Effective Low–Energy Gravitational Potential for Slow Fermions Coupled to Linearised Massive Gravity

A. N. Ivanov, M. Pitschmann, and M. Wellenzohn

1 Atominstitut, Technische Universität Wien, Stadionallee 2, A-1020 Wien, Austria
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We analyse the Dirac equation for slow fermions coupled to linearised massive gravity above the Minkowski background and derive the effective low–energy gravitational potential. The obtained results can be used in terrestrial laboratories for the detection of gravitational waves and fluxes of massive gravitons emitted by cosmological objects. We also calculate the neutron spin precession within linearised massive gravity, which in principle can be measured by neutron interferometers.

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I. INTRODUCTION

In this paper we analyse the low–energy reduction of Dirac fermions in interaction with linearised massive gravity. The theory of massive gravity starts with the pioneering paper by Fierz and Pauli [1], in which they carried out the analysis of the equations of motion of massive particles with spin 2. We shall call this graviton the Fierz–Pauli (FP) graviton. Very nice surveys of the subsequent development of this theory and gravitational theories beyond the cosmological standard model can be found in the papers by Hinterbichler [2] and by Joyce et al. [3], respectively. As has been pointed out by Brito, Cardoso and Pani [4], the motivations to investigate massive gravity are conceptual [2] and practical, related to the analysis of the influence of massive gravity on the dynamics of emission of gravitational waves by pulsars [5, 6] (see also [4]). The contribution of massive gravitons to gravitational wave emission leads to a deformation of the gravitational–wave signal during its journey from the source to observer. As usual there are at least two ways for investigating the deformation of the gravitational waveform caused by massive gravitons. They are [3]: i) a full non–linear simulation and ii) a slow–motion expansion or a perturbative expansion around some background. The later deals with a linear approximation of non–linear massive gravity above a background [4], which is valid at distances \( r > r_V \), where \( M \) is the mass of the source producing gravitons, \( M_{Pl} = 1/\sqrt{8\pi G_N} = 2.435 \times 10^{27} \text{ eV} \) is the reduced Planck mass defined in terms of the Newtonian gravitational coupling constant \( G_N \) [10] and \( m_g \) is the graviton mass.

The paper is organized as follows. In section II we follow [2, 3] and derive the action of linearised massive gravity above the Minkowski background by using the St¨ uckelberg trick to deal with the so–called van Dam–Veltman–Zakharov (vDVZ) discontinuity [11, 12]. In section III we analyse the Dirac equation for slow fermions in linearised massive gravity above the Minkowski background. In this section we follow the analysis of the Dirac equation in curved spacetime as carried out by Kostelecky [13]. However, in our analysis we neglect torsion. We calculate the spin connection and the Hamilton operator in terms of \( \tilde{h}_{\mu\nu} \) field, which can be identified with the field of the FP graviton including all shifts related to the St¨ uckelberg trick. In section IV we investigate the transformation properties of the Hamilton operator describing the interaction of fermions with a linearised massive gravitational field. In section V we derive the effective low–energy potential for slow fermions coupled to the fields of linearised massive gravity above the Minkowski background using Foldy–Wouthuysen transformations and write down the corresponding Schr¨ odinger–Pauli equation. In the restricted case of a static diagonal metric, calculated in the weak field approximation, we re-obtain our previous result for the effective low–energy potential published in [14]. We want to stress that in the Dirac Hamilton operator as well as in the effective low-energy potential of the Schrödinger–Pauli equation the field degrees of freedom appear exclusively in terms of \( \tilde{h}_{\mu\nu} \). In section VI we compare our results with those obtained by Gon¸ c ales, Obukhov, and Shapiro [15] as well as Quach [16], where similar calculations have been done for Dirac fermions in a gravitational–wave background. We calculate the neutron spin precession within linearised massive gravity. The phase–shift of the neutron wave function, caused by the neutron spin precession within linearised massive gravity, can in principle be measured by neutron interferometers [17]. In section VII we discuss the obtained results.

*Electronic address: ivanov@kph.tuwien.ac.at
†Electronic address: pitschmann@kph.tuwien.ac.at
‡Electronic address: max.wellenzohn@gmail.com
II. LINEARISED MASSIVE GRAVITY ABOVE MINKOWSKI BACKGROUND

We start with the action for linearised massive gravity\cite{2,3} coupled to Dirac fermions with mass $m$. For the gravitational field we use the standard Einstein–Hilbert action

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} R,$$

where $M_{\text{Pl}} = 1/\sqrt{8\pi G_N} = 2.435 \times 10^{23}$ eV is the reduced Planck mass, $G_N$ the Newtonian gravitational constant\cite{10}, $g_{\mu\nu}$ the metric tensor with signature $(+1, -1, -1, -1)$, $g = \det\{g_{\mu\nu}\}$ the determinant of the metric tensor, $R$ the Ricci scalar curvature given by\cite{18–20}

$$R = g^{\mu\lambda} R^\alpha_{\mu\alpha\lambda},$$

and $R^\alpha_{\mu\nu\lambda}$ the Riemann–Christoffel tensor defined by\cite{18–20}

$$R^\alpha_{\mu\nu\lambda} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\lambda} - \frac{\partial \Gamma^\alpha_{\mu\lambda}}{\partial x^\nu} + \Gamma^\alpha_{\mu\nu} \Gamma^\mu_{\nu\beta} - \Gamma^\alpha_{\mu\nu} \Gamma^\mu_{\nu\beta} + \Gamma^\alpha_{\mu\beta} \Gamma^\mu_{\nu\lambda} - \Gamma^\alpha_{\mu\beta} \Gamma^\mu_{\nu\lambda}.$$ \hspace{1cm} (3)

The affine connection $\Gamma^\alpha_{\mu\nu}$ is determined in terms of the Christoffel symbols\cite{18–20}

$$\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu}\} = \frac{1}{2} g^{\alpha\beta} \left( \frac{\partial g_{\beta\nu}}{\partial x^\mu} + \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right).$$ \hspace{1cm} (4)

Integrating by parts and using the properties of the affine connection and the metric tensor\cite{22} we arrive at the following expression for the Einstein–Hilbert action

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left( \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\mu\beta} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\nu\beta} + \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\lambda} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\lambda} \right).$$ \hspace{1cm} (5)

Next, we perform an expansion around a Minkowski background. For this, we set $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$ such that $g_{\mu\nu} g^{\mu\nu} = \delta_{\mu\nu}$, where $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are the Minkowski metric tensor and its inverse, respectively. To leading order we obtain

$$\sqrt{-g} g^{\mu\nu} \left( \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\mu\beta} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\nu\beta} + \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\lambda} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\lambda} \right) = \frac{1}{2} \left( \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \frac{\partial h^{\mu\nu}}{\partial x^\alpha} + \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \frac{\partial h^{\mu\nu}}{\partial x^\alpha} + \frac{1}{2} \frac{\partial h}{\partial x^\alpha} \frac{\partial h}{\partial x^\alpha} \right),$$ \hspace{1cm} (6)

where $h = \eta^{\mu\nu} h_{\mu\nu}$. The structure of Eq. (6) agrees well with Eq. (6.83) of Ref. [3]. Using Eq. (6) and adding the Pauli-Fierz mass term\cite{2,3}, we obtain the action for massive gravity

$$S_{\text{mg}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \left( \frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \frac{\partial h^{\mu\nu}}{\partial x^\alpha} + \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \frac{\partial h^{\mu\nu}}{\partial x^\alpha} + \frac{1}{2} \frac{\partial h}{\partial x^\alpha} \frac{\partial h}{\partial x^\alpha} - \frac{1}{2} \frac{m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2)}{\partial x^\alpha} \right),$$ \hspace{1cm} (7)

with graviton mass $m_g$. The main problem of this theory is that it does not agree with the massless theory after performing the limit $m_g \rightarrow 0$, which is known as the vDVZ discontinuity\cite{11,12}. It is useful to apply the so-called \textit{St"uckelberg trick}, related to the shift

