Asymptotic behaviour of weighted differential entropies in a Bayesian problem

Mark Kelbert * and Pavel Mozgunov †

International Laboratory of Stochastic Analysis and Its Applications
National Research University Higher School of Economics
Moscow, Russia

Department of Mathematics
Swansea University
Swansea, UK

Abstract

Consider a Bayesian problem of estimating of probability of success in a series of trials with binary outcomes. We study the asymptotic behaviour of weighted differential entropies for posterior probability density function (PDF) conditional on \( x \) successes after \( n \) trials, when \( n \to \infty \). In the first part of work Shannon’s differential entropy is considered in three particular cases: \( x \) is a proportion of \( n \); \( x \sim n^\beta \), where \( 0 < \beta < 1 \); either \( x \) or \( n - x \) is a constant. Then suppose that one is interested to know whether the coin is fair or not and for large \( n \) is interested in the true frequency. In other words, one wants to emphasize the parameter value \( p = 1/2 \). To do so the concept of weighted differential entropy introduced in [1] is used when the frequency \( \gamma \) is necessary to emphasize. It was found that the weight in suggested form does not change the asymptotic form of Shannon, Renyi, Tsallis and Fisher entropies, but change the constants. The main term in weighted Fisher Information is changed by some constant which depend on distance between the true frequency and the value we want to emphasize. In third part of paper we investigate the weighted version of Rao-Cramér inequality for the same Bayesian problem in three particular cases of weights.

AMS subject classification: 94A17, 62B10, 62C10

Key words: differential entropy, weighted differential entropy, Bernoulli random variable, Renyi entropy, Tsallis entropy, Fisher information

*Electronic address: mark.kelbert@gmail.com
†Electronic address: pmozgunov@gmail.com; corresponding author
1 Introduction

Consider a binary trial with unknown probability of success. Say, let a probability $p$ of a head in a coin tossing be uniformly distributed in interval $[0, 1]$. We are interested in a posterior distribution after $n$ throws, conditional on event that $x$ heads appeared. Denote by $S_n$ the total number of successes after $n$ throws, say $S_n = x = \sum_{i=1}^{n} x_i$, $x_i = 1$ in the case of head, 0 in the case of the tail. Here $x = (x_i, i = 1, ..., n)$ stands for the realization of throws.

The probability that after $n$ throws the exact sequence $x$ of heads and tails can be easily found:

$$P(\xi_1 = x_1, ..., \xi_n = x_n) = \int_0^1 p^x(1 - p)^{n-x} dp = \frac{1}{(n+1)(\binom{n}{x})}.$$ 

This implies that the posterior probability density function (PDF) of the number of $x$ heads after $n$ throws is uniform:

$$P(S_n = x) = \frac{1}{(n+1)}, x = 0, \ldots, n.$$ 

The posterior PDF given the information that after $n$ throws we observe $x$ heads takes the form

$$f_{p|S_n}(p|\xi_1 = x_1, ..., \xi_n = x_n) = (n+1)\binom{n}{x}p^x(1-p)^{n-x}. \quad (1.1)$$

Note that the PDF (1.1) has the followings conditional expectation:

$$E[p|S_n = x] = \frac{x + 1}{n + 2},$$

and conditional variance:

$$\text{Var}[p|S_n = x] = \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2}.$$ 

The first goal of our work is to study the differential entropy (DE): 

$$h_{\text{diff}}(f) = -\int_{\mathbb{R}} f(z) \log(f(z)) dz$$
of random variable (RV) $Z^{(n)}$ with PDF $f_{p|S_n}$ in the following special cases:

1. $x \sim \alpha n$, where $0 < \alpha < 1$
2. $x \sim n^\beta$, where $0 < \beta < 1$
3. $x$ or $n - x$ is a constant.

Let $Z^{(n)}_\alpha$ be a RV with posterior PDF $f^{(n)}_p$ obtained when $x \sim \alpha n$. We will demonstrate that the limiting distributions when $n \to \infty$ in the cases 1 and 2 are Gaussian. However, the asymptotic normality does not imply automatically the limiting form of differential entropy. In general the problem of taking the limits under the sign of entropy is rather delicate and was extensively studied in literature, cf., i.e., [4, 7]. In the third case the limiting distribution is not Gaussian, but still the asymptotic of differential entropy can be found explicitly.

Then suppose that one is interested to know whether the coin is fair or not and for large $n$ is interested in true frequency. So the goal of a statistical experiment in twofold: on the initial stage an experimenter is mainly concerns whether the coin is fair (i.e. $p = 1/2$) or not. As the size of a sample grows, he proceeds to estimating the true value of the parameter anyway. We want to quantify the differential entropy of this experiment taking into account its two sided objective. It seems that quantitative measure of information gain of this experiment is provided by the concept of weighted differential entropy [2, 1, 9, 10]. In our case $\phi(x)$ is a weight function that underline the importance of 0.5.

The goal of the second part of work is to study the weighted Shannon’s (1.2), Renyi (1.3), Tsallis (1.4) and Fisher (1.5) entropies [3]:

$$h_\phi(f) = - \int_R \phi^{(n)}(p) f(p) \log f(p) dp,$$

$$H_\phi^\nu(f) = \frac{1}{1 - \nu} \log \int_R \phi^{(n)}(z) (f(z))^\nu dz$$

$$S_q^\phi(f) = \frac{1}{q - 1} \left( 1 - \int_R \phi^{(n)}(z) (f(z))^q dz \right)$$

$$I_\phi(\theta) = \mathbb{E} \left( \phi^{(n)}(Z) \left( \frac{\partial}{\partial \theta} \log f(Z; \theta) \right)^2 \bigg| \theta \right)$$

where $Z = Z^{(n)}$ is a RV with PDF $f = f_{p|S_n}$ given in (1.1) and $\phi^{(n)}(p)$ is a weight function that underline the importance of some particular value. The following special cases are considered:

1. $\phi^{(n)}(p) = 1$
2. $\phi^{(n)}(p)$ depends both on $n$ and $p$
We will denote by $\gamma$ the frequency that we want to emphasize (the 0.5 in the example above). We assume that $\phi(x) \geq 0$ for all $x$. Choosing the weight function we adopt the following normalization rule:

$$\int_{\mathbb{R}} \phi^{(n)}(p)f^{(n)}(p)dp = 1 \quad (1.6)$$

It can be easily checked that if weight function $\phi^{(n)}(p)$ satisfies (1.6) then the Renyi weighted entropy (1.3) and Tsallis weighted entropy (1.4) tend to Shannon’s weighted entropy as $\nu \to 1$ and $q \to 1$ correspondingly.

Considering the goal of including the weight function - emphasizing some particular value, we consider the following weight function:

$$\phi^{(n)}(p) = \Lambda^{(n)}(\gamma)p^{(1-\gamma)\sqrt{n}}(1-p)^{(1-\gamma)\sqrt{n}}, \quad (1.7)$$

where $\Lambda^{(n)}(\gamma)$ is found from the normalizing condition (1.6) and is given explicitly in (3.1). This weight function is selected as a model example with a twofold goal to emphasize a particular value $\gamma$ for moderate $n$, while preserving the true frequency $p^\ast$.

In the third part we will introduce weighted variance of RV $Z$ with PDF $f_n(z)$ and weight function $\phi(z)$:

$$\mathbb{V}^{\phi}(Z) = \int_{0}^{1} (z - \mathbb{E}(Z))^2 \phi(z)f_n(z)dz$$

The goal of third part is to derive the asymptotic of the weighted version of Rao-Cramer bound for weighted variance in stated problem.

The motivation of this study is the following. Suppose one wants to declare that some coin is ideally fair, however, the ideal can not be reached and he supposes some interval of the length $\delta$ near 1/2. If the true frequency of some coin is far enough from 1/2 (i.e. [0.25; 0.3]) there is no large punishment for mistake because in any case after trial he supposes that it is unfair and generally there is no difference (in this statistical experiment) whether it is 0.25 or 0.3.

However, if one declares an interval of the same length as before which is close to 1/2 there is a very different punishment, i.e. [0.44; 0.49]. It is clear that in this case one declares that with some high probability the coin is almost fair, so price for mistake should be higher than for previous interval. We suggest that punishment should be proportional to number of trial required to archive the standard deviation $\sigma \sim \delta$. As standard deviations in the case $x = \alpha n$ behaves like $\frac{1}{\sqrt{n}}$, so for the accuracy $\delta = \frac{1}{n^\beta}$ near the value 1/2 we should set the punishment $n^{2\beta}$.

## 2 Asymptotic behaviour of Shannon’s DE

**Theorem 1.** Let $Z^{(n)}_\alpha$ be a RV with $f^{(n)}_\alpha - \text{conditional PDF after n trials given by (1.1), with } x = \alpha n$. 


