Differential Calculus on Quantum Spheres

Martin Welk
Universität Leipzig, Institut für Mathematik
Augustusplatz 9, 04109 Leipzig
Germany

Abstract

We study covariant differential calculus on the quantum spheres $S^{2N-1}_q$. Two classification results for covariant first order differential calculi are proved. As an important step towards a description of the noncommutative geometry of the quantum spheres, a framework of covariant differential calculus is established, including a particular first order calculus obtained by factorisation, higher order calculi and a symmetry concept.

1 Introduction

Quantum groups and quantum spaces are important examples of noncommutative geometric spaces. The description of differential calculus on them forms the fundament for an analysis of their geometric structure.

Quantum groups are the most advanced object of study. Covariant—and especially, bicovariant—differential calculi on quantum groups have been under investigation during the last years [7, 4, 5], and basic concepts of differential geometry on quantum groups have already been introduced, see e.g. [2]. There are also results concerning covariant differential calculi on several examples of quantum spaces, e.g. quantum vector spaces, and Podles' spheres [1].

Quantum homogeneous spaces are a class of quantum spaces which are in an especially close relation to quantum groups, therefore presenting themselves as a promising object for investigation.

In this paper, we study the quantum spheres $S^{2N-1}_q$ as introduced by Vaksman and Soibelman [6] as an example of a quantum homogeneous space.

We start by studying covariant first order differential calculi on these quantum spaces. We prove, as our main results on this topic, two classification theorems for first order calculi under slightly different selective constraints. The classification results hold for $N \geq 4$ but the differential calculi included
exist for \( N = 2 \) and \( N = 3 \), too. We point out the relations between the two sets of calculi.

Subsequently, we describe higher order differential calculus on the quantum spheres based on a particular first order differential calculus. Our approach uses ideas from the well-developed theory of bicovariant differential calculi on quantum groups, thereby providing us even with a symmetry concept for tensor products of differential forms linked with the higher order calculus. The framework of higher order differential calculus and symmetry is powerful enough to enable the introduction of basic concepts of noncommutative differential geometry on the quantum spheres \( S_q^{2N-1} \).

## 2 Quantum spaces

### 2.1 General definitions and conventions

We start by collecting some basic definitions on quantum spaces, mostly following the terminology as e.g. in [1].

Let \( \mathcal{A} \) be a Hopf algebra with comultiplication \( \Delta \) and counit \( \varepsilon \). A **quantum space** for \( \mathcal{A} \) is a pair \((X, \Delta_R)\) where \( X \) is a unital algebra and \( \Delta_R : X \to X \otimes \mathcal{A} \) a (right) coaction of \( \mathcal{A} \) on \( X \), i.e. an algebra homomorphism such that \((\Delta_R \otimes \text{id})\Delta_R = (\text{id} \otimes \Delta)\Delta_R \); \((\text{id} \otimes \varepsilon)\Delta_R = \text{id}\). \( X \) is called **quantum homogeneous space** for \( \mathcal{A} \) if there is an embedding \( \iota : X \to \mathcal{A} \) such that \( \Delta_R = \Delta \circ \iota \).

Throughout this paper, the dimension \( N \) of the underlying quantum group \( SU_q(N) \) is a natural number, \( N \geq 2 \). The deformation parameter \( q \) is a real number, \( q \notin \{-1, 0, 1\} \). We use the abbreviations \( Q = q - q^{-1} \), \( s_+ = \sum_{i=0}^{N-1} q^{2i} \), \( s'_+ = s_+ - 1 \).

We also need the R-matrices which are well-known from investigations on the quantum group \( SU_q(N) \), e.g. [3, 4]. Note that \( \hat{R} \) is an invertible \( N^2 \times N^2 \) matrix with the inverse \( \hat{R}^{-1} \), and that \( \hat{R} - \hat{R}^{-1} = QI \) with \( I \) being the \( N^2 \times N^2 \) unit matrix.

\[
\hat{R}_{ij}^{kl} = \begin{cases} 
1 & \text{for } i = l \neq k = j \\
q & \text{for } i = j = k = l \\
Q & \text{for } i = k < j = l \\
0 & \text{otherwise}
\end{cases} \quad \hat{R}^{-ij}_{kl} = \begin{cases} 
1 & \text{for } i = l \neq k = j \\
q^{-1} & \text{for } i = j = k = l \\
-Q & \text{for } i = k > j = l \\
0 & \text{otherwise}
\end{cases}
\]
The following matrices are derived from these fundamental \( R \)-matrices:

\[
\hat{R}_{ij}^{kl} = \hat{R}_{lk}^{ji}; \quad R_{ij}^{kl} = q^{2i-2l} \hat{R}_{ik}^{jl}; \quad \hat{R}_{ij}^{kl} = \hat{R}_{ki}^{lj};
\]

\[
\hat{R}^{-1}_{kl} = \hat{R}^{-1}_{kj}; \quad R^{-1}_{ij} = q^{2l-2i} \hat{R}^{-1}_{ik}^{jl}; \quad \hat{R}^{-1}_{ij} = \hat{R}^{-1}_{ki}^{lj}.
\]

### 2.2 The quantum spheres \( S^2_{q}^{2N-1} \)

Our object of study are the quantum spheres introduced by Vaksman and Soibelman \[6\] which we shall describe now.

Let \( X \) be the free unital algebra with a set of \( 2N \) generators \( \{z_i, z_i^* \mid i = 1, \ldots, N \} \) and defining relations

\[
\begin{align*}
  z_i z_j &= q z_j z_i \quad (1 \leq i < j \leq N) \\
  z_i^* z_j^* &= q^{-1} z_j^* z_i^* \quad (1 \leq i < j \leq N) \\
  z_i z_j^* &= q z_j^* z_i \quad (1 \leq i, j \leq N, \ i \neq j) \\
  \sum_{i=1}^{N} z_i z_i^* &= 1. \tag{2}
\end{align*}
\]

This algebra is made into a \( * \) algebra by letting \( (z_i)^* = z_i^*; (z_i^*)^* = z_i \). Then, \( X \) is called quantum sphere and denoted by \( S^2_{q}^{2N-1} \).

Using the \( R \)-matrices, the relations (1) can be rewritten as

\[
\hat{R}_{ij}^{kl} z_k z_l = q z_i z_j; \quad \hat{R}^{-1}_{ij}^{kl} z_k^* z_l^* = q^{-1} z_i^* z_j^*; \quad \hat{R}_{ij}^{kl} z_k^* z_l = q z_i^* z_j; \quad \hat{R}^{-1}_{ij}^{kl} z_k z_l^* = q^{-1} z_i z_j^*.
\]

The relations (2) imply

\[
\sum_{i=1}^{N} q^{-2i} z_i^* z_i = q^{-2} \quad \text{and} \quad z_i z_i^* - z_i^* z_i + qQ \sum_{k>i} z_k z_k^* = 0.
\]

Let \( u_i^j, 1 \leq i, j \leq N \) be the generators and \( S \) the antipode map of the quantum group \( SU_q(N) \) as defined in \[3\]. Then, by

\[
z_i = u_i^1, \quad z_i^* = (u_i^1)^* = S(u_i^1)
\]

an embedding of \( S^2_{q}^{2N-1} \) into \( SU_q(N) \) is given, making the quantum sphere into a quantum homogeneous space for \( SU_q(N) \) with the coaction

\[
\Delta_R(z_i) = \sum_{j=1}^{N} z_j \otimes u_i^j; \quad \Delta_R(z_i^*) = \sum_{j=1}^{N} z_j^* \otimes S(u_j^i).
\]
3 First order differential calculus

3.1 Basic definitions

First we recall important definitions concerning first order differential calculi, cf. [1].

A first order differential calculus on an algebra $X$ means a pair $(\Gamma, d)$ where $\Gamma$ is a bimodule over $X$ and $d : X \to \Gamma$ is a linear mapping which fulfils Leibniz’ rule $d(xy) = (dx)y + x(dy)$ for all $x, y \in X$, and $\Gamma = \text{Lin}\{xdy \mid x, y \in X\}$. The elements of $\Gamma$ are called one-forms.

A first order differential calculus $(\Gamma, d)$ on a quantum space $(X, \Delta_R)$ for $A$ is called (right) covariant if there is a linear mapping $\Phi_R : \Gamma \to \Gamma \otimes A$ with $(\Phi_R \otimes \text{id})\Phi_R = (\text{id} \otimes \Delta)\Phi_R; (\text{id} \otimes \varepsilon)\Phi_R = \text{id}; \Phi_R(x\omega y) = \Delta_R(x)\Phi_R(\omega)\Delta_R(y); \Phi_R(dx) = (d \otimes \text{id})\Delta_R(x)$ for all $x, y \in X, \omega \in \Gamma$. A one-form $\omega$ is called invariant if $\Delta_R(\omega) = \omega \otimes 1$.

If $X$ is a $*$ algebra then a first order differential calculus $(\Gamma, d)$ is called a $*$ calculus if $\sum_k x_k dy_k = 0$ implies $\sum_k d(y_k^*)x_k^* = 0$ for $x_k, y_k \in X$.

3.2 First order differential calculi on quantum spheres—Results

In this section, we give two classification results for covariant first order differential calculi on the quantum spheres $S^{2N-1}_q$ differing by the set of selective constraints used for classification.

Our first theorem gives a classification of covariant first order differential $*$ calculi on the quantum spheres with the constraint that the calculi be freely generated as left modules by the differentials of the generators of the quantum sphere.

Most of these calculi allow for a factorisation by an additional relation $\Omega_0 = 0$ where $\Omega_0$ is an invariant one-form of $\Gamma$, yielding first order differential $*$ calculi of a different kind. We shall see that a relation of this kind holds in the classical case, too. Therefore it is to be expected that differential calculi of this second type are more adequate to describe the noncommutative geometry of the quantum spheres than are the freely generated ones. This leads us to give also a direct classification of first order differential calculi of this second kind. We shall relax the selective constraint of our classification in this second case to give full account not only of $*$ calculi but of all covari-
ant first order differential calculi on $S_q^{2N-1}$ for which all relations in the left module $\Gamma$ are algebraically generated by one relation $\Omega_0 = 0$ where $\Omega_0$ is a fixed invariant one-form.

