A Star Product for Differential Forms on Symplectic Manifolds

Anthony Tagliaferro

1Department of Physics, University of California, Berkeley
2Theoretical Physics Group, LBNL, Berkeley, CA 94702, USA

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Abstract

We present a star product between differential forms to second order in the deformation parameter $\hbar$. The star product obtained is consistent with a graded differential Poisson algebra structure on a symplectic manifold. The form of the graded differential Poisson algebra requires the introduction of a connection with torsion on the manifold, and places various constraints upon it. The star product is given to second order in $\hbar^2$. 

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†Electronic address: atag@berkeley.edu
Since the early days of quantum mechanics, physicists have used star products to build noncommutative generalizations of commuting theories (see [4] for a historical treatment and the references therein). From describing theories on noncommutative spaces to providing alternate methods of quantization [1], star products and, more generally, the procedure of deformation quantization have given physicists a way to generalize commutative theories into noncommutative theories.

Deformation quantization begins with a manifold $M$ and the ring of smooth functions $C^\infty(M)$. The ring of commutative smooth functions $C^\infty(M)$ is deformed to a noncommutative ring of smooth functions, $C^\infty(M, *)[\hbar]$, where the deformation is parametrized by $\hbar$. We denote the noncommutative product of $f, g \in C^\infty(M, *)[\hbar]$ by the star product:

$$f \ast g = fg + \hbar\{f, g\} + \hbar^2 C_2(f, g) + O(\hbar^3)$$

where $\{f, g\} = \pi^{ij}\partial_i f \partial_j g$ is a Poisson bracket on $M$. Smooth functions $f, g \in C^\infty(M)$ are elements of the noncommutative algebra, and in fact, as a set $C^\infty(M) \subset C^\infty(M, *)[\hbar]$, but with a new product between elements. Although $C^\infty(M, *)[\hbar]$ is not commutative, it is required to be associative. Associativity places strong conditions on the form of the star product. The star product of functions is well known from the work of Kontsevich [7], Cattaneo and Felder [2], and many others, including [8]. We generalize the star product to act between differential forms, to $O(\hbar^2)$.

Gauge theories involve the differential form $A_\mu dx^\mu$, so to study gauge theories on noncommutative spaces, the star product must be extended to include differential forms. With this motivation, we deform the graded exterior algebra of differential forms, $\Omega^*(M)$, to a noncommutative graded exterior algebra $\Omega^*(M, *)[\hbar]$. Thinking of functions as 0-forms, we need to extend the star product to include forms of arbitrary degree:

$$\alpha \ast \beta = \alpha\beta + \hbar\{\alpha, \beta\} + \hbar^2 C_2(\alpha, \beta) + O(\hbar^3)$$

where $\{\alpha, \beta\}$ is the generalization of the Poisson bracket to differential forms, and $\alpha\beta$ denotes the wedge product\(^1\). The case of constant Poisson bivector, $\pi^{ij}$ is well understood, but we are interested in the case that $\pi^{ij}$ is not constant. Modulo some conditions to be discussed later, it is possible to define a consistent star product on the space of differential forms.

\(^1\) Unless otherwise specified, the product between two differential forms is the wedge product: $\alpha\beta = \alpha \wedge \beta$
In this note we give the explicit form of the star product for differential forms for nonconstant $\pi^{ij}$, to $O(\hbar^2)$. First we must generalize the Poisson bracket to differential forms. To do so, we write down the properties a graded differential Poisson algebra should satisfy, following [3] and [6]. After some motivation, we define an explicit graded Poisson bracket between arbitrary forms, and check that it satisfies the properties required of a graded differential Poisson bracket. We show that these properties place several conditions on our manifold (these conditions were also found in [3] and [6]). Lastly, we propose an $O(\hbar^2)$ product between differential forms and show that it satisfies the properties of a star product to that order. In particular, we show explicitly that our proposed product is associative, a crucial property of the star product.

II. PROPERTIES OF A DIFFERENTIAL POISSON ALGEBRA

The Poisson bracket between two functions, $\{f, g\}$, is ubiquitous in the theory of classical mechanics, and its properties are widely known (see [10] among many others). The basic properties are:

1. Skew-symmetry: $\{f, g\} = -\{g, f\}$

2. Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{g, f\}\} = 0$

3. Leibniz rule: $\{f, gh\} = \{f, g\} h + g \{f, h\}$.

Define the Poisson bivector $\pi^{ij}$ by:

$$\{f, g\} = \pi^{ij} \partial_i f \partial_j g. \quad (1)$$

Because the Poisson bracket obeys the Jacobi identity, $\pi^{ij}$ must satisfy the following condition:

$$\pi^{im} \partial_m \pi^{jk} + \pi^{jm} \partial_m \pi^{ki} + \pi^{km} \partial_m \pi^{ij} = 0. \quad (2)$$

Since functions are 0-forms, any Poisson bracket on the space of differential forms, $\Omega^*(M)$, must reduce to this simple Poisson bracket when restricted to 0-forms.

When a Poisson bracket is defined on a manifold, the manifold is called a Poisson manifold. If the Poisson bivector is invertible, its inverse, $\omega_{ij}$, is called the symplectic 2-form and the manifold a symplectic manifold. We use the convention:

$$\pi^{ij} \omega_{jk} = \delta^i_k$$
Equation (2) is equivalent to the condition \( d\omega = 0 \), where \( \omega = \frac{1}{2} \omega_{ij} dx^i dx^j \). For simplicity, we take the manifold \( M \) to be symplectic, but the results may generalize to the more general Poisson manifold case.

Following [3], we generalize the Poisson bracket to include differential forms. Denote by \( |\alpha| \) the degree of the form \( \alpha \). For all forms \( \alpha, \beta, \) and \( \gamma \), a graded differential Poisson algebra on a Poisson manifold must satisfy:

1. Bracket degree:
\[
|\{\alpha, \beta\}| = |\alpha| + |\beta| \tag{3}
\]

2. Graded symmetry:
\[
\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\} \tag{4}
\]

3. Graded product rule:
\[
\{\alpha, \beta \gamma\} = \{\alpha, \beta\} \gamma + (-1)^{|\alpha||\beta|}\beta\{\alpha, \gamma\} \tag{5}
\]

4. Leibniz rule for the exterior derivative, \( d \):
\[
d\{\alpha, \beta\} = \{d\alpha, \beta\} + (-1)^{|\alpha|}\{\alpha, d\beta\} \tag{6}
\]

5. Graded Jacobi identity:
\[
\{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha|(|\beta|+|\gamma|)}\{\beta, \{\gamma, \alpha\}\} + (-1)^{|\gamma|(|\alpha|+|\beta|)}\{\gamma, \{\alpha, \beta\}\} = 0 \tag{7}
\]

These properties naturally combine the defining characteristics of differential forms and the Poisson bracket. The Leibniz rule and the Jacobi identity place strong constraints on how the Poisson bracket acts on differential forms. As we show in the next section, Properties (1-5) uniquely determine the form of the Poisson bracket.

III. THE POISSON BRACKET

In this section, we motivate the form of the Poisson bracket between two arbitrary differential forms on a symplectic manifold. We start by using Properties (1-5) to determine the Poisson bracket between a function and an arbitrary differential form. Next, we introduce the most general Poisson bracket between two differential forms and discuss some of the conditions that arise. More detailed calculations are included in Appendix [B].
A. The Poisson Bracket between a function and a form

In this section, we show that for a symplectic manifold, there is a unique Poisson bracket consistent with the Poisson algebra. In particular, we show that the object \( \{ f, \cdot \} \) acts as a covariant derivative on the space of differential forms:

\[
\{ f, \alpha \} = \nabla_V \alpha
\]

To be a connection, \( \nabla_V \alpha \) must satisfy the following properties for \( f, g \) functions, \( V \) a vector field, and a form \( \alpha \):

1. \( \nabla_V (f) = V(f) \)
2. \( \nabla_V (f \alpha) = V(f) \alpha + f \nabla_V \alpha \)
3. \( \nabla_V (\alpha + \beta) = \nabla_V \alpha + \nabla_V \beta \)
4. \( \nabla_{(V+W)} \alpha = \nabla_V \alpha + \nabla_W \alpha \)
5. \( \nabla_V (\alpha \wedge \beta) = (\nabla_V \alpha) \wedge \beta + \alpha \wedge (\nabla_V \beta) \)
6. \( \nabla_{(gV)} \alpha = g \nabla_V \alpha. \)

The Poisson bracket \( \{ f, \cdot \} \) satisfies these properties with \( V^j = \pi^{ij} \partial_i f \). Multiplying \( V^j = \pi^{ij} \partial_i f = -\pi^{ji} \partial_i f \) by \( \omega_{jk} \), we have

\[
\partial_k f = \omega_{jk} V^j,
\]

which has as the solution

\[
f = \int_{x_0}^{x} \omega_{jk} V^j dx^k.
\]

Before checking that the bracket \( \{ f, \alpha \} \) is a covariant derivative of the second argument, note that it is a differential operator of order 1 in the first argument (e.g. \( \{ 1, \alpha \} = 0 \) and \( \{ fg, \alpha \} = \{ f, \alpha \} g + f \{ g, \alpha \} \)). This implies:

\[
\{ f, \alpha \} = (\partial_i f) \{ x^i, \alpha \}.
\]

A simple proof follows from Taylor expanding \( f \) inside the Poisson bracket and noticing that only the term linear in \( x^i \) does not vanish:

\[
\{ f, \alpha \} \bigg|_{x=x_0} = \lim_{x \to x_0} \left[ f(x_0), \alpha \right] + \left( \partial_i f(x_0) \right) \{ x^i - x_0^i, \alpha \} + \left( \frac{1}{2} \partial_i \partial_j f(x_0) \right) \{ (x^i - x_0^i)(x^j - x_0^j), \alpha \} + \ldots \right].
\]

From this, we see that \( \{ f, \alpha \} = V^i \omega_{ij} \{ x^j, \alpha \} \), which makes it easy to prove that \( \{ f, \cdot \} \) satisfies all the properties of a covariant derivative. Note that our arguments depend on the existence of \( \omega_{ij} \).

