Research Article

Hermite–Hadamard and Fractional Integral Inequalities for Interval-Valued Generalized $p$-Convex Function

Zhengbo Li,1 Kamran,2 Muhammad Sajid Zahoor,3 and Huma Akhtar3

1Moutai Institute, RenHuai 564500, China
2Department of Mathematics, Islamia College Peshawar, Peshawar, Khyber PakhtoonKhwa, Pakistan
3Department of Mathematics, University of Okara, Okara, Pakistan

Correspondence should be addressed to Kamran; kamrank44@gmail.com

Received 23 August 2020; Revised 7 September 2020; Accepted 25 September 2020; Published 21 October 2020

Academic Editor: Sunil Kumar

Copyright © 2020 Kamran et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In the present paper, the new interval-valued generalized $p$ convex functions are introduced. By using the notion of interval-valued generalized $p$ convex functions and some auxiliary results of interval analysis, new Hermite–Hadamard and Fejér type inequalities are proved. The established results are more generalized than existing results in the literature. Moreover, fractional integral inequality for this generalization is also established.

1. Introduction

The theory of interval analysis introduced in numerical analysis by Moore in [1] had rapid development in last few decades. In computational problems, one can program a computer to find interval that contains the exact answer to the problems. Also, interval analysis provides rigorous enclosure of solution to the model equation. Moreover, the interval analysis is widely used in chemical and structured engineering, economics, control circuitry design, robotics, beam physics, behavioral ecology, constraint satisfaction, computer graphics, signal processing, asteroid orbits and global optimization [2], neural network output optimization [3], and many others. For interesting fundamental results, we refer [2, 4–8] to the readers.

Since the convexity play a vital role not only in convex analysis but also in almost all branches of mathematics. The convexity in convex analysis are Jensen type, Hermite–Hadamard type, Fejér type, Ostrowski type, etc. For example, in [20], Nchama et al. used the Caputo-Fabrizio fractional integral and gave some new inequalities. For detailed applications of fractional calculus, we refer [21–28] to the readers and references therein.

In order to introduce the main definition of this paper, let us recall few generalizations of convexity present in the literature.

Definition 1 (see [17]). An interval $I_1$ is $p$-convex set, if for any $x_1, x_2 \in I_1$, $\alpha_1 \in [0, 1]$, we have

$$\alpha_1 x_1^p + (1 - \alpha_1) x_2^p \in I_1,$$

where $p = 2k_1 + 1$ or $p = (n_1/m_1)$, $n_1 = 2r_1 + 1$, $m_1 = 2t_1 + 1$, and $k_1, r_1, t_1 \in N$.

Definition 2 (see [17]). A mapping $f$ defined from a $p$-convex set $I_1$ to $\mathbb{R}$ is said to be $p$-convex function, if

$$f\left(\alpha_1 x_1^p + (1 - \alpha_1) x_2^p \right)^{(1/p)} \leq \alpha_1 f(x_1) + (1 - \alpha_1) f(x_2),$$

for each $x_1, x_2 \in I_1$ and $\alpha_1 \in [0, 1]$ hold.
Definition 3 (see [29]). The mapping \( f \) defined from \( I_1 \) to \( \mathbb{R} \) is said to be \( \eta \)-convex if
\[
 f(a_1 x_1 + (1 - a_1) x_2) \leq a_1 f(x_1) + (1 - a_1) f(x_2)
\] (3)
holds with respect to \( \eta: B_1 \times B_1 \rightarrow B_2 \) for appropriate \( B_1, B_2 \subseteq \mathbb{R} \), and for each \( x_1, x_2 \in I_1, \), \( a_1 \in [0, 1] \).

Definition 4 (see [29]). A mapping is nonnegatively homogeneous if \( \eta(ax_1, ax_2) = \eta(x_1, x_2) \) for each \( x_1, x_2 \in \mathbb{R} \) and \( a \geq 0 \).

Definition 5 (see [30]). A mapping \( f \) defined from a \( p \)-convex set \( I_1 \) to \( \mathbb{R} \) is said to be generalized \( p \)-convex function, if
\[
f\left(\left[a_1 x_1^p + (1 - a_1)x_2^p\right]^{1/p}\right) \leq a_1 f(x_1) + (1 - a_1) f(x_2)
\] (4)
holds for \( \eta: B_1 \times B_1 \rightarrow B_2 \) be a bifunction for appropriate \( B_1, B_2 \subseteq \mathbb{R} \), and for each \( x_1, x_2 \in I_1, \) and \( a_1 \in [0, 1] \).

