On Borsuk’s conjecture for two-distance sets

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Abstract

In this paper we answer Larman’s question on Borsuk’s conjecture for two-distance sets. We found a two-distance set consisting of 416 points on the unit sphere $S^{64} \subset \mathbb{R}^{65}$ which cannot be partitioned into 83 parts of smaller diameter. This also reduces the smallest dimension in which Borsuk’s conjecture is known to be false. Other examples of two-distance sets with large Borsuk’s numbers will be given.

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1 Introduction

For each $n \in \mathbb{N}$ the Borsuk number $b(n)$ is the minimal number such that any bounded set in $\mathbb{R}^n$ consisting of at least 2 points can be partitioned into $b(n)$ parts of smaller diameter. In 1933 Karol Borsuk [3] conjectured that $b(n) = n + 1$. The conjecture was disproved by Kahn and Kalai [10] who showed that in fact $b(n) > 1.2\sqrt{n}$ for large $n$. In particular, their construction implies that $b(n) > n + 1$ for $n = 1325$ and for all $n > 2014$. This result attracted substantial amount of attention from many mathematicians; see for example [1], [4], and [18]. Improvements on the smallest dimension $n$ such that $b(n) > n + 1$ were obtained by Nilli [14] $(n = 946)$, Raigorodskii [17] $(n = 561)$, Weißbach [19] $(n = 560)$, Hinrichs [8] $(n = 323)$, and Pikhurko [16] $(n = 321)$. Currently the best known result is that Borsuk’s conjecture is false for $n \geq 298$; see [9]. On the other hand, many related problems are still
unsolved. The Borsuk’s conjecture can be wrong even in dimension 4. Only the estimate $b(4) \leq 9$ is known; see [12].

In ’70s Larman asked if the Borsuk’s conjecture is true for two-distance sets; see also [11] and [18]. Denote by $b_2(n)$ the Borsuk number for two-distance sets in the dimension $n$, that is the minimal number such that any two-distance set in $\mathbb{R}^n$ can be partitioned into $b(n)$ parts of smaller diameter. The aim of this paper is to construct two-distance sets with large Borsuk’s numbers. Two basic constructions follow from Euclidean representations of $G_2(4)$ and $F_{i23}$ strongly regular graphs. First we prove

**Theorem 1.** There is a two-distance subset $\{x_1, \ldots, x_{416}\}$ of the unit sphere $S^{64} \subset \mathbb{R}^{65}$ such that $\langle x_i, x_j \rangle = 1/5$ or $-1/15$ for $i \neq j$ which cannot be partitioned into 83 parts of smaller diameter.

Hence $b(65) \geq b_2(65) \geq 84$. We also prove the following

**Theorem 2.** There is a two-distance subset $\{x_1, \ldots, x_{31671}\}$ of the unit sphere $S^{781}$ such that $\langle x_i, x_j \rangle = 1/10$ or $-1/80$ for $i \neq j$ which cannot be partitioned into 1376 parts of smaller diameter.

Then, using the configurations from Theorem 1 and Theorem 2 we prove

**Corollary 1.** For integers $n \geq 1$ and $k \geq 0$ we have

\begin{equation}
(1) \quad b_2(66n + k) \geq 84n + k + 1,
\end{equation}

and

\begin{equation}
(2) \quad b_2(783n + k) \geq 1377n + k + 1.
\end{equation}

Finally, using again the configuration from Theorem 2 we prove

**Corollary 2.** The following inequalities hold:

\begin{equation}
 b_2(781) \geq 1225, \quad b_2(780) \geq 1102, \quad \text{and} \quad b_2(779) \geq 1002.
\end{equation}

The paper is organized as follows. First, in Section 2 we describe Euclidean representations of a strongly regular graph by two-distance sets and then in Section 3 we prove our main results.
2 Eucledian representations of strongly regular graphs

A strongly regular graph $\Gamma$ with parameters $(v, k, \lambda, \mu)$ is an undirected regular graph on $v$ vertices of valency $k$ such that each pair of adjacent vertices has $\lambda$ common neighbors, and each pair of nonadjacent vertices has $\mu$ common neighbors. The adjacency matrix $A$ of $\Gamma$ has the following properties:

$$AJ = kJ$$

and

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where $I$ is the identity matrix and $J$ is the matrix with all entries equal to 1 of appropriate sizes. These conditions imply that

$$(3) \quad (v - k - 1)\mu = k(k - \lambda - 1).$$

Moreover, the matrix $A$ has only 3 eigenvalues: $k$ of multiplicity 1, one positive eigenvalue

$$r = \frac{1}{2} \left( \lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right)$$

of multiplicity

$$(4) \quad f = \frac{1}{2} \left( v - 1 - \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right),$$

and one negative eigenvalue

$$s = \frac{1}{2} \left( \lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right)$$

of multiplicity

$$g = \frac{1}{2} \left( v - 1 + \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right).$$
Clearly, both \( f \) and \( g \) must be integers. This together with (3) gives a collection of feasible parameters \((v, k, \lambda, \mu)\) for strongly regular graphs.