Although a general Poisson manifold, for which \( \pi^{ij} \) is not invertible, might have a different Poisson bracket, we do not consider this case.
Inserting the vector field $V$, the Poisson bracket for a function $f$ and a form $\alpha$ is,

$$\{ f, \alpha \} = \pi^{ij} \partial_i f \nabla_{e_j} \alpha$$

This form of the Poisson bracket agrees with the bracket put forth by [3] and [6]. The action of the connection on a basis 1-form may be expressed with the connection coefficients $\Gamma_{jk}^{i}$:

$$\nabla_{e_i} dx^k \equiv -\Gamma_{ij}^k dx^j.$$  

(8)

In general, this connection has torsion, so two connections can be defined from the same connection coefficients:

$$\Gamma_{i}^{k} = dx^\ell \Gamma_{\ell i}^{k} \quad \quad \tilde{\Gamma}_{i}^{k} = \Gamma_{i\ell}^{k} dx^\ell,$$

which act on 1-forms as:

$$\nabla_{e_i} dx^k = -\Gamma_{ij}^k dx^j = -\Gamma_{i i}^k \quad \quad \tilde{\nabla}_{e_i} dx^k = -\tilde{\Gamma}_{ij}^k dx^j = -\Gamma_{j i}^k,$$

where $\tilde{\Gamma}_{ij}^k = \Gamma_{ji}^k$ are the connection coefficients associated with $\tilde{\nabla}$. Notation is discussed in more detail in Appendix A.

The graded differential Poisson algebra structure imposes several conditions upon the connections that appear in this expression for the Poisson bracket. For example, for the Leibniz rule to hold, the Poisson bivector $\pi^{ij}$ must be covariantly constant under the connection $\tilde{\nabla}_i$. The Leibniz rule requires that:

$$d\{ x^i, x^j \} = \{ dx^i, x^j \} + \{ x^i, dx^j \}$$

$$d\pi^{ij} = \pi^{im} \Gamma_{mk}^{i} dx^k - \pi^{im} \Gamma_{mk}^{j} dx^k.$$

Using the definition of the Poisson bracket to calculate each of the quantities, we see that imposing the Leibniz rule requires the following condition on the connection,

$$d\{ x^i, x^j \} = \{ dx^i, x^j \} + \{ x^i, dx^j \}$$

$$d\pi^{ij} = \pi^{im} \Gamma_{mk}^{i} dx^k - \pi^{im} \Gamma_{mk}^{j} dx^k.$$

Using the definition of the Poisson bracket to calculate each of the quantities, we see that imposing the Leibniz rule requires the following condition on the connection,

$$\left( \tilde{\nabla}_{e_k} \pi \right)^{ij} = \partial_k \pi^{ij} + \Gamma_{mk}^{i} \pi^{mj} + \Gamma_{mk}^{j} \pi^{im} = 0.$$

(9)

Thus $\pi^{ij}$ is covariantly constant under $\tilde{\nabla}_{e_k}$, and, equivalently, $\omega_{ij}$ is covariantly constant under $\tilde{\nabla}_{e_k}$. We refer to $\tilde{\nabla}_{e_k}$ as a symplectic connection\(^2\), because it annihilates the symplectic 2-form.

One can use the symplectic condition (9) to rewrite the Jacobi identity (2) in terms of the torsion:

$$\pi^{im} \pi^{j\ell} \left( \Gamma_{\ell m}^{k} - \Gamma_{\ell m}^{k} \right) + \pi^{im} \pi^{k\ell} \left( \Gamma_{m}^{i} - \Gamma_{m}^{i} \right) + \pi^{km} \pi^{i\ell} \left( \Gamma_{\ell m}^{j} - \Gamma_{\ell m}^{j} \right) = 0$$

Note that while this relation shows that a torsionless connection identically satisfies the Jacobi identity, the Jacobi identity does not require the connection to be torsionless.

\(^2\) In the literature, symplectic connections are usually taken to be torsionless, but the connection defined here generally has torsion.
B. The Poisson bracket between two 1-forms

To find the form of the Poisson bracket for two forms, we use the Leibniz rule for basis 1-forms:

\[ \{dx^i, dx^j\} = d\{x^i, dx^j\}. \]

The Poisson bracket for a function and a 1-form gives:

\[ \{dx^i, dx^j\} = d(\{x^i, dx^j\}) = -\partial_a \pi^i \Gamma^j_{a \ell} dx^a dx^b - \pi^i \partial_a \Gamma^j_{\ell b} dx^a dx^b. \]

The symplectic condition of \( \tilde{\nabla}_{e_k} \) relates the derivative of \( \pi^{ij} \) to the connection coefficients \( \Gamma^k_{ij} \), which gives

\[ \{dx^i, dx^j\} = (\Gamma^i_{ma} \pi^{m \ell} + \Gamma^\ell_{ma} \pi^{im}) \Gamma^j_{\ell b} dx^a dx^b - \pi^i \partial_a \Gamma^j_{\ell b} dx^a dx^b = -\pi^i \partial_a \Gamma^j_{mb} - \Gamma^j_{\ell b} \Gamma^\ell_{ma}) dx^a dx^b + \pi^m \Gamma^i_{ma} \Gamma^j_{\ell b} dx^a dx^b. \]

The term in the parentheses can be rewritten in terms of the curvature \( \tilde{R}^{ij}_{mab} \) of the symplectic connection \( \tilde{\nabla}_{e_k} \) using the anti-symmetry of \( dx^a dx^b \):

\[ -\pi^i \partial_a \Gamma^j_{mb} - \Gamma^j_{\ell b} \Gamma^\ell_{ma}) dx^a dx^b = \frac{1}{2} \pi^i \tilde{R}^{ij}_{mab} dx^a dx^b \equiv \frac{1}{2} \tilde{R}^{ij}_{ab} dx^a dx^b. \]

The Poisson bracket between two basis 1-forms is then:

\[ \{dx^i, dx^j\} = \pi^{m \ell} \Gamma^i_{ma} \Gamma^j_{\ell b} dx^a dx^b - \frac{1}{2} \tilde{R}^{ij}_{mab} dx^a dx^b. \]

To find the expression in terms of two arbitrary 1-forms, not just in terms of the basis \( dx^i \), consider two 1-forms \( \alpha, \beta \) and use the product rule:

\[ \{\alpha, \beta\} = \{\alpha_i dx^i, \beta_j dx^j\} = \{\alpha_i dx^i, \alpha_j dx^j\} + \{\alpha_i dx^i, \beta_j dx^j\} + \{\alpha_i dx^i, \beta_j dx^j\} = \pi^{mn}(\partial_m \alpha_i - \Gamma^m_{ni} \alpha_l) dx^i (\partial_n \beta_j - \Gamma^k_{nj} \beta_k) dx^j - \frac{1}{2} \alpha_i \beta_j \tilde{R}^{ij}_{ab} dx^a dx^b. \]

After defining the 2-form

\[ \tilde{R}^{ij} = \frac{1}{2} \tilde{R}^{ij}_{ab} dx^a dx^b, \]

\[ \alpha \] Note that \( \tilde{R}^{ij}_{ab} \equiv \pi^{im} \tilde{R}^{ij}_{mab} \), which we often refer to as the curvature. Our notation is discussed further in Appendix A.
the last line gives:

\[ \{\alpha, \beta\} = \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta - \tilde{R}^{ij} \alpha_i \beta_j. \]

For later convenience, let us also introduce the interior product, \( i_{e_m} \), and rewrite the bracket between two 1-forms as:

\[ \{\alpha, \beta\} = \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta - \tilde{R}^{mn}(i_{e_m} \alpha)(i_{e_n} \beta) \quad |\alpha| = |\beta| = 1. \]

An important property of the curvature tensor \( \tilde{R}^{ij} \) is its symmetry in the upper two indices. The graded symmetry of the Poisson bracket requires \( \{dx^i, dx^j\} = \{dx^j, dx^i\} \). Using the Poisson bracket to calculate both sides, one finds:

\[ \pi^{m\ell} \Gamma_{ma}^i \Gamma_{\ell b}^j dx^a dx^b - \tilde{R}^{ij} = \pi^{m\ell} \Gamma_{ma}^j \Gamma_{\ell b}^i dx^a dx^b - \tilde{R}^{ji}. \]

The “\( \Gamma^i \Gamma^j \)" term is manifestly symmetric in \((i, j)\), from which it follows that \( \tilde{R}^{ij} \) ought to be symmetric in the upper two indices. We will soon show that \( \tilde{R}^{ij} \) is the curvature of a symplectic (Poisson) connection; that is \((\nabla_{e_i} \pi)^{jk} = 0\). Like in Riemannian geometry where preserving the metric causes these indices to be anti-symmetric, preserving the symplectic 2-form\(^5\) causes these indices to be symmetric. This calculation is performed in the appendix.

C. The Poisson bracket between two arbitrary forms

It is straightforward to generalize the Poisson bracket for two 1-forms to forms of arbitrary degree. We take the following ansatz for the Poisson bracket, where now \( \alpha, \beta \) are forms of arbitrary degree:

\[ \{\alpha, \beta\} = \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta + (-1)^{|\alpha|} \tilde{R}^{mn}(i_{e_m} \alpha)(i_{e_n} \beta). \]  \( (10) \)

As shown in Appendix B, requiring (10) to satisfy the properties of the graded differential Poisson bracket places the following conditions, also obtained in [3] and [6], on the connection coefficients \( \Gamma^i_{jk} \):

1. \( \nabla_{e_i} \) is symplectic:

\[ (\nabla_{e_i} \pi)^{ij} = 0 \]  \( (11) \)

\(^4\) The interior product maps \( p \)-forms into \((p - 1)\)-forms. Our conventions for the interior product are discussed in Appendix A.

