Neutrino Oscillations Induced by Two-loop Radiative Effects

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Abstract

Phenomena of neutrino oscillations are discussed on the basis of two-loop radiative neutrino mechanism. Neutrino mixings are experimentally suggested to be maximal in both atmospheric and solar neutrino oscillations. By using $L_{e} - L_{\mu} - L_{\tau}$ ($\equiv L'$)-conservation, which, however, only ensures the maximal solar neutrino mixing, we find that two-loop radiative mechanism dynamically generates the maximal atmospheric neutrino mixing and that the estimate of $\Delta m^2_{\odot}/\Delta m^2_{\text{atm}} \sim \epsilon m_e/m_{\tau}$ explains $\Delta m^2_{\odot}/\Delta m^2_{\text{atm}} \ll 1$ because of $m_e/m_{\tau} \ll 1$, where $\epsilon$ measures the breaking of the $L'$-conservation. Together with $\Delta m^2_{\text{atm}} \approx 3 \times 10^{-3}$ eV$^2$, this estimate yields $\Delta m^2_{\odot} \sim 10^{-7}$ eV$^2$ for $\epsilon \sim 0.1$, which corresponds to the LOW solution to the solar neutrino problem. Neutrino mass scale is given by $(16\pi^2)^{-2} m_e m_{\tau}/M$ ($M \sim 1$ TeV), which is of order 0.01 eV.

1 Introduction

Neutrino oscillations have been long recognized if neutrinos are massive particles [1]. Such oscillations in fact have been recently confirmed by the Super-Kamiokande collaboration [2] and have also been observed for solar neutrinos produced inside the Sun [3]. The recent report from the K2K collaboration [4] has further shown that the atmospheric neutrino oscillations are characterized by $\Delta m^2_{\text{atm}} \approx 3 \times 10^{-3}$ eV$^2$, which implies $\sim 5.5 \times 10^{-2}$ eV as neutrino masses. This tiny mass scale for neutrinos can be generated by radiative mechanisms, where the smallness originates from the smallness of radiative effects [7, 8]. Radiative mechanisms use $L=2$ interactions given by $\nu_{L}^{(i)}\nu_{L}^{(j)}$ for one-loop radiative effects [7, 8, 9, 10, 11] and by additional $\nu_{R}^{(i)}\nu_{R}^{(j)}$ for two-loop radiative effects [8, 12], where $i$ and $j$ denote three families ($i,j = 1,2,3$).

At the one-loop level, Zee [8] has presented the mechanism that utilizes a new standard Higgs scalar called $\phi'$ in addition to the standard Higgs scalar, $\phi$, both of which are $SU(2)_{L}$-doublets, and another
singly charged scalar called $h^+$, which is an $SU(2)_L$-singlet, with the coupling of $f_{[ij]}\nu^c_{L}^i\nu^c_{L}^j h^+$. The Fermi statistics forces $\nu^c_{L}^i\nu^c_{L}^j$ to be antisymmetrized with respect to the family indices. After the spontaneous breakdown of $SU(2)_L \times U(1)_Y$, an interaction of $\phi\phi' h^+$ yields the possible mixing of $h^+$ with $\phi^+$ characterized by the scale of $\mu$, which finally induces Majorana neutrino masses. Again, the Fermi statistics forces $\phi\phi'$ to be antisymmetrized with respect to the $SU(2)_L$-indices. Depicted in Figure 1(a) is the diagram for generating Majorana neutrino masses. The order-of-magnitude estimate gives the one-loop neutrino mass, $m_{1}^{1-\text{loop}}$, for $\nu_i-\nu_j$ to be:

$$m_{1}^{1-\text{loop}} \sim f_{[ij]} \frac{m_{\tau}^2}{16\pi^2 M^2 \mu},$$

for $\langle 0|\phi^0|0 \rangle \sim \langle 0|\phi'^0|0 \rangle$, where $M$ stands for the scale of the model, presumably of order 1 TeV. The factor of $16\pi^2$ in the denominator is specific to one-loop radiative corrections. This estimate turns out to be

$$m_{1}^{1-\text{loop}} \sim 2 \times 10^3 f_{[i\tau]} \left( \frac{\mu}{100 \text{ GeV}} \right) \text{ eV},$$

for $m_{\ell_i} = m_{\tau}$ ($j=\tau$). To obtain $m_{\nu} \sim 0.1$ eV, we require that

$$f_{[i\tau]} \sim 5 \times 10^{-5},$$

for $\mu \sim 100$ GeV. Therefore, to get tiny neutrino masses of order 0.1 eV, one has to give excessive suppression to the lepton-number violating $\nu\ell$-coupling.

