Approximation Algorithm for Sparsest $k$-Partitioning

Anand Louis *
Georgia Tech
anandl@gatech.edu

Konstantin Makarychev
Microsoft Research
komakary@microsoft.com

Abstract

Given a graph $G$, the sparsest-cut problem asks to find the set of vertices $S$ which has the least expansion defined as

$$\phi_G(S) \overset{\text{def}}{=} \frac{w(E(S, \bar{S}))}{\min\{w(S), w(\bar{S})\}},$$

where $w$ is the total edge weight of a subset. Here we study the natural generalization of this problem: given an integer $k$, compute a $k$-partition $\{P_1, \ldots, P_k\}$ of the vertex set so as to minimize

$$\phi^k_G(\{P_1, \ldots, P_k\}) \overset{\text{def}}{=} \max_i \phi_G(P_i).$$

Our main result is a polynomial time bi-criteria approximation algorithm which outputs a $(1 - \epsilon)k$-partition of the vertex set such that each piece has expansion at most $O(\epsilon^{1/(\sqrt{\log n \log k})})$ times $OPT$. We also study balanced versions of this problem.

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1 Introduction

The Sparsest Cut problem asks to find a subset $S$ of vertices of a given graph $G = (V, E)$ such that the total weight of edges leaving it is as small as possible compared to its size. This latter quantity, called expansion or conductance, is defined as:

$$\phi_G(S) = \frac{|E(S, \bar{S})|}{w(S)},$$

where $|E(S, \bar{S})|$ is the size of the cut $E(S, \bar{S})$ and $w(S) = \sum_{u \in S} w_u$ is the weight of vertices in $S$. Typically, the weight of each vertex $u$ is its degree $d_u$. The optimal value is called the expansion of the graph $G$, and is denoted by $\phi_G$:

$$\phi_G = \min_{S: w(S) \leq w(V)/2} \phi_G(S).$$

The Sparsest Cut problem has been highly influential in the study of algorithms and complexity. In their seminal work, Leighton and Rao [LR99] gave a $O(\log n)$ approximation algorithm for the problem. Later, Linial, London, Rabinovich [LLR95] and Aumann, Rabani [AR98] gave $O(\log n)$ approximation for Sparsest Cut with non-uniform demands and established a connection between the Sparsest Cut problem and embeddings of metric spaces into $\ell_2$. In a breakthrough result, Arora, Rao, and Vazirani [ARV04] gave $O(\sqrt{\log n})$ approximation algorithm for the problem. Given a graph with optimal sparsest cut $OPT = \phi_G$, their algorithm returns a set $S_A$ with $\phi_G(S_A) = O(\sqrt{\log n}OPT)$.

The fundamental Cheeger’s inequality (shown for graphs in [Alo86, AM85]) establishes a bound on expansion via the spectrum of the graph. In particular, it states that a natural spectral algorithm gives a solution of cost $\phi_G(S_A) = O(\sqrt{OPT})$. (Note that $OPT \leq 1$, so $\sqrt{OPT} \geq OPT$.)

Many extensions of this problem have been considered in the literature (see Section 1.1 for a brief survey). In this work, we study a very natural extension of the Sparsest Cut problem – the Sparsest $k$-Partitioning problem.

**Problem 1.1** (Sparsest $k$-Partitioning Problem). Given a graph $G = (V, E)$ and a parameter $k$, compute a partition $\{P_1, \ldots, P_k\}$ of $V$ into $k$ non-empty pieces so as to minimize

$$\phi^k_G(\{P_1, \ldots, P_k\}) = \max_i \phi_G(P_i).$$

The optimal value $OPT$ is called the $k$-sparsity and denoted by $\phi^k_G$.

The problem gained prominence because of its close connection to the graph spectrum. This connection was established and studied in the recent works [LGT12, LRTV12] which motivate the study of $\phi^k_G$ as a combinatorial analogue of $\lambda_k$ (the $k$-th smallest eigenvalue of the normalized Laplacian of the graph $G$). Lee, Gharan and Trevisan [LGT12] showed that there exists a $k$-partition $\{P_1, \ldots, P_k\}$ such that $\phi^k_G(\{P_1, \ldots, P_k\}) \leq O(k^3 \sqrt{\lambda_k})$. Louis, Raghavendra, Tetali, Vempala [LRTV12] showed that for any $k$ non-empty disjoint subset $S_1, \ldots, S_k$ of $V$, we have $\max_i \phi(S_i) \geq \Omega(\lambda_k)$. Moreover, they showed that for some absolute constant $c$, there exists $k$ disjoint non-empty sets $S_1, \ldots, S_k \subset V(G)$ such that $\max_i \phi(S_i) \leq O(\sqrt{\lambda_k} \log k)$. Lee et al. [LGT12] proved a similar result with $c = 1 + \varepsilon$ for any $\varepsilon > 0$. Note that in both these results, the sets $\{S_i : i \in [k]\}$ need not form a partition of the vertex set of the graph. As a by-product of our main result, we slightly strengthen the results above.
Proposition 1.2. Given a graph $G$ and a parameter $k$, 
\[
\lambda_k \leq \phi^k_G \leq O(\sqrt{\lambda(1+\varepsilon)k \log k})
\]
for every $\varepsilon > 0$. Here and below, $O_\varepsilon(f)$ denotes $O(poly(1/\varepsilon)f)$.

No approximation algorithm for $\phi^k_G$ with a multiplicative guarantee was known prior to our work. In this paper, we prove the following theorems.

**Theorem 1.3.** There exists a probabilistic polynomial-time algorithm that given an undirected graph $G = (V,E)$ with arbitrary vertex weights $w_u$ and parameters $k \in \mathbb{Z}^+$ ($k \geq 2$), $\varepsilon > 0$, outputs $k' \geq (1-\varepsilon)k$ partition such that each set has expansion at most $O_\varepsilon(\sqrt{\log n \log k OPT})$ w.h.p. Here $OPT$ is the cost of the optimal solution for the Sparsest $k$-Partitioning problem.

**Theorem 1.4.** There exists a probabilistic polynomial-time algorithm that given an undirected graph $G = (V,E)$ with weights $w_u = d_u$ ($d_u$ is the degree of the vertex $u$) and parameters $k \in \mathbb{N}$ ($k \geq 2$), $\varepsilon > 0$, outputs $k' \geq (1-\varepsilon)k$ partition such that each set has expansion at most $O_\varepsilon(\sqrt{OPT \log k})$ w.h.p.

**Remark 1.5.** Both theorems can be easily extended to edge-weighted graphs. W.l.o.g, we may assume that the weights of the edges are integers. The proofs of the theorems simply follow by introducing parallel edges: if $w_e \in \mathbb{Z}$ denotes the weight of an edge $e$, we replace $e$ with $w_e$ unweighted parallel edges. No changes are needed in the algorithm, and the algorithm still runs in polynomial time.

Note that for $k = 2$, Theorem 1.3 gives the same guarantee as that of Arora, Rao and Vazirani [ARV04] for Sparsest Cut and Theorem 1.4 gives the same guarantee as that of Cheeger’s inequality [AM85, Alo86] for Sparsest Cut. A direct corollary of the work of Raghavendra, Steurer and Tulsiani [RST12] is that Theorem 1.4 is optimal under the SSE hypothesis. We refer the reader to [RS10, RST12] for the statement and implications of the SSE hypothesis.

**SDP Relaxation.** The proofs of our main theorems go via an SDP relaxation of $\phi^k_G$ and a rounding algorithm for it. As a first attempt, one would try an assignment SDP à la Unique Games (as used in [Kho02, Tre08, CMM06a, CMM06b]), but such relaxations have a large integrality gap (see Appendix D). The main difficulty in constructing an integer programming formulation of sparsest $k$-partition is that we do not know the sizes of the sets in the optimal partition. We use a novel SDP relaxation which gets around this obstacle. In this SDP, we manage to encode a partitioning of the graph as well as a special measure on the vertices. This measure tells us how large every set must be. Roughly speaking, we expect that in the solution obtained by the algorithm, the measure of every set is approximately 1, irrespective of its size. We give a formal description of the SDP in Section 2.1.