Now, we present the concept of interval-valued generalized \( p \)-convex function.

Definition 6. A mapping \( f \) defined from a \( p \)-convex set \( I_1 \) to \( \mathbb{R} \) is said to be interval-valued generalized \( p \)-convex function, if
\[
f\left(\left[a_1 x_1^p + (1 - a_1)x_2^p\right]^{1/p}\right) \geq a_1 f(x_1) + (1 - a_1) f(x_2)
\] (5)
holds for \( \eta: B_1 \times B_1 \rightarrow B_2 \) be a bifunction for appropriate \( B_1, B_2 \subseteq \mathbb{R} \), and for each \( x_1, x_2 \in I_1, \) and \( a_1 \in [0, 1] \).

Here, for \( \bar{f} = f \) and \( p = 1 \), (5) is an \( \eta \)-convexity, for \( \bar{f} = f \) and \( \eta(x_1, x_2) = x_1 - x_2 \), (5) is \( p \)-convexity, and for \( p = 1 \) and \( \eta(x_1, x_2) = x_1 - x_2 \), (5) is classical convexity.

This article is in the direction of the concepts and some results discussed in [30], but now we use interval-valued generalized \( p \)-convex function instead of generalized \( p \)-convex function. After this introduction, in Section 2, we develop some basic properties of interval-valued generalized \( p \)-convex functions. In Section 3, we make some new inequalities like Hermite–Hadamard’s and Fejér type for interval-valued generalized \( p \)-convex functions.

2. Basic Results

Here, we derive some operations which preserves interval-valued generalized \( p \)-convex function.

Proposition 1. Let \( f_1 \) and \( f_2 \) be two interval-valued generalized \( p \)-convex functions:

1. If \( \eta \) is additive, then \( f_1 + f_2 \) is interval-valued generalized \( p \) convex.

2. If \( \eta \) is nonnegatively homogeneous, then \( \lambda f_1 \) is interval-valued generalized \( p \)-convex for any \( \lambda \geq 0 \).

Proof. The proof is straightforward. \( \square \)

Theorem 1. Let \( f: [r, s] \rightarrow R^+_I \) be an interval-valued function such that \( f(\lambda) = \left[ f(\lambda), \bar{f}(\lambda) \right] \), then \( f \in SX((\eta, p), [r, s], R^+_I) \) if \( f \in SV ((\eta, p), [r, s], R^+_I) \) and \( \bar{f} \in SX((\eta, p), [r, s], R^+_I) \).

Proof. Let \( f \in SX((\eta, p), [r, s], R^+_I) \), then for any \( x, y \in [r, s] \), \( \lambda \in (0, 1) \), we have
\[
f(y) + \eta(\lambda f(x), (1 - \lambda) y) \leq f(\lambda x^p + (1 - \lambda) y^p),
\] (6)
that is,
\[
\left[f(y) + \eta(\lambda f(x), (1 - \lambda) y)\right] \leq \left[f(\lambda x^p + (1 - \lambda) y^p)\right].
\] (7)

It follows that
\[
f(y) + \eta(\lambda f(x), (1 - \lambda) y) \leq \lambda f(x)^p + (1 - \lambda) y^p,
\] (8)
\[
f(y) + \eta(\lambda f(x), (1 - \lambda) y) \leq \lambda f(x)^p + (1 - \lambda) y^p.
\] (9)

Conversely, suppose that
\[
f \in SX((\eta, p), [r, s], R^+_I),
\]
\[
\bar{f} \in SV ((\eta, p), [r, s], R^+_I).
\] (10)

Then, it follows that \( f \in SX((\eta, p), [r, s], R^+_I) \). This completes the proof. \( \square \)

Theorem 2. Let \( f: [r, s] \rightarrow R^+_I \) be an interval-valued function such that \( f(\lambda) = \left[ f(\lambda), \bar{f}(\lambda) \right] \), then \( f \in SV((\eta, p), [r, s], R^+_I) \) if \( f \in SV ((\eta, p), [r, s], R^+_I) \) and \( \bar{f} \in SX((\eta, p), [r, s], R^+_I) \).

Proof. The proof is similar to that of Theorem 1. \( \square \)

3. Hermite–Hadamard-Type Inequality for Interval-Valued Generalized \( p \) Convex Function

In the following theorem, we present the Hermite–Hadamard type inequality for interval-valued generalized \( p \)-convex function.

