A REMARK ON NORM INFLATION FOR NONLINEAR WAVE EQUATIONS

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Abstract. In this note, we study the ill-posedness of nonlinear wave equations (NLW). Namely, we show that NLW experiences norm inflation at every initial data in negative Sobolev spaces. This result covers a gap left open in a paper of Christ, Colliander, and Tao (2003) and extends the result by Oh, Tzvetkov, and the second author (2019) to non-cubic integer nonlinearities. In particular, for some low dimensional cases, we obtain norm inflation above the scaling critical regularity.

1. Introduction

We consider the Cauchy problem of the following nonlinear wave equation (NLW):
\[
\begin{aligned}
\partial_t^2 u - \Delta u &= \pm u^k \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1),
\end{aligned}
\] (1.1)
where $\mathcal{M} = \mathbb{T}^d$ or $\mathbb{R}^d$ ($d \geq 1$) and $k \geq 2$ is an integer.

Our goal in this paper is to study ill-posedness of (1.1) in negative Sobolev spaces. In this regard, we recall the critical regularity associated to (1.1) posed on $\mathbb{R}^d$.

First, NLW (1.1) has the following scaling symmetry: given $\lambda > 0$, if $u$ solves (1.1), then $u_\lambda(t, x) = \lambda^{\frac{d}{2} - \frac{2}{k-1}} u(\lambda t, \lambda x)$ also solves (1.1) with rescaled initial data $\lambda^{\frac{d}{2}} u_0(\lambda x), u_1(\lambda x))$. This scaling leaves the $\dot{H}^{s_{\text{scaling}}(d,k)}(\mathbb{R}^d)$-norm invariant, where
\[
s_{\text{scaling}}(d,k) := \frac{d}{2} - \frac{2}{k-1}. \tag{1.2}
\]
Secondly, (1.1) is invariant under the Lorentz transformation (conformal symmetry), which gives rise to the critical regularity $s_{\text{conf}}(d,k) := \frac{d+1}{4} - \frac{1}{k-1}$; see [15]. In addition, we need the condition $s \geq 0$ in order for the nonlinearity to make sense as a distribution. Hence, the critical regularity of (1.1) is given by
\[
s_{\text{crit}}(d,k) = \min(s_{\text{scaling}}(d,k), s_{\text{conf}}(d,k), 0) = \min\left(\frac{d}{2} - \frac{2}{k-1}, \frac{d+1}{4} - \frac{1}{k-1}, 0\right). \tag{1.3}
\]

The purpose of the critical regularity for (1.1) on $\mathbb{R}^d$ is that we expect (local-in-time) well-posedness in $H^s(\mathbb{R}^d)$ when $s > s_{\text{crit}}(d,k)$ and ill-posedness, due to some instability, when $s < s_{\text{crit}}(d,k)$. This heuristic provided by (1.3) is also instrumental in the well-posedness theory of (1.1) on periodic domains $\mathcal{M} = \mathbb{T}^d$, despite the lack of scaling and conformal symmetries in this setting.

We now survey the well-posedness theory for (1.1), specifically restricting our attention to local-in-time results. Well-posedness of (1.1) above the critical regularity $s_{\text{crit}}(d,k)$ was studied in [9, 15, 10, 20]. Moreover, ill-posedness of (1.1) below the critical regularity has

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been studied in \cite{13,14,6,12,8,23,22,19}. In particular, Christ, Colliander, and Tao \cite{6} proved norm inflation for (1.1) on $\mathbb{R}^d$ when \footnote{They considered the nonlinearity $\pm |u|^{k-1}u$ instead of $\pm u^k$.} \(k \geq 2\) and
\[s_{\text{scaling}}(1,k) < s < \frac{1}{2} - \frac{1}{k},\]
and (ii) for either odd integer $k \geq 3$ or $k \geq k_0 + 1$ for integer $k_0 > \frac{d}{2}$ and
\[s \leq -\frac{d}{2} \quad \text{or} \quad 0 < s < s_{\text{scaling}}(d,k)\].

Applying the argument in \cite{6}, Corollary 7, which uses the finite speed of propagation for (1.1) to deduce norm inflation in dimension $d \geq 2$ from norm inflation in $d = 1$, the result of (ii) extends to norm inflation for any $k \geq 2$, $s \leq -\frac{1}{k}$, and $s < s_{\text{scaling}}(1,k)$. Here, norm inflation (at the trivial initial condition $(u_0, u_1) = (0, 0)$) means that given any $\varepsilon > 0$, there exists a solution $u_\varepsilon$ to (1.1) and $t_\varepsilon \in (0, \varepsilon)$ such that
\[\|(u_\varepsilon(0), \partial_t u_\varepsilon(0))\|_{H^s(M)} < \varepsilon \quad \text{and} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s(M)} > \varepsilon^{-1}, \tag{1.4}\]
where $H^s(M) := H^s(M) \times H^{s-1}(M)$. This phenomenon is a stronger notion of ill-posedness than the discontinuity of the solution map at zero. In particular, the result in \cite{6} leaves open the question of norm inflation for NLW (1.1) when
\[-\frac{1}{2} < s < \min(s_{\text{scaling}}(1,k), 0). \tag{1.5}\]

In the context of (1.1) on $\mathbb{T}^d$ for $0 < s < s_{\text{scaling}}(3,k)$, Xia \cite{23} generalized (1.4) to norm inflation based at general initial data (see (1.6) below). In \cite{19}, Oh, Tzvetkov, and the second author proved norm inflation at general initial data for the cubic NLW ($k = 3$) when $d \geq 2$ and $s < 0$. For the particular case $k = 3$ and $d \geq 2$, this result extends the norm inflation at zero in \cite{6} to norm inflation at general initial data.

Our aim in this paper is to prove norm inflation at general initial data for (1.1) in negative Sobolev spaces, thus filling the remaining gap left open in (1.3). The following is our main result.

