An analytical light distribution model in the optical system of a scintillation detector

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Abstract. The article describes an analytical light distribution model in the optical system of a scintillation detector. The model can be useful for scintillation detector development since it allows to make quick calculations with different parameters. Comparison of the analytical model and Geant4 calculation results has been done. The comparison of the analytical model calculation results and experimental measurements have been done. Both comparisons show model validity and a capability to be used in the research.

1. Introduction
One of the most important characteristic of the modern gamma-scanning equipment such as PET or SPECT is a spatial resolution which directly depends on detector design. Sizes, geometry, reflection surfaces of scintillation crystal, sizes and material of lightguide and light-sensitive elements location are parameters the right choice of which determines the main characteristics of the whole detector.

It is necessary to vary and select each of these parameters. Using mathematical modeling and then experiment verification is optimal for this problem. A widely used mathematical modeling method is Monte-Carlo method needs a lot of calculation power or a long time for calculation.

Because of this analytical light distribution model in optical system of scintillation detector has been developed (figure 1). The model allows to calculate a number of photons coming from point isotropic light source (scintillation) to square area (silicon photomultiplier as a light detector is considered). Also the model takes into account optical effects (refractions at interfaces, multiply reflections from crystal sides, light absorption in materials), allows to set scintillation crystal size and properties, number and properties of other materials (lightguide, optical glue etc.).

Also model can be used for gamma photoabsorption point estimation using, for example, Maximum likelihood estimation method [1].

Such problems have been solved in works [2], [3], but the developed models don’t take into account some optical effects. The work [2] hasn’t been taken interfaces refraction into account. There is no reflection from side faces contribution in work [3].

2. Assumptions in the model
The analytical model developed has next assumptions:

- Light source is isotropic.
- Light source is point.
Figure 1. The model scheme.

Figure 2. Number of optical photons falling on square area.

- Absolute absorption on the light detector.
- All materials are absolute homogeneous.
- The detector has a rectangle form. Other forms of crystal and lightguides are not considered.
- Multiple reflections only from the upper and side faces of the first material (scintillation crystal) are considered. For other materials only interfaces reflections are considered.

3. A number of optical photons falling on the square area

The base of the model is a function describes a number of optical photons falling on the square area. This function is derived from geometric optics. A point isotropic source which radiate $N_0$ optical photons and located at the point $(x_0, y_0, z_0)$ is considered (figure 2).

The square area has a linear size $a$ and coordinates $(x_d, y_d, z_d)$. The infinitesimal element of this area with square $dS_d$ is seen from the source at the elementary solid angle.

A number of optical photons falling on infinitesimal element of square area are defined by solid angle $d\Omega$ at which square area $dS_d$ is seen:

$$dN_{\text{point}} = \frac{d\Omega \cdot N_0}{4\pi};$$  \hspace{1cm} (1)

An elementary solid angle can be expressed as follows:

$$d\Omega = \frac{dS_d \cdot \cos \alpha}{R^2} = \frac{z_s \cdot dx_d \cdot dy_d}{\left( (z_s - z_d)^2 + (x_s - x_d)^2 + (y_s - y_d)^2 \right)^{3/2}};$$  \hspace{1cm} (2)

A full number of optical photons falling on the detector is whole area integral:
The equations system is composed. Each equation is a refraction law for corresponding interface (figure 4). Crosses interfaces of four rays from the light source to the light detector corners. So, it is needed to calculate trajectories (points where ray the light detector (if the light absorption in the material and reflection on the interfaces are neglected).

The reflection on all the interfaces is considered above. Effects on the interfaces (refraction and reflection) should be considered in the analytical model.

Refraction is considered by more a difficult way. The area which is a light detector image can be distinguished on the first interface (figure 3). The light from source passed through the area falls on the light detector (if the light absorption in the material and reflection on the interfaces are neglected).

Thus, knowing the light source (scintillation) coordinates, the square area (light detector) size and coordinates, the flash number of photons, it is possible to calculate how much optical photons fall on the square area using (3). It is assumed that the light source and the detector are inside homogeneous material and its borders don’t affect on the light distribution.

4. Interface reflection considering
The scintillation detector is often consist of not only crystal and light detector. There is an optical glue or grease, lightguide (in gamma-cameras Anger type) between the crystall and light detector. So optical effects on the interfaces (refraction and reflection) should be considered in the analytical model.

