ANALYSIS OF THE MOTION OF AN EXTRASOLAR PLANET IN A BINARY SYSTEM

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ABSTRACT

More than 10% of extrasolar planets (EPs) orbit in a binary or multiple stellar system. We investigated the motion of planets revolving in binary systems in the case of the three-body problem. We carried out an analysis of the motion of an EP revolving in a binary system with the following conditions: (1) a planet in a binary system revolves around one of the components (parent star); (2) the distance between the star’s components is greater than that between the parent star and the orbiting planet (ratio of the semi-major axes is a small parameter); and (3) the mass of the planet is smaller than the mass of the stars, but is not negligible. The Hamiltonian of the system without short periodic terms was used. We expanded the Hamiltonian in terms of the Legendre polynomial and truncated after the second-order term, depending on only one angular variable. In this case, the solution of the system was obtained and the qualitative analysis of the motion was produced. We have applied this theory to real EPs and compared to the numerical integration. Analyses of the possible regions of motion are presented. It is shown that stable and unstable motions of EPs are possible. We applied our calculations to two binary systems hosting an EP and calculated the possible values for their unknown orbital elements.

Key words: celestial mechanics – planets and satellites: dynamical evolution and stability – planets and satellites: individual (16 Cyb, HD 19994)

1. INTRODUCTION

More than half of all main-sequence stars reside in binary or multiple systems (Duquennoy & Mayor 1991). At least 10% of the discovered extrasolar planets (EPs) have been observed to be orbiting binary or multiple stellar systems (Schneider et al. 2011). Orbits in multiple stellar systems can have a miscellaneous, in many cases unbelievable and improbable, architecture.

We targeted the binary stellar systems that are hosting EPs. For these systems Dvorak (1986) defined three different possibilities for a stable planetary orbit: S-orbit (S-type)—the Satellite-type orbit, where an EP orbits one of the stars; P-orbit (P-type)—the Planet-type orbit, where an EP surrounds both stars; L-orbit (L-type)—the Libration-type orbit, where an EP is librating around one of the triangular Lagrangian points. We have focused on an S-type orbit in this paper. The EP’s motion in such a system was considered in the frame of the general three-body problem.

2. SETTING UP THE PROBLEM

The motion of an EP is investigated in the general three-body problem, where the planet in the binary system revolves around one of the components (a parent star) and the mass of the planet is much smaller than the masses of the stars, but is not negligible. The distance between the binary components is greater than between the parent star and the orbiting planet (the ratio of the semi-major axes is a small parameter). The motion is considered in the Jacobian coordinate system and the invariant plane is taken as the reference plane. We used the Delaunay canonical elements $L_i, G_i, H_i, l_i, g_i,$ and $h_i$ ($i = 1$ for the planet’s orbit, $i = 2$ for the distant star’s orbit). They can be expressed through the Keplerian elements as

$$L_i = \beta_i \sqrt{a_i}, \quad G_i = L_i \sqrt{1 - e_i^2}, \quad H_i = G_i \cos I_i,$$

$$l_i = M_i, \quad g_i = \omega_i, \quad h_i = \Omega_i,$$  

(1)

where

$$\begin{align*}
\beta_1 &= k \frac{m_0 m_1}{\sqrt{m_0 + m_1}} = k \mu_1, \\
\beta_2 &= k \frac{(m_0 + m_1) m_2}{\sqrt{m_0 + m_1 + m_2}} = k \mu_2.
\end{align*}$$  

(2)

In the previous expression, the notation is defined as follows: $m_0$ and $m_2$—the masses of the stars; $m_1$—the mass of the planet; $k$—the Gaussian constant; $a_i$—the semi-major axis; $e_i$—the eccentricity; $M_i$—the mean anomaly; $I_i, \omega_i,$ and $\Omega_i$—the angular variables to the observational plane; and $g_i$—the argument of the pericenter in the invariant plane (this plane is perpendicular to the angular momentum of the system). The eccentricities of the star’s and planet’s orbits can have any value from 0 to 1.

In general, the motion is defined by the masses of the components and by the six initial values of the Keplerian elements of the planet and the distant star. The solution of a task using the Hamiltonian without short-periodic terms was obtained in the hyper-elliptic integrals by the Hamilton–Jacobi method (Orlov & Solovaya 1988). The short-periodic terms are small and have no significant influence on the dynamic evolution of the system, the values of which are less than $10^{-3}$ which is less than the precision capabilities of the observations. The secular and essential long periodic terms are included in the Hamiltonian. The Hamiltonian expanded in terms of the Legendre polynomials and truncated after the second-order terms carries the following form:

$$F = \frac{\gamma_1}{2L_1^2} + \frac{\gamma_2}{L_2^2} - \frac{1}{16} \gamma_3 \frac{L_1^4}{L_2^2 G_2^2} (1 - 3q^2) (5 - 3q^2) - 15 (1 - q^2) (1 - q^3) \cos (2g_1),$$  