It is clear that in any covariant first order differential calculus $(\Gamma, d)$ on $S_q^{2N-1}$ the two one-forms

$$\Omega_+ = \sum_{i=1}^{N} z_i dz_i^*, \quad \Omega_- = \sum_{i=1}^{N} q^{-2i} z_i^* dz_i$$

are invariant and that any invariant one-form in $\Gamma$ is a linear combination of $\Omega_+$ and $\Omega_-.$

**Theorem 1** On $S_q^{2N-1},$ there exist first order differential $\ast$ calculi

- $(\Gamma, d) = (\Gamma_{\alpha\tau}, d)$ where $\alpha \in \mathbb{R} \setminus \{0, q^{-2}\}$ and $\tau \in \mathbb{R},$
- $(\Gamma, d) = (\Gamma'_{\alpha\omega}, d)$ where either $\alpha \in \mathbb{R} \setminus \{0, q^{-2}\}$ and $\omega \in \mathbb{R} \setminus \{0\},$ or $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and $\omega = q^4 \alpha \overline{\alpha},$
- $(\Gamma, d) = (\Gamma'_{\omega\psi}, d)$ where $\omega \in \mathbb{R} \setminus \{0\}$ and $\psi \in \mathbb{R},$
- $(\Gamma, d) = (\Gamma''_{q\tau}, d)$ where $q, \tau \in \mathbb{R} \setminus \{0\},$

which are covariant with respect to $SU_q(\mathbb{N})$ and for which $\{dz_i, dz_i^* \mid i = 1, \ldots, N\}$ is a free left module basis for $\Gamma,$ with their bimodule structure given by

$$\Gamma_{\alpha\tau}: \quad dz_k z_l = q a \hat{R}_{kl}^{-\alpha l} z_k dz_l + (q^2 \alpha - 1) z_k dz_l$$

$$+ q^2 \alpha^2 (1 - \lambda_k^\prime \tau) z_k z_l \Omega_+ + q^2 (1 - \lambda_k^\prime \tau) z_k z_l \Omega_-$$

$$dz_k dz_l^* = q^{-1} \alpha^{-1} R_{kl}^{st} z_k^* dz_l^* + (q^2 \alpha - 1) z_k^* dz_l^*$$

$$+(1 - \lambda_k^\prime \tau) z_k^* z_l^* \Omega_+ + \alpha^{-2} (1 - \lambda_k^\prime \tau) z_k^* z_l^* \Omega_-$$

$$dz_k z_l^* = q^{-1} \alpha^{-1} \hat{R}_{kl}^{st} z_k^* dz_l^* + (q^2 \alpha - 1) z_k^* dz_l^* - q^2 \alpha (1 - \lambda_k^\prime \tau) z_k^* z_l^* \Omega_+$$

$$- q^2 \alpha (1 - \lambda_k^\prime \tau) z_k^* z_l^* \Omega_-$$

$$dz_k^* dz_l^* = q a \hat{R}_{kl}^{st} z_k^* dz_l^* + (q^2 \alpha - 1) z_k^* dz_l^* - q^2 \alpha (1 - \lambda_k^\prime \tau) z_k^* z_l^* \Omega_+$$

$$- q^2 \alpha (1 - \lambda_k^\prime \tau) z_k^* z_l^* \Omega_-$$

$$\Gamma'_{\alpha\omega}: \quad dz_k z_l = q a \hat{R}_{kl}^{-\alpha l} z_k dz_l + (q^2 \alpha - 1) z_k dz_l$$

$$+ \omega z_k z_l \Omega_+ + (\alpha^{-1} \omega - q^2 (\alpha - 1)) z_k z_l \Omega_-$$

$$dz_k^* dz_l^* = q^{-1} \alpha^{-1} \hat{R}_{kl}^{st} z_k^* dz_l^* + (q^2 \alpha - 1) z_k^* dz_l^*$$

$$+(q^2 \alpha - 1) z_k^* z_l^* \Omega_+ + q^2 \omega^{-1} z_k^* z_l^* \Omega_-$$

$$+(q^2 \alpha - 1) z_k^* z_l^* \Omega_-$$
\[ d z_k \cdot z_l^* = q^{-1} \alpha^{-1} \tilde{R}^{-1} \tilde{R}^* z_k \cdot z_l^* + (q^2 \alpha \cdot 1) z_k \cdot d l^* \]
\[ -q^2 \alpha z_k \cdot z_l^* \Omega_+ - \alpha^{-1} z_k \cdot z_l^* \Omega_- \]
\[ d z_k \cdot z_l = q \tilde{R}^* z_k \cdot z_l^* + (q^{-2} \alpha^{-1} - 1) z_k \cdot d l^* \]
\[ -q^2 \alpha z_k \cdot z_l^* \Omega_+ - \alpha^{-1} z_k \cdot z_l^* \Omega_- \]

**Corollary 2** In \((\Gamma, \delta, d)\) with real \(\alpha\), there exists (up to scalar multiples) exactly one invariant one-form \(\Omega_+ + \alpha^{-1} \Omega_-\) that quasi-commutes with all \(x \in X\).

In \((\Gamma'''', \delta, d)\), there exist (up to scalar multiples) exactly two invariant one-forms that quasi-commute with all \(x \in X\). They are given by \(\Omega_+ + \lambda_1 \Omega_-\) and \(\Omega_+ + \lambda_2 \Omega_-\) where \(\lambda_{1,2}\) are the solutions of the quadratic equation
\[ q^{-4} \omega \lambda^2 - q^{-2} \omega \psi + 1 = 0. \]
In \((\Gamma''_{q_\tau}, d)\), with \(\tau / q \neq q^2\), there exist (up to scalar multiples) exactly two invariant one-forms that quasi-commute with all \(x \in X\). One of them is given by \(\Omega_+ + \lambda_1 \Omega_-\), \(\lambda_1 = q^2\tau / q\), while the other one is given by \(\Omega_+\), if \(s'_{+} q = 1\), or by \(\Omega_+ + \lambda_2 \Omega_-\), \(\lambda_2 = q^2(s'_{+} \tau - q^2) / (s'_{+} q - 1)\), otherwise.

In \((\Gamma''_{q_\tau}, d)\), with \(\tau = q^{-2} \neq s'_+^{-1}\), there exists (up to scalar multiples) exactly one invariant one-form, \(\Omega_+ + q^4 \Omega_-\), that quasi-commutes with all \(x \in X\).

In \((\Gamma''_{q_\tau}, d)\), with \(\tau = q^{-2} = s'_+^{-1}\), any invariant one-form quasi-commutes with all \(x \in X\).

**Proof** By direct calculation, one easily checks the quasi-commutation statements for the given invariant one-forms with \(z_m^{(*)}\).

By transforming \((\mu \Omega_+ + \lambda \Omega_-)z_m^{(*)}\), with variable coefficients \(\mu\), \(\lambda\), into a left module expression, we derive necessary conditions for \(\mu\) and \(\lambda\) which lead to the uniqueness assertions of the Corollary.

If, for a first order differential calculus \((\Gamma, d)\), the invariant one-form \(\Omega_0\) quasi-commutes with all \(x \in X\), the calculus \((\Gamma, d)\) allows for a factorisation by the additional relation \(\Omega_0 = 0\). Therefore, we obtain from the calculi of Theorem 1 new calculi \((\tilde{\Gamma}, d)\) for which \(\{d z_i, d z^*_i \mid i = 1, \ldots, N\}\) is no longer a free left module basis for \(\tilde{\Gamma}\).

A factorisation of this kind is quite natural since in the classical limit \((q = 1)\) we have \(\Omega_+ + \Omega_- = d \left(\sum_{i=1}^{N} z_i z_i^*\right) = 0\) anyway.

**Theorem 3** On \(S^2_{q}^{2N-1}\), there exist first order differential calculi

- \((\tilde{\Gamma}, d) = (\tilde{\Gamma}_\lambda, d)\) where \(\lambda \in \mathbb{C} \setminus \{0, q^2\}\),
- \((\tilde{\Gamma}, d) = (\tilde{\Gamma}_\lambda', d)\) where \(\lambda \in \mathbb{C} \setminus \{0\}\),
- \((\tilde{\Gamma}, d) = (\tilde{\Gamma}_\lambda'', d)\) where \(\lambda \in \mathbb{C} \cup \{\infty\}\),
- \((\tilde{\Gamma}, d) = (\tilde{\Gamma}_\lambda^*, d)\) where \(\lambda \in \mathbb{C} \setminus \{0, q^2\}\),
- \((\tilde{\Gamma}, d) = (\tilde{\Gamma}_\lambda^\bullet, d)\) where \(\lambda \in \mathbb{C} \setminus \{0, q^2\}\),

which are covariant with respect to \(\text{SU}_q(N)\) and for which all relations in the left module \(\tilde{\Gamma}\) are algebraically generated by one relation \(\Omega_+ + \lambda \Omega_- = 0\).
(if $\lambda \in \mathbb{C}$) or $\Omega_+ = 0$ (if $\lambda = \infty$), with the bimodule structure given by

\[ \hat{\Gamma}_\lambda : \quad \mathrm{d}z_k z_l = q^\lambda \hat{R}^{st}_{kl} z_s d z_t + (q^2 \lambda^2 - 1) z_k d z_l + q^2 \lambda (\lambda^2 - 1) z_k z_l \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^\lambda \hat{R}^{st}_{kl} z_s^* d z_t^* + (q^2 \lambda^2 - 1) z_k^* d z_l^* - (\lambda - 1) z_k^* z_l^* \Omega_+ \]
\[ \mathrm{d}z_k z_l = q^\lambda \hat{R}^{st}_{kl} z_s d z_t + (q^2 \lambda^2 - 1) z_k d z_l^* - (q^2 \lambda^2 - 1) z_k z_l^* \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^\lambda \hat{R}^{st}_{kl} z_s^* d z_t^* + (q^2 \lambda^2 - 1) z_k^* d z_l^* + (q^2 \lambda^2 - 1) z_k^* z_l^* \Omega_+ \]

\[ \hat{\Gamma}'_\lambda : \quad \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t - \lambda^{-1} (q^2 \lambda^2 - 1) z_k z_l \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* - q^{-2} \lambda (q^2 \lambda^2 - 1) z_k^* z_l^* \Omega_+ \]
\[ \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t + (q^2 \lambda^2 - 1) z_k z_l^* \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* + (q^2 \lambda^2 - 1) z_k^* z_l^* \Omega_+ \]

