On domain formation in non-instantaneous symmetry breaking phase transitions

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Abstract

In the framework of the Ginzburg-Landau harmonic potential approximation, we present a possible modeling of the time-dependence of the frequency of the order parameter mode suitable to account for the formation of correlated domains in non-instantaneous symmetry breaking phase transitions. An interesting spectrum of possibilities for the size and the life-time of these domains emerges, which appears to be consistent with the conclusions of other, differently grounded analysis.

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I. INTRODUCTION

In some recent papers\textsuperscript{1–3} much attention has been devoted to phase transitions in the so-called range of criticality, namely in the temperature range spanning between the critical temperature $T_C$, at which the phase transition ”starts”, and the Ginzburg temperature $T_G$, at which the broken symmetry phase becomes maximally stable. Long string formation in such a region has been hypothesized and its consequences have been investigated\textsuperscript{1}. Beyond the detailed phenomenological picture and the numerical simulations, the dynamic formation of non-homogeneous structures in the critical region seems to require further study. Our aim in this paper is to describe indeed a possible mechanism for the formation of domains in the process of non-instantaneous phase transitions characterized by time dependent order parameter. By resorting to known results in the quantum field theory (QFT) with spontaneous breakdown of symmetry (SBS), we also discuss, on the basis of a microscopic analysis, the assumption of temperature and time dependence of the phenomenological Ginzburg-Landau (GL) chemical potential.

To this aim let us first shortly summarize some known features of the spontaneously broken symmetry theory with non-constant order parameter.

We are interested in the case where a specific phase of the system at a certain time is characterized by a specific value of the order parameter. Changes in the order parameter, considered as a function of the temperature or of the time, or of both of them, are thus associated to a change of the system phase, i.e. to the process of phase transition. Here we closely follow ref.\textsuperscript{4}. The microscopic analysis may start considering, for simplicity, a quantum real scalar field $\phi(x)$ at $T \neq 0$ with Lagrangian and Hamiltonian given by

\[ L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4, \quad \lambda > 0, \quad (1.1) \]

\[ H = \int d^{D-1}x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \right\}, \quad (1.2) \]

respectively, where $D$ is the space-time dimensionality and $x = (x, t)$. The equation of motion is

\[ (-\partial^2 - \mu^2)\phi = \lambda \phi^3, \quad (1.3) \]

where the conventions $g^{00} = -g^{ii} = 1$ and $c = \hbar = 1$ have been adopted. In the variational approach, the free energy functional is given by the Bogoliubov inequality\textsuperscript{5}

\[ F \leq F_1 = F_0 + < H - H_0 >_0, \quad (1.4) \]

where $F = -KT \ln \text{Tr}[\exp(-\beta H)]$, with $\beta = \frac{1}{KT}$, is the free energy and the symbol $< >_0$ denotes the statistical average: $< A >_0 = \frac{1}{Z} \text{Tr}[\exp(-\beta H_0)A]$, with $Z = [\exp(-\beta H_0)]$. $H_0$ is the trial Hamiltonian. One introduces the variational parameter $\mu_0$ which in general can depend upon $x$ and $\beta$ and acts as the mass for the quantum field $\rho$. For a discussion on the relation between $\rho$ and the field $\phi$ see ref.\textsuperscript{6} where the detail of the variational computation is presented. Here we only recall that we have $\phi = \rho + v$, with the c-number $v$ being the non-vanishing expectation value (the order parameter) for the $\phi$ field:
\[ <0|\phi(x)|0> = v . \] (1.5)

Eq. (1.5) implies \( \mu^2 < 0 \) and expresses the condition of spontaneous breakdown of symmetry. A specific physical phase of the system corresponds to a specific value of \( v \). In the present paper we are interested in the case of a non-constant order parameter: \( v = v(x, \beta) \). In such a case, Eq. (1.5) tells us that the ground states or vacua may be labeled by the values of the order parameter: \( |0 > \equiv |0 >, |v > \). Thus different vacua (i.e. different phases) correspond to different values of the order parameter: \( |0 > \neq |0 >, \) with \( v' \equiv v(\beta'), v(\beta) \neq v(\beta') \) and \( \beta \neq \beta' \). In order to study the process of phase transition one thus searches for the equation describing the variations of the order parameter \( v(x, \beta) \). Such an equation may be understood as describing the "motion" over trajectories in the "space of the vacua" (the phases) of the system (also called the space of the representations of the CCR). Let us then observe that eq. (1.4) gives (we are assuming real \( v(x, \beta) \))

