Optimal Memory Scheme for Accelerated Consensus Over Multi-Agent Networks

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Abstract—The consensus over multi-agent networks can be accelerated by introducing agent’s memory to the control protocol. In this paper, a more general protocol with the node memory and the state deviation memory is designed. We aim to provide the optimal memory scheme to accelerate consensus. The contributions of this paper are three: (i) For the one-tap memory scheme, we demonstrate that the state deviation memory is useless for the optimal convergence. (ii) In the worst case, we prove that it is a vain to add any tap of the state deviation memory, and the one-tap node memory is sufficient to achieve the optimal convergence. (iii) We show that the two-tap state deviation memory is effective on some special networks, such as star networks. Numerical examples are listed to illustrate the validity and correctness of the obtained results.

Index Terms—Optimal memory, accelerated consensus, multi-agent networks, convergence rate.

I. INTRODUCTION

Consensus is a basic problem in distributed coordination control over multi-agent networks, which has been studied extensively in the last decades [1]–[4]. The usual idea to solve this problem is to use the local information to design a consensus protocol, so that the states of all agents can reach a common value over time. Due to the differences in network topologies, control strategies and system models, scholars study the problem of consensus from a multi-faceted perspective, including consensus in switching topology [1], [5], consensus with communication delays [6], [7], consensus with sampling data [9], [19], quantized consensus [10], [11], consensus in nonlinear systems [12], [13], etc.

Convergence rate is an important performance indicator of consensus. Optimizing the weight of the network [14]–[17] is an effective method to accelerate the consensus. Xiao et al. [14] concerned on how to choose the network weights to derive the fastest convergence rate, and gave the optimal constant edge weights to achieve the accelerated consensus. To improve the convergence rate, Kokiopoulou et al. [15] applied a polynomial filter on the network matrix to shape its spectrum. You et al. [16] revealed how the network affect the consensus of the discrete-time multi-agent system, and gave a lower bound of the optimal convergence rate. Apers et al. [17] proposed a preconditioner to optimize the edge weights of a given graph and cluster the eigenvalues towards better polynomial acceleration.

Some researchers consider time-varying control strategies [18]–[21] to improve the convergence rate. By analyzing the properties of Chebyshev polynomials, Montijano et al. [18] designed a fast and stable distributed consensus algorithm. Kibangou [19] considered the multi-agent system under time-varying control, and applied a matrix factorization approach to get the condition of finite-time consensus. Safavi and Khan [20] introduced an approach termed as successive nulling of eigenvalues under time-varying control, and proposed some necessary and sufficient conditions for the multi-agent system to achieve the finite-time consensus. Yi et al. [21] studied the fast consensus problem from the perspective of graph filtering, and provided some explicit formulas for the optimal convergence rate by using the period control strategy.

The method of using agent’s past information is also effective in improving the convergence rate [22]–[29]. Oreshkin et al. [22] proposed a short node memory scheme to accelerate the convergence in distributed averaging. Kia et al. [23] introduced agent’s memory into robust dynamic average consensus algorithms, and got the optimal convergence rate by using the method of root locus. Irofti [24] intentionally introduced agent’s past information in the control protocol to accelerate the consensus, and derived an optimized value of the convergence rate. Yi et al. [25] considered the control protocol with the node memory, and gave the optimal worst-case convergence rate for an uncertain graph set. These researches all uses agent’s own past state stored in memory to accelerate the consensus, which is called the node memory scheme in this paper. To improve the convergence rate, some researchers also utilize neighbours’ past information stored in memory, which is called the state deviation memory. Olshesky [26] added the state deviation memory to the consensus protocol, and get the linear convergence related to the number of nodes. Bu et al. [27] considered a consensus protocol with the state deviation memory, and provided a convergence bound in the worst case. Moradian et al. [28] applied neighbours’ past information to achieve the fast consensus of a single integrator system, and determined the depth of memory that can improve the convergence rate. Pasolini et al. [29] proposed a general protocol with the state deviation memory, and formulated the optimal convergence rate when adding the one-tap memory.

