UNIFORM STABILITY IN A VECTORIAL FULL VON KÁRMÁN THERMOELASTIC SYSTEM WITH SOLENOIDAL DISSIPATION AND FREE BOUNDARY CONDITIONS

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Abstract. We will consider the full von Kármán thermoelastic system with free boundary conditions and dissipation imposed only on the in-plane displacement. It will be shown that the corresponding solutions are exponentially stable, though there is no mechanical dissipation on the vertical displacements. The main tools used are: (i) partial analyticity of the linearized semigroup and (ii) trace estimates which exploit the hidden regularity harvested from partial analyticity.

1. Introduction. We consider the vectorial Von Kármán full system, which models the nonlinear oscillations of a thin plate, taking into account thermal effects and in-plane displacements. This is a fundamental PDE model of nonlinear elasticity which accounts for large displacements [35, 9] and, as such, has received a lot of attention in both engineering and mathematical literature [11]. One particularly attractive aspect of the von Kármán system is that it serves as a prototype for describing the dynamics of thin shells – the latter ubiquitous in technological applications [17, 8]. In spite of this, several fundamental issues are still unresolved, particularly in the dynamic case – for example, wellposedness of weak solutions (specifically uniqueness) and their long time asymptotic behavior. This paper’s goal is to address some of these.

It is known [10] that thermal effects provide some stabilizing effect on plate dynamics. However, this effect is more pronounced in scalar models of plates where in-plane displacements are neglected [14]. Instead, vectorial structures provide for a different mathematical landscape. In-plane accelerations display poor stability properties, even with thermal effects. This is a direct consequence of the fact that elastodynamic waves in dimensions higher than one are not uniformly stabilized by adding thermal dissipation [10, 15]. Thus, the vectorial von Kármán system (2-D model) with thermal effects is not uniformly stable. This phenomenon should be contrasted with scalar von Kármán nonlinear plates subject to thermal effects, where such stability is intrinsic. In the vectorial case, other modes of dissipation

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are necessary. We shall obtain uniform asymptotic stability of weak solutions under a weak solenoidal dissipation on the elastic waves.

Another hurdle in dealing with vectorial structures is the effect of nonlinearity on the dynamics. For scalar von Kármán plates (with large displacements) nonlinear terms have been shown to provide locally Lipschitz effects with respect to the topology generated by weak solutions (finite energy). The latter was accomplished by compensated compactness methods which led to sharp regularity of the Airy stress function\[12, 13\]. In vectorial models, this is no longer valid, as there is no Airy stress function decoupling the system. This clearly underscores fundamental issues of wellposedness – particularly uniqueness – of finite energy solutions. In the case of one dimensional von Kármán beams, detailed analysis has been carried out in [19]. However, this analysis breaks down in two dimensions. Thus, the preponderant issues challenging the model under consideration are the following: the lack of sufficient dissipation and the loss of regularity due to nonlinear effects. The degree of the challenge strongly depends on boundary conditions imposed. While some results have been proven for clamped plates (see section 2), the case of free or partially free boundary conditions is a different mathematical challenge.

The novelty of the present paper is therefore twofold: (1) we consider a plate with free boundary, so that we impose free boundary conditions [20] which take into account the bending moments and shear forces on the edge of the plate, and (2) the regularizing effects of rotational inertia are not accounted for. It turns out that the presence of free boundary conditions in plate models provides for drastic phenomenological and mathematical effects on the treatment of plate dynamics [20]. Mathematically, the critical difference lies in the fact that the square root of the linear generator corresponding to the plate does not recognize any boundary conditions. This leads to technical issues in integrating by parts where boundary terms (above the energy level) appear in the estimates. However, a technique which incorporates microlocal estimates on the boundary has proved successful in the linear theory [28]. These estimates will prove critical in the present treatment. In addition, the lack of regularizing effects due to rotational inertia leads to a major issue in the treatment of nonlinearity associated with the full von Kármán model. The nonlinear term corresponding to finite energy solutions is no longer in the energy space with the loss of 1 + \varepsilon derivative, rather that \varepsilon – as in the case when rotational inertia are accounted for in the model. As it is well known, this loss of regularity is critical for uniqueness properties of weak solutions. While in the case of incremental \varepsilon loss of differentiability, methods such as compensated compactness may be applied [12, 16], non-incremental loss is another matter and often leads to the loss of uniqueness.

In the present paper we shall resort to thermal effects which are shown to provide the so-called “partial” analyticity of the linearization. This property allows us to derive a string of estimates which are helpful in handling superlinear terms in the model and also the above energy-level boundary traces which result from the free boundary. Solenoidal dissipation introduces the necessity of estimating normal derivatives of elastic waves. While in the case of clamped plates such a property follows from “hidden regularity” of the plate [6], this is no longer true for free plates. Free boundary conditions do not satisfy the Lopatinski condition [32, 34, 27] which is responsible for hidden regularity. In our situation, however, we will be able to prove “partial hidden regularity” – which will suffice for the given purpose. Thus,
“partial analyticity” on vertical displacements and “partial hidden regularity” on horizontal displacements are the mathematical properties responsible for the proof. We shall elaborate more on this later after introducing the model, which we present below.

The variables \( u = (u_1, u_2) \) and \( w \) represent the in-plane and vertical displacements of a plate on a two-dimensional domain \( \Omega \) with sufficiently smooth boundary \( \Gamma = \Gamma_0 \cup \Gamma_1 \) with \( \Gamma_0 \cap \Gamma_1 = \emptyset \). The thermal effects are represented by the two thermal strain resultants \( \theta \) and \( \varphi \). Specifically, \( \theta \) is the average of the thermal moment through the thickness of the plate, and thus can be thought of as primarily affecting the vertical displacement. The variable \( \varphi \) is the average of thermal stress and thus interacts with the in-plane displacement [20].

We consider the following system of equations.

\[
\begin{align*}
\mathbf{u}_{tt} + b \text{curl } \mathbf{q}_t - \text{div} \left(C[\varepsilon(\mathbf{u}) + f(\nabla w)]\right) + \nabla \varphi &= 0 \quad \text{in } \Omega \times (0, \infty) \\
\frac{\partial}{\partial \mathbf{t}} \Delta w - \text{div} \left(C[\varepsilon(\mathbf{u}) + \frac{1}{2}(\nabla w)\mathbf{\nabla} w + \varphi \nabla w + \Delta \theta)\right) + \Delta \theta &= 0 \quad \text{in } \Omega \times (0, \infty) \\
\varphi_t - \Delta \varphi + \text{div } \mathbf{u}_t - \nabla w \cdot \nabla w_t &= 0 \\
\theta_t - \Delta \theta - \Delta w_t &= 0
\end{align*}
\] (1a)

with Dirichlet boundary conditions for the displacements on the portion of the boundary \( \Gamma_0 \):

\[
\mathbf{u} = \mathbf{0}, \quad w = 0, \quad \nabla w = \mathbf{0} \quad \text{on } \Gamma_0 \times (0, \infty) \quad (1b)
\]

the following conditions on the portion of the boundary \( \Gamma_1 \):

\[
\begin{align*}
\mathbf{u} &= \mathbf{0} \\
D[\Delta w + (1 - \mu) B_1 w] + \theta &= 0 \quad \text{on } \Gamma_1 \times (0, \infty) \\
D[\frac{\partial}{\partial \mathbf{n}} \Delta w + (1 - \mu) B_2 w] - C[\varepsilon(\mathbf{u}) + \frac{1}{2}(\nabla w)\mathbf{\nabla} w + \Delta \theta)\mathbf{n} \nabla w + \frac{\partial}{\partial \mathbf{n}} \theta &= 0
\end{align*}
\] (1c)

thermal boundary conditions of the form

\[
\varphi = 0; \quad \frac{\partial}{\partial \mathbf{n}} \theta + \lambda_1 \theta = 0 \quad \text{on } \Gamma \times (0, \infty) \quad (1d)
\]

and the following initial conditions:

\[
\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}_t(0) = \mathbf{u}_1, \quad w(0) = w_0, \quad w_t(0) = w_1, \quad \varphi(0) = \varphi_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega \quad (1e)
\]

Here \( \mathbf{n} = (n_1, n_2) \) is the outward unit normal direction to the boundary \( \Gamma \), \( \tau \) denotes tangential vector to the boundary. \( b, \lambda_1, \mu, \) and \( D \) are positive constants. The constant \( b \) is a damping coefficient, as our model features an internal damping acting only on the solenoidal portion \( q \) of the in-plane displacements – that is, we decompose \( \mathbf{u} = \nabla p + \text{curl } \mathbf{q} \). \( D \) represents the flexural rigidity, and we use \( \mu \) to represent Poisson’s modulus. Note that \( 0 < \mu < \frac{1}{2} \).

The fourth-order tensor \( C \) is given by

\[
C \varepsilon \equiv \frac{E}{(1 - 2\mu)(1 + \mu)} \left[\mu \text{trace } \varepsilon \mathbf{I} + (1 - 2\mu)\varepsilon\right]
\]

where the strain tensor is given by \( \varepsilon(\mathbf{u}) \equiv \frac{1}{2}(\mathbf{D}\mathbf{u} + \mathbf{D}^T \mathbf{u}) \). It can be shown that the tensor \( C \) is symmetric and strictly positive. The function \( f \) is given by

\[
f(s) \equiv \frac{1}{2} s \otimes s
\]
for $s \in \mathbb{R}^2$. The boundary operators $B_1$ and $B_2$ that appear in the free boundary conditions on $\Gamma_1$ are given by

$$B_1 w \equiv -\left(\frac{\partial}{\partial \tau}\right)^2 w - k(\nu) \frac{\partial}{\partial \nu} w,$$
$$B_2 w \equiv \frac{\partial}{\partial \tau} \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau} w \quad (2)$$

where $k(\nu)$ denotes the curvature given by $k(\nu) = \text{div } \nu$.

The operator $B_1$ models the bending moments and $B_2$, the shear forces. The expressions in (2) are shown in reference [29], Appendix 3c to be equivalent to the ones given in the original model [20].

**Remark 1.** Rotational inertia in the von Kármán model are accounted for by adding the term $-\gamma \Delta w_{tt}$, $\gamma > 0$ to the second equation in (1a). The addition of this term provides a regularizing effect by increasing the regularity of the vertical velocity by one derivative. In such a case, the effect of nonlinear term is less drastic and becomes incremental. This is to say, that loss of differentiability with respect to the finite energy solution is just $\epsilon$. (See the explanation at the end of Section 3.)

Throughout this paper we use the spaces $H^1_{\Gamma_0}(\Omega)$ and $H^2_{\Gamma_0}(\Omega)$ which denote the usual Sobolev spaces that incorporate zero boundary conditions on $\Gamma_0$; that is:

$$H^i_{\Gamma_0}(\Omega) \equiv \{ u \in H^i(\Omega) | \text{trace of } u = 0 \text{ on } \Gamma_0 \}, \quad i = 1, 2,$$

with

$$||| u |||_{H^1_{\Gamma_0}(\Omega)} = ||\nabla u||_{L^2(\Omega)}.$$

We shall also use the following notation for Sobolev norms:

$$|u|_{\alpha, \Omega} \equiv |u|_{H^\alpha(\Omega)}, \quad |u|_{\alpha, \Gamma} \equiv |u|_{H^\alpha(\Gamma)}$$

and inner products

$$(u,v)_\Omega \equiv (u,v)_{L^2(\Omega)}, \quad (u,v)_\Gamma = (u,v)_{L^2(\Gamma)}$$

The following wellposedness result for weak solutions will be proved in Section 4.

**Theorem 1.1** (Wellposedness of weak solutions). For any $T > 0$, for any initial data

$$u_0, u_1, \theta_0, \varphi_0 \in [H^1_{\Gamma_0}(\Omega)]^2 \times [L^2(\Omega)]^2 \times L^2(\Omega) \times L^2(\Omega)$$

$$w_0, w_1 \in H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \quad (3)$$

there exists a unique “finite energy” [see below], global in time solution of (1a)

$$(u, w, \theta, \varphi) \in C(0,T; [H^1(\Omega)]^2 \times H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega))$$

$$(u_t, w_t) \in C(0,T; [L^2(\Omega)]^2 \times L^2(\Omega))$$

The weak solution depends continuously (in the above topology) on the initial data. In addition, weak solutions display the following regularity.

$$w \in W^{\alpha,2}(0,T; H^{3-2\alpha}(\Omega)), \alpha \in [0, 1]$$

As expected, assuming more regular initial data one obtains solutions with higher regularity.

**Theorem 1.2** (Strong solutions). Assume

$$u_0, u_1, \theta_0, \varphi_0 \in [H^2(\Omega)]^2 \times [H^1(\Omega)]^2 \times H^2(\Omega) \times H^2(\Omega)$$

$$w_0, w_1 \in H^4(\Omega) \times H^2(\Omega) \quad (4)$$

subject to appropriate compatibility conditions on the boundary. Then one obtains

$$(u, w, \theta, \varphi) \in C(0,T; [H^2(\Omega)]^2 \times H^4(\Omega) \times H^2(\Omega) \times H^2(\Omega))$$
\[(u_t, w_t) \in C(0, T; [H^2(\Omega)]^2 \times H^2(\Omega))\]

where \(T\) can be taken arbitrary.

