HYPERCONTRACTIVITY OF THE BOHNENBLUST-HILLE INEQUALITY FOR POLYNOMIALS AND MULTIDIMENSIONAL BOHR RADII

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Abstract. In 1931 Bohnenblust and Hille proved that for each \( m \)-homogeneous polynomial \( \sum_{|\alpha|=m} a_\alpha z^\alpha \) on \( \mathbb{C}^n \) the \( \ell^{\frac{4m}{3}} \)-norm of its coefficients is bounded from above by a constant \( C_m \) (depending only on the degree \( m \)) times the sup norm of the polynomial on the polydisc \( D^n \). We prove that this inequality is hypercontractive in the sense that the optimal constant \( C_m \) is \( \leq C_m^\prime \) where \( C_m^\prime \geq 1 \) is an absolute constant.

From this we derive that the Bohr radius \( K_n \) of the \( n \)-dimensional polydisc in \( \mathbb{C}^n \) is up to an absolute constant \( \geq \log (n/n) \); this result was independently and with a different proof discovered by Ortega-Cerdà, Ounaïes and Seip in [25]. An alternative approach even allows to prove that the Bohr radius \( K_p^n \), \( 1 \leq p \leq \infty \) of the unit ball of \( \ell^p_n \), is asymptotically \( \geq (\log (n/n))^{1-1/\min(p,2)} \). This shows that the upper bounds for \( K_p^n \) given by Boas and Khavinson from [5] are optimal.

1. Introduction and Main Results

In 1930 Littlewood proved the following (innocent looking) inequality which is nowadays often cited as Littlewood’s 4/3-inequality: For every bilinear form \( A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \) we have

\[
\left( \sum_{i,j} |A(e_i, e_j)|^{4/3} \right)^{3/4} \leq \sqrt{2} \sup_{x,y \in D^n} |A(x,y)|,
\]

and the exponent 4/3 is optimal; here as usual \( D \) denotes the open unit disc in \( \mathbb{C} \). It seems that Bohnenblust and Hille in 1931 immediately realized the importance of this results (and the techniques used in its proof) for the study of lower bounds for the maximal width \( T \) of the strip of uniform but nonabsolute convergence of Dirichlet series \( \sum a_n \frac{1}{n^s} \). Bohr in 1913 in his article [7] had shown that \( T \leq 1/2 \), and the in the years following the question whether this estimate was optimal or not became well known under the name “Bohr’s absolute convergence problem”. Closings a long story Bohnenblust-Hille in their ingenious article [6] proved that in fact \( T = 1/2 \).

The crucial step in their solution is formed by an \( m \)-linear version of Littlewood’s result together with its symmetrization for polynomials: For each \( m \) there is a constant \( C_m \geq 1 \) such that for each \( n \) and for each \( m \)-linear
mapping $A : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ we have

$$\left( \sum_{i_1, \ldots, i_m} |A(e_{i_1}, \ldots, e_{i_m})|^\frac{2m}{m+1} \right)^\frac{m+1}{2m} \leq C_m \sup_{x \in \mathbb{D}^m} |A(x_1, \ldots, x_m)|,$$

and again the exponent $\frac{2m}{m+1}$ is optimal. Moreover, if $C_m$ stands for the best constant, then the original proof gives that $C_m \leq m^{(m+1)/(2m)}2^{(m-1)/2}$. This inequality was forgotten for long time and re-discovered by Davie [10] and Kaijser [23], see also [3]; their proofs are (slightly) different from the original one and give the better constant

$$C_m \leq \sqrt{2}^{m-1}.$$ 

In order to solve Bohr’s “absolute convergence problem” Bohnenblust and Hille in fact needed a symmetric version of (1.1). They used (or better invented) polarization and deduced from (1.1) that for each $m$ there is some constant $D_m \geq 1$ such that for each $n$ and for each $m$-homogeneous polynomial $\sum_{|\alpha| = m} a_\alpha z^\alpha$ on $\mathbb{C}^n$

$$\left( \sum_{|\alpha| = m} |a_\alpha|^\frac{2m}{m+1} \right)^\frac{m+1}{2m} \leq D_m \sup_{z \in \mathbb{D}^n} \left| \sum_{|\alpha| = m} a_\alpha z^\alpha \right|;$$

and again they showed through a highly non trivial argument that the exponent $\frac{2m}{m+1}$ can not be improved. A nowadays standard argument allows to deduce from (1.2) and an estimate for the polarization constant of $\ell_\infty$ due to Harris [22] that

$$D_m \leq (\sqrt{2})^{m-1} \frac{m^{m/2}(m+1)^{m+1}}{2^m (m!)^{m+1/2m}}$$

(see e.g. [15, Section 4]), and using Sawa’s Khinchine type inequality for Steinhaus variables Queffélec in [27, Theorem III-1] gets the slightly better estimate:

$$D_m \leq \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} \frac{m^{m/2}(m+1)^{m+1}}{2^m (m!)^{m+1/2m}}.$$ 

Our first main result is the following substantial improvement. We show that the Bohnenblust-Hille inequality (1.3) for polynomials in fact is supercontractive in the sense that its best constant $D_m$ for some absolute constant $C \geq 1$ satisfies $D_m \leq C^m$.

**Theorem 1.1.** There is a $C \geq 1$ such that for all $m, n$

$$\left( \sum_{|\alpha| = m} |a_\alpha|^\frac{2m}{m+1} \right)^\frac{m+1}{2m} \leq C^m \sup_{z \in \mathbb{D}^n} \left| \sum_{|\alpha| = m} a_\alpha z^\alpha \right|,$$

where $\sum_{|\alpha| = m} a_\alpha z^\alpha$ is an $m$-homogeneous polynomial on $\mathbb{C}^n$. 
Let us indicate that this result (see section 3 for the proof) has some far reaching consequences. Given an n-dimensional Banach space $X_n = (\mathbb{C}^n, \| \cdot \|)$ for which the $e_k$’s form a 1-unconditional basis, we use this result to estimate n-dimensional Bohr radii of the open unit ball $B_{X_n}$ in $X_n$, and to estimate unconditional basis constant $\chi_{\text{mon}}(\mathcal{P}(^mX_n))$ of the monomials $z^\alpha$ in the Banach space $\mathcal{P}(^mX_n)$ of all $m$-homogeneous polynomials.

Recall that the Bohr radius $K(B_{X_n})$ of the open unit ball $B_{X_n}$ (a Reinhardt domain) is the infimum of all $r \geq 0$ such that for each holomorphic function $f = \sum \alpha a_\alpha z^\alpha$ on $B_{X_n}$ we have

$$\sup_{z \in B_{X_n}} |\sum \alpha a_\alpha z^\alpha| \leq \sup_{z \in B_{X_n}} |\sum \alpha a_\alpha z^\alpha|.$$  

The unconditional basis constant $\chi_{\text{mon}}(\mathcal{P}(^mX_n))$ of the monomials $z^\alpha$ in $\mathcal{P}(^mX_n)$ by definition is the best constant $C \geq 1$ such that for every $m$-homogeneous polynomial $\sum |\alpha| = m a_\alpha z^\alpha$ on $\mathbb{C}^n$ and any choice of scalars $\varepsilon_\alpha$ with $|\varepsilon_\alpha| \leq 1$ we have

$$\sup_{z \in B_{X_n}} |\sum |\alpha| = m \varepsilon_\alpha a_\alpha z^\alpha| \leq C \sup_{z \in B_{X_n}} |\sum |\alpha| = m a_\alpha z^\alpha|.$$  

