Research Article

On Wiener Polarity Index and Wiener Index of Certain Triangular Networks

Mr. Adnan,1,2 Syed Ahtsham Ul Haq Bokhary©,1 and Muhammad Imran©3

1Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan
2Allama Iqbal Open University, Islamabad, Pakistan
3Department of Mathematical Sciences, United Arab Emirates University, P. O. Box, Al Ain 15551, UAE

Correspondence should be addressed to Syed Ahtsham Ul Haq Bokhary; sihtsham@gmail.com

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A topological index of graph $G$ is a numerical quantity which describes its topology. If it is applied to the molecular structure of chemical compounds, it reflects the theoretical properties of the chemical compounds. A number of topological indices have been introduced so far by different researchers. The Wiener index is one of the oldest molecular topological indices defined by Wiener. The Wiener index number reflects the index boiling points of alkane molecules. Quantitative structure activity relationships (QSAR) showed that it is also correlated with other quantities including the parameters of its critical point, density, surface tension, viscosity of its liquid phase, and the van der Waals surface area of the molecule. The Wiener polarity index has been introduced by Wiener and known to be related to the cluster coefficient of networks. In this paper, the Wiener polarity index $W_p(G)$ and Wiener index $W(G)$ of certain triangular networks are computed by using graph-theoretic analysis, combinatorial computing, and vertex-dividing technology.

1. Introduction

The Wiener index is originally the first and most studied topological index (see for details in [1]). It was the first molecular topological index that was used in chemistry. Since then, a lot of indices were introduced that relate the topological indices to different physical properties, and some of the recent results can be found in [3–6]. Wiener shows that the Wiener index number is closely correlated with the boiling points of alkane molecules [2]. Later work on quantitative structure activity relationships showed that it is also correlated with other quantities including the parameters of its critical point [7], the density, surface tension, and viscosity of its liquid phase [8], and the van der Waals surface area of the molecule [9].

Mathematically, the Wiener index is sum of all the distances between every vertex of the graph, denoted by $W(G)$, and is

$$W(G) = \sum_{p,q \in V(G)} d(p,q). \quad (1)$$

Later on, Wiener introduced another descriptor known as Wiener polarity index that is known to be related to the cluster coefficient of networks. The Wiener polarity index is denoted by $W_p(G)$ and is defined as the number of unordered pairs of vertices that are at distance 3 in $G$. That is,

$$W_p(G) = |\{ (p,q) | d_G(p,q) = 3, p, q \in V(G) \}|. \quad (2)$$

In organic compounds, say paraffin, the Wiener polarity index is the number of pairs of carbon atoms which are separated by three carbon-carbon bonds. Based on the Wiener index and the Wiener polarity index, the formula

$$t_B = aW(G) + bW_p(G) + c \quad (3)$$

was used to calculate the boiling points $t_B$ of the paraffins, where $a$, $b$, and $c$ are constants for a given isomeric group.
By using the Wiener polarity index, Lukovits and Linert demonstrated quantitative structure-property relationships in a series of acyclic and cycle-containing hydrocarbons in [10]. Hosoya in [11] found a physical-chemical interpretation of \( W_p(G) \). Actually, the Wiener polarity index of many kinds of graphs is studied, such as trees [12], unicyclic and bicyclic graphs [13], hexagonal systems, fullerenes, and polyphenylene chains [14], and lattice networks [15]. For more results on the Wiener polarity index, we refer some recent papers [16–19] and the survey paper [20].

2. The Wiener Polarity Index and Wiener Index of Networks Obtained from Triangular Mesh

The graph of the triangular mesh network, denoted by \( T_n \), is obtained inductively by the triangulation of the graph \( T_{n-1} \). The procedure to construct this network is as follows:

(i) Consider a basic graph \( T_3 \) which is a cycle \( C_3 \) of length 3.

(ii) Subdivide each edge of \( T_3 \), and then join them to form a triangle; the resulting graph is \( T_4 \).

(iii) Continuing in this way, construct a graph \( T_n \) from \( T_{n-1} \) by subdividing each edge of \( T_{n-1} \) and then connect them to form triangles.

(iv) The graph \( T_n \) has \( n \) vertices on each of its side.

The graph of triangular mesh network \( T_5 \) is shown in Figure 1.

The vertices and edges of \( T_n \) are defined as follows:
\[
V(T_n) = \{x_{lm}; 1 \leq l \leq n, 1 \leq m \leq l\},
\]
\[
E(T_n) = \{x_{lm}, x_{l+1,m}; 1 \leq l \leq n-1, 1 \leq m \leq l\} \cup \{x_{1m}, x_{l+1,1}, m + 1; 1 \leq l \leq n-1, 2 \leq m \leq l+1\}.
\]

The count of vertices of the graph \( T_n \) is \( n(n+1)/2 \) and edges of \( T_n \) is \( 3n(n-1)/2 \).

Furthermore, we partition the vertex set \( V(T_n) \) as follows:
\[
V(T_n) = \bigcup_{l=1}^n V_l,
\]
where \( V_l = \{x_{lm}; 1 \leq l \leq n, 1 \leq m \leq l\} \).

Thus, the graph is divided into \( n \) sets. This will help us in calculating the Wiener and Wiener polarity indices of \( T_n \).

The main result of this chapter is proved in the following.

**Theorem 1.** \( W_p(T_n) = 9(n-2)(n-3)/2 \), for \( n \geq 3 \).

**Proof.** Now, we find a number of pair \((p, q)\) of vertices of \( T_n \) which are connected through a path of length 3.

However,
\[
W_p(V_l) = \left| \{ (p, q) \mid d(p, q) = 3; p, q \in V_l \} \right|
\]
\[
W_p(V_l, V_m) = \left| \{ (p, q) \mid d(p, q) = 3; p \in V_l, q \in V_m \} \right|
\]
is the cardinality of the set of vertices in \( V_m \) that are at distance 3 from \( V_l \). From the construction of \( T_n \), it is important to note that there is no vertex \( p \in V_l \) and \( q \in V_m \) such that \( d(p, q) = 3 \) where \( l, m \in \{1, 2, 3\} \). It implies that for \( x \in V_l \) and \( y \in V_m \), we have 4 cases to consider.

\[
W_p(T_n) = \sum_{l=4}^{l=n} W_p(V_l) + \sum_{l=4}^{l=n} W_p(V_l, V_{l-1}) + \sum_{l=4}^{l=n} W_p(V_l, V_{l-2}) + \sum_{l=4}^{l=n} W_p(V_l, V_{l-3}).
\]

**Case 1.** Let \( x_{lm} \in V_l \), where \( 4 \leq l \leq n \). If \( d(x, y) < 3 \), then \( |l-m| \leq 3 \); then, for each \( l \), there is only one vertex \( x_{lm,3} \) which is at distance 3 from \( x_{lm} \).

