WELL-POSEDNESS OF NONISENTRONIC EULER EQUATIONS WITH PHYSICAL VACUUM

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ABSTRACT. We consider the local well-posedness of the one-dimensional nonisentropic Euler equations with moving physical vacuum boundary condition. The physical vacuum singularity requires the sound speed to be scaled as the square root of the distance to the vacuum boundary. The main difficulty lies in the fact that the system of hyperbolic conservation laws becomes characteristic and degenerate at the vacuum boundary. Our proof is based on an approximation of the Euler equations by a degenerate parabolic regularization obtained from a specific choice of a degenerate artificial viscosity term. Then we construct the solutions to this degenerate parabolic problem and establish the estimates that are uniform with respect to the artificial viscosity parameter. Solutions to the compressible Euler equations are obtained as the limit of the vanishing artificial viscosity. Different from the isentropic case [7,12], our momentum equation of conservation laws has an extra term that leads to some extra terms in the energy function and causes more difficulties even for the case of $\gamma = 2$. Moreover, we deal with this free boundary problem starting from the general cases of $2 \leq \gamma < 3$ and $1 < \gamma < 2$ instead of only emphasizing the isentropic case of $\gamma = 2$ in [7,12,14].

1. Introduction

We are concerned with the one-dimensional compressible flow moving inside a dynamic vacuum boundary governed by the following nonisentropic Euler equations with initial and free boundary conditions:

$$\begin{align*}
\rho_t + (\rho u)_{\eta} &= 0, & \text{in } I(t), \\
(\rho u)_t + (\rho u^2 + p)_{\eta} &= 0, & \text{in } I(t), \\
S_t + uS_{\eta} &= 0, & \text{in } I(t), \\
(\rho, u, S)|_{t=0} &= (\rho_0(\eta), u_0(\eta), S_0(\eta)), & \text{on } I(0), \\
\rho &= 0, & \text{on } \Gamma(t),
\end{align*}$$

(1.1)

where $t > 0$ is the time variable, $\eta$ is the space variable, $u$ represents the Eulerian velocity, and $p, \rho, S$ stand for the pressure function, the density of gas, the entropy function, respectively; the open bounded interval $I(t) \subset \mathbb{R}$ denotes the evolving domain occupied by the gas, and $I(0) = I = \{\eta \in \mathbb{R} : 0 < \eta < 1\}$ is the initial spatial domain; $\Gamma(t) := \partial I(t)$ denotes the moving vacuum boundary, and $v(\Gamma(t))$ is the velocity of $\Gamma(t)$ satisfying $v(\Gamma(t)) = u$, which means that the vacuum boundary is moving with the fluid velocity. The equation

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of state \( p(\rho, S) \) is given by
\[
p = C_\gamma \rho^{\gamma} e^S, \quad \gamma > 1,
\]
where \( C_\gamma \) is the adiabatic constant that is set to 1 in this paper, and \( \gamma \) is the adiabatic gas exponent. The density \( \rho \) satisfies the following conditions:
\[
\rho(\eta, t) > 0 \text{ in } I(t) \quad \text{and} \quad \rho(\eta, t) = 0 \text{ on } \Gamma(t).
\]
(1.3)
The equations (1.1)–(1.3) are the conservation laws of mass, momentum, and entropy, respectively. (1.1) is the initial condition for the density, velocity, and entropy. The boundary condition (1.1) states that the density must vanish along the vacuum boundary.

To understand the behavior of a solution near the vacuum \( \rho = 0 \), similarly to [25] we take
\[
\frac{c^2}{\rho} = p'(\rho, S) = \gamma \rho^{\gamma - 1} e^S,
\]
and rewrite (1.1)–(1.2) in terms of \( u \) and \( c \) as
\[
\begin{cases}
(c^2)_t + (\gamma - 1)c^2 u e^{-S} + u(c^2)_\eta = 0, \\
u_t + uu_\eta + \frac{1}{\gamma - 1}(c^2)_\eta + \frac{c^2}{\gamma} S_\eta = 0.
\end{cases}
\]
(1.4)
The trajectory of the free boundary \( \Gamma(\eta, t) = \{ (\eta, t) : \rho(\eta, t) > 0 \} \cap \{ (\eta, t) : \rho(\eta, t) = 0 \} \) coincides with the gas particle path
\[
\frac{d\eta(t)}{dt} = u(\eta(t), t).
\]
Thus, (1.4) on \( \Gamma(\eta, t) \) becomes
\[
\frac{du}{dt} = -\frac{1}{\gamma - 1}(c^2)_\eta.
\]
Generally, the acceleration \( du/dt \) of \( \Gamma(\eta, t) \) would be finite, hence we have
\[
0 < \left| \frac{\partial c^2(\Gamma(\eta, t), t)}{\partial \eta} \right| < \infty, \quad \text{on} \quad \Gamma(\eta, t),
\]
(1.5)
which defines a physical vacuum boundary condition (or singularity). Since \( \rho_0 > 0 \) in \( I \), (1.5) implies that for some positive constant \( C \) and \( \eta \in I \) near the vacuum boundary \( \Gamma = \partial I \),
\[
\rho_0^{\gamma - 1} \geq C \text{dist}(\eta, \Gamma).
\]
(1.6)
Equivalently, the physical vacuum condition (1.5) implies that for some \( \kappa > 0 \),
\[
0 < C_\kappa \leq \left| \frac{\partial \rho_0^{\gamma - 1}(\eta)}{\partial \eta} \right| < \infty \quad \text{if} \quad \text{dist}(\eta, \partial I) \leq \kappa,
\]
(1.7)
and
\[
\rho_0^{\gamma - 1}(\eta) \geq C_\kappa > 0 \quad \text{if} \quad \text{dist}(\eta, \partial I) \geq \kappa,
\]
(1.8)
for a constant \( C_\kappa \) depending on \( \kappa \).

The mathematical analysis of vacuum states dates back to Lin [22] and Liu-Smoller [26] for the isentropic gas dynamics. The definition of physical vacuum was motivated by the case of Euler equations with damping studied in [25,28], and we refer the reader to [23,25,29,34] for more discussions. Despite its physical importance, even the local existence theory
of smooth solutions featuring the physical vacuum boundary has not been completed yet. This is because the vacuum condition (1.6) implies that the hyperbolic system becomes degenerate at the vacuum boundary, then the classical theory of hyperbolic systems cannot be applied; and the characteristic speeds of Euler equations are $u \pm c$, thus they become singular with infinite spatial derivatives at the vacuum boundary [28]. When the data is compactly supported, there are three ways of looking at the problem.

The first consists of solving the Euler equations in the whole space and requiring that the system (1.1) holds in the sense of distribution for all $x \in \mathbb{R}^d$ and $t \in [0, T]$. This is in particular the strategy used to construct global weak solutions [3, 9, 24].

The second consists of symmetrizing the system first and then obtaining the local existence using the theory of symmetric hyperbolic system. Again the symmetrized form has to be solved in the whole space (see for instances [19, 32, 33] for classical systems and [11, 21] for relativistic systems).

The third consists of requiring the Euler equations to hold on the set $(x, t) : \rho(x, t) > 0$ and writing an equation for the vacuum boundary $\Gamma$ that is a free boundary. Here the vacuum boundary is part of the unknowns. In this case, an appropriate boundary condition at vacuum is necessary. There are several different cases according to the initial behavior of the sound speed $c$. For simplicity, let the origin be the initial vacuum contact point and let $c \sim |x|^h$. When $h \geq 1$, namely the initial contact with vacuum is sufficiently smooth, Liu and Yang [27] constructed the local-in-time solutions to the Euler equations with damping by using the energy method based on the adaptation of the theory of a symmetric hyperbolic system. They also proved that $c^2$ cannot be smooth across $\Gamma$ after a finite time. By the condition (1.5), we note that in these regimes there is no acceleration along the vacuum boundary. For $0 < h < 1$, the initial contact to vacuum is only Hölder-continuous. In particular, the corresponding behavior to $h = \frac{1}{2}$ is the case of physical vacuum [4, 7, 8, 14]. For $0 < h < \frac{1}{2}$ and $\frac{1}{2} < h < 1$, its boundary behavior is believed to be ill-posed [16].

The case $h = 0$ occurs when there is no continuous initial contact of the density with vacuum. It can be considered as either a Cauchy problem or a free boundary problem. An example of a Cauchy problem when $h = 0$ is the Riemann problem for genuinely discontinuous initial data (see for instance [2, 10]). An example of a free boundary problem when $h = 0$ is the work by Lindblad [22], where the density is positive at the vacuum boundary. Jang-Masmoudi [14, 18] gave a rigorous and detailed proof to the existence theory in a series of papers. For the one-dimensional isentropic compressible gas with physical vacuum boundary condition, to overcome the degeneracy difficulty of propagation speed, in [14] they proposed a new formulation so that some energy estimates can be closed in the appropriate space, and moreover, they proved that the vacuum boundary behavior is preserved on some short time interval. They also investigated the multidimensional compressible gas flow with vacuum boundary [15]. In [17], results on free boundary problems were reviewed and some related open problems were proposed. Meanwhile, they extended their research to the compressible Navier-Stokes-Poisson system of spherically symmetric isentropic flows, and established the local-in-time well-posedness of strong solutions to the vacuum free boundary problem [18]. Countand-Shkoller [7, 8] also did many works on this free boundary problem. In [7], they adopted a different approach from Jiang-Masmoudi [14] viewing the initial density function $\rho_0$ as a parameter, thus the isentropic compressible system becomes one momentum conservation equation, then
they used the vanishing viscosity method to establish the local existence to the isentropic one-dimensional compressible flow with physical vacuum. Recently, they also established a priori estimates for the free boundary problem of the three-dimensional compressible Euler equations [8]. This technique proposed by Countand-Shkoller in [7] has been applied to many other systems of degenerate and characteristic hyperbolic systems of conservation laws. For example, Luo-Xin-Zeng [30] and Gu-Lei [12] recently extended this method to the spherically symmetric system and one-dimensional compressible Euler-Poisson equations with moving physical vacuum boundary condition. Besides the local existence theory of physical vacuum states, there are also some works on the long time behavior. Based on self-similar behavior, via Darcy’s law, Liu conjectured [25] that solutions of Euler equations with damping should behave asymptotically like the solutions of the porous media equation. This problem was studied by Huang, Marcati and Pan [13] in the framework of entropy solutions where the method of compensated compactness yields a global weak solution in $L^\infty$, and later by Luo and Zeng [31] by tracking the vacuum boundary.

In this paper, we deal with the nonisentropic compressible Euler equations with physical vacuum condition. Compared with the results already obtained in [7][12][14], there are three novel features. The first new feature is that our momentum equation has an extra term $\rho_S S_x$, because the pressure function depends on not only the density function $\rho(x, t)$ but also the entropy function $S(x, t)$ that is equal to $S_0(x)$, here $S_0(x)$ stands for the initial entropy function (in Lagrangian coordinate, the entropy function satisfies $S_t = 0$). To overcome this difficulty, in the energy function we will introduce more terms $\sum_{j=1}^2 |\omega^{2+\mu} \partial_x \delta_{x} S_{\omega, t}^j v(\cdot, t)|^2_0 + |\omega^{1/2+\mu} \partial_t v' (\cdot, t)|^2_0$ and add a new artificial viscous term $\epsilon (\omega^{2+2\mu} v' e^{S_0})'$, with $\beta' = \frac{\beta}{\rho_0}$, compared with the isentropic case. To close the energy estimates, besides the similar estimates to the isentropic case, we also need to deal with these additional terms and the term $\epsilon (\omega^{2+2\mu} e^{S_0})' v'$ from the artificial viscous term. In fact, even for the case of $\gamma = 2$, the energy function also has more terms than the isentropic case.

The second new feature is that we will treat general $\gamma$ ($2 \geq \gamma < 3$ and $1 < \gamma < 2$) from the beginning instead of only emphasizing the isentropic case $\gamma = 2$ as in [7][12][14]. Thus, we will face some new mathematical difficulties. For example, when $2 \leq \gamma < 3$, we will deal with $|\frac{1}{\sqrt{1-\gamma}}| L^\beta$ (c.f. (4.29) in [10]) instead of $|\frac{1}{\rho_0}| L^\beta$ (c.f. (6.33) in [7] and (6.45) in [12]) for the case of $\gamma = 2$ in the process of energy estimates, where determining the proper value for $\beta$ is technically more difficult than that for $\beta'$ due to $-\frac{1}{4} < \mu \leq 0$.

The third new feature comes from the case $1 < \gamma < 2$, since the momentum equation can be written as the following equation with the distance function $\omega = \rho_0^{\gamma-1}$ as a parameter:

$$\omega^{-1} v_t + \left( \frac{\rho_0^{\gamma} e^{S_0}}{(\omega')^{\gamma}} \right)_x = 0,$$

thus we know that the coefficient of $v_t$ will degenerate fast as $\gamma \to 1$ near the vacuum boundary. To obtain the $H^2$ norm of $v$ (and thus the $C^2$ norm of $v$) for small $\gamma - 1$, from the Hardy type embedding inequality (2.13), the higher energy function $E(t)$ defined in
\[
\|v\|^2 \leq \|v\|^2_{L^2} - (\frac{1}{2} + \mu) \leq C \sum_{i=0}^{l+1} \left| \omega^{1/2+\mu} \partial^i_v \right|^2_0 \leq C \tilde{E}, \quad l = 3 + 2 \left[ \frac{1}{2} + \frac{1}{\gamma - 1} \right],
\]
indicating the order of derivatives \( l \to \infty \) as \( \gamma \to 1 \). For example, different from the case of \( 2 < \gamma < 3 \), the estimates of higher order spatial derivatives for the case \( \gamma = \frac{3}{2} \) are much more difficult, for which we use (4.31) to close the energy estimates and the details can be found in [5.15]-[5.19]. Finally, we also present the proof of uniqueness for the general case \( 1 < \gamma < 3 \).

