Propagators of the Jaynes-Cummings model in open systems

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Abstract

We present a propagator formalism to investigate the scattering of photons by a cavity QED system that consists of a single two-level atom dressed by a leaky optical cavity field. We establish a diagrammatic method to construct the propagator analytically. This allows us to determine the quantum state of the scattered photons for an arbitrary incident photon packet. As an application, we explicitly solve the problem of a single-photon packet scattered by an initially excited atom.

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I. INTRODUCTION

The engineering of novel quantum states of photons is a topic of interest fundamentally and for applications. As is well known in atomic and particle physics, exotic states could result from scattering processes. This suggests light scattering by suitable quantum systems can be an important tool for state engineering. A common and yet realistic situation in quantum optics involves the scattering of photon wave packets, comprising a few photons, from atoms and other photons situated inside an optical cavity. The atoms and the cavity field together form a cavity QED system that scatters incident photon packets (Fig. 1). The main question is: What is the quantum state of the scattered photons? As the cornerstone of a series of investigations of this topic, we establish in the present paper a propagator method to describe the scattering of photon packets from a cavity where a two-level atom, either in its ground or excited state, is placed with the companion of some quasi-mode photons. After the interaction, the input photons and the quasi-mode photons as well leak out of the cavity and become the output state of the system. The propagator method proposed here is capable to extract the details of the output state and thus form a powerful tool to analyze the physical content underlying the scattering process.

In addition to its potential applications in the study of novel quantum states, the model considered here is in fact a generalized variant of the Jaynes-Cummings model (JCM) [1], which has been a fundamental model in the realm of photon-atom interaction ever since its first introduction [2]. Not only does it provide the simplest description of an atom interacting with quantized fields in optical cavities, JCM is also an important tool for controlling quantum states [3]. One prominent generalization of the JCM is the inclusion of the leakage effect of the cavity, which is inevitable in any real experimental setup. Indeed recent studies in cavity QED have also emphasized the important role of dissipation effects in quantum information devices [4].

In conventional approaches to JCM in a leaky cavity it is customary to consider the space outside the cavity — the environment — as a Markovian bath, followed by solving
the master equation of the reduced density matrix of the cavity field \[5–9\]. Thus, energy inside the cavity flows unidirectionally to the environment, which plays the role of a sink of energy and the information on the exact state of the environment is reckoned unimportant and is lost during the evolution.

The model discussed in this paper distinguishes itself from other dissipative JCM’s in that the cavity and its environment play an equal role during the interaction. These two parts, constituting the “universe” in consideration, communicate their information through partially transmitting walls dividing them. Photons inside the cavity, described in terms of discrete quasi-modes, interact with the atom and eventually leak out of the cavity. On the other hand, incident photons characterized by continuous wave number enter the cavity, participate in the interactions with the atom, and leave the cavity in its final state as well. More importantly, these two kinds of photons lose their respective identities after entering (or returning to) the environment and interfere to form novel photon states. It is for this reason that we devote the current paper to a proper formalism describing this generalized JCM model.

To this end, we present here a propagator method that solves directly the quantum state of the whole amalgamated system, consisting of the atom, the cavity and the environment. Rather than discrete cavity modes, the atom now couples with the continuous field modes of the whole system \[10–13\]. In this sense, the artificial separation between the cavity and its environment is completely eliminated, and the effect of dissipation can be treated in a more rigorous and fundamental way. Once the propagator of the system is known, the exact details of the output photons can be obtained immediately without further ado. This knowledge is bound to be important for issues related to quantum information and quantum measurement.

The pure state approach has previously been used to study resonance fluorescence \[14\] and spontaneous atomic decay in a vacuum cavity, and exact solutions have been found \[11,15–17\]. The major objective in the present paper is to formulate a comprehensive theory studying the dynamical response of a two-level atom under the influence of photons inside
the cavity and from the environment. In a recent paper by the authors [18], a less general case with the input photons restricted to the cavity mode (i.e., quasi-mode) was considered and solved by method of Laplace transform. This restriction is now removed in this paper. By evaluating the propagator of the system, whose construction is based on a diagrammatic expansion method, one can readily handle situations with an arbitrary initial state. Under the rotating-wave-approximation, the full propagator is a block diagonal matrix with a sequence of $2 \times 2$ matrices forming its diagonal block. Each of these matrices is a propagator of the system in a subspace categorized by the total excitation number $N$ (to be defined rigorously in Sect. IV), providing a natural classification of relevant propagators. A construction method is established here to express the $N$-excitation propagators in terms of those of the lower excitations. Thus, the full propagator of the system can be obtained in a systematic way, paving the way for further investigation into the quantum state of the output photons.

This paper is organized as follows: In Sec. II, we describe the system under study, its normal modes, the atom-field interactions and obtain the Hamiltonian. An introduction on the propagator method and the related diagrammatic expansion will be given in Sec. III. The general solution for the propagator is derived in Sec. IV. We then apply the propagator for $N = 2$ to study the scattering of a single-photon packet from a two-level atom in its excited state in Sec. V. We draw our conclusion in Sec. VI.

II. THE MODEL

A. The cavity and continuous modes

We consider here a two-sided Fabry-Perot cavity in one-dimensional space. The two transmitting mirrors (hereafter referred to as mirrors R and L) are placed at $x = \pm L/2$. The set of continuous modes of such a Fabry-Perot cavity is known in the literature [10,13]. In the following, we will briefly sketch the main result for reference.

The two mirrors enclosing the cavity are modeled by two thin dielectric slabs of thickness
and refractive indices \( n_\alpha (\alpha = \text{L,R}) \). In the limit \( l \to 0 \) and \( n_\alpha \to \infty \) such that \( n_\alpha^2 l \to \mu_\alpha \) is finite, the corresponding complex amplitude reflection \( r_\alpha \) and transmission coefficients \( t_\alpha \) (\( \alpha = \text{L,R} \)) are in turn expressed as

\[
\begin{align*}
  r_\alpha &= \frac{ik\mu_\alpha}{2 - ik\mu_\alpha}, \\
  t_\alpha &= \frac{2}{2 - ik\mu_\alpha},
\end{align*}
\]

where \( k \) is the wave number.