(3)

where the coefficients $\gamma_1, \gamma_2,$ and $\gamma_3$ depend on mass as

$$\gamma_1 = \frac{\beta_1^4}{\mu_1}, \quad \gamma_2 = \frac{\beta_2^4}{\mu_2}, \quad \gamma_3 = k^2 \mu_1 \mu_2 \beta_3 \beta_4,$$  

(4)

and $\gamma_i = \frac{\beta_i^4}{\mu_i}$, where $\mu_i$ is the mass of the $i$th body and $\beta_i$ is the mean Keplerian element.
So, as the Hamiltonian does not depend on \( I_s, g_2, \) and \( h, \) the integral (11) is
\[
W = \varepsilon (t - t_0) + A_1 I_1 + A_2 I_2 + A_4 g_2 + A_5 h + W_1 (g_1),
\]
where \( \varepsilon \) is a new constant and \( W_1 (g_1) \) is a new function which we must define. When we substituted the function \( W_1 (g_1) \), defined by Equations (14) to (12), we found the equation to satisfy the function \( W_1 : \)
\[
\varepsilon + \Phi (g_1, A_1, A_2, W'_1 (g_1), A_4, A_5) = 0,
\]
where \( W'_1 (g_1) \) is the derivation. Using Equation (3), we rewrote the previous equation as
\[
\varepsilon = \gamma_1 \frac{W_2}{2A_1} + \gamma_2 \frac{W_2}{2A_2} - \frac{1}{16} \gamma_3 A_4^2 \frac{A_4}{A_1} A_3,
\]
where the constant \( A_3 \) has the following form:
\[
A_3 = \left[ 1 - 3 \frac{(A_1^2 - A_2^2 - W_1^2)}{4 A_1 W_1^3} \right] \left( 5 - \frac{3 W_2}{W_1^2} \right) - 15 \left[ 1 - \frac{(A_1^2 - A_2^2 - W_1^2)}{4 A_1 W_1^3} \right] \left( 1 - \frac{W_2}{W_1} \right) \cos (2g_1).
\]

Equation (17) is an ordinary differential equation of the first order. If the solution is found, then the general solution of the canonical system is
\[
L_1 = A_1, \quad B_1 = \frac{\partial \varepsilon}{\partial A_1} (t - t_0) + I_1 + \frac{\partial W_1}{\partial A_1},
\]
\[
L_2 = A_2, \quad B_2 = \frac{\partial \varepsilon}{\partial A_2} (t - t_0) + I_2 + \frac{\partial W_1}{\partial A_2},
\]
\[
G_1 = \frac{\partial W_1}{\partial g_1}, \quad B_3 = \frac{\partial \varepsilon}{\partial A_3} (t - t_0) + \frac{\partial W_1}{\partial A_3},
\]
\[
G_2 = A_4, \quad B_4 = \frac{\partial \varepsilon}{\partial A_4} (t - t_0) + g_2 + \frac{\partial W_1}{\partial A_4},
\]
\[
c = A_5, \quad B_5 = \frac{\partial \varepsilon}{\partial A_5} (t - t_0) + h + \frac{\partial W_1}{\partial A_5}.
\]

In the third line of system (18), one can see the dependence between \( \xi \) and time \( t, \) where
\[
\xi = \frac{W_1^2}{A_1^3} = \frac{G_1^2}{A_1^3}.
\]

After the differentiation and the algebraic operations, we obtained the following equation connecting \( \xi \) and \( t: \)
\[
\frac{1}{12} \gamma n^2 \int_{\xi_1}^\xi \frac{d\xi}{\sqrt{A}} = B_3 + \frac{1}{16} \gamma m'' (1 - e_i^2) n_1 (t - t_0),
\]
where \( m'' = n_2 / n_1, n_1 \) and \( n_2 \) are the mean motions of the planet and the distant star, \( \gamma = m'/(m_0 + m_1 + m_2), \) and \( A_1 \) and \( B_3 \) are the constants of integration.

In this approximation, we obtained an exact solution. We used this solution as an intermediary orbit for the planet’s motion in which the second-order perturbations are included.
The expressions for \( a_1 \) and \( a_2 \) do not contain secular terms, and hence are restricted in their time evolution.

The change in eccentricity of a planet is described by the expression \( e_1 = \sqrt{1-\xi} \). When the eccentricity of a planet’s orbit can change by almost as much as one, the planet’s pericenter is near or beyond the Roche limit, and large perturbations and tidal forces lead to the destruction of the planet. Such a system will be dynamically unstable.

On the left side of Equation (20) under the integral in the denominator is the square root from the polynomial of the fifth order. This polynomial can be presented as the product of the two polynomials of the second and third orders, \( \Delta = f_2(\xi) f_3(\xi) \), where \( \xi = 1 - e_1^2 \), and \( e_1 \) is the eccentricity of the planet’s orbit. The determination of the regions of motion is possible when the roots of the equations \( f_2(\xi) = 0 \) and \( f_3(\xi) = 0 \) are found and the signs of the functions are defined.

