RECONSTRUCTION FOR THE TIME-DEPENDENT COEFFICIENTS OF A QUASILINEAR DYNAMICAL SCHRÖDINGER EQUATION

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ABSTRACT. We study an inverse problem related to the dynamical Schrödinger equation in a bounded domain of $\mathbb{R}^n, n \geq 2$. Since the concerned non-linear Schrödinger equation possesses a trivial solution, we linearize the equation around the trivial solution. Demonstrating the well-posedness of the direct problem under appropriate conditions on initial and boundary data, it is observed that the solution admits $\varepsilon$-expansion. By taking into account the fact that the terms $\mathcal{O}(|\nabla u(t,x)|^2)$ are negligible in this context, we shall reconstruct the time-dependent coefficients such as electric potential and vector-valued function associated with quadratic non-linearity from the knowledge of input-output map using the geometric optics solution and Fourier inversion.

1. Introduction

In this paper, we address an inverse problem associated with non-linear dynamical Schrödinger equation. More precisely, we consider the following initial boundary value problem (IBVP):

\[
\begin{aligned}
&i\partial_t u(t,x) + \Delta u(t,x) + q(t,x)u(t,x) = \nabla \cdot \mathbf{J}(t,x,\nabla u), \quad (t,x) \in \Omega_T,
\end{aligned}
\]

\[
\begin{aligned}
&u(0,x) = \phi(x), \quad x \in \Omega,
\end{aligned}
\]

\[
\begin{aligned}
&u(t,x) = f(t,x), \quad (t,x) \in \Sigma := (0,T) \times \partial \Omega,
\end{aligned}
\]

where $\Omega \subset \mathbb{R}^n \ (n \geq 2)$ is a bounded and simply connected domain with smooth boundary $\partial \Omega$. Given time $T > 0$, we denote $\Omega_T := (0,T) \times \Omega$. Furthermore, $\varepsilon > 0$ is a small parameter and $\nabla := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$ represents the gradient operator with respect to the spatial variable $x := (x_1, x_2, \ldots, x_n)$. In (1.1), real-valued time-dependent bounded function $q(t,x)$ represents the electric potential and we assume that $q \in C^\infty(\Omega_T)$.

The vector-valued function $\mathbf{J}(t,x,\nabla u)$ in (1.1) is given by

\[
\mathbf{J}(t,x,\nabla u) := \mathbf{P}(t,x,\nabla u) + \mathbf{R}(t,x,\nabla u)
\]

in which the function $\mathbf{P}(t,x,\xi)$ is described by

\[
\mathbf{P}(t,x,\xi) := |\xi|^2 \mathbf{b}(t,x), \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n,
\]

with $|\xi|^2$ defined using the complex conjugate as

\[
|\xi|^2 = \sum_{j=1}^n \xi_j \xi_j.
\]

and $\mathbf{b} \in C^\infty([0,T] \times \overline{\Omega})$ where $\overline{\Omega} := \Omega \cup \partial \Omega$. We denote a set of a Fréchet space $E$-valued $C^\infty$ functions over $[0,T]$ flat at $t = 0$ by $C^\infty_{[0,T]}([0,T]; E)$ and we assume that $\mathbf{R}(t,x,\xi) \in C^\infty_{[0,T]}((0,T]; C^\infty(\overline{\Omega} \times H))$, where for a constant $h > 0$, the set $H$ is defined by $H := \{ \xi \in \mathbb{C}^n : |\xi| \leq h \}$. Moreover, there exists a positive constant $C > 0$ such that

\[
|\partial^\alpha_x \nabla^\beta_y \mathbf{R}(t,x,\xi)| \leq C|\xi|^{3-|\alpha|},
\]

holds for all multi-indices $\alpha$ and $\beta$ with $|\alpha| \leq 3$. In order to state well-posedness results, we define the following Sobolev space for two non-negative real numbers $r$ and $s$

\[
H^{r,s}(\Sigma) = L^2(0,T; H^r(\partial \Omega)) \cap L^2(0,T; H^{s}(\partial \Omega))
\]

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equipped with the norm 
\[ ||f||_{H^{r,s}(\Sigma)} = ||f||_{L^2(0,T;H^r(\partial\Omega))} + ||f||_{H^r(0,T;L^2(\partial\Omega))}. \]
We also set 
\[ H_0^{r,s}(\Sigma) = \{ f \in H^{r,s}(\Sigma) : f(0,\cdot) = \partial_t f(0,\cdot) = \cdots = \partial_t^{r-1} f(0,\cdot) = 0 \text{ on } \partial\Omega \}. \]

Let \( r \geq \lceil \frac{n}{2} \rceil + 3 \). Then there exists \( \varepsilon_0 = \varepsilon_0(h, r, T) \) such that \([72]\) has a unique solution (see section 2 for more details)
\[ u \in X_r := X_r([0,T]) := \bigcap_{k=0}^r C^k([0,T];H^{r-k}(\Omega)) \]
in lieu with \( H^0(\Omega) = L^2(\Omega) \), satisfying the compatibility condition of order \( r - 1 \) and \( 0 < \varepsilon < \varepsilon_0 \) for any \( \phi \in H^r_0(\Omega) \cap H^{r+1}(\Omega) \) and \( f \in H^{r+1}_0(\Sigma) \). Moreover, we define the norm on \( X_r \) as
\[ ||u||_{X_r([0,T])} := \sup_{t \in [0,T]} \left( \sum_{k=0}^r \left( \int_{H^{r-k}(\Omega)} |u^{(k)}(t,\cdot)|^2 \right)^{1/2} \right), \]

where \( u^{(k)} := \frac{\partial^k u}{\partial t^k} \).

In view of the unique solvability of the IBVP \([72]\), we define the input-output map \( \Lambda_q, \overline{\nabla} = \Lambda_q, \overline{\nabla} (\varepsilon \phi, \varepsilon f) \) associated with the dynamical Schrödinger equation as follows:
\[ \Lambda_q, \overline{\nabla} (\varepsilon \phi, \varepsilon f) := \left( u^{\phi, f}|_{t=T}, \partial_t u^{\phi, f} - \nu(x) \cdot \overline{\nabla} (t, x, \nabla u^{\phi, f}) \right)_{\Sigma} \]
for any \((\phi, f) \in (H^r_0(\Omega) \cap H^{r+1}(\Omega)) \times H^{r+1}_0(\Sigma)\), where \( 0 < \varepsilon < \varepsilon_0 \) and \( u^{\phi, f}(t, x) \) is the solution of IBVP \([72]\) and \( \nu \) is the unit normal vector to \( \partial\Omega \) at \( x \in \partial\Omega \) directed into the exterior of \( \Omega \).

In this paper, we are interested in the following inverse problem associated with the quasilinear dynamical Schrödinger equation.

**Inverse problem:** Indentify the electric potential \( q = q(t,x) \) and the quadratic nonlinearity \( \overline{\nabla} = \overline{\nabla} (t, x, \nabla u) \) from the knowledge of input-output map \( \Lambda_q, \overline{\nabla} \) defined in \([15]\).

In recent years, there is a growing interest in recovering coefficients associated with quasilinear equations from the knowledge of boundary observations. For instance, the coefficients of a quasilinear elliptic equation in divergence form are reconstructed from the the Dirichlet to Neumann (DN) map defined on the boundary (see \([10]\) for details). In \([11]\), similar analysis has been carried out for the perturbed weighted \( p \)-Laplacian equation to recover the coefficients from prescribed DN-map.

Significant progress has also been made to recover the time-dependent coefficients appearing in the quasilinear equations, for instance, refer to \([15, 25, 32]\) for quasilinear parabolic equation, and \([20, 35]\) and references therein for hyperbolic equations. In case of magnetic Schrödinger equation, the problem of recovering coefficients has attracted the attention of several researchers in recent years. For instance, in \([7]\), inverse problem of determining the magnetic field and time-dependent electric potential from the knowledge of DN-map was considered in three or higher dimension. In \([17]\), the author dealt with a problem of determining time-dependent electromagnetic potentials appearing in the Schrödinger equation using geometric optics construction. Further in this direction, inverse problems related to the linear dynamical Schrödinger equation is considered and corresponding stability estimates are obtained (one can refer to \([11, 13, 35, 39]\) and references cited therein for details).

Choulli et al. \([14]\) considered the problem of recovering the time-dependent scalar potential associated with the Schrödinger equation from boundary measurements in an unbounded domain. Moreover, determination of electric or magnetic potential appearing in the linear dynamical Schrödinger equation in Riemannian geometry have also been studied (see \([2, 6, 17, 31, 34]\) and references therein). However, reconstructions of the time-dependent coefficients for the nonlinear equations have not reached the same stage of maturity as linear problems. Recently in \([35]\), the determination of time-dependent coefficients in a semi-linear dynamical Schrödinger equation is considered where they showed that the electric potential and coefficient of non-linearity can be determined from the knowledge of source-to-solution map. Inverse initial boundary value problem for a quasilinear hyperbolic equation in second or higher dimension was studied (refer to \([39]\), in which unique determination of time-dependent coefficients is proved from the measurement given by input-output map.
Theorem 2.1. The electric potential $q(t, x)$ and the coefficient $\mathbf{b}(t, x)$ of the quadratic part of the vector-valued function $\mathbf{f}(t, x, \nabla u)$ can be constructively determined from the knowledge of input-output map $\Lambda_{\mathbf{q}, \mathbf{f}}(\epsilon \phi, \epsilon f)$ measured for all $(\phi, f) \in (H^r_0(\Omega) \cap H^{r+1}(\Omega)) \times H^r_0(\Sigma)$. 

To the best of our knowledge, this is the first result which deals with the recovery of time-dependent coefficients appearing in a non-linear dynamical Schrödinger equation from the knowledge of boundary measurements of the solution. Recently in [35], the problem of recovery of time-dependent coefficients is considered but they recovered these coefficients from the knowledge of source-to-solution map while we recovered from the knowledge of solutions measured on the boundary.

The remaining part of the article is organized as follows. In Section 2, we will analyze the $\epsilon$-expansion of the initial-boundary value problem (1.1) and use the resulting expansion to linearize the input-output map $\Lambda_{\mathbf{q}, \mathbf{f}}$. Furthermore, we study the well-posedness of problem (1.1) and demonstrate that the solution admits $\epsilon$-expansion. In Section 3, we reconstruct the electric potential $q(t, x)$ and the coefficient of nonlinearity $\mathbf{b}(t, x)$ from the knowledge of input-output map $\Lambda_{\mathbf{q}, \mathbf{f}}$.

2. $\epsilon$-expansion of the solution of IBVP

The aim of this section is to demonstrate the unique solvability of magnetic Schrödinger equation (1.1) via $\epsilon$-expansion technique. This expansion technique is quite instrumental in the hyperbolic and elliptic problems (see [38, 39]) for nonlinear wave equation and (see [10, 29]) for quasilinear elliptic equation. However, we hereby impose the technique in a nonlinear dynamical Schrödinger equation where the treatment differs significantly and hence, the complete details will be provided for existence, uniqueness and regularity of the solution to equation (1.1).