**Remark 2.** The result claimed by Theorem 1.2 is proved by fairly standard (by now) methods. Local existence/uniqueness can be proved by a fixed point argument while global bounds are obtained by propagating a-priori bounds from the energy level to higher derivatives. The details are similar to those in [21, 22]. The main issue is, of course, Hadamard wellposedness of weak solutions.

In order to proceed, we define the energy functional associated with the model (1).

\[
E(t) = E_k(t) + E_p(t)
\]

where the kinetic energy of the system is given by

\[
E_k(t) = |u_t|^2_{0, \Omega} + |w_t|^2_{0, \Omega}
\]

and the potential energy is given by

\[
E_p(t) = (CN(u, w), N(u, w))_\Omega + a(w, w) + |\varphi|^2_{0, \Omega} + |\theta|^2_{0, \Omega}
\]

Here the stress resultant \(N(u, w)\) is given by

\[
N(u, w) = \varepsilon(u) + f(\nabla w)
\]

and

\[
a(w, z) \equiv D \int_\Omega [w_{x,x}z_{x,x} + w_{y,y}z_{y,y} + \mu w_{x,x}z_{y,y} + \mu w_{y,y}z_{x,x} + 2(1-\mu)w_{x,y}z_{x,y}] d\Omega + l \int_{\Gamma_1} wz \, d\Gamma_1
\]

It is well known that \(E_p(t)\) is topologically equivalent to the \([H^1(\Omega)]^2 \times H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)\) topology. In particular, one can derive the following useful inequalities from Korn’s inequality and Sobolev’s imbeddings:

\[
|u|_{H^1(\Omega)} \leq C|[N(u, w)|_{L^2(\Omega)} + |\nabla w|^2_{L^2(\Omega)}] \\
\leq C|[N(u, w)|_{L^2(\Omega)} + |w|^2_{H^2(\Omega)}]
\]

The following energy inequality (Lemma 4.2) can be justified for all weak solutions. (We have equality for strong solutions).

\[
E(t) + 2 \int_s^t \lambda_1 |\theta|^2_{0, t} \, dt + 2 \int_s^t (b|\text{curl } q_t|^2_{0, \Omega} + |\nabla \varphi|^2_{0, \Omega} + |\nabla \theta|^2_{0, \Omega}) \, dt \leq E(s)
\]

This inequality shows that the energy of the system is nonincreasing. The questions we ask next are the following: does it go to zero? If so, at which rate? The model introduced features coupling of mechanical equations with thermal equations. In particular, the vertical displacements \(w\) are left undissipated. Thus the question becomes whether dissipation introduced by thermal variables together with solenoidal dissipation will suffice to dissipate mechanical variables \(u, w\) and, consequently, drive the energy to zero. Looking at the model equations and treating the energy function as a Lyapunov function, one can see that \(\frac{d}{dt} E(t) = 0\) would produce

\[
\theta = 0, \quad \phi = 0, \quad \text{curl } q_t = 0 \text{ in } \Omega \\
\Delta w_t = 0 \text{ in } \Omega, \quad w_t = 0, \quad \nabla w_t = 0 \text{ on } \Gamma_0
\]

Thus, if \(\Gamma_0\) is of a nonzero measure, by elliptic continuation property we obtain \(w_t \equiv 0\) in \(\Omega\). This produces \(\text{div } u_t = 0\) in \(\Omega\). The above information together...
with the representation \( \text{curl } \mathbf{q}_t = \mathbf{u}_t + \nabla A_D^{-1} \text{div } \mathbf{u}_t \), where \( A_D \) denotes \(-\Delta\) with zero Dirichlet data, implies that \( u_t \equiv 0 \) in \( \Omega \). Thus, the energy function becomes a strong Lyapunov function and asymptotic behavior is characterized by stationary dynamics with free boundary conditions:

\[
div(C[\varepsilon(\mathbf{u}) + f(\nabla w)]) = 0
\]

\[
D\Delta^2 w - \text{div}(C[\varepsilon(\mathbf{u}) + f(\nabla w)])\nabla w = 0 \quad \text{in } \Omega \times (0, \infty)
\]

(9)

with clamped boundary conditions for the displacements on the portion of the boundary \( \Gamma_0 \):

\[
\mathbf{u} = 0, \ w = 0, \ \nabla w = 0 \quad \text{on } \Gamma_0 \times (0, \infty)
\]

(10)

the following boundary conditions on the portion of the boundary \( \Gamma_1 \):

\[
\begin{cases}
\mathbf{u} = 0 \\
D[\Delta w + (1 - \mu)B_1 w] = 0 \\
D[\frac{d}{dt}\Delta w + (1 - \mu)B_2 w] - C[\varepsilon(\mathbf{u}) + f(\nabla w)]\nu\nabla w = 0
\end{cases}
\quad \text{on } \Gamma_1 \times (0, \infty)
\]

(11)

which then gives

\[
a(w, w) + ||N(\mathbf{u}, w)||^2_{L_2(\Omega)} = 0
\]

hence \( a(w, w) = 0 \) and \( N(\mathbf{u}, w) = 0 \) Since \( \text{mes}(\Gamma_0) > 0 \) we obtain \( w \equiv 0 \) and \( \varepsilon(\mathbf{u}) = 0 \) in \( \Omega \). The latter, by Korn’s inequality, allows us to conclude that \( \mathbf{u} \equiv \mathbf{0} \) in \( \Omega \). This (still at this stage formal argument) shows that the energy function converges asymptotically to zero when \( \text{mes}(\Gamma_0) > 0 \). The above considerations provide motivation for the main result of this paper, which is convergence of the energy function to zero exponentially fast. Our main result reads:

**Theorem 1.3 (Uniform Exponential Decay Rates).** Let \( \mathbf{u}, w, \theta, \) and \( \varphi \) be a weak solution to the original system where we assume that \( \text{mes}(\Gamma_0) > 0 \). Then, the following estimate holds:

\[
E(t) \leq C(E(0))e^{-\omega t}
\]

for \( t \geq 0 \), where the constant \( \omega > 0 \) may depend on \( E(0) \).

**Remark 3.**

1. It is known that the dynamic system of elasticity with thermal conduction is not uniformly stable when \( \dim \Omega > 1 \) [10, 15]. Thus, the dissipation only on the solenoidal component of velocity appears to be a minimal amount of dissipation needed to asymptotically stabilize the in-plane waves.

2. Whether the condition \( \text{mes}(\Gamma_0) > 0 \) is necessary is an open question. However, vertical displacements are not subjected to any form of dissipation in our model. Thus, a requirement that these displacements are fixed on a portion of the boundary seems reasonable in the case of free boundary conditions. However, an independent study of necessity of this condition is warranted. The fact that even strong stability (without uniformity) requires such condition may indicate indispensability of this requirement.

3. Another interesting question is whether one could show independence of the decay rates on rotational inertia. Specifically, if we added the rotational term \( \gamma \Delta w_{tt} \) to the equation for \( w \), could we show that decay rates \( \omega, \gamma \) were independent of the value of \( \gamma \). This seems unlikely given that the presence or absence of this term changes the underlying dynamics of the system.
2. **Background and literature.** This model of the nonlinear von Kármán plate that includes thermal effects was derived in [20, 3]. Exponential stability was shown first in the one-dimensional case [6], where a Lyapunov method was used. The one-dimensional case is rather special due to the fact that linear system of dynamic elasticity with thermal effects is exponentially stable [10]. This is no longer the case in higher dimensions. The two-dimensional case was then considered in a series of works [4, 5]. In [4], the authors considered solenoidal damping on a slightly simpler model without the nonlinear coupling term on the $\varphi$ equation. In that model, the resulting equation for $\varphi$ is $\varphi_t = \Delta \varphi - \text{div } u_t$. Thus, the thermal equations are coupled to the bending equations in a linear manner only. The authors also considered two cases of plate equation – with and without rotational inertia term ($\gamma > 0$ and $\gamma = 0$). The presence or absence of this term changes the underlying dynamics of the PDE model so that the two cases often need to be considered separately. In this reference, well-posedness and regularity results were proved for both cases: a significant source of difficulty, particularly at the level of uniqueness of weak solutions. This result was proved using an adaptation of Sedenko’s technique[33] which was originally developed in the context of Vlasov equations. For the case of $\gamma = 0$, an important role was played by the analyticity of the semigroup generated by the linear part of the equation corresponding to $w$. Then, the authors were able to prove stabilizibility by use of multiplier techniques.

In [5], the authors generalized the model by including the nonlinear coupling in the thermal equation. This complicates the analysis substantially because the nonlinear coupling introduces unbounded (with respect to energy level) terms in the thermal equation. This requires a different treatment in the following areas: uniqueness of weak solutions, stability analysis, and well-posedness for the case $\gamma = 0$. We note that both [4] and [5] deal with the case of clamped (Dirichlet) boundary conditions. In addition, this model includes an internal damping term which acts only on the solenoidal part of the displacement vector. It is well-understood that the thermal dissipation in the two-dimensional system of linear elasticity provides only strong, and not exponential stability [15], so it necessary to have some source of dissipation in the $u$-variable to guarantee the decay rates of the energy function.

A related model that arrives at exponential decay is the thermoelastic von Kármán system considered in [22]. In this work, there is no internal damping assumed on the plate. Dissipative effects are provided by the coupling with the heat equations, but as mentioned above, some sort of mechanical dissipation is required as well to achieve exponential decay rates. In [22], a nonlinear boundary dissipation is imposed on the velocity of the trace of $u$. In addition, the author considers the challenging case of free boundary conditions. Under these conditions she derives wellposedness of regular and weak solutions, and uniform decay rates for the energy function. She considers the case of $\gamma = 0$, so that rotational terms are not included in the model. This provides a challenge because the rotational inertia term provides a regularizing effect for the velocity of the component $w$. This effect was critical in other works, such as [21], which treats the case of the von Kármán system with $\gamma > 0$ and *without* thermoelastic effects. Instead of the methods used in [21], the $\gamma = 0$ requires use of the analyticity of the underlying linear system of thermoelasticity with free boundary conditions[28].

Finally, we mention several recent works which consider Von Kármán models with free boundary conditions [26, 25]. In [26], the authors are interested in the long-time behavior and theory of global attractors for a full vectorial von Kármán
plate system with thermal effects. The system considered is related to ours, but there is no dissipation assumed on the boundary or on the in-plane displacements. Instead, the model includes the presence of forces exerted on the body of the plate by some nonlinear elastic foundation. The authors include detailed assumptions on the form of these external forcing terms which are used to prove the existence of a compact global attractor. In [25], the authors generalize this plate model to a shallow shell model.

In the current work, we assume that in-plane displacements are not dissipated on the boundary (as in [22]), but rather the only mode of mechanical dissipation is solenoidal damping of horizontal displacements. The vertical displacements are left free and non-dissipated. We are considering the $\gamma = 0$ case, where rotational inertia is neglected – thus not contributing to regularizing effects. The main challenges are (i) the treatment of spurious boundary traces arising from higher order boundary conditions, (ii) handling superlinearity (unstructured) of von Kármán model without relying on additional regularity secured by rotational inertia. A critical role is played by developments in trace regularity for thermoelastic plates with free boundary conditions.

3. Preliminaries. We begin with giving a proper meaning of weak solutions corresponding to the original model. This can be done either via variational analysis or via semigroup framework. Since both approaches will be used in the sequel, we present the relevant formulations.

3.1. Variational formulation. The system (1a) admits the following variational form, where test functions $\xi \in H^1(\Omega) \times H^1(\Omega)$, $\varphi \in H^2(\Omega) \cup H^1_0(\Omega)$, $\eta \in H^1_0(\Omega)$, $\delta \in H^1(\Omega)$:

\[
\begin{align*}
(u_{tt}, \xi)_\Omega + (b \text{ curl } q_t, \xi)_\Omega + (C[\varepsilon(u) + f(\nabla w)], \varepsilon(\xi))_\Omega \\
\quad - (C[\varepsilon(u) + f(\nabla w)]\nu, \xi)_\Gamma + (\nabla \varphi, \xi)_\Omega &= 0 \quad (12a) \\
(w_{tt}, \psi)_\Omega + a(w, \psi) + (\theta, \frac{\partial}{\partial \nu} \psi)_\Gamma_1 + (C[\varepsilon(u) + f(\nabla w)], \nabla w \otimes \nabla \psi)_\Omega \\
\quad + (\varphi \nabla w, \nabla \psi)_\Omega - (\nabla \theta, \nabla \psi)_\Omega &= 0 \quad (12b) \\
(\varphi_t, \eta)_\Omega + (\nabla \varphi, \nabla \eta)_\Omega - (u_t, \nabla \eta)_\Omega - (\nabla w \cdot \nabla w_t, \eta)_\Omega &= 0 \quad (12c) \\
(\theta_t, \delta)_\Omega + (\nabla \theta, \nabla \delta)_\Omega + \lambda_1(\theta, \delta)_\Gamma_1 - (\frac{\partial}{\partial \nu} w_t, \delta)_\Gamma + (\nabla w_t, \nabla \delta)_\Omega &= 0 \quad (12d)
\end{align*}
\]

Note that we have used the boundary conditions that are satisfied on $\Gamma$. Further details of some tensor calculations appear in [22].

