Non-equilibrium Thermodynamics of Spacetime

Christopher Eling¹, Raf Guedens², and Ted Jacobson¹

¹ Department of Physics, University of Maryland
College Park, MD 20742-4111 USA and
² High Energy and Elementary Particle Division, University of Crete,
P.O. Box 2208, GR-710 03 Heraklion, Crete, GREECE

It has previously been shown that the Einstein equation can be derived from the requirement that the Clausius relation \( dS = \delta Q/T \) hold for all local acceleration horizons through each spacetime point, where \( dS \) is one quarter the horizon area change in Planck units, and \( \delta Q \) and \( T \) are the energy flux across the horizon and Unruh temperature seen by an accelerating observer just inside the horizon. Here we show that a curvature correction to the entropy that is polynomial in the Ricci scalar requires a non-equilibrium treatment. The corresponding field equation is derived from the entropy balance relation \( dS = \delta Q/T + dS \), where \( dS \) is a bulk viscosity entropy production term that we determine by imposing energy-momentum conservation. Entropy production can also be included in pure Einstein theory by allowing for shear viscosity of the horizon.

I. INTRODUCTION

A profound connection between gravitation and thermodynamics was first suggested by the discovery in the 1970’s of black hole entropy [¹], the four laws of classical black hole mechanics [²], and Hawking radiation [³]. But it is rather mysterious that the Einstein equation, a hyperbolic second order partial differential equation for the spacetime metric, has a predisposition to thermodynamic behavior. A decade ago one of us proposed [⁴] to explain this connection by reversing the logic, using the assumed proportionality of entropy and horizon area for all local acceleration horizons (called there local Rindler horizons) to derive the Einstein equation as an equilibrium equation of state. This derivation suggests the idea that gravitation on a macroscopic scale is a manifestation of thermodynamics of the vacuum.

Does this thermodynamic derivation of the Einstein equation indicate something deep, or is it a case of “assuming the answer”, or a superficial consequence of the assumptions, or perhaps just an accident? We address this question here by investigating whether and how the derivation can be generalized to allow for the higher curvature terms expected in the field equation on the grounds of effective field theory [⁵]. One might guess that this could be done simply by allowing for curvature to enter the ansatz for horizon entropy. The purpose of this article is to investigate whether this is so. We find that entropy dependence on the Ricci scalar can indeed be accommodated, but it requires a change of setting from equilibrium to non-equilibrium thermodynamics.

We begin by reviewing the two hypotheses on which the derivation of the Einstein equation of state given in Ref. [⁶] is based. Next we present that derivation, and then generalize it to allow for dependence of the entropy on the Ricci scalar. We end with several comments.

The motivating idea is that the origin of the thermodynamic behavior of black holes is to be traced to the thermal nature of the Minkowski vacuum. The vacuum is the ground state of the generator of time translations, but it is thermal with respect to any generator of Lorentz boosts. More precisely, restricted to the “Rindler wedge” \( x > |t| \) (using Minkowski coordinates) the vacuum density matrix for a relativistic quantum field has the form of the canonical ensemble (Gibbs state) \( \rho = Z^{-1} \exp(-H_B/T) \), where \( H_B \) is the boost Hamiltonian and the “temperature” is \( T = \hbar/2\pi \). This “temperature” does not have dimensions of energy, because the boost \( H_B \) generates translations of a dimensionless hyperbolic angle, rather than of time. When re-scaled to generate proper time translations on the world-line of a uniformly accelerated observer this becomes the Unruh temperature \( T_U = \hbar a/2\pi \), where \( a \) is the acceleration.

The past boundary \( x = t < 0 \) of the Rindler wedge is a lightlike hyperplane in spacetime, forming a causal boundary or horizon for the past of the “bifurcation plane” \( x = t = 0 \). The state behind the horizon is hidden to outside observers with access only to a spatial slice bounded by this plane. For such observers the relevant state is the density matrix \( \rho \). The entropy of \( \rho \) is infinite in standard quantum field theory, but when UV regulated it is proportional to the area of the bifurcation plane and depends on the number and nature of the quantum fields [⁶]. The similarity to Bekenstein-Hawking black hole entropy, which is universally given by the horizon area divided by \( 4\hbar G \), motivates our first hypothesis: we suppose that in quantum spacetime—whatever that is—there is a universal entropy density \( \alpha \) per unit horizon area. We imagine that \( \alpha \) indeed depends on the number and nature of quantum fields, if any such freedom exists in the underlying fundamental theory.

