Percolation in a Multifractal

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Abstract

We investigate percolation phenomena in multifractal objects that are built in a simple way. In these objects the multifractality comes directly from the geometrical tiling. We identify some differences between percolation in the proposed multifractals and in a regular lattice. There are basically two sources of these differences. The first is related with the coordination number that changes along the multifractal. The second comes from the way the weight of each cell in the multifractal affects the percolation cluster. We use many samples of finite size lattices and draw the histogram of percolating lattices against site occupation probability. Depending on a parameter characterizing the multifractal and the lattice size, the histogram can have two peaks. We observe that the percolation threshold for the multifractal is lower than one for the square lattice. We compute the fractal dimension of the percolating cluster and the critical exponent $\beta$. Despite the topological differences, we find that the percolation in a multifractal support is in the same universality class of the standard percolation.

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1- INTRODUCTION

Percolation theory has been used in several fields such as chemistry, epidemics, science of materials, transport of fluids in porous media, branched polymers, and econophysics [1, 2, 3, 4, 5, 6, 7]. The original percolation model based on a square lattice has been extended to several kinds of regular and random lattices, to continuous media where the objects overlap in space, and to other complex systems [8, 9, 10, 11]. In this work we generalize the percolation theory to cover an even broader range of complex systems. We devised an approach to investigate how percolation occurs in a support that is itself a multifractal. For this purpose we have constructed an easy assembling multifractal immersed in a 2D space.

Our work is inspired in the modeling of geophysical natural objects that show multifractal properties [12, 13, 14, 15]. The model can be applied to transport of fluid in a multifractal porous media such as sedimentary strata. Oil reservoirs are possible candidates to be modeled in such a way since that the measurement of some physical quantities in well-logs show multifractal behavior [16, 17]. Despite the potential applications inspiration, this problem is important by itself in the scientific context. The study of percolation phenomena in multifractal lattices is relevant in Statistical Physics specially when the size of the blocks and their number of neighbors can vary.

In order to make this analysis we create a multifractal object that can be used as a toy model and a laboratory for percolation theory. An important characteristic of this object is that its topologic properties (e.g. number of neighbors of each block) change over the object. In the reference [18] an algorithm that has some resemblance with ours is used. That multifractal is built from the partition of the square, but the object has a trivial topology. Besides, the object used in [18] is aleatory and ours is deterministic. Although both models present multifractality, our model has the following differences: it shows a non-trivial topology, we can determine analytically its spectrum of fractal dimensions, it generalizes the square lattice, and it shows simplicity in the construction.

The multifractal object we have developed is a natural generalization of the regular square lattice once we consider the algorithmic point of view. The algorithm that generates a square lattice with $2^n \times 2^n$ cells starting with a square of fixed size is the following. We begin with a $L \times L$ square and we cut if in 4 identical pieces (cells). At each step all the cells are equally divided in 4 parts using vertical and horizontal segments. This process produces a lattice as a partition of the square. The multifractal we create is also a partition of the square, but the ratio we divide the cells is different from $L/2$. The parameter characterizing the multifractal, $\rho$, is related to the ratio of this division.

What makes this problem new and appealing to physics is the following. The support of the percolation clusters is composed by subsets of different fractal dimensions. It is important to know how these different subsets are connected and how they participate in the conducting process. There are intriguing features in the network due to the fact
that all cells have rectangular shape but the area and the number of neighbors can vary, forming an exotic tiling.

In section 2 we present the multifractal object that we construct to study percolation, and we analyze how its multifractal partition maps into the square lattice. In section 3 we expose the algorithm we use to estimate percolation threshold and derive the multifractal spectrum of the multifractal object. In section 4 we show numerical results and discuss the histograms of percolating lattices versus occupation probability. Finally in section 5 we summarize the main differences between percolation in a regular lattice and in a multifractal support.

2- THE MULTIFRACTAL OBJECT $Q_{mf}$

The central object of our analysis is a multifractal object that we call $Q_{mf}$. Before defining it we enumerate some of its properties.

- $Q_{mf}$ is a multifractal, it means, $Q_{mf}$ has an infinite number of $k$-subsets each one with a distinct fractal dimension $D_k$.
- It is possible to determine analytically the spectrum of all $D_k$.
- The sum of all the families of $k$-subsets fills the square. This fact enables us to study its percolation properties using procedures similar to the ones applied in the site percolation in the square lattice.
- The algorithm of construction of $Q_{mf}$ has just one parameter $\rho$.
- For the special choice $\rho = 1$ the object $Q_{mf}$ degenerates into the square lattice. In this case we compare our results with the square lattice site percolation.
- The object $Q_{mf}$ shows self-affinity or self-scaling depending on the region of the object.
- Finally, the algorithm for construction of $Q_{mf}$ is simple and it is of easy implementation in the computer.

