The state–sum invariants for virtual knots

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Abstract

We construct the new non-trivial state–sum invariants for virtual knots and links by a generalization of the powerful Carter–Saito–Jelsovsky–Kamada–Langford theorem for classical knots. The main result of this work is based on cohomology quandle theory and colorings of virtual knot and link diagrams by quandle elements.

Keywords: Virtual knots, virtual links, quandle, invariant state–sum, cohomology of quandle.

1. Introduction

The theory of virtual knots discovered and described by Kauffman [1] arises from the study of knots in thickened surfaces, which is the natural generalization of classical knot theory. At the present time, this theory is well–developed and has many applications in other fields of mathematics. For instance, virtual knots play an important role in combinatorics of Gauss diagrams and codes [1] [2]. Also, virtual knots relate to the theory of virtual manifolds and the generalization of Vitten–Reshetikhin–Turaev invariants [3].

Virtual analogues of knot invariants are in the focus of many research dedicated to virtual knot theory [1]. The purpose of this paper is to construct new non–trivial state–sum invariants of virtual knots and links, which are obtained by a generalization of Carter–Saito–Jelsovsky–Kamada–Langford theorem for classical knots [4].

1.1. Kauffman theory of virtual knots and links. Similarly to the classical knots there is the diagram knot technique and the complete analogue of the classical Reidemeister theorem for virtual knots [1], [2], which play the central role in virtual knot theory. We define diagrams for virtual knots or links as it is defined in classical knot or link theory, however in comparison with the classical case some intersections of virtual knot or link diagrams might be virtual (see Fig.1).

![Figure 1. The classical and virtual (in the middle) intersections.](image)

According to the virtual Reidemeister theorem [1], [2] we consider two virtual knot diagrams equivalent, if one of them can be transformed to another by isotopy and a finite sequence of classical Reidemeister moves (see Fig.2) and four additional virtual Reidemeister moves (see Fig.3).
1.2. The Carter–Saito–Jelsovsky–Kamada–Langford theorem for classical knots and links. In this subsection we briefly examine the construction of a powerful family of knot invariants by Carter, Saito et al. [4]. Their approach is based on the following algebraic objects (see [4], [5], [6] for more details):

**Definition 1.1.** A quandle is a set equipped with a binary operation $*$ satisfying the following axioms:

1. $\forall a \in G : a * a = a$;
2. $\forall a, b \in G \exists ! x \in G : x * a = b$;
3. $\forall a, b, c \in G : (a * b) * c = (a * c) * (b * c)$.

Also we have to recall the definition of a quandle 2–cocycle [4], [5], [6]:

**Definition 1.2.** A 2–cocycle $\phi$ is a map $\phi : G \times G \to R$ (here $R$ is an Abelian ring) which satisfies the following identities:

4. $\forall a \in G : \phi(a, a) = 1$;
(5) \[
\forall a, b, c \in G : \phi(a, b)\phi(a \ast b, c) = \phi(a, c)\phi(a \ast c, b \ast c).
\]

A 2–cocycle \(\phi\) is called a coboundary 2–cocycle if the following holds:

(6) \[
\phi = \psi(x)\psi^{-1}(x \ast y),
\]

here \(\psi\) is a map \(\psi : G \to R\).

2–cocycles \(\phi\) and \(\phi'\) are called cohomologous if the following holds:

(7) \[
\phi = \phi\phi',
\]

here \(\phi\) is a coboundary 2–cocycle.

Now we are ready to describe the Carter–Saito et al. construction for classical knots and links invariants. Let us fix a quandle \(G\), some of its 2–cocycle \(\phi\) and consider an oriented diagram of a knot (or a link). Then we color its arcs with elements of \(G\). We will say that its arc coloring is possible if the following two rules (see Fig.4) are satisfied for each diagram intersection:

![Diagram](image)

**Figure 4.** The coloring rules.

**Theorem 1.3.** [4] Let us consider a knot (or a link) with a given oriented diagram \(D\). Let \(C\) be a set of all possible colorings of the diagram \(D\). For each possible coloring we define the weights of diagram intersections by the 2–cocycle \(\phi\) as it is shown in Fig.4. Then the following state–sum function is an invariant of the knot (or the link) with the diagram \(D\):

\[
Z(D, \phi) = \sum_C \prod \phi(x, y)^{\sigma t},
\]

where the product is taken over all intersections of the diagram \(D\) and the sum is taken over the set \(C\), signs \(\sigma = -1, 1\) are selected as it is shown in Fig.4, \(t\) is a formal parameter.

**Proof.** For completeness, we provide the sketch of the proof (for more details see [4]):

1. Consider the first Reidemeister move. Using the first quandle axiom (1) and the first 2–cocycle identity (4) we conclude that the state–sum function is preserved under the first Reidemeister move (see Fig.5).

2. By the same way using the second quandle axiom (2) we conclude that the state–sum function is preserved under the second Reidemeister move (see Fig.5).

