We investigate the dynamical behavior of the Gross–Pitaevskii equation for a Bose–Einstein condensate trapped in a generic spherical potential. A four-dimensional setup were found, where the Thomas–Fermi approximation constrains the parameter space of solutions. We show that for values of the trap shape parameter equal or smaller than 2 the system seems to be stable. However, when the shape of the potential is bigger that 2, the instability of the system grows when the shape parameter grows, indicating that large values of the aforementioned parameter can be related to a lost in the number of particles from the condensed state. This fact can be used also to show that the stability conditions of the condensate are highly sensitive to the shape of the external potential.

I. INTRODUCTION

The phenomena of Bose–Einstein condensation [1] is one of the most remarkable N–body quantum phenomena that can be observed at macroscopic scales. Since its laboratory observation Bose-Einstein condensation of dilute atomic gases has stimulated an enormous amount of related work [2]. We found mathematical issues related to Bose–Einstein condensates [3], several theoretical and heuristic aspects [4, 5], also as test tools in gravitational physics [6], or even to model dark matter in the universe [4, 5]. The study of its associated thermodynamic properties is naturally also a very pertinent aspect [6, 8–13]. The experimental understanding of Bose–Einstein condensates can be achieved in several alkali atoms which allows the use of various atomic systems, for instance, $^{87}\text{Rb}$, $^{23}\text{Na}$, which have repulsive interactions within the condensate [10, 14]. Conversely, $^{7}\text{Li}$ and $^{85}\text{Rb}$ that have attractive interactions [10, 14]. The low energy properties and specifically the stability properties can be characterized through the s–wave scattering length. These stability conditions of the condensate trapped in magnetic traps has been extensively analyzed from the numerical and analytical point of view [14]. It is seems that the condensate was predicted to be (meta)stable in magnetic traps, when the number of atoms is below some critical number. In other words, the possible values of the s–wave scattering length are intimately linked to the stability of the system in these circumstances. However, there are at least one more parameter linked to such stability, the specific form of the trapping potential. The form of this potential is strongly related to the macroscopic behavior of the condensate, and clearly the properties of the condensed cloud must be highly sensitive to its shape. Then, it is interesting analyze the stability of the cloud when one varies the shape of the trap besides the value of the scattering length.

Additionally, the dynamical system formalism [15] is a mathematical tool that is implemented usually in cosmology [16] in order to rewrite the evolution equations as a plane-autonomous systems and to analyze the stability of them. The results of the latter can give some additional information of the physics related to the system behavior which can be hided in the corresponding critical points of the system.

Therefore, we analyze the stability conditions of the Bose–Einstein condensate trapped in a generic potential from the dynamical systems point of view, in order to analyze the stability conditions in such a scenario. As far as we know, this mathematical tool has been never used in this context. Thus, the analysis of stability condition as an autonomous system over the condensate in a generic potential can be helpfully in order to find the most favorable scenarios in which the system actually acquire more stable configurations.

The outline of the paper is as follows: in Section II we describe the main properties of the Gross–Pitaevskii equation and of the corresponding generic trapping potential. Also, some issues regarded the Thomas–Fermi
approximation are revised. Section III is dedicated to the dynamical analysis of the cases in which a Bose–Einstein condensate is endowed with a generic potential. We first show that there are four cases of interest, related to the values of the corresponding shape parameter $s$. To have a complete picture of the solution for the Gross–Pitaevskii equation in a Thomas–Fermi approximation, these equations are written as an autonomous system, and we study its critical points and general trajectories in the phase space of the resulting dynamical variables. Finally, Section IV is devoted to final comments about the general properties of the Bose–Einstein condensate trapped in a generic potential.

II. THE GENERIC POTENTIAL AND THE GROSS–PITAEVSKII EQUATION

Let us start with the time dependent Gross–Pitaevskii equation in 3–dimensions

$$i\hbar \partial_t \psi(r,t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(r,t) + V(r) \psi(r,t) + U_0 |\psi(r,t)|^2 \psi(r,t),$$

where $m$ is the atomic mass and $U_0 = 4\pi\hbar^2a_s/m$ describes the interaction between the particles in the condensate with $a_s$ the $s$–wave scattering length. Its validity is based on the condition that the $s$–wave scattering length be much smaller than the average distance between atoms and that the number of atoms in the condensate be much larger than one. The Gross–Pitaevskii equation can be used, at temperatures below the condensation temperature, to explore the macroscopic behavior of the system, characterized by variations of the order parameter $\psi(r,t)$ over distances larger than the mean distance between atoms.

