Algebraic structures on graph associahedra

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**Abstract**
M. Carr and S. Devadoss introduced in ‘Coxeter complexes and graph associahedra’ [Topology Appl. 153 (1-2) (2006) 2155–2168] the notion of tubing on a finite simple graph \(\Gamma\), in the context of configuration spaces on the Hilbert plane. To any finite simple graph \(\Gamma\) they associated a finite partially ordered set, whose elements are the tubings of \(\Gamma\) and whose geometric realization is a convex polytope \(\mathcal{K}_\Gamma\), the graph-associahedron. For the complete graphs they recovered permutahedra, for linear graphs they got Stasheff’s associahedra, while for the empty graph they obtained the standard simplexes. The goal of the present work is to give an algebraic description of graph associahedra. We introduce a substitution operation on tubings, which allows us to describe the set of faces of graph-associahedra as a free object, spanned by the set of all connected simple graphs, under operations given via connected subgraphs. The boundary maps of graph-associahedra define natural derivations in this context. Along the way, we introduce a topological interpretation of the graph tubings and our new operations. In the last section, we show that substitution of tubings may be understood in the context of M. Batanin and M. Markl’s operadic categories.

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INTRODUCTION

Non-symmetric operads are algebraic structures defined by an infinite number of operations. They may be described, using natural substitution of planar rooted trees, as algebras over the monad of planar trees in a category of graded vector spaces (see, for instance, [18, 29] and [25]). This notion was generalized by V. Dotsenko and A. Koroshkin in [11], who defined another monad in the category of graded vector spaces; the algebras over this monad are called shuffle operads and play an important role in the description of Gröbner basis for operads. In [24], the notion of permutad was introduced, it describes a subclass of shuffle operads, using substitution on surjective maps between finite sets, instead of planar rooted trees.

The notion of operad has been generalized recently by M. Batanin and M. Markl in [3], where they introduced operadic categories, and defined operads in this context. Their construction covers a large range of algebraic structures which are not operads in the strict sense, but which are obtained by composing objects of certain type. As shown in [4], it is possible to pass from operadic categories Feynman categories, defined by R. Kaufman and B. Ward in [21].

The object of our work is graph associahedra. In [7] (and also see [9]), M. Carr and S. Devadoss introduced the notion of tubing on a finite simple graph $\Gamma$, and defined, for a fixed graph $\Gamma$ with $n$ nodes, an order on the set $\text{Tub}(\Gamma)$, of tubings of $\Gamma$. They proved that the geometric realization of $\text{Tub}(\Gamma)$ is always a convex polytope, whose dimension is $n - 1$.

1. When $\Gamma$ is the path, or linear graph $L_n$, tubings in $L_n$ are in a one-to-one correspondence with planar rooted trees, and their geometric realization is Stasheff associahedron of dimension $n - 1$.

2. When $\Gamma$ is the complete graph $K_n$, the tubings on $K_n$ describe the surjective maps between finite sets. In this case the polytope $\mathcal{K}K_n$ is the permutohedron.

3. When $\Gamma$ is the empty graph $\mathcal{C}_n$, with no edges, the polytope $\mathcal{K}\mathcal{C}_n$ is the standard simplex $\Delta^{n-1}$.

4. When $\Gamma$ is the cyclic graph $\mathcal{C}_{y_n}$, the M. Carr and S. Devadoss polytope is the cyclohedron.

The vector spaces spanned by the faces of certain families of graph associahedra have interesting algebraic structures. The space spanned by the faces of Stasheff associahedra, as well as the one spanned by the faces of permutohedra, have natural structures of graded conilpotent Hopf algebras, which have been widely studied (see, for instance, [1, 2, 8, 19, 23, 26]). It is well known that the space spanned by the faces of cyclohedra has a natural structure of a Hopf module over the Hopf algebra spanned by the faces of the Stasheff associahedra (see [16, 20]), and that the space spanned by the faces of standard simplices has a natural structure of free trialgebra spanned by one element, as proved in [8] and [23].

All these results concern graph associahedra of certain families of graphs, which satisfy very particular conditions. For instance, subgraphs of complete (respectively, linear) graphs, as well as their reconnected complements, are complete (respectively, linear) graphs. These features do not appear to be frequently found in families of graphs.

However, it is possible to show that the space spanned by all the faces of graph associahedra has a natural structure of pre-Lie coalgebra. When restricted to planar trees, this pre-Lie coproduct is the dual of a pre-Lie product defined by M. Gerstenhaber in [17], who defined it as a sum of binary products called graftings. Graftings are examples of substitution on trees: A substitution at an internal vertex of a tree $t$ consists in replacing the vertex by another tree with the same arity, and the result is a new tree which has the same number of leaves than $t$, but increases the number of internal vertices.
Substitution of planar trees appears in the description of the faces of Stasheff associahedra, whose boundary map is a derivation for the products $\circ_i$. From an algebraic point of view, the vector space spanned by the faces of Stasheff associahedra is the free non-symmetric operad generated by one element in each degree $n$, for $n \geqslant 0$, where the point in dimension 0 corresponds to the binary product of the operad.

Using the definition of symmetric operads described in [18], it is possible to associate to any planar tree a functor from the category of graded vector spaces to the category of vector spaces, and this functor induces an endofunctor $P$ of the category $\text{gVect}_K$ of graded vector spaces over a field $K$. The functor $P$ is a monad for the monoidal product given by composition, and the natural transformation $P \circ P \to P$ is defined in terms of the substitution on vertices of trees. A non-symmetric operad is an algebra over the monad $P$.

The notion of substitution arises quite naturally in the more general context of graph associahedra. Tubings are collections of connected subgraphs, called tubes, of a graph which satisfy certain compatibility conditions. It is possible to insert a tubing in a tube of another tubing, which extends the notion of replacing a vertex of a tree by another tree. Copying the construction of the functor $P$, we obtain an endofunctor $G$ of the category of graded vector spaces, but this functor is not a monad because composition is not always defined and there is no unit.

Our goal is to show that it is possible to give a general notion of substitution in the framework of Carr and Devadoss’s graph associahedra, in such a way that its restriction to linear graphs gives the monad defining non-symmetric operads and the restriction to complete graphs gives permutads.

These substitutions completely describe graph associahedra on connected simple finite graphs. The vector space $\text{Tub}$, spanned by the faces of all graph associahedra on connected graphs, is a free algebra on the vector space spanned by all finite connected simple graphs with a total order on its nodes, under binary operations $\circ_{\Gamma,t}$, where $\Gamma$ is a simple finite connected graph and $t$ is a tube of $\Gamma$, under certain relations. We construct a strict operadic category $\mathcal{O}_{CD}$ in the sense of M. Batanin and M. Markl, whose objects are ordered finite sets of tubings. Substitution on graph associahedra gives the operadic composition in this framework.

For certain families of graphs, tubings may be described as leveled trees of different types, as described in [12, 14, 15] and [5]. In these cases, substitution may be recovered as the grafting of trees.

Most of our work is done on graph associahedra of connected graphs, even if certain constructions may also be performed on any simple finite graph, not necessarily connected. The main example of graph associahedra for a non-connected graph is graph associahedra of the empty graph $C_n$, with $n$ nodes and no edge. In this case, the polytope $K C_n$ is the standard simplex of dimension $n - 1$. In [8], F. Chapoton defined a non-symmetric differential operad $\text{Trias}'$, and showed that the differential vector space spanned by the faces of standard simplices may be described as the free $\text{Trias}'$ algebra over the vector space of dimension one. We show that the vector space spanned by the faces of graph associahedra of finite simple graphs (not necessarily connected) is the free $\text{Trias}'$ algebra generated by $\text{Tub}$.

In the first section of the paper we recall the main definitions of graph associahedra and set the notations that we need in the sequel. We introduce a new interpretation of tubings as finite topologies which respect the graph structure. The last subsection is devoted to the description of the pre-Lie coproduct defined on the space of faces of graph associahedra.

Section 2 contains the definition of substitution on tubings, and its main properties, such as associativity.

Any finite simple graph may be obtained from the complete graph, removing some edges. At the level of tubings, it means that any tubing of a finite simple graph is obtained by the restriction
of some tubing on the complete graph with the same number of nodes. In Section 2.3, we study how substitution behaves with restriction.

In the third section, we give a complete description of graph associahedra in terms of generators and relations. We prove that the vector space $\text{Tub}$, spanned by tubings on connected simple finite graphs, is generated by the set of all connected simple finite graphs under the operations $\sigma_{\Gamma, i}$, for $i \in \text{Tub}(\Gamma)$. We also introduce a signed version of the products $\sigma_{\Gamma, i}$ in such a way that the boundary map of graph associahedra is determined by the condition of being a derivation for them.

In Section 4 we prove that the vector space spanned by the set of all tubings on finite simple graphs (not necessarily connected), denoted $\text{DTub}$, is equipped with the structure of a Trias’ algebra and that it is the free Trias’ algebra spanned by the vector space $\text{Tub}$.

Finally, in Section 5, we show that the restriction of substitution to linear graphs gives grafting and substitution on planar trees, while when we consider the substitution on the family of all complete graphs, we get the composition of surjective maps, as described in [24]. We recall M. Batanin and M. Markl’s definition of operadic category and construct the operadic category $\mathcal{O}_{CD}$ whose objects are finite families of tubings. We prove that the substitution of tubings gives the appropriate framework to define $\mathcal{O}_{CD}$ operads. Permutads were studied by M. Markl in [28] as colored operads, and also in the largest context of the operadic category $\text{Per}$. When we restrict substitution of graph associahedra to complete graphs, we obtain permutohedra, the full subcategory of $\mathcal{O}_{CD}$ whose objects are tubings on complete graphs is the category $\text{Per}$.

All our work deals with finite graphs, equipped with a total order on its set of nodes. However, the Carr and Devadoss polytope $\mathcal{K}\Gamma$ does not depend on the way of numbering the nodes of a graph $\Gamma$. We need the total order on the set of nodes of a graph $\Gamma$ to define the boundary map of $\mathcal{K}\Gamma$ in terms of reconnected complements and substitution on graphs, and in the definition of the operadic category $\mathcal{O}_{CD}$.

1 | PRELIMINARIES ON GRAPH ASSOCIAHEDRA

1.1 | Graphs

All the vector spaces considered in the present work are over a field $\mathbb{K}$. For any set $X$, we denote by $\mathbb{K}[X]$ the vector space spanned by $X$. For any positive integer $n$, we denote by $[n]$ the set $\{1, ..., n\}$. The graphs $\Gamma$ we deal with in the paper, except those in Section 4, are simple finite and connected, we also assume that their sets of nodes are totally ordered.

Let $\Gamma$ be a graph whose set of nodes $\text{Nod}(\Gamma)$ is finite and totally ordered, we denote by $\text{Edg}(\Gamma)$ the set of its edges. Any edge $e \in \text{Edg}(\Gamma)$ is identified with its (unordered) pair of extremities $e = \{v, w\}$.

Note that for a graph $\Gamma$ with $n$ nodes, the set of its nodes $\text{Nod}(\Gamma)$ is totally order if, and only if, there exists a bijective map from $\text{Nod}(\Gamma)$ to the set $[n]$ which preserves the order. So, we may identify the sets $\text{Nod}(\Gamma)$ and $[n]$. We denote by $\Gamma_{\emptyset}$ the graph whose set of nodes is the empty set.

**Definition 1.1.1.** For any simple graph $\Gamma$ and any subset $s$ of $\text{Nod}(\Gamma)$, the graph induced by $s$ is the maximal subgraph of $\Gamma$ whose set of nodes is $s$. We denote it by $\Gamma_s$.

**Notation 1.1.2.** Let $\Gamma$ be a graph with $\text{Nod}(\Gamma) = [n]$. For any subset $s = \{j_1 < \cdots < j_k\} \subseteq [n]$ and any integer $m \in \mathbb{Z}$ such that $-j_1 < m \leq n - j_k$, we denote by $s + m$ the subset $\{j_1 + m, ..., j_k + m\} \subseteq [n]$. 
For any simple graph $\Gamma$ satisfying that its set of nodes is totally ordered, the order of $\text{Nod}(\Gamma)$ induces a total order on any subset $s \subseteq \text{Nod}(\Gamma)$. Therefore, the graph $\Gamma_s$ is also simple finite and the set of its nodes is totally ordered.

**Definition 1.1.3.** A tube is a set of nodes $t$ of $\Gamma$ satisfying that the subgraph $\Gamma_t$ of $\Gamma$ is connected. When $\Gamma$ is connected, the set of all nodes of $\Gamma$ determines a tube called the universal tube of $\Gamma$, denoted $t_\Gamma$.

For any tube $t \in \Gamma$, the reconnected complement $\Gamma_t^*$ is the graph whose set of nodes is $\text{Nod}(\Gamma) \setminus \{t\}$, and whose edges are determined by the pairs of nodes $\{v, w\}$ satisfying one of the following conditions:

- (a) the edge $\{v, w\}$ belongs to $\text{Edg}(\Gamma)$,
- (b) there exist nodes $u$ and $u'$ in $t$ such that $\{v, u\}$ and $\{w, u'\}$ belong to $\text{Edg}(\Gamma)$.

**Remark 1.1.4.** If $\text{Nod}(\Gamma)$ is totally ordered, then the order of $\text{Nod}(\Gamma)$ induces a total order on the set of nodes of the reconnected complement $\Gamma_t^*$. Therefore, whenever $\Gamma$ is a simple finite and connected graph, whose set of nodes is totally ordered, both graphs $\Gamma_t$ and $\Gamma_t^*$ are also simple finite and connected, and their sets set of nodes are totally ordered, for any tube $t$ in $\Gamma$.

**Example 1.1.5.** Let $\Gamma$ be the graph

![Graph](image)

where $t = \{3, 4, 5\}$. Here, from left to right, are $\Gamma_t$, its renumbered version, $\Gamma_t^*$ and its renumbered version:

![Renumbered Graphs](image)

When $\Gamma = \Gamma_1 \coprod \cdots \coprod \Gamma_k$ is the disjoint union of connected graphs $\Gamma_i$, for $1 \leqslant i \leqslant k$, and $t$ is a tube of $\Gamma_{i_0}$, the reconnected complement $\Gamma_t^*$ is the graph $\Gamma_1 \coprod \cdots \coprod (\Gamma_{i_0})_t^* \coprod \cdots \coprod \Gamma_k$.

### 1.2 Basic constructions on Carr and Devadoss’ graph associahedra

We give a brief description of graph associahedra as introduced by M. Carr and S. Devadoss in [7]; for a complete description of their construction and further details we refer to their work.

**Definition 1.2.1.** Let $t$ and $t'$ be two different tubes in a simple finite graph $\Gamma$. We say that:

1. $t$ and $t'$ are nested if $t \subseteq t'$. 
(2) \( t \) and \( t' \) are far apart if \( t \cup t' \) is not a tube in \( \Gamma \), that is, the induced subgraph of the union is not connected (equivalently none of the nodes of \( t \) are adjacent to a node of \( t' \)).

(3) \( t \) and \( t' \) are compatible if they are either nested or far apart.

When \( t \) and \( t' \) are not nested, and \( t \cup t' \) is connected in \( \Gamma \) we say that \( t \) and \( t' \) are linked. If \( t \) and \( t' \) are linked there exists an edge \( e \in \text{Edg}(\Gamma) \) which has one extremity in \( t \) and the other one in \( t' \).

Definition 1.2.2. Let \( \Gamma \) be a simple connected finite graph. A tubing in \( \Gamma \) is a non-empty family \( T = \{t_i\}_{1 \leq i \leq k} \) of tubes, which contains the universal tube \( t_\Gamma \), and such that every pair of tubes in \( T \) is compatible.

We assume that \( \text{Tub}(\Gamma_\emptyset) \) has a unique element, the empty tubing \( T_\emptyset \).

A tubing of a connected graph covers the set of nodes of that graph by virtue of containing the universal tube, but no collection of proper tubes can cover the set of nodes. A tubing \( T \) in a connected graph \( \Gamma \) such that \( \text{Nod}(\Gamma) = [n] \), contains at most \( n \) tubes.

A tubing with \( k \) tubes is called a \( k \)-tubing on \( \Gamma \), for \( 1 \leq k \leq n \). We denote by \( \mathbb{L}(T) \) the number of tubes of a tubing \( T \).

We have many examples pictured throughout, in which tubes are shown as circled subgraphs. For simplicity the universal tube is not shown in the pictures of tubings, just assumed to be included. The only example pictured of a non-tubing is in Remark 1.2.4.

Notation 1.2.3. Let \( \text{Tub}(\Gamma) \) denote the set of all tubings of a connected graph \( \Gamma \).

(1) For any tubing \( T \in \text{Tub}(\Gamma) \), we denote by \( \overline{T} := T \setminus \{t_\Gamma\} \) the set of proper tubes of \( T \).

(2) A maximal tube of a tubing \( T \in \text{Tub}(\Gamma) \) is a tube \( t \in \overline{T} \) which is not contained in any other tube of \( \overline{T} \). We denote by \( \text{Max}(T) \) the set of maximal tubes of \( T \).

(3) In order to simplify notation, we denote by \( T_\Gamma \) the tubing whose unique tube is \( t_\Gamma \).

(4) We say that two tubings \( T \) and \( T' \) of \( \Gamma \) are compatible if the union of \( T \) and \( T' \) is also a tubing of \( \Gamma \).

Remark 1.2.4. Let \( T \) be a tubing in a graph \( \Gamma \).

(1) For any tube \( t \) in \( \Gamma \), the restriction of \( T \) to \( \Gamma_t \) is either a tubing of \( \Gamma_t \) or the empty tube. We denote by \( T|_t \) the union of the restriction of \( T \) to \( t \) and the universal tube \( t_\Gamma \). When \( t \in T \), the tubes of \( T|_t \) are all the tubes \( t' \in T \) such that \( t' \subseteq t \).

(2) For any tube \( t \) in \( \Gamma \) such that \( \{T, t\} \) is a tubing in \( \Gamma \), \( T \) induces a tubing \( T^*_t \) in the reconnected complement \( \Gamma^*_t \). A tube \( t' \) of \( T^*_t \) is either a tube \( t' \) of \( T \) such that \( t' \cap t = \emptyset \), or \( t' \) is the restriction of a tube which contains \( t \) to the set of nodes which do not belong to \( t \).