Theorem 3. Let \( f: I \rightarrow R \) be an interval-valued generalized \( p \)-convex function for \( \xi_1, \xi_2 \in I \) with condition \( \xi_1 < \xi_2 \), then we obtain the following inequality:
implies
\[
\frac{p}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} x^{p-1} f(x) \, dx \\
\geq f(\xi_1) + f(\xi_2) + \frac{1}{4} [\eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1))].
\]

Proof. Take \( u^p = t\xi_1^p + (1-t)\xi_2^p \) and \( v^p = (1-t)\xi_1^p + t\xi_2^p \), it implies
\[
\frac{\xi_1^p + \xi_2^p}{2} = \frac{u^p + v^p}{2} - \frac{t}{2}(\xi_1^p - \xi_2^p)
\]
So,
\[
f\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} = f\left(\frac{u^p + v^p}{2} \right)^{\frac{1}{p}}.
\]
By definition of interval-valued generalized \( p \) convex functions, we have
\[
\left[ f\left(\frac{\xi_1^p + \xi_2^p}{2}\right) , f\left(\frac{\xi_1^p + \xi_2^p}{2}\right) \right],
\]
\[
\geq \left[ f\left(\frac{(1-t)\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} + \frac{1}{2} \eta\left(f\left(\frac{(1-t)\xi_1^p + (1-t)\xi_2^p}{2}\right)^{\frac{1}{p}} , f\left((1-t)\xi_1^p + \xi_2^p\right)^{\frac{1}{p}}\right),
\right.
\]
\[
\left. \int f((1-t)\xi_1^p + t\xi_2^p)^{\frac{1}{p}} + \frac{1}{2} \eta\left( f\left((1-t)\xi_1^p + t\xi_2^p\right)^{\frac{1}{p}} , f\left((1-t)\xi_1^p + (1-t)\xi_2^p\right)^{\frac{1}{p}} \right) \right].
\]
It follows that
\[
f\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} \leq f\left(\frac{(1-t)\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}},
\]
\[
+ \frac{1}{2} \eta\left( f\left(\frac{(1-t)\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} , f\left((1-t)\xi_1^p + \xi_2^p\right)^{\frac{1}{p}}\right),
\]
\[
f\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} \geq f\left(\frac{(1-t)\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}},
\]
\[
+ \frac{1}{2} \eta\left( f\left((1-t)\xi_1^p + \xi_2^p\right)^{\frac{1}{p}} , f\left((1-t)\xi_1^p + (1-t)\xi_2^p\right)^{\frac{1}{p}}\right).
\]
Integrating (17) with respect to “x” on [0, 1], we get
\[
f\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} \leq f\left(\frac{(1-t)\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}},
\]
\[
+ \frac{1}{2} \int_{\xi_1}^{\xi_2} f(x)^{\frac{1}{p}} \eta\left( f(x)^{\frac{1}{p}} , f\left((1-t)\xi_1^p + \xi_2^p\right)^{\frac{1}{p}}\right) dx,
\]
\[
f\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}} \geq f\left(\frac{(1-t)\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}},
\]
\[
+ \frac{1}{2} \int_{\xi_1}^{\xi_2} f(x)^{\frac{1}{p}} \eta\left( f(x)^{\frac{1}{p}} , f\left((1-t)\xi_1^p + \xi_2^p\right)^{\frac{1}{p}}\right) dx.
\]
Now,

\[
\int_{\xi_1}^{\xi_2} x^{p-1} f(x)dx = \frac{\xi_2^p - \xi_1^p}{p} \int_0^1 f \left( t\xi_1^p + (1-t)\xi_2^p \right)^{(1/p)} dt,
\]

which implies

\[
\frac{\xi_2^p - \xi_1^p}{p} \int_0^1 \frac{1}{2} \left( \frac{\xi_1^p + \xi_2^p}{2} \right)^{(1/p)} dt + \frac{1}{2} \int_0^1 \eta \left( f \left( t\xi_1^p + (1-t)\xi_2^p \right)^{(1/p)}, \right. \\
\left. \frac{1}{2} \left( \frac{\xi_1^p + \xi_2^p}{2} \right)^{(1/p)} \right) dt,
\]\n
which implies

\[
\frac{\xi_2^p - \xi_1^p}{p} \int_0^1 f(x)dx \leq f(\xi_1) + \int_0^1 t \eta(f(\xi_1), f(\xi_2)) dt.
\]

