Directed Polymers on a Factorized Disorder Landscape

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We study the Directed Polymer model subject to a particular form of disorder, \( \eta(x, t) = \eta_x(x)\eta_T(t) \), recently proposed in biological applications. We find that two new universality classes arise, depending on the the lattice geometry. Using an intermediate model linking the two different orientations continuously, we find that there is a phase transition separating two distinct scaling phases. For both phases we get a reasonable understanding of the nature and values of the exponents, corroborated with numerical results.

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In their study of DNA-sequence alignments, Hwa and Lässig consider the mapping of the problem of sequence recognition and matching to a more standard (in statistical mechanics) model of directed polymers (DP) subject to a special disorder \( \eta \). They establish, both analytically and numerically, the equivalence of the two problems. In their study of DNA-sequence alignments, Hwa and Lässig consider the mapping of the problem of sequence recognition and matching to a more standard (in statistical mechanics) model of directed polymers (DP) subject to a special disorder \( \eta \). They establish, both analytically and numerically, the equivalence of the two problems.

Indeed, we recall the reader to Hwa and Lässig’s work \( \eta \). Naively one may expect that this disorder distribution should give no relevant, since

\[ \eta(x, t) = \eta_x(x)\eta_T(t), \]  

where \( \{\eta_x(x)\} \) and \( \{\eta_T(t)\} \) are independent unbiased (i.e. with zero average) random variables. For the biological motivations to consider this kind of disorder we refer the reader to Hwa and Lässig’s work \( \eta \). Naively one may expect that this disorder distribution should give no difference with respect to the usual uncorrelated disorder. Indeed correlations may appear non-relevant, since

\[ \overline{\eta(x_1, t_1)\eta(x_2, t_2)} = \overline{\eta_x(x_1)\eta_x(x_2)\eta_T(t_1)\eta_T(t_2)} \]  

is zero if \( x_1 \neq x_2 \) and \( t_1 \neq t_2 \), as for the usual uncorrelated disorder. However, the count of the total number of disorder degrees of freedom suggests that the model should be very different. On a 2-dimensional square lattice of linear size \( L \) (\( L^2 \) sites) there are only \( 2L \) degrees of freedom, versus \( L^2 \) degrees of freedom for the usual site disorder. In other contexts this difference leads to relevant and important consequences. Indeed, we recall the Mattis model of spin-glasses \( \eta \), where spin coupling disorder is replaced by a product of disorders, and frustration effects disappear.

In this Letter we systematically study the effects of such a factorized disorder on the properties of directed polymers, both with numerical simulations and with theoretical arguments. We find that two new universality classes emerge, depending on the orientation of the lattice. We then show that it is possible to obtain both universality classes on a same system by varying a parameter, a signature of a phase transition. Finally, we give explanations for the difference between our results and Hwa and Lässig’s ones \( \eta \). Equ. (1) defines the model on a rectangular lattice with a special disorder defined along the \((1,0)\) and \((0,1)\) directions. We can also define the model on the diagonal \((1,1)\) and \((-1,1)\) directions, i.e. using the so-called diamond lattice. In the latter case Eq. (1) becomes

\[ \eta(x, t) = \eta(t + x)\eta(t - x) \]  

as was originally studied in \( \eta \). First we study numerically the model on a diamond lattice, eq. (1). As for the usual study of the ground state of directed polymers (i.e. temperature \( T = 0 \)), we are interested in the scaling properties of the polymers as a function of their length. In particular we consider the transverse wandering fluctuations of the polymer, \(<|\delta x|>\sim t^\nu\) where angular brackets indicate the average over different realizations of disorder (as usual in quenched disorder problems). Also, the ground state energy fluctuations are of interest, \(<|\delta E|>\sim t^{\omega}\).

We always use disorder taken from uniform distributions over compact supports, in general in the interval \([-1/2,1/2]\). We discuss later the effects of biased distributions.

A summary of the exponents obtained is given in Table 1.

For the \( d = 1 + 1 \) diamond lattice we observe indeed the scaling of \( \nu = 2/3 \), consistent with the standard disorder result \( \eta \), see Fig. 3. Yet, the ground state energy fluctuations scale with an exponent \( \omega = 1/2 \) different from the standard \( \omega = 1/3 \). In order to appreciate the full mechanism of the model, and to understand where the difference comes from, we study the time evolution of the energy profiles on diamond lattice strips of various width \( L \). It corresponds to the growth of a KPZ-like \( \eta \) surface in discrete space.