The rest of the paper is organized as follows. In Section 2, we introduce the Lagrangian coordinates to transform the free boundary problem to a fixed boundary problem, and we provide some useful inequalities including the Sobolev embedding inequalities and state our main result. In Section 3, we first present a degenerate parabolic approximation with viscosity \( \varepsilon \) to the compressible Euler equations, then use a fixed point theorem to solve this approximate problem. In Section 4 and Section 5, we will prove some uniform estimates independent of \( \varepsilon \) for \( 2 \leq \gamma < 3 \) and \( 1 < \gamma < 2 \) respectively. Then we take the limit as \( \varepsilon \to 0 \) to obtain the solution of the compressible Euler equations and hence establish the local existence theorem. In Section 6, we will prove the main result, i.e., Theorem 2.1.

2. Preliminaries and main result

In this section, we provide some preliminaries and state the main result.

2.1. Lagrangian formulation. The system (1.1) is in the Eulerian coordinates \((\eta, t)\). We first rewrite it in the Lagrangian variables \((x, t)\). Let \( \eta(x, t) \) denote the position of the gas particle \( x \) at time \( t \), then
\[
\partial_t \eta = u(\eta(x, t), t) \quad \text{for} \quad t > 0 \quad \text{and} \quad \eta(x, 0) = x.
\]
Set the Lagrangian velocity, Lagrangian density and entropy as the following,
\[
v(x, t) = u(\eta(x, t), t), \quad f(x, t) = \rho(\eta(x, t), t), \quad \tilde{S}(x, t) = S(\eta(x, t), t).
\]
Then
\[
\begin{align*}
\frac{\partial v}{\partial t} &= u\eta_t + u_t = u\eta u + u_t, \\
\frac{\partial f}{\partial t} &= \rho\eta_t + \rho_t = \rho\eta u + \rho_t, \\
\frac{\partial \tilde{S}}{\partial t} &= S\eta_t + S_t = S\eta u + S_t,
\end{align*}
\]
Using (2.1), the Lagrangian version of equations (1.1)_1 - (1.1)_3 can be written on the fixed reference domain \( I \) as
\[
\begin{cases}
f_t + fv_x/\eta_x = 0, \\
f v_t + (f^\gamma \exp \tilde{S})_x/\eta_x = 0, \\
\tilde{S}_t = 0.
\end{cases}
\]
From (2.2)_3, we have
\[
\tilde{S}(x, t) = S_0(x).
\]
Thus, (2.2) becomes
\[
\begin{cases}
    f_t + f v_x / \eta_x = 0, \\
    f v_t + (f^\gamma e^{S_0})_x / \eta_x = 0, \\
    (f, v, \eta)|_{t=0} = (\rho_0, u_0, x).
\end{cases}
\]
(2.4)

From (2.4), we know that
\[
(f \eta)_t = f \eta_x + f \eta_x t = f \eta_x + f v_x = 0,
\]
which implies
\[f = \rho(\eta(x, t), t) = \rho_0 / \eta_x.\]

Hence, the initial density function \( \rho_0 \) can be viewed as a parameter in the Euler equations. Thus, the problem (2.4) can be rewritten as
\[
\begin{cases}
    \rho_0 v_t + \left( \frac{\rho_0^\gamma e^{S_0}}{\eta_x} \right)_x = 0, \quad \text{in } I \times (0, T], \\
    (\eta, v) = (x, u_0), \quad \text{in } I \times \{t = 0\}, \\
    \rho_0^{\gamma - 1} = 0, \quad \text{on } \Gamma,
\end{cases}
\]
(2.5)

with \( \rho_0^{\gamma - 1} \geq C \text{dist}(x, \Gamma) \) for \( x \) near \( \Gamma \). In the following, we adopt the notation
\[
\omega = \rho_0^{\gamma - 1}.
\]
(2.6)

It is obvious that for \( \gamma = 2 \), (2.6) leads to \( \omega = \rho_0 \).

As we have already noted above, the initial domain \( I \subset \mathbb{R} \) at time \( t = 0 \) is given by
\[I = (0, 1),\]
(2.7)
and the initial boundary points are given by \( \Gamma = \partial I = \{0, 1\} \).

2.2. Embedding and interpolation inequalities. For integers \( k \geq 0 \), we define the Sobolev space \( H^k(I) \) to be the completion of \( C^\infty(I) \) under the norm
\[
||u||_k := \left( \sum_{a \leq k} \int_I |D^a u(x)|^2 dx \right)^{\frac{1}{2}}.
\]
For a real number \( s \geq 0 \), the Sobolev spaces \( H^s(I) \) and the norms \( || \cdot ||_s \) are defined by interpolation. We use \( H^1_0(I) \) to denote the subspace of \( H^1(I) \) consisting of those functions \( u(x) \) that vanish at \( x = 0 \) and \( x = 1 \).

We denote by \( || \cdot ||_0 \) the \( L^2 \) norm. We first review some useful embedding and interpolation inequalities. For Sobolev spaces, one has
\[
||u||_{L^r} \leq C ||u||_{\frac{1}{2}}, \quad 1 < r < \infty.
\]
(2.8)
We will also make use of the standard interpolation inequality [1]:
\[
||u||_s \leq C ||u||^{1-\theta}_{s_0} ||u||^\theta_{s_1},
\]
(2.9)
with \( 0 \leq s_0 \leq s \leq s_1 \) and \( s = (1 - \theta)s_0 + \theta s_1 \), in particular, some useful inequalities in this paper are
\[
||u||_{\frac{3}{4}} \leq C ||u||^{1/2}_{1/2} ||u||^{1/2}_1, \quad ||u||_{1/2} \leq C ||u||^2_{1/2} ||u||^2_1.
\]
(2.10)
For simplicity, we denote by $\| \cdot \|_\infty$ the $L^\infty$ norm, then
\[
\| u \|_\infty \leq C_p \| u \|_1. \tag{2.11}
\]
Using (2.10), one has
\[
\| u \|_\infty \leq C_p \| u \|_{3/4} \leq C \| u \|^{1/2}_{1/2} \| u \|^{1/2}_1 \leq C \| u \|^{1/4}_0 \| u \|^{3/4}_1. \tag{2.12}
\]

Let $\omega$ denote the distance function to the boundary $\Gamma$, and let $p = 1, 2$. For any $a > 0$ and nonnegative integer $b$, the weighted Sobolev space $H^{a,b}(I)$ is given by
\[
H^{a,b}(I) := \{ \omega^{a/2} F \in L^2(I) : \int_I \omega(x)^a |D^k F(x)|^2 \, dx < \infty, \ 0 \leq k \leq b \}
\]
with the norm
\[
\| F \|^{a,b}(I) := \sum_{k=0}^b \int \omega^a |D^k F|^2 \, dx.
\]
Then for any $b > \frac{a}{2}$, one has the following embedding [20]:
\[
H^{a,b}(I) \hookrightarrow H^{b-a/2}(I). \tag{2.13}
\]

For the estimates on the higher order spatial derivatives of $v$, we introduce the following lemma.

**Lemma 2.1** (Lemma 1 in [5]). Let $\varepsilon > 0$ and $g \in L^\infty(0,T;H^s(I))$ be given, and let $f \in H^1(0,T;H^s(I))$ be such that
\[
f + \frac{\varepsilon}{\gamma} f_t = g \quad \text{in } (0,T) \times I.
\]
Then,
\[
\| f \|_{L^\infty(0,T;H^s(I))} \leq C \max\{ \| f(0) \|_s, \| g \|_{L^\infty(0,T;H^s(I))} \}.
\]

2.3. Higher-order energy functions and main result. To close the energy estimates and state the main theorem, we define the energy functions for the two cases of $2 \leq \gamma < 3$ and $1 < \gamma < 2$, respectively. We consider the following higher-order energy functions:

**Case I:** $2 \leq \gamma < 3$.

\[
\tilde{E}(t) = \left\| \omega^{1+\mu} \partial_t^4 v^\prime(\cdot,t) \right\|_0^2 + \left\| \omega^{1+\mu} \partial_t^4 v(\cdot,t) \right\|_0^2
\]
\[
+ \sum_{j=1}^2 \left\{ \left\| \omega^{3/2+\mu} \partial_t^{5-2j} \partial_x^{j+1} v(\cdot,t) \right\|_0^2 + \sum_{i=1}^j \left\| \omega^{1/2+\mu} \partial_t^{5-2j} \partial_x^{i} \partial_x^{j+1} v(\cdot,t) \right\|_0^2 \right\}
\]
\[
+ \sum_{j=1}^2 \left\{ \left\| \omega^{2+\mu} \partial_t^{1-2j} \partial_x^{j+2} v(\cdot,t) \right\|_0^2 + \sum_{i=-1}^j \left\| \omega^{1+\mu} \partial_t^{1-2j} \partial_x^{i+1} \partial_x^{j+2} v(\cdot,t) \right\|_0^2 \right\} \tag{2.14}
\]
where $-\frac{2}{3} < \mu = \frac{2-\gamma}{2(\gamma-1)} \leq 0$. 

Case II: \(1 < \gamma < 2\).

\[
\tilde{E}(t) = ||\omega^{1+\mu} \partial_t^l v'(-, t)||^2_0 + ||\omega^{1+\mu} \partial_t^j \partial_x^i v(-, t)||^2_0 \\
+ \sum_{j=1}^{i+1} \left\{ ||\omega^{3/2+\mu} \partial_t^{j-2j} \partial_x^{j+1} v(-, t)||^2_0 + \sum_{i=1}^{j} ||\omega^{1/2+\mu} \partial_t^{j+1-2j} \partial_x^i v(-, t)||^2_0 \right\} \\
+ \sum_{j=1}^{i-1} \left\{ ||\omega^{2+\mu} \partial_t^{j-2j} \partial_x^j v(-, t)||^2_0 + \sum_{i=1}^{j} ||\omega^{1+\mu} \partial_t^{j+1-2j} \partial_x^i v(-, t)||^2_0 \right\},
\]

(2.15)

where

\[
\mu = \frac{2 - \gamma}{2(\gamma - 1)} \geq 0, \quad l = 3 + 2\lceil 1/2 + \mu \rceil,
\]

and \(\lceil \cdot \rceil\) is the ceiling function defined for any real number \(q \geq 0\) as

\[
\lceil q \rceil := \min\{m : m \geq q, m \text{ is an integer}\}.
\]

**Remark 2.1.** From the definitions of \(\mu, l\) in the case II, we find that \(\gamma \to 1\) implies \(l \to \infty\). Thus, for \(1 < \gamma < 2\), the high order derivatives will be needed, which will cause technical difficulties compared with the case of \(2 \leq \gamma < 3\).

**Remark 2.2.** Even when \(\gamma = 2\), the higher-order energy function in Case I is different from the isentropic case in [7][12], and the former contains the latter. In the nonisentropic case, some additional terms \(\sum_{j=1}^{2} ||\rho_0^{j} \partial_t^{j-2} \partial_x^j v(-, t)||^2_0 \sum_{i=1}^{2} ||\rho_0^{1/2} \partial_t^i v'||^2_0\) will appear due to the additional term \(pS_S x\) even for \(\gamma = 2\).

In fact, when \(\gamma = 2\), the energy function for the isentropic case in [7][12] is

\[
E_v(t) = \sum_{s=0}^{4} ||\partial_t^s v(-, t)||^2_{2-s/2} + \sum_{s=0}^{2} ||\rho_0 \partial_t^s v(-, t)||^2_{2-s} \\
+ ||\rho_0^{3/2} \partial_t^s \partial_x^s v(-, t)||^2_{0} + ||\rho_0^{3/2} \partial_t^2 \partial_x^2 v(-, t)||^2_{0} \\
+ ||\rho_0^{1/2} \partial_t^s \partial_x^s v(-, t)||^2_{0} + ||\rho_0^{1/2} \partial_t^2 \partial_x^2 v(-, t)||^2_{0},
\]

(2.16)

Since \(\gamma = 2\) implies \(\mu = 0\) and \(\omega = \rho_0\), the energy function in Case I is

\[
\tilde{E}(t) = ||\rho_0 \partial_t^l v'(-, t)||^2_0 + ||\rho_0 \partial_t^j \partial_x^i v(-, t)||^2_0 \\
+ \sum_{j=1}^{2} \left\{ ||\rho_0^{3/2} \partial_t^{j-2} \partial_x^j v(-, t)||^2_0 + \sum_{i=1}^{j} ||\rho_0^{1/2} \partial_t^{j+1-2} \partial_x^i v(-, t)||^2_0 \right\} \\
+ \sum_{j=1}^{2} \left\{ ||\rho_0^{2} \partial_t^{j-2} \partial_x^j v(-, t)||^2_0 + \sum_{i=1}^{j} ||\rho_0^{2} \partial_t^{j+1-2} \partial_x^i v(-, t)||^2_0 \right\},
\]

(2.17)

From the weighted Sobolev embedding inequality (2.13), we know that

\[
||\rho_0 \partial_t^l v'||^2_0 \geq ||\partial_t^l v||^2_{l-1} = ||\partial_t^l v||^2_0,
\]
We now state our main result as follows.

**Theorem 2.1.** Given the initial data \((\rho_0, u_0, S_0)\) with \(E(0) < \infty, \rho_0(x) > 0\) in \(I\), if the physical vacuum condition (1.6) holds for \(\rho_0\), and

\[
\underline{S} \leq S_0(x) \leq \bar{S},
\]

in \(I\) for some positive constants \(\underline{S}\) and \(\bar{S}\), there exists a unique solution to (2.5) (and hence (1.1)) on \([0, T]\) for some \(T > 0\) sufficiently small, such that,

\[
\sup_{t \in [0, T]} E(t) \leq P(E(0)),
\]

where \(P\) is some polynomial function of its argument.

We note that we shall use \(P(\cdot)\) to denote a generic polynomial function of its argument, and \(P\) will change from line to line with no explicit expressions necessarily given in the paper.