We define $Q_{mf}$ through the following algorithm. We start with a square of linear size $L$ and choose a parameter $0 < \rho < 1$, where $\rho = \frac{r}{s}$ for $r$ and $s$ integers. In the first step, $n = 1$, the square is cut in two pieces of area $\frac{r}{s+r} = \frac{1}{1+\rho}$ and $\frac{s}{s+r} = \frac{\rho}{1+\rho}$ by a vertical line (we use $L^2$ units). In other words, the square is cut according to a given $\rho$. This step is shown in figure 1(a), in this figure we use $\rho = \frac{r}{s} = \frac{2}{3}$.

In the second step, $n = 2$, we cut the two rectangles of figure 1(a) by the same $\rho$, but using two horizontal lines as shown in figure 1(b). This partition of the square generates
four rectangular blocks: the smallest one of area \((\frac{\rho}{1+\rho})^2\), two of them of area \(\frac{\rho}{(1+\rho)^2}\) and the largest one of area \((\frac{1}{1+\rho})^2\), in the figure \(\rho > 0.5\).

The third step, \(n = 3\), is shown in figure (c) and the fourth step, \(n = 4\), in (d). As observed in figure, at level \(n = 4\) there are \(2^4\) blocks and the distribution of areas among the blocks follows the binomial law:

\[
1 = \left(\frac{\rho}{1+\rho}\right)^4 + 4\left(\frac{\rho}{1+\rho}\right)^3\left(\frac{1}{1+\rho}\right) + 6\left(\frac{\rho}{1+\rho}\right)^2\left(\frac{1}{1+\rho}\right)^2 + 4\left(\frac{\rho}{1+\rho}\right)\left(\frac{1}{1+\rho}\right)^3 + \left(\frac{1}{1+\rho}\right)^4. \tag{1}
\]

We call the elements with the same area as a \(k\)-set. In the case \(n = 4\) we have five \(k\)-sets.

At step \(n\) the square has \(2^{n-1}\) line segments, \((n+1)\) \(k\)-sets and \(2^n\) blocks. The partition of the area \(A = 1\) (using \(L^2\) units) of the square in different blocks follows the binomial rule:

\[
A = \sum_{k=0}^{n} C_n^k \left(\frac{\rho}{1+\rho}\right)^k \left(\frac{1}{1+\rho}\right)^{n-k} = \left(\frac{1+\rho}{1+\rho}\right)^n = 1. \tag{2}
\]

As \(n \to \infty\) each \(k\)-set (a subset made of cells of same area) determines a monofractal whose dimension we calculate in the next section. The ensemble of all \(k\)-sets engenders the multifractal object \(Q_{mf}\).

3- THE ALGORITHM OF PERCOLATION AND THE MULTIFRACTAL SPECTRUM

In this section we show the algorithm used to study the percolation properties of \(Q_{mf}\) and the analytic derivation of its spectrum of fractal dimensions. The estimation of the spectrum, \(D_k\), is performed using the box counting method \[23\] whose measure elements came from the percolation algorithm.

The concept of the percolation algorithm for \(Q_{mf}\) consists in mapping it into the square lattice. The square lattice should be large enough that each line segment of \(Q_{mf}\) coincides with a line of the lattice. Therefore we consider that the square lattice is more finely divided than \(Q_{mf}\). In this way all blocks of the multifractal are composed by a finite number of cells of the square lattice.

To explain the percolation algorithm we suppose that \(Q_{mf}\) construction is at step \(n\). We proceed the percolation algorithm by choosing at random one among the \(2^n\) blocks of \(Q_{mf}\). Once a block is chosen all the cells in the square lattice corresponding to this block are considered as occupied. Each time a block of \(Q_{mf}\) is chosen the algorithm check if the occupied cells at the underlying lattice are connected in such a way to form an infinite percolation cluster. The algorithm to check the percolation is similar to the one used in \[19, 20, 21, 22\].

