Variational study of the $\nu = 1$ quantum Hall ferromagnet in the presence of spin-orbit interaction

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We investigate the $\nu = 1$ quantum Hall ferromagnet in the presence of spin-orbit coupling of the Rashba or Dresselhaus type by means of Hartree-Fock-typed variational states. In the presence of Rashba (Dresselhaus) spin-orbit coupling the fully spin-polarized quantum Hall state is always unstable resulting in a reduction of the spin polarization if the product of the particle charge $q$ and the effective $g$-factor is positive (negative). In all other cases an alternative variational state with $O(2)$ symmetry and finite in-plane spin components is lower in energy than the fully spin-polarized state for large enough spin-orbit interaction. The phase diagram resulting from these considerations differs qualitatively from earlier studies.

I. INTRODUCTION

In the recent years the emerging field of spintronics$^{2,3}$ has generated an intense interest in effects of spin-orbit interaction in low-dimensional semiconductor heterostructures.

On the level of effective Hamiltonians arising from $\vec{k} \cdot \vec{p}$ theory, the most important effects of spin-orbit interaction in low-dimensional geometry are described by the Rashba term$^4$ and the (linear) Dresselhaus term$^5$. These effective contributions to the single-particle Hamiltonian stem from the structure inversion asymmetry of the heterostructure (such as a quantum well) and the bulk-inversion asymmetry of the semiconductor material, respectively. In recent years practical manifestations of these kinds of spin-orbit interactions have been investigated intensively, see e.g. Refs.$^6$–$^29$.

Another important topic in low-dimensional semiconductor heterostructures is the field of quantum Hall ferromagnets$^{30,31}$. This class of systems includes electron spin ferromagnets as realized by monolayers at filling factor $\nu = 1$, but also bilayer Quantum Hall systems involving a layer (pseudo-)spin$^32$. Bilayer quantum Hall systems at total filling factor $\nu = 1$ have attracted particular interest very recently due to the spectacular tunneling experiments by Spielman et al.$^{33}$. In such systems the layer spin is involved in the most interesting effects such as the spontaneous phase coherence between the layers, while the electron spin is assumed to be completely aligned along the magnetic field and is therefore not of significance. This is different in bilayer systems at total filling factor $\nu = 2$ where both the layer and the electron spin degree of freedom create a rich phase diagram$^{34–37}$. Yet another type of quantum Hall ferromagnetism occurs in monolayer systems if single particle states in different Landau level with different spins are tuned to energetic coincidence as it can be done by tilting the magnetic field$^{38,39}$. An important connection between electron spin quantum Hall ferromagnetism in monolayers and spin-orbit effects arises from studies of the dependence of the effective $g$-factor on the lattice constant of the semiconductor material which can be varied by applying external pressure$^{39–42}$.

In the present work we examine the ground state of a quantum Hall monolayer at filling factor $\nu = 1$ in the presence of spin-orbit coupling of either the Rashba or the Dresselhaus term. Depending on the type of spin-orbit interaction, the sign of the charge of the particles and the sign of their $g$-factor we find different kinds of instabilities of the conventional fully spin-polarized ferromagnetic quantum Hall ground state. In detail our results are to some extent in conflict with earlier findings$^{15}$.

The paper is organized as follows. In section II we review the single-particle states of charged particles in two-dimensional layers in the presence of a perpendicular magnetic field and spin-orbit interaction of the aforementioned type$^{8,10}$. In section III we present our variational approach with several technical details given in the two appendices. We close with a discussion of our results in section IV.

II. SINGLE-PARTICLE STATES

We consider a spin-$\frac{1}{2}$-particle of charge$^{43} q = \mp|e|$ and effective mass $m$ moving in a two-dimensional ($xy$)-plane provided by a semiconductor quantum well. The particle is subject to spin-orbit interaction and to a perpendicular magnetic field $\vec{B} = B\vec{e}_z = \nabla \times \vec{A}$ which couples to the orbital and the spin degree of freedom. The Hamiltonian reads using standard notation

$$\mathcal{H} = \frac{1}{2m} (\vec{p} - \frac{q}{e} \vec{A})^2 + \frac{1}{2} \mu_B B \sigma^z + \mathcal{H}_{so}$$  \hspace{1cm} (1)

where $g$ is the effective $g$-factor of the particle and $\mu_B = |e|\hbar/(2mc)$ the Bohr magneton with $m_0$ being the bare electron mass. In a semiconductor heterostructure such as a quantum well the spin-orbit coupling in the conduction band has, for appropriate growth geometry, two relevant contributions, $\mathcal{H}_{so} = \mathcal{H}_R + \mathcal{H}_D$ with

$$\mathcal{H}_R = \alpha (\pi_x \sigma^y - \pi_y \sigma^x)$$  \hspace{1cm} (2)

