ON A WAVE EQUATION WITH SINGULAR DISSIPATION

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Abstract. In this paper we consider a singular wave equation with distributional and more singular non-distributional coefficients and develop tools and techniques for the phase-space analysis of such problems. In particular we provide a detailed analysis for the interaction of singularities of solutions with strong singularities of the coefficient in a model problem of recent interest.

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1. Introduction

In a recent paper, Munoz, Ruzhansky and Tokmagambetov [9] investigated a particular wave model with singular dissipation arising from acoustic problems. They considered the Cauchy problem

\[ u_{tt} - \Delta u + \frac{b'(t)}{b(t)} u_t = 0 \]

\[ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \]
where $b$ is a piecewise continuous and positive function allowing in particular for jumps and in consequence a non-distributional singular coefficient in Cauchy problem. They considered the solution concept of very weak solutions of this singular problem and showed that this problem is well-posed in this very weak sense. Moreover, they numerically observed in one space dimension a very interesting phenomenon, namely the appearance of a new wave after the singular time travelling in the opposite direction to the main one.

The aim of this paper is two-fold. On the one hand we consider this model and carry out a detailed phase space analysis for families of regularised problems in order to describe the behaviour of the very weak solution in the vicinity of the singular time. This will allow us to show that the numerically observed partial reflection of wave packets at the singular time is really appearing and to calculate the partial reflection indices in terms of the jump of the coefficient. On the other hand this is a model study to develop tools and techniques to treat more general singular hyperbolic problems within the framework of very weak solutions and to provide a symbolic calculus framework for analysing singularities of such solutions.

2. The notion of very weak solutions

We will recall some basic concepts on the notion of very weak solutions for singular problems and comment on their relation to other solution concepts like weak solutions and Colombeau solutions. The concept was introduced by Garetto and Ruzhansky in [6] and further developed in a series of papers with different co-authors, [12], [13], [9], [11], [8] in order to show a wide applicability. The basic idea is as follows. Instead of considering the singular equation itself, one considers a family of regularised equations depending on a regularisation parameter and investigates the behaviour of the family of solutions as the regularisation parameter tends to zero.

Treating distributions and more singular objects as families of regularised objects has a long history. In order to provide a neat solution for the multiplication problem for distributions (on the background of Schwartz [15] famous impossibility result) Colombeau [2] proposed to consider more general algebras of nets of regularised objects modulo negligible nets

$$\mathcal{E}^\infty(\Omega)/\mathcal{N}^\infty(\Omega)$$

(2.1)

where $\mathcal{E}^\infty(\Omega)$ denotes all functions $(0, 1] \ni \epsilon \mapsto f_\epsilon \in C^\infty(\Omega)$ being moderate in the sense that

$$\sup_{x \in K} |\partial^\alpha f_\epsilon(x)| = O(\epsilon^{N-|\alpha|})$$

(2.2)

for some $N$ depending on $K \Subset \Omega$ and the multi-index $\alpha$ and similarly $\mathcal{N}^\infty(\Omega)$ the space of negligible nets satisfying the estimate for any number $N$. Convolution with Friedrichs mollifiers yields an embedding of both smooth functions $C^\infty(\Omega)$ and distributions $\mathcal{D}'(\Omega)$ into this algebra extending in particular multiplication. For more details, see Oberguggenberger [10].

This approach has a serious drawback as the multiplication in these algebras is only consistent with the multiplication of smooth functions and hence in general not consistent with the algebra structure of continuous or measurable functions. This is in particular problematic when applying this concept to well-posedness issues of singular partial differential equations, where the natural spaces are usually of low regularity. To overcome consistency issues, in [6] and later in [12], [13] Ruzhansky...
and his co-authors introduced a different concept of moderateness and negligibility based on natural norms associated to the problem under consideration.

For hyperbolic partial differential equations it seems natural to consider solutions of finite energy and the modification in the approach would be to call a family of solutions moderate if the energy satisfies a polynomial bound with respect to the regularisation parameter, while negligible nets are such that the energy is smaller than any power of the regularisation parameter. Thus the notion of very weak solutions depends on the equation under consideration (in contrast to distributional and Colombeau solutions, but similar to weak or mild solutions).

To make this precise let us define the notion of very weak solutions for wave equations with singular time-dependent coefficients

\[
\partial_t^2 u - a(t) \Delta u + 2b(t) \partial_t u + m^2(t) u = f
\]

for a given singular right-hand side \( f \) and singular coefficients \( a, b, m \). We say that a net \( \epsilon \mapsto u_\epsilon \in C^\infty([0,T]; H^1) \) is a very weak solution of energy type, if there are moderate regularisations \( a_\epsilon, b_\epsilon, m_\epsilon \in E^\infty([0,T]) \) of the coefficients and a \( C^\infty([0,T]; L^2) \)-moderate regularisation of the right-hand side \( f \) in the sense that

\[
\sup_{t \in [0,T]} \| \partial_t^k f_\epsilon(t, \cdot) \|_{L^2} = O(\epsilon^{N-k})
\]

for some number \( N \), such that \( \partial_t^2 u_\epsilon - a_\epsilon(t) \Delta u_\epsilon + 2b_\epsilon(t) \partial_t u_\epsilon + m_\epsilon^2(t) u_\epsilon = f_\epsilon \) holds true for any \( \epsilon > 0 \) and \( u_\epsilon \) itself is \( C^\infty([0,T]; H^1) \)-moderate in the sense that

\[
\sup_{t \in [0,T]} \| \partial_t^k u_\epsilon(t, \cdot) \|_{H^1} = O(\epsilon^{N-k})
\]

holds true for some \( N \).

Based on results from [17] and [16] it was shown in [9] that the model example we will consider later on is well-posed in this very weak sense and that the very weak solution is independent of the choice of the regularising family. For the general singular wave model with singular speed and mass term see [4].

3. OUR MODEL PROBLEM AND GENERAL STRATEGY

We consider the Cauchy problem

\[
bracket{\begin{align*}
\partial_t u + b(t) u_t &= 0 \\
\partial_t u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x)
\end{align*}}
\]

where \( b \) is a piecewise smooth and piecewise continuous function. We are interested in solutions close to a singularity of the coefficient and hence, without loss of generality, we assume that \( b \) has exactly one jump at \( t = 1 \). In particular we require that the limits

\[
b(1_{\pm 0}) = \lim_{t \to 1_{\pm 0}} b(t)
\]

exist for the function itself and also its derivatives. Thus we ask for \( b \) to satisfy the following two assumptions.

(H1): There exists a strictly positive number \( b_0 \) such that \( b(t) \geq b_0 > 0 \).

(H2): \( b \in C^\infty([-\infty, 1]) \cap C^\infty([1, +\infty]), \) having a jump at \( t = 1 \).

In contrast to [9] we do not require \( b \) to be monotonically increasing. Thus, we will not make use of any sign properties of the coefficient later on.
3.1. **Notation.** Throughout this paper we will use the following conventions and symbols.

- We denote the height of the jump of $b$ at $t = 1$ by $h = b(1_+) - b(1_-)$ and denote $H = \frac{b(1_+)}{b(1_+)}$.
- We write $f \lesssim g$ for two functions $f$ and $g$ on the same domain if there exists a positive constant $C$ such that $f \leq Cg$.
- We denote by $\Phi \in C_0^\infty(\mathbb{R})$ a fixed non-negative, continuous and symmetric function, such that $\Phi(-t) = \Phi(t)$, $\text{supp } \Phi = [-K', K']$ (3.3) holds true. We further assume that $\Phi$ is differentiable outside the origin and that
  \[ \Phi(t)^2 \lesssim \begin{cases} \Phi'(t), & t < 0, \\ -\Phi'(t), & t > 0 \end{cases} \] (3.4)
  holds true. This function will play an important role in the definition of zones and symbol classes and will be referred to as the shape function.
- We denote by $\psi \in C_0^\infty(\mathbb{R})$ a fixed non-negative and symmetric mollifier such that $\psi(-t) = \psi(t)$, $\text{supp } \psi = [-K, K]$ and $\int \psi(t) \, dt = 1$ (3.5) with $0 < K \leq K'$ describing the size of its support. We further require that derivatives of $\psi$ are bounded by powers of the shape function $\Phi$, i.e.
  \[ |\partial^k_t \psi(t)| \lesssim \Phi(t)^k \] (3.6)
  for any number $k \in \mathbb{N}$.
- The identity matrix will be denoted by $I$. Furthermore for any square matrix $A$ we denote by $||A||$ its Euclidean matrix norm.

3.2. **Regularisation of the problem.** In order to consider very weak solutions of our model problem, we solve families of regularised problems using the regularisations

\[ b_\epsilon(t) = b * \psi_\epsilon(t) \quad b'_\epsilon(t) = b' * \psi_\epsilon(t) = b * \psi'_\epsilon(t) \] (3.7)

in terms of the mollifier $\psi_\epsilon(t) = \epsilon^{-1} \psi(\epsilon^{-1} t)$ and with $\epsilon \in (0, 1]$. This gives rise to the family of Cauchy problems

\[ u_{tt} - \Delta u + \frac{b'_\epsilon(t)}{b_\epsilon(t)} u_t = 0 \]
\[ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \] (3.8)

parameterised by $\epsilon \in (0, 1]$. Our approach is based on a detailed phase space analysis for this family of problems treating $\epsilon$ as an additional variable of the extended phase space. For this we will introduce two zones and apply a diagonalisation based technique to extract leading order terms in each of them. For details on the diagonalisation procedure and its use in a related singular context we refer to [5] or [18], for a broader discussion of the techniques used see [14].
As the coefficients of (3.18) depend on \( t \) only, we apply a partial Fourier transform with respect to the spatial variables and, thus, reduce consideration the ordinary differential equation

\[
\ddot{u}_t + |\xi|^2 \dot{u} + \frac{b'_\epsilon(t)}{b_\epsilon(t)} \dot{u}_t = 0 \\
\dot{u}(0, \xi) = \tilde{u}_0(\xi), \quad \ddot{u}_t(0, \xi) = \tilde{u}_1(\xi)
\]  

(3.9)

parameterised by both \( \epsilon \in (0, 1) \) and \( \xi \in \mathbb{R}^n \). We construct its solutions for \( t \in [0, 2] \) and investigate the limiting behaviour of solutions as \( \epsilon \to 0 \). To write the equation in system form we introduce the micro-energy

\[
U(t, \xi, \epsilon) = \begin{pmatrix} |\xi| \dot{u} \\ D_t \dot{u} \end{pmatrix}
\]

(3.10)

where \( D_t = -i\partial_t \) denotes the Fourier derivative such that (3.9) rewrites as

\[
D_t U(t, \xi, \epsilon) = \begin{pmatrix} 0 & |\xi| \\ |\xi| & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & i\partial_\epsilon(t) \end{pmatrix} U(t, \xi, \epsilon)
\]

(3.11)

where we used the notation \( \partial_\epsilon(t) = \frac{b'_\epsilon(t)}{b_\epsilon(t)} \) for the net of dissipation coefficients. Denoting the coefficient matrices arising in this system by

\[
A(\xi) = \begin{pmatrix} 0 & |\xi| \\ |\xi| & 0 \end{pmatrix} \quad \text{and} \quad B(t, \epsilon) = \begin{pmatrix} 0 & 0 \\ 0 & i\partial_\epsilon(t) \end{pmatrix}
\]

(3.12)

we see that depending on the values \(|\xi|, \epsilon \) and \( t \) either the matrix \( A(\xi) \) is dominant or the matrix \( B(t, \epsilon) \). If \( A(\xi) \) is dominant, we apply a standard hyperbolic approach and diagonalise the system. If \( B(t, \epsilon) \) is dominant, we use a transformation of variables to reduce consideration to a model equation describing the behaviour close to the singularity.