3. EQUATIONS OF THE SECOND AND THIRD ORDER

We investigated the roots of the equations \( f_2(\xi) = 0 \) and \( f_3(\xi) = 0 \) with the following forms:

\[
f_2(\xi) = \xi^2 - 2 \left( \xi^2 + 3 \bar{G}_2^2 \right) \xi + \left( \xi^2 - \bar{G}_2^2 \right)^2 + \frac{2}{3} (10 + A_3) \bar{G}_2^2
\]

(21)

and

\[
f_3(\xi) = \xi^3 - \left( 2\xi^2 + \bar{G}_2^2 + \frac{5}{4} \right) \xi^2 + \left[ \frac{5}{2} \left( \xi^2 + \bar{G}_2^2 \right) + \left( \xi^2 - \bar{G}_2^2 \right)^2 \right.
- \frac{1}{6} \bar{G}_2^2 (10 + A_3) \left. \right] \xi - \frac{5}{4} \left( \xi^2 - \bar{G}_2^2 \right)^2,
\]

(22)

where

\[
\bar{c} = \frac{c}{L_1}, \quad \bar{G}_2 > 1,
\]

\[
\bar{G}_2 = \frac{G_2}{L_1} = \frac{\beta_2}{\beta_1} \sqrt{\frac{a_2 (1 - e_2^2)}{a_1}.}
\]

(23)

and

\[
A_3 = 2 - 6n_0^2 q_0^2 - 6 \left( 1 - n_0^2 \right) [2 - 5 \left( 1 - q_0^2 \right) \sin^2 g_{10}].
\]

(24)

We note that

\[
n_0 = \sqrt{1 - e_{10}^2},
\]

(25)

where \( e_{10} \) is the initial value of the eccentricity of the planet, \( q_0 \) is the initial value of the cosine of the mutual inclination between the orbits of the planet and the distant star, and \( g_{10} \) is the initial value of the argument of the pericenter of the planet’s orbit in the invariable plane.

If the orbit is assumed to be elliptic, \( 0 < e_1 < 1 \), then \( 0 < \xi < 1 \). The value of \( \xi = 1 \) corresponds to a circular motion. Such a case was not considered in this paper.

To determine the boundaries of regions of possible motion it is necessary to find the roots of Equations (21) and (22) and to define the signs of the function in the interval between the roots.

3.1. The Equation of the Second Order

We rewrote equation of the second order (21) in the following form:

\[
f_2(\xi) = \xi^2 - 2 \left( \xi^2 + 3 \bar{G}_2^2 \right) \xi + \left( \xi^2 - \bar{G}_2^2 \right)^2 + \frac{2}{3} (10 + A_3) \bar{G}_2^2
\]

\[+ \left[ -1 + 2 \left( \xi^2 + 3 \bar{G}_2^2 \right) + \frac{2}{3} \bar{G}_2^2 \bar{h} \right],
\]

(26)

where \( \bar{h} \) is difference between values \( A_3 \) and \( A_{3\text{crit}} \):

\[
\bar{h} = A_3 - A_{3\text{crit}}.
\]

(27)

Then

\[
2\bar{G}_2^2 \bar{h} = 3(1 - q^2) \left[ 1 - 8\bar{G}_2^2 - 4G_2 q\eta - \eta^2\right.
\]

\[+ 20\bar{G}_2^2 \left( 1 - q^2 \right) \sin^2 g_1 \].

(28)

We named the constant \( A_3 \) as \( A_{3\text{crit}} \) when the square equation has a root equal to one. We found \( A_{3\text{crit}} \) when \( f_2(\xi) = 0 \) and \( \xi = 1 \).

We refer to the roots Equation (26) as \( \epsilon_1 \) and \( \epsilon_2 \), and \( \epsilon_1 < \epsilon_2 \). The coefficient of \( \xi \) is always less than zero. This equation has no negative roots and the free term is also greater than zero. The roots are

\[
\epsilon_{1,2} = \left( \xi^2 + 3 \bar{G}_2^2 \right) \pm \sqrt{\left( \xi^2 + 3 \bar{G}_2^2 + 1 \right)^2 - \frac{2}{3} \bar{G}_2^2 \bar{h}},
\]

(29)

and \( \epsilon_2 \gg 1 \) always. The derivation \((d\epsilon_1)/(d\bar{h})\) is always positive, so \( \epsilon_1 \) is a growing function of \( \bar{h} \). When \( \bar{h} = 0 \) then \( \epsilon_1 = 1 \), and we conclude that when \( \bar{h} < 0 \) then \( \epsilon_1 < 1 \), and when \( \bar{h} > 0 \) then \( \epsilon_1 > 1 \). To define the branches of the parabola, we found the extreme of the function that is valid for \( \xi = \bar{c}^2 + 3\bar{G}_2^2 \).

The value of \( f_2(\xi) < 0 \) and the branches of this parabola are directed up.

