Stability of solutions to abstract differential equations

A.G. Ramm
Department of Mathematics
Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract
A sufficient condition for asymptotic stability of the zero solution to an abstract nonlinear evolution problem is given. The governing equation is \( \dot{u} = A(t)u + F(t, u) \), where \( A(t) \) is a bounded linear operator in Hilbert space \( H \) and \( F(t, u) \) is a nonlinear operator, \( \|F(t, u)\| \leq c_0\|u\|^{1+p}, \) \( p = \text{const} > 0, \) \( c_0 = \text{const} > 0. \) It is not assumed that the spectrum \( \sigma := \sigma(A(t)) \) of \( A(t) \) lies in the fixed halfplane \( \Re z \leq -\kappa, \) where \( \kappa > 0 \) does not depend on \( t. \) As \( t \to \infty \) the spectrum of \( A(t) \) is allowed to tend to the imaginary axis.

MSC: 34G20; 447J05; 47J35

Key words: dynamical systems; stability; asymptotic stability

1 Introduction
Let \( H \) be a Hilbert space. Consider the problem

\[
\dot{u} = A(t)u + F(t, u), \quad t \geq 0,
\]

\[ u(0) = u_0, \]

where \( \dot{u} = \frac{du}{dt} \) is the strong derivative, \( A(t) \) is a linear closed densely defined in \( H \) operator with the domain \( D(A) \), independent of \( t, \) \( u_0 \in D(A) \). We assume that \( F(t, u) \) is a nonlinear mapping, locally Lipschitz with respect to \( u, \) and satisfying the following inequality

\[ \|F(t, u)\| \leq c_0\|u\|^{1+p}, \quad p > 0, \quad c_0 > 0, \]
where \( p \) and \( c_0 \) are constants. We also assume that

\[
\text{Re}(Au, u) \leq -\gamma(t)\|u\|^2, \quad \forall u \in D(A),
\]

where

\[
\gamma(t) > 0, \quad \lim_{t \to \infty} \gamma(t) = 0, \quad (5)
\]

\[
\gamma(t) = \frac{b_1}{(b_0 + t)^d}, \quad d = \text{const} \in (0, 1),
\]

\( b_0 \) and \( b_1 \) are positive constants. Assumptions (5) are satisfied by the function (6). However, our method can be applied to many other \( \gamma(t) \) satisfying assumptions (5).

**Definition 1.** The zero solution to equation (1) is called Lyapunov stable if for any \( \epsilon > 0 \), sufficiently small, there exists a \( \delta = \delta(\epsilon) > 0 \), such that if \( \|u_0\| < \delta \), then the solution to problem (1) exists on \([0, \infty)\) and \( \|u(t)\| \leq \epsilon \). If, in addition,

\[
\lim_{t \to \infty} \|u(t)\| = 0,
\]

then the zero solution is asymptotically stable.

Basic results on the Lyapunov stability of the solutions to (1) one finds in [1]-[4], and in many other books and papers. In [4] these results are established under the assumption that the operator \( A(t) \) is bounded, \( D(A) = H \), and \( A(t) \) has property \( B(\nu, N) \). This means ([4], p.178) that every solution to the equation

\[
\dot{u} = A(t)u
\]

satisfies the estimate

\[
\|u(t)\| \leq Ne^{-\nu(t-s)}\|u(s)\|, \quad t \geq s \geq 0,
\]

where \( N > 0 \) and \( \nu > 0 \) are some constants. The quantity

\[
\kappa := \lim_{t \to \infty} \frac{\ln \|u(t)\|}{t}
\]

is called the exponent of growth of \( u(t) \). If \( \Sigma \) is the set of \( \kappa \) for all solutions to (8), then

\[
\kappa_s := \sup_{\kappa \in \Sigma} \kappa
\]

is called senior exponent of growth of solutions to (8). The general exponent \( \kappa_g \) is defined as

\[
\kappa_g := \inf \rho,
\]
where $\rho$ is the exponent in the inequality
\[
\|u(t)\| \leq Ne^{\rho(t-s)}\|u(s)\|, \quad t \geq s \geq 0.
\] (13)

