New non-compact Calabi–Yau metrics in $D = 6$

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Received 13 December 2009, in final form 16 December 2009
Published 18 June 2010
Online at stacks.iop.org/CQG/27/155013

Abstract
A method for constructing explicit Calabi–Yau metrics in six dimensions in terms of an initial hyperkahler structure is presented. The equations to solve are nonlinear in general, but become linear when the objects describing the metric depend only on one complex coordinate of the hyperkahler four-dimensional space and its complex conjugate. This situation in particular gives a dual description of D6-branes wrapping a complex 1-cycle inside the hyperkahler space, Fayyazuddin (2007 Class. Quantum Grav. 24 3151). The present work generalizes the construction given in that reference. But the explicit solutions we present correspond to the nonlinear problem. This is a nonlinear equation with respect to two variables which, with the help of some specific anzatz, is reduced to a nonlinear equation with a single variable solvable in terms of elliptic functions. In these terms we construct an infinite family of non-compact Calabi–Yau metrics.

PACS number: 11.25.Uv

1. Introduction
The development of the subject of Calabi–Yau (CY) manifolds is an illustrative example of the interplay between algebraic geometry and string theory. On one hand, CY spaces are interpreted as internal spaces of string and $M$-theory giving supersymmetric field theories after compactification. In fact, CY threefolds may provide compactifications which are more realistic than the ones corresponding to other Ricci-flat manifolds such as $G_2$ holonomy spaces, for which the generation of chiral matter and non-Abelian gauge symmetries seems harder (but not impossible) to achieve. On the other hand, string theory compactifications stimulated several new trends in the algebro-geometrical aspects of CY spaces; an example is the subject of mirror symmetry.

By definition a CY manifold is a compact Kahler $n$-dimensional manifold with vanishing first Chern class. The Yau proof of the Calabi conjecture implies that these manifolds admit a Ricci-flat metric, and their holonomy is reduced from $SO(2n)$ to $SU(n)$ [1]. Although these compact Ricci-flat metrics do exist, no explicit expression has been found. For the
non-compact case, the definition usually adopted is that a CY manifold is a Ricci-flat Kahler manifold, which also implies that the holonomy is reduced to $SU(n)$ or to a smaller subgroup. In this case, several explicit metrics are known. One of the oldest examples is the asymptotically conical metrics presented in [2]. Other interesting examples have been found in [3, 4]. The last ones possess curvature singularities, but for some of them the contribution to the gravitational action is finite and this makes it plausible that they may be extended to a six-dimensional gravitational instanton. In fact, some of these solutions were identified as the asymptotic form of the so-called generalized Bando–Kobayashi–Tian–Yau (BKTY) metrics [5, 6], which are by construction Calabi–Yau. More examples were found recently in [11–18] and higher dimensional ones in [19, 20]; some of these metrics possess conical singularities but in some cases these singularities have been resolved to give complete metrics.

A relatively new achievement in the subject is the characterization of the supergravity backgrounds corresponding to D6-branes wrapping a complex submanifold inside a four-dimensional hyperkahler space. It was shown by Fayyazuddin in [9] that these D6-backgrounds are described in terms of a linear system of equations and the uplift to 11 dimensions results in a purely geometrical background of the form $M_{1,4} \times Y_6$ where $Y_6$ is a Calabi–Yau space. The Ricci-flat Kahler metric on $Y_6$ is therefore determined in terms of this linear system. The present work generalizes this idea and is organized as follows. In section 2 the geometry of certain Calabi–Yau spaces in six dimensions with an isometry preserving the metric and the full $SU(3)$ structure is characterized. In addition, a class of these Calabi–Yau threefolds constructed in terms of an initial hyperkahler structure is presented and the nonlinear system describing these geometries is written explicitly. It is also shown that this nonlinear system reduces to the linear one found in [9] when the objects describing the geometry depend on a single complex coordinate and its complex conjugate and also that, in accordance with [9], when the initial hyperkahler metric is the flat one on $\mathbb{R}^4$ the resulting CY is the direct sum of $\mathbb{R}^2$ and the multi-centered hyperkahler 4-metrics [21]. These metrics are of holonomy $SU(2)$ which is a subgroup of $SU(3)$. Sections 3 and 4 improve all these situations by dealing with the full nonlinear problem associated with the flat hyperkahler structure. Particular solutions are constructed and it is shown that the corresponding metrics are of holonomy exactly $SU(3)$. In addition, an infinite family of non-compact Calabi–Yau metrics is presented.

2. Calabi–Yau metrics with an isometry preserving the $SU(3)$ structure

2.1. The general form of the $SU(3)$ structure

In this subsection a large family of Calabi–Yau (CY) manifolds with an isometry group of codimension 1 will be characterized. It will be assumed that the Killing vector $V$ corresponding to this isometry preserves not only the metric, but the full $SU(3)$ structure. It will be convenient to give an operative definition of CY manifolds in six dimensions first, for more details see for instance [25, 26]. Roughly speaking, a Calabi–Yau manifold $M_6$ is a Kahler manifold, thus sympletic, which in addition admits a Ricci-flat metric $g_6$. This definition means in particular that there exists an endomorphism of the tangent space $J : TM_6 \to TM_6$ such that $J^2 = -I_6$ and for which $g_6(X, JY) = g_6(JX, Y)$ being $X$ and $Y$ the arbitrary vector fields. It is commonly said that the metric $g_6$ is Hermitian with respect to $J$ and the tensor $(g_6)_{\mu\nu}J^\nu_\alpha$ is skew symmetric; therefore, locally it defines a 2-form

$$\omega_6 = \frac{1}{2}(g_6)_{\mu\nu}J^\nu_\alpha dx^\mu \wedge dx^\nu. \tag{2.1}$$

