ON A SUBFACTOR CONSTRUCTION OF A FACTOR NON-ANTIISOMORPHIC TO ITSELF

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Abstract. We define a $\mathbb{Z}_3$-kernel $\alpha$ on $\mathcal{L}\left(\mathbf{F}_{11}\right)$ and a $\mathbb{Z}_3$-kernel $\beta$ on the hyper-finite factor $R$, which have conjugate obstruction to lifting. Hence, $\alpha \otimes \beta$ can be perturbed by an inner automorphism to produce an action $\gamma$ on $\mathcal{L}\left(\mathbf{F}_{11}\right) \otimes R_0$. The aim of this paper is to show that the factor $\mathcal{M} = \left(\mathcal{L}\left(\mathbf{F}_{11}\right) \otimes R_0\right) \rtimes \mathbb{Z}_3$, which is similar to Connes’s example of a $\text{II}_1$ factor non-antiisomorphic to itself, is the enveloping algebra of an inclusion of $\text{II}_1$ factors $A \subset B$. Here $A$ is a free group factor and $B$ is isomorphic to the crossed product $A \rtimes_\theta \mathbb{Z}_3$, where $\theta$ is a $\mathbb{Z}_3$-kernel of $A$ with non-trivial obstruction to lifting. By using an argument due to Connes, which involves the invariant $\chi(\mathcal{M})$, we show that $\mathcal{M}$ is not anti-isomorphic to itself. Furthermore, we prove that for one of the generator of $\chi(\mathcal{M})$, which we will denote by $\sigma$, the Jones invariant $\kappa(\sigma)$ is equal to $e^{\frac{2\pi i}{9}}$.

1. Introduction

A von Neumann algebra $\mathcal{M}$ is anti-isomorphic to itself if there exists a vector space isomorphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ with the properties $\Phi(x^*) = \Phi(x)^*$ and $\Phi(xy) = \Phi(y)\Phi(x)$ for every $x, y \in \mathcal{M}$. This is equivalent to saying that $\mathcal{M}$ is isomorphic to its conjugate algebra $\mathcal{M}^c$ (defined in Section 6).

With his examples of a $\text{II}_1$ factor non-antiisomorphic to itself (cf. [4]), A. Connes gave in the ’70s an answer to a crucial problem posed by F. Murray and J. von Neumann a few decades earlier, concerning the possibility of realizing every $\text{II}_1$ factor as the left regular representation of a discrete group. His example was obtained from the $\text{II}_1$ factor $\mathcal{L}(\mathbf{F}_4) \otimes R$, where $R$ denotes the hyperfinite $\text{II}_1$ factor, using a crossed product construction with a $\mathbb{Z}_3$-action. After the innovative work on subfactors done by V. Jones in the ’80s (see [11]), it was a natural question to ask whether Connes’s factor could be obtained through a subfactor construction of finite index. Although extensive work ([2], [4], and [10]) has been done by both Connes and Jones on examples of $\text{II}_1$ and $\text{III}_\lambda$ factors non-antiisomorphic to itself, it does not seems that this problem has been addressed before and there is little literature on the subject.

In this paper we provide a positive answer to this question by giving an explicit subfactor construction for our example of a $\text{II}_1$ factor non-antiisomorphic to itself. Our model is a variation of Connes’s example ([3] and [4]), since we utilize in our approach the recently developed theory of interpolated free group factors ([13] and [Dv2]). The $\text{II}_1$ factor $\mathcal{M}$ we are going to study is constructed from the tensor product

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\end{itemize}
of the interpolated free group factor $\mathcal{L}\left(F_{11}^{3}\right)$ and the hyperfinite II$_1$ factor $R$. We use two $\mathbb{Z}_3$-kernels, $\alpha \in \text{Aut}\left(\mathcal{L}\left(F_{11}^{3}\right)\right)$ and $\beta \in \text{Aut}(R)$, which have conjugate obstructions to lifting, to generate an action of $\mathbb{Z}_3$ on $\mathcal{L}\left(F_{11}^{3}\right) \otimes R$. The action is given, up to an inner automorphism, by $\alpha \otimes \beta$, and the factor $\mathcal{M}$ is equal to the crossed product $\left(\mathcal{L}\left(F_{11}^{3}\right) \otimes R\right) \rtimes_{\gamma} \mathbb{Z}_3$ (cf. Section 4).

The main result of this paper is Theorem 4.4, where we show that $\mathcal{M}$ is the enveloping algebra of an inclusion $A \subset B$ of interpolated free group factors. Here $A$ is isomorphic to $\mathcal{L}\left(F_{35}^{27}\right)$ (Proposition 4.3), and $B$ is equal to the crossed product $A \rtimes_{\theta} \mathbb{Z}_9$, for a $\mathbb{Z}_9$-action $\theta$ of $A$ with outer period 3, and obstruction $e^{2\pi i/9}$ to lifting. The proof is based on Voiculescu’s random matrix model for circular and semicircular elements introduced in [17]. An analogous argument has been used by F. Radulescu in [14], to prove that a variation of the example given by Jones of a II$_1$ factor with Connes invariant equal to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, has a subfactor construction. We also show in Section 5 that the Connes invariant of our factor $\mathcal{M}$ is equal to $\mathbb{Z}_9$, a result announced by Connes in [3]. This invariant, which is defined for every factor $M$ with a separable predual, was introduced by Connes in [4]. It consists of an abelian subgroup of the group of outer automorphisms, and it is denoted by $\chi(M)$. It is an important tool for distinguishing factors, since it is an isomorphism invariant of the factor $M$. It is trivial for some of the most common II$_1$ factors, like the interpolated free group factors and the hyperfinite II$_1$ factor, as well as for any tensor product of these factors. However, a crossed product construction yields in general a non-trivial $\chi(M)$. To compute the Connes invariant of the factor $\mathcal{M} = \left(\mathcal{L}\left(F_{11}^{3}\right) \otimes R_0\right) \rtimes_{\gamma} \mathbb{Z}_3$, we use the short exact sequence described by Connes in [4].

In addition, we show that if $\sigma$ denotes the generator of $\chi(\mathcal{M})$ described in Remark 5.4 then the invariant $\kappa(\sigma)$, introduced by Jones in [10] is equal to $e^{2\pi i/9}$. The Jones invariant is defined for any element $\theta$ of $\chi(M)$, where $M$ is a II$_1$ factor without non-trivial hypercentral sequences, and it consists of a complex number of modulus one. It is a finer invariant than $\chi(M)$ and it is constant on the conjugacy class of $\theta$ in the group of outer automorphisms. Moreover, it behaves nicely with respect to antiautomorphisms of $M$, in the sense that conjugation by an anti automorphism changes $\kappa(\theta)$ by complex conjugation (see [10] for details).

Lastly, in Section 6 we use an argument of Connes [3] to show that $\mathcal{M} = \left(\mathcal{L}\left(F_{11}^{3}\right) \otimes R_0\right) \rtimes_{\gamma} \mathbb{Z}_3$ is not anti-isomorphic to itself. The two main ingredients of this argument are the uniqueness (up to inner automorphism) of the decomposition of $\gamma$ into the product of an approximately inner automorphism and a centrally trivial automorphism, and the fact that the unique subgroup of order 3 in $\chi(\mathcal{M})$ is generated by the dual action $\hat{\gamma}$ on $\mathcal{M} = \left(\mathcal{L}\left(F_{11}^{3}\right) \otimes R_0\right) \rtimes_{\gamma} \mathbb{Z}_3$. Using this decomposition we obtain a canonical way to associate to the II$_1$ factor $\mathcal{M}$ a complex number, the obstruction to lifting of the approximately inner automorphism in the
decomposition of $\gamma$, which is invariant under isomorphism, and distinguishes $M$ from its conjugate algebra $M^c$.

2. Definitions

Let $M$ be a factor with separable predual. Denote by $\text{Aut}(M)$ the group of automorphisms of $M$ endowed with the $u$-topology, i.e., a sequence of automorphisms $\alpha_n$ converges to $\alpha$ if and only if $\|\phi \circ \alpha_n - \phi \circ \alpha\| \to 0$ for all $\phi \in M_*$.

The definition of the Connes invariant $\chi(M)$ involves three normal subgroups of the group of automorphisms $\text{Aut}(M)$. For a unitary $u$ in $M$ denote by $\text{Ad}_M(u)$ the inner automorphism of $M$ defined by $\text{Ad}_M(u)(x) = uxu^*$, for any $x$ in $M$. Let $\text{Int}(M)$ be the subgroup of $\text{Aut}(M)$ formed by all inner automorphisms. Then $\text{Int}(M)$ is normal in $\text{Aut}(M)$ since $\alpha \text{Ad}_M(u)\alpha^{-1} = \text{Ad}_M \alpha(u)$ for every $\alpha \in \text{Aut}(M)$. Let $\overline{\text{Int}}(M)$ denote the closure of the group $\text{Int}(M)$ in the $u$-topology. The group $\text{Int}(M)$ of inner automorphisms and the group $\overline{\text{Int}}(M)$ of approximately inner automorphisms are two of the groups involved in the definition of the Connes invariant.

We restrict now our attention to $\text{II}_1$ factors. Recall that if $\tau$ denotes the unique faithful trace on $M$, then $M$ inherits an $L^2$-norm from the inclusion $M \subset L^2(M)$, given by $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in M$. Note also that for a $\text{II}_1$ factor $M$, the $u$-topology on $\text{Aut}(M)$ is equivalent to the topology for which a sequence of automorphisms $\alpha_n$ on $M$ converges to $\alpha$ iff $\lim_{n \to \infty} \|\alpha_n(x) - \alpha(x)\|_2 \to 0$. Since our study of automorphisms is simplified in the $\text{II}_1$ case, we will always assume that our factors are $\text{II}_1$, unless otherwise specified.