**Theorem 1.1.** Given $d \in \mathbb{N}$, let $M = \mathbb{R}^d$ or $\mathbb{T}^d$. Suppose that $k \geq 2$ is an integer and $s < 0$. Fix $(u_0, u_1) \in H^s(M)$. Then, given any $\varepsilon > 0$, there exist a solution $u_\varepsilon$ to (1.1) on $M$ and $t_\varepsilon \in (0, \varepsilon)$ such that
\[\|(u_\varepsilon(0), \partial_t u_\varepsilon(0)) - (u_0, u_1)\|_{H^s(M)} < \varepsilon \quad \text{and} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s(M)} > \varepsilon^{-1}. \tag{1.6}\]

Theorem 1.1 thus closes the remaining gap in (1.5) and, in the case $s < 0$ and $k \neq 3$ (in view of \cite{19}), extends the result in \cite{6} to norm inflation based at any initial condition. When $(u_0, u_1) = (0,0)$, Theorem 1.1 is reduced to the usual norm inflation at zero initial data stated in (1.3). As a corollary to Theorem 1.1, we obtain that the solution map to (1.1): $(u_0, u_1) \in H^s(M) \mapsto (u, \partial_t u) \in C([-T, T]; H^s(M))$ is discontinuous everywhere in $H^s(M)$, for $s < 0$.

Currently, there are two approaches to proving norm inflation for Cauchy problems. The first is the approach used in \cite{6} which is based on studying low-to-high energy transfer in the associated dispersionless (ODE) model and scaling analysis. By avoiding the scaling analysis, Burq and Tzvetkov \cite{31} proved norm inflation as in (1.4) for the cubic NLW on three-dimensional compact Riemannian manifolds when $0 < s < s_{\text{scaling}}(3,3)$. In particular,
the argument in [23] is also based on this method. The second method is a Fourier analytic approach introduced by Bejenaru and Tao [2] and developed further by Iwabuchi and Ogawa [8]; see also [11, 17]. Our proof of Theorem 1.1 uses this method and follows the presentation by Oh [17], which we now briefly describe. We begin with a reduction: we may assume the initial data \((u_0, u_1)\) are sufficiently regular by a density argument. The key idea is to express a solution \(u_\varepsilon\) in \([1,1]\) in terms of a power series expansion in the initial data and to show that one of the terms in the expansion dominates all the others. More specifically, we write \(u_\varepsilon\) as the following power series expansion:

\[
  u_\varepsilon = \sum_{j=0}^{\infty} \Xi_j(u_\varepsilon(0), \partial_t u_\varepsilon(0)),
\]

where \(\{\Xi_j\}_{j=0}^{\infty}\) are multilinear operators in the linear solution \(S(t)(u_\varepsilon(0), \partial_t u_\varepsilon(0))\) of (increasing) degree \(kj + 1\). They are precisely the successive new terms added to a Picard iteration expansion of \(u_\varepsilon\). We define the initial data for \(u_\varepsilon\) by

\[
  (u_\varepsilon(0), \partial_t u_\varepsilon(0)) = (u_0, u_1) + (\phi_{0,\varepsilon}, \phi_{1,\varepsilon}),
\]

where the perturbations \((\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\) are chosen so that:

(i) \((\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\) converges to \((0,0)\) in \(H^s(M)\), as \(\varepsilon \to 0\),

(ii) there exists times \(t_\varepsilon \to 0\) as \(\varepsilon \to \infty\) such that the second Picard iterate \(\Xi_1(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\) dominates in \((1.7)\); namely,

\[
  \|u_\varepsilon(t_\varepsilon)\|_{H^s(M)} \asymp \|\Xi_1(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})(t_\varepsilon)\|_{H^s(M)} \to \infty,
\]

as \(\varepsilon \to \infty\).

These ingredients then lead to norm inflation based at \((u_0, u_1)\) as in \((1.6)\). The mechanism responsible for the instability in (ii) is the high-to-low transfer of energy, which is specifically exploited by the choice of \((\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\). We note that although we work in rough topologies, the functions \(u_\varepsilon\) are smooth and hence there is no issue in making sense of the power series expansion \((1.7)\). In [17], the operators \(\Xi_j\) are indexed using trees, which allows to directly treat the nonlinear estimates without an induction.

As it is based on exploiting high-to-low energy transfer in the nonlinearity, the Fourier analytic approach works well in negative Sobolev spaces. Indeed, for the case of nonlinear Schrödinger equations (NLS), this method was used in \([8, 11, 17]\) to fill a similar gap left in \([6]\) of norm inflation for NLS in negative Sobolev spaces. However, it does rely on the translation invariance of the underlying space \(M\), making it unsuitable for the case of more general domains. See also \([16, 4, 1, 5]\) for ill-posedness results of NLS.

For \(k \in \{2, 3, 4\}\) in \(d = 1\) and \(k = 2\) in \(d = 2, 3\), Theorem 1.1 yields norm inflation at general initial data above the scaling critical regularity \(s_{scaling}(d,k)\) defined in \((1.2)\). This phenomenon of norm inflation above the scaling critical regularity has also been observed for the cubic fractional NLS \([7]\) and quadratic NLS \([8, 11, 18]\). In this regime, it is essential to exploit resonant interactions in the nonlinearity. In the aforementioned papers, the choice of the initial data \((\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\) in \((1.8)\) (with \((u_0, u_1) = (0,0)\)) only activates (nearly) resonant contributions in the second Picard iterate \(\Xi_1(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\). However, for the case of NLW \((1.1)\), our analysis of the second Picard iterate is more subtle since our choice of perturbation \((\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\) requires us to also handle nonresonant contributions. To show that the resonant part is dominant, we need to take the existence time a bit longer. See Proposition 3.4.

We also note that the argument in Corollary 7.1 is not applicable for deducing norm inflation at general initial in dimensions \(d \geq 2\) from norm inflation at general initial data...
in dimension $d = 1$. Thus, we cannot simply deduce Theorem 1.1 from the corresponding result in one dimension.

**Remark 1.2.** By a straightforward modification, the same norm inflation result as in Theorem 1.1 holds for the following nonlinear Klein-Gordon equation:

$$
\begin{cases}
\partial_t^2 u - \Delta u + u = \pm u^k, \\
(u, \partial_t u)|_{t=0} = (u_0, u_1),
\end{cases}
\quad (t, x) \in \mathbb{R} \times \mathcal{M}.
$$

See Remark 3.5 for a further discussion.

**Remark 1.3.** Theorem 1.1 completes the ill-posedness theory for NLW (1.1) in negative regularities $s < 0$. The situation however is not complete in positive regularities. In particular, for $d \geq 2$ and $0 < s < s_{\text{conf}}(d,k)$, the focusing NLW (corresponding to the $+$ sign in (1.1)) on $\mathbb{R}^d$ has explicit solutions with arbitrarily small $H^s(\mathbb{R}^d)$-norm which blow-up in arbitrarily small time; see [15] and [21, Exercise 3.67]. In contrast, it is not known if there are similarly behaving blow-up solutions in the defocusing case (the $-$ sign in (1.1)) when $s_{\text{scaling}}(d,k) < s < s_{\text{conf}}(d,k)$.

