A COMBINATORIAL APPROACH TO QUANTIFICATION OF LIE ALGEBRAS

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Abstract. We propose a notion of a quantum universal enveloping algebra for an arbitrary Lie algebra defined by generators and relations which is based on the quantum Lie operation concept. This enveloping algebra has a PBW basis that admits the Kashiwara crystallization. We describe all skew primitive elements of the quantum universal enveloping algebra for the classical nilpotent algebras of the infinite series defined by the Serre relations and prove that the set of PBW-generators for each of these enveloping algebras coincides with the Lalonde–Ram basis of the ground Lie algebra with a skew commutator in place of the Lie operation. The similar statement is valid for Hall–Shirshov basis of any Lie algebra defined by one relation, but it is not so in general case.

1. Introduction

Quantum universal enveloping algebras appeared in the famous papers by Drinfeld [14] and Jimbo [17]. Since then a great deal of articles and number of monographs were devoted to their investigation. All of these researches are mainly concerned with a particular quantification of Lie algebras of the classical series. This is accounted for first by the fact that these Lie algebras have applications and visual interpretations in physical speculations, and then by the fact that a general, and commonly accepted as standard, notion of a quantum universal enveloping algebra is not elaborated yet (see a detailed discussion in [1, 31]).

In the present paper we propose a combinatorial solution of this problem by means of the quantum (Lie) operation concept [21, 23, 24]. In line with the main idea of our approach, the skew primitive elements must play the same role in quantum enveloping algebras as the primitive elements do in the classical case. By the Friedrichs criteria [12, 15, 30, 32, 33], the primitive elements form the ground Lie algebra in the classical case. For this reason we consider the space spanned by the skew primitive elements and equipped with the quantum operations as a quantum analogue of a Lie algebra.

In the second section we adduce the main notions and consider some examples. These examples, in particular, show that the Drinfeld–Jimbo enveloping algebra as well as its modifications are quantum enveloping algebras in our sense.

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In the third section with the help of the Heyneman–Radford theorem we introduce a notion of a *combinatorial rank* of a Hopf algebra generated by skew primitive semi-invariants. Then we define the quantum enveloping algebra of an *arbitrary rank* that slightly generalises the definitions given in the preceding section.

The basis construction problem for the quantum enveloping algebras is considered in the fourth section. We indicate two main methods for the construction of *PBW-generators*. One of them modifies the Hall–Shirshov basis construction process by means of replacing the Lie operation with a skew commutator. The set of the PBW-generators defined in this way, the values of *hard super-letters*, plays the same role as the basis of the ground Lie algebra does in the PBW theorem. At first glance it would seem reasonable to consider the \(k[G]\)-module generated by the values of hard super-letters as a quantum Lie algebra. However, this extremely important module falls far short of being uniquely defined. It essentially depends on the ordering of the main generators, their degrees, and it is almost never antipode stable. Also we have to note the following important fact. Our definition of the hard super-letter is not constructive and, of course, it cannot be constructive in general. The basis construction problem includes the word problem for Lie algebras defined by generators and relations, while the latter one has no general algorithmic solution (see \([4,7]\)).

The second method is connected with the Kashiwara crystallisation idea \([19,20]\) (see also a development in \([11,25]\)). M. Kashiwara has considered the main parameter \(q\) of the Drinfeld–Jimbo enveloping algebra as a temperature of some physical medium. When the temperature tend to zero, the medium crystallises. The PBW-generators must crystallise as well. In our case under this process no one limit quantum enveloping algebra appears since the existence conditions normally include equalities of the form \(\prod p_{ij} = 1\) (see \([23]\)). Nevertheless if we equate all quantification parameters to zero, the hard super-letters would form a new set of PBW-generators for the given quantum universal enveloping algebra. To put this another way, the PBW-basis defined by the super-letters admits the Kashiwara crystallisation.

In the fifth section we bring a way to construct a Groebner–Shirshov relations system for a quantum enveloping algebra. This system is related to the main skew primitive generators, and, according to the Diamond Lemma (see \([3,5,37]\)), it determines the crystal basis. The usefulness of the Groebner–Shirshov systems depends upon the fact that such a system not only defines a basis of an associative algebra, but it also provides a simple diminishing algorithm for expansion of elements on this basis (see, for example \([2]\)).

In the sixth section we adapt a well known method of triangular splitting to the *quantification with constants*. The original method appeared in studies of simple finite dimensional Lie algebras. Then it has been extended into the field of quantum algebra in a lot of publications (see, for example \([8,29,38]\)). By means of this method the investigation of the Drinfeld–Jimbo enveloping algebra amounts to a consideration of its positive and negative homogeneous components, *quantum Borel sub-algebras*.

In the seventh section we consider more thoroughly the quantum universal enveloping algebras of nilpotent algebras of the series \(A_n, B_n, C_n, D_n\) defined by the
Serre relations. We adduce first lists of all hard super-letters in the explicit form, then Groebner–Shirshov relations systems, and next spaces $L(U_P(g))$ spanned by the skew primitive elements (i.e. the Lie algebra quantifications $g_P$ proper). In all cases the lists of hard super-letters (but the hard super-letters themselves) turn out to be independent of the quantification parameters. This means that the PBW-generators result from the Hall–Shirshov basis of the ground Lie algebra by replacing the Lie operation with the skew commutator. The same is valid for the Groebner–Shirshov relations systems. Note that the Hall–Shirshov bases, under the name standard Lyndon bases, for the classical Lie series were constructed by P. Lalonde and A. Ram [26], while the Groebner–Shirshov systems of Lie relations were found by L.A. Bokut’ and A.A. Klein [6].

Furthermore, in all cases $g_P$ as a quantum Lie algebra (in our sense) proves to be very simple in structure. Either it is a coloured Lie super-algebra (provided that the main parameter $p_{11}$ equals 1), or values of all non-unary quantum operations equal zero on $g_P$. In particular, if $\text{char}(k) = 0$ and $p_{11}' \neq 1$ then the partial quantum operations may be defined on $g_P$, but all of them have zero values. Thus, in this case we have a reason to consider $U_P(g)$ as an algebra of ‘commutative’ quantum polynomials, since the universal enveloping algebra of a Lie algebra with zero bracket is the algebra of ordinary commutative polynomials. From this standpoint the Drinfeld–Jimbo enveloping algebra is a ‘quantum’ Weyl algebra of (skew) differential operators. Immediately afterwards a number of interesting questions appears. What is the structure of other algebras of ‘commutative’ quantum polynomials? Under what conditions are the quantum universal enveloping algebras of homogeneous components of other Kac–Moody algebras defined by the Gabber–Kac relations [16] the algebras of ‘commutative’ quantum polynomials? When do the PBW-generators result from a basis of the ground Lie algebra by means of replacing the Lie operation with the skew commutator? These and other questions we briefly discuss in the last section.

It is as well to bear in mind that the combinatorial approach is not free from flaws: the quantum universal enveloping algebra essentially depends on a combinatorial representation of the ground Lie algebra, i.e. a close connection with the abstract category of Lie algebras is lost.

2. Quantum enveloping algebras

Recall that a variable $x$ is called a quantum variable if an element $g_x$ of a fixed Abelian group $G$ and a character $\chi^x \in G^*$ are associated with it. A noncommutative polynomial in quantum variables is called a quantum operation if all of its values in all Hopf algebras are skew primitive provided that every variable $x$ has a value $x = a$ such that

$$\Delta(a) = a \otimes 1 + g_x \otimes a, \quad g^{-1}ag = \chi^x(g)a, \quad g \in G. \quad (1)$$

Let $x_1, \ldots, x_n$ be a set of quantum variables. For each word $u$ in $x_1, \ldots, x_n$ we denote by $g_u$ an element of $G$ that appears from $u$ by replacing of all $x_i$ with $g_{x_i}$. In the same way we denote by $\chi^u$ a character that appears from $u$ by replacing
of all $x_i$ with $\chi^{x_i}$. Thus on the free algebra $k\langle x_1, \ldots, x_n \rangle$ a grading by the group $G \times G^*$ is defined. For each pair of homogeneous elements $u, v$ we fix the denotations $p_{uv} = \chi^{x}(g_u) = p(u, v)$.

The quantum operation can be defined equivalently as a $G \times 1$-homogeneous polynomial that has only primitive values in all braided bigraded Hopf algebras (by the formula skew commutator on the set of graded homogeneous noncommutative polynomials). Let us define a bilinear deg $\chi$.

These brackets satisfy the following Jacobi and skew differential identities:

$$[u, v, w] = [u, [v, w]] + p_{uv}^{-1}[[u, w], v] + (p_{uv} - p_{uv}^{-1})[u, w] \cdot v;$$

$$[[u, v], w] = [u, [v, w]] + p_{uv}[[u, w], v] + p_{uv}(p_{uv} - 1)v \cdot [u, w];$$

$$[u, v \cdot w] = [u, v] \cdot w + p_{uv}v \cdot [u, w]; \quad [u \cdot v, w] = p_{uv}[u, w] \cdot v + u \cdot [v, w],$$

where by the dot we denote the usual multiplication. It is easy to see that the following conditional restricted identities are valid as well

$$[u, v^n] = [[\ldots [u, v], v] \ldots], \quad [v^n, u] = [v, [\ldots [v, u] \ldots]],$$

provided that $p_{uv}$ is a primitive $t$-th root of unit, and $n = t$ or $n = t l^k$ in the case of characteristic $l > 0$.

Suppose that a Lie algebra $g$ is defined by the generators $x_1, \ldots, x_n$ and the relations $f_i = 0$. Let us convert the generators into quantum variables. For this associate to them elements of $G \times G^*$ in arbitrary way. Let $P = ||p_{ij}||$, $p_{ij} = \chi^{x_i}(g_{x_j})$ be the quantification matrix.

**Definition 2.1.** A braided quantum enveloping algebra is a braided bigraded Hopf algebra $U_k^B(g)$ defined by the variables $x_1, \ldots, x_n$ and the relations $f_i = 0$, where the Lie operation is replaced with $[B]$, provided that in this way $f_i$ are converted into the quantum operations $f^*_i$. The coproduct and the commutation relations in the tensor product are defined by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i,$$ 

$$(x_i \otimes x_j) \cdot (x_k \otimes x_m) = (\chi^{x_i}(x_j))^{-1} x_i x_k \otimes x_j x_m.$$
Definition 2.2. A simple quantification or a quantum universal enveloping algebra of \( \mathfrak{g} \) is an algebra \( U_P(\mathfrak{g}) \) that is isomorphic to the skew group algebra

\[ U_P(\mathfrak{g}) = U_P^b(\mathfrak{g}) \ast G, \]

where the group action and the coproduct are defined by

\[ g^{-1}x_ig = \chi^{x_i}(g)x_i, \quad \Delta(x_i) = x_i \otimes 1 + gx_i \otimes x_i, \quad \Delta(g) = g \otimes g. \]  

(9)

(10)

Definition 2.3. A quantification with constants is a simple quantification where additionally some generators \( x_i \) associated to the trivial character are replaced with the constants \( \alpha_i(1 - gx_i) \).

The formulae (10) and (11) correctly define the coproduct since by definition of the quantum operation \( \Delta(f_j^*) = f_j^* \otimes 1 + g \otimes f_j^* \) in the case of ordinary Hopf algebras and \( \Delta(f_j^*) = f_j^* \otimes 1 + 1 \otimes f_j^* \) in the braided case.

We have to note that the defined quantifications essentially depend on the combinatorial representation of the Lie algebra. For example, an additional relation \([x_1, x_1] = 0\) does not change the Lie algebra. At the same time if \( \chi^{x_1}(g_1) = -1 \) then this relation admits the quantification and yields a nontrivial relation for the quantum enveloping algebra, \( 2x_1^2 = 0 \).

Example 1. Suppose that the Lie algebra is defined by a system of constitution homogeneous relations. If the characters \( \chi^i \) are such that \( p_{ij}p_{ji} = 1 \) for all \( i, j \) then the skew commutator itself is a quantum operation. Therefore on replacing the Lie operation all relations become quantum operations as well. This means that the braided enveloping algebra is the universal enveloping algebra \( U(\mathfrak{g}^{col}) \) of the coloured Lie super-algebra which is defined by the same relations as the given Lie algebra is. The simple quantification appears as the Radford biproduct \( U(\mathfrak{g}^{col}) \ast k[G] \) or, equivalently, as the universal \( G \)-enveloping algebra of the coloured Lie super-algebra \( \mathfrak{g}^{col} \) (see [35] or [21], Example 1.9).

Example 2. Suppose that the Lie algebra \( \mathfrak{g} \) is defined by the generators \( x_1, \ldots, x_n \) and the system of nil relations

\[ x_j(adx_i)^{n_{ij}} = 0, \quad 1 \leq i \neq j \leq n. \]

(11)

Usually instead of the matrix of degrees (without the main diagonal) \(|n_{ij}|\) the matrix \( A = |a_{ij}| \), \( a_{ij} = 1 - n_{ij} \) is considered. The Coxeter graph \( \Gamma(A) \) is associated to every such a matrix. This graph has the vertices \( 1, \ldots, n \), where the vertex \( i \) is connected by \( a_{ij}a_{ji} \) edges with the vertex \( j \).

If \( a_{ij} = 0 \) then the relation \( x_jadx_i = 0 \) is in the list (11), and the relation \( x_i(adx_j)^{n_{ij}} = 0 \) is a consequence of it. The skew commutator \([x_j, x_i]\) is a quantum operation if and only if \( p_{ij}p_{ji} = 1 \). Under this condition we have \([x_i, x_j] = -p_{ij}[x_j, x_i]\). Therefore both in the given Lie algebra and in its quantification one may replace the relation \( x_i(adx_j)^{n_{ij}} = 0 \) with \( x_iadx_j = 0 \). In other words, without loss of generality, we may suppose that \( a_{ij} = 0 \leftrightarrow a_{ji} = 0 \). By the Gabber-Kac theorem (11) we get that the algebra \( \mathfrak{g} \) is the positive homogeneous component \( \mathfrak{g}_1^+ \) of a Kac-Moody algebra \( \mathfrak{g}_1 \).
Theorem 6.1 [21] describes the conditions for a homogeneous polynomial in two variables which is linear in one of them to be a quantum operation. From this theorem we have the following corollary.

**Corollary 2.4.** If \( n_{ij} \) is a simple number or unit and in the former case \( p_{ii} \) is not a primitive \( n_{ij} \)-th root of unit, then the relation (11) admits a quantification if and only if \( p_{ij}p_{ji} = p_{n_{ij}} \).

Theorem 6.1 [21] provides no essential restrictions on the non-diagonal parameters \( p_{ij} \) : if the matrix \( P \) correctly defines a quantification of (11) then for every set \( Z = \{ z_{ij} | z_{ij} z_{ji} = z_{ii} = 1 \} \) the following matrix does as well:

\[
P_Z = \{ p_{ij} z_{ij} | p_{ij} \in P, z_{ij} \in Z \}. \tag{12}
\]

**Example 3.** Let \( G \) be freely generated by \( g_1, \ldots, g_n \) and \( A \) be a generalised Cartan matrix symmetrised by \( d_1, \ldots, d_n \), while the characters are defined by \( p_{ij} = q^{-d_{ij}} \). In this case the simple quantification is the positive component of the Drinfeld–Jimbo enveloping algebra together with the group-like elements, \( U_P(g) = U^+_q(g) \ast G \). By means of an arbitrary deformation (12) one may define a ‘colouring’ of \( U_q^+(g) \ast G \).

The braided enveloping algebra equals \( U^+_q(g) \) where the coproduct and braiding are defined by [7] and [8] with the coefficient \( q^{d_{ij}} \). The formula (12) correctly defines its ‘colouring’ as well.

**Example 4.** If in the example above we complete the set of quantum variables by the new ones \( x^{-1}, \ldots, x^{-n}; z_1, \ldots, z_n \) such that

\[
\chi^x = (\chi^x)^{-1}, \quad g_{x^{-}} = g_x, \quad \chi^z = \text{id}, \quad g_{z_i} = g^2_i, \quad \tag{13}
\]

then by [21, Theorem 6.1] the Gabber–Kac relations (2), (3), and \([e_i, f_j] = \delta_{ij} h_i \) (see [16, Theorem 2]) under the identification \( e_i = x_i, \quad f_i = x_i^{-1}, \quad h_i = z_i \) admit the quantification with constants \( z_i = \varepsilon_i (1 - q^2_i). \) (Informally we may consider the obtained quantification as one of the Kac–Moody algebra identifying \( g_i \) with \( q^{h_i} \), where the rest of the Kac–Moody algebra relations, \([h_i, e_j] = a_{ij} e_i, [h_i, f_j] = -a_{ij} f_j, \) is quantified to the \( G \)-action: \( q^{-1} x_i \ast q^? \) \( g_j = q^{d_{ij} a_{ij}} x_i \)). This quantification coincides with the Drinfeld–Jimbo one under a suitable choice of \( x_i, x_i^{-1}, \) and \( \varepsilon_i \) depending up the particular definition of \( U_q(g) \) :

- \( x_i = E_i, \quad g_i = K_i, \quad x_i^{-1} = F_i K_i, \quad p_{ij} = v^{-d_{ij}}, \quad \varepsilon_i = (v^{-d_i} - v^{d_i})^{-1}; \)
- \( x_i = E_i, \quad g_i = \tilde{K}_i, \quad x_i^{-1} = F_i \tilde{K}_i, \quad p_{ij} = v^{-d_{ij}} \), \( \varepsilon_i = (v^{-1} - v_i)^{-1}; \)
- \( \varepsilon_i = (q_i - q_i^3)^{-1}; \)
- \( x_i = f_i, \quad g_i = t_i, \quad x_i^{-1} = \hat{t}_i f_i, \quad p_{ij} = q^{(h_i, a_i)} \), \( \varepsilon_i = (q_i - q_i)^{-1}; \)
- \( x_i = E_i \tilde{K}_i, \quad g_i = K_i^{-1}, \quad x_i^{-1} = F_i K_i, \quad p_{ij} = q^{-d_{ij} a_{ij}}, \quad \varepsilon_i = (1 - q^{d_i})^{-1}. \)

By (13) the brackets \([x_i, x_j^{-}]\) are quantum operations only if \( p_{ij} = p_{ji} \). So in this case the ‘colourings’ (12) may be only black-white, \( z_{ij} = \pm 1 \).