3.2. Semigroup formulation. We introduce the operators that provide an abstract semigroup representation of the model (1).

\[
A_0 : [L_2(\Omega)]^2 \to [L_2(\Omega)]^2; \quad A_0 u \equiv -\text{div } C\varepsilon(u); \quad D(A_0) \equiv [H^2(\Omega)]^2 \cap [H^1_0(\Omega)]^2
\]

\[
A : L_2(\Omega) \to L_2(\Omega); \quad Aw \equiv D\Delta^2 w; \quad D(A) \equiv \{ w \in H^4(\Omega); \ w = \nabla w = 0 \text{ on } \Gamma_0, \ \Delta w + (1 - \mu)B_1 w = 0, \ \frac{\partial}{\partial \nu} \Delta w + (1 - \mu)B_2 w = 0 \text{ on } \Gamma_1 \}
\]
Thus we can write the system (13) as
\[ D[\Delta v_1 + (1 - \mu)B_1 v_1] = g, \quad \frac{\partial}{\partial \nu} \Delta v_1 + (1 - \mu)B_2 v_1 = 0 \text{ on } \Gamma_1; \]
\[ D[\Delta v_2 + (1 - \mu)B_1 v_2] = 0, \quad \frac{\partial}{\partial \nu} \Delta v_2 + (1 - \mu)B_2 v_2 = g \text{ on } \Gamma_1 \]
\[ A_D : L_2(\Omega) \to L_2(\Omega); \quad A_D v = -\Delta v, \quad D(A_D) = H^2(\Omega) \cap H_0^1(\Omega) \]
\[ D : [L_2(\Omega)]^2 \to [L_2(\Omega)]^2; \quad D u = u + \nabla A_D^{-1} \text{div} u \]
\[ A_N : L_2(\Omega) \to L_2(\Omega); \quad A_N v = -\Delta v, \quad D(A_N) = \left\{ v \in H^2(\Omega) : \frac{\partial}{\partial \nu} v + \lambda_1 v = 0 \text{ on } \Gamma \right\} \]

Using these definitions we can rewrite the system (1) as
\[ u_{tt} + bD u_t + A_0 u + \nabla \varphi = f_1(w) \] (13a)
\[ w_{tt} + A w + AG_1 \theta|_{\Gamma_1} + AG_2 \frac{\partial}{\partial \nu} \theta - A_N \theta = f_2(u, w, \varphi) \] (13b)
\[ \varphi_t + A_D \varphi + \text{div} u_t = f_3(w, w_t) \] (13c)
\[ \theta_t + A_N \theta - \Delta w_t = 0 \] (13d)

with the nonlinear forcing terms given by
\[ f_1(w) = \text{div} C f(\nabla w) \] (13e)
\[ f_2(u, w, \varphi) = \text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w] \] (13f)
\[ f_3(w, w_t) = \nabla w \nabla w_t \] (13g)

Denoting
\[ A_1 : H_1 \to H_1; \quad A_1 = \begin{bmatrix} 0 & I & 0 \\ -A_D & -bD & -\nabla \\ 0 & -\text{div} & -A_D \end{bmatrix} \] (14)

where
\[ H_1 \equiv [D(A_0)]^2 \times [L_2(\Omega)]^2 \times L_2(\Omega) \sim [H_0^1(\Omega)]^2 \times [L_2(\Omega)]^2 \times L_2(\Omega) \]
\[ D(A_1) = \{ (u_1, u_2, \varphi) \in [D(A_0)]^2 \times [D(A_0)]^2 \times D(A_D) \} \]
\[ A_2 : H_2 \to H_2; \quad A_2 = \begin{bmatrix} 0 & I & 0 \\ -A & 0 & A_N - AG_1 (|_{\Gamma_1}) - AG_2 \frac{\partial}{\partial \nu} \\ 0 & \Delta & -A_N \end{bmatrix} \] (15)

where
\[ H_2 \equiv D(A^2) \times L_2(\Omega) \times L_2(\Omega) \sim H_0^2(\Omega) \times L_2(\Omega) \times L_2(\Omega) \]
\[ D(A_2) = \{ (w_1, w_2, \theta) \in D(A) \times D(A^2) \times D(A_N) \} \]

Thus we can write the system (13) as
\[ \frac{d}{dt} \begin{bmatrix} u \\ u_t \\ \varphi \end{bmatrix} = A_1 \begin{bmatrix} u \\ u_t \\ \varphi \end{bmatrix} + \begin{bmatrix} 0 \\ f_1(w) \\ f_3(w, w_t) \end{bmatrix} \] (16a)
\[ \frac{d}{dt} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = A_2 \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ f_2(u, w, \varphi) \end{bmatrix} \] (16b)

Here \( H_0^2(\Omega) \) is the usual Sobolev space which consists of \( H^2(\Omega) \) functions with zero traces and normal derivatives on the portion of the boundary \( \Gamma_0 \).
It is known [23] that the linear modes generated by $A_1$ and $A_2$ lead to linear semi-groups of contractions. This is due to maximal dissipativity of the pairs $(A_1, H^1_0)$ and $(A_2, H^2)$. However, the nonlinear terms given by $f_1$ and $f_2$ are no longer in the state space. To wit: with $w \in H^2(\Omega)$ one has $f_1(w) \in H^{-\epsilon}(\Omega)$ (with $\epsilon > 0$) – a consequence of the fact that $H^1$ is not a multiplier on $L^2$ (in two dimensions). Similarly $f_2(w)$ should be looked upon as an element of $[H^1(\Omega)]^2$ that displays a loss of $\epsilon$ derivative. This follows from $(\nabla w_1 \cdot \nabla w, \phi) = - (w_1, \Delta \phi + \nabla \phi \cdot \nabla w)$ and the fact that both terms $\Delta \phi$ and $\nabla \phi \cdot \nabla w$ are products of an $L^2$ function with an $H^1$ function, where the latter fails to be a multiplier by a factor of $\epsilon$. Clearly the highest loss of regularity is displayed by $f_2$. $N(u, w)\nabla w$ is in $H^{-\epsilon}$ and taking the divergence gives $f_2(u, w, \phi) \in H^{-1-\epsilon}(\Omega)$. The above calculations also reveal why the rotational model with $w_\ell \in H^1(\Omega)$ will give only an $\epsilon$ loss in $f_2$ (see Remark 1). Incremental losses can be handled by compensated compactness methods, but this is not the case with the higher loss occurred in $f_2$. This simply emphasizes a need for finding a suitable framework to handle nonlinear terms, both at the level of wellposedness and stabilization.

4. Wellposedness and regularity – Proof of Theorem 1.1.

Proof of Theorem 1.1. The proof of well-posedness (particularly of uniqueness) of weak solutions corresponding to the model (1) must account for the difficulties encountered in dealing with nonlinear terms which can be labeled as “supercritical”. This will be done by relying in a critical way on a relatively recent result of analyticity associated with the linear model corresponding to vertical displacements[2]. Then, we exploit the semigroup framework in order to take the advantage of the higher regularity of the variable $w$ (due to the analyticity of the corresponding linearized thermoelastic plate equation) to prove uniqueness and also continuous dependence on the initial data.

4.1. Nonlinear Galerkin method – existence of finite dimensional energy solutions.

Lemma 4.1 (Existence of Finite Dimensional Energy Solutions). For any $T > 0$, for any initial data

$$u_0, u_1, \theta_0, \varphi_0 \in [H^1_{1,0}(\Omega)]^2 \times [L^2_{\Gamma_0}(\Omega)]^2 \times L^2(\Omega) \times L^2(\Omega)$$

$$w_0, w_1 \in H^2_{1,0}(\Omega) \times L^2(\Omega)$$

there exists a finite energy, global in time solution of (1a)

$$(u, w, \theta, \varphi) \in C(0, T; [H^1(\Omega)]^2 \times H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega))$$

$$(u_\ell, w_\ell) \in C(0, T; [L^2(\Omega)]^2 \times L^2(\Omega))$$

Proof. The proof relies on a fairly standard nonlinear Galerkin method. Let $h$ denote a parameter tending to zero and let $U_h, W_h$ be finite dimensional subspaces of $H^2(\Omega) \cap H^1_0(\Omega), H^3(\Omega) \cap H^2_0(\Omega)$. The following approximation properties are imposed on the elements of $U_h, W_h$:

(i) for any $u \in H^s(\Omega)$ and in $H^s(\Omega) \cap H^2_0(\Omega)$ when $s > \frac{1}{2}$, there exists $\xi \in U_h$, so that

$$|u - \xi|_{s, \Omega} \to 0 \quad \text{when} \quad h \to 0, \quad 0 \leq s \leq 2$$
(ii) for any \( w \in H^s(\Omega) \cap H^{1,s}_0(\Omega) \) and in \( H^s(\Omega) \cap H^{2,s}_0(\Omega) \) when \( s > \frac{3}{2} \), there exists \( \psi \in \mathcal{W}_h \) so that
\[
|w - \psi|_{s,\Omega} \to 0 \quad \text{when} \quad h \to 0, \quad 0 \leq s \leq 3
\]

The space of splines, for example, satisfies the above properties. We further denote:
\[
\mathcal{U}_h = [U_h]^2 \times U_h, \quad \mathcal{V}_h = [U_h]^2 \times \mathcal{W}_h \times U_h \times \mathcal{U}_h
\]
damping operator \( D \in \mathcal{L}([L^2(\Omega)]^2) \) defined by \( D \mathbf{u} = u + \nabla A_D^{-1} \text{div} \mathbf{u} \) and consider the following semidiscrete approximation to the problem (12):

Given functions \((\mathbf{u}_h(t), w_h(t), \varphi_h(t), \theta_h(t)) \in \mathcal{V}_h\), we need to find functions \((\mathbf{u}_h(t), w_h(t), \varphi_h(t), \theta_h(t)) \in \mathcal{V}_h\) such that
\[
\mathbf{u}_h(0) = \mathbf{u}_{h,0}, \quad \mathbf{u}_{h,t}(0) = \mathbf{u}_{h,1}, \quad w_h(0) = w_{h,0}, \quad w_{h,t}(0) = w_{h,1}, \quad \varphi_h(0) = \varphi_{h,0}, \quad \theta_h(0) = \theta_{h,0}
\]
and
\[
(\mathbf{u}_{h,t}, \xi) + (b \mathbf{D} \mathbf{u}_{h,t}, \xi) + (CN(\mathbf{u}_h, w_h)](\xi)]_{\Omega} - (CN(\mathbf{u}_h, w_h]_\nu, \xi)_{\Omega} + (\nabla \varphi_{h,t}, \xi)_{\Omega} = 0 \quad (17a)
\]

\[
(\varphi_{h,t}, \eta) + (\nabla \varphi_{h, t}, \nabla \eta)_{\Omega} - (\mathbf{u}_{h,t}, \nabla \eta)_{\Omega} - (\nabla w_h \cdot \nabla w_{h, t}, \eta)_{\Omega} = 0 \quad (17c)
\]

\[
(\theta_{h,t}, \delta) + (\nabla \theta_{h, t}, \nabla \delta)_{\Omega} + \lambda_1 (\theta_h, \delta)_{\Gamma_1} - \frac{\partial}{\partial t} w_{h,t}, \delta)_{\Gamma} + (\nabla w_{h,t}, \nabla \delta)_{\Omega} = 0 \quad (17d)
\]

for all \((\xi, \psi, \eta, \delta) \in \mathcal{V}_h\). Global existence and uniqueness of the semidiscrete solution follows from the fact that the nonlinear terms are locally Lipschitz on \( \mathcal{V}_h \) and the \textit{a-priori} energy bound

\[
E_h(t) + 2 \int_0^t \lambda_1 |\theta_h|^2_{\partial t} \ dt + 2 \int_0^t (b |\mathbf{D} \mathbf{u}_{h,t}|_{\partial t}^2 + |\nabla \varphi_{h,t}|^2_{\partial t, \Omega} + |\nabla \theta_{h,t}|^2_{\partial t, \Omega}) \ dt = E_h(0) \quad (18)
\]

where
\[
E_h(t) = |\mathbf{u}_{h,t}|^2_{\partial t, \Omega} + |w_{h,t}|^2_{\partial t, \Omega} + (CN(\mathbf{u}_h, w_h), N(\mathbf{u}_h, w_h))_{\Omega} + a(w_h, w_h) + |\varphi_{h,t}|^2_{\partial t, \Omega} + |\theta_{h,t}|^2_{\partial t, \Omega}
\]

The bound (18) is derived using (17) with \( \xi = \mathbf{u}_h, \psi = w_h, \eta = \varphi_h, \) and \( \delta = \theta_h \), integrating by parts, and applying the Divergence theorem. This bound, together with the fact that the solutions \((\mathbf{u}_h, w_h, \varphi_h, \theta_h) \in C^0([0, T), \mathcal{V}_h)\) for some \( T > 0 \) allows us to conclude that finite-dimensional solutions to (17) exist and are unique. Then, based on (18) and the regularity of \( N(\mathbf{u}, w) \) we can extract convergent subsequences (denoted by the same symbol) such that
\[
\mathbf{u}_h \to \mathbf{u} \text{ weakly* in } L_\infty(0, T; [H^1(\Omega)]^2), \quad \mathbf{u}_{h,t} \to \mathbf{u}_t \text{ weakly* in } L_\infty(0, T; [L_2(\Omega)]^2),
\]
\[
w_h \to w \text{ weakly* in } L_\infty(0, T; H^2(\Omega)), \quad w_{h,t} \to w_t \text{ weakly* in } L_\infty(0, T; H^1(\Omega)),
\]
\[
\varphi_h \to \varphi \text{ weakly* in } L_\infty(0, T; H^1(\Omega)), \quad \theta_h \to \theta \text{ weakly* in } L_\infty(0, T; H^1(\Omega)),
\]
As in [4], this convergence and weak continuity of the von Kármán nonlinearity (30) allow us to pass with the limit on the original semidiscrete model (17). In order to pass the limit on the nonlinear terms in equation (17c), we need to apply the form
\[
\nabla w_h \cdot \nabla w_{h,t} = \frac{1}{2} \frac{d}{dt} |\nabla w_h|^2
\]
and use test functions which are regular in time. Thus
the limit points ũ, w, ϕ, θ satisfy the weak variational form of the original equation. In addition, the following regularity is inherited from the a-priori energy bound (18):

\[ u \in B(0, T; [H^1_0(\Omega)]^2), \quad u_t \in B(0, T; [L^2(\Omega)]^2) \]  

\[ w \in B(0, T; H^2_0(\Omega)), \quad w_t \in B(0, T; L^2(\Omega)) \]  

\[ \varphi \in L^2(0, T; H^1_0(\Omega)) \cap C(0, T; L^2(\Omega)) \]  

\[ \theta \in L^2(0, T; H^1(\Omega)) \cap C(0, T; L^2(\Omega)) \]

where the notation \( B(0, T; X) \) denotes a space of functions uniformly bounded on \([0, T]\) with the values in \( X \). In fact, \( B(0, T; X) \) could be replaced by weak continuity denoted by \( C^w(0, T; X) \).