\[ F_1(v(x, \beta)) = \int d^{D-1}x[\partial_0 v]^2 - \mathcal{L}_{\text{eff}}(v(x, \beta), \partial_\mu v(x, \beta)) \] (1.6)

with

\[ \mathcal{L}_{\text{eff}} = \frac{1}{2} \partial_\mu v \partial_\mu v - U_{\text{eff}}(v) \] (1.7)

\[ U_{\text{eff}}(v) = U_{\text{cl}}(v) + F_0(\mu_0^2) - \frac{3}{4} \lambda \rho^2 >_0 \] (1.8)

\[ U_{\text{cl}}(v) = \frac{1}{2} \mu^2 v^2 + \frac{1}{4} \lambda v^4 . \] (1.9)

Note that \( U_{\text{eff}}(v) \) includes not only the classical terms but also the quantum and the thermal contributions. We remark that the free energy \( F_1 \) plays the rôle of the energy for the c-number field \( v(x, \beta) \) whose Lagrangian is \( \mathcal{L}_{\text{eff}} \): in this way the dynamics for the quantum field \( \phi \) manifests as the dynamics for the classical field describing the order parameter \( v(x, \beta) \). Eq. (1.6) and \( U_{\text{eff}}(v) \) are the generalized Ginzburg-Landau functional and potential, respectively. The Euler-Lagrange equation of motion for the field \( v(x, \beta) \) is deduced from \( \mathcal{L}_{\text{eff}} \) through the action principle:

\[ (-\partial^2 + m^2)v(x, \beta) = \lambda v^3(x, \beta) , \] (1.10)

with

\[ m^2 = |\mu^2| - 3\lambda \rho^2 >_0 \] (1.11)

and therefore \( m^2 \) is temperature dependent: \( m^2 = m^2(\beta) \). The variational equation \( \frac{\partial F_1}{\partial \mu_0} = 0 \) gives the variational mass parameter

\[ \mu_0^2(x, \beta) = \mu^2 + 3\lambda v^2(x, \beta) + 3\lambda \rho^2 >_0 . \] (1.12)

Eq. (1.10) describes the evolution of the order parameter, namely the evolution of the system in the "space of the representations of the CCR", each representation being labeled...
by \( v(x, \beta) \), and therefore the evolution through the different phases of the system (phase transitions). Eq.(1.11) is a consistency relation between the "inner" dynamics described by \( \mathcal{L} \) (eq.(1.1)) and the dynamics in the space of the representations described by \( \mathcal{L}_{\text{eff}} \) (eq.(1.7)). Notice that the "mass term" in eq.(1.10) has the "wrong" sign with respect to a Klein-Gordon equation for a physical particle field: \(-m^2\) is interpreted as the GL chemical potential and negative chemical potential signals the realization of SBS. Temperature dependence in the mass term is usually introduced by hand. Here, through eq.(1.11) we see the microscopic origin of the temperature dependence of the chemical potential in the GL potential.

In the case of one space dimension one may obtain the stationary kink solution\(^6\). When a gauge field is present and \( \phi \) is a complex field one has the vortex solution in two space dimensions. In theories with SO(3) and SU(2) symmetry groups and three space dimensions one has the monopole and the sphaleron solutions\(^10\). A popular framework where the formation of defects in the process of non-equilibrium symmetry breaking phase transition is studied is the Kibble-Zurek one\(^11\).

**II. THE GINZBURG REGIME**

Having presented the essential outline of the framework of our discussion, we now turn our attention to the transition processes occurring in a finite span of time (see, for example\(^1\)) in which new phenomena, such as the formation of exponentially decaying strings, are expected to occur. In these processes the transition starts at the critical temperature \( T_C \) and, after a certain lapse of time, the maximally stable configuration is attained at the so-called "Ginzburg temperature"\(^3\) \( T_G \) \( (T_G < T_C) \). Between \( T_C \) and \( T_G \) the system is said to be in the critical regime\(^1\).