Here $x^\mu$ is a local choice of coordinates for $M_6$. The endomorphism $J$ is called an almost complex structure. If the Nijenhuis tensor

$$N(X, Y) = [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY]$$

is skew symmetric, then $J$ is said to be an integrable almost complex structure; in this case it defines a complex structure on $M_6$.
vanish identically for any choice of X and Y, then the tensor J will be called a complex structure and \(M_6\) a complex manifold. This is the case for a CY manifold. The Newlander–Nirenberg theorem states that there is an atlas of charts for \(M_6\) which are open subsets in \(\mathbb{C}^6\), in such a way that the transition maps are holomorphic functions. These local charts are parameterized by complex coordinates \((z_i, \bar{z}_i)\) with \(i = 1, 2, 3\) for which the complex structure looks like
\[
J_1^i = -J_2^i = i\delta_i^j, \quad J_3^i = J_4^i = 0, \tag{2.2}
\]
and for which the metric and the 2-form (2.1) are expressed as follows:
\[
g_6 = (g_6)_J dz_i \otimes d\bar{z}_j, \tag{2.3}
\]
\[
\omega_6 = \frac{1}{2} (g_6)_J dz_i \wedge d\bar{z}_j. \tag{2.4}
\]
The form (2.4) is called of type \((1, 1)\) with respect to \(J\), while a generic 2-form containing only terms of the form \((dz_i \wedge d\bar{z}_j)\) or \((d\bar{z}_i \wedge d\bar{z}_j)\) will be called of type \((2, 0)\) or \((0, 2)\), respectively. In addition a Calabi–Yau manifold is symplectic with respect to \(\omega_6\), in other words \(d\omega_6 = 0\). A complex manifold which is symplectic with respect to (2.1) is known as the Kahler manifold; thus, CY spaces are all Kahler. The Kahler condition itself implies that the holonomy is reduced from \(SO(6)\) to \(U(3)\). Furthermore, the fact that \(g_6\) is Ricci-flat is equivalent to the existence of a 3-form
\[
\Psi = \psi_\pm + i\psi_- \tag{2.5}
\]
of type \((3, 0)\) with respect to \(J\), satisfying the following compatibility conditions [7]:
\[
\omega_6 \wedge \psi_\pm = 0, \quad \psi_+ \wedge \psi_- = \frac{1}{2} \omega_6 \wedge \omega_6 \cong dV(g_6), \tag{2.6}
\]
in such a way that
\[
d\psi_+ = d\psi_- = 0. \tag{2.7}
\]
In formula (2.6) \(dV(g_6)\) denote the volume form of \(g_6\). In the situations described in (2.7) the holonomy is further reduced from \(U(3)\) to \(SU(3)\); thus, CY manifolds are \(SU(3)\) holonomy. The converse of these statements is also true; that is, for any Ricci-flat Kahler metric in \(D = 6\) there will exist an \(SU(3)\) structure \((\omega_6, \Psi)\) satisfying (2.6) and also
\[
d\omega_6 = d\psi_+ = d\psi_- = 0. \tag{2.8}
\]
The knowledge \(SU(3)\) structure determines univocally metric \(g_6\). In fact, the task to find complex coordinates for a given CY manifold may not be simple, but there always exists a tetrad basis \(e^a\) with \(a = 1, \ldots, 6\) for which the \(SU(3)\) structure is expressed as
\[
\omega_6 = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6, \quad \psi_+ = (e^1 \wedge e^4 + e^2 \wedge e^3) \wedge e^5 + (e^1 \wedge e^3 + e^4 \wedge e^5) \wedge e^6, \quad \psi_- = -(e^1 \wedge e^3 + e^4 \wedge e^5) \wedge e^6 + (e^1 \wedge e^4 + e^2 \wedge e^5) \wedge e^6, \tag{2.9}
\]
and for which the metric is \(g_6 = e^a \otimes e^a\), where the Einstein summation is understood.

The description given just above collects general facts about CY manifolds. In the following we will assume that our CY manifold \(M_6\) is equipped with a metric \(g_6\) in such a way that there is a Killing vector \(V\) preserving \(g_6\) and the whole \(SU(3)\) structure \((\omega_6, \psi_+, \psi_-)\).

For this situation there exists a local coordinate system \((\alpha, x^i)\) with \(i = 1, \ldots, 5\) for which \(V = \partial_\alpha\) and for which the metric tensor \(g_6\) takes the following form:
\[
g_6 = \frac{(dx^i + A^i_\alpha)^2}{H^2} + H g_5, \tag{2.10}
\]
\[\text{1} \quad \text{In fact, a vector } V \text{ preserving the whole } SU(3) \text{ structure automatically preserves } g_6.\]
where the function $H$, the 1-form $A$ and the metric tensor $g_5$ are independent on the coordinate $\alpha$. Thus, these objects live in a five-dimensional space which we denote by $M_5$. The metric $g_5$ appearing in (2.10) can be expressed as $g_5 = e^a \otimes e^a$ with $a = 1, \ldots, 5$ for some basis of $\alpha$-independent 1-forms $e^a$. Then, if $V$ also preserves the $SU(3)$ structure (as we are assuming) one has the decomposition

$$\omega_6 = \omega_1 + \frac{1}{\sqrt{H}} e^5 \wedge (d\alpha + A),$$  \hfill (2.11)  

$$\psi_+ = H^{3/2} \omega_3 \wedge e^5 + \omega_2 \wedge (d\alpha + A),$$  \hfill (2.12)  

$$\psi_- = -H^{3/2} \omega_2 \wedge e^5 + \omega_3 \wedge (d\alpha + A).$$  \hfill (2.13)  

Here the 1-form $e^5$ and the 2-forms $\omega_2$ and $\omega_3$ are by definition

$$e^5 = i_V \omega_6,$$  \hfill (2.14)  

$$\omega_2 = i_V \psi_+, \quad \omega_3 = i_V \psi_-,$$  \hfill (2.15)  

with $i_V$ denoting the contraction with the vector field $V$. In fact the triplet of forms $\omega_i$ may be represented as $\omega_i = e^i \wedge e^j + \epsilon_{ijk} e^k \wedge e^l$ with $i = 1, 2, 3$ for some convenient choice of $e^a$. By assumption $V$ preserves (2.11)–(2.13) and therefore $e^5$ and $\omega_i$ do not depend on the coordinate $\alpha$; i.e. they are also defined in $M_5$.

The task now is to derive the consequences of the CY relations (2.6) and (2.8) for the generic anzatz (2.10)–(2.13). From (2.6) it follows immediately that

$$\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_3 = \omega_2 \wedge \omega_3 = 0,$$  \hfill (2.16)  

$$\omega_1 \wedge \omega_1 = H^2 \omega_2 \wedge \omega_2 = H^2 \omega_3 \wedge \omega_3.$$  \hfill (2.17)  

Moreover (2.14) together with an elementary formula in differential geometry implies that

$$d_5 \left( \frac{e^5}{\sqrt{H}} \right) = \xi_{\beta} \omega_6 - i_{\beta} d\omega_6.$$  \hfill (2.18)  

Here $d_5 = \partial_\alpha$ and $\xi_{\beta}$ is the Lie derivate along the vector $\partial_\beta$. But the vector $\partial_y$, by assumption, which preserves $\omega_6$ and $\omega_6$ is closed; thus, the right-hand side of the last expression vanishes and

$$d_5 \left( \frac{e^5}{\sqrt{H}} \right) = 0.$$  \hfill (2.19)  

