ASYMPTOTIC FORMATION AND ORBITAL STABILITY OF PHASE-LOCKED STATES IN KURAMOTO–LOHE TYPE SYNCHRONIZATION MODELS ON LIE GROUPS

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Abstract. It is known that the Kuramoto model has a critical coupling strength above which phase-locked states exist, and, by the work of Choi, Ha, Jung, and Kim (2012), that these phase-locked states are orbitally stable. This property of admitting orbitally stable phase-locked states is preserved under the nonabelian generalizations of the Kuramoto model pioneered by Lohe (2009). We provide a framework for understanding these phenomena, by formulating the aforementioned models as special cases of a general ODE model describing populations of particles on Lie groups with pairwise attractive Kuramoto-type interactions. This general model can be used in producing many specific models with orbitally stable nonabelian phase-locked states.

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1. Introduction

1.1. The Kuramoto model and its nonabelian generalizations. To motivate the paper, we begin with a description of previous works; those who are curious of our main results may skip to subsection 1.2.

Starting with Huygens’s observation in the mid-seventeenth century of the asynchronization of two pendulum clocks hanging on a common bar, synchronous phenomena have often been observed in our nature and have been reported in scientific literature; see Pikovsky, Rosenblum, and Kurths [26] for a comprehensive survey. However, systematic approaches to synchronization based
on mathematical models were only relatively recently made by A. Winfree [30] and Y. Kuramoto [21, 22] in the 1960s and 1970s, respectively. After Winfree and Kuramoto’s works, many phenomenological and mechanical synchronization models were proposed in the biology, engineering, and statistical physics communities. Amongst other models, in this paper we begin our discussion with Kuramoto’s model, which often serves as a prototype model in the study of synchronization.

The **Kuramoto model**, introduced in [21], is a simple first-order dynamical system for the synchronization of an ensemble of weakly coupled phase oscillators. In [21] it was shown that the corresponding mean-field kinetic equation exhibits a phase-transition from the disordered state to the ordered state, as the coupling strength increases from zero to some large value; this led to an extensive investigation of the Kuramoto model, such as [1, 2, 3, 4, 5, 6, 10, 11, 13, 17, 18, 28, 29], to name a few. In this paper, we focus on the Kuramoto model without passing the system size to infinity; the phase-transition is still present in this finitary setting.

More precisely, the Kuramoto model is the Cauchy problem for the real variables \( \{ \theta_i(t) \}_{i=1}^N \) given by the nonlinear system of ordinary differential equations

\[
\begin{align*}
\dot{\theta}_i(t) &= \omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t)) \\
\theta_i(0) &= \theta_0^i, \quad i = 1, \ldots, N,
\end{align*}
\]

where \( \omega_1, \ldots, \omega_N \in \mathbb{R} \) are the intrinsic frequencies, \( \kappa > 0 \) is the coupling strength, and \( \theta_0^1, \ldots, \theta_0^N \in \mathbb{R} \) are the initial data. Existence and global uniqueness of solutions follow from the Cauchy-Lipschitz theory.

In nonlinear dynamical systems theory, one of the central problems other than well-posedness is to understand the large-time dynamics such as existence of equilibria or limit-cycles and their stability. Thus, the first step in this direction is to identify the possible limiting configurations of the given dynamical system, and the next step is to study the stability of the limiting configurations.

In order to do this, it is worth observing the following conservation law for (1.1):

\[
\frac{d}{dt} \left( \sum_{i=1}^{N} \theta_i(t) - t \sum_{i=1}^{N} \nu_i \right) = 0, \quad t > 0.
\]

Thus, if \( \sum_{i=1}^{N} \nu_i \neq 0 \), then the Kuramoto model (1.1) cannot have an equilibrium. However, the Kuramoto model has a related Galilean invariance, namely (1.1) is invariant under Galilean transformations of the form

\[
\begin{align*}
\theta'_i &= \theta_i - \nu t - \vartheta, \quad \nu'_i = \nu_i - \nu, \quad i = 1, \ldots, N,
\end{align*}
\]

for \( \nu, \vartheta \in \mathbb{R} \). In particular, if we take

\[
\nu = \frac{1}{N} \sum_{j=1}^{N} \nu_j, \quad \vartheta = 0,
\]

then the conservation law (1.2) now becomes conservation of total phase:

\[
\sum_{i=1}^{N} \theta'_i = \sum_{i=1}^{N} \theta^0_i.
\]

Thus it makes sense to discuss equilibria of the transformed variables \( \theta'_i \). In terms of the pre-transform variables \( \theta_i \), an equilibrium of \( \theta'_i \) means an equilibrium relative to a frame rotating with velocity given by the average phase velocity \( \frac{1}{N} \sum_{i=1}^{N} \nu_i \). By the conservation law (1.2), this is equivalent to a relative equilibrium, namely a state where the relative phases \( \theta_i - \theta_j \) are constant. We call these relative equilibria phase-locked states, and we call convergence to such equilibria asymptotic phase-locking.
Definition 1.1 (\cite{3,5,17,28,29}). Let \( \{ \theta_i(t) \}_{i=1}^N \) be a solution to the Kuramoto model (1.1).

1. \( \{ \theta_i(t) \}_{i=1}^N \) is a phase-locked state of (1.1) if all the relative phase differences are constant:
\[
\theta_i(t) - \theta_j(t) = \theta_i(0) - \theta_j(0), \quad t \geq 0, \quad 1 \leq i, j \leq N.
\]

2. \( \{ \theta_i(t) \}_{i=1}^N \) exhibits asymptotic phase-locking if the relative phase differences converge as \( t \to \infty \):
\[
\exists \lim_{t \to \infty} (\theta_i(t) - \theta_j(t)), \quad i, j = 1, \ldots, N.
\]

Remark 1.1. It is easy to see that asymptotic phase-locking implies that the relative frequencies must tend to zero asymptotically:
\[
\lim_{t \to \infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0, \quad i, j = 1, \ldots, N.
\]
This is sometimes used as the definition for asymptotic phase-locking.

The existence and stability of phase-locked states are important topics in the nonlinear dynamics of the Kuramoto model, as discussed in \cite{1,2,4,6,11}.

Heuristically, as the coupling strength \( \kappa \) gets larger and larger, it should become “easier” for the system (1.1) to admit phase-locked states and exhibit asymptotic phase-locking. Indeed, the interaction term \( \sin(\theta_i - \theta_j) \) is pairwise attractive, and so if the coupling strength \( \kappa \) is large enough compared to the differences \( \nu_i - \nu_j \) in the intrinsic frequencies, the pairwise interaction terms \( \sin(\theta_k - \theta_l) \) should dominate in the expression \( \dot{\theta}_i - \dot{\theta}_j \), leading to an “ease” of the existence of phase-locked states and an “ease” of approaching them.

This heuristic argument may be quantified into the following precise theorems. First, when the \( \nu_i \)'s are identical, we have the following theorem.

Theorem 1.1 (\cite{13, Theorem 3.1}). Suppose that the coupling strength and natural frequencies satisfy
\[
\kappa > 0, \quad \nu_i = \nu_j, \quad i, j = 1, \ldots, N,
\]
and that the initial configuration satisfies
\[
D^0 := \max_{i,j=1,\ldots,N} |\theta^0_i - \theta^0_j| < \pi.
\]
Then
\[
\max_{i,j=1,\ldots,N} |\theta_i(t) - \theta_j(t)| \leq D^0 \exp \left( -\frac{\kappa t \sin D^0}{D^0} \right), \quad t \geq 0.
\]

Second, when the \( \nu_i \)'s are nonidentical, we have the following theorem.

Theorem 1.2 (\cite{4,10,11}). Suppose that the coupling strength \( \kappa \) and an auxiliary parameter \( \kappa_e \) satisfy
\[
\kappa > \kappa_e > \max_{i,j=1,\ldots,N} |\nu_i - \nu_j| > 0
\]
and let \( \{ \theta_i(t) \}_{i=1}^N \) be a solution to (1.1) such that
\[
\max_{i,j=1,\ldots,N} |\theta^0_i - \theta^0_j| < \pi - \arcsin \left( \frac{\max_{i,j=1,\ldots,N} |\nu_i - \nu_j|}{\kappa_e} \right).
\]
Then, the following assertions hold:

1. The phase diameter is bounded in the sense that there exists a finite time \( T \geq 0 \) such that
\[
\max_{i,j=1,\ldots,N} |\theta_i(t) - \theta_j(t)| \leq \arcsin \left( \frac{\max_{i,j=1,\ldots,N} |\nu_i - \nu_j|}{\kappa} \right), \quad \text{for } t \geq T.
\]
that the Kuramoto model is the Lipschitz theory again guarantees uniqueness and global existence of solutions. It can be checked that the Hermitian conjugate, and phase-locked states are solutions and their stability. In the nonabelian setting, one can see that the correct definition for \((1.5)\) is to understand the large-time dynamics, namely the existence of equilibria or limit-cycles and their stability. In the nonabelian setting, one can see that the correct definition for phase-locked states are solutions \(\{U_i(t)\}_{i=1}^N\) such that the nonabelian ratios \(U_iU_j^\dagger\) are constant, or

(2) Asymptotic phase-locking occurs at an exponential rate: there exist positive constants \(C_0\) and \(\Lambda = O(\kappa)\) such that

\[
D(\tilde{\Theta}(t)) \leq C_0 \exp(-\Lambda t), \quad \text{for } t \geq 0.
\]

