A SURVEY ON THE DDVV CONJECTURE

GE JIANQUAN AND TANG ZIZHOU

Dedicated to Professor John C. Wood on his 60th birthday

1. Introduction

Let $M^n$ be an immersed submanifold of a real space form $N^{n+m}(c)$ of constant sectional curvature $c$. Let $R$ (resp. $\bar{R}$) be the Riemann curvature tensor of $M^n$ (resp. $N^{n+m}(c)$), $h$ the second fundamental form, $A_\xi$ the shape-operator associated to a normal vector field $\xi$, and $R^\perp$ the curvature tensor of the normal connection. The equations of Gauss and Ricci are given by

$\langle R(X,Y)Z,T\rangle = \langle \bar{R}(X,Y)Z,T\rangle + \langle h(X,T), h(Y,Z) \rangle - \langle h(X,Z), h(Y,T) \rangle,$

$\langle R^\perp(X,Y)\xi, \eta \rangle = \langle R\bar{R}(X,Y)\xi, \eta \rangle + \langle [A_\xi, A_\eta]X, Y \rangle,$

for tangent vectors $X, Y, Z, T$ and normal vectors $\xi, \eta$.

Let $\{e_1, ..., e_n\}$ (resp. $\{\xi_1, ..., \xi_m\}$) be an orthonormal basis of $T_pM$ (resp. $T^\perp_pM$).

The normalized scalar curvature $\rho$ and normal scalar curvature $\rho^\perp$ of $M^n$ at $p$ are defined by

$\rho = \frac{2}{n(n-1)} \sum_{1<i<j} \langle R(e_i, e_j)e_j, e_i \rangle,$

$\rho^\perp = \frac{2}{n(n-1)} \left( \sum_{1<i<j} \sum_{1=r<s} \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{\frac{1}{2}} = \frac{2}{n(n-1)} |R^\perp|.$

Let $H = \frac{1}{n} Tr(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$ be the mean curvature vector field.

Theorem 1.1. ([W], [GR]) Let $M^2$ be an immersed surface of a real space form $N^{2+m}(c)$. Then

$\rho + \rho^\perp \leq |H|^2 + c$

at every point $p$ of $M^2$, with equality if and only if the ellipse of curvature is a circle.
This inequality was proved by Wintgen [W] in 1979 for surfaces in 4 dimensional Euclidean space and by Guadelupe and Rodriguez [GR] in 1983 for surfaces in arbitrary space form. By applying this inequality, they derived many rigidity results about immersions of surfaces including some results proved by Barbosa [B], Yau [Y1], [Y2], Ruh [R]. Since the equality in the inequality holds if and only if the ellipse of curvature is a circle, and the property that the ellipse is a circle is a conformal invariant, surfaces in $\mathbb{R}^4$ that attain equality everywhere could be stereographically projected from those in $S^4$ (for instance see do Carmo and Wallach [dCW]). In [GR], they also remarked that Atiyah and Lawson (unpublished) had shown that an immersed surface in $S^4$ has the ellipse always a circle if and only if the canonical lift of the immersion map into the bundle of almost complex structures of $S^4$ (i.e., $CP^3$) is holomorphic (also see Theorem 5 in [BFLPP]). One can also consider surfaces with circular ellipse of curvature in general Riemannian manifolds other than real space forms, in which case circular ellipse of curvature also corresponds to the equality condition of some similar inequality involving relative curvatures. For this subject on surfaces with circular ellipse of curvature, we refer to [BFLPP], [DT1], [C] and references therein.

In 1996, Chen [Cby] proved the following similar but more elementary inequality for submanifolds in real space forms (also see Suceavă [Su], Ge [G1]):

$$\rho \leq |H|^2 + c.$$ 

To study such inequalities relating intrinsic and extrinsic curvature invariants, in 1999, De Smet, Dillen, Verstraelen and Vrancken [DDVV] defined the normal scalar curvature for high dimensional submanifolds and posed a conjecture as generalization of the previous Theorem (which is now called the DDVV conjecture):

**Conjecture 1.2.** ([DDVV]) Let $M^n$ be an immersed submanifold of a real space form $N^{n+m}(c)$. Then

$$\rho + \rho^\perp \leq |H|^2 + c.$$ 

The conjecture was proved for $m = 2$ by them, where also some partial classification results were obtained in case equality holds at every point.

