The Maximal A-regular Submodule of Module

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Abstract. Let \( R \) be commutative ring with identity and all module are (left) unitary \( R \)-module. An \( R \)-module \( M \) is said to be almost regular (for short A-regular) module if every submodule is almost pure (for short A-pure) submodule of \( M \). In this paper we show that each unitary \( R \)-module has unique maximal A-regular submodule which is denoted by \( L(M) \), means a submodule of \( M \) which contains every A-regular submodule of \( M \). Wei proved that if \( M \) is an \( R \)-module and \( N \) is a submodule of \( M \), then \( L(N) = N \cap L(M) \).

Keyword: regular module, almost pure submodule, almost regular module, maximal regular submodule.

1. Introduction

Let \( R \) be a commutative ring with identity and all modules are (left) unitary \( R \)-module. Unless otherwise stated. Recall that an element \( r \in R \) is said to be regular if there exist \( t \in R \) such that \( rtr = r \); a ring \( R \) is called regular if and only if each element of \( R \) is regular. An ideal \( I \) of a ring \( R \) is regular if each of its elements is regular in \( R \); indeed, a regular ideal \( J \) of \( R \) is itself a regular ring [1]. Brown and McCoy proved in [1] that each ring \( R \) contains a unique maximal regular ideal \( M(R) \) which satisfies the well-known radical properties. The ideal \( M(R) \) is called the regular radical of \( R \). The concept of regularity was extended to modules in several ways and in [2] the notion of Fi-regular modules (in the sense of Fieldhouse [3]) was generalized to GF-regular modules. Let \( A \) be an \( R \)-module, an element \( a \in A \) is said to be GF-regular if for each \( r \in R \) there exist \( t \in R \) and a positive integer \( n \) such that \( r^n tr^n a = r^n a \). An \( R \)-module \( A \) is called GF-regular if and only if all its elements are GF-regular; in [2] that each module contains a “unique maximal GF-regular submodule”. An \( R \)-module \( M \) is said to be A-regular module if for each nonzero element \( x \) of \( M \) and for each \( r \in J(R) \), there exist \( t \in R \) such that \( rx = rtrx \). An \( R \)-module \( N \) is called A-regular if every submodule of \( N \) is A-pure. Also, the concept of A-pure submodule has been introduced. A submodule \( N \) of an \( R \)-module \( M \) is called an A-pure if \( N \cap J(R)M = J(R)N \), i.e. for an ideal \( I \) of a ring \( R \) is said to be A-pure if \( I \cap J(R) = J(R)I \). where \( J(R) \) is the Jacobson radical of \( M \) [4]. In this paper, we show that each module contain a “unique maximal A-regular submodule,” which we dented by \( L(M) \), and we show that \( L(M) \) satisfies some but not all of the usual radical properties Among other results over a principal ideal ring \( R \) in this paper we have:

- Let \( M \) be a \( R \)-module and \( N \) be a submodule of \( M \), then \( L(N) = N \cap L(M) \).
- Let \( M \) be an \( R \)-module, then \( L(R)M \subseteq L(M) \).
- If \( M \) is A-regular \( R \)-module, then \( J(M) = J(R).J(M) \) if and only if \( J(M) = J(R).M \).
2. The Maximal A-regular submodule of Module

In this paper is devoted to study the maximal A-regular submodule. Some results are analogous to that of maximal regular submodule which was introduced and studied in [5], the show that each unitary R-module \( B \) contains a unique maximal regular submodule is denoted \( M (B) \) and satisfied some of the properties of radicals. Many results of maximal regular submodules are generalized to maximal A-regular submodules.

**Definition (2.1):** Let \( M \) be an \( R \)-module. We define the maximal A-regular submodule of \( M \) denoted by \( L(M) \) (if exist) is the submodule containing every A-regular submodule of \( M \), that is an A-regular submodule which not contain properly in any A-regular submodule.

Remark and Examples (2.2):

1. If \( M = r \), then \( L(M) \) is an ideal of \( R \).
2. It is clear \( M \) is A-regular \( R \)-module if and only if \( L(M) = M \).
3. The module \( Z \) as \( Z \)-module, then \( L(Z) = Z \) by remark (2).
4. The module \( Q \) as \( Z \)-module is not A-regular, hence \( L(Q) = 0 \), suppose not, the \( L(Q) = A \) for some submodule \( A \) of \( Q \) implies that \( A \) is A-regular as \( Z \)-module, take any element \( x \in A \), \( x = \frac{a}{b} \) where \( a \) and \( b \) are two non-zero element in \( Z \).