Due to the local-in-time nature of the analysis in this paper, the sign of the nonlinearity in (1.1) does not play any role. Hence, we only consider the $+$ sign in the following. Moreover, in view of the time reversibility of the equation, we focus only on positive times.

## 2. Power series expansion indexed by trees

In this section, we show the well-posedness in the Fourier-Lebesgue space and exploit power series expansions. First, we introduce some notations. Given $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathcal{M})$ by the norm:

$$
\|f\|_{\mathcal{F}L^{s,p}(\mathcal{M})} = \|\langle \xi \rangle^s \hat{f}\|_{L^p(\hat{\mathcal{M}})},
$$

where $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\hat{\mathcal{M}}$ denotes the Pontryagin dual of $\mathcal{M}$, i.e.,

$$
\hat{\mathcal{M}} = \begin{cases} \mathbb{R}^d & \text{if } \mathcal{M} = \mathbb{R}^d, \\
\mathbb{Z}^d & \text{if } \mathcal{M} = \mathbb{T}^d. 
\end{cases}
$$

When $\hat{\mathcal{M}} = \mathbb{Z}^d$, we endow it with the counting measure. We also define

$$
\mathcal{F}L^{s,p}(\mathcal{M}) := \mathcal{F}L^{s,p}(\mathcal{M}) \times \mathcal{F}L^{s-1,p}(\mathcal{M}),
$$

and, for convenience, write $\mathcal{F}L^p(\mathcal{M}) := \mathcal{F}L^{0,p}(\mathcal{M})$ and $\mathcal{F}\mathcal{F}L^p(\mathcal{M}) := \mathcal{F}\mathcal{F}L^{0,p}(\mathcal{M})$.

Let $S(t)$ denote the linear wave propagator:

$$
S(t)(\bar{u}_0) = S(t)(u_0, u_1) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1
$$

and let $\mathcal{I}$ denote the $k$-linear Duhamel operator

$$
\mathcal{I}[u_1, \ldots, u_k](t) := \int_0^t \sin((t-t')|\nabla|) \left[ \prod_{j=1}^k u_j(t') \right] dt'.
$$

Writing $\mathcal{I}^k[u] = \mathcal{I}[u, \ldots, u]$, we have the following Duhamel formulation of (1.1):

$$
u(t) = S(t)(\bar{u}_0) + \mathcal{I}^k[u](t).
$$
We use the convention
\[
\sin(t(0)) \frac{1}{|0|} = t. \tag{2.5}
\]
For \(0 \leq t \leq 1\), we have
\[
\|S(t)\tilde{u}_0\|_{H^s} \leq \|u_0\|_{H^s} + t\|u_1\|_{H^{s-1}} \leq \|\tilde{u}_0\|_{H^s}. \tag{2.6}
\]
Second, we recall the following definitions and terminology used in \cite{17} to describe \(k\)-ary trees.

**Definition 2.1.** (i) Given a partially ordered set \(\mathcal{T}\) with partial order \(\leq\), we say that \(b \in \mathcal{T}\) with \(b \leq a\) and \(b \neq a\) is a child of \(a \in \mathcal{T}\), if \(b \leq c \leq a\) implies either \(c = a\) or \(c = b\). If the latter condition holds, we also say that \(a\) is the parent of \(b\).

(ii) A tree \(\mathcal{T}\) is a finite partially ordered set, satisfying the following properties:
- Let \(a_1, a_2, a_3, a_4 \in \mathcal{T}\). If \(a_4 \leq a_2 \leq a_1 \) and \(a_4 \leq a_3 \leq a_1\), then we have \(a_2 \leq a_3\) or \(a_3 \leq a_2\).
- A node \(a \in \mathcal{T}\) is called terminal, if it has no child. A non-terminal node \(a \in \mathcal{T}\) is a node with exactly \(k\) children,
- There exists a maximal element \(r \in \mathcal{T}\) (called the root node) such that \(a \leq r\) for all \(a \in \mathcal{T}\),
- \(\mathcal{T}\) consists of the disjoint union of \(\mathcal{T}^0\) and \(\mathcal{T}^\infty\), where \(\mathcal{T}^0\) and \(\mathcal{T}^\infty\) denote the collections of non-terminal nodes and terminal nodes, respectively.

(iii) Let \(\mathcal{T}(j)\) denote the set of all trees with \(j\) non-terminal nodes.

Note that a given \(k\)-ary tree \(\mathcal{T} \in \mathcal{T}(j)\) has \(kj + 1\) nodes. This follows from the fact that the number of non-terminal and terminal nodes of \(\mathcal{T}\) are \(j\) and \((k - 1)j + 1\) respectively, where \(j \in \mathbb{N} \cup \{0\}\).

We have the following basic combinatorial property for \(k\)-ary trees. The proof is a straightforward adaptation of the one in \cite{17} Lemma 2.3 for ternary trees \((k = 3)\).

**Lemma 2.2.** There exists a constant \(C_0 > 0\) such that
\[
|\mathcal{T}(j)| \leq C_0^j. \tag{2.7}
\]

For fixed \(\tilde{\phi} \in \mathcal{FL}^1(\mathcal{M})\), we associate to a given tree \(\mathcal{T} \in \mathcal{T}(j)\), a space-time distribution \(\psi_\phi(\mathcal{T}) \in D'(\mathcal{O}(0, T] \times \mathcal{M})\) as follows: we replace a non-terminal node by the Duhamel integral operator \(\mathcal{I}\) with its \(k\) arguments as children and we replace all terminal nodes by the linear solution \(S(t)\tilde{\phi}\). We then define
\[
\Xi_j(\tilde{\phi}) = \sum_{\mathcal{T} \in \mathcal{T}(j)} \psi_\phi(\mathcal{T}). \tag{2.8}
\]

For example,
\[
\Xi_0(\tilde{\phi}) = S(t)\tilde{\phi} \quad \text{and} \quad \Xi_1(\tilde{\phi}) = \mathcal{I}[S(t)\tilde{\phi}, \ldots, S(t)\tilde{\phi}].
\]

The multilinear operators \(\Xi_j\) satisfy the following estimates. We use short-hand notations such as \(C_T\mathcal{F}L^p = C([0, T]; \mathcal{F}L^p(\mathcal{M}))\) for \(T > 0\).

