FINITENESS AND EXISTENCE OF ATTRACTORS AND REPELLERS ON SECTIONAL HYPERBOLIC SETS

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ABSTRACT. We study small perturbations of a sectional hyperbolic set of a vector field on a compact manifold. Indeed, we obtain an upper bound for the number of attractors and repellers that can arise from these perturbations. Moreover, no repeller can arise if the unperturbed set has singularities, is connected and consists of nonwandering points.

1. Introduction. The dynamical systems describe different properties about the evolution of initial states, asymptotic behavior and relationships between system’s elements. However, most of these systems’ behavior might be very complex, therefore, finding the link between them becomes a difficult task. It is well known that many of these properties come from physics phenomena. In the sixties, some definitions appeared that tried to explain these behaviors and properties, such as attractors and repellers. These concepts are well known and play a fundamental role in the dynamical systems theory. They have received some mathematical interpretations, such as turbulence that appears in the classical paper [25] which, simultaneously, provides the existence of attractors for particular vector fields. Since a repeller is an attractor for the reverse flow, it is clear that this result provides the existence for repellers too. Thereby, stressing the importance of attractors, we highlight the classical construction of the geometric Lorenz models [1], [12]. They provide a wide range of results and research in the theory of dynamical systems, particularly hyperbolic and sectional hyperbolic theories on three-dimensional manifolds. The study of sectional hyperbolic attractors for higher dimensional flows is, however, mostly open.

The objective of this paper is to study two problems related to the sectional hyperbolic sets for flows, that are, how many attractors and repellers can arise from small perturbations and, also, the possible appearance of repellers from small perturbations.

The motivations come from the previous result in dimension three [3], [18] in which an upper bound in terms of the number of singularities is obtained but for transitive or nonwandering flows. Another motivation is the recent paper by the

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author [16] where the same results for transitive flows were obtained in higher dimensions.

A further motivation is the well known Anomalous Anosov flow [11] beside [7], which are examples of connected sectional hyperbolic sets (with or without singularities) containing nontrivial repellors.

Here we will remove both transitivity and nonwandering hypotheses in order to obtain the finitude in a robust way of attractors and repellors for higher dimensional sectional hyperbolic sets. Moreover, we prove the non-existence of repellors for any perturbation of a connected sectional hyperbolic sets (with singularities) contained in the nonwandering set.

Let us state our results in a more precise way.

Consider a compact manifold $M$ of dimension $n \geq 3$ with a Riemannian structure $\| \cdot \|$ (a compact n-manifold for short). We denote by $\partial M$ the boundary of $M$. Let $\mathcal{X}^1(M)$ be the space of $C^1$ vector fields in $M$ endowed with the $C^1$ topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary $\partial M$ and denote by $X_t$ the flow of $X$, $t \in \mathbb{R}$. The maximal invariant set of $X$ is defined by

$$M(X) = \bigcap_{t \geq 0} X_t(M).$$

Notice that $M(X) = M$ in the boundaryless case $\partial M = \emptyset$. A subset $\Lambda$ is called invariant if $X_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$. We denote by $m(L)$ the minimum norm of a linear operator $L$, i.e., $m(L) = \inf_{v \neq 0} \frac{\|Lv\|}{\|v\|}$.

**Definition 1.1.** A compact invariant set $\Lambda$ of $X$ is partially hyperbolic if there is a continuous invariant splitting $T_\Lambda M = E^s \oplus E^c$ such that the following properties hold for some positive constants $C, \lambda$:

1. $E^s$ is contracting, i.e., $\| DX_t(x) \big|_{E^s_x} \| \leq C e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
2. $E^s$ dominates $E^c$, i.e., $\frac{\| DX_t(x) \big|_{E^s_x} \|}{m(DX_t(x) \big|_{E^c_x})} \leq C e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

We say the central subbundle $E^c_x$ of $\Lambda$ is sectionally expanding if

$$\dim(E^c_x) \geq 2 \quad \text{and} \quad |\det(DX_t(x) \big|_{L_x})| \geq C^{-1} e^{\lambda t}, \text{ for all } x \in \Lambda, \quad t > 0$$

and all two-dimensional subspace $L_x$ of $E^c_x$. Here $\det(DX_t(x) \big|_{L_x})$ denotes the jacobian of $DX_t(x)$ along $L_x$.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

**Definition 1.2.** A sectional hyperbolic set is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is sectionally expanding.

The $\omega$-limit set of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \to \infty} X_{t_n}(p)$ for some sequence $t_n \to \infty$. The $\alpha$-limit set of $p \in M$ is the set $\alpha_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \to \infty} X_{t_n}(p)$ for some sequence $t_n \to -\infty$. The non-wandering set of $X$ is the set $\Omega(X)$ of points $p \in M$ such that for every neighborhood $U$ of $p$ and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. Given $\Lambda \subset M$ compact, we say that $\Lambda$ is invariant if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We also say that $\Lambda$ is transitive if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; singular if it contains a singularity and attracting if $\Lambda = \cap_{t > 0} X_t(U)$ for some compact neighborhood $U$ of $\Lambda$. This neighborhood is often called isolating block. It is well known that the isolating block $U$ can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all
$t > 0$. An attractor is a transitive attracting set. An attractor is nontrivial if it is not a closed orbit. A repelling is an attracting for the time reversed vector field $-X$ and a repeller is a transitive repelling set.

With these definitions we can state our main results.

**Main Theorem 1.** For every sectional hyperbolic set $\Lambda$ of a vector field $X$ on a compact manifold there are neighborhoods $U$ of $X$, $U$ of $\Lambda$ and $n_0 \in \mathbb{N}$ such that

$$\# \{ L \subset U : L \text{ is an attractor or repeller of } Y \in U \} \leq n_0.$$  

**Main Theorem 2.** Let $X$ be a $C^1$ vector field with singularities of a compact manifold $M$, $n \geq 3$, $X \in \chi^1(M)$. Let $\Lambda \subset M$ be a connected sectional hyperbolic set of $X$. If $\Lambda \subset \Omega(X)$, then there are neighborhoods $U \subset \chi^1(M)$ of $X$ and $U \subset M$ of $\Lambda$ such that if $Y \in U$, $Y$ has no repeller in $U$.

Let us state the following corollaries of our results. Recall that a sectional Anosov flow is a vector field whose maximal invariant set is sectional hyperbolic [17].

**Corollary 1.** For every sectional Anosov flow of a compact manifold there are a neighborhood $U \in \chi^1(M)$ and $n_0 \in \mathbb{N}$ such that

$$\# \{ L \text{ is an attractor or repeller of } Y \in U \} \leq n_0.$$  

Main Theorem 1 also provides a corollary related to one of the Bonatti’s conjectures [6]. Indeed, Palis conjectured in [21] that generic diffeomorphisms far from homoclinic tangencies have only finitely many sinks and sources. Such a conjecture is true in the surface case by Pujals-Sambarino [24].

For the three-dimensional case obstructions appear and therefore is reasonable to assume some hypotheses in different scenarios. In particular, [19] gives an approach to this conjecture in the (open) set of $C^1$ vector fields on a closed three-manifold which can not be $C^1$ approximated by ones exhibiting infinitely many sinks or sources. Thus, in this specific scenario is proved that a generic vector field exhibits a finite number of attractors whose basins of attraction form an open dense set.

More recently, Bonatti stated a slightly stronger conjecture, namely, generic diffeomorphisms that are far from homoclinic tangencies have only finitely many attractors and repellers [6].

In our view, it is natural to consider even the flow version of Bonatti’s conjecture, namely, if a generic flow far from homoclinic tangencies has only a finite number of attractors and repellers.

In this direction, [9] announced in the $C^1$ topology that every three dimensional flow can be accumulated by robustly singular hyperbolic flows, or by flows with homoclinic tangencies. In addition, every chain recurrent classes of generic three dimensional flows far from tangencies is hyperbolic or is a singular hyperbolic (sectional hyperbolic in dimension three) attractor or repeller, that beside [19] is related with the finiteness of attractors on a particular scenario.

By using the main theorem, we can have the following corollary. Also, we remark that, by using [9], one knows that generic vector field that is away from homoclinic tangency is singular hyperbolic, hence is star. Then by using [19] Theorem B], one knows that the vector field is singular Axiom A without cycle. By the definition of singular Axiom A without cycle in [19], there are finitely many attractors.

