Some Cobweb Posets Digraphs’ Elementary Properties and Questions

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Abstract: A digraph that represents reasonably a scheduling problem should have no cycles i.e. it should be DAG i.e. a directed acyclic graph. Here down we shall deal with special kind of graded DAGs named KoDAGs. For their definition and first primary properties see [1], where natural join of di-bi graphs (directed bi-parted graphs) and their corresponding adjacency matrices is defined and then applied to investigate cobweb posets and their Hasse digraphs called KoDAGs. In this report we extend the notion of cobweb poset while delivering some elementary consequences of the description and observations established in [1].

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1 Introduction to the subject

It is now a Wiki important knowledge that an incidence structure is a triple \( C = (P, L, I) \) where \( P \) is a set of points, \( L \) is a set of lines and \( I \subseteq P \times L \) is the incidence relation. \( (I = P \times L \text{ for KoDAGs}) \) (compare: \( V = P \cup L, P \cap L = \emptyset \); \( P = \) black vertices = points, \( L = \) white vertices = lines).

The elements of \( I \) are called flags. If \((p, l) \in I\) we say that point \( p \) "lies on" line \( l \). The relation \( I \) is equivalently defined by its bipartite digraph \( G(I) \). The relation \( I \) and its bipartite digraph \( G(I) \) are equivalently defined by theirs biadjacency matrix. The example of thus efficiently coded finite geometries include such popular examples as Fano plane - a coding portrait of the distinguished composition algebra of John T. Graves octonions (1843), a friend of William Hamilton, who called them octaves [2].

The incidence matrix of an incidence structure \( C \) is a biadjacency matrix of the Levi graph of the \( C \) structure.

The biadjacency matrix of a finite bipartite graph \( G \) with \( n \) black vertices and \( m \) white vertices is an \( n \times m \) matrix where the entry \( a_{ij} \) is the number of edges joining black vertex \( i \) and white vertex \( j \). In the special case of a finite, undirected, simple bipartite graph, the biadjacency matrix is a Boolean \((0,1)\)-matrix.

The adjacency matrix \( A \) of a bipartite graph with the reduced adjacency or
-under synonymous substitution- the biadjacency Boolean matrix $B$ is given by

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

The adjacency matrix $A$ of a bipartite digraph $\overrightarrow{K_{k,l}}$ (see: [1]) coded via its reduced adjacency or biadjacency Boolean matrix $B$ is according to [1] defined by

$$A = \begin{pmatrix} 0_{k,k} & B(k \times l) \\ 0_{l,k} & 0_{l,l} \end{pmatrix}, \text{ where } k = |P|, \ l = |L|$$

**Example 1**

![Diagram](image1)

**Figure 1:**

![Diagram](image2)

**Figure 2:**

Fig.1 displays (upside down way with respect to drawings in [1]) the bipartite digraph $\overrightarrow{K_{2,3}}$. It is obviously Ferrers dim 1 digraph [1]. Fig.2 displays the bipartite sub-digraph of the $K$-digraph $\overrightarrow{K_{2,3}}$. It is not Ferrers dim 1 digraph. What is its Ferrers dimension? Adjoin minimal number of arcs in the Fig.2 in order to get Ferrers dim 1 digraph, bi-partite, of course.

The adjacency matrices coding digraphs from the example above are shown below.

$$A_{KoDAG} = \begin{bmatrix} O_{2 \times 2} & I(2 \times 3) \\ O_{3 \times 2} & O_{3 \times 3} \end{bmatrix} \text{ i.e. } A_{KoDAG} = \begin{pmatrix} 0_{2,2} & 111 \\ 0_{3,3} & 0_{3,3} \end{pmatrix},$$

$$A_{not-cobweb} = \begin{pmatrix} 0_{2,2} & 101 \\ 0_{3,3} & 0_{3,3} \end{pmatrix}$$

where $O_{s \times s}$ stays for $(k \times m)$ zero matrix while $I(s \times k)$ stays for $(s \times k)$ matrix of ones i.e. $[I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k.$

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Here above in the Example 1 we are led implicitly to the notion of an extended cobweb poset as compared to [1] and references therein. For associated poset - see [1].

Definition 1 (extended cobweb poset - naturally graded)
Let \( D = (\Phi, \prec) \) be a transitive irreducible digraph. Let \( n \in N \cup \{\infty\} \). Let \( D \) be a natural join \( D = \sum_{i=0}^{n} B_i \) of Ferrers dim 1 bi-partite subdigraphs \( B_i \) of di-bicliques \( K_{k,k+1} = (\Phi_k \cup \Phi_{k+1}, \Phi_k \times \Phi_{k+1}), n \in N \cup \{\infty\} \). The poset \( \Pi(D) \) associated to this graded digraph \( D = (\Phi, \prec) \) is called the extended cobweb poset or just cobweb, as a colloquial abbreviation.

