Online Dynamic Trajectory Optimization and Control for a Quadruped Robot

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Abstract—Legged robot locomotion requires the planning of stable reference trajectories, especially while traversing uneven terrain. The proposed trajectory optimization framework is capable of generating dynamically stable base and footstep trajectories for multiple steps. The locomotion task can be defined with contact locations, base motion or both, making the algorithm suitable for multiple scenarios (e.g., presence of moving obstacles). The planner uses a simplified momentum-based task space model for the robot dynamics, allowing computation times that are fast enough for online replanning. This fast planning capability also enables the quadruped to accommodate for drift and environmental changes. The algorithm is tested on simulation and a real robot across multiple scenarios, which includes uneven terrain, stairs and moving obstacles. The results show that the planner is capable of generating stable trajectories in the real robot even when a box of 15 cm height is placed in front of its path at the last moment.

I. INTRODUCTION

Legged robots can traverse uneven terrains that are not suitable for wheeled robots. However, legged locomotion on uneven terrain is a challenging task, especially as base motion and footstep are coupled in terms of kinematics and dynamics. Moreover, locomotion tasks are constrained by the contact dynamics such as unilateral force and friction cone constraints. These challenges make the generation of dynamically feasible motion trajectories an important part of the locomotion problem.

Nonlinear trajectory optimization (NLOPT) is an attractive approach for motion planning and control of legged robots. High level tasks and system dynamics can be represented using cost functions and constraints [1]. The formulation of the system dynamics and locomotion parameters is particularly important for the performance of the NLOPT. While whole-body dynamics models enable joint reference signal generation and better physics consistency [2], a locomotion task can be described in a more straightforward way with reduced dynamic models [3]. The dynamic model can be simplified further and contact forces and angular momentum dynamics can be excluded from the problem [4]. Contact location, contact timings and motion can be optimized jointly, effectively increasing the search space of the NLOPT [5].

Yet, the existing frameworks find limited use in real-world uneven terrain conditions with changing environments. The algorithms in [12] and [10] have only been implemented on flat terrain. In [13], non-coplanar terrain locomotion is achieved, but the motion was planned offline. These limitations arise from the fact that NLOPT based legged locomotion frameworks depend on good warm starts, carefully tuned task specific cost functions and stable and reachable reference states to act as attractors [14]. Due to the intuition and computation time need, satisfying these conditions in online replanning remains a challenge. Among successful real-world implementations of non-coplanar quadruped locomotion, a 3D mass model with nonlinear zero moment point (ZMP) constraints is used to generate base trajectories, while footsteps are optimized after every base motion optimization [4]. In [15], base motion and footsteps are optimized sequentially at every step and a foothold score map is used to find feasible footstep positions. In both papers, online footstep optimization increased the reachability, yet individual contact
forces were not part of the decision variables and contact stability could not be guaranteed. [13] used phase-based end-effector parameterization to generate optimal base, feet and contact timing trajectories with a 16 footstep horizon. The trajectory was not replanned during motion, which limits the capability of adapting to changes in the environment and state drifts. In [8], both coupled and uncoupled planning were discussed and implemented using a CoP preview model, where coupled planning required significantly longer computation times, but better motion capability.

There are also methods that do not utilize numerical optimization. In [16], a generalized inverted pendulum model based on the analysis of the potential energy surface in human locomotion is used to generate non-coplanar biped trajectories and was extended to quadruped trajectory planning in [17]. In [18], a stabilising foot placement planner is embedded in the whole body task space controller, where the horizontal CoM states of the biped are not controlled, but are the result of the stabilized system dynamics. However, these methods do not consider friction cone constraints that are critical to stabilise locomotion upon uneven terrain. Constrained nonlinear trajectory optimization can generate stable trajectories without such assumptions, where the constraints ensure the dynamic feasibility of the problem and cost functions define a high level locomotion task [5]. Once the decision variables and constraints are selected, this problem can be defined as a Nonlinear Optimal Control Problem (NLOPT) to be solved with a NLOPT solver.