At the two-loop level, additional suppression arises. In addition to $h^+$, a doubly charged $k^{++}$-scalar is required to realize the mechanism of the Zee-Babu type \cite{Zee:1980ai} and $k^{++}$ couples to a right-handed charged lepton pair via $\ell_R^i \ell_R^j k^{++}$ with coupling strength of $f_{[ij]}$. Using a possible coupling of this new $k^{++}$ with $h^+$ via $h^+ h^+ k^{++}$, we can find interactions corresponding to Figure 1(b). The order-of-magnitude estimate gives the two-loop neutrino mass, $m_{2}^{2-\text{loop}}$, for $\nu_i-\nu_k$ to be:

$$m_{2}^{2-\text{loop}} \sim f_{[ij]} f_{[j'j]} f_{[k'k]} \frac{m_{\ell_i} m_{\ell_j'}}{(16\pi^2)^2 M^2 \mu}.$$

The factor of $(16\pi^2)^2$ in the denominator is specific to two-loop radiative corrections. This estimate turns out to yield

$$m_{2}^{2-\text{loop}} \sim 10 f_{[i\tau]} f_{[\tau\tau]} f_{[j\tau]} \left( \frac{\mu}{100 \text{ GeV}} \right) \text{ eV},$$
for $m_{ij,ij'} = m_\tau$ ($j,j'=\tau$). To obtain $m_\nu \sim 0.1$ eV, thanks to the extra loop-factor of $16\pi^2$, we only require that

$$f_{[\nu]} \sim 0.1,$$

for $f_{[\tau\tau]} \sim 1$ and $\mu \sim 100$ GeV. Therefore, the two-loop radiative neutrino masses can be of order of 0.1 eV without excessive suppression for relevant couplings [13].

2 Bimaximal Mixing

The observed pattern of neutrino oscillations is consistent with the pattern arising from the requirement of the conservation of the new quantum number $L'_e - L'_\mu - L'_\tau$ ($\equiv L'$) [14]. The $U(1)_{L'}$ symmetry based on the $L'$-conservation can be used to describe the bimaximal mixing scheme for neutrino oscillations [15, 16]. However, the $L'$-conservation itself only ensures the maximal solar neutrino mixing but does not determine the atmospheric neutrino mixing angle. In fact, in the one-loop radiative mechanism, fine-tuning of lepton-number violating couplings is necessary to yield bimaximal mixing for atmospheric neutrino oscillations.

In the one-loop radiative mechanism, we have known the form of the neutrino mass matrix, which is given by

$$M_\nu \propto \begin{pmatrix} 0 & f_{[\nu]\nu} m_\nu^2 & f_{[\tau\nu]} m_\tau^2 \\ 0 & f_{[\nu]\tau} m_\nu^2 & f_{[\tau\tau]} m_\tau^2 \\ 0 & 0 & 0 \end{pmatrix} \bigg|_{sym} \Rightarrow \begin{pmatrix} 0 & \sim 1 & \sim 1 \\ \varv & \epsilon (\ll 1) & 0 \end{pmatrix} m,$$

where $m$ stands for the neutrino mass scale. The bimaximal mixing is realized if the couplings satisfy

$$f_{[\nu]\nu} m_\nu^2 = f_{[\tau\nu]} m_\tau^2 \Rightarrow f_{[\nu]\nu} \gg f_{[\nu]\tau} \gg f_{[\nu\tau]} \approx 0,$$

indicating the fine-tuning of the couplings $f$’s. This fine-tuning is referred to as “inverse hierarchy in the couplings”, namely, $f_{[\nu]\nu} \gg f_{[\nu]\tau} \gg f_{[\nu\tau]}$. The $L'$-conservation gives $f_{[\nu\tau]}=0$. Its tiny breaking effect characterized by the parameter, $\epsilon$, produces tiny solar neutrino oscillations.

