On the \( n \)-ary algebras, semigroups and their universal covers.

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Abstract

For any \( n \)-ary associative algebra we construct a \( \mathbb{Z}_{n-1} \) graded algebra, which is a universal object containing the \( n \)-ary algebra as a subspace of elements of degree 1. Similar construction is carried out for semigroups.

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1 Introduction

Recently there has been some interest in the studies of generalized algebraic structures, in particular, linear spaces, which are equipped with \( n \)-ary products. Such objects are natural generalizations of algebras. The simplest nontrivial examples, ternary structures, have attracted the attention due to some physical motivation \([MV, VK, K]\), but has also been a intense subject of studies in mathematics (see for instance \([Gn, Br]\) and references therein). Let us remind here briefly the basic definition.

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1.1 Definition. An \( n \)-ary algebra \( A \) is a linear space with a linear map \( m : A^\otimes n \to A \). We shall say that \( A \) is an associative\(^1\) \( n \)-ary algebra if the composition of any \( 2n-1 \) elements is uniquely defined, i.e.:

\[
m(id \otimes \cdots \otimes id \otimes m(i) \otimes id \otimes \cdots \otimes id) = m(id \otimes \cdots \otimes id \otimes m(j) \otimes id \otimes \cdots \otimes id)
\]

for all \( 1 \leq i, j \leq n - 1 \), where \( m(i) \) denotes that \( m \) is on the \( i \)-th position in the tensor product.

Of course, having a \( n \)-ary structure one may easily construct a \( (2n-1) \)-ary one on the top of it and, especially, every ordinary algebra gives rise to \( n \)-ary structures (associative or not).

In this note we shall study the inverted problem: of finding an algebra, which has a subspace stable under the \( n \)-ary multiplication. We shall demonstrate that all \( n \)-ary associative algebraic structures are of this form and we shall prove the universality of the constructed object.

A natural examples of such objects come from \( \mathbb{Z}_{n-1} \)-graded algebras and the subspaces of elements of degree 1 (which are, of course, stable under \( n \)-multiplication). We shall show that the constructed algebra, which covers a given \( n \)-ary algebra, is \( n-1 \)-graded.

Our results open new possibilities for studies of \( n \)-ary objects as well as graded algebras and we shall indicate few of them.

2 Universal \( \mathbb{Z}_{n-1} \)-graded algebras for \( n \)-ary algebras.

Let \( A \) be an \( n \)-ary algebra. For a while we do not assume anything more about \( m \). Let us take \( T(A) \), the tensor algebra of \( A \) and the natural inclusion map \( i : A \to T(A) \). So far, we have embedded \( A \) (as a linear space) in a \( \mathbb{Z} \)-graded algebra but only as a linear space. The following proposition allows us to construct the embedding as an \( n \)-ary homomorphism.

2.1 Proposition. Let \( \mathcal{I} \) in \( T(A) \) be a two-sided ideal generated by the elements of the form: \( a_1 \otimes \cdots \otimes a_n - (a_1 a_2 \cdots a_n) \). Let \( \mathcal{O}(A) = T(A)/\mathcal{I} \) be the quotient algebra and \( \xi : T(A) \to \mathcal{O}(A) \) the corresponding canonical projection.

Then, if the \( n \)-ary algebra \( A \) is associative, the restriction of the projection map \( \xi : A \to \mathcal{O}(A) \) is an injective homomorphism of \( n \)-ary algebras.

\(^1\)This notion of associativity is sometimes denoted as full associativity.
Proof: Suppose for a while that $\xi$ is injective. Then $\xi(A)$ has a natural $n$-ary algebraic structure obtained by taking the the product in $T(A)$ and projecting it again to $O(A)$. From the construction of the ideal $I$ and projection $\xi$ it is obvious that the product will be in $\xi(A)$ and that $\xi$ is an homomorphism between $A$ and $\xi(A)$.

Now, suppose that $\xi$ is not injective. Then, there exists an element $a \in A$, which belongs to the ideal $I$. Of course $a$ cannot belong to the linear span of generators, therefore it must be of the form $a = \sum x \otimes y \otimes z$, where $y$ is in the linear span of the generators and $x, z$ are of the form $b_1 \otimes \cdots \otimes b_l$, $b_i \in A$ (case $l = 0$ included). Next, we can look at the expression at a fixed degree. Obviously, since $a$ is of degree 1, elements of all other degrees must add up to zero. Moreover, we could concentrate our efforts only on elements of degree 1 modulo $n-1$.