Note that the requirement for \( \{T, t\} \) to be a tubing in \( \Gamma \) is necessary for the definition of the tubing \( T^*_t \). Consider, for instance, the graph

![Graph with tubing](image)

with the tube \( t = \{2, 4\} \). The tubing \( T = \{\{1\}, \{3, 5\}, \{1, 2, 3, 4, 5\} \} \) does not induce a tubing on \( \Gamma^*_t \).
Definition 1.2.5. Let $\Gamma$ be a finite simple graph and let $T$ be a tubing of $\Gamma$. The height of $T$ is the maximal integer $h$ such that there exists a collection of tubes $\emptyset \subset t_1 \subset t_2 \subset \cdots \subset t_h \subset t_{\Gamma}$ with $t_i \in T$, for $1 \leq i \leq h$. We denote it by $h(T)$.

1.2.1 Topological interpretation

We show here that the tubings of connected graphs are precisely topological bases which (1) are comprised of tubes and (2) generate a topology on nodes which respects the connectivity of the graph, and its reconnected complements. Recall that a space is topologically connected if there are no two disjoint open sets which cover the space. Recall that a subspace topology is formed by all intersections of any open set with the subspace. Recall that a topological basis on a set $X$ is a collection of subsets of $X$, which covers $X$, and such that for any two basis elements their intersection $I$ is itself covered by basis elements that are subsets of $I$. A basis on $X$ generates a unique topology on $X$.

Let $T$ be a collection of tubes on a connected graph $\Gamma$.

Theorem 1.2.7. $T$ is a tubing on $\Gamma$ if, and only if, $T$ is a topological basis on $\text{Nod}(\Gamma)$ such that the following connectivity condition holds: For all pairs $\{v, v'\}$ of nodes, if $\{v, v'\}$ is an edge in $\Gamma$ or in $\Gamma^*_t$ for some tube $t \in T$, then $\{v, v'\}$ is connected as a subspace in the topology generated by $T$.

Proof. First we show that if $T$ is a basis obeying our connectivity condition, then $T$ is a tubing. We use strong induction on the number of nodes in $\Gamma$. The case for one node is trivial. We assume the implication holds for $k < n$ nodes, and prove the case of $\Gamma$ with $n$ nodes. To show that the connectivity condition implies compatibility of tubes, we show the contrapositive: If tubes $s, t \in T$ are incompatible, then there is an edge $\{v, v'\}$ (either in $\Gamma$ or a reconnected complement) that is disconnected as a subspace. There are two cases when $s, t \in T$ are incompatible:

(i) $s$ and $t$ are linked. Taking the nodes that are connected by the linking edge to be $u \in s$ and $v' \in t$, we see that $\{v', v'\}$ has the sets $\{v\}$ and $\{v'\}$ in its subspace topology, so it is disconnected.

(ii) $s$ and $t$ are intersecting (not nested). Then since $T$ is a basis, $u = s \cap t$ must be covered by subsets of $u$ that are tubes in $T$. We may assume that $u$ is connected without loss of generality (when $u$ is not connected we may choose instead any single connected component of $u$). Thus the tubes in $T|_u$ form a topological basis for $u$, which inherits the connectivity condition. Thus by the inductive assumption, these tubes in $u$ will be compatible. Therefore $u$ itself must be a tube in $T$ in order to achieve a cover. Then letting $v \in s - t$ be connected by an edge to $u$ and $v' \in t - s$ be connected by an edge to $u$, we have that $\{v, v'\}$ is an edge in $\Gamma^*_u$ and simultaneously that the subspace topology of $\{v, v'\}$ is disconnected.

Therefore our basis elements of $T$ are tubes which are compatible. Since $T$ is a basis, the tubes cover $\text{Nod}(\Gamma)$, but since they are compatible this means the universal tube must be included as an element of $T$, so our basis is a tubing.

Finally we show the other implication: If $T$ is a tubing then $T$ is a topological basis that meets our connectivity condition. First, $T$ covers $\text{Nod}(\Gamma)$ since the universal tube is in $T$ by definition. Second, pairwise intersections of tubes are covered by the tubes inside them—since the only intersections of tubes are empty or (nested) tubes in $T$. We show that this topological basis obeys the connectivity condition. Again we prove the contrapositive: We show that if the connectivity condition does not hold, then there is a pair of incompatible tubes. Again there are two cases:
(i) Let \( \{v, v'\} \) be an edge in \( \Gamma \) and assume that the subspace of those two nodes is disconnected. Therefore there must be a pair of disjoint tubes in \( T \), one containing \( v \) and the other \( v' \), which are thus linked, so incompatible.

(ii) Assume \( \{v, v'\} \) is not an edge in \( \Gamma \), but \( \{v, v'\} \) is an edge of \( \Gamma_t^\ast \) for some tube \( t \). Therefore both \( v \) and \( v' \) are connected by an edge to \( t \). Also assume that the subspace of those two nodes is disconnected. Then there must be a pair of tubes in \( T \) one containing \( v \) and the other \( v' \) (but neither containing both nodes). If either tube in the pair is incompatible with \( t \), we are done. If not, then both tubes must contain \( t \), and thus are intersecting but not nested, so incompatible with each other.

For a finite topological space, being connected implies being path-connected. Therefore we have the easy corollary to Theorem 1.2.7:

**Corollary 1.2.8.** If \( T \) is a tubing on \( \Gamma \), then any pair of nodes connected by a path (a sequence of edges in the graph \( \Gamma \)) are also connected by a topological path (continuous map from \([0,1]\) to \( \text{Nod}(\Gamma) \)) in the topology generated by \( T \). Moreover, given the graphical path, there exists a topological path whose range is precisely the nodes in the graphical path.

When a graph is not connected, but rather contains multiple connected components, there are two existing definitions of tubing. In this paper, we follow [7] by requiring some of the connected components to not be themselves elements of the tubing:

**Definition 1.2.9.** Let \( \Gamma = \Gamma_1 \coprod \cdots \coprod \Gamma_k \) be a non-connected simple graph, where \( \Gamma_i \) is a connected component, for \( 1 \leq i \leq k \) and \( 2 \leq k \). A tubing \( T \) of \( \Gamma \) is a list \( (T_1, \ldots, T_k) \), satisfying that:

1. there exists a subset \( \{i_1 < \cdots < i_s\} \subseteq [k] \), with \( 1 \leq s \leq k \), such that \( T_{i_j} = \overline{W}_{i_j} \), with \( W_{i_j} \in \text{Tub}(\Gamma_{i_j}) \);
2. for \( i \notin \{i_1, \ldots, i_s\} \), \( T_i \in \text{Tub}(\Gamma_i) \).

**Remark 1.2.10.** Let \( \Gamma \) be a graph.

1. For any tube \( t \) in \( \Gamma \) and any tubing \( T \in \text{Tub}(\Gamma_t) \), the inclusion the \( \Gamma_t \hookrightarrow \Gamma \) defines a tubing of \( \Gamma \). A tube in this tubing is either the image of some tube \( s \in T \), or the universal tube \( t_{\Gamma_t} \).
2. For any subgraph \( \Omega \) of \( \Gamma \), satisfying that \( \text{Nod}(\Omega) = \text{Nod}(\Gamma) \), there exists a natural surjective map \( \text{res}^\Gamma_{\Omega} : \text{Tub}(\Gamma) \twoheadrightarrow \text{Tub}(\Omega) \). Given three graphs \( \Theta, \Omega \) and \( \Gamma \) with the same number of nodes and such that \( \text{Edg}(\Theta) \subseteq \text{Edg}(\Omega) \subseteq \text{Edg}(\Gamma) \), it is immediate to verify that the composition \( \text{res}^\Omega_{\Theta} \circ \text{res}^\Gamma_{\Omega} \) coincides with the application \( \text{res}^\Gamma_{\Theta} \).

For instance, if \( \Gamma = K_3 \) and \( \Omega = L_3 \) is the linear graph, as pictured above,

\[
\begin{array}{ccc}
K_3 & 3 & L_3 \\
1 & 2 & 1 \quad 2 \quad 3
\end{array}
\]

then the tubings \( T = \{\{1\}, \{1, 3\}, \{1, 2, 3\}\} \) and \( T' = \{\{3\}, \{1, 3\}, \{1, 2, 3\}\} \) on \( K_3 \) both map to the same tubing: \( \{\{1\}, \{3\}, \{1, 2, 3\}\} \) on \( L_3 \).
Definition 1.2.11. For any graph $\Gamma$, the set $\text{Tub}(\Gamma)$ of tubings of $\Gamma$ is partially ordered by the relation:

$$T \leq T' \text{ if } T \text{ is obtained from } T' \text{ by adding compatible tubes.}$$

In [7], M. Carr and S. Devadoss proved that for any simple finite graph $\Gamma$, the partially ordered set of tubings is equivalent, as a poset, to the face poset of a simple polytope of dimension $n - 1$. A simple polytope of dimension $n - 1$ is defined to have exactly $n - 1$ edges adjacent to each vertex. The vertices of the polytope $\mathcal{K}_\Gamma$ coincide with the subset $\text{MTub}(\Gamma)$ of minimal tubings, while the universal tubing $T_\Gamma$ corresponds to the $(n-1)$-cell of $\mathcal{K}_\Gamma$.

1.3 Graph associahedra for linear graphs and for complete graphs

We describe the tubings of two families of connected simple finite graphs: the linear graph $L_n$ and the complete graph $K_n$, for $n \geq 1$.

a) Linear graphs and planar rooted trees. For $n \geq 1$, the linear graph $L_n$ is the simple finite connected graph whose set of nodes is $[n]$ and whose set of edges is $\text{Edg}(L_n) = \{(i, i + 1) \mid 1 \leq i < n\}$. We assume $L_0 := \Gamma_\emptyset$.

Any tube $t$ in $L_n$ is uniquely determined by a sequence of consecutive integers $t = \{i + 1, \ldots, i + r\}$, for some $0 \leq i < n$ and $1 \leq r \leq n - i$. The graph $(L_n)_t$ is the linear graph $L_r$, while the graph $(L_n)^*_t$ is the linear graph $L_{n-r}$. To describe the tubings of $L_n$ we need some basic definitions on planar rooted trees.

For $n \geq 2$, let $\mathcal{T}_r_n$ be the set of planar rooted trees with $n$ leaves satisfying that each vertex has a least two incoming edges. For $n = 1$, $\mathcal{T}_r_1$ contains a unique element $c_1$, the tree with no vertex and one leaf. We denote by $c_n \in \mathcal{T}_r_n$ the planar tree which has a unique vertex, the root, for $n \geq 2$.

Notation 1.3.1. Given two planar trees $w \in \mathcal{T}_r_n$ and $u \in \mathcal{T}_r_m$, with the leaves of $w$ numbered, from left to right, by $1, 2, \ldots, n$. The element $w \circ_i u \in \mathcal{T}_r_{n+m-1}$ is the tree obtained by joining the root of $u$ to the $i$-th leaf of $w$, for $1 \leq i \leq n$.

Remark 1.3.2. Let $w \in \mathcal{T}_r_n$ be a planar rooted tree, with $n \geq 2$. If $w \neq c_n$, there exist a unique pair of integers $1 \leq s \leq r$ and a unique family of planar rooted trees $w^1, \ldots, w^s$ satisfying that

$$w = (((c_r \circ_{i_1} w^1) \circ_{i_2} w^2) \ldots) \circ_{i_s} w^s,$$

where $w^i \in \mathcal{T}_r_{n_i}$, with $n_j \geq 2$, for a unique family of integers $i_1, \ldots, i_s$ satisfying that $1 \leq i_1 < i_1 + n_1 \leq i_2 < i_2 + n_2 \leq \ldots \leq i_s < i_s + n_s \leq n$.

For any internal vertex $a$ of $w$, let $w_a$ denote the subtree of $w$ whose root is $a$. If $w_a \in \mathcal{T}_r_m$, there exist a unique integer $1 \leq i \leq n - m + 1$ and a unique tree $v_a \in \mathcal{T}_r_{n-m+1}$ such that $w = v_a \circ_i w_a$.

Definition 1.3.3. Let $w$ be an element in $\mathcal{T}_r_n$ and let $a$ be an internal vertex of $w$, with $k$ incoming edges. For a tree $u \in \mathcal{T}_r_k$, the substitution of $u$ in $w$ at the vertex $a$ is the tree $w \circ_a u \in \mathcal{T}_r_n$ obtained by replacing the subtree $w_a$ in $w$ by the tree

$$w \circ_a u := v_a \circ_i (((u \circ_{i_1} w^1) \circ_{i_2} w^2) \ldots) \circ_{i_s} w^s),$$
where \( w = v_a o_1 w_a \), and \( w_a = (((c_k o_1 w^1) o_{i_2} w^2) \ldots) o_{i_s} w^s \), for \( 0 \leq s \leq k \) and unique subtrees \( w^1, \ldots, w^s \) of \( w_a \).

Note that for \( w_a = c_k \) and \( w = v_a o_i c_k \), we have \( w o_a u = v_a o_i u \).

For \( n \geq 1 \), we define the map \( \rho_n \) from the set \( \text{Tub}(L_n) \) to the set \( \mathcal{T}r_{n+1} \) of planar rooted trees with \( n+1 \) leaves, recursively on the height \( h(T) \) of \( T \in \text{Tub}(L_n) \).

If \( h(T) = 0 \), then \( T = T_{L_n} \) and \( \rho_n(T_{L_n}) := c_{n+1} \), for \( n \geq 0 \), where \( L_0 = \emptyset \).

If \( h(T) \geq 1 \), then \( \text{Maxt}(T) = \{t^1, \ldots, t^s\} \), for some \( s \geq 1 \). As the tubes \( t^i \) and \( t^j \) are far apart for \( i \neq j \), we assume that \( t^j = \{i_j + 1, \ldots, i_j + r_j\} \), for \( 1 \leq j \leq s \), are such that \( 0 \leq i_1 < i_1 + r_1 < i_2 < \ldots < i_s < i_s + r_s \leq n \).

As \( h(T|t^i) \leq h - 1 \), for \( 1 \leq i \leq s \), the element \( \rho^{-1}_{r_i}(T|t^i) \in \mathcal{T}r_{r_i+1} \) is well defined, for \( 1 \leq i \leq s \).

The tree \( \rho_n(T) \) is
\[
\rho_n(T) := (((c_{n-r+1} o_{i_1+1} \rho_{r_1}(T|t^1)) o_{i_2+1} \rho_{r_2}(T|t^2)) \ldots) o_{i_s+1} \rho_{r_s}(T|t^s),
\]
where \( r = r_1 + \ldots + r_s \).

**Lemma 1.3.4.** The map \( \rho_n \) is bijective, for \( n \geq 1 \).

**Proof.** We proceed by induction on \( n \). For \( n = 0, 1 \) the result is immediate.

For \( n \geq 2 \), we assume that \( \rho_r \) is bijective and its inverse \( \rho^{-1}_r \) is defined for any \( 1 \leq r < n \). Clearly, \( \rho^{-1}_{n+1}(c_{n+1}) = T_{L_n} \) is the universal tubing.

Given a planar rooted tree \( w \in \mathcal{T}r_{n+1} \) different from \( c_{n+1} \). Remark 1.3.2 states that \( w = (((c_r o_{i_1} w^1) o_{i_2} w^2) \ldots) o_{i_s} w^s \), where \( w^j \in \mathcal{T}r_{r_n} \), for \( 1 \leq j \leq s \).

By a recursive argument, we assume that \( \rho^{-1}_{n-1}(w^l) \in \text{Tub}(L_{n-1}) \) is defined, for \( 1 \leq l \leq s \).

Let \( r_l := n_l - 1 \) and \( j_l := i_l - 1 \), for \( 1 \leq l \leq s \). We have that \( 0 \leq j_1 < j_1 + r_1 < j_2 < \ldots < j_s + r_s \leq n \) and that \( n + 1 = r_1 + \ldots + r_s + r \).

For \( 1 \leq l \leq s \), let \( t^j := \{i_j + 1, \ldots, i_j + r_j\} \subseteq \text{Nod}(L_n) \). Define \( \rho^{-1}_n(w) \) as the unique tubing of \( L_n \) satisfying that \( \text{Maxt}(\rho^{-1}_n(w)) = \{t^1, \ldots, t^s\} \) and that \( T|t^l = \rho^{-1}_r(w^l) \), for \( 1 \leq l \leq s \).

A straightforward calculation shows that \( \rho^{-1}_n \) is the inverse map of \( \rho_n \), which ends the proof. \( \square \)

In [7], M. Carr and S. Devadoss showed that the polytope \( \mathcal{KL}_n \) is precisely the associahedron.

**Remark 1.3.5.** Given \( T \in \text{Tub}(L_n) \) an easy recursive argument shows that there exists a bijective map between \( T \) and the internal vertices of \( \rho_n(T) \) which maps any \( t \in T \) to an internal vertex \( a_t \), in such a way that the number of incoming edges of \( a_t \) minus one coincides with the number of nodes in \( t \) which do not belong to any other tube \( t' \not\subset t \) in \( T \).

b) **Complete graphs and surjective maps** For the complete graph \( K_n \), with set of nodes \( [n] \) and edges \( \text{Edg}(K_n) = \{(i, j) \mid 1 \leq i < j \leq n\} \), any pair of nodes is an edge. Therefore, any tubing \( T \) of \( K_n \) may be identified with a sequence of subsets \( t^1 \not\subset t^2 \not\subset \ldots \not\subset t^r = t_{K_n} \).

As shown in [9], there exists a natural bijection between tubings of \( K_n \) and surjective maps \( x : [n] \longrightarrow [r] \), for \( 1 \leq r \leq n \), given by:
\[
x_{t^j}(i) = j, \text{ for } i \in t^j \setminus t^{j-1},
\]
for \( T = \{t^1 \not\subset t^2 \not\subset \ldots \not\subset t^r\} \), where \( t^0 = \emptyset \).
The faces of dimension $k$ of the permutohedron $\mathcal{K}_n$ correspond to the tubings $T \in \Tub(K_n)$ with $L(T) = n - k$.

In this case, the polytope $\mathcal{K}_n$ is the permutohedron of dimension $n - 1$.

Any simple graph $\Gamma$ with $n$ nodes, may be obtained from $K_n$ by eliminating some edges. The surjective map $\text{res}_T^{K_n}$ describes the way faces of the permutohedron are contracted in order to obtain the polytope $\mathcal{K}_T$.

The following result is the original characterization of faces of the graph associahedra from Carr and Devadoss (see Theorem 2.9 of [7]).