The last group of automorphisms involved in the definition of the Connes invariant is formed by the centrally trivial automorphisms of $M$.

Given $a, b \in M$ set $[a,b] = ab - ba$. We say that a bounded sequence $(x_n)_{n \geq 0}$ in $M$ is central if $\lim_{n \to \infty} \|[x_n, y]\|_2 = 0$ for every $y \in M$.

**Definition 2.1.** An automorphism $\alpha \in \text{Aut}(M)$ is centrally trivial if for any central sequence $(x_n)$ in $M$ we have $\lim_{n \to \infty} \|\alpha(x_n) - x_n\|_2 = 0$.

We denote by $\text{Ct}(M)$ the set of centrally trivial automorphisms of $M$, which is a normal subgroup of $\text{Aut}(M)$.

Let $\text{Out}(M) = \frac{\text{Aut}(M)}{\text{Int}(M)}$ be the group of outer automorphisms of $M$, and denote by $\xi : \text{Aut}(M) \to \text{Out}(M)$ the quotient map. The Connes invariant was introduced by Connes in [4] (see also [3]).

**Definition 2.2.** Let $M$ be a $\text{II}_1$ factor with separable predual. The Connes invariant of $M$ is the abelian group

$$\chi(M) = \frac{\text{Ct}(M) \cap \overline{\text{Int}}(M)}{\text{Int}(M)} \subset \text{Out}(M).$$

Note that the hyperfinite $\text{II}_1$ factor $R$ has trivial Connes invariant since $\text{Ct}(R) = \text{Int}(R)$.
Next we want to define a particular class of central sequences, the hypercentral sequences.

**Definition 2.3.** A central sequence \((x_n)\) is hypercentral if \(\lim_{n \to \infty} \|[x_n, y_n]\| = 0\) for every central sequence \((y_n)\) in \(M\).

Let \(\omega\) be a free ultrafilter over \(\mathbb{N}\) and \(M\) a \(\text{II}_1\) factor. Denote by \(\ell^\infty(\mathbb{N}, M)\) the algebra of bounded sequences in \(M\), and by \(C^{\omega}\) the subalgebra of the bounded sequence \((x_n)_{n \in \mathbb{N}}\) in \(M\) with \(\lim_{n \to \omega} \|[x_n, y]\| = 0\), for all \(y \in M\). Let \(\mathcal{J}_{\omega}\) be the subalgebra of \(\ell^\infty(\mathbb{N}, M)\) consisting of the sequences for which \(\lim_{n \to \omega} \|x_n\| = 0\). Set

\[
M^{\omega} = \frac{\ell^\infty(\mathbb{N}, M)}{\mathcal{J}_{\omega}} \quad \text{and} \quad M_{\omega} = \frac{C^{\omega}}{\mathcal{J}_{\omega} \cap C^{\omega}}
\]

Then, \(M^{\omega}\) and \(M_{\omega}\) are finite von Neumann algebras and

\[
M_{\omega} = M^{\omega} \cap M'.
\]

**Remark 2.4.** By [12] the existence of non-trivial hypercentral sequences is equivalent to the non-triviality of the center of \(M_{\omega}\) for some (and then for all) free ultrafilter \(\omega\).

**Definition 2.5.** A \(\mathbb{Z}_k\)-kernel on a von Neumann algebra \(M\) is an automorphism \(\alpha \in \text{Aut}(M)\) such that there exists a unitary \(U\) in \(M\) with the property \(\alpha^k = \text{Ad}_M U\).

Note that if \(\alpha\) is a \(\mathbb{Z}_k\)-kernel then \(\alpha(U) = \lambda U\), for \(\lambda\) a \(k\)-th root of unity. If \(\lambda \neq 1\) we say that \(\alpha\) has obstruction \(\lambda\) to lifting, meaning that the homomorphism \(\varphi : \mathbb{Z}_k \to \text{Out}(M)\) defined by \(\varphi(1) = [\alpha]\) cannot be lifted to an homomorphism \(\Phi : \mathbb{Z}_k \to \text{Aut}(M)\) with \(\Phi(1) = \alpha\).

We conclude this section by defining an invariant \(\varpi(\theta)\) for any element \(\theta\) in \(\chi(M)\).

**Definition 2.6.** Let \(M\) be a \(\text{II}_1\) factor without non-trivial hypercentral sequences and take \(\theta \in \chi(M)\). Let \(\phi\) be an automorphism in \(\text{Ct}(M) \cap \text{Int}(M)\) with \(\xi(\phi) = \theta\), and \(u_n\) a sequence of unitaries such that \(\phi = \lim_{n \to \infty} \text{Ad} u_n\). Since the sequence \((u_n^* \phi(u_n))_{n \geq 0}\) is hypercentral, there exists a sequence of scalars \((\lambda_n)_{n \geq 0}\) with the properties that \(\lim_{n \to \infty} \|\phi(u_n) - \lambda_n u_n\| = 0\), and \((\lambda_n)_{n \geq 0}\) converges to some \(\lambda_\phi \in \mathbb{T}\) (Lemmas 2.1 and 2.2 in [10]). Hence,

\[
\lim_{n \to \infty} \|\phi(u_n) - \lambda_\phi u_n\| = 0
\]

and the Jones invariant is \(\varpi(\theta) = \lambda_\phi\).

Jones proved in [10] that this definition makes sense (i.e. \(\varpi(\theta)\) does not depend on the choice of \(\phi\) or \(u_n\)) and that \(\varpi\) is a conjugacy invariant, meaning that if \(\alpha, \beta\) belong to \(\text{Ct}(M) \cap \text{Int}(M)\) and there exists \(\psi \in \text{Aut}(M)\) such that \(\psi \alpha \psi^{-1} = \beta\), then \(\varpi(\xi(\alpha)) = \varpi(\xi(\beta))\).

3. Preliminaries

Let \(M\) be a factor with separable predual. \(M\) is said to be full if \(\text{Int}(M)\) is closed in \(\text{Aut}(M)\) with respect to the \(u\)-topology. Obviously all type \(I\) factors are full, while the hyperfinite factor \(R\) provides an example of a \(\text{II}_1\) factor which is not full since \(\text{Int}(R) = \text{Aut}(R) \neq \text{Int}(R)\).
Remark 3.1. For an arbitrary factor, being full is equivalent to having no non-trivial central sequence (see [1]).

The following result, due to Connes, is an easy consequence of Lemma 4.3.3 in [15] and Corollary 3.6 in [2]. Some of the arguments used in the proof can be found in [9].

Lemma 3.2. Let $G$ be a discrete group containing a non-abelian free group and let $\tau$ be the usual trace on $L(G)$. Then $L(G)$ is full.

Proof Set $E = G - \{e\}$, where $e$ denotes the identity element in $G$. Let $g_1, g_2$ be two generators of the free group and $F = \{g \in E \mid g = g_1\tilde{g}, \tilde{g} \in E\}$. Take $x \in L(G)$. Then $x$ can be expressed as $x = \sum_{g \in G} \lambda_g \delta_g$ and the function $f : G \to \mathbb{C}$ defined by $f(g) = \lambda_g$ belongs to $l^2(G)$.

For such $f$ we have that
\[
\sum_{g \in E} |f(g)|^2 = \|x - \tau(x)\|_2^2 \quad \text{and} \quad \sum_{g \in G} |f(g)g_i^{-1} - f(g)|^2 = \|[x, \delta_{g_i}]\|_2^2.
\]

Now if $(x_n)$ is a central sequence in $L(G)$ then $\|[x_n, \delta_{g_i}]\| \to 0$ for $n \to \infty$, so we can apply Lemma 4.3.3 in [15] and conclude that
\[
\lim_{n \to \infty} \|x_n - \tau(x_n)\|_2 = 0 \tag{2}
\]

Let $\alpha$ be any automorphism in $\text{Int}(L(G))$ and choose a sequence of unitaries $(u_n)_n$ such that $\alpha = \lim_{n \to \infty} \text{Ad}(u_n)$. Since $(u_n^*u_{n+1})_{n \geq 0}$ is a central sequence in $L(G)$, by [2] there exists $\lambda_n \in \mathbb{T}$ such that
\[
\|u_n^*u_{n+1} - \lambda_n 1\|_2 < \frac{1}{2^n}.
\]

Set $v_n = \left(\prod_{i=1}^{n} \lambda_n\right) u_{n+1}$. Then $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence so it converges to some $t$ in $L(G)$. Since
\[
\text{Ad}(t) = \lim_{n \to \infty} \text{Ad}(v_n) = \lim_{n \to \infty} \text{Ad}(u_{n+1}) = \alpha
\]
we have that $\alpha \in \text{Int}(L(G))$.

Using the following remark we can conclude that not only the free group factors are full, but also the interpolated ones.

Remark 3.3. If $N \subseteq M$ is an inclusion of II$_1$ factors and $p$ is a projection in $N$, then $p(N' \cap M)p = pN'p \cap pMp$.

One inclusion of the previous remark is obvious. The other one is proved using the following argument due to S. Popa. Take any element $z \in pN'p \cap pMp$ so that $z = x'p$ with $x' \in N'$. Let $q$ a maximal projection in $N$ with the properties that $p \leq q \leq 1$ and $x'q \in M$. To show that $q = 1$, suppose that $1 - q \neq 0$. Then $(1 - q)Np \neq 0$, so using the polar decomposition of a non-zero element in $(1 - q)Np$ we can find
0 ≠ v ∈ N such that v*v ≤ p and vv* ≤ 1 − q. Thus x'vv* = vx*v = vpx'pv* ∈ M
and x'(q + vv*) ∈ M, contradicting the maximality of q.

Taking N = A and M = A^ω in the previous remark, and using [1] we obtain that
the compression of a full factor is also full.