(3) The emergent phase-locked state is unique up to \(U(1)\)-symmetry, and is ordered according to the ordering of their natural frequencies. Namely, if \(\{\tilde{\theta}_i(t)\}_{i=1}^N\) were another solution to \((1.1)\) with

\[
\max_{i,j=1,\ldots,N} |\tilde{\theta}_i - \tilde{\theta}_j| < \pi - \arcsin\left(\frac{\max_{i,j=1,\ldots,N} |\nu_i - \nu_j|}{\kappa_c}\right),
\]

then there exists \(\phi \in \mathbb{R}\) such that

\[
\lim_{t \to \infty} \left(\theta_i(t) - \tilde{\theta}_i(t)\right) = \phi, \quad i = 1, \ldots, N.
\]

Also, there are universal constants \(U\) and \(L\) such that for any indices \(i,j\) with \(\nu_i \geq \nu_j\),

\[
\sin^{-1}\left(\frac{\nu_i - \nu_j}{\kappa U}\right) \leq \lim_{t \to \infty} \left(\theta_i(t) - \theta_j(t)\right) \leq \sin^{-1}\left(\frac{\nu_i - \nu_j}{\kappa L}\right).
\]

Remark 1.2. The threshold \(\max_{i,j=1,\ldots,N} |\nu_i - \nu_j|\) in Theorem 1.2 is equivalent up to universal constant factors to the actual threshold for the existence of phase-locked states. More precisely, by [28], for each fixed \(N \geq 2\) and \(\{\nu_i\}_{1}^{N}\) there exists \(\kappa_c(\{\nu_i\}_{1}^{N}) \geq 0\) such that \((1.1)\) admits a phase-locked state if and only if \(\kappa \geq \kappa_c(\{\nu_i\}_{1}^{N})\); this \(\kappa_c(\{\nu_i\}_{1}^{N})\) satisfies the following:

\[
\frac{N}{2(N-1)} \max_{i,j=1,\ldots,N} |\nu_i - \nu_j| \leq \kappa_c(\{\nu_i\}_{1}^{N}) \leq \max_{i,j=1,\ldots,N} |\nu_i - \nu_j|.
\]

This concludes our discussion of the Kuramoto model; to recapitulate, in the case of identical oscillators, the Kuramoto model completely synchronizes in the sense of Theorem 1.1, and in the case of nonidentical oscillators, the Kuramoto model admits an orbitally stable phase-locked state in the sense of Theorem 1.2. The purpose of this paper is to obtain a better understanding of Theorems 1.1 and 1.2 by proving them in a more general setting.

Lohe [23, 24] suggested a nonabelian generalization of the Kuramoto model to the unitary groups \(U(d)\) as follows: the Lohe model describes the dynamics of \(\{U_i(t)\}_{i=1}^N \subset U(d)\) as

\[
\begin{align*}
\dot{U_i}U_i^\dagger &= iH_i + \frac{\kappa}{2N} \sum_{j=1}^{N} \left( U_jU_i^\dagger - U_iU_j^\dagger \right), \quad i = 1, \ldots, N, \\
U_i(0) &= U_i^0,
\end{align*}
\]

where \(\kappa > 0\) is again the coupling strength, the Hermitian matrix \(H_i\) is the Hamiltonian of \(U_i\), \(\dagger\) denotes the Hermitian conjugate, and \(\{U_i^0\}_{i=1}^N \subset U(d)\) is the initial data. The usual Cauchy-Lipschitz theory again guarantees uniqueness and global existence of solutions. It can be checked that the Kuramoto model is the \(U(1)\) case of the Lohe model \((1.5)\).

Model \((1.5)\) is a mathematical model for quantum synchronization [24]: consider a quantum network [20] consisting of \(N\) quantum nodes and quantum channels connecting all possible pairs of quantum nodes. Each quantum node can be viewed as a component of a physical system interacting via quantum channels. For instance, atoms at nodes can have effect spin-spin interactions generated by a single photon pulse traveling along the channels (see [20] for a detailed explanation). Lohe [23, 24] proposed the model \((1.5)\) for synchronization over such quantum networks and studied the existence of phase-locked states of the proposed model based on numerical simulations.

As explained earlier, once well-posedness is out of the way, the central task in studying model \((1.5)\) is to understand the large-time dynamics, namely the existence of equilibria or limit-cycles and their stability. In the nonabelian setting, one can see that [23, 24] the correct definition for phase-locked states are solutions \(\{U_i(t)\}_{i=1}^N\) such that the nonabelian ratios \(U_iU_j^\dagger\) are constant, or
equivalently right-invariant flows of the form $U_i = U_i^\infty e^{-iMt}$, where $U_i^\infty \in U(d)$, $i = 1, \ldots, N$ are unitary matrices, and $\Lambda$ is a constant $d \times d$ Hermitian matrix. Solving the corresponding nonlinear algebraic system, one can see that $U_i^\infty$, $i = 1, \ldots, N$, and $\Lambda$ must satisfy the following algebraic constraints:

$$U_i^\infty \Lambda U_i^\infty† = H_i - \frac{ik}{2N} \sum_{k=1}^N \left(U_k^\infty U_k^\infty† - U_k^\infty U_i^\infty†\right), \quad i = 1, \ldots, N.$$  

The orbital stability of these phase-locked states roughly means that if $\{V_i\}_{i=1}^N$ is another solution with similar initial conditions, then the limiting values of $U_iU_j^†$ and $V_iV_j^†$ are the same and that $V_i^†U_i$ converges to a value common for all $i$. Note that the stability issue can be studied in an a priori setting without knowing the existence of phase-locked states and implies the uniqueness of phase-locked states.

Lohe [23] numerically observed a nonlinear version of Theorem 1.2 for the $\mathbb{U}(2)$ model, and the author and Ha [16] gave a rigorous proof of a nonlinear version of Theorems 1.1 and 1.2 for all unitary groups $U(d)$. These theorems are stated as follows.

**Theorem 1.3 (16, Theorem 1).** Suppose that the coupling strength $\kappa$ and Hermitian matrices $H_i$, $i = 1, \ldots, N$, satisfy

$$\kappa > 0, \quad H_i = H_j, \quad i, j = 1, \ldots, N,$$

and suppose the initial data $\{U_i^0\}_{i=1}^N$ satisfies

$$D^0 := \max_{i,j=1,\ldots,N} \|U_i^0 - U_j^0\|_{\text{Frob}} < \sqrt{2},$$

where $\| \cdot \|_{\text{Frob}}$ denotes the Frobenius norm. Then

$$\max_{i,j=1,\ldots,N} \|U_i(t) - U_j(t)\|_{\text{Frob}} \leq \sqrt{\frac{2(D^0)^2}{(D^0)^2 + (2 - (D^0)^2)e^{2\kappa t}}}.$$  

**Theorem 1.4 (16, Theorem 3).** Suppose that the coupling strength $\kappa$ and Hermitian matrices $H_i$, $i = 1, \ldots, N$, satisfy

$$\kappa > \frac{54}{17} \max_{i,j=1,\ldots,N} \|H_i - H_j\|_{\text{Frob}},$$

and the initial data $\{U_i^0\}_{i=1}^N$ satisfies

$$\max_{i,j=1,\ldots,N} \|U_i^0 - U_j^0\|_{\text{Frob}} < \sqrt{2}.$$  

Then the following assertions are true.

1. The flow $\{U_i(t)\}_{i=1}^N$ achieves asymptotic phase-locking, i.e.,

$$\lim_{t \to \infty} U_i(t)U_j(t)^†$$

converges exponentially fast.

2. There exists a phase-locked state $\{V_i\}$ and a unitary matrix $L \in U(d)$ such that

$$\max_{i,j=1,\ldots,N} \|V_i - V_j\|_{\text{Frob}} < \sqrt{2}, \quad \lim_{t \to \infty} \|U_i - V_iL\|_{\text{Frob}} = 0,$$

the convergence happening exponentially fast.

3. Phase-locked states $\{V_i\}$ with $\max_{i,j=1,\ldots,N} \|V_i - V_j\|_{\text{Frob}} < \sqrt{2}$ are unique up to right-multiplication, that is, if $\{W_i\}$ is another phase-locked state with $\max_{i,j=1,\ldots,N} \|W_i - W_j\|_{\text{Frob}} < \sqrt{2}$, then there exists a unitary matrix $L \in U(d)$ such that $W_i = V_iL$.  

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Remark 1.3. One can see from Theorem 1.4 that if \( \{U_i(t)\}_{i=1}^N \) and \( \{\tilde{U}_i(t)\}_{i=1}^N \) are two solutions to (1.5) with
\[
\kappa > \frac{54}{17} \max_{i,j=1,\cdots,N} \|H_i - H_j\|_{\text{Frob}}, \quad \|U_i^0 - U_j^0\|_{\text{Frob}}, \quad \|\tilde{U}_i^0 - \tilde{U}_j^0\|_{\text{Frob}} < \sqrt{2}, \quad i,j = 1,\cdots,N,
\]
that the following orbital stability statements hold:
1. \( \lim_{t \to \infty} U_i(t)U_j(t)^\dagger = \lim_{t \to \infty} \tilde{U}_i(t)\tilde{U}_j(t)^\dagger, \quad i,j = 1,\cdots,N, \)
2. there exists a unitary matrix \( L_\infty \in U(d) \) such that \( \lim_{t \to \infty} U_i(t)^\dagger \tilde{U}_i(t) = L_\infty. \)

Theorems 1.3 and 1.4 can be viewed as \( U(d) \)-versions of Theorems 1.1 and 1.2.