The behavior of the energy profile fluctuations \( \Delta E(L, t) = \langle (E(x, t) - E(x, t))^2 \rangle^{1/2} \) (over-bars represent averages over one single disorder realization) is governed by three exponents. Initially \( \Delta E(L, t) \) grows with time as \( \Delta E(L, t) \sim t^{\beta} \). Then it saturates to a constant value, dependent on the size \( L \) of the system as
\[ \Delta E(L, t) \sim L^x. \]  The characteristic time \( \tau \) separating the two regimes scales with the size \( L \) of the system as \( \tau \sim L^2 \), and \( z \) is called dynamical exponent. Scaling consistency imposes \( z = \chi/\beta \). In standard DP-KPZ theory, the surface exponents and the polymer ones are related by \( \beta = 1/\nu \) and \( \beta = \omega \). Moreover, in standard KPZ theory, the law \( \omega + \chi = 2 \) holds, due to galilean invariance.

The scaling analysis of our simulations gives \( \chi = 1/2 \), as in standard KPZ theory, and \( \beta = 1/2 \). From scaling consistency we have \( z = \chi/\beta = 1 \), which is in agreement with numerical simulations, but that does not satisfy neither the relation \( z = 1/\nu = 3/2 \), nor \( z + \chi = 2 \). The reason behind the above scaling behavior, namely \( z = 1 \) instead of \( z = 1/\nu = 3/2 \), is due to the particular structure of the disorder landscape. Indeed, the diagonal \((\pm 1, 1)\) factorization of disorder imposes a space-time linear proportionality relation, corresponding to a dynamical exponent \( z = 1 \). We therefore find that the usual relation \( z = 1/\nu \) breaks down. Let us now try to explain the 1/2 value of the \( \chi \) and \( \omega = \beta \) exponents. Indeed the \( \omega = 1/2 \) could come from an uncorrelated random walk, whereas \( \chi = 1/2 \) is consistent with the standard result. In order to solve this ambiguity we have analyzed the energy profile evolution for a 2 + 1 dimensional diamond lattice. Again we find \( z = 1 \), strengthening the time-space proportionality interpretation, and \( \beta = \chi = 0.40 \pm 0.01 \), nicely consistent with the standard result [3]. We conclude therefore that the asymptotic behavior is KPZ dominated. At variance with standard KPZ, the approach to the stationary state changes, governed by a ballistic dynamical exponent \( z = 1 \).

Let us now consider the square lattice with disorder defined from Eq. (1). The directed polymer scaling results are shown in Fig. 2. The exponents are consistent with \( \nu = 2/3 \), as in the diamond case, and \( \omega = 2/3 \). From energy profile evolution, we find the exponents \( \beta = 2/3 \) and \( z = 3/2 \) (consistent respectively with \( \beta = \omega \) and \( z = 1/\nu \)). The roughness exponent is then \( \chi = z \beta = 1 \), in agreement with numerical results. The absence of any space-time relation imposed by the disorder structure leaves therefore the usual \( z = 1/\nu \) relation unchanged.

We explain the value 2/3 of the \( \nu \) and \( \omega \) exponents using an argument generalized from columnar disorder theory [3]. Indeed, as emerges from Fig. 3, after an initial wandering, polymers tend to localize in a very narrow region of the lattice, in general consisting of a pair of columns. The corresponding two \( \eta_x(x) \) values have opposite signs and their absolute values are extremely close to 1/2. Polymers are then attracted to such pairs, because they can jump from one column to the other depending on the sign of \( \eta_x(t) \), always choosing the negative product \( \eta_x(t) \eta_x(t) \) in their quest for low energies. We can therefore think that polymers search the best pairs available. The relevant feature of the system is therefore the effective pair energy probability distribution \( P(\epsilon) \). From its knowledge, using variational arguments, it is possible to obtain an analytical estimate of the exponents. Indeed, since the polymer always chooses the negative product, as long as we are interested in an effective pair energy distribution we can disregard the \( \eta_x(t) \) energies, just remembering that the chosen \( \eta_x(x) \) energies are extremal \( (i.e., \) as close as possible to \( \pm 1/2 \)). Moreover a non-restrictive extra condition is that the two \( \eta_x(x) \) energies are equal in absolute value. This is numerically well approximated, since the two energies are both extremely close, in absolute value, to 1/2.

