Fermionization of Bosons in a Flat Band

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We distinguish the role played by the Chern-Simons gauge field in lattices with flat band from a condensate. That is, the system favors a chiral spin-liquid state that spontaneously breaks time-reversal symmetry (TRS).

Current understanding of such a system is that the hard-core bosons can avoid paying any cost of interaction energy by forming spatially separated localized states which is possible due to the presence of a flat band in the Kagome lattice. Such a state can persist up to lattice filling of $\nu = 1/9$, beyond which the system is faced with a choice between populating higher energy bands (see Fig. 1), and letting the bosons still reside in the flat band and pay the interaction cost due to overlap. The former choice would result in condensation of bosons (represented by (blue) dots on $E_2$ in Fig. 1) to the $\Gamma$ point of the Brillouin zone, leading to the supersolid state whose chemical potential grows as $\mu \sim (\nu - 1/9)$, up to logarithmic prefactors. Such a supersolid state has been predicted as a mean-field theory for weakly interacting bosons at lattice fillings above $1/9$ in Ref. [31].

The latter choice essentially remains unexplored. We suggest that if interactions fermionize the bosons, the system can still save energy ($E_{GS} < \nu - 1/9$) by continuing to populate the flat band up to $\nu = 1/3$. Interestingly enough, we show that the scaling of the chemical potential of the fermionized system with particle density is insensitive to the filling fraction $1/9$, which is a critical value for condensed bosons. This fact suggests that the velocity distribution measured in time of flight experiments is a reliable probe that can distinguish between the two states above $1/9$ filling.

The fermionization of hard-core bosons relies on attaching a Chern-Simons (CS) phase, $\Lambda_{\{r\}}$, to the many-body wavefunction: $|\Phi_B\rangle = e^{i\Lambda_{\{r\}}}|\Phi_F\rangle$ (where $\{r\}$ denotes the set of coordinates of the particles, $\Psi_F$ denotes the fermionic wavefunction, and $\Phi_B$ denotes the bosonic wavefunction). While this technique was used extensively to describe fractional quantum Hall (FQH) states, it has also been applied to spin-orbit coupled bosons, bosons in honeycomb lattice, and bosons living on a moat. It has been shown that fermionization can stabilize topological spin ordering, high-temperature superconductivity, and even a chiral spin-liquid in a

A quantum spin-liquid is one of the sought after states in a strongly interacting spin system. A recent neutron scattering experiment on herbertsmithite[1] reported the first detection of a spin-liquid phase. While the characterization of this state has been a matter of debate[2], this experiment indicates that the detection of the various spin-liquid states is not far away. Some manifestations of this state include a gapless Dirac (4-spinor) spin liquid state coupled to a U(1) gauge field[3, 4], a gapped $Z_2$ spin liquid state[5, 6], a chiral spin liquid[7, 8] some of which can also be gapless[9]. Such states exhibit absence of rotational symmetry breaking and, as such, do not stabilize any long-range magnetic order. Their collective low-energy excitations support fractionalized statistics, which can be classified using topological quantum field theory with various symmetry properties. Variety of techniques have been used in the literature to identify and study the properties of such states[10–18] with many of the early and current attempts focusing on 2D triangular and honeycomb lattices[16, 23].

Amongst the numerous quantum spin-liquid candidates[24–27], the spin-1/2 Heisenberg magnet on a Kagome lattice stands out as a fascinating system that is believed to give rise to a variety of spin-liquid phases[28, 29]. If the lattice is sparsely populated by strongly interacting bosons (also referred to as hard-core bosons which avoid multiple occupancies of a single site), the system is equivalent to an XY model with the $z$-directional magnetic field term, $H_{mag} = \sum_{\mathbf{r}} \mu S_{\mathbf{r}}^z$. The field strength $\mu$ maps on to the chemical potential of hard-core bosons. These bosons do not condense because of the degeneracy of the condensate wave functions, which arises from the flat band in the Kagome lattice. In the XY model, this translates to the absence of magnetic order. One is thus interested in learning about phases that can be stabilized in such a system. In this letter, we demonstrate the use of a technique that fermionizes hard-core bosons to find a chiral spin-liquid state as the energetically most favorable candidate ground state of interacting spins on a Kagome lattice, which also spontaneously breaks time-reversal symmetry (TRS).

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moat band\[40\]. In the latter case, the role of magnetic frustration can be mapped to the degeneracy of the lowest single-particle states of the fermions (the moat-band).

In this letter, we devise a scheme to fermionize bosons in a non-Bravais lattice in general and apply it to the case of Kagome lattice. We prescribe a mean-field approximation (MFA) that can model a state that spontaneously breaks TRS and hence has non-zero flux per unit cell. In our MFA, the flux distribution within the unit cell is such that the flat band is preserved upon the TRS breaking. This feature is unique to our prescription and unlike other attempts in literature to tackle a similar problem.\[6\] \[11\] \[12\]. We set-up a variational approach in the continuum limit to demonstrate that our state has the lowest energy, potentially up to $\nu = 1/3$. We also compare the differences in frustration due to moats and flat-bands and finally discuss the interesting avenues of research this approach motivates for future work.

