On the Fractal Structure of the Universe

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Abstract. Despite the observational evidence that the Universe appears hierarchically structured up to a distance of at least 30 Mpc/h (and possibly up to 100 Mpc/h), the fractal paradigm has not yet been recognized by the majority of cosmologists today. In this work we provide a brief overview of the recent observational and theoretical advances relevant to the question of the global cosmic structure and present some simple calculations which indicate how the hierarchical structure may pass over to the homogeneous Universe at very large scale. We show that the fractal structure may be derived from the moderately nonuniform matter distribution. We address a number of epistemological questions relevant to a general outlook of the Cosmos at large too.
1 Introduction

Hierarchical models of the Universe appear as old as the cosmology. The first attempt to describe an infinite cosmos was made by Anaxagoras, who conceived the world built according to the principle: *everything contains everything* (see, e.g. [1]), what is tantamount to assuming that each part of the physical world resembles the entire cosmos. This idea was taken up by a number of European cosmologists, from Kant to the present day ones (see, e.g. [2]). The issue of the actual structure of our Cosmos appears controversial, the views going from one to the other extreme. The standard paradigm is that the Universe is spatially homogeneous and the inhomogeneities observed appear local disturbances, which smear out on sufficiently large scale. The other extreme, at least formally, would be the assumption that the Universe has a hierarchical, fractal structure and the observational evidence of the large-scale homogeneity reflects our present day inability to discern the higher-level hierarchical layers. Somewhere between these two end points lies the view that we have observational evidence about first two layers, galaxy clusters and superclusters, while the question concerning possible higher levels is to be left to the future observations.

The possible hierarchical structuring of the Universe bears questions which go beyond the cosmological issue. The concept of fractality appears sufficiently general to be restricted to the mere arrangement of the galaxies in the observable Cosmos. The very question of the cosmic structure brings in the old puzzle of the nature of the space and time, which intrigued cosmologists from the ancient time of Democritus, Aristotle to the modern naturalists, like Leibnitz, Berkeley, Mach, Einstein etc. In his recent work Roscoe argues that if one starts with the matter distribution in the Universe as the primary construct, prior to the determination of the space-time manifold, one arrives at the fractal structure, with the fractal dimension $D_f = 2$ [3]. The underlying principle of the fractal structure, the scale invariance, appears a powerful tool for examining fundamental properties on the microscopic level, like Schrödinger equation, as shown by Brenig ([4]). Staying on the cosmic scale, Abdalla et al ([5]) show that looking at the Cosmos within the past light cone, fractal structure appears even if the homogeneous distribution is assumed *a priory* (see, also [8]).

The concept of fractality appears all but simple one. On the phenomenological level, observational evidence of a possible hierarchical structuring at large cosmic scale is difficult to confirm, for a number of reasons. First, even if the overall structuring is following a hierarchical pattern one may hardly expect to see an elegant, clear geometrical picture. The question of galaxy distribution is essentially statis-
tical one and one needs to do much elaborate study of the astronomical catalogues in order to infer the underlying regularities (see, e.g. [6]). Second, the galaxy distribution may follow a more complicated pattern than simple fractal one, like the multifractal ansatz, which may be described as a non-uniform fractal distribution (see, e.g. [7], but see, also [9]). Another ingredient to be accounted for in studying the large scale structure are cosmic voids, which imply their own scaling features (see, e.g. [10],[12]). Two main issues in studying the large scale cosmic structures within the fractal paradigm are (i) what is the exact value of the fractal dimension $D_f$ and (ii) what is the value of the scale factor $\lambda_0$, separating the region with a clear fractal distribution and the outer cosmic region, where the galaxy distribution is definitely homogeneous (see, e.g. [11], [12]).

2 The fractal dimension

Accurate estimate of the cosmic fractal dimension appears of the crucial importance for several reasons. First, if different from $D_f = 3$ (fractal dimension must be less than the physical one, for topological reasons) the universe is endowed with a nontrivial structure, different from both uniform (homogeneous) and irregular inhomogeneity. Second, if the actual fractal dimension turns out to be an integer, it is possible to infer the geometry associated with the dimension [13]. In particular, if it turns out that $D_f = 2$ this would signal the presence of the holographic structure ("fractal holography") [14]. Recently, a particular interest in studying the actual metric of the observable universe by determining the matter distribution has arisen (see, e.g. [15], [16]).