**Main Theorem** [1] and the result announced in [9] immediately gives the following statement.

**Corollary 2.** Generic three-dimensional flows that are far from homoclinic tangencies have only finitely many attractors and repellers.
2. Preliminaries. In this section, we recall some results on sectional hyperbolic sets, and we obtain some useful results for the main theorems.

In first place, we recall the standard definition of hyperbolic set.

Definition 2.1. A compact invariant set $\Lambda$ of $X$ is hyperbolic if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^\chi \oplus E^u$ and positive constants $C, \lambda$ such that

- $E^\chi$ is the vector field’s direction over $\Lambda$.
- $E^s$ is contracting, i.e., $\|[DX_t(x)]|_{E^s_x}|| \leq C e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- $E^u$ is expanding, i.e., $\|[DX_t(x)]|_{E^u_x}|| \leq C e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

A closed orbit is hyperbolic if it is also hyperbolic, as a compact invariant set. An attractor is hyperbolic if it is also a hyperbolic set.

Let $\Lambda$ be a sectional hyperbolic set of a $C^1$ vector field $X$ of $M$. Henceforth, we denote by $Sing(X)$ the set of singularities of the vector field $X$ and by $Cl(A)$ the closure of $A, A \subset M$.

The following two results examining the sectional hyperbolic splitting $T_\Lambda M = E^\chi_\Lambda \oplus E^s_\Lambda$ of a sectional hyperbolic set $\Lambda$ of $X \in \mathcal{X}^1(M)$ appear in [20] for the three-dimensional case and in [4] for the higher dimensional case.

Lemma 2.2. Let $\Lambda$ be a sectional hyperbolic set of a $C^1$ vector field $X$ of $M$. Then, there are a neighborhood $U \subset \mathcal{X}^1(M)$ of $X$ and a neighborhood $U \subset M$ of $\Lambda$ such that if $Y \in U$, every nonempty, compact, non singular, invariant set $H$ of $Y$ in $U$ is hyperbolic saddle-type (i.e. $E^s \neq 0$ and $E^u \neq 0$).

Theorem 2.3. Let $\Lambda$ be a sectional hyperbolic set of a $C^1$ vector field $X$ of $M$. If $\sigma \in Sing(X) \cap \Lambda$, then $\Lambda \cap W^u_\Lambda(\sigma) = \{\sigma\}$.

Proof. We begin by proving two claims.

Claim 1. If $x \in (M(X) \setminus Sing(X))$, then $X(x) \notin E^s_x$.

Proof. Suppose by contradiction that there is $x_0 \in (M(X) \setminus Sing(X))$ such that $X(x_0) \in E^s_{x_0}$. Then, $X(x) \in E^s_x$ for every $x$ in the orbit of $x_0$ since $E^s_{M(X)}$ is invariant.

So $X(x) \in E^s_x$ for every $x \in \alpha(x_0)$ by continuity. (1)

It follows that $\omega(x)$ is a singularity for all $x \in \alpha(x_0)$. In particular, $\alpha(x_0)$ contains a singularity $\sigma$ which is necessary hyperbolic of saddle-type.

Now we have two cases: $\alpha(x_0) = \{\sigma\}$ or not.

If $\alpha(x_0) = \{\sigma\}$ then $x_0 \in W^u(\sigma)$. For all $t \in \mathbb{R}$ define the unitary vector

$$v^t = \frac{DX_t(x_0)(X(x_0))}{\|[DX_t(x_0)(X(x_0))]|}.$$  

It follows that

$$v^t \in TX_t(x_0)W^u(\sigma) \cap E^s_{X_t(x_0)}, \; \forall t \in \mathbb{R}. $$

Take a sequence $t_n \to \infty$ such that the sequence $v^{-t_n}$ converges to $v^\infty$ (say). Clearly $v^\infty$ is an unitary vector and, since $X_{-t_n}(x_0) \to \sigma$ and $E^s$ is continuous we obtain

$$v^\infty \in T_\sigma W^u(\sigma) \cap E^s_\sigma.$$  

Therefore $v^\infty$ is an unitary vector which is simultaneously expanded and contracted by $DX_t(\sigma)$, which is a contradiction. This contradiction shows the result in the first case.
For the second, we assume \( \alpha(x_0) \neq \{\sigma\} \). Then, \((W^u(\sigma) \setminus \{\sigma\}) \cap \alpha(x_0) \neq \emptyset \). Pick \( x_1 \in (W^u(\sigma) \setminus \{\sigma\}) \cap \alpha(x_0) \). It follows from [Lemma 2.2](#) that \( X(x_1) \in E^s_x \) and then we get a contradiction as in the first case replacing \( x_0 \) by \( x_1 \). This contradiction proves the claim.

**Claim 2.** If \( \sigma \in \text{Sing}(X) \), then \( M(X) \cap W^{ss}(\sigma) = \{\sigma\} \).

**Proof.** Take \( x \in W^{ss}(\sigma) \setminus \{\sigma\} \). Then, \( E^s_x = T_x W^{ss}(\sigma) \). Moreover, since \( W^{ss}(\sigma) \) is an invariant, we obtain \( X(x) \in T_x W^{ss}(\sigma) \). We conclude that \( X(x) \in E^s_x \) for all \( x \in W^{ss}(\sigma) \) and now Claim (1) applies.

Next we explain briefly how to obtain sectional hyperbolic sets nearby \( \Lambda \) from vector fields close to \( X \). Fix a neighborhood \( U \) with compact closure of \( \Lambda \) as in Lemma [2.2](#). Define

\[
\Lambda_X = \cap_{t \in \mathbb{R}} X_t(\text{Cl}(U)).
\]

Note that \( \Lambda_X \) is sectional hyperbolic and \( \Lambda \subset \Lambda_X \). Likewise, if \( Y \) is a \( C^1 \) vector field close to \( X \), we define the continuation

\[
\Lambda_Y = \cap_{t \in \mathbb{R}} Y_t(\text{Cl}(U)).
\]

### 2.1. Lorenz-like singularity and singular cross-section

Let \( M \) be a compact \( n \)-manifold, \( n \geq 3 \). Fix \( X \in \mathcal{X}^1(M) \), inwardly transverse to the boundary \( \partial M \). We denote by \( X_t \) the flow of \( X \), \( t \in \mathbb{R} \).

By using the standard definition of hyperbolic set (Definition [2.1](#)), it follows from the stable manifold theory [13](#) that if \( p \) belongs to a hyperbolic set \( \Lambda \), then the following sets

\[
W^s_X(p) = \{ x : d(X_t(x), X_t(p)) \to 0, t \to \infty \}
\]

and \( W^u_X(p) = \{ x : d(X_t(x), X_t(p)) \to 0, t \to -\infty \} \)

are \( C^1 \) immersed submanifolds of \( M \), which are tangent at \( p \) to the subspaces \( E^s_p \) and \( E^u_p \) of \( T_p M \), respectively. Similarly,

\[
W^s_{X}(p) = \bigcup_{t \in \mathbb{R}} W^s_{X}(X_t(p))
\]

and

\[
W^u_{X}(p) = \bigcup_{t \in \mathbb{R}} W^u_{X}(X_t(p))
\]

are also \( C^1 \) immersed submanifolds tangent to \( E^s_p \oplus E^X_p \) and \( E^u_p \oplus E^u_p \) at \( p \), respectively. Moreover, for every \( \epsilon > 0 \) we have that

\[
W^s_X(p, \epsilon) = \{ x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \geq 0 \}
\]

and

\[
W^u_X(p, \epsilon) = \{ x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \leq 0 \}
\]

are closed neighborhoods of \( p \) in \( W^s_X(p) \) and \( W^u_X(p) \), respectively.

For simplicity, given \( \epsilon > 0 \), the above submanifolds will be denoted by \( W^s_{X}(p), W^u_{X}(p), W^s_{X}(p) \) and \( W^u_{X}(p) \) respectively.