Fermionization in a non-Bravais lattice: The many-body ground state energy, $E_{\text{GS}}$, for the hard-core bosons on a Bravais lattice can be written as $E_{\text{GS}} = \langle \Phi_B | H(\{r\}) | \Phi_B \rangle = \langle \Psi_F | e^{-iA_1(r)} H(\{r\}) e^{iA_1(r)} | \Psi_F \rangle = \langle \Psi_F | H(\{r\}, A_{\{r\}}) | \Psi_F \rangle$, where $A_{\{r\}} = \kappa \sum_{r \in \Omega} \theta_{rr'}, \kappa$ is an odd integer, and $A \equiv \partial_r \Lambda$ and enters the Hamiltonian as $-i\partial_r \rightarrow -i\partial_r + A$. While $| \Psi_F \rangle$ is still undetermined, a variational estimate, $E$, can be postulated for $E_{\text{GS}}$ by constructing a Slater determinant from the single particle states, $\psi_n(r)$, of $H^{\text{MFA}}(r, A_L(r))$. Here $H^{\text{MFA}}$ is obtained from $H(\{r\}, A_{\{r\}})$ under a MFA that converts the non-local $A_{\{r\}}$ to a local $A_L(r)$. In terms of the magnetic field created by the vector potential $A$, the MFA amounts to setting $B(r) \equiv \nabla \times A_{\{r\}} = \sum_i 2\pi \delta(r - r_i)$ to a constant field $B \propto \nu$, spread uniformly across the whole system. Thus $A_L(r)$ is essentially a potential that produces a Maxwellian field, where the flux associated with a region is proportional to its area as opposed the particle number density enclosed in that area.

![FIG. 2: A unit cell of the Kagome lattice. (b) The unit cell redrawn with a shift. This explicitly shows that the cell area includes 3 atoms. The dashed lines are the internal bonds chosen to not include any atoms in the triangular loop. All the particles, and hence the flux, is contained within the hexagon. (c) Flux attachment within the unit cell of the Kagome lattice. $A_L$ denotes the phase accumulated by $A_L(r)$ while traversing the direction $a_i$ such that $A_1 = A_2 + A_3$.](image)

For a non-Bravais lattice, neither the MFA, nor the construction of a variational many-body state is straightforward. The MFA is characterized by the choice of flux distribution within the unit cell. The simplest choice would be to have a distribution where the flux is proportional to the area of the loops present in the unit cell. However, to distinguish the CS field from the Maxwell-type field, we note that flux distribution within the unit cell is such that the flux associated with certain triangular loops is zero\[36\] \[41\]. This is seen by redrawing the unit cell as in Fig. 2B. To account for this modulation in out MFA, one must introduce an intra-unit cell flux $\phi_C$ in addition to the flux $\phi_T$, generated from $A_L(r)$. This introduction is model dependent and will be demonstrated below for the Kagome lattice (see Fig. 2C).

![FIG. 3: Extended scheme for the flux attachment. Note that $\phi_T$ grows to account for the area law, where as $\phi_C$ is the same in every unit cell. The flux through the hexagon and the unit cell is $8\phi_T$ and through any triangle is $\phi_T - \phi_C$. In our MFA $\phi_T = \phi_C$.](image)

The Hamiltonian matrix in a non-Bravais lattice has a rank, $n$, that is the number of atoms in the basis of the lattice. An $N$-body wavefunction can be denoted as $\Psi^{[n]}_{\{\alpha\}}(\{r\}) \equiv \Psi^{a_1,a_2,...,a_N}_{n_1,n_2,...,n_N}(r_1, r_2, ... , r_N)$, where a given coordinate $r_i$ can describe the the wavefunction component $a_i \in \{1,2,...,n\}$ in quantum state $n_i$. Correspondingly, the N-body Hamiltonian acquires the form.
We have thus devised a way to remove the index dependence (by constructing a Slater determinant)\cite{42}. This ambiguity can be masked under a projection technique we describe below. This technique also allows us to effectively consider a single component Hamiltonian whose \( N \)-body fermionic wavefunction can be constructed using the conventional rules.