The reflection on interface between \( i \) and \( i + 1 \) material is considered by equation:

\[
K_i = \frac{1}{2} \left[ \left( \frac{\sin (\psi_i - \phi_i)}{\sin (\psi_i + \phi_i)} \right)^2 + \left( \frac{\tan (\psi_i - \phi_i)}{\tan (\psi_i + \phi_i)} \right)^2 \right];
\]  

where
- \( K_i \) – reflection coefficient on interface between \( i \) and \( i + 1 \) material;
- \( \phi_i \) – angle of incidence on interface between \( i \) and \( i + 1 \) material;
- \( \psi_i \) – angle of refraction on interface between \( i \) and \( i + 1 \) material.

The reflection on all the interfaces is considered above.

5. Refraction considering
Refraction is considered by more a difficult way. The area which is a light detector image can be distinguished on the first interface (figure 3). The light from source passed through the area falls on the light detector (if the light absorption in the material and reflection on the interfaces are neglected).

If the position and size of the light detector image on the first interface is determined, the refraction on the all other interfaces can be considered. So, it is needed to calculate trajectories (points where ray crosses interfaces) of four rays from the light source to the light detector corners.

To obtain a trajectory of one ray from the light source to some known point of the light detector equations system is composed. Each equation is a refraction law for corresponding interface (figure 4):
Thus there are $2^k$ are unknown. The point with index $k$ can’t be solved.

\begin{align}
\{ n_i \cdot \sin \theta_i &= n_{i+1} \cdot \sin \theta_{i+1}; \\
\cdots & \\
\{ n_i \cdot \sin \theta_1 &= n_2 \cdot \sin \theta_2; \\
\cdots &
\end{align}

(7)

Or expressing angles sinus by coordinates:

\begin{align}
\begin{cases}
    n_1 \cdot \frac{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}} \\
    \cdots \\
    n_i \cdot \frac{\sqrt{(x_{i-1}-x_i)^2+(y_{i-1}-y_i)^2}}{\sqrt{(x_{i-1}-x_i)^2+(y_{i-1}-y_i)^2+(z_{i-1}-z_i)^2}} \\
    \cdots \\
    n_{k-1} \cdot \frac{\sqrt{(x_{k-2}-x_{k-1})^2+(y_{k-2}-y_{k-1})^2}}{\sqrt{(x_{k-2}-x_{k-1})^2+(y_{k-2}-y_{k-1})^2+(z_{k-2}-z_{k-1})^2}} \\
\end{cases} &= n_k \cdot \frac{\sqrt{(x_{k-1}-x_k)^2+(y_{k-1}-y_k)^2}}{\sqrt{(x_{k-1}-x_k)^2+(y_{k-1}-y_k)^2+(z_{k-1}-z_k)^2}};
\end{align}

(8)

In this equation coordinates of the ray light and the interfaces intersections points $x_i, y_i (i = 1 : (k-1))$ are unknown. The point with index $k$ is a known point on the light detector. Coordinates $z_i$ are known because of they are interfaces coordinates, but width of each material is set in the model geometry definition. Thus there are $2(k-1)$ unknown and $k-1$ equations in system (8). The system obtained can’t be solved.
To determine a trajectory number of unknown in the system (8) is needed to be reduced. Transition to polar coordinates has been done for this. And it is useful to choose a new coordinate system in the way when \( r \) coordinate of source equals zero. Plane contains a ray trajectory is considered (figure 5). Coordinate \( \phi \) is constant at the plane.

The system of equations (8) in polar coordinates \((r, \phi, z)\) has a form:

\[
\begin{align*}
\frac{n_1}{\sqrt{r_1^2 + (z_i - z_1)^2}} &= \frac{n_2}{\sqrt{(r_2 - r_1)^2 + (z_2 - z_1)^2}}, \\
\vdots \\
\frac{n_i}{\sqrt{(r_i - r_{i-1})^2 + (z_i - z_{i-1})^2}} &= \frac{n_{i+1}}{\sqrt{(r_{i+1} - r_i)^2 + (z_{i+1} - z_i)^2}}, \\
\vdots \\
\frac{n_{k-1}}{\sqrt{(r_{k-1} - r_{k-2})^2 + (z_{k-1} - z_{k-2})^2}} &= \frac{n_k}{\sqrt{(r_k - r_{k-1})^2 + (z_k - z_{k-1})^2}}.
\end{align*}
\]

\(\text{(9)}\)

Thus \( r_i \) \((i = 1 : (k - 1))\) are unknown. A number of unknowns and number of equations are the same now and the system (9) can be solved using a numerical method (for example using Levenberg-Marquardt algorithm). The result of the system (9) solving is points which can be used for the trajectory construction. Also the obtained points are used for determining path length of the light ray inside of each optical environment. It is necessary for considering the light absorption in materials.

As a result, the algorithm of refraction considering looks like:

(i) Consider 4 rays from source to detector corners and for each:
   (a) Transit into the polar coordinate system.
   (b) Find the ray trajectory by solving equations system (9).
   (c) Return back to the rectangular coordinate system.