**Lemma 2.3.** There exists \(C > 0\) such that the following hold: Given \(\tilde{\phi} \in \mathcal{FL}^1(\mathcal{M}), j \in \mathbb{N}, \tilde{\psi} \in \mathcal{FL}^1(\mathcal{M}) \cap H^0(\mathcal{M}), \) and \(0 < T \leq 1\), we have
\[
\|\Xi_j(\tilde{\phi})(T)\|_{\mathcal{F}L^1} \leq C_T 2^{2j} \|\tilde{\phi}\|_{\mathcal{F}L^1}^{(k-1)j+1}, \tag{2.9}
\]
\[
\|\Xi_j(\tilde{\psi})(T)\|_{\mathcal{F}L^\infty} \leq C_T 2^{2j} \|\tilde{\psi}\|_{\mathcal{F}L^1}^{(k-1)j-1} \|\tilde{\psi}\|_{H^0}^2. \tag{2.10}
\]
Proof. For \( \vec{\phi} = (\phi_0, \phi_1) \), we have from (2.2), (2.3), and \( 0 < t < T \leq 1 \),
\[
\| S(t)(\vec{\phi}) \|_{C_T F L^1} \leq \| \phi_0 \|_{F L^1} + T \| \phi_1 \|_{F L^{-1,1}} \leq \| \vec{\phi} \|_{\overline{F L^1}}.
\] (2.11)
As \( |\sin t| \leq |t| \) for every \( t \in \mathbb{R} \), (2.3) and the algebra property of \( F L^1(M) \) imply
\[
\| I[u_1, \ldots, u_k] \|_{C_T F L^1} \leq \left( \int_0^T (T - t) dt \right) \prod_{j=1}^k \| u_j \|_{C_T F L^1} \leq C T^2 \prod_{j=1}^k \| u_j \|_{C_T F L^1}.
\] (2.12)
For a fixed \( T \in T(j) \), \( \Psi_{\vec{\phi}}(T) \) is essentially \( j \) iterated compositions of the operator \( I^k[S(t)\vec{\phi}] \), with \( (k-1)j + 1 \) terms \( S(t)\vec{\phi} \). Hence, (2.9) follows from (2.8), (2.7), (2.12), and (2.11). Likewise, (2.10) follows similarly in addition to using Young’s inequality. \( \square \)

We now justify the power series expansion for solutions to (2.4).

**Lemma 2.4.** Let \( k \geq 2 \) be an integer and \( M > 0 \). Then, for any \( 0 < T \ll \min(M^{-\frac{k-1}{2}}, 1) \) and \( \vec{u}_0 \in \overline{F L^1(M)} \) with \( \| \vec{u}_0 \|_{\overline{F L^1}} \leq M \), the following holds:

(i) There exists a unique solution \( u \in C([0, T]; F L^1(M)) \) satisfying \( (u, \partial_t u)|_{t=0} = \vec{u}_0 \) to (2.4).

(ii) The solution \( u \) in (i) may be expressed as
\[
u = \sum_{j=0}^{\infty} \Xi_j(\vec{u}_0) = \sum_{j=0}^{\infty} \sum_{T \in T(j)} \Psi_{\vec{u}_0}(T),
\] (2.13)
where the series converges absolutely in \( C([0, T]; F L^1(M)) \).

**Proof.** We begin with (i). We define
\[
\Gamma[u(t)] := S(t)(\vec{u}_0) + I^k[u(t)].
\]
Then, (2.12) implies
\[
\| \Gamma[u(t)] \|_{C_T F L^1} \leq \| \vec{u}_0 \|_{\overline{F L^1}} + C T^2 \| u \|_{C_T F L^1}.
\]
Thus, for \( 0 < T \leq 1 \) such that \( C T^2 M^{k-1} \ll 1 \), \( \Gamma \) maps the ball \( B_{2M} := \{ v \in C([0, T]; F L^1(M)) : \| v \|_{C_T F L^1} \leq 2M \} \) into itself. In view of the multilinearity of \( I \), we may reduce \( T \) further to ensure that \( \Gamma \) is in fact a strict contraction on \( B_{2M} \). The contraction mapping theorem and an a posteriori continuity argument completes (i). We move onto verifying (ii). We fix \( 0 < T \leq 1 \) such that \( C T^2 M^{k-1} \ll 1 \) and fix \( \varepsilon > 0 \). From (2.9), we see that the sum in (2.13) converges absolutely in \( C([0, T]; F L^1(M)) \) and hence there exists an integer \( J_1 \geq 0 \) such that
\[
\| U - U_{J_1} \|_{C_T F L^1} < \frac{\varepsilon}{3},
\] (2.14)
where
\[
U_{J} := \sum_{j=0}^{J} \Xi_j(\vec{u}_0), \quad \text{and} \quad U := \sum_{j=0}^{\infty} \Xi_j(\vec{u}_0).
\]
In particular, \( U, U_{J} \in B_{2M} \) for any \( J \in \mathbb{N} \). From (i), \( \Gamma \) is continuous from \( B_{2M} \) into itself and hence there exists an integer \( J_2 \geq 0 \) such that
\[
\| \Gamma[U_{J_2}] - \Gamma[U] \|_{C_T F L^1} < \frac{\varepsilon}{3},
\] (2.15)
Now for a fixed integer $J \geq 1$, we consider the difference $U_J - \Gamma[U_J]$. We have

$$U_J - \Gamma[U_J] = \sum_{j=1}^{J} \Xi_j(\vec{u}_0) - \sum_{0 \leq j_1, \ldots, j_k \leq J} I[\Xi_{j_1}(\vec{u}_0), \ldots, \Xi_{j_k}(\vec{u}_0)]$$

$$= - \sum_{\ell=J+1}^{k J} \sum_{0 \leq j_1, \ldots, j_k \leq J \atop j_1 + \cdots + j_k = \ell-1} I[\Xi_{j_1}(\vec{u}_0), \ldots, \Xi_{j_k}(\vec{u}_0)].$$