\(^5\) Or equivalently, the Poisson bivector.
2. $\pi^{ij}$ satisfies the Jacobi Identity:

\[ (\pi^{ab}\partial_b\pi^{mn} + \pi^{mb}\partial_b\pi^{na} + \pi^{nb}\partial_b\pi^{am}) = 0 \]  

(12)

3. The connection $\nabla_{e_i}$ has vanishing curvature:

\[ [\nabla_{e_m}, \nabla_{e_n}]_{\alpha} = 0 \]

(13)

4. The curvature $\tilde{R}^{mn}$ is covariantly constant under $\nabla_{e_a}$:

\[ (\nabla_{e_a}\tilde{R})^{mn} = 0 \]

(14)

5. $\tilde{R}^{mn}$ satisfies\(^6\):

\[ \tilde{R}^{ab}(i_{e_a}\tilde{R}^{mn}) + \tilde{R}^{mb}(i_{e_b}\tilde{R}^{na}) + \tilde{R}^{mb}(i_{e_b}\tilde{R}^{am}) = 0 \]

(15)

Equation (11) comes from requiring the Leibniz rule hold, and Equations (12,15) come from requiring that the Jacobi identity hold. These conditions, however, are not independent. Consider the following relation:

\[
d \left( \{x^i, \{dx^j, dx^k\} \} + \{dx^i, \{dx^j, x^k\}\} \right) = \{dx^i, \{dx^j, dx^k\}\} \]

After some reorganization, this equation relates (15) to the other conditions:

\[
\tilde{R}^{ib}(i_{e_b}\tilde{R}^{jk}) + \tilde{R}^{jb}(i_{e_b}\tilde{R}^{ki}) + \tilde{R}^{ab}(i_{e_a}\tilde{R}^{ij}) = \\
- (\pi^{ab}\partial_b\pi^{mn} + \pi^{mb}\partial_b\pi^{na} + \pi^{nb}\partial_b\pi^{am}) \nabla_{e_a} dx^i \nabla_{e_m} dx^j \nabla_{e_n} dx^k \\
- \pi^{ab}\pi^{mn} (\nabla_{e_a} \nabla_{e_m} dx^i \nabla_{e_n} dx^j \nabla_{e_m} dx^k + \nabla_{e_n} dx^i [\nabla_{e_a}, \nabla_{e_m}] dx^j \nabla_{e_m} dx^k + \nabla_{e_a} dx^i \nabla_{e_m} dx^j [\nabla_{e_n}, \nabla_{e_m}] dx^k) \\
+ \pi^{ab} \left( \nabla_{e_a} \tilde{R} \right)^{jk} \nabla_{e_a} dx^i + \pi^{ab} \left( \nabla_{e_a} \tilde{R} \right)^{ki} \nabla_{e_a} dx^j + \pi^{ab} \left( \nabla_{e_a} \tilde{R} \right)^{ij} \nabla_{e_a} dx^k \\
+ d \left( (\pi^{ib}\partial_b\pi^{mn} + \pi^{mb}\partial_b\pi^{ni} + \pi^{nb}\partial_b\pi^{im}) \nabla_{e_m} dx^i \nabla_{e_n} dx^k \right) \\
+ d \left( \pi^{nb}\pi^{mi} (\nabla_{e_a} \nabla_{e_m} dx^i \nabla_{e_n} dx^k + \nabla_{e_a} dx^i [\nabla_{e_n}, \nabla_{e_m}] dx^k) \right) \\
- d \left( \pi^{ib} \left( \nabla_{e_a} \tilde{R} \right)^{jk} \right) + ... 
\]

(16)

\(^6\) In tensor product notation, the classical Yang-Baxter (CYB) equation is given by $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$, where $[]$ is the matrix commutator. Taking into account that $\tilde{R}^{ab}_{0a}$ is symmetric in the two upper indices and antisymmetric in the two lower indices, one can verify that (15) implies that $\tilde{R}^{ab}_{0a}$ satisfies the CYB equation.
where the ellipses represents terms proportional to \((\partial_k \pi^{ij} + \pi^{ia} \Gamma^j_{ak} + \pi^{aj} \Gamma^i_{ak})\) and derivatives thereof. We have not written these terms since they vanish using the symplectic condition (11). Once one requires (12), (13), and (14), relation (16) implies:

\[
\tilde{R}^{ib} \left( i_{e_b} \tilde{R}^{jk} \right) + \tilde{R}^{j b} \left( i_{e_b} \tilde{R}^{ki} \right) + \tilde{R}^{k b} \left( i_{e_b} \tilde{R}^{ij} \right) = 0.
\]

This shows that (15) is implied by the other constraints. Similar relations hold between Equations 12-14. However, none of these constraints can be completely expressed in terms of the others.

If a connection exists that satisfies all of these properties on the manifold, we have given expressions for the Poisson bracket between two arbitrary differential forms. This bracket is the only possible bracket between differential forms on a symplectic manifold.

IV. THE STAR PRODUCT

In deformation quantization, star products replace the point-wise multiplication of functions on a Poisson manifold. For an overview of the subject, see [1] and [4]. The space of smooth functions, \(C^\infty(M)\), is replaced with \(C^\infty(M,\ast)[[\hbar]]\). As a set, \(C^\infty(M,\ast)[[\hbar]]\) consists of formal (infinite) polynomials in \(\hbar\) with the coefficients belonging to \(C^\infty(M)\):

\[
f(x) = f_0(x) + \sum_{n=1}^{\infty} \hbar^n f_n(x).
\]

Functions in \(C^\infty(M,\ast)[[\hbar]]\) are multiplied with the star product. In the limit \(\hbar \to 0\), the space \(C^\infty(M,\ast)[[\hbar]]\) becomes \(C^\infty(M)\), and the star product reduces to the usual point-wise multiplication of functions. Kontsevich [7] showed that the star product exists for all Poisson manifolds and gave a procedure to obtain the formula to all orders. The star product is written explicitly to \(O(\hbar^2)\) in [7], while more recently [8] gave the formula to \(O(\hbar^4)\). To \(O(\hbar^2)\), the star product is:

\[
f \ast g = fg + \hbar \{f, g\} + \hbar^2 \left( \frac{1}{2} \pi^{ij} \pi^{mn} \partial_i \partial_m f \partial_j \partial_n g + \frac{1}{3} \pi^{ia} \partial_a \pi^{mn} \left( \partial_i \partial_m f \partial_n g - \partial_m f \partial_i \partial_n g \right) \right).
\]

Two star products, \(\ast_1\) and \(\ast_2\), are considered equivalent if there exists a linear operator \(T\) of the form:

\[
T = id + \sum_{n=1}^{\infty} \hbar^n T_n
\]

such that

\[
T (f \ast_2 g) = T (f) \ast_1 T (g).
\]
A. The Star Product of Differential Forms

To enlarge the definition of the star product to include differential forms, we deform $\Omega^*(M)$ to $\Omega^*(M,*)[[h]]$, which, as a set, consists of formal power series in $h$ with coefficients in $\Omega^*(M)$:

$$\alpha = \alpha_0 + \sum_{n=1}^{\infty} h^n \alpha_n.$$  

For $\alpha, \beta \in \Omega^0(M,*)[[h]] = C^\infty(M,*)[[h]]$, we require that the star product of forms coincides with the usual star product of functions. A product of differential forms in $\Omega^*(M,*)[[h]]$ is a “star product” if it satisfies the following properties for $\alpha, \beta, \gamma \in \Omega^*(M,*)[[h]]$:

1. The product takes the form:

$$\alpha \ast b = \alpha \beta + \sum_{n=1}^{\infty} h^n C_n(\alpha, \beta)$$

where the $C_n$ are bilinear differential operators.

2. The product is associative: $(\alpha \ast \beta) \ast \gamma = \alpha \ast (\beta \ast \gamma)$

3. The order $h$ term is Poisson bracket for forms: $C_1(\alpha, \beta) = \{\alpha, \beta\}$

4. The constant function, 1, is the identity: $1 \ast \alpha = \alpha \ast 1 = \alpha$

5. The terms $C_n$ have a generalized Moyal symmetry:

$$C_n(\alpha, \beta) = (-1)^{||\alpha||+n} C_n(\beta, \alpha).$$

We want to find a star product which satisfies Properties 1-5 to $O(h^2)$ for two arbitrary forms $\alpha$ and $\beta$. We propose the following product:

$$\alpha \ast \beta = \alpha \beta + h C_1(\alpha, \beta) + h^2 \left[ C_2^\nabla(\alpha, \beta) + C_2^R(\alpha, \beta) \right] + \ldots$$

where we have:

$$C_1(\alpha, \beta) \equiv \{\alpha, \beta\} = \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta + (-1)^{||\alpha||} \tilde R^{mn}(i_{e_m} \alpha)(i_{e_n} \beta)$$

$$C_2^\nabla(\alpha, \beta) = \frac{1}{2} \pi^{ij} \pi^{mn} \nabla_{e_i} \nabla_{e_j} \alpha \nabla_{e_m} \nabla_{e_n} \beta + \frac{1}{3} \pi^{ia} \partial_a \pi^{mn} \left( \nabla_{e_i} \nabla_{e_m} \alpha \nabla_{e_n} \beta - \nabla_{e_m} \alpha \nabla_{e_i} \nabla_{e_n} \beta \right)$$

$$C_2^R(\alpha, \beta) = -\frac{1}{2} \tilde R^{ij} \tilde R^{mn} \left[ (i_{e_i} i_{e_m} \alpha)(i_{e_j} i_{e_n} \beta) \right] - \frac{1}{3} \tilde R^{ij} (i_{e_j} \tilde R^{mn}) \left[ (-1)^{||\alpha||} (i_{e_i} i_{e_m} \alpha)(i_{e_n} \beta) + (i_{e_m} \alpha)(i_{e_i} i_{e_n} \beta) \right]$$

$$+ (-1)^{||\alpha||} \pi^{ij} \tilde R^{mn} (i_{e_m} \nabla_{e_i} \alpha)(i_{e_n} \nabla_{e_j} \beta).$$
We write the $O(\hbar^2)$ term in this way for later convenience.