3.2. Equation of the Third Order

We rewrote Equation (22) using \( \bar{h} \) in the form

\[
f_3(\xi) = \xi^3 - \frac{1}{4} \left( 5 + 8\bar{c}^2 + 4\bar{G}_2^2 \right) \xi^2
\]

\[+ \left[ \frac{1}{4} + 2\bar{c}^2 + \bar{G}_2^2 + \frac{5}{4} \left( \xi^2 - \bar{G}_2^2 \right)^2 \right.
\]

\[- \frac{1}{6} \bar{G}_2^2 \bar{h}] \xi - \frac{5}{4} \left( \xi^2 - \bar{G}_2^2 \right)^2,
\]

(30)

then multiplied this equation by four and identified in its left part the term \( \xi f_2(\xi) \). It is possible to rewrite Equation (30) to take the following form:

\[
f_3(\xi) = 5(1 - \xi) \left[ (\bar{c} + \bar{G}_2)^2 - \xi \right] \left[ \xi - (\bar{c} - \bar{G}_2)^2 \right] - \xi f_2(\xi).
\]

(31)

To determine the qualitative characteristics of the motion, it is necessary to define where the roots of the equation of the third order \( f_3(\xi) = 0 \) are located in relation to the points of the axes:

\[
\xi = 0, \quad \xi = \epsilon_1, \quad \xi = 1, \quad \xi = \epsilon_2.
\]

(32)
1. If $\bar{h} < 0$, then we have:

1. $\xi = 0$, $f_1(0) < 0$.
2. $\xi = \epsilon_1$, $f_2(\epsilon_1) > 0$.
3. $\xi = 1$, $f_3(1) > 0$.
4. $\xi = \epsilon_2$, $f_3(\epsilon_2) > 0$.

From the Sturm theorem (e.g., Dörrie 1965), it follows that between zero and $\epsilon_1$ lies at least one root of the third-order equation.

2. If $\bar{h} > 0$, then we have:

1. $\xi = 0$, $f_3(0) \leq 0$.
2. $\xi = 1$, $f_3(1) < 0$.
3. $\xi = \epsilon_1$, $f_3(\epsilon_1) < 0$.
4. $\xi = \epsilon_2$, $f_3(\epsilon_2) > 0$.

The behaviors of the roots of the second- and third-order equations depend on the sign of $\bar{h}$ and the following validities:

1. for $\bar{h} < 0$, $0 \leq \epsilon_3 < \epsilon_1 < 1$, $\epsilon_4 < \epsilon_3 < \epsilon_2$ is valid.
2. for $\bar{h} > 0$, $0 \leq \epsilon_3 < \epsilon_4 < 1$, $\epsilon_3 < \epsilon_1 < \epsilon_2$ is valid.

We have two roots that are less than one, therefore we identified the roots in ascending order as $\xi_1, \xi_2, \xi_3, \xi_4$, and $\xi_5$. This means that $\epsilon_1$ can vary from $\epsilon_{1\min} = 1 - \xi_2$ to $\epsilon_{1\max} = 1 - \xi_1$. We show the behavior of these functions for the hypothetical values $\bar{e}$, $\bar{c}_2$, and $\bar{h}$ in Figure 1 for $\bar{h} < 0$ and in Figure 2 for $\bar{h} > 0$.

4. THE INVESTIGATION OF THE VALUE $\bar{h}$

Consider the value of $\bar{h}$, which has the following form:

$$
\bar{h} = \frac{3(1 - \eta^2)}{2G_2} \left[ 1 - 8G_2^2 q \eta - \eta^2 \right] + 20G_2^2 (1 - q^2) \sin^2 g_1. \tag{33}
$$

When the value of $g_1 = 0$, then the value of $\bar{h}$ reaches the minimum

$$
\bar{h}_{\min} = \frac{3(1 - \eta^2)}{2G_2} \left[ 1 - 8G_2^2 q \eta - \eta^2 \right]. \tag{34}
$$

The value of the $\bar{h}_{\min}$ is always negative. In this case, we have two roots valued less than one, one root of the equation of the third order and one root of the square equation. If $g_1 = \pi/2$, then $\bar{h}$ has a maximum value

$$
\bar{h}_{\max} = \frac{3(1 - \eta^2)}{2G_2} \left[ 1 - \eta^2 - 4G_2 q \eta + 12G_2^2 - 20G_2^2 q^2 \right]. \tag{35}
$$

The value of $\bar{h}_{\max}$ can either be negative or positive.