The proposed trajectory optimization framework removes the restrictions of a decoupled approach, and enables fast online re-planning for dynamic quadruped motion that can cope with terrain alterations, drifts and low frequency disturbances. The planner takes a gait pattern and kinematic tasks as inputs and returns reference trajectories for base and feet to be passed to the controller. The algorithm works as a standalone planner on convex terrain by discovering feasible base and feet trajectories and achieves collision-free foot placement. On the other hand, for non-convex terrains, a footstep planner or heuristics is required to select the contact surfaces and nominal contact locations for the optimizer. Body and foot plans are passed to a Projected Inverse Dynamics Controller (PIDC) in Task Space (TS-PIDC) [19] for whole-body control. The impedance behaviour of end-effectors are controlled in the null-space of the contact kinematic constraints, while the contact forces are constrained with higher priority [20]. Although the trajectories are optimized over a reduced dynamic model, TS-PIDC further guarantees the physical constraints through the use of whole body dynamics while tracking the Cartesian space trajectories.

The proposed planner uses Constrained Sequential Linear Quadratic Programming (SLQ) [21] with multiple shooting to solve the NLOPT. We used the ADRL Control Toolbox (CT) [22] to encode the nonlinear problem and the High Performance Interior Point Method (HPIPM) solver [23] to solve the constrained linear quadratic sub-problem. The HPIPM solver exploits the sparsity of the problem for better computation performance and supports inequality constraints through an interior-point method.

The contributions of this paper come from combining the following benefits in a single trajectory optimization and control framework:

- Coupled base and footstep trajectory generation for fast online planning of multiple footsteps without warm-start.
- Obstacle-free contact location discovery on non-coplanar terrain.
- Real world implementation on dynamic non-coplanar terrain that may change during run-time.

II. CONSTRAINED SEQUENTIAL LINEAR QUADRATIC PROGRAMMING

We formulate trajectory optimization as a nonlinear optimal control problem as follows,

\[
\min_{x,u} \phi(x_{f_f}) + \sum_{t_0}^{t_f} L_i(x,u,t) \\
\text{s.t.} \quad \dot{x} = f_i(x,u), \\
g_i(x,u,t) = 0, \quad h_i(x,u,t) \geq 0, \\
x \in \mathcal{X}, \quad u \in \mathcal{U}
\]  

(1)

where \(x,u\) are state and control input vectors respectively, \(f_i\) is system dynamics, \(\phi, L\) are final and running costs, and \(g_i, h_i\) are phase equality and inequality constraints. Details of each variable and constraint equation are given in the section 3 and 4.
The problem in Eq. (1) is solved with SLQ, where system dynamics are linearized around the result of the previous solution and the cost function is quadratically approximated if not already quadratic. The resulting problem is Linear Quadratic in terms of state and input increments $\delta x$, $\delta u$ and is solved sequentially with a constrained QP solver [23].

III. QUADRUPEL MODEL

The reduced momentum model we implement is formulated in task space variables. This formulation reduces the nonlinearity that is introduced by the joint space variables in the system dynamics and its constraints, simplifying the computational complexity of the algorithm and avoiding unneeded calculations.

A. Dynamics Model

The reduced momentum dynamics model for a single rigid body system with contacts is as follows:

\[
mp\ddot{p} = \sum_{i=1}^{k} \lambda_i + mg \tag{2a}
\]

\[
I_b\ddot{\omega}_b + \omega_b \times I_b\dot{\omega}_b = \sum_{i=1}^{k} (r_i - p) \times \lambda_i \tag{2b}
\]

\[
\dot{\theta} = R(\theta)\omega_b \tag{2c}
\]

\[
\dot{r}_i = v_i \tag{2d}
\]