**1.1 Related Work**

The Small Set Expansion problem (SSE) asks to find a set $S$ of weight at most $w(S)/k$ (where $k \geq 2$ is a parameter) with the smallest expansion $\phi_G(S)$. This problem got a lot of attention recently, partially because of the observed connection with the Unique Games Conjecture [RS10, ABS10] and because of a new Small Set Expansion Conjecture of Raghavendra and Steurer [RS10].
Raghavendra, Steurer, and Tetali [RST10] gave an algorithm with a Cheeger–type approximation guarantee of $O(\sqrt{OPT \log k})$ for this problem. Bansal et al. [BFK+11] gave a $O(\sqrt{\log n \log k})$ approximation algorithm (i.e., $\phi_G(S_A) \leq O(\sqrt{\log n \log k}OPT)$). The min-sum version of graph multi-partitioning has also been studied extensively, see e.g. [KVV04, AR06, KNS09, LRTV11].

Comparison to Previous Work Bansal et al. [BFK+11] studied the problem of partitioning the graph in $k$ equal pieces while minimizing the largest edge boundary of the piece (Min Max Graph Partitioning). They give a bi-criteria approximation algorithm where each set in the partition is of size at most $2n/k$ while approximating its edge boundary to within a $O(\sqrt{\log n \log k})$ factor of $OPT$. This problem is somewhat related to ours. However, the crucial difference is that the optimal solution to our problem may contain sets of very different sizes: large and small. This makes their algorithm and SDP relaxation non applicable in our settings. Since the aim of Min Max Graph Partitioning is to make all edge boundaries small, the algorithm of Bansal et al. [BFK+11] may sometimes add very small sets $P_i$ to the partition being output. Such sets can have large expansion inspite of having a small edge boundary. The main challenge in Min-Max Graph Partitioning is to find sets that (a) are of size at most $n/k$; and (b) cover all vertices (without these conditions, Min Max Graph Partitioning admits a simple constant factor approximation). In some sense, we need to ensure that the sets are not too small (rather than not too large), and hence the expansion is small. As Bansal et al., we also need to cover all vertices, but this is a relatively easy task in our case. In fact, we first drop this condition altogether and find a collection of disjoint non-expanding sets; then we transform these sets into a partitioning. We also note that our algorithm solves the SDP relaxation only once. This is again in contrast with [BFK+11], where the SDP relaxation is actually just a relaxation for the SSE problem (and not for Min Max Graph Partitioning!). So the Min Max Graph Partitioning algorithm has to solve the SDP relaxation at least once for each set in the partitioning (in fact, $O(\log n)$ times).

As we note above a natural assignment SDP relaxation has a large integrality gap (see Appendix D). To round our new SDP (see Section 2.1), one can try to adopt the rounding algorithms of Lee et al. [LGT12] and Louis et al. [LRTV12]. However, these algorithms could only possibly give an approximation guarantee of the form $O(\sqrt{OPT \log k})$. To get rid of the square root, we need to embed the SDP solution from $\ell_2^k$ to $\ell_2$. This step distorts the vectors, so that they no longer satisfy SDP constraints and no longer have properties required by these algorithms.

1.2 Extensions

Our SDP formulation and rounding algorithm can be used to solve other problems as well. Consider the balanced version of Sparsest $k$-Partition.

**Problem 1.6** (Balanced Sparsest $k$-Partitioning Problem). Given a graph $G = (V,E)$ and a parameter $k$, compute a partition $\{P_1, \ldots, P_k\}$ of $V$ into $k$ non-empty pieces each of weight $w(G)/k$ so as to minimize

$$\phi^k_G(\{P_1, \ldots, P_k\}) = \max_i \phi_G(P_i).$$

Using our techniques, we can prove the following theorems.

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1 Both [LRTV12, LGT12] construct an embedding of the graph into $\mathbb{R}^k$ as a first step. The proofs of their main theorems can be viewed as an algorithm to round these vectors into sets.
Theorem 1.7. There exists a probabilistic polynomial-time algorithm that given an undirected graph \( G = (V, E) \) with arbitrary vertex weights \( w_u \) and parameters \( k \in \mathbb{N} \) \((k \geq 2)\), \( \varepsilon > 0 \), outputs \( k' \geq (1 - \varepsilon)k \) disjoint sets (not necessarily a partition) such that the weight of each set is in the range \([w(G)/(2k), (1 + \varepsilon)w(G)/k]\), and the expansion of each set is at most \( O_\varepsilon \left( \sqrt{\log n \log k \cdot OPT} \right) \) w.h.p.

Theorem 1.8. There exists a probabilistic polynomial-time algorithm that given an undirected graph \( G = (V, E) \) with weights \( w_u = d_u \) \((d_u \) is the degree of the vertex \( u)\) and parameters \( k \in \mathbb{N} \) \((k \geq 2)\), \( \varepsilon > 0 \), outputs \( k' \geq (1 - \varepsilon)k \) disjoint sets (not necessarily a partition) such that the weight of each set is in the range \([w(G)/(2k), (1 + \varepsilon)w(G)/k]\), and the expansion of each set is at most \( O_\varepsilon \left( \sqrt{OPT \log k} \right) \) w.h.p.

Note that the algorithms above return \( k' \) disjoint sets that do not have to cover all vertices. The proofs of these theorems are similar to the proofs of our main results – Theorem 1.3 and Theorem 1.4. We refer the reader to Section 2.6 for more details. In fact, the assumption that all sets in the optimal solution have the same size makes the balanced problem much simpler. Theorem 1.7 also follows (possibly with slightly worse guarantees) from the result of Krauthgamer, Naor, and Schwartz [KNS09], who gave a bi-criteria \( O(\sqrt{\log n \log k}) \) approximation algorithm for the \( k \)-Balanced Partitioning Problem (with the “min-sum” objective).

1.3 Organization

We prove Theorem 1.3 in Section 2.4. We present the SDP relaxation of sparsest \( k \)-partition in Section 2.1 and the main rounding algorithm in Section 2.4. We prove Theorem 1.4 in Appendix A. We prove Proposition 1.2 in Appendix C.

2 Main Algorithm

We first prove a slightly weaker result. We give an algorithm that finds at least \((1 - \varepsilon)k\) disjoint sets each with expansion at most \( O_\varepsilon \left( \sqrt{\log n \log k \cdot OPT} \right) \). Note that we do not require that these sets cover all vertices in \( V \).

Theorem 2.1. There exists a probabilistic polynomial-time algorithm that given an undirected graph \( G \) and parameters \( k \in \mathbb{N} \) \((k \geq 2)\), \( \varepsilon > 0 \), outputs \( k' \geq (1 - \varepsilon)k \) disjoint sets \( P_1, \ldots, P_k' \) such that

\[
\mathbb{E} \left[ \max_i \phi(S_i) \right] \leq O_\varepsilon \left( \sqrt{\log n \log k \cdot OPT} \right),
\]

where \( OPT \) is the cost of the optimal sparsest \( k \)-partitioning of \( G \).

Then, in Section 3, we show how using \( k' \geq (1 - \varepsilon)k \) such sets, we can find a partitioning of \( V \) into \( k'' \geq (1 - 2\varepsilon)k \) sets with each set having expansion at most \( O_\varepsilon \left( \sqrt{\log n \log k \cdot OPT} \right) \).