Similarly,

\[
\frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} f(x)dx \leq f(\xi_1) + \int_0^1 t \eta(f(\xi_1), f(\xi_2)) dt.
\]

Adding (21) and (22), we obtain

\[
\frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} f(x)dx \leq f(\xi_1) + \int_0^1 t \eta(f(\xi_1), f(\xi_2)) dt + \frac{1}{4} \left[ \eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1)) \right].
\]

Now, integrating (18) with respect to “x” on [0, 1], we get

\[
\int_{\xi_1}^{\xi_2} x^{p-1} f(x)dx = \frac{\xi_2^p - \xi_1^p}{p} \int_0^1 f \left( t\xi_1^p + (1-t)\xi_2^p \right)^{(1/p)} dt,
\]

which implies

\[
\frac{\xi_2^p - \xi_1^p}{p} \int_0^1 f(x)dx \leq f(\xi_1) + \int_0^1 t \eta(f(\xi_1), f(\xi_2)) dt.
\]

Similarly,

\[
\frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} f(x)dx \leq f(\xi_1) + \int_0^1 t \eta(f(\xi_1), f(\xi_2)) dt.
\]

Adding (26) and (27), we obtain
\[ \frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} f(x) \, dx \geq \frac{\mathcal{F}(\xi_1) + \mathcal{F}(\xi_2)}{2} + \frac{1}{4} \left[ \eta(\mathcal{F}(\xi_1), \mathcal{F}(\xi_2)) \right]. \]

Combining (20) and (21), we obtain
\[ \int \left( \frac{\xi_2^p + \xi_1^p}{2} \right)^{(1/p)} \left( \frac{p}{2(\xi_2^p - \xi_1^p)} \right) \int_{\xi_1}^{\xi_2} x^{p-1} \eta \left( f(\xi_2^p + \xi_1^p - x^p), f(x) \right) \, dx, \]
\[ \leq \frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} f(x) \, dx, \]
\[ \leq \frac{f(\xi_1) + f(\xi_2)}{2} + \frac{1}{4} \left[ \eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1)) \right]. \]

Equations (29) and (31) follows:
\[ \left[ f \left( \frac{\xi_2^p + \xi_1^p}{2} \right) \right]^{(1/p)} \left( \frac{p}{2(\xi_2^p - \xi_1^p)} \right) \int_{\xi_1}^{\xi_2} x^{p-1} \eta \left( f(\xi_2^p + \xi_1^p - x^p), f(x) \right) \, dx, \]
\[ \leq \frac{f(\xi_1) + f(\xi_2)}{2} + \frac{1}{4} \left[ \eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1)) \right]. \]

which completely follows (11) \hfill \square

Remark 1. By putting \( \mathcal{F} = f \) and \( p = 1 \), (11) becomes Hermite–Hadamard type inequality for \( \eta \)-convexity [18].

Remark 2. By putting \( \mathcal{F} = f \) and \( \eta(\xi_1, \xi_2) = \xi_1 - \xi_2 \) in (11), we obtain Hermite–Hadamard type inequality for \( p \)-convexity [17].

Remark 3. By putting \( \mathcal{F} = f \), \( p = 1 \) and \( \eta(\xi_1, \xi_2) = \xi_1 - \xi_2 \) in (11), we get classical Hermite–Hadamard type inequality for convex functions.

Example 1. Consider \( \eta(x, y) = x - y, [\xi_1, \xi_2] = [-1, 1] \) and \( f: [r, s] \rightarrow \mathbb{R}^+ \) be defined by \( f(\lambda) = [\lambda^p, 4 - e^{\lambda^p}] \) with \( p \) as an odd number, then we have
\[ f \left( \frac{\xi_2^p + \xi_1^p}{2} \right) \left( \frac{p}{2(\xi_2^p - \xi_1^p)} \right) \int_{\xi_1}^{\xi_2} x^{p-1} \eta \left( f(\xi_2^p + \xi_1^p - x^p), f(x) \right) \, dx, \]
\[ \leq \frac{f(\xi_1) + f(\xi_2)}{2} + \frac{1}{4} \left[ \eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1)) \right]. \]

