Determination of the principal coordinates in solving the problem of the vertical dynamics of the vehicle using the method of mathematical modeling

V E Gozbenko\textsuperscript{1,2}, S K Kargapol'tsev\textsuperscript{2}, B O Kuznetsov\textsuperscript{2}, A I Karlina\textsuperscript{3}, Yu I Karlina\textsuperscript{3}

\textsuperscript{1} Angarsk State Technical University, 60 Chaykovskoy Street, Angarsk, 668535, Russia
\textsuperscript{2} Irkutsk State Transport University, 15 Chernyshevsky Street, Irkutsk, 664074, Russia
\textsuperscript{3} Irkutsk National Research Technical University, 83 Lermontov Street, Irkutsk, 664074, Russia

E-mail: irgups-journal@yandex.ru

Abstract. Attention to the theory of rolling stock oscillations is due primarily to the fact that oscillatory processes, which inevitably arise as a result of driving along a usually uneven road, degrade almost all the basic properties of rolling stock. The article considers oscillations of a four-axle vehicle with a double spring suspension. The study of oscillations with a finite number of degrees of freedom is simplified if we introduce the principal coordinates of this system. To simplify the finding of the principal coordinates, free and forced oscillations of the sprung parts of the vehicle are investigated. It is assumed that the body of the vehicle has two degrees of freedom: lateral motion and wabbling; bouncing and pitching of trolleys will be neglected. The total number of degrees of freedom of the model is two. Having set-up the kinetic and potential energy and using the Lagrange equations, a system of differential equations was obtained. Consideration of forced oscillations of a system with two degrees of freedom is greatly simplified when moving to the principal coordinates. The problem of the vertical dynamics of the rolling stock is simplified in the transition to the principal coordinates. The resulting differential equations of free and forced oscillations of the system in the principal coordinates are two independent second-order linear differential equations, which greatly simplifies their solution.

1. Introduction

Attention to the theory of rolling stock oscillations is due primarily to the fact that oscillatory processes, which inevitably arise as a result of driving along a usually uneven road, degrade almost all the basic properties of rolling stock.

The forward movement of the rolling stock is accompanied by fluctuations. Arising as a result of the interaction of the crew and the way, oscillatory processes worsen the conditions of transportation of passengers and traction-dynamic qualities of the rolling stock, accelerate the wear of its equipment and the way, cause additional energy consumption for movement.

Fluctuations significantly affect the choice of parameters of the chassis and body, as well as determine the smoothness and stability of the crew when it moves.
The study of fluctuations of a rolling stock is one of the main problems of the dynamics of the crew. On the basis of these studies, the most rational parameters of elastic suspension are selected, as well as the constructive solution, the location and fastening of the equipment, ensuring stable safe movement and the required smoothness of the rolling stock. The results of studies of oscillations are also initial in identifying the causes of rolling stock noise and possible ways to reduce its level.

2. Construction of a mathematical model
Let us consider the oscillations of a four-axle vehicle with double spring suspension (Figure 1). The study of oscillations with a finite number of degrees of freedom is simplified if we introduce the principal coordinates of this system.

To study the vibrations of the sprung parts of the vehicle, the following designations are used:
- $m_k, m_{t1}, m_{t2}$ – the mass of the body and trolleys, respectively;
- $I_k$ – the moment of inertia of the body when pitching;
- $c_{11}, c_{12}$ – the vertical stiffness of the central suspension of the trolley;
- $c_{21}, c_{22}, c_{31}, c_{32}$ – the vertical stiffness of the axle suspension of the wheelset;
- $z_k, z_{t1}, z_{t2}$ – the current vertical displacements of the center of gravity, respectively, of the body, the first and second carriages;
- $\varphi_k$ – the angular displacement of the body;
- $L_1 + L_2$ – the body base.

