Abstract. We survey some of our old results given in [CE95] and [CE10] and present some new ones in the last three sections.

1. Introduction and Summary of Results.

1.1. For the convenience of the reader we survey below material that was developed in [CE95] and [CE10].

Let \( v \) be an indeterminate and \( k \) a field of characteristic zero. Let \( U \) be the quantized enveloping algebra defined over \( k(v) \) with generators \( K^\pm, E, F \) and relations

\[
[E, F] = \frac{K - K^{-1}}{v - v^{-1}}, \quad KEK^{-1} = v^2 E \quad \text{and} \quad KFK^{-1} = v^{-2} F.
\]

Let \( U^0 \) be the subalgebra generated by \( K^\pm \) and let \( B \) be the subalgebra generated by \( U^0 \) and \( E \). More precisely we are following the notation given in [CE10] where we take \( I = \{i\}, \ i \cdot i = 2, \ Y = Z[I] \cong Z, \ X = \text{hom}(Z[I], Z) \cong Z, \ F = F_i, \ E = E_i, \) and \( K = K_i \).

Let \( R \) be the power series ring in \( T^{-1} \) with coefficients in \( k(v) \) i.e.

\[
R = k(v)[[T^{-1}]] := \lim_{\leftarrow} k(v)[[T, T^{-1}] / (T - 1)^i]. \quad (1.1)
\]

Set \( K \) equal to the field of fractions of \( R \). Let \( s \) be the involution of \( R \) induced by \( T \to T^{-1} \), i.e. the involution that sends \( T \) to \( T^{-1} = 1/(1 + (T - 1)) = \sum_{i \geq 0} (-1)^i (T - 1)^i \). Let the subscript \( R \) denote the extension of scalars from \( k(v) \) to \( R \), e.g. \( U_R = R \otimes_{k(v)} U \). For any representation \((\pi, A)\) of \( U_R \) we can twist the representation in two ways by composing with automorphisms of \( U_R \). The first is \( \pi \circ (s \otimes 1) \) while the second is \( \pi \circ (1 \otimes \Theta) \) for any automorphism \( \Theta \) of \( U \). We designate the corresponding \( U_R \)-modules by \( A_s \) and \( A_\Theta \).

Twisting the action by both \( s \) and \( \Theta \) we obtain the composite \((A^s)^\Theta = (A^\Theta)^s\) which we denote by \( A^{s\Theta} \).

Let \( m \) denote the homomorphism of \( U^0_R \) onto \( R \) with \( m(K) = T \). For \( \lambda \in Z \) let \( m + \lambda \) denote the homomorphism of \( U^0 \) to \( R \) with \( (m + \lambda)(K) = Tv^\lambda \). We use the additive notation \( m + \lambda \) to indicate that this map originated in the classical setting from an addition of two algebra homomorphisms. It however is not a sum of two homomorphisms but rather a product. Let \( R_{m+\lambda} \) be the corresponding \( R_B \)-module and define the Verma module

\[
\mathcal{R}M(m + \lambda) = U_R \otimes_{U^0_B} R_{m+\lambda}. \quad (1.2)
\]

Let \( \rho_1 : U \to U \) be the algebra isomorphism determined by the assignment

\[
\rho_1(E) = -vF, \quad \rho_1(F) = -v^{-1} E, \quad \rho_1(K) = K^{-1}
\]

(1.3)
for all \( i \in I \) and \( \mu \in Y \). Define also an algebra anti-automorphism \( \rho : \mathbf{U} \to \mathbf{U} \) by
\[
\rho(E) = vKF, \quad \rho(F) = vK^{-1}E, \quad \rho(K_\mu) = K_\mu.
\]
These maps are related through the antipode \( S \) of \( \mathbf{U} \) by \( \rho = \rho_1S \).

For \( \mathbf{U}_R \)-modules \( M, N \) and \( \mathcal{F} \), let \( \mathcal{P}(M, N) \) and \( \mathcal{P}(M, N, \mathcal{F}) \) denote the space of \( R \)-bilinear maps of \( M \times N \) to \( R \) and \( \mathcal{F} \) respectively, with the following invariance condition:
\[
\sum x_1 \star \phi(Sx_3 \cdot a, \phi(x_2)b) = \epsilon(x)\phi(a, b)
\]
where \( \Delta \otimes 1 \circ \Delta(x) = \sum x_1 \otimes x_2 \otimes x_3 \) and \( \epsilon : \mathbf{U} \to k(v) \) is the counit. If we let \( \text{hom}_{\mathbf{U}_R}(A, B) \) denote the set of module \( \mathbf{U}_R \)-module homomorphisms, then one can check on generators of \( \mathbf{U}_R \) that \( \mathcal{P}(M, N, \mathcal{F}) \cong \text{hom}_{\mathbf{U}_R}(M \otimes R N^{\rho_1}, R\mathcal{F}^\rho) \) (see [Jan96, 3.10.6]). Let \( \mathcal{P}(N) = \mathcal{P}(N, N) \) denote the \( R \)-module of invariant forms on \( N \).

For the rest of the introduction we let \( M \) denote the \( \mathbf{U}_R \) Verma module with highest weight \(Tv^{-1} \) i.e. \( M = M(m) \) and let \( \mathcal{F} \) be any finite dimensional \( \mathbf{U} \)-module. A natural parameterization for \( \mathcal{P}(M \otimes R \mathcal{F}) \) was given in [CE95]. Fix an invariant form \( \phi_M \) on \( M \) normalized as in (??). For each \( \mathbf{U}_R \)-module homomorphism \( \beta : R\mathcal{F} \otimes R \mathcal{F}^{\rho_1} \to \mathbf{U}_R \) define what we call the induced form \( \chi_{\beta,\phi_M} \) by the formula, for \( e, f \in R \mathcal{F} \), \( m, n \in M \),
\[
\chi_{\beta,\phi_M}(m \otimes e, n \otimes f) = \phi_M(m, \beta(e \otimes f) \star n).
\]

1.1.1. Proposition ([CE10]). Suppose \( \beta : R\mathcal{F} \otimes R \mathcal{F}^{\rho_1} \to \mathbf{U}_R \) is a module homomorphism with \( \mathbf{U}_R \) having the adjoint action. Then \( M \otimes R \mathcal{F} \) decomposes as the \( \chi_{\beta,\phi_M} \)-orthogonal sum of indecomposable \( \mathbf{U}_R \)-modules.

This last result has a number of intriguing consequences which are formulated in the context of induced filtrations. For any \( R \)-module \( B \) set \( \overline{B} = B/(T - 1) \cdot B \) and for any filtration \( \mathcal{B} = B_0 \supset B_1 \supset \ldots \supset B_r \), let \( \overline{B} = \overline{B_0} \supset \overline{B_1} \supset \ldots \supset \overline{B_r} \) be the induced filtration of \( \mathcal{B} \), with \( \overline{B_i} = (\overline{B_i} + (T - 1) \cdot B)/(T - 1) \cdot B \).

Now an invariant form \( \chi \) on an \( \mathbf{U}_R \)-module \( B \) gives a filtration on \( B \) by setting
\[
B_i = \{ v \in B | \chi(v, B) \subset (T - 1)^i \cdot R \}.
\]

1.1.2. Proposition ([CE10]). Suppose \( \mathcal{F} \) is a finite dimensional \( \mathbf{U} \)-module and \( \phi \) is an invariant form on \( M \otimes R \mathcal{F} \). Let \( M \otimes R \mathcal{F} = B_0 \supset B_1 \supset \ldots \supset B_r \) be the filtration (1.6) and \( \overline{B_0} \supset \overline{B_1} \supset \ldots \supset \overline{B_r} \) the induced filtration on \( M \otimes R \mathcal{F} \).

(1) The \( \mathbf{U} \)-module \( \overline{B_i}/\overline{B_{i+1}} \) is finite dimensional for \( i \) odd.

(2) The \( \mathbf{U} \)-module \( \overline{B_i}/\overline{B_{i+1}} \) is both free and cofree as a \( k(v)[F] \)-module, for \( i \) even.

A final result is cast in the language of hereditary filtrations which we now describe. Let \( \mathcal{E} \) denote the simple two dimensional \( \mathbf{U} \)-module with highest weight \( v \) and set \( P = M \otimes R \mathcal{E} \). Then \( P \) is isomorphic to the basic module \( P_1 = P(m + 1) \) as defined through the equations (??). Set \( M_{\pm} = M(m \pm 1) \). The construction of \( P \) gives inclusions and a short exact sequence:
\[
(T - 1)M_+ \oplus (T - 1)M_- \subset P \subset M_+ \oplus M_- \quad 0 \to (T - 1)M_+ \to P \to M_- \to 0.
\]

Let \( \mathcal{F} \) be any finite dimensional \( \mathbf{U} \)-module and set
\[
\mathbf{A} = (T - 1) \cdot M_+ \otimes R \mathcal{F}, \quad \mathbf{B} = (T - 1) \cdot M_- \otimes R \mathcal{F}, \quad \mathbf{D} = M_- \otimes R \mathcal{F}, \quad \mathbf{P} = P \otimes R \mathcal{F}.
\]
Then (1.8), gives inclusions and the short exact sequence:

\[ A \oplus B \subset P \subset (T - 1)^{-1}(A \oplus B), \quad 0 \rightarrow A \rightarrow P \xrightarrow{P} D \rightarrow 0. \quad (1.10) \]

Now fix an induced invariant form \( \chi = \chi_{\beta, \phi} \) on \( P \) where \( \beta \) is a homomorphism of \( R F \otimes R F \rho_1 \) into \( U_R \) and \( \phi_P \) is an invariant form on \( P \). Filter \( A, B \) and \( P \) using (1.7) and (1.9). Then we say that \( \chi \) gives a hereditary filtration if, for all \( i \),

\[ A_i \cap B = B_i. \quad (1.11) \]

For any weight module \( N \) for \( U \), let \( N_{\leq h} \) denote the span of the weight spaces with weights \( t \leq h \). We say that \( \chi \) gives a weakly hereditary filtration if, for some \( h \) and for all \( i \),

\[ A_i \cap B = B_i. \]

There is an action of the Weyl \( \{1, s\} \) group of \( \mathfrak{sl}_2 \) on the space of forms \( \chi_{\beta, \phi} \) where the lifted form \( \chi^s_{\beta, \phi} \) satisfies \( \chi^s_{\beta, \phi} = \chi_{s \beta, \phi} \) (see [CE10, Theorem 38]). We say that \( \chi \) is even if \( \chi^s_{\beta, \phi} = \chi_{\beta, \phi} \) and odd if \( \chi^s_{\beta, \phi} = -\chi_{\beta, \phi} \). Our aim is to prove our conjecture that

1.1.3. Conjecture. Suppose \( \chi \) is either even or odd. Then \( \chi \) gives a weakly hereditary filtration.

2. \( q \)-Calculus.