Put \( z = x^p \) and simplify, we get
\[ \frac{\int_{\xi_1}^{\xi_2} x^{p-1} f(x) \, dx}{\xi_2 - \xi_1} = \frac{1}{2} [0, 8 - (e^1 - e^{-1})] \]

\[ = \left[ 0, 4 - \frac{(e^1 - e^{-1})}{2} \right]. \]

\[ \begin{align*}
\frac{f(\xi_1) + f(\xi_2)}{2} &+ \frac{1}{4} \left[ \eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1)) \right], \\
&= \left[ \frac{\xi_1^p, \xi_2^p - e^{\eta}}{2} + \frac{\xi_2^p, \xi_1^p - e^{\eta}}{2} + \frac{1}{4} \left[ f(\xi_1) - f(\xi_2) \right] \\
&+ \left[ f(\xi_2) - f(a) \right], \\
&= \left[ -1, 4 - e^{-1} \right] \cup \left[ 1, 4 - e^1 \right] = \left[ 0, 4 - \frac{(e^1 + e^{-1})}{2} \right].
\end{align*} \]

(35)

Combining (32), (34), and (35), we get

\[ [0, 3] \supseteq \left[ 0, 4 - \frac{(e^1 - e^{-1})}{2} \right] \supseteq \left[ 0, 4 - \frac{(e^1 + e^{-1})}{2} \right]. \]

(36)

4. Fejér-Type Inequality for Interval-Valued Generalized p Convex Function

Now, we develop Fejér type inequality for interval-valued generalized p convex functions.

**Theorem 4.** Let \( f \) and \( g \) be nonnegative interval-valued generalized \( p \) convex functions \( \xi_1, \xi_2 \in I \xi_1 < \xi_2 \) such that \( f, g \in L_1[\xi_1, \xi_2] \), then

\[ \frac{p}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} x^{p-1} f(x)g(x) \, dx \geq M(\xi_1, \xi_2) + \frac{1}{2} N(\xi_1, \xi_2), \]

(37)

where

\[ M(\xi_1, \xi_2) = f(\xi_2)g(\xi_2) + \frac{1}{4} \eta(f(\xi_1), f(\xi_2))\eta(g(\xi_1), g(\xi_2)), \]

\[ N(\xi_1, \xi_2) = f(\xi_2)\eta(g(\xi_1), g(\xi_2)) + g(\xi_2)\eta(f(\xi_1), f(\xi_2)). \]

(38)

**Proof.** Since \( f \) and \( g \) are interval-valued generalized \( p \) convex functions, we have

\[ f\left( \left[ t\xi_1^p + (1-t)\xi_2^p \right]^{(1/p)} \right) \geq f(\xi_2) + t\eta(f(\xi_1), f(\xi_2)), \]

\[ g\left( \left[ t\xi_1^p + (1-t)\xi_2^p \right]^{(1/p)} \right) \geq g(\xi_2) + t\eta(g(\xi_1), g(\xi_2)), \]

(39)

for all \( t \in [\xi_1, \xi_2] \). Since \( f \) and \( g \) are nonnegative,
\[
\frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} f(x) g(x) dx,
\]
\[
\leq f(\xi_2) g(\xi_2) + \frac{1}{2} f(\xi_2) \eta(g(\xi_1), g(\xi_2))
\]
\[
+ \frac{1}{2} g(\xi_2) \eta(f(\xi_1), f(\xi_2)),
\]
\[
+ \frac{1}{3} \eta(f(\xi_1), f(\xi_2)) \eta(g(\xi_1), g(\xi_2)).
\]
Integrating (43) over (0, 1), we get
\[
\int_0^1 \mathcal{I}(t\xi_1^p + (1-t)\xi_2^p)^{(1/p)} \mathcal{g}(t\xi_1^p + (1-t)\xi_2^p)^{(1/p)}) dt,
\]
\[
\geq \int_0^1 \mathcal{I}(\xi_2) \mathcal{g}(\xi_2) dt + \int_0^1 t\mathcal{I}(\xi_2) \eta(\mathcal{g}(\xi_1), \mathcal{g}(\xi_2)) dt
\]
\[
+ \int_0^1 t\mathcal{g}(\xi_2) \eta(\mathcal{I}(\xi_1), \mathcal{I}(\xi_2)) dt,
\]
\[
+ \int_0^1 t^2 \eta(\mathcal{I}(\xi_1), \mathcal{I}(\xi_2)) \eta(\mathcal{I}(\xi_1), \mathcal{I}(\xi_2)) dt.
\]
(46)

Then, we obtain the inequality (37).