where $m$ is the total mass of the robot, $g$ is the gravitational acceleration, $I_b$ is the torso inertia tensor in base frame, $i$ is the leg index and $k$ is the total number of legs. The states $p, \theta$ are the base position and orientation in XYZ Euler angles, $\dot{p}, \omega_b$ are the base linear velocity and angular velocity in base frame and $r_i$ is the position of foot $i$ in base frame as well. As inputs to the system, we use $\lambda_i$ as the contact force vector and $v_i$ as foot velocity vector. $R(\theta)$ is the rotation matrix that transforms base frame to world frame and $\dot{\theta}$ is the derivative of the base orientation Euler angles. The state and control input vectors of the momentum model are as follows,

\[
x = \begin{bmatrix} p \\ \dot{p} \\ \omega_b \\ r_i \end{bmatrix}, \quad u = \begin{bmatrix} \lambda_i \\ v_i \end{bmatrix} \tag{3}
\]

In previous works [24][12], contact forces and joint space velocities have been set as control inputs and the generated trajectories were tracked using a joint space inverse dynamics controller. In our case while both state and control input trajectories are generated with SLQ, only the former in task space is tracked with TS-PIDC. Use of task space variables removes the extra nonlinearity introduced through the joint space variables from the trajectory optimization, as the joint torques are calculated at higher frequency through TS-PIDC for the optimized end effector trajectory, desired task space impedance behaviour and satisfaction of contact force constraints.

B. Constraints

The input constraints of the dynamic model in the optimization problem are unilateral and linearized friction cone and feet velocity constraints (4a) - (4c) for active contacts and zero contact force (4d) for inactive contacts. The state constraints are contact position constraint (4e).

\[
b_u, \lambda_i \geq 0 \quad \text{(stance leg)} \tag{4a}
\]

\[
U \lambda_i \leq 0 \quad \text{(stance leg)} \tag{4b}
\]

\[
\dot{r}_i = 0 \quad \text{(stance leg)} \tag{4c}
\]

\[
\lambda_i = 0 \quad \text{(swing leg)} \tag{4d}
\]

\[
c_i(r_i) = 0 \quad \text{(stance leg)} \tag{4e}
\]

The function $c_i(r_i)$ represents the contact surface equation. $b_u$ is the upper bound of the contact force box constraint. $U$ is the linearized friction cone inequality constraint.

The unilateral and friction cone constraints in (4a)- (4b) enforce contact force stability, whereas end-effector velocity constraint (4c) prevents movements of a foot in stance phase and (4d) ensures no contact force is generated by a swing leg. (4e) constrains the end effector to lie on a contact surface $c_i(r_i)$. Note that kinematic reachability is not presented as a hard constraint, but as a quadratic cost term in Section IV-A.

IV. QUADRUPEL LOCOMOTION

The proposed planner assumes that a feasible base and footstep trajectory exist around a nominal base and footstep position sequence, and the momentum generated by the legs are zero. Given a nominal base and footstep sequence with fixed timings, a constrained trajectory optimization problem is formulated. The contact planes of the nominal footstep sequence are used to generate contact surface constraints. Then, the footstep, base sequence and an analytical cost function are used to define the task. The solution of this problem yields dynamically feasible base and feet trajectories that are in the vicinity of the nominal sequences.

The locomotion task is described with a gait pattern, contact planes, running and final cost. By adjusting the final cost weight of base and footsteps, the user can decide which one will be predominant to describe the task. An overview of trajectory optimization framework and controller interface are presented in Fig. 2 and described in Algorithm 1.

A. Trajectory Optimization

In our reduced momentum dynamics model formulation, feet dynamics are modelled as a first order system in Cartesian space. Continuous swing leg trajectories are generated via a fifth order polynomial spline, with position boundary conditions set by the solution of the optimisation and a desired swing height.

