Threshold solutions for the 3d cubic-quintic NLS

Alex H. Ardila\textsuperscript{a} and Jason Murphy\textsuperscript{b}

\textsuperscript{a}Universidade Federal de Minas Gerais, ICEx-UFMG, Belo Horizonte, MG, Brazil; \textsuperscript{b}Missouri University of Science & Technology, Rolla, MO, USA

ABSTRACT
We study the cubic-quintic NLS in three space dimensions. It is known that scattering holds for solutions with mass-energy in a region corresponding to positive virial, the boundary of which is delineated both by ground state solitons and by certain rescalings thereof. We classify the possible behaviors of solutions on the part of the boundary attained solely by solitons. In particular, we show that non-soliton solutions either scatter in both time directions or coincide (modulo symmetries) with a special solution, which scatters in one time direction and converges exponentially to the soliton in the other.

1. Introduction

We study the cubic-quintic nonlinear Schrödinger equation (NLS)

\[
\begin{cases}
(i \partial_t + \Delta)u = -|u|^2 u + |u|^4 u, \\
u(0) = u_0 \in H^1(\mathbb{R}^3),
\end{cases}
\tag{NLS}
\]

where \(u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}\). This is the Hamiltonian evolution corresponding to the energy

\[
E(u) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 - \frac{1}{4} |u|^4 + \frac{1}{6} |u|^6 \, dx.
\]

Solutions to (NLS) additionally preserve the mass, defined by

\[
M(u) = \int_{\mathbb{R}^3} |u|^2 \, dx.
\]

Using the conservation of mass and energy, one can show that (NLS) is globally well-posed in \(H^1\), with solutions remaining uniformly bounded in \(H^1\) over time (see [1]). Our interest is in the long-time behavior of solutions to (NLS), with a focus on the question of scattering, i.e., whether an initial condition \(u_0 \in H^1(\mathbb{R}^3)\) leads to a solution \(u\) obeying

\[
\lim_{t \to \pm \infty} ||u(t) - e^{it\Delta}u_z||_{H^1} = 0 \quad \text{for some} \quad u_z \in H^1.
\tag{1.1}
\]
This problem was studied in [2], which identified a scattering region in the mass-energy plane corresponding to the region of positive virial. The boundary of this region is realized in part by (non-scattering) soliton solutions, but also in part by certain rescalings of solitons (which are not themselves soliton solutions to (NLS)). The behavior of solutions in the part of the boundary not realized by any soliton was further studied in [3], which established scattering in an open neighborhood of this part of the boundary (and, in particular, across the ‘virial threshold’). Our interest in this paper is to study the behavior of solutions with mass-energy belonging to the part of the boundary realized exclusively by solitons. We classify all possible behaviors: other than the soliton (corresponding to zero virial), solutions either scatter as $t \to \pm \infty$ or coincide (modulo symmetries) with a special solution, which scatters in one direction and converges exponentially to the soliton in the other. All three possible behaviors do occur. Our result is closely related to the threshold classifications appearing in works such as [4–9]; essentially, we obtain the usual classification, but with all blowup behavior removed.

To state our main result precisely, we first need to introduce the variational problem from [2] that is used to define the scattering region. Writing $V$ for the virial functional,
\[ V(f) = ||f||^2_{H^1} + ||f||^6_{L^6} - \frac{3}{4} ||f||^4_{L^4}, \]
we set
\[ \epsilon(m) := \inf \{ E(f) : f \in H^1(\mathbb{R}^3), \quad M(f) = m \quad \text{and} \quad V(f) = 0 \}. \]

The open set $K \subset \mathbb{R}^2$ is then defined by
\[ K := \{ (m,e) : 0 < m < m_2 \quad \text{and} \quad 0 < e < \epsilon(m) \}, \]
where is $m_2 = M(Q_1)$, with $Q_1$ an optimizer of a certain Gagliardo–Nirenberg–Hölder inequality (cf. Theorem 2.4 below with $\alpha = 1$). The scattering result of [2] can then be stated as follows:

**Theorem 1.1 (Scattering with positive virial [2]).** If $(M(u_0),E(u_0)) \in K$, then the solution to (NLS) is global and scatters as $t \to \pm \infty$.

The region $K$ is represented in Figure 1, imported from [3].

It was shown in [2] that the boundary of $K$ (at mass $m \in (m_0,m_2)$) is attained either at a soliton or a rescaled soliton. Here the (ground state) solitons refer to the unique,
non-negative, radially symmetric solutions $P_{\omega} \in H^1$ to
\[- \omega P_{\omega} + \Delta P_{\omega} = -P_{\omega}^3 + P_{\omega}^5, \quad \omega \in \left(0, \frac{3}{16}\right),\] (1.4)
while the rescaled solitons $R_{\omega}$ are given by
\[R_{\omega}(x) := \sqrt{\frac{1 + \beta(\omega)}{4\beta(\omega)}} P_{\omega}\left(\frac{3(1 + \beta(\omega))}{4\sqrt{3\beta(\omega)}} x\right), \quad \text{with} \quad \beta(\omega) := \frac{||P_{\omega}||_{L^6}^6}{||\nabla P_{\omega}||_{L^2}^2}.
\]
It was further shown in [2] that the left-most part of $\partial K$ is attained only by rescaled solitons, while the the right-most part is attained only by solitons. In fact, it is conjectured (and supported by numerics) that there exists a mass $m_1$ dividing the boundary into the rescaled soliton part and soliton part. It is also natural to conjecture that for a given mass, the corresponding soliton or rescaled soliton delineating the boundary is unique; we will work under this assumption below.

The behavior of solutions around the left-most part of $\partial K$ was studied in [3], where the following result was obtained:

**Theorem 1.2.** (Crossing the virial threshold [3]). There exists an open set $B \subset \mathbb{R}^2$ containing $K$ such that all solutions with mass-energy in $B$ scatter as $t \to \pm \infty$. The set $B$ satisfies (i) there exists $m^* > m_0$ so that $(0, m^*) \times (0, \infty) \subset B$, and (ii) any $(m, e) \in \partial K$ not achieved by a soliton belongs to $B$.

Our interest in this paper is the behavior of solutions on the right part of $\partial K$.

**Definition 1.3.** We let $\partial K_s$ denote the set of $(m, e) \in \partial K$ with $e > 0$ such that:

(i) $(m, e) = (M(P_{\omega}), E(P_{\omega}))$ for some unique $\omega \in \left(0, \frac{3}{16}\right)$;

(ii) $(m, e) \neq (M(R_{\omega}), E(R_{\omega}))$ for any $\omega \in \left(0, \frac{3}{16}\right)$.

As described above, the numerics from [3] suggest that $\partial K_s$ coincides with the portion of $\partial K$ with $m \in (m_1, m_2)$. The existence in (i) and nonexistence in (ii) are guaranteed for all $m$ sufficiently close to $m_2$ by the results of [2]; however, the uniqueness in (i) must be taken as an assumption throughout the paper.

Our first result is the existence of a solution scattering in one time direction and converging exponentially to $P_{\omega}$ in the other.

**Theorem 1.4.** If $(m, e) = (M(P_{\omega}), E(P_{\omega})) \in \partial K_s$, then there exists a global solution $\mathcal{G}_{\omega}$ of (NLS) with $(M(\mathcal{G}_{\omega}), E(\mathcal{G}_{\omega})) = (m, e)$ satisfying the following:

- $V(\mathcal{G}_{\omega}(t)) > 0$ for all $t \in \mathbb{R}$;
- $\mathcal{G}_{\omega}$ scatters as $t \to -\infty$;
- there exist constants $C > 0$ and $c > 0$ such that
  \[\|\mathcal{G}_{\omega}(t) - e^{i\omega t}P_{\omega}\|_{H^1} \leq Ce^{-ct} \quad \text{for all} \quad t \geq 0.\]

Using the solution $\mathcal{G}_{\omega}$ in Theorem 1.4, we can classify all possible behaviors for solutions with mass-energy in $\partial K_s$. 
Theorem 1.5. Threshold behaviors on $\partial K_s$ Suppose $u$ is a solution to (NLS) with $(M(u), E(u)) = (M(P_\omega), E(P_\omega)) \in \partial K_s$.

(i) If $V(u|_{t=0}) = 0$, then $u(t) = e^{itP_\omega}$ up to the symmetries of the equation.

(ii) If $V(u|_{t=0}) > 0$, then either $u$ scatters in both time directions or $u = G_\omega$ up to the symmetries of the equation.

Theorems 1.4 and 1.5 are similar to the threshold classification results appearing in works on pure power NLS (see e.g., [4, 7]); essentially, we obtain the standard classification with all blowup behavior removed. As mentioned above, all of the behaviors described in Theorem 1.5 actually do occur; in particular, we demonstrate the existence of scattering solutions in Proposition B.1. Our results do not apply to the case of zero energy solutions; indeed, one can show that scattering is not possible in this case (see Remark B.2).

In the rest of the introduction, let us briefly describe the organization of the paper and the strategy of proof for Theorems 1.4 and Theorem 1.5.

In Section 3, we analyze the spectrum of the operator arising from the linearization of (NLS) around the ground state. In Section 4, we then carry out the modulation analysis around the ground state. In Section 5, we use prove that solutions with $(M(u), E(u)) \in \partial K_s$ that do not scatter forward in time must converge exponentially to a ground state as $t \to \infty$ modulo symmetries. This involves some standard concentration-compactness techniques to establish compactness of the orbit, combined with the modulation analysis and the (modulated) virial estimate.

Section 6 contains the existence and uniqueness of solutions converging exponentially to the ground state (which ultimately yields the solution $G_\omega$ appearing in Theorem 1.4). Using the spectral properties of the linearized operator, we first construct suitable approximate solutions, which we then upgrade to true solutions via a fixed point argument. Section 7 is then devoted to the proof of a rigidity-type result, namely, that solutions converging exponentially the ground state must coincide with one of the solutions constructed in Section 6 (or the ground state itself). We further show that (modulo time translation and rotation), all of the solutions constructed in Section 6 (other than the ground state) are in fact that same.

With all of the preceding results in hand, we can quickly complete the proofs of the main results in Section 8. The appendix then contains the proof of a technical lemma from Section 3, as well as a demonstration of the existence of scattering solutions on $\partial K_s$.

Notation. We write $A \lesssim B$ or $B \gtrsim A$ to denote $A \leq CB$ for some positive constant $C$. We also write $A \sim B$ when $A \leq B \leq A$. For a function $u : I \times \mathbb{R}^3 \to \mathbb{C}$, $I \subset \mathbb{R}$, we write
\[ \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^3)} = \|u(t)\|_{L^q_x(\mathbb{R}^3)} \|u(t)\|_{L^r_x(I)} \]
with $1 \leq q \leq r \leq \infty$. We say that the pair $(q, r)$ is admissible when $2 \leq q, r \leq \infty$ and $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$. We write $\langle \nabla \rangle = (1 - \Delta)^{1/2}$ and we define the Sobolev norms
\[ \|u\|_{H^s(\mathbb{R}^3)} := \|\langle \nabla \rangle^s u\|_{L^2_x(\mathbb{R}^3)}. \]

If $\alpha(t)$ and $\beta(t)$ are two positive functions of $t$, we write $\alpha = O(\beta)$ when $\alpha \leq C\beta$ for some constant $C > 0$, and $\alpha \sim \beta$ when $\alpha = O(\beta)$ and $\beta = O(\alpha)$. 


2. Preliminaries

In this section, we record some preliminary results that will be used throughout the paper.

2.1. Well-posedness and stability

We first recall the Strichartz estimates for the linear Schrödinger equation. We call a pair \((q, r)\) admissible if \(2 \leq q, r \leq \infty\) and \(\frac{2}{q} + \frac{3}{r} = \frac{3}{2}\).

**Lemma 2.1.** (Strichartz estimates [10–12]). Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be admissible. Then the solution \(u\) to the equation \((i\partial_t + \Delta)u = F\) with \(u|_{t_0} = u_0\) satisfies

\[
\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^\tilde{q}_t L^\tilde{r}_x(I \times \mathbb{R}^3)} + \|F\|_{L^\tilde{q}_t L^\tilde{r}_x(I \times \mathbb{R}^3)},
\]

for any interval \(t_0 \in I \subset \mathbb{R}\).

We also have the following global well-posedness result from [1].

**Theorem 2.2.** (Global well-posedness [1]). For any \(u_0 \in H^1(\mathbb{R}^3)\), the solution to (NLS) is global and obeys \(u \in C(\mathbb{R}; H^1(\mathbb{R}^3))\). Moreover, the mass, energy, and momentum are conserved, where the momentum is defined by

\[
\mathcal{Z}(u(t)) := \int_{\mathbb{R}^3} 2\text{Im}(\overline{u(t, x)} \nabla u(t, x)) \, dx.
\]

For an interval \(I \subset \mathbb{R}\), we define

\[
\|u\|_{S^p(I)} := \sup\{\|u\|_{L^\tilde{q}_t L^\tilde{r}_x(I \times \mathbb{R}^3)} : (q, r) \text{ is admissible}\}.
\]

We then have the following stability result (cf. [2]):

**Lemma 2.3.** (Stability). Let \(I \subset \mathbb{R}\) be a time interval containing \(t_0\). Suppose that \(\tilde{u} : I \times \mathbb{R}^3 \to \mathbb{C}\) solves

\[
(i\partial_t + \Delta)\tilde{u} = |\tilde{u}|^4 \tilde{u} - |\tilde{u}|^2 \tilde{u} + \varepsilon, \quad \tilde{u}(t_0) = \tilde{u}_0
\]

for some function \(\varepsilon : I \times \mathbb{R}^3 \to \mathbb{C}\). Let \(u_0 \in H^1(\mathbb{R}^3)\) and suppose

\[
\|\tilde{u}\|_{L^\infty_t H^1_x(I \times \mathbb{R}^3)} \leq A, \\
\|\tilde{u}\|_{L^{10}_t H^1_x(I \times \mathbb{R}^3)} \leq L
\]

for some \(A, L > 0\). There exists \(\varepsilon_0 = \varepsilon_0(A, L) > 0\) such that if \(0 < \varepsilon < \varepsilon_0\) and

\[
\|u_0 - \tilde{u}_0\|_{H^1_x} \leq \varepsilon, \\
\|\nabla \varepsilon\|_{L^\infty_x(I \times \mathbb{R}^3)} \leq \varepsilon,
\]

then there exists a unique solution \(u\) to (NLS) with \(u(t_0) = u_0\) satisfying

\[
\|\nabla(u - \tilde{u})\|_{S^p(I)} \leq C(A, L)\varepsilon.
\]
2.2. Variational analysis

We record here some of the variational analysis that played a central role in [2]. As in that work, we utilize the notation

$$
\beta(\omega) := \frac{\|P_\omega\|_{L^6}^6}{\|\nabla P_\omega\|_{L^2}^2}.
$$

We first have the following:

**Theorem 2.4.** ($\alpha$-Gagliardo-Nirenberg-Hölder inequality [2]). Let $\alpha \in (0, \infty)$ and

$$
C_\alpha^{-1} := \inf_{f \in H^1 \setminus \{0\}} \frac{\|f\|_{L^2}^4 + \|f\|_{L^6}^6}{\|f\|_{L^2}^2 \|f\|_{L^6}^4 \|\nabla f\|_{L^2}^2}.
$$

Then $C_\alpha \in (0, \infty)$ and the infimum (2.2) is attained by a function of the form $f(x) = \beta P_\omega(r(x - x_0))$, where $P_\omega$ is a radial positive solution to the stationary problem (1.4) with $\beta(P_\omega) = \alpha$, $\lambda \in \mathbb{R}$, $r > 0$ and $x_0 \in \mathbb{R}^3$.

**Remark 2.5.** The optimal constant $C_1$ in (2.2) is given by

$$
C_1 = \frac{8}{3} \frac{1}{\|Q_1\|_{L^2}^2},
$$

where $Q_1$ denotes a ground state solution optimizing (2.2). In particular, we have

$$
\|f\|_{L^2}^4 \leq \frac{8}{3} \frac{\|f\|_{L^2}^2 \|f\|_{L^6}^6}{\|Q_1\|_{L^2}^2 \|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2}.
$$

Moreover, using Young’s inequality we get

$$
E(f) \geq \frac{1}{2} \|f\|_{L^2}^2 + \frac{1}{6} \|f\|_{L^6}^6 - \frac{2}{3} \frac{\|f\|_{L^2}^2 \|f\|_{L^6}^6}{\|Q_1\|_{L^2}^2 \|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2} \geq \left(1 - \frac{\|f\|_{L^2}^2}{\|Q_1\|_{L^2}^2}\right) \left[\frac{1}{2} \|f\|_{L^2}^2 + \frac{1}{6} \|f\|_{L^6}^6\right].
$$

(2.3)

The ground states $P_\omega$ satisfy the following identities:

$$
\|P_\omega\|_{L^2}^2 = \frac{\beta(\omega) + 1}{3\omega} \|\nabla P_\omega\|_{L^2}^2, \quad \|P_\omega\|_{L^6}^6 = \frac{4[\beta(\omega) + 1]}{3} \|\nabla P_\omega\|_{L^2}^2, \quad \|P_\omega\|_{L^6}^6 = \beta(\omega) \|\nabla P_\omega\|_{L^2}^2 \quad \text{and} \quad E(P_\omega) = \frac{[1 - \beta(\omega)]}{6} \|\nabla P_\omega\|_{L^2}^2.
$$

(2.4)

**Lemma 2.6.** Let $u_0 \in H^1(\mathbb{R}^3)$ satisfy $(M(u_0), E(u_0)) \in \partial K_\delta$ and $V(u_0) > 0$. Then the corresponding solution $u(t)$ of (NLS) is uniformly bounded in $H^1(\mathbb{R}^3)$ and $V(u(t)) > 0$ for all $t \in \mathbb{R}$.

**Proof.** First, notice that

$$
\int_{\mathbb{R}^3} \frac{1}{2} \|\nabla u\|^2 + \frac{1}{6} |u|^2 \left(|u|^2 - \frac{3}{4}\right)^2 \, dx = E(u) + \frac{3}{32} M(u).
$$

This implies that $\|u(t)\|_{L^2}^2 + \|u(t)\|_{L^6}^6 \leq M(u_0) + E(u_0)$ for all $t \in \mathbb{R}$. 