**Projection to a band:** We note that a single particle state is indexed as \( \psi_{n,k}^a(r) \). Indices \( a_i \) and \( n_i \) are necessary to account for the non-Bravais nature of the lattice. Index \( k \) reflects the crystal translational symmetry which is independent of the non-Bravais nature and allows us to write \( \psi_{n,k}^a(r) = \int \frac{d^2r'}{r'} R_n^a(r - r')\phi_k(r') \), \( \phi_k(r) \) is a solution to the characteristic equation of \( \hat{H}_{ab}(r) \), and \( R_n^a(r) \) is the Fourier transform of the eigenvectors of \( \hat{H}_{ab}(k) \). The normalization condition is enforced by requiring \( \int_{r_1} r_2 \sum_r \phi_k^*(r_2) R_n^a(r - r_1) R_n^a(r - r_2)\phi_k(r_2) = 1 \). The many-body wavefunction can then be constructed as

\[
\psi_{n,k}^a(r) = \int_{\{r'\}} R_n^a(r_1 - r'_1) \cdots R_n^a(r_N - r'_N)\phi_k(\{r'\}).
\]

where \( \phi_k(\{r\}) \) denotes the N-body wavefunction formed out of the quantum states \( \{k\} \) and coordinates \( \{r\} \). The energy expectation value of band \( n \) and quantum states \( \{k\} \) is given by

\[
E_n(\{k\}) = \sum_{\{a\},\{b\}} \int_{\{r\}} \psi_{n,k}^a(\{r\}) \hat{H}_{ab}(\{r\})\psi_{n,k}^b(\{r\}).
\]

Using Eq. \ref{eq:1}, together with the normalization condition, we can show that

\[
E_n(k) = \int_{\{r'\}} \phi_{\{k\}}^*(\{r'\}) E_n(\{r'\}, \{r''\})\phi_{\{k\}}(\{r''\}).
\]

\[
E_n(\{r'\}, \{r''\}) = \sum_{\{a\},\{b\}} \int_{\{r\}} \prod_i R_n^a_i(r'_i - r_i) \hat{H}_{ab}(\{r\}) \prod_j R_n^b_j(r_j - r''_j).
\]

We have thus devised a way to remove the index dependence of \( \hat{H}_{ab}(r) \) and map it to a single component energy function \( E_n(\{r'\}, \{r''\}) \) with single component wavefunction \( \phi_{\{k\}}(\{r\}) \). The fermionization of \( \phi_{\{k\}}(\{r\}) \) leads to:

\[
\phi_{\{k\}}(\{r\}) = e^{iA(r)} \psi_{\{k\}}(\{r\}),
\]

where \( \psi_{\{k\}}(\{r\}) \) is a Slater determinant. Thus the fermionized version of Eq. \ref{eq:6} is achieved by

\[
\phi_{\{k\}}(\{r\}) \rightarrow \psi_{\{k\}}(\{r\}), \quad \text{and} \quad E_n(\{r'\}, \{r''\}) \rightarrow e^{-iA(r')} E_n(\{r'\}, \{r''\}) e^{iA(r')}.
\]

We note that while \( \hat{H}(\{r\}) \) is entirely the property of the underlying lattice, the construction of \( E_n \) and the choice of \( \psi \) is a variational knob available to us. In what follows, we demonstrate that hard core bosons on Kagome lattice prefer a state with spontaneously broken TRS.

**Kagome Lattice and the many-body wavefunction:** Let the lattice be populated by hard-core bosons at every site with filling fraction \( \nu \). Within our MFA this provides non-zero flux at each site, spontaneously breaking TRS. The flux distribution is such that all the flux \( (3 \times 2\pi \nu) \) is concentrated through the hexagon (Fig. \ref{fig:4}). To achieve this, one has to introduce two fluxes \( \phi_T \) [which arises from a Maxwellian \( A_L(r) \)], and \( \phi_C \) (which modulates the intra unit cell distribution). In the absence of any external field, the CS field requires \( \phi_C = \phi_T = 3\pi \nu /4 \). The ‘Hofstadter’ spectrum of this system is shown in Fig. \ref{fig:4}(a) and is contrasted with the same obtained under a flux distribution different from what we impose by CS field. The property of the CS flux distribution seems to be that (i) the spectrum is unique up to \( \nu = 1/3 \) at which point the flux through the unit cell is \( 2\pi \); (ii) the lowest energy band is still flat.

![FIG. 4: Comparison of the Kagome energy spectrum in a CS field for two different flux distributions (a) and (b). The flat band is still flat.](image-url)

As prescribed above, to compute the energy of the variational state, we choose \( R_n^a \) and \( \psi \) to be the eigenstates of the \( \hat{H}_{MFA}(r, A_L) \). This ansatz for the wavefunction provides us with an estimate for \( E_{GS} \) using Eqs. \ref{eq:4}-\ref{eq:5}. This is expected to account for the non-local nature of \( A_L(\{r\}) \) and provide corrections to the many-body energy computed within MFA (Fig. \ref{fig:3}). We shall explicitly
Ground-state energy beyond MFA: We are interested in the stability of the ground state by computing corrections to the energy computed in Fig. 4. Having found the wavefunction of a gapped isolated flat band, we can use Eqs. (3)–(6) for $n = 1$. We observe that because of the non-dispersing nature of the isolated flat band, the effect of the non-local $A_{\mathbf{r}}$ drops out and the many-body ground state energy is still the same ($E = -2J_1$) as computed in MFA. This is not true for other cases of fermionization [33], e.g. when there is a moat band.

