A review of Space-time conservation element solution element (CESE) schemes

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Abstract: The space-time conservation element and solution element (CESE) method is one of the recent advancements in CFD. This paper attempts to review various schemes along with their limitations starting with a non-dissipative a-ε scheme. The following a-ε-a scheme has an added numerical dissipation term. This numerical dissipation is capable of damping out numerical instabilities that arise only from the smooth region of the solution but fails to suppress numerical wiggles. The a-ε-α-β scheme is augmented with the ability to suppress the wiggles introduced due to discontinuities, using another added term to the solution. Hence a-ε-α-β scheme is capable of capturing both small disturbances and sharp discontinuities simultaneously. Its advantage over a-ε scheme as well as the suppression of numerical wiggle at the discontinuity will be demonstrated using Sod’s shock tube problem. All of these CESE schemes described so far become excessively diffusive at low Courant number and Mach number. Hence the construction of a Courant number insensitive scheme which has a weighted average sense. Its advantage over a-ε-α-β in terms of stability will be explained further in the paper using Sod’s shock tube problem as an example and its ability to resolve contact discontinuities without any ad-hoc parameter over CFL number ranging from one to close to 0.001

1. Introduction

The space-time conservation element and solution element method is a multidimensional method for solving conservation equations (1). This approach is based on a unified treatment of space and time where both the flow variable and its spatial derivative are treated as independent variables to be solved for simultaneously. The evaluation of dependent variable gradients is done using element edges and corresponding neighbouring solution elements while keeping the flux integration procedure same. The new approach enforces both local and global space-time flux conservation without the need for interpolating or extrapolating. The development of the CESE method is based on the integral form of space-time flux conservation for numerical discretization using local discrete variables rather than global variables (1).

Consider 1D unsteady Euler equations of a perfect gas in its non-dimensional form

\[
\frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} = 0, \quad m = 1, 2, 3
\]  

(1.1)
Where m = 1, 2, 3 represents mass, momentum and energy equations respectively. Let \( p, v, \rho \) and \( \gamma \) be the static pressure, velocity, mass density and constant specific heat ratio respectively.

\[
\begin{align*}
\mathbf{u}_m &= \begin{bmatrix} \rho \\ \rho v \\ p/(\gamma - 1) + \frac{1}{2} \rho v^2 \end{bmatrix} \\
\mathbf{f}_m &= \begin{bmatrix} u_2 \\ (\gamma - 1) u_3 + \frac{1}{2} (3 - \gamma) (u_2)^2 / u_1 \\ \gamma u_2 u_3 / u_1 - \frac{1}{2} (\gamma - 1) (u_2)^3 / (u_1)^2 \end{bmatrix} \\
\end{align*}
\] (1.2)

Assuming the physical solution is smooth Eq. (1.1) is the differential form of a more fundamental integral conservation law.

\[
\oint_{\partial V} \mathbf{h}_m \cdot d\mathbf{\mathbf{s}} = 0 \quad m = 1, 2, 3
\] (1.3)

Where \( \mathbf{h}_m = (\mathbf{f}_m, \mathbf{u}_m), m = 1, 2, 3, \) are the space-time mass momentum and energy density vectors respectively and \( \partial V \) is the boundary of an arbitrary region \( V \) in space-time 2D Euclidean space. Let \( \Omega \) denote the set of all mesh points in figure 1, then for each \( (j, n) \in \Omega \) solution element \( \text{SE}(j, n) \) is defined as the immediate neighbourhood of a horizontal and vertical segment around mesh point \( (j, n) \) shown in figure 1 (2). The exact size of the neighbourhood doesn’t matter until Eq. (1.3) takes a complicated form involving source terms, then the \( \text{SE}_s \) will fill up the computational domain entirely such that volume integral can be modelled properly (3).

