Simulations of liquid crystals in Poiseuille flow

Colin Denniston\textsuperscript{1}, Enzo Orlandini\textsuperscript{2}, and J.M. Yeomans\textsuperscript{1}

\textsuperscript{1} Dept. of Physics, Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP
\textsuperscript{2} INFM-Dipartimento di Fisica, Università di Padova, 1-35131 Padova, Italy

(March 21, 2022)

Abstract

Lattice Boltzmann simulations are used to explore the behavior of liquid crystals subject to Poiseuille flow. In the nematic regime at low shear rates we find two possible steady state configurations of the director field. The selected state depends on both the shear rate and the history of the sample. For both director configurations there is clear evidence of shear-thinning, a decrease in the viscosity with increasing shear rate. Moreover, at very high shear rates or when the order parameter is large, the system transforms to a log-rolling state with boundary layers that may exhibit oscillatory behavior.

I. INTRODUCTION

As suggested by their name, liquid crystals are fundamentally a liquid and hence hydrodynamics can be very important to their behavior. Moreover, because they comprise rod-like molecules the properties of their flow can be much richer than that seen in simple fluids. This is because the translational motion of the fluid is coupled to the inner orientational motion of the molecules. As a result the flow disturbs the alignment. Conversely changing the alignment will almost invariably induce a flow.

This coupling can have important practical consequences in applications of liquid crystals. Back-flow can be an important limiting factor in determining the rate at which a liquid crystal display device can be switched\cite{1}. The ordering induced by a shear flow could
potentially be beneficial if understood and controlled or can be the origin of mechanical instabilities under processing conditions [2,3].

In addition to technologically relevant issues, there are also a number of fundamental scientific questions which can be addressed in the context of liquid crystals. It is known that shear flow can induce a phase transition from the isotropic (disordered) to nematic (liquid crystalline ordered) phase [4–6]. The resulting non-equilibrium phase diagram includes a line of first-order transitions ending in a critical point. Techniques for examining and classifying such non-equilibrium phase transitions are still in need of further testing and development. In addition, a large number of structured fluids, like a liquid crystal, have order parameters that have tensor characteristics. Methods for studying such systems are still somewhat lacking and need to be explored.

Although the effects of hydrodynamics are well known, it is usually very difficult to fully incorporate these effects into a calculation. This has been due to the complexity of the nine coupled non-linear partial differential equations which describe the system (the order parameter is a symmetric traceless tensor with five components, the Navier-Stokes equation has three components, and the continuity equation is scalar). While the final results may appear simple it can often be an imposing task to derive them analytically from the underlying equations. From a numerical viewpoint accurate simulations can be difficult as the solutions can contain topological defects requiring resolution of length scales much smaller than those associated with hydrodynamic flows.

In an attempt to overcome these problems we have developed a modified lattice Boltzmann algorithm to simulate liquid crystals. The lattice Boltzmann algorithm, which includes hydrodynamics, makes use of a physical analogue (from statistical mechanics) to map the partial differential equations onto equations for the evolution of two probability distributions. Moments of these distributions are then related to the physical variables of interest. Among the advantages of this reformulation is that the resulting equations are much simpler to model and possess greater numerical stability than, say, a traditional finite-difference scheme applied to the original equations.
In this paper we apply the lattice Boltzmann method to the study of Poiseuille flow in liquid crystals. The paper is organized as follows. First, we describe the model of liquid crystal hydrodynamics proposed by Beris and Edwards [7]. This contains both the Ericksen-Leslie and the Doi models as limiting cases. Second, we outline the main features of the lattice Boltzmann algorithm used to simulate the model. Results are presented to show the effect of the flow on the director orientation and how this in turn leads to both shear thinning and hysteresis. We also examine the transition to the log-rolling state.