Hierarchical model allows for the cosmic accelerated expansion too, as demonstrated by a number of authors (see, e.g. [20]), and the presence of dark matter (see, e.g. [21]). Inclusion of the scaling symmetry, via renormalisation group, completes the collection of the symmetry principles, which serve as the most fundamental background of the cosmological models [22].

Actual research, both observational and theoretical, revolves around the particular $D_f = 2$ value, for several reasons, besides the possible holographic fractality. As shown by Charlier [17], if Cosmos has $D_f \geq 2$ both Olbers’(blazing sky) and Neumann-Seeliger’s (gravitational) paradoxes are resolved [18]. Also, systems with such distribution have a compact projection onto a plane, like clouds shadows on Earth surface. In particular, hierarchical cosmic structure provides an isotropic projection to the observers situated at an occupied point in the Universe, like our planet. That all observational evidences point towards $D_f = 2$ appears a significant empirical fact. This evidence, however, although does not rule out the hierarchical model, can not decide between two paradigms - homogeneous and hierarchical ones - either.
Determining the fractal dimension and the scale where the fractal structure goes over to the homogeneous one appears all but easy task for the observational cosmology. Also, the usual procedures, based on the standard statistical method fail from the start, since they assume an approximately homogeneous distribution, which allows for making use of such quantities like the average density etc. By such an approach hierarchical structure is ruled out from the beginning and the methodology requires a more general approach (see, e.g. [19]). Here we propose a simple model which allows the fractal structure to go gradually to the homogeneous distribution. In the following chapter we present the model and in the last chapter a general discussion of the results and the prospects of the approach are given.

3 Calculations

3.1 The model

Assuming the galaxies equal mass points in the cosmic space, the most convenient way to infer the actual matter distribution from an occupied site (like our Milky Way) is to follow the mass (more precisely, the number of galaxies) as it increases with the radius \( R \) of a sphere centred on the observer. The formula used takes the form

\[
N_r = ar^{D_f}, R_{\text{min}} \leq r \leq R_{\text{max}},
\]

where \( a \) is constant and \( D_F \) is the fractal dimension. If \( D_F = 3 \) we have the usual formula for the uniform density. On the other hand, for \( D_f < 3 \) nonuniform distribution is present. As we mentioned before \( D_f = 2 \) corresponds to a two-dimensional "space", as if the matter is uniformly distributed on the spherical surfaces. The question arises then as of the possible and/or actual values the fractal dimension \( D_F \) assumes. One may put the question into a number of various forms.

(i) Is there a unique value of \( D_f \) for any possible scale (that is for \( R_{\text{max}} \to \infty \))? 
(ii) If \( R_{\text{max}} \) is finite, what is its value?

Recent observations have shown that for \( R_{\text{max}}^{(1)} = 10\,\text{Mps/h} \) one has for the fractal dimension \( D_f \approx 1.2 \) [23], [24], whereas for \( R_{\text{max}}^{(2)} = 100\,\text{Mps/h} \) we saw that \( D_f \approx 2 \). These data indicate that the fractal dimension may be an increasing function of the scale, which eventually reaches its maximum value \( D_f = 3 \). We therefore consider it appropriate to construct an analytical expression for the fractal dimension, which reproduces approximately the observed data.

\[
N = b\left(1 + \frac{R_1}{r} + \frac{R_2^2}{r^2}\right)r^3,
\]

\( 4 \)
where $N$ is the number of galaxies within a sphere of radius $r$, $b$ is a constant, and $R_1 = R_{\text{max}}^{(1)}$ and $R_2 = R_{\text{max}}^{(2)}$ are bordering values between scales with different fractal dimensions.

We consider now three scale regions separately.

(i) Innermost region:

$$\frac{R_2}{r} \leq \left( \frac{R_1}{r} \right)^2 = \left( \frac{R_2}{10r} \right)^2,$$

From (2) we have

$$N = bR_1^3r, \quad r \leq \frac{R_2}{100} = \frac{R_1}{10},$$

what provides $D_f = 1$.

(ii) Intermediate region

$$R_2 \geq r \geq R_1 = \frac{R_2}{10},$$

From (2) we obtain

$$N = bR_2^3r^2, \quad r \leq \frac{R_2}{100} = \frac{R_1}{10},$$

with $D_f = 2$.

(iii) Finally, in the outmost space

$$r \gg R_2 > R_1,$$

we have according to (2)

$$N = br^3,$$

and the "space" becomes finally Euclidian, with $D_f = 3$.