2.1. Definitions. As many before us have done, we define

\[
\begin{align*}
[m] & := \frac{v^m - v^{-m}}{v - v^{-1}}, \\
[m]! & := [m] \cdot [m - 1] \cdots [1] \\
[0]! & := 1 \\
\frac{[m]}{[n]} & = \frac{[m]!}{[n]! [m-n]!} \quad \text{for} \quad n \leq m \\
[m]_n & = \begin{cases} 
0 & \text{if} \quad m < n \quad \text{or} \quad n < 0, \\
1 & \text{if} \quad n = m \quad \text{or} \quad m = 0.
\end{cases}
\end{align*}
\]

For \( r \in \mathbb{Z} \) define

\[ [T; r] := \frac{v^r T - v^{-r} T^{-1}}{v - v^{-1}}, \quad (2.1) \]

\[ [T; r]_{(j)} := [T; r] [T; r - 1] \cdots [T; r - j + 1], \quad [T; r]_{(j)} := [T; r + 1] \cdots [T; r + j] \quad \text{if} \quad j > 0, \]

\[ [T; r]_{(0)} := [T; r]^{(0)} := 1, \]

\[ [\frac{T}{j}; r] := \begin{cases} 
[T; r]_{(j)} / [j]! & \text{if} \quad j \geq 0 \\
0 & \text{if} \quad j < 0.
\end{cases} \quad (2.3) \]

Observe that

\[ [T; r]_{(j)} = [T; r + j]_{(j)}. \quad (2.4) \]
we have used the notation $T$ homomorphisms above from their module homomorphism counterparts we will sometimes and for $\lambda \geq 0$ and $[T; \lambda+1]_{(k)}$ is invertible provided and $\lambda + 1 > k$ or $\lambda < 0$ ($k \geq 0$). In fact

$$[r]![T; r]^{-1}_{(r)} \cong T^0 \mod (T - 1).$$

Indeed the map sending $T \mapsto 1$ defines a surjection of $R$ onto $k(v)$ and under this map $[T; r]_{(r)} \mapsto [r]!$. Moreover for $k \neq 0,$

$$[T; k]^{-1} = T[k]^{-1} \left( \frac{1}{1 - (T - 1) \frac{v^k}{1-v^k}} \right) \left( \frac{1}{1 + (T - 1) \frac{v^k}{1+v^k}} \right)$$

and the two fractions on the right can be written as power series in $T - 1$ with 1 as their leading coefficient.

2.2. Identities. Two useful formulae for us will be

$$\begin{bmatrix} s - u \\ r \end{bmatrix} = \sum_p (-1)^p v^{p(s-u-r+1)+ru} \begin{bmatrix} u \\ p \end{bmatrix} \begin{bmatrix} s - p \\ r - p \end{bmatrix}$$

(2.7)

$$\begin{bmatrix} u + v + r - 1 \\ r \end{bmatrix} = \sum_p v^{p(u+v)-ru} \begin{bmatrix} u + p - 1 \\ p \end{bmatrix} \begin{bmatrix} v + r - p - 1 \\ r - p \end{bmatrix}$$

(2.8)

which come from [Ma93], equations 1.160a and 1.161a respectively.

3. $U_i$ Automorphisms and Intertwining Maps.

3.1. Following Lusztig, [Lus93, Chapter 5], we let $\mathcal{C}'$ denote the category whose objects are $\mathbb{Z}$-graded $U$-modules $M = \oplus_{n \in \mathbb{Z}} M^n$ such that

(i) $E,F$ act locally nilpotently on $M$,
(ii) $Km = v^m m$ for all $m \in M^n$.

Fix $e = \pm 1$ and let $M \in \mathcal{C}'$. Define Lusztig’s automorphisms $T'_i, T''_i : M \rightarrow M$ by

$$T'_i(m) := \sum_{a,b,c,a-b+c=n} (-1)^b v^{(a-c)} F(a) E(b) F(c) m,$$

(3.1)

and

$$T''_i(m) := \sum_{a,b,c,a-b+c=n} (-1)^b v^{(a-c)} E(a) F(b) E(c) m$$

(3.2)

for $m \in M^n$. In the above $E(a) := E^n/[a]!$ is the $a$-th divided power of $E$.

Lusztig defined automorphisms $T'_e$ and $T''_e$ on $U$ by

$$T'_e(E(p)) = (-1)^p v^{p(p-1)} K^{p} F(p), \quad T'_e(F(p)) = (-1)^p v^{-p(p-1)} E(p) K^{-p}$$

and

$$T''_e(E(p)) = (-1)^p v^{p(p-1)} F(p) K^{-p}, \quad T''_e(F(p)) = (-1)^p v^{-p(p-1)} K^{p} E(p).$$

One can check on generators that $\rho_1 \circ T'_{-1} = T'_{-1} \circ \rho_1$. In order to distinguish the algebra homomorphisms above from their module homomorphism counterparts we will sometimes use the notation $T'_{e,mod}$ and $T''_{e,mod}$ to denote the later. If $M$ is in $\mathcal{C}'$, $x \in U$ and $m \in M$, then we have
\[ \Theta(x \cdot m) = \Theta(x)\Theta m \] 

for \( \Theta = T'_{i,e} \) or \( \Theta = T''_{i,e} \) (see \cite{Lus93}, 37.1.2). The last identity can be interpreted to say that \( \Theta \) and \( \Theta \otimes s \) are intertwining maps:

\[ \Theta : M \to M^\Theta \quad \Theta \otimes s : R M \to R M^\Theta \otimes s. \] 

To simplify notation we shall sometimes write \( s\Theta \) in place of \( \Theta \otimes s \).

We now describe the explicit action of \( \Theta \) on \( M \).

3.1.1. Lemma. (\cite{Lus88} Prop.5.2.2). Let \( m \geq 0 \) and \( j, h \in [0, m] \) be such that \( j + h = m \).

(a) If \( \eta \in M^m \) is such that \( E\eta = 0 \), then \( T'_{i,e}(F^{(j)}\eta) = (-1)^j v^{e(jh+j)} F^{(h)} \eta \).

(b) If \( \zeta \in M^{-m} \) is such that \( F\zeta = 0 \), then \( T''(E^{(j)}\zeta) = (-1)^j v^{e(jh+j)} E^{(h)} \zeta \).

Let \( F(U) \) denote the ad-locally finite submodule of \( U \). We know from \cite{JL92} that \( F(U) \) is tensor product of harmonic elements \( \mathcal{H} \) and the center \( Z(U) \). Here \( \mathcal{H} = \oplus_{m \in \mathbb{Z}} \mathcal{H}_{2m} \) and \( \mathcal{H}_{2m} = \text{ad} U(EK^{-1}) \).

There is another category that we will need and it is defined as follows: Let \( M \) be a \( U_R \)-module. One says that \( M \) is \( R U^0 \)-semisimple if \( M \) is the direct sum of \( R \)-modules \( M^\mu \) where \( K \) acts by \( T v^\mu \), \( \mu \in \mathbb{Z} \); i.e. by weight \( m + \mu \). Then \( \mathcal{C}_R \) denotes the category of \( U_R \)-modules \( M \) for which \( E \) acts locally nilpotently and \( M \) is \( R U^0 \)-semisimple.

For \( M \) and \( N \) two objects in \( \mathcal{C}' \) or one of them is in \( R \mathcal{C} \), Lusztig defined the linear map \( L : M \otimes N \to M \otimes N \) given by

\[ L(x \otimes y) = \sum_n (-1)^n v^{n(n-1)/2} \{ n \} F^{(n)} x \otimes E^{(n)} y \] 

where \( \{ n \} := \prod_{a=1}^n (v^a - v^{-a}) \) and \( \{ 0 \} := 1 \). One can show

\[ L^{-1}(x \otimes y) = \sum_n v^{n(n-1)/2} \{ n \} F^{(n)} x \otimes E^{(n)} y. \] 

3.1.2. Lemma. (\cite{Lus93}). Let \( M \) and \( N \) be two objects in \( \mathcal{C}' \). Then \( T''_1 L(z) = (T''_1 \otimes T''_1)(z) \) for all \( z \in M \otimes N \).

3.1.3. Lemma. Let \( M \) be a module in \( \mathcal{C}_R \) and \( N \) a module in \( \mathcal{C}' \). Then for \( x \in M^t \) and \( y \in N^s \) we have

\[ FL(x \otimes y) = L(x \otimes Fy + v^sFx \otimes y) \]

\[ EL(x \otimes y) = L(Ex \otimes y + v^{-t}T^{-1}x \otimes Ey). \]

3.1.4. Corollary. Let \( M \) be a module in \( \mathcal{C}_R \) and \( \mathcal{E} \) a module in \( \mathcal{C}' \). Then \( L \) defines an isomorphism of the \( U \)-module \( M^{T^1} \otimes \mathcal{E}^{T^1} \) onto \( (M \otimes \mathcal{E})^{T^1} \).

Corollary 3.1.4 through the use of Lemma 3.1.2 however one must take into account that \( T''_E \) may not be defined on \( M \).

Set

\[ \mathcal{L}^{-1} = \sum_p (-1)^p v^{3(p-1)/2} \{ p \} E^{(p)} K^p F^{(p)}, \] 

(3.6)
and
\[
\mathcal{L} = \sum_p v^{-\frac{3(p-1)}{2}} \{p\} E^{(p)} K^{-p} F^{(p)}.
\]

and note that \(\mathcal{L}\) and \(\mathcal{L}^{-1}\) are well defined operators on lowest weight modules.

3.1.5. Lemma. Suppose that \(M\) and \(N\) be highest weight modules with \(\psi_M : M^{s_{T^{-1}}} \to M\), \(\psi_N : N^{s_{T^{-1}}} \to N\) homomorphisms and \(\phi\) a \(\rho\)-invariant form on \(M \times N\). Then
\[
\phi \circ (\psi_M \otimes \psi_N) \circ L^{-1} = \phi \circ (\psi_M \otimes \psi_N) \circ (\mathcal{L}^{-1} \otimes 1).
\]

3.1.6. Lemma. \(T'_{-1}(u)\mathcal{L}^{-1} = \mathcal{L}^{-1}T'_1(u)\) as operators on \(M\) for all \(u \in R\).

4. Invariant Forms and Liftings

4.1. Elements of \(\mathbb{P}(M, N)\) are called invariant pairings and for \(M = N\), set \(\mathbb{P}(M) = \mathbb{P}(M, M)\) and call the elements invariant forms on \(M\).

4.1.1. Lemma. Let \(M\) and \(N\) be finite dimensional \(\mathbb{U}_R\)-modules in \(\mathcal{C}'\) and \(\phi\) an invariant pairing. Then, for \(m \in M, n \in N\), \(\phi(T_i^\nu m, T_i^\nu n) = \phi(m, n)\).