Our algorithm works in several phases. First, it solves the SDP relaxation, which we present in Section 2.1. Then it transforms all vectors to unit vectors and defines a measure \( \mu(\cdot) \) on vertices of the graph. We give the details of this transformation in Section 2.2. Succeeding this, in the main phase, the algorithm samples many independent orthogonal separators \( S_1, \ldots, S_T \) and then extracts \( k' \geq (1 - \varepsilon)k \) disjoint subsets from them. We describe this phase in Section 2.4. Finally, the algorithm merges some of these sets with the left over vertices to obtain a \( k'' \geq (1 - \varepsilon)k' \) partition. We describe this phase in Section 2.6.
2.1 SDP Relaxation

We employ a novel SDP relaxation for the sparest $k$-partition problem. The main challenge in writing an SDP relaxation is that we do not know the sizes of the sets in advance, so we cannot write standard spreading constraints or spreading constraints used in the paper of Bansal et al. [BFK+11]. For each vertex $u$, we introduce a vector $\bar{u}$. In the integral solution corresponding to the optimal partitioning $P_1, \ldots, P_k$, each vector $\bar{u}$ has $k$ coordinates, one for every set $P_i$:

$$\bar{u}(i) = \begin{cases} \frac{1}{\sqrt{w(P_i)}} & \text{if } u \in P_i; \\ 0 & \text{otherwise.} \end{cases}$$

Observe, that the integral solution satisfies two crucial properties: for each set $P_i$,

$$\sum_{u \in P_i} w_u \|\bar{u}\|^2 = \sum_{u \in P_i} \frac{w_u}{w(P_i)} = 1, \quad (1)$$

and for every vertex $u \in P_i$,

$$\sum_{v \in V} w_v \langle \bar{u}, \bar{v} \rangle = \sum_{v \in P_i} \frac{w_v}{w(P_i)} + \sum_{v \notin P_i} 0 = 1. \quad (2)$$

Equation (1) gives us a way to measure sets. Given a set of vectors $\{\bar{u}\}$, we define a measure $\mu(\cdot)$ on vertices as follows

$$\mu(S) = \sum_{u \in S} w_u \|\bar{u}\|^2. \quad (3)$$

For the intended solution, we have $\mu(P_i) = 1$, and hence $\mu(V) = k$. This is the first constraint we add to the SDP:

$$\mu(V) \equiv \sum_{u \in V} w_u \|\bar{u}\|^2 = k.$$

From Equation (2), we get a spreading constraint:

$$\sum_{v \in V} w_v \langle \bar{u}, \bar{v} \rangle = 1.$$

We also add $\ell_2^2$ triangle inequalities to the SDP. It is easy to check that they are satisfied in the intended solution (since they are satisfied for each coordinate).

Finally, we need to write the objective function that measures the expansion of the sets. In the intended solution, if $u, v \in P_i$ (for some $i$), then $\bar{u} = \bar{v}$, and $\|\bar{u} - \bar{v}\|^2 = 0$. If $u \in P_i$ and $v \in P_j$ (for $i \neq j$), then

$$\|\bar{u} - \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 = 1/w(P_i) + 1/w(P_j).$$

Hence,

$$\frac{1}{k} \sum_{(u,v) \in E} \|\bar{u} - \bar{v}\|^2 = \frac{1}{k} \sum_{i<j} \sum_{u,v \in P_i \cap P_j} \left( \frac{1}{w(P_i)} + \frac{1}{w(P_j)} \right) = \frac{1}{k} \sum_{i} \frac{|E(P_i, V \setminus P_i)|}{w(P_i)} = \frac{1}{k} \sum_{i} \phi_G(P_i) \leq \text{OPT}. \quad (4)$$

We get the following SDP relaxation for the problem.
\[
\min \frac{1}{k} \sum_{(u,v) \in E} \| \bar{u} - \bar{v} \|^2 \\
\sum_{u \in V} w_u \| \bar{u} \|^2 = k \\
\sum_{v \in V} w_v \langle \bar{u}, \bar{v} \rangle = 1 \quad \forall u \in V \\
\| \bar{u} - \bar{x} \|^2 + \| \bar{x} - \bar{v} \|^2 \geq \| \bar{u} - \bar{v} \|^2 \quad \forall u, v, x \in V \\
0 \leq \langle \bar{u}, \bar{v} \rangle \leq \| \bar{u} \|^2 \quad \forall u, v \in V
\]

Figure 1: SDP Relaxation for Sparsest \(k\)-Partition

2.2 Normalization

After the algorithm solves the SDP 1, we define the measure \(\mu\) using Equation (3), and “normalize” all vectors using a transformation \(\psi\) from the paper of Chlamtac, Makarychev and Makarychev [CMM06b]. The transformation \(\psi\) defines the inner products between \(\psi(\bar{u})\) and \(\psi(\bar{v})\) as follows (all vectors \(\bar{u}\) are nonzero in our SDP relaxation):

\[
\langle \psi(\bar{u}), \psi(\bar{v}) \rangle = \frac{\langle \bar{u}, \bar{v} \rangle}{\max\{\| \bar{u} \|^2, \| \bar{v} \|^2\}}.
\]

This uniquely defines vectors \(\psi(\bar{u})\) (up to an isometry of \(\ell_2\)). Chlamtac, Makarychev and Makarychev showed that the image \(\psi(X)\) of any \(\ell_2^2\) space \(X\) is an \(\ell_2^2\) space, and the following conditions hold.

- For all non-zero vectors \(\bar{u} \in X\), \(\| \psi(\bar{u}) \|^2 = 1\).
- For all non-zero vectors \(u, v \in X\),

\[
\| \psi(\bar{u}) - \psi(\bar{v}) \|^2 \leq \frac{2\| \bar{u} - \bar{v} \|^2}{\max\{\| \bar{u} \|^2, \| \bar{v} \|^2\}}.
\]

2.3 Orthogonal Separators

Our algorithm uses the notion of orthogonal separators introduced by Chlamtac, Makarychev, and Makarychev [CMM06b]. Let \(X\) be an \(\ell_2^2\) space. We say that a distribution over subsets of \(X\) is a \(k\)-orthogonal separator of \(X\) with distortion \(D\), probability scale \(\alpha > 0\) and separation threshold \(\beta < 1\), if the following conditions hold for \(S \subset X\) chosen according to this distribution:

1. For all \(\bar{u} \in X\), \(\mathbb{P}[\bar{u} \in S] = \alpha \| \bar{u} \|^2\).
2. For all \(\bar{u}, \bar{v} \in X\) with \(\langle \bar{u}, \bar{v} \rangle \leq \beta \max\{\| \bar{u} \|^2, \| \bar{v} \|^2\}\),

\[
\mathbb{P}[\bar{u} \in S \text{ and } \bar{v} \in S] \leq \frac{\alpha \min\{\| \bar{u} \|^2, \| \bar{v} \|^2\}}{k}.
\]
Algorithm 2.3.

1. Solve SDP 1 and obtain vectors \{\overline{u}\}.
2. Compute normalized vectors \psi(\overline{u})\), and define the measure \mu(\cdot) (see Section 2.2 and Eq. Equation (3)).
3. Sample \(T = 2n/\alpha\) independent \((12k/\varepsilon)\)-orthogonal separators \(S_1, \ldots, S_T\) for vectors \psi(\overline{u}) \((u \in V)\) with separation threshold \(\beta = 1 - \varepsilon/4\).
4. For each \(i\), define \(S'_i\) as follows:
   \[S'_i = \begin{cases} S_i & \text{if } \mu(S_i) \leq 1 + \varepsilon/2; \\ \emptyset & \text{otherwise.} \end{cases}\]
5. For each \(i\), let \(S''_i = S'_i \setminus (\cup_{t=1}^{i-1} S'_t)\) be the set of yet uncovered vertices in \(S'_i\).
6. For each \(i\), set \(P_i = \{u \in S''_i : \|\overline{u}\|^2 \geq r_i\}\), where the parameter \(r_i\) is chosen to minimize the expansion \(\phi_G(P_i)\) of the set \(P_i\).
7. Output \((1 - \varepsilon)k\) non-empty sets \(P_i\) with the smallest expansion \(\phi_G(P_i)\).