On the other hand, in the two-loop radiative mechanism, we will find the mass matrix [12] given by

$$M_\nu \propto \begin{pmatrix} 0 & f_{[\tau\tau]\nu} m_\nu m_\tau & f_{[\tau\nu]} f_{[\nu\tau]} m_\tau^2 \\ f_{[\nu\nu]} f_{[\nu\tau]} m_\nu m_\tau & f_{[\nu\nu]} f_{[\nu\tau]} m_\nu^2 & f_{[\nu\nu]} f_{[\nu\tau]} m_\nu^2 \\ f_{[\tau\tau]} f_{[\nu\tau]} m_\tau^2 & f_{[\nu\nu]} f_{[\nu\tau]} m_\nu^2 & f_{[\nu\nu]} f_{[\nu\tau]} m_\nu^2 \end{pmatrix} \bigg|_{sym} \Rightarrow \begin{pmatrix} 0 & \sim 1 & \sim 1 \\ \varv & \epsilon (\ll 1) & \epsilon' \end{pmatrix} m.$$

The bimaximal structure is reproduced if

$$f_{[\tau\tau]} f_{[\nu\tau]} m_\nu m_\tau = f_{[\tau\tau]} f_{[\nu\tau]} m_\nu m_\tau \Rightarrow f_{[\nu\nu]} = f_{[\nu\tau]}.$$

Therefore, no hierarchy in the couplings is necessary. The breaking of the $L'$-conservation gives the suppressed entries, $\epsilon, \epsilon', \epsilon''$, proportional to $m_\nu^2$. Therefore, we observe that

$$\Delta m^2_{3\nu}/\Delta m^2_{atm} \propto m_\nu/m_\tau,$$

which dynamically guarantees $\Delta m^2_{atm} \gg \Delta m^2_{3\nu}$ because of $m_\tau \gg m_\nu$.

In radiative mechanisms, the hierarchy of $\Delta m^2_{atm} \gg \Delta m^2_{3\nu}$ can also be ascribed to the generic smallness of two-loop radiative effects over one-loop radiative effects [18]. Therefore, we have in hands two dynamical reasons for $\Delta m^2_{atm} \gg \Delta m^2_{3\nu}$:

$$\frac{\Delta m_{3\nu}^2}{\Delta m_{atm}^2} \ll 1 \text{ because } \begin{cases} 2 \text{- loop/1 - loop} & \ll 1 \\ m_\nu/m_\tau & \ll 1 \end{cases}.$$
Table 1: $L$ and $L'$ quantum numbers.

| Fields | $(\nu_{eL}, e_L^c)$, $e_R^c$ | $(\nu_{iL}, \ell_{iL}^c)$, $\ell_{iR}^c$ | $\phi$ | $h^+$ | $k^{++}$ | $k'^{++}$ |
|--------|-------------------------------|-------------------------------|-------|------|--------|--------|
| $L$    | 1                             | 1                             | 0     | -2   | -2     | -2     |
| $L'$   | 1                             | -1                            | 0     | 0    | 0      | -2     |

3 Two-loop Radiative Neutrino Masses

Interactions that we introduce can be classified by the ordinary lepton number ($L$) and $L'$-number of particles, which are listed in the Table 1. The new ingredients that are not contained in the standard model are the $SU(2)_L$-singlet scalars, $h^+$ and $k^{++}$. We have further employed an additional $k'^{++}$ to be denoted by $k'^{++}$ in order to import the $L'$-breaking. The $L$- and $L'$-quantum number of $k'^{++}$ is also listed in Table 1. Extra $L$- and $L'$-conserving Yukawa interactions are given by

\[
\begin{align*}
&f_{(e)} \left( \nu_{eL} \ell_{L}^c - \nu_{e}^c e_L^c \right) h^+, \\
&f_{(e)} e_R^c e_R k^{++}, \\
&\frac{1}{2} f_{(e)} e_R^c e_R k'^{++}.
\end{align*}
\]

(13)

An $L$-breaking but $L'$-conserving interaction is specified by

\[
\mu_0 h^+ h'^{++},
\]

(14)

where $\mu_0$ represents a mass scale. An $L'$-breaking interaction is activated by $k'^{++}$ via

\[
\mu_b h^+ h'^{++},
\]

(15)

where $\mu_b$ represents a breaking scale of the $L'$-conservation.