Our purpose is to model the time dependence of \( m^2(\beta) \) during the critical regime evolution, i.e. for transitions lasting a finite time interval where the formation of domains is allowed.

Following ref.\(^1\), we will focus our attention on the harmonic limit which is obtained by considering only the (linear) l.h.s. of eq.(1.10). Of course, this introduces a strong constraint and it does not accurately describe the behavior of the system. However, it gives enough reliable information on the critical regime behavior\(^1\). We notice that in such an approximation \( \mu_0^2 = -m^2 \).

Let us now consider the expansion of the \( \nu \)-field into partial waves

\[
v(x, \beta) = \sum_k \{u_k(t, \beta)e^{ikx} + u_k^\dagger(t, \beta)e^{-ikx}\}.
\]

(2.1)

Then, in the harmonic potential approximation, the GL equation (1.10) gives for each \( k \)-mode \((k \equiv \sqrt{k^2})\):

\[
\ddot{u}_k(t) + (k^2 - m^2)u_k = 0,
\]

(2.2)

i.e. it leads to the equations for the parametric oscillator modes \( u_k \)\(^13\) (see also\(^14\)). Let us denote the frequency of the \( k \)-mode by

\[
M_k(t) = \sqrt{k^2 - m^2(\beta, t)}.
\]

(2.3)
which is required to be real for each $k$ (in this paper we are not interested in fact in unstable modes corresponding to imaginary frequency; for a recent discussion on these modes and spinodal decomposition see, e.g., ref. 3 and references therein quoted). In full generality, we assume that $m^2$ as well as $\beta$ may depend on time. We remark that the reality condition on $M_k(t)$ turns out to be a condition on the $k$-modes propagation. The reality condition in fact is satisfied provided at each $t$, during the critical regime time interval, it is

$$k^2 \geq m^2(\beta, t),$$

(2.4)

for each $k$-mode. Here and in the following, for shortness we use the notation $\beta \equiv \beta(t)$. Let $t = 0$ and $t = \tau$ denote the times at which the critical regime starts and ends, respectively.

For a given $k$, eq. (2.4) will not hold for any $t$: of course, it holds up to a time $t = \tau_k$ after which $m^2$ is larger than $k^2$. Such a $\tau_k$ represents the maximal propagation time of that $k$ mode. The value of $\tau_k$ is given when the explicit form of $m^2$ is assigned. In the following we consider two possibilities for the time-dependence of $m(\beta, t)$. The first one can lead to large correlation domains, and thus possibly excludes defect formation. The second one does not allow very large domains and therefore it allows defect formation. Our first model choice is

$$a) \quad m^2(\beta, t) = m_0^2 \left( e^{2f(\beta(t), t)} - 1 \right),$$

(2.5)

with $t = 0$ assumed to correspond to the minimum of $m^2$ (we assume $f(0) = 0$ in the example of Figure 1 and in the following). Notice that although at the transition temperature infinite correlation length is allowed, the corresponding mode has only a limited time for propagating. So the formation of domains, i.e. the ‘effective causal horizon’ can be inside the system (domain formation) or outside (single domain) according to the time occurring for reaching the boundaries of the system is longer or shorter than the allowed propagation time. So, the dimensions to which the domains can expand depend on the rate between the speed at which the correlation can propagate, at a certain time, and the correlation length ($\lambda_k \propto (m(\beta, t))^{-1}$) at that time.

In the present case, each $k$-mode can propagate for a span of time $0 \leq t \leq \tau_k$. From eqs. (2.4) and (2.5) we obtain:

$$f(\beta(\tau_k), \tau_k) = \ln \left( \frac{\sqrt{k^2 + m_0^2}}{m_0} \right) \propto \ln \left( \frac{\mathcal{E}_k}{\mathcal{E}_0} \right),$$

(2.6)

where $\mathcal{E}_k$ and $\mathcal{E}_0$ are the $k$-mode energy for non-zero and zero $k$, respectively. The equilibrium time at which $T = T_G$ is $\tau \geq \tau_k$ for any $k$.