$$\mathcal{H}_D = \beta (\pi_x \sigma^x - \pi_y \sigma^y)$$  \hspace{1cm} (3)
where we have introduced the kinetic momentum \( \vec{p} = \vec{p} - eA \). The first term \( \mathcal{H}_R \) is the Rashba spin-orbit coupling arising from the structure inversion asymmetry of the quantum well, and the second contribution is the (linear) Dresselhaus term which stems from the bulk-inversion asymmetry of the semiconductor material. The coefficient \( \beta \) of the Dresselhaus term is fully determined by the geometry of the heterostructure while the Rashba coefficient \( \alpha \) can be varied by an electric field across the well\(^{15}\). We note that the Rashba Hamiltonian has an SU(2) symmetry (under simultaneous rotations of kinetic momentum and spin), while the symmetry group of the Dresselhaus term is SU(1,1).

Defining the usual bosonic operators

\[
a = \frac{1}{\sqrt{2}} \ell (\pi_x + i \delta \pi_y) \quad , \quad a^+ = (a^+) \quad \tag{4}
\]

with \([a, a^+] = 1\), \(\delta = \text{sgn}(qB)\) and \(\ell = \sqrt{\hbar c / |qB|}\) being the magnetic length, the Hamiltonian reads

\[
\mathcal{H} = \hbar \omega_c (a^+ a + \frac{1}{2}) + \frac{1}{2} g \mu_B B \sigma^z + \mathcal{H}_{so} \quad \tag{5}
\]

Here \(\omega_c = |qB|/(mc)\) is the cyclotron frequency, and the spin-orbit contributions take the form

\[
\mathcal{H}_R = \begin{cases} 
\frac{1}{\sqrt{2}} \alpha \frac{\hbar}{\ell} (a \sigma^- - a^+ \sigma^+) & \delta = +1 \\
\frac{1}{\sqrt{2}} \beta \frac{\hbar}{\ell} (a^+ \sigma - a \sigma^-) & \delta = -1 
\end{cases} \quad \tag{6}
\]

\[
\mathcal{H}_D = \begin{cases} 
\frac{1}{\sqrt{2}} \beta \frac{\hbar}{\ell} (a \sigma^+ + a^+ \sigma^-) & \delta = +1 \\
\frac{1}{\sqrt{2}} \beta \frac{\hbar}{\ell} (a \sigma^- + a^+ \sigma^+) & \delta = -1 
\end{cases} \quad \tag{7}
\]

with \(\sigma^\pm = \sigma^x \pm i \sigma^y\). The operators \(a\) and \(a^+\) connect different Landau levels. Another set of important operators is given in terms of the components of the center \(\vec{r}_0 = (x_0, y_0)\) of the classical orbital motion and read

\[
b = \frac{1}{\sqrt{2}} \frac{1}{\ell} (x_0 - i \delta y_0) \quad , \quad b^+ = (b^+) \quad \tag{8}
\]

These operators fulfil \([b, b^+] = 1\), commute with \(a\), \(a^+\) and connect different orbital states within a given Landau level. Since the Hamiltonian (including the spin-orbit part) can be expressed in terms of \(a\) and \(a^+\) only, its eigenstates have the same Landau level degeneracy as in the absence of spin-orbit coupling.

In the presence of both Rashba and the Dresselhaus term, the spin-orbit interaction couples all states in all Landau levels, and an analytical solution to the full problem is unknown\(^9,10\). Therefore we shall restrict ourselves to the case where either only Rashba or Dresselhaus spin-orbit coupling is present, and the Hamiltonian commutes with the operator \(L = a^+ a + \delta \sigma^z / 2\) which can be used to classify eigenstates. Fixing a certain intra-Landau-level quantum number, we denote by \(|n, \sigma\rangle = ((a^+)^n / \sqrt{n!})|0, \sigma\rangle\) a state in the \(n\)-th Landau level with spin direction \(\sigma \in \{\uparrow, \downarrow\}\). Without loss of generality we discuss the case of Rashba coupling with \(\delta > 0\). Then \(|0, \uparrow\rangle\) is an eigenstate with energy \(\epsilon_0 = (\hbar \omega_c + g \mu_B B) / 2\) and \(L = -1/2\). All other eigenstates are of the form\(^9,10\)

\[
|n, \pm\rangle = u_n^\pm |n, \uparrow\rangle + v_n^\pm |n - 1, \downarrow\rangle \quad \tag{9}
\]

with \(L = n - 1/2, n > 0\), and energy

\[
\epsilon_n^\pm = \hbar \omega_c n \pm \sqrt{2n\alpha^2 \hbar \omega_c + \frac{1}{4} (\hbar \omega_c + g \mu_B B)^2} \quad \tag{10}
\]

and the amplitudes parametrizing the eigenstates read

\[
u_n^\pm = \pm i \text{sgn}(\alpha) \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{2n\alpha^2 \hbar \omega_c + \frac{1}{4} (\hbar \omega_c + g \mu_B B)^2} \right) \quad \tag{11}
\]

\[
u_n^\mp = \pm i \text{sgn}(\alpha) \left( \frac{1}{2} \mp \frac{1}{2} \sqrt{2n\alpha^2 \hbar \omega_c + \frac{1}{4} (\hbar \omega_c + g \mu_B B)^2} \right) \quad \tag{12}
\]

The single-particle eigenstates for the case \(\delta < 0\) and/or Dresselhaus instead of Rashba coupling can be obtained by obvious modifications of the above expressions; in figure 1 we give a schematic overview of the coupling of Landau levels due to the two different types of spin-orbit interaction. Note that the above solution does not require the specification of a gauge for the vector potential creating the magnetic field.