Figure 2: Algorithm for generating \(k' \geq (1 - \varepsilon)k\) disjoint sets \(P_i\).

3. For all \(u, v \in X\)
   \[\mathbb{P}[I_S(\overline{u}) \neq I_S(\overline{v})] \leq \alpha D\|\overline{u} - \overline{v}\|^2.\]

Here \(I_S\) is the indicator function\(^2\) of the set \(S\).

Theorem 2.2 ([CMM06b, BFK+11]). There exists a polynomial-time randomized algorithm that given a set of vectors \(X\), a parameter \(k\), and \(\beta < 1\) generates a \(k\)-orthogonal separator with distortion \(D = O_\beta (\sqrt{\log |X| \log k})\) and scale \(\alpha \geq 1/p(|X|)\) for some polynomial \(p\).

In the algorithm, we sample orthogonal separators from the set of normalized vectors \(\{\psi(\overline{u}) : u \in V\}\). For simplicity of exposition we assume that an orthogonal separator \(S\) contains not vectors \(\overline{u}\), but the corresponding vertices. That is, for an orthogonal separator \(\tilde{S}\), we consider the set of vertices \(S = \{u \in V : \psi(\overline{u}) \in \tilde{S}\}\).

2.4 Algorithm

We give an algorithm for generating \(k' \geq (1 - \varepsilon)k\) disjoint sets \(P_i\) in Figure 2.

\(^2\)I.e., \(I_S(\overline{u}) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \overline{u} \in S, \\ 0 & \text{otherwise.} \end{cases}\)
2.5 Properties of Sets \( S_i'' \)

We prove that (a) the edge boundaries of the sets \( S_i'' \) are small; and (b) the sets \( S_i'' \) form a partition of \( V \) w.h.p. The following lemma makes these statements precise.

**Lemma 2.4.** For a set \( S \subset V \), define

\[
\nu(S) = \sum_{(u,v) \in E(S \setminus V \setminus S)} \|\bar{u}\|^2 + \sum_{(u,v) \in E \setminus S} (\|\bar{u}\|^2 - \|\bar{v}\|^2).
\]  

(4)

Then, sets \( S_i'' \) satisfy the following conditions:

(a) \( \mathbb{E} \left[ \sum_i \nu(S_i'') \right] \leq (8D + 1)k \cdot SDP \)

where \( D = O_\varepsilon(\sqrt{\log n \log k}) \) is the distortion of \((12k/\varepsilon)\)-orthogonal separator, and \( SDP \) is the value of the SDP solution.

(b) All sets \( S_i'' \) are disjoint; and

\[
\mathbb{P} \left[ \mu(\cup S_i'') = k \right] \geq 1 - ne^{-n}.
\]

**Proof.** (a) Let \( E_{cut} \) be the set of edges cut by the partitioning \( S_1'', \ldots, S_T'', V \setminus (\cup S_i'') \). Observe, that each cut edge \((u, v)\) contributes \( \|\bar{u}\|^2 + \|\bar{v}\|^2 \) to the sum \( \sum \nu(S_i'') \), and each uncut edge contributes \( \|\bar{u}\|^2 - \|\bar{v}\|^2 \), or 0. Hence,

\[
\mathbb{E} \left[ \sum_i \nu(S_i'') \right] \leq \mathbb{E} \left[ \sum_{(u,v) \in E_{cut}} (\|\bar{u}\|^2 + \|\bar{v}\|^2) \right] + \sum_{(u,v) \in E} (\|\bar{u}\|^2 - \|\bar{v}\|^2)
\]

The second term is bounded by

\[
\sum_{(u,v) \in E} \|\bar{u} - \bar{v}\|^2 = k \cdot SDP,
\]

since

\[
\|\bar{u}\|^2 - \|\bar{v}\|^2 = \|\bar{u} - \bar{v}\|^2 - 2(\|\bar{v}\|^2 - \langle \bar{u}, \bar{v} \rangle) \leq \|\bar{u} - \bar{v}\|^2.
\]

The inequality follows from the SDP constraint \( \|\bar{v}\|^2 \geq \langle \bar{u}, \bar{v} \rangle \). We now bound the first term. To do so we need the following lemma.

**Lemma 2.5.** For every vertex \( u \in V \) and \( i \in \{1, \ldots, T\} \), we have \( \mathbb{P} [u \in S_i'] \geq \alpha/2 \).

We give the proof of Lemma 2.5 after we finish the proof of Lemma 2.4. Let us estimate the probability that an edge \((u, v)\) is cut. Let \( U_t = \cup_{i \leq t} S_i' \) be the set of vertices covered by the first \( t \)}
sets $S_i'$. Note, that $S_i'' = S_i' \setminus U_{i-1}$. We say that the edge $(u, v)$ is cut by the set $S_i'$, if $S_i'$ is the first set containing $u$ or $v$, and it contains only one of these vertices. Then,

$$
P[(u, v) \in E_{\text{cut}}] = \sum_{i=1}^{T} P[(u, v) \text{ is cut by } S_i']
$$

$$
= \sum_{i=1}^{T} P[u, v \notin U_{i-1} \text{ and } I_{S_i}(u) \neq I_{S_i}(v)]
$$

$$
\leq \sum_{i=1}^{T} P[u \notin U_{i-1} \text{ and } I_{S_i}(u) \neq I_{S_i}(v)]
$$

$$
= \sum_{i=1}^{T} P[u \notin U_{i-1}] P[I_{S_i}(u) \neq I_{S_i}(v)].
$$

Now, by Lemma 2.5, $P[u \notin U_{i-1}] \leq (1 - \alpha/2)^{i-1}$, and, by Property 3 of orthogonal separators,

$$
P[I_{S_i}(u) \neq I_{S_i}(v)] \leq \alpha D \|\psi(u) - \psi(v)\|^2
$$

$$
\leq \frac{2\alpha D \|\bar{u} - \bar{v}\|^2}{\max\{\|\bar{u}\|^2, \|\bar{v}\|^2\}}.
$$

Thus (using $\sum_i (1 - \alpha/2)^i \leq 2/\alpha$),

$$
P[(u, v) \in E_{\text{cut}}] \leq \frac{4D \|\bar{u} - \bar{v}\|^2}{\max\{\|\bar{u}\|^2, \|\bar{v}\|^2\}}.
$$

We are almost done,

$$
\mathbb{E}\left[\sum_{(u, v) \in E_{\text{cut}}} (\|\bar{u}\|^2 + \|\bar{v}\|^2)\right] = \sum_{(u, v) \in E} P[(u, v) \in E_{\text{cut}}] (\|\bar{u}\|^2 + \|\bar{v}\|^2)
$$

$$
\leq \sum_{(u, v) \in E} \frac{4D \|\bar{u} - \bar{v}\|^2}{\max\{\|\bar{u}\|^2, \|\bar{v}\|^2\}} \cdot (\|\bar{u}\|^2 + \|\bar{v}\|^2)
$$

$$
\leq \sum_{(u, v) \in E} 8D \|\bar{u} - \bar{v}\|^2 = 8kD \cdot SDP.
$$

Thus we get that

$$
\mathbb{E}\left[\sum_{i} \nu(S_i'')\right] \leq (8D + 1)k \cdot SDP.
$$

(b) The sets $S_i''$ are disjoint by definition. By Lemma 2.5, the probability that a vertex is not covered by any set $S_i$ is $(1 - \alpha/2)^T = (1 - \alpha/2)^{2n/\alpha} < e^{-n}$. So with probability at least $1 - ne^{-n}$ all vertices are covered.

