Algebraic Semantics for the Logic of Proofs

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Abstract. We present algebraic semantics for the classical logic of proofs based on Boolean algebras. We also extend the language of the logic of proofs in order to have a Boolean structure on proof terms and equality predicate on terms. Moreover, the completeness theorem and certain generalizations of Stone’s representation theorem are obtained for all proposed algebras.

Keywords: Logic of proofs, Algebraic semantics, Completeness, Representation theorem

1 Introduction

Justification logics are modal-like logics that provide a framework for reasoning about epistemic justifications (see [4, 5, 19]). The language of justification logics extends the language of propositional logic by proof terms and expressions of the form $t : A$, with the intended meaning “$t$ is a justification for $A$” or “$t$ is a proof for $A$”. The logic of proofs LP is the first introduced logic in the family of justification logics due to Artemov ([1, 2]). The logic of proofs is a justification counterpart of modal logic S4. Other modal logics have justification counterparts too (cf. [9, 10, 13, 16, 17, 23, 28]).

Various semantics have been proposed for the logic of proofs: arithmetical semantics ([2]), Mkrtychev models ([21]), Fitting possible world models ([12]), modular models ([3, 18]), subset models ([20]), game semantics ([27]), etc. In this paper, we aim to propose an algebraic semantics for the logic of proofs. The only known algebraic semantics for justification logics is due to Baur and Studer [6, 7] and Pischke [25].

Baur and Studer ([6, 7]) impose a semiring structure on evidence by adding axioms of semirings (on proof terms) to a basic justification logic. The resulted logic, called SE, is proved to be sound and complete with respect to semiring models (a semiring equipped with an interpretation function, an evidence relation, and a truth assignment). In the justification logic framework, it is common to relativize logics with a set of justified axioms called constant specification. There are various kinds of constant specifications, some of them are used to show the internalization property: every theorem is justified by a proof term\textsuperscript{3}. In the logic SE no constant specification is mentioned in its standard form. However, the role of constant specifications is transferred to the set of assumptions.

Pischke ([25]) introduces algebraic variants of Mkrtychev models, Fitting models, and subset models for various intermediate justification logics. Since the main focus of [25] is on intuitionistic and intermediate justification logics, these algebraic variants are defined over Heyting algebras.

In this paper, we present algebraic semantics for the logic of proofs based on Boolean algebra. In modal logic, a modal algebra $A$ is a Boolean algebra equipped with the operator $\Box : A \to A$ that satisfies certain conditions. The standard method of proving completeness of modal logics with respect to modal algebras is to construct an algebra, called the Tarski-Lindenbaum algebra, out of formulas of the logic.

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\textsuperscript{3} In fact, the structure of this term reflects exactly the structure of the axiomatic proof of the theorem.
The proof of well-definedness of the operator $\Box$ in the Tarski-Lindenbaum algebra follows from the fact that the regularity rule is admissible in all normal modal logics:

$$\varphi \leftrightarrow \psi \quad \Box \varphi \leftrightarrow \Box \psi \quad \text{Reg.}$$

Since the justification counterpart of this rule, namely

$$\varphi \leftrightarrow \psi \quad t : \varphi \leftrightarrow t : \psi \quad \text{JReg},$$

is not admissible in justification logics, we use operators on formulas instead of operators on the carrier of algebras. In Section 3 we present algebras for the logic of proofs, called full $\text{LP}$ algebras, and we prove the completeness theorem. We also establish a generalization of Stone’s representation theorem and show that the logic of proofs is complete with respect to set algebras. The full $\text{LP}$ algebras are similar to algebraic Mkrtychev models of Pischke in [25].

In Section 5 we extend the language of $\text{LP}$ to have a Boolean structure on proof terms. We also add the equality predicate (on proof terms) to the language. We then add axioms of Boolean algebra for terms to axioms of $\text{LP}$. The resulting logic is denoted by $\text{LP}^B$. Algebras for $\text{LP}^B$, called full $\text{LP}^B$ algebras, contain two extensions of Boolean algebras: one for proof terms and the other for formulas. Then completeness theorem and a generalization of Stone’s representation theorem are proved. The results of Section 5 on full $\text{LP}^B$ algebras are comparable to the work of Baur and Studer in ([6, 7]) on semiring modals, although $\text{LP}^B$ algebras are defined slightly different from semiring models and further a Boolean algebra on proof terms are used instead of a semiring structure.

Finally, in Section 6 we consider the polynomial structure of proof terms and present an alternative class of algebras for $\text{LP}^B$, called polynomial algebras. Again the completeness theorem and a representation theorem with respect to polynomial algebras are proved.

2 Justification logics

The language of justification logics is an extension of the language of propositional logic by the formulas of the form $t : F$, where $F$ is a formula and $t$ is a term. Proof terms or justification terms (or terms for short) are built up from (proof) variables $x, y, z \ldots$ and (proof) constants $a, b, c, \ldots$ using several operators depending on the logic: (binary) application ‘·’, (binary) sum ‘+’, and (unary) verifier ‘!’. Proof terms are called proof polynomials in [1][2].

The binary operator ‘+’ combines two justifications: $s + t$ is a justification for everything justified by $s$ or by $t$. The binary operator ‘·’ is used to internalize modus ponens: if $s$ is a justification for $A \rightarrow B$ and $t$ is a justification for $A$, then $s \cdot t$ is a justification for $B$. The unary operator ‘!’ is a verifier: if $t$ is a justification for $A$, then this fact can be verified by the justification ‘!t’.

Proof term and formulas are constructed by the following mutual grammar:

$$s, t ::= c \mid x \mid s + t \mid s \cdot t \mid !t,$$

$$\varphi, \psi ::= p \mid \bot \mid \neg \varphi \mid \varphi \lor \psi \mid t : \varphi,$$

where $c \in \text{Const}$, $x \in \text{Var}$, and $p \in \text{Prop}$. Other connectives $\land, \rightarrow, \leftrightarrow$ are defined as usual. In particular, $\varphi \rightarrow \psi ::= \neg \varphi \lor \psi$.

Let $\text{Tm}$ and $\text{Fm}$ denote the set of all terms and the set of all formulas of $\text{LP}$ respectively.

We now begin with describing the axiom schemes and rules of justification logics. The set of axiom schemes of the logic of proofs $\text{LP}_\emptyset$ is:

PL1. $\varphi \rightarrow (\varphi \lor \psi)$,

PL2. $(\varphi \lor \psi) \rightarrow (\psi \lor \varphi)$,
PL3. \( \varphi \lor \varphi \to \varphi \),

PL4. \((\varphi \to \psi) \to (\chi \lor \varphi \to \chi \lor \psi)\),

PL5. \( \perp \to \varphi \),

Appl. \( s : (\varphi \to \psi) \to (t : \varphi \to (s \cdot t) : \psi) \),

Sum. \( s : \varphi \lor t : \varphi \to (s + t) : \varphi \)

jT. \( t : \varphi \to \varphi \),

j4. \( t : \varphi \to \neg t : t : \varphi \).

The only rule of inference of \( \text{LP}_\emptyset \) is:

**MP. Modus Ponens,**

\[
\frac{\varphi \quad \varphi \to \psi}{\psi}
\]

Given a justification logic \( \text{JL} \), a **constant specification** \( \text{CS} \) for \( \text{JL} \) is a set of formulas of the form \( c : \varphi \), where \( c \) is a proof constant and \( \varphi \) is an axiom instance of \( \text{JL} \). A constant specification \( \text{CS} \) is called **axiomatically appropriate** provided for every axiom instance \( \varphi \) of \( \text{JL} \) there exists a proof constant \( c \) such that \( c : \varphi \in \text{CS} \). The total constant specification \( \text{TCS} \) is defined as follows:

\[
\text{TCS} = \{ c : \varphi \mid c \text{ is a proof constant and } \varphi \text{ is an axiom instance} \}.
\]

Note that \( \text{TCS} \) is axiomatically appropriate.

Given a constant specification \( \text{CS} \) for \( \text{JL} \), the justification logic \( \text{JL}_{\text{CS}} \) is an extension of \( \text{JL}_\emptyset \) that has the formulas of \( \text{CS} \) as extra axioms. From now on when we write \( \text{JL}_{\text{CS}} \) we mean that \( \text{CS} \) is a constant specification for \( \text{JL} \).

Non-empty constant specifications are used to show the internalization property for the logic of proofs. This property simulates the necessitation rule from normal modal logic.

**Lemma 2.1 (Internalization).** Suppose that \( \text{CS} \) is an axiomatically appropriate constant specification. If \( \vdash_{\text{LP}_{\text{CS}}} \varphi \), then there is a term \( t \in \text{Tm} \) such that \( \vdash_{\text{LP}_{\text{CS}}} t : \varphi \).

**Proof.** The proof is by induction on the derivation of \( \varphi \). We have two base cases:

- If \( \varphi \) is an axiom instance of \( \text{LP}_\emptyset \). Then, since \( \text{CS} \) is axiomatically appropriate, there is a proof constant \( c \) such that \( c : \varphi \in \text{CS} \). Thus, put \( t := c \).
- If \( \varphi = c : \psi \in \text{CS} \), then using axiom j4 and MP we get \( \vdash_{\text{LP}_{\text{CS}}} c : c : \psi \). Thus, put \( t := c \).

For the induction step suppose \( \varphi \) is obtained by Modus Ponens from \( \psi \to \varphi \) and \( \psi \). By the induction hypothesis, there are terms \( r \) and \( s \) such that \( \vdash_{\text{LP}_{\text{CS}}} r : (\psi \to \varphi) \) and \( \vdash_{\text{LP}_{\text{CS}}} s : \psi \). Then put \( t := r \cdot s \) and use the axiom jK to obtain \( \vdash_{\text{LP}_{\text{CS}}} r \cdot s : \varphi \).

**Lemma 2.2 (Lifting).** Suppose that \( \text{CS} \) is an axiomatically appropriate constant specification. If

\[
\psi_1, \ldots, \psi_n \vdash_{\text{LP}_{\text{CS}}} \varphi,
\]

then there is a term \( t(\bar{x}) \in \text{Tm} \) such that

\[
x_1 : \psi_1, \ldots, x_n : \psi_n \vdash_{\text{LP}_{\text{CS}}} t(\bar{x}) : \varphi,
\]

where \( \bar{x} \) denotes \( x_1, \ldots, x_n \).

**Proof.** The proof is similar to the proof of Lemma 2.1. The only new case is that \( \varphi = \psi_i \), for some \( 1 \leq i \leq n \). In this case put \( t := x_i \).

**Example 2.3.** Here is a proof of \( p \to p \) in \( \text{LP}_\emptyset \). In the following proof, suppose that \( p \lor p \to p \), \( p \to p \lor p \), and \( ((p \lor p) \to p) \to ((p \to (p \lor p)) \to (p \to p)) \) are axiom instances of propositional logic.
1. \((p \lor p) \rightarrow p\) \(\rightarrow (p \rightarrow (p \lor p)) \rightarrow (p \rightarrow p)\),

axiom instance PL4

2. \(p \lor p \rightarrow p\),

axiom instance PL3

3. \(p \rightarrow p \lor p\),

axiom instance PL1

4. \((p \rightarrow (p \lor p)) \rightarrow (p \rightarrow p)\),

from 1 and 3 by MP

5. \(p \rightarrow p\).

from 2 and 4 by MP

Using the proof of the internalization lemma, we assign a proof term to each line of the above proof. For \(a, b, c \in \text{Const}\) we have:

1'. \(a : [(p \lor p) \rightarrow p] \rightarrow ((p \rightarrow (p \lor p)) \rightarrow (p \rightarrow p))\),

from TCS5

2'. \(b : (p \lor p \rightarrow p)\),

from TCS

3'. \(c : (p \rightarrow p \lor p)\),

from TCS

4'. \(a \cdot b : [(p \rightarrow (p \lor p)) \rightarrow (p \rightarrow p)]\),

from 1 and 3 by MP

5'. \((a \cdot b) \cdot c : (p \rightarrow p)\).

from 2 and 4 by MP

Thus, we showed that \(\vdash_{\text{LP}_{\text{CS}}} (a \cdot b) \cdot c : (p \rightarrow p)\). Note that the proof term \(t = (a \cdot b) \cdot c\) includes all the information that exist in the proof of \(p \rightarrow p\). Three proof constants \(a, b, c\) in \(t\) show that three axiom instances are used in the proof (see the axioms used in the steps 1'–3' above), and the two application operators “\(\cdot\)” in \(t\) show that two applications of the rule MP are used in the proof.

In the remaining of this section, we recall a well known semantics for the logic of proofs were originally introduced by Mkrtychev in [21] (see also the basic modular models of [18] for the logic of proofs).

**Definition 2.4.** An \(\text{LP}_{\text{CS}}\) model \(\mathcal{M} = (E, V)\) consists of an evidence function \(E : Tm \times Fm \rightarrow \{0, 1\}\) and a valuation \(V : \text{Prop} \rightarrow \{0, 1\}\). The valuation \(V\) is extended to all formulas \(\tilde{V} : Fm \rightarrow \{0, 1\}\) as follows:

\[
\tilde{V}(p) = V(p), \\
\tilde{V}(\bot) = 0, \\
\tilde{V}(\neg \varphi) = 1 - \tilde{V}(\varphi), \\
\tilde{V}(\varphi \lor \psi) = \max(\tilde{V}(\varphi), \tilde{V}(\psi)), \\
\tilde{V}(t : \varphi) = E(t, \varphi).
\]

The evidence function \(E\) should satisfy the following conditions:

**Appl.** \(\min(E(s, \varphi \rightarrow \psi), E(t, \varphi)) \leq E(s \cdot t, \psi),\)

**Sum.** \(\max(E(s, \varphi), E(t, \varphi)) \leq E(s + t, \varphi),\)

**jT.** \(E(t, \varphi) \leq \tilde{V}(\varphi),\)

**j4.** \(E(t, \varphi) \leq \tilde{E}(t, t : \varphi),\)

**CS** \(E(c, \varphi) = 1, \text{ for } c : \varphi \in \text{CS}.\)

A formula \(\varphi\) is \(\text{LP}_{\text{CS}}\)-valid, denoted by \(\vdash_{\text{LP}_{\text{CS}}} \varphi\), if \(\tilde{V}(\varphi) = 1\) for every \(\text{LP}_{\text{CS}}\) model \(\mathcal{M} = (E, V)\).