II. THE EQUATIONS OF MOTION FOR LIQUID CRYSTAL HYDRODYNAMICS

We follow the formulation of liquid crystal hydrodynamics described by Beris and Edwards [7]. The continuum equations of motion are written in terms of a tensor order parameter \( Q \). The advantage of this approach is that it includes both the isotropic (\( Q = 0 \)) and the nematic (\( Q \neq 0 \)) phases and allows an order parameter of variable magnitude within the latter. Hence it is possible to explore the effect of flow on the phase transition between the two states. Moreover the hydrodynamics of topological defects (point defects in two dimensions) are naturally included in the equations. The Beris–Edwards equations reduce to the Ericksen–Leslie formulation [8] of nematodynamics for a uniaxial nematic in the absence of defects.

We describe the equilibrium properties of the liquid crystal by the Landau–de Gennes free energy [8]

\[
\mathcal{F} = \int d^3r \left\{ \frac{a}{2} Q_{\alpha\beta}^2 - \frac{b}{3} Q_{\alpha\beta} Q_{\beta\gamma} Q_{\gamma\alpha} + \frac{c}{4} (Q_{\alpha\beta}^2)^2 + \frac{\kappa}{2} (\partial_\alpha Q_{\beta\lambda})^2 \right\},
\]

where Greek indices represent Cartesian directions and a sum over repeated indices is assumed. This free energy describes a first-order transition from the isotropic to the nematic phase. Note that, for simplicity, we are working within the one elastic constant approximation. Three elastic constants are needed to fully characterize the nematic phase [8].

The equation of motion for the nematic order parameter is

\[
(\partial_t + \vec{u} \cdot \nabla) Q - \mathbf{S}(\mathbf{W}, Q) = \Gamma \mathbf{H}
\]
where $\Gamma$ is a collective rotational diffusion constant. The first term on the left-hand side of equation (II.2) is the material derivative describing the usual time dependence of a quantity advected by a fluid with velocity $\vec{u}$. This is generalized by a second term

$$S(W, Q) = (\xi D + \Omega)(Q + I/3) + (Q + I/3)(\xi D - \Omega) - 2\xi(Q + I/3)\text{Tr}(QW)$$  \tag{II.3}$$

where $D = (W + W^T)/2$ and $\Omega = (W - W^T)/2$ are the symmetric part and the anti-symmetric part respectively of the velocity gradient tensor $W_{\alpha\beta} = \partial_\beta u_\alpha$. $S(W, Q)$ appears in the equation of motion because the rod-like shape of the molecules means that the order parameter distribution can be both rotated and stretched by flow gradients. $\xi$ is a constant which depends on the molecular details of a given liquid crystal.

The term on the right-hand side of equation (II.2) describes the relaxation of the order parameter towards the minimum of the free energy. The molecular field $H$ which provides the driving force is related to the derivative of the free energy by

$$H = -\left\{ \frac{\delta F}{\delta Q} - (I/3)\text{Tr}\frac{\delta F}{\delta Q} \right\}$$

$$= -aQ + b(Q^2 - (I/3)\text{Tr}Q^2) - c\text{Tr}Q^2 + \kappa\nabla^2 Q.$$  \tag{II.4}$$

The fluid, of density $\rho$ obeys the continuity

$$\partial_t \rho + \partial_\alpha \rho u_\alpha = 0$$  \tag{II.5}$$

and the Navier-Stokes equations

$$\rho\partial_t u_\alpha + \rho u_\beta \partial_\beta u_\alpha = \partial_\beta \tau_{\alpha\beta} + \partial_\beta \sigma_{\alpha\beta}$$

$$+ \frac{\rho\tau_f}{3}(\partial_\beta((\delta_{\alpha\beta} - 3\partial_\rho P_0)\partial_\gamma u_\gamma + \partial_\alpha u_\beta + \partial_\beta u_\alpha)).$$  \tag{II.6}$$

where $\tau_f$ is related to the viscosity and $P_0$ is the pressure,

$$P_0 = \rho T - \frac{\kappa}{2}(\nabla Q)^2.$$  \tag{II.7}$$
The form of this equation is not dissimilar to that for a simple fluid. However the details of the stress tensor reflect the additional complications of liquid crystal hydrodynamics. There is a symmetric contribution

