GEOMETRY OF FOLDED HYPERCOMPLEX STRUCTURES

ROGER BIELAWSKI AND CAROLIN PETERNELL

Abstract. We investigate the geometry of the Kodaira moduli space $M$ of sections of $\pi : Z \rightarrow \mathbb{P}^1$, the normal bundle of which is allowed to jump from $\mathcal{O}(1)^n$ to $\mathcal{O}(1)^{n-2m} \oplus \mathcal{O}(2)^m \oplus \mathcal{O}^m$. In particular, we identify the natural assumptions which guarantee that the Obata connection of the hypercomplex part of $M$ extends to a logarithmic connection on $M$.

1. Introduction

It is well known that a hypercomplex structure on a smooth manifold $M$ can be encoded in the twistor space, which is a complex manifold $Z$ fibring over $\mathbb{P}^1$ and equipped with an antiholomorphic involution $\sigma$ covering the antipodal map. The manifold $M$ is recovered as the parameter space of $\sigma$-invariant sections with normal bundle isomorphic to $\mathcal{O}(1)^{\oplus n}$ ($n = \dim \mathbb{C}M$). If we start with an arbitrary complex manifold $Z$ equipped with a holomorphic submersion $\pi : Z \rightarrow \mathbb{P}^1$ and an involution $\sigma$, then the corresponding component of the Kodaira moduli space of sections of $\pi$ will typically also contain sections with other normal bundles $\bigoplus_{i=1}^n \mathcal{O}(k_i)$. This will be in particular the case, if we try to compactify $M$ by first compactifying its twistor space and then considering the closure of $M$ in the (real) Douady space of this compactification. A simple but instructive example is $Z = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O})$, where the resulting manifold of real sections is $\mathbb{R}\mathbb{P}^1$, compactifying the flat hypercomplex manifold $\mathbb{R}^4$ with its twistor space $\mathcal{O}(1) \oplus \mathcal{O}(1)$. The normal bundle of sections contained in the exceptional divisor $Z_\infty = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O} \oplus \mathcal{O}(2)$.

This kind of jumping normal bundle attracted recently attention in the case of 4-dimensional hyperkähler manifolds [3, 4]. One speaks of folded hyperkähler metrics. The aim of this paper is to describe the natural geometry in the hypercomplex case. More precisely, we are interested in the differential geometry of the (smooth) parameter space $M$ of sections of $\pi : Z \rightarrow \mathbb{P}^1$ with normal bundle $N$ isomorphic $\bigoplus_{i=1}^n \mathcal{O}(k_i)$, where each $k_i \geq 0$. We shall discuss only the purely holomorphic case, i.e. we are interested in all sections, not just $\sigma$-invariant. Choosing an appropriate $\sigma$ allows one to carry over all results to hypercomplex or split hypercomplex manifolds.

Our particular object of interest is the (holomorphic) Obata connection $\nabla$, i.e. the unique torsion free connection preserving the hypercomplex (or, rather, the biquaternionic, i.e. complexified hypercomplex) structure. This is defined on the open subset $U$ of $M$ corresponding to the sections with normal bundle isomorphic to $\mathcal{O}(1)^{\oplus n}$. The general twistor machinery (see, e.g. [1]) implies that $\nabla$ extends to a first order differential operator $D$ on sections of certain vector bundle defined over all of $M$. Our point of view is to regard $D$ as a particular type of meromorphic

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connection with polar set $\Delta = M \setminus U$. In general, this meromorphic Obata connection can have a double pole along $\Delta$. We show, however, that in the case when $M$ arises from a (partial) compactification of the twistor space of a hypercomplex manifold, the meromorphic Obata connection has a simple pole, and in fact it is then a logarithmic connection.

2. Logarithmic hypercomplex structures

Let $Z$ be a complex manifold of dimension $n + 1$ and $\pi : Z \to \mathbb{P}^1$ a surjective holomorphic submersion. We are interested in the (necessarily smooth) parameter space $M$ of sections of $\pi$ with normal bundle $N$ isomorphic to $\bigoplus_{i=1}^n \mathcal{O}(k_i)$, where $k_i \in \{0, 1, 2\}$ and $n = \sum k_i$. Its dimension (as long as it is nonempty) is $2n$ and we consider a connected component $M$ which contains a section with normal bundle isomorphic to $\mathcal{O}(1)^\oplus n$.