In the perfect analogy the Kang quantification [18] of the generalised Kac–Moody algebras [9] is a quantification in our sense as well.

3. **Combinatorial rank**
By the above definitions the quantum enveloping algebras (with or without constants) are character Hopf algebras (see [21, Definition 1.2]). In this section by means of a combinatorial rank notion we identify the quantum enveloping algebras in the class of character Hopf algebras.

Let $H$ be a character Hopf algebra generated by $a_1, \ldots, a_n$:

$$
\Delta(a_i) = a_i \otimes 1 + g a_i \otimes a_i, \quad g^{-1}a_i g = \chi^{a_i}(g)a_i, \quad g \in G.
$$

Let us associate a quantum variable $x_i$ with the parameters $(\chi^{a_i}, g a_i)$ to $a_i$. Denote by $G\langle X \rangle$ the free enveloping algebra defined by the quantum variables $x_1, \ldots, x_n$. (see [21, Sec. 3] under denotation $H\langle X \rangle$).

The map $x_i \to a_i$ has an extension to a homomorphism of Hopf algebras $\varphi : G\langle X \rangle \to H$. Denote by $I$ the kernel of this homomorphism. If $I \neq 0$ then by the Heyneman–Radford theorem (see, for example [34, pages 65–71]), the Hopf ideal $I$ has a non-zero skew primitive element. Let $I_1$ be an ideal generated by all skew primitive elements of $I$. Clearly $I_1$ is a Hopf ideal as well. Now consider the Hopf ideal $I/I_1$ of the quotient Hopf algebra $G\langle X \rangle/I_1$. This ideal also has non-zero skew primitive elements (provided $I_1 \neq I$). Denote by $I_2/I_1$ the ideal generated by all skew primitive elements of $I/I_1$, where $I_2$ is its preimage with respect to the projection $G\langle X \rangle \to G\langle X \rangle/I_1$. Continuing the process we will find a strictly increasing, finite or infinite, chain of Hopf ideals of $G\langle X \rangle$:

$$
0 = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots.
$$

**Definition 3.1.** The length of (15) is called a **combinatorial rank** of $H$.

By definition, the combinatorial rank of any quantum enveloping algebra (with constants) equals one. In the case of zero characteristic the inverse statement is valid as well.

**Theorem 3.2.** Each character Hopf algebra of the combinatorial rank 1 over a field of zero characteristic is isomorphic to a quantum enveloping algebra with constants of a Lie algebra.

**Proof.** By definition, $I$ is generated by skew primitive elements. These elements as noncommutative polynomials are the quantum operations. Consider one of them, say $f$. Let us decompose $f$ into a sum of homogeneous components $f = \sum f_i$. All positive components belongs to $k\langle X \rangle$ and they are the quantum operations themselves, while the constant component has the form $\alpha(1 - g), g \in G$ (see [21, Sec. 3 and Prop. 3.3]). If $\alpha \neq 0$ then we introduce a new quantum variable $z_f$ with the parameters $(id, g)$. Each $f_i$ has a representation through the skew commutator. Indeed, by [21, Theorem 7.5] the complete linearization $f_i^{lin}$ of $f_i$ has the required representation. By the identification of variables in a suitable way in $f_i^{lin}$ we get the required representation for $f_i$ multiplied by a natural number, $m_i f_i = f_i^\parallel$.

Now consider a Lie algebra $g$ defined by the generators $x_i, z_f$ and the relations

$$
\sum m_i^{-1} f_i^\parallel + z_f = 0,
$$

with the Lie multiplication in place of the skew commutator. It is clear that $H$ is the quantification with constants of $g$. 
In the same way one may introduce the notion of the combinatorial rank for the braided bigraded Hopf algebras. In this case all braided quantum enveloping algebras are of rank 1, and all braided bigraded algebras of rank 1 are the braided quantification of some Lie algebras.

Now we are ready to define a quantification of arbitrary rank. For this in the definitions of the above section it is necessary to change the requirement that all $f_i$ are quantum operations with the following condition.

The set $F$ splits in a union $F = \cup_{j=1}^n F_j$ such that $F_1$ consists of quantum operations; the set $F_2$ consists of skew primitive elements of $G(X|F_1^*)$; the set $F_3$ consists of skew primitive elements of $G(X|F_1^*, F_2^*)$, and so on.

The quantum enveloping algebras of an arbitrary rank are character Hopf algebras also. But it is not clear if any character Hopf algebra is a quantification of some rank of a suitable Lie algebra. It is so if the Hopf algebra is homogeneous and the ground field has a zero characteristic (to appear). Also it is not clear if there exist character Hopf algebras, or braided bigraded Hopf algebras, of infinite combinatorial rank; while it is easy to see that $\cup I_n = I$. Also it is possible to show that $F_1$ always contains all relations of a minimal constitution in $F$. For example, each of (11) is of a minimal constitution in (11). Therefore the quantification of arbitrary rank with the identification $g_i = \exp(h_i)$ of any (generalized) Kac–Moody algebra $g$, or its nilpotent component $g^+$, is always a quantification in the sense of the above section.

4. PBW-generators and crystallisation

The next result yields a PBW basis for the quantum enveloping algebras.

THEOREM 4.1. Every character Hopf algebra $H$ has a linearly ordered set of constitution homogeneous elements $U = \{u_i \mid i \in I\}$ such that the set of all products $gu_1^{n_1}u_2^{n_2}\cdots u_m^{n_m}$, where $g \in G$, $u_1 < u_2 < \ldots < u_m$, $0 \leq n_i < h(i)$ forms a basis of $H$. Here if $p_{ii}^g \overset{df}{=} p_{u_iu_i}$ is not a root of unity then $h(i) = \infty$; if $p_{ii} = 1$ then either $h(i) = \infty$ or $h(i) = l$ is the characteristic of the ground field; if $p_{ii}$ is a primitive $t$-th root of unity, $t \neq 1$, then $h(i) = t$.

The set $U$ is referred to as a set of PBW-generators of $H$. This theorem easily follows from [22] Theorem 2]. Let us recall necessary notions.

Let $a_1, \ldots, a_n$ be a set of skew primitive generators of $H$, and let $x_i$ be the associated quantum variables. Consider the lexicographical ordering of all words in $x_1 > x_2 > \ldots > x_n$. A non-empty word $u$ is called standard if $vw > wv$ for each decomposition $u = vw$ with non-empty $v, w$. The following properties are well known (see, for example [10, 13, 27, 36, 37]).

1s. A word $u$ is standard if and only if it is greater than each of its ends.

2s. Every standard word starts with a maximal letter that it has.

3s. Each word $c$ has a unique representation $c = u_1^{n_1}u_2^{n_2}\cdots u_k^{n_k}$, where $u_1 < u_2 < \cdots < u_k$ are standard words (the Lyndon theorem).

4s. If $u, v$ are different standard words and $u^n$ contains $v^k$ as a sub-word, $u^n = cv^kd$, then $u$ itself contains $v^k$ as a sub-word, $u = bv^ke$. 
The set of \textit{standard nonassociative} words is defined as the smallest set $SL$ that contains all variables $x_i$ and satisfies the following properties.

1) If $[u] = [[v][w]] \in SL$ then $[v], [w] \in SL$, and $v > w$ are standard.
2) If $[u] = [[[v_1][v_2]][w]] \in SL$ then $v_2 \leq w$.

The following statements are valid as well.

5s. Every standard word has the only alignment of brackets such that the appeared nonassociative word is standard (the Shirshov theorem [36]).

6s. The factors $v, w$ of the nonassociative decomposition $[u] = [[v][w]]$ are the standard words such that $u = vw$ and $v$ has the minimal length ($[37]$).

\textbf{Definition 4.2.} A \textit{super-letter} is a polynomial that equals a nonassociative standard word where the brackets mean (2). A \textit{super-word} is a word in super-letters. By 5s every standard word $u$ defines a super-letter $[u]$.

Let $D$ be a linearly ordered Abelian additive group. Suppose that some positive $D$-degrees $d_1, \ldots, d_n \in D$ are associated to $x_1, \ldots, x_n$. We define the degree of a word to be equal to $m_1d_1 + \ldots + m_nd_n$ where $(m_1, \ldots, m_n)$ is the constitution of the word. The order and the degree on the super-letters are defined in the following way: $[u] > [v] \iff u > v; D([u]) = D(u)$.

\textbf{Definition 4.3.} A super-letter $[u]$ is called \textit{hard in $H$} provided that its value in $H$ is not a linear combination of values of super-words of the same degree in less than $[u]$ super-letters and $G$-super-words of a lesser degree.

\textbf{Definition 4.4.} We say that a \textit{height} of a super-letter $[u]$ of degree $d$ equals $h = h([u])$ if $h$ is the smallest number such that: first $p_{uu}$ is a primitive $t$-th root of unity and either $h = t$ or $h = tl^{r}$, where $l = \text{char}(k)$; and then the value in $H$ of $[u]^h$ is a linear combination of super-words of degree $hd$ in less than $[u]$ super-letters and $G$-super-words of a lesser degree. If there exists no such number then the height equals infinity.

Clearly, if the algebra $H$ is $D$-homogeneous then one may omit the underlined parts of the above definitions.

\textbf{Theorem 4.5.} ([22, Theorem 2]). \textit{The set of all values in $H$ of all $G$-super-words $W$ in the hard super-letters $[u_i]$,}

$$W = g[u_1]^{n_1}[u_2]^{n_2} \ldots [u_m]^{n_m},$$

\textit{where $g \in G$, $u_1 < u_2 < \ldots < u_m$, $n_i < h([u_i])$ is a basis of $H$.}

In order to find the set of PBW-generators it is necessary first to include in $U$ the values of all hard super-letters, then for each hard super-letter $[u]$ of a finite height, $h([u]) = tl^k$, to add the values of $[u]^t, [u]^{tl}, \ldots [u]^{tl^{(k-1)}}$, and next for each hard super-letter of infinite height such that $p_{uu}$ is a primitive $t$-th root of unity to add the value of $[u]^t$.

Obviously the set of PBW-generators plays the same role as the basis of the Lie algebra in the PBW theorem does. Nevertheless the $k[G]$-bimodule generated by the PBW-generators is not uniquely defined. It depends on the ordering of the main
generators, the $D$-degree, and under the action of antipode it transforms to a different bimodule of PBW-generators $k[G]S(U)$.

Another way to construct PBW-generators is connected with the M.Kashiwara crystallisation idea [19, 20]. M.Kashiwara considered the main parameter of the Drinfeld–Jimbo enveloping algebra as the temperature of some physical medium. When the temperature tends to zero the medium crystallises. By this means the ‘crystal’ bases must appear. If we replace $p_{ij}$ with zero then $[u,v]$ turns into $uv$, while $[u]$ turns into $u$.

**Lemma 4.6.** (Bases Crystallisation). Under the above crystallisation the set of PBW-generators constructed in Theorem 3.1 turns into another set of PBW-generators.

**Proof.** See [22, Corollary 1].

**Lemma 4.7.** (Super-letters Crystallisation). A super-letter $[u]$ is hard in $H$ if and only if the value of $u$ is not a linear combination of lesser words of the same degree and $G$-words of a lesser degree.

**Proof.** See [22, Corollary 2].

**Lemma 4.8.** Let $B$ be a set of the super-letters containing $x_1, \ldots, x_n$. If each pair $[u], [v] \in B$, $u > v$ satisfies one of the following conditions

1) $[[u][v]]$ is not a standard nonassociative word;
2) the super-letter $[[u][v]]$ is not hard in $H$;
3) $[[u][v]] \in B$,

then the set $B$ includes all hard in $H$ super-letters.

**Proof.** Let $[w]$ be a hard super-letter of minimal degree such that $[w] \notin B$. Then $[w] = [[u][v]], u > v$ where $[u], [v]$ are hard super-letters. Indeed, if $[u]$ is not hard then by Lemma 4.7 we have $u = \sum \alpha_i u_i + S$, where $u_i < u$ and $D(u_i) = D(u)$, $D(S) < D(u)$. We have $w = \sum \alpha_i u_i v + S v$, where $u_i v < w$. Therefore by Lemma 4.7, the super-letter $[w] = [uv]$ can not be hard in $H$. Contradiction. Similarly, if $[v]$ is not hard then $v = \sum \alpha_i v_i + S$, $v_i < v$, $D(v_i) = D(v)$, $D(S) < D(v)$. Therefore $wv = \sum \alpha_i w_i + S v$, $w_i < w$, and again $[w]$ can not be hard.

Thus, according to the choice of $[w]$, we get $[u], [v] \in B$. Since this pair satisfies neither condition 1) nor 2), the condition 3), $[uv] \notin B$, holds.

**Lemma 4.9.** If $T \in H$ is a skew primitive element then

$$T = [u]^h + \sum \alpha_i W_i + \sum \beta_j g_j W'_j$$

where $[u]$ is a hard super-letter, $W_i$ are basis super-words in super-letters less than $[u]$, $D(W_i) = hD([u])$, $D(W'_j) < hD([u])$. Here if $p_{uu}$ is not a root of unity then $h = 1$; if $p_{uu}$ is a primitive $t$-th root of unity then $h = 1$, or $h = t$, or $h = tl^k$, where $l$ is the characteristic.

**Proof.** Consider an expansion of $T$ in terms of the basis (10)

$$T = \alpha gU + \sum_{i=1}^{k} \gamma_i g_i W_i + W', \quad \alpha \neq 0,$$

(17)
where \( gU, gW_i \) are different basis elements of maximal degree, and \( U \) is one of the biggest words among \( U, W_i \) with respect to the lexicographic ordering of words in the super-letters. On basis expansion of tensors, the element \( \Delta(T) - T \otimes 1 - g_i \otimes T \) has only one tensor of the form \( gU \otimes \ldots \) and this tensor equals \( gU \otimes \alpha(g - 1) \). Therefore \( g = 1 \) and one may apply \([22]\), Lemma 13.

5. Groebner–Shirshov relations systems

Let \( x_1, \ldots, x_n \) be variables that have positive degrees \( d_1, \ldots, d_n \in D \). Recall that a Hall ordering of words in \( x_1, \ldots, x_n \) is an order when the words are compared firstly by the degree and then words of the same degree are compared by means of the lexicographic ordering. Consider a set of relations

\[
 w_i = f_i, \quad i \in I, \tag{18}
\]

where \( w_i \) is a word and \( f_i \) is a linear combination of Hall lesser words. The system \([18]\) is said to be closed under compositions or a Groebner–Shirshov relations system if first none of \( w_i \) contains \( w_j, i \neq j \in I \) as a sub-word, and then for each pair of words \( w_k, w_j \) such that some non-empty terminal of \( w_k \) coincides with an onset of \( w_j \), that is \( w_k = w_k'w, w_j = vw_j' \), the difference (a composition) \( f_kw_j' - w_k'f_j \) can be reduced to zero in the free algebra by means of a sequence of one sided substitutions \( w_i \rightarrow f_i, i \in I \).

**Lemma 5.1. (Diamond Lemma \([3, 5, 27]\)).** If the system \([18]\) is closed under compositions then the words that have none of \( w_i \) as sub-words form a basis of the algebra \( H \) defined by \([18]\).

If none of the words \( w_i \) has sub-words \( w_j, j \neq i \), then the converse statement is valid as well. Indeed, any composition by means of substitutions \( w_i \rightarrow f_i \) can be reduced to a linear combination of words that have no sub-words \( w_i \). Since \( f_iw_j' - w_k'f_j = (f_i - w_i)w_j' - w_k'f_j \), this linear combination equals zero in \( H \). Therefore all the coefficients have to be zero.

Since Bases Crystallisation Lemma provides the basis that consists of words, the above note gives a way to construct the Groebner–Shirshov relations system for any quantum enveloping algebra.