Recent developments mostly based on the following translation of the conjecture established by Dillen, Fastenakels and Veken [DFV].

**Theorem 1.3.** ([DFV]) Conjecture 1.2 is true for submanifolds of dimension $n$ and codimension $m$ if for every set $\{B_1, \ldots, B_m\}$ of real symmetric $(n \times n)$-matrices with trace zero the following inequality holds:

$$\sum_{r,s=1}^m \| [B_r, B_s] \|^2 \leq \left( \sum_{r=1}^m \| B_r \|^2 \right)^2.$$
In fact, putting \( B_r = A_{\xi_r} - \langle H, \xi_r \rangle \text{id} \), and using Gauss and Ricci equations, we find

\[
|H|^2 - \rho + c = \frac{1}{n(n-1)} \sum_{r=1}^{m} \|B_r\|^2,
\]

\[
\rho^\perp = \frac{1}{n(n-1)} \left( \sum_{r,s=1}^{m} \|[B_r, B_s]\|^2 \right)^{\frac{1}{2}}.
\]

When \( m = 2 \), such kind of matrix inequality in Theorem 1.3 was already studied by Simons [S] and Chern [Css] in proving the well-known scalar curvature pinching theorem for minimal submanifolds in spheres.

## 2. Recent developments

For convenience, we denote both inequalities in Conjecture 1.2 and Theorem 1.3 by \( P(n, m) \). Thus, \( P(2, m) \) and \( P(n, 2) \) were proved in [W], [GR] and [DDVV] respectively as mentioned before. We’d like to list the recent developments as follows:

- In 2004, Dillen, Haesen, Petrović-Torgašev and Verstraelen [DHTV] proved the \( P(n, 2) \) of Lorentz type, i.e., for Lorentzian manifold embedded locally and isometrically in a pseudo-Euclidean space with codimension 2.
- A weaker version of the conjecture and also a special case when the submanifold is either H-umbilical Lagrangian in \( \mathbb{C}^n \) or ultra-minimal in \( \mathbb{C}^4 \) were proved in the same paper as Theorem 1.3 by Dillen, Fastenakels and Veken [DFV].
- The first nontrivial case \( P(3, m) \) was proved by Choi and Lu [CL] in 2006, where they also classified all 3 dimensional minimal submanifolds satisfying the equality everywhere and compared this conjecture with the comass problem in calibrated geometry.
- Another nontrivial case \( P(n, 3) \) was proved by Lu [L1] in 2007, where he also discussed the relationship between DDVV conjecture and pinching theorems of scalar curvature for minimal submanifolds in spheres.
- Besides the introduction of some recent developments, Lu [L2] discussed some applications of DDVV conjecture and its relation with a conjecture of Böttcher and Wenzel [BW] in random matrix theory.
- In 2007, the whole conjecture was proved by Lu [L3] and the authors [GT] independently by quite different methods. The equality condition (locally) was also determined completely by us (see later).
- Lu [L3] also observed some interesting applications of this conjecture to the Simons type pinching theorems (see Simons [S]) and established a new pinching theorem, in view of which he made a conjecture about the gap theorem of Peng-Terng type (see Peng and Terng [PT], Chern, do Carmo and Kobayashi [CCK]).
in high codimensions. There’s also a proof of the conjecture of Böttcher and Wenzel [BW] in [L3].

- By the same method (but a little more complicated) as [GT], Ge [G2] obtained a similar optimal inequality like \( P(n, m) \) for real skew-symmetric matrices (see later).
- Some partial results on classification of submanifolds when the equality in the DDVV inequality holds everywhere were proved by Dajczer and Florit [DF], Dajczer and Tojeiro [DT1], [DT2]. In particular, [DT2] provided a parametric construction in terms of minimal surfaces of the Euclidean submanifolds of codimension two and arbitrary dimension that attain equality everywhere in the DDVV inequality. As they remarked that by the pointwise equality condition determined in [GT], their results provide a complete classification of all non-minimal submanifolds (of arbitrary codimension) of dimension at least four that attain equality in the DDVV inequality and whose first normal spaces have dimension two everywhere. It is a very interesting problem to study the remaining cases.

We now state the main theorems in [GT] and [G2] as follows.

