Entropy of entangled three-level atoms interacting with entangled cavity fields: entanglement swapping

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The dynamics of an entangled atomic system in a partial interaction with entangled cavity fields, characterizing an entanglement swapping, have been studied through the use of Von Neuman entropy. We consider the interaction via two-photon process given by a full microscopical Hamiltonian approach. The explicit expression of the entropy is obtained, wherefrom we estimated the largest period. The numerical simulation of the entropy of the entangled atomic and cavity systems shows that its time evolution presents multi-periodicity. The effects of detuning parameter on the period and the amplitude of the entropy are also discussed.

INTRODUCTION

Like the Shannon entropy measures the uncertainty associated with a classical probability distribution, the Von Neuman entropy describes a quantum state via density operators [1]. It is a very useful measure of the purity of quantum states and contains all moments of the density operator. Therefore, the entropy can be used as a measure of the entanglement degree of quantum states and has been extensively used to study the interaction between light field and atoms via Jaynes-Cummings model [2,3]. On the other hand, even though the entropy given an estimate of the entanglement degree, but it is not sufficient, such as the negativity or the concurrence. It is a very useful measure of the purity of quantum states and contains all moments of the density operator. Therefore, the entropy can be used as a measure of the entanglement degree of quantum states and has been extensively used to study the interaction between light field and atoms via Jaynes-Cummings model [2,3].

Some numerical simulations and discussions are given in section 4. We finally present our conclusions in section 5.

OVERVIEW OF THE MODEL

In this paper, we use the so-called ‘full microscopic Hamiltonian approach’ (FMHA) [18,19] to describe a three-level atom in $\Xi$-configuration interacting with a single mode of a cavity-field via two-photon Jaynes-Cummings model. In the absence of a driven field upon the atom, the Hamiltonian (interaction picture) that describes the atom-field interaction is given by

$$H_I = h\Omega (|g\rangle\langle f|e^{-i\delta t} + a^\dagger |f\rangle\langle e|e^{i\delta t}) + h\Omega (|e\rangle\langle f|g|e^{i\delta t} + a^\dagger |g\rangle\langle f|e^{-i\delta t}),$$

where $g_1$ and $g_2$ stand for the one-photon coupling constant with respect to the transitions $|e\rangle \leftrightarrow |f\rangle$ and $|f\rangle \leftrightarrow |g\rangle$ respectively. $\delta$ is detuning parameter defined as:

$$\delta = \Omega - (\omega_e - \omega_f) = (\omega_f - \omega_g) - \Omega,$$

where $\Omega$ denotes the cavity-field frequency and $\omega_e, \omega_f$ as well as $\omega_g$ are the frequencies associated with the atomic levels $|e\rangle, |f\rangle$ and $|g\rangle$, respectively. Fig.1 is a schematic representation of the atomic levels.

The state of the composed atom-field system can be written as

$$|\psi(t)\rangle = \sum_n [C_{e,n}(t)|e, n\rangle + C_{f,n}(t)|f, n\rangle + C_{g,n}(t)|g, n\rangle],$$

or

$$|\psi(t)\rangle = \sum_n [C_{e,n}(t)|e, n\rangle + C_{f,n}(t)|f, n\rangle + C_{g,n}(t)|g, n\rangle].$$
Solving the time dependent Schrödinger equation with Eqs.(1) and (3) we obtain the probability amplitudes

$$C_{e,n}(t) = \left[ \frac{g_1^2(n+1)}{\Lambda_n \alpha_n^2} \gamma_n(t) + 1 \right] C_e C_n$$

$$-i \frac{g_1 \sqrt{n+1}}{\Lambda_n} \sin(\Lambda_n t) e^{-\frac{i \delta}{2} t} C_{f,n+1}$$

$$+ \left[ \frac{g_1 g_2 \sqrt{n+1}(n+2)}{\Lambda_n \alpha_n^2} \right] \gamma_n(t) C_{g,n+2} + \cdots$$

$$C_{f,n+1}(t) = -i \frac{g_1 \sqrt{n+1}}{\Lambda_n} \sin(\Lambda_n t) e^{\frac{i \delta}{2} t} C_{e,n}$$

$$+ \left[ \frac{g_1 g_2 \sqrt{n+1}(n+2)}{\Lambda_n \alpha_n^2} \right] \gamma_n(t) + 1 \right] C_{g,n+2} + \cdots$$

where

$$\gamma_n(t) = \left[ \Lambda_n \cos(\Lambda_n t) + i \frac{\delta}{2} \sin(\Lambda_n t) - \Lambda_n e^{\frac{i \delta}{2} t} \right] e^{-\frac{i \delta}{2} t}$$

$$\Lambda_n = \sqrt{\frac{\delta^2}{4} + \alpha_n^2}$$

$$\alpha_n = \sqrt{g_1^2(n+1) + g_2^2(n+2)}$$

Now, let us suppose two atoms (1 and 2) and two cavities (3 and 4) initially in the states respectively