The last relation can be integrated, at least locally, to obtain

$$e^5 = \sqrt{H} \, dy$$  \hfill (2.20)  

with $y$ being some function of the coordinates $x^i$ parameterizing $M_5$, which is the momentum map of the isometry. At least locally, one can take the function $y$ defined in (2.20) as one of the coordinates, which leads to the decomposition $M_5 = M_4 \times R_y$ and $d_6 = d_4 + \partial_y dy$. The metric (2.10) in these coordinates becomes

$$g_6 = \frac{(d\alpha + A)^2}{H^2} + H^2 \, dy^2 + g(y),$$  \hfill (2.21)  

where $g(y)$ will be determined below under certain additional assumptions. An analogous calculation taking into account (2.15) shows that

$$d_5 \omega_2 = d_5 \omega_3 = 0.$$  \hfill (2.22)
In the remaining part of the paper it will be assumed that the forms $\omega_i$ are defined on $M_4$ and that they depend on $y$ as a parameter. We are not sure if this is the most general case, but is the one that we were able to deal with. Formally, this means that

$$\omega_i(X, \partial y) = 0.$$  \hfill (2.23)

But the closure of $\omega_2$ and $\omega_3$ (2.22) together with the decomposition $d_3 = d_4 + \partial y$ and (2.23) implies that $\omega_2$ and $\omega_3$ are $y$ independent. Thus, only $\omega_1$ is allowed to depend on $y$. In addition $g(y)$ can be interpreted as a four-dimensional metric depending on $y$ as a parameter. Taking into account these considerations, the $SU(3)$ structure (2.11) and (2.13) takes the following form:

$$\omega_6 = \omega_1(y) + dy \wedge (d\alpha + A),$$

$$\psi_+ = H^2 \omega_3 \wedge dy + \omega_2 \wedge (d\alpha + A),$$

$$\psi_- = -H^2 \omega_2 \wedge dy + \omega_3 \wedge (d\alpha + A).$$  \hfill (2.24)

The next task will be to find specific examples of this type of Calabi–Yau structures.

### 2.2. Solutions related to hyperkahler four-dimensional spaces

The method we will employ in order to find particular Calabi–Yau structures of the form (2.24) is to start with a hyperkahler structure $\tilde{\omega}_i$ defined over a 4-manifold $M_4$. As is well known, a four-dimensional hyperkahler manifold is a Kahler one which admits a Ricci-flat metric $g_4$. This automatically implies that the holonomy is in $SU(2)$. In fact, these manifolds can be considered as four-dimensional CY spaces; they are complex with respect to some endomorphism $J_1$ of the tangent space and that the corresponding form $\tilde{\omega}_1$ is of type $(1,1)$ with respect to $J_1$ and closed, i.e. $d\tilde{\omega}_1 = 0$. The reduction of the holonomy to $SU(2)$ implies that the existence of a complex 2-form $\tilde{\omega}_2 + i\tilde{\omega}_3$ is of type $(2,0)$ with respect to $J_1$ and so

$$\tilde{\omega}_1 \wedge \tilde{\omega}_2 = \tilde{\omega}_1 \wedge \tilde{\omega}_3 = 0,$$  \hfill (2.25)

and which, in addition, satisfies

$$d\tilde{\omega}_2 = d\tilde{\omega}_3 = 0.$$  \hfill (2.26)

The form $\tilde{\omega}_2 + i\tilde{\omega}_3$ is in fact the analog of the 3-form $\Psi_1$ defined in (2.5) for the four-dimensional case. The conditions stated above imply that all the endomorphisms $J_i$ defined through the usual relation

$$g_4(X, J_i Y) = \tilde{\omega}_i(X, Y),$$  \hfill (2.28)

are all complex structures and that $M_4$ is Kahler with respect to any $\tilde{\omega}_i$. In fact any endomorphism of the form

$$J = a_1 J_1 + a_2 J_2 + a_3 J_3,$$

$$a_1^2 + a_2^2 + a_3^2 = 1,$$  \hfill (2.29)

is a complex structure as well, so there is an $S^2$-bundle of complex structures for any hyperkahler manifold. The triplet of Kahler forms $\tilde{\omega}_i$ is enough to determine the hyperkahler metric $g_4$ univocally, as in the six-dimensional case.

Given a hyperkahler structure $\tilde{\omega}_i$, we deform $\tilde{\omega}_1$ to a $y$-dependent 2-form

$$\omega_1(y) = \tilde{\omega}_1 - d\alpha dy G, \quad d\alpha = d_2.$$  \hfill (2.30)
while keeping $\tilde{\omega}_2$ and $\tilde{\omega}_3$ intact. Here we have introduced the operator $d^c = J_1 d$. In expression (2.30) $G$ denotes an unknown function which varies on $M_4$ and depends also on $y$. In terms of this anzatz the SU(3) structure (2.24) takes the following form:

$$
\omega_0 = \tilde{\omega}_1 - d_4 d_4^* G + d y \wedge (d \alpha + A),
$$

$$
\psi_+ = H^2 \tilde{\omega}_3 \wedge d y + \tilde{\omega}_2 \wedge (d \alpha + A),
$$

$$
\psi_- = -H^2 \tilde{\omega}_2 \wedge d y + \tilde{\omega}_3 \wedge (d \alpha + A).
$$

The reason for the choice (2.30) for $\omega_1(y)$ is simple to explain. The form $\omega_1$ in (2.30) is of type (1,1) with respect to the complex coordinates which diagonalize $J_1$, and the term $d_4 d_4^* G$ is also of this type. Thus, the deformed form $\omega_1(y)$ is of type (1,1) as well. As the form $\tilde{\omega}_2 + i \tilde{\omega}_3$ is kept intact and is of type (2,0) the fundamental condition (2.16) is identically satisfied. Additionally (2.30) is closed with respect to $d_4$, and this will simplify considerably the analysis given below.