![Figure 1. The computational scheme of oscillations of a vehicle with a two-stage spring suspension.](image)

To simplify finding the principal coordinates, we investigate free and forced oscillations of the sprung parts of the vehicle. It is assumed that the body of the vehicle has two degrees of freedom: lateral motion and wabbling; bouncing and pitching of trolleys will be neglected. The total number of degrees of freedom of the model is two (Figure 2) [1-4].

Setting-up the kinetic and potential energy and using the Lagrange equations, we obtain a system of differential equations:

\[
\begin{align*}
(m_k + m_{t1}) \ddot{z}_k + (c_{11} + c_{12}) z_k + (c_{11} L_1 - c_{12} L_2) \varphi_k = 0; \\
I_k \ddot{\varphi}_k + (c_{11} L_1^2 + c_{12} L_2^2) \varphi_k + (c_{11} L_1 - c_{12} L_2) z_k = 0.
\end{align*}
\]
To simplify the system (1), we assume that

\[ A_{11} = m_k, \quad C_{11} = c_{11} + c_{12}, \quad C_{12} = C_{21} = c_{11}L_1 - c_{12}L_2, \]

\[ A_{22} = I_k, \quad C_{22} = c_{11}L_1^2 + c_{12}L_2^2. \]

Then the system (1) takes the form:

\[
\begin{align*}
A_{11} \ddot{z}_k + C_{11} \dot{z}_k + C_{12} \dot{\varphi}_k &= 0; \\
A_{22} \ddot{\varphi}_k + C_{21} \dot{z}_k + C_{22} \dot{\varphi}_k &= 0.
\end{align*}
\]

(2)

The introduction of the principal coordinates greatly simplifies the study of vibrations. The relationship between the generalized coordinates \( z_k \) and \( \varphi_k \), and the principal coordinates \( q_1 \) and \( q_2 \) can be expressed as:

\[
\begin{align*}
z_k &= q_1 + q_2; \\
\varphi_k &= \mu_1 q_1 + \mu_2 q_2,
\end{align*}
\]

(3)

Let us seek a solution to the system of differential equations (2) in the form of \( z_k = A_1 \sin(kt + \alpha) \), \( \varphi_k = A_2 \sin(kt + \alpha) \). Substituting in (2) and discarding the multiplier \( \sin(kt + \alpha) \), we get:

\[
\begin{align*}
A_1 \left(C_{11} - A_1 k^2\right) + A_2 \left(C_{12}\right) &= 0; \\
A_1 \left(C_{12}\right) + A_2 \left(C_{22} - A_2 k^2\right) &= 0
\end{align*}
\]

(4)

This system has a solution that is different from a trivial \( A_i = 0 \), \( i = 1, 2 \) if its determinant is zero. Thus, we arrive at a characteristic (frequency) equation \( \Delta\left(k^2\right) = 0 \).

Considering the frequencies to be definite and different in size, we get:

\[
k_1^2 = \frac{A_{11} C_{22} + A_{22} C_{11} + \sqrt{A_{11}^2 C_{22}^2 - 2 A_{11} A_{22} C_{22} + 4 A_{11} A_{22} C_{12}^2 + A_{22}^2 C_{11}^2}}{2 A_{11} A_{22}},
\]

\[
k_2^2 = \frac{A_{11} C_{22} + A_{22} C_{11} - \sqrt{A_{11}^2 C_{22}^2 - 2 A_{11} A_{22} C_{22} + 4 A_{11} A_{22} C_{12}^2 + A_{22}^2 C_{11}^2}}{2 A_{11} A_{22}}.
\]

We substitute the root \( k_1^2 \), found from the characteristic equation, into system (4). Since the determinant \( \Delta\left(k_1^2\right) \) is zero, then in the system (4) there will be only one equation:
\[ \left( C_{11} - A_1 k^2 \right) + \frac{A_2}{A_1} C_{12} = 0. \]  

(5)