   If we take an ideal \( n > 0 \) of where \( n \) is greater than one then by the same argument of (every non zero cyclic submodule of the \( Z \)-module \( Q \) is not A-pure submodule) of the non-zero cyclic submodule generated by \( a/b \) is not A-pure in \( A \), \( J(r) \cap A \cap \frac{a}{b} \neq J(r) \) which is a contradiction since \( A \) is A-regular.

5. If \( n \) is an A-regular submodule of an \( R \)-module \( M \), then it is not necessary that \( N \) is an A-pure submodule of \( M \). For example, consider the module \( Z_8 \) as \( Z \)-module. It is easily to see that the regular submodule of \( Z_8 \) are \( \{ 0, 2, 4, 6 \} \equiv Z_4 \) is A-regular but \( N \) is not A-pure submodule of \( Z_8 \), see remark (3.2) (1). Moreover the submodule \( \{ 0, 4 \} \equiv Z_2 \) is A-regular module and hence the maximal A-regular module of \( Z_8 \) is \( L(Z_8) = \{ 0, 4 \} \).

6. Let \( M \) be an \( R \)-module and \( L'(M) \) be the maximal regular submodule of \( M \) then it may be \( L'(M) \neq L(M) \).

**Proposition (2.3):** Let \( M \) be an \( R \)-module and \( N \) be a submodule of \( M \), then \( L(N) \cap Li(M) \).

Proof: Let \( N \) be a submodule of \( M \), then \( L(iN) \subseteq iN \) and \( L(iM) \subseteq iM \) implies \( L(iN) \subseteq iN \cap L(iM) \).

Let \( x \in iN \cap iM \), then \( x \in N \) and \( x \in iM \), thus for each \( r \in f(R) \), \( rx = rtx \) for some \( t \in R \) by proposition [4]. Then \( x \in L(iN) \) implies \( Ni \cap Li(M) \subseteq L(Ni) \). Thus \( L(Ni) = Ni \cap Li(M) \).

**Proposition (2.4):** Let \( M \) and \( N \) be an \( R \)-homomorphism and \( f: M \rightarrow N \) be an \( R \)-homomorphism, then \( f(L(M)) \subseteq L(N) \).

Proof: Let \( f: M \rightarrow N \) be an \( R \)-homomorphism and \( y \in f(L(M)) \), then \( y = f(x) \) where \( 0 \neq x \in L(M) \) implies for each \( r \in f(R) \), \( rx = rtx \) for some \( t \in R \) by proposition [4]. Hence \( f(rx) = f(rtx) = rtf(x) \). That is \( ry = rty \), therefore \( y \in f(L(N)) \).

**Proposition (2.5):** Let \( M = M_1 \oplus M_2 \) be an \( R \)-module, where \( M_i \) be a submodule of \( M \) for each \( i = 1, 2 \), then \( L(M_1 \oplus M_2) = L(M_1) \oplus L(M_2) \).

Proof: Let \( (x, y) \in L(M_1 \oplus M_2) \) where \( x \in M_1 \) and \( y \in M_2 \). Then for each \( r \in f(R) \), \( r(x, y) = rtr(x, y) \) for some \( t \in R \) implies \( rx = rtx \) and \( ry = rty \). Therefore \( x \in L(M_1) \) and \( y \in L(M_2) \), hence \( (x, y) \in L(M_1) \oplus L(M_2) \). For the reverse inclusion, let \( (x, y) \in L(M_1) \oplus L(M_2) \), \( x \in L(M_1) \) and \( y \in L(M_2) \). Then for each \( r \in f(R) \), \( rx = rtrx \) and \( ry = rty \) for some \( t_1, t_2 \in R \). If we choose \( t = t_1 + t_2 - rt_1 t_2 \) then easily to see that \( rx = rtx \) and
where for every finitely generated R-module M, if I M = M, implies M = < 0 >, [7].

**Lemma (2.11):** Let r ∈ R, such that J (r) = < r >. If N is finitely generated A-pure submodule of an R-module M such that N ⊆ < r > M, then M = < 0 >.