One has
\[
\kappa_s \leq \kappa_g,
\] (14)
and the case $\kappa_s < \kappa_g$ can occur (the Perron’s example, see [4], p.177). If $\kappa_g < 0$ then the zero solution to (8) is Lyapunov asymptotically stable. If $A(t) = A$ does not depend on $t$ and $A$ is a bounded linear operator, then $\kappa_g < 0$ if and only if the spectrum of $A$, denoted $\sigma(A)$, lies in the halfplane $\text{Re} z \leq \kappa_g < 0$. In this case
\[
\|e^{At}\| \leq N_0 e^{\kappa_g t},
\] (15)
and if $\|F(t,u)\| \leq q\|u\|, \ t \geq 0, \ \|u\| < \rho$, and $q < \frac{\kappa_g}{N_0}$, then equation (1) has negative general exponent also, so the zero solution to equation (1) is Lyapunov asymptotically stable (1, p.403).

If $A = A(t)$, and for any solution to (5) estimate (9) holds with $\nu > 0$, and if (3) holds, then for any solution to (1) with $\|u_0\| \leq \delta$ and $\delta > 0$ sufficiently small, estimate (10) holds with a different $N = N_1$ and $\nu = \nu_1$, $0 < \nu_1 \leq \nu$ (see [4], p.414). This means that the zero solution to (1) is asymptotically stable under the above assumptions.

The basic new result of our work, Theorem 1 in Section 2, generalizes the above results to the case when the assumption $\kappa_g < 0$ is not valid. We allow the spectrum $\sigma(A(t))$ to approach imaginary axis as $t \to \infty$. This is the principally new generalization of the classical Lyapunov-Krein theory. If $\cap$ is the complex plane and $l$ is the imaginary axis, then we assume that $\sigma(A(t)) \subset \cap$ for every $t \geq 0$, but we allow $\lim_{t \to \infty} d(\sigma(A(t)), l) = 0$, where $d(\sigma, l)$ is the distance between two sets $\sigma$ and $l$. The new stability result is formulated in Theorem 1. In Lemma 1 an auxiliary result is formulated. A proof of Lemma 1 differs in details from the one in [7]. In Section 2 Theorem 1 and Lemma 1 are formulated. In Section 3 proofs are given. In Section 4 examples of applications of our method are given.

2 Formulation of the results

Lemma 1. Let the inequality
\[
\dot{g}(t) \leq -\gamma(t)g(t) + a(t)g^{1+p}(t) + \beta(t),
\] (16)
hold for $t \in [0, T)$, where $g(t) \geq 0$ has finite derivative from the right at every point $t$ at which $g(t)$ is defined, $\gamma(t) \geq 0$, $a(t) \geq 0$ and $\beta(t) \geq 0$ are continuous on $\mathbb{R}_+ := [0, \infty)$ functions, and $p = \text{const} > 0$. Assume that there exists a $\mu(t) \in C^1[0, \infty)$, $\mu(t) > 0$, $\dot{\mu}(t) \geq 0$, such that

\[ a(t)[\mu(t)]^{-1-p} + \beta(t) \leq \mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)], \quad t \geq 0, \quad (17) \]

\[ \mu(0)g(0) < 1. \quad (18) \]

Then $g(t)$ exists for all $t \in [0, \infty)$ and

\[ 0 \leq g(t) < \mu^{-1}(t), \quad \forall t \geq 0. \quad (19) \]

**Theorem 1.** Assume that conditions (11)-(16) hold and $b_1 > 0$ is sufficiently large. Then the zero solution to (1) is asymptotically stable for any fixed initial data $u(0)$.