**Remark 3.4.** Let A be a II_1 factor with separable predual and p a projection in A.
Then A is full if and only if pAp is full. In particular, any interpolated free group
factor \( \mathcal{L}(F_t) \), with \( t > 1 \), is full.

**Proposition 3.5.** Let \( \mathcal{L}(F_t) \), for \( t ∈ \mathbb{R} \) and \( t > 1 \), be any interpolated free group
factor, and denote by \( R \) the hyperfinite II_1 factor. Then \( \mathcal{L}(F_t) ⊗ R \) has no non-trivial
hypercentral sequences.

**Proof** We start by proving the result for \( \mathcal{L}(G) ⊗ R \), where G is a discrete group
containing a non-abelian free group. First we want to show that any central sequence
in \( \mathcal{L}(G) ⊗ R \) has the form \( (1 ⊗ x_n)_{n ≥ 0} \), for a central sequence \( (x_n)_{n ≥ 0} \) in R.

Denote by \( g_i \), for \( i = 1, 2 \) the generators of \( F_2 ⊆ G \). By the proof of Lemma
\( \mathcal{L}(G) \) satisfies the hypothesis of Lemma 2.11 in [1] with \( Q_1 = \mathcal{L}(G) \), \( Q_2 = R \)
and \( b_k = \delta_{g_k} \). Therefore, we can apply the above mentioned lemma to any central
sequence \( (X_n)_{n ≥ 0} \) in \( \mathcal{L}(G) ⊗ R \) to obtain that \( \lim_{n → ∞} ||X_n - (τ ⊗ 1)(X_n)||_2 = 0 \). Since
\( (τ ⊗ 1)(X_n) ∈ \mathbb{C} ⊗ R \), this implies that \( X_n = 1 ⊗ x_n \) for a central sequence \( (x_n)_{n ≥ 0} \) in R.

Now suppose \( (Y_n)_{n ≥ 0} \) is a hypercentral sequence in \( \mathcal{L}(G) ⊗ R \). Since \( (Y_n) \) is central
it has the form \( Y_n = 1 ⊗ y_n \), for a hypercentral sequence \( (y_n) \) in R. So we need only
to prove that R has no non-trivial hypercentral sequences.

This follows immediately from Remark 2.4 and Theorem 15.15 in [8].

In the case of the factor \( \mathcal{L}(F_t) ⊗ R \), we can find an integer \( k > 1 \) and a projection
p in \( \mathcal{L}(F_k) ⊗ R \) such that \( \mathcal{L}(F_t) ⊗ R ≅ p(\mathcal{L}(F_k) ⊗ R)p \). Obviously p belongs to
\( (\mathcal{L}(F_k) ⊗ R)^ω \). Therefore, by Remark 2.4 it is enough to show that given a II_1 factor
M and a projection \( p ∈ M'_ω \), \( (pMp)'_ω \) is a factor if and only if \( M'_ω \) is a factor. This is
an immediate consequence of the equality \( (pMp)'_ω = pM'_ω p \).

\[ \square \]

**Lemma 3.6.** If \( α ∈ \text{Ct}(\mathcal{L}(F_t) ⊗ R) \) then \( α = \text{Ad } z(ν ⊗ \text{id}) \), for some unitary z ∈ \( \mathcal{L}(F_t) ⊗ R \) and an automorphism ν of \( \mathcal{L}(F_t) \).

**Proof** Let \( (K_n)_{n ∈ \mathbb{N}} \) be an increasing sequence of finite dimensional subfactors of
R generating R, and denote by \( R_n = K_n^0 \cap R \) the relative commutant of \( K_n \) in R.

Set \( L_n = 1 ⊗ R_n ⊆ \mathcal{L}(F_t) ⊗ R \). Then there exists an \( n_0 \) such that for all \( x ∈ L_{n_0} \),
\( ||x|| ≤ 1 \) one has \( ||α(x) − x||_2 ≥ 1/2. \) In fact, otherwise it would exist a uniform bounded
sequence \( (x_n) \), \( x_n ∈ L_n \) and \( ||x_n|| ≤ 1 \), such that \( ||α(x_n) − x_n||_2 > 1/2 \). But \( (x_n) \) is a
central sequence in \( \mathcal{L}(F_t) ⊗ R \) because for each m and \( n ≥ m \), \( x_n \) commutes with
\( \mathcal{L}(F_t) ⊗ K_m \), so we get a contradiction.

By Lemma 3.3 in [2], up to inner automorphism, \( α \) is of the form \( α_1 ⊗ 1_{R_{n_0}} \) where
\( α_1 \) is an automorphism of \( \mathcal{L}(F_t) ⊗ K_{n_0} \). Set \( F = 1 ⊗ K_{n_0} \). Then F is a type I
subfactor of \( \mathcal{L}(F_t) ⊗ K_{n_0} \).
Applying [Lemma 3.11, 2] to \(\alpha_1\) we obtain that \(\alpha_1|_{1\otimes \mathcal{K}_{n_0}} = \text{Ad} V|_{1\otimes \mathcal{K}_{n_0}}\). This implies that \(\alpha = \text{Ad} \, z(\nu \otimes 1)\) for some automorphism \(\nu\) of \(\mathcal{L}(\mathcal{F}_1)\).

4. THE SUBFACTOR CONSTRUCTION OF INTERPOLATED FREE GROUP FACTORS

In this section we use Voiculescu’s random matrix model for free group algebras \([17]\), to show that the crossed product

\[
\left( \mathcal{L} \left( \mathcal{F}^{1/3} \right) \otimes \mathbb{R} \right) \rtimes \gamma \mathbb{Z}_3
\]

implies that \(\alpha\) by enveloping algebra of an inclusion of interpolated free group factors \(A \subset B\).

For this purpose we first give an explicit construction of \(\left( \mathcal{L} \left( \mathcal{F}^{1/3} \right) \otimes \mathbb{R} \right) \rtimes \gamma \mathbb{Z}_3\), by providing models for the II\(_1\) factors \(\mathcal{L} \left( \mathcal{F}^{1/3} \right)\) and \(R\).

Let \(\{X_1, X_2, X_3\}\) be a free semicircular family and \(u = \sum_{j=1}^{3} e^{\frac{2\pi i j}{3}} e_j\) a unitary whose spectral projections \(\{e_j\}_{j=1}^{3}\) have trace \(\frac{1}{3}\). Assume also that \(\{u\}''\) is free with respect to \(\{X_1, X_2, X_3\}''\). Then \(\mathcal{L} \left( \mathcal{F}^{1/3} \right)\) can be thought as the von Neumann algebra generated by \(\{X_1, X_2, X_3, u\}\) as in \([17]\) and \([13]\).

The model for \(R\) is outlined in the following lemma. It is analogous to the construction given in the case of \(\mathbb{Z}_2\) by Rădulescu \([14]\). We include it here for the sake of completeness.

**Lemma 4.1.** Given a von Neumann algebra \(M\), let \((U_k)_{k \in \mathbb{Z}}\) be a family of unitaries in \(M\) of order 9. Assume that each \(U_k\) has a decomposition of the form \(U_k = \sum_{j=1}^{9} e^{\frac{2\pi ij}{9}} U_{k,j}\), where each spectral projection \(U_{k,j}\) has trace \(\frac{1}{9}\). Let \(g = \sum_{j=1}^{3} e^{\frac{2\pi ij}{3}} g_j\) be a unitary in \(M\) of order 3 whose spectral projections \(\{g_j\}_{j=1}^{3}\) have trace \(\frac{1}{3}\). Suppose that the following relations hold between the \(U_k\)’s and \(g\):

(i) \(U_k g U_k^* = e^{\frac{2\pi i}{9}} g\) if \(k = 0, -1\), while \(U_k g U_k^* = g\) if \(k \in \mathbb{Z} \setminus \{0, -1\}\)

(ii) \(U_k U_{k+1} U_k^* = e^{\frac{2\pi i}{9}} U_{k+1}\), for \(k \in \mathbb{Z}\),

(iii) \(U_i U_j = U_j U_i\) if \(|i - j| \geq 2\).

The algebra generated by the \(U_k\)’s and \(g\) is endowed with a trace defined by \(\tau(m) = 0\), for each non-trivial monomial \(m\) in these unitaries. Set

\[ R_{-1} = \{gU_0^3, U_1, U_2, \ldots\}'' \subset \{g, U_0, U_1, U_2, \ldots\}'' = R_0. \]

This defines an inclusion of type II\(_1\) factors of index 9, such that \(R_{-1} \cap R_0 = \{g\}''\).

Let \(\theta = \text{Ad}_{R_{-1}}(U_0)\). Then \(\theta\) is an outer automorphism of \(R_{-1}\) of order 9 with outer invariant \((3, e^{\frac{2\pi i}{9}})\). Moreover, \(R_0\) is equal to the crossed product \(R_{-1} \rtimes_\theta \mathbb{Z}_9\).

Also, the Jones tower for the inclusion \(R_{-1} \subset R_0\) is given by

\[ R_{-1} \subset R_0 \subset R_1 \subset \cdots \subset R_{k-1} \subset R_k \subset R_{k+1} \subset \cdots, \]

where \(R_k = \{gU_0^3 \cdots U_{-k}^3, U_{-k}, U_{-k+1}, \ldots\}''\) for \(k \geq 1\).
The properties of the family \((U_k)_{k \in \mathbb{Z}}\) and of the unitary \(g\) imply immediately that \(U_0 x U_0^* \in R_{-1}\) for every \(x \in R_{-1}\). Therefore, \(\theta = \text{Ad}(U_0)\) defines an automorphism of \(R_{-1}\). Since \(g\) commutes with \(R_{-1}\), \(\theta^g = \text{Ad}_{R_{-1}}(U_0^3) = \text{Ad}_{R_{-1}}(g U_0^3)\) belongs to \(\text{Int}(R_{-1})\). Moreover \(\theta(g U_0^3) = e^{\frac{2 \pi i}{3}} g U_0^3\), so \(\theta\) has outer invariant \((3, e^{\frac{2 \pi i}{3}})\).