We shall start with the following convention:

$$v(t, x) = O(\epsilon^3) \iff \|v\|_{X_{\mu}(0, T)} = O(\epsilon^3).$$

Theorem 1.1. The electric potential $q(t, x)$ and the coefficient $\mathbf{b}(t, x)$ of the quadratic part of the vector-valued function $\mathbf{f}(t, x, \nabla u)$ can be constructively determined from the knowledge of input-output map $\Lambda_{\mathbf{q}, \mathbf{f}}(\epsilon \phi, \epsilon f)$ measured for all $(\phi, f) \in (H^r_0(\Omega) \cap H^{r+1}(\Omega)) \times H^r_0(\Sigma)$. Then for given $T > 0$, there exists $\epsilon_0 = \epsilon_0(h, r, T)$ such that (1.1) has a unique solution $u(t, x) \in X_r([0, T])$ for any $0 < \epsilon < \epsilon_0$, where $h$ and $X_r$ are defined in Section 1.

Furthermore, $u(t, x)$ admits $\epsilon$-expansion

$$u(t, x) = \epsilon u_1(t, x) + \epsilon^2 u_2(t, x) + O(\epsilon^3)$$

(2.1)

where $\epsilon \to 0$ and $u_1(t, x)$ is a solution to the homogeneous equation:

$$\begin{align*}
\begin{cases}
i \partial_t u_1(t, x) + \Delta u_1(t, x) + q(t) u_1(t, x) = 0, & \quad (t, x) \in \Omega T, \\
u_1(0, x) = \phi(x), & \quad x \in \Omega, \\
u_1(t, x) = f(t, x), & \quad (t, x) \in \Sigma;
\end{cases}
\end{align*}$$

(2.2)

and $u_2(t, x)$ is a solution to the non-homogeneous equation:

$$\begin{align*}
\begin{cases}
i \partial_t u_2(t, x) + \Delta u_2(t, x) + q(t) u_2(t, x) = \nabla_x \cdot \left( |\nabla_x u_1(t, x)|^2 \mathbf{b}(t, x) \right), & \quad (t, x) \in \Omega T, \\
u_2(0, x) = 0, & \quad x \in \Omega, \\
u_2(t, x) = 0, & \quad (t, x) \in \Sigma.
\end{cases}
\end{align*}$$

(2.3)

2.1. Well-posedness: Hereby, we describe the well-posedness of initial-boundary value problems (2.2) and (2.3).

Theorem 2.2. Let $T > 0$. Assume that $q \in W^{1, \infty}(\Omega_T)$, $\phi \in H^1_0(\Omega) \cap H^2(\Omega)$, and $f \in H^{1, 2}_0(\Sigma)$. Then, there exists a unique solution $u_1 \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ of the dynamical Schrödinger equation (2.2). Moreover, $\partial_{\nu} u_1 \in L^2(\Sigma)$ and the following estimate holds:

$$\|u_1(t, \cdot)\|_{H^1(\Omega)} + \|\partial_{\nu} u_1\|_{L^2(\Sigma)} \leq C(\|\phi\|_{H^2(\Omega)} + \|f\|_{H^{1, 2}(\Sigma)})$$

(2.4)

with the positive constant $C = C(T, \|q\|_{W^{1, \infty}(\Omega_T)})$. 

The following result is our main contribution in this paper: 

Theorem 2.1. Let $\epsilon$ there exists $\epsilon \to 0$ and $u_1(t, x)$ is a solution to the homogeneous equation:

$$\begin{align*}
\begin{cases}
i \partial_t u_1(t, x) + \Delta u_1(t, x) + q(t) u_1(t, x) = 0, & \quad (t, x) \in \Omega T, \\
u_1(0, x) = \phi(x), & \quad x \in \Omega, \\
u_1(t, x) = f(t, x), & \quad (t, x) \in \Sigma;
\end{cases}
\end{align*}$$

and $u_2(t, x)$ is a solution to the non-homogeneous equation:

$$\begin{align*}
\begin{cases}
i \partial_t u_2(t, x) + \Delta u_2(t, x) + q(t) u_2(t, x) = \nabla_x \cdot \left( |\nabla_x u_1(t, x)|^2 \mathbf{b}(t, x) \right), & \quad (t, x) \in \Omega T, \\
u_2(0, x) = 0, & \quad x \in \Omega, \\
u_2(t, x) = 0, & \quad (t, x) \in \Sigma.
\end{cases}
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$$\|u_1(t, \cdot)\|_{H^1(\Omega)} + \|\partial_{\nu} u_1\|_{L^2(\Sigma)} \leq C(\|\phi\|_{H^2(\Omega)} + \|f\|_{H^{1, 2}(\Sigma)})$$

(2.4)

with the positive constant $C = C(T, \|q\|_{W^{1, \infty}(\Omega_T)})$. 

The proof of Theorem 2.2 follows by decomposing the solution \( u_1 \) of (2.2) as \( u_1 = v + w \), where \( v \) and \( w \) satisfy the following equations respectively:

\[
\begin{align*}
&i\partial_t v + \Delta v = 0, & (t, x) &\in \Omega_T, \\
v(0, x) = 0, & x &\in \Omega, \\
v(t, x) = f(t, x), & (t, x) &\in \Sigma; \\
&i\partial_t w + \Delta w + q(t, x)w = -q(t, x)v, & (t, x) &\in \Omega_T \\
w(0, x) = 0, & x &\in \Omega, \\
w(t, x) = 0, & (t, x) &\in \Sigma.
\end{align*}
\]

Afterwards, it can be seen from [4, Appendix A] that

\[ v \in C^1(0, T; H^1(\Omega)) \quad \text{and} \quad \|v\|_{C^1(0, T; H^1(\Omega))} \leq C \|f\|_{H^{1,2}(\Sigma)}. \]

In addition, we also have

\[ \partial_\nu v \in L^2(\Sigma) \quad \text{and} \quad \|\partial_\nu v\|_{L^2(\Sigma)} \leq C \|f\|_{H^{1,2}(\Sigma)}. \]

Furthermore, following the argument from [7, Theorem 1.1] we can show that

\[ w \in C(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \cap C^1(0, T; L^2(\Omega)) \quad \text{and} \quad \|w(t, \cdot)\|_{H^{1,2}(\Omega)} \leq C(\|\phi\|_{H^2(\Omega)} + \|f\|_{H^{1,2}(\Sigma)}) \]

along with

\[ \partial_\nu w \in L^2(\Sigma) \quad \text{and} \quad \|\partial_\nu w\|_{L^2(\Sigma)} \leq C(\|\phi\|_{H^2(\Omega)} + \|f\|_{H^{1,2}(\Sigma)}). \]

Hence the estimate (2.4) is obtained by combining the above estimates. Furthermore, we have the following regularity theorem.

**Theorem 2.3.** Let \( T > 0 \) and \( q \in C^\infty(\Omega_T) \). Assume that \( \phi \in H^1_0(\Omega) \cap H^{r+1}(\Omega) \) and \( f \in H^{r, r+1}_{0, r}(\Sigma) \) with \( r \in \mathbb{N} \). Then the equation (2.2) admits a unique solution

\[ u_1 \in \bigcap_{k=0}^r C^k(0, T; H^{r-k}(\Omega)). \]

The proof of regularity results presented in Theorem 2.3 follows from the decomposition of \( u \) into \( v \) and \( w \) as above. Afterwards, one can refer to [30, Theorem 12.1] in order to obtain the required regularity for \( w \), and similar approach as in [3, Lemma A.4] for the regularity of \( v \).

To demonstrate the well-posedness of IBVP (2.2), we define

\[ H^{r,s}(\Omega_T) = L^2(0, T; H^r(\Omega)) \cap H^s(0, T; L^2(\Omega)) \]

equipped with the norm

\[ \|v\|_{H^{r,s}(\Omega_T)} = \|v\|_{L^2(0, T; H^r(\Omega))} + \|v\|_{H^s(0, T; L^2(\Omega))}. \]

We state the following result for general non-homogeneous (source) term \( F = F(t, x) \) (cf. [30, Lemma 2.3])

**Theorem 2.4.** Consider the following IBVP:

\[
\begin{align*}
i\partial_t v(t, x) + \Delta v(t, x) + q(t, x)v(t, x) &= F(t, x), & (t, x) &\in \Omega_T, \\
v(0, x) &= 0, & x &\in \Omega, \\
v(t, x) &= 0, & (t, x) &\in \Sigma
\end{align*}
\]

with \( q \in W^{1,\infty}(\Omega_T) \) and \( F \in H^1(0, T; L^2(\Omega)) \), with \( F(0, \cdot) = 0 \) a.e. in \( \Omega \). Then there exists a unique solution

\[ v \in C(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \cap C^1(0, T; L^2(\Omega)) \]

of (2.5). In addition, there holds

\[ \|v\|_{C(0, T; H^2(\Omega))} + \|v\|_{C^1(0, T; L^2(\Omega))} \leq C \|F\|_{H^1(0, T; L^2(\Omega))} \]

with the positive constant \( C = C(T, \Omega, \|q\|_{W^{1,\infty}(\Omega_T)}). \)

In a similar manner as in Theorem 2.4 we can deduce the following regularity result:
Theorem 2.5. Suppose that \( q \in C^\infty(\Omega_T) \) and \( F \in H^{r-1,r}(\Omega_T), \ r \in \mathbb{N}, \) and \( F \) satisfies the compatibility condition of order \( r - 1 \) at \( t = 0 \) a.e. in \( \Omega. \) Then there exists a unique solution
\[
v \in \bigcap_{k=0}^{r} C^k(0,T;H_{0}^{r-k}(\Omega))
\]
of (2.5).

The proof of Theorem 2.5 follows from [36, Theorem 12.2] using parabolic regularization of (2.5). One can also refer to [39, Lemma 2.3] in which Faedo-Galerkin technique is applied. As a consequence, well-posedness of the IBVP (2.3) follows from Theorem 2.5 under the appropriate assumptions mentioned therein.

Corollary 2.6. Under the hypothesis of Theorem 2.2 and Theorem 2.5, equation (2.3) possess an unique solution \( u_2 \) and moreover, \( \partial_t u_2 \in L^2(\Sigma) \) with the estimate
\[
\| \partial_t u_2 \|_{L^2(\Sigma)} \leq C(||\phi||_{H_1^1(\Omega_T)}) + ||f||_{H^1(\Sigma)}).
\]

To demonstrate the above estimate, one can follow the similar approach presented in [7, Appendix A].

Remark 2.7. The DN-map \( \Lambda_\vartheta \) defined in (1.5) is well-defined under the hypothesis of Theorem 2.2 and 2.5. Furthermore, DN-map acts as a bounded linear map from \( H^2(\Omega) \times H^1(\Sigma) \) to \( H^1(\Omega) \times L^2(\Sigma). \)

Hereby, our aim is to prove Theorem 2.1 through the following Proposition 2.8 and several lemmas described below.