Let \( H \) be a character Hopf algebra generated by skew primitive semi-invariants \( a_1, \ldots, a_n \) (or a braided bigraded Hopf algebra generated by grading homogeneous primitive elements \( a_1, \ldots, a_n \)) and let \( x_1, \ldots, x_n \) be the related quantum variables. A non-hard in \( H \) super-letter \([w]\) is referred to as a minimal one if first \( w \) has no proper standard sub-words that define non-hard super-letters, and then \( w \) has no sub-words \( u^h \), where \([u]\) is a hard super-letter of the height \( h \).

By the Super-letters Crystallisation Lemma, for every minimal non-hard in \( H \) super-letter \([w]\) we may write a relation in \( H \)

\[
 w = \sum \alpha_i w_i + \sum \beta_j g_j w_j, \tag{19}
\]

where \( w_j, w_i < w \) in the Hall sense, \( D(w_i) = D(w), D(w_j) < D(w) \). In the same way if \([u]\) is a hard in \( H \) super-letter of a finite height \( h \) then

\[
 u^h = \sum \alpha_i u_i + \sum \beta_j g_j u_j, \tag{20}
\]
where $u_j, u_i < u^h$ in the Hall sense, $D(u_i) = hD(u), \ D(u_j) < hD(u)$. The relations (14) and the group operation provide the relations

\[ x_ig = \chi^{x_i}(g)x_i, \quad g_1g_2 = g_3. \]  

(21)

**Theorem 5.2.** The set of relations (13), (20), and (21) forms a Groebner–Shirshov system that defines $H$. The basis determined by this system in Diamond Lemma coincides with the crystal basis.

**Proof.** The property 4s implies that none of the left hand sides of (19), (20), (21) contains another one as a sub-word. Therefore by the Bases Crystallisation Lemma it is sufficient to show that the set of all words $c$ determined in the Diamond Lemma coincides with the crystal basis. By 3s we have $c = u_1^{n_1}u_2^{n_2} \cdots u_k^{n_k}$, where $u_1 < \ldots < u_k$ is a sequence of standard words. Every word $u_i$ define a hard super-letter $[u_i]$ since in the opposite case $u_i$, and therefore $c$, contains a sub-word $w$ that defines a minimal non-hard super-letter $[w]$. In the same way $n_i$ does not exceed the height of $[u_i]$. \(\Box\)

**Lemma 5.3.** In terms of Lemma 4.8 the set of all super-letters $[[u][v]]$ that satisfy the condition 2) contains all minimal non-hard super-letters, but non-hard generators $x_i$.

**Proof.** If $[w]$ is a minimal non-hard super-letter then $[w] = [[u][v]]$, where $[u], [v]$ are hard super-letters. By Lemma 4.8 we have $[u], [v] \in B$, while $[[u][v]]$ neither satisfies 1) nor 3). \(\Box\)

6. Quantification with constants

By means of the Diamond Lemma in some instances the investigation of a quantification with constants can be reduced to one of a simple quantification.

Let $H_1 = \langle x_1, \ldots, x_k \rangle | F_1 \rangle$ be a character Hopf algebra defined by the quantum variables $x_1, \ldots, x_k$ and the grading homogeneous relations $\{f = 0 : f \in F_1\}$, while $H_2 = \langle x_{k+1}, \ldots, x_n \rangle | F_2 \rangle$ is a character Hopf algebra defined by the quantum variables $x_{k+1}, \ldots, x_n$ and the grading homogeneous relations $\{h = 0 : h \in F_2\}$. Consider the algebra $H = \langle x_1, \ldots, x_n \rangle | F_1, F_2, F_3 \rangle$, where $F_3$ is the following system of relations with constants

\[ [x_i, x_j] = \alpha_{ij}(1 - g_ig_j), \quad 1 \leq i < j \leq n. \]  

(22)

If the conditions below are met then the character Hopf algebra structure on $H$ is uniquely determined:

\[ p_{ij}p_{ji} = 1, \quad 1 \leq i \leq k < j \leq n; \quad \chi^{x_i}\chi^{x_j} \neq 1 \implies \alpha_{ij} = 0. \]  

(23)

Indeed, in this case the difference $w_{ij}$ between the left and right hand sides of (22) is a skew primitive semi-invariant of the free enveloping algebra $G\langle x_1, \ldots, x_n \rangle$. Consider the ideals of relations $I_1 = \text{id}(F_1)$ and $I_2 = \text{id}(F_2)$ of $H_1$ and $H_2$ respectively. They are, in the present context, Hopf ideals of $G\langle x_1, \ldots, x_k \rangle$ and $G\langle x_{k+1}, \ldots, x_n \rangle$, respectively. Therefore $V = I_1 + I_2 + \sum k w_{ij}$ is an antipode stable coideal of $G\langle X \rangle$. Consequently the ideal generated by $V$ is a Hopf ideal. It remains to note that this ideal is generated in $G\langle X \rangle$ by $w_{ij}$ and $F_1, F_2$. 
Lemma 6.1. Every hard in $H$ super-letter belongs to either $H_1$ or $H_2$, and it is hard in the related algebra.

Proof. If a standard word contains at least one of the letters $x_i$, $i \leq k$ then it has to start with one of them (see s2). If this word contains a letter $x_j$, $j > k$ then it has a sub word of the form $x_i x_j$, $i \leq k < j$. Therefore by Lemma 4.7 and relations (22) this word defines a non-hard super-letter.

The converse statement is not universally true. In order to formulate the necessary and sufficient conditions let us define partial skew derivatives:

$$(x_j)_i' = (x_i)_j' = \alpha_{ij}(1 - g_i g_j), \quad i \leq k < j;$$

$$(v \cdot w)_i' = (v)_i' \cdot v + p(x_i, v) v \cdot (w)_i', \quad i \leq k, \quad v, w \in G(x_{k+1}, \ldots, x_n);$$

$$(u \cdot v)_j' = p(v, x_j)(u)_j' \cdot u + v \cdot (u)_j', \quad j > k, \quad u, v \in G(x_1, \ldots, x_k). \quad (24)$$

Lemma 6.2. All hard in $H_1$ or $H_2$ super-letters are hard in $H$ if and only if $(h)_i' = 0$ in $H_2$ for all $i \leq k$, $h \in F_2$ and $(f)_j' = 0$ in $H_1$ for all $j > k$, $f \in F_1$. If these conditions are met then

$$H \cong H_2 \otimes_{k[G]} H_1$$

as $k[G]$-bimodules and the space generated by the skew primitive elements of $H$ equals the sum of these spaces for $H_1$ and $H_2$.

Proof. By (3) and (24) the following equalities are valid in $H$:

$$0 = [x_i, h] = (h)_i'; \quad 0 = [f, x_j] = (f)_j', \quad i \leq k < j. \quad (26)$$

If all hard in $H_1$ or $H_2$ super-letters are hard in $H$ then $H_1, H_2$ are sub-algebras of $H$. So (26) proves the necessity of the lemma conditions.

Conversely. Let us consider an algebra $R$ defined by the generators $g \in G$, $x_1, \ldots, x_n$ and the relations (21), (22). Evidently this system is closed under the compositions. Therefore by Diamond Lemma the set of words $gvw$ forms a basis of $R$ where $g \in G$; $v$ is a word in $x_j$, $j > k$; and $w$ is a word in $x_i$, $i \leq k$. In other words $R$ as a bimodule over $k[G]$ has a decomposition

$$R = G\langle x_{k+1}, \ldots, x_n \rangle \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle. \quad (27)$$

Let us show that the two sided ideal of $R$ generated by $F_2$ coincides with the right ideal $I_2 R = I_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle$. It will suffice to show that $I_2 R$ admits left multiplication by $x_i$, $i \leq k$. If $v$ is a word in $x_{k+1}, \ldots, x_k$, $h \in F_2$, $r \in R$ then $x_i vr = [x_i, vh]r + p(x_i, vh)v x_i h$. The second term belongs to $I_2 R$, while the first one can be rewritten by (3): $[x_i, v]h + p(x_i, v)v[x_i, h]$. Both of these addends belong to $I_2 R$ since $[x_i, v] = (v)'_i \in G\langle x_{k+1}, \ldots, x_n \rangle$ and $[x_i, h] = (h)'_i \in I_2$.

Furthermore, consider a quotient algebra $R_1 = R/I_2 R$:

$$R_1 = (G\langle x_{k+1}, \ldots, x_n \rangle \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle)/(I_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle) = H_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle,$$

where the equality means the natural isomorphism of $k[G]$-bimodules.
Along similar lines, the left ideal $R_1 I_1 = H_2 \otimes \mathbb{k}[G] I_1$ of this quotient algebra coincides with the two sided ideal generated by $F_1$. Therefore

$$H = R_1/R_1 I_1 = H_2 \otimes \mathbb{k}[G] G\langle x_1, \ldots, x_k \rangle / H_2 \otimes \mathbb{k}[G] I_1 = H_2 \otimes \mathbb{k}[G] H_1.$$ 

Thus the monotonous restricted $G$-words in hard in $H_1$ or $H_2$ super-letters form a basis of $H$. This, in particular, proves the first statement.

Now let $T = \sum \alpha_t g_t V_t W_t$ be the basis decomposition of a skew primitive element, $g_t \in G$, $V_t \in H_2$, $W_t \in H_1$, $\alpha_t \neq 0$. We have to show that for each $t$ one of the super-words $V_t$ or $W_t$ is empty. Suppose that it is not so. Among the addends with non-empty $V_t$, $W_t$ we choose the largest one in the Hall sense, say $g_s V_s W_s$. Under the basis decomposition of $\Delta(T) - T \otimes 1 - g(T) \otimes T$ the term $\alpha_s g_s g(V_s) W_s \otimes g_s V_s$ appears and cannot be cancelled with other. Indeed, since the coproduct is homogeneous (see [22, Lemma 9]) and since under the basis decomposition the super-words are decreased (see [22, Lemma 7]) the product $\alpha_s (g_s \otimes g_s) \Delta(V_s) \Delta(W_s)$ has the only term of the above type. By the same reasons $\alpha_t (g_t \otimes g_t) \Delta(V_t) \Delta(W_t)$ has a term of the above type only if $V_t \geq V_s$ and $W_t \geq W_s$ with respect to the Hall ordering of the set of all super-words. However, by the choice of $s$, we have $D(V_t W_t) \geq D(V_t W_t)$.

Hence $D(V_t) = D(V_s)$ and $D(W_t) = D(W_s)$. In particular $V_t$ is not a proper onset of $V_s$. Therefore $V_t = V_s$ since otherwise the inequality $V_t > V_s$ yields a contradiction $V_t W_t > V_s W_s$. The inequality $W_t > W_s$ get the same contradiction. Therefore $V_t = V_s$ and $W_t = W_s$, in which case $g_t g(V_t) W_t \otimes g_t V_t = g_s g(V_s) W_s \otimes g_s V_s$. Thus $g_t = g_s$ and $t = s$. 

\[\square\] 

7. Quantification of the classical series

In this section we apply the above general results to the infinite series $A_n$, $B_n$, $C_n$, $D_n$ of nilpotent Lie algebras defined by the Serre relations \(((11))$. Let $\mathfrak{g}$ be any such Lie algebra.

**Lemma 7.1.** If a standard word $u$ has no sub words of the type

$$x_i^s x_j x_i^m, \text{ where } s + m = 1 - a_{ij} \tag{28}$$

then $[u]$ is a hard in $U_P(\mathfrak{g})$ super-letter.

**Proof.** Let $R$ be defined by the generators $x_1, \ldots, x_n$ and the relations

$$x_i^s x_j x_i^m = 0, \text{ where } s + m = 1 - a_{ij}. \tag{29}$$

Clearly \((29)\) implies \((11)\) with the skew commutator in place of the Lie operation. Therefore $R$ is a homomorphic image of $U_P(\mathfrak{g})$. The system \((23)\) is closed under compositions since a composition of monomial relations always has the form $0 = 0$.

Let $u$ have no sub-words \((28)\). If $[u]$ is not hard then, by the Super-letters Crystalisation Lemma, $u$ is a linear combination of lesser words in $U_P(\mathfrak{g})$. Therefore $u$ is a linear combination of lesser words in $R$ as well. This contradicts the fact that $u$ belongs to the Groebner–Shirshov basis of $R$, since every word either belongs to this basis or equals zero in $R$. 

\[\square\]
Theorem A\textsubscript{n}. Suppose that \( \mathfrak{g} \) is of the type A\textsubscript{n}, and \( p_{ii} \neq -1 \). Denote by \( B \) the set of the super-letters given below:

\[
[u_{km}] \triangleq [x_kx_{k+1} \ldots x_m], \quad 1 \leq k \leq m \leq n.
\]

The following statements are valid.

1. The values of \([u_{km}]\) in \( U_P(\mathfrak{g}) \) form a PBW-generators set.
2. Each of the super-letters \((30)\) has infinite height in \( U_P(\mathfrak{g}) \).
3. The values of all non-hard in \( U_P(\mathfrak{g}) \) super-letters equal zero.
4. The following relations with \((31)\) form the Groebner–Shirshov relations system that determines the crystal basis of \( U_P(\mathfrak{g}) \):

\[
[u_0] \triangleq [x_kx_m] = 0, \quad 1 \leq k < m - 1 < n; \\
[u_1] \triangleq [x_kx_{k+1} \ldots x_{m-1}x_m] = 0, \quad 1 \leq k < m < n; \\
[u_2] \triangleq [x_kx_{k+1} \ldots x_{m-1}x_mx_{m+1} \ldots x_n] = 0, \quad 1 \leq k \leq m < n.
\]

5. If \( p_{11} \neq 1 \) then the generators \( x_i \), the constants \( 1 - g, g \in G \), and, in the case that \( p_{11} \) is a primitive \( t \)-th root of 1, the elements \( x_i^t, x_i^{tk} \) form a basis of \( \mathfrak{g}_P = L(U_P(\mathfrak{g})) \). Here \( l \) is the characteristic of the ground field.
6. If \( p_{11} = 1 \) then the elements \((30)\) and, in the case \( l > 0 \), their \( l^k \)-th powers, together with \( 1 - g, g \in G \) form a basis of \( \mathfrak{g}_P \).

By Corollary 2.4 the relations \((11)\) with a Cartan matrix \( A \) of type A\textsubscript{n} admit a quantification if and only if

\[
p_{ii} = p_{11}, \quad p_{ii+1}p_{i+1i} = p_{11}^2; \quad p_{ij}p_{ji} = 1, \quad i - j > 1.
\]

In this case the quantified relations \((11)\) take up the form

\[
x_i^2x_{i+1} = p_{ii+1}(1 + p_{i+1i})x_{i+1}x_i - p_{ii+1}^2x_{i+1}^2x_i, \quad (33)
\]

\[
x_i^2x_{i+1} = p_{ii+1}(1 + p_{i+1i})x_{i+1}x_i - p_{ii+1}^2p_{ii}x_{i+1}x_i^2, \quad (34)
\]

\[
x_i^2x_{i+1} = p_{ii+1}(1 + p_{i+1i})x_{i+1}x_i - p_{ii+1}^2x_{i+1}^2x_i, \quad (35)
\]

Let us introduce a congruence \( u \equiv_k v \) on \( G(X) \). This congruence means that the value of \( u - v \) in \( U_P^k(\mathfrak{g}) \) belongs to the subspace generated by values of all words with the initial letters \( x_i, i \geq k \). Clearly, this congruence admits right multiplication by arbitrary polynomials as well as left multiplication by the independent of \( x_k \)-1 ones (see \((33)\)). For example, by \((33)\) and \((34)\) we have

\[
x_i^2x_{i+1} \equiv_{i+1} 0; \quad x_i^2x_{i+1}x_i \equiv_{i+1} \alpha x_i^2x_{i+1}, \quad \alpha \neq 0.
\]

**Lemma 7.2.** If \( y = x_i \), \( m + 1 \neq i > k \) or \( y = x_i^2 \), \( m + 1 = i > k \) then \( u_{km} \equiv_{k+1} 0 \).

**Proof.** Let \( y = x_i^2 \), \( m + 1 > k \). By \((34)\) and \((33)\) we have that \( u_{km}y = u_{k+1m}x_m^2x_{m+1} \equiv_{m+1} 0 \). If \( y = x_i \) and \( m + 1 \neq i > k \) then we get \( u_{km}y = \alpha u_{k+1m}x_i^2x_{i+1}x_i \equiv_{i+1} 0 \) by the above case. \( \square \)
LEMMA 7.3. The brackets in $[u_{km}]$ are left-ordered, $[u_{km}] = [x_k[u_{k+1m}]].$

Proof. The statement immediately follows from the properties 6s and 2s.

LEMMA 7.4. If a nonassociative word $[[u_{km}][u_{rs}]]$ is standard then $k = m \leq r$; or $r = k + 1, m \geq s$; or $r = k, m < s$.

