} = 0 \);
  - broadcasts the message \( m^{(3)}_i = \left( 3, \sigma^{(3)}_i, \operatorname{sig}^{(3)}_i\left( v^{(3)}_i \right), \operatorname{sig}^{(3)}_i\left( H\left( \Theta^{(3)}_i \right) \right) \right) \).

**STEP s** \( 4 \leq s, s - 1 \equiv 0 \mod 3 \) (Coin-Fixed-To-0 step and starting broadcast of Coin-Fixed-To-1 step in m-dimensional BBA)

- Each user \( i \) collects the messages received from the players active during step \( s - 1 \);
- after waiting an amount of time \( t^{(s)} = t^{(s-1)} + 2\lambda \), the user \( i \) starts building the vector \( v^{(s)}_i \) performing the following operations:
  - verifies the ENDING CONDITION;
  - sets \( v^{(s)}_{i,c} = v^{(s-1)}_{i,c} \) for all \( 1 \leq c \leq m \) such that \( f_{i,c} = 1 \);
  - performs the FINALIZATION CHECK 0;
  - performs the FINALIZATION CHECK 1;
- if \( i \notin P^{(s)} \), the user \( i \) ends the execution of step \( s \);
- if \( i \in P^{(s)} \), the player \( i \) does the following:
  - completes the vector \( v^{(s)}_i \) depending on the vectors \( v^{(s-1)}_j \) included in the valid messages it has received, in particular, for each component \( c \) such that \( f_{i,c} = 0 \):
    * if \( \#^{(s-1)}_i (1, c) \geq t_R \), then \( i \) sets \( v^{(s)}_{i,c} = 1 \);
    * else \( i \) sets \( v^{(s)}_{i,c} = 0 \);
  - computes the vector \( \Theta^{(s)}_i \) such that \( \Theta^{(s)}_{i,c} = \perp \) when \( v^{(s)}_{i,c} = 1 \), \( \Theta^{(s)}_{i,c} = O_{i,c} \) when \( v^{(s)}_{i,c} = 0 \).
3.2 Cob Protocol description

- broadcasts the message \( m_i^{(s)} = \left( s, \sigma_i^{(s)}, \text{sig}_i^{(s)} \left( v_i^{(s)} \right), \text{sig}_i^{(s)} \left( H \left( \Theta_i^{(s)} \right) \right) \right). \)

**STEP s 5 \leq s, s - 1 \equiv 1 \mod 3**  (Coin-Fixed-To-1 step and starting broadcast of Coin-Genuinely-Flipped step in \( m \)-dimensional BBA)

- Each user \( i \) collects the messages received from the players active during step \( s - 1 \);
- after waiting an amount of time \( t^{(s)} = t^{(s-1)} + 2 \lambda \), the user \( i \) starts building the vector \( v_i^{(s)} \) performing the following operations:
  - verifies the ENDING CONDITION;
  - sets \( v_{i,c}^{(s)} = v_{i,c}^{(s-1)} \) for all \( 1 \leq c \leq m \) such that \( f_{i,c} = 1 \);
  - performs the FINALIZATION CHECK 0;
  - performs the FINALIZATION CHECK 1;
- if \( i \not\in P^{(s)} \), the user \( i \) ends the execution of step \( s \);
- if \( i \in P^{(s)} \), the player \( i \) does the following:
  - completes the vector \( v_i^{(s)} \) depending on the vectors \( v_j^{(s-1)} \) included in the valid messages it has received, in particular, for each component \( c \) such that \( f_{i,c} = 0 \):
    * if \( \#_{i^{(s-1)}}(0,c) \geq t_B \), then \( i \) sets \( v_{i,c}^{(s)} = 0 \);
    * else \( i \) sets \( v_{i,c}^{(s)} = 1 \);
  - computes the vector \( \Theta_i^{(s)} \) such that \( \Theta_i^{(s)} = \bot \) when \( v_{i,c}^{(s)} = 1 \), \( \Theta_i^{(s)} = O_{i,c} \) when \( v_{i,c}^{(s)} = 0 \);
  - broadcasts the message \( m_i^{(s)} = \left( s, \sigma_i^{(s)}, \text{sig}_i^{(s)} \left( v_i^{(s)} \right), \text{sig}_i^{(s)} \left( H \left( \Theta_i^{(s)} \right) \right) \right). \)

**STEP s 6 \leq s, s - 1 \equiv 2 \mod 3**  (Coin-Genuinely-Flipped step and starting broadcast of Coin-Fixed-To-0 step in \( m \)-dimensional BBA)

- Each user \( i \) collects the messages received from the players active during step \( s - 1 \);
- after waiting an amount of time \( t^{(s)} = t^{(s-1)} + 2 \lambda \), the user \( i \) starts building the vector \( v_i^{(s)} \) performing the following operations:
  - verifies the ENDING CONDITION;
  - sets \( v_{i,c}^{(s)} = v_{i,c}^{(s-1)} \) for all \( 1 \leq c \leq m \) such that \( f_{i,c} = 1 \);
  - performs the FINALIZATION CHECK 0;
  - performs the FINALIZATION CHECK 1;
• if $i \notin \mathcal{P}(s)$, the user $i$ ends the execution of step $s$;

• if $i \in \mathcal{P}(s)$, the player $i$ does the following:
  
  - completes the vector $v_j^{(s)}$ depending on the vectors $v_j^{(s-1)}$ included in the valid messages it has received, in particular, for each component $c$ such that $f_{i,c} = 0$:
    
    * if $\#(i^{(s-1)}) (0, c) \geq t_B$, then $i$ sets $v_{i,c}^{(s)} = 0$;
    
    * if $\#(i^{(s-1)}) (1, c) \geq t_B$, then $i$ sets $v_{i,c}^{(s)} = 1$;
    
    * otherwise, letting $\mathcal{P}_i^{(s-1)} \subseteq \mathcal{P}^{(s-1)}$ be the set of players who sent $i$ a valid message in the previous step, then $i$ sets $v_{i,c}^{(s)} = k_c$, where $k = H \left( \min_{j \in \mathcal{P}_i^{(s-1)}} H (\Theta_j^{(s-1)}) \right)$;

  we will refer to the player whose hashed credential is minimal from $i$’s point of view as the coin flipper selected by $i$ during step $s$;

  - computes the vector $\Theta_i^{(s)}$ such that $\Theta_i^{(s)} \parallel = 1$ when $v_{i,c}^{(s)} = 1$, $\Theta_i^{(s)} = O_i,c$ when $v_{i,c}^{(s)} = 0$;

  - broadcasts the message $m_i^{(s)} = (s, \sigma_i^{(s)}, \mathbf{sig}_i^{(s)} (v_j^{(s)}), \mathbf{sig}_i^{(s)} (H (\Theta_i^{(s)})))$.

---

**ENDING CONDITION**

If, while user $i$ waits for the end of the current step (step $s$), there exist a string $\theta \in \{0,1\}^d$ and a step $s'$ such that:

• $4 \leq s'$ with $s' - 1 \equiv 0 \mod 3$;

• user $i$ has received at least $t_B$ messages $m_j^{(s'-1)}$ containing the signature of $s' - 1 \parallel \theta = s' - 1 \parallel H (\Theta_j^{(s'-1)})$ and at least $t_B$ messages $m_j^{(s')}$ containing the signature of $s' \parallel \theta = s' \parallel H (\Theta_j^{(s')})$;

then $i$ can build its certificate $C_i$, diffuse it in the network and terminate its execution of the protocol.