Using (2.12), (2.9), and crudely estimating the sums, we obtain

$$\|U_J - \Gamma[U_J]\|_{C_T F L^1} \leq C T^2 \sum_{\ell=J+1}^{k J} \sum_{0 \leq j_1, \ldots, j_k \leq J \atop j_1 + \cdots + j_k = \ell-1} \prod_{m=1}^{k} \|\Xi_{j_m}(\vec{u}_0)\|_{C_T F L^1}$$

$$\leq C T^2 M^k J^k \sum_{\ell=J+1}^{\infty} (C T^2 M^{k-1})^{\ell-1}$$

$$\leq C M J^k (C T^2 M^{k-1})^J.$$

Thus, there exists an integer $J_3 \geq 0$ such that

$$\|U_{J_3} - \Gamma[U_{J_3}]\|_{C_T F L^1} < \frac{\varepsilon}{3}. \quad (2.16)$$

With $J := \max_{t=1,2,3} J_t$, (2.14), (2.15), and (2.16) imply

$$\|U - \Gamma[U]\|_{C_T F L^1} \leq \|U - U_J\|_{C_T F L^1} + \|U_J - \Gamma[U_J]\|_{C_T F L^1} + \|\Gamma[U_J] - \Gamma[U]\|_{C_T F L^1} < \varepsilon.$$

Thus, $U = \Gamma[U]$ and, by uniqueness, we conclude $u = U$. \hfill $\square$

## 3. Norm inflation for NLW

In this section, we present the proof of Theorem 1.1 by establishing the following proposition. We use $\mathcal{S}(\mathcal{M})$ to denote the Schwarz space of functions if $\mathcal{M} = \mathbb{R}^d$ or the space of $C^\infty$ functions if $\mathcal{M} = \mathbb{T}^d$.

**Proposition 3.1.** Let $\mathcal{M} = \mathbb{R}^d$ or $\mathbb{T}^d$, $k \geq 2$ be an integer, $s < 0$, and fix $u_0, u_1 \in \mathcal{S}(\mathcal{M})$. Then, for any $n \in \mathbb{N}$, there exists a smooth solution $u_n$ to the NLW (1.1) and $t_n \in (0, \frac{1}{n})$ such that

$$\|(u_n(0), \partial_t u_n(0)) - (u_0, u_1)\|_{H^s(\mathcal{M})} < \frac{1}{n} \quad \text{and} \quad \|u_n(t_n)\|_{H^s(\mathcal{M})} > n. \quad (3.1)$$

From density and diagonal arguments, Theorem 1.1 follows from Proposition 3.1. See [23, 17] for the details.

Thus, the remaining part of this paper is devoted to the proof of Proposition 3.1. In the following, we fix $\vec{u}_0 = (u_0, u_1)$ with $u_0, u_1 \in \mathcal{S}(\mathcal{M})$. In Subsection 3.1, we prove multilinear estimates for each term in the power series expansion. Moreover, by observing high-to-low energy transfer and resonant interaction, we show a crucial lower bound for the first multilinear term. We then present the proof of Proposition 3.1 in Subsection 3.2.
3.1. **Multilinear estimates.** In this subsection, we state the multilinear estimates on $\Xi_j$. Moreover, we show that the first multilinear term $\Xi_1$ is the leading part in the power series expansion in negative Sobolev spaces.

Let $\chi_K$ denote the indicator function of a subset $K \subset \hat{\mathcal{M}}$, where $\hat{\mathcal{M}}$ is as in (2.1). Set $e_1 := (1, 0, \ldots, 0) \in \hat{\mathcal{M}}$. Given $n \in \mathbb{N}$, let $N = N(n) \gg 1$ to be chosen later. We set $\tilde{\phi}_n = (\phi_{0,n}, 0)$ by

$$\tilde{\phi}_{0,n} = R\chi_{\Omega},$$

where $R = R(N) \geq 1$,

$$\Omega = \bigcup_{\eta \in \Sigma} (\eta + Q_A),$$

$Q_A = [-\frac{A}{2}, \frac{A}{2})^d$, $A = A(N) \gg 1$, and

$$\Sigma = \{-2Ne_1, -Ne_1, Ne_1, 2Ne_1\}.$$  (3.4)

We require $N, R$, and $A$ to be chosen so that

$$||\tilde{u}_0||_{\overline{F^1 L^1}} \ll RA^d \quad \text{and} \quad A \ll N,$$

where the last condition ensures that $\Omega$ in (3.3) is a disjoint union. Notice that (3.2) and (3.3) imply

$$||\tilde{\phi}_n||_{\overline{F^1 L^1}} = ||\phi_{0,n}||_{F^1 L^1} \sim RA^d \quad \text{and} \quad ||\tilde{\phi}_n||_{H^s} \sim RA^d N^s,$$

for any $s \in \mathbb{R}$. We define $\tilde{u}_{0,n} := \tilde{u}_0 + \tilde{\phi}_n$. For each $n \in \mathbb{N}$, Lemma 2.4 implies that there exists a unique solution $u_n \in C([0, T]; \mathcal{F}L^1(\mathcal{M}))$ to (2.4) with $(u_n, \partial_t u_n)_{t=0} = \tilde{u}_{0,n}$ and admitting the power series expansion

$$u_n = \sum_{j=0}^{\infty} \Xi_j(\tilde{u}_{0,n}) = \sum_{j=0}^{\infty} \Xi_j(\tilde{u}_0 + \tilde{\phi}_n)$$

(3.7)

on $[0, T]$, provided

$$0 < T \ll (||\tilde{u}_0||_{\overline{F^1 L^1}} + RA^d)^{-\frac{k-1}{k}}.$$  (3.8)

We now state some key estimates for the multilinear expressions $\Xi_j(\tilde{u}_{0,n})$.

**Lemma 3.2.** For any $j \in \mathbb{N}$, the following estimates hold:

$$||\tilde{u}_{0,n} - \tilde{u}_0||_{H^s} \lesssim RA^d N^s,$$

$$||\Xi_0(\tilde{u}_{0,n})(t)||_{H^s} \lesssim 1 + RA^d N^s,$$

$$||\Xi_1(\tilde{u}_{0,n})(t) - \Xi_1(\tilde{\phi}_n)(t)||_{H^s} \lesssim t^2 ||\tilde{u}_0||_{H^0} R^{k-1} A^{d(k-1)}$$

(3.11)

$$||\Xi_j(\tilde{u}_{0,n})(t)||_{H^s} \lesssim C^j t^{2j} (RA^d)^{(k-1)j} (||\tilde{u}_0||_{H^0} + Rf_s(A)),$$

(3.12)

where

$$f_s(A) := \begin{cases} 
1 & \text{if } s < -\frac{d}{2}, \\
(\log A)^{\frac{d}{2}} & \text{if } s = -\frac{d}{2}, \\
A^\frac{d}{2} + s & \text{if } s > -\frac{d}{2}
\end{cases}$$

and $0 < t \leq 1$.  


