Accuracy of Fifth-Order Improved Runge-Kutta Method for Handling Hyperchaotic Finance Systems

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Abstract: A new alternative technique named fifth order Improved Runge-Kutta (IRK5) method is applied to compute a hyperchaotic finance system, which needs less number of function evaluations per time step. This method obtains a solution in terms of a rapidly convergent infinite power series compared to fourth order Runge-Kutta (RK4) and fifth order Diagonally Implicit Runge-Kutta (DIRK5) methods. Moreover, the residual error of the IRK5 solution is defined and computed for each time interval. Via the computing of the residual error we observe that the accuracy of the IRK5 method compared to DIRK5.

1. Introduction

Economic and financial systems are very complicated linear systems related to humans and contain many complex factors. The chaotic phenomenon in the economy was discovered in 1985, a huge impact imposed on today's leading economists. Chaos appeared in the financial system during the economic crisis. The chaotic phenomenon in the economy was discovered in 1985, a huge impact imposed on today's leading economists. Chaos appeared in the financial system during the economic crisis. Because this global economic crisis can cause great depression, this very chaotic financial system reflects the financial phenomenon. Therefore the actual background and source of the chaotic financial system has become hyperchaotic since 2007 [1].

Yu et al. [1], Cai et al. [2] and Wu & Chen [3] have reported a dynamic financial model consisting of four sub-blocks: production, money, stock, and labor, and expressed as three first-order differential equations. The model illustrates the time variation of three state variables: interest rate, x, investment demand, y, and price index, z. The factors that influence changes in x mainly come from two aspects: first, the contradiction of the investment market, that is, the surplus between investment and savings, and second, the structural adjustment of prices. The rate of change y is proportional to the level of investment, and is proportional to the inversion with investment costs and interest rates. Changes in z, on the one hand, are controlled by contradictions between supply and demand on the commercial market, and on the other hand, influenced by inflation rates. By choosing the appropriate coordinates and setting the right dimensions for each state variable, Yu, et al. [1], Cai, et al. [2] and Wu & Chen [3] offer a simplified financial model [4] as
\[
\begin{align*}
\frac{dx}{dt} &= z + (y - a)x + w \\
\frac{dy}{dt} &= 1 - by - x^2 \\
\frac{dz}{dt} &= -x - cz \\
\frac{dw}{dt} &= -dxy - kw
\end{align*}
\] (1)

where \( a \geq 0 \) is the amount of savings, \( b \geq 0 \) is investment cost per year, and \( c \geq 0 \) is the elasticity of commercial market demand. \( a, b, c, d, k \) are positive constant parameters. To generate new hyperchaos, we specify \( a = 0.9, b = 0.2, c = 1.5, d = 0.2 \), and choose \( k \) as the governing parameter according to the two criteria as mentioned by Rossler in 1996. Yu et al. [1], Cai et al. [2] and Wu & Chen [3] study the dynamic character of this system under a combination of different parameters. Due to changes in interest rates and the emergence of many investment channels, the amount of savings has changed enormously and has had very far-reaching effects on economic development, so it has important theoretical and practical significance for studying the effects of changes in the amount of savings in the overall financial system.

Discovery accurate and efficient methods for solving Hyperchaotic system has been an active research undertaking. Exact solutions of the Hyperchaotic system cannot be found easily, thus analytical and numerical methods must be used. One such technique yielding numerical solutions is called fifth order Improved Runge-Kutta Method (IRK5) that proposed by Rabiei and Ismail in 2012 [5] which has a number of function evaluations than the RK methods while maintaining the same order of local accuracy. Rabiei and Ismail [5] developed the IRK5 method for solving ordinary differential equations. The method used only five stages. The IRK methods arise from the classical RK methods, can also be considered as a special class of two-step methods. That is, the approximate solution is calculated using the values of \( y_n \) and \( y_{n+1} \). Our method introduces the new term of \( k_{ij} \), which are calculated from \( k_i (i > 2) \), in the previous step. The scheme proposed herein has a lower. Hussain et al. [6] has successful solved directly solving a special third-order ordinary differential equation using fourth order improved Runge-Kutta Method (IRK4) by comparing with RK4. Rabiei et al. [7] illustrate the efficiency of IRK5 method for handling Volterra Integro-Differential Equation.

In this paper, the accuracy of IRK5 method will be shown by solving hyperchaotic finance system. Comparing with fifth order diagonally implicit Runge-Kutta which introduced by Ababneh, et al. [8] and RK4 methods for various values \( k \).

2. Fifth Order Improved Runge Kutta Method

Runge-Kutta method is one of the most popular one-step method. An accelerated RK4 is derived purposely for solving autonomous first order ODEs where the stage or function evaluation involved is of the form

\[
k_i = f \left( y \left( x_n + h \sum_{i=0}^{4} a_{i-1} k_{i-1} \right) \right),
\] (5)

where \( k_i \) is a function of \( y \) only and the term involved \( k_{i-1} \). There are two improvement used here, the first one is the function \( f \) is not autonomous, thus the method is not specific for \( y' = f(y(x)) \), but it can be used for solving both the autonomous equations as well as the more general differential equations \( y' = f(x, y(x)) \).