For the collision resistance of $H$ we can assume that there is a vector $\Theta \in \mathcal{V}$ such that $\theta = H (\Theta)$, i.e. $i$ has collected $2t_B$ signatures that refer to the same vector $\Theta$. Lemma 4.3 states that in this case $i$ has received, for each $1 \leq c \leq m$, at least a message $m_j^{(2)}$ with $v_j^{(2)} = \Theta_c$, hence it can reconstruct the vector $\Theta$. Let $\mathcal{P}_i^{(s-1)} \subseteq \mathcal{P}^{(s-1)}$ and $\mathcal{P}_i^{(s)} \subseteq \mathcal{P}^{(s)}$ be the the sets of players (each of cardinality at least $t_B$) that have sent messages with the signature of $H (\Theta)$ as stated above, then $i$ can build its certificate as:

$$C_i = \left( \Theta, s, \left\{ \left( \mathbf{sig}_j^{(s-1)} (H (\Theta)), \sigma_j^{(s-1)} \right) \right\}_{j \in \mathcal{P}_i^{(s-1)}}, \left\{ \left( \mathbf{sig}_j^{(s)} (H (\Theta)), \sigma_j^{(s)} \right) \right\}_{j \in \mathcal{P}_i^{(s)}} \right).$$  \hspace{1cm} (4)
FINALIZATION CHECK 0

Let \( s' \) be a step such that \( 4 \leq s' \leq s \) and \( s' - 1 \equiv 0 \mod 3 \) (that is, step \( s' \) is a Coin-Fixed-To-0 step).

For each component \( c \in \{1, \ldots, m\} \) such that \( f_{i,c} = 0 \), if, considering the vectors \( v_j^{(s'-1)} \) contained in the valid messages \( m_j^{(s'-1)} \) received by \( i \), we have that \( \#(s'-1)_i(0,c) \geq t_H \), then \( i \) sets:

- \( v_{i,c}^{(s)} = 0 \);
- \( f_{i,c} = 1 \).

FINALIZATION CHECK 1

Let \( s' \) be a step such that \( 4 \leq s' \leq s \) and \( s' - 1 \equiv 1 \mod 3 \) (that is, step \( s' \) is a Coin-Fixed-To-1 step).

For each component \( c \in \{1, \ldots, m\} \) such that \( f_{i,c} = 0 \), if, considering the vectors \( v_j^{(s'-1)} \) contained in the valid messages \( m_j^{(s'-1)} \) received by \( i \), we have that \( \#(s'-1)_i(1,c) \geq t_H \), then \( i \) sets:

- \( v_{i,c}^{(s)} = 1 \);
- \( f_{i,c} = 1 \).

User \( i \) keeps following the protocol instruction until the ending conditions are satisfied and \( i \) is able to build a certificate \( C_i \). A certificate \( C_i \) contains the vector \( \Theta \) on which the network has reached agreement, alongside a set of digital signatures for \( H(\Theta) \) together with a proof that those who signed were indeed players of a specific step. In particular \( C_i \) contains at least \( t_H \) signatures from players of a Coin-Genuinely-Flipped step \( s \) and \( t_H \) signatures from players of the subsequent subsequent Coin-Fixed-To-0 step \( s + 1 \).

4 Security Analysis

In this section we will state and prove some properties of the Cob protocol. In particular we will prove that the honest nodes in the network agree on the same vector \( \Theta \) at the end of the protocol execution, and we will give a probabilistic upper bound on the time required to terminate a protocol run.

**Theorem 4.1.** *(Main Theorem)* Given an instance of the Cob protocol described in Section 3.2, the following properties about each protocol execution hold with overwhelming probability:
1. if $0 \leq \ell \leq m$ is the number of ambiguous components, then we have that $T \leq \Omega + 2\Lambda + (7 + 6\chi_{\ell,2})\lambda$, where $\chi_{\ell,2}$ is the random variable described in [7];

2. all honest users agree on the same vector $\Theta$ and know it in the interval $[T, T + \lambda]$.

**Remark 1.** When there are no ambiguous components, i.e. all honest nodes at the beginning of the protocol build the same vector, then $\chi_{\ell,2} = 0$, so $T \leq \Omega + 2\Lambda + 7\lambda$;

### 4.1 Preliminary Results

In order to prove our Main Theorem 4.1, we need some preliminary lemmas and propositions.

**Lemma 4.2.** For each step $s \geq 1$ of a protocol run we have:

1. if $i \in P^{(s)}$ is honest, then $\beta_{i}^{(s)} \in [t^{(s)}, t^{(s)} + \lambda]$;

2. if $i \in P^{(s)}$ is honest, then by time $\beta_{i}^{(s)}$ it has received all messages sent by all honest players $j \in \mathcal{HP}^{(s)}$ for all steps $s' < s$;

3. for each step $\bar{s} > s$, fixing a component $1 \leq c \leq m$, with overwhelming probability there do not exist two players $i, i' \in P^{(s)}$ such that:
   
   - $i$ has received at least than $t_{H}$ messages $m_{j,c}^{(s)}$ advertising $v_{j,c}^{(s)} = b$;
   - $i'$ has received at least than $t_{H}$ messages $m_{j,c}^{(s)}$ advertising $v_{j,c}^{(s)} = b'$ with $b' \neq b$.

   *Note that for step 2 and 3 we have $b, b' \in V_{c}$, while for the next steps $b, b' \in \{0, 1\}$.*

**Proof.** Property 1 holds as a consequence of the network assumptions regarding the same speed clocks delay. In fact we know that each user $i$ starts its protocol execution at a time $\alpha_{i} \in [0, \lambda]$ and waits for time $t^{(s)}$ before acting and then ending its step. This means that $\beta_{i}^{(s)} = \alpha_{i} + t^{(s)} \in [t^{(s)}, t^{(s)} + \lambda]$.

Property 2 holds by the definition of the protocol, noticing that $t^{(s)} \geq t^{(s')}$ for all $s' < s$. If $s = 2, 3$, then for all $s' < s$, $i \in P^{(s)}$, we have that:

$$\beta_{i}^{(s)} \geq t^{(s)} = t^{(s-1)} + \lambda + \Lambda \geq t^{(s')} + \lambda + \Lambda \geq \beta_{i}^{(s')} + \Lambda,$$

(5)

since the honest players $j \in \mathcal{HP}^{(s')}$ send their messages of step $s'$ at time $\beta_{j}^{(s')}$ and the messages reach all honest users in at most $\Lambda$ time, then player $i$ has received all the messages from honest players of the previous steps.

If $s \geq 4$, then:

$$\beta_{i}^{(s)} \geq t^{(s)} = t^{(s-1)} + 2\lambda \geq t^{(s')} + \lambda + \Lambda \geq \beta_{j}^{(s')} + \Lambda,$$

(6)

since each honest player $j \in \mathcal{HP}^{(s')}$ sends its message of step $s'$ at time $\beta_{j}^{(s')}$, then it will reach all honest players by time $\beta_{j}^{(s')} + \Lambda \leq \beta_{i}^{(s)}$. **
Finally we prove Property 3. Let us assume for sake of contradiction that the two players $i, i'$ and the two values $b, b'$ of Property 3 do exist. Note that each malicious player $j \in MP^s$ may have signed both a vector $v^s_j = b$ and another vector $v'^s_j = b'$, but all honest players have signed exactly one vector, hence their $c$-th component is unequivocal.