For $g_1 = \pi/2$ we rewrote Equations (21) and (22) in the following forms:

$$
f_2(\xi) = (\xi - \eta_0^2) \left( \xi - \eta_0^2 - 4G_2 \eta q_0 + 8G_2^2 \right) + 4G_2^2 (1 - 5q_0^2) (1 - \eta_0^4) + 16G_2^2 \tag{36}
$$

and

$$
f_3(\xi) = (\xi - \eta_0^2) \left[ (\xi - \frac{5}{4}) \left( \xi - \eta_0^2 \right) - 3G_2^2 \xi \right]. \tag{37}
$$

When one of the roots of Equation (37) carries the value of $\xi = \eta_0^2$, the other roots have to satisfy the equation

$$
(\xi - \frac{5}{4}) (\xi - \eta_0^2) - 3G_2^2 \xi = 0. \tag{38}
$$

For $\bar{h}_{\max}$, the initial value $\eta_0^2$ is one of the boundary limits of the change in the value of $\xi$. We should establish that $\xi = \eta_0^2$ is the least root and the second root has a value less than one. We therefore substitute the value $\xi = \eta_0^2$ in Equation (38). If the obtained expression is negative, then $\eta_0^2$ should become the second root in Equation (37), if positive, then it should become the least root. The left part of Equation (38) for $\xi = \eta_0^2$ is

$$
2 \left[ 5q_0^2 - 3\eta_0^2 + \frac{1}{G_2} \eta_0 \eta q_0 (5 - 4\eta_0^2) \right]. \tag{39}
$$

EQUATING THIS EXPRESSION TO ZERO AND SOLVING THE OBTAINED EQUATION CONCERNING $q_0$, WE FIND THE VALUE FOR $q_0$ OF WHICH $\eta_0^2$ IS THE ROOT OF EQUATION (39). So

$$
q = \frac{4\eta_0^2 q_0^2 - 5 \pm \sqrt{60G_2^2 + (5 - 4\eta_0^2)^2}}{10G_2}. \tag{40}
$$

We denote the roots of Equation (40) as $q_0$ for the minus sign before the root term and the $q_{02}$ for the plus sign. If the value of $q_0$ lies within $q_0 < q_0 < q_{02}$, expression (39) is negative, and the value of the root lies between the other two roots. If either of the conditions $q_0 < q_{01}$ or $q_0 > q_{02}$ is valid, then $\xi = \eta_0^2$ is the least root of Equation (38).
In this case, $e_{\text{max}} = \sqrt{1 - \eta_0^2}$ and the maximum value of the eccentricity of the planet’s orbit cannot exceed the initial value of the eccentricity. The orbit of the EP may be dynamically stable.

When the starting value of the cosine of the mutual inclination is $q_0 = -\eta_0/2G_2$, then from Equation (6), $c^2 - G_2^2 \approx 0$ and $e_1 \to 1$.

5. APPLICATION OF THE THEORY ON REAL EXTRA-SOLAR PLANETS

Knowledge of the six pairs of Keplerian elements of the orbits of the system allows us to investigate the character of the evolution of the planet’s orbit and the possible conditions of stability. They may be presented by the orbital parameters, which we obtain from the analytical theory. They are the angle of the mutual inclination between orbits of the planet and star, the angular moment of the star, and the maximum value of the eccentricity of the planet’s orbit. The growth of the eccentricity of the orbit could lead to the destruction of the orbit in pericenter from tidal forces.

In the catalog of EPs, the data for the longitude of the ascending node and the value of inclination are generally absent. The existing observational techniques do not unambiguously allow estimation of these two elements. Our theory allows us to define a range of possible values for these unknown elements, by which the planet’s eccentricity does not increase. In our application of the theory, we have selected two binary stellar systems, with a planet revolving around one of the components. The first is system 16 Cyg with an interesting planet 16 Cyg Bb, which has the argument of pericenter close to 90°. The second system is HD 19994.

5.1. 16 Cyg

16 Cyg A (HD 186408) and 16 Cyg B (HD 186427) are both members of a well-known wide binary system with stars of spectral types G1.5V and G3V. For the calculations, we used orbital elements used by Hauser & Marcy (1999), which were published in the Sixth Catalogue of Orbits of Visual Binary Stars (Mason & Hartkopf 2001). The semi-major axis $a_2$ was calculated to be 754.53 AU using the values 21.41 pc (Fuhrmann et al. 1998) for the distance and 35′/242 for the angular separation. We made a revision calculation for this semi-major axis using the third Keplerian law and values for the orbital period, which has the argument of pericenter close to 90°. The second system is HD 19994.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Star 16 Cyg A} & \textbf{Planet 16 Cyg Bb} \\
\hline
Mass (\(M_{\odot}\)) & 1.53 \\
Mass (\(M_{\text{Jup}} \times \sin I_1\)) & 1.68 ± 0.07 \\
Semi-major axis (AU) & 754.53 ± 0.07 \\
Eccentricity & 0.863 ± 0.689 \\
Inclination & 135° ± 1° \\
Ascending node & 313°/44 \\
Argument of pericenter & 26.6° ± 2°/1 \\
Period & 13512.7 yr ± 799.5 days \\
\hline
\end{tabular}
\caption{Initial Orbital Elements of the System 16 Cyg}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Prograde Orbit} & \textbf{Retrograde Orbit} & \\
\hline
Mass (\(M_{\text{Jup}}\)) & 2.38 ± 0.04 & 2.38 ± 0.04 \\
Semi-major axis (AU) & 1.693 & 1.693 \\
Eccentricity & 0.689 ± 0.011 & 0.689 ± 0.011 \\
Inclination & 45° ± 1° & 135° ± 1° \\
Argument of pericenter & 83.4° ± 2°/1 & 83.4° ± 2°/1 \\
Period (days) & 799.5 & 799.5 \\
\hline
\end{tabular}
\caption{Our Proposal for Possible Orbital Elements for Planet 16 Cyg Bb}
\end{table}