The tangent space $T_m M$ at any $m \in M$ is canonically isomorphic to $H^0(s_m, N)$, where $s_m$ is the section corresponding to $m$. Similarly, we have a rank $n$ bundle $E$ over $M$, the fibre of which is $H^0(s_m, N(-1))$, where $N(-1) = N \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-1)$. The multiplication map $H^0(N(-1)) \otimes H^0(\mathcal{O}(1)) \to H^0(N)$ induces a homomorphism

$$\alpha : E \otimes \mathbb{C}^2 \to TM,$$

which is an isomorphism at any $m$ with $N_{s_m/\mathbb{Z}} \simeq \mathcal{O}(1)^n$. It follows that the subset of $M$ consisting of sections with other normal bundles is a divisor $\Delta$ in $M$. We shall assume throughout that the set of singular points of $\Delta$ has codimension 2 in $M$ (in particular $\Delta$ is reduced). This means that the normal bundle of a section corresponding to a smooth point of $\Delta$ is isomorphic to $\mathcal{O}(1)^{n-2} \oplus \mathcal{O}(2) \oplus \mathcal{O}$.

Observe also that $\alpha$ is injective on each subbundle of the form $E \otimes v$, where $v$ is a fixed nonzero vector in $\mathbb{C}^2$. The image $D_v$ of the subbundle $E \otimes v$ is an integrable distribution on $TM$ (sections of $\pi$ vanishing at the zero of $v \in H^0(\mathcal{O}(1))$) and we recover $Z$ as the space of leaves of the distribution $D$ on $M \times \mathbb{P}^1$ given by $D|_{M \times \{v\}} = D_v$.

**Remark 2.1.** We can also define $E$ as the kernel of the evaluation map $H^0(N) \otimes \mathcal{O}_{\mathbb{P}^1} \to N$ (which is what we do in [2]), i.e.

$$0 \to E_m \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{A} H^0(N) \otimes \mathcal{O}_{\mathbb{P}^1} \to N \to 0.$$

We obtain again a map $\alpha : E \otimes \mathbb{C}^2 \to TM$ by restricting $A$ to each subspace of the form $E \otimes \langle v \rangle$, $v \in \mathbb{C}^2$. But then the above sequence identifies $H^0(N(-1))$ with $E_m \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$. Thus, viewing $\alpha$ as the multiplication map $H^0(N(-1)) \otimes H^0(\mathcal{O}(1)) \to H^0(N)$ means that we have implicitly identified $H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$ with $\mathbb{C}$. Such an identification yields also a choice of a symplectic form on $H^0(\mathcal{O}(1))$ within its conformal class, i.e. an identification of $\mathbb{C}^2$ with $(\mathbb{C}^2)^*$.

There is another useful point of view on the bundle $E$ and the homomorphism $\alpha$. Start with a complex manifold $M$ and a divisor $\Delta$ satisfying the above smoothness assumption. Suppose that we are given a codimension 1 distribution $\mathcal{V}$ on the smooth locus $\Delta_{\text{reg}}$ of $\Delta$. We define $TM(-\mathcal{V})$ to be the sheaf of germs of holomorphic vector fields $X$ on $M$ such that $X_x \in \mathcal{V}_x$ for any $x \in \Delta_{\text{reg}}$. If the sheaf $TM(-\mathcal{V})$ is locally free, i.e. a vector bundle $F$, then we obtain a homomorphism $\alpha : F \to TM$ from the inclusion $TM(-\mathcal{V}) \subset TM$ (and $\mathcal{V} = \text{Im} \alpha$). In the case of folded hypercomplex structures $F \simeq E \otimes \mathbb{C}^2$, and the action of $\text{Mat}_2(\mathbb{C})$ gives an action of complexified quaternions on $TM(-\mathcal{V})$. 
In the above setting, the case of particular interest is $\mathcal{V} = T\Delta_{\text{reg}}$. Vector fields in $TM(-\mathcal{V})$ are called then logarithmic and $TM(-\mathcal{V})$ is denoted by $TM(-\log \Delta)$ [8]. Another way to characterise logarithmic vector fields is via the condition $X.z \in (z)$, where $z = 0$ is the local equation of $\Delta$. This shows, in particular, that the subsheaf $TM(-\log \Delta)$ is closed under the Lie bracket.

**Definition 2.2.** Let $M$ be a complex manifold and $\Delta$ a divisor in $M$ such that its set of singular points is of codimension 2 in $M$. A logarithmic biquaternionic structure on $M$ is an action of $\text{Mat}_2(\mathbb{C})$ on $TM(-\log \Delta)$ such that the Nijenhuis tensor of each $A \in \text{Mat}_2(\mathbb{C})$ vanishes.

**Remark 2.3.** The same definition can be used for real manifolds and we can speak of logarithmic hypercomplex or logarithmic split hypercomplex structures.