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial A_\mu / \partial x^\nu + \partial A_\nu / \partial x^\mu,$$ \hspace{1cm} (8)

where $A_\mu$ is a new vector field\cite{2,3}. Such a trick has been applied for the first time in the context of massive gravity in\cite{8}. Since the shift Eq. (8) is equivalent to an infinitesimal local shift in coordinates $x^\mu \rightarrow x^\mu - A^\mu (x)$, the terms corresponding to the Ricci scalar in Eq. (7) are invariant under the shift Eq. (8). After performing this shift, the action of massive gravity takes the form

$$S_{\text{mg}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \left( \frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \frac{\partial h^{\mu\nu}}{\partial x^\alpha} + \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \frac{\partial h^{\mu\nu}}{\partial x^\alpha} + \frac{1}{2} \frac{\partial h}{\partial x^\alpha} \frac{\partial h}{\partial x^\alpha} - \frac{1}{2} \frac{m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2)}{\partial x^\alpha} \right) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \left( h^{\mu\nu} \frac{\partial A_\mu}{\partial x^\nu} + h \frac{\partial A_\nu}{\partial x^\mu} \right),$$ \hspace{1cm} (9)

The next step is to perform the shift

$$A_\mu \rightarrow A_\mu - \frac{\partial \varphi}{\partial x^\mu}.$$ \hspace{1cm} (10)
This gives

\[
S_{\text{mgr}} = \frac{M_p^2}{4} \int d^4x \left\{ \frac{1}{2} \left( \frac{\partial h_{\mu\nu}}{\partial x_\alpha} \frac{\partial h^{\mu\nu}}{\partial x_\alpha} - \frac{\partial h_{\mu\alpha}}{\partial x_\nu} \frac{\partial h^{\mu\alpha}}{\partial x_\nu} + \frac{\partial h_{\mu\nu}}{\partial x_\mu} \frac{\partial h_{\mu\nu}}{\partial x_\nu} - \frac{1}{2} \frac{\partial h}{\partial x_\alpha} \frac{\partial h}{\partial x_\alpha} - \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right) + 2m_g^2 \right\},
\]

where we have integrated by parts in the last term. Following [2, 3], we make rescalings of the fields

\[
h_{\mu\nu} \rightarrow \frac{2}{M_p} h_{\mu\nu}, \quad A_\mu \rightarrow \frac{1}{M_p} m_g A_\mu, \quad \varphi \rightarrow \frac{1}{M_p} m_g^2 \varphi,
\]

where \(d\) is a dimensionless parameter to be determined later and all fields have the same dimension \([h_{\mu\nu}] = [A_\mu] = [\varphi] = eV\). Hence, we arrive at the action

\[
S_{\text{mgr}} = \int d^4x \left\{ \frac{1}{2} \left( \frac{\partial h_{\mu\nu}}{\partial x_\alpha} \frac{\partial h^{\mu\nu}}{\partial x_\alpha} - \frac{\partial h_{\mu\alpha}}{\partial x_\nu} \frac{\partial h^{\mu\alpha}}{\partial x_\nu} + \frac{\partial h_{\mu\nu}}{\partial x_\mu} \frac{\partial h_{\mu\nu}}{\partial x_\nu} - \frac{1}{2} \frac{\partial h}{\partial x_\alpha} \frac{\partial h}{\partial x_\alpha} - \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2m_g \left( h_{\mu\nu} \frac{\partial A_\mu}{\partial x_\nu} - h \frac{\partial A_\mu}{\partial x_\mu} - \frac{1}{2} m_g \left( h_{\mu\nu} h^{\mu\nu} - h^2 \right) \right) \right\}.
\]

Following [2, 3], we perform the shift

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + d \varphi \eta_{\mu\nu}.
\]

Substituting Eq. (14) into Eq. (13) we obtain the following action

\[
S_{\text{mgr}} = \int d^4x \left\{ \frac{1}{2} \left( \frac{\partial h_{\mu\nu}}{\partial x_\alpha} \frac{\partial h^{\mu\nu}}{\partial x_\alpha} - \frac{\partial h_{\mu\alpha}}{\partial x_\nu} \frac{\partial h^{\mu\alpha}}{\partial x_\nu} + \frac{\partial h_{\mu\nu}}{\partial x_\mu} \frac{\partial h_{\mu\nu}}{\partial x_\nu} - \frac{1}{2} \frac{\partial h}{\partial x_\alpha} \frac{\partial h}{\partial x_\alpha} - \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \sqrt{2} m_g \left( h_{\mu\nu} \frac{\partial A_\mu}{\partial x_\nu} - h \frac{\partial A_\mu}{\partial x_\mu} + 3d^2 \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\alpha} + 3d m_g^2 \varphi (h + 2d \varphi) + 3 \sqrt{2} dm_g^2 \frac{\partial A_\mu}{\partial x_\mu} \right) \right\}.
\]

The problem of action Eq. (15) is that the mass term of the scalar particle \(\varphi\) has incorrect sign. According to [2], this problem can be remedied via gauge fixing, which furthermore leads to the diagonalisation of action Eq. (15). Hence, we add the terms

\[
S_{\text{gf1}} = \int d^4x \left( \frac{\partial h_{\mu\alpha}}{\partial x_\nu} - \frac{1}{\sqrt{2}} \frac{\partial h_{\mu\nu}}{\partial x_\alpha} - \frac{1}{\sqrt{2}} m_g A_\alpha \right) \left( \frac{\partial h^{\mu\alpha}}{\partial x_\nu} - \frac{1}{\sqrt{2}} \frac{\partial h^{\mu\nu}}{\partial x_\alpha} - \frac{1}{\sqrt{2}} m_g A_\alpha \right),
\]

and

\[
S_{\text{gf2}} = -\frac{1}{2} \int d^4x \left( \frac{\partial A_\mu}{\partial x_\nu} + \frac{1}{\sqrt{2}} m_g h + 3d \sqrt{2} m_g \varphi \right)^2,
\]

fixing the gauge this way. Hence, we arrive at the diagonal action

\[
S_{\text{mgr}} = \int d^4x \left\{ \frac{1}{2} \left( \frac{\partial h_{\mu\nu}}{\partial x_\alpha} \frac{\partial h^{\mu\nu}}{\partial x_\alpha} - \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right) - \frac{1}{4} \frac{\partial h}{\partial x_\alpha} \frac{\partial h}{\partial x_\alpha} + \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right\}.
\]

In order to obtain the correct kinetic term of the massive scalar field we set the parameter \(3d^2 = 1/2\), finally arriving at

\[
S_{\text{mgr}} = \int d^4x \left\{ \frac{1}{4} \left( \frac{\partial h_{\mu\nu}}{\partial x_\alpha} \frac{\partial h^{\mu\nu}}{\partial x_\alpha} - \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right) + \frac{1}{2} \frac{\partial h}{\partial x_\alpha} \frac{\partial h}{\partial x_\alpha} + \frac{1}{2} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right\}.
\]