**Theorem 2.1.** (L3, GT) Let \( B_1, \ldots, B_m \) be \((n \times n)\) real symmetric matrices. Then

\[
\sum_{r,s=1}^{m} \| [B_r, B_s] \|^2 \leq \left( \sum_{r=1}^{m} \| B_r \|^2 \right)^2,
\]

where the equality holds if and only if under some \( O(n) \times O(m) \) action\(^1\) all matrices \( B_r \) are zero except the following 2 matrices:

\[
H_1 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad H_2 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

where \( \mu \geq 0 \) is a real number.

**Theorem 2.2.** (L3, GT) Let \( f : M^n \rightarrow N^{n+m}(c) \) be an isometric immersion. Then

\[
\rho + \rho^\perp \leq |H|^2 + c,
\]

\(^1\) A \((P, R) \in O(n) \times O(m)\) acts on \((B_1, \ldots, B_m)\) by: \((P, R) \cdot (B_1, \ldots, B_m) := (P^t B_1 P, \ldots, P^t B_m P) R.\)
where the equality holds at some point \( p \in M \) if and only if there exist an orthonormal basis \( \{e_1, ..., e_n\} \) of \( T_pM \) and an orthonormal basis \( \{\xi_1, ..., \xi_m\} \) of \( T^\perp_p M \), such that

\[
A_{\xi_1} = \begin{pmatrix}
\lambda_1 + \mu & 0 & 0 & \cdots & 0 \\
0 & \lambda_1 - \mu & 0 & \cdots & 0 \\
0 & 0 & \lambda_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_1
\end{pmatrix}, \quad A_{\xi_2} = \begin{pmatrix}
\lambda_2 & \mu & 0 & \cdots & 0 \\
\mu & \lambda_2 & 0 & \cdots & 0 \\
0 & 0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_2
\end{pmatrix},
\]

\( A_{\xi_3} = \lambda_3 I_n \) and all other shape operators \( A_{\xi_r} = 0 \), where \( \mu, \lambda_1, \lambda_2, \lambda_3 \) are real numbers.

**Theorem 2.3.** \([G2]\) Let \( B_1, ..., B_m \) be \((n \times n)\) real skew-symmetric matrices. Then

(i) for \( n = 3 \),

\[
\sum_{r,s=1}^m \| [B_r, B_s] \|^2 \leq \frac{1}{3} \left( \sum_{r=1}^m \| B_r \|^2 \right)^2,
\]

where the equality holds if and only if under some \( O(n) \times O(m) \) action all matrices \( B_r \) are zero except the following 3 matrices:

\[
C_1 := \begin{pmatrix}
0 & \lambda & 0 \\
-\lambda & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad C_2 := \begin{pmatrix}
0 & 0 & \lambda \\
0 & 0 & 0 \\
-\lambda & 0 & 0
\end{pmatrix}, \quad C_3 := \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \lambda \\
0 & -\lambda & 0
\end{pmatrix},
\]

where \( \lambda \geq 0 \) is a real number;

(ii) for \( n \geq 4 \),

\[
\sum_{r,s=1}^m \| [B_r, B_s] \|^2 \leq \frac{2}{3} \left( \sum_{r=1}^m \| B_r \|^2 \right)^2,
\]

where the equality holds if and only if under some \( O(n) \times O(m) \) action all matrices \( B_r \) are zero except the following 3 matrices: \( \text{diag}(D_1, 0), \text{diag}(D_2, 0), \text{diag}(D_3, 0) \), where \( 0 \in M(n - 4) \) is a zero matrix, \( \lambda \geq 0 \) is a real number,

\[
D_1 := \begin{pmatrix}
0 & \lambda & 0 & 0 \\
-\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda \\
0 & 0 & -\lambda & 0
\end{pmatrix}, \quad D_2 := \begin{pmatrix}
0 & 0 & \lambda & 0 \\
0 & 0 & -\lambda & 0 \\
-\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0
\end{pmatrix}, \quad D_3 := \begin{pmatrix}
0 & 0 & 0 & \lambda \\
0 & 0 & \lambda & 0 \\
0 & -\lambda & 0 & 0 \\
-\lambda & 0 & 0 & 0
\end{pmatrix}.
\]

3. Sketch of our method

Due to the success of \([G2]\) in finding and proving optimal inequalities involving norms of commutators or Lie brackets by the method of \([GT]\), we’d like to give a sketch of the proof of Theorem 2.1 to conclude this survey. It goes in 3 major steps as follows:

(1) Translation of the inequality into a polynomial inequality
Denote by $SM(n)$ the $N := \frac{n(n+1)}{2}$-dimensional space of real symmetric $(n \times n)$ matrices. Let $\{E_{ij}\}_{1 \leq i \leq j \leq n}$ be the standard orthonormal basis of $SM(n)$. Taking the letter order for the indices set $S := \{(i,j)|1 \leq i \leq j \leq n\}$, we identify $S$ with $\{1,...,N\}$. Then there’s a unique $(N \times m)$ matrix $B$ such that $(B_1,...,B_m) = (E_1,...,E_N)B$.