The second star in the 16 Cyg binary system, 16 Cyg B (parent star), with a mass of 1.01 ± 0.04 \(M_{\odot}\) (Fuhrmann et al. 1998), is known to host a giant gas planet 16 Cyg Bb with taxonomy class 2J0.2W7 (Plávalová 2012). The EP has a minimum mass of \(M_{\text{Jup}} \times \sin I_1\) = 1.68 ± 0.07 \(M_{\text{Jup}}\) in an orbit with high eccentricity \(e_1 = 0.689\) (Wittenmyer et al. 2007). We took the initial conditions for investigating this planet from The Extrasolar Planets Encyclopaedia (Schneider et al. 2011). The values of the inclination and ascending node for the planet are unknown. Shown in Table 1 are the orbital elements for the planet and distant star.

We made a revision calculation for the semi-major axis of the planet using the third Keplerian law and values for the orbital period of the planet, which is 799.5 days. We know the minimum mass of the planet, \(m_1 = 1.68 M_{\text{Jup}}\). We varied the elements \(I_1\) from 0° to 180°, 1° at a time and \(\Omega_1\) from 0° to 360°, also for each degree. We found the value of the planet’s inclination \(I_1\) for which the maximum value of the planet’s eccentricity \(e_1\) is close to the initial value 0.689.

According to our calculation, we received the value 1.693 AU for the semi-major axis, 2.38 \(M_{\text{Jup}}\) for the mass, and 45° or 135° for the inclination of the planet. These values are presented in Table 2 and were used in our next calculations.

The value of the ascending node of the planet’s orbit \(\Omega_1\) is unknown. However, the orientation of the orbit is defined by the node, therefore the different outcomes of the dynamical evolution can be found. We varied the ascending node of the planet from 0° to 360°, by step 1°. We found the range of values of \(\Omega_1\) for which the planet’s orbit would be stable or unstable.

For the stability criteria, we used the Roche limit. This is the minimum distance a planet can approach its parent star without being torn apart by tidal forces. For the calculation of the Roche limit \(d_R\) in our paper, we used the equation published by Eggleton (1983):

\[ d_R = \frac{0.49 \mu^{\frac{2}{3}}}{0.6 \mu^{\frac{1}{3}} + \ln \left(1 + \mu^{\frac{1}{3}}\right)}, \]

where \(\mu = m_1/m_0\).

The results of our calculations are presented in Figures 3 and 5. If the ascending node is \(\Omega_1 \in (249°, 374\°)\) for prograde motion, \(I_1 = 45°\), and \(\Omega_1 \in (73°, 198°)\) for retrograde motion, \(I_1 = 135°\), the planet reaches the Roche limit in its pericenter \((d_R = 0.063 \text{ AU})\) with an eccentricity of \(e_1 \geq 0.963\). In such a
pericenter, large perturbations and tidal forces drastically affect the planet and lead to its destruction.

For the values $I_1 = 45^\circ$ and $\Omega_1 = 135^\circ$ the maximum and minimum values of eccentricity $e_1$ are equal to the initial value $e_1 = 0.689$. In this case, $q_0 = -0.999 < q_0 = -0.561$. With these elements, the planet’s orbit would be stable.

When the eccentricity of the planetary orbit grows close to the Roche limit, large perturbations in the planet’s pericenter are affected. In such cases, measurements based on point mass bodies can be inaccurate, and a dynamic theory for real dimensional bodies must be used instead.

5.2. HD 19994

The binary system HD 19994 (94 Ceti, ADS 2406 AB) contains a A component: a yellow-white dwarf with a mass of $1.34 \, M_{\odot}$, and a B component: a red dwarf with a mass of $0.37 \, M_{\odot}$. For the distant star we used Keplerian elements published by Hale (1994). To define its semi-major axis $a_2$, we used two methods. First, we derived this value using stellar parallax, where the distance was 22.38 pc (Schneider et al. 2011) and the angular separation was $6\arcsec 77$ (Hale 1994), which resulted in a semi-major axis of 151.51 AU. Second, with the application of Keplerian law, using the values from Table 3, we calculated the value to be 151.37 AU. The difference between these two values is negligible. We applied the value $a_2 = 151.51$ AU in the following calculations.

The planet HD 19994 b was discovered in 2000 (Queloz et al. 2001) and is orbiting the A component. The taxonomy class is 2J0.2G3 (Plávalová 2012) and the minimum mass is 1.68 $M_{\text{Jup}}$. This planet is orbiting with a semi-major axis of 1.42 AU with quite a high eccentricity of $0.3 \pm 0.04$. For our calculations we used the orbital elements published by Mayor et al. (2004). In Table 3, the initial Kepler orbital elements for the planet and its distant star are shown.