It is important to note that all fields \(h_{\mu\nu}, A_\mu\) and \(\varphi\) have masses equal to \(m_g\). Now we may proceed to analyse the gravitational interaction of slow fermions within linearised massive gravity.
Inverting relation Eq. (22) we obtain
\[ g_{\alpha\beta} = \frac{1}{2} \bar{\psi}(x) \gamma^\mu(x) \gamma^\nu(x) \gamma^{\mu\nu}(x) \psi(x) - m \bar{\psi}(x) \psi(x), \]
where \( m \) is the fermion mass, \( \gamma^\mu(x) \) are the Dirac matrices in curved spacetime satisfying the anticommutation relations
\[ \gamma^\mu(x) \gamma^\nu(x) + \gamma^\nu(x) \gamma^\mu(x) = 2g^{\mu\nu}(x), \]
and \( g_{\mu\nu} \) is the covariant derivative. For the definition of the Dirac matrices \( \gamma^\mu(x) \) and covariant derivative \( D_\mu \) we follow \[13\] (see also Eq. (28)). However, unlike Kostelecky \[13\] our analysis is carried out for vanishing torsion. In order to derive the low–energy approximation of the Dirac equation in curved spacetime, we introduce a set of vierbein fields \( e_\mu^\alpha(x) \) at each spacetime point \( x \) defined by
\[ dx^\alpha = e_\mu^\alpha(x) dx^\mu. \]

In terms of these vierbein fields slow fermions couple to the gravitational field. The vierbein fields \( e_\mu^\alpha(x) \) are related to the metric tensor \( g_{\mu\nu}(x) \) by
\[ ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = \eta_{\alpha\beta} \left( e_\mu^\alpha(x) dx^\mu \right) \left( e_\nu^\beta(x) dx^\nu \right) = \left( \eta_{\alpha\beta} e_\mu^\alpha(x) e_\nu^\beta(x) \right) dx^\mu dx^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu, \]
where \( \eta_{\alpha\beta} \) is the metric tensor in Minkowski spacetime. This gives
\[ g_{\mu\nu}(x) = \eta_{\alpha\beta} e_\mu^\alpha(x) e_\nu^\beta(x) = e_\mu^\alpha(x) e_{\alpha\mu}(x). \]

Hence, the vierbein fields may be viewed as the square root of the metric tensor \( g_{\mu\nu}(x) \) in the sense of a matrix equation \[23\]. Inverting relation Eq. (22) we obtain
\[ \eta_{\alpha\beta} = g_{\mu\nu}(x) e_\alpha^\mu(x) e_\beta^\nu(x) = e_\alpha^\mu(x) e_{\mu\beta}(x). \]

The following important relations hold
\[ e_\alpha^\mu(x) e_\beta^\nu(x) = \delta_\alpha^\beta, \]
\[ e_\alpha^\mu(x) e_{\mu\beta}(x) = \delta_\beta^\nu, \]
which are useful for the derivation of the Dirac equation and calculation of the Dirac Hamilton operator. In terms of the vierbein fields the Dirac matrices \( \gamma^\mu(x) \) are given by
\[ \gamma^\mu(x) = e_\alpha^\mu(x) \gamma^\alpha, \]
where \( \gamma^\alpha \) are the standard Dirac matrices in Minkowski spacetime \[24\]. The covariant derivative \( D_\mu \) we define as \[13\]
\[ D_\mu \psi(x) = \partial_\mu \psi(x) - \Gamma_\mu(x) \psi(x), \]
where \( \Gamma_\mu(x) \) is the spin affine connection, which can be expressed in terms of the spin connection \( \omega_{\mu\alpha\beta}(x) \) \[13\]
\[ \Gamma_\mu(x) = \frac{i}{4} \omega_{\mu\alpha\beta}(x) \sigma^{\alpha\beta}, \]

where \( \sigma^{\alpha\beta} = (i/2)(\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha) \) and \( \gamma^\beta \Gamma_\mu^\beta(x) \gamma^{\hat{\alpha}} = -\Gamma_\mu(x) \). The spin connection \( \omega_{\mu\alpha\beta}(x) \) is related to the vierbein fields and the affine connection as follows \[13\]
\[ \omega_{\mu\alpha\beta}(x) = -\eta_{\alpha\hat{\beta}} \left( \partial_\mu e_\hat{\alpha}^\hat{\beta}(x) - \Gamma^\alpha_{\mu\nu}(x) e_\alpha^\beta(x) \right) e_\beta^\gamma(x), \]
The integrand of the action Eq. (20) is hermitian. Integrating by parts we transcribe the action Eq. (20) into the form

\[
S_{\psi} = \int d^4x \sqrt{-g} \left\{ i e_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \partial_\mu \psi(x) - \frac{i}{2} e_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \Gamma_\mu(x) \psi(x) + \frac{i}{2} \epsilon_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \Gamma^\mu_\nu(x) \gamma^\nu_\lambda \psi(x) \\
+ \frac{i}{2} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} e_\lambda^\mu(x) \right) \bar{\psi}(x) \gamma^\lambda \psi(x) - m \bar{\psi}(x) \psi(x) \right\}.
\]

Rewriting the fourth term in the integrand

\[
\frac{i}{2} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} e_\lambda^\mu(x) \right) \gamma^\lambda = - \frac{i}{2} \omega_{\lambda_\alpha_\beta}(x) e^\mu_\alpha(x) \eta^{\lambda_\beta} \gamma^\alpha,
\]

the action Eq. (31) can be transformed into the form

\[
S_{\psi} = \int d^4x \sqrt{-g} \left\{ i e_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \partial_\mu \psi(x) - \frac{i}{2} e_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \Gamma_\mu(x) \psi(x) + \frac{i}{2} \epsilon_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \Gamma^\mu_\nu(x) \gamma^\nu_\lambda \psi(x) \\
- \frac{i}{2} \omega_{\lambda_\alpha_\beta}(x) e^\mu_\alpha(x) \eta^{\lambda_\beta} \bar{\psi}(x) \gamma^\alpha \psi(x) - m \bar{\psi}(x) \psi(x) \right\},
\]

respectively

\[
S_{\psi} = \int d^4x \sqrt{-g} \left\{ i e_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \partial_\mu \psi(x) - \frac{i}{2} \omega_{\lambda_\alpha_\beta}(x) e^\mu_\alpha(x) \bar{\psi}(x) \left( \eta^{\lambda_\beta} \gamma^\alpha + \frac{i}{4} \left\{ \sigma^{\lambda_\beta}, \gamma^\alpha \right\} \right) \psi(x) - m \bar{\psi}(x) \psi(x) \right\}.
\]

Using the following relations for Dirac matrices

\[
\gamma^\lambda \sigma^{\alpha_\beta} = i \left( \eta^{\lambda_\alpha} \gamma^\beta - \eta^{\beta_\lambda} \gamma^\alpha \right) - \epsilon^{\lambda_\alpha_\beta_\gamma} \gamma^\gamma, \\
\sigma^{\alpha_\beta} \gamma^\lambda = i \left( \eta^{\alpha_\beta} \gamma^\gamma - \eta^{\gamma_\alpha} \gamma^\beta \right) - \epsilon^{\alpha_\beta_\gamma_\delta} \gamma^\delta, \\
\left\{ \sigma^{\alpha_\beta}, \gamma^\lambda \right\} = -2 \epsilon^{\lambda_\alpha_\beta_\gamma} \gamma^\gamma,
\]

where \( \epsilon^{\lambda_\alpha_\beta_\gamma} \) is the Levi-Civita tensor (\( \epsilon^{0123} = +1 \)), we get

\[
S_{\psi} = \int d^4x \sqrt{-g} \left\{ i e_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \partial_\mu \psi(x) - \frac{i}{2} \omega_{\lambda_\alpha_\beta}(x) e^\mu_\alpha(x) \bar{\psi}(x) \left( \eta^{\lambda_\beta} \gamma^\alpha - \frac{i}{2} \epsilon^{\lambda_\alpha_\beta_\gamma} \gamma^\gamma \right) \psi(x) - m \bar{\psi}(x) \psi(x) \right\}.
\]