Since \( l \leq m \leq l-3 \),
\[
W_p(V_l) = \left| \{ (p, q) \mid d(p, q) = 3; p, q \in V_l \} \right| = l - 3
\]
\[
\sum_{l=4}^{l=n} W_p(V_l) = \sum_{l=4}^{l=n} (l - 3) = \frac{(n - 3)(n - 2)}{2}.
\]

**Case 2.** Let \( u \in V_l \) and \( v \in V_{l-1} \) where \( 4 \leq l \leq n \). In this case, there are \( 2l - 6 \) vertices in \( V_{l-1} \) that are at distance 3 from \( V_l \) for each \( i \). Since \( i \leq m \leq n - 3 \),
\[
W_p(V_l, V_{l-1}) = 2l - 6
\]
\[
\sum_{l=4}^{l=n} W_p(V_l, V_{l-1}) = \sum_{l=4}^{l=n} 2l - 6 = (n - 2)(n - 3).
\]
Case 3. Let \( u \in V_I \) and \( v \in V_{I-2} \) where \( 4 \leq l \leq n \); in this case, there are \( 2l - 6 \) vertices in \( V_{I-2} \) that are at distance 3 from \( V_I \) for each \( i \). Since \( i \leq m \leq n - 3 \),

\[
W_p(V_I, V_{I-2}) = 2l - 6
\]

\[
\sum_{l=4}^{n-I} W_p(V_I, V_{I-2}) = \sum_{l=4}^{n-I} (2l - 6) = (n - 2)(n - 3).
\]

Case 4. Let \( u \in V_I \) and \( v \in V_{I-2} \) where \( 4 \leq l \leq n \). In this case, there are \( 4l + 6 \) vertices in \( V_{I-2} \) that are at distance 3 from \( V_I \) for each \( i \). Since \( i \leq m \leq n - 3 \),

\[
W_p(V_I, V_{I-3}) = 4l + 6
\]

\[
\sum_{l=4}^{n-I} W_p(V_I, V_{I-3}) = \sum_{l=4}^{n-I} (4l + 6) = 2(n - 2)(n - 3).
\]

Putting equations (6) and (7) and (31) and (9) in (5), we get

\[
W_p(T_n) = \frac{(n - 2)(n - 3)}{2} + (n - 2)(n - 3) + (n - 2)(n - 3)
\]

\[
+ 2(n - 2)(n - 3)
\]

\[
= \frac{9(n - 2)(n - 3)}{2}.
\]

In the next result, the Wiener index of the graph \( T_n \) is computed.

**Theorem 2.** \( W(T_n) = W(T_{n-1}) + n^4 - n^2/4 \).

**Proof.** Let \( W(V_n, T_{n-1}) \) be the distance between the vertices of \( V_n \) and \( T_{n-1} \), where \( V_n = \{x_{n,1}, x_{n,2}, x_{n,3}, \ldots, x_{n,n}\} \). For \( x_{i,m} \in V(T_{n-1}) \),

\[
W(V_n, T_{n-1}) = \sum_{\theta=1}^{\theta=n} W(x_{n,\theta}, T_{n-1}) + W(V_n).
\]

This can be computed by finding the distance between each vertex \( x_{n,m} \) from the vertices of \( T_{n-1} \). These distances are listed in the following.

For \( 2 \leq \theta \leq n \) and \( 1 \leq m \leq \theta - 1 \),

\[
d(x_{n,\theta}, x_{i,m}) = \begin{cases} n - 1, & \text{for } m \leq l + m - \theta - 1, \\ \theta - m, & \text{for } n + m - \theta \leq l \leq n - 1. \end{cases}
\]

For \( 1 \leq \theta \leq n \) and \( \theta \leq m \leq n \),

\[
d(x_{n,\theta}, x_{i,m}) = \begin{cases} n + m - l - \theta & \text{for } m \leq l \leq n - 1. \end{cases}
\]

Thus, we get

\[
\sum_{\theta=1}^{\theta=n} \sum_{m=1}^{m=\theta-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m})
\]

\[
= \sum_{\theta=1}^{\theta=n} \sum_{m=1}^{m=\theta-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m}) + \sum_{\theta=1}^{\theta=n} \sum_{m=\theta}^{m=\theta} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m})
\]

\[
= \sum_{\theta=1}^{\theta=n} \sum_{m=1}^{m=\theta-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m}) + \sum_{\theta=2}^{\theta=\theta} \sum_{m=1}^{m=\theta} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m})
\]

\[
+ \sum_{\theta=1}^{\theta=n} \sum_{m=\theta}^{m=\theta} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m})
\]

\[
= \sum_{\theta=2}^{\theta=\theta} \sum_{m=1}^{m=\theta-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m}) + \sum_{\theta=2}^{\theta=\theta} \sum_{m=\theta}^{m=\theta} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m})
\]

\[
+ \sum_{\theta=1}^{\theta=n} \sum_{m=\theta}^{m=\theta} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, x_{i,m})
\]

\[
= \sum_{\theta=2}^{\theta=\theta} \sum_{m=1}^{m=\theta-1} (n - l) + \sum_{\theta=2}^{\theta=\theta} \sum_{m=\theta}^{m=\theta} \sum_{l=n+m-\theta} (\theta - m)
\]

\[
+ \sum_{\theta=1}^{\theta=n} \sum_{m=\theta}^{m=\theta} (n + m - l - \theta)
\]

\[
= \frac{1}{12}[n^4 - 2n^3 - n^2 + 2n] + \frac{1}{12}[n^4 - n^2] + \frac{1}{12}[n^4 - n^2],
\]

\[
W(V_n) = \sum_{\theta=1}^{\theta=n} W(x_{n,\theta}) = \sum_{m=1}^{m=\theta-1} \sum_{l=m}^{l=n-1} m = \frac{1}{6}[n^3 - n].
\]
By replacing the values of $\sum_{a=1}^{n} W(x_{n,a}, T_{n-1})$ and $W(V_n)$ in equation (11), we get

$$W(V_n, T_{n-1}) = \frac{1}{12} \left[ n^4 - 2n^3 + 2n - n^2 \right] + \frac{1}{12} \left[ n^4 - n^2 \right] + \frac{1}{6} \left[ n^3 - n \right].$$