On the other hand, suppose by contradiction that \( V(u(t_0)) = 0 \) for some \( t_0 \in \mathbb{R} \). Since \((M(u(t_0)), E(u(t_0))) \in \partial K_s\), by the variational characterization given in [2, Theorem 5.6] we infer that \( u(t_0, x) = e^{it_0}p_\omega(x + x_0) \) for some \( \omega \in (0, \frac{3}{16}) \), \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^3 \). By uniqueness of the solution, we have \( V(u_0) = 0 \), which is a contradiction. \( \square \)

**Remark 2.7.** Let \( f \in H^1(\mathbb{R}^3) \). We observe that if

\[
E(f) = E(P_\omega), \quad M(f) = M(P_\omega) \quad \text{and} \quad ||\nabla f||_{L^2}^2 = ||\nabla P_\omega||_{L^2}^2, \tag{2.5}
\]

then \( f(x) = e^{it_0}p_\omega(x - x_0) \) for some \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^3 \). Indeed, by (2.5) we deduce

\[
\int_{\mathbb{R}^3} \frac{\omega}{2} |f|^2 + \frac{1}{6} |f|^6 - \frac{1}{4} |f|^4 \, dx = \int_{\mathbb{R}^3} \frac{\omega}{2} |P_\omega|^2 + \frac{1}{6} |P_\omega|^6 - \frac{1}{4} |P_\omega|^4 \, dx.
\]

Since \( ||\nabla f||_{L^2}^2 = ||\nabla P_\omega||_{L^2}^2 \), the variational characterization of \( P_\omega \) given in [2, (2.8)] and Theorem 2.2 in [2] (uniqueness of solitons) guarantees that \( f \) must agree with \( P_\omega \) up to translation and a phase rotation.

In particular, we see that if

\[
E(u_0), \quad M(u_0) = M(P_\omega) \quad \text{and} \quad V(u_0) > 0,
\]

then either

\[
||\nabla u(t)||_{L^2}^2 < ||\nabla P_\omega||_{L^2}^2 \quad \forall t \in \mathbb{R} \quad \text{or} \quad ||\nabla u(t)||_{L^2}^2 > ||\nabla P_\omega||_{L^2}^2 \quad \forall t \in \mathbb{R},
\]

where \( u(t) \) is the corresponding solution to (NLS) with initial data \( u_0 \).

**Lemma 2.8.** Suppose \((m, e) \in \partial K_s\). If the sequence \( \{f_n\} \subset H^1(\mathbb{R}^3) \) obeys \( M(f_n) = m, E(f_n) = e \) and \( V(f_n) \to 0 \) as \( n \to \infty \), then there exists \( \theta_n \in \mathbb{R} \) and \( x_n \in \mathbb{R}^3 \) such that

\[
||f_n - e^{it_0}p_\omega(\cdot - x_n)||_{H^1} \to 0
\]
as \( n \to \infty \), where \( \omega \in (0, \frac{3}{16}) \) is unique.

**Proof.** Following the same argument developed in the proof of [3, Lemma 3.2], by Theorem 5.6 and \((m, e) \in \partial K_s\) we obtain that there exist a unique \( \omega \in (0, \frac{3}{16}) \), \( \theta_n \) and \( x_n \) such that

\[
e^{-it_0}f_n(x + x_n) \to p_\omega(x) \quad \text{in} \quad H^1(\mathbb{R}^3).
\]

\( \square \)

### 2.3. Linear profile decomposition

The following appears as [2, Theorem 7.5].

**Proposition 2.9.** Let \( \{f_n\} \) be a bounded sequence in \( H^1(\mathbb{R}^3) \). The following holds up to a subsequence:

There exist \( J^* \in \{0, 1, 2, \ldots \} \cup \{ \infty \} \), non-zero profiles \( \{\psi_j\}_{j=1}^{J^*} \subset \dot{H}^1(\mathbb{R}^3) \) and parameters

\[
\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, 1], \quad \{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \quad \text{and} \quad \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3
\]

such that

\[
\lim_{n \to \infty} \left( f_n(x) - \sum_{j=1}^{J^*} \lambda_n \psi_j(x - t_n x_n) \right) = 0 \quad \text{in} \quad H^1(\mathbb{R}^3).
\]
so that for each finite $1 \leq J \leq J^*$, we can write

$$f_n = \sum_{j=1}^{J} \psi_n^j + W_n^J,$$

(2.6)

with

$$\psi_n^j(x) := \begin{cases} 
\left[ e^{it_n \Delta} \psi^j \right](x - x_n^j), & \text{if } \lambda_n^j \equiv 1, \\
(\lambda_n^j)^{-1} \left[ e^{it_n \Delta} P_{\geq (\lambda_n^j)^{-1}} \psi^j \right]\left(\frac{x - x_n^j}{\lambda_n^j}\right), & \text{if } \lambda_n^j \to 0,
\end{cases}$$

(2.7)

for some $0 < \theta < 1$, satisfying the following statements:

- $\lambda_n^j \equiv 1$ or $\lambda_n^j \to 0$ and $t_n^j \equiv 0$ or $t_n^j \to \pm \infty$,
- if $\lambda_n^j \equiv 1$ then $\{\psi_n^j\}_{j=1}^J \subset L^2_x(\mathbb{R}^3)$

for each $j$. Moreover, we have:

- Smallness of the reminder:

$$\lim_{J \to J^*} \limsup_{n \to \infty} \| e^{it_n \Delta} W_n^J \|_{L_x^1(\mathbb{R} \times \mathbb{R}^3)} = 0.$$

(2.8)

- Weak convergence property:

$$e^{-it_n \Delta} \left[ (\lambda_n^j)^{-1} W_n^J (\lambda_n^j x + x_n^j) \right] \to 0 \text{ in } H_x^1, \text{ for all } 1 \leq j \leq J.$$

(2.9)

- Asymptotic Pythagorean expansions:

$$\sup_{J} \lim_{n \to \infty} \left[ M(f_n) - \sum_{j=1}^{J} M(\psi_n^j) - M(W_n^J) \right] = 0,$$

(2.10)

$$\sup_{J} \lim_{n \to \infty} \left[ E(f_n) - \sum_{j=1}^{J} E(\psi_n^j) - E_a(W_n^J) \right] = 0.$$

(2.11)

- Asymptotic orthogonality: for all $1 \leq j \neq k \leq J^*$

$$\lim_{n \to \infty} \left[ \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{|\lambda_n^j|^2}{\lambda_n^j \lambda_n^k} + \frac{|\lambda_n^k|^2}{\lambda_n^j \lambda_n^k} \right] = \infty.$$

(2.12)

2.4. Virial identity

Given $R \geq 1$, we define

$$w_R(x) = R^2 \phi \left( \frac{x}{R} \right) \quad \text{and} \quad w_{\infty}(x) = |x|^2,$$

where $\phi$ is a real-valued and radial function satisfying
\[ \phi(x) = \begin{cases} |x|^2, & |x| \leq 1 \\ 0, & |x| \geq 2, \end{cases} \text{ with } |\partial_x^2 \phi(x)| \leq |x|^{2-|x|} \]
and \( \partial_t \phi \geq 0 \). Here, \( \partial_t \) denotes the radial derivative.

We also define the functional
\[ \mathcal{P}_R[u] = 2 \Im \int_{\mathbb{R}^3} \bar{u} \nabla u \cdot \nabla w_R \, dx. \]

**Lemma 2.10.** Let \( R \in [1, \infty] \). Let \( u(t) \) be a solution of \((NLS)\). Then we have
\[ \frac{d}{dt} \mathcal{P}_R[u] = F_R[u(t)], \]
where
\[ F_R[u] = \int_{\mathbb{R}^3} 4 \text{Re} \bar{u} \partial_r \delta_{jk}[w_R] - |u|^4 \Delta w_R + \frac{4}{3} |u|^6 \Delta w_R - \Delta \Delta w_R |u|^2 \, dx. \]
Note that if \( R = \infty \), then we have \( F_\infty[u] = 8V(u) \) (cf. \((1.2)\)).

**Lemma 2.11.** Let \( R \in [1, \infty] \), \( \theta \in \mathbb{R} \) and \( y \in \mathbb{R} \). Then we have
\[ F_R[e^{i\theta} P_{\omega}(\cdot - y)] = 0. \]

**Lemma 2.12.** Let \( u \) be the solution to \((NLS)\) on an interval \( I_0 \). Let \( R \in [1, \infty] \), \( \chi : I_0 \to \mathbb{R} \), \( \theta : I_0 \to \mathbb{R} \), \( y : I_0 \to \mathbb{R} \). Then for all \( t \in \mathbb{R} \),
\[ \frac{d}{dt} \mathcal{P}_R[u] = F_\infty[u(t)] \\
+ F_R[u(t)] - F_\infty[u(t)] \\
- \chi(t) \left\{ F_R[e^{i\theta(t)} P_{\omega}(\cdot - y(t))] - F_\infty[e^{i\theta(t)} P_{\omega}(\cdot - y(t))] \right\}. \]

**3. Spectral properties of the linearized operator**

Let \( u \) be a solution to \((NLS)\), and define \( h = h_1 + ih_2 \) via
\[ h(t, x) := e^{-i\omega t} u(t, x) - P_{\omega}(x). \]
Then, defining the operators \( L_\pm \) (acting on \( L^2(\mathbb{R}^3; \mathbb{R}) \)) via
\[ L_+ h_1 = -\Delta h_1 + \omega h_1 - 3P_{\omega}^2 h_1 + 5P_{\omega}^4 h_1, \]
\[ L_- h_2 = -\Delta h_2 + \omega h_2 - P_{\omega}^2 h_2 + P_{\omega}^4 h_2, \]
and setting
\[ R(h) = |h|^2 h - h|h|^4 + P_{\omega}(2|h|^2 + h^2 - 2|h|^2 h^2 - 3|h|^4) \\
- 2P_{\omega}^2 \left( \frac{1}{2} h^3 + 3|h|^2 h + \frac{3}{2} |h|^2 h \right) - 2P_{\omega}^3 \left( 2|h|^2 + \frac{5}{2} h^2 + \frac{1}{2} (h)^2 \right), \]
we find that $h$ satisfies the equation

$$\partial_t h + L h = i R(h), \quad \text{where} \quad L := \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}. \quad (3.2)$$

The spectra of $L_+$ and $L_-$ consist of essential spectrum in $[\omega, \infty)$ and of a finite number of eigenvalues in $(-\infty, \tilde{\omega})$ for all $\tilde{\omega} < \omega$. Since $L_- P_\omega = 0$ with $P_\omega > 0$, it follows that $\ker\{L_-\} = \text{span}\{P_\omega\}$. In particular, by the Min-Max characterization of eigenvalues we have there exists $\eta > 0$ such that

$$\langle L_- v, v \rangle \geq \eta \|v\|_{L^2}^2 \quad \text{for} \quad v \in H^1(\mathbb{R}^3) \quad \text{with} \quad \langle v, P_\omega \rangle_{L^2} = 0. \quad (3.3)$$

On the other hand, $L_+$ has only one negative eigenvalue $-\lambda_1$ with a corresponding eigenfunction $e_1 \in H^2(\mathbb{R}^3)$ (cf. [2, Theorem 2.2]). We assume that $\|e_1\|_{L^2} = 1$. The second eigenvalue is 0 and

$$\ker\{L_+\} = \text{span}\{\partial_j P_\omega : j = 1, 2, 3\}.$$

In particular, the space $H^1(\mathbb{R}^3)$ can be decomposed into

$$H^1(\mathbb{R}^3) = \text{span}\{e_1\} \oplus \text{span}\{\partial_j P_\omega : j = 1, 2, 3\} \oplus E_+,$$  

where $L_+$ defines a positive definite quadratic form on $E_+$. Notice that in the direct sum (3.4) the spaces are mutually orthogonal with the inner product of $L^2(\mathbb{R}^3)$.

We denote by $\mathcal{F}(g, h)$ the bilinear symmetric form

$$\mathcal{F}(g, h) := \frac{1}{2} \langle L_+ g_1, h_1 \rangle + \frac{1}{2} \langle L_- g_2, h_2 \rangle, \quad (3.5)$$

where $g = g_1 + ig_2$ and $h = h_1 + ih_2$. We denote $\mathcal{F}(h, h)$ by $\mathcal{F}(h)$, i.e.,

$$\mathcal{F}(h) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla h|^2 dx + \frac{1}{2} P_\omega^2 (5 \lambda_1^2 + \lambda_2^2) - \frac{1}{2} P_\omega^2 (3 \lambda_1^2 + \lambda_2^2) + \frac{\omega}{2} |h|^2 \ dx. \quad (3.6)$$

Note that

$$\mathcal{F}(i P_\omega, h) = \mathcal{F}(\partial_j P_\omega, h) = 0 \quad \text{for all} \quad h \in H^1(\mathbb{R}^3). \quad (3.7)$$

We have the following result.

**Lemma 3.1.** There exists $C > 0$ such that for every $h = h_1 + ih_2 \in H^1(\mathbb{R}^3)$ satisfying

$$\langle h_1, \partial_1 P_\omega \rangle = \langle h_1, \partial_2 P_\omega \rangle = \langle h_1, \partial_3 P_\omega \rangle = \langle h_2, P_\omega \rangle = 0 \quad (3.8)$$

and either

$$\langle h_1, L_+ (x \cdot \nabla P_\omega) \rangle = \langle h_1, -2 \Delta P_\omega \rangle = 0; \quad (3.9)$$

or

$$\langle h_1, L_+ (x \cdot \nabla P_\omega) \rangle \geq 0; \quad (3.10)$$

or

$$\langle h_1, L_+ (P_\omega) \rangle \geq 0, \quad (3.11)$$

then we have

$$\mathcal{F}(h) \geq C \|h\|_{H^1}^2.$$
Moreover, by using the fact that \( L \) is positive definite and \( \Delta P_{\omega} = 0 \), this yields
\[
\frac{1}{2} \langle f, L(x \cdot \nabla P_{\omega}) \rangle = \langle f, -\Delta P_{\omega} \rangle = \langle f, \partial_{\omega} P_{\omega} \rangle = 0 \quad \text{for all } j = 1, 2, 3, \tag{3.12}
\]
we have
\[
\langle L_+ f, f \rangle > 0. \tag{3.13}
\]
Indeed, setting \( g_{\omega} := \frac{1}{2} x \cdot \nabla P_{\omega} \), it follows from straightforward calculations (see [2, Table 2.1]) that
\[
L_+ g_{\omega} = -\Delta P_{\omega} \quad \text{and} \quad \langle L_+ g_{\omega}, g_{\omega} \rangle = -\frac{1}{2} \|g_{\omega}\|_{H^1}^2 < 0.
\]
Notice also that \( (g_{\omega}, \partial_{\omega} P_{\omega})_{L^2} = 0 \) for all \( j = 1, 2, 3 \). By the decomposition (3.4), and (3.12) we infer that there exist \( a, b \in \mathbb{R} \) and \( \psi, \zeta \in E_+ \) so that
\[
f = ae_1 + \psi \quad \text{and} \quad g_{\omega} = be_1 + \zeta.
\]
As \( \langle L_+ g_{\omega}, g_{\omega} \rangle < 0 \), it follows that \( b \neq 0 \).

If \( a = 0 \), then it is clear that \( \langle L_+ f, f \rangle = \langle L_+ \psi, \psi \rangle > 0 \) (recall that \( f \neq 0 \)). Suppose instead that \( a \neq 0 \). As \( L_+ \) defines a positive definite quadratic form on \( E_+ \), we have the Cauchy–Schwartz inequality
\[
\langle L_+ \psi, \zeta \rangle^2 \leq \langle L_+ \psi, \psi \rangle \langle L_+ \zeta, \zeta \rangle,
\]
which implies
\[
\langle L_+ f, f \rangle = -a^2 \lambda_1 + \langle L_+ \psi, \psi \rangle \geq -a^2 \lambda_1 + \frac{\langle L_+ \psi, \zeta \rangle^2}{\langle L_+ \psi, \psi \rangle} \tag{3.14}
\]
Moreover, by using the fact that \( L_+ g_{\omega} = -\Delta P_{\omega} \) and the orthogonality condition \( \langle f, \Delta P_{\omega} \rangle = 0 \), we have
\[
0 = -\langle f, \Delta P_{\omega} \rangle = \langle f, L_+ g_{\omega} \rangle = -ab \lambda_1 + \langle L_+ \psi, \zeta \rangle.
\]
Thus, \( \langle L_+ \psi, \zeta \rangle = ab \lambda_1 \), which implies
\[
-a^2 \lambda_1 + \frac{\langle L_+ \psi, \zeta \rangle^2}{\langle L_+ \psi, \psi \rangle} = -a^2 \lambda_1 + \frac{a^2 b^2 \lambda_1^2}{\langle L_+ \psi, \psi \rangle}
= -a^2 \lambda_1 + \frac{\langle L_+ \psi, \psi \rangle}{\langle L_+ \psi, \psi \rangle} > 0.
\]
Combined with (3.14), this yields \( \langle L_+ f, f \rangle > 0 \).

Next, we show that under conditions (3.12) there exists \( C_+ > 0 \) such that
\[
\langle L_+ f, f \rangle \geq C_+ \|f\|_{H^1}^2.
\]
Suppose instead that there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3) \) such that \( \langle f_n, \Delta P_{\omega} \rangle = \langle f_n, \partial_{\omega} P_{\omega} \rangle = 0 \) for all \( j = 1, 2, 3 \),
\[
\|\nabla f_n\|_{L^2}^2 + \omega \|f_n\|_{L^2}^2 = 1 \quad \text{and} \quad \langle L_+ f_n, f_n \rangle \to 0
\]
as \( n \to \infty \). It is clear that \( \{f_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^3) \). Therefore, there exists \( f \) such that \( f_n \to f \) in \( H^1(\mathbb{R}^3) \). Notice that by the exponential decay of \( P_{\omega} \) we get as \( n \to \infty \)
\[
5 \int_{\mathbb{R}^3} P_0^4 f_n^2 \, dx - 3 \int_{\mathbb{R}^3} P_0^2 f_n^2 \, dx \to 5 \int_{\mathbb{R}^3} P_0^4 f^2 \, dx - 3 \int_{\mathbb{R}^3} P_0^2 f^2 \, dx,
\]
(3.15)

\[
\langle f, \Delta P_0 \rangle = \langle f, \partial_j P_0 \rangle = 0 \quad \text{for all } j = 1, 2, 3.
\]
(3.16)

In particular, by weak lower semi-continuity of the \(H^1(\mathbb{R}^3)\)-norm we obtain
\[
\langle L_+ f, f \rangle \leq \liminf_{n \to \infty} \langle L_+ f_n, f_n \rangle = 0.
\]
(3.17)

Combining (3.16), (3.17), and (3.13), we see that \(f \equiv 0\). However,
\[
3 \int_{\mathbb{R}^3} P_0^2 f^2 - 5 \int_{\mathbb{R}^3} P_0^4 f^2 = 1 - \liminf_{n \to \infty} \langle L_+ f_n, f_n \rangle = 1,
\]
which thus yields a contradiction. Therefore, \(\langle L_+ f, f \rangle \geq C_+ \| f \|_{H^1}^2\) for some constant \(C_+ > 0\).

On the other hand, by (3.3) and using an argument similar to the above we infer that there exists \(C_- > 0\) such that \(\langle L_- f, f \rangle \geq C_- \| f \|_{H^1}^2\). Therefore, for \(h = h_1 + i h_2 \in H^1(\mathbb{R}^3)\) we have
\[
\mathcal{F}(h) \geq \frac{C_+ + C_-}{2} \| f \|_{H^1}^2.
\]

Next, assume that \(\langle h_1, L_+ (x \cdot \nabla P_0) \rangle \geq 0\). The proof follows using the same argument (with some obvious modifications) used above.