The lowest single-particle states are given by \(|0, \uparrow\rangle\) and \(|1, -\rangle\) which will be of particular interest in the following. The latter state is lower in energy than the first one, i.e. \(\epsilon_1^- - \epsilon_0 < 0\), if

\[
-g \mu_B B < 2\alpha^2 m \quad \tag{13}
\]

This condition involves the Zeeman energy \(\Delta_z = -g \mu_B B\) and the Rashba energy scale \(\alpha^2 m\), but remarkably not the cyclotron energy \(\hbar \omega_c\). Provided that \(\hbar \omega_c > g \mu_B B\) the above condition is not only a sufficient but also a necessary criterion for \(\epsilon_1^- - \epsilon_0 < 0\). The inequality (13) will be of crucial importance in the following section.

### III. Variational Approach to the \(\nu = 1\) Quantum Hall Ferromagnet

We now investigate variational \emph{ansätze} for the ground state of the two-dimensional electron gas at filling factor \(\nu = 1\) in the presence of Coulomb interaction and spin-orbit coupling. Assuming the ground state to be spatially homogeneous, we consider as variational states Slater determinants consisting of single particle states having each a different intra-Landau-level quantum number, and all these quantum numbers are covered leading to a filling factor of unity.
A. Rashba coupling with $qq < 0$, or Dresselhaus coupling with $qq > 0$

Let us first consider variational states appropriate for particles being subject to Rashba spin-orbit interaction and having charge $q$ and g-factor $g$ fulfilling $qq < 0$, or, alternatively, Dresselhaus coupling with $qq > 0$. To be specific we investigate electrons ($q = -|e| < 0$) with positive g-factor in the presence of Rashba coupling. All other cases can be derived from this one by obvious modifications.

Without loss of generality we choose the magnetic field to point in the negative z-direction, $\vec{B} = -|B|\hat{e}_z$, i.e. $\delta > 0$. Since the g-factor is positive the Zeeman term favors the electron spin to align antiparallel to the magnetic field, i.e. in the positive z-direction. Therefore we consider as a variational ansatz for the ground state the Slater determinant constructed from all single-particle states of the form (cf. Eq. (9)),

$$p(0, \uparrow, m) + r|1, -, m\rangle$$

where the variational parameters $p$ and $r$ are subject to the normalization condition $|p|^2 + |r|^2 = 1$. $m$ is some intra-Landau-level index parametrizing all degenerate single-particle states of the above form. Thus our variational state is a Slater determinant built up from all these single-particle states yielding a filling factor of $\nu = 1$.

We now study the energy of our variational state in the presence of Coulomb interaction including a neutralizing background. With the results of appendix A one finds for the energy per particle

$$\varepsilon^{(1)}_{\text{var}} = \varepsilon^\uparrow_{fp} + (\varepsilon^\uparrow_0 - \varepsilon_0)|r|^2$$

$$+ \left[\frac{\varepsilon^2}{\kappa l}\sqrt[8]{\frac{1}{8}}|u^-|^2 \left(\frac{1}{2}|u_1|^2 + 2|v_1|^2\right)\right]|r|^4$$

(15)

where $\kappa$ is the dielectric constant of the semiconductor material and the coefficients $u_1$ and $v_1$ are given by Eqs. (11),(12).

$$\varepsilon^\uparrow_{fp} = \frac{1}{2}(\hbar \omega_c + g\mu_B B) - \frac{\varepsilon^2}{\kappa l}\sqrt[8]{\frac{\pi}{8}}$$

(16)

is the energy per particle of the variational state at $r = 0$, i.e. the $\nu = 1$ ferromagnetic quantum Hall state lying purely in the lowest Landau level with all spins pointing along the positive z-axis (which is the preferred direction for $gB < 0$).