$$|\psi\rangle_{12} = \alpha_1 |g\rangle_1 |e\rangle_2 + \beta_1 |e\rangle_1 |g\rangle_2$$

$$|\psi\rangle_{34} = \alpha_2 |2\rangle_3 |0\rangle_4 + \beta_2 |0\rangle_3 |2\rangle_4$$

where |0⟩ and |2⟩ indicate the vacuum and two-photon states in the cavities, respectively. Then, the atom 2 is sent to the cavity 3 during an interacting time t. Finally, the total system state develops into

$$|\Psi\rangle_{1234} = \alpha_1 \alpha_2 |g\rangle_1 |e\rangle_2 |2\rangle_3$$

$$+ \alpha_1 \beta_2 |g\rangle_1 |e\rangle_2 |0\rangle_4$$

$$+ \alpha_1 \beta_2 |g\rangle_1 |e\rangle_2 |2\rangle_4$$

$$+ \alpha_2 \beta_1 |2\rangle_3 |0\rangle_4$$

$$+ \alpha_2 \beta_1 |2\rangle_3 |2\rangle_4$$

$$\beta_1 \beta_2 |1\rangle_2 |4\rangle_1 |0\rangle_3$$

$$\beta_1 \beta_2 |1\rangle_2 |4\rangle_1 |0\rangle_3,$$

where $C_{y,m}^x(x,y = e,f,g)$ and $(n,m = 0,1,2)$ denotes the amplitude probability given by equation (4-6) considering the initial state in (y,m) with x and y representing the atomic state, but n and m the field state. For example, from equation (4) and (5) we have

$$C_{e,2}^x(t) = 1 + \frac{3g_1^2 \gamma_2(t)}{\alpha_n^2}$$

$$C_{f,1}^x(t) = -i \frac{\sqrt{2} g_2 \sin(\Lambda_0 t)}{\alpha_n} e^{i \delta t}.$$

In the next section, we shall study the entropy of the system in this state.

ENTROPY

After the entanglement swapping, the density operator of the whole system is written in the form

$$\rho = |\Psi\rangle_{1234} \langle \Psi|.$$ (16)

It is easy to see from Eqs.(11) and (12) that both atomic and cavity field systems are in pure states at initial moment and independent of each other. Therefore, the entropy of the whole system of field and atom is zero and remains constant. On the other hand, according to Araki-Lieb inequality [20], the entropies of the atomic system and the cavity system satisfy

$$|S_a - S_f| \leq S \leq |S_a + S_f|.$$ (17)
Since S = 0 at any moment t > 0, entropy of the cavity system equals that of the atomic system. So, in the following we only need to calculate the entropy of the atomic system. This can be easily done. Taking partial trace of ρ with cavity 3 and 4, we obtain the reduced density operator of the two atom system

\[
\rho_{12} = |\alpha_2\beta_1 C_{e_0}^{(g_2)}(t)|^2 [\rho_{e_i, e_j} + |\alpha_2\beta_1 C_{f_1}^{(g_2)}(t)|^2 |e_i, f_1 \rangle \langle f_1, e_j | + |\beta_1|^2 |\beta_2|^2 + |\alpha_2 C_{g_2}^{(g_2)}(t)|^2] |e_i, g_2 \rangle \langle g_2, e_j |
\]

Using orthogonal basis |a, b⟩ = |a⟩_1 |b⟩_2 (a, b = e, f, g), we obtain a 9 × 9 matrix expression of ρ_{12} with following non-zero elements:

\[
\begin{align*}
\rho_{gg, gg} &= |\alpha_1|^2 |\beta_2 C_{g_0}^{(e_0)}(t)|^2 + |\alpha_2 C_{g_2}^{(g_2)}(t)|^2, \\
\rho_{gf, gf} &= |\alpha_1|^2 |\beta_2 C_{f_1}^{(e_0)}(t)|^2 + |\alpha_2 C_{f_3}^{(g_2)}(t)|^2, \\
\rho_{ge, eg} &= |\alpha_1|^2 |\beta_2 C_{e_0}^{(e_0)}(t)|^2 + |\alpha_2 C_{e_2}^{(g_2)}(t)|^2, \\
\rho_{ge, eg} &= |\alpha_1|^2 |\beta_2 C_{e_0}^{(e_0)}(t)|^2 + |\alpha_2 C_{e_2}^{(g_2)}(t)|^2, \\
\rho_{eg, eg} &= |\alpha_1|^2 |\beta_2 C_{e_0}^{(e_0)}(t)|^2 + |\alpha_2 C_{e_2}^{(g_2)}(t)|^2, \\
\rho_{ef, ef} &= |\alpha_2\beta_1 C_{e_0}^{(g_2)}(t)|^2, \\
\rho_{ee, ee} &= |\alpha_2\beta_1 C_{e_0}^{(g_2)}(t)|^2.
\end{align*}
\]