$$\sigma_{\alpha\beta} = -P_0 \delta_{\alpha\beta} - \xi H_{\alpha\gamma}(Q_{\gamma\beta} + \frac{1}{3} \delta_{\gamma\beta})$$

$$- \xi (Q_{\alpha\gamma} + \frac{1}{3} \delta_{\alpha\gamma}) H_{\gamma\beta} + 2\xi (Q_{\alpha\beta} + \frac{1}{3} \delta_{\alpha\beta}) Q_{\gamma\delta} H_{\gamma\delta} - \partial_{\beta} Q_{\gamma\nu} \frac{\delta F}{\delta \alpha Q_{\gamma\nu}}$$

(II.8)

and an antisymmetric contribution

$$\tau_{\alpha\beta} = Q_{\alpha\gamma} H_{\gamma\beta} - H_{\alpha\gamma} Q_{\gamma\beta}.$$ 

(II.9)

A detailed account of the theories of liquid crystal hydrodynamics can be found in the book by Beris and Edwards [7].

III. A LATTEC BOLTZMANN ALGORITHM FOR LIQUID CRYSTAL HYDRODYNAMICS

We now define a lattice Boltzmann algorithm which solves the hydrodynamic equations of motion of a liquid crystal (II.2), (II.5), and (II.6). Lattice Boltzmann algorithms are defined in terms of a set of continuous variables, usefully termed partial distribution functions, which move on a lattice in discrete space and time. For a simple fluid a single set of partial distribution functions which sum on each site to give the density is needed. For liquid crystal hydrodynamics this must be supplemented by a second set, which are tensor variables, and which are related to the tensor order parameter $Q$.

We define two distribution functions, the scalars $f_i(\vec{x})$ and the symmetric traceless tensors $G_i(\vec{x})$ on each lattice site $\vec{x}$. Each $f_i$, $G_i$ is associated with a lattice vector $\vec{e}_i$. We choose a nine-velocity model on a square lattice with velocity vectors $\vec{e}_i = (\pm 1, 0), (0, \pm 1), (\pm \sqrt{2}, \pm \sqrt{2}), (0, 0)$. Physical variables are defined as moments of the distribution function

$$\rho = \sum_i f_i, \quad \rho u_\alpha = \sum_i f_i e_{i\alpha}, \quad Q = \sum_i G_i.$$ (III.10)
The distribution functions evolve in a time step $\Delta t$ according to

$$f_i(\vec{x} + \vec{e}_i \Delta t, t + \Delta t) - f_i(\vec{x}, t) = \frac{\Delta t}{2} \left[ C_{f_i}(\vec{x}, t, \{f_i\}) + C_{f_i}(\vec{x} + \vec{e}_i \Delta t, t + \Delta t, \{f^*_i\}) \right],$$

(III.11)

$$G_i(\vec{x} + \vec{e}_i \Delta t, t + \Delta t) - G_i(\vec{x}, t) = \frac{\Delta t}{2} \left[ C_{G_i}(\vec{x}, t, \{G_i\}) + C_{G_i}(\vec{x} + \vec{e}_i \Delta t, t + \Delta t, \{G^*_i\}) \right],$$

(III.12)

The left-hand side of these equations represents free streaming with velocity $\vec{e}_i$, while the right-hand side is a collision step which allows the distribution to relax towards equilibrium. $f^*_i$ and $G^*_i$ are first order approximations to $f_i(\vec{x} + \vec{e}_i \Delta t, t + \Delta t)$ and $G_i(\vec{x} + \vec{e}_i \Delta t, t + \Delta t)$ respectively. They are obtained from equations (III.11) and (III.12) but with $f^*_i$ and $G^*_i$ set to $f_i$ and $G_i$. Discretizing in this way, which is similar to a predictor-corrector scheme, has the advantages that lattice viscosity terms are eliminated to second order and that the stability of the scheme is improved.