Returning back to the lattice problem, we note that attaching a flux of $p/q$ to a unit cell causes the BZ to fold over $q$-times. We have just proven that a CS flux distribution ($p \neq 0$) retains the flat band of the system at $p = 0$. A remarkable consequence of this is that for any $q$ the degeneracy of the flat band is always the same as the system with no flux attachment. For a Kagome lattice with $N$ unit cells ($3N$ atoms), $N$ states correspond to the flat band. Even though the flux attachment changes with $p$, we can now conclude that the flat band can remain occupied up to $p = 1/3$ (this state is depicted in Fig. 3).

We take note of the fact that the spinless fermionic description, where no spatial symmetry is broken, naturally implies lack any spin-ordering ordering in the language of hard-core bosons. Thus, fermionization is a natural tool that can be used to describe a spin-liquid state in strongly interacting bosons. Further, since our MFA breaks TRS, we expect the spin-liquid state to be chiral. At this stage, we are able to conclude that strongly interacting bosons in a Kagome lattice favor a chiral spin-liquid state that spontaneously breaks TRS.

The phase modulation of Fig. 4(b) corresponds to a Chern insulator with staggered flux $\phi_\Delta = -3\pi/4$ threading corner equilateral triangles of the unit cell with $-2\phi_\Delta$ threading the hexagon, superimposed with the uniform flux, $6\pi\nu$. Notice that, apart from the external flux, any finite modulation, $\phi_\Delta$, opens a gap at the band-touching points and defines Chern numbers for each of the three bands. The lowest band, in this case, will have a Chern number $C = 1$, which cannot be altered unless the system undergoes a phase transition with the closing of the gap. Thus the lattice gauge theory of the chiral spin-liquid outlined above can be regarded as such topologically nontrivial Chern insulator coupled to the fluctuating Chern-Simons gauge field. Because of the topological nature of the Chern insulator, Fermion fluctuations here will give rise to an additional Chern-Simons term in the low-energy effective action giving a Chern-Simons theory defined by a “K matrix” with $K = 2$. This implies that the vortex excitations in this system have fractional statistics with statistical angle $\theta = \pi/2$ corresponding to electrons, flat band wavefunctions in the presence of CS field are also inherently localized. To impose analyticity of the wavefunction, we postulate $f_i(\mathbf{r}) = e^{-(r-R_i)^2/2\ell_{CS}^2}$, where $R_i$ is the center of the localized state, and $\ell_{CS}$ is an undermined localization length scale in the theory.

**FIG. 5:** Ground state energy of non-interacting fermions in Kagome lattice subject to the CS flux at mean-field level. Energy begins to rise after $\nu = 1/3$ filling.
semions.

Summary of results: We prescribed a general scheme to construct the N-body wavefunction and compute the ground state energy of hard-core bosons in a non-Bravais lattice using fermonicization. Using the example of the Kagome lattice, we showed that a CS type flux attachment, which grows with the filling fraction $\nu$, can retain the massive degeneracy of system’s original electronic structure. As a result the system prefers a spontaneously TRS broken chiral spin-liquid state. We proved our variational hypothesis for the ground state beyond a mean-field type ansatz in the continuum limit of the Kagome lattice. The implications of the state can be probed by studying velocity distribution of an expanding gas using ultra-cold atoms in optical lattices.

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[41] Whenever two lattice hops can be attained by a single hop, the CS flux enclosed within these hops is zero. The proof of this work is in a manuscript currently in progress.

This property was already utilized in [36].

[42] See Sec. V of supplementary material.
[43] See Sec. III of supplementary material.
[44] See Sec. I through III of supplementary material.
[45] See Sec. IV of supplementary material.
SUPPLEMENTARY MATERIAL FOR FERMIONIZATION OF BOSONS IN A FLAT BAND

MOMENTUM SPACE TO REAL SPACE WAVEFUNCTIONS

The problem of hard core bosons on a lattice can be studied by looking at the spin-1/2 XY model with Hamiltonian

\[ H = \sum_m J_m \sum_{r,n} S_+^r S_-^{r+r_{mn}} + \text{h.c.} \] (13)

Here \( S^\pm = S_x \pm iS_y \) are the spin-1/2 operators; the index \( n \) scans all the neighbors at distance \( m \); \( r_{mn} \) is the vector to the \((m,n)\)th nearest neighbor. The choice of lattice is reflected in the choice of \( r_{mn} \). One choice of the phase attachment that accomplishes the CS transformation is \[32–34\]

\[ S_+^r = c_+^r e^{i\Lambda_r}, \quad S_-^r = e^{-i\Lambda_r} c_+^r, \] (14)

where \( c_+^r \) is a fermionic creation operator, and

\[ \Lambda_r \equiv \kappa \sum_{r \neq r'} \theta_{rr'} c_+^{r'} c_+^r. \] (15)