Finally, assume that \(\langle h_1, L_+ (P_0 \omega) \rangle \geq 0\). In this case, setting \(g_\omega := P_0 \omega\) we have (see [2, Table 2.1])
\[
L_+ g_\omega = 4P_0^5 - 2P_0^3, \quad \langle L_+ g_\omega, g_\omega \rangle = \frac{4}{3} (\beta(\omega) - 2) \| P_0 \|_{H^1}^2 < 0,
\]
and \((g_\omega, \partial_j P_0)_{L^2} = 0\) for all \(j = 1, 2, 3\). Now, by applying the same argument as above to \(g_\omega = P_0 \omega\), we find that \(\mathcal{F}(h) \geq C \| h \|_{H^1}^2\) holds for all \(f \in H^1(\mathbb{R}^3)\).

**Corollary 3.2.** There exists \(C > 0\) such that for every \(h = h_1 + i h_2 \in H^1(\mathbb{R}^3)\) satisfying
\[
\langle h_1, \partial_1 P_0 \rangle = \langle h_1, \partial_2 P_0 \rangle = \langle h_1, \partial_3 P_0 \rangle = \langle h_2, P_0 \rangle = 0 \quad (3.18)
\]
and
\[
\left\langle h_1, L_+ \left( x \cdot \nabla P_0 + \frac{3}{2} P_0 \right) \right\rangle = 0, \quad (3.19)
\]
we have
\[
\mathcal{F}(h) \geq C \| h \|_{H^1}^2.
\]

**Proof.** Consider \(h = h_1 + i h_2\). By (3.19) we have
\[
\langle h_1, L_+ (x \cdot \nabla P_0) \rangle + \langle h_1, L_+ \left( \frac{3}{2} P_0 \right) \rangle = 0. \quad (3.20)
\]
Now, if \(\langle h_1, L_p(x \cdot \nabla P_\omega) \rangle \geq 0\), then (3.8) (recall (3.18)) and (3.10) hold. Lemma 3.1 implies that \(\mathcal{F}(h) \geq C||h||^2_{H^1}\) for some constant \(C > 0\).

On the other hand, if \(\langle h_1, L_+(\frac{h}{2} P_\omega) \rangle < 0\), then by (3.20) we infer that \(\langle h_1, L_+(\frac{h}{2} P_\omega) \rangle \geq 0\), which implies that (3.8) and (3.11) hold. Again, by Lemma 3.1 we find \(\mathcal{F}(h) \geq C||h||^2_{H^1}\) for some constant \(C > 0\).

The following lemma gives the general structure of the spectrum of the operator \(L\). The proof of this result is given in Appendix A.

**Lemma 3.3.** The operator \(L\) defined on \(L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) with domain \(H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\) has two simple eigenfunctions \(e_+\) and \(e_-\) in \(S(\mathbb{R}^3)\) with real eigenvalues \(\pm \lambda_1\) with \(\lambda_1 > 0\). Moreover

\[
\sigma(L) \cap \mathbb{R} = \{-\lambda_1, 0, \lambda_1\}.
\]

The essential spectrum of \(L\) is \(\{i\xi : \xi \in \mathbb{R}, |\xi| \geq \omega\}\), and the kernel is

\[
\text{ker}\{L\} = \text{span}\{\partial_1 P_\omega, \partial_2 P_\omega, \partial_3 P_\omega, iP_\omega\}.
\]

We set

\[
e_1 := \text{Re} e_+ \quad \text{and} \quad e_2 := \text{Im} e_+,
\]

so that \(L_+ e_1 = \lambda_1 e_2\) and \(L_- e_2 = -\lambda_1 e_1\).

The following lemma shows that the linearized energy \(\mathcal{F}\) is coercive in the set \(Y^\perp\) of \(h \in H^1(\mathbb{R}^3)\) satisfying the orthogonality relations (3.21) and (3.22).

**Lemma 3.4.** There exists \(C > 0\) such that for every \(h = h_1 + ih_2 \in H^1(\mathbb{R}^3)\) satisfying

\[
\langle h_1, \partial_1 P_\omega \rangle = \langle h_1, \partial_2 P_\omega \rangle = \langle h_1, \partial_3 P_\omega \rangle = \langle h_2, P_\omega \rangle = 0,
\]

(3.21)

\[
\langle e_1, h_2 \rangle = \langle e_2, h_1 \rangle = 0,
\]

(3.22)

we have

\[
\mathcal{F}(h) \geq C||h||^2_{H^1}.
\]

**Proof.** The proof is similar to that of [7, Proposition 2.7].

We first show that if \(h = h_1 + ih_2 \in H^1(\mathbb{R}^3) \setminus \{0\}\) satisfies (3.21) and (3.22), then \(\mathcal{F}(h) > 0\). Suppose instead that there exists nonzero \(g \in H^1(\mathbb{R}^3)\) such that

\[
\mathcal{F}(g) \leq 0, \quad \langle e_1, g_2 \rangle = \langle e_2, g_1 \rangle = \langle g_2, P_\omega \rangle = \langle g_1, \partial_j P_\omega \rangle = 0, \quad j = 1, 2, 3.
\]

(3.23)

We now note that (cf. Remarks 3.5 and 3.6)

\[
\mathcal{F}(e_+, g) = 0, \quad \mathcal{F}(e_-, g) = 0, \quad \mathcal{F}(e_+, e_+) = 0 \quad \text{and} \quad \mathcal{F}(e_+, e_-) \neq 0.
\]

(3.24)

We also recall that

\[
\mathcal{F}(iP_\omega) = \mathcal{F}(\partial_j P_\omega) = 0 \quad \text{for all} \quad j = 1, 2, 3.
\]

(3.25)

Now we set

\[
E_\pm := \text{span}\{\partial_1 P_\omega, \partial_2 P_\omega, \partial_3 P_\omega, iP_\omega, e_\pm, g\}
\]
Combining (3.23), (3.24) and (3.25) we get $\mathcal{F}(h) \leq 0$ for all $h \in E_-$. Similarly, by using (3.23), (3.24) and (3.25) one can show that $\dim_{\mathbb{R}} E_- = 6$ (see [7, Proposition 2.7] for more details). However, Lemma 3.1 shows that $\mathcal{F}$ is definite positive on a codimension 5 subspace of $H^1(\mathbb{R}^3)$, which is a contradiction.

Finally, the same argument given in the second part of the proof of Lemma 3.1 shows that $\mathcal{F}(h) \geq C \|h\|^2_{H^1}$ for all $h \in H^1(\mathbb{R}^3)$.

**Remark 3.5.** A direct computation shows that for any $f, g \in H^1$,

$$\mathcal{F}(e_-) = 0, \quad \mathcal{F}(P_0) = \frac{4}{3} [\beta(\omega) - 2] \|\nabla P_0\|^2_{L^2},$$

$$\mathcal{F}(h, g) = \mathcal{F}(g, h), \quad \mathcal{F}(Lh, g) = -\mathcal{F}(h, Lg).$$

**Remark 3.6.** We have that $\mathcal{F}(e_+, e_-) \neq 0$. Indeed, suppose instead that $\mathcal{F}(e_+, e_-) = 0$. Consider

$$h \in \text{span}\{iP_0, e_+, e_-, \partial_1 P_0, \partial_2 P_0, \partial_3 P_0 \},$$

which is of codimension 6. Then by Remark 3.5 we see that $\mathcal{F}(h) = 0$, which is a contradiction because $\mathcal{F}$ is positive on a codimension 5 subspace (see Proposition 3.1).

**Remark 3.7.** We have

$$\int_{\mathbb{R}^3} \Delta P_0 e_1 dx \neq 0.$$

Indeed, suppose instead that $\langle \Delta P_0, e_1 \rangle = 0$. Recall that

$$L_+ g_0 = -\Delta P_0 \quad \text{and} \quad \langle L_+ g_0, g_0 \rangle = -\frac{1}{2} \|g_0\|^2_{H^1} < 0,$$

where $g_0 := \frac{1}{2} x \cdot \nabla P_0$. Notice that $\langle L_+ (g_0), e_1 \rangle = \lambda_1 \langle g_0, e_2 \rangle$. On the other hand, $\langle L_+ (g_0), e_1 \rangle = \langle -\Delta P_0, e_1 \rangle = 0$. Therefore, $\langle g_0, e_2 \rangle = 0$. Then, by Lemma 3.4 we infer that $\mathcal{F}(g_0) > 0$, contradicting that $\mathcal{F}(g_0) = \langle L_+ g_0, g_0 \rangle < 0$.

Finally, we record the following identity, which follows from direct computation:

**Lemma 3.8.** Let $h \in H^1(\mathbb{R}^3)$ and assume $E(P_0 + h) = E(P_0)$ and $M(P_0 + h) = M(P_0)$. Then

$$\mathcal{F}(h) = \int_{\mathbb{R}^3} P_0 (|h|^2 h_1 - |h|^4 h_1) dx - \frac{1}{2} \int_{\mathbb{R}^3} P_0^2 (|h|^4 + 4|h|^2 h_1^2) dx$$

$$- \frac{1}{3} \int_{\mathbb{R}^3} P_0^3 (6|h|^2 h_1 + 4h_1^3) dx + \frac{1}{4} \int_{\mathbb{R}^3} |h|^4 dx - \frac{1}{6} \int_{\mathbb{R}^3} |h|^6 dx.$$

### 4. Modulation analysis

Throughout this section, we fix $(m, e) = (M(P_0), E(P_0)) \in \partial \mathcal{K}_s$ and a solution $u$ to (NLS) satisfying
\[(M(u), E(u)) = (m, e) \quad \text{and} \quad V(u_0) > 0.\]

We denote
\[\delta(t) := V(u(t)) \quad \text{for} \quad t \in \mathbb{R},\]
where \(V\) is the virial functional (1.2). By Lemma 2.6 we have
\[\delta(t) = V(u(t)) > 0, \quad \text{for all} \quad t \in \mathbb{R}.\] (4.1)

We let \(\delta_0 > 0\) be a small parameter and define the open set
\[I_0 = \{ t \in [0, \infty) : \delta(t) < \delta_0 \}.
\]
We will prove the following.

**Proposition 4.1.** For \(\delta_0 > 0\) sufficiently small, there exist functions \(\theta : I_0 \to \mathbb{R}, \; x : I_0 \to \mathbb{R}, \; y : I_0 \to \mathbb{R}^3, \; \text{and} \; h : I_0 \to H^1\) such that we can write
\[e^{-i\omega t}u(t, \cdot + y(t)) = e^{i\theta(t)}[(1 + x(t))P_\omega + h(t)] \quad \text{for all} \quad t \in I_0,\] (4.2)
with
\[|y'(t)| + |\theta'(t)| + |x'(t)| \leq \delta(t) \sim |x(t)| \sim ||h(t)||_{H^1}.\] (4.3)

Using the implicit theorem and the variational characterization of \(P_\omega\) in Lemma 2.8, we can obtain the following orthogonal decomposition. We will omit the proof, as it is essentially the same given in [7, Lemma 4.1].

**Lemma 4.2.** There exist \(\delta_0 > 0\), a positive function \(\varepsilon(\delta)\) defined for \(0 < \delta \leq \delta_0\) and functions \(\sigma : I_0 \to \mathbb{R}\) and \(y : I_0 \to \mathbb{R}^3\) such that if \(\delta(t) < \delta_0\), then
\[||u(t) - e^{i\sigma(t)}P_\omega(\cdot - y(t))||_{H^1} \leq \varepsilon(\delta).\] (4.4)

The mapping \(u \mapsto (\sigma, y)\) is \(C^1\) and \(\varepsilon(\delta) \to 0\) as \(\delta \to 0\). The functions \(\sigma(\cdot)\) and \(y(\cdot)\) are chosen to impose the following orthogonality conditions: for \(j \in \{1, 2, 3\},\)
\[\text{Im}(e^{i\sigma(t)}P_\omega(\cdot - y(t)), u(t)) = \text{Re}(e^{i\sigma(t)}\partial_j P_\omega(\cdot - y(t)), u(t)) = 0.\] (4.5)

With \((\sigma(t), y(t))\) as in Lemma 4.2, we write
\[e^{-i\theta(t)}[e^{-i\omega t}u(t, x + y(t))] = (1 + x(t))P_\omega(x) + h(t, x),\] (4.6)
where \(\theta(t) := \sigma(t) - \omega t\) and
\[x(t) = \frac{\langle L_+ \left(x \cdot \nabla P_\omega + \frac{3}{2} P_\omega \right), e^{-i\theta(t)}e^{-i\omega t}u(t, \cdot + y(t)) \rangle}{\langle L_+ \left(x \cdot \nabla P_\omega + \frac{3}{2} P_\omega \right), P_\omega \rangle} - 1.\]

We then have the following orthogonally conditions for \(h\): for \(j \in \{1, 2, 3\},\)
\[\text{Im}(h(t), P_\omega) = \text{Re}\left(h(t), L_+ \left(x \cdot \nabla P_\omega + \frac{3}{2} P_\omega \right) \right) = 0.\] (4.7)

We also note that \(\langle L_+ \left(x \cdot \nabla P_\omega + \frac{3}{2} P_\omega \right), P_\omega \rangle = 2[\beta(\omega) - 1]||\nabla P_\omega||_{L^2}^2.\)

The following lemma relates the parameters \(\delta(t), x(t), \text{and} \ h(t).\)
Lemma 4.3. For all \( t \in I_0 \),
\[
\delta(t) \sim |x(t)| \sim ||h(t)||_{H^1}. \tag{4.8}
\]

Proof. We expand the virial functional around \( P_{o_0} \) (recalling that \( V(P_{o_0}) = 0 \)) to obtain
\[
\delta(t) = V(P_{o_0} + xP_{o_0} + h) - V(P_{o_0})
\]
\[
= \langle -2AP_{o_0} + 6P^5_{o_0} - 3P^3_{o_0}, x(t)P_{o_0} + h_1(t) \rangle + O(x^2 + ||h||_{H^1}^2)
\]
\[
= \langle L_p \left( x \cdot \nabla P_{o_0} + \frac{3}{2} P_{o_0} \right), x(t)P_{o_0} + h_1(t) \rangle + O(x^2 + ||h||_{H^1}^2),
\]
where we have used \(-2AP_{o_0} + 6P^5_{o_0} - 3P^3_{o_0} = L_p \left( x \cdot \nabla P_{o_0} + \frac{3}{2} P_{o_0} \right)\). By (4.7) we infer that
\[
\delta(t) = \langle L_p \left( x \cdot \nabla P_{o_0} + \frac{3}{2} P_{o_0} \right), P_{o_0} \rangle x(t) + O(x^2 + ||h||_{H^1}^2). \tag{4.9}
\]

On the other hand, by using Lemma 3.8 we can write (recalling that \( F(P_{o_0}) < 0 \), cf. Remark 3.5):
\[
F(h) = x^2[-F(P_{o_0})] + 2x[-F(P_{o_0}, h_1)] + O(x^2 + ||h||_{H^1}^3). \tag{4.10}
\]
In particular, as \( F(h) \leq ||h||_{H^1}^2 \) we see that \( |x| = O(||h||_{H^1}) \). Moreover, by Corollary 3.2 (recalling the orthogonally conditions for \( h, (4.7) \)) and (4.10) we infer that \( ||h||_{H^1} = O(|x|), \) i.e., \( ||h||_{H^1} \sim |x| \).

Finally, from (4.9) we see that
\[
\delta(t) = \langle L_p \left( x \cdot \nabla P_{o_0} + \frac{3}{2} P_{o_0} \right), P_{o_0} \rangle x(t) + O(x^2),
\]
which implies that \( \delta \sim |x| \). Therefore, \( ||h||_{H^1} \sim |x| \sim \delta \). \( \square \)

Lemma 4.4. Let \((\sigma(t), y(t))\) be as in Lemma 4.2 and \( h(t), 0(t) \) and \( x(t) \) be as in (4.6). Then, taking a smaller \( \delta_0 \) if necessary, we have
\[
|y'(t)| + |\theta'(t)| + |x'(t)| \leq \delta(t).
\]

Proof. With Lemma 4.3 in hand, the proof is very similar to that of [7, Lemma 4.3] (in particular, the key idea is to differentiate the orthogonality relations). We omit the details. \( \square \)

Proposition 4.1 now follows immediately from Lemmas 4.2, 4.3 and 4.4.

To close this section, we record one final lemma, which shows how control over \( \delta(t) \) guarantees exponential closeness to the orbit of \( P_{o_0} \).

Lemma 4.5. Suppose that there exist \( a, b > 0 \) so that
\[
\int_t^\infty \delta(s) \, ds \leq ae^{-bt} \quad \text{for all} \quad t \geq 0. \tag{4.11}
\]
Then \( \lim_{t \to \infty} \delta(t) = 0 \). Moreover, there exist \( \theta_0 \in \mathbb{R}, y_0 \in \mathbb{R}^3 \) and \( c, C > 0 \) such that
\[
||u(t) - e^{ib\theta} e^{i\theta(t) P_{o_0}} \cdot (\cdot - y_0)||_{H^1} \leq Ce^{-ct}.
\]
Proof. With Proposition 4.1 in hand, the proof is essentially the same as that of [7, Lemma 4.4]. □

5. Concentration compactness

In this section, we show that the solutions to (NLS) with \((M(u), E(u)) = (M(P_\omega), E(P_\omega)) \in \partial K_s)\) that do not scatter forward in time, must converge exponentially to \(P_\omega\) as \(t \to \infty\) up to the symmetries of the equation.

Proposition 5.1. Let \(u\) be a solution of (NLS) satisfying
\[
(M(u), E(u)) = (M(P_\omega), E(P_\omega)) \in \partial K_s \quad \text{with} \quad V(u_0) > 0, \quad (5.1)
\]
\[
||u||_{L^{10}_{t,x}(0,\infty) \times \mathbb{R}^3} = \infty. \quad (5.2)
\]
Then there exist \(\theta \in \mathbb{R}, y_0 \in \mathbb{R}^3\) and \(c, C > 0\) such that
\[
||u(t) - e^{i\theta} e^{i\omega t} P_\omega(\cdot - y_0)||_{H^1} \leq Ce^{-ct}. \quad (5.3)
\]

As a corollary, we will obtain the following:

Corollary 5.2. There is no solution to (NLS) satisfying (5.1) and
\[
||u||_{L^{10}_{t,x}(0,\infty) \times \mathbb{R}^3} = ||u||_{L^{10}_{t,x}(-\infty,0) \times \mathbb{R}^3} = \infty. \quad (5.3)
\]

The first step in the proof of Proposition 5.1 is to establish compactness for nonscattering solutions.

Lemma 5.3. Suppose that \(u(t)\) satisfies (5.1) and (5.2). Then there exists \(x_0 : [0, \infty) \to \mathbb{R}^3\) such that
\[
\{u(t, \cdot + x_0(t)) : t \in [0, \infty)\} \quad \text{is pre-compact in} \quad H^1(\mathbb{R}^3). \quad (5.4)
\]

Proof. First note that by Lemma 2.6, the solution \(u(t)\) is uniformly bounded in \(H^1(\mathbb{R}^3)\) and \(V(u(t)) > 0\) for all \(t \in \mathbb{R}\).