The proof of soundness and completeness theorems is given in [21] and [18].

**Theorem 2.5.** Let \(CS\) be a constant specification for \(\text{LP}\). Then, \(\vdash_{\text{LP}_{\text{CS}}} \varphi\iff \vdash_{\text{LP}_{\text{CS}}} \varphi\).

We give the preliminary definitions of Boolean, modal, and regular algebras in the next section. Then, in Section [18] we present algebraic semantics for \(\text{LP}\).
3 Regular algebras

Throughout the paper we only consider classical logics, and so we deal with Boolean algebras (see [22] for a more detailed exposition). A Boolean algebra \((A, 0, \ominus, \oplus)\) (where the constant 0 is the least element and the operators \(\ominus\) and \(\oplus\) give the complement of an element and join of two elements respectively) is a structure in which the following laws hold (here \(1 := \ominus 0\) is the greatest element, and \(a \otimes b := (\ominus (\ominus a) \oplus (\ominus b))\) gives the meet of two elements): commutative and associative laws for join and meet, distributive laws both for meet over join and for join over meet, and the following special laws:

\[
\begin{align*}
    a \oplus 0 &= a, \\
    a \otimes 1 &= a, \\
    a \oplus (\ominus a) &= 1, \\
    a \otimes (\ominus a) &= 0.
\end{align*}
\]

The implication is defined as \(a \Rightarrow b := (\ominus a) \oplus b\). The order \(\leq\) on a Boolean algebra is defined as follows:

\[
a \leq b \text{ iff } a \Rightarrow b = 1 \text{ (iff } a \oplus b = b \text{ iff } a \otimes b = a).\]

Since the logic of proofs \(\text{LP}\) is a justification counterpart of modal logic \(\text{S4}\), let us recall \(\text{S4}\) algebras first. An \(\text{S4}\) algebra (or an interior algebra) is a tuple \((A, 0, \ominus, \oplus, \Box)\) such that \((A, 0, \ominus, \oplus)\) is a Boolean algebra, and the operator \(\Box : A \to A\) satisfies the following equations:

\[
\begin{align*}
    - \Box(1) &= 1, \\
    - \Box(a \otimes b) &= \Box a \otimes \Box b, \\
    - \Box \Box a &= \Box a, \\
    - a \otimes a &= a.
\end{align*}
\]

The completeness proof of modal logics with respect to modal algebras follows from the fact that the regularity rule is admissible in all normal modal logics (cf. [11, 8]):

\[
\phi \leftrightarrow \psi \quad \Box \phi \leftrightarrow \Box \psi \quad \text{Reg}
\]

Likewise we extend the logic of proofs with a justification counterpart of this rule. Let \(\text{HLP}_0\) be an extension of \(\text{LP}_0\) by the following justification regularity rule:

\[
\phi \leftrightarrow \psi \quad t : \phi \leftrightarrow t : \psi \quad \text{JReg}
\]

A justification logic in which the justification regularity rule \(\text{JReg}\) is admissible is called regular.

**Definition 3.1 (\(\text{HLP}_0\) algebra).** An \(\text{HLP}_0\) algebra is a tuple \(\mathcal{A} = (A, 0, \ominus, \oplus, \Box_t)_{t \in Tm}\) such that \((A, 0, \ominus, \oplus)\) is a Boolean algebra with operators \(\Box_t : A \to A\) satisfying the following conditions. For all \(a, b \in A\) and all \(s, t \in Tm\):

- **A-Appl.** \(\Box_s(a \Rightarrow b) \otimes \Box_t(a) \leq \Box_{s+t}(b)\),
- **A-Sum.** \(\Box_s(a) \oplus \Box_t(a) \leq \Box_{s+t}(a)\),
- **A-jT.** \(\Box_t(a) \leq a\),
- **A-j4.** \(\Box_t(a) \leq \Box_{\Box_t}(a)\).

A regular algebra is a tuple \(\mathcal{A} = (A, 0, \ominus, \oplus, \Box_t)_{t \in Tm}\) such that \((A, 0, \ominus, \oplus)\) is a Boolean algebra with operators \(\Box_t : A \to A\) satisfying conditions A-Appl and A-Sum.\(^4\)

\(^4\) Note that every \(\text{HLP}_0\) algebra is a regular algebra.
Definition 3.2 (Valuation). Let $A$ be a regular algebra. A valuation on $A$ is a function $\theta : \text{Prop} \rightarrow A$. The assignment $\tilde{\theta} : Fm \rightarrow A$ on $A$ is an extension of $\theta$ defined as follows

$$\tilde{\theta}(p) = \theta(p),$$
$$\tilde{\theta}(\bot) = 0,$$
$$\tilde{\theta}(\neg \varphi) = \ominus \tilde{\theta}(\varphi),$$
$$\tilde{\theta}(\varphi \lor \psi) = \tilde{\theta}(\varphi) \oplus \tilde{\theta}(\psi),$$
$$\tilde{\theta}(t : \varphi) = \Box_t (\tilde{\theta}(\varphi)).$$

Note that from the above conditions one can obtain the following:

$$\tilde{\theta}(\varphi \rightarrow \psi) = \tilde{\theta}(\varphi) \Rightarrow \tilde{\theta}(\psi).$$

Next we give a general definition for validity which will be used throughout the paper.

Definition 3.3 (Validity). Let $A$ be an algebra and let $\nabla$ be a set of certain distinguished elements of $A$. The set $(A, \nabla)$ is called a matrix. If $A$ is a regular algebra, then $(A, \nabla)$ is called a regular matrix.

- A formula $\varphi$ is true in the matrix $(A, \nabla)$, denoted by $(A, \nabla) \models \varphi$, if $\tilde{\theta}(\varphi) \in \nabla$ for every valuation $\theta$ on $A$.
- A formula $\varphi$ is valid in a class $C$ of matrices, denoted by $C \models \varphi$, if $(A, \nabla) \models \varphi$ for every algebra $(A, \nabla) \in C$.
- A logic $L$ is characterized by a class $C$ of matrices if for every formula $\varphi : \vdash_L \varphi$ iff $C \models \varphi$.

The class of all $\text{HLP}_\emptyset$ algebras with singleton $\nabla$ is denoted by $A^N$ (the superscript $N$ denotes the use of singleton $\nabla$ for all algebras in the class).

Notation: We shall often deal with matrices $(A, \nabla)$ in which $\nabla = \{1_A\}$, where $1_A$ denotes the unit of $A$. In this case instead of $(A, \nabla) \models \varphi$ we write $A \models \varphi$. Thus, $A \models \varphi$ means that $\tilde{\theta}(\varphi) = 1_A$, for every valuation $\theta$ on $A$.

Now we use the so called Tarski-Lindenbaum algebra to prove completeness.

Definition 3.4. For $\varphi \in Fm$, let

$$[\varphi] := \{ \psi \mid \vdash_{\text{HLP}_\emptyset} \varphi \leftrightarrow \psi \}.$$

and let

$$[Fm] := \{ [\varphi] \mid \varphi \in Fm \}.$$

The Tarski-Lindenbaum algebra $A_{\text{HLP}_\emptyset} := ([Fm], 0, \ominus, \oplus, \Box_t)_{t \in Tm}$ for $\text{HLP}_\emptyset$ is defined as follows:

$$0 := [\bot],$$
$$\ominus [\varphi] := [-\varphi],$$
$$[\varphi] \oplus [\psi] := [\varphi \lor \psi],$$
$$\Box_t ([\varphi]) := [t : \varphi].$$

Let $\nabla = \{[\top]\}$.

Observe that

$$\vdash \varphi \rightarrow \psi \iff [\varphi] \leq [\psi],$$

and

$$[\varphi] \Rightarrow [\psi] = [\varphi \rightarrow \psi].$$
Lemma 3.5. The Tarski-Lindenbaum algebra $A_{HLP_0}$ is an HLP$_0$ algebra.

Proof. We show that the operators $\Box_t$, for $t \in Tm$, are well-defined. Checking the rest of the conditions is straightforward.

Let $t \in Tm$ be a fixed term. If $[\varphi] = [\psi]$, then $\vdash_{HLP_0} \varphi \leftrightarrow \psi$. By the rule $JReg$, $\vdash_{HLP_0} t : \varphi \leftrightarrow t : \psi$, and hence $[t : \varphi] = [t : \psi]$. Therefore, $\Box_t([\varphi]) = \Box_t([\psi])$. □

Theorem 3.6 (Soundness and completeness). $\vdash_{HLP_0} \varphi$ iff $A^N \models \varphi$.

Proof. Soundness is straightforward. For completeness define the valuation $\theta$ as follows

$$\theta(p) := [p], \quad \text{for } p \in Prop.$$

Now it is easy to prove the Truth Lemma: for every formula $\varphi$

$$\check{\theta}(\varphi) = [\varphi].$$

The proof is by induction on the complexity of $\varphi$. The proof of the case $\varphi = t : \psi$ follows from Definition 3.2 and Definition 3.3 and the proof of other cases are standard.

Finally, completeness follows easily from the Truth Lemma. If $\not\vdash_{HLP_0} \varphi$, then $\check{\theta}(\varphi) = [\varphi] \notin \nabla = \{[\top]\}$, and hence $(A_{HLP_0}, \nabla) \not\models \varphi$. □

Theorem 3.6 shows that HLP$_0$ is characterized by the class $A^N$, i.e. the class of all HLP$_0$ matrices with singleton $\nabla$.

Theorem 3.7. If a justification logic $JL$ is characterized by a class of regular matrices $C$ with singleton $\nabla$ then the rule $JReg$ is admissible in $JL$.

Proof. Suppose that $\vdash_{JL} \varphi \leftrightarrow \psi$. Then, $C \models \varphi \leftrightarrow \psi$. Thus, for every $A \in C$ and for every valuation $\theta$ on $A$, we have $\check{\theta}(\varphi) = \check{\theta}(\psi)$. Thus, $\Box_t(\check{\theta}(\varphi)) = \Box_t(\check{\theta}(\psi))$, for every $t$. Hence, $\check{\theta}(t : \varphi) = \check{\theta}(t : \psi)$, for every $\theta$. Thus, $C \models t : \varphi \leftrightarrow t : \psi$, and hence $\vdash_{JL} t : \varphi \leftrightarrow t : \psi$. □

Let us consider the rule $JReg$. This rule says that if $\varphi \leftrightarrow \psi$ is a theorem, then every proof $t$ of $\varphi$ is a proof of $\psi$ and vice versa. For instance, let $\varphi$ be $p \rightarrow p$, and $\psi$ be $\bot \rightarrow q$. We know that $\varphi \leftrightarrow \psi$ is a theorem of propositional logic. It should be obvious that in this example, a proof of $\varphi$ could be different from that of $\psi$ and it is not the case that every proof of $\varphi$ is a proof of $\psi$ and vice versa. For example, both $(a \cdot b) \cdot c : (p \rightarrow p)$ (see Example 3.3) and $d : (ot \rightarrow q)$, for some $d \in Const$, are provable in $LP_{TCS}$ but $(a \cdot b) \cdot c$ and $d$ are different proof terms, and one can show that $(a \cdot b) \cdot c : (\bot \rightarrow q)$ is not a theorem of $LP_{TCS}$.

It is noteworthy that a mono-agent version of $JReg$ is used in the axiomatic formulation of some of the relevant justification logics of [29]. One might argue that for an equivalence to be valid in a relevant logic there must be some connection between the two equivalent propositions, and thus $JReg$ is expected to be admissible in this setting. Nonetheless, it is not still clear why two equivalent propositions should be known for the same reason. It seems that an argument in favor of $JReg$ in a relevant logic depends on a sensible notion of ‘connection’, and thus this issue has yet to be investigated more.

Since $JReg$ is not admissible in the standard justification logics, in order to present algebraic semantics for the logic of proofs $LP$ we consider two possibilities:

1. $\nabla$ is not singleton.
2. Replacing the operators $\Box_t : A \rightarrow A$ with an alternate.

In the rest of this section we consider the first possibility. We show that the logic of proofs $LP_0$ is characterized by the class of all HLP$_0$ matrices when $\nabla$ is not singleton. This class is denoted by $A^{\inf}$.  

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5 Even since $\varphi$ is a theorem of paraconsistent logics but $\psi$ is not, it seems that the ‘informal justifications’ of $\varphi$ and $\psi$ are different too.
Definition 3.8. The Tarski-Lindenbaum algebra
\[ A^{\text{inf}}_{\LP_0} := ([Fm], 0, \sqcap, \oplus, \Box_{t})_{t \in Tm}, \]
is defined similar to Definition 3.4 with the difference that now the class of a formula is defined as follows:
\[ [\varphi] := \{ \psi \mid (\forall n \geq 0)(\forall s_1, \ldots, s_n \in Tm) \vdash_{\LP_0} s_n : \cdots : s_1 : \varphi \leftrightarrow s_n : \cdots : s_1 : \psi \}. \]
Let \( \nabla' = \{ [\varphi] \mid \vdash_{\LP_0} \varphi \} \).

Lemma 3.9. The Tarski-Lindenbaum algebra \( A^{\text{inf}}_{\LP_0} \) is an \( \LP_0 \) algebra.

Proof. We show that the operators \( \Box_{t} \), for \( t \in Tm \), are well-defined. Checking the rest of the conditions is straightforward.