### 3.3. Zones

To make use of different leading terms we use the following definition of zones. For a later to be fixed zone constant \( N \) we define the hyperbolic zone

\[
Z_{\text{hyp}}(N) = \{(t, \xi, \epsilon) \in [0, 2] \times \mathbb{R}^n \times (0, 1) : |\xi| \geq N[\Phi_\epsilon(t - 1) + 1] \}, \quad (3.13)
\]

where \( \Phi_\epsilon(t) = \epsilon^{-1} \Phi(\epsilon^{-1}t) \) is defined in terms of the function \( \Phi \) from Section 3.1.

The singular zone

\[
Z_{\text{sing}}(N) = \{(t, \xi, \epsilon) \in [0, 2] \times \mathbb{R}^n \times (0, 1) : N < |\xi| \leq N[\Phi_\epsilon(t - 1) + 1] \} \quad (3.14)
\]

is used to investigate the vicinity of the jump of the coefficient while the remaining bounded frequencies

\[
Z_{\text{bd}}(N) = \{(t, \xi, \epsilon) \in [0, 2] \times \mathbb{R}^n \times (0, 1) : |\xi| \leq N \}
\]

(3.15)

will be dealt with later by a simple argument. The common boundary of the hyperbolic and the singular zone will be denoted by \( (t_{\xi_i}(\epsilon))_{i=1,2} \) and defined implicitly by the equation

\[
|\xi| = N[\Phi_\epsilon(t_{\xi_i} - 1) + 1] \quad (3.16)
\]

for \( \xi \) satisfying \( N < |\xi| \leq N(\epsilon^{-1}\Phi(0) + 1) \) and with the convention that \( t_{\xi_i} \) is the solution branch for \( t < 1 \) and \( t_{\xi_2} \) when \( t > 1 \). The zones are depicted in Figure [ ].

We will omit writing down the \( \epsilon \)-dependence later on to shorten notation.

The singular zone \( Z_{\text{sing}}(N) \) is better understood in the variables \( \Lambda = \epsilon|\xi| \) and \( \tau = \epsilon^{-1}(t - 1) \). Then the definition of the singular zone rewrites as

\[
Z_{\text{sing}}(N) = \{ (\tau, \Lambda, \epsilon) : N\epsilon < \Lambda \leq N\Phi(\tau) + N\epsilon \}
\]

(3.17)
and is morally independent of the choice of $\epsilon$. We will use these singular variables when discussing the solutions of the regularised problem in the singular zone. For convenience the zone is depicted in Figure 2 using these variables. We will also use a notation for the zone-boundary and denote it by $\tau_{\Lambda_1}(\epsilon)$ and $\tau_{\Lambda_2}(\epsilon)$.

Our strategy is as follows. Within the hyperbolic zone we will apply a diagonalisation procedure taking care of the $\epsilon$-dependence of the transformation matrices and all appearing symbols in an appropriate way. This allows to construct the fundamental solution of the parameter-dependent family \((3.8)\) within the hyperbolic zone and to investigate its limiting behaviour as $\epsilon \rightarrow 0$. Within the singular zone, we transform the problem to the singular variables and construct its fundamental solution.
solution as power series in $\Lambda$ with $\tau, \epsilon$-dependent coefficients and again study the limiting behaviour of this solution as $\epsilon \to 0$.

Remark 3.1. We note that in coordinates $(t, \xi)$, (Figure 1), the point $C(\epsilon)$ tends to $\infty$ when $\epsilon$ tends to 0 and that $t_{\min}$ and $t_{\max}$ depend on $\epsilon$ and tend to $1_-0$ and $1_+0$ when $\epsilon \to 0$, respectively.

Remark 3.2. The interval $[1 - \epsilon K, 1 + \epsilon K]$ (see Figure 1) is the support of $\psi_\epsilon(t-1)$. The lines $t = 1 - \epsilon K$ and $t = 1 + \epsilon K$ divide the hyperbolic zone into two parts, one with $|t - 1| > \epsilon K$ and another one with $|t - 1| < \epsilon K$. The last one is of minor interest since the points $A(\epsilon) = N\epsilon^{-1}\Phi(K) + N$ and $B(\epsilon)$ tend to infinity when $\epsilon$ tends to 0.

3.4. Regular faces of the zones. The hyperbolic zone $Z_{hyp}(N)$ and the zone of bounded frequencies $Z_{bd}(N)$ have a boundary on which $\epsilon \to 0$. This will be of importance later on when relating our representation of very weak solutions with the standard theory for smooth coefficients for $t \neq 1$.

We refer to the two parts $\{(t, \xi, 0) \mid |\xi| > N, \ t \neq 1\}$ as the regular face of $Z_{hyp}(N)$ and the set $\{(t, \xi, 0) \mid |\xi| \leq N\}$ as the regular face of $Z_{bd}(N)$. The singular zone does not have a regular face.

4. Representation of solutions

4.1. Some useful lemmata. The nets $\partial_\epsilon(t) = b_\epsilon'(t)/b_\epsilon(t)$, $b_\epsilon(t)$ and its derivatives $b_\epsilon^{(k)}(t)$ defined in terms of (3.7) satisfy the following inequalities.

Lemma 4.1. The estimates

$$|\partial_\epsilon^k b_\epsilon(t)| \leq C_{1, k} [\Phi_\epsilon(t-1) + 1]^k$$
$$|\partial_\epsilon^k \partial_\epsilon(t)| \leq C_{2, k} [\Phi_\epsilon(t-1) + 1]^{k+1}$$

(4.1)

hold true for all $k \geq 0$ and all $t \in [0, 2]$ and $\epsilon \in (0, 1]$.

Proof. The second estimate will follow from the first one, so we concentrate on the first. For $k = 0$ we use that by Assumption (H1) and (H2) and the positivity of the mollifier $\psi$

$$0 < b_0 \leq b_\epsilon(t) = \int_{-\infty}^{\infty} b(t - \epsilon s)\psi(s)ds \leq \max_{s \in [-K, 1+K]} b(s).$$

(4.2)

For $k = 1$ we apply integration by parts. As

$$b_\epsilon'(t) = \epsilon^{-2} \int_{-\infty}^{\infty} b(s)\psi'(\epsilon^{-1}(t - s))ds$$
$$= \epsilon^{-2} \int_{-\infty}^{1} b(s)\psi'(\epsilon^{-1}(t - s))ds + \int_{1}^{\infty} b(s)\psi'(\epsilon^{-1}(t - s))ds$$
$$= \epsilon^{-1}\psi(\epsilon^{-1}(t - 1))[b(1_0) - b(1_{+0})]$$
$$- \epsilon^{-1} \int_{-\infty}^{1} b'(s)\psi(\epsilon^{-1}(t - s))ds + \int_{1}^{\infty} b'(s)\psi(\epsilon^{-1}(t - s))ds$$

we obtain

$$|b_\epsilon'(t)| \leq |b|\psi_\epsilon(t-1) + \sup_{s \in [-K, 1+K]} |b'(s)| \leq C_{1} (\Phi_\epsilon(t-1) + 1)$$

(4.4)
using the bound \( \psi \lesssim \Phi \). For higher \( k \) we have to apply several steps of integration by parts. For \( k \geq 2 \) we obtain by induction
\[
\partial_t^k b_s(t) = \sum_{\ell=0}^{k-1} (-1)^{k-\ell-1} \left[ b^{(k-\ell-1)}(1_-,0) - b^{(k-\ell-1)}(1_+,0) \right] \partial_t^\ell \psi_s(t-1)
\]
\[
+ (-1)^k \int_{-\infty}^1 b^{(k)}(s) \psi_s(t-s) ds + \int_1^\infty b^{(k)}(s) \psi_s(t-s) ds \] \hspace{1cm} (4.5)

Again the remaining integrals can be estimated by uniform bounds on the derivatives of \( b \) outside the singularity and the statement follows from
\[
|\partial_t^k b_s(t)| \leq \sum_{\ell=0}^{k-1} \left| b^{(k-\ell-1)}(1_-,0) - b^{(k-\ell-1)}(1_+,0) \right| |\partial_t^\ell \psi_s(t)|
\]
\[
+ \sup_{s \in [-\epsilon K,1) \cup [1,1+K]} |b^{(k)}(s)|
\]
\[
\leq C_k (\Phi_s(t) + 1)^k
\]
by estimating \( |\partial_t^k \psi(t)| \leq C \Phi^k(t) \).

Finally, the estimate for derivatives of \( \partial_s(t) \) follow from applying the quotient rule and using the uniform lower bound \( b_0 \leq b_s(t) \) for estimating the denominator. \( \square \)

**Lemma 4.2.** The estimates
\[
|\partial_t^k b_s(t) - \partial_t^k b(t)| \lesssim \epsilon
\]
hold true for all \( k \geq 0 \) uniformly in \( \epsilon \in (0,1] \) and \( t \) satisfying \( |t-1| > \epsilon K \).

**Proof.** Let \( |t-1| > \epsilon K \). For \( k = 0 \) we use
\[
b_s(t) - b(t) = \int_{-\infty}^\infty b(s) \psi_s(t-s) ds - b(t) = \int_{[t-\epsilon K,t+\epsilon K]} (b(s) - b(t)) \psi_s(t-s) ds \] \hspace{1cm} (4.8)

using that \( \int \psi(s) ds = 1 \) and that \( \text{supp} \psi_s = [-\epsilon K, \epsilon K] \). Hence,
\[
|b_s(t) - b(t)| \leq \int_{[t-\epsilon K,t+\epsilon K]} |b(s) - b(t)| \psi_s(t-s) ds. \] \hspace{1cm} (4.9)

As the range of integration does not contain 1 we can use the differentiability of \( b \) to estimate
\[
|b(s) - b(t)| \leq |s-t| \sup_{\theta \in [t-\epsilon K,t+\epsilon K]} |b'(\theta)| = M |s-t| \] \hspace{1cm} (4.10)
for \( s \in [t-\epsilon K,t+\epsilon K] \). Therefore
\[
|b_s(t) - b(t)| \leq \epsilon M K. \] \hspace{1cm} (4.11)

For \( k \geq 1 \) the argumentation is similar using
\[
\partial_t^k b_s(t) - \partial_t^k b(t) = \int_{[t-\epsilon K,t+\epsilon K]} (\partial_t^k b(s) - \partial_t^k b(t)) \psi_s(t-s) ds \] \hspace{1cm} (4.12)

with the corresponding bound on the derivatives of \( b \) on the interval of integration. \( \square \)

These two technical lemmas are the model behaviours for our symbol classes and the key estimates for the boundary behaviour at regular faces of the zones.
4.2. Treatment in the hyperbolic zone.