Asymptotic estimates for unconditional basis constants of spaces of $m$-homogeneous polynomials on $X_n = \ell^n_p$ were given in [11, Theorem 3]; as usual $\ell^n_p$, $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, stands for $\mathbb{C}^n$ together with the $p$-norm $\|z\|_p := (\sum_k |z_k|^p)^{1/p}$ (with the obvious modification for $p = \infty$). These results were improved in [13, Lemma 3.1] where it is shown that

$$\chi_{\text{mon}}(\mathcal{P}(^m\ell^n_p)) \leq C^m n^{(m-1)(1-\frac{1}{\min(p,2)})},$$  

$C \geq 1$ some absolute constant. Our second main result is:

**Theorem 1.2.** There is a constant $C \geq 1$ such that for each $1 \leq p \leq \infty$ and all $m, n$

$$\chi_{\text{mon}}(\mathcal{P}(^m\ell^n_p)) \leq C^m \left(1 + \frac{n}{m}\right)^{(m-1)(1-\frac{1}{\min(p,2)})}.$$  

During the preparation of this manuscript we were informed that for $p = \infty$ and $n > m^2 > 1$ this result has been obtained independently and with a substantially different proof by Ortega-Cerdà, Ounaïes and Seip in their very recent article [25, Theorem 1]. There it is presented as an upper estimate of the Sidon constant for the index set of nonzero $m$-homogeneous polynomials in $n$ complex variables (see also (1.4) and (1.5) below for equivalent formulations). Several **remarks on Theorem 1.1** follow:

(1) Let us first indicate how for $p = \infty$ the preceding theorem can be deduced as an immediate consequence of the hypercontractivity of the constant in Theorem 1.1: Clearly we have

$$\chi_{\text{mon}}(\mathcal{P}(^m\ell^n_\infty)) = \sup\{ \sum |\alpha|: \sup_{z \in \mathbb{C}^n} |\sum |\alpha| = m a_\alpha z^\alpha| \leq 1 \},$$  

$$\sup\{ \sum |\alpha|: \sup_{z \in \mathbb{C}^n} |\sum |\alpha| = m a_\alpha z^\alpha| \leq 1 \},$$
hence by Hölder’s inequality for each polynomial $\sum_{|\alpha|=m} a_{\alpha} z^\alpha$
\[
\sum_{|\alpha|=m} |a_\alpha| \leq \left( \sum_{|\alpha|=m} 1 \right)^{m-1/2m} \left( \sum_{|\alpha|=m} |a_\alpha|^{2m/(m+1)} \right)^{m/(2m)}.
\]

But then Theorem 1.1 and a straightforward calculation using Stirling’s formula (see also (2.1)) as desired show that there is a constant $C \geq 1$ such that for all $m$-homogeneous polynomials $\sum_{|\alpha|=m} a_{\alpha} z^\alpha$ on $\mathbb{C}^n$ we have
\[(1.4) \quad \sum_{|\alpha|=m} |a_\alpha| \leq C m \left( 1 + \frac{n}{m} \right)^{m-1/2m} \sup_{z \in \mathbb{D}^n} |\sum_{|\alpha|=m} a_\alpha z^\alpha|.
\]

(2) From [13, Lemma 3.2] we know that there is some constant $C \geq 1$ such for each Banach space $X_n = (\mathbb{C}^n, \| \cdot \|)$ for which the $e_k$’s form a 1-unconditional basis and each $m$, $\chi_{\text{mon}}(P(m X_n)) \leq \chi_{\text{mon}}(P(m \ell_\infty^n))$. Hence, once in Theorem 1.2 the case $p = \infty$ is proved, the case $2 \leq p$ follows.

(3) Moreover, for $2 \leq p$ Theorem 1.2 is optimal in the following sense: Given $2 \leq p \leq \infty$, we have
\[(1.5) \quad \chi_{\text{mon}}(P(m \ell_p^n)) \sim \begin{cases} 
\frac{1}{\sqrt{m-1}} \sqrt[2m-1]{n^{m-1}} & \text{if } n > m, \\
1 & \text{if } m \geq n,
\end{cases}
\]
where $A_{mn} \sim B_{mn}$ means that there is some constant $C \geq 1$ such that for every $m, n$ we have $1/C^m A_{mn} \leq B_{mn} \leq C^m A_{mn}$; indeed, this follows from an easy calculation since by a probabilistic estimate from [14, (4.4)] we know that for each such $p$ there is some constant $d_p > 0$ such that for every $m, n$
\[
\frac{\sqrt[2m]{n^{m-1}}}{d_p \sqrt[3]{m^{1/2}}} \leq \chi_{\text{mon}}(P(m \ell_p^n)).
\]

(4) The case $p \leq 2$ in Theorem 1.2 needs a different approach of independent interest. This approach improves ideas from [11], will be given in section 6 based on the results from the sections 4 and 5, and does still cover the case $p \geq 2$. Invariants from local Banach space theory as Gordon-Lewis and projection constants are involved.

Let us finally turn to multidimensional Bohr radii. In [14, Theorem 2.2] a basic link between Bohr radii and unconditional basis constants is given: For every $n$-dimensional Banach space $X_n = (\mathbb{C}^n, \| \cdot \|)$ for which the $e_k$’s form a 1-unconditional basis we have
\[(1.6) \quad \frac{1}{3 R(X_n)} \leq K(B_{X_n}) \leq \min \left( \frac{1}{3}, \frac{1}{R(X_n)} \right),
\]
where $R(X_n) := \sup_m \chi_{\text{mon}}(P(m X_n))^{1/m}$. This means that estimates for unconditional basis constants of $m$-homogeneous polynomials always lead
to estimates for multidimensional Bohr radii. For $n = 1$ we obtain Bohr’s famous power series theorem

$$K(\mathbb{D}) = \frac{1}{3}$$

from [9], and hence (1.6) can be seen as an abstract extension of Bohr’s theorem (let us remark that Bohr discovered his power series theorem in the context of the above mentioned “absolute convergence problem”).

By results of Aizenberg, Boas, Dineen, Khavinson, Timoney and ourselves from [1], [4], [5], [13], [18] there is a constant $C \geq 1$ such that for all $1 \leq p \leq \infty$ and all $n$

$$1 \leq K(B_{\ell_p^n}) \leq C \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min(p, 2)}}.$$

Our third main result is the following improvement:

**Theorem 1.3.** There is a constant $C > 0$ such that for each $1 \leq p \leq \infty$ and all $n$

$$1 \leq K(B_{\ell_p^n}) \leq C \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min(p, 2)}}.$$

The proof is an almost immediate consequence of the basic link from (1.6) and Theorem 1.2, see section 6. As pointed out above the case $p = \infty$ also follows from (1.4) (which is itself an immediate consequence of Theorem 1.1, see above).

Let us again emphasize that in Theorem 1.3 (as in Theorem 1.2) the most important case $p = \infty$ was observed independently and through a substantially different proof by Ortega-Cerdà, Ounaïes and Seip in their very recent article [25, Theorem 2].

### 2. More preliminaries

We use standard notation and notions from (local) Banach space theory, as presented e.g. in [12], [16], [24] or [30]. All considered Banach spaces $X$ are assumed to be complex. We denote their open unit balls by $B_X$ and their duals by $X^*$. The Minkowski spaces $\ell_p^n$ were already defined in the introduction.