After all shifts and rescalings, carried out in section II below the expansion of the metric tensor \( g_{\mu\nu} \) above Minkowski spacetime acquires the form

\[
g_{\mu\nu} = \eta_{\mu\nu} + 2 \tilde{h}_{\mu\nu},
\]

where \( \tilde{h}_{\mu\nu} \) is defined by \((d = \pm 1/\sqrt{6})\)

\[
\tilde{h}_{\mu\nu} = \frac{1}{M_P^2} \left( h_{\mu\nu} + \frac{1}{\sqrt{2m_\phi}} \left( \frac{\partial A_\mu}{\partial x^\rho} + \frac{\partial A_\rho}{\partial x^\mu} \right) - \frac{2d}{m_\phi} \frac{\partial^2 \varphi}{\partial x^\rho \partial x^\mu} + d \varphi \right) \eta_{\mu\nu}.
\]

The vierbein fields are given by

\[
e^\alpha_\mu = \delta^\alpha_\mu + \tilde{h}^\alpha_\mu, \quad e^\nu_\beta = \delta^\nu_\beta - \tilde{h}^\nu_\beta.
\]

We would like to note that in comparison with the metric decomposition \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) as used in section II below Eq. (5), in the decomposition \( g_{\mu\nu} = \eta_{\mu\nu} + 2 \tilde{h}_{\mu\nu} \), the weak gravitational field enters with a factor 2. This is done in order to use the vierbein fields Eq. (30) without additional factors of 1/2 in front of \( \tilde{h}^\alpha_\mu \) and \( \tilde{h}^\nu_\beta \). To linear order the vierbein fields Eq. (30) satisfy the relations in Eq. (20). The spin connection \( \omega_{\mu_\alpha_\beta} \), calculated to linear order, reads

\[
\omega^\alpha_{\mu_\alpha_\beta} = \frac{\partial h^\alpha_{\mu_\mu}}{\partial x^\beta} - \frac{\partial h^\beta_{\mu_\mu}}{\partial x^\alpha}.
\]
Plugging it into Eq. (36) we arrive at the action

$$S_\psi = \int d^4x \sqrt{-g} \left\{ ie^\mu_\lambda(x) \bar{\psi}(x) \gamma^\lambda \partial_\mu \psi(x) - \frac{i}{2} \left( \partial H_0 - \partial x^\alpha \right) \bar{\psi}(x) \gamma^\alpha \gamma^\lambda \psi(x) - m \bar{\psi}(x) \psi(x) \right\}. \quad (41)$$

The Dirac equation in the standard form is

$$i \frac{\partial \psi}{\partial t} = H \psi, \quad (42)$$

where $H = H_0 + \delta H$ is the Dirac Hamilton operator for fermions with mass $m$; $H_0 = \gamma^i m - i \gamma^0 \gamma^j \bar{\psi} \gamma^j \psi$ is the Hamilton operator for free fermions and $\delta H$ describes the interaction of fermions with linearised massive gravity

$$\delta H = \tilde{h}_0 \gamma^0 \psi(x) - i \gamma^0 \gamma^j \tilde{h}_0 \gamma^j \frac{\partial}{\partial x^i} + \frac{i}{2} \gamma^0 \gamma^i \left( \frac{\partial \tilde{h}_0}{\partial x^\mu} - \frac{\partial h_0}{\partial x^\mu} \right) + i \gamma^0 \gamma^j \tilde{h}_0 \frac{\partial}{\partial t} \psi(x) + i \tilde{h}_0 \frac{\partial}{\partial x^i}, \quad (43)$$

where we have dropped the hat on the indices and have kept only the linear order contributions of $\tilde{h}_{\mu \nu}$.

**IV. NON–UNITARY TRANSFORMATIONS OF WAVE FUNCTIONS AND HAMILTON OPERATOR**

It is well-known that the Hamilton operator of a Dirac particle in curved spacetime is not Hermitian [22, 23]. The corresponding Hermitian Hamilton operator may be obtained by means of a non–unitary transformation of its wave function [23] (see also [22, 23] and [14]):

$$\psi(x) = (\sqrt{-g} e^0)_{-1/2} \psi(x). \quad (44)$$

The Hermitian Hamilton operator $H'$ is related to the non–Hermitian $H$ by

$$H' = (\sqrt{-g} e^0)_{-1/2} H (\sqrt{-g} e^0)_{-1/2} - i (\sqrt{-g} e^0)_{-1/2} \frac{\partial}{\partial t} (\sqrt{-g} e^0)_{-1/2} \quad (45).$$

Keeping only linear contributions in the $\tilde{h}_{\mu \nu}$–expansion and using $\sqrt{-g} e^0 = 1 + (\tilde{h} - \tilde{h}_0)$ we have to perform the non–unitary transformation of the wave function of the Dirac particle

$$\psi(x) = (1 - \frac{1}{2} (\tilde{h} - \tilde{h}_0)) \psi(x), \quad (46)$$

and arrive at the Hamilton operator

$$\delta H' = \delta H - \frac{1}{2} [H_0, (\tilde{h} - \tilde{h}_0)] + \frac{i}{2} \frac{\partial}{\partial t} (\tilde{h} - \tilde{h}_0). \quad (47)$$

After the evaluation of the commutator

$$-\frac{1}{2} [H_0, (\tilde{h} - \tilde{h}_0)] = \frac{i}{2} \gamma^0 \gamma^j \frac{\partial}{\partial x^i} (\tilde{h} - \tilde{h}_0), \quad (48)$$

we obtain

$$\delta H' = \tilde{h}_0 \gamma^0 \psi(x) - i \gamma^0 \gamma^j \tilde{h}_0 \gamma^j \frac{\partial}{\partial x^i} + \frac{i}{2} \gamma^0 \gamma^j \left( \frac{\partial \tilde{h}_0}{\partial x^\mu} - \frac{\partial h_0}{\partial x^\mu} \right) + i \gamma^0 \gamma^j \tilde{h}_0 \frac{\partial}{\partial t} \psi(x) + i \tilde{h}_0 \frac{\partial}{\partial x^i}, \quad (49)$$

The Hamilton operator Eq. (49) may also be transcribed into the form

$$\delta H' = \tilde{h}_0 \gamma^0 \psi(x) - i \gamma^0 \gamma^j \tilde{h}_0 \gamma^j \frac{\partial}{\partial x^i} + \frac{i}{2} \gamma^0 \gamma^j \left( \frac{\partial \tilde{h}_0}{\partial x^\mu} - \frac{\partial h_0}{\partial x^\mu} \right) + i \gamma^0 \gamma^j \tilde{h}_0 \frac{\partial}{\partial t} \psi(x) + i \tilde{h}_0 \frac{\partial}{\partial x^i} \quad (50)$$
One can show that the Hamilton operator Eq. (50) is hermitian. Introducing the notation
\[ Q'_\nu = \tilde{h}_0^\nu \delta^\nu_{\nu'} - \tilde{h}'_\nu, \quad Q_\nu := \frac{\partial Q'_\nu}{\partial x'^\nu} = \frac{\partial \tilde{h}_0^\nu}{\partial x'^\nu} - \frac{\partial \tilde{h}'_\nu}{\partial x'^\nu}. \] (51)
allows to transcribe the Hamilton operator Eq. (50) into the form
\[ \delta H' = \tilde{h}_0^0 \gamma^0 m - i \gamma^0 \gamma^j Q^k \frac{\partial}{\partial x^j} - i \frac{\gamma}{2} \gamma^0 \gamma^j Q_j + i \frac{\gamma}{2} \gamma^0 \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t} + i \frac{\gamma}{2} \gamma^0 \gamma^j \tilde{h}'_j \frac{\partial}{\partial t}, \] (52)
which is convenient for the derivation of the effective low–energy gravitational potential.