Observe that for a logarithmic biquaternionic structure the leaves of the distribution $D_v = \alpha(E \otimes v)$ on $\Delta$ are contained in $\Delta$, i.e. the image of $\Delta$ in each fibre of the twistor space has codimension 1. In other words $Z$ is a (partial) compactification of the twistor space of a hypercomplex manifold. More precisely:

**Proposition 2.4.** The following two conditions are equivalent:

(i) $\text{Im } \alpha_x = T_x\Delta$ for each $x \in \Delta_{\text{reg}}$;
(ii) for each $\zeta \in \mathbb{P}^1$, the map $\Delta \to \pi^{-1}(\zeta)$, given by intersecting a section with the fibre, maps a neighbourhood of each point $x \in \Delta_{\text{reg}}$ onto an $(n-1)$-dimensional submanifold.

**Proof.** Let $f$ denote the map $M \to \pi^{-1}(\zeta)$, given by intersecting a section with the fibre. For an $x \in \Delta_{\text{reg}}$, we have

$$\text{Im } \alpha_x = H^0(\mathcal{O}(1)^{n-2} \oplus \mathcal{O}(2)) \subset H^0(N) \cong TuM.$$  

Thus $df(\text{Im } \alpha_x)$ is an $n-1$-dimensional subspace for any $x \in \Delta_{\text{reg}}$. Since $f$ is a submersion at $x$, the condition $\text{Im } \alpha_x = T_x\Delta$ implies now that the $f(\Delta_{\text{reg}})$ is an immersed $(n-1)$-dimensional submanifold. Conversely, suppose that the condition (ii) holds. Then the image $f(U)$ of a neighbourhood $U$ of $x \in \Delta_{\text{reg}}$ is a codimension 1 submanifold $Z_0$ of $Z$. It follows that $T_xU \cong H^0(N_{s/Z_0})$, where $s$ is the section corresponding to $x$. Suppose that $N_{s/Z_0} \cong \bigoplus_{i=1}^{n-1} \mathcal{O}(k_i)$. Given the injection $N_{s/Z_0} \to N_{s/Z}$, we have (after reordering the $k_i$) $k_1 \leq 2$ and $k_2, \ldots, k_{n-1} \leq 1$. Since $H^1(N_{s/Z_0}) = 0$, we have $\dim \Delta_{\text{reg}} = h^0(N_{s/Z_0})$ and therefore $\sum_{i=1}^{n-1} (k_i + 1) = 2n-1$. Thus $(2 - k_1) + \sum_{i=2}^{n-1} (1 - k_i) = 0$ and since each summand is nonnegative, we conclude that $N_{s/Z_0} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2}$. Thus $T_x\Delta = \text{Im } \alpha_x$. \hfill $\Box$

**Remark 2.5.** There certainly exist natural and interesting folded hypercomplex structures, which are not logarithmic. For example, we have shown in [8] that this is the case for the natural folded hypercomplex (or rather biquaternionic) structure on the Hilbert scheme of rational quartics in $\mathbb{P}^3 \setminus \mathbb{P}^1$.

### 3. The meromorphic Obata connection

The Obata connection of a hypercomplex manifold is the unique torsion-free connection with respect to which the hypercomplex structure is parallel. From the twistor point of view it is obtained via the Ward transform [3] [4]. We now wish to discuss an extension of the Obata connection to a folded hypercomplex manifold.
Let $Z, M, \Delta, E$ and $\alpha$ be all as in the previous section. We consider the double fibration

$$M \xrightarrow{T} M \times \mathbb{P}^1 \rightarrow Z.$$  

The normal bundle $N$ of any section of $\pi$ is isomorphic to the vertical tangent bundle $T_{\pi}Z = \text{Ker} d\pi$ restricted to the section, and, consequently, the (holomorphic) tangent bundle $TM$ can be viewed as the Ward transform of $T_{\pi}Z$, i.e. $TM \simeq \tau_* T_{\pi}Z$. Similarly, the bundle $E$ is the Ward transform of $T_{\pi}Z \otimes \pi^* \mathcal{O}(-1)$.

In [2] §2 we have identified the algebraic condition satisfied by the differential operator produced by the Ward transform from any $M$-uniform vector bundle $F$ on $Z$. In our situation, we can state the results for $F = T_{\pi}Z(-1)$ as:

**Proposition 3.1.** The bundle $E$ is equipped with a first order differential operator $D : E \rightarrow E^* \otimes TM$ which satisfies $D(fs) = \sigma(df \otimes s) + fDs$, where $\sigma$ (which is the principal symbol of $D$) is the composition of the following two maps

\[
\begin{array}{c}
T^*M \otimes E \xrightarrow{\alpha \otimes 1} E^* \otimes \mathbb{C}^2 \otimes E \xrightarrow{1 \otimes \alpha} E^* \otimes TM
\end{array}
\]

(where $\mathbb{C}^2 \simeq (\mathbb{C}^2)^*$ as explained in Remark [2]). \hfill \Box

**Remark 3.2.** On $M \setminus \Delta$ $\sigma$ is invertible and $\sigma^{-1} \circ D$ is the standard hyperholomorphic connection on $E$, i.e. its tensor product with the standard flat connection on $\mathbb{C}^2$ is the (holomorphic) Obata connection on $M \setminus \Delta$.