The variational energy (15) becomes smaller than $\varepsilon^\uparrow_{fp}$ for certain values of $|r|$ if and only if the coefficient $(\varepsilon^\uparrow_0 - \varepsilon_0)$ of the quadratic term is negative which is equivalent to the condition (13). In this case the minimizing value for $|r|$ is given by

$$|r|^2 = \min \left\{1, \frac{|\varepsilon^\uparrow_0 - \varepsilon_0|}{\frac{\varepsilon^2}{\kappa l}\sqrt[8]{\frac{\pi}{8}}|u^-|^2 \left(\frac{1}{2}|u_1|^2 + 2|v_1|^2\right)}\right\}$$

(17)

with a variational ground state energy of (for $|r| < 1$)

$$\varepsilon^{(1)}_{\text{var}} = \varepsilon^\uparrow_{fp} - \frac{1}{2} \frac{\varepsilon^2}{\kappa l}\sqrt[8]{\frac{\pi}{8}}|u^-|^2 \left(\frac{1}{2}|u_1|^2 + 2|v_1|^2\right)$$

(18)

Thus we have found a variational state being lower in energy than the conventional quantum Hall ferromagnet (having an energy per particle of $\varepsilon^\uparrow_{fp}$).

The variational ground state has a uniform spin density with a reduced z-component and non-zero in-plane components. The spin expectation values per particle read

$$\langle s^z \rangle = \frac{\hbar}{2} \left((1 - |r|^2) + |r|^2 (|u_1|^2 - |v_1|^2)\right)$$

(19)

$$\langle s^\uparrow \rangle = \hbar \rho^r v_1$$

(20)

The expectation values of the in-plane components depend on the relative phase between $p$ and $r$ which does not influence the energy. Thus the variational ground state reflects an O(2) symmetry. A similar instability of the conventional ferromagnetic quantum Hall state (for pure Rashba coupling) was also found recently by Falko and Iordanskii\textsuperscript{15} who studied perturbative expansions of the thermodynamic potential in a path integral formulation. However, these authors find a different condition for the instability of the conventional ferromagnetic state which reads in the notation used here

$$| - g\mu_B B + \alpha^2 m | < 2 \alpha^2 m \frac{\varepsilon^2}{\kappa l}\sqrt[8]{\frac{\pi}{8}}$$

(21)

This differs, especially due to the magnetic field dependence of the right hand side, qualitatively and quantitatively from our result (13). In particular, for appropriate parameters the results of Ref.\textsuperscript{15} predict the stability of the conventional ferromagnetic state while our variational approach rigorously establishes the existence of a state lower in energy.

Quantum wells with a positive electron g-factor can be realized in terms of biased GaAs/AlGaAs structures\textsuperscript{44-46}. In figure 1 we have plotted the difference $\delta \varepsilon^{(1)}$ of the minimum variational energy $\varepsilon^{(1)}_{\text{var}}$ per particle and $\varepsilon^\uparrow_{fp}$, as a function of $|B|$ for electrons with an effective mass of 0.2 times the bare electron mass $m_0$, $g = 1$, Rashba energy $\alpha^2 m = 0.5\text{meV}$, and a dielectric constant of $\kappa = 10.0$. For this choice of parameters this quantity is, for magnetic fields $|B|$ between 2 and 5 Tesla, of order 0.5K which should be resolvable by experimental cooling techniques.

Figure 2 shows spin expectation values as a function of $|B|$ for the same system parameters as in figure 1. As seen, the in-plane spin components can be of substantial magnitude.

The variational ansätze studied so far are Slater determinants consisting of single-particle states being linear combinations of two different states: One of the states
lies completely in the lowest Landau level and is not perturbed by the spin-orbit coupling (in the above example \(0, \uparrow, m\)) and another state closest in energy which is modified by the spin-orbit interaction and has a contribution from the first excited Landau level (in the above example \(1, -, m\)). The spin direction of the unperturbed state is determined by \(\delta = \text{sgn}(qB)\) and the type of spin-orbit coupling. In all combinations investigated here of the sign of \(q\) and the type of spin-orbit coupling, the spin of the unperturbed state points into the direction favored by the Zeeman coupling. Therefore our ansatz does not frustrate the Zeeman coupling. In the following subsection we will study the opposite cases.

B. Rashba coupling with \(qq > 0\), or Dresselhaus coupling with \(qq < 0\)

We now investigate the case of Rashba spin-orbit coupling with \(qq > 0\) or, alternatively, Dresselhaus coupling with \(qq < 0\). In these cases the variational ansatz studied in the previous subsection frustrates the Zeeman coupling, and an alternative variational state (possibly competing with the first one) becomes appropriate.