Let \( V \) be the set of vertices \( \Gamma \). Consider columns \( \{y_i : i \in V\} \) of the matrix \( A - sI \) and put \( x_i := z_i/\|z_i\|, \ i \in V \). Note that while the vectors \( x_i \) lie in \( \mathbb{R}^v \), they span at most a \( f \)-dimensional vector space. Thus for convenience we consider them to lie in \( \mathbb{R}^f \). By easy calculations
\[
\langle x_i, x_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ p, & \text{if } i \text{ and } j \text{ are adjacent,} \\ q, & \text{otherwise,} \end{cases}
\]
where
\[
p = \frac{\lambda - 2s - \beta}{s^2 + k - \beta}, \quad q = \frac{\mu - \beta}{s^2 + k - \beta}, \quad \beta = \frac{1}{v}(s^2 + k(\lambda - 2s) + (v - k - 1)\mu).
\]
Denote by \( \Gamma_f \) the configuration \( x_i, i \in V \). Similarly, we can define the configuration \( \Gamma_g \) in \( \mathbb{R}^g \). The configurations \( \Gamma_f \) and \( \Gamma_g \) were also considered in [6] and have many other fascinating properties. For example, they are spherical 2-designs.

3 Proof of main results

For any vertex \( v \in V \) of a strongly regular graph \( \Gamma \), let \( N(v) \) be the set of all neighbors of \( v \) and let \( N'(v) \) be the set of non-neighbors of \( v \), i.e. \( N'(v) = V \setminus (\{v\} \cup N(v)) \).

Proof of Theorem 1. We consider the configuration \( \Gamma_f \) of the well-known strongly regular graph \( \Gamma = G_2(4) \) with parameters \((416, 100, 36, 20)\). By [6] we have that \( f = 65 \). Moreover, \( p = 1/5 \) and \( q = -1/15 \). Therefore the diameter of \( \Gamma_f \) is the distance between \( x_i \) and \( x_j \) where \( i \) and \( j \) are non-adjacent. Hence, the configuration cannot be partitioned into less than \( v/m \) parts, where \( m \) is the size of the largest clique in \( \Gamma \). To prove Theorem 1
it is enough to show that $G_2(4)$ has no 6-clique. Now we use the following result \cite{5}.

\textbf{Theorem A.}

(i) For each $u \in V$ the subgraph of $\Gamma$ induced on $N(u)$ is a strongly regular graph with parameters $(100, 36, 14, 12)$ (the Hall-Janko graph). In other words the Hall-Janko graph is the first subconstituent of $\Gamma$.

(ii) The first subconstituent of the Hall-Janko graph is the $U_3(3)$ strongly regular graph with parameters $(36, 14, 4, 6)$.

(iii) The first subconstituent of $U_3(3)$ is a graph on 14 vertices of regularity 4 (the co-Heawood graph).

(iv) The co-Heawood graph has no triangles.

Parts (i)-(iii) are folklore. They follows from D.G. Higman’s theory of rank 3 permutation groups (see also \cite{7} and \cite{13}). Part (iv) follows from the fact that the co-Heawood graph is a subgraph of the Gewirtz graph with parameters $(56, 10, 0, 2)$; see also \cite{2}.

Now, for vertices $u, v, w \in V$ forming a triangle, (i)-(iii) implies that

$$|N(u) \cap N(v) \cap N(w)| = 14.$$  

Moreover, the subgraph induced on $N(u) \cap N(v) \cap N(w)$ is the co-Heawood graph. Therefore by (iv) the maximal cliques in $\Gamma$ are of size 5.

\textbf{Proof of Theorem 2.} Consider the configuration $\Gamma_f$ of the $Fi_{23}$ graph with parameters $(31671, 3510, 693, 351)$. We have $f = 782$, $p = 1/10$, and $q = -1/80$. Hence, the diameter of $\Gamma_f$ is the distance between non adjacent vertices. Similarly to the proof of Theorem 1 $\Gamma_f$ cannot be partitioned into less than $v/m$ parts, where $m$ is the size of the largest clique in $\Gamma$. We will use the well-known fact (see \cite{15}) that the first subconstituent of $\Gamma$ is the strongly regular graph with parameters $(3510, 693, 180, 126)$ and the second subconstituent of $\Gamma$ is the strongly regular graph $G$ with parameters $(693, 180, 51, 45)$. Now we will estimate from above the size of a clique in $G$. To this end consider the complement graph $\tilde{G}$ having parameters $(693, 512, 376, 384)$. For the configuration $\tilde{G}_f$, we have that $f = 440,$
\( p = 1/64 \), and \( q = -1/20 \). Therefore, the size of a clique \( K \) in \( G \) cannot be larger than 21. Otherwise the vector
\[
\sum_{i \in K} x_i, \quad x_i \in \bar{G}_f,
\]
is of negative norm. Thus, the size of a clique in \( \Gamma \) is not larger than 23 and hence \( \Gamma_f \) cannot be partitioned into less than \( 31671/23 = 1377 \) parts of smaller diameter.