A second possibility to model $m^2$ is:

$$b) \quad m^2(\beta, t) = m_0^2 e^{2f(\beta(t), t)}.$$

(2.7)

In this case, a cut-off exists for the correlation length, $L \propto m_0^{-1}$ (in the example of Figure 2, it corresponds to the time $t = 0$). In this case the propagation time is implicitly given by:

$$f(\beta(\tau_k), \tau_k) = \ln \left( \frac{k}{m_0} \right) \propto \ln \left( \frac{L}{\xi} \right),$$

(2.8)
\( f(\beta(t), \tau_k) \) resembling the commonly called string tension. In eq. (2.8) \( \xi \) is the correlation length corresponding to the \( k \)-mode propagation. In this case the reality condition acts as an intrinsic infrared cut-off since small \( k \) values are excluded, due to eq. (2.4), by the non-zero minimum value of \( m^2 \). This means that infinitely long wave-lengths are actually precluded i.e. only domains of finite size can be obtained. Finally (phase) transitions through different vacuum states (which would be unitarily inequivalent vacua in the infinite volume limit) at a given \( t \) are possible. This is consistent with the fact that the system is indeed in the middle of a phase transition process (it is in the critical regime). At the end of the critical regime the correlation may extend over domains of linear size of the order of \( \xi_k \propto (m(\beta_G, \tau))^{-1} \).

In conclusion we see that the model choice \( a \) differs from the model choice \( b \) in the fact that the case \( a \) allows the formation of large correlation domains (infrared modes are allowed); in the case \( b \), on the contrary, only finite-size domains can be formed. Due to the fact that defects may be thought as the normal symmetric phase region trapped in between correlated (ordered) domains, we see that model \( a \) may exclude defect formation if, as mentioned above, \( k \)-mode life-times are longer than the ones needed to reach the system boundaries.

We may then specify the function \( f \) by assuming that it is positive, monotonically growing in time from \( t = 0 \) to \( t = \tau \), at which the Ginzburg temperature is reached.

A possible analytic expression for \( f(\beta(t), t) \) is

\[
f(\beta(t), t) = \frac{at}{bt^2 + c},
\]

where \( a, b, c \) are (positive) parameters chosen so to guarantee the correct dimensions and the correct behavior in time and in temperature (see below for the relations between the various time scales and the ratios among \( a, b, c \)): \( T \) decreases from \( T_C \) to \( T_G \) as time grows from \( t = 0 \) to \( t = \tau \); \( f(\beta(t), t) \) is positive in the critical region (and even for \( t > \tau \), i.e. in the SBS region). The equilibrium time scale is given by \( \tau^2 = \frac{a}{b} \). Notice that in order to obtain this result, the variations in time of \( \beta(t) \) have been assumed small, according to the picture of a slow (non-instantaneous) transition (here we refer to slow transitions as those for which \( \tau \) is large; for fixed parameter \( a \), large \( \tau \) also means large \( \tau_Q \), see below).

The behavior of \( m^2(\beta, t) \) corresponding to (2.9) is shown in Figure 1 (case \( a \)) and in Figure 2 (case \( b \)). The maximum of \( m^2(\beta) \) has to be identified with the minimum of the potential \( \frac{\partial \xi_{ef}}{\partial \nu} = 0 = m(\beta)^2 v(x, \beta) + \lambda g^3(x, \beta) \) (cf. eq. (1.1)), so it corresponds to \( T = T_G \). The negative time region in the Figures 1 and 2 corresponds to \( T > T_G \), so there \( f \) should be negative.

Finally, the present treatment allows to recover the known results on the number of defects, i.e. \( n_{def} \propto (\tau_0/\tau_Q)^{\frac{3}{4}} \) (see ref. for a recent review). In fact, recalling that the equilibrium time \( \tau \) is given by \( \tau = \sqrt{c/b} \) it is possible to introduce the time scales: \( \tau_Q = c/a\lambda \) and \( \tau_0 = a\lambda/b \), with \( \lambda \) arbitrary constant and \( a, b, c \) introduced in eq. (2.9). Thus \( \tau = \sqrt{\tau_Q \tau_0} \), which also says that for fixed \( a \) (and \( b \)) large \( \tau \) also means large \( \tau_Q \). We now observe that, at the first order approximation, one has