**Remark 4.** If we put \( \mathcal{I} = f \), \( p = 1 \) and \( \eta(x, y) = x - y \) in (37), then it reduces to classical convex functions.

### 5. Fractional Hermite–Hadamard-Type Inequalities for Interval-Valued Generalized \( p \) Convex Functions

The fractional inequalities has applications in every field of science and engineering. The new fractional integral inequalities in analysis are always appreciable. Because of the wide applications of Hermite–Hadamard-type inequalities and fractional integrals, many researchers extended their studies to Hermite–Hadamard-type inequality involving fractional integral inequalities. For fractional integral inequalities for interval-valued function, we suggest the reader to refer [31, 32].

**Definition 7** (see [33–35]). Let \( \phi \in L[a, b] \). The right-hand side and left-hand side Riemann-Liouville fractional integral of order \( \alpha > 0 \) with \( b > a > 0 \) are defined by
**Definition 8.** Let \( \rho \in \mathbb{R}/0 \). A function \( w: [a, b] \subset (0, \infty) \rightarrow \mathbb{R} \) is said to be \( p \)-symmetric with respect to \([a^p + b^p/2])^{(1/p)} \) if \( w(x) = w([a^p + b^p - x^p])^{(1/p)} \) and holds for all \( x, y \) such that \( x \in [a, b] \).

Following lemma will help us in obtaining our fractional integrals inequalities which can be found in [36].

**Lemma 1.** Let \( p \in (\mathbb{R}/0), \alpha > 0 \) and \( w: [a, b] \subset (0, \infty) \rightarrow \mathbb{R} \) be a generalized \( p \)-convex function with \( p \geq 0 \) and \( \eta \) is bounded above by \( M_\eta \). Take \( x = (ka^p + (1-k)b^p)^{(1/p)} \) and \( y = (kb^p + (1-k)a^p)^{(1/p)} \). Since

\[
\phi\left(\frac{x^p + y^p}{2}\right) - M_\eta \geq \frac{\phi(x) + \phi(y)}{2} + M_\eta,
\]

(53)

Multiplying both sides of (54) by \( k^{a-1} \) and then integrating the resulting inequality with respect to \( k \) over \([0, (1/2)]\), we obtain

\[
\int_{0}^{(1/2)} \phi\left(\frac{a^p + b^p}{2}\right)^{(1/p)} k^{a-1} \, dk - \int_{0}^{(1/2)} M_\eta k^{a-1} \, dk,
\]

(55)

By definition of RiemannLiouville integrable function with \( g(x) = x^{(1/p)} \), we obtain

\[
\phi\left(\frac{a^p + b^p}{2}\right)^{(1/p)} - M_\eta \geq \frac{\Gamma(a + 1)}{b^p - a^p}\left[\frac{\int_{(a+\nu/2)}^{(1/2)} \, dk}{\left(\Gamma(a + 1)\right)^{2\nu}}\right] \cdot \Gamma(a + 1) \left[\frac{\int_{(a+\nu/2)}^{(1/2)} \, dk}{\left(\Gamma(a + 1)\right)^{2\nu}}\right] \cdot \Gamma(a + 1) \left[\frac{\int_{(a+\nu/2)}^{(1/2)} \, dk}{\left(\Gamma(a + 1)\right)^{2\nu}}\right],
\]

(56)

which is the left-hand side of theorem (56).
To prove the right-hand side, we take $x = (ka^p + (1 - k)b^p)^{(1/p)}$ and $y = (kb^p + (1 - k)a^p)^{(1/p)}$:

\[
\phi(ka^p + (1 - k)b^p)^{(1/p)} \geq \phi(b) + k\eta(\phi(a), \phi(b)), \quad (57)
\]

\[
\phi(kb^p + (1 - k)a^p)^{(1/p)} \geq \phi(a) + k\eta(\phi(b), \phi(a)). \quad (58)
\]

Adding the (57) and (58) and multiplying the resulting inequality with $2k^{\alpha-1}$ and integrating with respect to $k$ over $[0, 1/2$), we obtain

\[
2 \int_0^{(1/2)} \phi(kb^p + (1 - k)a^p)^{(1/p)} k^{\alpha-1} dk 
+ 2 \int_0^{(1/2)} \phi(kb^p + (1 - k)a^p)^{(1/p)} k^{\alpha-1} dk,
\]

\[
\geq 2 \int_0^{(1/2)} \phi(a) k^{\alpha-1} dk + 2 \int_0^{(1/2)} \phi(b) k^{\alpha-1} dk,
\]

\[
+ \frac{(\eta(\phi(b), \phi(a)) + \eta(\phi(a), \phi(a)))}{2^\alpha(\alpha + 1)}.
\]