It was suggested by Lohe [23] that the model (1.5) can be extended to other matrix groups. Here is a direct quotation from [23, p.5]:

\[ \cdots \text{There also exist non-Abelian Kuramoto models for the symplectic groups, with defining equations similar to (1.5), and hence for all classical compact Lie groups.} \]

By choosing an indefinite metric one may extend these models to the noncompact case, for which trajectories are unbounded, however, we do not discuss synchronization or other properties of these particular models here.

In [15], the author, Ha, and Ko interpreted this as a model on \( GL_n(\mathbb{C}) \) as follows:
\[
(1.6) \quad \begin{cases} 
X_iX_i^{-1} = H_i + \frac{\kappa}{2N} \sum_{j=1}^N \left( X_jX_i^{-1} - X_iX_j^{-1} \right), \\
X_i(0) = X_i^0,
\end{cases} \tag{1.6}
\]
where \( H_i \in M_{n,n}(\mathbb{C}) \) are \( n \times n \) complex matrices and \( X_i^0 \in GL_n(\mathbb{C}) \) are the initial data, and characterized various matrix Lie groups, such as \( U(d), O(m,n), SO(m,n), SP(d), SL_2(\mathbb{R}) \), on which this model defines a closed trajectory. (More precisely, if a matrix Lie group \( G \leq GL_n(\mathbb{C}) \) has Lie algebra \( g \) satisfying the closure condition
\[
X - X^{-1} \in g, \quad \forall X \in G,
\]
then with \( H_i \in g \) and \( X_i^0 \in G \) the local solution \( \{X_i(t)\}_{i=1}^N \) to (1.6) satisfies \( X_i(t) \in G \) for its time of existence.) Note that the Cauchy-Lipschitz theory now only guarantees the uniqueness and local existence of solutions to (1.6), for \( GL_n(\mathbb{C}) \) is noncompact. The following example illustrates finite-time blowup.

Example 1.1 ([15, Remark 2.3]). \textbf{We have the divergent initial condition}
\[
n = N = 2, \quad H_1 = H_2 = 0, \quad X_1^0 = \text{diag}(1,1), \quad X_2^0 = \text{diag}(-0.5,-2).
\]
Indeed, the solution takes the form
\[
X_1 = \text{diag}(x_1,x_1^{-1}), \quad X_2 = \text{diag}(x_2,x_2^{-1}),
\]
and \( x_1/x_2 \) solves
\[
\frac{d}{dt}(x_1/x_2) = \kappa \left( 1 - (x_1/x_2)^2 \right), \quad x_1(0)/x_2(0) = -2
\]
and hence diverges to \(-\infty\) in finite time.

Thus, care is needed in arguing global existence of solutions for special initial data.

One can define phase-locked states of (1.6) to be solutions \( \{X_i(t)\}_{i=1}^N \) such that \( X_i(t)X_j(t)^{-1} \) is constant, and can see that they are right-invariant flows of the form \( X_i(t) = X_i^\infty e^{\Lambda t} \), where \( X_i^\infty \in GL_n(\mathbb{C}), \ i = 1,\cdots,N, \) and \( \Lambda \in M_{n,n}(\mathbb{C}) \) satisfy
\[
X_i^\infty \Lambda (X_i^\infty)^{-1} = H_i + \frac{\kappa}{2N} \sum_{j=1}^N \left[ X_j^\infty (X_i^\infty)^{-1} - X_i^\infty (X_j^\infty)^{-1} \right], \quad i = 1,\cdots,N.
\]
The work [15] was able to establish a nonlinear version of Theorems 1.1 and 1.2 for the model (1.6), as follows. First, for identical oscillators, we have the following theorem.

**Theorem 1.5** ([15, Proposition 4.1, Theorems 4.1 and 4.2]). Let the coupling strength and intrinsic Hamiltonians satisfy

$$\kappa > 0, \quad H_i = H \in M_{n,n}(\mathbb{C}), \quad i = 1, \cdots, N,$$

and suppose the initial data satisfies

$$D^0 := \|X_i^0(X_j^0)^{-1} - I_n\|_{\text{Frob}} < 1.$$

Then

1. (Global existence of solutions) There is a global solution \( \{X_i(t)\}_{i=1}^N \) to (1.6).
2. (Solution operator splitting) \( \tilde{X}_i(t) = e^{-Ht}X_i(t) \) is a solution to (1.6) with \( H_i = 0, \tilde{X}_i(0) = X_i^0 \).
3. (Exponential synchronization) We have

$$\|e^{-Ht}X_i(t)X_j(t)^{-1}e^{Ht} - I_n\|_{\text{Frob}} \leq \frac{D^0}{(1 - D^0)e^{\kappa t} + D^0}.$$

In particular, if \( \kappa > 2\|H\|_{\text{Frob}} \), then \( X_i(t)X_j(t)^{-1} \to I_n \) exponentially.

Second, for nonidentical oscillators, we have the following theorem.

**Theorem 1.6** ([15, Theorems 5.1, 5.2]). Suppose that the coupling strength \( \kappa \) and initial data \( \{X_i^0\}_{i=1}^N \) satisfy

$$\kappa > (6 + 4\sqrt{2}) \max_{i=1,\cdots,N} \|H_i\|_{\text{Frob}}, \quad \|X_i^0(X_j^0)^{-1} - I_d\|_{\text{Frob}} < \sqrt{2} - 1, \quad i, j = 1, \cdots, N.$$

Then the following assertions hold.

1. The solution \( \{X_i(t)\}_{i=1,\cdots,N,t \geq 0} \) to (1.6) exists for all \( t \geq 0 \).
2. The solution \( \{X_i(t)\}_{i=1,\cdots,N,t \geq 0} \) achieves asymptotic phase-locking in the sense that the limit \( \lim_{t \to \infty} X_i(t)X_j(t)^{-1} \) exists for all \( i, j = 1, \cdots, N \). This convergence is exponentially fast.
3. There exists a phase-locked state \( \{Y_i(t)\}_{i=1,\cdots,N,t \geq 0} \) with

$$\|Y_iY_j^{-1} - I_d\|_{\text{Frob}} < \sqrt{2} - 1,$$

and it is unique up to right-multiplication: if \( \{Z_i(t)\}_{i=1,\cdots,N,t \geq 0} \) is another phase-locked state with

$$\|Z_iZ_j^{-1} - I_d\|_{\text{Frob}} < \sqrt{2} - 1,$$

then there exists some \( S \in GL_n(\mathbb{C}) \) such that

$$Z_i = Y_iS, \quad i = 1, \cdots, N.$$

Moreover, under the stronger assumption

$$\kappa > \frac{16}{7} (6 + 4\sqrt{2}) \max_{i=1,\cdots,N} \|H_i\|_{\text{Frob}},$$

we have the following assertion.

4. For the above phase-locked state \( \{Y_i(t)\}_{i=1,\cdots,N,t \geq 0} \) there exists some \( R \in GL_n(\mathbb{C}) \) such that

$$\lim_{t \to \infty} Y_i(t)^{-1}X_i(t) = R.$$

This convergence is exponentially fast.
All in all, Theorems 1.5 and 1.6 give analogues of Theorems 1.1 and 1.2 for system (1.6). Compared to the previous Theorems 1.1, 1.2, 1.3, 1.4 we start to see that the threshold for $\kappa$ for synchronous behavior start to depend on the maximum absolute magnitude $\max_{i=1,\ldots,N} \|H_i\|_{\text{Frob}}$, instead of the maximum relative magnitude $\max_{i,j=1,\ldots,N} \|H_i - H_j\|_{\text{Frob}}$. This is because the right-invariant quantity $\|XY^{-1} - I_d\|_{\text{Frob}}$, which serves as a proxy for a metric, is not left-invariant.

Given this line of reasoning so far, it is natural to imagine whether there is a version of the Kuramoto and Lohe models on general Lie groups and whether it still enjoys the stability properties outlined so far. We will now present an abstraction of the ideas presented so far in the setting of Lie groups.

1.2. The generalized Kuramoto–Lohe model and our main theorems. Let $G$ be a Lie group with Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. We wish to model the behavior of particles $\{X_i(t)\}_{i=1}^N$ on $G$. As before, each particle $X_i$ has an intrinsic Hamiltonian $H_i \in \mathfrak{g}$, and, under the absence of other particles, has equation of motion

$$\dot{X}_i = (dR_{X_i})_{e}(H_i),$$

where $e \in G$ is the identity element of $G$, and for $g \in G$, $R_g : G \to G$ denotes right multiplication $R_g(x) := xg$ (and similarly $L_g : G \to G$ denotes left multiplication $L_g(x) := gx$). The interaction between particles $X_i$ and $X_j$ will happen via their relative phases $X_jX_i^{-1}$, and in order to properly define the equation of motion, we will need a function $\phi : G \to \mathfrak{g}$ to choose the direction $\phi(X_jX_i^{-1}) \in \mathfrak{g}$ that $X_i$ `heads toward' $X_j$.

Thus, we define the generalized Kuramoto–Lohe model on $G$ as the Cauchy problem

$$\dot{X}_i = (dR_{X_i})_{e}(H_i + \frac{\kappa}{N} \sum_{j=1}^N (\phi(X_jX_i^{-1}))), \quad i = 1, \ldots, N,$$

for the solution $\{X_1, \ldots, X_N\}$, where $X_i^0, \ldots, X_N^0 \in G$ are the initial data, $H_1, \ldots, H_N \in \mathfrak{g}$ are the intrinsic Hamiltonians, and $\kappa > 0$ is the coupling strength. Also, $\phi : G \to \mathfrak{g}$ is a map describing the pairwise interaction of the oscillators and satisfies

(H) $\phi$ is $C^1$, $\phi(e) = 0$, $(d\phi)_e : \mathfrak{g} \to \mathfrak{g}$ has only (complex) eigenvalues with positive real parts.