A proposed IRK method in this paper with \( s \)-stage for solving (1) has the form [5]:

\[
y_{n+1} = (1 - \alpha) y_n + \alpha y_{n-1} + h \left( b_1 k_1 - b_{-1} k_{-1} + \sum_{i=1}^{s} b_i (k_i - k_{i-1}) \right).
\] (6)

For \( 0 \leq \alpha \leq 1, 1 \leq n \leq N - 1 \), where

\[
\begin{align*}
k_1 &= f(x_n, y_n) \\
k_{-1} &= f(x_{n-1}, y_{n-1}) \\
k_i &= f(x_n + q_i h, y_n + h \sum_{j=0}^{i-1} a_{ij} k_j), 2 \leq i \leq s \\
k_{-i} &= f(x_n + c_i h, y_n + h \sum_{j=0}^{i-1} a_{ij} k_{-j}), 2 \leq i \leq s
\end{align*}
\]
For $c_i \in [0,1], i = 2, \ldots, s$ and $f$ depends on both $x$ and $y$ while $k_i$ and $k_{-i}$ depend on the values of $k_i$ and $k_{-i}$ for $j = 1, \ldots, i - 1$. Here $s$ is the number of function evaluations performed at each step and increases with the order of local accuracy of the IRK method. In each step we only need to evaluate the values of $k_1, k_2, \ldots$, while $k_{-1}, k_{-2}, \ldots$ are calculated from the previous step. Note that IRK method is not self-starting therefore a one-step method must provide the approximate solution of $y_i$ at the first step. The one-step method must be of appropriate order to ensure that the difference $y_1 - y(x_1)$ is order of $p$ or higher. In this paper, without loss of generality we derived the method with $\alpha = 0$, so the fifth order IRK method can be represented as follows

$$y_{n+1} = y_n + h \left( b_1k_1 - b_{-1}k_{-1} + \sum_{i=1}^{s} b_i(k_i - k_{i-1}) \right).$$  \hspace{1cm} (7)

For $0 \leq \alpha \leq 1$, $1 \leq n \leq N - 1$, where

$$k_1 = f(x_n, y_n)$$
$$k_{-1} = f(x_{n-1}, y_{n-1})$$
$$k_i = f(x_n + c_ih, y_n + h \sum_{j=1}^{i-1} a_ijk_j), \ 2 \leq i \leq s$$
$$k_{-i} = f(x_n + c_ih, y_n + h \sum_{j=1}^{i-1} a_ijk_{-j}), \ 2 \leq i \leq s$$

In this work, we apply IRK5 and DIRK5 to obtain the numerical analytical solution for the hyperchaotic finance system as in [4]

$$\frac{dx}{dt} = z + (y - a)x + w$$  \hspace{1cm} (9)
$$\frac{dy}{dt} = 1 - by - x^2$$  \hspace{1cm} (10)
$$\frac{dz}{dt} = -x - cw$$  \hspace{1cm} (11)
$$\frac{dw}{dt} = -dxy - kw$$  \hspace{1cm} (12)

where initial condition $x(0) = 5, y(0) = 2, z(0) = -6$ and $w(0) = 4$ with parameter $a = 0.9, b = 0.2, c = 1.5, d = 0.2$ and $k > 0$. Cao [4] considered initial condition caused the system in Eqs. (1)-(4) is strange attractor on $(x,y,z) -$space, $(x,y,w) -$space, $(y,z,w) -$space, and $(x,w,z)$-space, respectively.

3. Result And Discussions

We solve four initial value problems using the new fourth order IRK5 method. The numerical results are compared with other existing methods to show the efficiency and the accuracy of the IRK5 method. The following explicit RK4 danDIRK5 methods are chosen for numerical comparison. Numerical results illustrate in absolute error for various values $k$ is chosen as $k = 0.17, k = 0.5$ and $k = 1.0$, see Figure 1-8. Figure 1-8 displays absolute error of $x$ as interest rate, $y$ as investment demand, $z$ as price and $w$ as the average profit margin for various values $k$ is chosen as $k = 0.17, k = 0.5$ and $k = 1.0$ DIRK5 which compared RK4. As Increasing value of $k$, absolute error of all variables approach $x$-axis. Figure 9-11 shows residual error of the hyperchaotic system using IRK5, DIRK5 and RK4 methods. Figure 9 express the accuracy DIRK5 method only for $0 \leq t \leq 4$, whereas Figure 10 shows the accuracy DIRK5 method more longer than DIRK5 solution, accurate for $0 \leq t \leq 25$. In the end, Figure 11 displays the accuracy DIRK5 method more longer than DIRK5 solution, accurate for $0 \leq t \leq 35$. From Figure 9-11, it shows that IRK5 more accurate that DIRK5 methods.
Figure 1. Absolute error of DIRK5 for $x$ (interest rate) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$

Figure 2. Absolute error of IRK5 for $x$ (interest rate) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$

Figure 3. Absolute error of DIRK5 for $y$ (investment demand) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$
Figure 4. Absolute error of IRK5 for $y$ (investment demand) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$

Figure 5. Absolute error of DIRK5 for $z$ (price) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$

Figure 6. Absolute error of IRK5 for $z$ (price) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$
Figure 7. Absolute error of DIRK5 for $w$ (average profit margin) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$

Figure 8. Absolute error of IRK5 for $w$ (average profit margin) with various values $k$, namely $k = 0.17, k = 0.5$ and $k = 1.0$
Figure 9. Residual error of method; (a) DIRK5 in $0 \leq t \leq 4$, (b) IRK5 in $0 \leq t \leq 25$ and (c) RK4 in $0 \leq t \leq 35$ with $k = 0.17$

4. Conclusions

In this paper, we considered numerical solution of hyperchaotic finance system by using Improved Runge-Kutta method of order five with 5 stages (IRK5). Hyperchaotic finance system solve by IRK5 and DIRK5 method and numerical results were compared with the existing RK4 method. Numerical results showed that IRK5 using five stages achieved better accuracy compared to DIRK5. Therefore, IRK5 with higher accuracy and less computational cost is an efficient method for solving hyperchaotic system.

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