In our formulation, we use quadratic cost functions. This cost formulation has convergence and tuning benefits and its derivatives are immediately known, which reduces the computational complexity [25]. The cost functions we use in the optimization are in the form of:

\[
L(t) = \ddot{x}_d^T Q_d \ddot{x}_d + \dot{\theta}^T Q_\theta \dot{\theta} + \ddot{u}^T R \ddot{u} \tag{5}
\]
\[ \phi(x_{t_f}) = \tilde{x}_{d_f}^T Q_{d_f} \tilde{x}_{d_f} \]  

where \[ \tilde{x}_d = x_d - x(t), \quad \tilde{x}_h = x_h - x(t) \]

\[ \tilde{x}_{d_f} = x_{d_f} - x(t_f), \quad \tilde{u} = u(t) \]

\[ Q_d = \text{diag}(Q_{\text{base}}, Q_{\text{footstep}}) \]

\[ R = \text{diag}(R_{\text{contact}}, R_{\text{velocity}}) \]

The reference states \( x_d \) and \( x_{d_f} \) represent the desired intermediate and final base pose, twist and footsteps. \( Q_d \) is the quadratic state cost weight matrix. \( Q_{\text{base}} \) and \( Q_{\text{footstep}} \) are the corresponding weight terms of the base and footstep tasks. The running and final state cost weight matrices \( Q_d \) and \( Q_{d_f} \) have the same form, with different weights. This representation provides a straightforward way to describe a high-level task.

Input cost weights \( R_{\text{contact}} \) and \( R_{\text{velocity}} \) are for the contact force and footstep velocity. In our approach constrained contact forces are used to enforce dynamic feasibility. They are not tracked by the inverse dynamics controller, yet penalizing them provides numerical stability to the optimization.

We formulate the reachability cost as a linear least-square interpolation between a default stance and a sloped stance, where \( x_h \) is the state reference and \( Q_h \) is the weight matrix (see the Appendix). This suboptimal, but convex approximation reduces the amount of computation by removing their numerical gradient calculations. It should be noted that all quadratic cost weight matrices are diagonal except \( Q_h \).

By adjusting the state cost weights \( Q_{\text{base}}, Q_{\text{footstep}} \), one can determine if the task is defined predominantly with base pose, twist or footstep positions. With high \( Q_{\text{base}} \) low \( Q_{\text{footstep}} \), the optimizer finds footstep positions that comply with the base task and vice versa. In our implementations, the desired \( \theta \) and \( \omega_b \) are zero. The resulting orientation state, including the Yaw (Euler Y) angle in Figure 8 is due to the reachability cost terms \( Q_h \) and \( x_h \).

B. Nominal Base Pose and Footstep Position

Nominal base and footstep sequence \( X_d \) is a set of points that guide the trajectory optimizer to a feasible solution. \( X_d \) is encoded to the problem as final cost reference \( Q_{d_f} \) in Eq. 6. We generated \( X_d \) using a similar approach as in [15], where a geometrical approach with a default stance and footstep length is used to generate nominal footstrokes towards the goal. The difference in our method is, we used contact surface planes as constraint (4c) and the nominal footholds as final cost reference to simultaneously optimize base and footstep trajectories, rather than searching the region around the nominal foothold and optimizing footsteps and base pose sequentially. If the nominal foothold is not in contact with the environment, it is projected on the surface plane in vertical direction.

Since we used surface constraints with analytical gradients, instead of scoring the terrain cells for feasibility, our method relies on heuristics to provide clearance margins between the footsteps and non-convex surface edges.

C. Gait Sequence

A gait phase is described by its stance configuration \( e_i \in \{ \text{Left Front (LF), Right Front (RF), Left Hind (LH), Right Hind (RH), On Four (F)} \} \) and the corresponding duration \( d_i \). Three contact configurations are represented with swing leg IDs \{LF,RF,LH,RH\} and four with F. Phase configuration and duration sequences \( E \) and \( D \) form a gait sequence as seen in Fig. 3. In cyclic gait, each phase is revisited periodically. \( E \) and \( D \) are determined by using the findings in [26] and intuition.