It is well known from the stability theory for hyperbolic sets, that we can fix a neighborhood \( U \subset M \) of \( \Lambda \), a neighborhood \( \mathcal{U} \subset \mathcal{X}^1(M) \) of \( X \) and \( \epsilon > 0 \) such that every hyperbolic set \( H \) in \( U \) of every vector field \( Y \) in \( \mathcal{U} \) satisfies that its local stable and instable manifold

\[
W^s_Y(x, \epsilon) \text{ and } W^u_Y(x, \epsilon)
\]

have uniform size \( \epsilon \) for all \( x \in H \).  (2)

There is also a stable manifold theorem in the case when \( \Lambda \) is a sectional hyperbolic set. Indeed, if we denote by \( T_{\Lambda} M = E^s_\Lambda \oplus E^u_\Lambda \) the corresponding sectional hyperbolic splitting over \( \Lambda \), we assert that there exists such contracting foliation on a small neighborhood \( U \) of \( \Lambda \). Note that this extended foliation is not necessarily invariant, and we can only ensure the invariance if this one, is at least, an attracting
set \([2]\). This extension is carried out as follows: first, we can choose cone fields on \(U\) and we consider the space of tangent foliations to the cone fields. Given a point \(x \in U\), whenever the positive orbit remain within to \(U\), for example \(t = 1\), we can use the derivate map \(DX_{-1}(x)\). This map sends the leaf at \(X_{-1}(x)\) inside of cone \(C_{X_{-1}(x)}\) to the leaf at \(x\) inside of cone \(C_x\), contracting the angle and stretching the tangent vectors to the initial foliation. Then, we can apply fiber contraction \([3]\). Now, by using the Fiber Contraction Theorem \([4]\) the foliation arises. Thus, we have from \([5]\) that the contracting subbundle \(E^s_\lambda\) can be extended to a contracting subbundle \(E^s_\nu\) in \(M\) (not necessarily invariant).

Moreover, such an extension by construction is tangent to a continuous foliation denoted by \(W^{ss}\) (or \(W^{ss}_X\) to indicate dependence on \(X\)). By adding the flow direction to \(W^{ss}\) we obtain a continuous foliation \(W^s\) (or \(W^s_X\)) now tangent to \(E^u_\nu \oplus E^s_\nu\).

Unlike the hyperbolic case \(W^s\) may have singularities, all of which being the leaves \(W^{ss}(\sigma)\) passing through the singularities \(\sigma\) of \(X\).

It turns out that every singularity \(\sigma\) of a sectional hyperbolic set \(\Lambda\) satisfies \(W^{ss}_X(\sigma) \subset W^s_X(\sigma)\). Furthermore, there are two possibilities for such a singularity, namely, either \(\dim(W^{ss}_X(\sigma)) = \dim(W^s_X(\sigma))\) (and so \(W^{ss}_X(\sigma) = W^s_X(\sigma)\)) or \(\dim(W^{ss}_X(\sigma)) = \dim(W^s_X(\sigma)) + 1\). In the later case we call it Lorenz-like according to the following definition.

**Definition 2.4.** Let \(\Lambda\) be a sectional hyperbolic set of a \(C^1\) vector field \(X\) of \(M\).

We say that a singularity \(\sigma\) of \(\Lambda\) is **Lorenz-like** if \(\dim(W^s_X(\sigma)) = \dim(W^{ss}_X(\sigma)) + 1\).

Hereafter, we will denote \(\dim(W^{ss}_X(\sigma)) = s\), \(\dim(W^u_X(\sigma)) = u\) and therefore \(\dim(W^s_X(\sigma)) = s + 1\) by definition. Moreover, \(W^{ss}_X(\sigma)\) separates \(W^s_{loc}(\sigma)\) in two connected components which we will denote by \(W^s_{loc}^{+,c}(\sigma)\) and \(W^s_{loc}^{+,b}(\sigma)\), respectively (in this way \(W^{s,+}(\cdot)\) and \(W^{s,b}(\cdot)\) denote top and bottom stable components).

### 2.1.1. Singular cross-sections.

Given \(X_t\) the \(C^1\) flow generated by the vector field \(X\) on a compact \(n\)-manifold \(M\), and \(\sigma\) a Lorenz-like singularity of \(X_t\), we shall define singular cross-section in the higher dimensional context.

We start by recalling the two-dimensional singular cross section construction in order to explain carefully the geometry of the singular cross section for the higher dimensional case. Since \(\sigma\) is a Lorenz-like singularity of the vector field \(X\), it is hyperbolic. Then, we recall that the definition for three-dimensional case says: a singularity \(\sigma\) for \(X\) is **Lorenz-like** if it has three real eigenvalues

\[
\lambda_3, \lambda_2, \lambda_1 \text{ with } \lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.
\]

By hyperbolicity, there exist the stable and unstable manifold, \(W^s_X(\sigma)\) and \(W^u_X(\sigma)\) respectively. Also, these manifolds are tangent at \(\sigma\) to the eigenspace associated to the set of eigenvalues \(\{\lambda_2, \lambda_3\}\) and \(\{\lambda_1\}\) respectively \([13]\). Then, note that \(\dim(W^s_X(\sigma)) = 2\) and \(\dim(W^u_X(\sigma)) = 1\).

A further invariant manifold, the strong stable manifold \(W^{ss}_X(\sigma)\), is well defined and tangent at \(\sigma\) to the eigenspace associated the \(\{\lambda_2\}\). Consequently \(\dim(W^{ss}_X(\sigma)) = 1\).

It is well known that we can take a linearizing coordinate system \((x_1, x_2, x_3)\) in a neighborhood of the singularity \(\sigma\). Note that \(W^{ss}_X(\sigma)\) separates \(W^s_X(\sigma)\) in two connected components (two semi-planes), namely, the top and the bottom ones. In the top component we consider a cross-section of \(X_t\) together with a curve \(l_{\sigma}\). Here, it arises the classical square \([-1, 1] \times [-1, 1]\) that supports the definition of singular cross section.

In fact, the singular cross section will be the compact manifold diffeomorphic to
$I^2 = [-1,1] \times [-1,1] \subset \mathbb{R}^3$. Also, note that the curve $l_\sigma$ is the intersection between the top semi-plane and the square $[-1,1] \times [-1,1]$.

Now, by using the above information beside the dimension of the strong stable and unstable manifold (fundamental idea), we will start our construction for the higher dimensional case.

First, we will denote a cross-section by $\Sigma$ and its boundary by $\partial \Sigma$. In this direction, we defined the hypercube by $I^k = [-1,1]^k$ and it will be submanifold of dimension $k$ on $\mathbb{R}^n$, with $k \in \mathbb{N}$.

Thus, we begin by considering $B^u[0,1] \approx I^u$ and $B^{ss}[0,1] \approx I^s$, where $B^{ss}[0,1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^{ss}(\sigma))} = \mathbb{R}^s$ and $B^u[0,1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^{ss}(\sigma))} = \mathbb{R}^{n-s-1} = \mathbb{R}^n$.

**Definition 2.5.** A singular cross-section of a Lorenz-like singularity $\sigma$ consists of a pair of submanifolds $\Sigma^t, \Sigma^b$, where $\Sigma^t, \Sigma^b$ are cross-sections (top and bottom) such that

$$\Sigma^t$$ is transversal to $W_{loc}^{ss}(\sigma)$ and $\Sigma^b$ is transversal to $W_{loc}^{ss}(\sigma)$.

By simplicity, we will indicate by “$*$” either $t$ (for top) or $b$ (for bottom), and we will specify it if necessary.

Note that every singular cross-section contains a pair of singular submanifolds $l^t, l^b$ defined as the intersection of the local stable manifold of $\sigma$ with $\Sigma^t, \Sigma^b$ respectively and, additionally, $\dim(l^*) = \dim(W^{ss}(\sigma))$ where $* = t, b$.

The positive flow lines of $X_t$ starting at $\Sigma^t \cup \Sigma^b \setminus (l^t \cup l^b)$ exit a small neighborhood of $\sigma$ passing through the cusp region (cross-section). The positive orbit starting at $l^t \cup l^b$ goes directly to $\sigma$.

Thus, each component of a singular cross-section $\Sigma^*$ will be a hypercube of dimension $(n-1)$, i.e., diffeomorphic to $B^u[0,1] \times B^{ss}[0,1]$. Let $f : B^u[0,1] \times B^{ss}[0,1] \to \Sigma^*$ be the diffeomorphism, such that

$$f(\{0\} \times B^{ss}[0,1]) = l^*$$

and $\{0\} = 0 \in \mathbb{R}^u$.