Note that if \(\phi(\eta, \eta) = 1\), then for \(0 \leq j \leq \nu\), the proof above shows that
\[
\phi(F^{(j)}\eta, F^{(j)}\eta) = v^{j^2-j} \left[\nu\right]
\]

For the proof of some future results we must be explicit about the definition of \(f\mathcal{R}\). Recall a \(\mathbb{U}\)-module \(M\) is said to be integrable if for any \(m \in M\) and all \(i \in I\), there exists a positive integer \(N\) such that \(\mathcal{E}^{(n)}_i m = 0 = \mathcal{F}^{(n)}_i m\) for all \(n \geq N\), and \(M = \bigoplus_{\lambda \in X} M^\lambda\) where for any \(\mu \in Y, \lambda \in X\) and \(m \in M^\lambda\) one has \(K_{\mu, m} = v^{(\mu, \lambda, m)}\). Let \(\mathbb{U}_0^\infty\) denote the set of units of \(\mathbb{U}_0\) and let \(f : X \times X \to \mathbb{U}_0^\infty\) be a function such that
\[
f(\zeta + \nu, \zeta' + \nu') = f(\zeta, \zeta')v^{-\sum \nu_i(i, i+1) - \sum \nu'_i(i, i+1) - \nu \cdot \nu'} K_{\nu}
\]
for all \(\zeta, \zeta' \in X\) and all \(\nu, \nu' \in X\) (see [Lus93 32.1.3]). Here \(K_{\nu} = \prod \bar{K}_{(i+1)\nu_i}\).

4.1.2. Theorem. ([Lus93 32.1.5]). If \(\mathcal{E}\) is an integrable \(\mathbb{U}_R\) module and \(A \in \mathbb{C}_R\), then for each \(f\) satisfying (3.1.1), there exists an isomorphism \(f\mathcal{R} : A \otimes \mathcal{E} \to \mathcal{E} \otimes A\).

The map \(\tau : A \otimes B \to B \otimes A\) for any two modules \(A\) and \(B\) denotes the twist map \(\tau(a \otimes b) = b \otimes a\). Define \(\prod_f : \text{End}_R(R\mathcal{E} \otimes_R \mathcal{F} \otimes_R M)\) by \(\prod_f(e \otimes e' \otimes m) = f(\lambda, \lambda')e \otimes e' \otimes m\) for \(m \in M^\lambda\) and \(e \otimes e' \in (\mathcal{E} \otimes \mathcal{F})^\lambda\). Lastly we define \(\chi \in \text{End}_R(R\mathcal{E} \otimes_R \mathcal{F} \otimes_R M)\) by
\[
\chi(e \otimes e' \otimes m) = \sum \sum p_{b, b'} b^{-1} (e \otimes e') \otimes b^+ m
\]
where \(p_{b, b'} = p_{\nu_i, \nu_i} \in \mathbb{R}\), and \(B_{\nu}\) is a subset of \(f\). Then \(f\mathcal{R}\) is defined to be equal to \(\chi \circ \prod_f \circ \tau\). The proof that it is an \(\mathbb{U}\)-module homomorphism is almost exactly the same as in [Lus93], 32.1.5, or [Jan96], 3.14, which the exception that one must take into account that \(M\) is in the category \(\mathbb{C}_R\) instead of \(\mathbb{C}'\).

Let \(\mathcal{E}\) and \(\mathcal{F}\) be finite dimensional \(\mathbb{U}\)-modules and \(\tau : \mathcal{E} \otimes_R \mathcal{F}^{p_1} \to \mathbb{U}\), a \(\mathbb{U}\)-module homomorphism into \(\mathbb{U}\), where \(\mathbb{U}\) is a module under the adjoint action. Suppose \(\phi\) is a
pairing of $M$ and $N$. Define $\psi_{\tau,\phi}$ to be the invariant pairing of $M \otimes_R E$ and $N \otimes_R F$ defined by the formula, for $e \in E$, $f \in F$, $m \in M$, and $n \in N$,

$$\psi_{\tau,\phi}(m \otimes e, n \otimes f) = \phi(m, \tau(e \otimes f) \ast n),$$

(4.3)

Here $F^{p_1}$ is a twist of the representation $F$ by $p_1$. We call the pairing $\psi_{\tau,\phi}$ the pairing induced by $\tau$ and $\phi$. In the cases when $M, N$ and $\phi$ are fixed we write $\psi_{\tau}$ in place of $\psi_{\tau,\phi}$ and say this pairing is induced by $\tau$.

Let us check that $\psi_{\tau,\phi}$ indeed is a $\rho$-invariant pairing: For $x \in U$

$$\sum \psi_{\tau,\phi}(S(x(2))(m \otimes e), \rho(x(1))(n \otimes f))$$

$$= \sum \phi(S(x(4))m, \tau(S(x(3))e \otimes \rho(x(2))f) \ast (\rho(x(1))n))$$

$$= \sum \phi(S(x(4))m, \rho(x(3))\rho_1(\tau(e \otimes f))\rho(x(1))S(x(2)))n)$$

$$= \sum \phi(S(x(2))m, \rho(x(1))(\tau(e \otimes f) \ast n)))$$

$$= e(x)\phi(m, \tau(e \otimes f) \ast n),$$

The first equality is due to the act that

$$S \otimes S \circ \Delta = \tau \circ \Delta \circ S$$

(4.4)

where $\tau : U \otimes U \to U \otimes U$ is the twist map and the fact that $\rho \otimes \rho \circ \Delta = \Delta \circ \rho$. The second equality follows from the definition of $\psi_{\tau,\phi}$. The third equality is obtained from (4.4) and the fact that $\rho$ is an anti-automorphism. The last equality is due to the assumption that $\phi$ is $\rho$-invariant.

4.2. A result from [CE95] shows that in the setting of Verma modules the collection of maps $\tau$ is a natural set of parameters for invariant forms.

4.2.1. Proposition ([CE95]). Suppose $U$ is of finite type and $E$ and $F$ are finite dimensional $U$-modules. Let $M$ be an $U$-Verma module and $\phi$ the Shapovalov form on $M$. Suppose the Shapovalov form on $M$ is nondegenerate. Then every invariant pairing of $M \otimes_R E$ and $M \otimes_R F$ is induced by $\phi$.

4.3.

4.3.1. Theorem. ([CE95] Lifting Theorem]). Let $A$ and $B$ be modules in $C_R$ and $\phi \in P_R(A, B)$. Then $\phi$ uniquely determines an invariant form $\phi_F \in P_R(A_F, B_F)$ which is determined by the following properties:

1. $\phi_F$ vanishes on the subspaces $\iota A \times B_F$ and $A_F \times \iota B$.

2. For each $\mu \in \mathbb{Z}$ with $\mu + 1 = r \in \mathbb{N}$, and any vectors $a \in A$ and $b \in B$ both of weight $m + \epsilon \mu$ with $\epsilon \in \{1, s\}$ and $E \ a = E \ b = 0$,

$$\phi_F(F^{-1}a, F^{-1}b) = v^{-r+1} \frac{\mu e[T; 0]}{1 e[T; r - 1]} \phi(a, b).$$

(4.5)

4.3.2. Proposition. The form $\phi_F$ induces an $\rho$-invariant bilinear map on $A_\pi \times B_\pi$ which we denote by $\phi_\pi$. 

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4.4. At times the subscript notation for lifted forms will be inconvenient and so we shall also use the symbol \( \text{loc} \) for the localization of both forms and modules. We write \( \text{loc}(\phi) \) and \( \text{loc}(A) \) in place of \( \phi_{F} \) and \( A_{F} \).

For invariant forms we find that induction and localization commute in the following sense.

5. QUANTUM CLEBSCH-GORDAN DECOMPOSITION

5.1. Basis and Symmetries. For \( m \in \mathbb{Z} \), let \( \mathcal{F}_{m} \) denote the finite dimensional irreducible module of highest weight \( v^{m} \) with highest weight vector \( u^{(m)} \). For \( k \) any non-negative integer set \( u^{(m)}_{k} = F^{(k)}u^{(m)} \) and \( u^{(m)}_{-1} = 0 \).

In particular
\[
\theta^{-1}(u^{(m)}_{j}) = T^{\theta}_{1}(u^{(m)}_{j}) = (-1)^{m-j}v^{(m-j)(j+1)}u^{(m)}_{m-j}
\]
and
\[
K^{p}u^{(m)}_{j} = v^{p(m-2j)}u^{(m)}_{j}, \quad F^{(p)}u^{(m)}_{j} = \left[ \begin{array}{c} p+j \\ j \end{array} \right] u^{(m)}_{j+p}, \quad E^{(p)}u^{(m)}_{j} = \left[ \begin{array}{c} m+p-j \\ p \end{array} \right] u^{(m)}_{j-p}.
\]

(5.1)

5.1.1. Lemma. [Clebsch-Gordan, CE10] For any two non-negative integers \( m \geq n \), there is an isomorphism of \( U \)-modules
\[
\mathcal{F}_{m+n} \oplus \mathcal{F}_{m+n-2} \oplus \cdots \oplus \mathcal{F}_{m-n} \cong \mathcal{F}_{m} \otimes \mathcal{F}_{n}.
\]
Moreover the isomorphism is defined on highest weight vectors by
\[
\phi(u^{(m+n-2p)}) = \sum_{k=0}^{p} (-1)^{k} \frac{(n-p+k)!(m-k)!}{(n-p)!(m)!} v^{(k-p)(m-p-k+1)}u^{(m)}_{k} \otimes u^{(n)}_{p-k}.
\]

5.1.2. Lemma. The map \( \varphi : \mathcal{F}^{m}_{m} \to \mathcal{F}_{m} \) given by \( \varphi(u^{(m)}_{k}) = (-v)^{-k}u^{(m)}_{m-k} \) is an isomorphism.

5.1.3. Corollary. Let \( m \geq n \) be two non-negative integers. Then there is an isomorphism of \( U \)-modules
\[
\mathcal{F}_{m+n} \oplus \mathcal{F}_{m+n-2} \oplus \cdots \oplus \mathcal{F}_{m-n} \cong \mathcal{F}_{m} \otimes \mathcal{F}^{n}_{n}.
\]
Moreover the isomorphism is defined on highest weight vectors by
\[
\phi(u^{(m+n-2p)}) = \sum_{k=0}^{p} (-1)^{n-p} \frac{(n-p+k)!(m-k)!}{(n-p)!(m)!} v^{k(2+m-k)+n-2p+mp+p^{2}}u^{(m)}_{k} \otimes u^{(n)}_{n-p+k}
\]
\[
= (-1)^{n-p}v^{p(n+2p-k)} \sum_{k=0}^{p} v^{k(2+m-k)} \left[ \begin{array}{c} n-p+k \\ k \end{array} \right] \left[ \begin{array}{c} m \\ k \end{array} \right]^{-1} u^{(m)}_{k} \otimes u^{(n)}_{n-p+k}.
\]

(the action on the second factor \( u^{(n)}_{k} \) is twisted by the automorphism \( \rho_{1} \)).