Proposition 2.8. Let us consider that the solution \( u(x,t) \) of (1.1) assumes the form
\[
(2.7)
\]
where \( u_1 \) and \( u_2 \) are the solutions to the IBVP (2.2) and (2.3) respectively. Consequently, we look to derive the equation for \( w(t,x) \). Then \( w(t,x) \) satisfies
\[
\begin{cases}
\begin{align*}
&i\partial_t w - A(w)w = \varepsilon \overline{G}(t,x,\varepsilon \nabla w) \cdot \nabla w + \varepsilon F(t,x,\nabla u_1, \nabla u_2), \quad (t,x) \in \Omega_T, \\
&w(0,x) = 0, \quad x \in \Omega, \\
&w(t,x) = 0, \quad (t,x) \in \partial \Omega_T;
\end{align*}
\end{cases}
\]
where the following notations are being incorporated:
\[
\begin{align*}
&\Lambda_\vartheta := -\Delta w - q(t,x)w + \varepsilon E(t,x,\nabla w) \cdot \nabla w; \\
&E(t,x,\nabla w) := [2\varepsilon (\overrightarrow{b} \otimes \nabla u_1) + 2\varepsilon^2 (\overrightarrow{b} \otimes \nabla u_2) + 2\varepsilon^2 (\overrightarrow{b} \otimes \nabla u_2) + \varepsilon K(t,x,\varepsilon \nabla w) + \varepsilon^2 K(t,x,\varepsilon \nabla w) \nabla w]; \\
&F(t,x,\nabla u_1, \nabla u_2) := 2\varepsilon \nabla \cdot (\nabla u_1 \cdot \nabla u_2) + \varepsilon^2 \nabla \cdot (|\nabla u_2|^2 \overrightarrow{b}) + \frac{1}{\varepsilon} \nabla \cdot \overline{R}(t,x,\varepsilon \nabla u_1 + \varepsilon^2 \nabla u_2); \\
&\overline{G}(t,x,\nabla w) \cdot \nabla w := 2\varepsilon \nabla \cdot (\overrightarrow{b} \otimes \nabla u_1 \cdot \nabla w) + 2\varepsilon \nabla \cdot (\overrightarrow{b} \otimes \nabla u_1 \cdot \nabla u_2) + \varepsilon^2 \nabla \cdot (\nabla \cdot \overrightarrow{b}) (\nabla \cdot \nabla w) \\
&\quad + 2\varepsilon \nabla \cdot (\nabla \cdot \overrightarrow{b}) (\nabla \cdot \nabla u_2) + \varepsilon (\nabla \cdot (K(t,x,\varepsilon \nabla w)) \cdot \nabla w.
\end{align*}
\]
\[
(2.8)
\]
Proof. Since we seek for a solution \( u(x,t) \) of (1.1) of the form (2.7), we substitute \( u(t,x) \) from (2.7) in (1.1) to obtain
\[
\begin{cases}
\begin{align*}
&i\partial_t w(t,x) + \Delta w(t,x) + q(t,x)w(t,x) = \frac{1}{\varepsilon^2} \nabla \cdot \overline{R}(t,x,\varepsilon \nabla u_1 + \varepsilon^2 \nabla u_2 + \varepsilon^2 w) \\
&\quad + 2\varepsilon \nabla \cdot (|\nabla u_1|^2 \overrightarrow{b}) + 2\varepsilon \nabla \cdot (|\nabla u_2|^2 \overrightarrow{b}) + 2\varepsilon \nabla \cdot (\nabla u_1 \cdot \nabla u_2) \overrightarrow{b}) + 2\varepsilon \nabla \cdot (\nabla u_2 \cdot \nabla w) \overrightarrow{b}), \quad (t,x) \in \Omega_T, \\
&w(0,x) = 0, \quad x \in \Omega, \\
w(t,x) = 0, \quad (t,x) \in \partial \Omega_T;
\end{align*}
\end{cases}
\]
where we have taken into account (2.2) and (2.3). We re-write the equation for \( w(t,x) \) in (2.10) as
\[
\begin{align*}
i \partial_t w(t,x) + \Delta w(t,x) + q(t,x)w(t,x) = T_1 + T_2 + T_3 + T_4,
\end{align*}
\]
where \( T_i, \ i = 1, 2, 3, 4 \) represent the right side terms in the same order. As a result,
\[
T_1 := \frac{1}{\varepsilon^2} \nabla \cdot \overline{R}(t,x,\varepsilon \nabla u_1 + \varepsilon^2 \nabla u_2 + \varepsilon^2 w)
\]
Lemma 2.9. Operator properties:

Let us consider that \( U_M > 0 \) with \( Z = \{0, 1\} \) and (2.10) to (2.12): Well-posedness for the semilinear Schrödinger equation.

2.2. We start by introducing the operator \( L \) for a given function \( U(t) \)

\[
Lw := i\partial_tw - A(U)w. \tag{2.14}
\]

In the following lemma, we derive the coercivity and continuous dependence estimates involving the operator \( A(U)(\cdot) \), which play an important role in establishing the well-posedness of (2.10).

Lemma 2.9. (Properties of the operator \( A(U)(\cdot) \))

Let us consider that \( U \in Z(M) \) and \( \varepsilon \in [0, \varepsilon_0) \) for a fixed \( \varepsilon_0 > 0 \). Then \( A(U)w \) has the following properties:
(1) For any \( w \in H^k_0(\Omega) \cap H^{k+1}(\Omega) \) and \( t \in [0, T] \),
\[
\|w\|_{H^{k+1}(\Omega)} \leq C_1(\|w\|_{H^{k-1}(\Omega)} + \|A(U_1)w\|_{H^{k-1}(\Omega)}), \quad k = 1, 2, \ldots, m - 1,
\]
where \( C_1 \) is a positive constant.

(2) Coercivity estimate: There exist positive constants \( C_b \) and \( C_c \) such that
\[
-\langle A(U_1)w, w \rangle + C_b\|w\|^2_{L^2(\Omega)} \geq C_c\|w\|^2_{H^1(\Omega)}, \quad t \in [0, T],
\]
where \( w \in H^1_0(\Omega) \).

(3) Continuous dependence: Let \( U_1, U_2 \in H^1(\Omega) \) with \( \|U_1\|_{H^1(\Omega)}, \|U_2\|_{H^1(\Omega)} \leq M \). There holds:
\[
\| (A(U_1) - A(U_2))w \|_{L^2(\Omega)} \leq \varepsilon \sigma(M, \varepsilon) \cdot \|\nabla U_1 - \nabla U_2\|_{L^2(\Omega)}, \quad t \in [0, T], \quad w \in Z(M);
\]
where \( \sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function and \( \mathbb{R}^+ \) is the set of non-negative real numbers.

Proof. For the properties (1) and (2), one can use the elliptic regularity argument \([39]\). For the property (3), we have
\[
\|A(U_1)w - A(U_2)w\|_{L^2(\Omega)} = \|\varepsilon(E(t, x, \nabla U_1) - E(t, x, \nabla U_2)) \cdot \nabla^2 w\|_{L^2(\Omega)} \\
\leq 2\varepsilon^2 \| (\nabla U_1 - \nabla U_2) \cdot \nabla^2 w \|_{L^2(\Omega)} \\
+ \varepsilon \|K(t, x, \varepsilon \nabla U_1) \cdot \nabla^2 w - K(t, x, \varepsilon \nabla U_2) \cdot \nabla^2 w\|_{L^2(\Omega)} \\
+ \varepsilon^2 \| (K(t, x, \varepsilon \nabla U_1) \nabla U_1) \cdot \nabla^2 w = (K(t, x, \varepsilon \nabla U_2) \nabla U_2) \cdot \nabla^2 w\|_{L^2(\Omega)} \\
=: I_1 + I_2 + I_3.
\]
We obtain the following estimates on \( I_1, I_2 \) and \( I_3 \):
\[
I_1 = 2\varepsilon^2 \left\| (\nabla U_1 - \nabla U_2) \cdot \nabla^2 w \right\|_{L^2(\Omega)} \\
\leq 2\varepsilon^2 \| \nabla \|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)} \|\nabla(U_1 - U_2)\|_{L^2(\Omega)};
\]
\[
I_2 = \varepsilon \|K(t, x, \varepsilon \nabla U_1) \cdot \nabla^2 w - K(t, x, \varepsilon \nabla U_2) \cdot \nabla^2 w\|_{L^2(\Omega)} \\
= \varepsilon \left\| \int_0^1 \{ \nabla (\varepsilon^2 \nabla U_1 + \varepsilon^2 \nabla U_2 + \theta \varepsilon^2 \nabla U_1) - \nabla (\varepsilon^2 \nabla U_2 + \theta \varepsilon^2 \nabla U_2) \} \, d\theta \cdot \nabla^2 w \right\|_{L^2(\Omega)} \\
\leq C_1(M, \varepsilon) \|\nabla(U_1 - U_2)\|^2_{L^2(\Omega)};
\]
\[
I_3 = \varepsilon^2 \| (K(t, x, \varepsilon \nabla U_1) \nabla U_1) \cdot \nabla^2 w - (K(t, x, \varepsilon \nabla U_2) \nabla U_2) \cdot \nabla^2 w\|_{L^2(\Omega)} \\
\leq C_2(M, \varepsilon) \|\nabla(U_1 - U_2)\|_{L^2(\Omega)},
\]
where we have taken into account the estimate \([1, 4]\) and the constants \( C_1 \) and \( C_2 \) depend on \( M \) and \( \varepsilon \) and finally, in the estimates \( I_2 \) and \( I_3 \), we have incorporated \( \|U_i\|_{H^1(\Omega)} \leq M, \quad i = 1, 2 \).

We seek an appropriate estimate of the function \( G(t, x, \nabla w) \). The resulting estimate will be required to analyze the well-posedness of (2.10). In the following lemma, we analyze the estimate of \( G(t, x, \nabla w) \):

Lemma 2.10. [Estimate of \( G(t, x, \nabla w) \)]
Let us assume that \( r \geq \left[ \frac{n}{2} \right] + 3 \). Then for any \( w \in Z(M) \), there holds
\[
\|G(t, x, \nabla w)\|_{H^{r-1}(\Omega)} \leq C \left( 1 + \|w\|_{H^{r-1}(\Omega)} \right), \quad t \in [0, T],
\]
where the constant \( C > 0 \) is depending only on \( M \).

To demonstrate the estimate \((2.18)\) in Lemma 2.10, the following result will be quite instrumental.