There are two independent sets of modes for the entire system (polarization is ignored in the current study), namely, the left-propagating modes \( u_\text{L}(k,x) \) and the right-propagating modes \( u_\text{R}(k,x) \), which are given by [10,13]:

\[
\begin{align*}
  u_\text{L}(k,x) &= \begin{cases} 
    e^{ikx} + R_\text{L}(k)e^{-ikx} & -\infty < x < -L/2 \\
    I_\text{L}(k)e^{ikx} + J_\text{L}(k)e^{-ikx} & -L/2 < x < L/2 \\
    T_\text{L}(k)e^{ikx} & L/2 < x < \infty
  \end{cases} \\
  u_\text{R}(k,x) &= \begin{cases} 
    T_\text{R}(k)e^{-ikx} & -\infty < x < -L/2 \\
    I_\text{R}(k)e^{-ikx} + J_\text{R}(k)e^{ikx} & -L/2 < x < L/2 \\
    e^{-ikx} + R_\text{R}(k)e^{ikx} & L/2 < x < \infty
  \end{cases}
\end{align*}
\]

where

\[
\begin{align*}
  R_\text{L}(k) &= \left\{ r_\text{L}e^{-ikL} + r_\text{R}e^{ikL+2i\arg t_\text{L}} \right\}/D(k), \\
  I_\text{L}(k) &= t_\text{L}/D(k), \\
  J_\text{L}(k) &= t_\text{L}r_\text{R}e^{ikL}/D(k), \\
  T_\text{L}(k) &= T_\text{R}(k) = t_\text{L}t_\text{R}/D(k),
\end{align*}
\]

with

\[
D(k) = 1 - r_\text{L}r_\text{R}e^{2ikL}.
\]

Likewise, \( I_\text{R}(k) \), \( J_\text{R}(k) \) and \( R_\text{R}(k) \) can be obtained by interchanging the roles of L and R in the above equations. The left-luminating modes are shown in Fig. 2. The mode functions
\( u_L(k, x) \) and \( u_R(k, x) \) \((k \geq 0)\) together form a complete orthonormal set in \(-\infty \leq x \leq \infty\), satisfying the orthonormal condition:

\[
\int_{-\infty}^{\infty} n(x) u_\alpha(k, x) u_\beta^*(k', x) dx = 2\pi \delta_{\alpha \beta} \delta(k - k'),
\]

(2.10)

where \(\alpha, \beta = \text{L, R}\). The quasi-mode frequencies, \(\bar{k}_m \) \((m = 1, 2, 3, \ldots)\), defined by the zeros of \(D(k)\), are given explicitly by

\[
\bar{k}_m = \frac{i}{2L} \ln(r_L r_R) + \frac{m\pi}{L}.
\]

(2.11)

For real-valued \(k\) close to a quasi-mode frequency, i.e., \(|k - \bar{k}_m| \ll \pi/L\), we have

\[
D(k) = -2iL(\Delta k + i\kappa_c),
\]

(2.12)

where \(\Delta k = k - k_c\), with

\[
k_c = \frac{m\pi}{L} - \frac{1}{2L} \text{Im}[\ln(r_L r_R)],
\]

(2.13)

\[
\kappa_c = -\frac{1}{2L} \text{Re}[\ln(r_L r_R)].
\]

(2.14)

Here \(\kappa_c\) is the decay rate of the cavity.

**B. The two-level atom and interaction Hamiltonian**

Consider a system of a two-level atom placed inside the cavity described above (or other leaky cavities in general) at \(x = x_0\). The ground-state energy of the atom is arbitrarily taken as zero, while the excited-state energy is \(\omega_A\) in units of \(\hbar = c = 1\). The full Hamiltonian of the system in the rotating-wave-approximation is given by

\[
H = \omega_A \sigma_+ \sigma_- + \int_{0}^{\infty} k(a_{kL}^\dagger a_{kL} + a_{kR}^\dagger a_{kR}) dk
+ \int_{0}^{\infty} \{[g_L(k)a_{kL} + g_R(k)a_{kR}] \sigma_+ + \text{h.c.}\} dk,
\]

(2.15)

where \(a_{k\alpha}\) and \(a_{k\alpha}^\dagger\) \((\alpha = \text{L, R})\) are the annihilation and creation operators of the \(k\alpha\)-mode photon, \(\sigma_{\pm}\) are the pseudo-spin flip operators of the atom. They satisfy the usual commutation relations: \([a_{k\alpha}, a_{k'\alpha'}^\dagger] = \delta(k - k')\delta_{\alpha \alpha'}\), \([a_{k\alpha}, a_{k'\alpha'}] = [a_{k\alpha}^\dagger, a_{k'\alpha'}^\dagger] = [a_{k\alpha}, \sigma_{\pm}^\dagger, \sigma_{\pm}] = \ldots \)
0, and \{\sigma_-, \sigma_+\} = 1. The coupling constant \(g_\alpha(k)\) of the atom with the \(k\alpha\)-mode photon depends on the dipole moment of the atom and is proportional to \(u_\alpha(k, x_0)\). Here we are particularly interested in the case where the transition frequency of the atom is close to one of the resonance frequencies of the cavity, say, \(k_c\). Hence, only those continuous modes with frequencies near \(k_c\) have significant interactions with the atom and in this single-mode approximation, \(g_\alpha(k) \propto (\Delta k + i\kappa_c)^{-1}\).

Despite that the atom ostensibly couples with both the R and L modes, it is always possible to use a unitary transformation to redefine the photon modes so that the atom interacts with only one set of modes. For concreteness, we consider in the present paper a symmetric cavity with identical mirrors and the atom being situated at its center. However, it can be proved that generalizations to cases with dissimilar mirrors, arbitrary atomic position and one-sided cavity are straightforward. For this specific model, we have \(u_L(k, x_0) = u_R(k, x_0), g_L(k) = g_R(k)\). Accordingly, one can define a new basis of photons by the unitary transformation

\[
a_{k+} = \frac{1}{\sqrt{2}}(a_{kL} + a_{kR}),
\]

\[
a_{k-} = \frac{1}{\sqrt{2}}(a_{kL} - a_{kR}).
\]

It is readily observed that the “−” modes do not couple to the atom. In the following discussion, we will ignore the “−” modes and will focus on the evolution of the “+” modes, with the “+” index being suppressed. Hence, the full Hamiltonian reduces to

\[
H = \omega_A \sigma_+ \sigma_- + \int_0^\infty k a_k^\dagger a_k dk + \int_0^\infty [g(k)a_k \sigma_+ + g^*(k)a_k^\dagger \sigma_-] dk,
\]

where \(a_k\) here actually denotes \(a_{k+}\) and \(g(k) = \sqrt{2}g_R(k)\).

III. PROPAGATOR AND FEYNMAN DIAGRAMS

The dynamical response of a system with a Hamiltonian \(H\) is governed by the retarded Green’s function that satisfies the equation:
\[
\left( i \frac{d}{dt} - H \right) K_+(t, t') = \delta(t - t'), \quad (3.1)
\]
and is null for \( t < t' \). Obviously, \( K_+ \) can be given explicitly by
\[
K_+(t, t') = e^{-iH(t-t')}\theta(t-t'),
\]
where \( \theta(x) \) is the Heaviside step function. Accordingly, its Fourier transform, \( G(\omega) \), defined by the relation
\[
G(\omega) = -i \int_{-\infty}^{\infty} K_+(t, 0)e^{i\omega t} dt \quad (3.2)
\]
and termed the retarded propagator \( G(\omega) \), can be expressed symbolically as
\[
G(\omega) = \frac{1}{\omega - H}, \quad (3.3)
\]
where the prescription \( \omega = \lim_{\epsilon \to 0^+} (\omega + i\epsilon) \) is assumed hereafter.