Although the aforementioned results have great improvement for the convergence rate of consensus, it’s worth noting that there still leave some problems that have not been considered. For example, in the consensus protocol, introducing the node memory or the state deviation memory can accelerate the
Lemma 1. For any connected undirected network $G$, its Laplacian matrix $L$ is positive-semidefinite, and has the decomposition $L = VA^T$, where $A = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ and $V = \begin{bmatrix} v_1, v_2, \ldots, v_N \end{bmatrix} \in \mathbb{R}^{N \times N}$ are unitary. Zero is a single eigenvalue of $L$, and the corresponding eigenvector is $v_1 = \frac{1}{\sqrt{N}} \mathbf{1}$, where $\mathbf{1}$ denotes the vector of all ones.

Lemma 2. The consensus of system (4) is achieved only if $\sum_{m=0}^{M} \theta_m = 0$.

Proof. When $k \to \infty$, the stationary solutions of (4) satisfies
\[ \bar{x}_1 = \bar{x}[(1+\theta_0)I-\varepsilon_0L]+\bar{x} \sum_{m=1}^{M} (\theta_m I-\varepsilon_mL)\mathbf{1}. \] (5)

Note that $L\mathbf{1} = 0$. The stationary solutions in equation (5) are kept only if $\sum_{m=0}^{M} \theta_m = 0$. Thus, the condition $\sum_{m=0}^{M} \theta_m = 0$ is required to ensure that the consensus can be reached. □
Perform the graph Fourier transform \( \hat{x}_i(k) = v_i^T x(k), i = 1, \ldots, N \), and get
\[
\hat{x}_i(k+1) = (1 + \theta_0 - \varepsilon_0 \lambda_i)\hat{x}_i(k) + \sum_{m=1}^{M} (\theta_m - \varepsilon_m \lambda_i)\hat{x}_i(k-m).
\]
(6)

Then the agent’s state signal can be analyzed in the graph spectral domain.

**Lemma 3.** Assume that \( \sum_{m=0}^{M} \theta_m = 0 \). The consensus of system (4) is achieved if and only if \( \lim_{k \to \infty} \hat{x}_i(k) = 0 \) holds for any \( i \in \{2, 3, \ldots, N\} \).

**Proof.** For a connected graph, \( \lambda_1 = 0 \) and
\[
\hat{x}_i(k+1) = \hat{x}_i(k) + \sum_{m=0}^{M} \theta_m \hat{x}_i(k-m).
\]
Since \( \hat{x}_i(-M) = \cdots = \hat{x}_i(-1) = \hat{x}_i(0) \) and \( \sum_{m=0}^{M} \theta_m = 0 \), we have
\[
\hat{x}_1(k) = \hat{x}_1(0), \quad \forall k \geq 0.
\]
It follows that
\[
v_i^T \hat{x}_1(k) = v_i^T \hat{x}_1(0) = \frac{1}{N} 1_N^T x(0) = \bar{x} 1.
\]
Then the state of system (3) can be written as
\[
\lim_{k \to \infty} x(k) = \bar{x} 1 + \lim_{k \to \infty} \sum_{i=2}^{N} v_i \hat{x}_i(k).
\]
(7)

Substituting \( \lim_{k \to \infty} \hat{x}_i(k) = 0, i = 2, \ldots, N \) into (7), the sufficiency can be proved directly. Suppose that there is a scalar \( j \in \{2, \ldots, N\} \) that satisfies \( \lim_{k \to \infty} \hat{x}_j(k) = \delta \neq 0 \). Since \( \langle v_i, v_j \rangle = 0 \) holds for any \( i \neq j \), then \( \lim_{k \to \infty} x(k) - \bar{x} 1 \neq 0 \). This contradiction proves the necessity.

Denote \( y_i(k) = [\hat{x}_i(k), \hat{x}_i(k-1), \ldots, \hat{x}_i(k-M)]^T \). The problem of consensus is transformed into the simultaneous stability problem of \( N-1 \) systems
\[
y_i(k+1) = \Gamma(\lambda_i) y_i(k), i = 2, 3, \ldots, N,
\]
(8)
where
\[
\Gamma(\lambda_i) = \begin{bmatrix}
1 + \theta_0 - \varepsilon_0 \lambda_i & \theta_1 - \varepsilon_1 \lambda_i & \cdots & \theta_M - \varepsilon_M \lambda_i \\
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]
(9)

**Definition 2.** Assume that the system (4) on a given network \( G \). Define the convergence rate of the consensus as \( r_M = \max_{i \in \{2, \ldots, N\}} \rho(\Gamma(\lambda_i)) \), where \( \rho(\cdot) \) denotes the spectral radius.