Given the deformed structure (2.30), the compatibility condition (2.17) implies that

$$
(\tilde{\omega}_1 - d_4 d_4^* G) \wedge (\tilde{\omega}_1 - d_4 d_4^* G) = H^2 \tilde{\omega}_2 \wedge \tilde{\omega}_2.
$$

But the wedge products appearing in the last equality are all proportional to the volume form $dV(g_4)$ of the initial hyperkahler metric $g_4$; therefore, the relation

$$
(\tilde{\omega}_1 - d_4 d_4^* G) \wedge (\tilde{\omega}_1 - d_4 d_4^* G) = M(G) \tilde{\omega}_1 \wedge \tilde{\omega}_1
$$

defines a nonlinear expression $M(G)$ involving $G$. In the following we will loosely call $M(G)$ a nonlinear operator, although the correct spelling is that $M(G)$ is the action of a nonlinear operator over $G$. We will use this terminology in order to simplify the vocabulary, when there is no place for confusion. Taking into account the last two formulas (2.32) and (2.33) together with (2.26) it follows that

$$
M(G) = H^2.
$$

The CY condition (2.8) applied to (2.31) imposes further constraints. The closure of $\omega_0$

$$
d\omega_0 = d\omega_1 - d y \wedge (d_4 d_4^* G + d_4 A) = 0
$$

together with the Kahler condition $d\omega_1 = 0$ implies that

$$
A = -d_4^* G_g,
$$

up to a gauge transformation that can be absorbed by a redefinition of the coordinate $\alpha$ in (2.31). Furthermore, the closure of $\psi_{+}$ gives the following equations:

$$
\tilde{\omega}_2 \wedge d_4 A = \tilde{\omega}_3 \wedge d_4 A = 0,
$$

$$
\tilde{\omega}_3 \wedge d_4 H^2 \wedge d y + \tilde{\omega}_2 \wedge d y \wedge A_g = 0.
$$

But condition (2.36) is identically satisfied by use of (2.35) and therefore redundant, as

$$
\tilde{\omega}_2 \wedge d_4 A = -\tilde{\omega}_2 \wedge d_4 d_4^* G_g = 0,
$$

because $\omega_2$ contain only terms of type (2,0) and (0,2) with respect to $J_1$ while $d_4 d_4^* G_g$ is purely of type (1,1). Instead (2.37) give new constraints. The 2-forms $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are related by

$$
\tilde{\omega}_2(X, Y) = \tilde{\omega}_3(J_1 X, Y),
$$

and it is a general fact that an equation of the form

$$
\tilde{\omega}_2 \wedge \alpha + \beta \wedge \tilde{\omega}_3 = 0
$$
is solved when
\[ \alpha = J_1 \beta. \]
Taking into account this and the second (2.36) we conclude that
\[ d_4^2 H^2 = -\partial_4 A. \]  
(2.39)
From (2.39) combined with (2.35) it follows that
\[ G_{yy} = H^2. \]  
(2.40)
From the last equation together with (2.34) it is obtained that a nonlinear differential equation determined the function \( G \), namely
\[ G_{yy} = M(G). \]  
(2.41)
This is the fundamental equation we will solve along the text. Finally, we have checked that the closure of \( \psi_- \) gives no new constraints, but we omit this analysis. By introducing (2.37) and (2.40) into (2.31) it follows that the \( SU(3) \) structure is
\[ \omega_6 = \bar{\omega}_1 - d_4 d_4^* G + dy \wedge (d\alpha - d_4^* G_y), \]
\[ \psi_+ = G_{yy} \bar{\omega}_1 + dy \wedge \bar{\omega}_2 \wedge (d\alpha - d_4^* G_y), \]
\[ \psi_- = -G_{yy} \bar{\omega}_2 \wedge dy + \bar{\omega}_1 \wedge (d\alpha - d_4^* G_y), \]
(2.42)
which can be checked to be closed. The generic form of the six-dimensional Calabi–Yau metric corresponding to this structure is given by
\[ g_6 = g_4(y) + G_{yy} dy^2 + \frac{(d\alpha - d_4^* G_y)^2}{G_{yy}}, \]
(2.43)
where \( g_4(y) \) is the Kahler four-dimensional metric corresponding to the deformed Kahler structure \( \omega_1(y) = \bar{\omega}_1 - d_4 d_4^* G \). Note that \( g_4(y) \) depends on \( y \) as a parameter; in other words, it is a four-dimensional Kahler metric which varies as \( y \) takes different values.

2.3. The Fayyazuddin linearization
It is important to remark that the family of \( SU(3) \) structures (2.42) and (2.43) found above are completely determined in terms of a single function \( G \) and derivates; therefore, the task to find our CY metrics has been reduced to solve a single equation (2.41) defining \( G \). This is a nonlinear equation and the general solution is not known, but it can be solved in some specific examples. We will now focus our attention, in particular, to solutions in this equation. The source of the nonlinearity of the operator \( M(G) \) defined in (2.33) and (2.41) is the quadratic term
\[ Q(G) = d_4 d_4^* G \wedge d_4 d_4^* G; \]
(2.44)
therefore, the operator \( M(G) \) will reduce to a linear one if \( Q(G) \) vanish. This will be the case when the function \( G \) is of the form \( G = G(w, \overline{w}) \) where \( w = f(z_1, z_2) \) is a holomorphic function of the coordinates \( (z_1, z_2) \) which diagonalize the complex structure \( J_1 \) [8]. This affirmation may be justified as follows. From the formula
\[ dd^c G = 2i G_{\overline{w} \overline{J}} dz_1 \wedge d\overline{w}_1, \]
(2.45)
it follows that (2.44) is
\[ Q(G) = -8(G_{11}G_{22} - G_{12}G_{21}) dz_1 \wedge d\overline{w}_1 \wedge dz_2 \wedge d\overline{w}_2. \]
(2.46)
But the functional dependence \( G = G(w, \overline{w}) \) implies that
\[ G_{\overline{w} \overline{J}} = w/\overline{w} G_w \overline{w}, \]
and by inserting this into (2.46) gives \( Q(G) = 0 \). This result may be paraphrased as follows. If the function \( G \) depends only on two complex coordinates \((w, \varphi)\), then the quantity \( dd^c G \) is essentially a 2-form in two dimensions; therefore, the wedge product (2.44) vanishes identically.

The situation described above is essentially the one considered by Fayyazuddin in [9] and if suitable boundary conditions are imposed, the resulting metrics give a dual description of D6-branes wrapping a complex submanifold in a hyperkahler manifold. A simple example is obtained when the initial hyperkahler structure is the flat metric on \( R^4 \)

\[
g_4 = dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2,
\]

(2.47)

with the Kahler triplet which is expressed in the complex form as

\[
\tilde{\omega}_1 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2),
\]

(2.48)

\[
\tilde{\omega}_2 + i\tilde{\omega}_3 = dz_1 \wedge d\bar{z}_2.
\]

(2.49)

The volume 4-form for (2.47) in complex coordinates is simply

\[
\tilde{\omega}_1 \wedge \tilde{\omega}_1 = -\frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.
\]

(2.50)

In these terms relation (2.33) for the flat metric in \( R^4 \) takes the following form:

\[
(\tilde{\omega}_1 - dd^c G) \wedge (\tilde{\omega}_1 - dd^c G) = -\frac{1}{2} M(G) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.
\]

(2.51)