Moreover, the following Dissipativity inequality follows immediately from the usual lower semicontinuity argument.

**Lemma 4.2** (Dissipativity Inequality). Let \((u, w, \varphi, \theta)\) be a finite energy solution to the system (1a)–(1e). Then for any \( s \leq t \),

\[
E(t) + 2 \int_s^t \lambda_1 |\theta|^2_{0,1} \, dt + 2 \int_s^t (b |\text{curl} \, q|^2_{0,\Omega} + |\nabla \varphi|^2_{0,\Omega} + |\nabla \theta|^2_{0,\Omega}) \, dt \leq E(s)
\]  

(21)

Here we note that having equality instead of inequality in (21) and also uniqueness of weak solutions would allow us to prove that solutions are strongly continuous in time with the Hadamard wellposedness property. Thus, the corresponding system would generate nonlinear semigroup and the underlying dynamical system. We shall work toward this goal.

**Remark 4.** Equality in (21) is of course valid for regular (strong solutions) corresponding to smooth initial data. The latter can be shown by formal integration by parts with standard energy multipliers \((u_t, w_t, \theta, \phi)\). However, at this stage of the analysis we do not have norm convergence in order to pass with the limit on energy level terms. Such a procedure will be justified later with the help of the partial regularity enjoyed by the vertical component of the plate.

4.2. **Partial analyticity.** Applying the dissipation inequality (42) along with the regularity (in negative Sobolev’s spaces) of higher time derivatives allows us to improve regularity in (20) to weak continuity in time denoted by \( C^w \). Thus we have that weak solutions satisfy

\[ u \in C^w(0, T; [H^1_0(\Omega)]^2) \cap C^1_w(0, T; [L^2(\Omega)]^2) \]  

\[ w \in C^w(0, T; H^2_0(\Omega)) \cap C^1_w(0, T; L^2(\Omega)) \]  

\[ \varphi \in L^2(0, T; H^1_0(\Omega)) \cap C(0, T; L^2(\Omega)) \]  

\[ \theta \in L^2(0, T; H^1(\Omega)) \cap C(0, T; L^2(\Omega)) \]

However, for our model, the variable \( w \) in fact displays a higher regularity, due to the underlying analyticity of the linear thermoelastic plate. The analyticity of thermoelastic plates has long been known for standard boundary conditions (clamped, simply supported) [31]. However, the same property with free boundary conditions requires a very different approach – due to persistence of unbounded boundary traces in the estimates – which in turn result from higher order boundary conditions. This issue has been resolved more recently [28]. In order to proceed we define
Let the initial data be of finite energy, that is, Lemma 4.3.
the finite energy space
\[ H_1 \times H_2 = [H^0_0(\Omega)]^2 \times [L_2(\Omega)]^2 \times L_2(\Omega) \times H^1_{1,0}(\Omega) \times L_2(\Omega) \times L_2(\Omega) \]

**Lemma 4.3.** Let the initial data be of finite energy, that is, \((u_0, u_1, \varphi_0, w_0, w_1, \theta_0) \in H_1 \times H_2\). Then
\[(w, w_1) \in L_2(0, T; H^3(\Omega) \times H^1(\Omega))\]
for all \(T > 0\). Moreover, the following estimate holds:
\[
\int_0^T |w|^2_{3, \Omega} + |w_1|^2_{1, \Omega} \, dt \leq CE(0) + C(E(0)) \int_0^T |w|^2_{0, \Omega} \, dt
\]
where the constants do not depend on \(T\).

**Proof.** We start by noting some of the properties of the operators \(A, G_1, G_2\) and \(A_2\) defined in section 3.2. The key point is that the operator \(A_2\) generates an analytic and exponentially stable semigroup on \(H_2 = H^2_{1,0}(\Omega) \times L_2(\Omega) \times L_2(\Omega)\)-see [28]. Moreover, \(A_2\) is m-dissipative and \(A_2^{-1}\) is bounded on \(H_2\). We have
\[
D(A_2) = D(A) \times D(A^{\frac{1}{2}}) \times D(A_N)
\]
\[
D(A^{\frac{1}{2}}) \subset D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})
\]
so that
\[
[D(A^{\frac{1}{2}})]' \supset [D(A^{\frac{1}{2}})]' \times [D(A^{\frac{1}{2}})]' \times [D(A^{\frac{1}{2}})]'\]
Application of Green’s Theorem [29] gives
\[
G_2^*Av = -v|_{\Gamma_1}; \quad G_1^*Av = -\frac{\partial}{\partial \nu}v \bigg|_{\Gamma_1}; \quad v \in H^2(\Omega)
\]
Now, using the abstract model (16b), we will write the solution to the plate component with the semigroup formula.
\[
\begin{bmatrix}
w(t) \\
w_1(t) \\
\theta(t)
\end{bmatrix} = e^{A_2^t} \begin{bmatrix}
w_0 \\
w_1 \\
\theta_0
\end{bmatrix}
+ \int_0^t e^{A_2(t-s)} \begin{bmatrix}
\quad 0 \\
\quad 0 \\
\quad 0
\end{bmatrix} \begin{bmatrix}
\nabla(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w]
\end{bmatrix} \, ds
\]
We next obtain
\[
A^{\frac{1}{2}}_2 \begin{bmatrix}
w(t) \\
w_1(t) \\
\theta(t)
\end{bmatrix} = A^{\frac{1}{2}}_2 e^{A_2^t} \begin{bmatrix}
w_0 \\
w_1 \\
\theta_0
\end{bmatrix}
+ \int_0^t A_2 e^{A_2(t-s)} A^{-\frac{1}{2}}_2 \begin{bmatrix}
\quad 0 \\
\quad 0 \\
\quad 0
\end{bmatrix} \begin{bmatrix}
\nabla(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w]
\end{bmatrix} \, ds
\]
Because of the analyticity of \(A_2\), we can apply the following singular integral estimates [7, 29]:
\[
\int_0^t A_2 e^{A_2(t-s)} f(s) \, ds \bigg|_{L_2(0, T; H_2)} \leq C |f|_{L_2(0, T; H_2)} \]
\[
|A_2^{\alpha} e^{A_2 t} x|_{L_2(0, T; H_2)} \leq C |x|_{H_2}; \quad \alpha \leq \frac{1}{2}
\]
where we are able to allow the critical exponent of $\alpha = \frac{1}{2}$ due to the fact that $A_2$ is m-dissipative and invertible\cite{7}. Thus we have

\[
\begin{bmatrix}
\frac{1}{2} A_2 & \frac{1}{2} A_2 \\
\frac{1}{2} A_2 & \frac{1}{2} A_2
\end{bmatrix}
\begin{bmatrix}
w(t) \\
w(t)
\end{bmatrix}

\leq C
\begin{bmatrix}
w_0 \\
w_1
\end{bmatrix}

\quad \text{in } L^2(0,T;H^2)

+ C \left[ A_2^{-\frac{1}{2}} \left[ \frac{1}{2} \right] \int_0^T \left[ \text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w] \right] \right] \quad \text{in } L^2(0,T;H^2)

\tag{29}
\]

Using (24) gives

\[
\left[ A_2^{-\frac{1}{2}} \left[ \frac{1}{2} \right] \int_0^T \left[ \text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w] \right] \right] \leq C \left[ \text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w] \right] \quad \text{in } L^2(0,T;H^2)

\tag{30}
\]

Sobolev’s imbeddings $H^{2+\epsilon}(\Omega) \subset L_\infty(\Omega)$ and equation (25) means that with any $\xi \in H^1(\Omega)$,

\[
\text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w]\xi_\Omega

= - (CN(u, w)\nabla w + \varphi \nabla w, \nabla \xi)_{\Omega}

\leq |(CN(u, w)\nabla w, \nabla \xi)_{\Omega}| + |(\varphi \nabla w, \nabla \xi)_{\Omega}|

\leq C |\xi|_{1,\Omega} (|N(u, w)|_{0,\Omega} |w|_{2+\epsilon,\Omega} + |\varphi|_{0,\Omega} |w|_{2+\epsilon,\Omega})

\tag{31}
\]

This allows us to conclude that

\[
|\text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w]|_{-1,\Omega}

\leq C (|N(u, w)|_{0,\Omega} + |\varphi|_{0,\Omega}) |w|_{2+\epsilon,\Omega}

\tag{32}
\]

hence

\[
\left[ A_2^{-\frac{1}{2}} \left[ \frac{1}{2} \right] \int_0^T \left[ \text{div}(CN(u, w)\nabla w + \varphi \nabla w) + AG_2[CN(u, w)\nu \nabla w] \right] \right] \leq C (|N(u, w)|_{0,\Omega} + |\varphi|_{0,\Omega}) |w|_{2+\epsilon,\Omega}

\tag{33}
\]

Using the interpolation inequality $|w|_{s+\alpha,\Omega} \leq |w|_{0,\Omega}^{\frac{s}{s+\alpha}} |w|_{t,\Omega}^{\frac{t}{s+\alpha}}$ for $0 \leq \alpha \leq 1$, Young’s inequality, the definition of $D(A_2^{\frac{1}{2}})$, (29), (33), and the dissipation equality (42), we have that

\[
\int_0^T |w|_{3,\Omega}^2 + |w|_{1,\Omega}^2 \, dt \leq CE(0) + C(E(0)) \int_0^T |w|_{2+\epsilon,\Omega}^2 \, dt

\leq CE(0) + \epsilon \int_0^T |w|_{3,\Omega}^2 \, dt + C_\epsilon(E(0)) \int_0^T |w|_{0,\Omega}^2 \, dt

\]

Taking $\epsilon$ small gives the result (23). \qed
4.3. **Uniqueness of weak solutions.** Assume \((u_1, w_1, \varphi_1, \theta_1)\) and \((u_2, w_2, \varphi_2, \theta_2)\) are two solutions of finite energy of the system (1). Consider \(\tilde{u} = u_1 - u_2, \tilde{w} = w_1 - w_2, \tilde{\varphi} = \varphi_1 - \varphi_2\) and \(\tilde{\theta} = \theta_1 - \theta_2\). Then the system (16) becomes, in variables \(\tilde{u}, \tilde{w}, \tilde{\varphi}, \tilde{\theta}\),

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} \tilde{u} \\ \tilde{u}_t \\ \tilde{\varphi} \\ \tilde{\varphi}_t \\ \tilde{\theta} \end{bmatrix} &= A_1 \begin{bmatrix} \tilde{u} \\ \tilde{u}_t \\ \tilde{\varphi} \\ \tilde{\varphi}_t \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ f_1(\tilde{w}) \\ f_3(\tilde{w}, \tilde{w}_t) \end{bmatrix} \\
\frac{d}{dt} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\varphi} \\ \tilde{\varphi}_t \\ \tilde{\theta} \end{bmatrix} &= A_2 \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\varphi} \\ \tilde{\varphi}_t \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ f_2(\tilde{u}, \tilde{w}, \tilde{\varphi}) \\ 0 \end{bmatrix}
\end{align*}
\]

(34a) (34b)

where we can rewrite the forcing expressions as

\[
\begin{align*}
f_1(\tilde{w}) &= \text{div} \ C[f(\nabla w_1 - f(\nabla w_2))] \\
f_2(\tilde{u}, \tilde{w}, \tilde{\varphi}) &= \text{div}[CN(\mathbf{u}_1, w_1)\nabla w_1 - CN(\mathbf{u}_2, w_2)\nabla w_2 + \tilde{\varphi}\nabla w_1 + \varphi_2 \nabla \tilde{w}] \\
&\quad + AG2(CN(\mathbf{u}_1, w_1)\nu\nabla w_1 - CN(\mathbf{u}_2, w_2)\nu\nabla w_2)
\end{align*}
\]