Proof. The proofs of (3.9) and (3.10) follow immediately from $\tilde{\phi}_n = \tilde{u}_{0,n} - \tilde{u}_0$, (3.6), and (2.6). By the multilinearity of $I$, we have

$$
\Xi_1(\tilde{u}_{0,n})(t) - \Xi_1(\tilde{\phi}_n)(t) = \sum_{\tilde{\psi}_1, \ldots, \tilde{\psi}_k} I[S(t)\tilde{\psi}_1, \ldots, S(t)\tilde{\psi}_k],
$$

(3.13)

where the sum is over all choices of $\tilde{\psi}_j \in \{\tilde{u}_0, \tilde{\phi}_n\}$ with at least one appearance of $\tilde{u}_0$. Since $s < 0$, (3.13), Young’s inequality, (2.6), and (2.11) imply

$$
\|\Xi_1(\tilde{u}_{0,n})(t) - \Xi_1(\tilde{\phi}_n)(t)\|_{H^s} \leq \sum_{\tilde{\psi}_1, \ldots, \tilde{\psi}_k} \|I[S(t)\tilde{\psi}_1, \ldots, S(t)\tilde{\psi}_k]\|_{L^2} \lesssim C^2\|\tilde{u}_0\|_{H^0}(\|\tilde{u}_0\|_{\mathcal{F}L^1}^{k-1} + \|\tilde{\phi}_n\|_{\mathcal{F}L^1}^{k-1}).
$$

Using (3.5) and (3.6), we obtain (3.11).

We now prove (3.12). By the triangle inequality, we have

$$
\|\Xi_j(\tilde{u}_{0,n})(t)\|_{H^s} \leq \|\Xi_j(\tilde{u}_{0,n})(t) - \Xi_j(\tilde{\phi}_n)(t)\|_{H^s} + \|\Xi_j(\tilde{\phi}_n)(t)\|_{H^s},
$$

(3.14)

and thus we reduce to proving estimates for the two terms on the right hand side of the above.

From (3.2) and (3.3), supp $\mathcal{F}[S(t)\tilde{\phi}_n]$ is contained within at most four disjoint cubes of volume approximately $A^d$. Thus, for each fixed $T \in T(j)$, the support of $\mathcal{F}[\Psi_{\tilde{\phi}_n}(T)]$ is contained in at most $4^{(k-1)j+1}$ cubes of volume approximately $A^d$. As $s < 0$, $\langle \xi \rangle^s$ is decreasing in $|\xi|$ and using (2.10), (2.6), (2.11), and (3.6), we have

$$
\|\Xi_j(\tilde{\phi}_n)(t)\|_{H^s} \lesssim \|\langle \xi \rangle^s\|_{L^2(\text{supp } \mathcal{F}[\Psi_{\tilde{\phi}_n}(T)])}\|\Xi_j(\tilde{\phi}_n)(t)\|_{\mathcal{F}L^\infty} \lesssim Cj^i\|\langle \xi \rangle^s\|_{L^2(C_1Q_A)}t^{2j}(RA^d)^{(k-1)j-1}(RA^d)^2 \lesssim Cj^i t^{2j}(RA^d)^{(k-1)j}Rf_s(A).
$$

(3.15)

Meanwhile, by considerations similar to (3.13), we have

$$
\|\Xi_j(\tilde{u}_{0,n})(t) - \Xi_j(\tilde{\phi}_n)(t)\|_{H^s} \leq \|\Xi_j(\tilde{u}_{0,n})(t) - \Xi_j(\tilde{\phi}_n)(t)\|_{L^2} \lesssim Cj^i t^{2j}\|\tilde{u}_0\|_{H^0}(\|\tilde{u}_0\|_{\mathcal{F}L^1}^{(k-1)j} + \|\tilde{\phi}_n\|_{\mathcal{F}L^1}^{(k-1)j}) \lesssim Cj^i t^{2j}\|\tilde{u}_0\|_{H^0}(RA^d)^{(k-1)j}.
$$

(3.16)

Thus, (3.12) follows from (3.14), (3.15), and (3.16).

We now recall the following bounds on convolutions of characteristic functions of cubes:

Lemma 3.3. For any $a, b, \xi \in \hat{\mathcal{M}}$ and $A \geq 1$, we have

$$
c_d A^d \chi_{a+b+Q_A}(\xi) \leq \chi_{a+Q_A} * \chi_{b+Q_A}(\xi) \leq C_d A^d \chi_{a+b+Q_{2A}}(\xi).
$$

(3.17)

In the following proposition, we identify that the first multilinear term in the Picard expansion is culpable for the instability in Proposition 3.1.

Proposition 3.4. Let $k \geq 2$ be an integer and $s < 0$. Let $\tilde{\phi}_n$ be as in (3.2). Then, for $(AN)^{-\frac{1}{2}} \ll T \ll A^{-1}$, we have

$$
\|\Xi_1(\tilde{\phi}_n)(T)\|_{H^s} \lesssim T^2 R^k A^{d(k-1)} f_s(A).
$$

(3.18)
Proof. To simplify notation, we write
\[ \Gamma := \left\{ (\xi_1, \ldots, \xi_k) \in \mathcal{M}^k : \sum_{j=1}^{k} \xi_j = \xi \right\} \quad \text{and} \quad d\xi := d\xi_1 \cdots d\xi_{k-1}. \]