By definition of Riemann-Liouville integrable function, we get

\[
\frac{2\Gamma(\alpha)}{(b^p - a^p)^{2\alpha+1}} \left[ \int_{(a^p+b^p/2)}^a \phi \rho g(b^p) + \int_{(a^p+b^p/2)}^b \phi \rho g(a^p) \right] 
\geq \frac{\phi(a) + \phi(b)}{2} + 2N_\eta.
\]

Rearranging the above inequality, we get the right-hand side:

\[
\frac{\Gamma(\alpha + 1)}{(b^p - a^p)^{2\alpha+1}} \left[ \int_{(a^p+b^p/2)}^a \phi \rho g(b^p) + \int_{(a^p+b^p/2)}^b \phi \rho g(a^p) \right] 
\geq \frac{\phi(a) + \phi(b)}{2} + 2N_\eta.
\]

This completes the proof. □

**Remark 5.** If we put $p = 1, \eta(x, y) = x - y$, and $\phi = \overline{\phi}$, then we will get Hermite–Hadamard-type inequality for fractional function for classical convex function [37].

### 6. Conclusions

The convex functions and fractional calculus play an important role in applied sciences [38–43]. Here, the new interval-valued generalized convex functions are introduced. By using the notion of interval-valued generalized $p$ convex functions and some auxiliary results of interval analysis, some new Hermite–Hadamard- and Fejér-type inequalities are presented. Our results can be considered as generalization of many existing results. Moreover, fractional integral inequality for this generalization is also established.

**Data Availability**

The data used to support the article are available within the article.

**Conflicts of Interest**

The authors declare that do not have any conflicts of interest.

**Authors’ Contributions**

All the authors contributed equally to this paper.

**Acknowledgments**

This research was supported in part by the Higher Education Commission, Pakistan.

**References**

[1] R. E. Moore, *Interval Analysis*, Prentice-Hall, Inc., Englewood Cliffs, NJ, USA, 1966.

[2] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, PA, USA, 2009.

[3] E. D. Weerd, Q. P. Chu, and J. A. Mulder, “Neural network output optimization using interval analysis,” *IEEE Transactions on Neural Networks*, vol. 20, no. 4, pp. 638–653, 2009.

[4] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, and M. D. Jiménez-Gamero, “Calculus for interval-valued functions using generalized Hukuhara derivative and applications,” *Fuzzy Sets and Systems*, vol. 219, pp. 49–67, 2013.

[5] Y. Chalco-Cano, G. N. Silva, and A. Rufián-Lizana, “On the Newton method for solving fuzzy optimization problems,” *Fuzzy Sets and Systems*, vol. 272, pp. 60–69, 2015.

[6] T. M. Costa, H. Bouwmeester, W. A. Lodwick, and C. Lavor, “Calculating the possible conformations arising from uncertainty in the molecular distance geometry problem using constraint interval analysis,” *Information Sciences*, vol. 415-416, pp. 41–52, 2017.

[7] T. M. Costa, Y. Chalco-Cano, W. A. Lodwick, and G. N. Silva, “Generalized interval vector spaces and interval optimization,” *Information Sciences*, vol. 311, pp. 74–85, 2015.

[8] R. Osuna-Gómez, Y. Chalco-Cano, B. Hernández-Jiménez, and G. Ruiz-Garzón, “Optimality conditions for generalized differentiable interval-valued functions,” *Information Sciences*, vol. 321, pp. 136–146, 2015.

[9] L. Fejér, “Üdie fourierreihen, II,” *Math. Naturwiss. Anz Ungar. Akad. Wiss*, vol. 24, pp. 369–390, 1906.

[10] S. M. Aslani, M. R. Delavar, and S. M. Vaezpour, “Inequalities of Fejér type related to generalized convex functions with applications,” *International Journal of Analysis and Applications*, vol. 16, no. 1, pp. 38–49, 2018.

[11] S. S. Dragomir, J. Pecaric, and L. E. Persson, “Some inequalities of Hadamard type,” *Soochow Journal of Mathematics*, vol. 21, pp. 335–341, 1995.