It is clear that this model generalizes the Kuramoto model (1.1) and the Lohe matrix models (1.5) and (1.6). Indeed, for the Kuramoto model we take $G = \mathbb{S}^1$ and interaction function $\phi(\theta) = \sin \theta$ (we are considering the dynamics on the universal cover $\mathbb{R}$ of $\mathbb{S}^1$). The unitary Lohe model (1.5) is the case $G = U(d)$, $\phi(U) = \frac{1}{d}(U - U^\dagger)$, and the matrix Lohe model (1.6) is the case $G = GL_n(\mathbb{C})$, $\phi(X) = \frac{1}{d}(X - X^{-1})$ of the generalized Kuramoto–Lohe model.

DeVille [9] considered the following model on $GL_n(\mathbb{C})$:

$$\dot{X}_iX_i^{-1} = H_i + \frac{1}{d} \sum_{j=1}^N \gamma_{ij} \left(f(X_jX_i^{-1}) - f(X_iX_j^{-1})\right).$$

For functions $f$ satisfying (H), DeVille’s model also is of the form (1.7).

The standard Cauchy–Lipschitz theory guarantees the existence of a local $C^1$-solution to (1.7). However, global existence of a solution may fail if the initial data are far apart, as demonstrated in Example 1.1.

Note that, for $g \in G$ and $J$ in the center of $\mathfrak{g}$, the system (1.7) admits the Galilean transformation

$$X_i^0 \mapsto X_i^0 g, \quad X_i(t) \mapsto X_i(t)ge^{Ht}, \quad H_i \mapsto H_i + J.$$

Therefore, it is natural to consider right-invariant metrics on $G$ for the purposes of analyzing (1.7), and to consider relative equilibria of (1.7) as the solutions for which the ratios $X_iX_j^{-1}$ are constant.
More precisely, endow \( g \) with a Euclidean norm \( | \cdot | \), extend this to a right-invariant Riemannian metric on \( G \), and let \( d : G \times G \to [0, \infty) \) be the corresponding right-invariant geodesic distance. Also, we will call relative equilibria phase-locked states, as defined below.

**Definition 1.2.** A solution \( \{X_i\}_{i=1}^N \) to (1.7) is a phase-locked state if and only if
\[
X_i(t) X_j(t)^{-1} \text{ is constant in } t \text{ for each } i, j = 1, \cdots, N.
\]

We may again characterize the phase-locked states as follows. This is essentially due to Lohe [23].

**Proposition 1.1** (Characterization of phase-locked states, [23, p.6]). A solution \( \{X_i\}_{i=1}^N \) to (1.7) is a phase-locked state if and only if
\[
X_i(t) X_j(t)^{-1} \text{ is constant in } t \text{ for each } i, j = 1, \cdots, N, \quad \text{and } \Lambda \in g \text{ satisfy}
\]
(1.9)
\[
\text{Ad}_{X_i} \Lambda = H_i + \frac{\kappa}{N} \sum_{j=1}^N \phi(X_j^\infty (X_i^\infty)^{-1}), \quad i = 1, \cdots, N,
\]
where \( \text{Ad}_g = (dL_g)_{g^{-1}} \circ (dR_{g^{-1}})_e \) is the adjoint.

The main theorems of this paper are as follows. The first main theorem says that if the intrinsic Hamiltonians are identical, then a unique global solution exists and exhibits complete synchronization.

**Theorem 1.7.** There exist constants \( C, c, c_1, c_2 > 0 \) depending on \( G, g, \) and \( \phi \) such that the following is true.

1. Let the coupling strength \( \kappa \) and intrinsic Hamiltonians \( H_i \) satisfy
\[
\kappa > c |H|, \quad H_i = H, \quad i = 1, \cdots, N,
\]
for some fixed \( H \in g \), and let the initial condition \( \{X_i^0\}_{i=1}^N \) satisfy
\[
d(X_i^0, X_j^0) < c_1, \quad i, j = 1, \cdots, N.
\]
Then the following statements hold.
(a) (Global existence of solutions) There is a global solution \( \{X_i(t)\}_{i=1}^N \) to (1.7).
(b) (Exponential synchronization) We have
\[
d(X_i(t), X_j(t)) \leq Ce^{-c_2 \kappa t}, \quad i, j = 1, \cdots, N.
\]

2. Suppose that the interaction function \( \phi \) satisfies
\[
\text{Ad}_g \phi(h) = \phi(ghg^{-1}), \quad g, h \in G.
\]
Let the coupling strength \( \kappa \) and intrinsic Hamiltonians \( H_i \) satisfy
\[
\kappa > 0, \quad H_i = H, \quad i = 1, \cdots, N,
\]
for some fixed \( H \in g \), and the initial condition \( \{X_i^0\}_{i=1}^N \) satisfy
\[
d(X_i^0, X_j^0) < c_1.
\]
Then the following statements hold.
(a) (Global existence of solutions) There is a global solution \( \{X_i(t)\}_{i=1}^N \) to (1.7).
(b) (Solution operator splitting) \( \dot{X}_i(t) = \exp(-Ht)X_i(t) \) is a solution to (1.7) with \( H_i = 0, \dot{X}_i(0) = X_i^0 \).
(c) (Exponential synchronization) We have
\[
d(\exp(-Ht)X_i(t), \exp(-Ht)X_j(t)) \leq Ce^{-c_2 \kappa t}, \quad i, j = 1, \cdots, N.
\]
The second main theorem says that if the coupling strength $\kappa$ is sufficiently large compared to the magnitude of the Hamiltonians $|H_i|$, then phase-locked states, with each particle being sufficiently close together (measured by $d$), exist and are unique up to right-multiplication. Also, if the initial data are sufficiently close together, then a global solution exists and exponentially converges towards a right-multiplication of the aforementioned phase-locked state.

**Theorem 1.8.** There exist constants $C, c, c_1, c_2 > 0$ depending on $G, g$, and $\phi$ such that if

\[(1.10) \quad \kappa > c \|H\|_\infty, \quad \|H\|_\infty := \max_{i=1,\ldots,N} |H_i|,\]

then the following holds.

1. **(Existence and uniqueness of a phase-locked state)** There exists a phase-locked state $\{X_i^\infty \exp(\Lambda t)\}_{i=1}^N$, with $X_i^\infty \in G, i = 1, \ldots, N$, $\Lambda \in g$ satisfying (1.9), such that

\[d(X_i^\infty, X_j^\infty) \leq \frac{C \|H\|_\infty}{\kappa}.\]

This is unique up to right-multiplication in a larger set, i.e., if $\{X_i(t)\}_{i=1}^N$ is a phase-locked state with

\[d(X_i^0, X_j^0) < c_1, \quad \forall i, j = 1, \ldots, N,\]

then $X_i(t) = X_i^\infty \exp(\Lambda t)g, i = 1, \ldots, N$, for some fixed $g \in G$. (This larger set will be the basin of attraction for this phase-locked state.)

Now let $\{X_i(t)\}_{i=1}^N$ be a solution to (1.4) with initial data $\{X_i^0\}_{i=1}^N$ satisfying

\[d(X_i^0, X_j^0) < c_1, \quad \forall i, j = 1, \ldots, N.\]

2. **(Global existence and local stability)** There exists a unique global solution $X(t) = \{X_i(t)\}_{i=1}^N$ to (1.7), with

\[d(X_i(t), X_j(t)) < c_1, \quad t \geq 0, \quad \forall i, j = 1, \ldots, N,\]

and

\[\limsup_{t \to \infty} d(X_i(t), X_j(t)) \leq \frac{C \|H\|_\infty}{\kappa}, \quad t \geq 0, \quad \forall i, j = 1, \ldots, N.\]

3. **(Asymptotic phase-locking)** We have

\[\lim_{t \to \infty} X_i(t)X_j(t)^{-1} = X_i^\infty(X_j^\infty)^{-1},\]

the convergence rate being $O_G(e^{-\kappa c_2 t})$.

4. **(Orbital stability)** The solution $X(t) = \{X_i(t)\}_{i=1}^N$ exponentially converges to a right-multiplication of the phase-locked state of (1), i.e., there is a phase-locked state $\{Z_i^\infty \exp(Mt)\}_{i=1}^N$ such that

\[d(X_i(t), Z_i^\infty \exp(Mt)) = O_G, \quad t \geq 0.\]

5. **(Exponential synchronization of the normalized speeds)** We have

\[(dL_{X_i(t)})_{eX_i(t)} - M = O_G, \quad t \geq 1, \ldots, N.\]

\[\text{We will use the following (standard) asymptotic notation. For } P, Q > 0, P = O(Q) \text{ and } Q = \Omega(P) \text{ mean that } P \leq KQ \text{ for a universal constant } K \in (0, \infty).\]

\[\text{If we need to allow for dependence on parameters, we indicate this by subscripts. For example, in the presence of auxiliary parameters } \psi, \xi, \text{ the notations } P = O_{\psi,\xi}(Q) \text{ and } Q = \Omega_{\psi,\xi}(P) \text{ mean that } P \leq K(\psi,\xi)Q \text{ where } K(\psi,\xi) \in (0, \infty) \text{ may depend only on } \psi \text{ and } \xi.\]
Theorem 1.7 is a generalization of Theorems 1.1, 1.3, and 1.5 and Theorem 1.8 is a generalization of Theorems 1.2, 1.4, and 1.6. Theorem 1.7 and part (2) of Theorem 1.8 are relatively unsurprising: as the particles are pairwise attractive, if they are initially close together, then they should stay close together for all times, guaranteeing global existence, and should attract each other as much as they can. The author finds part (4) of Theorem 1.8, the orbital stability statement, to be somewhat surprising: it only requires (H), namely that \( \phi \) is \( C^1 \) and that it is locally attractive in the sense that \((d\phi)_e\) has only eigenvalues with positive real parts. Note that higher order considerations, such as \( C^2 \) regularity of \( \phi \) or curvature properties of \( G \), are irrelevant.