Similarly, an orthogonal matrix $Q \in SO(N)$ determines $N$ real symmetric $(n \times n)$ matrices $\{\hat{Q}_1,...,\hat{Q}_N\}$ by $(\hat{Q}_1,...,\hat{Q}_N) = (E_1,...,E_N)Q$.

Since $BB^t$ is a semi-positive definite matrix in $SM(N)$, there exists an orthogonal matrix $Q \in SO(N)$ such that $BB^t = Q \text{diag}(x_1,...,x_N)Q^t$ with $x_\alpha \geq 0, 1 \leq \alpha \leq N$. Then direct calculations show that

$$m \sum_{r=1}^m \|B_r\|^2 = \|B\|^2 = \sum_{\alpha=1}^N x_\alpha,$$

$$m \sum_{r,s=1}^m \|[B_r,B_s]\|^2 = \sum_{\alpha,\beta=1}^N x_\alpha x_\beta \|[\hat{Q}_\alpha,\hat{Q}_\beta]\|^2.$$

Now the problem is changed into proving the following polynomial inequality:

$$f_Q(x) := F(x,Q) := \sum_{\alpha,\beta=1}^N x_\alpha x_\beta \|[\hat{Q}_\alpha,\hat{Q}_\beta]\|^2 - \left(\sum_{\alpha=1}^N x_\alpha\right)^2 \leq 0,$$

for all $x \in \mathbb{R}_+^N = \{0 \neq x = (x_1,...,x_N) \in \mathbb{R}^N|x_\alpha \geq 0, 1 \leq \alpha \leq N\}$ and $Q \in SO(N)$.

(2) Approximate process

Since $f_Q(x)$ is a homogeneous polynomial, we need only to consider the inequality on the subset $\triangle := \{x \in \mathbb{R}_+^N|\sum_\alpha x_\alpha = 1\}$ and show that

$$G := \{Q \in SO(N)|f_Q(x) \leq 0, \forall x \in \triangle\} = SO(N).$$

By an observation to the case when $Q = I_N$, we find it may happen to be that $f_Q(x) < 0$ for all $x$ in the interior of $\triangle$, which inspires us to consider the following approximate process.

For any sufficiently small $\varepsilon > 0$, put $\triangle_\varepsilon := \{x \in \triangle|x_\alpha \geq \varepsilon, 1 \leq \alpha \leq N\}$ and $G_\varepsilon := \{Q \in SO(N)|f_Q(x) < 0, \forall x \in \triangle_\varepsilon\}$. Try to prove

$$G_\varepsilon = SO(N).$$

Finally,

$$G = \lim_{\varepsilon \to 0} G_\varepsilon = SO(N).$$

(3) Continuity method and a priori estimate
Since $SO(N)$ is connected, we can use continuity method which consists of the following three steps:

**step 1:** $I_N \in G_\varepsilon$, (thus $G_\varepsilon \neq \emptyset$),

**step 2:** $G_\varepsilon$ is open in $SO(N)$,

**step 3:** $G_\varepsilon$ is closed in $SO(N)$.

Step 1 and Step 2 can be easily verified, while step 3 needs the following a priori estimate which is the main difficulty in the whole proof and indicates the idea of the proof of the equality case.

**a priori estimate:** Suppose $f_Q(x) \leq 0$, for every $x \in \Delta_\varepsilon$. Then $f_Q(x) < 0$, for every $x \in \Delta_\varepsilon$.

The proof of the a priori estimate depends on a sequence of lemmas mostly depending on the concrete properties of those coefficients of Lie brackets of a given basis of the space in problem. Once one knows more about those coefficients (structural constants when the space is a Lie algebra), one will probably find and prove such type optimal inequality by our method.

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School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875

E-mail address: jqge@bnu.edu.cn

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875

E-mail address: zztang@mx.cei.gov.cn