**Theorem 1.3.6.** For any graph $\Gamma$ and any proper tube $t$ in it, the facet of $\mathcal{K}_\Gamma$ corresponding to the tubing $\{t\}$ is isomorphic to the product $\mathcal{K}_\Gamma_t \times \mathcal{K}_{\Gamma^*}$.

We refer to [7, 9] and [10] for more descriptions of the polytopes $\mathcal{K}_\Gamma$.

### 1.4 The pre-Lie coproduct on tubings

We denote by $\Tub$ the $\mathbb{K}$-vector space spanned by all the pairs $(\Gamma, T)$, where $\Gamma$ is a finite connected simple graph, equipped with a total order on the its set of nodes, and $T$ is a tubing of $\Gamma$.

The *degree* of an element $(\Gamma, T)$ is the number of nodes of the graph $\Gamma$. Let $\Tub_n$ be the vector space spanned by homogeneous elements of degree $n$, for $n \geq 1$. We denote by $\Tub^+$ the direct sum $\mathbb{K} \oplus \Tub$, where $\mathbb{K}$ is the subspace of degree 0.

Let us recall the definition of right pre-Lie algebra (also called right symmetric algebra). For a complete description of the properties and examples of this type of algebras we refer to D. Burde’s work [6].

**Definition 1.4.1.** A right pre-Lie, or right symmetric, algebra is a vector space $S$ equipped with a binary product $\cdot : S \otimes S \rightarrow S$, satisfying that:

\[ x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z, \]

for any elements $x, y, z \in S$.

Clearly, any associative algebra is a right pre-Lie algebra. Moreover, the binary operation $[x, y] := x \cdot y - y \cdot x$ defines a Lie bracket on $S$.

Dualizing the notion of right pre-Lie algebra, we get that a right pre-Lie coalgebra is a vector space $C$ equipped with a coproduct $\Delta_\cdot : V \rightarrow V \otimes V$ satisfying the relation:

\[ (\text{Id}_C \otimes \Delta_\cdot - \Delta_\cdot \otimes \text{Id}_C) \circ \Delta_\cdot (x) = (\tau \otimes \text{Id}_C) \circ (\text{Id}_C \otimes \Delta_\cdot - \Delta_\cdot \otimes \text{Id}_C) \circ \Delta_\cdot (x), \]

for any $x \in C$, where $\tau$ denotes the twisting $\tau(x \otimes y) := y \otimes x$ on $C \otimes C$.

Theorem 1.3.6 induces the following definition.

**Definition 1.4.2.** Let $T$ be a tubing of a connected graph $\Gamma$, define $\Delta_\cdot(T) \in \Tub^+ \otimes \Tub^+$ as follows:

\[ \Delta_\cdot(T) := \sum_{i \in I} T_i \otimes T_i^*, \]
where the sum is taken over all the tubes $t$ in $T$ and on the empty tube, which is identified with the unit $1_\mathbb{K}$ of $\mathbb{K}$. We define completely the coproduct $\Delta_\ast$ on the vector space $\text{Tub}^+$ by setting that $\Delta_\ast(1_\mathbb{K}) = 1_\mathbb{K} \otimes 1_\mathbb{K}$, and extending it by linearity.

**Lemma 1.4.3.** The coproduct $\Delta_\ast$ defines a right pre-Lie coalgebra structure on $\text{Tub}$.

**Proof.** It is easy to see that

$$(\text{Id}_C \otimes \Delta_\ast - \Delta_\ast \otimes \text{Id}_C) \circ \Delta_\ast(T) = \sum_{t_1, t_2 \in T} T|_{t_1} \otimes T|_{t_2} \otimes T_{\{t_1, t_2\}}^\ast,$$

where the sum is taken over all pairs of non-empty tubes $(t_1, t_2)$ of $T$ satisfying that $t_1 \cap t_2$ is empty.

If the pair $(t_1, t_2)$ satisfies that $t_1 \cap t_2 = \emptyset$, then the pair $(t_2, t_1)$ also satisfies the condition, which implies the result. \qed

## 2 | SUBSTITUTION ON TUBINGS

Let $\text{gVect}_\mathbb{K}$ denote the category of graded vector spaces over $\mathbb{K}$. In [18] the authors introduced algebraic operads as algebras over a monad $\mathcal{P}$ in the category of endofunctors of the category $\text{gVect}_\mathbb{K}$. The functor is defined in terms of rooted trees, in such a way that the natural transformation $\mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$, where $\circ$ denotes the composition, is described by the insertion of a tree $u$ in an internal vertex of a tree $w$.

Non-symmetric operads are obtained in the same way, replacing rooted trees by planar rooted ones (see [29] and [25]).

In Definition 1.3.3 we described the insertion of a tree $u$ in an internal vertex of another $w$. Our aim is to extend the definition of substitution to the more general context of graph associahedra.

### 2.1 Basic definitions

We want to formalize the notion of composing tubes, into a tubing. In order to do that properly we need some previous results.

**Notation 2.1.1.** Let $T \in \text{Tub}(\Gamma)$ be a tubing of a graph $\Gamma$ and let $t \in T$ be a tube. For any tubing $S$ of $\Gamma|_t$ which is compatible with $T|_t$, the union of the sets of tubes of $T$ and of $S$ is a tubing of $\Gamma$. We denote it by $T \circ_t S$.

Let $\Gamma$ be a simple finite graph. For any pair of tubes $t$ and $t'$ in $\Gamma$, we have that either $t' \subseteq t$ or $t' \setminus t$ is a tube in the reconnected complement $\Gamma|_t^\ast$.

**Notation 2.1.2.** For any pair of tubes $t$ and $t'$ in $\Gamma$, we denote by $(\Gamma|_t^\ast)|_{t'}$ the graph

$$(\Gamma|_t^\ast)|_{t'} := \begin{cases} 
\Gamma|_t^\ast, & \text{for } t' \subseteq t, \\
(\Gamma|_{t'}^\ast), & \text{otherwise}.
\end{cases}$$
**Lemma 2.1.3.** For any pair of tubes $t$ and $t'$ in $\Gamma$, we have that $(\Gamma^*_t)^* = (\Gamma^*_t')^*$.  

**Proof.** The sets of nodes of the graphs $(\Gamma^*_t)^*$ and $(\Gamma^*_t')^*$ are both equal to $\text{Nod}(\Gamma) \setminus (t \cup t')$, where $\cup$ denotes the union.

On the other hand, for any pair of nodes $v$ and $w$ in $\text{Nod}(\Gamma) \setminus (t \cup t')$, the pair $(v, w)$ is an edge of $(\Gamma^*_t)^*$ if, and only if, at least one of the following situations is fulfilled:

(i) $(v, w)$ is an edge in $\Gamma$,
(ii) there exist edges $(v, v')$ and $(w, w')$ in $\Gamma$ such that either $v'$ and $w'$ belong to $t$, or $v'$ and $w'$ belong to $t'$,
(iii) there exist an edge $(u, z)$ in $\Gamma$ such that $u \in t$ and $z \in t'$, and edges $(v, v')$ and $(w, w')$ in $\Gamma$ satisfying that either $v' \in t$ and $w' \in t'$, or $v' \in t'$ and $w' \in t$.

Note that, exchanging $t$ and $t'$, we get that the edges of $(\Gamma^*_t')^*$ are determined in the same way, so both graphs coincide. □

**Definition 2.1.4.** Let $t^1, \ldots, t^k$ be a collection of tubes in a graph $\Gamma$ satisfying that $t^i$ and $t^j$ are far apart whenever $i \neq j$, for $1 \leq i, j \leq k$. The graph $\Gamma^*_{t^1, \ldots, t^k}$ is the reconnected complement $(((\Gamma^*_t)^*)^*)^*_{t^k}$.

**Notation 2.1.5.** For any tubing $T$ of $\Gamma$ such that $\text{Max}_t(T) = \{t^1, \ldots, t^m\}$, we denote by $\Gamma^*_{T}$ the graph $\Gamma^*_{t^1, \ldots, t^k}$.

Note that, for any tubing $T$ of $\Gamma$, the tubing induced by $T$ on $\Gamma^*_{T}$ is just the universal tubing $T_{\Gamma^*_{T}}$.

**Definition 2.1.6.** Let $T \in \text{Tub}(\Gamma)$ be a tubing of a graph $\Gamma$ with $\text{Max}_t(T) = \{t^1, \ldots, t^m\}$. For a tubing $S$ of $\Gamma^*_{T}$, define the substitution of $S$ in $T$, denoted $T \cdot_{T} S$, to be the set of tubes $s'$ in $\Gamma$ satisfying one of the following conditions:

(i) $s' \in T$;
(ii) $s' \in S$ is such that $s'$ and $t$ are far apart as tubes in $\Gamma$, for any tube $t \in T$;
(iii) $s' = s \cup t^i \cup \ldots \cup t^k$, where $s \in S$ and $\{t^i, \ldots, t^k\}$ is the subset of all the tubes in $\text{Max}_t(T)$ satisfying that $s$ and $t^i$ are not far apart, for $1 \leq j \leq k$.

**Lemma 2.1.7.** If $S$ is a tubing of $\Gamma^*_{T}$, then the set $T \cdot_{T} S$ is a tubing of $\Gamma$, for any tubing $T \in \text{Tub}(\Gamma)$.

**Proof.** Assume that $\text{Max}_t(T) = \{t^1, \ldots, t^m\}$, we proceed by induction on $m$.

For $m = 1$, consider a pair of tubes $s$ and $s'$ in $T \cdot_{T} S$, we must consider the following two cases:

1. For $s' \in T$, we have that $s' \subseteq t^1$,
   (a) if $s \in T$, then $s$ and $s'$ are compatible because $T$ is a tubing,
   (b) if $s \in S$ and $s$ and $t^1$ are far apart, then $s$ and $s'$ are compatible because they are far apart,
   (c) if $s = s_1 \cup t^1$, with $s_1 \in S$ then $s' \subseteq s$ and the tubes $s$ and $s'$ are compatible.
2. For $s \not\in T$ and $s' \not\in T$, we have to consider three cases:
   (a) If $s$ and $t^1$ are far apart, and $s'$ and $t^1$ are also far apart in $\Gamma$, then $s, s' \in S$ are compatible.
   (b) If $s = s_1 \cup t^1$ and $s' \in S$, then either $s' \subseteq s_1$ or $s'$ and $s_1$ are far apart. As $s'$ and $t^1$ are far apart, we get that $s$ and $s'$ are compatible.
(c) If \( s = s_1 \cup t^1 \) and \( s' = s'_1 \cup t^1 \), with \( s_1, s'_1 \in S \), then \( s_1 \) and \( s'_1 \) are not far apart in \( \Gamma^n \). Therefore, as \( S \in \text{Tub}(\Gamma^n) \), either \( s_1 \subseteq s'_1 \) or \( s'_1 \subseteq s \), which implies that either \( s \subseteq s' \) or \( s' \subseteq s \) in \( \Gamma \), and that \( s \) and \( s' \) are compatible.

For \( m \geq 2 \), let \( T' := \{ t \in T \mid t \not\subseteq t^m \} \). The recursive argument states that \( T' \ast \Gamma S \) is a tubing, which induces a tubing \( S' \in \text{Tub}(\Gamma^n \upharpoonright t \cdot T) \).

Let \( W = T_{\Gamma} \circ_{t^m} T \upharpoonright t \cdot T \). \( W \) has a unique maximal tube \( t^m \). As any tube in \( T' \) and \( t^m \) are far apart, we get that \( T \ast \Gamma S = W \ast \Gamma S' \). Applying the previous argument, for \( |\text{Maxt}(W)| = 1 \), we get that \( W \ast \Gamma S' = T \ast \Gamma S \) is a tubing, which ends the proof.

\[ \text{□} \]

**Definition 2.1.8.** Let \( T \) be a tubing of \( \Gamma \). For any tube \( t \in T \) and any tubing \( S \in \text{Tub}(\Gamma^n \upharpoonright t \cdot T) \), the \( t\)-\textit{substitution} of \( S \) in \( T \) is the tubing \( T \circ_{t} (T \upharpoonright t, \cdot, T) \) on \( \Gamma \). We denote it simply by \( \gamma_{t}(T; S) \).

When \( \Gamma \) is connected and \( t = t_{\Gamma} \), we get \( \gamma_{t_{\Gamma}}(T; S) = T \ast \Gamma S \). Lemma 2.1.7 shows that substitution is well-defined.

**Example 2.1.9.**

(1) Let \( \Gamma \) be the linear graph \( L_4 \). Consider the tubings \( T = \{ \{2\}, \{4\}, \{1,2,3,4\} \} \) of \( L_4 \) and \( S = \{ \{1\}, \{1,2\} \} \) of \( \Gamma \).

\[ \begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array} \]

\[ \begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array} \]

The tubing \( \gamma_{L_4}(T; S) \) is:

(2) Here are some of the other operations on an arbitrary graph:
(3) Here is the single substitution on that graph:

(4) For the complete graph $\Gamma = K_6$. The tubings $T = \{\{1, 3, 4, 6\}, \{1, 4\}, t_{K_6}\}$ and $S = \{\{2\}, \{1, 2\}\}$ in $(\Gamma_{\{1,3,4,6\}})^*_{T_{\{1,3,4,6\}}} = K_2$, are shown below, followed by the substitution $\gamma_{\{1,3,4,6\}}(T; S)$:

As there exists a bijection between the set $\text{Tub}(K_6)$ of tubings of the complete graph $K_6$ and the set

$$ \text{ST}_6 := \{x : [6] \rightarrow [r] | 1 \leq r \leq 6 \text{ and } x \text{ is surjective}\}, $$

we may describe the substitution in terms of surjective maps, as in [24].
If we draw the surjections determined by $T$ and $S$ as:

then we get that $\gamma_{\{1,3,4,6\}}(T;S) = (1,4,3,1,4,2)$ is given by:

### 2.2 Associativity of substitution

As in the case of substitution on planar rooted trees, given a tubing $T$ of $\Gamma$, and a family of tubings $S_t \in \text{Tub}(\Gamma_t)$, with $t \in T$, it is possible to define the complete substitution $\gamma(T;\{S_t\}_{t \in T})$.

We begin with the following result, whose proof is immediate.

**Lemma 2.2.1.** Let $T$ be a tubing of $\Gamma$, and let $t$ and $t'$ be a pair of different tubes in $T$. For tubings $S \in \text{Tub}(\Gamma_t)$ and $S' \in \text{Tub}(\Gamma_{t'})$ such that $S$ is compatible with $T|_t$ and $S'$ is compatible with $T \circ_t S$, we have that:

1. $S$ is compatible with $T \circ_{t'} S'$,
2. $(T \circ_t S) \circ_{t'} S' = (T \circ_{t'} S') \circ_t S$.

**Proposition 2.2.2.** Let $T$ be a tubing on $\Gamma$. For any pair of tubes $t, t' \in T$ and any pair of tubings $S \in \text{Tub}(\Gamma_t)$ and $S' \in \text{Tub}(\Gamma_{t'})$, we have that:

1. $t'$ is a tube of $\gamma_t(T;S)$ and $t$ is a tube of $\gamma_{t'}(T;S')$,
2. $\gamma_{t'}(\gamma_t(T;S);S') = \gamma_t(\gamma_{t'}(T;S');S)$.

**Proof.** The first assertion is evident. For the second item, we have to consider the following cases:

(i) if $t \cap t' = \emptyset$, then $s \cap t' = \emptyset = s' \cap t$ for any tubes $s \in S$ and $s' \in S'$. As there does not exist an edge $\{v, w\}$ with $v \in t$ and $w \in t'$, we get that $t$ and $s'$ are far apart and $t'$ and $s$ are far apart, for any tubes $s' \in S'$ and $s \in S$, which implies the result.

(ii) Suppose that $t \cap t' \neq \emptyset$. We may assume, without loss of generality, that $t' \subseteq t$. In this case, $\text{Max}_t(\gamma_t(T;S)|_{t'}) = \text{Max}_t(T|_{t'})$ and $t'$ is contained in some maximal tube of $T|_{t'}$ (eventually it is a maximal tube). So, the tubes added by $S'$ to $\gamma_t(T;S)$ are contained in $t'$ and appear in $\gamma_{t'}(T;S')$. In the same way, $S'$ does not add tubes to $\gamma_t(T;S)$ or to $S$, which proves the equality.
**Definition 2.2.3.** Let \( T \) be a tubing of \( \Gamma \) with \( \mathbb{L}(T) = k + 1 \), that is, \( T = \{ t^0 = t^1, t^2, \ldots, t^k \} \). For any family of tubings \( S^i \in \text{Tub}(\Gamma)^{\mathbb{L}(T)} \), \( 0 \leq i \leq k \), the substitution \( \gamma(T; S^0, \ldots, S^k) \) of \( S^0, \ldots, S^k \) in \( T \) is defined recursively as follows:

(a) For \( T = T^1 \Gamma \) and \( S \in \text{Tub}(\Gamma) \), let \( \gamma(T; S) := S \),

(b) For \( T = \{ t^0 = t^1, t^2, \ldots, t^k \} \),

\[
\gamma(T; S^0, \ldots, S^k) := \gamma_{t^k}(\gamma(T; S^0, \ldots, S^{k-1}); S^k).
\]

Proposition 2.2.2 implies that \( \gamma(T; S^0, \ldots, S^k) \) is well defined.

**Remark 2.2.4.** Note that the substitution of tubings \( S^i \) into a tubing \( T \) of a graph \( \Gamma \) results in a tubing \( T' = \gamma(T; S^0, \ldots, S^k) \) that contains \( T \). This implies two results:

1. The tubing \( T' \) resulting from substitution into \( T \) corresponds to a sub-face of the face corresponding to \( T \) on the graph associahedron \( \mathcal{K}\Gamma \). This is implied by Theorem 1.3.6.

2. The resulting tubing \( T' \) is a basis for a topology which refines the topology generated by \( T \). Moreover, the new topology still has the property that nodes connected by edges are path connected in the topology, as predicted by Corollary 1.2.8. However, there may be pairs of nodes in the topology generated by \( T \) that are connected by a path whose range is only those two nodes; but for which, in the topology generated by \( T' \), any topological path that connects that pair of nodes has a range that includes additional nodes. For instance, see the nodes 6 and 8 in the following Example 2.2.5.