This after simplification implies

$$W(V_n, T_{n-1}) = \frac{1}{4} \left[ n^4 - n^2 \right].$$

However, the Wiener index of $T_n$ is $W(T_n) = W(T_{n-1}) + W(V_n, T_{n-1})$. Therefore,

$$W(T_n) = W(T_{n-1}) + \frac{1}{4} \left[ n^4 - n^2 \right].$$

3. The Wiener Polarity Index and Wiener Index of the Equilateral Triangular Tetra Sheet

This section will start with the definitions and properties of the equilateral triangular tetra sheet network. The graph of equilateral triangular tetra sheet network denoted by $ET_n$ is obtained from the graph of triangular mesh network by replacing each triangle by the complete graph obtained from the graph of triangular mesh network by replacing each triangle by the complete graph $K_4$. This can be done by inserting a vertex in each triangle of the graph $T_n$ and then connecting all the adjacent vertices to form $K_4$. These new vertices will be denoted by $u_i$ and $w_j$, where $1 \leq i \leq n - 1$ and $1 \leq j \leq n - 2$.

The order and size of the graph $ET(n)$ are

$$|V(ETn)| = \frac{(3n^2 - 3n + 2)}{2},$$

$$|E(ET(n))| = \frac{(9n^2 - 15n + 6)}{2}. $$

The graph of equilateral triangular tetra sheet is shown in Figure 2.

In order to compute the Wiener and Wiener polarity indices, we want to find the distance between each pair of vertices of $ET_n$. For this purpose, we define partition of the vertex set as follows: $V(ET_n) = \bigcup_{l=1}^{n} V_l \cup \bigcup_{i=1}^{l-1} U_l \cup \bigcup_{i=l}^{n-2} W_l$, where $V_l = \{x_{lm} | 1 \leq m \leq l \leq n \}$, $U_l = \{u_{lm} | 1 \leq m \leq l \leq n - 1 \}$, and $W_l = \{w_{lm} | 1 \leq m \leq l \leq n - 2 \}$.

Furthermore, define $V'_l = V(ET_n) \setminus V_l$, $U'_l = V(ET_n) \setminus \{V_l \cup U_l \}$, and $W'_l = V(ET_n) \setminus \{V_l \cup U_l \cup U'_{l+1} \cup W_l \}$.

In the next result, the Wiener polarity index of the graph $ET_n$ is computed.

Theorem 3. For $n \geq 3$, $W_p(ET_n) = 63n^2 - 357n + 504/2$.

Proof. In order to find the Wiener polarity index, we have to compute all those pairs of vertices that are at distance 3 to each other. Since the vertex set of the graph $ET_n$ is divided into three parts, we first find the number of such pairs in each possible set. Define $W_p(A, B)$ as the set of those vertices of $A$ that are at distance 3 from the vertices of $B$. For simplicity, $W_p(A, A) = W_p(A)$. Thus, we have

$$W_p(V_l) = \{(v, v') | d(v, v') = 3; v, v' \in V_l \},$$

$$W_p(U_l) = \{(u, u') | d(u, u') = 3; u, u' \in U_l \},$$

$$W_p(W_l) = \{(w, w') | d(w, w') = 3; w, w' \in W_l \},$$

$$W_p(V_l, V'_l) = \{(v, x) | d(v, x) = 3; v \in V_l, x \in V'_l; 4 \leq l \leq n \},$$

$$W_p(U_l, U'_l) = \{(u, x) | d(u, x) = 3; u \in U_l, x \in U'_l; 3 \leq l \leq n - 1 \},$$

$$W_p(W_l, W'_l) = \{(w, x) | d(w, x) = 3; w \in W_l, x \in W'_l; 2 \leq l \leq n - 2 \},$$

$$W_p(ET(n)) = \sum_{l=1}^{n} (W_p(V_l) + W_p(V_l, V'_l)) + \sum_{l=3}^{n-1} (W_p(U_l) + W_p(U_l, U'_l)) + \sum_{l=1}^{n-2} (W_p(W_l) + W_p(W_l, W'_l)).$$

![Figure 2: The graph of equilateral triangular tetra sheet.](image-url)
For simplicity, we compute the three factors separately:

(i) Let \( v \in V_i \) and \( x \in V_i' \). From the construction of \( ET_n \), there does not exist any \( x \in V_{i-\theta} \), \( 4 \leq \theta \leq n-1 \). From Theorem 1 and equations (22) and (23), we get after simplification

\[
W_p(V_i, V_i') = W_p(V_i, U_{i-1}) + W_p(V_i, W_{i-2}) + W_p(V_i, V_{i-1}) + W_p(V_i, U_{i-2}) + W_p(V_i, U_{i-3}) + W_p(V_i, W_{i-4}).
\]  

(ii) If \( x \in U_{i-1} \cup W_{i-2} \cup U_{i-3} \cup W_{i-4} \), then there are \( 2l - 6 \) vertices that are at distance 3 from \( V_i \). Thus,

\[
W_p(V_i, U_{i-1}) = W_p(V_i, W_{i-2}) = W_p(V_i, U_{i-2}) = W_p(V_i, W_{i-3}) = 2l - 6
\]

\[
\sum_{l=4}^{\infty} W_p(V_i, U_{i-1}) = \sum_{l=4}^{\infty} W_p(V_i, W_{i-2}) = \sum_{l=4}^{\infty} W_p(V_i, U_{i-2}) = \sum_{l=4}^{\infty} W_p(V_i, W_{i-3}) = \sum_{l=4}^{\infty} 2l - 6
\]  

\[
= (n - 2) (n - 3).
\]

(iii) Let \( u \in U_{i-1} \) and \( x \in U_{i-\theta} \). From the construction of \( ET_n \), there does not exist any \( x \in U_{i-\theta} \), \( 4 \leq \theta \leq n-1 \). From Theorem 1 and equations (22) and (23), we get after simplification

\[
\sum_{l=4}^{\infty} (W_p(V_i) + W_p(V_i, V_i')) = 3(n - 3)(4n - 9).
\]  

(iii) Let \( u \in U_{i-1} \) and \( x \in U_{i-1} \). From the construction of \( ET_n \), there does not exist any \( x \in U_{i-\theta} \), \( 4 \leq \theta \leq n-1 \). From Theorem 1 and equations (22) and (23), we get after simplification

\[
W_p(U_{i-1}, U_{i-1}') = W_p(U_{i-1}, W_{i-2}) + W_p(U_{i-1}, V_{i-1}) + W_p(U_{i-1}, U_{i-2}) + W_p(U_{i-1}, W_{i-3}) + W_p(U_{i-1}, V_{i-2}) + W_p(U_{i-1}, U_{i-3}) + W_p(U_{i-1}, W_{i-4}) + W_p(U_{i-1}, V_{i-3}).
\]
We compute each of these factors as follows:

(iv) If \( x \in W_{l-2} \cup V_{l-1} \cup U_{l-2} \cup W_{l-3} \cup V_{l-2} \), then there are \( 2l - 4 \) vertices \( x \) that are at distance 3 from \( U_{l-1} \).