(34c) (34d)

A standard energy method on (34a) and estimates allow us to conclude

\[
\left\| \begin{bmatrix} \tilde{u}(t) \\ \tilde{u}_t(t) \\ \tilde{\varphi}(t) \end{bmatrix} \right\|_{H^1} + \left[ \int_0^t \left( |\nabla \tilde{\varphi}|^2_{0, \Omega} + b|D\tilde{u}_t|^2_{0, \Omega} \right) ds \right]^{\frac{1}{2}} \leq C \int_0^t |f_1|_{0, \Omega} ds + C \left[ \int_0^t |A_D^{-\frac{\delta}{2}} f_3|^2_{0, \Omega} ds \right]^{\frac{1}{2}}
\]

(35)

We need to estimate the norms of \(f_i\). Following [5] we can derive that

\[
\begin{align*}
|f_1(\tilde{w})|_{0, \Omega} &\leq C|\tilde{w}|_{2, \Omega}(w_1 + w_2)|_{3, \Omega} \\
A_D^{-\frac{\delta}{2}} f_3(\tilde{w}, \tilde{w}_t)_{0, \Omega} &\leq \epsilon |\tilde{w}_t|_{0, \Omega} |w_2|_{3, \Omega} + |\tilde{w}|_{2, \Omega} |w_1, t|_{1, \Omega} \\
&\quad + C|\tilde{w}_t|_{0, \Omega} |w_2|_{2, \Omega} + |\tilde{w}|_{2, \Omega} |w_1, t|_{0, \Omega}
\end{align*}
\]

(36) (37)

Applying these to (35) and estimating gives

\[
\left\| \begin{bmatrix} \tilde{u}(t) \\ \tilde{u}_t(t) \\ \tilde{\varphi}(t) \end{bmatrix} \right\|_{H^1} \leq C(r(E(0)) \left[ \int_0^t |\tilde{w}(s)|^2_{0, \Omega} ds + \int_0^t |\tilde{w}_t|^2_{0, \Omega} ds \right]^{\frac{1}{2}}
\]

\[
+ \epsilon C(E(0)) |\tilde{w}_t|_{L^\infty(0, t; L^2(\Omega))} + |\tilde{w}|_{L^\infty(0, t; H^2(\Omega))} \]

(38)

On the other hand, to estimate the energy of the plate components we can proceed similarly to (33).
where \( \rho \) can be taken arbitrarily small. The analyticity of \( e^{A_2 t} \) and (39) give
\[
\begin{bmatrix}
\dot{\tilde{w}} \\
\dot{\tilde{\theta}}
\end{bmatrix}_H 
\leq \left| \int_0^t A_2^{1+\rho} e^{A_2 (t-s)} A_2^{-1-\rho} \begin{bmatrix}
0 \\
0
\end{bmatrix} \cdot f_2(\tilde{u}, \tilde{w}, \tilde{\varphi}) \right| ds
\leq C(E(0)) \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\rho}} \left[ |\tilde{u}(s)|_{1,\Omega} + |\tilde{w}(s)|_{2,\Omega} + |\tilde{\varphi}(s)|_{0,\Omega} \right] ds
\leq C(E(0)) \left[ \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\rho}} ds \right]^{\frac{1}{2}} \cdot \left[ \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\rho}} \left[ |\tilde{u}(s)|_{1,\Omega}^2 + |\tilde{w}(s)|_{2,\Omega}^2 + |\tilde{\varphi}(s)|_{0,\Omega}^2 \right] ds \right]^2
\]
From (38) and (40), taking \( \epsilon \) suitably small yields
\[
|\tilde{u}(t)|_{1,\Omega}^2 + |\tilde{\varphi}(t)|_{0,\Omega}^2 + |\tilde{u}|_{0,\Omega}^2 + |\tilde{\varphi}(t)|_{0,\Omega}^2 + \tilde{\theta}(t)|_{0,\Omega}^2 
\leq C_\epsilon(E(0)) \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\rho}} \left[ |\tilde{u}(s)|_{1,\Omega} + |\tilde{w}(s)|_{2,\Omega} + |\tilde{\varphi}(s)|_{0,\Omega} \right] ds
\]
This inequality and Gronwall’s inequality with \( L_1 \) kernel yield that
\[
\tilde{u} = 0, \quad \tilde{w} = \tilde{\varphi} = \tilde{\theta} = 0
\]
so the solutions must be unique, as claimed in Theorem 1.1.

The same argument as above, when applied to two solutions with different initial data, provides continuous dependence of solutions in strong topology on the initial data. Taking into consideration that energy equality holds for strong solutions (see Remark 4), continuous dependence on initial data allows to replace the energy inequality by the corresponding energy equality. Thus we obtain:

**Corollary 1 (Dissipativity Equality).** Let \((u, w, \varphi, \theta)\) be a finite energy solution to the system (1a)-(1e). Then for any \( s \leq t \)
\[
E(t) + 2 \int_s^t \lambda_1 |\theta|_{0,1,\Omega}^2 dt + 2 \int_s^t \left[ b |\text{curl } q|_{0,\Omega}^2 + |\nabla \varphi|_{0,\Omega}^2 + |\nabla \theta|_{0,\Omega}^2 \right] dt = E(s) \]

The proof of Theorem 1.1 is thus complete. \( \square \)

The fact that the energy equality (42) is valid for weak solutions is the key in obtaining stabilizability estimates.

5. **Stabilizability – Proof of Theorem 1.3.** The proof of uniform decay rates for the energy function proceeds through the classical main steps which involve inverse type of estimates – recovering potential and kinetic energy from the damping. However, due to the strong nonlinearity and high order boundary conditions, such recovery estimates will be “polluted” by higher order terms which can not be topologically absorbed by the energy (as expected from a standard approach). Thus, our challenge will be to devise appropriate estimates in order to handle such terms. Here is the first property: boundary regularity for in-plane displacements which does not follow from the regularity enjoyed by the state space.
5.1. Hidden boundary regularity of in plane displacements. We begin with a “hidden regularity” result which will be proved to hold for horizontal displacements $u$. Such a result is by now standard for the wave equation with Dirichlet boundary data [24]; it is also known for the linear dynamic system of elasticity [2] and for the von Kármán plate with clamped boundary conditions [4]. However, it does fail for vertical displacements with free boundary conditions (as considered in this paper). Thus, the arguments of [4] are no longer applicable. In order to overcome the difficulty we shall use the method developed in [2] along with partial analyticity developed in the previous section.

Lemma 5.1. Let $y \equiv (u, w, \theta, \phi)$ be a weak solution corresponding to the original model (1) with the initial energy $E(0)$. Then, for every $T > 0$ and for every $\epsilon > 0$ there exists a constant $C > 0$ such that the following boundary regularity regularity holds:

$$
\int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 \leq CTE(0) + \epsilon E^{3/2}(0) + C_\epsilon(E(0)) \int_0^T |w|^2_{\delta, \Omega} \tag{43}
$$

where $C_\epsilon(E(0))$ is a continuous and increasing function in $E(0)$.

Proof. This type of “hidden regularity” estimate has been proved in [4], however in the case of plate with clamped boundary conditions. The argument in that paper breaks down when one has free boundary conditions imposed. These introduce terms which cannot be handled – because the plate with free boundary conditions does not satisfy the Lopatinski condition, hence hidden regularity fails there [34]. However, our result pertains only to hidden regularity of horizontal displacements, hence the validity of “partial” hidden regularity is by no means precluded. In order to exhibit this phenomenon our starting point are the calculations in [2] performed for linear system of dynamic elasticity with zero boundary conditions:

$$
\int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 \leq CE(0) + C \int_0^T |u|_{0, \Omega}^2 + |\varepsilon(u)|_{0, \Omega}^2
$$

$$
+ C \int_0^T \int_{\Omega} |\text{div}[f(\nabla w)]\nu D\mathbf{u}|dxdt + C \int_0^T \int_{\Omega} \left| \nabla \phi D\mathbf{u}\nu \right| dxdt \tag{44}
$$

On the other hand,

$$
\int_0^T \int_{\Omega} |\text{div}[f(\nabla w)]\nu D\mathbf{u}|dxdt \leq C \int_0^T |\Delta w \nabla w|_{0, \Omega} |\mathbf{u}|_{1, \Omega} dt
$$

and by (42)

$$
\leq CE^{1/2}(0) \int_0^T |\Delta w \nabla w|_{0, \Omega}
$$

By Sobolev’s embeddings in 2-D, interpolation inequality and Young’s inequality we have

$$
|\Delta w \nabla w|_{0, \Omega} \leq C|w|_{2, \Omega} |w|_{W^{1,\infty}(\Omega)} \leq C|w|_{2, \Omega} |w|_{2+\epsilon, \Omega}
$$

$$
\leq C|w|_{2, \Omega} \left[ |w|_{3, \Omega} + C_\epsilon |w|_{2, \Omega} \right] \leq C_\epsilon |w|_{2, \Omega}^2 + \epsilon |w|_{3, \Omega}^2 \tag{45}
$$

and by Lemma 4.3

$$
\int_0^T |\Delta w \nabla w|_{0, \Omega} \leq C_\epsilon \int_0^T |w|_{3, \Omega}^2 + C_\epsilon \int_0^T |w|_{2, \Omega}^2 \leq \epsilon CE(0) + C_\epsilon(E(0)) \int_0^T |w|_{0, \Omega}^2 \tag{46}
$$
Putting everything together and applying inequality (42) leads to
\[
\int_0^T \int_\Omega |\text{div}(f(\nabla w))\nu Du| dx dt \leq \epsilon E^{3/2}(0) + C_\epsilon(E(0)) \int_0^T |w|_{2,\Omega}^2 dt
\] (47)
Noting also that by using, again, (42) we obtain
\[
\int_0^T \int_\Omega |\nabla \phi Du| \leq C_\epsilon \int_0^T |\phi|_{1,\Omega}^2 + \epsilon \int_0^T |u|_{1,\Omega}^2 dt \leq CE(0) + TE(0)
\] (48)
Collecting (47), (48) and (44) which then concludes the proof. \( \square \)

5.2. Recovery of potential and kinetic energy. We begin by building an estimate for the recovery of kinetic energies. This will be done by applying several nonlocal multipliers to the equations involved. Since the calculations involve regularity higher than that of weak solutions, we shall perform computations initially on strong solutions –see Theorem 1.2. A density argument will allow us to extend the estimates to all weak solutions.

**Lemma 5.2.** Estimate for \( |u_t|_{0,\Omega}^2 \). Let \( u, w, \theta, \varphi \) be a regular solution to (1). Then for any \( T > 0 \), any \( \epsilon, \epsilon_1 \) there exists constants \( C_\epsilon, C_{\epsilon_1} \), such that the following estimate holds.

\[
\int_0^T |u_t|_{0,\Omega}^2 \leq CE(0) + \epsilon_1 \int_0^T E_p(t) dt + CE(0) \int_0^T |w|_{2,\Omega}^2 dt + \epsilon \int_0^T |\nabla \phi|_{1,\Omega}^2 dt + C_{\epsilon,\epsilon_1}(E(0)) \int_0^T (|\varphi|_{1,\Omega}^2 + |\text{curl} q_t|_{0,\Omega}^2) dt
\] (49)

**Proof.** We apply nonlocal multiplier \( \nabla A_D^{-1} \varphi \) to the first equation in (1a).
\[
\int_0^T \left( u_{tt} + b \text{curl} q_t - \text{div}(C[\epsilon(u) + f(\nabla w)]) + \nabla \varphi, \nabla A_D^{-1} \varphi \right)_\Omega dt = 0
\] (50)
use the facts that \( A_D^{-1} \Delta y = -y + Dy|_\Gamma \), \( u = \nabla p + \text{curl} q \), and \( \text{div curl} \psi = 0 \) for any vector \( \psi \). So, for the first term:
\[
\int_0^T \left( u_{tt}, \nabla A_D^{-1} \varphi \right)_\Omega dt = \left. (u_t, \nabla A_D^{-1} \varphi) \right|_0^T - \int_0^T \left( u_t, \nabla A_D^{-1} \varphi \right)_\Omega dt
\]
\[
= \left( u_t, \nabla A_D^{-1} \varphi \right)_\Omega \left. \right|_0^T - \int_0^T \left( u_t, \nabla A_D^{-1} (\Delta \varphi - \text{div} u_t + \nabla w \cdot \nabla w_t) \right)_\Omega dt
\]
\[
= \left( u_t, \nabla A_D^{-1} \varphi \right)_\Omega \left. \right|_0^T + \int_0^T \left( u_t, \nabla (\varphi - D\varphi|_\Gamma) \right)_\Omega dt
\]
\[
+ \int_0^T \left( u_t, \nabla A_D^{-1} (\text{div} (\nabla p_t + \text{curl} q_t)) \right)_\Omega dt
\]
\[
- \int_0^T \left( u_t, \nabla A_D^{-1} (\nabla w \cdot \nabla w_t) \right)_\Omega dt
\]
Finally, after using where we take \( q \) above can be taken arbitrarily small on the strength of Sobolev's embeddings in 2-D. Similarly

\[
\begin{align*}
&= (u_t, \nabla A_D^{-1} \varphi)_{\Omega}^T + \int_0^T (u_t, (\nabla \varphi - \nabla p_t))_{\Omega} dt \\
&\quad - \int_0^T (u_t, \nabla A_D^{-1} (\nabla w \cdot \nabla w_t))_{\Omega} dt \\
&= (u_t, \nabla A_D^{-1} \varphi)_{\Omega}^T + \int_0^T (\nabla p_t + \text{curl } q_t, (\nabla \varphi - \nabla p_t))_{\Omega} dt \\
&\quad - \int_0^T (u_t, \nabla A_D^{-1} (\nabla w \cdot \nabla w_t))_{\Omega} dt \\
&= (u_t, \nabla A_D^{-1} \varphi)_{\Omega}^T + \int_0^T (\nabla p_t, \nabla \varphi)_{\Omega} dt - \int_0^T |\nabla p_t|_{0,\Omega}^2 dt \\
&\quad - \int_0^T (u_t \cdot \nu, A_D^{-1}(\nabla w \cdot \nabla w_t))_{\Gamma} dt \\
&\quad + \int_0^T (\text{div } u_t, A_D^{-1}(\nabla w \cdot \nabla w_t))_{\Omega} dt
\end{align*}
\]