4.2.1. Symbol classes and their properties. For the treatment within the hyperbolic zone symbol classes and their basic calculus properties are used.

Definition 4.3 (Symbol classes). Let \( N > 0 \) be fixed and \( \Phi \) as in Section 3.1.

(i) We say that a function

\[
\phi \in C^\infty([0, 2] \times \mathbb{R}^n \times (0, 1])
\]

belongs to the hyperbolic symbol class \( \mathcal{S}_{N,\Phi}\{m_1, m_2\} \) if it satisfies the estimates

\[
|\partial_t^k \partial_\xi^\alpha \phi(t, \xi, \epsilon)| \leq C_{k,\alpha} |\Phi_\epsilon(t - 1) + 1|^{m_2 + k} |\xi|^{m_1 - |\alpha|}
\]

uniformly within \( Z_{\text{hyp}}(N) \) for all non-negative integers \( k \in \mathbb{N}_0 \) and all multi-indices \( \alpha \in \mathbb{N}_0^n \) together with the existence of the limits

\[
a(t, \xi, 0) = \lim_{\epsilon \to 0} a(t, \xi, \epsilon), \quad t \neq 1,
\]

at the regular face of the zone satisfying the estimates

\[
|\xi|^{\alpha - m_1} |\partial_t^k \partial_\xi^\alpha \phi(t, \xi, 0)| \leq C'_{k,\alpha},
\]

\[
|\xi|^{\alpha - m_1} \left| \partial_t^k \partial_\xi^\alpha \left( a(t, \xi, \epsilon) - a(t, \xi, 0) \right) \right| \leq C''_{k,\alpha} \epsilon
\]

the latter one uniformly on \( |t - 1| \geq \epsilon K \).

(ii) We say that a matrix-valued function \( A \) belongs to \( \mathcal{S}_{N,\Phi}\{m_1, m_2\} \) if all its entries belongs to the scalar-valued symbol class \( \mathcal{S}_{N,\Phi}\{m_1, m_2\} \).

Example 4.1. Due to Lemma 4.1 we know that the regularising families \( b_\epsilon \) and \( \delta_\epsilon \) satisfy

\[
(b_\epsilon) \in \mathcal{S}_{N,\Phi}\{0, 0\}, \quad (\delta_\epsilon) \in \mathcal{S}_{N,\Phi}\{0, 1\}
\]

for any zone constant \( N > 0 \). Similarly \( |\xi|^{\alpha} \) is a symbol from \( \mathcal{S}_{N,\Phi}\{1, 0\} \) for any admissible \( \Phi \) and \( N > 0 \).

Remark 4.2. The boundary behaviour of symbols given by (4.16) corresponds to a characterisation of symbol classes defined on the regular face \( \{(t, \xi, 0) \mid |\xi| \geq N\} \) of \( Z_{\text{hyp}}(N) \) with symbol estimates uniform with respect to \( t \).

Increasing the zone constant \( N \) makes the hyperbolic zone \( Z_{\text{hyp}}(N) \) smaller and thus the symbol class \( \mathcal{S}_{N,\Phi}\{m_1, m_2\} \) larger. We will make use of this fact later on by choosing \( N \) sufficiently large in order to guarantee smallness of some terms. We will omit the indices \( N \) and \( \Phi \) to simplify notation.

Proposition 4.4 (Properties of symbol classes). For any fixed \( N > 0 \) and admissible \( \Phi \) the following statements hold true:

1. \( \mathcal{S}\{m_1, m_2\} \) is a vector space.
2. \( \mathcal{S}\{m_1, m_2\} \subset \mathcal{S}\{m_1 + \ell_1, m_2 - \ell_2\} \) for all \( \ell_1 \geq \ell_2 \geq 0 \).
3. If \( f \in \mathcal{S}\{m_1, m_2\} \) and \( g \in \mathcal{S}\{m_1', m_2'\} \) then \( f \cdot g \in \mathcal{S}\{m_1 + m_1', m_2 + m_2'\} \).
4. If \( f \in \mathcal{S}\{m_1, m_2\} \) then \( \partial_\xi^\alpha f \in \mathcal{S}\{m_1, m_2 + k\} \) and \( \partial_t^k f \in \mathcal{S}\{m_1 - |\alpha|, m_2\} \).
5. If \( f \in \mathcal{S}\{m_1, 0\} \) satisfies \( |f(t, \xi, \epsilon)| \geq c|\xi|^{m_1} \) for a positive constant \( c \), then inverse of the symbol satisfies \( 1/f \in \mathcal{S}\{-m_1, 0\} \).
Proof. Properties (1) and (4) follow immediately from the definition of the symbol classes. For (3) we apply the product rule for derivatives to derive the symbol estimate. The boundary behaviour (4.17) follows by

\[ f(t, \xi, \epsilon)g(t, \xi, \epsilon) - f(t, \xi, 0)g(t, \xi, 0) = (f(t, \xi, \epsilon) - f(t, \xi, 0))g(t, \xi, \epsilon) + f(t, \xi, \epsilon) (g(t, \xi, \epsilon) - g(t, \xi, 0)) \]

and combining this with the product rule and both estimates (4.16) and (4.17) from the symbol classes for each the factors on the right. To prove (2) we use the definition of the hyperbolic zone \( Z_{\text{hyp}}(N) \) in the form

\[ |\Phi_\epsilon(t - 1) + 1|^{-\ell_2} |\xi|^{\ell_1} \geq N^{\ell_1} [\Phi_\epsilon(t - 1) + 1]^{\ell_1 - \ell_2} \geq N^{\ell_1} \]

and conclude that symbol estimates from \( S\{m_1, m_2\} \) imply symbol estimates from \( S\{m_1 + \ell_1, m_2 - \ell_2\} \). It remains to prove (5). Here we use Faà di Bruno’s formula \([7],[3]\) (\( \text{II} \)) and write

\[ \frac{\partial^k}{\partial \xi^k} \left( \frac{1}{f(t, \xi, \epsilon)} \right) = \sum_{\ell = 1}^{k + |\alpha|} \sum_{j_1 + \cdots + j_\ell = k} C_{k, \alpha, j_1, \cdots, j_\ell} \frac{\partial_{\xi_{j_1}}^{\alpha_{j_1}} f \cdots \partial_{\xi_{j_\ell}}^{\alpha_{j_\ell}} f}{f^{\ell + 1}} \]

where \( C_{k, \alpha, j_1, \cdots, j_\ell} \) are constants depending on the order of the derivatives. Each term in the last sum can be estimated in the following way. As \( f \in S\{m_1, 0\} \) property (4) implies for \( i = 1, \ldots, \ell \) that

\[ |\partial_{\xi_i}^{\alpha_i} f| \leq C_{j_i, \alpha_i} |\Phi_\epsilon(t - 1) + 1|^{j_i} |\xi|^{m_1 + |\alpha_i|} \]

Therefore

\[ |\partial_{\xi_i}^{\alpha_i} f \cdots \partial_{\xi_\ell}^{\alpha_\ell} f| \lesssim |\Phi_\epsilon(t - 1) + 1|^{j_1 + \cdots + j_\ell} |\xi|^{m_1 + \cdots + |\alpha_i|} \]

and using the condition \(|f(t, \xi, \epsilon)| > c|\xi|^{m_1}\) we obtain

\[ \left| \frac{\partial_{\xi_i}^{\alpha_i} f \cdots \partial_{\xi_\ell}^{\alpha_\ell} f}{f^{\ell + 1}} \right| \lesssim \frac{|\Phi_\epsilon(t - 1) + 1|^{j_1 + \cdots + j_\ell} |\xi|^{m_1 + \cdots + |\alpha_i|}}{|\xi|^{m_1(t + 1)}} \]

\[ \lesssim |\Phi_\epsilon(t - 1) + 1|^{j_1 + \cdots + j_\ell} |\xi|^{m_1(t + 1) - m_1 + |\alpha_i|} \]

\[ \lesssim |\Phi_\epsilon(t - 1) + 1|^{k} |\xi|^{-m_1 + |\alpha_i|}. \]

Summing all these terms yields the desired estimate. The boundary estimate follows on similar lines. \( \square \)

These symbol classes and in particular the embeddings

\[ S\{-1, 2\} \hookrightarrow S\{0, 1\} \hookrightarrow S\{1, 0\} \]

will be of importance for the treatment within the hyperbolic zone. The gain of decay in \( |\xi| \) will be paid for by a loss of point-wise control in the \( t \)-variable near the singularity. What we gain are integrability properties and improved limits at the regular face.

**Proposition 4.5.** Within the hyperbolic zone \( Z_{\text{hyp}}(N) \),

1. symbols from \( S\{0, 0\} \) are uniformly bounded,
2. symbols from \( S\{0, 1\} \) are uniformly integrable with respect to \( t \),
(3) symbols \( a \in \mathcal{S}\{-1,2\} \) satisfy
\[
\int_0^t |a(\theta, \xi, \epsilon)| d\theta \leq C|\xi|^{-1} [\Phi_\epsilon(t - 1) + 1] 
\]  
(4.26)
for all \( 0 < t \leq t\xi_1 \), and
\[
\int_t^2 |a(\theta, \xi, \epsilon)| d\theta \leq C|\xi|^{-1} [\Phi_\epsilon(t - 1) + 1] 
\]  
(4.27)
for all \( t\xi_2 \leq t \leq 2 \).