As before and without loss of generality we concentrate on Rashba coupling of electrons \((q < 0)\) in a magnetic field pointing in the negative \(z\)-direction \((\delta > 0)\). Since the \(g\)-factor is by assumption negative \((qq > 0)\) our variational ansatz is a Slater determinant built up from single-particle states of the form

\[
s|0, \downarrow, m\rangle + t|1, \uparrow, m\rangle
\]

with variational parameters \(s\) and \(t\) restricted by \(|s|^2 + |t|^2 = 1\), and \(m\) is again some intra-Landau-level index. Choosing \(s^* = i\text{sgn}(\alpha)s^*t\) the variational ground state energy per particle is given by

\[
\varepsilon_{\text{var}}^{(2)} = \varepsilon_{fp}^\downarrow + (\hbar\omega_c + g\mu_B B)|t|^2 - \frac{4|\alpha|\hbar}{\sqrt{2}\kappa_l}\sqrt{\frac{3}{8}}|t|\sqrt{1 - |t|^2} + \frac{\varepsilon_{fp}^\uparrow}{\kappa_l}\sqrt{\frac{3}{8}}|t|^2 + O(|t|^3)
\]

with \(\varepsilon_{fp}^\downarrow, \varepsilon_{fp}^\uparrow\) being the energy per particle of the conventional ferromagnetic state in the absence of spin-orbit coupling. Since the coefficient of the term linear in \(|t|\) is negative (for the above choice of phases) the minimum \(\varepsilon_{\text{var}}^{(2)}\) is always smaller than \(\varepsilon_{fp}^\downarrow\). Note that this observation is independent of the sign of the \(g\)-factor and occurs also for the cases studied in the previous subsection. This shows that the fully spin-polarized quantum Hall state in the lowest Landau level is strictly speaking always unstable in the presence of spin-orbit coupling.

This holds also if both Rashba and Dresselhaus coupling are present. Within the ansatz (22) for instance the Dresselhaus term does not contribute to the energy expectation value at all. Therefore this variational ansatz might not be optimal for this more general case but still yields a variational energy lower than \(\varepsilon_{fp}^\downarrow\). However, in the cases investigated in the previous subsection IIIA the ansatz (14) usually gives lower energies than (22) for realistic system parameters fulfilling the inequality (13). In the remainder of this subsection we shall concentrate again on the case of Rashba spin-orbit coupling with \(qq > 0\).

In the variational state (22) the \(z\)-component of the spin per particle is reduced and given by

\[
\langle s^\downarrow \rangle = \frac{\hbar}{2} \left(1 - 2|t|^2\right)
\]

The in-plane spin components identically vanish within the above variational state. This might appear as an artifact of the ansatz used here and could be altered if the other spin direction in the lowest Landau level is also taken into account. Such a generalized variational state is studied in appendix B. It turns out that this generalized ansatz does not lead to variational energies lower than obtained so far if the cyclotron energy \(\hbar\omega\) is larger in magnitude than the Zeeman splitting \(\Delta_z = -g\mu_B B\). This is usually the case in semiconductors. Therefore, within our variational approach, the only effect of spin orbit coupling under the conditions discussed in this subsection is to reduce the magnetization of the quantum Hall ground state but not to alter its direction. The case considered here (electrons with negative \(g\)-factor and Rashba coupling) includes the important case of conduction band electrons in the III-V semiconductors GaAs, InAs, and InSb. For InAs a typical value for the Rashba energy \(\alpha^2m\) is 0.5meV\(^1\). With this number and the material parameters for InAs we find the reduction of the magnetization according to Eq. (25) (for the minimizing value of \(t\)) to be a few percent at typical fields \(|B|\) of a few Tesla. However, the reduction becomes more pronounced with decreasing modulus of the \(g\)-factor.

IV. DISCUSSION

We have studied the effect of spin-orbit coupling on the ground state of a \(\nu = 1\) quantum Hall monolayer using variational Hartree-Fock-typed states, which are Slater determinants consisting of linear combinations of low-lying single-particle states.

In the case of Rashba coupling and the product of the charge \(q\) and the effective \(g\)-factor \(q\) of the particles being negative we find an instability of the spin-polarized ferromagnetic state toward a state with \(O(2)\) symmetry. The same result is valid for the formally equivalent case of Dresselhaus coupling with \(qq > 0\). These results are the similar to the recent findings by Falko and Iordanskii\(^1\) (for pure Rashba or Dresselhaus coupling), although we
obtain a qualitatively and quantitatively different phase boundary between both types of states.

For the opposite cases (Rashba coupling with \(gq > 0\), or Dresselhaus coupling with \(gq < 0\)) we have used a variational ansatz which involves all relevant low-lying single-particle states: the two states in the lowest Landau level for the two spin directions, and the states in the first excited Landau level with the the appropriate spin direction such that this state is coupled to the lowest Landau level by the spin-orbit interaction. We therefore believe that this variational state captures the essential ground state properties. As a result, we do not find an instability toward an \(O(2)\) symmetric ground state with finite in-plane spin components, but just a (typically small) reduction of the magnetization with its direction being unaltered. Moreover, this instability always occurs and does not depend on other system parameters. These results are further important differences from the findings of Ref.\(^{15}\), where, depending on system parameters, a deviation of the magnetization from the field direction in the ground state was predicted. The reduction of the magnetization increases with decreasing modulus of the Zeeman splitting. Therefore spin-orbit effects can be a part of the explanation for recent experimental data by Zhitomirsky et al.\(^{42}\), where a quantum Hall state at \(\nu=1\) with incomplete spin polarization was reported for small g-factors.