To establish a diagrammatic expansion for \( G(\omega) \), we separate the Hamiltonian in the form:
\[
H = H_0 + V,
\]
where
\[
H_0 = \omega_A \sigma_+ \sigma_- + \int_0^\infty k a_k^\dagger a_k dk \quad (3.4)
\]
is the free atom-field Hamiltonian, and
\[
V = \int_0^\infty [g(k)a_k^\dagger \sigma_- + g^*(k)a_k^\dagger \sigma_-]dk \quad (3.5)
\]
represents the interaction between the atom and the field. Hence, the propagator can formally be expanded in a power series of \( V \), yielding
\[
\frac{1}{\omega - H_0 - V} = \frac{1}{\omega - H_0} + \frac{1}{\omega - H_0} V \frac{1}{\omega - H_0} + \frac{1}{\omega - H_0} V \frac{1}{\omega - H_0} V \frac{1}{\omega - H_0} + \ldots.
\]

The propagator so defined is an operator. In the energy-eigenstate basis, the transition amplitudes are given by the matrix elements of the propagator, also referred to as “propagator” hereafter. The main purpose of this paper is to calculate these amplitudes by the associated Feynman diagrams as illustrated in Fig. 4. The basic construction rules and interpretations of these diagrams are specified as follows:
1. **External and Internal Lines:** Photons are represented by wavy lines labelled by their momenta $k$. Atoms in the excited and the ground states are respectively represented by solid and dashed lines.

2. **Vertex Factors:** Each vertex contributes a factor $g(k)$ (photon absorption) or $g^*(k)$ (photon emission) to the associated amplitude.

3. **Free Propagators:** Each segment in between successive vertices contributes a factor $(\omega - E)^{-1}$ to the associated amplitude, where $E$ is the energy of the free Hamiltonian (i.e., $H_0$) in this segment.

4. **Integrate Over Internal Momenta.** For each internal momentum $k$, write down a factor $dk$ and integrate.

**IV. EVALUATION OF THE PROPAGATORS**

In this section, we evaluate the matrix elements of the propagator using the diagrammatic rules stated in Sec. [III]. Owing to the rotating-wave-approximation, it is readily observed from Eq. (2.18) that the total excitation number

$$N = \sigma_+ \sigma_- + \int_0^\infty a_k^\dagger a_k dk$$

is a constant of motion, resulting in vanishing propagators from an initial state to a final state with different excitation numbers. Therefore, the full propagator can be represented by an infinite sequence of $2 \times 2$ matrices, each characterized by its excitation number $N$. Hereafter we will, for convenience, define an energy eigenstate of $H_0$ with an excitation number $N$ by:

$$|p; k_1, k_2, \cdots k_{N-p}\rangle = \frac{1}{\sqrt{(N-p)!}} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_{N-p}}^\dagger \sigma_+^p |0; \phi\rangle \equiv |p; K_{N-p}\rangle,$$

where $p = 0(1)$ if the atom is in its ground (excited) state, and $|\phi\rangle$ is the vacuum-field state. The factor $1/\sqrt{(N-p)!}$ is introduced here to take care of the multiple-count of the bosonic states in integrations and $|p; k_1, k_2, \cdots k_{N-p}\rangle$ is not necessarily normalized to unity.
In terms of this notation the four propagators with excitation number \( N \) are

\[
G_{pq}^{(N)}(\omega; \mathbf{K}_{N-p}, \mathbf{K}'_{N-q}) \equiv \langle p; \mathbf{K}_{N-p} | \frac{1}{\omega - H} | q; \mathbf{K}'_{N-q} \rangle ,
\]

(4.3)

where \( p, q = 0, 1 \). As an example, the propagator of zero excitation number, governing the propagation of a ground-state atom in vacuum, is trivially given by

\[
G_{00}^{(0)}(\omega; \phi, \phi) = \frac{1}{\omega} ,
\]

(4.4)

for there is only one diagram, namely, the free propagation of the collective vacuum state.

A. Quasi-mode propagators

Before proceeding to explicit evaluation of general propagators, we introduce here the concepts of quasi-mode photon states and quasi-mode propagators. A normalized quasi-mode single-photon state is defined by

\[
|1_c\rangle = a_{c}^\dagger |\phi\rangle ,
\]

(4.5)

where \( a_c^\dagger \) is the effective creation operator for the quasi-mode:

\[
a_{c}^\dagger = \frac{1}{\sqrt{\lambda}} \int_0^{+\infty} dk \ g^*(k) \ a_{k}^\dagger .
\]

(4.6)

The quantity \( \lambda \) in the normalization constant is the coupling strength defined by

\[
\lambda = \int_0^{+\infty} g(k) g^*(k) dk .
\]

(4.7)

Physically speaking, the state \( |1_c\rangle \) is, in a perturbative sense and also in the single-mode-approximation, the cavity field set up by the atom during de-excitation process. Similarly, the atom-field state with \( N \) excitations, where there are \( N - p \) quasi-mode photons and \( p \) atomic excitation, is defined by

\[
|p; (N - p)_c\rangle \equiv \frac{1}{\sqrt{(N - p)!}} (a_{c}^\dagger)^{(N-p)} a_{p}^\dagger |0; \phi\rangle .
\]

(4.8)

We therefore accordingly define the \( N \)-excitation quasi-mode propagator by
\[ \Phi^{(N)}_{pq}(\omega) = \langle p; (N - p)c \mid \frac{1}{\omega - H} \mid q; (N - q)c \rangle . \] (4.9)

From the Hamiltonian given in Eq. (2.18), the importance of the quasi-mode propagator is readily clear. Aside from the input and output photons, those present in the intermediate states are all quasi-mode photons. Thus, these \textit{quasi-mode propagators form the backbone of our theory from which other propagators can be derived.}

From definition (4.9), it is obvious that the vacuum propagator in Eq. (4.4) is the simplest quasi-mode propagator: \( \Phi^{(0)}_{00}(\omega) = G^{(0)}_{00}(\omega; \phi, \phi) = \omega^{-1} \). We begin with the quasi-mode propagator of single excitation

\[ \Phi^{(1)}_{11}(\omega) = \langle 1; \phi \mid \frac{1}{\omega - H} \mid 1; \phi \rangle . \] (4.10)

All relevant diagrams are shown in Fig. 4 and the physical picture indicated by the diagrams is clear. The excited-state atom may freely propagate in vacuum, possibly followed by equal numbers of emissions and absorptions of quasi-mode photons, and exits in its excited state. In Fig. 4, bold-wavy lines are used to represent intermediate quasi-mode photon state, to distinguish it from the input and output normal-mode states.