In addition formula (2.45) obtained previously gives the linear term

\[
2dd^c G \wedge \tilde{\omega}_1 = 4(G_{1T} + G_{2T}) \tilde{\omega}_1 \wedge \tilde{\omega}_1,
\]

and also the following quadratic one

\[
Q(G) = dd^c G \wedge dd^c G = 16(G_{1T}G_{2T} - G_{1T}G_{2T}) \tilde{\omega}_1 \wedge \tilde{\omega}_1.
\]

By use of the last two identities it is obtained from (2.51) the following expression for \( M(G) \) :

\[
M(G) = 1 - 4(G_{1T} + G_{2T}) + 16(G_{1T}G_{2T} - G_{1T}G_{2T}).
\]

(2.52)

The fundamental equation defining the Calabi–Yau geometry is then obtained from (2.41) and (2.52); the result is

\[
G_{yy} = 1 - 4(G_{1T} + G_{2T}) + 16(G_{1T}G_{2T} - G_{1T}G_{2T}).
\]

(2.53)

The next problem is to find particular solutions of (2.53). As was discussed above if one assumes that the solution depends on one complex coordinate \( w = f(z_1) \), then the nonlinear term in (2.53) vanishes identically and the resulting equation is

\[
G_{yy} = 1 - 4(G_{1T} + G_{2T}).
\]

(2.54)

This is a Laplace equation in five dimensions parameterized by \((y, z_1, z_2, \bar{z}_1, \bar{z}_2)\). We may derivate (2.54) twice with respect to \( y \) which gives a Laplace equation for the quantity \( V = G_{yy} \) defining the radius of the ‘circle’ in (2.43). This equation is

\[
V_{yy} + 4(V_{1T} + V_{2T}) = 0.
\]

(2.55)

In the following we choose \( w = z_1 \). A solution with good behavior at infinite is obtained when we put constant density charge at the planes \( z_1 = c_i \); in other words, the right-hand side of (2.55) is zero up to a delta term of the form \( \delta\rho = \sum_{i=1}^{N} q \delta(z_1 - c_i) \delta(\bar{z}_1 - \bar{c}_i) \delta(y - a_i) \). The resulting electrostatic potential is

\[
V = 1 + \sum_{i=1}^{N} \frac{q}{\sqrt{|z_1^1 - c_i|^2 + (y - a_i)^2}}.
\]

(2.56)
and it is commonly said that the D6-branes are wrapped on the $z_2$ plane. We can also find the Kahler potential $G$ explicitly through the relation $V = G_{yy}$ by a double integration

$$G = 2q \sum_{i=1}^{N} [(y - a_i) \ln \left\{ (y - a_i) + \sqrt{(y - a_i)^2 + |z^1 - c_i|^2} \right\} - \sqrt{(y - a_i)^2 + |z^1 - c_i|^2}].$$

(2.57)

The metric $g_4(y)$ corresponding to (2.57) is given by

$$g_{1\bar{1}} = (1 + G_{11}) dz_1 \otimes d\bar{z}_1 + (1 + G_{22}) dz_2 \otimes d\bar{z}_2 = V dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2.$$  

(2.58)

The full Calabi–Yau metric (2.43) takes the following form:

$$g_6 = dz_2 \otimes d\bar{z}_2 + V(dy^2 + dz_1 \otimes d\bar{z}_1) + \frac{1}{V} (d\psi + A_1 dz_1 + A_{\bar{1}} d\bar{z}_1)^2,$$  

(2.59)

where $A$ is calculated by use of (2.35) and the result satisfies

$$\nabla \times A = \pm \nabla V.$$  

(2.60)

Therefore, (2.59) is the direct sum of a flat metric in $R^2 \simeq C$ and a general Gibbons–Hawking metric in dimension 4 [21]. As the Gibbons–Hawking metrics are hyperkahler, the whole metric (2.59) is of holonomy SU(2), which is a subgroup of SU(3). Our aim in the following is to improve this situation and find Calabi–Yau metrics of holonomy exactly SU(3).

3. Non-trivial Calabi–Yau extensions of the flat metric

As we have shown above, the Fayyazuddin linearization is easy to perform for the flat hyperkahler metric on $R^4$. For other hyperkahler structures the task is much harder, and we did not manage to find an explicit solution in those cases. The problem is that when the initial hyperkahler structure has a non-trivial curvature, the laplace-type equation (2.54) is generalized to some equation which is schematically

$$g^{\bar{7}} G_{\bar{7}} + G_{\bar{7}y} = 1,$$  

(3.61)

up to constants which are irrelevant for our discussion. Here $g^{\bar{7}}$ is the inverse metric of the initial hyperkahler structure. After writing this equation explicitly one assumes a functional dependence of the form $G(w, \overline{w})$ with some guess $w = f(z_i)$ and then tries to solve it. The existence of solutions of (3.61) of the form $G(w, \overline{w})$ is ensured only if

$$g^{\bar{7}} w_1 \overline{w}_1 = L(w, \overline{w}),$$  

(3.62)

for our guess $w$, $L(w, \overline{w})$ being a function which depends only on $(w, \overline{w})$. But the inverse metric $g^{\bar{7}}$ of the initial hyperkahler structure depends in general on two complex variables, say $(w, z_2)$ and their complex conjugates. Therefore, it may be very hard to find a clever guess in order to assure that the left-hand side of (3.62) is $z_2$ independent. That is the main technical problem to find Calabi–Yau metrics by means of the Fayyazuddin linearization.

The difficulty described above should not be interpreted as a no-go theorem for the Fayyazuddin linearization. In fact we suspect that non-trivial solutions may be found for some hyperkahler manifolds. In fact, it has been pointed out in [22] that the supergravity solution describing a D6-branes wrapping the $S^2$ in Eguchi–Hanson gravitational instanton is described by the resolved conifold metric [2]. Arguably the Fayyazuddin linearization may work for this instanton, we hope to elaborate this point in future, at the moment we have no answer.
The last alternative is to assume that $G$ do not satisfy constraint (3.62). In this case the equation to solve is the nonlinear one (2.41) and this may be even a harder task. The facts explained above seem to be discouraging when one attempts to find a six-dimensional metric with holonomy exactly $SU(3)$ by our methods. But in the remaining part of this work it will be shown that some solutions can be found explicitly. The idea behind is to tackle the full nonlinear equation (2.41) for the flat hyperkahler structure on $R^4$. In fact, we worked out explicitly this equation in (2.53). Particular solutions will be found below and it will be shown that some of them correspond to non-compact Ricci-flat Kahler metrics with holonomy exactly $SU(3)$.