\[ \hat{\Gamma}''_\lambda, \lambda \not\in \{0, \infty\} : \]
\[ \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* \]
\[ \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t + q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* - q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ \]

\[ \hat{\Gamma}'''_0 : \quad \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* \]
\[ \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t - q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ + q^{2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* - q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ + q^{2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ \]

\[ \hat{\Gamma}''''_0 : \quad \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* \]
\[ \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t - q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ + q^{2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* - q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ + q^{2} \lambda^{-1} (q^2 \lambda^2 - 1) q^{2k} \delta_{kl} \Omega_+ \]

\[ \hat{\Gamma}''_\lambda : \quad \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t - \lambda^{-1} (q^2 \lambda^2 - 1) z_k z_l \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* - q^{-2} \lambda^{-1} (q^2 \lambda^2 - 1) z_k^* z_l^* \Omega_+ \]
\[ \mathrm{d}z_k z_l = q^{-1} \hat{R}^{st}_{kl} z_s d z_t - \lambda^{-1} (q^2 \lambda^2 - 1) z_k z_l \Omega_+ \]
\[ \mathrm{d}z_k^* z_l^* = q^{-1} \hat{R}^{st}_{kl} z_s^* d z_t^* - \lambda^{-1} (q^2 \lambda^2 - 1) z_k^* z_l^* \Omega_+ \]
If $N \geq 4$, any first order differential calculus $(\tilde{\Gamma}, d)$ on $S^2_{q^{2N-1}}$ which is covariant with respect to $SU_q(N)$ and for which all relations in the left module $\tilde{\Gamma}$ are algebraically generated by one relation $\Omega_0 = 0$ with invariant $\Omega_0 \in \tilde{\Gamma}$ is isomorphic to one of the calculi $(\tilde{\Gamma}_\lambda, d), (\tilde{\Gamma}_\lambda', d), (\tilde{\Gamma}_{\lambda,}\, d), (\tilde{\Gamma}_{\lambda,}'', d)$. 

The differential calculi $(\tilde{\Gamma}_\lambda, d)$ for $\lambda \in \mathbb{R} \setminus \{0, q^{-2}\}$, $(\tilde{\Gamma}_\lambda', d)$ for $\lambda \in \mathbb{R} \setminus \{0\}$, and $(\tilde{\Gamma}_{\lambda,}\, d)$ for $\lambda \in \mathbb{R} \cup \{\infty\}$ are $\ast$ calculi.

If $N \geq 4$, any first order differential $\ast$ calculus $(\tilde{\Gamma}, d)$ on $S^2_{q^{2N-1}}$ which is covariant with respect to $SU_q(N)$ and for which all relations in the left module $\tilde{\Gamma}$ are algebraically generated by one relation $\Omega_0 = 0$ ($\Omega_0 \in \tilde{\Gamma}$ invariant), is isomorphic to one of these calculi.

**Corollary 4** The calculi $(\tilde{\Gamma}_\lambda, d)$ are inner, with

$$dx = \Omega x - x \Omega \text{ for all } x \in X$$

for $\Omega := (q^{-2}\lambda - 1)^{-1}$. None of the other calculi described in Theorem 3 is inner.

**Proof** In each of the calculi of the Theorem, all invariant one-forms are scalar multiples of one invariant element $\Omega_+$ or $\Omega_-$. The statement for $(\tilde{\Gamma}_\lambda, d)$ is proved by calculation. The assertion for the remaining calculi follows from the fact that in each of them the non-zero invariant one-forms (which are scalar multiples of one element, $\Omega_+$ or $\Omega_-$) quasi-commute with the algebra generators from at least one of the sets $\{z_i \mid i = 1, \ldots, N\}$ or $\{z_i^* \mid i = 1, \ldots, N\}$.

Now we describe the factorisation of the freely generated first order differential $\ast$ calculi from Theorem 3 by the relations $\Omega_0 = 0$ where $\Omega_0$ are the quasi-commuting invariant one-forms according to Corollary 2.

**Corollary 5** From the first order differential $\ast$ calculi from Theorem 3, the following first order differential $\ast$ calculi are obtained by factorisation:

- for any $\alpha \in \mathbb{R} \setminus \{0, q^{-2}\}$,

$$\Gamma_\alpha/(\Omega_+ + \alpha^{-1}\Omega_-) = \Gamma_\alpha'//(\Omega_+ + \alpha^{-1}\Omega_-) = \tilde{\Gamma}_{\alpha^{-1}};$$

9
• for any \( \omega \in \mathbb{R} \setminus \{0\} \), \( \psi \in \mathbb{R} \), and \( \lambda_{1,2} \) as in Corollary \[ \text{Corollary 2}, \]
\[
\Gamma''_{\omega\psi}/\left(\Omega_{+} + \lambda_{1}\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda_{1}} \quad \text{and} \quad \Gamma''_{\omega\psi}/\left(\Omega_{+} + \lambda_{2}\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda_{2}};
\]

• for any \( \varrho \in \mathbb{R} \setminus \{0, s_{+}^{-1}\} \), \( \tau \in \mathbb{R} \setminus \{0\} \), and \( \lambda_{1,2} \) as in Corollary \[ \text{Corollary 2} \]
(\( \lambda_{1} = \lambda_{2} = q^{4} \) if \( \tau = q^{2} \varrho \)),
\[
\Gamma'''_{\varrho\tau}/\left(\Omega_{+} + \lambda_{1}\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda_{1}} \quad \text{and} \quad \Gamma'''_{\varrho\tau}/\left(\Omega_{+} + \lambda_{2}\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda_{2}};
\]

• for \( \varrho = s_{+}^{-1} \) and any \( \tau \in \mathbb{R} \setminus \{0, q^{2}s_{+}^{-1}\} \), and \( \lambda_{1} \) as in Corollary \[ \text{Corollary 2} \]
\[
\Gamma'''_{\varrho\tau}/\left(\Omega_{+} + \lambda_{1}\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda_{1}} \quad \text{and} \quad \Gamma'''_{\varrho\tau}/\left(\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda_{1}};
\]

• for \( \varrho = q^{-2}\tau = s_{+}^{-1} \), and any \( \lambda \in \mathbb{R} \),
\[
\Gamma'''_{\varrho\tau}/\left(\Omega_{+} + \lambda\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda}; \quad \Gamma'''_{\varrho\tau}/\left(\Omega_{-}\right) = \tilde{\Gamma}'_{\lambda}.
\]

**Proof** By eliminating \( \Omega_{-} \) (\( \Omega_{+} \) for \( \Gamma'''_{\varrho\tau} \) with \( \lambda_{2} = 0 \)) from the equations describing the bimodule structure of \( \Gamma_{\alpha\tau}, \Gamma'_{\alpha\omega}, \Gamma''_{\omega\psi}, \Gamma'''_{\varrho\tau} \), one obtains the corresponding equations for the factorised calculi.

Let us now characterise one particular \( * \) calculus which is of particular importance for our further considerations. It is the calculus \( \tilde{\Gamma}_{1} \) with the relation
\[
\Omega_{+} + \Omega_{-} = \sum_{i=1}^{N} z_{i}dz_{i}^{*} + \sum_{i=1}^{N} q^{-2i}z_{i}^{*}dz_{i} = 0. \quad (4)
\]

The bimodule structure of \( \tilde{\Gamma}_{1} \) is given by
\[
\begin{align*}
dz_{k}z_{l} &= q\hat{R}^{st}_{kl}z_{s}dz_{t}, \\
\dz_{k}^{*}z_{l}^{*} &= q^{-1}\hat{R}^{-st}_{kl}z_{s}^{*}dz_{t}^{*} \\
\dz_{k}z_{l}^{*} &= q^{-1}\hat{R}^{-st}_{kl}z_{s}dz_{t}^{*} + qQz_{k}dz_{l}z_{l}^{*}\Omega \\
\dz_{k}^{*}z_{l} &= q\hat{R}^{st}_{kl}z_{s}dz_{t}^{*} - q^{-1}Qz_{k}^{*}dz_{l} + Q^{2}z_{k}^{*}z_{l}\Omega,
\end{align*}
\]
where \( \Omega = -qQ^{-1}\Omega_{+} = qQ^{-1}\Omega_{-} \).

Our considerations concerning higher order differential calculus and symmetry in section \[ \text{Section 4} \] will be based on this calculus because it has two essential properties which are fulfilled simultaneously only by this calculus.
• First, $\tilde{\Gamma}_1$ is inner, as stated in Corollary 4. We have
\[ dx = \Omega x - x\Omega \] for all $x \in X$. \hfill (6)

• Second, this calculus decomposes into subcalculi on the “homomorphic” and “antiholomorphic” subalgebras of $S_{q^2N-1}$, i.e. the subalgebras generated by $\{ z_i \mid i = 1, \ldots, N \}$ and $\{ z_i^* \mid i = 1, \ldots, N \}$, resp.

To end this section, we just mention three further differential * calculi from Theorem 3 for which the bimodule structure takes a simpler form. These are

- the calculus $\tilde{\Gamma}_q^{N+2}$, displaying the simplest bimodule structure of all calculi:
\[
\begin{align*}
\text{dz}_kz_l &= q^{-1}\hat{R}_{kl}^{st}z_sdz_l; \\
\text{dz}_kz_l^* &= q\hat{R}_{kl}^{st}z_s^*dz_l^*; \\
\text{dz}_k^*z_l^* &= q\hat{R}_{kl}^{st}z_s^*dz_l^*; \\
\text{dz}_k^*z_l &= q^{-1}\hat{R}_{kl}^{st}z_sdz_l.
\end{align*}
\]

- the calculus $\tilde{\Gamma}_q^4$, with
\[
\begin{align*}
\text{dz}_kz_l &= q^{-1}\hat{R}_{kl}^{st}z_sdz_l; \\
\text{dz}_kz_l^* &= q\hat{R}_{kl}^{st}z_s^*dz_l - q^{-1}Qz_kz_l\Omega_+; \\
\text{dz}_k^*z_l^* &= q\hat{R}_{kl}^{st}z_s^*dz_l^* - q^{-1}Qz_k^*z_l\Omega_+.
\end{align*}
\]

- the calculus $\tilde{\Gamma}_q^3$ which is also obtained from $\tilde{\Gamma}_\lambda$ in the limit $\lambda \to q^2$. Its bimodule structure is
\[
\begin{align*}
\text{dz}_kz_l &= q^{-1}\hat{R}_{kl}^{st}z_sdz_l - q^{-1}Qz_kz_l\Omega_+; \\
\text{dz}_kz_l^* &= q\hat{R}_{kl}^{st}z_s^*dz_l; \\
\text{dz}_k^*z_l^* &= q\hat{R}_{kl}^{st}z_s^*dz_l^* - Qz_k^*z_l\Omega_+; \\
\text{dz}_k^*z_l &= q^{-1}\hat{R}_{kl}^{st}z_sdz_l.
\end{align*}
\]

None of these calculi is inner. The first two of them decompose into subcalculi on the holomorphic and antiholomorphic subalgebras.