Thus,

\[
W_p(U_{l-1}, W_{l-2}) = W_p(U_{l-1}, V_{l-1}) = W_p(U_{l-1}, U_{l-2}) = W_p(U_{l-1}, W_{l-3}) = W_p(U_{l-1}, V_{l-2}) = 2l - 4,
\]

\[
\sum_{l=3}^{l=n-1} W_p(U_{l-1}, W_{l-2}) = \sum_{l=3}^{l=n-1} W_p(U_{l-1}, V_{l-1}) = \sum_{l=3}^{l=n-1} W_p(U_{l-1}, U_{l-2}) = \sum_{l=3}^{l=n-1} W_p(U_{l-1}, W_{l-3}) = \sum_{l=3}^{l=n-1} W_p(U_{l-1}, V_{l-2}) = (26)
\]

If \( x \in U_{l-3} \cup V_{l-3} \), then there are \( 3l - 6 \) vertices of \( x \) that are at distance 3 from \( U_{l-1} \).

\[
W_p(U_{l-1}, U_{l-3}) = W_p(U_{l-1}, V_{l-3}) = 2l - 6
\]

\[
\sum_{l=3}^{l=n-1} W_p(U_{l-1}, U_{l-3}) = \sum_{l=3}^{l=n-1} W_p(U_{l-1}, V_{l-3}) = \sum_{l=3}^{l=n-1} 2l - 6 = \frac{1}{2} [3n^2 - 15n + 18].
\] (27)

If \( x \in W_{l-4} \), then there are \( 2l - 6 \) vertices that are at distance 3 from \( U_{l-1} \).

\[
W_p(U_{l-1}, W_{l-4}) = 2l - 6
\]

\[
\sum_{l=4}^{l=n-1} W_p(U_{l-1}, W_{l-4}) = \sum_{l=4}^{l=n-1} 2l - 6 = n^2 - 7n + 12.
\] (28)

For every \( u \in U_{l-1} \), there are \( l - 2 \) \( u \in U_{l-1} \) that are at distance 3 from \( u \).

\[
W_p(U_{l-1}) = l - 2
\]

\[
\sum_{l=3}^{l=n-1} W_p(U_{l-1}) = \sum_{l=3}^{l=n-1} l - 2 = \frac{n^2 - 5n + 6}{2}.
\] (29)

Substituting each of these values in the second factor, we get after simplification

\[
\sum_{l=4}^{l=n-1} \left( W_p(U_{l-1}) + W_p(U_{l-1}, U_{l-1}) \right) = \left( \frac{19}{2} n - 21 \right) (n - 3).
\] (30)

Let \( \omega \in W_{l-2} \) and \( x \in W_{l-2}' \). From the construction of \( ET_n \), there does not exist any \( x \in U_{l-6}, 4 \leq \theta \leq n - 1 \) s.t. \( d(u, x) = 3 \). This implies that \( x \in \{ V_{l-1} \cup V_{l-2} \cup U_{l-2} \cup U_{l-3} \cup W_{l-3} \cup W_{l-4} \} \), and we get

\[
W_p(W_{l-2}, W_{l-2}') = W_p(W_{l-2}, V_{l-1}) + W_p(W_{l-2}, U_{l-2}) + W_p(W_{l-2}, W_{l-3}) + W_p(W_{l-2}, V_{l-2}) + W_p(W_{l-2}, U_{l-3}) + W_p(W_{l-2}, W_{l-4}) + W_p(W_{l-2}, V_{l-3}).
\] (31)
We compute each of the factors in the following.

\[ W_p(W_{l-2}, V_{l-1}) = W_p(W_{l-2}, U_{l-2}) = W_p(W_{l-2}, W_{l-3}) = W_p(W_{l-2}, V_{l-2}) = 2l - 4, \]

\[ \sum_{l=3}^{l=2} W_p(W_{l-2}, V_{l-1}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, U_{l-2}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, W_{l-3}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, V_{l-2}) \]

(32)

\[ (2l - 4) = n^2 - 7n + 12. \]

If \( x \in U_{l-3} \cup V_{l-3} \), then there are \( 4l - 6 \) vertices that are at distance 3 from \( W_{l-2} \). This follows that

\[ W_p(W_{l-2}, U_{l-3}) = W_p(W_{l-2}, V_{l-3}) = 4l - 6, \]

\[ \sum_{l=3}^{l=2} W_p(W_{l-2}, U_{l-3}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, V_{l-3}) = \sum_{l=3}^{l=2} 4l - 6 = [2n^2 - 12n + 18]. \]

(33)

If \( x \in W_{l-4} \), then there are \( 3l - 6 \) vertices that are at distance 3 from \( W_{l-2} \). This follows that

\[ W_p(W_{l-2}, W_{l-4}) = 3l - 6 \]

\[ \sum_{l=3}^{l=2} W_p(W_{l-2}, W_{l-4}) = \sum_{l=3}^{l=2} 3l - 6 = \frac{3n^2 - 21n + 36}{2}. \]

(34)

For every \( w \in W_{l-2} \), there are \( l - 2 \) vertices \( x \) in \( W_{l-1} \) that are at distance 3 from \( w \).

\[ W_p(W_{l-1}) = l - 2, \]

\[ \sum_{l=3}^{l=2} W_p(W_{l-1}) = \sum_{l=3}^{l=2} l - 2 = \frac{n^2 - 7n + 12}{2}. \]

(35)

Replace all these values in the third factor, and we get after simplification

\[ \sum_{l=4}^{l=2} (W_p(W_{l-2}) + W_p(W_{l-2}, W_{l-2}')) = 10n^2 - 66n + 108. \]

(36)