### 3. The degenerate parabolic approximation of the system

For convenience, we write \(\gamma' = \frac{\beta}{\nu_0}\). Now for \(\epsilon > 0\), we consider the following nonlinear degenerate parabolic approximation of (2.5):

\[
\rho_0 v_t + \left( \frac{\rho_0 e^{S_0}}{(\eta')^\gamma} \right)' = \epsilon (\rho_0 e^{S_0})' \quad \text{in} \quad I \times [0, T].
\]

(3.1)
With $\rho_0^{\gamma-1} = \omega \geq C\text{dist}(x, \Gamma)$ for $x \in I$ near $\partial I$, (3.1) with the initial and boundary conditions becomes
\[
\begin{cases}
\omega^{1+2\mu} v_t + \left( \frac{\omega^{2+2\mu} e^{S_0}}{(\eta')^\gamma} \right)' = \varepsilon(\omega^{2+2\mu} v'e^{S_0})', & \text{in } I \times [0, T], \\
(v, \eta) = (u_0, x), & \text{in } I \times \{t = 0\}, \\
\rho_0 = 0, & \text{on } \Gamma,
\end{cases}
\tag{3.2}
\]
where $\mu = \frac{2-\gamma}{2(\gamma-1)}$. Compared with the isentropic case [7, 12], this nonlinear equation has an additional term $\varepsilon\omega^{2+2\mu} v'e^{S_0} S'_0$, which will cause more technical difficulties in the following proofs and computations.

Given $u_0$ and $\rho_0$ and using the fact that $\eta(x, 0) = x, \eta'(x, 0) = 1$ we can compute the quantity $v_{t|t=0}$ for the degenerate parabolic $\varepsilon-$problem (3.2) by using (3.1):
\[
v_{t|t=0} = \left. \left( -\frac{1}{\rho_0} \left( \left( \frac{\rho_0}{\eta'} \right)^{\gamma} e^{S_0} \right)' + \varepsilon \frac{1}{\rho_0} (\rho_0^{\gamma-1} S'_0) \right) \right|_{t=0} \tag{3.3}
\]
Similarly, for all $k \in \mathbb{N}$,
\[
u_k := \partial_t^k v|_{t=0} = \partial_t^{k-1} \left( -\frac{1}{\rho_0} \left( \left( \frac{\rho_0}{\eta'} \right)^{\gamma} e^{S_0} \right)' + \varepsilon \frac{1}{\rho_0} (\rho_0^{\gamma-1} S'_0) \right)|_{t=0}. \tag{3.4}
\]
These formulas show that each $\partial_t^k v|_{t=0}$ is a function of spatial derivatives of $u_0$ and $\rho_0$.

As in [6, 7, 12], the linearized problem (3.2) is
\[
\begin{cases}
\omega^{1+2\mu} v_t + \left( \frac{\omega^{2+2\mu} e^{S_0}}{(\bar{\eta})'\gamma} \right)' = \varepsilon(\omega^{2+2\mu} v'e^{S_0})', & \text{in } I \times [0, T], \\
(v, \eta) = (u_0, x), & \text{in } I \times \{t = 0\}, \\
\rho_0 = 0, & \text{on } \Gamma,
\end{cases}
\tag{3.2}
\]
where $\bar{\eta} = x + \int_0^t \bar{v}(x, \tau) d\tau$ and $\bar{v}$ is given, one can easily show the existence and the uniqueness of the solution $v^\varepsilon$ to the above degenerate parabolic problem (3.2) in a time interval $[0, T^\varepsilon]$ with sufficiently smooth initial data, using the fixed point argument. Henceforth, we assume that $T^\varepsilon > 0$ is sufficiently small such that, independent of the choice of $v^\varepsilon$,
\[
\bar{\eta}^\varepsilon(x, t) = x + \int_0^t \bar{v}^\varepsilon(x, s) ds \tag{3.5}
\]
is injective for $t \in [0, T^\varepsilon]$, and $\frac{1}{2} \leq \bar{\eta}'(x, t) \leq \frac{3}{2}$ for $t \in [0, T^\varepsilon]$ and $x \in \bar{I}$.

We will then perform energy estimates on this solution to show that the time of existence, in fact, does not depend on $\varepsilon$, and moreover the a priori estimates for this sequence of solutions are also uniform in $\varepsilon$. The existence of solution to the compressible Euler equations (1.1) is then obtained as the weak limit of $\varepsilon \to 0$ of the sequence of solutions to (3.2).
4. Uniform estimates of $v^\varepsilon$ for $2 \leq \gamma < 3$

Our objective in this section is to prove the uniform estimates of $v^\varepsilon$ for $2 \leq \gamma < 3$. For the sake of notational convenience, we omit the superscript $\varepsilon$. We first give some analysis on $\hat{E}(t)$ in (2.14) in order to establish the desired estimates. From the weighted Sobolev embedding inequality (2.13) and $-\frac{1}{4} \leq \mu \leq 0$ as $2 \leq \gamma < 3$, we have, for the first term in (2.14),

$$\left| \omega^{1+\mu} \partial_t^4 v' \right|^2 \geq \left| \partial_t^4 v \right|^2_{1-\mu} \geq \left| \partial_t^4 v \right|^2_0,$$

(4.1)

for the first term of the second line in (2.14),

$$\sum_{j=1}^{2} \left| \omega^{3/2+\mu} \partial_t^{5-2j} \partial_x^{j+2} v(\cdot, t) \right|^2_0 \geq \left| \partial_t^{5-2j} v(\cdot, t) \right|^2_{j+2-3/2-\mu} \geq \sum_{j=1}^{2} \left| \partial_t^{5-2j} v \right|^2_{j-1/2},$$

(4.2)

and for the first term of the third line in (2.14),

$$\sum_{j=1}^{2} \left| \omega^{2+\mu} \partial_t^{1-2j} \partial_x^{j+2} v(\cdot, t) \right|^2_0 \geq \sum_{j=1}^{2} \left| \partial_t^{1-2j} v \right|^2_{j+2-2-\mu} \geq \sum_{j=1}^{2} \left| \partial_t^{1-2j} v \right|^2_j.$$  

(4.3)

Thus, (4.1) - (4.3) lead to

$$\hat{E}(t) \geq \sum_{s=0}^{4} \left| \partial_t^s v \right|_{2-s/2}^2.$$

(4.4)

4.1. Some $\varepsilon$–independent energy estimates on the $\partial_t^k$–problem.

**Proposition 4.1.** For $2 \leq \gamma < 3$, there exists a constant $\alpha \in (0, 1)$, such that one has the following $\varepsilon$–independent energy estimate for $\partial_t^5$–problem of (3.2)$_1$:

$$\left| \omega^{1/2+\mu} \partial_t^5 v \right|_{0+}^2 + \left| \omega^{1+\mu} \partial_t^4 v' \right|_{0}^2 + \left| \omega^{1+\mu} \partial_t^4 v \right|_{0}^2 + \varepsilon \int_0^t \left| \omega^{1+\mu} \partial_t^5 v' \right|_{0}^2 \leq \hat{E}^\alpha \left( \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right),$$

(4.5)

with $\hat{M}_0 = P(\hat{E}(0))$ and $P(\cdot)$ some polynomial function.

**Proof.** We take the fifth partial derivative $\partial_t^5$ in (3.2) and multiply it by $\partial_t^5 v$, after integrating by parts, we have the following identity:

$$\frac{1}{2} \frac{d}{dt} \int_1 \omega^{1+2\mu} |\partial_t^5 v|^2 - \int_1 \partial_t^5 \left( \frac{\omega^{2+2\mu}}{(\eta')^\gamma} \right) \partial_t^5 v' e^{S_0} + \varepsilon \int_1 \omega^{2+2\mu} |\partial_t^5 v'|^2 e^{S_0} = 0,$$

(4.6)
Hence, substituting (4.7) into (4.6), integrating it from 0 to \(t\), we see that the second term on the right-hand side of (4.6) can be written as

\[
- \int_I \partial_t^5 \left( \frac{\omega^{2+2\mu}}{(\eta')^\gamma} \right) \partial_t^5 v e^S_0 = \gamma \int_I \frac{\omega^{2+2\mu}}{(\eta')^{\gamma+1}} \partial_t^4 v' \partial_t^5 v e^S_0
\]

\[
+ \sum_{\alpha=1}^4 c_\alpha \int_I \omega^{2+2\mu} \partial_t^{4-\alpha} v' \partial_t^5 v e^S_0
\]

\[
= \frac{\gamma}{2} \frac{d}{dt} \int_I \frac{\omega^{2+2\mu}}{(\eta')^{\gamma+1}} |\partial_t^2 v'|^2 e^S_0
\]

\[
+ \frac{(\gamma + 1) \gamma}{2} \int_I \frac{\omega^{2+2\mu}}{((\eta')^{\gamma+2} v' |\partial_t^4 v'|^2 e^S_0
\]

\[
+ \sum_{\alpha=1}^4 c_\alpha \int_I \partial_t^{4-\alpha} v' \omega^{2+2\mu} \partial_t^5 v e^S_0.
\]

Hence, substituting (4.7) into (4.6), integrating it from 0 to \(t\), we find that

\[
\frac{1}{2} \int_I \omega^{1+2\mu} |\partial_t^5 v|^2 + \frac{\gamma}{2} \int_I |\partial_t^4 v'|^2 \omega^{2+2\mu} + \frac{\gamma}{2} \int_I \frac{\omega^{2+2\mu}}{(\eta')^{\gamma+1}} |\partial_t^2 v'|^2 e^S_0
\]

\[
= \frac{1}{2} \int_I \omega^{1+2\mu} |\partial_t^5 v_0|^2 + \frac{\gamma}{2} \int_I \frac{\omega^{2+2\mu}}{(\eta')^{\gamma+1}} |\partial_t^2 v_0'|^2 e^S_0
\]

\[
- \frac{(\gamma + 1) \gamma}{2} \int_0^t \int_I \frac{\omega^{2+2\mu}}{((\eta')^{\gamma+2}) v' |\partial_t^4 v'|^2 e^S_0
\]

\[
- \sum_{\alpha=1}^4 c_\alpha \int_0^t \int_I \partial_t^{4-\alpha} v' \omega^{2+2\mu} \partial_t^5 v e^S_0
\]

\[
= \sum_{i=1}^4 I_i.
\]

It is easy to verify that \(I_1, I_2\) can be controlled by \(\hat{M}_0\). Now we estimate \(I_3, I_4\) on the right hand-side of (6.4),

\[
I_3 = - \frac{(\gamma + 1) \gamma}{2} \int_0^t \int_I \frac{\omega^{2+2\mu}}{((\eta')^{\gamma+2}) v' |\partial_t^4 v'|^2 e^S_0 \leq C \int_0^t \|v'\|_\infty \|\omega^{1+\mu} \partial_t^4 v'|_0^2
\]

\[
\leq C \int_0^t \|v'\|_2 \|\omega^{1+\mu} \partial_t^4 v'|_0^2 \leq C t P \left( \sup_{[0,t]} \hat{E} \right),
\]
Similarly, we have used (2.20), (4.1) and $1/2 \leq \eta' \leq 3/2$ . Then, using integration by part in time, we have

$$I_4 = -\sum_{i=1}^{4} c_i \int_{t}^{T} \int_{I} \partial_t^i \left( \frac{1}{(\eta')^{i+1}} \partial_t^{4-i} \nu' \omega^{2+2\mu} \partial_t^4 v' e^{S_0} \right) dt,$$

$$= \int_{t}^{T} \int_{I} \left( \sum_{i=1}^{4} c_i \partial_t^i \left( \frac{1}{(\eta')^{i+1}} \partial_t^{4-i} \nu' \right) \omega^{2+2\mu} \partial_t^4 v' e^{S_0} \right) dt,$$

$$+ \int_{t}^{T} \sum_{i=1}^{4} c_i \partial_t^i \left( \frac{1}{(\eta')^{i+1}} \partial_t^{4-i} \nu' \omega^{2+2\mu} \partial_t^4 v' \right) |_{t=0}^{t} e^{S_0}.$$  

where

$$J := \int_{t}^{T} \sum_{i=1}^{10} J_i dt, \quad \text{and} \quad J_i = \int_{I} R(\eta') \partial_t^{2+2\mu} \partial_t^4 v' dx,$$

the terms $j_i, i = 1, 2, \cdots, 10$ are the functions of $\nu', \partial_t^k \nu', k = 1, 2, 3, 4$, as the following:

$$j_1 = \partial_t \nu' \nu', \quad j_2 = \partial_t^3 \nu'(\nu')^2, \quad j_3 = \partial_t^3 \nu' \partial_t \nu', \quad j_4 = \partial_t^2 \nu' \partial_t \nu', \quad j_5 = (\partial_t^2 \nu')^2,$$

$$j_6 = \partial_t^2 \nu'(\nu')^3, \quad j_7 = (\partial_t \nu')^3, \quad j_8 = (\partial_t \nu')^2 (\nu')^2, \quad j_9 = \partial_t \nu'(\nu')^4, \quad j_{10} = (\nu')^6,$$

where $R(\eta')$ denotes some power functions of $\eta'$. We first note

$$|J_1| \leq C ||\nu'||_{\infty} ||R(\eta')||_{1/\infty} ||\omega^{1+\mu} \partial_t^4 \nu'||_{0} \leq CP \left( \sup_{[0,T]} \hat{E} \right),$$

(4.11)

where we have used (2.12) and (4.4), which means that

$$||\nu'||_{\infty} \leq C ||\nu'||_{1/4} ||\nu'||_{4/3} \leq C \hat{E}.$$  

For $J_2$, we have

$$|J_2| \leq ||R(\eta')||_{\infty} ||\omega^{1+\mu} \partial_t^3 \nu'||_{0} ||\omega^{1/2} ||\nu'||_{\infty} ||\omega^{1+\mu} \partial_t^4 \nu'||_{0} \leq CP \left( \sup_{[0,T]} \hat{E} \right).$$

Similarly, we have

$$|J_3| \leq ||R(\eta')||_{\infty} ||\omega^{1+\mu} \partial_t^3 \nu'||_{L^1} ||\partial_t \nu'||_{L^4} \leq C \left( ||\omega^{1+\mu} \partial_t^3 \nu'||_{1/2} ||\partial_t \nu'||_{4/3} \right) \leq CP \left( \sup_{[0,T]} \hat{E} \right),$$

where we have used the physical vacuum condition (1.7) and

$$||\omega^{1/2} \partial_x (\omega^{1+\mu} \partial_t^3 \nu')||_{0} \leq ||\omega^{3/2+\mu} \partial_t^3 \nu'||_{0} + ||\omega^{1/2+\mu} \omega' \partial_t^3 \nu'||_{0} \leq \hat{E}.$$  