We denote by $\text{gl}(X)$ the Gordon-Lewis constant of a Banach space $X$ (see section 4 for the definition), by $\lambda(X)$ the projection constant (see section 5 for the definition), and by $d(X, Y)$ the Banach-Mazur distance between the Banach spaces $X$ and $Y$. The $1$–summing norm of a (linear and bounded) operator $T : X \to Y$ is denoted by $\pi_1(T)$ (we recall this definition in section 3). A Schauder basis $(x_n)$ of a Banach space $X$ is said to be unconditional if there is a constant $c \geq 0$ such that $\| \sum_{k=1}^n |\alpha_k| x_k \| \leq c \| \sum_{k=1}^n \alpha_k x_k \|$ for all $n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. In this case, the best constant $c$ is denoted by $\chi((x_n))$ and called the unconditional basis constant of $(x_n)$. Moreover, the infimum over all possible constants $\chi(x_n)$ is the unconditional basis constant $\chi(X)$.
of \( X \). We will often consider Banach spaces \( X = (\mathbb{C}^n, \| \cdot \|) \) such that the standard unit vectors \( e_k, 1 \leq k \leq n \) form a \( 1 \)-unconditional basis. Then the \( e_k \)'s also form a \( 1 \)-unconditional basis of the dual space \( X^* \).

For the metric theory of tensor products we refer to [12], and for the metric theory of symmetric tensor products and spaces of polynomials to [17] and [20]. If \( X = (\mathbb{C}^n, \| \cdot \|) \) is a Banach space and \( m \in \mathbb{N} \), then \( \mathcal{P}^{(m)}X \) stands for the Banach space of all \( m \)-homogeneous polynomials \( p(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha, \ z \in \mathbb{C}^n \), together with the norm \( \|p\|_{\mathcal{P}^{(m)}X} := \sup_{\|z\| \leq 1} |p(z)| \). The unconditional basis constant of all monomials \( z^\alpha, |\alpha| = m \), is denoted by \( \chi_{\text{mon}}(\mathcal{P}^{(m)}X) \). We identify \( \mathcal{P}^{(m)}X \) with the space \( \mathcal{L}_s^{(m)}X \) of symmetric \( m \)-linear forms, which is a subspace of \( \mathcal{L}^{(m)}X \), the space of \( m \)-linear forms. From the polarization formula we get

\[
\|p\|_{\mathcal{P}^{(m)}X} \leq \|p\|_{\mathcal{L}_s^{(m)}X} \leq \frac{m^m}{m!} \|p\|_{\mathcal{P}^{(m)}X}.
\]

Sometimes it will be more convenient to think in terms of (symmetric) tensor products instead of spaces of polynomials. For a vector space \( X \) we denote by \( \otimes^{(m)}X \) the \( m \)th full tensor product, and by \( \otimes^{m,s}X \) the \( m \)th symmetric tensor product. Recall that \( \otimes^{m,s}X \) can be identified with the image of the symmetrization operator

\[
S_m : \quad \otimes^{(m)}X \rightarrow \otimes^{m,s}X, \quad y_1 \otimes \ldots \otimes y_m \mapsto \frac{1}{m!} \sum_{\sigma \in \Pi_m} y_{\sigma(1)} \otimes \ldots \otimes y_{\sigma(m)},
\]

where \( \Pi_m \) stands for the group of permutations of \( \{1, \ldots, m\} \); note that the symmetrization operator in fact is a projector. We will often use the fact that there is some absolute constant \( C \geq 1 \) such that for any \( n, m \)

\[
\dim \otimes^{m,s}C^n = \sum_{|\alpha|=m} 1 = \left( \begin{array}{c} n + m - 1 \\ n - 1 \end{array} \right) \leq C^m \left( 1 + \frac{n}{m} \right)^m;
\]

this follows by an easy calculation using Stirling’s formula.

Recall the notation for injective and projective full and symmetric tensor products of Banach spaces (we follow [20]): We write \( \otimes^{(m)}_\alpha X \) for the \( m \)th full tensor product endowed with the injective norm \( \alpha = \varepsilon \) or projective norm \( \alpha = \pi \). Moreover, we write \( \otimes^{m,s}_\alpha X \) for the \( m \)th symmetric tensor product of \( X \) endowed with the symmetric injective norm \( \varepsilon_s \) or symmetric projective norm \( \pi_s \), respectively. If \( \alpha = \varepsilon \) or \( \pi \), then by \( \otimes^{m,s}_\alpha X \) we mean the \( m \)th symmetric tensor product equipped with \( \alpha \)-norm induced by \( \otimes^{m,s}_\alpha X \). For \( z \in \otimes^{m}X \) we have by the polarization formulas (see e.g. [20, pp. 165,167])

\[
\varepsilon_s(S(z)) \leq \varepsilon(S(z)) \leq \varepsilon(z) \text{ and } \varepsilon(S(z)) \leq \frac{m^m}{m!} \varepsilon_s(S(z)),
\]

\[
\pi(S(z)) \leq \pi_s(S(z)) \leq \frac{m^m}{m!} \pi(z) \text{ and } \pi(S(z)) \leq \pi_s(S(z)).
\]
The symmetrization operator $S_m : \otimes^m_\alpha X \to \otimes^m_\alpha X$ is a norm 1 projection onto $\otimes^m_\alpha X$, and in particular $S_m : \otimes^m_\alpha X \to \otimes^m_\alpha X$ is a projector onto $\otimes^m_\alpha X$ of norm 1 for $\alpha = \varepsilon$ and of norm $\leq \frac{m^m}{m!}$ for $\alpha = \pi$.

Let us fix some useful index sets: For natural numbers $m, n$ we define $M(m, n) := \{i = (i_1, \ldots, i_m) : i_1, \ldots, i_m \in \{1, \ldots, n\}\}$ and $J(m, n) := \{j = (j_1, \ldots, j_m) : j_1 \leq \ldots \leq j_m\}$. We will consider the following equivalence relation for multi-indices $i, j \in M(m, n)$: $i \sim j \iff \exists \sigma \in \Pi_m$ such that $i_{\sigma(k)} = j_k$ for every $k = 1, \ldots, m$. The class of equivalence defined by $i$ is denoted by $[i]$. Also we denote by $|i| := \text{card}[i]$ the cardinal of $[i]$. Note that for each $i \in M(m, n)$ there is a unique $j \in J(m, n)$ with $|i| = |j|$. Moreover, for elements $x_1, \ldots, x_m$ in a vector space $X$ and $i \in M(m, n)$ define $x_i := x_{i_1} \otimes \cdots \otimes x_{i_m} \in \otimes^m X$. In this context the following elementary lemma from [11, Lemma 1] will be used frequently:

**Lemma 2.1.** Let $m \in \mathbb{N}$ and $X$ a finite dimensional vector space with a basis $(x_k)_{k=1}^n$. Denote the orthogonal basis of the algebraic dual $X^*$ of $X$ by $(x^*_k)_{k=1}^n$, i.e. $x^*_i(x_k) = \delta_{ik}$. Then $(S(x_j))_{j \in J(m, n)}$ is a basis of $\otimes^m X$ and $(|j|S(x^*_j))_{j \in J(m, n)}$ is its orthogonal basis in $\otimes^m X^*$. Moreover, we have

$$S(x_i) = \frac{1}{|i|} \sum_{j \in [i]} x_j \text{ for all } i \in M(m, n).$$

There is a one-to-one correspondence between $J(m, n)$ and $\Lambda(m, n) = \{\alpha \in \mathbb{N}_0^n : |\alpha| = m\}$: If $j \in J(m, n)$ there is an associated multi-index $\alpha$ given by $\alpha_r = |\{k : j_k = r\}|$ (i.e. $\alpha_1$ is the number of 1’s in $j$, $\alpha_2$ is the number of 2’s, …,), and conversely, if $\alpha \in \Lambda(m, n)$, then the associated index is given by $j = (1, \alpha_1, 1, 2, \alpha_2, 2, \ldots) \in J(m, n)$. We have

$$|j| = m!/\alpha!. $$

Moreover, identifying $z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m} = S(e_j)$ we have

$$\chi_{\text{mon}}(P^{(m)}X) = \chi((S(e_j))_{j \in J(m, n)} : \otimes^m X^*).$$