(a) Standardized $Z^{(n)}_{\alpha}$ weakly converges to standard Gaussian RV:

$$\sqrt{n} \frac{Z^{(n)}_{\alpha} - \alpha}{\sqrt{\alpha(1-\alpha)}} \Rightarrow \mathcal{N}(0,1).$$

(b) The differential entropy of $Z^{(n)}_{\alpha}$ converges to differential entropy of Gaussian RV:

$$\exists \lim_{n \to \infty} \left[ h_{\text{diff}}(f^{(n)}_{\alpha}) - \frac{1}{2} \log \frac{2\pi e[\alpha(1-\alpha)]}{n} \right] = 0, \quad 0 < \alpha < 1.$$ 

Proof. (a) We proceed by the method of characteristic functions, and establish that:

$$\mathbb{E}[e^{it\frac{Z^{(n)}_{\alpha} - \alpha}{\sqrt{\alpha(1-\alpha)}}}] \to e^{-t^2/2} \quad (2.1)$$

Denote by

$$\Lambda = \int_0^1 e^{it\frac{\sqrt{\alpha(1-\alpha)}p}{\sqrt{n}}} f^{(n)}_{\alpha}(p) dp = (n+1) \left( \frac{n}{x} \right) e^{it\frac{\sqrt{\alpha(1-\alpha)}p}{\sqrt{n}}} \int_0^1 e^{it\frac{\sqrt{\alpha(1-\alpha)}p}{\sqrt{n}}} p^x (1-p)^{n-x} dp$$

and consider the integral:

$$I = \int_0^1 e^{n(it\frac{p}{\sqrt{\alpha(1-\alpha)n} +\alpha \log(p) + (1-\alpha) \log(1-p))}} dp \quad (2.2)$$

Let $f(p) = 1$ and $g(p) = it\frac{p}{\sqrt{\alpha(1-\alpha)n} + \alpha \log(p) + (1-\alpha) \log(1-p)}$.

The integrand in (2.2) has a narrow sharp peak, and the integral is completely dominated by the maximum of $g(p)$. It can be studied by the saddle point method [5]:

$$I \simeq e^{ng(p^*)} \sqrt{\frac{2\pi}{-ng''(p^*)}} (f(p^*) + O(1/n)). \quad (2.3)$$

Finding the point of maximum of $g(p)$, we get that only one root lies in the interval $[0, 1]$:

$$p^* \simeq \alpha + it\frac{\sqrt{(1-\alpha)\alpha}}{\sqrt{n}} + O(1/n),$$

$$\Lambda \simeq e^{-t^2(n+1)} \left( \frac{n}{x} \right) (p^*)^x (1-p^*)^{n-x} \sqrt{\frac{2\pi}{-ng''(p^*)}}.$$

Next, by Stirling’s formula:

$$(n+1) \left( \frac{n}{x} \right) \simeq (n+1) \frac{n^n}{x^x (n-x)^{n-x}} \sqrt{\frac{n}{2\pi x(n-x)}}.$$

So, the straightforward computations yields:

$$(p^*)^x (1-p^*)^{n-x} \simeq \alpha^x (1-\alpha)^{(n-x)} e^{it\sqrt{(1-\alpha)\alpha n + \frac{(1-\alpha)\alpha^2}{2}} - it\sqrt{(1-\alpha)\alpha n + \frac{\alpha^2}{2}}}$$

$$= e^{\frac{t^2}{2}} \left( \frac{x}{n} \right)^x \left( \frac{n - x}{n} \right)^{n-x}.$$
Note that the next term in asymptotic of $\log(p^*)$ (as well as $\log(1-p^*)$) is decaying after multiplication of $\alpha n$ and $(1-\alpha)n$, correspondingly.

We have:

$$\Lambda \simeq e^{-t^2} \frac{(n+1)n^n}{x^n(n-x)^{n-x}} e^{\frac{t^2}{2}} \left( \frac{n}{n} \right)^{n-x} \sqrt{\frac{2\pi(n-x)}{n^3}} \simeq e^{-t^2}$$

This proves (2.1) and the Gaussian limit theorem.

(b) Consider the differential entropy:

$$h_{\text{diff}}(f_{\alpha}^{(n)}) = -\left( \log \left( \frac{(n+1)}{x} \right) + (n+1) x I_1 + (n+1) \left( \frac{n}{n} \right) (n-x) I_2 \right), \quad (2.4)$$

where

$$I_1 = \int_0^1 p^* (1-p)^{n-x} \log(p) dp \quad (2.5)$$

$$I_2 = \int_0^1 p^* (1-p)^{n-x} \log(1-p) dp. \quad (2.6)$$

The integrals $I_1$ and $I_2$ can be computed explicitly [6]:

$$\int_0^1 x^{\mu-1} (1-x^r)^{\nu-1} \log(x) dx = \frac{1}{r^2} B \left( \frac{\mu}{r}, \nu \right) \left( \psi \left( \frac{\mu}{r} \right) - \psi \left( \frac{\mu}{r} + \nu \right) \right), \quad (2.7)$$

where $\psi(x)$ - digamma function, and $B(x,y)$ - Beta-function.

Applying (2.7) for integral $I_1$, we get:

$$U_1 = (n+1) \left( \frac{n}{x} \right) x I_1 = -x (\psi(n+2) - \psi(x+1)).$$

Similarly, for the second integral $I_2$, with substitution $z = (1-x)$, we obtain:

$$U_2 = (n+1) \left( \frac{n}{x} \right) (n-x) I_2 = -(n-x) (\psi(n+2) - \psi(n+x+1)).$$

After summation of these two integrals and using the asymptotic for digamma function, we obtain:

$$U_1 + U_2 \simeq x \log(x) - n \log(n) + (n-x) \log(n-x) - \frac{1}{2}.$$ 

Next, we apply the Stirling formula to the first term in (2.4):

$$U_0 = \log \left( \frac{(n+1)}{x} \right) \simeq n \log(n) - x \log(x) - (n-x) \log(n-x) +$$

$$+ \frac{1}{2} \log(n) - \frac{1}{2} \log(\alpha) - \frac{1}{2} \log(1-\alpha) - \log(\sqrt{2\pi}) + \frac{1}{n}.$$ 

So, we obtain the asymptotic of the differential entropy:

$$h_{\text{diff}}(f_{\alpha}^{(n)}) \simeq \frac{1}{2} \log \frac{2\pi e [\alpha(1-\alpha)]}{n} - \frac{1}{n}. \quad (2.8)$$
The differential entropy of $\eta$ of

$$\mathbb{V}[\eta_j = x] = \frac{(x + 1)(n_x + 1)}{(n + 3)(n + 2)} \sim \frac{\alpha(1 - \alpha)}{n}$$

Proof. (a) In this case, it is more convenient to proceed by the method of moments. Consider RV $\eta_n = n^{1 - \beta/2}(Z - n^{-\beta - 1})$, where $Z$ has PDF \[1.1] \text{ and compute all moments of } \eta_n. \text{ First, } \mathbb{E}(\eta_n) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ because } \mathbb{E}(Z) \simeq n^{\beta - 1} + \frac{1}{n}. \text{ Next, we check that}

$$\mathbb{E}(\eta_n^k) = n^{k - \frac{nk}{2}}(1 - n^{-\beta})^{k - k(1 - n^{-\beta - 1})}k \quad 2F_1[-k, n^{-\beta}; n + 2; n^{-1 - \beta}]$$

where $2F_1[-k, n^{-\beta}; n + 2; n^{-1 - \beta}]$ is hypergeometric function, which, in this case, is just a polynomial:

$$2F_1[-k, n^{-\beta}; n + 2; n^{-1 - \beta}] = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \frac{(n + 1)_n}{(n + 2)_n} n^{n(1 - \beta)}$$

where $(q)_n = q(q + 1)...(q + n - 1)$ for $n > 0$ and 1 for $n = 0$. Consider asymptotic of terms separately:

$$n^{k - \frac{nk}{2}}(1 - n^{-\beta})^{k - k(1 - n^{-\beta - 1})^k} \simeq O(n^{k/2})$$

and

$$2F_1[-k, n^{-\beta}; n + 2; n^{-1 - \beta}] \simeq O(n^{-[0.5 + 0.5k]})$$

where $[k]$ is the integer part of $k$. For $k$ odd:

$$n^{k(1 - \beta/2)}\mathbb{E}(Z - n^{-\beta - 1})^k = O(n^{k/2})O(n^{-[0.5 + 0.5k]}) \simeq O(n^{-\beta/2}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

For $k$ even:

$$n^{k(1 - \beta/2)}\mathbb{E}(Z - n^{-\beta - 1})^k = O(n^{k/2})O(n^{-[0.5 + 0.5k]}) \simeq O(1).$$

We see that every even central moment tends to a constant which is the coefficient in front of term $n^{-[0.5 + 0.5k]})$ in hypergeometric function. For $k$ even, we have:

$$n^{k(1 - \beta/2)}\mathbb{E}(Z - n^{-\beta - 1})^k \rightarrow (k - 1)!.$$
These imply that RV \( \eta_n \) weakly converges to standard Gaussian RV.

(b) As in Theorem 1 consider the differential entropy:

\[
h_{\text{diff}}(f_{\beta}^{(n)}) = - \left( \log \left( \frac{(n+1)}{x} \right) + (n+1) \left( \frac{n}{x} \right) x I_1 + (n+1) \left( \frac{n}{x} \right) (n-x) I_2 \right) = -(U_0 + U_1 + U_2),
\]

where \( I_1 \) and \( I_2 \) are defined in (2.5) and (2.6) and can be computed explicitly by (2.7).