Using (2.13), we have

$$||\omega^{1+\mu} \partial_t^3 \nu'||_{1/2} = ||\omega^{1+\mu} \partial_t^3 \nu'||_{1/2} \leq ||\omega^{1/2} \partial_x (\omega^{1+\mu} \partial_t^3 \nu')||_{0} \leq \hat{E},$$

(4.12)

and similarly,

$$|J_4| \leq C ||R(\eta')||_{1/\infty} ||\omega^{1+\mu} \partial_t^2 \nu'||_{L^1} ||\partial_t \nu'||_{L^4} ||\nu'||_{\infty} ||\omega^{1+\mu} \partial_t^4 \nu'||_{0} \leq M_0 + CtP \left( \sup_{[0,T]} \hat{E} \right),$$
where we have used the fact
\[ \|\omega^{1+\mu}\partial^2_t v'\|_{1/2} = \|\omega^{1+\mu}v'_{2} + \int_0^t \omega^{1+\mu}\partial^3_t v'\|_{1/2}. \]
Similarly, \( J_5 \) can be estimated as
\[ |J_5| \leq C \| R(\eta') \|_\infty \|\omega^{1+\mu}\partial^2_t v'\|_{L^4}^2 \|\omega^{1+\mu}\partial^4_t v'\|_{0} \leq \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right). \]
For \( J_6 \), we have
\[ |J_6| \leq C \| R(\eta') \|_\infty \|\omega^{1+\mu}\partial^2_t v'\|_{0} \|v'\|_{3}^3 \|\omega^{1+\mu}\partial^4_t v'\|_{0} \leq C t P \left( \sup_{[0,t]} \hat{E} \right), \]
and
\[ |J_7| \leq C \| R(\eta') \|_\infty \|\partial_t v'\|_{L^6}^2 \|v'\| \|\omega^{1+\mu}\partial^4_t v'\|_{0} \leq C \left( \sup_{[0,t]} \hat{E} \right), \]
\[ |J_8| \leq C \| R(\eta') \|_\infty \|\omega^{1/2+\mu}\partial_t v'\| \|\omega^{1/2+\mu}\partial^4_t v'\|_{0} \leq C \left( \sup_{[0,t]} \hat{E} \right), \]
and
\[ |J_9| \leq C \| R(\eta') \|_\infty \|\omega_{0}^{1/2+\mu}\partial_t v'\|_{1/2} \|\omega^{1+\mu}\partial^4_t v'\|_{0} \leq C \left( \sup_{[0,t]} \hat{E} \right). \]
Next, we treat the second term on the right-hand side of (4.10), beginning with the case of \( i = 1 \). We see that for \( \delta > 0 \),
\[ \int_I \frac{1}{\eta^{(\gamma+1)}} \partial^3_t v' \omega^{2+2\mu}\partial^4_t v' \|_0 = \int_I \left( R(\eta') \partial^3_t v' \omega^{2+2\mu}\partial^4_t v' \right)(t) \]
\[ - \int_I \left( R(\eta') \partial^3_t v' \omega^{2+2\mu}\partial^4_t v' \right)(0), \]
and
\[ \int_I \left( R(\eta') \partial^3_t v' \omega^{2+2\mu}\partial^4_t v' \right)(t) \leq C \|v'\|_\infty \|\omega^{1+\mu}\partial^3_t v'\|_0 \|\omega^{1+\mu}\partial^4_t v'\|_0 \]
\[ \leq C \hat{E}^{3/4} \left( M_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right). \]
The case when \( i = 2, 3, 4 \) can be estimated in the same fashion. Using (2.19) again, there exists a constant \( \alpha \), such that all the estimates (1.8)-(1.13) together yield
\[ \|\omega^{1/2+\mu}\partial^5 v\|^2 + \|\omega^{1+\mu}\partial^4_t v'\|^2 + \varepsilon \int_0^t \|\omega^{1+\mu}\partial^5_t v'\|^2 \]
\[ \leq \hat{E}^\alpha \left( M_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right), \quad 0 < \alpha < 1. \]
Since \( \partial_t^l v = u_l + \int_0^t \partial_t^{l+1} v \), we have
\[
||\omega^{1+\mu} \partial_t^l v||_0^2 \leq \tilde{M}_0 + C \epsilon ||\omega||_\infty ||\omega t^{1/2+\mu} \partial_t^{l+1} v||^2_0.
\]

Letting \( l = 4 \) and using first term in the energy estimate (4.14), we have
\[
||\omega^{1+\mu} \partial_t^4 v||_0^2 \leq \hat{E}^\alpha \left( \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right).
\]
(4.15)

Thus, the proof of Proposition 4.1 is complete. \( \square \)

Using the same argument as proving Proposition 4.1, we can consider the \( \epsilon \)-independent energy estimates for the \( \partial_t^4 \)-problem and \( \partial_t \)-problem of (3.2) and obtain the following estimates:

**Proposition 4.2.** For \( 2 \leq \gamma < 3 \), there exists a constant \( \alpha \in (0, 1) \), such that one has the following energy estimates uniform in \( \epsilon \):
\[
||\omega t^{1/2+\mu} \partial_t^2 v'||_0^2 + ||\omega^{1+\mu} \partial_t^2 v'||_0^2 + ||\omega^{1+\mu} \partial_t v'||_0^2 + \epsilon \int_0^t ||\omega^{1+\mu} \partial_t v'||_0^2 + \epsilon \int_0^t ||\omega^{1+\mu} \partial_t v'||_0^2 \leq \hat{E}^\alpha \left( \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right),
\]

and
\[
||\omega^{1+\mu} \partial_t v||_0^2 + ||\omega^{1+\mu} v'||_0^2 + ||\omega^{1+\mu} v||_0^2 + \epsilon \int_0^t ||\omega^{1+\mu} \partial_t v'||_0^2 \leq \hat{E}^\alpha \left( \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right).\]

**4.2. Estimates of higher order spatial derivatives.** Having obtained the uniform energy estimates in Propositions 4.1, 4.2, we can begin our estimates of higher spatial derivatives. We consider \( \partial_t^{k+1} \)-problem of (3.2):
\[
(\omega^{2+2\mu} \partial_t^k v' e S_0)' - \frac{\epsilon}{\gamma} \partial_t (\omega^{2+2\mu} \partial_t^k v' e S_0)' = g, \tag{4.16}
\]

where
\[
g = -\frac{1}{\gamma} \omega^{1+2\mu} \partial_t^2 v + \sum_{\alpha=1}^k \left( c_{\alpha} \omega^{2+2\mu} \partial_t^\alpha \frac{1}{(\eta')^{\gamma+1}} \partial_t^{k-\alpha} v' e S_0 \right)' + \left[ \left( 1 - \frac{1}{(\eta')^{\gamma+1}} \right) (\omega^{2+2\mu} \partial_t^k v' e S_0)' \right] - (\gamma + 1) \frac{\omega^{2+2\mu} \partial_t^k v' e S_0''}{(\eta')^{\gamma+2}}.
\]

Applying Lemma 2.1 directly, we have
\[
||\omega^{2+2\mu} \partial_t^k v'||_0 \leq C \max \{ ||g||_0, ||f(0)||_0 \}.
\]

However, this estimate is not good enough to obtain the corresponding estimates with weights of \( \omega^{1/2+\mu}, \omega^{3/2+\mu}, \omega^{1+\mu}, \omega^{2+\mu} \) in the energy function \( \hat{E}(t) \). To obtain the desired estimates, we shall reduce (4.16) to some new equations of the form in Lemma 2.1 by multiplying (4.16) by some suitable multipliers.
Proposition 4.3. For $2 \leq \gamma < 3$, there exists a constant $\alpha \in (0, 1)$, such that one has the following estimates:

\[
\sup_{[0,t]} \left( |\omega^{1/2+\mu} \partial_t^3 v'|^2 + |\omega^{3/2+\mu} \partial_t^3 v''|^2 \right) \leq C \left( \hat{M}_0 + CTP \left( \sup_{[0,t]} \hat{E} \right) \right) + CTP \left( \sup_{[0,t]} \hat{E} \right),
\]

\[\text{(4.17)}\]

and

\[
\sup_{[0,t]} \left( |\omega^{1/2+\mu} \partial_t v'|^2 + |\omega^{3/2+\mu} \partial_t v''|^2 \right) \leq C \left( \hat{E}^{\alpha} + 1 \right) \left( \hat{M}_0 + CTP \left( \sup_{[0,t]} \hat{E} \right) \right) + CTP \left( \sup_{[0,t]} \hat{E} \right),
\]

\[\text{(4.18)}\]

Proof. First, choosing $k = 3$ in (4.16), and multiplying (4.16) both sides by $\omega^{-(1/2+\mu)}$, we have

\[
\omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \newline - \frac{\varepsilon}{\gamma} \partial_t \left[ \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \right] \\
= - \frac{3\gamma}{\gamma+1} \omega^{2+2\mu} \partial_t^3 v - \frac{1}{\eta'} \partial_t^3 v e^{S_0} \newline + \sum_{i=1}^{3} c_i \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \\
+ \omega^{-(1/2+\mu)} \left[ \left( 1 \frac{1}{\eta'} \right) \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \right].
\]

Using Lemma 2.1 and fundamental theorem of calculus for the terms on the right-hand side of (4.19), we obtain that for any $t \in [0, T^\varepsilon]$,

\[
\sup_{[0,t]} \left| \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \right|_0 \leq C \sup_{[0,t]} \left| \omega^{1/2+\mu} \partial_t^3 v \right|_0 + C \sup_{[0,t]} \left| \omega^{3/2+\mu} \partial_t^3 v' e^{S_0} \right|_0 \\
+ C \sup_{[0,t]} \sum_{i=1}^{3} \left| \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \right|_0 \\
+ C \sup_{[0,t]} \left| \omega^{-(1/2+\mu)} \left[ \left( 1 \frac{1}{\eta'} \right) \left( \omega^{2+2\mu} \partial_t^3 v' e^{S_0} \right)' \right] \right|_0 \\
= \sum_{i=5}^{8} I_i.
\]

\[\text{(4.20)}\]

We estimate each term on the right-hand side of (4.20). For the first term, noting that $-\frac{1}{4} < \mu \leq 0$, as $2 \leq \gamma < 3$, and using the first term of the estimate (4.14), one has, for
each \( t \in [0, T_\varepsilon] \),

\[
I_5 = \sup_{[0,t]} \left| \omega^{1/2+\mu} \partial^5_t v \right|_0 \leq C \hat{E}^{n/2} \left( \hat{M}_0 + CtP \left( \sup_{[0,\bar{t}]} \hat{E} \right) \right). 
\]

(4.21)

The second term \( I_6 \) on the right-hand side of (4.20) can be estimated as

\[
I_6 = \left| \frac{\omega^{3/2+\mu} \partial^3_t \omega' \eta'' e_{S_0}}{(\eta')^{\gamma+2}} \right|_0 \leq C \left| \omega^{3/2+\mu} \partial^3_t \omega' \right|_0 \left| \int_0^t v'' \right|_0 
\]

\[
\leq Ct \left( \left| \omega^{1/2+\mu} \omega' \partial^3_t v' \right|_0 + \left| \omega^{3/2+\mu} \partial^3_t \omega'' \right|_0 \right) \left| v'' \right|_0 \leq CtP \left( \sup_{[0,t]} \hat{E} \right). 
\]

(4.22)

For the third term, we have,

\[
I_7 = \sum_{i=1}^{3} \left| \left( \frac{1}{(\eta')^{\gamma+1}} \partial^3_t - i v' e^{S_0} \right) \right|_0 
\]

\[
\leq C \left| \omega^{3/2+\mu} (v')^3 v'' \right|_0 + C \left| \omega^{3/2+\mu} \left( |\partial_t v' v''|^2 + |\partial_t v' v'' v''| \right) \right|_0 
\]

\[
+ C \left| \omega^{3/2+\mu} \left( |\partial^2_t v' v''| + |\partial^2_t v' v'' v''| \right) \right|_0 + C \left| \omega^{3/2+\mu} \partial_t v' \partial_t v'' \right|_0 
\]

\[
+ C \left| \left( \omega^{1/2+\mu} \omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu} \eta'' \right) (v')^4 \right|_0 
\]

\[
+ C \left| \left( \omega^{1/2+\mu} \omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu} \eta'' \right) |\partial_t v' (v')^2| \right|_0 
\]

\[
+ C \left| \left( \omega^{1/2+\mu} \omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu} \eta'' \right) |\partial^2_t v' (v')^2| \right|_0 
\]

\[
= \sum_{i=1}^{8} I_{7i}, 
\]

(4.23)

where we have used the fact \( 1/2 \leq \eta' \leq 3/2 \) and (2.20). To obtain the desired estimates, we shall use the following form of estimates:

\[
\sum_{j=1}^{2} \left| \left| \omega^{3/2+\mu} \partial^2_t - j \partial^2_x j \partial^2_t + 1 \right| v \right|_0^2 + \sum_{i=1}^{j} \left| \left| \omega^{1/2+\mu} \partial^2_t - j \partial^2_x j \partial^2_t + 1 \right| v \right|_0^2 + \sum_{i=1}^{j} \left| \left| \omega^{1/2+\mu} \partial^2_t - j \partial^2_x j \partial^2_t + 1 \right| v \right|_0^2 
\]

\[
\leq C \sum_{j=1}^{2} \left| \omega^{3/2+\mu} u^{(j+1)}_{4-2j} \right| + \left| \omega^{3/2+\mu} \partial^4 - j \partial^2_x j \partial^4 + 1 \right| v \right|_0^2 
\]

\[
+ \sum_{i=1}^{j} \left| \left| \omega^{1/2+\mu} u^{(i)}_{4-2j} \right| \right| + \left| \omega^{1/2+\mu} \partial^4 - j \partial^2_x j \partial^4 + 1 \right| v \right|_0^2 
\]

\[
+ \sum_{i=1}^{j} \left| \left| \omega^{1/2+\mu} u^{(i)}_{4-2j} \right| \right| + \left| \omega^{1/2+\mu} \partial^4 - j \partial^2_x j \partial^4 + 1 \right| v \right|_0^2 
\]

\[
\leq \hat{M}_0 + CtP \left( \sup_{[0,\bar{t}]} \hat{E} \right). 
\]