Here \( \theta_{rr'} \) denotes the argument of \( r - r' \), and \( \kappa \) is an odd integer. This modifies the Hamiltonian to

\[ H = \sum_m J_m \sum_{r,n} c_+^r e^{i\Lambda_r} c_+^{r+r_{mn}} c_+^{r_{mn}} + \text{h.c.}, \] (16)

where \( \Lambda_{r_1,r_2} \equiv \Lambda_{r_1} - \Lambda_{r_2} \) evaluated along the line joining \( r_1 \) and \( r_2 \). It is the analog of the accumulated phase \( \int_{r_2}^{r_1} A \cdot dl \) in a lattice. Geometrically, \( \Lambda_{r_1,r_2} \) is the sum of the angles subtended by the vector \( r_1 - r_2 \) at every other site (located at \( r' \)), weighted by the occupation probability at that site.

From here on, for definiteness, we will work on a Kagome lattice with the first neighbor hoppings from here on.

To achieve this we set \( m = 1 \), and let \( n \) scan from 1 through 4 for each of the three atoms in the unit cell. We then perform a mean-field approximation (MFA) where \( c_+^r \rightarrow \langle c_+^r c_+^r \rangle = n_r = \nu \), the filling fraction in the lattice. The nature of our mean-field approximation is detailed in the main text, and must be distinguished from other works in the literature\[11, 12\].

Prior to implementing the MFA, we quickly review the Kagome Hamiltonian at the single particle level and find the wavefunctions of the flat band in the low energy limit. This will set us up to tackle the scenario with the CS flux distribution.

The lattice Hamiltonian from Eq. 16 (without \( \Lambda \)) can be written in \( k \)-space as

\[ H = \sum_k \bar{\Psi}_k^\dagger \mathcal{H}_k \bar{\Psi}_k, \] (17)
In the Hamiltonian to rotate as

\[ \tilde{\Psi}_k = (c_{a,k}^d, c_{b,k}^d, c_{c,k}^d) \]

The annihilation operators are given by

\[ \tilde{c}_{x,r} = \sum_k \tilde{c}_{x,k} e^{i k \cdot r}, \]

such that \( x \in \{a, b, c\} \). The vector \( r \) only runs over lattice translations and not internal bonds. Lastly,

\[
\tilde{H}_k = J_1 \begin{pmatrix}
0 & (1 + e^{i k \cdot a_2}) & (1 + e^{i k \cdot a_3}) \\
(1 + e^{-i k \cdot a_2}) & 0 & (1 + e^{i k \cdot a_3}) \\
(1 + e^{-i k \cdot a_3}) & (1 + e^{-i k \cdot a_3}) & 0
\end{pmatrix},
\]

where

\[ a_1 = a(1, 0); \quad a_2 = a \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right); \quad a_3 = a \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right). \]

Note that in addition to the lattice translation vectors \( a_1 \) and \( a_2 \), we have introduced \( a_3 = a_1 - a_2 \). It will be useful to perform a gauge transformation: \( H_k = M^\dagger \tilde{H}_k M \) and \( \Psi_k = M^\dagger \tilde{\Psi}_k \) where \( M^\dagger = \text{diag}(1, e^{-i k \cdot a_2/2}, e^{-i k \cdot a_3/2}) \) such that

\[
H_k = J_1 \begin{pmatrix}
0 & H_2 & H_1 \\
H_2 & 0 & H_3 \\
H_1 & H_3 & 0
\end{pmatrix},
\]

and \( H_i \equiv (e^{i k \cdot a_2/2} + e^{-i k \cdot a_3/2}) \). The resulting characteristic equation to find the eigenvalues is

\[
\left( \frac{E}{J_1} \right)^3 - \frac{E}{J_1} (H_1^2 + H_2^2 + H_3^2) - 2H_1H_2H_3 = 0.
\]

The eigenvalues are \( E_1 = -2J_1 \) and \( E_{2/3} = J_1 \left( -1 \mp \sqrt{1 + H_1H_2H_3} \right) \). Note that \( E_1 \) is independent of any parameters in the Hamiltonian and hence dispersionless. The wavefunction corresponding to this flat band is

\[
\Psi_1^k = \frac{1}{N_1} \begin{pmatrix}
e^{i(k \cdot a_1 + k \cdot a_2)/2} \sin \frac{k a_2}{2} \\
e^{-i(k \cdot a_1 - k \cdot a_2)/2} \sin \frac{k a_2}{2} \\
e^{-i(k \cdot a_1 - k \cdot a_2)/2} \sin \frac{k a_2}{2}
\end{pmatrix},
\]

where \( N_1^2 = 4 \left( \sin^2(k \cdot a_1/2) + \sin^2(k \cdot a_2/2) + \sin^2(k \cdot a_3/2) \right) \).