Thus, as long as we define any model in the form of (1.7), by choosing a Lie group \( G \) and an interaction function \( \phi : G \to \mathfrak{g} \) to satisfy (H), this model will automatically have the stability properties of Theorem 1.8. This grants us significant liberty in defining locally well-behaved synchronization models: we only need to choose a Lie group \( G \) and an interaction function \( \phi : G \to \mathfrak{g} \) satisfying (H). For example, it was suggested by an anonymous referee for the previous paper [10] that we consider the model on \( G = SU(d), SL_d(\mathbb{C}), SL_d(\mathbb{R}) \) given by

\[
(1.11) \quad \dot{X}_i X_i^{-1} = H_i + \frac{\kappa}{2N} \sum_{j=1}^{N} \left( X_j X_j^{-1} X_i X_i^{-1} - \frac{1}{d} \text{Tr}[X_j X_i^{-1}] I_d + \frac{1}{d} \text{Tr}[X_i X_j^{-1}] I_d \right).
\]

One can check that (H) holds for this interaction function, so that Theorem 1.8 holds for (1.11).

Here is an intuitive heuristic as to why Theorem 1.8 should be true. If the particles are concentrated locally, say around the identity \( e \in G \), then by setting \( X_i = \exp(Y_i) \quad i = 1, \ldots, N \), we can linearize the governing equation (1.7) to

\[
(1.12) \quad \dot{Y}_i = H_i + \frac{\kappa}{N} \sum_{k=1}^{N} (d\phi)_e(Y_k - Y_i), \quad i = 1, \ldots, N.
\]

This is an exactly solvable linear differential equation. In particular, the dynamics for the differences \( Y_i - Y_j \) is simple:

\[
\dot{Y}_i - \dot{Y}_j = H_i - H_j - \kappa (d\phi)_e(Y_i - Y_j), \quad i, j = 1, \ldots, N.
\]

As \((d\phi)_e\) has only eigenvalues with positive real parts, if we make \( \kappa \) large enough compared to \(|H_i - H_j|\), then \( Y_i - Y_j \) should converge exponentially to a fixed value.

Thus, the proof of Theorem 1.8 for the nonlinear model (1.7) will be a “nonlinearization” of the above heuristic argument, i.e., we will check that the nonlinear effects are controllable compared to the dominant linear term. One caveat of this nonlinearization is that, as mentioned after Theorem 1.6, we will need to require that \( \kappa \) is large compared to the \(|H_i|'s\) as in (1.10), instead of requiring \( \kappa \) to be large compared to the \(|H_i - H_j|'s\). There will also be the complication that the \( X_i \)'s may not stay in the neighborhood of a single point for all time, so we will work with the logarithms of the ratios \( X_i X_j^{-1} \), which will stay close to the identity \( e \in G \).

### 1.3. Directions for further research.

One reason for considering generalizations of the Kuramoto model (1.11) should come from a search for new phenomena not observed in the Kuramoto model. Thus, since Theorem 1.8 is true not only for the Kuramoto model but also for its Lie group generalizations in the form of (1.7), the merit of studying such models in the form of (1.7) will have to come from the behavior of the model when the particles are not extremely concentrated. For example, DeVille [9] presents several special solutions to (1.8) and shows that it has multiple attractors under rich network topologies, while Ritchie, Lohe, and Williams [27] present a detailed study of the global solutions of the special case \( G = SO(1, 1) \).
In the proof of Theorem 1.8 one proves the statements (2), (3), (1), (4), and (5) in this order, with possibly smaller \(c_1\) and larger \(c\) (and consequently larger \(\kappa\)) along the way. Thus there might be a sequence of phase transitions

\[X_i X_j^{-1} \text{ bounded } \Rightarrow X_i X_j^{-1} \text{ converges } \Rightarrow X_i \rightarrow X_j^\infty \exp(At),\]

as we increase the coupling strength \(\kappa\). It would be interesting to see whether the first phase-transition actually occurs. That is, for small \(\kappa\), the solutions might stay bounded from each other with their ratios not converging, possibly exhibiting chaotic behavior. Observing such ‘synchronous chaos’ in a prototypical model such as (1.5) would be interesting.

**Question 1.1** (Synchronous chaos). Is it possible that in a semi-local regime with intermediate coupling strength \(\kappa\):

\[c_1' < \max_{i,j=1,\ldots,N} \frac{d(X_i^0, X_j^0)}{c_1}, \quad c\|H\|_\infty < \kappa < c'\|H\|_\infty,\]

with different constants \(c_1'\) and \(c'\), \(X_i X_j^{-1}\) stay bounded yet fail to converge? Does this occur for the model (1.5)?

See Remark 3.3 for another formulation of this question.

We remark that some models which are qualitatively different from the Kuramoto model can still be written in the form (1.7) if we abandon hypothesis (H). For example, the Cucker-Smale model [7, 8]

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} \psi(|x_i - x_j|)(v_j - v_i),
\end{align*}
\]

(1.13)

where \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}\) is a function, is, up to a uniform translation in the \(x\)-variable, in the form of (1.7) with \((x, v) \in G = T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d\), with the interaction function \(\phi\) now failing to have only eigenvalues with positive real parts, and fails to be smooth if \(\psi\) possesses a singularity at 0.

Another example is the Swarmlator model proposed by O’Keeffe, Hong, and Strogatz [25]:

\[
\begin{align*}
\dot{x}_i &= v_i + \frac{1}{N} \sum_{j=1}^{N} [I_{\text{att}}(x_j - x_i)F(\theta_j - \theta_i) - I_{\text{rep}}(x_j - x_i)], \\
\dot{\theta}_i &= \omega_i + \frac{1}{N} \sum_{j=1}^{N} H_{\text{att}}(\theta_j - \theta_i)G(x_j - x_i),
\end{align*}
\]

(1.14)

This is again in the form of (1.7) with \((x, \theta) \in G = \mathbb{R}^d \times \mathbb{S}^1\), but with \(\phi\) failing to satisfy (H).

Thus it is natural to pose the following question.

**Question 1.2.** What is the behavior of the model (1.7) for different \(\phi\)?

1. Let \(\phi : G \rightarrow \mathfrak{g}\) be smooth but not satisfy (H). To what extent can the local behavior of (1.7) be explained by the behavior of the linearized model (1.12)?

2. Let \(G = \mathbb{R}^d\) and let \(\phi : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}^d\) possess a singularity at the origin. Are there certain types of singularities of \(\phi\) for which solutions to (1.7) near the origin exhibit nice behavior?

Another drawback of Theorem 1.8 is a possible deterioration of the statements for large dimensions. In many cases, we have a family of Lie groups, such as the family of unitary groups \(\{U(d)\}_{d=1}^\infty\), whose dimension grows to infinity. In this case, the size of the stability basin of Theorem 1.8 compared to the whole group diminishes exponentially as the dimension parameter grows to infinity. Thus, in practice, it would be desirable to obtain dimension-independent bounds for Theorem 1.8. This is possible for the Lohe model (1.5), by using the operator norm instead of the Frobenius norm, as can be checked by modifying the arguments of 16 in a straightforward manner.

**Question 1.3.** For incarnations of the model (1.7) other than (1.5), is it possible to obtain dimension-independent versions of Theorem 1.8?
It would also be interesting to study infinite limits such as the mean-field limit, as performed for the Lohe model \([13]\) by Golse and Ha \([12]\), or to study stochastic variations, as performed for the Lohe matrix model \([16]\) by Kim and Kim \([19]\). One could also study the effects of frustrations as in the author’s joint work with Ha, Kim, and Park \([14]\).

The rest of this paper is organized as follows. We recall basic Lie group theory and establish some notation in Section 2 and then we prove Theorem 1.8 in Section 3.

2. Notation and preliminaries on Lie groups and Lie algebras

We establish some notation about our Lie group \((G,g)\) and recall some standard facts from the literature.

As \((d\phi)_e : g \rightarrow g\) has only eigenvalues with positive real parts, we may assume that the right-invariant Riemannian metric on \(G\), which is given as an inner product \(\langle , \rangle\) on \(g\), with corresponding norm \(|\cdot|\) = \(\langle , \rangle^{1/2}\), is so that for some \(\lambda > 0\),

\[
\langle v, (d\phi)_e(v) \rangle \geq \lambda |v|^2, \quad \forall v \in g.
\]

(This change of Riemannian metric only distorts the geodesic metric of \(G\) by a constant factor, and thus does not affect the validity of Theorem 1.8.) We define, for \(r > 0\), the ball of radius \(r\)

\[B_r = \{ v \in g : |v| < r \}.
\]

The Lie group exponential map \(\exp : g \rightarrow G\) is a local diffeomorphism at \(0 \in g\), with \((d\exp)_0 = \text{id}_g\). We may choose a radius \(r > 0\) depending on \(G\) so that \(\exp : B_r \rightarrow \exp(B_r)\) is a diffeomorphism, and so that consequently the logarithm \(\log : \exp(B(0, r)) \rightarrow B_r\) is well defined and smooth; we will use this as a coordinate chart for \(G\) in this paper.