We shall perform the proof in two steps. First, note that since $a \in I$ then the part of $\sum x \otimes y \otimes z$ of degree 0 must be of the form $a = \sum_i (a^{(i)}_1 \cdots a^{(i)}_n)$. Immediately we get that in the part of the sum of degree $n$ we must have $\sum_i -(a^{(i)}_1 \otimes \cdots \otimes a^{(i)}_n)$. But the latter do not add up to zero (in fact we can safely assume that all are linearly independent), so there must be further components contributing to the sum of all elements of degree $n$. These can only come from the expressions of the type $\sum x \otimes y \otimes z$, with $y$ being a generator of $I$ and $x, z$ such that their degrees add up to $n-1$. This means that for each $i$ there is a $k$ that $a^{(i)}_k$ is again a product of $n$ elements: $a^{(i)}_k = (a^{(i)}_1)_{k_1} \cdots (a^{(i)}_n)_{k_n}$.

We can now repeat the entire procedure step by step going from $n$ to $2n-1$ and further on. At each step either the sum of the elements of given degree vanishes or we can still go up. However, since our sum is finite there is a maximum degree of it and our procedure must stop at a given moment. Then we have the following situation: in each step we have decomposed one of the elements of $A$ as a product of $n$ elements. Such refining goes on until at a certain degree we arrive at a sum of the type:

$$\sum a_1 \otimes a_2 \otimes \cdots \otimes (a^n_1 a^n_2 \cdots a^n_k) \otimes \cdots \otimes a_p,$$

where $p = m(n-1) + 1$ for certain $m$. However, we know that for term of the above type in the sum the following tensor product:

$$a_1 \otimes a_2 \otimes \cdots \otimes a^n_1 \otimes a^n_2 \otimes \cdots \otimes a^n_k \otimes \cdots \otimes a_p,$$

must be the same.

Then we can see that from the original expression for $a = \sum (a^{(i)}_1 \cdots a^{(i)}_n)$ we have obtained that it could be rewritten as a sum of products of elements...
of $\mathcal{A}$ of length $n$, $2n-1$ up to $m(n-1)+1$ such that for sums of equal lengths the terms differ only in the order in which the product is taken and the constant coefficients in front of them add up to 0. So, if our algebra was $n$-ary associative then the order of product does not matter and the sum vanishes, so $a = 0$ and the map $\xi$ is injective.

**Proof:** Clearly, it is sufficient to study the structure of the ideal $\mathcal{I}$. Let us note that its generators are all of degree 1 mod $n-1$ and therefore $\mathcal{I}$ becomes naturally a $\mathbb{Z}_{n-1}$-graded subalgebra of $T(\mathcal{A})$ (we just take the degree of the elements of $\mathcal{I}$ to be degree in $T(\mathcal{A})$ mod $n-1$).

Then, the quotient has again a natural $\mathbb{Z}_{n-1}$-graded structure. We have constructed an algebra, which contains the $n$-ary algebra as a subspace stable under $n$-multiplication. Since in our construction we have used the tensor algebra, which carries a natural $\mathbb{Z}$-grading, we might ask whether $O\mathcal{A}$ is a graded algebra:

**2.2 Corollary.** The algebra $O(\mathcal{A})$ is a $\mathbb{Z}_{n-1}$ graded associative algebra and the subspace of elements of grade 1 is $n$-ary isomorphic with $\mathcal{A}$.

**2.3 Corollary.** Suppose now that $\mathcal{B}$, $\mathcal{B} \subset \mathcal{A}$ is a $n$-ary associative subalgebra of $\mathcal{A}$. Then $O(\mathcal{B})$ is a $n$-1 graded subalgebra of $O(\mathcal{A})$.

The proof of the corollary is simple: first one observes that $T(\mathcal{B}) \subset T(\mathcal{A})$ and, moreover $\mathcal{I}_B \subset \mathcal{I}_A$. Then the appropriate relation for the inclusion of $(n-1)$-graded algebras follows.

Now, we shall state the main proposition, which concerns the universality property of the constructed object.

**2.4 Proposition.** Let $M$ be a $\mathbb{Z}_{n-1}$-graded associative algebra, and let $M_1$ denote the space of all its elements of degree 1, which is then an $n$-ary associative algebra. If there exists an a $n$-ary homomorphism $\rho : \mathcal{A} \rightarrow M_1 \subset M$ then there exist a unique homomorphism of $\mathbb{Z}_{n-1}$-graded algebras $\hat{\rho} : O(\mathcal{A}) \rightarrow M$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\rho} & M \\
\downarrow & & \downarrow \\
O(\mathcal{A}) & \xrightarrow{\hat{\rho}} & M
\end{array}
\]
**Proof:** First, note that the map $\rho$ is linear. Let us extend it in a natural way to $T(A)$ by taking:

$$T(\rho)(a_1 \otimes \cdots \otimes a_k) = \rho(a_1) \cdots \rho(a_k).$$

Now the only thing is to check that $T(\rho)$ vanishes on the ideal $I$, but due to the fact that $\rho$ is an $n$-ary homomorphism this is obvious. Similarly, by construction it is also clear that $\hat{\rho}$ preserves the grading. \[\blacksquare\]

What we have shown are two important facts: first, that every $n$-ary associative algebra could be identified with the space of elements of degree 1 of some $\mathbb{Z}_{n-1}$-graded algebra, second that the latter is universal, i.e. for every $\mathbb{Z}_{n-1}$-graded algebra in which our $n$-ary structure is embedded there exists a homomorphism between them.

Therefore the above lemma solves the problem posed in the introduction: not only we know that every $n$-ary algebra can be embedded in a normal (binary) algebra but we know how to find such objects.

2.5 Corollary. Let $A, M$ and $\rho : A \to M$ be as defined in the universality proposition. Let us call $\hat{M}$ the subalgebra of $M$ generated by the image of $\rho$. Then $\hat{M}$ is isomorphic (as a $\mathbb{Z}_{n-1}$-graded algebra) to $O(A)/I$ for some ideal $I \subset O(A)$.

2.6 Corollary. Let $A$ and $B$ be $n$-ary associative algebras and $\phi : A \to B$ a homomorphism. Then there exists a unique homomorphism of $\mathbb{Z}_{n-1}$-graded algebras $O(\phi) : O(A) \to O(B)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\xi \downarrow & & \downarrow \xi \\
O(A) & \xrightarrow{O(\phi)} & O(B)
\end{array}
\]

where, $\xi$ denotes the embedding of $A, B$ into $O(A), O(B)$, respectively. Again, the proof of both corollaries is a simple consequence of the universality lemma.

3 Examples.

In the previous section we have learned how to construct a $\mathbb{Z}_{n-1}$-graded universal envelope algebra of the $n$-ary algebra $A$. A simple example of ternary ($n = 3$) structure would illustrate the problem.
3.1 Example. Let us take a ternary algebra $A$ of the odd degree elements of the exterior tensor algebra of $\mathbb{R}^n$ (with the product obtained from the exterior product). Then its $\mathbb{Z}_2$-graded universal enveloping algebra is equal $A \oplus A_1 \otimes A$, where $A_1$ denotes the linear span of the generators of $A$.

The constructed universal embedding is rather big - note that in the above example we do not recover the exterior algebra over $\mathbb{R}^n$ we have started with but a space much larger. The following observation suggests the way to restrict it:

3.2 Observation. Let $A$ be an associative $n$-ary algebra, $\mathcal{O}(A)$ its $\mathbb{Z}_{n-1}$-graded envelopping algebra $i : A \to \mathcal{O}(A)$ and $I \subset \mathcal{O}(A)$ be a graded ideal which does not intersect the image of $A$. Then $A$ can be embedded into $\mathcal{O}(A)/I$.

Of course, out of the mentioned class of ideals with this property we can always find a maximal ideal, then we shall obtain the minimal $\mathbb{Z}_{n-1}$-graded algebras which contain $A$.

The maximal ideal, however, could be too big and in fact we might not recover the original algebra we have started with. Let us illustrate it with an example.

3.3 Example. For the above simple ternary example we might choose the ideal generated by the symmetric part of the tensor product $A_1 \otimes A_1$. Note, that for an even $n$ this is not a maximal ideal: we can add to it an element, which is of the form $e_1 \otimes e_2 e_3 \cdots e_n$, as it is annihilated by the action of the entire algebra from both sides and clearly does not belong to our previously chosen ideal.

For our original choice we recover as the quotient the original $\mathbb{Z}_2$-graded exterior algebra and in the case of the extended (maximal) ideal we would have the product of $n$ generators $e_1 \otimes e_2 e_3 \cdots e_n$ vanishing.

4 Semigroups

A vast class of algebras come from groups, constructed as groups algebras and, similarly, $\mathbb{Z}_n$-graded algebras can originate from $\mathbb{Z}_n$-graded semigroups. Let us propose a definition.

4.1 Definition. Let $\mathcal{G}$ be a set with a map $m : \mathcal{G}^n \to \mathcal{G}$, then we shall call it an $n$-semigroup.
Of course, there is a difficulty in extending this notion, for instance, by introducing a unit element in such a way that it does not reduce the n-product to the binary one.

Similarly as in the case of algebras some nontrivial examples of n-semigroups come from \( \mathbb{Z}_{n-1} \)-graded groups (groups with a given homomorphism on \( \mathbb{Z}_{n-1} \)) by considering the inverse image of 1.

Now, we can pose the following question: can every n-ary associative semigroup be embedded into a semigroup and, what are the conditions that allow it to be embedded in a group?

**4.2 Proposition.** If \( \mathcal{M} \) is a associative n-ary semigroup then there exists a \( \mathbb{Z}_{n-1} \)-graded semigroup \( \mathcal{O}(\mathcal{M}) \) such that \( i : \mathcal{M} \rightarrow \mathcal{O}(\mathcal{M}) \) is an isomorphism between \( \mathcal{M} \) and the subspace of elements of degree 1.

Moreover the construction is universal in the following sense: for every \( \mathbb{Z}_{n}-\)graded semigroup \( N \) and a homomorphism \( \rho : \mathcal{M} \rightarrow N \) there exist a homomorphism \( \hat{\rho} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\rho} & N \\
\downarrow{i} & & \downarrow{\hat{\rho}} \\
\mathcal{O}(\mathcal{M}) & \xrightarrow{} & \\
\end{array}
\]

**Proof:** Although the subject of the lemma is similar as in the case of algebras the problem is slightly more complicated. Again we begin by constructing the direct sum \( T(\mathcal{M}) \) of \( \mathcal{M}^k \) (cartesian product of \( k \) copies of \( \mathcal{M} \)), for \( k > 0 \). We introduce the product in this space as the standard cartesian product. Now we may introduce a relation in \( T(\mathcal{M}) \) - we say that two elements are related if the product of all their components are equal to each other:

\[ g_1 \times \cdots \times g_k \sim h_1 \times \cdots \times h_l \iff g_1g_2\cdots g_k = h_1h_2\cdots h_k. \]

This relation is clearly an equivalence relation (note that it makes sense only for \( k, l = r_{k,l}(n - 1) + 1 \) and is well-defined due to associativity). We extend it that it agrees with the product, i.e. for any \( x, y, w, z \in T(\mathcal{M}) \) and \( x \sim y \) we postulate \( z \times x \times w \sim z \times y \times w \).

Therefore we might introduce the quotient of \( T(\mathcal{M}) \) by the relation \( \sim \) and transport the product to the quotient \( \mathcal{O}(\mathcal{M}) = T(\mathcal{M})/\sim \).

The map \( i : \mathcal{M} \rightarrow \mathcal{O}(\mathcal{M}) \) is clearly injective (from the definition no elements of \( \mathcal{M} \) could be related among themselves) and preserving the n-ary product. What remains to be checked is the universality property and \( \mathbb{Z}_{n-1} \)-grading.
This, however, almost exactly copies the idea of the proof for \(n\)-ary algebras.

Let us notice that we were able to proof the correspondence between \(n\)-semigroups and semigroups and the construction cannot tell us whether the universal object (or the \(n\)-semigroups we started from) is related with a monoid or a group. We shall come to this problem later.