Obviously, any monomial in \(R_0\) can be written using only one occurrence of \(U_0\) to some power because of the relations between the generators of \(R_0\). Moreover, by definition, the trace on the algebra generated by the \(\{U_k\}_{k \in \mathbb{Z}}\) and \(g\) (and therefore on its subalgebra \(R_0\)) is compatible with the usual trace defined on the crossed product \(R_{-1} \rtimes_{\theta} \mathbb{Z}_9\), so that \(R_0 = (R_{-1} \cup \{U_0\})'' = R_{-1} \rtimes_{\theta} \mathbb{Z}_9\).

To show that \(R_{-1}' \cap R_0 \subset \{g\}''\), write any element \(x \in R_{-1}' \cap R_0\) as \(x = \sum_{k=0}^{8} a_k U_0^k\). It is easy to check that \(x\) belongs to \(R_{-1}'\) if and only if \(a_0 \in \mathbb{C}\), \(a_3\) and \(a_6\) are multiples of \(g^2 U_0^6\) and \(g U_0^3\), respectively, and all the other \(a_k\)'s are zero. The other inclusion follows immediately from the relations verified by the \(U_k\)'s and \(g\), thus \(R_{-1}' \cap R_0 = \{g\}''\).

Note that \(\text{Ad}_{R_0}(U_{-1})(U_0) = e^{\frac{2 \pi i}{3}} U_0\), while \(\text{Ad}_{R_0}(U_{-1})(x) = x\) for all \(x \in R_{-1}\). Hence, \(\text{Ad}_{R_0}(U_{-1})\) implements the dual action of \(\mathbb{Z}_9\) on the crossed product \(R_{-1} \rtimes_{\theta} \mathbb{Z}_9\), and the next step in the Jones tower for the inclusion \(R_{-1} \subset R_0\) is given by

\[
R_1 = \{U_{-1}, g, U_0, U_1, \ldots\}''.
\]

Similarly, the other steps in the Jones constructions are obtained by adding the unitaries \(U_{-2}, U_{-3}, \ldots\), so that the \(k\)-th step is given by

\[
R_k = \{U_{-k}, U_{-k+1}, \ldots, U_{-1}, g, U_0, U_1, \ldots\}''.
\]

Observe that to construct unitaries with the properties in the statement it is enough to consider the Jones tower for an inclusion of the form \(R \subset R \rtimes_{\beta} \mathbb{Z}_9\), where \(\beta\) is an automorphism of order 9 with outer invariant \((3, e^{\frac{2 \pi i}{3}})\). We choose the unitary \(g\) of order \(3\) between the elements of the relative commutant. The unitaries implementing the crossed product in the successive steps of the basic construction will satisfy the desired relations.

The next step is to give a concrete realization of the crossed product \(\mathcal{M} = \left(\mathcal{L}(\mathbf{F}_U) \otimes R_0\right) \rtimes_{\gamma} \mathbb{Z}_3\), with \(\gamma\) a \(\mathbb{Z}_3\)-action on \(\mathcal{L}(\mathbf{F}_U) \otimes R_0\).

Using the model \(\mathcal{L}(\mathbf{F}_U) = \{X_1, X_2, X_3, u\}''\), where the \(X_i\)'s are semicircular elements, \(u\) is a unitary of order 3 and \(\{X_i, u \mid i = 1 \ldots 3\}\) is a free family, we define the automorphism \(\alpha\) on \(\mathcal{L}(\mathbf{F}_U)\) by:

\[
\alpha(X_i) = X_{i+1}, \text{ for } i = 1, 2
\]

\[
\alpha(X_3) = u X_1 u^*, \text{ and}
\]

\[
\alpha(u) = e^{\frac{2 \pi i}{3}} u.
\]

Since \(\alpha^3 = \text{Ad } u\), \(\alpha\) is a \(\mathbb{Z}_3\)-kernel with obstruction \(e^{\frac{2 \pi i}{3}}\) to lifting.
For the automorphism $\beta$ on the hyperfinite II$_1$ factor we use the model of Lemma 4.1: $R \cong R_0 = \{g, U_0, U_1, \ldots \}$ and $\beta = \text{Ad}_{R_0}(U_{-1})$, with $\beta^3 = \text{Ad}_{R_0}(g)$ and $\beta(g) = e^{-\frac{2\pi i}{3}} g$.

Observe that $\alpha \otimes \beta \in \text{Aut} \left( \mathfrak{L} \left( \mathfrak{F}_{\frac{1}{3}} \right) \otimes R_0 \right)$ has outer period 3 and obstruction to lifting 1, so it can be perturbed by an inner automorphism to obtain a $\mathbb{Z}_3$-action on $\mathfrak{L} \left( \mathfrak{F}_{\frac{1}{3}} \right) \otimes R_0$. This action is defined by

$$\gamma = \left( \text{Ad}_{\mathfrak{L} \left( \mathfrak{F}_{\frac{1}{3}} \right) \otimes R_0} W \right) (\alpha \otimes \beta),$$

where $W$ is any cube root of $u^* \otimes g^*$ which is fixed by the automorphism $\alpha \otimes \beta$. For example, if $\delta = e^{\frac{2\pi i}{3}}$ and $\{e_i\}_{i=1}^3, \{g_j\}_{j=1}^3$ denote the spectral projections of $u$ and $g$, respectively, take

$$W = \delta E_1 + \delta^2 E_2 + E_3,$$  \hspace{1cm} (3)

where

$$E_l = \sum_{i,j=1, \ldots, 3 : i+j \equiv l \text{ mod } 3} e_i \otimes g_j, \text{ for } l = 1, \ldots, 3.  \hspace{1cm} (4)$$

Note that $\alpha$ acts on the spectral projections of $u$ as $\alpha(e_i) = e_{i-1}$ for $i = 2, 3$ and $\alpha(e_1) = e_3$, while $\beta$ acts on the spectral projections of $g$ as $\beta(g_j) = g_{j+1}$ for $j = 1, 2$ and $\beta(g_3) = g_1$. Hence, $\alpha \otimes \beta$ fixes $W$.

**Observation 4.2.** Note that $W$ belongs to the center of the fixed point algebra of $\alpha \otimes \beta$ since for any element $z$ in the fixed point algebra we have

$$z(u \otimes g) = (\alpha \otimes \beta)^3(z(u \otimes g)) = \text{Ad}(u \otimes g)(z(u \otimes g)) = (u \otimes g)z.$$

We can now prove our main theorem, using an argument similar to the one used by Rădulescu in [13]. We show that $\mathcal{M} = \mathfrak{L} \left( \mathfrak{F}_{\frac{1}{3}} \right) \otimes R_0 \rtimes_\gamma \mathbb{Z}_3$ is the enveloping algebra of an inclusion $A \subset B$ of interpolated free group factors. We divide the proof into two parts, first proving that $A$ is isomorphic to the interpolated free group factor $\mathfrak{L} \left( \mathfrak{F}_{\frac{1}{3}} \right)$.

**Proposition 4.3.** Let $v$ be the unitary implementing the crossed product $\mathcal{M} = \mathfrak{L} \left( \mathfrak{F}_{\frac{1}{3}} \right) \otimes R_0 \rtimes_\gamma \mathbb{Z}_3$. Consider the von Neumann subalgebra

$$A = \{X_i \otimes 1, u \otimes 1, 1 \otimes g, v | i = 1, \ldots, 3\}'' \subset \mathcal{M},$$

endowed with a trace with respect to which $\{X_i \otimes 1, u \otimes 1 | i = 1, \ldots, 3\}$ is a free family, $\{X_i \otimes 1\}_{i=1}^3$ are semicircular elements, and $u \otimes 1, 1 \otimes g, v$ are unitaries of order 3 with spectral projections of trace $\frac{1}{3}$.

Moreover, assume that the following relations are satisfied by the elements of $A$

1. $[1 \otimes g, X_i \otimes 1] = 0$ for $i = 1, \ldots, 3$, and $[1 \otimes g, u \otimes 1] = 0$.
2. $v(u \otimes 1) = e^{\frac{2\pi i}{3}} (u \otimes 1)v$
3. $v(1 \otimes g) = e^{-\frac{2\pi i}{3}} (1 \otimes g)v$
iv) $\text{Ad } v(X_i \otimes 1) = \text{Ad } W \circ \alpha(X_i \otimes 1)$, where $W = e^{2\pi i} E_1 + e^{4\pi i} E_2 + E_3$ as in (3).

For any monomial $m$ in the variables $\{X_i \otimes 1, u \otimes 1, 1 \otimes g | i = 1, \ldots, 3\}$, suppose that the trace $\tau$ on the algebra $A$ verifies the following properties:

(v) $\tau(mv^k) = 0$ for $k = 1, 2$,

(vi) the trace of $m$ in $A$ coincides with its trace as element of the von Neumann algebra $\{X_i, u | i = 1, \ldots, 3\}'' \otimes \{g\}''$.

Under these conditions $A$ is isomorphic to $\mathcal{L}(\mathbb{F}_{35}^{27})$.

Proof First we realize the algebra $A$ in terms of random matrices [17] and then use Voiculescu’s free probability theory to show that $A$ is an interpolated free group factor. The random matrix model we give is a subalgebra of the algebra of $9 \times 9$ matrices with entries in a von Neumann algebra.