**Example 2.2.5.** Here is an example on a given graph \( \Gamma \) with labeled tubing \( T = \{ t^1, t^2, t^3, t^4 \} = \{ t^1, \{ 3, 4, 5, 6, 7, 8 \}, \{ 6, 7, 8 \}, \{ 3 \}, \{ 8 \} \} \). In this case restricting \( \Gamma \) to the tube \( t^1 \) and then forming the reconnected complement with respect to the tubes inside of \( t^1 \) results in the path on two nodes. Restricting \( \Gamma \) to the tube \( t^2 \) and then forming the reconnected complement with respect to the tubes inside of \( t^2 \) also results in the path on two nodes. This highlights the need to have a labeling of the tubes in \( T \) so as to match the tubings \( S^i \) to their respective destinations.
2.2.6 Substitution on complete graphs

We proved that tubings on complete graphs are in one-to-one correspondence with surjective maps. Let us give an example of how substitution acts on surjective maps.

Consider the complete graph $K_8$ with the tubing as follows:

As tubings in $K_8$

\[ T = \{ t^0, t^1, t^2, t^3 \} = \{ t_{K_8}, \{3\}, \{2, 3, 6, 7\}, \{2, 3, 5, 6, 7, 8\} \} \]

As surjective maps

\[ x_T = (4, 2, 1, 4, 3, 2, 2, 3) \]

As tubings in $K_8$

\[ S^0 = \{\{2\}, \{1, 2\}\} \]

\[ S^1 = T_0 \text{ and } S^2 = \{\{2\}, \{1, 2\}, t_{K_3}\} \]

\[ S^3 = \{\{1\}, \{1, 2\}\} \]

As surjective maps

\[ x_{S^0} = (2, 1) \]

\[ x_{S^1} = (1) \text{ and } x_{S^2} = (2, 1, 3) \]

\[ x_{S^3} = (1, 2) \]
the substitution $\gamma(T; S^0, S^1, S^2, S^3)$ is given by:

$$\gamma(T; S^0, S^1, S^2, S^3) = \{t_K^8, \{3\}, \{3, 6\}, \{2, 3, 6\}, \{2, 3, 6, 7\}, \{2, 3, 5, 6, 7\}, \{2, 3, 5, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8\}\}.$$

**Notation 2.2.7.** Let $T$ be a tubing of $\Gamma$ and let $s$ be a tube in $\Gamma_s$. Let $\bar{s}$ denotes the tube $\bar{s} := s \cup \{t^1, \ldots, t^r\}$, where $t^1, \ldots, t^r$ are the tubes in Maxt($T$) satisfying that $s \cup t^j$ is connected in $\Gamma$, for $1 \leq j \leq r$. In other words, $\bar{s}$ is the minimal tube satisfying that $s \subseteq \bar{s}$ and $\{T, \bar{s}\}$ belongs to Tub($\Gamma$).

**Theorem 2.2.8** (Associativity of substitution). Let $T$ be a tubing of $\Gamma$ and let $t$ be a tube in $T$. Given a tubing $S \in$ Tub($\Gamma, T$) and a proper tube $s \in S$, we have that:

1. the graphs $((\Gamma_s, s, S)_{(T, s)})^*_{\bar{s}}$ and $((\Gamma_{\bar{s}})^*_{\bar{s}})_{(T, s)}$ are equal.
2. for any tubing $W$ of $((\Gamma_s, s, S)_{(T, s)})^*_{\bar{s}}$, the tubing $\gamma_s(S; W)$ is a tubing of $((\Gamma_s, s, S)_{(T, s)})^*_{\bar{s}}$, which satisfies that

$$\gamma_t(T, \gamma_s(S, W)) = \gamma_s(\gamma_t(T, S), W).$$

**Proof.** For the first assertion, the nodes of the graph $((\Gamma_s, s, S)_{(T, s)})^*_{\bar{s}}$ are the nodes of $s$ which do not belong to any other tube of $S$, which coincide with the nodes of $((\Gamma_{\bar{s}})^*_{\bar{s}})_{(T, s)}$. As we work on the subgraph determined by $s$, it suffices to prove the result for $t = t^t$, or equivalently that

$$((\Gamma_s, s, S)_{(T, s)})^*_{\bar{s}} = ((\Gamma_{\bar{s}})^*_{\bar{s}})_{(T, s)}.$$

for any tubing $S$ compatible with $T$ and $s \in S$.

The edges of $((\Gamma_s, s, S)_{(T, s)})^*_{\bar{s}}$ are either:

1. edges $(u, v)$ of $\Gamma_s$ satisfying that for any $s' \in S$, with $s' \subsetneq s$, $s'$ does not contain both nodes $u, v$,
(ii) the pairs of nodes $u, v$ in $\Gamma_s$ satisfying that there exists a tube $t'$ in $T$ such that $\{u, t'\}$ and $\{v, t'\}$ are connected in $\Gamma$,

(iii) the pairs of nodes $u, v$ in $\Gamma_s$ satisfying that there exists a tube $s' \subset s$ in $S$ such that $s' \not\subset s$ and $\{u, s'\}$ and $\{v, s'\}$ are connected in $\Gamma^\text{in}_\Gamma$.

The maximal tubes of $\gamma_{t|_T}(T, S)_s$ are either tubes $s' \subset s$ such that $s' \not\subset s$, or tubes $t^i_j \in \text{Max}_T(T)$ which are not far apart from $s$, or connected unions $s^i \cup t^i_1 \cup \cdots \cup t^i_l$, for some $s^i \not\subset s$ in $S$ and $t^i_1, \ldots, t^i_l \in \text{Max}_T(T)$.

Therefore the edges of $(\Gamma^\text{in}_\Gamma)_s$, are:

(iv) the edges of $\Gamma$ which have both extremes in $s$ and in no other tube $s' \not\subset s$ in $S$,

(v) the pairs of nodes $u, v$ in $s$ satisfying that there exists a tube $t^i_j \in \text{Max}_T(T)$ in $S$ with $\{u, t^i_j\}$ and $\{v, t^i_j\}$ are connected, $t'$ in $T$,

(vi) the pairs of nodes $u, v$ in $s$ satisfying that there exist tubes $s' \subset s$ in $S$ and $t^i_j \in \text{Max}_T(T)$ such that $\{u, s', t^i_j\}$ and $\{v, s', t^i_j\}$ are connected.

For any node $u \in S$ and any tube $s' \not\subset s$ in $S$, we have that $\{u, s'\}$ is connected in $\Gamma^\text{in}_\Gamma$ when either $\{u, s'\}$ is connected in $\Gamma$ or there exists $t^i_j \in \text{Max}_T(T)$ such that $\{u, s', t^i_j\}$ is connected in $\Gamma$. So, the edges of the graphs $(\Gamma^\text{in}_\Gamma)_S$ and $(\Gamma^\text{in}_\Gamma)_T$ are the same, which ends the proof.

To prove the second assertion, note that a tube $\bar{w}$ belongs to $\gamma_{s}(S; W)$ if either $\bar{w} \in S$, or it is of the form $\bar{w} = w \cup \{s^1_i, \ldots, s^r_i\}$, for $w \in W$ and $s^1_i, \ldots, s^r_i$ the maximal tubes in $S|_s$ satisfying that $w \cup s^i_1$ is connected for $1 \leq j \leq r$.

In a similar way, a tube $t'$ in $\Gamma$ belongs to $\gamma_{t}(T; \gamma_{s}(S; W))$ whenever it fulfills one of the following conditions:

- $t' \in T$,
- $t' = \bar{w} = w \cup \{t^1_j, \ldots, t^k_j\}$ for some $w \in \gamma_{s}(S; W)$, where $t^1_j, \ldots, t^k_j$ are the maximal tubes of $T|_t$ which are linked to $w$.

The tubing $\gamma_{t}(T; \gamma_{s}(S; W))$ is the collection of tubes $u$ in $\Gamma$ satisfying one of the following conditions:

1. $u \in T$, or
2. $u \not\subset t$ and $u = s' = s' \cup \{t^1_j, \ldots, t^k_j\}$, for some tube $s' \not\subset s$ in $S$,
3. $u \subset s$ and $u = \bar{v} = w \cup \{s^1_i, \ldots, s^r_i\} \cup \{t^1_j, \ldots, t^k_j\}$ for some tube $w \in W$, where $s^1_i, \ldots, s^r_i$ are the tubes in $\text{Max}_T(S|_s)$ linked to $w$, and $t^1_j, \ldots, t^k_j$ are the maximal tubes of $T|_t$ linked to $w$.

On the other hand, a tube $\bar{w} \in \gamma_{s}(\gamma_{t}(T, S), W)$ is either a tube of $\gamma_{t}(T, S)$ or a tube of the form $\bar{w} = w \cup \{u^1_i, \ldots, u^p_i\}$, where $u^1_i, \ldots, u^p_i$ are the maximal tubes of $\gamma_{t}(T, S)$ contained in $\bar{s}$ which are linked to $w$.

But the elements of $\text{Max}_T(\gamma_{t}(T, S))$ contained in $\bar{s}$ are the maximal tubes of $T|_t$ which are not linked to any $s \in S$ and the tubes of the form $\bar{s}'$, for $s \in S$.

Therefore, a tube $u \in \gamma_{s}(\gamma_{t}(T, S), W)$ is a tube in $\Gamma$ satisfying one of the following conditions:

- $u \in T$,
- $u = \bar{s}'$, for some $s \in S$,
(c) \( u = w \cup \{s^i_1, \ldots, s^i_r\} \cup \{t^j_1, \ldots, t^j_k\} \) for some tube \( w \in W \), where \( s^i_1, \ldots, s^i_r \) are the tubes in \( \text{Maxt}(S|_e) \) linked to \( w \), and \( t^j_1, \ldots, t^j_k \) are maximal tubes of \( T|_e \) linked to \( w \) but not linked to any \( s' \in S \),

which ends the proof.

Let \( T = \{t_1 = t^0_1, \ldots, t^k_1\} \) be a tubing in \( \Gamma \), and let \( S^i = \{s^{0i}_1, \ldots, s^{li}_i\} \) be a family of tubings \( S^i \in \text{Tub}((\Gamma, t^i_1)_{T|_e}^*), \) for \( 0 \leq i \leq k \). Theorem 2.2.8 implies that the substitution \( \gamma \) is associative in the following way.

Suppose that \( W^j_i \in \text{Tub}((\Gamma, t^i_1)_{T|_e}^*), \) is a collection of tubings, for each pair \((i, j)\) with \( 0 \leq j \leq i \). We have that:

\[
\gamma(\gamma(\gamma(T; S_0, \ldots, S_k); W_0^0, \ldots, W_0^k); W_1^0, \ldots, W_1^k) = \gamma(T; \gamma(\gamma(t^0_1), \gamma(S_0)|_{t^0_1}; W_0^0, \ldots, W_1^0), \ldots, \gamma(\gamma(t^k_1), \gamma(S_k)|_{t^k_1}; W_0^k, \ldots, W_1^k)).
\]

Notation 2.2.9. For any connected graph \( \Gamma \) and any tube \( t \) in \( \Gamma \), we denote by \( \{t\} \in \text{Tub}((\Gamma, t)) \) the tubing whose unique tubes are \( t \).

Proposition 2.2.10. For any connected graph \( \Gamma \), a tubing \( T \in \text{Tub}((\Gamma, t)) \) may be obtained from \((\Gamma, \{t\})\) applying substitutions of type \( \gamma_{t^i}(\{t\}) \).

Proof. Let \( T \) be a tubing of \( \Gamma \) with \( T = \{t^1, \ldots, t^k\} \). We use a recursive argument on \( k \).

If \( k = 1 \), then \( T = \gamma_{t^1}(\{t\}) \).

If \( k \geq 2 \), we may assume that \( t^k \) is a maximal tube in \( T \). By a recursive argument, \( T' := \{t^1, \ldots, t^{k-1}\} \) is obtained as \( \gamma_{t^1}(\gamma_{t^2}(\gamma_{t^3}(\gamma_{t^4}(\gamma_{t^5}(\{t\}, \{t^1\}), \{t^2\}), \ldots, \{t^k\})) \).

Let \( S' := \{t^k\} \in \text{Tub}((\Gamma_{-t'}, t^k)_{T'}^*), \) that is, \( S' \) is the union of the universal tube of \( \Gamma_{-t'}^* \) and \( t^k \). Clearly \( T = \gamma_{t^1}(T', \{t^k\}) \).

Example 2.2.11. The tubing \( T \) is obtained as

\[
T = \gamma_{t^1}(\gamma_{t^2}(\gamma_{t^3}(\gamma_{t^4}(\gamma_{t^5}(\gamma_{t^6}(\gamma_{t^7}(\gamma_{t^8}(\{t\}, \{2\}), \{7\}), \{3\}), \{5, 6\}).
\]

2.3 | Relation with the restriction

We want to show how does substitution behaves with the restriction of a tubing to a subgraph. In the whole subsection we assume that \( \Gamma \) is a connected simple graph, with its set of nodes totally ordered, and that \( \Omega \) is a subgraph of \( \Gamma \) having the same ordered set of nodes.

The proof of the following result is straightforward.
Lemma 2.3.1. Let \( t \) be a tube in \( T \). Using the same notations as in Remark 1.2.10, we have that:

1. \( \Omega^*_{\text{res}_\Omega(t)} \) is a subgraph of \( \Gamma^*_T \) with the same set of nodes,
2. for any tubing \( S \) of \( \Gamma^*_T \), \( S \) induces a tubing of \( \Omega^*_{\text{res}_\Omega(t)} \), denoted \( \text{res}_\Omega^*(S)_T \).

The following lemma shows that, to a certain extent, substitution commutes with restriction.

Lemma 2.3.2. For any pair of tubings \( T \in \text{Tub}(\Gamma) \) and \( S \in \text{Tub}(\Gamma^*_T) \), we have that:

1. the graph \( \Omega^*_{\text{res}_\Omega(t)} \) is a subgraph of \( \Gamma^*_T \) with the same set of nodes,
2. \( \text{rest}_\Omega^*(\gamma_T(t, S)) = \gamma_{\text{res}_\Omega}(T, S) \),

where \( \hat{T} := \text{rest}_\Omega^*(T) \) and \( \hat{S} := \text{rest}_\Omega^*(S) \).

Proof. The first assertion is easily verified.

A tube \( \hat{t} \in \gamma_{\text{res}_\Omega}(T, S) \) is a tube in \( \Omega \) which satisfies one of the following conditions:

1. \( \hat{t} \) is a connected component of \( \text{rest}_\Omega^*(t) \), for some \( t \in T \),
2. \( \hat{t} \) is of the form \( \hat{s} \cup \{\hat{t}^1, \ldots, \hat{t}^r\} \), where \( \hat{s} \) is a connected component of \( \text{rest}_\Omega^*(s) \), for some \( s \in \hat{S} \), and \( \hat{t}^1, \ldots, \hat{t}^r \) are the maximal tubes in \( \text{rest}_\Omega^*(T) \) which are linked to \( \hat{s} \) in \( \Omega \).

For any \( s \in \hat{S} \), the tube induced by \( s \) in \( \gamma_{\text{res}_\Omega}(T, S) \) is \( \hat{s} \cup \{t^1, \ldots, t^m\} \), where \( t^1, \ldots, t^m \) are the maximal proper tubes of \( T \) which are linked to \( s \). The restriction of \( \hat{s} \cup \{t^1, \ldots, t^m\} \) to \( \Omega \) is a union of tubes \( \{s_i\}_{1 \leq i \leq k} \). Each tube \( s_i \), for \( 1 \leq i \leq k \), is of one of the following types:

1. \( s_i \) is a tube with all its nodes in a maximal tubing \( t^{j_i} \) of \( T \), and in this case \( s_i \in \text{rest}_\Omega^*(T) \), because the maximal tubings of \( T \) are not linked.
2. If \( s_i \) contains some node in \( \Gamma^*_T \), then \( s_i \) cannot contain two disjoint tubes \( \hat{s} \) and \( \hat{s}' \) of \( \hat{S} \) because in this case both tubes must be linked to the same maximal proper tube of \( T \), and therefore they cannot be disjoint in \( \Gamma^*_T \).

So, there exists a unique tube \( \hat{s} \in \hat{S} \) such that \( s_i = \hat{s} \cup \{\hat{t}^{j_1}, \ldots, \hat{t}^{j_r}\} \), where \( \hat{t}^{j_l} \in \hat{T} \) is a connected component of the restriction of a maximal tube \( t^{j_l} \) of \( T \) to \( \Omega \), for \( 1 \leq l \leq r \), such that \( \hat{t}^{j_l} \) is linked to \( \hat{s} \), and \( \{t^{j_1}, \ldots, t^{j_r}\} \subseteq \{t^1, \ldots, t^m\} \).

The last paragraph shows that the tubes of \( \text{rest}_\Omega^*(\gamma_T(t, S)) \) are the same than the tubes of \( \gamma_{\text{res}_\Omega}(T, \hat{S}) \), which ends the proof. \( \square \)

Applying Lemma 2.3.2 and a recursive argument on \( k \), we get the following result.

Proposition 2.3.3. For any \( T \in \text{Tub}(\Gamma) \), any tube \( t \in T \) and any tubing \( S \in \text{Tub}(\Gamma^*_T) \), the restriction satisfies the equality:

\[
\gamma(\text{res}_\Omega^*(T); \hat{S}^0, \ldots, \hat{S}^k) = \text{res}_\Omega^*(\gamma_t(T, S)),
\]

where the tube \( t \) induces a tubing \( \text{res}_\Omega^*(t) = \{t_1, \ldots, t_k\} \), with \( t_i \cap t_j = \emptyset \) for \( i \neq j \), and \( \hat{S}^i \) is the tubing induced by \( S \) on the reconnected complement \( (\Omega^*_T)_{\text{res}_\Omega^*(T)} \).

3 | **ALGEBRAIC DESCRIPTION OF GRAPH ASSOCIAHEDRA**

Let CGraph be the set of all finite connected simple graphs, equipped with a total order on the set of its nodes, and let Tub be the vector space spanned by the graded set

\[ \bigcup_{n \geq 1} \bigcup_{\Gamma \in \text{CGraph}} \text{Tub}(\Gamma) \cdot \nolimits_{\text{Nod}(\Gamma) = [n]} \]

The \( t \)-substitution \( \gamma_t : \text{Tub}(\Gamma) \times \text{Tub}(\Gamma_t) \to \text{Tub}(\Gamma) \), is not graded. We introduce graded binary products \( \circ_{\Gamma,t} \) on the vector space Tub.

**3.1 Graph associahedra described by operations and relations**

Let

\[ \text{Tub}_n := \bigoplus_{\Gamma \in \text{CGraph}} \text{Tub}(\Gamma) \]

for \( n \geq 0 \). In the following definition we introduce a family of graded products \( \circ_{\Gamma,t} \), where \( \Gamma \) is a connected finite simple graph and \( t \) is a tube of \( \Gamma \).