Note that $(\tilde{\Gamma}_q^4, d) \equiv (\tilde{\Gamma}_q^4, d)$ is the only isomorphy of two calculi from Theorem 3.

3.3 Proof of the classification theorems

3.3.1 Ansatz obtained by morphisms of tensor products

We turn now to prove the Theorems 1 and 3. Our approach is based on an investigation of intertwining mappings for the corepresentations of the
quantum group $A = SU_q(N)$ on the quantum spheres $X = S_q^{2N-1}$. By a similar approach covariant first order differential calculi on Podleś’ spheres $S_q^{2c}$ have been classified by Apel and Schmüdgen [1]. Note that $q$ is not a root of unity; so the representation theory is similar as in the classical case $q = 1$.

Let $V(k)$ be the vector space of all $k$-th order polynomials in the generators $z_i, z_i^*$ of $X$. By the relations (1) and (2), some of these polynomials are identified with polynomials of lower order. They form a vector subspace in $V(k)$. Let $\tilde{V}(k)$ denote the complement of this subspace in $V(k)$. From the coaction $\Delta_R$ we then obtain corepresentations $\pi(k)$ of the quantum group $A = SU_q(N)$ on $X = S_q^{2N-1}$, with $\pi(k) : \tilde{V}(k) \to \tilde{V}(k) \otimes A$.

Here, $\pi(1)$ is the following sum of two irreducible corepresentations, namely the fundamental representation $u$ of $SU_q(N)$ and its contragredient $u^c$.

In order to find covariant first order differential calculi on $X$, intertwining mappings $T \in \text{Mor}(\pi(1) \otimes \pi(1), \pi(k) \otimes \pi(1))$ have to be investigated.

To this goal, the direct sum decompositions of the tensor products $\pi(k) \otimes \pi(1)$, $k = 1, 2, \ldots$, are calculated. This can be accomplished e.g. using Young tableaux; note that $\pi(k+1)$ is obtained from $\pi(k) \otimes \pi(1)$ by removing certain direct summands according to the commutation relations (1).

Intertwining mappings $T \in \text{Mor}(\pi(1) \otimes \pi(1), \pi(k) \otimes \pi(1))$ must correspond to identical direct summands occurring in both the decompositions of $\pi(k) \otimes \pi(1)$, and $\pi(1) \otimes \pi(1)$. If $N \geq 4$, such common summands exist only for $k = 1$ (trivial) and $k = 3$; they lead to morphisms from $\tilde{V}(1) \otimes \tilde{V}(1)$ to $\tilde{V}(1) \otimes \tilde{V}(1)$ and $\tilde{V}(3) \otimes \tilde{V}(1)$ which are listed below. If $N = 2$ or $N = 3$, there are additional morphisms for $k = N - 1$, so the completeness statements in both theorems are guaranteed only for $N \geq 4$.

The resulting morphisms from $\tilde{V}(1) \otimes \tilde{V}(1)$ to $X \otimes \tilde{V}(1)$ are given by

\[
\begin{align*}
    z_k \otimes z_l &\mapsto z_k \otimes z_l, \quad z_k \otimes z_l \mapsto \tilde{R}^{-st}_{kl} z_s \otimes z_t \\
    z_k^* \otimes z_l^* &\mapsto z_k^* \otimes z_l^*, \quad z_k^* \otimes z_l^* \mapsto \tilde{R}^{st}_{kl} z_s^* \otimes z_t^* \\
    z_k \otimes z_l^* &\mapsto z_k \otimes z_l^*, \quad z_k \otimes z_l^* \mapsto \tilde{R}^{-st}_{kl} z_s \otimes z_t \\
    z_k^* \otimes z_l &\mapsto z_k^* \otimes z_l, \quad z_k^* \otimes z_l \mapsto \tilde{R}^{st}_{kl} z_s^* \otimes z_t^* \\
    z_k \otimes z_l^* &\mapsto \delta_{kl} q^{2k} \Psi^+_s, \quad z_k \otimes z_l^* \mapsto \delta_{kl} q^{2k} \Psi^-_s \\
    z_k^* \otimes z_l &\mapsto \delta_{kl} \Psi^+_s, \quad z_k^* \otimes z_l \mapsto \delta_{kl} \Psi^-_s
\end{align*}
\]
that are freely generated as left modules, but needs additional justification in arguments. This yields no difficulty as long as we deal with differential calculi.

Coefficient comparison in the left module \( \Gamma \) is essential for the following arguments. This leads us to the following ansatz for the bimodule structure of any covariant differential calculus on \( S_q^{2N-1} \) which is freely generated as a left module by \( \{ dz_i, dz_i^* \mid i = 1, \ldots, N \} \):

\[
dz_k z_l = a_1 \hat{R}^{-st}_{kl} z_s \dz_t + b_1 z_k \dz_l + c_1 z_k z_l \Omega_+ + e_1 z_k z_l \Omega_-
\]

\[
dz_k^* z_l^* = a_2 \hat{R}^{st}_{kl} z_s^* \dz_t^* + b_2 z_k^* \dz_l^* + c_2 z_k^* z_l^* \Omega_+ + e_2 z_k^* z_l^* \Omega_-
\]

\[
dz_k z_l^* = a_3 \hat{R}^{-st}_{kl} z_s \dz_t + b_3 z_k \dz_l^* + c_3 z_k z_l^* \Omega_+ + d_3 q^{2k} \delta_{kl} \Omega_-
\]

\[
dz_k^* z_l = a_4 \hat{R}^{st}_{kl} z_s^* \dz_t^* + b_4 z_k^* \dz_l + c_4 z_k^* z_l \Omega_+ + d_4 \delta_{kl} \Omega_-
\]

where \( \Omega_+ = \sum_{i=1}^{N} z_i \otimes z_i^* \), \( \Omega_- = \sum_{i=1}^{N} q^{-2i} z_i^* \otimes z_i \).

This leads us to the following ansatz for the bimodule structure of any covariant differential calculus on \( S_q^{2N-1} \) which is freely generated as a left module by \( \{ dz_i, dz_i^* \mid i = 1, \ldots, N \} \):

\[
dz_k z_l = a_1 \hat{R}^{-st}_{kl} z_s \dz_t + b_1 z_k \dz_l + c_1 z_k z_l \Omega_+ + e_1 z_k z_l \Omega_-
\]

\[
dz_k^* z_l^* = a_2 \hat{R}^{st}_{kl} z_s^* \dz_t^* + b_2 z_k^* \dz_l^* + c_2 z_k^* z_l^* \Omega_+ + e_2 z_k^* z_l^* \Omega_-
\]

\[
dz_k z_l^* = a_3 \hat{R}^{-st}_{kl} z_s \dz_t + b_3 z_k \dz_l^* + c_3 z_k z_l^* \Omega_+ + d_3 q^{2k} \delta_{kl} \Omega_-
\]

\[
dz_k^* z_l = a_4 \hat{R}^{st}_{kl} z_s^* \dz_t^* + b_4 z_k^* \dz_l + c_4 z_k^* z_l \Omega_+ + d_4 \delta_{kl} \Omega_-\]

with \( \Omega_+ \) and \( \Omega_- \) as defined in section 3.2. Here, the 20 variables \( a_\nu, b_\nu, c_\nu, d_\nu', e_\nu, f_\nu' \) (\( \nu = 1, 2, 3, 4; \nu' = 3, 4 \)) denote unknown (complex) coefficients.

For the case of covariant first order differential calculi with one relation \( \Omega_0 = 0 \), \( \Omega_0 \) invariant, the ansatz simplifies to

\[
dz_k z_l = a_1 \hat{R}^{-st}_{kl} z_s \dz_t + b_1 z_k \dz_l + c_1 z_k z_l \Omega_1
\]

\[
dz_k^* z_l^* = a_2 \hat{R}^{st}_{kl} z_s^* \dz_t^* + b_2 z_k^* \dz_l^* + c_2 z_k^* z_l^* \Omega_1
\]

\[
dz_k z_l^* = a_3 \hat{R}^{-st}_{kl} z_s \dz_t + b_3 z_k \dz_l^* + c_3 z_k z_l^* \Omega_1 + d_3 q^{2k} \delta_{kl} \Omega_1
\]

\[
dz_k^* z_l = a_4 \hat{R}^{st}_{kl} z_s^* \dz_t^* + b_4 z_k^* \dz_l + c_4 z_k^* z_l \Omega_1 + d_4 \delta_{kl} \Omega_1\]

where \( \Omega_1 \) is an invariant one-form linearly independent on \( \Omega_0 \).

3.3.2 Conditions for the coefficients of the ansatz

The defining relations of \( S_q^{2N-1} \) together with the properties required for a covariant first order differential \( * \) calculus can now be used to compile a system of necessary conditions for the mapping \( d : X \rightarrow \Gamma \), and thereby for the coefficients of [7] and [8].

Coefficient comparison in the left module \( \Gamma \) is essential for the following arguments. This yields no difficulty as long as we deal with differential calculi that are freely generated as left modules, but needs additional justification in
case of the setting of Theorem 3. Now our second classification constraint requires any relation between one-forms to be generated algebraically by $\Omega_0 = 0$, where $\Omega_0$ is a linear combination of $\Omega_+^\ast$ and $\Omega_-^\ast$; thus, any relation between one-forms possibly obstructing coefficient comparison needs to involve at least $N$ of the generators. In fact, none of the coefficient comparisons done in the following involves more than 3 independent generators $dz_i$ or $dz_i^\ast$, so the validity of the classification is guaranteed for $N \geq 4$.