Solving this equation, we get:

\[ \frac{A_1^{(1)}}{A_1^{(0)}} = - \frac{C_{11} - A_1 k_1^2}{C_{12}} = \mu_1 \]  

(6)

\[ \frac{A_2^{(1)}}{A_2^{(0)}} = - \frac{C_{11} - A_1 k_2^2}{C_{12}} = \mu_2. \]  

(7)

In order to establish the correspondence between the coefficients of inertia \( A_1, A_2 \) and \( a_1, a_2 \), as well as the stiffness coefficients \( C_{11}, C_{12}, C_{22} \) and \( c_1, c_2 \), we substitute into the expression \( T \) and \( II \), calculated in generalized coordinates, their values (3) and comparing with the values \( T = \frac{1}{2} \left( a_1 q_1^2 + a_2 q_2^2 \right) \) \( II = \frac{1}{2} \left( c_1 q_1^2 + c_2 q_2^2 \right) \), we obtain the formula for calculating the coefficients of inertia \( a_1 \) and \( a_2 \), as well as the stiffness coefficients \( c_1 \) and \( c_2 \):

\[
\begin{align*}
\frac{a_1}{A_1} &= A_{11} + A_{22} \mu_1^2, \\
\frac{a_2}{A_2} &= A_{11} + A_{22} \mu_2^2, \\
\frac{c_1}{C_{11}} &= 2C_{12} \mu_1 + C_{22} \mu_1^2, \\
\frac{c_2}{C_{11}} &= 2C_{12} \mu_2 + C_{22} \mu_2^2.
\end{align*}
\]  

(8)

Usually, when solving specific problems, it is difficult to pre-determine the parameters, which are the principal coordinates of the system. Therefore, choosing the values that determine the position of the system in a most simple way as the generalized coordinates, the frequencies of the main oscillations \( k_1 \) and \( k_2 \) are calculated using the equations (4), and then the distribution coefficients \( \mu_1 \) and \( \mu_2 \) are found using the formulas (6), (7). Since \( z_k = q_1 + q_2 \) \( \varphi_k = \mu_1 q_1 + \mu_2 q_2 \), then:

\[ q_1 = \frac{-\mu_2 z_k}{\mu_1 - \mu_2} \]  

(9)

Thus, the equations of motion in the principal coordinates \( q_1, q_2 \) will take the form:

\[
\begin{align*}
\frac{a_1}{A_1} q_1 + c_1 q_1 &= 0, \\
\frac{a_2}{A_2} q_2 + c_2 q_2 &= 0.
\end{align*}
\]  

(10)

The obtained differential equations of free oscillations of the system in the principal coordinates (10) are two independent linear differential equations of second order. The general solution of these equations is:

\[
\begin{align*}
q_1 &= C_1 \sin (k_1 t + a_1), \\
q_2 &= C_2 \sin (k_1 t + a_2),
\end{align*}
\]  

(11)

where \( C_1, C_2, a_1 \) and \( a_2 \) are the integration constants determined from the initial conditions [5] with:

\[ z_k = z_{k0}^1; \quad \dot{z}_k = \dot{z}_{k0}^1; \quad \varphi_k = \varphi_{k0}^1; \quad \dot{\varphi}_k = \dot{\varphi}_{k0}^1, \]  

then
\[ q_1 = q_{10} \quad \dot{q}_1 = \dot{q}_{10} \quad q_2 = q_{20} \quad \dot{q}_2 = \dot{q}_{20}, \]

where
\[ q_{10} = \frac{\Phi_{k0} - \mu_1 z_{k0}}{\mu_1 - \mu_2} \quad \dot{q}_{10} = \frac{\mu_1 z_{k0} - \Phi_{k0}}{\mu_1 - \mu_2} \quad q_{20} = \frac{\mu_2 z_{k0} - \Phi_{k0}}{\mu_1 - \mu_2} \quad \dot{q}_{20} = \frac{\mu_2 z_{k0} - \Phi_{k0}}{\mu_1 - \mu_2}. \]