It remains to prove Lemma 2.5.
Proof of Lemma 2.5. We adopt a slightly modified argument from the paper of Bansal et al. [BFK+11] (Theorem 2.1, arXiv). If \( u \in S_i \), then \( u \in S'_i \) unless \( \mu(S_i) > 1 + \varepsilon/2 \), hence
\[
\mathbb{P}[u \in S'_i] = \mathbb{P}[u \in S_i] (1 - \mathbb{P}[\mu(S_i) > 1 + \varepsilon/2 \mid u \in S_i])
= \alpha(1 - \mathbb{P}[\mu(S_i) > 1 + \varepsilon/2 \mid u \in S_i]).
\]
Here, we used that \( \mathbb{P}[u \in S_i] = \alpha \|\psi(\bar{u})\|^2 = \alpha \) (see Property 1 of orthogonal separators). We need to show that \( \mathbb{P}[\mu(S_i) > 1 + \varepsilon/2 \mid u \in S_i] \leq 1/2 \). Let us define the sets \( A_u \) and \( B_u \) as follows.
\[
A_u = \{ v \in V : \langle \psi(\bar{u}), \psi(\bar{v}) \rangle \geq \beta \}
\]
and
\[
B_u = \{ v \in V : \langle \psi(\bar{u}), \psi(\bar{v}) \rangle < \beta \}.
\]
Now,
\[
\mu(A_u) = \sum_{v \in A_u} w_v \|\bar{v}\|^2 \leq \frac{1}{\beta} \sum_{v \in V} w_v \|\bar{v}\|^2 \langle \psi(\bar{u}), \psi(\bar{v}) \rangle = \frac{1}{\beta} \sum_{v \in V} w_v \|\bar{v}\|^2 \frac{\langle \bar{u}, \bar{v} \rangle}{\max\{\|\bar{v}\|^2, \|\bar{v}\|^2\}}
\]
\[
\leq \frac{1}{\beta} \sum_{v \in V} w_v \langle \bar{u}, \bar{v} \rangle \leq \frac{1}{\beta} \leq 1 + \frac{\varepsilon}{3}.
\]
Equality “\( \diamond \)” follows from the SDP constraint \( \sum_{v \in V} w_v \langle \bar{u}, \bar{v} \rangle = 1 \). For any \( v \in B_u \), we have \( \langle \psi(\bar{u}), \psi(\bar{v}) \rangle < \beta \). Hence, by Property 2 of orthogonal separators,
\[
\mathbb{P}[u \in S_i \mid u \in S_i] \leq \frac{\varepsilon}{12k}
\]
Therefore,
\[
\mathbb{E}[\mu(S_i \cap B_u) \mid u \in S_i] \leq \frac{\varepsilon \mu(B_u)}{12k} \leq \frac{\varepsilon \mu(V)}{12k} = \frac{\varepsilon}{12}.
\]
By Markov’s inequality, \( \mathbb{P}[\mu(S_i \cap B_u) \geq \varepsilon/6 \mid u \in S_i] \leq 1/2 \). Since \( \mu(S_i) = \mu(S_i \cap A_u) + \mu(S_i \cap B_u) \), we get \( \mathbb{P}[\mu(S_i) \geq 1 + \varepsilon/2 \mid u \in S_i] \leq 1/2 \).

2.6 End of Proof

We are ready to finish the analysis of Algorithm 2.3 and prove Theorem 2.1 and Theorem 1.7.

Proofs of Theorem 2.1 and Theorem 1.7. We first prove Theorem 2.1, then we slightly modify Algorithm 2.3 and prove Theorem 1.7.

I. We show that Algorithm 2.3 outputs sets satisfying conditions of Theorem 2.1. The sets \( S'_i \) are disjoint (see Lemma 2.4), thus sets \( P_i \) are also disjoint. We now need to prove that among sets \( P_i \) obtained at Step 6 of the algorithm, there are at least \((1 - \varepsilon)k\) sets with expansion less than \( O_{\varepsilon}(\sqrt{\log n \log k OPT}) \) (in expectation).

Let \( Z = \frac{1}{k} \sum_i \nu(S'_i) \). By Lemma 2.4 we have,
\[
\mathbb{E}[Z] \leq (8D + 1) OPT
\]
and $S''_i$ form a partition\footnote{With an exponentially small probability the sets $S''_i$ do not cover all the vertices. In this unlikely event, the algorithm may output an arbitrary partition.} of $V$. We throw away all empty sets $S''_i$, and set $\lambda_i = \mu(S''_i)/k$. Then $\sum_i \lambda_i = 1$, and

$$Z = \frac{1}{k} \sum_i \nu(S''_i) = \sum_i \lambda_i \cdot \frac{\nu(S''_i)}{\mu(S''_i)}.$$  

Define $I = \{ i : \nu(S''_i)/\mu(S''_i) \leq 3Z/\varepsilon \}$. By Markov’s inequality (we can think of $\lambda_i$ as the weight of $i$),

$$\sum_{i \in I} \lambda_i \geq 1 - \varepsilon/2.$$  

(5)

Since each $\lambda_i$ satisfies

$$\lambda_i = \mu(S''_i)/k \leq (1 + \varepsilon/2)/k$$

the set $I$ has at least $(1 - \varepsilon/2)k/(1 + \varepsilon/2) \geq (1 - \varepsilon)k$ elements.

Fix an $i \in I$. Since $i \in I$, we have

$$\nu(S''_i) \leq 3Z/\varepsilon \cdot \mu(S''_i).$$

Let $R = \max \{ \| \bar{u} \|^2 : u \in S''_i \}$. For a random $r \in (0, R)$ and $L_r = \{ u \in S''_i : \| \bar{u} \|^2 \geq r \}$, we have

$$\mathbb{E}[w(L_r)] = \mu(S''_i)/R$$

(6)

as each $u$ belongs to $L_r$ with probability $\| \bar{u} \|^2/R$ and

$$\mathbb{E}[E(L_r, V \setminus L_r)] = \nu(S''_i)/R$$

(since an edge in $S''_i \times S''_i$ is cut with probability $\| \bar{u} \|^2 - \| \bar{v} \|^2)/R$; and an edge $(u, v)$ with $u \in S''_i$ and $v \notin S''_i$ is cut with probability $\| \bar{u} \|^2$ if and only if $u \in L_r$; compare with Definition 4). Therefore,

$$\mathbb{E}[|E(L_r, V \setminus L_r)|] = \frac{\nu(S''_i)}{R} \leq \frac{3Z}{\varepsilon} \cdot \frac{\mu(S''_i)}{R} = \frac{3Z}{\varepsilon} \cdot \mathbb{E}[w(L_r)].$$

For some $r^*$, we get

$$|E(L_{r^*}, V \setminus L_{r^*})| \leq 3Z/\varepsilon \cdot w(L_{r^*}).$$

By definition, $\phi_G(P_i) = \min_r \phi_G(L_r)$, thus

$$\phi_G(P_i) = \frac{|E(P_i, V \setminus P_i)|}{w(P_i)} \leq \frac{3Z}{\varepsilon}.$$  

We showed that there are at least $|I| \geq (1-\varepsilon)k$ sets $P_i$ with expansion at most $3Z/\varepsilon$. Therefore, the expansion of the sets returned by the algorithm is at most $3Z/\varepsilon$. This finishes the proof, since $\mathbb{E}[3Z/\varepsilon] = O_\varepsilon(\sqrt{\log n \log k}) OPT$.

II. To prove Theorem 1.7, we need to modify the algorithm. For simplicity, we rescale all weights $w_u$ and assume that $w(G) = k$. Then our goal is to find $k'$ disjoint sets $P_i$ of weight in the range $[1/2, 1 + \varepsilon]$ each. Since all sets in the optimal solution to the $k$-Balanced Sparsest
Partitioning Problem have weight 1, we add the SDP constraint that all vectors \( \bar{u} \) have length 1 (see Section 2.1): for all \( u \in V \):
\[
||\bar{u}||^2 = 1.
\]

The intended solution satisfies this constraint. We also change the way the algorithm picks the parameters \( r_i \). The algorithm chooses \( r_i \) so as to minimize the expansion \( \phi_G(P_i) \) subject to an additional constraint \( \mu(P_i) \geq (1 - \varepsilon/2)\mu(S''_i) \). Finally, once the algorithm obtains sets \( P_i \), it greedily merges sets of weight at most 1/2. The rest of the algorithm is the same as Algorithm 2.3.