Lemma 2.11. (\([39]\) Lemma 2.6)
Let \( r \in \mathbb{Z} \) such that \( r \geq \left[ \frac{n}{2} \right] + 3 \) and \( \kappa > 0 \) be the Sobolev embedding
\[
H^{[n/2]+1} \hookrightarrow C^0(\Omega)
\]
constant. For a given \( C^{r-1} \) function \( f(t, x; z) \) on \( \mathcal{D} = \{(t, x, z) \in [0, T] \times \Omega \times \mathbb{C} : |z| \leq \kappa M \} \), we have
\[
\|f(t, x; z)\|_{H^{r-1}(\Omega)} \leq C_{r-1}M_{r-1}\left\{1 + (1 + \|z(t)\|_{H^{r-2}(\Omega)}\|z(t)\|_{H^{r-1}(\Omega)} \right\}, \quad t \in [0, T],
\]
(2.19)
where the constant $C_{r-1}$ is depending on $r-1$ and

$$M_{r-1} := \max_{|\beta| \leq r-1} \sup_{|x| \leq 1} \left| \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) \beta f(t, x, z) \right|.$$  

**Proof of the Lemma 2.10** By the Lemma 2.11 and the fact that $\partial^\alpha w \in X_{p-|\alpha|}$ for $w \in X_p$ for $p \in \mathbb{N} \cup \{0\}$ and multi-index $|\alpha| \leq r$, we obtain

$$\|G(t, \cdot, \nabla w)\|_{H^{r-1}(\Omega)} \leq C_{r-1}M_{r-1} \left\{ 1 + (1 + \|w(t)\|_{H^{r-2}(\Omega)}) \|w(t)\|_{H^{r-1}(\Omega)} \right\}.$$  

Further, we observe that

$$1 + (1 + \|w(t)\|_{H^{r-2}(\Omega)}) \|w(t)\|_{H^{r-1}(\Omega)} \leq 1 + (1 + \|w(t)\|_{H^{r-2}(\Omega)})(1 + \|w(t)\|_{H^{r-1}(\Omega)}) = 1 + (1 + \|w(t)\|_{H^{r-1}(\Omega)})^{r-1} \leq C(1 + \|w(t)\|_{H^{r-1}(\Omega)}),$$

where $C$ is a positive constant. Hence the result follows. $\square$

**Theorem 2.12.** Let $r \in \mathbb{N}$ such that $r \geq \left[ \frac{n}{2} \right] + 3$. Then there exists $\varepsilon_1 > 0$ such that semilinear IBVP 2.11 has a unique solution $w_{\text{sem}} \in Z(M)$ for each $\varepsilon \in (0, \varepsilon_1)$. Furthermore,

$$\|w_{\text{sem}}\|_{X_r} \leq \varepsilon C(M, \varepsilon_1) \exp(KT), \hspace{1em} \varepsilon \in (0, \varepsilon_1),$$  

where $K = K(M, \varepsilon)$ is a positive constant.

To demonstrate the result in Theorem 2.12 we require a suitable energy estimate which will be provided by the following lemma (cf. [39, Section 2]):

**Lemma 2.13.** Consider the IBVP:

$$\begin{align*}
L[v] &= S, \hspace{1em} (t, x) \in \Omega_T, \\
v(0, x) &= 0, \hspace{1em} x \in \Omega, \\
v(t, x) &= 0, \hspace{1em} (t, x) \in \partial \Omega_T,
\end{align*}$$

where the operator $L$ is defined by (2.13) and $S \in X_{r-1}$. Moreover, we assume that $S$ satisfies the compatibility condition of order $r-1$ at $t = 0$ a.e. in $\Omega$. Then

(i) there exists a unique solution $v \in X_r$ to (2.21);

(ii) there holds the following energy estimate:

$$\|v(t)\|_{H^r(\Omega)}^2 \leq C_r \int_0^T \|S(t)\|^2_{H^{r-1}(\Omega)} dt, \hspace{1em} t \in [0, T],$$

where the general constant $C_r$ depending on $r$.

**Proof of Theorem 2.12** Existence of solution (2.11): We represent the equation (2.11) as follows:

$$\begin{align*}
L[w^{\text{sem}}] &= \varepsilon G(t, x, \nabla w^{\text{sem}}), \hspace{1em} (t, x) \in Q_T, \\
w^{\text{sem}}(0, x) &= 0, \hspace{1em} x \in \Omega, \\
w^{\text{sem}}(t, x) &= 0, \hspace{1em} (t, x) \in \partial \Omega_T.
\end{align*}$$

We introduce the sequence of functions $\{w_j^{\text{sem}}\}$ by

$$\begin{align*}
L[w_1^{\text{sem}}] &= \varepsilon G(t, x, 0); \\
L[w_2^{\text{sem}}] &= \varepsilon G(t, x, \nabla w_1^{\text{sem}}); \\
&\vdots \\
L[w_j^{\text{sem}}] &= \varepsilon G(t, x, \nabla w_{j-1}^{\text{sem}}), \hspace{1em} j = 2, 3, \ldots.
\end{align*}$$

Our aim is to show that for any small $\varepsilon > 0$,

$$w_j^{\text{sem}} \in X_r \text{ and } \sup_{t \in [0, T]} \|w_j^{\text{sem}}\|_{H^r(\Omega)} \leq M, \hspace{1em} j = 1, 2, 3, \ldots.$$
Since it follows from (2.12) that
\[ G(t, x, 0) = F(t, x, \nabla u_1, \nabla u_2) \]
and as a consequence, for \( \varepsilon \) small enough
\[
\sup_{t \in [0, T]} \| w_j^{sem}(t) \|_{H^r(\Omega)} \leq M.
\]
Furthermore, due to the following fact: if
\[
\sup_{t \in [0, T]} \| w_j^{sem}(t) \|_{H^r(\Omega)} \leq M,
\]
then we have for \( t \in [0, T] \)
\[
\| w_j^{sem}(t) \|_{H^r(\Omega)}^2 \leq \varepsilon^2 \int_0^T \| G(t, x, \nabla w_j^{sem}) \|_{H^{-1}(\Omega)}^2 \, dt
\]
\[
\leq \varepsilon^2 C \int_0^T \left( 1 + \| w_j^{sem}(t) \|_{H^r(\Omega)}^{r-1} \right)^2 \, dt,
\]
and subsequently, if we choose \( \varepsilon \) small enough such that it satisfies
\[
\varepsilon \leq \min \left\{ \frac{M}{\sqrt{CT(1 + M^{r-1})}}, \frac{1}{2} \right\},
\]
then it results into (by using induction principle)
\[
\sup_{t \in [0, T]} \| w_j^{sem}(t) \|_{H^r(\Omega)} \leq M, \quad j \geq 2.
\]
Hence \( \{ w_j \} \) is bounded and observe that
\[
L[w_{j+1}^{sem} - w_j^{sem}] = \varepsilon \left \{ G(t, x, \nabla w_{j+1}^{sem}) - G(t, x, \nabla w_j^{sem}) \right \}
\]
\[
= \varepsilon \left \{ \int_0^1 \nabla q G \left( t, x, \nabla w_{j+1}^{sem} + \theta \nabla (w_j^{sem} - w_{j+1}^{sem}) \right) \, d\theta \right \} \cdot \nabla (w_j^{sem} - w_{j+1}^{sem}).
\]
Thanks to the Lemma 2.13 and Lemma 2.10 we have
\[
\| w_{j+1}^{sem}(t) - w_j^{sem}(t) \|_{H^r(\Omega)}^2 \leq C \varepsilon^2 \int_0^T \| w_j^{sem}(s) - w_{j-1}^{sem}(s) \|_{H^r(\Omega)}^2 \, ds, \quad t \in [0, T],
\]
and subsequently, by the choice of \( \varepsilon \), it is evident that \( \{ w_j^{sem}(t) \} \) is a Cauchy sequence with respect to the norm \( \sup_{t \in [0, T]} \| \cdot \|_{H^r(\Omega)} \). Let \( w^{sem}(t) \) be the limit of the Cauchy sequence \( \{ w_j^{sem}(t) \} \). As a consequence, by standard regularity argument, we can demonstrate that \( w^{sem} \in X_r \) and \( w^{sem} \) satisfies the equation (2.23). Hence the existence of solution of (2.23) follows.

**Uniqueness of the solution (2.11):** We prove the estimate (2.21). We fix \( t \in (0, T) \) and we consider the equation (2.24) by taking inner product with \( w^{sem}(t, \cdot) \) to obtain
\[
i \left \langle \partial_t w^{sem}(t, \cdot), w^{sem}(t, \cdot) \right \rangle_{L^2(\Omega)} - \left \langle A(U) w^{sem}(t, \cdot), w^{sem}(t, \cdot) \right \rangle_{L^2(\Omega)}
\]
\[
= \left \langle \varepsilon F(t, x, \nabla w^{sem}), w^{sem}(t, \cdot) \right \rangle_{L^2(\Omega)}.
\]
Taking the imaginary part of both sides in the above identity we obtain
\[
\frac{d}{ds} \| w^{sem}(s, \cdot) \|_{L^2(\Omega)}^2 = 2 \, \text{Im} \left \langle \varepsilon F(t, x, \nabla w^{sem}), w^{sem}(s, \cdot) \right \rangle_{L^2(\Omega)}, \quad s \in (0, T).
\]
Integrating the above identity over \( (0, t) \) resulting into
\[
\| w^{sem}(t, \cdot) \|_{L^2(\Omega)}^2 = 2 \, \text{Im} \int_0^t \left \langle \varepsilon F(t, x, \nabla w^{sem}), w^{sem}(s, \cdot) \right \rangle_{L^2(\Omega)} \, ds
\]
\[
\leq 2 \int_0^t \left \langle \varepsilon F(t, x, \nabla w^{sem}), w^{sem}(s, \cdot) \right \rangle_{L^2(\Omega)} \, ds.
\]
since $w_{sem}(0, \cdot) = 0$. Taking the real part in equation (2.24), we get
\[
\left\langle A(U)w_{sem}(t, \cdot), w_{sem}(t, \cdot) \right\rangle_{L^2(\Omega)} = -\left| \left\langle \varepsilon F(t, x, \nabla w_{sem}), w_{sem}(t, \cdot) \right\rangle_{L^2(\Omega)} \right|.
\]

Thanks to the coercivity estimate (2.16), we have the following inequality
\[
\|w_{sem}(t, \cdot)\|_{H^1(\Omega)}^2 \leq C \left( \|\varepsilon F(t, x, \nabla w_{sem}), w_{sem}(t, \cdot)\|_{L^2(\Omega)} \right).
\]

The quadratic term $|\nabla w_{sem}(s, \cdot)|^2$ present in $F$ is estimated using Sobolev embedding as follows: for any $s \in [0, T],$
\[
\int_\Omega |\nabla w_{sem}(s)|^2 |w(s)| \, dx \leq \sup_\Omega |\nabla w_{sem}(s)| \cdot \|w_{sem}(s)\|_{L^2(\Omega)} \cdot \|\nabla w_{sem}(s)\|_{L^2(\Omega)} \leq C \left( \|w_{sem}(s)\|_{L^2(\Omega)}^2 + \|w_{sem}(s)\|_{H^1(\Omega)}^2 \right).
\]

Combining the estimates (2.25), (2.26) and Lemma 2.9 yield the following estimate for sufficiently small $\varepsilon > 0$
\[
\|w_{sem}(t, \cdot)\|_{H^1(\Omega)}^2 + \|w_{sem}(t, \cdot)\|_{L^2(\Omega)}^2 \leq C_1 \varepsilon^2 + C_2 \int_0^t \left( \|w_{sem}(s, \cdot)\|_{H^1(\Omega)}^2 + \|w_{sem}(s, \cdot)\|_{L^2(\Omega)}^2 \right) \, ds
\]
for any $t \in [0, T]$. In (2.28), $C_1$ is a general positive constant and the constant $C_2$ is also positive. In addition, $C_2$ is bounded with respect to $\varepsilon$.