If \( x \in V_{l-1} \cap U_{l-2} \cup W_{l-3} \cup V_{l-2} \), then there are \( 2l - 4 \) vertices that are at distance 3 from \( W_{l-2} \). This follows that

\[ W_p(W_{l-2}, V_{l-1}) = W_p(W_{l-2}, U_{l-2}) = W_p(W_{l-2}, W_{l-3}) = W_p(W_{l-2}, V_{l-2}) = 2l - 4, \]

\[ \sum_{l=3}^{l=2} W_p(W_{l-2}, V_{l-1}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, U_{l-2}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, W_{l-3}) = \sum_{l=3}^{l=2} W_p(W_{l-2}, V_{l-2}) \]

(32)

\[ (2l - 4) = n^2 - 7n + 12. \]

By combining the values of all three factors in equation (20), we found that the Wiener polarity index of the graph \( ET_n \) is

\[ W_p(ET_n) = \frac{63n^2 - 357n + 504}{2}. \]

(37)

**Theorem 4.** \( W(ET_n) = W(ET_{n-1}) + 1/12[28n^4 - 86n^3 + 134n^2 - 172n + 120]. \)

**Proof.** Let \( W(V_n, ET_{n_0}) \) be the distance between the vertices of \( V_n \) from itself and from \( U_{n-1}, W_{n-2} \), and \( ET_{n-1} \), where \( V_n = \{x_n, x_{n_2}, x_{n_3}, \ldots, x_{n_R}\}, U_{n-1} = \{u_{n_1}, u_{n_2}, u_{n_3}, \ldots, u_{n_{n-1}}\}, W_{n-2} = \{w_{n_1}, w_{n_2}, w_{n_3}, \ldots, w_{n_{n-2}}\}. \) Thus, for any vertex in \( V_n \), we have

\[ W(V_n, ET_{n_0}) = \sum_{\theta=1}^{\theta=0} W(x_n, \theta, ET_{n-1}) + \sum_{\theta=1}^{\theta=0} W(x_n, \theta, U_{n-1}) \]

\[ + \sum_{\theta=1}^{\theta=0} W(x_n, \theta, W_{n-2}) + W(V_n). \]

(38)
This can further reduce to the following equation:

\[
W(V_{n}, ET_{m}) = \sum_{\theta=m}^{\theta=n} W(x_{\theta}, \cup_{l=1}^{\theta=n-l} V_{l}) + \sum_{\theta=m}^{\theta=n} W(x_{\theta}, \cup_{l=1}^{\theta=n-2} U_{l})
\]

\[
+ \sum_{\theta=1}^{\theta=n} W(x_{\theta}, \cup_{l=1}^{\theta=n-3} W_{l}) + \sum_{\theta=1}^{\theta=n} W(x_{\theta}).
\]

(39)

We compute each of the factors in equation (39) separately.

The first factor is computed with the help of following distances:

\[
\sum_{\theta=m}^{\theta=n} W(x_{\theta}, \cup_{l=1}^{\theta=n-l} V_{l}) = \sum_{\theta=2}^{\theta=n-1} \sum_{m=1}^{\theta-1} \sum_{l=m}^{\theta-n-l} d(x_{\theta}, x_{l:m})
\]

\[
= \sum_{\theta=m}^{\theta=m+n-1} \sum_{m=1}^{\theta-n} \sum_{l=m}^{\theta-n-l} d(x_{\theta}, x_{l:m}) + \sum_{\theta=2}^{\theta=n} \sum_{m=\theta}^{\theta-1} \sum_{l=m}^{\theta-n-m} d(x_{\theta}, x_{l:m})
\]

\[
= \sum_{\theta=2}^{\theta=n} \sum_{m=1}^{\theta-1} \sum_{l=m}^{\theta-n-l} d(x_{\theta}, x_{l:m}) + \sum_{\theta=m}^{\theta=n} \sum_{m=1}^{\theta-n} \sum_{l=m}^{\theta-n-l} d(x_{\theta}, x_{l:m})
\]

(42)

For \(2 \leq \theta \leq n\) and \(1 \leq m \leq \theta - 1\),

\[
d(x_{\theta}, x_{l:m}) = \begin{cases} n - l, & \text{for } m \leq l \leq n + m - \theta - 1, \\ \theta - m, & \text{for } n + m - \theta \leq l \leq n - 1. \end{cases}
\]

(40)

For \(1 \leq \theta \leq n\) and \(\theta \leq m \leq n\),

\[
d(x_{\theta}, x_{l:m}) = \begin{cases} n + m - l - \theta, & \text{for } m \leq l \leq n - 1. \end{cases}
\]

(41)

Thus, we get

\[
\sum_{\theta=1}^{\theta=n} W(x_{\theta}).
\]

The second factor is computed with the help of following distances:

For \(2 \leq \theta \leq n\) and \(1 \leq m \leq \theta - 1\),

\[
d(x_{\theta}, U_{l:m}) = \begin{cases} n - l, & \text{for } m \leq l \leq n + m - \theta - 1, \\ \theta - m, & \text{for } n + m - \theta \leq l \leq n - 1. \end{cases}
\]

(43)
For \( 1 \leq \theta \leq n \) and \( \theta \leq m \leq n \),

\[
d(x_{n,\theta}, u_{l,m}) = \begin{cases} 
  n + m - \theta, & \text{for } m \leq l \leq n - 1, \\
  \theta - m, & \text{for } n + m - \theta - 1 \leq l \leq n - 2.
\end{cases}
\]

The third factor is computed with the help of following distances:

For \( 2 \leq \theta \leq n - 1 \) and \( 1 \leq m \leq \theta - 1 \),

\[
d(x_{n,\theta}, u_{l,m}) = \begin{cases} 
  n - l + 1, & \text{for } m \leq l \leq n + m - \theta - 2, \\
  \theta - m, & \text{for } n + m - \theta - 1 \leq l \leq n - 2.
\end{cases}
\]
\[ d(x_{n,\theta}, u_{l,m}) = \begin{cases} n + m - l - \theta & \text{for } m \leq l \leq n - 2, \\ \theta n \end{cases} \]

\[
 \sum_{\theta=1}^{\theta n} W(x_{n,\theta} \cup_{l=1}^{l=m-1} W_l) = \sum_{\theta=1}^{\theta n} m \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) \\
= \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) \\
= \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) \\
+ \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) \\
+ \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) \\
+ \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} d(x_{n,\theta}, u_{l,m}) \\
= \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} (n - l + 1) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} (n - m) \\
= \sum_{\theta=2}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} (n - l + 1) + \sum_{\theta=1}^{\theta n} \sum_{m=1}^{l=m-1} \sum_{l=m}^{l=n-1} (n - m) \\
= \frac{1}{12} \left[ n^4 - 6n^3 + 11n^2 - 6n \right] + \frac{1}{12} \left[ n^4 - 4n^3 + 5n^2 - 2n \right] \\
+ \frac{1}{12} \left[ 4n^3 - 12n^2 + 8n \right] + \frac{1}{12} \left[ n^4 - 4n^3 - n^2 + 2n \right] \\
= \frac{1}{12} \left[ 3n^4 - 8n^3 + 3n^2 + 2n \right]. \\
\]