Sometimes when we are in need we shall distinguish by name the complete cobwebs (i.e. cobwebs represented by KoDAGs) from the overall family of cobwebs (the extended cobweb posets as introduced above).

Colligate with Levi graph of an incidence structure. Each incidence structure \( C \) corresponds to a bipartite graph called Levi graph or incidence graph with a given black and white vertex coloring where black vertices correspond to points and white vertices correspond to lines of \( C \) and the edges correspond to flags.

Question 1
Is the natural join operation technique as started in [1] applicable to sequences of Levi graphs of an incidence structures somehow?

In the case of graded digraphs with the finite set of minimal elements we have what follows (Observation 7 in [1]).

Observation 1 Consider bi-partite digraphs’ chain obtained from the di-bicliques’ chain via deleting or no arcs making thus [if deleting arcs] some or all of the di-bicliques \( K_{k,k+1} \) not di-bicliques; denote them as \( G_k \). Let \( B_k = B(G_k) \) denotes their biadjacency Boolean matrices correspondingly. Then for any such \( F \)-denominated chain [hence any chain ] of bi-partite digraphs \( G_k \) the general formula is:

\[
B(\sum_{i=1}^{n} G_i) \equiv B[\sum_{i=1}^{n} A(G_i)] = \oplus_{i=1}^{n} B[A(G_i)] \equiv \text{diag}(B_1, B_2, ..., B_n) =
\begin{bmatrix}
B_1 & B_2 \\
& B_3 \\
& & ... \\
& & & B_n
\end{bmatrix}
\]

\( n \in N \cup \{\infty\} \).

Comment 1 (not only notation matter)
Let us denote by \( \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) the di-bicliques denominated by subsequent levels \( \Phi_k, \Phi_{k+1} \) of the graded \( F \)-poset \( P(D) = (\Phi, \leq) \) i.e. levels \( \Phi_k, \Phi_{k+1} \) of its cover relation graded digraph \( D = (\Phi, \prec) \) i.e. Hasse diagram (see notation in the authors and others papers quoted in [1]). Then one may conditionally
approve the following identification if necessary natural join condition [1] is
imPLICIT within this identification.

\[ B \left( \oplus_{k=1}^{n} \langle \Phi_k \to \Phi_{k+1} \rangle \right) = B \left( \bigcup_{k=1}^{n} \langle \Phi_k \to \Phi_{k+1} \rangle \right) \]

if the conditioned set sum of digraphs concerns an ordered digraphs’s pair satisfying natural join condition [1] what makes such a conditioned set sum of vertices and simultaneously the set sum of disjoint arcs \( E_k , E_{k+1} \) families non commutative. Note that this what just has been said is exactly the reason of

\[ B(G_1 \oplus \rightarrow G_2) = B(G_1 \cup G_2) = B(G_1) \oplus B(G_2). \]

2 On number of finite cobwebs an related questions

2.1 Two schemes and a Question

Before we deal with questions ”‘how many”’ let us jot first two schemes of two statements which may be simultaneously referred to relations, their digraphs or corresponding adjacency matrices. Secondly comes an elementary question without giving an answer.

\[ (\text{Ferrers dim 1}) \oplus \rightarrow (\text{Ferrers dim 1}) = (\text{Ferrers dim 1}). \]

(Obvious: use \( 2 \times 2 \) permutation sub-matrix forbidding i.e. \( 2 \times 2 \) permutation sub-matrix disqualification criterion)

\[ \text{Ferrers} \oplus \rightarrow \text{Ferrers} = \text{Ferrers}. \]

See Observation 3 in [1] and note that resulting biadjacency matrices neither contain any of two \( 2 \times 2 \) permutation matrices. Nota bene the Observation 3 from [1] follows from the above obvious statements.

**Question 2**
For biadjacency matrices \( B(G_1) = B_1 \) and \( B(G_2) = B_2 \) of bipartite digraphs \( G_1 \) and \( G_2 \) we have the matrix exponential rule

\[ \exp[B_1 \oplus B_2] = \exp[B_1] \otimes \exp[B_2], \]

where \( \otimes \) stays for the Kronecker product.

Let \( F \) be any natural number valued sequence. Let \( A_F \) denotes the Hasse matrix of the \( F \)-denominated cobweb poset \( \langle \Phi, \leq \rangle \) [1]. The \( \zeta \) matrix is then the geometrical series in \( A_F \): \( \zeta = (1 - A_F)^{-1}\mathbb{C} \). (Recommended: consult the remark from page 12 in [1] on \( \zeta = \exp[A] \) in the cases of the Boolean poset \( 2^N \) and the Ferrand-Zeckendorf poset of finite subsets of \( F = N \) without two consecutive elements [3]). The **Question 2** is: find the rule if any for

\[ \zeta_{B_1 \oplus B_2} = (1 - B_1 \oplus B_2)^{-1}\mathbb{C} =? \]
2.2 How many