Afterwards, the Gronwall’s inequality along with the estimate (2.28) yield the estimate (2.20). The uniqueness of the solution of (2.11) immediately follows from the estimate (2.20). Hence the Theorem 2.10 follows.

\textbf{Proof of Theorem 2.10} To establish the well-posedness described of the initial boundary value problem (2.11) in the Proposition 2.8, we use the idea of fixed point argument. We have observed in the Lemma 2.9 that given $U \in Z(M)$, there exists a unique solution $w$ of (2.11) and $w \in Z(M)$ provided $\varepsilon$ is small enough. This fact ensures that the following mapping
\[
T: Z(M) \to Z(M)
\]
for any $t \in [0, T]$. In (2.11), $w$ is the solution of (2.11), is well-defined. We will prove that the map $T$ is a contraction map.

Let $U_1$ and $U_2$ be such that $T(U_i) = w_i$, $i = 1, 2$, where $w_i$ is the solution of semi-linear equation (2.11) for $U_i \in Z(M)$. Let us define
\[
W := w_1 - w_2, \quad V := U_1 - U_2.
\]
Then the variable $W$ will satisfy the following initial boundary value problem
\[
\left\{
\begin{array}{l}
    i \partial_t W - B(U_1)W = (B(U_1) - B(U_2))w_2 = \varepsilon \overline{G}(t, x, \nabla w_1) \cdot \nabla W \\
    + \varepsilon \left( \overline{G}(t, x, \nabla w_1) - \overline{G}(t, x, \nabla w_2) \right) \cdot \nabla w_2, \quad (t, x) \in \Omega_T \\
\end{array}
\right.
\]
(2.29)

Multiplying (2.20) by $2W$ and integrate over $[0, t] \times \Omega$, we obtain
\[
\frac{1}{2} \|W(t)\|_{L^2(\Omega)}^2 = 2 \int_0^t \left\langle (B(U_1) - B(U_2))w_2, W \right\rangle \, d\tau + 2 \int_0^t \left\langle (B(U_1))w_2, W \right\rangle \, d\tau + 2 \varepsilon \int_0^t \left\langle (\overline{G}(t, x, \nabla w_1) \cdot \nabla W, W \right\rangle \, d\tau =: I_1 + I_2 + I_3 + I_4.
\]

Considering the imaginary part on both sides, we have
\[
\|W(t)\|_{L^2(\Omega)}^2 = \text{Im} \left( I_1 \right) + \text{Im} \left( I_2 \right) + \text{Im} \left( I_3 \right) + \text{Im} \left( I_4 \right) \\
\leq |I_1| + |I_2| + |I_3| + |I_4|.
\]
Since \( w_2 \in Z(M) \), the continuous dependence estimate in (2.17) implies
\[
\| (B(U_1) - B(U_2)) w_2 \|_{L^2(\Omega)} \leq \varepsilon \sigma(M, \varepsilon) \| \nabla V \|_{L^2(\Omega)}. \tag{2.30}
\]
As a consequence, the term \( I_2 \) can be estimated as follows:
\[
I_2 = 2 \int_0^t \left\langle (B(U_1) - B(U_2)) w_2, W \right\rangle \, d\tau
\leq C \varepsilon^2 \sup_{t \in [0, T]} \| V \|^2_{L^2(\Omega)} + 2 \int_0^t \| W(s) \|^2_{L^2(\Omega)} \, ds.
\]
Afterwards, we shall estimate \( I_3 \) and \( I_4 \) as follows:
\[
I_3 = 2 \varepsilon \int_0^t \left\langle \overrightarrow{G}(t, x, \nabla w_1) \cdot \nabla W, W \right\rangle \, d\tau
= 2 \int_0^t \left\langle 2 \varepsilon \nabla^2 u_1 \cdot \overrightarrow{b} \otimes \nabla W \right\rangle + 2 \varepsilon (\nabla \cdot \overrightarrow{b}) (\nabla u_1 \cdot \nabla W)
+ \varepsilon^2 (\nabla \cdot \overrightarrow{b}) (\nabla w_1 \cdot \nabla W) + 2 \varepsilon^2 \nabla^2 u_2 \cdot (\overrightarrow{b} \otimes \nabla W)
+ 2 \varepsilon^2 (\nabla \cdot \overrightarrow{b}) (\nabla u_2 \cdot \nabla W) + \varepsilon (\nabla \cdot K) \cdot \nabla W, W \right\rangle \, d\tau
\leq K \varepsilon \int_0^t \left( \| \nabla W \|^2_{L^2(\Omega)} + \| W \|^2_{L^2(\Omega)} \right) \, d\tau,
\]
and
\[
I_4 = \left\langle \overrightarrow{G}(t, x, \nabla w_1) - \overrightarrow{G}(t, x, \nabla w_2), \nabla w_2 \right\rangle \, d\tau
= 2 \varepsilon \int_0^t \left\langle \varepsilon^2 (\nabla \cdot \overrightarrow{b}) (\nabla w_1 \cdot \nabla w_2) - \varepsilon^2 (\nabla \cdot \overrightarrow{b}) (\nabla w_2 \cdot \nabla w_2)
+ \varepsilon (\nabla \cdot K) \cdot \nabla w_2 - \varepsilon (\nabla \cdot K) \cdot \nabla w_2, W \right\rangle \, d\tau
\leq C \varepsilon^2 + 2 \int_0^t \| W(s) \|^2_{L^2(\Omega)} \, d\tau,
\]
where we have used the following estimate
\[
\nabla \cdot (\varepsilon K(t, x, \varepsilon \nabla w_1)) - \nabla \cdot (\varepsilon K(t, x, \varepsilon \nabla w_2))
= \nabla \cdot \left( \int_0^1 \left\{ \nabla \overrightarrow{R}(t, x, \varepsilon \nabla u_1 + \varepsilon^2 \nabla u_2 + \theta \varepsilon^2 \nabla w_1) - \nabla \overrightarrow{R}(t, x, \varepsilon \nabla u_1 + \varepsilon^2 \nabla u_2 + \theta \varepsilon^2 \nabla w_2) \right\} \, d\theta \right)
\leq C \varepsilon^2.
\]
Combining the above estimates and using coercivity property, we have
\[
\| W(t) \|^2_{L^2(\Omega)} \leq \varepsilon^2 C \sup_{t \in [0, T]} \| V(t) \|^2_{H^1(\Omega)} + K \varepsilon \int_0^t \| W(\tau) \|^2_{H^1(\Omega)} \, d\tau \tag{2.31}
\]
with a constant \( K_\varepsilon > 0 \) bounded with respect to \( \varepsilon \) and a general constant \( C > 0 \).

We introduce a metric \( \rho \) in the space \( Z(M) \) as follows:
\[
\rho(f, g) := \max_{t \in [0, T]} \{ \| f(t) - g(t) \|_{H^1(\Omega)} \}.
\]
Finally using the Gronwall’s inequality, we have
\[
\rho(T(U_1), T(U_2)) \leq \varepsilon C e^{K_\varepsilon T} \rho(U_1, U_2)
\]
with the constant \( C > 0 \). As a result, we assert that the map \( T \) is a contraction mapping for \( \varepsilon > 0 \) small enough. We conclude that for each \( \varepsilon > 0 \) small enough, there exists \( w \in Z(M) \) such that
\[
T(w) = w, \quad \| w \|_{X_M} = O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{2.32}
\]
Hence the Theorem 2.1 follows. \qed
3. Proof of Theorem

In this section our aim is to give the reconstruction formula for determining the coefficients \(q(t, x)\) and \(b(t, x)\) from the knowledge of input-output map \(\Lambda_q, \tilde{f}(\varepsilon \phi, \varepsilon f)\) for all \(\phi\) and \(f\). Based on the analysis presented in the earlier section, we consider the \(\varepsilon\)-expansion of the solution to (3.1) as

\[
u(t, x) = \varepsilon u_1(t, x) + \varepsilon^2 u_2(t, x) + O(\varepsilon^3),
\]

where \(u_1(t, x)\) and \(u_2(t, x)\) satisfy the equations (2.2) and (2.3) respectively.

To begin with, we linearize the input-output map incorporating the representation of \(u\) in (3.1).

\[
\Lambda_{q, \tilde{f}}(\varepsilon \phi, \varepsilon f) = \left[u_{1|_{t=T}}\left[\partial_x \nu - \varepsilon \tilde{f}(t, x, \nabla_x u_1)\right]_{\partial \Omega_T}\right]
\]

\[
= \left[\varepsilon^2 u + \varepsilon^2 u_2 + O(\varepsilon^3)\right]_{t=T} \partial_x \left[\varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)\right]_{\partial \Omega_T}
\]

\[
- \left[\nu \cdot \tilde{f}(t, x, \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3))\right]_{\partial \Omega_T}
\]

\[
:= \varepsilon \tilde{g}_1 + \varepsilon^2 \tilde{g}_2 + O(\varepsilon^3),
\]

where \(\tilde{g}_1\) and \(\tilde{g}_2\) are given by

\[
\tilde{g}_1 := \left[u_{1|_{t=T}}\left[\partial_x \nu\right]_{\partial \Omega_T}\right]
\]

\[
\tilde{g}_2 := \left[u_{2|_{t=T}}\left[\partial_x \nu - \varepsilon \tilde{f}(t, x)|\nabla_x u_1|^2\right]_{\partial \Omega_T}\right].
\]

Now let us define

\[
\Lambda_q(\phi, f) := \left[u_{1|_{t=T}}\left[\partial_x \nu\right]_{\partial \Omega_T}\right] = \text{known}
\]

\[
\Lambda_\tilde{b}(\phi, f) := \left[u_{2|_{t=T}}\left[\partial_x \nu - \varepsilon \tilde{b}(t, x)|\nabla_x u_1|^2\right]_{\partial \Omega_T}\right] = \text{known}.
\]

Now that the map \(\Lambda_q(\phi, f)\) defined by (3.4) is input-output map for (2.2). Using the knowledge of \(\Lambda_q(\phi, f)\) given for all \((\phi, f) \in (H^s_0(\Omega) \cap H^{r+1}(\Omega)) \times H^s_0(\Omega)\) in subsection 3.1, we give the reconstruction formula for determining \(q(t, x)\) in \(\Omega_T\).