The fourth factor is

\[ W(V_n) = \sum_{\theta=1}^{\theta n} W(x_{n,\theta}) = \sum_{l=1}^{l=m-l} \sum_{m=1}^{m=n-l} m = \frac{1}{6} \left[ n^3 - n \right]. \]

Putting equations (42), (44), (47), and (48) in (39), we get

\[ W(U_{n-1}, ET_{n}) = \sum_{\theta=1}^{\theta n} W(u_{n-1,\theta}, ET_{n}) + \sum_{\theta=1}^{\theta n} W(u_{n-1,\theta}, W_{n-2}) + W(U_{n-1}), \]

\[ W(U_{n-1}, ET_{n}) = \sum_{\theta=1}^{\theta n} W(u_{n-1,\theta} \cup_{l=1}^{l=m-2} W_l) + \sum_{\theta=1}^{\theta n} W(u_{n-1,\theta} \cup_{l=1}^{l=m-1} V_l) + \sum_{\theta=1}^{\theta n} W(u_{n-1,\theta} \cup_{l=1}^{l=m-3} U_l) + \sum_{\theta=1}^{\theta n} W(u_{n-1,\theta}). \]

Again, we compute each of the factors separately.

The first factor is computed with the help of following distances:

For \( 2 \leq \theta \leq n - 2 \) and \( 1 \leq m \leq \theta - 1 \),

\[ d(u_{n,\theta}, u_{l,m}) = \begin{cases} n - l - 1, & \text{for } m \leq l \leq n + m - \theta - 2, \\ \theta - m + 1, & \text{for } n + m - \theta - 1 \leq l \leq n - 2. \end{cases} \]
\[ d(u_{n, \theta}, w_{l,m}) = \begin{cases} n + m - l - \theta & \text{for } m \leq l \leq n - 2, \\
 \end{cases} \]

\[
\sum_{\theta=1}^{\theta=n-1} \sum_{m=1}^{m=n-1} \sum_{l=1}^{l=n-2} W(u_{n-1, \theta}, \bigcup_{l=1}^{l=n-2} W_l) = \sum_{\theta=1}^{\theta=n-1} \sum_{m=1}^{m=n-1} \sum_{l=1}^{l=n-2} d(u_{n-1, \theta}, w_{l,m})
\]

\[
= \sum_{\theta=2}^{\theta=n-2} \sum_{m=1}^{m=n-2} \sum_{l=1}^{l=n-2} d(u_{n-1, \theta}, w_{l,m}) + \sum_{\theta=1}^{\theta=n-1} \sum_{m=1}^{m=n-2} \sum_{l=1}^{l=n-2} d(u_{n-1, \theta}, w_{l,m})
\]

\[
= \sum_{\theta=2}^{\theta=n-2} \sum_{m=1}^{m=n-2} \sum_{l=1}^{l=n-2} d(u_{n-1, \theta}, w_{l,m}) + \sum_{\theta=1}^{\theta=n-2} \sum_{m=1}^{m=n-1} \sum_{l=1}^{l=m-\theta-2} d(u_{n-1, \theta}, w_{l,m})
\]

\[
= \frac{1}{12} \left[ n^4 - 6n^3 + 11n^2 - 6n \right] + \frac{1}{12} \left[ n^4 - 2n^3 - n^2 + 2n \right]
\]

\[
+ \frac{1}{12} \left[ n^4 - 2n^3 - n^2 + 2n \right]
\]

\[
= \frac{1}{12} \left[ 3n^4 - 10n^3 + 9n^2 - 2n \right].
\]

The second factor is computed with the help of following distances:

\[
\begin{aligned}
d(u_{n, \theta}, w_{l,m}) = \begin{cases} n - l, & \text{for } m \leq l \leq n + m - \theta - 2, \\
\theta - m + 1, & \text{for } n + m - \theta - 1 \leq l \leq n - 2.
\end{cases}
\end{aligned}
\]
\[
d(d_{n,\theta}, w_{l,m}) = \begin{cases} 
  n + m - l - \theta, & \text{for } m \leq l \leq n - 2, \\
  n - l, & \text{for } m \leq n + m - \theta - 1, \\
  \theta - m + 1, & \text{for } n + m - \theta \leq l \leq n - 1.
\end{cases}
\]

The third factor is computed with the help of following distances:

For 2 \leq \theta \leq n - 1 and 1 \leq m \leq \theta - 1,

\[
d(d_{n,\theta}, w_{l,m}) = \begin{cases} 
  n + m - l - \theta, & \text{for } m \leq l \leq n - 2, \\
  n - l, & \text{for } m \leq n + m - \theta - 1, \\
  \theta - m + 1, & \text{for } n + m - \theta \leq l \leq n - 1.
\end{cases}
\]
\[
\sum_{\theta=1}^{\theta=m-1} W(u_{n-1,\theta} \cup \bigcup_{l=1}^{l=m-1} V_i) = \sum_{\theta=1}^{\theta=m-1} \sum_{l=1}^{l=m-1} d(u_{n-1,\theta}, x_{lm}) = \\
\sum_{\theta=1}^{\theta=m-1} \sum_{l=1}^{l=m-1} d(u_{n-1,\theta}, x_{lm}) + \sum_{\theta=1}^{\theta=m-1} \sum_{l=1}^{l=m-1} d(u_{n-1,\theta}, x_{lm}) + \sum_{\theta=1}^{\theta=m-1} \sum_{l=1}^{l=m-1} d(u_{n-1,\theta}, x_{lm})
\]

Again, we find each factor separately.

