Flow characteristics in a crowded transport model

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Abstract

The aim of this paper is to discuss the appropriate modelling of in- and outflow boundary conditions for nonlinear drift-diffusion models for the transport of particles including size exclusion and their effect on the behaviour of solutions. We use a derivation from a microscopic asymmetric exclusion process and its extension to particles entering or leaving on the boundaries. This leads to specific Robin-type boundary conditions for inflow and outflow, respectively. For the stationary equation we prove the existence of solutions in a suitable set-up. Moreover, we investigate the flow characteristics for a small diffusion parameter \(\varepsilon\), which yields the occurrence of a maximal current phase in addition to well-known one-sided boundary layer effects for linear drift-diffusion problems. In a 1D set-up we provide rigorous estimates in terms of \(\varepsilon\), which confirm three different phases. Finally, we derive a numerical approach to solve the problem also in multiple dimensions.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Transport phenomena of crowded particles and their mathematical modelling have received considerable attention recently, driven by a variety of important applications in biology and

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social sciences, e.g. transport of ions and macromolecules through channels and nanopores (see [5, 14, 15, 17, 23]), cargo transport by molecular motors on microtubuli (see [9, 28, 35]), collective cell migration (see [16, 27, 34, 37]), tumour growth (see [25, 38]) or dynamics of human pedestrians (see [10, 24, 36]). Such applications naturally lead to questions related to boundary (or interface) conditions restricting in- or outflow of particles, and the resulting characteristics of flow. In ion channels the characteristics are relations between bath concentrations (boundary values) and current, and in pedestrian motion one is interested in flow and evacuation properties depending on exit doors, and the movement of cargo between microtubuli respectively delivery to the desired site as similar boundary conditions. A variety of computational investigations of such issues have been performed, partly with additional complications such as electrostatic interactions, chemotaxis, or herding. Such simulations can give hints on the flow behaviour, but it becomes difficult to understand the causes and asymptotic regimes for certain effects. Therefore we investigate in detail a canonical simple model with in- and outflow boundary conditions in this paper. The model we use is derived from the paradigmatic (totally) asymmetric exclusion process (T)ASEP. In this model, particles located on a 1D lattice can jump, with a certain probability, to a neighbouring cell (left or right), if the target cell is empty. Totally asymptotic refers to case when particles can only jump in one direction, see [8]. We consider such a model with appropriate modifications to include realistic in- and outflow boundaries. In a simple 1D set-up, this model was investigated recently in [40] including stochastic entrance and exit conditions, exhibiting three different phases of behaviour. We will take a continuum limit of that model and verify that these three phases are still present under the same conditions on parameters and demonstrate how the model generalizes to multiple dimensions and multiple species going in potentially different directions. The (formal) continuum limit naturally leads to the case of nonlinear convection dominating the diffusion, hence the limit of diffusion tending to zero is natural, and indeed the appearing boundary layers are separating the different phases.

Let us mention that the study of continuum limits of microscopic particle models with size exclusion effects is a very timely research topic. The majority of the rigorous analysis is however carried out for closed systems, i.e. under no-flux boundary conditions or on the whole space, where such systems possess a gradient flow structure that can be well exploited either with transport metrics (see [2, 7, 29]) or with entropy dissipation techniques (see [4, 6]). The case of non-closed systems has been studied at the continuum level mainly for Dirichlet boundary conditions, where at least the modelling is obvious. In the case of general inflow and outflow conditions the modelling of boundary conditions needs to be adapted to the specific approach for deriving continuum equations (see [18]), which seems to have been ignored by most authors in the past. Moreover, the case of non-equilibrium boundary conditions poses additional challenges on the analysis, in particular existence proofs for stationary solutions cannot be carried out any more by explicit computations or energy minimization arguments. Nonetheless, some arguments can still benefit from the underlying gradient flow structure in the energy, in particular a transformation to dual variables (also called entropy variables) is quite beneficial for existence proofs, since it yields maximum principles that do not hold for the original variables (see [4, 5, 26]). In this paper we will use similar ideas and extend them from Dirichlet to inflow and outflow boundary conditions.

This paper is organized as follows: in section 2 we present the model for several species and give more details about the nonlinear boundary conditions. In section 3 we present existence proofs for a single species. We separately treat the cases when the velocity field is either a given divergence free vector field or the gradient of a potential function. In section 4 we investigate the behaviour for a small diffusion coefficient and compare this to the results presented in [40]. Finally, in section 5, we introduce a discontinuous Galerkin scheme and present examples in one and two spatial dimensions.
2. Modelling

Crowding models based on (totally) asymmetric exclusion processes as well as their mean-field continuum limits have gained strong attention recently (see e.g. [32, 33] and the references above). The main paradigm is to model jumps of particles on a discrete lattice with jump probabilities consisting of unoriented parts (diffusion) and oriented drifts (transport). The exclusion is incorporated by avoiding jumps to an occupied cell. Using standard continuum limits (rescaling time and space to have grid sizes and typical waiting times converge to zero) as well as simple mean-field closure assumptions, which can also be made rigorous (see [20]), one obtains partial differential equations of the form

\[
\partial_t \rho + \nabla \cdot j = 0, \quad j = -D \nabla \rho + \rho(1 - \rho)u,
\]

for \( x \in \Omega \subset \mathbb{R}^n, t > 0 \) and \( \rho = \rho(x, t) \) and where \( u : \mathbb{R}^n \to \mathbb{R}^n \) is a given velocity field. We mention that with similar arguments as in [25, 38], this equation can also be derived from standard continuum fluid mechanical models adding a congestion constraint. In the previously well-investigated case of a potential field \( u = -\nabla V \) for some \( V : \mathbb{R}^n \to \mathbb{R} \) (see [4]), the equation can be recast in gradient form

\[
\partial_t \rho = \nabla \cdot (D(1 - \rho)\rho \nabla (\partial_j E[\rho])),
\]

with the entropy functional

\[
E[\rho] = \int_{\Omega} (\rho \log \rho + (1 - \rho) \log(1 - \rho) - \rho V) \, dx.
\]

The above differential equation has been studied in detail with potential fields and no-flux boundary conditions, also for more than one active species, where one obtains a system which is indeed a gradient flow and stationary solutions can be characterised as minimisers of the entropy at fixed mass (see [4]). In many practical applications different boundary conditions and non-zero flow is of fundamental importance however. In [5] the case of mixed no-flux and Dirichlet boundary conditions has been studied in a model for charged particles coupled with the Poisson equation. Here we want to focus on in- and outflow boundaries, as recently also used in 1D stochastic models [40].

2.1. Boundary conditions

We assume that the boundary \( \partial \Omega \) of our domain is subdivided into three parts: inflow \( \Gamma \), outflow \( \Sigma \) and insulating \( \partial \Omega \setminus (\Gamma \cup \Sigma) \) with \( \Gamma \cap \Sigma = \emptyset \). We assume that particles enter the domain \( \Omega \) on \( \Gamma \) with rate \( \alpha > 0 \) and exit at \( \Sigma \) with rate \( \beta > 0 \). On the remaining part of the boundary, we impose no flux conditions. Without exclusion principle, this would simply mean in the continuum that the normal flux equals \( \alpha \) at the entrance. Modelling volume exclusion in the discrete setting means that the particle can only enter a grid cell adjacent to the boundary if it is empty. Hence, the probability of entering is modified to \( \alpha(1 - \rho) \), and we deduce the boundary conditions

\[
-j \cdot n = \alpha(1 - \rho) \quad \text{on } \Gamma,
\]

\[
j \cdot n = \beta \rho \quad \text{on } \Sigma,
\]

\[
j \cdot n = 0 \quad \text{on } \partial \Omega \setminus (\Gamma \cup \Sigma).
\]
Note the negative sign in front of the normal flux at the entrance since we use the convention of a normal oriented outward. The boundary condition at the inflow boundary can be rewritten as

\[ D(1 - \rho) \rho \nabla \left( \log \frac{\rho}{(1 - \rho)} \right) \cdot n = (1 - \rho)(\alpha + \rho u \cdot n), \]

which clarifies the role of the normal velocity at the inflow boundary. The inflow rate can balance the normal velocity only if \( u \cdot n \leq 0 \). On the other hand we will have \( \rho \leq 1 \) and thus, balancing only for \( \alpha \leq 1 \). At the outflow boundary we have

\[ -D(1 - \rho) \rho \nabla \left( \log \frac{\rho}{(1 - \rho)} \right) \cdot n = \rho(\beta - (1 - \rho)u \cdot n), \]

hence \( u \cdot n \geq 0 \) is needed for balancing.