(4.24)
For $I_{71}$, using (4.12), we have
\[
I_{71} = \| \omega^{3/2+\mu} (v')^3 v'' \|_0 \leq \| \omega^{3/2+\mu} v'' \|_{L^6} \leq C \| \omega^{3/2+\mu} v'' \|_1 \| v' \|^{3/2}_1 = \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right), \tag{4.25}
\]
where we have used the physical vacuum condition (1.7) and (4.24), as well as the following two estimates:
\[
\| \omega^{3/2+\mu} v'' \|_\infty \leq \| \omega^{3/2+\mu} v'' \|_1 \leq C \| \omega^{1/2+\mu} \omega' v'' + \omega^{3/2+\mu} v'' \|_0 \leq \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]
and
\[
\| v' \|_{1/2} \leq \| v' \| + \int_0^t \| \partial_t v' \|_{1/2} \leq C \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right). \tag{4.26}
\]

For $I_{72}$, using (4.26), we have
\[
I_{72} = \| \omega^{3/2+\mu} (\partial_t v'' (v')^2 + \partial_t v' v'') \|_0 \leq C \| \omega^{3/2+\mu} \partial_t v'' \|_4 \| v' \|^{1/2}_4 + C \| \omega^{3/2+\mu} v'' \|_{L^6} \| \partial_t v' \|_0 \| v' \|_\infty \leq C \| \omega^{3/2+\mu} \partial_t v'' \|_1 \leq \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]
where we used (4.24) to derive the following estimates
\[
\| \omega^{3/2+\mu} \partial_t v'' \|_{1/2} \leq C \| \omega^{1/2+\mu} \partial_t v'' \|_0 \leq \tilde{M}_0 + t P \left( \sup_{[0,t]} \hat{E} \right),
\]
and used the physical vacuum condition (1.7) to get
\[
\| \omega^{3/2+\mu} \partial_t v'' \|_1 = \| \omega^{1/2+\mu} \omega' \partial_t v'' + \omega^{3/2+\mu} \partial_t v''' \|_0 \leq C \hat{E}^{1/4}.
\]

Using (4.24) to estimate $\| \omega^{3/2+\mu} \partial_t v'' \|_0$, $\| \omega^{1/2+\mu} \partial_t v' \|_0$, and $\| \omega^{1/2+\mu} v'' \|_0$, one has,
\[
I_{73} = \| \omega^{3/2+\mu} (\partial_t^2 v'' v' + \partial_t^2 v' v'') \|_0 \leq C \| \omega^{3/2+\mu} \partial_t^2 v'' \|_0 \| v' \|_\infty + C \| \omega^{1/2+\mu} \partial_t^2 v' \|_0 \| \omega v'' \|_\infty \leq (\hat{E}^{3/8} + 1) \left( \tilde{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right),
\]
where we have used the fact
\[
\| \omega'' \|_\infty \leq C \| \omega \|^{-\mu/4} \| \omega^{1+\mu} u'' \|_{L^0}^{1/4} \| v'' + \omega v''' \|^{3/4} \\
\leq C \| \omega^{1/2+\mu} u'' \|_{L^0}^{1/4} \left( \| v'' \|_{L^0}^{3/4} + \| \omega \|^{-3\mu/4} \| \omega^{1+\mu} u''' \|_{L^0}^{3/4} \right) \\
\leq \hat{E}^{3/8} \left( \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right).
\]

For \( I_{74} \), using (4.27) and (4.24) to estimate \( \| \omega^{1+\mu} \partial_t v'' \|_0 \), we have
\[
I_{74} = \| (\omega^{3/2+\mu} \partial_t v''') (v') \|_0 \leq C \| \omega^{3/2+\mu} \partial_t v'' \|_\infty \| \partial_t v' \|_0 \\
\leq C \| \omega^{3/2+\mu} \partial_t v'' \|_0^{1/4} \| \omega^{3/2+\mu} \partial_t v'' \|_1^{3/4} \| \partial_t v' \|_0 \\
\leq C \| \omega \|^{1/8} \| \omega^{1+\mu} \partial_t v'' \|_0^{1/4} \| \omega^{1/2+\mu} \partial_t v'' + \omega^{3/2} \partial_t v'' \|_0^{3/4} \| \partial_t v' \|_0 \\
\leq \hat{E}^{2} \left( \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right) \right).
\]

For \( I_{75} \), using (4.26) to estimate \( \| v' \|_{L^8} \), we have
\[
I_{75} = \| (\omega^{1/2+\mu} \omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu} \eta'') (v') \|_0 \\
\leq C \| \omega^{1/2+\mu} \|_\infty \| v' \|_{L^8}^{1/4} + C \left\| \int_0^t \omega^{1+\mu} v'' \right\|_0 \| v' \|_{L^\infty}^{4/3} \| \omega \|^{3/2} \\
\leq \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]

Similarly, using (4.26) and (4.27) again, one has
\[
I_{76} = \| (\omega^{1/2+\mu} \omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu} \eta'') \partial_t v' (v')^2 \|_0 \\
\leq C \| \omega^{1/2+\mu} \partial_t v' \|_{L^4} \| v' \|_{L^4} \| \partial_t v' \|_{L^4} \| v' \|_{L^\infty} \| v' \|_{L^\infty} \\
\leq \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]

where we have used (2.13), the method similar to (4.12) to deal with \( \| \omega^{1/2+\mu} \partial_t u' \|_{1/2} \), and (4.24) to estimate \( \| \omega^{1+\mu} \partial_t v' \|_0 \), as well as
\[
\| \omega^{1/2+\mu} \partial_t v' \|_{L^1} \leq C \| \omega^{1/2+\mu} \partial_t v' \|_{1/2} \leq C \| \omega^{1+\mu} \partial_t v' + \omega^{1+\mu} \partial_t v'' \|_0 \\
\leq \hat{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]

and
\[
\left\| \int_0^t \omega^{3/2+\mu} v'' \right\|_{L^4} \leq \left\| \int_0^t \omega^{3/2+\mu} v'' \right\|_1 \leq \left\| \int_0^t (\omega^{1/2+\mu} \omega' v'' + \omega^{3/2+\mu} v''') \right\|_0 \leq C t P \left( \sup_{[0,t]} \hat{E} \right).
\]
For $I_{77}$, using (4.24) to deal with $||\omega^{1/2+\mu}\partial_t^2 v'||_0$ and (4.26), we have

$$I_{77} = ||(\omega^{1/2+\mu}\omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu}\eta'')\partial_t^2 v'||_0$$

$$\leq C||\omega^{1/2+\mu}\partial_t^2 v'||_0||v'||_\infty (1 + ||\int_0^t v''|_0)$$

$$\leq \hat{E}^{1/4} \left( \hat{M}_0 + C \hat{P} \left( \sup_{[0,\tilde{t}]} \hat{E} \right) \right) + C \hat{P} \left( \sup_{[0,\tilde{t}]} \hat{E} \right).$$

For $I_{78}$, using (4.27), we have

$$I_{78} = ||(\omega^{1/2+\mu}\omega' + \omega^{3/2+\mu} + \omega^{3/2+\mu}\eta'')(\partial_t v')^2||_0$$

$$\leq C||\omega^{1/2+\mu}\partial_t v'||_\infty||\partial_t v'||_0 + C||\omega^{3/2+\mu}\partial_t v'||_\infty||\partial_t v'||_0$$

$$+ C \left( ||\omega^{3/2+\mu}\partial_t v'||_1 + ||\omega^{3/2+\mu}\partial_t v'||_1 \right)||\partial_t v'||_0$$

$$\leq C \left( ||\omega^{3/2+\mu}\partial_t v'||_1 + ||\omega^{3/2+\mu}\partial_t v'||_1 \right)||\partial_t v'||_0$$

$$\leq \left( \hat{E}^{1/4} + 1 \right) \left( \hat{M}_0 + C \hat{P} \left( \sup_{[0,\tilde{t}]} \hat{E} \right) \right) + C \hat{P} \left( \sup_{[0,\tilde{t}]} \hat{E} \right),$$

where $\alpha_0 < \alpha < 1$ with $\frac{3}{4} \leq \alpha_0 = \frac{3}{4(1+\mu)} = \frac{3}{2} \left( 1 - \frac{1}{\gamma} \right) < 1$ as $2 \leq \gamma < 3$ and $\frac{1}{\gamma} = \frac{3}{2} - \frac{3}{4\alpha}$, we have used the following estimate:

$$\left\| \left( \omega^{1/2+\mu}\partial_t v' \right)' \right\|_{L^r} \leq C \left\| \frac{\partial_t v'}{\omega^{1/2+\mu}} \right\|_{L^r} + C \left\| \omega^{1/2+\mu}\partial_t v'' \right\|_0$$

$$\leq C \left\| \frac{1}{\omega^{1/2-\mu}} \right\|_{L^\beta} ||\partial_t v'||_{L^\theta} + C ||\omega^{1/2+\mu}\partial_t v''||_0$$

$$\leq C \left\| \frac{1}{\omega^{1/2-\mu}} \right\|_{L^\beta} ||\partial_t v'||_{1/2} + C ||\omega||_{L^\theta} ||\omega^{1/2+\mu}\partial_t v''||_0,$$

where, to ensure $\left\| \frac{1}{\omega^{1/2-\mu}} \right\|_{L^\beta}$ be meaningful, we need $\beta \left( \frac{1}{2} - \mu \right) < 1$, so we choose $\frac{1}{\beta} \in \left( \frac{1}{2} - \mu, \frac{1}{\gamma} \right)$, and $\frac{\beta}{\gamma} = \frac{1}{\beta} + \frac{1}{\beta'}$, with $\beta' > 1$ and $||\partial_t v'||_{L^\beta} \leq ||\partial_t v'||_{1/2}$ in (2.24). This analysis is different from the isentropic case of $\gamma = 2$. Substituting (4.24)-(4.28) into (4.23), we have

$$I_7 \leq \left( \hat{E}^{1/4} + 1 \right) \left( \hat{M}_0 + C \hat{P} \left( \sup_{[0,\tilde{t}]} \hat{E} \right) \right) + C \hat{P} \left( \sup_{[0,\tilde{t}]} \hat{E} \right).$$
Due to
\[
1 - \frac{1}{\eta^{(\gamma + 1)}} = (\gamma + 1) \int_0^t \frac{v'}{\eta^{(\gamma + 2)}},
\]
the last term on the right-hand side of (4.20) can be estimated as
\[
I_8 = \left\| \omega^{-(1/2 + \mu)} \left[ \left( 1 - \frac{1}{(\eta^2)^{\gamma + 1}} \right) \left( \omega^{2 + 2\mu} \partial_t^2 v' e^{S_0} \right) \right] \right\|_0
\leq C \left\| (\gamma + 1) \int_0^t \frac{v'}{(\eta^2)^{\gamma + 2}} \left\| \omega^{1/2 + \mu} \partial_t^3 v' \right\|_0 \omega' \right\|_\infty + \left\| \omega^{1/2 + \mu} \partial_t^3 v'' \right\|_0 \left\| \omega' \right\|_\infty^{1/2}
\]
(4.32)

Substituting (4.21), (4.22), (4.30) and (4.32) into (4.20) leads to
\[
\sup_{t \in [0, \tau]} \left\| \omega^{-(1/2 + \mu)} \left( \omega^{2 + 2\mu} \partial_t^3 v' e^{S_0} \right) \right\|_0^2
\leq C \left( \hat{E}^n/2 + 1 \right)^2 \left( \dot{M}_0 + C t \left( \sup_{[0, \tau]} \hat{E} \right) \right) + C t \left( \sup_{[0, \tau]} \hat{E} \right).
\]
(4.33)

Expanding the left-hand side of (4.33) and using physical vacuum condition (1.7), we have
\[
\sup_{t \in [0, \tau]} \left\| \omega^{-(1/2 + \mu)} \left( \omega^{2 + 2\mu} \partial_t^3 v' e^{S_0} \right) \right\|_0^2 \geq 4(1 + \mu)^2 \left\| \omega^{1/2 + \mu} \omega' \partial_t^3 v' e^{S_0} \right\|_0^2 + \left\| \omega^{3/2 + \mu} \partial_t^3 v'' e^{S_0} \right\|_0^2
\]
\[
+ \left\| \omega^{3/2 + \mu} \partial_t^3 v' e^{S_0} S_0' \omega^{-\mu} \right\|_0^2
+ 4(1 + \mu) \int_t^0 \omega^{2 + 2\mu} \partial_t^3 v' \partial_t^3 v'' \omega' \exp 2S_0
dataction
+ 4(1 + \mu) \int_t^0 \omega^{2 + 2\mu} \partial_t^3 v' \partial_t^3 v'' \omega' \exp 2S_0 S_0'
dataction
+ 2 \int_t^0 \omega^{3 + 2\mu} \partial_t^3 v' \partial_t^3 v'' \exp 2S_0
\geq C \left\| \omega^{1/2 + \mu} \partial_t^3 v' \right\|_0^2 + C \left\| \omega^{3/2 + \mu} \partial_t^3 v'' \right\|_0^2
\]
\[
-M_0 - C t \left( \sup_{[0, \tau]} \hat{E} \right),
\]
and thus, by (4.33),
\[
\sup_{[0, \tau]} \left( \left\| \omega^{1/2 + \mu} \partial_t^3 v' \right\|_0^2 + \left\| \omega^{3/2 + \mu} \partial_t^3 v'' \right\|_0^2 \right)
\leq C \left( \hat{E}^n + 1 \right) \left( \dot{M}_0 + C t \left( \sup_{[0, \tau]} \hat{E} \right) \right) + C t \left( \sup_{[0, \tau]} \hat{E} \right).
\]
(4.34)
Choosing the multiplier \( \omega^{-(1/2+\mu)} \) and letting the first term \( \omega^{1+2\mu}\partial_t v' \) in \( g \) to be \( \omega^{1+2\mu}\partial_t v' \), using the same method as that of proving (1.17), we can prove (1.18).