The continuum limit can be obtained by studying the Hamiltonian around the Γ-point. We shall restrict the terms in the Hamiltonian to \( O(k^2) \). This will result in \( H_i = 2 - k_i^2 \), where \( k_i \equiv k \cdot a_i \). The resulting characteristic equation is

\[
\frac{E}{J_1} \left( \frac{E}{J_1} + 2 \right) \left[ \left( \frac{E}{J_1} + 2 \right) \left( \frac{E}{J_1} - 4 \right) + \frac{3}{2} k^2 \right] = 0.
\]

It can be shown that \( E = 0 \) is not a non-trivial solution. The eigenvalues are thus \( E_1 = -2J_1, \ E_{2/3} = J_1 \left( 1 \mp \sqrt{1 - k^2/6} \right) \). And the flat band wavefunction [up to \( O(k^2) \)] is

\[
\Psi_1^k = \sqrt{\frac{2}{3k^2}} \begin{pmatrix}
k_3 \\
-k_1 \\
k_2
\end{pmatrix} = \sqrt{\frac{2}{3}} \begin{pmatrix}
\cos \left( \frac{\theta_k + \pi}{3} \right) \\
\cos \left( \theta_k + \pi \right) \\
\cos \left( \frac{\theta_k - \pi}{3} \right)
\end{pmatrix}.
\]

Note that the flat band wavefunction has the property that a rotation of \( \pi/3 \) causes the weights on the sub lattice to rotate as \( a \to b \to c \to a \) and causes the wavefunction to acquire a phase of \( \pi \). Since the Kagome lattice in invariant under \( a \to b \to c \to a \), the wavefunction acquires a negative sign under a \( C_6 \) rotation thus the fermionic ground state possesses an \( f \)-wave symmetry. This property is also obeyed by the localized state discussed in Ref. [30]. Since this property is maintained by any \( k \)-state, it suggests that any fermionic state with a filling fraction \( \nu < 1/3 \) also has this character.
Real space wavefunctions

The Bloch solution, allows us to write down the solution in real space as

\[ \psi_k^a(r) = \tilde{k}_1 \phi_k(r), \]
\[ \psi_k^b(r) = -\tilde{k}_1 \phi_k(r), \]
\[ \psi_k^c(r) = \tilde{k}_2 \phi_k(r), \]

(24)

where \( \tilde{k}_i \equiv -i \mathbf{a}_i \cdot \partial_r \), and \( \phi_k(r) \) is a function that solves the characteristic equation of the Kagome Hamiltonian. Because of the structure of Eq. (22), we see that \( \phi_k(r) = e^{i \mathbf{k} \cdot \mathbf{r}} \).

THE CHERN-SIMONS FLUX AND THE COVARIANT MOMENTUM

As introduced in the main text, our MFA introduces two fluxes \( \phi_C \) and \( \phi_T \) (which are eventually set equal). While \( \phi_C \) is simply imposed onto the model, \( \phi_T \), which grows with area (Maxwell-type), can be thought of as originating from a vector potential \( \mathbf{A}_L(r) \). \( \mathbf{A}_L(r) \) is the one that is to be used in creating the covariant momentum \( \mathbf{p} = -i \partial_r + \mathbf{A}_L(r) \). The corresponding translation operators have the following properties:

\[
p_1 = p_2 + p_3,
\]
\[
e^{-ip_2} e^{-ip_3} e^{ip_1} = e^{-i \phi_T} = e^{-ip_1} e^{ip_2} e^{ip_3},
\]
\[
e^{-ip_3} e^{-ip_2} e^{ip_1} = e^{i \phi_T} = e^{-ip_1} e^{ip_2} e^{ip_3},
\]
\[
e^{-ip_1} e^{ip_3} e^{ip_2} = e^{-2i \phi_T},
\]
\[
e^{-2ip_1} e^{2ip_3} e^{-2ip_2} = e^{-8i \phi_T} \) (area of the unit cell),
\]
\[
e^{ip_3} e^{ip_1} e^{ip_2} = e^{3i \phi_T} e^{2ip_1},
\]
\[
e^{ip_2} e^{ip_3} e^{ip_1} = e^{3i \phi_T} e^{2ip_1},
\]
\[
e^{ip_2} e^{ip_3} e^{ip_1} e^{-ip_2} = e^{6i \phi_T} \) (area of the hexagon).