As before, we apply the Stirling formula for \( U_0 \):

\[
U_0 \simeq n \log(n) - x \log(x) - (n-x) \log(n-x) + \log(n) + \frac{1}{n}
\]

\[
+ \frac{1}{2} \left( \log(n^\beta) - \log(1 - n^{\beta-1}) \right) - \frac{1}{2} \log(2\pi)
\]

and estimate \( U_1 + U_2 \) as follows:

\[
U_1 + U_2 \simeq x \log(x) - n \log(n) + (n-x) \log(n-x) - \frac{1}{2}.
\]

So, we proved that differential entropy has the following form:

\[
h_{\text{diff}}(f_{\beta}^{(n)}) \simeq \frac{1}{2} \log \frac{2\pi e (1 - n^{\beta-1})}{n^{2-\beta}} - \frac{1}{n}.
\]

(2.10)

Remark 2. The first term in equation above is nothing else but the differential entropy of Gaussian RV with variance \( \sigma^2 = \frac{1-n^{\beta-1}}{n^{2-\beta}} \), which agrees with expression for variance:

\[
\mathbb{V}[p|S_n = x] = \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2} \sim \frac{1-n^{\beta-1}}{n^{2-\beta}}.
\]

Theorem 3. Let the \( f_x^{(n)} \) be posterior PDF given in (1.1) when \( x \) is fixed, and the \( f_y^{(n)} \) be posterior PDF given in (1.1) when \( y = n-x \) is fixed. Denote by \( H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k} \) harmonic series and by \( \gamma \) the Euler-Mascheroni constant. We have:

(a) \( \exists \lim_{n \to \infty} \left[ h_{\text{diff}}(f_x^{(n)}) + \log(n) \right] = x + \sum_{i=1}^{x-1} \log(x-i) - x(H_x - \gamma). \)

(b) \( \exists \lim_{n \to \infty} \left[ h_{\text{diff}}(f_y^{(n)}) + \log(n) \right] = y + \sum_{i=1}^{y-1} \log(y-i) - y(H_y - \gamma) + 1. \)

Proof. (a) Consider the differential entropy: \( h(f^{(n)}) = -(U_0 + U_1 + U_2) \), where \( U_0 \), \( U_1 \) and \( U_2 \) are defined above. Applying the Stirling formula for \( U_0 \):

\[
U_1 \simeq \log(n) + \frac{1}{n} + \frac{x}{2n} - \log(x!) + x \log(n).
\]
Next, we compute $U_1 + U_2$ as before. The only difference will be in asymptotic of digamma functions:

$$\psi(n + 2) \simeq \log(n) + \frac{1}{2n} + \frac{1}{n+1}, \quad \psi(n - x + 1) \simeq \log(n) + \frac{1/2 - x}{2n}, \quad \text{and} \quad \psi(x + 1) = H_x - \gamma,$$

where $H_x$ is harmonic series and $\gamma$ stands for Euler-Mascheroni constant. So, the differential entropy has the form:

$$h_{\text{diff}}(f^{(n)}_f) \simeq -\log(n) - \frac{1}{n} + \frac{x^2}{2n} + x(\log(n + 2) - \gamma) + \sum_{i=1}^{x-1} \log(x - i).$$

(b) In a similar way we compute $h_{\text{diff}}(f^{(n)}_y)$, where $y = n - x$ is a constant. The asymptotic of digamma function is given as follows:

$$\psi(n - x + 1) = H_{n-x} - \gamma,$$

and the final result for differential entropy is:

$$h_{\text{diff}}(f^{(n)}_y) \simeq -\log(n) - \frac{1}{n} + y - y(\log(n + 2) - \gamma) + \sum_{i=0}^{y-1} \log(y - i) + 1.$$

\[\square\]

### 3 Asymptotic of weighted entropies

The normalizing constant in the weight function (1.7) is found from the condition (1.6). We obtain that:

$$\Lambda^{(n)}(\gamma) = \frac{\Gamma(x + 1)\Gamma(n - x + 1)\Gamma(n + 2 + \sqrt{n})}{\Gamma(x + \gamma\sqrt{n} + 1)\Gamma(n - x + 1 + \sqrt{n} - \gamma\sqrt{n})\Gamma(n + 2)}. \quad (3.1)$$

We denote by $\psi^{(0)}(x) = \psi(x)$ and by $\psi^{(1)}(x)$ the digamma function and its first derivative respectively.\n
$$\psi^{(n)}(x) = \frac{\partial^{n+1}}{\partial x^{n+1}} \log(\Gamma(x)) \quad (3.2)$$

In further calculations we will need the asymptotic of these functions:

$$\psi(x) \simeq \log(x) - \frac{1}{2x} + O(1/x^2), \quad \text{as} \quad x \to \infty$$

$$\psi^{(1)}(x) \simeq \frac{1}{x} + \frac{1}{2x^2} + O(1/x^3), \quad \text{as} \quad x \to \infty$$

**Proposition 1.** Let $Z^{(n)}$ be a RV with $f^{(n)}$ - conditional PDF after $n$ trials given by (1.7), $h^{\phi}(f^{(n)}_{\alpha})$ - the weighted Shannon entropy of $Z^{(n)}$ given in (1.3). When $x = \alpha n$ $(0 < \alpha < 1)$ and the weight function $\phi^{(n)}(p)$ is given in (1.7)

$$\exists \lim_{n \to \infty} \left(h^{\phi}(f^{(n)}_{\alpha}) - \frac{1}{2} \log \left(\frac{2\pi e\alpha(1 - \alpha)}{n}\right)\right) = \frac{(\alpha - \gamma)^2}{2\alpha(1 - \alpha)}. \quad (3.3)$$

If the $\alpha = \gamma$ then the asymptotic of $h^{\phi}(f)$ is exactly the asymptotic of differential Shannon’s entropy with $\phi^{(n)}(p) = 1$. 

9
Proof. The Shannon differential entropy of PDF \( f^{(n)}(p) = f(p) \) given in (1.1) and weight function \( \phi^{(n)}(p) \) given in (1.2) takes the form:

\[
h^\phi(f) = \log \left[ (n + 1) \left( \frac{n}{x} \right) \right] + x \int_0^1 \log(p)\phi^{(n)}(p)f(p)dp + (n-x)\int_0^1 \log(1-p)\phi^{(n)}(p)f(p)dp
\]

The integrals can be computed explicitly \([6]\) (page 552):

\[
\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\log(x)dx = \frac{1}{r^2} B\left(\frac{\mu}{r},\nu\right) \left( \psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r} + \nu\right) \right),
\]

Applying this formula for integral, we get:

\[
\int_0^1 \log(p)\phi^{(n)}(p)f(p)dp = \psi(x + z + 1) - \psi(n + \sqrt{n} + 2), \text{ where } z = \gamma \sqrt{n} \text{ and } \psi(x) \text{ is a digamma function.}
\]

\[
\int_0^1 \log(1-p)\phi^{(n)}(p)f(p)dp = \psi(n - x + \sqrt{n} - z + 1) - \psi(n + \sqrt{n} + 2)
\]

So we have that

\[
h^\phi(f) = \log \left[ (n + 1) \left( \frac{n}{x} \right) \right] + x\psi(x + z + 1) + (n-x)\psi(n - x + \sqrt{n} - z + 1) - n\psi(n + \sqrt{n} + 2).
\]

By Stirling’s formula we have that for \( x = n\alpha \):

\[
\log \left[ (n + 1) \left( \frac{n}{x} \right) \right] = n\log(n) - x\log(x - (n-x)\log(n-x) + \frac{1}{2}\log(n) - \frac{1}{2}\log(\alpha) - \frac{1}{2}\log(1 - \alpha) - \log(\sqrt{2\pi} + \frac{1}{n})
\]

Using the asymptotic for digamma function

\[
\psi(x + z + 1) \approx \log(x) + \frac{\gamma \sqrt{n}}{x} + \frac{\alpha - \gamma^2}{2\alpha x}
\]

\[
\psi(n - x + \sqrt{n} - z + 1) \approx \log(n) + \frac{(1 - \gamma)\sqrt{n}}{n-x} + \frac{2\gamma - \gamma^2 - \alpha}{2(1 - \alpha)(n-x)}
\]

\[
\psi(n + \sqrt{n} + 2) = \log(n) + \frac{\sqrt{n}}{n} + \frac{1}{n}
\]

we get:

\[
h^\phi(f^{(n)}) \approx \frac{1}{2}\log \frac{2\pi e [\alpha(1 - \alpha)]}{n} + \frac{(\alpha - \gamma^2)}{2\alpha(1 - \alpha)}
\]

The first term in (3.3) is differential entropy with weight \( \phi \equiv 1 \) of Gaussian RV. Moreover, note that the asymptotic of the weighted entropy exceeds classical entropy studied above. The only difference is constant, which tend to zero if \( \gamma \to \alpha \). \qed

Theorem 4. Let \( Z^{(n)} \) be a RV with \( f^{(n)} \) - conditional PDF after \( n \) given by (1.1) and with weighted Renyi differential entropy \( H_\nu(f^{(n)}) \) given in (1.3).