Let \( t \in Tm \) be a fixed term. If \( [\varphi] = [\psi] \), then by the definition of the class of formulas in Definition 3.8 we get for all \( n \geq 0 \) and for all \( s_1, \ldots, s_n \in Tm \):
\[ \vdash_{\LP_0} s_n : \cdots : s_1 : t : \varphi \leftrightarrow s_n : \cdots : s_1 : t : \psi. \]

Hence \( [t : \varphi] = [t : \psi] \). Therefore, \( \Box_{t}([\varphi]) = \Box_{t}([\psi]) \). \( \square \)

Theorem 3.10 (Soundness and completeness). \( \vdash_{\LP_0} \varphi \iff A^{\text{inf}}_{\LP_0} \models \varphi. \)

Proof. Similar to the proof of Theorem 3.6. Just note that if \( \not\vdash_{\LP_0} \varphi \), then \( \tilde{\theta}(\varphi) = [\varphi] \notin \nabla' \), and hence \( (A^{\text{inf}}_{\LP_0}, \nabla') \not\models \varphi. \) \( \square \)

It is worth nothing that considering \( \nabla' \) as the set of all theorems is not a viable solution. In fact this theorem conveys nothing but the fact that \( \LP_0 \) is closed under substitution (cf. [11] page 195). Hence in the rest of the paper, we study the second case where the operators \( \Box_{t} \) on the algebra are replaced by functions on the formulas.

4 Full \( \LP_{CS} \) algebras

As mention before, the rule \( JReg \) is not plausible in justification logics. However, since this rule is essential to have well-defined operators in the Tarski-Lindenbaum algebras, in this section (and also the next sections) by considering functions on formulas instead of operators on the algebra we overcome this issue.

Definition 4.1. A pre \( \LP_{CS} \) algebra is a tuple \( \mathcal{A} = (A, 0, \sqcap, \oplus, \Box_{t})_{t \in Tm} \) such that \( (A, 0, \sqcap, \oplus) \) is a Boolean algebra, and for each \( t \in Tm \), \( \Box_{t} : Fm \rightarrow A \) is a function satisfying the following conditions. For all \( \varphi, \psi \in Fm \), all \( s, t \in Tm \), and all \( c \in \text{Const} \):

- \( \text{Al-Apply-} \LP_{CS} \). \( \Box_{s}(\varphi \to \psi) \otimes \Box_{t}((\varphi) \leq \Box_{s+t}(\psi) \),
- \( \text{Al-Sum-} \LP_{CS} \). \( \Box_{s}(\varphi) \oplus \Box_{t}((\varphi) \leq \Box_{s+t}(\varphi) \),
- \( \text{Al-J4-} \LP_{CS} \). \( \Box_{t}(\varphi) \leq \Box_{t}(t : \varphi) \),
- \( \text{Al-CS} \). \( \Box_{c}(\varphi) = 1, \text{ for } c : \varphi \in CS \).

Let \( T \) be a set. In what follows we call \( \mathcal{A} = (A, 0, \sqcap, \oplus, \Box_{t})_{t \in T} \) an \( \LP_{CS} \) algebra over \( T \). One may justify this terminology by considering an \( \LP_{CS} \) algebra \( \mathcal{A} = (A, 0, \sqcap, \oplus, \Box_{t})_{t \in T} \) as a Boolean algebra \( (A, 0, \sqcap, \oplus) \) endowed with a family of relations \( \Box_{t} \), parametrized by elements \( t \in T \). In the next sections, we consider various algebraic structures on the set \( T \).
**Definition 4.2.** Given a valuation $\theta : \text{Prop} \rightarrow A$, the assignment $\tilde{\theta} : \text{Fm} \rightarrow A$ on $A$ is defined as in Definition 3.2 with the following difference:

$$\tilde{\theta}(t : \varphi) = \Box_t(\varphi).$$

Validity is defined as in Definition 3.3.

**Definition 4.3.** A full LP\textsubscript{CS} algebra $A = (A, 0, \ominus, \oplus, \Box_t)_{t \in T_m}$ is a pre LP\textsubscript{CS} algebra that satisfies the following condition. For all $\varphi \in \text{Fm}$, all $t \in T_m$, and all valuation $\theta : \text{Prop} \rightarrow A$:

$$\Box_t(\varphi) \leq \tilde{\theta}(\varphi).$$

The class of all full LP\textsubscript{CS} algebras with singleton $\nabla = \{1\}$ is denoted by $A_{\text{full}}^{\text{LP}\text{CS}}$.

**Example 4.4.** Let $2 = ((0, 1), 0, \ominus, \max)$ be the Boolean algebra of truth values, where $\ominus$ is defined by $\ominus a = 1 - a$. Here, the join of two elements is given by maximum (max), and it is easy to show that the meet of two elements is given by minimum (min). Given a constant specification $\text{CS}$ for LP, we construct a full LP\textsubscript{CS} algebra based on the Boolean algebra $2$. For every $t \in T_m$, define $\Box_t$ by induction on the complexity of $t$ as follows. For every $\varphi \in \text{Fm}$:

$$\Box_c(\varphi) := 0,$$

$$\Box_c(\varphi) := \begin{cases} 1 & \text{if } c : \varphi \in \text{CS}, \\ 0 & \text{otherwise}. \end{cases}$$

$$\Box_{s \cdot t}(\varphi) := \max \{ \min(\Box_s(\psi \rightarrow \varphi), \Box_t(\psi)) \mid \psi \in \text{Fm} \},$$

$$\Box_{s + t}(\varphi) := \max(\Box_s(\varphi), \Box_t(\varphi)),$$

$$\Box_t(\varphi) := \begin{cases} \Box_t(\psi) & \text{if } \varphi = t : \psi, \\ 0 & \text{otherwise}. \end{cases}$$

Let $2_{\text{LP}\text{CS}} = ((0, 1), 0, \ominus, \max, \Box_t)_{t \in T_m}$. It is easy to show that $2_{\text{LP}\text{CS}}$ is a pre LP\textsubscript{CS} algebra. Now we show that $2_{\text{LP}\text{CS}}$ is a full LP\textsubscript{CS} algebra. To this end, by induction on the complexity of the term $t$ we show that:

$$\Box_t(\varphi) \leq \tilde{\theta}(\varphi).$$

We only check the case $t = s \cdot r$ (the proof for other cases is simpler). By the induction hypothesis, we have

$$\Box_s(\psi \rightarrow \varphi) \leq \tilde{\theta}(\psi \rightarrow \varphi) \quad \text{and} \quad \Box_t(\psi) \leq \tilde{\theta}(\psi).$$

Thus,

$$\min(\Box_s(\psi \rightarrow \varphi), \Box_t(\psi)) \leq \min(\tilde{\theta}(\psi \rightarrow \varphi), \tilde{\theta}(\psi)),$$

and hence

$$\min(\Box_s(\psi \rightarrow \varphi), \Box_t(\psi)) \leq \tilde{\theta}(\varphi).$$

By taking maximum over all formulas $\psi$, we get

$$\Box_{s \cdot t}(\varphi) \leq \tilde{\theta}(\varphi).$$

**Lemma 4.5.** The regularity rule

$$\frac{\varphi \leftrightarrow \psi}{t : \varphi \leftrightarrow t : \psi} \quad J\text{Reg}$$

is not validity preserving in the class of full LP\textsubscript{CS} algebras, i.e. there exists $\varphi, \psi \in \text{Fm}$ and $t \in T_m$ such that $A_{\text{full}}^{\text{LP}\text{CS}} \models \varphi \leftrightarrow \psi$ but $A_{\text{full}}^{\text{LP}\text{CS}} \not\models t : \varphi \leftrightarrow t : \psi$.  

9
Proof. Let \( CS = \{ c : (p \land p \rightarrow p) \} \) be a constant specification for \( LP \). Consider the full \( LP_{CS} \) algebra \( 2_{LP_{CS}} \) from Example 4.4. It is easy to show that \( A_{LP_{CS}}^{\text{full}} \models (p \land p \rightarrow p) \leftrightarrow \top \). On the other hand, we observe that \( \Box_c (p \land p \rightarrow p) = 1 \) and \( \Box_c (\top) = 0 \). Thus, given an arbitrary valuation \( \theta \), we have \( \tilde{\theta}(c : (p \land p \rightarrow p)) \neq \tilde{\theta}(c : \top) \). Hence, \( A_{LP_{CS}}^{\text{full}} \not\models c : (p \land p \rightarrow p) \leftrightarrow c : \top \). □

In what follows, we prove completeness theorem with respect to the classes \( A_{LP_{CS}}^{\text{full}} \) and we show a representation theorem for \( LP_{CS} \).

4.1 Completeness of \( LP_{CS} \)
Completeness is proved by the standard method of defining the Tarski-Lindenbaum algebra for \( LP_{CS} \).

Definition 4.6. For \( \varphi \in Fm \), let \( [\varphi] := \{ \psi \mid \vdash_{LP_{CS}} \varphi \leftrightarrow \psi \} \), and let \( [Fm] := \{ [\varphi] \mid \varphi \in Fm \} \).

The Tarski-Lindenbaum algebra
\( A_{LP_{CS}}^{f} := ([Fm], 0, \ominus, \oplus, \Box_t)_{t \in Tm} \),
for \( LP_{CS} \) is defined as follows:

\[
0 := \top,
\ominus[\varphi] := \neg[\varphi],
[\varphi] \oplus [\psi] := [\varphi \lor \psi],
\Box_t (\varphi) := [t : \varphi].
\]

Note that \( 1 = [\top] \).

Lemma 4.7. The Tarski-Lindenbaum algebra \( A_{LP_{CS}}^{f} \) is a pre \( LP_{CS} \) algebra.

Proof. The proof is straightforward. We only verify the condition \( Al-CS \). Suppose that \( c : \varphi \in CS \). Then, \( \Box_c (\varphi) = [c : \varphi] = [\top] = 1 \). □

Theorem 4.8 (Soundness and completeness). \( \vdash_{LP_{CS}} \varphi \iff A_{LP_{CS}}^{\text{full}} \models \varphi \).

Proof. Similar to the proof of Theorem 3.6 Define the valuation \( \theta \) as follows

\[
\theta(p) := [p], \quad \text{for } p \in \text{Prop}.
\]

Now it is easy to prove the Truth Lemma: for every formula \( \varphi \)

\[
\tilde{\theta}(\varphi) = [\varphi].
\]

The proof is by induction on the complexity of \( \varphi \). The proof of the case \( \varphi = t : \psi \) is as follows:

\[
\tilde{\theta}(t : \psi) = \Box_t (\psi) = [t : \psi].
\]

Next we show that the Tarski-Lindenbaum algebra \( A_{LP_{CS}}^{f} \) satisfies \( Al-jT \)-\( LP_{CS} \), and hence is a full \( LP_{CS} \) algebra:

\[
\Box_t (\varphi) = [t : \varphi] \leq [\varphi] = \tilde{\theta}(\varphi).
\]

Finally, completeness follows easily from the Truth Lemma. □
4.2 Representation theorem for full \( \mathbf{LP}_{\mathbf{CS}} \) algebras

In this section we show a representation theorem for full \( \mathbf{LP}_{\mathbf{CS}} \) algebras. We first recall that the power set algebra of a set \( A \) is the structure

\[
\mathfrak{P}(A) = (\mathcal{P}(A), \emptyset, \setminus, \cup),
\]

where \( \mathcal{P}(A) \) is the power set of \( A \), \( \setminus \) is the complement relative to \( A \). A set algebra is a subalgebra of a power set algebra.

We now recall a classical result in propositional logic that will be used in the sequel (cf. [8]).

**Theorem 4.9 (Stone Representation Theorem).** Every Boolean algebra is isomorphic to a set algebra.

Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are two full \( \mathbf{LP}_{\mathbf{CS}} \) algebras. We say that \( f \) is an isomorphism between \( \mathcal{A} \) and \( \mathcal{B} \) provided that

\[
f(\square^A_t(\varphi)) = \square^B_t(\varphi),
\]

for every \( \varphi \in \text{Fm} \).

Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are isomorphic Boolean algebras via an isomorphism \( f \). Then if \( \mathcal{A} \) is a full \( \mathbf{LP}_{\mathbf{CS}} \) algebra with functions \( \square^A_t \), define \( \square^B_t \) on the algebra \( \mathcal{B} \) as follows:

\[
\square^B_t(\varphi) := f(\square^A_t(\varphi)). \tag{1}
\]

Now we show that \( \mathcal{A} \) and \( \mathcal{B} \) are isomorphic as full \( \mathbf{LP}_{\mathbf{CS}} \) algebras.

**Lemma 4.10.** Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are two isomorphic Boolean algebras via isomorphism \( f \). Then if \( \mathcal{A} \) is a full \( \mathbf{LP}_{\mathbf{CS}} \) algebra with functions \( \square^A_t \), then \( \mathcal{B} \) is a full \( \mathbf{LP}_{\mathbf{CS}} \) algebra with functions \( \square^B_t \) defined in (1). Moreover, \( f \) is an isomorphism of full \( \mathbf{LP}_{\mathbf{CS}} \) algebras.

**Proof.** First note that since \( f \) is an isomorphism between \( \mathcal{A} \) and \( \mathcal{B} \), for every \( a, b \in \mathcal{A} \) we have

\[
a \leq b \quad \text{iff} \quad f(a) \leq f(b) \tag{2}
\]

In order to show that \( \mathcal{B} \) is a full \( \mathbf{LP}_{\mathbf{CS}} \) algebra we only check the conditions \( \text{Al} \)-\text{Appl} and \( \text{Al} \)-\text{jT} from Definition 4.1 (other cases can be verified similarly).