For 1 \leq \theta \leq n - 2 and 1 \leq m \leq \theta,

\[
d(u_{n,\theta}, w_{l,m}) = \begin{cases} 
    n + m - l - \theta & \text{for } m \leq l \leq n - 1, \\
    \theta - m + 1 & \text{for } n + m - \theta - 1 \leq l \leq n - 1.
\end{cases}
\]
\[ d(u_{n, \theta}, w_{l,m}) = \begin{cases} n + m - l - \theta & \text{for } m \leq l \leq n - 1, \\ \theta - m + 1, & \text{for } n + m - \theta - 1 \leq l \leq n - 1. \end{cases} \] (62)

For \( 1 \leq \theta \leq n - 1 \),

\[
d(u_{n, \theta}, w_{l,m}) = \begin{cases} n - l, & \text{for } 1 \leq \theta \leq n - 2 \text{ and } 1 \leq m \leq \theta - 1, \\ d(u_{n, \theta}, w_{l,m}) = \begin{cases} n - l & \text{for } \theta \leq l \leq n - 2. \end{cases} \end{cases} \] (63)

The second factor is computed with the help of following distances:
For $1 \leq \theta \leq n - 2$ and $\theta \leq m \leq n - 2$,

$$d(u_{n,\theta}, w_{lm}) = \begin{cases} n + m - l - \theta & \text{for } m \leq l \leq n - 1, \\ \sum_{\theta=1}^{m-2} W\left( u_{n-\theta, \theta} \cup \left[ l=n-2 \right] \right) & \text{for } m \leq \theta - 1, \end{cases}$$

$$= \sum_{\theta=1}^{m-2} \sum_{l=m}^{n-2} d(u_{n-\theta, \theta}, u_{lm}) + \sum_{\theta=1}^{m-2} \sum_{l=m}^{n-2} d(u_{n-\theta, \theta}, u_{lm})$$

$$= \sum_{\theta=1}^{m-2} \sum_{l=m}^{n-2} \left( n - l \right) + \sum_{\theta=1}^{m-2} \sum_{l=m}^{n-2} \left( \theta - m + 1 \right)$$

$$= \frac{1}{12} \left[ n^4 - 4n^3 - n^2 + 16n - 12 \right] + \frac{1}{12} \left[ n^4 - 6n^3 + 11n^2 - 6n \right]$$

$$+ \frac{1}{12} \left[ 2n^3 - 14n + 12 \right] + \frac{1}{12} \left[ n^4 - 6n^3 + 11n^2 - 6n \right]$$

$$= \frac{1}{12} \left[ 3n^4 - 14n^3 + 21n^2 - 10n \right].$$

The third factor is computed with the help of following distances:

$$d(u_{n,\theta}, w_{lm}) = \begin{cases} n - l - 1, & \text{for } m \leq l \leq n + m - \theta - 3, \\ \theta - m - 1, & \text{for } n + m - \theta - 2 \leq l \leq n - 3. \end{cases}$$

For $2 \leq \theta \leq n - 2$ and $1 \leq m \leq \theta - 1$,
\[ d(u_{n,\theta}, w_{l,m}) = \begin{cases} n + m - l - \theta - 1 & \text{for } m \leq n - 3, \\ \sum_{\theta=1}^{\theta=n-2} W(w_{n-2,\theta}, \bigcup_{l=1}^{\theta=n-2} W_l) = \sum_{\theta=1}^{\theta=n-2} \sum_{m=1}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) \\ \sum_{\theta=1}^{\theta=n-2} \sum_{m=1}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) + \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) \\ \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) + \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) + \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) \\ \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} (n - l - 1) + \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} (\theta - m - 1) + \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} (n + m - l - \theta - 1) \\ \frac{1}{12} \left[n^4 - 8n^3 + 17n^2 + 2n - 24\right] + \frac{1}{12} \left[n^4 - 6n^3 + 11n^2 - 6n\right] + \frac{1}{12} \left[2n^4 - 20n^3 + 82n^2 - 160n + 120\right] \right.

The fourth factor is

\[ W(W_{n-2}) = \sum_{\theta=1}^{\theta=n-2} W(w_{n-2,\theta}) = \sum_{\theta=1}^{\theta=n-2} \sum_{m=\theta}^{\theta=n-3} \sum_{l=m}^{\theta=n-3} d(w_{n-2,\theta}, w_{l,m}) = \frac{1}{12} \left[2n^3 - 6n^2 - 8n + 24\right]. \tag{67} \]

Putting equations (61), (64), (66), and (67) in (59), we get

\[ W(w_{n-2,ET_n}) = \frac{1}{12} \left[10n^4 - 56n^3 + 134n^2 - 184n + 120\right]. \tag{68} \]

Now, the Wiener index of the graph ET_n is \( W(ET_n) = W(ET_{n-1} + W(V_n, ET_n) + W(U_{n-1}, ET_n) + W(W_{n-2}, E T_n)) \). Hence, by using (49), (58), and (68), we get \( W(ET_n) = W(ET_{n-1}) + \frac{1}{12}[28n^4 - 86n^3 + 134n^2 - 172n + 120] \).

4. The Wiener Polarity Index of the Graph Derived from Hexagonal Networks

The graphs derived from hexagonal networks are finite subgraphs of the triangular grid. In this section, Wiener polarity index of the graph derived from the hexagonal network is computed.

The graph of hexagonal network of dimension \( n \) is denoted by \( HX_n \) (Figure 3). The graph contains \( 3n^2 - 3n + 1 \) vertices and \( 9n^2 - 15n + 6 \) edges, where \( n \) is the number of
vertices on one side of the hexagon [5]. There is only one vertex \( v \) which is at distance \( n - 1 \) from every other corner vertices. This vertex is said to be the center of \( HX_n \) and is represented by \( O \).

**Theorem 5.** \( W_p(HX_n) = 27n^2 - 81n + 48 \).

**Proof.** In order to find the vertex PI index of \( HX_n \), firstly, we will divide the graph into two parts by drawing a line passing through the central vertex and parallel to \( x \)-axis. Now, extend the upper and lower part of the graph \( HX_n \) to form triangular mesh networks \( T_{2n-1} \) and \( T'_{2n-1} \) of dimension \( 2n - 1 \), respectively. Define the vertex set of \( T_{2n-1} \) and \( T'_{2n-1} \) as follows: \( V_I = \{ x_{im}: 1 \leq i \leq 2n - 1, 1 \leq m \leq l \} \) and \( V'_I = \{ x_{im}' : 1 \leq i \leq 2n - 1, 1 \leq m \leq l \} \).