As with the first system, we revised the calculation for the semi-major axis of the planet using Keplerian law. With the values listed in Table 3, we valued the planet’s semi-major axis at $a_1 = 1.427$ AU. If we varied the mass of the planet to 5 $M_{\text{Jup}}$, then we would get a value of 1.428 AU. We decided to use the value 1.427 AU for the semi-major axis and 1.68 $M_{\text{Jup}}$ for the mass of the planet in our calculations.

As in the case of 16 Cyg b, we varied the elements $I_1$ from $0^\circ$ to $180^\circ$, $1^\circ$ at a time, and $\Omega_1$ from $0^\circ$ to $360^\circ$, also for each degree. We found the value of the planet’s inclination $I_1$ for which the maximum value of the planet’s eccentricity $e_1$ is close to the initial value 0.300.

According to our calculation, we received the value $a_1 = 1.427$ AU for the semi-major axis, $1.86 \, M_{\text{Jup}}$ for the mass, and $65^\circ$ or $115^\circ$ for the inclination of the planet. These values are presented in Table 4 and were used for our next calculations.

The results of our calculations are presented in Figures 4 and 6. If the ascending node is $\Omega_1 \in (146^\circ, 177^\circ)$ or $\Omega_1 \in (350^\circ, 383^\circ)$ for prograde motion, $I_1 = 65^\circ$, and $\Omega_1 \in (171^\circ, 202^\circ)$ or $\Omega_1 \in (325^\circ, 358^\circ)$ for retrograde motion, $I_1 = 115^\circ$; the planet reaches the Roche limit in its pericenter ($d_R = 0.052$ AU) with an eccentricity of $e_1 \geq 0.963$. In such a pericenter, large perturbations and tidal forces drastically affect the planet and lead to its destruction.

For the values $I_1 = 65^\circ$ and $\Omega_1 = 263^\circ$ the maximum and minimum values of the eccentricity $e_1$ are equal to the initial value $e_1 = 0.300$. In this case, $q_0 = -0.999 < q_0 = -0.739$. With these elements, the planet’s orbit would be stable.

6. COMPARISON OF OUR THEORETICAL RESULTS WITH NUMERICAL INTEGRATION

For confirmation of the obtained analytic results, we compared them with the results of numerical integration. The equations of the motion of the systems were numerically integrated from the initial date 2000 January 1, through $8 \times 10^6$ yr, using...
Figure 4. Planet HD 19994 b. The evolution of the maximum value of the planet’s eccentricity $e_1$ and the cosine of the mutual inclination between the planet’s orbit and the distant star’s orbit $q$ vs. the ascending node of the planet $Ω_1$ for the prograde, $I_1 = 65°$, and retrograde, $I_1 = 115°$, planet orbits. The Roche limit is plotted by a dashed line.

Figure 5. Planet 16 Cyg Bb. The evolution of the planet’s eccentricity $e_1$ and the pericenter distance $r_p$ over $8 \times 10^6$ yr. The curves are the result of the numerical integration. For $I_1 = 45°$ and $Ω_1 = 20°$ or $I_1 = 45°$ and $Ω_1 = 134°$, the planet’s orbit does not reach the Roche limit. For $I_1 = 45°$ and $Ω_1 = 285°$, the planet stays within the Roche limit. The boundaries of the gray zones for all three cases were computed from our theory. For $Ω_1 = 263°$, the gray zone is comparable with the line. The Roche limit is plotted by a dashed line.

Figure 6. Planet HD 19994 b. The evolution of the planet’s eccentricity $e_1$ and the pericenter distance $r_p$ over $4 \times 10^6$ yr. The curves are the result of the numerical integration. For $I_1 = 65°$ and $Ω_1 = 30°$ or $I_1 = 65°$ and $Ω_1 = 263°$, the planet’s orbit does not reach the Roche limit. For $I_1 = 65°$ and $Ω_1 = 165°$, the planet stays within the Roche limit. The boundaries of the gray zones for all three cases were computed from our theory. For $Ω_1 = 263°$, the gray zone is comparable with the line. The Roche limit is plotted by a dashed line.

For HD 19994 b (see Figure 6) we used as the initial values $i_1 = 65°$ and $Ω_1 = 30°$, $Ω_1 = 165°$, and $Ω_1 = 263°$. As in the case of 16 Cyg Bb, the results of the numerical integration are the same as the results we obtained by the analytical theory.

We also obtained identical results from the numerical integration and analytical theory for the retrograde orbits of both systems.

7. COMPARISON WITH PREVIOUS WORKS

In our investigation, we used the methods of classical celestial mechanics, developed by Hamilton. We used the Principle of Determinacy (PD; Lidov 2010): if the initial conditions of each object in a mechanical system are defined at any instant of time, then their further behavior is defined expressly.

Note that Hamilton’s equations are of the first order in the time derivative, which makes them more convenient for computation. The Hamiltonian (Equation (3)) of systems (9) and (10), without short-periodic terms, permits the solution in which the secular and the long-periodic terms are taken into account in the intermediate orbit and allows the close approach of the EP to the star to be established.

