A Kind of New Coupled Model for Rossby Waves in Two Layers Fluid

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ABSTRACT In this article, a new model for Rossby waves in two layers fluid is studied. Form the dimensionless baroclinic quasi-geostrophic vortex equations include exogenous and dissipative, the new (2 + 1)-dimensional coupled ZK-mZK equations are established by multiscale analysis and perturbation method. Based on the semi-inverse and Agrawal’s method, the time-fractional coupled ZK-mZK equations are derived. Then, Lie symmetries and conservation laws of time-fractional equations are analyzed. Finally, the exact and numerical solutions of the time-fractional coupled ZK-mZK equations are obtained by the Jacobi elliptic function expansion method and alternative variational iteration method. The relative errors between solutions show that the alternative variational iteration method gives a high-precision numerical solution. Further, propagation of Rossby waves in two layers fluid is affected by time, order of fractional derivative and coefficient of coupling term.

INDEX TERMS Two layers fluid, Rossby waves, time-fractional coupled ZK-mZK equations, alternative variational iteration method.

I. INTRODUCTION

Many large-scale fluids in nature and engineering, such as the atmosphere and oceans, are stratified. The study of stratified flow plays an important role in national economic construction, people’s life and environmental protection [1]–[3]. The existence of solitary waves in hydrodynamics has been known over a century. If there are waves in stratified fluid, they will interact with each other. Therefore, it is of practical significance to establish the coupling model in the stratified fluid. Compared with the single equation of low dimension, the coupled equations of high dimension are more practical in the actual marine-atmosphere system [4]–[6]. The coupled equations models are also used in many scientific fields, such as fluid dynamics, aerodynamics and mass transport, but it mostly describes processes with weak perturbations.

As a matter of fact, the problems we studied often do not entirely meet ideal situation. Therefore, the effect of friction dissipation and viscous dissipation should be considered. The application of dissipative effect in the atmospheric dynamics and oceanic dynamics is more obvious. So, the shallow water equation is still too rough as the starting equation in the treatment of such problems [7], [8]. In this article, we only consider the large scale shallow water equations with dissipation in the horizontal direction [9]. In the future, we will study the shallow water equation with dissipation in both the horizontal and vertical directions.

Because of the complexity of nature world, many of the systems we studied are non-conservative [10]. In 1996, the fractional derivative term was introduced into the functional by Riewe [11], [12] to obtain the non-conservative term needed in the differential equation. In the years since, fractional calculus has developed on the basis of nonlinear science [13]–[15]. Nowadays, it is more important to establish fractional derivative model for daily production and life. Conservation laws mean the mathematical formula in which the total amount of a physical quantity remains constant during the evolution of a physical system [16]–[18]. The Lie symmetries and conservation laws of nonlinear partial differential equations play important roles in the study of nonlinear physical phenomena.
The solution of partial differential equation is of great practical significance [19], [20]. Therefore, many different methods are used to calculate the exact and numerical solutions of the equation [21]–[23]. In order to reflect the dissipative effect more intuitively, the Jacobi elliptic function expansion method which can produce periodic solution can be chosen to derive the exact solution [24], [25]. Compared with exact solutions, numerical solutions have fewer restrictions on the equation and are allowed complex structures, but they have certain requirements on the selection of initial values. In 1998, Chinese mathematician He proposed the variational iterative method (VIM) for solving fractional differential equations. This method has been applied to the solution of strongly nonlinear equations. Some ameliorative measures are put forward to improve the convergence speed and extend the convergence interval of the solution of VIM series. Variational iterative method has been widely adopted by many scholars after years of research [26], [27].

The structure of the paper is as follows: In Section 2, we consider the Rossby waves in two layers fluid with dissipation and exogenous. Form the dimensionless baroclinic quasi-geostrophic vortex equations including exogenous and dissipative, the new coupled ZK-mZK equations are yielded by using the method of multiscale expansion and disturbance analysis. As the the Agrawal method, the semi-inverse method, and the fractional variational principle used, the time-fractional coupled ZK-mZK equations are obtained in section 3. In section 4, according to the Lie group theory, the generators are obtained, and then the conservation laws of the equations can be established. In Section 5, the exact and numerical solutions of fractional equations are separately given by using the Jacobi elliptic function expansion method and alternative variational iteration method. Especially, the relative errors between the exact solution and numerical solution are analyzed. In addition, we analyze and discuss the effect of time, order of fractional derivative, and coefficient of coupling term. Finally, the full text is summarized.

II. DERIVATION OF THE COUPLED ZK-mZK EQUATIONS

Stratified flow has multiple layers of fluid. To facilitate the study, we first simplified the model to a two-layer fluid. A two-layer fluid can be divided into upper and lower layers. If there are waves in two layers fluid, they will interact with each other. In this article, we study the dimensionless baroclinic quasi-geostrophic vortex equations including exogenous and dissipative in two layers fluid as following

\[
\frac{\partial}{\partial t} + \frac{\partial \psi_A}{\partial x} - \frac{\partial \psi_D}{\partial y} \left[ \psi_{Ax} + \psi_{Ay} \right] + F_1(\psi_B - \psi_A) + f \frac{\partial}{\partial y} \left( \frac{\rho_B}{s} \frac{\partial \psi_B}{\partial z} \right) = 0,
\]

\[
\frac{\partial}{\partial t} + \frac{\partial \psi_D}{\partial x} - \frac{\partial \psi_B}{\partial y} \left[ \psi_{Bx} + \psi_{By} \right] + F_2(\psi_A - \psi_B) + f \frac{\partial}{\partial y} \left( \frac{\rho_A}{s} \frac{\partial \psi_A}{\partial z} \right) = 0,
\]

where \( \psi_A \) and \( \psi_B \) are the stream functions of upper layer and lower layer fluid respectively, \( F_1 \) and \( F_2 \) represent the weak coupling coefficients between two layers fluid respectively, \( f \) is the Coriolis parameter, \( s = N^2/f \), \( N \) is a physical quantity that measures the level of layer stability, and stands for Brunt-Vaisala frequency, \( \rho_s = \rho_d(z) \) for density. The boundary condition is \( z = 0 \). Then (1) can be expanded to

\[
\psi_{Ax} + \psi_{Ay} + \frac{f \rho_B}{s \rho_d} \psi_{Az} + \frac{f}{s} \psi_{Azz} + F_1(\psi_{By} - \psi_{Ay}) - \psi_{Ax} + \psi_{Ay} + \psi_{Ax} + \psi_{Ay} + F_1(\psi_{Bx} - \psi_{Ax}) + \frac{f \rho_A}{s \rho_d} \psi_{Ax} + \frac{f}{s} \psi_{Az} = 0,
\]

\[
\psi_{Bx} + \psi_{By} + \frac{f \rho_B}{s \rho_d} \psi_{Bz} + \frac{f}{s} \psi_{Bzz} = 0.
\]

Next, in order to obtain the coupled ZK-mZK equations, we separate the stream functions \( \psi_A \) and \( \psi_B \) into the basic stream functions and the disturbance stream functions. The form of the stream functions are as follows

\[
\psi_A = \phi_A(y) + \phi_1(x, y, t) = (U_A_0 + c_0 y) + \phi_A(x, y, t),
\]

\[
\psi_B = \phi_B(y) + \phi_2(x, y, t) = (U_B_0 + c_0 y) + \phi_B(x, y, t).
\]

Consider that the coupling between the two layers fluid is weak, and the rotation effect of the Earth is smaller. Therefore we set transformations as

\[
F_1 = \varepsilon F_A, \quad F_2 = \varepsilon F_B, \quad f = \varepsilon^2 f,
\]

where \( \varepsilon \) is a small parameter. Taking the \( x \)-directional long-wave approximation, we also introduce the stretched variables as

\[
X = \varepsilon(x - c_0 t), \quad Y = \varepsilon y, \quad T = \varepsilon^2 t.
\]

Then, we expend the perturbation stream functions into the following form

\[
\phi_A = \varepsilon \phi_A(X, Y, y, z, T) + \varepsilon^2 \phi_A(X, Y, y, z, T) + \varepsilon^3 \phi_A(X, Y, y, z, T) + o(\varepsilon^4),
\]

\[
\phi_B = \varepsilon \phi_B(X, Y, y, z, T) + \varepsilon^2 \phi_B(X, Y, y, z, T) + \varepsilon^3 \phi_B(X, Y, y, z, T) + o(\varepsilon^4).
\]

Substitute (3), (4), (5), and (6) into (2). Equations for the coefficient on the parameter \( \varepsilon^2 \) are obtained as follows

\[
\varepsilon^2 : \begin{cases} \phi_{Ax} \phi_{Axy} - c_0 \phi_{Ax} \phi_{Ay} - \phi_{A0x} \phi_{Axy} = 0, \\ \phi_{Bx} \phi_{Byy} - c_0 \phi_{Bx} \phi_{By} - \phi_{B0x} \phi_{Bxy} = 0. \end{cases}
\]

Integrating with respect to \( X \) and \( Y \), we get a set of equations that are related only to \( y \),

\[
\phi_A U_{A0y} - U_{A0y} \phi_A_{y1} = 0, \quad \phi_B U_{B0y} - U_{B0y} \phi_B_{y1} = 0.
\]
Form the expression in (8), we assume that functions \( \phi_{A_i} \) and \( \phi_{B_i} \) have the form of separate variables and can be rewritten as the following special form

\[
\phi_{A_i} = A_i(X, Y, T)B_i(y, z) \equiv A_iB_i,
\]
\[
\phi_{B_i} = A_2(X, Y, T)B_2(y, z) \equiv A_2B_2.
\]  

Equations (8) are equal to

\[
U_{A_{0y}} = \frac{B_{1y}}{B_1} U_{A_0},
\]
\[
U_{B_{0y}} = \frac{B_{2y}}{B_2} U_{B_0}.
\]  

We obtained the following equations for the coefficient on the parameter \( \varepsilon^3 \)

\[
\varepsilon^3 : 
\begin{cases} 
F_A c_0 \phi_{AX} - 2c_0 \phi_{AXY} - c_0 \phi_{A2XY} - F_A c_0 \phi_{BX} \\
+ \phi_{AX} \phi_{B0y} + \phi_{AXY} \phi_{A1y} + F_A \phi_{B0y} \\
- F_A \phi_{A0} + \phi_{AXY} + \phi_{A0y} - \phi_{AX} \phi_{A0y} \\
+ 2\phi_{AXY} + F_A \phi_{B1x} - F_A \phi_{A0x} = 0, \\
F_B c_0 \phi_{BX} - 2c_0 \phi_{BXY} - c_0 \phi_{B2XY} - F_B c_0 \phi_{AX} \\
+ \phi_{BX} \phi_{B0y} + \phi_{BXY} \phi_{A1y} + F_B \phi_{B0y} \\
- F_B \phi_{B0} + \phi_{BXY} + \phi_{B0y} - \phi_{BX} \phi_{B0y} \\
+ 2\phi_{BXY} + F_B \phi_{A1x} - F_B \phi_{B0x} = 0.
\end{cases}
\]  

