SPECTRAL ISOPERIMETRIC INEQUALITY FOR THE $\delta'$-INTERACTION ON A CONTOUR

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ABSTRACT. We consider the problem of geometric optimization for the lowest eigenvalue of the two-dimensional Schrödinger operator with an attractive $\delta'$-interaction of a fixed strength, the support of which is a $C^2$-smooth contour. Under the constraint of a fixed length of the contour, we prove that the lowest eigenvalue is maximized by the circle. The proof relies on the min-max principle and the method of parallel coordinates.

1. Introduction

1.1. The state of the art and motivation

The question of optimizing shapes in spectral theory is a rich subject with many applications and deep mathematical insights; see the monographs [H-1, H-2] and the references therein. In this note, we consider the problem of shape optimization for the lowest eigenvalue of the two-dimensional Schrödinger operator with a $\delta'$-interaction supported on a closed contour in $\mathbb{R}^2$. This problem can be regarded as a counterpart of the analysis performed in [EHL06] for $\delta$-interactions.

In the recent years, the investigation of Schrödinger operators with $\delta'$-interactions supported on hypersurfaces became a topic of permanent interest – see, e.g., [BGLL15, BEL14, BLL13, EJ13, EKh15, EKh18, JL16, MPS16]. The Hamiltonians with $\delta'$-interactions and some of their generalizations appear, for example, in the study of photonic crystals [FK96a, FK96b] and in the analysis of the Dirac operator with scalar shell interactions [HOP18]. The boundary condition corresponding to the $\delta'$-interaction arises in the asymptotic analysis of a class of structured thin Neumann obstacles [DFZ18, H70]. Finally, the same boundary condition pops up in the computational spectral theory; see [Da99] and the references therein.

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The proofs in [EHL06] and in related optimization problems for singular interactions on hypersurfaces [AMV16, BFK+17, E05, EL17, EL18, L18] rely on the Birman-Schwinger principle, which can also be viewed as a boundary integral reformulation of the spectral problem. In this note, we do not pass to any boundary integral reformulation. Instead, we combine the min-max principle and the method of parallel coordinates on the level of the quadratic form for the Hamiltonian, in the spirit of the recent analysis for the Robin Laplacian [AFK17, FK15, KL, KL18, KL17]. Our main motivation is to show that this approach initially developed for the Robin Laplacian can also be adapted for a much wider class of optimization problems involving surface interactions. The convenience of this alternative method is particularly visible for δ′-interactions, because the operator arising in the corresponding Birman-Schwinger principle (cf. [BLL13, Rem. 3.9]) is more involved than for δ-interactions.

1.2. Schrödinger operator with a δ′-interaction on a contour

In order to define the Hamiltonian, we need to introduce some notation. In what follows we consider a bounded, simply connected, $C^2$-smooth domain $\Omega_+ \subset \mathbb{R}^2$, whose boundary will be denoted by $\Sigma = \partial \Omega_+$. The complement $\Omega_- := \mathbb{R}^2 \setminus \Omega_+$ of $\Omega_+$ is an unbounded exterior domain with the same boundary $\Sigma$. For a function $u \in L^2(\mathbb{R}^2)$ we set $u_\pm := u|_{\Omega_\pm}$. We also introduce the first order $L^2$-based Sobolev space on $\mathbb{R}^2 \setminus \Sigma$ as follows

$$H^1(\mathbb{R}^2 \setminus \Sigma) := H^1(\Omega_+) \oplus H^1(\Omega_-),$$

where $H^1(\Omega_\pm)$ are the conventional first-order $L^2$-based Sobolev spaces on $\Omega_\pm$.

Given a real number $\omega > 0$, we consider the spectral problem for the self-adjoint operator $H_{\omega, \Sigma}$ corresponding via the first representation theorem to the closed, densely defined, symmetric, and semi-bounded quadratic form in $L^2(\mathbb{R}^2)$,

$$h_{\omega, \Sigma}[u] = \|\nabla_{\mathbb{R}^2 \setminus \Sigma} u\|_{L^2(\mathbb{R}^2; \mathbb{C}^2)}^2 - \omega \|[u]_{\Sigma}\|_{L^2(\Sigma)}^2, \quad \text{dom } h_{\omega, \Sigma} = H^1(\mathbb{R}^2 \setminus \Sigma),$$

where $\nabla_{\mathbb{R}^2 \setminus \Sigma} u := \nabla u_+ \oplus \nabla u_-$ and $[u]_{\Sigma} := u_+|_{\Sigma} - u_-|_{\Sigma}$ denotes the jump of the trace of $u$ on $\Sigma$; cf. [BEL14, Sec. 3.2]. The operator $H_{\omega, \Sigma}$ is usually called the Schrödinger operator with the δ′-interaction of strength $\omega$ supported on $\Sigma$. It acts as the minus Laplacian on functions satisfying the transmission boundary condition of δ′-type on the interface $\Sigma$