Our second hypothesis concerns the relation between this entropy and the flux of boost energy across the Rindler horizon. Under a small perturbation of any Gibbs state at temperature \( T \), the variation of the entropy is related to the variation of the mean energy by \( \delta S = \delta E/T \). When the energy is “heat” this identity expresses the Clausius relation \( dS = \delta Q/T \). The detailed nature of energy that flows into a Rindler wedge cannot be examined by the outside observers, hence it
can be considered as them as heat has flowed into
the thermal system behind the horizon. This motivates
our second hypothesis: the Clausius relation holds for all
local causal horizons (defined precisely below) in a suffi-
ciently small neighborhood of the bifurcation plane, with
\( \delta Q \) interpreted as the mean flux of boost energy across
the horizon and \( dS \) as the difference of area for the wedge
with and without the boost energy flux. This hypothesis
is inconsistent with the assumption of a fixed, flat spacetime,
since a Rindler horizon has fixed area, but it is
consistent with the horizon-focusing effects of spacetime
curvature provided the Einstein equation holds \( e \).

We define a local causal horizon at a point \( p \) as follows:
choose a spacelike 2-surface patch \( B \) including \( p \),
and all subsequent integrals are taken over a short seg-
ment of a thin pencil of horizon generators centered
on the one that terminates at \( p \). Using the relation
\( \chi^a = -\lambda k^a \) (which holds on the generator through \( p \) up to order \( O(x^3) \)) and \( T = h/2\pi \) we thus have
\[
\frac{\delta Q}{T} = (2\pi/h) \int T_{ab}k^a k^b(-\lambda) d\lambda d^2A.
\]  
(2)

The entropy change \( \delta S = \alpha \delta A \) is determined by the
area change of the horizon,
\[
\delta A = \int \theta d\lambda d^2A,
\]  
(3)

where \( \theta = d(ln d^2A)/d\lambda \) is the expansion of the congru-
ence of null geodesics generating the horizon. Using the
Raychaudhuri equation
\[
\frac{d\theta}{d\lambda} = \frac{1}{2} \theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b
\]  
(4)

and the assumed vanishing of \( \theta \) and \( \sigma_{ab} \) at \( p \) we have
\[
\theta = -\lambda R_{ab}k^a k^b + O(\lambda^2).
\]  
(5)

The entropy change is thus given to lowest order in \( \lambda \) by
\[
\delta S = \alpha \int R_{ab}k^a k^b(-\lambda) d\lambda d^2A.
\]  
(6)

If we now require that \( \delta S = \delta Q/T \) hold for all local
Rindler horizons through all points \( p \), we infer that the
integrands of (2) and (6) must match for all null vectors
\( k^a \). The integrands are both first order in \( \lambda \), and equality
of the coefficients of \( \lambda \) implies the relation
\[
R_{ab} + \Phi g_{ab} = (2\pi/h\alpha) T_{ab}
\]  
(7)

where \( \Phi \) is a so far undetermined function. To deter-
mine \( \Phi \) we require that the matter stress tensor is
divergence free, corresponding to the usual local conser-
vation of matter energy. Taking the divergence of both sides of (7) and using the contracted Bianchi identity
\( R_{ab}^a = \frac{1}{3} R_b \) we then find that \( \Phi = -\frac{1}{2} R - \Lambda \),
where \( \Lambda \) is a constant. Therefore (7) corresponds to the Ein-
stein equation \( R_{ab} - \frac{1}{2} R_{ab} - \Lambda_{ab} = 8\pi GT_{ab} \) with (un-
determined) cosmological constant \( \Lambda \) and with Newton’s
constant determined by the universal entropy density \( \alpha \),
\[
G = (4\hbar\alpha)^{-1}.
\]  
(8)

Note that the entropy area density is thus universally
\( (4\hbar G)^{-1} \), no matter what is the nature and number
of quantum fields, in agreement with the Bekenstein-
Hawking black hole entropy. This completes our review
of Ref. [8].

We now seek the thermodynamic equation of state if
the entropy density is taken to be \( \alpha \) times a function
\( f(R) = 1 + O(R) \) of the Ricci scalar, instead of a constant
\( \alpha \) as before. In this case the entropy change is given in
analogy with (3) by
\[
\delta S = \alpha \int (\theta f + \dot{f}) d\lambda d^2A,
\]  
(9)
where the overdot signifies derivative with respect to $\lambda$. If the expansion $\theta$ vanishes at $p$ then the integrand of (9) at $p$ is $\dot{\hat{f}} = f'(R)k^aR_{,a}$, which is generally non-zero. This cannot match the $\delta Q/T$ integrand which is of order $\lambda$. Thus the Clausius relation implies that $\theta(p)$ must be non-vanishing, so as to cancel off the derivative of $f(12)$. That is, we must have

$$ (\theta f + \dot{\hat{f}})(p) = 0. \tag{10} $$

This means that the causal horizon must be defined as the boundary of the past of a 2-surface $\mathcal{B}$ that is warped at $p$ such that (10) is satisfied. This does not coincide with the approximate Killing horizon as closely as when the expansion vanishes. Nevertheless, the approximate Killing vector is still related to the tangent of the horizon generator through $p$ by $\chi^a = -\lambda k^a$, up to the $O(x^3)$ ambiguity of $\chi^a$.