Therefore, at least $t_h - MP^s$ of the messages received by $i$ advertising $b$ must come from a set $H$ of honest players, and $t_h - MP^s$ must come from another set $H'$ of honest players with $H$ and $H'$ disjoint sets. Note that the messages advertising different values in the $c$-th component must be distinct messages, this means that we are considering at least $2t_h$ distinct messages.

Let $M$ be this set of at least $t_h$ messages collected by $i$ and $M'$ the analogous set of messages collected by $i'$, then:

$$2t_h \leq |M| + |M'| \leq |H| + |MP^s| + |H'| + |MP^s| \leq |HP^s| + 2|MP^s| < 2t_h$$  \hspace{1cm} (7)

where the last inequality holds with overwhelming probability thanks to the assumptions of Section 2.3. This is a contradiction, therefore such players $i$ and $i'$ do not exist.

Now we prove a lemma that justifies the construction of a certificate as described in Section 3.2.

**Lemma 4.3.** If a user $i$ builds a certificate $C_i$ for the vector $\Theta$, then, for each $1 \leq c \leq m$, $i$ has received at least one step 2 message from $j \in HP^s$ with $v^{(2)}_{j,c} = \Theta_c$.

**Proof.** Note that from the assumptions of Section 2.3 on the number of malicious players we have that $|MP^s| < \frac{2t_h - |HP^s|}{2} < \frac{2t_h - t_h}{2} = \frac{t_h}{2}$, so at least one of the signatures in the certificate $C_i$ must come from an honest player $k \in HP^s$. Then $k$ must have received, during step 3, at least $\frac{t_h}{2}$ messages for $\Theta_c$ in $c$-th component. Again, since $|MP^s| < \frac{t_h}{2}$, at least one of them must come from an honest player $j \in HP^{(2)}$ and, according to Item 2 of Lemma 4.2, his message must have reached also $i$ within time $\beta_i^{(3)}$.

To conclude, note that $\beta_i^{(3)}$ is the ending time of step 3 for player $i$ and it is before any possible certificate production time. 

4.2 Component-Wise Finalization

In this section we prove some properties of the finalization of a single component, distinguishing between ambiguous and unambiguous ones.

4.2.1 Unambiguous Components

**Proposition 4.4.** (c-Agreement on Unambiguous Components) Let $c$ be an unambiguous component, then the following happens with overwhelming probability:

- all honest users have their $c$-th component finalized by step 5 (and in particular there is $c$-agreement on the vectors $\Theta^{(s)}_i$ for all $s \geq 5$);
- $T_c \leq t(5) + \lambda$.

**Proof.** Note that every honest player $i \in HP^s$ starts its step $s$ at time $\alpha_i \in [0, \lambda]$. Now we analyse the protocol step by step.
STEP 1 Since component $c$ is unambiguous, then for a certain value $x \in V_c$ each honest player $i \in \text{HP}^{(1)}$ will build a vector $v_{i}^{(1)}$ with $v_{i,c}^{(1)} = x$. Then $i$ will propagate its message $m_{i}^{(1)}$ at time $\beta_{i}^{(1)} = \alpha_i + \Omega$.

STEP 2 When an honest player $i \in \text{HP}^{(2)}$ stops waiting at time $\beta_{i}^{(2)} = \alpha_i + t^{(2)}$, $i$ has received all step 1 messages sent by the other honest players.

By our assumptions we have, with overwhelming probability, $|\text{HP}^{(1)}| > t_{H}$, hence more than $t_{H}$ step 1 messages $m_{i}^{(1)}$ that $i$ has received contain a vector $v_{j}^{(1)}$ with $v_{j,c}^{(1)} = x$. Then, whether $x = \bot$ or $x \neq \bot$, player $i$ builds a vector $v_{i}^{(2)}$ with $v_{i,c}^{(2)} = x$ and broadcasts the message $m_{i}^{(2)}$ containing the digital signature of this vector.

STEP 3 When an honest player $i \in \text{HP}^{(3)}$ stops waiting at time $\beta_{i}^{(3)} = \alpha_i + t^{(3)}$, $i$ has received all step 2 messages from all the honest players.

Since the vectors in their messages $m_{i}^{(2)}$ have $v_{j,c}^{(2)} = x$, and with overwhelming probability $|\text{HP}^{(2)}| > t_{H}$, then player $i$ will set $(O_{i,c}, g_{i,c}) = (x, 2)$ if $x \neq \bot$, $(O_{i,c}, g_{i,c}) = (x, 0)$ if $x = \bot$. So, $i$ will build the vector $v_{i}^{(3)}$ with $v_{i,c}^{(3)} = 0$ if $x \neq \bot$ or $v_{i,c}^{(3)} = 1$ if $x = \bot$, and broadcast its message $m_{i}^{(3)}$.

STEP 4 When an honest player $i \in \text{HP}^{(4)}$ stops waiting at time $\beta_{i}^{(4)} = \alpha_i + t^{(4)}$, $i$ has received all step 3 messages from all the honest players. We now consider separately two cases:

- $x \neq \bot$, in this case player $i$ enters the FINALIZATION CHECK 0 $(4 - 1 \equiv 0 \mod 3)$ and since the number of honest players is $|\text{HP}^{(3)}| \geq t_{H}$ with overwhelming probability, player $i$ sets $v_{i,c}^{(4)} = 0$ and $f_{i,c} = 1$. This means that all honest players have finalized the $c$-th component of the vector and they will get $\Theta_{i,c}^{(4)} = x$ for all $s \geq 4$.

- $x = \bot$, in this case player $i$ will neither enter the ENDING CONDITION nor any FINALIZATION CHECK. It will build a vector $v_{i}^{(4)}$ such that $v_{i,c}^{(4)} = 1$ since with overwhelming probability $|\text{HP}^{(3)}| > t_{H}$. Player $i$ will broadcast its message $m_{i}^{(4)}$ containing the digital signature of $v_{i}^{(4)}$.

Thus $c$-agreement has been reached if $x \neq \bot$, otherwise we will see that it will be reached in the next step.

STEP 5 When an honest player $i \in \text{HP}^{(5)}$ stops waiting at time $\beta_{i}^{(5)} = \alpha_i + t^{(5)}$, $i$ has received all step 4 messages from all the honest players. Again, we consider two cases:

- $x \neq \bot$, in this case $c$-agreement on vectors $\Theta_{i,c}^{(s)}$ has already been reached, and the $c$-th component has already been finalized by the honest players.

- $x = \bot$, in this case, $i$ has received with overwhelming probability at least $t_{H}$ messages from all the other honest players $j \in \text{HP}^{(4)}$ containing the digital signature of a vector $v_{j}^{(4)}$ with $v_{j,c}^{(4)} = 1$. Then $i$ enters the FINALIZATION CHECK 1 and sets $v_{i,c}^{(5)} = 1$ and $f_{i,c} = 1$. This means that all honest players have finalized the $c$-th component of the vector and they will get $\Theta_{i,c}^{(s)} = \bot$ for all $s \geq 5$. 

Since the malicious players are less than $t_H$, they will not be able to produce the number of messages required to mislead the honest players. We have seen that by the end of step 5 all honest users have finalized the $c$-th component (and they are in agreement with each other), so we have that $T_c \leq t^{(5)} + \lambda$.

### 4.2.2 Ambiguous Components

Let us now tackle the more difficult case of ambiguous components. We first deal with the simpler sub-case when no honest player sets $g_{i,c} = 2$ during step 3, then we complete the analysis thanks to four lemmas that prove the properties we need.