3. Basic properties

Due to the boundary conditions there is obviously no mass conservation in the system. However, there is still a natural balance condition between in- and outflow, i.e. if \( \rho \) solves (2.1) then

\[ \partial_t \int_\Omega \rho \, dx = \int_\Sigma \beta \rho \, d\sigma - \int_\Gamma \alpha(1 - \rho) \, d\sigma, \quad (3.1) \]

where \( \alpha \) and \( \beta \) denote the in- and outflow rate for \( \rho \), respectively. In the stationary case the two integrals need to balance, which implies an interesting coupling in the balance conditions (via \( (1 - \rho) \)). Note also that in an evacuation case, i.e. \( \Sigma = \emptyset \), the mass in the system is monotonically decreasing as it is expected. Finally, we state the following assumptions for later use.

**Assumption 3.1.** (A1) \( \Omega \subset \mathbb{R}^n \) bounded, \( n = 1, 2, 3 \) with boundary \( \partial \Omega \) of class \( C^2 \). Furthermore, let \( \Gamma \) and \( \Sigma \) be closed subsets of \( \partial \Omega \) such that \( \Gamma \cap \Sigma = \emptyset \).

(A2) \( 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, \) and \( \min\{\alpha, \beta\} < 1 \).

(A3) \( u \in [W^{1,\infty}(\Omega)]^n \) such that \( \nabla \cdot u = 0, u \cdot n = -1 \) on \( \Gamma \), \( u \cdot n = 1 \) on \( \Sigma \), and \( u \cdot n = 0 \) on \( \partial \Omega \setminus (\Gamma \cup \Sigma) \).

(A3’) \( V \in H^1(\Omega) \).

**Remark 3.2.** Let us briefly comment on the assumptions: (A1) is needed to obtain enough regularity for the linearized problems employed in the existence proof. Assumption (A3) and in particular the restriction of the unit normal derivative can be explained by our modelling: since we expect particles to enter the domain at \( \Gamma \) and to exit at \( \Sigma \) it is reasonable to assume that the velocity field in normal direction points inwards (i.e. \( u \cdot n \leq 0 \) at \( \Gamma \)) at the entrances and outwards (i.e. \( u \cdot n \geq 0 \) at \( \Sigma \)) at the exits. Our analysis can be carried out with these assumption instead of enforcing (A3), however at the expense that some expressions become lengthy since we always have consider the inf and sup of \( u \) on the boundary. See also remark 3.7 for further details.

3.1. Existence of stationary solutions

We shall present two different proofs, one for a given velocity vector field \( u \) and a second one where \( u = \nabla V \) for some potential \( V \), in which case we can employ a transformation to
so-called ‘entropy variables’. Note that we used a diffusion coefficient equal to one, but all results of this section remain true for an arbitrary value $D > 0$ and even for regular spatially varying coefficients. This is important for the next section, where we study the natural case of a small diffusion coefficient.

**Definition 3.3 (Weak solution).** For $u$ satisfying assumption (A3) we say that a function $\rho \in W^{1,\infty}(\Omega)$ is a weak solution to the equation

$$\nabla \cdot (-\nabla \rho + \rho (1 - \rho) u) = 0, \quad (3.2)$$

supplemented with the boundary conditions (2.4)–(2.6) if the identity

$$\int_\Omega \nabla \rho \cdot \nabla \varphi - \rho (1 - \rho) u \cdot \nabla \varphi \, dx - \int_\Gamma \alpha (1 - \rho) \varphi \, d\sigma + \int_\Sigma \beta \rho \varphi = 0 \quad (3.3)$$

holds for all $\varphi \in H^1(\Omega)$.

**Theorem 3.4 (Incompressible case).** Let the assumptions (A1)–(A3) hold and also $0 < \alpha, \beta < 1$. Then, the equation

$$\nabla \cdot (-\nabla \rho + \rho (1 - \rho) u) = 0,$$

supplemented with the boundary conditions (2.4)–(2.6) has at least one weak solution $\rho$ in the sense of definition 3.3. In addition, $\rho \in W^{1,\infty}(\Omega)$ and the bounds

$$\min \{\alpha, 1 - \beta\} \leq \rho(x) \leq \max \{\alpha, 1 - \beta\} \quad (3.4)$$

hold.

**Proof.** The proof is based on Schauder’s fixed-point theorem. We define the set

$$\mathcal{M} = \left\{ \rho \in W^{1,\infty}(\Omega) \mid \min \{\alpha, 1 - \beta\} \leq \rho \leq \max \{\alpha, 1 - \beta\}, \|\rho\|_{W^{1,\infty}(\Omega)} \leq C_M \right\}$$

with a constant $C$ that will be specified below and will depend only on $\|u\|_{L^\infty(\Omega)}$, $\alpha$, $\beta$ and $\Omega$. We consider $\mathcal{M}$ as a closed set in the strong topology of $W^{1,\infty}(\Omega)$. For given $\bar{\rho} \in \mathcal{M}$, we now consider the linearized problem

$$-\nabla \cdot \nabla \bar{\rho} + (1 - 2\bar{\rho}) \nabla \cdot u = 0, \quad (3.5)$$

supplemented with the boundary conditions (remember (A3))

$$\nabla \rho \cdot n + (1 - \bar{\rho}) \rho = \alpha (1 - \bar{\rho}), \quad \text{on } \Gamma, \quad (3.6)$$

$$\nabla \rho \cdot n + \bar{\rho} \rho = (1 - \bar{\rho}) \bar{\rho}, \quad \text{on } \Sigma, \quad (3.7)$$

$$\nabla \rho \cdot n = 0, \quad \text{on } \partial \Omega \setminus (\Gamma \cup \Sigma). \quad (3.8)$$

Note that we linearized the boundary conditions differently on $\Gamma$ and $\Sigma$, this will be crucial later on. Due to our assumptions on $\alpha$ and $\beta$ we have that both $\bar{\rho}$ and $1 - \bar{\rho}$ are uniformly greater than zero. Thus we have true Robin conditions on part of the boundary. Since furthermore $\bar{\rho} \in W^{1,\infty}(\Omega)$, we can apply [12, theorem 3.28] which yields the existence of a unique solution $\rho \in W^{2,4}(\Omega)$ with the $a priori$ estimate.
\[ \|\rho\|_{W^{1,4}(\Omega)} \leq C(\|\alpha(1 - \bar{\rho})\|_{W^{1,4}(\Gamma')} + \|((1 - \beta)\bar{\rho})\|_{W^{1,4}(\Sigma')}). \] (3.9)

Note that our definition of the set \( \mathcal{M} \) ensures that all terms on the right hand side are uniformly bounded since the trace theorem for Sobolev spaces ([21, theorem 1.5.1.2]) implies
\[ \|\alpha(1 - \bar{\rho})\|_{W^{1,4}(\Gamma')} + \|((1 - \beta)\bar{\rho})\|_{W^{1,4}(\Sigma')} \leq C_2\|\bar{\rho}\|_{W^{1,\infty}(\Omega)}. \] (3.10)

This allows us to define the operator \( S : \mathcal{M} \to W^{2,4}(\Omega) \) that maps \( \bar{\rho} \) to the solution of (3.5)–(3.8). In order to apply Schauder’s fixed-point theorem, we have to prove that \( S \) is self-mapping from \( \mathcal{M} \) to \( \mathcal{M} \), continuous and compact.