(a) When both \( (x) \) and \( (n-x) \) tend to infinity as \( n \to \infty \) in the case \( \phi^{(n)}(p) = 1 \),

\[
\lim_{n \to \infty} \left( H_\nu(f^{(n)}) - \frac{1}{2}\log \frac{2\pi x(n-x)}{n^3} \right) = -\frac{\log(\nu)}{2(1 - \nu)}.
\]

For any fixed \( n \) when \( \nu \to 1 \) Renyi’s differential entropy of \( Z^{(n)} \) tends to Shannon’s differential entropy of \( Z^{(n)} \).
(b) When \( x = \alpha n \) (0 < \( \alpha < 1 \)) and the weighted function is given in (1.3),

\[
\lim_{n \to \infty} \left( H_\nu^{\phi}(f_\alpha^{(n)}) - \frac{1}{2} \log \frac{2\pi \alpha(1-\alpha)}{n} \right) = -\frac{\log(\nu)}{2(1-\nu)} + \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)\nu}.
\] (3.6)

For any fixed \( n \) the Renyi weighted differential entropy tends to Shannon’s weighted differential entropy RV with PDF given in (1.1) as \( \nu \to 1 \).

Proof. (a) In this case \( \phi^{(n)}(p) \equiv 1 \), so the Renyi entropy have the form:

\[
(1-\nu)H_\nu(f) = \log \int_0^1 (f(p))^{\nu} \, dp = \nu \log \left[ (n+1) \binom{n}{x} \right] + \log \left[ \int_0^1 \nu^{\nu x} (1-p)^{\nu(n-x)} \right] = U_0 + U_1
\]

By Stirling formula:

\[
U_0 = \nu \log \left[ (n+1) \binom{n}{x} \right] \approx \nu n \log(n) - \nu x \log(x) - \nu (n-x) \log(n-x) + \nu \log(n) + \frac{\nu}{2} \log(n) - \frac{\nu}{2} \log(x) - \frac{\nu}{2} \log(n-x) - \frac{\nu}{2} \log(2\pi)
\]

Consider the integral:

\[
\int_0^1 \nu^{\nu x} (1-p)^{\nu(n-x)} = B(\nu x+1, \nu(n-x)+1) = \frac{\Gamma(\nu x+1)\Gamma(\nu(n-x)+1)}{\Gamma(\nu n+2)}
\]

So by Stirling formula again:

\[
U_1 = \log \left[ \frac{\Gamma(\nu x+1)\Gamma(\nu(n-x)+1)}{\Gamma(\nu n+2)} \right] \approx
\]

\[
\nu x \log(\nu) + \nu x \log(x) - \nu x + \frac{1}{2} \log(\nu) + \frac{1}{2} \log(x) + \frac{1}{2} \log(2\pi)
\]

\[
+ \nu (n-x) \log(\nu) + \nu (n-x) \log(n-x) - \nu (n-x) + \frac{1}{2} \log(\nu) + \frac{1}{2} \log(n-x) + \frac{1}{2} \log(2\pi)
\]

\[
- [\nu n \log(n) + \nu n \log(n) - \nu n + \frac{1}{2} \log(\nu) + \frac{1}{2} \log(n) + \frac{1}{2} \log(2\pi)] - \log(\nu) - \log(n)
\]

We obtain:

\[
U_0 + U_1 \approx \frac{1-\nu}{2} \log(x) + \frac{1-\nu}{2} \log(n-x) + \frac{1-\nu}{2} \log(2\pi) - \frac{1}{2} \log(\nu) + \nu \log(n) - \log(n) - \frac{1-\nu}{2} \log(n) =
\]

\[
= (1-\nu) \left[ -\frac{\log(n)}{2} + \log(x) + \log(n-x) + \log(2\pi) - 2\log(n) \right] - \frac{1}{2} \log(\nu)
\]

\[
= \frac{1}{2} \log \left( \frac{2\pi x(n-x)}{n^3} \right) - \frac{1}{2} \log(\nu)
\]

So we have that:

\[
H_\nu(f) \approx \frac{1}{2} \log \left( \frac{2\pi x(n-x)}{n^3} \right) - \frac{\log(\nu)}{2(1-\nu)}. \tag{3.7}
\]

note that it tends to Renyi differential entropy of Gaussian RV as \( n \to \infty \).

Taking the limit when \( \nu \to 1 \) and applying L’Hopital’s rule we get that:

\[
H_{\nu \to 1}(f) = \lim_{\nu \to 1} H_\nu(f) \approx \frac{1}{2} \log \left( \frac{2\pi x(n-x)}{n^3} \right). \tag{3.8}
\]

For example, when \( x = \alpha n, 0 < \alpha < 1 \) the Renyi entropy:
where the first term is Shannon’s entropy of Gaussian RV with corresponding variance.

Or similarly when \( x = n^\beta, 0 < \beta < 1 \) the Renyi entropy:

\[
H_{\nu \rightarrow 1}(f) \simeq \frac{1}{2} \log \frac{2 \pi e [\alpha(1-\alpha)]}{n},
\]

where the first term is Shannon’s differential entropy of Gaussian RV with variance \( \sigma^2 = \frac{1 - n^{\beta-1}}{n^{2-\beta}} \).

(b) In this case when \( \phi^{(n)}(p) \) is given in (3.1) and \( x = \alpha n \), the weighted Renyi entropy has the form:

\[
H_{\nu}^\psi(f) = \frac{1}{1 - \nu} \log \int_0^1 \phi^{(n)}(p) (f(p))^\nu \, dp
\]

\[
\int_0^1 \phi^{(n)}(p) (f(p))^\nu \, dp = U_1 U_2 U_3,
\]

where

\[
U_1 = \frac{\Gamma(\nu x + \gamma \sqrt{n} + 1) \Gamma(\nu(n-x) + (1-\gamma) \sqrt{n} + 1)}{\Gamma(\nu n + \sqrt{n} + 2) \Gamma(x+z+1) \Gamma(n-x+\sqrt{n}-z+1)}
\]

\[
\log(U_1) \simeq \nu x \log(x) + z \log(x) + \frac{1}{2} \log(x) + \nu(n-x) \log(n-x) + (\sqrt{n} - z) \log(n-x) + \frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\nu) + \frac{1}{2} \log(n-x) - \nu \log(n) - \sqrt{n} \log(n) - \frac{1}{2} \log(n) - \log(n) + \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha) \nu} + \frac{1}{2} \log \left( \frac{2\pi \alpha(1-\alpha)}{\nu} \right)
\]

\[
\log(U_2) \simeq \nu \log(n) - \nu x \log(x) - \nu(n-x) \log(n-x) + \nu \log(n) + \frac{\nu}{2} \log(n) - \log(n) - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\nu) + \frac{\nu}{2} \log(n) - \log(n) - \log(n-x) - \log(n-x) - \nu \log(n-x) - (\sqrt{n}-z) \log(n-x) + \frac{1}{2} \log(n-x) - \log(2\pi) - \log(n-x)
\]

Taking all parts together, we obtain that

\[
H_{\nu}^\psi(f) = \frac{1}{2} \log \frac{2\pi \alpha(1-\alpha)}{n} - \frac{\log(\nu)}{2(1-\nu)} + \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)(1-\nu)} \left( \frac{1}{\nu} - 1 \right) \tag{3.9}
\]

Taking the limit when \( \nu \rightarrow 1 \) and applying L’Hopital’s rule we get that:

\[
H_{\nu}^\psi(f) = \lim_{\nu \rightarrow 1} H_{\nu}(f) \simeq \frac{1}{2} \log \frac{2 \pi e [\alpha(1-\alpha)]}{n} + \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)} \tag{3.10}
\]

So the weighted Renyi entropy tends to Shannon’s weighted entropy as \( \nu \rightarrow 1 \). \qed
Proposition 2. For any continuous random variable $X$ with PDF $f(x)$ and for any non-negative weight function $\phi(x)$ which satisfies condition (1.7) and such that
\[ \int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \log(f(x)) \, dx < \infty, \]
the weighted Renyi differential entropy $H_\nu^\phi(f)$ is a non-increasing function of $\nu$ and
\[ \frac{\partial}{\partial \nu} H_\nu^\phi(f) = - \frac{1}{(1 - \nu)^2} \int_{\mathbb{R}} z(x) \log \frac{z(x)}{\phi(x)f(x)} \, dx, \quad (3.11) \]
where
\[ z(x) = \frac{\phi(x)(f(x))^{\nu}}{\int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx}. \quad (3.13) \]
Similarly, the Tsallis weighted entropy $S_q^\phi(f)$ given in (1.4) is a non-increasing function of $q$.