Proof. By definition, $u_{km} > u_{rs}$ if and only if either $k < r$; or $k = r, m < s$. If $k = m$ then $u_{km} = x_k$ and $m \leq r$. If $k \neq m$ then $[u_{km}] = [x_k[u_{k+1m}]].$ Therefore $u_{k+1m} \leq u_{rs}$, i.e. either $k+1 > r$; or $k+1 = r$ and $m \geq s.$ The former case contradicts $k < r$ while the latter one does $k = r$. Thus only the possibilities set in the lemma remain.

LEMMA 7.5. If $[w] = [[u_{km}][u_{rs}]], n \geq 1$ is a standard nonassociative word then the constitution of $[w]^h$ does not equal the constitution of any super-word in less than $[w]$ super-letters from $B$.

Proof. The inequalities at the last column of the following tableaux are valid for all $[u] \in B$ that are less than the super-letters located on the same row, where as above $\deg(u)$ means the degree of $u$ in $x_i$.

$$
\begin{align*}
[x_ku_{k+1}] & \quad \deg_k(u) \leq \deg_{k+1}(u); \\
[x_ku_{rs}] & \quad k \neq r \leq k+1 \quad \deg_k(u) \leq \deg_{k+1}(u); \\
u_{km}u_{k+1} & \quad m \geq s \quad \deg_k(u) \leq \deg_{m+1}(u); \\
u_{km}u_{ks} & \quad m < s \quad \deg_k(u) \leq \deg_{m+1}(u). \\
\end{align*}
$$

If all super-letters of a super-word $U$ satisfy one of these inequalities then $U$ does as well. Clearly, no one of the super-letters in the first column satisfies the degree inequality on the same row. Finally, by Lemma 7.3 the first column contains all standard nonassociative words of the type $[[u_{km}][u_{rs}]].$

LEMMA 7.6. If $p_{11} \neq 1$ then the values of $[u_{km}]^h$, $k < m$, $h \geq 1$ are not skew primitive, in particular they are non-zero.

Proof. The sub-algebra generated by $x_2, \ldots, x_n$ is defined by the Cartan matrix of the type $A_{n-1}$. This allows us to use induction on $n$. If $n = 1$ then the lemma is correct in the sense that $[u_{km}]^h = x_1^k \neq 0$.

Let $n > 1$. If $k > 1$ then we may use the inductive supposition directly. Consider the decomposition $\Delta([u_{1m}]) = \sum u^{(1)} \otimes u^{(2)}$. Since

$$
[u_{1m}] = x_1[u_{2m}] - p(x_1, u_{2m})[u_{2m}]x_1,
$$

we have

$$
\Delta([u_{1m}]) = (x_1 \otimes 1 + g_1 \otimes x_1)\Delta([u_{2m}]) - p(x_1, u_{2m})\Delta([u_{2m}]) (x_1 \otimes 1 + g_1 \otimes x_1).$$

Therefore the sum of all tensors $u^{(1)} \otimes u^{(2)}$ with $\deg(u^{(2)}) = 1$, $\deg(u^{(2)}) = 0$, $k > 1$ has the form $\varepsilon g_1[u_{2m}] \otimes x_1$, where $\varepsilon = 1 - p(x_1, u_{2m})p(u_{2m}, x_1)$ since $[u_{2m}]g_1 = p(u_{2m}, x_1)g_1[u_{2m}]$. By $[32]$ we have $p_{ij}p_{ji} = 1$ for $i - 1 > j$. Therefore $\varepsilon = 1 - p_{12}p_{21} = 1 - p_{11}^1 \neq 0.$
This implies that in the decomposition $\Delta([u_{1m}]^h) = \sum v^{(1)} \otimes v^{(2)}$ the sum of all tensors $v^{(1)} \otimes v^{(2)}$ with $\deg_1(v^{(1)}) = h$, $\deg_k(v^{(2)}) = 0$, $k > 1$ equals $\varepsilon^h[u_{2m}]^h \otimes x_1^h$. Thus $[u_{1m}]^h$ is not skew primitive in $U_P(\mathfrak{g})$.

Proof of Theorem A. Let us show firstly that $B$ satisfies the conditions of Lemma 4.8. By the Super-letter Crystallisation Lemma $[w] = ([u_{km}][u_{rs}])$ is non-hard if the value of $u_{km}u_{rs}$ is a linear combination of lesser words. For $k = m$, $r = k + 1$ we have $[w] = [u_{ks}] \in B$. If $k = m$, $r > k + 1$ then the word $x_k u_{rs}$ can be diminished by (34) or (35). If $k \neq m$ then by Lemma 7.4 the word $u_{km}u_{rs}$ has a sub-word of the type $u_1$ or $u_2$. Thus we need show only that the values in $U_P(\mathfrak{g})$ of $u_1$ and $u_2$ are linear combinations of lesser words.

The word $u_1$ has such a representation by Lemma 7.2. Consider the word $u_2$. Let us show by downward induction on $k$ that

$$u_{km}u_{km+1} \equiv_{k+1} \gamma u_{km+1}u_{km}, \quad \gamma \neq 0. \quad (41)$$

If $k = m$ then one may use (34) with $i = k$. Let $k < m$. Let us transpose the second letter $x_k$ of $u_2$ as far to the left as possible by (35). We get

$$u_2 = \alpha x_k x_{k+1} x_k x_{k+2} \cdots x_m x_{k+1} \cdots x_{m+1}, \quad \alpha \neq 0.$$

By (34) we have

$$u_2 \equiv_{k+1} \beta x_k^2 x_{k+1} x_k x_{k+2} \cdots x_m x_{k+1} \cdots x_{m+1}, \quad \beta \neq 0.$$

Let us apply the inductive supposition to the word in the parentheses. Since $x_i$, $i > k + 1$ commutes with $x_k^2$ according to the formulae (33), we get

$$u_2 \equiv_{k+1} \gamma x_k^2 x_{k+1} x_k x_{k+2} \cdots x_m x_{k+1} \cdots x_{m+1}.$$

Now it remains to replace the underlined sub-word according to (34) and then to transpose the second letter $x_k$ to its former position by (35).

Note that for the diminishing of $u_1$, $u_2$ we did not use, and we could not use, the relation $[x_{n-1} x_n^2] = 0$ since $\deg_n(u_1) \leq 1$, $\deg_n(u_2) \leq 1$.

Thus $B$ satisfies the conditions of Lemma 1.3. Since none of $[u_{km}]$ has sub-words (28), Lemmas 7.1 and 4.8 show that the first statement is correct.

If $[u_{km}]$ has a finite height $h$ then the value of the polynomial $[u_{km}]^h$ in $U_P(\mathfrak{g})$ is a linear combination of words in hard super-letters that are less than $[u_{km}]$. However by Lemma 7.3 this linear combination is trivial, $[u_{km}]^h = 0$, since the defining relations are homogeneous. By Lemma 7.6 the second statement is correct for $p_{11} \neq 1$.

Similarly consider the skew primitive elements. Since both the defining relations and the coproduct are homogeneous, all the homogeneous components of a skew primitive element are skew primitive itself. Therefore it remains to describe all skew primitive elements homogeneous in each $x_i$. Let $T$ be such an element. By Lemma 4.9 we have

$$T = [u]^h + \sum \alpha_i W_i,$$

where $[u]$ is a hard super-letter, $u = u_{km}$, and $W_i$ are super-words in less than $[u]$ super-letters from $B$. By the homogeneity all $W_i$ have the same constitution as $[u_{km}]^h$ does. However by Lemma 7.3 there exist no such super-words. This means that the
only case possible is \( T = [u_{km}]^{1 \text{h}} \). Thus, by Lemma 7.6 the fifth statement is valid as well.

If \( p_{11} = 1 \) then \( p_{ij}p_{ji} = p_{ii} = 1 \) for all \( i, j \). So we are under the conditions of Example 1, that is \( U^b_P(\mathfrak{g}) \) is the universal enveloping algebra of the colour Lie algebra \( \mathfrak{g}^{\text{col}} \). Further, \([u_{km}] \in \mathfrak{g}^{\text{col}} \) and \([u_{km}] \) are linearly independent in \( \mathfrak{g}^{\text{col}} \) since they are hard super-letters and no one of them can be a linear combination of the lesser ones. Let us complete \( B \) to a homogeneous basis \( B' \) of \( \mathfrak{g}^{\text{col}} \). Then by the PBW theorem for the colour Lie algebras the products \( b_1^{rk} \cdots b_k^{rk} \), \( b_1 < \cdots < b_k \) form a basis of \( U(\mathfrak{g}^{\text{col}}) = U^b_P(\mathfrak{g}) \). However, the monotonous restricted words in \( B \) form a basis of \( U^b_P(\mathfrak{g}) \) also. Thus \( B' = B \) and all hard super-letters have the infinite height.

In particular, we get that the second statement is valid in complete extent. Moreover, if \( p_{11} = 1 \) then \( p(u_{km}, u_{km}) = 1 \), thus for \( l = 0 \) all homogeneous skew primitive elements became exhausted by \([u_{km}] \), while for \( l > 0 \) the powers \([u_{km}]^{l \text{h}} \) are added to them (of course, here \( l \neq 2 \) since \(-1 \neq p_{ii} = 1 \)).

So we have proved all statements, but the third and fourth ones. These statements will follow Theorem 5.3 and Lemma 7.6 if we prove that all non-hard super-letters \([[u_{km}][u_{rs}] \] equal zero in \( U_P(\mathfrak{g}) \). By the homogeneous definition, \([[u_{km}][u_{rs}] \] is a linear combination of super-words in lesser hard super-letters. However, by Lemma 7.9, there exist no such super-words of the same constitution. Therefore, by the homogeneity, the above linear combination equals zero.

**Theorem B.** Let \( \mathfrak{g} \) be of the type \( B_n \), and \( p_{ii} \neq -1, 1 \leq i < n \), \( p_{mn}^{[3]} \neq 0 \). Denote by \( B \) the set of the super-letters given below:

\[
[u_{km}] \overset{\text{df}}{=} [x_k x_{k+1} \ldots x_m], \quad 1 \leq k \leq m \leq n; \\
[w_{km}] \overset{\text{df}}{=} [x_k x_{k+1} \ldots x_n \cdot x_{n-1} \ldots x_m], \quad 1 \leq k < m \leq n.
\]  

(42)

The following statements are valid.

1. The values of (12) in \( U_P(\mathfrak{g}) \) form the PBW-generators set.
2. Every super-letter \([u] \in B \) has infinite height in \( U_P(\mathfrak{g}) \).
3. The relations (21) with the following ones form a Groebner–Shirshov system that determines the crystal basis of \( U_P(\mathfrak{g}) \).

\[
[u_0] \overset{\text{df}}{=} [x_k x_m] = 0, \quad 1 \leq k < m - 1 < n; \\
[u_1] \overset{\text{df}}{=} [u_{km} x_{k+1}] = 0, \quad 1 \leq k < m \leq n, \quad k \neq n - 1; \\
[u_2] \overset{\text{df}}{=} [u_{km} u_{km+1}] = 0, \quad 1 \leq k \leq m < n; \\
[u_3] \overset{\text{df}}{=} [w_{km} x_{k+1}] = 0, \quad 1 \leq k < m \leq n, \quad k \neq m - 2; \\
[u_4] \overset{\text{df}}{=} [w_{kk+1} x_{k+2}] = 0, \quad 1 \leq k < n - 1; \\
[u_5] \overset{\text{df}}{=} [w_{km} w_{km-1}] = 0, \quad 1 \leq k < m - 1 \leq n - 1; \\
[u_6] \overset{\text{df}}{=} [u_{km}^2 x_n] = 0, \quad 1 \leq k < n.
\]  

(43)

4. If \( p_{11} \neq 1 \) then the generators \( x_i \) and their powers \( x_i^l, x_i^{mk} \), such that \( p_{ii} \) is a primitive \( l \)-th root of \( 1 \), together with the constants \( 1 - g, g \in G \) form a basis of \( \mathfrak{g}_P = L(U_P(\mathfrak{g})) \). Here \( l \) is the characteristic of the ground field.
5. If \(p_{nm} = p_{11} = 1\) then the elements \([12]\) and, for \(l > 0\), their \(l^k\)-th powers, together with \(1 - g, g \in G\) form a basis of \(\mathfrak{g}_P\). If \(p_{nn} = -p_{11} = -1\) then \([u_{kn}]^2, [u_{kn}]^{2^k}\) are added to them.

Recall that in the case \(B_n\) the algebra \(U_P^r(\mathfrak{g})\) is defined by \(\{33\}, \{34\}, \{35\}\) where in \(\{33\}\) the last relation, \(i = n - 1\), is replaced with

\[
x_{n-1}x_n^3 = p_{n-1n}p_{nn}x_nx_{n-1}x_n^2 - p_{n-1n}p_{nn}p_{nn}p_{nn}x_nx_{n-1}x_n + p_{n-1n}p_{nn}x_n^3x_{n-1}.
\]

(44)

By Corollary \(\{2.4\}\) we get the existence conditions

\[
p_{ii} = p_{11}, \ p_{ii+1}p_{i+i} = p_{11}^{-1} = p_{nn}^{-2}; \ 1 \leq i \leq n - 1; \ p_{ij}p_{ji} = 1, \ i - j > 1.
\]

(45)

The relations \(\{33\}\) and \(\{14\}\) show that

\[
x_{i+1}x_{i+1}^2 \equiv_{i+1} 0, \ i < n - 1; \ x_{n-1}x_n^3 \equiv_n 0,
\]

(46)

while the relations \(\{34\}\) imply

\[
x_{i+1}x_{i+1} \equiv_{i+1} \alpha x_i^2x_{i+1}, \ \alpha \neq 0.
\]

(47)

By means of these relations and \(\{35\}, \{14\}\) we have

\[
x_{n-2}x_{n-1}x_n^2x_{n-1}x_n \equiv_{n-1} 0.
\]

(48)

**Lemma 7.7.** The brackets in \([u_{km}]\) are set by the recurrence formulae:

\[
[w_{km}] = [x_k[w_{k+1}m]], \quad \text{if } 1 \leq k < m - 1 < n;
\]

\[
[w_{kk+1}] = [[w_{kk+2}x_{k+1}], \quad \text{if } 1 \leq k < n.
\]

(49)

Here by the definition \(w_{k+1} = u_{kn}\).

**Proof.** It is enough to use the property 6s and then 1s and 2s. 

**Lemma 7.8.** The nonassociative word \([[u_{km}][w_{rs}]\]) is standard only in the following two cases: 1) \(s \geq m > k + 1 = r\); 2) \(s < m, \ r = k\).

**Proof.** If \([[u_{km}][w_{rs}]\]) is standard then \(w_{km} > w_{rs}\) and by \(\{49\}\) either \(w_{k+1} \leq w_{rs}\), or \(m = k + 1\) and \(x_{k+1} \leq w_{rs}\). The inequality \(w_{km} > w_{rs}\) is correct only in two cases: \(k < r\) or \(k = r, m > s\). We get four possibilities: 1) \(k < r, \ k < m - 1, \ w_{k+1} \leq w_{rs}\); 2) \(k < r, \ m = k + 1, \ x_{k+1} \leq w_{rs}\); 3) \(k = r, \ m > s, \ k < m - 1, \ w_{k+1} \leq w_{rs}\); 4) \(k = r, \ m > s, \ m = k + 1, \ x_{k+1} \leq w_{rs}\). Only the first and third ones are consistent since in the second case \(x_{k+1} \leq w_{rs}\) implies \(k + 1 > r\), while in the fourth case \(r < s\) and \(k = r < s < m = k + 1\). If now we decode \(w_{k+1} \leq w_{rs}\) in the first and third cases, we get the two possibilities mentioned in the lemma. 

**Lemma 7.9.** The nonassociative word \([[u_{km}][w_{rs}]\]) is standard only in the following two cases: 1) \(k = r\); 2) \(k = m < r\).

**Proof.** The inequality \(u_{km} > w_{rs}\) means \(k \leq r\). Since \([u_{km}] = [x_k[u_{k+1}m]\), for \(k \neq m\) we get \(u_{k+1} \leq w_{rs}\), so \(k + 1 > r\) and \(k = r\). If \(k = m \neq r\) then \(x_m > w_{rs}\) and \(m < r\).
Lemma 7.10. The nonassociative word \([w_{kn}][u_{rs}]\) is standard only in the following two cases: 1) \(r = k + 1 < m\); 2) \(r = k + 1 = m = s\).

Proof. The inequality \(w_{kn} > u_{rs}\) implies \(r > k\). If \(k < m - 1\) then by the first formula (13) we have \(w_{k+1m} \leq u_{rs}\) that is equivalent to \(k + 1 \geq r\). Therefore \(r = k + 1 < m\). If \(k = m - 1\) then by the second formula (14) we get \(x_{k+1} \leq u_{rs}\), i.e. either \(k + 1 > r\) or \(k + 1 = r = s\). The former case contradicts \(r > k\) while the latter one is mentioned in the lemma.

Lemma 7.11. If \([u], [v] \in B\) then one of the statements below is correct.
1) \([u][v]\) is not a standard nonassociative word;
2) \(w\) contains a sub-word of one of the types \(u_0, u_1, u_2, u_3, u_4, u_5, u_6\);
3) \([u][v] \in B\).