Now, if $\Sigma$ is a singular cross-section of $\Lambda$ contained in $U$ as above, we shall construct a foliation $\mathcal{F}$ on $\Sigma$ by projecting $\mathcal{F}^{ss}$ onto $\Sigma$, i.e., an induced foliation. Then, given $I \subset \mathbb{R}$ and $S \subset M$ we define $X_t(D) = \{X_t(x) : (t, x) \in I \times D\}$. For $\epsilon > 0$ and $x \in U$ we define

$$\mathcal{F}_{x,\epsilon}^{ss} = X_{[\epsilon,\epsilon]}(W_{loc}^{ss}(x))$$.

By choosing a regular point $x$ (i.e. $X(x) = 0$), we obtain that $\mathcal{F}_{x,\epsilon}^s$, is a $(s+1)$-dimensional submanifold of $M$.

We can note that by construction each point on the singular cross-section is regular, and so, the above remark applies for all $x \in \Sigma$. Thus, given $x \in \Sigma$ we define $F_{x,\epsilon} = \mathcal{F}_{x,\epsilon}^{ss} \cap \Sigma$. Then, by replacing $\Sigma$ by a small section around $l^t \cup l^b$ if necessary, we can assume that the vertical boundary of $\Sigma$ is formed by type leaves $\mathcal{F}_{x,\epsilon}$. Hence, since $\Sigma$ is compact and formed by regular points, we can find $\epsilon > 0$ such that if $\mathcal{F}_x = \mathcal{F}_{x,\epsilon}$, then the family

$$\mathcal{F} = \{\mathcal{F}_x : x \in \Sigma\}$$

is a continuous $s$-dimensional foliation of $\Sigma$ such that the curves in $l^t \cup l^b$ are leaves of $\mathcal{F}$.

**Definition 2.6.** We say that a submanifold $c$ of $\Sigma$ is a $k$-surface if it is the image of a $C^1$ injective map $c : Dom(c) \subset \mathbb{R}^k \to \Sigma$, with $Dom(c)$ being $I^k$ and $k \leq n-1$. 
For simplicity, hereafter $c$ stands the image of this one map. A $k$-surface $c$ is vertical if it is the graph of a $C^1$ map $g : I^{n-k-1} \to I^k$, i.e., $c = \{(y, y) : y \in I^k\} \subset \Sigma$.

**Definition 2.7.** A continuous foliation $\mathcal{F}$ on a component $\Sigma^*$ of $\Sigma$ is called vertical if its leaves are vertical $s$-surfaces and $\partial^v \Sigma \subset \mathcal{F}$, where $s = \text{dim}(B^{ss}[0,1])$.

It follows from the above definition that the leaves $L$ of a vertical foliation $\mathcal{F}$ are vertical $s$-surfaces, hence differentiable ones. In particular, the tangent space $T_xL$ is well defined for all $x \in L$.

**Remark 1.** Note that, given a singular cross-section $\Sigma$ equipped with a vertical foliation $\mathcal{F}$, one has that $\text{dim}(\mathcal{F}) = \text{dim}(B^{ss}[0,1]) = s$, and each leaf $L$ of $\mathcal{F}$ has the same dimension of $W^{ss}(\sigma)$, being $\sigma$ the Lorenz-like singularity associated to $\Sigma$.

2.1.2. **Boundary of singular cross-sections.** Now, given a singular cross-section $\Sigma$, for each one of its components, we define the boundary as

$$
\partial \Sigma^* = \partial^h \Sigma^* \cup \partial^v \Sigma^*,
$$

where:

$\partial^h \Sigma^* = \{\text{union of the boundary submanifolds which are transverse to } l^*\}$ and

$\partial^v \Sigma^* = \{\text{union of the boundary submanifolds which are parallel to } l^*\}$.

From this decomposition and by construction we can provide its general form, given by

$$
\partial^h \Sigma^* = (I^u \times [\bigcup_{j=0}^{s-1}(I^j \times \{1\} \times I^{s-j-1})]) \bigcup (I^u \times [\bigcup_{j=0}^{s-1}(I^j \times \{1\} \times I^{s-j-1})])
$$

$$
\partial^v \Sigma^* = (|\bigcup_{j=0}^{u-1}(I^j \times \{1\} \times I^{u-j-1})| \times I^s) \bigcup (|\bigcup_{j=0}^{u-1}(I^j \times \{1\} \times I^{u-j-1})| \times I^s),
$$

where $I^0 \times I = I$.

In fact, we can observe as the above definitions are extending the concept of singular cross-section for the higher dimensional case. These definitions are outstanding for the next results. Firstly, the Figure 1 shows the well known three dimensional case, with a 2-dimensional singular cross-section that allows only one 1-dimensional stable foliation.
Finiteness and Existence on Sectional Hyperbolic Sets

\[ \dim(l^*) = \dim(W^{ss}(\sigma)) = 1 \]

Secondly, the Figure 2 shows the 4-dimensional case, where it is exhibiting two different cases for the stable foliation of a 3-dimensional singular cross section. Then, the new singular cross section definition allows different stable foliations and this in turn allows to construct different singular cross-sections of Lorenz-like singularities.

Hereafter, we denote \( \Sigma^* = B^u[0, 1] \times B^{ss}[0, 1] \).

**Definition 2.8.** Let \( A \) and \( B \) be compact sets of \( M \), and let \( d(\cdot, \cdot) \) be a metric on \( M \). Define the Hausdorff distance between \( A \) and \( B \) by

\[
d_H(A, B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.
\]

We define \( K(M) = \{ A \subset M \mid A \text{ is compact} \} \).

**Remark 2.** We have that \( d_H \) is a metric on \( K(M) \) and the metric space \( (K(M), d_H) \) is compact by Blaschke’s selection theorem.

**Proposition 1.** Let \( \Lambda \) be a sectional hyperbolic set of a \( C^1 \) vector field \( X \) of \( M \). Let \( \sigma \) be a Lorenz-like singularity of \( X \) in \( \Lambda \). Let \( Y^n \) be a sequence of vector fields converging to \( X \) in the \( C^1 \) topology. Then, there are a neighborhood \( U \subset M \) of \( \Lambda \), a singular-cross section \( \Sigma^t, \Sigma^b \) of \( \sigma \) in \( M \) and \( N \in \mathbb{N} \) enough large such that for every \( n \geq N \) one has

\[(\Lambda_{Y^n}) \cap (\partial^h\Sigma^t \cup \partial^h\Sigma^b) = \emptyset.\]

**Proof.** We fix the neighborhood \( U \) of \( \Lambda \) as in Lemma 2.2.

By using the Hausdorff’s metric and by Remark 2 we have that there exists a subsequence of sectional hyperbolic sets in \( (\Lambda_{Y^n})_{n \in \mathbb{N}} \) that converges to a compact invariant set in \( Cl(U) \). Without loss of generality, we say that the sequence itself converges to a compact invariant set in \( Cl(U) \), i.e.,

\[
\Lambda_{Y^n} \xrightarrow{h} \tilde{\Lambda},
\]

where \( \tilde{\Lambda} \) is compact invariant set and “\( \xrightarrow{h} \)” denotes the convergence on the Hausdorff distance.

Since \( \Lambda_{Y^n} \xrightarrow{h} \tilde{\Lambda}, \) given \( \epsilon > 0 \), there exists \( N_1 \in \mathbb{N} \) such that if \( n \geq N_1 \) one has

\[
d_H(\Lambda_{Y^n}, \tilde{\Lambda}) < \frac{\epsilon}{4}. \tag{3}
\]
Let us prove that \( \overline{\Lambda} = \Lambda_X \). Firstly, for \( m \in \mathbb{N} \), every vector field \( Y^m \) and \( X \), we define the following sets

\[
\Lambda^m = \cap_{|t| \leq m} Y^m(\text{CL}(U)) \quad \text{and} \quad \Lambda_X^m = \cap_{|t| \leq m} X_t(\text{CL}(U)).
\]

By construction we have that \( \Lambda^m \overset{h}{\rightarrow} \Lambda_Y^m \) and \( \Lambda_X^m \overset{h}{\rightarrow} \Lambda_X \) if \( m \to \infty \).