Given \(\tau_n \to \infty\), we will first show that there exists a subsequence in \(n\) and \(x_n \in \mathbb{R}^3\) such that
\[
\{u(\tau_n, x + x_n)\} \quad \text{converges in} \quad H^1(\mathbb{R}^3). \quad (5.5)
\]
We apply the profile decomposition Proposition 2.9 to the sequence \(u(\tau_n)\) to obtain
\[
u_n := u(\tau_n) = \sum_{j=1}^J \psi_n^j + W_n^j, \quad (5.6)
\]
with all of the properties stated in Proposition 2.9.

If \(J' = 0\), then
\[
||e^{it\Delta} u(\tau_n)||_{L^{10}_{t,x}(0,\infty) \times \mathbb{R}^3} \to 0 \quad \text{as} \quad n \to \infty.
\]
In particular, by using the stability result (Lemma 2.3) with \( \tilde{u}_n = e^{it_n}u(\tau_n) \) and \( u_n = u(\tau_n) \), we obtain

\[
||u(t + \tau_n)||_{L^\infty([0, \infty) \times \mathbb{R}^3)} = ||u||_{L^\infty([\tau_n, \infty) \times \mathbb{R}^3)} \lesssim 1,
\]

for large \( n \), which is a contradiction. Therefore \( J^* \geq 1 \).

Now suppose that \( J^* \geq 2 \). By decoupling of mass and energy, we have

\[
\lim_{n \to \infty} \left[ \sum_{j=1}^J M(\psi^j_n) + M(W^j_n) \right] = \lim_{n \to \infty} M(u_n) = M(P_{\infty}),
\]

and

\[
\lim_{n \to \infty} \left[ \sum_{j=1}^J E(\psi^j_n) + E(W^j_n) \right] = \lim_{n \to \infty} E(u_n) = E(P_{\infty}).
\]

for any \( 0 \leq J \leq J^* \).

By assumption, we have \( E(P_{\infty}) > 0 \). We claim that there exists \( \delta > 0 \) such that for all \( n \) large and \( 1 \leq j \leq J \),

\[
M(\psi^j_n) \leq M(P_{\infty}) - \delta \quad \text{and} \quad E(\psi^j_n) \leq E(P_{\infty}) - \delta.
\]

In particular, by the strict monotonicity of \( m \mapsto E(m) \), (5.9) implies that \((M(\psi^j_n), E(\psi^j_n)) \in K\) (the scattering region) for all \( n \) large and \( 1 \leq j \leq J \).

**Proof** of (5.9). First, suppose that \( \lambda^j_n \equiv 1 \). Then there exists \( \delta > 0 \) such that \( M(\psi^j_n) \leq M(P_{\infty}) - \delta \). In particular, as \( M(P_{\infty}) \leq M(Q_1) \), by (2.3) we obtain

\[
E(\psi^j_n) \geq \delta ||\psi^j||_{H^1}^2,
\]

which implies (5.9). Now suppose instead \( \lambda^j_n \to 0 \) as \( n \to \infty \). By Bernstein’s inequality and a changes of variables we see that \( M(\psi^j_n) \leq (\lambda^j_n)^{1-\theta}||\psi^j||_{H^1}^2 \). As \( \lambda^j_n \to 0 \), we find that there exists \( \delta > 0 \) such that \( M(\psi^j_n) \leq M(P_{\infty}) - \delta \) for \( n \) large. Moreover, using

\[
||\psi^j_n||_{H^1}^2 = ||P_{(\lambda^j_n)^{-\theta}}\psi^j||_{H^1} \to ||\psi^j||_{H^1}^2 \quad \text{as} \quad n \to \infty
\]
together with (2.3), we have that

\[
\liminf_{n \to \infty} E(\psi^j_n) \geq \delta \liminf_{n \to \infty} ||\psi^j_n||_{H^1}^2 \geq ||\psi^j||_{H^1}^2 \quad \text{for} \quad 1 \leq j \leq J,
\]

which yields (5.9). \( \square \)

Now we define solutions \( \psi^j \) to (NLS) as follows:

(i) If \( \lambda^j_n \equiv 1 \) and \( t^j_n \to \pm \infty \) as \( n \to \infty \), we define \( \psi^j \) to be the global solution to (NLS) which scatters to \( e^{it}u \) when \( t \to \pm \infty \). Note that

\[
\lim_{n \to \infty} ||\psi^j(t^j_n) - \psi^j_n||_{H^1} = 0.
\]

In particular, since (5.9) holds, by Theorem 1.1 we obtain global space-time bounds for the solution \( \psi^j \).
(ii) If \( \lambda_n^j \equiv 1 \) and \( t_n^j = 0 \), we define \( v^j \) to be the global solution to \((\text{NLS})\) with the initial data \( v^j(0) = \psi^j \). Since \( (M(\psi^j), E(\psi^j)) \in K \) (see (5.9)), Theorem 1.1 implies that the solution \( v^j \) is a global solution with finite scattering size.

(iii) If \( \lambda_n^j \to 0 \) as \( n \to \infty \), we define \( v_n^j \) to be the solution to \((\text{NLS})\) with the initial data \( v_n^j(0) = \psi_n^j \) established in [2, Proposition 8.3].

In the cases (i) and (ii) we define the global solutions to \((\text{NLS})\),

\[
v_n^j(t, x) := v(t + t_n^j, x - x_n^j).
\]

By (5.9) and inequality (2.3) we get \( ||v_n^j(t)||^2_{H^1} \leq \delta E(v_n^j) \). Thus, persistence of regularity (see Lemma 6.2 in [2]) implies

\[
||v_n^j||_{L_t^z(\mathbb{R} \times \mathbb{R}^3)} + ||\nabla v_n^j||_{L_t^z(\mathbb{R} \times \mathbb{R}^3)} \leq \delta E(v_n^j)^{\frac{1}{2}}.
\] (5.10)

Similarly, we get

\[
||v_n^j||_{L_t^{\infty}(\mathbb{R} \times \mathbb{R}^3)} \leq \delta M(v_n^j)^{\frac{1}{2}}.
\] (5.11)

Moreover, an argument similar to the one above shows that

\[
||W_n^j||^2_{H^1} \leq \delta E(W_n^j).
\] (5.12)

We now define approximate solutions to \((\text{NLS})\) by

\[
u_n^j(t, x) = \sum_{j=1}^{J} v_n^j(t, x) + e^{it\Delta} W_n^j,
\]

and we use perturbation argument to obtain a contradiction to (5.2). We will verify that for \( n \) and \( J \) large, \( u_n^j \) is an approximate solution to \((\text{NLS})\) with global finite space-time bounds. Indeed, with estimates (5.10), (5.11) and (5.12) in hand, we can repeat the argument of [2, Lemmas 9.3, 9.4 and 9.5] to show that

\[
\lim_{n \to \infty} ||u_n^j(0) - u_n(0)||_{H^1} = 0,
\] (5.13)

\[
\sup_j \limsup_{n \to \infty} \left[ ||u_n^j||_{L_t^{10}(\mathbb{R} \times \mathbb{R}^3)} + ||u_n^j||_{L_t^\infty H^1(\mathbb{R} \times \mathbb{R}^3)} \right] \leq \delta, 1,
\] (5.14)

\[
\lim_{j \to \infty} \limsup_{n \to \infty} \left[ ||\nabla [i \partial_t u_n^j + \Delta u_n^j - F(u_n^j)]||_{L_t^{\infty}(\mathbb{R} \times \mathbb{R}^3)} = 0,
\] (5.15)

where \( F(z) = |z|^4z - |z|^2z \). Thus, an application of the stability result (Lemma 2.3) then yields that \( u \) satisfies finite space-time bounds globally in time, i.e., \( u \in L_t^{10}(\mathbb{R} \times \mathbb{R}^3) \), contradicting (5.2) and so \( J^* \geq 2 \) cannot occur.

From the analysis above we obtain that \( J^* = 1 \). Therefore,

\[
u_n = u(t_n) = \psi_n + W_n
\]

with \( W_n \to 0 \) weakly in \( \dot{H}^1(\mathbb{R}^3) \) (cf. (2.9)). To prove the proposition we need to show that \( W_n \to 0 \) strongly in \( H^1(\mathbb{R}^3) \), \( \lambda_n \equiv 1 \) and \( t_n \equiv 0 \). First we show that \( W_n \to 0 \) in
Recall that $E(P_0) > 0$ by assumption. If $W_n \to 0$ in $\dot{H}^1(\mathbb{R}^3)$, as $M(W_n) \leq M(P_0) < M(Q_1)$, then by inequality (2.3) we infer that $\liminf_{n \to \infty} E(W_n) > 0$ and one can show by the previous argument that $u$ scatters in $H^1(\mathbb{R}^3)$. Therefore, $W_n \to 0$ in $\dot{H}^1(\mathbb{R}^3)$. In particular, $\lim_{n \to \infty} E(W_n) = 0$ and $\lim_{n \to \infty} E(\psi_n) = E(P_0)$ (see (5.8)). Moreover, if $M(\psi_n) \leq M(P_0) - \delta$ for some $\delta > 0$ and $n$ large, then since $m \to E(m)$ is strictly decreasing we infer that (again by the above argument) $u$ scatters in $H^1(\mathbb{R}^3)$, contradicting (5.2). Thus, $\lim_{n \to \infty} M(\psi_n) = M(P_0)$ and $\lim_{n \to \infty} M(W_n) = 0$.

If $\hat{\lambda}_n \to 0$ as $n \to \infty$, by using Proposition 8.3 in [2] we see that the unique global solution with initial data $v_n(0) = \psi_n$ satisfies $\|v_n\|_{L^p_t([0, \infty) \times \mathbb{R}^3)} \leq C(\|\psi_n\|)$ for $n$ sufficiently large. As

$$
\lim_{n \to \infty} \|u_n - v_n(0)\|_{H^1} = \lim_{n \to \infty} \|W_n\|_{H^1} = 0,
$$

applying the stability result Lemma 2.3 we obtain a contradiction to (5.2).

Now we preclude the possibility that $t_n \to \pm \infty$. If $t_n \to \infty$ as $n \to \infty$ (say), then by Strichartz estimates, monotone convergence theorem we infer that

$$
\|e^{itA}u_n\|_{L^6_t(L^6_x(\mathbb{R}^3) \times \mathbb{R}^3)} \leq \|e^{itA}\psi_n\|_{L^6_t(L^6_x(\mathbb{R}^3) \times \mathbb{R}^3)} + \|e^{itA}W_n\|_{L^6_t(L^6_x(\mathbb{R}^3) \times \mathbb{R}^3)},
$$

$$
\leq \|e^{itA}\phi\|_{L^6_t(L^6_x([t_n, \infty) \times \mathbb{R}^3))} + \|W_n\|_{H^1} \to 0.
$$

Applying the stability result once more, we reach a contradiction.

Finally, by (5.5) and using the same argument developed in [13] we deduce that there exist a function $x_0(t)$ such that (5.4) holds, which finally completes the proof of the proposition. □

Next, the same argument as in [14, Lemma 4.2] yields the following lemma.

**Lemma 5.4.** If $\delta_0 > 0$ is sufficiently small, then there exists $C > 0$ such that

$$
|x_0(t) - y(t)| < C \quad \text{for } t \in I_0,
$$

where the parameter $y(t)$ is given in Proposition 4.1.

Using Lemma 5.4, we can define the modified spatial center $x(t)$ via

$$
x(t) = \begin{cases} 
  x_0(t) & t \in [0, \infty) \setminus I_0, \\
  y(t) & t \in I_0.
\end{cases} \quad (5.16)
$$

In this case, we still have that

$$
\{u(t, \cdot + x(t))\} \quad \text{is pre-compact in } H^1(\mathbb{R}). \quad (5.17)
$$

Next, we use the minimality property of $\partial K$ to obtain control over the motion of the spatial center $x(t)$:

**Lemma 5.5.** Let $u(t)$ be the solution in Lemma 5.3 with $x(t)$ defined in (5.16). Then the conserved momentum $Z(u(t)) = \int_{\mathbb{R}^3} 2 \Im \bar{u}(t) \nabla u(t) \, dx$ is zero. Moreover,

$$
\frac{|x(t)|}{t} \to 0 \quad \text{as } t \to \infty. \quad (5.18)
$$

**Proof.** Assume that $Z(u) \neq 0$. We then define

$$
w(t, x) = e^{ix \cdot \xi_0} e^{-it|\xi_0|^2} u(t, x - 2\xi_0 t),
$$
where $\xi_0 = -\mathcal{Z}(u)/(2M(u))$. As $E(P_\omega) > 0$ and 
\[ M(w) = M(u) = M(P_\omega) < M(Q_1), \]
the inequality (2.3) implies that $E(w) > 0$. Moreover.
\[ E(w) = E(u) - \frac{1}{2} \frac{|\mathcal{Z}(u)|^8}{M(u)} < E(u) = E(P_\omega). \]

In particular, $(M(w), E(w)) \in K$. Theorem 1.1 implies that $w$ scatters in $H^1(\mathbb{R}^3)$, which is a contradiction because 
\[ ||w||_{L^6_t(L^\infty_x(\mathbb{R}^3))} = ||u||_{L^6_t(L^\infty_x(\mathbb{R}^3))} = \infty. \]

Finally, the proof of (5.18) is identical to the proof of Proposition 10.2 in [2].

Next, we use the localized virial argument to show that $\delta(t_n) \to 0$ along some sequence $t_n \to \infty$. This gives a preliminary indication of the convergence of $u$ to $P_\omega$.

**Lemma 5.6.** Suppose $u$ is a solution of (NLS) satisfying the assumptions of Proposition 5.1. Then there exists a sequence $t_n \to \infty$ such that
\[ \lim_{n \to \infty} \delta(t_n) = 0. \]

**Proof.** We use a localized virial argument. First, recall $\delta(t) = V(u(t))$. By Lemma 2.10 we can write
\[ \frac{d}{dt} P_R(u(t)) = 8V(u(t)) + [F_R(u(t)) - F_\infty(u(t))]. \quad (5.19) \]

Moreover,
\[ |P_R(u)| \leq R ||u||_{L^8_t H^1_x} \leq P_\omega R \quad (5.20) \]

and
\[ |F_R(u(t)) - F_\infty(u(t))| = O \left( \int_{|x| > R} \left[ |\nabla u|^2 + |u|^6 + |u|^4 + |u|^2 \right] dx \right). \quad (5.21) \]

By compactness in $H^1(\mathbb{R}^3)$ (see (5.4)), for $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that
\[ \sup_{t \in \mathbb{R}} \int_{|x-x(t)| > C(\varepsilon)} \left[ |\nabla u|^2 + |u|^6 + |u|^4 + |u|^2 \right] (t,x) dx \leq \varepsilon. \]

As $|x(t)| = o(t)$ as $t \to \infty$ (cf. Lemma 5.5), it follows that there exists $T_0(\varepsilon)$ such that
\[ |x(t)| \leq \varepsilon t \quad \text{for all } t \geq T_0(\varepsilon). \]

Given $T > T_0(\varepsilon)$, we put
\[ R := C(\varepsilon) + \sup_{t \in [T_0(\varepsilon), T]} |x(t)|. \]

As $\{x : |x| \geq R\} \subset \{x : |x-x(t)| \geq C(\varepsilon)\}$ for $t \in [T_0(\varepsilon), T]$, by (5.21) we get
\[ |F_R(u(t)) - F_\infty(u(t))| \leq C \varepsilon \quad \text{for } t \in [T_0(\varepsilon), T]. \quad (5.22) \]
Combining this with (5.19), (5.20) and integrating on $[0, T]$ we see that
\[
\int_0^T V(u(t)) \, dt = \int_0^{T_0(\varepsilon)} V(u(t)) \, dt + \int_{T_0(\varepsilon)}^T V(u(t)) \, dt
\leq \int_0^{T_0(\varepsilon)} V(u(t)) \, dt + \tilde{C} \left[ \varepsilon T + C(\varepsilon) + \sup_{t \in [T_0(\varepsilon), T]} |x(t)| \right].
\]

Since $\sup_{t \in [T_0(\varepsilon), T]} |x(t)| \leq \varepsilon T$, we derive
\[
\frac{1}{T} \int_0^T V(u(t)) \, dt \leq \frac{1}{T} \int_0^{T_0(\varepsilon)} V(u(t)) \, dt + \tilde{C} \varepsilon + \frac{\tilde{C} C(\varepsilon)}{T}.
\]
Taking to the limit $T \to \infty$ and then $\varepsilon \to 0$ we find
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T V(u(t)) \, dt = 0.
\]
In particular, there exists a sequence $t_n \to \infty$ and $V(u(t_n)) \to 0$ as $n \to \infty$. \qed

Next, we use a more refined version of the localized virial estimate (taking the modulation analysis into account) to obtain a better estimate on $\delta(t)$.

**Lemma 5.7.** Suppose $u$ is a solution of (NLS) satisfying the assumptions of Proposition 5.1 with $x(t)$ defined in (5.16). Then there exists a positive constant $C$ such that for any interval $[t_1, t_2] \subset [0, \infty)$ we have
\[
\int_{t_1}^{t_2} \delta(t) \, dt \leq C \left[ 1 + \sup_{t \in [t_1, t_2]} |x(t)| \right] \{ \delta(t_1) + \delta(t_2) \}. \tag{5.23}
\]

**Proof.** Let $\delta_1 \in (0, \delta_0)$ be sufficiently small (cf. Proposition 4.1). We use the localized virial identity in Lemma 2.12 with the function $\chi(t)$ satisfying
\[
\chi(t) = \begin{cases} 
1 & \delta(t) < \delta_1, \\
0 & \delta(t) \geq \delta_1.
\end{cases}
\]
Let $R > 1$ to be specified later. Notice that we can write
\[
\frac{d}{dt} \mathcal{P}_R[u(t)] = 8\delta(t) + \sigma(t) \tag{5.24}
\]
where
\[
\sigma(t) = \begin{cases} 
F_R[u(t)] - F_\infty[u(t)] & \text{if } \delta(t) \geq \delta_1, \\
F[u(t)] - F_\infty[u(t)] - \Gamma[u(t)] & \text{if } \delta(t) < \delta_1,
\end{cases} \tag{5.25}
\]
with
\[
\Gamma(t) = F_R \left[ e^{i\alpha t} e^{i\theta(t)} P_{\omega}(\cdot - y(t)) \right] - F_\infty \left[ e^{i\alpha t} e^{i\theta(t)} P_{\omega}(\cdot - y(t)) \right]. \tag{5.26}
\]
We will show the following:

(i) **Claim I.** Fix $R > 1$. We have

$$|\mathcal{P}_R[u(t)]| \leq \frac{R}{\delta_1} \delta(t_j) \quad \text{if } \delta(t_j) \geq \delta_1 \text{ for } j = 1, 2,$$

$$|\mathcal{P}_R[u(t)]| \leq R\delta(t) \quad \text{if } \delta(t) < \delta_1 \text{ for } j = 1, 2.$$  

(ii) **Claim II.** For $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ so that if $R = \rho_\varepsilon + \sup_{t \in [t_1, t_2]} |x(t)|$, then

$$|\sigma(t)| \leq \varepsilon \quad \text{uniformly for } t \in [t_1, t_2] \text{ and } \delta(t) \geq \delta_1,$$

$$|\sigma(t)| \leq \varepsilon \delta(t) \quad \text{uniformly for } t \in [t_1, t_2] \text{ and } \delta(t) < \delta_1.$$  

By using (5.27), (5.28), (5.29), (5.30) and applying the fundamental theorem of calculus to (5.24) over $[t_1, t_2]$ we get

$$\int_{t_1}^{t_2} \delta(t)dt \leq \frac{1}{\delta_1} \left[ \rho_\varepsilon + \sup_{t \in [t_1, t_2]} |x(t)| \right] \left( \delta(t_1) + \delta(t_2) \right) + \left( \frac{\varepsilon}{\delta_1} + \varepsilon \right) \int_{t_1}^{t_2} \delta(t)dt,$$

Thus, choosing $\varepsilon = \varepsilon(\delta_1) > 0$ sufficiently small we get (5.23).

