On locally constructible spheres and balls

by

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1. Introduction

Ambjørn, Boulatov, Durhuus, Jonsson and others have worked to develop a 3-dimensional analogue of the simplicial quantum gravity theory, as provided for two dimensions by Regge [42]. (See [3] and [43] for surveys.) The discretized version of quantum gravity considers simplicial complexes instead of smooth manifolds; the metric properties are artificially introduced by assigning length $a$ to any edge. (This approach is due to Wein- garten [47] and known as “theory of dynamical triangulations”.) A crucial path integral over metrics, the “partition function for gravity”, is then defined via a weighted sum over all triangulated manifolds of fixed topology. In three dimensions, the whole model is convergent only if the number of triangulated 3-spheres with $N$ facets grows not faster than $C^N$, for some constant $C$. But does this hold? How many simplicial spheres are there with $N$ facets, for $N$ large?

This crucial question still represents a major open problem, which was put into the spotlight also by Gromov [19, pp. 156-157]. Its 2-dimensional analogue, however, was answered a long time ago by Tutte [45], [46], who proved that there are asymptotically fewer than $(16/3\sqrt{3})^N$ combinatorial types of triangulated 2-spheres. (By Steinitz’ theorem, cf. [49, Lecture 4], this quantity equivalently counts the maximal planar maps on $n\geq 4$ vertices, which have $N=2n-4$ faces, and also the combinatorial types of simplicial 3-dimensional polytopes with $N$ facets.)

In the following, the adjective “simplicial” will often be omitted when dealing with balls, spheres or manifolds, as all the regular cell complexes and polyhedral complexes that we consider are simplicial.

B. B. was supported by DFG via the Berlin Mathematical School, G. M. Z. was partially supported by DFG. Both authors are supported by ERC Advanced Grant No. 247029 “SDModels”.
Why are 2-spheres “not so many”? Every combinatorial type of triangulation of the 2-sphere can be generated as follows (see Figure 1): First for some even \(N \geq 4\) build a tree of \(N\) triangles (which combinatorially is the same thing as a triangulation of an \((N+2)\)-gon), and then glue edges according to a complete matching of the boundary edges. A necessary condition in order to obtain a 2-sphere is that such a matching is planar. Planar matchings and triangulations of \((N+2)\)-gons are both enumerated by the Catalan number \(C_{N+2}\), and since the Catalan numbers satisfy a polynomial bound

\[
C_N = \frac{1}{N+1} \binom{2N}{N} < 4^N,
\]

we get an exponential upper bound for the number of triangulations.

Neither this simple argument nor Tutte’s precise count can be easily extended to higher dimensions. Indeed, we have to deal with three different problems when trying to extend results or methods from dimension 2 to dimension 3:

(i) Many combinatorial types of simplicial 3-spheres are not realizable as boundaries of convex 4-polytopes; thus, even though we observe below that there are only exponentially many simplicial 4-polytopes with \(N\) facets, the 3-spheres could still be more numerous.

(ii) The counts of combinatorial types according to the number \(n\) of vertices and according to the number \(N\) of facets are not equivalent any more. We have \(3n - 10 \leq N \leq \frac{1}{2} n(n-3)\) by the lower (resp. upper) bound theorem for simplicial 3-spheres. We know that there are more than \(2^n \sqrt{n}\) 3-spheres [30], [40], but less than \(2^{20n \log n}\) types of 4-polytopes with \(n\) vertices [1], [17], yet this does not answer the question for a count in terms of the number \(N\) of facets.

(iii) While it is still true that there are only exponentially many “trees of \(N\) tetrahedra”, the matchings that can be used to glue 3-spheres are not planar any more; thus, they could be more than exponentially many. If, on the other hand, we restrict ourselves to “local gluings”, we generate only a limited family of 3-spheres, as we will show below.