For \( \text{Al} \)-\text{Appl} \( \mathbf{LP}_{\mathbf{CS}} \), suppose that \( \varphi, \psi \in \text{Fm} \) and \( s, t \in \text{Tm} \):

\[
\square^B_t(\varphi \rightarrow \psi) \otimes \square^B_t(\varphi) =
\]
\[
f(\square^A_s(\varphi \rightarrow \psi) \otimes f(\square^A_t(\varphi))) =
\]
\[
f(\square^A_s(\varphi \rightarrow \psi) \otimes \square^A_t(\varphi)) \leq
\]
\[
f(\square^A_s(\psi)) = \square^B_s(\psi).
\]

For \( \text{Al} \)-\text{jT} \( \mathbf{LP}_{\mathbf{CS}} \), suppose that \( \varphi \in \text{Fm} \), \( t \in \text{Tm} \) and \( \theta : \text{Prop} \rightarrow B \):

\[
\square^B_t(\varphi) = f(\square^A_t(\varphi)) \leq f(f^{-1}(\tilde{\theta}(\varphi))) = \tilde{\theta}(\varphi).
\]

The above inequality follows from the fact that \( f \) is an isomorphism and \( f^{-1} \circ \tilde{\theta} : \text{Fm} \rightarrow A \) is an assignment on \( \mathcal{A} \).

A set algebra enriched with the functions \( \square_t \), for all \( t \in \text{Tm} \), is called a full \( \mathbf{LP}_{\mathbf{CS}} \) set algebra.

**Theorem 4.11 (Representation Theorem).** Every full \( \mathbf{LP}_{\mathbf{CS}} \) algebra is isomorphic to a full \( \mathbf{LP}_{\mathbf{CS}} \) set algebra.
\textbf{Theorem 4.12.} $\text{LPC}_S$ is complete with respect to the class of all full $\text{LPC}_S$ set algebras.

\textit{Proof.} If $\not\models \varphi$, then by Theorem 4.8 there is a full $\text{LPC}_S$ algebra $A$ and valuation $\theta : \text{Prop} \to A$ such that $A \not\models \varphi$, i.e $\theta(\varphi) \neq 1_A$. By the Representation Theorem 4.11 there is an isomorphism $f$ and a full $\text{LPC}_S$ set algebra $B = (B, \ldots)$ isomorphic to $A$. Define the valuation $\theta' : \text{Prop} \to B$ as follows:

$$\theta'(p) := f(\theta(p)).$$

It is easy to prove that

$$\theta'(\varphi) = f(\theta(\varphi)),$$

for every formula $\varphi$. Finally we get

$$\theta'(\varphi) = f(\theta(\varphi)) \neq 1_B$$

Thus, $B \not\models_{\theta'} \varphi$. \hfill $\square$

\subsection{4.3 Binary $\text{LP}_S$ algebras}

The full algebra $2_{\text{LP}_S}$ of Example 4.4 is an example of full $\text{LPC}_S$ algebras whose carrier is the two elements set $\{0, 1\}$. It is known that propositional logic is complete with respect to the Boolean algebra of truth values $2$ (cf. [8]). Next we define a new class of $\text{LPC}_S$ algebras whose Boolean part is the Boolean algebra of truth values. Then, we establish a completeness theorem with respect to this new class of $\text{LPC}_S$ algebras.

\textbf{Definition 4.13.} A binary $\text{LP}_S$ algebra is a tuple $A = (\{0, 1\}, 0, \cdot, \max, \square, \theta)_{t \in T_m}$ such that the structure $(\{0, 1\}, 0, \cdot, \max)$ is the Boolean algebra defined in Example 4.4 for each $t \in T_m$, $\square : Fm \to \{0, 1\}$ is a function, and $\theta : \text{Prop} \to \{0, 1\}$ is a valuation. The function $\theta$ is extended to the assignment $\theta : Fm \to \{0, 1\}$ as in Definition 4.2. The functions $\square_t$ should satisfy the following conditions. For all $\varphi, \psi \in Fm$, all $s, t \in T_m$, and all $c \in \text{Const}$:

\begin{itemize}
  \item $\text{Al-Appal-}\text{LP}_S$. $\square_s(\varphi \to \psi) \otimes \square_t(\varphi) \leq \square_{s \cdot t}(\psi)$,
  \item $\text{Al-Sum-}\text{LP}_S$. $\square_s(\varphi) \otimes \square_t(\varphi) \leq \square_{s + t}(\varphi)$,
  \item $\text{Al-J4-}\text{LP}_S$. $\square_t(\varphi) \leq \square_{t \cdot t}(t : \varphi)$,
  \item $\text{Al-CS}$. $\square_c(\varphi) = 1$, for $c : \varphi \in \text{CS}$,
  \item $\text{Al-JT-}\text{LP}_S$. $\square_t(\varphi) \leq \tilde{\theta}(\varphi)$.
\end{itemize}

Let $\mathcal{A}^2_{\text{LP}_S}$ denote the class of all binary $\text{LP}_S$ algebras with $\nabla = \{1\}$.

In order to show completeness with respect to the class $\mathcal{A}^2_{\text{LP}_S}$, we first show that every $\text{LPC}_S$ model is equivalent to a binary $\text{LPC}_S$ algebra, and vice versa. Then, completeness of $\text{LPC}_S$ with respect to $\mathcal{A}^2_{\text{LP}_S}$ follows from completeness of $\text{LPC}_S$ models. We need the following auxiliary lemma.

\textbf{Lemma 4.14.} Let $M = (E, \nabla)$ be an $\text{LPC}_S$ model and $A = (\{0, 1\}, 0, \cdot, \max, \square, \theta)_{t \in T_m}$ be a binary $\text{LPC}_S$ algebra. If $\nabla = \theta$ and for all $t \in T_m$ and all $\varphi \in Fm$:

$$E(t, \varphi) = \square_t(\varphi),$$

then, $M$ and $A$ are equivalent, i.e. for all $\varphi \in Fm$:

$$\tilde{\nabla}(\varphi) = \tilde{\theta}(\varphi).$$

\hfill $\square$
Proof. The proof is by induction on the complexity of \( \varphi \). The base case follows from \( \mathcal{V} = \theta \). For the induction step we only check the case where \( \varphi = t : \psi \): 
\[
\tilde{V}(t : \psi) = \tilde{E}(t, \psi) = \square_t(\psi) = \tilde{\theta}(t : \psi).
\]
Note that in the above equations the second equality follows from (3).

\[\blacksquare\]

Lemma 4.15. For every \( \text{LP}_{CS} \) model there is an equivalent binary \( \text{LP}_{CS} \) algebra, and vice versa.

Proof. Suppose that \( \mathcal{M} = (\mathcal{E}, \mathcal{V}) \) is an \( \text{LP}_{CS} \) model. Define the binary \( \text{LP}_{CS} \) algebra 
\[
\mathcal{A} = (\{0, 1\}, 0, - , \max, \square_t, t)_{t \in \text{Tm}}
\]
as follows: Let \( \theta := \mathcal{V} \), and for every \( t \in \text{Tm} \) let \( \square_t \) be defined on formulas as (3). Then, by Lemma 4.14, \( \mathcal{M} \) and \( \mathcal{A} \) are equivalent. The proof for the converse direction is similar.

\[\blacksquare\]

Lemma 4.16. \( \models_{\text{LP}_{CS}} \varphi \) iff \( \mathcal{A}_{\text{LP}_{CS}}^{\varphi} \models \varphi \).

Proof. Follows from Lemma 4.15.

\[\blacksquare\]

Theorem 4.17 (Soundness and completeness). \( \vdash_{\text{LP}_{CS}} \varphi \) iff \( \mathcal{A}_{\text{LP}_{CS}}^{\varphi} \models \varphi \).

Proof. Follows from Theorem 2.5 and Lemma 4.16.

\[\blacksquare\]

5 LP-Algebras over arbitrary term Boolean algebras

In this section we impose a Boolean structure on proof terms of LP. The resulting logic is an extension of LP and is denoted by \( \text{LP}^B \).

Proof term and formulas of \( \text{LP}^B \) are constructed by the following mutual grammars:
\[
t, s ::= c | x | 0 | - t | t + s | t \cdot s | \mathcal{I},
\]
\[
\varphi, \psi ::= p | t \approx s | \bot | - \varphi | \varphi \lor \psi | t : \varphi.
\]
where \( c \in \text{Const} \), \( x \in \text{Var} \), and \( p \in \text{Prop} \). Define \( 1 := \neg 0 \) and \( s \circ t := -(\neg s + \neg t) \). Let \( \text{Term} \) and \( \text{Form} \) denote the set of all terms and the set of all formulas of \( \text{LP}^B \) respectively.

We briefly give a possible meaning of the Boolean operators on proof terms. Let \( s + t \) prove (or justify) everything that \( s \) or \( t \) proves (or justifies), and let \( - t \) prove (or justify) the negation of everything that \( t \) proves (or justifies). For example, let \( x \) denote “my observation of the table in front of me”. Thus \( x \) justifies the proposition “there is a table in front of me”. Then, for an arbitrary \( y \), the term \( x + y \) justifies again the proposition “there is a table in front of me”, and \( - x \) justifies, say, the absence of a table in front of me. As we will observe later \( 1 \) proves (or justifies) every theorem, and thus \( 0 \) proves (or justifies) the negation of theorems.

Note that we use constants from \( \text{Const} \) as proofs or justifications for axioms, and since \( 0 \) does not have this role it is not considered as a proof constant. Moreover, we keep use \( \cdot \) as a term operator formalizing the modus ponens rule, while use \( \circ \) as the dual of \( + \). Here, the term operator \( + \) is used for the combination of two justifications (as in the ordinary justification logics) as well as for the join operator in Boolean algebras of terms.

Let’s make the above argument more precise. In the following theorem we show how to construct a Boolean algebra out of the set of all finite sequences of formulas of a given logic \( L \).

Theorem 5.1. Given a logic \( L \) formulated as a Hilbert system, there is a Boolean algebra associated to the proofs of \( L \).
Proof. Consider a logic $L$ formulated as a Hilbert system. A proof of $\varphi$ from a set of assumptions $\Gamma$ in the logic $L$ is a finite sequence of formulas, say $(\varphi_1, \varphi_2, \ldots, \varphi_n)$, such that each formula $\varphi_i$ in the sequence is either an axiom instance of $L$ or is in $\Gamma$ or is obtained from previous formulas using rules, and in addition $\varphi_n = \varphi$. We consider only single-conclusion proofs here, i.e. every proof is of only one formula. Let $Pr$ be the set of all proofs from assumptions in $L$. Given $\sigma \in Pr$, we write $\sigma^L_\varphi$ if $\sigma$ is a proof of $\varphi$ using the set of assumptions $\Gamma$. Note that the sequence $(\varphi)$ denotes a proof of $\varphi$ from the assumption $\varphi$ (i.e. a proof for the sequent $\varphi \vdash_L \varphi$). For $\sigma^L_\varphi \in Pr$, let

$$[\sigma^L_\varphi] := \{ \tau^A_\psi \in Pr \mid \vdash_L \varphi \leftrightarrow \psi \}.$$ 

Let $[Pr] := \{ [\sigma^L_\varphi] \mid \sigma^L_\varphi \in Pr \}$. Now we define the Boolean operators on $[Pr]$ as follows:

$$
0_{Pr} := [(\bot)], \\
-\cdot_{Pr}[\sigma^L_\varphi] := [(-\varphi)], \\
[\sigma^L_\varphi] + \cdot_{Pr}[\tau^A_\psi] := [(\varphi \lor \psi)].
$$

Then, one can observe that $1_{Pr}$ is $[(\top)]$, and $[\sigma_\varphi] \odot \cdot_{Pr}[\tau_{\psi}]$ is $[(\varphi \land \psi)]$.

Now it is straightforward to show that $A_{Pr} = ([Pr], 0_{Pr}, -\cdot_{Pr}, +\cdot_{Pr})$ is a Boolean algebra. \qed

In the above proof note that if $\psi$ is an axiom of $L$, then $[(\psi)]$ plays a role similar to a proof constant, and if $\chi$ is a formula which is not a theorem of $L$, then $[(\chi)]$ plays the same role as a proof variable which has been already replaced by a proof of $\chi$ (from the assumption $\chi$). The Boolean algebra $A_{Pr}$ gives a generic example of Boolean algebras on terms in the sense that for every $L$ (formulated as a Hilbert system) one can construct a Boolean algebra associated to the proofs of $L$.

Remark 5.2. We continue the proof of Theorem 5.1 by defining the term operators $\cdot$ and $+$ on $A_{Pr}$. To this end we assume that the logic $L$ satisfies the Lifting Lemma 2.2. Define $\cdot_{Pr}$ as follows:

$$[\sigma^L_\varphi] \cdot_{Pr} [\tau^A_\psi] := \begin{cases} 
[\omega_{\varphi_1 \land \psi}] & \text{if } \varphi = \varphi_1 \rightarrow \varphi_2 \text{ and } \psi = \varphi_1 \\
0_{Pr} & \text{Otherwise.}
\end{cases}$$

In order to define $1_{Pr}$, we use the Lifting Lemma. Suppose that $\Gamma \vdash \varphi$ and $\sigma$ denotes this proof. By the Lifting Lemma we get

$$\vec{x} : \Gamma \vdash t(\vec{x}) : \varphi,$$

for some term $t(\vec{x}) \in Tm$ (here by $\vec{x} : \Gamma$ we mean $x_1 : \psi_1, \ldots, x_n : \psi_n$ where $\Gamma = \{ \psi_1, \ldots, \psi_n \}$). Now one possible definition of $1_{Pr}$ is as follows:

$$!_{Pr}[\sigma^L_\varphi] := [\tau_{t(\vec{x}) \varphi}],$$

where $\tau$ denotes the proof of the sequent $\vec{x} : \Gamma \vdash t(\vec{x}) : \varphi$. For example, if $\sigma = (p, p \rightarrow p \lor q, q)$, then

$$!_{Pr}[\sigma] = [(x : p, c : (p \rightarrow p \lor q), c \cdot x : q)],$$

where $x \in \text{Var}$ and $c \in \text{Const}$.