Properties $1$ and $3.5$ are manifestly satisfied by (17); however we leave the calculation that shows (17) is associative (Property 2) to Appendix C because the calculation is quite involved. However, constructing (17) for the product to this order was relatively straightforward. The condition of associativity at order $\hbar^2$ requires that for $f$ a function and $\alpha, \beta$ forms:

$$fC_2(\alpha, \beta) - C_2(f\alpha, \beta) + C_2(f, \alpha\beta) - C_2(f, \alpha)\beta = \{\{f, \alpha\}, \beta\} - \{f, \{\alpha, \beta\}\}. \quad (18)$$

The right-hand side of (18) is calculable using the Poisson bracket (10). The star product between a function and a form to any order can be obtained by replacing ordinary derivatives in the star product of functions with covariant derivatives. This prescription gives $C_2(f, \alpha\beta)$ and $C_2(f, \alpha)$ in (18), so constructing the ansatz for $C_2(\alpha, \beta)$ simplifies considerably. Using the same methods, constructing an ansatz for the star product to $O(\hbar^3)$ is similarly straightforward, although verifying the associativity of the product is laborous.

B. Changing Coordinates

As can easily be seen from the form of the star product, it behaves very badly under a change in coordinates. Say we start with coordinates $x^i$ and we want to switch to coordinates $y^\mu$. First we write the $x^i$ as functions of $y^\mu$, e.g. $x^i(y^\mu)$. Then, we see how the basis vectors and basis 1-forms behave under this change of coordinates:

$$e_i = \frac{\partial y^\mu}{\partial x^i} e_\mu \quad \text{and} \quad dx^i = \frac{\partial x^i}{\partial y^\mu} dy^\mu$$

Where $\partial y^\mu/\partial x^i$ is the matrix inverse of $\partial x^i/\partial y^\mu$. We will also use the fact that:

$$\frac{\partial}{\partial y^\mu} \left( \frac{\partial y^\nu}{\partial x^i} \right) = - \frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial y^\nu}{\partial x^i} \frac{\partial y^\lambda}{\partial x^j}.$$  

When we plug these expressions into the above formula for the star product, we get the same terms, except now in the $y$-coordinate basis, plus additional terms. To be precise, after a coordinate change, we get:

$$\alpha \ast' \beta = \alpha\beta + \hbar C_1(\alpha, \beta) + \hbar^2 \left[ C_2^\Sigma(\alpha, \beta) + C_2^\bar{R}(\alpha, \beta) \right]$$

$$+ \frac{\hbar^2}{6} \left[ -\frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial y^\lambda}{\partial x^i} \pi_\mu \pi_\nu \pi_\lambda \pi_\tau \pi_\nu (\nabla e_\rho e_\sigma \alpha \nabla e_\lambda \beta + \nabla e_\lambda \alpha \nabla e_\nu \beta + 2 \nabla e_\mu \nabla e_\lambda \alpha \nabla e_\nu \beta + 2 \nabla e_\mu \alpha \nabla e_\nu \nabla e_\lambda \beta)$$

$$+ \left( \frac{7}{2} \frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial^2 y^\rho}{\partial y^\lambda \partial y^\tau} \frac{\partial^2 y^\sigma}{\partial x^j} \pi_\mu \pi_\nu \pi_\lambda \pi_\tau + 2 \frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial^2 y^\rho}{\partial y^\lambda \partial y^\tau} \frac{\partial^2 y^\sigma}{\partial x^j} \frac{\partial y^\rho}{\partial x^j} \pi_\mu \pi_\nu \pi_\lambda \pi_\tau \right) \left( \nabla e_\rho e_\sigma \alpha \nabla e_\alpha \beta + \nabla e_\alpha \alpha \nabla e_\rho \beta \right) \right]$$

$$+ \frac{2}{\partial y^\mu \partial y^\nu} \frac{\partial^2 x^i}{\partial x^j} \pi_\mu \pi_\nu \pi_\lambda \pi_\sigma \left( \nabla e_\rho e_\sigma \alpha \nabla e_\rho \beta + \nabla e_\rho \alpha \nabla e_\sigma \beta \right)$$
Where $C_1(\alpha, \beta)$, $C_\nabla^2(\alpha, \beta)$, and $C_R^2(\alpha, \beta)$ take the same form, except now in the $y^\mu$ coordinates; e.g. $C_1(\alpha, \beta) = \pi^{\mu\nu} \nabla_{e_\mu} \alpha \nabla_{e_\nu} \beta + (-1)^{|\alpha|} \tilde{R}^{\mu\nu}(i_{e_\alpha} \alpha)(i_{e_\beta} \beta)$. As we can see, this is not the form we want for the star product, and the difference appears at order $\hbar^2$. It turns out that this formula for the star product is actually equivalent (in the sense discussed above) to the original formula for the star product. To see this, we will need to construct a differential operator $T$ such that
\[
T(\alpha \ast \beta) = T(\alpha) \ast' T(\beta) \quad \Rightarrow \quad \alpha \ast \beta = T^{-1}(T(\alpha) \ast' T(\beta))
\]
We need to make changes at order $\hbar^2$, so take $T$ to be of the form $T = 1 + \hbar^2 T_2 + \ldots$ where $T_2$ is the differential operator that contributes to $T$ at order $\hbar^2$. Then the formula above implies that:
\[
\alpha \ast \beta = \alpha \ast' \beta - \hbar^2 T_2(\alpha \beta) + \hbar^2 T_2(\alpha) \beta + \hbar^2 \alpha T_2(\beta) + \mathcal{O}(\hbar^3).
\]
It turns that the operator $T_2$ required to eliminate the unwanted terms is given by:
\[
T_2(\alpha) = \frac{1}{6} \left[ -\frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial y^\lambda}{\partial x^i} \pi^{\mu\rho} \pi^{\nu\sigma} \nabla_{e_\mu} \nabla_{e_\nu} \nabla_{e_\rho} \nabla_{e_\sigma} + \frac{7}{2} \frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial y^\rho}{\partial x^i} \frac{\partial y^\sigma}{\partial x^j} \pi^{\mu\lambda} \pi^{\nu\sigma} \nabla_{e_\mu} \nabla_{e_\rho} \nabla_{e_\sigma} + 2 \frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial^2 x^j}{\partial y^\lambda \partial y^\tau} \frac{\partial y^\rho}{\partial x^i} \frac{\partial y^\sigma}{\partial x^j} \pi^{\mu\lambda \nu\sigma} \nabla_{e_\mu} \nabla_{e_\rho} \nabla_{e_\sigma} + 2 \frac{\partial^2 x^i}{\partial y^\mu \partial y^\nu} \frac{\partial y^\rho}{\partial x^i} \pi^{\mu\lambda} \pi^{\nu\rho} \nabla_{e_\mu} \nabla_{e_\rho} \nabla_{e_\sigma} + \mathcal{O}(\hbar^3) \right] \alpha
\]
Though not very illuminating, it is comforting to know that even though the product appears to be badly behaved under a change in coordinates, there is a transformation that brings it back into the form we started out with.

V. DISCUSSION

The graded differential Poisson algebra on a symplectic manifold was initially introduced by [3] and [6]. These papers contain an explicit form of the bracket for 1-forms. In this note, we have generalized the explicit form of the graded Poisson bracket for all degrees of forms, and checked that the form satisfies all the required properties. We found that for the Poisson bracket to satisfy the properties required of a graded differential Poisson algebra, the symplectic manifold must satisfy three new conditions\textsuperscript{7}, also found by [3] and [6]:

1. $\nabla_{e_\ell}$ is symplectic: $\nabla_{e_\ell} \pi^{ij} = 0$

2. $R^i_{mab}$ is flat: $[\nabla_{e_m}, \nabla_{e_n}]\alpha = 0$

\textsuperscript{7} We already have $\pi^{ij} = -\pi^{ji}$ and $\pi^{im} \partial_m \pi^{jk} + \pi^{jm} \partial_m \pi^{ki} + \pi^{km} \partial_m \pi^{ij} = 0$ from the Poisson algebra of functions.
3. $\tilde{R}^{ij}$ is covariantly constant: $\left(\nabla_{e_k}\tilde{R}\right)^{ij} = 0$

These conditions come from requiring that the graded Poisson bracket satisfy the Leibniz rule (1) and the Jacobi identity (2-3). They are not independent.

We then use this graded differential Poisson bracket in our presentation of a non-commutative deformation (the star product) of the graded differential algebra of forms to $O(h^2)$. The star product we present is:

$$\alpha \ast \beta = \alpha \beta + hC_1(\alpha, \beta) + h^2C_2(\alpha, \beta),$$

where we have:

$$C_1(\alpha, \beta) \equiv \{\alpha, \beta\} = \pi^{mn}\nabla_{e_m}\alpha\nabla_{e_n}\beta + (-1)^{|\alpha|}\tilde{R}^{mn}(ie_m\alpha)(ie_n\beta)$$

$$C_2(\alpha, \beta) = C_2^R(\alpha, \beta) + C_2^\nabla(\alpha, \beta)$$

$$C_2^\nabla(\alpha, \beta) = \frac{1}{2}\pi^{ij}\pi^{mn} \nabla_{e_i}\nabla_{e_m}\alpha\nabla_{e_j}\nabla_{e_n}\beta + \frac{1}{3}\pi^{ia}\partial_a\pi^{mn} (\nabla_{e_i}\nabla_{e_m}\alpha\nabla_{e_n}\beta - \nabla_{e_m}\alpha\nabla_{e_i}\nabla_{e_n}\beta)$$

$$C_2^R(\alpha, \beta) = \frac{1}{2}\tilde{R}^{ij}\tilde{R}^{mn} \left[(ie_i\epsilon_{e_m}\alpha)(ie_j\epsilon_{e_n}\beta)\right] - \frac{1}{3}\tilde{R}^{it}(ie_i\tilde{R}^{mn}) \left[(-1)^{|\alpha|}(ie_i\epsilon_{e_m}\alpha)(ie_n\beta) + (ie_m\alpha)(ie_i\epsilon_{e_n}\beta)\right]$$

$$+ (-1)^{|\alpha|}\pi^{ij}\tilde{R}^{mn}(ie_m\nabla_{e_i}\alpha)(ie_n\nabla_{e_j}\beta),$$

and we verify that this star product satisfies all the necessary properties, such as associativity, to $O(h^2)$.