$$\partial_{\nu_+} u_+|_{\Sigma} = -\partial_{\nu_-} u_-|_{\Sigma} = \omega [u]|_{\Sigma},$$

where $\partial_{\nu_+} u_\pm|_{\Sigma}$ denotes the trace onto $\Sigma$ of the normal derivative of $u_\pm$ with the normal vector $\nu_\pm$ at the boundary of $\Omega_\pm$ pointing outwards; see Section 2 for more details.
Recall that the essential spectrum of $H_{\omega, \Sigma}$ coincides with the set $[0, \infty)$ and that its negative discrete spectrum is known to be non-empty; see Proposition 2 below. By $\lambda_1^\omega(\Sigma)$ we denote the spectral threshold of $H_{\omega, \Sigma}$, which is an isolated negative eigenvalue.

1.3. The main result

The aim of this note is to demonstrate that $\lambda_1^\omega(\Sigma)$ is maximized by the circle $C \subset \mathbb{R}^2$, among all contours of a fixed length. A precise formulation of this statement is the content of the following theorem.

**Theorem 1.** For any $\omega > 0$, one has

$$\max_{|\Sigma|=L} \lambda_1^\omega(\Sigma) = \lambda_1^\omega(C),$$

where $C \subset \mathbb{R}^2$ is a circle of a given length $L > 0$ and the maximum is taken over all $C^2$-contours of length $L$.

The proof of Theorem 1 relies on the min-max principle and the method of parallel coordinates. The latter method has been proposed in [PW61] by L. E. Payne and H. F. Weinberger in order to obtain inequalities being reverse to the celebrated Faber-Krahn inequality [F23, K24] with some geometrically-induced corrections. Recently it has been observed that this method is very efficient in the proofs of isoperimetric inequalities for the lowest eigenvalue of the Robin Laplacian on bounded [AFK17, FK15] and exterior [KL18, KL17] domains with an ‘attractive’ boundary condition. In the present paper we adapt this approach for the case of a bounded domain and its exterior coupled via the transmission boundary condition (1.2) of $\delta'$-type.

Organisation of the paper

In Section 2 we recall the known spectral properties of $H_{\omega, \Sigma}$ that are needed in this paper. Section 3 is devoted to the spectral analysis of $H_{\omega, C}$ with the interaction supported on a circle $C$. The method of parallel coordinates is briefly outlined in Section 4. Theorem 1 is proven in Section 5. The paper is concluded by Section 6 containing a discussion of the obtained results and their possible extensions.

2. The spectral problem for the $\delta'$-interaction supported on a closed contour

Recall that we consider a bounded, simply connected, $C^2$-smooth domain $\Omega_+ \subset \mathbb{R}^2$ with the boundary $\Sigma = \partial \Omega_+$ and with the complement $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega_+}$. 
Recall also that for a function $u \in L^2(\mathbb{R}^2)$, we set $u_{\pm} := u|_{\Omega_{\pm}}$. At the same time, the (attractive) coupling strength $\omega$ is a fixed positive number.

We are interested in the spectral properties of the self-adjoint operator $H_{\omega,\Sigma}$ in $L^2(\mathbb{R}^2)$ introduced via the first representation theorem [K, Thm. VI 2.1] as associated with the closed, densely defined, symmetric, and semi-bounded quadratic form $h_{\omega,\Sigma}$ defined in (1.1); see [BEL14, Sec. 3.2] and also [BLL13, Sec. 3.3 and Prop. 3.15].

We would like to warn the reader that in the majority of the papers on $\delta'$-interactions not $\omega$ itself, but its inverse $\beta := \omega^{-1}$ is called the strength of the interaction. This tradition goes back to papers on point $\delta'$-interaction on the real line; see [AGHH] and the references therein. Preserving this tradition for $\delta'$-interactions on hypersurfaces can be physically motivated, but leads to a technical mathematical inconvenience, which we would like to avoid.

Let us add a few words about the explicit characterisation of the operator $H_{\omega,\Sigma}$. The domain of $H_{\omega,\Sigma}$ consists of functions $u \in H^1(\mathbb{R}^2 \setminus \Sigma)$, which satisfy $\Delta u_{\pm} \in L^2(\Omega_{\pm})$ in the sense of distributions and the $\delta'$-type boundary condition (1.2) on $\Sigma$ in the sense of traces. Moreover, for any $u \in \text{dom} H_{\omega,\Sigma}$ we have $H_{\omega,\Sigma} u = (-\Delta u_{\pm}) \oplus (-\Delta u_{\pm})$. The reader may consult [BEL14, Sec. 3.2 and Thm. 3.3], where it is shown that the operator characterised above is indeed the self-adjoint operator representing the quadratic form $h_{\omega,\Sigma}$ in (1.1). It is worth mentioning that $C^2$-smoothness of $\Sigma$ is not needed to define the operator $H_{\omega,\Sigma}$, but it is important for the method of parallel coordinates used in the proof of Theorem 1.