**Proposition 4.5** (*c*-Agreement on Very Ambiguous Components). Let $c$ be an ambiguous component and assume that all honest step 3 players set $g_{i,c} < 2$, then with overwhelming probability we have that:

- all honest users have their $c$-th component finalized (in particular will be in $c$-agreement on the vector $\Theta_i^{(s)}$) in step 5, setting $\Theta_i^{(s)} = \bot$;
- $T_c \leq t^{(5)} + \lambda$.

**Proof.** By definition of the protocol Cob, for each $i \in HP^{(3)}$, $i$ sets $v_{i,3} = 1$, since $g_{i,c} < 2$. This means that the honest step 3 players start in agreement on the component $c$ of the vector $v_{i}^{(3)}$. They may not be in agreement on the vector component $O_{i,c}$, but this does not matter. In fact, during step 4 no honest player is able to finalize component $c$ by collecting more than $t_H$ messages with $v_{k,c} = 0$, since $|MP^{(3)}| < t_H$. In the same way even if the honest players have received more than $t_H$ valid messages with $v_{k,c} = 1$ they will not finalize the component $c$ because $4 - 1 \not\equiv 1$ mod 3. Anyway, each honest step 4 player $i$ has received all honest messages, hence more than $t_H$ advertising $v_{k,c}^{(3)} = 1$. This means that $i$ will create a message $\omega_i^{(4)}$ with $v_{i,c}^{(4)} = 1$.

During step 5, which is a Coin-Fixed-To-1 step, each honest user $i$ will receive all messages by other honest players before $\beta_i^{(5)}$, hence with overwhelming probability $i$ will receive $t_H$ messages for $v_{k,c}^{(4)} = 1$. Thus $i$ will enter the FINALIZATION CHECK 1 and will finalize the $c$-th component by setting $v_{i,c}^{(5)} = 1$ and $f_{i,c} = 1$.

This means that the honest users reach $c$-agreement on $\Theta_i^{(s)} = \bot$ by the end of step 5 (which happens in the interval $[t^{(5)}, t^{(5)} + \lambda]$), and it will hold for all $s \geq 5$. Note that in this case, a strong disagreement among the players results in the component to be set to $\bot$ as it happens in Proposition 4.4 when every honest players starts with $\bot$.

**Proposition 4.6** (*c*-Agreement on Ambiguous Components). Let $c$ be an ambiguous component and assume that there exists a player $i \in HP^{(3)}$ which sets $g_{i,c} = 2$, then:

- all honest users have their $c$-th component finalized (in particular will be in $c$-agreement on the vector $\Theta_i^{(s)}$) within step $3L + 5$;
- $T_c \leq t^{(3L+5)} + \lambda$. 

\[\square\]
Where \( L \) is a random variable representing the number of Bernoulli trials needed to see the output 1, when each trial outputs 1 with probability \( \frac{1}{2} \).

In order to prove Proposition 4.6 we need the results stated in the following four lemmas.

**Lemma 4.7.** Under the assumption of Proposition 4.6, we show that the following properties hold:

1. \( g_{j,c} \geq 1 \) for all \( j \in \text{HP}^{(3)} \);
2. there is a value \( x \in V_c \) such that \( O_{j,c} = x \) for all \( j \in \text{HP}^{(3)} \).

**Proof.** Since player \( i \in \text{HP}^{(3)} \) is honest and sets \( g_{i,c}^{(3)} = 2 \), then:

1. \( i \) sets \( O_{i,c} = x \) since it has received more than \( t_H \) messages \( m^{(2)}_k \) advertising \( v^{(2)}_{k,c} = x \). By Property 3 of Lemma 4.2 we know that no honest player \( j \in \text{HP}^{(3)} \) has received \( t_H \) messages \( m^{(2)}_k \) for \( v^{(2)}_{k,c} = x' \neq x \), hence if \( g_{j,c}^{(3)} = 2 \) it must be \( O_{j,c} = x \).

As showed in the proof of Lemma 4.3, we have that with overwhelming probability \( |\text{MP}^{(s)}| < \frac{t_H}{2} \), so we can state that more than \( \frac{t_H}{2} \) honest players must have signed for \( x \). Therefore, if \( g_{j,c}^{(3)} < 2 \) then \( g_{j,c}^{(3)} = 1 \), and Property 1 holds.

2. We now show that, even if \( j \) sets \( g_{j,c} = 1 \), it will set \( O_{j,c} = x \). In fact, there can not exist a value \( x' \neq \perp \) and \( x' \neq x \) such that \( j \) has received also more than \( \frac{t_H}{2} \) step 2 messages \( m^{(2)}_k \) with \( v^{(2)}_{k,c} = x' \). For the sake of contradiction, we suppose that these messages exist; many of them may come from malicious players in \( \text{MP}^{(2)} \), but at least one of them must come from an honest player \( p \in \text{HP}^{(2)} \). This means that \( p \) has received \( t_H \) step 1 messages \( m^{(1)}_k \) with \( v^{(1)}_{k,c} = x' \). Since we have seen that some other honest step 2 players have signed a step 2 message advertising \( v^{(2)}_{k,c} = x \), this implies that they have seen \( t_H \) step 1 messages with \( v^{(1)}_{k,c} = x \), which by Lemma 4.2 is a contradiction. This means that Property 2 holds.

\( \square \)

**Lemma 4.8.** Under the assumptions of Proposition 4.6 we have that \( c \)-agreement on the vector \( \Theta \) is reached when \( c \)-agreement is reached on the bit vector \( \nu^{(1)} \).

**Proof.** Since the honest player \( i \in \text{HP}^{(3)} \) sets \( g_{i,c} = 2 \), it will set \( \nu_{i,c} = 0 \), therefore it is possible that \( c \)-agreement is reached on 0. By the analysis of Property 2 of Lemma 4.7 the honest users may not have an agreement on their \( \nu^{(3)}_{i,c} \) at the end of step 3 but they will have an agreement on \( O^{(3)}_{i,c} = x \). This means that, by the definition of Cob Protocol, when \( c \)-agreement is reached on the bit vector (either on 0 or 1), then it will also be reached on \( x \) in \( \Theta \) (respectively on \( x \) or \( \perp \)).

\( \square \)

**Remark 2.** This property is true also in the general case, but we have explicitly proved it only under the assumptions of Proposition 4.6.

**Lemma 4.9.** Being \( c \) a component on which agreement among the honest user does not hold, at every Coin-Genuinely-Flipped step \( c \)-agreement is reached with probability at least \( \frac{1}{2} \).
Proof. Assuming that $c$-agreement is not reached at the beginning of a Coin-Genuinely-Flipped step $s$ where $s \geq 6$, $s - 1 \equiv 2 \mod 3$, let an honest player $i \in HP(s)$ be in the condition that it must flip the coin during such step.

This means that the player $i$ has not received more than $t_{H_i}$ messages for a bit $b \in \{0,1\}$ in component $c$, so $i$ selects its own coin flipper $\ell^i$, and $i$ will set $v_{i,c}^{(s)} = b_i = H \left( H \left( r_{i,c}^{(s-1)} \right) \right)_c$, where $H (K)_c$ is the $c$-th bit of $H (K)$.

Note that, by Lemma 4.2, if an honest player has seen more than $t_{H_i}$ messages for the same bit $b$ in component $c$, then no honest player has seen more than $t_{H_i}$ messages for $1 - b$ in the same component. This means that $i$ will be in $c$-agreement with the honest players who did not flip the coin only if $b_i = b$, and this happens with probability $\frac{1}{2}$ with the Random Oracle assumption for the hash function $H$.