3. The third Reidemeister move remains to be examined. Similarly, using the second and the third quandle axiom (2), (3) and the second 2–cocycle identity (4) we conclude that the state–sum function is preserved under the third Reidemeister move (see Fig.6). \(\square\)
Remark 1.4. Formally, other cases of the orientation and the position of the arcs of diagram $D$ are not analyzed, but they might be checked by the same fashion.

Remark 1.5. An analogue of the Carter–Saito et al. theorem is also holds for knots and links in $\mathbb{RP}^3$ [8]. For the definition of knots and links in $\mathbb{RP}^3$ see [9], [10].

2. Generalization of the Carter–Saito–Jelsovsky–Kamada–Langford theorem for virtual knots

In this section we turn to the main result of our article. Consider a virtual knot (or a virtual link) with an oriented diagram $D$. Then fix a finite quandle $G$, its 2–cocycle $\phi$ and an automorphism $f$ of the quandle $G$.

Remark 2.1. Here as usual a quandle automorphism is a bijection $f : G \to G$ which preserves a quandle operation $\forall a, b \in G : f(a \ast b) = f(a) \ast f(b)$.

Let us color the arcs of a diagram $D$ with the elements of the quandle $G$. We call the coloring possible, if it satisfies the rules demonstrated in Fig.4 and Fig.7. For each classical intersection we define the weight that is obtained by the 2–cocycle $\phi$ as it is shown in Fig.4. Finally, for each virtual intersection we define the weight is equal to 1 as it is shown in Fig.7.
Remark 2.2. The natural coloring rule for virtual diagrams, which we use in this paper, probably were firstly introduced in the work [7].

Figure 7. The coloring rules for virtual intersection.

Now we describe how the coloring of a diagram changes under the first three virtual moves (see Fig.8). By simple brute force it is not difficult to consider all possible orientations and obtain the following proposition:

Figure 8. The first three virtual moves.

Proposition 2.3. Consider a virtual knot or a link with a diagram D. Then the state–sum function \( Z(D, \phi, f) = \sum_{C} \prod_{\sigma} \phi(x, y)^{\sigma} \) is the invariant of this virtual knot (or link) under the classical Reidemeister moves and the first three virtual Reidemeister moves. Where the product is taken over all classical intersections of the given diagram D, and the sum is taken over the set of all possible colorings C, \( \sigma = -1, 1 \) is selected as shown in Fig.4.

Now we present how the coloring of a virtual knot (or a link) diagram changes under the move \( \Omega'_{3} \) (see Fig.9) to obtain the following theorem:
Theorem 2.4. 1. The function
\[ Z_1(D, \phi, f) = \prod_C \prod \phi(x, y)^{\sigma t} \]
is the invariant of a virtual knot (or a virtual link) with a given diagram \( D \) under all classical and virtual moves, where the second product is taken over all classical intersections of the given diagram, and the first product is taken over the set of all possible colorings \( C \). We call the function \( Z_1 \) a state-weight.

2. Consider a fixed automorphism \( f \) of a quandle \( G \) and its fixed 2-cocycle \( \phi \). Then if \( f \) and \( \phi \) are aligned with each other – it means that the following holds: \( \forall a, b \in G : \phi(a, b) = \phi(f(a), f(b)) \), the function
\[ Z_2(D, \phi, f) = \sum_C \prod \phi(x, y)^{\sigma t} \]
is the invariant of a virtual knot (or link) with a diagram \( D \) under all classical and virtual moves, where the product is taken over all classical intersections of the diagram \( D \), and the sum is taken over the set of all possible colorings \( C \). We call the function \( Z_2 \) a state-sum.

3. If 2-cocycles \( \phi \) and \( \phi' \) are cohomologous (see 7), then \( Z_1(D, \phi, f) = Z_1(D, \phi', f) \).

Proof. 1. It suffices to prove that \( Z_1 \) is invariant under \( \Omega'_3 \). Values of \( Z_1 \) before and after \( \Omega'_3 \) move are different on factors which are equal to \( P_1 := \prod \phi(x_i, y_i) \) and \( P_2 := \prod \phi(f(x_i), f(y_i)) \) correspondingly.

Let us consider \( A \) – an arbitrary possible coloring of \( D \), define the coloring \( f(A) \) as follows: if in the coloring \( A \) an arc is colored by an element \( a \), then in the coloring
The function $Z_2(D, g)$ is the same arc is colored by the element $f(a)$. As soon as $f$ is an automorphism we conclude that if $A$ is a possible coloring of $D$, then $f(A)$ is a possible coloring of $D$ too. Moreover, it is easy to see that $f$ acts on the set of all possible collorings as a permutation. So, we obtain that $P_1$ and $P_2$ are different only on an order of factors, that completes the proof.