Proof. The proof results from Lemmas 7.4, 7.8, 7.9, 7.10.

Lemma 7.12. If a super-word \(W\) equals one of the super-letters \([u_1] - [u_0]\) or \([u_{kn}]^h\), \([w_{kn}]^h\), \(h \geq 1\) then its constitution does not equal the constitution of any super-word in less than \(W\) super-letters from \(B\).

Proof. The proof is akin to Lemma 7.5 with the following tableaux:
\[
\begin{align*}
[u_{km}], [u_{km}x_{k+1}], [u_{km}u_{km+1}] & \quad \deg_k(u) \leq \deg_m(u); \\
[w_{km}], [w_{km}x_{k+1}], [w_{km}w_{km-1}] & \quad 2\deg_k(u) \leq \deg_m(u); \\
[w_{kn+1}x_{k+2}] & \quad \deg_k(u) = 0; \\
[u_{kn}^2x_n] & \quad \deg_k(u) \leq \deg_n(u). \\
\end{align*}
\]

Lemma 7.13. If \(y = x_i\), \(m - 1 \neq i > k\) or \(y = x_i^2\), \(m - 1 = i > k\) then
\[
w_{kn}y \equiv_{k+1} 0.
\]

Proof. If \(i < m - 1\) then by means of (33) it is possible to permute \(y\) to the left beyond \(x_n^2\) and use Lemma 7.2 with \(m' = n - 1\). If \(y = x_i^2\), \(m - 1 = i > k\) then by the above case, \(i < m - 1\), we get
\[
w_{km}y = w_{km+1}x_mx_{m-1}^2 = w_{km+1}x_m(x_mx_{m-1} + \beta x_{m-1}x_m) \equiv_{k+1} 0,
\]
where for \(m = n\) by definition \(w_{kn+1} = u_{kn}\), and \(u_{kn}x_{n-1} \equiv_{n-1} 0\).

If \(y = x_i, i = m > k\) then for \(m = n\) one may use the second equality (16). For \(m < n\) we have \(w_{km}y = w_{km+1}y_1\) where \(y_1 = x_m^2\). Therefore for \(k < n - 1\) we may use (52) with \(m + 1\) in place of \(m\). For \(k = n - 1\) we have \(w_{km}x_n = x_nx_{n-1}^3 \equiv_n 0\).

Finally, if \(y = x_i, i > m > k\) then by (33) we have \(w_{km}y = \alpha w_{ki+1}x_i^2x_{i-1}x_i \cdot v\). For \(i = n\) one may use (15), while for \(i < n\), changing the underlined word according to (33), we may use the above considered cases: \(m' - 1 = i'\), where \(m' = i + 1, i' = i\); and \(i' < m' - 1\), where \(m' = i + 1, i' = i - 1\).

Another interesting relation appears if we multiply (14) by \(x_{n-1}\) from the left and subtract (34) with \(i = n - 1\) multiplied from the right by \(x_n^2\):
\[
x_{n-1}x_nx_{n-1}x_n^2 \equiv_n \alpha x_{n-1}x_nx_{n-1}x_n.
\]
in which case \( \alpha = p_{n-1}p_n^{[3]} \neq 0 \).

**Lemma 7.14.** For \( k < s < m \leq n \) the following relation is valid.

\[
w_{km}w_{ks} \equiv_{k+1} \varepsilon w_{ks}w_{km}, \quad \varepsilon \neq 0.
\] (54)

**Proof.** Let us use downward induction on \( k \). For this we first transpose the second letter \( x_k \) of \( w_{km}w_{ks} \) as far to the left as possible by means of (53), and then change the onset \( x_k x_{k+1} x_k \) according to (47). We get

\[
w_{km}w_{ks} \equiv_{k+1} \alpha x_k^2 (w_{k+1} w_{k+1} s), \quad \alpha \neq 0.
\] (55)

For \( k + 1 < s \) we apply the inductive supposition to the word in the parentheses and then by (17) and (55) transpose \( x_k \) to its former position.

The case \( k + 1 = s \), the basis of the induction on \( k \), we prove by downward induction on \( s \).

Let \( k + 1 = s = n - 1 \). Then \( m = n \). Let us show firstly that

\[
x_{n-1} \overline{x_{n-1}^2 x_n} x_{n-1} x_{n-1} \equiv_n \alpha x_{n-1}^2 x_n^2 x_{n-1}^2 + \beta x_{n-1} x_n x_{n-1}^3, \quad \alpha \neq 0.
\] (56)

For this in the left hand side we transpose the first letter \( x_n \) by means of (53) to the penultimate position, and then replace the ending \( x_n^3 x_{n-1} \) by (47). We get a linear combination of three words. One of them equals the second word of (56), while two other have the following forms.

\[
x_{n-1} x_n x_{n-1} x_n x_{n-1}^3 x_n^2, \quad x_{n-1} x_n x_{n-1} x_n x_{n-1}^3 x_n.
\]

The former word by (54) transforms into the form (56). The latter one, after the application of (53) and the replacing of \( x_{n-1} x_n x_{n-1} \) by (54), will have an additional term \( x_{n-1}^3 x_n \), to which it is possible to apply (17). The direct calculation of the coefficients shows that \( \alpha = p_{n-1} p_n^{[3]} \neq 0 \).

Now let us multiply (56) by \( x_n^2 \) from the left and use (54) with \( i = n - 2 \). We get that \( w_{n-2} w_{n-2}w_{n-2}1 \) with respect to \( \equiv_{n-1} \) equals

\[
\gamma x_{n-2} x_{n-2} x_{n-2}^2 x_{n-2} x_{n-2}^2 x_{n-2}^2 x_{n-2}^2 + \delta x_{n-2} x_{n-1} x_n x_{n-2} x_{n-2} x_{n-1}^3, \quad \gamma \neq 0.
\] (57)

Let us apply (17) and then (17) and (17) to the second word. We get that this word with respect to \( \equiv_{n-1} \) equals zero. The first word after application of (54) takes up the form

\[
\varepsilon w_{n-2} w_{n-2} + \varepsilon' w_{n-2} x_{n-1}^2 x_{n-2}^2 x_{n-2}^2, \quad \varepsilon \neq 0.
\]

Thus, by Lemma 7.13, the basis of the induction on \( s \) is proved.

Let us carry out the inductive step. Let \( k + 1 = s < n - 1 \). If \( m > s + 1 = k + 2 \) then by the inductive supposition on \( s \) we may write

\[
w_{km}w_{ks} = (w_{km} w_{kk} + 2)^k x_{k+1} \equiv_{k+1} \alpha w_{kk} w_{km} x_{k+1} = \beta w_{kk} x_{k+1} x_{k+1} x_{k+1} u_{k+1} w_{k+1}.
\] (58)

Taking into account (51) we may neglect the words starting with \( x_{k+1}, x_{k+2} \) while transforming the underlined part:

\[
x_{k} x_{k+1} x_{k+2} x_{k+1} \equiv \gamma x_{k+1} x_{k+1} x_{k+1} x_{k+1} \equiv \delta x_{k+1} x_{k+1} x_{k+1} x_{k+2}.
\] (59)
In this way (58) is transformed into (54).

If \( m = s + 1 = k + 2 < n \) then the relation (54) takes up the form

\[
wxmwxk ≡ k+1 αx^2 w_{k+1k+2w_{k+1k+3}} x_{k+2x_{k+1}}.
\]

Let us apply the inductive supposition with \( k = k + 1 \), \( s = k + 2 \), \( m' = k + 3 \) to the word in the parentheses. We get

\[
wxmwxk ≡ k+1 αx^2 w_{k+1k+3w_{k+1k+3}x_{k+2x_{k+1}}}.
\]

or after an evident replacement

\[
wxmwxk ≡ k+1 αx^2 w_{k+1k+2w_{k+1k+2}} x_{k+2x_{k+1}} + δx^2 w_{k+1k+3x_{k+1}x_{k+2}}.
\]

In both terms we may transpose one letter \( x_k \) to its former position by means of (17) and (35). We get

\[
wxmwxk ≡ k+1 γx^2 w_{kk+1x_{k+2}} + δ'w_{kk+3x_{k+1}x_{k+2}}.
\]

(60)

It is possible to apply (54) with \( m' = k + 3 \), \( s' = k + 1 \) to the first term since the case \( m > s + 1 \) is completely considered. Therefore it is enough to show that the second term equals zero with respect to \( k+1 \). When we transpose the third letter \( x_{k+1} \) as far to the left as possible we get the word

\[
wx_{kk+3x_{kk+1}x_{kk+2}} x_{kk+3x_{kk+2}}.
\]

(61)

Taking into account (24) we may neglect the words starting with \( x_{k+1} \) while transforming the underlined part:

\[
x_{kk+1x_{kk+2x_{kk+2}}} ≡ x_{kk+2x_{kk+2}} x_{kk+1x_{kk+2}} ≡ x_{kk+2x_{kk+1}x_{kk+1}}.
\]

(62)

Therefore the word (61) equals \( w_{kk+1} w_{kk+3x_{kk+2}} \) with respect to \( k+1 \) and it remains only to apply Lemma 7.13 twice.

\[\square\]

**Lemma 7.15.** The set \( B \) satisfies the conditions of Lemma 4.8.

**Proof.** By Lemmas 7.11 and 4.7 it is sufficient to show that in \( U^b_p(g) \) all words of the form \( u_0, \ldots, u_6 \) are linear combinations of lesser ones. The words \( u_0 \) are diminished by (54). The words \( u_1, u_2 \) have been presented in this way, without using \([x_{n-1}x_n^2] = 0\), in the proof of the above theorem. The relation (24) shows that \( u_3 \equiv k+1 0, u_4 \equiv k+1 0 \).

Lemma 7.14 with \( s = m - 1 \) yields the necessary representation for \( u_5 \).

Let us prove by downward induction on \( k \) that

\[
u_6 \equiv u_{kn}^2 x_n ≡ k+1 \varepsilon u_{kn} x_n u_{kn}, \ \varepsilon \neq 0.
\]

For \( k = n - 1 \) this equality takes up the form (58). Let \( k < n - 1 \). Let us transpose the second letter \( x_k \) of \( u_{kn}^2 x_n \) as far to the left as possible by means of (15) and then apply (33). We get

\[
u_{kn}^2 x_n \equiv k+1 αx^2 (u_{k+1n} x_n), \ \alpha \neq 0.
\]

We may apply the inductive supposition to the term in the parentheses and then by (33), (33) transpose one of \( x_k \)’s to its former position.

\[\square\]

**Lemma 7.16.** If \( p_{11} \neq 1 \) then the values of polynomials \([v]^h\), where \( [v] \in B, v \neq x_i \)

\(h \geq 1\) are not skew primitive, in particular, they are non-zero.
Proof. Note that for \( n > 2 \) the sub-algebra generated by \( x_2, \ldots, x_n \) is defined by the Cartan matrix of the type \( B_{n-1} \). This allows us to carry out the induction on \( n \) with additional supposition that the statements 1 and 2 of Theorem \( B_n \) are valid for lesser values of \( n \). It is convenient formally consider the sub-algebras \( \langle x_i \rangle \) as algebras of the type \( B_1 \). In this case for \( n = 1 \) the lemma and the statements 1 and 2 are correct in the evident way. If \( v \) starts with \( x_k \neq x_1 \) then we may directly use the inductive supposition. If \( v = u_{1m} \), one may literally repeat the arguments of Lemma \( \ref{7.6} \) starting at the formula \( \ref{63} \). Let \( v = \sum \). If \( m > 2 \) then by Lemma \( \ref{7.7} \) we have \( w_{1m} = [x_1[w_{2m}]] \). This provides a possibility to repeat the same arguments of Lemma \( \ref{7.6} \) with \( w \) in place of \( u \).

Consider the last case \( v = w_{12} \). By Lemma \( \ref{7.7} \) we have
\[
[w_{12}] = [w_{13}]x_2 - p(w_{13}, x_2)x_2[w_{13}],
\]
\[
[w_{13}] = x_1[w_{23}] - p(x_1, w_{23})[w_{23}]x_1.
\]

Applying the coproduct first to \( \ref{64} \) then to \( \ref{63} \) we may find the sum \( \sum \) of all tensors \( \sum \) of the type \( B \) with additional supposition that the statements 1 and 2 of Theorem \ref{7.6} (as \( \ref{40} \)):
\[
\sum = \varepsilon g_1[w_{23}]x_1(x_2 \otimes 1) - p(w_{13}, x_2)(x_2 \otimes 1)(\varepsilon g_1[w_{23}]x_1) = \varepsilon g_1([w_{23}]x_2 - p(w_{13}, x_2)p(x_2, x_1)x_2[w_{23}]^{\otimes}x_1.
\]

For \( n > 2 \), taking into account first the bicharacter property of \( p \), then the equality \( x_2[w_{23}] = x_2[w_{23}] - p(x_2, w_{23})[w_{23}]x_2 \), and next the following relations \( p_{ji}p_{ji} = 1 \), \( i - j > 1 \); \( p_{11} = p_{12}p_{21} = p_{22} = p_{32}p_{32} \), we may write
\[
\sum = \varepsilon g_1(-p(w_{13}, x_2)p_{21}[x_2w_{23}] + (1 - p_{11}^{-1})[w_{23}]x_2^{\otimes}x_1.
\]

Consider the last case of this tensor on applying the inductive supposition. Note that \( x_2w_{23} \) is a standard word and \( [x_2w_{23}] = x_2[w_{23}] \). This super-letter is non-hard in \( U_p(g) \) since \( x_2w_{23} \) contains the sub-word \( x_2^2 \). Thus \( x_2w_{23} \) is a linear combination of monotonous non-decreasing super-words in lesser super-letters. Among these super-words there is no \( w_{23} \cdot x_2 \) since \( x_2 > x_2w_{23} \). On the other hand, \( w_{23} \cdot x_2 \) is a monotonous non-decreasing super-word and hence its value in \( U_p(g) \) is a basis element. Therefore for \( n > 2 \) the left hand side \( W \) of \( \sum \) is non-zero.

For \( n = 2 \), by the definition \( w_{23} = x_2, w_{13} = x_1x_2 \), and the equality \( \ref{63} \) takes up the form \( \sum = \varepsilon g_1(1 - p_{12}p_{21})x_2^2 \otimes x_1 \). Since \( 1 \neq p_{11}^{-1} = p_{12}p_{21} = p_{22} = 1 - p_{22}^{-1} \neq 0 \). Therefore in this case \( \sum \neq 0 \) as well.

By \[22\], Corollary 10] the sub-algebra generated by \( x_2, \ldots, x_n \) has no zero divisors. In particular \( W^h \neq 0 \) and \( \sum^h \neq 0 \) in any case.

It remains to note that for \( n > 1 \) the sum of all tensors \( w(1) \otimes w(2) \) of \( \Delta([w_{12}]^h) \) such that \( \deg_1(w(2)) = h \), \( \deg_k(w(2)) = 0 \), \( k > 1 \) equals \( \sum^h \), hence \( [w_{12}]^h \) can not be skew-primitive.

Proof of Theorem \( B_n \). Since none of \( u_{km} \), \( w_{km} \) contains sub-words \( \ref{28} \), Lemmas \[13, 7.1, 1.8 \] imply the first statement.

If \( \nu \in B \) is of finite height then by Lemma \[12 \] and the homogeneous version of Definition \[4.4 \] we have \( \nu^h = 0 \). For \( p_{11} \neq 1 \) this contradicts Lemma \[7.10 \].
Along similar lines, by Lemma 4.9, every skew primitive homogeneous element has the form $[u]^h$. This, together with Lemma 7.10, proves the fourth statement and, for $p_{11} \neq 1$, the second one too.

If $p_{11} = 1$ then by (45) we have $p_{nn}^2 = 1$, $p_{ii} = 1$, $i < n$. Besides, $p_{ij}p_{ji} = 1$ for all $i, j$. This means that the skew commutator is a quantum operation. Hence all elements of $B$ are skew primitive. In the case $p_{nn} = 1$ these elements span a colour Lie algebra, while in the case $p_{nn} = -1$ they span a colour Lie super-algebra. Now as in Theorem $A_n$, we may use the PBW-theorem for the colour Lie super-algebras.

The third statement will follow Theorem 5.2 and Lemmas 5.3, 7.11 if we prove that all super-letters (43) are zero in $U_P(g)$. We have already proved that these super-letters are non-hard. Therefore it remains to use the homogeneous version of Definition 4.3 and Lemma 7.12.

**Theorem C_n.** Suppose that $g$ is of the type $C_n$, and $p_{ii} \neq -1$, $1 \leq i \leq n$, $p_{n-1n-1}^3 \neq 0$. Denote by $B$ the set of the following super-letters:

$$
[u_{km}] \overset{df}{=} [x_k x_{k+1} \ldots x_m], \quad 1 \leq k \leq m \leq n;
$$

$$
[v_{km}] \overset{df}{=} [x_k x_{k+1} \ldots x_n \cdot x_{n-1} \ldots x_m], \quad 1 \leq k < m < n; \tag{67}
$$

$$
[v_k] \overset{df}{=} [u_{k-1} u_k], \quad 1 \leq k < n.
$$

The statements given below are valid.