### 3.2 The origin of fractal structure

Much attention has been devoted to conceiving a mechanism responsible for forming the hierarchical structure of the observable universe (see, e.g. [1]). Generally, one may envisage two principal ways of forming fractal structure: (i) breaking apart a primordial cosmic objects (so-called structuring from above), and (ii) lumping together initial smaller units into ever higher order systems (forming from below). Numerical research of forming the fractal structure from a homogeneous self-gravitating
cosmic matter have been carried out (see, e.g. [25]). The common rationale for forming (any) structure with galaxies as units is the presence of the Newtonian force of the universal attraction.

Within the model of an expanding Universe (Hubble flow), two competing effects are operative. Hubble flow tends to preserve (assumed) initial, presumably homogeneous distribution, whereas mutual gravitational forces force the galaxies two come together and form clusters etc. The later may join each other to form superclusters, and the process may continue (in principle) indefinitely (the so-called bottom-up mechanism, see, e.g. [26]. Of course, with the assumption that the age of the Universe is finite, there is no (cosmic) time for forming too many hierarchical levels, and the net result of clustering would be the present existence of finite number of hierarchical levels, as observed.

Generally, therefore, we take Newton gravitation as an destabilizing factor, whereas Hubble flow reduce the inevitable instability of the system with attractive forces only.

We examine here possibility of deriving the hierarchical structure within the Newtonian gravitational dynamics applied to the approximately homogeneous statistical ensemble of galaxies.

We start with nearly homogeneously distributed statistical ensemble of the galaxies with constant total mass $m$ and mass density distribution $\rho$ within a sphere

$$\rho_m^H = \frac{3m}{4\pi r^3},$$

(9)

In the lowest, zero approximation order, with a completely continuous and homogeneous distribution of $m/V$, absolute value of the classical Newtonian gravitational force at the unit mass on the sphere surface equals

$$F = \frac{GM}{r^2} = \frac{G\rho_m^H V}{r^2},$$

(10)

Now, we take higher order approximation, which assumes discrete and not quite homogeneous distribution of $m$ over $V$. It, formally, can be done by the second order Taylor expansion (with absolute value of any term and finite value of the variation $\delta r$ in (10) of $\rho_m^H/r^2$ over $r$, with $V$ fixed. We have

$$\langle F \rangle = G\rho_m^H \frac{V}{r^2} + G\rho_m^H V \left| \frac{d}{dr} \left( \frac{1}{r^2} \right) \right| \delta r + \frac{1}{2} G\rho_m^H V \left| \frac{d^2}{dr^2} \left( \frac{1}{r^2} \right) \right| \delta r^2,$$

(11)

$$\langle F \rangle = G\rho_m^H \frac{V}{r^2} + 2G\rho_m^H \delta r \frac{V}{r^3} + 3G\rho_m^H \delta r \frac{V}{r^4},$$

(12)

$$\langle F \rangle = G\rho_m^H \left[ 1 + 2 \frac{\delta r}{r} + 3 \left( \frac{\delta r}{r} \right)^2 \right] \frac{V}{r^2},$$

(13)
From 13 follows

\[ \rho_m = \rho_m^H[1 + 2\frac{\delta r}{r} + 3(\frac{\delta r}{r})^2] \]  \hspace{1cm} (14)

with the form compatible with (2).

4 Conclusion

Unlike the standard cosmological paradigm, which asserts that the universe is homogeneous and isotropic, fractal paradigm is based on the so-called Conditional Cosmological Principle, as formulated by Mandelbrot, which asserts that Cosmos appears isotropic from any occupied point of space. This principle appears less restrictive and more in accordance with general epistemology of science, which is particularly evident in the case of Quantum Mechanics and definitely less metaphysical. The principle treats the observer and the object on equal footing, what appears more acceptable than the usual ”God’s eye” perspective.

We have shown that partial fractal symmetry characteristic for (2) can be simply obtained by classical, Newtonian gravitational dynamics, applied to the nearly homogeneous statistical ensemble of the galaxies. It practically means that partial fractal distribution appears a part of the standard cosmological model [11].

We have defined the partial fractal symmetry and have demonstrated that it is neither totally global fractal (usual fractal) nor totally local fractal (multi-fractal) symmetry. But, for discretely different domains of the cosmic distances this partial fractal symmetry can be effectively approximated by different global fractal symmetric functions with corresponding discretely different fractal dimensions. In this way, on the one hand, partial fractal symmetry appears in a satisfactory agreement with astronomic data which include practically all cosmic scales. We have shown that given partial fractal symmetry can be obtained by classical, Newtonian gravitational dynamics applied on the nearly homogeneous statistical ensemble of the galaxies. In this sense, we show that partial fractal cosmology appears a part of the standard cosmological model.