5.1.4. Lemma. CE10. For \( 0 \leq k \leq \min\{n-p, m+n-2p\} \),
\[
\left[ \begin{array}{cc} m & n \\ 0 & p + k \end{array} \right] = v^{-p(m-p+1)} \left[ \begin{array}{c} p + k \\ p \end{array} \right]
\]

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and for \( \max\{0, m - p\} \leq k \leq m + n - 2p \)

\[
\begin{bmatrix}
m & n & m + n - 2p \\
m & p + k - m & k \\
\end{bmatrix} \\
= v^p(p - 1) - m(m + n - p - k) \sum_{l=0}^{\min\{p,m\}} (-1)^l v^{(1+p+n-2p+k)} \begin{bmatrix} n-p+l \\ p + k - m \end{bmatrix}^{p + k - m} \\
= (-1)^p v^{(p-m)(m+n)+mk} \begin{bmatrix} m + n - p - k \\ p \end{bmatrix}, \quad \text{if } n \leq m.
\]

Consider now the \( \rho \)-invariant forms (4.1) on \( \mathcal{F}_m \) and \( \mathcal{F}_n \), both denoted by \((,)\), normalized so that their highest weight vectors have norm 1. Define the symmetric invariant bilinear form on \( \mathcal{F}_m \otimes \mathcal{F}_n \) given by the tensor product of the two forms (the resulting pairing is \( \rho \)-invariant). In this case

\[
(u^{(m+n-2p)} , u^{(m+n-2p)}) = \sum_{k=0}^{p} \left( \frac{n-p+k}{[n-p][m]} \right) v^{(k-p)(m-p-k+1)} \left( u^{(m)}_{k} , u^{(n)}_{k} \right) (u^{(m)}_{p-k} , u^{(n)}_{p-k}) \\
= \sum_{k=0}^{p} \left( \frac{n-p+k}{[n-p][m]} \right) v^{2(k-p)(m-p-k+1)+k^2-mk+(p-k)^2-n(p-k)} \begin{bmatrix} m \\ p \end{bmatrix} \\
= \sum_{k=0}^{p} \left( \frac{n-p+k}{[n-p][m]} \right) v^{k(2+m+n-2p)+p(3p-2-2m-n)} \\
= \sum_{k=0}^{p} \frac{n}{[n-p][m][k][p-k]} v^{k(2+m+n-2p)} \\
= v^{p(2p-2m-n)} \frac{n}{[n-p][m][p]} \sum_{k=0}^{p} \frac{n-p+k}{[n-p][m][p-k]} v^{k(2+m+n-2p)},
\]

where we have used formula (2.8).

The same proof that gave us (4.1) now implies

\[
(u^{(m+n-2p)} , u^{(m+n-2p)}) = v^{p(2p-2m-1)-(m+n-2p-k)} \begin{bmatrix} m+n-p+1 \\ p \end{bmatrix} \begin{bmatrix} m+n-2p \\ k \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix}^{-1}. 
\]

5.1.5. Proposition [CE10]. (i) The basis \( \{u^{(m+n-2p)}_k\} \) of \( \mathcal{F}_m \otimes \mathcal{F}_n \) is orthogonal.

(ii). For \( 0 \leq i \leq m \), and \( 0 \leq j \leq n \),

\[
u^{i+j-22i} \sum_p v^{(1+m-n)p} \begin{bmatrix} m \\ n \\ i+j-p \end{bmatrix}^{-1} u^{(m+n-2p)} = v^{m+n-2j} \begin{bmatrix} m \\ n \\ i \end{bmatrix} \begin{bmatrix} m+n-2p \\ i+j-p \end{bmatrix} u^{(m+n-2p)}_i. 
\]
In $\mathcal{F}_m \otimes \mathcal{F}_n^{\rho_1}$ (recall $\phi(u_j^{(n)}) = (-v)^{-j}u_{n-j}^{(n)}$)

$$u_i^{(m)} \otimes u_j^{(n)} = (-1)^j v^{i(2j-n)+m-n-j(1+m)} \times \sum_p v^{(1+m-n-p)p} \begin{bmatrix} m \ n \ m \ p \ i \ n-j \ i+n-j-p \end{bmatrix} \begin{bmatrix} m+n-p+1 \ n \ p \ i+n-j-p \end{bmatrix} u_{i+n-j-p}^{(m+n-2p)}.$$  \hspace{1cm} (5.2)

In particular

$$u^{(m)} \otimes u_j^{(n)} = (-1)^j v^{mn-j(1+m)} \sum_{p=0}^{\min\{n-j,m+j\}} v^{-np} \begin{bmatrix} m+n-p+1 \ n \ p \ m+n-2p \ n-j \ p \ i+n-j-p \end{bmatrix} u_{n-j-p}^{(m+n-2p)}.$$  \hspace{1cm} (5.3)

as

$$\begin{bmatrix} m \ n \ m+n-2p \ 0 \ p+r \ r \end{bmatrix} = v^{(-p)(m-p+1)} \begin{bmatrix} p+r \ p \end{bmatrix}.$$  \hspace{1cm} for $0 \leq n-j-p \leq m+n-2p$, i.e. for $p \leq \{n-j,m+j\}$

$$u^{(m)}_m \otimes u_j^{(n)} = (-1)^j v^{m(m+j-n)-j} \times \sum_p v^{(1+m-n-p)p} \begin{bmatrix} n \ j \ p \ m \ n-j \ m+n-j-p \ m+n-2p \ n \ m+n-j-p \end{bmatrix} u_{m+n-j-p}^{(m+n-2p)} \hspace{1cm} (5.4)$$

$$= (-1)^j v^{-j} \times \sum_p (-1)^p v^{(1+m-p)p} \begin{bmatrix} n \ j \ p \ m \ n-j \ m+n-j-p \ m+n-2p \ n \ m+n-j-p \end{bmatrix} u_{m+n-j-p}^{(m+n-2p)} \hspace{1cm} (5.5)$$

if $m \geq n$, as

$$\begin{bmatrix} m \ n \ m+n-2p \ m \ p+k \ m+n-2p \ k \end{bmatrix} = (-1)^{p} v^{(p-m)(m+n)+mk} \begin{bmatrix} m+n-p-k \ p \end{bmatrix}.$$  \hspace{1cm} (5.6)

6. **Basis and the Intertwining map $\mathcal{L}$**

6.1. **A Basis.** For $s \geq 1$, and any lowest weight vector $\eta$ of weight $Tv^{\lambda+\rho}$, set

$$F^{-k}\eta := T_{-1}(F^{(k)})\eta = v^{k(k-1)}F^{-k}K^k[K;-1]^{(k)}\eta = v^{k(\lambda+k)}T^{k}[T;\lambda]^{(k)}F^{-k}\eta$$  \hspace{1cm} (6.1)

as

$$T_{-1}'(E^{(k)}) = (-1)^k v^{-k(k-1)}K^{-k}F^{(k)}, \hspace{1cm} T_{-1}'(F^{(k)}) = (-1)^k v^{k(k-1)}E^{(k)}K^k.$$
6.1.1. **Lemma** ([CE10]). Suppose \( r, s \in \mathbb{Z}, \ s > r \geq 0, \ \zeta \) is a highest weight vector of weight \( Tv^\lambda \rho \) and \( \eta \) is a lowest weight vector of weight \( Tv^{\lambda+\rho} \). Then
\[
E^{(r)} F^{(s)} \zeta = \left[ T; \lambda - 1 + r - s \atop r \right] F^{(s-r)} \zeta, \quad F^{(r)} F^{(s)} \zeta = \left[ r + s \atop r \right] F^{(s-r)} \zeta, \tag{6.2}
\]
and
\[
F^{(r)} F^{(-s)} \eta = v^{r(\lambda+2s-r)} T^r \left[ T; \lambda + s \atop r \right] F^{(r-s)} \eta, \tag{6.3}
\]
\[
E^{(r)} F^{(-s)} \eta = (-1)^r v^{-r(\lambda+r+2s)} T^{-r} \left[ r + s \atop r \right] F^{(-r-s)} \eta. \tag{6.4}
\]

Define indexing sets \( I_\lambda \) and \( I_{-\lambda} \) by \( I_\lambda = \{ n - 2, n - 4, \ldots \}, \ I_{-\lambda} = \{ -n, -n - 2, \ldots \} \). One should compare the previous result with

6.1.2. **Lemma.** ([CE10] 2.2) Now for integers \( j \in I_\lambda \) (resp. \( I_{-\lambda} \)) set \( k_j = \frac{n-2-j}{2} \) and \( l_j = \frac{-n-j}{2} \) and define basis vectors for \( M(m+l) \) and \( M(m-l) \) by \( v_j = F^{k_j} \otimes 1_{m+\lambda} \) and \( v_{j,s} = F^{l_j} \otimes 1_{m-\lambda-\rho} \). The action of \( U_R \) is given by
\[
K v_j = T v^{\lambda-1-2k_j} v_j, \quad F w_{\lambda,j} = w_{\lambda,j-2} \tag{6.5}
\]
\[
K v_{j,s} = T v^{-\lambda-1-2l_j} v_{j,s} \quad F w_{\lambda,j} = w_{\lambda,j-2} \tag{6.6}
\]
\[
E v_j = [k_j][T; -l_j] w_{\lambda,j+2}, \quad E v_j = [l_j][T; -k_j] w_{-\lambda,j+2}. \tag{6.7}
\]

6.1.3. **Corollary.** For \( k \geq 0 \) and \( 0 \leq j \leq n \) we have
\[
\mathcal{L}(F^{(k)} \eta \otimes u_{j-n}^{(n)})
= \sum_j \sum_{q=j-n}^{n-j+k} \sum_{p=0}^k (-1)^{t-q} v^{-3p(p-1)/2+p(2j+2p-n)} \{p\}
\times v^{t(-1+2j-2k-n-\lambda)+q(1+4k-p-2q+2\lambda)}
\times T^{2q-t} \left[ p+t+j \atop j \right] \left[ T; \lambda + k \atop t \right] \left[ k-q \atop t \right] \left[ n+q-j \atop p+q-t \right] F^{(q-k)} \eta \otimes u_{j-q}^{(n)}.
\]

6.2. The articles [CE95] and [CE10] study noncommutative localization of highest weight modules. This article may be viewed as an extension of what was begun there. For any \( U \)-module \( A \) let \( A_F \) denote the localization of \( A \) with respect to the multiplicative set in \( U \) generated by \( F \). If \( F \) acts without torsion on \( A \) (we shall assume this throughout) then \( A \) injects into \( A_F \) and we have the short exact sequence of \( U \)-modules: \( 0 \to A \to A_F \to A_\pi \to 0 \).

7. **Maps into the Harmonics.**

7.1. **Harmonics.** We know from [AL92] that \( F(U) \cong \mathcal{H} \otimes Z(U) \) where \( \mathcal{H} = \oplus_{n \in \mathbb{N}} \mathcal{L}_{2n} \) is the space of harmonics, and \( \mathcal{L}_{2n} \cong F_{2n} \). We would now like to give an explicit basis of \( \mathcal{L}_{2n} \).