![A staggered space time mesh and SE \((j, n)\)](image)

For any \( (x, t) \in \text{SE}(j, n), \mathbf{u}_m(x, t), \mathbf{f}_m(x, t) \) and \( \mathbf{h}_m(x, t) \) are approximated by \( \mathbf{u}_m^n(x, t; j, n), \mathbf{f}_m^n(x, t; j, n) \) and \( \mathbf{h}_m^n(x, t; j, n) \) respectively defined as

\[
\begin{align*}
\mathbf{u}_m^n(x, t; j, n) &= (\mathbf{u}_m)_j^n + (\mathbf{u}_{mx})_j^n (x - x_j) + (\mathbf{u}_{mt})_j^n (t - t_j) \\
\mathbf{f}_m^n(x, t; j, n) &= (\mathbf{f}_m)_j^n + (\mathbf{f}_{mx})_j^n (x - x_j) + (\mathbf{f}_{mt})_j^n (t - t_j) \\
\end{align*}
\] (1.4)

also as \( \mathbf{h}_m^n = (\mathbf{f}_m^n, \mathbf{u}_m^n) \) then,

\[
\mathbf{h}_m^n(x, t; j, n) = (\mathbf{f}_m^n(x, t; j, n), \mathbf{u}_m^n(x, t; j, n)), \quad m = 1, 2, 3
\] (1.6)

Where, \( (\mathbf{f}_{mx})_j^n \) and \( (\mathbf{f}_{mt})_j^n \) are the numerical analogues for the values of \( \partial \mathbf{f}_m / \partial x \) and \( \partial \mathbf{f}_m / \partial t \) respectively defined as

\[
\frac{\partial \mathbf{f}_m}{\partial x} = \sum_{k=1}^{3} f_{m,k} \frac{\partial \mathbf{u}_k}{\partial x} \quad \text{and} \quad \frac{\partial \mathbf{f}_m}{\partial t} = \sum_{k=1}^{3} f_{m,k} \frac{\partial \mathbf{u}_k}{\partial t} \quad m, k = 1, 2, 3
\] (1.7)

and

\[
\partial f_{m,k} \stackrel{df}{=} \frac{\partial \mathbf{f}_m}{\partial \mathbf{u}_k}, \quad m, k = 1, 2, 3
\] (1.8)
Thus \((u_m)^j\) and \((u_{mx})^j\) are the two independent marching variables that will be solved for where\((f_{mx})^j\), \((f_m)^j\) and \((u_m)^j\) are the dependent variables defined using Eqs. (1.1), (1.7) and (1.8). Let the two dimensional Euclidean space \(E_2\) be divided into non overlapping regions referred to as conservation elements \((CE)\) with its top right or top right vertex as the mesh point \((j, n) \in \Omega\) defined as \(CE_+(j, n)\) containing \(AD\) and \(AC\) and \(CE_-(j, n)\) containing the line segments \(AB\) and \(AC\). Then the discrete form of equation (1.3) can be written as

\[
\oint_{s(CE_+(j, n))} \vec{n} \cdot d\vec{s} = 0
\]

(1.9)

At each \((j, n) \in \Omega\), Eq. (1.9) provides the two required conditions to solve the two independent marching variables. For \(CE_+(j, n)\) the line segments \(AB\) and \(AC\) belongs to \(SE(j, n)\) thus the flux leaving through these boundaries will be evaluated using Eqs. (1.4), (1.5), and (1.6) with \((x, t) \in SE(j, n)\).

By substituting Eq.(1.7) into Eq.(1.10) and considering \(F^+ = \frac{(\Delta t/\Delta x)}{f_{mk}}\), \(m, k = 1, 2, 3\), and assuming the existence of the inverse of matrix \([I - (F^+)^2]\) for all \((j, n) \in \Omega\).

\[
(u_m)^j = \frac{1}{2} \left[ \left( (I - F^+) s_+ \right)_{j+1/2}^{n-1/2} + \left( (I + F^+) s_- \right)_{j-1/2}^{n-1/2} \right]
\]

(1.11)

where

\[
(s_+)^{-1/2} \overset{\text{def}}{=} \left[ u_m - \left( I + F^+ \right) u_{mx} \right]_{j+1/2}^{n-1/2}
\]