The lowest spectral point of $H_{\omega,\Sigma}$ can be characterised by the min-max principle [RS-IV, Sec. XIII.1] as follows

$$
\lambda_1^\omega(\Sigma) = \inf_{u \in H^1(\mathbb{R}^2 \setminus \Sigma)} \frac{h_{\omega,\Sigma}[u]}{\|u\|^2_{L^2(\mathbb{R}^2)}}.
$$

It is not surprising that the operator $H_{\omega,\Sigma}$ has a non-empty essential spectrum. In fact, one can show that $H_{\omega,\Sigma}$ is a compact perturbation in the sense of resolvent differences of the free Laplacian on $\mathbb{R}^2$ and thus the essential spectrum coincides with the positive semi-axis. Using the characteristic function of $\Omega_{\pm}$ as a test function for (2.1) one gets that the negative discrete spectrum of $H_{\omega,\Sigma}$ is non-empty. More specifically, we have the following statement.

**Proposition 2.** For all $\omega > 0$, the following hold.

(i) The essential spectrum of $H_{\omega,\Sigma}$ is characterized as follows $\sigma_{\text{ess}}(H_{\omega,\Sigma}) = [0, \infty)$.

(ii) The negative discrete spectrum of $H_{\omega,\Sigma}$ is non-empty.

A proof of (i) in the above proposition can be found in [BEL14, Thm. 4.2 (ii)], see also [BLL13, Thm. 3.14 (i)]. A proof of (ii) is contained in [BEL14, Thm. 4.4].
Some further properties of the discrete spectrum of $H_{\omega, \Sigma}$ are investigated in or follow from [BLL13, EJ13]. Note that by [BLL13, Thm. 3.14(ii)] the negative discrete spectrum of $H_{\omega, \Sigma}$ is finite for $C^\infty$-smooth $\Sigma$ and it can be shown in a similar way that the discrete spectrum persists to be finite for $C^2$-smooth $\Sigma$.

Taking that the spectral threshold of $H_{\omega, \Sigma}$ is a negative discrete eigenvalue into account, we can slightly modify the characterisation of $\lambda_1^\omega(\Sigma)$ given in (2.1) as follows:

\begin{equation}
\lambda_1^\omega(\Sigma) = \inf_{u \in H^1(\mathbb{R}^2 \setminus \Sigma), \ h_{\omega, \Sigma}[u] < 0} \frac{h_{\omega, \Sigma}[u]}{\|u\|^2_{L^2(\mathbb{R}^2)}}. \tag{2.2}
\end{equation}

3. The spectral problem for the $\delta'$-interaction supported on a circle

In this section we consider the lowest eigenvalue for the operator $H_{\omega, c}$ with the $\delta'$-interaction of strength $\omega > 0$ supported on a circle $C = C_R \subset \mathbb{R}^2$ of radius $R > 0$. Our primary interest concerns the dependence of this eigenvalue on the radius $R$. For the sake of convenience, we introduce the polar coordinates $(r, \theta)$, whose pole coincides with the center of $C$. Note also that the circle $C$ splits the Euclidean plane $\mathbb{R}^2$ into the disk $D_+ = \{x \in \mathbb{R}^2: |x| < R\}$ and its exterior $D_- = \{x \in \mathbb{R}^2: |x| > R\}$.

**Proposition 3.** Let $C = C_R \subset \mathbb{R}^2$ be a circle of radius $R > 0$. Let $\lambda_1^\omega(C_R) = -k^2 < 0$ and $u_1 \in H^1(\mathbb{R}^2 \setminus C)$ be, respectively, the lowest eigenvalue and a corresponding eigenfunction of $H_{\omega, c}$. Then the following hold.

\begin{enumerate}[(i)]
  \item The value $k > 0$ is the unique positive solution of the equation
    \[ k^2 RI_1(kR)K_1(kR) = \omega. \]
  \item The function $(0, \infty) \ni R \mapsto \lambda_1^\omega(C_R)$ is continuous, increasing, and
    \[ \lim_{R \to 0^+} \lambda_1^\omega(C_R) = -\infty \quad \text{and} \quad \lim_{R \to \infty} \lambda_1^\omega(C_R) = -4\omega^2. \]
  \item The ground-state $u_1$ is radial and can be expressed in polar coordinates $(r, \theta)$ as
    \begin{equation}
    u_1(r, \theta) = \begin{cases} 
    K_1(kR)I_0(kr), & r < R, \\
    -I_1(kR)K_0(kr), & r > R.
    \end{cases} \tag{3.1}
    \end{equation}
\end{enumerate}