The results of this paper may be generalized in two directions. Using the method outlined in Section IV, it should be possible (although tedious) to find the star product between differential forms to $O(h^3)$.

It may also be possible to generalize the star product to Poisson manifolds, where $\pi^{ij}$ is not invertible. Many of the arguments go through unchanged. However, the form of the star product between a function and a form is not guaranteed in this case, and the conditions on the connection $\Gamma^k_{ij}$ are more complicated (e.g. $\pi^{ia}\pi^{jb}R^k_{mab} = 0$ instead of $R^k_{mab} = 0$).

Finally, we can apply the star product between differential forms to physics. Applications include considering more general star products in deformation quantization, to studying gauge theories on noncommutative spaces, to generalizing the Seiberg-Witten map.
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APPENDIX A: NOTATION

1. Basis Vectors and Basis 1-forms

Vectors can be thought of as linear differential operators acting on smooth functions over a manifold. In a coordinate chart spanned by $x^1, \ldots, x^n$, there is a natural set of basis vectors, $e_i$, given by:

$$e_i(f) \equiv \frac{\partial f}{\partial x^i} \quad (= \partial_i f)$$

The 1-forms are dual to these vectors and can be thought of as linear operators turning a vector field into a function. They are spanned by basis 1-forms $dx^i$ that are defined by:

$$dx^i(e_j) = \delta^i_j$$

2. The Interior Product

The interior product is a map $i_V : \Omega^n(M) \rightarrow \Omega^{n-1}(M)$. For an $n$-form the interior product is given by the formula:

$$(i_V \omega)(X_1, \ldots, X_{n-1}) = \omega(V, X_1, \ldots, X_{n-1})$$

We are concerned with the interior product of a basis vector acting on a differential form in this paper. We will typically take the interior product with respect to the basis vectors $e_i$. 
Using this notation, here are a few of the properties of the interior product:

\[ \alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} \]

\[ i_{e_m} \alpha = \frac{1}{(p-1)!} \alpha_{m i_2 \cdots i_p} dx^{i_2} \cdots dx^{i_p} \]

\[ i_{e_m} (\alpha \wedge \beta) = (i_{e_m} \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (i_{e_m} \beta) \]

\[ i_{e_m} i_{e_n} \alpha = -i_{e_n} i_{e_m} \alpha \]

In essence, the interior product does the following:

\[ i_{e_m} dx^k = \delta^k_m. \]

It maps 1-forms to numbers and it satisfies the graded product rule as given above. This is the natural tool to use when describing the Poisson bracket of higher degree forms.

3. The connections \( \nabla_{e_i}, \bar{\nabla}_{e_i} \)

We use the convention that covariant derivatives only act nontrivially on basis 1-forms, \( dx^i \) and basis vectors \( e_i \). They act like partial derivatives on the coefficients of 1-forms (and other tensors), regardless of the index structure. For instance:

\[ \nabla_{e_i} \alpha_k = \partial_i \alpha_k, \quad \text{while} \quad \nabla_{e_i} dx^k = -\Gamma^k_{ij} dx^j. \]

Combined, these give the expected result,

\[ \nabla_{e_i} (\alpha_k dx^k) = (\partial_i \alpha_k) dx^k + \alpha_k (-\Gamma^k_{ij} dx^j) = (\partial_i \alpha_j - \Gamma^k_{ij} \alpha_k) dx^j. \]

Note that the details of the calculation differ from the convention for covariant derivatives widely used by physicists. What physicists typically call the “covariant derivative” is reproduced by the so-called covariant differential:

\[ \nabla \alpha = (\nabla_{e_i} \alpha) \otimes dx^i. \]

We have explicitly written the full vector \( e_i \) as the argument of the covariant derivative, \( \nabla_{e_i} \), rather than using the conventional \( \nabla_i \) to avoid any potential confusion.

As stated in the text, we have defined the two connections as:

\[ \nabla_{e_i} dx^k = -\Gamma^k_{ij} dx^j \quad \bar{\nabla}_{e_i} dx^k = -\bar{\Gamma}^k_{ij} dx^j = -\Gamma^k_{ji} dx^j. \]
The curvature for these two connections are given as:

\[ [\nabla_{e_a}, \nabla_{e_b}] dx^i = R^i_{mab} dx^m = \left( \partial_a \Gamma^i_{bm} - \partial_b \Gamma^i_{am} + \Gamma^\ell_i \Gamma^i_{bm} - \Gamma^i_\ell \Gamma^\ell_{am} \right) dx^m \]

\[ [\tilde{\nabla}_{e_a}, \tilde{\nabla}_{e_b}] dx^i = \tilde{R}^i_{mab} dx^m = \left( \partial_a \Gamma^i_{mb} - \partial_b \Gamma^i_{ma} + \Gamma^\ell_i \Gamma^i_{mb} - \Gamma^i_\ell \Gamma^\ell_{ma} \right) dx^m \]

Without torsion, these two expressions would be equal, but since the torsion is non-zero, these two curvatures are different from each other. In particular, one of these curvatures is zero while the other is non-vanishing.

We also define

\[ \tilde{R}^{ij}_{ab} \equiv \pi^{im} \tilde{R}^j_{mab} \]

where we raised one of the lower indices of the curvature tensor with the Poisson bivector \( \pi \). We often also refer to \( \tilde{R}^{ij}_{ab} \) as the curvature.

When the covariant derivative acts on mixed tensors, we use parentheses as a shorthand notation. For example, \((\nabla_{e_c} \tilde{R})^{ij}\) is the \((i, j)\) component of

\[ \nabla_{e_c} \tilde{R} = \frac{1}{2} \left[ \partial_c \tilde{R}^{ij}_{ab} + \Gamma^i_c \Gamma^j_b \tilde{R}^{ij}_{ab} - \Gamma^i_c \Gamma^j_b \tilde{R}^{ij}_{ab} - \Gamma^i_\ell \Gamma^j_b \tilde{R}^{ij}_{a\ell} \right] dx^a \wedge dx^b \otimes \partial_i \otimes \partial_j. \]

In particular, we have the following identity:

\[ \nabla_{e_c} \left[ \tilde{R}^{mn}(i_{em} \alpha)(i_{en} \beta) \right] = (\nabla_{e_c} \tilde{R})^{mn}(i_{em} \alpha)(i_{en} \beta) + \tilde{R}^{mn}(i_{em} \nabla_{e_c} \alpha)(i_{en} \beta) + \tilde{R}^{mn}(i_{em} \alpha)(i_{en} \nabla_{e_c} \beta) \]

4. **Showing \( \tilde{R}^{ij} \) is Symmetric in the upper two indices**

First, remember that the connection \( \tilde{\nabla}_{e_c} \) preserves the Poisson bivector:

\[ (\tilde{\nabla}_{e_c} \pi)^{jk} = \partial_j \pi^{ik} + \tilde{\Gamma}^{ij}_{kl} \pi^{lk} + \tilde{\Gamma}^{ij}_{k\ell} \pi^{j\ell} = 0 \]

Next, we manipulate the curvature tensor, using the above relation:

\[ \tilde{R}^{ij}_{ab} \equiv \pi^{il} \tilde{R}^{j}_{lab} = \pi^{il} \partial_a \tilde{\Gamma}^j_{lb} + \pi^{il} \tilde{\Gamma}^m_{am} \tilde{\Gamma}^j_{bl} - (a \leftrightarrow b) \]

\[ = \left[ \partial_a \left( \pi^{il} \tilde{\Gamma}^j_{lb} \right) - \partial_a \pi^{il} \tilde{\Gamma}^j_{lb} \right] + \tilde{\Gamma}^m_{am} \left( \pi^{il} \tilde{\Gamma}^j_{bl} \right) - (a \leftrightarrow b) \]

\[ = \partial_a \left( -\partial_b \pi^{ij} - \pi^{ij} \tilde{\Gamma}^m_{bl} \right) - \partial_a \pi^{il} \tilde{\Gamma}^j_{lb} + \tilde{\Gamma}^j_{am} \left( -\partial_b \pi^{im} - \pi^{im} \tilde{\Gamma}^j_{bl} \right) - (a \leftrightarrow b) \]
Notice that \( \partial_a \partial_b \pi^{ij} \) and \( (\partial_a \pi^{ij} \tilde{\Gamma}^{i}_{\ell} + \partial_b \pi^{im} \tilde{\Gamma}^{i}_{am}) \) vanish when we anti-symmetrize the \( a, b \) indices. Continuing,

\[
\tilde{R}^{ij}_{ab} = -\partial_a \pi^{ij} \tilde{\Gamma}^{i}_{\ell} - \pi^{ij} \partial_a \tilde{\Gamma}^{i}_{\ell} - \left( \pi^{tm} \tilde{\Gamma}^{i}_{am} \right) \tilde{\Gamma}^{i}_{b\ell} - (a \leftrightarrow b) \\
= -\partial_a \pi^{ij} \tilde{\Gamma}^{i}_{\ell} - \pi^{ij} \partial_a \tilde{\Gamma}^{i}_{\ell} - \left( -\partial_a \pi^{ij} - \pi^{mj} \tilde{\Gamma}^{i}_{am} \right) \tilde{\Gamma}^{i}_{b\ell} - (a \leftrightarrow b) \\
= -\pi^{mj} \left( \partial_a \tilde{\Gamma}^{b}_{m} - \tilde{\Gamma}^{i}_{b\ell} \tilde{\Gamma}^{i}_{am} \right) - (a \leftrightarrow b) \\
= +\pi^{jm} \tilde{R}^{ji}_{mab} = \tilde{R}^{ji}_{ab}
\]

The change in sign comes from switching the indices in \( \pi^{mj} \), which is anti-symmetric. This is identical to the proof that the Levi-Civita connection is anti-symmetric in the same indices, except for this last step where instead of having a symmetric metric, we have an anti-symmetric Poisson bivector.