7.1.1. **Lemma.** For \( n \in \mathbb{N} \), we can take
\[
\mathcal{H}_{2n} = \oplus_{p=0}^n k(v) \text{ad} F^{(p)}(E^{(n)} K^{-n}), \tag{7.1}
\]
Proposition. Suppose \( \eta \) is a lowest weight vector of weight \( T^{\lambda + \rho} \). Then

\[
\rho_1 \left( \beta_{2r}^{m,n} (u_i^{(m)} \otimes u_j^{(n)}) \right) = (-1)^j v^i (2j-n)+mn-j(1+m)+(1+\frac{m-3n}{2}+r)(\frac{m+n}{2}-r)
\]

\[
\times \sum_{l=0}^{i-j+n-m+r} (-1)^r l v^l (r+j-i+\frac{m-n}{2}+1) \left[ \begin{array}{ccc} \lambda + l + c \\ r \end{array} \right] \left[ \begin{array}{cc} T; \lambda + l + c \\ r \end{array} \right] \left[ \begin{array}{cc} l + c \\ \lambda + l \end{array} \right] F^{(i-j+n-m-r)} \eta
\]

and

\[
\rho_1 \left( T_{-1} \beta_{2r}^{m,n} (u_i^{(m)} \otimes u_j^{(n)}) \right) = (-1)^j v^i (2j-n)+mn-j(1+m)+(1+\frac{m-3n}{2}+r)(\frac{m+n}{2}-r)
\]

\[
\times \sum_{l=0}^{i-j+n-m+r} (-1)^r l v^l (r+j-i+\frac{m-n}{2}+1) \left[ \begin{array}{ccc} T; \lambda + l + c \\ r \end{array} \right] \left[ \begin{array}{cc} T; \lambda + l + c - l \\ r \end{array} \right] \left[ \begin{array}{cc} r + c - l \\ \lambda + l \end{array} \right] F^{(i-j+n-m-r)} \eta
\]
8. Symmetry Properties of Induced Forms

8.1. Twisted action of $R$. We shall twist by an automorphism of $\mathbf{U}_R$ in the setting of $\mathbf{U}_R$-modules. Let $\Theta$ be an automorphism of $\mathbf{U}_R$. Then for any $\mathbf{U}_R$-module $\mathbf{E}$ define a new $\mathbf{U}_R$-module $\mathbf{E}$ with set equal to that of $\mathbf{E}$ and action given by: for $e \in \mathbf{E}$ and $x \in \mathbf{U}_R$ the action of $x$ on $e$ equals $\Theta(x)e$. For any $\mathbf{U}_R$-module $A$, let $A^{s\Theta}$ denote the module with action $\delta_i$ on $A^{s\Theta}$ defined as follows: For $a \in A$ and $x \in \mathbf{U}_R$,

$$x\delta_i a = s\Theta(x) a.$$ 

8.1.1. Lemma. Suppose $\phi$ is a $\rho$-invariant $R$-valued pairing of $\mathbf{U}_R$-modules $A$ and $B$. Then $s \circ \phi$ is a $\rho$-invariant pairing of $A^s$ and $B^s$ as well as $A^{sT_v}$ and $B^{sT_v}$. Also $\phi$ itself is $\rho$-invariant pairing of these two pairs taking values in the $R$-module $R^s$.

8.1.2. Lemma. Let $m$ be a lowest weight vector in $M_\pi$ of weight $Tv$ and $\Psi(m)$ a highest weight vector in $M$ of weight $Tv^{-1}$. The map $\Psi : (M_\pi)^{s\Theta} \to M$ given by

$$\Psi(F^{-k}m) = \frac{(-1)^{k}k^{-k^2}T^k}{[k]!(T; -1)_{(k)}} F^k \Psi(m)$$

for $k \geq 0$ is an isomorphism.

The above can be rewritten as

$$\Psi(F^{(-k)}m) = F^{(k)} \Psi(m).$$

Set

$$\Psi := \Psi \otimes s\Theta \circ L^{-1} : (M_\pi \otimes \mathcal{E})^{s\Theta} \to M \otimes \mathcal{E}. \quad (8.1)$$

Define $t \in \text{End}(\mathbb{P}(M \otimes \mathcal{E}, N \otimes \mathcal{F}))$ by

$$t(\chi)(\Psi(a), \Psi(b)) := s \circ \chi_\pi \circ L(a \otimes b) \quad (8.2)$$

for $a \in (M \otimes \mathcal{E})_\pi$, $b \in (N \otimes \mathcal{F})_\pi$ and $\chi \in \mathbb{P}(M \otimes \mathcal{E}, N \otimes \mathcal{F})$. On the left hand side one is to consider $a \in (M \otimes \mathcal{E})_\pi^\theta$, and $b \in (N \otimes \mathcal{F})_\pi^\theta$ and on the right hand side $a \in (M \otimes \mathcal{E})_\pi$, $b \in (N \otimes \mathcal{F})_\pi^\rho$. We can view $L : (M \otimes \mathcal{E})_\pi^\theta \otimes (N \otimes \mathcal{F})_\pi^\theta \rho \to ((M \otimes \mathcal{E})_\pi \otimes (N \otimes \mathcal{F})_\pi^\rho)^{s\Theta}$ as a module isomorphism (see Lemma ?? and Corollary 3.14). Then $s \circ \chi_\pi \circ L : (M \otimes \mathcal{E})_\pi^\theta \otimes (N \otimes \mathcal{F})_\pi^\theta \rho \to R$ is a module homomorphism. More explicitly we can show that $t(\chi) \in \mathbb{P}(M \otimes \mathcal{E}, N \otimes \mathcal{F})$ by the following calculation for $x \in U$:

$$\sum \chi^\#(S(x_2)\Psi(a), \rho(x_1)\Psi(b)) = \sum \chi^\#(\Psi(\theta S(x_2)a), \Psi(\theta \rho(x_1)b))$$

$$= \sum s \circ \chi_\pi \circ L(\theta S(x_2)a \otimes \theta \rho(x_1)b)$$

$$= s \circ \chi_\pi(\theta S(x)L(a \otimes b))$$

$$= e(x)s \circ \chi_\pi \circ L(a \otimes b)$$

$$= e(x)\chi^\#(\Psi(a), \Psi(b))$$
where the third equality is from Corollary 3.1.4 and the fourth equality is due to the fact that $\chi_\pi$ is $\rho$-invariant. Note that as a linear map $L \in \text{End} ((M \otimes \mathcal{E})_\pi \otimes (N \otimes \mathcal{F})_\pi)$ is well defined as $F$ acts locally nilpotently on $(M \otimes \mathcal{E})_\pi$.

8.2. For any homomorphism $\beta$ of $\mathcal{E}^{s\Theta} \otimes_R (\mathcal{F}^{s\Theta})^{\rho_1}$ into $F(U)$, define another such $t(\beta)$ by the formula

$$t(\beta) = s\Theta^{-1} \circ \beta \circ L \circ (s\Theta \otimes s\Theta)$$

where the $\Theta^{-1}$ on the left is assumed to be the module homomorphism defined on $F(U)$.

8.2.1. Theorem. Let $\lambda = 0$. Suppose $\chi$ is the induced pairing $\chi_{\beta,\phi}$ as defined in (4.3), $F_m$ and $F_n$ are the $X$-graded finite dimensional $U$-modules given in §5.1.1 and $\phi$ is a $U$-invariant pairing satisfying $s \circ \phi \pi \circ L = \phi \circ (\Psi \otimes \Psi)$. Then

$$t(\chi_{\beta,\phi}) = \chi_{\beta,\phi}. \quad (8.4)$$

8.3. Fix a finite dimensional $U$-module $F$ with highest weight $v^\lambda$ and let $M$ be the Verma module of highest weight $Tv^{-1}$. Recall from [CE95 §2], the modules $P(m + \lambda) := P_{m+\lambda}$. Then we have the decomposition $M \otimes F = \sum P(m+i)$ where the sum is over the nonnegative weights of $F$ and by convention we set $P(0) = M(m)$. Set $P_i = P(m+i)$ and following the notation of [E,3.6] let $Z_i$ equal the set of integers with the opposite parity to $i$. For $j \in Z_i$, set $z_j^i = v_j + v_j^s$. Then for $i \in \mathbb{N}^*$, the set $\{[T;0]v_j : j \in Z_i\} \cup \{z_j^i : j \in Z_i\}$ is an $R$ basis for the localization $P_{i,F}$. Also the action of $RU_i$ is given by the formulas in Lemma 6.1.2 as well as the formulas:

$$K_\mu z_j = T \langle \mu, s\lambda - 2\rho \rangle z_j, \quad Fz_j = z_{j-2} \quad \text{and} \quad Ez_j = [l_j]T; -k_j]z_{j+2} + [k_j - l_j][T; 0]w_{\lambda,j+2}. \quad (8.5, 8.6)$$

The corresponding picture for $P_n$ of weight vectors for $P_n$ is given by
Fix a positive weight \( v^r \) of \( \mathcal{F} \) and let \( P = P_v \). Set \( \mathcal{L} \) equal to the \( m - r \)th weight space of \( P \). Then \( \mathcal{L} \) is a free rank two \( R \)-module with basis \( \{ z_{u-1}, [T; 0] w_{r-u-1} \} \). Define an \( s \)-linear map \( \Gamma \) on \( \mathcal{L} \) and constants \( a_{\pm r} \) by the formula:

\[
\Gamma([T; 0] w_{r-u-1}) = \Psi_F([T; 0] w_{r-u+1}) = a_{r} [T; 0] w_{r-u-1}. \tag{8.7}
\]

This \( s \)-linear map \( \Gamma \) is the mechanism by which we analyze the symmetries which arise through the exchange of \( \mathcal{L} \cap ([T; 0] \cdot M(m + r)) \) and \( \mathcal{L} \cap ([T; 0] \cdot M(m - r)) \). The following is a fundamental calculation for all which follows. Set \( \overline{\Gamma} = \overline{[r]!} \). 

8.3.1. **Lemma.** Let \( \epsilon = \pm 1 \). For \( a, b \in \mathcal{L} \cap M(m + \epsilon r) \), we have:

\[
\chi^n(\Gamma a, \Gamma b) = \frac{1}{T_\epsilon [r]!} \frac{\epsilon}{[r]! [T-\epsilon; r]} s \chi(a, b) \quad \text{and} \quad \chi^n(\Gamma a, \Gamma b) = u_\epsilon \, s \chi(a, b),
\]

where \( u_\epsilon \) is a unit and \( u_\epsilon \equiv 1 \mod (T - 1) \).

8.4. Now we recall to the delicate calculation of the constants \( a_{\pm r} \).

8.4.1. **Lemma (CE10).** We may choose a basis for \( P_v \), satisfying the relations (8.5) and (8.6), dependent only on the cycle \( \Psi \), and such that the constants \( a_{\pm r} \) are uniquely determined by the three relations:

\[
a_{-r} = s \, a_r, \quad a_r^2 = \frac{1}{[r]! [T-1; r](r)} \quad \text{and} \quad a_r \equiv -1 \mod T - 1. \tag{8.8}
\]

8.4.2. **Corollary.** For \( \epsilon = \pm \),

\[
\overline{\Gamma}([T; 0] w_{r-\epsilon u-1}) = w_{r, \epsilon}[T; 0] w_{r-\epsilon u-1},
\]

where \( w_{r, \epsilon} \) is the unit determined by conditions:

\[
w_{r, \epsilon}^2 = \frac{[r]!}{[T-\epsilon; r]} \quad \text{and} \quad w_{r, \epsilon} \equiv -1 - \epsilon \alpha(r)(T - 1) \mod (T - 1)^2. \tag{8.9}
\]

Moreover \( \overline{\Gamma} \) induces a \( \mathbb{C} \)-linear map on \( \mathcal{L} / (T - 1) \cdot \mathcal{L} \) given by the matrix

\[
\begin{pmatrix}
1 & -\alpha(r) & 0 \\
0 & 1 & (1 + T) \\
0 & 1 & 1
\end{pmatrix} \tag{8.10}
\]

where \( \alpha(r) = -\sum_{s=1}^{r} \frac{v^s + u^s}{v^s - u^s} \). Moreover, if \( x_\epsilon \in M(m + \epsilon r) \) and \( [T; 0] \cdot x_\epsilon \) is an \( R \)-basis vector for \( \mathcal{L} \cap M(m + \epsilon r) \) then \( \{ [T; 0] \cdot x_\epsilon, x_\epsilon + \overline{\Gamma} x_\epsilon \} \) is an \( R \)-basis for \( \mathcal{L} \) and \( x_\epsilon + \overline{\Gamma} x_\epsilon \) generates the \( \mathcal{U}_R \)-submodule \( P_v \).