**Proof.** In view of the radial symmetry of the problem, the eigenfunction $u_1 \in L^2(\mathbb{R}^2)$ must necessarily be radially symmetric as well. Therefore, in polar coordinates $(r, \theta)$ we have $u_1(r, \theta) = \psi(r)$. Using this simple observation we see
that $\lambda^2(\mathcal{C}) = -k^2 < 0$ if, and only if, the following ordinary differential spectral problem

\[
\begin{align*}
-\rho^{-1}[\rho\psi'(\rho)]' &= -k^2\psi(\rho) \quad \text{for } \rho \in \mathbb{R}_+ \setminus \{R\}, \\
\psi'(R^-) &= \psi'(R^+) = \omega[\psi(R^-) - \psi(R^+)], \\
\lim_{\rho \to \infty} \psi(\rho) &= \psi'(0) = 0,
\end{align*}
\]

possesses a solution $(\psi, k)$ with $\psi \neq 0$, $k > 0$; cf. [AFK17, Sec. 2] and [KL18, Sec. 3]. Observe that the general solution of the differential equation in (3.2) with $k > 0$ is given by

\[
\psi(\rho) = \begin{cases} 
A_+ K_0(k\rho) + B_+ I_0(k\rho), & r < R, \\
A_- K_0(k\rho) + B_- I_0(k\rho), & r > R,
\end{cases}
\]

where $A_\pm, B_\pm \in \mathbb{C}$ are some coefficients and $K_0(\cdot), I_0(\cdot)$ are the modified Bessel functions of zero order. Taking into account the boundary conditions at infinity and at the origin from (3.2) and using the behaviour of $K_0(x)$ and $I_0(x)$ and of their derivatives for large [AS64, 9.7.1-4] and small [AS64, 9.6.7-9] values of $x$ we conclude that $A_+ = B_- = 0$. Thus, the expression for $\psi$ simplifies to

\[
\psi(\rho) = \begin{cases} 
B_+ I_0(k\rho), & r < R, \\
A_- K_0(k\rho), & r > R,
\end{cases}
\]

where the constants $B_+$ and $A_-$ must not both be zero to get a non-trivial solution. Differentiating $\psi$ with respect to $r$, we find

\[
\psi'(\rho) = \begin{cases} 
kB_+ I_1(k\rho), & r < R, \\
-kA_- K_1(k\rho), & r > R.
\end{cases}
\]

Thus, the boundary condition in (3.2) at the point $r = R$ yields the requirement

\[
\begin{align*}
B_+ I_1(kR) &= -A_- K_1(kR), \\
\omega(B_+ I_0(kR) - A_- K_0(kR)) &= B_+ k I_1(kR).
\end{align*}
\]

This linear system of equations can be simplified as

\[
\begin{align*}
A_- K_1(kR) + B_+ I_1(kR) &= 0, \\
-A_- \omega K_0(kR) + B_+ (\omega I_0(kR) - k I_1(kR)) &= 0.
\end{align*}
\]

The existence of a non-trivial solution for the system above is equivalent to vanishing of the underlying determinant, which gives us a scalar equation on $k$

\[
\omega [K_1(kR) I_0(kR) + I_1(kR) K_0(kR)] - k K_1(kR) I_1(kR) = 0.
\]
Provided \( k > 0 \) is a solution of (3.4), the vector \((A_-, B_+)\) is a solution of the system (3.3) and, hence, the expression (3.1) for the ground-state \( u_1 \) immediately follows.

Furthermore, using the identity \( K_1(x)I_0(x) + I_1(x)K_0(x) = x^{-1} \) (see [AS64, 9.6.15]) we simplify (3.4) as

\[
(3.5) \quad k^2RK_1(kR)I_1(kR) = \omega.
\]

Consider now the \( C^\infty \)-smooth function \( F(x) := xK_1(x)I_1(x) \) on \((0, \infty)\) in more detail. The analysis in [HW74, Prop 7.2] implies that \( F'(x) > 0 \) and \( \lim_{x \to 0^+} F(x) = 0, \lim_{x \to \infty} F(x) = \frac{1}{2} \). Hence, the function \( G(k) := kF(kR) \) in the left-hand side of (3.5) is strictly increasing in \( k \) and satisfies \( \lim_{k \to 0^+} G(k) = 0, \lim_{k \to \infty} G(k) = \infty, G(k) < \frac{k}{2} \). Therefore, the equation (3.5) possesses a unique positive solution \( k_* = k_*(R) > 0 \) satisfying the bounds

\[
(3.6) \quad 2\omega < k_*(R) < \frac{\omega}{F(2\omega R)}.
\]

Consequently, we get \( \lim_{R \to \infty} k_*(R) = 2\omega \) and, hence, \( \lim_{R \to \infty} \lambda_1^\omega(C_R) = -4\omega^2 \).