From Equation (6) and Equation (7), we get
\[
\mathbb{E}_w[w(L_r)] \geq \frac{\varepsilon^2}{6Z} \mathbb{E}_r[|E(L_r, V \setminus L_r)|] + \frac{(1 - \varepsilon/2)\mu(S''_i)}{R} \geq \max \left\{ \frac{\varepsilon^2}{6Z} \mathbb{E}_r[|E(L_r, V \setminus L_r)|], \frac{(1 - \varepsilon/2)\mu(S''_i)}{R} \right\}.
\]

Since \( ||\bar{u}||^2 = 1 \) for all \( u \in V \), we have \( R = 1 \) and \( \mu(L_r) = w(L_r) \). Therefore,
\[
\mathbb{E}_w[w(L_r)] \geq \max \left\{ \frac{\varepsilon^2}{6Z} \mathbb{E}_r[|E(L_r, V \setminus L_r)|], (1 - \varepsilon/2)\mu(S''_i) \right\},
\]
and for some \( r^* \),
\[
w(L_r^*) \geq \frac{\varepsilon^2}{6Z} \mathbb{E}_r[|E(L_r^*, V \setminus L_r^*)|];
\]
\[
\mu(L_r^*) \geq (1 - \varepsilon/2)\mu(S''_i).
\]

Consequently, we get
\[
\phi_G(P_i) \leq \phi_G(L_r^*) \leq \frac{6Z}{\varepsilon^2}.
\]

Now, recall, that by (5), \( \sum_{i \in I} \lambda_i \geq 1 - \varepsilon/2 \). Hence,
\[
\sum_{i \in I} w(P_i) = \sum_{i \in I} \mu(P_i) \geq (1 - \varepsilon/2) \sum_{i \in I} \mu(S''_i) = (1 - \varepsilon/2) \sum_{i \in I} k\lambda_i \geq (1 - \varepsilon)k.
\]

We showed that the algorithm gets sets \( P_i \) satisfying the following properties: (a) the expansion \( \phi_G(P_i) \leq \frac{6Z}{\varepsilon^2} \); (b) \( w(P_i) \leq (1 + \varepsilon/2) \) and (c) \( \sum_i w(P_i) \geq (1 - \varepsilon)k \). To get sets of weight in the range \([1/2, 1 + \varepsilon]\) the algorithm greedily merges sets \( P_i \) of weight at most 1/2 and obtains a collection of new sets, which we denote by \( Q_i \). The algorithm outputs all sets \( Q_i \) with weight at least 1/2.

Note that for any two disjoint sets \( A \) and \( B \), \( \phi_G(A \cup B) \leq \max\{ \phi_G(A), \phi_G(B) \} \). So \( \phi_G(Q_i) \leq \max j \phi_G(P_j) \leq \frac{6Z}{\varepsilon^2} \). All sets \( Q_i \) but possibly one have weight at least 1/2. So the weight of sets \( Q_i \) output by the algorithm is at least \( (1 - \varepsilon)k - 1/2 \). The maximum weight of sets \( Q_i \) is \( 1 + \varepsilon/2 \), so the number of sets \( Q_i \) is at least
\[
\left\lceil \frac{(1 - \varepsilon)k - 1/2}{1 - \varepsilon/2} \right\rceil \geq \left\lceil (1 - 2\varepsilon)k - 1/2 \right\rceil \geq \left\lceil (1 - 4\varepsilon)k \right\rceil.
\]

To verify the last inequality check two cases: if \( 2\varepsilon k \geq 1/2 \), then \( (1 - 2\varepsilon)k - 1/2 \geq (1 - 4\varepsilon)k \); if \( 2\varepsilon k < 1/2 \), then \( \left\lceil (1 - 2\varepsilon)k - 1/2 \right\rceil = k \). This finishes the proof. \( \square \)
3 From Disjoint Sets to Partitioning

We now show how given $k' \geq (1 - \varepsilon)$ sets $P_1, \ldots, P_{k'}$, we can obtain a true partitioning $P'_1, \ldots, P'_{k''}$ of $V$.

**Proof of Theorem 1.3.** To get the desired partitioning, we first run Algorithm 2.3 several times (say, $n$) to obtain disjoint non-empty sets $P_1, \ldots, P_{k'}$ that satisfy $\max_i \phi_G(P_i) \leq O(\varepsilon \sqrt{\log n \log k}) \text{OPT}$ w.h.p. Let $Z = \max_i \phi_G(P_i)$. We sort sets $P_i$ by weight $w(P_i)$. We output the smallest $k'' = \lfloor (1 - \varepsilon) k' \rfloor$ sets $P_i$, and the compliment set $P' = V \setminus (\cup_{1 \leq i \leq k''} P_i)$.

Since sets $P_i$ are disjoint and non-empty, the first $k''$ sets $P_i$ and the set $P'$ are also disjoint and non-empty. Moreover, $\phi_G(P_i) \leq Z$, so we only need to show that $\phi_G(P') \leq O(\varepsilon Z)$. Note, that $w(P') \geq \varepsilon w(V)$, since $P'$ contains vertices in the $\lceil k \varepsilon \rceil$ largest sets $P_i$ and all vertices not covered by sets $P_i$. Then,

$$E(P', V \setminus P') = \cup_{1 \leq i \leq k''} E(P', P_i) \subset \cup_{1 \leq i \leq k''} E(P_i, V \setminus P_i).$$

So

$$\phi_G(P') = \frac{|E(P', V \setminus P')|}{w(P')} \leq \sum_{i=1}^{k''} \frac{E(P_i, V \setminus P_i)}{w(P')} \leq \sum_{i=1}^{k''} \frac{w(P_i) \phi_G(P_i)}{\varepsilon w(V)} \leq \sum_{i=1}^{k''} \frac{w(P_i) Z}{\varepsilon w(V)} \leq Z \frac{w(V)}{\varepsilon w(V)} = \frac{Z}{\varepsilon}.$$

This concludes the proof. \qed

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A Proof of Theorem 1.4

The proof of Theorem 1.4 is almost the same as the proof of Theorem 1.3. The only difference is that we need to replace orthogonal separators with a slightly different variant of orthogonal separators (implicitly defined in [CMM06b]).

**Orthogonal Separators with $\ell_2$ distortion.** Let $X$ be a set of unit vectors in $\ell_2$. We say that a distribution over subsets of $X$ is a $k$-orthogonal separator of $X$ with $\ell_2$ distortion $D$, probability scale $\alpha > 0$ and separation threshold $\beta < 1$, if the following conditions hold for $S \subset X$ chosen according to this distribution:

1. For all $\bar{u} \in X$, $\mathbb{P}[\bar{u} \in S] = \alpha$.
2. For all $\bar{u}, \bar{v} \in X$ with $\langle \bar{u}, \bar{v} \rangle \leq \beta \max\{\Vert \bar{u} \Vert^2, \Vert \bar{v} \Vert^2\}$,
   $$\mathbb{P}[\bar{u} \in S \text{ and } \bar{v} \in S] \leq \frac{\alpha}{k}.$$  
3. For all $u, v \in X$,
   $$\mathbb{P}[I_S(\bar{u}) \neq I_S(\bar{v})] \leq \alpha D \Vert \bar{u} - \bar{v} \Vert.$$

**Theorem A.1** ([CMM06b]). *There exists a polynomial-time randomized algorithm that given a set of unit vectors $X$, a parameter $k$, and $\beta < 1$ generates a $k$-orthogonal separator with $\ell_2$ distortion $D = O_{\beta}(\sqrt{\log k})$ and scale $\alpha \geq 1/n$."