The natural frequencies \( k_1 \) and \( k_2 \) of vibrations of the system in the principal coordinates are determined from equations (10) using the following formulas:
\[ k_1 = \sqrt{\frac{c_1}{a_1}} \quad \text{and} \quad k_2 = \sqrt{\frac{c_2}{a_2}} \quad (12) \]

The arbitrarily chosen generalized coordinates are the principal coordinates of the system if the coefficients \( A_{12} \) and \( C_{12} \) are zero in the expressions for the kinetic and potential energies of the system.

Of particular importance is the use of the principal coordinates in the study of forced oscillations of the system \([6-13]\). Let us consider the forced oscillations of a system with two degrees of freedom. In this case, the system is affected by perturbing forces, which are some specified functions of time \( t \). We assume that the generalized perturbing forces are simple harmonic functions of time, having the same frequency \( p \) and phase \( \delta \), i.e.:
\[ f_1 = H_1 \sin(pt + \delta); \quad f_2 = H_2 \sin(pt + \delta). \quad (13) \]

Based on (2), the differential equations of forced oscillations of this system are:
\[ \begin{cases} A_{11} \ddot{x}_k + C_{11} \dot{x}_k + C_{12} \dot{q}_k = H_1 \sin(pt + \delta); \\ A_{22} \ddot{q}_k + C_{22} \dot{q}_k + C_{21} \dot{x}_k = H_2 \sin(pt + \delta). \end{cases} \quad (14) \]

The general solution of this system of differential equations is the sum of the general solution of the corresponding system of homogeneous equations, i.e. system (2), and private solution of system (14). The first solution found above remains to determine the particular solution.

3. **Passing to main coordinates**

Consideration of forced oscillations of a system with two degrees of freedom is greatly simplified when passing to the principal coordinates. By definition of generalized forces, the elementary work of perturbing forces on a possible displacement of the system can be represented as:
\[ \delta W = f_1 \delta \dot{x}_k + f_2 \delta \dot{q}_k. \]

Based on (3):
\[ \begin{cases} \delta \dot{x}_k = \delta q_1 + \delta q_2; \\ \delta \dot{q}_k = \delta \mu_1 q_1 + \delta \mu_2 q_2. \end{cases} \]

Then:
\[ \delta W = f_1 (\delta q_1 + \delta q_2) + f_2 (\delta \mu_1 q_1 + \delta \mu_2 q_2) \]
or
\[ \delta W = (f_1 + \mu_1 f_2) \delta q_1 + (f_1 + \mu_2 f_2) \delta q_2. \]

Therefore, the generalized forces corresponding to the principal coordinates will be:
\[ \begin{cases} f_1' = f_1 + \mu_1 f_2 = (H_1 + \mu_1 H_2) \sin(pt + \delta); \\ f_2' = f_1 + \mu_2 f_2 = (H_1 + \mu_2 H_2) \sin(pt + \delta). \end{cases} \quad (15) \]

The differential equations of forced oscillations of the system in principal coordinates are as follows:
\[
\begin{align*}
\ddot{q}_1 + k_1 q_1 &= \frac{H_1 + \mu_1 H_2}{a_1} \sin(pt + \delta), \\
\ddot{q}_2 + k_2 q_2 &= \frac{H_1 + \mu_2 H_2}{a_2} \sin(pt + \delta).
\end{align*}
\] (16)

The problem is reduced to the integration of two independent differential equations. Solutions of the differential equations (16) can be represented in the form:

\[
\begin{align*}
q_1 &= q_{10} \cos k_1 t + \frac{\dot{q}_{10}}{k_1} \sin k_1 t + \frac{1}{a_1 k_1} \int \left( H_1(\xi) + \mu_1 H_2(\xi) \right) \sin k_1 (t - \xi) d\xi, \\
q_2 &= q_{20} \cos k_2 t + \frac{\dot{q}_{20}}{k_2} \sin k_2 t + \frac{1}{a_2 k_2} \int \left( H_1(\xi) + \mu_2 H_2(\xi) \right) \sin k_2 (t - \xi) d\xi.
\end{align*}
\] (17)