\[
e^{g(x)} - g(x) \approx e^{g(x_0)} - g(x_0), \quad \text{if} \quad \left. \frac{\partial g(x)}{\partial x} \right|_{x_0} = 0,
\]

so, in our case, \( e^{2f(\tau)} = 2f(\tau) + e^{2f(0)} \).
The number of defects can be finally obtained. Our choice is to consider the defect number at $T_G$, namely when the system gets enough stabilized (in some sense we count the defects that are still present at the end of the critical regime). In the case $a)$ it is:

$$n_{\text{def}} \propto m^2(\beta_G, \tau) \approx 2m_0^2 f(\tau) = m_0^2 \frac{\tau}{\lambda \tau_Q} \propto \sqrt{\frac{\tau_0}{\tau_Q}}.$$ (2.11)

Similarly, in the case $b)$:

$$n_{\text{def}} \propto m^2(\beta_G, \tau) \approx m_0^2 (2f(\tau) + 1) \approx m_0^2 \frac{\tau}{\lambda \tau_Q} \text{ for large } \tau \text{ (slow transitions)} \quad (2.12)$$

As observed above, for fixed $a$ (and $b$) large $\tau$ also means large $\tau_Q$. We note that the form of $m^2$ is a good approximation at the Ginzburg temperature in our models, since, as already observed, at $T_G$ it corresponds to the minimum of the potential, which is where we want to know the defect number. It is in fact interesting that Zurek’s results are there also recovered.

### III. FINAL REMARKS

It is interesting to write $M_k^2(t)$ as

$$M_k^2(\Lambda_k(t)) = M_k^2(0) e^{-2\Lambda_k(t)}, \quad \text{(3.1)}$$

$$e^{-2\Lambda_k(t)} = e^{2f(\beta(t),t)} \frac{\sinh(f(\tau_k) - f(\beta(t),t))}{\sinh(f(\tau_k))}. \quad \text{(3.2)}$$

where $f(0) = 0$ has been used. Eq.(3.2) shows that $\Lambda_k(t) \geq 0$ for $0 \leq t \leq \tau_k$, $\Lambda_k(0) = 0$ and $\Lambda_k(\tau_k) = \infty$. Since $M_k(\Lambda_k(\tau_k)) = 0$ we see that $\Lambda_k(t)$ acts as a life-time, say with $\Lambda_k(t) \propto s_k$, for the $k$-mode: each $k$-mode "lives" with a proper time $s_k$, i.e. it is born when $s_k$ is zero and it dies for $s_k \to \infty$. The "lives" of the $k$-modes for growing $k$ are shown in Figure 3, where $\Lambda_k(t)$ is plotted versus $t$ for different values of $k$. The blowing up values are reached in correspondence of the $\tau_k$ values for each $k$-mode. Only the modes satisfying the reality condition are present at a certain time $t$, being the other ones decayed. In this way the causal horizon sets up. Figure 3 and Eqs. (3.1) and (3.2) show that modes with larger $k$ have a longer life with reference to time $t$.

Since, on the other hand, longer wave-lengths correspond to lower $k$, we see that domains with a specific spectrum of $k$-modes components may coexist, some of them disappearing before, some other ones persisting longer in dependence of the number in the spectrum of the smaller or larger $k$ components, respectively. In general, the boundaries of larger size domains are thus expected to be less persistent than those of smaller size domains. This fits with the observation that "critical regime has little effect over the small scale dynamics", thus allowing the survival of localized defects (such as vortex strings).

In conclusion, we have studied, in the framework of the GL harmonic potential approximation, the non-equilibrium symmetry breaking phase transition processes occurring in a finite time interval. We have presented a possible modeling of the time-dependence of the frequency of the order parameter modes suitable to account for the formation of correlated domains. An interesting spectrum of possibilities for the size and the life-time of these domains emerges, which appears to be consistent with the conclusions of other, differently grounded analysis.
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FIGURES

FIG. 1. Case a) Behavior in time of $m^2(\beta, t)$
FIG. 2. **Case b)** Behavior in time of $m^2(\beta, t)$

![Graph](image1)

FIG. 3. Life-time for different values of $k$

![Graph](image2)