The trapping potential in Eq. (2.1) is given by

$$V(r) = A \left(\frac{r}{a_0}\right)^s,$$

this is the so–called generic potential where, for simplicity, we assume spherical symmetry (see for instance Refs. [17,19] and references therein for more general potentials). In the generic potential (2.2) we have $A = \hbar \omega_0/2$ and $a_0 = \sqrt{\hbar/m\omega_0}$, which are characteristic scales of energy and length associated with the trap [20]. The parameter $s$ depicts the shape of the trapping potential, which in our approach will play a relevant role in the dynamics. Notice that if we set $s = 2$ in the generic potential (2.2), we recover the more common harmonic trap. These type of potentials are known as generic spherical potentials. It is quite interesting to analyze the behavior of the condensate when the shape of the trap varies as a power of the radial coordinate for several reasons: first, these class of trapping potentials could be useful in order to cool the system with an adiabatic procedure, by changing the shape of the trap [4]. Second, all the relevant thermodynamical properties associated with the system, e.g., the condensation temperature, the number of particles within the condensate, the corresponding density, the entropy, etc., explicitly exhibits a sensitive trap–dependence, i.e., the macroscopic behavior of the condensate is highly sensitive to the shape of the external trapping potential. In this sense, it is relevant to analyze the stability of the system when this generic potential is present.

In order to obtain a dimensionless version of the Gross–Pitaevskii equation (2.1), let us define, as usual, the following dimensionless variables in function of the characteristic scales associated with the system [21]

$$\bar{t} = \omega t, \quad \bar{r} = \frac{r}{a_0},$$

$$\bar{\psi}(\bar{r}, \bar{t}) = a_0^{3/2} \psi(r,t), \quad \bar{V}(\bar{r}) = \frac{V(r)}{2A}.$$  (2.3)

By using the dimensionless variables defined above, the Gross–Pitaevskii equation in 3–dimensions becomes

$$i\bar{\partial}_t \bar{\psi}(\bar{r}, \bar{t}) = -\frac{1}{2} \nabla^2 \bar{\psi}(\bar{r}, \bar{t}) + \bar{V}(\bar{r}) \bar{\psi}(\bar{r}, \bar{t}) + \beta |\bar{\psi}(\bar{r}, \bar{t})|^2 \bar{\psi}(\bar{r}, \bar{t}),$$

where we have defined $\beta = U_0/\hbar \omega_0 a_0^3$. We restrict our analysis to positive values of the corresponding self–interaction potential since positive scattering lengths provides repulsive interactions among the particles.

Consequently, the time independent dimensionless Gross–Pitaevskii equation can be obtained by written the dimensionless order parameter $\bar{\psi}(\bar{r}, \bar{t})$ as follows

$$\bar{\psi}(\bar{r}, \bar{t}) = e^{-i\bar{\mu} \bar{t}} \bar{\phi}(\bar{r}),$$

where we have defined the dimensionless chemical potential as $\bar{\mu} = \mu/\hbar \omega_0$. Inserting (2.5) into (2.4), this leads to the following equation

$$\frac{\partial^2 \bar{\phi}(\bar{r})}{\partial \bar{r}^2} = 2 \left[\bar{V}(\bar{r}) - \bar{\mu}\right] \bar{\phi}(\bar{r}) + 2\beta \bar{\phi}^3(\bar{r}).$$

An analytical and formal solution associated with our system can be obtained by neglecting the kinetic energy term from the very beginning in Eq. (2.6), this is the so–called Thomas–Fermi approximation [10], with the result

$$n(\bar{r}) = |\bar{\phi}(\bar{r})|^2 = \frac{\bar{\mu} - \bar{V}(\bar{r})}{\beta},$$

while $n(r) = 0$ outside of this region [1]. Here, $n(\bar{r}) = 1$ We are able to estimate the spatial extend $\bar{r}_e$ of the cloud in the dimensionless variables, through the condition $\bar{\mu} = \bar{V}(\bar{r})$, which in our case is given by

$$\bar{r}_e = (2\bar{\mu})^{1/s}.$$  (2.8)

This spatial extend fixes the regime of validity of the Thomas–Fermi approximation and consequently, also fixes the validity of the analysis followed in this paper.
\[ |\phi(\bar{r})|^2 \] depicts the density of particles within the condensate in the Thomas–Fermi approximation.