With \( u = 0 \) the terms on \( \Gamma \) equal zero. Now we have:

\[
(u_t, \nabla A_D^{-1} \varphi)_{\Omega}^T \leq CE(0) + CE(T)
\]

\[
\int_0^T (\nabla p_t, \nabla \varphi)_{\Omega} dt \leq \epsilon \int_0^T |\nabla p_t|_{0,\Omega}^2 dt + C_\epsilon \int_0^T |\nabla \varphi|_{0,\Omega}^2 dt
\]

For the nonlinear term we have (noting \( p_t = 0 \) on \( \Gamma \) and \( p_t = A_D^{-1} \text{div } u_t \) ):

\[
(\text{div } u_t, A_D^{-1}(\nabla w \cdot \nabla w_t))_{\Omega} = (\text{div}(\nabla p_t), A_D^{-1}(\nabla w \cdot \nabla w_t))_{\Omega} \leq CE(0) + CE(T)
\]

\[
\int_0^T (\nabla p_t, \nabla \varphi)_{\Omega} dt \leq (\text{div } u_t, A_D^{-1}(\nabla w \cdot \nabla w_t))_{\Omega}
\]

\[
(p_t \Delta w, w_t)_{\Omega} \leq |w_t|_{0,\Omega} |p_t \Delta w|_{0,\Omega}
\]

\[
\leq |w_t|_{0,\Omega} |p_t|_{L_{\infty}(\Omega)} |\nabla w|_{L_{\infty}(\Omega)}
\]

\[
\leq C |w_t|_{0,\Omega} |p_t|_{L_{\infty}(\Omega)} |\nabla w|_{1+\rho,\Omega}
\]

Similarly

\[
(p_t \Delta w, w_t)_{\Omega} \leq |w_t|_{0,\Omega} |p_t \Delta w|_{0,\Omega}
\]

\[
\leq |p_t|_{L_{\infty}(\Omega)} |\Delta w|_{L_{\infty}(\Omega)} |w_t|_{0,\Omega}
\]

\[
\leq C |w_t|_{0,\Omega} |p_t|_{1,\Omega} |\Delta w|_{\rho,\Omega}
\]

where we take \( q \to \infty \), and \( 2q \to \rho \). The constant \( \rho \) above can be taken arbitrarily small on the strength of Sobolev's embeddings in 2-D.
interpolation and the Gagliardo-Nirenberg inequality, we have
\[ \rho \leq \epsilon_1 |\nabla p_i|_{0,\Omega} + \epsilon_2 |p_i|_{1,\Omega} \]
\[ + C_{\epsilon_1, \epsilon_2} |w_i|_{0,\Omega} (|\Delta w_i|_{\rho,\Omega} + |\nabla w_i|_{1,\rho,\Omega}) \]
\[ \leq \epsilon_3 |\nabla p_i|_{0,\Omega}^2 + C_{\epsilon_3} |w_i|_{0,\Omega}^2 |w_i|_{2,\rho,\Omega} \]
where \( \rho \) can be taken arbitrary small. Thus we have
\[ \int_0^T (\text{div } \mathbf{u}_t, A_D^{-1} (\nabla w \cdot \nabla w_i))_{0,\Omega} \, dt \leq \epsilon_3 \int_0^T |\nabla p_i|_{0,\Omega}^2 \, dt + C_{\epsilon_3} E(0) \int_0^T |w_i|_{2,\rho,\Omega}^2 \, dt \]
(60)
So, using inequalities (51),(52) and (60) gives
\[ \int_0^T (\mathbf{u}_{tt}, \nabla A_D^{-1} \varphi)_{0,\Omega} \, dt \leq C[E(0) + E(T)] + \]
\[ \epsilon_3 \int_0^T |\nabla p_i|_{0,\Omega}^2 \, dt + C_{\epsilon_3} E(0) \int_0^T |w_i|_{2,\rho,\Omega}^2 \, dt \]
(61)
and taking \( \epsilon + \epsilon_3 \leq \frac{1}{2} \) we have
\[ \int_0^T (\mathbf{u}_{tt}, \nabla A_D^{-1} \varphi)_{0,\Omega} \, dt \leq C[E(0) + E(T)] - \frac{1}{2} \int_0^T |\nabla p_i|_{0,\Omega}^2 \, dt + C \int_0^T |\nabla \varphi|_{0,\Omega}^2 \, dt \]
\[ + CE(0) \int_0^T |w_i|_{2,\rho,\Omega}^2 \, dt \]
(62)
which gives
\[ \int_0^T |\nabla p_i|^2 \, dt \leq CE(0) + C \int_0^T |\varphi|^2_{\Omega,\Omega} \, dt \]
\[ + CE(0) \int_0^T |w_i|^2_{2,\rho,\Omega} \, dt - \int_0^T (\mathbf{u}_{tt}, \nabla A_D^{-1} \varphi)_{0,\Omega} \, dt \]
(63)
For the last term in (63) we use the equation (50). This leads to the estimates
\[ \int_0^T (b \text{curl } \mathbf{q}_t, \nabla A_D^{-1} \varphi)_{0,\Omega} \, dt = b \int_0^T (\text{curl } \mathbf{q}_t, A_D^{-1} \varphi)_{\Gamma} - (\text{div } \text{curl } \mathbf{q}_t, A_D^{-1} \varphi)_{0,\Omega} \, dt = 0 \]
(64)
\[ - \int_0^T (\text{div } (C\varepsilon(\mathbf{u}) + f(\nabla w)), \nabla A_D^{-1} \varphi)_{0,\Omega} \, dt = \]
\[ - \int_0^T (C\varepsilon(\mathbf{u}) + f(\nabla w))_\Gamma \, dt + \int_0^T (C\varepsilon(\mathbf{u}) + f(\nabla w), \varepsilon(\nabla A_D^{-1} \varphi))_{\Omega,\Omega} \]
(65)
\[
\int_0^T (\text{div}(C[u] + f(vv)), (\nabla A_D^{-1} \varphi))_\Omega dt \leq \int_0^T (C[u] + f(vv))[x, \nabla A_D^{-1} \varphi]_\Gamma dt \\
+ \epsilon_1 \int_0^T \|N(u, v)\|^2_{\Omega} + C_\epsilon \int_0^T |\varphi|^2_{\Omega} dt \\
\leq \epsilon_1 \int_0^T \|N(u, v)\|^2_{\Omega} dt + \epsilon_2 \int_0^T \|Du\|^2_{\Omega} dt^{1/2} \left[ \int_0^T |\nabla A_D^{-1} \varphi|_{\Gamma_1}^2 dt \right]^{1/2} \\
+ \epsilon_3 \int_0^T |\nabla w|^2_{\Omega} |\nabla A_D^{-1} \varphi|_{\Gamma_1} + C_\epsilon \int_0^T |\varphi|^2_{\Omega} dt \\
\leq \epsilon_1 \int_0^T \|N(u, v)\|^2_{\Omega} dt + \epsilon_2 \int_0^T \|Du\|^2_{\Omega} dt \\
+ E^{1/2} \int_0^T \|w\|^2_{\Omega} |\varphi|_{\Omega_1} + C_\epsilon \int_0^T |\varphi|^2_{\Omega} dt \\
\leq \epsilon_1 \int_0^T \|N(u, v)\|^2_{\Omega} dt + \epsilon_2 \int_0^T \|Du\|^2_{\Omega} dt \\
+ \epsilon_3 \int_0^T |w|^2_{\Omega} + C_\epsilon E(0) \int_0^T |\varphi|^2_{\Omega} + C_\epsilon \int_0^T |\varphi|^2_{\Omega} dt \quad (66)
\]

where we have used: \( H^{1/2}(\Gamma) \subset L_2(\Gamma) \), trace theorem and \( A_D^{-1} \in L(L_2(\Omega) \to H^2(\Omega)) \). Finally, since \( \varphi|_\Gamma = 0 \), by Divergence Theorem

\[
\int_0^T (\nabla \varphi, \nabla A_D^{-1} \varphi)_\Omega dt \leq \int_0^T (\nabla \varphi, \nabla A_D^{-1} \varphi)_\Gamma - \int_0^T (\varphi, \text{div}(\nabla A_D^{-1} \varphi))_\Omega dt \\
\leq \int_0^T \|\varphi\|^2_{\Omega} dt \leq C \int_0^T |\varphi|^2_{\Omega} dt \\
(67)
\]

since \( \nabla A_D^{-1} : L_2(\Omega) \to H^1(\Omega) \) is bounded. Inserting the estimates from (64), (66), and (67) into (63) and (50) gives

\[
\int_0^T \|w_t\|^2 \leq CE(0) + CE(0) \int_0^T |w|^2_{\Omega} + \epsilon \int_0^T \|N(u, v)\|^2_{\Omega} + |w|^2_{\Omega} \\
+ \epsilon_2 \int_0^T |Du|^2_{\Omega} + C_\epsilon \|1 + E(0)\| \int_0^T |\varphi|^2_{\Omega} dt \\
(68)
\]

Combining (62), (64), (66) and (67) and using the fact that \( u_t = \nabla p_t + \text{curl} \ q_t \) gives the estimate in Lemma 5.2.

\[\square\]

**Lemma 5.3.** Estimate for kinetic energy of \( |w_t|^2_\Omega \) Let \( u, w, \theta, \varphi \) be a regular solution to (1). Then for any \( T > 0 \) we have the following estimate:

\[
\int_0^T |w_t|^2_\Omega dt \leq CE(0) + C \int_0^T |w_t|^2_{\Gamma_1} dt + \epsilon \int_0^T |\Delta w|^2_{\Gamma_1} dt \\
+ \epsilon_2 \int_0^T E_p(t) dt + C_\epsilon \epsilon_2 E(0) \int_0^T (|\varphi|^2_{\Omega_1} + |\theta|^2_{\Omega_1}) dt \quad (69)
\]
Proof. We apply the multiplier $A_D^{-1} \theta$ to the plate equation. This operator multiplier for the thermoelastic plate equations was introduced in [1]. Here we recall

$$A_D : L_2(\Omega) \rightarrow L_2(\Omega), \quad A_D \equiv -\Delta; \quad D(A_D) \equiv H^2(\Omega) \cap H_0^1(\Omega)$$

$$D : H^{-1/2}(\Gamma) \rightarrow L_2(\Omega), \quad v = Dg \text{ iff } \Delta v = 0 \text{ in } \Omega, \quad v = g \text{ on } \Gamma$$

Multiplying by $A_D^{-1} \theta$ and integrating by parts gives

$$\begin{align*}
(w_t, A_D^{-1} \theta)_\Omega &+ a(w, A_D^{-1} \theta) + \left( D(\frac{\partial}{\partial \nu} \Delta w + (1 - \mu)B_2 w), A_D^{-1} \theta \right)_\Gamma \\
&= -\left( D(\Delta w + (1 - \mu)B_1 w), \frac{\partial}{\partial \nu} A_D^{-1} \theta \right)_\Gamma - (C[\varepsilon(u) + f(\nabla w)]\nu \nabla w, A_D^{-1} \theta)_\Gamma \\
&\quad+ (\varphi \nabla w, \nabla A_D^{-1} \theta)_\Omega + (\frac{\partial}{\partial \nu} \theta, A_D^{-1} \theta)_\Gamma - (\nabla \theta, \nabla A_D^{-1} \theta)_\Omega = 0 \quad (70)
\end{align*}$$

Applying boundary conditions and integrating in time gives

$$\begin{align*}
\int_0^T [(w_t, A_D^{-1} \theta)_\Omega &+ a(w, A_D^{-1} \theta) + (C[\varepsilon(u) + f(\nabla w)]\nu \nabla w)_\Omega \\
&- (\Delta w, \frac{\partial}{\partial \nu} A_D^{-1} \theta)_\Gamma + (\theta, \frac{\partial}{\partial \nu} A_D^{-1} \theta)_\Gamma \\
&+ (\varphi \nabla w, \nabla A_D^{-1} \theta)_\Omega - (\nabla \theta, \nabla A_D^{-1} \theta)_\Omega] dt = 0 \quad (71)
\end{align*}$$