The Baker–Campbell–Hausdorff formula describes the group operation on this coordinate chart. Given \(v, w \in g\), with \(|v|, |w|\) sufficiently small, if we define

\[
u := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{r_1+s_1>0} \sum_{s_1} \sum_{r_m+s_m>0} \frac{[v^{r_1} w^{s_1} v^{r_2} w^{s_2} \ldots v^{r_m} w^{s_m}]}{\left(\sum_{j=1}^{m} (r_j + s_j)\right) \cdot \prod_{i=1}^{m} r_i! s_i!} = v + w + a(v, w),
\]

(2.2)

where we have used the notation

\[\left[g^{r_1} h^{s_1} \ldots g^{r_m} h^{s_m}\right] = \left[g, [g, \ldots [g, h, h, \ldots h], \ldots [g, h, h, \ldots h]] \ldots]\]

and \(a(v, w)\) denotes the sum of the terms of second or higher degree, then the power series defining \(u\) is absolutely convergent, and satisfies

\[\exp(u) = \exp(v) \exp(w).
\]

We may assume that the radius \(r > 0\) is sufficiently small so that for \(u, v \in B_r\), the Baker–Campbell–Hausdorff formula for \(u\) and \(v\) converges absolutely, with the map \(a : B_r \times B_r \rightarrow g\) being smooth with bounded derivatives:

\[
a(u, v) = O_G(|u| |v|), \quad a(u, v) - a(u', v') = O_G(|u| |v| + |v'| + |v'|) |u-u'|, \quad \forall u, v, u', v' \in B_r.
\]

We denote the adjoint map \(\text{Ad} : G \rightarrow \text{Aut}(g)\), which is defined to be \(\text{Ad}_g = (dL_g)_{g^{-1}} \circ (dR_g)_{e} = (dR_{g^{-1}})_{g} \circ (dL_g)_{e}\) for \(g \in G\). The derivative of the adjoint map, \(\text{Ad} : g \rightarrow \text{Der}(g)\), is given by

\[\text{Ad}_{\exp(v)} = e^{ad_v}, \quad \forall v \in g,
\]

(2.4)
where the right-hand side denotes the element of $\text{Aut}(\mathfrak{g})$ given by
\[
e^{\text{ad}_v} := \sum_{m=0}^{\infty} \frac{1}{m!} (\text{ad}_v)^m.
\]

We will need to take derivatives under the exponential map. If $Y : I \to \mathfrak{g}$ is a differentiable curve in $\mathfrak{g}$ defined on some interval $I \subset \mathbb{R}$, then it is well-known that
\[
\frac{d}{dt} \exp(Y) = (dR_{\exp(Y)}) e^{\text{ad}_Y} \left( \frac{dY}{dt} \right),
\]
where, on the right-hand side, the operator $e^{\text{ad}_Y} - 1$ is defined using the power series expansion for $e^z - 1$. By functional calculus, if $r > 0$ is sufficiently small and $v \in B_r$, then the operator $e^{\text{ad}_v} - 1$ is invertible with the well-defined inverse $\text{ad}_v$. Thus, if $Y(t) \in B_r$ for all $t \in I$, then denoting $X = \exp(Y)$, we have
\[
\frac{d}{dt} \log(X) = \text{ad}_Y e^{\text{ad}_Y - 1} \left( dR_X - 1 \right) X \left( \frac{dX}{dt} \right),
\]
Again by functional calculus, if $r > 0$ is sufficiently small, we have
\[
\frac{\text{ad}_v}{e^{\text{ad}_v} - 1}, \frac{\text{ad}_w}{e^{\text{ad}_w} - 1} = \text{id}_{\mathfrak{g}} + O(|v|), \quad v \in B_r,
\]
and
\[
\frac{\text{ad}_v}{e^{\text{ad}_v} - 1} - \frac{\text{ad}_w}{e^{\text{ad}_w} - 1}, \frac{\text{ad}_v}{1 - e^{-\text{ad}_v} - \text{ad}_w} - \frac{\text{ad}_w}{1 - e^{-\text{ad}_w}} = O(|v - w|), \quad v, w \in B_r.
\]

We will later use (2.7) and (2.8) to control (2.6).

### 3. PROOF OF THEOREMS 1.7 AND 1.8

In this section we prove Theorems 1.7 and 1.8.

Theorem 1.7 is proven directly assuming Theorem 1.8.

Theorem 1.8 is proven by two reductions. First, we will simplify Theorem 1.8 into the more dynamically tractable statement of Theorem 3.1. Then, to prove Theorem 3.1, instead of working with the system (1.7) on the Lie group $G$, we will work with the system (3.6) on the Lie algebra $\mathfrak{g}$ of the logarithms of the ratios of the particles. Proving Theorem 3.1 on the Lie group $G$ reduces to proving the corresponding Theorem 3.2 on the Lie algebra $\mathfrak{g}$. We will then prove Theorem 3.2 by introducing two Lyapunov functionals and establishing Gronwall inequalities for them.

#### 3.1. Proof of Theorem 1.7 assuming Theorem 1.8

(1) This is a direct consequence of Theorem 1.8 (2) and (3).

(2) Statement (b) is a straightforward computation. Statements (a) and (c) are immediate from Theorem 1.8 (2) and (3) for $\{X_i(t)\}_{i=1}^N$.

#### 3.2. Reducing Theorem 1.8 to Theorem 3.1

Now we begin the proof of Theorem 1.8. We first remark on the order of constants. We will first select $C$ and $c_2$ depending on $G$, $\mathfrak{g}$, and $\phi$, then select $c_1$ sufficiently small depending on $C$ and $c_2$, then select $c$ sufficiently large depending on $C$, $c_2$, and $c_1$.

We first reduce Theorem 1.8 to the following theorem.
Theorem 3.1. There exist a universal constant $c_3 > 0$, a sufficiently large constant $c > 0$ and a sufficiently small constant $c_1 > 0$ depending on $G$, $g$, and $\phi$ such that if
\[ \kappa > c\|H\|_\infty, \]
then the following holds.

(1) (Local stability) Let \{\textbf{X}_i(t)\}_{i=1}^N be a solution to (1.7) with initial data \{\textbf{X}_i^0\}_{i=1}^N satisfying
\[ d(\textbf{X}_i^0, \textbf{X}_j^0) < c_1, \quad \forall i, j = 1, \ldots, N. \]
Then there exists a unique global solution \textbf{X}(t) = \{\textbf{X}_i(t)\}_{i=1}^N to (1.7), with
\[ d(\textbf{X}(t), \textbf{X}_j(t)) < c_1, \quad t \geq 0, \quad \forall i, j = 1, \ldots, N, \]
and
\[ \limsup_{t \to \infty} d(\textbf{X}(t), \textbf{X}_j(t)) \leq \frac{c_2\|H\|_\infty}{\kappa\lambda}, \quad t \geq 0, \quad \forall i, j = 1, \ldots, N. \]

(2) (Convergence of two flows) Let \{\textbf{\tilde{X}}_i^0\}_{i=1}^N be a solution to (1.7) with initial data \{\textbf{\tilde{X}}_i^0\}_{i=1}^N satisfying
\[ d(\textbf{\tilde{X}}_i^0, \textbf{\tilde{X}}_j^0) < c_1, \quad \forall i, j = 1, \ldots, N. \]
Then
\[ d(\textbf{X}(t), \textbf{\tilde{X}}(t))^{-1, \textbf{X}}(t)\textbf{\tilde{X}}(t)^{-1}) = O_G(d(\textbf{X}_i^0(\textbf{X}_j^0)^{-1}, \textbf{\tilde{X}}_i^0(\textbf{\tilde{X}}_j^0)^{-1})e^{\kappa\lambda t/3}). \]

Proof of Theorem 1.8 assuming Theorem 3.1. We prove Theorem 1.8 in the following order.

(2) We note that Theorem 1.8 (2) is just Theorem 3.1 (1).

(1)+(3) We prove statements (1) and (3) together.

Let \{\textbf{\tilde{X}}_i\}_{i=1}^N be a solution to (1.7) with initial data \{\textbf{\tilde{X}}_i^0\}_{i=1}^N satisfying
\[ d(\textbf{\tilde{X}}_i^0, \textbf{\tilde{X}}_j^0) < c_1, \quad \forall i, j = 1, \ldots, N. \]
By Theorem 3.1 (2), we know that
\[ d(\textbf{X}(t), \textbf{\tilde{X}}(t))^{-1, \textbf{X}}(t)\textbf{\tilde{X}}(t)^{-1}) \leq C e^{\kappa\lambda t/3}, \]
for some constant $C$ depending on $G$. By considering the time-delayed solution \textbf{\tilde{X}}(t) = \textbf{X}(t+T) to (1.7), Theorem 3.1 (1) tells us that (3.1) is true, so (3.2) tells us that \{\textbf{X}_i(t)\textbf{X}_j(t)^{-1}\}_{t \in \mathbb{N}} is a Cauchy sequence and hence is exponentially convergent. By considering the time-delayed solutions \textbf{\tilde{X}}(t) = \textbf{X}(t+T) for $0 \leq T \leq 1$, again Theorem 3.1 (1) tells us that (3.1) is true, so we have by (3.2) that
\[ \sup_{0 \leq T \leq 1} d(\textbf{X}_i(n)\textbf{X}_j(n)^{-1}, \textbf{X}_i(n+T)\textbf{X}_j(n+T)^{-1}) \leq C e^{\kappa\lambda T/3}, \]
which tells us that \{\textbf{X}_i(t)\textbf{X}_j(t)^{-1}\}_{t \geq 0} is convergent to some value \textbf{X}_i^\infty, \textbf{X}_j^\infty, \textbf{X}_i^\infty \textbf{X}_j^\infty, i, j = 1, \ldots, N.