**Self-mapping:** Equation (3.5) satisfies a maximum principle with vanishing normal derivative on \( \partial \Omega \setminus (\Gamma \cup \Sigma) \), and thus (by Hopf’s maximum principle) \( \rho \) attains its maximum on \( \Gamma \cup \Sigma \). We have to distinguish the following cases:

- \( \rho \) attains its maximum on \( \Gamma \) and thus \( 0 \leq \nabla \cdot n = (\alpha - \rho)(1 - \bar{\rho}). \) Since by assumption \( (1 - \bar{\rho}) \geq 0 \), this implies \( \rho \leq \alpha \).
- \( \rho \) attains its maximum on \( \Sigma \) we conclude \( 0 \geq -\nabla \cdot n = (\beta - (1 - \rho))\bar{\rho} \) which, since \( \bar{\rho} \geq 0 \) yields \( \rho \leq 1 - \beta \).

If \( \rho \) attains its minimum on the boundary we use the same arguments to conclude \( \alpha \leq \rho \) and \( 1 - \beta \leq \rho \). Finally, the \( W^{1,\infty} \)-bound on \( \rho \) (with \( C_4 = C_2 \)) is a consequence of (3.9) and (3.10).

**Continuity:** To show continuity of \( S \) we take a sequence \( \rho_k \) in \( \mathcal{M} \) such that
\[ \|\rho_k - \bar{\rho}\|_{W^{1,\infty}(\Omega)} \to 0. \]

We have to show that the sequence \( \rho_k = S(\rho_k) \) converges to \( \rho = S(\bar{\rho}) \). Subtracting the equations for \( \rho_k \) and \( \rho \), we have
\[ -\nabla \cdot \nabla (\rho - \rho_k) + (1 - 2\bar{\rho})\nabla (\rho - \rho_k) \cdot u = 2(\rho - \rho_k)\nabla \rho_k \cdot u, \] (3.11)

with boundary conditions
\[ \nabla (\rho - \rho_k) \cdot n + (1 - \rho)(\rho - \rho_k) = (\alpha - \rho)(\rho - \rho_k), \quad \text{on } \Gamma, \]
\[ \nabla (\rho - \rho_k) \cdot n + \bar{\rho}(\rho - \rho_k) = (\beta + \rho)(\rho - \rho_k), \quad \text{on } \Sigma, \]
\[ \nabla (\rho - \rho_k) \cdot n = 0, \quad \text{on } \partial \Omega \setminus (\Gamma \cup \Sigma). \]

Since \( \rho \) and \( \rho_k \) are uniformly bounded on \( W^{1,\infty}(\Omega) \), all assumptions of theorem [12, theorem 3.28] are satisfied and we obtain the *a priori* estimate
\[ \|\rho - \rho_k\|_{W^{2,4}(\Omega)} \leq C(\|2(\bar{\rho} - \rho_k)\nabla \rho \cdot u\|_{L^4(\Omega)} + \|((\beta + \rho)(\rho - \rho_k))\|_{W^{1,4}(\Sigma')}) \] (3.12)

Since \( \rho_k \to \rho \) in \( W^{2,4}(\Omega) \), and thus in particular in every \( L^p \) and by the trace theorem also in \( W^{3/2,4}(\Gamma) \) and \( W^{3/2,4}(\Gamma) \), the right hand side of (3.12) tends to zero as \( k \to \infty \). The fact that \( W^{2,4}(\Omega) \) is continuously embedded into \( W^{1,\infty}(\Omega) \) yields the continuity of \( S \).

**Compactness:** The compactness of the operator \( S \) follows from the fact that the embedding \( W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \) is compact for \( n \leq 3 \), see [1, theorem 6.2]. This completes the proof. \( \square \)
Next we treat the potential case \( u = \nabla V \), where we obtain the following

**Theorem 3.5 (Potential case).** Let the assumptions (A1), (A2) and (A3') hold. Then, the equation
\[
\nabla \cdot (-\nabla \rho + \rho(1 - \rho)\nabla V) = 0, \quad x \in \Omega,
\]
supplemented with the boundary conditions (2.4)–(2.6) has at least one weak solution \( u \in H^1(\Omega) \cap L^\infty(\Omega) \) in the sense of definition 3.3 such that \( 0 \leq \rho \leq 1 \).

If additionally \( \Delta V = 0 \), \( \partial_n V = -1 \) on \( \Gamma \), \( \partial_n V = 1 \) on \( \Sigma \) holds, the solution \( \rho \) satisfies the maximum principle
\[
\min\{\alpha, 1 - \beta\} \leq \rho(x) \leq \max\{\alpha, 1 - \beta\}.
\]

**Proof.** Our proof is based on an approximation procedure, applied to the equation in entropy variables which are defined as the variation of the entropy functional (2.3) with respect to the density \( \rho \).
\[
\psi := \partial E[\rho] = \log \rho - \log(1 - \rho) - V.
\]

We apply this transformation to (3.13), (2.4)–(2.6) and define
\[
A(\psi, V) := \frac{e^{\psi + V}}{(1 + e^{\psi + V})^2}.
\]

We then obtain the nonlinear equation
\[
-\nabla \cdot (A(\psi, V)\nabla \psi) = 0,
\]
supplemented with the boundary conditions
\[
A\nabla \psi \cdot n = \begin{cases} 
\frac{\alpha}{1 + e^{\psi + V}}, & \text{on } \Gamma, \\
-\frac{\beta e^{\psi + V}}{1 + e^{\psi + V}}, & \text{on } \Sigma, \\
0, & \text{on } \partial \Omega \setminus (\Gamma \cup \Sigma).
\end{cases}
\]

We will now apply an approximation procedure to this equation and proceed in several steps. **Existence for an auxiliary problem:** For \( \delta > 0 \) and with \( A_\delta(\psi, V) := A(\psi, V) + \delta \) we consider the regularised problem
\[
-\nabla \cdot (A_\delta(\psi^\delta, V)\nabla \psi^\delta) + \delta \psi^\delta = 0,
\]
and \( A_\delta \nabla \psi^\delta \cdot n = \begin{cases} 
\frac{\alpha}{1 + e^{\psi^\delta + V}}, & \text{on } \Gamma, \\
-\frac{\beta e^{\psi^\delta + V}}{1 + e^{\psi^\delta + V}}, & \text{on } \Sigma, \\
0, & \text{on } \partial \Omega \setminus (\Gamma \cup \Sigma).
\end{cases}
\]

To prove existence of a weak solution to (3.17)–(3.18), we use a fixed-point argument. For given \( \tilde{\psi} \in L^2(\Omega) \) we define \( A_\delta(x) = A_\delta(\tilde{\psi}(x), V(x)) \) for a.e. \( x \in \Omega \) which yields the linear equation...
\[-\nabla \cdot (\tilde{A}_\delta \nabla \tilde{\psi}^\delta) + \delta \tilde{\psi}^\delta = 0, \tag{3.19}\]

subject to the nonlinear boundary conditions

\[
\tilde{A}_\delta \nabla \tilde{\psi}^\delta \cdot n = \begin{cases} 
\alpha \frac{1}{1 + e^{\psi^\delta + V}}, & \text{on } \Gamma, \\
-\beta \frac{e^{\psi^\delta + V}}{1 + e^{\psi^\delta + V}}, & \text{on } \Sigma, \\
0, & \text{on } \partial \Omega \setminus (\Gamma \cup \Sigma).
\end{cases} \tag{3.20}\]

The corresponding weak formulation is the Euler–Lagrange equation to the nonlinear minimisation problem for the energy functional

\[
E(\tilde{\psi}^\delta) = \frac{1}{2} \int_\Omega (\tilde{A}_\delta |\nabla \tilde{\psi}^\delta|^2 + \delta |\tilde{\psi}^\delta|^2) \, dx - \alpha \int_\Gamma F(\tilde{\psi}^\delta, V) \, d\sigma + \beta \int_\Sigma G(\tilde{\psi}^\delta, V) \, d\sigma,
\]

where \(F\) and \(G\) are chosen such that

\[
\partial_\psi F(\psi, V) = \frac{1}{1 + e^{\psi^\delta + V}}, \quad \partial_\psi G(\psi, V) = \frac{e^{\psi^\delta + V}}{1 + e^{\psi^\delta + V}}. \tag{3.21}\]