Finally we mention the following isometric equalities which will be used frequently: For every finite dimensional Banach space $X$ we have

$$\otimes^m X^* = (\otimes^m X)^* \text{ and } \otimes^{m,s} X^* = (\otimes^m_{\varepsilon} X)^*,$$

as well as the identifications

$$\otimes^m X^* = \mathcal{L}(m X), \quad (x_1^* \otimes \cdots \otimes x_m^*) \sim [x_1 \otimes \cdots \otimes x_m \sim \prod_k x_k^*(x_k)],$$

$$\otimes^{m,s} X^* = \mathcal{L}_s(m X) = P^{(m)}X, \quad x^* \otimes \cdots \otimes x^* \sim [x \sim x^*(x)^m].$$
3. A FUNDAMENTAL ESTIMATE AND THE PROOF OF THEOREM 1.1

Recall that the 1-summing norm of a linear operator $T : X \to Y$ (between finite dimensional Banach spaces) is given by

$$\pi_1(T) := \sup \left\{ \sum_{i=1}^n \|Tx_i\| : \sum_{i=1}^n \lambda_i x_i \leq 1, n \in \mathbb{N}, |\lambda_i| \leq 1 \right\};$$

it is well known that

$$\pi_1(T) = \sup_n \|\text{id} \otimes T : \ell_1^n \otimes_\varepsilon X \to \ell_1^n \otimes_\pi Y = \ell_1^n(Y)\|$$

(see e.g. [12] or [16]). Define for $m$ and $n$ the canonical mapping

$$T : \mathcal{P}(\ell_\infty^m) \to \ell_2^{(n+m-1)\setminus(n-1)}$$

The fundamental tool of the whole paper is an estimate for the 1-summing norm of $T$. The proof is modelled along the proof of Theorem 3.2 from the phd-thesis of F. Bayart [2] which itself is based on a hypercontractivity result of A. Bonami [7].

**Lemma 3.1.** For each $m$ and $n$ the operator defined in (3.2) satisfies

$$\pi_1(T : \mathcal{P}(\ell_\infty^m) \to \ell_2^{(n+m-1)\setminus(n-1)}) \leq \sqrt{2^m}.$$ 

**Proof.** Let $\mu$ the normalized Lebesgue measure on the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\mu^n := \otimes_{k=1}^n \mu$ the product measure on the $n$-dimensional torus $\mathbb{T}^n$. It is well known that the $\pi_1$-norm of the canonical inclusion $L_\infty(\mu^n) \hookrightarrow L_1(\mu^n)$ equals 1 (see e.g. [12] or [16]). Since $\mathcal{P}(\ell_\infty^m)$ is an isometric subspace of $L_\infty(\mu^n)$ (maximum modulus theorem), it remains to show that for every $m$-homogeneous polynomial $P(z) = \sum_{|\alpha| = m} a_\alpha z^\alpha$ on $\mathbb{C}^n$

$$\left(\sum_{|\alpha| = m} |a_\alpha|^2\right)^{\frac{1}{2}} = \left\| \sum_{|\alpha| = m} a_\alpha z^\alpha \right\|_{L_2(\mu^n)} \leq \sqrt{2^m} \left\| \sum_{|\alpha| = m} a_\alpha z^\alpha \right\|_{L_1(\mu^n)}$$

(the first equality is a consequence of the orthogonality of the monomials in $L_2(\mu^n)$). Now we follow precisely the proof of F. Bayart [2, Theorem 3.2].

A result of A. Bonami [7, Theorem III 7] states that for every polynomial

$$\sum_{\nu=0}^m a_\nu z^\nu$$

in one complex variable

$$\left\| \sum_{\nu=0}^m \frac{1}{\sqrt{2}} a_\nu z^\nu \right\|_{L_2(\mu)} \leq \left\| \sum_{\nu=0}^m a_\nu z^\nu \right\|_{L_1(\mu)}.$$
But then we conclude with the continuous Minkowski inequality that

$$\frac{1}{\sqrt{2^m}} \left( \int \left| \sum_{|\alpha|=m} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} \right|^2 d\mu^n(z) \right)^{\frac{1}{2}}$$

$$= \left( \int \int \left| P\left(\frac{z_1}{\sqrt{2}}, \cdots, \frac{z_n}{\sqrt{2}}\right) \right|^2 d\mu(z_n) d\mu^{n-1}(z_1, \cdots, z_{n-1}) \right)^{\frac{1}{2}}$$

$$\leq \left( \int \left( \int \left| P\left(\frac{z_1}{\sqrt{2}}, \cdots, \frac{z_{n-1}}{\sqrt{2}}, z_n\right) \right|^2 d\mu(z_n) \right)^{\frac{1}{2}} d\mu^{n-1}(z_1, \cdots, z_{n-1}) \right)^{\frac{1}{2}}$$

$$\leq \int \left( \int \left| P\left(\frac{z_1}{\sqrt{2}}, \cdots, \frac{z_{n-1}}{\sqrt{2}}, z_n\right) \right|^2 d\mu(z_n) \right)^{\frac{1}{2}} d\mu(z_n).$$

The same argument applied to the other coordinates $z_{n-1}, \cdots, z_1$ gives then as desired

$$\frac{1}{\sqrt{2^m}} \left( \int \left| P(z) \right|^2 d\mu^n(z) \right)^{\frac{1}{2}} \leq \int \left| P(z) \right| d\mu^n(z).$$

□

For the proof of Theorem 1.1 we will need another lemma due to Blei [3]:

For all families $(c_i)_{i \in M(m,n)}$ of complex numbers

$$(3.3) \quad \left( \sum_{i \in M(m,n)} |c_i|^{2m+1} \right)^{\frac{1}{2m+1}} \leq \prod_{1 \leq k \leq m} \left( \sum_{i \in M(m-1,n)} \left( \sum_{i^k \in M(m-1,n)} |c_i|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{m}};$$

here the following notation is used

$$\sum_{i^k \in M(m-1,n)} := \sum_{i_1, \cdots, i_{k-1}, i_{k+1}, \cdots, i_m = 1}^{n}$$

Finally we are ready to give the proof of Theorem 1.1: Again we use the representation

$$\mathcal{P}^{(m \ell_\infty^n)} = \otimes_{\ell_\infty^n}^{m,s} \ell_1^n.$$  

Step 1. Define

$$I : \otimes_{\ell_\infty^n}^{m,s} \ell_1^n, r_i = r_j \text{ for } [i] = [j] \quad \rightarrow \quad \ell_2(M(m,n)).$$
We show that \( \pi_1(I) \leq \sqrt{2}^m \). Indeed, from the preceding lemma and (2.2) we get that the \( \pi_1 \)-norm of the map

\[
T : \bigotimes_\varepsilon^{m,s} \ell_1^n \rightarrow \ell_2(J(m,n))
\]

is \( \leq \sqrt{2}^m \). Note that for

\[
\sum_{i \in M(m,n)} r_i e_i \in \bigotimes_\varepsilon^{m,s} \ell_1^n \quad \text{with} \quad r_i = r_j \quad \text{for} \quad [i] = [j],
\]

we have (see 2.1)

\[
\sum_{i \in M(m,n)} r_i e_i = \sum_{j \in J(m,n)} [j] r_j S(e_j),
\]

hence the \( \pi_1 \)-norm of (the same) map

\[
K : \bigotimes_\varepsilon^{m,s} \ell_1^n \rightarrow \ell_2(J(m,n))
\]

is \( \leq \sqrt{2}^m \). Now consider

\[
J : \ell_2(J(m,n)) \rightarrow \ell_2(M(m,n))
\]

\[
(\lambda_j)_{j \in J(m,n)} \mapsto \left( \frac{\lambda_j}{\sqrt{|j|}} \right)_{j \in J(m,n)}.
\]

Then \( J \) is an isometry. Since \( I = J \circ K \) we obtain as desired \( \pi_1(I) \leq \sqrt{2}^m \).