For completeness we sketch the proof of this lemma in Section B. Algorithm 2.3' is the same as Algorithm 2.3 except that at Step 3, it samples orthogonal separators with $\ell_2$ distortion $O_\epsilon(\sqrt{\log k})$ using Theorem A.1. The proof of Theorem 1.3 goes through for the new algorithm essentially as is. The only statement we need to take care of is Lemma 2.4 (a). We prove the following bound on $\mathbb{E}[\sum_i \nu(S''_i)]$.

**Lemma A.2.** The sets $S''_i$ satisfy the following condition: $\mathbb{E}[\sum_i \nu(S''_i)] \leq (8D+1)k \cdot \sqrt{SDP}$, where $D = O_\epsilon(\sqrt{\log k})$ is the $\ell_2$ distortion of $(12k/\epsilon)$-orthogonal separator, and $SDP$ is the value of the SDP solution.
Proof. Let $E_{\text{cut}}$ be the set of edges cut by the partitioning $S''_1, \ldots, S''_T, V \setminus (\cup S''_i)$. As before (in Lemma 2.4), we have

$$\mathbb{E} \left[ \sum_i \nu(S''_i) \right] \leq \mathbb{E} \left[ \sum_{(u,v) \in E_{\text{cut}}} (\|\bar{u}\|^2 + \|\bar{v}\|^2) \right] + \sum_{(u,v) \in E} \|\bar{u}\|^2 - \|\bar{v}\|^2 \]

$$

$$\leq \mathbb{E} \left[ \sum_{(u,v) \in E_{\text{cut}}} (\|\bar{u}\|^2 + \|\bar{v}\|^2) \right] + k \text{SDP}. $$

We now bound the first term. Estimate the probability that an edge $(u,v)$ is cut. Let $U_t = \cup_{i \leq t} S'_i$ be the set of vertices covered by the first $t$ sets $S'_i$. Note, that $S''_i = S'_i \setminus U_{i-1}$. We say that the edge $(u,v)$ is cut by the set $S'_t$, if $S'_t$ is the first set containing $u$ or $v$, and it contains only one of these vertices. Then,

$$\mathbb{P}[(u,v) \in E_{\text{cut}}] = \sum_i \mathbb{P}[(u,v) \text{ is cut by } S'_i]$$

$$= \sum_i \mathbb{P}[u, v \notin U_{i-1} \text{ and } I_{S'_i}(u) \neq I_{S'_i}(v)]$$

$$\leq \sum_i \mathbb{P}[u \notin U_{i-1} \text{ and } I_{S'_i}(u) \neq I_{S'_i}(v)]$$

$$= \sum_i \mathbb{P}[u \notin U_{i-1}] \mathbb{P}[I_{S'_i}(u) \neq I_{S'_i}(v)].$$

Now, by Lemma 2.5, $\mathbb{P}[u \notin U_{i-1}] \leq (1 - \alpha/2)^{i-1}$, and, by Property 3 of $\ell_2$ orthogonal separators,

$$\mathbb{P}[I_{S'_i}(u) \neq I_{S'_i}(v)] \leq \alpha D \|\psi(u) - \psi(v)\| \leq \alpha D \sqrt{2} \|\bar{u} - \bar{v}\| \cdot \max\{\|\bar{u}\|, \|\bar{v}\|\}. $$

Thus,

$$\mathbb{P}[(u,v) \in E_{\text{cut}}] \leq \frac{2\sqrt{2} D \|\bar{u} - \bar{v}\|}{\max\{\|\bar{u}\|, \|\bar{v}\|\}}.$$ 

Now, the proof deviates from the proof of Lemma 2.4:

$$\mathbb{E} \left[ \sum_{(u,v) \in E_{\text{cut}}} (\|\bar{u}\|^2 + \|\bar{v}\|^2) \right] = \sum_{(u,v) \in E} \mathbb{P}[(u,v) \in E_{\text{cut}}] (\|\bar{u}\|^2 + \|\bar{v}\|^2)$$

$$\leq \sum_{(u,v) \in E} \frac{2\sqrt{2} D \|\bar{u} - \bar{v}\|}{\max\{\|\bar{u}\|, \|\bar{v}\|\}} \cdot (\|\bar{u}\|^2 + \|\bar{v}\|^2)$$

$$\leq 2\sqrt{2} D \sum_{(u,v) \in E} \|\bar{u} - \bar{v}\| \cdot (\|\bar{u}\| + \|\bar{v}\|).$$

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By Cauchy–Schwarz,

\[
2\sqrt{2} D \sum_{(u,v) \in E} \|\bar{u} - \bar{v}\| \cdot (\|\bar{u}\| + \|\bar{v}\|) \leq 2\sqrt{2} D \left( \sum_{(u,v) \in E} \|\bar{u} - \bar{v}\|^2 \right)^{1/2} \left( \sum_{(u,v) \in E} (\|\bar{u}\| + \|\bar{v}\|)^2 \right)^{1/2} \\
\leq 4 D \left( \sum_{(u,v) \in E} \|\bar{u} - \bar{v}\|^2 \right)^{1/2} \left( \sum_{(u,v) \in E} \|\bar{u}\|^2 + \|\bar{v}\|^2 \right)^{1/2} \\
= 4 D \left( k \text{SDP} \right)^{1/2} \left( \sum_{(u,v) \in E} d_u \|\bar{u}\|^2 \right)^{1/2}.
\]

Recall, that in Theorem 1.4, we assume that the weight of every vertex \(w_u\) equals its degree \(d_u\). Hence, \(\sum_{(u,v) \in E} d_u \|\bar{u}\|^2 = \sqrt{\mu(V)} = \sqrt{k}\). We get,

\[
E \left[ \sum_{(u,v) \in E_{cut}} (\|\bar{u}\|^2 + \|\bar{v}\|^2) \right] \leq 4 D k \sqrt{\text{SDP}}.
\]

Since \(\text{SDP} \leq \text{OPT} = \phi_k^G \leq 1\) (here we use that \(d_u = w_u\), \(\text{SDP} \leq \sqrt{\text{SDP}}\), and

\[
E \left[ \sum_i \nu(S'_i) \right] \leq 8 D k \sqrt{\text{SDP}} + k \text{SDP} \leq (8 D + 1) k \sqrt{\text{SDP}}.
\]

This concludes the proof.

\[\Box\]

### B Orthogonal Separators with \(\ell_2\) Distortion

In this section, we sketch the proof of Theorem A.1 which is proven in [CMM06b] as part of Lemma 4.9. Let us fix some notation. Let \(\Phi(t)\) be the probability that the standard \(N(0,1)\) Gaussian variable is greater than \(t\). We will use the following easy lemma from [MM12].

**Lemma B.1 (Lemma 2.1. in [MM12]).** For every \(t > 0\) and \(\beta \in (0,1]\), we have

\[
\Phi(\beta t) \leq \Phi(t)^{\beta^2}.
\]

We now describe an algorithm for \(m\)-orthogonal separators with \(\ell_2\) distortion (see Appendix A). Let \(\beta < 1\) be the separation threshold. Assume w.l.o.g. that all vectors \(\bar{u}\) lie in \(\mathbb{R}^n\). Fix \(m' = m^{1+\beta}\) and \(t = \Phi^{-1}(1/m')\) (i.e., \(t\) such that \(\Phi(t) = 1/m'\)). Sample a random Gaussian \(n\) dimensional vector \(\gamma\) in \(\mathbb{R}^n\). Return the set

\[
S = \{ \bar{u} : \langle \bar{u}, \gamma \rangle \geq t \}.
\]

We claim that \(S\) is an \(m\)-orthogonal separator with \(\ell_2\) distortion \(O(\sqrt{\log m})\) and scale \(\alpha = 1/m'\). We now verify the conditions of orthogonal separators with \(\ell_2\) distortion.