Let $D$ be a von Neumann algebra with a finite trace $\tilde{\tau}$ which contains a family of free elements $\{a_i\}_{i=1}^{18}$, with the property that the elements $\{a_i\}_{i=1}^9$ are semicircular, while the others ones are circular. Denote by $(e_{ij})_{i,j=1,\ldots,9}$ the canonical system of matrix units in $M_9(\mathbb{C})$. Set $\epsilon = e^{2\pi i/3}$. Using the same notation as before for the spectral projections of $u$ and $g$, we set

$$e_i \otimes 1 = \sum_{\substack{j=1,\ldots,9 \mod 3 \atop j \equiv i \mod 3}} e_{jj} \in M_9(\mathbb{C}) \subset D \otimes M_9(\mathbb{C}), \text{ for } i = 1, \ldots, 3,$$

and

$$1 \otimes g_1 = e_{11} + e_{55} + e_{99},$$

$$1 \otimes g_2 = e_{33} + e_{44} + e_{88},$$

$$1 \otimes g_3 = e_{22} + e_{66} + e_{77}.$$

Therefore, $u \otimes 1 = \epsilon (e_1 \otimes 1) + \epsilon^2 (e_2 \otimes 1) + e_3$ and $1 \otimes g = \epsilon (1 \otimes g_1) + \epsilon^2 (1 \otimes g_2) + 1 \otimes g_3$ can be written in matrix notation as

$$u \otimes 1 = \begin{pmatrix}
\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \epsilon^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
Moreover, set
\[ v = (e_{12} + e_{23} + e_{31}) + (e_{45} + e_{56} + e_{64}) + (e_{78} + e_{89} + e_{97}), \]
i.e.,
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}.
\]
Note that the unitaries $u \otimes 1, 1 \otimes g$, and $v$ generate a copy of $M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \subseteq M_9(\mathbb{C})$. In addition, with this choice of $u \otimes 1, 1 \otimes g$ and $v$ we obtain that
\[
W = \delta^2 Id \oplus Id \oplus \delta Id,
\]
where $\delta = e^{\frac{2\pi i}{9}}$ and $Id$ denotes the identity of $M_3(\mathbb{C})$. In matrix notation
\[
\begin{pmatrix}
\delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta 
\end{pmatrix}
\]
Furthermore, $u \otimes 1, 1 \otimes g$ and $v$ satisfy the required conditions
\[
(u \otimes 1)(1 \otimes g) = (1 \otimes g)(u \otimes 1),
\]
\[
v(u \otimes 1) = \epsilon (u \otimes 1)v,
\]
\[
v(1 \otimes g) = \epsilon (1 \otimes g)v.
\]
Denote by $\text{Re}(a) = \frac{1}{2}(a + a^*)$, for $a \in D \otimes M_9(\mathbb{C})$. We set:

$$X_1 \otimes 1 = a_1 \otimes e_{11} + a_2 \otimes e_{22} + a_3 \otimes e_{33} + a_4 \otimes e_{44} + a_5 \otimes e_{55} + a_6 \otimes e_{66} + a_7 \otimes e_{77} + a_8 \otimes e_{88} + a_9 \otimes e_{99} + 2\text{Re}(a_{10} \otimes e_{15}) + 2\text{Re}(a_{11} \otimes e_{19}) + 2\text{Re}(a_{12} \otimes e_{26}) + 2\text{Re}(a_{13} \otimes e_{27}) + 2\text{Re}(a_{14} \otimes e_{34}) + 2\text{Re}(a_{15} \otimes e_{38}) + 2\text{Re}(a_{16} \otimes e_{48}) + 2\text{Re}(a_{17} \otimes e_{59}) + 2\text{Re}(a_{18} \otimes e_{67})$$

To find $X_2$ and $X_3$, use the relations:

$$X_2 \otimes 1 = \text{Ad}(W^*v)(X_1 \otimes 1) \quad \text{and} \quad X_3 \otimes 1 = \text{Ad}(W^*v)(X_2 \otimes 1)$$

Thus, using matrix notation we have:

$$X_1 \otimes 1 = \begin{pmatrix}
    a_1 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & a_{11} \\
    0 & a_2 & 0 & 0 & 0 & a_{12} & a_{13} & 0 & 0 \\
    0 & 0 & a_3 & a_{14} & 0 & 0 & 0 & a_{15} & 0 \\
    0 & 0 & a_{14} & a_4 & 0 & 0 & 0 & a_{16} & 0 \\
    a_{10} & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & a_{17} \\
    0 & a_{12} & 0 & 0 & 0 & a_6 & a_{18} & 0 & 0 \\
    0 & a_{13} & 0 & 0 & 0 & a_{18} & a_7 & 0 & 0 \\
    0 & 0 & a_{15} & a_{16} & 0 & 0 & 0 & a_{18} & 0 \\
    a_{11} & 0 & 0 & 0 & a_{17} & 0 & 0 & 0 & a_9 \\
\end{pmatrix}$$

$$X_2 \otimes 1 = \begin{pmatrix}
    a_2 & 0 & 0 & 0 & \delta^2 a_{12} & 0 & 0 & 0 & \delta a_{13} \\
    0 & a_3 & 0 & 0 & 0 & \delta^2 a_{14} & \delta a_{15} & 0 & 0 \\
    0 & 0 & a_1 & \delta^2 a_{10} & 0 & 0 & 0 & \delta a_{11} & 0 \\
    0 & 0 & \delta^2 a_{10} & a_5 & 0 & 0 & 0 & \delta a_{16} & 0 \\
    \delta^2 a_{12} & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & \delta a_{18} \\
    0 & \delta^2 a_{14} & 0 & 0 & 0 & a_4 & \delta a_{16} & 0 & 0 \\
    0 & \delta a_{15} & 0 & 0 & 0 & \delta a_{16} & a_8 & 0 & 0 \\
    0 & 0 & \delta a_{11} & \delta a_{17} & 0 & 0 & 0 & \delta a_{18} & 0 \\
    \delta a_{13} & 0 & 0 & 0 & \delta a_{18} & 0 & 0 & 0 & a_7 \\
\end{pmatrix}$$

$$X_3 \otimes 1 = \begin{pmatrix}
    a_3 & 0 & 0 & 0 & \delta^4 a_{14} & 0 & 0 & 0 & \delta^2 a_{15} \\
    0 & a_1 & 0 & 0 & 0 & \delta^4 a_{12} & \delta^2 a_{11} & 0 & 0 \\
    0 & 0 & a_2 & \delta^4 a_{12} & 0 & 0 & 0 & \delta^2 a_{13} & 0 \\
    0 & 0 & \delta^4 a_{12} & a_6 & 0 & 0 & 0 & \delta^2 a_{18} & 0 \\
    \delta^4 a_{14} & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & \delta^2 a_{16} \\
    0 & \delta^4 a_{12} & 0 & 0 & 0 & a_5 & \delta^2 a_{17} & 0 & 0 \\
    0 & \delta^2 a_{11} & 0 & 0 & 0 & \delta^2 a_{17} & a_9 & 0 & 0 \\
    0 & 0 & \delta^2 a_{13} & \delta^2 a_{18} & 0 & 0 & 0 & a_7 & 0 \\
    \delta^2 a_{15} & 0 & 0 & 0 & \delta^2 a_{16} & 0 & 0 & 0 & a_8 \\
\end{pmatrix}$$

Obviously $X_1 \otimes 1, X_2 \otimes 1, X_3 \otimes 1$ commute with $1 \otimes g$. Moreover, the assumption that the family $\{a_i\}_{i=1,\ldots,18}$ is free implies that $\{X_i \otimes 1, u \otimes 1 | i = 1, \ldots, 3\}$ is a free family with respect to the unique normalized trace on $D \otimes M_9(\mathbb{C})$. In addition, $\{X_i \otimes 1, u \otimes 1 | i = 1, \ldots, 3\}''$ and $\{g\}''$ are independent with respect to this trace.
To show that the normalized trace of $D \otimes M_0(\mathbb{C})$ has the properties (v) and (vi) of the statement, let $m$ be any monomial in the variables $\{X_i \otimes 1, u \otimes 1, 1 \otimes g | i = 1, \ldots, 3\}$, and consider the product $v^k m$ with $k = 1, 2$. Since $m$ commutes with $1 \otimes g$, it must be of the form

$$m = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & * & 0 & 0 & 0 & 0 & 0 & *
\end{pmatrix}$$

If we multiply $m$ by $v$ or $v^2$, we obtain a matrix with zero on the diagonal and a few non-zero entries outside the diagonal. This implies that $v^k m$ has zero trace for $k = 1, 2$. Thus, we have built a matrix model for $A$ which satisfies the conditions of the statement.

One can easily check that $A$ is a factor. In order to prove that $A$ is an interpolated free group factor we reduce $A$ by one of the spectral projections of $g$, and show that the new factor we obtain is an interpolated free group factor. Reduce $A$ by $g_3 = 1 \otimes e_{22} + 1 \otimes e_{66} + 1 \otimes e_{77}$, which has trace $\frac{1}{3}$.

