ARITHMETICITY OF THE 4 MONODROMY GROUPS ASSOCIATED TO THE CALABI-YAU THREEFOLDS

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Abstract. In [11], we show that 3 of the 14 monodromy groups associated to the Calabi-Yau threefolds, are arithmetic. Brav-Thomson (in [3]) show that 7 of the remaining 11, are thin. In this article, we settle the arithmeticity problem for the 14 monodromy groups associated to the Calabi-Yau threefolds, by showing that, the remaining 4 monodromy groups are arithmetic.

1. Introduction

For \( \theta = z \frac{d}{dz} \), we write the differential operator

\[
D = D(\alpha; \beta) = D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) \\
= (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n)
\]

for \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{C} \), and consider the hypergeometric differential equation

\[
D(\alpha; \beta)w = 0
\]

on \( \mathbb{P}^1(\mathbb{C}) \) with regular singularities at the points 0, 1 and \( \infty \), and regular elsewhere.

The fundamental group of \( \mathbb{P}^1(\mathbb{C})\setminus\{0, 1, \infty\} \) acts on the solution space of the differential equation (1), and we get a representation \( M(\alpha; \beta) \) of \( \pi_1(\mathbb{P}^1(\mathbb{C})\setminus\{0, 1, \infty\}) \) inside \( \text{GL}_n(\mathbb{C}) \), called monodromy; and the monodromy group of the hypergeometric equation (1) is the image of this map, i.e. the subgroup of \( \text{GL}_n(\mathbb{C}) \) generated by the monodromy matrices \( M(\alpha; \beta)(h_0), M(\alpha; \beta)(h_1), M(\alpha; \beta)(h_\infty) \), where \( h_0, h_1, h_\infty \) (loops around 0, 1, \( \infty \) resp.) are the generators of \( \pi_1(\mathbb{P}^1(\mathbb{C})\setminus\{0, 1, \infty\}) \) with a single relation \( h_\infty h_1 h_0 = 1 \).

By a theorem of Levelt ([9]; cf. [2, Theorem 3.5]), if \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C} \) such that \( \alpha_j - \beta_k \notin \mathbb{Z} \), for all \( j, k = 1, 2, \ldots, n \), then the monodromy group of the hypergeometric differential equation (1)
is (up to conjugation in $\text{GL}_n(\mathbb{C})$) a subgroup of $\text{GL}_n(\mathbb{C})$ generated by the companion matrices $A$ and $B$ of

$$f(X) = \prod_{j=1}^{n}(X - e^{2\pi i \alpha_j}) \quad \text{and} \quad g(X) = \prod_{j=1}^{n}(X - e^{2\pi i \beta_j})$$

resp., and the monodromy is defined by $h_\infty \mapsto A$, $h_0 \mapsto B^{-1}$, $h_1 \mapsto A^{-1}B$.

Therefore, to study the monodromy groups of $n$-order hypergeometric differential equations, it is enough to study the subgroups of $\text{GL}_n(\mathbb{C})$ generated by the companion matrices of pairs of degree $n$ polynomials in $\mathbb{C}[X]$, which do not have any common root in $\mathbb{C}$. Here we concentrate only on the case $n = 4$ and $\alpha, \beta \in \mathbb{Q}^4$.

Let $f, g \in \mathbb{Z}[X]$ be a pair of degree four polynomials which are product of cyclotomic polynomials, do not have any common root in $\mathbb{C}$, $f(0) = g(0) = 1$ and form a primitive pair i.e., $f(X) \neq f_1(X^k)$ and $g(X) \neq g_1(X^k)$, for any $k \geq 2$ and $f_1, g_1 \in \mathbb{Z}[X]$. Now, form the companion matrices $A, B$ of $f, g$ resp., and consider the subgroup $\Gamma \subset \text{SL}_4(\mathbb{Z})$ generated by $A$ and $B$. It follows from [2] that $\Gamma$ preserves a non-degenerate integral symplectic form $\Omega$ on $\mathbb{Z}^4$ and $\Gamma \subset \text{Sp}_4(\Omega)(\mathbb{Z})$ is Zariski dense. The case when $\Gamma \subset \text{Sp}_4(\Omega)(\mathbb{Z})$ has finite index, it is called arithmetic, and thin in other case [10].