The long-periodic stability of planets in a binary system was investigated by Holman & Wiegert (1999). The planets are taken to be test particles moving in the field of an eccentric binary system. This study investigated the orbital stability numerically, with the elliptic restricted three-body problem. We made a comparison with their results concerning planet 16 Cyg Bb. We used the elements of this planet for the application of our theory. Holman & Wiegert (1999) integrated differential equations of the motion using $Ω_1 = 0°$ and obtained instability between $10^7$ and $10^8$ yr. We varied this element from $0°$ till $360°$ and also obtained instability around $Ω_1 = 0°$.

The secular dynamics of massless particles orbiting a central star and perturbed by a secondary star component with high eccentricity have been investigated Giuppone et al. (2011). They used the Lie series perturbation scheme restricted to the second order in the small parameter. Their results have shown that
the second-order secular dynamic reproduces the behavior of a planet with good precision.

We eliminated the short-periodic terms by von Zeipel’s method in the general three-body problem. This method allows us to estimate values for the short-periodic terms; these values are less than $\pm 10^{-3}$. The short-periodic terms are small and do not influence the evolution and stability. Our condition of stability is valid at any time interval. Our theory may also be used for EPs with large masses.

In the Sun–asteroid–Jupiter problem, the Hamiltonian describing the motion of the massless asteroid in the heliocentric coordinates was used by Innanen et al. (1997). They used the so-called Koziæ mechanism, which shows the behavior of the eccentricity as a function of the initial inclination. In the restricted three-body problem, averaging over the mean anomalies, the perturbation function contains the integrals

$$\sqrt{1 - e^2} \sin i = \text{const.}$$

and

$$e^2 \left( \frac{2}{5} - \sin^2 i \sin^2 \omega \right) = \text{const.}$$  \hspace{1cm} (42)

These were obtained at nearly the same time by Lidov (1962) and Koziæ (1962). The integrals permit us to execute qualitative analysis of the family of the phase trajectories.

In our case, the connection between the eccentricity, the inclination, and the argument of pericenter is more complex, depends on all Keplerian elements, and is defined by Equation (33). It allows us to arrive at a conclusion similar to the results of Innanen et al. (1997). The increase in eccentricity occurs even if the third body is very distant and the perturbation is small.

The timescale needed to observe this phenomenon is quite lengthy, for example, the sudden increase in the eccentricity of Neptune is observed at 100 Myr. (Innanen et al. 1997). However, what is clear is that instability only occurs when the mutual inclination of a planet’s orbit and the distant star’s orbit is high. So we suppose, for the stability of a planet’s orbit, the following conditions are necessary: the initial value of the cosine of the mutual inclination $q_0 < q_{01}$ or $q_0 > q_{02}$, where $q_{01}$ and $q_{02}$ are roots of expression (40) and in this case $\bar{\tau} - \bar{G}_2 > 0$.

8. CONCLUSION

We have shown that an EP revolving in a binary system around one of the components may have stable or unstable orbits. The conditions for stable motion depend on the orbital parameters, which can be calculated from the formulae of the previous sections. There are the angle of the mutual inclination between orbits, the angular momentum of the distant star, and the maximum value of the eccentricity of the planet’s orbit. When the value of the planet’s eccentricity is close to one, in the pericenter, the planet reaches the Roche limit. The tidal forces of the star destroy the planet. We have suggested the possible values for the unknown elements with which the orbit of the planet would remain stable.

The results of our calculations are presented in Figures 1–6. The results of the numerical integration support the results obtained by the analytical theory.

For 16 Cyg Bb, there are three possible regions of stability. First, for the prograde orbit $I_1 = 45^\circ$ and the ascending node $\Omega_1 \in [14^\circ, 249^\circ]$. Second, for the retrograde orbit $I_1 = 135^\circ$ and the ascending node $\Omega_1 \in [0^\circ, 74^\circ]$ or $\Omega_1 \in [198^\circ, 360^\circ]$. We proposed the planet’s mass to be $2.38 \pm 0.04 \, M_{\text{Jup}}$ and the value of its semi-major axis equal to 1.693 AU, for which the third Keplerian law is valid.

For the second system, HD19994, the values of inclination and the ascending node of the planet for which the motion is stable are: for the prograde motion $I_1 = 65^\circ$ and the ascending node $\Omega_1 \in [23^\circ, 146^\circ]$ or $\Omega_1 \in [177^\circ, 350^\circ]$; for the retrograde motion $I_1 = 115^\circ$ and the ascending node $\Omega_1 \in [202^\circ, 325^\circ]$ or $\Omega_1 \in [358^\circ, 0^\circ]$ or $\Omega_1 \in [0^\circ, 171^\circ]$. For the planet’s mass, we proposed the value of $1.86 \pm 0.04 \, M_{\text{Jup}}$ and for the planet’s semi-major axis, a value of 1.427 AU. These values are in accordance with the third Keplerian law.

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