Proof. (1) is obvious from the definition of the symbol class.
(2) If \( f \in \mathcal{S}\{0,1\} \) then it satisfies the point-wise estimate
\[
|f(t, \xi, \epsilon)| \leq C[\Phi_\epsilon(t - 1) + 1] 
\]  
(4.28)
and therefore after integrating over \( t \in [0, t\xi_1(\epsilon)] \) (or similarly over \( t \in [t\xi_2(\epsilon), 2] \))
\[
\int_0^{t\xi_1} |f(t, \xi, \epsilon)| ds|dt \leq C \int_0^{t\xi_1} \Phi_\epsilon(t - 1) dt + C \int_0^{t\xi_1} dt
\]
\[
= C \int_{-\epsilon^{-1}}^{-\epsilon^{-1}(t\xi_1)} \Phi(\tau) d\tau + C \left( 1 + \int_{-\epsilon^{-1}}^{0} \Phi(\tau) d\tau \right) 
\]  
(4.29)
for any fixed \( \epsilon \in (0,1) \) and \( \xi \in \mathbb{R}^n \).
(3) If \( a \in \mathcal{S}\{-1,2\} \) then it satisfies the point-wise estimate
\[
|a(t, \xi, \epsilon)| \leq C|\xi|^{-1} [\Phi_\epsilon(t - 1) + 1]^2 
\]  
(4.30)
and the only new term to be treated is the one arising from the square of the shape function. This can be estimated by means of \( \text{(3.4)} \) for \( t < 1 \) as
\[
\int_0^t \Phi_\epsilon(\theta)^2 d\theta = \epsilon^{-1} \int_{-\epsilon^{-1}}^{-\epsilon^{-1}(t - 1)} \Phi(\tau)^2 d\tau \leq C\epsilon^{-1} \int_{-\epsilon^{-1}}^{-\epsilon^{-1}(t - 1)} \Phi'(\tau) d\tau
\]
\[
\leq C\epsilon^{-1}\Phi(-\epsilon^{-1}(t - 1)) - C\epsilon^{-1}\Phi(-\epsilon^{-1}) \leq C\Phi_\epsilon(t - 1) 
\]  
(4.31)
and similarly for the case \( t > 1 \). \( \square \)

4.2.2. Transformations. Within the hyperbolic zone we apply transformations to our system in order to extract precise information about the behaviour of its fundamental solution. Recall that \( \text{(3.11)} \) is of the form \( D_t U = (A + B)U \) with \( A \in \mathcal{S}\{1,0\} \) and \( B \in \mathcal{S}\{0,1\} \). Using the diagonaliser of the principal part \( A \)
\[
M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{with inverse } M^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} 
\]  
(4.32)
we can rewrite \( A \) as
\[
A(\xi) = MD(\xi)M^{-1} 
\]  
(4.33)
with \( D(\xi) = \text{diag}(|\xi|, -|\xi|) \). Hence, writing \( V(t, \xi, \epsilon) := M^{-1}U(t, \xi, \epsilon) \) system \( \text{(3.11)} \) rewrites as
\[
D_t V(t, \xi, \epsilon) = |D(\xi) + R(t, \epsilon)| V(t, \xi, \epsilon) 
\]  
(4.34)
with a remainder given by
\[
R(t, \epsilon) = M^{-1}B(t, \epsilon)M = \frac{i}{2} D_\epsilon(t) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{S}\{0,1\}. 
\]  
(4.35)
Our aim is to further improve the remainder within the hyperbolic hierarchy \( \text{(4.25)} \). This allows to extract more detailed information on the propagation of singularities.
close to the singularity later on. For this we follow [14] construct transformation matrices \( N_k(t, \xi, \epsilon) \) transforming the system (4.34) into a new system with an updated diagonal part and an improved remainder. The construction is done in such a way that the operator identity

\[
(D_t - D(\xi) - R(t, \epsilon)) N_k(t, \xi, \epsilon) = N_k(t, \xi, \epsilon) (D_t - D_k(t, \xi, \epsilon) - R_k(t, \xi, \epsilon))
\]  

(4.36)

holds true for \( k \geq 1 \) and

- the matrix-valued symbols \( D_k(t, \xi, \epsilon) \) are given by
  \[
  D_k(t, \xi, \epsilon) = D(\xi) + F^{(0)}(t, \xi, \epsilon) + \cdots + F^{(k-1)}(t, \xi, \epsilon)
  \]
  (4.37)
  with diagonal \( F^{(j)}(t, \xi, \epsilon) \in S\{-j, j + 1\} \);
- the transformation matrices \( N_k(t, \xi, \epsilon) \) are of the form
  \[
  N_k(t, \xi, \epsilon) = I + N^{(1)}(t, \xi, \epsilon) + \cdots + N^{(k)}(t, \xi, \epsilon)
  \]
  (4.38)
  with \( N^{(j)}(t, \xi, \epsilon) \in S\{-j, j\} \);
- the remainder satisfies \( R_k(t, \xi, \epsilon) \in S\{-k, k + 1\} \).

We give the construction for \( k = 1 \) in full detail. In this case (4.36) simplifies modulo \( S\{-1, 2\} \) to the commutator equation

\[
\left[D(\xi), N^{(1)}(t, \xi, \epsilon)\right] = F^{(0)}(t, \xi, \epsilon) - R(t, \epsilon).
\]  

(4.39)

As the diagonal part of the commutator vanishes, we set

\[
F^{(0)}(t, \epsilon) = \text{diag} R(t, \epsilon) = \frac{i}{2} \partial_k(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S\{0, 1\}
\]

(4.40)

and determine the off-diagonal entries of

\[
N^{(1)}(t, \xi, \epsilon) = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}
\]

(4.41)

from (4.39) as

\[
n_{12} = -\frac{i}{4|\xi|} \partial_k(t) \quad \text{and} \quad n_{21} = \frac{i}{4|\xi|} \partial_k(t).
\]

The diagonal entries are chosen as zero, hence

\[
N^{(1)}(t, \xi, \epsilon) = \frac{i}{4 |\xi|} \partial_k(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in S \{-1, 1\}.
\]

(4.42)

The transformation matrix \( N_1(t, \xi, \epsilon) = I + N^{(1)}(t, \xi, \epsilon) \) is invertible provided that the zone constant is chosen large enough.

**Proposition 4.6.** Assume Hypotheses (H1) and (H2). Then, there exists a matrix \( N^{(1)}(t, \xi, \epsilon) \in S \{-1, 1\} \) and a diagonal matrix \( F^{(0)}(t, \epsilon) \in S\{0, 1\} \) such that the identity (4.36) is satisfied with a remainder \( R_1(t, \xi, \epsilon) \in S\{-1, 2\} \). Moreover, we can find a zone constant \( N \) such that the transformation matrix \( N_1(t, \xi, \epsilon) = I + N^{(1)}(t, \xi, \epsilon) \) is invertible in \( Z_{\text{hyp}}(N) \) and \( N_1(t, \xi, \epsilon)^{-1} \in S\{0, 0\} \).

**Proof.** It remains to show the invertibility of \( N_1(t, \xi, \epsilon) \). Indeed, by (4.42) it follows that

\[
\det N_1 = 1 - \frac{\partial_k^2(t)}{16 |\xi|^2}
\]

(4.43)

and by Lemma 4.1

\[
\frac{\partial_k^2(t)}{16 |\xi|^2} \leq \frac{[c_1 \Phi(t - 1) + c_2]^2}{16 |\xi|^2}.
\]  

(4.44)
Hence, by choosing the zone constant $N$ large enough such that
\[ c_1 \psi_e(t - 1) + c_2 \leq N \psi_e(t - 1) + N \]  
the invertibility follows. By the calculus rules of Proposition 4.4 we also conclude $N^{-1}_1 \in S\{0, 0\}$.

The matrices $N^{(1)}(t, \xi, \epsilon) \in S\{-1, 1\}$ and $F^{(0)}(t, \epsilon) \in S\{0, 1\}$ are already constructed in such a way that (4.36) holds true with remainder
\[ R_1(t, \xi, \epsilon) = N_1(t, \xi, \epsilon)^{-1} \left( R(t, \epsilon) N^{(1)}(t, \xi, \epsilon) - D_1 N^{(1)}(t, \xi, \epsilon) \right. 
\[ \left. - N^{(1)}(t, \xi, \epsilon) F^{(0)}(t, \epsilon) \right) \in S\{-1, 2\} \]  
and the statement is proven. \qed

Remark 4.3. Taking limits $\epsilon \to 0$ at the regular faces of $Z_{hyp}(N)$ the diagonalisation procedure yields in particular the transformations used to construct representations of solutions in the case of smooth coefficients. In particular the limits
\[ N^{(1)}(t, \xi, 0) = \lim_{\epsilon \to 0} N^{(1)}(t, \xi, \epsilon) \]  
exist for $t \neq 1$ and $|\xi| > N$ and satisfy
\[ \|N_1(t, \xi, \epsilon) - N_1(t, \xi, 0)\| = \|N^{(1)}(t, \xi, \epsilon) - N^{(1)}(t, \xi, 0)\| \leq C \epsilon |\xi|^{-1}, \]  
and as the inverse $N_1(t, \xi, \epsilon)^{-1}$ can be written as Neumann series, we know that $N_1(t, \xi, \epsilon)^{-1} - I \in S\{-1, 1\}$ and consequently
\[ \|N_1(t, \xi, \epsilon)^{-1} - N_1(t, \xi, 0)^{-1}\| \leq C \epsilon |\xi|^{-1}. \]  
Similarly the limit $R_1(t, \xi, 0) = \lim_{\epsilon \to 0} R_1(t, \xi, \epsilon)$ satisfies
\[ \|R_1(t, \xi, \epsilon) - R_1(t, \xi, 0)\| \leq C \epsilon |\xi|^{-1}. \]  
This will allow to relate the construction of fundamental solutions for the regularised family to the fundamental solution of the original problem outside the singularity.

4.2.3. Fundamental solution to the diagonalized system. We now fix the zone constant $N$ large enough to guarantee that $N_1(t, \xi, \epsilon)$ is uniformly invertible within the hyperbolic zone $Z_{hyp}(N)$. Then for $V$ solving (4.34), the transformed function
\[ V_1(t, \xi, \epsilon) = N_1(t, \xi, \epsilon)^{-1} V(t, \xi, \epsilon) \]  
satisfies due to (4.36)
\[ D_1 V_1(t, \xi) = \left( D(\xi) + F^{(0)}(t, \xi, \epsilon) + R_1(t, \xi, \epsilon) \right) V_1(t, \xi) \]  
with diagonal matrix $F^{(0)} \in S\{0, 1\}$ given by (4.46) and a remainder $R_1(t, \xi, \epsilon) \in S\{-1, 2\}$ specified by (4.46). We construct its fundamental solution.