We now take particular examples where $f = (X - 1)^4$ (i.e. the local monodromy is maximally unipotent at $\infty$ and $\alpha = (0, 0, 0, 0)$). Then, it turns out (for a reference, see [1], [4] and [5]) that, this monodromy is same as the monodromy of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$ on certain pieces of $H^3$ of the fibre of a family $\{Y_t : t \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}\}$ of Calabi-Yau threefolds, provided $f, g$ satisfy the conditions that, $f(X) = (X - 1)^4$ and $g(X)$ is the product of cyclotomic polynomials such that $g(1) \neq 0$, $f(0) = g(0) = 1$, and $f, g$ form a primitive pair. There are precisely $14$ such examples, which have been listed in [1], [4], [5], [11] and Table 1.1 of this article. It is then of interest to know whether the associated monodromy group $\Gamma$ is arithmetic.

In [11], we prove the arithmeticity of $3$ monodromy groups associated to Examples 1, 11, 13 of Table 1.1. Brav-Thomas (in [3]) prove the thinness of $7$ monodromy groups associated to Examples 2, 4, 6, 8, 9, 12, 14 of Table 1.1. In this article, we prove the arithmeticity of the remaining $4$ monodromy groups, for which, the parameters are $\alpha = (0, 0, 0, 0)$, $\beta = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})$; these are Examples 4, 8, 10, 11 of [11] and Examples 3, 10, 5, 7 of Table 1.1 (cf. [11] Table 5.3). In fact, we get the following theorem:
Theorem 1.1. The monodromy groups associated to the Calabi-Yau threefolds, for which, the parameters are \( \alpha = (0, 0, 0); \beta = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{3}{4}) \), (\frac{1}{3}, \frac{2}{3}, \frac{3}{4}), (\frac{1}{3}, \frac{1}{3}, \frac{5}{6}) \), are arithmetic.

For a better reference, we list here the pairs of polynomials \( f, g \) and the parameters \( \alpha, \beta \), which correspond to the 14 monodromy groups associated to the Calabi-Yau threefolds; in the following list,
\[ \alpha = (0, 0, 0) \text{ i.e. } f(X) = (X - 1)^4 = X^4 - 4X^3 + 6X^2 - 4X + 1 : \]

| No. | \( g(X) \) | \( \beta \) | \( f(X) - g(X) \) | Arithmetic |
|-----|-------------|-------------|----------------|------------|
| 1   | \( X^4 - 2X^3 + 3X^2 - 2X + 1 \) | \( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \) | \( -2X^3 + 3X^2 - 2X \) | Yes, [1] |
| 2*  | \( X^4 + 4X^3 + 6X^2 + 4X + 1 \) | \( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \) | \( -8X^3 - 8X \) | No, [2] |
| 3   | \( X^4 + 2X^3 + 3X^2 + 2X + 1 \) | \( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \) | \( -6X^3 + 3X^2 - 6X \) | Yes |
| 4*  | \( X^4 + 3X^3 + 4X^2 + 3X + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -7X^3 + 2X^2 - 7X \) | No, [3] |
| 5   | \( X^4 + 2X^2 + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -4X^3 + 4X^2 - 4X \) | Yes |
| 6*  | \( X^4 + 2X^3 + 2X^2 + 2X + 1 \) | \( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \) | \( -6X^3 + 4X^2 - 6X \) | No, [4] |
| 7   | \( X^4 + X^3 + 2X^2 + X + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -5X^3 + 4X^2 - 5X \) | Yes |
| 8*  | \( X^4 + X^3 + X^2 + X + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -5X^3 + 5X^2 - 5X \) | No, [3] |
| 9*  | \( X^4 + X^3 + X + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -5X^3 + 6X^2 - 5X \) | No, [3] |
| 10  | \( X^4 + X^2 + 1 \) | \( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \) | \( -4X^3 + 5X^2 - 4X \) | Yes |
| 11  | \( X^4 - X^3 + 2X^2 - X + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -3X^3 + 4X^2 - 3X \) | Yes, [1] |
| 12* | \( X^4 + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -4X^3 + 6X^2 - 4X \) | Yes |
| 13  | \( X^4 - X^3 + X^2 - X + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -3X^3 + 5X^2 - 3X \) | Yes, [1] |
| 14* | \( X^4 - X^2 + 1 \) | \( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \) | \( -4X^3 + 7X^2 - 4X \) | No, [4] |