Substituting (9) and (10) into (11) and integrating with respect to \( X \), we get

\[
2U_{A_0} (B_{1y} - B_1 \partial_y) \phi_{A_2} + B_1 [(B_1 B_{1yy} - B_1 B_{1yy}) \\
- B_1 B_{1yy} A_1^2 - 4U_{A_0} B_1 A_1 Y_1] \\
- 2F_A U_{A_0} B_2 A_2 + 2F_A U_{B_0} B_1 = 0,
\]
\[
2U_{B_0} (B_{2yy} - B_2 \partial_y) \phi_{B_2} + B_2 [(B_2 B_{2yy} - B_2 B_{2yy}) \\
- B_2 B_{2yy} A_2^2 - 4U_{B_0} B_2 A_2 Y_2] \\
- 2F_B U_{B_0} B_1 A_1 + 2F_B U_{A_0} B_2 = 0.
\]  

Form the expression in (12), we take the form of separate variables of \( \phi_{A_2} \) and \( \phi_{B_2} \) as the following form

\[
\phi_{A_2} = (m_1 A_2^2 + m_2 A_1 Y_1 + m_3 A_1 + m_4 A_2) B_1, \\
\phi_{B_2} = (m_5 A_2^2 + m_6 A_2 Y_2 + m_7 A_2 + m_8 A_1) B_2,
\]  

where \( m_i (i = 1, 2, \ldots, 8) \) can be determined later. Substituting (13) into (14), we can obtain

\[
m_{1y} = \frac{n_1}{B_1}, \\
m_{2y} = \frac{n_2}{B_1}, \\
m_{3y} = \frac{n_3}{B_1}, \\
m_{5y} = \frac{n_5}{B_2}, \\
m_{6y} = \frac{n_6}{B_2}, \\
m_{7y} = \frac{n_7}{B_2}, \\
m_{8y} = \frac{n_8}{B_2},
\]

Substituting (9), (10), and (13) into (15), we have a set of identities. In the general process of dealing with such identities, \( \phi_{A_1} \) and \( \phi_{B_1} \) would be set to zero. But this approach may lead to inconsistencies. To avoid this kind of situation, simple and ambiguous methods are usually used in previous research discussions, e.g., integrating of variable \( y \) from \( y_1 \) to \( y_2 \). But in this article, we do not take \( \phi_{A_1} = \phi_{B_1} = 0 \) nor to integrate of variable \( y \) from \( y_1 \) to \( y_2 \).

In order to obtain the coupled ZK-mZK equations for \( A_1 \) and \( A_2 \) in this article, we only give out one potential alternative of \( \phi_{A_1} \) and \( \phi_{B_1} \) as

\[
\phi_{A_1} = r_1 A_1 + r_2 A_2 + r_3 A_1^2 + r_4 A_2^2 + r_5 A_1 A_2 \\
+ r_6 A_1 + r_7 A_1 Y_1 + r_8 A_2 Y_2 \\
+ r_9 (A_1 Y_2 + r_10 A_1 Y_1 + r_11 A_2 Y_1 Y_2), \\
\phi_{B_1} = s_1 A_2 + s_2 A_1 + s_3 A_2^2 + s_4 A_1^2 + s_5 A_1 A_2 \\
+ s_6 A_2^2 + s_7 A_2 Y_2 + s_8 A_1 Y_1 \\
+ s_9 (A_1 Y_2 + s_{10} A_2 Y_2 + s_{11} A_2 Y_2 Y_2),
\]

where

\[
r_i = B_1 \int \frac{1}{B_1 (y')^2} \int R_i(y')dy'd'y'' \hspace{1cm} (i = 1, 2, \ldots, 11), \\
s_i = B_2 \int \frac{1}{B_2 (y'')^2} \int S_i(y'')dy'd'y'' \hspace{1cm} (i = 1, 2, \ldots, 11),
\]

and \( R_i (i = 1, 2, \ldots, 11) \), \( S_i (i = 1, 2, \ldots, 11) \) are in Appendix A. By calculation, the terms containing \( y \) can be eliminated, and the equations with \( X \) and \( T \) can be obtained. We get that \( A_1 \) and \( A_2 \) are suitable for the relationship of the
following coupled ZK-mZK system:
\[ A_{1T} + \alpha_{11}A_{1X} + \alpha_{12}(A_{1}^{3})_{X} + \alpha_{13}(A_{1}A_{2})_{X} + \alpha_{14}(A_{1}^{3})_{X} + \alpha_{15}A_{1XXX} + \alpha_{16}A_{1XYY} = 0, \]
\[ A_{2T} + \alpha_{21}A_{2X} + \alpha_{22}(A_{2}^{3})_{X} + \alpha_{23}(A_{1}A_{2})_{X} + \alpha_{24}(A_{2}^{3})_{X} + \alpha_{25}A_{2XXX} + \alpha_{26}A_{2XYY} = 0, \] (17)
where constants \( \alpha_{ij}(i = 1, 2, \ldots, 6, j = 1, 2) \) are arbitrable in \( R_{0} (i = 1, 2, \ldots, 11) \) and \( S_{0} (i = 1, 2, \ldots, 11) \).

Equations (17) are the two-dimensional coupled ZK-mZK (Zakharov-Kuznetsov-modified Zakharov-Kuznetsov) equations. Compared with the original mZK equation, coupled ZK-mZK equations are a system of equations with weak coupling terms \( \alpha_{13}(A_{1}A_{2})_{X} \), \( \alpha_{23}(A_{1}A_{2})_{X} \) and compared with the general ZK system, coupled ZK-mZK equations have both coupling terms \( \alpha_{13}(A_{1}A_{2})_{X} \), \( \alpha_{23}(A_{1}A_{2})_{X} \) and high-order nonlinear terms \( \alpha_{14}(A_{1}^{3})_{X} \), \( \alpha_{24}(A_{2}^{3})_{X} \). It can be seen from the coefficient relationship of the equations that the coupling constants have effects on the coefficients of terms of \( A_{1X} \), \( (A_{1}^{3})_{X} \), \( (A_{1}A_{2})_{X} \), \( (A_{2}^{3})_{X} \), \( (A_{1}A_{2})_{X} \), but have no effect on the coefficients of terms of \( (A_{1}^{3})_{X} \), \( (A_{2}^{3})_{X} \). And we can see that the coupling terms are up to the square terms, and the highest degree terms are not coupled.

III. THE TIME-FRACTIONAL COUPLED ZK-mZK EQUATIONS

In order to build a more suitable model to describe the non-conservative system, we extend the integral derivatives equations (17) to the field of fractional derivatives. Take integral derivatives equations as a special case of fractional derivatives equations. In this section, we will derive the time-fractional coupled ZK-mZK equations by the semi-inverse method, the variational method and Agrawal's method.

In the first place, we introduce two potential functions \( U(X, Y, T) \) and \( V(X, Y, T) \), where \( A_{1} = U_{X}, A_{2} = V_{X} \). Substitute potential functions into (17), and the potential equations of the coupled ZK-mZK equations can be written as
\[ U_{XT} + \alpha_{11}U_{XX} + \alpha_{12}(U_{X}^{3})_{X} + \alpha_{13}(U_{X}V_{X})_{X} + \alpha_{14}(U_{X}^{3})_{X} + \alpha_{15}U_{XXX} + \alpha_{16}U_{XYY} = 0, \]
\[ V_{XT} + \alpha_{21}V_{XX} + \alpha_{22}(V_{X}^{3})_{X} + \alpha_{23}(U_{X}V_{X})_{X} + \alpha_{24}(V_{X}^{3})_{X} + \alpha_{25}V_{XXX} + \alpha_{26}V_{XYY} = 0. \] (18)

Next, in order to obtain the Lagrangian equations of the coupled ZK-mZK equations, we have written the functional of (18) as
\[ J(U, V) = \int\int\int_{\Omega} dX dY dT \left[ -m_{1}U_{XT} + m_{2}a_{11}U_{XX} + \frac{m_{3}a_{12}(U_{X}^{3})_{X} + m_{4}a_{13}(U_{X}V_{X})_{X} + m_{5}a_{14}(U_{X}^{3})_{X} + m_{6}a_{15}U_{XXX} + m_{7}a_{16}U_{XYY} + U_{11}V_{XT} + m_{2}a_{21}V_{XX} + m_{3}a_{22}(V_{X}^{3})_{X} + m_{4}a_{23}(U_{X}V_{X})_{X} + m_{5}a_{24}(V_{X}^{3})_{X} + m_{6}a_{25}V_{XXX} + m_{7}a_{26}V_{XYY} \right], \] (19)
where \( \Omega = R \times R \times T^{*}, \) and \( m_{i}, n_{j}(i = 1, \ldots, 9) \) are Lagrange coefficients which will be determined later. For the function in the first parenthesis in (19), integration by parts of \( U \), and for the function in the second parenthesis, integration by parts of \( V \), and taking \( U_{X} |_{r} = U_{XXX} |_{r} = U_{XYY} |_{r} = U_{TT} |_{r} = V_{X} |_{r} = V_{XXX} |_{r} = V_{XYY} |_{r} = V_{TT} |_{r} = 0 \), the functional can be rewritten as
\[ J(U, V) = \int\int\int_{\Omega} dX dY dT \left[ -m_{1}U_{XT} + m_{2}a_{11}U_{XX} - m_{3}a_{12}U_{X}^{3} - m_{4}a_{13}U_{X}V_{X} - m_{5}a_{14}U_{X}^{3} + m_{6}a_{15}U_{XXX} + m_{7}a_{16}U_{XYY} \right] + m_{2}a_{21}V_{XX} + m_{3}a_{22}V_{X}^{3} + m_{4}a_{23}U_{X}V_{X} + m_{5}a_{24}V_{X}^{3} + m_{6}a_{25}V_{XXX} + m_{7}a_{26}V_{XYY} \]. (20)