(i) If $\sum z^{(s)}_{k_i} z^{(s)}_{l_i} = 0$ is one of the (homogeneous) defining relations (1), $d \left( \sum z^{(s)}_{k_i} z^{(s)}_{l_i} \right)$ must vanish. Using Leibniz’ rule and (7), this expression can be written as an element of the left $X$ module generated by $dz_i$ and $dz_i^\ast$. Since $\{dz_i, dz_i^\ast\}$ is required to be a left module basis, coefficient comparison can be applied.

(ii) If $\sum z^{(s)}_{k_i} z^{(s)}_{l_i} = 0$ is, again, one of the relations (1) and $dz_{m}^{(s)}$ any of the $2N$ bimodule generators, the expression $dz_{m}^{(s)} \sum z^{(s)}_{k_i} z^{(s)}_{l_i}$ must vanish. Use again (7) to write this as a left module expression and compare coefficients.

(iii) The same procedure can be applied to

$$dz_{m}^{(s)} - dz_{m}^{(s)} \left( \sum_i z_i z_i^\ast \right) \quad \text{and} \quad q^{-2}dz_{m}^{(s)} - dz_{m}^{(s)} \left( \sum_i q^{-2i}z_i^\ast z_i \right) \quad (9)$$

both of which must be zero because of (2).

(iv) From the $\ast$ calculus requirement one can infer $\Omega_+ + \Omega_+^\ast = 0$. From the definition which gives explicitly a left module expression for $\Omega_+$ one obtains by the $\ast$ requirement a right module expression for $\Omega_+^\ast$. Rewrite $\Omega_+^\ast$ to a left module expression using (7) and compare coefficients.

Even more conditions can be derived from the $\ast$ calculus requirement by taking some expression like $z_k^{(s)} d z_l^{(s)}$, apply $\ast$, (7), $\ast$, and (7) again, and compare coefficients with identity.

3.3.3 The case of freely generated $\ast$ calculi

Exploiting conditions from the list given in the preceding section, we obtain a system of equations for $a_\nu, b_\nu, c_\nu, d_\nu', e_\nu, f_\nu'$ of the ansatz (7) which is listed
below. Note that the coefficient comparisons have been done completely for (i) and (ii) but only in part for (iii) and (iv). Complex conjugates occur in equations (33)–(37) which are results of calculus conditions.

\[
\begin{align*}
    a_1 &= b_1, & c_3 &= c_4; & a_1a_3 &= 1 \\
    a_2 &= q(b_2 + 1); & d_3 &= q^{-2}d_4; & a_2a_4 &= 1 \\
    a_3 &= q(b_2 + 1); & e_3 &= e_4; & qa_4 + c_3 + q^2s_4d_3 &= 0 \\
    a_4 &= q^{-1}(b_3 + 1); & f_3 &= q^{-2}f_4; & qa_3 + c_3 + q^2s_4f_3 &= 0
\end{align*}
\]

\[
\begin{align*}
    b_4(c_3 + b_3) + q^{-2}b_1c_3 &= 0 \quad (11) \\
    (q^{-1}a_1 - qa_4 + b_1 + q^{-2}c_1)b_3 + b_2c_1 &= 0 \quad (12) \\
    q^{-2}a_2b_3 + b_2(Qa_2 + b_2 + c_2) &= 0 \quad (13) \\
    b_4c_1 + b_1(-Qa_1 + b_1 + q^{-2}c_1) &= 0 \quad (14) \\
    b_3(b_4 + q^{-2}d_4) + b_2c_4 &= 0 \quad (15) \\
    (qa_2 + b_2 + c_2 - q^{-1}a_3)b_1 + q^{-2}b_1c_2 &= 0 \quad (16) \\
    (-q^{-1}a_2 + b_2 + q^{-1}a_3 + q^{-2}e_3)b_3 + b_2c_3 &= 0 \quad (17) \\
    (-qa_1 + b_1 + qa_4 + c_4)b_4 + q^{-2}b_1c_4 &= 0 \quad (18)
\end{align*}
\]

\[
\begin{align*}
    (q^{-1}a_3 + q^{-2}e_3 - qa_2 - c_2)c_1 - q^{-2}(c_3 + q^2d_3)e_1 + b_3c_4 + (qa_4 + c_4 + d_4 - q^{-1}a_1 - b_1)c_3 &= 0 \quad (19) \\
    (b_3 + e_3)c_4 + c_3f_4 - b_1c_3 - c_1e_2 - e_1f_3 &= 0 \quad (20) \\
    b_4c_3 - b_2c_4 + q^{-2}(c_3 + q^2d_3)e_4 - d_4c_2 - q^{-2}c_1e_2 &= 0 \quad (21) \\
    (qa_4 + c_4 - q^{-1}a_1 - q^{-2}e_1)c_2 - c_2(e_4 + f_4) + (q^{-1}a_3 + q^{-2}e_3 + f_3 - q^{-1}a_2 - b_2)e_4 &= 0 \quad (22) \\
    (-q^{-1}a_1 + qa_4 + c_4 + d_4)d_3 + q^{-2}c_1f_3 &= 0 \quad (23) \\
    (e_4 + f_4)d_3 + q^{-2}e_1f_3 &= 0 \quad (24) \\
    c_2d_4 + q^{-2}(c_3 + d_3)f_4 &= 0 \quad (25) \\
    e_2d_4 + q^{-2}(-q^3a_2 + qa_3 + e_3 + f_3)f_4 &= 0 \quad (26) \\
    (Qa_4 + c_4 + d_4)d_4 + q^{-2}c_1f_4 &= 0 \quad (27) \\
    (e_4 + f_4)d_4 + q^{-2}(qa_1 + e_1 - qa_4)f_4 &= 0 \quad (28) \\
    (qa_2 - qa_3 + c_2)d_3 + q^{-2}(c_3 + q^2d_3)f_3 &= 0 \quad (29) \\
    e_2d_3 + q^{-2}(-q^2Qa_3 + e_3 + q^2f_3)f_3 &= 0 \quad (30) \\
    b_4d_3 + q^{-2}b_1f_3 &= 0 \quad (31) \\
    b_2d_4 + q^{-2}b_3f_4 &= 0 \quad (32) \\
    a_3b_4 + a_3(b_3 - c_3b_4 - q^{-2}c_3b_1) &= 0 \quad (33)
\end{align*}
\]
Let now

\[ a_4 b_3 + a_4 (b_4 - c_4 b_2 - q^{-2} c_4 b_3) = 0 \]  \tag{34}

\[ q a_4 c_3 + e_4 ((b_4 - c_4 b_2 - q^{-2} c_4 b_3)) \]

\[ - (q a_4 + c_4 + d_4) (c_4 (a_4 e_2 + c_2) + q^{-2} c_4 (c_3 + q^2 d_3)) \]

\[ - q^{-2} c_1 (c_4 e_2 + q^{-2} c_4 (a_4 + e_3 + q^2 f_3)) = 0 \]  \tag{35}

\[ (q^{2N+1} a_4 + b_4 - c_4 b_2 - q^{-2} c_4 b_3) d_3 - d_3 = 0 \]  \tag{36}

\[ (q^{2N+1} a_4 + b_4 - c_4 b_2 - q^{-2} c_4 b_3) f_3 - f_3 = 0 \]  \tag{37}

We stress that this (partially redundant) system of equations gives a set of necessary conditions which is not a priori complete since not all required properties of a first order covariant differential * calculus have been fully exploited. So, for any set of coefficients solving this system, it remains still necessary to prove that all required properties are satisfied.

In order to solve this system of equations, we observe first that \((23)-(25)\) together with \((10)\) imply that

\[ d_3 = d_4 = 0 \quad \text{if and only if} \quad f_3 = f_4 = 0. \]  \tag{38}

Furthermore, equation \((33)\) together with \((10)\) implies that

\[ b_1 = 0 \quad \text{if and only if} \quad b_2 = 0. \]  \tag{39}

Note that \((33)\), unlike \((38)\), depends on the * calculus requirement.

Assume now that \(b_1 \neq 0\) and \(d_3 \neq 0\). In this case, we can express \(a_\nu, b_\nu, c_\nu, d_\nu, e_\nu, f_\nu\) (\(\nu = 1, 2, 3, 4; \ \nu' = 3, 4\)) in terms of only two parameters \(\alpha := q^{-1} a_1\) and \(\tau := f_3\), using the equations \((10), (31), \) and \((32)\). Subsequently, \(c_1, c_2, e_1, \) and \(e_2\) are also expressed in terms of these parameters by using \((23)-(25)\).

Let now \(d_3 = 0\). Then the equations \((31)\) and \((32)\) don’t lead to any additional restriction. Instead, we use \((10), (23)-(25), (34)\), and \((33)\) to express \(a_\nu, b_\nu, c_\nu, e_\nu\) by \(\alpha = q^{-1} a_1, \ \omega = c_1\) and \(\psi = c_2 + \alpha^{-1}\). Then, for \(b_1 \neq 0\) equation \((13)\) implies \(\psi = q^{2}\alpha^{-1} + 1\), yielding the bimodule structure of \((\Gamma_{a\omega}^\nu, d)\). For \(b_1 = 0\), \((\Gamma_{a\psi}^\nu, d)\) is obtained.

Finally, if \(b_1 = 0\) and \(d_3 \neq 0\), the coefficients \(a_\nu\) and \(b_\nu\) are already determined while \(c_\nu, e_\nu, d_\nu, f_\nu\) \((\nu = 1, 2, 3, 4; \ \nu' = 3, 4\) depend on two parameters \(q = -q^2 d_3, \ \tau = -f_3\) by equations \((11), (23)-(26)\), yielding the bimodule structure of \((\Gamma_{a\tau}^{\nu}', d)\).