Secondly, it follows from the \([1]\) that there exists \( M = M_1(n) \in \mathbb{N} \) and \( M_2 \in \mathbb{N} \) such that if \( m \geq M_1 \) and \( k \geq M_2 \) then

\[
d_H(\Lambda^m, \Lambda_Y^m) < \frac{\epsilon}{4} \quad \text{and} \quad d_H(\Lambda_X^m, \Lambda_X^k) < \frac{\epsilon}{4}.
\]

Since \( Y^m \overset{\epsilon}{\rightarrow} X \), given \( m \in \mathbb{N} \), there exists \( N_2(m) \in \mathbb{N} \) such that if \( n \geq N_2(m) \) one has

\[
\Rightarrow \quad d_H(\cap_{|t| \leq m} Y^m(\text{CL}(U)), \cap_{|t| \leq m} X_t(\text{CL}(U))) < \frac{\epsilon}{4} \quad \Rightarrow \quad d_H(\Lambda^m, \Lambda_X^m) < \frac{\epsilon}{4}.
\]

Let \( M = \max \{ M_1(N_1), M_2 \} \in \mathbb{N} \) and \( N = \max \{ N_1, N_2(M(N_1)) \} \). Thus, by using \([3],[5] \) and \([6] \) we have

\[
d_H(\overline{\Lambda}, \Lambda_X) \leq d_H(\Lambda, \Lambda_Y^m) + d_H(\Lambda_Y^m, \Lambda_X) \leq d_H(\Lambda, \Lambda_Y^m) + d_H(\Lambda_Y^m, \Lambda_Y^N) + d_H(\Lambda_Y^N, \Lambda_X) + d_H(\Lambda_X^M, \Lambda_X) < \epsilon,
\]

and it follows that \( \overline{\Lambda} = \Lambda_X \).

Since \( \Lambda_X \) is sectional hyperbolic, by using the Theorem \([2,3] \) for \( \Lambda_X \), the equality implies that the negative orbit of every point in \( W^s_X(\sigma) \setminus \{ \sigma \} \) does not intersect \( \text{CL}(U) \).

Now, as in \([16] \) Lemma 3, for \( \epsilon > 0 \) let us fix a fundamental domain \( D_\epsilon \) of the vector field’s flow \( X_t \) restricted to the local stable manifold \( W^s_{\text{loc}}(\sigma) \) \([22] \) (homeomorphic to the sphere \( s \)-dimensional). Then, we define a \( \tilde{D} \), cross section of \( X \) such that \( W^s_{\text{loc}}(\sigma) \cap \tilde{D} = D_\epsilon \). It follows that \( \tilde{D} \) is a \((n-1)\)-cylinder, and by considering a system coordinated \((x,s)\) with \( x \in D_\epsilon \) and \( s \in I^n \) we can construct a family of singular cross-sections \( \Sigma^s_\delta \), \( \Sigma^b_\delta \) (for all \( \delta \in [-\epsilon, \epsilon] \)) by setting

\[
\Sigma^s_\delta = \{ (x,s) \in \tilde{D}_\epsilon : x \in S_\delta, s \in I^n \},
\]

\[
\Sigma^b_\delta = \{ (x,s) \in \tilde{D}_\epsilon : x \in S_{-\delta}, s \in I^n \}.
\]

Hence, by using the Theorem \([2,3] \) and this construction, we can set a singular-cross section \( \Sigma^s, \Sigma^b \) nearby \( \sigma \) such that

\[
\Lambda_X \cap (\partial^s \Sigma^s \cup \partial^b \Sigma^b) = \emptyset.
\]

Since \( \Lambda_X \) is maximal invariant of \( \text{CL}(U) \) and the boundary of \( \Sigma^s, \Sigma^b \) is compact we can find \( T > 0 \) such that

\[
X_T(\text{CL}(U)) \cap (\partial^s \Sigma^s \cup \partial^b \Sigma^b) = \emptyset.
\]

Thus, there exists \( N \in \mathbb{N} \) such that for every \( n \geq N \) one has

\[
Y^m_T(\text{CL}(U)) \cap (\partial^s \Sigma^s \cup \partial^b \Sigma^b) = \emptyset.
\]

The result follows since \( \Lambda_Y^m \subset Y^m_T(\text{CL}(U)) \).

\[\square\]

**Remark 3.** We can observe that if there is \( x \in \Lambda \setminus W^{ss}(\sigma) \) such that \( \sigma \in \omega_X(x) \subset \Lambda \), then \( \sigma \) is Lorenz-like and satisfies the Theorem \([2,3],[3],[15] \). It shows that the outstanding dynamic of the system is found around of the Lorenz-like singularities.
Corollary 3. Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in X^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Let $\sigma$ be a singularity of $X$ in $\Lambda$. Then, there is a neighborhood $V$ of $W^{ss}(\sigma) \setminus \{\sigma\}$ such that

$$\Lambda \cap V = \emptyset.$$ 

Proof: The equality in Theorem 2.3 implies that the negative orbit of every point in $W^{ss}(\sigma) \setminus \{\sigma\}$ does not intersect $\Lambda$. Given $x \in (W^{ss}(\sigma) \setminus \{\sigma\})$, we denote by $l_x$ the distance of $x$ to $\Lambda$, i.e., $l_x = d(x, \Lambda)$. Then, we define the neighborhood $V$ as follows

$$V = \bigcup_{x \in W^{ss}(\sigma) \setminus \{\sigma\}} B(x, \frac{l_x}{2}).$$

Thus, by construction $V$ satisfies that $\Lambda \cap V = \emptyset$ and the proof follows. 

3. Finiteness and existence. We start by recalling some useful definitions to prove the lemmas and the propositions that provide very important properties on sectional hyperbolic sets, that in our case support the main theorems’ proofs.

Let $O = \{X_t(x) : t \in \mathbb{R}\}$ be the orbit of $X$ through $x$, then the stable and unstable manifolds of $O$ defined by

$$W^s(O) = \bigcup_{x \in O} W^{ss}(x),$$

and

$$W^u(O) = \bigcup_{x \in O} W^{uu}(x)$$

are $C^1$ submanifolds tangent to the subbundles $E^s_x \oplus E^u_x$ and $E^u_x \oplus E^s_x$, respectively.

A homoclinic orbit of a hyperbolic periodic orbit $O$ is an orbit in $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_qM = T_qW^s(O) + T_qW^u(O)$ for some (and hence all) point $q \in \gamma$, then we say that $\gamma$ is a transverse homoclinic orbit of $O$.

Definition 3.1. The homoclinic class $H(O)$ of a hyperbolic periodic orbit $O$ is the closure of the union of the transverse homoclinic orbits of $O$. We say that an invariant set $L$ is a homoclinic class if $L = H(O)$ for some hyperbolic periodic orbit $O$.

Recall that $Sing(X)$ denotes the set of singularities of the vector field $X$ and $Cl(A)$ denotes the closure of $A$, $A \subset M$. Moreover, for $\delta > 0$ we define $B_\delta(A) = \{x \in M : d(x, A) < \delta\}$, where $d(\cdot, \cdot)$ is the metric in $M$.

Lemma 3.2. Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $X \in X^1(M)$. Let $\Lambda \subset M$ be a hyperbolic set of $X$. Then, there are a neighborhood $U \subset X^1(M)$ of $X$, a neighborhood $U \subset M$ of $\Lambda$ and $n_0 \in \mathbb{N}$ such that

$$\# \{L \subset U : L \text{ is homoclinic class of } Y \in U\} \leq n_0$$

for every vector field $Y \in U$.

Proof: We prove the Lemma by contradiction. So, we suppose that there exists a sequence of vector fields $(X^n)_{n \in \mathbb{N}} \subset U$, $X^n \overset{C^1}{\to} X$ and such that

$$\# \{L \subset U : L \text{ is homoclinic class of } X^n\} \geq n.$$ 

It is well known [14] that the periodic orbits are dense in $L \subset \Lambda^n = \Lambda_X^n$, for all $n \in \mathbb{N}$. Recall that the homoclinic classes are pairwise disjoint on hyperbolic sets.

Let us consider $\eta > 0$ such that $0 < \eta < \frac{\epsilon}{2}$, where $\epsilon > 0$ is given by [2]. Let $\bigcup_{x \in Cl(U)} B(x, \frac{\eta}{2})$ be the collection of open balls of radius $\frac{\eta}{2}$ covering $Cl(U)$.