According to the Feynman rules, \( \Phi^{(1)}_{11}(\omega) \) is given by an infinite series:

\[ \Phi^{(1)}_{11}(\omega) = \frac{1}{\omega - \omega_A} \left( 1 + \frac{\zeta}{\omega - \omega_A} + \frac{\zeta^2}{(\omega - \omega_A)^2} + \frac{\zeta^3}{(\omega - \omega_A)^3} + \cdots \right) , \] (4.11)

where

\[ \zeta(\omega) = \int_0^{+\infty} \frac{g(k)g^*(k)}{\omega - k} dk . \] (4.12)

Adopting the single-mode approximation and, as usual, extending the lower limit of all the \( k \)-integrations from 0 to \(-\infty\), we find

\[ \zeta(\omega) = \frac{\lambda}{\omega - k_c + i\kappa_c} . \] (4.13)

Therefore, Eq. (4.11) can be expressed in the closed form

\[ \Phi^{(1)}_{11}(\omega) = \frac{A_{+}^{(1)}}{\omega - \Omega_{+}^{(1)}} + \frac{A_{-}^{(1)}}{\omega - \Omega_{-}^{(1)}} , \] (4.14)
where
\[
A_{\pm}^{(N)} = \frac{1}{2} \left[ 1 \pm \frac{(\omega_A - k_c + i \kappa_c)/2}{\sqrt{(\omega_A - k_c + i \kappa_c)^2/4 + N\lambda}} \right], \tag{4.15}
\]
\[
\Omega_{\pm}^{(N)} = \frac{\omega_A}{2} + \left( N - \frac{1}{2} \right) (k_c - i \kappa_c) \pm \sqrt{\left( \frac{\omega_A - k_c + i \kappa_c}{2} \right)^2 + N\lambda}, \tag{4.16}
\]
for \( N = 1, 2, 3, \ldots \). It is then obvious that \( \sqrt{N\lambda} \) essentially plays the role of the \( N \)-photon Rabi frequency.

We have derived in the Appendix the quasi-mode propagator for an arbitrary excitation number \( N \). They are given by:
\[
\Phi_{11}^{(N)}(\omega) = \frac{A_{+}^{(N)}}{\omega - \Omega_{+}^{(N)}} + \frac{A_{-}^{(N)}}{\omega - \Omega_{-}^{(N)}}, \tag{4.17}
\]
\[
\Phi_{00}^{(N)}(\omega) = \frac{1 - A_{+}^{(N)}}{\omega - \Omega_{+}^{(N)}} + \frac{1 - A_{-}^{(N)}}{\omega - \Omega_{-}^{(N)}}, \tag{4.18}
\]
\[
\Phi_{01}^{(N)}(\omega) = \frac{\sqrt{N\lambda}}{\omega - \omega_A - (N - 1)(k_c - i \kappa_c)} \Phi_{00}^{(N)}(\omega), \tag{4.19}
\]
\[
\Phi_{10}^{(N)}(\omega) = \Phi_{01}^{(N)}(\omega). \tag{4.20}
\]

**B. Propagators of single excitation \( (N = 1) \)**

With the help of the quasi-mode propagators, we can derive the simplest propagators for the case \( N = 1 \) to manifest the techniques in calculations aided by the diagrammatic scheme. We begin with the propagator \( G_{11}^{(1)}(\omega; \phi, \phi) = \langle 1; \phi | (\omega - H)^{-1} | 1; \phi \rangle \), which coincides with the first-order quasi-mode propagator given in Eq. (4.14), i.e.,
\[
G_{11}^{(1)}(\omega; \phi, \phi) = \Phi_{11}^{(1)}(\omega). \tag{4.21}
\]

One can, of course, follow the same route in the deviation of \( G_{11}^{(1)}(\omega; \phi, \phi) \) to obtain the other three propagators of single excitation. However, it is clear that once a particular \( N \)-excitation propagator is derived, other members in the same class can be easily obtained in an alternative way by relating them to the one already obtained, as shown in Fig. 3. For
example, from Fig. 5 (a), the propagator \( G^{(1)}_{01}(\omega; k, \phi) \) can be related to \( G^{(1)}_{11}(\omega; \phi, \phi) \) by:

\[
G^{(1)}_{01}(\omega; k, \phi) = \frac{g^*(k)}{\omega - k} G^{(1)}_{11}(\omega; \phi, \phi). \tag{4.22}
\]

Likewise, we can show from Fig. 5 (b) that the propagator

\[
G^{(1)}_{10}(\omega; \phi, k) = \langle 1; \phi| \frac{1}{\omega - H} |0; k\rangle = \frac{g(k)}{\omega - k} G^{(1)}_{11}(\omega; \phi, \phi). \tag{4.23}
\]

Eqs. (4.22) and (4.23) also reveal a useful relation of the propagators, namely, the propagator \( G^{(N)}_{pq} (\omega; K_N - p, K'_N - q) \) can be obtained from \( G^{(N)}_{qp} (\omega; K'_N - q, K_N - p) \) by simply replacing each \( g(k) \) with \( g^*(k) \), and vice versa, which is obvious from the diagrammatic scheme and the Feynman rules.

Finally, for the scattering of a photon from the \( k' \)-th mode to the \( k \)-th mode by the ground-state atom, the corresponding propagator is \( G^{(1)}_{00}(\omega; k, k') = \langle 0; k| (\omega - H)^{-1} |0; k'\rangle \).

Similarly, from Fig. 5 (c), we have

\[
G^{(1)}_{00}(\omega; k, k') = \frac{\delta(k - k')}{\omega - k'} + \frac{g(k)g^*(k')}{(\omega - k)(\omega - k')} G^{(1)}_{11}(\omega; \phi, \phi). \tag{4.24}
\]

C. Construction of propagators of general excitation number \( N \)

In general, an \( N \)-excitation propagator can be calculated from those of lower excitation numbers. We will develop here a systematic approach to evaluate the propagators with \( N \geq 2 \).

First of all, in some diagrams there may exist “spectator photons” that do not interact with the atom in the whole evolution. These diagrams are said to be factorizable (or unlinked), and can be straightforwardly related to propagators of lower excitation numbers in a way to be stated explicitly in the following discussion. We therefore focus mainly on the linked diagrams that all the input and output photons take part in the interactions. These absorptions and emissions can take place in any order, resulting in different physical processes.
For each linked diagram we label it with a set of momenta, \( S \equiv \{ \tilde{k}_1, \tilde{k}_2, \cdots, \tilde{k}_{2N-p-q} \} \), comprising the time-ordered momenta of the photons created or annihilated in the process. Here \( p \) and \( q \) are integers defined in Eq. (4.3). The elements of \( S \) are taken from \( N-p \) output photons in the set \( K_{N-p} \) and \( N-q \) input photons in the set \( K'_{N-q} \), with no repetitions. In other words, a particular \( S \) represents a particular sequence of the \( N-q \) absorptions of the input photons and \( N-p \) emissions of the output photons. For a given \( S \), we introduce a quantity \( L^{(N)}_{pq}(\omega; S) \) to denote the corresponding contributions to the propagator. The symbol “\( L \)” is used here to refer to “linked” diagrams. We note here that this convention should be accompanied by a final symmetrization of the propagator with respect to \( K_{N-p} \) and \( K'_{N-q} \).

There are at most only three kinds of photons present in any segment of the Feynman diagrams, namely, the input and output photons in continuous modes, and the quasi-mode photons. Between the \( 2N-p-q \) vertices associated with the sequence defined by \( S \), the atom only interacts with the quasi-mode photons. Therefore the evolution is governed by the quasi-mode propagators derived. This explains the importance of the quasi-mode propagators discussed in Sec. (IV A).