Since the area of the horizon is changing at $p$, it seems at first that the “system” does not approach an equilibrium state at $p$, even though the entropy is instantaneously stationary there. However, the relevant notion of time here is the Killing flow. The relation between the affine parameter $\lambda$ and Killing parameter $v$ on a Rindler horizon is $\lambda = -\exp(-v)$, so the point $p$ occurs at infinite Killing time. Even if $\theta(p) \neq 0$, the rate of change of area with respect to Killing time vanishes like $\sim \exp(-v)$ as $p$ is approached. This can be considered an approach to equilibrium. In the special case $\theta(p) = 0$ considered previously, the expansion vanishes at twice this rate, i.e. as $\sim \exp(-2v)$. The slower decay rate in the general case suggests that equilibrium thermodynamics may not apply, so that the Clausius relation may not hold. Instead we may have $dS > \delta Q/T$, or more precisely the entropy balance relation

$$ dS = \delta Q/T + d_iS, \tag{11} $$

where $d_iS$ is the entropy developed internally in the system as a result of being out of equilibrium. The entropy production rate vanishes at $p$, since that is an equilibrium point, so we expect the rate is of order $\lambda$.

To extract the $O(\lambda)$ term in the integrand of (9) we differentiate with respect to $\lambda$ and use (10),

$$ \frac{d}{d\lambda}(\theta f + \dot{\hat{f}})_{\lambda=0} = \dot{\theta} f - f^{-1} \dot{f}^2 + \dot{\hat{f}}. \tag{12} $$

Using the Raychaudhuri equation (4) and the geodesic equation $k_a k^b = 0$ this takes the form

$$ -k^a k^b (f R_{ab} - f_{,ab} + f^{-1} f_{,a} f_{,b}) - \frac{1}{2} f \theta^2. \tag{13} $$

Were there no entropy production $d_iS$, the Clausius relation would imply that (13) must be equal to the coefficient of $\lambda$ in the heat flux integrand of (12) for all null vectors $k^a$. It would follow that in place of (10) we have

$$ f R_{ab} - f_{,ab} + \frac{3}{2} f^{-1} f_{,a} f_{,b} + \Psi g_{ab} = (2\pi/h\alpha) T_{ab}, \tag{14} $$

where (10) has been used to re-express the $\theta^2$ term and $\Psi$ is a so far undetermined function. We now show that this is inconsistent with energy conservation.

We require, as before, that the matter stress tensor is divergence free, so the divergence of the left hand side of (14) must vanish. Using the contracted Bianchi identity, the commutator of covariant derivatives $2\nabla^c [_{ab}] = R_{abc} c^{,d}$, and defining $\mathcal{L}$ by $f = d\mathcal{L}/dR$, we find

$$ (f R_{ab} - f_{,ab})^{,a} = \left( \frac{1}{2} \mathcal{L} - f \right)_{,b}. \tag{15} $$

Thus we must have

$$ \Psi = \square f - \frac{1}{2} \mathcal{L} - \Theta, \tag{16} $$

where the gradient of $\Theta$ matches the divergence of the remaining term in (14),

$$ \Theta_{,b} = \frac{3}{2} f f_{,a} f_{,b})^{,a}. \tag{17} $$

This reveals a contradiction, however, since the right hand side of (17) is generally not the gradient of a scalar.

We propose that this contradiction with energy-momentum conservation is resolved by the entropy production term $d_iS$ in (11). Examination of (13) shows that the problematic term would be canceled if we set

$$ d_iS = \int \sigma d\lambda d^2A \tag{18} $$

with entropy production density

$$ \sigma = -\frac{3}{2} \alpha f^{-1} \dot{f}^2 \lambda = -\frac{3}{2} \alpha f \theta^2 \lambda \tag{19} $$

(using (10)). In terms of the expansion with respect to Killing parameter, $\theta = \theta(d\lambda/dv)$, we have

$$ \sigma d\lambda = \frac{3}{2} \alpha f \theta^2 d\lambda. \tag{20} $$

This is just like the entropy production term for a fluid at temperature $T$ due to a bulk viscosity $\eta = (3/2)\alpha f T$. Putting $T$ equal to the boost temperature $h/2\pi$ yields $\eta = 3\alpha f/4\pi$. With this for $d_iS$, equation (11) at $O(\lambda)$ implies the equation of state

$$ f R_{ab} - f_{,ab} + (\square f - \frac{1}{2} \mathcal{L}) g_{ab} = (2\pi/h\alpha)T_{ab}. \tag{21} $$

This coincides with the equation of motion arising from the Lagrangian $(\alpha f/4\pi)\mathcal{L}(R)$ for which the entropy density of a stationary black hole horizon is $\alpha f(R)$. The thermodynamic equation of state is therefore again consistent with the Lagrangian field equation, as in the pure GR case. We conclude with a number of remarks.