The Rashba coupling with \(gq > 0\) covers in particular the important case of conduction band electrons in III-V semiconductors such as GaAs, InAs, and InSb. Thus, our results indicating only a small reduction in magnetization can be seen as good news with respect to proposals to use integer quantum Hall systems as sources of spin-polarized electrons in experiments related to spintronics and quantum information processing\(^{47,48}\).

It is interesting to speculate how spin-orbit interactions might affect other types of quantum Hall ferromagnets\(^{30}\). In the important case of bilayers at total filling factor \(\nu = 1\) the electron spins are assumed to be polarized by the magnetic field, while the layer pseudospin forms an easy-plane typed ferromagnetic ground state showing a very robust spontaneous symmetry breaking\(^{33}\). In this case we do not expect the spin-orbit coupling to the electron spin to contribute substantially to those physical properties. The situation is different for bilayer systems at total filling factor \(\nu = 2\), where both the electron and the layer spin are involved in various phase transitions\(^{34–37}\). In this case we expect the spin-orbit interaction to even enrich the phase diagram of the system. Similarly, spin-orbit coupling might also alter the phase transitions predicted in monolayers when two different Landau level with different spin directions are tuned to energetic coincidence\(^{38,31}\).

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APPENDIX A: THE COULOMB ENERGY OF THE VARIATIONAL STATES

The Coulomb energy of the variational states investigated in this paper can be obtained via evaluating the pair distribution function for many-body Slater determinants \(|\Psi\rangle\) constructed from all possible single-particle states of the form

\[
a|0, \uparrow, m\rangle + b|0, \downarrow, m\rangle + c|1, \uparrow, m\rangle
\]

with \(|a|^2 + |b|^2 + |c|^2 = 1\). Here \(m\) is some intra-Landau-level index, and \(|1, \uparrow, m\rangle = a^\dagger|0, \uparrow, m\rangle\). \(|\Psi\rangle\) contains all single-particle states of this type leading to a filling factor of unity. To compute the pair distribution function

\[
g(\vec{r}_1 - \vec{r}_2) = \langle \Psi | \sum_{i \neq j} \delta(\vec{r}_1 - \hat{\vec{r}}_i) \delta(\vec{r}_2 - \hat{\vec{r}}_j) |\Psi\rangle
\]

it is convenient to work in the symmetric gauge \(\vec{A} = B(-y, x, 0)/2\) assuming (without loss of generality) \(\delta > 0\) with orbital part of the wave functions given by (suppressing the spin index)

\[
\psi_{0,m}(z) := \langle \vec{r}|0, m\rangle = \frac{1}{\sqrt{2\pi\ell^2m!}} \left( \frac{z}{\sqrt{2}\ell} \right)^m \exp \left( -\frac{|z|^2}{4\ell^2} \right)
\]

\[
\psi_{1,m-1}(z) := \langle \vec{r}|1, m\rangle = \frac{-i}{\sqrt{2\pi\ell^2m!}} \left( \frac{z}{\sqrt{2}\ell} \right)^{m-1} \cdot \left( m - \frac{|z|^2}{2\ell^2} \right) \exp \left( -\frac{|z|^2}{4\ell^2} \right)
\]

where \(z = x + iy\) and the second subscript in the wave functions denotes the eigenvalue of the angular momentum \(M = \delta h(b^+b - a^+a)\).

A straightforward calculation leads to the following expression for the pair distribution function

\[
g(\vec{r}) = \frac{1}{(2\pi\ell^2)^2} \left[ 1 - \left( 1 + |c|^4 \right) \left( 1 - \frac{r^2}{2\ell^2} \right)^2 - 1 \right] - 2|b|^2|c|^2 \left( 1 - \frac{r^2}{2\ell^2} \right) + 2\Re \left\{ a^2 e^{i2x^2 - y^2 + 2ixy} \right\} \cdot \exp \left( -\frac{r^2}{2\ell^2} \right)
\]

To obtain this result we have used the relations
\[\sum_{m=0}^{\infty} \psi_{0,m}^*(z_1)\psi_{0,m}(z_2) = \frac{1}{2\pi \ell^2} \exp\left(\frac{z_1^* z_2 - |z_1|^2 + |z_2|^2}{4\ell^2}\right)\]  
(A6)

\[\sum_{m=0}^{\infty} \psi_{0,m}^*(z_1)\psi_{1,m-1}(z_2) = \frac{-i}{2\pi \ell^2} \sqrt{2/\ell} \exp\left(\frac{z_1^* z_2 - |z_1|^2 + |z_2|^2}{4\ell^2}\right)\]  
(A7)

\[\sum_{m=0}^{\infty} \psi_{1,m-1}^*(z_1)\psi_{1,m-1}(z_2) = \frac{1}{2\pi \ell^2} \left(1 - \frac{|z_1-z_2|^2}{2\ell^2}\right) \exp\left(\frac{z_1^* z_2 - |z_1|^2 + |z_2|^2}{4\ell^2}\right)\]  
(A8)

The first term in the rectangular brackets in Eq. (A5) is the Hartree contribution to the pair distribution function, while the expression proportional to the exponential is the Fock term. Note that for \(\nu = 0\) the state \(|\Psi\rangle\) is just the usual \(\nu = 1\) ferromagnet in the lowest Landau level with its spontaneous spin polarization parametrized by the coefficients \(b\) and \(c\), and the pair distribution function reduces to its well-known expression for this case.