1. The values of the super-letters (67) in $U_P(g)$ form the PBW-generators set.
2. Each of these super-letters has the infinite height in $U_P(g)$.
3. The following relations with (21) form a Groebner–Shirshov system that determines the crystal basis of $U_P(g)$.

$$
[u_0] \overset{df}{=} [x_k x_m] = 0, \quad 1 \leq k < m - 1 < n;
$$

$$
[u_1] \overset{df}{=} [u_{km} x_{k+1}] = 0, \quad 1 \leq k \leq m \leq n, \quad (k, m) \neq (n - 2, n);
$$

$$
[u_2] \overset{df}{=} [u_{km} u_{km+1}] = 0, \quad 1 \leq k \leq m < n - 1;
$$

$$
[w_3] \overset{df}{=} [v_{km} x_{k+1}] = 0, \quad 1 \leq k < m < n, \quad k \neq m - 2;
$$

$$
[w_4] \overset{df}{=} [v_{km} v_{km+1}] = 0, \quad 1 \leq k < m - 1 < n;
$$

$$
[w_5] \overset{df}{=} [v_{km} u_{km+1}] = 0, \quad 1 \leq k < m - 1 \leq n - 1;
$$

$$
[w_6] \overset{df}{=} [u_{k-1} x_{k-1} x_n] = 0, \quad 1 \leq k < n.
$$

4. If $p_{11} \neq 1$ then the generators $x_i$ and their powers $x_i^t, x_i^{lk}$, such that $p_{ii}$ is a primitive $t$-th root of 1 with the constants $1 - g, g \in G$ form a basis of $g_P = L(U_P(g))$. Here $l$ is the characteristic of the ground field.

5. If $p_{11} = 1$ then the elements (67), and in the case of prime characteristic $l$ theirs $l^k$-th powers, together with the constants $1 - g, g \in G$ form a basis of $g_P$.

In the case $C_n$ the algebra $U_P^h(g)$ is defined by the same relations (33), (34), (35), where in (34) the last relation, $i = n - 1$, is replaced with

$$
x_{n-1}^3 x_n = p_{n-1n}^3 p_{n-1n-1}^3 x_{n-1}^2 x_n x_{n-1} +
-p_{n-1n}^2 p_{n-1n-1}^2 x_{n-1}^2 x_n x_{n-1} + p_{n-1n}^3 p_{n-1n-1}^3 x_n x_{n-1}^3. \tag{69}
$$
By Corollary 2.4 we get the existence conditions

\[ p_n = p_{11}, \quad p_{i-1}p_{i-1} = p_{11}^{-1}, \quad 1 < i < n, \]
\[ p_{n-1}p_{n-1} = p_{11}^{-1} = p_{n-1}^{-2}; \quad p_{ij}p_{ji} = 1, \quad i - j > 1. \] (70)

Therefore the following relations are correct

\[ x_i x_{i+1} \equiv i+1 0, \quad 1 \leq i < n; \] (71)
\[ x_i x_{i+1} x_i \equiv i+1 \alpha x_i x_{i+1}, \quad 1 \leq i < n - 1, \quad \alpha \neq 0; \] (72)
\[ x_{n-1} x_n x_{n-1} \equiv_n \alpha x_{n-1}^2 x_n + \beta x_{n-1}^2 x_{n-1}, \quad \alpha, \beta \neq 0. \] (73)

The left multiplication by \( x_{n-2} \) of the last relation implies

\[ x_{n-2} x_n x_{n-1} x_n^2 \equiv_n 1 0. \] (74)

**Lemma 7.17.** The brackets in \([v_{km}], [v_k] \) are set according to the following recurrence formulae, where by the definition \( v_{kn} = u_{kn} \).

\[
[v_{km}] = [x_k[v_{k+1}]], \quad \text{if } 1 \leq k < m - 1 < n - 1; \\
v_{kk+1} = [v_{kk+2} x_{k+1}], \quad \text{if } 1 \leq k < n - 1; \\
[v_k] = [u_{k+1}] u_{kn}], \quad \text{if } 1 \leq k < n. \] (75)

**Proof.** It is enough to use the properties 6s, 1s and 2s. \( \square \)

**Lemma 7.18.** If \([u], [v] \in B \) then one of the following statements is valid.

1) \([u][v] \) is not a standard nonassociative word;
2) \( uv \) contains a sub-word of one of the types \( u_0, u_1, u_2, w_3, w_4, w_5, w_6; \)
3) \([u][v] \in B \).

**Proof.** The first two formulae (75) coincide with (19) up to replacement of \( v \) with \( w \) provided \( k + 1 \neq n > m \). Obviously for \( m < n \) the inequality \( v_{km} > v_{rs} \) is equivalent to \( w_{km} > w_{rs} \), while \( v_{km} > v_{rs} \) is equivalent to \( w_{km} > w_{rs} \). Hence Lemmas (7.8), (7.9), (7.10) are still valid under the replacement of \( w \) with \( v \):

\[
[v_{km}] [v_{rs}] \text{ is standard } \iff \ s \geq m > k + 1 = r \lor (s < m \& r = k); \\
[u_{km}] [v_{rs}] \text{ is standard } \iff \ k = r \lor k = m < r; \\
[v_{km}] [u_{rs}] \text{ is standard } \iff \ r = k + 1 < m \lor r = k + 1 = m = s. \] (76)

Further, \( v_k > v_r \) if and only if \( k < r \), and under this condition \([u_{km}][v_r] \) is not standard since \( u_{kn} > u_{n-1} u_{rn} \).

In a similar manner \( v_k > u_{rm} \) is equivalent to \( k < r \), while \( v_k > u_{rm} \) is equivalent to \( k \leq r \). Therefore none of the words \([v_k][u_{rm}] \), \([v_k][v_{rm}] \) is standard since \( u_{kn} > u_{rm} \) and \( u_{kn} > v_{rm} \), respectively.

For the remaining two cases we have only two possibilities

\[
[u_{km}] [v_r] \text{ is standard } \iff \ r = k \leq m < n; \\
[v_{km}] [v_r] \text{ is standard } \iff \ r = k + 1 \& k < m - 1. \] (77)

The treatment in turn of the eight possibilities (76), (77) proves the lemma. \( \square \)
Lemma 7.19. If a super-word $W$ equals one of the super-letters $[v]^h$, $[v] \in B$, $h \geq 1$, then its constitution does not equal the constitution of any word in less then $W$ super-letters from $B$.

Proof. The proof is akin to Lemma 7.5 with the following tableaux:

\[
\begin{align*}
[u_{km}]^h, [u_{km}u_{km+1}], [u_{km}u_{km+1}] \\
[v_{km}]^h, [v_{km}u_{km}] \quad & \deg_k(u) \leq \deg_{m+1}(u); \\
[v_{km}]^h, [v_{km}u_{km-1}] \quad & 2\deg_k(u) \leq \deg_{m-1}(u); \\
[v_{kk+1}x_{k+1}] \quad & \deg_k(u) = 0; \\
[v_k]^h \quad & \deg_k(u) \leq \deg_n(u); \\
[u_{kn-1}x_n] \quad & \deg_k(u) \leq 2\deg_n(u).
\end{align*}
\]

\[\Box\]

Lemma 7.20. If $y = x_i$, $m - 1 \neq i > k$ or $y = x_i^2$, $m - 1 = i > k$ then
\[v_{km}y \equiv_{k+1} 0.\]

Proof. For $i < m - 1$, we may transpose $y$ by means of (83) to the left across $x_n^2$ and then use Lemma 7.2 with $m' = n - 1$.

If $y = x_i^2$, $m - 1 = i > k$ then by the above case, $i < m - 1$, we get
\[v_{km}y = v_{km+1}x_{m}x_{m-1}^2 = v_{km+1}x_{m-1}(x_{m}x_{m-1} + \beta x_{m-1}x_{m}) \equiv_{k+1} 0,\]

where by definition $v_{kn} = u_{kn}$ and $u_{kn}x_{n-2} \equiv_{n-2} 0$, while $n - 2 = i > k$.

If $y = x_i$, $i = m > k$ then for $m = n - 1$ we may use the inequality \((74)\), while for $m < n - 1$ we have $v_{km}y = v_{km+1}y_1$ where $y_1 = x_m^2$. Hence we may use \((80)\) replacing $m$ by $m + 1$.

If $y = x_i$, $i > m > k$ then by \((83)\) we get $v_{km}y = \alpha v_{ki+1}x_i x_{i-1}x_i \cdot w$. Changing the underlined by \((83)\), we may apply the previously considered cases: $m' - 1 = i'$, where $m' = i + 1$, $i' = i$; and $i' < m' - 1$, where $m' = i + 1$, $i' = i - 1$. \[\Box\]

If we multiply \((83)\) by $x_n$ from the right and subtract \((83)\) with $i = n - 1$ multiplied from the left by $x_{n-1}^2$, then by means of $p_{n-1n-1}^2 = p_{n-1n-1} = p_{nn}^1$ we get
\[x_{n-1}^2x_nx_{n-1}x_n \equiv_{n} p_{n-1n}(p_{n-1n-1}x_{n-1}x_{n-1}x_n - p_{n-1n}x_{n-1}^2x_n^2) \equiv_{n} p_{n-1n}(3).
\]

Let us first multiply this relation by $x_{n-2}^2$ from the left and then apply \((83)\) to the underlined sub-word. Taking into account the relation $x_{n-2}^2x_{n-1}^2 \equiv_{n-1} 0$, we get that the left hand side of the multiplied \((81)\) equals $p_{n-1n}p_{nn}(1 + p_{nn})^{-1}x_{n-2}^2x_{n-1}x_{n-1}$ up to $\equiv_{n-1}$, i.e. it is proportional to the second term of the right hand side. As a result the relation below with $\alpha = p_{n-1n-1}^1(1 + p_{nn}) \neq 0$ is correct.
\[x_{n-2}^2x_{n-1}^2x_{n-1}x_n \equiv_{n-1} \alpha x_{n-2}^2x_{n-1}x_n x_{n-1}x_n.
\]

\[\Box\]

Lemma 7.21. If $k < s < m \leq n$ and as above $v_{kn} = u_{kn}$ then
\[v_{km}v_{ks} \equiv_{k+1} \varepsilon v_{ks}v_{km}, \varepsilon \neq 0,\]
Proof. Let us use downward induction on $k$. For this we first transpose the second letter $x_k$ of $v_{km}v_{ks}$ as far to the left as possible by means of (82), and then change the onset $x_kx_{k+1}x_k$ according to (72). We get

$$v_{km}v_{ks} \equiv_{k+1} \alpha x_k^2 (v_{k+1m}v_{k+1s}), \quad \alpha \neq 0. \quad (84)$$

For $k+1 < s$ we may apply the inductive supposition to the word in the parentheses, and then transpose $x_k$ to its former position by (72), (83).

For $k + 1 = s$ we will use downward induction on $s$.

Let $k + 1 = s = n - 1$. In this case $m = n$ and (84) becomes:

$$v_{n-2n}v_{n-2n-1} \equiv_{n-1} \beta x_{n-2}^2 \left(x_{n-1}x_{n-1}x_{n}x_{n-1}\right).$$

Let us replace the underlined part according to (83). Since $x_{n-2}^2x_{n-1}x_{n}^2 \equiv_n 0$, we may continue by (82):

$$\equiv_{n-1} \beta_1 x_{n-2}^2x_{n-1}^2 x_{n-1} \equiv_{n-1} \beta_2 x_{n-1} x_{n} x_{n-1} \equiv_{n-1} \beta_3 x_{n-2} x_{n-1} x_{n}^2 \equiv_{n-1} \beta_4 x_{n-2} x_{n-1} x_{n} x_{n-2} x_{n-1}^2 x_n.$$

With the help of (83) we get

$$= \varepsilon v_{n-2n} v_{n-2n} + \beta v_{n-2n} x_{n-1} x_{n} x_{n-1} x_{n-2} x_{n}, \quad \varepsilon \neq 0.$$

By (83) and (71) we see that the second term equals zero up to $\equiv_{n-1}$.

The inductive step on $s$ coincides the inductive step on $s$ in Lemma 7.14 up to replacing both the citations of Lemma 7.13 with the citations of Lemma 7.20 and $w$ with $v$.

**Lemma 7.22.** The set $B$ satisfies the Lemma 4.8 conditions.

Proof. According to the Super-letter Crystallisation Lemma and Lemma 7.11 it is sufficient to show that words of the form $u_0, u_1, u_2, w_3, w_4, w_5, w_6$ are linear combinations of lesser words in $U_P(\mathfrak{g})$. The words $u_0$ are diminished by (83). The words $u_1, u_2$ have been diminished in Theorem $A_0$, since in the case $C_0$ the words $u_2$ are independent of $x_n$, while $u_1$ depends on $x_n$ only if $u_1 = x_{n-1} x_n^2$. The relation (79) shows that $w_3 \equiv_{k+1} 0$, $w_4 \equiv_{k+1} 0$. Lemma 7.21 with $s = m - 1$ gives the required representation for $u_5$.  

Consider the words $w_6$. For $k = n - 1$ the relation (69) defines the required decomposition. Let $k < n - 1$. Since $x_1, \ldots, x_{n-1}$ generate a sub-algebra of the type $A_{n-1}$, the crystal decomposition of $u_{kn-2}^3 x_{n-1}$ has the form

$$u_{kn-2}^3 x_{n-1} = \sum \alpha u_{m_1 s_1} u_{m_2 s_2} \cdots u_{m_t s_t}, \quad (85)$$

where $u_{m_1 s_1} \leq u_{m_2 s_2} \leq \cdots \leq u_{m_t s_t}$, that is $m_1 \geq m_2 \geq \cdots \geq m_t$ and $s_i \geq s_{i+1}$ if $m_i = m_{i+1}$ In particular, if $m_1 = k$ then $m_2 = \ldots = m_t = k$ and, due to the homogeneity, $t = 3$, $s_1 = n - 1$, $s_2 = s_3 = n - 2$. Therefore

$$u_{kn-2}^3 x_{n-1} \equiv_{k+1} \varepsilon u_{kn-1} u_{kn-2}^2. \quad (86)$$

Along similar lines, the following relations are valid as well

$$u_{kn-2}^3 x_{n-1}^2 \equiv_{k+1} \mu u_{kn-1} u_{kn-2}, \quad u_{kn-2}^2 x_{n-1}^3 \equiv_{k+1} 0. \quad (87)$$
Now let us multiply \((33)\) with \(i = n - 2\) by \(x_{n-1}\) from the right, and then add to the result the same relation multiplied by \(p_{n-2n-1}(1 + p_{n-1n-1})x_{n-1}\) from the left. We get the following relation with \(\alpha = p_{n-2n-1}p_{n-1n-1} \neq 0\).

\[
x_{n-2}x_{n-1}^3 = \alpha x_{n-1}^2 x_{n-2} + \beta x_{n-1}^3 x_{n-2},
\]

Further, we may write

\[
u_{k_{n-1}}^3 = \beta_1 u_{k_{n-2}} u_{k_{n-3}} x_{n-1} x_{n-2} x_{n-1} u_{k_{n-1}}, \quad \beta_1 \neq 0,
\]

where for \(k = n - 2\) the term \(u_{k_{n-3}}\) is absent. Let us apply \((33)\) with \(i = n - 2\) to the underlined word. Since \(u_{k_{n-2}} u_{k_{n-3}} x_{n-1} \equiv_{n-1} 0\), we have got

\[
u_{k_{n-1}}^3 \equiv_{n-1} \beta_2 u_{k_{n-2}}^2 u_{k_{n-3}} x_{n-1}^2 x_{n-2} x_{n-1}.
\]

Let us apply \((33)\). Taking into account the second of \((87)\) we get

\[
u_{k_{n-1}}^3 \equiv_{k+1} \beta_3 u_{k_{n-2}}^3 x_{n-1}^3.
\]

Let us multiply this relation from the right by \(x_n\). By \((89)\) we have

\[
u_{k_{n-1}}^3 x_n \equiv_{k+1} \alpha u_{k_{n-2}}^2 x_{n-1} x_n x_{n-1}^2 + \beta u_{k_{n-2}} x_{n-1}^2 x_{n-1} x_n.
\]

By means of \((80)\) and \((87)\) we have got

\[
u_{k_{n-1}}^3 x_n \equiv_{k+1} \alpha_1 u_{k_{n-1}} x_n u_{k_{n-2}} x_{n-1} + \beta_1 u_{k_{n-1}} x_n u_{k_{n-2}} x_{n-1},
\]

and both of these words are less than \(u_{k_{n-1}}^3 x_n\). \(\square\)

**Lemma 7.23.** If \(p_{11} \neq 1\) then the values of \([v]^h\), where \([v] \in B, v \neq x_1, h \geq 1\) are not skew primitive. In particular they are non-zero.