1. Given an entropy functional of the macroscopic variables of an ordinary thermodynamic system, one can normally derive the equation of state from the Clausius relation together with the first law of thermodynamics. In our case the first law was not explicitly invoked. However, to fix the trace part of the field equation we required that the energy-momentum tensor of matter is divergence...
2. It is common in near-equilibrium thermodynamics to deduce the general form of entropy production terms quadratic in the gradients of state variables, but the coefficients of those terms are phenomenological and depend on details of the microphysics \[\ref{1}]. It may therefore seem puzzling that here we deduce also the bulk viscosity \(\eta\). However, \(\eta\) is precisely \(3\hbar/4\pi\) times the entropy density \(\alpha f\), and \(\alpha\) is purely phenomenological in our derivation.

3. In pure GR the bulk viscosity appears to become \(3\hbar\alpha/4\pi\), but this is not correct since \(\theta(p) = 0\), so there is no \(O(\lambda)\) entropy production term in the integrand of \(\ref{1}\). By contrast, for globally defined black hole horizons it has been shown \(\ref{1}\) that the bulk viscosity is negative and equal to \(-1/16\pi G = -\hbar\alpha/4\pi\).

4. In the presence of curvature contributions to the entropy, we had no choice but to use a non-equilibrium entropy balance relation, due to the nonzero expansion at \(p\). Could a non-equilibrium description be used voluntarily also for pure GR, where the entropy is just the area? We cannot allow for expansion at \(p\), since without the derivative of \(f\) to balance the expansion, there would be a first order term in the entropy change \(\delta S\) not matched by the heat flux term. But how about allowing for shear at \(p\)? We initially set the shear to zero on the grounds that it was required by equilibrium at \(p\). This is not valid however, since the shear defined with respect to Killing parameter would in any case vanish. The consequence of nonzero shear at \(p\) is to introduce into \(\ref{1}\) a term of the form \(\lambda\sigma_{ab}a^{ab}\). This additional term can be written in terms of derivatives of \(k^a\), which can be independently chosen at \(p\). The \(k^a\) parts of the Clausius relation imply the Einstein equation as above, but the \(\partial k^a\) part is satisfied only if the shear vanishes. However, nonzero shear at \(p\) is allowed if we include an internal entropy production term \(\alpha a^{ab}\sigma_{ab}\) corresponding \(\ref{1}\) at temperature \(T(=\hbar/2\pi)\) to a shear viscosity \(T\alpha/2 = \hbar\alpha/4\pi = 1/16\pi G\), just as for a black hole horizon \(\ref{1}\).

5. Under what conditions are the higher curvature terms meaningful in the thermodynamic interpretation? Let the entropy expansion coefficients \(\beta\) be defined by \(f(R) = 1 + \beta_1 R + \beta_2 R^2 + \cdots\), and let the curvature length scale at \(p\) be set by \(L_c\). Then \(\sigma_{ab}\) in \(\ref{1}\) is \(O(\beta_1/L_c^2)\) smaller than \(\sigma_{ab}\) and the \(O(R^2)\) part of \(fR_{ab}\) which are smaller than the Einstein term by the same factor.

Acknowledgments

We would like to thank J.R. Dorfman, S. Hayward, and A. Roura for helpful discussions. This research was supported in part by the NSF under grant PHY-0300710.

References

[1] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[2] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
[3] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)].
[4] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995) [arXiv:gr-qc/9504004].
[5] C. P. Burgess, Living Rev. Rel. 7, 5 (2004) [arXiv:gr-qc/0311082].
[6] J. J. Bisognano and E. H. Wichmann, J. Math. Phys. 16, 985 (1975).
[7] W. G. Unruh and N. Weiss, Phys. Rev. D 29, 1656 (1984).
[8] L. Bombelli, R. K. Koul, J. H. Lee and R. D. Sorkin, Phys. Rev. D 34, 373 (1986).
[9] S. R. de Groot and P. Mazur, Non-equilibrium Thermodynamics (North-Holland, 1962).
[10] See, for example, T. Jacobson, G. Kang and R. C. Myers, Phys. Rev. D 49, 6587 (1994) [arXiv:gr-qc/9312023].
[11] R. H. Price and K. S. Thorne, Phys. Rev. D 33, 915 (1986), and references therein.
[12] This point was missed in Ref. [1], where it was therefore incorrectly asserted that the derivation of the equation of state goes through without modification.