This paper focuses on designing the control parameters \( \varepsilon_0, \ldots, \varepsilon_M \) and \( \theta_0, \ldots, \theta_M \) to achieve the consensus with fast convergence speed, and then obtain the optimal memory scheme.

**III. Optimal Short Memory Scheme on a Given Network**

This section aims to analyze the optimal short memory scheme on a given network.

**Lemma 4.** (Schur Complement [37]) Given a matrix
\[
W = \begin{bmatrix}
W_1 & W_2 \\
W_3 & W_4
\end{bmatrix},
\]
with nonsingular \( W_1 \in \mathbb{R}^{n \times n}, W_2 \in \mathbb{R}^{n \times \mu}, W_3 \in \mathbb{R}^{\mu \times n}, \) and \( W_4 \in \mathbb{R}^{\mu \times \mu} \). Then \( \det W = \det W_1 \cdot \det (W_4 - W_3 W_1^{-1} W_2) \).

**Lemma 5.** (Jury Stability Criterion [32]) Given a second-order polynomial \( d(z) = z^2 + a_1 z + a_0 \). The roots of \( d(z) = 0 \) are all in the unit circle if and only if
\[
d(1) > 0, d(-1) > 0, |a_0| < 1.
\]

Define \( p(z, \lambda_i) = \det(zI - \Gamma(\lambda_i)) \). Using the method of Schur complement, the characteristic polynomial of \( \Gamma(\lambda_i) \) can be easily calculated as
\[
p(z, \lambda_i) = z^{M+1} + (\varepsilon_0 \lambda_i - 1 - \theta_0)z^M + \sum_{m=1}^{M} (\varepsilon_m \lambda_i - \theta_m)z^{M-m}.
\]
(11)

Then the problem of accelerated consensus can be converted into the optimization problem
\[
\min_{\varepsilon_0, \ldots, \varepsilon_M, \theta_0, \ldots, \theta_M} r_M
\]
subject to
\[
p(z, \lambda_i) = 0, \quad i = 2, \ldots, N.
\]
(12)

The optimization problem (12) is difficult to solve especially when \( M \) is large. The optimal convergence rate with \( M = 0 \) is \( r_0^* = \frac{\lambda_N - \lambda_1}{\lambda_N + \lambda_1} \), which has been solved in [14]. In this section, we explore the optimal convergence rate when \( M = 1 \).

**Theorem 1.** Consider the consensus of the system (4) on a connected network \( G \). The optimal convergence rate of \( M = 1 \) is
\[
r_1^* = \sqrt{\frac{\lambda_N - \sqrt{\lambda_2}}{\lambda_N + \sqrt{\lambda_2}}},
\]
(13)
with the optimal control parameters
\[
\varepsilon_0^* = \frac{4}{(\sqrt{\lambda_N} + \sqrt{\lambda_2})^2}, \quad \varepsilon_1^* = 0,
\]
\[
\theta_0^* = \frac{2}{(\sqrt{\lambda_N} + \sqrt{\lambda_2})^2}, \quad \theta_1^* = -\theta_0.
\]
(14)