Using the implicit function theorem [KP, Thm. 3.3.1] we find that \( (0, \infty) \ni R \mapsto k_*(R) \) is a \( C^\infty \)-smooth function, whose derivative satisfies

\[
k'_*(R)F(k_*(R)R) + (k_*(R) + RK'_*(R))F'(k_*(R)R) = 0.
\]

The above equation yields

\[
k'_*(R) = -\frac{k_*(R)F'(k_*(R)R)}{F(k_*(R)R) + RF'(k_*(R)R)} < 0.
\]

Hence, the function \( R \mapsto \lambda_1^\omega(C_R) = -(k_*(R))^2 \) is increasing.

Finally, using the characteristic function \( \chi_{D_+} \in H^1(\mathbb{R}^2 \setminus C) \) of the disk \( D_+ \) as a test function we get

\[
\lambda_1^\omega(C_R) \leq \frac{\| \{ \omega, \chi \}_{L^2(\mathbb{R}^2)} \}}{\| \chi_{D_+} \|_{L^2(\mathbb{R}^2)}} = \frac{-2\pi R\omega}{\pi R^2} = -\frac{2\omega}{R} \to -\infty, \quad \text{as} \ R \to 0^+.
\]

4. The method of parallel coordinates

In this section we briefly recall the method of parallel coordinates. We follow the modern presentation in [S01] with an adjustment of notation. Further details and proofs can be found in the classical papers [F41, H64], see also the monograph [Ba80] and the references therein.

First, we introduce the distance-functions on the domains \( \Omega_\pm \) as

\[
\rho_\pm : \Omega_\pm \to \mathbb{R}_+, \quad \rho_\pm(x) := \text{dist}(x, \Sigma).
\]
The functions \( \rho_{\pm} \) are Lipschitz continuous with the Lipschitz constant \( = 1 \)

\[
|\rho_{\pm}(x) - \rho_{\pm}(y)| \leq |x - y|, \quad \forall x, y \in \Omega_{\pm}.
\]  

(4.1)

For the convenience of the reader we will show (4.1) for \( \Omega_- \). Without loss of generality we suppose that \( \rho_-(x) \geq \rho_-(y) \). Let \( z \in \Sigma \) be such that \( \rho_-(y) = |y - z| \). Hence, we obtain that \( \rho_-(x) \leq |x - z| \). Thus, we get

\[
|\rho_-(x) - \rho_-(y)| \leq |x - z| - |y - z| \leq |x - y|,
\]

where the last step follows from the triangle inequality in \( \mathbb{R}^2 \).

Furthermore, we introduce the in-radii of \( \Omega_{\pm} \) by

\[
R_{\pm} := \sup_{x \in \Omega_{\pm}} \rho_{\pm}(x).
\]

The in-radius of \( \Omega_+ \) is thus the radius of the largest disk in \( \mathbb{R}^2 \) that can be inscribed into \( \Omega_+ \), and due to the standard well-known isoperimetric inequality

\[
|\Sigma|^2 \geq 4\pi|\Omega_+|
\]

we get

\[
R_+ \leq R, \quad \text{where} \quad R = \frac{L}{2\pi}.
\]  

(4.2)

On the other hand, we obviously have \( R_- = \infty \).

Finally, we introduce the following auxiliary functions

\[
L_{\pm}: [0, R_{\pm}] \to \mathbb{R}_+, \quad L_{\pm}(t) := |\{x \in \Omega_{\pm}: \rho_{\pm}(x) = t\}|,
\]

\[
A_{\pm}: [0, R_{\pm}] \to [0, |\Omega_{\pm}|], \quad A_{\pm}(t) := \{|x \in \Omega_{\pm}: \rho_{\pm}(x) < t\}.
\]  

(4.3)

Clearly, \( L_{\pm}(0) = L \) and \( A_+(R_+) = |\Omega_+| \). The value \( A_{\pm}(t) \) is simply the area of the sub-domain of \( \Omega_{\pm} \), which consists of the points located at the distance less that \( t \) from its boundary \( \Sigma \). On the other hand, \( L_{\pm}(t) \) is the length of the corresponding level set of the function \( \rho_{\pm} \).

Some analytic properties of the functions in (4.3) are summarized in the following proposition.

**Proposition 4.** [S01, App. 1, Prop. A.1], [Ba80, Chap. I, Sec. 3.6] Let the functions \( A_{\pm} \) and \( L_{\pm} \) be as in (4.3). Then the following hold.