Therefore 7 of the 14 monodromy groups associated to the Calabi-Yau threefolds, are arithmetic and other 7 are thin.

For the pairs in Theorem 1.1, we explicitly compute, up to scalar multiples, the symplectic form \( \Omega \) on \( \mathbb{Q}^4 \) and get a basis \( \{ \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \} \) of \( \mathbb{Q}^4 \), satisfying the following: \( \Omega(\epsilon_i, \epsilon_i) \neq 0 \) for \( i = 1, 2 \), and \( \Omega(\epsilon_1, \epsilon_2) = \Omega(\epsilon_1, \epsilon_3) = \Omega(\epsilon_2, \epsilon_4) = 0 \). It can be shown easily that, with respect to the symplectic basis \( \{ \epsilon_1, \epsilon_2, \epsilon_4 \} \) of \( \mathbb{Q}^4 \), the group of diagonal matrices in \( \text{Sp}_4(\Omega) \) form a torus \( T \), the group of upper (resp. lower) triangular matrices in \( \text{Sp}_4(\Omega) \) form a Borel subgroup \( B \) (resp. \( B^- \), opposite to \( B \)), and the group of unipotent upper (resp. lower) triangular matrices in \( \text{Sp}_4(\Omega) \) form the unipotent radical \( U \) (resp. \( U^- \), opposed to \( U \)) of \( B \) (resp. \( B^- \)).

It follows from [12] that, if \( B \) is a Borel subgroup of \( \text{Sp}_4 \), \( U \) is the unipotent radical of \( B \), and \( \Gamma \) is a Zariski dense subgroup of \( \text{Sp}_4(\mathbb{Z}) \) such that \( \Gamma \cap U(\mathbb{Z}) \) is of finite index in \( U(\mathbb{Z}) \), then \( \Gamma \) is of finite index in
Therefore our strategy is to show that the monodromy groups \( \Gamma \) associated to the pairs in Theorem 1.1 intersect the group \( U(\mathbb{Z}) \) of unipotent upper (or lower) triangular matrices in \( \text{Sp}_4(\Omega)(\mathbb{Z}) \), in a finite index subgroup of \( U(\mathbb{Z}) \) i.e., \( \Gamma \cap U(\mathbb{Z}) \) is of finite index in \( U(\mathbb{Z}) \).

**Acknowledgements**

I thank Jörg Hofmann for introducing me the program “Maple” to do the matrix computations; Maple was very helpful for the computations in this paper. I am grateful to Professor T. N. Venkataramana for introducing me the criterion to prove the arithmeticity; the strategy to prove the arithmeticity in this article is same as in our joint paper [11]. I thank him for his encouragement and constant support. I am also grateful to Professor Duco van Straten and Professor Wadim Zudilin for their encouragement. I thank Institut für Mathematik, Johannes Gutenberg-Universität for the postdoctoral fellowship, and for very pleasant hospitality.

**2. Proof of Theorem 1.1**

We will first compute the symplectic form \( \Omega \) preserved by the monodromy group \( \Gamma \), then show that there exists a basis \( \{\epsilon_1, \epsilon_2, \epsilon_1^*, \epsilon_2^*\} \) of \( \mathbb{Q}^4 \) with respect to which

\[
\Omega = \begin{pmatrix}
0 & 0 & 0 & \lambda_1 \\
0 & 0 & \lambda_2 & 0 \\
0 & -\lambda_2 & 0 & 0 \\
-\lambda_1 & 0 & 0 & 0
\end{pmatrix}
\]

where \( \Omega(\epsilon_i, \epsilon_i^*) = \lambda_i \in \mathbb{Q}^*, \forall 1 \leq i \leq 2 \).