**Proof.** When \( M = 1 \), the system can be written as
\[
x(k+1) = [(1 + \theta_0)I - \varepsilon_0 L]x(k) + (\theta_1 I - \varepsilon_1 L)x(k-1).
\]
(15)
The consensus is achieved if and only if the roots of
\[
p(z, \lambda_2) = z^2 + (\varepsilon_0 \lambda_2 - 1 - \theta_0)z + \varepsilon_1 \lambda_2 + \theta_0 = 0,
\]
\[
p(z, \lambda_N) = z^2 + (\varepsilon_0 \lambda_N - 1 - \theta_0)z + \varepsilon_1 \lambda_N + \theta_0 = 0
\]
are in the unit circle. Let \( z = r \tilde{z} \) in (15). Then the roots of (15) are in the circle with radius \( r \) if and only if the roots of
\[
r^2 \tilde{z}^2 + (\varepsilon_0 \lambda_2 - \theta_0 - 1)r \tilde{z} + \varepsilon_1 \lambda_2 + \theta_0 = 0,
\]
\[
r^2 \tilde{z}^2 + (\varepsilon_0 \lambda_N - \theta_0 - 1)r \tilde{z} + \varepsilon_1 \lambda_N + \theta_0 = 0.
\]
(16)
are in the unit circle. According to the Jury stability criterion, the roots of (16) are in or on the unit circle if and only if
\[
\begin{align*}
    r^2 + (\varepsilon_0 \lambda - \theta_0 - 1)r + \varepsilon_1 \lambda_2 + \theta_0 & \geq 0 \quad (17a) \\
    r^2 + (\varepsilon_0 \lambda_2 - \theta_0 - 1)r + \varepsilon_1 \lambda_3 + \theta_0 & \geq 0 \quad (17b) \\
    r^2 - (\varepsilon_0 \lambda - \theta_0 - 1)r + \varepsilon_1 \lambda_2 + \theta_0 & \geq 0 \quad (17c) \\
    r^2 - (\varepsilon_0 \lambda_2 - \theta_0 - 1)r + \varepsilon_1 \lambda_3 + \theta_0 & \geq 0 \quad (17d) \\
    r^2 - \varepsilon_1 \lambda_2 - \theta_0 & \geq 0 \quad (17e) \\
    r^2 - \varepsilon_1 \lambda_3 - \theta_0 & \geq 0. \quad (17f)
\end{align*}
\]

To eliminate \(\theta_0\), we multiply \(1 - r\) to (17e) and add (17a), and give
\[
r \lambda_2 (\varepsilon_0 + \varepsilon_1) - r (r-1)^2 \geq 0. \quad (18)
\]

Similarly, multiply \(1+r\) to (17f) and add (17d), give
\[
-r \lambda_N (\varepsilon_0 + \varepsilon_1) + r (r+1)^2 \geq 0. \quad (19)
\]

To eliminate \(\varepsilon_0\) and \(\varepsilon_1\), we first multiply \(\lambda_N\) to (18) and \(\lambda_2\) to (19), and have
\[
\begin{align*}
    r \lambda_2 \lambda_N (\varepsilon_0 + \varepsilon_1) - \lambda_N r (r-1)^2 & \geq 0, \quad (20) \\
    -r \lambda_2 \lambda_N (\varepsilon_0 + \varepsilon_1) + \lambda_2^2 r (r+1)^2 & \geq 0. \quad (21)
\end{align*}
\]

Then add (20) and (21), get
\[
\lambda_2 (r+1)^2 - \lambda_N (r-1)^2 \geq 0.
\]

It follows that
\[
r \geq \frac{\sqrt{\lambda_2} - \sqrt{\lambda_N}}{\sqrt{\lambda_2} + \sqrt{\lambda_N}}. \quad (22)
\]

The optimal solution \(r = r^*_1\) is obtained if and only if
\[
\begin{bmatrix}
    \lambda_2 r \\
    -\lambda_N r \\
    0
\end{bmatrix}
\begin{bmatrix}
    \lambda_2 r \\
    -\lambda_N r \\
    0
\end{bmatrix}
\begin{bmatrix}
    \lambda_2 r \\
    -\lambda_N r \\
    0
\end{bmatrix}
\begin{bmatrix}
    1-r \\
    1+r \\
    -1
\end{bmatrix}
\begin{bmatrix}
    \varepsilon_0 \\
    \varepsilon_1 \\
    \theta_0
\end{bmatrix}
= 
\begin{bmatrix}
    -r^2 + r \\
    -r^2 - r \\
    -r^2
\end{bmatrix}. \quad (23)
\]

The solution of equation (23) is given by (14). It is verified that the remaining constraints in (17) are satisfied. This completes the proof. \(\square\)

**Remark 1.** The optimal convergence rate and the corresponding control parameters are related to the eigenratio \(\lambda_2/\lambda_N\) of the Laplacian matrix. The larger eigenratio corresponds to the better network connectivity, which leads to the faster convergence of consensus.

**Remark 2.** By utilizing time-varying control without memory, [27] proposed that the lower bound of the optimal convergence rate is \(\sqrt{\lambda_2 - \lambda_N}/\sqrt{\lambda_2 + \lambda_N}\). This means that the optimal convergence rate of the constant control scheme with the one-tap memory has reached the limit value of the convergence rate of the time-varying control scheme without memory.