Then $g_3 A g_3$ is generated by:

(i) $a_2 \otimes e_{22} + 2\text{Re}(a_{12} \otimes e_{26}) + 2\text{Re}(a_{13} \otimes e_{27}) + a_6 \otimes e_{66} + a_7 \otimes e_{77}$

(ii) $a_3 \otimes e_{22} + 2\text{Re}(\delta a_{14} \otimes e_{26}) + 2\text{Re}(\delta a_{15} \otimes e_{27}) + a_4 \otimes e_{66} + a_8 \otimes e_{77}$

(iii) $a_1 \otimes e_{22} + 2\text{Re}(\delta a_{10} \otimes e_{26}) + 2\text{Re}(\delta a_{11} \otimes e_{27}) + a_5 \otimes e_{66} + a_9 \otimes e_{77}$

(iv) $\epsilon^2 \otimes e_{22} + 1 \otimes e_{66} + \epsilon \otimes e_{77}$,

Thus, by the Voiculescu’s random matrix model [18], $g_3 A g_3$ is isomorphic to the free group factor $\mathcal{L}(F_3 * \mathbb{Z}_3) \cong \mathcal{L}(F_{14}^\theta)$. This implies that $A$ is also an interpolated free group factor, and using the well-known formula for reduced factors ([7], [Dy2] or [13]) we get that $A = \mathcal{L}(F_{14}^\theta)$.

\begin{theorem}
Set
$$A = \{X_i \otimes 1, u \otimes 1, 1 \otimes g, v | i = 1 \ldots 3\}'' \subset B = (A \cup \{1 \otimes U_0\}'')'.''
$$
in $\mathcal{M} = \left(\mathcal{L}\left(F_{14}^\theta\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$. Then $A$ is isomorphic to the interpolated free group factor $\mathcal{L}\left(F_{14}^\theta\right)$, and $B$ is the crossed product of $A$ by a $\mathbb{Z}_3$-action $\theta$ on $A$, with outer invariant is $(3, e^{2\pi i 3})$. Furthermore, $\mathcal{M}$ is the enveloping algebra in the basic construction for the inclusion $A \subset B$.
\end{theorem}
Proof First we want to describe how $\text{Ad}(1 \otimes U_0)$ acts on the subalgebra $A$ of $\mathcal{M}$. Obviously

$$\text{Ad}(1 \otimes U_0)(X_i \otimes 1) = X_i \otimes 1 \quad \text{for } i = 1, \ldots, 3,$$

$$\text{Ad}(1 \otimes U_0)(u \otimes 1) = u \otimes 1$$

and

$$\text{Ad}(1 \otimes U_0)(1 \otimes g) = e^{-\frac{2\pi i}{3}}(1 \otimes g).$$

In addition, if $\{E_i\}_{i=1}^3$ are the projections defined in (4), then

$$\text{Ad}(1 \otimes U_0)(E_i) = E_{i+1} \text{ for } i = 1, \ldots, 3,$$

where $i+1$ is taken mod 3. Using (3) and (5), together with the relation $\gamma(1 \otimes U_0) = \delta \text{Ad}(1 \otimes U_0)$, for $\delta = e^{\frac{2\pi i}{3}}$, we obtain

$$\text{Ad}(1 \otimes U_0)(v) = (1 \otimes U_0)(v(1 \otimes U_0^*)v^*)v = \bar{\delta} \text{Ad}(1 \otimes U_0)(W)W^*v$$

$$= \bar{\delta}(E_1 + \delta E_2 + \delta^2 E_3)W^*v = \bar{\delta}^2(E_1 + E_2 + e^{-\frac{2\pi i}{3}}E_3)v.$$

It follows that $\text{Ad}(1 \otimes U_0)$ leaves $A$ invariant. Furthermore, $\text{Ad}_A(1 \otimes U_0)^3$ acts identically on $\{X_i \otimes 1, u \otimes 1, 1 \otimes g \mid i = 1, \ldots, 3\}$, while $\text{Ad}_A(1 \otimes U_0)^3(v) = e^{-\frac{2\pi i}{3}}v$.

Set $\theta = \text{Ad}_A(1 \otimes U_0)$. Then, using the fact that $\text{Ad}_A(1 \otimes g^*)$ acts identically on $\{X_i \otimes 1, u \otimes 1, 1 \otimes g \mid i = 1, \ldots, 3\}$, and $\text{Ad} v(1 \otimes g) = e^{-\frac{2\pi i}{3}}(1 \otimes g)$, we conclude that

$$\theta^3 = \text{Ad}_A(1 \otimes g^*) \quad \text{and} \quad \theta(1 \otimes g^*) = e^{\frac{2\pi i}{3}}(1 \otimes g^*).$$

Thus $\theta$ is a $Z_3$-kernel on $A$ with obstruction $e^{\frac{2\pi i}{3}}$ to lifting.

To complete the proof that $B = A \rtimes_{\theta} Z_3$ we need to check that any monomial $m$ in the variables $\{X_i \otimes 1, u \otimes 1, 1 \otimes g, 1 \otimes U_0, v \mid i = 1, \ldots, 3\}$ contains only one occurrence of $1 \otimes U_0$ to some power. We also need to verify that any monomial containing $1 \otimes U_0$ has zero trace.

Since the crossed product $\left( \mathcal{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0 \right) \rtimes_{\gamma} Z_3$ is implemented by the unitary $v$, any monomial in the variables $\{X_i \otimes 1, u \otimes 1, 1 \otimes g, 1 \otimes U_0, v \mid i = 1, \ldots, 3\}$ has the form $v^k m$, where $m$ is an element in $\mathcal{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0$ and $k = 0, 1, 2$. Because $1 \otimes U_0$ commutes with the elements of $\mathcal{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes 1$ and $\text{Ad}(1 \otimes U_0)(1 \otimes g) = e^{-\frac{2\pi i}{3}}(1 \otimes g)$, it follows that $1 \otimes U_0$ can appear at most once in $m$ with some power. Thus, any monomial in $B$ can be written using only one occurrence of $1 \otimes U_0$.

Furthermore, by the definition of the trace on the crossed product, any monomial of the form $v^k m$, for $k = 1, 2$, has zero trace in $B$. Therefore, it is enough to compute the trace of any monomial $m$ in $\mathcal{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0$ containing $1 \otimes U_0$. Because of the definition of the trace on $R_0$ (Lemma 4.11) and hence on $\mathcal{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0$, we can conclude immediately that the trace of $m$ is zero. Therefore $B = A \rtimes_{\theta} Z_3$.

In addition, using an argument similar to the one used to build the model for $R$ (Lemma 4.11), one can easily verify that the unitaries $1 \otimes U_1, 1 \otimes U_2, \ldots$ implement the consecutive terms in the iterated basic construction of $A \subset A \rtimes_{\theta} Z_3$. This implies that $\left( \mathcal{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0 \right) \rtimes_{\gamma} Z_3$ is the enveloping algebra of the above inclusion of factors. 

$\blacksquare$
5. The Connes invariant of the crossed product \( \mathcal{L} \left( F_{11} \right) \otimes R_0 \rtimes_\gamma \mathbb{Z}_4 \)

The arguments in this section are analogous to the ones used by Jones in [10] to compute the Connes invariant for his example of a factor which is anti-isomorphic to itself but has no involutory antiautomorphism. The definition of \( K, K^\perp \) and \( L \) given below are due to Connes (see [4]).

Let \( N \) be a II\(_1\) factor without non-trivial hypercentral sequences and \( G \) a finite subgroup of \( \text{Aut}(N) \) such that \( G \cap \text{Int}(N) = \{Id\} \). Set \( K = G \cap \text{Ct}(N) \) and \( K^\perp = \{f : G \to \mathbb{T} \mid f \text{ is a homomorphism and } f|_K \equiv 1\} \).

Let \( \xi : \text{Aut}(N) \to \text{Out}(N) \) be the usual quotient map and \( \text{Fint} \) be the subgroup \( \{\text{Ad}_N u \mid u \in N \text{ is fixed by } G\} \subseteq \text{Aut}(N) \). Denote by \( \text{Fint}^\text{cl} \) its closure in \( \text{Aut}(N) \) with respect to the pointwise weak topology, and by \( G \ltimes \text{Ct}(N) \) the subgroup of \( \text{Aut}(N) \) generated by \( G \cup \text{Ct}(N) \). Set \( L = \xi((G \ltimes \text{Ct}(N)) \cap \text{Fint}^\text{cl}) \subseteq \text{Out}(N) \).

Connes showed in [4] that there exists an exact sequence

\[
0 \to K^\perp \xrightarrow{\partial} \chi(W^*(N,G)) \xrightarrow{\Pi} L \to 0
\]

where \( M = W^*(N,G) \) denotes the crossed product implemented by the action of \( G \) on \( N \). We briefly describe the maps \( \partial \) and \( \Pi \) in the exact sequence above. Given an element \( x \in M \), write it as \( \sum_{g \in G} a_g u_g \), with \( a_g \in N \) and \( \text{Ad}_M u_g|_N = g \). For each \( \eta : G \to \mathbb{T} \) in \( K^\perp \), define the map \( \Delta(\eta) : M \to M \) by

\[
\Delta(\eta) \left( \sum_{g \in G} a_g u_g \right) = \sum_{g \in G} \eta(g) a_g u_g.
\]

Then \( \Delta(\eta) \) belongs to \( \text{Ct}(M) \cap \text{Int}(M) \), and \( \partial = \xi \circ \Delta \) is the desired map.

To see how \( \Pi \) acts on \( \chi(M) \), for any element \( \sigma \in \chi(M) \) choose an automorphism \( \alpha \in \text{Ct}(M) \cap \text{Int}(M) \) such that \( \xi(\alpha) = \sigma \).

The hypothesis \( G \cap \text{Int}(N) = \{Id\} \), implies that there exists a sequence of unitaries \( (u_n)_{n \geq 0} \) in \( N \), which are fixed by \( G \), and a unitary \( z \) in \( M \) such that \( \alpha = \text{Ad}_M z \lim_{n \to \infty} \text{Ad}_N u_n \) (Corollary 6 and Lemma 2 in [9], or Lemma 15.42 in [8]). Set \( \psi_\sigma \text{ Ad}(z^*)\alpha|_N \in \text{Aut}(N) \).

One can show that \( \psi_\sigma \in \text{Fint}^\text{cl} \cap (G \ltimes \text{Ct}(N)) \), and that the composition map \( \xi \circ \psi_\sigma \) does not depend on the choice of \( \alpha \), but only on the class \( \sigma = \xi(\alpha) \). Therefore, the map \( \Pi : \chi(M) \to L \) given by \( \Pi(\sigma) = \xi(\psi_\sigma) \) is well defined.