It can be checked easily that the diagonal matrices in \( \text{Sp}_4(\Omega) \) form a maximal torus \( T \) i.e.,

\[
T = \left\{ \begin{pmatrix}
t_1 & 0 & 0 & 0 \\
0 & t_2 & 0 & 0 \\
0 & 0 & t_2^{-1} & 0 \\
0 & 0 & 0 & t_1^{-1}
\end{pmatrix} : t_i \in \mathbb{C}^*, \forall 1 \leq i \leq 2 \right\}
\]

is a maximal torus in \( \text{Sp}_4(\Omega) \). Once we fix a maximal torus \( T \) in \( \text{Sp}_4(\Omega) \), one may compute the root system \( \Phi \) for \( \text{Sp}_4(\Omega) \). If we denote by \( t_i, \ldots \)
the character of $T$ defined by

$$
\begin{pmatrix}
t_1 & 0 & 0 & 0 \\
0 & t_2 & 0 & 0 \\
0 & 0 & t_2^{-1} & 0 \\
0 & 0 & 0 & t_1^{-1}
\end{pmatrix} \mapsto t_i, \quad \text{for } i = 1, 2,
$$

then the roots are

$$\Phi = \{t_1^2, t_1t_2, t_1^{-1}t_2, t_2, t_1^{-2}, t_1^{-1}t_2, t_1t_2, t_2^2\}.$$

If we fix a set of simple roots $\Delta = \{t_1t_2^{-1}, t_2^2\}$, then the set of positive roots $\Phi^+ = \{t_1^2, t_1t_2, t_1^{-1}t_2, t_2^2\}$ and the group of upper (resp. lower) triangular matrices in $\text{Sp}_4(\Omega)$ form a Borel subgroup $B$ (resp. $B^-$, opposite to $B$), and the group of unipotent upper (resp. lower) triangular matrices in $\text{Sp}_4(\Omega)$ form the unipotent radical $U$ (resp. $U^-$, opposed to $U$) of $B$ (resp. $B^-$).

Now, to show that the group $\Gamma$, with respect to the basis $\{\epsilon_1, \epsilon_2, \epsilon_3^*, \epsilon_4^*\}$ of $\mathbb{Q}^4$, intersects $U(\mathbb{Z})$ (resp. $U^-(\mathbb{Z})$) in a finite index subgroup of $U(\mathbb{Z})$ (resp. $U^-(\mathbb{Z})$), it is enough to show that $\Gamma$ contains non-trivial unipotent elements corresponding to each of the positive (resp. negative) roots.

We will compute the symplectic form $\Omega$, the basis $\{\epsilon_1, \epsilon_2, \epsilon_3^*, \epsilon_4^*\}$ and the positive (or negative) root group elements $P, x, y, z \in \Gamma$ (for notation, see the proof below), and the proof of arithmeticity of $\Gamma$ follows from [12] (cf. [13] Theorem 3.5). We will write the descriptions only for the pair $\alpha = (0, 0, 0, 0)$, $\beta = (\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4})$ in Subsection 2.1, and write directly the form $\Omega$, the basis $\{\epsilon_1, \epsilon_2, \epsilon_3^*, \epsilon_4^*\}$, the elements $P, x, y, z \in \Gamma$ for other pairs, and the proof follows by the same descriptions as in Subsection 2.1.

2.1. Arithmeticity of the monodromy group associated to the pair $\alpha = (0, 0, 0, 0)$, $\beta = (\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4})$. This is Example 11 of [11, 4] and Example 7 of Table 1.1 (cf. [11] Table 5.3). In this case

$$f(X) = X^4 - 4X^3 + 6X^2 - 4X + 1, \quad g(X) = X^4 + X^3 + 2X^2 + X + 1;$$

and $f(X) - g(X) = -5X^3 + 4X^2 - 5X$.