**Remark 3.** Note that \(\varepsilon_1^* = 0\) in (14). This means that it is not necessary to add the state deviation memory when \(M = 1\). The optimal convergence rate with the one-tap state deviation memory proposed in [29] is \(\sqrt{\lambda_2 - \lambda_N}/\sqrt{\lambda_2 + \lambda_N}\), which also corroborates our analysis.

The following two statements are worth noting. (i) If only the state deviation memory is used, the convergence rate can be improved as \(M\) increases [29]. (ii) If only the node memory is used, the convergence rate cannot be improved for any \(M > 1\) in the worst case [25]. Then a question arises spontaneously: can the convergence rate of the general memory scheme in this paper be improved as \(M\) increases? This question is explored in the next section.

**IV. The worst-case optimal memory scheme and a special case**

This section proposes the optimal memory scheme in the worst case, and introduces a special case on star networks.

**Lemma 6.** (Routh-Hurwitz Criterion [33]) Given a third-order polynomial \(d(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3\). The roots of \(d(s) = 0\) are all in the open left half plane (the polynomial \(d(s)\) is stable) if and only if
\[
a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 - a_0 a_3 > 0.
\]

Assume that the system (4) on the uncertain network \(G_{[\alpha, \beta]}\), where \(G_{[\alpha, \beta]}\) represents all the networks with the nonzero eigenvalues of \(L\) in the interval \([\alpha, \beta]\).

Define the worst-case convergence rate of the consensus as
\[
\gamma_M = \sup_{G \in G_{[\alpha, \beta]}} r_M. \quad (24)
\]

**Lemma 7.** The worst-case convergence rate satisfies
\[
\gamma_M = \sup_{G \in G_{[\alpha, \beta]}} p(\Gamma(t)) = \sup_{G \in G_{[\alpha, \beta]}} \max_{\gamma \in [1, 2, \ldots, M+1]} |z_\gamma(\lambda)|, \quad (25)
\]
where \(z_\gamma(\lambda)\) denotes the root of \(p(z, \lambda) = 0\), and \(p(z, \lambda)\) is defined in (11).

From Theorem 1, the optimal worst-case convergence rate of \(M = 1\) is
\[
\gamma_1^* = \sqrt{\beta - \alpha}/\sqrt{\beta + \alpha}, \quad (26)
\]
and the corresponding control parameters are
\[
\varepsilon_0^* = \frac{4}{(\sqrt{\beta} + \sqrt{\alpha})}, \quad \varepsilon_1^* = 0, \quad \theta_0^* = \left(\frac{4}{(\sqrt{\beta} + \sqrt{\alpha})}\right)^2, \quad \theta_1^* = -\theta_0. \quad (27)
\]

Before giving the result, we make some settings. Let
\[
p(rz, \lambda) = r^{M+1} z^{M+1} + (\varepsilon_0 \lambda - 1 - \theta_0) r^M z^M + \sum_{m=1}^M (\varepsilon_m \lambda - \theta_m) r^{M-m} z^{M-m}.
\]

Set up the feedback system as shown in Fig.1, where
\[
P(rz, \lambda) = \frac{(rz)^{-1}}{1 - (rz)^{-1}} \lambda, \quad C(rz) = \frac{\sum_{m=0}^M \varepsilon_m r^{-m} z^{-m}}{1 - \sum_{m=1}^M \theta_j r^{-m} z^{-m}}.
\]

The transfer function of the closed-loop system is
\[
G(rz, \lambda) = \frac{P(rz, \lambda)}{1 + P(rz, \lambda) C(rz)} = \frac{\lambda^M}{p(rz, \lambda)}.
\]
Next, by converting the problem of fast consensus to the robust stabilization of the feedback system, we give the optimal worst-case convergence rate of any $M \geq 1$ as follows.

**Theorem 2.** Consider the consensus of system (4) on the uncertain network $G_{(\alpha, \beta)}$. The optimal worst-case convergence rate is
\[ \gamma^*_M = \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}}, \quad (28) \]
and a set of optimal control parameters is given by (27).