For all calculi, * conditions imply that the parameters be real, except for \((\Gamma_{a\omega}^{\nu}, d)\) where \(\alpha\) can take non-real values if the additional condition \(\omega = q^4 \alpha \overline{\alpha}\) is fulfilled.
One checks that \((\Gamma_\alpha \tau \omega, d), (\Gamma'_\. \omega \psi, d), (\Gamma'\'. \omega \psi, d), (\Gamma''\'. \omega \psi, d)\) satisfy all requirements for a covariant first order differential \(*\) calculus: Covariance is guaranteed by the ansatz \(\#\). The first two groups of conditions in the list above resulting from the relations \(\#\) are completely encoded in the system of equations, so these are also fulfilled. Finally, it is checked by direct calculations that the elements \(\$\) vanish and the \(*\) calculus property is satisfied. This completes the proof of Theorem \(1\).

\[\text{3.3.4 Calculi with the relation } \Omega_0 = 0\]

We shall now use again the conditions of types \([i]\) \([ii]\) in order to specify the possible sets of coefficients in \(\#\) for first order differential calculi with one relation

\[\Omega_+ + \lambda \Omega_- = 0, \quad \lambda \neq 0. \tag{40}\]

Much as in the case of freely generated calculi, we evaluate the conditions \([i]\), observing now the additional condition \(\#\), to obtain the equations

\[
\begin{align*}
  a_1 &= q^{-1}(b_1 + 1); \quad a_3 = q(b_4 + 1); \quad c_3 = c_4; \quad a_1a_3 = 1; \\
  a_2 &= q(b_2 + 1); \quad a_4 = q^{-1}(b_3 + 1); \quad d_3 = q^{-2}N_4; \quad a_2a_4 = 1; \\
  -q\lambda^{-1}a_5 + b_3 + c_3 + q^2s_4d_3 &= -1. \tag{41}
\end{align*}
\]

The condition \(\#\) implies \((\Omega_+ + \lambda \Omega_-)z_m^{(s)} = 0\) which leads to

\[
\begin{align*}
  q^{-2}\lambda b_1 + b_4 &= 0 \tag{43} \\
  q^{-2}\lambda^{-1}b_2 + b_3 &= 0 \tag{44} \\
  (qa_4 + c_4 + d_4 - q^{-1}a_1) + q^{-2}\lambda c_1 &= 0 \tag{45} \\
  (qa_2 + c_2 - q^{-1}a_3) + q^{-2}\lambda(c_3 + q^2d_3) &= 0. \tag{46}
\end{align*}
\]

From \((\#)\), \((\#)\) it follows by means of \(\#\) that

\[
\begin{align*}
  (a_1 &= q\lambda^{-1} \quad \text{or} \quad a_1 = q^{-1}) \tag{47} \\
  (a_2 &= q^{-1}\lambda \quad \text{or} \quad a_2 = q). \tag{48}
\end{align*}
\]

By virtue of \((\#)\), \((\#)\), \((\#)\) and \((\#)\), \(c_\mu\) can be expressed in terms of \(a_1, a_2, d_3\). Evaluation of conditions of type \(\#\) then yields that \(d_3\) has to be zero, except if \(a_1 = q^{-1}\) and \(a_2 = q\). In the latter case, \(d_3\) can still take the values 0 or \(q^{-2}s_4^{-1}(q^4\lambda^{-1} - 1)\). By combination, we obtain five cases which give the differential calculi \((\tilde{\Gamma}_\lambda, d), (\tilde{\Gamma}'_\lambda, d), (\tilde{\Gamma}''_\lambda, d), (\tilde{\Gamma}'''_\lambda, d) \quad (\lambda \neq 0, \infty), (\tilde{\Gamma}'_\lambda, d), (\tilde{\Gamma}''_\lambda, d)\) from Theorem \(\#\), resp.
Similar considerations based on the relations \( \Omega_+ = 0 \) or \( \Omega_- = 0 \) instead of (40) lead to \((\hat{\Gamma}_\infty', d)\) and \((\hat{\Gamma}_0', d)\), resp.

Again, one checks all required properties to establish that all of these are first order differential calculi, and that only the three series \((\hat{\Gamma}_\lambda, d)\), \((\hat{\Gamma}_\lambda', d)\), \((\hat{\Gamma}_\lambda'', d)\) for real \( \lambda \) are * calculi. Thus, Theorem 3 is proved.

\section{Higher order calculus and symmetry}

\subsection{Higher order differential calculi}

In order to describe the noncommutative differential geometry of quantum spaces, it is insufficient to have only first order differential calculus since basic concepts of differential geometry require at least second order differential forms. This causes us to turn our attention to higher order differential calculi on quantum homogeneous spaces.

Our definitions for higher order differential calculi on quantum homogeneous spaces follow those given in [7] for the bicovariant case in the quantum group setting.

Let \( X \) a quantum homogeneous space for the quantum group \( A \). Let \((\Gamma, d)\) be a covariant first order differential calculus on \( X \).

Then, a covariant higher order differential calculus on \( X \) is a pair \((\Gamma^\wedge, d)\) consisting of a graded algebra \( \Gamma^\wedge \) with multiplication denoted by \( \wedge \) and a linear mapping \( d : \Gamma^\wedge \to \Gamma^\wedge \) such that the following conditions are fulfilled:

(i) The degree 0 and degree 1 components of \( \Gamma^\wedge \) are isomorphic to \( X \) and \( \Gamma \), respectively (they will be identified with \( X \) and \( \Gamma \) in the following).

(ii) The mapping \( d \) increases the degree by 1, and \( d \) extends the differential \( d : X \to \Gamma \) from the first order differential calculus \((\Gamma, d)\).

(iii) The mapping \( d \) is a graded derivative, i.e. it fulfils the graded Leibniz' rule

\[ d(\vartheta_1 \wedge \vartheta_2) = d\vartheta_1 \wedge \vartheta_2 + (-1)^d \vartheta_1 \wedge d\vartheta_2 \]

for any \( \vartheta_1, \vartheta_2 \in \Gamma^\wedge \) (with \( d \) being the degree of \( \vartheta_1 \)). For any \( \vartheta \in \Gamma^\wedge \),

\[ d(d\vartheta) = 0. \]
(iv) The covariance map $\Phi_R : \Gamma \to \Gamma \otimes A$ from the first order differential calculus can be extended to a map $\Phi_R^\wedge : \Gamma^\wedge \to \Gamma^\wedge \otimes A$ making $\Gamma^\wedge$ into a covariant $X$-bimodule.

If the underlying first order calculus $(\Gamma, d)$ is a $*$ calculus, then there is an induced $*$ structure on $(\Gamma^\wedge, d)$, and $d(\vartheta^*) = (d\vartheta)^*$ is fulfilled for all $\vartheta \in \Gamma^\wedge$.

### 4.2 Higher order differential calculi on quantum spheres

Now we turn again to study the quantum spheres $S_q^{2N-1}$. Since our main interest is to provide a framework of covariant differential calculus appropriate to describe the noncommutative geometry of the quantum spheres, our considerations should be based on one of the differential calculi from Theorem 3 that are not freely generated (as left modules) by the generator set $\{dz_i, dz_i^* \mid i = 1, \ldots, N\}$ but carry an additional relation of the type $\Omega_+ + \lambda \Omega_- = 0$, as holds in the classical case $q = 1$ with $\lambda = 1$.

We restrict our considerations from now on to the differential calculus $(\Gamma, d) \equiv (\tilde{\Gamma}, d)$ with the relation 4 and bimodule structure 5 given in 3.2. Since we want to transfer ideas and techniques from the theory of bicovariant differential calculi on quantum groups, it is essential to work with an inner calculus since all bicovariant calculi are inner in the quantum group case; on the other hand, for many of the calculations done in this section we need the decomposition of $\tilde{\Gamma}$ into subcalculi on the holomorphic and antiholomorphic subalgebras of $X = S_q^{2N-1}$.

As a consequence of (4), we have the relation $d\Omega_+ + d\Omega_- = 0$ or

$$
\sum_{i=1}^N dz_i \wedge dz_i^* + \sum_{i=1}^N q^{-2i}dz_i^* \wedge dz_i = 0
$$

in any higher order differential calculus extending $(\Gamma, d)$. The bimodule structure of $(\Gamma, d)$ implies

$$
d\Omega = \Omega \wedge \Omega
$$

(remember that $\Omega = qQ^{-1}\Omega_-$, thus $d\Omega = qQ^{-1}d\Omega_-$), and the following commutation relations for the generators $dz_i, dz_i^*$:

$$
\begin{align*}
0 &= dz_k \wedge dz_l + q\tilde{R}_{kl}^{st}dz_s \wedge dz_t \\
0 &= dz_k^* \wedge dz_l^* + q^{-1}\tilde{R}_{kl}^{st}dz_s^* \wedge dz_t^* \\
0 &= q^{-3}\tilde{R}_{kl}^{st}(dz_s^* \wedge dz_t + Q^2z_s^*dz_t \wedge \Omega) \\
&\quad + (dz_k \wedge dz_l^* + Q^2z_kdz_l^* \wedge \Omega) + Q^2(Q^2 + 1)z_kz_l^*d\Omega.
\end{align*}
$$

19
There is a universal higher order differential calculus \((\Gamma^\wedge_u, d)\) which is generated as an algebra by the \(4N\) elements \(z_i, z^*_i, dz_i, dz^*_i\) subject to the relations (1), (2), (4), (5), (49), (50), (51).

The calculus \((\Gamma^\wedge_u, d)\) is not an “inner” calculus in the sense of the differential mapping \(d\) being generated by a graded commutator. So by imposing this as an additional condition, i.e.

\[
d\vartheta = \Omega \wedge \vartheta - (-1)^d \vartheta \wedge \Omega, \quad d = \text{degree of } \vartheta, \tag{52}
\]

for all \(\vartheta\), we obtain a smaller higher order differential calculus which we will denote by \((\Gamma^\wedge_*, d)\). Obviously, (49) and (52) together imply

\[d\Omega = 0.\]

**Proposition 6** In \((\Gamma^\wedge_*, d)\), all differential forms of degree \(2N - 1\) or higher vanish.

*If \(N \geq 3\), there is one differential \((2N - 2)\)-form which generates the set of all \((2N - 2)\)-forms as a left module.*

For the proof we need to consider differential forms

\[\Theta = dz_{j_1} \wedge \ldots \wedge dz_{j_\mu} \wedge dz^*_{k_1} \wedge \ldots \wedge dz^*_{k_\nu}\]

with \(1 \leq j_1 < \ldots < j_\mu \leq N\), \(1 \leq k_1 < \ldots < k_\nu \leq N\). These elements generate \(\Gamma^\wedge_*\) as a left module.