### III. DYNAMICAL SYSTEM STRUCTURE

Our goal is to study the evolution of the Gross–Pitaevskii equation (2.6) in which the condensate is endowed with the generic trapping potential (2.2). As in the standard case, it is possible to perform a dynamical study of the model so that its relevant solutions are easily unveiled [22,24]. It is important to mention that the Thomas–Fermi approximation (2.7) will fix a constraint for the parameter space in where the solutions of the system are possible. In order to construct a dynamical system for a Bose–Einstein condensate trapped in the generic potential (2.2), the first step is to introduce a set of conveniently chosen variables which may allow rewriting the evolution equation as an autonomous system subject to the constraint arising from the Thomas–Fermi approximation. We choose the following variables

\[
x \equiv \tilde{\phi}', \quad y \equiv \frac{\tilde{\rho} s}{2} - \bar{\mu}, \quad z \equiv \tilde{\phi}, \quad w \equiv \frac{s}{2 \bar{r}} \left( \frac{\tilde{\rho} s}{\bar{r}} \right), \quad (3.1)
\]

where the prime denote derivatives with respect to \( \bar{r} \). These variables render the Thomas–Fermi approximation constraint set by:

\[
F(y, z) := \kappa |z|^2 = -2y, \quad (3.2)
\]

where \( \kappa \equiv 2\beta \). We shall restrict ourselves to the parameter space delimited by this hyperbolic region, which contains the solutions in this approximation for the Gross–Pitaevskii equation. Combining (2.6) and (3.1), the equations of motion read

\[
x' = 2y z + \kappa z^3, \quad (3.3a)
\]
\[
y' = w, \quad (3.3b)
\]
\[
z' = x, \quad (3.3c)
\]
\[
w' = \frac{s(s-1)}{y^2} (y + \bar{\mu}). \quad (3.3d)
\]

The stability of the points is investigated through linear perturbations around the critical values of the form \( x = x_0 + \delta x \), \( y = y_0 + \delta y \), and \( z = z_0 + \delta z \). The equations of motion (3.3) can be written as \( x' = f(x) \), which upon linearization reads

\[
u' = M\delta x, \quad M_{ij} = \frac{\partial f_i}{\partial x_j} \bigg|_{x_0}, \quad (3.4)
\]

where \( M \) is called the linearization matrix. The eigenvalues \( m \) of the matrix \( M \) determine the stability of the critical points, whereas the eigenvectors determine the principal directions of the perturbations. In general, if \( \text{Re}(m) < 0 \) (\( \text{Re}(m) > 0 \)) the critical point is called stable (unstable). We should study the perturbations of the three dynamical variables \((x, y, z)\), but remember that they are not all independent because they are bond together by the Thomas–Fermi approximation (2.7), and the same happens to their perturbations. This constraint defines a two dimensional surface \((3.2)\) upon which lie all physically relevant phase space trajectories (solutions), and then we will be interested on perturbations lying also on the constraint surface. In other words, perturbations which are perpendicular to the Thomas–Fermi constraint surface should be taken away from the analysis. Next, we list the eigenvalues of the stability matrix \( M \) for each of the critical point with the following ansatz at first order in the perturbations:

\[
x = x_c + u, \quad y = y_c + v, \quad z = z_c + \omega, \quad (3.5)
\]

where

\[
\xi' = M \xi, \quad \rightarrow \left( \begin{array}{c} u' \\ v' \\ \omega' \end{array} \right) = M \left( \begin{array}{c} u \\ v \\ \omega \end{array} \right) \quad (3.6)
\]

is the system to resolve and with solutions

\[
u = u_+ e^{m_+ \bar{r}} + u_- e^{m_- \bar{r}}, \quad (3.7a)
\]
\[
v = v_+ e^{m_+ \bar{r}} + v_- e^{m_- \bar{r}}, \quad (3.7b)
\]
\[
\omega = \omega_+ e^{m_+ \bar{r}} + \omega_- e^{m_- \bar{r}}, \quad (3.7c)
\]

where \([m_+, m_-]\) are the eigenvalues of the system and \( \xi \) the eigenvectors and \([u_+, u_-, v_+, v_-, w_+, w_-]\) are arbitrary constants which are determined by the initial conditions.