3.1. Reconstruction of electric potential. Our aim is to reconstruct \(q(t, x)\) from \(\Lambda_q(\phi, f)\) for all \((\phi, f) \in (H^s_0(\Omega) \cap H^{r+1}(\Omega)) \times H^s_0(\Omega)\) and the equation (2.2). We multiply the equation (2.2) by \(v(t, x)\), where \(v(t, x)\) is assumed to be the solution of

\[
-i \partial_t v + \Delta v = 0
\]

and integrate over \(\Omega_T\) to obtain

\[
\int_0^T \int_\Omega \left[i \partial_t u_1(t, x) + \Delta u_1(t, x) + q(t, x)u_1(t, x)\right] v(t, x) \, dx \, dt = 0.
\]

Now integration by parts yields

\[
\int_\Omega i[u_1(T, x)v(T, x) - u_1(0, x)v(0, x)] \, dx - \int_0^T \int_\Omega i \partial_t v \, dx \, dt + \int_0^T \int_\Omega \nabla v \, dx \, dt
\]

\[
+ \int_0^T \int_{\partial \Omega_T} (\partial_x u_1 v - \partial_x v u_1) \, ds \, dt + \int_0^T \int_\Omega q(t, x)u_1 v \, dx \, dt = 0
\]

which results into

\[
\int_0^T \int_\Omega q(t, x)u_1(t, x) v(t, x) \, dx \, dt = - \int_0^T \int_\Omega i u_1(T, x)v(T, x) \, dx - \int_0^T \int_\Omega i u_1(0, x)v(0, x) \, dx
\]

\[
- \int_0^T \int_{\partial \Omega_T} \partial_x u_1(t, x)v(t, x) \, ds \, dt - \int_0^T \int_{\partial \Omega_T} \partial_x v(t, x) u_1(t, x) \, ds \, dt.
\]
In the right hand side of (3.7), first and third terms are known from the measurement whereas second and fourth terms are known thanks to the input data. As a consequence,
\[
\int_0^T \int_\Omega q(t, x)u_1(t, x)v(t, x) \, dx \, dt = \text{known},
\] (3.8)
where \( v(t, x) \) satisfies the PDE (3.6).

In order to reconstruct \( q(t, x) \) from (3.8), we shall choose geometric optics solution for \( u_1 \) and \( v \).

Observe that
\[
(i\partial_t + \Delta)[e^{-i(\lambda^2 t + \lambda x \cdot \omega)} A_1(t, x)] = e^{-i(\lambda^2 t + \lambda x \cdot \omega)} \{ i\partial_t A_1 + \Delta A_1 - 2i\lambda \omega \cdot \nabla_x A_1 \},
\]
where \( \omega \) is an unit vector. By equating coefficient of \( \lambda \) to zero, we obtain the transport equation
\[
\omega \cdot \nabla_x A_1 = 0.
\] (3.9)

As a consequence, we seek the solution \( u_1 \) of the form
\[
u^1 \equiv \int_0^\infty \int_{\Gamma_\tau} q(t, x) v(t, x) \, dx \, dt = \text{known},
\]
where
\[
u_1 = \int_0^\infty \int_{\Gamma_\tau} q(t, x) v(t, x) \, dx \, dt \equiv \text{known}.
\]

Using (5), we choose
\[
\lambda = \frac{R_1}{\| \nabla_R \|_{L^2} + \| \nabla_x R_1 \|_{L^2}} \leq C.
\] (3.12)

In the similar manner, we recall the adjoint equation
\[
-i\partial_t v + \Delta v = 0 \text{ in } \Omega_T.
\] (3.13)

We seek for the solution \( v(t, x) \) of (3.13) in the following form:
\[
v(t, x) = e^{i(\lambda^2 t + \lambda x \cdot \omega)} A_2(t, x) + R_2(t, x).
\] (3.14)

It is straightforward to derive that we shall choose \( A_2(t, x) \) such that
\[
\omega \cdot \nabla_x A_2 = 0.
\] (3.15)

As a consequence, we will consider
\[
A_2(t, x) = e^{i(\tau t + \xi \cdot x)},
\] (3.16)
where \( \xi \cdot \omega = 0 \). Moreover, using (5) the following estimates can be obtained
\[
\| R_2 \|_{L^2} \leq \frac{C}{\lambda}, \quad \| \nabla_x R_2 \|_{L^2} \leq C
\]
for some positive constant \( C \) independent of \( \lambda \). Hence we choose the geometric optics solution \( v(t, x) \) as
\[
v(t, x) = e^{i(\lambda^2 t \lambda x \cdot \omega)} \left( e^{i(\tau t \xi \cdot x)} + R_2(t, x) \right)
\] (3.17)
with \( \omega \perp \xi \) and
\[
\lambda \| R_2 \|_{L^2(\Omega_T)} + \| R_2 \|_{L^2(\Omega_T)} \leq C.
\] (3.18)

Substituting the expressions for \( u_1(t, x) \) from (3.11) and for \( v(t, x) \) from (3.17) in (3.8), we obtain
\[
\int_0^T \int_\Omega q(t, x) \left( e^{i(\tau t \xi \cdot x)} + R_2(t, x) \right) (1 + R_1(t, x)) \, dx \, dt = 0
\]
which after simplification gives
\[
\int_0^T \int_\Omega q(t, x) \left( e^{i(\tau t \xi \cdot x)} + R_1(t, x) + R_2(t, x) + R_1(t, x) R_2(t, x) \right) \, dx \, dt = \text{known}
\]
for all \( \xi \perp \omega \). Now using (3.12) and (3.18) and letting \( \lambda \to \infty \), we get
\[
\int_0^T \int_\Omega q(t, x) e^{i(\tau t \xi \cdot x)} \, dx \, dt = \text{known}, \quad \text{for all } \xi \perp \omega.
\]
Consequently, we have
\[ \int_{\mathbb{R}^{1+n}} \chi_{\Omega_{r}}(t, x)q(t, x)e^{i(\tau t+\xi x)} \, dx \, dt = \text{known}. \]
As a result, we have the following
\[ \chi_{\Omega_{r}}q(\tau, \xi) = \text{known}, \quad \text{for all } (\tau, \xi) \in \mathbb{R}^{1+n} \tag{3.19} \]
which results into \( \chi_{\Omega_{r}}(t, x)q(t, x) \) is known for all \((t, x) \in \mathbb{R}^{1+n}\). Thus we conclude that \( q(t, x) \) is known for all \((t, x) \in \Omega_{r}\).

3.2. Reconstruction of the coefficient of quadratic nonlinearity. We begin this subsection by proving the following lemma which will be helpful in giving the reconstruction formula in determining \( b(t, x) \) for \((t, x) \in \Omega_{T}\).

**Lemma 3.1.** Let \( n \geq 2 \) and \( N \geq \frac{n}{2} + 2 \). Then there exist \((v_{j})_{1 \leq j \leq n}\) solutions to \( L_{q}v := -i\partial_{t}v + \Delta v + q(t, x)v = 0 \), such that \( v_{j} \in L^{2}((0, T); H^{N}(\Omega)) \) for \( 1 \leq j \leq n \) and \( \det \left( \begin{pmatrix} \partial_{v_{j}}(t, x) \end{pmatrix} \right)_{1 \leq i, j \leq n} \neq 0, \) a.e. in \( \Omega_{T} \).

**Proof.** We follow the arguments similar to the one used in [29] for elliptic case and in [59] for the wave equation. Let us begin with choosing \( \omega_{j} \in S^{n-1} \) for \( 1 \leq j \leq n \) such that the set \( \{ \omega_{1}, \omega_{2}, \ldots, \omega_{n} \} \) is a linearly independent subset of \( \mathbb{R}^{n} \). Also let us extend the potential \( q(t, x) \) to function in \( C_{0}^{\infty}(\mathbb{R}^{1+n}) \). Now for \( 1 \leq j \leq n \), we construct the asymptotic solutions \( v_{j}(t, x) \) to \( L_{q}v = 0 \) in \( \mathbb{R}^{1+n} \), taking the following form
\[ v_{j}(t, x) = e^{i(\lambda^{2}t+\lambda x\omega)} \sum_{k=0}^{N} \frac{A_{jk}(t, x)}{(2i\lambda)^{k}} + R_{j}(t, x), \quad \text{with } N > \frac{n}{2} + 2 \text{ and } \lambda >> 1. \tag{3.20} \]
It is easy to see that
\[ L_{q}v_{j} = e^{i(\lambda^{2}t+\lambda x\omega)} \left[ (L_{q} + 2i\lambda L_{j}) \left( A_{j0}(t, x) + \frac{A_{j1}(t, x)}{2i\lambda} + \frac{A_{j2}(t, x)}{(2i\lambda)^{2}} + \cdots \right) + \frac{A_{jN}(t, x)}{(2i\lambda)^{N}} + e^{i(\lambda^{2}t+\lambda x\omega)}R_{j}(t, x) \right] \]
where \( L_{j} := \omega_{j} \cdot \nabla_{x} \) is the transport operator. Arranging the terms in descending power of \( \lambda \), we get
\[ 2i\lambda L_{j}A_{j0} + (L_{j}A_{j1} + L_{q}A_{j0}) + \frac{1}{(2i\lambda)^{N+2}} (L_{j}A_{jN}, A_{jN} - 1 + L_{q}A_{jN} + e^{-i(\lambda^{2}t+\lambda x\omega)}L_{q}R_{j} = 0. \]
Comparing the various powers of \( \lambda \), we have the transport equations for \( A_{jk} \) for \( 0 \leq k \leq N \) given by
\[ L_{j}A_{j0}(t, x) = 0 \]
\[ L_{j}A_{jk}(t, x) = -L_{q}A_{j,k-1}(t, x), \quad 1 \leq k \leq N. \tag{3.21} \]
We take \( A_{j0} = 1 \), as a solution to first equation in \( \{3.21\} \) and using this we find \( A_{jk}(t, x) \) for \( 1 \leq k \leq N \) by solving the second equation in \( \{3.21\} \). Using \( A_{jN}(t, x) \) we take \( R_{j}(t, x) \) as the solution to the following equation
\[ \begin{cases} L_{q}R_{j}(t, x) = -\frac{1}{(2i\lambda)^{N}} e^{i(\lambda^{2}t+\lambda x\omega)}L_{q}A_{jN}(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^{n} \\ R_{j}(0, x) = 0. \end{cases} \tag{3.22} \]
After solving the Cauchy problem given by \( \{3.22\} \) for \( R_{j} \), we get that \( R_{j} \in C((0, \infty); H^{N}(\mathbb{R}^{n})) \). Now after restricting \( R_{j} \) to \( \Omega_{T} \) and using the Sobolev embedding theorem, we have that \( R_{j} \in W^{1, \infty}(\Omega_{T}) \) and \( \| R_{j} \|_{W^{1, \infty}(\Omega_{T})} \leq C \) for some constant \( C > 0 \) independent of \( \lambda \). Now let us define \( \alpha_{j}(t, x) \) and \( \tilde{R}_{j}(t, x) \) by
\[ \alpha_{j}(t, x) := e^{i(\lambda^{2}t+\lambda x\omega)} \quad \text{and} \quad \tilde{R}_{j}(t, x) := e^{i(\lambda^{2}t+\lambda x\omega)} \sum_{k=1}^{N} \frac{A_{jk}(t, x)}{(2i\lambda)^{k}} + R_{j}(t, x). \]
Using these in (3.20), we get
\[ v_j(t, x) = \alpha_j(t, x) + \widetilde{R}_j(t, x) \quad \text{and} \quad \nabla_x v_j = i\lambda \alpha_j(t, x) \omega_j + \nabla_x \widetilde{R}_j(t, x). \]