where ρ_{ab, cd}(a, b, c, d = e, f, g) denotes ρ_{|ab⟩_{12} |cd⟩_{12}}. This density matrix ρ has six non-zero eigenvalues:

\[
\begin{align*}
\lambda_1 &= |\alpha_2\beta_1 C_{e_0}^{(e_0)}(t)|^2, \\
\lambda_2 &= |\alpha_2\beta_1 C_{f_1}^{(g_2)}(t)|^2, \\
\lambda_3 &= |\alpha_1|^2 \left[ |\beta_2 C_{f_1}^{(e_0)}(t)|^2 + |\alpha_2 C_{f_3}^{(g_2)}(t)|^2 \right], \\
\lambda_4 &= |\alpha_1|^2 \left[ |\beta_2 C_{g_2}^{(e_0)}(t)|^2 + |\alpha_2 C_{g_2}^{(g_2)}(t)|^2 \right], \\
\lambda_5 &= \frac{1}{2} |\alpha_1\beta_2 C_{e_0}^{(e_0)}(t)|^2 + |\alpha_1\alpha_2 C_{e_2}^{(g_2)}(t)|^2 + |\alpha_2\beta_1 C_{e_2}^{(g_2)}(t)|^2 + |\beta_1\beta_2|^2 - | \eta(t) |, \\
\lambda_6 &= \frac{1}{2} |\alpha_1\beta_2 C_{e_0}^{(e_0)}(t)|^2 + |\alpha_1\alpha_2 C_{e_0}^{(g_2)}(t)|^2 + |\alpha_2\beta_1 C_{e_0}^{(g_2)}(t)|^2 + |\beta_1\beta_2|^2 + | \eta(t) |.
\end{align*}
\]

FIG. 2: Time evolution of the entropy of two three-level atoms interacting with a single-mode of a cavity field with α_1 = α_2 = β_1 = β_2 = 1/√2 and g_1 = g_2 = g = 1MHz, δ = 0.

where

\[
\eta(t) = \left \{ \left[ |\alpha_1|^2 \left( |\beta_2 C_{e_0}^{(e_0)}(t)|^2 + |\alpha_2 C_{e_0}^{(e_0)}(t)|^2 \right) \right] - |\beta_1|^2 \left( |\alpha_2 C_{g_2}^{(g_2)}(t)|^2 + |\beta_2|^2 \right)^2 + 4|\alpha_1\beta_2|^2 \left( |\beta_2|^2 C_{e_0}^{(e_0)}(t) + |\alpha_2|^2 C_{e_0}^{(e_0)}(t) C_{g_2}^{(g_2)}(t) \right) \right \} / \sqrt{2} (21)
\]

Finally, the entropies of two atomic system and field are given by

\[
S_f(t) = S_o(t) = -\text{Tr}[\rho_{12} \log(\rho_{12}(t))] = - \sum_{i=1}^{6} \lambda_i \log(\lambda_i) (22)
\]

in the above formula, logarithms are taken to base two, as usual.

**NUMERICAL RESULTS AND DISCUSSION**

To reveal the properties of the entropy of the atomic and field systems, we numerically calculated the entropy according to Eqs.(20)-(22) with various values of the detuning δ. The results are shown in Figs. 2-8. These plots show the
Following interesting features. First, when $\delta$ is zero or small, the time evolution of the entropy oscillates with time $t$ but not explicitly shows periodicity. For larger $\delta$, the periodicity of the time evolution of the entropy is more and more obvious. This situation is similar to the case of a single three-level atom interacting with single cavity field [11]. Second, for large $\delta$, the periodicity showed by the time evolution of the entropy is multiple which means in a larger period the evolution of the entropy further periodically shows variety modes. This is different from the case of a single three-level atom interacting with single-mode or two-mode cavity field [11,19] and can be attributed to entanglements of atom-atom, cavity-cavity and atom-cavity. Furthermore, even though we can not find the explicit formula of the period, we can approximately estimate from Figs. 2-8 for larger $\delta$, with the maximum periods of 316$\mu$s, 630$\mu$s, 941$\mu$s and 1257$\mu$s corresponding to $\delta = 50g, 100g, 150g, 200g$, respectively. In addition, these data show the period linearly increased when $\delta$ increased. Third, Figs. 2-8 also show that the maximum values of the entropy decreased when $\delta$ increased. This is natural because the larger detuning $\delta$ means that atomic system is weakly coupled to cavity fields, therefore, the degree of the entanglement is weakened.
CONCLUDING REMARKS

Using FMHA and two-photon Jaynes-Cummings model, we have found the analytical expression of the reduced density matrix for two entangled atoms after an entanglement swapping with two cavity fields. The explicit formula of the entropy for atomic and field systems are also obtained. It is a distinct feature that the entropy evolution of atomic system and fields exhibits multi-periodicity. Furthermore, maximum period linearly increased as $\delta$ increased for large detuning $\delta$.

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