8.5. To verify the correct choice of sign for the third identity we shall need some preliminary lemmas. Let \( M' \) denote the span of all the weight subspaces of \( M \) other than the highest weight space. Let \( \delta \) denote the projection of \( M \otimes_R \mathcal{F} \) onto \( w_{0,1} \otimes \mathcal{F} \) with kernel \( M' \otimes_R \mathcal{F} \). Define constants \( c_+ \) by the relations: \( \delta(w_{r,-r-1}) \equiv c_+ w_{0,-1} \otimes F^{(k)} f_n \mod M' \otimes_R \mathcal{F} \) and \( \delta(w_{r,-r-1}) \equiv c_- w_{0,-1} \otimes F^{(0)} f_n \mod M' \otimes_R \mathcal{F} \) where \( n - 2k - 1 = r - 1 \) and \( n - 2l - 1 = -r - 1 \). For any integer \( t \) set \( z_t = w_{0,-1} \otimes F^{(t)} f_n \). In a similar fashion define the projection \( \delta' \) of \( M' \otimes_R \mathcal{F} \) onto \( w_{0,1} \otimes \mathcal{F} \) with kernel \( M' \otimes_R \mathcal{F} \) and \( M' \) equal to the span of all weight subspaces in \( M_f \) for weights other than \( m + 1 \).
8.5.1. **Lemma (CE10).** Set
\[ A_t = \frac{[n-k+t](t)}{[t]!} T^t v^{-t^2} \]
then
\[ w_{r,r-1} = c_+ \sum_{0 \leq t \leq k} A_t w_{0,-1-2t} \otimes F^{(k-t)} f_n. \]

**Proof.** Note that \( 0 = E \cdot w_{r,r-1} \) and solve the recursion relations in \( A_t. \)

\[ \square \]

8.5.2. **Lemma (CE10).**
\[ \delta^v(w_{r,r+1}) = c_+ \frac{v^{(n-k)(k+1)} [T-1; 2k - n - 1](k)}{[T-1; k](k)} w_{0,1} \otimes F^{(k)} f_n, \] (8.11)
\[ -\frac{c_-}{[l]!} \equiv \frac{c_+}{[k]!} \mod T - 1 \text{ and } \delta(\Psi(w_{r,r+1})) \equiv -c_- \frac{1}{[r-1]!} z_l \mod (T - 1) \cdot P_r. \] (8.12)

We now return to the proof of the congruence. Since \( \Psi(w_{r,r-1}) = -a_r w_{r-r-1} \) we can calculate the constant \( a_r \) as the ratio of \( \delta(\Psi(w_{r,r-1})) \) and \( \delta(w_{r-r-1}). \) From (4.4) we find the ratio is congruent to \(-\frac{1}{[r-1]!} \mod T - 1. \) This completes the proof of Lemma 4.5.

8.6. Recall from [3] the category \( R \mathcal{C}_i \) and note that any module \( N \) in the category is the direct sum of generalized eigenspaces for the Casimir element \([CE10]\) in the sense that \( N = \sum N^{(\pm r)} \) where the sum is over \( \mathbb{N} \) and \( N^{(\pm r)} \) contains all highest weight vectors in \( N \) with weights \( m+r-1 \) and \( m-r-1. \) Note that \( N^{(\pm r)} \) need not be generated by its highest weight vectors. The decomposition in (4.2), \( M \otimes R \mathcal{F} = \sum P_i \) where the sum is over the nonnegative weights of \( \mathcal{F}_R \) is such a decomposition. In this case \( (M \otimes R \mathcal{F})^{(\pm i)} = P_i. \) The Casimir element \( \Omega_0 \) of \( R \mathcal{U} \) by
\[ \Omega_0 = FE + \frac{vK_i - 2 + v^{-1}K^{-1}}{(v - v^{-1})^2}. \] (8.13)

Let \( N^{(r)} \) (resp. \( N^{(-r)} \)) denote the submodule of \( N \) where the Casimir element acts by the scalar
\[ c(\lambda) = \frac{v^{r-1}T - 2 + v^{-r+1}T^{-1}}{(v - v^{-1})^2} = \frac{[\sqrt{T}; (r - 1)/2]^2}{(v - v^{-1})^2} \] (8.14)
\[ c(s\lambda) = \frac{v^{-r+1}T - 2 + v^{r-1}T^{-1}}{(v - v^{-1})^2} = \frac{[\sqrt{T}; (-r + 1)/2]^2}{(v - v^{-1})^2}. \] (8.15)

8.7. We now turn to the general case where \( \mathcal{F} \) is a finite dimensional \( R \mathcal{U}_\mathcal{F} \)-module but not necessarily irreducible. We extend the definition of the \( s \)-linear maps \( \Gamma \) and \( \Gamma \) defined in (??) as follows. Decompose \( M \otimes R \mathcal{F} \) into generalized eigenspaces for the Casimir \( (M \otimes R \mathcal{F})^{(\pm r)} \) and let \( \mathcal{L}^r \) denote the \( m - r - 1 \) weight subspace of \( (M \otimes R \mathcal{F})^{(\pm r)} \). Then set \( \mathcal{L} = \sum \mathcal{L}^r. \) Decompose \( \mathcal{F} = \sum \mathcal{F}_j \) into irreducible \( R \mathcal{U}_i \) modules. Then \( M \otimes R \mathcal{F} = \sum M \otimes R \mathcal{F}_j \) and so we obtain \( s \)-linear extensions also denoted \( \Gamma \) and \( \Gamma \) from \( \mathcal{L} \cap \sum (M \otimes R \mathcal{F}_j) \) to all of \( \mathcal{L}. \)

8.7.1. **Proposition (CE10).** Suppose \( \phi \) is any invariant form on \( M \otimes R \mathcal{F} \) with \( \phi = \pm \phi^2. \) Let \( \{w_j, j \in J\} \) be an \( R \)-basis for the highest weight space of \( (M \otimes R \mathcal{F})^{(0)} \) and \( \{u_i, i \in I\} \) a basis of weight vectors for the \( E \)-invariant weight spaces of weight \( m+t \) for \( t < -1. \) Set \( M_j \) equal to the \( R \mathcal{U}_i \)-module generated by \( u_j \) and \( Q_t \) the \( R \mathcal{U}_i \)-module generated by \( (T - 1)^{-1}(u_i + \Gamma(u_i)). \)
Then $M \otimes_R \mathcal{F} = \sum_j M_j \oplus \sum_i Q_i$ where each $M_j \cong M(m)$ and if $U_i$ has weight $m - t$, then $Q_i \cong P(m + t)$. Moreover, if the basis vectors $w_j$ and $u_i$ are $\phi$-orthogonal then the sum is an orthogonal sum of $\mathcal{R}_{U_i}$-modules.

8.7.2. Proposition (CE10). Suppose $\phi$ is any invariant form on $M \otimes_R \mathcal{F}$ with $\phi = \pm \phi^\sharp$. Then $M \otimes_R \mathcal{F}$ admits an orthogonal decomposition with each summand an indecomposable $\mathcal{R}$-module and isomorphic to $M$ or some $P(m + t)$ for $t \in \mathbb{N}^*$.

Proof. If $R$ is a discrete valuation ring we may choose an orthogonal $R$-basis for the free $R$-module $\mathfrak{L} \cap (F_R \otimes M)^{(\ast)}$. $\square$

9. Filtrations

9.1. We continue with the notation of the previous section. So $\phi$ is an invariant form on $M \otimes_R \mathcal{F} = \sum_i P_i$. For any $R$-module $B$ set $\overline{B} = B/(T - 1) \cdot B$ and for any filtration $B = B_0 \supset B_1 \supset \ldots \supset B_r$, let $\overline{B} = \overline{B}_0 \supset \overline{B}_1 \supset \ldots \supset \overline{B}_r$ be the induced filtration of $\overline{B}$, with $\overline{B}_i = (B_i + (T - 1) \cdot B)/(T - 1) \cdot B$.

Now $\phi$ induces a filtration on $F_R \otimes M$ by

$$(M \otimes_R \mathcal{F})^i = \{v \in M \otimes_R \mathcal{F} | \phi(v, M \otimes_R \mathcal{F}) \subset (T - 1)^i \cdot R\}. \quad (9.1)$$

9.2. Let $P = P_r$ and $\mathfrak{L}$ equal to the $m - r$th weight subspace of $P$. Suppose $P = P_0 \supset \cdots \supset P_r = 0$ is a filtration. Then choose constants $a, b$ and $c$ so that $a$ is maximal with $P = P_a \supset b \geq a$ maximal with $P_{a+1}/P_b$ finite dimensional if such exist and otherwise set $b = a$ and $c$ maximal with $P_c \neq 0$. We say that the filtration is of type $(a, b, c)$.

9.2.1. Lemma. Set $\phi(w_{r-r-1}, w_{r-r-1}) = p$ and $\phi(w_{r-r-1}, w_{r-r-1}) = q$. Then on $\mathfrak{L}$, $\phi$ is represented with respect to the basis $\{w_{r-r-1}, (T - 1)^{-1}(w_{r-r-1} + w_{r-r-1})\}$ by the matrix:

$$\begin{pmatrix}
    p & (T - 1)^{-1}p \\
    (T - 1)^{-1}p & (T - 1)^{-2}(p + q)
\end{pmatrix}$$

with determinant $(T - 1)^{-2}pq$. Suppose $p$ has order $d$ (i.e. $(T - 1)^d$ divides $p$ and $(T - 1)^{d+1}$ does not) and $q$ has order $d'$. Then if $d \neq d'$, the filtration is of type $(\text{min}\{d, d'\} - 2, d - 1, \max\{d, d'\})$ and if $d = d'$, the filtration is either of type $(d - 1, d - 1, d - 1)$ or type $(d - 2, d - 1, d)$ depending as $p + q \equiv 0 \mod (T - 1)^{d+1}$ or not.

9.3. Suppose $\phi^\sharp = \phi$ (resp. $-\phi$). In the first case we say $\phi$ is $\mathbb{Z}_2$-invariant and in the second skew invariant.