### 3 Proofs

**Proof of Lemma 7.** Let $v(t) := g(t)\int_0^t \gamma(s)ds := g(t)q(t)$. Then (16) yields

\[ \dot{v}(t) \leq q(t)a(t)q^{-1+p}(t)v^{1+p}(t) + q(t)\beta(t), \quad v(0) = g(0), \quad t > 0. \quad (20) \]

We do not assume a priori that $v(t)$ is defined for all $t \geq 0$. This conclusion will follow from our proof. Denote $\eta(t) := q(t)\mu^{-1}(t)$, $\eta(0) = \mu^{-1}(0) > g(0)$. Using (18) and (20), one gets

\[ \dot{v}(0) \leq a(0)v^{1+p}(0) + \beta(0) \leq \mu^{-1}(0)[\gamma(0) - \dot{\mu}(0)\mu^{-1}(0)] = \dot{\eta}(0). \quad (21) \]

Since $v(0) = g(0) < \eta(0) = \mu^{-1}(0)$ by (18), and $\dot{v}(0) \leq \dot{\eta}(0)$, it follows that

\[ v(t) < \eta(t), \quad 0 \leq t < \tau, \quad (22) \]

where $\tau > 0$ is the right end of the maximal interval on which $v(t) < \eta(t)$, i.e., $\tau = \sup\{t : v(t) < \eta(t)\}$. Let us prove that $\tau = \infty$. Note that if (22) holds, then

\[ \dot{v}(t) \leq \dot{\eta}(t), \quad 0 \leq t < \tau. \quad (23) \]

Indeed, using (17) and (20) one obtains

\[ \dot{v}(t) = q(t)(\dot{g} + \gamma g) \leq q(t)\mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)] = \dot{\eta}(t), \quad (24) \]

as claimed. If $\tau < \infty$, then (22) and (23) imply

\[ v(\tau - 0) - v(0) \leq \eta(\tau - 0) - \eta(0). \quad (25) \]
Since \( \eta(t) \in C^1[0, \infty) \) by definition, inequality (25) implies that \( v(\tau-0) < \infty \) and, since \( v(0) = g(0) < \mu^{-1}(0) = \eta(0) \), so that \( v(0) < \eta(0) \), one gets
\[
v(\tau-0) < \eta(\tau-0) < \infty.
\] (26)

Inequality \( (26) \) implies that \( \tau = \infty \), because \( \tau \) is the maximal interval \([0, \tau)\) of the existence of \( v \), and if \( \tau < \infty \) is the right end of the maximal interval of the existence of \( v \) then \( \lim_{t\to\tau-0}v(t) = \infty \), which contradicts (26). Thus, \( \tau = \infty \) and, therefore, \( T = \infty \).

Proof of Theorem 1. Let \( \|u(t)\|=g(t) \). Multiply (1) by \( u(t) \), take the real part, and get
\[
g(t)\dot{g}(t) \leq -\gamma g^2(t) + c_0 g^{2+p}(t).
\] (27)

Since \( g \geq 0 \), inequality (27) is equivalent to
\[
\dot{g}(t) \leq -\gamma(t)g(t) + c_0 g^{1+p}(t).
\] (28)

If \( g(t) > 0 \), then (28) is obviously equivalent to (27). If \( g(t) = 0 \) \( \forall t \in \Delta \), where \( \Delta \subset \mathbb{R}_+ \) is an open set, then \( u(t) = 0 \) \( \forall t \in \Delta \), so \( u(t) = 0 \) \( \forall t \geq 0 \) by the uniqueness of the solution to the Cauchy problem for equation (11).

This uniqueness holds due to the assumed local Lipschitz condition for \( F \). If \( g(t_0)=0 \), but \( g(t) \neq 0 \) for \( (t_0, t_0+\delta) \) for some \( \delta > 0 \), then one divides (27) by \( g(t) \) for \( t \in (t_0, t_0+\delta) \), then one passes to the limit \( t \to t_0+0 \) and gets (28) at \( t = t_0 \). Let us explain the meaning of \( \dot{g}(t_0) \) at a point where \( u(t_0) = 0 \). The function \( \dot{u}(t) \) is continuous and it is known that \( \frac{d\|u(t)\|}{dt} \leq \|\dot{u}(t)\| \). We define
\[
\dot{g}(t_0) = \lim_{s\to+0}\|u(t_0+s)\|s^{-1}.
\]
This limit exists and is equal to \( \|\dot{u}(t_0)\| \).