The results of the analysis of the dynamical system (3.3), their critical points and its stability properties, are summarized in Table I. At this point we are interested in where the trapping potential, (i.e for different values of \( s \)), can set a stable (or unstable) scenario. To proceed with these ideas, we follow the cases related to the Bose–Einstein condensate physics. Notice that if we set \( s = 0 \) in the generic potential (2.2), we obtain a free gas. For \( s = 1 \) a linear trap and for \( s = 2 \) the usual harmonic oscillator. Additionally, when \( s \to \infty \), we obtain the homogeneous system trapped in a box.

Let us now analyze the stability conditions for some potentials of interest.

- Case \( s = 0 \), i.e \( \bar{V} = \text{const} \). The autonomous system is reduced to the following bidimensional system:

\[
x' = 2\alpha_1 z + \kappa z^3, \quad z' = x, \quad (3.8)
\]

where \( \alpha_1 = 1/2 - \bar{\mu} \). The real critical point for this system is \((x_c = 0, z_c = 0)\) and

\[
M_{ij} = \left( \begin{array}{cc} 0 & 1 \\ 2\alpha_1 + 3\kappa z^2 & 0 \end{array} \right) \bigg|_{(x_c, z_c)} = \left( \begin{array}{cc} 0 & 1 \\ 2\alpha_1 & 0 \end{array} \right), \quad (3.9)
\]

with eigenvalues of \( m = \pm \sqrt{2\alpha_1} \) and eigenvectors of \( \xi = (\pm 1/2\sqrt{2/\alpha_1}, 1) \). The constraint is fixed by \(|z|^2 = -2\alpha_1/\kappa \). The trajectories in Figure I
FIG. 1. For the case \( s = 0 \). Left: Vector field plot for \( m_+ \). Middle: Vector field plot for \( m_- \). Right: Eigenvector plot for the cases \( \xi_+ \) (green lines) and \( \xi_- \) (red lines).

for the case \( \xi_- \) of \( m_- \) initially start at infinite-distance away, move toward and eventually converge at the critical point. The trajectories that represent the eigenvector \( \xi_+ \) of \( m_+ \) move exactly the opposite way: start at the critical point then diverge to infinite-distance out. As a result, the critical point for the case results in a saddle point, which is always unstable. When \( s = 0 \) the potential becomes a constant that can be reabsorbed in the dimensionless chemical potential. This type of potential may not be of physical relevance. However, we have included this case for mathematical reasons. In this scenario, the system is unstable, which makes sense since there is no confinement region.

- Case \( s = 1 \), i.e \( \bar{V} = \bar{r}/2 \), which corresponds to the linear trapping potential. The autonomous system is reduced to the following set of equations:

\[
\begin{align*}
  x' &= 2yz + \kappa z^3 = 2\left(\frac{\bar{r}}{2} - \bar{\mu}\right) z + \kappa z^3, \\
  y' &= 1/2, \quad z' = x, \\
  w' &= 1 
\end{align*}
\]  

(3.10)

with two critical points at \((x_{c1} = 0, y_{c1} = y, z_{c1} = 0)\) and \((x_{c2} = 0, y_{c2} = -\kappa z^2/2, z_{c2} = z)\) and

\[
\mathcal{M}_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 2z & 0 & 0 \\ 2y + 3\kappa z^2 & 0 & 0 \end{pmatrix}_{(x_{c1}, y_{c1}, z_{c1})} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2\left(\frac{\bar{r}}{2} - \bar{\mu}\right) & 0 & 0 \end{pmatrix},
\]