(ii) Find the center and size of light detector image on the first interface.

(iii) Using the equation for number of optical photons on the square area (3) determine how much photons fall on the detector image. The same number of optical photons (if materials absorption and interfaces reflection are neglected).

6. Considering light absorption in materials

Light absorption in materials is considered by the attenuation law:

\[
N_{\text{att},i} = N_i \cdot e^{-L_i / \lambda_i.}
\]

\(\text{(10)}\)

where \( N_i \) – number of photons which falls on the material; \( N_{\text{att},i} \) – number of photons which falls out of the material; \( L_i \) – path length in the material; \( \lambda_i \) – the material attenuation length; \( i \) – the material number.

The infinitely number of light rays pass through the square area image on each interface. So the assumption that each ray path length is not very different from path length of the ray falling to the center of the light detector is considered. Therefore the model doesn’t precisely considers refraction when linear size of detector’s optical system is great in comparison with its height.

7. Horizontal multireflections at the side facets of a scintillation crystal considering

Light reflection from the side faces is considered using virtual sources method. Virtual images of source which are mirror reflections from the side facets of scintillation crystal are created. A number of photons irradiated from the virtual sources is calculated from the reflection coefficient of side facets. As mentioned above it is assumed that the reflection coefficient of the side facets is independent from the incidence angle. One reflection is considered in this way.
Virtual sources are reflected from side facets of scintillation crystal to consider multiple reflections (figure 6).

On the figure marked:

- by green – real source position;
- by blue – real sources derived from 3 multireflections;
- by red – crystal facets.

![Figure 6. Light reflection from the side facets of the scintillation crystal considering.](image)

A number of optical photons falling on the light detector is calculated using the equation:

$$N_{sq} = \sum_{v=0}^{M} \sum_{h=0}^{M} \left( N_{point}(x_h, y_v, z) \cdot K_{refl, side}^{v+h} \right);$$ (11)

where

$$x_h = \begin{cases} x_s, & h = 0; \\ 2x_e - x_s, & h = 1; \\ 2x_l - x_s, & h = 2; \\ 2x_e - x_{h-3}, & h \neq 0, 1, h \text{ – odd}; \\ 2x_l - x_{h-1}, & h \neq 2, h \text{ – even}. \\ \end{cases}$$ (12)

- $N_{sq}$ – number of optical photons falling on light detector from all sources at one plate (without vertical reflections);
- $x_e$ – crystal right facet coordinate;
- $x_l$ – crystal left facet coordinate;
- $x_s$ – real source coordinate;
- $M$ - number of multireflections taken into account.

Values of $y_v$ are similar to (12).

8. Vertical multireflections considering

The plane contains a real light source and virtual sources is obtained while considering horizontal multiple reflections. It is necessary to reflect the obtained plane of sources respecting to the upper and lower crystal facet to consider vertical (along $z$ axis) and mixed (i.e. from side and upper facet in different variations) multiple reflections (figure 7).
It can be described by the equation:

\[ N_{total} = \sum_{q=0}^{T} \left[ N_{sq}(z+2qh_{cr}) \cdot K_{refl,up}^{q} + +N_{sq}(2h_{all}+2qh_{cr}-z) \cdot K_{refl,up}^{(q+1)} \right] \]  \hspace{1cm} (13)

where

- \( N_{total} \) – total number of photons which are falling on light detector;
- \( h_{cr} \) – crystal height;
- \( h_{all} \) – total height of all detector optical system (assuming \( z = 0 \) is coordinate of lower facet of lower material);
- \( K_{refl,up} \) – upper facet reflection coefficient.

Thus several plates are obtained each of which consists of point sources irradiating different number of optical photons. The contribution of each point source can be estimated using a function (3) which describes how many optical photons from the point source fall on the square area.

9. Comparing with Geant4
To verify the analytical model developed comparison of modeling results obtained from Monte-Carlo method using Geant4 [4] and calculation results using the analytical model was done.

The scintillation detector was considered as follows (figure 8): crystal CsI(Tl) (refraction index 1.79) with size 50x50x10 mm, glass with the same length and width but 1 mm height under the crystal (refraction index 1.52), optical grease 0.1 mm height (refraction index 1.51), silicon photomultiplier (SiPM) inside epoxy (refraction index 1.54) with 3x3 mm sensitive area. Three different cases of the reflection surfaces were modeled:

- All facets have total light absorption;
- All facets have mirror reflection with 0.95 reflection index;
- Side facets have total light absorption and upper facet has mirror reflection with 0.95 reflection index.

The crystal facet joined to glass always had no covering (reflecting or absorbing).