(1.12)

\[
(s_-)^{-1/2} \overset{\text{def}}{=} \left[ u_m + \left( I - F^+ \right) u_{mx} \right]_{j-1/2}^{n-1/2}
\]

(1.13)

and

\[
(u_{mx})^n = \frac{\Delta x}{4} (u_{mx})_j^n \quad m = 1, 2, 3
\]

(1.14)

2. 1D Euler Schemes

2.1 The simplified Euler a scheme

The Euler a scheme solves two system of linear equations simultaneously resulting in it becoming locally implicit for each \((j, n) \in \Omega\), but approximating some expressions a simplified version that is completely explicit can be developed (4). The expression for the derivative of flow variable was defined using Eqs (1.12) and (1.13)

\[
(u_{mx})^n_j = \left( u_{mx}^a \right)_j^n \overset{\text{def}}{=} \frac{1}{2} \left[ (s_+)^{n-1/2} - (s_-)^{n-1/2} \right]
\]

(2.1)

With the assumptions

\[
\left( (I - F^+) \right)^{n-1/2} \overset{\text{def}}{=} \left[ (I - F^+) \right]_{j+1/2}^{n-1/2}, \quad \left( (I + F^+) \right)^{-1} \overset{\text{def}}{=} \left[ (I + F^+) \right]_{j-1/2}^{n-1/2}
\]

(2.2)
The marching algorithm defined by Eqs. (1.11) and (2.1) is referred to as the simplified Euler \( a \) scheme. The superscript \( "a" \) in \( \mathbf{u}_k^{a+} \) is introduced to notify that the Eq. (2.1) is valid for simplified Euler scheme. The explicit simplified Euler \( a \) scheme is generally unstable but this scheme can be extended to become a simplified Euler \( a-\varepsilon \) scheme which is stable over a larger domain (1).

2.2 The Euler \( a-\varepsilon \) scheme and simplified Euler \( a-\varepsilon \) scheme

The Euler \( a-\varepsilon \) scheme is formed by defining the expression for the derivative of flow variable in a different manner than that for Eq. (2.1) using Eq. (2.3), \( (\mathbf{u}_m^{a+})_{j\pm 1/2}^n \) can be interpreted as Taylor series approximation of \( \mathbf{u}_m^n \) at \( (j \pm 1/2, n) \). Thus \( (\mathbf{u}_m^{a+})_{j\pm 1/2}^n \) is a central difference approximation of \( \partial \mathbf{u}_m^n / \partial x \), normalized by the same factor \( \Delta x / 4 \) that is used in Eq. (1.14) and superscript ‘c’ represents the central difference nature of the term. Then using Eq. (2.4) the Euler \( a-\varepsilon \) scheme is defined as

\[
(\mathbf{u}_m^{a+})_j^n = \left( \mathbf{u}_m^{a+} \right)_j^n + 2\varepsilon \left( \mathbf{u}_m^{c+} - \mathbf{u}_m^{a+} \right)_j^n
\]

Where \( \varepsilon \) is a real number, the first part of the expression is a non-dissipative part that originates from non-dissipative \( a \) scheme and the second part is the dissipative part which is the difference between the central difference term and the non-dissipative term whose magnitude can be controlled using the parameter \( \varepsilon \). The scheme reduces to \( a \) scheme when \( \varepsilon = 0 \), and a central difference scheme with \( (\mathbf{u}_m^{c+})_{j\pm 1/2}^n \) is only slightly less stable than its original form (5). The simplified Euler \( a-\varepsilon \) scheme is stable over a much larger range \( 0 \leq \varepsilon \leq 1 \) and \( v_j^n < 1 \) for all \( (j, n) \in \Omega \), where \( v_j^n \) is the local Courant number.