Step 2. We show that

\[
\sum_{k=1}^n \left( \sum_{i^k \in M(m,n)} \left( \sqrt{|(i_{0}, i_{1}, \ldots, i_{m})|} |\lambda_{(i_{0}, i_{1}, \ldots, i_{m})}| \right)^2 \right)^{1/2}
\]

\[
\leq \sqrt{2}^m \sqrt{m + 1} \varepsilon \left( \sum_{i \in M(m+1,n)} \lambda_i e_i, \right)
\]

where \( 0 \leq k \leq m \) and \((\lambda_i)_{i \in M(m+1,n)}\) is a family of complex numbers for which \( \lambda_{(i_0, i_1, \ldots, i_m)} = \lambda_{(j_0, j_1, \ldots, j_m)} \) for \([i_0, i_1, \ldots, i_m] = [j_0, j_1, \ldots, j_m]\). From now on we denote \((i_{0}, i_{1}, \ldots, i_{m}) =: (i_0, i) \in M(m+1,n)\), and have hence \( \lambda_{(i_0, i)} = \lambda_{(k_0, k)} \) if \([i_0, i] = [(k_0, k)]\). If we consider \( \bigotimes_\varepsilon^{m+1,s} \ell_1^n \) as a subspace of \( \ell^n \otimes_\varepsilon \bigotimes_\varepsilon^{m,s} \ell_1^n \), then Step 1 and (3.1) imply that the (operator) norm of the mapping

\[
\bigotimes_\varepsilon^{m+1,s} \rightarrow \ell_1^n \otimes_\varepsilon \bigotimes_\varepsilon^{m,s} \ell_1^n \rightarrow \ell_1^n \otimes_\pi \ell_2(M(m, n)) = \ell_1^n (\ell_2(M(m, n))),
\]

which assigns to every

\[
\sum_{(i_0, i) \in M(m+1,n)} \lambda_{(i_0, i)} e_{(i_0, i)} = \sum_{i_0=1}^n e_{i_0} \otimes \sum_{i \in M(m,n)} \lambda_{(i_0, i)} e_i \in \ell_1^n \otimes_\varepsilon \bigotimes_\varepsilon^{m,s} \ell_1^n
\]

the element

\[
((\sqrt{|i|} \lambda_{(i_0, i)})_{i \in M(m,n)})_{1 \leq i_0 \leq n} \in \ell_1^n (\ell_2(M(m, n))),
\]
is $\leq \sqrt{2}^m$. But this means precisely that
\[
\sum_{i_0=1}^{n} \left( \sum_{i \in M(m,n)} \left( \sqrt{|i||\lambda_{(i_0,i)}}| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2}^m \epsilon \left( \sum_{(i_0,i) \in M(m+1,n)} \lambda_{(i_0,i)} e_{(i_0,i)} \right).
\]
Since
\[
\frac{|(i_0, i)|}{|i|} = \frac{m+1}{|\{\nu \in \{1, \ldots, m\} : i_\nu = i_0\}| + 1} \leq m + 1,
\]
we get
\[
\sum_{i_0=1}^{n} \left( \sum_{i \in M(m,n)} \left( \sqrt{|(i_0, i)||\lambda_{(i_0,i)}}| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2}^m \sqrt{m+1} \epsilon \left( \sum_{(i_0,i) \in M(m+1,n)} \lambda_{(i_0,i)} e_{(i_0,i)} \right).
\]
Clearly, we can apply this inequality also to the other coordinates $i_1, \ldots, i_m$, and hence we obtain as desired for all $0 \leq k \leq m$ and all $\lambda_i \in M(m+1,n)$ with $\lambda_i \lambda_j$ for $[i] = [j]$ that
\[
\sum_{i_k=1}^{n} \left( \sum_{i^k \in M(m,n)} \left( \sqrt{|i||\lambda_i|} \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2}^m \sqrt{m+1} \epsilon \left( \sum_{i \in M(m+1,n)} \lambda_i e_i \right).
\]
Step 3. Blei’s inequality (3.3) applied to preceding inequality from Step 2 (for $m-1$ instead of $m$) implies that
\[
\left( \sum_{i \in M(m,n)} \left( \sqrt{|i||\lambda_i|} \right)^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \sqrt{2}^{m-1} \sqrt{m} \epsilon \left( \sum_{i \in M(m,n)} \lambda_i e_i \right).
\]
Step 4. Finally we show for all families $(\lambda_j)_{j \in J(m,n)}$ of complex numbers that
\[
\left( \sum_{j \in J(m,n)} |\lambda_j|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \sqrt{2}^{m-1} \sqrt{m} \epsilon \left( \sum_{j \in J(m,n)} \lambda_j S(e_j) \right),
\]
and this finishes the proof of Theorem 1.1: From Step 3 applied to $\tilde{\lambda}_i := \frac{\lambda_i}{\sqrt{|i|}}$, $i \in [j]$, we obtain
\[
\left( \sum_{j \in J(m,n)} |\lambda_j|^{2m/(m+1)} \right)^{m+1} = \left( \sum_{j \in J(m,n)} \sum_{i \in [j]} |j|^{-1} \left( \sqrt{|j|} \frac{|\lambda_j|}{|j|} \right)^{2m/(m+1)} \right)^{m+1}
\]

\[
\leq \left( \sum_{j \in J(m,n)} \sum_{i \in [j]} \left( \sqrt{|i|} \frac{|\lambda_i|}{|i|} \right)^{2m/(m+1)} \right)^{m+1}
\]

\[
= \left( \sum_{i \in M(m,n)} \left( \sqrt{|i|} \frac{|\lambda_i|}{|i|} \right)^{2m/(m+1)} \right)^{m+1}
\]

\[
\leq \sqrt{2}^{m-1} \sqrt{m} \varepsilon \left( \sum_{i \in M(m,n)} \lambda_i e_i \right)
\]

\[
= \sqrt{2}^{m-1} \sqrt{m} \varepsilon \left( \sum_{j \in J(m,n)} \lambda_j S(e_j) \right)
\]

\[
\leq \sqrt{2}^{m-1} \sqrt{m} m^m \varepsilon \left( \sum_{j \in J(m,n)} \lambda_j S(e_j) \right)
\]

Since there obviously is some constant \( C \geq 1 \) such that \( \sqrt{2}^{m-1} \sqrt{m} m^m \varepsilon \leq C^m \) for all \( m \), the proof is complete. \( \square \)

4. **Gordon-Lewis and unconditional basis constants**

A Banach space invariant very closely related to unconditional basis constants is the Gordon-Lewis constant invented in the classical paper [21]. A Banach space \( X \) is said to have the Gordon-Lewis property if every 1-summing operator \( T : X \to \ell_2 \) allows a factorization \( T : X \overset{R}{\to} L_1(\mu) \overset{S}{\to} \ell_2 \) (\( \mu \) some measure, \( R \) and \( S \) operators). In this case, there is a constant \( c \geq 0 \) such that \( \gamma_1(T) := \inf \|R\|\|S\| \leq c \gamma_1(T) \) for all \( T : X \to \ell_2 \), and the best such \( c \) is called the Gordon-Lewis constant of \( X \) and denoted by \( gl(X) \).