Then, the main argument used to prove both parts of the Proposition is stated in the following lemma:

**Lemma 7** If \(\Theta\) is chosen as above and if the index sets \(J = \{j_1, \ldots, j_\mu\}\) and \(K = \{k_1, \ldots, k_\nu\}\) fulfil \(J \cup K = \{1, \ldots, N\}\) and \(J \cap K \neq \emptyset\), then \(\Theta = 0\).

**Proof of the Lemma** Note that all \(dz_i\) quasi-commute, as do all \(dz^*_i\). Therefore we can use the substitution

\[dz_j \wedge dz^*_j = - \sum_{i \neq j} dz_j \wedge dz^*_j\]

(resulting from \(d\Omega = 0\)) to rewrite \(\Theta\) as a sum of \(N - 1\) members each of which contains one of the expressions \(dz_i \wedge dz_i, dz^*_i \wedge dz^*_i\) with \(1 \leq i \leq N\).
Proof of the Proposition  It is easily seen that each differential form of degree $2N - 1$ is a linear combination (with coefficients from $S_q^{2N-1}$) of forms of type $\Theta$ fulfilling the additional index set condition, and thus vanishes. This proves the first part of the Proposition.

Moreover, for any differential $(2N-2)$-form $\Theta$ of the above type it can be seen that the index sets $J$ and $K$ either fulfil the index set condition leading to $\Theta = 0$, or $J = K = \{1, \ldots, N\} \setminus \{j\}$ with a single index $j \in \{1, \ldots, N\}$. But all $\Theta$'s with the latter property are transformed into (scalar) multiples of each other just by the same substitution as in the proof of the lemma. This completes the proof of the second part.

4.3 A symmetry concept for quantum spaces

For quantum groups, Woronowicz \cite{7} has described a construction extending a bicovariant first order differential calculus $(\Gamma, d)$ to a bicovariant higher order differential calculus $(\Gamma^\wedge, d)$ by antisymmetrisation. For the antisymmetrisation procedure, a bimodule homomorphism of the tensor product $\Gamma \otimes \Gamma$ is required $\sigma : \Gamma \otimes \Gamma \rightarrow \Gamma \otimes \Gamma$, which needs to fulfil the following braid equation on the three-fold tensor product $\Gamma \otimes \Gamma \otimes \Gamma$:

$$((\sigma \otimes \text{id}) \circ (\text{id} \otimes \sigma) \circ (\sigma \otimes \text{id}) \equiv (\text{id} \otimes \sigma) \circ (\sigma \otimes \text{id}) \circ (\text{id} \otimes \sigma)$$

or, in short, $\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$. Here, $\sigma_{i,i+1}$ denote actions of $\sigma$ on the $i$-th and $(i+1)$-th components of a multiple tensor product.

The antisymmetrisation procedure then works as follows: For any permutation $p$ of $\{1, \ldots, k\}$ let $\ell(p)$ denote the length of $p$, i.e. the number of inversions in $p$. Then $p$ has a decomposition $p = t_{i_1} \circ t_{i_2} \circ \cdots \circ t_{i_\ell}$, where $1 \leq i_1, \ldots, i_\ell \leq p - 1$, and $t_i$ means the permutation of length 1 (transposition) that exchanges the $i$-th and $(i+1)$-th elements. Let $\sigma^p = \sigma_{i_1,i_1+1} \circ \sigma_{i_2,i_2+1} \circ \cdots \circ \sigma_{i_\ell,i_\ell+1}$. Because of the braid relation (53), $\sigma^p$ is independent on the choice of the decomposition of $p$ and thereby well-defined. By $A_k = \sum\limits_p (-1)^{\ell(p)} \sigma^p$ an antisymmetriser on the $k$-fold tensor product $\Gamma^{\otimes k}$ is defined. The $k$-th degree component of the higher order differential algebra $\Gamma^\wedge k$, and the higher order differential algebra $\Gamma^\wedge$ are obtained by

$$\Gamma^\wedge k = \Gamma^{\otimes k} / \ker A_k; \quad \Gamma^\wedge = \sum\limits_{k=0}^{\infty} \Gamma^\wedge k.$$ 

\footnote{Here and in the following tensor products of differential modules are always meant to be tensor products over the corresponding quantum group or space, i.e. $\Gamma \otimes \Gamma$ means $\Gamma \otimes_A \Gamma$ or $\Gamma \otimes_X \Gamma$, and so on.

21
Since \( \sigma \) encodes a symmetry in \( \Gamma^\otimes \) which is also relevant for the noncommutative geometry of the quantum group, we want to transfer Woronowicz’s construction to our setting of a quantum homogeneous space with (only one-sided) covariant first order differential calculus.

We start with a definition for a symmetry homomorphism which is already adapted for a slightly generalised situation, admitting factorisation of the underlying tensor product—we shall need this later.

**Definition 1** Let \( \mathcal{A} \) be a quantum group and \( X \) a quantum space for \( \mathcal{A} \). Let \((\Gamma, d)\) be a first order differential calculus for \( X \) which is covariant w. r. t. \( \mathcal{A} \), and \( M \) a sub-bimodule in \( \Gamma \otimes \Gamma \). Let \( \Gamma^\otimes 2_M := (\Gamma \otimes \Gamma)/M \).

Then, a *symmetry homomorphism* for \( \Gamma^\otimes 2_M \) is a bimodule homomorphism \( \sigma : \Gamma^\otimes 2_M \to \Gamma^\otimes 2_M \) for which the braid equation (53) is fulfilled in \( \Gamma^\otimes 3_M := (\Gamma \otimes \Gamma \otimes \Gamma)/M_3 \) where \( M_3 \) is the closure under \( \sigma_{12} \) and \( \sigma_{23} \) of the subbimodule generated by \( M \otimes X + X \otimes M \) in \( \Gamma \otimes \Gamma \otimes \Gamma \).

**4.4 Symmetry on quantum spheres**

We are interested in symmetry homomorphisms which lead to non-trivial higher order differential calculi. By a non-trivial higher order differential calculus we shall mean, in the following statements, a higher order differential calculus in which the differential 2-forms \( dz_k \wedge dz_l \) and \( dz^*_k \wedge dz^*_l \) for \( k \neq l \) are nonzero.

First we let \( M = \{0\} \), thereby seeking symmetry homomorphisms on \( \Gamma \otimes \Gamma \).

**Proposition 8** On \( \Gamma \otimes \Gamma \), there is no bimodule homomorphism \( \sigma \) leading to non-trivial higher order differential calculus.

**Proof** Assume there is a bimodule homomorphism \( \sigma \) on \( \Gamma \otimes \Gamma \) leading to non-trivial higher order differential calculus. The latter requires \( \sigma \) to fulfil (among others) the following equations:

\[
(\sigma - \text{id})(dz_k \otimes dz_l + q\hat{R}^s_{kl}dz_s \otimes dz_l) = 0
\]

\[
(\sigma - \text{id})(dz^*_k \otimes dz^*_l + q^{-1}\hat{R}^{-st}_{kl}dz^*_s \otimes dz^*_l) = 0.
\]

Equation (54) implies that \( \sigma \) is given on elements of the form \( dz_k \otimes dz_l \) by
one of the following three equations:

\[ \sigma(dz_k \otimes dz_l) = dz_k \otimes dz_l \quad \text{(A1)} \]

or \[ \sigma(dz_k \otimes dz_l) = q^{-1} R_{k,l}^{st} dz_s \otimes dz_t \quad \text{(A2)} \]

or \[ \sigma(dz_k \otimes dz_l) = q R_{k,l}^{st} dz_s \otimes dz_t \quad \text{(A3)} \]

equally it follows from equation (55) that, on elements of the form \( dz_k^* \otimes dz_l^* \), the homomorphism \( \sigma \) is given by one of the three equations:

\[ \sigma(dz_k^* \otimes dz_l^*) = dz_k^* \otimes dz_l^* \quad \text{(B1)} \]

or \[ \sigma(dz_k^* \otimes dz_l^*) = q R_{k,l}^{st} dz_s^* \otimes dz_t^* \quad \text{(B2)} \]

or \[ \sigma(dz_k^* \otimes dz_l^*) = q^{-1} R_{k,l}^{st} dz_s^* \otimes dz_t^* \quad \text{(B3)} \]

Now since (A1) or (B1) would annihilate any 2-form \( dz_k \wedge dz_l, dz_k^* \wedge dz_l^* \), resp., in the higher order differential calculus, only the cases (A2) and (A3), (B2) and (B3), resp., need to be considered. By the identities

\[ \Omega \otimes dz_k^* = -qQ^{-1} \sum_{i=1}^{N} z_i dz_i^* \otimes dz_k, \]

\[ \Omega \otimes dz_k = qQ^{-1} \sum_{i=1}^{N} q^{-2i} z_i^* dz_i \otimes dz_k, \]

and the required left module homomorphism property of \( \sigma \), expressions for \( \sigma(\Omega \otimes dz_k^* \otimes dz_l^*) \) are obtained which depend on the respective case conditions (A2) or (A3), (B2) or (B3). From those we infer, by a similar decomposition of \( \Omega \) in the second tensor factor, and the right module homomorphism requirement of \( \sigma \), the following expressions for \( \sigma(\Omega \otimes \Omega) \):

\[ \sigma(\Omega \otimes \Omega) = \Omega \otimes \Omega + qQ^{-1} \left( \sum_{i=1}^{N} dz_i \otimes dz_i^* - q^{-2} \sum_{i=1}^{N} q^{-2i} dz_i^* \otimes dz_i \right) \]

\( \text{(cases (A2), (B3))} \)

\[ \sigma(\Omega \otimes \Omega) = \Omega \otimes \Omega + q^2 Q^{-1} \left( \sum_{i=1}^{N} dz_i \otimes dz_i^* - q^{-2} \sum_{i=1}^{N} q^{-2i} dz_i^* \otimes dz_i \right) \]

\( \text{(cases (A3), (B2))} \)

Since both expressions differ, our case distinction reduces from four to two possible combinations of cases, (A2)/(B3) and (A3)/(B2).