We take all the terms in turn. Using the fact that $A_D^{-1} \Delta u = -u + Du|_\Gamma; \; u \in H^2(\Omega)$ and substituting the $\theta$-equation in (1a) allows us to say that

$$\int_0^T (w_t, A_D^{-1} \theta_t)_\Omega dt = \int_0^T \left[ -(w_t, \theta)_\Omega + (w_t, D\theta|_\Gamma)_\Omega - |w_t|_{0, \Omega}^2 + (w_t, Dw_t|_\Gamma)_\Omega \right] dt \quad (72)$$

The last term in (72) is estimated via regularity of Dirichlet map as follows:

$$(w_t, Dw_t|_\Gamma)_\Omega \leq C|w_t|_{0, \Omega}|w_t|_{-1/2, \Gamma_1} \leq \epsilon_0 |w_t|_{0, \Omega}^2 + C_\epsilon |w_t|_{-1/2, \Gamma_1}^2 \quad (73)$$

The next terms, using trace theory and elliptic regularity of $A_D^{-1}$, give:

$$\begin{align*}
\int_0^T a(w, A_D^{-1} \theta) dt &\leq \epsilon \int_0^T |w|_{2, \Omega}^2 dt + C_\epsilon \int_0^T |\theta|_{0, \Omega}^2 dt \quad (74) \\
\int_0^T [(\nabla \theta, \nabla A_D^{-1} \theta)_\Omega + (\theta, \frac{\partial}{\partial \nu} A_D^{-1} \theta)_\Gamma] dt &\leq C \int_0^T |\theta|_{1, \Omega}^2 dt \quad (75) \\
\int_0^T (D\Delta w, \frac{\partial}{\partial \nu} A_D^{-1} \theta)_\Gamma dt &\leq \epsilon \int_0^T |\Delta w|_{-1/2, \Gamma_1}^2 dt + C_\epsilon \int_0^T |\theta|_{0, \Omega}^2 dt \quad (76)
\end{align*}$$
Using Holder’s inequality, Sobolev embeddings, the dissipation equality (42) and the regularity of \( A^{-1}_D \) give:

\[
\int_0^T (CN(u, w), \nabla A^{-1}_D \theta \otimes \nabla w)_{\Omega} dt \leq \epsilon \int_0^T |N(u, w)|^2_{0, \Omega} dt + C \epsilon \int_0^T |\nabla A^{-1}_D \theta \otimes \nabla w|^2_{0, \Omega} dt
\]

\[
\leq \epsilon \int_0^T |N(u, w)|^2_{0, \Omega} dt + C \epsilon \int_0^T |\nabla w|^2_{L^4(\Omega)} |\nabla A^{-1}_D \theta|^2_{L^4(\Omega)} dt
\]

\[
\leq \epsilon \int_0^T |N(u, w)|^2_{0, \Omega} dt + C \epsilon \int_0^T |w|^2_{L^2(\Omega)} |\theta|^2_{\Omega, \Omega} dt
\]

\[
\leq \epsilon \int_0^T |N(u, w)|^2_{0, \Omega} dt + C \epsilon \int_0^T |\theta|^2_{\Omega, \Omega} dt \quad (77)
\]

\[
\int_0^T (\varphi \nabla w, \nabla A^{-1}_D \theta)_{\Omega} dt \leq \int_0^T |\varphi \nabla w|_{0, \Omega} |\nabla A^{-1}_D \theta|_{0, \Omega} dt
\]

\[
\leq \int_0^T |\varphi|_{L^4(\Omega)} |\nabla w|_{L^4(\Omega)} |\nabla A^{-1}_D \theta|_{0, \Omega} dt
\]

\[
\leq C \int_0^T |\varphi|_{1, \Omega} |w|_{2, \Omega} |\theta|_{1, \Omega} dt
\]

\[
\leq C E(0) \int_0^T (|\varphi|^2_{1, \Omega} + |\theta|^2_{1, \Omega}) \, dt \quad (78)
\]

Taking in (73) \( \epsilon_0 = 1/2 \) and combining expressions (71)-(78) gives the estimate of Lemma 5.3 for the kinetic energy of the plate.

Lemma 5.2 and Lemma 5.3 cumulatively give the estimate for the kinetic energy of the plate:

**Lemma 5.4.** With reference to strong solutions in (1a) we have with any \( \rho > 0 \).

\[
\int_0^T E_k(t) dt \leq CE(0) + CE(0) \int_0^T |w|^2_{2+\rho, \Omega} dt + \epsilon_1 \int_0^T E_p(t) dt
\]

\[
+ \epsilon \int_0^T \left| \frac{\partial u}{\partial \nu} \right|^2_{0, \Gamma_1} dt + C \int_0^T |w_t|^2_{-1/2, \Gamma_1} dt + \epsilon \int_0^T |\Delta w|^2_{-1/2, \Gamma_0} dt
\]

\[
+ C_{\rho, \epsilon_1, \epsilon_2} E(0) \int_0^T (|\varphi|^2_{1, \Omega} + |\theta|^2_{1, \Omega} + |\text{curl} \, q_0|^2_{0, \Omega}) dt \quad (79)
\]

**Lemma 5.5.** Estimate for potential energy. Let \( u, w, \theta, \varphi \) be a regular solution to (1). Then for any \( T > 0 \) we have the following estimate:

\[
\frac{1}{4} \int_0^T E_p(t) dt \leq CE(0) + CE(T) + \int_0^T E_k(t) dt
\]

\[
+ CE(0) \int_0^T (|\varphi|^2_{1, \Omega} + |\theta|^2_{1, \Omega} + |\text{curl} \, q_0|^2_{0, \Omega}) \quad (80)
\]
Proof. We apply the multipliers \( u \) and \( \frac{1}{2}w \) to the wave and plate equation, respectively, integrate by parts, and apply the boundary conditions. This gives

\[
(u_t, u)_{\Omega}^0 - \int_0^T |u|_{0, \Omega}^2 dt + \int_0^T [(CN(u, w)\varepsilon(u))_{\Omega} + b(curl \ q_t, u)_{\Omega} + (\nabla \varphi, u)_{\Omega}] dt = 0 \quad (81)
\]

and

\[
\frac{1}{2}(w_t, w)_{\Omega}^T - \frac{1}{2} \int_0^T |w_t|_{0, \Omega}^2 dt + \int_0^T \left[ \frac{1}{2}a(w, w) + \frac{1}{2}(CN(u, w)\nabla w \otimes \nabla w)_{\Omega} \right] dt + \frac{1}{2} \int_0^T [(\varphi \nabla w, \nabla w)_{\Omega} - (\nabla \theta, \nabla w)_{\Omega}] dt = 0 \quad (82)
\]

Noting that the \( u \)-component of the potential energy equals

\[
(CN(u, w)N(u, w))_{\Omega} = (CN(u, w)\varepsilon(u))_{\Omega} + \frac{1}{2}(CN(u, w)\nabla w \otimes \nabla w)_{\Omega}
\]

we add together equations (81) and (82) which results in

\[
\int_0^T [\frac{1}{2}a(w, w) + (CN(u, w), N(u, w))_{\Omega}] dt = \int_0^T [||u||_{0}^2 + \frac{1}{2}w_t|_{0}^2] dt - (u_t, u)_{\Omega}^T - \frac{1}{2}(w_t, w)_{\Omega}^T - \int_0^T [b(curl \ q_t, u)_{\Omega} + \frac{1}{2}(\varphi \nabla w, \nabla w)_{\Omega}] dt + \int_0^T [\frac{1}{2}(\nabla \theta, \nabla w)_{\Omega} - (\nabla \varphi, u)_{\Omega}] dt \quad (83)
\]

We use the following estimates:

\[
\int_0^T |b(curl \ q_t, u)|_{\Omega} dt \leq \epsilon \int_0^T |u|_{0, \Omega}^2 dt + C_{\epsilon} \int_0^T |curl \ q_t|_{0, \Omega}^2 dt \quad (84)
\]

\[
\int_0^T (\varphi \nabla w, \nabla w)_{\Omega} dt \leq C \int_0^T |\varphi|_{0, \Omega} |w|_{2, \Omega}^2 dt \leq \epsilon C \int_0^T |w|_{2, \Omega}^2 dt + C_{\epsilon} \int_0^T |w|_{2, \Omega}^2 |\varphi|_{0, \Omega}^2 dt \quad (85)
\]

\[
\leq \epsilon C \int_0^T E_p(t) dt + C_{\epsilon} E(0) \int_0^T |\varphi|_{0, \Omega}^2 dt \quad (86)
\]

\[
\int_0^T \frac{1}{2}(\nabla \theta, \nabla w)_{\Omega} - (\nabla \varphi, u)_{\Omega} \leq \epsilon \int_0^T (|u|_{0, \Omega}^2 + |w|_{1, \Omega}^2) dt \quad (87)
\]

\[
+ C_{\epsilon} \int_0^T ||\nabla \varphi|_{0, \Omega}^2 + |\nabla \theta|_{0, \Omega}^2 dt \quad (88)
\]

\[
\leq \epsilon \int_0^T E_p(t) dt + C_{\epsilon} \int_0^T ||\nabla \varphi|_{0, \Omega}^2 + |\nabla \theta|_{0, \Omega}^2 dt \quad (89)
\]
Applying (84)-(90) to equation (83) and estimating gives

\[
\frac{1}{2} \int_{0}^{T} E_p(t) \, dt \leq \int_{0}^{T} \left( |u_{1,0,\Omega}|^2 + \frac{1}{2} |w_{1,0,\Omega}|^2 \right) \, dt + \frac{1}{2} \|u_{1,0,\Omega}\|^{T}_{0} + \frac{1}{2} \|w_{1,0,\Omega}\|^{T}_{0} + \epsilon \int_{0}^{T} |u_{1,0,\Omega}|^2 \, dt + \frac{1}{2} C \int_{0}^{T} E_p(t) \, dt + \frac{1}{2} C E(0) \int_{0}^{T} |\varphi_{0,\Omega}|^2 \, dt
\]

\[
+ \epsilon \int_{0}^{T} E_p(t) \, dt + C \epsilon \int_{0}^{T} |\varphi_{1,\Omega}|^2 + |\theta_{1,\Omega}|^2 + |\text{curl} \, q_{0,\Omega}|^2 \, dt \quad (91)
\]

which implies the result of Lemma 5.5.

\[\square\]

**Lemma 5.6. Total Energy Estimate** Let \( u, w, \theta, \varphi \) be a regular solution to (1a). Then for any \( T > 0 \) we have the following estimate:

\[
\int_{0}^{T} E(t) \, dt = \int_{0}^{T} (E_k(t) + E_p(t)) \, dt \leq C[E(0) + E(T)]
\]

\[
+ C \int_{0}^{T} \left( |w_{t}|_{-1/2, \Gamma}^2 + |\Delta w|_{-1/2, \Gamma}^2 + \epsilon \left| \frac{\partial}{\partial \nu} u_{1,0,\Gamma} \right|^2 \right) \, dt
\]

\[
+ C E(0) \int_{0}^{T} |w|_{2+\rho,\Omega}^2 + C \epsilon \int_{0}^{T} |\varphi_{1,\Omega}|^2 + |\theta_{1,\Omega}|^2 + |\text{curl} \, q_{0,\Omega}|^2 \, dt \quad (92)
\]

**Proof.** In order to derive (92) from the estimates in Lemmas 5.4 and 5.5, we combine the two estimates with suitable rescaling and appropriate choice of the parameter \( \epsilon_1 \) in (69) \( (\epsilon_1 = \frac{1}{2}) \). \[\square\]

**5.3. Absorption of extraneous terms.** Our next goal is to obtain the estimate without higher order terms \( H^{2+\rho}(\Omega) \) and without boundary terms. In order to accomplish this, we shall invoke our two fundamental regularity results: partial hidden regularity (Lemma 5.1) and partial analyticity (Lemma 4.3).

**Lemma 5.7. Absorption of Boundary Traces and Higher-order Term** Let \( u, w, \theta, \varphi \) be a regular solution to (1a). Then for any \( T > 0 \) we have the following estimate:

\[
\int_{0}^{T} E(t) \, dt \leq C[E(0)][E(0) + E(T)] + CE(0)^2 + CE(0)^2 \int_{0}^{T} |w|_{0,\Omega}^2 \, dt
\]

\[
+ C_T(E(0)) \int_{0}^{T} |\varphi_{1,\Omega}|^2 + |\theta_{1,\Omega}|^2 + |\text{curl} \, q_{0,\Omega}|^2 \, dt \quad (93)
\]

**Proof.** Partial analyticity gives – a posteriori via trace theory – the estimate for the boundary terms of vertical displacement:

\[
\int_{0}^{T} |w_{t}|_{1/2, \Gamma}^2 + |\Delta w|_{1/2, \Gamma}^2 \, dt \leq C \int_{0}^{T} |w_{1,1,\Omega}^2 + |w_{3,\Omega}|^2 \, dt
\]

\[
\leq C E(0) + CE(0) \int_{0}^{T} |w|_{0,\Omega}^2 \, dt
\]

The higher order term is handled in a similar way by partial analyticity:

\[
E(0) \int_{0}^{T} |w|_{2+\rho,\Omega}^2 \leq CE^2(0) + CE^2(0) \int_{0}^{T} |w|_{0,\Omega}^2 \, dt
\]
With respect to the boundary traces, the delicate point here is due to the fact that hidden regularity for \( \frac{\partial}{\partial \nu} u \) is estimated by \( T E(0) \). This term will ruin the game. However, we have \( \epsilon \) in front of this normal derivative which allows us to choose \( \epsilon T \leq 1 \) – with the price being paid that the term involving damping will be now rescaled with \( T \).

\[
\epsilon \left| \frac{\partial}{\partial \nu} u \right|^2|_{0, \Gamma_1} dt \leq \epsilon C T E(0) + \epsilon \epsilon_2 E^{3/2}(0) + \epsilon C_{\epsilon_2} \int_0^T |w|^2_{0, \Omega}
\]

With the above estimates inserted into (92) and accounting once more for the dissipativity relation, we derive Lemma 5.7. \( \square \)

Our next task is to absorb the lower-order terms that appear in equation (93).