**Proof:** Since N ⊆ < r > M, then by lemma (2.9), we get N = < r > N, then by Nakayama’s lemma, M = < 0 >.

**Proposition (2.12):** Let M be an R-module. If L (M) is A-pure submodule of M, then L (M) ∩ J (r) M = = < 0 >.

**Proof:** Let xi ∈ L (M) ∩ J (r) M, then x ∈ L (M) and x ∈ J (r) M. Let N be the cyclic submodule generated by x. Thus N ⊆ L (i M), but L (M) is A-regular R-module. Hence N is A-pure in L (i M), since L (M) is A-pure in M implies N is A-pure in M by proposition (2.9). But N is finitely generated, x ∈ J (r) M and N = (x) then N ⊆ J (r) M, therefore by lemma (2.11) N = 0. That is L (M) ∩ J (r) M = = < 0 >.

Recall that the trace ideal of an R-module M denoted by T is defined to be T = \( \sum_{f \in \text{Hom}(M, R)} f(M) \). Clearly T is an ideal of R, [8].
If M is regular R-module. And the trace ideal $T = R$, then Ris regular. For Ai-regular Ri-module we have the following:

**Proposition (2.13):** Let M be A-regular R-module. If the trace ideal $T = R$, then Ris Ai-regular.

**Proof:** Since $M$ is Ai-regular, then $L_i(M) = M_i$, hence $f(M) = f(L_i(M)) \subseteq (L(R)$ by proposition (2.4) where $f \in Hom(M, R)$. Thus $R = T = \sum_{f \in Hom(M, R)} f(M) \subseteq L(R)$. Therefore $R$ is A-regular.

**Proposition (2.14):** Let M be A-regular R-module, then $L(M) = M$. Thus $L(M) = L(M) = < 0 >$.

**Remark (2.15):** For any R-module $M$, $L(M) = < 0 >$ in general. For example, the module $Z$ as Z-module, then $L(M) = L(M)$.

Recall that a submodule $N$ of a R-module is said to be stable if $f(N) \subseteq N$ for each R-homomorphism $f: N \rightarrow M$, [9].

**Proposition (2.16):** For any R-module M, then $L(M)$ is stable submodule of M.

**Proof:** Let $f \in Hom_R(L(M))$. By Proposition (2.4), $f(L(M)) \subseteq L(M)$. But $L(f)(L(M)) = L(M)$ since $L(M)$ is A-regular. Thus $f(L(M)) = f(L(L(M))) \subseteq L(M)$. Therefore $L(M)$ is stable submodule.

Recall that non-zero submodule $N$ of an R-module $M$ is said to be dense in $M$ if $N$ generates $M$, that is $M = \sum_{f \in Hom(N, M)} f(N)$, [10].

**Proposition (2.17):**

Let $M$ be an $R_i$-module and $L(M)$ be a dense submodule in $M$, then $M$ is A-regular module.

**Proof:** Since $L(M)$ is dense in $M = \sum_{f \in Hom(L(M), M)} f(L(M))$. But $L(M)$ is stable submodule of $M$ by the previous Proposition (2.16), thus $f(L(M)) \subseteq L(M)$ implies $\sum_{f \in Hom(L(M), M)} f(L(M)) \subseteq L(M)$. Then $L_i = L_i(M)$, therefore $M$ is A-regular.

Recall that the Jacobson Radical of an R-module $M$ that denoted by $J(M)$ is defined to be $J(M) = \sum_{i}$ of all small submodules of $M$.

A submodule $N$ of an R-module $M$ is called small submodule of $M$ if for any submodule $L$ of $M$ such that $M = N + L$ implies $L = M$. It is well-known that if $M$ is finitely generated, then $J(M)$ is small submodule of $M$, [11, p.218].

**Proposition (2.18):** Let $M$ be a finitely generated R-module and $L(M) + J(M) = M$ implies $M$ is A-regular.

**Proof:** Since $M$ is finitely generated, then $J(M)$ is small submodule of $M$, but $L(M) + J(M)$, therefore $L_i(M) = M$ and hence $M$ is A-regular.

**Remark (2.19):** For any R-module $M$, $L(M) + J(M) \neq M$ in general. For example, the module $Z$ as Z-module where $L(Z) + J(Z) = < 2 > + < 2 > = < 2 > \neq Z$.

Recall that a submodule $N$ of an R-module $M$ is called an essential submodule of $M$ if foreach submodule $L$ of $M$ with $N \cap L = 0$ implies $L = 0$, [6].

We have the following:

**Proposition (2.20):** Let $M$ be an R-module and $N$ be an essential submodule of $M$. If $L(N) = 0$, then $L(M) = 0$.

**Proof:** Since $L(N) = N_i \cap L(M)$ by Proposition (2.3). Then $0 = N \cap L(M)$.

But $N$ is an essential submodule of $M$, thus $L(M) = 0$.

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