It remains to establish the above claims.

**Proof of Claim I.** Assume that $\delta(t_j) \geq \delta_1$. Notice that

$$|\mathcal{P}_R[u(t)]| \leq R ||u||_{L^\infty H^1} \leq p_\omega \frac{R}{\delta_1} \delta(t_j)$$

which implies (5.27). Now, if $\delta(t_j) < \delta_1$, then as $P_\omega$ is real valued, we write

$$P_\omega(t) = e^{i(\theta(t) + \omega t)} P_\omega(\cdot - y(t))$$

and obtain

$$|\mathcal{P}_R[u(t)]| = |2 \text{ Im} \int_{\mathbb{R}^3} \nabla w_R \cdot [\bar{u}(t_j) \nabla u(t_j) - \bar{P}_\omega(t_j) \nabla P_\omega(t_j)] dx|$$

$$\leq R \left[ ||u||_{L^\infty H^1} + ||P_\omega||_{H^1} \right] ||u(t_j) - P_\omega(t_j)||_{H^1}$$

$$\leq p_\omega R \delta(t_j).$$

where in the last line we have used estimate (4.3) in Proposition 4.1. \qed

**Proof of Claim II.** Suppose that $\delta(t) \geq \delta_1$. Using compactness (see (5.17)) we obtain that for each $\varepsilon > 0$, there exists a positive constant $\rho_\varepsilon = \rho(\varepsilon) > 0$ such that

$$\int_{|x-x(t)| > \rho_\varepsilon} \left| \nabla u(t, x) \right|^2 + |u(t, x)|^2 + |u(t, x)|^6 + |u(t, x)|^6 dx < \varepsilon.$$  

We put

$$R := \rho_\varepsilon + \sup_{t \in [t_1, t_2]} |x(t)|.$$  

Note that $\{|x| \geq R\} \subset \{|x-x(t)| \geq \rho_\varepsilon\}$ for all $t \in [t_1, t_2]$. Then the same argument developed in the proof of Lemma 5.6 shows that (see (5.22))
Proof. We follow [7, Lemma 6.8] and divide the proof into three steps.

Step 1. There exists a constant $C > 0$ such that

$$|x(t) - x(s)| \leq C \int_{t_1}^{t_2} \delta(t) \, dt,$$

for all $t_1, t_2 > 0$ with $t_1 + 1 \leq t_2$.

Proof. We follow [7, Lemma 6.8] and divide the proof into three steps.

Step 1. There exists a constant $C > 0$ such that

$$|x(t) - x(s)| \leq C \quad \text{for all} \quad t, s \geq 0 \quad \text{such that} \quad |t - s| \leq 2. \quad (5.35)$$

The proof of (5.35) is the same as the one given in [6, Lemma 3.10, Step 1].

Step 2. Let $\delta_0 > 0$ be as Proposition 4.1. We will show that there exists $\delta > 0$ such that either
for all $T \geq 0$, either
\[
\inf_{t \in [T, T+2]} \delta(t) \geq \delta_* \quad \text{or} \quad \sup_{t \in [T, T+2]} \delta(t) < \delta_0.
\] (5.36)

Assume instead that there exist $t_n^* \geq 0$ and two sequences $t_n, t_n' \in [t_n^*, t_n^* + 2]$ such that

\[
\delta(t_n) \to 0 \quad \text{and} \quad \delta(t_n') \geq \delta_1 \quad \text{as} \quad n \to \infty,
\] (5.37)

\[
t_n' - t_n \to t^* \in [-2, 2].
\] (5.38)

By (5.17), we deduce that there exits $\varphi \in H^1(\mathbb{R})$ such that

\[
\{u(t_n', \cdot + x(t_n))\} \to \varphi \quad \text{strongly in} \quad H^1(\mathbb{R}^3) \quad \text{as} \quad n \to \infty.
\] (5.39)

In particular, as $\delta(t_n) \to 0$ and $(M(u), E(u)) = (M(P_{\omega}), E(P_{\omega}))$, Lemma 2.8 implies that $\varphi(x) = e^{i\theta_0}P_{\omega}(x - x_0)$ for some $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$. Notice also that the solution to (NLS) with initial data $\varphi$ is $f(t, x) = e^{i(t+\theta_0)}P_{\omega}(x - x_0)$. Thus, by the continuity of the flow and (5.38) we infer that

\[
\delta(t_n') \to V(e^{i(t' + \theta_0)}P_{\omega}(x - x_0)) = 0,
\]

which contradicts (5.37). This completes the proof of (5.35).

Step 3. Conclusion. We prove (5.34) with an additional condition that $t_2 \leq t_1 + 2$. By (5.36) we have two cases:

(i) If $\inf_{t \in [t_1, t_2]} \delta(t) \geq \delta_*$ holds, then (5.34) follows immediately by applying (5.35).

(ii) On the other hand, if $\sup_{t \in [t_1, t_2]} \delta(t) < \delta_1$ holds, then as $x(t) = y(t)$ for all $t \in I_0$, from (4.3) we have $|x'(t)| \leq C\delta(t)$ for all $t \in I_0$. Applying the fundamental theorem of calculus we get (5.34).

Finally we may remove the assumption $t_2 \leq t_1 + 2$ by dividing the interval $[t_1, t_2]$ into intervals of length at least 1 and at most 2 and combining together the inequalities in (i) and (ii).

**Proof of Proposition 5.1.** Combining Lemmas 5.6, 5.7 and 5.8, and using the same argument developed in [7, Proposition 6.1], we can show that $|x(t)|$ is bounded on $[0, \infty)$. Briefly, we use the standard localized virial to find a sequence $t_n \to \infty$ with $\delta(t_n) \to 0$. Using the modulated virial and the fact that the integral of $\delta$ controls the variation of $x(\cdot)$, we can obtain that $|x(t)| \leq |x(t_n)|$ for all $t \geq t_n$ for some sufficiently large $N$.

Applying Lemma 5.7, we then obtain that there exists $C > 0$ so that

\[
\int_T^s \delta(t) \, dt \leq C\{\delta(T) + \delta(s)\}
\]

with $[T, s] \subset [0, \infty]$. Applying this with a sequence $t_n \to \infty$ such that $\delta(t_n) \to 0$, we find that $\int_T^\infty \delta(t) \, dt \leq C\delta(T)$ for all $T \geq 0$. Gronwall’s lemma then implies that there exists $\alpha, \beta > 0$ such that

\[
\int_T^\infty \delta(t) \, dt \leq \alpha e^{-\beta T}.
\]

The desired convergence then follows from Lemma 4.5.

\[\square\]
Proof of Corollary 5.2. Assume that \( u \) satisfies (5.1) and (5.3). Arguing as above, we can construct \( x(t) \) such that \( \{ u(t + x(t)) : t \in \mathbb{R} \} \) is pre-compact in \( H^1 \). Furthermore, we can prove that \( x(t) \) is bounded and
\[
\lim_{t \to -\infty} \delta(t) = \lim_{t \to \infty} \delta(t) = 0.
\]
Modifying the proof of Lemma 5.7, one obtains
\[
\int_{-n}^{n} \delta(t) \, dt \leq C(\delta(n) + \delta(-n)) \quad \text{for all } n \in \mathbb{N}.
\]
Sending \( n \to \infty \), we obtain \( V(u(t)) = \delta(t) = 0 \), contradicting (5.1).

6. Construction of local stable solutions

In this section, we establish the existence and uniqueness of the solution converging exponentially to the soliton \( P_{\infty} \).

We begin with the construction of some approximate solutions to the linearized equation (3.2). The proof is similar to that of [6, Proposition 6.3], so it will suffice to sketch the argument. We recall the notation for the eigenfunctions and eigenvalues of \( L \) introduced in Section 3 (see e.g., Lemma 3.3).

Proposition 6.1. Let \( a \in \mathbb{R} \). There exist \( \{ g_j^a \}_{j \geq 1} \) in \( S(\mathbb{R}^3) \) such that the following holds: writing \( g_1^a = ae_+ \) and
\[
W_k^a(t,x) := \sum_{j=1}^{k} e^{-j\lambda_j t} g_j^a(x) \quad \text{for } k \geq 1,
\]
we have
\[
\partial_t W_k^a + LW_k^a = iR(W_k^a) + O(e^{-(k+1)\lambda_1 t}) \quad \text{in } S(\mathbb{R}^3) \quad \text{as} \quad t \to \infty, \quad (6.1)
\]
where the nonlinear terms are defined in (3.1).

Proof. We prove this proposition by induction. To simplify notation, we omit most superscripts.

We define \( g_1 = ae_+ \) and \( W_1(t,x) := e^{-\lambda_1 t} g_1(x) \). It is clear that
\[
\partial_t W_1 + LW_1 - iR(W_1) = -R(W_1) = -iR(e^{-\lambda_1 t} g_1),
\]
Since \( iR(e^{-\lambda_1 t} g_1) = O(e^{-2\lambda_1 t}) \) we obtain (6.1) for \( k = 1 \).

Let \( k \geq 1 \). We assume that \( g_1, g_2, \ldots, g_k \) and the corresponding \( W_k \) satisfying (6.1) have been constructed. Using the explicit expression for \( R \), we can write
\[
R(W_k) = \sum_{j=2}^{5k} e^{-j\lambda_1 t} \psi_{j,k} \quad \text{with } \psi_{j,k} \in S(\mathbb{R}^3).
\]
Combining this with (6.1) we deduce that as \( t \to \infty \), we have
\[
\partial_t W_k + LW_k = iR(W_k) + e^{-(k+1)\lambda_1 t} V_{k+1} + O(e^{-(k+2)\lambda_1 t}) \quad \text{in } S(\mathbb{R}^3) \quad (6.2)
\]

Proof. We prove this proposition by induction. To simplify notation, we omit most superscripts.

We define \( g_1 = ae_+ \) and \( W_1(t,x) := e^{-\lambda_1 t} g_1(x) \). It is clear that
\[
\partial_t W_1 + LW_1 - iR(W_1) = -R(W_1) = -iR(e^{-\lambda_1 t} g_1),
\]
Since \( iR(e^{-\lambda_1 t} g_1) = O(e^{-2\lambda_1 t}) \) we obtain (6.1) for \( k = 1 \).

Let \( k \geq 1 \). We assume that \( g_1, g_2, \ldots, g_k \) and the corresponding \( W_k \) satisfying (6.1) have been constructed. Using the explicit expression for \( R \), we can write
\[
R(W_k) = \sum_{j=2}^{5k} e^{-j\lambda_1 t} \psi_{j,k} \quad \text{with } \psi_{j,k} \in S(\mathbb{R}^3).
\]
Combining this with (6.1) we deduce that as \( t \to \infty \), we have
\[
\partial_t W_k + LW_k = iR(W_k) + e^{-(k+1)\lambda_1 t} V_{k+1} + O(e^{-(k+2)\lambda_1 t}) \quad \text{in } S(\mathbb{R}^3) \quad (6.2)
\]
for some $V_{k+1} \in S(\mathbb{R}^3)$. Noting that $(k+1)\lambda_i$ is not in the spectrum of $\mathcal{L}$ (recall that $k \geq 1$), we define

$$g_{k+1} := -(\mathcal{L} - (k+1)\lambda_i)^{-1}V_{k+1}.$$ 

As $V_{k+1} \in S(\mathbb{R}^3)$, it follows that $g_{k+1} \in S(\mathbb{R}^3)$ (see Remark 7.2 in [6]). Now, we set

$$W_{k+1}(t,x) := W_k(t,x) + e^{-(k+1)\lambda_i t}g_{k+1}(x).$$

By definition of $W_{k+1}$ and (6.2), we get

$$\begin{align*}
\partial_t W_{k+1} + \mathcal{L}W_{k+1} - iR(W_{k+1})
&= iR(W_k) - iR(W_{k+1}) + O(e^{-(k+2)\lambda_i t}) \quad \text{in } S(\mathbb{R}^3).
\end{align*}$$

(6.3)

Since $W_j = O(e^{-\lambda_i t})$ for $j = 1, 2, \ldots k$ and $W_k - W_{k+1} = O(e^{-(k+1)\lambda_i t})$ in $S(\mathbb{R}^3)$ as $t \to \infty$, it follows that $iR(W_k) - iR(W_{k+1}) = O(e^{-(k+2)\lambda_i t})$ in $S(\mathbb{R}^3)$ as $t \to \infty$ (cf. (5.32)). Therefore, using the expansion (6.3) we have that (6.1) holds at $k+1$, which completes the induction. \hfill \Box

We will use the approximate solutions $W_k(t)$ to construct a true solution to (3.2); however, we first need to collect a few technical lemmas.

The linearized Equation (3.2) may be written as a Schrödinger equation

$$i\partial_t h + \Delta h - \omega h + [\Lambda_1(h) + iR_1(h)] + [\Lambda_2(h) + iR_2(h)] = 0,$$

(6.4)

where

$$\begin{align*}
\Lambda_1(h) &= 2P_{\omega}^2 h + P_{\omega}^2 \bar{h}, \\
\Lambda_2(h) &= -3P_{\omega}^4 h - 2P_{\omega}^4 \bar{h}, \\
R_1(h) &= |h|^2 h + P_{\omega}[2|h|^2 h + h^2], \\
R_2(h) &= -h|h|^4 + P_{\omega}[-2|h|^2 |h|^2 - 3|h|^4] - 2P_{\omega}^3 \left[\frac{h^2}{2} + 3|h|^2 h + \frac{3}{2} |h|^2 \bar{h}\right] \\
&\quad - 2P_{\omega}^3 \left[2|h|^2 + \frac{5}{2} h^2 + \frac{1}{2} (\bar{h})^2\right].
\end{align*}$$

We first have the following nonlinear estimates:

**Lemma 6.2.** Let $I$ be a finite interval of length $|I|$. Then, there exists $\varepsilon > 0$ and a constant $C$ independent of $I$ such that

$$||R_1(f) - R_1(g)||_{L_{x,t}^{2,\varepsilon}} \leq C||f - g||_{L_{t,x}^{2,\varepsilon}} \left(||f||_{L_{x,t}^{\varepsilon}L_{x,t}^{\varepsilon}} + 1\right)||f||_{L_{x,t}^{2,\varepsilon}} \left(||g||_{L_{t,x}^{\varepsilon}L_{t,x}^{\varepsilon}} + 1\right)||g||_{L_{t,x}^{2,\varepsilon}}$$

(6.5)

and
\[ \| R_2(f) - R_2(g) \|_{L_t^\infty H_x^s} \leq C(1 + |I|^2) \| f - g \|_{L_{tx}^{10}} \left[ \| g \|_{L_t^2 H_x^s} + \| f \|_{L_t^2 H_x^s} \right] \left( 1 + \sum_{j=1}^3 \| f \|_{L_{tx}^{10}}^j + \| g \|_{L_{tx}^{10}}^j \right) \] (6.6)

Moreover,
\[ C(1 + |I|^2) \| f - g \|_{L_t^2 H_x^s} \sum_{j=1}^3 \| f \|_{L_{tx}^{10}}^j, \]

where all spacetime norms are over \( I \times \mathbb{R}^3 \). In particular, we have
\[ \| R_1(f) \|_{L_t^\infty H_x^s} \leq C \| f \|_{L_t^2 H_x^s} \left[ 1 + \| f \|_{L_t^\infty X}^{10} \right] \] (6.7)
\[ \| R_2(f) \|_{L_t^\infty H_x^s} \leq C(1 + |I|^2) \| f \|_{L_t^2 H_x^s} \sum_{j=1}^4 \| f \|_{L_{tx}^{10}}^j. \]

Moreover,
\[ \| \Lambda_1(f) \|_{L_t^\infty H_x^s} \leq C |I|^2 \| f \|_{L_t^2 H_x^s}, \]
\[ \| \Lambda_2(f) \|_{L_t^\infty H_x^s} \leq C |I|^2 \| f \|_{L_t^2 H_x^s}. \] (6.8)

**Proof.** Let \( F(z) = |z|^p z \) for \( p \geq 1 \). We have the following pointwise estimate:
\[ |F(f) - F(g)| \leq |f - g|(|f|^p + |g|^p), \] (6.9)
\[ |\nabla F(f) - \nabla F(g)| \leq |f - g|(|f|^{p-1} + |g|^{p-1})(|\nabla f| + |\nabla g|) \] (6.10)
\[ + |\nabla f - \nabla g|(|f|^p + |g|^p). \] (6.11)

Moreover, using Hölder, we have
\[ \| fgh \|_{L_t^\infty H_x^s} \leq \| f \|_{L_{tx}^{10}} \| g \|_{L_t^\infty X} \| h \|_{L_t^{\infty} L_x^2}, \] (6.12)
\[ \| fghuv \|_{L_t^\infty H_x^s} \leq \| f \|_{L_{tx}^{10}} \| g \|_{L_t^{10}} \| h \|_{L_t^{10}} \| u \|_{L_x^{10}} \| v \|_{L_x^{10}}. \] (6.13)

Combining (6.9)–(6.13), we obtain (6.5)–(6.8). \( \square \)

Next, we record a useful integral summation argument from [6].