We derive the first order variational equations of functional equation (20) by using the variational method. Variational equations can be expressed as
\[ 2m_{1}U_{XT} + 2m_{2}a_{11}U_{XX} + 3m_{3}a_{12}(U_{X}^{3})_{X} + 2m_{4}a_{13}(U_{X}V_{X})_{X} + 4m_{5}a_{14}(U_{X}^{3})_{X} + 2m_{6}a_{15}U_{XXX} + 2m_{7}a_{16}U_{XYY} = 0, \]
\[ 2n_{1}V_{XT} + 2m_{2}a_{21}V_{XX} + 3m_{3}a_{22}(V_{X}^{3})_{X} + 2m_{4}a_{23}(U_{X}V_{X})_{X} + 4m_{5}a_{24}(V_{X}^{3})_{X} + 2m_{6}a_{25}V_{XXX} + 2m_{7}a_{26}V_{XYY} = 0. \] (21)

By the semi-inverse method, it is obvious that (18) and (21) are equal. So, the Lagrange coefficients can be obtained, that is \( m_{1} = m_{2} = m_{4} = m_{5} = m_{7} = n_{1} = n_{2} = n_{4} = n_{5} = n_{7} = 1/2, m_{3} = n_{3} = 1/3, m_{5} = n_{5} = 1/4. \) Substituting the Lagrange coefficients into (21), we get Lagrangian forms as
\[ I_{1} = -\frac{1}{2}U_{X}U_{T} - \frac{1}{2}a_{11}U_{X}^{2} - \frac{1}{3}a_{12}U_{X}^{3} - \frac{1}{2}a_{13}U_{X}V_{X} - \frac{1}{4}a_{14}U_{X}^{4} + \frac{1}{2}a_{15}U_{XXX} + \frac{1}{2}a_{16}U_{XYY}, \]
\[ I_{2} = -\frac{1}{2}V_{X}V_{T} - \frac{1}{2}a_{21}V_{X}^{2} - \frac{1}{3}a_{22}V_{X}^{3} - \frac{1}{2}a_{23}U_{X}V_{X} - \frac{1}{4}a_{24}V_{X}^{4} + \frac{1}{2}a_{25}U_{XXX} + \frac{1}{2}a_{26}V_{XYY}. \] (22)

At the moment, we have the Lagrangian equations for integral derivatives. Then, we use Agrawal’s method to extend the Lagrangian equations for integral derivatives to the Lagrangian equations for fractional derivatives. The Lagrangian forms of the time-fractional coupled ZK-mZK equations are given as
\[ F_{1} = -\frac{1}{2}D^{\alpha}_{t}U_{X}U_{T} - \frac{1}{2}a_{11}U_{X}^{2} - \frac{1}{3}a_{12}U_{X}^{3} - \frac{1}{4}a_{14}U_{X}^{4} + \frac{1}{2}a_{15}U_{XXX} + \frac{1}{2}a_{16}U_{XYY}, \]
\[ F_{2} = -\frac{1}{2}D^{\alpha}_{t}V_{X}V_{T} - \frac{1}{2}a_{21}V_{X}^{2} - \frac{1}{3}a_{22}V_{X}^{3} - \frac{1}{2}a_{23}U_{X}V_{X} - \frac{1}{4}a_{24}V_{X}^{4} + \frac{1}{2}a_{25}U_{XXX} + \frac{1}{2}a_{26}V_{XYY}. \] (23)
where $D^a_T$ is the operator of fractional derivative, $T$ is independent variable, $\alpha$ is the order of the derivative and $\alpha$ can be a fraction. Similar to the integral derivatives equations, the functional of the time-fractional coupled ZK-mZK equations is

$$J_{F_1,F_2}(U, V) = \int \int_{\Omega} dX dY (dT)^\alpha (F_1 + F_2). \tag{24}$$

Using the variational method of Agrawal’s method, we have

$$\delta J_{F_1,F_2}(U, V) = \int \int_{\Omega} dX dY (dT)^\alpha \left[ \{D^a_T - \frac{\partial F_1}{\partial dT_u} U \} \delta U + \frac{\partial^2 F_1}{\partial U \partial \bar{T}} \delta U + \frac{\partial^2 F_1}{\partial U \partial \bar{Y}} \delta U \right]. \tag{25}$$

Take $\delta J_{F_1,F_2}(U, V) = 0$. For the function in the first parenthesis in (25), derive the first order variational equations of $U$, and for the function in the second parenthesis, derive the first order variational equations of $V$. The Euler-Lagrange equations for the time-fractional coupled ZK-mZK equations can be expressed as

$$D^a_T \left( \frac{\partial F_1}{\partial dT_u} \right) - \frac{\partial F_1}{\partial U} \frac{\partial}{\partial X} - \frac{\partial F_1}{\partial \bar{U}} \frac{\partial}{\partial \bar{X}} - \frac{\partial^2 F_1}{\partial U \partial \bar{U}} \frac{\partial^2}{\partial X \partial \bar{X}} = 0,$$

$$D^a_T \left( \frac{\partial F_2}{\partial dT_v} \right) - \frac{\partial F_2}{\partial V} \frac{\partial}{\partial Y} - \frac{\partial F_2}{\partial \bar{V}} \frac{\partial}{\partial \bar{Y}} - \frac{\partial^2 F_2}{\partial V \partial \bar{V}} \frac{\partial^2}{\partial \bar{Y} \partial \bar{Y}} = 0. \tag{26}$$

The last step is to substitute the expressions for $F_1, F_2$ given by (23) into (26), and we get

$$D^a_T U_X + \alpha_1 U_{XX} + \alpha_1 U_{X}^2 + \alpha_2 U_{XX} + \alpha_2 U_{Y}^2 = 0,$$

$$D^a_T V_Y + \alpha_1 V_{YY} + \alpha_1 V_{Y}^2 + \alpha_2 V_{XX} + \alpha_2 V_{Y}^2 = 0. \tag{27}$$

Take potential functions $U_X(X, Y, T) = A_1(X, Y, T)$ and $V_X(X, Y, T) = A_2(X, Y, T)$, and the final solutions are

$$D^a_T A_1 + \alpha_1 A_{1X} + \alpha_1 A_{1}^2 + \alpha_2 A_{1XX} + \alpha_2 A_{1}^2 = 0,$$

$$D^a_T A_2 + \alpha_1 A_{2X} + \alpha_1 A_{2}^2 + \alpha_2 A_{2XX} + \alpha_2 A_{2}^2 = 0. \tag{28}$$

Equations (28) are the time-fractional coupled ZK-mZK equations. When $\alpha = 1$, they are equal to (17). This new set of fractional derivative equations is more suitable to describe the non-conservative system. It provides more possibilities for future research.

### IV. CONSERVATION LAWS OF THE TIME-FRACTIONAL COUPLED ZK-mZK EQUATIONS

#### A. LIE SYMMETRY ANALYSIS

We assume that (28) are invariant under a one parameter Lie group of point transformations in the following form

$$\tilde{X} = X + \epsilon \xi(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{Y} = Y + \epsilon \eta(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{T} = T + \epsilon \tau(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{A_1} = A_1 + \epsilon \phi_1(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{A_2} = A_2 + \epsilon \phi_2(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{A_1} = A_1 + \epsilon \phi_1(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{A_2} = A_2 + \epsilon \phi_2(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{A_1} = A_1 + \epsilon \phi_1(X, Y, T, A_1, A_2) + O(\epsilon^2),$$

$$\tilde{A_2} = A_2 + \epsilon \phi_2(X, Y, T, A_1, A_2) + O(\epsilon^2). \tag{29}$$

where $\xi, \eta, \tau, \phi_1, \phi_2$ are infinitesimal functions, $\phi_1^{XXX}, \phi_1^{XYY}, \phi_2^{XXX}, \phi_2^{XYY}, \phi_1^{T}, \phi_2^{T}$ are the prolongations of infinitesimal functions which are defined as

$$\phi_1^{XXX} = D_X(\phi_1^{XXX}) - A_1XXX + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta),$$

$$\phi_2^{XXX} = D_X(\phi_2^{XXX}) - A_2XXX + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta).$$

$$\phi_1^{XYY} = D_X(\phi_1^{XYY}) - A_1XY + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta),$$

$$\phi_2^{XYY} = D_X(\phi_2^{XYY}) - A_2XY + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta).$$

$$\phi_1^{XXX} = D_X(\phi_1^{XXX}) - A_1XXX + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta),$$

$$\phi_2^{XXX} = D_X(\phi_2^{XXX}) - A_2XXX + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta).$$

$$\phi_1^{XYY} = D_X(\phi_1^{XYY}) - A_1XY + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta),$$

$$\phi_2^{XYY} = D_X(\phi_2^{XYY}) - A_2XY + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta).$$

$$\phi_1^{XXX} = D_X(\phi_1^{XXX}) - A_1XXX + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta),$$

$$\phi_2^{XXX} = D_X(\phi_2^{XXX}) - A_2XXX + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta).$$

$$\phi_1^{XYY} = D_X(\phi_1^{XYY}) - A_1XY + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta),$$

$$\phi_2^{XYY} = D_X(\phi_2^{XYY}) - A_2XY + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta) + A_{1Y}D_X(\eta).$$
where $D_T$, $D_X$, and $D_Y$ are the total derivative operators as follows

$$D_T = \frac{\partial}{\partial T} + A_{1T} \frac{\partial}{\partial A_1} + A_{2T} \frac{\partial}{\partial A_2} + A_{1XT} \frac{\partial}{\partial A_{1X}} + A_{1YT} \frac{\partial}{\partial A_{1Y}} + A_{1YT} \frac{\partial}{\partial A_{1Y}} + A_{2XT} \frac{\partial}{\partial A_{2X}} + A_{2YT} \frac{\partial}{\partial A_{2Y}} + \cdots,$$

$$D_X = \frac{\partial}{\partial X} + A_{1X} \frac{\partial}{\partial A_1} + A_{2X} \frac{\partial}{\partial A_2} + A_{1XX} \frac{\partial}{\partial A_{1XX}} + A_{1XY} \frac{\partial}{\partial A_{1XY}} + A_{2XY} \frac{\partial}{\partial A_{2XY}} + \cdots,$$

$$D_Y = \frac{\partial}{\partial Y} + A_{1Y} \frac{\partial}{\partial A_1} + A_{2Y} \frac{\partial}{\partial A_2} + A_{1Y} \frac{\partial}{\partial A_{1Y}} + A_{2Y} \frac{\partial}{\partial A_{2Y}} + \cdots. \quad (31)$$

Apply the generalized Leibnitz rule as follows

$$D_T^n[f(T)g(T)] = \sum_{n=0}^{\infty} [C_n D_T^{n-\alpha} f(T) D_T^n g(T)], \quad (32)$$

where $C_n = \left(\frac{-(-1)^{n-1} \Gamma(n-\alpha)}{\Gamma(n)}\right)$, and $\alpha > 0$. The chain rule for compound function which is defined as

$$\frac{d^m f(T)}{dT^m} = \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{k}{r} \left[1 - g(T)^{k-r}\right] \frac{d^k f(T)}{dT^k}. \quad (33)$$