Let $A$ and $B$ be the companion matrices of $f(X)$ and $g(X)$ resp., and let $C = A^{-1}B$. Then

$$A = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 4
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & -1
\end{pmatrix}, \quad C = A^{-1}B = \begin{pmatrix}
1 & 0 & 0 & -5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -5 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$
Let $\Gamma = \langle A, B \rangle$ be the subgroup of $\text{SL}_4(\mathbb{Z})$ generated by $A$ and $B$.

The invariant symplectic form. It follows form [2] that $\Gamma$ preserves a non-degenerate symplectic form $\Omega$ on $\mathbb{Q}^4$, which is integral on $\mathbb{Z}^4$ and the Zariski closure of $\Gamma$ is $\text{Sp}_4(\Omega)$. Let us denote $\Omega(v_1, v_2)$ by $v_1.v_2$, for any pairs of vectors $v_1, v_2 \in \mathbb{Q}^4$.

We compute here the form $\Omega$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{Q}^4$ over $\mathbb{Q}$ and $v = -5e_1 + 4e_2 - 5e_3$ which is the last column vector of $C - I_4$, where $I_4$ is the $4 \times 4$ identity matrix. Since $C$ preserves the form $\Omega$, for $1 \leq i \leq 3$, we get

$$e_i.e_4 = e_i.(-5e_1 + 4e_2 - 5e_3 + e_4)$$
$$= e_i.(v + e_4)$$
$$= e_i.v + e_ie_4.$$

This implies that

$$e_i.v = 0 \quad \text{for} \quad 1 \leq i \leq 3. \quad (2)$$

That is, $v$ is $\Omega$- orthogonal to the vectors $e_1, e_2, e_3$ and $e_4.v \neq 0$ (since $\Omega$ is non-degenerate). Since $B$ preserves $\Omega$, we get

$$e_1.e_2 = e_2.e_3 = e_3.e_4 = e_4.(-e_1 - e_2 - 2e_3 - e_4)$$
$$= e_1.e_4 + e_2.e_4 + 2e_3.e_4.$$

This implies that

$$e_3.e_4 = -e_1.e_4 - e_2.e_4. \quad (3)$$

It now follows from (2), (3) and the invariance of $\Omega$ under $B$, that

$$e_1.e_3 = \frac{4}{5}e_1.e_2 = e_2.e_4 \quad (4)$$

and

$$e_1.e_4 = -\frac{9}{5}e_1.e_2. \quad (5)$$

We now get from (3), (4) and (5), up to scalar multiples, the matrix form of $\Omega$:

$$\begin{pmatrix}
0 & 1 & 4/5 & -9/5 \\
-1 & 0 & 1 & 4/5 \\
-4/5 & -1 & 0 & 1 \\
9/5 & -4/5 & -1 & 0
\end{pmatrix}.$$
Proof of the arithmeticity of $\Gamma$. By an easy computation we get $\epsilon_1 = e_1 + e_3$, $\epsilon_2 = -5e_1 + 4e_2 - 5e_3$, $\epsilon_2^* = -9e_1 + 4e_2 - 5e_3 + 4e_4$ and $\epsilon_1^* = e_1 + e_2$ form a basis of $\mathbb{Q}^4$ over $\mathbb{Q}$, with respect to which

$$\Omega = \begin{pmatrix}
0 & 0 & 0 & -4/5 \\
0 & 0 & \frac{144}{5} & 0 \\
0 & -\frac{144}{5} & 0 & 0 \\
4/5 & 0 & 0 & 0
\end{pmatrix}.$$ 

Let

$$P = C = A^{-1}B, \quad Q = B^{-3}CB^3, \quad R = B^3CB^{-3}.$$ 

It can be checked easily that with respect to the basis $\{\epsilon_1, \epsilon_2, \epsilon_2^*, \epsilon_1^*\}$, the $P, Q, R$ have, resp., the matrix form

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -4 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & -72 & 0 & -36 \\
0 & 1 & 0 & 0 \\
0 & -4 & 1 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$ 