References

[1] P. Grujić, ”The Concept Of Fractal Cosmos: I. Anaxagoras’ Cosmology”, Serb. Astron. J. 163 (2001) 21-34.

[2] P. Grujić, ”The Concept Of Fractal Cosmos: II. Modern Cosmology”, Serb. Astron. J. 165 (2002) 45-65.

[3] D. F. Roscoe, ”Via Aristotle, Leibnitz, Berkeley and Mach to necessarily fractal large-scale structure in the Universe”, arXiv:astro-ph/ 0802.2889v1 20 Feb 2008
[4] L. Brenig, "Is quantum mechanics based on an invariance principle?", J. Phys. A: Math. Teor. 40 (2007) 4567-4584.

[5] E. Abdalla, R. Mohayee and M. Ribeiro, "Scale invariance in a perturbed Einstein-deSitter cosmology", Fractals, Vol. 9, No. 4 (2001) 451-462.

[6] Y. V. Baryshev, F. S. Labini, M. Montuori, L. Pietronero and P. Teerikorpi, "On the Fractal Structure of Galaxy Distribution and its Implications for Cosmology", Fractals, Vol. 6, No. 3 (1998) 231-243.

[7] J. Galte and A. Dominguez, "Scaling Laws in the Cosmic Structure and Renormalization Group", arXiv:astro-ph/0610886v1 30 Oct 2006.

[8] M. B. Ribeiro, "Cosmological Distances and Fractal Statistics of Galaxy Distribution", Astron. Astrophys., vol. 429, Nr. 1, (2005) 65-74.

[9] J. L. McCauley, "The Galaxy Distribution Homogeneous, Fractal, or Neither?", Fractals, Vol. 6, No. 2 (1998) 109-119.

[10] J. Gaite and Alvaro Dominguez, "Scaling Laws in the Cosmic Structure and Renormalization Group", ArXiv:astro-ph/0610886v1 30 Oct 2006.

[11] Y. V. Baryshev, and P. Teerikorpi, "Fractal Approach to Large-Scale Galaxy Distribution", ArXiv:astro-ph/0505185v1 10 May 2005.

[12] Y. V. Baryshev, "Conceptual Problems of Fractal Cosmology", ArXiv:astro-ph/9912074v1 3 Dec 1999.

[13] T. Hui-Ching Lu and C. Hellaby, "Obtaining the space metric from cosmological observations", Class. Quantum Grav., 24 (2007) 4107-4131.

[14] J. R. Mureika, "Fractal Holography: a geometric re-interpretation of cosmological large scale structure", ArXiv:gr-qc/0609001v2 17 May 2007.

[15] F. Pompilio and M. Montuori, "An inhomogeneous fractal cosmological model", ArXiv:astro-ph/0111534v1 28 Nov 2001.

[16] A. K. Mittal and D. Lohiya, "From Fractal Cosmography to Fractal Cosmology", ArXiv:astro-ph/0104370v2 27 Apr 2001.

[17] C. V. Charlier, "How an infinite world may be built up", Ark. Mat. Astr. Fys. 16 (1922) 1.

[18] A. Gabrielli, F. Sylos Labini, M. Joyce and L. Pietronero, Statistical Physics for Cosmic Structures,(Springer, Berlin, 2005).
[19] F. Sylos Labini, N. L. Vasilyev, L. Pietronero, and Y. V. Baryshev, "The large scale structure of the galaxy distribution", xx

[20] P. Grujic, "A Simple Newtonian Model for the Fractal Accelerating Universe", Astrophys. Space Sci., 295 (2004) 363-374.

[21] F. Sylos Labini, "Structure and correlations in the large scale universe", Studying Nature Through Centuries, S. Ninkovic and P. Grujic (Eds.), Publ. Astron. Obs. Belgrade, No 85,(2009) 137-147.

[22] P. Grujic, "Some Epistemic Questions of Cosmology", Found. Sci., 12, No 1 (2007) 39-83.

[23] M. Davies, P. J. E. Peebles, Ap. J. 267(1983) 465

[24] P. H. Coleman, L. Pietronero, R. H. Sanders, Astron. Astrophys. 200 (1988) L32

[25] F. Combes, "Fractal structures driven by self-gravity", Celest. Mech. Dyn. Astron., 73 (1999) 91-127.

[26] N. Yoshida, "Structure Formation in the Early Universe", Advanced Sciences Letters, 2009 (to be published); arXv:0906.4372v1[astro-ph.CO]