(i) \( A_{\pm} \) is continuous, locally Lipschitz, and increasing.

(ii) \( A_+(t) = L_+(t) > 0 \) for almost every \( t \in [0, R_+] \).

(iii) \( L_+(t) \leq L - 2\pi t \) and \( L_-(t) \leq L + 2\pi t \).

Further, let \( \psi_+ \in C^\infty([0, R_+]) \) and \( \psi_- \in C^\infty_0([0, \infty)) \) be arbitrary and real-valued. Due to the properties of \( A_{\pm} \) stated in Proposition 4 (i), there exist Lipschitz continuous functions \( \phi_+ : [0, |\Omega_+|] \to \mathbb{R} \) and \( \phi_- : [0, \infty) \to \mathbb{R} \) satisfying

\[
\psi_+|_{[0,R_+]} = \phi_+ \circ A_+ \quad \text{and} \quad \psi_- = \phi_- \circ A_-. \]  

(4.4)
Consider now the test function
\[ u = (\phi_+ \circ A_+ \circ \rho_+) \oplus (\phi_- \circ A_- \circ \rho_-). \]

Lipschitz continuity of \( \phi_\pm \), Proposition 4 (i) and (4.1) imply that \( u \in H^1(\mathbb{R}^2 \setminus \Sigma) \).

Employing the parallel coordinates together with the co-area formula (see [S01, Eq. 30] for more details) and applying further (4.2), (4.4) we get
\[
\|\nabla_{\mathbb{R}^2 \setminus \Sigma} u\|_{L^2(\mathbb{R}^2 \setminus \Sigma)}^2 = \|\nabla u_+\|_{L^2(\Omega_+ \cap C^2)}^2 + \|\nabla u_-\|_{L^2(\Omega_- \cap C^2)}^2
\]
\[
= \int_0^{R_+} |\phi'_+ (A_+ (t))|^2 (A'_+ (t))^3 dt + \int_0^{\infty} |\phi'_- (A_- (t))|^2 (A'_- (t))^3 dt
\]
\[
(4.5)
\]
\[
= \int_0^{R_+} |\psi'_+ (t)|^2 (A'_+ (t)) \text{dt} + \int_0^{\infty} |\psi'_- (t)|^2 (A'_- (t)) \text{dt}
\]
\[
\leq \int_0^{R} |\psi'_+ (t)|^2 (L - 2\pi t) \text{dt} + \int_0^{\infty} |\psi'_- (t)|^2 (L + 2\pi t) \text{dt},
\]
where Proposition 4 (ii), (iii) was used in the last step. Following the same steps (cf. [S01, App. 1]) we also get
\[
\|u\|_{L^2(\mathbb{R}^2)}^2 = \int_0^{R_+} |\phi'_+ (A_+ (t))|^2 (A'_+ (t)) \text{dt} + \int_0^{\infty} |\phi'_- (A_- (t))|^2 (A'_- (t)) \text{dt}
\]
\[
(4.6)
\]
\[
\leq \int_0^{R} |\psi'_+ (t)|^2 (L - 2\pi t) \text{dt} + \int_0^{\infty} |\psi'_- (t)|^2 (L + 2\pi t) \text{dt}.
\]

Let us focus on the jump of the trace of \( u \) onto \( \Sigma \). It is easy to see that for any \( x \in \Sigma \) we have \([u]_\Sigma (x) = \psi_+ (0) - \psi_- (0)\). Hence, we obtain
\[
(4.7) \quad \|\|[u]_\Sigma\|_{L^2(\Sigma)}^2 = L |\psi_+ (0) - \psi_- (0)|^2.
\]

5. **Proof of Theorem 1**

We are now able to conclude the proof of Theorem 1. The argument will be split into two steps.

**Step 1.** On this step, we make several preliminary constructions. First, we define the sub-space of \( H^1(\mathbb{R}^2 \setminus \mathcal{C}) \) as
\[
\mathcal{L} := \{ w = w_+ \oplus w_- \in C^\infty (\overline{D_+}) \oplus C_0^\infty (\overline{D_-}) : \partial_\theta w = 0 \}.
\]
Notice that for any \( w \in \mathcal{L} \) there exist functions \( \psi_+ \in C^\infty ([0, R]) \) and \( \psi_- \in C_0^\infty ([0, \infty)) \) satisfying \( w_+ (r, \theta) = \psi_+ (R - r) \) and \( w_- (r, \theta) = \psi_- (r - R) \). Next, we point out that the ground-state \( u_1 \in H^1(\mathbb{R}^2 \setminus \mathcal{C}) \) of \( H_{\omega, \epsilon} \) given in (3.1) belongs to the closure of \( \mathcal{L} \) in the norm of \( H^1(\mathbb{R}^2 \setminus \mathcal{C}) \); i.e. there exists a sequence \( (w_n)_n \in \mathcal{L} \) such that
\[
(5.1) \quad \|w_n - u_1\|_{H^1(\mathbb{R}^2 \setminus \mathcal{C})} \to 0, \quad n \to \infty.
\]
Finally, we define the linear mapping $V: \mathcal{L} \to H^1(\mathbb{R}^2 \setminus \Sigma)$ by

$$(Vw)(x) := \begin{cases} \psi_+(\rho_+(x)), & x \in \Omega_+, \\ \psi_-(\rho_-(x)), & x \in \Omega_- \end{cases}$$