**Proof of Corollary 1.** Let us first prove (1) for \( k = 0 \). Fix \( n \in \mathbb{N} \) and put \( m = 66n \). Consider the following coordinate representation of a vector \( y \in \mathbb{R}^m \):
\[
y = (y_1, \ldots, y_n|a_1, \ldots, a_n),
\]
where \( y_k \in \mathbb{R}^{65} \) and \( a_k \in \mathbb{R} \), \( k = 1, \ldots, n \). Now we take the following set of unit vectors in \( \mathbb{R}^m \):
\[
Y = \{ v_{ik}, i = 1, \ldots, 416, k = 1, \ldots, n \},
\]

where each \( v_{ik} \) has only two nonzero coordinates \( y_k \) and \( a_k \), and vectors \( x_i \) are such as in Theorem 1. Clearly, \( \langle v_{ik}, v_{jl} \rangle = 0 \) if \( k \neq l \). Moreover,
\[
\langle v_{ik}, v_{jk} \rangle = \begin{cases} 
1, & \text{if } i = j, \\
1/4, & \text{if } i \text{ and } j \text{ are adjacent}, \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore, \( Y \) is a two-distance set consisting of 416n vectors. Now, by Theorem 1, this set cannot be partitioned into less than 84n parts of smaller diameter. Adding the vector \( v \) which is on the distance \( \sqrt{2} \) to each vector of \( Y \)
\[
v = (0, \ldots, 0 | \alpha, \ldots, \alpha), \quad \alpha = 1 + \sqrt{1 + 16n} / 4n
\]
(\( \alpha \) is a solution of the equation \((\alpha - 1/4)^2 + (n - 1)\alpha^2 = 17/16\)) we obtain that \( b_2(66n) \geq 84n + 1 \). Finally we note that all these 416n + 1 vectors are on the same distance \( R \) to the vector \((0, \ldots, 0 | \gamma, \ldots, \gamma)\), where
\[
\gamma = \frac{\alpha}{4n\alpha - 1} \quad \text{and} \quad R = \frac{4\sqrt{n}}{\sqrt{16n + 1}} < 1
\]
\( \gamma \) is a solution of the equation \((\gamma - 1/4)^2 + (n - 1)\gamma^2 + 15/16 = n(\alpha - \gamma)^2\).

Hence we can add a new vector on the diameter distance \( \sqrt{2} \) to each of these 416n + 1 vectors to get a new set of 416n + 2 vectors in \( \mathbb{R}^{m+1} \) providing that \( b_2(m + 1) \geq 84n + 2 \). We can also rescale this new set to be on the sphere \( S^m \). Now inductive application of this procedure immediately gives us (1).

This procedure was also described in [9, Lemma 9]. Similarly, Theorem 2 implies (2).

\[ \square \]

**Proof of Corollary 2.** Let \( \Gamma \) be the \( Fi_{23} \) graph. For a vertex \( u \in V \), consider the subset \( \{ x_i : i \in N'(u) \} \) of the configuration \( \Gamma_f \). This subset lies in the hyperplane \( \langle x_u, x \rangle = -1/80 \) and consists of 31671 - 3510 - 1 = 28160 vectors. Hence, \( b_2(781) > [28160/23] = 1224 \).

Similarly, for adjacent vertices \( u \) and \( v \), the subset \( \{ x_i : i \in N'(u) \cap N'(v) \} \) consists of 31671 - 2 \times 3510 + 693 = 25344 vectors. This subset lies in the hyperplane \( \{ x \in \mathbb{R}^{782} : \langle x_u, x \rangle = -1/80 \text{ and } \langle x_v, x \rangle = -1/80 \} \), and provides that \( b_2(780) > [25344/23] = 1101 \).

Finally, consider a subset \( \{ x_i : i \in N'(u) \cap N'(v) \cap N'(w) \} \) such that vertices \( u, v, w \) form a triangle. This subset consists of 31671 - 3 \times 3510 + 3 \times 693 - 180 = 23040 vectors, and provides that \( b_2(779) > [23040/23] = 1001 \).

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