**Definition 3.1.1.** Let \( \Gamma \) be a graph. The binary operation \( \circ_{\Gamma,t} \) is partially defined on Tub as follows:

\[
S \circ_{\Gamma,t} W := \gamma_t(\Gamma) \circ_{\Gamma,t} S, W,
\]

for any pair of tubing \( S \in \text{Tub}(\Gamma_t) \) and \( W \in \text{Tub}(\Gamma_t) \), where \( T \circ_{\Gamma,t} S \) is the tubing of \( \Gamma \) described in Definition 2.1.6, whose tubes are the tubes of \( S \) and the universal tube \( t_{\Gamma} \).

**Example 3.1.2.** For

As \( t \) and \{7\}, and \( t \) and \{1, 2, 3\}, are not far apart, we get that \( T \circ_{\Gamma,t} W \) is the tubing
Remark 3.1.3. Given two finite connected simple graphs \( \Omega \) and \( \Phi \) and a pair of tubings \( S \in \text{Tub}(\Omega) \) and \( T \in \text{Tub}(\Phi) \) define the product

\[
S \ast T := \sum_{(\Gamma, t\Gamma)} S \circ_{\Gamma, t'} T,
\]

where the sum is taken over all the pairs \((\Gamma, t\Gamma)\), with \( \Gamma \in \text{CGraph} \) and \( t\Gamma \) a tube in \( \Gamma \), satisfying that \( \Gamma_{t\Gamma} = \Omega \) and \( \Gamma^*_{t\Gamma} = \Phi \). The product \( \ast \) is the dual of the pre-Lie coproduct \( \Delta \) described in Definition 1.4.2.

**Proposition 3.1.4.** The operations \( \circ_{\Gamma, t} \) satisfy the following relations:

1. For two tubes \( t \) and \( t' \) in a graph \( \Gamma \), which are far apart,

\[
T_2 \circ_{\Gamma, t'} (T_1 \circ_{\Gamma^*_{t'}, \tilde{t}} S) = T_1 \circ_{\Gamma, t} (T_2 \circ_{\Gamma^*_{t}, \tilde{t}} S),
\]

for any tubings \( T_1 \in \text{Tub}(\Gamma_t) \), \( T_2 \in \text{Tub}(\Gamma_{t'}) \) and \( S \in \text{Tub}(\Gamma^*_{t'}) \).

2. For two tubes \( t' \not\subseteq t \) in a graph \( \Gamma \),

\[
(T_2 \circ_{\Gamma, t'} T_1) \circ_{\Gamma, t} S = T_2 \circ_{\Gamma, t'} (T_1 \circ_{\Gamma^*_{t'}, \tilde{t}} S),
\]

for \( T_1 \in \text{Tub}(\Gamma^*_{t'}) \), \( T_2 \in \text{Tub}(\Gamma_{t'}) \) and \( S \in \text{Tub}(\Gamma^*_{t'}) \), where \( \tilde{t} \) denotes the tube induced by \( t \) in \( \Gamma^*_{t'} \).

**Proof.** Recall that \( t \) and \( t' \) are not connected, so \( \Gamma^*_{[t, t']} = (\Gamma^*_t)^{t'} = (\Gamma^*_t)^{t'} \). It is immediate to verify that:

\[
\gamma_{t}(T_1 \circ_{\Gamma^*_{t'}} (T_2, \gamma_{t^*_{t'}} (T_1, S))) = \gamma_{t}(T_1 \circ_{\Gamma^*_{t'}} (T_2, \gamma_{t^*_{t'}} (T_1, S))),
\]

which proves the first point.

Suppose that \( t' \not\subseteq t \). A tube \( w \) in \( \Gamma \) belongs to \( \gamma_{t}(T_2, \gamma_{t^*_{t'}} (T_1, S)) \) if, and only if, it fulfills one of the following conditions:

(i) \( w \) is a tube of \( T_2 \), or \( w \) a tube of \( T_1 \) such that \( w \) and \( t' \) are far apart, or \( w \) is a tube of \( S \) such that \( w \) and \( t \) are far apart,

(ii) \( w = \tilde{w} \cup t' \), with \( \tilde{w} \in T_1 \) such that \( \tilde{w} \cup t' \) is connected,

(iii) \( w = \tilde{w} \cup t \), with \( \tilde{w} \in S \) such that \( \tilde{w} \cup t \) is connected.

It is not difficult to see that the tubes \( \gamma_{t^*_{t'}} (T_2, T_1, S) \) are exactly the connected subgraphs of \( \Gamma \) which satisfy the same conditions. So, we get

\[
(T_2 \circ_{\Gamma, t'} T_1) \circ_{\Gamma, t} S = T_2 \circ_{\Gamma, t'} (T_1 \circ_{\Gamma^*_{t'}, \tilde{t}} S),
\]

which ends the proof. \( \square \)
Definition 3.1.5. A $G$ algebra over a field $\mathbb{k}$ is a graded vector space $G = \bigoplus_{\Gamma} G_{\Gamma}$, where the direct sum is taken over the set of all finite connected simple graphs whose set of notes is $[n]$ for some $n \in \mathbb{N}$, equipped with binary operations $\circ_{\Gamma,t} : G_{\Gamma_t} \otimes G_{\Gamma_{t'}} \to G_{\Gamma}$, for any tube $t$ in $\Gamma$ satisfying the conditions of Proposition 3.1.4,

(1) For two tubes $t$ and $t'$ in a graph $\Gamma$, which are far apart,

$$W \circ_{\Gamma, t'} (T \circ_{\Gamma^*, t'} S) = T \circ_{\Gamma, t} (W \circ_{\Gamma^*, t} S),$$

for $T \in G_{\Gamma_t}$, $W \in G_{\Gamma_{t'}}$ and $S \in G_{\Gamma^*_{t,t'}}$.

(2) For two tubes $t' \subsetneq t$ in a graph $\Gamma$,

$$(W \circ_{\Gamma, t'} T) \circ_{\Gamma, t} S = W \circ_{\Gamma, t'} (T \circ_{\Gamma^*, t'} S),$$

for $T \in G_{\Gamma_{t'}}$, $W \in G_{\Gamma_t}$ and $S \in G_{\Gamma^*_{t,t'}}$, where $\bar{t}$ denotes the tube induced by $t$ in $\Gamma^*_{t,t'}$.

In order to prove our main result, we need to define a total order on the set of tubings of a tube.

Definition 3.1.6. Let $T$ be a tubing of $\Gamma$. The map $\mathcal{R}_T : \{t \in T\} \to [L(T)]$ is defined, recursively, as follows:

(1) If $L(T) = 1$, then $T = T_{\Gamma}$. In this case, $\mathcal{R}_T(t_{\Gamma}) := 1$.

(2) Suppose that $L(T) > 1$. Let $X = \{t^1, \ldots, t^r\}$ be the set of tubes in $T$ which do not contain another tube of $T$, ordered in such a way that the minimal node of $t^i$ is smaller than the minimal node of $t^{i+1}$, for $1 \leq i < r$. In this case $\mathcal{R}_T(t_i) = i$, for $1 \leq i \leq r$.

(3) Consider the reconnected complement $\Gamma^*_{t^1, \ldots, t^r} := (((\Gamma^*_{t_1})^*_{t_2})^*_{t_3})^*_{t^r}$ and let $T_1$ be the tubing induced by $T$ on $\Gamma^*_{t^1, \ldots, t^r}$. As $L(T_1) < L(T)$, we may assume that $\mathcal{R}_{T_1}$ is defined. We define

$$\mathcal{R}_T(t) := \mathcal{R}_{T_1}(t) + r,$$

for $t \notin X_1$.

Notation 3.1.7. Let $T$ be a tubing of $\Gamma$. By Definition 3.1.6 the set of tubes of $T$ is equipped with a total order $\leq_T$, defined by:

(1) $t \leq_T s$ whenever $\mathcal{R}_T(t) \leq \mathcal{R}_T(s)$,

in the totally ordered set $[L(T)] = \{1 < 2 < \cdots < L(T)\}$.

Define $GColl$ to be the category whose objects are collections of sets $X = \{X_\Gamma\}_{\Gamma \in CGraph}$, indexed by the elements of CGraph. A morphism $f : X \to Y$ is a collection of maps $f_\Gamma : X_\Gamma \to Y_\Gamma$, for $\Gamma \in CGraph$.

Let $1_{GC}$ be the object of $GColl$ such that $1_{GC, \Gamma}$ is the set with a unique element $1_{\Gamma}$, for any $\Gamma \in CGraph$. Given an object $X$ in $GColl$, there exists a unique morphism $X \to 1_{GC}$, which maps all the elements of $X_\Gamma$ to the element $1_{\Gamma}$, for $\Gamma \in CGraph$. The object $1_{GC}$ is a terminal object in $GColl$. 
For any base field \( \mathbb{K} \), consider the forgetful functor \( \mathcal{O} : \mathcal{G}\text{-}\text{alg}_{\mathbb{K}} \to \text{GColl} \), where \( \mathcal{G}\text{-}\text{alg}_{\mathbb{K}} \) denotes the category of \( \mathcal{G} \) algebras over \( \mathbb{K} \). The functor \( \mathcal{O} \) admits a left adjoint functor \( \mathcal{F}_\mathcal{G} \). For any object \( X \) in \( \text{GColl} \), the free \( \mathcal{G} \) algebra spanned by \( X \) is the image of \( X \) under the functor \( \mathcal{F}_\mathcal{G} \).

**Theorem 3.1.8.** The vector space \( \text{Tub} \), equipped with the partially defined binary operations \( \circ_{\Gamma,t} \), for any \( \Gamma \in \text{CGraph} \) and any tube \( t \) in \( \Gamma \), is the free \( \mathcal{G} \) algebra spanned by the terminal object \( 1_{\text{GC}} \) of the category \( \text{GColl} \).

**Proof.** Proposition 3.1.4 implies that \( \text{Tub} \), equipped with the products \( \circ_{\Gamma,t} \), is a \( \mathcal{G} \) algebra.

Clearly, there exists a canonical morphism \( \iota : 1_{\text{GC}} \to \mathcal{F}(\text{Tub}) \) which maps the unique element \( 1_{\Gamma} \) of \( 1_{\text{GC}} \) to the universal tubing \( T_{\Gamma} \).

We need to prove that, given a \( \mathcal{G} \) algebra \( (G, \cdot_{\Gamma,t}) \) and a morphism \( f \) from \( 1_{\text{GC}} \) to \( \mathcal{O}(G) \) in \( \text{GColl} \), there exists a unique morphism of \( \mathcal{G} \) algebras \( f : \text{Tub} \to G \) satisfying that \( \mathcal{O}(f) \circ \iota = f \).

Let \( S \in \text{Tub}(\Gamma) \) be a tubing of \( \Gamma \in \text{CGraph} \).

From Definition 3.1.6 the tubes of \( S \) are totally ordered, so we may assume that \( S = \{s^1 <_S \cdots <_S s^k <_S t_{\Gamma}\} \) in such a way that whenever \( s^i \nsubseteq t_{\Gamma} \), we have that \( i < j \). In this case, \( S \) may be written in a unique way as

\[
S := T_{\Gamma,s^1} \circ_{\Gamma,s^1} (T_{(\Omega^2)s^2} \circ_{\Omega^2,s^2}(\cdots \circ_{\Omega^{k-1},s^{k-1}}(T_{(\Omega^k)s^k} \circ_{\Omega^k,s^k} T_{\Omega^{k+1}})))),
\]

where \( \Omega^l = \Gamma^*, \ldots, \Gamma^2, \Gamma^1 \), for \( 2 \leq l \leq k + 1 \).

The unique linear map \( f : \text{Tub} \to G \) satisfying that \( \mathcal{O}(f) \circ \iota = f \) is defined by:

\[
f(S) := f(1_{\Gamma,s^1}) \circ_{\Gamma,s^1} (f(1_{(\Omega^2)s^2}) \circ_{\Omega^2,s^2}(\cdots \circ_{\Omega^{k-1},s^{k-1}}(f(1_{(\Omega^k)s^k}) \circ_{\Omega^k,s^k} f(1_{\Omega^{k+1}})))).
\]

Let \( \Gamma \) be an element of \( \text{CGraph} \) and \( t \) be a tube in \( \Gamma \). To show that \( f \) is a \( \mathcal{G} \) algebra map, we have to prove that

\[
f(T \circ_{\Gamma,t} S) = f(T) \circ_{\Gamma,t} f(S),
\]

for any pair of tubings \( T \in \text{Tub}(\Gamma_t) \) and \( S \in \text{Tub}(\Gamma_s) \).

Assume that \( S = \{s^1 <_S \cdots <_S s^k <_S t_{\Gamma}\} \). We apply a recursive argument on the number of tubes \( \mathcal{L}(T) \) of \( T \).

If \( \mathcal{L}(T) = 1 \), then \( T = T_{\Gamma,t} \) and \( T \circ_{\Gamma,t} S = \{\hat{s}^1, \ldots, \hat{s}^l, t, \hat{s}^{l+1}, \ldots, \hat{s}^k, t_{\Gamma}\} \), for some \( 0 \leq l \leq k \) such that

(i) \( \hat{s}^i := \begin{cases} s^i, & \text{for } s^i \text{ and } t \text{ far apart in } \Gamma, \\ s^i \cup t, & \text{otherwise.} \end{cases} \)

(ii) \( \hat{s}^i <_{\mathcal{L},S} \cdots <_{\mathcal{L},S} \hat{s}^l <_{\mathcal{L},S} t <_{\mathcal{L},S} \hat{s}^{l+1} <_{\mathcal{L},S} \cdots <_{\mathcal{L},S} \hat{s}^k \),

(iii) if \( s^i \) and \( t \) are far apart in \( \Gamma \), for \( 1 \leq i \leq l \). For \( l < j \leq k \), either the tube \( \hat{s}^k \) and \( t \) are far apart, or \( t \nsubseteq \hat{s}^k \).

Proposition 3.1.4 implies that

\[
T \circ_{\Gamma,t} S = T_{\Gamma,t} \circ_{\Gamma,t} (T_{\Omega^1,s^1} \circ_{\Omega^1,s^1}(T_{(\Omega^2)_{s^1}} \circ_{\Omega^2,s^1}(\cdots \circ_{\Omega^{k-1},s^{k-1}}(T_{(\Omega^k)s^k} \circ_{\Omega^k,s^k} T_{\Omega^{k+1}})))).
\]
where \( \Omega^{i+1} := \begin{cases} \Gamma^{x}_{t, \hat{s}_1, \ldots, \hat{s}_i}, & \text{for } l < i \leq k + 1 \\ \Gamma^{x}_{\hat{s}_1, \ldots, \hat{s}_i}, & \text{for } 2 < i \leq l \end{cases} \), and \( \Omega := \Gamma^{x}_{\hat{s}_1, \ldots, \hat{s}_l} \).

So,

\[
\mathbf{f}(T_{\Gamma, t} S) = f(1_{\Gamma_{t, 1}}) \circ_{\Gamma, \hat{s}_1} \cdots \circ_{\Omega^{x}, \hat{s}_l} (f(1_{\Gamma_{t, 1}}) \circ_{\Omega^{x}, \hat{s}_1} \cdots \circ_{\Omega^{x}, \hat{s}_l} f(1_{\Omega^{x}_{t, 1}})).
\]

As \( G \) is a \( \mathbb{G} \) algebra, and \( \hat{s}_j \) and \( t \) are far apart in \( \Gamma \) for \( 1 \leq j \leq l \), applying condition (1) of Definition 3.1.5, we get that

\[
\mathbf{f}(T_{\Gamma, t} S) = f(1_{\Gamma_{t, 1}}) \circ_{\Gamma, \hat{s}_1} (f(1_{\Gamma_{t, 1}}) \circ_{\Gamma_{t, 1}} \cdots \circ_{\Gamma_{t, 1}} f(1_{\Omega^{x}_{t, 1}})).
\]

Proposition 3.1.4 implies that

\[
T_{\Gamma, t} S = (T_{\Gamma_{t, 1}} \circ_{\Gamma_{t, 1}} T' \circ_{T'} (T_{\Gamma_{t, 1}} \circ_{T'} T' \circ_{T'} T' \circ_{T'} T' \circ_{T'} T')).
\]

Thus, using the proof for \( L(T_{\Gamma_{t, 1}}) = 1 \), we get that

\[
\mathbf{f}(T_{\Gamma, t} S) = \mathbf{f}(T_{\Gamma_{t, 1}}) \circ_{\Gamma_{t, 1}} \mathbf{f}(T' \circ_{T'} (T_{\Gamma_{t, 1}} \circ_{T'} T' \circ_{T'} T')) = \mathbf{f}(T_{\Gamma_{t, 1}}) \circ_{\Gamma_{t, 1}} \mathbf{f}(T') \circ_{\Gamma_{t, 1}} \mathbf{f}(S).
\]

Again, as \( G \) is a \( \mathbb{G} \) algebra, a recursive argument together with condition (2) of Definition 3.1.5 and the equality already proved for \( L(T_{\Gamma_{t, 1}}) = 1 \) imply that

\[
\mathbf{f}(T_{\Gamma, t} S) = \mathbf{f}(T_{\Gamma_{t, 1}}) \circ_{\Gamma_{t, 1}} \mathbf{f}(T') \circ_{\Gamma_{t, 1}} \mathbf{f}(S) = \mathbf{f}(T_{\Gamma_{t, 1}}) \circ_{\Gamma_{t, 1}} \mathbf{f}(T) \circ_{\Gamma_{t, 1}} \mathbf{f}(S) = \mathbf{f}(T_{\Gamma_{t, 1}}) \circ_{\Gamma_{t, 1}} \mathbf{f}(S).
\]

So, the unique linear map \( \mathbf{f} : \text{Tub} \rightarrow G \) satisfying that \( \mathcal{O}(\mathbf{f}) \circ \mathbf{t} = \mathbf{f} \) is a morphism of \( \mathbb{G} \) algebras, which ends the proof.
3.2 The differential structure on Tub

The aim of the present subsection is to give a description of graph associahedra boundary map in terms of the products $\circ_{\Gamma, t}$.

**Notation 3.2.1.** Let $T$ be a tubing of a graph $\Gamma$, we denote by $||T||$ the dimension of the face of the polytope $K\Gamma$ determined by $T$. That is, $||T|| = |\text{Nod}(\Gamma)| - L(T)$.