Therefore, all honest players in $HP(s)$ will be in agreement with probability $\frac{1}{2}$. Actually, this is true if the coin flipper is an honest player, in fact in this case $b_i$ can be assumed to be randomly chosen and globally shared among the honest players flipping the coin. If the player with minimal hashed credential is a malicious player, then we cannot say much about the probability distribution of the output of the bit extraction, since some players may not have seen its message.

However, with our assumptions on the common reference string $r$, we can state that, with probability $h$, the coin flipper will be honest, and in this case all honest players will be in agreement with probability $\frac{1}{2}$.

Combining these two independent probabilities we get that with probability at least $\frac{2}{3}$ the honest players reach $c$-agreement every time they enter a Coin-Genuinely-Flipped step. Note that they will finalize this component within the following 2 steps: in $s + 1$ if $b_i = 0$, in $s + 2$ if $b_i = 1$.

Lemma 4.10. Under the assumptions of Proposition 4.6, we have that:

1. being $E$ the event “there exists a step $\hat{s} \geq 4$ such that, for the first time, some user $i \in PK$ (either malicious or honest) should finalize its $c$-th component of vector $v_i^{(s)}$; $E$ happens with probability 1;

2. $T_c \leq t^{(\hat{s}+3)} + \lambda$ and $c$-agreement is reached in step $\hat{s}$ on the same value finalized by $i$ (however, the $c$-th component might be finalized 3 steps later).

Proof. 1. As proven in Lemma 4.9, if $c$-agreement is not reached by the honest users, at every Coin-Genuinely-Flipped step they will reach it with probability at least $\frac{2}{3} > \frac{1}{3}$.

Therefore, once every 3 steps the $c$-th component will be finalized with probability greater than $\frac{2}{3}$, therefore the probability that the event $E$ happens converges to 1.

2. Step $\hat{s}$ is the first step in which a user $i$ can finalize the $c$-th component. By the construction of the protocol, this happens in two possible ways:

- $E_a$: $i$ is able to collect or generate (and then propagate) at least $t_{H_i}$ valid messages $m_k^{(s'-1)}$ with $r_{k,c}^{(s'-1)} = 0$, $4 \leq s' \leq \hat{s}$, and $s' - 1 \equiv 0 \mod 3$;
- $E_b$: $i$ is able to collect or generate (and then propagate) at least $t_{H_i}$ valid messages $m_k^{(s'-1)}$ with $r_{k,c}^{(s'-1)} = 1$, $4 \leq s' \leq \hat{s}$, and $s' - 1 \equiv 1 \mod 3$;
Because the messages produced during step \( s' - 1 \) by honest players are received by every user before they are done waiting in step \( s' \), and because the adversary receives everything no later than the honest users, without loss of generality we can assume that \( s' = \hat{s} \), and that the user \( i \) is malicious.

For any step \( s \geq 4 \), every honest player \( i \in \mathcal{HP}^{(s)} \) who has waited time \( t^{(s)} \) has received all honest step \( s - 1 \) messages (thanks to Lemma 4.2), and all honest players in \( \mathcal{HP}^{(s)} \) have set \( O_{i,c} = x \) (according to Lemma 4.7).

We now consider step \( \hat{s} \) and examine 4 exhaustive ways in which event \( E \) may happen.

**Case 2.1.a:** event \( E_a \) happens and there is an honest user \( i' \in \mathcal{PK} \) who should also finalize the \( c \)-th component.

In this case, we have \( \hat{s} - 1 \equiv 0 \mod 3 \), hence Step \( \hat{s} \) is a Coin-Fixed-To-0 step. By assumption, \( i' \) has received at least \( t_\mathcal{H} \) valid step \( \hat{s} - 1 \) messages \( \mathcal{M}_k^{(s'-1)} \) with \( v^{(s'-1)}_{k,c} = 0 \).

Thus \( s' \) finalizes its component \( c \) setting \( v_{i',c}^{(s')} = 0 \) and \( f_{i,c} = 1 \).

Now we show that any other honest user \( i \) has either finalized its \( c \)-th component, setting \( v_{i,c}^{(s')} = 0 \) and \( f_{i,c} = 1 \), or has set \( v_{i,c}^{(s')} = 0 \) without finalizing such component.

Because step \( \hat{s} \) is the first time any player \( i \) should finalize component \( c \) of the vector \( v_i^{(s)} \), there does not exist a Coin-Fixed-To-1 step \( s' < \hat{s} \) (hence \( s' - 1 \equiv 1 \mod 3 \)) such that \( \mathcal{H} \) players have signed \( v_{i,c}^{(s'-1)} = 1 \). Accordingly, no online user in \( \mathcal{PK} \) finalizes the vector component \( c \) in step \( s' \) setting \( v_{i,c}^{(s')} = 1 \). Moreover, if an honest user has waited for a time \( \hat{t}^{(s')} \), then it must have received all step \( \hat{s} - 1 \) messages from honest players, and (considering the messages received by \( i' \)) at least \( t_\mathcal{H} - \lfloor |\mathcal{HP}^{(s'-1)}| \rfloor \geq 1 \) must have \( v_{k,c}^{(s'-1)} = 0 \).

According to Property 4 of Lemma 4.2, an honest player \( i \) cannot collect \( t_\mathcal{H} \) messages with \( v_{k,c}^{(s'-1)} = 1 \), therefore it sets \( v_{i,c}^{(s')} = 0 \).

For step \( \hat{s} + 1 \), since user \( i' \) has helped propagating the messages that have let it finalize the \( c \)-th component on or before time \( \alpha_{i'} + t^{(s)} \), then on or before time \( \beta_{i'}^{(s+1)} \) each honest user \( i \) has received at least \( t_\mathcal{H} \) valid \( \hat{s} - 1 \) messages for the bit 0. In fact, even if some of the \( t_\mathcal{H} \) messages received by \( i' \) were not broadcast in time by a malicious user, within time \( \beta_{i'}^{(s+1)} \) they have reached \( i \), for all \( i \in \mathcal{PK} \). This is true because user \( i' \) received the messages within time \( \beta_{i'}^{(s)} \) and helped propagating them, hence within time \( \beta_{i'}^{(s)} + \lambda \) they have reached all honest players, and for all honest players \( i \in \mathcal{HP}^{(s+1)} \) we have:

\[
\beta_{i'}^{(s+1)} \geq t^{(s)} + 2\lambda \geq \alpha_{i'} + t^{(s)} + \lambda = \beta_{i'}^{(s)} + \lambda.
\] (8)

Furthermore, honest players will not end step \( \hat{s} + 1 \) before receiving those step \( \hat{s} - 1 \) messages, because there do not exist other \( t_\mathcal{H} \) valid step \( s' - 1 \) messages for 1 in the component \( c \) with \( s' - 1 \equiv 1 \mod 3 \) and \( 5 \leq s' < \hat{s} + 1 \), by the definition of Step \( \hat{s} \) in assumption \( E_a \) (step \( \hat{s} \) is the first step in which a user should finalize component \( c \)). In particular, step \( \hat{s} + 1 \) itself is a Coin-Fixed-To-1 step, but no honest player has propagated during step \( \hat{s} \) a message for 1 (as we have shown they have reached \( \epsilon \)-agreement on 0), and \( |\mathcal{HP}^{(s)}| < t_\mathcal{H} \). Thus all honest users finalize their \( c \)-th component, setting \( v_{i,c}^{(s+1)} = 0 \) and \( f_{i,c} = 1 \).