**Proof.** The roots of $p(z, \lambda) = 0$ are in the circle with radius $r$ if and only if the polynomial $p(rz, \lambda)$ is stable. This means that the system with the transfer function $G(rz, \alpha)$ needs to be stable for any $\lambda \in [\alpha, \beta]$ to ensure the worst-case convergence rate. The statement that $G(rz, \alpha)$ is stable for any $\lambda \in [\alpha, \beta]$ can be regarded as the problem of gain margin optimization \[24\] (Chapter 11), that is, a stable $G(rz, \alpha)$ with the gain margin $k_{\text{sup}} \geq \beta/\alpha$. Then the gain margin of $P(rz, \alpha)$ needs to satisfy $k_{\text{sup}} \geq \beta/\alpha$. Note that the optimal gain margin of $P(rz, \alpha) = \frac{(rz)^{N-1}}{1-(rz)^{N}}\alpha$ has been given by $k_{\text{sup}} = \frac{(1+r)^2}{1-r}$ \[23\] (Lemma 5). It follows from \[\frac{(1+r)^2}{1-r} \geq \beta/\alpha\] that $r \geq \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}}$. A set of control parameters to reach the optimal worst-case convergence rate $r = \gamma^*_M$ is given by (27). \[\square\]

**Remark 4.** Note that $\gamma^*_M = \gamma^*_1$ for any $M \geq 1$. This means that, in the worst case, the one-tap node memory is sufficient to achieve the optimal convergence, and the state deviation memory in \[29\] is unnecessary.

It is worth noting that \[25\] has proposed that the convergence rate of the two-tap node memory scheme cannot be improved under any given network. However, we find that the two-tap state deviation memory is effective on some special networks. In particular, on star networks, an explicit formula for the convergence rate with $M = 2$ is given as follows.

**Theorem 3.** Consider the MAS (1) on star networks with $N$ nodes under the control protocol (2). Denote $M = 2$ and $\mu = \frac{N-1}{N+1}$. The following conclusions hold.

(i) If the control parameters are set as
\[\varepsilon_0 = \frac{6\mu}{N-1}, \varepsilon_1 = 0, \varepsilon_2 = \frac{2(\tilde{r}^2)^3}{N-1}, \theta_0 = \frac{8(\tilde{r}^2)^2}{(\tilde{r}^2)^2 + 3}, \theta_1 = -3(\tilde{r}^2)^2, \theta_2 = 3(\tilde{r}^2)^2 - \frac{8(\tilde{r}^2)^2}{(\tilde{r}^2)^2 + 3}, \quad (29)\]
the consensus is achieved with the convergence rate
\[\tilde{r}_2 = \mu + \left( \frac{\sqrt{3}i - 1}{2} \right)^\frac{1}{4} \left( \mu^3 - \mu - \sqrt{(\mu^3 - \mu)^2 + (1 - \mu^2)^2} \right)^\frac{1}{2}, \quad (30)\]

(ii) The convergence rate of $M = 2$ is better than the optimal convergence rate of $M = 1$, i.e., $\tilde{r}_2 < r^*_1$.