**APPENDIX B: GRADED DIFFERENTIAL POISSON BRACKET PROPERTIES**

We verify the ansatz of Equation (10) satisfies the properties of a graded differential Poisson bracket, Equation (37). To save space on the page, we will abbreviate \( \nabla_{e_m} \) as \( \nabla_{m} \) and we will abbreviate \( i_{e_m} \) as \( i_{m} \). Otherwise, equations would not fit on the page.

1. **Bracket degree, \(|\{\alpha, \beta\}| = |\alpha| + |\beta|\):**

The covariant derivative does not change the degree of the differential form, \(|\nabla_{a}\alpha| = |\alpha|\), while the interior product does, \(|i_{m}\alpha| = |\alpha| - 1\). The first term of the Poisson bracket has the proper degree:

\[
|\nabla_{a}\alpha \nabla_{b}\beta| = |\nabla_{a}\alpha||\nabla_{b}\beta| = |\alpha| + |\beta|
\]

as does the second term:

\[
|R^{mn}(i_{m}\alpha)(i_{n}\beta)| = |R^{mn}| + |i_{m}\alpha| + |i_{n}\beta| = 2 + (|\alpha| - 1) + (|\beta| - 1) = |\alpha| + |\beta|.
\]

2. **Graded Symmetry, \(\{\alpha, \beta\} = (-1)^{|\alpha||\beta|} - 1\{\beta, \alpha\}\):**

The first term in the Poisson bracket has the proper graded symmetry by inspection:

\[
\pi^{mn} \nabla_{m}\alpha \nabla_{n}\beta = \pi^{mn} (-1)^{|\alpha||\beta|} \nabla_{n}\beta \nabla_{m}\alpha = (-1)^{(|\alpha||\beta|) - 1) \pi^{mn} \nabla_{n}\beta \nabla_{m}\alpha.
\]
The additional minus sign comes from the antisymmetry of the Poisson bivector: \( \pi^{nm} = -\pi^{mn} \).

The second term in the bracket also has the proper grading:

\[
(-1)^{|\alpha|} \tilde{R}^{mn}(i_m \alpha)(i_n \beta) = (-1)^{|\alpha|} \tilde{R}^{mn} (-1)^{(|\alpha|-1)(|\beta|-1)} (i_n \beta)(i_m \alpha)
\]

\[
= (-1)^{|\alpha|+|\beta|-1} \tilde{R}^{mn} (i_m \alpha)(i_n \beta)
\]

\[
= (-1)^{|\alpha|+|\beta|-1} \left[ (-1)^{|\beta|} \tilde{R}^{mn} (i_n \beta)(i_m \alpha) \right].
\]

In the last line, we use the fact that \( \tilde{R}^{mn} \) is symmetric in the upper two indices.

3. **Graded Product**, \( \{ \alpha \land \beta, \gamma \} = \alpha \land \{ \beta, \gamma \} + (-1)^{|\beta||\gamma|} \{ \alpha, \gamma \} \land \beta \):

To show that the Poisson bracket satisfies the graded product rule, consider:

\[
\{ \alpha \land \beta, \gamma \} = \pi^{mn} \nabla_m (\alpha \beta) \nabla_n \gamma + (-1)^{|\alpha|+|\beta|} \tilde{R}^{mn} i_m (\alpha \beta) i_n \gamma
\]

\[
= \pi^{mn} [(\nabla_m \alpha) \beta + \alpha (\nabla_m \beta)] \nabla_n \gamma + (-1)^{|\alpha|+|\beta|} \tilde{R}^{mn} \left[ (i_m \alpha) \beta + (-1)^{|\alpha|} \alpha (i_m \beta) \right] i_n \gamma
\]

\[
= \alpha \left[ \pi^{mn} \nabla_m \beta \nabla_n \gamma + (-1)^{|\beta|} \tilde{R}^{mn} i_m \beta i_n \gamma \right] + (-1)^{|\beta||\gamma|} \left[ \pi^{mn} \nabla_m \alpha \nabla_n \gamma + (-1)^{|\alpha|} \tilde{R}^{mn} i_m \alpha i_n \gamma \right] \beta
\]

where we use \( \beta \gamma = (-1)^{|\beta||\gamma|} \gamma \beta \) and \( |i_m \beta| = |\beta| - 1 \) to go from the second line to the third line.

4. **Leibniz Rule**, \( d\{ \alpha, \beta \} = \{ d\alpha, \beta \} + (-1)^{|\alpha|} \{ \alpha, d\beta \} \):

To verify that our bracket satisfies the Liebniz rule, we take the differential of our Poisson bracket directly:

\[
d\{ \alpha, \beta \} = d \left[ \pi^{ij} \nabla_i \alpha \nabla_j \beta + (-1)^{|\alpha|} \tilde{R}^{ij} (i_i \alpha) (i_j \beta) \right]
\]

\[
= d \pi^{ij} \nabla_i \alpha \nabla_j \beta + \pi^{ij} \left[ (d \nabla_i \alpha) \nabla_j \beta + (-1)^{|\alpha|} \nabla_i \alpha (d \nabla_j \beta) \right]
\]

\[
+ (-1)^{|\alpha|} \left[ d \tilde{R}^{ij} (i_i \alpha) (i_j \beta) + \tilde{R}^{ij} \left( (d \nabla_i \alpha) (i_j \beta) + (-1)^{|\alpha|-1} (i_i \alpha) (d i_j \beta) \right) \right].
\]

Note that \( \mathcal{L}_j = di_j + i_j d \), where \( \mathcal{L} \) is the Lie derivative, and \( \mathcal{L}_j \gamma = \partial_j \gamma \) for a form \( \gamma \). Also, the covariant derivative can be rewritten as \( \nabla_i \gamma = \partial_i \gamma - \Gamma^k_i (i_k \gamma) \). Combining these two observations

\[8\] We take \( \partial_k \alpha \) to mean:

\[ \partial_k \alpha = \frac{1}{p!} (\partial_k \alpha_{i_1 \cdots i_p}) dx^{i_1} \cdots dx^{i_p} \]
Thus, for the Poisson bracket to satisfy the Leibniz rule for all forms, one must apply the constraint
\[
\pi^{ij} d\nabla_j\gamma = \pi^{ij} \left[ \nabla_j d\gamma + \tilde{\Gamma}_j^k \partial_k \gamma - d\tilde{\Gamma}_j^k (i_k \gamma) \right] = \pi^{ij} \nabla_j d\gamma + \pi^{ij} \tilde{\Gamma}_j^k \nabla_k \gamma - \tilde{R}^{ij} (i_j \gamma)
\]

Rewriting \(d\{\alpha, \beta\}\) using this identities,
\[
d\{\alpha, \beta\} = d\pi^{ij} \nabla_i \alpha \nabla_j \beta + \left( \pi^{ij} \nabla_i d\alpha + \pi^{ij} \tilde{\Gamma}_i^k \nabla_k \alpha + \tilde{R}^{ij} (i_i \alpha) \right) \nabla_j \beta + \left( -1 \right)^{|\alpha|} \partial_i \alpha \left( \pi^{ij} \nabla_j d\beta + \pi^{ij} \tilde{\Gamma}_j^k (i_k \beta) - \tilde{R}^{ij} (i_j \beta) \right)
\]
\[
+ \left( -1 \right)^{|\alpha|} \tilde{R}^{ij} \left( (\partial_j \alpha - (i_j d\alpha)) (i_j \beta) + \left( -1 \right)^{|\alpha|-1} (i_i \alpha) (\partial_j \beta - i_j d\beta) \right) + \left( -1 \right)^{|\alpha|} d\tilde{R}^{ij} (i_i \alpha) (i_j \beta)
\]

Let us inspect \(d\tilde{R}^{ij} + \tilde{R}^{ik} \tilde{\Gamma}_k^j + \tilde{R}^{kj} \tilde{\Gamma}_k^i\). Recall that \(\tilde{R}^{ij} = \pi^{ik} \left( d\tilde{\Gamma}_k^j + \tilde{\Gamma}_k^j \right) \). Then, with a bit of rewriting,
\[
d\tilde{R}^{ij} + \tilde{R}^{ik} \tilde{\Gamma}_k^j + \tilde{R}^{kj} \tilde{\Gamma}_k^i = \left( d\pi^{ik} + \pi^{ik} \tilde{\Gamma}_k^j + \pi^{ik} \tilde{\Gamma}_k^j \right) \tilde{\nabla} \alpha \nabla_j \beta + \left( -1 \right)^{|\alpha|} \tilde{R}^{mn} (i_i \alpha) (i_m \beta)
\]

Using this result, \(d\{\alpha, \beta\}\) becomes:
\[
d\{\alpha, \beta\} = \{d\alpha, \beta\} + \left( -1 \right)^{|\alpha|} \{\alpha, d\beta\} + \left( d\pi^{ij} + \pi^{ij} \tilde{\Gamma}_i^k + \pi^{ik} \tilde{\Gamma}_k^j \right) \left[ \nabla_i \alpha \nabla_j \beta + \left( -1 \right)^{|\alpha|} \tilde{R}^{mn} (i_i \alpha) (i_m \beta) \right]
\]