For the chain rule (33), when $f(T) = 1$, we can get

$$D_T^n \phi_i = \frac{\partial^n \phi_i}{\partial T^n} + \phi_{iA_1} \frac{\partial^n A_1}{\partial T^n} + \phi_{iA_2} \frac{\partial^n A_2}{\partial T^n} + A_{1T} \frac{\partial^n \phi_{iA_1}}{\partial T^n} + A_{2T} \frac{\partial^n \phi_{iA_2}}{\partial T^n} + \sum_{n=1}^{\infty} \left(\alpha \frac{\partial^n \phi_{iA_1}}{\partial T^n} \right) \left[D_T^{-n} A_1\right] + \sum_{n=1}^{\infty} \left(\alpha \frac{\partial^n \phi_{iA_2}}{\partial T^n} \right) \left[D_T^{-n} A_2\right] + R_{ai}, \quad (34)$$

where $i = 1, 2, \text{and}$

$$R_{ai} = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \sum_{r=0}^{k} \left[\left(\alpha \frac{\partial^n \phi_{iA_1}}{\partial T^n} \right) \left[D_T^{-n} A_1\right] + \sum_{n=1}^{\infty} \left(\alpha \frac{\partial^n \phi_{iA_2}}{\partial T^n} \right) \left[D_T^{-n} A_2\right]\right] \frac{T^n - \alpha}{\Gamma(n + 1 - \alpha)} \frac{\partial^m}{\partial T^m} \left(\frac{1}{m!}\right) (-A_1)^r \left[\frac{1}{m!}\right] (-A_2)^r \left[\frac{1}{m!}\right]. \quad (35)$$

So, when $i = 1, j = 2$ or $i = 2, j = 1$, (34) can be written as

$$\phi_{iA} = \frac{\partial^n \phi_i}{\partial T^n} + (\phi_{iA_1} - \alpha A_{1T}) \frac{\partial^n A_1}{\partial T^n} + \phi_{iA_2} \frac{\partial^n A_2}{\partial T^n} - A_{1T} \frac{\partial^n \phi_{iA_1}}{\partial T^n} - A_{2T} \frac{\partial^n \phi_{iA_2}}{\partial T^n} + \sum_{n=1}^{\infty} \left(\alpha \frac{\partial^n \phi_{iA_1}}{\partial T^n} \right) \left[D_T^{-n} A_1\right] + \sum_{n=1}^{\infty} \left(\alpha \frac{\partial^n \phi_{iA_2}}{\partial T^n} \right) \left[D_T^{-n} A_2\right] + R_{ai}. \quad (36)$$

The infinitesimal generator $V$ can be defined as

$$V = \xi \frac{\partial}{\partial X} + \eta \frac{\partial}{\partial Y} + \tau \frac{\partial}{\partial T} + \phi_1 \frac{\partial}{\partial A_1} + \phi_2 \frac{\partial}{\partial A_2}. \quad (37)$$

Under the infinitesimal transformations, the invariance of the system (28) lead to the invariance condition as follows

$$P_j^{(n)} V(\Delta_1, \Delta_2)|_{\Delta_1=0, \Delta_2=0} = 0, \quad n = 1, 2, 3, \ldots,$$

$$\Delta_1 = D_T^2 A_1 + f_{11} A_{1X} + f_{12} A_{1Y} + f_{13} A_{1Z} + f_{14} \phi_1 + f_{15} A_{1XX} + f_{16} A_{1XY} + \cdot \cdot \cdot,$$

$$\Delta_2 = D_T^2 A_2 + f_{21} A_{2X} + f_{22} A_{2Y} + f_{23} A_{2Z} + f_{24} \phi_2 + f_{25} A_{2XX} + f_{26} A_{2XY} + \cdot \cdot \cdot. \quad (38)$$

According to (37) and (38), we can gain the following invariance criterion

$$\phi_{1A} + f_{11} A_{1X} + f_{12} A_{1Y} + f_{13} A_{1Z} + f_{14} \phi_1 + f_{15} A_{1XX} + f_{16} A_{1XY} + \cdot \cdot \cdot = 0,$$

$$\phi_{2A} + f_{21} A_{2X} + f_{22} A_{2Y} + f_{23} A_{2Z} + f_{24} \phi_2 + f_{25} A_{2XX} + f_{26} A_{2XY} + \cdot \cdot \cdot = 0. \quad (39)$$

Substituting (30), (31), and (36) into (39), by unifying the similar terms of $A$ and its derivative, we can get a bunch of equations about the coefficients. When $f_{11} = f_{12}$, by solving the equations, a set of Lie algebra of point symmetries will be obtained as follows

$$\xi = C_1 + C_4 aX + 2C_4 aT, \quad \eta = C_2 + C_4 dY, \quad \tau = C_3 + 3C_4 T, \quad \phi_1 = 2C_4 A_1, \quad \phi_2 = 2C_4 A_2. \quad (40)$$

Hence, a series of Lie algebra of point symmetries can be written as

$$V_1 = \frac{\partial}{\partial X}, \quad V_2 = \frac{\partial}{\partial Y}, \quad V_3 = \frac{\partial}{\partial T}, \quad V_4 = (\alpha X + 2\alpha T) \frac{\partial}{\partial X} + \alpha Y \frac{\partial}{\partial Y} + 3\alpha \frac{\partial}{\partial T} + 2A_1 \frac{\partial}{\partial A_1} + 2A_2 \frac{\partial}{\partial A_2}. \quad (41)$$
B. CONSERVATION LAWS

We obtain Lie symmetry generator above. According to it, we will discuss conservation laws of the time-fractional coupled ZK-mZK equations in this section. We know that the conserved vectors satisfy conservation equation in the following form

\[ D_T(C^T) + D_X(C^X) + D_Y(C^Y) = 0, \quad (42) \]

where \( C^T, C^X, \) and \( C^Y \) are conserved vectors.

A formal Lagrangian for the time-fractional coupled ZK-mZK equations can be presented as follows

\[ \mathcal{L} = \theta_1[D_\alpha^T A_1 + \alpha_{11} A_{1X} + \alpha_{12} (A_1^2)_X + \alpha_{13} A_1 (A_2)_X + \alpha_{14} (A_3)_X + \alpha_{15} A_{1XXX} + \alpha_{16} A_{1YYY}] + \theta_2[D_\alpha^T A_2 + \alpha_{21} A_{2X} + \alpha_{22} (A_2^2)_X + \alpha_{23} A_2 (A_3)_X + \alpha_{24} (A_4)_X + \alpha_{25} A_{2XXX} + \alpha_{26} A_{2YYY}], \quad (43) \]

where \( \theta_1 = \theta_1(X, Y, T) \) and \( \theta_2 = \theta_2(X, Y, T) \) are new dependent variables. According to the formal Lagrangian, an action integral is defined as the following form

\[ \int \int dxdy (dt)^\mu \mathcal{L}(X, Y, T, A_1, D_\alpha^T A_1, A_{1X}, A_{1XXX}, A_{1YYY}, A_2, D_\alpha^T A_2, A_{2X}, A_{2XXX}, A_{2YYY}, \theta_1(X, Y, T), \theta_2(X, Y, T)). \quad (44) \]

So, we can get the adjoint equations of (28) as Euler-Lagrange equations

\[ F_1^* = \frac{\delta \mathcal{L}}{\delta A_1} = 0, \quad \text{and} \quad F_2^* = \frac{\delta \mathcal{L}}{\delta A_2} = 0, \quad (45) \]

where \( \frac{\delta}{\delta A_1} \) and \( \frac{\delta}{\delta A_2} \) are the Euler-Lagrange operators. They are defined as

\[ \frac{\delta}{\delta A_1} = \frac{\partial}{\partial A_1} + (D_\alpha^T)^* \frac{\partial}{\partial D_\alpha^T A_1} + D_X \frac{\partial}{\partial A_{1X}}, \]

\[ -D_\alpha^T \frac{\partial}{\partial A_{1XXX}} - D_X D_\alpha^T \frac{\partial}{\partial A_{1YYY}}, \]

\[ \frac{\delta}{\delta A_2} = \frac{\partial}{\partial A_2} + (D_\alpha^T)^* \frac{\partial}{\partial D_\alpha^T A_2} + D_X \frac{\partial}{\partial A_{2X}}, \]

\[ -D_\alpha^T \frac{\partial}{\partial A_{2XXX}} - D_X D_\alpha^T \frac{\partial}{\partial A_{2YYY}}, \]

where \((D_\alpha^T)^*\) is the adjoint operators of the Riemann-Liouville fractional differential operator \( D_\alpha^T \), which is given by

\[ (D_\alpha^T)^* = (-1)^n \int_0^\alpha (D_\alpha^T)^{-\alpha} (D_\alpha^T)^\alpha = \frac{\xi}{T} D^\alpha_p, \quad (46) \]

where \( D^\alpha_p \) is the right-sided fractional integral operator of order \( p - \alpha \). And \( D_\alpha^T \) is the right-sided fractional differential operator of order \( \alpha \). So, the adjoint equations (45) can be rewritten as

\[ F_1 = (D_\alpha^T)^* \theta_1 + [\theta_2 (2 \alpha_{22} A_{2X} + \alpha_{23} A_{2X} + 3 \alpha_{14} A_1 A_{1X}) + \theta_2 (2 \alpha_{23} A_{2X} + \alpha_{24} A_{2X} + \alpha_{25} A_{2XXX} + \alpha_{26} A_{2YYY})], \]

\[ + \theta_2 (2 \alpha_{23} A_{2X} + \alpha_{24} A_{2X} + \alpha_{25} A_{2XXX} + \alpha_{26} A_{2YYY})], \quad \theta_1(X, Y, T), \theta_2(X, Y, T)). \]

\[ F_2 = (D_\alpha^T)^* \theta_2 + [\theta_2 (2 \alpha_{22} A_{2X} + \alpha_{23} A_{2X} + 3 \alpha_{24} A_{2X} + \alpha_{25} A_{2XXX} + \alpha_{26} A_{2YYY})], \]

\[ + \theta_2 (2 \alpha_{23} A_{2X} + \alpha_{24} A_{2X} + \alpha_{25} A_{2XXX} + \alpha_{26} A_{2YYY})], \quad \theta_2(X, Y, T), \theta_1(X, Y, T)). \]