Restricting \( \xi \) to \( 4^{-1} e_1 + Q_4 \) and using \( A \ll N \), \( (3.2) \), and product-to-sum formulas, we have
\[ F[\Xi_1(\phi_n)(T)](\xi) = \int_0^T \frac{\sin((T-t)|\xi|)}{|\xi|} F[(S(t)(\phi_{0,n}, \phi_{1,n}))^k] dt \]
\[ = \int_0^T \frac{\sin((T-t)|\xi|)}{|\xi|} \prod_{m=1}^{k} \cos(t|\xi_m|) \phi_{0,n}(\xi_m) d\xi dt \]
\[ = R^k \int_0^T \frac{\sin((T-t)|\xi|)}{|\xi|} \prod_{m=1}^{k} \cos(t|\xi_m|) \chi(\xi_m) d\xi dt \]
\[ = R^k 2^k \sum_{(\eta_1, \ldots, \eta_k) \in \Sigma^k} \sum_{(\eta_1 + \cdots + \eta_k = 0)} \int_0^T \frac{\sin((T-t)|\xi|)}{|\xi|} \left( t \sum_{j=1}^{k} \varepsilon_j |\xi_j| \right) \prod_{j=1}^{k} \chi_{\eta_j + Q_A}(\xi_j) d\xi dt. \]

For each fixed \( \eta := (\eta_1, \ldots, \eta_k) \in \Sigma^k \), we split the inner summation into two parts:
\[ \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{-1,1\}^k} = \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in S_1(\eta)} + \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in S_2(\eta)} \]
where
\[ S_1(\eta) := \left\{ (\varepsilon_1, \ldots, \varepsilon_k) \in \{-1,1\}^k : \sum_{j=1}^{k} \varepsilon_j |\eta_j| = 0 \right\}, \]
\[ S_2(\eta) := \{-1,1\}^k \setminus S_1(\eta) \]
and we write
\[ F[\Xi_1(\phi_n)(T)](\xi) = R^k 2^k \sum_{\eta = (\eta_1, \ldots, \eta_k) \in \Sigma^k} \left( I_1(\eta, \xi, T) + I_2(\eta, \xi, T) \right). \quad (3.19) \]

From \( (3.4) \), the set \( S_1(\eta) \) is non-empty. For fixed \( (\varepsilon_1, \ldots, \varepsilon_k) \in S_1(\eta) \) and \( \xi_j \in \eta_j + Q_A \),
\[ \left| \sum_{j=1}^{k} \varepsilon_j |\xi_j| \right| \leq \left| \sum_{j=1}^{k} \varepsilon_j |\eta_j| \right| + \sum_{j=1}^{k} |\xi_j - \eta_j| \lesssim A, \quad (3.20) \]
since \( A \ll N \). Then, it follows from \( (3.20) \) that
\[ \cos \left( t \sum_{j=1}^{k} \varepsilon_j |\xi_j| \right) \geq \frac{1}{2} \quad (3.21) \]
for \( 0 < t < T \ll A^{-1} \). Moreover, we have
\[ \frac{\sin((T-t)|\xi|)}{|\xi|} \gtrsim T - t \quad (3.22) \]
for $0 < t < T \ll A^{-1}$ and $|\xi| \lesssim A$. Using (3.19), (3.17), (3.21), and (3.22), we obtain

$$I_1(\eta, \xi, T) \gtrsim \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in S_1(\eta)} \int_0^T (T - t) dt \int_{\Gamma} \prod_{j=1}^k \chi_{\eta_j + Q_A}(\xi_j) d\xi$$

$$\gtrsim T^2 A^{d(k-1)} \chi_{Q_A}(\xi)$$

and hence

$$\frac{R^k}{2^k} \sum_{\eta=(\eta_1, \ldots, \eta_k) \in \Sigma^k_{\eta_1 + \cdots + \eta_k = 0}} I_1(\eta, \xi, T) \gtrsim T^2 R^k A^{d(k-1)} \chi_{Q_A}(\xi)$$

(3.23)

for $1 \leq A \ll N$ and $0 < T \ll A^{-1}$.

We now turn to the contribution from $I_2(\eta, \xi, T)$. We observe that for each fixed $\eta = (\eta_1, \ldots, \eta_k), (\varepsilon_1, \ldots, \varepsilon_k) \in S_2(\eta)$, and $\xi_j \in \eta_j + Q_A$,

$$\left| \sum_{j=1}^k \varepsilon_j |\xi_j| \right| \sim N.$$

(3.24)

In view of $A \ll N$, the upper bound is obvious. For the lower bound, the reverse triangle inequality yields $||\xi_j| - |\eta_j|| \leq |\xi_j - \eta_j| \lesssim A$ for $\xi_j \in \eta_j + Q_A$ and hence, we have

$$\sum_{j=1}^k \varepsilon_j |\xi_j| = \sum_{j=1}^k \varepsilon_j |\eta_j| + \sum_{j=1}^k \varepsilon_j (|\xi_j| - |\eta_j|) = \sum_{j=1}^k \varepsilon_j |\eta_j| + O(A).$$

It follows from (3.24) that

$$\left| \sum_{j=1}^k \varepsilon_j |\eta_j| \right| \geq N$$

for $(\varepsilon_1, \ldots, \varepsilon_k) \in S_2(\eta)$, which verifies (3.24). We therefore have

$$I_2(\eta, \xi, T) = \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in S_2(\eta)} \frac{1}{2|\xi|} \int_{\Gamma} \int_0^T \left[ \sin \left( T|\xi| - t \left( |\xi| - \sum_{j=1}^k \varepsilon_j |\xi_j| \right) \right) 
+ \sin \left( T|\xi| - t \left( |\xi| + \sum_{j=1}^k \varepsilon_j |\xi_j| \right) \right) \right] dt \prod_{j=1}^k \chi_{\eta_j + Q_A}(\xi_j) d\xi.$$

Recalling $\xi \in \frac{4}{3} e_1 + Q_\frac{4}{3}$ and using (3.24), (3.17), and $A \ll N$, we obtain

$$|I_2(\eta, \xi, T)| \lesssim 2^k A^{-1} N^{-1} A^{d(k-1)} \chi_{Q_A}(\xi),$$

which implies

$$\left| \frac{R^k}{2^k} \sum_{\eta=(\eta_1, \ldots, \eta_k) \in \Sigma^k_{\eta_1 + \cdots + \eta_k = 0}} I_2(\eta, \xi, T) \right| \lesssim A^{-1} N^{-1} R^k A^{d(k-1)} \chi_{Q_A}(\xi).$$

(3.25)
Returning to (3.19) and using (3.23), (3.25) and imposing \( T^2AN \gg 1 \), we obtain

\[
\|\Xi_1(\tilde{\phi}_n)(T)\|_{H^s} \geq \|\langle \xi \rangle^s F[\Xi_1(\tilde{\phi}_n)](T)\|_{L^2_\xi(\mathbb{R}^+)} \geq (T^2 - A^{-1}N^{-1})R^kA^{d(k-1)}f_s(A) \geq T^2R^kA^{d(k-1)}f_s(A),
\]

which shows (3.18). \( \square \)

**Remark 3.5.** The same result as in Proposition 3.4 is valid for (1.9). Indeed, since the linear solution of (1.9) is written as

\[
\eta_{|T_n} = (\eta_{|T_1}, \ldots, \eta_{|T_k}) \in S_1(\eta), \quad \xi_{|T_n} = (\xi_{|T_1}, \ldots, \xi_{|T_k}) \in S_1(\xi),\]

there is a unique solution \( u_{|T_n} \). Hence, from the same argument as in the proof of Proposition 3.4, the first multilinear term in the Picard expansion for (1.9) satisfies (3.18).