We are going to use the obvious fact that for two Banach spaces \( X, Y \)

\[
(4.1) \quad gl(X) \leq d(X, Y) gl(Y).
\]

A fundamental tool for the study of unconditionality in Banach spaces is the Gordon-Lewis inequality from [21] (see also [16, 17.7]): For every unconditional basis \( (x_i) \) of a (complex) Banach space \( X \) we have

\[
(4.2) \quad gl(X) \leq 2 \chi((x_i)).
\]

We now follow a cycle of ideas invented in [26, 28] and which was later applied to spaces of \( m \)-homogeneous polynomials in [11]. Given a Banach space \( X_n = (C^n, \| \cdot \|) \) for which the \( e_k \)'s form a 1-unconditional basis, for Banach spaces \( \mathcal{P}(m X_n) \) the converse of the Gordon-Lewis inequality holds true; the main difference to [11, Theorem 1] is the hypercontractivity of the constant.
**Proposition 4.1.** There are constants $C \geq 1$ such that for each Banach space $X_n = (\mathbb{C}^n, \| \cdot \|)$ for which the $e_k$’s form a 1-unconditional basis, we have

$$\chi_{\text{mon}}(\mathcal{P}^m X_n) \leq C_m \text{gl}(\mathcal{P}^m X_n).$$

We prefer to prove this result in terms of symmetric tensor products; again we use the representation $\mathcal{P}^m X_n = \otimes_{\varepsilon}^{m,s} X_n$ (see (2.4)). In the following $\alpha$ will always be either the projective tensor norm $\pi$ or the injective tensor norm $\varepsilon$, and $\alpha_s$ stands either for the symmetric projective tensor norm $\pi_s$ or the symmetric injective tensor norm $\varepsilon_s$. Moreover, we put $\pi^* = \varepsilon$ and $\varepsilon^* = \pi$, as well as $\pi^*_s = \varepsilon_s$ and $\varepsilon^*_s = \pi_s$ (see (2.4)). The following result is a reformulation of the preceding one with a more precise constant.

**Proposition 4.2.** Let $X$ be a Banach space with the 1-unconditional basis $(x_k)_{k=1}^n$, and let $\alpha_s$ be either $\pi_s$ or $\varepsilon_s$. Then

$$\chi((S(x_i))_{i \in J(n,m)}; \otimes_{\alpha_s}^{m,s} X) \leq \left(\frac{m_m}{m!}\right)^2 2^m \text{gl}(\otimes_{\alpha_s}^{m,s} X).$$

Again we divide the proof into several steps. The first is [11, Lemma 4] which we repeat for the sake of completeness.

**Lemma 4.3.** Let $Y$ be a finite dimensional Banach space with a basis $(y_i)_{i=1}^n$ and orthogonal basis $(y_j^*)_{j=1}^n$. Suppose that there exist constants $M_1, M_2 \geq 1$ such that for every choice of $\lambda, \mu \in \mathbb{C}^n$ the diagonal mappings

$$D_\lambda : \sum_{i=1}^n a_i y_i \mapsto (\lambda_i a_i)_{i=1}^n, \quad D_\mu : \sum_{j=1}^n a_j y_j^* \mapsto (\mu_j a_j)_{j=1}^n$$

satisfy

$$\pi_1(D_\lambda) \leq M_1 \| \sum_{i=1}^n \lambda_i y_i^* \| y^*, \quad \pi_1(D_\mu) \leq M_2 \| \sum_{j=1}^n \mu_j y_j \| y.$$ 

Then

$$\chi((y_i)) \leq M_1 M_2 \text{gl}(Y).$$

The next four lemmata show how to control these diagonal operators in case of symmetric tensor products/spaces of $m$-homogeneous polynomials.

**Lemma 4.4.** Let $X$ be a Banach space, and $(x_k)_{k=1}^n$ a 1-unconditional basis. Then we have for all families $(\tilde{c}_j)_{j \in M(n,m)}$ of complex numbers that the diagonal operator

$$D_{\tilde{c}} : \otimes_{\alpha}^m X \rightarrow \mathcal{L}^{(m,h_m)}$$

has (operator) norm $\leq \alpha^* \left( \sum_{j \in M(n,m)} \| \tilde{c}_j x_j \| \right)$.

**Proof.** Define for $z = (z^{(1)}, \cdots, z^{(m)}) \in B_{\text{in}}^{m}$

$$T_z := \otimes_{k=1}^m T_z^{(k)} : \otimes_{\alpha}^m X \rightarrow \otimes_{\alpha}^m X$$



$$x_{j_1} \otimes \cdots \otimes x_{j_m} \mapsto (z^{(1)}_{j_1} x_{j_1}) \otimes \cdots \otimes (z^{(m)}_{j_m} x_{j_m}).$$
Since $(x_k)_{k=1}^n$ is a 1-unconditional basis, we know that $\|T_z\| \leq 1$. But then we obtain with the mapping property of $\alpha$ for all $z^{(1)}, \ldots, z^{(m)} \in B_{\ell^\infty}$ that

\[
\left| D_c \left( \sum_{j \in M(m,n)} \tilde{\lambda}_j x_j \right) (z^{(1)}, \ldots, z^{(m)}) \right| = \left| \sum_{j \in M(m,n)} \tilde{\lambda}_j x_j z^{(1)}_{j_1} \cdots z^{(m)}_{j_m} \right|
\]

\[
\leq \alpha \left( \sum_{j \in M(m,n)} (\tilde{\lambda}_j z^{(1)}_{j_1} \cdots z^{(m)}_{j_m}) x_j \right) \alpha^* \left( \sum_{j \in M(m,n)} \tilde{c}_j x_j^* \right)
\]

\[
= \alpha \left( T_z \left( \sum_{j \in M(m,n)} \tilde{\lambda}_j x_j \right) \right) \alpha^* \left( \sum_{j \in M(m,n)} \tilde{c}_j x_j^* \right)
\]

\[
\leq \alpha \left( \sum_{j \in M(m,n)} \tilde{\lambda}_j x_j \right) \alpha^* \left( \sum_{j \in M(m,n)} \tilde{c}_j x_j^* \right),
\]

which clearly implies as desired that $\|D_c\| \leq \alpha^* \left( \sum_{j \in M(m,n)} \tilde{c}_j x_j^* \right).$ □

We proceed with a symmetric version of this lemma.

**Lemma 4.5.** Let $X$ be a Banach space with a 1-unconditional basis $(x_k)_{k=1}^n$. Then for every family $(c_i)_{i \in J(m,n)}$ of complex numbers the diagonal operator

\[
D_c : \bigotimes_{\alpha_s}^{m,s} X \rightarrow \mathcal{L}_{s}^{(m,\ell^\infty)}(S(x_i)) \rightarrow \{ (z^{(1)}, \ldots, z^{(m)}), \mapsto c_i \frac{1}{|i|} \sum_{j \in [i]} z^{(1)}_{j_1} \cdots z^{(m)}_{j_m} \}
\]

has norm $\leq \frac{m^m}{m!} \alpha_s^* \left( \sum_{i \in J(m,n)} c_i |i| S(x_i^*) \right)$.
Proof. Take \( \sum_{i \in J(m,n)} \lambda_i S(x_i) \in \otimes^{m,s} X \), and apply the preceding Lemma to \( \tilde{\lambda}_j := \lambda_i |i| \) and \( \tilde{c}_j := c_i, j \in [i] \). Then

\[
\left| \left[ D_c \left( \sum_{i \in J(m,n)} \lambda_i S(x_i) \right) \right] \left( z^{(1)}, \ldots, z^{(m)} \right) \right|
\]

\[
= \left| \sum_{i \in J(m,n)} \lambda_i c_i \frac{1}{|i|} \sum_{j \in [i]} z_{j_1}^{(1)} \cdots z_{j_m}^{(m)} \right|
\]

\[
\leq \alpha \left( \sum_{i \in J(m,n)} \sum_{j \in [i]} \tilde{\lambda}_i \tilde{c}_j x_j \right)^* \left( \sum_{i \in J(m,n)} \sum_{j \in [i]} \tilde{c}_j \tilde{\lambda}_i z_j^{(1)} \cdots z_j^{(m)} \right)
\]

\[
= \alpha \left( \sum_{i \in J(m,n)} \lambda_i S(x_i) \right)^* \left( \sum_{i \in J(m,n)} c_i \alpha_i \right)^* \left( \sum_{i \in J(m,n)} \lambda_i S(x_i) \right)
\]

\[
\leq \frac{m^m}{m!} \alpha_s \left( \sum_{i \in J(m,n)} \lambda_i S(x_i) \right)^* \left( \sum_{i \in J(m,n)} c_i \alpha_i \right),
\]

where the latter inequality follows from (2.2) and (2.3). \( \square \)

The last lemma needed for the proof of Proposition 4.2 is an immediate consequence of the preceding one and our fundamental estimate from Lemma 3.1.