It is found that \( L^{(N)}_{pq}(\omega; S) \) can be written in the following compact form

\[
L^{(N)}_{pq}(\omega; S) = \left( \prod_{i=1}^{N-q} g(\tilde{k}'_i) \right) \left( \prod_{i=1}^{N-p} g^*(\tilde{k}_i) \right) \left( \prod_{i=0}^{2N-p-q} \Phi^{(N_i)}(\omega - E_i) \right). \tag{4.25}
\]

The physical picture of the above equation is clear. \( N-q \) times of absorptions of input photons and \( N-q \) times of emissions of output photons must take place at some time, with amplitude given by the first part of the equation. These events split the whole process into \( 2N-p-q+1 \) segments. In a particular segment, the atom interacts only with the quasi-mode photons. This means that the propagator of this segment is governed by the quasi-mode propagators derived in the last section. The presence of continuous mode spectator photons in the \( i \)-th segment has the sole effect of shifting the frequency \( \omega \) by \( E_i \), which is the total
energy of the continuous-mode photons present in the \(i\)-th segment.\(^1\)

While not explicitly shown, the \(L_{pq}^{(N)}(\omega; S)\) in Eq. (4.23) depends on \(S\) through the terms \(N_i, p_i, q_i\) and \(E_i\).

Summing all diagrams corresponding to all possible \(S\), we have

\[
\Lambda_{pq}^{(N)}(\omega; K_{N-p}, K'_{N-q}) \equiv \sum_S L_{pq}^{(N)}(\omega; S),
\]

which is the propagator that includes all linked diagrams only.

To include the unlinked diagrams, we go back to the case that some of the input and output photons act as spectators in the whole process. There may be one spectator, which, without loss of generality, can be assumed to be \(k_{N-p}(k'_{N-q})\). The propagation of the remaining system is governed by \(\Lambda^{(N-1)}_{pq}\). Similarly we can have two photons acting as spectators, which are assumed to be \(k_{N-p}(k'_{N-q})\) and \(k_{N-p-1}(k'_{N-q-1})\), and so on. The maximum possible number of spectator photons is \(M = \min\{N-p, N-q\}\). Hence we have

\[
G_{pq}^{(N)}(\omega; K_{N-p}, K'_{N-q}) = \sum_{\text{sym}} \sum_{j=0}^{M} \left[ \prod_{l=0}^{j-1} \delta(k_{N-p-l} - k'_{N-q-l}) \right] \\
\times \Lambda_{pq}^{(N-j)}(\omega - \sum_{l=0}^{j-1} k_{N-p-l}; K_{N-j-p}, K'_{N-j-q}) \right].
\]

In the above equation, \(\sum_{\text{sym}}\) denotes the symmetrization of the expression with respect to the input and output photons, and \(\Lambda\) is given by Eq. (4.26).

\(^1\)For example, in Fig. 3, consider the first stage of the second term in the summation, the excited atom interacts with quasi-mode photon while the input normal-mode photon \(k'_1\) acts as spectator photon. The evolution of the atom plus quasi-mode would be governed by the the single excitation quasi-mode propagator \(\Phi_{11}^{(1)}(\omega)\) in the absence of \(k'_1\). However when \(k'_1\) is present, the propagator of the whole system in this segment becomes \(\Phi_{11}^{(1)}(\omega - k'_1)\).
D. Example: Propagators with excitation number $N = 2$

The general expression of the propagators is simple if the excitation number $N$ is not large. In this section we will derive the four propagators of excitation number $N = 2$.

Consider first the propagator $G^{(2)}_{11}(\omega; k_1, k'_1) \equiv \langle 1; k_1 | (\omega - H)^{-1} | 1; k'_1 \rangle$. According to Eq. (4.27), we have

$$G^{(2)}_{11}(\omega; k_1, k'_1) = \Lambda^{(2)}_{11}(\omega; k_1, k'_1) + \delta(k_1 - k'_1) \Lambda^{(1)}_{11}(\omega - k_1; \phi, \phi),$$

(4.28)

where

$$\Lambda^{(2)}_{11}(\omega; k_1, k'_1) = \sum_S \mathcal{L}^{(2)}_{11}(\omega; S),$$

(4.29)

with $S = \{k_1, k'_1\}$ or $\{k'_1, k_1\}$. One can readily show that (see Fig. 6)

$$\mathcal{L}^{(2)}_{11}(\omega; S = \{k_1, k'_1\}) = g(k'_1)g^*(k_1)\Phi^{(1)}_{10}(\omega - k_1)\Phi^{(2)}_{01}(\omega - k'_1)$$

(4.30)

and

$$\mathcal{L}^{(2)}_{11}(\omega; S = \{k'_1, k_1\}) = g(k'_1)g^*(k_1)\Phi^{(1)}_{11}(\omega - k_1)\Phi^{(0)}_{00}(\omega - k_1 - k'_1)\Phi^{(1)}_{11}(\omega - k'_1).$$

(4.31)

The total propagator is hence

$$G^{(2)}_{11}(\omega; k_1, k'_1) = \Phi^{(1)}_{11}(\omega - k_1) \left\{ \delta(k_1 - k'_1) + g(k'_1)g^*(k_1)\Phi^{(1)}_{11}(\omega - k'_1) \right\}.$$

(4.32)

The other three propagators of the same excitation number can be evaluated similarly. However, as mentioned in Sec. [IVB], these propagators can be obtained immediately from their relations with $G^{(2)}_{11}$. For example, the propagator $G^{(2)}_{01}(\omega; k_1k_2, k'_1) = \langle 0; k_1, k_2 | (\omega - H)^{-1} | 1; k'_1 \rangle$ is given by (see Fig. 7 (a))

$$G^{(2)}_{01}(\omega; k_1k_2, k'_1) = \frac{1}{\omega - k_1 - k_2} [g^*(k_1)G^{(2)}_{11}(\omega; k_2, k'_1) + g^*(k_2)G^{(2)}_{11}(\omega; k_1, k'_1)].$$

(4.33)
Likewise, we can obtain the remaining two propagators with $N = 2$ from Figs. 7(b) and (c), and the results are stated below:

\[
G^{(2)}_{10}(\omega; k_1, k'_1 k'_2) = \langle 1; k_1 | \frac{1}{\omega - H} | 0; k'_1, k'_2 \rangle = \frac{1}{\omega - k'_1 - k'_2} \left[ g(k'_1) G^{(2)}_{11}(\omega; k_1, k'_2) + g(k'_2) G^{(2)}_{11}(\omega; k_1, k'_1) \right].
\]