Yukawa interactions, then, take the form of

\[
-L_Y = \sum_{i=e, \mu, \tau} \int f_{(e)} \bar{\psi}_{L}^\dagger \phi \psi_R^c + \sum_{i=\mu, \tau} \left( f_{(e)} \bar{\psi}_{L}^\dagger \psi_R^c \phi h^+ + f_{(e)} \bar{\psi}_{L}^\dagger \psi_R^c \phi k'^{++} \right) \\
+ \frac{1}{2} f_{(e)} \bar{\psi}_{R}^\dagger \psi_R^c e_R k'^{++} + (\text{h.c.}),
\]

(16)

and Higgs interactions are described by self-Hermitian terms composed of $\varphi \varphi^\dagger$ ($\varphi = \phi, h^+, k^{++}, k'^{++}$) and by the non-self-Hermitian terms in

\[
V_0 = \mu_0 h^+ h'^{++} + (\text{h.c.}).
\]

(17)

This coupling softly breaks the $L$-conservation but preserves the $L'$-conservation. To account for solar neutrino oscillations, the breaking of the $L'$-conservation should be included and is assumed to be furnished by

\[
V_b = \mu_b h^+ h'^{++} + (\text{h.c.}).
\]

(18)

Neutrino masses are generated by interactions corresponding to the diagrams depicted in Figure 2(a,b). The resulting Majorana neutrino mass matrix is given by

\[
M_\nu = \begin{pmatrix} 0 & m_{e\mu} & m_{e\tau} \\ m_{e\mu} & \delta_{\mu\mu} & \delta_{\mu\tau} \\ m_{e\tau} & \delta_{\mu\tau} & \delta_{\tau\tau} \end{pmatrix}.
\]

(19)
Here, the bimaximal structure is controlled by
\[ m_{ei} \approx -2f_{[er]}f_{[ee]}f_{[\tau\tau]} m_{ee} m_{\mu} \mu_0 \left[ \frac{1}{16\pi^2} \ln \left( \frac{m_{k}^2}{m_{h}^2} \right) \right]^2 (i = \mu, \tau), \tag{20} \]
where the product of \( m_e \) and \( m_\tau \) appears. This is because the exchanged leptons are \( e \) and \( \tau \) as can be seen from Figure 2(a). Tiny splitting is induced by
\[ \delta_{ij} \approx -f_{[ei]}f_{[e\tau]}f_{[ee]} m_{ee} m_{\mu} \mu_0 \left[ \frac{1}{16\pi^2} \ln \left( \frac{m_{k}^2}{m_{h}^2} \right) \right]^2, \tag{21} \]
where \( m_e^2 \) appears because the exchanged leptons are both \( e \) and \( e \) as can be seen from Figure 2(b). These expressions, Eqs. (20) and (21), are subject to the approximation of \( m_{k,k'} \gg (\text{other mass squared}) \).

These expressions are described by these mass parameters:
\[ \Delta m_{atm}^2 = m_{e\mu}^2 + m_{e\tau}^2 (\equiv m_{\nu}^2), \quad \Delta m_\odot^2 = 4m_\nu\delta m, \tag{22} \]
where
\[ \delta m = \frac{1}{2} \left| \delta_{\mu \mu} \cos^2 \theta_\nu + 2\delta_{\mu \tau} \cos \theta_\nu \sin \theta_\nu + \delta_{\tau \tau} \sin^2 \theta_\nu \right| \tag{23} \]
with
\[ \cos \theta_\nu = m_{ee}/m_{\nu}, \quad \sin \theta_\nu = m_{e\tau}/m_{\nu}. \tag{24} \]

It is thus found that (nearly) bimaximal mixing is reproduced by requiring
\[ f_{[ei]} \approx f_{[e\tau]}, \tag{25} \]
yielding \( \sin 2\theta_\nu \approx 1 \). Tiny mass-splitting \( \Delta m_{atm}^2 \gg \Delta m_\odot^2 \) is ensured by the mass-hierarchy:
\[ m_\tau \gg m_e. \tag{26} \]

As a result, we obtain an estimate of the ratio:
\[ \frac{\Delta m_\odot^2}{\Delta m_{atm}^2} \approx \frac{\mu_0}{\mu_{h} m_{e} m_{\mu}^2 m_{\tau}^2}. \tag{27} \]
From this estimate, we find that
\[
\Delta m_{\odot}^2 \sim 3 \times 10^{-4} \frac{H_0}{\mu_0} \Delta m_{\text{atm}}^2 \left( m_k^2 \sim m_{k'}^2 \right)
\]
\[
\Rightarrow \Delta m_{\odot}^2 \sim 3 \times 10^{-5} \Delta m_{\text{atm}}^2 \left( \mu_b \sim \mu_0/10 \right)
\]
\[
\Rightarrow \Delta m_{\odot}^2 \sim 10^{-7} \text{eV}^2 \left( \Delta m_{\text{atm}}^2 \sim 3 \times 10^{-3} \text{eV}^2 \right). \tag{28}
\]