**Lemma 6.3.** Let \( a_0 > 0, t_0 > 0, p \in [1, \infty), E \) a normed vector space, and \( f \in L_t^{p} (\mathbb{R}^3; E) \). Suppose that there exist \( \tau_0 > 0 \) and \( C_0 > 0 \) with
\[ \| f \|_{L_t^{p}(t,t+\tau_0)} \leq C_0 e^{-a_0 t} \text{ for all } t \geq t_0. \]

Then for all \( t \geq t_0, \)
\[ \| f \|_{L_t^{p}(t,\infty)} \leq \frac{C_0 e^{-a_0 t}}{1 - e^{-a_0 \tau_0}}. \]

We now construct true solutions to (NLS) that are close to the soliton as \( t \to \infty \).
Proposition 6.4. Let $a \in \mathbb{R}$. There exist $k_0 > 0$ and $t_k \geq 0$ such that for any $k \geq k_0$, there exists a solution $W^a$ to (NLS) such that for $t \geq t_k$, we have
\[
\|W^a(t) - U^a_k(t)\|_{H^1} + \|W^a - U^a_k\|_{L^2 H^1 \cap L^1_t H^{1,2}((t, \infty) \times \mathbb{R})} \leq e^{-(k+1)\lambda_1 t},
\] (6.14)
where
\[
U^a_k(t) := e^{i\lambda t}(P_0 + W^a_k(t)).
\]
In addition, $W^a$ is the unique solution to equation (NLS) satisfying (6.14) for large $t$. Finally, $W^a$ is independent of $k$ and satisfies for large $t$,
\[
\|W^a(t) - e^{i\lambda t}P_0 - ae^{i\lambda t}e^{-\lambda_1 t}e_+\|_{H^1} \leq e^{-\frac{1}{2}\lambda_1 t}.
\] (6.15)

Remark 6.5. Let $a \neq 0$. Then $V(W^a(t)) > 0$ for all $t \in \mathbb{R}$. Indeed, by (6.15), conservation of mass and energy, we first note that
\[
M(W^a) = M(P_0) \quad \text{and} \quad E(W^a) = E(P_0).
\]
Now suppose that there exists $t_0$ such that $V(W^a(t_0)) = 0$. As $M(W^a(t_0)) = M(P_0)$ and $E(W^a(t_0)) = E(P_0)$, the variational characterization from [2] and uniqueness for (NLS) implies that $W^a(t) = e^{i\theta}e^{i\lambda t}P_0(\cdot - y_0)$ for some $\theta \in \mathbb{R}$ and $y_0 \in \mathbb{R}^3$. In this case, (6.15) shows that $\theta = 0$ and $y_0 = 0$, so that $W^a(t) = e^{i\lambda t}P_0$. Then (6.15) and the fact that $a \neq 0$ imply
\[
\|e_+\|_{H^1} \leq e^{-\frac{1}{2}\lambda_1 t},
\]
for large $t > 0$, which is a contradiction.

Proof. Define
\[
\Lambda(h) = 2P^a_0 h + P^a_0 \bar{h} - 3P^a_0 h - 2P^a_0 \bar{h},
\]
and recall the functions $W^a_k$ constructed in Proposition 6.1, which satisfy
\[
\varepsilon_k := \partial_t W^a_k + \mathcal{L}W^a_k - iR(W^a_k) = O(e^{-(k+1)\lambda_1 t}) \quad \text{in} \quad \mathcal{S}([\mathbb{R}^3]).
\] (6.16)
We wish to construct a suitable solution to
\[
i\partial_t v + \Delta v - \omega v = -\Lambda(W^a_k + v) + \Lambda(W^a_k) + R(W^a_k + v) - R(W^a_k) - \varepsilon_k.
\] (6.17)
Indeed, this equation may equivalently be written as
\[
\partial_t v + \mathcal{L}v = -iR(W^a_k + v) + iR(W^a_k) - \varepsilon_k,
\]
from which we can deduce that $W^a_k + v$ solves (3.2). In particular the desired solution to (NLS) may be defined as
\[
W^a(t) = e^{i\lambda t}P_0 + e^{i\lambda t}[W^a_k(t) + v(t)].
\]
We construct the solution to (6.17) via a fixed point argument. We define the operator
\[
[\Phi v](t) := -\int_t^\infty e^{i(t-s)\Delta}[-\Lambda(v(s)) + R(W^a_k(s) + v(s)) - R(W^a_k(s)) + \varepsilon_k(s)] \, ds\quad (6.18)
\]
and the spaces
\[
X(t, \infty) := L^2_t H^{1,6}_x \cap L^1_t H^{1,6}_x((t, \infty) \times \mathbb{R}^3),
\]
\[
N(t, \infty) := L^\infty_t H^{1,6}_x((t, \infty) \times \mathbb{R}^3) + L^{\frac{5}{3}}_t H^{1,6}_x((t, \infty) \times \mathbb{R}^3).
\]

We now fix \(k\) and \(t_k\). We will show that the map \(\Phi\) defined above is a contraction on the Banach space
\[
B^k := \left\{ v \in E^k, ||v||_{E^k} \leq 1 \right\},
\]
where
\[
E^k := \left\{ v \in C([t_k, \infty)) \cap X(t_k, \infty), ||v||_{E^k} < \infty \right\},
\]
\[
||v||_{E^k} = \sup_{t \geq t_k} e^{(k+\frac{1}{2})\lambda t} \left[ ||v(t)||_{H^1_t} + ||v||_{X(t, \infty)} \right].
\]

By Strichartz, for \(v, u \in B^k\), we have
\[
||\Phi v(t)||_{H^1_t} + ||\Phi v||_{X(t, \infty)} \\
\leq C^* \left[ ||\Lambda(v)||_{N(t, \infty)} + ||R(W^a_k + v) - R(W^a_k)||_{N(t, \infty)} + ||\bar{v}_k||_{N(t, \infty)} \right], \quad (6.19)
\]
\[
||\Phi v(t) - \Phi u(t)||_{H^1_t} + ||\Phi v - \Phi u||_{X(t, \infty)} \\
\leq C^* \left[ ||\Lambda(v - u)||_{N(t, \infty)} + ||R(W^a_k + v) - R(W^a_k + u)||_{N(t, \infty)} + ||\bar{v}_k||_{N(t, \infty)} \right]. \quad (6.20)
\]

Here \(C^*\) encodes the various constants appearing in the Strichartz estimates.

We now need the following:

**Claim 6.6.** Let \(v, u \in E^k\). There exists \(k_0 > 0\) such that for \(k \geq k_0\), we have
\[
||\Lambda(u - v)||_{N(t, \infty)} \leq \frac{1}{4C^*} e^{-(k+\frac{1}{2})\lambda t} ||u - v||_{E^k}, \quad (6.21)
\]
\[
||R(W^a_k + v) - R(W^a_k + u)||_{N(t, \infty)} \leq C_k e^{-(k+\frac{1}{2})\lambda t} ||u - v||_{E^k}, \quad (6.22)
\]
\[
||\bar{v}_k||_{N(t, \infty)} \leq C_k e^{-(k+1)\lambda t}, \quad (6.23)
\]
for all \(t \geq t_k\), where the constant \(C_k\) depends only on \(k\).

**Proof of Claim 6.6.** We first estimate (6.21). Let \(\tau_0 > 0\). Using (6.8), we have
\[
||\Lambda(u - v)||_{N(t, t + \tau_0)} \leq C_1 \tau_0^2 e^{-(k+\frac{1}{2})\lambda t} ||u - v||_{E^k}.
\]

We then obtain (6.21) from Claim 6.6 for \(k \geq k_0\) by choosing \(\tau_0\) and \(k_0\) appropriately.

Now we show (6.22). Recall that by construction \(||W^a_k(t)||_{L^1 \cap X(t, \infty)} \leq C_k e^{-\lambda t}\) (cf. Proposition 6.1). By (6.5) and (6.6) (and recalling that \(I = [t, t + 1]\)), we deduce that for \(v, u \in E^k\),

\[
||\Lambda(u - v)||_{N(t, \infty)} \leq C_1 \tau_0^2 e^{-(k+\frac{1}{2})\lambda t} ||u - v||_{E^k}.
\]
\[ ||R(W^u_k + v) - R(W^u_k + u)||_{N(t,T+t)} \leq C_k e^{-\lambda_1 t} ||u - v||_{H^1_x (\mathbb{R}^3)} \]
\[ \leq C_k e^{-(k+\frac{1}{2})t} ||u - v||_{L^6_x (\mathbb{R}^3)} , \]

where the constant \( C_{k,2} \) depends only on \( k \). Thus, Lemma 6.3 implies (6.22).

Finally, the estimate (6.23) is a direct consequence of (6.16).

Now let \( k \geq k_0 \), where \( k_0 \) is defined in Claim 6.6. Combining (6.19), (6.20), (6.21), (6.22) and (6.23) we get for all \( t \geq t_k \),
\[ ||\Phi v||_{E^k} \leq \left( \frac{1}{4} + C^* C_k e^{-\lambda_1 t_k} + C^* C_k e^{-\frac{1}{2} \lambda_1 t_k} \right) , \]
\[ ||\Phi v - \Phi u||_{E^k} \leq ||u - v||_{E^k} \left( \frac{1}{4} + C^* C_k e^{-\lambda_1 t_k} \right) , \]

so that \( \Phi \) is a contraction provided \( t_k \) is chosen sufficiently large. Therefore, for \( k \geq k_0 \), equation (NLS) has a unique solution \( W^u \) satisfying the estimate (6.14) for \( t \geq t_k \). Note also that all of the above still remains valid for larger \( t_k \); in particular, the uniqueness still holds in the class of solution of (NLS) satisfying (6.14) for \( t > t'_k \) with \( t'_k \) is a real number larger than \( t_k \). Moreover, by the uniqueness in the fixed point argument, one can also show that \( W^u \) does not depend on \( k \) (cf. [6, Proposition 6.3]).

Finally, we prove (6.15). By construction we have \( ||\Phi v||_{H^1} \leq C e^{-(k+\frac{1}{2}) \lambda_1 t} \). As \( e^{it\alpha} v = W^u - U^u_k \) and \( v = \Phi v \), we obtain
\[ ||W^u(t) - U^u_k(t)||_{H^1} \leq C e^{-(k+\frac{1}{2}) \lambda_1 t} . \]
This, together with the fact that \( U^u_k = e^{it\alpha} P_{\omega} + ae^{it\alpha} e^{-\lambda_1 t} e_+ + O(e^{-2\lambda_1 t}) \) in \( S(\mathbb{R}^3) \), yields (6.15).

\[ \square \]

7. A uniqueness result

Our first main goal in this section is to establish the following:

Proposition 7.1. Let \( (M(P_{\omega}), E(P_{\omega})) \in \partial K_s \). If \( u \) is a solution to (NLS) satisfying
\[ E(u) = E(P_{\omega}), M(u) = M(P_{\omega}), \text{ and } ||u - e^{it\alpha} P_{\omega}||_{H^1} \leq C e^{-\alpha t} \]
for some \( C, c > 0 \), then there exists unique \( a \in \mathbb{R} \) such that \( u = W^a \), where \( W^a \) is the solution of (NLS) constructed in Proposition 6.4.

We begin with a lemma.

Lemma 7.2. Let \( v \) be a solution of (6.4) with
\[ ||v(t)||_{H^1_x} \leq C e^{-\epsilon_0 t} \]
for some constants \( C > 0 \) and \( \epsilon_0 > 0 \). Then for any admissible pair \( (q, r) \) we have for \( t \) large
\[ ||v||_{L^q_x L^r_x((t,\infty) \times \mathbb{R}^3)} + ||v||_{L^q_x H^1_x((t,\infty) \times \mathbb{R}^3)} \leq C e^{-\epsilon_0 t} . \]
Proof. Given $t > 0$ and $\tau \in (0, 1)$, let us write

$$H(t) = \|v\|_{L^2_t H^1_x \cap L^{10}_t H^{1/2}_x((t,t+\tau)\times\mathbb{R}^3)}.$$  

By Strichartz and Lemma 6.2, we have

$$H(t) \leq K\left\{\|v(t)\|_{H^1_x} + [H(t)]^2 + [H(t)]^5 + \tau^2 H(t)\right\}$$  \hspace{1cm} (7.4)

for some $\alpha > 0$. Using (7.2), a continuity argument implies that there exists $\tau > 0$ such that

$$\|v\|_{L^2_t H^1_x \cap L^{10}_t H^{1/2}_x((t,t+\tau)\times\mathbb{R}^3)} \leq Ce^{-c_0 t}$$  \hspace{1cm} (7.5)

for large $t$. Lemma 6.3 and Sobolev embedding then yield

$$\|v\|_{L^2_t H^1_x \cap L^{10}_t H^{1/2}_x((t,t+\tau)\times\mathbb{R}^3)} \leq Ce^{-c_0 t}.$$  \hspace{1cm} \Box

Let $t_0 \geq 0$ and suppose that we have functions

$$v \in C^0\left([t_0, \infty), H^1_x(\mathbb{R}^3)\right), \quad g_1, g_2 \in C^0\left([t_0, \infty), L^2_t(\mathbb{R}^3)\right)$$

satisfying the following:

$$g_1 \in L^2_t H^{1/2}_x \left([t_0, \infty) \times \mathbb{R}^3\right), \quad g_2 \in L^{10}_t H^{1/2}_x \left([t_0, \infty) \times \mathbb{R}^3\right),$$

and

$$\partial_t v + \mathcal{L}v = g_1 + g_2, \quad (t, x) \in [t_0, \infty) \times \mathbb{R}^3,$$  \hspace{1cm} (7.6)

$$\|v(t)\|_{H^1_x} \leq Ce^{-c_1 t},$$  \hspace{1cm} (7.7)

$$\|g_1 + g_2\|_{L^2_t(\mathbb{R}^3)} + \|g_1\|_{L^2_t H^{1/2}_x \left([t, \infty) \times \mathbb{R}^3\right)} + \|g_2\|_{L^{10}_t H^{1/2}_x \left([t, \infty) \times \mathbb{R}^3\right)} \leq Ce^{-c_2 t}$$  \hspace{1cm} (7.8)

for all $t \geq t_0$, where $0 < c_1 < c_2$.

Using Strichartz estimates, (6.8), and Lemma 6.3, we can obtain the following result.

Lemma 7.3. Under the above assumptions (7.6), (7.7) and (7.8) with $0 < c_1 < c_2$, we have

$$\|v\|_{L^2_t H^{1/2}_x ([t, \infty) \times \mathbb{R}^3)} \leq Ce^{-c_1 t}$$  \hspace{1cm} (7.9)

for any admissible pair $(q, r)$.

In what follows, we adopt the following notation: given $c > 0$, we denote by $c^-$ a positive number arbitrary close to $c$ and such that $0 < c^- < c$. Recall that $\lambda_1 > 0$ denotes the eigenvalue of the linearized operator, as introduced in Lemma 3.3.

Lemma 7.4. Consider $v$, $g_1$ and $g_2$ satisfying assumptions (7.6), (7.7), (7.8) with parameters $0 < c_1 < c_2$. Then for any admissible pair $(q, r)$, we have:
(i) If \( \lambda_1 \not\in [c_1, c_2] \), then
\[
\|v(t)\|_{H^1} + \|v\|_{L^2_0 H^r_x((t,\infty) \times \mathbb{R}^3)} \leq Ce^{-c_1 t}. \tag{7.10}
\]

(ii) If \( \lambda_1 \in [c_1, c_2] \), then there exists \( a \in \mathbb{R} \) such that
\[
\|v(t) - ae^{-\lambda_1 t} e_{+}\|_{H^1} + \|v - ae^{-\lambda_1 t} e_{+}\|_{L^2_0 H^r_x((t,\infty) \times \mathbb{R}^3)} \leq Ce^{-c_2 t}. \tag{7.11}
\]

**Proof.** We closely follow the argument in [6, Proposition 5.9] and [7, Lemma 7.2]. Let \( Y^\perp \) be the set of \( h \in H^1(\mathbb{R}^3) \) satisfying the orthogonality relations (3.21) and (3.22). We decompose \( v \) as
\[
v(t) = \alpha_+(t)e_{+} + \alpha_-(t)e_{-} + \sum_{j=0}^{3} \beta_j(t) P_{o,j} + v^\perp(t), \tag{7.12}
\]
where \( v^\perp(t) \in Y^\perp \) and
\[
P_{o,0} = \frac{i P_{o}}{\|P_o\|_{L^2}}, \quad P_{o,j} = \frac{\partial_j P_{o}}{\|\partial_j P_{o}\|_{L^2}} \quad \text{for} \quad j = 1, 2, 3.
\]

By Remark 3.6, we can normalize the eigenfunctions \( e_{\pm} \) so that \( \mathcal{F}(e_{+}, e_{-}) = 1 \) (recall the notation \( \mathcal{F} \) from (3.5)). Moreover, from Remark 3.5 and definition of \( Y^\perp \) we have that
\[
\alpha_+(t) = \mathcal{F}(v(t), e_{-}), \quad \alpha_-(t) = \mathcal{F}(v(t), e_{+}),
\]
\[
\beta_j(t) = (v(t) - \alpha_+(t)e_{+} - \alpha_-(t)e_{-}, P_{o,j})_{L^2} \quad \text{for} \quad j = 0, 1, 2, 3.
\]

**Step 1. Decay estimates.** By condition (7.8) and following the same argument developed in [7, Lemma 7.2, Step 1], one can show that
\[
|\alpha_-'(t) - \lambda_1 \alpha_-(t)| \leq Ce^{-c_1 t}, \quad |\alpha_+'(t) + \lambda_1 \alpha_+(t)| \leq Ce^{-c_2 t}, \tag{7.13}
\]
\[
|\beta_j'(t)| \leq C(|v^\perp(t)|_{L^2} + e^{-c_1 t}), \tag{7.14}
\]
and
\[
|\alpha_+(t)| \leq e^{-c_2 t} \quad \text{if} \quad c_2 \leq \lambda_1 \tag{7.15}
\]
\[
|\alpha_+(t) - ae^{-\lambda_1 t}| \leq e^{-c_1 t} \quad \text{if} \quad c_2 > \lambda_1, \tag{7.16}
\]
where
\[
a := \lim_{t \to \infty} e^{\lambda_1 t} \alpha_+(t). \tag{7.17}
\]

Next, we show that
\[
\mathcal{F}(v(t)) \leq Ce^{-(c_1 + c_2)t}. \tag{7.18}
\]

Indeed, notice that, since \( \mathcal{F}(\mathcal{L}v, v) = 0 \) (cf. Remark 3.5) we have
\[
\frac{d}{dt} \mathcal{F}(v(t)) = 2\mathcal{F}(\partial_t v(t), v(t)) = -2\mathcal{F}(\mathcal{L}v, v) + 2\mathcal{F}(g_1 + g_2, v) = 2\mathcal{F}(g_1 + g_2, v). \tag{7.19}
\]
Moreover, by the definition of the quadratic form $\mathcal{F}$, we have for any time-interval $|I|$ with $|I| < \infty$,
\[
\int_I |\mathcal{F}(g_1(t) + g_2(t), v(t))| \, dt = \left\| |g_1| + \frac{1}{2} |v| \right\|_{L_t^2 H^\alpha_x^1} + \left\| |g_2| + \frac{1}{2} |v| \right\|_{L_t^{10/9} H_x^{13/9}}^2 + \left\| |g_1| + g_2 \right\|_{L_t^{\infty} L_x^2} \left\| |v| \right\|_{L_t^{10} H_x^{13}}.
\]  
(7.20)
where all spacetime norms are over $I \times \mathbb{R}^3$. Thus, by conditions (7.7)-(7.8) and Lemma 7.3, we get
\[
\int_t^{t+1} |\mathcal{F}(g_1(t) + g_2(t), v(t))| \, dt \leq Ce^{-(c_1+c_2)t}.
\]  
In this case, Lemma 6.3 implies that
\[
\int_t^\infty |\mathcal{F}(g_1(t) + g_2(t), v(t))| \, dt \leq Ce^{-(c_1+c_2)t}.
\]  
As $|\mathcal{F}(v(t))| \leq \left\| v(t) \right\|_{H^1}^2 \to 0$ as $t \to \infty$ (cf. (7.7)), from (7.19) and inequality above we have
\[
|\mathcal{F}(v(t))| \leq \int_t^\infty |\mathcal{F}(g_1(t) + g_2(t), v(t))| \, dt \leq Ce^{-(c_1+c_2)t}.
\]  
This proves (7.18).