Note that for given \(V\), the function \(F(\cdot, V)\) is concave (and thus \(-F\) is convex) while \(G(\cdot, V)\) is convex, since their second derivatives are non-positive and non-negative, respectively. Furthermore, by definition we have that \(\tilde{A}_\delta \in L^\infty(\Omega)\), uniformly with \(\delta \leq \tilde{A}_\delta(\alpha) \leq \delta + 1/4\). Thus \(E(\tilde{\psi}^\delta)\) is coercive with respect to the \(H^1\)-norm and due to its convexity we conclude (see [19, section 8.2, theorems 2 and 3]) the existence of a unique minimiser \(\tilde{\psi}^\delta \in H^1(\Omega)\), which is by definition a weak solution to (3.19). Furthermore, since \(G(\tilde{\psi}^\delta, V) \geq 0\) and \(F(\tilde{\psi}^\delta, V) < \infty\), we infer

\[
\int_\Omega |\nabla \tilde{\psi}^\delta|^2 + |\tilde{\psi}^\delta|^2 \, dx \leq \frac{2}{\delta} \left( E(\tilde{\psi}^\delta) + \alpha \int_\Gamma F(\tilde{\psi}^\delta, V) \, d\sigma \right) \leq \frac{2}{\delta} \left( E(0) + \alpha \int_\Gamma F(\tilde{\psi}^\delta, V) \, d\sigma \right). \tag{3.22}\]

From (3.21) we know that the derivative of \(F\) with respect to its first argument is bounded between 0 and 1. Thus, we can estimate \(F\) from above by the identity and obtain

\[
\frac{2}{\delta} \left( E(0) + \alpha \int_\Gamma F(\tilde{\psi}^\delta, V) \, d\sigma \right) \leq \frac{C}{\delta} \left( E(0) + \alpha \int_\Gamma \tilde{\psi}^\delta \, d\sigma \right).
\]

Using the usual trace inequality for \(H^1\)-functions ([19, section 5.5, theorem 1]) and Cauchy’s inequality with \(c\) (see [19, appendix B.2]), the last term on the right hand side can be estimated by a constant that can be made arbitrarily small. Thus we can subtract this term on both sides of the inequality (absorb it in the left side of (3.22)) which yields

\[
\int_\Omega |\nabla \tilde{\psi}^\delta|^2 + |\tilde{\psi}^\delta|^2 \leq C_M. \tag{3.23}\]

Here the constant \(C_M\) depends on the geometry, \(\alpha, \beta, \delta\). For fixed \(\delta\) this implies that \(\tilde{\psi}^\delta\) is uniformly bounded in \(H^1(\Omega)\). This result, together with the compactness of the embedding \(H^1(\Omega) \hookrightarrow L^2(\Omega)\) allows us to define the compact, selfmapping nonlinear operator \(K : \mathcal{M} \to \mathcal{M}\).
with \( \mathcal{M} := \{ \psi \in L^2(\Omega) \mid \| \psi \|_{H^1(\Omega)} \leq C_M \} \) which maps \( \tilde{\psi} \) to the solution of (3.19)–(3.20). To apply Schauder’s fixed point theorem it remains to show its continuity. We consider a sequence \( \tilde{\psi}_n \) that converges to \( \tilde{\psi} \) in \( L^2(\Omega) \). This yields a sequence \( \tilde{\psi}_n \in H^1(\Omega) \) having a weak limit. Since \( A_\delta \) is uniformly bounded in \( L^\infty(\Omega) \), an application of Lebesgue’s theorem yields \( A_\delta(\tilde{\psi}_n) \to A_\delta(\tilde{\psi}) \) in \( L^p(\Omega) \) for any \( p < \infty \). Since \( \nabla V \in [L^2(\Omega)]^n \) only, an additional approximation of the test functions in \( W^{1,\infty}(\Omega) \) allows us to pass to the limit in the weak formulation of (3.13). Uniqueness of the weak solution (due to the convexity of the Energy \( E \)) thus implies \( K(\tilde{\psi}_n) \to K(\tilde{\psi}) \) in \( L^2(\Omega) \) and we conclude the existence of a solution \( \psi^\delta \) to the auxiliary problem (3.19)–(3.20).

Limit \( \delta \to 0 \): To this end, we define

\[
\rho^\delta = \frac{e^{\psi^\delta + V}}{1 + e^{\psi^\delta + V}}.
\]

Choosing the test function \( \varphi = \psi^\delta \), we obtain that \( \rho^\delta \in H^1(\Omega) \) satisfies

\[
0 = \int_{\Omega} (\nabla \rho^\delta \cdot \nabla \psi^\delta - \rho^\delta (1 - \rho^\delta) \nabla V \cdot \nabla \psi^\delta) \, dx + \delta \int_{\Omega} |\nabla \psi|^2 + (\psi^\delta)^2 \, dx
\]

\[
- \alpha \int_{\Gamma} (1 - \rho^\delta) \psi^\delta \, d\sigma + \beta \int_{\Sigma} \rho^\delta \psi^\delta \, d\sigma.
\]

(3.24)

Our aim is to derive a priori estimates for \( \rho^\delta \) by estimating each term in (3.24) separately, noting that \( \nabla \psi^\delta = \frac{1}{\rho^\delta (1 - \rho^\delta)} \nabla \rho^\delta - \nabla V \). For the first term we have

\[
\int_{\Omega} \frac{|\nabla \rho^\delta|^2}{\rho^\delta (1 - \rho^\delta)} \, dx \geq \frac{1}{2} \int_{\Omega} \frac{|\nabla \rho^\delta|^2}{\rho^\delta (1 - \rho^\delta)} \, dx - \int_{\Omega} \rho^\delta (1 - \rho^\delta) |\nabla V|^2 \, dx \geq 2 \int_{\Omega} |\nabla \rho^\delta|^2 \, dx - \frac{1}{4} \int_{\Omega} |\nabla V|^2 \, dx,
\]

where we used Cauchy’s inequality to estimate the mixed term and the fact that \( \rho^\delta (1 - \rho^\delta) \leq 1/4 \).

For the second we write

\[
- \alpha \int_{\Gamma} (1 - \rho^\delta) \psi^\delta \, d\sigma = \alpha \int_{\Gamma} \left( (1 - \rho^\delta) \log \frac{1 - \rho^\delta}{\rho^\delta} + 2 \rho^\delta - 1 \right) \, d\sigma + \alpha \int_{\Gamma} \left( (1 - \rho^\delta) V + 1 - 2 \rho^\delta \right) \, d\sigma.
\]

The first term in this equation is a Kullback–Leibler distance and thus non-negative. As \( V \in H^1(\Omega) \), the trace theorem yields \( V|_{\partial \Omega} \in L^2(\partial \Omega) \) and since, by definition \( 0 \leq \rho^\delta \leq 1 \), the second term is bounded. The same arguments yield that also the third term in (3.24) can be split into a non-negative and a bounded term. Summarising, we obtain

\[
\int_{\Omega} |\nabla \rho^\delta|^2 \, dx \leq \frac{1}{8} \int_{\Omega} |\nabla V|^2 \, dx + \alpha \int_{\Gamma} \left( (1 - \rho^\delta) V + 1 - 2 \rho^\delta \right) \, d\sigma - \beta \int_{\Sigma} (\rho^\delta V - 1 + 2 \rho^\delta) \, d\sigma.
\]

These estimates yield a a priori bound for \( \rho^\delta \) in \( H^1(\Omega) \). Due to \( 0 \leq \rho^\delta \leq 1 \), we additionally obtain strong convergence in every \( L^p, p < \infty \) which allows us to pass to the limit in the weak formulation, using again a standard approximation argument for the test functions in \( W^{1,\infty}(\Omega) \).
To prove the bounds (3.4), we recall the additional assumptions $\Delta V = 0$, $\partial_\nu V = -1$ on $\Gamma$, $\partial_\nu V = 1$ on $\Sigma$. Since for $\Delta V = 0$ the equation (3.13) fulfills a maximum principle, and the assertion (3.14) follows by similar arguments as in the proof of theorem 3.4.