Since $Cl(U)$ is a compact neighborhood of $\Lambda$, this one admit a finite sub-coverage, i.e., there exists some sub-collection consisting only of finitely many open balls of radius $\frac{\eta}{2}$ which also covers $Cl(U)$. We denote this finite number by $n_0$. 
It follows from definition of homoclinic class and (2) that given points $p_1$ and $p_2$ of $L$, if $d(p_1, p_2) < \eta$ then

$$W_X^u(p_1, \epsilon) \cap W_X^u(p_2, \epsilon) \neq \emptyset. \quad (7)$$

In particular, if we choose $N \in \mathbb{N}$ such that $N > n_0$, we have that the sequence's element $X^N$ exhibits more than $N$ homoclinic classes in $Cl(U)$. Since $Cl(U)$ is covered by $n_0$ balls and $N > n_0$, it states that $X^N$ exhibits at least two homoclinic classes contained at same $\frac{\delta}{2}$-ball. We denote by $L_1^N$ and $L_2^N$ these homoclinic classes.

Since $L_1^N$ and $L_2^N$ are homoclinic classes, there are periodic points $p^1$ and $p^2$ of $L_1^N$ and $L_2^N$ respectively satisfying (7), and it states that $p^1$ and $p^2$ belongs to the same homoclinic class. Therefore, it shows that $L_1^N = L_2^N$.

Then, we obtain that $X^N$ exhibits finite homoclinic classes in $U$. This is a contradiction and the proof follows.

\section*{4. Attractor part}

The following lemma will be useful for the proof of finiteness related with attractors on sectional hyperbolic sets.

We start by recalling some useful and fundamental properties of attractors and Lorenz-like singularities. In particular, recall that an attractor contains the unstable manifold of all its points. Also, recall that given a Lorenz-like singularity there is a singular cross section for it. These properties are a fundamental tool for the next result, and for this reason, we will give a global idea about the next technical proof of finiteness. Thus, note that if the attractor intersects the section transversal of the singularity, then the singularity consequently will be contained in the attractor. So, it provides the finiteness. Otherwise, if not, one can consider the distance between the stable manifold of the singularity and the attractor in the cross section, and this one will be positive. Note that this distance is minimal. However, by using the unstable manifold of the points in common between the attractor and the singular cross section, we can found points still inside which one obtains that the distance becomes to be less, that is a contradiction. The proof follows this way.

\begin{proposition}
Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Then, there are neighborhoods $U$ of $X$, $U$ of $\Lambda$ and $n_0 \in \mathbb{N}$ such that

$$\#\{A \subset U : A \text{ is an attractor of } Y \in U\} \leq n_0.$$

\end{proposition}

\begin{proof}
First, we fix the neighborhood $U$ of $\Lambda$ as in Lemma 2.2. We prove the proposition by contradiction. So, we suppose that for $n \in \mathbb{N}$, one has that for all neighborhood $U$ of $X$ there exists $Y \in U$ such that

$$\#\{A \subset U : A \text{ is an attractor of } Y \in U\} \geq n.$$  

Then, we consider a sequence of vector fields $(X^n)_{n \in \mathbb{N}}$, such that $X^n \subset X$ and each $X^n$ is exhibiting at least $n$ attractors in $U$. Therefore, by choosing an attractor $A^n$ of $X^n$, we have a sequence of attractors $(A^n)_{n \in \mathbb{N}}$ in $U$. By compactness we can suppose that the attractors are non-singular, since the singularities are isolated.

We assert that the sequence $(A^n)_{n \in \mathbb{N}}$ of attractors accumulates on the singularities of $X$, otherwise if $Sing(X) \cap (\bigcap_{N > 0} Cl(\cup_{n \geq N} A^n)) = \emptyset$, then it could exist $\delta > 0$, such that $B_{\delta}(Sing(X)) \cap (\cup_{n \in \mathbb{N}} A^n) = \emptyset$.

Thus, we define

$$H = \bigcap_{t \in \mathbb{R}} X_t (U \setminus B_{\delta/2}(Sing(X))). \quad (8)$$

\end{proof}
By definition $H$ is an invariant set and $Sing(X) \cap H = \emptyset$. Additionally, $H$ is a compact set as $\Lambda$ is too, and therefore $H$ is a nonempty compact set [13], [10]. It follows from the Lemma 2.2 that $H$ is a hyperbolic set and by using the Lemma 3.2 there is $n_0 \in \mathbb{N}$ such that the sequence of attractors is bounded by $n_0$, that is a contradiction.

Then, the sequence $(A^n)_{n \in \mathbb{N}}$ of attractors accumulates on the singularities of $X$, i.e., $Sing(X) \cap (\bigcap_{N > 0} Cl((\cup_{m \geq N} A^n))) = \emptyset$. Thus, there exists $\sigma \in U$ such that $\sigma \in Sing(X) \cap (\bigcap_{N > 0} Cl((\cup_{m \geq N} A^n)))$.

The subbundle $E^s$ of $\Lambda$ extends to a contracting subbundle on the whole $U$ and we take a continuous extension of $E^s$ in $U$ (not necessarily invariant). We have that this extension persists by small perturbations of $X$ [13] and we denote the splitting by $E^{s,n} \oplus E^{c,n}$, where $E^{s,0} \oplus E^{c,0} = E^s \oplus E^c$.

On the other hand, it is well known that a hyperbolic singularity $\sigma$ of a $C^1$ vector field $X$ has a continuation $\sigma(Y)$ for every $C^1$ vector field $Y$ close to $X$. In fact, this continuation is done by using the standard Function Implicit Theorem [22] [Page 3]. Also, it exhibits a smooth variation of $W^s_\sigma(\sigma(Y))$ with respect to $Y$ close to $X$.

Since $X^n \to X$, we have that $\Sigma$ is a singular cross section of $X^n$ too. Thus, in order to simplify the notation and by using the Function Implicit Theorem, we can suppose $\sigma = \sigma(X^n)$ and $l^t \cup l^b \subset W^s_{X,n}(\sigma)$ for all $n$.

Recall that a singular cross-section is a pair of cross-sections of the Lorenz-like singularity, and we denote ones by $\Sigma^t$ (top) and $\Sigma^b$ (bottom), or by simplicity we denote $\Sigma^*$ any one them $(*=t,b)$. As before we fix a coordinate system $(x, y) = (x^*, y^*)$ in $\Sigma^*$ with $(*=t,b)$ and such that $\Sigma^* = B^u[0,1] \times B^s[0,1]$ and $l^* = \{0\} \times B^u[0,1]$ with respect to $(x, y)$.

In order to obtain a refinement on the singular cross-section, since each component $\Sigma^* = B^u[0,1] \times B^s[0,1]$, we will set up a family of singular cross-sections as follows: given $0 < \Delta \leq 1$ small, for each component, we define $\Sigma^s, \Delta = B^u[0,\Delta] \times B^s[0,1]$, such that

$$l^* \subset \Sigma^{s,\Delta} \subset \Sigma^*, \text{i.e.,}$$

$$(l^* = \{0\} \times B^s[0,1]) \subset (\Sigma^{s,\Delta} = B^u[0,\Delta] \times B^s[0,1]) \subset (\Sigma^* = B^u[0,1] \times B^s[0,1]),$$

where we fix a coordinate system $(x^*, y^*)$ in $\Sigma^*$ $(*=t,b)$. We will assume that $\Sigma^* = \Sigma^{s,1}$.

Denote by $\Pi^*: \Sigma^* \to B^u[0,1]$ the projection, where $\Pi^*(x,y) = x$ and for $\Delta > 0$ we define $\Sigma^s, \Delta = B^u[0,\Delta] \times B^s[0,1]$. Thus, we have that $\Sigma^{s,\Delta} = \Sigma^t, \Delta \cup \Sigma^b, \Delta$.