D. Control

The result of the trajectory optimization is the input of TS-PIDC. The task space control law is

\[ F = h_c + \Lambda_c \ddot{x}_d - D_d \dot{x} - K_d x \]  

where \( \Lambda_c \) is the constraint-consistent operational space inertia matrix, \( h_c \) are the non-linear terms, \( D_d \) and \( K_d \) are desired damping and stiffness matrices, \( \ddot{x}_d \) is the desired
acceleration, \( \dot{x} \) and \( \ddot{x} \) are velocity and position tracking errors. The joint torque commands for trajectory tracking can be derived from Eq. (7)

\[
\tau_m = (PB)^{+}\mathbf{J}_x^{T}F
\]

where \( \tau_m \) is the motion space torques, \( P \) is the contact kinematic constraint null-space projector, \( B \) is the selection matrix for floating base systems, \( \mathbf{J}_x \) is the Jacobian matrix that maps external forces/torques to joint space. The final torque commands are the result of a QP optimization considering dynamic constraints plus \( \tau_m \) as follows

\[
\tau = \tau^* + (\mathbf{I} - N_s)\tau_m
\]

where \( N_s \) is the null space projector of the swing leg, \( \tau^* \) is the optimized torque command based on the torso motion. Further details can be found in [19].

V. Results

An ANYmal quadruped robot [27] has been used for both real experiments and simulations. A VICON motion capture system has been used for measuring the global positions of the robot and the obstacle. The planner and controller run on separate on-board computers and communicate via ROS network. A video of the experimental results can be found in https://bit.ly/3gyccU6.

The planner is set with a two footstep horizon on flat terrain forward walking motion; \( E = [F, RH, F, RF, F] \) and \( D = [0.3, 0.3, 0.25, 0.3, 0.3] \), 1.55 seconds trajectory, with 0.02 seconds sampling time. The constraints (4a)-(4c) are active for their respective phases. Average iteration time for \( 10^4 \) tests is 18,340 milliseconds on a consumer grade laptop with 2.80GHz CPU and it takes 4 iterations for convergence without warm-start.

In real-time simulations and real-world experiments, the planner computes base and feet trajectories for the next two steps using the latest state estimation, at every contact event. Then, the two step plan with timestamp is sent to the 400Hz TS-PIDC controller that runs on a second on-board computer. In our experiments only the first step from every planned trajectory is tracked by the controller before receiving a new input from the planner. Although the second step of the plan is not used at the controller level, it plays a fundamental role in the trajectory optimization by acting as a statically stable attractor, which provides a zero velocity terminal cost without requiring the robot to have a zero velocity at the end of each step. This can be seen in Figure 4.

We tested the planner and controller on several cases in simulation other than flat terrains, namely: relocated box climbing, stair climbing (see video), gap crossing, obstacle avoiding footstep discovery and on the real robot, a relocated box (see video). We did not use any warm start, but instead we used quadratic costs with nominal final state references to guide the solution towards the desired tasks.

A. Base and Swing Leg Motion

In simulation, stable base motions can be realised by penalizing the velocities and normal directions of the contact forces. On real robot uneven terrain locomotion, additional intermediate position costs were added to move the base closer to support polygon. This makes the local optimum solution more reliant on the reference states, but they are needed to plan feasible trajectories due to model inaccuracies, tracking errors and occasional slips. Increasing the planning frequency is a possible solution, but the computation time remains a limiting factor. In Figure 5 and 6 the discontinuities between the consequent first plans are caused by the tracking errors and time spent during the trajectory optimization. The delay in the issuing a new plan implied that the TS-PIDC tracked the second step plan for a brief time before receiving the new step plan. The foot swing height is set prior to a task. For uneven terrain tasks, higher swing foot heights are used to be conservative against collisions with the environment. Also for uneven terrain, terms such as minimum clearance to a the edges and maximum distance between the feet are introduced. Although the analytical reachability cost itself encourages the latter, we have observed the benefits of these additional heuristics.