(3.11)

\[
\mathcal{M}_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 2z & 0 & 0 \\ 2y + 3\kappa z^2 & 0 & 0 \end{pmatrix}_{(x_{c2}, y_{c2}, z_{c2})} = \begin{pmatrix} 0 & 0 & 1 \\ 2z & 0 & 0 \\ 2\kappa z^2 & 0 & 0 \end{pmatrix},
\]

(3.12)

with eigenvalues

\[ m_{c1} = (0, \pm \sqrt{2y}), \quad m_{c2} = (0, \pm \sqrt{2\kappa z}), \]

(3.13)

and eigenvectors of

\[ \xi_{c1} = \{(0, 1, 0), (\pm \sqrt{2y}/2, 0, 1)\}, \]

\[ \xi_{c2} = \{(0, 1, 0), (\pm \sqrt{2\kappa z}/(2z), (\kappa z)^{-1}, 1)\}. \]

(3.14)

The constraint is fixed by the curve \(|z|^2 = (2\bar{\mu} - \bar{r})/\kappa\). The instability happens only along the eigenvectors corresponding to \(\pm \sqrt{2y}\) and \(\pm \sqrt{2\kappa z}\), which are saddle points. The first eigenvalues are null, and then the system is indifferent under perturbations along the constraint curve for this case. The behavior of the latter is showed in Figure 2.

- Case \( s = 2 \), i.e \( \bar{V} = \bar{r}^2/2 \), which corresponds to the more common harmonic oscillator potential. The autonomous system is reduced in this case to the system:

\[
\begin{align*}
  x' &= 2yz + \kappa z^3 = 2\left(\frac{\bar{r}^2}{2} - \bar{\mu}\right) z + \kappa z^3, \\
  y' &= w, \quad z' = x, \quad w' = 1.
\end{align*}
\]

(3.15)

with two critical points at \((x_{c1} = 0, y_{c1} = y, z_{c1} = 0)\) and \((x_{c2} = 0, y_{c2} = -\kappa z^2/2, z_{c2} = z)\) and

\[
\mathcal{M}_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 2z & 0 & 0 \\ 2y + 3\kappa z^2 & 0 & 0 \end{pmatrix}_{(x_{c1}, y_{c1}, z_{c1})} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2\left(\frac{\bar{r}^2}{2} - \bar{\mu}\right) & 0 & 0 \end{pmatrix},
\]

(3.16)
FIG. 2. For the case $s = 1$. Eigenvector plot for the cases $\xi_{\pm}$ (3.14) with vector fields for $m_{\pm}$ (3.13).

$0, w_{c1} = 0$ and $(x_{c2} = 0, y_{c2} = -(\kappa z^2)/2, z_{c2} = z, w_{c2} = 0$ and

$$M_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2z & 0 & 0 & 0 \\ 2y + 3\kappa z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{(x_{c1}, y_{c1}, z_{c1}, w_{c1})} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2y & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(3.16)

$$M_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2z & 0 & 0 & 0 \\ 2y + 3\kappa z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{(x_{c2}, y_{c2}, z_{c2}, w_{c2})} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2z & 0 & 0 & 0 \\ 2\kappa z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(3.17)

with eigenvalues

$m_{c1} = (0, 0, \pm \sqrt{2}y), \quad m_{c2} = (0, 0, \pm \sqrt{2\kappa z}), \quad (3.18)$

and eigenvectors of

$$\xi_{c1} = \{(0, 0, 0, 1), (\pm \sqrt{2}y/2, 0, 1, 0)\},$$

$$\xi_{c2} = \{(0, 0, 0, 1), (\kappa z, \pm \sqrt{2\kappa z}, \pm \sqrt{2\kappa^3 z^2}, 1)\}. \quad (3.19)$$

The constraint is fixed by the curve $|z|^2 = (2\mu - \bar{r}^2)/\kappa$. For the first critical point we observe that only the non-vanishes eigenvalues, which points in $z$-direction gives the information about the stability of this point. In such case, when $\bar{r} > 2\mu$ ($\bar{r} < 2\mu$) the point will be unstable (stable). For the case $s = 2$, or the usual harmonic trap, the system seems to be stable in all cases. In other words, note that in our approach the Thomas–Fermi approximation fixes the permissible values for the spatial extend of the cloud as $\bar{r} = (2\mu)^{1/s}$, this means that for all $\bar{r}$ of interest $\bar{r} < (2\mu)^{1/s}$, and consequently $\bar{r} < 2\mu$ is always satisfied, leading to a stable scenario.