The collision operators are taken to have the form of a single relaxation time Boltzmann equation, together with a forcing term,

$$C_{f_i}(\vec{x}, t, \{f_i\}) = -\frac{1}{\tau_f} (f_i(\vec{x}, t) - f^{eq}_i(\vec{x}, t, \{f_i\})) + p_i(\vec{x}, t, \{f_i\}),$$

(III.13)

$$C_{G_i}(\vec{x}, t, \{G_i\}) = -\frac{1}{\tau_G} (G_i(\vec{x}, t) - G^{eq}_i(\vec{x}, t, \{G_i\})) + M_i(\vec{x}, t, \{G_i\}).$$

(III.14)

The form of the equations of motion and of thermodynamic equilibrium follow from the choice of the moments of the equilibrium distributions $f^{eq}_i$ and $G^{eq}_i$ and the driving terms $p_i$ and $M_i$. $f^{eq}_i$ is constrained by

$$\sum_i f^{eq}_i = \rho, \quad \sum_i f^{eq}_i e_{i\alpha} = \rho u_{\alpha}, \quad \sum_i f^{eq}_i e_{i\alpha} e_{i\beta} = -\sigma_{\alpha\beta} + \rho u_{\alpha} u_{\beta}$$

(III.15)

where the zeroth and first moments are chosen to impose conservation of mass and momentum. The second moment of $f^{eq}$ controls the symmetric part of the stress tensor, whereas the moments of $p_i$
\[
\sum_i p_i = 0, \quad \sum_i p_i e_{i\alpha} = \partial_\beta \tau_{\alpha\beta}, \quad \sum_i p_i e_{i\alpha} e_{i\beta} = 0 \tag{III.16}
\]

impose the antisymmetric part of the stress tensor. For the equilibrium of the order parameter distribution we choose

\[
\sum_i G_{i\alpha}^e = Q, \quad \sum_i G_{i\alpha}^e e_{i\alpha} = Q u_\alpha, \quad \sum_i G_{i\alpha}^e e_{i\alpha} e_{i\beta} = Q u_\alpha u_\beta. \tag{III.17}
\]

This ensures that the order parameter is convected with the flow. Finally the evolution of the order parameter towards the minimum of the free energy is most conveniently modeled by choosing

\[
\sum_i M_i = \Gamma H(Q) + S(W, Q), \quad \sum_i M_i e_{i\alpha} = (\sum_i M_i) u_\alpha. \tag{III.18}
\]

Conditions (III.13)–(III.18) can be satisfied as is usual in lattice Boltzmann schemes by writing the equilibrium distribution functions and forcing terms as polynomial expansions in the velocity \[9\].

Taking the continuum limit of equations (III.11) and (III.12) and performing a Chapman-Enskog expansion leads to the equations of motion of liquid crystal hydrodynamics (II.2), (II.3), and (II.6).

IV. POISEUILLE FLOW

We now consider Poiseuille flow in a liquid crystal. Poiseuille flow is the flow between two non-slip boundaries at \(y = 0\) and \(y = h\), driven by a pressure gradient or body force. In a simple fluid this geometry leads to a parabolic profile of velocities \[10\]

\[
u_x = -\frac{1}{2\eta} (\rho b_f) y (y - h) \tag{IV.19}
\]

where \(\eta\) is the viscosity, \((\rho b_f)\) is an applied body force, and \(u_x\) is the velocity along the channel. A simple example is gravity-driven flow.

In an experiment the volumetric flow rate

\[
Z = \int_0^h u_x dy = \frac{1}{12\eta} (\rho b_f) h^3 \tag{IV.20}
\]
can be measured as a function of the applied body force (or pressure gradient) to obtain
the viscosity $\eta$. If the material being studied is homogeneous, then the viscosity should be a
unique function of the effective shear rate, $Z/h^2$. This can be used as a working definition of
a simple fluid. If the fluid is not homogeneous, then the relation between $\eta$ and the effective
shear rate $Z/h^2$ will depend on geometrical factors such as the channel width $h$.