**Proof.** Note that for \(n > 3\) the algebra generated by \(x_2, \ldots, x_n\) is a sub-algebra of the type \(C_{n-1}\). Therefore we may use induction on \(n\) with additional supposition that the theorem statements 1 and 2 are valid for the lesser values of \(n\). We will formally consider the sub-algebra generated by \(x_{n-1}, x_n\) as an algebra of the type \(C_2\), and the sub-algebra generated by \(x_n\) as an algebra of type \(C_1\). In this case for \(n = 1\) the present lemma and the statements 1 and 2 are valid in obvious way.

If the first letter \(x_k\) of \(v\) is less than \(x_1\) then we may use the inductive supposition directly. If \(v = u_{1m}\) then one may literally repeat arguments of Lemma 7.6 starting at \((89)\).

If \(v = v_{1m}\) and \(n > 3\) then we may repeat arguments of Lemma 7.16 starting at \((33)\) up to replacing \(w\) with \(v\). For \(n = 3\) in these arguments the formula \((60)\) assumes the form

\[
\Sigma = \varepsilon g_1 (-p(v_{13}, x_2)p_{21}[x_{2}^2 x_3] + (1 - p_{11}^{-1})[x_2 x_3] \cdot x_2) \otimes x_1.
\]

Therefore the left component of the tensor \(\Sigma\) is a non-zero linear combination of the basis elements. For \(n = 2\) the set \(B\) has no elements \(v_{1m}\) at all.

Consider the last case, \(v = v_1 = [u_{1n-1}^2 x_n]\). Let \(S_k\) be the sum of all tensors of \(\Delta([u_{kn}]) = \sum u^{(1)} \otimes u^{(2)}\) with \(\text{deg}_{n}(w^{(1)}) = 1\), \(\text{deg}_{k}(w^{(1)}) = 0\), \(k < n\). Evidently
$S_n = x_n \otimes 1$. Let us show by downward induction on $k$ that $S_k = (1 - p_{11}^{-1}) g(u_{kn} x_n \otimes [u_{kn-1}]$ at $k < n$. We have

$$
\Delta([u_{kn}]) = \Delta(x_k) \Delta([u_{k+1n}]) - p(x_k, u_{k+1n}) \Delta([u_{k+1n}]) \Delta(x_k).
$$

(94)

Consequently,

$$
S_k = (g_k \otimes x_k) S_{k+1} - p(x_k, u_{k+1n}) S_{k+1} (g_k \otimes x_k).
$$

(95)

This implies the required formula since by (70) at $k < n - 1$ we have

$$
p(x_k, u_{k+1n}) p(x_n, x_k) = p(x_k, u_{k+1n-1}),
$$

while at $k = n - 1$ we have $p(x_{n-1}, x_n) p(x_n, x_{n-1}) = p_{11}^{-1}$.

In a similar manner, consider the sum $S$ of all tensors of $\Delta([u_{kn} x_n]) = \sum w^{(1)} \otimes w^{(2)} with \deg_\alpha(w^{(1)}) = 1, \deg_\beta(w^{(1)}) = 0, at i < n$.

$$
\Delta([u_{1n-1} [u_{1n}]) = \Delta([u_{1n-1}]) \Delta([u_{1n}]) - p(u_{1n-1}, u_{1n}) \Delta([u_{1n-1}]) \Delta([u_{1n-1}]).
$$

(96)

Since we now $S_1$, we may calculate $S$ :

$$
S = (g(u_{1n-1}) \otimes [u_{1n-1}]) S_1 - p(u_{1n-1}, u_{1n}) S_1 (g(u_{1n-1}) \otimes [u_{1n-1}]) = (1 - p_{11}^{-1}) g(u_{2n-1} x_n \otimes (1 - p(u_{1n-1}, u_{1n}) p(x_n, u_{1n-1})) [u_{1n-1}]^2.
$$

(97)

By (70), using the bicharacter property of $p$, we have

$$
1 - p(u_{1n-1}, u_{1n}) p(x_n, u_{1n-1}) = 1 - p(u_{1n-1}, u_{1n}) p_{n-1n-1} p_{n-1n-1} = 1 - p_{n-1n-1} p_{n-1n-1} = 1 - p_{11}^{-1} \neq 0.
$$

Because of this, $S \neq 0$ and the sum of all tensors $w^{(1)} \otimes w^{(2)} with \deg_\alpha(w^{(1)}) = h, \deg_\beta(w^{(1)}) = 0, k < n of the basis decomposition of $\Delta([v_k])$ equals $S^h \neq 0. Therefore $[v_k]^h$ is not skew primitive. 

Proof of Theorem $C_n$. For the first statement it will suffice to prove that all super-letters (77) are hard in $U_P(\mathfrak{g})$. Since none of $u_{km}, v_{km}$ contains a sub-word (28), Lemma 7.1 implies that $[u_{km}], [v_{km}]$ are hard.

If $[v_k]$ is not hard then, by the homogeneous version of Definition 4.3, its value is a polynomial in lesser hard super-letters. In line with Lemmas 7.22 and 4.3, all hard super-letters belong to $B$. Therefore, by Lemma 7.19, $[v_k] = 0$. Since $\deg_\gamma(v_k) = 1$ and $\deg_\gamma(v_k) = 2$, the equality $[v_k] = 0$ is valid in the algebra $C'$ which is defined by all relations of $U_P(\mathfrak{g})$, but ones of degree greater than 1 in $x_n$ and ones of degree greater than 2 in $x_{n-1}$, that is in the algebra defined by (33), (34) with $\gamma < n - 1, and (35)$. These relations do not reverse the order of $x_{n-1}$ and $x_n$ in monomials since none of them has both $x_{n-1}$ and $x_n$. This implies that the sum of all monomials of $[v_k] = [u_{kn-1}] \cdot [u_{kn}] - p(u_{kn-1}, u_{kn}) [u_{kn}] \cdot [u_{kn-1}]$ in which $x_n$ is prefixed to $x_{n-1}$ equals zero in $C'$, that is $[u_{kn}] \cdot [u_{kn-1}] = 0$. Especially, this equality is valid in $U_P(\mathfrak{g})$. Since, by Theorem 4.3, the super-word $[u_{kn}] \cdot [u_{kn-1}]$ is a basis element, the first statement is proved.

If $[v] \in B$ is of finite height then, by Lemma 7.19 and the homogeneous version of Definition 4.4, we have $[v]^{11} = 0$. For $p_{11} \neq 1$ this contradicts Lemma 7.23. In a similar
manner, according to Lemma [4.9], every skew primitive homogeneous element has the form \([u]^p\). This, together with Lemma [7.23], proves the fourth statement and, for \(p_{11} \neq 1\), the second one too. If \(p_{11} = 1\) then according to (70) we have \(p_{ii} = p_{ij}p_{ji} = 1\) at all \(i, j\). In particular, the skew commutator is a quantum operation. Hence all elements of \(B\) are skew primitive. These elements span a colour Lie algebra. Now, as in Theorem \(A_n\), we may use the coloured PBW theorem.

The third statement will follow from Theorem 5.2 and Lemmas 5.3, 7.18 provided we note that all super-letters (68) are zero in \(U_P(\mathfrak{g})\). We have proved already that these super-letters are non-hard. So it remains to use first the homogeneous version of Definition 4.3 and then Lemma 7.20.

**Theorem \(D_n\).** Let \(g\) be of the type \(D_n\), and \(p_{ii} \neq -1, 1 \leq i \leq n\). Denote by \(B\) the set of the following super-letters:

\[
\begin{align*}
[u_{km}] & \overset{df}{=} [x_kx_{k+1} \ldots x_m], & 1 \leq k \leq m < n; \\
e [e_{km}] & \overset{df}{=} [x_kx_{k+1} \ldots x_{n-2} \cdot x_nx_{n-1} \ldots x_m], & 1 \leq k < m \leq n, \\
e [e_{n-1n}] & \overset{df}{=} x_n.
\end{align*}
\]

The statements given below are valid.

1. The values of (88) in \(U_P(\mathfrak{g})\) form the PBW-generators set.
2. Each of the super-letters (88) has infinite height in \(U_P(\mathfrak{g})\).
3. The relations (21) together with the following ones form a Groebner–Shirshov system that determines the crystal basis of \(U_P(\mathfrak{g})\).

\[
\begin{align*}
[u_0] & \overset{df}{=} [x_kx_m] = 0, & 1 \leq k < m - 1 < n, \ (k, m) \neq (n-2, n); \\
[u_1] & \overset{df}{=} [u_{km}x_{k+1}] = 0, & 1 \leq k < m < n; \\
[u'_1] & \overset{df}{=} [x_{n-2}x_n^2] = 0, \\
[u_2] & \overset{df}{=} [u_{km}u_{k+1}] = 0, & 1 \leq k \leq m < n - 1; \\
[v_3] & \overset{df}{=} [e_{km}x_{k+1}] = 0, & 1 \leq k < m \leq n, \ n - 1 \neq k \neq m - 2; \\
v_4 & \overset{df}{=} [e_{kk+1}x_{k+2}] = 0, & 1 \leq k < n - 2; \\
v'_4 & \overset{df}{=} [e_{n-3}x_n] = 0, \\
v_5 & \overset{df}{=} [e_{km}e_{k+1}] = 0, & 1 \leq k < m - 1 \leq n - 1; \\
v_6 & \overset{df}{=} [u_{km}e_n] = 0, & 1 \leq k \leq m < n, \ n - 2 \leq m.
\end{align*}
\]

4. If \(p_{11} \neq 1\), then the generators \(x_i\), their powers \(x_i^t, x_i^{uk}\), such that \(p_{ii}\) is a primitive \(t\)-th root of 1, together with the constants \(1 - g, g \in G\) form a basis of \(\mathfrak{g}_P = L(U_P(\mathfrak{g}))\). Here \(l = \text{char}(k)\).

5. If \(p_{11} = 1\), then the elements of \(B\) and, for \(l > 0\), their \(l^k\)-th powers together with the constants \(1 - g, g \in G\) form a basis of \(\mathfrak{g}_P\).

In the case \(D_n\) the algebra \(U_P^b(\mathfrak{g})\) can be defined by the condition that the sub-algebras \(U_{n-1}\) and \(U_n\) generated, respectively, by \(x_1, \ldots, x_{n-1}\) and \(x_1, \ldots, x_{n-2}, x_{n-1}' = \ldots, x_{n-1} = x_{n-1}'^t = \ldots, x_{n-1}'' = \ldots = x_{n-1}^{(t)} = x_{n-1}^{(t+1)} = \ldots\)
Lemma 7.25. The brackets in (98) are set up by the recurrence formulae
\[
\begin{align*}
[e_{km}] &= [x_k[e_{k+1}m]], & \text{if } 1 \leq k < m-1 < n, \ k \neq n-1; \\
[e_{kk+1}] &= [[e_{kk+2}x_{k+1}], & \text{if } 1 \leq k < n-1.
\end{align*}
\]

Proof. It is enough to use the properties 6s, 1s, and 2s.

Lemma 7.26. If a super-word \(W\) equals one of the super-letters (93) or \([v]^h, [v] \in B, h \geq 1\) then its constitution does not equal the constitution of any super-word in less than \(W\) super-letters from \(B\).

Proof. The proof is similar to the one of Lemma 7.3 with the tableaux
\[
\begin{align*}
[u_{km}]^h, & \quad [u_{km}x_{k+1}], & \quad [u_{km}u_{km+1}] & \quad \deg_k(u) \leq \deg_{m+1}(u); \\
[e_{km}]^h, & \quad [e_{km}x_{km+1}], & \quad [e_{km}e_{km+1}] & \quad m < n, \ 2\deg_k(u) \leq \deg_{m-1}(u); \\
[e_{kn}]^h, & \quad [e_{kn}x_{k+1}], & \quad [e_{kn}e_{kn+1}] & \quad \deg_k(u) \leq \deg_{m-1}(u); \\
[e_{kk+1}x_{k+2}] & \quad \deg_k(u) = 0; \\
e_{n-3n-2x_n} & \quad \deg_{n-3}(u) = 0; \\
u_{km-2}e_{kn} & \quad \deg_k(u) \leq \deg_{n-1}(u) + \deg_n(u); \\
u_{km-1}e_{kn} & \quad \deg_k(u) \leq \deg_n(u).
\end{align*}
\]

Lemma 7.27. If \(y = x_i, \ m - 1 \neq i > k \) or \(y = x_i^2, \ m - 1 = i > k \) then
\[
e_{km}y \equiv_{k+1} 0.
\]
Proof. If \( i < m - 1, m \neq n, \) or \( m = n, i < n - 2 \), then with the help of (35) and (100) it is possible to permute \( y \) to the left beyond \( x_n \) and then to use Lemma 7.2 for \( U_{n-1} \).

If \( m = n, i = n - 2 \) then we may use Lemma 7.2 for \( U_n \).

If \( y = x_i^2, m - 1 = i > k \) then for \( m < n \) by the above case we get

\[
e_{km}y = e_{km+1}x_m^2x_{m-1}^2 = e_{km+1}x_m^2(\alpha x_{m-1}x_m + \beta x_{m-1}x_m) \equiv_{k+1} 0.
\]

(106)

For \( m = n \) we have \( e_{kn}x_{n-1}^2 = \alpha u_{kn-2}x_{n-1}^2 \equiv_{n-1} 0 \) since the underlined part belongs to \( U_{n-1} \).

If \( y = x_i, i = m > k \) then for \( m = n \) we may use Lemma 7.2 applied to \( U_n \); for \( m = n - 1 \) we may use the same lemma applied to \( U_{n-1} \) provided that beforehand we permute \( x_n \) with \( y \) by (100); for \( m < n - 1 \) we may first rewrite \( e_{km}y = e_{km+1}y_1 \), where \( y_1 = x_m^2 \), and then use (100) with \( m + 1 \) in place of \( m \).

If \( y = x_i, i > m > k \) then for \( i < n \) we have \( e_{km}y = \alpha e_{ki+1}x_i x_{i-1} x_i \cdot v \). Replacing the underlined word by (33) in \( U_{n-1} \), we may use the previously considered cases: \( m' - 1 = i' \), where \( m' = i + 1, i' = i \); and \( i' < m' - 1 \), where \( m' = i + 1, i' = i - 1 \). For \( i = n \), and \( m = n - 1 \) we have \( e_{kn-1}x_n = \alpha u_{kn-2}x_n^2x_{n-1} \) and one may apply Lemma 7.2 to \( U_n \). Finally, for \( i = n \) and \( m < n - 1 \) we get

\[
e_{km}x_n = \beta_1 u_{kn-2}x_n x_{n-1} x_n x_{n-1} x_n \cdot v = \beta_2 u_{kn-2}x_n x_{n-1} x_n x_{n-1} x_n \cdot v = \frac{\beta_3 u_{kn-2}x_n x_{n-1} x_n x_{n-2}^2 x_n \cdot v + \beta_4 u_{kn-2}x_n x_{n-1} x_n x_{n-2}^2 x_n \cdot v.}{\}

One may apply first Lemma 7.2 for \( U_{n-1} \) to the underlined sub-word of the first term, and then, after (100), Lemma 7.2 for \( U_n \) to the second term.

\[\square\]

Lemma 7.28. If \( k < s < m \leq n \) then \( e_{km}e_{ks} \equiv_{k+1} \varepsilon e_{ks}e_{km}, \varepsilon \neq 0. \)

Proof. Let us carry out downward induction on \( k \). The largest value of \( k \) equals \( n - 2 \). In this case \( s = n - 1, m = n \) and we have

\[
x_{n-2}x_n \cdot x_{n-2}x_n x_{n-1} \equiv x_{n-2}x_n x_{n-1} x_n x_{n-1} x_n = \alpha x_{n-2}x_n x_{n-1} x_n \equiv_{n-1} \beta x_{n-2}x_n x_{n-1} x_n x_{n-2}^2 x_n \equiv_{n} \varepsilon x_{n-1} x_n \cdot x_{n-1} x_n x_{n-2} x_n.
\]

(107)

Let us first transpose the second letter \( x_k \) of \( e_{km}e_{ks} \) as far to the left as possible by (33), and then replace the onset \( x_k x_{k+1} x_k \) by (36). We get

\[
e_{km}e_{ks} \equiv_{k+1} \alpha x_{k}^2(e_{km+1} e_{ks} x_k), \quad \alpha \neq 0.
\]

(108)

For \( k + 1 < s \) it suffices to apply the inductive supposition to the word in the parentheses and then by (37) and (35) to put \( x_k \) to the proper place.

For \( k + 1 = s \) one may use downward induction on \( s \). The basis of this induction, \( s = n - 1 \), has been proved, see (107). For \( k < n - 3 \) the inductive step on \( s \) coincides with the one of Lemma 7.14 with \( e \) in place of \( u \) since in this case the active variables \( x_k, x_{k+1} \) \( q \)-commute with \( x_n \). If \( k = n - 3 \) then in consideration of Lemma 7.14 the variable \( x_{k+1} = x_{n-2} \) is transposed across \( x_n \) twice: in (38) and in the second word of (39).
In (58) with $k = n - 3$ we have $s = n - 2$, $m = n$; and (38) becomes
\[
e_n e_{n-3n-2} \equiv_n -2 \beta e_n - e_{n-3n-2x_n-2x_n} x_n-2.
\]
(109)
In view of Lemma 7.27, we may transform the underlined part in $U_n$ neglecting the words starting with $x_{n-2}$ and $x_n$ in much the same way as in (60), with $x_n$ in place of $x_{k+1}$. So (109) reduces to the required form.