- Case $s = 3$, i.e $\bar{V} = \bar{r}^3/2$, which corresponds to a cubic potential. The autonomous system is reduced to the system:

$$x' = 2yz + \kappa z^2 = 2\left(\frac{\bar{r}^3}{2} - \bar{\mu}\right)z + \kappa z^3,$$

$$y' = w = \frac{3}{2}\bar{r}^2, \quad z' = x, \quad w' = \frac{6}{\bar{r}^2}(y + \bar{\mu}). \quad (3.20)$$

with three critical points at $(x_{c1} = 0, y_{c1} = -\bar{\mu}, z_{c1} = 0, w_{c1} = 0)$ and $(x_{c(2,3)} = 0, y_{c(2,3)} = -\bar{\mu}, z_{c(2,3)} = \pm \sqrt{\frac{2\mu}{\pi}}, w_{c2} = 0$ and
with eigenvalues

\[ m_{c_1} = (\sqrt{6/\bar{r}}, -\sqrt{6/\bar{r}}, \sqrt{-2\bar{\mu}}, -\sqrt{-2\bar{\mu}}), \quad m_{(c_2, c_3)} = (\sqrt{6/\bar{r}}, -\sqrt{6/\bar{r}}, 2\sqrt{\bar{\mu}}, -2\sqrt{\bar{\mu}}) \]  

(3.23)

and eigenvectors of

\[ \xi_{c_1} = \{ (1/\sqrt{-2\bar{\mu}}, -1/\sqrt{-2\bar{\mu}}, 0, 0), (0, 0, \sqrt{6/\bar{r}}, -\sqrt{6/\bar{r}}), (1, 1, 0, 0), (0, 0, 1, 1) \}, \]

\[ \xi_{(c_2, c_3)} = \{ (0, 0, \pm \{ (-3 + 2\bar{\mu})/2\bar{\bar{r}} \mp \sqrt{\bar{\bar{r}}} \sqrt{\bar{\bar{r}}}/\bar{\bar{r}} \}, (0, 0, \pm \{ \sqrt{\bar{\bar{r}}} (3 + 2\bar{\mu})/2\bar{\bar{r}} \}, (1, 1, 1, 1) \}. \]  

(3.24)

The constraint is fixed by the curve |z|^2 = (2\bar{\bar{r}} - r^3)/\kappa. To have a Bose–Einstein condensate the values of \( \bar{\bar{r}} > 0 \), i.e. the first energy state is a non-vanishing constant. Then, for the first critical point the trajectories stay in an elliptical traces. However, with each revolution, their distance from \( c_1 \) grow/decay exponentially according to the term e^{n_i \bar{\bar{r}}} \) (see Eq. (3.7)). Therefore, the 4-dim phase portrait shows trajectories that spiral away from the critical point to infinite-distant away (when \( m_{c_1} > 0 \)). Or trajectories that spiral toward, and converge to the critical point (when \( m_{c_1} < 0 \)). As a result, when \( \bar{\bar{r}} > 0 \), \( c_1 \) is a spiral point, which it is asymptotically stable if \( m_{c_1} < 0 \) and it is unstable if \( m_{c_1} > 0 \).

- Case \( s = 4 \), i.e. \( \bar{\bar{V}} = r^4/2 \), which corresponds to the quartic potential. The autonomous system is now reduced to the following system of equations:

\[ x' = 2yz + \kappa z^3 = (r^4 - 2\bar{\mu})z + \kappa z^3, \]

\[ y' = w = 2\bar{\bar{r}}, \quad z' = x, \]

\[ w' = 12 \bar{\bar{r}}^2 (y + \bar{\bar{\mu}}) = 6\bar{\bar{r}}^2. \]  