[12] S. S. Dragomir, “Refinements of the Hermite-Hadamard inequality for convex functions,” *Tamkang Journal of Mathematics*, vol. 32, no. 1, pp. 1–7, 2001.

[13] F. C. Mitroi and C. I. Spiridon, “Refinements of Hermite-Hadamard inequality on simplices,” *Mathematical Reports (Bucuresti)*, vol. 15, no. 65, pp. 69–78, 2013.

[14] M. Raisouli and S. S. Dragomir, “Refining recursively the Hermite-Hadamard inequality on a simplex,” *Bulletin of the
M. A. Noor, K. I. Noor, and M. U. Awan, “Hermite–Hadamard Inequalities for modified $h$-convex functions,” TJMM, vol. 6, no. 5, 2014.

J. E. Pecaric, F. Proschan, and Y. L. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, Boston, MA, USA, 1992.

K. S. Zhang and J. P. Wan, “$P$-convex functions and their properties,” Pure and Applied Mathematics Number, vol. 1, no. 23, pp. 130–133, 2007.

M. R. Delavar and S. S. Dragomir, “On $h$-convexity,” Journal of Mathematical Inequalities Number, vol. 1, no. 20, p. 203, 2017.

S. Varošanec, “On $h$-convexity,” Journal of Mathematical Analysis and Applications, vol. 326, pp. 303–311, 2007.

S. Mehmoood, G. Farid, and G. Farid, “Fractional integrals inequalities for exponentially $m$-convex functions,” Open Journal of Mathematics Sciences, vol. 4, no. 1, pp. 78–85, 2020.

S. Kumar, K. S. Nisar, R. Kumar, C. Cattani, and B. Samet, “A new Rabotnov fractional-exponential function-based fractional derivative for diffusion equation under external force,” Mathematical Methods in the Applied Sciences, vol. 43, no. 7, pp. 4460–4471, 2020.

Z. Odibat and S. Kumar, “A robust computational algorithm of homotopy asymptotic method for solving systems of fractional differential equations,” Journal of Computational and Nonlinear Dynamics, vol. 14, no. 8, 2019.

S. Kumar, S. Ghosh, B. Samet, and E. F. D. Goufo, “An analysis for heat equations arises in diffusion process using new Yang–Abdel-Aty–Cattani fractional operator,” Mathematical Methods in the Applied Sciences, vol. 43, no. 9, pp. 6062–6080, 2020.

M. M. A. Khater, R. A. M. Attia, A.-H. Abdel-Aty, W. Alharbi, and D. Lu, “Abundant analytical and numerical solutions of the fractional microbiological densities model in bacteria cell as a result of diffusion mechanisms,” Chaos, Solitons & Fractals, vol. 136, Article ID 109824, 2020.

S. Owyed, M. A. Abdou, A. H. Abdel-Aty, A. A. Ibrahim, R. Nekhili, and D. Baleanu, “New optical soliton solutions of space-time fractional nonlinear dynamics of microtubes via three integration schemes,” Journal of Intelligent & Fuzzy Systems, vol. 38, no. 6, pp. 1–8, 2020.

M. M. Khater, C. Park, A. H. Abdel-Aty, R. A. Attia, and D. Lu, “On new computational and numerical solutions of the modified ZakharovKuznetsov equation arising in electrical engineering,” Alexandria Engineering Journal, vol. 59, no. 3, pp. 1099–1105, 2020.

M. A. Abdou, S. Owyed, A. Abdel-Aty, B. M. Raffah, and S. Abdel-Khalik, “Optical soliton solutions for a space-time fractional perturbed nonlinear Schrödinger equation arising in quantum physics,” Results in Physics, vol. 16, Article ID 102895, 2020.

L. Qian, R. A. M. Attia, Y. Qiu, D. Lu, and M. M. A. Khater, “On Breather and Cuspon waves solutions for the generalized higher-order nonlinear Schrodinger equation with light-wave pomulgation in an optical fiber,” International Journal for Numerical Methods in Engineering, vol. 1, no. 2, pp. 101–110, 2019.

M. E. Gordji, M. Rostamian, and M. Delasen, “On $\rho$-convex functions,” Journal of Mathematical Inequalities, vol. 10, no. 1, pp. 173–183, 2016.

C. Y. Jung, M. S. Saleem, W. Nazeer, M. S. Zahoor, A. Latif, and S. M. Kang, “Unification of Generalized and $p$-convexity,” Journal of Function Spaces, vol. 23, pp. 1–6, 2020.