V. FOLDY–WOUTHUYSEN TRANSFORMATIONS AND LOW–ENERGY APPROXIMATION OF THE DIRAC EQUATION

For the derivation of the low–energy approximation of the Dirac equation Eq. (49) with the Hamilton operator \( H' = H_0 + \delta H' \), where \( \delta H' \) is given by Eq. (52), we employ the Foldy–Wouthuysen (FW) transformation [27]. The aim of the FW transformation is obtain the rigorous low-energy limit by using unitary transformations [27]. First, we obtain
\[ H_1 = e^{+iS_1} H' e^{-iS_1} - i e^{iS_1} \frac{\partial}{\partial t} e^{-iS_1} \]
\[ = H' - \frac{\partial S_1}{\partial t} + i \left[ S_1, H' - \frac{1}{2} \frac{\partial S_1}{\partial t} \right] + \frac{i^2}{2} \left[ S_1, \left[ S_1, H' - \frac{1}{3} \frac{\partial S_1}{\partial t} \right] \right] + \ldots . \] (53)
Following [27], we take the operator \( S_1 \) in the following form
\[ S_1 = -i \frac{\gamma}{2m} \gamma^0 \gamma^j \frac{\partial}{\partial x^j} - i \frac{\gamma}{2m} \gamma^0 \gamma^j Q^k \frac{\partial}{\partial x^j} - i \frac{\gamma}{2m} \gamma^0 \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t} \]
\[ = -i \frac{\gamma}{2m} \gamma^j \frac{\partial}{\partial x^j} - \frac{1}{2m} \gamma^j Q^k \frac{\partial}{\partial x^j} - \frac{1}{2m} \gamma^j Q_j + \frac{1}{2m} \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t}, \] (54)
The time derivative of \( S_1 \) and the commutators in Eq. (53) are equal to
\[ \frac{\partial S_1}{\partial t} = -\frac{1}{2m} \gamma^j Q^k \frac{\partial}{\partial x^j} - i \frac{\gamma}{2m} \gamma^j Q_j + i \frac{\gamma}{2m} \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t}, \] (55)
respectively
\[ i \left[ S_1, H' - \frac{1}{2} \frac{\partial S_1}{\partial t} \right] = i \frac{\gamma}{2m} \gamma^j \frac{\partial}{\partial x^j} + i \gamma^0 \gamma^j Q^k \frac{\partial}{\partial x^j} + i \gamma^0 \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t} - \frac{\gamma}{m} \Delta \]
\[ + \frac{1}{2m} \gamma^j Q^k \frac{\partial}{\partial x^j} + \frac{1}{2m} \gamma^j Q_j + \frac{\gamma}{2m} \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t}, \]
\[ + \frac{1}{2m} \gamma^j Q^k \frac{\partial}{\partial x^j} + \frac{1}{2m} \gamma^j Q_j + \frac{\gamma}{2m} \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t}, \]
\[ - \frac{1}{2m} \gamma^j Q^k \frac{\partial}{\partial x^j} - \frac{1}{2m} \gamma^j Q_j + \frac{\gamma}{2m} \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t}, \]
\[ + \frac{1}{2m} \gamma^j Q^k \frac{\partial}{\partial x^j} + \frac{1}{2m} \gamma^j Q_j + \frac{\gamma}{2m} \gamma^j \tilde{h}_0^j \frac{\partial}{\partial t}. \] (56)
and finally
\[
\frac{i^2}{2} \left[ S_1, [S_1, H' - \frac{1}{3} \frac{\partial S_1}{\partial t}] \right] = \gamma^0 \frac{1}{2m} \Delta + \gamma^0 \frac{1}{2m} \nabla h_0 \cdot \nabla + \gamma^0 \frac{1}{8m} \Delta h_0 + \gamma^0 \frac{1}{2m} \nabla h_0 \cdot \nabla + \gamma^0 \frac{1}{4m} \Sigma - (\nabla h_0^\text{T} \times \nabla)
\]
\[
- \gamma^0 \frac{1}{2m} Q_j^k \frac{\partial^2}{\partial x_j \partial x_k} - \gamma^0 \frac{1}{2m} Q_j^k \frac{\partial}{\partial x_j} \nabla + \gamma^0 \frac{1}{4m} \Sigma_m \frac{\partial Q_j^k}{\partial x^m} \frac{\partial}{\partial x^l}
\]
\[
- \gamma^0 \frac{1}{2m} Q_j^k \frac{\partial}{\partial x^l} - \gamma^0 \frac{1}{4m} \Sigma_m \frac{\partial Q_j^k}{\partial x^m} \frac{\partial}{\partial x^l}
\]
\[
+ \gamma^0 \frac{1}{2m} h_0^\text{T} \frac{\partial^2}{\partial x_j \partial x_k} + \gamma^0 \frac{1}{2m} h_0^\text{T} \frac{\partial}{\partial x_j} \nabla + \gamma^0 \frac{1}{4m} \Sigma_m \frac{\partial h_0^\text{T}}{\partial x^m} \frac{\partial}{\partial x^l}.
\]

(57)

Here \( \Sigma_1 = -\Sigma^1 = -(\bar{\Sigma})_1 \), etc. and \( \bar{\Sigma} = \gamma^0 \gamma^5 \Sigma \) is the standard diagonal matrix with elements \((\bar{\sigma}, \bar{\sigma})\) and \( \bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli 2 \times 2 matrices \cite{22}. For the Hamilton operator \( H_1 \) we obtain the following expression

\[
H_1 = \gamma^0 m - \gamma^0 \frac{1}{2m} \Delta + \gamma^0 \frac{1}{2m} h_0^\text{T} \cdot \nabla + \gamma^0 \frac{1}{8m} \nabla h_0^\text{T} + \gamma^0 \frac{1}{2m} \nabla h_0^\text{T} \Delta + \gamma^0 \frac{1}{4m} \Sigma \cdot (\nabla h_0^\text{T} \times \nabla)
\]
\[
+ \gamma^0 \frac{1}{m} Q_j^k \frac{\partial^2}{\partial x_j \partial x_k} + \gamma^0 \frac{1}{2m} Q_j^k \frac{\partial}{\partial x_j} \nabla + \gamma^0 \frac{1}{2m} \Sigma_m \frac{\partial Q_j^k}{\partial x^m} \frac{\partial}{\partial x^l}
\]
\[
+ \gamma^0 \frac{1}{2m} Q_j^k \frac{\partial}{\partial x^l} + \gamma^0 \frac{1}{4m} \Sigma_m \frac{\partial Q_j^k}{\partial x^m} \frac{\partial}{\partial x^l}
\]
\[
- \gamma^0 \frac{1}{m} h_0^\text{T} \frac{\partial^2}{\partial x_j \partial x_k} - \gamma^0 \frac{1}{2m} h_0^\text{T} \frac{\partial}{\partial x_j} \nabla + \gamma^0 \frac{1}{4m} \Sigma_m \frac{\partial h_0^\text{T}}{\partial x^m} \frac{\partial}{\partial x^l}
\]
\[
+ \frac{i}{2} \gamma^0 \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} + \frac{i}{2} \gamma^0 \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} + \frac{1}{2m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} + \frac{1}{4m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} - \frac{1}{2m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k}
\]
\[
+ \frac{i}{2m} \gamma^j \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k} + \frac{i}{4m} \gamma^j \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k} + \frac{i}{2m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k}.
\]