Proof. We need to show that
\[ \frac{\partial}{\partial \nu} H_\nu^\phi(f) \leq 0. \]
\[ \frac{\partial}{\partial \nu} H_\nu^\phi(f) = \frac{\log \int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx}{(1 - \nu)^2} + \frac{\int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \log(f(x)) \, dx}{(1 - \nu) \int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx} = I_1 + I_2 \quad (3.12) \]
Denote
\[ z(x) = \frac{\phi(x)(f(x))^{\nu}}{\int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx}. \]
Note that $z(x) \geq 0$ for any $x$ and
\[ \int_{\mathbb{R}} z(x) \, dx = 1 \]
Let $Q_1 = \int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx$ and $Q_2 = \log \int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx$.
Using the substitution (3.13)
\[ Q_2 = \log(\phi(x)) + \nu \log(f(x)) - \log(z(x)). \quad (3.14) \]
We have that
\[ I_2 = \frac{1}{1 - \nu} \int_{\mathbb{R}} z(x) \log(f(x)) \, dx = \frac{1}{1 - \nu} \int_{\mathbb{R}} z(x) \log(f(x)) \, dx \]
\[ I_1 + I_2 = \frac{1}{(1 - \nu)^2} \left( \log \int_{\mathbb{R}} \phi(x)(f(x))^{\nu} \, dx + (1 - \nu) \int_{\mathbb{R}} z(x) \log(f(x)) \, dx \right) = \frac{1}{(1 - \nu)^2} I_3 \]
By substitution $\log(f(x))$ using (3.14) we get:
\[ I_3 = Q_2 + (1 - \nu) \left( \frac{Q_2}{\nu} + \frac{1}{\nu} \int_{\mathbb{R}} z(x) \log(z(x)) \, dx - \frac{1}{\nu} \int_{\mathbb{R}} z(x) \log(\phi(x)) \, dx \right) = \]
\[ \int_{\mathbb{R}} z(x) \log(z(x)) \, dx - \frac{1}{\nu} \int_{\mathbb{R}} z(x) \log(\phi(x)) \, dx = \]
\[
\frac{Q_2}{\nu} + \frac{1}{\nu} \int_{\mathbb{R}} z(x) \log(z(x)) dx - \int_{\mathbb{R}} z(x) \log(z(x)) dx + \int_{\mathbb{R}} z(x) \log(\phi(x)) dx - \frac{1}{\nu} \int_{\mathbb{R}} z(x) \log(\phi(x)) dx
\]

Applying \(3.14\) again we get that\(I_3 = \int_{\mathbb{R}} z(x) \log(f(x)) dx - \int_{\mathbb{R}} z(x) \log(z(x)) dx + \int_{\mathbb{R}} z(x) \log(\phi(x)) dx = - \int_{\mathbb{R}} z(x) \log \left( \frac{z(x)}{\phi(x)f(x)} \right) dx\)

We obtain that
\[
- \frac{\partial}{\partial \nu} H_\nu^\phi(f) = \frac{1}{(1 - \nu)^2} \int_{\mathbb{R}} z(x) \log \left( \frac{z(x)}{\phi(x)f(x)} \right) dx = \frac{1}{(1 - \nu)^2} \mathbb{D}_{KL}(z||f). \tag{3.15}
\]

Here \(\mathbb{D}_{KL}(z||f)\) is Kullback–Leibler divergence between \(z\) and \(f\) which is always non-negative. Due to conditions \(\phi(x)f(x) \geq 0\) and \(1.6\), \(\phi(x)f(x)\) is itself a PDF:

\[
\int_{\mathbb{R}} \phi(x)f(x) dx = 1
\]

Similarly, one can show that Tsallis weighted differential entropy given in \(1.4\) is non-increasing function of \(q\). So, the result follows. \(\Box\)

**Theorem 5.** Let \(Z^{(n)}\) be a RV with \(f^{(n)}\) - conditional PDF after \(n\) trials given by \(1.1\) with the weighted Tsallis differential entropy \(S_q(f^{(n)})\) given in \(1.4\).

(a) When both \((x)\) and \((n - x)\) tend to infinity as \(n \to \infty\) and \(\phi^{(n)}(p) = 1\),

\[
\exists \lim_{n \to \infty} \left( S_q(f^{(n)}) - \frac{1}{q - 1} \left( 1 - \frac{1}{\sqrt{q}} \left( \frac{2\pi x(n - x)}{n^3} \right)^{\frac{1}{2}} \right) \right) = 0. \tag{3.16}
\]

For any fixed \(n\) the Tsallis differential entropy tends to Shannon’s differential entropy as \(q \to 1\).

(b) When \(x = \alpha n\) and the weight function \(\phi^{(n)}(p)\) given in \(1.4\)

\[
\exists \lim_{n \to \infty} \left( S_q^\phi(f^{(n)}) - \frac{1}{q - 1} \left( 1 - \frac{1}{\sqrt{q}} \left( \frac{2\pi \alpha(1 - \alpha)}{n} \right)^{\frac{1}{2}} \exp \left( \frac{(\alpha - \gamma)^2(1 - q)}{2\alpha(1 - \alpha)q} \right) \right) \right) = 0 \tag{3.17}
\]

The weighted Tsallis differential entropy tends to Shannon’s weighted differential entropy RV with PDF given in \(1.1\) as \(q \to 1\).

**Remark 3.** It can be seen from Theorem 4(a) and Theorem 5(a) that for large \(n\) Renyi’s entropy and Tsallis’s entropy (for \(\phi \equiv 1\) "behaves" like respective entropies of Gaussian RV with variance \(\sigma^2 = \frac{x(n-x)}{n^2}\).

**Proof.** (a) In this case \(\phi^{(n)}(p) \equiv 1\), the Tsallis entropy have the form:

\[
S_q(f) = \frac{1}{q - 1} \left( 1 - \int_{0}^{1} (f(p))^q \ dp \right) = \frac{1}{q - 1} \left( 1 - \int_{0}^{1} \left( (n + 1) \left( \frac{n}{x} \right) p^x(1 - p)^{n-x} \right)^q \ dp \right)
\]

It was shown above that

\[
\log \int_{0}^{1} (f(p))^q \ dp \simeq \frac{1 - q}{2} \log \left( \frac{2\pi x(n - x)}{n^3} \right) - \frac{1}{2} \log(q)
\]
So we have that
\[ V_0 = \int_0^1 (f(p))^q \, dp \approx \frac{1}{\sqrt{q}} \left( \frac{2\pi x(n-x)}{n^3} \right)^{\frac{1}{2}-q} \]

We straightforwardly obtain that
\[ S_q(f) \approx \frac{1}{q-1} \left( 1 - \frac{1}{\sqrt{q}} \left( \frac{2\pi x(n-x)}{n^3} \right)^{\frac{1}{2}-q} \right) \]  

(3.18)

Note that \( V_0 \to 1 \) when \( q \to 1 \), applying L’Hospital’s rule we get that:
\[ \lim_{q \to 1} S_q(f) = S_1(f) \approx \frac{1}{2} \log \left( \frac{2\pi x(n-x)}{n^3} \right) \]  

(3.19)

The first term in expression above is nothing else but Shannon’s differential entropy of Gaussian RV.

(b) In this case when \( \phi^{(n)}(p) \) is given in (1.7) the Tsallis entropy have the form:
\[ S_\phi(f) = \frac{1}{q-1} \left( 1 - \int_0^1 \phi^{(n)}(p) (f(p))^q \, dp \right) \]

Using that \( x = \alpha n \) and by Stirling’s formula, it was shown above that
\[
\log \left[ \int_0^1 \phi^{(n)}(p) (f(p))^q \, dp \right] \approx \frac{1-q}{2} \log \frac{2\pi \alpha(1-\alpha)}{n} - \frac{\log(q)}{2} + \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)} \left( \frac{1}{q-1} \right)
\]

So we have:
\[ V_1 = \int_0^1 \phi^{(n)}(p) (f(p))^q \, dp \approx \frac{1}{\sqrt{q}} \left( \frac{2\pi \alpha(1-\alpha)}{n} \right)^{\frac{1}{2}-q} \exp \left( \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)} \left( \frac{1}{q-1} \right) \right) \]

(3.20)

Weighted Tsallis entropy:
\[ S^\phi_q(f) \approx \frac{1}{q-1} \left( 1 - \frac{1}{\sqrt{q}} \left( \frac{2\pi \alpha(1-\alpha)}{n} \right)^{\frac{1}{2}-q} \exp \left( \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)} \left( \frac{1}{q-1} \right) \right) \right) \]

Note that \( V_0 \to 1 \) when \( q \to 1 \), applying L’Hospital’s rule we get that:
\[ S_1^\phi(f) = \lim_{q \to 1} S^\phi_q(f) \approx \frac{1}{2} \log \left( \frac{2\pi e\alpha(1-\alpha)}{n} \right) + \frac{(\alpha - \gamma)^2}{2\alpha(1-\alpha)}. \]  

(3.21)

Then the weighted Tsallis entropy tends to weighted Shannon’s differential entropy when \( q \to 1 \).

\[ \square \]

**Theorem 6.** Let \( Z^{(n)} \) be a RV with \( f_\alpha^{(n)} \) - conditional PDF after \( n \) trials given by (1.1), when \( x = \alpha n \) (0 \(<\alpha<1\)) and \( I(f_\alpha^{(n)}) \) is the weighted Fisher information of \( Z^{(n)} \) given in (1.5):

(a) When \( \phi^{(n)}(p) = 1 \),
\[
\exists \lim_{n \to \infty} \left[ I(f_\alpha^{(n)}) - \left( \frac{1}{\alpha(1-\alpha)} \right) n \right] = -\frac{2\alpha^2 - 2\alpha + 1}{2\alpha^2(1-\alpha)^2}. \]  

(3.22)
(b) When \( \phi^{(n)}(p) \) is given in (1.7):

\[
\exists \lim_{n \to \infty} \left[ I^\phi(f^{(n)}_\alpha) - \left( \frac{1}{\alpha(1 - \alpha)} + \frac{(\alpha - \gamma)^2}{(1 - \alpha)^2 \alpha^2} \right) n - B(\alpha, \gamma) \sqrt{n} \right] = C(\alpha, \gamma), \tag{3.23}
\]

where \( B(\alpha, \gamma) \) and \( C(\alpha, \gamma) \) are constants which depend only on \( \alpha \) and \( \gamma \) and are given in (3.29) and (3.30) respectively.