3.1. The ‘radial’ anzatz

A simple guess for solving the fundamental nonlinear equation (2.53) or equivalently

$$G_{yy} - 1 + 4(G_{11} + G_{22}) - 16(G_{11}G_{22} - G_{12}G_{21}) = 0 \quad (3.63)$$

is to take $G$ as a function of $y$ and the ‘radius’ $u = r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$. In this situation the Fayyazuddin linearization does not work, as $G$ is not of the form $G(w, \bar{w})$ with $w$ being a holomorphic function of $z_i$. For this anzatz for $G$ the following identities are easy to prove:

$$G_i = z_i G_u, \quad G_{\bar{1}} = z_1 G_u, \quad G_{\bar{2}} = z_2 G_u + z_1 z_2 G_{uu}. \quad (3.64)$$

from where it is obtained that

$$G_{11} + G_{22} = 2G_u + u G_{uu},$$

and also that

$$G_{11} G_{22} - G_{12} G_{21} = (G_u + z_1 z_2 G_{uu})(G_u + z_2 z_2 G_{uu}) - z_1 z_1 z_2 z_2 G_{uu} = (G_u)^2 + u G_{uu} G_u.$$

With the help of the last equalities equation (3.63) becomes

$$G_{yy} - 1 - 16(G_u)^2 - 16u G_u G_{uu} + 8G_u + 4u G_{uu} = 0, \quad (3.65)$$

or, by rearranging terms

$$G_{yy} - 8(2G_u - 1)(G_u + u G_{uu}) - 4u G_{uu} = 0. \quad (3.66)$$

A simple solution of (3.66) is found by further postulating the following quadratic expression for $G$:

$$G = \frac{a}{2} u^2 + bu + c, \quad (3.67)$$

in which the coefficients $a$, $b$ and $c$ are assumed to be the functions of $y$. By introducing this anzatz into (3.66) it is obtained that

$$\frac{a_{yy}}{2} u^2 + b_{yy} u + c_{yy} - 8(2au + 2b - 1)(2au + b) - 4au - 1 = 0. \quad (3.68)$$

This is a polynomial expression in $u$ and it will be identically zero if its coefficients vanish identically. This leads to three equations which can be written in the following form:

$$a_{yy} - 64a^2 = 0, \quad (b - \frac{1}{4})_{yy} - 48(b - \frac{1}{4})a = 0, \quad c_{yy} - 16(b - \frac{1}{4})b - 1 = 0.$$

By introducing the quantity $\xi = b - 1/4$ the last system can be expressed as

$$a_{yy} - 64a^2 = 0, \quad \xi_{yy} - 48\xi a = 0, \quad c_{yy} - 16\xi^2 = 0. \quad (3.69)$$

The solutions of the system (3.69) will determine a family of Calabi–Yau geometries.
3.2. The general solution of the defining equations

The interesting fact about the procedure implemented above is that we have reduced a nonlinear equation in two variables \((y, u)\), namely (3.63), to a nonlinear system in one variable \(y\) with three unknown functions \((a(y), b(y), c(y))\) namely. The unique nonlinear equation in (3.69) is the first one defining \(a(y)\) and this turns out to be very easy to solve in terms of elliptic functions. Once this equation is solved, the remaining ones are straightforward. The first equation (3.69) can easily be integrated to give

\[
a_y = \pm \sqrt{m + \frac{128a^3}{3}}.
\]  

(3.70)

with \(m\) being a parameter with arbitrary values. We will consider two cases separately. For \(m > 0\) equation (3.70) implies that

\[
y(a) = \pm \frac{1}{\sqrt{m}} \int_a^{\infty} \frac{dv}{\sqrt{1 + \frac{128v^3}{3m}}} + y_0.
\]

(3.71)

With the help of the formula

\[
\int_{-\infty}^{c} \frac{dx}{\sqrt{1 - x^3}} = \frac{1}{\sqrt{3}} F(\phi, k),
\]

(3.72)

found in section 1.2.70 of [10] it follows after a short calculation that

\[
y(a) = \pm \frac{1}{\sqrt{3m}} \left(\frac{3m}{128}\right)^{\frac{1}{3}} \sqrt{F(\phi(a), k) + y_0},
\]

(3.73)

\[
\phi(a) = \cos^{-1}\left(\frac{(128)^{\frac{1}{3}}a + (3m)^{\frac{1}{3}} - (3m)^{\frac{1}{3}}\sqrt{3}}{(128)^{\frac{1}{3}}a + (3m)^{\frac{1}{3}} + (3m)^{\frac{1}{3}}\sqrt{3}}\right), \quad k = \sin \frac{5\pi}{12}.
\]

Here \(F(\phi, k)\) is an elliptic function of the first kind. Formula (3.72) gives \(y\) as a function of \(a\). Note that the range of \(y\) is finite, as the function \(F\) take values in a finite range. The range of \(a\) is bounded from below but not from above, that is

\[
-1 < \frac{128a^3}{3m},
\]

(3.74)

and it follows that \(a\) can take positive and negative values as well. When \(m < 0\), the relation \(y(a)\) is given by

\[
y(a) = \pm \frac{1}{\sqrt{3}} \int_a^{\infty} \frac{dv}{\sqrt{128v^3 - 1}} + y_0,
\]

(3.75)

where \(s\) is defined as \(s = -m\). As for the case corresponding to \(m > 0\), the value of \(a\) is bounded from below and not from above

\[
1 < \frac{128a^3}{3s},
\]

(3.76)

but the difference is that \(a\) can only take positive values. The solution in this case is given by

\[
y(a) = \pm \frac{1}{\sqrt{3}} \left(\frac{3s}{128}\right)^{\frac{1}{3}} \sqrt{F(\psi(a), k) + y_0},
\]

(3.77)

\[
\psi(a) = \cos^{-1}\left(\frac{(128)^{\frac{1}{3}}a + (3s)^{\frac{1}{3}} + (3s)^{\frac{1}{3}}\sqrt{3}}{(128)^{\frac{1}{3}}a + (3s)^{\frac{1}{3}} - (3s)^{\frac{1}{3}}\sqrt{3}}\right), \quad k = \sin \frac{k}{12}.
\]

(3.78)

Formulas (3.72) and (3.75) are the full solution of the first equation (3.69).
The second equation (3.69) gives $\xi$ as a function of $y$, but it seems to be a bit more problematic, as to solve it requires to invert (3.72) or (3.75) and to express $a$ as a function of $y$. Instead of doing this it is more direct to calculate $\xi$ as a function of $a$, which implicitly gives it as a function of $y$ by (3.72) or (3.75). From an elementary chain rule using (3.70) it is easy to show that for an arbitrary function $f(y)$ of the variable $y$, the following identities are true:

$$f_y = a_y f_a = \pm \sqrt{m + \frac{128}{3} a^3 f_a}, \quad f_{yy} = \left( m + \frac{128}{3} a^3 \right) f_{aa} + 64 a^2 f_a.$$