Choose
\[
\mu(t) = \mu(0)e^{\frac{1}{2}\int_0^t \gamma(s)ds}, \quad \mu^{-1}(t) = \gamma(t)/2.
\] (29)

Remark 1. Note that \( \lim_{t\to\infty} \mu(t) = \infty \) if and only if \( \int_0^\infty \gamma(t)dt = \infty \). If \( \lim_{t\to\infty} \mu(t) = \infty \), then \( \lim_{t\to\infty} \|u(t)\| = 0 \). Under the assumption (6) one has \( \int_0^\infty \gamma(t)dt = \infty \), and we use this to derive some results about asymptotic stability.

Condition (18) is satisfied if
\[
\mu(0) < [g(0)]^{-1},
\] (30)
and we choose \( \mu(0) \) so that this inequality holds. Using (29), one sees that inequality (17) is satisfied if
\[
2c_0 \mu^{-p}(0) \leq \gamma(t)e^{\frac{1}{2}\int_0^t \gamma(s)ds}, \quad \forall t \geq 0.
\] (31)
Inequality (31) is satisfied if
\[ 2c_0\mu^{-p}(0) \leq \gamma(0), \] (32)
provided that
\[ \gamma(0) \leq \gamma(t)e^{\int_0^t \gamma(s)ds} \quad \forall t \geq 0. \] (33)
Let us first use assumption (6) with \(d \in (0,1)\):
\[ \int_0^t \gamma(s)ds = b_1 \left( b_0 + t \right)^{1-d} - b_0^{1-d} \frac{1}{1-d}, \quad 0 < d < 1. \] (34)
In this case \(\gamma(0) = b_1 b_0^{-d}\), and inequality (33) holds if
\[ 2d < pb_1 b_0^{1-d}. \] (35)
Inequality (35) is a sufficient condition for the function on the right of (33) to have non-negative derivative for all \(t \geq 0\), i.e., to be monotonically growing on \([0, \infty)\), if \(\gamma(t)\) is defined in (6). Conditions (32) and (35) hold if
\[ 2c_0\mu^{-p}(0) \leq b_1 b_0^{-d} \quad \text{and} \quad 2d < pb_1 b_0^{1-d}. \] (36)
For any fixed four parameters \(d, c_0, p,\) and \(\mu(0) < [g(0)]^{-1}\), where \(d \in (0,1)\), \(c_0 > 0, p > 0,\) and \(\mu(0) > 0\), inequalities (36) can be satisfied by choosing sufficiently large \(b_1 > 0\). With the choice of \(\mu(t)\), given in (29), and the parameters \(\mu(0), b_0\) and \(b_1\), chosen as above, one obtains inequality (19):
\[ 0 \leq g(t) < \frac{e^{-b_1 \ln \left( \frac{b_0 + t}{b_0} \right)^{1-d} b_0^{1-d}}}{\mu(0)}, \quad d \in (0,1). \] (37)
Since \(g(t) = \|u(t)\|\), inequality (37) implies asymptotic stability of the zero solution to equation (1) for any initial value of \(u_0\), that is global asymptotic stability. Moreover, (37) gives a rate of convergence of \(\|u(t)\|\) to zero as \(t \to \infty\).
Consider now the case \(d = 1, \gamma(t) = b_1(b_0 + t)^{-1}\),
\[ \int_0^t \gamma(s)ds = b_1 \ln \frac{b_0 + t}{b_0}, \quad e^{b_1 \gamma(s)ds} = \left( \frac{b_0 + t}{b_0} \right)^{b_1}. \] (38)
In this case the choice of \(\mu(t)\) in (29) yields
\[ \mu(t) = \mu(0) \left( \frac{b_0 + t}{b_0} \right)^{b_1/2}. \] (39)
Choose \( \mu(0) \) so that (30) holds, and fix it. Then inequality (31) holds if
\[
2c_0\mu^{-p}(0) \leq \frac{b_1}{b_0 + t} \left( \frac{(b_0 + t)^{b_1p}}{b_0^2} \right), \quad \forall t \geq 0.
\] (40)

Choose \( b_1 \) so that
\[
b_1p > 2, \quad p > 0.
\] (41)

Then (40) holds if and only if it holds for \( t = 0 \), that is:
\[
2c_0\mu^{-p}(0) \leq \frac{b_1}{b_0}.
\] (42)

Inequality (42) is satisfied if either \( b_1 \) is chosen sufficiently large for any fixed \( b_0 \), or \( b_0 \) is chosen sufficiently small for any fixed \( b_1 > 2p^{-1} \) (see (41)). In either case one concludes that the zero solution to equation (11) is globally asymptotically stable.