Theorem 4.7. Assume Hypotheses (H1) and (H2). Then the fundamental solution $E_1(t, s, \xi, \epsilon)$ to the transformed system (4.52) can be represented by
\[ E_1(t, s, \xi, \epsilon) = \sqrt{\frac{b_\epsilon(s)}{b_\epsilon(t)}} E_0(t, s, \xi) Q(t, s, \xi, \epsilon) \]  
for $[s, t] \times \{(\xi, \epsilon)\} \subset Z_{hyp}(N)$, where
(1) the factor $\sqrt{\frac{b_\epsilon(s)}{b_\epsilon(t)}}$ describes the main influence of the dissipation term,
(2) the matrix $E_0(t, s, \xi)$ is the fundamental solution of the hyperbolic principal part $\mathcal{D}_t - D(\xi)$ and given by

$$E_0(t, s, \xi) = \begin{pmatrix} \exp(i(t - s)|\xi|) & 0 \\ 0 & \exp(-i(t - s)|\xi|) \end{pmatrix}$$  \hspace{1cm} (4.54)

(3) the matrix $Q(t, s, \xi, \epsilon)$ is uniformly bounded

$$\|Q(t, s, \xi, \epsilon)\| \leq \exp\left(\int_s^t \|R_1(\tau, \xi, \epsilon)\|d\tau\right),$$  \hspace{1cm} (4.55)

uniformly invertible within the hyperbolic zone due to

$$|\det Q(t, s, \xi, \epsilon)| \geq \exp\left(\int_s^t \|R_1(\tau, \xi, \epsilon)\|d\tau\right)$$  \hspace{1cm} (4.56)

and has the precise behaviour for large $|\xi|$ determined by the identity matrix

$$\|Q(t, s, \xi, \epsilon) - I\| \leq \int_s^t \|R_1(t_1, \xi, \epsilon)\| \exp\left(\int_{t_1}^t \|R_1(\tau, \xi, \epsilon)\|d\tau\right)^{dt_1},$$  \hspace{1cm} (4.57)

Proof: We consider first $\mathcal{D}_t - D(\xi) - F^0$, i.e. the main diagonal part of the transformed system (4.52). Its fundamental solution is given by

$$E_0(t, s, \xi, \epsilon) = \sqrt{b(s)} \frac{b(s)}{b(t)} E_0(t, s, \xi, \epsilon)$$  \hspace{1cm} (4.58)

where $E_0(t, s, \xi, \epsilon)$ is the fundamental solution to $\mathcal{D}_t - D(\xi)$ given by (4.54). For the fundamental solution to the system (4.52) we use an ansatz in the form

$$E_1(t, s, \xi, \epsilon) = \sqrt{b(s)} \frac{b(s)}{b(t)} E_0(t, s, \xi, \epsilon) Q(t, s, \xi, \epsilon)$$  \hspace{1cm} (4.59)

for a still to be determined matrix $Q(t, s, \xi, \epsilon)$. A simple calculations shows that $Q(t, s, \xi, \epsilon)$ must solve

$$\mathcal{D}_t Q(t, s, \xi, \epsilon) = \mathcal{R}(t, s, \xi, \epsilon) Q(t, s, \xi, \epsilon), \hspace{0.5cm} Q(s, s, \xi, \epsilon) = I$$  \hspace{1cm} (4.60)

with coefficient matrix

$$\mathcal{R}(t, s, \xi, \epsilon) = E_0(s, t, \xi) R_1(t, \xi, \epsilon) E_0(t, s, \xi)$$  \hspace{1cm} (4.61)

determined by the remainder $R_1$ and the fundamental solution of the hyperbolic principal part $E_0$. The solution $Q$ can thus be represented in terms of Peano-Baker series

$$Q(t, s, \xi, \epsilon) = I + \sum_{k=1}^{\infty} \int_s^t \mathcal{R}(t_1, s, \xi, \epsilon) \int_s^{t_1} \mathcal{R}(t_2, s, \xi, \epsilon) \int_s^{t_2} \cdots \int_s^{t_{k-1}} \mathcal{R}(t_k, s, \xi, \epsilon) dt_k \cdots dt_2 dt_1$$  \hspace{1cm} (4.62)

and it remains to provide estimates based on this series representation. As $E_0$ is unitary, we obtain from the symbol estimate of the remainder $R_1$

$$\|\mathcal{R}(t, s, \xi, \epsilon)\| = \|R_1(t, \xi, \epsilon)\|$$

$$\leq C|\xi|^{-1} (\Phi_\epsilon(t - 1) + 1)^2 \leq \frac{C}{N} (\Phi_\epsilon(t - 1) + 1)$$  \hspace{1cm} (4.63)
and thus it follows that
\[ \|Q(t, s, \xi, \epsilon)\| \leq \exp \left( \int_s^t \|R_1(\tau, \xi, \epsilon)\| d\tau \right) \leq \exp(C/N), \] (4.64)
and together with
\[ \det Q(t, s, \xi, \epsilon) = \exp \left( \int_s^t \text{trace } R_1(\tau, \xi, \epsilon) d\tau \right) \] (4.65)
the uniform invertibility of \( Q \) follows. Furthermore, by using (3.4)
\[ \|Q(t, s, \xi, \epsilon) - I\| \leq C|\xi|^{-1}\Phi(1) + C/N \] (4.66)
and the main contribution of \( Q \) for large \(|\xi|\) is given by the identity matrix. □

4.2.4. Fundamental solution to the original system. After obtaining the fundamental solution to the transformed system (4.52) we go back to the original problem (3.11) and obtain in the hyperbolic zone the representation
\[ E_{\text{hyp}}(t, s, \xi, \epsilon) = \sqrt{\frac{b_{\epsilon}(s)}{b_{\epsilon}(t)}} MN_1(t, \xi, \epsilon)E_0(t, s, \xi, \epsilon) Q(t, s, \xi, \epsilon) N^{-1}(s, \xi, \epsilon) M^{-1} \] (4.67)
for the fundamental solution. We will briefly discuss its limiting behaviour as \( \epsilon \to 0 \) for fixed \( s < t < 1 \) or \( 1 < s < t \). As \( E_0(t, s, \xi) \) is independent of \( \epsilon \) and the transformation matrix \( N_1(t, \xi, \epsilon) \) is already estimated by (4.48), this boils down to considering \( Q(t, s, \xi, \epsilon) \).

Lemma 4.8. The limit
\[ Q(t, s, \xi, 0) = \lim_{\epsilon \to 0} Q(t, s, \xi, \epsilon) \] (4.68)
exists for fixed \( s < t < 1 \) and all fixed \( 1 < s < t \), is uniformly bounded and invertible and satisfies the estimate
\[ \|Q(t, s, \xi, \epsilon) - Q(t, s, \xi, 0)\| \lesssim \epsilon|\xi|^{-1} \] (4.69)
holds true for all \([s, t] \times \{\xi, \epsilon\}\) \( \subset Z_{\text{hyp}} \) with the condition \( \min\{t-1, |s-1|\} \geq \epsilon K \).

Proof. We use (4.50) in combination with (4.62) and consider
\[ Q(t, s, \xi, 0) = 1 + \sum_{k=1}^{\infty} \int_s^t R(t_1, s, \xi, 0) \int_s^{t_1} R(t_2, s, \xi, 0) \int_s^{t_2} \cdots \int_s^{t_{k-1}} R(t_k, s, \xi, 0) dt_1 \cdots dt_{k-1} \] (4.70)
defined in terms of
\[ R(t, s, \xi, 0) = E_0(t, s, \xi) R_1(t, \xi, 0) E_0(t, s, \xi) \] (4.71)
with
\[ \|R(t, s, \xi, 0)\| = \|R_1(t, \xi, 0)\| \leq C|\xi|^{-1} \] (4.72)
uniformly in \( 0 < s < t < 1 \) or \( 1 < s < t < 2 \) and \(|\xi| > N\). It thus follows that \( Q(t, s, \xi, 0) \) is uniformly bounded and uniformly invertible. To estimate the difference between \( Q(t, s, \xi, \epsilon) \) and \( Q(t, s, \xi, 0) \) we use a perturbation argument based on the estimate
\[ \|R(t, s, \xi, \epsilon) - R(t, s, \xi, 0)\| = \|R_1(t, \xi, \epsilon) - R_1(t, \xi, 0)\| \leq C\epsilon|\xi|^{-1} \] (4.73)
for $|t - 1| \geq \epsilon K$ following from (4.50). Differentiating
\begin{equation}
Q(t, s, \xi, \epsilon) = Q(t, s, \xi, 0) \Xi(t, s, \xi, \epsilon)
\end{equation}
yields for $\Xi(t, s, \xi, \epsilon)$ the equation

\begin{equation}
D_t \Xi(t, s, \xi, \epsilon) = Q(t, s, \xi, 0) (R(t, s, \xi, \epsilon) - R(t, s, \xi, 0)) Q(s, t, \xi, 0) \Xi(t, s, \xi, \epsilon)
\end{equation}

with initial condition $\Xi(s, s, \xi, \epsilon) = 1$ and coefficient matrix estimated by

\begin{equation}
\|Q(t, s, \xi, 0) (R(t, s, \xi, \epsilon) - R(t, s, \xi, 0)) Q(s, t, \xi, 0)\| \leq C\epsilon|\xi|^{-1}
\end{equation}

Therefore, using the representation of $\Xi(t, s, \xi, \epsilon)$ in terms of Peano–Baker series we obtain the estimate

\begin{equation}
\|\Xi(t, s, \xi, \epsilon)\| \leq \exp(C\epsilon|\xi|^{-1}|t - s|) = 1 + O(\epsilon|\xi|^{-1})
\end{equation}

uniform with respect to $t$ and $s$ with $|t - 1|, |s - 1| \geq \epsilon K$ and thus the desired statement follows.

**Proposition 4.9.** The estimate

\begin{equation}
\|\xi_{\text{hyp}}(t, s, \xi, \epsilon) - \xi_{\text{hyp}}(t, s, \xi, 0)\| \lesssim \epsilon
\end{equation}

holds true for all $[s, t] \times \{ (\xi, \epsilon) \} \subset Z_{\text{hyp}}$ with the condition $\min\{ |t - 1|, |s - 1| \} \geq \epsilon K$.

**Proof.** The proof follows directly from the representation (4.67) of $\xi_{\text{hyp}}(t, s, \xi, \epsilon)$ combined with analogous formula for the limit $\xi_{\text{hyp}}(t, s, \xi, 0) = \lim_{\epsilon \to 0} \xi_{\text{hyp}}(t, s, \xi, \epsilon)$. As all terms in (4.67) are uniformly bounded within the hyperbolic zone we obtain

\begin{equation}
\|\xi_{\text{hyp}}(t, s, \xi, \epsilon) - \xi_{\text{hyp}}(t, s, \xi, 0)\| \lesssim \left\| \frac{b_\epsilon(s)}{b_\epsilon(t)} - \frac{b_\epsilon(0)}{b_\epsilon(0)} \right\|
\end{equation}

\begin{equation}
+ \|N_1(t, \xi, \epsilon) - N_1(t, \xi, 0)\|
\end{equation}

\begin{equation}
+ \|Q(t, s, \xi, \epsilon) - Q(t, s, \xi, 0)\|
\end{equation}

\begin{equation}
+ \|N_1(s, \xi, \epsilon)^{-1} - N_1(s, \xi, 0)^{-1}\|
\end{equation}

Each of the last three differences appearing on the right hand side can be estimated by $\epsilon|\xi|^{-1}$. This is due to estimate (4.48) for $N_1(t, \xi, \epsilon)$ and (4.49) for $N_1(s, \xi, \epsilon)^{-1}$, due to estimate (4.69) for $Q(t, s, \xi, \epsilon)$. Furthermore by Proposition 4.2 for $b_\epsilon(s)$ and $b_\epsilon(t)$ we know that the first difference is estimated by $\epsilon$. The desired estimate for the fundamental solution follows.