**Proof.** (i) For a star network with $N$ nodes, the eigenvalues of its Laplacian matrix are
\[\lambda_i = \begin{cases} 0, & i = 1 \\ 1, & 2 \leq i \leq N - 1 \\ N, & i = N \end{cases}. \]
The consensus is achieved if and only if the roots of
\[p(z, \lambda_i) = z^3 + (\varepsilon_0 \lambda_i - 1 - \theta_0)z^2 + (\varepsilon_1 \lambda_i + \theta_0 + \theta_2)z + \varepsilon_2 \lambda_i - \theta_2 = 0, \quad i = 2, N, \quad (31)\]
are in the unit circle. Let $z = \frac{\tilde{r}_2 + 1}{\sqrt{\tilde{r}_2}}$ in (31), and denote
\[f_0^{(i)} = \tilde{r}_2^3 + (\varepsilon_0 \lambda_i - 1 - \theta_0)\tilde{r}_2^2 + (\varepsilon_1 \lambda_i + \theta_0 + \theta_2)\tilde{r}_2 + \varepsilon_2 \lambda_i - \theta_2, \quad (i)\]
\[f_1^{(i)} = 3\tilde{r}_2^3 + (\varepsilon_0 \lambda_i - 1 - \theta_0)\tilde{r}_2^2 - (\varepsilon_1 \lambda_i + \theta_0 + \theta_2)\tilde{r}_2 - 3(\varepsilon_2 \lambda_i - \theta_2), \quad (i)\]
\[f_2^{(i)} = 3\tilde{r}_2^3 - (\varepsilon_0 \lambda_i - 1 - \theta_0)\tilde{r}_2^2 - (\varepsilon_1 \lambda_i + \theta_0 + \theta_2)\tilde{r}_2 + 3(\varepsilon_2 \lambda_i - \theta_2), \quad (i)\]
\[f_3^{(i)} = \tilde{r}_2^3 - (\varepsilon_0 \lambda_i - 1 - \theta_0)\tilde{r}_2^2 + (\varepsilon_1 \lambda_i + \theta_0 + \theta_2)\tilde{r}_2 - (\varepsilon_2 \lambda_i - \theta_2), \quad (i)\]
Then the roots of $p(z, \lambda_i) = 0$ are in the circle with radius $r$ if and only if the polynomial
\[p^{(i)}(s, \lambda_i) = f_0^{(i)} s^3 + f_1^{(i)} s^2 + f_2^{(i)} s + f_3^{(i)} \quad (33)\]
is stable. Denote $g^{(i)} = f_1^{(i)} f_3^{(i)} - f_0^{(i)} f_2^{(i)}$. According to the Routh stability criterion, the polynomial (33) is stable or marginally stable, if and only if
\[f_0^{(i)} \geq 0, f_1^{(i)} \geq 0, f_2^{(i)} \geq 0, f_3^{(i)} \geq 0, g^{(i)} \geq 0, i = 2, N. \quad (34)\]
Perform the operations
\[
\begin{align*}
3 \times f_0^{(2)} &+ \frac{1}{2} \times f_1^{(2)} + \frac{1}{4} \times f_1^{(N)} + \frac{1}{4} \times f_2^{(2)} + \frac{1}{2} \times f_2^{(N)} + \frac{3}{4} \times f_3^{(N)},
\end{align*}
\]
get $(1 - N)r^2 \varepsilon_0 + 6r^3 \geq 0$. It follows that $\varepsilon_0 \leq \frac{6r}{N-1}$. The parameter $\varepsilon_0$ can be taken to the upper bound $\frac{6r}{N-1}$ when
\[f_0^{(2)} = 0, f_1^{(2)} = 0, f_1^{(N)} = 0, f_2^{(2)} = 0, f_2^{(N)} = 0, f_3^{(N)} = 0. \quad (35)\]
Let the cubic polynomial
\[h(r) = r^3 + \frac{N-1}{N+1}r^2 + 3r - \frac{N-1}{N+1}. \quad (36)\]
It is calculated by (35) that the control parameters satisfies (29) and $h(r) = 0$. Applying the Cardano’s formula \[35\], the sole real root of the equation $h(r) = 0$ in the interval $r \in (0, 1)$ is given by (30). It is verified that the remaining constraints
\[f_0^{(N)} > 0, f_2^{(N)} > 0, g^{(2)} = g^{(N)} = 0 \]
are satisfied by using the control parameters (29). This completes the proof.

(ii) Note that the derivative of the function $h(r)$ satisfies
\[h'(r) = 3r^2 - 6\frac{N-1}{N+1}r + 3 > 3r^2 - 6r + 3 = 3(r-1)^2 > 0. \]
Thus, $h(r)$ is increasing monotonically. Substitute $r = r_1^* = \frac{\sqrt{N} - 1}{\sqrt{N} + 1}$ into $h(r)$, get

$$h(r_1^*) = \frac{8(\sqrt{N} - 1)N}{(\sqrt{N} + 1)^3(N + 1)} > 0.$$ 

It follows from $h(\tilde{r}_2) = 0 < h(r_1^*)$ that $\tilde{r}_2 < r_1^*$. □

It follows from Theorem 3 that on star networks, the consensus with the two-tap memory can be achieved faster. In fact, the consensus with the two-tap memory is not only accelerated on star networks. For example, applying the control parameters (29), the maximum modulus root of $p(z, \lambda) = 0$, $\lambda \in [1, 5]$ is shown in Fig. 2. It can be observed from Fig. 2 that the consensus with the two-tap memory is accelerated on some special networks, where the nonzero eigenvalues of its Laplacian matrix are in the interval $\lambda \in [1, 1.28] \cup [4.72, 5]$.

**V. NUMERICAL EXAMPLES**

In this section, some examples are listed to verify the validity and correctness of the proposed results.