On the basis of last section, we get infinitesimal symmetry of (28). In order to get the conservation laws, we assume that the Lie characteristic functions \( W_i (i = 1, 2) \) are as follows

\[ W_i = \phi_i - \tau A_{iT} - \xi A_{iX} - \eta A_{iY}, \quad i = 1, 2. \quad (48) \]

Applying on the \( V_j (j = 1, 2, 3, 4) \) of the symmetry (41), we have

\[ W_{11} = -A_{1X}, \quad W_{12} = -A_{1Y}, \quad W_{13} = -A_{1T}, \]

\[ W_{14} = 2 A_{1} - (\alpha X + 2 \alpha_{11} T) A_{1X} - \alpha Y A_{1Y} - 3 T A_{1T}, \]

\[ W_{21} = -A_{2X}, \quad W_{22} = -A_{2Y}, \quad W_{23} = -A_{2T}, \]

\[ W_{24} = 2 A_{2} - (\alpha X + 2 \alpha_{11} T) A_{2X} - \alpha Y A_{2Y} - 3 T A_{2T}. \quad (49) \]

By using the Riemann-Liouville fractional derivative, the component of conserved vectors of (28) are defined as

\[ C^T = \tau \mathcal{L} + \sum_{k=0}^{n-1} (-1)^k D_{\alpha}^{\alpha-1-k} (W_{1m}) D_{\alpha}^k \frac{\partial \mathcal{L}}{\partial D_{\alpha}^k A_1}, \]

\[ - (-1)^n J(W_{1m}, D_{\alpha}^n A_1) \]

\[ + \sum_{k=0}^{n-1} (-1)^k D_{\alpha}^{\alpha-1-k} (W_{2m}) D_{\alpha}^k \frac{\partial \mathcal{L}}{\partial D_{\alpha}^k A_2}, \]

\[ C^X = \xi \mathcal{L} + W_{1m} \left[ \frac{\partial \mathcal{L}}{\partial A_{1X}} - D_{\alpha}^1 \left( \frac{\partial \mathcal{L}}{\partial A_{1Xi}} \right) - \cdots \right], \]

\[ + D_{\alpha}(W_{1m}) \left[ \frac{\partial \mathcal{L}}{\partial A_{1Xj}} - D_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial A_{1Xij}} \right) - \cdots \right], \]

\[ + W_{2m} \left[ \frac{\partial \mathcal{L}}{\partial A_{2X}} - D_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial A_{2Xi}} \right) - \cdots \right], \]

\[ + D_{\alpha}(W_{2m}) \left[ \frac{\partial \mathcal{L}}{\partial A_{2Xj}} - D_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial A_{2Xij}} \right) - \cdots \right] + \cdots, \]

\[ C^Y = \eta \mathcal{L} + W_{1m} \left[ \frac{\partial \mathcal{L}}{\partial A_{1Y}} - D_{\alpha}^1 \left( \frac{\partial \mathcal{L}}{\partial A_{1Yi}} \right) - \cdots \right], \]

\[ + D_{\alpha}(W_{1m}) \left[ \frac{\partial \mathcal{L}}{\partial A_{1Yj}} - D_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial A_{1Yij}} \right) - \cdots \right], \]

\[ + W_{2m} \left[ \frac{\partial \mathcal{L}}{\partial A_{2Y}} - D_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial A_{2Yi}} \right) - \cdots \right], \]

\[ + D_{\alpha}(W_{2m}) \left[ \frac{\partial \mathcal{L}}{\partial A_{2Yj}} - D_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial A_{2Yij}} \right) - \cdots \right] + \cdots, \]

where \( \frac{\partial \mathcal{L}}{\partial A_{1X}} \) and \( \frac{\partial \mathcal{L}}{\partial A_{1Xj}} \) are the right-sided fractional integral operator of order \( n - \alpha \). And \( \frac{\partial \mathcal{L}}{\partial A_{1X}} \) is the right-sided fractional differential operator of order \( \alpha \). So, the adjoint equations (45) can be integrated as

\[ J(a, b) = \frac{1}{\Gamma(n - \alpha)} \int_0^a \int_0^b \frac{d\mu dT}{T^{\alpha+1-n}} \left[ \frac{f(T, X, Y) g(\mu, X, Y)}{(\mu - T)^{\alpha+1-n}} \right]. \quad (50) \]
When $m = 4$, we can have the following components of conserved vectors
\[
C^T = \theta_1 D_T^{-1} W_{14} + J(W_{14}, \theta_{1T}) \\
+ \theta_2 D_T^{-1} W_{24} + J(W_{24}, \theta_{2T}).
\]
\[
C^X = W_{14}[D_2^{2}(\theta_1) + D_2^2(\theta_1) + \theta_{11} + 2\alpha_1 A_1 + 3\alpha_{14} A_1^3 + 2\theta_{23} A_2] + \theta_1 D_1^2 (W_{14}) - D_X(W_{14})[D_X(\theta_1)] - D_Y(W_{14})[D_Y(\theta_1)] \\
+ \theta_1 D_Y^2(W_{14}) + W_{24}[D_2^2(\theta_2) + D_2^2(\theta_2)] + \theta_1 D_1^2(\theta_{11}) + 2\theta_{23} A_2 + 2\alpha_2 A_2 + \alpha_2 A_2 + 3\alpha_{24} A_2^3 \\
+ D_X(W_{24})[D_X(\theta_2)] - D_Y(W_{24})[D_Y(\theta_2)],
\]
\[
C^Y = W_{14}[D_X D_Y(\theta_1)] - D_X(W_{14})[D_Y(\theta_1)] - D_Y(W_{14})[D_X(\theta_1)] \\
+ W_{24}[D_X D_Y(\theta_2)] - D_X(W_{24})[D_Y(\theta_2)] \\
- D_Y(W_{24})[D_X(\theta_2)] + \theta_2 D_2^2 (W_{24}).
\] (51)

Equations (41) are Lie algebra of point symmetries of the time-fractional coupled ZK-mZK equations and (51) are conservation laws of the time-fractional coupled ZK-mZK equations. Lie symmetries and conservation laws are one of the properties of fractional partial differential equations. The forms of Lie symmetries and conservation laws play an important role in the analysis of the stability of equations and the solution of some special structures. However, in the process of literature review, it is found that the conservation laws of fractional order equations are rarely mentioned.

V. SOLUTIONS OF THE TIME-FRACTIONAL COUPLED ZK-mZK EQUATIONS

In this section, two methods will be used to solve the exact and numerical solutions of (28). Firstly, we calculate the exact solution by applying the Jacobi elliptic function expansion method. Next, we will derive the asymptotic numerical solution of the system by referring to ÁVIM. Then, the relative errors between the numerical solution obtained by the ÁVIM method and the exact solution obtained by the Jacobi elliptic function expansion method at different points are compared. In the end, one set of numerical solutions is taken as an example to analyze the influence of time, order of fractional derivative and coefficient of coupling term.

A. EXACT SOLUTIONS OF THE TIME-FRACTIONAL COUPLED ZK-mZK EQUATIONS

We apply the Jacobi elliptic function expansion method to calculating the exact solutions. In the first place, we use the wave variable as follows:
\[
A_1(X, Y, T) = A_1(\xi), \\
A_2(X, Y, T) = A_2(\xi), \\
\xi = k(X + lY - n \frac{T^a}{\Gamma(1 + \alpha)}),
\] (52)

where $k$, $l$, and $n$ are undetermined positive parameters, and $n$ is the velocity of propagation. So, (28) simplify to polynomials of $A_1, A_2$, and its total derivatives
\[
-kan_2 + ka_1 A_1 + 2ka_2 A_2 + 2ka_{14} A_1 A_2 + 3ka_{14} A_1^2 A_2 \\
+ k^3 A_1 A_1'' + k^3 A_1 A_2'' + 3k^3 A_2 A_2'' = 0.
\] (53)

In order to cancel out higher order nonlinear term $A_1^2 A_1', A_2^2 A_2'$ and higher derivative terms $A_1''', A_2'''$, the functions are assumed to have solutions of the form
\[
A_1(\xi) = a_0 + a_1 Z(\xi), \\
A_2(\xi) = b_0 + b_1 Z(\xi),
\] (54)

where $Z(\xi)^2 = a + b Z^2(\xi) + c Z^4(\xi),$ (55)

and $a, b, c, a_0, a_1, b_0, b_1$ are constants which can be determined later.

Substituting (54) along with (55) into (53) and collecting all the coefficients of $Z^i(\xi) (i = 0, 1, 2, \ldots)$, then setting these coefficients to zero, we yield a set of algebraic equations, which can be solved to find the values of $k, l, a, b, c, a_0, a_1, b_0, b_1$. By calculation we can obtain the following sets of solutions.

Case I. If we set $a = 1, b = -(1 + m^2), c = m^2$, we get
\[
n = \alpha_{11} + 2a_{12} a_0 + \alpha_{13}(\frac{a_0 b_1}{a_1} + b_0) + 3a_{14} a_0^2 \\
- k^2(\alpha_{15} + \alpha_{16})(1 + m^2), \quad l = 1, \\
a_0 = -\frac{\alpha_{12} a_1 + \alpha_{13} b_1}{3\alpha_{14} a_1}, \quad a_1 = \frac{\sqrt{-2(\alpha_{15} + \alpha_{16})m k}}{\alpha_{14}}, \\
b_0 = -\frac{\alpha_{22} b_1 + \alpha_{23} a_1}{3\alpha_{24} b_1}, \quad b_1 = \frac{\sqrt{-2(\alpha_{25} + \alpha_{26})m k}}{\alpha_{24}}.
\]