(4.34)

\[
G^{(2)}_{00}(\omega; k_1 k_2, k'_1 k'_2) = \langle 0; k_1, k_2 | \frac{1}{\omega - H} | 0; k'_1, k'_2 \rangle = \frac{1}{\omega - k'_1 - k'_2} \left[ \delta(k'_1 - k_1) \delta(k'_2 - k_2) + \delta(k'_1 - k_2) \delta(k'_2 - k_1) \right]
\]

\[+ \frac{1}{\omega - k'_1 - k'_2} \left[ g(k'_1) G^{(2)}_{01}(\omega; k_1 k_2, k'_2) + g(k'_2) G^{(2)}_{01}(\omega; k_1 k_2, k'_1) \right].
\]

(4.35)

\[\]

V. APPLICATION: SINGLE-ATOM SINGLE-PHOTON SCATTERING

As an application of our method, we study the scattering of a single-photon wave packet by an excited two-level atom inside the cavity. In particular, we investigate how the spectral width of the incident photon affects the outcome of stimulated emission process. To begin, we consider the incident photon initially prepared in the “+” modes as defined in Eq. (2.16). Hence the initial state is given by

\[
|\psi(t = 0)\rangle = \int dk' C(k') a_{k'}^\dag |1; \phi\rangle,
\]

(5.1)

where $C(k')$ is the photon amplitude. At a later time $t$, the state becomes

\[
|\psi(t)\rangle = \int dk B(k, t)a_k^\dag |1; \phi\rangle + \frac{1}{\sqrt{2}} \iint dk_1 dk_2 C(k_1, k_2, t)a_{k_1}^\dag a_{k_2}^\dag |0; \phi\rangle,
\]

(5.2)

where the two-photon amplitude $C(k_1, k_2) = C(k_2, k_1)$ satisfies the normalization condition,

\[
\iint |C(k_1, k_2)|^2 dk_1 dk_2 = 1.
\]

(5.3)

For simplicity, we will only consider the resonance case: $k_c = \omega_A$.

The long time state $|\psi(t \to \infty)\rangle$ is determined by the asymptotic behavior of $C(k_1, k_2, t)$. Utilizing the propagator obtained in Eq. (4.33), we have

\[\]
\[
\lim_{t \to \infty} C(k_1, k_2, t) = \lim_{t \to \infty} \frac{i}{2\pi} \int d\omega e^{-i\omega t} \int dk' G_{10}^{(2)}(\omega; k_1 k_2, k') C(k').
\] (5.4)

The explicit form is given by

\[
C(k_1, k_2, t \to \infty) \to \frac{1}{\sqrt{2}} e^{-i(k_1+k_2)t} \left\{ g^*(k_1)\Phi_{11}^{(1)}(k_1)C(k_2) + g^*(k_1)g^*(k_2) \right.
\]
\[
\times \left[ \Phi_{11}^{(1)}(k_1)I_1(k_1, k_2) + \frac{\lambda}{k_1 - k_c + i\kappa_{in}}\Phi_{11}^{(1)}(k_1)\Phi_{11}^{(2)}(k_1 + k_2)I_2(k_1, k_2) \right] \right\}
\]
\[
+ \{k_1 \leftrightarrow k_2\},
\] (5.5)

where

\[
I_1(k_1, k_2) = \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} \frac{g(k')\Phi_{11}^{(1)}(k_1 + k_2 - k')}{k_1 - k' + i\delta} C(k')dk',
\] (5.6)

\[
I_2(k_1, k_2) = \int_{-\infty}^{\infty} \frac{g(k')}{k_1 + k_2 - k' - k_c + i\kappa_{in}}\Phi_{11}^{(1)}(k_1 + k_2 - k')C(k')dk',
\] (5.7)

and \(\{k_1 \leftrightarrow k_2\}\) denotes the previous expression with \(k_1\) and \(k_2\) interchanged. It should be noted that the contour of the \(I_1\)-integration should be closed in the lower half-plane.

The main advantage of our formalism is that it can handle any state of the incident photon. As an example, we consider that \(C(k')\) is a lorentzian with a peak frequency equal to the cavity resonance frequency, i.e.,

\[
C(k') = \sqrt{\frac{\kappa_{in}}{\pi}} \frac{1}{k' - k_c + i\kappa_{in}}.
\] (5.8)

Here the width \(\kappa_{in}\) is the spectral width of the incident photon. The pole is located in the lower-half-plane in order to ensure that the atom can only “feel” the photon for \(t \geq 0\).

In Fig. 8 we show the contour-plot of \(|C(k_1, k_2)|^2\) for \(\lambda = 0.1\kappa_c^2\), with various spectral widths of incident photons. When \(\kappa_{in} \gg \gamma_{sp}\) (Fig. 8 (a)), the input photon has a very short pulse duration compared with the decay time of the atom. Therefore the incident photon is incapable of having sufficient interactions with the atom. As a result, the output state is approximately a direct product state of the input photon and the spontaneously-decayed photon of the atom, which corresponds to a “cross” shape in \(|C(k_1, k_2)|^2\). A similar effect can be seen if the incident photon has a narrow width such that \(\kappa_{in} \ll \gamma_{sp}\) (Fig. 8 (d)). In this case, the input photon has a very long duration and so it participates in the interaction.
mainly after the atom has reached the ground state, causing no interference with the photon emitted from spontaneous decay.

However, interesting features show up when $\kappa\text{in}$ is the same order as $\gamma_{\text{sp}}$ (Figs. 8 (b) and (c)). We see that the final two-photon amplitude is drastically different. For example, in Figs. 8 (b), there is an unexpected dip at the center and peaks at $\Delta k \approx \gamma_{\text{sp}}$. In other words, although the frequencies of the input photon and the photon emitted in spontaneous decay both peak at $k_c$, it is very unlikely to have two photons with frequencies around $k_c$ in the output. Instead, the peak frequency has been shifted to $\Delta k_1, \Delta k_2 \approx \lambda/\kappa_c$.

To understand the interference effects shown in Figs. 8 (b) and (c), we identify the contributions of relevant diagrams associated with the propagator. Fig. 9 (a) shows the contributions solely from the unlinked diagram in which the incident photon does not participate in the interactions at all. Obviously, the corresponding two-photon amplitude disagrees with the exact one. However, a much better agreement can be achieved if we include just the lowest order linked diagram given in Fig. 9 (b). The interference between the linked diagram and unlinked diagram produces a two-photon amplitude that is almost the same as the exact one (Fig. 9(c)).

VI. CONCLUSION

In this paper, we have developed a diagrammatic scheme to construct the propagator governing the interaction between an atom and photons in a leaky cavity. Under the assumption that the frequency dependence of atom-field coupling is a lorentzian, we found that the perturbation series is summable and so exact solution can be obtained for any excitation number $N$. The propagator provides an analytical tool to investigate the cavity QED effect for the photon scattering problem. In particular, the quantum state of output photons in continuous modes can be determined explicitly.

Our results are illustrated by the derivation of propagators with excitation number $N = 2$, which are then applied to study the scattering of a single-photon wave packet from an
excited atom. We found that the spectral width of the incident photon can significantly modify the final two-photon amplitudes. This occurs when $\kappa_{\text{in}}$ matches the cavity modified atom decay rate. Our calculations show that two output photons are entangled in the sense that their spectrum displays nontrivial correlation. With the aid of Feynman diagrams, we have identified the essential processes causing the interference. However, more detailed investigations are needed for a thorough understanding of the rich features in cavity QED assisted photon scattering problems.