In these terms the second (3.69) is rewritten as

$$\left( m + \frac{128}{3} a^3 \right) \xi_{aa} + 64 a^2 \xi_a - 48 a \xi = 0. \quad (3.76)$$

By further introducing the natural variable

$$z = -\frac{128 a^3}{3m}, \quad (3.77)$$

it follows after some calculation that the second equation (3.69) is

$$z(1 - z) \xi_{zz} + \left( \frac{7}{6} z - \frac{5}{6} \right) \xi_z + \frac{1}{8} \xi = 0. \quad (3.78)$$

This is a hypergeometric equation, namely an equation of the form

$$z(1 - z) \xi_{zz} + \left[ n - (p + q + 1)z \right] \xi_z - pq \xi = 0,$$

with the following values for the constants $n, p$ and $q$:

$$p = \frac{1}{12} - \frac{\sqrt{19}}{12}, \quad q = \frac{1}{12} + \frac{\sqrt{19}}{12}, \quad n = \frac{2}{3}.$$

The solutions of the hypergeometric equation are well known. This equation has three regular singular points at $z = 0$, $z = 1$ and $z \to \infty$. Once the functions $a$ and $b$ are determined, the third equation (3.69) is not difficult to solve. This equation gives $c_{yy}$ as a function of $y$ and it is straightforward to find $c(y)$ by a double integration. Nevertheless, as will be seen below, this calculation is irrelevant and only the value of $c_{yy}$ is of importance.

### 3.3. Local form of the Calabi–Yau metric

After solving the system defining the functions $a(y)$, $b(y)$ and $c(y)$ determining our Calabi–Yau geometries, the next problem is to find the local form of the generic CY metric (2.43) for this case. The Kahler form of the base 4-metric is given by

$$\omega(y) = \ddbar{\omega} - dd^c G = \frac{i}{2} (\delta_i j - 4G_{ij}) dz_i \wedge d\bar{z}_j$$

$$= \frac{i}{2} (1 - 4G_{ui} - 4z_i \bar{z}_i G_{uu}) dz_i \wedge d\bar{z}_j - 2i G_{uu}(\bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 dz_2 \wedge d\bar{z}_1).$$

(3.80)

This expression may be simplified by use of (3.67) together with the definition $\xi = b - 1/4$ to give

$$\omega(y) = -\frac{i}{2} (4au + 4\xi + 4az_i \bar{z}_i) dz_i \wedge d\bar{z}_j - 2a(\bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 dz_2 \wedge d\bar{z}_1).$$

(3.81)

The Kahler 4-metric corresponding to (3.81) is simply

$$g_4(y) = -(4au + 4\xi + 4az_i \bar{z}_i) dz_i \otimes d\bar{z}_j - 4a(\bar{z}_1 z_2 dz_1 \otimes d\bar{z}_2 + \bar{z}_2 z_1 dz_2 \otimes d\bar{z}_1).$$

(3.82)
It is important to remark that when \( r \simeq 0 \) one has \( u \simeq 0 \) and \( z_i \overline{z}_j \simeq 0 \), and therefore (3.82) behaves as
\[
g_4(y) \simeq -4\xi (dz_1 \otimes d\overline{z}_1 + dz_2 \otimes d\overline{z}_2). \tag{3.83}
\]
Clearly (3.83) is, up to an \( a \)-dependent conformal factor, the flat metric on \( R^4 \). Therefore, near \( r = 0 \) no conical singularity appears. A more handy expression for (3.82) can be obtained by introducing the polar coordinates \((r, \theta, \psi, \phi)\) for \( R^4 \)
\[
z_1 = r \cos \frac{\theta}{2} \exp \left( \frac{i\psi + i\phi}{2} \right), \tag{3.84}
\]
\[
z_2 = r \sin \frac{\theta}{2} \exp \left( -\frac{i\psi - i\phi}{2} \right). \tag{3.85}
\]
The expression for metric (3.82) in these coordinates is
\[
g_4(y) = -\frac{4}{r^4} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) \tag{3.86}
\]
with \( \sigma_i \) being the usual \( SO(3) \) Maurer–Cartan forms
\[
\sigma_1 = \frac{1}{2} \left( \sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi \right),
\]
\[
\sigma_2 = \frac{1}{2} \left( -\cos \psi \, d\theta - \sin \theta \sin \psi \, d\phi \right),
\]
\[
\sigma_3 = \frac{1}{2} \left( d\psi + \cos \theta \, d\phi \right),
\]
for which the angles, \( a \) priori, take values in the following interval:
\[
0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi. \tag{3.87}
\]
Clearly, when \( r \simeq 0 \), metric (3.86) is approximated by
\[
g_4(y) = -4(2ar^2 + \xi)(dr^2 + r^2 \sigma_1^2) - 4(ar^2 + \xi)r^2 \left( \sigma_1^2 + \sigma_2^2 \right), \tag{3.88}
\]
which gives another check that non-conical singularity appears near \( r = 0 \). On the other hand, the fiber metric of the general Calabi–Yau metric (2.43) is determined in terms of \( G_{yy} \), namely
\[
G_{yy} = \frac{a_{yy}}{2} u^2 + b_{yy} u + c_{yy}. \tag{3.89}
\]
The explicit values of the derivatives appearing in (3.90) are obtained by use of (3.69); the result is
\[
a_{yy} = 64a^2, \quad b_{yy} = 48a\xi, \tag{3.90}
\]
\[
c_{yy} = 16 \left( \xi - \frac{1}{4} \right) \left( \xi + \frac{1}{4} \right) + 1 = 16\xi^2. \tag{3.91}
\]
Inserting expressions (3.91)–(3.92) into (3.90) gives
\[
G_{yy} = 32(au)^2 + 48\xi(au) + 16\xi^2, \tag{3.92}
\]
which can be expressed more neatly as
\[
G_{yy} = 16(2au + \xi)(au + \xi). \tag{3.93}
\]
Finally, the connection \( A \) defining our bundle is given by
\[
A = d^c G_y = G_{yu} d^c u = i(a_y u + b_y) (\overline{z}_1 dz_1 - z_1 d\overline{z}_1 + \overline{z}_2 dz_2 - z_2 d\overline{z}_2), \tag{3.94}
\]
where we have used (2.2) to calculate the action of \( d^c \) over \((z_i, \overline{z}_i)\). By use of parameterizations (3.84) and (3.85) we may re-express (3.95) as
\[
A = (a_y r^2 + \xi y) r^2 \sigma_3, \tag{3.96}
\]
where the relation \( b_\nu = \xi_\nu \) has been used. Taking into account the general local form for our Ricci-flat Kahler metrics (2.43) together with formulas (3.72), (3.75), (3.78), (3.86), (3.94) and (3.96) it follows that the local form of the Calabi–Yau metrics we are looking for is given by

\[
g_6 = 4(2ar^2 + \xi)(dr^2 + r^2\sigma_3^2) + 4(ar^2 + \xi)r^2(\sigma_1^2 + \sigma_2^2) + \frac{16(2au + \xi)(au + \xi)}{m - \frac{128a^3}{3}} da^2
+ \frac{1}{16(2au + \xi)(au + \xi)} \left( da + \sqrt{m - \frac{128a^3}{3}(r^2 + \xi_3)r^2\sigma_3} \right)^2,
\]