### 3.2. Proof of Proposition 3.1

In order to prove Proposition 3.1 it suffices to show, given \( n \in \mathbb{N} \), the following hold:

(i) \( RA^dN^s \ll \frac{1}{n} \),

(ii) \( T^2R^{k-1}A^{d(k-1)} \ll 1 \),

(iii) \( T^2R^kA^{d(k-1)}f_s(A) \gg n \),

(iv) \( T^2R^kA^{d(k-1)}f_s(A) \gg T^4R^{2k-1}A^{2d(k-1)}f_s(A) \),

(v) \( (AN)^{-\frac{1}{2}} \ll T \ll \min \left( A^{-1}, \frac{1}{n} \right) \),

(vi) \( \|\tilde{u}_0\|_{\dot{H}^0} \ll Rf_s(A), \quad A \ll N, \quad \|\tilde{u}_0\|_{\dot{F}\dot{L}^1} \ll RA^d \)

for some particular choices of \( A, R, T, \) and \( N \) all depending on \( n \). Notice that (iv) is satisfied automatically once (ii) is satisfied. The conditions (ii) and the last of (vi) ensure that the power series expansion (3.7) is valid on \([0,T]\), where \( T \) must satisfy (3.8).

We now indicate how establishing (i) through (vi) suffices to prove Proposition 3.1. When (ii) and (vi) hold, it follows from (3.2), (3.3), (3.5), (3.6), and Lemma 2.4 that for each \( n \in \mathbb{N} \), there is a unique solution \( u_n \in C([0,T_n]; \mathcal{F}\dot{L}^1(\mathcal{M})) \) to (2.4) with \( (u_n, \partial_t u_n)|_{t=0} = \tilde{u}_{0,n} \).
and such that the power series expansion (3.7) converges on \([0, T_n]\). By (3.9), condition (i) ensures the first expression in (3.1). From (3.12), (ii), and (vi), we have

\[
\| \sum_{j=2}^{\infty} \Xi_j(\vec{u}_{0,n})(T) \|_{H^s} \lesssim R f_s(A) \sum_{j=2}^{\infty} (CT^2 R^{k-1} A^{d(k-1)})^j
\]

(3.26)

Then, from Proposition 3.4 (thus requiring (v)), (3.10), (3.11), (3.26), (iii), (iv), and (vi), we have

\[
\| u_n(T) \|_{H^s} \geq \| \Xi_1(\vec{\phi}_n)(T) \|_{H^s} - \| \Xi_0(\vec{u}_{0,n}) \|_{H^s} - \| \sum_{j=2}^{\infty} \Xi_j(\vec{u}_{0,n})(T) \|_{H^s}
\]

\[
\lesssim T^2 R^k A^{d(k-1)} f_s(A) - (1 + RA^{\frac{s}{2}} N^s) - T^2 \| \vec{u}_0 \|_{\mathcal{H}} R^{k-1} A^{d(k-1)}
\]

\[
\sim T^2 R^k A^{d(k-1)} f_s(A) \gg n,
\]

which establishes the second expression in (3.1) and hence Proposition 3.1. It remains to show, given \(n \in \mathbb{N}\), we can choose \(A, R, T\) depending on \(N\) and then \(N = N(n) \gg 1\) so that conditions (i) through (vi) hold.

- **Case 1:** \(-\frac{1}{k-1} \leq s < 0\).

We choose

\[
A = 10, \quad R = N^{-s-\delta}, \quad T = N^{\frac{k-1}{2}(s+\frac{\delta}{2})},
\]

for \(0 < \delta \ll 1\) sufficiently small so that

\[-s > \frac{k + 1}{2} \delta.
\]

Then, we have

\[
RA^{\frac{s}{2}} N^s \sim N^{-\delta} \ll \frac{1}{n},
\]

\[
T^2 R^{k-1} A^{d(k-1)} \sim N^{-\frac{k-1}{2} \delta} \ll 1,
\]

\[
T^2 R^k A^{d(k-1)} f_s(A) \sim N^{-s-\frac{k+1}{2} \delta} \gg n,
\]

\[
TA \sim N^{-\frac{k-1}{2}(s+\frac{\delta}{2})} \ll \frac{1}{n},
\]

\[
T^2 AN \sim N^{(k-1)(s+\frac{1}{2} + \frac{k}{2k-1})} \gg 1,
\]

since \(k \geq 2\) and \(-\frac{1}{k-1} \leq s < 0\).

- **Case 2:** \(s = -\frac{1}{k-1}\).

This case follows from Case 1 with \(s = -\frac{1}{k-1}\). More precisely, we choose

\[
A = 10, \quad R = N^{\frac{1}{k-1}-\delta}, \quad T = N^{-\frac{1}{2} + \frac{k+1}{4k-1} \delta},
\]
for $0 < \delta < \frac{2}{k-1}$. Then, we obtain

\[ R^{A_d} N^s \sim N^{s + \frac{1}{k-1} - \delta} \ll \frac{1}{n}, \]

\[ T^2 R^{k-1} A^{d(k-1)} \sim N^{-\frac{1}{k-1} + \frac{k-1}{4} \delta} \ll 1, \]

\[ T^2 R^k A^{d(k-1)} f_s(A) \sim N^{\frac{1}{k-1} + \frac{1}{4} \delta} \gg n, \]

\[ T A \sim N^{-\frac{1}{4} + \frac{k-1}{4} \delta} \ll \frac{1}{n}, \]

\[ T^2 A N \sim N^{\frac{k-1}{4} \delta} \gg 1, \]

since $k \geq 2$ and $s < -\frac{1}{k-1}$.

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