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APPENDIX A: DETAILS OF THE DERIVATION OF QUASI-MODE PROPAGATORS

The main idea of the evaluation of the propagators is to move all the annihilation and creation operators to the right and left side of the perturbation series, employing the commutation relations, followed by noticing the fact that $a_k|\phi\rangle = \langle \phi | a_k^\dagger = 0$ and $\sigma_-^2 = \sigma_+^2 = 0$.

The commutation relations between $a_c$, $a_c^\dagger$ with $H_0$ and $V$ are:

\begin{align}
[a_c^\dagger, V] &= -\sqrt{\lambda} \sigma_+,
\end{align}

\begin{align}
[a_c, V] &= \sqrt{\lambda} \sigma_-,
\end{align}

\begin{align}
\frac{1}{\omega - H_0} a_c^\dagger &= \int_{-\infty}^{\infty} dkg(k) a_k^\dagger \frac{1}{\omega - k - H_0},
\end{align}

\begin{align}
\frac{1}{\omega - H_0} a_c &= \int_{-\infty}^{\infty} dkg(k) \frac{1}{\omega - k - H_0} a_k.
\end{align}
Consider the quasi-mode propagator

$$\Phi_{00}^{(N)}(\omega) = \langle 0; N_c | \frac{1}{\omega - H} | 0; N_c \rangle. \quad (A5)$$

The atom is initially at its ground state with $N$ photons in quasi-mode. The propagator measures the probability that the system remains in the same state finally.

The interaction-free part can be evaluated by considering Eq. (A3), which gives

$$\langle 0; N \phi | \frac{1}{\omega - H} V (\omega - H) | 0; \phi \rangle = \frac{1}{N! \lambda N} \prod_{i=1}^{N} \int g(k_i) dk_i a_{k_i}^\dagger \int g^*(k_i) dk_i a_{k_i} | 0; \phi \rangle \frac{1}{\omega - \sum_{i=1}^{N} k_i}$$

$$= \frac{1}{\omega - N(k_c - i\kappa_c)}. \quad (A6)$$

Paths with an odd number of interactions have zero contributions. And from the commutation relations, it is readily shown that

$$V \frac{1}{\omega - H_0} | 0; N_c \rangle = \frac{1}{\lambda^{N/2} \sqrt{N!}} \prod_{j \neq i} \int dk_j g^*(k_j) a_{k_j}^\dagger \sigma_+ | 0; \phi \rangle \sum_{i} \frac{\lambda}{\omega - k_j + i\kappa_c - \sum_{j \neq i} k_j}$$

and similarly

$$\left( V \frac{1}{\omega - H_0} \right)^2 | 0; N_c \rangle = \frac{1}{\lambda^{N/2} \sqrt{N!}} \sum_{i} \left( \prod_{j \neq i} \int dk_j g^*(k_j) a_{k_j}^\dagger \right) \int dk g^*(k) a_k^\dagger | 0; \phi \rangle$$

$$\times \frac{\lambda}{(\omega - \sum_{j \neq i} k_j - k_c + i\kappa_c)(\omega - \sum_{j \neq i} k_j - k_a)}$$

$$= \frac{N \lambda}{\lambda^{N/2} \sqrt{N!}} \left( \prod_{j=1}^{N} \int dk_j g^*(k_j) a_{k_j}^\dagger | 0; \phi \rangle \right)$$

$$\times \frac{1}{(\omega - k_c + i\kappa_c - \sum_{j=1}^{N-1} k_j)(\omega - k_a - \sum_{i=1}^{N-1} k_j)}.$$ \quad (A8)

Hence

$$\langle 0; N_c | \frac{1}{\omega - H} V \frac{1}{\omega - H} V \frac{1}{\omega - H} | 0; N_c \rangle$$

$$= \frac{N \lambda}{[\omega - N(k_c - i\kappa_c)]^2 [\omega - k_a - (N-1)(k_c - i\kappa_c)]}. \quad (A9)$$
It can be shown that the contributions of different paths with even times of interactions,

\[
\langle 0; N_c | \frac{1}{\omega - H_0} \left( \frac{V}{\omega - H_0} \right)^i | 0; N_c \rangle = \frac{1}{[\omega - N(k_c - i\kappa_c)]^{i/2 + 1}} \left[ \frac{N\lambda}{\omega - \omega_A - (N - 1)(k_c - i\kappa_c)} \right]^{i/2}.
\]  
(A10)

Hence

\[
\Phi^{(N)}_{00}(\omega) = \frac{1}{\omega - N(k_c - i\kappa_c)} \left( 1 + \zeta + \zeta^2 + \cdots \right)
\]

\[
= \frac{1}{[\omega - N(k_c - i\kappa_c)] (1 - \zeta)},
\]  
(A11)

where

\[
\zeta = \frac{N\lambda}{[\omega - N(k_c - i\kappa_c)] [\omega - \omega_A - (N - 1)(k_c - i\kappa_c)]}.
\]  
(A12)

Similar to Eq. (4.14), the propagator can be written as a sum of two lorentzians,

\[
\Phi^{(N)}_{00}(\omega) = \frac{\omega - \omega_A - (N - 1)(k_c - i\kappa_c)}{[\omega - N(k_c - i\kappa_c)] [\omega - \omega_A - (N - 1)(k_c - i\kappa_c)] - N\lambda}
\]

\[
= \frac{1 - A^{(N)}_+}{\omega - \Omega^{(N)}_+} + \frac{1 - A^{(N)}_-}{\omega - \Omega^{(N)}_-},
\]  
(A13)

where

\[
A^{(N)}_\pm = \frac{1}{2} \left[ 1 \pm \frac{(\omega_A - k_c + i\kappa_c)/2}{\sqrt{((\omega_A - k_c + i\kappa_c)^2)/4 + N\lambda}} \right],
\]  
(A14)

\[
\Omega^{(N)}_\pm = \frac{\omega_A}{2} + \left( N - \frac{1}{2} \right) (k_c - i\kappa_c) \pm \sqrt{\left( \frac{(\omega_A - k_c + i\kappa_c)^2}{2} \right)^2 + N\lambda}.
\]  
(A15)

By substituting \( N = 0 \), we have \( A^{(0)}_+ = 0, A^{(0)}_- = 1, \Omega^{(0)}_+ = \omega_A - k_c + i\kappa_c, \Omega^{(0)}_- = 0 \), and the propagator reduces to

\[
\Phi^{(0)}_{00}(\omega) = \frac{1}{\omega}.
\]  
(A16)
b. $\Phi^{(N)}_{01}(\omega)$