In particular, a nematic liquid crystal has internal structure which can lead to inhomoge-
neities. The internal structure is strongly affected by the presence of the flow field. In
a simple shear flow the director in a nematic liquid crystal will try to align with the flow
direction at an angle $\theta$ given by

$$\xi \cos 2\theta = \frac{3q}{2 + q}, \quad (IV.21)$$

where $q$ is the amplitude of the order parameter (equal to $3/2$ times the magnitude of the
largest eigenvalue of $Q$) [4]. However this condition is not compatible with fixed boundary
conditions at the walls of the channel and therefore the direction of the director field will be
determined by a balance of competing factors. In addition note that because $\xi < 1$ there will
be no solution to equation (IV.21) as $q$ approaches one. When no solution is available the
director tumbles in the flow. An examination of the different regimes of tumbling under shear
(but imposing a constant shear gradient across the sample rather than solving Navier-Stokes
equations with shear boundary conditions) is given in Ref. [11].

In order to make it easier to compare our results to a wider array of experimental
situations, we will plot all quantities in dimensionless numbers. First, note that $1/\Gamma$ has
units of viscosity and $\kappa$ has units of force. One can then form a velocity scale as $\kappa \Gamma/h$.
Using this velocity scale one can construct the Erickson number $Er = u_xh/(\kappa \Gamma)$ (normally,
in the Erickson-Leslie theory, the Ericksen number is defined using a viscosity. However, it
can be shown that $\Gamma$ sets a scale for the contribution to the measured viscosity from the
liquid crystal order [4], which is what we are interested in here). The Ericksen number for
the flows we examine is in the range $10^2$ to $10^4$. For comparison, the Ericksen number for
the flows in the experiments described in Reference [13] is in the range $10^2$ to $10^5$. A useful
The time scale is $1/(p_0\Gamma)$ where $p_0$ is the ambient pressure ($p_0 = 1 \text{ atm}$ in all our simulations). All the simulations are performed in two spatial dimensions, while the order parameter is in three (i.e. the director can point out of the plane).

The remainder of the paper is organized as follows. We first examine the order parameter configurations which are possible in different flow regimes. Then the corresponding effective viscosity is measured as a function of flow rate. We find shear thinning, and that the viscosity is not a unique function of the effective shear rate, in good agreement with experiments. Applying a scaling suggested by Ericksen [12] we find that the data can be collapsed onto a single curve with two branches. The transition to the log-rolling state, where the director points out of the shear plane, is then briefly examined.

A. Order parameter profile

Distortions in the nematic state are induced by the laminar Poiseuille flow. Two possible configurations of the director field, for the case with strong normal anchoring at the boundaries, are shown in Figure 1. In case (a) the system was initialized in the nematic phase with the director uniform and perpendicular to the boundary before the flow was turned on. In case (b) the system was kept in the isotropic phase for the first 4000 time steps to establish the flow and then quenched into the nematic phase. Otherwise all parameters were identical. It is clear that the two systems have chosen different compatibility conditions at the center of the flow. Configuration (a) is described in the book by deGennes [8]. Case (b) does not appear to have been previously considered in the context of flow. It is, however, very similar to the equilibrium director configuration expected in a cylindrical tube with normal anchoring boundary conditions.

To understand these configurations note that in both cases there are three distinct regions of the flow. First, away from the walls and the center, the director is oriented at the angle given by equation (IV.21). Approaching the wall, the director must change its configuration to match the boundary condition (normal anchoring). The distance $e_1$ over which this
occurs is determined by a competition between the elastic energy and the shear stress at the boundary and can be estimated as

\[ e_1 \sim \left( \frac{K}{\eta s_1} \right)^{1/2} \]  

(IV.22)

where \( K \) is an elastic constant, \( \eta \) a viscosity and \( s_1 \) is the shear rate at the boundary [8]. In terms of the parameters of our model this becomes

\[ e_1 \sim \left( \frac{\kappa \Gamma}{s_1} \right)^{1/2} \]  

(IV.23)

as the polymer viscosities can be shown to be proportional to \( 1/\Gamma \) [7].