**Definition 3.2.2.** Let $\Gamma$ be a graph in $\text{CGraph}$.

1. For a tube $t = \{i_1 < \cdots < i_{|t|}\}$ in $\Gamma$, where $|t|$ denotes the number of nodes in $t$, the element $\sigma_t$ is the permutation whose image is

$$\sigma_t := (i_1, \ldots, i_{|t|}, j_1, \ldots, j_{n-|t|}),$$

where $\text{Nod}(\Gamma^*_t) = [n] \setminus \text{Nod}(t) = \{j_1 < \cdots < j_{n-|t|}\}$.

2. Let $t$ be a tube in $\Gamma$. For any tubing $W \in \text{Tub}(\Gamma^*_t)$, define the integer $\alpha(t, W)$ as

   (i) $\alpha(t, W) := 1$, for $t \cup w$ connected in $\Gamma$, for some $w \in \text{Max}(W)$,
   (ii) $\alpha(t, W) := (-1)^{|t|+1}(1+|w_1|+\cdots+|w_r|+r)$, otherwise

where $w_1, \ldots, w_r$ are the tubes in $\text{Max}(W)$ satisfying that the minimal node of $w_i$ is smaller than the minimal node of $t$ in $\Gamma$. When $W = \{t\}$, we denote $\alpha(t, W)$ simply by $\alpha(t, t')$.

3. The binary operation $\cdot_{(\Gamma, t)}$ is partially defined on $\text{Tub}$ as follows:

$$S \cdot_{(\Gamma, t)} W := \alpha(t, W) S \circ_{\Gamma, t} W,$$

for any pair of tubings $S \in \text{Tub}(\Gamma_t)$ and $W \in \text{Tub}(\Gamma^*_t)$, where $T_{\Gamma} \circ_t S$ is the tubing of $\Gamma$ described in Definition 2.1.6, whose tubes are the tubes of $S$ and the universal tube $t_{\Gamma}$.

The product $\cdot_{(\Gamma, t)}$ is just a signed version of $\circ_{\Gamma, t}$.

**Definition 3.2.3.** Let $\Gamma$ be a finite connected simple graph, define the map $\partial : \mathbb{K}[\text{Tub}(\Gamma)] \rightarrow \mathbb{K}[\text{Tub}(\Gamma)]$, where $\mathbb{K}[\text{Tub}(\Gamma)]$ denotes the vector space spanned by the set of tubings of $\Gamma$, as the unique $\mathbb{K}$-linear endomorphism satisfying:

1. $\partial(T_{\Gamma}) = \sum_{t \in \Gamma} (-1)^{|t|} \text{sgn}_n(\sigma_t)(t_{\Gamma}, t)$, where the sum is taken over all the tubes $t$ in $\Gamma$ different from the universal tube $t_{\Gamma}$, while the permutation $\sigma_t$ was introduced at Definition 3.1.1 and the sgn denotes the usual signature of a permutation in $\Sigma_n$, where $\text{Nod}(\Gamma) = n$.

2. Define, for any tube $t$ in $\Gamma$ and any pair of tubings $T \in \text{Tub}(\Gamma_t)$ and $S \in \text{Tub}(\Gamma^*_t)$,

$$\partial(T \cdot_{(\Gamma, t)} S) = \partial(T) \cdot_{(\Gamma, t)} S + (-1)^{|T||T|} T \cdot_{(\Gamma, t)} \partial(S),$$

where $||T|| = n - L(T)$ for $T \in \text{Tub}(\Gamma)$ and $\text{Nod}(\Gamma) = n$.

Theorem 3.1.8 shows that there exists a unique linear map $\partial$ satisfying both conditions. Note that $||\partial(T)|| = ||T|| - 1$. Also note that if $T$ has the maximal number of tubes (so $T$ has dimension 0) then $\partial(T) = 0$. 

Here is an example of calculating the boundary recursively via Definition 3.2.3.

\[ \partial = \partial \]

\[ \partial \circ \partial = 0 \]

**Proposition 3.2.4.** The homomorphism \( \partial \) satisfies that the composition \( \partial \circ \partial = 0 \).

**Proof.** Let \( T \in \text{Tub}(\Gamma) \) be a \( k \)-tubing. We proceed by recursion on \( \mathcal{L}(T) = k \).

For \( \mathcal{L}(T) = 1 \), we have that \( T = T_{\Gamma} \) is the tubing whose unique tube is the universal one. We have that:

\[ \partial^2(T_{\Gamma}) = \sum_{t \subseteq T} (-1)^{|t|} \text{sgn}_n(\sigma_t) \partial(\{t\}) \]

But \( \{t\} = T_{\Gamma, t} \star (\Gamma, t) T_{\Gamma, t}^* \), so

\[ \partial(\{t\}) = \sum_{t' \not\subseteq t} (-1)^{|t'|} \text{sgn}_n(\sigma_{t'}) \{t'\} \star (\Gamma, t) T_{\Gamma, t}^* + \]

\[ (-1)^{|t| - 1} \sum_{t' \not\subseteq T_{\Gamma, t}^*} (-1)^{|t'|} \text{sgn}_{n - |t|}(\sigma_{t'}) \{t\} \star (\Gamma, t') \{t'\} \]

So, in \( \partial^2(T_{\Gamma}) \) we get two different class of 3-tubings:

1. the tubings \( \{t, t', t_{\Gamma}\} \), where \( t \) and \( t' \) are far apart,
2. the tubings \( \{t, t', t_{\Gamma}\} \), where \( t' \not\subseteq t \).

Let us compute the coefficient of the tubing \( \{t, t', t_{\Gamma}\} \) in \( \partial^2(T_{\Gamma}) \) for both cases:

1. In the first case, we get that the coefficient of \( \{t, t', t_{\Gamma}\} \) is:
   
   \( ( -1 )^{|t|} \text{sgn}_n(\sigma_t) ( -1 )^{ |t| - 1 } ( -1 )^{ |t'|} \text{sgn}_{n - |t|}(\sigma_{t'}) \alpha(t, t') \), coming from \( \partial(\{t\}) \),
\[ (ii) \quad (-1)^{|t'|} \text{sgn}(\sigma_{t'}) (\sigma_{t'}) (-1)^{|t'|-1} (-1)^{|t|} \text{sgn}(\sigma_{t}) (-1)^{|t'|} \alpha(t', t), \] 

coming from \( \partial(\{t'\}) \).

It is immediate to verify that, when \( t \) and \( t' \) are far apart, we get that
\[
\text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) = (-1)^{|t'|} \text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) \text{sgn}_{n-\text{vert}}(\sigma_{t'}).
\]

So, the coefficient of \( \{t, t'\} \) in \( \partial^2(T_\Gamma) \) is:
\[
-\text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) \left[ (-1)^{|t'|} \alpha(t, \{t'\}) + (-1)^{|t|} (-1)^{|t'|} \alpha(t', \{t\}) \right],
\]

and either \( \alpha(t, \{t'\}) = 1 \) and \( \alpha(t', \{t\}) = (-1)^{|t|+|t'|+1} \), or
\[ \alpha(t, \{t'\}) = (-1)^{|t|+|t'|+1} \text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) \text{sgn}_{n-\text{vert}}(\sigma_{t'}).
\]

In both cases \((-1)^{|t'|} \alpha(t, \{t'\}) + (-1)^{|t|} (-1)^{|t'|} \alpha(t', \{t\}) = 0\), which ends the proof.

(2) When \( t' \subset t \), the tube \( \{t, t'\} \) appears twice in the computation of \( \partial^2(T_\Gamma) \):

(i) in \( \partial(\{t\}) \), its coefficient is
\[
(-1)^{|t|} \text{sgn}_n(\sigma_{t'}) (-1)^{|t'|} \text{sgn}_n(\sigma_{t'}),
\]

(ii) in \( \partial(\{t'\}) \), its coefficient is
\[
(-1)^{|t'|} \text{sgn}_n(\sigma_{t'}) (-1)^{|t'|-1} (-1)^{|t|} \text{sgn}_n(\sigma_{t'}). \]

It is easily seen that \( \text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) = \text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) \text{sgn}_{n-\text{vert}}(\sigma_{t'}). \)

So, the coefficient of \( \{t, t'\} \) in \( \partial^2(T_\Gamma) \) is
\[
\text{sgn}_n(\sigma_{t'}) \text{sgn}_n(\sigma_{t'}) \left[ (-1)^{|t'|+|t'|} + (-1)^{|t|+|t'|+1} \right] = 0,
\]

and we get that \( \partial^2(T_\Gamma) = 0 \).

For \( \mathbb{L}(T) \geq 2 \), we get that \( T = (-1)^p T_{\{t\}} T^*_t \), for some tube \( t \in T, t \neq t_\Gamma \) and some \( p \in \{0, 1\} \).

The number of tubes of \( T_{\{t\}} \) and the number of tubes of \( T^*_t \) are both smaller than \( \mathbb{L}(T) \), so a recursive argument on \( ||T|| \) and the second condition of Definition 3.2.3 imply that \( \partial^2(T) = 0 \). \( \square \)

### 4 | ALGEBRAIC STRUCTURES ON DTub

Up to this section, all the graphs considered are finite, simple and connected. The vector space \( \text{Tub} \) is the vector space spanned by all tubings of connected simple finite graphs. In the present section, we study the vector space \( \text{DTub} \) generated by tubings of graphs which are not necessarily connected, but disjoint unions of finite connected graphs.

The simplest example of this type of graphs is the empty graph \( C_n \) with \( n \) nodes and no edge. In \([7]\), M. Carr and S. Devadoss proved that the graph associahedra \( \mathcal{K}C_{n+1} \) is standard simplex of dimension \( n \). In \([8]\), F. Chapoton introduced the differential operad Trias', whose free object on one element is the vector space spanned by all the faces of standard simplexes, equipped with the usual boundary map, and three binary products. Our goal is to define a largest Trias' algebra structure on the faces of graph associahedra.
4.1 The graded vector space DTub

**Definition 4.1.1.** For any pair of simple finite graphs $\Gamma$ and $\Omega$ satisfying that $\text{Nod}(\Gamma) = [n]$ and $\text{Nod}(\Omega) = [m]$, the graph $\Gamma \sqcup \Omega$ is the disjoint union of $\Gamma$ and $\Omega$, with a bijective map $\lambda : \text{Nod}(\Gamma \sqcup \Omega) \rightarrow [n + m]$ which satisfies that:

1. the restrictions of $\lambda$ to $\Gamma$ and to $\Omega$ preserve the order of the nodes,
2. $\lambda(u) < \lambda(v)$, for any pair $u \in \text{Nod}(\Gamma)$ and $v \in \text{Nod}(\Omega)$.

For $n \geq 1$, let $\text{Graph}_n$ be the set of all simple finite graphs $\Gamma$ satisfying that $\text{Nod}(\Gamma) = [n]$ and that $\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_r$, with $\Gamma_i \in \text{CGraph}$ for $1 \leq i \leq r$ and $r \geq 1$. The graded set $\text{Graph}$ is the union $\bigcup_{n \geq 1} \text{Graph}_n$.

We denote by $\text{DTub}$ the graded $\mathbb{K}$-vector space spanned by the set $\text{Graph}$ of simply finite, not necessarily connected, graphs.

**Notation 4.1.2.** Let $\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_r$ be a graph, where $\Gamma_i$ is connected for $1 \leq i \leq r$, and let $T$ be a tubing of $\Gamma$. For $r = 1$, $T$ is the set of all proper tubes of $T$. For $r > 1$, $T^c$ denotes the union of $T$ and the set $\{t_{\Gamma_i} \mid 1 \leq i \leq r\}$ of all universal tubes of the graphs $\Gamma_i$, for $1 \leq i \leq r$.

For $r > 1$, and any tubing $T = T_1 \sqcup \ldots \sqcup T_r \in \text{Tub}(\Gamma)$, there exists at least one integer $1 \leq k \leq r$ such that $T_k$ does not contain the universal tube $t_{\Gamma_k}$ of $\Gamma_k$. Denote by $\text{def}(T)$ the number of positive integers $1 \leq k \leq r$ such that $T_k$ does not contain the universal tube $t_{\Gamma_k}$.

4.2 DTub as the free Trias’ algebra spanned by Tub

We begin recalling the definition of Trias’ algebras. Our definition simplifies the signs of F. Chapoton’s original one, but it is easy to see that the definitions are equivalent.

**Definition 4.2.1.** A Trias’ algebra is a graded differential vector space $(V, d)$ equipped with three associative products $\triangleright$, $\times$ and $\lhd$, which satisfy the following relations:

\begin{enumerate}
  \item $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z)$,
  \item $(x \lhd y) \triangleright z = x \triangleright (y \lhd z)$,
  \item $(x \lhd y) \triangleright z = x \triangleright (y \triangleright z)$,
  \item $(x \lhd y) \times z = x \times (y \triangleright z)$,
  \item $(x \times y) \triangleright z = x \times (y \lhd z)$,
  \item $(x \times y) \lhd z = 0 = x \times (y \triangleright z)$,
  \item $(x \times y) \lhd z = 0 = x \times (y \times z)$,
  \item $d(x \times y) = d(x) \times y + (-1)^{|x|}x \times d(y) + (-1)^{|x|}(x \triangleright y - y \lhd x)$,
  \item $d(x \triangleright y) = d(x) \triangleright y + (-1)^{|x|}x \lhd d(y)$,
  \item $d(x \lhd y) = d(x) \lhd y + (-1)^{|x|}x \triangleright d(y)$,
\end{enumerate}

for $x, y, z \in D$, where $|x|$ is the degree of an element $x \in V$, the degree of the products $\triangleright$ and $\lhd$ is 0, the product $\times$ is of degree 1 and the differential $d$ is of degree $-1$.

We define a Trias’ algebra structure on $\text{DTub}$ as follows.
Definition 4.2.2. Let $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_r$ and $\Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_s$ be two graphs such that $\Gamma_i$ and $\Omega_j$ are connected, for $1 \leq i \leq r$ and $1 \leq j \leq s$. For any pair of tubings $T \in \text{Tub}(\Gamma)$ and $S \in \text{Tub}(\Omega)$, define in $\text{Tub}(\Gamma \sqcup \Omega)$ the tubings:

- **(a)** $T \vdash S := \begin{cases} T^c \sqcup S & \text{for } s = 1 \text{ and } \text{def}(T) \leq 1, \\ T^c \sqcup S & \text{for } s > 1 \text{ and } \text{def}(T) \leq 1, \\ 0 & \text{otherwise}, \end{cases}$

- **(b)** $T \dashv S := \begin{cases} T \sqcup S^c & \text{for } r = 1 \text{ and } \text{def}(S) \leq 1, \\ T \sqcup S^c & \text{for } r > 1 \text{ and } \text{def}(S) \leq 1, \\ 0 & \text{otherwise}, \end{cases}$

- **(c)** $T \times S := \begin{cases} (\overline{T} \sqcup \overline{S}) & \text{for } r = s = 1, \\ T \sqcup S & \text{for } r = 1 \text{ and } s > 1, \\ T \sqcup S & \text{for } r > 1 \text{ and } s = 1, \\ T \sqcup S & \text{for } r > 1 \text{ and } s > 1. \end{cases}$

The products are extended to all the vector space $\text{DTub}$ by linearity.

**Proposition 4.2.3.** The vector space $\text{DTub}$ is the free object spanned by the vector space $\text{Tub}$ and the products described in Definition 4.2.2 with the relations (i)–(vii) of Definition 4.2.1.

**Proof.** A straightforward calculation shows that $(\text{DTub}, \vdash, \times, \dashv)$ satisfies the relations (i)–(vii) of Definition 4.2.1. On the other hand, as proved in [8], the free Trias' algebra spanned by a set $X$ has a basis of the form

$$(q, x),$$

where $q = q_1 \ldots q_n \in \{0, 1\}^n$ and $x = x_1 \otimes \cdots \otimes x_n \in X^n$, for $1 \leq i \leq n$,

where at least one $q_i$ is equal to 0. Moreover, the products are given by

- **(i)** $(q, x) \vdash (p, y) := \begin{cases} 0, & \text{for at least two } q_i = 0, \\ (1 \ldots 1 p, x \otimes y), & \text{otherwise} \end{cases}$

- **(ii)** $(q, x) \times (p, y) := (qp, x \otimes y),$

- **(iii)** $(q, x) \dashv (p, y) := \begin{cases} 0, & \text{for at least two } p_i = 0, \\ (q1 \ldots 1, x \otimes y), & \text{otherwise} \end{cases}$

for $x = x_1 \otimes \cdots \otimes x_n \in X^n$, $y = y_1 \otimes \cdots \otimes y_m \in X^m$, $q = q_1 \ldots q_n$, $p = p_1 \ldots p_m$ and $qp := p_1 \ldots p_n q_1 \ldots q_m \in \{0, 1\}^{n+m}$, for $n, m \geq 1$.

The vector space $\text{DTub}$ has a basis

$$\{T_1 \sqcup \cdots \sqcup T_r \mid T_i \in \text{Tub}(\Gamma_i), \text{ for } \Gamma_i \in \text{CGraph}, 1 \leq i \leq r\}.$$

The map

$$T_1 \sqcup \cdots \sqcup T_r \longmapsto (q_1 \ldots q_r, T_1^c \otimes \cdots \otimes T_r^c),$$
where

\[ q_i := \begin{cases} 
0, & \text{for } T_i = \overline{T}_i, \\
1, & \text{for } T_i = T^c_i,
\end{cases} \]

induces a linear isomorphism from \( \text{DTub} \) to the underlying vector space of the free Trias’ algebra spanned by the set \( \bigcup_{\Gamma \in \text{CGraph}} \text{Tub}(\Gamma) \). Moreover, applying Definition 4.2.2 it is immediate to see that this isomorphism is a Trias’ algebra isomorphism.

**Definition 4.2.4.** For a tubing \( T = T_1 \sqcup \ldots \sqcup T_r \) of the graph \( \Gamma_1 \sqcup \ldots \sqcup \Gamma_r \), define the weight of \( T \) as the integer \( |T| := \sum_{i=1}^r (n_i - L(T_i)) - 1 \), for \( \text{Nod}(\Gamma_i) = n_i \), for \( 1 \leq i \leq r \).

When \( r = 1 \), \( T \), we assume that \( T_1 = \overline{T}_1 \), so \( |T_1| = ||T^c_1|| \). Therefore, we get the following result.

**Theorem 4.2.5.** The vector space \( \text{DTub} \), equipped with the products \( \vdash, \lnot \) and \( \times \) and the differential map \( d \), is the free Trias’ algebra generated by the differential graded space \( (\text{Tub}, \partial) \).