Thus, for the Poisson bracket to satisfy the Leibniz rule for all forms, one must apply the constraint on the connection coefficients that \(\tilde{\nabla}_k\) that \(\tilde{\nabla}_k\) is symplectic (Equation 2). Then,
\[
d\{\alpha, \beta\} = \{d\alpha, \beta\} + \left( -1 \right)^{|\alpha|} \{\alpha, d\beta\}.
\]

5. Graded Jacobi Identity, \(\{\alpha, \{\beta, \gamma\}\} + \left( -1 \right)^{|\alpha|(|\beta|+|\gamma|)} \{\beta, \{\gamma, \alpha\}\} + \left( -1 \right)^{|\alpha|+|\beta|+|\gamma|} \{\gamma, \{\alpha, \beta\}\} = 0:\)

Next, we check that the ansatz for the Poisson bracket satisfies the graded Jacobi Identity. We use the Poisson bracket to calculate \(\{\alpha, \{\beta, \gamma\}\}\), and after simplifying, we have:
\[
\{\alpha, \{\beta, \gamma\}\} = \pi^{ab} \partial_b \pi^{mn} \nabla_a \alpha \nabla_m \beta \nabla_n \gamma + \pi^{ab} \pi^{mn} \left( \nabla_a \alpha \nabla_b \nabla_m \beta \nabla_n \gamma + \nabla_a \alpha \nabla_m \beta \nabla_b \nabla_n \gamma \right)
\]
\[
+ \pi^{ab} \tilde{R}^{mn} \left[ \left( -1 \right)^{|\beta|} \left( \nabla_a \alpha \right) \left( i_m \nabla_b \beta \right) \left( i_n \gamma \right) + \left( -1 \right)^{|\beta|} \left( \nabla_a \alpha \right) \left( i_m \beta \right) \left( i_n \nabla_b \gamma \right) \right]
\]
\[
+ \left( -1 \right)^{|\alpha|+|\beta|} \left( i_m \alpha \right) \left( i_n \nabla_a \beta \right) \left( \nabla_b \gamma \right) + \left( -1 \right)^{|\alpha|+|\beta|} \left( i_m \alpha \right) \left( \nabla_a \beta \right) \left( i_n \nabla_b \gamma \right)
\]
\[
+ \left( -1 \right)^{|\alpha|+|\beta|} \tilde{R}^{ab} \tilde{R}^{mn} \left[ \left( i_a \alpha \right) \left( i_b \nabla_m \beta \right) \left( i_n \gamma \right) + \left( -1 \right)^{|\beta|-1} \left( i_a \alpha \right) \left( i_m \beta \right) \left( i_b i_n \gamma \right) \right]
\]
\[
+ \left( -1 \right)^{|\beta|} \pi^{ab} \left( \nabla_b \tilde{R} \right)^{mn} \nabla_a \alpha \left( i_m \beta \right) \left( i_n \gamma \right) + \left( -1 \right)^{|\beta|-1} \tilde{R}^{ab} \tilde{R}^{mn} \left( i_a \alpha \right) \left( i_m \beta \right) \left( i_n \gamma \right).
\]
Cycling through $\alpha, \beta,$ and $\gamma$ we find that $\{\{a, \{\beta, \gamma\}\}\} + (-1)^{|a||\beta|+|\gamma|}\{\beta, \{\gamma, \alpha\}\} + (-1)^{|\gamma|(|\beta|+|\gamma|)}\{\gamma, \{\alpha, \beta\}\}$ does not automatically vanish. Instead:

\[
\{\{a, \beta, \gamma\}\} + (-1)^{|a||\beta|+|\gamma|}\{\beta, \gamma, \alpha\} + (-1)^{|\gamma|(|\alpha|+|\beta|)}\{\gamma, \alpha, \beta\} = \\
(\pi^{ab}\partial_b\pi^{mn} + \pi^{mn}\partial_b\pi^{na} + \pi^{nb}\partial_b\pi^{am})\nabla_a\nabla_m\beta\nabla_n\gamma \\
+ \pi^{ab}\pi^{mn}\left([\nabla_a, \nabla_m]\beta\right)\nabla_b\nabla_n\gamma + \nabla_n\alpha([\nabla_a, \nabla_m]\beta)\nabla_b\gamma + \nabla_b\alpha\nabla_n\beta([\nabla_a, \nabla_m]\gamma) \\
+ (-1)^{|\beta|}\pi^{ab}(\nabla_i\tilde{R})^{mn}\left[\nabla_a\alpha(i_{m}\beta)(i_{n}\gamma) + (-1)^{|i_{m}\alpha|}i_{n}\beta(i_{m}\gamma) + (-1)^{|i_{m}\alpha|+|\beta|}(i_{m}\alpha)(i_{n}\beta)\nabla_a\gamma\right] \\
+ (-1)^{|\beta|-1}\left[\tilde{R}^{ab}(i_{b}\tilde{R}^{mn}) + \tilde{R}^{mb}(i_{b}\tilde{R}^{na}) + \tilde{R}^{nb}(i_{b}\tilde{R}^{am})\right](i_{a}\alpha)(i_{m}\beta)(i_{n}\gamma).
\]

This gives the condition that the bivector $\pi^{ij}$ is a Poisson bivector as well as several new conditions on the connection coefficients $\Gamma^{k}_{ij}$. In particular we find:

1. $\pi^{ij}$ satisfies the Jacobi Identity:
\[
(\pi^{ab}\partial_b\pi^{mn} + \pi^{mn}\partial_b\pi^{na} + \pi^{nb}\partial_b\pi^{am}) = 0
\]

2. The connection $\nabla_i$ has vanishing curvature:
\[
[\nabla_m, \nabla_n]\alpha = 0
\]

3. The curvature $\tilde{R}^{mn}$ is covariantly constant under $\nabla_a$:
\[
(\nabla_a\tilde{R})^{mn} = 0
\]

4. $\tilde{R}^{mn}$ satisfies:
\[
\tilde{R}^{ab}(i_{b}\tilde{R}^{mn}) + \tilde{R}^{mb}(i_{b}\tilde{R}^{na}) + \tilde{R}^{nb}(i_{b}\tilde{R}^{am}) = 0
\]

These conditions, however, are not independent. This is discussed in Section III C.

**APPENDIX C: VERIFYING THE PRODUCT ANSATZ IS A STAR PRODUCT**

The associativity condition for the star product is:
\[
\alpha * (\beta * \gamma) - (\alpha * \beta) * \gamma = 0. \tag{C1}
\]

To $O(\hbar^2)$, we have
\[
\alpha * (\beta * \gamma) = \alpha\beta\gamma + \hbar(\{\{\alpha, \beta, \gamma\}\} + (\{\alpha, \{\beta, \gamma\}\})) + \hbar^2(\{\{\alpha, \{\beta, \gamma\}\}\} + \alpha C_2(\beta, \gamma) + C_2(\alpha, \beta\gamma)) \\
(\alpha * \beta) * \gamma = \alpha\beta\gamma + \hbar(\{\{\alpha, \beta\}\} + \{\alpha\beta, \gamma\}) + \hbar^2(\{\{\alpha, \beta\}\} + C_2(\alpha, \beta\gamma) + C_2(\alpha, \beta\gamma)).
\]
The associativity condition is trivially satisfied at $\mathcal{O}(1)$. At $\mathcal{O}(h)$, we must use the graded product rule of the Poisson bracket:

$$\{\alpha \beta, \gamma\} = \alpha \{\beta, \gamma\} + (-1)^{|\beta||\gamma|}\{\alpha, \gamma\}\beta.$$

Checking the associativity condition at order $h$, we find:

$$\alpha \ast (\beta \ast \gamma)|_h - (\alpha \ast \beta) \ast \gamma = a\{\beta, \gamma\} + \{\alpha, \beta\} \gamma - \{\alpha, \beta, \gamma\}$$

$$= a\{\beta, \gamma\} + \{\alpha, \beta\} \gamma + (-1)^{|\alpha||\beta|}\{\alpha, \beta, \gamma\} - \{\alpha, \beta\} \gamma - \{\alpha, \beta, \gamma\} + (-1)^{|\beta||\gamma|}\{\alpha, \gamma\}\beta,$$

where the last line makes it evident that the associativity condition is satisfied.

At $\mathcal{O}(h^2)$, the calculation rapidly becomes more complicated. The form of Equation (10) at $\mathcal{O}(h^2)$ gives:

$$\{\alpha, \{\beta, \gamma\}\} + aC_2(\beta, \gamma) + C_2(\alpha, \beta\gamma) - \{\{\alpha, \beta\}, \gamma\} - C_2(\alpha, \beta) \gamma - C_2(\alpha, \beta, \gamma) = 0.$$

For brevity, we rewrite this condition as:

$$\delta C_2(\alpha, \beta, \gamma) \equiv aC_2(\beta, \gamma) - C_2(\alpha, \beta, \gamma) + C_2(\alpha, \beta) \gamma - C_2(\alpha, \beta, \gamma) = \{\{\alpha, \beta\}, \gamma\} - \{\alpha, \{\beta, \gamma\}\},$$

where $\delta$ is called the Hochschild coboundary operator. We do not go into details of Hochschild cohomology, but this is a useful shorthand notation.