9.3.1. Corollary. Suppose $\phi^\sharp = \phi$ (resp. $-\phi$) and other notation is as in (5.2). Then if $d$ is even, $(P_r, \phi)$ has a filtration of type $(d - 2, d - 1, d)$ (resp. $(d - 1, d - 1, d - 1)$) and if $d$ is odd $(P_r, \phi)$ has a filtration of type $(d - 1, d - 1, d - 1)$ (resp. $(d - 2, d - 1, d)$).

9.4. Corollary. Suppose $\mathcal{F}$ is a finite dimensional $\mathcal{U}_i$-module and $\phi$ is an invariant form on $M \otimes_R \mathcal{F}$. Assume $\phi^\sharp = \phi$ (resp. $-\phi$) and let $M \otimes_R \mathcal{F} = B_0 \supset B_1 \supset \ldots \supset B_r$ be the filtration (5.1.1) and $\overline{B}_0 \supset \overline{B}_1 \supset \ldots \supset \overline{B}_r$, the induced filtration on $\overline{M} \otimes_R \overline{\mathcal{F}}$. Then

1. (i) The $\mathcal{U}_i$-module $\overline{B}/\overline{B}^{i+1}$ is finite dimensional for $i$ odd (resp. even).
2. (ii) The $\mathcal{U}_i$-module $\overline{B}/\overline{B}^{i+1}$ is both free and cofree as a $k(v)[F]$-module, for $i$ even (resp. odd).

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9.5. The symmetry of $\mathbb{Z}_2$-invariant and skew forms can be expressed in another form. Define the Jantzen sum of a filtration to be $\sum_i \text{character}(\mathcal{B}_i)$.

9.5.1. Corollary. Let notation and assumptions be as in (5.4). Then in both cases the Jantzen sum of the filtration is invariant by the Weyl group action on the characters which exchanges the characters of Verma modules $M(t)$ and $M(-t)$ for all $t \in \mathbb{Z}$.

10. Filtrations and Wall-crossing

10.1. Here we begin the study of the relationships of wall-crossing and the theory of induced and $\mathbb{Z}_2$-invariant forms. As in section four set $M$ equal to the Verma module with highest weight $Tv^{-1}$; i.e. $M = M(m)$. Then $\Theta^\ast \Theta$ is isomorphic to $M$ itself and so we may choose a cycle $\Psi : M_F \to M$ which induces the isomorphism. For $a \in M, x \in U_R$, $\Psi(sT_{i-1}^r(x) \cdot a) = x \cdot \Psi(a)$. Recall that $\Psi$ is not $R$ linear due to the role of $s$ in the previous formula. Let $\mathcal{E}$ denote the simple two dimensional $U_i$-module and set $P = M \otimes R \mathcal{E}$. Then $P$ is isomorphic to the basis module $P_1 = P(m + 1)$ as defined in (8.5) and (8.6). Set $M_{\pm} = M(m \pm 1)$. Then the construction of $P$ gives inclusions and a short exact sequence:

$$(T - 1) \cdot M_+ \oplus (T - 1) \cdot M_- \subset P \subset M_+ \oplus M_-, \quad 0 \to (T - 1) \cdot M_+ \to P \to M_- \to 0. \quad (10.1)$$

Let $\mathcal{F}$ be a finite dimensional $U_i$-module and set $\Lambda = (T - 1) \cdot M_+ \otimes_R \mathcal{F}$, $\Lambda = (T - 1) \cdot M_- \otimes_R \mathcal{F}$, $\Pi = P \otimes_R \mathcal{F}$. Then (6.1.1) gives:

$$\Lambda \oplus \Lambda \subset \Pi \subset (T - 1)^{-1}(\Lambda \oplus \Lambda), \quad 0 \to \Lambda \to \Pi \xrightarrow{\Pi} \Lambda \to 0 \quad (10.2)$$

10.2. We now turn to the study of the $s$-linear map $\Gamma$ defined in (4.2.2) and derived from the cycle $\Psi_\mathcal{F} = \Theta_\mathcal{F} \otimes \Theta_\mathcal{E} \otimes \Psi$. Decompose $\mathcal{F}$ into generalized eigenspaces for the Casimir $P(\pm r)$ and let $\mathcal{Q}^r$ denote the $Tv^{-r-1}$ weight subspace of $P(\pm r)$. Recall $\mathbb{L} = \sum \mathcal{Q}_i$. From (4.2.2) we obtain an $s$-linear map $\Gamma : \mathbb{L} \to \mathbb{L}$.

10.2.1. Lemma. We have the following

i). $\Gamma$ restricts to an $s$-linear involutive isomorphism $\Gamma : \mathbb{L} \cap \Lambda \cong \mathbb{L} \cap \Lambda$.

ii). $\Gamma$ induces the identity map $T - 1$; i.e. $\Gamma(e) \equiv e \mod (T - 1) \cdot \mathbb{L}$.

iii). Suppose $\phi$ is any $\mathbb{Z}_2$-invariant (resp. skew invariant) form on $\mathcal{F}$. Then for all $e \in \mathbb{L}$,

$$\phi(\Gamma e, \Gamma e) = s \phi(e, e) \quad (\text{resp.} \quad \phi(\Gamma e, \Gamma e) = -s \phi(e, e)).$$

10.3. As before for any $R \mathcal{U}_i$-module $N$ we let $\overline{N}$ equal the quotient $N/T \cdot N$. From (10.2) we obtain inclusions and a short exact sequence:

$$\overline{\Lambda} \subset \overline{\mathcal{F}}, \quad \overline{\mathcal{B}} \subset \overline{\mathcal{F}}, \quad 0 \to \overline{\Lambda} \to \overline{\mathcal{F}} \xrightarrow{\Pi} \overline{\mathcal{B}} \to 0. \quad (10.3)$$

Now fix an invariant form $\phi$ on $\mathcal{F}$ and let superscripts denote the filtrations on $\mathcal{F}, \Lambda, \mathcal{B}$ and $\mathcal{D}$ induced by $\phi$ and the restrictions of $\phi$ to $\mathcal{F} \times \Lambda, \mathcal{B} \times \mathcal{B}$ and $\mathcal{D} \times \mathcal{D}$ respectively. Let superscripts on $\overline{\Lambda}, \overline{\mathcal{B}}, \overline{\mathcal{D}}$ and $\overline{\mathcal{F}}$ give the filtrations obtained by projecting. So $\overline{\mathbb{L}} = \Lambda^i/(T - 1) \cdot \Lambda \cap \Lambda^i$, etc.

The $R \mathcal{U}_i$-module $\overline{\mathcal{F}}$ is of course also an $\mathcal{U}_i$-module and it has simple socle $M(-1)$. So we find that the inclusions $M_+ \hookrightarrow P$ induce an inclusion $\overline{M}_- \hookrightarrow \overline{M}_+$. In turn we obtain the inclusion $\overline{\mathcal{B}} \hookrightarrow \overline{\Lambda}$.

10.3.1. Proposition. Let $\pi$ denote the natural map $\pi : \mathcal{F} \to \overline{\mathcal{F}}$ and suppose $\phi$ is any $\mathbb{Z}_2$-invariant (resp. skew invariant) form on $\mathcal{F}$. Then $\pi(\mathbb{L} \cap \Lambda) = \pi(\mathbb{L} \cap \mathcal{B})$ and for each $i$, $\Gamma$ induces an $s$-isometry (resp. skew $s$-isometry) of $\mathbb{L} \cap \Lambda^i$ onto $\mathbb{L} \cap \mathcal{B}^i$. 

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10.3.2. Corollary. For all \(i\), \(\mathbb{A}^i \cap \pi(L) = \mathbb{B}^i \cap \pi(L)\).

For any form \(\phi\) on \(P\) we say \(\phi\) is \textit{weakly hereditary} whenever the identity of the corollary holds. We say \(\phi\) is \textit{hereditary} if for all \(i\), \(\mathbb{A} \cap \mathbb{B} = \mathbb{B}^i\).

10.4. The relationship between the filtrations of \(\mathbb{A}\) and \(\mathbb{P}\) is more delicate than that between \(\mathbb{A}\) and \(\mathbb{B}\).

10.4.1. Lemma. Assume that \(\phi\) is induced from an invariant form on \(P\). Then \(\mathbb{A}\) and \(\mathbb{B}\) are orthogonal submodules, and \(\mathbb{A}^i \subset \mathbb{P}^{i-1}\), \(\mathbb{B}^i \subset \mathbb{P}^{i-1}\), and \(\Pi(\mathbb{P}^i) \subset \mathbb{D}^{i-1}\). Also \(\mathbb{D}^i = (T - 1)^{-1} \mathbb{B}^{i+2}\).

10.4.2. Lemma. Assume that \(\phi\) is induced from an invariant form on \(P\) and is \(\mathbb{Z}_2\)-invariant or skew invariant. Then

\[
\mathbb{L} \cap \mathbb{D}^i \subset \Pi(\mathbb{P}^i) \quad \text{and} \quad \mathbb{L} \cap \mathbb{P}^i \cap \mathbb{A} \subset \mathbb{A}^i.
\]

11. Equivalence Classes of Forms

11.1. In this section we describe explicitly all the equivalence classes of invariant forms on \(B \otimes \mathcal{E}_R\) where \(\mathcal{E}\) is the irreducible two dimensional \(R\mathcal{U}\)-module and \(B\) is one of the indecomposable modules \(M(U_R, m + b), \ b \in \mathbb{Z}\) or \(P(U_R, m + b), \ b \in \mathbb{N}^*\). Two invariant forms \(\chi\) and \(\chi^\rho\) on a \(U_R\)-module \(A\) are equivalent if there exists an \(U_R\)-module automorphism \(\kappa : A \to A\) with \(\chi(a, b) = \chi^\rho(\kappa a, \kappa b)\).

For \(n \in \mathbb{N}\) let \(\phi_{\pm n}\) denote the Shapovalov form on \(M(RU_i, m \pm n)\) normalized by the identities:

\[\phi_n(v_{n, -n-1}, v_{n, -n-1}) = 1 = \phi_{-n}(v_{n, -n-1}, v_{n, -n-1}).\] (11.1)

\textbf{Caution:} This is not the obvious normalization. But we will find it to be the most convenient.

11.1.1. Lemma. The equivalence classes of invariant forms on \(M(U_R, m+n)\) (resp. \(M(U_R, m-n)\)) are represented by the forms \(T^r \cdot \phi_n, \ r \in \mathbb{N}^*\) (resp. \(T^r \cdot \phi_{-n}, \ r \in \mathbb{N}\)).

11.2. The indecomposable module \(P_n = P(U_R, m + n)\) was constructed as a submodule \(P_n \subset M_n \oplus M_{-n}\) where we set \(M_n = M(U_R, m + n)\) and \(M_{-n} = M(U_R, m - n)\). Therefore any invariant form on \(P\) is the restriction of the orthogonal sum of a form on \(M_n\) and one on \(M_{-n}\). For scalars \(q\) and \(r\) let \(\phi_{n,q,r} = q\phi_n \oplus r\phi_{-n}\) denote such an orthogonal sum of forms.

11.2.1. Lemma. The equivalence classes of invariant forms on \(P_n\) are represented by the degenerate forms: \(\phi_{n,0,0,0,0}\) and \(\phi_{n,0,0,0,0}\), \(b \in \mathbb{N}\) and the nondegenerate forms: \(\phi_{n,T^k, uT^k}\) where \(u\) is nonzero complex number and \(k, l \in \mathbb{N}\).