Let us define \( \epsilon \) such that the energies of the two \( \eta_x(x) \) of the pair are \( \pm (1/2 - \epsilon) \). Then \( \epsilon \in [0, 1] \). The probability that the pair energies are closer than \( \epsilon \) to \( \pm 1/2 \) is

\[
\int_{-\epsilon}^{\epsilon} P(\epsilon) d\epsilon = \left[ \int_{-\epsilon}^{\epsilon} P(\epsilon) d\epsilon \right] \left[ \int_{-\epsilon}^{\epsilon} d\epsilon \right] = \epsilon^2 \quad (4)
\]

leading to a probability distribution for \( \epsilon \) that is \( P(\epsilon) = 2\epsilon \). \( Q(\epsilon) \) is then the relevant pair effective distribution.

It is then possible to apply a variational argument to give a prediction on the exponent \( \nu \). Indeed, the polymer wanders initially in a region of characteristic width \( R \), in search of the optimal (pair) energy. Once it has found the optimal pair, it localizes there up to a time \( t \). The global energy can therefore be evaluated as

\[
E = \gamma R + (t - R)\epsilon_{min} \quad (5)
\]

The optimal energy available in a region of width \( R \) can be obtained from the statistics of the extrema. Indeed, the probability \( P_m(\epsilon) \) of the smaller energy among \( R \) energies distributed according to \( P(\epsilon) = (\mu + 1)\epsilon^\mu \) in the interval \([0, 1]\) is

\[
Q_m(\epsilon) = R Q(\epsilon) \left[ 1 - \int_{0}^{\epsilon} P(\epsilon) d\epsilon \right]^{R-1} = \quad (6)
\]

\[
R(\mu + 1)\epsilon^\mu (1 - \epsilon^{\mu+1})^{R-1} \quad (7)
\]

Then the characteristic extremal energy \( \epsilon_{min} \) is

\[
\epsilon_{min} = \int_{0}^{1} \epsilon Q_m(\epsilon) d\epsilon = R B \left[ R - 1, \frac{\mu + 2}{\mu + 1} \right] \quad , (8)
\]

where \( B(x, y) \) is the Euler Beta function. After some manipulation, the scaling behavior of \( \epsilon_{min} \) is found to be \( \epsilon_{min} \sim R^{-1/(\mu + 1)} \) for large \( R \). Inserting this result in (5) and optimizing with respect to \( R \), we find the scaling behavior \( R \sim t^{(\mu + 1)/(\mu + 2)} \) and therefore \( \nu = (\mu + 1)/(\mu + 2) \); in our case \( \mu = 1 \) and therefore \( \nu = 2/3 \), as from numerical simulations. Also, the energy \( E \) scales as \( R \) and therefore \( \omega = \nu \), again in agreement with numerical simulations.

We can also change the probability distribution of the energies along the \( x \) and \( t \) directions. As long as both energies can take on positive and negative values, and the
polymer does not loose any energy wandering through different columns, the universality class does not change. It does not even change if the energies $\eta_X(x)$ take only positive values: the polymers continue to localize on pairs of energies, to maximize the energy gain when the $\eta_T(t)$ energy is negative and to minimize the energy loss when $\eta_T(t)$ is positive. The only different universality class comes when the $\eta_T(t)$ energies assume only positive values: in that case there is no more any convenience for the polymers to bind to a pair, and localization takes place over a single optimal $\eta_X(x)$ energy. Being the energy distribution flat ($\mu = 0$), the exponents are $\nu = \omega = 1/2$, in agreement with numerical results. If instead there is an energy cost to change column, then a transition as a function of the bias in the distributions takes place between the two scaling regimes $\nu = 1/2$ and $\nu = 2/3$. The full phase diagram of the latter situation will be published elsewhere.

This polymer problem defined by (1) is indeed sensitive to the chosen disorder distribution. For instance, using Gaussians distributions, the variational procedure predicts a linear relation between $R$ and $t$ (apart from logarithmic corrections (3)), confirmed by our numerical simulations.