So, we have proven that, for all \( s \geq \hat{s} + 1 \), the honest users set \( v_{i,c}^{(s)} = 0 \) and \( f_{i,c} = 1 \) (i.e. they finalize the component \( c \) within the end of step \( \hat{s} + 1 \)). We have already seen
that $\exists x \in V_c$ such that $O_{i,c} = x$ for every honest user $i$, so they will set $\Theta_{i,c}^{(s)} = x$ and therefore $c$-agreement reached on $\Theta_{i,c}^{(s)}$ for all $s \geq \hat{s} + 1$.

**Case 2.1.b:** event $E_b$ happens and there is an honest user $i' \in PK$ who should also finalize $c$-th component.

In this case we have $\hat{s} - 1 \equiv 1 \mod 3$, then step $\hat{s}$ is a Coin-Fixed-To-1 step. The analysis is similar to Case 2.1.a and we will omit many details.

As in the previous case, $i'$ must have received $t_b$ valid step $\hat{s} - 1$ messages with $v_{k,c}^{(\hat{s} - 1)} = 1$. Again, by the definition of step $\hat{s}$, there does not exist a step $s'$, with $4 \leq s' \leq \hat{s}$ and $s' - 1 \equiv 0 \mod 3$, where at least $t_b$ players have signed a message with $v_{k,c}^{s' - 1} = 0$.

Thus, $i'$ finalizes the $c$-th component and sets $v_{i',c}^{(\hat{s})} = 1$ and $f_{i',c} = 1$. Moreover, any other honest user $i \in PK$ has either finalized its $c$-th component if it has received $t_b$ messages with $v_{k,c}^{(\hat{s} - 1)} = 1$ or has set $v_{i,c}^{(\hat{s})} = 1$ and broadcast its $m_i^{(\hat{s})}$ message. Since $i'$ has helped propagating the step $\hat{s} - 1$ messages it has received by time $\alpha_{i'} + t^{(\hat{s})}$, all honest users finalize the $c$-th component during step $\hat{s} + 1$.

Again, they will reach $c$-agreement over $\bot$ on $\Theta_{i,c}^{(s)}$ for all $s \geq \hat{s}$, they will set $v_{i,c}^{(\hat{s} + 1)} = 1$ and $f_{i,c} = 1$, finalizing the component $c$ within the end of step $\hat{s} + 1$.

**Case 2.2.a:** event $E_a$ happens and there does not exist an honest user $i' \in PK$ who should also finalize $c$-th component.

In this case, note that $\hat{i}$ could have received or generated $t_b$ step $\hat{s} - 1$ messages with $v_{k,c}^{(\hat{s} - 1)} = 0$. However, the malicious users may not help propagating those messages, so we cannot conclude that the honest users will receive them after time $\lambda$. In fact, $|H^{(\hat{s})}|$ of those messages may be from malicious players, who did not propagate their messages at all and only sent them to the other malicious players cooperating with them.

Therefore, the honest users will wait for time $t^{(\hat{s})}$ without finalizing component $c$. However, by Property 4 of Lemma 4.2, they will not see more than $t_b$ of the messages received with $v_{k,c}^{(\hat{s} - 1)} = 1$, again because with overwhelming probability $|H^{(\hat{s})}| + 2|H^{(\hat{s} - 1)}| < 2t_b$. Since step $\hat{s}$ is a Coin-Fixed-To-0 step, every honest player $i \in H^{(\hat{s})}$ thus sets $v_{i,c}^{(\hat{s})} = 0$ and propagates its message at time $\alpha_i + t^{(\hat{s})}$.

During step $\hat{s} + 1$, which is a Coin-Fixed-To-1 step, two things may happen:

1 an honest user receives the $t_b$ messages received by $\hat{i}$ (who decided to propagate them and let it finalize component $c$): in this case the situation is similar to Case 2.1.a, and every honest user $i$ will finalize its $c$-th component within time $\alpha_i + t^{(\hat{s} + 1)} + \lambda$;
2 the honest users will receive at least $|H^{(\hat{s})}| > t_b$ with overwhelming probability) messages with $v_{k,c}^{(\hat{s})} = 0$ from the honest players. Then they propagate their messages with $v_{i,c}^{(\hat{s} + 1)} = 0$ but do not finalize since step $\hat{s} + 1$ is not a Coin-Fixed-To-0 step.

In this case, in step $\hat{s} + 2$ which is a Coin-Genuinely-Flipped step, two things may happen:

2.1. if $\hat{i}$ broadcasts the step $\hat{s} - 1$ messages that let it finalize its $c$-th component, then the honest users will finalize their $c$-th component as well setting $v_{i,c}^{(\hat{s} + 2)} = 0$ and $f_{i,c} = 1$, hence reaching $c$-agreement over $\Theta_{i,c}^{(s)} = x$ for all $s \geq \hat{s} + 2$ within time $\alpha_i + t^{(\hat{s} + 2)}$;
2.2. otherwise all honest users have received all step \( s \) + 1 messages from the honest players with \( v_{i,c}^{(s+1)} = 0 \). Again they are more than \( t_H \), so the honest users set \( v_{i,c}^{(s+2)} = 0 \) without flipping the coin. Again, they do not finalize component \( c \) since \( s + 2 \) is not a Coin-Fixed-To-0 step so they just broadcast their step \( s + 2 \) messages.

Finally, step \( s + 3 \) is a Coin-Fixed-To-0 step, so everyone will receive at least \( t_H \) messages with \( v_{i}^{(s+2)} = 0 \) where \( i \in \mathcal{H}\mathcal{P}^{(s+2)} \). Then all honest users \( k \) at time \( \alpha_k + f^{(s+3)} \) can finalize component \( c \) setting \( v_{k,c}^{(s+3)} = 0 \) and \( f_{k,c} = 1 \) and hence reach c-agreement over \( \Theta_{i}^{(s)} = x \) for all \( s \geq \hat{s} + 3 \).

Depending on how \( i \) and in general the malicious users behave, some users may finalize the component \( c \) within the end of step \( s \) (with \( s \in \{\hat{s}, \hat{s} + 1, \hat{s} + 2\} \)) using step \( \hat{s} - 1 \) messages, or within the end of step \( \hat{s} + 3 \) with step \( \hat{s} + 2 \) messages. It does not matter since c-agreement is reached anyway over \( \Theta_{i}^{(s)} \) for all \( s \geq \hat{s} \), the component \( c \) is finalized setting \( f_{i,c} = 1 \) within step \( \hat{s} + 3 \), and \( v_{i,c}^{(s)} = 0 \) for every step \( s \) such that \( s \geq \hat{s} + 3 \).

**Case 2.2.b:** event \( E:b \) happens and there does not exist an honest user \( i' \in \text{PK} \) who should also finalize \( c \)-th component.

The analysis in this case is similar to Case 2.1.b and Case 2.2.a, thus many details have been omitted.

We know that \( i \) has collected or generated at least \( t_H \) step \( \hat{s} - 1 \) messages with \( v_{k,c}^{(\hat{s} - 1)} = 1 \) and \( \hat{s} - 1 \equiv 1 \mod 3 \) (hence \( \hat{s} \) is a Coin-Fixed-To-1 step) and that no honest player could have seen more than \( t_H \) messages for 0. Thus each honest player \( i \in \mathcal{H}\mathcal{P}^{(\hat{s})} \) sets \( v_{i,c}^{(\hat{s})} = 1 \) and propagates its message \( m_{i}^{(\hat{s})} \) at time \( \alpha_{i} + f^{(\hat{s})} \). Similar to Case 2.2.a, within 3 steps user \( i \) will finalize their \( c \)-th component.