Notation for this subsection.
Consider natural number $N$ composition $N = f_1 + f_2 + ... + f_k$ where $0 < f_1, f_2, ..., f_k \leq N$. The compositions’ type $\vec{k} = (f_1, f_2, ..., f_k)$ labels compositions of the chosen natural number $N$, where $N = |V|$ labels on its own the partial graded orders $P_N = (V, \leq)$ with $N$ points (vertices) and the partition $V = \bigcup_{r=1}^{k} V_r$, $V_r \cap V_s = \emptyset$ for $r \neq s$, $f_r = |V_r|$ and $r, s = 1, ..., k$, $k = 1, ..., N$. The partial order $\leq$ is the subset according to $\subseteq V_1 \times V_2 \times ... \times V_k$. The symbol $\{ \binom{N}{k} \}$ denotes the array of Stirling numbers of the second kind.

Obvious from obvious and Questions
Number of all $k$-tuples for any $k$-block ordered partition $< V_1, V_2, V_k >$ equals to

$$|V_1| \cdot |V_2| \cdot \ldots \cdot |V_k| = \prod_{r=1}^{k} V_r.$$ 

The number of all complete cobweb posets $P_N = (V, \leq)$ with $|V| = N$ elements is equal to $T_N$ = the number of ordered partitions of $V$. - Why? Note: The number of ordered partitions of $\langle f_1, f_2, ..., f_k \rangle = \vec{k}$ type is equal to $\binom{f_1,n}{f_2,\ldots,f_k} = \binom{f_1,f_2,\ldots,f_k}{f_1,f_2,\ldots,f_k}$. Thereby: the number of all complete cobweb posets $P_N = (V, \leq)$ of $\langle f_1, f_2, ..., f_k \rangle = \vec{k}$ type is equal to $\binom{f_r}{f_1,f_2,\ldots,f_k}$ where $f_r = |V_r|$ and $r = 1, ..., k$ for all particular $k$ type $k$-block ordered partitions $\bigcup_{r=1}^{k} V_r = V$. Altogether:

2.2.1. The number $\text{Cob}^c(N, \vec{k})$ of all complete of the type $\langle f_1, f_2, ..., f_k \rangle = \vec{k}$ cobweb posets $P_N$ is given by:

$$\text{Cob}^c(N, \vec{k}) = \binom{N}{f_1,n} \cdot N = f_1 + f_2 + \ldots + f_k , 0 < f_1, f_2, ..., f_k \leq N,$$

2.2.2. The number $\text{Cob}^c(N, k)$ of all complete $k$-level cobweb posets $P_N$ reads

$$\text{Cob}^c(N, k) = \sum_{f_1+f_2+\ldots+f_k=N \atop 0 < f_1, ... , f_k \leq N} \binom{N}{f_1, f_2, ..., f_k} = k! \binom{N}{k} = \sum_{r=0}^{k} (-1)^{N-k} k! \binom{N}{r}.$$ 

$\text{Cob}^c(N, k)$ = number of surjections $f : V \mapsto [k]$.

2.2.3. The number $\text{Cob}^c(N)$ of all complete cobweb posets $P_N$ is then the sum:

$$\text{Cob}^c(N) = \sum_{k=1}^{N} k! \binom{N}{k}.$$ 

$\text{Cob}^c(N) = T_N$ = the number of ordered partitions of $V$.

2.2.4. The number $K(N, \vec{k})$ of all $k$-ary relations of the given $\vec{k}$ type is:
\[ K(N, \overline{k}) = 2^{\prod_{r=1}^{k} V_r} - 1, \]

where

\[ N = f_1 + f_2 + ... + f_k, 0 < f_1, f_2, ..., f_k \leq N, f_r = |V_r| \text{ and } r = 1, ..., k \text{ while } k = 1, ..., N. \]

2.2.5. For the number \( K(N) \) of all type \( k \)-ary relations, \( k = 1, ..., N \) we then have

\[ K(N) = \sum_{f_1 + f_2 + ... + f_k = N, 0 < f_1, f_2, ..., f_k \leq N} [2^{f_1 \cdot f_2 \cdot ... \cdot f_k} - 1]. \]

2.2.6. Question 3. The number of all \( k \)-level graded posets \( P_N = \langle V, \leq \rangle \) with \( |V| = N \) elements where the partial order \( \leq \) is the subset of Cartesian product: \( \leq \subseteq V_1 \times V_2 \times ... \times V_k \) and where \( f_r = |V_r|, r = 1, ..., k \) and \( k = 1, ..., N \) while \( N = f_1 + f_2 + ... + f_k, 0 < f_1, f_2, ..., f_k \leq N \) ... equals ?

2.2.7. Question 4. The number of all graded posets \( P_N = \langle V, \leq \rangle \) with \( |V| = N \) elements for any type \( \overline{k}, k = 1, ..., N \) ... equals ?

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