It is easy to see that \( V(T_{2n-1}) \cap V(T'_{2n-1}) = V_{2n-1} \) and \( d(x_{im}, x_{im}') > 3 \) for \( i < 2n - 3 \). This implies that the Wiener polarity index of \( HX_n \) can now be written in the following form:

\[
W_p(HX_n) = 2(\{ W_p(T_{2n-1}) - W_p(HX_n, T_{n-1}) - W_p(T_{n-1}) \}) + W_p(V_{2n-2}, V_{2n-2}')
\]

\[
+ W_p(V_{2n-2}, V_{2n-2}') + W_p(V_{2n-3}, V_{2n-3}') - W_p(V_{2n-1}).
\]

From Theorem 1, we know that \( W_p(T_{n}) = 9(n^2 - 5n + 6)/2 \). This implies that

\[
W_p(T_{2n-1}) = 18n^2 - 63n + 54.
\]

Now, we calculate the terms \( W_p(V_{2n-2}, V_{2n-2}') \), \( W_p(V_{2n-2}, V_{2n-2}') \), and \( W_p(V_{2n-3}, V_{2n-3}') \), which are equal to the number of vertices of the lower triangle that are at distance 3 from the vertices of the upper triangle. However, for every \( v \in V_{2n-2} \) and \( v' \in V_{2n-2}' \), \( |(v, v')|d(v, v') = 3| = 4n - 8 \). Therefore,

\[
W_p(V_{2n-2}, V_{2n-2}') = 4n - 8.
\]

Similarly, for every \( v \in V_{2n-2} \) and \( v' \in V_{2n-3}' \), we have

\[
|(v, v')|d(v, v') = 3| = 8n - 14.
\]

And, for every \( v \in V_{2n-3} \) and \( v' \in V_{2n-2}' \), we have

\[
|(v, v')|d(v, v') = 3| = 8n - 14.
\]

This implies that

\[
W_p(V_{2n-2}, V_{2n-3}) = 4n - 8,
\]

\[
W_p(V_{2n-2}, V_{2n-2}') = 4n - 8.
\]
Table 1: Comparison of Wiener and Wiener polarity indices of the graph of triangular mesh network.

| n  | $W_p(T_n)$ | $W(T_n)$ |
|----|------------|----------|
| 2  | 0          | 3        |
| 3  | 0          | 39       |
| 4  | 9          | 159      |
| 5  | 27         | 459      |
| 6  | 54         | 1089     |
| 7  | 90         | 2265     |
| 8  | 135        | 4281     |
| 9  | 189        | 7521     |
| 10 | 252        | 12471    |
| 11 | 324        | 19731    |
| 12 | 405        | 30027    |
| 13 | 495        | 44223    |
| 14 | 594        | 63333    |
| 15 | 702        | 88533    |
| 16 | 819        | 121173   |
| 17 | 945        | 162789   |
| 18 | 1080       | 215115   |
| 19 | 1214       | 280095   |
| 20 | 1377       | 359895   |

Table 2: Comparison of Wiener and Wiener polarity indices of $ET_n$.

| n  | $W_p(ETT_n)$ | $W(ETT_n)$ |
|----|--------------|------------|
| 4  | 42           | 339        |
| 5  | 147          | 1119       |
| 6  | 315          | 2921       |
| 7  | 546          | 6522       |
| 8  | 840          | 13020      |
| 9  | 1197         | 23890      |
| 10 | 1617         | 41040      |
| 11 | 2100         | 66867      |
| 12 | 2646         | 104313     |
| 13 | 3255         | 156921     |
| 14 | 3927         | 228891     |
| 15 | 4662         | 325136     |
| 16 | 5460         | 451338     |
| 17 | 6321         | 614004     |
| 18 | 7245         | 820522     |
| 19 | 8232         | 1079217    |
| 20 | 9282         | 1399407    |

Figure 4: Comparison of Wiener and Wiener polarity indices of graph $T_n$.

Figure 5: Comparison of Wiener and Wiener polarity index of the equilateral triangular tetra sheet.
The term $W_p(HX_n, T_{n-1})$ is the cardinality of set of vertices of $T_{n-1}$ that are connected through a path of length 3 from the vertices of $HX_n$. It is easy to see that

$$W_p(HX_n, T_{n-1}) = \sum_{l=n}^{l=n+2} W_p(V_1, V_{l-1}) + \sum_{l=n}^{l=n+1} W_p(V_1, V_{l-2}) + W_p(V_n, V_{n-3})$$

$$= \sum_{l=n}^{l=n+2} W_p(V_1, V_{l-1}) + \sum_{l=n}^{l=n+1} W_p(V_1, V_{l-2}) + W_p(V_n, V_{n-1})$$

$$= \sum_{l=n}^{l=n+2} (4l - 12) + \sum_{l=n}^{l=n+1} (2l - 6 + 2n - 6)$$

$$= \sum_{l=n}^{l=n+2} (4l - 12) + \sum_{l=n}^{l=n+1} (2l - 6) + 2n - 6$$

$$= 18n - 40. \quad (74)$$

The distance between the set of vertices of the set $V_{2n-1}$ is equal to $W_p(V_{2n-1})$ and it is easy to see that

$$W_p(V_{2n-1}) = 2n - 4. \quad (75)$$

Now, by replacing the values of all these factors in equation (69) and simplifying, we get

$$W_p(HX_n) = 2(W_p(T_{2n-1}) - W_p(HX_n, T_{n-1}) - W_p(T_{n-1})) + W_p(V_{2n-1}, V_{2n-2})$$

$$+ W_p(V_{2n-2}, V_{2n-3}) + W_p(V_{2n-3}, V_{2n-2}) - W_p(V_{2n-1})$$

$$= 27n^2 - 81n + 48. \quad (76)$$

5. Conclusion

First of all, we will present comparison between two topological indices analytically and graphically.

5.1. Comparison of Wiener and Wiener Polarity Indices of the Triangular Mesh Network. The comparison between the Wiener and Wiener polarity indices of triangular mesh network $T_n$ for different values of $n$ is shown in Table 1. The values show that the Wiener index increases rapidly compared to Wiener polarity index as $n$ increases. The graphical representation of both indices is also presented. In Figure 4, the black curve denotes the behavior of the Wiener polarity index and red line shows the behavior of the Wiener index.

5.2. Comparison of Wiener and Wiener Polarity Indices of the Graph of Equilateral Triangular Tetra Sheet Networks. The comparison between the Wiener and Wiener polarity indices of equilateral triangular tetra sheet network $ET_n$ for different values of $n$ is shown in Table 2. The values show that the Wiener index increases rapidly compared to Wiener polarity index as $n$ increases. The graphical representation of both indices is also presented. In Figure 5, the black curve denotes the behavior of Wiener polarity index and red line shows the behavior of Wiener index.

Data Availability

No data were used for this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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