Similarly, multiplying (4.16) both sides by \( \omega^{-\mu} \) and replacing the first term \( \omega^{1+2\mu}\partial_t^k v \) in \( g \) by \( \omega^{1+2\mu}\partial_t^k v', \omega^{1+2\mu}\partial_t^2 v' \), respectively, one has

**Proposition 4.4.** For \( 2 \leq \gamma < 3 \), there exists some \( \alpha \in (0,1) \), such that the following estimate holds,

\[
\sup_{[0,t]} \left( ||\omega^{1+\mu}\partial_t^2 v'||_0^2 + ||\omega^{2+\mu}\partial_t^2 v''||_0^2 + ||\omega^{2+\mu}\partial_t^2 v'''||_0^2 + ||\omega^{1+\mu}\partial_t^2 v'||_0^2 + ||\omega^{1+\mu}v'||_0^2 \right)
\]

\[
\leq C\left( \tilde{E}^\alpha + 1 \right) \left( \tilde{M}_0 + tP\left( \sup_{[0,\tilde{t}]} \tilde{E} \right) \right) + CtP\left( \sup_{[0,\tilde{t}]} \tilde{E} \right).
\]

5. Uniform estimates of \( v^\varepsilon \) for \( 1 < \gamma < 2 \)

In this section, we shall prove the uniform estimates for the case \( 1 < \gamma < 2 \). It should be noted that the value of \( \gamma \) determines the rate of degeneracy near the vacuum boundary, since \( \rho_0 \) appears as the coefficient of \( \partial_t v \) in (2.5) and the physical vacuum condition indicates that

\[
\rho_0(x) \sim \text{dist}(x, \partial I)^{-\frac{1}{2}}, \quad x \to \partial I.
\]

Thus, the smaller the value of \( \gamma \) is, the more degenerate the equation (2.5) near the vacuum boundary. We also need higher-order derivatives in the energy function to control the \( H^2 \)-norm of \( v \) (and thus the \( C^1 \)-norm of \( v \)) for smaller \( \gamma \). Since from the embedding inequality (2.13), the higher energy function \( \tilde{E}(t) \) defined in (2.15) for \( 1 < \gamma < 2 \) implies that

\[
||v||_2^2 \leq ||v||_2^{2-\gamma} \leq C \sum_{i=0}^{l+1} ||\omega^{1/2+\mu}\partial_t^{i+1} v||_0^2 \leq C \tilde{E}, \quad l = 3 + 2\left[ \frac{1}{2} + \mu \right],
\]

which indicates that the high-order energy function \( \tilde{E} \) is suitable for the study of the physical vacuum problem (2.5) when \( \gamma \in (1,2) \), and as \( \gamma \to 1 \) (which means \( \mu \to \infty \)) the estimate of \( ||v||_2^2 \) needs infinite higher-order derivatives.

5.1. Energy estimates. In order to obtain a series of estimates independent of \( \varepsilon \), we first need some energy estimates as in Propositions 4.1-4.2.

**Proposition 5.1.** For \( \gamma \in (1,2) \), we have the following \( \varepsilon \)-independent energy estimate for \( \partial_t^{l+1} \)-problem of (3.2),

\[
||\omega^{1/2+\mu}\partial_t^{l+1} v||_0^2 + ||\omega^{1+\mu}\partial_t^l v'||_0^2 + ||\omega^{1+\mu}\partial_t^l v'||_0^2 + \varepsilon \int_0^t ||\omega^{1+\mu}\partial_t^{l+1} v'||_0^2
\]

\[
\leq \tilde{E}^\alpha \left( \tilde{M}_0 + tP\left( \sup_{[0,\tilde{t}]} \tilde{E} \right) \right),
\]

with \( \tilde{M}_0 = P(\tilde{E}(0)) \) and \( P(\cdot) \) some polynomial function.
Proof. Similar to the derivation of (4.6), we first take \((l+1)\)th time derivative of equation (3.2), then multiply it by \(\partial_t^{l+1} v\) and integrate this resulting equation with respect to time and space to get

\[
\frac{1}{2} \frac{d}{dt} \int_I \omega^{1+2\mu} |\partial_t^{l+1} v|^2 - \int_I \partial_t^{l+1} \left( \frac{\omega^{2+2\mu}}{(\eta')^\gamma} \right) \partial_t^{l+1} v' e^{S_0} + \varepsilon \int_I \omega^{1+\mu} |\partial_t^{l+1} v'|^2 e^{S_0} = 0. \tag{5.1}
\]

By the similar way in (4.7) to deal with the second item on the left-hand side of (5.1), we get

\[
\frac{1}{2} \int_I \omega^{1+2\mu} |\partial_t^{l+1} v|^2 + \frac{\gamma}{2} \int_I \frac{\omega^{2+2\mu}}{(\eta')^\gamma+1} |\partial_t^{l} v|^2 e^{S_0} + \varepsilon \int_0^t \int_I \omega^{1+\mu} |\partial_t^{l+1} v'|^2 e^{S_0} \\
= \frac{1}{2} \int_I \omega^{1+2\mu} |\partial_t^{l+1} v_0|^2 + \frac{\gamma}{2} \int_I \frac{\omega^{2+2\mu}}{(\eta')^\gamma+1} |\partial_t^{l} v_0|^2 e^{S_0} \\
- \frac{\gamma + 1}{\gamma} \int_0^t \int_I \omega^{2+2\mu} \frac{v'}{(\eta')^\gamma+2} |\partial_t^{l-1} v'|^2 e^{S_0} \\
- \sum_{i=1}^l c_i \int_0^t \int_I \omega^{2+2\mu} \frac{1}{(\eta')^\gamma+1} |\partial_t^{l-i} v' \partial_t^{l+1} v'| e^{S_0} \\
= \sum_{i=1}^4 I_i.
\]

It is obvious that \(I_1, I_2\) can be controlled by \(\tilde{M}_0\). Now we estimate \(I_3\) and \(I_4\). Similar to (4.9), we have

\[
I_3 = -\frac{\gamma}{\gamma + 1} \int_0^t \frac{\omega^{2+2\mu} v'}{(\eta')^\gamma+2} |\partial_t^{l} v'|^2 e^{S_0} \\
\leq C \int_0^t ||v'||_{\infty} ||\omega^{1+\mu} \partial_t^{l} v'||^2_0 \leq C \int_0^t ||v'||_2 ||\omega^{1+\mu} \partial_t^{l} v'||^2_0 \\
\leq CtP \left( \sup_{[0,t]} \tilde{E} \right).
\tag{5.3}
\]

Using integrating by parts in time, we have

\[
I_4 = -\sum_{i=1}^l c_i \int_0^t \int_I \omega^{2+2\mu} \frac{1}{(\eta')^\gamma+1} |\partial_t^{l-i} v' \partial_t^{l+1} v'| e^{S_0} \\
= \sum_{i=1}^l c_i \int_0^t \int_I \omega^{2+2\mu} \frac{1}{(\eta')^\gamma+1} |\partial_t^{l-i} v' \partial_t^{l+1} v'| e^{S_0} \\
- \sum_{i=1}^l c_i \int_0^t \int_I \omega^{2+2\mu} \frac{1}{(\eta')^\gamma+1} |\partial_t^{l-i} v' \partial_t^{l+1} v'| e^{S_0} |^t_0 \\
= I_{41} + I_{42}.
\tag{5.4}
\]
Proposition 5.2. For following elliptic estimates. Now we prove the energy estimate:

We note that (5.7) also implies the inequality (4.4) which will be usually used in the

To show clearly the idea of proving the estimate (5.5), we will take 1 < γ = \frac{3}{2} < 2 as an example in the Subsection 5.2. Using the similar method to (4.13), for 0 < α < 1, we have

where \( \tilde{M}_0 = P(\tilde{E}(0)) \). Substituting (5.3)-(5.6) into (5.2), and using the same derivation as (4.13), we can prove Proposition 5.1.

5.2. The case of γ = \frac{3}{2}. When γ = \frac{3}{2}, then \( \mu = \frac{1}{2}, l = 5 \). The higher-order energy function \( \tilde{E}(t) \) in (2.13) is

We note that (5.7) also implies the inequality (4.14) which will be usually used in the following elliptic estimates. Now we prove the energy estimate:

**Proposition 5.2.** For γ = \frac{3}{2}, there exists a constant α ∈ (0,1), such that one has the following ε-independent energy estimate on the \( \partial_t^6 \)-problem of (3.2),

where \( \tilde{M}_0 = P(\tilde{E}(0)) \) and \( P(\cdot) \) some polynomial function.

**Proof.** In the above energy estimate for Proposition 5.1, except for the term \( I_{41} \) in (5.5), the others are the same as the proof of Proposition 5.1. Now we focus on the estimates of...
$I_{41}$. After detailed computations, $I_{41}$ is equal to the sum of the following terms:

$$J := \int_0^t \sum_{i=1}^{14} J_i dt, \quad \text{with} \quad J_i = \int_I R(y') j_i \omega^{2+2\mu} \partial_1 v' dx,$$

where the terms $j_i, i = 1, 2, \ldots, 14$ are the functions of $v', \partial_1^k v', k = 1, 2, 3, 4$ of the following form:

- $j_1 = v'$,
- $j_2 = \partial_1 v'(v')^2$,
- $j_3 = \partial_1 v' \partial_t v'$,
- $j_4 = \partial_1^2 v'(v')^3$,
- $j_5 = \partial_1^2 v' \partial_t v'$,
- $j_6 = \partial_1^3 v' \partial_1^2 v'$,
- $j_7 = \partial_1^2 v'(v')^4$,
- $j_8 = \partial_1^2 v' \partial_t v'(v')^2$,
- $j_9 = \partial_t v'(v')^5$,
- $j_{10} = \partial_1 v'(v')^5$,
- $j_{11} = (\partial_t v')^2(v')^3$,
- $j_{12} = (\partial_t v')^2 \partial^2_1 v'$,
- $j_{13} = (v')^7$,
- $j_{14} = (\partial_1^2 v')^2 v'$.

For $J_1$, we have

$$|J_1| \leq C \omega^{1+\mu} \partial_1^5 v' \|0 \|v'| \|\|_\infty \leq CP \left( \sup_{0,t} \hat{E} \right).$$

For $J_2, J_3, J_4$ we have

- $|J_2| \leq C \omega^{1+\mu} \partial_1^5 v' \|0 \|v'| \|\|_\infty \leq CP \left( \sup_{0,t} \hat{E} \right)$,
- $|J_3| \leq C \omega^{1+\mu} \partial_1^5 v' \|0 \|v'| \|\|_\infty \leq CP \left( \sup_{0,t} \hat{E} \right)$,
- $|J_4| \leq C \omega^{1+\mu} \partial_1^5 v' \|0 \|v'| \|\|_\infty \leq CP \left( \sup_{0,t} \hat{E} \right)$.

For $J_5$, we have

$$|J_5| \leq C \omega^{1+\mu} \partial_1^5 v' \|0 \|v'| \|\|_\infty \leq C \omega^{1+\mu} \partial_1^5 v' \|0 \|v'| \|\|_\infty \leq \hat{M}_0 + C t P \left( \sup_{0,t} \hat{E} \right),$$

where we have used the similar method to (4.12) to deal with $||\omega^{1+\mu} \partial^2_1 v'||_{1/2}$. Similarly, we have

- $|J_6| \leq C \omega^{1+\mu} \partial_1^2 v' \|0 \|v'| \|\|_\infty \leq \hat{M}_0 + C t P \left( \sup_{0,t} \hat{E} \right)$,
- $|J_7| \leq C \omega^{1+\mu} \partial_1^2 v' \|0 \|v'| \|\|_\infty \leq \hat{M}_0 + C t P \left( \sup_{0,t} \hat{E} \right)$,
- $|J_8| \leq C \omega^{1+\mu} \partial_1^2 v' \|0 \|v'| \|\|_\infty \leq \hat{M}_0 + C t P \left( \sup_{0,t} \hat{E} \right)$,
where we have used the fact
\[ \omega^{1+\mu} \partial_t^2 v' = \omega^{1+\mu} u_2' + \int_0^t \omega^{1+\mu} \partial_t^3 v', \]
and the similar method to \((1.12)\) to deal with \( \| \omega^{1+\mu} \partial_t^2 v' \|_{1/2} \).