The point isotropic light source was placed on the different positions along X axis at fixed \( x, y \) coordinates (\( z = 0 \) was corresponded to light detector plane). The distribution of 10 000 000 photons were modeled using Geant4. Several positions with different \( z \) coordinate of source were considered.

Dependences of photons number on the detector versus distance between the light source and light detector at \( z = 10.3 \) mm coordinate of source are shown at figure 9, 10. Curves form of dependency are
Figure 8. Scheme of scintillation detector modeling.

Figure 9. A number of optical photons falling in the light detector versus distance to source. All the crystal facets have mirror reflection with 0.95 index.

Figure 10. A number of optical photons falling in the light detector versus distance to source. All the crystal facets have total absorption.

Table 1. The maximal difference between Geant4 modeling results (statistics used: 10 000 000 optical photons) and the analytical model developed relatively to Geant4.

| distance coordinate, mm | 10.3 | 9.3 | 8.3 | 6.3 |
|-------------------------|------|-----|-----|-----|
| All the facets have total absorption | 6.2% | 5.7% | 5.5% | 11.8% |
| The upper facet has mirror reflection | 4.5% | 7.2% | 7.2% | 9.0% |
| All the facets have mirror reflection | 2.7% | 2.8% | 1.9% | 3.2% |

the same for Geant4 and the analytical model. Maximum differences between Geant4 results and the analytical model results are shown at table 1.

The analytical model concurs with Geant4 numerically. The case when all reflecting surfaces have mirror reflectance is most precisely. Computer with CPU Intel Core i7 4770 was used for calculations. The analytical model algorithm was programmed on C++. Geant4 model used 5 thread and the analytical model - only one thread. Besides, the time needed for the calculations for the analytical model was ~ 0.07 seconds. Time needed for the calculations for the same detector using Geant4 was ~ 70 seconds. I.e. speed increase about 1000 times is obtained even without optimization with negligible accuracy losing.
10. Comparing to the experiment

The comparison of the analytical model calculations and experimental measurements was done. The experiment scheme is shown at figure 11.

Gamma source Co-57 was placed in the collimator with 1 mm hole diameter and 45 mm height. A scintillation detector was placed at 4 mm distance from collimator. The scintillation detector was consisted of: crystal CsI(Tl) with 20x20x10 mm size, glass with 1 mm height and silicon photomultiplier Sensl MicroFC-30035 [5]. Metal dusting drifted on the upper (nearest to gamma source) facet was provided mirror reflection. The side facets have black covering but in spite of this its reflected light was less than the upper facet. The sensitive area of SiPM was covered by epoxy. The optical grease was placed between the glass and epoxy.

The collimator with source were moved along X axis with fixed coordinate y. Mean SiPM signal amplitude were calculated at each source position.

The analytical model describes optical photons propagation in the optical system of scintillation detector. A photoabsorption point of each gamma was unknown in the experiment, so mean value (based on exponential law of gamma attenuation) of z coordinate 9.3 mm were used. Geometry of the scintillation detector were set according to the real detector in the experiment. Other parameters were chosen as follows:

- Crystal refraction index $n_{\text{crystal}} = 1.79$;
- Glass refraction index $n_{\text{glass}} = 1.52$;
- Optical grease refraction index $n_{\text{opt.grease}} = 1.51$;
- Epoxy refraction index $n_{\text{epoxy}} = 1.54$;
- Upper facet reflection index $K_{\text{refl.up}} = 0.9$;
- Side facets reflection index $K_{\text{refl.side}} = 0.5$;

Comparison of the results obtained is shown at figure 12.

Amplitude signal versus source position derived from the analytical model concurs with experimental within error.
11. Conclusion
The analytical model of the light distribution in the scintillation detector optical system has been developed. The model considers optical effects which allow to set a different scintillation detector configuration. The model correlates with Geant4 and allows to reduce the calculation time. Besides, the model developed agrees with the experiment within error.

In the future it is possible to improve model’s accuracy, to add diffuse reflection facets covering.

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References
[1] H. H. Barrett, C. J. H William 2009 Maximum-likelihood methods for processing signals from gamma-ray detectors *IEEE Transactions on Nuclear Science*, 56 72535
[2] Zhi Li, G. Vandersteen, P. Bruyndonckx 2009 3D nonlinear least squares position estimation in a monolithic scintillator block, *IEEE Nuclear science symposium conference record*
[3] M. Galasso, C. Borrazzo, A. Fabbri 2015 A scintillation light radial distribution model for monolithic crystal gamma cameras *Nuclear instruments and methods in physics research A*, 786 40-6.
[4] Geant4 official web page, https://geant4.web.cern.ch/geant4/
[5] C-Series Datasheet, Sensl, http://sensl.com/downloads/ds/DS-MicroCseries.pdf