(58)

For the derivation of the Hamilton operator Eq. (58) we have neglected contributions of order \( 1/m^2 \). Since there are still so-called odd-operators left, we have to perform another FW transformation

\[
H_2 = e^{iS_2} H_1 e^{-iS_2} - i \Sigma S_2 \frac{\partial}{\partial t} e^{-iS_2}
\]
\[
H_1 - \frac{\partial S_2}{\partial t} + i \left[ S_2, H_1 - \frac{1}{3} \frac{\partial S_2}{\partial t} \right] + \frac{i^2}{2} \left[ S_2, \left[ S_2, H_1 - \frac{1}{3} \frac{\partial S_2}{\partial t} \right] \right] + \ldots,
\]

(59)

where the operator \( S_2 \) is equal to

\[
S_2 = -\frac{i}{2m} \gamma^0 \left( \frac{1}{2} \gamma^0 \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} + \gamma^0 \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} + \frac{1}{2m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} + \frac{1}{4m} \gamma^j \frac{\partial^2 h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} - \frac{1}{2m} \gamma^j \frac{\partial^2 h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k}
\]
\[
+ \frac{1}{2m} \gamma^j \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x_k} - \frac{1}{4m} \gamma^j \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x_k} \right)
\]
\[
= \frac{1}{4m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} + \frac{1}{2m} \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} - \frac{i}{4m^2} \gamma^0 \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} - \frac{i}{8m^2} \gamma^0 \gamma^j \frac{\partial^2 h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k} + \frac{i}{4m^2} \gamma^0 \gamma^j \frac{\partial h_0^\text{T}}{\partial x^j} \frac{\partial}{\partial x_k}
\]
\[
- \frac{i}{4m^2} \gamma^0 \gamma^j \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x_k} - \frac{i}{8m^2} \gamma^0 \gamma^j \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x_k}.
\]

(60)
Keeping only contributions of order 1/m for the time derivative of $S_2$ and the commutators we obtain the following expressions

$$\frac{\partial S_2}{\partial t} = \frac{1}{4m} \gamma \cdot \nabla \frac{\partial \tilde{h}_0}{\partial t} + \frac{1}{2m} \frac{\partial \tilde{h}_0}{\partial t} \gamma \cdot \nabla,$$

$$i \left[ S_2, H_1 - \frac{1}{2} \frac{\partial S_2}{\partial t} \right] = -\frac{i}{2} \gamma \cdot \nabla \frac{\partial \tilde{h}_0}{\partial t} - i \gamma \cdot \nabla \tilde{h}_0 \frac{\partial}{\partial x^j} - \frac{1}{2m} \gamma \frac{\partial \tilde{h}_0}{\partial x^j} \frac{\partial}{\partial x^k} - \frac{1}{4m} \gamma \frac{\partial^2 \tilde{h}_0}{\partial x^j \partial x^k} + \frac{1}{2m} \gamma \frac{\partial \tilde{h}_0}{\partial t} \frac{\partial}{\partial t}.$$

$$\frac{i^2}{2} \left[ S_2, \left[ S_2, H_1 - \frac{1}{2} \frac{\partial S_2}{\partial t} \right] \right] = 0. \quad (61)$$

Hence, after two FW transformations the effective Hamilton operator takes the form

$$H_2 = \gamma \cdot m - \gamma \cdot \frac{1}{2m} \Delta + \gamma \cdot \tilde{h}_0 \frac{\partial}{\partial t} + \gamma \cdot \frac{1}{8m} \Delta \tilde{h}_0 + \gamma \cdot \frac{1}{2m} \tilde{h}_0 \Delta + \gamma \cdot \frac{i}{4m} \tilde{\Sigma} \cdot (\nabla \tilde{h}_0 \times \nabla),$$

$$+ \gamma \cdot \frac{1}{2m} Q_{kj} \frac{\partial^2}{\partial x^k \partial x_j} + \gamma \cdot \frac{1}{2m} \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k} + \gamma \cdot \frac{1}{2m} \epsilon^{jkm} \Sigma_m \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k}$$

$$+ \gamma \cdot \frac{1}{2m} Q_{kj} \frac{\partial}{\partial x_j} + \gamma \cdot \frac{1}{2m} \frac{\partial Q_j^k}{\partial x^j} + \gamma \cdot \frac{1}{4m} \epsilon^{jkm} \Sigma_m \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k}$$

$$- \gamma \cdot \frac{1}{4m} \tilde{\Sigma} \cdot \nabla \frac{\partial \tilde{h}_0}{\partial t} - \gamma \cdot \frac{1}{2m} \frac{\partial \tilde{h}_0}{\partial t} \gamma \cdot \nabla + \frac{i}{2m} \frac{\partial \tilde{h}_0}{\partial t} i \tilde{\Sigma} \cdot \nabla. \quad (62)$$

The remaining odd-operators may be deleted by the final FW transformation

$$H_3 = e^{iS_3} H_2 e^{-iS_3} - i e^{iS_3} \frac{\partial}{\partial t} e^{-iS_3}$$

$$= H_2 - \frac{\partial S_3}{\partial t} + i \left[ S_3, H_2 - \frac{1}{2} \frac{\partial S_3}{\partial t} \right] + \frac{i^2}{2} \left[ S_3, \left[ S_3, H_2 - \frac{1}{3} \frac{\partial S_3}{\partial t} \right] \right] + \ldots, \quad (63)$$

with the operator $S_3$ given by

$$S_3 = -\frac{i}{2m} \gamma \cdot \frac{1}{4m} \nabla \frac{\partial \tilde{h}_0}{\partial t} - \frac{1}{2m} \frac{\partial \tilde{h}_0}{\partial t} \gamma \cdot \nabla + \frac{1}{8m^2} \gamma \cdot \frac{\partial \tilde{h}_0}{\partial t} \gamma \cdot \nabla + \frac{1}{2m} \frac{\partial \tilde{h}_0}{\partial t} i \gamma \cdot \gamma \cdot \nabla. \quad (64)$$

Again, keeping only contributions of order 1/m for the time derivative of $S_3$ and the commutators we obtain the following expressions

$$\frac{\partial S_3}{\partial t} = 0,$$

$$i \left[ S_3, H_2 - \frac{1}{2} \frac{\partial S_3}{\partial t} \right] = \frac{1}{4m} \gamma \cdot \nabla \frac{\partial \tilde{h}_0}{\partial t} + \frac{1}{2m} \frac{\partial \tilde{h}_0}{\partial t} \gamma \cdot \nabla,$$

$$\frac{i^2}{2} \left[ S_3, \left[ S_3, H_2 - \frac{1}{3} \frac{\partial S_3}{\partial t} \right] \right] = 0. \quad (65)$$