Theorem 1 is proved. \( \square \)

4 Additional results. Examples

Example 1. Consider two equations:
\[
\dot{u}(t) = Au(t), \quad (43)
\]
\[
\dot{v}(t) = Av(t) + B(t)v(t), \quad t \geq 0, \quad (44)
\]
where \( A \) and \( B(t) \) are bounded linear operators in \( H \), \( A \) does not depend on \( t \), and
\[
\int_0^\infty \|B(t)\|dt < \infty. \quad (45)
\]

We assume that all the solutions to (43) are bounded. Then by the Banach-Steinhaus theorem the following inequality holds:
\[
\sup_{t \geq 0} \|e^{tA}\| \leq c < \infty. \quad (46)
\]

This implies Lyapunov’s stability of the zero solution to (43), and the inclusion \( \sigma(A) \subset \cap := \{ z : \ Re(z) \leq 0 \} \), which implies \( Re(Au, u) \leq 0 \ \forall u \in H \). A well-known result is (see, e.g., [2]):

If (45) and (46) hold then the zero solution to (44) is Lyapunov stable.

The usual proof (see [2], where \( H = \mathbb{R}^n \)) is based on the Gronwall inequality. We give a new simple proof based on Lemma 1. Let \( g(t) := \|v(t)\| \).
Multiply (44) by \( u \), take the real part and use the inequality \( \text{Re}(Av, v) \leq 0 \) to get: \( g\dot{g} \leq \|B(t)\|g^2(t) \), \( t \geq 0 \). Using the inequalities \( g(t) \geq 0 \) and (45), one obtains

\[
\dot{g}(t) \leq \|B(t)\|g(t), \quad g(t) \leq g(0)e^{\int_0^t \|B(s)\|ds} := c_1 g(0). \tag{47}
\]

Therefore, the zero solution to (44) is Lyapunov stable. Moreover, since \( |\dot{g}(t)| \in L^1(\mathbb{R}_+) \), it follows that there exists the finite limit: \( \lim_{t \to \infty} \|v(t)\| := V. \)

**Example 2.** Consider a theorem of N. Levinson in \( \mathbb{R}^n \) (see [6] and [5], pp. 159-164):

If (45) and (46) hold, then for every solution \( v \) to (44) one can find a solution \( u \) to (43) such that

\[
\lim_{t \to \infty} \|u(t) - v(t)\| = 0. \tag{48}
\]

We give a new short proof of a generalization of this theorem to an infinite-dimensional Hilbert space \( H \). If (15) and (16) hold, then, as we have proved in Example 1, \( \sup_{t \geq 0} \|v(t)\| < \infty \), \( \sup_{t \geq 0} \|u(t)\| < \infty \). If \( u(0) = u_0 \), then \( u(t) = e^{tA}u_0 \) solves (43). Let \( v(t) \) solve the equation

\[
v(t) = e^{tA}u_0 - \int_t^\infty e^{(t-s)A}B(s)v(s)ds. \tag{49}
\]

A simple calculation shows that \( v(t) \) solves (44) and

\[
\|v(t) - u(t)\| \leq \int_t^\infty \|e^{(t-s)A}\|\|B(s)\|\|v(s)\|ds \leq C \int_t^\infty \|B(s)\|ds \to 0, \quad t \to \infty,
\]

where

\[
C = \sup_{t \geq 0} \|e^{tA}\| \sup_{t \geq 0} \|v(t)\| < \infty.
\]

The generalization of Levinson’s theorem for \( H \) is proved. \( \square \)

Equation (49) is uniquely solvable in \( H \) by iterations for all sufficiently large \( t \) because for such \( t \) the norm of the integral operator in (49) is less than one. The unique solution to (49) for sufficiently large \( t \) defines uniquely the solution \( v \) to (44) which satisfies (48).

**Remark 2.** Our methods are applicable to the equation (1) with a force term: \( \dot{u} = A(t)u + F(t, u) + f(t) \).
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