Here \( p_i = \mathbf{p} \cdot \mathbf{a}_i \). We further have the following commutation relations for \( i, j \in \{1, 2, 3\} \):

\[
[k_i, k_j] = 0,
\]
\[
[p_1, p_2] = -2i \phi_T,
\]
\[
[p_2, p_3] = 2i \phi_T,
\]
\[
[p_3, p_1] = -2i \phi_T.
\]

(25)

The flux \( \phi_C \) is introduced to account for the internal modulation and is incorporated directly in the Hamiltonian as shown in Eq (9) of the main text. This is necessary because the continuum limit is obtained from the Bloch solution, \( \phi_C \), a property of the unit cell itself, cannot be accounted for by introducing a position dependent gauge field like \( \mathbf{A}_L(r) \).

The translational operator on a lattice taking a fermion from \( \mathbf{r}_1 \) to \( \mathbf{r}_2 \) is \( T_{\mathbf{r}_2 \mathbf{r}_1} = c_{\mathbf{r}_2}^\dagger c_{\mathbf{r}_1} \). For a triangular loop \( L_T : a \rightarrow b \rightarrow c \), it follows that \( T(L_T) = T_{\mathbf{r}_a \mathbf{r}_c} T_{\mathbf{r}_c \mathbf{r}_b} T_{\mathbf{r}_b \mathbf{r}_a} \) is the same as \( T(L_T') = T_{\mathbf{r}_a \mathbf{r}_b} T_{\mathbf{r}_b \mathbf{r}_c} T_{\mathbf{r}_c \mathbf{r}_a} \). If we couple the fermions to a Maxwell-type gauge field where the flux grows with the area, then \( T_{\mathbf{r}_2 \mathbf{r}_1} \rightarrow e^{i A^M_{\mathbf{r}_2 \mathbf{r}_1}} c_{\mathbf{r}_2}^\dagger c_{\mathbf{r}_1} \) and

\[
T^M(L_T) = e^{i \phi_T} T(L_T)
\]
\[
T^M(L_T') = e^{-i \phi_T} T(L_T')
\]

(26)

\[
\phi_T = \Lambda^M_{\mathbf{r}_a \mathbf{r}_c} + \Lambda^M_{\mathbf{r}_c \mathbf{r}_b} + \Lambda^M_{\mathbf{r}_b \mathbf{r}_a} = \int \mathbf{A} \cdot d\mathbf{l}.
\]

Similarly, we may consider the hexagonal loop which yields \( T^M(L_H) = e^{i \phi_T} T(L_H) \).

On implementing the CS flux as shown in Figs. 2c and 3 of the main text, we see that

\[
T^{CS}(L_T) = e^{i(\phi_T - \phi_C)} T(L_T) = T(L_T),
\]
\[
T^{CS}(L_H) = e^{i(6 \phi_T + 2 \phi_C)} T(L_T) = e^{8i \phi_T} T(L_T).
\]

(27)
The last equality is obtained by setting $\phi_T = \phi_C$. Equating the total flux through the unit cell $8\phi_T$ to $6\pi\nu$, we arrive at the relation

$$\phi_T = \phi_C = 3\pi\nu/4.$$  \hspace{1cm} (28)

**THE FLAT BAND WAVEFUNCTION IN A KAGOME LATTICE WITH CS FLUX**

Before deriving the case with the CS flux, we explicitly derive Eq. (24). This is informative and the derivation with the CS flux follows similar lines. Plugging the flat band eigenvalue to the Hamiltonian we see that the flat band wavefunction components $\psi^a$ satisfy

$$k_2(k_2\psi^a - k_3\psi^c) = 0,$$

$$k_3(k_3\psi^b + k_1\psi^a) = 0,$$

$$k_1(k_1\psi^c + k_2\psi^b) = 0.$$  \hspace{1cm} (29)

It is useful note that other equations can be generated using $a \rightarrow b \rightarrow c \rightarrow a; \{H_1, H_1^*\} \rightarrow \{H_2^*, H_2\}, \{H_2, H_2^*\} \rightarrow \{H_3, H_3^*\}, \{H_3, H_3^*\} \rightarrow \{H_1^*, H_1\};$ and $k_1 \rightarrow k_2, k_2 \rightarrow -k_3, k_3 \rightarrow k_1$. This implies that $k_1(\psi^a + \psi^b + \psi^c) = \text{const}$. Since $k \rightarrow -i\partial_k$, normalizability over the whole space not only requires $\text{const} = 0$, but $\psi^a + \psi^b + \psi^c = 0$. The only combination that satisfies the Hamiltonian is then given by Eq. (24).

It is sometimes inconvenient to have the components of the wavefunctions expressed as operators. To remedy this the action of the operator $\hat{R}_r$ can be implemented by convoluting with the Greens’ function of the operator $R(r - r')$.

Thus

$$\hat{R}_r f(r) = \int_{r'} R(r - r') f(r').$$  \hspace{1cm} (30)

If $\hat{R}_r = -i\partial_r$, then $R(r - r') = -i\partial_r \delta(r - r') = i\partial_r \delta(r - r')$.