Finally, we obtain the low–energy reduction of the Dirac Hamilton operator for slow fermions

$$H_3 = \gamma \cdot m - \gamma \cdot \frac{1}{2m} \Delta + \gamma \cdot \tilde{h}_0 \frac{\partial}{\partial t} + \gamma \cdot \frac{1}{8m} \Delta \tilde{h}_0 + \gamma \cdot \frac{1}{2m} \tilde{h}_0 \Delta + \gamma \cdot \frac{i}{4m} \tilde{\Sigma} \cdot (\nabla \tilde{h}_0 \times \nabla),$$

$$+ \gamma \cdot \frac{1}{2m} Q_{kj} \frac{\partial^2}{\partial x^k \partial x_j} + \gamma \cdot \frac{1}{2m} \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k} + \gamma \cdot \frac{1}{2m} \epsilon^{jkm} \Sigma_m \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k}$$

$$+ \gamma \cdot \frac{1}{2m} Q_{kj} \frac{\partial}{\partial x_j} + \gamma \cdot \frac{1}{2m} \frac{\partial Q_j^k}{\partial x^j} + \gamma \cdot \frac{1}{4m} \epsilon^{jkm} \Sigma_m \frac{\partial Q_j^k}{\partial x^j} \frac{\partial}{\partial x^k}$$

$$- \gamma \cdot \frac{1}{4m} \tilde{\Sigma} \cdot \nabla \frac{\partial \tilde{h}_0}{\partial t} - \gamma \cdot \frac{1}{2m} \frac{\partial \tilde{h}_0}{\partial t} \gamma \cdot \nabla + \frac{i}{2m} \frac{\partial \tilde{h}_0}{\partial t} i \tilde{\Sigma} \cdot \nabla. \quad (66)$$
Following the standard procedure [27] (see also [11] for further details), we skip intermediate calculations and arrive at the Schrödinger–Pauli equation for the large components of the Dirac wave function of slow fermions

\[ i \frac{\partial \Psi(t, \vec{r})}{\partial t} = \left( -\frac{1}{2m} \Delta + \Phi_{\text{mgr}} \right) \Psi(t, \vec{r}), \tag{67} \]

where \( \Psi(t, \vec{r}) \) is the large component of the Dirac wave function and \( \Phi_{\text{mgr}} \) the effective low–energy potential caused by massive gravity

\[
\Phi_{\text{mgr}} = \frac{1}{2m} \frac{\partial^2 \tilde{h}_i^0}{\partial x^i \partial x^j} - \frac{1}{4m} \frac{\partial^2 \tilde{h}_j^0}{\partial x^j \partial x^k} - \frac{1}{2m} \frac{\partial^2 \tilde{h}_k^0}{\partial x^k \partial x^l} - i \frac{1}{2} \varepsilon^{jkm} \sigma_m \frac{\partial \tilde{h}_k^0}{\partial x^j} \frac{\partial \tilde{h}_l^0}{\partial x^l}.
\]

Above, we have made use of \( \sigma_1 = -\sigma^1 = (\sigma_x) \) etc. and have used for \( Q_j^a \) and \( Q_j \) the corresponding expressions in terms of \( \tilde{h}_i^0 \) in Eq. (71). Some terms of order \( O(1) \) in the large \( m \)–expansion appear in the effective low–energy potential Eq. (68) due to the removal of the mass term in the Hamilton operator Eq. (66) by means of the fermion wave function transformation with the phase factor \( e^{-imt} \) (see [14] for details). We would like to note that setting \( \tilde{h}_i^0 = U, \tilde{h}_j^0 = -\gamma U \delta_j^0 \) and \( \tilde{h}_j^0 = 0 \) we arrive at the effective gravitational potential given by Eq. (68) of Ref. [14].

VI. MASSIVE GRAVITON, SCALAR AND VECTOR FIELDS CAUSED BY MASSIVE POINT–LIKE BODY WITH MASS \( M \)

Here we would like to compare our results with those obtained by Gonçalves, Obukhov, and Shapiro [15] as well as Quach [16]. The main distinction lies in the fact that the results, obtained by these authors, are not related to linearised massive gravity but instead are derived for massless gravitons. The effective low–energy Hamilton operator Eq. (37) of Ref. [15] is derived for gravitational fields (gravitational waves) in spacetimes with metric

\[ ds^2 = -dt^2 + dx^2 + (1 - 2u) dy^2 + (1 + 2v) dz^2 - 2u dy dz - 2u dz dy, \tag{69} \]

for \( u = 0 \) and \( v = v(t - x) \). Apart from the different signature the metric Eq. (69) is, of course, a partial case of the metric used for the derivation of the effective low–energy potential Eq. (68). In Ref. [16] the metric Eq. (69) with the replacement \( v \rightarrow f \) has been used for the derivation of the effective low–energy Hamilton operator Eq. (21). Switching off the electromagnetic field, taken into account in Refs. [15],[16], the effective low–energy potential \( \Phi_{\text{eff}} \), which can be obtained from Eq. (37) of Ref. [15] and Eq. (21) of Ref. [16], is given by

\[
\Phi_{\text{eff}} = -\frac{1}{m} u T^{ab} \frac{\partial^2}{\partial x^a \partial x^b} - i \frac{1}{2} \varepsilon^{abc} \sigma_a \frac{\partial v}{\partial x^b} \frac{\partial \Phi_{\text{eff}}}{\partial x^c}, \tag{70} \]

where \( T = \text{diag}(0, -1, +1) \) [13],[16]. The potential \( \Phi_{\text{eff}} \) does not reproduce the effective low–energy potential \( \Phi_{\text{mgr}} \) given in Eq. (68). It can be compared only with the part of Eq. (68) equal to

\[
\delta \Phi_{\text{mgr}} = -\frac{1}{m} \frac{\partial^2 \tilde{h}_j^0}{\partial x^j \partial x^i} - i \frac{1}{2} \varepsilon^{jkm} \sigma_m \frac{\partial \tilde{h}_k^0}{\partial x^j} \frac{\partial \tilde{h}_l^0}{\partial x^l}, \tag{71} \]

for a corresponding choice of \( \tilde{h}_j^0 \). Because of a specific choice of the metric Eq. (69) with functions \( u = 0 \) and \( v = v(t - x) \), describing a gravitational wave propagating along \( x \) axis and having one polarization state [14], the results in [15] and [16], cannot be applied to experimental investigations of linearised massive gravity induced by massive bodies for slow Dirac fermions, in contrast to our results which allow for that. As an example, we discuss the gravitational field of a point–like mass \( M \). Following [2] we take as source a point–like mass \( M \) with energy–momentum tensor

\[ T^{(M)}_{\mu \nu} = M \delta^{(3)}(\vec{r}) \delta(\vec{r}), \tag{72} \]
where $\delta^{(3)}(\vec{r})$ is the Dirac $\delta$ function. For the experimental analysis we consider static solutions of the equations of motion, which are given by

\[(\Delta - m_g^2) h^{(M)}_{\mu\nu}(\vec{r}) = \frac{M}{M_{Pl}} \left( \delta^{(3)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \right) \delta^{(3)}(\vec{r}),\]

\[(\Delta - m_g^2) A^{(M)}_{\alpha}(\vec{r}) = 0,\]

\[(\Delta - m_g^2) \varphi^{(M)}(\vec{r}) = d \frac{M}{M_{Pl}} \delta^{(3)}(\vec{r}),\]