**Step 2.** Using the inequalities (4.5), (4.6) and the identity (4.7), we obtain from the min-max principle (2.2) that

$$\lambda_1^\omega(\Sigma) \leq \inf_{w \in \mathcal{L} : \mathcal{h}_\omega, \Sigma[w] < 0} \frac{\mathcal{h}_\omega, \Sigma[Vw]}{\|Vw\|_{L^2(\mathbb{R}^2)}^2} \leq \inf_{w \in \mathcal{L} : \mathcal{h}_\omega, \Sigma[w] < 0} \frac{\mathcal{h}_\omega, \Sigma[u_1]}{\|u_1\|_{L^2(\mathbb{R}^2)}^2} = \lambda_1^\omega(\mathcal{E}),$$

where the property (5.1) was used in the last but one step.

### 6. Discussion

The same technique can be used to reprove the optimization result in [EHL06] on $\delta$-interactions without making use of the Birman-Schwinger principle. In fact, the method seems to be applicable for a larger sub-class of general four-parametric boundary conditions, considered in [ER16]. One has only to ensure that the lowest spectral point is indeed a negative eigenvalue and that the corresponding ground-state is real-valued and radially symmetric for the case of the interaction supported on a circle.

For the moment, it is unclear how to prove a counterpart of Theorem 1 and whether it is true or not under the constraint of a fixed area. In contrast to the case of the Robin Laplacian on an exterior domain [KL18, KL17], this result does not follow from the corresponding inequality under the constraint of a fixed perimeter, because the lowest eigenvalue for the $\delta'$-interaction supported on a circle is not a decreasing, but an increasing function of its radius; see Proposition 3. The same problem arises for the Robin Laplacian on a bounded domain with a negative boundary parameter [AFK17, FK15].

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References

[AS64] M. S. Abramowitz and I. A. Stegun, eds., Handbook of mathematical functions, Dover, New York, 1964.

[AGHH] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, Solvable models in quantum mechanics. With an appendix by Pavel Exner. 2nd revised ed. Providence, AMS Chelsea Publishing, 2005.

[AFK17] P. R. S. Antunes, P. Freitas, and D. Krejčířík, Bounds and extremal domains for Robin eigenvalues with negative boundary parameter, Adv. Calc. Var. 10 (2017), 357–380.

[AMV16] N. Arrizabalaga, A. Mas, and L. Vega, An isoperimetric-type inequality for electrostatic shell interactions for Dirac operators, Commun. Math. Phys. 344 (2016), 483–505.

[Ba80] C. Bandle, Isoperimetric inequalities and applications, Monographs and Studies in Mathematics, Pitman, 1980.

[BEL14] J. Behrndt, P. Exner, and V. Lotoreichik, Schrödinger operators with δ and δ′-interactions on Lipschitz surfaces and chromatic numbers of associated partitions, Rev. Math. Phys. 26 (2014), 1450015.

[BFK+17] J. Behrndt, R. L. Frank, C. Kühn, V. Lotoreichik, and J. Rohleder, Spectral theory for Schrödinger operators with δ-interactions supported on curves in $\mathbb{R}^3$, Ann. Henri Poincaré, 18 (2017), 1305–1347.

[BGLL15] J. Behrndt, G. Grubb, M. Langer, and V. Lotoreichik, Spectral asymptotics for resolvent differences of elliptic operators with δ and δ′-interactions on hypersurfaces, J. Spectr. Theory. 5 (2015), 697–729.

[BLL13] J. Behrndt, M. Langer, and V. Lotoreichik, Schrödinger operators with δ and δ′-potentials supported on hypersurfaces, Ann. Henri Poincaré 14 (2013), 385–423.

[DFZ18] G. Dal Maso, G. Franzina, and D. Zucco, Transmission conditions obtained by homogenisation, to appear in Nonlinear Anal., arXiv:1805.01736.

[Da99] E. B. Davies, ICMS lecture notes on computational spectral theory, in: Spectral theory and geometry. Proceedings of the ICMS instructional conference, Edinburgh, UK, 1998. Cambridge University Press, Lond. Math. Soc. Lect. Note Ser. 273 (1999), 76–94.