**Lemma 4.6.** Let \((x_k)_{k=1}^n\) be a 1-unconditional basis of the Banach space \(X\). Then for every family \((c_i)_{i \in J(m,n)}\) of complex numbers the diagonal operator

\[
D_c : \otimes_{\alpha_s}^{m,s} X \rightarrow \ell_2 \left( J(m,n) \right)
\]

\[
\sum_{i \in J(m,n)} \lambda_i S(x_i) \rightarrow (\lambda_i c_i)_{i \in J(m,n)}
\]

satisfies

\[
\pi_1(D_c) \leq \frac{m^m}{m!} \sqrt{2} \alpha_s \left( \sum_{i \in J(m,n)} c_i \alpha_i \right).
\]

Note now finally that Lemma 4.3, the preceding Lemma 4.6 and Lemma 2.1 together yield

\[
\chi \left( (S(x_i))_{i \in J(m,n)} ; \otimes_{\alpha_s}^{m,s} X \right) \leq \left( \frac{m^m}{m!} \right)^2 2^m \text{gl}(\otimes_{\alpha_s}^{m,s} X),
\]

which completes the proof of Proposition 4.2 (which was nothing else than a tensor product formulation of the main result of this section, Proposition 4.1).
5. GORDON-LEWIS CONSTANTS AND PROJECTION CONSTANTS

Recall that the projection constant of a finite dimensional Banach space $X$ is defined to be

$$\lambda(X) = \sup \{ \lambda(I(X), Z) : I : X \hookrightarrow Z \text{ an isometric embedding into } Z \} ,$$

where for a subspace $Y$ of a Banach space $Z$ the relative projection constant $\lambda(Y, Z)$ is the infimum of all $\|P\|$ taken with respect to all projections $P$ onto $Z$. We will use the well known estimates (see [30, 9.12]).

(5.1) $$\lambda(X) \leq \sqrt{\dim X} ,$$

and also the obvious fact that

(5.2) $$\lambda(X) \leq d(X, Y) \lambda(Y) ,$$

The main purpose of this section is to prove the following proposition which in combination with Proposition 4.1 allows to estimate unconditional basis constants of symmetric tensor products/spaces of $m$- homogeneous polynomials.

**Proposition 5.1.** Let $X$ be a Banach space with a 1-unconditional basis $(x_k)_{k=1}^n$. Then for every $m \geq 2$ we have

1. $\text{gl}(\otimes_{\varepsilon}^{m,s} X) \leq 2\lambda(\otimes_{\varepsilon}^{m-1,s} X)$
2. $\text{gl}(\otimes_{\varepsilon}^{m,s} X) \leq 2(\frac{m}{m!})^2 \lambda(\otimes_{\varepsilon}^{m-1,s} X)$.

Note that the projection constant of the polynomials appears with degree $m - 1$ whereas the Gordon-Lewis constant is taken with respect to all polynomials of degree $m$. The trick which makes this possible is isolated in the following lemma.

**Lemma 5.2.** Let $X$ be a finite dimensional Banach space and $m \in \mathbb{N}$. Then

$$\text{gl}(\otimes_{\varepsilon}^{m+1,s} X) \leq \sup_{N} \text{gl}(X \otimes_{\varepsilon} \ell^N_\infty) \lambda(\otimes_{\varepsilon}^{m,s} X).$$

**Proof.** Step 1. Let $\varepsilon > 0$ be arbitrary. We map $\otimes_{\varepsilon}^{m,s} X$ onto a subspace $Y$ of $\ell^N_\infty$ with $d(\otimes_{\varepsilon}^{m,s} X, Y) \leq 1 + \varepsilon$ such that there is a projector $P : \ell^N_\infty \to \ell^N_\infty$ onto this subspace with $\|P\| \leq \lambda(Y) + \varepsilon$. Then

$id \otimes P : X \otimes_{\varepsilon} \ell^N_\infty \to X \otimes_{\varepsilon} Y$

is a projector with the same norm. Hence $\text{gl}(X \otimes_{\varepsilon} Y) \leq \|P\| \text{gl}(X \otimes_{\varepsilon} \ell^N_\infty)$. Since $\varepsilon > 0$ was arbitrary, we get

$$\text{gl}(X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m,s} X)) \leq \lambda(\otimes_{\varepsilon}^{m,s} X) \sup_{N} \text{gl}(X \otimes_{\varepsilon} \ell^N_\infty).$$

Step 2. Since the injective norm respects isometric subspaces, $X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m,s} X)$
is an isometric subspace of $X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m} X) = \otimes_{\varepsilon}^{m+1} X$. Because of

$$\otimes_{\varepsilon}^{m+1} X = \text{span}\{\otimes_{\varepsilon}^{m+1} x : x \in X\}$$

we see that $\otimes_{\varepsilon}^{m+1} X$ is an isometric subspace of $X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m} X)$. Consider now the norm 1 projection $S_{m+1} : \otimes_{\varepsilon}^{m+1} X \to \otimes_{\varepsilon}^{m+1} X$ onto $\otimes_{\varepsilon}^{m+1} X$ (see (2.2)). Clearly, if this map is restricted to $X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m} X)$, then we obtain a norm 1 projection $X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m} X) \to X \otimes_{\varepsilon} (\otimes_{\varepsilon}^{m} X)$ onto $\otimes_{\varepsilon}^{m+1} X$. This finally implies

$$\text{gl}(\otimes_{\varepsilon}^{m+1} X) \leq \text{gl}(X \otimes (\otimes_{\varepsilon}^{m} X))$$

which together with Step 1 leads to the conclusion. 

Now the proof of Proposition 5.1 is easy: The unconditional basis constant of $X \otimes_{\varepsilon} \ell_{N}^{1}$ is 1 (see e.g. [28, Lemma 5]), hence the Gordon-Lewis constant of this space is $\leq 2$ by (4.2). To get the first inequality we apply the preceding Lemma. For the second inequality recall that we have $d(\otimes_{\varepsilon}^{m} X, \otimes_{\varepsilon}^{m} X) \leq \frac{m^{m}}{m!}$ (see (2.2)). Hence we obtain from (4.1) and (5.2) that

$$\text{gl}(\otimes_{\varepsilon}^{m} X) \leq \frac{m^{m}}{m!} \text{gl}(\otimes_{\varepsilon}^{m} X)$$

We remark that we already here get an alternative proof of Theorem 1.2 in the case $p \geq 2$ (recall that this case was already proved on the basis of Theorem 1.1): By the propositions 4.2 and 5.1 as well as (5.1) and (2.1) we have that

$$\chi((S(x_{j}))_{j \in (m,n)}; \otimes_{\varepsilon}^{m} X) \leq C^{m} \text{gl}(\otimes_{\varepsilon}^{m} X) \leq C^{m}(1 + \frac{n}{m-1}) \frac{m}{m-1}.$$ 