For these cases, expressions for \( \sigma(dz_k^* \otimes dz_l) \) and \( \sigma(dz_k \otimes dz_l^*) \) can be calculated from \( \sigma(\Omega \otimes dz_k^*) \) using (1), (2), and the bimodule homomorphism.
requirement for \( \sigma \); but finally it turns out that the braid relation is violated. This is demonstrated by considering one particular element of \( \Gamma \otimes \Gamma \otimes \Gamma \); namely, we have

\[
(\sigma_{12}\sigma_{23}\sigma_{12} - \sigma_{23}\sigma_{12}\sigma_{23}) \left( \sum_{i=1}^{N} \Omega \otimes dz_i \otimes dz_i^* \right) \neq 0.
\]

This implies that even under the conditions (A2)/(B3) or (A3)/(B2) no symmetry homomorphism for \( \Gamma \otimes \Gamma \) is obtained.

**Lemma 9** In \( \Gamma \otimes \Gamma \), the elements

\[
\sum_{i=1}^{N} q^{-2i}dz_i^* \otimes dz_i - q^{2i} \sum_{i=1}^{N} dz_i \otimes dz_i^* \quad \text{and} \quad \sum_{i=1}^{N} q^{-2i}dz_i^* \otimes dz_i + Q^2 \Omega \otimes \Omega \quad (56)
\]

generate a sub-bimodule \( M_0 \) as a left module.

**Proof** Both of these elements quasi-commute with all \( x \in X \).

Define

\[
\Gamma_0^{\otimes 2} := (\Gamma \otimes \Gamma)/M_0.
\]

**Proposition 10** There are exactly two symmetry homomorphisms on \( \Gamma_0^{\otimes 2} \) leading to non-trivial higher order differential calculi. They are inverse to each other.

One of them is the bimodule homomorphism \( \sigma \) defined by

\[
\begin{align*}
\sigma(dz_k \otimes dz_l) &= q^{-1} \tilde{R}_{kl}^{st} (dz_s \otimes dz_l) \\
\sigma(dz_k^* \otimes dz_l^*) &= q^{-1} \tilde{R}_{kl}^{st} (dz_s^* \otimes dz_l^*) \\
\sigma(dz_k \otimes dz_l^*) &= q^{-3} \tilde{R}_{kl}^{st} (dz_s^* \otimes dz_l + Q^2 z_s z_l^* dz_l \otimes \Omega) \\
&\quad + q^{-1} Q dz_k \otimes dz_l^* \\
&\quad - Q^2 z_l \Omega \otimes dz_l^* - q^{-3} Q^3 z_k z_l^* \Omega \otimes \Omega \\
\sigma(dz_k^* \otimes dz_l) &= q \tilde{R}_{kl}^{st} (dz_s \otimes dz_l^*) + Q^2 z_s dz_l^* \otimes \Omega) \\
&\quad - q^{-2} Q^2 z_k^* \Omega \otimes dz_l + q Q^3 z_k^* z_l \Omega \otimes \Omega.
\end{align*}
\]

**Proof** Since the factorisation of the tensor product brings about no change for the parts of the tensor product which are generated only by \( dz_i \) resp. only
by $dz^*_k$, we can start as in the proof of Prop. 8 and obtain the same possible cases (A2), (A3) for $\sigma(dz_k \otimes dz_l)$, and (B1), (B3) for $\sigma(dz^*_k \otimes dz^*_l)$ as before. The subsequent calculations for $\sigma(\Omega \otimes dz^*_k)$ and $\sigma(\Omega \otimes \Omega)$ remain valid, too; but the resulting expressions are now simplified since the invariant elements \([50]\) are zero in $\Gamma^\otimes 2$. In particular, our argument used above to rule out the combinations (A2)/(B2) and (A3)/(B3) fails since we have now in all cases

$$\sigma(\Omega \otimes \Omega) = \Omega \otimes \Omega.$$ 

To deal, therefore, with (A3)/(B3) (the argument is quite the same for the other case) we calculate expressions for $\sigma(dz_k \otimes dz^*_l)$ and $\sigma(dz^*_k \otimes dz_l)$ just as done for (A2)/(B3) in the proof of Prop. 8. Then we obtain, for any $k \neq l$,

$$\begin{align*}
\left(\sigma - \text{id}\right)(qdz_l \otimes dz_k^* + q^{-2}dz_k^* \otimes dz_l) &= q\Omega dz_l \otimes dz_k^* + qQ^2dz_l dz_k^* \otimes \Omega + q^{-2}Q^2dz_k^* dz_l \otimes \Omega \\
&= -Qdz_l \otimes dz_k^* - q^{-1}Qdz_k^* \otimes dz_l - Q^2dz_k^* dz_l \otimes \Omega + qQ^2(z_l \otimes dz_k^*) - q^{-2}Q^2(z_k^* \otimes dz_l).
\end{align*}$$

Here, the left-hand side should vanish for any $\sigma$ leading to non-trivial higher order differential calculus, but the right-hand side does not.

So we are once more left with (A2)/(B3) and (A3)/(B2). By calculating $\sigma(dz_k \otimes dz^*_l)$ and $\sigma(dz^*_k \otimes dz_l)$ for (A2)/(B3), we find the last two equations of the Proposition.

Consider now the left module homomorphism $\sigma_L : \Gamma \otimes \Gamma \rightarrow \Gamma \otimes \Gamma$ and the right module homomorphism $\sigma_R : \Gamma \otimes \Gamma \rightarrow \Gamma \otimes \Gamma$ defined by the equations \([57]\). In order to prove that the bimodule homomorphism $\sigma$ is well-defined one checks that

(i) the subbimodule $M_0$ of $\Gamma \otimes \Gamma$ is invariant under both $\sigma_L$ and $\sigma_R$;

(ii) for any $a \in \Gamma \otimes \Gamma$, $\sigma_L(a) - \sigma_R(a) \in M_0$.

For the second part it is sufficient to consider expressions of the type $dz^*_j \otimes dz^*_k \cdot z_l$, e.g. for $dz_j \otimes dz_k \cdot z_l$ (the simplest case) we have

$$\begin{align*}
\sigma_L(dz_j \otimes dz_k \cdot z_l) &= q^2\bar{R}^{uv}_{js}R_{ki}^{vt}z_u \sigma_L(dz_v \otimes dz_t) = q\bar{R}^{uv}_{js}R_{ki}^{vt}z_u dz_a \otimes dz_b \\
\sigma_R(dz_j \otimes dz_k \cdot z_l) &= q^{-1}\bar{R}^{vt}_{jk}dz_s \otimes dz_t \cdot z_l = q\bar{R}^{uv}_{js}R_{ki}^{vt}z_u dz_a \otimes dz_b.
\end{align*}$$

Finally, the braid relation \([53]\) has to be proved for $\sigma$. Because of \([3] \text{ and } [23] \text{ on} \)
the elements $dz_k^{(*)} \otimes \Omega \otimes dz_l^{(*)}$. To give again the simplest case,

$$
\sigma_{12}\sigma_{23}\sigma_{12}(dz_k \otimes \Omega \otimes dz_l) = \sigma_{23}\sigma_{12}\sigma_{23}(dz_k \otimes \Omega \otimes dz_l)
$$

$$
= q^{-4}Q\hat{R}_{kl}^s\Omega \otimes dz_s \otimes dz_t + q^{-3}\hat{R}_{kl}^sdz_s \otimes \Omega \otimes dz_t
$$

Another symmetry homomorphism $\sigma'$ is obtained by applying the same procedure to the case (A3)/(B2). From the defining equations for $\sigma$ and $\sigma'$ it can be seen that $\sigma \circ \sigma' \equiv \sigma' \circ \sigma \equiv \text{id}$. Obviously, $\sigma \neq \sigma'$.

As a consequence of Proposition 10, we are now able to transfer Woronowicz's antisymmetrisation construction to the quantum spheres. Let $\Gamma_0^\otimes := \Gamma^\otimes / I_0$ where $I_0$ is the closure under actions of $\sigma_{i,i+1}$ of the two-sided ideal in $\Gamma^\otimes$ generated by the elements $\Gamma_0$. The antisymmetriser $A_k$ is now defined on the $k$-th order component of $\Gamma_0^\otimes$ which is subsequently factorised by the kernel of $A_k$ to obtain $\Gamma^\wedge k$ from which by direct summation the differential algebra $\Gamma^\wedge$ is formed. From the properties of $\sigma$ it is seen that in $\Gamma^\wedge$ the relations of $\Gamma_0^\wedge$ are satisfied. Thus, $\Gamma^\wedge$ is either $\Gamma_0^\wedge$ or a factor algebra of it.

Acknowledgment The work on this paper has been supported by a grant of the Studienstiftung des deutschen Volkes.

References

[1] Apel, J.; Schm"udgen, K.: Classification of three dimensional covariant differential calculi on Podles' quantum spheres and on related spaces.—Lett. Math. Phys. 32 (1994), 25–36.

[2] Heckenberger, I.; Schm"udgen, K.: Levi-Civita Connections on the Quantum Groups $\text{SL}_q(N)$, $\text{O}_q(N)$ and $\text{Sp}_q(N)$.—Commun. Math. Phys. (1997).

[3] Reshetikhin, N. Yu.; Takhtajan, L. A.; Faddeev, L. D.: Kvantovanie grupp Li i algebr Li.—Algebra i analiz 1 (1989), 178–206. (English translation: Leningrad Journal of Mathematics, 1 (1990), 178–206.)

[4] Schm"udgen, K.; Sch"uler, A.: Classification of b covariant differential calculi on quantum groups of type A, B, C and D.—Commun. Math. Phys. 167, 635–670.
[5] Schm"udgen, K.; Sch"uler, A.: Classification of bicovariant differential calculi on quantum groups.—Commun. Math. Phys. 170, 315–335.

[6] Vaksman, L. L.; Soibelman, Ya. S.: Algebra funkciy na kvantovoy gruppe $SU(n + 1)$ i ne"chetnomernie kvantovie sferi.—Algebra i analiz 2 (1990) n. 5, 101–120. (English translation: Leningrad Journal of Mathematics 2 (1990).)

[7] Woronowicz, S. L.: Differential calculus on compact matrix pseudogroups (quantum groups).—Commun. Math. Phys. 122 (1989), 125–170.

Erratum and revision remark: In the first version of this paper (as of Feb., 1998), the classification of freely generated first order differential $\ast$ calculi was collapsed by an error in the system of equations used in the proof of the theorem.—The second classification result, dealing with differential calculi with one relation between invariant elements, is new in the revised version.