B. Contact Forces

Contact forces are subject to high frequency changes, thus, contact forces from the trajectory optimisation are not tracked by the low-level controller. The TS-PIDC optimizes the contact forces via a constrained QP. In flat terrains, contact forces in \( z \) direction have a unique solution for a base state in the reduced momentum model. For this reason, simulation contact forces are similar to the trajectory optimisation contact forces in \( z \) direction as seen in Fig. 7.

C. Uneven Terrain Locomotion

1) Relocated Box Climbing: The box in both the simulation and real world experiment demonstrated is 15cm in height. In this scenario, the box is relocated to an arbitrary position on the movement direction of the robot at an arbitrary time to test the online optimization capability of the robot. As demonstrated in the simulation, the planner can generate a new feasible trajectory even when the previous trajectory would violate the new contact surface constraints. In the real world experiment, a VICON Tracker system is used for the global localization of the robot base and the movable box. To demonstrate the effectiveness of online replanning, the box is moved in front of the robot during locomotion. When and where the box would be moved is not predetermined.

2) Gap Crossing with Height Difference: The gap crossing scenario shown in Figure 4 consists of two planes 30cm apart where the second plane is 10cm higher. Our method treats the gap as a rectangular box volume not to be assigned any contact on, otherwise it is treated as a climbing down and up task. When a nominal footstep is assigned on a gap, it is checked if it covers more than half of the gap. If it can, then the footstep is shifted forward to the second plane, otherwise it is shifted backwards to the first plane. Base position and yaw angle orientation states for the 14 footstep gap crossing with height difference task are shown in Figure 8.
D. Footstep Position Discovery and Obstacle Avoidance

By setting the $Q_{\text{footstep}}$ lower than the $Q_{\text{base}}$, we defined a base task and let the optimizer converge to footstep positions that are locally optimal to the base task on flat terrain. We provided infeasible nominal footsteps, yet the optimizer converged to feasible footsteps that satisfy the base task. We then introduced a 10 cm diameter spherical obstacle as an inequality constraint and the resulting footsteps successfully avoided stepping on it. We used 4 step horizon in the footstep discovery and obstacle avoidance task. The base and footstep trajectories are shown in Fig. 9 and simulation is presented in the video.

E. Limitations and Future Work

In the test scenarios, four feet support phase durations are less than 400 ms and the solver needs to converge to a feasible solution within this time window. The main reason behind the choice of number of footstep in the horizon is that the convergence of a two footstep trajectory without warm-start takes roughly 100 ms on the on-board PC of the robot, depending on the gait phase durations and terrain constraints. Convergence for higher number of footstep require more iterations with increased computation time per iteration.

In our method, the duration and equation of the switching constraints change with the moving horizon. Thus, the nonlinear optimal control problem is reinitialized each time the robot makes a new contact. Eliminating the computation cost of the reinitialization and exploiting the multi-thread capability of the solver can reduce computation times and allow online optimization for larger number of footsteps. Increasing the number of footsteps per horizon or using a database for final state cost reference can improve the capability of exploiting momentum of the system and enable handling more challenging tasks.

Reachability is represented as an analytical cost function in the proposed optimization method. Although such a cost function requires less computation and is less sensitive to initial guess, it can perform poorly when a foot in the final phase is far from the 2D least squares (LS) fit plane of contacts. A nonconvex reachability cost may perform better if provided with an adequate initial guess [15].

The main limitation with the footstep discovery is that the contact planes are selected through the vertical projection of
the nominal footsteps and/or heuristics. Introducing contact planes as decision variables would significantly increase the complexity of the problem and require good initialisation to solve. Thus, the obstacle avoiding footstep position discovery is implemented for a convex terrain with a convex obstacle. A sampling-based high level motion planner such as Humanoid Path Planner (HPP) [28] can be used to generate the foothold and base sequence $E, X$ for the Algorithm 1.