**Lemma 5.8. Absorption of Lower-order Terms** Let \( u, w, \varphi, \theta \) be a solution to the system and \( T > 0 \) be large enough. Assume that \( \Gamma_0 \neq \emptyset \). Then there exists \( C_T(E(0)) \) so that

\[
\text{lot}(u, w) \leq C_T(E(0)) \int_0^T \| \varphi \|^2_{1, \Omega} + |\theta|^2_{1, \Omega} + |\text{curl } q|^2_{0, \Omega} dt
\]

**Proof.** The proof relies on Unique Continuation Principle and Compactness. While Compactness is obvious from the structure of the lower-order terms, unique continuation is more involved due to the structure of the free boundary conditions. Here we shall see that the fact that \( \Gamma_0 \) is nonempty plays an important role. To wit, we argue by contradiction. We will take a sequence of initial data \( (u_n(0), u_{t,n}(0), \varphi_n(0), w_n(0), w_{t,n}(0), \theta_n(0)) \) such that the initial energy of these data, \( E_n(0) \), is bounded by a constant \( C < 1 \). By the well-posedness result, Theorem 1.1, we know that the corresponding solutions must be bounded in the energy space, and their energy \( E_n(t) \) is bounded uniformly in \( n \). However, if (94) is not true, then there exists at least one of these sequences so that

\[
\frac{\text{lot}(u_n, w_n)}{\int_0^T (\| \varphi_n \|^2_{1, \Omega} + |\theta_n|^2_{1, \Omega} + |\text{curl } q_n|^2_{0, \Omega}) dt} \to \infty
\]

Since \( E_n(t) \) is bounded, we have that \( E_n(t) \leq M \) so

\[
\int_0^T (\| \varphi_n \|^2_{1, \Omega} + |\theta_n|^2_{1, \Omega} + |\text{curl } q_n|^2_{0, \Omega}) dt \to 0
\]

\[
\int_0^T (|u_n|^2_{1, \Omega} + |u_{t,n}|^2_{0, \Omega} + |\varphi_n|^2_{0, \Omega} + |w_n|^2_{2, \Omega} + |w_{t,n}|^2_{0, \Omega} + |\theta_n|^2_{0, \Omega}) dt \leq C
\]

These bounds and the compactness of Sobolev’s embeddings allow us to extract a subsequence, indexed by the same symbol, such that

\[
w_n \to w \text{ weakly}^* \text{ in } L_\infty(0, T; H^2(\Omega))
\]

\[
w_{n,t} \to w_t \text{ weakly}^* \text{ in } L_\infty(0, T; L^2(\Omega))
\]

\[
u_n \to u \text{ weakly}^* \text{ in } L_\infty(0, T; [H^1(\Omega)]^2)
\]

\[
u_{n,t} \to u_t \text{ weakly}^* \text{ in } L_\infty(0, T; [L^2(\Omega)]^2)
\]

\[
\theta_n \to 0 \text{ in } L^2(0, T; H^1(\Omega))
\]

\[
\varphi_n \to 0 \text{ in } L^2(0, T; H^1(\Omega))
\]

\[
\text{curl } q_n \to 0 \text{ in } L^2(0, T; H^2(\Omega))
\]

\[
\text{curl } q_{n,t} \to 0 \text{ in } L^2(0, T; L^2(\Omega))
\]

\[
\text{lot}(u_n, w_n) \to \text{lot}(u, w)
\]
This convergence and the weak continuity of the von Kármán nonlinearity allow us to pass the limit on the original equation and deduce that \( u, w \) satisfy

\[
\begin{cases}
  u_{tt} - \text{div} \left( C[\varepsilon(u) + f(\nabla w)] \right) = 0 \\
  w_{tt} + D\Delta^2 w - \text{div} \left( C[\varepsilon(u) + f(\nabla w)] \nabla w \right) = 0 \\
  \text{div} \ u - \nabla w \cdot \nabla w_t = 0 \\
  \Delta w_t = 0
\end{cases}
\text{in } \Omega \times (0, \infty) \tag{100a}
\]

with clamped boundary conditions on \( \Gamma_0 \):

\[
u = 0, \quad w = 0, \quad \nabla w = 0 \text{ on } \Gamma_0 \times (0, \infty) \tag{100b}
\]

and the following boundary conditions on \( \Gamma_1 \):

\[
\begin{cases}
  u = 0 \\
  D[\Delta w + (1 - \mu)B_1 w] = 0 \text{ on } \Gamma_1 \times (0, \infty) \tag{100c}
\end{cases}
\]

By using the fourth equation of (100a) and the boundary conditions on \( \Gamma_0 \) which imply \( w_t = \frac{\partial}{\partial \nu} w_t = 0 \) we get the following system with \( \bar{w} = w_t \):

\[
\begin{cases}
  \Delta \bar{w} = 0 \text{ in } \Omega \times (0, \infty) \\
  \bar{w} = \frac{\partial}{\partial \nu} \bar{w} = 0 \text{ on } \Gamma_0 \times (0, \infty) \tag{101}
\end{cases}
\]

By Holmgren’s unique continuation result for elliptic systems, (101) implies that \( \bar{w} = 0 \). Feeding \( w_t \equiv 0 \) back into \( \text{div} \ u_t - \nabla w \cdot \nabla w_t = 0 \) gives us that \( \text{div} \ u_t = 0 \). However we already know that \( D u_t = \text{curl} \ q_t = 0 \) so since \( u_t = \nabla p_t + \text{curl} \ q_t \), we get \( D u_t = \Delta p_t = 0 \) on \( \Omega \) with \( p_t = 0 \) on \( \Gamma \). Thus we can conclude \( p_t = 0 \), which implies \( u_t \equiv 0 \). Putting this back into the equations (100a) yields \( w \equiv 0 \), \( u \equiv 0 \). Thus \( \text{lot}(u, w) = 0 \) and we have that by (98)

\[
\text{lot}(u_n, w_n) \to 0 \tag{102}
\]

Let us now rescale by defining

\[
c_n^2 \equiv \text{lot}(u_n, w_n) \tag{103}
\]

Without loss of generality we can assume that \( c_n > 0 \) for larger \( n \) and define the scaled sequence with a tilde symbol as

\[
\tilde{w}_n \equiv \frac{w_n}{c_n}, \quad \tilde{u}_n \equiv \frac{u_n}{c_n}, \quad \tilde{\theta}_n \equiv \frac{\theta_n}{c_n}, \quad \tilde{\varphi}_n \equiv \frac{\varphi_n}{c_n} \tag{104}
\]

Clearly, from the definition of \( c_n \) and (95) we have

\[
\text{lot}(\tilde{u}_n, \tilde{w}_n) = 1 \int_0^T \left[ |\tilde{\varphi}_n|^2_{1, \Omega} + |\tilde{\theta}_n|^2_{2, \Omega} + |D \tilde{u}_n|_{0, \Omega}^2 \right] dt \to 0 \tag{105}
\]

We wish to again extract a subsequence but we need to show that the energy terms \( \tilde{u}, \tilde{w} \), etc are uniformly bounded. We can divide (92) and the dissipation equality (42) by \( c_n^2 \), and apply the relations (105) to get

\[
\int_0^T \left[ |\tilde{u}_n|_{1, \Omega}^2 + |\tilde{u}_t|_{0, \Omega}^2 + |\tilde{\varphi}_n|^2_{2, \Omega} + |\tilde{\varphi}_t|_{0, \Omega}^2 \right] dt \leq C(E_n(0)) \tag{106}
\]
which will allow us to extract a convergent subsequence (again still indexed by \( n \)) so that

\[
\begin{align*}
\tilde{w}_n & \to \tilde{w} \text{ weakly in } L_\infty(0, T; H^2(\Omega)) \\
\tilde{w}_{n,t} & \to \tilde{w}_t \text{ weakly in } L_\infty(0, T; L_2(\Omega)) \\
\tilde{u}_n & \to \tilde{u} \text{ weakly in } L_\infty(0, T; [H^1(\Omega)]^2) \\
\tilde{u}_{n,t} & \to \tilde{u}_t \text{ weakly in } L_\infty(0, T; [L_2(\Omega)]^2) \\
\text{lot}(\tilde{u}_n, \tilde{w}_n) & \to \text{lot}(\tilde{u}, \tilde{w})
\end{align*}
\]  

(107)

Thus we can see that it must be the case that

\[ \text{lot}(\tilde{u}, \tilde{w}) = 1 \]  

(108)

Now, the system of equations we have for \( \tilde{u}_n, \tilde{w}_n \) in \( \Omega \times (0, \infty) \) is

\[
\begin{cases}
\tilde{u}_{n,tt} - \text{div} (C\varepsilon(\tilde{u}_n)) + \nabla \tilde{\varphi}_n = 0 \\
\tilde{w}_{n,tt} + D\Delta^2 \tilde{w}_n - \text{div} (C\varepsilon(\tilde{u}_n))\nabla \tilde{w}_n + \tilde{\varphi}_n \nabla \tilde{w}_n + \Delta \tilde{\theta}_n = 0 \\
\tilde{\varphi}_n - \Delta \tilde{\varphi}_n + \text{div} \tilde{u}_n - \nabla \tilde{w}_n \cdot \nabla \tilde{w}_{n,t} = 0 \\
\tilde{\theta}_n - \Delta \tilde{\theta}_n - \Delta \tilde{w}_{n,t} = 0
\end{cases}
\]

(109a)

with clamped boundary conditions for the displacements on the portion of the boundary \( \Gamma_0 \times (0, \infty) \):

\[
\tilde{u}_n = 0, \quad \tilde{w}_n = 0, \quad \nabla \tilde{w}_n = 0
\]

(109b)

the following boundary conditions on the portion of the boundary \( \Gamma_1 \times (0, \infty) \):

\[
\begin{cases}
D[\Delta \tilde{w}_n + (1 - \mu) B_1 \tilde{w}_n] + \tilde{\theta}_n = 0 \\
D[\frac{\partial}{\partial \nu} \Delta \tilde{w}_n + (1 - \mu) B_2 \tilde{w}_n] - C\varepsilon(\tilde{u}_n) + f(\nabla \tilde{w}_n) |\nabla \tilde{w}_n + \frac{\partial}{\partial \nu} \tilde{\theta}_n = 0
\end{cases}
\]

(109c)

We pass the limit and use (98),(107) and (102) to derive the system

\[
\begin{cases}
\tilde{u}_{tt} - \text{div} (C\varepsilon(\tilde{u})) = 0 \\
\tilde{w}_{tt} + D\Delta^2 \tilde{w} = 0 \\
\text{div} \tilde{u}_t = 0 \quad \text{in } \Omega \times (0, \infty) \\
\Delta \tilde{w}_t = 0
\end{cases}
\]

(110a)

\[
\tilde{u}_n = 0, \quad \tilde{w}_n = 0, \quad \nabla \tilde{w}_n = 0 \quad \text{on } \Gamma_0 \times (0, \infty)
\]

(110b)

\[
\begin{cases}
\tilde{u} = 0 \\
D[\Delta \tilde{w} + (1 - \mu) B_1 \tilde{w}] = 0 \quad \text{on } \Gamma_1 \times (0, \infty) \\
D[\frac{\partial}{\partial \nu} \Delta \tilde{w} + (1 - \mu) B_2 \tilde{w}] = 0
\end{cases}
\]

(110c)

As before, we obtain \( \tilde{u}_t \equiv 0, \tilde{w}_t \equiv 0 \), and feeding this back into (110) we obtain \( \tilde{u} \equiv 0, \tilde{w} \equiv 0 \). This furnishes the contradiction to equation (108), showing that the claim of Lemma 5.8 must be true.

Finally, to finish off the proof of stability, we use Lemma 5.8 in inequality (93) to derive that

\[
\int_0^T E(t) \, dt \leq C(E(0))[E(0) + E(T)] + C_T(E(0)) \int_0^T [\|\varphi\|^2_{L_2,\Omega} + \|	heta\|^2_{L_2,\Omega} + |\text{curl } q_t|^2_{L_0,\Omega}] \, dt
\]

(111)

applying the dissipation equality gives that

\[
2TE(T) \leq [2C(E(0)) + C_T(E(0))]E(0) + [2C(E(0)) - C_T(E(0))]E(T)
\]

(112)
or

$$E(T) \leq \frac{2C(E(0)) + C_T(E(0))}{2T - 2C(E(0)) + C_T(E(0))} E(0) = \rho E(0)$$

(113)

with $\rho < 1$ for $T > 2C(E(0))$. Standard nonlinear semigroup results then give the exponential rates of Theorem 1.3 for regular solutions. Then, the uniqueness results established in Theorem 1.1 allows us to extend the decay rates to all weak solutions [18].

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