Now consider the matrix
\[
A(t, x, \lambda) := \begin{pmatrix}
\frac{\partial v_i}{\partial x_j}
\end{pmatrix}_{1 \leq i, j \leq n} = \begin{bmatrix}
  \lambda \alpha_1(t, x) \omega_1 + \partial_t R_1 & i\lambda \alpha_1(t, x) \omega_1 + \partial_x R_1 \\
  \lambda \alpha_2(t, x) \omega_2 + \partial_t R_2 & i\lambda \alpha_2(t, x) \omega_2 + \partial_x R_2 \\
  : & : \\
  \lambda \alpha_n(t, x) \omega_n + \partial_t R_n & i\lambda \alpha_n(t, x) \omega_n + \partial_x R_n
\end{bmatrix}
\]
where \( \omega_{ij} \) denote the \( j \)’th component in \( \omega_i \in \mathbb{R}^n \) and \( \partial_i := \frac{\partial}{\partial x_i} \) for \( 1 \leq i \leq n \). Our aim is to show that \( \text{Det}(A(t, x, \lambda)) \neq 0 \) for almost everywhere in \( \Omega_T \) for \( \lambda >> 1 \). Now using the fact that \( \| \nabla_x \tilde{R}_j \|_{L^\infty(\Omega_T)} \leq C \) for some constant \( C > 0 \) independent of \( \lambda \), we get
\[
\lim_{\lambda \to \infty} \left| \frac{\text{Det}(A(t, x, \lambda))}{\lambda^n} \right| = \text{Det} \begin{bmatrix}
  \alpha_1(t, x) \omega_1 & \alpha_1(t, x) \omega_1 \\
  \alpha_2(t, x) \omega_2 & \alpha_2(t, x) \omega_2 \\
  : & : \\
  \alpha_n(t, x) \omega_n & \alpha_n(t, x) \omega_n
\end{bmatrix} \neq 0, \quad \text{as } \lambda \to \infty \text{ in } L^2(\Omega_T).
\]
This gives us
\[
\frac{\text{Det}(A(t, x, \lambda))}{\lambda^n} \to \text{Det} \begin{bmatrix}
  \alpha_1(t, x) \omega_1 & \alpha_1(t, x) \omega_1 \\
  \alpha_2(t, x) \omega_2 & \alpha_2(t, x) \omega_2 \\
  : & : \\
  \alpha_n(t, x) \omega_n & \alpha_n(t, x) \omega_n
\end{bmatrix} \neq 0, \quad \text{pointwise for a.e. } (t, x) \in \Omega_T.
\]
Due to the convergence of \( \text{Det}(A(t, x, \lambda)) \) in \( L^2(\Omega_T) \) as \( \lambda \to \infty \), we can have a subsequence of \( \text{Det}(A(t, x, \lambda)) \) still denote the same such that
\[
\lim_{\lambda \to \infty} \frac{\text{Det}(A(t, x, \lambda))}{\lambda^n} = \text{Det} \begin{bmatrix}
  \alpha_1(t, x) \omega_1 & \alpha_1(t, x) \omega_1 \\
  \alpha_2(t, x) \omega_2 & \alpha_2(t, x) \omega_2 \\
  : & : \\
  \alpha_n(t, x) \omega_n & \alpha_n(t, x) \omega_n
\end{bmatrix} \neq 0, \quad \text{pointwise for a.e. } (t, x) \in \Omega_T.
\]
Hence, \( \text{Det}(A(t, x, \lambda)) \neq 0 \) for \( \lambda >> 1 \), a.e. \( (t, x) \in \Omega_T \). Thus, we have that \( \nabla_x v_1(t, x), \nabla_x v_2(t, x), \cdots, \nabla_x v_n(t, x) \) are linearly independent for a.e. \( (t, x) \in \Omega_T \). The proof of lemma is complete.

**Reconstruction for \( \tilde{q}(t, x) \):** Since \( q(t, x) \) is known in \( \Omega_T \), hence we consider \( u_1^{\tilde{q}, f} \) is a solution to (2.2). Consequently, \( u_2(t, x) \) satisfies
\[
\begin{cases}
  i\partial_t u_2(t, x) + \Delta u_2(t, x) + q(t, x) u_2(t, x) = \nabla \cdot (\tilde{b}(t, x) \nabla u_1^{\tilde{q}, f}(t, x)^2), & (t, x) \in \Omega_T, \\
  u_2(0, x) = 0, & x \in \Omega, \\
  u_2(t, x) = 0, & \text{on } \partial \Omega_T.
\end{cases}
\]
By multiplying (3.23) by \( v(t, x) \) where \( v(t, x) \) satisfies
\[
- i\partial_t v(t, x) + \Delta v(t, x) + q(t, x) v(t, x) = 0, \quad (t, x) \in \Omega_T
\]
and integrate over \( \Omega_T \), we obtain
\[
\begin{align*}
  - \int_0^T \int_\Omega i\partial_t v(t, x) u_2(t, x) \, dx dt + \int_0^T \int_\Omega \Delta u_2(t, x) v(t, x) \, dx dt + \int_0^T \int_\Omega q(t, x) u_2(t, x) v(t, x) \, dx dt \\
  = \int_0^T \int_\Omega \nabla \cdot (\tilde{b}(t, x) \nabla u_1^{\tilde{q}, f}(t, x)^2) v(t, x) \, dx dt.
\end{align*}
\]
Further, integration by parts yields
\[
\begin{align*}
  \int_0^T \int_\Omega \left( - i\partial_t v(t, x) u_2(t, x) + \Delta v(t, x) u_2(t, x) + q(t, x) u_2(t, x) v(t, x) \right) \, dx dt \\
  + \int_0^T \int_{\partial \Omega} (\partial_n u_2 - \partial_n v u_2) (t, x) \, dS_x dt = - \int_0^T \int_\Omega \tilde{b}(t, x) \cdot \nabla_x v(t, x) |\nabla_x u_1^{\tilde{q}, f}(t, x)|^2 \, dx dt.
\end{align*}
\]
\[ + \int_0^T \int_{\partial \Omega} \nu(x) \cdot \overrightarrow{b}(t, x) |\nabla_x u_1^{\phi, f}(t, x)|^2 v(t, x) \ dS_x \ dt \]

which results into
\[ \int_0^T \int_{\Omega} \left( \overrightarrow{b}(t, x) \cdot \nabla_x v(t, x) \right) |\nabla_x u_1^{\phi, f}(t, x)|^2 \ dx \ dt = \int_0^T \int_{\Omega} \nu(x) \cdot \overrightarrow{b}(t, x) |\nabla_x u_1^{\phi, f}(t, x)|^2 v(t, x) \ dS_x \ dt \]
\[ - \int_0^T \int_{\partial \Omega} \partial_n u_2(t, x) v(t, x) \ dS_d \ dt. \]

As a consequence,
\[ \int_0^T \int_{\Omega} \left( \overrightarrow{b}(t, x) \cdot \nabla_x v(t, x) \right) |\nabla_x u_1^{\phi, f}(t, x)|^2 \ dx \ dt = \text{known} \]

for all \( v(t, x) \) satisfying equation (3.24). Now choosing \( u_1^{\phi_1, f_1}, f_1 \) the solutions to (2.2) when \( \phi = \phi_1, f = f_1 \) and \( f = f_1 \pm f_2 \) and subtracting the two sets of equations, we get
\[ \int_0^T \int_{\Omega} \left( \overrightarrow{b}(t, x) \cdot \nabla_x v(t, x) \right) \nabla_x u_1^{\phi_1, f_1} \cdot \nabla_x u_2^{\phi_2, f_2}(t, x) \ dx \ dt = \text{known} \]

for all \( v(t, x) \) solution to (3.24) and \( u_1^{\phi_1, f_1} \) solutions to (2.2) when \( \phi = \phi_i \) and \( f = f_i \) for \( i = 1, 2 \). Now denote \( B_v(t, x) := \chi_{\Omega_T}(t, x) \overrightarrow{b}(t, x) \cdot \nabla_x v(t, x) \) where \( \chi_{\Omega_T} \) is the characteristic function of \( \Omega_T \). Using this in previous equation, we get
\[ \int_{\mathbb{R}^{+\infty}} B_v(t, x) \nabla_x u_1^{\phi_1, f_1} \cdot \nabla_x u_2^{\phi_2, f_2}(t, x) \ dx \ dt = \text{known} \]

(3.25) for all \( u_1^{\phi_1, f_1} \) for \( i = 1, 2 \), solutions to (2.2) when \( \phi = \phi_i \) and \( f = f_i \). Now choosing
\[ u_1^{\phi_1, f_1}(t, x) = e^{-i(\lambda^T \tau + \lambda x \cdot \omega)} \left( e^{i(\tau t + \xi x)} + R_1(t, x) \right) \]
\[ u_1^{\phi_2, f_2}(t, x) = e^{-i(\lambda^T \tau + \lambda x \cdot \omega)} \left( 1 + R_2(t, x) \right) \]

where \( \xi \perp \omega \) and \( R_i \) satisfies \( \|R_i\|_{L^2(\Omega_T)} \leq C \) for some constant \( C \) independent of \( \lambda \). Using these expressions of \( u_1^{\phi_1, f_1}(t, x) \) and \( u_1^{\phi_2, f_2}(t, x) \) in (3.24), we have
\[ \int_{\mathbb{R}^{+\infty}} B_v(t, x) \left( e^{i(\tau t + \xi x)} i \xi \cdot \nabla_x R_2 + \nabla_x R_1 \cdot \nabla_x R_2 + i \lambda \omega \cdot \nabla_x R_1 + i \lambda R_2 \omega \cdot \nabla_x R_1 - i \lambda e^{i(\tau t + \xi x)} \omega \cdot \nabla_x R_2 \right. \]
\[ + \lambda^2 e^{i(\tau t + \xi x)} + \lambda^2 R_2 \right) \ dx \ dt = \text{known} \]

for all \( \lambda > 0, \tau \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \) such that \( \xi \cdot \omega = 0 \). Now dividing by \( \lambda^2 \) and taking \( \lambda \to \infty \), we get
\[ \int_{\mathbb{R}^{+\infty}} B_v(t, x) e^{-i(\tau t + \xi x)} \ dx \ dt = \text{known} \]