Using \((2.3)\), \( J_9, J_{10}, J_{11}, J_{12}, J_{13} \) can be estimated as
\[
\begin{align*}
|J_9| &\leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| \partial_t v' \|_{L^4} \| v' \|_{L^4} \| v' \|_{L^4}^4 \leq \tilde{M}_0 + CP \left( \sup_{[0,t]} \tilde{E} \right), \\
|J_{10}| &\leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| \partial_t v' \|_{L^6} \| v' \|_{L^6}^4 \leq \tilde{M}_0 + CP \left( \sup_{[0,t]} \tilde{E} \right), \\
|J_{11}| &\leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| \partial_t v' \|_{L^6} \| v' \|_{L^6}^3 \leq \tilde{M}_0 + CP \left( \sup_{[0,t]} \tilde{E} \right), \\
|J_{12}| &\leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| \partial_t v' \|_{L^6}^2 \| \omega^{1+\mu} \partial_t^2 v' \|_{L^1} \leq \tilde{M}_0 + CP \left( \sup_{[0,t]} \tilde{E} \right), \\
|J_{13}| &\leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| v' \|_0 \| v' \|_0^6 \leq CP \left( \sup_{[0,t]} \tilde{E} \right).
\end{align*}
\]

For \( J_{14} \), we have
\[
|J_{14}| \leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| \omega^{1+\mu} \partial_t^2 v' \|_{L^6} \| v' \|_{L^6} \| \partial_t v' \|_0 \\
\leq C \| \omega^{1+\mu} \partial_t^5 v' \|_0 \| \omega^{1+\mu} \partial_t^2 v' \|_{L^6} + \omega^{1+\mu} \partial_t^2 v' \|_0 \| v' \|_{L^6} \| \partial_t v' \|_0 \\
\leq CP \left( \sup_{[0,t]} \tilde{E} \right).
\]

Thus, using the same argument as Proposition \((4.1)\) for \( 0 < \alpha < 1 \), we obtain \((5.8)\). \( \square \)

Similarly, we have

\textbf{Proposition 5.3.} For \( \gamma = \frac{3}{2} \), there exists a constant \( \alpha \in (0,1) \), such that one has the following \( \varepsilon \)-dependent energy estimates:
\[
\begin{align*}
\left( \omega^{1+\mu} \partial_t^3 v' \right)_0^2 + \left( \omega^{1+\mu} \partial_t^3 v' \right)_0^2 + \| \omega^{1+\mu} \partial_t^3 v' \|_0 + \varepsilon \int_0^t \left( \omega^{1+\mu} \partial_t^4 v' \right)_0^2 \\
\leq \tilde{E}^\alpha \left( \tilde{M}_0 + CP \left( \sup_{[0,t]} \tilde{E} \right) \right),
\end{align*}
\]

\[
\begin{align*}
\left( \omega^{1+\mu} \partial_t^2 v' \right)_0^2 + \left( \omega^{1+\mu} \partial_t v' \right)_0^2 + \| \omega^{1+\mu} \partial_t v' \|_0 + \varepsilon \int_0^t \left( \omega^{1+\mu} \partial_t^2 v' \right)_0^2 \\
\leq \tilde{E}^\alpha \left( \tilde{M}_0 + CP \left( \sup_{[0,t]} \tilde{E} \right) \right).
\end{align*}
\]

5.3. Estimates of higher order spatial derivatives for \( \gamma = \frac{3}{2} \). Based on the energy estimate \((5.8)\), by elliptic estimates we can derive the estimates of the higher order spatial derivatives associated with the weights.
Proposition 5.4. For $\gamma = \frac{3}{2}$, there exists a constant $\alpha \in (0, 1)$, such that one has the following estimates:

\[
\sup_{[0,t]} \left( \left| \omega^{1/2+\mu} \partial_t^4 v' \right|_0^2 + \left| \omega^{3/2+\mu} \partial_t^4 v''' \right|_0^2 \right) \leq C \left( \tilde{E}^\alpha + 1 \right) \left( \tilde{M}_0 + C \mathcal{P} \left( \sup_{[0,t]} \tilde{E} \right) \right), \tag{5.11}
\]

\[
\sup_{[0,t]} \left( \left| \omega^{1/2+\mu} \partial_t^2 v'' \right|_0^2 + \left| \omega^{1/2+\mu} \partial_t^2 v'' \right|_0^2 + \left| \omega^{1/2+\mu} v'' \right|_0^2 + \left| \omega^{1/2+\mu} v'' \right|_0^2 \right)
+ \left| \omega^{1/2+\mu} v''' \right|_0^2 + \left| \omega^{3/2+\mu} \partial_t^4 v' \right|_0^2 \leq C \left( \tilde{E}^\alpha + 1 \right) \left( \tilde{M}_0 + C \mathcal{P} \left( \sup_{[0,t]} \tilde{E} \right) \right).
\tag{5.12}
\]

**Proof.** Choosing $k = 4$ in (4.16), using the first term of (5.8), multiplying (4.16) both sides by $\omega^{-1/2+\mu}$, then, using Lemma 2.4 and fundamental theorem of calculus, we obtain that, for any $t \in [0, T^\varepsilon]$,

\[
\sup_{[0,t]} \left| \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^4 v' e^S \right)' \right|_0^2 \leq C \sup_{[0,t]} \left| \omega^{1/2+\mu} \partial_t^0 v \right|_0^2
+ C \sup_{[0,t]} \left| \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^\alpha \frac{1}{(\eta')^\gamma+1} \partial_t^{\alpha-\alpha} v' e^S \right)' \right|_0^2
+ C \sup_{[0,t]} \left| \omega^{-(1/2+\mu)} \left( 1 - \frac{1}{(\eta')^\gamma+1} \right) \left( \omega^{2+2\mu} \partial_t^4 v' e^S \right)' \right|_0^2
+ C \sup_{[0,t]} \left| \omega^{3/2+\mu} \partial_t^4 v' e^S \right|_0^2 \leq \sum_{i=1}^4 I_i.
\tag{5.13}
\]

We next estimate each term on the right-hand side of (5.13). For the first term, we use the estimate (5.8) to obtain, for each $t \in [0, T_k]$,

\[
I_1 = \sup_{[0,t]} \left| \omega^{1/2+\mu} \partial_t^0 v \right|_0 \leq \tilde{E}^\alpha \left( \tilde{M}_0 + C \mathcal{P} \left( \sup_{[0,t]} \tilde{E} \right) \right).
\tag{5.14}
\]

The remaining terms will be estimated by using the definition of the energy function $\tilde{E}$. For the second term, from (4.31), after detailed computations, we have

\[
I_2 = \sum_{\alpha=1}^4 \left| \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^\alpha \frac{1}{(\eta')^\gamma+1} \partial_t^{\alpha-\alpha} v' e^S \right)' \right|_0^2
= \sum_{\alpha=1}^4 \left| - \frac{1}{\gamma+1} \omega^{-(1/2+\mu)} \left( \omega^{2+2\mu} \partial_t^\alpha \int_0^t \left( \frac{\partial_t^{\alpha-\alpha} v' e^S}{(\eta')^\gamma+1} \right) \partial_t^{\alpha-\alpha} v' e^S \right)' \right|_0^2 \leq \sum_{i=1}^{75} K_i,
\tag{5.15}
\]
where we have used the fact $1/2 \leq \eta' \leq 3/2$ and $S \leq S'_0 \leq \overline{S}$. For $i = 1, 2, \cdots, 17$, $K_i = C \| (\omega_1^{1/2+\mu} + \omega_3^{3/2+\mu}) k_i \|_0$ with

\[
k_1 = \frac{\partial^3 v}{\partial t^3 v} \int_0^t (v')^2, \quad k_2 = \frac{\partial^3 v}{\partial t^3 v} \int_0^t \partial_t v', \quad k_3 = \frac{\partial^2 v}{\partial t^2 v} \int_0^t (v')^3, \quad k_4 = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v' \partial_t v', \\
k_5 = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^2 v', \quad k_6 = \partial_t v' \int_0^t (v')^4, \quad k_7 = \frac{\partial_t v}{\partial t v} \int_0^t (v')^2 \partial_t v', \quad k_8 = \partial_t v' \int_0^t (\partial_t v')^2, \\
k_9 = \partial_t v' \int_0^t v' \partial_t^2 v', \quad k_{10} = \partial_t v' \int_0^t \partial_t^3 v', \quad k_{11} = \frac{v'}{\partial_t v} \int_0^t (v')^5, \quad k_{12} = \frac{v'}{\partial_t v} \int_0^t (v')^3 \partial_t v', \\
k_{13} = \frac{v'}{\partial_t v} \int_0^t (v')^2 \partial_t^2 v', \quad k_{14} = \frac{v'}{\partial_t v} \int_0^t (\partial_t v')^2, \quad k_{15} = \frac{v'}{\partial_t v} \int_0^t \partial_t v' \partial_t^2 v', \quad k_{16} = \frac{v}{\partial_t v} \int_0^t v' \partial_t^3 v', \\
k_{17} = \frac{v}{\partial_t v} \int_0^t \partial_t^4 v'.
\]

For $i = 18, 19, \cdots, 58$, $K_i = C \| \omega_3^{3/2+\mu} k_i \|_0$ with

\[
k_{18} = \frac{\partial^3 v}{\partial t^3 v} \int_0^t v' v'', \quad k_{19} = \frac{\partial^3 v}{\partial t^3 v} \int_0^t \partial_t v'', \quad k_{20} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t (v')^2 v'', \quad k_{21} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' \partial_t v', \\
k_{22} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^2 v'', \quad k_{23} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v' \partial_t v''', \quad k_{24} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^3 v''', \quad k_{25} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t (v')^3 v''', \\
k_{26} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' \partial_t v''', \quad k_{27} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v' \partial_t v''''', \quad k_{28} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t v' \partial_t v''''', \quad k_{29} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^2 v' \partial_t v''''', \\
k_{30} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' \partial_t v''''', \quad k_{31} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^2 v'''', \quad k_{32} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t (v')^4 v'''', \quad k_{33} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t (v')^2 v''''', \quad k_{34} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''''', \quad k_{35} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v' \partial_t v''''', \quad k_{36} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t v' \partial_t v''''', \quad k_{37} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''''', \quad k_{38} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''''', \quad k_{39} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''''', \\
k_{40} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''''', \quad k_{41} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''''', \quad k_{42} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^3 v'''', \quad k_{43} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^4 v''''', \\
k_{44} = \frac{\partial^3 v}{\partial t^3 v} \int_0^t (v')^2, \quad k_{45} = \frac{\partial^3 v}{\partial t^3 v} \int_0^t \partial_t v', \quad k_{46} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t (v')^3, \quad k_{47} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t v'' v''', \quad k_{48} = \frac{\partial^2 v}{\partial t^2 v} \int_0^t \partial_t^2 v'', \quad k_{49} = \frac{\partial v}{\partial t v} \int_0^t (v')^4, \quad k_{50} = \frac{\partial v}{\partial t v} \int_0^t (v')^2 \partial_t v', \quad k_{51} = \frac{\partial v}{\partial t v} \int_0^t (\partial_t v')^2, \\
k_{52} = \frac{\partial v}{\partial t v} \int_0^t v' \partial_t^2 v', \quad k_{53} = \frac{\partial v}{\partial t v} \int_0^t \partial_t^3 v', \quad k_{54} = \frac{\partial v}{\partial t v} \int_0^t (v')^2 (\partial_t v')^2, \quad k_{55} = \frac{\partial v}{\partial t v} \int_0^t (v')^2 \partial_t^2 v', \quad k_{56} = \frac{\partial v}{\partial t v} \int_0^t \partial_t v' \partial_t^2 v', \quad k_{57} = \frac{\partial v}{\partial t v} \int_0^t v' \partial_t^3 v', \quad k_{58} = \frac{\partial v}{\partial t v} \int_0^t \partial_t^4 v'.
\]
For $i = 59, 60, \cdots, 75$, $K_i = C \| \omega^{3/2+\mu} k_i \|_0$ with

\[
k_{59} = \partial_t^3 v' \int_0^t (v')^2 \eta'' \, dt, \quad k_{60} = \partial_t^3 v' \int_0^t (v')^2 \eta'' \, dt, \quad k_{61} = \partial_t^3 v' \int_0^t (v')^3 \eta'' \, dt, \quad k_{62} = \partial_t^3 v' \int_0^t \partial_v v' \eta'' \, dt,
\]
\[
k_{63} = \partial_t^3 v' \int_0^t (\partial_t^2 v')^2 \eta'' \, dt, \quad k_{64} = \partial_t^3 v' \int_0^t (v')^4 \eta'' \, dt, \quad k_{65} = \partial_t v' \int_0^t (v')^2 \eta'' \, dt, \quad k_{66} = \partial_t v' \int_0^t (\partial_t v')^2 \eta'' \, dt,
\]
\[
k_{67} = \partial_t v' \int_0^t v' \partial_t^2 v' \eta'' \, dt, \quad k_{68} = \partial_t v' \int_0^t \partial_t^3 v' \eta'' \, dt, \quad k_{69} = \partial_t v' \int_0^t (v')^3 \eta'' \, dt, \quad k_{70} = \partial_t v' \int_0^t (v')^5 \eta'' \, dt,
\]
\[
k_{71} = \partial_t^3 v' \int_0^t \partial_t v' \eta'' \, dt, \quad k_{72} = \partial_t^3 v' \int_0^t (v')^2 \partial_t v' \eta'' \, dt, \quad k_{73} = \partial_t^3 v' \int_0^t \partial_t v' \partial_t^2 v' \eta'' \, dt, \quad k_{74} = \partial_t^3 v' \int_0^t v' \partial_t^3 v' \eta'' \, dt,
\]
\[
k_{75} = \partial_t^3 v' \int_0^t \partial_t^3 v' \eta''.
\]

Before deriving the estimates of $K_i, i = 1, 2, \cdots, 75$, similar to (4.24), one has

\[
\sum_{j=1}^{3} \left\{ \| \omega^{3/2+\mu} \partial_t^{5-2j} \partial_x^{j+1} v \|_0^2 + \sum_{i=1}^{j} \| \omega^{1/2+\mu} \partial_t^{5-2j} \partial_x^i v \|_0^2 + \sum_{i=-1}^{j} \| \omega^{1/2+\mu} \partial_t^{5-2j} \partial_x^i v \|_0^2 \right\}
\]
\[
\leq \bar{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right).
\]

(5.16)

For $K_1, K_2, K_4, K_5$, using the estimates (5.16), we have

\[
K_1 = C \left\| (\omega^{1/2+\mu} + \omega^{3/2+\mu}) \partial_t^3 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \partial_t^3 v' \right\|_0 \int_0^t \| v' \|_\infty^2 \leq \bar{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]
\[
K_2 = C \left\| (\omega^{1/2+\mu} + \omega^{3/2+\mu}) \partial_t^3 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \partial_t^3 v' \right\|_0 \int_0^t \| v' \|_\infty \leq C \bar{E}^{1/4} \left( \bar{M}_0 + t P \left( \sup_{[0,t]} \hat{E} \right) \right),
\]
\[
K_4 = C \left\| (\omega^{1/2+\mu} + \omega^{3/2+\mu}) \partial_t^2 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_0 \int_0^t \| v' \|_\infty \leq \bar{M}_0 + C t P \left( \sup_{[0,t]} \hat{E} \right),
\]
\[
K_5 = C \left\| (\omega^{1/2+\mu} + \omega^{3/2+\mu}) \partial_t^2 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_\infty \int_0^t \| \partial_t^2 v' \|_0 \leq C t P \left( \sup_{[0,t]} \hat{E} \right),
\]