To show that \( \Pi \) is surjective, let \( \mu \) be any element in \( L \) and denote by \( \alpha_\mu \in \text{Fint}^\text{cl} \cap (G \ltimes \text{Ct}(N)) \) a representative of \( \mu \), i.e. \( \xi(\alpha_\mu) = \mu \). \( \alpha_\mu \) commutes with \( G \) since it is the limit of automorphisms with this property. Hence, the map \( \beta_\mu \) defined by

\[
\beta_\mu \left( \sum_{g \in G} a_g u_g \right) = \sum_{g \in G} \alpha_\mu(a_g) u_g.
\]

is an automorphism of \( M \). In addition, we have that \( \beta_\mu \in \text{Ct}(M) \cap \text{Int}(M) \) and \( \Pi(\xi(\beta_\mu)) = \mu \).
Remark 5.1 (Jones). Note that if \( u_n \) is a sequence of unitaries left invariant by \( G \), with the property \( \alpha_n = \lim_{n \to \infty} \text{Ad} \, u_n \) in \( \text{Aut}(N) \), and \( \beta_n \) is the map defined in (6), then

\[
\alpha_n = \lim_{n \to \infty} u_n^* \beta_n(u_n) = \lim_{n \to \infty} u_n^* \alpha_n(u_n).
\]

Our next goal is to show that if \( N = \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \) and \( G \) is the subgroup of \( \text{Aut} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) \) generated by \( \gamma = \text{Ad} W(\alpha \otimes \beta) \), then \( \chi(\mathcal{M}) \cong \mathbb{Z}_9 \), as it was observed by Connes in [3].

For all the rest of this paper we will identify \( G = \langle \gamma \rangle \) with \( \mathbb{Z}_3 \). Note that by Proposition 3.5, \( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \) has no non-trivial hypercentral sequences, so in order to use the exactness of the Connes sequence described above, we only need to show that \( \mathbb{Z}_3 \cap \text{Int} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) = \{ Id \} \). Obviously it is enough to check that \( \gamma \not\in \text{Int} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) \).

By [5, Corollary 3.3] if \( \alpha \otimes \beta \in \text{Int} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) \) then \( \alpha \in \text{Int} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \right) \) and \( \beta \in \text{Int}(R) \). But \( \mathcal{L} \left( \frac{F_{11}}{9} \right) \) is full (Remark 3.4) and \( \alpha \not\in \text{Int} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \right) \), thus we can conclude that \( \mathbb{Z}_3 \cap \text{Int} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) = \{ Id \} \).

Hence, for the factor \( \mathcal{M} = \mathcal{L} \left( \frac{F_{11}}{9} \otimes R_0 \right) \), the sequence

\[
0 \to K^\perp \xrightarrow{\beta} \chi(\mathcal{M}) \xrightarrow{\Pi} L \to 0
\]

is exact (4). In order to compute \( \chi(\mathcal{M}) \) we first show that

\[
K^\perp = \mathbb{Z}_3 \quad \text{and} \quad L = \mathbb{Z}_3.
\]

Lemma 5.2. The group \( K = \mathbb{Z}_3 \cap \text{Ct} \left( \mathcal{L} \left( \frac{F_{11}}{9} \otimes R_0 \right) \right) \) is trivial.

Proof Obviously it is enough to show that the automorphism \( \gamma = \text{Ad} W(\alpha \otimes \beta) \) is not in \( \text{Ct} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) \). We have already shown in the proof of Lemma 3.5 that any central sequence in \( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \) has the form \((1 \otimes x_n)_{n \in \mathbb{N}}\) for a central sequence \((x_n)_{n \in \mathbb{N}}\) in \( R_0 \). It follows that \( \gamma \in \text{Ct} \left( \mathcal{L} \left( \frac{F_{11}}{9} \right) \otimes R_0 \right) \) if and only if \( \beta \in \text{Ct}(R_0) \). Since for \( \epsilon = e^{\frac{2\pi i}{3}} \), \( \beta \) is outer conjugate to the automorphism \( s_3^\epsilon \) described by Connes in [6], and \( s_3^\epsilon \) does not belong to \( \text{Ct}(R_0) \) [6, Proposition 1.6], we conclude that \( \beta \not\in \text{Ct}(R_0) \).

Lemma 5.3. The group \( L \) is isomorphic to \( \mathbb{Z}_3 \), and a generator of \( L \) is given by

\[
\mu = \xi \left( \text{Id} \otimes \left( \text{Ad} \, U_0^* \beta \right) \right),
\]

where \( U_0 \) is the unitary in \( R_0 \) defined in Lemma 4.1, such that \( \beta(U_0) = e^{\frac{2\pi i}{3}} U_0 \).
Proof We need to show that the automorphism $Id \otimes (\text{Ad } U_0^* \beta)$ belongs to $(\mathbb{Z}_3 \vee \text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right) ) \cap \overline{\text{Fin}}$. To prove that it is in $\mathbb{Z}_3 \vee \text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right), \text{multiply } Id \otimes (\text{Ad } U_0^* \beta) \text{ by } \gamma^{-1} = (\text{Ad } W(\alpha \otimes \beta))^{-1} \text{ to obtain the automorphism }$

$$\text{Ad } (W^*(1 \otimes U_0^*)) (\alpha^{-1} \otimes Id),$$

which is in $\text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right)$, since the central sequences in $\mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0$ have the form $1 \otimes x_n$, with $x_n$ central in $R_0$. Thus $Id \otimes (\text{Ad } U_0^* \beta) \in \mathbb{Z}_3 \vee \text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right).

Next we want to show that $Id \otimes (\text{Ad } U_0^* \beta) \in \overline{\text{Fin}}$. Therefore, we need to exhibit a sequence $(\tilde{u}_n)$ of unitaries in $\mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0$, that are invariant with respect to $\gamma$, and satisfy $Id \otimes (\text{Ad } U_0^* \beta) = \lim_{n \to \infty} \text{Ad } \tilde{u}_n$.

Observe that the sequence $(x_n)_{n \in \mathbb{N}}$ of unitaries in $R_0$ given by

$$x_n = \begin{cases} U_0 U_1^* U_2^* U_3^* \ldots U_n^* , & \text{if } n \text{ is odd}, \\ U_0 U_1^* U_2^* U_3^* \ldots U_n , & \text{if } n \text{ is even} \end{cases}$$

has the properties

$$\beta = \lim_{n \to \infty} \text{Ad } x_n \quad \text{and} \quad \beta(x_n) = e^{\frac{2\pi}{3}} x_n.$$

Define $u_n = U_0^* x_n$. Obviously

$$Id \otimes (\text{Ad } U_0^* \beta) = \lim_{n \to \infty} \text{Ad } (1 \otimes u_n) \quad \text{and} \quad \beta(u_n) = u_n$$

so that

$$(\alpha \otimes \beta)(1 \otimes u_n) = 1 \otimes u_n. \quad (7)$$

In addition, from Observation 4.2 it follows that

$$\gamma(1 \otimes u_n) = \text{Ad } W(\alpha \otimes \beta)(1 \otimes u_n) = 1 \otimes u_n.$$ 

Thus,

$$Id \otimes (\text{Ad } U_0^* \beta) \in (\mathbb{Z}_3 \vee \text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right) ) \cap \overline{\text{Fin}}.$$

Lastly we want to prove that the order 3 element $\mu = \xi (Id \otimes (\text{Ad } U_0^* \beta))$ generates $L$. Thus we need to show that any element $\varphi$ in $(\mathbb{Z}_3 \vee \text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right) ) \cap \overline{\text{Fin}}, \text{is of the form } \mu^n \text{Ad } w, \text{for some unitary } w \text{ in } \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \text{ and } n = 0, \ldots, 2. \text{ Since } \varphi \in \mathbb{Z}_3 \vee \text{Ct} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \right), \text{there exists } k \in \{0, 1, 2\} \text{ such that } \varphi \gamma^k \text{ is centrally trivial. By Proposition 3.6, it follows that there exists a unitary } z \in \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0 \text{ and an automorphism } \nu \in \text{Aut} \left( \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \right) \text{ such that } \varphi \gamma^n \text{ = Ad } z(\nu \otimes id). \text{ Therefore, }$

$$\varphi = \text{Ad } x(\nu \alpha^{-n} \otimes \beta^{-n}),$$

with $x = z(\nu \otimes Id)(W^n)^* \in \mathcal{L} \left( \mathbf{F}_{\frac{3}{2}} \right) \otimes R_0$. 

ON A SUBFACTOR CONSTRUCTION 17
Since \( \varphi \in \overline{\text{Fin}} \subseteq \text{Int} \left( \mathfrak{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0 \right) \) and \( \mathfrak{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \) is full (Remark 3.4), by [5, Corollary 3.3] there exists a unitary \( w \) in \( \mathfrak{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \) such that \( \nu \alpha^{-n} = \text{Ad} w \). This implies that \( \varphi = \text{Ad} x' \left( \text{Id} \otimes \beta^{-n} \right) \), where \( x' = x (w \otimes 1) \). Thus \( \varphi \) differs from a power of \( \text{Id} \otimes (\text{Ad} U_0^* \beta) \) only by an inner automorphism. Hence \( \text{Id} \otimes (\text{Ad} U_0^* \beta) \) generates \( L \).