The pair distribution function \(g(\vec{r})\) contains non-isotropic contributions (i.e. terms are not functions of \(r = \sqrt{x^2 + y^2}\) only). This is due to the fact that the single-particle states (A1) involve \((a \neq 0 \neq c)\) superpositions of states with the same spin but different orbital angular momentum. However, these non-isotropic terms do not contribute to integrals of the form \(\int d^2 r f(r)g(\vec{r})\). In particular, for the Coulomb interaction energy per particle in the presence of a neutralizing background one finds

\[\varepsilon_c = -\frac{e^2}{\kappa \ell} \sqrt{\frac{\pi}{8}} \left(1 - \frac{1}{2}|c|^2 \left(\frac{1}{2}|c|^2 + 2|b|^2\right)\right)\]  
(A9)

**APPENDIX B: THE GENERALIZED VARIATIONAL ANSATZ**

In this appendix we discuss a generalized variational ansatz where the single-particle states are arbitrary linear combinations (for a given intra-Landau level quantum number) of both states in the lowest Landau level (for both spin directions) and the state in the first excited Landau level with the appropriate spin direction coupled by the spin-orbit interaction to the lowest Landau level. We again concentrate on the case of Rashba coupling of electrons \((q = -|c| < 0)\) in a magnetic field \(\vec{B} = -|B|\vec{e}_z\), i.e. \(\delta > 0\). In this case our variational many-body state is a Slater determinant consisting of single particle states of the form (cf. Eq. (A1))

\[a|0, \uparrow, m\rangle + b|0, \downarrow, m\rangle + c|1, \uparrow, m\rangle\]  
(B1)

with \(|a|^2 + |b|^2 + |c|^2 = 1\). Choosing \(b^* c = \text{sgn}(\alpha)|b^* c|\) the variational ground state energy per particle is given by

\[\varepsilon_{\text{var}} = \varepsilon_{fp} + \Delta_c|b|^2 + \hbar\omega_c|c|^2 - \frac{4|\alpha|\hbar}{\sqrt{2\ell}} |b||c| \]  
\[\quad + \frac{e^2}{\kappa \ell} \sqrt{\frac{\pi}{8}} \left(\frac{1}{2}|c|^2 + 2|b|^2\right)\]  
(B2)

where \(\varepsilon_{fp}\) is given by Eq. (16) and we have introduced the Zeeman splitting \(\Delta_c = -g\mu_B B = g\mu_B|B|\). We now search for the minimum of the above variational energy under the normalization restriction \(|b|^2 + |c|^2 \leq 1\). This minimum can either lie on the edge of the allowed range \((|b|^2 + |c|^2 = 1)\) or in its interior. In the latter case the stationarity condition \((\partial\varepsilon_{\text{var}}/\partial|b|) = (\partial\varepsilon_{\text{var}}/\partial|c|) = 0\) holds from which one finds the relations

\[|b| = \frac{4|\alpha|\hbar}{\sqrt{4\ell}} |c| \]  
(B3)

\[2\Delta_c|b|^2 - \hbar\omega_c|c|^2 - \frac{e^2}{\kappa \ell} \sqrt{\frac{\pi}{8}} |c|^4 = 0\]  
(B4)

These equations have the obvious solution \(|b| = |c| = 0\) corresponding to a conventional ferromagnetic quantum Hall state. As seen in subsection III B this is not an energy minimum. For \(|b| \neq 0 \neq |c|\) inserting (B3) into (B4) yields a third-order polynomial equation for \(x := |c|^2\),

\[x^3 + rx^2 + sx + t = 0\]  
(B5)

with

\[r = \frac{2(\hbar\omega_c + \Delta_c)}{4\ell} \sqrt{\frac{3}{\pi}}\]  
(B6)

\[s = \frac{\Delta_c (4\hbar\omega_c + \Delta_c)}{(4\ell)^{3/2}} \frac{\sqrt{3}}{\pi}\]  
(B7)

\[t = \frac{2\hbar\omega_c \Delta_c (\Delta_c - 2\alpha^2 m)}{(4\ell)^{3/2}} \frac{\sqrt{3}}{\pi}\]  
(B8)