**Example 1.** In this example, the convergence performance of different consensus algorithms with or without memory is compared. The compared algorithms are: (i) the best constant gain scheme (BC) proposed in [14], (ii) accelerated consensus algorithm with the state deviation memory (SDMem) proposed in [29], (iii) the optimal one-tap memory scheme (OptMem) in this paper. Randomly generate a network $G_1$ with 9 nodes, as shown in Fig. 3. It can be calculated that $\lambda_2 = 0.8835, \lambda_N = 7.1716$. Table I lists the optimal convergence rate and corresponding control parameters of each algorithm. Generate the initial state of each agent in the interval $[-10, 10]$. In order to facilitate the comparison of the convergence speed, the definition of $\epsilon$-convergence time [36] is introduced:

$$T(\epsilon) = \min \left\{ k^* : \frac{\|x(k) - \bar{x}1\|_{\infty}}{\|x(0) - \bar{x}1\|_{\infty}} \leq \epsilon \quad \forall k \geq k^*, \forall x(0) \neq \bar{x}1 \right\}.$$ 

Set the error threshold as $\epsilon = 10^{-5}$. Fig. 4 shows the $\epsilon$-convergence time of each consensus algorithm. It can be observed that the convergence speed of the optimal one-tap memory scheme proposed in this paper is faster than that of SDMem with $M = 1$, and even faster than that of SDMem with $M = 2$. In addition, the convergence speed of the memoryless BC is the slowest.

**TABLE I: The optimal convergence rate and corresponding control parameters under different algorithms**

|                | OptMem with $M = 1$ | BC [14] | SDMem [29] with $M = 1$ | SDMem [29] with $M = 2$ |
|----------------|---------------------|---------|--------------------------|--------------------------|
| $r_1^*$        | 0.4804              | 0.7806  | 0.6402                   | 0.5582                   |
| $\epsilon_0^*$ | 0.3056              | 0.2483  | 0.3180                   | 0.3226                   |
| $\epsilon_1^*$ | 0 N/A               | 0.0571  | 0.0789                   | 0.0112                   |
| $\theta_0^*$   | 0.2308              | N/A     | N/A                      | N/A                      |
| $\theta_1^*$   | -0.2308             | N/A     | N/A                      | N/A                      |

Fig. 2: The maximum modulus root of $p(z, \lambda) = 0$

**Example 2.** In this example, the proposed algorithm is verified on large-scale networks. Consider 80 random connected net-
is also optimal on large-scale networks. It can be seen from Fig. 5 that the algorithm proposed is also optimal on large-scale networks.

Example 3. This example demonstrates the effectiveness of the control strategy in Theorem 3. Table II lists the convergence rate $\tilde{r}_1$ and $\tilde{r}_2$ on the star network with different numbers of nodes. Consider $N = 10$, and randomly generate the initial state of each agent in the interval $[−10, 10]$. The convergence of consensus is shown in Fig. 6. It can be observed that on the star network, the consensus with the two-tap memory scheme can be reached more quickly than that with the one-tap memory scheme.

TABLE II: Convergence rate on the star network

| Graph number | Convergence rate | Convergence rate |
|--------------|------------------|------------------|
| $N = 5$      | $0.3820$         | $0.2620$         |
| $N = 10$     | $0.5195$         | $0.3660$         |
| $N = 20$     | $0.6345$         | $0.4616$         |
| $N = 50$     | $0.7522$         | $0.5730$         |
| $N = 100$    | $0.8182$         | $0.6455$         |

VI. CONCLUSION

This paper has proposed a more general control protocol with both the node memory and the state deviation memory to accelerate the consensus over multi-agent networks. The optimal convergence rate with the one-tap memory has been formulated based on the Jury stability criterion. The state deviation memory has been pointed out to be useless for the optimal convergence under the one-tap memory scheme. By transforming the optimization problem of the convergence rate into the robust stabilization of the feedback system, it has been proved that in the worst case, the one-tap node memory scheme is sufficient to achieve the optimal convergence, and adding any tap of the state deviation memory is unnecessary. Moreover, the two-tap state deviation memory has been found to be effective on some special networks. Specially, for star networks, an optimized explicit convergence rate with the two-tap memory scheme has been given. Numerical examples have demonstrated the validity and correctness of the obtained results.

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