A computation shows that if

$$E = Q^{-1}R, \quad F = Q^{-1}RQP^{-1}, \quad G = [E, F] = EFE^{-1}F^{-1}, \quad x = [G, E],$$

$$y = x^{-36}E^{1152}, \quad u = G^{82944}y^{1152}, \quad z = u^{-1152}x^{63403237638144},$$

then

$$\begin{pmatrix}
1 & -72 & 0 & -36 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & -288 & -36 \\
0 & 1 & -4 & 0 \\
0 & -4 & 17 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & 1152 & 288 & 1728 \\
0 & 1 & 0 & -8 \\
0 & 0 & 1 & 32 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 23887872 & -63403237638144 \\
0 & 1 & 0 & -663552 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & 0 & -1152 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & -82944 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -2304 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$
The elements $P, x, y, z$ patently generate the positive root groups of $\text{Sp}_4(\Omega)$. Therefore if $B$ is the Borel subgroup of $\text{Sp}_4(\Omega)(\mathbb{Q})$ preserving the full flag 

$$\{0\} \subset \mathbb{Q}e_1 \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}e_2^* \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}e_2^* \oplus \mathbb{Q}e_1^* = \mathbb{Q}^4$$

and $U$ its unipotent radical, then $\Gamma \cap U(\mathbb{Z})$ is of finite index in $U(\mathbb{Z})$. Since $\Gamma$ is Zariski dense in $\text{Sp}_4(\Omega)$ by [2], it follows from [12] that $\Gamma$ is an arithmetic subgroup of $\text{Sp}_4(\Omega)(\mathbb{Z})$. □

2.2. **Arithmeticity of the monodromy group associated to the pair** $\alpha = (0, 0, 0, 0), \beta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$. This is Example 4 of [11, 4] and Example 3 of Table 1.1 (cf. [11, Table 5.3]). In this case

$$f(X) = X^4 - 4X^3 + 6X^2 - 4X + 1, \quad g(X) = X^4 + 2X^3 + 3X^2 + 2X + 1;$$

and $f(X) - g(X) = -6X^3 + 3X^2 - 6X$.

Let $A$ and $B$ be the companion matrices of $f(X)$ and $g(X)$ resp., and let $C = A^{-1}B$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \quad C = A^{-1}B = \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\Gamma = \langle A, B \rangle$ be the subgroup of $\text{SL}_4(\mathbb{Z})$ generated by $A$ and $B$.

**The invariant symplectic form.** Using the same method as in Subsection 2.1 we get the matrix form of

$$\Omega = \begin{pmatrix} 0 & 1 & 1/2 & -3 \\ -1 & 0 & 1 & 1/2 \\ -1/2 & -1 & 0 & 1 \\ 3 & -1/2 & -1 & 0 \end{pmatrix}.$$
Proof of the arithmeticity of $\Gamma$. By an easy computation we get
$$\epsilon_1 = \epsilon_1 + \epsilon_3, \quad \epsilon_2 = -6\epsilon_1 + 3\epsilon_2 - 6\epsilon_3, \quad \epsilon_2^* = 24\epsilon_1 + 12\epsilon_2 + 12\epsilon_3 - 3\epsilon_4$$
and $\epsilon_1^* = \epsilon_1 + 2\epsilon_2$ form a basis of $\mathbb{Q}^4$ over $\mathbb{Q}$, with respect to which
$$\Omega = \begin{pmatrix}
0 & 0 & 0 & -1/2 \\
0 & 0 & -\frac{s_1}{2} & 0 \\
0 & \frac{s_1}{2} & 0 & 0 \\
1/2 & 0 & 0 & 0
\end{pmatrix}.$$ 

Let
$$P = C = A^{-1}B, \quad Q = B^{-4}CB^4, \quad S = A^{-1}CA.$$ 

It can be checked easily that with respect to the basis $\{\epsilon_1, \epsilon_2, \epsilon_2^*, \epsilon_1^*\}$, the $P, Q, S$ have, resp., the matrix form
$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 3 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2 & 12 & 1 & 0 \\
27 & -162 & 0 & 1
\end{pmatrix}.$$ 