The evaluation of \( (\mathbf{u}_m^{a+})_{j\pm 1/2}^n \) is required for the implementation of Euler \( a-\varepsilon \) scheme thus making it locally implicit except the case where \( \varepsilon = 1/2 \). To make the scheme explicit \( (\mathbf{u}_m^{a+})_{j\pm 1/2}^n \) is replaced by \( (\mathbf{u}_m^{a+})_{j\pm 1/2}^n \) defined earlier in Eq. (2.1) Thus using Eq.(1.11) and this new explicit variant simplified Euler \( a-\varepsilon \) scheme is defined having same algebraic structure as that of Euler \( a-\varepsilon \) scheme. The simplified Euler \( a-\varepsilon \) scheme is only slightly less stable than its original form (5). The \( a-\varepsilon \) scheme becomes progressively diffusive as the value of \( \varepsilon \) from its lower bound to 1.

2.3 Euler \( a-\varepsilon-\alpha-\beta \) scheme
The numerical dissipation introduced by the second term of Euler \( a-\varepsilon \) scheme is equipped to effectively control the numerical instabilities but is less effective in case of numerical wiggles arising due to a discontinuity. For effectively suppressing the numerical wiggles \( a-\varepsilon-\alpha-\beta \) scheme was proposed (5) and the derivative of flow variable was defined as

\[
(\mathbf{u}_m^{a+})_j^n = (\mathbf{u}_m^{a+})_j^n + 2\varepsilon (\mathbf{u}_m^{c+} - \mathbf{u}_m^{a+})_j^n + \beta (\mathbf{u}_m^{w+} - \mathbf{u}_m^{c+})_j^n
\]

where

\[
\mathbf{u}_m^{w+} = \mathbf{u}_m^{c+} + \alpha \mathbf{u}_m^{c+} + \mathbf{u}_m^{c+} - \mathbf{u}_m^{a+} + \mathbf{u}_m^{c+} = \mathbf{u}_m^{c+} + \mathbf{u}_m^{c+} - \mathbf{u}_m^{a+}
\]

and

\[
(\mathbf{u}_m^{c+})_j^n \text{ defined as } \pm \frac{1}{2} (\mathbf{u}_m^{n} - \mathbf{u}_m^{n})
\]
Also it can be shown that \((u_{mx}^c)^n_j\) is the normal average of \((u_{mx}^c)^n_j\) and \((u_{mx}^c)^n_j\). Thus the new term that’s added to the earlier expression for the derivative is the difference between the weighted average and the normal average of \((u_{mx}^c)^n_j\) and \((u_{mx}^c)^n_j\) and its numerical dissipation is effective in suppressing the numerical wiggles. Also as the magnitude of both \((u_{mx}^c)^n_j\) and \((u_{mx}^c)^n_j\) are more or less same, the weighted average is almost equal to the normal average which causes only slight effects in the smooth part of the solution. Thus the two types of mutually complementing numerical dissipations are equipped to handle both sharp discontinuities as well as small disturbances (6).

The values of parameters generally taken as \(\varepsilon = 1/2, \alpha = (1, 2)\) and \(\beta = 1\) are good enough for an accurate solution for the range \(0.1 \leq \nu^2 \leq 1\), this will be known as Euler \(a-\alpha\) scheme. The parameters \(\varepsilon\) and \(\beta\) can be taken as functions of local variables and mesh providing more flexibility (5).

As discussed earlier the \(a-\varepsilon\) scheme is only stable over range \(a\) of epsilon values \((0.03 \leq \varepsilon \leq 1)\), the behaviour will be shown using Sod’s shock tube problem (7). The domain was divided in 100 parts with \(\Delta x = 0.01\) and a time size \(\Delta t = 4 \times 10^{-3}\) was taken into consideration corresponding to Courant number (CFL = 0.88). The numerical results at time \(t=0.2\) sec for three different epsilon values \((\varepsilon = 0.01, 0.1, 0.5)\) are shown in figure 2(a) along with their comparison with analytical result. It can be observed that as the epsilon value goes below the stability limit the numerical solution deteriorates quickly and for the epsilon values in the stability range there are numerical wiggles near the discontinuity. In figure 2(b) a comparison between Euler \(a-\varepsilon\) scheme \((\varepsilon = 0.5)\) and Euler \(a-\varepsilon-\alpha-\beta\) scheme is shown with same domain discretization and a time size \(\Delta t = 4 \times 10^{-3}\) corresponding to Courant number (CFL = 0.88). It can be observed that the added term is effective in suppressing the numerical wiggles and causes only slight effects in the smooth part of the solution.