Now it remains to verify the existence of a phase-locked state \textbf{X}_i^\infty \exp(\Lambda t) satisfying (1), and that \textbf{X}_ij = \textbf{X}_i^\infty(\textbf{X}_j^\infty)^{-1}.

Note that, as \[ \frac{d}{dt} \textbf{X}_i = -(dL_{X_i})e(dR_{X_i})_X \textbf{\tilde{X}}, \]
\[ \frac{d}{dt} \textbf{X}_i^{-1} = -(dL_{X_i})e \left( H_i + \frac{\kappa}{N} \sum_{k=1}^N \phi(X_k X_i^{-1}) \right). \]
Now from (1.9), we have

\[
\frac{d}{dt} X_i x_j^{-1} = \left( dR_{x_j} \right) x_i \left( \frac{d}{dt} x_i \right) + \left( dL_{x_i} \right) x_j^{-1} \left( \frac{d}{dt} x_j^{-1} \right)
\]

\[
= \left( dR_{x_j} x_i \right) e \left( H_i + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k x_i \right) \right) - \left( dL_{x_i} x_j^{-1} \right) e \left( H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k x_j^{-1} \right) \right)
\]

Since the ratios \( X_i x_j^{-1} \) converge, both sides of (3.3) must converge to zero. Hence

\[
\frac{d}{dt} X_i x_j^{-1} = \left( dR_{x_j} x_i \right) e \left( H_i + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k x_i \right) \right) - \left( dL_{x_i} x_j^{-1} \right) e \left( H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k x_j^{-1} \right) \right), \quad i, j = 1, \ldots, N.
\]

Now, by continuity, we must have \( X_i^{\infty} = e \) and \( X_j^{\infty} X_k^{\infty} = X_k^{\infty}, \quad i, j, k = 1, \ldots, N. \) Therefore if we set

\[
X_i^{\infty} := X_i^{\infty}, \quad i = 1, \ldots, N,
\]

then \( X_{ij} = X_i^{\infty} x_i^{-1} \), and (3.3) becomes

\[
\left( \text{Ad} X_i^{\infty} \right)^{-1} \left( H_i + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k^{\infty} x_i^{\infty} \right) \right) = \left( \text{Ad} X_j^{\infty} \right)^{-1} \left( H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k^{\infty} x_j^{\infty} \right) \right), \quad i, j = 1, \ldots, N.
\]

Thus we can define the common value \( \Lambda \in g \) as

\[
\Lambda = \left( \text{Ad} X_i^{\infty} \right)^{-1} \left( H_i + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \left( X_k^{\infty} x_i^{\infty} \right) \right), \quad i = 1, \ldots, N.
\]

Then \( \Lambda \) and \( X_i^{\infty} \) satisfy (1.9), so by Proposition 1.1 \( \{ X_i^{\infty} x_i^{\infty} \exp(\Lambda t) \}_{i=1}^{N} \) is indeed a phase-locked state.

We have just proved (3) and the existence statement of (1). It remains to verify the uniqueness statement of (1).

Let \( \{ \tilde{X}_i^{\infty} x_i^{\infty} \exp(\tilde{\Lambda} t) \}_{i=1}^{N} \) be another phase-locked state, with

\[
d(\tilde{X}_i^{\infty} x_i^{\infty}, \tilde{X}_j^{\infty} x_j^{\infty}) < c_1.
\]

By Theorem 5.1 (2), we have

\[
X_i^{\infty} x_j^{\infty} = \tilde{X}_i^{\infty} x_j^{\infty}, \quad i, j = 1, \ldots, N.
\]

So \( \tilde{X}_i^{\infty} x_j^{\infty} \) is also a phase-locked state, and if we set \( g \) as this common value, we have

\[
\tilde{X}_i^{\infty} = X_i^{\infty} g, \quad i = 1, \ldots, N.
\]

Now from (1.9), we have \( \text{Ad} X_i^{\infty} \Lambda = \text{Ad} X_i^{\infty} g \tilde{\Lambda} \), so

\[
\tilde{\Lambda} = \text{Ad}_{g^{-1}} \Lambda.
\]

Therefore

\[
\tilde{X}_i^{\infty} x_i^{\infty} \exp(\tilde{\Lambda} t) = X_i^{\infty} g x_i^{\infty} \exp(\text{Ad}_{g^{-1}} \Lambda t) = X_i^{\infty} \exp(\Lambda t) g, \quad i = 1, \ldots, N, t \geq 0,
\]

as desired.
By (3), we may define the quantities

$$
\Lambda_i = H_i + \frac{\kappa}{N} \sum_{j=1}^{N} \phi(\lim_{t \to \infty} X_j(t)X_i(t)^{-1}), \quad i = 1, \cdots, N.
$$

(By the discussion of the proof of (3), $$\Lambda_i = \text{Ad}_{X_i} \Lambda_i$$.) By (2), we have $$|\Lambda_i| \leq \text{O}_G(\|H\|_{\infty})$$ for some constant $$C$$.

Now define

$$Z_i(t) = \exp(-\Lambda_i t)X_i(t), \quad i = 1, \cdots, N.$$

Then by [2.1],

$$
(dR_{Z_i^{-1}}) Z_i \dot{Z_i} = e^{-\text{ad}_{\Lambda_i} t} \left( H_i + \frac{\kappa}{N} \sum_{j=1}^{N} \phi(X_jX_i^{-1}) - \Lambda_i \right) = \text{O}_G(\|H\|_{\infty}) e^{-\kappa t/3},
$$

so if $$\kappa > \text{O}_G(\|H\|_{\infty}/\lambda)$$, $$Z_i$$ exponentially converges, say to $$Z_i^\infty$$.

Now define

$$d(X_i, \exp(\Lambda_i t)Z_i^\infty) = d(\exp(\Lambda_i t)Z_i \exp(-\Lambda_i t), \exp(\Lambda_i t)Z_i^\infty \exp(-\Lambda_i t))$$

and the first inequality follows from right-invariance of $$d$$, and the first inequality follows from [2.4] ($$\|\text{ad}_{\Lambda_i}\|$$ is the operator norm). As $$\exp(\Lambda_i t)Z_i^\infty = Z_i^\infty \exp(\text{Ad}_{Z_i^{-1}}(\Lambda_i) t)$$, we have

$$d(Z_i^\infty \exp(\text{Ad}_{Z_i^{-1}}(\Lambda_i) t)(Z_j^\infty \exp(\text{Ad}_{Z_j^{-1}}(\Lambda_j) t))^{-1}, \lim_{t \to \infty} X_i(t)X_j(t)^{-1})$$

$$\leq d(\exp(\Lambda_i t)Z_i^\infty (\exp(\Lambda_j t)Z_j^\infty)^{-1}, \exp(\Lambda_i t)Z_i^\infty X_j(t)^{-1})$$

$$+ d(\exp(\Lambda_i t)Z_i^\infty X_j(t)^{-1}, X_i(t)X_j(t)^{-1})$$

$$+ d(X_i(t)X_j(t)^{-1}, \lim_{t \to \infty} X_i(t)X_j(t)^{-1})$$

$$\leq e^{\|\text{ad}_{\Lambda_i}\|} \|\text{Ad}_{Z_i^{-1}}\| d((\exp(\Lambda_j t)Z_j^\infty)^{-1}, X_j(t)^{-1})$$

$$+ d(\exp(\Lambda_j t)Z_j^\infty, X_i(t)) + d(X_i(t)X_j(t)^{-1}, \lim_{t \to \infty} X_i(t)X_j(t)^{-1})$$

$$\leq e^{\|\text{ad}_{\Lambda_i}\|} \|\text{Ad}_{Z_i^{-1}}\| e^{\|\text{ad}_{\Lambda_j}\|} \|\text{Ad}_{Z_j}\| d(\exp(\Lambda_j t)Z_j^\infty, X_i(t))$$

$$+ d(\exp(\Lambda_j t)Z_j^\infty, X_i(t)) + d(X_i(t)X_j(t)^{-1}, \lim_{t \to \infty} X_i(t)X_j(t)^{-1}).$$

The last two terms clearly converge to zero, while the first term is $$\text{O}_G(\|H\|_{\infty}) e^{-\kappa t/3}$$ and thus converges to zero with large $$\kappa$$. Thus we have the convergence

$$Z_i^\infty \exp(\text{Ad}_{Z_i^{-1}}(\Lambda_i) t)(Z_j^\infty \exp(\text{Ad}_{Z_j^{-1}}(\Lambda_j) t))^{-1} \to \lim_{t \to \infty} X_i(t)X_j(t)^{-1}, \quad t \to \infty,$$

for all $$i, j = 1, \cdots, N$$, which is only possible if $$Z_i^\infty (Z_j^\infty)^{-1} = \lim_{t \to \infty} X_i(t)X_j(t)^{-1}$$ and $$\text{Ad}_{Z_i^{-1}}(\Lambda_i) = \text{Ad}_{Z_j^{-1}}(\Lambda_j)$$, $$i, j = 1, \cdots, N$$. If we denote this latter common value by $$M \in \mathfrak{g}$$, then $$\text{Ad}_{Z_i^{-1}} M = \Lambda_i = H_i + \frac{\kappa}{N} \sum_{j=1}^{N} \phi(Z_j^\infty (Z_i^\infty)^{-1})$$ satisfies [1.9]. Therefore $$\{Z_i^\infty \exp(M t)\}_{i=1}^{N}$$ is a phase-locked state such that $$d(X_i(t), Z_i^\infty \exp(M t))$$ converges exponentially to zero.
Indeed, the only if direction is trivial, and conversely if the global solution to (3.5) exists if and only if a global solution to (1.7) exists. A standard argument (using the fact that the right-hand side of (1.7) is bounded for all time).