According to the formulas (3) we now return to the original unknowns \( z_K \) and \( \varphi_K \). Then we obtain the general solution of the system of differential equations (14):

\[
\begin{align*}
z_K(t) &= q_1 + q_2; \\
\varphi_K(t) &= \mu_1 q_1 + \mu_2 q_2.
\end{align*}
\] (18)

containing four arbitrary constants [6, 14] \( \dot{q}_1 = \dot{q}_{10}; \dot{q}_1 = \dot{q}_{10}; q_2 = q_{20}; \dot{q}_2 = \dot{q}_{20} \), which must be determined by the initial values of the generalized coordinates \( z_K = z_{K_0}; \varphi_K = \varphi_{K_0} \) and generalized speeds \( \dot{z}_K = \dot{z}_K; \dot{\varphi}_K = \dot{\varphi}_K \).

4. Conclusion

The article considers oscillations of a four-axle vehicle with a double spring suspension. The study of oscillations with a finite number of degrees of freedom is simplified if we introduce the principal coordinates of this system. To simplify the finding of the principal coordinates, free and forced oscillations of the sprung parts of the vehicle are investigated.

It is assumed that the body of the vehicle has two degrees of freedom: lateral motion and wabbling; bouncing and pitching of trolleys will be neglected. The total number of degrees of freedom of the model is two. Having set-up the kinetic and potential energy and using the Lagrange equations, a system of differential equations was obtained. Consideration of forced oscillations of a system with two degrees of freedom is greatly simplified when moving to the principal coordinates.

The task of the vertical dynamics of the rolling stock is simplified in the transition to the principal coordinates. The resulting differential equations of free and forced oscillations of the system in the principal coordinates are two independent second-order linear differential equations, which greatly simplifies their solution.

References
[1] Balanovskii A E, Huy V V 2017 Letters on Materials 7(2) 175-179
[2] Balanovskii A E, Van Huy V 2018 Journal of Friction and Wear 39(4) 311-318
[3] Medvedev S I, Nezhivlyak A E, Grechneva M V, Balanovsky A E, Ivakin V L 2015 Welding International 29(8) 643-649
[4] Balanovskii A E 2018 High Temperature 56(4) 486-495
[5] Verigo M F 1988 Dynamics of cars. Lecture notes (Moscow: Transport) 174 p
[6] Vershinsky S V, Danilov V N, Khusidov V D 1991 Dynamics of the car (Moscow: Transport) 360 p
[7] Garg V K, Dukkipati R V 1988 Dynamics of Railway Vehicle Systems (Moscow: Transport) 391 p
[8] Nikolaev V A 2003 Development of methods for analytical design of quasi-invariant systems of spring suspension of railway carriages (Omsk: Omsk State Transport University) 371 p

[9] Gozbenko V E, Kargapoltsev S K, Minaev N V, Karlina A I 2016 *International Journal of Applied Engineering Research* **11** 23 pp 11132-11136

[10] Khomenko A P, Gozbenko V E, Kargapoltsev S K, Minaev N V, Karlina A I 2017 *International Journal of Applied Engineering Research* T. 12 23 pp 13773-13778

[11] Balanovskiy A E 2017 *Welding International* **31**(6) 467-476

[12] Balanovskii A E 2016 *High Temperature* **54**(5) 627-631.

[13] Tsisovski T 2001 Improving the control systems of oscillations of the rolling stock of railways: a dissertation for the degree of Doctor of Technical Sciences: 05.22.07. (Moscow: Russian State Open Technical Transport University)

[14] Balanovskii A E 2018 *High Temperature* **56**(1) 1-9