(73)

where we have used $6d^2 = 1$. Taking into account the gauge conditions

\[\frac{\partial h_{\mu\nu}^{(M)}}{\partial x_\nu} - \frac{1}{2} \frac{\partial h^{(M)}}{\partial x^\alpha} - \frac{1}{\sqrt{2}} m_g A^{(M)}_{\alpha} = 0,\]

\[\frac{\partial A_{\mu}^{(M)}}{\partial x_\mu} + \frac{1}{\sqrt{2}} m_g h^{(M)} + 3d\sqrt{2} m_g \phi^{(M)} = 0,\]

(74)

we adduce the solutions

\[h^{(M)}_{\mu\nu}(\vec{r}) = - \frac{M}{M_{Pl}} \left( \delta^{(3)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \right) \frac{1}{4\pi} \frac{e^{-m_g r}}{r},\]

\[A_{\alpha}^{(M)}(\vec{r}) = 0,\]

\[\varphi^{(M)}(\vec{r}) = - d \frac{M}{4\pi} \frac{e^{-m_g r}}{r}.\]

(75)

The massive gravitational field $\tilde{h}_{\mu\nu}^{(M)}$, which couples to slow neutrons, is equal to

\[\tilde{h}_{\mu\nu}^{(M)} = - \frac{M}{M_{Pl}} \left( \delta^{(3)}_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} + \frac{1}{m_g^2} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \right) \frac{1}{4\pi} \frac{e^{-m_g r}}{r},\]

(76)

and obeys the constraint $\tilde{h}^{(M)} = 0$. One can see that the gravitational field $\tilde{h}_{ij}^{(M)}$ becomes singular in the limit of a vanishing graviton mass $m_g \to 0$. FollowingArkani–Hamed, Georgi, and Schwartz (see also [8]) such a singularity is due to the restricted applicability of linearised massive gravity only for distances much larger than the Vainshtein radius $r_V = \sqrt{M/m_g^2 M_{Pl}^2}$, i.e. $r \gg r_V = \sqrt{M/m_g^2 M_{Pl}^2}$. In terms of the Vainshtein radius the gravitational field $\tilde{h}_{\mu\nu}^{(M)}$ reads

\[\tilde{h}_{\mu\nu}^{(M)} = - r_V \left( m_g^2 \delta^{(3)}_{\mu\nu} - \frac{1}{3} \left( m_g^2 \eta_{\mu\nu} + \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \right) \right) \frac{1}{4\pi} \frac{e^{-m_g r}}{r}.\]

(77)

Clearly, such an analysis of static gravitational fields generated by massive bodies in terms of their gravitational interactions with slow fermions is not possible by using the results, obtained in [15, 16].

Following [15, 16] and using the effective low–energy potential Eq. (68) we calculate the neutron spin precession in the gravitational field within linearised massive gravity

\[\frac{d \vec{S}}{dt} = \Omega_{\text{ingr}} \times \vec{S},\]

(78)

where $\vec{S} = \frac{1}{2} \vec{\sigma}$ is the neutron spin operator and $\Omega_{\text{ingr}}$ is the angular velocity operator of the neutron spin precession equal to

\[\Omega_{\text{ingr}}^a = \varepsilon^{abc} \left( \frac{\partial \tilde{h}^b_k}{\partial x^c} - \frac{i}{2m} \frac{\partial \tilde{h}^a_k}{\partial x^c} \frac{\partial}{\partial t} + \frac{i}{m} \frac{\partial \tilde{h}^a_k}{\partial x^c} \frac{\partial}{\partial t} + \frac{i}{m} \frac{\partial \tilde{h}^a_k}{\partial x^c} \frac{\partial}{\partial t} + \frac{i}{2m} \frac{\partial \tilde{h}^a_k}{\partial x^c} \frac{\partial}{\partial t} \right).\]

(79)

The phase-shift of the neutron wave function, induced by the effective low–energy operator $\Phi_{\text{spin}} = \tilde{\Omega}_{\text{ingr}} \cdot \vec{S}$ can in principle be measured by neutron interferometers [17, 34–45]. Using Eq. (70) we may calculate the angular velocity operator of the neutron spin precession

\[\Omega_{\text{gr}}^a = \varepsilon^{abc} \frac{i}{m} \frac{\partial v}{\partial x^b} T^d_e \frac{\partial}{\partial x^d},\]

(80)

which can be compared with only one term in the angular velocity operator $\Omega_{\text{ingr}}$ in Eq. (79).

Thus, apart from non–vanishing electromagnetic field contributions, which are taken into account in [15, 16], we may argue that the effective low–energy gravitational potential given in Eq. (68) is a generalization of the corresponding results in [15] and [16].
VII. CONCLUSION

We have analysed the Dirac equation for slow fermions within linearised massive gravity \[2, 3\]. We treat massive gravity as a linear expansion above the Minkowski background. According to \[8\] (see also \[9\]), such a theory is applicable to observable phenomena at distances much larger than the Vainshtein radius \( r \gg r_V \), where \( r_V = \sqrt{M/m_g M^2_{Pl}} \), and \( M \) is a gravitating mass. We would like to emphasize that unlike Vainshtein \[9\], who claimed that the radius of applicability of linearised massive gravity is \( r_V = 5 \sqrt{M/m_g^4 M^2_{Pl}} \), we have found that the radius of applicability is \( r_V = \sqrt{M/m_g^2 M^2_{Pl}} \) instead, which is in complete agreement with the analysis by Arkani–Hamed, Georgi, and Schwartz \[8\].

Using Foldy–Wouthuysen transformations we have derived the effective low–energy gravitational potential for slow fermions coupled to the fields of linearised massive gravity. We have used a version of linearised massive gravity employing Stückelberg tricks in which the Fierz–Pauli (FP) massive graviton \( \tilde{h}_{\mu\nu} \) with mass \( m_g \) is decomposed into a linear superposition of a tensor \( h_{\mu\nu} \), vector \( A_\mu \) and scalar field \( \varphi \). The slow fermions couple to these fields only in terms of the FP field \( \tilde{h}_{\mu\nu} \), which are functions of time and spatial coordinates. This implies that the results obtained in this paper can be used for the analysis of the interaction of slow fermions in the terrestrial laboratories coupled to gravitational waves, which are emitted by cosmological objects, within the context of massive gravity \[4\]. This might allow to understand the role of massive gravitons in the dynamics of the evolution of the Universe \[28–31\] and instabilities of black holes \[32\]–[33].

Finally, we have compared our results with the results, obtained by Gonçalves, Obukhov, and Shapiro \[15\] and as well as by Quach \[16\]. We have shown that the effective low–energy potentials, calculated in \[15, 16\], are obtained for massless gravitons and, hence, have no relation to linearised massive gravity. These potentials, obtained as functionals of the metric Eq. (69) for \( u = 0 \) and \( v = v(t - x) \), describing a gravitational wave with one polarization state and propagating along \( x \) axis, cannot describe interactions of static gravitational fields of massive bodies and their effect on slow Dirac fermions. Since a massive graviton has five polarization states \[2\], one might assert that the potentials, derived in \[15, 16\], can only partly account for the effects, which might be analysed in our approach. Hence, one may conclude that the effective low–energy potential Eq. (68) is a generalization to the effective low–energy potentials calculated in \[15, 16\].

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