(3.97)

where we made the redefinitions \( a \rightarrow -a \) and \( \xi \rightarrow -\xi \). The curvature tensor is irreducible for all the cases described in (3.97); therefore, these metrics are of holonomy exactly \( SU(3) \).

4. Global analysis of the new CY metrics

It will be convenient to give a consistency check about the correctness of (3.97). If we assume intuitively that the radius \( r \) has length dimensions \([r] = L\) and that the metric has dimensions \( L^2\), then clearly the quantity

\( ar^2 + \xi, \)

appearing in the component \( g_{rr} \), should be dimensionless. This will be the case if \([a] = L^{-2}\). Under this assumption all the components of the Kahler base 4-metric of (3.97) are of dimensions \([L]^2\). Let us see if this is true for the fiber components. The component \( g_{\gamma\gamma} \) is clearly dimensionless and therefore we should have \([\gamma] = L\). On the other hand, the relation between \( \gamma \) and \( a \) can be read off from (3.70) and is

\([\gamma] = [a]^{-\frac{1}{2}} = L, \)

which confirms what we postulated. Moreover the fiber contains the term

\((ar^2 + \xi) r^2, \)

and it is seen that

\[ [(ar^2 + \xi) r^2] = L^2 [\gamma]^{-1} = L. \]

Therefore, if \([a] = L\) the whole metric has dimensions \( L^2\), which give a consistency check for (3.97).

Let us also note that metrics (3.97) can never be compact if \( K = \partial_\alpha \) is a global isometry. This is because a Killing vector satisfies the equation

\[- \square K_i + R_{ij} K^j = 0, \]

and by multiplying this equation by \( K_i \) and integrating over the manifold gives that

\[ \int_{M_6} (|\nabla_i K_j|^2 - R_{ij} K^i K^j). \]

Here the first term is obtained by integration by parts and there is a boundary term which vanishes if \( M_6 \) is compact. If the manifold is Ricci flat, then \(|\nabla_i K_j|^2 = 0\) pointwise in the manifold, and such vectors do not exist in a compact manifold.

In order to make a global analysis of metric (3.97) there are several options to consider, depending the signature of (3.97) and the sign of the parameter \( m \) defined in (3.70). For \( m > 0 \) the function \( a \) may take positive or negative values, as shown in (3.73), with the maximal value \( a_0 \) given 128a^3_0 = 3m. In general the range of \( u = r^2 \) will be constrained unless \( a > 0 \) and \( \xi > 0 \); in this case the range of \( r \) is \((0, \infty)\). Clearly there will be a power law singularity.
in the metric when \((2au + \xi)(au + \xi) = 0\). In principle one may choose the hypergeometric function \(\xi(a)\) in such a way that it is always positive and never zero in the range of values of \(a\). It seems that in this case this singularity will be absent. The problem with this argument is that the metric (3.97) does not have an asymptotic region where its is locally flat, so it should have a singularity, like a delta in the curvature acting as a cosmological constant. Therefore, even if we manage to construct such a strictly positive function \(\xi(a)\) in our interval this does not mean that the singularity is not present, but that we are using a bad coordinate system. For instance, the metric \(dr^2 + r^2 d\theta^2\) is singular, but with a parametrization \(r = x + 1/x\) the singularity would not be seen for any positive \(x\) value. Arguably, the same situation is happening here. Therefore, none of the constructed metrics should be geodesically complete.

5. Discussion

To conclude we remark the most salient features of the present work. We have developed a method to construct six-dimensional Calabi–Yau metrics in terms of an initial four-dimensional hyperkahler structure generalizing the situations considered in [9]; the last ones correspond to D6-branes with two directions inside a complex submanifold of the hyperkahler space and one direction transversal to it parameterized with a coordinate \(y\). In all these cases the resulting CY metrics possess a Killing vector which preserves the whole \(SU(3)\) structure.

It should be remarked that in general the CY metrics we described are not fibered over the initial hyperkahler metric. This situation in fact occurs in [9] and the physicist explanation is that when the D6-branes wrap the complex submanifold, their back-reaction deforms the initial hyperkahler geometry. The same consideration follows here and the resulting CY metrics are fibered over a four-dimensional base which is just Kahler and depends on the coordinate \(y\) as a parameter. In other words it is Kahler for any fixed value of \(y\).

We also showed that the entire geometry is described in terms of a single function \(G\) satisfying a nonlinear quadratic differential equation (2.41). That this function is enough to determine the full geometry is evident, as the whole \(SU(3)\) structure (2.42) is defined purely in terms of \(G\) and derivates. The function \(G\) depends on \(y\) and in fact can be interpreted as the Kahler potential for the base four-dimensional metric for any fixed value of \(y\). We studied the nonlinear problem associated with the flat hyperkahler metric in \(R^4\) and we found some particular solutions, which allowed us to construct a family of CY examples with power law singularities.

An interesting open problem is the one arising by the fact that the supergravity solution describing a D6-branes wrapping the \(S^2\) in Eguchi–Hanson gravitational instanton is described by the resolved conifold metric [22–24]. As the Eguchi–Hanson metric is hyperkahler, the method developed here may be suitable to make this link explicit. In particular it may be one of the situations for which the Fayyazuddin linearization works properly. We hope to find an answer to these questions in the near future.

Acknowledgments

The completion of the present work would not be possible without important advice given by S Cherkis who also participated at early stages of this project. Also I sincerely acknowledge H Lu who pointed out a crucial numerical error in the initial form of this work. I have also benefited from short but stimulating discussions with J Vazquez-Poritz, M Leston and C Saemann. This work is supported by the Science Foundation Ireland grant no 06/RFP/MAT050.
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