We provided minimal amount of reference state for the optimisation. Running cost state reference is time invariant and both running and final cost state references come from the nominal base position and footsteps. In this case, having a horizon of two or more steps is critical. Shorter horizons need more sophisticated, time variant reference states and providing these reference states itself becomes a trajectory optimisation problem. As an example, a QP based controller is an MPC with zero horizon length that needs a nontrivial reference state to reach a high level goal.

Our framework has limited push recovery capacity due to the short prediction horizon and optimizing only at feet touch downs. Reduced computation time can allow intermediate trajectory optimizations during the swing phases and improve this aspect by generating reactive footstep motions similar to the capture point method [29]. Alternatively, the divergence between the actual state and the state trajectory can be used as a stability metric to trigger a capture point algorithm in a finite state machine fashion. In fact, achieving uneven terrain locomotion and push recovery behavior within the same trajectory optimization framework remains an open problem.

**VI. Conclusion**

In this paper, we propose a framework for online dynamic trajectory optimization and control for quadruped robots. The online trajectory optimization is based on optimizing base and footstep trajectories simultaneously around a nominal sequence. The nominal sequence both serves as a goal and a statically stable attractor for the constrained optimizer to find dynamically feasible trajectories. The trajectory optimizer runs at every foot touchdown for online replanning. A task space projected inverse dynamics controller tracks base and feet trajectories and may sacrifice tracking performance to satisfy physical constraints when necessary. The application of the framework on non-coplanar terrain is demonstrated in simulation and real world experiments. Future work will focus on increasing the computation efficiency to optimize for longer horizons online and enable intermediate optimizations during swing phase.

**APPENDIX**

**A. QP form from LS**

The reachability term is the result of the LS solution of the interpolation between a nominal posture and an altered posture, as shown in Figure 10. In Figure 10, $h_n = \overline{OC}^2$.

![Fig. 10. Geometric representation of the reachability approximation. Nominal posture is dashed black and the altered posture solid red body. N and C are the body frames of the nominal and altered posture and O is the world frame.](image-url)
We write our LS problem in quadratic cost function form

\[ w_x = x_2 - x_1 \] and \( w_y = y_2 - y_1 \) and \( w_z = z_2 - z_1 \) are nominal x, y distances between the feet.

\[ q_x = \tan(\alpha_y) \cdot h_n / h_c, \quad q_y = \tan(\alpha_x) \cdot h_n / h_c \]

\[ r_x = \frac{1}{2} \alpha_y / h_c, \quad r_y = \frac{1}{2} \alpha_x / h_c \]

Linear set of equations:

\[
\begin{align*}
p_x &= \frac{3}{4} \sum_{i=0}^{3} r_{ix} + \frac{1}{2} d_x (r_0 - r_1 + r_2 + r_3)
\end{align*}
\]

\[
\begin{align*}
p_y &= \frac{3}{4} \sum_{i=0}^{3} r_{iy} + \frac{1}{2} d_y (r_0 - r_1 + r_2 - r_3)
\end{align*}
\]

\[
\begin{align*}
p_z &= h_n + \frac{3}{4} \sum_{k=0}^{3} r_{iz}
\end{align*}
\]

\[
\begin{align*}
\theta_x &= q_x (r_0 - r_1 + r_2 - r_3)
\end{align*}
\]

\[
\begin{align*}
\theta_y &= q_y (r_0 - r_1 + r_2 + r_3)
\end{align*}
\]

\[
\begin{align*}
\theta_z &= 0
\end{align*}
\]

Equation set (10) is represented as \( Ax = b \) and solved via LS \( \frac{1}{2}||Ax - b|| \). The intermediate quadratic cost function for the linear optimal control problem is in the form of

\[
L = (x - x_{ref})^T Q_h (x - x_h)
\]

\[
= x^T Q_h x - 2x^T Q_h x_h
\]

We write our LS problem in quadratic cost function form

\[
Q_h = A^T A, \quad x_h = Q_h^+ A^T b
\]

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