**4.3. Treatment in the singular zone.** Now we consider equation (3.9) within the singular zone. In order to describe its fundamental solution we use the substitution $\tau = \epsilon^{-1}(t - 1)$ and replace the parameter $|\xi|$ by $\Lambda = \epsilon|\xi|$. Then the equation (3.9) rewrites as

\begin{equation}
\dot{u}_\tau + \Lambda^2 \dot{u} + \beta_\epsilon(\tau) u_\tau = 0,
\end{equation}

where

\begin{equation}
\beta_\epsilon(\tau) = \epsilon \partial_\tau (1 + \epsilon \tau) = \frac{\int_{-\infty}^{\infty} b(1 + \epsilon(\tau - \theta))\psi'(\theta)d\theta}{\int_{-\infty}^{\infty} b(1 + \epsilon(\tau - \theta))\psi(\theta)d\theta}.
\end{equation}

We recall here that the singular zone rewrites in the new coordinates

\begin{equation}
Z_{\text{sing}}(N) = \{ (\tau, \Lambda, \epsilon) : \Lambda \leq N \Phi(\tau) + N\epsilon \}
\end{equation}

such that the interval to solve our equation on is given by $[\tau_{\Lambda_1}(\epsilon), \tau_{\Lambda_2}(\epsilon)]$ with implicitly defined endpoints given by $\Lambda = N \Phi(\tau_{\Lambda}) + \epsilon$. 

4.3.1. **System form.** Reformulating our equation as system in

\[ U(\tau, \Lambda, \epsilon) = \left( \begin{array}{c} \Lambda \dot{u} \\ \partial_\tau \dot{u} \end{array} \right) \]  

(4.83)
yields

\[ \partial_\tau U(\tau, \Lambda, \epsilon) = \left[ \begin{array}{cc} 0 & 0 \\ 0 & -\beta_\epsilon(\tau) \end{array} \right] + \left[ \begin{array}{cc} 0 & \Lambda \\ -\Lambda & 0 \end{array} \right] U(\tau, \Lambda, \epsilon), \]  

(4.84)
where now the first matrix \( A(\tau, \epsilon) = \text{diag}(0, -\beta_\epsilon(\tau)) \) is treated as the dominant part and the second matrix

\[ \left( \begin{array}{cc} 0 & \Lambda \\ -\Lambda & 0 \end{array} \right) = \Lambda \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) = \Lambda J \]  

(4.85)
plays the role of the remainder.

**Proposition 4.10.** Under the assumptions (H1) and (H2) for the function \( b_\epsilon(\tau) = b_0(\tau) + \mathcal{O}(\epsilon) \)

(4.86)
holds true uniformly with respect to \( \tau \) and with \( b_0(\tau) \) given by

\[ b_0(\tau) = \frac{h_\psi(\tau)}{h \int_{-K}^{K+} \psi(\theta) d\theta + b(1+0)} = \frac{h \psi(\tau)}{b(1+0) - h \int_\tau^K \psi(\theta) d\theta} \]  

(4.87)
in terms of \( h = b(1+0) - b(1-0) \) the jump of \( b \) at \( t = 1 \).

**Remark 4.4.** In the (4.87) the numerator is the derivative of the denominator with respect to \( \tau \). We also see that \( b_0(\tau) \) is compactly supported with \( \text{supp } b_0 = [-K, K] \).

**Proof.** The statement follows by considering both numerator and denominator of the representation (4.81) separately. First,

\[ \int_{-\infty}^{\infty} b(1 + \epsilon(\tau - \theta)) \psi'(\theta) d\theta = \int_{-\infty}^{\tau} b(1 + \epsilon(\tau - \theta)) \psi'(\theta) d\theta \]  

\[ + \int_{\tau}^{+\infty} b(1 + \epsilon(\tau - \theta)) \psi'(\theta) d\theta \]  

\[ = h \psi(\tau) + \epsilon \int_{-\infty}^{\tau} b'(1 + \epsilon(\tau - \theta)) \psi(\theta) d\theta \]  

\[ + \epsilon \int_{\tau}^{+\infty} b'(1 + \epsilon(\tau - \theta)) \psi(\theta) d\theta \]  

(4.88)
using integrating by parts and the fact that \( b' \) is bounded on both \([0, 1]\) and \([1, 2]\).

Similarly, we obtain for the denominator

\[ \left| \int_{-\infty}^{\tau} b(1 + \epsilon(\tau - \theta)) \psi(\theta) d\theta - \int_{-\infty}^{\tau} b(1+0) \psi(\theta) d\theta - \int_{\tau}^{+\infty} b(1-0) \psi(\theta) d\theta \right| \]  

\[ \leq \int_{-\infty}^{\tau} |b(1 + \epsilon(\tau - \theta)) - b(1+0)| \psi(\theta) d\theta \]  

\[ + \int_{\tau}^{+\infty} |b(1 + \epsilon(\tau - \theta)) - b(1-0)| \psi(\theta) d\theta \]  

(4.89)
\[ \leq \int_{-K}^{\tau} C_1 \epsilon |\tau - \theta| \psi(\theta) d\theta + \int_{\tau}^{K} C_2 \epsilon |\tau - \theta| \psi(\theta) d\theta \]
where for the last line we applied the mean value theorem to the function \( b \) using \( C_1 = \sup_{s \in [1,2]} |b'(s)| \) and \( C_2 = \sup_{s \in [0,1]} |b'(s)| \). Hence
\[
\int_{-\infty}^{\infty} b(1 + \epsilon (\tau - \theta)) \psi(\theta) d\theta = b(1+0) \int_{-\infty}^{\tau} \psi(\theta) d\theta + b(1-0) \int_{\tau}^{+\infty} \psi(\theta) d\theta + O(\epsilon) \quad (4.90)
\]
and therefore by combining (4.88) and (4.90) the desired statement follows. \( \square \)

4.3.2. Construction of the fundamental solution in the singular zone.
In the following we want to derive properties of the fundamental solution to (4.84). The strategy is again to use a perturbation argument to incorporate the remainder terms. Note, that in singular variables both \( \tau \) and \( \Lambda \) stay bounded within \( Z_{\text{sing}}(N) \) and our main interest is in the characterisation of the solution for \( \Lambda \to 0 \) and \( \epsilon \to 0 \).

**Theorem 4.11.** The fundamental solution to the system (4.84) can be represented by
\[
E_{\text{sing}}(\tau, \theta, \Lambda, \epsilon) = F(\tau, \theta, \epsilon) G(\tau, \theta, \Lambda, \epsilon) \quad (4.91)
\]
for \( \theta, \tau \times \{(\Lambda, \epsilon)\} \subset Z_{\text{sing}}(N) \), \( \theta < \tau \), where

1. \( F(\tau, \theta, \epsilon) \) is the fundamental solution to the main part \( \partial_\tau - A(\tau, \epsilon) \) given by
\[
F(\tau, \theta, \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \exp (- \int_\theta^\tau \beta_\epsilon(\tau) d\tau) \end{pmatrix} \quad (4.92)
\]
with
\[
\exp \left( \int_\theta^\tau \beta_\epsilon(\tau) d\tau \right) = \frac{h\Theta(\tau) + b(1-0)}{h\Theta(\theta) + b(1-0)} (1 + O(\epsilon)) \quad (4.93)
\]
given in terms of the smoothed Heaviside function
\[
\Theta(\tau) = \int_{-\infty}^\tau \psi(\vartheta) d\vartheta = 1 - \int_{\tau}^{\infty} \psi(\vartheta) d\vartheta \quad (4.94)
\]
and the height of the jump \( h = b(1+0) - b(1-0) \).

2. The matrix \( G(\tau, \theta, \Lambda, \epsilon) \) is given as power series
\[
G(\tau, \theta, \Lambda, \epsilon) = I + \sum_{k=1}^{\infty} \Lambda^k G_k(\tau, \theta, \epsilon) \quad (4.95)
\]
with coefficients \( G_k \) satisfying
\[
||G_k(\tau, \theta, \epsilon)|| \leq \frac{C_k |\tau - \theta|^k}{k!} \quad (4.96)
\]
uniformly in \( k \) and the occurring variables.

**Proof.** The representation for \( F \) follows by integrating the main diagonal part in equation (4.84). Using the explicit form of \( \beta_0(\tau) \) from (4.87) in combination with \( \beta_\epsilon(\tau) = \beta_0(\tau) + O(\epsilon) \), we obtain (4.93).

We make the ansatz
\[
E_{\text{sing}}(\tau, \theta, \Lambda, \epsilon) = F(\tau, \theta, \epsilon) G(\tau, \theta, \Lambda, \epsilon) \quad (4.97)
\]
for the fundamental solution to the system (4.84). Then by construction
\[
\partial_\tau G(\tau, \theta, \Lambda, \epsilon) = \Lambda \bar{F}(\tau, \theta, \epsilon) G(\tau, \theta, \Lambda, \epsilon), \quad G(\theta, \theta, \Lambda, \epsilon) = I \quad (4.98)
\]
where the coefficient matrix satisfies

$$\tilde{F}(\tau, \theta, \epsilon) = F(\tau, \theta, \epsilon)JF(\theta, \tau, \epsilon)$$

\[
\begin{pmatrix}
0 & \exp \left( - \int_{\theta}^{\tau} \beta_{\epsilon}(\tau) \, d\tau \right) \\
- \exp \left( - \int_{\theta}^{\tau} \beta_{\epsilon}(\tau) \, d\tau \right) & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & \exp \left( - \int_{\theta}^{\tau} \beta_{0}(\tau) \, d\tau \right) \\
- \exp \left( - \int_{\theta}^{\tau} \beta_{0}(\tau) \, d\tau \right) & 0
\end{pmatrix} (1 + O(\epsilon)).
\]

In particular we obtain the uniform bound

$$\|\tilde{F}(\tau, \theta, \epsilon)\| \leq C$$

independent of \(\tau, \theta\) and \(\epsilon\). Writing the solution to (4.98) by the Peano-Baker series we obtain the representation (4.95)

$$G(\tau, \theta, \Lambda, \epsilon) = I + \sum_{k=1}^{\infty} \Lambda^{k} \int_{\theta}^{\tau} \int_{\theta}^{\tau_{1}} \int_{\theta}^{\tau_{2}} \cdots \int_{\theta}^{\tau_{k-1}} \tilde{F}(\tau_{k}, \theta, \epsilon) \, d\tau_{k} \cdots d\tau_{2} d\tau_{1}$$

as power series in \(\Lambda\) with coefficients

$$G_{k}(\tau, \theta, \epsilon) = \int_{\theta}^{\tau} \int_{\theta}^{\tau_{1}} \int_{\theta}^{\tau_{2}} \cdots \int_{\theta}^{\tau_{k-1}} \tilde{F}(\tau_{k}, \theta, \epsilon) \, d\tau_{k} \cdots d\tau_{2} d\tau_{1}.$$

Combining this with (4.100) concludes the proof. 

\[\square\]

Remark 4.5. By plugging in the zone boundaries \(\theta = \tau_{A_{1}}\) and \(\tau = \tau_{A_{2}}\) for given \(N \epsilon < \Lambda < N \Phi(0)\) the above formulas simplify due to

$$\exp \left( - \int_{\tau_{A_{1}}}^{\tau_{A_{2}}} \beta_{\epsilon}(\tau) \, d\tau \right) = \frac{b(1-a)}{b(1+0)} \left(1 + O(\epsilon)\right)$$

and hence using that \(G(\tau, \theta, \Lambda, \epsilon) - I = O(\Lambda)\) uniform with respect to \(\epsilon, \tau\) and \(\theta\) we conclude

$$E_{\text{sing}}(\tau_{A_{2}}, \tau_{A_{1}}, \Lambda, \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{b(1-a)}{b(1+0)} \end{pmatrix} \left(1 + O(\epsilon) + O(\Lambda)\right).$$

4.3.3. Limiting behaviour of the fundamental solution in the singular zone. We want to describe the behaviour of the fundamental solution \(E_{\text{sing}}(\tau_{A_{2}}, \tau_{A_{1}}, \Lambda, \epsilon)\) as \(\epsilon \to 0\) for fixed \(\Lambda\). By (4.93) we already know that the limit

$$\mathcal{F}(\tau, \theta, 0) = \lim_{\epsilon \to 0} \mathcal{F}(\tau, \theta, \epsilon)$$

exists and satisfies

$$\|\mathcal{F}(\tau, \theta, \epsilon) - \mathcal{F}(\tau, \theta, 0)\| \leq C \epsilon$$

uniform in \(\tau_{A_{1}} \leq \theta < \tau \leq \tau_{A_{2}}\). In a next step we consider the limiting behaviour of the power series \(G\) and in particular its coefficients \(G_{k}\).