**Proof.** The result is a straightforward consequence of Proposition 4.2.3.

For any connected graph \( \Gamma \), a tubing \( T \in \text{Tub}(\Gamma) \) is identified with the element \( \overline{T} \in \text{DTub} \).

From Proposition 4.2.3, there exists a unique endomorphism \( d \) of \( \text{DTub} \) which satisfies relations (viii)-(x) of Definition 4.2.1, and such that its restriction \( d|_{\text{Tub}} \) to the subspace \( \text{Tub} \) coincides with the differential map \( \partial \), described in Definition 3.2.3.

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## 5 AN OPERADIC CATEGORY ASSOCIATED TO GRAPH ASSOCIAHEDRA

Algebraic operads are described in [18] (see also [25]) as algebras over the monad of trees.

### 5.1 Substitution on trees

Lemma 1.3.4 shows that there exists a bijective map \( \rho_n \) between the tubings of the lineal graph \( L_n \) and the set \( \mathcal{T}_{n+1} \) of planar rooted trees with \( n + 1 \) leaves. We want to see that the application \( \rho_n \) maps the product \( \gamma_t \) (respectively, \( o_{T, t} \)) to the grafting of trees (respectively, to the substitution of trees at one internal vertex) as described in Notation 1.3.1 and Definition 1.3.3.

**Proposition 5.1.1.** For \( n \geq 1 \) and any tube \( t = \{i+1, \ldots, i+r\} \) in \( L_n \), the bijective map \( \rho_n : \text{Tub}(L_n) \to \mathcal{T}_{n+1} \) satisfies that

1. for any pair of tubings \( S \in \text{Tub}(L_r) \) and \( W \in \text{Tub}((L_n)^\tau) = \text{Tub}(L_{n-r}) \), the tree \( \rho_n(S \circ_{L_{n-r}} W) \) is equal to

\[
\begin{cases} 
\rho_{n-r}(W) \circ_{i+1} \rho_t(S), & \text{for } t \text{ and } w \text{ far apart, } f \text{ for all } w \in W, \\
\rho_{n-r}(W) \circ_{i+1} \rho_t(S), & \text{otherwise.}
\end{cases}
\]

---
\[ \rho_n(\gamma_t(T;S)) = \rho_n(T) \circ_{a_t} \rho_r(S), \] for any tubing \( T \) of \( L_n \) and any \( S \in \text{Tub}(L_n)^r_T \), where \( a_t \) is the internal vertex of \( \rho_n(T) \) associated to the tube \( t \).

**Proof.** For the first point, let \( T \) be a tubing, with \( \text{Maxt}(T) = \{t^1, \ldots, t^m\} \), with \( t^i = \{j^i + 1, \ldots, j^i + r^i\} \) for \( 0 \leq j_1 < j_1 + r_1 < \cdots < j_m < j_m + r_m \leq n \). We have that
\[ \rho_n(T) = (c_{n-r+1} \circ_{j_1+1} \rho_{r_1}(T|t^1)) \circ_{j_2+1} \ldots \circ_{j_m+1} \rho_{r_m}(T|t^m), \]
where \( r = r_1 + \cdots + r_m \).

Therefore, for any \( 1 \leq i \leq m \), we get
\[ \rho_n(T) = \rho_n(T^i) \circ_{j_i+1} \rho_{r_i}(T|t^i). \]

Given a tube \( t = \{i + 1, \ldots, i + r\} \), and tubings \( S \in \text{Tub}(L_r) \) and \( W \in \text{Tub}(L_{n-r}) \) we proceed by induction on the number \( p \) of tubes \( w \in W \) satisfying that \( w \cap \{i, i + 1\} \neq \emptyset \).

For \( p = 0 \), we have that any tube \( w = \{l + 1, \ldots, l + q\} \in W \) satisfies that either \( l + q < i \) or \( i < l \).

In this case, \( t \in \text{Maxt}(\gamma_{t_n}(T_{L_n} \circ_1 S, W)) \), and \( \gamma_{t_n}(T_{L_n} \circ_1 S, W)^*_i = W \). Therefore, we have that
\[ \rho_n(\gamma_{t_n}(T_{L_n} \circ_1 S, W)) = \rho_n(W) \circ_{i+1} \rho_r(S) \]
which proves the result because \( S \circ_{t_n} W = \alpha(t, W)(-1)^r \gamma_{t_n}(T_{L_n} \circ_1 S, W). \)

For \( p > 0 \), there exists a unique tube \( w^0 \in \text{Maxt}(W) \) such that \( w^0 \cap \{i, i + 1\} \neq \emptyset \). Assume that \( w^0 = \{l + 1, \ldots, l + q\} \), with \( i \leq l + q \) and \( l \leq i \).

Define the tube \( \bar{t} := \{i - l + 1, \ldots, i + r\} \subseteq L_{q+r} \). The set of tubes
\[ W|_{w^0} := \{w - l \mid w \in W \text{ and } w \subseteq w^0\} \]
is a tubing in \( L_q = (L_{q+r})^*_\bar{t} \), where \( w - l := \{k - l + 1, \ldots, k - l + p\} \) for \( w = \{k + 1, \ldots, k + p\} \).

Note that \( \bar{t} = t_{L_q} \) and that \( S \in \text{Tub}(L_r) = \text{Tub}(L_{q+r})|_{\bar{t}} \).

Therefore the tubing \( \gamma_{t_{L_q+r}}(T_{L_{q+r}} \circ S, W|_{w^0}) \) is a tubing on \( L_{q+r} = (L_n)|_{t'} \), for \( t' := \{l + 1, \ldots, l + q + r\} \), which satisfies that

(i) the number of tubes of \( w' \in W|_{w^0} \) such that \( w' \cap \{i - l, i - l + 1\} \neq \emptyset \) is \( p - 1 \);

(ii) for all tubes \( w \in W^*_w = L_{n-(q+r)} \), the intersection \( w \cap \{l, l + 1\} \) is empty;

(iii) \( \gamma_{t_n}(T_{L_n} \circ_1 S, W) = \gamma_{t_{L_q+r}}(T_{L_{q+r}} \circ_1 S, W|_{w^0}), W^*_w). \)

From the proof of the result for \( p = 0 \), we get
\[ \rho_n(\gamma_{t_{L_q+r}}(T_{L_{q+r}} \circ_1 S, W)) = \rho_n(W|_{w^0}) \circ_{i+1} \rho_{q+r}(\gamma_{t_{L_q+r}}(T_{L_{q+r}} \circ_1 S, W|_{w^0})). \]

Applying a recursive argument, we obtain that
\[ \rho_{q+r}(\gamma_{t_{L_q+r}}(T_{L_{q+r}} \circ_1 S, W|_{w^0})) = \rho_q(W|_{w^0}) \circ_{i-1} \rho_r(S). \]
Now, given planar rooted trees $u_1, u_2$ and $u_3$, the grafting of trees satisfies the relation

$$u_1 \circ_l (u_2 \circ_{l-i} u_3) = (u_1 \circ_l u_2) \circ_l u_3,$$

where $u_i \in \mathcal{T}_{r_{n_i}}$, for $i = 1, 2, 3$, $1 \leq l \leq n_1$ and $l + 1 \leq i \leq l + n_2$.

From the equality above, we get

$$\rho_n(\gamma_{L_{n}}(T_{L_{n}} \circ_l S, W)) = (\rho_{n-(q+r)}(W^u_{\omega^u}) \cdot i_{l+1} \rho_q(W_{\omega^u})) \circ_l \rho_r(S),$$

where $\rho_{n-(q+r)}(W^u_{\omega^u}) \cdot i_{l+1} \rho_q(W_{\omega^u}) = \rho_{n-r}(W)$, which implies that

$$\rho_n(\gamma_{L_{n}}(T_{L_{n}} \circ_l S, W)) = \rho_{n-r}(W) \circ_l \rho_r(S).$$

For the signs, we have that $t$ and $w$ are far apart in $L_n$, for all $w \in W$, if and only if $p = 0$.

For the second point, given a tubing $T$ of $L_n$, a tube $t = \{i + 1, \ldots, i + r\}$ in $T$ and a tubing $S \in \text{Tub}(L_r)$, we have

$$\gamma_t(T, S) = \gamma_{L_{n}}(T_{L_{n}} \circ_t \gamma_{L_r}(T|_t, S), T^t_s).$$

Suppose that $\text{Max}(T|_t) = \{t^1, \ldots, t^k\}$, where $t^i = \{j_i + 1, \ldots, j_i + l_i\}$ for $i \leq k$, and that $0 \leq j_1 < j_1 + l_1 < j_2 < \cdots < j_k < j_k + l_k \leq r$.

Applying the first point and a recursive argument on $k$, we get that $\rho_T(\gamma_{L_r}(T|_t, S))$ is equal to

$$((\rho_{r-1}(S) \circ_{j_1+1} \rho_{l_1}(T|_{t^1})) \circ_{j_2+1} \rho_{l_2}(T|_{t^2})) \cdots \circ_{j_k+1} \rho_{l_k}(T|_{t^k}),$$

where $l = l_1 + \cdots + l_k$.

Again, the first point of the proposition implies that

$$\rho_n(\gamma_t(T, S)) = \rho_{n-r}(T^t_s) \circ_{i+1} \rho_r(\gamma_{L_r}(T|_t, S)),$$

which ends the proof, because $a_i$ is precisely the vertex $i + 1$ of the tree $\rho_n(T)$.

Let $L = \bigoplus_{n \geq 0} \mathbb{K}[\text{Tub}(L_n)]$ be the vector space spanned by the tubings on linear graphs, where $L_0$ is the empty graph and $\mathbb{K}[\text{Tub}(L_0)] = \mathbb{K}$. As any tube $t$ in $L_n$ is of the form $t_{i,r} = \{i + 1, \ldots, i + r\}$, Theorem 3.1.8 implies that $L$ is generated by the set of all graphs $\{L_n\}_{n \geq 0}$ and the binary products

$$\circ_{i}^{n,r} : \mathbb{K}[\text{Tub}(L_r)] \otimes \mathbb{K}[\text{Tub}(L_{n-r})] \rightarrow \mathbb{K}[\text{Tub}(L_n)],$$

for $0 \leq i < i + r \leq n$, under the relations

(a) $T_2 \circ_{i}^{j+1,r} T_1 \circ_{i}^{n-s,r} S = \max(\alpha(t, S), \alpha(t', S)) T_1 \circ_{i}^{n,r} (T_2 \circ_{j-r}^{n-r,s} S)$, for $i + r < j$, $T_1 \in \text{Tub}(L_r)$, $T_2 \in \text{Tub}(L_s)$ and $S \in \text{Tub}(L_{n-r+s})$,

(b) $T_2 \circ_{j-r}^{i,s} T_1 \circ_{i}^{n,r} S = T_2 \circ_{j-r}^{i,s} T_1 \circ_{i}^{n-r,s} S$, for $i < j < s < i + r$, $T_1 \in \text{Tub}(L_{r-s})$, $T_2 \in \text{Tub}(L_s)$ and $S \in \text{Tub}(L_{n-r-s})$,

where $t = \{i + 1, \ldots, i + r\}$ and $t' = \{j + 1, \ldots, j + s\}$ are tubes in $L_n$.

For complete graphs we may perform a similar construction. For any $n \geq 1$, a tube $t \in K_n$ is a subset $\{i_1 < \cdots < i_r\} \subseteq [n]$, the graph $(K_n)_t = K_r$ and the reconnected complement is $(K_n)_t^* = K_{n-r}$.
Definition 5.1.2. For $1 \leq r \leq n$, an $(r, n-r)$-shuffle is a permutation of $n$ elements $\sigma \in \Sigma_n$ satisfying that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r + 1) < \cdots < \sigma(n)$.

Using the notations of Section 1.3, to any tube $t = \{i_1 < \cdots < i_r\}$ in $K_n$, we associate the permutation $\sigma_t = (i_1, \ldots, i_r, j_1, \ldots, j_{n-r})$, where $\{n\} \setminus t = \{j_1 < \cdots < j_{n-r}\}$. The map $t \mapsto \sigma_t$ defines a one-to-one correspondence between the set of tubes of the complete graph $K_n$ and the set of shuffles $\text{Sh}(n) := \bigcup_{r=1}^{n} \text{Sh}(r, n-r)$.

Note that there do not exist two tubes in $K_n$ which are far apart.

Let $T \in \text{Tub}(K_n)$ be the tubing $T = \{t^1 \subsetneq t^2 \subsetneq \cdots \subsetneq t^r \subsetneq [n]\}$. Suppose that $t^i \setminus t^{i-1} = \{j_1 < \cdots < j_k\}$ and let $S \in \text{Tub}(K_k)$. An easy calculation shows that

$$x_{T,i(T,S)}(l) = \begin{cases} x_T(l), & \text{for } x_T(l) < i, \\ x_S(i_l) + i - 1, & \text{for } x_T(l) = i, \\ x_T(l) + \mathbb{L}(S) - 1, & \text{for } x_T(l) > i, \end{cases}$$

which shows that, for any tube $t \in T$, the surjective map $x_{T,i(T,S)}$ is obtained by replacing the inverse image of $i$ under $x_T$ by $x_S$, as described in the description of permutads given in [24]. We illustrate this substitution in Section 2.2.6.

On the other hand, for any $i \in [n]$ and any pair of tubings $S \in \text{Tub}(K_r)$ and $W \in \text{Tub}(K_{n-r})$, the result above implies that

$$x_{S \circ K_n,t}W(l) = \begin{cases} x_S(l), & \text{for } l = j_i \in t, \\ x_T(i_l) + \mathbb{L}(S), & \text{for } l = k_i \in [n] \setminus \{t\}, \end{cases}$$

where $t = \{i_1 < \cdots < i_r\}$ and $[n] \setminus \{t\} = \{k_1 < \cdots < k_{n-r}\}$. That is,

$$x_{S \circ K_n,t}W = (x_S \times x_T) \cdot \sigma_t^{-1},$$

where $x_S \times x_T = (x_S(1), \ldots, x_S(r), x_T(1) + r, \ldots, x_T(n-r) + r)$ is the surjective map whose image is the concatenation of $x_S$ with $x_T$, and $\cdot$ denotes the composition as surjective maps.

Let $K$ be the vector space $K := \bigoplus_{n \geq 0} \mathbb{K}[\text{Tub}(K_n)]$. The formulas above show that the restriction of substitution to $K$ gives substitution on the set of surjective maps, as described in [24] and [28].

5.2 \quad Monad of trees and non-symmetric operads

Let $g\text{Vect}_\mathbb{K}$ denote the category of graded vector spaces over $\mathbb{K}$.

Given a planar rooted tree $w$ and a graded vector space $V$, the vector space $w(V)$ is

$$w(V) := \bigotimes_{a \in \text{Vert}(w)} V_{|a|},$$

where $\text{Vert}(w)$ denotes the set of internal vertices of the tree $w$ and $|a|$ is the number of inputs of the vertex $a$. 
The construction above defines a functor \( \mathcal{P} : \text{gVect}_\mathcal{K} \rightarrow \text{gVect}_\mathcal{K} \), given by:

\[
\mathcal{P}(V) = \bigoplus_{n \geq 0} \bigoplus_{|\text{Lea}(w)| = n} w(V),
\]

where \( \text{Lea}(w) \) denotes the number of leaves of a planar rooted tree \( w \).

The functor \( \mathcal{P} \) is a monad in the category of endofunctors of \( \text{gVect}_\mathcal{K} \), equipped with the composition \( \circ \) (see [18, 22, 25, 29, 30]). The natural transformation \( \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P} \) is given by substitution on vertices, as described in Definition 1.3.3, while the natural transformation \( \text{Id} \rightarrow \mathcal{P} \) assigns to a graded vector space \( V \) the morphism \( V \rightarrow \mathcal{P}(V) \) which maps an element \( v \in V_n \) to the corolla \( c_n \) with its root colored by \( v \).

A non-symmetric operad is a unital \( \mathcal{P} \) algebra, that is graded vector space \( \mathcal{A} \) with linear maps \( \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{A} \) and \( \mathcal{K} \rightarrow \mathcal{A}_1 \), compatible with the monad structure.

In [24], another monad in the category of endofunctors of \( \text{gVect}_\mathcal{K} \) was introduced, where planar rooted trees are replaced by surjective maps.

Motivated by the previous construction, we also may define a functor \( \mathcal{G} : \text{gVect}_\mathcal{K} \rightarrow \text{gVect}_\mathcal{K} \) as follows:

1. for any tube \( t \) is a tubing \( T \) in \( \Gamma \), define the \textit{arity} of \( t \) as the number of nodes in \( t \) which do not belong to any other tube \( t' \in T_{|t} \) plus one. We denote the arity of \( t \) by \( \text{ar}(t) \).
2. For any tubing \( T \) of \( \Gamma \) and any graded vector space \( V \), let \( V_T := \bigotimes_{t \in T} V_{\text{ar}(t)} \).
3. Define

\[
\mathcal{G}(V)_{n+1} := \bigoplus_{|\text{Nod}(\Gamma)| = n} \left( \bigoplus_{T \in \text{Tub}(\Gamma)} V_T \right).
\]

However, we do not get a monad structure due to:

1. \( \mathcal{G} \) is not unital, because there exist many finite simple connected graphs with \( n \) vertices, for any fixed \( n \geq 3 \),
2. the composition \( \mathcal{G} \circ \mathcal{G} \) is not always defined. For a graph \( \Gamma \) and a tubing \( T \in \text{Tub}(\Gamma) \), we get that

\[
\mathcal{G}(\mathcal{G}(V))_T = \bigotimes_{t \in T} \left( \bigoplus_{|\text{Nod}(\Omega_t)| = \text{ar}(t) - 1} \left( \bigoplus_{S_t \in \text{Tub}(\Omega_t)} V_{S_t} \right) \right).
\]

But, in order to apply substitution and get an element in \( \mathcal{G}(V) \), we need that \( \Omega_t = (\Gamma_t)^{\text{Max}(T_{|t})} \), for any \( t \in T \).

In [3], M. Batanin and M. Markl introduced the notion of operadic category, which allows them to provide a large generalization of operads (see [22, 29, 30] and [25]). In [27], M. Markl defined an operadic category whose operads are precisely permutads.