Given any first order differential operator $D : E \rightarrow F$ between (sections of) vector bundles on a manifold $M$, with symbol $\sigma : E \otimes T^*M \rightarrow F$, we can “tensor” it with any connection $\nabla$ on a vector bundle $W$ over $M$:

$$(D \otimes_\sigma \nabla)(e \otimes w) = D(e) \otimes w + (\sigma \otimes 1)(e \otimes \nabla w).$$

The symbol of this new operator is $\sigma \otimes 1$. We can do this for our operator $D$ and the flat connection on $\mathbb{C}^2$. We obtain a differential operator $\tilde{D} : E \otimes \mathbb{C}^2 \rightarrow E^* \otimes TM \otimes \mathbb{C}^2$ which extends the Obata connection.

**Remark 3.3.** The results claimed by Pantillie [7] would imply that the Obata connection extends to a differential operator satisfying $\tilde{D}(fs) = \alpha^*(df) \otimes s + f\tilde{D}s$, but we have trouble following his arguments (in particular the second last paragraph in the proof of his Theorem 2.1).

We can view $\sigma^{-1} \circ D$ as a meromorphic connection on $E$, with polar set $\Delta$. Similarly the Obata connection on $M \setminus \Delta$ can be viewed as a meromorphic connection on $TM$ with polar set $\Delta$. It follows from Proposition [3.1] that $\sigma^{-1}$ generally has a double pole along $\Delta$ and, hence, so does $\sigma^{-1} \circ D$. We shall now show that for a logarithmic hypercomplex structure (Definition [2.2]), the pole becomes simple.

Let $z = 0$ be the local equation of $\Delta$. The meromorphic connection $\sigma^{-1} \circ D$ has a simple pole if $\lim_{z \to 0} z^2 \sigma^{-1} \circ D = 0$. Let us trivialise locally $E$, so that $\alpha$ is an endomorphism of the trivial bundle. We can then write $z = \text{det } \alpha$, and owing to Proposition [3.1] we have:

$$z^2 \sigma^{-1}(De) = ((\alpha^*)_{\text{adj}} \otimes 1)(1 \otimes \alpha_{\text{adj}})(De),$$

where the subscript “adj” denotes the classical adjoint. Thus, we can conclude:

**Lemma 3.4.** The meromorphic connection $\sigma^{-1} \circ D$ has a simple pole along $\Delta$ provided that $De|_x \in E^*_x \otimes \text{Im } \alpha_x$ for any $x \in \Delta_{\text{reg}}$ and any local section $e$ of $E$. If this is the case, then the residue of $\sigma^{-1} \circ D$ belongs to $\text{Ker } \alpha^* \otimes \text{End } E$. \hfill \Box
Proposition 3.5. Suppose that $\text{Im} \alpha_x = T_x \Delta$ for each $x \in \Delta_{\text{reg}}$. Then the condition of the above lemma is satisfied.

Proof. Proposition 2.4 implies that points of $\Delta_{\text{reg}}$ correspond to sections of $\pi : Z \to \mathbb{P}^1$ contained in a codimension 1 submanifold $Z'$ of $Z$. The differential operator $D$ is obtained by the push-forward of the flat relative connection $\nabla_\eta$ on $\eta^* T\pi Z(-1)$, i.e. of the exterior derivative in the vertical directions of the projection $\eta : M \times \mathbb{P}^1 \to Z$.

It follows that, over $\Delta_{\text{reg}}$, $D$ restricts to an operator $D'$ defined in the same way as $D$, but with $Z$ replaced by $Z'$. This means that $D'$ takes values in $E \otimes T\Delta_{\text{reg}}$. □

Recall that a meromorphic connection on a vector bundle $E$ is called logarithmic, if it has a simple pole along $\Delta = \{ z = 0 \}$ and its residue is of the form $Adz$, where $A \in \text{End} E$. Thus, under the assumption of the last proposition, $\sigma^{-1} \circ D$ is a logarithmic connection. We easily conclude:

Corollary 3.6. Suppose that the equivalent conditions of Proposition 2.4 are satisfied. Then the holomorphic Obata connection on $M \setminus \Delta$ extends to a logarithmic connection on $M$.  □

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Institut für Differentialgeometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany