Minimal velocity bound for Schrödinger-type operator with fractional powers

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Abstract

It is known in scattering theory that the minimal velocity bound plays a conclusive role in proving the asymptotic completeness of the wave operators. In this study, we prove the minimal velocity bound and other important estimates for the two-body Schrödinger-type operator with fractional powers. We assume that the pairwise potential functions belong to broad classes that include long-range decay and Coulomb-type local singularities. Our estimates are expected to be applied to prove the asymptotic completeness for the fractional Schrödinger-type operators in various (not only short-range but also long-range and N-body) situations.

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1 Introduction

For $s \geq 0$, let us define the function

$$
\Psi_\rho(s) = (s + 1)^\rho - 1
$$

(1.1)

with $0 < \rho \leq 1$. The free dynamics that we consider in this paper are denoted by the symbol $\Psi_\rho$ of the Laplacian

$$
\Psi_\rho(-\Delta) = \Psi_\rho(|D|^2)
$$

(1.2)

as a self-adjoint operator acting on $L^2(\mathbb{R}^n)$, where $D$ is the momentum operator $D = -i\nabla = -\sqrt{-1}(\partial_1, \ldots, \partial_n)$ with $\partial_j = \partial_{x_j}$ for $1 \leq j \leq n$. More precisely, $\Psi_\rho(|D|^2)$ is defined by the Fourier multiplier

$$
\Psi_\rho(|D|^2) \phi(x) = \mathcal{F}^*\Psi_\rho(|\xi|^2) \mathcal{F} \phi(x) = \frac{1}{(2\pi)^n} \int \int_{\mathbb{R}^{2n}} e^{-i(x-y)\cdot\xi} \Psi_\rho(|\xi|^2) \phi(y) dy d\xi
$$

(1.3)
for \( \phi \in H^{2\rho}(\mathbb{R}^n) \), which is the Sobolev space with order \( 2\rho \), where \( \mathcal{F} \) and \( \mathcal{F}^* \) respectively denote the Fourier transform on \( L^2(\mathbb{R}^n) \) and its inverse. In particular, when \( \rho = 1 \), \( \Psi_1(|D|^2) \) is coincident with \(-\Delta\) itself and when \( \rho = 1/2 \), \( \Psi_{1/2}(|D|^2) \) represents \( \sqrt{-\Delta + 1} - 1 \), which is, as is well known, the massive relativistic Schrödinger operator. \( \Psi_\rho(|D|^2) \) is therefore the generalized Schrödinger-type operator in this sense. The full Hamiltonian \( H_\rho \) is perturbated by the pairwise interaction potential \( V = V(x) \) that is a multiplication operator of the function \( V : \mathbb{R}^n \to \mathbb{R} \); i.e.,

\[
H_\rho = \Psi_\rho(|D|^2) + V.
\]

In our study, we treat the general potentials that belong to class \( \text{es} \) as broadly as possible. To prove the main theorem, the minimal velocity bound in Theorem 1.4 and other theorems in Section 3 and 4, we assume that the value of \( V \) vanishes for \( |x| \to \infty \), where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). In contrast, to prove the maximal velocity bound in Theorem 2.3, it is sufficient to assume the weaker conditions that guarantee only the self-adjointness of \( H_\rho \). Further details are stated in Assumptions 1.1 and 2.1.

In the following Assumption 1.1, the bracket of \( x \) have the usual definition \( \langle x \rangle = \sqrt{1 + |x|^2} \). \( A \lesssim B \) means that there exists a constant \( C > 0 \) such that the inequality \( A \leq CB \) holds. If emphasizing the dependence of \( \alpha \) on \( C = C_\alpha \), we write \( A \lessapprox \alpha B \).

**Assumption 1.1.** \( V = V(x) \) is a real-valued-function and decomposes into the sum of three parts:

\[
V = V_{\text{sing}} + V_{\text{short}} + V_{\text{long}},
\]

where these real-valued functions \( V_{\text{sing}}, V_{\text{short}} \) and \( V_{\text{long}} \) satisfy respective conditions.

- \( V_{\text{sing}} = V_{\text{sing}}(x) \) satisfies that, for \( \gamma_{\text{sing}} > 1 \), \( \langle x \rangle^{\gamma_{\text{sing}}} V_{\text{sing}} \) belongs to \( L^p(\mathbb{R}^n) \), where \( p = 2 \) if \( n < 4\rho \) and \( p > n/(2\rho) \) if \( n \geq 4\rho \).
- \( V_{\text{short}} = V_{\text{short}}(x) \) is bounded on \( \mathbb{R}^n \) and has the spatial decay

\[
|V_{\text{short}}(x)| \lessapprox \langle x \rangle^{-\gamma_{\text{short}}},
\]

where \( \gamma_{\text{short}} > 1 \).

- \( V_{\text{long}} = V_{\text{long}}(x) \) belongs to \( C^1(\mathbb{R}^n) \) and, for the multi-indices \( \beta \) with \( 0 \leq |\beta| \leq 1 \), has the spatial decay

\[
|\partial^\beta_x V_{\text{long}}(x)| \lessapprox_\beta \langle x \rangle^{-\gamma_{\text{long}} - |\beta|},
\]

where \( \gamma_{\text{long}} > 0 \).

**Remark 1.2.** Although \( V_{\text{short}} \) and \( V_{\text{long}} \) are bounded functions, \( V_{\text{sing}} \) is not always bounded. We therefore have to consider the self-adjointness of \( H_\rho \). We provide the proof later in Proposition 1.5. In the case where \( 0 < \rho \leq 1/4 \), we cannot
assume that \( \langle x \rangle \gamma V_{\text{sing}} \) belongs to \( L^2(\mathbb{R}^n) \) because the dimension \( n \) has to satisfy \( n < 4\rho \leq 1 \). In the case where \( 3/4 < \rho < 1 \), \( V_{\text{sing}} \) satisfies that \( \langle x \rangle \gamma V_{\text{sing}} \) belongs to \( L^p(\mathbb{R}^n) \), where \( p=2 \) if \( n \leq 3 \) and \( p > n/(2\rho) \) if \( n \geq 4 \). These conditions are almost the same as the self-adjointness of the standard Schrödinger operator, \(-\Delta\) and perturbational potentials; however, we cannot admit that \( n/2 \leq p \leq n/(2\rho) \) in our case even though \( n \geq 5 \).

**Remark 1.3.** If \( 3/4 < \rho \leq 1 \) and \( n = 3 \), the Coulomb-type local singularity, for \( \kappa \in \mathbb{R} \),
\[
V_{\text{sing}}(x) = \kappa |x|^{-1} F (|x| \leq 1)
\]
is admitted, where \( F(\cdots) \) denotes the characteristic function of the set \( \{\cdots\} \). Practically speaking,
\[
\int_{|x|\leq 1} |x|^{-2} dx = \omega_n \int_0^1 r^{-3+n} dr
\]
is bounded for \( n = 3 \) and \([1.8]\) belongs to \( L^2(\mathbb{R}^3) \), where \( \omega_n \) is the surface area of the \( n \)-dimensional unit sphere. If \( 1/2 < \rho \leq 3/4 \), \([1.8]\) does not belong to \( L^2(\mathbb{R}^n) \) for \( n = 1 \) and \( 2 \) because \([1.9]\) is the divergent integral. In this case, when choosing \( p \) such that \( n/(2\rho) < p < n \),
\[
\int_{|x|\leq 1} |x|^{-p} dx = \omega_n \int_0^1 r^{-p+n-1} dr
\]
is bounded. Therefore, when \( 1/2 < \rho \leq 3/4 \), \([1.8]\) belongs to \( L^p(\mathbb{R}^n) \) and is admitted for all \( n \geq 3 \). If \( 0 < \rho \leq 1/2 \), we cannot admit \([1.8]\) no matter what the dimension is. However,
\[
V_{\text{sing}}(x) = \kappa |x|^{-1+\epsilon} F (|x| \leq 1)
\]
with \( \epsilon > 0 \) that satisfies \( 1 - 2\rho < \epsilon < 1 \) is admitted for all \( n \geq 1 \). This is because we can take \( p \) that satisfies \( n/(2\rho) < p < n/(1-\epsilon) \), and
\[
\int_{|x|\leq 1} |x|^{-(1+\epsilon)} r dx = \omega_n \int_0^1 r^{-(1+\epsilon)p+n-1} dr
\]
is bounded for \((1 - \epsilon)p + n - 1 > -1\) which is equivalent to \( p < n/(1-\epsilon) \). This implies that \([1.11]\) belongs to \( L^p(\mathbb{R}^n) \).

The main result of this paper is the propagation estimate that has the following integral-form. In scattering theory, we often refer to this estimate as the minimal velocity bound. We denote the pure point spectrum of \( H_\rho \) by \( \sigma_{pp}(H_\rho) \).
Theorem 1.4. **Minimal velocity bound.** Let $f \in C_0^\infty((0, \infty))$ satisfy $\text{supp } f \cap \sigma_{pp}(H_\rho) = \emptyset$ and $\theta_0 > 0$ be sufficiently small. Then, the inequality
\[
\int_1^\infty \left\| F\left(\frac{|x|}{2t} \leq \theta_0\right) f(H_\rho) e^{-itH_\rho} \phi \right\|^2 \frac{dt}{t} \lesssim \|\phi\|^2_{L^2(\mathbb{R}^n)} \tag{1.13}
\]
holds for $\phi \in L^2(\mathbb{R}^n)$.

This propagation estimate (1.13) is powerful and if $V$ has the short-range parts only, the asymptotic completeness of the wave operators can be obtained immediately. With regard to the long-range case, if we construct some type of modification of the wave operators, Theorem 1.4 can also be applied to the proof of the asymptotic completeness of the modified wave operators. Moreover, it is known that the propagation estimates of the integral-form are available to the $N$-body case. As we state below, although there are some results of the minimal velocity bound with the integral-form for the standard Schrödinger time evolution, the case for the Schrödinger-type operator with the general fractional powers has not been discussed up until our study. Of course, in the fractional powers, the relativistic quantum case $\rho = \frac{1}{2}$ is most important physically. However, the both cases where $0 < \rho < \frac{1}{2}$ and $\frac{1}{2} < \rho < 1$ are of mathematical interest and challenging. For instance, in the case where $1/2 < \rho < 1$, $\Psi_\rho(|D|^2) \langle D \rangle$ is not bounded and we have to make full use of the energy cut off. When $0 < \rho < 1/2$, $\Psi_\rho(|D|^2) \langle D \rangle$ is bounded, while $\langle D \rangle (H_\rho)^{-1}$ is not bounded and this difficulty affects parts of our discussions.

In section 4 we prove the Mourre estimate in Theorem 4.3. In our proof of Theorem 1.4, the Mourre inequality also fulfills a crucial role. In Mourre theory, it is important to find a conjugate operator. We employ the choice $A_\rho$ (see (4.27)) and prove the isolatedness and finite multiplicity of $\sigma_{pp}(H_\rho) \setminus \{0\}$ in Corollary 4.4 using the Mourre inequality.

It seems that the minimal velocity bound with the integral-form was first obtained by Sigal and Soffer [18, Theorem 4.2] for the long-range and $N$-body Schrödinger operator. We currently refer to the works of Dereziński and Gérard [3, Propositions 4.4.7 and 6.6.8] and Isozaki [12, Theorems 2.38 and 3.11], which explain in detail the method of reaching the minimal velocity bound for the standard Schrödinger operators in the cases of two- to $N$-body. In the same manner as for the standard Schrödinger case, in proving Theorem 1.4 we need the maximal velocity bound in Theorem 2.3 and the middle velocity bound in Theorem 3.1. The maximal velocity bound with the integral-form was first proved by [18 Theorem 4.3]. The middle velocity bound with the integral-form was first proved by Graf [6, Theorem 4.3] for the short-range $N$-body Schrödinger operator. Meanwhile, the minimal velocity bound with pointwise-form initiated by Skibsted [19] and Gérard [4] is also an important estimate with which to prove the asymptotic
completeness. The pointwise-form of the conjugate operator was developed by Hunziker, Sigal and Soffer [8] and Richard [16] in the abstract settings.

Scattering theory for the Schrödinger-type operator with fractional powers has been studied. Gire [5] considered general functions of the Laplacian that included the relativistic Schrödinger operator and discussed the asymptotic completeness for the short-range potentials by investigating the semigroup differences. Kitada [13, 14] constructed long-range scattering theory for the fractional Laplacian \((-\Delta)^\rho\) with \(1/2 \leq \rho \leq 1\) adopting the Enss method and smooth perturbation theory. Ishida [9] studied inverse scattering for \((-\Delta)^\rho\) with \(1/2 < \rho \leq 1\).

It is noteworthy that although it was only the case of the massless relativistic Schrödinger operator \(\sqrt{-\Delta}\), Soffer [20] obtained the integral-form minimal velocity bound by using its pointwise maximal velocity bound. Ishida and Wada [11] considered non-local Schrödinger operators that included the Bernstein functions of the Laplacian, and they decided the threshold between short- and long-range decay conditions of the potential functions by providing a counter-example such that the wave operators did not exist.

At the end of this section, we prove the self-adjointness of \(H^\rho\). By virtue of the Kato–Rellich theorem (Reed and Simon [23, Theorem X.12]) and following Proposition 1.5, if \(V\) satisfies Assumption 1.1, then \(H^\rho = \Psi^\rho (|D|^2) + V\) is essentially self-adjoint with the core \(C_0^\infty(\mathbb{R}^n)\). The original idea of this proof for the standard Schrödinger operator \(-\Delta + V\) is explained in [23, Theorem X.20] and [12, Lemma 1.9].

**Proposition 1.5.** Suppose the real-valued function \(\hat{V}_{\text{sing}} = \hat{V}_{\text{sing}}(x)\) satisfies that \(\hat{V}_{\text{sing}}\) belongs to \(L^p(\mathbb{R}^n)\), where \(p = 2\) if \(n < 4\rho\) and \(p > n/(2\rho)\) if \(n \geq 4\rho\). Then, for any \(\epsilon > 0\) and \(\phi \in C_0^\infty(\mathbb{R}^n)\), there exists a constant \(C_\epsilon > 0\) such that

\[
\|\hat{V}_{\text{sing}} \phi\|_{L^2(\mathbb{R}^n)} \leq \epsilon \|\Psi^\rho (|D|^2) \phi\|_{L^2(\mathbb{R}^n)} + C_\epsilon \|\phi\|_{L^2(\mathbb{R}^n)} \tag{1.14}
\]

holds.

**Proof of Proposition 1.5.** Let \(0 < \delta < 1/2\). We note that

\[
\int_{\mathbb{R}^n} \frac{d\xi}{\{1 + \delta \Psi^\rho (|\xi|^2)\}^p} = \omega_n \delta^{-n/(2\rho)} \int_0^\infty \frac{\eta^{n-1}d\eta}{\{1 - \delta + (\delta^{1/\rho} + \eta^2)^{2p}\}^p}
\]

\[
\leq \omega_n \delta^{-n/(2\rho)} \left\{2^p + \int_1^\infty \eta^{n-1-2\rho p}d\eta \right\} \tag{1.15}
\]

with a changing variable \(\eta = \delta^{1/(2\rho)}|\xi|\), and that

\[
\left\|\left\{1 + \delta \Psi^\rho (|\xi|^2)\right\}^{-1}\right\|_{L^p(\mathbb{R}_\xi^n)} \lesssim \delta^{-n/(2\rho p)} \tag{1.16}
\]
because \( n \geq 1 - 2\rho \rho < -1 \). If \( n < 4\rho \), we express \( \phi \in C_0^\infty(\mathbb{R}^n) \) by
\[
\phi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\}^{-1} \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\} \mathcal{F} \phi(\xi) d\xi \tag{1.17}
\]
and there exists \( C > 0 \) such that
\[
|\phi(x)| \lesssim \left\| \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\}^{-1} \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\} \mathcal{F} \phi \right\|_{L^2(\mathbb{R}^n)}^2 \lesssim C \left( \delta^{1-n/(4\rho)} \left\| \Psi_\rho \left( |D|^2 \right) \phi \right\|_{L^2} + \delta^{-n/(4\rho)} \left\| \phi \right\|_{L^2} \right), \tag{1.18}
\]
using the Schwarz inequality and (1.16) for \( p = 2 \). If making \( \delta \) small such that \( C\delta^{1-n/(4\rho)} \left\| \hat{V}_{\text{sing}} \right\|_{L^2} \leq \epsilon \), then (1.18) and
\[
\left\| \hat{V}_{\text{sing}} \phi \right\|_{L^2} \leq \left\| \hat{V}_{\text{sing}} \right\|_{L^2} \sup_{x \in \mathbb{R}^n} |\phi(x)| \tag{1.19}
\]
implies (1.12). We next assume that \( n \geq 4\rho \) and \( p > n/(2\rho) \). For \( q_1 = 2p/(p-2) \), by the Hölder inequality,
\[
\left\| \hat{V}_{\text{sing}} \phi \right\|_{L^2} \leq \left\| \hat{V}_{\text{sing}} \right\|_{L^{q_1}} \left\| \phi \right\|_{L^{q_1}} \tag{1.20}
\]
holds. For \( q_2 = q_1/(q_1 - 1) = 2p/(p+2) \), by the Hausdorff–Young inequality [23, Theorem IX.8], we have
\[
\left\| \phi \right\|_{L^{q_1}} \leq (2\pi)^{n(1/2 - 1/q_2)} \left\| \mathcal{F} \phi \right\|_{L^{q_2}} \tag{1.21}
\]
noting that \( q_1 > 2 \) and \( 1 < q_2 < 2 \). Using the Hölder inequality and (1.16) again, we have
\[
\left\| \mathcal{F} \phi \right\|_{L^{q_2}} \leq \left\| \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\}^{-1} \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\} \mathcal{F} \phi \right\|_{L^p(\mathbb{R}^n)} \left\| \left\{ 1 + \delta \Psi_\rho \left( |\xi|^2 \right) \right\} \mathcal{F} \phi \right\|_{L^2(\mathbb{R}^n)} \lesssim \delta^{1-n/(2\rho p)} \left\| \Psi_\rho \left( |D|^2 \right) \phi \right\|_{L^2} + \delta^{-n/(2\rho p)} \left\| \phi \right\|_{L^2}. \tag{1.22}
\]
This completes the proof. \( \square \)

## 2 Maximal velocity bound

In this section, we prove the propagation estimate for the high-velocity region in Theorem 2.3, which is needed for the proof of Theorem 3.1 in the next section. We often refer to this estimate as the maximal velocity bound. If we prove Theorem 2.3 only, the bounded parts of the potential function \( V_{\text{short}} + V_{\text{long}} \) do not necessarily disappear for \( |x| \to \infty \), and the singular part \( V_{\text{sing}} \) can decay far more slowly. Throughout this section, instead of Assumption 1.1, we assume the following Assumption 2.1.
Assumption 2.1. $V = V(x)$ is a real-valued function and decomposes into the sum of two parts:

$$V = \hat{V}_{\text{sing}} + V_{\text{bdd}},$$

(2.1)

where $\hat{V}_{\text{sing}} = \hat{V}_{\text{sing}}(x)$ satisfies the conditions in Proposition 1.5 while $V_{\text{bdd}} = V_{\text{bdd}}(x)$ belongs to $L^\infty(\mathbb{R}^n)$.

Remark 2.2. In the case where $\rho = 1$ (i.e., the standard Schrödinger operator case), $V_{\text{bdd}}$ in (2.1) can be replaced with $\tilde{V}_{\text{sing}} = \tilde{V}_{\text{sing}}(x)$ that belongs to $L^2_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$\tilde{V}_{\text{sing}}(x) \gtrsim -\langle x \rangle^2$$

(2.2)

by applying the Kato distributional inequality [23, Theorems X.27] and Faris-Lavine theorem [23, Theorems X.38]. This means that the potential function $V$ can be allowed to grow in $x$ to prove Theorem 2.3 only. We note that $\text{Domain}(-\Delta + V)$ does not always coincide with $H^2(\mathbb{R}^n)$ in this case.

Under Assumption 2.1, $H_\rho = \Psi_\rho(|D|^2) + V$ is self-adjoint by Proposition 1.5. We here note again that if $V$ satisfies Assumption 1.1, $V$ also satisfies Assumption 2.1. The maximal velocity bound is stated as the following theorem. The corresponding propagation estimate for the standard two-body Schrödinger operator is detailed in [3, Proposition 4.2.1] and [12, Theorem 2.31].

**Theorem 2.3. Maximal velocity bound.** Take $f \in C^\infty_0(\mathbb{R})$ arbitrarily. There exists $\Theta > 0$ such that, for any $\theta > \Theta$ and $\phi \in L^2(\mathbb{R}^n)$,

$$\int_1^\infty \left\| F \left( \Theta \leq \frac{|x|}{2t} \leq \theta \right) f(H_\rho)e^{-itH_\rho}\phi \right\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \lesssim \|\phi\|^2_{L^2(\mathbb{R}^n)}$$

(2.3)

holds.

We provide preparations in advance of the proof of Theorem 2.3. To analyze $\Psi_\rho(|D|^2)$ as a function of the Laplacian, we make efficient use of the almost analytic extension and commutator expansions. We thus extend the domain of $\Psi_\rho(s)$ to a full real axis and employ the function $\Psi_\rho \in C^\infty(\mathbb{R})$ such that

$$\Psi_\rho(s) = \begin{cases} 
(s + 1)^\rho - 1 & \text{if } s \geq 0, \\
0 & \text{if } s \leq -1
\end{cases}$$

(2.4)

for $0 < \rho < 1$. This $\Psi_\rho$ satisfies, for all $k \in \mathbb{N} \cup \{0\}$,

$$\left| \frac{d^k}{ds^k}\Psi_\rho(s) \right| \lesssim_k (s)^{\rho-k}$$

(2.5)
on \( \mathbb{R} \). We therefore find a function \( \tilde{\Psi}_\rho \in C^\infty(\mathbb{C}) \), called an almost analytic extension of \( \Psi_\rho \) \[3\] Propositions C.2.1 and C.2.2; i.e., \( \tilde{\Psi}_\rho \) with

\[
\text{supp } \tilde{\Psi}_\rho \subset \{ z \in \mathbb{C} \mid |\text{Im}z| \lesssim \langle \text{Re}z \rangle \}
\]

(2.6)
satisfies that \( \tilde{\Psi}_\rho(s) = \Psi_\rho(s) \) for \( s \in \mathbb{R} \) and that

\[
|\tilde{\partial}_z \tilde{\Psi}_\rho(z)| \lesssim_N |\text{Im}z|^{N}(z)^{\rho - 1 - N}
\]

(2.7)
for \( N \in \mathbb{N} \), where \( \tilde{\partial}_z = (\partial_{\text{Re}z} + i\partial_{\text{Im}z})/2 \). One of the most effective applications of the almost analytic extension is the Helffer–Sjöstrand formula originated by \[7\] Proposition 7.2] (see also \[12\] Theorem 1.17). Unfortunately, we cannot apply this formula to \( \Psi_\rho(|D|^2) \) directly because \( \rho > 0 \). However, when \( 0 < \rho < 1 \), we can consider commutator expansions with a function of \( x \) by applying the Helffer–Sjöstrand formula to \( \Psi_\rho/(1 + s) \) instead of \( \Psi_\rho \). The more general settings of the commutator expansions are referred to \[3\] Lemma C.3.1 and \[12\] Definition 4.11.

**Lemma 2.4.** Suppose \( 0 < \rho < 1 \) and put \( \Phi_\rho(s) = \Psi_\rho(s)/(1 + s) \). For a smooth function \( \chi = \chi(x) \) such that its all derivatives are bounded, the commutator \( [\Psi_\rho(|D|^2), \chi] \) has the expansions

\[
[\Psi_\rho (|D|^2), \chi] = [|D|^2, \chi] \Psi_\rho' (|D|^2) \\
+ \frac{1}{2\pi i} \int_{\mathbb{C}} (\tilde{\partial}_z \tilde{\Psi}_\rho)(z) (z - |D|^2)^{-1} [|D|^2, [|D|^2, \chi]] (z - |D|^2)^{-2} dz \wedge d\bar{z},
\]

(2.8)
and

\[
[\Psi_\rho (|D|^2), \chi] = \Psi_\rho' (|D|^2) [|D|^2, \chi] \\
- \frac{1}{2\pi i} \int_{\mathbb{C}} (\tilde{\partial}_z \tilde{\Psi}_\rho)(z) (z - |D|^2)^{-2} [|D|^2, [|D|^2, \chi]] (z - |D|^2)^{-1} dz \wedge d\bar{z},
\]

(2.9)
where \( dz \wedge d\bar{z} = -2i\text{Re}z \wedge \text{dIm}z \) is the two-dimensional Lebesgue measure and \( \Psi_\rho' \) denotes \( d\Psi_\rho/ds \).

**Remark 2.5.** Right-hand sides of (2.8) and (2.9) are operators on \( H^{2\rho}(\mathbb{R}^n) \) because the integral terms are bounded by (2.11) and

\[
[|D|^2, \chi] = -iD \cdot \nabla \chi - i\nabla \cdot D = -2iD \cdot \nabla \chi + \Delta \chi = -2i\nabla \chi \cdot D - \Delta \chi
\]

(2.10)
holds on \( H^2(\mathbb{R}^n) \).

**Proof of Lemma 2.4.** We prove the formula (2.8) only. We first note that

\[
\int_{\mathbb{C}} |\tilde{\partial}_z \tilde{\Psi}_\rho(z)| \left\| (z - |D|^2)^{-1} [|D|^2, [|D|^2, \chi]] (z - |D|^2)^{-2} \right\| |dz \wedge d\bar{z}| < \infty,
\]

(2.11)
where we denote the operator norm on $L^2(\mathbb{R}^n)$ by $\| \cdot \|$. This is seen as follows. By the basic inequality

$$\sup_{\lambda \in \mathbb{R}} \frac{\langle \lambda \rangle^{q_1}}{|z - \lambda|^{q_2}} \lesssim_{q_1, q_2} \frac{\langle z \rangle^{q_1}}{|\text{Im } z|^{q_2}}$$

(2.12)

for $q_2 > 0$ and $0 \leq q_1 \leq q_2$, we have

$$\left\| (z - |D|^2)^{-1} [ [D]^2, [D]^2, \chi] (z - |D|^2)^{-2} \right\| \lesssim |\text{Im } z|^{-3} \langle z \rangle$$

(2.13)

for $z \in \mathbb{C} \setminus \mathbb{R}$. Inequality (2.12) will be used often in our proof. Therefore, the left-hand side of (2.11) is bounded because, for $\rho < 1$,

$$\int_{\mathbb{C}} \langle z \rangle^{\rho - 3} |dz \wedge d\bar{z}| < \infty$$

(2.14)

by (2.7) with $N = 3$. We now will prove (2.8). Because

$$\left| \frac{d^k}{ds^k} \Phi_{\rho}(s) \right| \lesssim_k \langle s \rangle^{\rho - 1 - k}$$

(2.15)

holds for $k \in \mathbb{N} \cup \{0\}$, an almost analytic extension $\tilde{\Phi}_{\rho} \in C^\infty(\mathbb{C})$ has the estimate

$$|\partial_{\bar{z}} \tilde{\Phi}_{\rho}(z)| \lesssim_N |\text{Im } z|^N \langle z \rangle^{\rho - 2 - N}$$

(2.16)

for any $N \in \mathbb{N}$. According to the Helffer–Sjöstrand formula, $\Phi_{\rho}(|D|^2)$ is expressed as

$$\Phi_{\rho}(|D|^2) = \frac{1}{2\pi i} \int_{\mathbb{C}} (\partial_{\bar{z}} \tilde{\Phi}_{\rho})(z) (z - |D|^2)^{-1} d\bar{z} \wedge dz.$$  

(2.17)

We therefore compute

$$[\Phi_{\rho}(|D|^2), \chi] = \frac{1}{2\pi i} \int_{\mathbb{C}} (\partial_{\bar{z}} \tilde{\Phi}_{\rho})(z) (z - |D|^2)^{-1} [ [D]^2, \chi] (z - |D|^2)^{-1} d\bar{z} \wedge dz + \frac{1}{2\pi i} \int_{\mathbb{C}} (\partial_{\bar{z}} \tilde{\Phi}_{\rho})(z) (z - |D|^2)^{-1} [ [D]^2, [D]^2, \chi] (z - |D|^2)^{-2} d\bar{z} \wedge dz.$$  

(2.18)

Incidentally, from the definition of $\Phi_{\rho}$,

$$[\Phi_{\rho}(|D|^2), \chi] = [\Psi_{\rho}(|D|^2), \chi] \langle D \rangle^{-2} - \Phi_{\rho}(|D|^2) [ [D]^2, \chi] \langle D \rangle^{-2}$$

(2.19)

and

$$\Phi'_{\rho}(|D|^2) = \Psi'_{\rho}(|D|^2) \langle D \rangle^{-2} - \Phi_{\rho}(|D|^2) \langle D \rangle^{-2}$$

(2.20)
hold. Combining (2.18), (2.19), and (2.20), we have
\[
\begin{align*}
\left[\Psi_\rho (|D|^2), \chi\right] \langle D \rangle^{-2} &= \left[|D|^2, \chi\right] \Psi_\rho' (|D|^2) \langle D \rangle^{-2} + \left[\Phi_\rho (|D|^2), \left[|D|^2, \chi\right]\right] \langle D \rangle^{-2} \\
&+ \frac{1}{2\pi i} \int_C (\bar{\partial}_z \Phi_\rho)(z) \left(z - |D|^2\right)^{-1} \left[|D|^2, \left[|D|^2, \chi\right]\right] \left(z - |D|^2\right)^{-1} dz \wedge d\bar{z}.
\end{align*}
\] (2.21)

This equation implies
\[
\begin{align*}
\left[\Psi_\rho (|D|^2), \chi\right] &= \left[|D|^2, \chi\right] \Psi_\rho' (|D|^2) \\
&+ \frac{1}{2\pi i} \int_C (\bar{\partial}_z \Phi_\rho)(z)(1 + z) \left(z - |D|^2\right)^{-1} \left[|D|^2, \left[|D|^2, \chi\right]\right] \left(z - |D|^2\right)^{-1} dz \wedge d\bar{z},
\end{align*}
\] (2.22)

noting that
\[
\begin{align*}
\left[\Phi_\rho (|D|^2), \left[|D|^2, \chi\right]\right] \\
= \frac{1}{2\pi i} \int_C (\bar{\partial}_z \Phi_\rho)(z) \left(z - |D|^2\right)^{-1} \left[|D|^2, \left[|D|^2, \chi\right]\right] \left(z - |D|^2\right)^{-1} dz \wedge d\bar{z}
\end{align*}
\] (2.23)

by the Helffer–Sjöstrand formula and that \( \langle D \rangle^2 = -(z - |D|^2) + 1 + z \). Because \( \Phi_\rho(z)(1 + z) \) corresponds with one of the almost analytic extensions of \( \Psi_\rho \), we have
\[
(\bar{\partial}_z \Phi_\rho)(z)(1 + z) = \bar{\partial}_z \left\{ \Phi_\rho(z)(1 + z) \right\} = (\bar{\partial}_z \Psi_\rho)(z). \] (2.24)

(2.22) and (2.24) imply (2.8). \( \square \)

We will use the following notations frequently. The Heisenberg derivative of a time-dependent operator \( P(t) \) associated with an operator \( Q \) is
\[
\mathbb{D}_Q P(t) = \frac{d}{dt} P(t) + i [Q, P(t)].
\] (2.25)

If \( P \) is time-independent, \( \mathbb{D}_Q P \) is \( i [Q, P] \). \( P(t) = \mathcal{O}(t^\nu) \) means that \( P(t) \) is the bounded operator and that \( \| P(t) \| \lesssim t^\nu \) for \( \nu \in \mathbb{R} \). The Hermitian conjugate \( hc \) is defined by \( Q + hc = Q + Q^* \), where \( Q^* \) is the formal adjoint of \( Q \).

**Proof of Theorem 2.3.** Let \( \chi \in C_0^\infty (\mathbb{R}) \) satisfy that \( \chi(s) = 1 \) if \( \Theta/2 \leq s \leq 2\theta \) and \( \chi(s) = 0 \) if \( s \leq \Theta/3 \) for \( 0 < \Theta < \theta \), where the size of \( \Theta \) is to be determined below. Put \( X(s) = \int_{-\infty}^s \chi(\tau)^2 d\tau \) and
\[
\mathcal{L}(t) = f(H_\rho)X \left( \frac{|x|}{2t} \right) f(H_\rho), \] (2.26)
according to [31 Proposition 4.2.1] and [12 Theorem 2.31]. Clearly, \( \mathcal{L}(t) = \mathcal{O}(1) \).

We first give the proof for the case where \( \rho < 1 \). Using (2.9), we compute
\[
\begin{align*}
&i \left[ \Psi_\rho \left( |D|^2 \right), X \left( \frac{|x|}{2t} \right) \right] = \Psi_\rho \left( |D|^2 \right) \left\{ \frac{1}{2t} D \cdot \frac{x}{|x|} \chi \left( \frac{|x|}{2t} \right) + hc \right\} + \mathcal{O} \left( t^{-2} \right) \\
&= \frac{1}{2t} \Psi_\rho \left( |D|^2 \right) D \cdot \frac{x}{|x|} \chi \left( \frac{|x|}{2t} \right)^2 + hc + \mathcal{O} \left( t^{-2} \right). 
\end{align*}
\]

We here adopted the estimate
\[
\left[ \Psi_\rho \left( |D|^2 \right), \chi \left( \frac{|x|}{2t} \right)^2 \frac{x}{|x|} \cdot D \right] = \mathcal{O} \left( t^{-1} \right),
\]
using the Helffer–Sjöstrand formula directly with
\[
\left\| (z - |D|^2)^{-1} \left[ |D|^2, \chi \left( \frac{|x|}{2t} \right)^2 \frac{x}{|x|} \cdot D \right] (z - |D|^2)^{-1} \right\| \lesssim t^{-1} |\text{Im} z|^{-2} \langle z \rangle \quad \text{(2.29)}
\]
and \(|\partial_z \tilde{\Psi}'(z)| \lesssim |\text{Im} z|^2 \langle z \rangle^{-4}\). Therefore, from (2.27), we have
\[
\begin{align*}
&\mathbb{D}_{\Psi_\rho(|D|^2)} X \left( \frac{|x|}{2t} \right) = - \frac{|x|}{2t^2} \chi \left( \frac{|x|}{2t} \right)^2 \\
&+ \frac{1}{2t} \Psi_\rho \left( |D|^2 \right) D \cdot \frac{x}{|x|} \chi \left( \frac{|x|}{2t} \right)^2 + hc + \mathcal{O} \left( t^{-2} \right). 
\end{align*}
\]

We take \( g \in C_0^\infty(\mathbb{R}) \) such that \( f = fg \) and compute
\[
\begin{align*}
f(H_\rho) \Psi_\rho \left( |D|^2 \right) D \cdot \frac{x}{|x|} \chi \left( \frac{|x|}{2t} \right)^2 f(H_\rho) \\
= f(H_\rho) \chi \left( \frac{|x|}{2t} \right) g(H_\rho) \Psi_\rho \left( |D|^2 \right) D \cdot \frac{x}{|x|} \chi \left( \frac{|x|}{2t} \right) f(H_\rho) + I_1(t) + I_2(t). 
\end{align*}
\]

We defined \( I_1 \) and \( I_2 \) in (2.31) by
\[
\begin{align*}
I_1(t) &= f(H_\rho) \sum_{j=1}^n \left[ \Psi_\rho \left( |D|^2 \right) D_j, \chi \left( \frac{|x|}{2t} \right) \right] \frac{x_j}{|x|} \chi \left( \frac{|x|}{2t} \right) f(H_\rho), \\
I_2(t) &= f(H_\rho) \left[ g(H_\rho), \chi \left( \frac{|x|}{2t} \right) \right] \Psi_\rho \left( |D|^2 \right) D \cdot \frac{x}{|x|} \chi \left( \frac{|x|}{2t} \right) f(H_\rho),
\end{align*}
\]
where \( D_j \) is the \( j \)th component of \( D \). Making the same computation as (2.28) yields

\[
\begin{aligned}
\left[ \Psi'_\rho \left( |D|^2 \right) D_j, \chi \left( \frac{|x|}{2t} \right) \right] \\
= \Psi'_\rho \left( |D|^2 \right) \mathcal{O} \left( t^{-1} \right) + \left[ \Psi'_\rho \left( |D|^2 \right), \chi \left( \frac{|x|}{2t} \right) \right] D_j = \mathcal{O} \left( t^{-1} \right)
\end{aligned}
\]

(2.34)

for \( 1 \leq j \leq n \) and \( I_1(t) = \mathcal{O}(t^{-1}) \) holds. We here note that \( \langle \Psi_\rho(|D|^2) \rangle (H_\rho)^{-1} \) is bounded by virtue of Proposition 1.5 and the Kato–Rellich theorem. We write

\[
\begin{aligned}
\Psi'_\rho \left( |D|^2 \right) \langle D \rangle (z - H_\rho)^{-1} \\
= \Psi'_\rho \left( |D|^2 \right) \langle D \rangle \langle \Psi_\rho \left( |D|^2 \right) \rangle^{-1} \langle \Psi_\rho \left( |D|^2 \right) \rangle \langle H_\rho \rangle^{-1} \langle H_\rho \rangle (z - H_\rho)^{-1},
\end{aligned}
\]

(2.35)

and then estimate

\[
\| \Psi'_\rho \left( |D|^2 \right) \langle D \rangle (z - H_\rho)^{-1} \| \lesssim |\text{Im} z|^{-1}(z),
\]

(2.36)

if \( \rho > 1/2 \). If \( 0 < \rho \leq 1/2 \), the right-hand side of (2.36) can be replaced with just \( |\text{Im} z|^{-1} \) because \( \Psi'_\rho(|D|^2) \langle D \rangle \) is bounded. By virtue of the Helffer–Sjöstrand formula, (2.27), and (2.36), we have

\[
\begin{aligned}
\left[ g(H_\rho), \chi \left( \frac{|x|}{2t} \right) \right] &= \frac{1}{2\pi i} \int_{\mathbb{C}} (\tilde{\partial}_z \tilde{g})(z) (z - H_\rho)^{-1} \\
&\times \left[ \Psi_\rho \left( |D|^2 \right), \chi \left( \frac{|x|}{2t} \right) \right] (z - H_\rho)^{-1} d\bar{z} = \mathcal{O} \left( t^{-1} \right),
\end{aligned}
\]

(2.37)

noting that an almost analytic extension \( \tilde{g} \) is compactly supported in \( \mathbb{C} \). Because we know \( \Psi'_\rho(|D|^2) D \cdot (x/|x|) \chi(|x|/(2t)) f(H_\rho) = \mathcal{O}(1) \) even though \( \rho > 1/2 \) from (2.31) and boundedness of \( \Psi'_\rho(|D|^2) D f(H_\rho) \), we have \( I_2(t) = \mathcal{O}(t^{-1}) \). It follows from (2.30) and (2.31) that

\[
\mathbb{D}_{H_\rho} \mathcal{L}(t) = f(H_\rho) \left\{ \mathbb{D}_{\Psi_\rho(|D|^2)} X \left( \frac{|x|}{2t} \right) \right\} f(H_\rho)
\]

\[
\leq -\frac{1}{t} \left( \frac{\Theta}{3} - C \right) f(H_\rho) \chi \left( \frac{|x|}{2t} \right)^2 f(H_\rho) + \mathcal{O} \left( t^{-2} \right),
\]

(2.38)

where we put \( C = \| g(H_\rho) \Psi'_\rho(|D|^2) D \cdot x/|x| \| \) and choose \( \Theta \) such that \( \Theta/3 - C > 0 \). This implies (2.3) (by [3] Lemma B.4.1 for example). The proof in the case where \( \rho = 1 \) is given by simply replacing \( \Psi'_\rho \) with 1 in the proof above (see also [3] Proposition 4.2.1 or [12] Theorem 2.31). In particular, the commutator calculation is simpler than that of \( \rho < 1 \) because \( \Psi_1(|D|^2) = |D|^2 \). \( \square \)
3 Middle velocity bound

In Sections 3 and 4, we assume that the potential function $V$ satisfies Assumption 1.1. In this section, we focus on proving Theorem 3.1 that is the propagation estimate in the mid-range velocity region. This estimate is needed for the proof of Theorem 1.4. The corresponding propagation estimate for the standard two-body Schrödinger operator is given in [3, Proposition 4.4.3] and [12, Theorem 2.36]. To withdraw the time decay in the middle region, we have to add the factor $\Psi'_{\rho}(|D|^2)D - x/(2t)$ that comes from the Hamilton canonical equation $\nabla_{D}H_{\rho} = dx/dt$. Thus, $\Psi'_{\rho}(|D|^2)D$ and $x/(2t)$ are close asymptotically. In this context, the propagation estimate with regard to the solution of the Hamilton–Jacobi equation was investigated by [17].

Theorem 3.1. Middle velocity bound. For any $0 < \theta_1 < \theta_2$ and $f \in C^\infty_0(\mathbb{R})$, the inequality

$$
\int_1^\infty \left| F\left( \frac{x}{2t} \right) \right|^2 \left( \frac{x}{2t} \right) f(H_{\rho}) e^{-itH_{\rho}} \phi \right|_{L^2(\mathbb{R}^n)} d\frac{dt}{t} \lesssim \|\phi\|_{L^2(\mathbb{R}^n)}^2
$$

holds for $\phi \in L^2(\mathbb{R}^n)$.

We provide preparations before proving Theorem 3.1. Let $r \in C^\infty(\mathbb{R})$ satisfy that $r(s) = \theta^2/4$ if $s < \theta^2/4$ and $r(s) = s/2$ if $s > \theta^2$ for $0 < \theta < \theta_1$, and that $r', r'' \geq 0$ where $r'' = d^2r/ds^2$. Putting $R(x) = r(|x|^2)$, we have $R(x) = \theta^2/4$ if $|x| < \theta/2$ and $R(x) = |x|^2/2$ if $|x| > \theta$ holds. We also note that

$$
y \cdot (\nabla^2 R)(x)y = 4r''(|x|^2)(x \cdot y)^2 + 2r'(|x|^2)|y|^2 \geq 0
$$

holds for any $y \in \mathbb{R}^n$, where $\nabla^2 R$ is the Hessian matrix of $R$. The original idea of this function $R$ comes from [3] and [12]. We set $\mathcal{M}(t)$ such that

$$
\mathcal{M}(t) = \frac{1}{2} \left\{ \Psi'_{\rho}(|D|^2)D - \frac{x}{2t} \right\} \cdot (\nabla R) \left( \frac{x}{2t} \right) + \text{hc} + R \left( \frac{x}{2t} \right).
$$

We first suppose that $\rho < 1$ and the case where $\rho = 1$ is given the end of the proof of Theorem 3.1.

Lemma 3.2. Under the notations above,

$$
\mathbb{D}_{\Psi'_{\rho}(|D|^2)}\mathcal{M}(t) = \frac{1}{t} \left\{ \Psi'_{\rho}(|D|^2)D - \frac{x}{2t} \right\} \cdot (\nabla^2 R) \left( \frac{x}{2t} \right) \left\{ \Psi'_{\rho}(|D|^2)D - \frac{x}{2t} \right\} + \mathcal{O}(t^{-2})
$$

holds.
Proof of Lemma 3.2. In this proof, we use the following commutator notations

\[ \text{ad}_2 [P, Q] = [P, [P, Q]], \quad \text{ad}_3 [P, Q] = [P, [P, [P, Q]]] \quad (3.5) \]

for the operators \( P \) and \( Q \). By the same computation with (2.30), we have

\[ \mathcal{D}_{\Psi, \rho} (t) \left( \frac{x}{2t} \right) \frac{x}{2t} \left\{ \Psi' (|D|^2) D - \frac{x}{2t} \right\} \cdot (\nabla R) \left( \frac{x}{2t} \right) + h c + \mathcal{O} \left( t^{-2} \right). \quad (3.6) \]

It follows from (3.6) and

\[ \mathcal{D}_{\Psi, \rho} (t) \left( \frac{x}{2t} \right) \frac{x}{2t} = \frac{1}{t} \left\{ \Psi' (|D|^2) D - \frac{x}{2t} \right\}, \quad (3.7) \]

that

\[ \mathcal{D}_{\Psi, \rho} (t) \left( \frac{x}{2t} \right) \mathcal{T} (t) = \frac{1}{2} \left\{ \Psi' (|D|^2) D - \frac{x}{2t} \right\} \cdot \mathcal{D}_{\Psi, \rho} (t) (\nabla R) \left( \frac{x}{2t} \right) + h c + \mathcal{O} \left( t^{-2} \right). \quad (3.8) \]

Using (2.8), we compute, for \( 1 \leq j \leq n \),

\[ \i [\Psi, \rho (|D|^2), (\partial_j R) \left( \frac{x}{2t} \right)] = \frac{1}{t} (\nabla \partial_j R) \left( \frac{x}{2t} \right) \cdot \Psi' (|D|^2) D + B_j^L (t) + \Gamma_j^L (t), \quad (3.9) \]

where

\[ B_j^L (t) = -\frac{i}{4t^2} (\Delta \partial_j R) \left( \frac{x}{2t} \right) \Psi' (|D|^2), \quad (3.10) \]

\[ \Gamma_j^L (t) = \frac{1}{2\pi} \int_C (\partial_z \tilde{\Psi}_\rho) (z) \left( z - |D|^2 \right)^{-1} \times \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \left( z - |D|^2 \right)^{-2} d\bar{z} \land dz. \quad (3.11) \]

Obviously, \( B_j^L (t) = \mathcal{O} (t^{-2}) \) and \( \Gamma_j^L (t) = \mathcal{O} (t^{-2}) \) hold. At the same time, using (2.9), we obtain another expression

\[ \i [\Psi', \rho (|D|^2), (\partial_j R) \left( \frac{x}{2t} \right)] = \frac{1}{t} \Psi' (|D|^2) D \cdot (\nabla \partial_j R) \left( \frac{x}{2t} \right) - B_j^R (t) - \Gamma_j^R (t), \quad (3.12) \]

where \( B_j^R (t) = -B_j^L (t)^* \) and

\[ \Gamma_j^R (t) = \frac{1}{2\pi} \int_C (\partial_z \tilde{\Psi}_\rho) (z) \left( z - |D|^2 \right)^{-2} \times \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \left( z - |D|^2 \right)^{-1} d\bar{z} \land dz. \quad (3.13) \]
with $\Gamma^R_j(t) = \mathcal{O}(t^{-2})$. Combining (3.8), (3.9), and (3.12), we have

$$\mathbb{D}_{\Psi_{\rho}(|D|^2)} \mathcal{M}(t) = \frac{1}{t} \left\{ \Psi^\prime_{\rho}(|D|^2) \right\} \cdot (\nabla^2 R) \left( \frac{x}{2t} \right) \left\{ \Psi^\prime_{\rho}(|D|^2) D - \frac{x}{2t} \right\} + \frac{1}{2} \left\{ \Psi^\prime_{\rho}(|D|^2) D - \frac{x}{2t} \right\} \cdot \left\{ B^L(t) + \Gamma^L(t) \right\} - \frac{1}{2} \left\{ B^R(t) + \Gamma^R(t) \right\} \cdot \left\{ \Psi^\prime_{\rho}(|D|^2) D - \frac{x}{2t} \right\} + \mathcal{O}(t^{-2}). \quad (3.14)$$

We here defined $B^L(t) = (B^L_1(t), \ldots, B^L_n(t))$. $B^R$, $\Gamma^L$, and $\Gamma^R$ have the same definitions. It is clear that

$$\frac{x}{t} \cdot B^L(t) = \mathcal{O}(t^{-2}) \quad (3.15)$$

and that

$$\Psi^\prime_{\rho}(|D|^2) D \cdot B^L(t) - B^R(t) \cdot \Psi^\prime_{\rho}(|D|^2) D = -\frac{1}{8t^3} \Psi^\prime_{\rho}(|D|^2) (\Delta^2 R) \left( \frac{x}{2t} \right) \Psi^\prime_{\rho}(|D|^2) = \mathcal{O}(t^{-3}). \quad (3.16)$$

By calculating the commutator $x_j/t$ and $(z - |D|^2)^{-1}$, we have

$$\left\| \frac{x}{t} \left( z - |D|^2 \right)^{-1} \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] (z - |D|^2)^{-2} \right\| \lesssim t^{-2} |\text{Im} z|^{-3}(z) + t^{-3} |\text{Im} z|^{-4}\langle z \rangle^{3/2}. \quad (3.17)$$

This implies that

$$\frac{x}{t} \cdot \Gamma^L(t) = \mathcal{O}(t^{-2}). \quad (3.18)$$

In the same way, we have

$$\Gamma^R(t) \cdot \frac{x}{t} = \mathcal{O}(t^{-2}). \quad (3.19)$$

If $\rho \leq 1/2$, clearly

$$\Psi^\prime_{\rho}(|D|^2) D \cdot \Gamma^L(t) = \mathcal{O}(t^{-2}) \quad (3.20)$$

because $\Psi^\prime_{\rho}(|D|^2) \langle D \rangle$ is bounded. Moreover, even in the case of $1/2 < \rho < 3/4$, (3.20) holds because

$$\left\| \Psi^\prime_{\rho}(|D|^2) \langle D \rangle \left( z - |D|^2 \right)^{-1} \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] (z - |D|^2)^{-2} \right\| \lesssim t^{-2} |\text{Im} z|^{-3}(z)^{\rho+1/2}. \quad (3.21)$$

Similarly, for $\rho < 3/4$, we have

$$\Gamma^R(t) \cdot \Psi^\prime_{\rho}(|D|^2) D = \mathcal{O}(t^{-2}). \quad (3.22)$$
However, instead of (3.20) and (3.22), we can have the shaper estimate

\[ \Psi_\rho' (|D|^2) D \cdot \Gamma^L(t) - \Gamma^R(t) \cdot \Psi_\rho' (|D|^2) D = \mathcal{O} (t^{-3}) \]  

(3.23)

for all \(0 < \rho < 1\) as follows. From the definitions \(\Gamma^L\) and \(\Gamma^R\), we denote

\[ \Psi_\rho' (|D|^2) D \cdot \Gamma^L(t) - \Gamma^R(t) \cdot \Psi_\rho' (|D|^2) D = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\bar{z}} \bar{\Psi}_\rho)(z) (z - |D|^2)^{-1} \sum_{j=1}^{n} Z_{j, z}(t) (z - |D|^2)^{-1} \, dz \wedge d\bar{z}. \]  

(3.24)

We here put

\[ Z_{j, z}(t) = \Psi_\rho' (|D|^2) D_j \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] (z - |D|^2)^{-1} \]

(3.25)

\[ - (z - |D|^2)^{-1} \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \Psi_\rho' (|D|^2) D_j. \]

We further put \(Z_{1j, z}\) and \(Z_{2j, z}\) by \(Z_{j, z} = Z_{1j, z} + Z_{2j, z}\) such that

\[ Z_{1j, z}(t) = - (z - |D|^2)^{-1} \text{ad}_3 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] (z - |D|^2)^{-1} \Psi_\rho' (|D|^2) D_j, \]

(3.26)

\[ Z_{2j, z}(t) = \left[ \Psi_\rho' (|D|^2) D_j, \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right] (z - |D|^2)^{-1}. \]

(3.27)

We have, from the direct calculation of the commutator,

\[ \| (z - |D|^2)^{-1} Z_{1j, z}(t) (z - |D|^2)^{-1} \| \lesssim t^{-3} |\text{Im} z|^{-4} (z)^{\rho + 1} \]  

(3.28)

and

\[ \int_{\mathbb{C}} (\partial_{\bar{z}} \bar{\Psi}_\rho)(z) (z - |D|^2)^{-1} Z_{1j, z}(t) (z - |D|^2)^{-1} \, dz \wedge d\bar{z} = \mathcal{O} (t^{-3}). \]  

(3.29)

As for \(Z_{2j, z}\), we write

\[ Z_{2j, z}(t) = \{ \Lambda_{1j}(t) + \Lambda_{2j}(t) \} (z - |D|^2)^{-1}, \]  

(3.30)

using the terms

\[ \Lambda_{1j}(t) = \Psi_\rho' (|D|^2) \left[ D_j, \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right], \]

(3.31)

\[ \Lambda_{2j}(t) = \left[ \Psi_\rho' (|D|^2), \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right] D_j. \]

(3.32)

We compute directly

\[ \| (z - |D|^2)^{-1} \Lambda_{1j}(t) (z - |D|^2)^{-2} \| \lesssim t^{-3} |\text{Im} z|^{-3} (z)^{\rho}. \]  

(3.33)
\( \Lambda_{2j} \) is written such that
\[
\Lambda_{2j}(t) = \frac{1}{2\pi i} \int_{\mathbb{C}} (\bar{\partial}_z \tilde{\Psi}_\rho) (z) (z - |D|^2)^{-1} \text{ad}_3 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right]
\times (z - |D|^2)^{-1} D_j d\bar{z} \tag{3.34}
\]
by the Helffer–Sjöstrand formula. The commutator above becomes
\[
\text{ad}_3 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] = \sum_{k=1}^{n} \text{ad}_2 \left[ D_k, \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right] + 2 \sum_{k=1}^{n} \left[ D_k, \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right] D_k. \tag{3.35}
\]
Inserting the estimates
\[
\left\| (z - |D|^2)^{-1} \text{ad}_2 \left[ D_k, \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right] (z - |D|^2)^{-1} \right\| \lesssim t^{-4} |\text{Im} z|^{-2}(z), \tag{3.36}
\]
\[
\left\| (z - |D|^2)^{-1} \left[ D_k, \text{ad}_2 \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right] (z - |D|^2)^{-1} \right\| \lesssim t^{-3} |\text{Im} z|^{-3}(z) \tag{3.37}
\]
into (3.34), we have
\[
\left\| (z - |D|^2)^{-1} \Lambda_{2j}(t) (z - |D|^2)^{-1} \right\| \lesssim t^{-4} |\text{Im} z|^{-2}(z)^{1/2} + t^{-3} |\text{Im} z|^{-3}(z). \tag{3.38}
\]
From (3.33) and (3.38), we estimate
\[
\int_{\mathbb{C}} (\bar{\partial}_z \tilde{\Psi}_\rho) (z) (z - |D|^2)^{-1} Z_{2j}(t) (z - |D|^2)^{-1} d\bar{z} \wedge d\bar{z} = \mathcal{O} (t^{-3}). \tag{3.39}
\]
(3.29) and (3.39) imply (3.28). In summary, (3.14), (3.15), (3.16), (3.18), (3.19), and (3.23) yield (3.1).

**Proof of Theorem 3.1.** We take \( \chi_1 \in C^\infty(\mathbb{R}^n) \) such that \( \chi_1(s) = 1 \) if \( s < 2\theta \) and \( \chi_1(s) = 0 \) if \( s > 3\theta \), and we define the observable \( \mathcal{L}(t) \) by
\[
\mathcal{L}(t) = f(H_\rho) \chi_1 \left( \frac{|x|}{2t} \right) \mathcal{M}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho), \tag{3.40}
\]
according to [3] Proposition 4.4.3 and [12] Theorem 2.36. We know \( \mathcal{L}(t) = \mathcal{O}(1) \) because (3.31) holds for \( \chi_1 \). We now compute the Heisenberg derivative of \( \mathcal{L}(t) \) associated with \( H_\rho \),
\[
\mathcal{D}_{H_\rho} \mathcal{L}(t) = I_1(t) + I_2(t) + I_3(t). \tag{3.41}
\]
where

\[
I_1(t) = f(H_\rho) \left\{ \mathbb{D}_{\Psi_\rho(|D|^2)} \chi_1 \left( \frac{|x|}{2t} \right) \right\} \mathcal{M}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho) + h c, \quad (3.42)
\]

\[
I_2(t) = f(H_\rho) \chi_1 \left( \frac{|x|}{2t} \right) \left\{ \mathbb{D}_{\Psi_\rho(|D|^2)} \mathcal{M}(t) \right\} \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho), \quad (3.43)
\]

\[
I_3(t) = f(H_\rho) \chi_1 \left( \frac{|x|}{2t} \right) i[V, \mathcal{M}(t)] \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho). \quad (3.44)
\]

**Estimate for \( I_1 \).** The same computation with (3.9) and (3.12) give

\[
I_1(t) = \frac{1}{t} f(H_\rho) \left\{ \Psi'_\rho \left( |D|^2 \right) D - \frac{x}{2t} \right\} \cdot \frac{x}{|x|} \chi'_1 \left( \frac{|x|}{2t} \right) \mathcal{M}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho)
\]

\[
+ h c + \mathcal{O} \left( t^{-2} \right). \quad (3.45)
\]

Let \( \chi_2 \in C^\infty_0(\mathbb{R}) \) such that \( \chi_2(s) = 1 \) if \( 2\theta_2 < s < 3\theta_2 \) and \( \chi_2(s) = 0 \) if \( s < \theta_2 \) and \( s > 4\theta_2 \). We see that \( \chi_2 \) satisfies \( \chi'_1 = \chi_2^2 \chi_1 \). Let \( g \in C^\infty_0(\mathbb{R}) \) such that \( f = f g \). We compute

\[
f(H_\rho) \left\{ \Psi'_\rho \left( |D|^2 \right) D - \frac{x}{2t} \right\} \cdot \frac{x}{|x|} \chi'_1 \left( \frac{|x|}{2t} \right) \mathcal{M}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho)
\]

\[
= f(H_\rho) \chi_2 \left( \frac{|x|}{2t} \right) g(H_\rho) \left\{ \Psi'_\rho \left( |D|^2 \right) D - \frac{x}{2t} \right\} \cdot \frac{x}{|x|} \chi'_1 \left( \frac{|x|}{2t} \right)
\]

\[
\times \mathcal{M}(t) \chi_1 \left( \frac{|x|}{2t} \right) \chi_2 \left( \frac{|x|}{2t} \right) f(H_\rho) + \mathcal{O} \left( t^{-1} \right). \quad (3.46)
\]

We here used the commutator estimates (2.34) and (2.37). Because \( 4\rho - 2 < 2\rho \) and \( \Psi'_\rho \left( |D|^2 \right) D |D|^2 g(H_\rho) = \langle D \rangle ^{4\rho - 4} |D|^2 g(H_\rho) \) is bounded, we have

\[
I_1(t) = \frac{1}{t} f(H_\rho) \chi_2 \left( \frac{|x|}{2t} \right) \mathcal{O}(1) \chi_2 \left( \frac{|x|}{2t} \right) f(H_\rho) + \mathcal{O} \left( t^{-2} \right). \quad (3.47)
\]

If necessary, we can assume that \( \theta_2 \) is sufficiently large. By virtue of (3.47) and Theorem 2.3,

\[
\int_1^\infty \left\| \left( I_1(t) e^{-itH_\rho} \phi, e^{-itH_\rho} \phi \right) \right\|_{L^2} \, dt \lesssim \int_1^\infty \left\| \chi_2 \left( \frac{|x|}{2t} \right) f(H_\rho) e^{-itH_\rho} \phi \right\|_{L^2}^2 \, \frac{dt}{t} \lesssim \| \phi \|^2_{L^2}
\]

is obtained, where \( \langle \cdot, \cdot \rangle_{L^2} \) is the scalar product of \( L^2(\mathbb{R}^n) \).

**Estimate for \( I_2 \).** We take \( \chi \in C^\infty_0(\mathbb{R}) \) such that \( \chi(s) = 1 \) if \( \theta_1 \leq s \leq \theta_2 \) and \( \chi(s) = 0 \) if \( s < (\theta_1 + \theta)/2 \) and \( s > \theta_2 + (\theta_1 - \theta)/2 \). Noting that \( (\nabla^2 R)(x) = \text{Id}, \)
which is the identity matrix if $|x| \geq (\theta_1 + \theta)/2$, and that $\nabla^2 R$ is non-negative from (3.2), we have
\[
\left( \nabla^2 R \left( \frac{x}{2t} \right) \right) = \chi \left( \frac{|x|}{2t} \right) \nabla^2 R \left( \frac{x}{2t} \right) \chi \left( \frac{|x|}{2t} \right) \\
+ \sqrt{1 - \chi \left( \frac{|x|}{2t} \right)^2} \nabla^2 R \left( \frac{x}{2t} \right) \sqrt{1 - \chi \left( \frac{|x|}{2t} \right)^2} \geq \chi \left( \frac{|x|}{2t} \right)^2 \text{Id.} \tag{3.49}
\]
Using (2.34), (3.4), (3.49), and $\chi_1 \chi = \chi$, $I_2$ is estimated as
\[
I_2(t) \geq \frac{1}{t} f(H_{\rho}) \left\{ \Psi'_{\rho} (|D|^2) D - \frac{x}{2t} \right\} \\
\cdot \chi \left( \frac{|x|}{2t} \right)^2 \left\{ \Psi'_{\rho} (|D|^2) D - \frac{x}{2t} \right\} f(H_{\rho}) + O(t^{-2}). \tag{3.50}
\]
**Estimate for $I_3$.** It follows that $(\nabla V_{\text{long}})(x) \cdot (\nabla R)(x/(2t)) = O(t^{1-\gamma_{\text{long}}})$ by the condition (1.17) because $|x| \geq t\theta$ holds on the support of $(\partial_j R)(x/(2t))$ for all $1 \leq j \leq n$. We thus compute
\[
\left[ V_{\text{long}}, \Psi'_{\rho} (|D|^2) D \cdot (\nabla R) \left( \frac{x}{2t} \right) \right] \\
= \Psi'_{\rho} (|D|^2) \mathcal{O}(t^{-1-\gamma_{\text{long}}}) + \left[ V_{\text{long}}, \Psi'_{\rho} (|D|^2) \right] \left\{ (\nabla R) \left( \frac{x}{2t} \right) \cdot D - \frac{i}{2t} (\Delta R) \left( \frac{x}{2t} \right) \right\}. \tag{3.51}
\]
To apply the Helffer–Sjöstrand formula, we compute
\[
(z - |D|^2)^{-1} (\nabla R) \left( \frac{x}{2t} \right) \cdot D = (\nabla R) \left( \frac{x}{2t} \right) \cdot D \ (z - |D|^2)^{-1} \\
+ (z - |D|^2)^{-1} \left[ |D|^2, (\nabla R) \left( \frac{x}{2t} \right) \right] (z - |D|^2)^{-1}. \tag{3.52}
\]
Noting that $[|D|^2, V_{\text{long}}] = -i D \cdot \nabla V_{\text{long}} - i \nabla V_{\text{long}} \cdot D$, we have the estimate
\[
\left\| (z - |D|^2)^{-1} \left[ |D|^2, V_{\text{long}} \right] (\nabla R) \left( \frac{x}{2t} \right) \cdot D (z - |D|^2)^{-1} \right\| \\
\lesssim t^{-1-\gamma_{\text{long}}} |\text{Im} z|^{-2}(z) + t^{-2-\gamma_{\text{long}}} |\text{Im} z|^{-1}(z)^{1/2}, \tag{3.53}
\]
and, by
\[
\left[ |D|^2, (\nabla R) \left( \frac{x}{2t} \right) \cdot D \right] = -\frac{i}{t} (\nabla^2 R) \left( \frac{x}{2t} \right) D \cdot D - \frac{1}{4t^2} (\nabla \Delta R) \left( \frac{x}{2t} \right) \cdot D, \tag{3.54}
\]
we also have
\[
\left\| (z - |D|^2)^{-1} \left[ |D|^2, V_{\text{long}} \right] (z - |D|^2)^{-1} \left[ |D|^2, (\nabla R) \left( \frac{x}{2t} \right) \cdot D \right] (z - |D|^2)^{-1} \right\| \\
\lesssim t^{-2-\gamma_{\text{long}}} |\text{Im} z|^{-3}(z)^{3/2} + t^{-3-\gamma_{\text{long}}} |\text{Im} z|^{-3}(z) + t^{-2} |\text{Im} z|^{-4}(z)^2 + t^{-2} |\text{Im} z|^{-3}(z). \tag{3.55}
\]
We here computed the commutator \((z - |D|^2)^{-1}\) and \((\nabla^2 R)(x/(2t))D \cdot D\). \(3.53\) and \(3.55\) imply

\[
[V_{\text{long}}, \Psi'_\rho (|D|^2)] (\nabla R) \left( \frac{x}{2t} \right) \cdot D = O \left( t^{-1-\gamma_{\text{long}}} \right) + O \left( t^{-2} \right)
\]

by the Helffer–Sjöstrand formula. By the same computations, we have

\[
[V_{\text{long}}, \Psi'_\rho (|D|^2)] (\Delta R) \left( \frac{x}{2t} \right) = O \left( t^{-1-\gamma_{\text{long}}} \right) + O \left( t^{-2} \right).
\]

\(3.51\), \(3.56\), and \(3.57\) yield

\[
[V_{\text{long}}, \cdot \mathcal{M}(t)] = O \left( t^{-1-\gamma_{\text{long}}} \right) + O \left( t^{-2} \right).
\]

We put

\[
\mathcal{K}(t) = \frac{1}{2} \left\{ \Psi'_\rho (|D|^2) D - \frac{x}{2t} \right\} \cdot (\nabla R) \left( \frac{x}{2t} \right) + \text{hc.}
\]

Because we know that \(\langle x \rangle^{-\gamma_{\text{sing}}} V_{\text{sing}} \chi_1 (x/(2t)) f(H_\rho) = O(1)\) by Proposition \(1.3\) and \(2.37\) or \(3.12\), we have

\[
f(H_\rho) \chi_1 \left( \frac{|x|}{2t} \right) \left[ V_{\text{sing}}, \cdot \mathcal{M}(t) \right] \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho) = O(1) \langle x \rangle^{-\gamma_{\text{sing}}} \mathcal{K}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho) = O(1) \langle x \rangle^{-\gamma_{\text{sing}}} \mathcal{K}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho)\]

By computing the commutator \(\Psi'_\rho (|D|^2) D_j\) and \((\partial_j R)(x/(2t))\), we have

\[
\langle x \rangle^{-\gamma_{\text{sing}}} \Psi'_\rho (|D|^2) D_j \left( \partial_j R \right) \left( \frac{x}{2t} \right) = O(t^{-\gamma_{\text{sing}}} \Psi'_\rho (|D|^2) D_j)
\]

\[
+ O(t^{-1-\gamma_{\text{sing}}} \chi_1) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho)
\]

It follows from

\[
\left\| \langle x \rangle^{-\gamma_{\text{sing}}} \left( z - |D|^2 \right)^{-1} D_j \left[ |D|^2, (\partial_j R) \left( \frac{x}{2t} \right) \right] \right\| \langle z \rangle^{-|D|^2} \langle z \rangle^{-\gamma_{\text{sing}}} \langle z \rangle^{-\gamma_{\text{sing}}} \langle z \rangle^{-2} \langle z \rangle^{-2} \langle z \rangle^{-3} \langle z \rangle^{-2} \langle z \rangle^{-1/2}
\]

\[
\lesssim t^{-1-\gamma_{\text{sing}}} |\text{Im} z|^{-2} \langle z \rangle^{-2} |\text{Im} z|^{-2} \langle z \rangle^{-3} \langle z \rangle^{-2} \langle z \rangle^{-1/2}
\]

as in \(3.55\) that

\[
\langle x \rangle^{-\gamma_{\text{sing}}} D_j \left[ \Psi'_\rho (|D|^2), (\partial_j R) \left( \frac{x}{2t} \right) \right] = O \left( t^{-2} \right)
\]

by the Helffer–Sjöstrand formula again. \(3.61\) and \(3.63\) imply that

\[
\langle x \rangle^{-\gamma_{\text{sing}}} \mathcal{K}(t) \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho) = O \left( t^{-\gamma_{\text{sing}}} \right) + O \left( t^{-2} \right)
\]

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and that, from (3.60),

$$f(H_\rho) \chi_1 \left( \frac{|x|}{2t} \right) [V_{\text{sing}}, \mathcal{M}(t)] \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho) = O \left( t^{-\gamma_{\text{sing}}} \right) + O \left( t^{-2} \right). \quad (3.65)$$

We also have

$$f(H_\rho) \chi_1 \left( \frac{|x|}{2t} \right) [V_{\text{short}}, \mathcal{M}(t)] \chi_1 \left( \frac{|x|}{2t} \right) f(H_\rho) = O \left( t^{-\gamma_{\text{short}}} \right) + O \left( t^{-2} \right) \quad (3.66)$$

by replacing $\langle x \rangle^{-\gamma_{\text{sing}}}$ with $V_{\text{short}}$ in the computations above. By (3.58), (3.65), and (3.66), we have

$$I_3(t) = O \left( t^{-\min\{\gamma_{\text{sing}}, \gamma_{\text{short}}, 1+\gamma_{\text{long}}\}} \right). \quad (3.67)$$

We combine (3.50) and (3.67). There exists a constant $C > 0$ such that

$$\left\{ D_{H_\rho, \mathcal{L}(t)} e^{-itH_\rho \phi}, e^{-itH_\rho \phi} \right\}_{L^2} \geq \frac{1}{t} \left\| \chi \left( \frac{|x|}{2t} \right) \right\|^2 \left\{ \Psi'_{\rho} (|D|^2) D - \frac{x}{2|D|} \right\} f \left( H_\rho \right)e^{-itH_\rho \phi} \right\|_{L^2}^2 \quad (4.1)$$

motivated with which

$$i \left[ \Psi_{\rho} (|D|^2), \hat{A}_\rho \right] = 2\Psi'_{\rho} (|D|^2)^2 |D|^2 \quad (4.2)$$

holds. This completes our proof for the case $\rho < 1$ by virtue of (3.48) and $\min\{\gamma_{\text{sing}}, \gamma_{\text{short}}, 1+\gamma_{\text{long}}\} > 1$. In the case where $\rho = 1$, the proof is simpler (see [3, Proposition 4.4.3] or [12, Theorem 2.36]). Indeed, by replacing $\Psi'_{\rho}$ with 1, we omit many of the commutator calculations. In particular, (3.6) holds without the error term $O(t^{-2})$. We therefore explicitly have (3.4) replacing $O(t^{-2})$ with $-(\Delta^2 R)(x/(2t))/|16t^3|$. \qed

4 Minimal velocity bound

This section completes the proof of Theorem 1.4. Before giving the proof, we initially prepare the Mourre estimate of our version in Theorem 4.3 and prove the isolatedness and finite multiplicity of $\sigma_{\text{pp}}(H_\rho) \backslash \{0\}$ in Corollary 4.4. When we consider the Mourre estimate, how to choose a conjugate operator is the heart of matter. In our case, we first employ

$$\hat{A}_\rho = \frac{L}{2} \left\{ (D)^{2\rho-2} D \cdot x + x \cdot D (D)^{2\rho-2} \right\}$$

$$= \frac{1}{2} \left\{ \Psi_{\rho} (|D|^2) D \cdot x + x \cdot D \Psi_{\rho} (|D|^2) \right\} \quad (4.1)$$

motivated with which

$$i \left[ \Psi_{\rho} (|D|^2), \hat{A}_\rho \right] = 2\Psi'_{\rho} (|D|^2)^2 |D|^2 \quad (4.2)$$

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holds by a straightforward computation on $C^\infty_0(\mathbb{R}^n)$ and (4.2) is non-negative. The choice of conjugate operator is not unique. Indeed, if $1/2 \leq \rho \leq 1$, we can admit

$$A = \hat{A}_\rho = \frac{1}{2} (D \cdot x + x \cdot D)$$ (4.3)

(see Remarks 4.5 and 4.6) that works well for the standard Schrödinger operator.

The resolvent of $|D|^2$ was first introduced as the conjugate operator of the Mourre estimate by [24] such that

$$\frac{1}{2} \left\{ \langle D \rangle^{-2} D \cdot x + x \cdot D \langle D \rangle^{-2} \right\}$$ (4.4)

to consider the time-dependent Schrödinger operator

$$H(t) = -\Delta + V(t)$$ (4.5)

where $V(t) = V(t, x)$ had time-periodicity in $t$. Thereafter, [1] also treated the Hamiltonian (4.5) and introduced the resolvent of $D_t = -i/dt$ into the conjugate operator to relax the smoothness condition on $V$. Both [1] and [24] applied the Howland–Yajima method for the Floquet Hamiltonian $D_t + H(t)$. They estimated the commutators with the Floquet Hamiltonian and the conjugate operator, and their estimates were independent of the fractional operator.

We now begin with the self-adjointness of $\hat{A}_\rho$.

**Proposition 4.1.** $\hat{A}_\rho$ is essentially self-adjoint with the core $C^\infty_0(\mathbb{R}^n)$.

**Proof.** We define the operator $N_\rho$ by

$$N_\rho = \Psi'_\rho (|D|^2)^2 |D|^2 + |x|^2 + 1.$$ (4.6)

If $\rho > 1/2$, $N_\rho$ is self-adjoint on $H^{4\rho-2}(\mathbb{R}^n) \cap \text{Domain } |x|^2$. Whereas if $\rho \leq 1/2$, $\Psi'_\rho (|D|^2)^2 |D|^2$ is bounded and $N_\rho$ is self-adjoint on $\text{Domain } |x|^2$. We compute on $C^\infty_0(\mathbb{R}^n)$,

$$i \left[ \hat{A}_\rho, \Psi'_\rho (|D|^2)^2 |D|^2 \right] = -2 \left\{ 2 \Psi''_\rho (|D|^2) |D|^2 + \Psi'_\rho (|D|^2) \right\} \Psi'_\rho (|D|^2)^2 |D|^2$$

$$\lesssim \Psi''_\rho (|D|^2)^2 |D|^2$$ (4.7)

because $\Psi''_\rho (|D|^2) |D|^2$ and $\Psi'_\rho (|D|^2)$ are bounded. In the rest of this proof, we put $D_{\rho j} = \Psi'_\rho (|D|^2) D_j$ for simplicity. We thus compute, for $1 \leq j, k \leq n$,

$$i \left[ D_{\rho j} x_j + x_j D_{\rho j}, x_k^2 \right] = 2 x_j i \left[ D_{\rho j}, x_k \right] x_k + hc + i \left[ \left[ D_{\rho j}, x_k \right], x_k \right], x_j \right].$$ (4.8)

Because $[D_{\rho j}, x_k]$ and $[[D_{\rho j}, x_k], x_j]$ are bounded, (4.8) implies

$$i [\hat{A}_\rho, |x|^2] \lesssim |x|^2 + 1,$$ (4.9)
where we used the estimate
\[(x_i[D_{ij}, x_k] x_k\phi, \phi) \leq i [D_{ij}, x_k] \|x_j\phi\| \|x_k\phi\| \lesssim \|x_j\phi\|^2 + \|x_k\phi\|^2 \quad (4.10)\]
for $\phi \in C_0^\infty(\mathbb{R}^n)$. It follows from (4.7) and (4.9) that
\[i [\hat{A}_\rho, N_\rho] \lesssim N_\rho. \quad (4.11)\]
We next compute, noting that $[D_{ij}, x_j]$ does not depend on $x_j$,
\[(D_{ij} x_j + x_j D_{ij})^2 = 2D_{ij}^2 x_j^2 + 2x_j^2 D_{ij}^2 - 2D_{ij} [[D_{ij}, x_j], x_j] - 3 [D_{ij}, x_j]^2 \quad (4.12)\]
and we have
\[
2 \hat{A}_\rho^2 = \sum_{j=1}^n (D_{ij}^2 x_j^2 + x_j^2 D_{ij}^2) + \frac{1}{2} \sum_{j=1, k \neq j}^n (D_{ij} x_j + x_j D_{ij}) (D_{ik} x_k + x_k D_{ik})
\[
- \frac{1}{2} \sum_{j=1}^n \left\{ 2D_{ij} [[D_{ij}, x_j], x_j] + 3 [D_{ij}, x_k]^2 \right\}. \quad (4.13)\]
We here note that $D_{ij}[[D_{ij}, x_j], x_j]$ is bounded. We also compute
\[
N_\rho^2 \geq \sum_{j=1}^n (D_{ij}^2 x_j^2 + x_j^2 D_{ij}^2) + \sum_{j=1, k \neq j}^n (D_{ij}^2 + x_j^2) (D_{ik}^2 + x_k^2) + 1. \quad (4.14)\]
We have
\[
2 (D_{ij}^2 + x_j^2) (D_{ik}^2 + x_k^2) - (D_{ij} x_j + x_j D_{ij}) (D_{ik} x_k + x_k D_{ik})
= (D_{ij} D_{ik} - x_j x_k)^2 + (D_{ij} x_k - x_j D_{ik})^2
+ D_{ij}^2 D_{ik}^2 + x_j^2 x_k^2 + D_{ij}^2 x_k^2 + x_j^2 D_{ik}^2 + R_{jk} \geq R_{jk}, \quad (4.15)\]
where
\[
\sum_{j=1, k \neq j}^n R_{jk} = 4i \sum_{j=1, k \neq j}^n \left\{ x_j \Psi''_\rho (|D|^2) D_j D_k D_{ij} - D_{ij} \Psi''_\rho (|D|^2) D_j D_k x_k \right\}
= 4i \sum_{j=1, k \neq j}^n \left[ x_j, \Psi' (|D|^2) \Psi''_\rho (|D|^2) D_j D_k^2 \right] \quad (4.16)\]
is bounded. From (4.13), (4.14), (4.15), and (4.16), it follows that
\[\hat{A}_\rho^2 \lesssim N_\rho^2. \quad (4.17)\]
By (4.11) and (4.17), the Nelson commutator theorem [23, Theorem X.37] completes our proof. \hfill \Box

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Lemma 4.2. For $z \in \mathbb{C} \setminus \mathbb{R}$, the relation
\begin{equation}
\text{Domain}\langle x \rangle \subset \left\{ \phi \in \text{Domain} \hat{A}_\rho \mid (z - H_\rho)^{-1} \phi \in \text{Domain} \hat{A}_\rho \right\} \tag{4.18}
\end{equation}
holds.

Proof. We prove the domain property
\begin{equation}
(z - H_\rho)^{-1} \text{Domain}\langle x \rangle \subset \text{Domain} \hat{A}_\rho \tag{4.19}
\end{equation}
that is equivalent to (4.18). We first prove that
\begin{equation}
\langle x \rangle (z - H_\rho)^{-1} \langle x \rangle^{-1} \tag{4.20}
\end{equation}
is bounded. By the resolvent formula, we write
\begin{equation}
(z - H_\rho)^{-1} = \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} V (z - H_\rho)^{-1} + \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} \tag{4.21}
\end{equation}
It follows from
\begin{equation}
\left[ x_j, \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} \right] = 2i \Psi_\rho' (|D|^2) D_j \left\{ z - \Psi_\rho (|D|^2) \right\}^{-2} \tag{4.22}
\end{equation}
on Domain\langle x \rangle for $1 \leq j \leq n$ that
\begin{equation}
\langle x \rangle^\nu \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} \langle x \rangle^{-\nu} \tag{4.23}
\end{equation}
is bounded for $\nu \in \mathbb{R}$ by (4.22) and the complex interpolation derived from the Hadamard three-line theorem ([23], Appendix to IX.4). Because $\langle x \rangle (V_{\text{sing}} + V_{\text{short}})(z - H_\rho)^{-1}$ and (4.23) of $\nu = 1$ are bounded, to prove the boundedness of (4.20), it suffices to prove that $\langle x \rangle V_{\text{long}}(z - H_\rho)^{-1} \langle x \rangle^{-1}$ is bounded. Using the resolvent formula, we have
\begin{equation}
V_{\text{long}} (z - H_\rho)^{-1} = V_{\text{long}} \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} V_{\text{long}} (z - H_\rho)^{-1} + V_{\text{long}} \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1}. \tag{4.24}
\end{equation}
If $\gamma_{\text{long}} \geq 1/2$, writing $\langle x \rangle^{2\gamma_{\text{long}}} V_{\text{long}} \{ z - \Psi_\rho (|D|^2) \}^{-1} V_{\text{long}}$ such that
\begin{equation}
\langle x \rangle^{2\gamma_{\text{long}}} V_{\text{long}} \{ z - \Psi_\rho (|D|^2) \}^{-1} V_{\text{long}} = \langle x \rangle^{\gamma_{\text{long}}} V_{\text{long}} \langle x \rangle^{\gamma_{\text{long}}} \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} \langle x \rangle^{-\gamma_{\text{long}}} \langle x \rangle^{\gamma_{\text{long}}} V_{\text{long}}, \tag{4.25}
\end{equation}
we find that (4.25) is bounded by (4.23) of $\nu = \gamma_{\text{long}}$ and that $\langle x \rangle V_{\text{long}}(z - H_\rho)^{-1} \langle x \rangle^{-1}$ is bounded by (4.24). For the general $\gamma_{\text{long}} > 0$, we can take $N \in \mathbb{N}$ that satisfies $\gamma_{\text{long}} \geq 1/N > 0$ and iterate the above procedure $N - 1$ times. If $\rho \leq
1/2, the boundedness of (4.20) implies (4.19) immediately because \( \Psi'_{\rho}(|D|^2) \langle D \rangle \) is bounded and \( \hat{A}_\rho \) is closed. If \( \rho > 1/2 \), we can also prove that \( \langle x \rangle \langle D \rangle^{2\rho-1} (z - H_\rho)^{-1} \langle x \rangle^{-1} \) is bounded in the same way, noting that

\[
\langle x \rangle \langle D \rangle^{2\rho-1} \left\{ z - \Psi_{\rho} \left( |D|^2 \right) \right\}^{-1} \langle x \rangle^{-1}
\]

(4.26)
is bounded. We thus have (4.19) even for \( 1/2 \leq \rho \leq 1 \). In more detail, because \( \langle x \rangle (z - H_\rho)^{-1} \phi \in H^{2\rho-1}(\mathbb{R}^n) \) for \( \phi \in \text{Domain}(x) \), there exists a sequence \( \psi_k \in C_0^\infty(\mathbb{R}^n) \) such that \( \langle x \rangle \psi_k \to \langle x \rangle (z - H_\rho)^{-1} \phi \) as \( k \to \infty \) in \( H^{2\rho-1}(\mathbb{R}^n) \). We have \( A_\rho \psi_k \to \hat{A}_\rho (z - H_\rho)^{-1} \phi \) as \( k \to \infty \) and \( (z - H_\rho)^{-1} \phi \in \text{Domain} \hat{A}_\rho \) noting that \( \hat{A}_\rho \) is closed.

By Proposition 1.5, \( V \) is relatively compact associated with \( \Psi_{\rho}(|D|^2) \). This can be proved in the same way as in the standard Schrödinger case. Because the essential spectrum of \( \Psi_{\rho}(|D|^2) \) is \([0, \infty)\), the essential spectrum of \( H_\rho \) is also coincident with \([0, \infty)\) by virtue of the relative compactness of \( V \) and the Weyl theorem (23, Theorem XIII.14).

We now prove the Mourre estimate. However, it seems difficult that the commutator (4.2) extends on \( H^{2\rho}(\mathbb{R}^n) \cap \text{Domain} \hat{A}_\rho \) in the form sense. To overcome this difficulty, we give a modification in \( \hat{A}_\rho \) according to the original idea [15]. Let \( G_\rho \in C_0^\infty(\mathbb{R}) \) such that \( G_\rho(s) = \Psi'_{\rho}(s) \) on a some compact set of \( \mathbb{R} \). We define

\[
A_\rho = \frac{1}{2} \left\{ G_\rho \left( |D|^2 \right) D \cdot x + x \cdot G_\rho \left( |D|^2 \right) D \right\}.
\]

(4.27)

By the same way with Proposition 4.1, \( A_\rho \) is essentially self-adjoint with the core \( C_0^\infty(\mathbb{R}^n) \) and Lemma 4.2 also holds even for \( A_\rho \). In particular, it follows from the proof of Lemma 4.2 that

\[
2A_\rho \phi = G_\rho \left( |D|^2 \right) D \cdot (x \phi) + x \cdot \left\{ G_\rho \left( |D|^2 \right) D \phi \right\}
\]

(4.28)

for \( \phi \in \text{Domain}(x) \). This will be often used in the rest of our discussion.

**Theorem 4.3. Mourre estimate.** Let \( 0 < \lambda_1 < \lambda_2 \) and \( g \in C_0^\infty((\lambda_1, \lambda_2)) \). Assume that \( G_\rho(s) = \Psi'_{\rho}(s) \) if \( \Psi_{\rho}(s) \in \text{supp} \, g \). There exists a compact operator \( K \) such that

\[
g(H_\rho)i[H_\rho, A_\rho]_{-2\rho} g(H_\rho) \geq \frac{2\rho^2 \lambda_1}{(1 + \lambda_2 ^{1-\rho})^\rho} g(H_\rho)^2 + K
\]

(4.29)

holds, where the sense of the extended commutator \([H_\rho, A_\rho]_{-2\rho}\) is explained in the proof.
Proof. We first suppose that $\rho < 1$. By [15] Proposition II.1, the form commutator $i[\Psi_\rho(|D|^2), A_\rho]$ on $H^{2\rho}(\mathbb{R}^n) \cap \text{Domain } |x|^2$ is extended on $H^{2\rho}(\mathbb{R}^n) \cap \text{Domain } A_\rho$ and there exists the self-adjoint operator $i[\Psi_\rho(|D|^2), A_\rho]^0$ associated with the closed extension of $i[\Psi_\rho(|D|^2), A_\rho]$ (see also the proof of [15 Corollary I.3]). We therefore have

$$i[\Psi_\rho(|D|^2), A_\rho]^0 = 2G_\rho(|D|^2)|D|^2$$

that is a bounded operator. Using the fact that $\langle \rho \rangle$ and there exists the self-adjoint operator $i[\Psi_\rho(|D|^2), A_\rho]^0$, we estimate

$$\left| (A_\rho \phi, V_{\text{sing}} \psi)_{L^2} \right| = \left| \langle (x)^{-1} A_\rho \phi, \langle x \rangle V_{\text{sing}} \psi \rangle \right|_{L^2} \lesssim \| \phi \|_{L^2} \left\{ \epsilon \left\| (D)_{\rho} \psi \right\|_{L^2} + C_{\epsilon} \| \psi \|_{L^2} \right\} \lesssim \| \phi \|_{L^2} \left\| (D)_{\rho} \psi \right\|_{L^2}$$

and

$$\left| (A_\rho \phi, V_{\text{sing}} \psi)_{L^2} - (V_{\text{sing}} \phi, A_{\rho} \psi)_{L^2} \right| \lesssim \left\| (D)_{\rho} \phi \right\|_{L^2} \left\| (D)_{\rho} \psi \right\|_{L^2}$$

for $\phi, \psi \in H^{2\rho}(\mathbb{R}^n) \cap \text{Domain } A_\rho$. By the Riesz representation theorem ([23 Theorem II.4]) and Lemma [12] there exists a bounded operator $L_{V_{\text{sing}} A_\rho} : H^{2\rho}(\mathbb{R}^n) \rightarrow \mathcal{H}_{-2\rho} \simeq H^{2\rho}(\mathbb{R}^n)^*$ such that

$$\langle A_\rho \phi, V_{\text{sing}} \psi \rangle_{L^2} = \langle V_{\text{sing}} \phi, A_{\rho} \psi \rangle_{L^2} = \left\langle (D)^{-2\rho} L_{V_{\text{sing}} A_\rho} \phi, (D)^{2\rho} \psi \right\rangle_{L^2}.$$  

(4.33)

We note that $\mathcal{H}_{-2\rho}$ is the completion of

$$\left\{ \phi \in L^2(\mathbb{R}^n) \left| \int_{\mathbb{R}^n} \langle \xi \rangle^{-4\rho} |\mathcal{F} \phi(\xi)|^2 d\xi < \infty \right\}$$

(4.34)

that is regarded as the dual space of $H^{2\rho}(\mathbb{R}^n)$, and that the relation $H^{2\rho}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{H}_{-2\rho}$ holds. We denote $L_{V_{\text{sing}} A_\rho} = [V_{\text{sing}}, A_\rho]_{-2\rho}$ and (see also [12 Lemma 6.2] or the paragraphs below of [2 Theorem 6.2.10]). Similarly, we define $[V_{\text{short}}, A_\rho]_{-2\rho}$ by the estimate

$$\left| (A_\rho \phi, V_{\text{short}} \psi)_{L^2} \right| = \left| \langle (x)^{-1} A_\rho \phi, \langle x \rangle V_{\text{short}} \psi \rangle \right|_{L^2} \lesssim \| \phi \|_{L^2} \| \psi \|_{L^2}.$$  

(4.35)

In contrast with $V_{\text{sing}}$ and $V_{\text{short}}$, $V_{\text{long}}$ is differentiable. When $\rho < 1$, the commutator $[V_{\text{long}}, A_\rho]_{-2\rho}$ on Domain$(x)$ is extended to a compact operator on $L^2(\mathbb{R}^n)$ by the computations below (see (1.14), (4.35), and (4.36)). Therefore, by the extensions of the commutators,

$$[H_\rho, A_\rho]_{-2\rho} = [\Psi_\rho (|D|^2), A_\rho]_{-2\rho} + [V_{\text{sing}} + V_{\text{short}}, A_\rho]_{-2\rho} + [V_{\text{long}}, A_\rho]_{-2\rho}$$

$$= [\Psi_\rho (|D|^2), A_\rho]_{-2\rho} + [V_{\text{sing}} + V_{\text{short}}, A_\rho]_{-2\rho} + [V_{\text{long}}, A_\rho]$$

(4.36)
holds on $H^{2\rho}(\mathbb{R}^n)$ because $H^{2\rho}(\mathbb{R}^n) \cap \text{Domain } A_\rho$ is core for $H_\rho$, and the left-hand side of (4.29) is defined as the bounded operator on $L^2(\mathbb{R}^n)$. We note that
\begin{equation}
\begin{aligned}
&\imath \left[ \Psi_\rho (|D|^2), A_\rho \right]^0 \\
&= 2G_\rho (|D|^2) \left\{ G_\rho (|D|^2) \langle D \rangle^2 - \rho \right\} + 2G_\rho (|D|^2) \left\{ \rho - G_\rho (|D|^2) \right\}
\end{aligned}
\end{equation}
(4.37)
and that
\begin{equation}
\begin{aligned}
g \left( \Psi_\rho (|D|^2) \right) G_\rho (|D|^2) \left\{ \rho - G_\rho (|D|^2) \right\} g \left( \Psi_\rho (|D|^2) \right) \\
= \rho g \left( \Psi_\rho (|D|^2) \right) \Psi'_\rho (|D|^2) \left\{ 1 - \langle D \rangle^{2\rho - 2} \right\} g \left( \Psi_\rho (|D|^2) \right) \geq 0.
\end{aligned}
\end{equation}
(4.38)
We therefore have the inequality
\begin{equation}
\begin{aligned}
g \left( \Psi_\rho (|D|^2) \right) \imath \left[ \Psi_\rho (|D|^2), A_\rho \right]^0 g \left( \Psi_\rho (|D|^2) \right) \\
\geq 2\rho g \left( \Psi_\rho (|D|^2) \right) \Psi'_\rho (|D|^2) \Psi_\rho (|D|^2) g \left( \Psi_\rho (|D|^2) \right)
\end{aligned}
\end{equation}
(4.39)
holds. Because
\begin{equation}
g(H_\rho) - g \left( \Psi_\rho (|D|^2) \right) = \frac{1}{2\pi i} \int_C \partial_z \bar{g}(z) (z - H_\rho)^{-1} V \left\{ z - \Psi_\rho (|D|^2) \right\}^{-1} dz \wedge d\bar{z}
\end{equation}
(4.40)
is compact, there exists a compact operators $\hat{K}$ such that
\begin{equation}
g(H_\rho) \imath \left[ \Psi_\rho (|D|^2), A_\rho \right]^0 g(H_\rho) \\
\geq 2\rho g \left( \Psi_\rho (|D|^2) \right) \Psi'_\rho (|D|^2) \Psi_\rho (|D|^2) g \left( \Psi_\rho (|D|^2) \right) + \hat{K}
\end{equation}
(4.41)
on the right-hand side of (4.41), we used the relation $\Psi'_\rho = \rho/(1 + \Psi_\rho)^{(1-\rho)/\rho}$ and the inequality
\begin{equation}
\int_{\lambda_1}^{\lambda_2} \frac{\lambda}{(1 + \lambda)^{(1-\rho)/\rho}} g(\lambda) E_{\Psi_\rho(|D|^2)}(d\lambda) \geq \frac{\lambda_1}{(1 + \lambda_2)^{(1-\rho)/\rho}} g \left( \Psi_\rho (|D|^2) \right)^2,
\end{equation}
(4.42)
where $E_{\Psi_\rho(|D|^2)}$ is the spectral measure of $\Psi_\rho(|D|^2)$. Writing such that
\begin{equation}
g(H_\rho) [V_{\text{sing}} + V_{\text{short}}, A_\rho]_{-2\rho} g(H_\rho) = g(H_\rho) (V_{\text{sing}} + V_{\text{short}}) \langle x \rangle^{-1} A_\rho g(H_\rho) - \text{he},
\end{equation}
(4.43)
we find that (4.43) is compact because $\langle x \rangle (V_{\text{sing}} + V_{\text{short}}) \Psi_\rho(|D|^2))^{-1}$ is compact by Proposition 1.5 and (1.6). We also write
\begin{equation}
\begin{aligned}
[V_{\text{long}}, G_\rho (|D|^2) \cdot x] &= [V_{\text{long}}, G_\rho (|D|^2)] \cdot x + i G_\rho (|D|^2) \nabla V_{\text{long}} \cdot x
\end{aligned}
\end{equation}
(4.44)
on \( C_0^\infty(\mathbb{R}^n) \) that is a core of \( \text{Domain}(\langle x \rangle) \). We know that \( G_\rho([|D|^2] \nabla V_{\text{long}} \cdot x \) is compact by \( \text{(1.7)} \). The compactness of the commutator \([V_{\text{long}}, A_\rho]\) is obtained as follows. Noting that \((z - |D|^2)^{-1} [|D|^2, V_{\text{long}}](z - |D|^2)^{-1}\) is a compact operator, we compute

\[
(z - |D|^2)^{-1} i [|[D|^2, V_{\text{long}}]| z - |D|^2)^{-1} D \cdot x
= (z - |D|^2)^{-1} (D \cdot \nabla V_{\text{long}} + \nabla V_{\text{long}} \cdot D) D \cdot x (z - |D|^2)^{-1}
+ 2 (z - |D|^2)^{-1} [|[D|^2, V_{\text{long}}]| z - |D|^2)^{-2} |D|^2
\]

and estimate such that

\[
\left\| (z - |D|^2)^{-1} [|[D|^2, V_{\text{long}}]| z - |D|^2)^{-1} D \cdot x \right\| \lesssim |\text{Im} z|^{-2} \langle z \rangle + |\text{Im} z|^{-3} \langle z \rangle^{3/2}.
\]

By the Helffer–Sjöstrand formula, we find that \([V_{\text{long}}, A_\rho]\) is compact because \(x \cdot \nabla V_{\text{long}} \langle D \rangle^{-1}\) is also compact. From \( \text{(4.36)} \) and \( \text{(4.41)} \), we have \( \text{(4.29)} \) with a compact operator

\[
K = \hat{K} + \frac{2\lambda_1}{(1 + \lambda_2)^{(1-\rho)/\rho}} \left\{ g (\Psi_\rho ([|D|^2])^2 - g(H_\rho)^2 \right\} + g(H_\rho) i [V, A_\rho]_{-2\rho} g(H_\rho).
\]

The case of \( \rho = 1 \) is the traditional result given by \( \text{(1.15)} \). Because \( i[|D|^2, A] = 2|D|^2 = 2H_1 - 2V \) is obtained directly, we do not have to compute \( \text{(4.40)}, \text{(4.41)}, \) and \( \text{(4.42)} \). We only note that, although \([V_{\text{long}}, A] = ix \cdot \nabla V_{\text{long}}\) is not compact but just bounded, \(x \cdot \nabla V_{\text{long}} g(H_1)\) is compact.

The Mourre inequality \( \text{(4.29)} \) provides us detailed information on the eigenvalues of \( H_\rho \) as in Corollary \( \text{4.3} \) below. To investigate the singular continuous spectrum of \( H_\rho \), we have to prove the limiting absorption principle in Mourre theory. Many studies have investigated this topic, even for the \( N \)-body Schrödinger operator case (e.g., \( \text{[22, 2]} \) and \( \text{[21]} \)).

**Corollary 4.4.** Any point in \( \varphi_{pp}(H_\rho) \setminus \{0\} \) is isolated and its multiplicity is at most finite, and the only accumulation point of \( \varphi_{pp}(H_\rho) \) can be at zero.

**Proof.** We already know that

\[
|\langle A_\rho \phi, H_\rho \phi \rangle_{L^2} - \langle H_\rho \phi, A_\rho \phi \rangle_{L^2}| \lesssim \| (D)^{2\rho} \phi \|^2_{L^2} \lesssim \| (H_\rho) \phi \|^2_{L^2}
\]

holds for \( \phi \in H^{2\rho}(\mathbb{R}^n) \cap \text{Domain} A_\rho \) by the proof of Theorem \( \text{4.3} \). Lemma \( \text{4.2} \) and \( \text{1.48} \) imply that \( H_\rho \) belongs to the class \( C^1(A_\rho) \). The Mourre inequality \( \text{(1.29)} \) and virial theorem compete our proof (see \( \text{[2]} \) Theorem 6.2.10, Proposition 7.2.10, and Corollary 7.2.11)).
Remark 4.5. If $1/2 \leq \rho \leq 1$, we can choose (4.3) as the conjugate operator by virtue of [10] Theorems 3.4 and 3.10. In more details, the commutator $\Psi (|D|^2)$ and $A$ on $C_0^\infty (\mathbb{R}^n)$ can be extended to a self-adjoint operator

$$i \left[ \Psi (|D|^2), A \right]^0 = 2 \Psi' \left( |D|^2 \right) |D|^2.$$  

(4.49)

Noting that Lemma 4.2 holds for replacing $A_\rho$ with $A$ and

$$\left| (A \phi, V_{\text{sing}} \psi)_{L^2} \right| \lesssim \| \langle D \phi \rangle_{L^2} \| \| D \cdot \nabla \psi \|_{L^2}$$  

(4.50)

also holds for $\phi, \psi \in H^{2\rho}(\mathbb{R}^n) \cap \text{Domain} A$, the commutator $[H_\rho, A]_{-2\rho}$ is defined as a bounded operator from $H^{2\rho}(\mathbb{R}^n)$ to $\mathcal{H}_{2\rho}$. The shape of the Mourre estimate in this case is

$$g(H_\rho) i [H_\rho, A]_{-2\rho} g(H_\rho) \geq 2 \rho \lambda_1 g(H_\rho)^2 + g(H_\rho) \left\{ i [V_{\text{sing}} + V_{\text{short}}, A]_{-2\rho} + x \cdot \nabla V_{\text{long}} - 2 \rho V \right\} g(H_\rho)$$  

(4.51)

and the second term of the right-hand side is compact. If $0 < \rho < 1/2$, the commutator $H_\rho$ and $A_\rho$ can be extended to the map $H^1(\mathbb{R}^n)$ to $\mathcal{H}_{-1}$ by (4.50). However, unfortunately, the left-hand side of the Mourre inequality can not be defined because $H^1(\mathbb{R}^n) \subsetneq H^{2\rho}(\mathbb{R}^n)$. Meanwhile, if $V$ has the long-range part only (i.e., $V = V_{\text{long}}$), we can employ $A$ for all $0 < \rho \leq 1$ with the Mourre estimate

$$g(H_\rho) i [H_\rho, A]_{-2\rho} g(H_\rho) \geq 2 \rho \lambda_1 g(H_\rho)^2 + g(H_\rho) (ix \cdot \nabla V_{\text{long}} - 2 \rho V) g(H_\rho)$$  

(4.52)

by [10] Theorem 3.10.

We have everything arranged to prove the minimal velocity bound.

Proof of Theorem 1.4. As in the proofs before, we first assume that $\rho < 1$. Let $g \in C_0^\infty((0, \infty))$ satisfy $fg = f$. Let $\chi$ and $\chi_1$ that belong to $C_0^\infty(\mathbb{R})$ satisfy that $\chi(s) = 1$ if $|s| < \theta_0$ and $\chi(s) = 0$ if $|s| > 2\theta_0$, and that $\chi_1(s) = 1$ if $|s| < 2\theta_0$ and $\chi_1(s) = 0$ if $|s| > 3\theta_0$. The size of $\theta_0$ is to be determined later. According to [3] Proposition 4.4.7, and [12] Theorem 2.38, we define the observables $\mathcal{M}(t)$ and $\mathcal{L}(t)$ by

$$\mathcal{M}(t) = \frac{1}{2} \left\{ \Psi'(\left| D \right|^2) D - \frac{x}{2t} \right\} : \frac{x}{|x|} \chi' \left( \frac{|x|}{2t} \right) + \text{hc} + \chi \left( \frac{|x|}{2t} \right),$$  

(4.53)

$$\mathcal{L}(t) = f(H_\rho) \mathcal{M}(t) g(H_\rho) A_\rho \frac{1}{t} g(H_\rho) \mathcal{M}(t) f(H_\rho).$$  

(4.54)

Because $g(H_\rho) \text{Domain}(x) \subset \text{Domain} A_\rho$ holds and $A_\rho g(H_\rho)(x)^{-1}$ is a bounded operator as we proved in Lemma 4.2, $\mathcal{L}(t)$ is well-defined. By the supporting properties $\chi = \chi_1 \chi$ and $\chi' = \chi_1 \chi'$, we compute

$$\mathcal{M}(t) = \chi_1 \left( \frac{|x|}{2t} \right) \mathcal{M}(t) + B(t)$$  

(4.55)

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with
\[
B(t) = \frac{1}{2} \sum_{j=1}^{n} \left[ \Psi'_\rho \left( |D|^2 \right) D_j, \chi_1 \left( \frac{|x|}{2t} \right) \right] \frac{x_j}{|x|} x' \left( \frac{|x|}{2t} \right).
\] (4.56)

We already know \( B(t) = O(t^{-1}) \) from the computation (2.34). Moreover, by
\[
\left\| \frac{x_k}{t} (z - |D|^2)^{-1} D_j \left[ |D|^2, \chi_1 \left( \frac{|x|}{2t} \right) \right] (z - |D|^2)^{-1} \right\| \\
\lesssim t^{-1} |\text{Im}z|^{-2} \langle z \rangle + t^{-2} |\text{Im}z|^{-3} \langle z \rangle^{3/2}
\] (4.57)
and the Helffer–Sjöstrand formula, we find that
\[
\frac{x_k}{t} D_j \left[ \Psi'_\rho \left( |D|^2 \right) , \chi_1 \left( \frac{|x|}{2t} \right) \right] = O(t^{-1})
\] (4.58)
for \( 1 \leq j, k \leq n \). (4.58) and \( (x_k/t)[D_j, \chi_1(|x|/(2t))] \Psi'_\rho(|D|^2) = O(t^{-1}) \) yield
\[
\frac{x_k}{t} \left[ \Psi'_\rho \left( |D|^2 \right) D_j, \chi_1 \left( \frac{|x|}{2t} \right) \right] = O(t^{-1})
\] (4.59)
and
\[
\frac{x_k}{t} B(t) = O(t^{-1}).
\] (4.60)

We therefore have
\[
\frac{A_\rho}{t} g(H_\rho) \left( \frac{x}{2t} \right)^{-1} = O(1)
\] (4.61)
and \((A_\rho/t) g(H_\rho) \mathcal{M}(t) = O(1)\) from (4.55) and (4.60). This implies that \( \mathcal{L}(t) = O(1) \). We write
\[
\mathcal{D}_{H_\rho} \mathcal{L}(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t),
\] (4.62)
where
\[
I_1(t) = f(H_\rho) \left\{ \mathcal{D}_{\Psi'_\rho(|D|^2)} \mathcal{M}(t) \right\} g(H_\rho) \frac{A_\rho}{t} g(H_\rho) \mathcal{M}(t) f(H_\rho) + \text{hc},
\] (4.63)
\[
I_2(t) = f(H_\rho) i[V, \mathcal{M}(t)] g(H_\rho) \frac{A_\rho}{t} g(H_\rho) \mathcal{M}(t) f(H_\rho) + \text{hc},
\] (4.64)
\[
I_3(t) = -\frac{1}{t} f(H_\rho) \mathcal{M}(t) g(H_\rho) \frac{A_\rho}{t} g(H_\rho) \mathcal{M}(t) f(H_\rho),
\] (4.65)
\[
I_4(t) = -\frac{1}{t} f(H_\rho) \mathcal{M}(t) g(H_\rho) i[H_\rho, A_\rho]_{-2\rho} g(H_\rho) \mathcal{M}(t) f(H_\rho).
\] (4.66)

**Estimate for \( I_3 \) and \( I_4 \).** By the same computations as (2.37), (4.55), (4.60) and
\[
\mathcal{M}(t) = \mathcal{M}(t) \chi_1 \left( \frac{|x|}{2t} \right) + \frac{1}{2} \sum_{j=1}^{n} \chi' \left( \frac{|x|}{2t} \right) \frac{x_j}{|x|} \left[ \chi_1 \left( \frac{|x|}{2t} \right), \Psi'_\rho \left( |D|^2 \right) D_j \right],
\] (4.67)
By (2.8), we compute

\[ I_3(t) = -\frac{1}{t} f(H_\rho).\mathcal{M}(t)g(H_\rho)x_1 \left( \frac{|x|}{2t} \right) A_{\rho} t x_1 \left( \frac{|x|}{2t} \right) g(H_\rho).\mathcal{M}(t)f(H_\rho) + O(t^{-2}) \]  

(4.68)

We have, using \([G_\rho(D)D_j, x_1(|x|/(2t))] = O(t^{-1})\) similar to (2.34),

\[ g(H_\rho)x_1 \left( \frac{|x|}{2t} \right) A_{\rho} t x_1 \left( \frac{|x|}{2t} \right) g(H_\rho) \]

\[ = g(H_\rho)g_\rho(|D|^2) D \cdot \frac{x}{|x|} x_1 \left( \frac{|x|}{2t} \right) g(H_\rho) + \text{hc} + O(t^{-1}) \]

\[ \leq 2 \left\| g(H_\rho)g_\rho(|D|^2) D \cdot \frac{x}{|x|} x_1 \left( \frac{|x|}{2t} \right) \right\| \left\| x_1 \left( \frac{|x|}{2t} \right) g(H_\rho) \right\| + O(t^{-1}) \]  

(4.69)

and we then estimate

\[ I_3(t) \geq -\frac{\theta}{t} f(H_\rho).\mathcal{M}(t)^2 f(H_\rho) + O(t^{-2}) \]  

(4.70)

where we put \(\theta = 6\theta_0 \|g(H_\rho)g_\rho(|D|^2) D \cdot x/|x|\| \|g(H_\rho)\|\). We next estimate \(I_4\). It follows from (2.37) that

\[ [g(H_\rho), \mathcal{M}(t)] = \frac{1}{2} \left[ g(H_\rho), \Psi_\rho'(|D|^2) D \cdot \frac{x}{|x|} x_1 \left( \frac{|x|}{2t} \right) + \text{hc} \right] + O(t^{-1}) \]  

(4.71)

By (2.8), we compute

\[ \left[ \Psi_\rho'(|D|^2), \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} x_1^' \left( \frac{|x|}{2t} \right) \right] \]

\[ = \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} \left[ |D|^2, \frac{x_j}{|x|} x_1^' \left( \frac{|x|}{2t} \right) \right] \Psi_\rho'(|D|^2) + \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} O(t^{-2}) \]  

(4.72)

and

\[ \left\| (z - H_\rho)^{-1} \left[ \Psi_\rho'(|D|^2), \Psi_\rho'(|D|^2) D \cdot \frac{x}{|x|} x_1^' \left( \frac{|x|}{2t} \right) \right] (z - H_\rho)^{-1} \right\| \]

\[ \lesssim t^{-1} |\text{Im}z|^{-2} \langle z \rangle^2 \]  

(4.73)

recalling (2.35) and (2.36). We write the commutator such that

\[ V_{\text{sing}}, \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} x_1^' \left( \frac{|x|}{2t} \right) = V_{\text{sing}} \langle x \rangle^\gamma_{\text{sing}} O(t^{-\gamma_{\text{sing}}}) \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} \]

\[ + V_{\text{sing}} \left[ \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} x_1^' \left( \frac{|x|}{2t} \right) \right] - \Psi_\rho'(|D|^2) D_{\frac{x_j}{|x|}} O(t^{-\gamma_{\text{sing}}}) \langle x \rangle^\gamma_{\text{sing}} V_{\text{sing}}. \]  

(4.74)
We here used $\langle x \rangle^{-\gamma_{\text{sing}}'}(|x|/(2t)) = O(t^{-\gamma_{\text{sing}}'})$. By the same computations as \(3.63\), we have

$$\langle x \rangle^{-\gamma_{\text{sing}}} \left[ \Psi'_\rho'(|D|^2) D_j, \frac{x_j}{|x|} \chi_\rho' \left( |x| \frac{1}{2t} \right) \right] = O(t^{-2}). \quad (4.75)$$