Since we have a Boolean algebra on proof terms, we have an order $\leq$ on proof terms. For $s, t \in \text{Term}$:

$$s \leq t \text{ iff } s + t \approx t.$$  

Another intuitive way to define $+_{Pr}$ is $[\sigma^L_\varphi] +_{Pr} [\tau^A_\psi] := [\pi^{\psi \lor \varphi}_x]$, where $\pi^{\psi \lor \varphi}_x = \sigma^L_\varphi \sharp \tau^A_\psi (\psi \rightarrow \varphi \lor \psi, \varphi \lor \psi)$ and $\sharp$ denotes the concatenation of sequences. Since $[\pi^{\psi \lor \varphi}_x] = [(\varphi \lor \psi)]$, the two definitions are equivalent.
For example, in the algebra $A$, we have

\[ \sigma \Gamma \phi \leq \tau \Delta \psi \] iff

\[ \sigma \Gamma \phi + Pr[\tau \Delta \psi] = \tau \Delta \psi \] iff

\[ \vdash L \phi \vee \psi \leftrightarrow \psi \] iff

\[ \vdash L \phi \rightarrow \psi. \]

Next we present an axiomatic formulation for $LP^B$. The set of axioms of $LP^B_{\emptyset}$ is an extension of that of $LP^B_{\emptyset}$ with the following axioms of the Boolean algebra for terms:

- **B1.** $s + t \approx t + s$  
  $s \circ t \approx t \circ s$

- **B2.** $s + (t + u) \approx (s + t) + u$  
  $s \circ (t \circ u) \approx (s \circ t) \circ u$

- **B3.** $s + 0 \approx s$  
  $s \circ 1 \approx s$

- **B4.** $s + (-s) \approx 1$  
  $s \circ (-s) \approx 0$

- **B5.** $s + (t \circ u) \approx (s + t) \circ (s + u)$  
  $s \circ (t + u) \approx (s \circ t) + (s \circ u)$

and the following axioms for equality:

- **Eq1.** $t \approx t$
- **Eq2.** $s \approx t \wedge \varphi[x/s] \rightarrow \varphi[x/t]$,

where $\varphi[x/s]$ denotes the result of substitution of $s$ for $x$ in $\varphi$.

The definition of constant specification and axiomatically appropriate constant specification is similar to that given in Section 4 (just let $JL$ be $LP^B$). Given a constant specification $CS$ for $LP^B$, the axiomatic system $LP^B_{CS}$ is obtained by adding the formulas of $CS$ as new axioms to the axiomatic system $LP^B_{\emptyset}$. In addition, the rules of $LP^B_{CS}$ are $MP$ and the following rule:

**Int.**

\[ \varphi \]

\[ 1 : \varphi \]

The rule $Int$ says that $1$ is a proof of every theorem. In this respect, $1$ can be called a universal proof. The rule $Int$ is enough to show that the internalization property holds in $LP^B_{CS}$ for arbitrary (not necessarily axiomatically appropriate) constant specification $CS$.

**Lemma 5.3 (Internalization via the universal proof).** Given an arbitrary constant specification $CS$, if $\vdash_{LP^B_{CS}} \varphi$, then $\vdash_{LP^B_{CS}} 1 : \varphi$.

**Proof.** Follows immediately from the rule $Int$.  

However, in contrast to the Internalization Lemma 2.1 the structure of the proof of theorems are not reflected by the universal proof $1$. Thus, we next show that the standard Internalization Lemma also holds for $LP^B$.

**Lemma 5.4 (Internalization).** Suppose that $CS$ is an axiomatically appropriate constant specification for $LP^B$. If $\vdash_{LP^B_{CS}} \varphi$, then there is a term $t \in Term$ such that $\vdash_{LP^B_{CS}} t : \varphi$.

**Proof.** The proof is similar to that of Lemma 2.1. The only new case is when $\varphi$ is obtained by the rule $Int$. Suppose that $\varphi = 1 : \psi$ is obtained by the rule $Int$ from $\psi$. Then, using axiom j4 and $MP$, we get $1 : 1 : \psi$. Thus, put $t := 1$.

\[ 1 \circ 1 \approx 1 \]
Lemma 5.5. The following rule is admissible in $L_{CS}^B$:

$$
\frac{s \approx t}{s : \varphi \leftrightarrow t : \varphi} \quad \text{EqTm}
$$

Proof. Follows from axiom Eq2. \hfill \Box

We immediately get the following result.

Lemma 5.6. The following are theorems of $L_{CS}^B$:

1. $s + t : \varphi \leftrightarrow t + s : \varphi$
2. $s + (t + u) : \varphi \leftrightarrow (s + t) + u : \varphi$
3. $s + 0 : \varphi \leftrightarrow s : \varphi$
4. $s + (-s) : \varphi \leftrightarrow 1 : \varphi$
5. $s + (t \circ u) : \varphi \leftrightarrow (s + t) \circ (s + u) : \varphi$

Proof. Follows from the axioms of Boolean algebra, B1-B5, and the admissible rule EqTm. \hfill \Box

5.1 Full $L_{CS}^B$ algebras

Next we set semantics for the logic $L_{CS}^B$. Unlike the full $L_{CS}$ algebras that have been defined over the set of proof terms $Tm$, here we define algebras over certain extensions of a Boolean algebra. We begin with a Boolean algebra $(T, 0_T, -T, +T)$, and then we extend it by the operators $\cdot_T : T \times T \rightarrow T$ and $!_T : T \rightarrow T$ to $T$. The members of $T$ are considered as denotations of proof terms. Then, for every $\alpha \in T$ we have a function $\Box_{\alpha} : \text{Form} \rightarrow A$ in the algebra. Thus, the difference, as compared to full $L_P$ algebras in Section 3, is that now for every member of $\alpha \in T$ we have a function $\Box_{\alpha}$ in the algebra. As opposed to full $L_P$ algebras in which the number of functions $\Box_{t}$, for $t \in Tm$, were countably infinite, since $T$ may be finite, the number of functions $\Box_{\alpha}$ may be finite as well.

Definition 5.7. Let $T = (T, 0_T, -T, +T, \cdot_T, \cdot_T, !_T)$, where $(T, 0_T, -T, +T)$ is a Boolean algebra, and $\cdot_T$ and $!_T$ are two operators on $T$ such that

$$
\cdot_T : T \times T \rightarrow T \quad \text{and} \quad !_T : T \rightarrow T.
$$

Given a constant specification $CS$ for $L_{CS}^B$, a pre $L_{CS}^B$ algebra is a tuple $A = (A, 0, \ominus, \oplus, \Box_{\alpha}, I)_{\alpha \in T}$ such that $(A, 0, \ominus, \oplus)$ is a Boolean algebra, $I : \text{Const} \cup \{0\} \rightarrow T$ is an interpretation such that

$$I(0) = 0_T,
$$

and operators $\Box_{\alpha} : \text{Form} \rightarrow A$, for $\alpha \in T$, satisfy the following conditions. For all $a, b \in A$, and all $\alpha, \beta \in T$:

- $\text{Al-Apply-LP}^B_{CS}$: $\Box_{\alpha}(\varphi \rightarrow \psi) \otimes \Box_{\beta}(\varphi) \leq \Box_{\alpha \cdot \beta}(\psi)$,
- $\text{Al-Sum-LP}^B_{CS}$: $\Box_{\alpha}(\varphi) \oplus \Box_{\beta}(\varphi) \leq \Box_{\alpha + \beta}(\varphi)$,
- $\text{Al-1-LP}^B_{CS}$: $\Box_{1_T}(\varphi) = 1$, where $\Gamma_{\text{LP}^B_{CS}}$,
- $\text{Al-CS-LP}^B_{CS}$: $\Box_{I(0)}(\varphi) = 1$, where $c : \varphi \in CS$.

Definition 5.8. Given $T = (T, 0_T, -T, +T, \cdot_T, !_T)$ and a function $v : \text{Var} \rightarrow T$, for every $t \in \text{Term}$ define $t^v_T$ as follows:

1. $x^v_T := v(x)$ for $x \in \text{Var}$ and $c^v_T := I(c)$ for $c \in \text{Const} \cup \{0\}$,
2. $(s \cdot t)^v_T := (s^v_T) \cdot_T (t^v_T)$ where $\cdot \in \{+, \cdot\}$,
3. $(s t)^v_T := \cdot_T (s^v_T)$ where $\cdot \in \{-, !\}$.
Note that $t_f \in T$, for every $t \in \text{Term}$.

**Definition 5.9.** A valuation is a function $\theta : \text{Prop} \to A$. Given a function $v : \text{Var} \to T$, the assignment $\bar{\theta}_v : \text{Form} \to A$ on $A$ is defined as follows:

$$
\bar{\theta}_v(p) = \theta(p) \\
\bar{\theta}_v(\bot) = 0 \\
\bar{\theta}_v(\neg \varphi) = \ominus \bar{\theta}_v(\varphi) \\
\bar{\theta}_v(\varphi \lor \psi) = \bar{\theta}_v(\varphi) \oplus \bar{\theta}_v(\psi) \\
\bar{\theta}_v(t : \varphi) = \Box t_f(\varphi) \\
\bar{\theta}_v(s \approx t) = \begin{cases} 1 & \text{if } s_f = t_f \\ 0 & \text{if } s_f \neq t_f. \end{cases}
$$

**Definition 5.10.** A full $\text{LP}^B_{\text{CS}}$ algebra $A = (A, 0, \ominus, \oplus, \Box, I)_a \in T$ is a pre $\text{LP}^B_{\text{CS}}$ algebra that satisfies the following condition. For all $\varphi \in \text{Form}$, all $t \in \text{Term}$, all $a \in T$, all functions $v : \text{Var} \to T$, and all valuation $\theta : \text{Prop} \to A$:

- Al-j4-$\text{LP}^B_{\text{CS}}$. $\Box_a(\varphi) \leq \Box_{t \varphi}(t : \varphi)$,
- Al-jT-$\text{LP}^B_{\text{CS}}$. $\Box_a(\varphi) \leq \bar{\theta}_v(\varphi)$.

The class of all full $\text{LP}^B_{\text{CS}}$ algebras with singleton $\nabla = \{1\}$ is denoted by $A^\text{full}_{\text{LP}^B_{\text{CS}}}$.

**Definition 5.11** (Validity). Let $A = (A, 0, \ominus, \oplus, \Box, I)_a \in T$ be an $\text{LP}^B_{\theta}$ algebra (with $\nabla = \{1\}$) and $\varphi \in \text{Form}$.

- A formula $\varphi$ is true in the algebra $A$ under the valuation $\theta$, denoted by $A \models_{\theta} \varphi$, if $\bar{\theta}_v(\varphi) = 1$, for every function $v : \text{Var} \to T$.
- A formula $\varphi$ is true in the algebra $A$, denoted by $A \models \varphi$, if $A \models_{\theta} \varphi$ for every valuation $\theta$ on $A$.
- A formula $\varphi$ is valid in the class $A^\text{full}_{\text{LP}^B_{\text{CS}}}$, denoted by $A^\text{full}_{\text{LP}^B_{\text{CS}}} \models \varphi$, if $A \models \varphi$ for every algebra $A \in A^\text{full}_{\text{LP}^B_{\text{CS}}}$.

### 5.2 Completeness of $\text{LP}^B_{\text{CS}}$

In order to prove completeness of $\text{LP}^B_{\text{CS}}$ we construct two Tarski-Lindenbaum algebras; one out of $\text{Form}$ and the other out of $\text{Term}$.

**Definition 5.12.** Given $\varphi \in \text{Form}$ and $t \in \text{Term}$, let

$$
[\varphi] := \{ \psi \in \text{Form} \mid \text{LP}^B_{\text{CS}} \varphi \leftrightarrow \psi \},
$$

$$
[t] := \{ s \in \text{Term} \mid \text{LP}^B_{\text{CS}} s \approx t \}.
$$

First define $T = ([\text{Term}], 0_T, -_T, +_T, \cdot_T, !_T)$ as follows:

- $[\text{Term}] := \{ [t] \mid t \in \text{Term} \}$,
- $0_T := [a]$,
- $-_T[t] := [-t]$,
- $[s] +_T [t] := [s + t]$,
- $[s] \cdot_T [t] := [s \cdot t]$,
- $!_T[t] := [t]$. 

17
The Tarski-Lindenbaum algebra for $\text{LP}_{\text{CS}}^B$ is defined as follows:

$$A_{\text{LP}_{\text{CS}}^B} := ([\text{Form}], 0, \otimes, \oplus, [r], I)_{t \in \text{Term}},$$

where

$$[\text{Form}] := \{[\varphi] \mid \varphi \in \text{Form}\},$$

$$\Box_t(\varphi) := [t : \varphi],$$

$$I(c) := [c],$$

for $c \in \text{Const} \cup \{0\},$

and $0$, $\odot$, and $\oplus$ are defined similar to Definition 3.4. Let $\nabla = \{[\top]\}$.

**Lemma 5.13.** The Tarski-Lindenbaum algebra $A_{\text{LP}_{\text{CS}}^B}^I$ is a pre $\text{LP}_{\text{CS}}^B$ algebra.

**Proof.** We only check $Al$-Appl-$\text{LP}_{\text{CS}}^B$ and $Al$-1-$\text{LP}_{\text{CS}}^B$. The proof of other cases is simple.

- Condition $Al$-Appl-$\text{LP}_{\text{CS}}^B$.

$$\Box_{[s]}(\varphi \to \psi) \otimes \Box_{[t]}(\varphi) = [s : (\varphi \to \psi)] \otimes [t : \varphi]$$

$$= [s : (\varphi \to \psi) \land t : \varphi]$$

$$\leq [s \cdot t : \psi]$$

$$= \Box_{s \cdot t}(\psi)$$

$$= \Box_{[s] \cdot [t]}(\psi).$$

- Condition $Al$-1-$\text{LP}_{\text{CS}}^B$. Suppose that $\vdash \text{LP}_{\text{CS}}^B \varphi$. Then

$$\Box_{[1]}(\varphi) = [1 : \varphi] = [\top].$$

**Theorem 5.14 (Soundness and completeness).** $\vdash \text{LP}_{\text{CS}}^B \varphi$ iff $A_{\text{LP}_{\text{CS}}^B}^\text{full} \models \varphi$.