When a similar analysis is carried out for $H_{\text{CS}}^\text{MFA}$ with the CS flux attached, we end up with Eq. (29) but with $k \rightarrow p \equiv -i\partial_r + A_L(r)$ (only for $\phi_T = \phi_C$). The cyclic interchange also works the same way with $k \rightarrow p$, and with an addition of $\phi_C \rightarrow -\phi_C$. Just like before, normalizability will enforce that $\psi^a + \psi^b + \psi^c = 0$. This result is independent of the choice of gauge for writing down $A_L(r)$. Thus the flat band wavefunction can be written as

$$\psi^a_k(r) = \hat{p}_3\phi_i(r),$$

$$\psi^b_k(r) = -\hat{p}_1\phi_i(r),$$

$$\psi^c_k(r) = \hat{p}_2\phi_i(r),$$  \hspace{1cm} (31)

where $i$ is some index denoting the quantum state (which is no longer the momentum). It is worth noting that when this wavefunction is substituted back into the Hamiltonian equations for $E = -2J_1$, we get

$$0 = 2\psi^a + H_2 e^{i\phi_C} \psi^b + H_3 \psi^c + \mathcal{O}(p^3)$$

$$= 2p_3 + (2 - p_3^2)(1 + i\phi_C)(-p_1) + (2 - p_1^2)p_2 + \mathcal{O}(p^3)$$

$$= 2(p_3 + 2p_2 - p_1) - 2ip_1\phi_C + p_2p_1 - p_1^2p_2 + \mathcal{O}(p^3)$$

$$= -2ip_1(\phi_C - \phi_T) + \mathcal{O}(p^3)$$

$$= 0 + \mathcal{O}(p^3).$$  \hspace{1cm} (32)

Note that since our equations are derived correct to $\mathcal{O}(p^2)$, we conclude that the wavefunction guess in Eq. (31) is correct to $\mathcal{O}(p^2)$. 

We require $\int r \Psi^\dagger(r)\Psi(r) = 1$. From Eq. (12) of the main text and using the form of $f(r)$, we see that

$$N^2 = \int r \left[ \sum \{p_i^* f(r)\} \{p_i f(r)\} \right]$$

$$= \int r \left[ \sum \{k_i f(r)\}^2 + \{A_i f(r)\}^2 \right]$$

$$= \frac{3}{2} \int r \left[ (\partial_x f(r))^2 + (\partial_y f(r))^2 + A_L^2 f^2(r) \right]$$

$$= \frac{1}{2} \left[ 1 + gB^2l_1^4 \right] \left[ 1 + gB^2l_4^4 \right], \quad (33)$$

where $g$ is a gauge dependent constant factor. If $A_L$ is chosen in Landau gauge, $g = 1/2$. If $A_L$ is chosen in symmetric gauge, $g = 1/4$.

MULTI-COMPONENT SLATER DETERMINANT CONSTRUCTION OF MANY-BODY STATE

If the single particle states are given by $\psi^n_a(r)$, where $n$ denotes the band and $a$ denotes the component of the wavefunction, the most general construction of a fermionic $N$-body state is

$$\Psi_{\{s_i\}}(\{r\}) = M_{\{s_i\}}^{r_1 \ldots r_N} \times \frac{1}{\sqrt{N!}} \text{Det} \left[ \psi^{n_1, n_2, \ldots, n_N}_{s_1, s_2, \ldots, s_N}(r_1, r_2, \ldots, r_N) \right], \quad (34)$$

where $s, s_i \in \{a, b, c\}$, the repeated indices are summed over, and the Slater determinant is formed out of the indices $n_i$ and $r_i$. The tensor $M$ superposes the anti-symmetrized Slater determinants for various combinations of $\{s_i\}$. The constraints on $M$ are enforced by requiring the many-body wavefunction be normalized to unity, and that the permutation properties of $M$ must respect antisymmetrization.

The construction of this state formally allows for any choice of wavefunctions based on the choice of $M$. While this does provide a variational knob for multi-component Hamiltonians, the problem is often rendered intractable due to the size of the tensor $M$. In the scheme chosen in the main text, we have absorbed the choice of $M$ in to the construction of $E_n$. Due to the summation over indices $\{ab\}$ in the calculation of $E_n$, the degrees of freedom of $M$ are summed over. This means that different choices of $M$ changes the function $E_n$. Thus one may argue that a large dimensional parametric dependence of $M$ has been mapped to the functional dependence of $E_n$. In our example in the main text, the energy functional we chose corresponding to the Fourier transform of the $k$-dependent eigenvalue. The corresponding choice of $M$ is highly non-trivial as its components will have to depend on $k$. We thus advocate the use of the functional approach over the $M$-tensor approach.