(3.25)

with three critical points at \( (x_{c_1} = 0, y_{c_1} = -\bar{\bar{\mu}}, z_{c_1} = 0, w_{c_1} = 0) \) and \( (x_{(c_2, c_3)} = 0, y_{(c_2, c_3)} = -\bar{\bar{\mu}}, z_{(c_2, c_3)} = \pm \sqrt{2\bar{\bar{r}} \kappa}, w_{c_2} = 0 \) and
and eigenvectors of

\[\xi_{c_1} = \{(1/\sqrt{-2\mu}, -1/\sqrt{-2\mu}, 0, 0), (0, 0, 2\sqrt{3}/\bar{r}, -2\sqrt{3}/\bar{r}), (1, 1, 0, 0), (0, 0, 1, 1)\},\]

\[\xi_{c_2} = \{(0, 0, 0, 0), \pm([-3 + 2\mu\bar{r})\sqrt{2}/\bar{r}\sqrt{\mu/\kappa}), \pm(\sqrt{12/\bar{r}}\sqrt{1/\bar{r}}, -\sqrt{12/\bar{r}}\sqrt{1/\bar{r}}, \sqrt{4\mu}, -\sqrt{4\mu}), (0, 0, 0, \pm(\sqrt{4\mu}(-3 + 2\mu\bar{r})\sqrt{2}/(\bar{r}\sqrt{\mu/\kappa})), (0, 0, 0, 1, 1, 1, 1)\}.\] (3.29)

| TABLE I. Critical points for the autonomous system (3.3) |
|--------------|--------------|
| **Label**    | **Critical point** |
|              | \((x_c, y_c, z_c, w_c)\) | **Eigenvales** |
| A            | \((0, -\bar{\mu}^2, 0, 0)\) | \(\left\{\sqrt{\frac{\bar{r} - 3}{\bar{r}}}, -\sqrt{\frac{\bar{r} - 3}{\bar{r}}}, \sqrt{2\bar{\mu}}, -\sqrt{2\bar{\mu}}\right\}\) |
| B            | \((0, -\bar{\mu}^2, \pm\sqrt{2\bar{\mu}}, 0)\) | \(\left\{\sqrt{\frac{\bar{r} + 1}{\bar{r}}}, -\sqrt{\frac{\bar{r} + 1}{\bar{r}}}, \sqrt{2\bar{\mu}}, -\sqrt{2\bar{\mu}}\right\}\) |

The constraint is fixed by the curve \(|z|^2 = (2\bar{\mu} - \bar{r}^4)/\kappa\). As in the case \(s = 3\), for the first critical point the trajectories are elliptical. Therefore, we have the same 4-dimensional phase portrait with spiral trajectories when \(\bar{\mu} > 0\), \(c_1\) is a spiral point, which it is asymptotically stable if \(m_{c_1} < 0\) and it is unstable if \(m_{c_1} > 0\).

Furthermore, for the cubic potential \(s = 3\) and the quartic potential \(s = 4\) the stability analysis shows that these systems are unstable due to the presence of spiral trajectories. If we interpreted these trajectories as the velocity field of the particles in the condensate, we can immediately observe that for these cases the system is always unstable. Moreover, the functional form of the equations, the eigenvalues and the eigenvectors allows us to conclude that for \(s > 2\) the system is unstable. These facts can be interpreted as follows: when \(s\) grows, the number of particles in the ground state decreases and clearly the number of particles in the excited states increase. Also, the size of the cloud decreases.

In order to study the stability of the condensate when the shape of the trap varies, we have analyzed the stability conditions of a Bose–Einstein condensate trapped in a generic potential using dynamical systems.

We observe that when the shape parameter \(s\) grows, the system tends to be unstable, i.e., when the shape parameter fulfill the condition \(s > 2\). Basically, the perturbations deduced in the above sections are directly related to perturbations of the order parameter (see the system (3.4)), and consequently, related to fluctuations of the condensed state. This fact can be interpreted as follows: when \(s\) grows, the number of particles in the ground state decreases and clearly the number of particles in the excited states increase. Also, the size of the cloud decreases.

This formalism can extended to more general traps even time-dependent traps and rotating traps in order to analyze the stability of the system under more general conditions, where it would be interesting to show if the solutions are independent of the initial conditions. The latter will be reported elsewhere.

IV. DISCUSSION

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