1. For every \(\bar{u}\),

\[
P [\bar{u} \in S] = P [\langle \bar{u}, \gamma \rangle \geq t] = 1/m' \equiv \alpha.
\]

Here we used that \(\langle \bar{u}, \gamma \rangle\) is distributed as \(N(0,1)\), since \(\bar{u}\) is a unit vector.
2. For every $\bar{u}$ and $\bar{v}$ with $\langle \bar{u}, \bar{v} \rangle \leq \beta$,

$$\mathbb{P} [\bar{u}, \bar{v} \in S] = \mathbb{P} [\langle \bar{u}, \gamma \rangle \geq t \text{ and } \langle \bar{v}, \gamma \rangle \geq t] \leq \mathbb{P} [\langle \bar{u} + \bar{v}, \gamma \rangle \geq 2t].$$

Note that $\|\bar{u} + \bar{v}\| = \sqrt{2 + 2\langle \bar{u}, \bar{v} \rangle}$, hence $(\bar{u} + \bar{v})/\sqrt{2 + 2\langle \bar{u}, \bar{v} \rangle}$ is a unit vector. We have

$$\mathbb{P} [\bar{u}, \bar{v} \in S] \leq \mathbb{P} \left[ \left( \frac{\bar{u} + \bar{v}}{\sqrt{2 + 2\langle \bar{u}, \bar{v} \rangle}}, \gamma \right) \geq \frac{2t}{\sqrt{2 + 2\langle \bar{u}, \bar{v} \rangle}} \right] = \Phi \left( \frac{\sqrt{2t}}{\sqrt{1 + \langle \bar{u}, \bar{v} \rangle}} \right) \leq \Phi \left( \frac{\sqrt{2t}}{1 + \beta} \right) = \left( \frac{1}{m'} \right)^{\frac{2}{1 + \beta}} = \frac{1}{\alpha} \cdot \frac{1}{m'} = \frac{\alpha}{m}.$$

3. The third property directly follows from Lemma A.2. in [CMM06b].

We note that this proof gives probability scale $\alpha = m^{\frac{1 + \beta}{1 + \beta}}$. So, for some $\beta$, we may get $\alpha \ll 1/n$. However, it is easy to sample $\gamma$ in such a way that $\mathbb{P} [\langle \bar{u}, \gamma \rangle \geq 1/n]$ for every vector $\bar{u}$ in our set. To do so, we order vectors $\{\bar{u}\}$ in an arbitrary way: $\bar{u}_1, \ldots, \bar{u}_n$. Then, we pick a random index $i \in \{1, \ldots, n\}$, and sample a random Gaussian vector $\gamma'$ conditional on $\langle \bar{u}_i, \gamma' \rangle \geq t$. We set $S' = \{\bar{u} : \langle \bar{u}, \gamma' \rangle \geq t\}$ as in the algorithm above. Note that $\bar{u}_i$ always belongs to $S'$. We output $S'' = S'$ if $S'$ does not contain vectors $\bar{u}_1, \ldots, \bar{u}_{i-1}$; and we output $S'' = \emptyset$ otherwise. It is easy to verify that $\mathbb{P} [\bar{u} \in S''] = 1/n$ for every $\bar{u}$, and, furthermore, for every non-empty set $S^* \neq \emptyset$,

$$\mathbb{P} [S'' = S^*] = \frac{1}{\alpha n} \mathbb{P} [S = S^*],$$

where $S$ is the orthogonal separator from the proof above. So all properties of orthogonal separators hold for $S''$ with $\alpha' = \alpha/(\alpha n) = 1/n$.

## C Proof of Proposition 1.2

We restate Proposition 1.2 below.

**Proposition C.1.** Given a graph $G$ and a parameter $k$,

$$\lambda_k \leq \phi_G^k \leq O_\varepsilon \left( \sqrt{\lambda_{(1+\varepsilon)k} \log k} \right),$$

for every $\varepsilon > 0$.

**Proof.** [LGT12] show that there exist disjoint non-empty sets $P_1, \ldots, P_k$ that satisfy $\max_i \phi_G(P_i) \leq O_\varepsilon (\sqrt{\lambda_{(1+\varepsilon)k} \log k})$ for $k' \geq k(1+\varepsilon/2)$. Let $Z = \max_i \phi_G(P_i)$. We sort sets $P_i$ by weight $w(P_i)$. We output the smallest $k$ sets $P_i$, and the compliment set $P' = V \setminus (\cup_{1 \leq i \leq k} P_i)$.

Since sets $P_i$ are disjoint and non-empty, the first $k$ sets $P_i$ and the set $P'$ are also disjoint and non-empty. Moreover, $\phi_G(P_i) \leq Z$, so we only need to show that $\phi_G(P') \leq O_\varepsilon (Z)$. Note, that $w(P') \geq \varepsilon w(V)$, since $P'$ contains vertices in the $\lceil \varepsilon k \rceil$ largest sets $P_i$ and all vertices not covered by sets $P_i$. Then,

$$E(P', V \setminus P') = \cup_{i \leq k} E(P', P_i) \subset \cup_{i \leq k} E(P_i, V \setminus P_i).$$

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So

\[ \phi_G(P') = \frac{|E(P', V \setminus P')|}{w(P')} \leq \frac{\sum_{i=1}^{k} E(P_i, V \setminus P_i)}{w(P')} \]

\[ = \frac{\sum_{i=1}^{k} w(P_i)\phi_G(P_i)}{\varepsilon w(V)} \leq \frac{\sum_{i=1}^{k} w(P_i)Z}{\varepsilon w(V)} \]

\[ \leq \frac{Z w(V)}{\varepsilon w(V)} = \frac{Z}{\varepsilon}. \]

This concludes the proof. 

\[ \square \]

D Integrality Gap for the Assignment SDP

In this Section, we show that the standard Assignment SDP has high integrality gap.

\[
\begin{array}{ll}
\min \alpha \\
\sum_{(u,v) \in E} \|\bar{u}_i - \bar{v}_i\|^2 \leq \alpha \sum_{u \in V} w_u \|\bar{u}_i\|^2 & \forall i \in [k] \\
\sum_{i \in [k]} \|\bar{u}_i\|^2 = 1 \\
\langle \bar{u}_i, \bar{u}_j \rangle = 0 & \forall i \neq j \text{ and } \forall u \in V \\
\langle \sum_i \bar{u}_i, I \rangle = 1 \\
\|I\|^2 = 1
\end{array}
\]

Figure 3: Assignment SDP

**Proposition D.1.** SDP 3 has an unbounded integrality gap.

**Proof.** Consider the following infinite family of graphs \( \mathcal{G} = \{G_n : n \geq 0\} \). \( G_n \) consists of the two disjoint cliques of size \( C_1 = K_{\lfloor n/2 \rfloor} \) and \( C_2 = K_{\lceil n/2 \rceil} \). It is easy to see that for \( \phi^k(G_n) = \Omega(1) \) for \( k > 2 \).

For the sake of simplicity, let us assume that \( k \) is a multiple of 2. Let \( e_1, \ldots, e_{k/2} \) be the standard basis vectors. Consider the following vector solution to SDP 3.

\[ \bar{u}_i = \begin{cases} \\
\sqrt{\frac{2}{k}} e_i & \text{if } u \in C_1 \text{ and } i \leq k/2 \\
\sqrt{\frac{2}{k}} e_{(i-k/2)} & \text{if } u \in C_2 \text{ and } i > k/2 \\
0 & \text{otherwise}
\end{cases} \]
and

\[ I = \sqrt{\frac{2}{k}} \sum_{i=1}^{k/2} e_i. \]

It is easy to verify that this is a feasible solution with \( \alpha = 0 \). Therefore, SDP 3 has an unbounded integrality gap.