\section*{Remark 3.6.} Note that testing the weak form with the entropy variable $\psi^S$ results in estimates analogous to those that are obtained by the entropy dissipation method in the time dependent case. See [4, 26] for details.

\section*{Remark 3.7.} If instead of assumption (A3) we only impose the weaker conditions $u \cdot n \leq 0$ at $\Gamma$ and $u \cdot n \geq 0$ at $\Sigma$, the previous analysis still works, however the statements of the maximum principles become

$$\min \left\{ - \sup \left\{ \alpha \frac{\partial_\nu}{u \cdot n} \right\}, \inf_{x \in \Sigma} \left( 1 - \frac{\beta}{u \cdot n} \right) \right\} \leq \rho(x) \leq \max \left\{ - \inf_{x \in \Gamma} \frac{\alpha}{u \cdot n}, \sup_{x \in \Sigma} \left( 1 - \frac{\beta}{u \cdot n} \right) \right\},$$

i.e. one has to account for the values of the velocity field in normal direction.

\section*{4. Asymptotic unidirectional flow characteristics}

In the following we discuss in detail the flow properties of the single species model for small diffusion $D = \varepsilon \ll 1$, in particular for the stationary solution $\rho \in H^1(\Omega) \times L^\infty(\Omega)$ of

$$\nabla \cdot (-\varepsilon \nabla \rho + \rho (1 - \rho) u) = 0, \quad \text{in } \Omega$$

with boundary conditions (2.4)–(2.6). We are interested in the asymptotic behaviour as $\varepsilon \downarrow 0$, in particular the boundary layers and the asymptotic flow patterns, which we expect to be characterised by three different phases. For small $\varepsilon$, we expect boundary layers while the asymptotic density, i.e. the weak limit of $\rho$ as $\varepsilon \downarrow 0$, is constant in $\Omega$. Following the typical nomenclature as in [12], these can be categorized as:

- An \textit{influx-limited} phase with an asymptotically low density ($\rho < \frac{1}{2}$) corresponding to a density of outgoing particles on $\Sigma$ and with a boundary layer at the exit $\Sigma$.
- An \textit{outflux-limited} phase with an asymptotically high density ($\rho > \frac{1}{2}$) corresponding to a density of incoming particles on $\Gamma$, with a boundary layer at the entrance $\Gamma$ created by lower outflux rates.
- A \textit{maximal current} phase with asymptotic density $\rho = \frac{1}{2}$ and boundary layers both at $\Sigma$ and $\Gamma$, which occurs at high in- and outflow rates.

See also figure 2 for numerical examples of the three phases for positive but small $\varepsilon$. Note that a direct conclusion from the maximum principle (3.4) is the non-appearance of a maximal current phase if

$$\frac{1}{2} \notin \min \{ \alpha, (1 - \beta) \}, \max \{ \alpha, (1 - \beta) \}.$$

In this case the maximum principle implies that the densities are bounded away from $\frac{1}{2}$ uniformly in $\varepsilon$. 

\hfill 3537
4.1. Characterization of phases in the 1D flow

We now turn to the 1D case with constant velocity, where we can use a scaling of space and flow such that \( \Omega = [0, 1], u \equiv 1, \Gamma = \{0\}, \) and \( \Sigma = \{1\}. \) This setting corresponds exactly to the continuum limit of the setting in \([40]\), and we will rigorously show that indeed the same behaviour as in the TASEP with stochastic entrance and exit conditions— with transitions between the phases at exactly the same parameter values—appears for the continuum limit.

We remark that in one spatial dimension one can try to obtain the results of this section by analysing explicit solutions, which are available at least for special choices of \( j, \alpha \) and \( \beta. \) However, since we are aiming at an general understanding of the behaviour of our model— possibly also in higher space dimensions—we rather work with the differential equation itself. See also remark 4.5. For convenience we restate the 1D version of \((4.1), (2.4)–(2.6)\) as

\[
\rho_t - \epsilon \partial_x \rho + \partial_x (\rho (1 - \rho)) = 0 \quad \text{in } (0, 1),
\]

with boundary conditions

\[
\epsilon \partial_x \rho = (1 - \rho)(\rho - \alpha) \quad \text{at } x = 0,
\]

\[
\epsilon \partial_x \rho = \rho (1 - \rho - \beta) \quad \text{at } x = 1.
\]

A first result particular for the 1D case is the uniqueness of a solution:

**Proposition 4.1.** There exists exactly one weak solution \( \rho \in H^1(\Omega) \) of \((4.2)–(4.4).\)

**Proof.** Let \( \rho_1 \) and \( \rho_2 \) be two solutions, then \( w = \rho_1 - \rho_2 \) satisfies

\[
-\epsilon \partial_x w + \partial_x ((1 - \rho_1 - \rho_2)w) = 0
\]

with boundary conditions

\[
\epsilon \partial_x w = (1 - \rho_1 - \rho_2)w = -\alpha w \quad \text{at } x = 0,
\]

\[
-\epsilon \partial_x w + (1 - \rho_1 - \rho_2)w = \beta w \quad \text{at } x = 1.
\]

Now let \( V \in H^2([0, 1]) \) be such that \(-\epsilon \partial_x V = (1 - \rho_1 - \rho_2)\) and \( w = e^V v. \) Then, \( v \) is the weak solution of

\[
\partial_x (e^V \partial_x v) = 0
\]

in \( (0,1) \) with boundary conditions

\[
\epsilon \partial_x v = \alpha v \quad \text{at } x = 0,
\]

\[
\epsilon \partial_x v = -\beta v \quad \text{at } x = 1.
\]

Using the weak formulation of this boundary value problem with test function \( v \) implies

\[
\int_0^1 e^V v^2 \, dx + \alpha e^{(0)}v(0)^2 + \beta e^{(0)}v(1)^2 = 0,
\]

which yields \( v \equiv 0 \) and thus uniqueness of the solution. \( \square \)

We start our analysis of the flow properties with a simple calculation relating the difference of \( \rho \) to the constant state \( \frac{1}{2} \) to the boundary values:
Lemma 4.2. Let \( \rho \in H^1([0, 1]) \) be the unique weak solution of (4.2)–(4.4). Then the estimate
\[
\int_0^1 \left( \rho - \frac{1}{2} \right)^2 \, dx + \beta \rho(1) - \frac{1}{4} \leq \epsilon |1 - \alpha - \beta| \tag{4.5}
\]
holds.

Proof. Using the test function \( \varphi(x) = x \) in the weak form of (4.2) and adding and subtracting \( \frac{1}{2} \) we find
\[
\int_0^1 (\epsilon \partial_x \rho + (\rho - \frac{1}{2})^2) \, dx + \beta \rho(1) - \frac{1}{4} = 0.
\]
Further integrating the first term and using the a priori bounds from the maximum principle for the boundary values concludes the proof.

Lemma 4.2 will yield the desired asymptotic estimate if we can guarantee that \( \beta \rho(1) \geq \frac{1}{4} \), such that the second term on the left-hand side is nonnegative. Note also that in spatial dimension one the flux is constant, thus we find \( \beta \rho(1) = \alpha(1 - \rho(0)) \), i.e. the above result could equally be formulated in terms of \( \alpha \) respectively the inflow boundary value. To prove the latter under appropriate conditions is the objective of the next result:

Theorem 4.3 (Maximal current phase). Let \( \rho \in H^1([0, 1]) \) be the unique weak solution of (4.2)–(4.4) and let
\[\min\{\alpha, \beta\} \geq \frac{1}{2} \]
Then the estimate
\[
\int_0^1 \left( \rho - \frac{1}{2} \right)^2 \, dx \leq \epsilon |1 - \alpha - \beta| \tag{4.6}
\]
holds and furthermore, we have \( j \geq 1/4 \).