Then c-agreement is reached over \( \Theta_{i}^{(s)} \) for all \( s \geq \hat{s} \), and the component \( c \) is finalized setting \( f_{i,c} = 1 \) within step \( \hat{s} + 3 \) and \( v_{i,c}^{(s)} = 0 \) for all steps \( s, s \geq \hat{s} + 3 \).

Combining the four sub-cases, we obtain:

- \( T_c \leq f^{(s)} + \lambda \) in Case 2.1.a and Case 2.1.b;
- \( T_c \leq f^{(\hat{s} + 3)} + \lambda \) in Case 2.2.a and Case 2.2.b;

but we also have that they all are in agreement at the end of step \( \hat{s} \), and that c-agreement is reached over \( \Theta_{i}^{(s)} \) for all \( s \geq \hat{s} \).

Now we can prove Proposition 4.6.

**Proof of Proposition 4.6.** Given the results of Lemma 4.10, it remains to upper-bound \( \hat{s} \) and thus \( T_c \). We do that by considering how many times the Coin-Genuinely-Flipped steps are executed by at least one honest player.

If no honest player flips the coin in a Coin-Genuinely-Flipped step \( s \), it means that they all have received more than \( t_H \) messages with \( v_{k,c}^{(s-1)} = b \in \{0, 1\} \) and c-agreement has been reached, letting them finalize the component in at most 2 more protocol steps. Moreover if they reach c-agreement over 0, this means that they agree on the same value \( O_{k,c} \) to insert as \( c \)-th component of the vector.
4.2 Component-Wise Finalization

$\Theta_{s}$. Once $c$-agreement is reached in step $s$, the honest players will finalize the $c$-th component either in step $s + 1$ or step $s + 2$ depending on whether $b = 0$ or $b = 1$.

By Lemma 4.9, at every Coin-Genuinely-Flipped step $c$-agreement is reached with probability at least $\frac{1}{2}$, so we can compare this step to a Bernoulli trial that outputs 1 if $c$-agreement is reached. This means that, before step $\hat{s}$ (the first step in which a user can finalize the $c$-th component), the distribution of the number of times the Coin-Genuinely-Flipped steps are executed to finalize a component $c$ can be upper-bounded by to the random variable $L$, which we recall represents the number of Bernoulli trials needed to see a 1 when each trial gives 1 with probability $\frac{1}{2} > \frac{1}{3}$. Letting $s'$ be the last Coin-Genuinely-Flipped step before the finalization of the $c$-th component, then we have, by the protocol construction, $s' = 3 + 3L$.

Assuming that the adversary knows the outcome of $L$ in advance, when should the adversary make step $\hat{s}$ happen to maximize the delay of the finalization time $T_c$ of the $c$-th component by an honest user?

If $\hat{s} > s'$ (hence $\hat{s} = s' + 1$ or $\hat{s} = s' + 2$) then this means that we are in Case 2.1.a or Case 2.1.b of Lemma 4.10 since at the end of step $s'$ the honest players are already in agreement, so when a malicious player could finalize, also the honest users can, hence:

$$T_c \leq t^{(s') + 1} + \lambda \leq t^{(s' + 2)} + \lambda.$$

If $\hat{s} < s' - 3$, that is $\hat{s}$ is before the second to last Coin-Genuinely-Flipped step, then by the analysis of Case 2.2.a or Case 2.2.b we get:

$$T_c \leq t^{(\hat{s} + 3)} + \lambda \leq t^{(s')} + \lambda.$$

that is, the Adversary is making the agreement on component $c$ happen faster.

If $\hat{s} = s' - 1$ or $\hat{s} = s' - 2$, then $\hat{s}$ is the Coin-Fixed-To-0 or Coin-Fixed-To-1 step before $s'$. By the analysis of the 4 sub-cases we know that the honest players never flip the coin and finalize the $c$ component within the next two steps. Therefore:

$$T_c \leq t^{(\hat{s} + 3)} + \lambda \leq t^{(s' + 2)} + \lambda.$$

To summarize, no matter what $\hat{s}$ is, we have:

$$T_c \leq t^{(s' + 2)} + \lambda \leq t^{(3L + 5)} + \lambda,$$

which upper-bounds the time needed to reach agreement on the $c$-th vector component.

Now we will prove Theorem 4.1, using the results of the previous Lemmas and Propositions.

**Proof of Theorem 4.1.** We will prove that all the honest users will agree on the same $\Theta$, that no malicious user can build a valid certificate for a different $\tilde{\Theta}$, and that the honest users will be able to produce a certificate for $\Theta$ within time $t^{\left(\frac{5}{2} + 3\chi + \lambda\right)} + \lambda$. To conclude the proof it is sufficient to
4.2 Component-Wise Finalization

Note that, since \( t^{(s+1)} = t^{(s)} + 2\lambda \) for all \( s \geq 3 \), we have that:

\[
t^{(5+3\chi_{\ell, h})} = t^{(1)} + \sum_{s=2}^{5+3\chi_{\ell, h}} (t^{(s)} - t^{(s-1)})
\]

\[
= t^{(1)} + (t^{(2)} - t^{(1)}) + (t^{(3)} - t^{(2)}) + \sum_{s=4}^{5+3\chi_{\ell, h}} (t^{(s)} - t^{(s-1)})
\]

\[
= \Omega + \Lambda + \lambda + \sum_{s=4}^{5+3\chi_{\ell, h}} 2\lambda = \Omega + 2\Lambda + 2\lambda + \left(2 + 3\chi_{\ell, h}\right)2\lambda
\]

\[
= \Omega + 2\Lambda + \left(6 + 6\chi_{\ell, h}\right)\lambda.
\]

Let \( m - \ell \) be the number of unambiguous components, we have shown in Proposition 4.4 that these components will reach agreement in at most 5 steps. The same will happen for some of the \( \ell \) ambiguous components according to Proposition 4.5, while the others, as shown in Proposition 4.6, will reach agreement within a number of steps whose distribution is upper-bounded by the random variable \( 3L + 5 \).

The Cob Protocol runs until a certificate for a vector \( \Theta \in V \) is created, this happens no later than the moment in which every component is finalized by the honest players, which happens once \( c \)-agreement is reached on each of the \( \ell \) ambiguous components.

Since we have shown in Lemma 4.9 that with probability at least \( \frac{h}{2} \) the honest players will reach \( c \)-agreement on a single component, and once agreement is reached it is maintained for the whole protocol run, then agreement will be reached in at most \( \chi_{\ell, h} \) Coin-Genuinely-Flipped steps (accordingly to the analysis in Proposition 4.6, malicious users might speed the consensus process up) where \( \chi_{\ell, h} \) is the same random variable described in [7].

Every honest user will be able to obtain a certificate for a block at the end of the two steps following the \( \chi_{\ell, h} \)-th Coin-Genuinely-Flipped step according to Cob Protocol definition and the analysis performed up to now. Therefore it holds \( T \leq t^{(5+3\chi_{\ell, h})} + \lambda \).

Now we show that if a certificate is created for the first time in step \( s \) for a vector \( \Theta \in V \), then any certificate created will be for the same vector \( \Theta \).

Let step \( s \) be the first step in which a user \( k \) is able to collect a certificate \( C_k \) for \( \Theta \). We recall that, by the assumptions in Section 2.3, given two distinct players \( i, j \in Ps \) of the same step, it is negligible the probability that \( i \) collects \( \tau_B \) messages for a vector \( \Theta_i^{(s-1)} \) and \( j \) collects \( \tau_B \) messages for a distinct vector \( \Theta_j^{(s-1)} \).