**Lemma 4.12.** The limits

$$G_{k}(\tau, \theta, 0) = \lim_{\epsilon \to 0} G_{k}(\tau, \theta, \epsilon)$$

exist for all \(\tau_{A_{1}} \leq \theta < \tau \leq \tau_{A_{2}}\) and satisfy

$$\|G_{k}(\tau, \theta, \epsilon) - G_{k}(\tau, \theta, 0)\| \leq C \epsilon$$
Furthermore,
\[ G_1(\tau, \theta, 0) = \int_\theta^\tau \left( - \exp \left( \int_\theta^\tau \beta_0(\tau) d\tau \right) \exp \left( \int_\theta^\tau \beta_0(\tau) d\tau \right) \right) d\tau. \quad (4.109) \]

**Proof.** The limit behaviour of \( F \) implies that the limits
\[ \tilde{F}(\tau, \theta, 0) = \lim_{\epsilon \to 0} \tilde{F}(\tau, \theta, \epsilon) \quad (4.110) \]
exist and are uniformly bounded with respect to \( \tau_{\Lambda_1} \leq \theta < \tau \leq \tau_{\Lambda_2} \) and that therefore the functions
\[ G_k(\tau, \theta, 0) = \int_\tau^\tau \tilde{F}(\tau_1, \theta, 0) \int_\theta^\tau_1 \cdots \int_\theta^\tau_{k-1} \tilde{F}(\tau_k, \theta, 0) d\tau_k \cdots d\tau_1. \quad (4.111) \]
are good candidates to be considered for the limit behaviour of \( G_k \). For \( k = 1 \) the representation
\[ G_1(\tau, \theta, 0) = \int_\theta^\tau \tilde{F}(\tau_1, \theta, 0) d\tau_1 \quad (4.112) \]
corresponds directly to (4.109) due to the formula (4.99) for \( \tilde{F}(\tau, \theta, 0) \). Hence, using the analogue to (4.106) we obtain
\[ \|G_1(\tau, \theta, \epsilon) - G_1(\tau, \theta, 0)\| \leq \int_\theta^\tau \| \tilde{F}(\tau_1, \theta, \epsilon) - \tilde{F}(\tau_1, \theta, 0)\| d\tau_1 \lesssim C |\tau - \theta| \epsilon \quad (4.113) \]
and using that \(|\tau - \theta| \leq 2K'\) the first statement follows.

The estimate for \( G_k \) is obtained by telescoping the integral
\[ G_k(\tau, \theta, \epsilon) - G_k(\tau, \theta, 0) = \int_\theta^\tau (\tilde{F}(\tau_1, \theta, \epsilon) - \tilde{F}(\tau_1, \theta, 0)) \]
\[ \times \int_\theta^\tau_1 \tilde{F}(\tau_1, \theta, 0) \cdots \int_\theta^\tau_{k-1} \tilde{F}(\tau_k, \theta, 0) d\tau_k \cdots d\tau_1 \]
\[ + \int_\theta^\tau (\tilde{F}(\tau_1, \theta, \epsilon) - \tilde{F}(\tau_1, \theta, 0)) \]
\[ \times \int_\theta^\tau_1 (\tilde{F}(\tau_1, \theta, \epsilon) - \tilde{F}(\tau_1, \theta, 0)) \]
\[ \times \int_\theta^\tau_2 \tilde{F}(\tau_3, \theta, 0) \cdots \int_\theta^\tau_{k-1} \tilde{F}(\tau_k, \theta, 0) d\tau_k \cdots d\tau_1 \]
\[ + \cdots, \quad (4.114) \]
each term containing one difference more up to having \( k \) differences as integrands. Note, that this represents the difference \( G_k(\tau, \theta, \epsilon) - G_k(\tau, \theta, 0) \) in terms of differences \( \tilde{F}(\tau_1, \theta, \epsilon) - \tilde{F}(\tau_1, \theta, 0) \) and terms of the form \( G_{k-\ell}(\tau, \theta, 0) \) already estimated in the previous induction step. Hence
\[ \|G_k(\tau, \theta, \epsilon) - G_k(\tau, \theta, 0)\| \lesssim \sum_{\ell=1}^{k} \epsilon^\ell \lesssim \epsilon \quad (4.115) \]
and the lemma is proven. \( \square \)
4.4. **Bounded frequencies.** We will give some remarks concerning estimates for the fundamental solution for \(|\xi| \leq N\). Here it suffices to consider the system \(\text{(3.11)}\) in original form and to observe that its coefficient matrices have norm estimates \(\|A(\xi)\| \lesssim |\xi|\) and \(\|B(t,\epsilon)\| \lesssim 1 + \Phi_\epsilon(t - 1)\).

Representing its solution directly by Peano–Baker series yields
\[
\|E(t, s, \xi, \epsilon)\| \leq \exp \left( C \int_s^t (|\xi| + 1 + \Phi_\epsilon(\theta - 1)) \, d\theta \right) \leq \tilde{C}
\] (4.116)
using that \(\int_2^0 \Phi_\epsilon(t - 1) \, dt\) is independent of \(\epsilon\) and that both \(|\xi|\) as well as \(s, t\) are bounded.

**Remark 4.6.** Note that for dissipative problems the uniform boundedness of the fundamental solution follows already from the positivity of the coefficient of \(\text{(3.1)}\) in front of \(u_t\). For more general wave models this statement needs a proof and the above reasoning seems viable for this case too.

4.5. **Combining the bits.** We collect here the estimates obtained so far. As we are interested in the influence of the point singularity on the structure of the fundamental solution we consider \(t_1, t_2 \in [0, 2]\) with \(t_1 < 1 < t_2\) and look the fundamental solution to \(\text{(3.11)}\) for fixed \(\epsilon\) chosen sufficiently small. This is given by
\[
\mathcal{E}(t_2, t_1, \xi, \epsilon) = \mathcal{E}_{\text{hyp}}(t_2, t_{\xi_2}(\epsilon), \xi, \epsilon) T(\epsilon)^{-1} \mathcal{E}_{\text{sing}}(t_{\xi_2}(\epsilon), t_{\xi_1}(\epsilon), \epsilon|\xi|, \epsilon) T(\epsilon) \mathcal{E}_{\text{hyp}}(t_{\xi_1}(\epsilon), t_1, \xi, \epsilon) \tag{4.117}
\]
with \(T(\epsilon)\) the transformation matrix between the micro-energies used in the hyperbolic and the singular zone,
\[
T(\epsilon) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} = \epsilon I. \tag{4.118}
\]
Note that both of these matrices cancel each other and can therefore be neglected. As \(\epsilon\) tends to 0 we have that \(t_{\xi_1} \to 1_{-0}\) and \(t_{\xi_2} \to 1_{+0}\). So using estimates \(\text{(4.78)}\) and \(\text{(4.104)}\) we obtain for fixed \(\xi\)
\[
\lim_{\epsilon \to 0} \mathcal{E}(t_2, t_1, \xi, \epsilon) = \mathcal{E}_{\text{hyp}}(t_2, 1_{+0}, \xi, 0) \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \mathcal{E}_{\text{hyp}}(1_{-0}, t_1, \xi, 0), \tag{4.119}
\]
where \(H = \frac{b(1-\alpha)}{b(1+\alpha)}\) is given in terms of the jump of \(\log b\) at \(t = 1\).

5. **Results**

5.1. **Existence of very weak solutions.** Although in our model case the existence of very weak solutions was already established in \([9]\), we will show how to obtain this from the properties of fundamental solutions just constructed.

**Proposition 5.1.** For \(\epsilon \in (0, 1]\), \(0 \leq s < t \leq 2\) and \(|\xi| \geq N\) the fundamental solution to the system \(\text{(3.11)}\) is uniformly bounded,
\[
\|\mathcal{E}(t, s, \xi, \epsilon)\| \leq C. \tag{5.1}
\]

**Proof.** If \([s, t] \times \{(\xi, \epsilon)\} \subset Z_{\text{hyp}}(N)\), the result follows directly from the construction in the hyperbolic zone. So it remains to consider only situations where the time interval intersects with the singular zone \(Z_{\text{sing}}(N)\).
We focus on the situation where \((s, \xi, \epsilon) \in Z_{hyp}(N)\) and \((t, \xi, \epsilon) \in Z_{sing}(N)\), i.e., \(s < t_{\xi_1}(\epsilon) < 1\) and \(t_{\xi_1}(\epsilon) < t < t_{\xi_2}(\epsilon)\). Then the fundamental solution to system [3.11] is given by
\[
E(t, s, \xi, \epsilon) = T(\epsilon)^{-1}E_{sing}(\tau_{\xi_2}, \tau_{\xi_1}, \epsilon)E_{hyp}(t_{\xi_1}, t, \xi, \epsilon).
\] (5.2)

As the factors of \(\epsilon^{-1}\) and \(\epsilon\) arising from \(T(\epsilon)^{\pm 1}\) cancel out, it suffices to show the uniform boundedness of \(E_{hyp}(t, s, \xi, \epsilon)\) for \(s < t\) over the hyperbolic zone and of \(E_{sing}(\tau, \theta, \Lambda, \epsilon)\) for \(\theta < \tau\) over the singular zone (in singular variables).

Hence, it remains to collect the already proven boundedness results. For \(|\xi| \leq N\) (i.e. for \(\Lambda \leq N\epsilon\)) the uniform bound was shown in (??). For \(|\xi| > N\) and within the hyperbolic zone the boundedness follows from the representation (4.67) and the boundedness of each individual factor due to Theorem 4.7, while for the singular zone the representation of Theorem 4.11 gives a uniform bound on the fundamental solution based on the uniform boundedness of \(\tau_{\Lambda_1}\) and \(\tau_{\Lambda_2}\) with respect to both \(\epsilon\) and \(\Lambda\).