Hence after identifying $\mathcal{P}(m \ell_{p}) = \otimes_{\varepsilon}^{m} \ell_{q}^{m}$ with $1/p + 1/q = 1$, we conclude

$$\chi_{\text{mon}}(\mathcal{P}(m \ell_{p})^{m}) \leq C^{m} \text{gl}(\mathcal{P}(m-1) \ell_{p}^{m}) \leq C^{m}(1 + \frac{n}{m-1}) \frac{m-1}{m-1},$$

the statement of Theorem 1.2 in the case $p \geq 2$. But for $p \leq 2$ this estimate has to be improved, and we established this in the two final results of this section.
Lemma 5.3. For a given Banach space \( X := (\mathbb{C}^n, \| \cdot \|) \) define for each \( |\alpha| = m \)

\[ d_\alpha := \sup \{ |a_\alpha| : \sup_{z \in B_X} |\sum_{|\beta| = m} a_\beta z^\beta| \leq 1 \}. \]

Then

\[ \lambda(\mathcal{P}(mX)) \leq \sup_{\|z\| \leq 1} \sum_{|\alpha| = m} d_\alpha |z^\alpha| =: p(X). \]

Proof. Consider \( \mathcal{P}(mX) \) as a subspace of \( \ell_\infty(B_X) \). We construct a projector \( P : \ell_\infty(B_X) \to \mathcal{P}(mX) \) with norm \( \leq p(X) \). We use that the functionals \( k_\alpha : \mathcal{P}(mX) \to \mathbb{C}, \sum_{|\beta|=m} a_\beta z^\beta \mapsto a_\alpha \) have norm \( d_\alpha \). With the Hahn-Banach theorem we extend them to \( K_\alpha : \ell_\infty(B_X) \to \mathbb{C} \) with the same norm. Let now

\[ P : \ell_\infty(B_X) \to \mathcal{P}(mX), \ f \mapsto \sum_{|\alpha|=m} K_\alpha(f) z^\alpha. \]

Then \( P \) is a projector on \( \mathcal{P}(mX) \) and we have

\[
\begin{align*}
\|P(f)\|_\infty &= \|P(f)\|_{\mathcal{P}(mX)} = \sup_{\|z\|_X \leq 1} \sum_{|\alpha|=m} K_\alpha(f) z^\alpha |z^\alpha| \\
&\leq \sup_{\|z\|_X \leq 1} \sum_{|\alpha|=m} |K_\alpha(f)| |z^\alpha| \\
&\leq \sup_{\|z\|_X \leq 1} \sum_{|\alpha|=m} d_\alpha \|f\|_\infty |z^\alpha| \leq \|f\|_\infty p(X).
\end{align*}
\]

\( \square \)

We now follow the proof of [13, Lemma 3.3] in order to get the needed estimate for the projection constant of \( \mathcal{P}(m\ell_p^n) \).

Proposition 5.4. There is a \( C \geq 1 \) such that for all \( 1 \leq p \leq \infty \), all \( n, m \in \mathbb{N} \) we have

\[ \lambda(\mathcal{P}(m\ell_p^n)) \leq C m^{1 - \frac{1}{\min\{p,2\}}} \left(1 + \frac{n}{m}\right)^{m \left(1 - \frac{1}{\min\{p,2\}}\right)}. \]

Proof. The case \( p \geq 2 \) was already proved, see (5) or the remark after (1.4). For \( 1 \leq p \leq 2 \) we apply the preceding Lemma to \( X = \ell_p^n \). From the proof of [13, Lemma 3.3] we know that \( d_\alpha \leq e^{m/p} \left(\frac{m}{|\alpha|}\right)^{1/p} \), and hence by Hölder’s
inequality (with $\frac{1}{p} + \frac{1}{q} = 1$) and (2.1) for $z \in \mathbb{C}^n$

$$\sum_{|\alpha| = m} d_\alpha |z^\alpha| \leq e^{\frac{m}{\theta}} \sum_{|\alpha| = m} \left( \frac{m!}{\alpha!} \right)^{\frac{1}{\theta}} |z^\alpha|$$

$$\leq e^{\frac{m}{\theta}} \left( \sum_{|\alpha| = m} 1 \right)^{\frac{1}{q}} \left( \sum_{|\alpha| = m} \frac{m!}{\alpha!} (|z_1|^p, \cdots, |z_n|^p)^{\frac{m}{p}} \right)^{\frac{1}{q}}$$

$$\leq e^{\frac{m}{\theta}} c^m \frac{m}{m} \left(1 + \frac{n}{m}\right) \left( \sum_{k=1}^{n} |z_k|^p \right)^{\frac{m}{p}}$$

$$= e^{\frac{m}{\theta}} c^m \frac{m}{m} \left(1 + \frac{n}{m}\right) \frac{m}{m} \|z\|_{\ell_p^m}.$$ 

Thus

$$\lambda(\mathcal{P}(m_{\ell_p^m})) \leq e^{\frac{m}{\theta}} c^m \frac{m}{m} \left(1 + \frac{n}{m}\right)^{\frac{m}{m}}.$$

\[\square\]

6. Proofs of Theorem 1.2 and Theorem 1.3

All we have to do is to collect the results already shown in the preceding sections.

(1) **Proof of Theorem 1.2**: Fix $m, n$ and $1 \leq p \leq \infty$. We again identify $\mathcal{P}(m_{\ell_p^m}) = \otimes_{\mathfrak{S}_m} \ell_p^m$, where $1/p + 1/p = 1$. From Proposition 4.2 we know that

$$\chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^m})) \leq C_{m} \chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^m})),$$

hence we conclude from Proposition 5.1 that

$$\chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^m})) \leq C_{m} \lambda(\mathcal{P}(m_{\ell_p^m})),$$

and then finally by Proposition 5.4

$$\chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^m})) \leq C_{m} \left(1 + \frac{n}{m}\right)^{(m-1)(1-\frac{1}{\min(p,2)})};$$

(here the absolute constant $C$ is of course changing step by step). This gives Theorem 1.2. \[\square\]

(2) **Proof of Theorem 1.3**: Fix some Banach space $\ell_p^n$. From (1.6) we know that

$$\frac{1}{3 \sup_m \chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^n}))^{\frac{1}{m}}} \leq K(B_{\ell_p^n}),$$

hence by Theorem 1.2 there is some absolute constant $C \geq 1$ such that

$$\chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^n}))^{\frac{1}{m}} \leq C$$

whenever $m \geq n$

and

$$\chi_{\text{mon}}(\mathcal{P}(m_{\ell_p^n}))^{\frac{1}{m}} \leq C \left(\frac{m}{m}^{\frac{m-1}{m}}(1-\frac{1}{\min(p,2)})\right)$$

whenever $n > m$. 

Minimizing $mn^{1/m}$ for $n > m$ then proves Theorem 1.3. □

We finished with an improved definite version of [13, Remark 1] which in the context of unconditionality quantifies the “gap” between symmetric and full injective tensor products of $\ell_p$’s.

Remark 1. There is a constant $C > 0$ such that the following estimates hold for each $1 \leq p \leq \infty$ and $n$:

$$\frac{1}{n} \left( \frac{n}{\log n} \right)^{\max(1, \frac{1}{p})} \leq \sup_m \chi_{\text{mon}}(\otimes_s^{m,s} \ell_p^s)^{1/m} \leq C \left( \frac{n}{\log n} \right)^{\frac{1}{\max(1, \frac{1}{p})}}$$

$$\frac{1}{n} n^{\max(1, \frac{1}{p})} \leq \sup_m \chi_{\text{mon}}(\otimes_s^{m,s} \ell_p^s)^{1/m} \leq C n^{\frac{1}{\max(1, \frac{1}{p})}}.$$

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