5 Properties of \(n\)-ary algebras and \(n\)-semigroups

One of the consequences of the proven correspondence between \(n\)-ary objects and \(\mathbb{Z}_{n-1}\)-graded algebras is the possibility to translate several constructions proposed for \(n\)-ary algebras (semigroups).

We shall indicate here two problems: one of Hochschild homology, and another related with ternary semigroups.

5.1 Hochschild Homology of \(n\)-ary algebras

A generalization of Hochschild homology has been proposed in some papers (see \([MV,Gn]\)), some modification will also be discussed in \([Si]\). Let us remind the definition \([Gn]\):

5.1 Definition. Let \(A\) be an \(n\)-ary algebra, for an even \(n\). Let \(C_k\) denote \(A^{k(n-1)+1}\). Consider a map \(\delta_i^k: C_k \rightarrow C_{k-1}\) defined as follows:

\[
\delta_i(a_0 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_{i+n-1} \otimes \cdots \otimes a_{k(n-1)}) = \\
(a_0 \otimes \cdots \otimes (a_ia_{i+1} \cdots a_{i+n-1}) \otimes \cdots \otimes a_{k(n-1)}).
\]

(2)

Then \(d_k = \sum_i (-1)^i \delta_i^k: C_k \rightarrow C_{k-1}\) is a linear operator which satisfies \(d_k d_{k+1} = 0\). (For details and proof of this statement see \([Gn]\)).

Now, let us formulate the problem:

5.2 Problem. How does the above defined homology of the chain complex \((C^k(A), d)\) relate to the usual Hochschild homology of the universal cover of \(A\)? Is there a chain complex and homology of \(O(A)\) (also a generalized homology with a boundary satisfying \(d^N = 0\) for some \(N\), see \([DV]\) for details) such that Hochschild homology of \(A\) can be expressed in terms of this homology?
5.2 Ternary groups

By looking at the elements of degree 1 of a $\mathbb{Z}_{n-1}$-graded group it is obvious that they do not form a subgroup, nevertheless still some specific group operations do exist, for instance, for every element one can find $n-2$ elements such that their product gives the unit of the groups.

A very special situation occurs when we have $n = 3$ - then the operation is unique since the map $g \to g^{-1}$ does not change the degree of an element.

For this reason when we take this very simple case $n = 3$ we might be able to propose a definition of a ternary group:

5.3 Definition. A ternary group $\mathcal{G}$ is a set, with a ternary associative map $\mathcal{G}^3 \to \mathcal{G}$ and the inverse map $\mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$ such that:

\[ \forall g, h \in \mathcal{G} \quad gg^{-1}h = h = hgg^{-1} \]

Notice that for an ordinary group, the above statement is equivalent to the existence of a unit and an inverse. Here, however, we cannot draw the same conclusion. We begin with a very easy lemma.

5.4 Lemma. For a ternary group the map $h \to g^{-1}hg$ is an injective homomorphism.

Proof: It is clear that it is a morphism, the injectivity follows from the existence of its inverse $h \to ghg^{-1}$. Notice that unlike in the classical case this does not proof that it is surjective. Of course in the finite dimensional case (i.e. when cardinality of the set $G$ is finite). Now we can state the problem:

5.5 Problem. Are there non-trivial ternary groups (in the above sense), which cannot be embedded in usual ($\mathbb{Z}_2$-graded) groups?

6 Conclusions

In this paper we have shown that the problem of at least associative $n$-ary algebras can be reduced to the $n-1$-graded usual algebras. This has many implications: first of all, one can translate the result of studies of $n$-ary objects to the language of graded algebras, as we have suggested here in the case of Hochschild homology. Since our results concern only associative structures, one may try investigate whether analogous relations are present in the arbitrary case, of Leibniz, or Lie-type structures.
Another application comes from the opposite direction. For \( n \)-ary algebras one can have several notions of commutativity or generalizations of anticommutativity. An example of that is \( j \)-commutativity of ternary algebras, for \( j \) being a cubic root of unity, for any \( a, b, c \in A \) we impose:

\[
abc = j bca = j^2 cab.
\]

This relation can be easily translated for relations between elements of the universal (or any other) \( \mathbb{Z}_2 \)-graded covering of \( A \). This provides us with a new class of algebraic objects, with interesting commutation relations.

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