From \(4.74\) and \(4.75\), we estimate

$$\left\| (z - H_\rho)^{-1} \left[ V_{\text{sing}}, \Psi'_\rho'(|D|^2) \cdot \frac{x}{|x|} \chi_\rho' \left( |x| \frac{1}{2t} \right) \right] (z - H_\rho)^{-1} \right\| \lesssim t^{-\gamma_{\text{sing}}'} |\text{Im} z|^{-2}(\langle z \rangle^2 + t^{-2} |\text{Im} z|^{-2}(\langle z \rangle), \quad (4.76)$$

using \(2.36\) and \(\| (z - H_\rho)^{-1} V_{\text{sing}} \langle x \rangle^{-\gamma_{\text{sing}}} \| \lesssim |\text{Im} z|^{-1}(\langle z \rangle). \) Because $V_{\text{short}} \langle x \rangle^{-\gamma_{\text{short}}}$ is bounded by \(1.6\), we also estimate

$$\left\| (z - H_\rho)^{-1} \left[ V_{\text{short}}, \Psi'_\rho'(|D|^2) \cdot \frac{x}{|x|} \chi_\rho' \left( |x| \frac{1}{2t} \right) \right] (z - H_\rho)^{-1} \right\| \lesssim t^{-\gamma_{\text{short}}'} |\text{Im} z|^{-2}(\langle z \rangle) + t^{-2} |\text{Im} z|^{-2}. \quad (4.77)$$

Noting that an almost analytic extension of $g$ has compact support, from \(3.58\), \(4.71\), \(4.73\), \(4.76\), and \(4.77\), we have

$$[g(H_\rho), \mathcal{M}(t)] = O(t^{-\min\{\gamma_{\text{sing}}, \gamma_{\text{short}}', \gamma_{\text{long}}', 2\}}) + O(t^{-1}) = O(t^{-1}). \quad (4.78)$$

Incidentally, let $\lambda_1$ and $\lambda_2$ in Theorem \(4.3\) satisfy $(\lambda_1, \lambda_2) \cap \sigma_{pp}(H_\rho) = \emptyset$. For $\lambda_1 < \lambda < \lambda_2$, we take $0 < \delta < \min \{\lambda - \lambda_1, \lambda_2 - \lambda\}$. $\lambda \not\in \sigma_{pp}(H_\rho)$ is equivalent to the point spectral measure $E_{H_\rho}(\{\lambda\})$ being zero. This implies that $E_{H_\rho}(\lambda - \delta, \lambda + \delta)) \to 0$ as $\delta \to 0$ in the strong norm sense of $L^2(\mathbb{R}^n)$ and that, for the compact operator $K$ of \(4.29\), $E_{H_\rho}(\lambda - \delta, \lambda + \delta))K \to 0$ as $\delta \to 0$ in operator norm sense of $L^2(\mathbb{R}^n)$. Therefore, Theorem \(4.3\) yields

$$E_{H_\rho}(\lambda - \delta, \lambda + \delta)) i [H_\rho, A_\rho]_{-2\rho} E_{H_\rho}(\lambda - \delta, \lambda + \delta)) \geq \rho^2 \lambda_1 \left( \frac{1}{\lambda_2} \right)^{(1-\rho)/\rho} E_{H_\rho}(\lambda - \delta, \lambda + \delta)) \quad (4.79)$$

for a small $\delta > 0$. We assume that supp $g$ is sufficiently small without loss of generality because, if not, supp $g$ can be covered by $\bigcup_{k=1}^N \text{supp} g_k$ where supp $g_k$ is small (see the proof of \(3.1\) Proposition 4.4.7]). By virtue of \(4.78\) and \(4.79\), there exists $c = c_{ppg} > 0$ such that $I_4$ is estimated as

$$I_4(t) \geq \frac{c}{t} f(H_\rho) \mathcal{M}(t) g(H_\rho)^2 \mathcal{M}(t)f(H_\rho) = \frac{c}{t} f(H_\rho)^2 \mathcal{M}(t)^2 f(H_\rho) + O(t^{-2}). \quad (4.80)$$
We here choose \( \theta_0 > 0 \) which satisfies \( 0 < \theta < c \) noting the definition of \( \theta \), and put \( \mathcal{K}(t) \)

\[
\mathcal{K}(t) = \frac{1}{2} \{ \Psi'_\rho(|D|^2) D - \frac{x}{2t} \} \cdot \frac{x}{|x|} \chi' \left( \frac{|x|}{2t} \right) + \text{hc}
\]

(4.81)
as in (3.59). From (4.70) and (4.80), using the inequality

\[
\mathcal{M}(t)^2 \geq \chi \left( \frac{|x|}{2t} \right)^2 + \mathcal{K}(t)^2 - \left\{ 2\mathcal{K}(t)^2 + \frac{1}{2} \chi \left( \frac{|x|}{2t} \right)^2 \right\} = \frac{1}{2} \chi \left( \frac{|x|}{2t} \right)^2 - \mathcal{K}(t)^2,
\]

we have

\[
I_3(t) + I_4(t) \geq \frac{c - \theta}{2t} f(H_\rho) \chi \left( \frac{|x|}{2t} \right)^2 f(H_\rho) - \frac{c - \theta}{t} f(H_\rho) \mathcal{K}(t)^2 f(H_\rho) + \mathcal{O} \left( t^{-2} \right).
\]

(4.83)

We note that, by virtue of Theorem 3.1

\[
\int \frac{dt}{t} \left| \left( \mathcal{K}(t)^2 f(H_\rho) e^{-itH_\rho \phi}, f(H_\rho) e^{-itH_\rho \phi} \right)_{L^2} \right| \lesssim \| \phi \|^2_{L^2}
\]

(4.84)
holds because \( f(H_\rho) \mathcal{K}(t)^2 f(H_\rho) \) has the following shape

\[
f(H_\rho) \mathcal{K}(t)^2 f(H_\rho) = f(H_\rho) \left\{ \Psi'_\rho(|D|^2) D - \frac{x}{2t} \right\} \cdot \frac{x}{|x|} \chi' \left( \frac{|x|}{2t} \right) \\
\times \chi' \left( \frac{|x|}{2t} \right) \frac{x}{|x|} \cdot \left\{ \Psi'_\rho(|D|^2) D - \frac{x}{2t} \right\} f(H_\rho) + \mathcal{O} \left( t^{-1} \right),
\]

(4.85)
by (2.34).

**Estimate for \( I_2 \).** By (3.58), (4.76), and (4.77), replacing \((z - H_\rho)^{-1}\) by \( \langle H_\rho \rangle^{-1} \) in (4.76) and (4.77), we have

\[
\langle H_\rho \rangle^{-1} [V, \mathcal{M}(t)] \langle H_\rho \rangle^{-1} = \mathcal{O} \left( t^{-\min\{\gamma_\text{sing} + \gamma_{\text{short}, 1} + \gamma_{\text{long}, 2}\}} \right)
\]

(4.86)
and

\[
I_2(t) = \mathcal{O} \left( t^{-\min\{\gamma_\text{sing} + \gamma_{\text{short}, 1} + \gamma_{\text{long}, 2}\}} \right).
\]

(4.87)

**Estimate for \( I_1 \).** Put \( R(x) = \chi(|x|) \). Then, by the formula (3.4), \( I_1 \) is

\[
I_1(t) = \frac{1}{t} f(H_\rho) \left\{ \Psi'_\rho(|D|^2) D - \frac{x}{2t} \right\} \cdot \left( \nabla^2 R \right) \left( \frac{x}{2t} \right) \left\{ \Psi'_\rho(|D|^2) D - \frac{x}{2t} \right\} \\
\times g(H_\rho) \frac{A_\rho}{t} g(H_\rho) \mathcal{M}(t) f(H_\rho) + \text{hc} + \mathcal{O} \left( t^{-2} \right) = I_5(t) + I_6(t) + \mathcal{O}(t^{-2}),
\]

(4.88)
where we defined $I_5$ and $I_6$ by
\[
I_5(t) = \frac{1}{t} f(H_\rho) \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \} \cdot (\nabla^2 R) \left( \frac{x}{2t} \right) \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \}
\]
\[
\times g(H_\rho) \frac{A_\rho}{t} g(H_\rho) \chi \left( \frac{|x|}{2t} \right) \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \} \frac{x}{|x|} \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \}
\]
\[
I_6(t) = \frac{1}{t} f(H_\rho) \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \} \cdot (\nabla^2 R) \left( \frac{x}{2t} \right) \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \}
\]
\[
\times g(H_\rho) \frac{A_\rho}{t} g(H_\rho) \chi \left( \frac{|x|}{2t} \right) f(H_\rho) + \text{hc},
\]
\[
(4.89)
\]
\[
(4.90)
\]
using (2.31),
\[
\mathcal{M}(t) = \chi^\prime \left( \frac{|x|}{2t} \right) \frac{x}{|x|} \cdot \{ \Psi^\prime (|D|^2) D - \frac{x}{2t} \} + \chi \left( \frac{|x|}{2t} \right) + \mathcal{O}(t^{-1})
\]
\[
(4.91)
\]
and (4.61). Let $\chi_2 \in C_0^\infty((\theta_0/2, \infty))$ satisfy $\chi^\prime = \chi^\prime \chi_2$. We write $I_5$ such that
\[
I_5(t) = \frac{1}{t} f(H_\rho) \sum_{j=1}^{n} \{ \Psi^\prime (|D|^2) D_j - \frac{x_j}{2t} \} \chi_2 \left( \frac{|x|}{2t} \right)
\]
\[
\times \mathcal{O}(1) \sum_{k=1}^{n} \chi_2 \left( \frac{|x|}{2t} \right) \{ \Psi^\prime (|D|^2) D_k - \frac{x_k}{2t} \} f(H_\rho) + \mathcal{O}(t^{-2}),
\]
\[
(4.92)
\]
where we also used (4.61). We finally estimate $I_6$. By the same computations as (1.72), (1.73), (1.74), (1.75) and (1.76), we have
\[
\left[ \chi_2 \left( \frac{|x|}{2t} \right) \Psi^\prime (|D|^2) D_j, g(H_\rho) \right] = \mathcal{O} \left( t^{-\min\{\gamma_{\text{sing}}, \gamma_{\text{short}}, 1+\gamma_{\text{long}}, 2\} \} \right).
\]
\[
(4.93)
\]
We also have
\[
\left[ \chi_2 \left( \frac{|x|}{2t} \right) \frac{x_j}{t}, g(H_\rho) \right] = \mathcal{O}(t^{-1})
\]
\[
(4.94)
\]
by (2.31). (4.93) and (4.94) imply
\[
\left[ \chi_2 \left( \frac{|x|}{2t} \right) \{ \Psi^\prime (|D|^2) D_j - \frac{x_j}{2t} \}, g(H_\rho) \right] = \mathcal{O}(t^{-1}).
\]
\[
(4.95)
\]
We note that
\[
\left[ \chi_2 \left( \frac{|x|}{2t} \right) \frac{x_j}{t}, x_k \Psi^\prime (|D|^2) D_k \right] = \frac{x_k}{t} \left[ \chi_2 \left( \frac{|x|}{2t} \right) \frac{x_j}{t}, \Psi^\prime (|D|^2) D_k \right] = \mathcal{O} \left( t^{-1} \right)
\]
\[
(4.96)
\]
34
by (4.59) and that
\[ \chi_2 \left( \frac{|x|}{2t} \right) \Psi'_\rho (|D|^2) D_j, \frac{x_j}{t} \Psi'_\rho (|D|^2) D_k \]
\[ = \chi_2 \left( \frac{|x|}{2t} \right) O(t^{-1}) \Psi'_\rho (|D|^2) D_k + \frac{x_k}{t} O(t^{-1}) \Psi'_\rho (|D|^2) D_j \]  \hspace{1cm} (4.97)
for \( 1 \leq j, k \leq n \), where we used (4.59) again in the second term on the right-hand side of (4.97). From (4.96) and (4.97), we have
\[ \left[ \chi_2 \left( \frac{|x|}{2t} \right) \left\{ \Psi'_\rho (|D|^2) D_j - \frac{x_j}{2t} \right\}, \frac{A\rho}{t} \right] g(H_\rho) = O(t^{-1}) \]  \hspace{1cm} (4.98)
Clearly,
\[ \frac{A\rho}{t} g(H_\rho) \left[ \chi_2 \left( \frac{|x|}{2t} \right) \left\{ \Psi'_\rho (|D|^2) D_j - \frac{x_j}{2t} \right\}, \chi \left( \frac{|x|}{2t} \right) \right] = O(t^{-1}) \]  \hspace{1cm} (4.99)
holds by (2.34) and (4.61). Combining (4.95), (4.98) and (4.99), we have
\[ \langle \frac{x}{2t} \rangle^{-1} \left[ \chi_2 \left( \frac{|x|}{2t} \right) \left\{ \Psi'_\rho (|D|^2) D_j - \frac{x_j}{2t} \right\}, g(H_\rho) \frac{A\rho}{t} g(H_\rho) \chi \left( \frac{|x|}{2t} \right) \right] = O(t^{-1}) \]  \hspace{1cm} (4.100)
By (4.100), we find that \( I_6 \) has the estimate
\[ I_6(t) = \frac{1}{t} f(H_\rho) \sum_{j=1}^{n} \left\{ \Psi'_\rho (|D|^2) D_j - \frac{x_j}{2t} \right\} \chi_2 \left( \frac{|x|}{2t} \right) \]
\[ \times O(1) \sum_{k=1}^{n} \chi_2 \left( \frac{|x|}{2t} \right) \left\{ \Psi'_\rho (|D|^2) D_k - \frac{x_k}{2t} \right\} f(H_\rho) + O(t^{-2}) \]  \hspace{1cm} (4.101)
By virtue of Theorem 3.1, (4.88), (4.92) and (4.101),
\[ \int_{1}^{\infty} \left| (I_1(t)e^{-itH_\rho \phi}, e^{-itH_\rho \phi})_{L^2} \right| dt \lesssim \| \phi \|_{L^2}^2 \]  \hspace{1cm} (4.102)
is obtained.
There exists a constant \( C > 0 \) such that
\[ \left( \left\{ \mathcal{D}_t \mathcal{L}(t) \right\} e^{-itH_\rho \phi}, e^{-itH_\rho \phi} \right)_{L^2} \geq \frac{e - \theta}{2t} \left\| \chi \left( \frac{|x|}{2t} \right) f(H_\rho)e^{-itH_\rho \phi} \right\|_{L^2}^2 \]
\[ - \frac{e - \theta}{t} \left| \left( \mathcal{K}(t)^2 f(H_\rho)e^{-itH_\rho \phi}, f(H_\rho)e^{-itH_\rho \phi} \right)_{L^2} \right| \]
\[ - \left| (I_1(t)e^{-itH_\rho \phi}, e^{-itH_\rho \phi})_{L^2} \right| - C t^{- \min(\gamma_{\text{sing}},\gamma_{\text{short}})+1+\gamma_{\text{long}}+2} \| \phi \|_{L^2}^2 \]  \hspace{1cm} (4.103)
holds by (4.83) and (4.87). This completes our proof for $0 < \rho < 1$ by (4.84), (4.102), and $\min\{\gamma_{\text{sing}}, \gamma_{\text{short}}, 1 + \gamma_{\text{long}}, 2\} > 1$. In the case where $\rho = 1$, as in the proofs before, we simply replace $\Psi'_\rho$ by 1 and reduce many of the computations. For more details, see [3, Proposition 4.4.7] or [12, Theorem 2.38].

**Remark 4.6.** If $1/2 \leq \rho \leq 1$, we can prove Theorem 1.4 even by adopting the conjugate operator $A$ with some modifications to the proof presented above. In particular, $Ag(H_\rho)x^{-1}$ is a bounded operator because $(D)g(H_\rho)$ is a bounded operator.

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