![Figure 2(a)](image1.png)
![Figure 2(b)](image2.png)

Figure 2 (a) Comparison of Euler \(a-\varepsilon\) scheme with different epsilon values \((\varepsilon = 0.01, 0.1, 0.5)\) and CFL number = 0.88 (b) Comparison of Euler \(a-\varepsilon-\alpha-\beta\) scheme \((\alpha = 1, \varepsilon = 0.5, \beta = 1)\) with Euler \(a-\varepsilon\) scheme \((\varepsilon = 0.5)\) at CFL number = 0.88.
2.4 Courant number insensitive scheme

The a-ε-α-β scheme (α = 1, 2, ε = 0.5, β = 1) becomes highly diffusive in nature when |v| ≪ 1, to overcome this a Courant number insensitive scheme was proposed (8). Let \( P_+, M_+, P_- \) and \( M_- \) be points defined in the figure 2, as the value of |v| decreases from 1 to 0 point \( P_+ \) will move away from point (j + 1/2, n) and towards the point \( M_+ \), which is the midpoint of the line segment AC. Similarly point \( P_- \) will move away from point (j - 1/2, n) and towards the point \( M_- \), which is the midpoint of line segment AB.

\[
\text{Figure 3 Position of points } P_+, M_+, P_- \text{ and } M_-
\]

Thus using Figure 2 and first order Taylor approximation of \( u_m \)

\[
u_m(P^+) = (u_m - \left(1 + 2v_{mj}^n - (\tau_m)^n\right)u_{mx}^n)_{j+1/2}^{n-1/2}
\]

(2.9)

\[
u_m(P^-) = (u_m + \left(1 - 2v_{mj}^n - (\tau_m)^n\right)u_{mx}^n)_{j-1/2}^{n-1/2}
\]

(2.10)

Where \( v_m \), \( m = 1, 2, 3 \) is defined as the \( \frac{\Delta x}{\Delta t} \) * diag \((v, v - c, v + c)\), where c is the sonic speed and \( \tau_m \) is a parameter used in the figure 2 and defined as \( (\tau \geq \tau_0 (v^2)) \) (9). The mesh point (j, n) is the midpoint of line segment joining \( P_+ \) and \( P_- \), taking one sided difference approximation for \((u_{mx})_j^n\) using \( u_m(P^+) \) and \( u_m(P^-) \) it can be shown that

\[
\left(\frac{u_{mx}^n}{u_{mx}^n}\right)_{j}^{n} \equiv \frac{\Delta x}{4} \left(\frac{u_m(P^+)}{u_m(P^-)}\right)_j^n \left(1 + (\tau_m)^n\right)_j^n \Delta x / 2
\]

(2.11)

(2.12)

A new weighted average scheme is defined as

\[
\left(\frac{u_{mx}^n}{u_{mx}^n}\right)_{j}^{n} \equiv \frac{\Delta x}{4} \left[\left(\frac{s_m}{u_{mx}^n}\right)_j^{n} + \left(\frac{s_m}{u_{mx}^n}\right)_j^{n}ight]
\]

(2.13)

where \( (s_m)^n \) is

\[
(s)_j^n \equiv \frac{\left(\frac{u_{mx}^n}{u_{mx}^n}\right)_j^{n}}{\min\left(\left(\frac{u_{mx}^n}{u_{mx}^n}\right)_j^{n}, \left(\frac{u_{mx}^n}{u_{mx}^n}\right)_j^{n}\right)} - 1
\]

(2.14)