In this case, (28) have the solutions
\[
A_1 = -\frac{\alpha_{12}}{3\alpha_{14}} \mp \frac{\alpha_{13}}{3\alpha_{14}} \sqrt{\frac{\alpha_{16}(\alpha_{25} + \alpha_{26})}{(\alpha_{15} + \alpha_{16})\alpha_{24}}} \\
\pm \frac{-2(\alpha_{15} + \alpha_{16})m k}{\alpha_{14}} \sqrt{n(\xi)},
\]
\[
A_2 = -\frac{\alpha_{22}}{3\alpha_{24}} \mp \frac{\alpha_{23}}{3\alpha_{24}} \sqrt{\frac{(\alpha_{15} + \alpha_{16})\alpha_{24}}{(\alpha_{15} + \alpha_{16})\alpha_{24}}} \\
\pm \frac{2(\alpha_{25} + \alpha_{26})m k}{\alpha_{24}} \sqrt{n(\xi)},
\]

where $\xi = k(X + Y - (\alpha_{11} + 2a_{12} a_0 + \alpha_{13}(\frac{a_0 b_1}{a_1} + \alpha_{13} b_0) + 3a_{14} a_0^2 - k^2(\alpha_{15} + \alpha_{16})(1 + m^2)) \frac{T^a}{\Gamma(1 + \alpha)}$. In the particular case
where \( m \to 1, \) \( \text{sn}(\xi) \to \tanh(\xi), \) the solutions are:

\[
A_1 = -\frac{\alpha_{12}}{3\alpha_{14}} + \frac{\alpha_{13}}{3\alpha_{14}} \sqrt{\frac{\alpha_{14}(\alpha_{25} + \alpha_{26})}{(\alpha_{15} + \alpha_{16})\alpha_{24}}}
\]

\[
\pm \sqrt{-2(\alpha_{15} + \alpha_{16}) \alpha_{14}} k \tanh(\xi),
\]

\[
A_2 = -\frac{\alpha_{22}}{3\alpha_{24}} + \frac{\alpha_{23}}{3\alpha_{24}} \sqrt{\frac{(\alpha_{15} + \alpha_{16})\alpha_{24}}{\alpha_{14}(\alpha_{25} + \alpha_{26})}}
\]

\[
\pm \sqrt{-2(\alpha_{25} + \alpha_{26}) \alpha_{24}} k \tanh(\xi),
\]

where \( \xi = k(X + Y - (\alpha_{11} + 2\alpha_{12}a_0 + \alpha_{13} \frac{a_{bh_1}}{a_1} + \alpha_{13} b_0 + 3\alpha_{14}a_0^2 - 2\alpha_{15} + \alpha_{16})(2m^2 - 1))^{\frac{p}{(1+\gamma)}}. \)

**Case 2.** If we set \( a = 1 - m^2, b = 2m^2 - 1, \) \( c = -m^2, \) we get

\[
n = \alpha_{11} + 2\alpha_{12} a_0 + \alpha_{13} \frac{a_{bh_1}}{a_1} + \alpha_{13} b_0 + 3\alpha_{14}a_0^2 - 2k^2(\alpha_{15} + \alpha_{16})(2m^2 - 1),
\]

\[
l = 1,
\]

\[
a_0 = -\frac{\alpha_{12}a_1 + \alpha_{13}b_1}{3\alpha_{14}a_1}, \quad a_1 = \pm \sqrt{\frac{2(\alpha_{15} + \alpha_{16})}{\alpha_{14}} mk},
\]

\[
b_0 = -\frac{\alpha_{22}b_1 + \alpha_{23}a_1}{3\alpha_{24}b_1}, \quad b_1 = \pm \sqrt{\frac{2(\alpha_{25} + \alpha_{26})}{\alpha_{24}} mk}.
\]

In this case, (28) have the solutions

\[
A_1 = -\frac{\alpha_{12}}{3\alpha_{14}} + \frac{\alpha_{13}}{3\alpha_{14}} \sqrt{\frac{\alpha_{14}(\alpha_{25} + \alpha_{26})}{(\alpha_{15} + \alpha_{16})\alpha_{24}}}
\]

\[
\pm \sqrt{-2(\alpha_{15} + \alpha_{16}) \alpha_{14}} k \text{dn}(\xi),
\]

\[
A_2 = -\frac{\alpha_{22}}{3\alpha_{24}} + \frac{\alpha_{23}}{3\alpha_{24}} \sqrt{\frac{(\alpha_{15} + \alpha_{16})\alpha_{24}}{\alpha_{14}(\alpha_{25} + \alpha_{26})}}
\]

\[
\pm \sqrt{-2(\alpha_{25} + \alpha_{26}) \alpha_{24}} k \text{dn}(\xi),
\]

where \( \xi = k(X + Y - (\alpha_{11} + 2\alpha_{12}a_0 + \alpha_{13} \frac{a_{bh_1}}{a_1} + \alpha_{13} \frac{a_{bh_1}}{a_1} + \alpha_{13} b_0 + 3\alpha_{14}a_0^2 - 2\alpha_{15} + \alpha_{16})(2m^2 - 1))^{\frac{p}{(1+\gamma)}}. \)

In the particular case where \( m \to 1, \) \( \text{dn}(\xi) \to \text{sech}(\xi), \) the solutions are:

\[
A_1 = -\frac{\alpha_{12}}{3\alpha_{14}} + \frac{\alpha_{13}}{3\alpha_{14}} \sqrt{\frac{\alpha_{14}(\alpha_{25} + \alpha_{26})}{(\alpha_{15} + \alpha_{16})\alpha_{24}}}
\]

\[
\pm \sqrt{-2(\alpha_{15} + \alpha_{16}) \alpha_{14}} k \text{sech}(\xi),
\]

\[
A_2 = -\frac{\alpha_{22}}{3\alpha_{24}} + \frac{\alpha_{23}}{3\alpha_{24}} \sqrt{\frac{(\alpha_{15} + \alpha_{16})\alpha_{24}}{\alpha_{14}(\alpha_{25} + \alpha_{26})}}
\]

\[
\pm \sqrt{-2(\alpha_{25} + \alpha_{26}) \alpha_{24}} k \text{sech}(\xi),
\]

where \( \xi = k(X + Y - (\alpha_{11} + 2\alpha_{12}a_0 + \alpha_{13} \frac{a_{bh_1}}{a_1} + \alpha_{13} \frac{a_{bh_1}}{a_1} + \alpha_{13} b_0 + 3\alpha_{14}a_0^2 - 2\alpha_{15} + \alpha_{16})(2m^2 - 1))^{\frac{p}{(1+\gamma)}}. \)

In order to give a better understanding of the properties of Rossby waves, we take the appropriate parameters of the above exact solution and draw some three-dimensional diagrams. In the process of plotting, we find that (57) and (58) have the same tendency under the appropriate parameters. So we only plot Case 3.

It can be seen from the Fig. 1-4 that the fractional derivative may cause drastic changes in the form of the solution, or it may be more in line with the actual situation. In the subsequent process of numerical solution calculation,
in (58) when \( \alpha = 1 \).

![FIGURE 3. The plot of solution \( A_1 \) in (58) when \( \alpha = 1 \).](image1)

the deformation of solutions (58) with a more stable trend is adopted.

**B. SOLUTIONS OF THE TIME-FRACTIONAL COUPLED ZK-mZK EQUATIONS BY AVIM**

In this part, we will apply the alternative variational iteration method to determine the solution of the time-fractional coupled ZK-mZK equations. Considering (28) and applying the alternative variational iteration method, we construct the following correction functionals

\[
A_{1,k+1}(X, Y, T) = A_{1,k} + \int_0^t (d\tau)\beta_1(\tau)(D^\alpha_T A_{1,k}) + \alpha_{11}(A_{1,k})_X + \alpha_{12}(A^2_{1,k})_X + \alpha_{13}(A_{1,k} A_{2,k})_X + \alpha_{14}(A^3_{1,k})_X + \alpha_{15}(A_{1,k}XXX + \alpha_{16}(A_{1,k}XYZ),
\]

\[
A_{2,k+1}(X, Y, T) = A_{2,k} + \int_0^t (d\tau)\beta_2(\tau)(D^\alpha_T A_{2,k}) + \alpha_{21}(A_{2,k})_X + \alpha_{22}(A^2_{2,k})_X + \alpha_{23}(A_{1,k} A_{2,k})_X + \alpha_{24}(A^3_{2,k})_X + \alpha_{25}(A_{1,k}XXX + \alpha_{26}(A_{2,k}XYZ),
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the general Lagrange multiplier whose optimal value are found via variational theory, And \( A_1(X, Y, 0), A_2(X, Y, 0) \) are initial approximations. Make the above correction functionals stationary, and use the initial conditions:

\[
\delta A_{1,k+1}(X, Y, T) = \delta A_{1,k} + \delta \int_0^t (d\tau)\beta_1(\tau)(D^\alpha_T A_{1,k}) + \alpha_{11}(A_{1,k})_X + \alpha_{12}(A^2_{1,k})_X + \alpha_{13}(A_{1,k} A_{2,k})_X + \alpha_{14}(A^3_{1,k})_X + \alpha_{15}(A_{1,k}XXX + \alpha_{16}(A_{1,k}XYZ),
\]

\[
\delta A_{2,k+1}(X, Y, T) = \delta A_{2,k} + \delta \int_0^t (d\tau)\beta_2(\tau)(D^\alpha_T A_{2,k}) + \alpha_{21}(A_{2,k})_X + \alpha_{22}(A^2_{2,k})_X + \alpha_{23}(A_{1,k} A_{2,k})_X + \alpha_{24}(A^3_{2,k})_X + \alpha_{25}(A_{1,k}XXX + \alpha_{26}(A_{2,k}XYZ),
\]

we yield the Lagrange multiplier as the following forms

\[
\lambda_1 = -1, \quad \text{and} \quad \lambda_2 = -1.
\]

Therefore (59) can be rewritten as

\[
A_{1,k+1}(X, Y, T) = A_{1,k}(X, Y, 0) - \int_0^t (d\tau)\beta_1(\tau)(D^\alpha_T A_{1,k}) + \alpha_{11}(A_{1,k})_X + \alpha_{12}(A^2_{1,k})_X + \alpha_{13}(A_{1,k} A_{2,k})_X + \alpha_{14}(A^3_{1,k})_X + \alpha_{15}(A_{1,k}XXX + \alpha_{16}(A_{1,k}XYZ),
\]

\[
A_{2,k+1}(X, Y, T) = A_{2,k}(X, Y, 0) - \int_0^t (d\tau)\beta_2(\tau)(D^\alpha_T A_{2,k}) + \alpha_{21}(A_{2,k})_X + \alpha_{22}(A^2_{2,k})_X + \alpha_{23}(A_{1,k} A_{2,k})_X + \alpha_{24}(A^3_{2,k})_X + \alpha_{25}(A_{1,k}XXX + \alpha_{26}(A_{2,k}XYZ),
\]

In our work, based on the alternative variational iteration method, we can obtain the variational iteration solution:
be seen from Fig. 1-4 that the form of (58) is more stable.

The selection of initial value has a crucial influence on the solution if 0 exists such that 1 \leq \alpha \leq m:

\[ u(x, t) = \sum_{k=0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-\alpha x^2}}{2\pi \sqrt{\pi}} dx \, dx \, dt \]

which converges to the solution if 0 \leq \gamma \leq 1 exists such that \|v_{k+1}\| \leq \gamma \|v_k\| \forall k \in \mathbb{N} \cup 0.

The selection of initial value has a crucial influence on the solution of numerical solution. So, we choose the initial value of the numerical solution based on the exact solution. It can be seen from Fig. 1-4 that the form of (58) is more stable.