Proof. Using lemma 4.2 it suffices to show \( \beta \rho(1) \geq 1/4 \), which we carry out by contradiction. Assume \( \rho(1) = \frac{1}{4\beta} - \delta \) with \( \delta > 0 \). Since \( \beta \geq \frac{1}{2} \) and \( \alpha \geq \frac{1}{2} \) we conclude in particular
\[
\rho(1) = \frac{1}{4\beta} - \delta \leq \frac{1}{2} \quad \text{and} \quad \rho(0) = 1 - \frac{\beta}{\alpha} \rho(1) \geq 1 - \frac{1}{4\alpha} + \frac{\beta}{\alpha} \delta \geq \frac{1}{2} + \frac{\beta}{\alpha} \delta.
\]
Let \( H \) be a smooth monotone function such that
\[
H(0) = 0, \quad H(1) = 1, \quad \text{supp} (H') \subset \left( \frac{1}{2} - \gamma, \frac{1}{2} + \gamma \right)
\]
with \( \gamma < \min\{\delta, \frac{\beta}{\alpha} \delta\} \). Now we choose the test function \( \varphi = H(\rho) \) in the weak form of (4.2) again with \( \frac{1}{4} \) added and subtracted. Then we find
\[
\int_0^1 (\epsilon H'(\rho) |\partial_x \rho|^2 - H'(\rho) \rho(1 - \rho) \partial_x \rho) \, dx + \beta \rho(1)(H(\rho(1)) - H(\rho(0))) = 0.
\]
Using the nonnegativity of the first term and rewriting the second term yields

\[
\beta \rho(1) - \frac{1}{4} \left( H(\rho(1)) - H(\rho(0)) \right) \leq - \int_0^1 H'(\rho) \left( \rho - \frac{1}{2} \right)^2 \partial_x \rho \, dx = F(\rho(0)) - F(\rho(1)),
\]

where \( F \) satisfies \( F'(p) = H'(p)(\rho - \frac{1}{2})^2 \) and \( F(0) = 0 \). With the properties of \( H \) it is straightforward to see that

\[
H(\rho(1)) - H(\rho(0)) = 1, \quad F(\rho(1)) = 0, \quad F(\rho(0)) \leq 3\gamma^2.
\]

Hence, we conclude

\[-\delta = \beta \rho(1) - \frac{1}{4} \geq 3\gamma^2,
\]

which is a contradiction for \( \gamma \) sufficiently small. Since the flux is constant, the fact that \( j \geq 1/4 \) follows immediately from (4.4).

We remark that the strategy of proof of theorem 4.3 is reminiscent of entropy solution concepts for conservation laws and parabolic equations (see [22]), where roughly speaking the Heaviside function applied to \( \rho - c \) for arbitrary constant \( c \) multiplied with a nonnegative smooth function is used as a test function to define entropy inequalities. The function \( H \) in the above proof will indeed approximate the Heaviside function of \( \rho - \frac{1}{2} \) as \( \gamma \) tends to zero.

In the inflow- and outflow-limited case the analysis is easier, an estimate like in lemma 4.2 suffices:

**Theorem 4.4 (Inflow- and outflow limited phases).** Let \( \rho \in H^1([0, 1]) \) be the unique weak solution of (4.2)–(4.4) and let

\[ \max\{\alpha, \beta\} < \frac{1}{2}. \]

Then for \( \alpha < \beta \) the estimate

\[
\int_0^1 |\rho - \alpha| \, dx \leq \frac{1 - \alpha - \beta}{\beta - \alpha}
\]

holds, while for \( \alpha > \beta \) we have

\[
\int_0^1 |\rho - 1 + \beta| \, dx \leq \frac{1 - \alpha - \beta}{\alpha - \beta}.
\]

**Proof.** Note that the maximum principle implies \( \alpha \leq \rho \leq 1 - \beta \) in any of the two cases. We only detail the case \( \frac{1}{2} > \beta \geq \alpha \), as the other one is analogous. Using the test function \( \varphi(x) = 1 - x \) in the weak formulation and some rewriting we have

\[
0 = \int_0^1 \left( \epsilon \partial_x \rho + \rho^2 - \rho \right) \, dx + \alpha(1 - \rho(0)).
\]

With some rearranging and the bounds on \( \rho \) we have
\( \beta \int_0^1 (\rho - \alpha) \, dx \leq \int_0^1 (1 - \rho)(\rho - \alpha) \, dx \leq \epsilon (1 - \beta - \alpha) + \alpha \int_0^1 \rho \, dx - \alpha \rho(1). \)

Since \( \rho \geq \alpha \) we have

\[
(\beta - \alpha) \int_0^1 |\rho - \alpha| \, dx \leq \epsilon (1 - \beta - \alpha).
\]

Summing up, we have shown exactly the same behaviour for our the continuum model as [40] for the discrete TASEP.

**Remark 4.5 (Higher space dimensions).** In more than one spatial dimension, we can still obtain an expression analogous to lemma 4.2, namely

\[
\int_\Omega (\rho - \frac{1}{2})^2 \nabla V^2 - \int_\Gamma V \left[ \alpha(1 - \rho) + \frac{1}{4} \partial_\nu V \right] \, ds + \int_\Sigma V \left[ \beta \rho - \frac{1}{4} \partial_\nu V \right] \, ds = -\epsilon \int_{\Gamma \cup \Sigma} \rho \partial_\nu V \, ds
\]

In order to conclude the same behaviour as in the 1D case (i.e. theorem 4.3), one has to proof that

\[
\int_\Gamma V \left[ \alpha(1 - \rho) + \frac{1}{4} \partial_\nu V \right] \, ds \leq \int_\Sigma V \left[ \beta \rho - \frac{1}{4} \partial_\nu V \right] \, ds.
\]

A simple condition reminiscent of the 1D case would be a maximum principle to be fulfilled, namely

\[
\inf_{x \in \Sigma} \frac{1}{4\beta} \partial_\nu V \leq \rho \leq \sup_{x \in \Gamma} \left( 1 - \frac{1}{4\alpha} \partial_\nu V \right).
\]

An interesting feature of the multi-dimensional situation is that we can have inflow and outflow of different size, respectively \( \partial_\nu V \neq 0 \) on the no-flux boundary for \( \rho \). In this case we cannot expect the maximum principle to hold as also confirmed by numerical results below. In the other case the boundary integrals of the \( \cdot \cdot n \) and \( \partial_\nu V \) need to vanish, which means it suffices to show

\[
\int_\Gamma (V - c) \left[ \alpha(1 - \rho) + \frac{1}{4} \partial_\nu V \right] \, ds \leq \int_\Sigma (V - c) \left[ \beta \rho - \frac{1}{4} \partial_\nu V \right] \, ds.
\]

Hence, if the domain is such that we can choose \( c \) with \( V \leq c \) on \( \Gamma \) and \( V \geq c \) on \( \Sigma \) the maximum principle implies the same behaviour as in theorem 4.3. Although the integral arguments as a above do not yield the maximum principle, but only an integrated version, we conjecture it to be true in such situations, which is strongly supported by the numerical results of section 5.