A computation shows that if
$$G = Q^{-4}S, \quad H = PGP^{-1}, \quad x = [G, H], \quad y = x^{27}G^{1944}, \quad E = H^{1944}x^{27},$$
then
$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
27 & -162 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1944 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-3888 & 0 & 1 & 0 \\
0 & -314928 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
11664 & 1 & 0 & 0 \\
-3888 & 0 & 1 & 0 \\
0 & -314928 & -944784 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
-11664 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-3673320192 & 0 & 944784 & 1
\end{pmatrix}.$$
$z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
22674816 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1836660096 & 1 \\
\end{pmatrix}$

The elements $P, x, y, z$ patently generate the negative root groups of $\text{Sp}_4(\Omega)$, therefore $\Gamma$ intersects the group $U^{-}(\mathbb{Z})$ of unipotent lower triangular matrices in $\text{Sp}_4(\Omega)(\mathbb{Z})$, in a finite index subgroup of $U^{-}(\mathbb{Z})$ i.e., $\Gamma \cap U^{-}(\mathbb{Z})$ is of finite index in $U^{-}(\mathbb{Z})$. Since $\Gamma$ is Zariski dense in $\text{Sp}_4(\Omega)$ by [2], it follows from [12] that $\Gamma$ is an arithmetic subgroup of $\text{Sp}_4(\Omega)(\mathbb{Z})$.

2.3. Arithmeticality of the monodromy group associated to the pair $\alpha = (0, 0, 0, 0), \beta = (1, 2, 3, 4, 5, 6)$. This is Example 8 of [1, 4] and Example 10 of Table 1.1 (cf. [11, Table 5.3]). In this case $f(X) = X^4 - 4X^3 + 6X^2 - 4X + 1, \ g(X) = X^4 + X^2 + 1$.

and $f(X) - g(X) = -4X^3 + 5X^2 - 4X$.

Let $A$ and $B$ be the companion matrices of $f(X)$ and $g(X)$ resp., and let $C = A^{-1}B$. Then

$$A = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 4 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad C = A^{-1}B = \begin{pmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & 5 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.$$ 

Let $\Gamma = < A, B >$ be the subgroup of $\text{SL}_4(\mathbb{Z})$ generated by $A$ and $B$.

The invariant symplectic form. Using the same method as in Subsection 2.1 we get the matrix form of

$$\Omega = \begin{pmatrix}
0 & 1 & 5/4 & 0 \\
-1 & 0 & 1 & 5/4 \\
-5/4 & -1 & 0 & 1 \\
0 & -5/4 & -1 & 0 \\
\end{pmatrix}.$$ 

Proof of the arithmeticality of $\Gamma$. By an easy computation we get $\epsilon_1 = e_1, \epsilon_2 = -4e_1 + 5e_2 - 4e_3, \epsilon_3^* = 5e_1 + 4e_4$ and $\epsilon_1^* = e_1$ form a basis
of $\mathbb{Q}^4$ over $\mathbb{Q}$, with respect to which

$$
\Omega = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 9 & 0 \\
0 & -9 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

Let

$$
P = C = A^{-1}B, \quad Q = B^{-1}CB, \quad R = B^3CB^{-3}, \quad S = P^{25}R^{-4}.
$$

It can be checked easily that with respect to the basis $\{\epsilon_1, \epsilon_2, \epsilon^*_2, \epsilon^*_1\}$, the $P, Q, R, S$ have, resp., the matrix form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 9 \\
0 & 1 & 25 & 5 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & -45 & -9 \\
0 & 1 & 0 & -20 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 180 & 36 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 80 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

A computation shows that if

$$
G = QSQ^{-1}, \quad x = [S, G], \quad y = S^{28800}x^{-36}, \quad H = G^{5184000}y^{-180}, \quad z = H^{28800}x^{-38698326586624000},
$$

then

$$
G = \begin{pmatrix}
1 & 720 & 180 & 36 \\
0 & 1 & 0 & -20 \\
0 & 0 & 1 & 80 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad x = \begin{pmatrix}
1 & 0 & 0 & 28800 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
y = \begin{pmatrix}
1 & 0 & 5184000 & 0 \\
0 & 1 & 0 & -576000 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad H = \begin{pmatrix}
1 & 3732480000 & 0 & 386983526586624000 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 414720000 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
z = \begin{pmatrix}
1 & 107495424000000 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 11943936000000 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