In these circumstances, we start with initial conditions

\[ A_{10}(X, Y, T) = A_{11}(X, Y, T) = A_{20}(X, Y, 0) = A_{21}(X, Y, T) = c_1 \text{sech}(mX + nY), \]

\[ A_{22}(X, Y, T) = A_{23}(X, Y, 0) = c_2 \text{sech}(mX + nY). \]

By the above formulas (62), we can obtain a few first terms being calculated:

\[ A_{11}(X, Y, T) = c_1 \text{sech}(mX + nY) + \frac{c_{11}mT^\alpha}{\Gamma(1 + \alpha)} \times [6mq_1 - p_1] \text{sech}^4(mX + nY) - \alpha_{14}c_1^2 \text{sech}^2(mX + nY) - mq_1 + \alpha_{11}] \times \text{tanh}(mX + nY) \text{sech}(mX + nY), \]

where \( q_1 = \alpha_{15}m + \alpha_{16}n, q_2 = \alpha_{25}m + \alpha_{26}n, p_1 = \alpha_{12}c_1 + \alpha_{13}c_2, p_2 = \alpha_{23}c_1 + \alpha_{24}c_2. \) Due to the complexity of the structure of the numerical solutions, we give the form of A_{12} and A_{22} in Appendix B.

The higher-order approximation can be calculated to the appropriate order using the Maple or Mathematica package, where the infinite approximation leads to the exact solutions.

### C. RESULTS AND DISCUSSION

In order to judge the practicability, veracity and reliability of the methods proposed in this study, the relative errors are discussed between the numerical solution obtained by the alternative variational iteration method and the exact solution obtained by the Jacobi elliptic function expansion method.

Take the first equation of (58) as the exact solution, and calculate A_{14} as the numerical solution. We take \( \alpha = 0.3, \) \( \alpha = 0.7, \) and \( \alpha = 1, \) respectively, in Tab. 1-3 at different points of X and Y. When \( \alpha = 1, \) it means that time derivative is in the order of integers. From the tables, it is obvious to see that the numerical results of the time-fractional coupled ZK-mZK equations obtained by the alternative variational iteration method are satisfactory. This is because we

| \( X \) | \( T = 1 \) | \( T = 2 \) | \( T = 3 \) | \( T = 4 \) | \( T = 5 \) | \( T = 6 \) | \( T = 7 \) | \( T = 8 \) | \( T = 9 \) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 8.97E-04 | 8.64E-04 | 8.33E-04 | 7.98E-04 | 7.62E-04 | 7.27E-04 | 6.91E-04 | 6.57E-04 | 6.21E-04 |
| 1 | 6.96E-04 | 6.54E-04 | 6.23E-04 | 5.91E-04 | 5.62E-04 | 5.35E-04 | 5.09E-04 | 4.84E-04 | 4.60E-04 |
| 2 | 5.04E-04 | 4.72E-04 | 4.41E-04 | 4.11E-04 | 3.82E-04 | 3.55E-04 | 3.30E-04 | 3.06E-04 | 2.83E-04 |
| 3 | 3.74E-04 | 3.46E-04 | 3.20E-04 | 2.95E-04 | 2.72E-04 | 2.51E-04 | 2.32E-04 | 2.14E-04 | 1.98E-04 |
| 4 | 3.13E-04 | 2.87E-04 | 2.62E-04 | 2.39E-04 | 2.19E-04 | 1.99E-04 | 1.82E-04 | 1.66E-04 | 1.51E-04 |
| 5 | 2.78E-04 | 2.55E-04 | 2.35E-04 | 2.18E-04 | 2.03E-04 | 1.89E-04 | 1.77E-04 | 1.65E-04 | 1.54E-04 |
| 6 | 2.52E-04 | 2.31E-04 | 2.13E-04 | 1.99E-04 | 1.87E-04 | 1.76E-04 | 1.66E-04 | 1.58E-04 | 1.51E-04 |
| 7 | 2.34E-04 | 2.16E-04 | 2.00E-04 | 1.88E-04 | 1.78E-04 | 1.70E-04 | 1.63E-04 | 1.57E-04 | 1.52E-04 |

### TABLE 2. The relative errors between AVIM and the Jacobi elliptic function expansion method when \( Y = 0, \alpha = 0.7. \)
TABLE 3. The relative errors between AVIM and the Jacobi elliptic function expansion method when $Y = 0$, $\alpha = 1$.

| $X$ | $T = 1$       | $T = 2$       | $T = 3$       | $T = 4$       | $T = 5$       | $T = 6$       | $T = 7$       | $T = 8$       | $T = 9$       |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0   | 1.87E-04      | 1.87E-04      | 1.87E-04      | 1.87E-04      | 1.87E-04      | 1.87E-04      | 1.87E-04      | 1.87E-04      | 1.87E-04      |
| 1   | 2.05E-04      | 2.05E-04      | 2.05E-04      | 2.05E-04      | 2.05E-04      | 2.05E-04      | 2.05E-04      | 2.05E-04      | 2.05E-04      |
| 2   | 2.47E-04      | 2.47E-04      | 2.47E-04      | 2.47E-04      | 2.47E-04      | 2.47E-04      | 2.47E-04      | 2.47E-04      | 2.47E-04      |
| 3   | 2.71E-04      | 2.71E-04      | 2.71E-04      | 2.71E-04      | 2.71E-04      | 2.71E-04      | 2.71E-04      | 2.71E-04      | 2.71E-04      |
| 4   | 3.36E-04      | 3.36E-04      | 3.36E-04      | 3.36E-04      | 3.36E-04      | 3.36E-04      | 3.36E-04      | 3.36E-04      | 3.36E-04      |
| 5   | 4.04E-04      | 4.04E-04      | 4.04E-04      | 4.04E-04      | 4.04E-04      | 4.04E-04      | 4.04E-04      | 4.04E-04      | 4.04E-04      |
| 6   | 6.42E-04      | 6.42E-04      | 6.42E-04      | 6.42E-04      | 6.42E-04      | 6.42E-04      | 6.42E-04      | 6.42E-04      | 6.42E-04      |
| 7   | 9.72E-04      | 9.72E-04      | 9.72E-04      | 9.72E-04      | 9.72E-04      | 9.72E-04      | 9.72E-04      | 9.72E-04      | 9.72E-04      |
| 8   | 1.15E-03      | 1.15E-03      | 1.15E-03      | 1.15E-03      | 1.15E-03      | 1.15E-03      | 1.15E-03      | 1.15E-03      | 1.15E-03      |
| 9   | 1.34E-03      | 1.34E-03      | 1.34E-03      | 1.34E-03      | 1.34E-03      | 1.34E-03      | 1.34E-03      | 1.34E-03      | 1.34E-03      |
| 10  | 1.46E-03      | 1.46E-03      | 1.46E-03      | 1.46E-03      | 1.46E-03      | 1.46E-03      | 1.46E-03      | 1.46E-03      | 1.46E-03      |

The quasi-geostationary vortex equations are of great practical significance, and the solution of the model established in this article is instructive to the practical production and life in the future. Therefore, we select appropriate values for different parameters to study and discuss their variation trends.

In Fig. 5, three curves with different trends can be clearly seen. When $\alpha = 1$, $A_1 > 0$. When $\alpha = 0.5$, $A_1$ on the left half of the $X$-axis can take a negative value, but can not take a negative value when $X > 1$. And when $\alpha = 0.3$, $A_1$ can get negative values on both sides of the $X$-axis. That is to say, the order of fractional derivative will affect the number and height of peaks and troughs produced by the interaction. Although the fractional order model and integer order model have obvious differences in some places, all of them can describe natural phenomena. Of course, which model is better depends on the question being studied. In Fig. 6, the number and height of peaks and troughs also varies significantly with the time of interaction. Within a certain range, the number of peaks and troughs increases as the time of interaction increases. That is to say, in the short term, an already unstable wave becomes more unstable as the time of interaction increases.

Fig. 7 and Fig. 8 are the effect of the coefficients of the coupling terms of (28) on the interaction. The corresponding two sets of curves in Fig. 7 are obtained by taking symmetric values from the coupled ZK-mZK equations, that is, the interaction problems between the two fluid are completely symmetric. At this time, as the coupling term coefficient increases, the wave width will increase and the wave height will decrease. This is probably due to the fact...
that the wave widens and cancels out with another wave. The corresponding two sets of curves in Fig. 8 are obtained by the coupled ZK-mZK equations without symmetry, that is, the interaction problems between the two fluid are not symmetric. At this point, the coupling term coefficient of the large increase first, the coupling term coefficient of the small increase. This may be due to the large coupling term coefficient and the large influence in the interaction process. The relative positions of its peaks and troughs are not very different.

VI. CONCLUSION

In this article, the Rossby waves in two layers fluid with dissipation and exogenous is studied. Form the dimensionless baroclinic quasi-geostrophic vortex equations include exogenous and dissipative, a new (2 + 1)-dimensional coupled ZK-mZK equations is established. In the weakly coupled system, the coupling terms are generally linear, but in this article, the coupling terms we get are nonlinear terms. We can see that the coupling terms are up to the square terms, and the highest degree terms are not coupled. Then, we deduced the time-fractional coupled ZK-mZK equations and analyze its conservation laws, which provides a basis for the theoretical study of the Rossby waves in two layers fluid. In order to study the properties of solitary waves, we derived the exact solutions and the numerical solutions of the time-fractional coupled ZK-mZK equations. By comparing the relative errors of the numerical solutions and the exact solutions, the accuracy of the alternative variational iteration method can be judged. From the numerical results above, it can be seen that the alternative variational iteration method gives a high-precision numerical solution of the time-fractional coupled ZK-mZK equations. This suggests that the stability of the solution is greatly improved when the stable exact solution is taken as the initial value of the numerical solution. It can be seen from trends of variation in figures that the wave height, the wave width, and the number of peaks are changed when time, order of fractional derivative, and coefficient of coupling term changed. Especially, as the coupling term coefficient increases, the wave width will increase and the wave height will decrease, which has the instruction to the future actual production life.