Similarly at the center of the flow the director must turn to extrapolate between the directions on the two sides. This will happen over a distance

\[ e_2 \sim \left( \frac{\kappa \Gamma}{s_2} \right)^{1/2} \]  

(IV.24)

where \( s_2 \) is the shear rate at a distance \( e_2/2 \) from the center.

For very low (or zero) flow rates the boundary and central regions will join and the configuration (a) with the director perpendicular to the flow at the center will be preferred (this should be clear for the zero flow case where this configuration has no elastic energy). At high flow rates or for wider channels the configuration with the director aligned with the flow at the center will be stable. To see why this is so, consider the elastic energy in the central region. Outside the central region, the director makes an angle \( \pm \theta \) with the flow direction. As can be seen from equation (IV.21), this angle is always less than 45° and it will therefore cost less energy to go through 0° in the center than 90°. Hence the flow aligned configuration (b) is preferred. Note, however, that (a) can exist as a metastable state, given suitable initial conditions.

As the flow rate is increased the system can spontaneously change the director configurations at the center from perpendicular to parallel to the flow. This happens by nucleating a short lived defect at the center which “unzips” the director. Figure 2 shows the amplitude of the order parameter at the center of the flow as a function of time for a situation where
this happens. The sharp dip corresponds to the appearance of the defect. To correctly describe such a change of state clearly requires a formalism, such as that used here, where the amplitude of the order parameter can vary.

**B. Shear thinning**

The viscosity of a liquid crystal depends on its orientation with respect to the flow [8]. As the orientation of the director at the boundaries and center of the channel depends on the flow rate the viscosity will be a function of the flow rate. This dependence is shown in Figure 3. The apparent viscosity was calculated assuming equation (IV.19) holds (indeed the velocity profile does remain close to parabolic). This is equivalent to the procedure used in experiments [13] where equation (IV.20) is assumed to hold and the apparent viscosity is measured as a function of flow rate.

As expected the apparent viscosity also depends on the configuration of the director at the center of the flow and hence the viscosity curves in Figure 3 have two branches. As the flow and hence the strain rate is increased, the viscosity decreases: the fluid exhibits shear thinning. At high shear rates the curves approach each other as the boundary and central regions become smaller and the viscosity is dominated by the flow-aligned regions.

Ericksen has shown [12], based on a dimensional argument, that the apparent viscosity at a shear rate \( s \) can be written

\[
\eta_{app}(s) = \eta_{app}(0) f \left[ \frac{e_1}{h} \right] = \eta_{app}(0) f \left[ \left( \frac{\kappa \Gamma}{s_1} \right)^{1/2} \frac{1}{h} \right],
\]

(IV.25)

where the dimensionless function \( f \) depends on the ratio between various Leslie coefficients and the particular laminar flow under study. Scaling the data in this way, as shown in Figure 4, we find that the results from the two different channel width do indeed collapse onto a single curve, with two branches for the two different director configurations.
C. Log-rolling state

So far we have assumed that the director stays in the shear plane. This is not always the case. If no solution to equation (IV.21) exists, the director tends to move out of the plane to a log-rolling state. In this state the director is stationary and perpendicular to the shear plane. However, the eigenvector corresponding to the second largest eigenvalue of $Q$ (the sub-director), remains in the shear plane but oscillates (thus the “rolling” in log-rolling). As one approaches the boundary where the director is constrained to be in the shear plane (and perpendicular to the wall) the director must rotate back into the plane. In this regime, the director in the boundary layer may be stationary or may exhibit oscillatory states. An example of one of these dynamic flows is shown in Figure 5. The oscillation of the director in the boundary layer is plotted explicitly in Figure 6. There remain interesting questions about the nature of the transitions between different types of director behavior in shear flows.