**Step 2. Proof in the case when either $\lambda_1 \geq c_2$, or $\lambda_1 < c_2$ and $a = 0$.** Combining (7.13), (7.15) and (7.16) one can obtain (see [7, Lemma 7.2, Step 3])
\[
|\chi_+(t)| + |\dot{\chi}_+(t)| \leq Ce^{-c_2 t} \quad (7.21)
\]
\[
|\chi_-(t)| + |\dot{\chi}_-(t)| \leq Ce^{-c_2 t}. \quad (7.22)
\]
On the other hand, since $\mathcal{F}(e_+, v^\bot) = \mathcal{F}(e_-, v^\bot) = 0$ (recall $v^\bot \in Y^\bot$), $\mathcal{F}(e_+, e_-) = 1$, and $\mathcal{F}(e_+) = \mathcal{F}(e_-) = 0$ we obtain
\[
\mathcal{F}(v) = \mathcal{F}(v^\bot) + 2\chi_+ \chi_-.
\]
Thus, by (7.21), (7.22) and (7.18), Proposition 3.4 implies that
\[
\left\| v^\bot(t) \right\|_{H^1} \leq \sqrt{\left| \mathcal{F}(v^\bot) \right|} \leq Ce^{-(\frac{1}{2}c_1 + c_2)t}. \quad (7.23)
\]
Using (7.14) we can therefore obtain the following estimate of $\beta_j(t)$,
\[
|\beta_j(t)| \leq Ce^{-\frac{1}{2}(c_1 + c_2)t}, \quad \text{for } j = 0, 1, 2, 3. \quad (7.24)
\]
From the decomposition (7.12) and summing up estimates (7.21), (7.22), (7.23) and (7.24), we arrive at
\[
\left\| v(t) \right\|_{H^1} \leq Ce^{-(\frac{1}{2}c_1 + c_2)t}.
\]  
Therefore, $v$, $g_1$ and $g_2$ satisfies (7.6), (7.7) and (7.8) with $c_1$ replaced by $c_1^* = \frac{(c_1 + c_2)}{2}$. An iteration argument now yields
\[ ||v(t)||_{H^1} \leq Ce^{-\lambda_1 t}. \]  

(7.25)

In this case, using Lemma 7.3, we obtain the estimate (7.10)-(7.11).

**Step 3. Proof in the remaining cases.** If \( \lambda_1 < c_1 \), then we have \( \lambda_1 < c_2 \) and \( a = 0 \), and hence we obtain the estimate in (ii) with \( a = 0 \) using Step 2. Thus, it suffices to consider the case \( c_1 \leq \lambda_1 < c_2 \) and \( a \neq 0 \).

We set

\[ \nu(t) = v(t) - ae^{-\lambda_1 t}e_+. \]

Note that

\[ \partial_t \nu(t) + L \nu(t) = g_1(t) + g_2(t), \quad ||\nu(t)||_{H^1} \leq Ce^{-c_1 t}. \]

(7.25)

Let \( \tilde{x}_+(t) = F(\nu(t), e_-) \). From (7.16) we infer that

\[ |e^{i\lambda t} \tilde{x}_+(t)| \leq Ce^{-(c_2 - \lambda_1) t} \to 0 \quad \text{as} \quad t \to \infty. \]

Therefore, \( \tilde{x}_+(t), g_1 \) and \( g_2 \) satisfy all the assumptions of Step 2 (cf. (7.17)), hence, we have

\[ ||\nu(t) - ae^{-\lambda_1 t}e_+||_{H^1} = ||\nu(t)||_{H^1} \leq Ce^{-c_1 t}, \]

and Lemma 7.3 implies the estimate (7.11). \( \square \)

We will also need the following lemma, whose proof is standard.

**Lemma 7.5.** Let \( f, g \) functions in \( H^1(\mathbb{R}^3) \). Define \( R(\cdot) \) as in (3.1). Then

\[ ||R(f) - R(g)||_{L^5} \leq C||f - g||_{L^6}^5 ||f||_{H^1} + ||g||_{H^1} + ||f||_{H^1}^4 + ||g||_{H^1}^4. \]  

(7.26)

We turn now to the proof of Proposition 7.1.

**Proof of Proposition 7.1.** Let \( u = e^{it\Omega}(P_\omega + h) \) be a solution of (NLS) satisfying (7.1).

Note that \( h \) satisfies the linearized equation

\[ \partial_t h + Lh = iR_1(h) + iR_2(h). \]

In particular, \( h \) satisfies the Eq. (6.4).

**Step 1.** We will show that there exists \( a \in \mathbb{R} \) such that

\[ ||h(t) - ae^{-\lambda_1 t}e_+||_{H^1} + ||h(t) - ae^{-\lambda_2 t}e_+||_{L^5_xH^{1/2}([t,\infty) \times \mathbb{R}^3)} \leq Ce^{-2\lambda_1 t}. \]

(7.27)

First, we show that

\[ ||h(t)||_{H^1} \leq Ce^{-\lambda_1 t}. \]

(7.28)

and

\[ ||R_1(h) + R_2(h)||_{L^5_x} + ||R_1(h)||_{L^5_xH^{1/2}([t,\infty) \times \mathbb{R}^3)} + ||R_2(h)||_{L^5_xH^{1/2}([t,\infty) \times \mathbb{R}^3)} \leq Ce^{-2\lambda_1 t}. \]

(7.29)
Indeed, by (7.1) and Lemma 7.2 we obtain for any admissible pair \((q, r)\),
\[
\|h\|_{L^1_q((t, \infty) \times \mathbb{R}^3)} + \|h\|_{L^q_tH^r_x((t, \infty) \times \mathbb{R}^3)} \leq C e^{-ct}.
\] (7.30)

Thus, by Lemmas 7.5 and 6.2 we see that
\[
\|R_1(h) + R_2(h)\|_{L^6_t(\mathbb{R}^3)} \leq C e^{-2ct}.
\] (7.31)

\[
\|R_1(h)\|_{L^{20}_tH^\frac{20}{3}(t, \infty) \times \mathbb{R}^3)} + \|R_2(h)\|_{L^{10}_tH^\frac{10}{3}(t, \infty) \times \mathbb{R}^3)} \leq C e^{-2ct}.
\] (7.32)

Moreover, by using Lemma 6.3 and (7.32) we obtain
\[
\|R_1(h)\|_{L^{20}_tH^\frac{20}{3}(t, \infty) \times \mathbb{R}^3)} + \|R_2(h)\|_{L^{10}_tH^\frac{10}{3}(t, \infty) \times \mathbb{R}^3)} \leq C e^{-2ct}.
\] (7.33)

Thus \(h\) satisfies the conditions in Lemma 7.4 with \(g_1 = iR_1(h)\), \(g_2 = iR_2(h)\), \(c_1 = c\), and \(c_2 = 2c\). Therefore, we obtain
\[
\|h(t)\|_{H^1} \leq \mathcal{C}(e^{-\lambda_1 t} + e^{-\frac{c}{2} t}).
\]

If \(\frac{3}{2} c > \lambda_1\), it is clear that (7.28) holds. If not, by using (7.1) (with \(\frac{3}{2} c\) instead of \(c\)), Lemma 7.4 and an iteration argument gives the estimate (7.28). Moreover, combining Lemma 7.2, estimates (6.7) and (7.26), and Lemma 6.3 we get (7.29).

Thus, we can apply Lemma 7.4 to obtain (7.27).

**Step 2.** We will use the induction argument to show that there exists \(t_0 \geq 0\) such that for all \(m > 0\),
\[
\|u(t) - W^a(t)\|_{H^1_q} + \|u - W^a\|_{L^2_tH^6_x \cap L^{10}_tH^\frac{10}{3}(t, \infty) \times \mathbb{R}^3)} \leq e^{-mt} \quad \text{for all } t \geq t_0,
\] (7.34)

where \(W^a = e^{i\omega t}(P_\omega + h^a)\) (recall that \(W^a\) is the solution to \((\text{NLS})\) defined in Proposition 6.4). Indeed, combining (7.27) and estimate (6.14), we have that (7.34) holds with \(m = \frac{3}{2} \lambda_1\). Now we show that if (7.34) holds for some \(m = m_1 > \lambda_1\), then it also holds for \(m = m_1 + \lambda_1\). Indeed, recalling that \(u = e^{i\omega t}(P_\omega + h)\) and \(W^a = e^{i\omega t}(P_\omega + h^a)\), we note that \(h - h^a\) satisfies the equation
\[
\partial_t(h - h^a) + \mathcal{L}(h - h^a) = iR(h) - iR(h^a).
\]

Since by hypothesis we have
\[
\|h(t) - h^a(t)\|_{H^1_q} + \|h - h^a\|_{L^2_tH^6_x \cap L^{10}_tH^\frac{10}{3}(t, \infty) \times \mathbb{R}^3)} \leq e^{-m_1 t},
\]

it follows from (7.28) and (6.15), and Lemma 7.5,
\[
\|R(h) - R(h^a)\|_{L^\frac{6}{5}_x} \leq C e^{-(\lambda_1 + m_1) t}.
\]

Similarly, combining Lemmas 6.2, 6.3 and 7.2 we get
\[
\|R_1(h) - R_1(h^a)\|_{L^{20}_tH^\frac{10}{3}(t, \infty) \times \mathbb{R}^3)} + \|R_1(h) - R_1(h^a)\|_{L^{10}_tH^\frac{10}{3}(t, \infty) \times \mathbb{R}^3)} \leq C e^{-(\lambda_1 + m_1) t}.
\]
Thus, from Lemma 7.4 we obtain
\[
||h(t) - h^a(t)||_{H^1_x} + ||h - h^a||_{L^2_tH^{1,6}_x \cap L^{30}_tH^{1,\infty}_x((t,\infty) \times \mathbb{R}^3)} \leq e^{- \frac{m_1 + \frac{1}{2}}{2}} t,
\]
which implies (7.34) with \( m = m_1 + \frac{1}{2} \). Thus, estimate (7.34) follows by iteration for all \( m > 0 \).

Finally, we are ready to finish the proof of the proposition. Combining (7.34) with \( m = (k_0 + 1)\lambda_1 \), where \( k_0 \) is defined in Proposition 6.4, and (6.14) we find that
\[
||u(t) - U^a_{k_0}(t)||_{H^1_x} + ||u - U^a_{k_0}||_{L^2_tH^{1,6}_x \cap L^{30}_tH^{1,\infty}_x((t,\infty) \times \mathbb{R}^3)} \leq e^{-(k_0 + \frac{1}{2})\lambda_1 t},
\]
for large \( t \). By the uniqueness in Proposition 6.4, we finally obtain \( u = W^a \).

Our next goal is to show that if \( u \) is a solution as in Proposition 7.1 with positive virial, then \( u = W^a \) for some \( a \geq 0 \). We begin with the following lemma:

**Lemma 7.6.** Let \((M(P_o), E(P_o)) \in \partial K_s\). Suppose \( u_0 \in H^1(\mathbb{R}^3) \), with
\[
E(u_0) = E(P_o), \quad M(u_0) = M(P_o), \quad \text{and} \quad V(u_0) > 0.
\]
Then we have
\[
||\nabla u(t)||^2_{L^2} \leq ||\nabla P_o||^2_{L^2} \quad \text{for all} \quad t \in \mathbb{R},
\]
where \( u(t) \) is the corresponding solution to (NLS) with initial data \( u_0 \).

**Proof.** Using Corollary 5.2, we can assume that
\[
||u||_{L^1_{t,x}((-\infty,0) \times \mathbb{R}^3)} < \infty,
\]
i.e., the solution \( u(t) \) scatters for negative time. Therefore, there exists \( \psi_- \in H^1(\mathbb{R}^3) \) such that
\[
\lim_{t \to -\infty} ||u(t) - e^{it\Lambda} \psi_-||_{H^1} = 0.
\]
Note also that
\[
E(u(t)) + \frac{1}{2} \omega M(u(t)) = E(P_o) + \frac{1}{2} \omega M(P_o) = \frac{1}{3} ||\nabla P_o||^2_{L^2} \quad \text{for all} \quad t \in \mathbb{R}.
\]
Since \( u(t) - e^{it\Lambda} \psi_- \to 0 \) in \( H^1 \) and \( e^{it\Lambda} \psi_- \to 0 \) in \( L^4 \) as \( t \to -\infty \), we see that
\[
\lim_{t \to -\infty} ||\nabla u(t)||^2_{L^2} = ||\nabla \psi_-||^2_{L^2},
\]
and
\[
\frac{1}{2} ||\nabla \psi_-||^2_{L^2} + \frac{1}{2} \omega M(P_o) \leq \frac{1}{3} ||\nabla P_o||^2_{L^2},
\]
which implies that there exists \( t_0 \in (-\infty,0) \) such that
\[
||\nabla u(t_0)||^2_{L^2} \leq \frac{2}{3} ||\nabla P_o||^2_{L^2} < ||\nabla P_o||^2_{L^2}
\]
(recall \( \omega M(P_o) > 0 \)). Therefore, by Remark 2.7 we get \( ||\nabla u(t)||^2_{L^2} < ||\nabla P_o||^2_{L^2} \) for all \( t \in \mathbb{R} \).
Proposition 7.7. If $u$ satisfies the assumption (7.1) and $V(u_0) > 0$, then $u = W^a$ for some unique $a \geq 0$.

Proof. By Proposition 7.1, it is enough to show that there is no solution $W^a$ to (NLS) satisfying (6.15) with $a < 0$. Indeed, suppose $W^a$ were such a solution. Using (6.15), a direct calculation leads to

$$||\nabla W^a(t)||_{L^2}^2 = ||\nabla P_{\varnothing}||_{L^2}^2 + ae^{-\frac{3}{2}t} \int_{\mathbb{R}^3} \nabla P_{\varnothing}e_1 dx + O(e^{-\frac{3}{2}t}).$$

From Remark 3.7, we get $\int_{\mathbb{R}^3} \nabla P_{\varnothing}e_1 dx \neq 0$. Replacing $e_+$ by $-e_+$ if necessary, we can assume that

$$\int_{\mathbb{R}^3} \nabla P_{\varnothing}e_1 dx < 0.$$

Thus, if $a < 0$, then $||\nabla W^a(t)||_{L^2}^2 > ||\nabla P_{\varnothing}||_{L^2}^2$ for large $t$, which is a contradiction to Lemma 7.6. Therefore $a \geq 0$.

As a corollary of Propositions 7.1 and 7.7, we see that (modulo time translation and rotation) all of the functions $W^a$ with $a > 0$ are the same.

Corollary 7.8. Let $a > 0$. Then there exists $\theta_a, T_a \in \mathbb{R}$ such that

$$W^a = e^{i\theta_a} W^1(t - T_a).$$

Proof. Let $a > 0$ and choose $T_a$ such that $ae^{-\frac{3}{2}T_a} = 1$. Then by estimate (6.15) we get

$$||e^{-i\theta T_a} W^a(t + T_a) - e^{i\theta T_a} P_{\varnothing} - e^{i(\theta - \frac{3}{2})T_a} e_+||_{H^1} \leq e^{-\frac{3}{2}T_a}.$$  \hspace{1cm} (7.37)

Note that $e^{-i\theta T_a} W^a(t + T_a)$ satisfies the assumption of Proposition 7.1. Thus, there exists $\tilde{a} \geq 0$ such that

$$e^{-i\theta T_a} W^a(t + T_a) = W^{\tilde{a}}.$$

From (7.37) and Proposition 6.4 we infer that $\tilde{a} = 1$. Therefore,

$$W^a = e^{i\theta T_a} W^1(t - T_a).$$

8. Proofs of the main results

Finally, we are in a position to establish the main results, namely, Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.4. We let $(m, e) = (M(P_{\varnothing}), E(P_{\varnothing})) \in \partial \mathcal{K}_s$. We set

$$\mathcal{G}_\varnothing(t) := W^1(t),$$

where $W^1$ is the global solution to (NLS) defined in Proposition 6.4. By Remark 6.5 we see that $V(\mathcal{G}_\varnothing(t)) > 0$ for all $t \in \mathbb{R}$. Moreover, by Corollary 5.2 we obtain that the
solution $G_\omega(t)$ scatters for negative time. If not, applying the above arguments to the solution $\overline{G_\omega}(x,-t)$ we get the desired result. □

Proof of Theorem 1.5. Let $u$ be a solution to (NLS) such that $$(M(u), E(u)) = (m, e) \in \partial \mathcal{K}_s$$ and $V(u(0)) > 0$.

Suppose that $u$ does not scatter, i.e., $\|u\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{R}^3)} = \infty$. Then, replacing if necessary $u(t)$ by $\overline{u(-t)}$, Proposition 5.1 implies that there exist $\theta \in \mathbb{R}$, $y_0 \in \mathbb{R}^3$, $c$ and $C > 0$ such that

$$\|e^{-it}u(t, \cdot + y_0) - e^{i\omega t}P_\omega\|_{H^1} \leq Ce^{-ct}.$$ 

Now, Propositions 7.1 and 7.7 imply that $e^{-it}u(\cdot, \cdot + y_0) = W^a$ for some $a \geq 0$. As $V(W^a(t)) > 0$ for all $t \in \mathbb{R}$ (recall that $V(u(0)) > 0$), it follows that $a > 0$. But then, by Corollary 7.8 we know that there exist $\theta_a$, $T_a \in \mathbb{R}$ such that

$$e^{-it}u(t, \cdot + y_0) = W^a(t) = e^{i\theta_a}W^{a+1}(t-T_a),$$

which implies item (i).

As for item (ii), this follows from the variational characterization of $P_\omega$ established in [2]. □

ORCID

Alex H. Ardila http://orcid.org/0000-0002-3731-0892

References

[1] Zhang, X. (2006). On the Cauchy problem of 3-D energy-critical Schrödinger equations with subcritical perturbations. J. Differ. Eqn. 230(2):422–445. DOI: 10.1016/j.jde.2006.08.010.

[2] Killip, R., Oh, T., Pocovnicu, O., Vişan, M. (2017). Solitons and scattering for the cubic–quintic nonlinear Schrödinger equation on $\mathbb{R}^3$. Arch. Ration. Mech. Anal. 225(1):469–548. DOI: 10.1007/s00205-017-1109-0.