The resulting \( \Delta m_{\odot}^2 \) corresponds to the allowed region for the LOW solution to the solar neutrino problem. Since \( k^{++} \) and \( k'^{++} \) couple to the charged lepton pairs, these scalars produce extra contributions on the well-established low-energy phenomenology. In particular, we should consider effects from \( \mu^- \rightarrow e^- \gamma \), \( e^- e^- e^+ \), \( e^- e^- \rightarrow e^- e^- \) and \( \nu_\mu e^- \rightarrow \nu_\mu e^- \). The relevant constraints on the parameters associated with the scalars of \( h^+, k^{++} \) and \( k'^{++} \) are, thus, given by

1. \( \mu^- \rightarrow e^- e^- e^+ \) in Figure 3(a) and \( \mu^- \rightarrow e^- \gamma \) in Figure 3(b) \[20\] (forbidden by the \( L' \)-conservation), yielding

\[
\frac{\xi_{\{\nu_\mu\}} f_{\{\nu_\mu\}}}{m_{b}^2} < \left\{ \begin{array}{ll} 1.2 \times 10^{-10} \text{GeV}^{-2} \text{ from } B(\mu^- \rightarrow e^- e^- e^+) < 10^{-12} \text{[21]} \\
2.4 \times 10^{-8} \text{GeV}^{-2} \text{ from } B(\mu^- \rightarrow e^- \gamma) < 1.2 \times 10^{-11} \text{[22]} \end{array} \right.. \tag{29}
\]

\[2\] The constraints of Eqs.(12) and (13) in Ref.[19] should, respectively, be replaced by the corresponding bounds in the items 1, 3 and 4. Namely, \( f_{\{11,12\}} \) should read \( f_{\{11,12\}}/2 \) in Ref.[18].
where \( \bar{m}_k \sim m_k \sim m_{k'} \) and \( \xi \) estimated to be
\[
\xi \sim \frac{1}{16\pi^2} \frac{\mu_b\mu_0}{\bar{m}_k^2} (\ll 1)
\] (30)
reads the suppression due to the approximate \( L' \)-conservation,

2. \( \tau^- \to \mu^- e^- e^+ \) in Figure 4(a) and \( \tau^- \to \mu^- \gamma \) in Figure 4(b) (allowed by the \( L' \)-conservation), yielding
\[
\left| \frac{f_{(\tau\tau)} f_{(\tau\mu)}}{\bar{m}_k^2} \right| < \left\{ \begin{array}{l}
2.1 \times 10^{-7} \text{ GeV}^{-2} \text{ from } B(\tau^- \to \mu^- e^- e^+) < 1.7 \times 10^{-6} [21] \\
4.2 \times 10^{-6} \text{ GeV}^{-2} \text{ from } B(\tau^- \to \mu^- \gamma) < 1.1 \times 10^{-6} [21],
\end{array} \right.
\]
(31)

3. \( e^- e^- \to e^- e^- \) in Figure 5(a), yielding
\[
\left| \frac{f_{(ee)}}{m_{k'}} \right|^2 < 4.8 \times 10^{-5} \text{ GeV}^{-2},
\] (32)

4. \( \nu_\mu e^- \to \nu_\mu e^- \) in Figure 5(b), yielding
\[
\left| \frac{f_{(\nu\mu)}}{m_h} \right|^2 < 1.7 \times 10^{-6} \text{ GeV}^{-2}.
\] (33)

It should be noted that the leading contribution of \( h^+ \) to \( \mu^- \to e^- \gamma \), which gives the most stringent constraint on \( h^+ \), is forbidden by the \( U(1)_L \)-invariant coupling structure.

Typical parameter values are so chosen to satisfy these constraints:
\[
\begin{align*}
\begin{array}{l}
f_{(ee)} = f_{(\tau\tau)} \approx 2e \\
f_{(ee)} = f_{(\tau\tau)} \approx e \\
m_h \approx 350 \text{ GeV} \\
m_k = m_{k'} \approx 2 \text{ TeV} \\
\mu_0 \approx 1.5 \text{ TeV} \\
\mu_b \approx \mu_0/10
\end{array}
\end{align*}
\]
(34)

Figure 5: (a) \( e^- e^- \to e^- e^- \), (b) \( \nu_\mu e^- \to \nu_\mu e^- \).
We obtain the following numerical values:
\[
\begin{align*}
\Delta m^2_{\text{atm}} &\approx 2.4 \times 10^{-3} \text{ eV}^2, \\
\Delta m^2_{\odot} &\approx 10^{-7} \text{ eV}^2.
\end{align*}
\] (35)

Therefore, we in fact successfully explain phenomena of atmospheric and solar neutrino oscillations characterized by \(\Delta m^2_{\text{atm}} \approx 2.4 \times 10^{-3} \text{eV}^2\) and \(\Delta m^2_{\odot} \approx 10^{-7} \text{eV}^2\).