Note that if \( u_n \) is the sequence of unitaries defined in the previous lemma
\[
(\text{Id} \otimes (\text{Ad} U_0^* \beta))(1 \otimes u_n) = 1 \otimes U_0^* u_n U_0.
\]
But
\[
\lim_{n \to \infty} \text{Ad} u_n = \text{Ad} U_0^* \beta \quad \text{and} \quad \beta(U_0) = e^{\frac{2\pi i}{9}} U_0,
\]
so
\[
\lim_{n \to \infty} u_n^* U_0^* u_n = \text{Ad} U_0 (\beta^{-1}(U_0^*)) = e^{\frac{2\pi i}{9}} U_0^*
\]
and
\[
(1 \otimes u_n^*)(\text{Id} \otimes (\text{Ad} U_0^* \beta))(1 \otimes u_n) = e^{\frac{2\pi i}{9}}.
\]
Thus, in view of Remark 5.1 we obtain the following:

Remark 5.4. Let \( \mu = \xi \left( \text{Id} \otimes (\text{Ad} U_0^* \beta) \right) \) be the generator of \( L \) as in the previous lemma. If \( \beta \mu \in \text{Ct}(M) \cap \text{Int}(M) \) is the automorphism described in equation (6), and \( \sigma = \varepsilon(\beta \mu) \), then
\[
\kappa(\sigma) = e^{\frac{2\pi i}{9}}.
\]

Theorem 5.5. Let \( M \) denote the crossed product \( \left( \mathfrak{L} \left( \mathbf{F}_{\frac{11}{3}} \right) \otimes R_0 \right) \rtimes_{\gamma} \mathbb{Z}_3 \). Then \( \chi(M) = \mathbb{Z}_9 \).

Proof. By Connes [4] the sequence
\[
0 \to \mathbb{Z}_3 \to \chi(M) \to \mathbb{Z}_9 \to 0
\]
is exact. Moreover, according to (6), \( \mu = \xi \left( \text{Id} \otimes (\text{Ad} U_0^* \beta) \right) \) lifts to the element \( \sigma = \xi(\beta \mu) \) of \( \chi(M) \), where
\[
\beta \mu \left( \sum_{k=0}^{2} a_k v^k \right) = \sum_{k=0}^{2} (\text{Id} \otimes (\text{Ad} U_0^* \beta))(a_k)v^k.
\]
Since the only possibilities for \( \chi(M) \) are \( \mathbb{Z}_9 \) and \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), it is enough to show that \( \sigma^3 \neq 1 \), i.e. \( \beta \mu^3 \notin \text{Int}(M) \).

From the relations
\[
(\alpha \otimes \beta)(1 \otimes (U_0^*)^3 g) = e^{\frac{2\pi i}{9}} (1 \otimes (U_0^*)^3 g)
\]
and
\[
\text{Ad} W (1 \otimes (U_0^*)^3 g) = 1 \otimes (U_0^*)^3 g
\]
we obtain that
\[
\gamma(1 \otimes (U_0^*)^3 g) = e^{\frac{2\pi i}{9}} (1 \otimes (U_0^*)^3 g).
\]
Since $\text{Ad} v = \gamma$ on $\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0$, the last equality can be rewritten as $\text{Ad} v(1 \otimes (U_0^*)^3 g) = e^{2\pi i/3} (1 \otimes (U_0^*)^3 g)$ or
$$\text{Ad} (1 \otimes (U_0^*)^3 g)(v) = e^{2\pi i/3} v.$$ 
Thus, using the definition of $\beta_\mu$, the relation $\beta^3 = \text{Ad} g$, and (8) we obtain
$$\beta_\mu^3 \left( \sum_{k=0}^{2} a_k v^k \right) = \sum_{k=0}^{2} \left( \text{Id} \otimes (\text{Ad} (U_0^*)^3 \beta^3) \right)(a_k)v^k = \sum_{k=0}^{2} \text{Ad} (1 \otimes (U_0^*)^3 g)(a_k)v^k$$
$$= \text{Ad} (1 \otimes (U_0^*)^3 g) \left( \sum_{k=0}^{2} e^{\frac{2\pi i k}{3}} a_k v^k \right),$$
which implies that up to an inner automorphism $\beta_\mu^3$ is a dual action, so it is outer and $\chi(\mathcal{M}) \cong \mathbb{Z}_9$.

6. $(\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0) \rtimes_\gamma \mathbb{Z}_3$ is not anti-isomorphic to itself.

In this section we are going to show that $\mathcal{M} = (\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0) \rtimes_\gamma \mathbb{Z}_3$ is not anti-isomorphic to itself by using the dual action $\hat{\mathbb{Z}}_3 \to \text{Aut}(\mathcal{M})$, which gives rise to the only subgroup of order 3 in $\chi(\mathcal{M})$. This argument has been described by Connes in [3] and [4].

First of all note that the action $\gamma$ can be decomposed as
$$\gamma = \text{Ad} W \gamma_1 \gamma_2,$$
where $\gamma_1 \in \text{Ct} \left(\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0\right)$ and $\gamma_2 \in \text{Int} \left(\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0\right)$ and $W$ is a unitary in $\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0$.

In fact, $\gamma = \text{Ad} W(\alpha \otimes \text{Id})(\text{Id} \otimes \beta)$ and $\gamma_1 = \alpha \otimes \text{Id}$ is centrally trivial, since any central sequence in $\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0$ has the form $(1 \otimes x_n)$, for a central sequence $(x_n)$ in $R_0$. Furthermore in the proof of Lemma 5.3 we showed that $\beta = \lim_{n \to \infty} \text{Ad} x_n$ so that $\gamma_2 = \text{Id} \otimes \beta = \lim_{n \to \infty} \text{Ad} (1 \otimes x_n)$ belongs to $\text{Int} \left(\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0\right)$.

Note also that this decomposition of $\gamma$ into an approximately inner automorphism and a centrally trivial automorphism is unique up to inner automorphisms, since $\chi \left(\mathcal{L}\left(\mathbf{F}_{1/3}\right) \otimes R_0\right) = 1$.

Let $M$ be an arbitrary von Neumann algebra. Define the conjugate $M^c$ of $M$ as the algebra whose underlying vector space is the conjugate of $M$ (i.e. for $\lambda \in \mathbb{C}$ and $x \in M$ the product of $\lambda$ by $x$ in $M^c$ is equal to $\bar{\lambda}x$) and whose ring structure is the same as in $M$. The opposite $M^o$ of $M$ is by definition the algebra whose underlying vector space is the same as for $M$ while the product of $x$ by $y$ is equal to $yx$ instead of $xy$. $M^c$ and $M^o$ are clearly isomorphic through the map $x \to x^\ast$. For $\psi \in \text{Aut}(M)$ we denote by $\psi^c$ the automorphism of $M^c$ induced by $\psi$. 
For the convenience of the reader we detail here Connes’s argument to show that the factor \((L(F_{11}^3) \otimes R_0) \rtimes_{\gamma} \mathbb{Z}_3\) is not antiisomorphic to itself (see [3] and [4] for Connes’s argument).

**Theorem 6.1.** \(\mathcal{M} = (L(F_{11}^3) \otimes R_0) \rtimes_{\gamma} \mathbb{Z}_3\) is not anti–isomorphic to itself.

**Proof** We proved in the previous section (Theorem 5.5) that \(\chi(\mathcal{M}) \cong \mathbb{Z}_9\) and that the dual action \(\widehat{\gamma} : \widehat{\mathbb{Z}}_3 \to \text{Aut}(\mathcal{M})\) produces the only subgroup of order 3 in \(\chi(\mathcal{M})\), namely \(\langle \sigma^3 \rangle = \langle \xi(\beta^3) \rangle\).

Since \(\chi(\mathcal{M})\) is an invariant of the von Neumann algebra \(\mathcal{M}\), it follows that \(\widehat{\mathcal{M}} \rtimes \widehat{\mathcal{Z}}_3\) is an invariant of \(\mathcal{M}\). This implies that \(\mathcal{M} \rtimes \widehat{\mathcal{Z}}_3\) is an invariant of \(\mathcal{M}\), as it is the dual action \(\widehat{\gamma} : \mathbb{Z}_3 \to \text{Aut}(\mathcal{M} \rtimes \widehat{\mathcal{Z}}_3)\) of \(\widehat{\gamma}\).

Since, by Takesaki’s duality theory [Theorem 4.5, [16]],

\[
\mathcal{M} \rtimes \widehat{\mathcal{Z}}_3 \cong \left( L\left( F_{11}^3 \right) \otimes R_0 \right) \otimes B(\ell^2(\mathbb{Z}_3)),
\]

and in this identification the dual action \(\widehat{\gamma}\) of \(\widehat{\gamma}\) corresponds to the action \(\gamma \otimes \text{Ad}(\lambda(1)^*)\), where \(\lambda\) is the left regular representation of \(\mathbb{Z}_3\) on \(\ell^2(\mathbb{Z}_3)\), we conclude that \(\gamma\) is an invariant of \(\mathcal{M}\). Consequenty, the centrally trivial automorphism \(\gamma_1\) and the approximately inner automorphism \(\gamma_2\), which appear in the decomposition of \(\gamma\), are also invariants of \(\mathcal{M}\).

Observe also that in the above argument we have only used the abstract group \(\widehat{\mathbb{Z}}_3\) and the dual action defined on \(\mathcal{M} \rtimes \widehat{\mathcal{Z}}_3\). Hence we have found a canonical way to associate to the von Neumann algebra \(\mathcal{M}\) a scalar, equal to \(e^{2\pi i/3}\) in our case, which is invariant under isomorphisms.

Now if \(\mathcal{M} = \left( L\left( F_{11}^3 \right) \otimes R_0 \right) \rtimes_{\gamma} \mathbb{Z}_3\) was anti–isomorphic to itself, then \(\mathcal{M}\) and \(\mathcal{M}^c\) would be isomorphic. But the obstruction associated to \(\gamma_1^c\), and therefore to \(\mathcal{M}^c\) is equal to \(e^{-2\pi i/3}\).

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