Proof. (a) The Fisher information in the case \( \phi^{(n)}(p) = 1 \) and \( x = \alpha n \) takes the form:

\[
I(\alpha) = \mathbb{E} \left( \left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 \mid \alpha \right) = \int_0^1 \left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 f(p, \alpha) dp,
\]

where \( f = f^{(n)}_\alpha \).

\[
\log(f(p, \alpha)) = n \log(p) + (1 - \alpha)n \log(1 - p) + \log(n + 1)! - \log((n - x)!) - \log((n - x)!) \tag{3.24}
\]

\[
\left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 = n^2 \log^2(p) + n^2 \log(1 - p) + n^2 \psi^2(x - 1) + n^2 \psi^2(x + 1) - 2n^2 \log(p) \log(1 - p) + 2n^2 \log(p) \psi(x - 1) - 2n^2 \log(1 - p) \psi(x + 1) - 2n^2 \log(p) \psi(x - 1) + 2n^2 \log(1 - p) \psi(x + 1) - 2n^2 \psi(x + 1) \psi(n - x + 1).
\]

For the following computation we will need so following integrals:

\[
\int_0^1 \log^2(p)(1 - p)^{n-x} dp = \frac{\Gamma(n - x + 1)\Gamma(x + 1)}{\Gamma(n + 2)}(\psi(n + 2) - \psi(x + 1))^2 - \psi^{(1)}(n + 2) + \psi^{(1)}(x + 1),
\]

where \( \Gamma(x) \) is a Gamma function and \( \psi^{(1)}(x) \) is the first derivative of digamma function.

\[
\int_0^1 \log(p)(1 - p) p^x (1 - p)^{n-x} dp = \frac{\Gamma(n - x + 1)\Gamma(x + 1)}{\Gamma(n + 2)}(\psi(n + 2) - \psi(n - x + 1))^2 - \psi^{(1)}(n + 2) + \psi^{(1)}(n + x + 1) - \psi^{(1)}(n + 2)
\]

\[
\int_0^1 \log(p) p^x (1 - p)^{n-x} dp = \frac{\Gamma(n - x + 1)\Gamma(x + 1)}{\Gamma(n + 2)}(-\psi(n + 2) + \psi(x + 1))
\]

\[
\int_0^1 (1 - p) p^x (1 - p)^{n-x} dp = \frac{\Gamma(n - x + 1)\Gamma(x + 1)}{\Gamma(n + 2)}(-\psi(n + 2) + \psi(n - x + 1))
\]

So we have that:

\[
\int_0^1 \left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 f(p, \alpha) dp =
\]

\[
n^2(n + 1) \left( \frac{n}{x} \right) \frac{\Gamma(n - x + 1)\Gamma(x + 1)}{\Gamma(n + 2)}(\psi^2(n + 2) + \psi^2(x + 1) - 2\psi(n + 2) \psi(x + 1) - \psi^{(1)}(n + 2) + \psi^{(1)}(x + 1) + \psi^2(n + 2) + \psi^2(n - x + 1) - 2\psi(n + 2) \psi(n - x + 1) - \psi^{(1)}(n + 2) + \psi^{(1)}(n - x + 1) + \psi^2(n - x + 1) - 2\psi^2(n + 2) + 2\psi(n + 2) \psi(x + 1) + 2\psi(n -
Theorem 6 has the same form as in the classical problem of estimating \( p \) in binary trials.

\[
I(\alpha) = n^2(\psi^{(1)}(x + 1) + \psi^{(1)}(n - x + 1)).
\]  

(3.25)

Using the asymptotic for the digamma function we can rewrite:

\[
I(\alpha) \simeq \frac{1}{\alpha(1 - \alpha)} n - \frac{12\alpha^2 - 2\alpha + 1}{2\alpha^2(1 - \alpha)^2}.
\]  

(3.26)

Remark 4. When \( x = \alpha n \)

\[
\int_0^1 pf^{(n)} dp = \alpha + b_n(\alpha),
\]

where \( b_n(\alpha) \) is a bias.

\[
b_n(\alpha) \simeq \frac{1 - 2\alpha}{n}
\]

Note that

\[
\frac{\partial}{\partial \alpha} b_n(\alpha) \simeq -\frac{2}{n} \rightarrow 0
\]

as \( n \rightarrow \infty \). So, our estimate is asymptotically unbiased. Also note that the first term in Theorem \( \text{6} \) has the same form as in the classical problem of estimating \( p \) in a series of binary trials \( \frac{n}{p(1-p)}. \)

(b) The weighted Fisher Information in the case \( x = \alpha n \) (0 < \( \alpha < \) takes the following form:

\[
I^\phi(f) = \mathbb{E} \left( \phi^{(n)}(p) \left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 | \alpha \right) = \int_0^1 \phi^{(n)}(p) \left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 f(p, \alpha) dp
\]

where the \( \phi^{(n)}(p) \) is given in (1.7).

The second term under integral \( \left( \frac{\partial}{\partial \alpha} \log f(p; \alpha) \right)^2 \) can be found as before exactly.

Let

\[
W = \frac{\Gamma(n - x + 1 + \sqrt{n} - z)\Gamma(x + 1 + z)}{\Gamma(n + 2 + \sqrt{n})}.
\]

So in order to compute the weighted Fisher information we will need to compute following integrals.

\[
\int_0^1 \log^2(p)p^{\alpha+x}(1-p)^{n-x+\sqrt{n}+z} dp = W(\psi(n + 2 + \sqrt{n}) - \psi(x + z + 1))^2 - \psi^{(1)}(n + 2 + \sqrt{n}) + \psi^{(1)}(x + z + 1)
\]

\[
\int_0^1 \log(p)p^{\alpha+x}(1-p)^{n-x+\sqrt{n}+z} dp = W(\psi(n + 2 + \sqrt{n}) - \psi(n - x + 1 + \sqrt{n} - z))^2 - \psi^{(1)}(n + 2 + \sqrt{n}) + \psi^{(1)}(n - x + 1 + \sqrt{n} - z) - \psi^{(1)}(n + 2 + \sqrt{n})
\]

\[
\int_0^1 \log(p)\log(1-p)p^{\alpha+x}(1-p)^{n-x+\sqrt{n}+z} dp = W(\psi(n + 2 + \sqrt{n}) - \psi(x + 1 + z)) - \psi^{(1)}(n + 2 + \sqrt{n})
\]

\[
\int_0^1 \log(p)p^{\alpha+x}(1-p)^{n-x+\sqrt{n}-z} dp = W(-\psi(n + 2 + \sqrt{n}) + \psi(x + 1 + z)
\]
\[
\int_0^1 \log(1 - p)p^{z + x}(1 - p)^{n - x + \sqrt{n} - z} dp = W(-\psi(n + 2 + \sqrt{n}) + \psi(n - x + 1 + \sqrt{n} - z)
\]

Taking all parts together:
\[
I^\phi(f^{(n)}_\alpha) = n^2 (\psi^{(1)}(x + z + 1) + \psi^{(1)}(n - x + 1 + \sqrt{n} - z)) + \\
+n^2 [\psi(x + z + 1) - \psi(x + 1)]^2 + (\psi(n - x + 1 + \sqrt{n} - z) - \psi(n - x + 1))^2 + \\
+2n^2 \left[ (\psi(n - x + 1) - \psi(n - x + \sqrt{n} - z + 1))(\psi(x + z + 1) - \psi(x + 1)) \right]
\]

Using the asymptotic for the digamma function we can rewrite:
\[
I(\alpha) \simeq A(\alpha, \gamma)n + B(\alpha, \gamma)\sqrt{n} + C(\alpha, \gamma) + O\left(\frac{1}{\sqrt{n}}\right) \tag{3.27}
\]

where
\[
A(\alpha, \gamma) = \frac{1}{\alpha(1 - \alpha)} + \frac{(\alpha - \gamma)^2}{(1 - \alpha)^2 \alpha^2} \tag{3.28}
\]
\[
B(\alpha, \gamma) = \frac{2\alpha\gamma - \gamma - \alpha^2}{(1 - \alpha)^2 \alpha^2} + \frac{(\alpha - \gamma)^2}{(1 - \alpha)^3 \alpha^3} (\alpha(2\gamma - 1) - \gamma) \tag{3.29}
\]
\[
C(\alpha, \gamma) = \frac{\alpha - 2\alpha^4 - 2\gamma^2 + 6\alpha^3 + 3\alpha^3(2 + 4\gamma) - 3\alpha(1 + \gamma^2)}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} + \\
\frac{-2(1 - \alpha)^3 \alpha^3}{12(1 - \alpha)^4 \alpha^4} +
\]

A rôle of the weight function of form (4.1) results in appearance of the term of order \(\sqrt{n}\), but the main order, \(n\), remains the same. However, the coefficient in front of it is higher by \(\frac{(\alpha - \gamma)^2}{(1 - \alpha)^2 \alpha^2}\). Evidently, when the frequency of special interest is equal to the true frequency the leading term is the same as in Fisher Information with constant weight. Also note that the rate depends on the distance between \(\gamma\) and \(\alpha\) and when \(\gamma \rightarrow \alpha\) the only first terms remains.

\[\square\]

### 4 Weighted Rao-Cramér inequality

Let \(Z = Z_\alpha\) be a RV with a PDF (11) assuming that \(x = \alpha n\). As before consider the goal of including the weight function - emphasizing some particular value that does not depend on \(\alpha\).