Next, we move on to show that the two different scaling behaviors (namely, $z = 1$ and $z = 3/2$) are compatible on the same disorder configuration. Let us start from the square lattice: a polymer has three possibilities to go one step forward, namely, to the left, in front, to the right. In our simulations, we chose total isotropy, that is, neither of the three choices imply any energy loss, and the $z = 3/2$ result is recovered. We let then one of the directions, say the leftward one, be energetically unfavorable, with an associated energy cost $a$. Consequently the polymer end-points show a drift in time toward the right direction. Indeed, we find that, for very large values of $a$, the polymer average final position moves to the right linearly with time. The fluctuations of the end-points around their average position scale with time with exponent $\nu = 2/3$, and the energy fluctuations scale with exponent $\omega = 1/2$, as for the diamond lattice case. Indeed, looking at the scaling properties of the energy profile, we recover the full set of exponents typical of polymers with factorized disorder on the diamond lattice, Eq.(3). Strong anisotropy induces therefore a linear space-time relation (as emerges from the linear drift of the polymer end-points) that forces a $z = 1$ dynamical exponent. For small values of $a$ (down to the isotropic case $a = 0$), instead, we find the other full set of exponents, typical of disorder (1).

As Fig. 1 shows, by varying $a$ we find the signature of a phase transition between the two regimes $z = 3/2$ and $z = 1$. The three point sets are obtained from collapse plots of the scaling features of energy surfaces for different system sizes. Precisely, the values plotted come from the collapse of data of couples of strips. The values of the width pairs are given in the legend. The data are steeper with increasing strip width in the neighbor of $a = 0.25$. We expect $a = 0.25$ to be precisely the transition value of the anisotropic energy. Indeed, localization takes place when, on the average, it is convenient for polymers to localize even if it has to pay a price $a$: a localized polymer chooses the column of the two where it finds negative disorder energies. Since the sign of $\eta_X(x)$ is fixed, it finds a sequence of $n$ negative values on the same column if $\eta_T(t)$ keeps the same sign $n$ times, and this happens with probability $(1/2)^n$. The average length $\bar{n}$ of a sequence with the same sign is therefore

$$\bar{n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2 .$$

The average energy $|\eta_T(t)| = 1/4$ (the sign depends on the sign of $\eta_X(x)$). The absolute value of the energy $\eta_X(x)$ is extremely close to $1/2$ (again irrespective of the sign, which depends on the sign of $\eta_T(t)$ so that their product is negative). The average energy that a localized polymer gains against $a$ is therefore

$$\bar{c} = -\frac{1}{4} \cdot \frac{1}{2} \bar{n} = -\frac{1}{4} .$$

Therefore if $a < 1/4$ localization takes place. Such a transition was also alluded to in Ref. 2.

Summarizing, we have found that a factorized disorder distribution such as (1) or (3) leads to new universality classes, depending on the orientation of the lattice. We have been able to explain the origin of the characteristic exponents of these classes and to connect them to each other via an anisotropy driven phase transition.

A challenging issue is to connect these results to Hwa et al. 14, where the usual disorder universality class was obtained (3). Indeed, in their work they defined the disorder and the lattice in a slightly different fashion (see Refs. 13 for details). As we showed above, the polymer problem with factorized disorder is very sensitive to the details of the disorder and of the lattice. We believe that the reason underlying this difference resides in the presence, in their work, of some directions along which polymers do not feel any disorder (the gaps in the language of Refs. 13). Indeed, depending on the energy cost of these directions, we found indications of a phase transition between the standard disorder universality class and the diamond lattice one (see Table 3). Further work is needed in this direction, to explore the full phase diagram of the model.
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FIG. 1. Wandering and Energy Fluctuations for polymers on the diamond lattice; data are obtained after averaging over 10000 disorder realizations.

FIG. 2. Wandering and Energy Fluctuations for polymers on the diamond lattice; data are obtained after averaging over 10000 disorder realizations. Roughness of the energy profile on strips of different width $L$, again averages are over 10000 samples.

FIG. 3. Ground state polymers on a square lattice. Every polymer corresponds to a different disorder realization.

FIG. 4. Exponent $z$ vs. anisotropy energy $a$, as obtained from collapse plots of roughnesses for different strip widths.

| disorder | Diamond | Square | Standard |
|----------|---------|--------|----------|
| $\nu$    | $2/3$   | $2/3$  | $2/3$    |
| $\omega = \beta$ | $1/2$ | $2/3$  | $1/3$    |
| $z$      | $1$     | $3/2$  | $3/2$    |
| $\chi$   | $1/2$   | $1$    | $1/2$    |

TABLE I. Characteristic exponents for directed polymers and energy surfaces for the diamond lattice (disorder defined from Eq.(3)), square lattice (Eq.(1)), and for the standard disorder [3].