**Proof.** Soundness is straightforward. For completeness, define $\theta$ and $v$ as follows:

$$\theta(p) := [p], \quad \text{for } p \in \text{Prop},$$

$$v(x) := [x], \quad \text{for } x \in \text{Var},$$

and let $A_{\text{LP}_{\text{CS}}^B} := ([\text{Form}], 0, \otimes, \oplus, [r], I)_{t \in \text{Term}}$ be the Tarski-Lindenbaum algebra for $\text{LP}_{\text{CS}}^B$. Then, by induction on the complexity of $t$, one can show that

$$t_I = [t].$$

Then we show the Truth Lemma:

$$\tilde{\theta}_v(\varphi) = [\varphi].$$

The proof is by induction on the complexity of $\varphi$. We only check two cases:

- Case $\varphi = t : \psi$:

$$\tilde{\theta}_v(t : \varphi) = \Box_I(\varphi) = \Box_{[t]}(\varphi) = [t : \varphi].$$

- Case $\varphi = s \approx t$:

$\theta_v(s \approx t) = [\top]$ if $s \approx t$ iff $s \approx t$ if $\vdash \text{LP}_{\text{CS}}^B s \approx t$ iff $s \approx t = [\top]$. Thus, $\tilde{\theta}_v(s \approx t) = [s \approx t]$.

Again completeness follows from the Truth Lemma. □
5.3 Bi-representation theorem for full \( \text{LP}_{CS}^B \) algebras

In this section, we present a counterpart of the Stone’s representation theorem in this setting by means of certain isomorphisms for full \( \text{LP}_{CS}^B \)-algebras.

**Definition 5.15.** Let \( \mathcal{A} = (A, 0, \ominus, \oplus, \Box, I)_{\alpha \in T} \) and \( \mathcal{A}' = (A', 0', \ominus', \oplus', \Box', I')_{\alpha \in T'} \) be two full \( \text{LP}_{CS}^B \) algebras over \( T = (T, 0_T, -T, +T, \cdot_T, \mid_T) \) and \( T' = (T', 0_{T'}, -T', +T', \cdot_{T'}, \mid_{T'}) \) respectively. A bi-isomorphism between \( \mathcal{A} \) and \( \mathcal{A}' \) is a pair of Boolean isomorphisms \( (f, g) \) such that \( f : (A, 0, \ominus, \oplus) \to (A', 0', \ominus', \oplus') \) and \( g : (T, 0_T, -T, +T) \to (T', 0_{T'}, -T', +T') \) satisfying the following conditions:

\[
\begin{align*}
  f(\Box \alpha (\varphi)) &= \Box' g(\alpha) (\varphi), & \text{for } \alpha \in T \text{ and } \varphi \in \text{Form}, \\
g(I(c)) &= I'(c), & \text{for } c \in \text{Const} \cup \{0\}, \\
g(\alpha \cdot_T \beta) &= g(\alpha) \cdot_{T'} g(\beta), & \text{for } \alpha, \beta \in T, \\
g(\mid_T \alpha) &= \mid_{T'} g(\alpha), & \text{for } \alpha \in T.
\end{align*}
\]

**Definition 5.16.** Given two set algebras \( (A, \emptyset, \setminus, \cup) \) and \( (B, \emptyset, \setminus, \cup, \cdot, \beta, \lambda) \), a full \( \text{LP}_{CS}^B \) set algebra is a structure

\[ \mathcal{A} = (A, \emptyset, \setminus, \cup, \Box, I)_{\alpha \in B} \]

over \( B = (B, \emptyset, \setminus, \cup, \cdot, \beta, \lambda) \) satisfying the conditions of Definitions 5.7 and 5.10.

In contrast with the standard representation theorems we construct two isomorphisms for a given full \( \text{LP}_{CS}^B \) algebra \( \mathcal{A} \) over \( T \): One for the algebra \( \mathcal{A} \) and one for the algebra \( T \). This leads to a bi-isomorphism of full \( \text{LP}_{CS}^B \) algebras.

**Theorem 5.17 (Bi-Representation Theorem).** Every full \( \text{LP}_{CS}^B \) algebra is bi-isomorphic to a full \( \text{LP}_{CS}^B \) set algebra.

**Proof.** Suppose that \( \mathcal{A} = (A, 0, \ominus, \oplus, \Box, I)_{\alpha \in T} \) is a full \( \text{LP}_{CS}^B \) algebra over \( T = (T, 0_T, -T, +T, \cdot_T, \mid_T) \). Considering the Boolean algebra \( (T, 0_T, -T, +T) \), by the Stone representation theorem, there exists an isomorphism \( g \) from \( (T, 0_T, -T, +T) \) to a set algebra \( (P, \emptyset, \setminus, \cup) \). Next, we shall extend \( g \) to an isomorphism from \( T = (T, 0_T, -T, +T, \cdot_T, \mid_T) \) to an extension of \( (P, \emptyset, \setminus, \cup) \).

Let \( \alpha', \beta' \in P \). There are \( \alpha, \beta \in T \) such that \( g(\alpha) = \alpha' \) and \( g(\beta) = \beta' \). Now define term operators \( \cdot_P \) and \( \mid_P \) on \( P \) as follows:

\[
\begin{align*}
  \alpha' \cdot_P \beta' &:= g(\alpha \cdot_T \beta), \\
  \mid_P \alpha' &:= g(\mid_T \alpha).
\end{align*}
\]

Since \( \cdot_T \) and \( \mid_T \) are well-defined, it follows that \( \cdot_P \) and \( \mid_P \) are well-defined. Now it is obvious that \( g \) is an isomorphism from \( T = (T, 0_T, -T, +T, \cdot_T, \mid_T) \) to \( (P, \emptyset, \setminus, \cup, \cdot_P, \mid_P) \). Let \( P = (P, \emptyset, \setminus, \cup, \cdot_P, \mid_P) \).

Next, by the Stone representation theorem, the Boolean algebra \( (A, 0, \ominus, \oplus) \) is isomorphic to a set algebra \( (B, \emptyset, \setminus, \cup) \) via an isomorphism \( f \). Define \( \Box_{P} \) (for \( \alpha' \in P \)) and the interpretation \( I' \) as follows:

\[
\begin{align*}
  \Box_{P} (\alpha'(\varphi)) &= f(\Box \alpha (\varphi)), & \text{for } \alpha \in T, \\
  I'(c) &= g(I(c)), & \text{for } c \in \text{Const} \cup \{0\}.
\end{align*}
\]

Let \( \mathcal{B} = (B, \emptyset, \setminus, \cup, \Box_{P}, I')_{\alpha' \in P} \). Then, we show that \( \mathcal{B} \) is a full \( \text{LP}_{CS}^B \) algebra. To this end we only check conditions \( Al\text{-Appl}-\text{LP}_{CS}^B, Al\text{-CS}-\text{LP}_{CS}^B, Al\text{-j4}-\text{LP}_{CS}^B \) and \( Al\text{-jT}-\text{LP}_{CS}^B \) from Definitions 5.7 and 5.10 (other cases can be verified similarly).
For $Al$-$\text{App}-LP_{CS}^{B}$, suppose that $\alpha, \beta \in T$:
\[
\square_{g(\alpha)}(\varphi \to \psi) \otimes \square_{g(\beta)}(\varphi) =
\]
\[
f(\square_{\alpha}(\varphi \to \psi)) \otimes f(\square_{\beta}(\varphi)) =
\]
\[
f(\square_{\alpha}(\varphi \to \psi) \otimes \square_{\beta}(\varphi)) \leq
\]
\[
f(\square_{\alpha \cdot \tau \beta}(\psi)) = \square_{g(\alpha) \cdot p_{\tau} g(\beta)}(\psi).
\]

For $Al$-$\text{CS}-LP_{CS}^{B}$, suppose $c : \varphi \in \text{CS}$, then we have
\[
\square'_{I(\epsilon)}(\varphi) = \square'_{g(I(\epsilon))}(\varphi) = f(\square_{I(\epsilon)}(\varphi)) = f(1) = B.
\]

For $Al$-$j4$-$LP_{CS}^{B}$, suppose that $t \in \text{Term}$ is an arbitrary term. Then, for $\alpha \in T$, we have
\[
\square'_{g(\alpha)}(\varphi) = f(\square_{\alpha}(\varphi)) \leq f(\square_{\tau \alpha}(t : \varphi)) = \square'_{g(t(\tau \alpha))}(t : \varphi) = \square'_{p g(\alpha)}(t : \varphi).
\]

For $Al$-$jT$-$LP_{CS}^{B}$, we need some preliminary results. Given a function $v : \text{Var} \to T$, define the function $v' : \text{Var} \to P$ by
\[
v'(x) := g(v(x)).
\]

It is not difficult to show that for every $t \in \text{Term}$ we have
\[
t'^{v}_{I'} = g(t^{v}_{I}).
\]

The proof involves a routine induction on the complexity of $t$. The base cases, where $t \in \text{Var}$ or $t \in \text{Const} \cup \{0\}$, follows from (9) and (10). The proof for the induction steps is straightforward.

Now suppose that $\theta' : \text{Prop} \to B$ is an arbitrary valuation on $B$. Then, $\theta = f^{-1} \circ \theta'$ would be a valuation on $A$. Then, by induction on the complexity of the formula $\varphi$, one can show that
\[
f(\tilde{\theta}_{v}(\varphi)) = \tilde{\theta}'_{v'}(\varphi).
\]

The base case follows from $\theta = f^{-1} \circ \theta'$. For the induction step, we only show the case where $\varphi = t : \psi$. We have
\[
f(\tilde{\theta}_{v}(t : \psi)) = f(\square_{I'(\epsilon)}(\psi)) = \square'_{g(I'(\epsilon))}(\psi) = \square'_{I'}(\psi) = \tilde{\theta}'_{v'}(t : \psi).
\]

Thus, $f \circ \tilde{\theta}_{v} = \tilde{\theta}'_{v'}$. Finally, we verify the condition $Al$-$jT$-$LP_{CS}^{B}$. We have
\[
\square'_{g(\alpha)}(\varphi) = f(\square_{\alpha}(\varphi)) \leq f(\tilde{\theta}_{v}(\varphi)) = \tilde{\theta}'_{v'}(\varphi).
\]

The above inequality follows from the fact that $f$ is an isomorphism and $\square_{\alpha}(\varphi) \leq \tilde{\theta}_{v}(\varphi)$.

It is obvious that $B$ is a set algebra, and hence it is a full $LP_{CS}^{B}$ set algebra. \hfill $\square$

**Theorem 5.18.** $LP_{CS}^{B}$ is complete with respect to the class of all full $LP_{CS}^{B}$ set algebras.

**Proof.** Follows from the Bi-representation Theorem 5.17 and the Completeness Theorem 5.14 \hfill $\square$

### 6 LP$_{CS}$ algebras over arbitrary polynomial Boolean algebras

In this section, similar to Section 5, we assume that proof terms constitute a Boolean algebra. We give another algebraic semantics for the logic LP$^{B}$. In contrast to the results of Section 5 in this section we use a Boolean algebra on polynomials. More precisely, while in Section 5 the truth of a formula in an algebra is defined directly using functions $v : \text{Var} \to T$, where $T$ is a set of terms, in this section we make use of polynomials with coefficients in $T$. 

20
Let \( T = (T, 0_T, \neg T, +T, \cdot T, !T) \), where \((T, 0_T, \neg T, +T)\) is a Boolean algebra, and \( \cdot T : T \times T \rightarrow T \) and \( !T : T \rightarrow T \). Let \( T[\text{Var}] \) denote the set of polynomials with variables in \( \text{Var} \) and coefficients in \( T \). Two polynomials \( f, g \in T[\text{Var}] \) are equal, denoted by \( f \equiv g \), if and only if for every \( v : \text{Var} \rightarrow T \), \( v(f) = v(g) \), where by \( v(f) \) we mean replacing every occurrence of every variable in \( f \) by its image under \( v \). It follows from the equality defined above that the Boolean structure on \( T \) induces a Boolean structure on \( T[\text{Var}] \). Thus, we obtain a Boolean algebra \((T[\text{Var}], 0_T, \neg T[\text{Var}], +T[\text{Var}], \cdot T[\text{Var}], !T[\text{Var}] )\). Next we add the operators

\[ \cdot T[\text{Var}] : T[\text{Var}] \times T[\text{Var}] \rightarrow T[\text{Var}] \quad \text{and} \quad !T[\text{Var}] : T[\text{Var}] \rightarrow T[\text{Var}] \]

to this Boolean algebra in such a way that \( \cdot T[\text{Var}] \) and \( !T[\text{Var}] \) are extensions of \( \cdot T \) and \( !T \) respectively. For example, one can define \( \cdot T[\text{Var}] \) and \( !T[\text{Var}] \) as follows:

\[
\alpha \cdot T[\text{Var}] \beta := \begin{cases} 
\alpha \cdot T \beta, & \text{if } \alpha, \beta \in T, \\
\alpha, & \text{Otherwise},
\end{cases}
\]

(11)

and

\[
!T[\text{Var}] \alpha := \begin{cases} 
!T \alpha, & \text{if } \alpha \in T, \\
\alpha, & \text{Otherwise}.
\end{cases}
\]

(12)

By \( T[\text{Var}] \) we denote the structure

\[
(T[\text{Var}], 0_T, \neg T[\text{Var}], +T[\text{Var}], \cdot T[\text{Var}], !T[\text{Var}]),
\]

where \( \cdot T[\text{Var}] \) and \( !T[\text{Var}] \) are arbitrary extensions of \( \cdot T \) and \( !T \) respectively. From now on when we consider \( T[\text{Var}] \), for some \( T \), we assume that \( \cdot T[\text{Var}] \) and \( !T[\text{Var}] \) are extensions of \( \cdot T \) and \( !T \) respectively.