[E05] P. Exner, An isoperimetric problem for leaky loops and related mean-chord inequalities, J. Math. Phys. 46 (2005), 062105.

[EHL06] P. Exner, E. M. Harrell, and M. Loss, Inequalities for means of chords, with application to isoperimetric problems, Lett. Math. Phys. 75 (2006), 225–233.

[EJ13] P. Exner and M. Jex, Spectral asymptotics of a strong δ′-interaction on a planar loop, J. Phys. A: Math. Theor. 46 (2013), 345201.

[EKh15] P. Exner and A. Khrabustovskyi, On the spectrum of narrow Neumann waveguide with periodically distributed traps, J. Phys. A: Math. Theor. 48 (2015), 315301.

[EKh18] P. Exner and A. Khrabustovskyi, Gap control by singular Schrödinger operators in a periodically structured metamaterial, to appear in Zh. Mat. Fiz. Anal. Geom., arXiv:1802.07522.

[EL17] P. Exner and V. Lotoreichik, A spectral isoperimetric inequality for cones, Lett. Math. Phys. 107 (2017), 717–732.

[EL18] P. Exner and V. Lotoreichik, Optimization of the lowest eigenvalue for leaky star graphs, in the proceedings of the conference Mathematical Results in Quantum Physics (QMath13), arXiv:1701.06840.
[ER16] P. Exner and J. Rohleder, Generalized interactions supported on hypersurfaces, *J. Math. Phys.* 57 (2016), 041507, 23 p.

[F23] G. Faber, Beweis dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, *Sitz. bayer. Akad. Wiss.* (1923), 169–172.

[F41] F. Fiala, Les problèmes des isopérimètres sur les surfaces ouvertes à courbure positive, *Comm. Math. Helv.* 13 (1941), 293–346.

[FK96a] A. Figotin and P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. I. Scalar model, *SIAM J. Appl. Math.*, 56 (1996), 68–88.

[FK96b] A. Figotin and P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. II. Two-dimensional photonic crystals, *SIAM J. Appl. Math.* 56 (1996), 1561–1620.

[FK15] P. Freitas and D. Krejčířík, The first Robin eigenvalue with negative boundary parameter, *Adv. Math.* 280 (2015), 322–339.

[H64] P. Hartman, Geodesic parallel coordinates in the large, *Amer. J. Math.* 86 (1964), 705–727.

[HW74] P. Hartman and G. Watson, ‘Normal’ distribution functions on spheres and the modified Bessel functions, *Ann. Probab.* 2 (1974), 593–607.

[H-1] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, Birkhäuser, Basel, 2006.

[H-2] A. Henrot, *Shape optimization and spectral theory*, De Gruyter, Warsaw, 2017.

[HOP18] M. Holzmann, T. Ourmieres-Bonafos, and K. Pankrashkin, Dirac operators with Lorentz scalar shell interactions, *Rev. Math. Phys.* 30 (2018), 1850013, 46 pp.

[H70] E. Hruslov, On the Neumann boundary value problem in a domain with complicated boundary, *Mat. Sb.* 12 (1970), 553–571.

[JL16] M. Jex and V. Lotoreichik, On absence of bound states for weakly attractive $\delta'$-interactions supported on non-closed curves in $\mathbb{R}^2$, *J. Math. Phys.* 57 (2016), 022101.

[K] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[KP] S. Krantz and H. Parks, *The implicit function theorem. History, theory, and applications*, Birkhäuser/Springer, New York, 2013.

[KL18] D. Krejčířík and V. Lotoreichik, Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, *J. Convex Anal.* 25 (2018), 319–337.

[KL17] D. Krejčířík and V. Lotoreichik, Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, II: non-convex domains and higher dimensions, *submitted*, arXiv:1707.02269.

[K24] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, *Math. Ann.* 94 (1924), 97–100.

[KL] M. Khalile and V. Lotoreichik, *in preparation*.

[L18] V. Lotoreichik Spectral isoperimetric inequalities for singular interactions on open arcs, to appear in *Appl. Anal.*, arXiv:1609.07598.

[MPS16] A. Mantile, A. Posilicano, and M. Sini, Self-adjoint elliptic operators with boundary conditions on not closed hypersurfaces, *J. Differ. Equations* 261 (2016), 1–55.
[PW61] L. E. Payne and H. F. Weinberger, Some isoperimetric inequalities for membrane frequencies and torsional rigidity, J. Math. Anal. Appl. 2 (1961), 210–216.

[RS-IV] M. Reed and B. Simon, Methods of modern mathematical physics, IV. Analysis of operators, Academic Press, New York, 1978.

[S01] A. Savo, Lower bounds for the nodal length of eigenfunctions of the Laplacian, Ann. Glob. Anal. Geom. 16 (2001), 133–151.

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