In this section we will consider the weight function of the following form:
\[
\phi(\alpha, p) = \phi^{(n)}(\alpha, p) = \Lambda^{(n)}(\alpha, \gamma)\tilde{\phi}(p), \tag{4.1}
\]

where \(\Lambda^{(n)}(\gamma)\) is found from the normalizing condition (10) and \(\tilde{\phi}(p)\) is function that have a sharp peak near \(\gamma\) and does not depend on \(\alpha\). Denote by
\[
\kappa(\alpha) = \frac{1}{\Lambda^{(n)}(\alpha, \gamma)}
\]
So, the condition (1.6) takes the form:

\[ \int_{\mathbb{R}} \tilde{\phi}(p) f^{(n)}(\alpha, p) dp = \kappa(\alpha) \] (4.2)

We will consider three particular cases:

\[ \phi_1^{(n)}(p) = \Lambda_1^{(n)}(\alpha, \gamma) p^n (1 - p)^{1 - \gamma}, \] (4.3)

\[ \phi_2^{(n)}(p) = \Lambda_2^{(n)}(\alpha, \gamma) p^{\gamma} \sqrt{n} (1 - p)^{(1 - \gamma) \sqrt{n}}, \] (4.4)

and

\[ \phi_3^{(n)}(p) = \Lambda_3^{(n)}(\alpha, \gamma) p^{\gamma n} (1 - p)^{(1 - \gamma)n}. \] (4.5)

**Theorem 7.** Let \( Z_\alpha \) be a RV with a PDF (1.1) assuming that \( x = n \alpha \). Let \( \mathbb{V}^\phi(Z_\alpha) \) be the weighted variance of RV \( Z_\alpha \)

\[ \mathbb{V}^\phi(Z_\alpha) = \int_0^1 \phi(\alpha, p) (p - \mathbb{E}Z_\alpha)^2 f_n(\alpha, p) dp, \]

\( I^\phi(\alpha) \) - weighted Fisher information and \( \kappa(\alpha) = \frac{1}{\Lambda^{(n)}(\alpha, \gamma)} \). Then the following inequality hold:

\[ \mathbb{V}^\phi(Z_\alpha) \geq \frac{\left( \frac{\partial g(\alpha)}{\partial \alpha} - \frac{\kappa'(\alpha)}{\kappa(\alpha)} (p^* - g(\alpha)) \right)^2}{I^\phi(\alpha)} \] (4.6)

where

\[ g(\alpha) = \int_0^1 p \phi(\alpha, p) f_n(\alpha, p) dp \]

\[ p^* = \mathbb{E}Z_\alpha. \] (4.7)

Particularly,

(a) When weight function \( \phi(p) = \phi_1(p) \) is given in (4.3):

\[ \mathbb{V}^{\phi_1}(Z) \geq \frac{\alpha(1 - \alpha)}{n} + \frac{1 - 14\alpha + 18\alpha^2 + 2\gamma - 8\alpha\gamma + 2\gamma^2}{2n^2} + O \left( \frac{1}{n^{5/2}} \right) \] (4.8)

(b) When weight function \( \phi(p) = \phi_2(p) \) is given in (4.4):

\[ \mathbb{V}^{\phi_2}(Z) \geq \frac{\alpha(1 - \alpha)}{n} + \frac{(\alpha - \gamma)^2}{n} - \frac{-2\alpha + \alpha^2 + \gamma + 2\alpha\gamma - 2\gamma^2}{n^{3/2}} + O \left( \frac{1}{n^2} \right) \] (4.9)

(c) When weight function \( \phi(p) = \phi_3(p) \) is given in (4.5):

\[ \mathbb{V}^{\phi_3}(Z) \geq \frac{(\alpha - \gamma)^2}{4} + C_3(\alpha, \gamma) \frac{1}{n} + O \left( \frac{1}{n^{3/2}} \right) \] (4.10)

where \( C_3 \) is constant which depends only on \( \alpha \) and \( \gamma \).
Proof. Consider the following integral

\[ g(\alpha) = \int_0^1 p\phi(\alpha, p)f_n(\alpha, p)dp \quad (4.11) \]

Differentiating both sides in (4.11) and in (4.2) and multiplying the latter one by \( p^* \) where \( p^* = E(Z_\alpha) = \int_0^1 pf_n(\alpha, p)dp \)

\[ \int_0^1 p\phi(\alpha, p)\frac{\partial f}{\partial \alpha}dp - \frac{\kappa'(\alpha)}{\kappa^2(\alpha)} \int_0^1 p\phi(\alpha, p)dp = \frac{\partial g(\alpha)}{\partial \alpha} \]

\[ p^* \int_0^1 \phi(p)\frac{\partial f}{\partial \alpha}dp = \frac{\kappa'(\alpha)}{\kappa(\alpha)}p^* \]

Subtracting the one from another:

\[ \int_0^1 (p - p^*)\phi(p)\frac{\partial f}{\partial \alpha}dp = \frac{\partial g(\alpha)}{\partial \alpha} - \frac{\kappa'(\alpha)}{\kappa(\alpha)} (p^* - g(\alpha)) \]

Multiplying and dividing by \( \sqrt{f} \), squared and applying Cauchy–Schwarz inequality one can get that

\[ \int_0^1 (p - p^*)^2\phi(p)f_n(\alpha)dp \geq \left( \frac{\partial g(\alpha)}{\partial \alpha} - \frac{\kappa'(\alpha)}{\kappa(\alpha)} (p^* - g(\alpha)) \right)^2 \]

where \( I^\phi(\alpha) \) introduced below. So, we have that

\[ \forall^\phi(Z) \geq \left( \frac{\partial g(\alpha)}{\partial \alpha} - \frac{\kappa'(\alpha)}{\kappa(\alpha)} (p^* - g(\alpha)) \right)^2 \quad (4.12) \]

(a) Consider the following weight function:

\[ \phi_1^{(n)}(p) = \Lambda_1^{(n)}(\alpha, \gamma)p^\gamma(1 - p)^{1-\gamma}, \quad (4.13) \]

where \( \Lambda_1^{(n)}(\gamma) \) is found from the normalizing condition (1.6):

\[ \Lambda_1^{(n)}(\gamma) = \frac{\Gamma(x + 1)\Gamma(n - x + 1)\Gamma(n + 3)}{\Gamma(x + \gamma + 1)\Gamma(n - x + 2 - \gamma)\Gamma(n + 2)} \]

Note that only normalizing constant depends on \( n \), but the remainder do not contains \( n \) and \( \alpha \). For a given weight function (4.13) the Fisher information is equal:

\[ I^\phi(f_\alpha^{(n)}) = n^2 \left( \psi(x + \gamma + 1) + \psi(n - x + 1 - 1 + \gamma) \right) + n^2 \left( (\psi(x + \gamma + 1) - \psi(x + 1))^2 + (\psi(n - x + 1 + 1 - \gamma) - \psi(n - x + 1))^2 \right) + 2n^2 \left( (\psi(n - x + 1) - \psi(n - x + 1 + \gamma + 1) - \psi(x + 1) \right) \]

Denote by

\[ \kappa_1(\alpha) = \frac{1}{\Lambda_1^{(n)}(\alpha, \gamma)} \]
\[ \tilde{\phi}_1(p) = p^\gamma (1 - p)^{1 - \gamma} \]

For the weight function (4.13), integral in (4.11) can be found explicitly
\[
\int_0^1 p \phi_1(p) f_n(\alpha) \, dp = \frac{\Gamma(n + 3)}{\Gamma(x + \gamma + 1) \Gamma(n - x - \gamma + 2)} \int_0^1 p^{x + \gamma + 1} (1 - p)^{n - x + 1 - \gamma} \, dp
\]
\[
= \frac{\Gamma(n + 3) \Gamma(x + \gamma + 2)}{\Gamma(n + 4) \Gamma(x + \gamma + 1)} = g_1(\alpha)
\]