2. The proof is similar to the proof of the Carter–Saito et al. theorem for classical knots (see Fig.5, 6, 8 and 9).

3. Let us introduce an equivalence relation on the set $C$ of all possible collorings: two possible collorings $A_1$ and $A_2$ are equivalent if $A_2 = f^k(A_1)$ for some $k \in \mathbb{N}$. Since the quandle $G$ is finite, then there exists a minimal $n$ such that $id = f^n$, and therefore the introduced relation is the true equivalence relation. Denote by $C_A$ an equivalence class of a colloring $A$, using this notation we can rewrite $Z_1$ as follows:

$$Z_1(D, \phi, f) = \prod_{C_A} \left( \prod_{B \in C_A} \prod_{x, y} \phi(x, y)^{\sigma t} \right),$$

where the third product is taken over all classical intersections of $D$, the second product is taken over all possible collorings $B$ belonged to a selected equivalence class $C_A$ and the first – over all equivalence classes of possible collorings.

Now, we are ready to prove the third statement. Easy to see, that for this goal it suffices to show that $Z_1(D, \phi, f) = 1$ for any coboundary 2–cocycle $\phi(x, y) = \psi(x)\psi^{-1}(x*y)$ (see 6).

So, let us fix a coboundary 2–cocycle and consider a long virtual arc of $D$ (i.e. a part of the $D$ between two upper intersections which contains only virtual and bottom intersections, see Fig.10). By direct computation a contribution of a possible colloring $A$ to the $Z_1$ on the long virtual arc is equal to $\psi^{-1}(\omega)\psi(f^m(\omega))$ (here $m$ is some integer which depends only on number of virtual intersections crossed the long virtual arc). A contribution of all possible collorings $B \in C_A$ to the $Z_1$ on the long virtual arc is equal to

$$\psi^{-1}(\omega)\psi\left(f^m(\omega)\right)\psi^{-1}\left(f(\omega)\right)\psi\left(f^{m+1}(\omega)\right)\ldots\psi^{-1}\left(f^{\left|C_A\right|-1}(\omega)\right)\psi\left(f^{\left|C_A\right|-1+m}(\omega)\right).$$

Rewriting the last expression as

$$\psi^{-1}(\omega)\psi^{-1}\left(f(\omega)\right)\ldots\psi^{-1}\left(f^{\left|C_A\right|-1}(\omega)\right)\psi\left(f^m(\omega)\right)\psi\left(f^{m+1}(\omega)\right)\ldots\psi\left(f^{\left|C_A\right|-1+m}(\omega)\right)$$

and taking into account that $f$ acts on $C_A$ as a permutation, we obtain that this product is equal to 1, that completes the proof.

In conclusion of this section, we formulate the following theorem which are obtained as the simple corollary of the speculations above:

**Theorem 2.5.**

1. The function $Z_3(D, \phi) = \sum C \prod \phi(x, y)^{\sigma t}$ is the invariant of a virtual knot (or virtual link) under all classical and virtual moves, where the sum is taken over all automorphism of $G$.

2. The function

$$Z_4(D, \phi, f) = \sum_{C_A} \left( \prod_{B \in C_A} \prod_{x, y} \phi(x, y)^{\sigma t} \right),$$

where the second product is taken over all classical intersections of $D$, the first product is taken over all possible collorings $B$ belonged to a selected equivalence class $C_A$ and the sum – over all equivalence classes of possible collorings.

3. If 2–cocycles $\phi$ and $\phi'$ are cohomologous (see 7), then $Z_4(D, \phi, f) = Z_4(D, \phi', f)$. 


Consider the quandle \( Q = \mathbb{Z}_4 \) with the operation \( * : i * j = 2j - i \ (\text{mod} \ 4) \). We have the non-trivial 2–cocycle \( \phi : Q \to \mathbb{Z} \), which is defined as follows: \( \phi(a_1, b_1) = \phi(a_1, b_2) = t \) otherwise \( \phi = 1 \), where \( t \) is a generator under multiplication of \( \mathbb{Z} \), and \( a_1 = 0, b_1 = 1, a_2 = 2, b_2 = 3 \). It is easy to verify that the automorphism \( f(b) = b * a_1 \) is aligned with the given 2–cocycle \( \phi \).

Now we calculate the state–sum related with the quandle \( Q \) and the automorphism \( f \) for the links as shown in Fig. 11.

1. For the first example the state–sum is equal to \( 8(1 + t) \).
2. For the second example the state–sum is equal to \( 8 \).
3. For the third example the state–sum is equal to \( 4(1 + t) \).
4. For the fourth example the state–sum is equal to \( 4(1 + t) \).

Acknowledgments. The author is grateful to Dmitry V. Talalaev (Moscow State University, Faculty of Mechanics and Mathematics) for the support and fruitful discussions, Evgenii Pavlov (National Research University “Higher School of Economics”, Faculty of Mathematics) and Ammar Basheer (Ural Federal University, Ekaterinburg Department of Humanities) for careful reading of the article and helpful remarks.
The work was partially supported by the Basis foundation, the grant Leader (Math) 20-7-1-21-1.

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