For the case when \( (s_m)^n \) is very small as compared to 1 the scheme can be written as

\[
\left(\frac{u_{mx}^n}{u_{mx}^n}\right)_{j}^{n} \equiv \left(\frac{1 + \sigma^n(s_{m-n})^{a}}{1 + \sigma^n(s_{m-n})^{a}}\right) \left(\frac{u_{mx}^n}{u_{mx}^n}\right)_j^n \left(1 + (\tau_m)^n\right)_j^n \Delta x / 2 + \sigma^n_1 \left(\frac{u_{mx}^n}{u_{mx}^n}\right)_j^n \left(\frac{u_{mx}^n}{u_{mx}^n}\right)_j^n
\]

(2.15)
With $\alpha$ and $(\tau)_n^j$ taken to be a function of $\sigma_n$ and $(v_{\text{max}})_j^n$ respectively as

$$\alpha = \sigma_n^j \begin{array}{c} \text{def} \\
\end{array} \frac{\sigma_o}{(v_{\text{max}})_j^n} \quad \text{and} \quad (\tau)_m^n = \beta_m \left( \frac{v_{\text{max}}}{n} \right)_j^n, \quad (j, n) \in \Omega$$  \hspace{1cm} (2.16)

The value for $\beta_m \geq 1$ and $\sigma_o > 0$ can be pre-specified and are the order of 1. and $|v_{\text{max}}|_j^n$ is the local Courant number defined as the max$\{|(v_1)_j^n|, |(v_2)_j^n|, |(v_3)_j^n|\}$. The scheme defined using Eqs. (1.11) and (2.15) is known as Courant number insensitive scheme (CNIS) (8). The Courant number insensitive solutions presented in (6) and (10) are generated using the current scheme with $\sigma_o = 0.5$ and $\beta_m = 1, m = 1, 2, 3$.

As mentioned earlier Euler $a$-$\varepsilon$-$\alpha$-$\beta$ scheme becomes highly diffusive in nature when $|v| \ll 1$, this behaviour is shown in figure 4(a). Also the Courant number insensitive behaviour of new scheme will be shown using Sod’s shock tube problem (7). The domain was divided in 100 parts with $\Delta x = 0.01$ and two time sizes of $\Delta t = 4e^{-3}$ and $\Delta t = 4e^{-5}$ were taken into consideration corresponding to two different Courant numbers (CFL = 0.88 and CFL = 0.0088). The numerical results at time $t=0.2$sec for the two different Courant numbers are shown in figure 4(a) along with their comparison with analytical result. It can be observed that as the Courant number becomes very small as compared to 1, the $a$-$\varepsilon$-$\alpha$-$\beta$ scheme ($\alpha = 1, 2$) becomes highly diffusive in nature and the numerical result deteriorates near the discontinuity. In figure 3(b) the Courant number insensitive scheme is compared with that of $a$-$\varepsilon$-$\alpha$-$\beta$ scheme corresponding to Courant number = 0.0088. It is demonstrated that the Courant number insensitive scheme behaves doesn’t show a numerical diffusive behaviour for low Courant number.

![Figure 4](image_url)

Figure 4 (a) Comparison of Euler $a$-$\varepsilon$-$\alpha$-$\beta$ with different Courant number values (CFL = 0.88 and 0.0088) (b) CNIS (Courant number insensitive scheme with Euler $\alpha$ scheme at CFL = 0.0088)
3. Conclusion

The various factors controlling the numerical dissipation of the Courant number insensitive scheme and the weighted average nature of the scheme allows it to be stable over a wide range of Courant numbers. For a stable numerical solution of an unsteady non-linear problem, numerical dissipation is required in sufficient amount but too much dissipation causes deterioration in the accuracy of the numerical solution thus making it difficult to be reduced while maintaining the accuracy. The CESE scheme is inherently based on a core non-dissipative $a$-scheme and the different parameters controlling the dissipation for the Courant number insensitive scheme allows a proper control over the reduction of numerical dissipation if required maintaining accuracy. For higher dimension extensions for 2D and 3D Euler equations involves associated Jacobian matrices and they are diagonalizable using the same diagonalization matrix thus making it easier to extend the scheme to higher dimensions.

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