The aim of the present section is to introduce a category, whose objects are tubings on finite graphs, and to show that substitution on graph associahedra provides an example of Batanin and Markl’s strict operadic category.
5.3 | Operadic categories

We follow the definition of operadic category given by M. Batanin and M. Markl in [3], for a more detailed description, examples and applications we refer to their publication. In all the sections, \( C \) denotes a complete and cocomplete closed symmetric monoidal category, whose monoidal product is denoted by \( \otimes \) and whose unit is denoted \( I \).

**Definition 5.3.1.** The category \( sFSet \) is the category whose objects are the linearly ordered sets \([n] = \{1 < 2 < \cdots < n\}\), for \( n \geq 1 \), and whose arrows are map between finite sets, which do not necessarily preserve the order.

The terminal object of \( sFSet \) is \([1]\).

**Definition 5.3.2.** Given two maps \( f \in sFSet([n],[m]) \) and \( i \in sFSet([1],[m]) \) the \( i \)-th fiber of \( f \) on \( i \) is the pull back \( f^{-1}(i) \) of the diagram:

\[
\begin{array}{ccc}
f^{-1}(i) & \longrightarrow & [n] \\
\downarrow & & \downarrow \\
[1] & \longrightarrow & [m]
\end{array}
\]

where \( f^{-1}(i) \) is identified with the finite set \( \{j \mid 1 \leq j \leq n, \ f(j) = i\} \) renumbered using the order induced by the linear order of \([f^{-1}(i)]\).

Strict operadic categories are categories with a functor to \( sFSet \), having a family of terminal objects (one for each connected component of the category) and certain inverse images, which behave like pull-backs.

**Definition 5.3.3.** A strict operadic category is a category \( \mathcal{O} \) together with:

1. a fixed family of terminal objects \( U_c \), for each connected component \( c \in \pi_0(\mathcal{O}) \),
2. a cardinality functor \( | - | : \mathcal{O} \longrightarrow sFSet \),
3. an object \( f^{-1}(i) \) in \( \mathcal{O} \) such that \( |f^{-1}(i)| = |f|^{-1}(i) \), for every pair of homomorphisms \( f \in \mathcal{O}(T,S) \) and every element \( i \in |S| \),

satisfying that

- (i) \( \pi_0(\mathcal{O}) \) is small.
- (ii) \( |U_c| = |1| \), for any \( c \in \pi_0(\mathcal{O}) \).
- (iii) For any object \( T \) in \( \mathcal{O} \), the identity \( 1_T \in \mathcal{O}(T,T) \) satisfies that \( 1_T^{-1}(i) = U_{c_i} \), for all \( i \in |T| \) and some \( c_i \in \pi_0(\mathcal{O}) \).
- (iv) For any pair of morphisms \( f \in \mathcal{O}(T,S) \) and \( g \in \mathcal{O}(S,R) \), and every \( i \in |R| \), there exists \( f_i : (g \circ f)^{-1}(i) \longrightarrow g^{-1}(i) \), such that \( |f|_i \) is the natural map \( |g \circ f|^{-1}(i) \longrightarrow |g|^{-1}(i) \) in \( sFSet \). Moreover, the assignment \( \text{Fib}_i(g) := g^{-1}(i) \) gives a functor \( \text{Fib}_i : \mathcal{O}/R \longrightarrow \mathcal{O} \), which is the domain functor for \( R = U_c \).
- (v) For any pair of morphisms \( f \in \mathcal{O}(T,S) \) and \( g \in \mathcal{O}(S,R) \), and any \( j \in |S| \), we have that \( f^{-1}(j) = f_i^{-1}(j) \), where \( i = |g|(j) \).
- (vi) Given three homomorphisms \( f \in \mathcal{O}(T,S) \), \( g \in \mathcal{O}(S,Q) \) and \( h \in \mathcal{O}(Q,R) \), and \( i \in |R| \), the condition above states that there exist morphisms

\[
\begin{array}{ccc}
f_i^{-1}(j) & \longrightarrow & f_{i'}^{-1}(j) \\
\downarrow & & \downarrow \\
|g_i| & \longrightarrow & |g|_i
\end{array}
\]
(a) \((g \circ f)_i = g_i \circ f_i : (h \circ g \circ f)^{-1}(i) \rightarrow h^{-1}(i)\),
(b) \(f_i : (h \circ g \circ f)^{-1}(i) \rightarrow (h \circ g)^{-1}(i)\),
(c) \(g_i : (h \circ g)^{-1}(i) \rightarrow h^{-1}(i)\).

For \(j \in |Q|\), such that \(|h|(j) = i\), we have that \(g^{-1}(j) = (g_i)^{-1}(j)\) and \((g \circ f)^{-1}(j) = (g \circ f)_i^{-1}(j)\).

As \(|h^{-1}(i)| = |h|^{-1}(i)\) is the inverse image (reordered) in sFSet, we get that \(j \in |h^{-1}(i)| = |h|^{-1}(i)\).

So, there exists \((f_i)_j : (g \circ f)_i^{-1}(j) \rightarrow g_i^{-1}(j)\), the equality \(f_j = (f_i)_j\) is required.

**Definition 5.3.4.** Given an strict operadic category \(\mathcal{O}\) and a monoidal category \(C\), an \(\mathcal{O}\)-collection in \(C\) is a collection \(\{E(T)\}_{T \in \mathcal{O}}\) of objects of \(C\), indexed by the objects of the category \(\mathcal{O}\). For an \(\mathcal{O}\)-collection \(E\) in \(C\) and a homomorphism \(f : T \rightarrow S\) in \(\mathcal{O}\), the object \(E(f)\) in \(C\) is defined by

\[
E(f) := \bigotimes_{i \in |S|} E(f^{-1}(i)).
\]

Operadic categories provide a good framework to define operadic composition.

**Definition 5.3.5.** An \(\mathcal{O}\)-operad in \(C\) is an \(\mathcal{O}\)-collection \(\mathcal{P} = \{P(T)\}_{T \in \mathcal{O}}\) in \(C\), equipped with:

(a) units \(I \rightarrow P(U_c)\), for \(c \in \Pi_0(C)\),
(b) structure maps \(\mu(f) : P(S) \otimes P(f) \rightarrow P(T)\), for any \(f \in \mathcal{O}(T, S)\),

satisfying

(i) for any pair of homomorphisms \(f \in \mathcal{O}(T, S)\) and \(g \in \mathcal{O}(S, R)\), the following diagram commutes:

\[
\begin{array}{ccc}
\bigotimes_{i \in |R|} P(R) \otimes P(g) \otimes P(f_i) & \xrightarrow{\mu(g) \otimes \text{id}} & P(R) \otimes P(h) \\
\mu(f) & & \downarrow \mu(h) \\
P(R) \otimes P(f) & \xrightarrow{\mu(f)} & P(T),
\end{array}
\]

where the \(f_i\) s are the morphisms introduced in Definition 5.3.3.

(ii) The compositions

\[
P(T) \rightarrow \bigotimes_{i \in |T|} I \otimes P(T) \rightarrow \bigotimes_{i \in |T|} P(U_{c_i}) \otimes P(T) \xrightarrow{\mu(\text{id})} P(T)
\]

and

\[
P(T) \otimes I \rightarrow P(T) \otimes P(U_c) \xrightarrow{\mu(p_c)} P(T),
\]

where \(c := \pi_0(T)\) and \(p_c\) is the unique element of \(\mathcal{O}(T, U_c)\), are the identity.
5.4 | The category $\mathcal{O}_{CD}$

We want to describe the operadic category $\mathcal{O}_{CD}$ such that the substitution of tubings defined in Section 2 provides a natural example of $\mathcal{O}_{CD}$ operad. Our model is M. Markl's operadic category $\text{Per}$, described in [27].

**Definition 5.4.1.** Define the category $\mathcal{O}_{CD}$ as follows:

1. The objects of $\mathcal{O}_{CD}$ are pairs $(\Gamma, T)$, where $\Gamma$ is a connected simple finite graph and $T$ is a tubing of $\Gamma$.
2. The homomorphisms in $\mathcal{O}_{CD}$ are given by:

$$
\mathcal{O}_{CD}((\Gamma, T), (\Omega, S)) := \begin{cases} 
\emptyset, & \text{for } \Gamma \neq \Omega \text{ or } T \nsubseteq S, \\
\{t_{r,T,S}\}, & \text{for } \Gamma = \Omega \text{ and } T \subseteq S,
\end{cases}
$$

where $T \subseteq S$ is described in Definition 1.2.11, and means that $T$ is obtained from $S$ by adding compatible tubes.

Note that $\pi_0(\mathcal{O}_{CD}) = \text{CGraph}$, the set of all simple connected finite graphs equipped with a total order on the set of nodes, and the terminal objects of $\mathcal{O}_{CD}$ are $(\Gamma, T_{\Gamma})$, for $\Gamma \in \text{CGraph}$.

Let $\Gamma$ be a finite connected simple graph and let $T$ be a tubing of $\Gamma$. In Definition 3.1.6, we defined a bijective map $\mathcal{M}_T$ from the set of tubes on $T$ to the set $[L(T)]$.

The map which sends the object $(\Gamma, T)$ to the set $|(\Gamma, T)| := [L(T)]$ induces a functor $\mathcal{O}_{CD} \rightarrow s\text{FSet}$. We get that

$$|t_{r,T,S}|(k) = \mathcal{M}_S(t'),$$

where $k = \mathcal{M}_T(t)$ and $t'$ is the minimal tube in $S$ which contains $t$.

For $f = t_{r,T,S}$ and $i \in |(\Gamma, S)|$, there exists a unique tube $s \in S$ such that $\mathcal{M}_S(s) = i$. Define $f^{-1}(i)$ as:

$$f^{-1}(i) = \left( (\Gamma_s)_{(S\setminus s)}^*, (T|_s)_{(S\setminus s)}^* \right),$$

where $(\Gamma_s)_{(S\setminus s)}^*$ is the reconnected complement of $\Gamma_s$ by the set $\text{Maxt}(S|_s)$ of proper maximal tubes of $S|_s$, as described in Definition 2.1.4, and $(T|_s)_{(S\setminus s)}^*$ denotes the tubing induced by $T|_s$ on $(\Gamma_s)_{(S\setminus s)}^*$.

For example, consider

we get that

$$f^{-1}(1) = \begin{array}{c}
\begin{array}{c}
\text{original tubing}
\end{array}
\end{array} = f^{-1}(2) \quad f^{-1}(3) = \begin{array}{c}
\begin{array}{c}
\text{tubing induced by } T|_s
\end{array}
\end{array} \quad f^{-1}(4) = \begin{array}{c}
\begin{array}{c}
\text{reconnected complement}
\end{array}
\end{array}$$
Proposition 5.4.2. The category $\mathcal{O}_{CD}$ is a strict operadic category.

Proof. We know that $\pi_0(\mathcal{O}_{CD})$ is the set of all simply connected finite graphs whose set of nodes is totally ordered. For any simple connected graph $\Gamma \in \mathcal{O}_{CD}$, the terminal object $U_\Gamma$ is the pair $(\Gamma, T_\Gamma)$. Clearly, we have that $|\Gamma, T_\Gamma| = 1$.

Suppose that $(\Gamma, T)$ is an object of $\mathcal{O}_{CD}$, where $T$ has $k$ tubes. For any $1 \leq i \leq k$, we have that the identity map $1^{-1}_{(\Gamma, T)}(i)$ is

$$(\Gamma_t)_{(T_{l_i})}^\ast, (T_{l_i})_{(T_{l_i})}^\ast = (\Gamma_t)_{(T_{l_i})}^\ast, T_{(T_{l_i})}^\ast = U_{\Gamma_t}^\ast,$$

where $t_i$ is the unique tube in $T$ such that $\mathfrak{N}_T(t_i) = i$.

Let $\Gamma$ be a graph, and $T \leq S \leq R$ be three tubing of $\Gamma$. For an element $1 \leq i \leq |R|$, there exists a unique tube $r_i$ in $R$, satisfying that $\mathfrak{N}_R(r_i) = i$. As $T \leq S \leq R$, we get that $r_i$ belongs to $S$ and to $T$, and that $\mathfrak{N}_R(r_i) \leq \mathfrak{N}_S(r_i) \leq \mathfrak{N}_T(r_i)$.

We have that $\mathcal{O}_{CD}((\Gamma, T), (\Gamma, S)) = \{t_{\Gamma, T, S}\}$ and $\mathcal{O}_{CD}((\Gamma, S), (\Gamma, R)) = \{t_{\Gamma, S, R}\}$. It suffices to prove condition (iv) of Definition 5.3.3 for $f = t_{\Gamma, T, S}$ and $g = t_{\Gamma, S, R}$. In this case, we get that $g \circ f = t_{\Gamma, T, R}$.

So, for any $1 \leq i \leq |R|$, we get that $g^{-1}(i) = ((\Gamma \cdot r_i)^\ast, (S \cdot r_i)^\ast, (R \cdot r_i)^\ast)$, where $\mathfrak{N}_R(r_i) = i$.

As $T \leq S$, we have that $(T \cdot r_i)^\ast \leq (S \cdot r_i)^\ast$, so

$$f_i = (r_i)^\ast$$

When $\mathcal{O}_{CD}((\Gamma, T), (\Gamma, S)) = \emptyset$ or $\mathcal{O}_{CD}((\Gamma, S), (\Gamma, R)) = \emptyset$, the result is immediate because there do not exist two morphisms $f$ and $g$.

The assignment $\text{Fib}_i(g) := g^{-1}(i)$ gives a functor

$$\text{Fib}_i : \mathcal{O}_{CD}/(\Gamma, R) \longrightarrow \mathcal{O}_{CD},$$

which is the domain functor for $(\Gamma, R) = (\Gamma, T_\Gamma)$, for any graph $\Gamma$.

Let $j \in |S|$ be such that $|g(j)| = i$. There exists a unique tube $s_j \in S$ satisfying that $\mathfrak{N}_S(s_j) = j$, and that the minimal tube of $R$ which containing $s_j$ is $r_i$. Moreover, we have that $f^{-1}(j) = ((\Gamma \cdot s_j)^\ast, (T \cdot s_j)^\ast, (S \cdot s_j)^\ast)$.

On the other hand, $f_i : ((\Gamma \cdot r_i)^\ast, (S \cdot r_i)^\ast, (R \cdot r_i)^\ast, (S \cdot r_i)^\ast) \longrightarrow ((\Gamma \cdot s_j)^\ast, (S \cdot s_j)^\ast)$, is the map $t_{\Gamma, T, S}^\ast, t_{\Gamma, S, R}^\ast, t_{\Gamma, S, R}^\ast, t_{\Gamma, S, R}^\ast$.

Since $S \leq R$, $s_j \subseteq r_i$ and any tube $r$ of $R$ containing $s_j$ satisfies that $r_i \subseteq r$, we have that $((\Gamma \cdot s_j)^\ast, (S \cdot s_j)^\ast) = (\Gamma \cdot s_j)^\ast$.

The restriction of the tubing $(T \cdot r_i)^\ast$ to $(\Gamma \cdot s_j)^\ast$ coincides with the tubing $(T \cdot s_j)^\ast$. Therefore, $f_i^{-1}(j) = ((\Gamma \cdot s_j)^\ast, (T \cdot s_j)^\ast) = f^{-1}(j)$.

To prove the last condition of Definition 5.3.3, consider a graph $\Gamma$ and tubings $T \leq S \leq Q \leq R$ of $\Gamma$. We get that $f = t_{\Gamma, T, S}, g = t_{\Gamma, S, Q}$ and $h = t_{\Gamma, Q, R}$. If we do not have that $T \leq S \leq Q \leq R$, then the triple of morphisms $(f, g, h)$ does not exist.
Let $1 \leq j \leq |Q|$, there exists a unique tube $q_j$ in $Q$ such that $\mathfrak{N}_Q(q_j) = j$. If $|t_{\Gamma, Q, R}|(j) = i$, for some $1 \leq i \leq |R|$, then there exists a unique tube $r_i \in R$ such that $r_i$ is the minimal tube in $R$ which contains $q_j$.

We have that $f_i = t_{(r_i)_{R|R_i}, (T|_{r_i})_{R|R_i}, (S|_{r_i})_{R|R_i}}$ is

$$f_i : ((\Gamma_{r_i})_{R|R_i}^\ast, (T|_{r_i})_{R|R_i}^\ast, (S|_{r_i})_{R|R_i}^\ast) \rightarrow ((\Gamma_{r_i})_{R|R_i}^\ast, (T|_{r_i})_{R|R_i}^\ast, (S|_{r_i})_{R|R_i}^\ast)$$

and $h^{-1}(i) = ((\Gamma_{r_i})_{R|R_i}^\ast, (Q|_{r_i})_{R|R_i}^\ast)$.

As $j \in |h|^{-1}(i)$, the morphism $(f_i)_j$ is the restriction of $f_i$ to

$$(g \circ f)^{-1}(j) = ((\Gamma_{q_j})_{Q|q_j}^\ast, (T|_{q_j})_{Q|q_j}^\ast, (S|_{q_j})_{Q|q_j}^\ast)$$

So, $(f_i)_j = t_{(\Gamma_{q_j})_{Q|q_j}^\ast, (T|_{q_j})_{Q|q_j}^\ast, (S|_{q_j})_{Q|q_j}^\ast}$, which coincides with the definition of $f_j$. □

**Remark 5.4.3.** Consider the full subcategory $\mathcal{O}_K$ of $\mathcal{O}_{CD}$ whose objects are pairs $(K_n, T)$, where $K_n$ is the complete graph with $n$ nodes. As shown in Section 1.3, $T$ is given by a family of tubes $t^1 \subseteq t^2 \subseteq \cdots \subseteq t^{r-1} \subseteq [n]$.

Therefore, the tubing $T$ is identified with the map $f_T : [n] \mapsto [r]$, given by $f_T(i) = j$, where $j$ is the minimal integer satisfying that the node $i$ belongs to $t_j$ (that is, $i \notin t_{j-1}$ and $i \in t_j$).

Suppose that we have a morphisms $\varphi : (K_n, T) \rightarrow (K_n, S)$ in $\mathcal{O}_{CD}$, and that $S = \{s_1 \subseteq \cdots \subseteq s_{k-1} \subseteq [n]\}$. As $S \subseteq T$, we have that $k \leq r$.

The map $g : [r] \rightarrow [k]$, defined by $g(i) = j$, where $s_j$ is the minimal tube of $S$ containing $t_i$, for $1 \leq i \leq r$, is clearly surjective, and satisfies that $g \circ f_T = f_s$.

Therefore, the category $\mathcal{O}_K$ is equivalent to the category Per, defined in 14.4 of [4] and in [27].

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