In combination with the bound \(\epsilon^{-1} + |\xi|\) for the coefficient matrix of (3.11) we conclude the bounds
\[
\|D^k E(t, s, \xi, \epsilon)\| \leq C_k \epsilon^{-k} |\xi|^k
\] (5.3)
uniform in \(s < t\), in \(\epsilon > 0\) and \(\xi \in \mathbb{R}^n\).

**Corollary 5.2.** Let the net \((u_n)_{\epsilon \in [0, 1]}\) be solution to the Cauchy problem (3.8) for initial data \(u_0 \in H^1(\mathbb{R}^n)\) and \(u_1 \in L^2(\mathbb{R}^n)\). Then the estimate
\[
\|\partial_t^k u(t, \cdot)\|_{H^{-k}(\mathbb{R}^n)} + \|\partial_x^k u(t, \cdot)\|_{H^{1-k}(\mathbb{R}^n)} \leq C_k \epsilon^{-k}
\] (5.4)
holds true.

**Remark 5.1.** Note that the negative powers of \(\epsilon\) only appear for the solution at and after the singularity in \(t = 1\), the estimates hold true without the \(\epsilon\) for \(t < 1\).

### 5.2 Exceptional propagation of singularities

Now we want to prove the exceptional propagation of singularities already hinted at by the numerical experiments from [4]. For this we consider the model problem in one space dimension and use specially prepared initial data in the form of wave packets
\[
u_0(x) = e^{i \delta^{-1} \xi_0 \chi(x)},
\]
\[
u_1(x) = \partial_x \nu_0(x) = e^{i \delta^{-1} \xi_0 (i \xi_0 \delta^{-1} \chi(x) + \chi'(x))}
\] (5.5)
parameterized by a fixed frequency \(\xi_0 \in \mathbb{R} \setminus \{0\}\) and using a smooth rapidly decaying function \(\chi \in S(\mathbb{R})\) with sufficiently small Fourier support around the origin.

Applying a Fourier transform we see that
\[
|\xi| \hat{\nu}_0(\xi) \pm i \hat{\nu}_1(\xi) = \begin{cases} 0, & \pm \xi > 0 \\ \pm 2 \xi \hat{\chi}(\xi - \delta^{-1} \xi_0), & \pm \xi < 0 \end{cases}
\] (5.6)
Without loss of generality we can assume that \(\xi_0 > 0\) and \(\text{supp } \hat{\chi} \subset [-\xi_0/2, \xi_0/2]\).

Hence, for such initial data the initial datum \(U_0(\xi, \epsilon)\) to (3.11) satisfies
\[
M^{-1} U_0(\xi, \epsilon) = \sqrt{2} \begin{pmatrix} 0 \\ \xi \hat{\chi}(\xi - \delta^{-1} \xi_0) \end{pmatrix}
\] (5.7)
for the diagonaliser $M$ from (4.32). Let now $t < 1$. As $\mathcal{E}_0(t, s, \xi)\) is diagonal and $Q(t, s, \xi, \epsilon) - I$ as well as $N_1(t, s, \xi, \epsilon) - I$ are both bounded by $|\xi|^{-1}$ uniformly in $\epsilon > 0$ (small enough such that $(t, s, \xi, \epsilon) \in Z_{\text{hyp}}(N))$ and $s \in [0, t]$ we obtain that

$$V(t, \xi, \epsilon) = \sqrt{\frac{b(0)}{b(t)}} N_1(t, \xi, \epsilon) \mathcal{E}_0(t, 0, \xi) Q(t, 0, \xi, \epsilon) N_1(0, \xi, \epsilon)^{-1} M^{-1} U_0(\xi, \epsilon)$$

(5.8) is given by

$$V(t, \xi, \epsilon) = \sqrt{\frac{b(0)}{b(t)}} \sqrt{2} \left( e^{-i\epsilon \xi} \hat{\gamma}(\xi - \delta^{-1} \xi_0) \right) + O(1), \quad t < 1,$$

(5.9) for fixed $t$ and with a uniformly bounded remainder independent of the choice of $\delta$. This corresponds to a wave traveling to the right plus remainder terms with smaller norm. Note, that the first term behaves like $\delta^{-1}$ due to the support assumption made for $\hat{\gamma}$ and thus dominates the remainder term for choosing $\delta$ small enough.

In the following, we consider $t > 1$ and ask for the influence of the point singularity at time 1 on the behaviour of our net of solutions. If $\epsilon > 0$ is small enough such that $(t, \xi, \epsilon) \in Z_{\text{hyp}}(N)$ the solution is represented by

$$V(t, \xi, \epsilon) = \sqrt{\frac{b(0)b(1+\epsilon)}{b(1-\epsilon)b(t)}} N_1(t, \xi, \epsilon) \mathcal{E}_0(t, \xi, \epsilon) Q(t, \xi, \epsilon) N_1(\xi, \epsilon)^{-1}$$

$$\times M^{-1} T(\epsilon)^{-1} \mathcal{E}_{\text{sing}}(\tau_{\xi}, \tau_{\xi}, \epsilon|\xi|, \epsilon) T(\epsilon)^{-1} M$$

$$\times N_1(t, \xi, \epsilon) \mathcal{E}_0(t, 0, \xi) Q(t, 0, \xi, \epsilon) N_1(0, \xi, \epsilon)^{-1} M^{-1} U_0(\xi, \epsilon).$$

(5.10)

We again look at the main terms and estimates for remainders. In order to get the desired estimates we choose first the zone constant $N$ large enough to control non-diagonal terms appearing in the transformation matrices and in $Q$. This yields based on the symbol estimate for $N_1(t, \xi, \epsilon) - I$ and estimate (4.66) for $Q(t, s, \xi, \epsilon) - I$

$$V(t, \xi, \epsilon) = \sqrt{\frac{b(0)b(1+\epsilon)}{b(1-\epsilon)b(t)}} \frac{1}{\sqrt{2}} \left( e^{i(t-1) \xi} \begin{pmatrix} e^{-i(t-1)\xi} & 0 \\ 0 & e^{-i(t-1)\xi} \end{pmatrix} \right)$$

$$\times \begin{pmatrix} H + 1 & H - 1 \\ H - 1 & H + 1 \end{pmatrix} \left( e^{-i\epsilon \xi} \hat{\gamma}(\xi - \delta^{-1} \xi_0) \right) + O(\epsilon) + O(\epsilon|\xi|))$$

(5.11)

using in an essential way that the $T(\epsilon)$-terms cancel out, that $|t_{\xi}(\epsilon) - 1| \leq C \epsilon$

combined with

$$\|N_1(t, \xi, \epsilon) - I\| + \|N_1(0, \xi, \epsilon)^{-1} - I\| \leq C|\xi|^{-1}$$

$$\|Q(t, t_{\xi}(\epsilon), \xi, \epsilon) - I\| + \|Q(t_{\xi}(\epsilon), 0, \xi, \epsilon) - I\| \leq C/N$$

$$\|N_1(t, t_{\xi}(\epsilon), \xi, \epsilon)^{-1} - I\| + \|N_1(t_{\xi}(\epsilon), \xi, \epsilon) - I\| \leq C/N$$

(5.12)

due to (4.42), (4.66) and (3.10) and that

$$M^{-1} \mathcal{E}_{\text{sing}}(\tau_{\xi}, \tau_{\xi}, \epsilon|\xi|, \epsilon) M$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + O(\epsilon) + O(\epsilon|\xi|))$$

(5.13)
due to (4.104) and with $H = \frac{b(1)}{b(2)} \in (0, 1]$. As for our net of initial data $|\xi| \sim \delta^{-1}$, the second remainder term is of order $\epsilon \delta^{-1}$ and thus negligible for $\epsilon$ small enough and $\delta$ fixed.

To recover the solution $u(t, x)$ we have to multiply by the matrix $M$ and apply the inverse Fourier transform. Thus we obtain the following theorem.

**Theorem 5.3.** The very weak solution corresponding to the net of initial date (5.5) is described (up to terms small compared to the solution itself)

- by a wave travelling to the right for $t < 1$
- and by two waves travelling to the left and to the right for all $t > 1$.

**Remark 5.2.** The partial reflection of rays at the singularity is characterised by the matrix

$$
\frac{1}{2} \begin{pmatrix}
H + 1 & H - 1 \\
H - 1 & H + 1
\end{pmatrix}
$$

in terms of the jump of $\log b$ at $t = 1$.

Thus, if the coefficient $b$ has no jump and therefore $H = 1$ this matrix becomes the identity and for $t > 1$ only one wave propagates to the right. Hence, no reflected wave occurs.

If $b$ has a jump we can compare the amplitude of both travelling waves. For this we fix a sufficiently small $\delta > 0$ and write down the main terms of the travelling wave as

$$
u(t, x) = \sqrt{\frac{b(0)}{b(t)}} u(x - t), \quad 0 < t < 1
$$

and

$$
u(t, x) = \frac{H + 1}{2\sqrt{H}} \sqrt{\frac{b(0)}{b(t)}} u(x - t) + \frac{H - 1}{2\sqrt{H}} \sqrt{\frac{b(0)}{b(t)}} u(x - 2 + t), \quad t > 1.
$$

The first term corresponds to a wave continuing in the same direction but with amplitude multiplied by $\frac{H + 1}{2\sqrt{H}}$, while the second term gives the reflected part with amplitude multiplied by $\frac{H - 1}{2\sqrt{H}}$.

**Remark 5.3.** The related wave model

$$u_{tt} - \Delta u + \delta_1(t) u_t = 0
$$

with coefficient given by the Delta distribution supported in $t = 1$ appears almost as special case of treatment here in the paper. For the choice of $b(t) = 1/2$ for $0 \leq t < 1$ and $b(t) = 3/2$ for $1 < t \leq 2$ we obtain a closely related net of coefficients leading to $H = 1/3$ and a resulting transfer matrix at the singularity.

The true consideration of the above equation can be done on lines similar to the treatment provided here in the paper. This would lead to the (related) transfer matrix

$$
\frac{1}{2\epsilon} \begin{pmatrix}
1 + e & 1 - e \\
1 - e & 1 + e
\end{pmatrix}
$$

**Remark 5.4.** The arguments presented in this section for the case of one space dimension applies in a similar way to higher dimensions. The main reflected wave travels in opposite direction to the main one, lower order terms could propagate along cones emanating from the point of interaction of singularities.
6. Concluding remarks

We will conclude this article with some comments on the tools and techniques developed so far and mention some open problems and challenges.

(1) The symbol classes used in the treatment were adapted to one point singularity at $t = 1$. This can clearly be extended to treat point singularities at a finite number of times.

(2) Using the same symbol classes one can treat other wave models with time-dependent coefficients having point singularities of suitable strength. This corresponds to the models proposed in [3] and will be considered in detail in a forthcoming paper.

(3) A related problem are singular wave models with singularities depending on space and time. Here an adapted version of a full $\epsilon$-dependent pseudo-differential calculus has to be used in order to describe the propagation of singularities. It is not clear to us, whether for local singular variables allow a description of the scattering process of waves (and wave front sets) at such singularities.

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