The propagator

$$\Phi^{(N)}_{01}(\omega) = \langle 0; N_c | \frac{1}{\omega - H} | 1; (N - 1)_c \rangle$$  \hspace{1cm} (A17)

can be evaluated by the usual commutation relations, and noticing that

$$\frac{V}{\omega - H_0} \frac{1}{\sqrt{(N - 1)!}} (a^{\dagger}_{c})^{N-1} \sigma_+ | 0; \phi \rangle$$

$$= \frac{1}{\sqrt{N!} \lambda^{N/2}} \left( \prod_i \int d k_i g^*(k_i) a^\dagger_{k_i} | 0; \phi \rangle \right) \frac{\sqrt{N \lambda}}{\omega - \omega_A - \sum_{j=1}^{N-1} k_j}.$$  \hspace{1cm} (A18)

It can be proved that

$$\Phi^{(N)}_{01}(\omega) = \frac{\sqrt{N \lambda}}{\omega - \omega_A - (N - 1)(k_c - i \kappa_c)} \Phi^{(N)}_{00}(\omega)$$

$$= \frac{\sqrt{N \lambda}}{\Omega^{(N0)}_{+1} - \Omega^{(N0)}_{-1}} \left[ \frac{1}{\omega - \Omega^{(N0)}_{+1}} - \frac{1}{\omega - \Omega^{(N0)}_{-1}} \right].$$  \hspace{1cm} (A19)

c. $\Phi^{(N)}_{10}(\omega)$

The calculation of

$$\Phi^{(N)}_{10}(\omega) = \langle 1; (N - 1)_c | \frac{1}{\omega - H} | 0; N_c \rangle$$  \hspace{1cm} (A20)

is similar to that of $\Phi^{(N)}_{01}(\omega)$. In fact, the symmetries in the commutation relations result in the relation

$$\Phi^{(N)}_{10}(\omega) = \Phi^{(N)}_{01}(\omega).$$  \hspace{1cm} (A21)

The above equation can also be obtained immediately by replacing $g(k)$ with $g^*(k)$, which does not change the expression in this case since the quasi-mode propagators do not contain any free $g(k)$ or $g^*(k)$ terms.
The propagator

\[ \Phi_{11}^{(N)}(\omega) = \langle 1; (N - 1)c | \frac{1}{\omega - H} | 1; (N - 1)c \rangle \]  

(A22)

can be evaluated by similar derivations, yielding

\[ \langle 1; (N - 1)c | \frac{1}{\omega - H_0} | 1; (N - 1)c \rangle = \frac{1}{\omega - \omega_A - (N - 1)(k_c - i\kappa_c)}, \]  

(A23)

and for an even integer \( i \),

\[ \langle 1; (N - 1)c | \frac{1}{\omega - H_0} \left( \frac{V}{\omega - H_0} \right)^i | 1; (N - 1)c \rangle = \frac{N\lambda}{[\omega - \omega_A - (N - 1)(k_c - i\kappa_c)]^2} \langle 0; Nc | \frac{1}{\omega - H_0} \left( \frac{V}{\omega - H_0} \right)^{i-2} | 0; Nc \rangle. \]  

(A24)

Similarly all paths with odd number times of interactions have null contributions to the propagator. Hence, we have

\[ \Phi_{11}^{(N)}(\omega) = \frac{1}{\omega - \omega_A - (N - 1)(k_c - i\kappa_c)} \times \left\{ \frac{N\lambda}{[\omega - N(k_c - i\kappa_c)] [\omega - \omega_A - (N - 1)(k_c - i\kappa_c)] - N\lambda} \right\} \]

\[ = \frac{\omega - N(k_c - i\kappa_c)}{(\omega - \Omega^{(N)}_+)(\omega - \Omega^{(N)}_-)} \]

\[ = \frac{A^{(N)}_+}{\omega - \Omega^{(N)}_+} + \frac{A^{(N)}_-}{\omega - \Omega^{(N)}_-}. \]  

(A25)

A very simple relation exists between the \( e(\text{excited-state}) \rightarrow e \) and \( g(\text{ground-state}) \rightarrow g \) propagators,

\[ \Phi_{11}^{(N)}(\omega) = \frac{\omega - N(k_c - i\kappa_c)}{\omega - \omega_A - (N - 1)(k_c - i\kappa_c)} \Phi_{00}^{(N)}(\omega). \]  

(A26)

It can be verified that \( \Phi_{11}^{(N)}(\omega) \) reduces to that of Eq. (4.14) if we take \( N = 1 \).
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FIGURES

FIG. 1. A sketch of the system: A two-sided Fabry-Perot cavity with a two-level atom inside and partially reflecting mirrors at both ends. We only consider identical mirrors and the atom being locating at the center.

FIG. 2. The left-luminating modes, with lumination from the left, and transmitted waves only at the right. This set of modes are labeled by the subscript L and the positive wave-number \( k \). Together with the right-luminating modes, these continuous field modes form a complete and orthogonal set of the system.

FIG. 3. Basic components of the Feynman diagrams: (a) The total propagator from initial state \( \psi_i \) to final state \( \psi_f \). (b) Free propagation of a ground-state atom. (c) Free propagation of an excited-state atom. (d) Free propagation of a \( k \)-th mode photon. (e) An excited-state atom decays into ground state and emits a \( k \)-th mode photon. (f) A ground-state atom excited by a \( k \)-th mode photon and jumps to the excited state.

FIG. 4. The propagator \( \Phi_{11}^{(1)}(\omega) = G_{11}^{(1)}(\omega; \phi, \phi) \) can be expressed as the sum of an infinite series, corresponding to all the possible paths. Bold-wavy lines are used to represent intermediate quasi-mode photon state. This series is exactly summable.

FIG. 5. Relationship between the propagators of single-excitation. The \( G \)'s appear in different components are different because they are attached with different input and output legs.

FIG. 6. Feynman diagrams for \( G_{11}^{(2)}(\omega; k_1, k'_1) \). The first diagram is the unlinked diagram with a spectator photon. The second diagram corresponds to \( S = \{k_1, k'_1\} \), which means the atom emits the output photon prior to the absorption of the input. All other diagrams belong to the third group, corresponding to \( S = \{k'_1, k_1\} \).

FIG. 7. Relationship between the propagators with two excitations \( N = 2 \).
FIG. 8. Contour-plot of $|C(k_1, k_2)|^2$ for $\lambda = 0.1\kappa_c^2$, with four different widths of the input photon: (a) $\kappa_{in} = 10\gamma_{sp}$, (b) $\kappa_{in} = \gamma_{sp}$, (c) $\kappa_{in} = 0.5\gamma_{sp}$ and (d) $\kappa_{in} = 0.1\gamma_{sp}$. The axis labels are in unit of $\kappa_c$.

FIG. 9. The contributions of different Feynman diagrams for the case $\kappa_{in} = \gamma_{sp}$ in Fig. 8 (b): (a) The unlinked diagram. (b) The lowest order linked diagram plus the unlinked diagram. (c) All diagrams. The axis labels are in unit of $\kappa_c$. 
Incident photons
packet

Atoms + cavity photons

Output photons

Fig. 1   Chen
Fig. 2  Chen
Fig. 3  Chen
Fig. 4  Chen
Fig. 5   Chen
Fig. 6   Chen
Fig. 7 Chen
Fig. 8  Chen
Summation of all Feynman diagrams