**Definition 6.1.** Let \( T = (T, 0_T, \neg T, +T, \cdot T, !T) \), where \((T, 0_T, \neg T, +T)\) is a Boolean algebra, and \( \cdot T : T \times T \rightarrow T \) and \( !T : T \rightarrow T \), and let \( T[\text{Var}] = (T[\text{Var}], 0_T, \neg T[\text{Var}], +T[\text{Var}], \cdot T[\text{Var}], !T[\text{Var}] ) \). Given a constant specification \( \text{CS} \) for \( \text{LP}^B \), a pre polynomial \( \text{LP}_{\text{CS}}^B \) algebra

\[
\mathcal{A} = (A, 0, \emptyset, \oplus, \Box, I)_{\alpha \in T[\text{Var}]}
\]

is defined similar to pre \( \text{LP}_{\text{CS}}^B \) algebra (Definition [def:LP_CS]), where \( I : \text{Const} \cup \{0\} \rightarrow T \) is an interpretation such that \( I(0) = 0_T \), and operators \( \Box \alpha : \text{Form} \rightarrow A \), for \( \alpha \in T[\text{Var}] \), satisfy the following conditions. For all \( a, b \in A \), and all \( \alpha, \beta \in T[\text{Var}] \):

- **Al-App-LP^B CS**: \( \Box \alpha (\varphi \rightarrow \psi) \otimes \Box \beta (\varphi) \leq \Box \alpha \cdot T[\text{Var}] \beta (\psi) \),
- **Al-Sum-LP^B CS**: \( \Box \alpha (\varphi) \oplus \Box \beta (\varphi) \leq \Box \alpha + T[\text{Var}] \beta (\varphi) \),
- **Al-1-LP^B CS**: \( \Box 1_T (\varphi) = 1 \), where \( 1_{\text{LP}_{\text{CS}}^B} \varphi \),
- **Al-CS-LP^B CS**: \( \Box I(c) (\varphi) = 1 \), where \( c : \varphi \in \text{CS} \).

**Definition 6.2.** The interpretation \( I \) can be extended to \( \tilde{I} : \text{Term} \rightarrow T[\text{Var}] \) such that for every \( t \in \text{Term} \), \( \tilde{I}(t) \) is defined as follows:

1. \( \tilde{I}(x) := x \) for \( x \in \text{Var} \) and \( \tilde{I}(c) := I(c) \) for \( c \in \text{Const} \cup \{0\} \).
2. \( \tilde{I}(s \ast t) := \tilde{I}(s) \ast T[\text{Var}] \tilde{I}(t) \) where \( \ast \in \{+, \cdot\} \).
3. \( \tilde{I}(s \ast s) := \ast T[\text{Var}] \tilde{I}(s) \) where \( \ast \in \{-, !\} \).
Definition 6.3. A valuation is a function $\theta : \text{Prop} \to A$. The assignment $\tilde{\theta} : Fm \to A$ on $A = (A, 0, \ominus, \oplus, \Box, I)_{\alpha \in \mathcal{T}[\mathcal{V}_\mathcal{A}]}$ is defined as follows:

$$
\tilde{\theta}(p) = \theta(p),
\tilde{\theta}() = 0,
\tilde{\theta}(\varphi \rightarrow \psi) = \tilde{\theta}(\varphi) \Rightarrow \tilde{\theta}(\psi),
\tilde{\theta}(t : \varphi) = \Box \tilde{I}(t)(\varphi),
\tilde{\theta}(s \approx t) = \begin{cases} 1 & \text{if } \tilde{I}(s) = \tilde{I}(t) \\ 0 & \text{if } \tilde{I}(s) \neq \tilde{I}(t). \end{cases}
$$

Definition 6.4. A polynomial $\mathsf{LP}^B_{\mathcal{CS}}$ algebra $A = (A, 0, \ominus, \oplus, \Box, I)_{\alpha \in \mathcal{T}[\mathcal{V}_\mathcal{A}]}$ is a pre polynomial $\mathsf{LP}^B_{\mathcal{CS}}$ algebra that satisfies the following condition. For all $\varphi \in \text{Form}$, all $t \in \text{Term}$, all $\alpha \in \mathcal{T}[\mathcal{V}_\mathcal{A}]$, and all valuation $\theta : \text{Prop} \to A$:

\begin{align*}
\text{Al-j4-LP}^B_{\mathcal{CS}}. & \quad \Box \tilde{I}(t)(\varphi) \leq \bigtriangleup \mathcal{A}_t[t](t : \varphi), \\
\text{Al-jT-LP}^B_{\mathcal{CS}}. & \quad \Box \alpha(\varphi) \leq \tilde{\theta}(\varphi).
\end{align*}

The class of all polynomial $\mathsf{LP}^B_{\mathcal{CS}}$ algebras with singleton $\nabla = \{1\}$ is denoted by $\mathcal{A}_{\mathsf{LP}^B_{\mathcal{CS}}}^{\text{poly}}$.

Definition 6.5 (Validity). A formula $\varphi$ is true in the algebra $A = (A, 0, \ominus, \oplus, \Box, I)_{\alpha \in \mathcal{T}[\mathcal{V}_\mathcal{A}]}$, denoted by $A \models \varphi$, if $\tilde{\theta}(\varphi) = 1$ for every $\theta$. A formula $\varphi$ is valid in the class $\mathcal{A}_{\mathsf{LP}^B_{\mathcal{CS}}}^{\text{poly}}$, denoted by $\mathcal{A}_{\mathsf{LP}^B_{\mathcal{CS}}}^{\text{poly}} \models \varphi$, if $A \models \varphi$ for every algebra $A \in \mathcal{A}_{\mathsf{LP}^B_{\mathcal{CS}}}^{\text{poly}}$.

6.1 Completeness of $\mathsf{LP}^B_{\mathcal{CS}}$

Now we show that $\mathsf{LP}^B_{\mathcal{CS}}$ is characterized by the class $\mathcal{A}_{\mathsf{LP}^B_{\mathcal{CS}}}^{\text{poly}}$.

Definition 6.6. Given $\varphi \in \text{Form}$ and $t \in \text{Term}$, let

$$
[\varphi] := \{ \psi \in \text{Form} \mid \models_{\mathsf{LP}^B_{\mathcal{CS}}} \varphi \leftrightarrow \psi \},
[\mathcal{T}] := \{ s \in \text{Term} \mid \models_{\mathsf{LP}^B_{\mathcal{CS}}} s \approx t \}.
$$

First define $\mathcal{T} = ([\text{Term}], +_\tau, -_\tau, 0_\tau, \cdot_\tau, !_\tau)$ as follows:

$$
[\text{Term}] := \{ [t] \mid t \in \text{Term} \},
[s] +_\tau [t] := [s + t],
-\tau [t] := [-t],
0_\tau := [0],
[s] \cdot_\tau [t] := [s \cdot t],
!_\tau [t] := [t].
$$

Now consider the structure $[\text{Term}]$|$\mathcal{V}_\mathcal{A}$]. The definition of interpretation $I$ is similar to that is given in Definition 5.12 for the Tarski-Lindenbaum algebra of $\mathsf{LP}^B_{\mathcal{CS}}$, as follows:

$$
I(c) := [c], \text{ for } c \in \text{Const} \cup \{0\}.
$$

Note that the function $\tilde{I} : \text{Term} \to [\text{Term}]$|$\mathcal{V}_\mathcal{A}$] is surjective.
The Tarski-Lindenbaum algebra for $\mathbf{LP}_{\mathbf{CS}}^B$ is defined as follows:

$$\mathcal{A}_{\mathbf{LP}_B}^p := ([\text{Form}], 0, \oplus, \ominus, \sqcap, I)_{\alpha \in \text{Term}|\text{Var}}$$

where $\oplus, \ominus, \text{ and } 0$ are defined similar to Definition 3.4 and

$$[\text{Form}] := \{[\varphi] \mid \varphi \in \text{Form}\},$$

$$\square_{\alpha}(\varphi) := [t : \varphi], \quad \text{where } \alpha = \tilde{I}(t) \text{ for some } t \in \text{Term}.$$ 

Let $\nabla = \{[\top]\}$.

**Lemma 6.7.** Let $\mathcal{A}_{\mathbf{LP}_B}^p := ([\text{Form}], 0, \oplus, \ominus, \sqcap, I)_{\alpha \in \text{Term}|\text{Var}}$ be the Tarski-Lindenbaum algebra for $\mathbf{LP}_{\mathbf{CS}}^B$ and let $v : \text{Var} \to [\text{Term}]$ be a function. Then, for every $t \in \text{Term}$:

$$v(\tilde{I}(t)) = t^*_v.$$

Where $t^*_v$ is defined as in Definition 5.8.

**Proof.** The proof is by induction on the complexity of $t$. The base cases $t \in \text{Var}$ and $t \in \text{Const} \cup \{0\}$ are trivial. For the induction step we only show the case that $t = s$.

$$v(\tilde{I}(s)) = v[I_{\text{Var}}(\tilde{I}(s))] = I_{\text{Var}}[v(\tilde{I}(s))] = I_{\text{Var}}[v(s)] = (s^*)_v.$$ 

The third equation follows from the fact that $!_{\text{Var}}$ is an extension of $!_{\text{Form}}$ and $v(\tilde{I}(s)) \in [\text{Var}]$.

**Corollary 6.8.** The operators $\square_{\alpha}$, for each $\alpha \in [\text{Term}]|\text{Var}$, in $\mathcal{A}_{\mathbf{LP}_B}^p$ are well-defined.

**Proof.** We first show that $\square_{\alpha}$ is well-defined on $[\text{Term}]|\text{Var}$. Suppose that $s, t \in \tilde{I}^{-1}(\alpha)$, for some $s, t \in \text{Term}$. Then, $\tilde{I}(s) \equiv \tilde{I}(t)$. Hence, $v(\tilde{I}(s)) = v(\tilde{I}(t))$, for every $v : \text{Var} \to [\text{Term}]$. By Lemma 6.7 we get $s^*_v = t^*_v$, for every $v : \text{Var} \to [\text{Term}]$. Now define $v : \text{Var} \to [\text{Term}]$ as follows:

$$v(x) = [x].$$

For this function, we have $s^*_v = t^*_v$. By 13 in the proof of Theorem 5.14, we obtain $[s] = [t]$, and hence $\vdash_{\mathbf{LP}_B} s \equiv t$. Thus, $\vdash_{\mathbf{LP}_B} t : \varphi \leftrightarrow s : \varphi$, for every $\varphi \in \text{Form}$. Therefore $[t : \varphi] = [s : \varphi]$, and then $\square_{\tilde{I}(s)}(\varphi) = \square_{\tilde{I}(t)}(\varphi)$.

Let $\alpha = \tilde{I}(t)$, for some $t \in \text{Term}$. We now show that $\square_{\alpha}$ is well-defined on $\text{Form}$. Suppose that $\varphi = \psi$. Then, $\vdash_{\mathbf{LP}_B} t : \varphi \leftrightarrow t : \psi$. Thus, $[t : \varphi] = [t : \psi]$. Hence, $\square_{\alpha}(\varphi) = \square_{\alpha}(\psi)$. □

**Lemma 6.9.** The Tarski-Lindenbaum algebra $\mathcal{A}_{\mathbf{LP}_B}^p$ is a polynomial $\mathbf{LP}_{\mathbf{CS}}^B$ algebra.

**Proof.** We only check $\text{Al-App} \text{-} \mathbf{LP}_{\mathbf{CS}}^B$ and $\text{Al-1} \text{-} \mathbf{LP}_{\mathbf{CS}}^B$. The proof of other cases is simple.

- Condition $\text{Al-App} \text{-} \mathbf{LP}_{\mathbf{CS}}^B$.

$$\square_{\tilde{I}(s)}([\varphi] \Rightarrow [\psi]) \otimes \square_{\tilde{I}(t)}([\varphi])$$

$$= \square_{\tilde{I}(s)}([\varphi \Rightarrow \psi]) \otimes \square_{\tilde{I}(t)}([\varphi])$$

$$= [s : (\varphi \Rightarrow \psi)] \otimes [t : \varphi]$$

$$= [s : (\varphi \Rightarrow \psi) \land t : \varphi]$$

$$\leq [s \cdot t : \psi]$$

$$= \square_{\tilde{I}(s \cdot t)}[\psi]$$
– Condition \( \text{Al-1-LP}_{\text{CS}} \). Suppose that \( \vdash_{\text{LP}_{\text{CS}}} \varphi \). Thus, by the rule Int, \( \vdash_{\text{LP}_{\text{CS}}} 1 : \varphi \). Then
\[
\Box_{1_T}(\varphi) = \Box_{1_{[1]}}(\varphi) = [1 : \varphi] = [\top].
\]

\[\square_{1_T}(\varphi) = \square_{1_{[1]}}(\varphi) = [1 : \varphi] = [\top].\]

\(\square\)

**Theorem 6.10 (Soundness and completeness).** \( \vdash_{\text{LP}_{\text{CS}}} \varphi \) iff \( \mathcal{A}_{\text{LP}_{\text{CS}}}^{\text{poly}} \models \varphi \).

**Proof.** Soundness is straightforward. For completeness, define \( \theta \) as follows:
\[
\theta(p) := [p], \quad p \in \text{Prop}.
\]

Then we show the Truth Lemma. For every \( \varphi \in \text{Form} \):
\[
\bar{\theta}(\varphi) = [\varphi].
\]

The proof is by induction on the complexity of \( \varphi \). We only check two cases:

- Case \( \varphi = t : \psi \):
  \[
  \bar{\theta}(t : \varphi) = \Box_{I(t)}(\varphi) = [t : \varphi].
  \]

- Case \( \varphi = s \approx t \):
  Suppose that \( \bar{\theta}(s \approx t) = [\top] \). Then \( \bar{I}(s) \approx \bar{I}(t) \), and hence \( v(\bar{I}(s)) = v(\bar{I}(t)) \), for every \( v : \text{Var} \to [\text{Term}] \). Thus, by Lemma 6.7, we get \( s^\prime_I = t^\prime_I \), for every \( v : \text{Var} \to [\text{Term}] \). Then, for \( v(x) = [x] \) we obtain \( s^\prime_I = t^\prime_I \). By (5) in the proof of Theorem 5.14, we get \( s = t \), and hence \( s \approx t = [\top] \).