For \(\Delta_c > 0\) (corresponding to \(q > 0\), cf. subsection III A) \(r\) and \(s\) are both positive. Therefore the above equation can only have a positive solution if \(\Delta_c - 2\alpha^2 m < 0\), in accordance with our criterion (13) for the instability of the conventional ferromagnetic state. The latter result was obtained in subsection III A within a restricted ansatz where the ratio of \(c/b\) is fixed according to the single-particle amplitudes (11), (12). Our result here shows that the generalized variational state (A1) leads to the same stability region, i.e. qualitatively to the identical phase diagram. The more general ansatz (A1) might in principle give even lower energy minima than obtained before. However, this cannot alter the qualitative results and we shall not further discuss this issue here.
Let us now turn to the case $\Delta_z < 0$ (corresponding to $r < 0$, cf. subsection III B). With the standard substitution $x =: y - r/3$ one finds

$$y^3 + py + q = 0 \quad (B9)$$

with

$$p = -\frac{1}{(\frac{2 \sqrt{2}}{\sqrt{3}})^\frac{3}{2}} \left( \frac{\hbar \omega_c - \frac{1}{2} \Delta_z}{\alpha^2 m} \right)^2 \leq 0 \quad (B10)$$

$$q = \frac{16}{27} \left( \frac{\hbar \omega_c - \frac{1}{2} \Delta_z}{\alpha^2 m} \right)^3 - \frac{27}{8} \hbar \omega_c \Delta_z \left( 2 \alpha^2 m \right) \quad (B11)$$

Equation (B9) has exactly one real solution if and only if the discriminant

$$D = \left( \frac{q}{2} \right)^2 + \left( \frac{p}{3} \right)^3 = \left( \frac{2(\hbar \omega_c - \frac{1}{2} \Delta_z)}{\alpha^2 m} \right)^6 \left( 1 - \frac{27 \hbar \omega_c \Delta_z (2 \alpha^2 m)}{(\hbar \omega_c - \frac{1}{2} \Delta_z)^3} \right)^2 - 1 \quad (B12)$$

is positive which is obviously the case for $\Delta_z < 0$. Since the constant term $q$ in Eq. (B9) is also positive for $\Delta_z < 0$ this single real root has to be negative. It follows that Eq. (B9) does not have any positive solutions for $y$ if $\Delta_z < 0$. The physical solutions for $y$ have to lie in the interval $[r/3, 1 + r/3]$ and are ruled out if $r > 0$. The sign of $r$ is determined by the sign of $(\hbar \omega_c + \Delta_z)$. Therefore, to exclude the possibility of physical solutions to Eq. (B9) it is sufficient that the cyclotron energy $\hbar \omega$ is larger in magnitude than the Zeeman splitting $\Delta_z < 0$, which is usually the case. In fact, both quantities are proportional to $|B|$, and their ratio depends just on the effective mass and the effective g-factor of the electrons. In table I we have listed parameter values for for conduction band electrons in typical III-V semiconductors which show that $\hbar \omega_c/|\Delta_z|$ is larger than unity for these materials. Thus, for the physically important case of conduction band electrons in III-V semiconductors no physical stationary point of the variational energy exists except for the solution $c = b = 0$, which corresponds to an energy maximum. Therefore, the energy minimum must lie on the boundary defined by $|b|^2 + |c|^2 = 1$, and we step back to the restricted variational state used in subsection III B with $a = 0$ (cf. (A1)). This finding also shows that the ansatz (14) used in section III A (involving a non-zero coefficient $a$) when applied to the case of subsection III B cannot give a lower energy minimum than found by the ansatz (22).

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A particle with a positive charge $q = |e| > 0$ can formally be viewed as a hole in a semiconductor quantum well. Note, however, that the spin orbit interaction for holes in the valence band of most semiconductor materials is more complicated than for electrons in the conduction band whose effective spin-orbit interaction is described by (3). For a more complete theory of spin-orbit coupling in semiconductors including holes see, e.g., Refs.6–7. In the present work, however, we shall concentrate on the effective Hamiltonians (3) and discuss both cases given by the sign $gq$ of effective $g$-factor $g$ and charge $q$, since for conduction band electrons $g$ can take both signs$^{44–46}$. 

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FIG. 3. Spin expectation values per particle (in units of $\hbar$) as a function of $|B|$ for the same system parameters as in figure 1. The phase occurring in Eq. (20) has been adjusted such that $\langle s^x \rangle = 0$ and $\langle s^y \rangle \geq 0$.

|     | $m/m_0$ | $g$ | $\frac{\hbar \omega_c}{|\Delta_z|}$ |
|-----|---------|-----|-----------------------------------|
| GaAs | 0.067   | -0.44 |                                67.5 |
| InAs | 0.023   | -15   |                                 5.7 |
| InSb | 0.014   | -51   |                                 2.8 |

TABLE I. Effective masses (in units of the bare electron mass $m_0$) and g-factors for conduction band electrons in different III-V semiconductors. The last column shows that the ratio $\frac{\hbar \omega_c}{|\Delta_z|}$ is always larger than unity.