The elements $P, x, y, z$ patently generate the positive root groups of $\text{Sp}_4(\Omega)$. Since $\Gamma$ is Zariski dense in $\text{Sp}_4(\Omega)$ by [2], it follows from [12] that $\Gamma$ is an arithmetic subgroup of $\text{Sp}_4(\Omega)(\mathbb{Z})$. □
2.4. Arithmeticity of the monodromy group associated to the pair\(\alpha = (0, 0, 0, 0), \beta = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})\). This is Example 10 of \([1, 4]\) and Example 5 of Table [11] (cf. \([8]\) Table 5.3]). In this case

\[
f(X) = X^4 - 4X^3 + 6X^2 - 4X + 1, \quad g(X) = X^4 + 2X^2 + 1;
\]

and \(f(X) - g(X) = -4X^3 + 4X^2 - 4X\).

Let \(A\) and \(B\) be the companion matrices of \(f(X)\) and \(g(X)\) resp., and let \(C = A^{-1}B\). Then

\[
A = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 4
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 0
\end{pmatrix},
C = A^{-1}B = \begin{pmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Let \(\Gamma = \langle A, B \rangle\) be the subgroup of \(\text{SL}_4(\mathbb{Z})\) generated by \(A\) and \(B\).

The invariant symplectic form. Using the same method as in Subsection 2.1 we get the matrix form of

\[
\Omega = \begin{pmatrix}
0 & 1 & 1 & -1 \\
-1 & 0 & 1 & 1 \\
-1 & -1 & 0 & 1 \\
1 & -1 & -1 & 0
\end{pmatrix}.
\]

Proof of the arithmeticity of \(\Gamma\). By an easy computation we get

\[\epsilon_1 = e_2, \quad \epsilon_2 = -e_1 + e_2 - e_3, \quad \epsilon_2^* = 3e_1 + e_2 + 2e_3 + e_4\]

and \(\epsilon_1^* = 3e_1 + 2e_3\)

form a basis of \(\mathbb{Q}^4\) over \(\mathbb{Q}\), with respect to which

\[
\Omega = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

Let

\[P = C = A^{-1}B, \quad Q = B^{-5}CB^5, \quad R = B^3CB^{-3}, \quad S = P^{-1}RPQ^{-1}.
\]

It can be checked easily that with respect to the basis \(\{\epsilon_1, \epsilon_2, \epsilon_2^*, \epsilon_1^*\}\), the \(P, Q, R, S\) have, resp., the matrix form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -4 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 16 & -64 & -64 \\
0 & 15 & 64 & 64 \\
0 & -4 & 17 & 16 \\
0 & 0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 16 & 0 & -64 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 16 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
A computation shows that if
\[ E = P S P^{-1}, \quad x = [S, E], \quad y = S^{2048} x^{64}, \quad G = E^{2048} x^{64}, \quad H = G^{-1} y, \]
then
\[
E = \begin{pmatrix} 1 & 16 & -64 & -64 \\ 0 & 1 & 0 & 64 \\ 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 & 0 & 0 & 2048 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
\[
y = \begin{pmatrix} 1 & 32768 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 32768 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 32768 & -131072 & 0 \\ 0 & 1 & 0 & 131072 \\ 0 & 0 & 1 & 32768 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
\[
H = \begin{pmatrix} 1 & 0 & 131072 & 4594967296 \\ 0 & 1 & 0 & -131072 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & -268435456 & 0 \\ 0 & 1 & 0 & 268435456 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

The elements \(P, x, y, z\) patently generate the positive root groups of \(\text{Sp}_4(\Omega)\). Since \(\Gamma\) is Zariski dense in \(\text{Sp}_4(\Omega)\) by [2], it follows from [12] that \(\Gamma\) is an arithmetic subgroup of \(\text{Sp}_4(\Omega)(\mathbb{Z})\). □

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