(5.17)
For $K_7$, $K_8$, $K_9$

$$K_7 = C \left\| \left( \omega^{1/2+\mu} + \omega^{3/2+\mu} \right) \partial_t v' \int_0^t (v')^2 \partial_t v' \right\|_0 \leq C \int_0^t \left\| \omega^{1/2+\mu} \partial_t v' \right\|_{L^8} \left\| \partial_t v' \right\|_{L^8} \left\| v' \right\|_{L^8}^2$$

$$\leq C t P \left( \sup_{[0,t]} \bar{E} \right),$$

$$K_8 = C \left\| \left( \omega^{1/2+\mu} + \omega^{3/2+\mu} \right) \partial_t v' \int_0^t (\partial_t v')^2 \right\|_0 \leq C \int_0^t \left\| \omega^{1/2+\mu} \partial_t v' \right\|_{L^8} \left\| \partial_t v' \right\|_{L^8}^2$$

$$\leq C t P \left( \sup_{[0,t]} \bar{E} \right),$$

$$K_9 = \left\| \left( \omega^{1/2+\mu} + \omega^{3/2+\mu} \right) \partial_t v' \int_0^t \partial_t^2 v' v' \right\|_0 \leq C \int_0^t \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{\infty} \left\| \partial_t v' \right\|_0 \left\| v' \right\|_{\infty}$$

$$\leq C t P \left( \sup_{[0,t]} \bar{E} \right).$$

For $K_{10}$, using the same method as dealing with $I_{78}$ in (4.27) and (4.29), we have

$$K_{10} = \left\| \left( \omega^{1/2+\mu} + \omega^{3/2+\mu} \right) \partial_t v' \int_0^t \partial_t^3 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{L^8} \left\| \partial_t v' \right\|_0$$

$$\leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{3/4} \left\| \partial_t v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{1-\alpha} \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{L^8} \left\| \partial_t v' \right\|_0$$

$$\leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{1-\alpha} \left( \left\| \omega^{1-2} \partial_t^2 v' \right\|_{L^8} + \left\| \omega^{1/2+\mu} \partial_t^2 v'' \right\|_{L^8} \right) \left\| \partial_t v' \right\|_0$$

$$\leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{1-\alpha} \left( \frac{1}{\sqrt{\omega}} \left\| \partial_t^2 v' \right\|_{L^8}^\alpha \right) \left\| \partial_t v' \right\|_0$$

$$\leq C \left\| \omega^{1/2+\mu} \partial_t^2 v' \right\|_{1-\alpha} \left( \frac{1}{\sqrt{\omega}} \left\| \partial_t^2 v' \right\|_{L^8}^\alpha \right) \left\| \partial_t v' \right\|_0 \leq C \bar{E} \alpha.$$

To ensure $\frac{1}{2} \beta < 1$, here we choose $\frac{1}{2} \beta \in \left( \frac{1}{4}, \frac{1}{4} \right)$, and $\alpha_0 < \alpha < 1$ with $0 \leq \alpha_0 = \frac{3}{4(1+\mu)} < \frac{3}{4} < \frac{3}{4(1+\mu)}$ as $1 < \gamma < 2$ and $\frac{1}{2} = \frac{3}{2} - \frac{3}{4\gamma}$. Similarly,

$$K_{19} = \left\| \omega^{3/2+\mu} \partial_t^3 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} v'' \right\|_0 \left\| \omega^{1/2+\mu} \partial_t^3 v' \right\|_0 \leq C \bar{E} \alpha,$$

$$K_{53} = \left\| \omega^{3/2+\mu} \partial_t v'' \partial_t^2 v' \right\|_0 \leq C \left\| \omega^{1/2+\mu} \int_0^t \partial_t^2 v' \right\|_{L^8} \left\| \omega^{1/2+\mu} \partial_t v'' \right\|_0 \leq C \bar{E} \alpha.$$

For $K_{18}$, we have

$$K_{18} = \left\| \omega^{3/2+\mu} \partial_t^3 v' \int_0^t v'' v' \right\|_0 \leq C \int_0^t \left\| \omega^{3/2+\mu} \partial_t^3 v'' \right\|_{L^8} \left\| v'' \right\|_0 \left\| v' \right\|_0 \leq \tilde{M}_0 + C t P \left( \sup_{[0,t]} \bar{E} \right).$$

For $K_{28}$, we have

$$K_{28} = \left\| \omega^{3/2+\mu} \partial_t v' \partial_t v'' \right\|_0 \leq C \int_0^t \left\| \omega^{3/2+\mu} \partial_t v'' \right\|_{L^4} \left\| \partial_t v' \right\|_0 \leq \tilde{M}_0 + C t P \left( \sup_{[0,t]} \bar{E} \right).$$
By detailed analysis, we find that the estimates of $K_i, i = 3, 6, 11, 12, 13, 16, 17, 43, 44, 45, 49$ are the same as $K_1$; the estimates of $K_i, i = 14, 15, 38$, are the same as $K_4$; the estimates of $K_i, i = 45, 46, 47, 49, 50, 51, 52, 53$ are the same as $K_5$; the estimates of $K_i, i = 9, 23, 39, 40, 48, 50, 52$ are the same as $K_9$; the estimates of $K_i, i = 20, 21, 22, 24 – 27, 29 – 37, 39 – 42, 47, 54 – 58$ are the same as $K_{18}$, thus we omit them. Due to $\eta'' = \int_0^t \nu''$, similar to $K_{18}$, we also have

$$K_{99} = C \left\| \omega^{3/2+\mu} \partial_t^3 v' \right\|_0 \leq \int_0^t \left\| \omega^{3/2+\mu} \partial_t^3 v' \right\|_0 \leq \int_0^t \left\| \omega^{3/2+\mu} \partial_t^3 v' \right\|_0 \leq \tilde{M}_0 + CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right).$$

Similarly, we obtain the estimates of $K_i, i = 60 – 75$. Plugging $5.17 – 5.18$ into $5.15$, one has

$$I_2 \leq \tilde{E}^\alpha \left( \tilde{M}_0 + CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right) \right) + CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right).$$

From $4.31$, the third term $I_3$ on the right-hand side of $5.13$ can be controlled by

$$I_3 = \left\| \omega \left( 1 - \frac{1}{(\eta')^\gamma + 1} \right) \left( \omega^{2+2\mu} \partial_t^4 v' \right)^\gamma \right\|_0 \leq CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right).$$

The last term of $5.13$ can be estimated as

$$I_4 \leq C \left\| \omega^{2+\mu} \partial_t^4 v' \right\|_0 \leq CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right).$$

Substituting $5.14$, $5.19$ and $5.20$ into $5.13$, we have

$$\left\| \omega^{-\left(1/2+\mu\right)} \left( \omega^{2+2\mu} \partial_t^4 v' \right)^\gamma \right\|_0 \leq C \left( \tilde{E}^\alpha + 1 \right) \left( \tilde{M}_0 + tP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right) \right) + CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right).$$

Similarly to $4.33 – 4.34$, we have

$$\sup_{[0,\tilde{t}]} \left( \left\| \omega^{3/2+\mu} \partial_t^3 v' \right\|_0 + \left\| \omega^{3/2+\mu} \partial_t^3 v'' \right\|_0 \right) \leq C \left( 1 + \tilde{E}^\alpha \right) \left( \tilde{M}_0 + tP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right) \right) + CtP \left( \sup_{[0,\tilde{t}]} \tilde{E} \right).$$

Choosing the multipliers $\omega^{-\left(1/2+\mu\right)}$, and replacing the first term $\omega^{1/2+\mu} \partial_t^5 v$ on the right hand-side of $5.13$ by $\omega^{1/2+\mu} \partial_t^5 v, \omega^{1/2+\mu} \partial_t^5 v', \omega^{1/2+\mu} \partial_t^5 v'\gamma, \omega^{1/2+\mu} \partial_t^5 v'', \omega^{1/2+\mu} \partial_t^5 v''\gamma$, respectively, we can obtain the estimates $5.12$.\]

Furthermore, choosing the multiplier $\omega^{-\mu}$, and replacing $\omega^{1/2+\mu} \partial_t^5 v$ in $5.13$ by $\omega^{1/2+\mu} \partial_t^5 v, \omega^{1/2+\mu} \partial_t^5 v', \omega^{1/2+\mu} \partial_t^5 v''$, respectively, we have the following proposition:
Proposition 5.5. For $\gamma = \frac{3}{2}$, there exists a constant $\alpha \in (0, 1)$, such that one has the following estimate:

$$
\sup_{[0,t]} \left( \left| \omega^{1+\mu} \partial_t^2 v' \right|^2_0 + \left| \omega^{1+\mu} \partial_t^3 v'' \right|^2_0 + \left| \omega^{1+\mu} \partial_t^4 v''' \right|^2_0 \right) \\
+ \left| \omega^{2+\mu} \partial_t^2 v \right|^2_0 \leq C(\bar{E}^{\alpha} + 1) \left( \bar{M}_0 + CtP \left( \sup_{[0,t]} \bar{E} \right) \right).
$$

6. The proof of Theorem 2.1

We are now ready to finish the proof of Theorem 2.1 as follows.

From (6.15), the Propositions 4.1-4.4 for $2 \leq \gamma < 3$ and Propositions 5.2-5.4 for $\gamma = \frac{3}{2}$, one has

$$
\sup_{[0,T]} E(t) \leq M_0 + CtP \left( \sup_{[0,t]} E \right) + C \sup_{[0,t]} E^{\alpha} \left( M_0 + tP \left( \sup_{[0,t]} E \right) \right),
$$

where $M_0 = P(E(0))$, $P(\cdot)$ is some polynomial function, and $0 < \alpha < 1$. By Young's inequality and adjusting the constants, we obtain

$$
\sup_{[0,t]} E(t) \leq M_0 + CtP \left( \sup_{[0,t]} E \right).
$$

As in [5] this provides us with a time of existence $T_1$ independent of $\varepsilon$ and an energy estimate on the time interval $(0, T_1)$ independent of $\varepsilon$ of the form:

$$
\sup_{[0,t]} E(t) \leq 2M_0. \quad (6.1)
$$

By the $\varepsilon$-independent estimate (6.1), there exists a subsequence of $v^\varepsilon$ converging to $v$ in $L^2(0, T; H^2(I))$ with $\eta = x + \int_0^t v(x, s)ds$. The standard compactness arguments shows that $v$ is a solution to (2.5). Thus, we can prove Theorem 2.1 for $2 \leq \gamma < 3$ and $\gamma = \frac{3}{2}$. Moreover, in the above process, we find that the method can be extended to all the cases of $1 < \gamma < 2$, thus Theorem 2.1 can be proved for all the general case $1 < \gamma < 3$.

Now, we are ready to prove the uniqueness of solutions. For two solutions $v_1(x, t), v_2(x, t)$ satisfying Theorem 2.1 to the free-boundary problem of the compressible Euler equations (1.1), we want to prove $v_1(x, t) = v_2(x, t)$. In fact, from (2.20), there exits three positive constants $c_1, c_2, c_3$ such that

$$
c_1 \leq \partial_x \eta_i(x, t) \leq c_2, \quad |\partial_x v_i(x, t)| \leq c_3, \quad (x, t) \in [0, 1] \times [0, T], \quad i = 1, 2. \quad (6.2)
$$

We define $\delta v = v_1 - v_2$, then $\delta v$ satisfy the following equation:

$$
\omega^{1+2\mu} \partial_t \delta v + \left[ \omega^{2+2\mu} e^{S_0} \left( \frac{1}{(\eta'_1)^\gamma} - \frac{1}{(\eta'_2)^\gamma} \right) \right]' = 0. \quad (6.3)
$$
By considering the fifth differential version of (6.3), from the Proposition 5.1 we have the following equation for $\delta v$:

\[
\frac{1}{2} \int_I \omega^{1+2\mu} |\partial_t^5 \delta v|^2 + \frac{\gamma}{2} \int_I |\partial_t^4 \delta v|^2 \frac{\omega^{2+2\mu}}{(\eta'_1)^{\gamma+1}} e^{S_0} = -\frac{(\gamma+1)\gamma}{2} \int_0^t \int_I \omega^{2+2\mu} e^{S_0} \left( \partial_t^{4-\alpha} v'_1 \partial_t^5 v_1 - \partial_t^{4-\alpha} v'_2 \partial_t^5 v_2 \right) - \frac{\gamma}{2} \int_I \omega^{2+2\mu} e^{S_0} \left( \partial_t^{4-\alpha} v'_1 \partial_t^5 v_1 - \partial_t^{4-\alpha} v'_2 \partial_t^5 v_2 \right)
\]

where

\[
g = -\frac{1}{\gamma} \omega^{1+2\mu} \partial_t^{k+2} \delta v + \sum_{\alpha=1}^k \left( c_\alpha \omega^{2+2\mu} e^{S_0} \left( \partial_t^{4-\alpha} v'_1 - \partial_t^{4-\alpha} v'_2 \right) \right) + \left[ \left( 1 - \frac{1}{(\eta'_1)^{\gamma+1}} \right) \left( (\eta'_1)^{\gamma+1} \right) \left( \omega^{2+2\mu} \partial_t^{4-\alpha} v'_2 e^{S_0} \right) \right] + (\gamma+1) \omega^{2+2\mu} e^{S_0} \left( \partial_t^{4-\alpha} v'_1 \eta''_1 - \partial_t^{4-\alpha} v'_2 \eta''_2 \right).
\]

From $\eta'_i = \int_0^t v''_i dt$, using (6.2) to control $\eta'_i$, by the weighted embedding inequality (2.13) and repeating elliptic estimates, we have

\[
\sup_{t \in [0,T]} \sum_{s=0}^4 \left| \partial_t^s \delta v \right|^2 \leq C T P \left( \sup_{t \in [0,T]} \sum_{s=0}^4 \left| \partial_t^s \delta v \right|^2 \right),
\]

which shows that $\delta v = 0$.

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