4 Summary

We have discussed how neutrino oscillations arise from two loop-radiative mechanism, which exhibits

1. bimaximal mixing due to the \(L_e - L_\mu - L_\tau\) conservation via the coupling of \(e^-\tau^-k^{++}\),

2. dynamically induced tiny mass-splitting for solar neutrino oscillations due to the smallness of \(m_e^2\) via \(e^-e^-k^{++}\).

The interactions required to generate two-loop Majorana neutrino masses are specified by
\[
\begin{align*}
&\{ f_{\{ee\}} (\nu_{eL}^cL_e^i - \nu_{\tau L}^cL_\tau^i) h^+ \\
&\quad - \frac{1}{2} f_{\{ee\}} e_R^c \tilde{R}^c k^{++} \},
&\{ \mu_{0h^+} h^+ k^{++} \}
&\quad \cup \{ \mu_{h^+} h^+ k^{++} \}.
\end{align*}
\] (36)

The resulting mass scale for neutrino masses is determined by
\[
\frac{m_e m_e}{(16\pi^2)^2 m_k^2} \mu_0 \sim \frac{m_e m_e}{(16\pi^2)^2 m_k} \sim 10^{-2} \text{ eV}.
\] (37)

Thus, to obtain the neutrino mass of order of 0.01 eV is a natural consequence without fine-tuning of coupling parameters. And the hierarchy of \(\Delta m^2_{\text{atm}} \gg \Delta m^2_{\odot}\) is expressed by the estimate
\[
\Delta m^2_{\odot} \sim \frac{\mu_k}{\mu_0} \frac{m_e}{m_\tau} \frac{m_e^2}{m_{\mu}} \Delta m^2_{\text{atm}},
\] (38)

which ensures \(\Delta m^2_{\text{atm}} \gg \Delta m^2_{\odot}\) because of \(m_\tau \gg m_e\). This estimation yields the LOW solution to the solar neutrino problem.

It should be finally noted that

- since the \(L’\)-conservation forbids primary flavor-changing processes involving \(e^-\), the coupling strengths of \(h^+\) and \(k^{++}\) to leptons are not severely constrained and can be as large as \(\mathcal{O}(e)\),

- characteristic signatures of \(h^+\) include
\[
B(h^+ \to e^+ \mathcal{E}_R) \approx 2B(h^+ \to \mu^+ \mathcal{E}_R) \approx 2B(h^+ \to \tau^+ \mathcal{E}_R)
\]
(39)
since \(f_{[\mu]} \approx f_{[\tau]}\), which should be compared with \([12]\)
\[
B(h^+ \to e^+ \mathcal{E}_R) \approx B(h^+ \to \mu^+ \mathcal{E}_R) \gg B(h^+ \to \tau^+ \mathcal{E}_R)
\]
(40)
in the one-loop radiative mechanism with \(f_{[\mu]} \gg f_{[\tau]} \gg f_{[\nu]}\) \([17]\).

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\(^3\) One should be aware of higher-order contributions found by Lavoura in Ref.\(^{[12]}\). The (1,1)-entry of \(M_\nu\), which vanishes up to the two-loop level, is induced by the four-loop diagram shown in Figure\(^{[16]}\). The contributions are at most characterized by \(\delta \sim (16\pi^2)^{-1} \xi m^2_e/m_k^2\), which should be compared with \(m^2_e/m_k^2\). Our parameter-setting in Eq.\(^{[13]}\) gives \(\delta \sim 2m^2_e/m_k^2\), which turns out to be \(\mathcal{O}(m^2_e/m_k^2\)). Therefore, our estimate of Eq.\(^{[13]}\) remains valid to predict \(\Delta m^2_{\odot}\) from \(\Delta m^2_{\text{atm}}\).
Figure 6: Four loop-diagram for $\nu_e - \nu_e$.

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