For the estimation of the spectrum \(D_k\) of an object \(X\) we use the box counting method \[23\]. The object \(X\) is immersed in the plane of real numbers \(\mathbb{R}^2\), we use the trivial metric. Cover \(\mathbb{R}^2\) by just-touching square boxes of side length \(\epsilon\). Let \(N(X)\) denote the number of
square cells of side length $\epsilon$ which intersect $X$. If

$$D_X = \lim_{\epsilon \to 0} \frac{\log N(X)}{\log \frac{1}{\epsilon}} = \lim_{L \to \infty} \frac{\log N(X)}{\log L},$$

is finite, then $D_X$ is the dimension of the $X$.

In our case the object $X$ is a $k$-set. Remember that the $k$-set corresponds to a set of rectangles of the same area. For a $k$-set we have that $N_k$ is given by:

$$N_k = C_n^k s^k r^{(n-k)},$$

where $C_n^k$ is the binomial coefficient that express the number of elements $k$-type, and $s^k r^{(n-k)}$ is the area of each element of this set. If the square is partitioned $n$ times ($\frac{n}{2}$ horizontal cuts and $\frac{n}{2}$ vertical cuts) its size is $L = (s + r)^{n/2}$. Combining all this information we have for the fractal dimension of each $k$-set:

$$D_k = \lim_{n \to \infty} \frac{\log C_n^k s^k r^{(n-k)}}{\log (s + r)^{n/2}},$$

In the $r = s = 1$ case all subsets of $Q_{mf}$ are composed by elements of the same area, square cells. In this way the object is formed by a single subset with dimension:

$$D = \lim_{n \to \infty} \frac{\log (1 + 1)^n}{\log (1 + 1)^{n/2}} = 2,$$

This result is expected once in this particular case $Q_{mf}$ degenerates into the square lattice that has dimension 2.

In figure 2 we show the picture of $Q_{mf}$ for $\rho = \frac{2}{3}$. We have used $n = 12$, in (a) the full object is shown, in (b) a zoom of an internal square of the object is illustrated. We have used the same color to indicate the elements of a same $k$-set. The funny tilling depicted in the figure is common to $Q_{mf}$ with different values of $\rho$.

Figure 3 shows the spectrum of $D_k$ for $n = 400$ calculated from equation (5). The use of increasing $n$ does not change the shape of the curve, it only increases the number of $k$ and makes the curve appear more dense. We use $(s, r) = (2, 3)$ to illustrate the asymmetry of the distribution. The spectrum has a maximum close to $\rho n$. In this case $\frac{2}{3}400 \simeq 270$. It means that the majority of mass of the multifractal is concentrated in the $k$-sets around this value. The spectrum $D_k$ is typically asymmetric around its maximum. Only the case $(s, r) = (1, 1)$ is symmetric and the asymmetry of $D_k$ increases as $\frac{s}{s+r} \to 1$, which is related to the area distribution among the blocks as we shall see in the next chapter.

4- NUMERICAL SIMULATIONS

In this chapter we focus our attention on the numerical results obtained from the algorithm exposed above. We are interested mainly in analyzing the percolating properties
of $Q_{mf}$. Figure 4 (a) shows the histogram of percolating lattices versus the occupation probability $p$. The area under the histogram is normalized to unity. We use $n = 10$ and average the results over 40000 samples. We consider that a lattice percolates when it percolates from top to bottom or from left to right. The histogram of percolating lattices at both directions is similar but slightly shifted to the right. This shift is common in percolation (see the reference [20] for percolation in the square lattice).

We show in figure 4 (a) the results of simulations for the following values of $(s, r)$: $(1, 1)$, which degenerates into the square lattice; and $(2, 1), (4, 1), (6, 1)$ which correspond to truly multifractals. In this figure the histograms corresponding to $(2, 1), (4, 1)$ and $(6, 1)$ are shifted to the left compared to the histogram of $(1, 1)$. The peak of the histogram for $(1, 1)$ corresponds, as expected, to the square lattice size percolation threshold, $p_c = 0.597$, since this case matches exactly the square lattice. The other values of $p_c$ are shown in Table I.

The reason why $Q_{mf}$, for diverse $\rho$, shows roughly the same $p_c$ comes from the topology of the multifractal. The topology of a set of blocks is related with the coordination number, $c$, which is defined as the number of neighbors of each block [2]. $Q_{mf}$ has the property that $c$ changes along the object and with $\rho$. However, we compute the average coordination number $c_{ave}$. These results do not depend significantly on $\rho$, neither on $n$, the number of steps to build $Q_{mf}$, which determines the number of blocks. The value found, $c_{ave} = 5.436$, for the multifractal is close to the value of $c$ of the triangular percolation problem which has $c = 6$ and whose analytic percolation threshold is $p_c = 0.5$. The situation $(s, r) = (1, 1)$, the square lattice, shows trivially $c = 4$. Because the square lattice has a different $c$ it configures a particular situation compared to other $Q_{mf}$ and it shows a different $p_c$ as depicted in figure 4 (a).