From the Theorem 2.3 we have that $\Lambda \cap W^s_\sigma(\sigma) = \{\sigma\}$ and by using the Lemma 2.2 one has that $A^n$ is a hyperbolic attractor of type saddle of $X^n$ for all $n$. By Proposition 1 and by using the neighborhood $U$ of $\Lambda$, we can find $\Sigma^t, \Sigma^b$, singular-cross section for $\sigma$ in $U$ such that

$$(A_{X^n}) \cap (\partial^h \Sigma^t \cup \partial^b \Sigma^b) = \emptyset. \quad (9)$$

As the splitting $E^s \oplus E^c$ persists by small perturbations of $X$ [13], the dominance condition [Definition 1.1 (2)] together with [10] Proposition 2.2 imply that for $*=t,b$ one has

$$T_x \Sigma^* \cap (E^s_x \oplus E^c_x) = T_x l^*,$$

for all $x \in l^*$.

Denote by $\angle(E,F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle(T_x \Sigma^* \cap E^c_x, T_x l^*) > \rho,$$
for all $x \in l^* \ (s = t, b)$. In this way, since $E_{n}^{c,n} \to E^{c}$ as $n \to \infty$ we have for $n$ large enough that
\[ \angle(T_{x}X_{n}^{c},E_{x}^{c,n},T_{x}l^{*}) > \frac{\rho}{2}, \] (10)
for all $x \in l^{*} \ (s = t, b)$. The continuity of $E_{n}^{c,n}$ and (10) imply that there is $\Delta_{0} > 0$ such that for every $n$ large the line $F^{n}$ is transverse to $\Pi^{*}$. By this we mean that $F^{n}(z)$ is not tangent to the curves $(\Pi^{*})^{-1}(c)$, for every $c \in B^{n}[0, \Delta_{0}]$. Recall that $A^{n}$ is a hyperbolic attractor of type saddle of $X^{n}$ for all $n$ and the periodic orbits of $X^{n}$ in $A^{n}$ are dense in $A^{n}$ [23]. As $\sigma \in Cl \ (\bigcup_{n \in N} A^{n})$, we can find a sequence of periodic orbits $(O_{n})_{n \in N}$, such that $O_{n} \in A^{n}$ and accumulating on $\sigma$. It follows from the [16] Lemma 3 that there exists $n_{1} \in N$ such that either
\[ O_{n_{1}} \cap int(\Sigma^{t,\Delta_{0}}) \neq \emptyset \text{ or } O_{n_{1}} \cap int(\Sigma^{b,\Delta_{0}}) \neq \emptyset. \]
As $O_{n_{1}} \subset A_{n_{1}}$, we can conclude that either
\[ A^{n_{1}} \cap int(\Sigma^{t,\Delta_{0}}) \neq \emptyset \text{ or } A^{n_{1}} \cap int(\Sigma^{b,\Delta_{0}}) \neq \emptyset. \]
We shall assume that $A^{n_{1}} \cap int(\Sigma^{t,\Delta_{0}}) \neq \emptyset$ (Analogous proof for the case $s = b$). By [9] we have $A^{n_{1}} \cap \partial^{h}\Sigma^{t,\Delta_{0}} = \emptyset$ and by compactness we have that there is $p \in \Sigma^{t,\Delta_{0}} \cap A^{n_{1}}$ such that
\[ dist(\Pi^{t}(\Sigma^{t,\Delta_{0}} \cap A^{n_{1}}), 0) = dist(\Pi^{t}(p), 0), \]
where $dist$ denotes the distance in $B^{u}[0, \Delta_{0}]$. Note that $dist(\Pi^{t}(p), 0)$ is the minimum distance of $\Pi^{t}(\Sigma^{t,\Delta_{0}} \cap A^{n_{1}})$ to 0 in $B^{u}[0, \Delta_{0}]$. By domination [Definition 1.1-(2)], $T_{z}(W_{X^{n_{1}}}(p)) = E_{z}^{c,n_{1}}$ for every $z \in W_{X^{n_{1}}}(p)$ and hence, $dim(W_{X^{n_{1}}}(p)) = (n - s - 1)$. Next, we can ensure that
\[ T_{z}(W_{X^{n_{1}}}(p)) \cap T_{z}\Sigma^{t,\Delta_{0}} = E_{z}^{c,n_{1}} \cap T_{z}\Sigma^{t,\Delta_{0}} = F_{z}^{n_{1}}, \]
for every $z \in W_{X^{n_{1}}}(p) \cap \Sigma^{t,\Delta_{0}}$. First, note that the last equality shows that $W_{X^{n_{1}}}(p) \cap \Sigma^{t,\Delta_{0}}$ is transversal, and therefore there exists some compact submanifold inside of $W_{X^{n_{1}}}(p) \cap \Sigma^{t,\Delta_{0}}$. We denote this compact submanifold by $K^{n_{1}}$. Thus, by construction $p \in K^{n_{1}}$ and $K^{n_{1}}$ is tangent to $F^{n_{1}}$, since $K^{n_{1}} \subset W_{X^{n_{1}}}(p) \cap \Sigma^{t,\Delta_{0}}$. Then, this leads to that $K^{n_{1}}$ is transverse to $\Pi^{t}$ (i.e. $K^{n_{1}}$ is transverse to the curves $(\Pi^{t})^{-1}(c)$, for every $c \in B^{u}[0, \Delta_{0}])$ [16]. Let us denote the image of $K^{n_{1}}$ by the projection $\Pi^{t}$ in $B^{u}[0, \Delta_{0}]$ by $K_{1}^{n_{1}}$, i.e., $\Pi^{t}(K^{n_{1}}) = K_{1}^{n_{1}}$. Note that $K_{1}^{n_{1}} \subset B^{u}[0, \Delta_{0}]$ and $\Pi^{t}(p) \in int(K_{1}^{n_{1}})$. Since $dim(K_{1}^{n_{1}}) = dim(B^{u}[0, \Delta_{0}]) = (n - s - 1)$ there is $z_{0} \in K^{n_{1}}$ such that
\[ dist(\Pi^{t}(z_{0}), 0) < dist(\Pi^{t}(p), 0). \]
Recall that we have $A^{n_{1}} \cap \partial^{h}\Sigma^{t,\Delta_{0}} = \emptyset$ [9], $K^{n_{1}} \subset W_{X^{n_{1}}}(p)$ and $dim(K^{n_{1}}) = dim(B^{u}[0, \Delta_{0}])$ by construction. So, we conclude that $dist(\Pi^{t}(\Sigma^{t,\Delta_{0}} \cap A^{n_{1}}), 0) = 0$ and this one last equality implies that
\[ A^{n_{1}} \cap l^{t} \neq \emptyset. \]
Since $l^{t} \subset W_{X^{n_{1}}}(\sigma)$ and $A^{n_{1}}$ is a closed invariant set for $X^{n_{1}}$ we obtain that $\sigma \in A^{n_{1}}$. By hypothesis, $A^{n}$ is non-singular for all $n \in N$, so this leads to a contradiction and the proof follows.
5. Repeller part. The following lemmas will be useful for the finiteness and existence’s proof related with repellers on sectional hyperbolic sets.

**Lemma 5.1.** Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Let $Y^n$ be a sequence of vector fields converging to $X$ in the $C^1$ topology. Then, there is a neighborhood $U \subset M$ of $\Lambda$, such that if $R^n$ is a repeller of $Y^n$, $R^n \subset \Lambda_{Y^n}$, for each $n \in \mathbb{N}$, then the sequence $(R^n)_{n \in \mathbb{N}}$ of repellers does not accumulate on the singularities of $X$, i.e.,

$$Sing(X) \cap (\cap_{N>0} Cl(\cup_{m \geq N} R^n)) = \emptyset.$$  

**Proof.** Fix the neighborhood $U$ of $\Lambda$ as in Lemma 2.2. Assume by contradiction that

$$Sing(X) \cap (\cap_{N>0} Cl(\cup_{m \geq N} R^n)) \neq \emptyset.$$  

Then, there exists a subsequence $(x_{nk})_{k \in \mathbb{N}}$, with $x_{nk} \in R^{nk} \subset \Lambda_{Y^{nk}}$ for all $k \in \mathbb{N}$ and such that $x_{nk} \to \sigma, \sigma \in Sing(X)$. Without loss of generality, we can suppose that sequence itself converges to $\sigma$.