Then
\[
\frac{\partial g_1(\alpha)}{\partial \alpha} = n \frac{\Gamma(n + 3) \Gamma(x + \gamma + 2)}{\Gamma(n + 4) \Gamma(x + \gamma + 1)} (\psi(x + \gamma + 2) - \psi(x + \gamma + 1))
\] (4.16)

Differentiating \(\kappa(\alpha)\) we obtain that:
\[
\kappa'_1(\alpha) = n \left( \psi(n - x + 1) - \psi(n - x + 1 - \gamma + 1) + \psi(x + \gamma + 1) - \psi(x + 1) \right).
\] (4.17)

Also \(p^*\) is equal
\[
p^* = \frac{\Gamma(n + 2)}{\Gamma(x + 1) \Gamma(n - x + 1)} \int_0^1 p^{x + 1} (1 - p)^{n - x} \, dp = \frac{\Gamma(n + 2) \Gamma(x + 2)}{\Gamma(n + 3) \Gamma(x + 1)}
\] (4.18)

Plugging in (4.14), (4.15), (4.16), (4.17) and (4.18) in (4.12) we get that
\[
\mathbb{V}^{\phi_1}(Z) \geq \frac{\alpha(1 - \alpha)}{n} + \frac{1 - 14\alpha + 18\alpha^2 + 2\gamma - 8\alpha \gamma + 2\gamma^2}{2n^2} + O\left(\frac{1}{n^{5/2}}\right)
\] (4.19)

(b) Consider the following weight function:
\[
\phi_2^{(n)}(p) = \Lambda_2^{(n)}(\alpha, \gamma) p^\gamma \sqrt{n} (1 - p)^{(1 - \gamma) \sqrt{n}},
\] (4.20)

where as before \(\Lambda_2^{(n)}(\gamma)\) is found from the normalizing condition (1.6):
\[
\Lambda_2^{(n)}(\gamma) = \frac{\Gamma(x + 1) \Gamma(n - x + 1) \Gamma(n + 2 + \sqrt{n})}{\Gamma(x + \gamma \sqrt{n} + 1) \Gamma(n - x + 1 + \sqrt{n} - \gamma \sqrt{n}) \Gamma(n + 2)}.
\]

Note that only normalizing constant depends on \(n\) as well as the remainder. For a given weight function (4.20) the Fisher information is equal:
\[
I^{\phi_2_{(n)}} = n^2 (\psi^{(1)}(x + z + 1) + \psi^{(1)}(n - x + 1 + \sqrt{n} - z)) +
+n^2 \left[ (\psi(x + z + 1) - \psi(x + 1))^2 + (\psi(n - x + 1 + \sqrt{n} - z) - \psi(n - x + 1))^2 \right] +
+2n^2 \left[ (\psi(n - x + 1) - \psi(n - x + 1 + \sqrt{n} - z + 1)) (\psi(x + z + 1) - \psi(x + 1)) \right]
\] (4.21)

where \(z = \gamma n\).
For the weight function (4.20), integral in (4.11) equal
\[
\int_0^1 p\phi_2(p)f_n(\alpha)dp = \frac{\Gamma(n + 3)}{\Gamma(n + \gamma\sqrt{n} + 1)} \int_0^1 p^{x+\gamma\sqrt{n}+1}(1-p)^{n-x+\gamma\sqrt{n}}dp
\]
\[
= \frac{\Gamma(n + \gamma\sqrt{n} + 2)}{\Gamma(n + \gamma\sqrt{n} + 3)}
\]

Then
\[
\frac{\partial g_2(\alpha)}{\partial \alpha} = n \frac{\Gamma(n + \gamma\sqrt{n} + 2)}{\Gamma(n + \gamma\sqrt{n} + 3)} \frac{(\psi(x + \gamma\sqrt{n} + 2) - \psi(x + \gamma\sqrt{n} + 1))}{(\psi(x + \gamma\sqrt{n} + 2) - \psi(x + \gamma\sqrt{n} + 1))}
\]

Differentiating \(\kappa_2(\alpha)\) we obtain that:
\[
\frac{\kappa_2'(\alpha)}{\kappa_2(\alpha)} = n \left( \psi(n - x + 1) - \psi(n - x + 1 - \gamma\sqrt{n} + 1) + \psi(x + \gamma\sqrt{n} + 1) - \psi(x + 1) \right)
\]

Plugging in (4.21), (4.22), (4.23), (4.24) and (4.18) in (4.12) we get that
\[
\forall \phi_2(Z) \geq \frac{\alpha(1 - \alpha) + (\alpha - \gamma)^2}{n} + \frac{-2\alpha + \alpha^2 + \gamma + 2\alpha\gamma - 2\gamma^2}{n^{3/2}} + O \left( \frac{1}{n} \right)
\]

(c) Consider the following weight function:
\[
\phi_3^{(n)}(p) = \Lambda_3^{(n)}(\alpha, \gamma)p^{\gamma n}(1-p)^{(1-\gamma)n},
\]

where as before \(\Lambda_3^{(n)}(\gamma)\) is found from the normalizing condition (1.6):
\[
\Lambda_3^{(n)}(\gamma) = \frac{\Gamma(x + 1)\Gamma(n - x + 1)\Gamma(2n + 2)}{\Gamma(x + \gamma n + 1)\Gamma(2n - x + 1 - \gamma n)\Gamma(n + 2)}.
\]

Note that only normalizing constant depends on \(n\) as well as the remainder. Let \(y = \gamma n\) then the Fisher Information in this case is equal:
\[
I^{\phi_3}(f_\alpha^{(n)}) = n^2 \left( \psi^{(1)}(x + y + 1) + \psi^{(1)}(2n - x + 1 - y) \right)
+ n^2 \left[ (\psi(x + y + 1) - \psi(x + 1))^2 + (\psi(2n - x + 1 - y) - \psi(n - x + 1))^2 \right] + 2n^2 \left[ (\psi(n - x + 1) - \psi(2n - x - y + 1)) (\psi(x + y + 1) - \psi(x + 1)) \right].
\]

Note that unlike two cases above the differences in brackets do not tend to zero, i.e.,
\[
\psi(x + y + 1) - \psi(x + 1) = \log \left( \frac{\alpha + \gamma}{\alpha} \right) - \frac{\gamma}{2\alpha(\alpha + \gamma)n} + O \left( \frac{1}{n} \right)
\]

Using the asymptotic for digamma function one can obtain that:
\[
I^{\phi_3}(f_\alpha^{(n)}) = \left( \log^2 \frac{(1 - \alpha)(\alpha + \gamma)}{\alpha(2 - \alpha - \gamma)} \right) n^2 + C_1(\alpha, \gamma)n + C_2(\alpha, \gamma) + O \left( \frac{1}{n} \right)
\]
where \( C_1(\alpha, \gamma) \) and \( C_2(\alpha, \gamma) \) are constant that depends only on \( \alpha \) and \( \gamma \) and can be found explicitly. Also note that

\[
g_3(\alpha) = \int_0^1 p \phi_2(p) f_n(\alpha) dp = \frac{\Gamma(2n + 2)\Gamma(x + y + 2)}{\Gamma(2n + 3)\Gamma(x + y + 1)} = \frac{\alpha + \gamma}{2} + \frac{1 - \alpha - \gamma}{2n} + O\left(\frac{1}{n^2}\right)
\]

(4.29)

It is easy to see that in this case the \( g(\alpha) \) has different asymptotic comparing two cases above, so \( g(\alpha) - \mathbb{E}(Z_\alpha) \) does not tend to zero as before. Proceeding with the same computations as before we obtain that

\[
\psi^{\phi_3}(Z) \geq \frac{(\alpha - \gamma)^2}{4} + C_3(\alpha, \gamma)\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right)
\]

(4.30)

where \( C_3(\alpha, \gamma) \) is a constant that depend only of \( \alpha \) and \( \gamma \) and can be found explicitly. □

**Acknowledgement**

The article was prepared within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

**References**

[1] M. Belis, S. Guiasu, *A Quantitative and qualitative measure of information in cybernetic systems* (1968), IEEE Trans. on Inf. Th.,14, 593-594

[2] A. Clim, *Weighted entropy with application*, Analele Universitatii Bucurestica Matematica (2008), Anul LVII, 223-231.

[3] T.M. Cover, Thomas J.M., *Elements of Information Theory*, NY: Basic Books (2006)

[4] R.L. Dobrushin, *Passing to the limit under the sign of the information and entropy*, Theory Prob.Appl., (1960), 29-37

[5] M.V. Fedoruk, *Saddle Point Method*, Moscow: Nauka, 1977, 162–173

[6] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Product* (2007), Elsevier, page 552

[7] M. Kelbert, Yu. Suhov, *Continuity of mutual entropy in the large signal-to-noise ratio limit*, Stochastic Analysis (2010), Berlin: Springer, 281–299

[8] M. Kelbert, Yu. Suhov, *Information Theory and Coding by Example*, Cambridge: Cambridge University Press, 2013
[9] YU. SUHOV, SALIMEH YASAEI SEKEH, M. KELBERT, *Entropy-power inequality for weighted entropy*, arXiv:1502.02188 2015

[10] YU. SUHOV, S. YASAEI SEKEH, *Simple inequalities for weighted entropies*, arXiv:1409.4102 2015