V. DISCUSSION

We have extended lattice Boltzmann algorithms of multiphase flow to treat the case of a non-conserved, tensor order parameter. Hence it is possible to simulate the Beris–Edwards equations of liquid crystal hydrodynamics. These hold for the isotropic, uniaxial nematic, and biaxial nematic phases.

Using the approach to investigate Poiseuille flow in liquid crystals a rich phenomenology is apparent because of the coupling between the director and the flow fields. In weak shear, for strong normal anchoring at the boundaries, two states can be stabilized. For narrow tubes the director field at the center of the flow prefers to align perpendicular to the boundaries, for wider tubes it prefers to align along the flow direction. The effective viscosity decreases markedly with increases in the shear rate. This is in agreement with the shear thinning seen in experiments.

As the shear rate is increased the stable configuration changes to a log-rolling state where
the order parameter tumbles with the flow. Oscillations in the director orientation in the boundary layer are also observed. In experiments topological defects are nucleated at these shear rates, perhaps because of impurities [14, 15]. In the simulations a similar phenomenon may occur if fluctuations are added and work is in progress in this direction.

There are many avenues for further research opened up by the rich physics inherent in liquid crystal hydrodynamics and the generality of the Beris–Edwards equations. The possibility of numerical investigations will prove very helpful as the complexity of the equations makes analytic progress difficult. Of particular interest are the possibilities of exploring nonequilibrium phase transitions such as shear banding: the coexistence of states with different strain rates. In particular it will be possible to investigate the pathways by which shear bands form. Work is also in progress to investigate the effects of flow on the dynamics of topological defects.
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FIG. 1. Left: Two different director configurations for steady-state Poiseuille flow with the director in the plane. The boundary conditions are such that the director is perpendicular ($\theta = 90$) to the walls at the top and bottom of the figure and the flow is from left to right. For clarity, only the director on every third lattice point is displayed. Right: The corresponding fluid velocity $v_y$. The solid and dotted lines are for the director configurations (a) and (b) respectively. Velocities are scaled by $\kappa \Gamma / h$ to make them dimensionless.

FIG. 2. The amplitude of the order parameter $q$ at the center of the channel as a function of time $t$, measured in units of $1/(p_0 \Gamma)$, for a flow strong enough to cause the system to switch between the two states (a) and (b) in Figure 1. This occurs via a two-stage process. The distorted region at the center is very slowly compacted, then quickly 'unzipped' by a defect running along the channel.
FIG. 3. Apparent viscosity $\eta$ as a function of the average strain. (The solid curves correspond to channels twice as wide as the dotted curves.) Different branches correspond to the director perpendicular (upper branch) or parallel (lower branch) to the flow at the center of the channel. The strain rate is scaled by $p_0\Gamma$ and the viscosity by $1/\Gamma$ to make them dimensionless.

FIG. 4. Apparent viscosity as a function of the scaled variable $(h\sqrt{s})^{-1}$. ($\kappa$ and the viscosity coefficients are held the same for the two cases, only the channel width is changed.) Different branches correspond to the director perpendicular (upper branch) or parallel (lower branch) to the flow at the center. The units are the same as for Figure 3.
FIG. 5. Steady and oscillating states in Poiseuille flow. The lines represent the director orientation (eigenvector corresponding to largest eigenvalue of $Q$) projected down onto the $xy$-plane, and shading represents the amplitude of the order parameter (largest eigenvalue). Flow is from top to bottom, and the walls are at the left and right. At the walls, the director is aligned perpendicular to the boundary. (a) Configuration (b) from Figure 1. (b) Snapshots of an oscillating configuration where the central region is in the “log-rolling state” (director perpendicular to the plane) and the boundary region consists of a transition from a configuration in the shear plane to a “tumbling” (director rotating in the shear plane) and “kayaking” region (director rotating in and out of the plane) interfacing to the central log-rolling state.

FIG. 6. Component of the director in the direction of fluid flow as a function of time at a point in the oscillating boundary layer (see Figure 5).