\textit{Proof.} Recall the automorphism \(\kappa\) of \(P_n\) determined by the units \(u\) and \(v\) with \(u \equiv v \pmod{T}\), as in the proof of Lemma [11.2]. Then by \(\kappa\), we see that \(\phi_{n,q,r}\) and \(\phi_{n,u^2v^2x^2y^2}\) are equivalent. Choose integers \(k\) and \(l\) and units \(u_0\) and \(v_0\) with \(q = u_0t^k\) and \(r = v_0T^l\). Let \(c\) be the complex number which is the ratio of the constant term of \(v_0\) by that of \(u_0\). Then \(r = v_1cT^l\) with \(v_1\) a unit and \(U_0 \equiv v_1 \pmod{T}\). Choose square roots \(u\) and \(v\) with \(u^2 = u_0, v^2 = v_1\) and \(U \equiv v \pmod{T}\). Then with \(\kappa\) defined as above, we find that \(\phi_{n,q,r}\) is equivalent to \(\phi_{n,T^k, uT^k}\). \(\square\)
11.3. Suppose $\Psi$ is a cycle on $P_n$ and the basis is chosen as in Lemma and Corollary 4.3. Set $v_\pm = v_{\pm n,n-1}$. Then $\Gamma v_\pm = v_\mp$.

11.3.1. Lemma. Suppose $\chi$ is an invariant form on $P_n$ which is also $\mathbb{Z}_2$-invariant. Then, for some $q \in R$, $\chi = \phi_{n,q,sq}$ and $\chi$ is equivalent to one of the $\mathbb{Z}_2$-invariant forms $\phi_{n,Td,(-1)^dTd}$, for $d \in \mathbb{N}$.

Proof. Choose $q$ and $r$ with $\chi = \phi_{n,q,r}$. Then $\mathbb{Z}_2$-invariance gives $r = sq$. Write $q = uTd$ with $u$ a unit. Then $r = (-1)^d suTd$. Let $\kappa$ be the automorphism of $P_n$ which equals $u_1$ on $M(\mathbb{U}_R, m + n)$ and $su_1$ on $M(\mathbb{U}_R, m - n)$ where $u_1^2 = u$. Then via $\kappa$, $\phi_{n,q,r}$ is equivalent to $\phi_{n,Td,(-1)^dTd}$.

Let us consider the identity

$$\Theta \circ \beta(a \otimes b) = \beta(\Theta \otimes \Theta(b)) \quad (11.2)$$

Here $\Theta$ on the left is assumed to be the module homomorphism. The motivation for this comes from [Lus93, 5.3.4]. Let $v \in (\mathcal{E} \otimes \mathcal{F})^m$ be a highest weight vector, then $\beta(v) \in F(U)^m$ is a highest weight vector. Then

$$\beta(\Theta \circ (\Theta \otimes \Theta)(F(j)v)) = \beta(\Theta \otimes \Theta \otimes F(j)v)) = \beta((-1)^j v^{e(jh+j)} F(h)v) = (-1)^j v^{e(jh+j)} F(h) \beta(v) = T_{-1} \beta(v)$$

where $h + j = m$.

Proof. Fix $m \in \mathbb{N}^*$ and let $D$ denote the $T + b - 2m$ weight space in $\mathcal{E}_R \otimes M(\mathbb{U}_R, m + b)$. Set $\overline{v} = v_{b,b-1-2m}$. Then $D$ has basis $\{d_+, d_0\}$ where $d_+ = E \otimes F\overline{v}, d_- = FK \otimes \overline{v}$ and the form $\delta \otimes \phi_b$ on $D$ is given by:

$$\begin{pmatrix}
(-1) & 0 \\
0 & -m(T + b - m)
\end{pmatrix}
\phi_b(\overline{v}, \overline{v}). \quad (11.3)$$

Suppose $b \neq 0$. Then $\mathcal{E}_R \otimes M_b \equiv M_{b-1} \oplus M_{b+1}$ and if we look at the weight space $D$ for $m \neq b$ we conclude: $\delta \otimes \phi_b = q \phi_{b-1} + r \phi_{b+1}$ where both $q$ and $r$ equal $\phi_b(\overline{v}, \overline{v})$ times a unit. Inturn this gives the formula for $b \neq 0$. For $b = 0$ note that $\delta \otimes \phi_b$ is $\mathbb{Z}_2^2$-invariant and since $\mathcal{E}_R \otimes M_0 \equiv P(\mathbb{U}_R, m + 1), \delta \otimes \phi_0 = \phi_{1,sq}$ for some $q \in R$. Write $q = uT^d$ for some unit $u$. To determine the integer $d$ we need only compare the values of the forms on an $R$-basis vector for the highest weight space. But $\delta \otimes \phi_0(e_+ \otimes v_{0-1}, e_+ \otimes v_{0-1}) = -\phi_0(v_{0-1}, v_{0-1}) = -1$. □

12. Change of Coordinates for Induced Forms

12.1. We continue with the notation from the earlier sections.. Set $M$ (resp. $M_\pm$) equal to the $\mathbb{U}_R$-Verma module with highest weight $T - 1$ (resp. $T, T - 2$). Let $\phi$ and $\phi_\pm$ be the canonical invariant forms on these Verma modules. Let $\mathcal{E}$ (resp. $\mathcal{F}$) be an irreducible $\mathbb{U}_r$-module of dimension $d + 1$ (resp. $d + 2$). Then as $\mathcal{B}$-modules we have the short exact sequence:

$$0 \rightarrow \mathcal{E} \otimes R_T \rightarrow \mathcal{F} \otimes R_{T-1} \rightarrow R_{T-d-1} \rightarrow 0. \quad (12.1)$$

Inducing up to $\mathbb{U}_R$ we obtain:

$$0 \rightarrow \mathcal{E}_R \otimes M_+ \rightarrow \mathcal{F}_R \otimes M \rightarrow M(\mathbb{U}_R, T - d) \rightarrow 0. \quad (12.2)$$

From [] we know that restriction from $\mathcal{E}_R \otimes M(\mathbb{U}_R, T + b)$ to $\mathcal{E}_R \otimes R_{T+b-1}$ gives an isomorphism of $\mathbb{U}_R$-invariant forms on $\mathcal{E}_R \otimes M(\mathbb{U}_R, T + b)$ to $\mathbb{U}_0$-invariant forms on $\mathcal{E}_R \otimes R_{T+b-1}$. 20
Since (12.1) splits as an $U_0$-module, each $U_0$-invariant form on $\mathcal{E}_R \otimes R_T$ can be extended uniquely to $\mathcal{F}_R \otimes R_{T-1}$ with it equaling zero on the weight space for $T - d - 1$. With this convention we obtain a map $\phi_\mathcal{E}$ from invariant forms on $\mathcal{E}_R \otimes M_+$ to those on $\mathcal{F}_R \otimes M$.

12.2. Fix a nonnegative integer $d$ and suppose $\beta$ is an $U_R$-module homomorphism $\beta : \mathcal{E}_R^\otimes \otimes \mathcal{E}_R \to U_R$ with image equal to the irreducible with highest weight vector $E^{(d)}$ and normalized so that $\beta(f_d \otimes f_d) = \text{ad}(F^{(d)})(E^{(d)})$. Normalize the basis of $\mathcal{E}_R$ by setting $\mathcal{F}_d \cdot \mathcal{F}_d = \mathcal{F}_{d+2}$. Then $\Theta_\mathcal{E}(\mathcal{F}_{d-2}) = (-1)^d \mathcal{F}_{d+2}$.

12.2.1. Lemma. Let $\gamma$ be the invariant form on $M_b = M(U_R, T + b + 1)$ normalized by $\gamma(v_{b-1, b}, v_{b-1, b}) = 1$ and $\chi_{b, \gamma}$ the induced form on $\mathcal{E}_R \otimes M_b$. For $0 \leq j \leq d$, define complex constants by $\beta(\Theta_\mathcal{E} f_{d-2j} \otimes f_{d-2j}) = c_j \beta(f_d \otimes f_d)$ and define $p_d \in U_0$ by $p_d(T) = \prod_{1 \leq t \leq d} [T; d - 2t]$. Then with respect to the basis $\mathcal{F}_{d-2j} \otimes 1$, the restriction of the induced form $\chi_{b, \gamma}$ to $\mathcal{E}_R \otimes R_{T+b}$ is given by the diagonal matrix

$$\begin{pmatrix}
    c_0 \\
    \cdot \\
    \cdot \\
    c_d
\end{pmatrix} p_c(T + b). \quad (12.3)$$

Moreover, the constants have the symmetry: $c_{d-j} = (-1)^d c_j$, $0 \leq j \leq d$.

Proof. Let $\pi$ be the projection of $R U_i$ onto $U_0$ which is zero on $F \cdot U_i + U_i \cdot E$. First we evaluate $\pi \beta(f_d \otimes f_d)$. Since

$$\text{ad}(F^{(p)})(E^{(d)}) = E^{(d)} \quad (12.4)$$

by (5.1), we conclude using (6.2)

$$\text{ad}(F^{(d-2j)})(E^{(d-2j)}) = \begin{pmatrix} T; d - 2j \\ d \end{pmatrix} E^{(d-2j)}, \quad (12.5)$$

that

$$\beta \pi (\text{ad}(F^{(d-2j)})(E^{(d-2j)})) = \begin{pmatrix} T; d - 2j \\ d - 2j \end{pmatrix} \quad (12.6)$$

Now to obtain the matrix expression in (12.3) we observe

$$\chi_{b, \gamma}(\mathcal{F}_{d-2j} \otimes 1, \mathcal{F}_{d-2j} \otimes 1) = \beta(\Theta_\mathcal{E} \mathcal{F}_{d-2j} \otimes \mathcal{F}_{d-2j}) \cdot 1_{T+b} = c_j \prod_{1 \leq t \leq d} [T; d - 2t].$$

Finally regarding the symmetry, $\beta \circ (\Theta_\mathcal{E} \otimes \Theta_\mathcal{E}) = \Theta \circ \beta$. So by the basis given in §5.1 gives us

$$(-1)^d c_j \beta(\mathcal{F}_d \otimes \mathcal{F}_d) = \Theta_\mathcal{E}(\mathcal{F}_{d-2j} \otimes \mathcal{F}_{d-2j}) = \beta(\Theta_\mathcal{E} \mathcal{F}_{d-2j} \otimes \Theta_\mathcal{E} \mathcal{F}_{d-2j})$$

This gives the symmetry of the constants and completes the proof. \qed

Let $D$ denote the diagonal matrix with constants $c_j$ on the diagonal. Then Lemma 12.2.1 asserts that the induced form $\chi_{b, \gamma}$ is determined by the matrix $D \cdot p_d(T + b)$. We shall call $D$ the coefficient matrix associated to $\beta$ and $D \cdot p_d(T + b)$ the full matrix associated to the form $\chi_{b, \gamma}$.

We plan on describing the coefficient matrix for particular cases in a future publication.
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