In table I we show $p_c$ and the fractal dimension of the percolating cluster, $d_f$, for diverse $\rho$. We have done an average over 100000 samples and $n = 16$. The estimation of $d_f$ is done by the relation $M \sim L^{d_f}$ for the ‘mass’, $M$, of the percolation cluster, that means, the area of the cluster measured in units of the underlying square lattice, and $L$, the size of the underlying lattice. Based on the values of $d_f$ of table I we conclude that the percolation on a multifractal support (imbeded in two dimensions) belong to the same class of universality than the usual percolation in two-dimensions. The calculated value of $d_f$ for the $(6, 1)$ case is smaller compared to the others because of finite size effects. We discuss in detail this effect in the following paragraphs.

Percolation shows critical phenomena and several scaling relations are observed. The critical exponent $\beta$ is defined from the equation:

$$R_L \sim (p_c(L) - p_c)^\beta,$$

where $p_c$ is the exact occupation probability value in contrast to $p_c(L)$ which is the finite size value. The power-law (7) is verified for $p_c(L)$ obtained from $R_L$. The numerical
estimation of $\beta$ is based in equation (7) where $R_L$ is a key element of the analysis. For $Q_{mf}$ the probability $R_L$ is not a well behaved function of $p$ for low $L$ as we shall see in the next paragraphs. Actually, $R_L$ can show, depending on $\rho$, an inflection point at $p_c$ in this regime. However, in the case where $L \to \infty$ the scaling of $(p_c(L) - p_c)$ recovers an usual behavior. In this regime we find the same $\beta$ characteristic of two dimension case, $\beta = \frac{5}{36} = 0.13888$. We checked in our simulations that for $n = 18$, $\beta$ is around 5% of the exact value. The full set of values of $\beta$ is in table I.

| Table I  |
|----------|
| $(s, r)$ | (1,1) | (2,1) | (3,1) | (3,2) | (4,1) | (5,1) | (6,1) |
| $p_c$    | 0.593 | 0.527 | 0.526 | 0.526 | 0.525 | 0.525 | 0.530 |
| $d_f$    | 1.895 | 1.900 | 1.911 | 1.890 | 1.902 | 1.929 | 1.842 |
| $\beta$  | 0.127 | 0.128 | 0.140 | 0.141 | 0.141 | 0.118 | 0.109 |

It is worth to say that, despite small fluctuations in the values shown in the table, there is no tendency in the numbers. The conclusion we take from this data is that the errors are caused by finite size effects and low-number statistics.

The dispersion of the histogram changes significatively with $(s, r)$ as intuitively expected. To illustrate the changing in the width of the histogram of a generic $(s, r)$ multifractal we analyze the area of its blocks. At step $n$ of the construction of $Q_{mf}$ the largest element has the area $\frac{s^n}{(s+r)^n}$ and the smallest $\frac{r^n}{(s+r)^n}$ (using $L^2$ units). In this way the largest area ratio among blocks increases with $(\frac{s}{r})^n$. As the occupation probability, entering in the percolation algorithm, is in general proportional to the area of the blocks we expect that the width of the histograms in figure 4 (a) increases with $(\frac{s}{r})^n$. This increase in the dispersion is visualized clearly in the curves $(2, 1)$ and $(4, 1)$ of the figure.

The most singular curve in figure 4 (a) is $(6, 1)$ which shows clearly two peaks. We stress this point when we comment figure 5. Figure 4 (b) uses the same data of figure (a), but instead of the histogram of percolating lattices we show the cumulative sum, $R_L$. As $R_L$ is normalized, this parameter is also called the fraction of percolating lattices. As in 4 (a) the case $(s, r) = (1, 1)$, the square lattice, reproduces the results of literature 20. In this situation the lattice size, $L$, is $L = (s + r)^{10} = 1024$. For this special case the number of blocks is equal to the number of unit boxes covering the surface. The double peak case $(s, r) = (6, 1)$ shows an inflection point in the graphic of $R_L$ versus $p$. In the following figure we explore in detail this point.