### 4.2. Maximal current in one spatial dimension

In this section, we briefly discuss situations in which the maximal current phase occurs, using explicit solutions to the 1D equations (4.2)–(4.4). This approach allows us to clarify the role of the parameter \( \epsilon \) with respect to the phase diagram. In particular, we can show that for \( \epsilon > 0 \), maximal current can occur for values of \( \alpha, \beta \) that are strictly smaller than \( 1/2 \). Since in one
space dimension, the flux $j$ is constant, we can set $j = 1/4$ and integrate (4.2) and obtain the first order ordinary differential equation

$$-\partial_t \rho + \rho(1 - \rho) = \frac{1}{4},$$

which is also known as ‘logistic equation with harvesting’ in the context of population dynamics, see [3, 11]. Solving this equation subject to the boundary conditions elementary calculation shows that on of the following conditions on $\alpha$ and $\beta$ have to hold in order to obtain a continuous solution:

$$\frac{1}{2} \frac{1 + 2\varepsilon}{2 \varepsilon + 1} < \alpha < \frac{1}{2} \quad \text{and} \quad \beta = \frac{1}{2} \frac{4\alpha \varepsilon + 2\alpha - 1}{8\alpha \varepsilon + 2\alpha - 2\varepsilon - 1}. \quad (4.11)$$

$$\frac{1}{2} \frac{1 + 2\varepsilon}{2 \varepsilon + 1} < \beta < \frac{1}{2} \quad \text{and} \quad \alpha = \frac{1}{2} \frac{4\beta \varepsilon + 2\beta - 1}{8\beta \varepsilon + 2\beta - 2\varepsilon - 1}. \quad (4.12)$$

Interestingly, maximal flux is achieved for values of $\alpha, \beta < 1/2$ which is in contrast to the discrete model, [40]. To illustrate this, we depicted the changes in the phase diagram for different values of $\varepsilon$ in figure 1. In section 5.2 we present numerical results based on a discretization of (4.2)–(4.4) that confirm this observation.

5. Numerical solution

In this section we will describe the numerical method that we used and present some examples in one and two space dimensions. Our implementation is based on the discontinuous finite element method which is well-suited for convection dominated problems, see [13] and the references therein. We will not give any details regarding error estimates and the convergence of our algorithm which remains future work.
5.1. Setting and discontinuous Galerkin scheme

Let us recall some well-known notations and definitions, see [13]. We start by dividing our domain into elements which are triangles in two space dimensions and intervals in 1D. For simplicity, we shall only discuss the 2D case from now on. We cover the domain $\Omega \subset \mathbb{R}^2$ by a finite collection of triangles which we denote by $T_h$, where $h$ refers to the diameter of the largest triangle. Furthermore, we denote by $F$ the mesh faces which are characterised by one of the following two conditions:

1. Either, there are distinct triangles $T_1$ and $T_2$ such that $F = \partial T_1 \cap \partial T_2 = F$ is an interface,
2. or, there is $T \in T_h$ such that $F = \partial T \cap \partial \Omega = F$ is a boundary face.

We denote by $\mathcal{F}_h^i$ the set of all interfaces, $\mathcal{F}_h^b$ the boundary faces and by $\mathcal{F}_h$ the union of these two sets. Furthermore, $n_F$ is the normal vector of a facet, pointing outward. We always assume that the boundary segments $\Gamma$ and $\Sigma$ are given as unions of boundary faces. On $T_h$ we introduce the broken polynomial space

$$V_h = \{ v \in L^2(\Omega) : \forall T \in T_h \text{, } v|_T \in \mathcal{P}^1(T) \},$$

where $\mathcal{P}^1(T)$ denotes polynomials of degree one on $T$. For a scalar function $v$, smooth enough for the expression $v|_T$ for all $F \in \mathcal{F}$ to make sense, we define interface averages and jumps in the following way

$$[v]_F(x) := \frac{1}{2}(v|_T(x) + v|_T(x)), \text{ for a.e. } x \in F, \text{ (average)},$$

$$\llbracket v \rrbracket_F(x) := v|_T(x) - v|_T(x), \text{ for a.e. } x \in F, \text{ (jump).}$$

With these definitions at hand, we can state our discontinuous Galerkin scheme. Starting from the weak formulation of a linearized version of (2.1) we consider

$$\varepsilon \int_\Omega \nabla \rho \nabla \phi \, dx + \int_\Omega (1 - \tilde{\rho}) u \nabla \phi \, dx + \alpha \int_\Gamma \rho \phi \, ds + \beta \int_\Gamma \rho \phi \, ds = \alpha \int_\Gamma \phi \, ds, \quad \phi_h \in H^1(\Omega),$$

with $\tilde{\rho} \in H^1(\Omega) \cap L^\infty(\Omega)$ given. In order to obtain a discrete solution $\rho_h \in V_h$ we define the bilinear form

$$a(\rho_h, \phi_h; \tilde{\rho}) = a^{\text{upw}}(\rho_h, \phi_h) + a^{\text{swip}}(\rho_h, \phi_h; \tilde{\rho}),$$

with a symmetric weighted interior penalty method for the diffusion given by

$$a^{\text{swip}}(\rho_h, \phi_h) = \int_\Omega \varepsilon \nabla \rho_h \cdot \nabla \phi_h \, dx - \sum_{F \in \mathcal{F}_h} \int_F (\llbracket \nabla \rho_h \rrbracket \cdot n_F \llbracket \phi_h \rrbracket + \llbracket \rho_h \rrbracket (\llbracket \nabla \phi_h \rrbracket \cdot n_F) \, d\sigma$$

$$+ \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F \llbracket \rho_h \rrbracket (\llbracket \phi_h \rrbracket) \, d\sigma$$

and a upwind scheme for the advection part
\[ a^{uo}(\rho_h, \phi_h) = \int_{\Omega} -\rho_h((1 - \rho_h)u \cdot \nabla \phi_h) \, dx + \sum_{F \in \mathcal{F}_h} \int_F (1 - \rho_h)u \cdot n_F \{ (\rho_h) \{ (\phi_h) \} \} \, d\sigma \]

\[ + \sum_{F \in \mathcal{F}_h} \frac{1}{2}(1 - \rho_h)u \cdot n_F \int_F [\rho_h] \{ \phi_h \} \, d\sigma, \]

again with \( \rho_h \in V_h \) given. The local length scale \( h_F \) is defined as \( h_F = \frac{1}{2}(h_T + h_T) \), where \( T_1 \) and \( T_2 \) are the two triangles adjacent to face \( F \). In order to obtain a solution to the original nonlinear problem (2.1) we employ the following semi-implicit iteration scheme: for \( u_h^n \) given find \( u_h^{n+1} \in V_h \) s.t.

\[ (u_h^{n+1}, \phi_h) + \tau (a(u_h^{n+1}, \phi_h; u_h^n, \phi_h + a_F(u_h^{n+1}, \phi_h)) = (u_h^n, \phi_h) + f(\phi_h), \quad \forall \phi_h \in V_h, \quad (5.2) \]

with a relaxation parameter \( \tau > 0 \). In all experiments below we chose \( u_0 = 1/2 \) and \( \tau = 0.01 \). Thus in each step one has to solve the following system of linear equations

\[ (M + \tau A)u^{n+1} = (M u^n + \tau f), \]

where \( u \) denotes the vector of coefficient of \( u \) in the linear finite element basis, \( A \) is the matrix corresponding to the bilinear form \( (a + a_F) \), and \( M \) denotes the mass matrix. The vector \( f \) stems from the term \( f(\phi_h) \) on the rhs of (5.2) with \( u^n \) being the solution of the previous step. Note that this scheme can be interpreted as a semi-implicit time discretization of the parabolic version of (2.1) with time step size \( \tau \).

5.2. Results in one spatial dimension

In one space dimension, we used MATLAB to implement the scheme described above. We will present several examples in the following and consider the domain \( \Omega = [0, 1] \) discretized by \( n = 200 \) elements and with \( \Gamma = \{0\} \) and \( \Sigma = \{1\} \).

5.2.1. Different phases. First we present some examples to illustrate the occurrence of the three different phases (namely influx limited, outflux limited and maximal current) that are analysed in section 4 (and also in [40]). We performed simulations for \( \varepsilon = 0.1, 0.01, 0.001 \). For \( \alpha \) and \( \beta \) we chose the values 0.2, 0.4, 0.6 and 0.4, 0.2, 0.7, respectively. The numerical results confirm the predicted occurrence of three phases and the results are shown in figure 2.