To check the associativity condition for $C_2(\alpha, \beta)$, we use the graded Jacobi identity to rewrite $\{\{\alpha, \beta\}, \gamma\}$:

$$\{\{\alpha, \beta\}, \gamma\} = \{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha||\beta|+1}\{\beta, \{\alpha, \gamma\}\}.$$

Using the Poisson bracket (Equation (10)) to calculate $\{\{\alpha, \beta\}, \gamma\}$ and $\{\{\alpha, \beta\}, \gamma\}$,

$$(-1)^{|\alpha||\beta|+1}\{\beta, \{\alpha, \gamma\}\} = -\pi^{ab}\left[\partial_b \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta \nabla_{e_{\gamma}} + \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta \nabla_{e_{\gamma}} + \nabla_{e_m} \alpha \nabla_{e_n} \beta \nabla_{e_{\gamma}}\right]$$

$$+ \pi^{ab} \tilde{R}^{mn} \left[(-1)^{|\alpha|+|\beta|}(i_{e_m} \nabla_{e_n} \alpha)(\nabla_{e_{\beta}})(i_{e_{\gamma}}) + (-1)^{|\alpha|+|\beta|-1}(i_{e_m} \nabla_{e_n} \alpha)(\nabla_{e_{\beta}})(i_{e_{\gamma}})\right]$$

$$+ (-1)^{|\alpha|}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(\nabla_{e_{\gamma}}) + (-1)^{|\beta|-1}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(\nabla_{e_{\gamma}})\right]$$

$$- \tilde{R}^{ab} \tilde{R}^{mn} \left[(-1)^{|\beta|}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(i_{e_{\gamma}}) + (-1)^{|\alpha|}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(i_{e_{\gamma}})\right]$$

$$+ (-1)^{|\beta|-1} \tilde{R}^{ab}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(i_{e_{\gamma}}).$$

The three terms in the first line do not contain $\tilde{R}^{ij}$, while the remaining terms do contain $\tilde{R}^{ij}$.

This means that the associativity condition at $\mathcal{O}(h^2)$ separates into two pieces:

$$\delta C_2^\nabla(\alpha, \beta, \gamma) \equiv 2\pi^{ab}\left(\partial_b \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta \nabla_{e_{\gamma}} + \pi^{mn} \nabla_{e_m} \alpha \nabla_{e_n} \beta \nabla_{e_{\gamma}} + \nabla_{e_m} \alpha \nabla_{e_n} \beta \nabla_{e_{\gamma}}\right)$$

$$+ \pi^{ab} \tilde{R}^{mn}\left[(-1)^{|\alpha|+|\beta|}(i_{e_m} \nabla_{e_n} \alpha)(\nabla_{e_{\gamma}})(i_{e_{\beta}}) + (-1)^{|\alpha|+|\beta|-1}(i_{e_m} \nabla_{e_n} \alpha)(\nabla_{e_{\gamma}})(i_{e_{\beta}})\right]$$

$$+ (-1)^{|\alpha|}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(\nabla_{e_{\gamma}}) + (-1)^{|\beta|-1}(i_{e_m} \nabla_{e_n} \alpha)(i_{e_{\beta}})(\nabla_{e_{\gamma}})\right]$$

(C2)
and
\[
\delta C_2^R(\alpha, \beta, \gamma) \equiv \pi^{ab} \tilde{R}^{mn} \left[ (-1)^{|\alpha|+|\beta|} (i_{e_m} \nabla_{e_a} \alpha)(\nabla_{e_b} \beta)(i_{e_n} \nabla_{e_\gamma}) - (-1)^{|\alpha|+|\beta|} (i_{e_m} \alpha)(\nabla_{e_a} \beta)(i_{e_n} \nabla_{e_b} \gamma) \right. \\
+ (-1)^{|\alpha|} (i_{e_n} \nabla_{e_a} \alpha)(i_{e_m} \beta)(\nabla_{e_b} \gamma) + (-1)^{|\beta|-1} (\nabla_{e_a} \alpha)(i_{e_m} \beta)(i_{e_n} \nabla_{e_b} \gamma) \\
+ \tilde{R}^{ab} \tilde{R}^{mn} \left[ (-1)^{|\beta|} (i_{e_b} i_{e_m} \alpha)(i_{e_n} \beta)(i_{e_\gamma}) + (-1)^{|\alpha|-1} (i_{e_m} \alpha)(i_{e_n} \beta)(i_{e_b} i_{e_\gamma}) \right. \\
+ (-1)^{|\beta|-1} \tilde{R}^{ab} (i_{e_b} \tilde{R}^{mn}) (i_{e_m} \alpha)(i_{e_n} \beta)(i_{e_\gamma}).
\] (C3)

1. Associativity of $C_2^V$

To show that $C_2^V$ satisfies the associativity condition, we calculate the following quantity:
\[
C_2^V(\alpha, \beta, \gamma) - \alpha C_2^V(\beta, \gamma) = \frac{1}{2} \pi^{ij} \pi^{mn} \left( (\nabla_{e_i} \nabla_{e_m} \alpha) \beta + 2 \nabla_{e_i} \alpha \nabla_{e_m} \beta \right) \nabla_{e_j} \nabla_{e_n} \gamma \\
+ \frac{1}{3} \pi^{ia} \partial_\alpha \pi^{mn} \left[ \left( (\nabla_{e_i} \nabla_{e_m} \alpha) \beta + \nabla_{e_i} \alpha \nabla_{e_m} \beta + \nabla_{e_m} \alpha \nabla_{e_i} \beta \right) \nabla_{e_n} \gamma - (\nabla_{e_m} \alpha) \beta \nabla_{e_i} \nabla_{e_n} \gamma \right].
\]

This is found via a straightforward calculation. The quantity $C_2^V(\alpha, \beta, \gamma) - C_2^V(\alpha, \beta)\gamma$ can be calculated similarly, so that $\delta C_2^V$ is:
\[
\delta C_2^V(\alpha, \beta, \gamma) = \pi^{ij} \pi^{mn} \left( \nabla_{e_i} \nabla_{e_m} \alpha \right) \nabla_{e_j} \beta \nabla_{e_n} \gamma - \nabla_{e_i} \alpha \nabla_{e_m} \beta \left( \nabla_{e_j} \nabla_{e_n} \gamma \right) \\
- \frac{1}{3} \pi^{ij} \partial_j \pi^{mn} \left[ \nabla_{e_m} \alpha \left( \nabla_{e_i} \beta \nabla_{e_n} \gamma + \nabla_{e_n} \beta \nabla_{e_i} \gamma \right) + \left( \nabla_{e_i} \alpha \nabla_{e_m} \beta + \nabla_{e_m} \alpha \nabla_{e_i} \beta \right) \nabla_{e_n} \gamma \right]. \]

Notice that the second term can be rewritten as:
\[
-\frac{1}{3} \left[ \pi^{mj} \partial_j \pi^{in} + \pi^{nj} \partial_j \pi^{im} + \pi^{ij} \partial_j \pi^{mn} + \pi^{mj} \partial_j \pi^{in} \right] \nabla_{e_i} \alpha \nabla_{e_m} \beta \nabla_{e_n} \gamma = -\pi^{mj} \partial_j \pi^{in} \nabla_{e_i} \alpha \nabla_{e_m} \beta \nabla_{e_n} \gamma
\]

The terms in the middle combine via the Jacobi identity. Thus we have the following expression for $\delta C_2^V$:
\[
\delta C_2^V(\alpha, \beta, \gamma) = \pi^{ij} \pi^{mn} \left[ \nabla_{e_i} \nabla_{e_m} \alpha \right] \nabla_{e_j} \beta \nabla_{e_n} \gamma - \nabla_{e_i} \alpha \nabla_{e_m} \beta \left( \nabla_{e_j} \nabla_{e_n} \gamma \right) - \pi^{mj} \partial_j \pi^{ni} \nabla_{e_i} \alpha \nabla_{e_m} \beta \nabla_{e_n} \gamma.
\]

This is identical to the right-hand side of (C2), so $C_2^V$ is associative at order $\hbar^2$.

2. Associativity of $C_2^F$

Calculating the associativity $C_2^F$ in the same way, we construct $\delta C_2^F$ by finding:
\[
\alpha C_2^F(\beta, \gamma) = -\frac{1}{2} \tilde{R}^{ab} \tilde{R}^{mn} \alpha (i_{e_a} i_{e_m} \beta)(i_{e_b} i_{e_n} \gamma) - \frac{1}{3} \tilde{R}^{ab} (i_{e_b} \tilde{R}^{mn}) \left[ (-1)^{|\alpha|+|\beta|} \alpha (i_{e_a} i_{e_m} \beta)(i_{e_n} \gamma) + (-1)^{|\alpha|} \alpha (i_{e_m} \beta)(i_{e_b} i_{e_n} \gamma) \right. \\
+ (-1)^{|\beta|-1} \tilde{R}^{ab} (i_{e_b} \tilde{R}^{mn}) \alpha (i_{e_m} \nabla_{e_a} \beta)(i_{e_n} \nabla_{e_\gamma}).
\]
\[ C_2^\tilde{R}(\alpha, \beta; \gamma) = \frac{1}{2} R^{ab} \tilde{R}^{mn}(i_{e_a} i_{e_m} \alpha) \left( i_{e_b} i_{e_n} \beta \right) + (-1)^{1|\beta} i_{e_b} (i_{e_n} \gamma) + (-1)^{2|\beta} i_{e_b} (i_{e_n} \gamma) + \beta (i_{e_b} i_{e_n} \gamma) \]

We get similar equations for \( C_2^\tilde{R}(\alpha, \beta; \gamma) \) and \( C_2^\tilde{R}(\alpha, \beta; \gamma) \). Combining everything, we get:

\[ \delta C_2^\tilde{R}(\alpha, \beta; \gamma) = \frac{1}{2} R^{ab} \tilde{R}^{mn}(i_{e_a} i_{e_m} \alpha) \left( i_{e_b} i_{e_n} \beta \right) + 2(-1)^{1|\beta} i_{e_b} (i_{e_n} \gamma) + (-1)^{1|\beta} i_{e_b} (i_{e_n} \gamma) + \beta (i_{e_b} i_{e_n} \gamma) \]

This is identical to the right-hand side of (C3), so \( C_2^\tilde{R} \) is associative at order \( h^2 \). Combining the result for \( C_2^\tilde{R} \) with the result for \( C_2^\Sigma \), this calculation shows \( C_2 \) is associative at order \( h^2 \). Thus, the proposed product is associative, and therefore a star product.

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