Let $\epsilon > 0$ be given by (2). As $x_n \to \sigma$,

$$W^s_{Y^n}(x^n, \epsilon) \to W^s_X(\sigma, \epsilon)$$  

in the sense of $C^1$ manifolds [23]. Then for $N \in \mathbb{N}$ enough large, and by using the Corollary [3] we have that

$$W^s_{Y^n}(x^N, \epsilon) \cap V \neq \emptyset.$$  

By using the Hausdorff’s metric and by Remark (2) we have that the repellers sequence converges to a compact invariant set in $\Lambda_X$, i.e.,

$$R^n \overset{h}{\to} R,$$  

where $R$ is a compact invariant set. Thus, $R \subset \Lambda_X$.

Since that $W^s(x^n, \epsilon)$ is included in $R^n$, we have that the limit $W^s(\sigma, \epsilon)$ is included in $R$ (11) and (13).

By using (12) and (13), we obtain that $W^s(\sigma, \epsilon) \subset R \subset \Lambda_X$ and therefore $\Lambda_X \cap V \neq \emptyset$. This is a contradiction. \qed

5.1. Existence.

**Lemma 5.2.** Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a connected sectional hyperbolic set of $X$ with singularities. If $\Lambda \subset \Omega(X)$, then $\Lambda$ has no repellers.

**Proof.** We prove the lemma by contradiction. Since $\Lambda$ has singularities, it cannot be a repeller. Then, we assume that $\Lambda$ at least contains a repeller that will be denoted by $R$. It follows from definition of repeller that there exists $U$ isolant block negatively invariant of $R$, such that $R = \cap_{t \leq 0} X_t(U)$, where $U \subset \Lambda$.

Since $\Lambda$ a is connected set and $R \subset \Lambda$, we can pick $p \in \Lambda$ be such that $p \in int(U) \setminus R$. So, we assert that there exists time $\tau < 0$ such that $p \notin X_\tau(U)$ and therefore $X_{-\tau}(p) \notin U$. As $R$ is compact, there is an open neighborhood $V \subset int(U) \setminus R$ of $p$ such that $X_{-\tau}(V)$ is not contained in $U$. By construction $p \in V \subset \Omega(X)$, thus we can choose time $T > 0$ with $T > -\tau > 0$, such that $X_T(V) \cap V \neq \emptyset$. So, there exists $q \in X_T(V) \cap V$ and one has
6. **Proof of the main theorems.**

6.1. **Proof of main Theorem 1**

**Proof.** We prove the theorem by contradiction. Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Then, we suppose that there is a sequence of vector fields $(X^n)_{n \in \mathbb{N}} \subset \mathcal{X}^1(M)$, $X^n \supset X$ such that every vector field $X^n$ exhibits $n$ attractors or repellers, with $n > n_0$. It follows from the Definition 2.8 and Remark 3.2 that there are neighborhoods $U \subset \mathcal{X}^1(M)$ of $X$ and $U \subset M$ of $\Lambda$ such that the attractors in $U$ are finite for all vector field $Y$ in $U$. Thus, we are left to prove only for the repeller case. We denote by $R^n$ a repeller of $X^n$ in $\Lambda_{X^n}$. Since $\Lambda_{X^n}$ is arbitrarily close to $\Lambda_X$ and $R^n \in \Lambda_{X^n}$, $R^n$ is also arbitrarily close to $\Lambda_X$. Therefore, we can assume that $R^n$ belongs to $\Lambda_X$ for all $n$.

Let $(R^n)_{n \in \mathbb{N}}$ be the sequence of repellers contained in $\Lambda_X$. By using the Lemma 5.1 we have that

$$Sing(X) \cap (\cap_{N \geq 0} Cl(\cup_{m \geq N} R^n)) = \emptyset.$$  

Then, we can find $\delta > 0$, such that $B_\delta(Sing(X)) \cap (\cup_{n \in \mathbb{N}} R^n) = \emptyset$.

As in 3.1 we define $H = \cap_{n \in \mathbb{N}} X_i \setminus B_{\delta/2}(Sing(X))$.

It follows from the Lemma 5.2 that $H$ is a hyperbolic set and beside the Lemma 3.2 we have that there are neighborhoods $U \subset \mathcal{X}^1(M)$ of $X$, $U \subset M$ of $H$ and $n_1 \in \mathbb{N}$ such that

$$\#\{R \subset U : R \text{ is a repeller of } Y \in U\} \leq n_1 \leq n_0.$$  

Note that the last inequality holds for every vector field $Y \in U$, but this leads to a contradiction, since by hypothesis we have that

$$\#\{R \subset H : R \text{ is a repeller of } Y \in U\} \geq n > n_0.$$  

**□**

Before starting the proof of Main Theorem 2 recall the Definition 2.8 and Remark 3.2

6.2. **Proof of main Theorem 2**

**Proof.** We prove the theorem by contradiction. It follows from the Lemma 5.2 that $\Lambda$ has no repellers. Then we suppose that for all neighborhood $U$ of $\Lambda$ there exists a vector field $C^1$ close to $X$ exhibiting a repeller in $U$. Thus, we begin by considering a sequence of the vector fields $(X^n)_{n \in \mathbb{N}} \subset \mathcal{X}^1(M)$, with $X^n \supset X$ and each one is exhibiting a repeller $R^n$ in $U$.

So, using the Lemma 5.1 we have that $Sing(X) \cap (\cap_{N \geq 0} Cl(\cup_{m \geq N} R^n)) = \emptyset$. Then, we can find $\delta > 0$ such that $B_\delta(Sing(X)) \cap (\cup_{n \in \mathbb{N}} R^n) = \emptyset$. 

We define $H = \bigcap_{t \in \mathbb{R}} X_t \left( U \setminus B_{\frac{1}{2}}(\text{Sing}(X)) \right)$ and hence we can assume that $H$ is a hyperbolic set \[10\].

Beside definition of $H$ and from the Lemma \[5.1\] we can suppose that the sequence $(R^n)_{n \in \mathbb{N}}$ is contained in $U$.

By using the Hausdorff’s metric and by Remark \[2\] we have that the repellers’ sequence converges to a compact invariant set in $H$, i.e.,

$$ R^n \xrightarrow{n} R, $$

where $R$ is a compact invariant set and “$\xrightarrow{n}$” denotes the convergence on the Hausdorff distance. Therefore, $R \subset H \subset \Lambda$.

As $R^n$ is a hyperbolic set, we can choose a periodic point $p^n \in R^n$ for every $n \in \mathbb{N}$ and it follows that the sequence $(p^n)_{n \in \mathbb{N}}$ converges to a point $p \in R$. From the hyperbolicity $H$ and as $X^n \longrightarrow X$, we have that $W_{X,p}^{ss}(p^n, \epsilon) \longrightarrow W_{X,p}^{ss}(p, \epsilon)$ in the sense of $C^1$ submanifolds \[23\], where $\epsilon > 0$ is given by \[2\].

It is well known that the repeller sets contain the stable manifold of all its points. So, $W_{X,p}^{ss}(p^n, \epsilon) \subset R^n$ for all $n \in \mathbb{N}$ and by \[14\], we have that $W_{X,p}^{ss}(p, \epsilon) \subset W_{X,\Lambda}^{ss}(O_p) \subset R$, since $R$ is a compact invariant set.

From the above we obtain $\text{Cl}(W_{X,\Lambda}^{ss}(O_p)) \subset R$ and in particular $\text{Cl}(W_{X,\Lambda}^{ss}(O_p))$ is a hyperbolic set contained in $R \subset \Lambda \subset \text{Cl}(X)$, which can be used to construct a hyperbolic repeller inside $R$. Specifically, the $\alpha$-limit set $\alpha(W_{X,\Lambda}^{ss}(O_p))$ would be such a set and note that $\alpha(W_{X,\Lambda}^{ss}(O_p)) \subset \text{Cl}(W_{X,\Lambda}^{ss}(O_p)) \subset R$. Finally, there is a hyperbolic repeller contained in $\Lambda \subset \text{Cl}(X)$. This is a contradiction by Lemma \[5.2\] and the result follows.

6.3. Proof of the corollaries.

Proof of Corollary \[4\]. Since $X$ is a sectional Anosov flow, then its maximal invariant $M(X)$ is a sectional hyperbolic set for $X$. By using the Main Theorem \[1\] for $M(X)$ the proof follows.

Proof of Corollary \[3\]. From \[4\], we have that the sectional hyperbolic three-dimensional flows are open and dense in the open set of flows which are far from homoclinic tangencies. By using Main Theorem \[0\] we obtain finiteness of attractors and repellers and the proof follows.

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