5.2.2. Maximal current for \( \alpha < 1/2 \) or \( \beta < 1/2 \). Here we present numerical evidence for the results of section 4.2, namely the occurrence of the maximal flow phase for \( \alpha, \beta < 1/2 \). We chose \( \varepsilon = 0.01 \) which yields \( \frac{1}{2(2\varepsilon + 1)} = 0.4902 \). We chose \( \alpha = 0.4912 \) and by (4.11), the corresponding \( \beta \) is 0.603 773 585. To illustrate the case \( \beta < 1/2 \) we simply interchange the roles of \( \alpha \) and \( \beta \). Both results are depicted in figure 3 and confirm the results from section 4.2.

To further explore the behaviour of the flux, we used the discontinuous Galerkin scheme introduced above to numerically produce a phase diagram by sampling the values of \( \alpha, \beta \) from 0 to 1 with a stepsize of 0.01. For \( \varepsilon = 0.1 \) we compared the contour line \( j = 1/4 \) with (4.11) or (4.12), respectively, see figure 4.
Figure 2. Some results in one spatial dimension: $\epsilon = 0.1$ (blue), $\epsilon = 0.05$ (black), and $\epsilon = 0.01$ (red). Top left: $\alpha = 0.2$, $\beta = 0.4$, Top right: $\alpha = 0.4$, $\beta = 0.2$, Bottom left: $\alpha = 0.6$, $\beta = 0.7$.

Figure 3. For $\epsilon = 0.01$, the resulting density (left) and flux (right) is depicted for the values $\alpha = 0.4912$, $\beta = 0.6043$ (left) and $\alpha = 0.6043$, $\beta = 0.4912$ (right). The maximal flux $j = 1/4$ is observed in both cases.

Figure 4. For $\epsilon = 0.1$, the left picture shows a phase diagram generated using the discontinuous Galerkin method described above for values of $\alpha$, $\beta = 0, 0.001, 0.002, \ldots, 1$. On the right side, contour lines for several values of the flux $j$ are depicted. The line for $j = 1/4$ (blue line) is compared to the analytical results of section 4.2 (red circles). Compare also with figure 1.
Results in two spatial dimensions

In two spatial dimensions, we used the software package FeniCS, [30, 31] to implement the scheme described in section 5.1. We present several examples using the domain sketched in figure 5, i.e. a corridor of length 2 and height 1 with two entrances and two exits on each side. The upper entrance and exit are located at $0.65 < y < 0.85$, the lower ones at $0.15 < y < 0.35$. For each entrance, we have a different inflow rates $\alpha_i$, $i = 1, 2$, while we have $\beta_i$, $i = 1, 2$ for the exits.

5.3.1. Maximum principle. In all examples in this section, we use a velocity field given as the gradient of some potential. From theorem 3.5 we know that for general potentials $V$ we only have

$$\rho_0 \leq \rho \leq \lambda_1$$

while for $V$ satisfying the assumptions $\Delta V = 0$, $\partial_n V = -1$ on $\Gamma$, $\partial_n V = 1$ on $\Sigma$ we have that

$$\min \{\alpha, 1 - \beta\} \leq \rho(x) \leq \max \{\alpha, 1 - \beta\}.$$  

To illustrate this, let $V_m$ be given as the solution to the equation

$$-\Delta V_m = 0 \quad \text{in} \quad \Omega,$$

$$\partial_n V_m = -1 \quad \text{on} \quad \Gamma,$$

$$\partial_n V_m = 1 \quad \text{on} \quad \Sigma,$$

$$\partial_n V_m = 0 \quad \text{on} \quad \partial \Omega \setminus (\Gamma \cup \Sigma),$$

with the normalisation condition $\int_{\Gamma} u \, d\sigma(x) = C$. On the discrete level, we use a mixed method to discretise this equation in order to ensure that the condition $\nabla \cdot V_m = 0$ is fulfilled exactly on the discrete level. The normalisation constrained is achieved by setting an arbitrary boundary node to zero. The resulting velocity field $u_m = \nabla V_m$ is depicted in figure 5. Alternatively we chose $V_l(x) = x$ which yields $u_l = \nabla V_l = (1, 0)$. In our first example we then chose $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.4$ and $\beta_2 = 0.2$ and apply our scheme with $V = V_l$ and $V = V_m$, respectively. The results shown in figure 6 produce the expected behaviour, namely the maximum principle (3.4) only occurs for $V = V_m$. Furthermore, the results show that the asymmetric in- and outflow rates indeed some of the ‘particles’ entering at $\alpha_2$ move over two the exit at $\beta_1$. This indicates that the model may be able to also predict lane formation in the case with more than one active species, see also [36].

5.3.2. Additional maximum principle and size of entrance and exit. As detailed in remark 4.5, we conjecture that an additional maximum principle (namely $1/(4\beta) \leq \rho \leq 1 - 1/(4\alpha)$) olds in the case when the size of entrance and exit coincide, i.e. $|\Gamma| = |\Sigma|$. To support this conjecture, we performed simulations on a domain with a single entrance of diameter $d_{\text{entrance}}$ and exit of diameter $d_{\text{exit}}$, only, see figure 7. Again, the computational domain is $[0, 2] \times [0, 2] \subset \mathbb{R}^2$. Both doors

![Figure 5.](image)
are centred around \( y = 1/2 \). We performed three simulations (\( \varepsilon = 0.05, \alpha = 0.6, \beta = 0.7 \)), one with doors of equal size, one with a larger entrance and one with a larger exit. The results are:

| \( d_{\text{entrance}} \) | \( d_{\text{exit}} \) | \( 1/(4\beta) \) | \( 1 - 1/(4\alpha) \) | \( \min \rho \) | \( \max \rho \) |
|---|---|---|---|---|---|
| 0.2 | 0.2 | 0.357 | 0.583 | 0.363 | 0.573 |
| 0.2 | 0.15 | 0.357 | 0.583 | 0.463 | 0.766 |
| 0.15 | 0.2 | 0.357 | 0.583 | 0.205 | 0.481 |

This shows that the lower bound of the maximum principle (4.10) is violated in the case when the entrance is smaller than the exit while the upper bound fails to hold otherwise. These results clearly support our conjecture that \(|\Gamma| = |\Sigma|\) is essential for the additional maximum principle to hold.

5.3.3. High densities and obstacles. In a second example, we explored the situation of maximal flow by using the values \( \alpha_1 = 0.6, \alpha_2 = 0.9, \beta_1 = 0.9 \) and \( \beta_2 = 0.6 \) and \( V = V_l \). Note that in both cases we observe on the parts between the in- and outflow boundaries that the maximum principle of theorem 3.4 does not hold since \( u_0 \cdot n = 0 \) on the no-flux boundary. The results are shown in figure 8. Since this example shows that high densities can occur between the two exits, we modify the domain by adding an round obstacle in front of the doors as shown in figure 8. This is motivated by results from models for human crowd motions where
in some situations, an obstacle in front of the exits can improve the situation. Indeed, our
results show that the densities between the two exits decreases, however at the price of a large
density in front of the obstacle itself. Furthermore, the transition from high to low densities
observed in figure 6 is shifted towards the entrances.

6. Summary & outlook

In this paper we analyzed a model for crowded transport with a single active species. We
started by giving some details about the modelling then proceeding with two existence proofs
in the stationary case. Next we analysed the flow characteristic of our model in the case of
small diffusion. In one space dimension, we were able to recover three different phases that
were already observed in the stochastic model [40]. Further investigation showed however
that the continuous model can produce fluxes that exceed the value $j = 1/4$ which do not occur
on the discrete level. We concluded by presenting some numerical examples in one and two
spatial dimensions.

Our analysis and especially the numerical examples in two spatial dimensions suggest that
interesting phenomena can occur when dealing with more than one active species. Since each
species has its own in- and outflow rate, it is not clear whether one would again observe dif-
ferent phases, clearly separated by certain values of these parameters. Also, to prove existence
in the case $M > 1$ becomes much more involved. Regarding the numerical discretization, a
scheme that uses the reformulated problem in entropy variables might be an alternative to the
direct approach used here.

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