For the converse, from \( s \approx t = [\top] \) it follows that \( \vdash_{\text{LP}_{\text{CS}}} s \approx t \). By soundness theorem 5.14, we have \( s^\prime_I = t^\prime_I \) for every \( \text{LP}_{\text{CS}}^B \) algebra \( A \) over \( T \), every valuation \( \theta \), and every \( v : \text{Var} \to T \). Thus, for the Tarski-Lindenbaum algebra \( \mathcal{A}_{\text{LP}_{\text{CS}}}^p \) over \( [\text{Term}] \) (cf. Definition 5.12), we have \( s^\prime_I = t^\prime_I \), for every valuation \( \theta \) and every \( v : \text{Var} \to [\text{Term}] \). Again since the definition of interpretation \( I \) in the Tarski-Lindenbaum algebras \( \mathcal{A}_{\text{LP}_{\text{CS}}} \) and \( \mathcal{A}_{\text{LP}_{\text{CS}}}^p \) are the same, we get \( s^\prime_I = t^\prime_I \), for every \( v : \text{Var} \to [\text{Term}] \).

Hence, by Lemma 6.7, \( \bar{I}(s) \approx \bar{I}(t) \). Thus, \( \bar{\theta}(s \approx t) = [\top] \).

From the above two arguments, it follows that \( \bar{\theta}(s \approx t) = [s \approx t] \).

\(\square\)

### 6.2 Bi-representation theorem for full \( \text{LP}_{\text{CS}}^B \) algebras

At the end, we present a bi-representation theorem for polynomial \( \text{LP}_{\text{CS}}^B \) algebras similar to Section 5.3.

Given \( T = (T, 0_T, -T, +T, \cdot T, !T) \) and \( T' = (T', 0_{T'}, -T', +T', \cdot T', !T') \), note that every isomorphism \( g : T \to T' \) can be naturally extended to an isomorphism between the polynomial structures \( T[\text{Var}] \) and \( T'[\text{Var}] \) where

\[
g(x) := x, \quad \text{for every } x \in \text{Var}.
\]

By the abuse of notation we denote this extension again by \( g \).

**Definition 6.11.** Let \( A = (A, 0, \odot, \oplus, \ominus, 0, \Box, I)_{\alpha \in T[\text{Var}]} \) and \( A' = (A', 0', \odot', \oplus', \ominus', 0', \Box', I')_{\alpha \in T'[\text{Var}]} \) be polynomial \( \text{LP}_{\text{CS}}^B \) algebras over \( T[\text{Var}] \) and \( T'[\text{Var}] \) respectively, where \( T = (T, 0_T, -T, +T, \cdot T, !T) \) and \( T' = (T', 0_{T'}, -T', +T', \cdot T', !T') \). A bi-isomorphism between \( A \) and \( A' \) is a pair of Boolean isomorphisms \( (f, g) \) such that \( f : (A, 0, \odot, \oplus) \to (A', 0', \odot', \oplus') \) and \( g : (T, 0_T, -T, +T) \to (T', 0_{T'}, -T', +T') \) satisfying the following conditions:

\[
f(\Box_{\alpha}(\varphi)) = \Box_{\alpha}(f(\varphi)), \quad \text{for } \alpha \in T[\text{Var}] \text{ and } \varphi \in \text{Form},
\]

\[
g(I(c)) = I'(c), \quad \text{for } c \in \text{Const} \cup \{0\},
\]

\[
g(\alpha \cdot_T \beta) = g(\alpha) \cdot_{T'} g(\beta), \quad \text{for } \alpha, \beta \in T[\text{Var}],
\]

\[
g(\alpha \cdot_T^0 \beta) = g(\alpha) \cdot_{T'}^0 g(\beta), \quad \text{for } \alpha \in T[\text{Var}].
\]

Note that the definition of the interpretation \( I \) in the Tarski-Lindenbaum algebras \( \mathcal{A}_{\text{LP}_{\text{CS}}}^p \) (cf. Definition 5.12) and \( \mathcal{A}'_{\text{LP}_{\text{CS}}} \) are the same.
Definition 6.12. Given two set algebras \((A, \emptyset, \setminus, \cup)\) and \((B, \emptyset, \setminus, \cup)\), a polynomial \(\text{LP}^B_{\text{CS}}\) set algebra is a structure

\[ \mathcal{A} = (A, \emptyset, \setminus, \cup, [\alpha], I)_{\alpha \in B[\text{Var}]} \]

over \(B[\text{Var}]\), where \(B = (B, \emptyset, \setminus, \cup, \cdot, 1, \emptyset)\), satisfying the conditions of Definition 6.1.

Similar to Section 5.3, we construct two isomorphisms for a given polynomial \(\text{LP}^B_{\text{CS}}\) algebra \(\mathcal{A}\) over \(T[\text{Var}]\): One for the algebra \(\mathcal{A}\) and one for the polynomial structure \(T[\text{Var}]\). This leads to a bi-isomorphism of polynomial \(\text{LP}^B_{\text{CS}}\) algebras.

Theorem 6.13 (Bi-Representation Theorem). Every polynomial \(\text{LP}^B_{\text{CS}}\) algebra is bi-isomorphic to a polynomial \(\text{LP}^B_{\text{CS}}\) set algebra.

Proof. Suppose that \(\mathcal{A} = (A, \emptyset, \setminus, \cup, [\alpha], I)_{\alpha \in T[\text{Var}]}\) is a polynomial \(\text{LP}^B_{\text{CS}}\) algebra over \(T[\text{Var}]\), where \(T = (T, 0_T, -T, +T, \cdot, 1_T)\). Considering the Boolean algebra \((T, 0_T, -T, +T)\), by the Stone representation theorem, there exists an isomorphism \(g\) from \((T, 0_T, -T, +T)\) to a set algebra \(\mathcal{P} = (P, \emptyset, \setminus, \cup)\). Similar to the proof of Theorem 5.17, \(g\) is extended to an isomorphism from \(T[\text{Var}]\) to \(P_T[\text{Var}]\). As mentioned before, \(g\) can be further extended to an isomorphism from \(T[\text{Var}]\) to \(P_T[\text{Var}]\).

Next, by the Stone representation theorem, \((A, \emptyset, \setminus, \cup)\) is isomorphic to a set algebra \((B, \emptyset, \setminus, \cup)\) via isomorphism \(f\). Define \(\square'_{\alpha'}\) (for \(\alpha' \in P[\text{Var}]\)) and the interpretation \(\tilde{I}'\) as follows:

\[ \square'_{\alpha'}(\varphi) := f(\square_{\alpha}(\varphi)), \quad \text{for } \alpha \in T[\text{Var}]. \]  

\[ \tilde{I}'(c) := g(I(c)), \quad \text{for } c \in \text{Const} \cup \{0\}. \]  

Let \(B = (B, \emptyset, \setminus, \cup, \square'_{\alpha'}, I')_{\alpha' \in P[\text{Var}]}\). It is not difficult to show that for any \(t \in \text{Term}\):

\[ \tilde{I}'(t) = g(\tilde{I}(t)). \]  

The proof is by a simple induction on the complexity of \(t\). Then, we show that \(B\) is a polynomial \(\text{LP}^B_{\text{CS}}\) algebra. To this end we only check conditions \(\text{Al-}\text{Appl-}\text{LP}^B_{\text{CS}}, \text{Al-CS-}\text{LP}^B_{\text{CS}}, \text{Al-}\text{j4-}\text{LP}^B_{\text{CS}}\) and \(\text{Al-jT-}\text{LP}^B_{\text{CS}}\) from Definition 6.1 (other cases can be verified similarly).

For \(\text{Al-}\text{Appl-}\text{LP}^B_{\text{CS}}\), suppose that \(a, b \in A\):

\[ \square'_{\alpha'}(\varphi \rightarrow \psi) \otimes \square'_{\beta'}(\varphi) = f(\square_{\alpha}(\varphi) \rightarrow \psi) \otimes f(\square_{\beta}(\varphi)) = f(\square_{\alpha}(\varphi \rightarrow \psi) \otimes \square_{\beta}(\varphi)) \leq f(\square_{\alpha \cdot T[\text{Var}]\beta}(\psi)) = \square'_{g(\alpha) \cdot P_T[\text{Var}]g(\beta)}(\psi). \]

For \(\text{Al-CS-}\text{LP}^B_{\text{CS}}\), suppose \(c : \varphi \in \text{CS}\), then we have

\[ \square'_{\tilde{I}(c)}(\varphi) = \square'_{g(I(c))}(\varphi) = f(\tilde{I}(c)(\varphi)) = f(1) = B. \]

For \(\text{Al}\text{-j4-}\text{LP}^B_{\text{CS}}\), note that since \(g\) is an isomorphism from \(T[\text{Var}]\) to \(P_T[\text{Var}]\), for every \(t \in \text{Term}\) there is \(\alpha \in T[\text{Var}]\) such that \(g(\alpha) = \tilde{I}'(t)\). Furthermore, from (13) it follows that \(\tilde{I}(t) = g^{-1}(\tilde{I}'(t)) = g^{-1}(g(\alpha)) = \alpha\). Thus,

\[ \square'_{\tilde{I}(t)}(\varphi) = \square'_{g(\alpha)}(\varphi) = f(\square_{\alpha}(\varphi)) = f(\tilde{I}(t)(\varphi)) \leq f(\square_{T[\text{Var}]\tilde{I}(t)}(t : \varphi)) = f(\square_{T[\text{Var}]\alpha}(t : \varphi)) = \square'_{g(\alpha |_{T[\text{Var}]t})}(t : \varphi) = \square'_{P_T[\text{Var}]g(\alpha)}(t : \varphi). \]

25
For Al-jT-LP$^B_{CS}$, suppose that $\theta' : \text{Prop} \rightarrow B$ is a valuation on $\mathcal{B}$. Then, $\theta = f^{-1} \circ \theta'$ would be a valuation on $\mathcal{A}$. Thus, we have
\[ \Box_{g(\alpha)}(\varphi) = f(\Box_\alpha(\varphi)) \leq f(\check{\theta}(\varphi)) = \check{\theta'}(\varphi). \]

The above inequality follows from the fact that $f$ is an isomorphism and $\Box_\alpha(\varphi) \leq \check{\theta}(\varphi)$.

It is obvious that $\mathcal{B}$ is a polynomial LP$^B_{CS}$ set algebra. \hfill \Box

**Theorem 6.14.** LP$^B_{CS}$ is complete with respect to polynomial LP$^B_{CS}$ set algebras.

**Proof.** Follows from the Bi-representation Theorem 6.13 and the Completeness Theorem 6.10. \hfill \Box

**Conclusion**

In this paper we have introduced an algebraic semantics for the logic of proofs. These algebraic models are obtained from Boolean algebras by adding countably infinite functions on formulas. We have also extended the language of LP in a way that proof terms constitute a Boolean algebra. The language of the resulting logic LP$^B$ includes an equality predicate. We have proved completeness theorems and certain generalizations of Stone’s representation theorem for all of the aforementioned algebras.

Comparing modal algebras with LP algebras, we see that the class of LP algebras does not form a variety; in other words, the LP algebras are not equationally definable. While pre LP$^B_{CS}$ algebras form a variety in the above sense, the condition Al-jT-LP$^B_{CS}$ in full algebras cannot be expressed by an equation.

There remain many research problems deserving further study. For instance, a natural question to ask is whether it is possible to present regular algebras with a finite set $\forall$ for LP$^B_{CS}$. Another problem is to present algebraic models for other justification logics. Although it is straightforward to present algebraic semantics for fragments of LP, such as JT and J4, it would be interesting to extend the results of this paper to other justification logics, such as extensions of LP.

Another interesting direction to extend this work is to consider fuzzy justification logics. Various fuzzy justification logics have been introduced in the literature (see [14, 15, 24, 26]), however none of these logics have algebraic semantics. One might expect that LP algebras can be modified in order to construct fuzzy algebraic semantics, such as MV algebras, for a fuzzy version of LP.

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**References**

[1] Sergei Artemov. Operational modal logic. Technical Report Technical Report MSI 95–29, Cornell University, dec 1995.
[2] Sergei Artemov. Explicit Provability and Constructive Semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, mar 2001.
[3] Sergei Artemov. The Ontology of Justifications in the Logical Setting. *Studia Logica*, 100(1–2):17–30, apr 2012. doi:10.1007/s11225-012-9387-x.
[4] Sergei Artemov and Melvin Fitting. Justification Logic. In Edward N Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Fall 2012 edition, 2012. URL: [http://plato.stanford.edu/archives/fall2012/entries/logic-justification/](http://plato.stanford.edu/archives/fall2012/entries/logic-justification/).

To this end, it is enough to omit the condition Al-j4 to obtain JT-algebras, and omit the condition Al-jT to obtain J4-algebras.
[27] Bryan Renne. Propositional Games with Explicit Strategies. Technical Report TR–2005012, CUNY Ph.D. Program in Computer Science, oct 2005.

[28] Natalia Rubtsova. Evidence Reconstruction of Epistemic Modal Logic S5. In Dima Grigoriev, John Harrison, and Edward A Hirsch, editors, Computer Science– Theory and Applications, First International Computer Science Symposium in Russia, CSR 2006, St. Petersburg, Russia, June 8–12, 2006, Proceedings, volume 3967 of Lecture Notes in Computer Science, pages 313–321. Springer, 2006. doi:10.1007/11753728_32

[29] Shawn Standefer. Tracking reasons with extensions of relevant logics. Logic Journal of the IGPL, 27(4):543–569, jul 2019. URL: https://academic.oup.com/jigpal/article/27/4/543/5498855, doi:10.1093/JIGPAL/JZZ018