The most noticeable signature of percolation in the multifractal $Q_{mf}$ is the double peak observed for $(s, r) = (6, 1)$ in figure 5. In this figure the histograms of percolating lattices versus $p$ is plotted for diverse $n$ as indicated in the figure. The distance between the peaks decreases as $n$ increases. This picture indicates that the double peak is a phenomenon that is relevant for percolation in the multifractal, when $\rho$ is low, in the finite lattice size condition used in the simulation. From an analytic point of view the curve $(6, 1)$ in figure...
5 is different from curve (1,1). In curve (6,1) there are three extrema points while in the (1,1) case the curve shows a single maximum point. We conjecture that in the limit of \( n \to \infty \) these three points coalesce into a single one and all the curves show a similar behavior.

The two peaks in the histogram come from the huge difference among the area of the blocks of \( Q_{mf} \). For large \((\frac{s}{s+r})^n\) the area difference is so accentuated that we model the histogram of percolating lattices as a bimodal statistics. In the case of the largest block be chosen the multifractal easily percolates compared with the opposite possibility. To estimate the effect of the largest area block on the statistic we make Table II. The difference between the first peak at \( p_1 \) and the second one at \( p_2 \) is \( \Delta p_{max} \). In Table II we compare \( \Delta p_{max} \) with the fraction of the largest block over the total square area \((\frac{s}{s+r})^n\). This comparison is made for different steps in the construction of the multifractal \( n \), as \( n \) increases the area difference decreases as well as the distance between peaks. Table II shows a good agreement between both values, we conclude that the bimodal statistic is caused by the huge mass of the largest block.

**Table II**

| \( n \) | 8   | 10  | 12  | 14  | 16  | 18  |
|--------|-----|-----|-----|-----|-----|-----|
| \( \Delta p_{max} \) | 0.29 | 0.22 | 0.15 | 0.11 | 0.070 | 0.040 |
| \((\frac{s}{s+r})^n\) | 0.291 | 0.211 | 0.157 | 0.115 | 0.084 | 0.062 |

We notice, however, that the agreement between \( \Delta p_{max} \) and \((\frac{s}{s+r})^n\) decreases as \( n \) increases. We interpret the disagreement between the bimodal statistics hypothesis and the numerics for high \( n \) as the limit of the hypothesis. Actually, the largest block is not the only one that produces anisotropy in the multifractal, and as \( n \) increases this fact becomes more accentuated. For small \( n \) the large block can be taken as the main factor of anisotropy, and the bimodal statistics apply. Large \( n \) implies, however, truly multifractals and a more complex statistics should be used to treat the problem.

5- CONCLUSION

In this work we develop a multifractal object, \( Q_{mf} \) to study percolation in a multifractal support. Besides of \( Q_{mf} \) being a multifractal, it shows several interesting properties. The sum of all its fractal subsets fills a square and it is possible to determine the spectrum of its fractal dimensions. In addition, the algorithm that generates \( Q_{mf} \) has only one free parameter \( \rho \), and in the \( \rho = 1 \) case \( Q_{mf} \) becomes the square lattice.

We observe that percolation in a multifractal presents different features from percolation in a regular lattice. There are two reasons for that: the heterogeneous distribution of weight (area) among the blocks and the variation of the coordination number of the
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FIGURE LEGENDS

FIG. 1: The four initial steps in the formation of $Q_{mf}$. In (a) the vertical line cutting the square in two pieces of area ratio $\rho$. Two horizontal lines sharing the rectangles by the same ratio are depicted in (b). The third step is indicated in (c) and the fourth in (d). At each step the areas of the respective blocks are shown in figure.

FIG. 2: The figure shows the three views of the multifractal $Q_{mf}$, it is used $n = 12$, $(s, r) = (3, 2)$. In (a) we have the original picture, figure (b) is a zoom of the square indicated in (a).

FIG. 3: The spectrum of fractal dimensions $D_k$ of $Q_{mf}$ for $n = 400$ and $(p, q) = (3, 2)$. 
FIG. 4: In (a) is depicted the histogram of percolation lattices versus the occupation probability $p$ for the cases $(s, r) = (1, 1), (2, 1), (4, 1)$, and $(6, 1)$. The areas under the curve are normalized to unity. For the same $(s, r)$ is shown in (b) the graphic of the fraction of percolation lattices $R_L$ versus $p$. It is used 40000 lattices to make the average.

FIG. 5: The histogram of percolated lattices versus the occupation probability $p$ for several values lattice size. The graphic shows the double peak approaching each other as $n$ increases, in the figure $(s, r) = (1, 6)$ and $8 < n < 18$. It is used 40000 lattices to make the average.