for all \( \tau \in \mathbb{R} \) and \( \xi \in \omega^\perp \) for any \( \omega \in \mathbb{S}^{n-1} \). But \( \cup_{\omega \in \mathbb{S}^{n-1}} \{ \xi \in \mathbb{R}^n : \xi \in \omega^\perp \} = \mathbb{R}^n \) therefore we have that \( \tilde{B_v}(\tau, \xi) \) is known for \( (\tau, \xi) \in \mathbb{R}^{1+n} \). Hence by using the Fourier inversion, we have that \( \overrightarrow{b}(t, x) \cdot \nabla_x v(t, x) \) is known in \( \Omega_T \) for all \( v(t, x) \) solution to (3.24). Using Lemma 3.1 we choose \( v_1, v_2, v_3, \ldots, v_n \) solutions to (3.24) such that \( \nabla_x v_1, \nabla_x v_2, \ldots, \nabla_x v_n \) are linearly independent a.e. in \( \Omega_T \). Using these choices of \( v_j \) for \( 1 \leq j \leq n \), we get that \( \overrightarrow{b}(t, x) \cdot \nabla_x v_j(t, x) \) is known for a.e. \( (t, x) \in \Omega_T \) and for each \( 1 \leq j \leq n \). After denoting this known value by \( F_{v_j}(t, x) \), we get the following system of equations
\[
\begin{bmatrix}
\frac{\partial b_1}{\partial x_1} & \frac{\partial b_1}{\partial x_2} & \frac{\partial b_1}{\partial x_3} & \ldots & \frac{\partial b_1}{\partial x_n} \\
\frac{\partial b_2}{\partial x_1} & \frac{\partial b_2}{\partial x_2} & \frac{\partial b_2}{\partial x_3} & \ldots & \frac{\partial b_2}{\partial x_n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial b_n}{\partial x_1} & \frac{\partial b_n}{\partial x_2} & \frac{\partial b_n}{\partial x_3} & \ldots & \frac{\partial b_n}{\partial x_n}
\end{bmatrix}
\begin{bmatrix}
b_1(t, x) \\
b_2(t, x) \\
\vdots \\
b_n(t, x)
\end{bmatrix}
= \begin{bmatrix}
F_{v_1}(t, x) \\
F_{v_2}(t, x) \\
\vdots \\
F_{v_n}(t, x)
\end{bmatrix}, \text{ for a.e. } (t, x) \in \Omega_T.
\]
Now since the matrix $A(t, x) := \left( \begin{array}{c} \frac{\partial b}{\partial x_j} \\ \end{array} \right)_{1 \leq i, j \leq n}$ is invertible for a.e. $(t, x) \in \Omega_T$, therefore we obtain that
\[ \overrightarrow{b}(t, x) := A^{-1}(t, x) \overrightarrow{F}(t, x), \quad \text{for a.e. } (t, x) \in \Omega_T, \]
where
\[ \overrightarrow{F}(t, x) := \begin{bmatrix} F_{v_1}(t, x) \\ F_{v_2}(t, x) \\ \vdots \\ F_{v_n}(t, x) \end{bmatrix}. \]

This gives the reconstruction for $\overrightarrow{b}(t, x)$ for a.e. $(t, x) \in \Omega_T$ but $\overrightarrow{b} \in C^\infty(\Omega_T)$ therefore $\overrightarrow{b}(t, x)$ is known for all $(t, x) \in \Omega_T$. Thus the proof of Theorem 1.1 is complete.

REFERENCES

[1] S. A. Avdonin and M. I. Belishev, Dynamical inverse problem for the Schrödinger equation (BC-method), in *Proceedings of the St. Petersburg Mathematical Society, Amer. Math. Soc. Transl. Ser.*, 10, Amer. Math. Soc., Providence, RI, 2005, 1–14.
[2] L. Baudouin and J.-P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, *Inverse Problems*, 25 (2007), no. 3, 1327–1328.
[3] M. Bellassoued and D. Dos Santos Ferreira, Stable determination of coefficients in the dynamical anisotropic Schrödinger equation from the Dirichlet-to-Neumann map, *Inverse Problems*, 26 (2010), no. 12, 125010, 30pp.
[4] M. Bellassoued, Stable determination of coefficients in the dynamical Schrödinger equation in a magnetic field, *Inverse Problems*, 33 (2017), no. 5, 055009.
[5] M. Bellassoued, Y. Kian and E. Soccorsi, An inverse problem for the magnetic Schrödinger equation in infinite cylindrical domains, *Publ. Res. Inst. Math. Sci.*, 54 (2018), no. 4, 679-728.
[6] M. Bellassoued, I. Ben Aïcha and Z. Rezig, Stable determination of a vector field in a non-self-adjoint dynamical Schrödinger equation on Riemannian manifolds, *Math. Control Relat. Fields*, 11 (2021), no. 2, 403–431.
[7] I. Ben Aïcha, Stability estimate for an inverse problem for the Schrödinger equation in a magnetic field with time-dependent coefficient, *Journal of Mathematical Physics*, 58 (2017), no. 7, 071508.
[8] P. Caro and Y. Kian, Determination of convection terms and quasi-linearities appearing in diffusion equations, preprint, arXiv:1812.08495.
[9] C. I. Cârstea and A. Feizmohammadi, An inverse boundary value problem for certain anisotropic quasilinear elliptic equations, *J. Differential Equations*, 284 (2021), 318–349.
[10] C. I. Cârstea, G. Nakamura and M. Vashisth, Reconstruction for the coefficients of a quasilinear elliptic partial differential equation, *Applied Mathematics Letters*, 98 (2019), 121–127.
[11] C. I. Cârstea, and M. Kar, Recovery of coefficients for a weighted p-Laplacian perturbed by a linear second order term, *Inverse Problems*, 37 (2020), no. 1, 015013.
[12] C. M. Dafermos and W. J. Hrusa, Energy methods for quasilinear hyperbolic initial-boundary value problems. Applications to elastodynamics, *Archive for Rational Mechanics and Analysis*, 87 (1985), 267–292.
[13] M. Choulli and Y. Kian, Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map, *Math. Control Relat. Fields*, 5 (2015), no. 6, 4536–4558.
[14] M. Choulli and Y. Kian, Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map, Application to the determination of a nonlinear term, *J. Math. Pures Appl.*, 114 (2018), 235-261.
[15] H. Egger, J.-F. Pietschmann and M. Schlottbom, On the uniqueness of nonlinear diffusion coefficients in the presence of lower order terms, *Inverse Problems*, 33 (2017), 115005.
[16] G. Eskin, Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohm effect, *J. Math. Phys.*, 49, 2008, no.2, 022105, 18, 0022-2488.
[17] O. Imanuvilov and M. Yamamoto, Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data, *Journal of Inverse and Ill-Posed Problems*, 21 (2013), 85-108.
[18] V. Isakov, On an inverse hyperbolic problem with many boundary measurements, *Taylor & Francis*, 1991.
[19] V. Isakov, On uniqueness in inverse problems for semilinear parabolic equations, *Arch. Rat. Mech. Anal.*, 124 (1993), 1-12.
[20] V. Isakov, Uniqueness of recovery of some systems of semilinear partial differential equations, *Inverse Problems*, 17 (2001), 607-618.
[21] V. Isakov, Uniqueness of recovery of some quasilinear Partial differential equations, *Comm. PDE*, 26 (2001), 1947-1973.
[22] V. Isakov, Inverse Problems for Partial Differential Equations, Volume 127, Springer-Verlag, Berlin, Heidelberg, 2006.
[25] V. Isakov and A. Nachman, Global uniqueness for a two-dimensional semilinear elliptic inverse problem, *Trans. Amer. Math. Soc.*, 347 (1995), no. 9, 3375-3390.
[26] V. Isakov and J. Sylvester, Global uniqueness for a semi linear elliptic inverse problem, *Comm. Pure Appl. Math.*, 47 (1994), 1403-1410.
[27] B. Kaltenbacher and W. Rundell, The inverse problem of reconstructing reaction–diffusion systems, *Inverse Problems*, 36 (2020), 065011.
[28] B. Kaltenbacher and W. Rundell, On the simultaneous recovery of the conductivity and the nonlinear reaction term in a parabolic equation, *Inverse Problems and Imaging*, 14 (2020), 939-966.
[29] H. Kang and G. Nakamura, Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map, *Inverse Problems*, 18 (2002), no. 4, 1079-1088.
[30] Y. Kian and E. Soccorsi, Hölder stably determining the time-dependent electromagnetic potential of the Schrödinger equation, *SIAM J. Math. Anal.*, 51 (2019), no. 2, 627-647.
[31] Y. Kian and A. Tetlow, Hölder Stable Recovery of Time-Dependent Electromagnetic Potentials Appearing in a Dynamical Anisotropic Schrödinger Equation, *Inverse Probl. Imaging*, 14 (2020), no. 5, 819-839.
[32] Y. Kian and G. Uhlmann, Recovery of nonlinear terms for reaction diffusion equations from boundary measurements, *arXiv preprint arXiv:2011.06039*, 2020.
[33] Y. Kian and M. Yamamoto, Reconstruction and stable recovery of source terms and coefficients appearing in diffusion equations, *Inverse Problems*, 35 (2019), no. 11, 115006.
[34] K. Krupchyk and G. Uhlmann, Inverse problems for nonlinear magnetic Schrödinger equations on conformally transversally anisotropic manifolds, preprint, *arXiv:2009.05080*.
[35] M. Lassas, L. Oksanen, M. salo and A. Tret’yakov, Inverse problems for non-linear Schrödinger equations with time-dependent coefficients, preprint (2022) *arXiv:2201.05099*.
[36] J.-L. Lions, and E. Magenes, Non-homogeneous boundary value problems and applications. Vol. II. Translated from the French by P. Kenneth. Die Grundlehren der mathematische n Wissenschaften, Band 182, Springer-Verlag, New York-Heidelberg, 1972.
[37] J. C. Liu, Reconstruction Algorithms of an Inverse Coefficient Identification Problem for the Schrödinger Equation, *Mathematical Modelling and Analysis*, 22 (2017), no. 3, 352–372.
[38] G. Nakamura and M. Watanabe, An inverse boundary value problem for a nonlinear wave equation, *Inverse Problems and Imaging*, 2 (2008), no. 1, 121.
[39] G. Nakamura, M. Vashisth and M. Watanabe, Inverse Initial Boundary Value Problem for a Non-linear Hyperbolic Partial Differential Equation, *Inverse Problems*, 37 (2020), no. 1, 015012.
[40] Z. Sun, On a quasilinear inverse boundary value problem, *Math. Z.*, 221 (1996), 293-305.
[41] Z. Sun and G. Uhlmann, Inverse problems in quasilinear anisotropic media, *Amer. J. Math.*, 119 (1997), 771-799.

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