As before, we can assume that the user \( k \) is malicious and the certificate is made of step \( s - 2 \) and step \( s - 1 \) messages, where step \( s \) is a Coin-Fixed-To-1 step.

We distinguish two cases:

- **There is an honest user \( k' \) who also can collect a certificate \( C_{k'} \) for \( \Theta \) in step \( s \).**

  In this case \( k' \) has propagated the messages which let it certify the vector \( \Theta \), hence all honest users will be in possess of a certificate (possibly a different one) for the same vector \( \Theta \), so the honest users will agree on the same vector \( \Theta \). Also, the honest users will end the protocol...
execution, so there is no chance that another certificate is produced in the following steps since with overwhelming probability \( MP(s) < t_H \forall s \).

- **There is no honest user** \( k' \) **who also can collect a certificate** \( C_{k'} \) **for** \( \Theta \) **in step** \( s \).

In this case the honest users will keep executing the protocol until they can create a certificate for a block \( \Theta \) or until they receive from \( k \) the messages that allowed \( k \) to create the certificate in step \( s \).

We will show that if they do not receive the certificate from \( k \), then they will build a certificate for \( \Theta = \Theta \). This guarantees that honest users will agree on the same vector, since there are no two valid certificates around the network for two distinct vectors.

If the user \( k \) has built a certificate for \( \Theta \) in step \( s \), it means that \( k \) has received \( t_H \) messages from step \( s - 2 \) and step \( s - 1 \) for \( \Theta \). In particular, \( k \) collected \( t_H \) step \( s - 2 \) messages for 0 in every component \( c \) such that \( \Theta_c \neq \perp \). We will call \( I_0 = \{ c : \Theta_c \neq \perp, 1 \leq c \leq m \} \). By the assumptions in Section 2.3, no honest user has received at least \( t_H \) messages from step \( s - 2 \) that sponsors 1 in a component \( c \in I_0 \). This implies that all the honest players have sent in step \( s - 1 \) a message with 0 in each component \( c \in I_0 \). This brings all the honest users in \( c \)-agreement on such components and it will keep holding in the following steps.

We also know that the user \( k \) has received \( t_H \) messages from step \( s - 1 \) for \( \Theta \). This means that \( k \) has received at least \( t_H \) messages for 1 in every component \( c \) such that \( \Theta_c = \perp \). We will call \( I_1 = \{ c : \Theta_c = \perp, 1 \leq c \leq m \} \). Again, by the assumptions in Section 2.3, no honest user has received at least \( t_H \) messages from step \( s - 1 \) that sponsor 0 in a component \( c \in I_1 \). Therefore, all the honest players send, in step \( s \), a message with 1 in each component \( c \in I_1 \). This brings all the honest users in \( c \)-agreement on such components and it will keep holding in the following steps.

Now we note that \( I_0 \cup I_1 = \{ 1, \ldots, m \} \), hence all the honest users are in agreement on all vector components. Therefore, in step \( s + 3 \), which is again a Coin-Fixed-To-1 step, they will be able to build a certificate for the vector of relevant information \( \Theta \) using their messages of step \( s + 1 \) and step \( s + 2 \).

\[ \square \]

### 5 Conclusions

We presented the Cob protocol, an extension of the MBA protocol \([7]\) which allows the nodes of a wide asynchronous gossiping network to reach consensus on a vector of arbitrary values, working in parallel on each component.

This generalization widens the applications of the original protocol thanks to the sortition mechanism that limits the number of messages to be broadcast and processed at each step with many players, and the relaxed network assumptions which model real-case scenarios more closely. Notice that the protocol retains the leaderless approach of the MBA protocol, a democratic feature that is valued in permissionless distributed settings and thwarts attacks from malicious leaders. Moreover, it is also conserved the parallel approach that enhances efficiency in comparison to multiple executions of monodimensional protocols.
5.1 Future Works

An interesting application of the Cob protocol could be to be serving the coordination of a blockchain platform that implements sharding. In fact, the MBA Protocol would allow the nodes working on different shards to synchronize their operations. For example the platform could allocate to miners in each shard precise time-slots in which they can create new blocks, and relegate to the Cob protocol the job of assuring that time-limits are respected. This approach avoids expensive free-for-all proof of work competitions, enhancing the efficiency of block creation, and also foils malicious attempts to fragment the shards’ consensus by holding or delaying blocks. The Cob protocol would help the enforcement of the platform’s rules, creating an extremely regulated environment which gives the right conditions for a practical reconciliation of the transactions recorded on the shards.

5.1 Future Works

The Cob Protocol guarantees to reach consensus if the assumptions are met, and its leaderless and parallel approach maximizes the number of vector components that are finalized on a meaningful value (i.e. $\neq \perp$). However its execution is probabilistic, and although it halts with probability 1, the number of steps necessary to halt have only an upper bound in the form of a Bernoulli-like distribution.

An interesting research direction could focus on extending the protocol by introducing some termination steps, in order to have a fixed upper bound on its execution, which would benefit many concrete applications. Specifically, such an extension would see the protocol running normally up to a pre-determined number of steps, then, if the execution has not halted yet, the protocol starts a sequence of termination steps that guarantee to reach a consensus in a fixed number of steps. In this phase it is quite tricky to try to preserve as much meaningful agreement as possible: the trivial solution is to collapse the agreement on $\perp$ if consensus is not reached in time, but avoiding to do so has to account for a wide array of attacks with which malicious players could try to disrupt agreement.

References

[1] Vitalik Buterin and Virgil Griffith. Casper the friendly finality gadget. \textit{arXiv preprint arXiv:1710.09437}, 2017.

[2] Shiyu Cai. Analysis of committee selection mechanism in blockchain. \textit{arXiv preprint arXiv:1905.05079}, 2019.

[3] Miguel Castro, Barbara Liskov, et al. Practical byzantine fault tolerance. In \textit{OSDI}, volume 99, pages 173–186, 1999.

[4] Jing Chen and Silvio Micali. Algorand. \textit{arXiv preprint arXiv:1607.01341}, 2016.

[5] Jing Chen and Silvio Micali. Algorand: A secure and efficient distributed ledger. \textit{Theoretical Computer Science}, 777:155–183, 2019.

[6] Pesech Feldman and Silvio Micali. An optimal probabilistic protocol for synchronous byzantine agreement. \textit{SIAM Journal on Computing}, 26(4):873–933, 1997.

[7] Andrea Flamini, Riccardo Longo, and Alessio Meneghetti. Multidimensional byzantine agreement in a synchronous setting, 2021.
[8] Leslie Lamport, Robert Shostak, and Marshall Pease. The Byzantine Generals Problem, page 203–226. Association for Computing Machinery, New York, NY, USA, 2019.

[9] Alessio Meneghetti, Tommaso Parise, Massimiliano Sala, and Daniele Taufer. A survey on efficient parallelization of blockchain-based smart contracts. Annals of Emerging Technologies in Computing (AETiC), 3(5), 2019.

[10] Silvio Micali. Byzantine agreement, made trivial, 2016.

[11] Zilliqa Team et al. The zilliqa technical whitepaper. Retrieved Sept, 16:2019, 2017.

[12] Maofan Yin, Dahlia Malkhi, Michael K Reiter, Guy Golan Gueta, and Ittai Abraham. Hotstuff: Bft consensus in the lens of blockchain. arXiv preprint arXiv:1803.05069, 2018.