The second word of (60) with $k = n - 3$ assumes the form $e_n^2 e_{n-3n-2x_n-2x_n} = e_n^3 e_{n-3n-2x_n-2x_n} x_n-2 x_n-2$. By Lemma 7.2 applied to $U_n$, the underlined word is a linear combination of words starting with $x_{n-2}$ and $x_n$. However, by Lemma 7.27 both $e_n^3 e_{n-3n-2}$ and $e_n^3 e_{n-3n-2}$ equal zero up to $e_{n-2}$.

**Lemma 7.29.** The set $B$ satisfies the conditions of Lemma 4.8.

**Proof.** By Lemmas 7.26 and 7.4 one need show only that in $U_p(g)$ the words (99) are linear combinations of lesser ones. The words $v_6$ with $m = n - 2$, and $u_0$, $u_1$, $u_2$ have the required decomposition since they belong either to $U_{n-1}$ or to $U_n$. Lemma 7.27 shows that $v_3 \equiv_{k+1} 0$, $v_4 \equiv_{k+1} 0$, $v_4' \equiv_{k+1} 0$. Lemma 7.28 with $s = m - 1$ yields the required representation for $v_5$. Consider $v_6$ with $m = n - 1$. Let us prove by downward induction on $k$
that
\[
u_{k} \equiv_{k+1} \varepsilon e_{k} u_{k-1}, \quad \varepsilon \neq 0.
\]
For $k = n - 1$ this equality assumes the form (100). Let $k < n - 1$. Let us transpose the second letter $x_k$ of $v_{k-1} e_{k}$ to $x_k$ of the left as possible in $U_{n-1}$. After an application of (33) we get
\[
u_{k} \equiv_{k+1} \alpha x_{k}^2 (v_{k-1} e_{k+1} u_{k+1}), \quad \alpha \neq 0.
\]
It suffices to apply the inductive supposition to the term in the parentheses, and then by (33) and (35) for $U_n$ to move $x_k$ to the proper place.

**Lemma 7.30.** If $p_{11} \neq 1$ then the values of $[v]^b$, where $[v] \in B$, $v \neq x_i$, $h \geq 1$ are not skew primitive, in particular they are non-zero.

**Proof.** One need consider only super-letters that belong neither to $U_{n-1}$ nor to $U_n$. That is $e_{2m}$ with $m < n$. We use induction on $n$.

For $n = 3$ the algebra of the type $D_3$ reduces to the algebra of the type $A_3$ with a new ordering of variables $x_2 > x_1 > x_3$. Therefore we may use Theorem $A_n$ after the decomposition below of $e_{12}$ in the PBW-basis:
\[
[[x_1 x_3] x_2] = -p_{12} p_{32} [x_2 [x_1 x_3]] + \beta [x_1 x_3] x_2.
\]
Let $n > 3$. If $k > 1$ then the inductive supposition works. For $k = 1$, $m > 2$ we have $e_{1m} = [x_1 e_{2m}]$, and one may repeat the arguments of Lemma 7.6 with $e$ in place of $u$ starting at (39). If $m = 2$ then we may repeat the arguments of Lemma 7.16 with $e$ on place of $w$ starting at (63). □

**Proof of Theorem D.$n$.** For the first statement it will suffice to prove that all super-letters (38) are hard in $U_p(g)$.

Since none of $u_{km}$ contains sub-words (28), $[u_{km}]$ are hard.
Suppose \([e_{km}]\) is non-hard. By Lemmas 7.29 and 4.8 all hard super-letters belong to \(B\). Thus, by Lemma 7.26, we get \([e_{km}] = 0\). Since \(\text{deg}_n(e_{km}) = \text{deg}_{n-1}(e_{km}) = 1\), the equality \([e_{km}] = 0\) is also valid in the algebra \(D'\) defined by the same relations as \(U_P(g)\) is, but \([x_{n-2}x_n^n] = 0\) and \([x_{n-2}x_{n-1}^2] = 0\). Let us equate to zero all monomials in all the defining relations of \(D'\), but \([x_{n-1}x_n^n] = 0\). Consider the algebra \(R'\) defined by (100) and by the resulting system of monomial relations. It is easy to verify that the mentioned relations system \(\Sigma\) of \(R'\) is closed under the compositions. Since \(e_{km}\) contains none of leading words of \(\Sigma\), the super-letter \([e_{km}]\) is non-zero in \(R'\), and so in \(D'\) too. This contradiction proves the first statement.

If \([v]^h, [v] \in B\) is of finite height then by Lemma 7.26 and the homogeneous version of Definition 4.4 we have \([v]^h = 0\). For \(p_{11} \neq 1\) this contradicts Lemma 7.30. In a similar manner, by Lemma 4.9, every skew primitive homogeneous element has the form \([v]^h\). This, together with Lemma 7.30, proves both the fourth statement and the second one with \(p_{11} \neq 1\).

If \(p_{11} = 1\) then by (101) we have \(p_{ii} = p_{ij}p_{ji} = 1\) for all \(i, j\). This means that the skew commutator itself is a quantum operation. Hence all elements of \(B\) are skew-primitive. These elements span a colour Lie super-algebra. Now, as in Theorem A_n, one may use the PBW theorem for colour Lie super-algebras.

For the third statement it will suffice to show that all super-letters (99) are zero in \(U_P(g)\). We have proved already that they are non-hard. Therefore it remains to use the homogeneous version of Definition 4.3 and Lemma 7.24. \(\Box\)

8. Conclusion

We see that in all Theorems \(A_n-D_n\) the lists of hard super-letters are independent of the parameters \(p_{ij}\). This fact signifies that the Lalonde–Ram basis of the ground Lie algebra (see, [23], Figure 1) with the skew commutator in place of the Lie operation coincides with the set of all hard super-letters. It is very interesting to clarify how general this statement is. On the one hand, this does not hold without exception for all quantum enveloping algebras since in Theorems \(A_n-D_n\) a restriction does exist. If \(p_{ii} = -1, 1 \leq i < n, n > 2\) then it is easy to see by means of the Diamond Lemma that the sets of hard super-letters are infinite. On the other hand, this is not a specific property of Lie algebras defined by the Serre relations. By the Shirshov theorem [36] any relation can be reduced to a linear combination of standard nonassociative words.

Corollary 8.1. If \(g\) is defined by the only relation \(f = 0\), where \(f\) is a linear combination of standard nonassociative words, then the set of all hard in \(U_P(g)\) super-letters coincides with the Hall–Shirshov basis of \(g\) with the skew commutator in place of the Lie operation.

Proof. The only relation \(f^* = 0\) forms a Groebner–Shirshov system since, according to 1s, none of onsets of its leading word, say \(w\), coincides with a proper terminal of \(w\). Consequently, a super-letter \([u]\) is hard if and only if \(u\) does not contain \(w\) as a sub-word. We see that this criteria is independent of \(p_{ij}\) as well. \(\Box\)
Furthermore, the third statement of Theorem $A_n$ shows that $U_P^g(\mathfrak{g})$ can be defined by the following relations in the PBW-generators $X_u = [u]$.

\[
[X_u, X_v] = 0, \quad u > v, \quad [u][v] \notin B \quad [u][v] \in B.
\] (110)

This is an argument in favour of considering the super-letters PBW-generators $k[G]$-module as a quantum analogue of a Lie algebra. However in the cases $B_n, C_n, D_n$ the defining relations became more complicated. For example,

\[
B_n : \quad [[u_{k_n-1}[u_{k_n}]]] = \alpha[u_{k_n}]^2, \quad \alpha \neq 0 \text{ if } p_{nn} \neq 1;
\]

\[
C_n : \quad [[u_{k_n-2}[v_{k_n-1}]]] = \alpha[v_k] + \beta[u_{k_n}] \cdot [u_{k_n-1}], \quad \beta \neq 0 \text{ if } p_{11} \neq 1;
\]

\[
D_n : \quad [[u_{k_n-2}[e_{k_n-1}]]] = \alpha[e_{k_n}] \cdot [u_{k_n-1}], \quad \alpha \neq 0 \text{ if } p_{11} \neq \pm 1. \quad (111)
\]

It is far more interesting that for $p_{11} \neq 1$ the algebra $g_P$ turns out to be very simple in structure. Only unary quantum operations can be non-zero. Other ones may be defined, but due to the homogeneity their values equal zero. In particular, if $p_{11}^j \neq 0$ then without exception all quantum operations have zero values. This provides reason enough to consider $U_P^g(\mathfrak{g}) = U(\mathfrak{g}_P)$ as an algebra of ‘commutative’ quantum polynomials. Certainly it is very interesting to elucidate to what extent this statement is still retained for the quantum universal enveloping algebras of homogeneous components of other Kac–Moody algebras defined by the Gabber–Kac relations (2). Also it is interesting to investigate the structure of other ‘commutative’ quantum polynomial algebras. For example, one may note that if a semi-group generated by $p_{ij}^j p_{ji}$ does not contain 1, then $G(x_1, \ldots, x_n)$ itself is a ‘commutative’ quantum polynomial algebra merely since in this case there exists no non-zero quantum operation at all. In another extreme case when $p_{ij}^j p_{ji} = 1$ for all $i, j$, the ‘commutative’ quantum variables commute by $x_i x_j = p_{ij}^j x_j x_i$.

In a similar manner, the Drinfeld–Jimbo enveloping algebra can be considered as a ‘quantum’ Weyl algebra of (skew) differential operators (see Sec. 6). The resulting ‘quantum’ Weyl algebra is simple in the following sense.

**Corollary 8.2.** Let $g$ be a simple finite dimensional Lie algebra of the infinite series. If $q^{[m]} \neq 0, m \geq 2$ then every non-zero Hopf ideal $I$ of the Drinfeld–Jimbo enveloping algebra contains all generators $x_i, x_i^\pm$.

**Proof.** By the Heyneman–Radford theorem, the ideal $I$ has a non-zero skew primitive element, say $a$. According to Lemma 3.2 and Theorems $A_n$, $C_n$, the element $a$ is either a constant, $\alpha (1 - g)$, or proportional to one of the elements $x_i, x_i^-$. In the former case $I$ contains all $x_i$ with $\chi^i(g) \neq 1$ since $x_i a - \chi^i(g) a x_i = \alpha (1 - \chi^i(g)) x_i$. Here the equality $\chi^i(g) = 1$ can not be valid for all $i$ since $\chi^i(g_j) = q^{-a_{ii}}$ (see, Example 4 of Section 2) and the columns of the Cartan matrix are linearly independent. In the latter case (and now in the former one as well) we get $[x_i, x_i^-] = \varepsilon_i (1 - g_i^2) \in I$, i.e. as above $I$ contains all elements $y = x_i^\pm$ with $1 \neq \chi^y(g_i^2) = q^{a_{ii}}$. Since the Coxeter graph is connected, $I$ contains all $x_i, x_i^-$. \hfill \Box

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References

1. C. Bautista, ‘A Poinkare–Birkhoff–Witt theorem for generalized Lie color algebras’, Journal of Mathematical Physics 39 N7(1998) 3829–3843.
2. K.I. Beidar, W.S. Martindale III, and A.V. Mikhalev, Rings with Generalized Identities (Pure and Applies Mathematics 196, Marcel Dekker, New York–Basel–Hong Kong, 1996).
3. G.M. Bergman, ‘The diamond lemma for ring theory’, Adv. in Math. 29 N2(1978) 178–218.
4. L.A. Bokut’, ‘Unsolvability of the word problem and subalgebras of finitely presented Lie algebras’, Izv.Akad.Nauk. Ser. Mat. 36 N6(1972) 1173–1219.
5. L.A. Bokut’, ‘Imbeddings into simple associative algebras’, Algebra and Logic 15 N2(1976) 117–142.
6. L.A. Bokut’, and A.A. Klein, ‘Serre relations and Groebner–Shirshov bases for simple Lie algebras I, II’, International Journal of Algebra and Computation 6 N4(1996) 389–412.
7. L.A. Bokut’, and G.P. Kukin, Algoritmic and Combinatorial Algebra (Mathematics and Its Applications 255, Kluwer Academic Publishers, Dordrecht-Boston-London, 1994).
8. L.A. Bokut’, and P. Malcolmson, ‘Groebner bases for quantum enveloping algebras’, Israel Journal of Mathematics 96(1996) 97–113.
9. R. Borcherds, ‘Generalized Kac-Moody algebras’, Journal of Algebra, 11(1988) 501–512.
10. K.T. Chen, R.H. Fox, and R.C. Lyndon, ‘Free differential calculus IV, the quotient groups of the lower central series’, Ann. of Math. 68(1958) 81–95.
11. G. Clift, ‘Crystal bases and Young tableaux’, Journal of Algebra 202 N1(1998) 10–35.
12. P.M. Cohn, ‘Sur le crit`ere de Friedrichs pour les commutateur dans une alg`ebre associative libre’, C. r. Acad. sci. Paris 239 N13(1954) 743–745.
13. P.M. Cohn, Universal Algebra (Harper and Row, New-York, 1965).
14. V.G. Drinfeld, ‘Hopf algebras and the Yang–Baxter equation’, Soviet Math. Dokl., 32(1985) 254–258.
15. K.O. Friedrichs, ‘Mathematical aspects of the quantum theory of fields. V’, Communications in Pure and Applied Mathematics 6(1953) 1–72.
16. O.Gabber, and V. Kac, ‘On defining relations of certain infinite-dimensional Lie algebras’, Bulletin (New series) of the American Mathematical Society 5, N2(1981) 185–189.
17. M. Jimbo, ‘A q-difference analogue of U(g) and the Yang–Baxter equation’, Lett. Math. Phis. 10(1985) 63–69.
18. S.-J. Kang, ‘Quantum deformations of generalized Kac-Moody algebras and their modules’, Journal of Algebra, 175(1995) 1041–1066.
19. M. Kashiwara, ‘Crystallizing the q-analogue of universal enveloping algebras’, Comm. Math. Phis. 133(1990) 249–260.
20. M. Kashiwara, ‘On crystal bases of the q-analog of universal enveloping algebras’, Duke Mathematical Journal 63 N2(1991) 465–516.
21. V.K. Kharchenko, ‘An algebra of skew primitive elements’, Algebra and Logic 37 N2(1998) 101–126.
22. V.K. Kharchenko, ‘A quantum analogue of the Poincaré–Birkhoff–Witt theorem’, Algebra and Logic 38 N4(1999) 476–507; English translation 259–276.
23. V.K. Kharchenko, ‘An existence condition for multilinear quantum operations’, Journal of Algebra 217(1999) 188–228.
24. V.K. Kharchenko, ‘Character Hopf algebras and quantizations of Lie algebras’, Doklady Mathematics 60 N3(1999) 328–329.
25. A. Kuniba, K.C. Misra, M. Okado, T. Takagi, and J. Uchiyama, ‘Crystals for Demazure modules of classical affine Lie algebras’, Journal of Algebra 208(1998) 185–215.
26. M. Lalonde, and A. Ram, ‘Standard Lyndon bases of Lie algebras and enveloping algebras’, Trans. Amer. Math. Soc. 347 N5(1995) 1821–1830.
27. M. Lothaire, Combinatorics on words, (Encyclopedia of Mathematics and its Applications 17, Addison–Wesley Publ. Co. 1983).
28. G. Lusztig, ‘Quantum groups at roots of 1’, Geometria Dedicada 35, N1-3(1990) 89–113.
29. G. Lusztig, Introduction to Quantum Groups (Progress in Mathematics 10, Birkhauser Boston, 1993).
30. R.C. Lyndon, ‘A theorem of Friedrichs’, Michigan Mathematical Journal 3, N1(1955–1956) 27–29.
31. V. Lyubashenko, and A. Sudbery, ‘Generalized Lie algebras of type $A_n$', Journal of Mathematical Physics 39, N6(1998) 3487–3504.
32. W. Magnus, ‘On the exponential solution of differential equations for a linear operator’, Communications in Pure and Applied Mathematics 7(1954) 649–673.
33. J.W. Milnor and J.C. Moore, ‘On the structure of Hopf algebras’, Annals of Math. 81(1965) 211–264.
34. S. Montgomery, Hopf Algebras and Their Actions on Rings (CBMS 82, AMS, Providence, 1993).
35. D.E. Radford, ‘The structure of Hopf algebras with projection’, Journal of Algebra 92(1985) 322–347.
36. A.I. Shirshov, ‘On free Lie rings’, Matem. Sbornic 45(87) N2(1958) 113–122.
37. A.I. Shirshov, ‘Some algorithmic problems for Lie algebras’, Sibirskii Math. Journal 3 N2(1962) 292–296.
38. Yamane, ‘A Poincarè-Birkhoff-Witt theorem for quantized universal enveloping algebras of type $A_N$’, Publ. RIMS. Kyoto Univ. 25(1989) 503–520.

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