[3] Killip, R., Murphy, J., Visan, M. (2021). Cubic-quintic NLS: scattering beyond the virial threshold. SIAM J. Math. Anal. 53(5):5803–5812. DOI: 10.1137/20M1381824.

[4] Campos, L., Farah, L. G., Roudenko, S. (2022). Threshold solutions for the nonlinear Schrödinger equation. Rev. Mat. Iberoamericana. 38(5):1637–1708.

[5] Campos, L., Murphy, J. Threshold solutions for the intercritical inhomogeneous NLS. Preprint arXiv:2205.09714.

[6] Duyckaerts, T., Merle, F. (2009). Dynamic of threshold solutions for energy-critical NLS. GAFA Geom. Funct. Anal. 18(6):1787–1840. DOI: 10.1007/s00039-009-0707-x.

[7] Duyckaerts, T., Roudenko, S. (2010). Threshold solutions for the focusing 3D cubic Schrödinger equation. Rev. Mat. Iberoamericana. 26:1–56. DOI: 10.4171/RMI/592.

[8] Li, D., Zhang, X. (2010). Dynamics for the energy critical nonlinear wave equation in high dimensions. Trans. Am. Math. Soc. 363(3):1137–1160. DOI: 10.1090/S0002-9947-2010-04999-2.

[9] Yang, K., Zeng, C., Zhang, X. (2022). Dynamics of threshold solutions for energy critical NLS with inverse square potential. SIAM J. Math. Anal. 54(1):173–219. DOI: 10.1137/21M1406003.
Appendix A. proof of lemma 3.3

In this section, we provide the proof of Lemma 3.3. Noting that $\overline{L v} = -L(\overline{v})$, we find that if $\lambda_1 > 0$ is an eigenvalue of the operator $L$ with the eigenfunction $e_+$, then $-\lambda_1$ is also an eigenvalue of $L$ with eigenfunction $e_+ = e_+$. We put $e_1 = \text{Re} e_+$ and $e_2 = \text{Im} e_+$. To show the existence of $e_+$, we must study the system

$$\begin{align*}
L_+ e_1 &= \lambda_1 e_2, \\
-L_- e_2 &= \lambda_1 e_1.
\end{align*}$$  

(A.1)

Recall that $L_-$ is self-adjoint on $L^2$ and nonnegative (see (3.3)). Thus, we see that $L_-$ has a unique square root $(L_-)^{\frac{1}{2}}$ with domain $H^1$. Consider the operator $T$ on $L^2$ with domain $H^4$,

$$T = (L_-)^{\frac{1}{2}} (L_+) (L_-)^{\frac{1}{2}}.$$

As $P_\omega$ has exponential decay, it follows that $T$ is a relatively compact, self-adjoint, perturbation of $(-\Delta + \omega)^2$. Thus, by the Weyl Theorem, we know that $\sigma_{\text{ess}}(T) = [\omega, \infty)$. Now, suppose that there exists $g \in H^4$ such that

$$T g = -\lambda_2^2 g.$$  

(A.2)

Then, taking

$$e_1 := (L_-)^{\frac{1}{2}} g, \quad e_2 := \frac{1}{\lambda_1} (L_+) (L_-)^{\frac{1}{2}} g$$

we obtain a solution to (A.1), which implies the existence of the eigenfunction $e_+$.

Thus, to show the existence of $e_+$, we need to show that $T$ has at least one negative eigenvalue $-\lambda_2^2$. Indeed, we have the following:

**Lemma A.1.**

$$\Lambda(T) := \inf \{ (T g, g)_{L^2} : g \in H^4, \| g \|_{L^2} = 1 \} < 0.$$

**Proof.** We recall that $L_+$ has only one negative eigenvalue $-\lambda_1$ with a corresponding eigenfunction $Z \in H^2(\mathbb{R}^3)$ (it is radial and positive function). We define

$$\Phi = Z + \mu x \cdot \nabla P_\omega, \quad \text{with} \quad \mu = -\frac{(Z, P_\omega)_{L^2}}{(x \cdot \nabla P_\omega, P_\omega)_{L^2}} = \frac{(Z, P_\omega)_{L^2}}{\frac{2}{\beta} \| P_\omega \|_{L^2}^2}.$$  

Notice that $\mu > 0$ and $(\Phi, P_\omega)_{L^2} = 0$. 

References:

[10] Ginibre, J., Velo, G. (1992). Smoothing properties and retarded estimates for some dispersive evolution equations. *Commun. Math. Phys.* 144(1):163–188. DOI: 10.1007/BF02099195.

[11] Keel, M., Tao, T. (1998). Endpoint Strichartz estimates. *Am. J. Math.* 120(5):955–980. DOI: 10.1353/ajm.1998.0039.

[12] Strichartz, R. S. (1977). Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.* 44(3):705–774. DOI: 10.1215/S0012-7094-77-04430-1.

[13] Duyckaerts, T., Holmer, J., Roudenko, S. (2008). Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. *Math. Res. Lett.* 15(6):1233–1250. DOI: 10.4310/MRL.2008.v15.n6.a13.

[14] Murphy, J., Miao, C., Zheng, J. (2023). Threshold scattering for the focusing NLS with a repulsive potential. *Indiana Univ. Math. J.* 72:409–453.
Now we show that

$$N_0 := \int_{\mathbb{R}^3} L_+ (\Phi) \Phi dx < 0.$$  

Indeed, we easily see that

$$\langle L_+ (\Phi), \Phi \rangle = \langle L_+(Z), Z \rangle + \mu^2 \langle L_+(x \cdot \nabla P_\omega), x \cdot \nabla P_\omega \rangle - 2\mu \lambda_1 \langle Z, x \cdot \nabla P_\omega \rangle$$

(A.3)

We claim that \( \langle Z, x \cdot \nabla P_\omega \rangle \geq 0 \). Suppose by contradiction that \( \langle Z, x \cdot \nabla P_\omega \rangle < 0 \). Then we have

$$\langle Z, L_+(x \cdot \nabla P_\omega) \rangle = \langle L_+(Z), x \cdot \nabla P_\omega \rangle = -\lambda_1 \langle Z, x \cdot \nabla P_\omega \rangle > 0.$$  

This implies \( Z \) satisfies the condition (3.10) in Lemma 3.1. Thus, as \( \langle Z, \partial_\phi P_\omega \rangle = 0 \) for \( j = 1, 2, 3 \), by Lemma 3.1 we deduce that \( \langle L_+(Z), Z \rangle > 0 \), which is a contradiction. Having shown that \( \langle Z, x \cdot \nabla P_\omega \rangle \geq 0 \), by (A.3) we obtain that \( N_0 < 0 \).

Next, we have (recall that \( \langle \Phi, P_\omega \rangle_{L^2} = 0 \))

$$\langle (L_- + 1)\Phi, P_\omega \rangle_{L^2} = \langle \Phi, (L_- + 1)P_\omega \rangle_{L^2} = 0.$$  

Since

$$\text{Ran}(L_-)^\perp = \text{Ker}(L_-) = \text{span}\{P_\omega\},$$

it follows that \( (L_- + 1)\Phi \in \text{Ran}(L_-) \). Then for \( \varepsilon > 0 \) (which will be chosen later) there exists \( g_\varepsilon \in H^2 \) such that

$$||L_- g_\varepsilon - (L_- + 1)\Phi||_{L^2} < \varepsilon.$$  

(A.4)

We set \( G_\varepsilon := (L_- + 1)^{-1}L_- g_\varepsilon \). Note that \( G_\varepsilon \in H^2 \). By (A.4), we infer that

$$||G_\varepsilon - \Phi||_{H^2} \leq \varepsilon ||(L_- + 1)^{-1}||_{L^2 \to L^2},$$

which implies that there exists a constant \( C_0 > 0 \) such that

$$\left| \int_{\mathbb{R}^3} L_+ (G_\varepsilon)G_\varepsilon dx - \int_{\mathbb{R}^3} L_+ (\Phi)\Phi dx \right| \leq C_0 \varepsilon.$$  

Choosing \( \varepsilon = \frac{N_0}{2C_0} \), we see that \( (L_+ G_\varepsilon, G_\varepsilon) < 0 \).

Now, if \( G = (L_-)^{-2}G_\varepsilon \), then we have

$$\langle TG, G \rangle_{L^2} = \langle L_+ G_\varepsilon, G_\varepsilon \rangle_{L^2} < 0.$$  

Since \( \sigma_{\text{ess}}(T) = [\omega, +\infty) \), we conclude that the operator \( T \) has at least one negative eigenvalue.

Now, using the same argument developed in [6, Subsection 7.2.2] we deduce that \( \varepsilon_\pm \in S(\mathbb{R}^3) \).

We observe that discussions given in Section 3 show that

$$\ker(\mathcal{L}) = \text{span}\{\partial_1 P_\omega, \partial_2 P_\omega, \partial_3 P_\omega, iP_\omega\}.$$  

(A.5)

Moreover, by Lemma A.1 we see that \( \{\pm \lambda_1\} \subset \sigma(\mathcal{L}) \).

Finally, we will characterize the real spectrum of \( \mathcal{L} \). First, notice that

$$\mathcal{L} = JL, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad L := \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}.$$  

As the operator \( L \) is a compact perturbation of

$$\begin{pmatrix} -\Delta + \omega & 0 \\ 0 & -\Delta + \omega \end{pmatrix}$$

we find that \( \sigma_{\text{ess}}(L) = \{i\xi : \xi \in \mathbb{R}, |\xi| \geq \omega \} \). Consequently, \( \sigma(L) \cap (\mathbb{R} \setminus \{0\}) \) contains only eigenvalues. It remains to show \( \sigma(\mathcal{L}) \cap (\mathbb{R} \setminus \{0\}) = \{-\lambda_1, \lambda_1\} \).
Assume toward a contradiction that there instead exists $f \in H^2$ such that
\[
Lf = -\lambda_0 f,
\]
with $\lambda_0 \in \mathbb{R} \setminus \{0, -\lambda_1, \lambda_1\}$. Since $\mathcal{F}(\mathcal{L}g, h) = -\mathcal{F}(g, \mathcal{L}h)$ we see that
\[
(\lambda_1 + \lambda_0)\mathcal{F}(f, e_+) = (\lambda_1 - \lambda_0)\mathcal{F}(f, e_-) = 0 \quad \text{and} \quad \lambda_0 \mathcal{F}(f, f) = -\lambda_0 \mathcal{F}(f, f).
\]
Therefore,
\[
\mathcal{F}(f, e_+) = \mathcal{F}(f, e_-) = \mathcal{F}(f, f) = 0.
\]
Equivalently, we have
\[
(f, e_+)_{L^2} = (f, e_-)_{L^2} = 0.
\]
Next, we write
\[
f = i\beta_0 P_{a_0} + \sum_{j=1}^{3} \beta_j (\partial_j P_{a_0}) + g, \quad \beta_0 = \frac{(f, iP_{a_0})_{L^2}}{||P_{a_0}||^2_{L^2}}, \quad \beta_j = \frac{(f, \partial_j P_{a_0})_{L^2}}{||\partial_j P_{a_0}||^2_{L^2}},
\]
where $j = 1, \ldots, 3$ and $g \in Y^\perp$. By Remark 3.5 we get $\mathcal{F}(f, f) = \mathcal{F}(g, g)$. Then, Lemma 3.4 implies
\[
||g||_{H^1}^2 \leq \mathcal{F}(g, g) = \mathcal{F}(f, f) = 0.
\]
In particular, $g = 0$ and (see (A.5))
\[
\lambda_0 f = Lf = L(i\beta_0 P_{a_0} + \sum_{j=1}^{3} \beta_j (\partial_j P_{a_0})) = 0,
\]
which is a contradiction.

**Appendix B: scattering at the threshold**

In this section, we demonstrate the existence of scattering threshold solutions.

**Proposition B.1.** Assume $E(P_{a_0}) > 0$. There exists a solution $v$ of (NLS) such that $E(v) = E(P_{a_0})$, $M(v) = M(P_{a_0})$, $V(v_0) > 0$ and $||v||_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^3)} < \infty$. In particular, $v$ scatters in both time directions.

**Proof.** First we claim that there exists $\phi \in C^\infty_c(\mathbb{R}^3)$ so that
\[
E(\phi) = \frac{1}{2} E(P_{a_0}) \quad \text{and} \quad M(\phi) = \frac{1}{2} M(P_{a_0}). \quad (B.1)
\]
Indeed, consider $\psi \in C^\infty_c(\mathbb{R}^3)$ with $M(\psi) > 0$. We define $\psi_\lambda(x) = \lambda \psi(\lambda x)$, where $\lambda = \frac{2E(\psi)}{M(\psi)}$. Notice that $M(\psi_\lambda) = \frac{1}{\lambda} M(\psi)$. Now, consider $f_a(x) = a^2 f(ax)$, where $f(x) = \psi_\lambda(x)$ and $a > 0$. Then $M(f_a) = M(f) = \frac{1}{2} M(P_{a_0})$, and
\[
E(f_a) = \frac{a^2}{2} ||\nabla f||_{L^2}^2 + \frac{a^6}{6} ||f||_{L^6}^6 - \frac{a^3}{4} ||f||_{L^4}^4.
\]
Since $\lim_{a \to 0^+} E(f_a) = 0$ and $\lim_{a \to \infty} E(f_a) = \infty$, there exists $a_0 > 0$ such that $E(f_{a_0}) = \frac{1}{2} E(P_{a_0}) > 0$. Thus, writing $\phi(x) = f_{a_0}(x)$ we get (B.1).

By the claim above we see that there exist $\phi^1, \phi^2 \in C^\infty_c(\mathbb{R}^3)$ satisfying (B.1), with
\[
E(P_{a_0}) = E(\phi^1) + E(\phi^2) \quad \text{and} \quad M(P_{a_0}) = M(\phi^1) + M(\phi^2).
\]
Now consider sequences \( \{ x_n^1 \} \) and \( \{ x_n^2 \} \) in \( \mathbb{R}^3 \) with \( |x_n^1 - x_n^2| \to \infty \) as \( n \to \infty \). We set
\[
\phi_j^i(x) = \phi_j(x - x_n^i), \quad \text{for } j = 1, 2,
\]
Writing
\[
\phi_n(x) = \phi_n^1(x) + \phi_n^2(x),
\]
as \( |x_n^1 - x_n^2| \to \infty \) as \( n \to \infty \), we have
\[
E(\phi_n) = E(\phi_0) \quad \text{and} \quad M(\phi_n) = M(\phi_0)
\]
for all \( n \) large. Now, for \( j = 1, 2 \), we define \( \nu_j^i \) to be the global solution of (NLS) with the initial data \( \nu_j^i(0) = \phi_j^i \). Moreover, we define
\[
\nu_n^j(t, x) := \nu_j(t, x - x_n^i), \quad j = 1, 2.
\]
By Theorem 1.1, we infer that, for \( j = 1, 2 \),
\[
\|\nu_n^j\|_{L^\infty_t(L^2_x(\mathbb{R}^2 \times \mathbb{R}^2))} \leq p_n 1 \quad \text{for all } n.
\]
Moreover, by persistence of regularity we get (cf. [2, Lemma 6.2])
\[
\|\nu_n^j\|_{L^\infty_t(L^2_x(\mathbb{R}^2 \times \mathbb{R}^2))} + \|\nabla \nu_n^j\|_{L^\infty_t(L^2_x(\mathbb{R}^2 \times \mathbb{R}^2))} \leq p_n 1
\]
and
\[
\|\nu_n^j\|_{L^2_t(L^4_x(\mathbb{R}^2 \times \mathbb{R}^2))} \leq p_n 1
\]
for all \( n \).
Following the same argument developed in [2, Lemma 9.2] and using orthogonality condition \( |x_n^1 - x_n^2| \to \infty \) as \( n \to \infty \), we have that
\[
\lim_{n \to \infty} \left[ \|\nu_n^1 \nu_n^2\|_{L^2_{t,x}} + \|\nu_n^1 \nabla \nu_n^2\|_{L^2_{t,x}} + \|\nabla \nu_n^1 \nabla \nu_n^2\|_{L^2_{t,x}} + \|\nu_n^1 \nu_n^2\|_{L^2_{t,x}} \right] = 0.
\]
We now define the approximate solution to (NLS) by
\[
W_n(t) := \nu_n^1(t) + \nu_n^2(t).
\]
It is clear that (cf. (B.2))
\[
W_n(0) = \phi_n.
\]
Moreover, we have the following global space bound
\[
\limsup_{n \to \infty} \left[ \|W_n\|_{L^\infty_t(H^1_x(\mathbb{R}^2 \times \mathbb{R}^2))} + \|W_n\|_{L^\infty_t(L^4_x(\mathbb{R}^2 \times \mathbb{R}^2))} + \|W_n\|_{L^\infty_t(W_{4/3}^{1/2}(\mathbb{R}^2 \times \mathbb{R}^2))} \right] \leq p_n 1.
\]
and
\[
\limsup_{n \to \infty} \|\nabla i\partial_t W_n + \Delta W_n - F(W_n)\|_{L^\infty_t(\mathbb{R}^2 \times \mathbb{R}^2)} = 0,
\]
where \( F(z) = |z|^4 z - |z|^2 z \). Indeed, with (B.4), (B.5) and (B.6) in hands, the proof of (B.8) and (B.9) is essentially the same as in [2, Lemma 9.4] and [2, Lemma 9.5], respectively.

Let \( u_n \) be the corresponding solution to (NLS) with initial data \( \phi_n \), then the stability result [2, Proposition 6.3] implies that \( \|u_n\|_{L^\infty_t(L^2_x(\mathbb{R}^2 \times \mathbb{R}^2))} < \infty \) for \( n \) large. Moreover, by (B.3) we have \( E(u_n) = E(P_\infty) \) and \( M(u_n) = M(P_\infty) \). Note also that \( V(u_n(0)) > 0 \). Indeed, if \( V(u_n(0)) = 0 \), then \( u_n = P_\infty \) modulo symmetries, which is a contradiction.

We close with a remark about case \( E(P_\infty) = 0 \), which shows that the threshold behaviors will not be the same as in the case \( E(P_\infty) > 0 \).
Remark B.2. Assume $E(P_0) = E(P_\infty)$ and $M(u_0) = M(P_\infty)$, then the solution $u$ to (NLS) does not scatter in both time directions. Indeed, suppose by contradiction that there exists $\psi_+ \in H^1$ so that
\[
\lim_{t \to -\infty} ||u(t) - e^{it}\psi_+||_{H^1} = 0.
\]
As $e^{it}\psi_+ \to 0$ in $L^4$ as $t \to \infty$, we infer that
\[
\frac{1}{2} \left| \left| \nabla \psi_+ \right| \right|_{L^2}^2 \leq \lim_{t \to -\infty} E(u(t)) = E(P_\infty) = 0 \quad M(\psi_+) = \lim_{t \to -\infty} M(u(t)) = M(P_\infty) > 0,
\]
which is a contradiction. Therefore, $u$ does not scatter in positive time. A similar argument shows that $u$ does not scatter in negative time.