**APPENDIX A**

\[ b_1 = B_1B_{1yy} - B_{1y}B_{1yy}, \quad b_2 = B_2B_{2yy} - B_{2y}B_{2yy}, \]
\[ R_1 = \frac{B_1}{U_{A0}}[\alpha_{11}B_{1yy} - B_{1y}U_{A0y}f \rho s_c - B_{1yy}U_{A0}f \rho s_c - U_{A0y}F_{A0}m_2B_2 + U_{A0y}F_{A0}m_3B_1]. \]
\[ R_2 = \frac{B_1}{U_{A0}}[-U_{A0y}F_{A0}m_2B_1 + U_{A0y}F_{A0}m_4B_1]. \]
\[ R_3 = \frac{B_1}{U_{A0}}\left(\frac{U_{B0}}{2U_{A0}}(F_A B_2^2 + F_{B1}B_{1y})\right)
  + \alpha_{12}B_{1yy} + U_{A0y}F_{A0}m_1B_1 + \frac{1}{2}m_3B_1. \]
\[ R_4 = -F_{A0}m_5B_1B_2, \quad R_5 = \frac{B_1}{U_{A0}}[\alpha_{13}B_{1yy} + m_4B_1]. \]
\[ R_6 = \frac{B_1}{U_{A0}}[\alpha_{14}B_{1yy} + \frac{b_1B_1}{6U_{A0y}} - \frac{b_1B_1}{2U_{A0y}} + m_1B_1], \]
\[ R_7 = \frac{B_1}{U_{A0}}[-2U_{A0y}(\frac{n_3}{B_1} + m_3B_1y) + U_{B0y}F_{A0}m_2B_1]. \]
\[ R_8 = -2(n_4 + m_2B_{1yy})F_{A0}m_2B_1B_2, \]
\[ R_9 = \frac{B_1}{U_{A0}}[-2U_{A0y}(\frac{n_1}{B_1} + m_1B_1y) + \frac{1}{2}m_2B_1]. \]
\[ R_{10} = \frac{B_1}{U_{A0}}[\alpha_{15}B_{1yy} - B_1U_{A0y}]. \]
\[ R_{11} = \frac{B_1}{U_{A0}}[\alpha_{16}B_{1yy} - U_{A0y}B_1 + 2\frac{n_2}{B_1} + 2m_2B_1y]. \]
\[ S_1 = \frac{B_2}{U_{B0y}}[\alpha_{21}B_{2yy} - B_{2y}U_{B0y}f \rho s_c - B_{2yy}U_{B0}f \rho s_c - U_{B0y}F_{B0}m_4B_1 + U_{B0y}F_{B0}m_3B_2]. \]
\[ S_2 = \frac{B_2}{U_{B0y}}[-U_{B0y}F_{B0}m_3B_1 + U_{A0y}F_{B0}m_5B_2]. \]
\[ S_3 = \frac{B_2}{U_{B0y}}\left(\frac{U_{B0}}{2U_{B0y}}(F_B B_1^2 + F_{B2}B_{1y} - F_{B2}B_{2y})\right)
  + \alpha_{22}B_{2yy} + U_{A0y}F_{B0}m_3B_2 + \frac{1}{2}m_7B_2]. \]
\[ S_4 = -F_{B0}m_2B_1B_2, \quad S_5 = \frac{B_2}{U_{B0y}}[\alpha_{23}B_{2yy} + m_8B_2]. \]
\[ S_6 = \frac{B_2}{U_{B0y}}[\alpha_{24}B_{2yy} + \frac{b_2B_2}{B_2} - \frac{b_2B_2}{2U_{B0y}} + m_5B_2]. \]
\[ S_7 = \frac{B_2}{U_{B0y}}[-2U_{B0y}(\frac{n_7}{B_2} + m_7B_{2y}) + U_{A0y}F_{B0}m_6B_2]. \]
\[ S_8 = -2(n_8 + m_8B_{2yy})F_{B0}m_5B_1B_2, \]
\[ S_9 = \frac{B_2}{U_{B0y}}[-2U_{B0y}(\frac{n_7}{B_2} + m_7B_{2y}) + \frac{1}{2}m_6B_2]. \]
\[ S_{10} = \frac{B_2}{U_{B0y}}[\alpha_{25}B_{2yy} - B_2U_{B0y}]. \]
\[ S_{11} = \frac{B_2}{U_{B0y}}[\alpha_{26}B_{2yy} - U_{B0y}(B_2 + \frac{n_6}{B_2} + 2m_6B_{2y})]. \]
\[ A_{12} = c_1 \sech(\omega) + \frac{m T^\alpha}{\Gamma(1 + \alpha)} \left[ - (36, 0, 0) M_{16} \sech^7(\omega) + (354, 6, 42) M_{16} \sech^5(\omega) - (94, 8, 56) M_{16} \sech^3(\omega) + (6 M_{14} + (16, 0) M_{18}) \sech^2(\omega) + (8, 1, 5) M_{16} \sech(\omega) + (5 M_{14} + (88, 5) M_{18}) \right] \tan(\omega) + \frac{\alpha_{14} c_2^2 m T^{3\alpha}}{\Gamma(1 + 3\alpha) \Gamma(1 + \alpha)} \cdot \left[ \left( - 9 M_5 + (108, 0, 0) M_8 - (16, 0, 0) M_{13} \right) \sech^6(\omega) + (98, 14, 7) M_8 + (63, 7, 8) M_{13} \sech^4(\omega) + (3 M_6 - (42, 3) M_{10}) \sech^2(\omega) + (10, 10, 5) M_8 - (216, 12, 96) M_{13} \sech^2(\omega) \right] + (4, 2) M_{10} \sech(\omega) + (27, 6, 48) M_{13} \tan(\omega) \sech^2(\omega) + 2 m T^{2\alpha} \left[ (1 + 2\alpha^2) - (168, 0, 0) M_{17} \sech^6(\omega) + (16, 4, 1) M_{17} + (189, 0) M_{15} \sech^3(\omega) - (16 \cdot 12 - (256, 36, 9) M_{17}) \sech^4(\omega) - (222, 9) M_{15} \sech^3(\omega) + (54, 12, 3) M_{17} - 3 M_{12} \sech^2(\omega) + (48, 4) M_{15} \sech(\omega) \right] + \frac{\alpha_{14} c_2^2 m T^{4\alpha}}{\Gamma(1 + 4\alpha) \Gamma(1 + \alpha)^2} \cdot \left[ (4 M_1 - (72, 0, 0) M_3 + (432, 0, 0) M_7 - (864, 0, 0) M_{11}) \sech^{10}(\omega) + (11 M_2 - (132, 0, 0) M_4 + (396, 0) M_6) \sech^8(\omega) + (296, 3, 12) \cdot M_2 - (192, 2) M_6 \sech^7(\omega) - (136, 10, 40) \cdot M_2 - (786, 60, 240) M_7 + (1872, 360, 0) M_{11}) \cdot \sech^6(\omega) + (3 M_1 - (71, 17, 68) M_3 + (536, 172, 580) M_7 - (1308, 708, 72) M_{11}) \sech^6(\omega) + (8 M_2 + (475, 289) M_9 - (126, 5, 20) M_9 \sech^5(\omega) - \sech^4(\omega)) + (97, 167, 470) M_7 - (7, 7, 28) M_3 - (332, 414, 156) M_{11} + ((5, 55, 130) M_7 - (33, 69, 111) M_{11}) \cdot \sech^2(\omega) - \sech^3(\omega)^2 + (83, 284) M_9 + (62, 2, 8) M_4 + (4, 103) M_6 \sech(\omega) + (17, 3, 27) M_{11} \right], \]

where

\[ \omega = m X + n Y, \quad \sigma = \omega^{2\alpha} / \Gamma(1 + 2\alpha), \]

\[ q_1 = a_{15} m + a_{16} n, \quad q_2 = a_{25} m + a_{26} n, \]

\[ q_3 = 5 a_{12} a_{14} c_1^3 + a_{13} a_{24} c_1^3 + 8 a_{13} a_{14} c_1 c_2, \]

\[ q_4 = 5 a_{14} a_{23} c_1^3 + 8 a_{23} a_{24} c_1 c_2 + a_{22} a_{24} c_1^3, \]

\[ q_5 = 2 a_{12} c_1 c_2 + a_{13} c_1^2, \quad q_6 = a_{23} c_1^2 + 2 a_{22} c_1 c_2, \]

\[ p_1 = a_{12} c_1 + a_{13} c_2, \quad p_2 = a_{23} c_1 + a_{24} c_2, \]

\[ p_3 = a_{14} c_1 + a_{12} c_2, \quad p_4 = a_{24} c_1 + a_{22} c_2, \]

\[ M_1 = a_{14} c_1^2, \quad M_2 = (m a_{14} c_1 q_1, a_{14} a_{13} c_1 c_2), \quad M_4 = (m a_{14} c_1 p_1, c_2 p_1, a_{13} c_2 p_1^2), \]

\[ M_5 = a_{14} c_1^2, \quad M_7 = (m a_{14} c_1 q_1, m c_1 p_1, m q_1 p_1), \]

\[ M_8 = (m a_{14} c_1 q_1, c_1^2 p_1, p_1), \quad M_9 = (m^2 p_1 q_1, m p_1 q_1), \]

\[ M_{10} = (q_1 q_3 + q_2 q_1 a_{14}, a_{11} c_1 p_1^2), \]

\[ M_{11} = (m_1^3 q_1, m_2^3 q_1, m_1 m_2^2), \quad M_{12} = a_{14} c_1^2 q_1 p_1, \]

\[ M_{13} = (m^2 q_1^2, a_{11} m_1 q_1, a_{11}^2), \quad M_{14} = a_{14} c_1 q_1 p_2, \]

\[ M_{15} = (a_{12} c_1 q_2, a_{13} c_1 c_2, a_{14} c_1 m q_2), \quad M_{16} = (a_{14} c_1^2 q_2, c_1^2 q_1 p_1, a_{13} c_1 q_1 p_2), \]

\[ M_{17} = (a_{14} c_1^2 a_{11} q_1 p_1, a_{14} c_1^2 m^2 a_1 p_1, q_3), \]

\[ M_{18} = (a_{12} c_1^2 m_1 q_2, a_{13} c_1^2 m_1^2, a_{13} c_1^2 m_2 q_2). \]
\[ M_{33} = \begin{bmatrix} (m^2 q_2, a_2 m q_2, c^2 q_1) \end{bmatrix}^T, \quad M_{34} = a_2 a c_2 q_2 p_1, \]
\[ M_{35} = \begin{bmatrix} a_2^2 c_2 q_1, a_2 c_2 m q_1 \end{bmatrix}^T, \]
\[ M_{36} = \begin{bmatrix} a_2 c_2^2 q_1, a_2 c_2^2 p_3, a_2 c_2^2 c q_2 p_2 \end{bmatrix}^T, \]
\[ M_{37} = \begin{bmatrix} a_2^2 c_2^2 q_2, a_2^2 c_2^2 m^2 c^2 q_1 p_2, q_5 \end{bmatrix}^T, \]
\[ M_{38} = \begin{bmatrix} a_2^2 c_2^2 m^2 q_1, a_2 c_2^2 m^2 c_2^2 c q_2 m q_1 \end{bmatrix}^T. \]

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