Reducing from a model on the Lie group to a model on the Lie algebra. Our goal now is to prove Theorem 3.1. Theorem 3.1 roughly says that if the particles $X_i$ are initially close together, they will stay together for all time and will exhibit good stability behavior. However, we have little control on where the whole aggregate will head toward; it is possible that the entire population heads away from its initial position exponentially fast.

Thus, if we restrict our attention to the ratios $X_i X_j^{-1}$ of the particles, we should be able to analyze them in a small neighborhood of the identity.

From (3.3), we may formulate the Cauchy problem for the relative phases $X_i X_j^{-1}$.

\begin{align*}
\frac{d}{dt} (X_i X_j^{-1}) &= (dR_{X_j^{-1}}) e \left( H_i - Ad_{X_i X_j^{-1}} H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \phi(X_k X_j^{-1}) - Ad_{X_i X_j^{-1}} \phi(X_k X_j^{-1}) \right), \\
X_i X_j^{-1}(0) &= X_i^0 X_j^0, \quad i, j = 1, \ldots, N,
\end{align*}

Again, the standard Cauchy–Lipschitz theory guarantees the existence of a local $C^1$-solution to (3.5). Observe that a global solution to (1.7) exists if and only if a global solution to (3.5) exists. Indeed, the only if direction is trivial, and conversely if the $X_i X_j^{-1}$'s are globally defined, we may consider the right-hand side of (1.7) to be a time-dependent bounded $C^1$-forcing term, and so by a standard argument (using the fact that $G$ as a manifold is complete) the solutions $X_i(t)$ must exist for all time.

Assuming $X_i X_j^{-1} \in \exp(B_r)$ for all $i, j = 1, \ldots, N$, define $Y_{ij} = \log(X_i X_j^{-1})$. We will analyze the $Y_{ij}$'s on $\mathfrak{g}$ instead of the $X_i X_j^{-1}$'s on $G$; we are using $B_r$ as a coordinate chart for $\exp(B_r)$. Now, by (2.4) and (2.6), the $Y_{ij}$'s form the solution to the Cauchy problem

\begin{align*}
\dot{Y}_{ij} &= \frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} H_i - \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \left[ \frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} \phi \circ \exp(Y_{ki}) - \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} \phi \circ \exp(Y_{kj}) \right], \\
Y_{ij}(0) &= \log(X_i^0 X_j^0), \quad i, j = 1, \ldots, N.
\end{align*}
Once again, the standard Cauchy–Lipschitz theory guarantees the existence of a local $C^1$-solution to (3.3). This time, we can only observe that the existence of a global solution to (3.6) guarantees the existence of a global solution to (3.3), but not the other way around.

Note that, by the Baker-Campbell-Hausdorff formula, the $Y_{ij}$’s satisfy the compatibility equations
\begin{align}
Y_{ji} &= -Y_{ij}, \quad Y_{ik} = Y_{ij} + Y_{jk} + a(Y_{ij}, Y_{jk}), \quad i, j, k = 1, \ldots, N
\end{align}
for all times of their existence.

We now translate Theorem 3.1 into a statement about the $X_i$’s, into a statement about the $Y_{ij}$’s.

**Theorem 3.2.** There exist a universal constant $C > 0$, a sufficiently large constant $c > 0$ and a sufficiently small constant $c_1 > 0$ depending on $G$, $g$, and $\phi$ such that if
\begin{align}
\kappa > c\|H\|_{\infty}
\end{align}
then the following holds.

1. (Local stability) If $|Y_{ij}(0)| < c_1$, $\forall i, j = 1, \ldots, N$, then there exists a unique global solution $Y(t) = \{Y_{ij}(t)\}_{i,j=1}^N$ to (3.6) satisfying $|Y_{ij}(t)| < c_1$, $\forall i, j = 1, \ldots, N,$ and
\[\limsup_{t \to \infty} |Y_{ij}(t)| \leq \frac{C\|H\|_{\infty}}{\kappa \lambda}, \quad \forall i, j = 1, \ldots, N.\]

2. (Convergence of two flows) Let $\tilde{Y}(t) = \{\tilde{Y}_{ij}(t)\}_{i,j=1}^N$ be a solution to (3.6) with initial data $\tilde{Y}_0$ satisfying the compatibility equations (3.7) and
\[|\tilde{Y}_{ij}(0)| < c_1, \quad \forall i, j = 1, \ldots, N.\]

Then $|\tilde{Y}_{ij}(t) - Y_{ij}(t)| \leq |\tilde{Y}_{ij} - Y_{ij}| \cdot e^{-\kappa \lambda t/3}, \quad t \geq 0.$

It is clear that Theorem 3.2 implies Theorem 3.1 so our goal is now to prove Theorem 3.2.

### 3.4 Two Gronwall inequalities

The proof of Theorem 3.2 depends on two Gronwall inequalities for two Lyapunov functionals for $Y$, each respectively corresponding to statements (1) and (2) of Theorem 3.2.

First, we consider the Lyapunov functional
\begin{align}
\|Y\|_{\infty} := \max_{i,j=1,\ldots,N} |Y_{ij}|.
\end{align}

**Lemma 3.1 (First Gronwall inequality).** Let $Y(t) = \{Y_{ij}(t)\}_{i,j=1}^N$, $t \in [0, T)$, be a solution to (3.6), with $Y_{ij}(t) \in B_r$ for all $i, j = 1, \ldots, N$ and $t \in [0, T)$. Then
\begin{align}
\dot{Y}_{ij} &= -\kappa [d\phi_e(Y_{ij}) + g_{ij}(|Y|_{\infty})] + O(\|H\|_{\infty}), \quad i, j = 1, \ldots, N,
\end{align}
and thus
\begin{align}
\frac{d}{dt} \|Y\|_{\infty} \leq -\kappa [\lambda \|Y\|_{\infty} + g_{ij}(|Y|_{\infty})] + O(\|H\|_{\infty}),
\end{align}
where $\frac{d}{dt}$ denotes the upper right Dini derivative, and the little $o$ notation is with respect to $\|Y\|_{\infty} \to 0$ and has constants depending on $G$, $g$, and $\phi$.

**Remark 3.1.** Intuitively, since the original model (1.7) is locally attractive, it is expected and unsurprising that a measure such as $\|Y\|_{\infty}$ for the diameter of a concentrated population should stay small; the ‘first-order force’ $-\kappa d\phi_e(Y_{ij})$ in (3.10) represents this local attraction.
Proof of Lemma 3.7. By functional calculus \(2.7\),
\[
\frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}}-1} = \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} = \text{id}_g + O(|Y_{ij}|),
\]
and
\[
\phi \circ \exp(Y_{ij}) = d\phi_c(Y_{ij}) + o_{G,\phi}(|Y_{ij}|),
\]
so by (3.6)
\[
\dot{Y}_{ij} = O(\|H\|_{\infty}) + \frac{\kappa}{N} \sum_{k=1}^{N} [d\phi_c(Y_{ki}) - d\phi_c(Y_{kj}) + o_{G,\phi}(\|Y\|_{\infty})].
\]
Applying the Baker-Campbell-Hausdorff formula (3.7),
\[
Y_{ki} - Y_{kj} = Y_{ij} - a(Y_{jk}, Y_{ki}) = Y_{ij} + O(\|Y\|_{\infty}^2),
\]
so
\[
\dot{Y}_{ij} = O(\|H\|_{\infty}) - \kappa [\phi_c(Y_{ij}) + o_{G,\phi}(\|Y\|_{\infty})]
\]
which is (3.10). Now (3.11) follows from (3.10) by (2.1). □

This Gronwall inequality allows us to prove the first part of Theorem 3.2.

Proof of Theorem 3.2 (1). Let \(c_1\) be small enough so that \(|Y_{ij}| < c_1, \ i, j = 1, \ldots, N\), guarantees that \(Y_{ij} \in B_r\) and that the \(o(\|Y\|_{\infty})\) in (3.11) is smaller than \(\frac{\kappa}{2}\|Y\|_{\infty}\). Then (3.11) becomes
\[
\frac{d}{dt} \|Y(t)\|_{\infty} \leq -\frac{\kappa \lambda}{2} \|Y(t)\|_{\infty} + C \|H\|_{\infty}
\]
for some absolute constant \(C\), for all times \(t \geq 0\) such that \(\|Y(t)\| < c_1\). If \(\kappa\) is large enough so that
\[
\frac{2C \|H\|_{\infty}}{\kappa \lambda} < c_1,
\]
then by a standard exit-time argument, \(Y(t) < c_1\) for all \(t \geq 0\) and \(\limsup_{t \rightarrow \infty} \|Y(t)\|_{\infty} \leq \frac{2C \|H\|_{\infty}}{\kappa \lambda}\). □

Our second Lyapunov functional measures the maximal mismatch between the two flows.
\[\|Y - \tilde{Y}\|_{\infty} := \max_{i,j=1,\ldots, N} |Y_{ij} - \tilde{Y}_{ij}|.\]

Lemma 3.2 (Second Gronwall inequality). Let \(Y(t) = (Y_{ij}(t))_{i,j=1}^{N}\) and \(\tilde{Y}(t) = (\tilde{Y}_{ij}(t))_{i,j=1}^{N}\), \(t \in [0, T]\) be solutions to (3.6) with initial data \((Y_{ij})_{i,j=1}^{N}\) and \((\tilde{Y}_{ij})_{i,j=1}^{N}\), respectively, with \(Y_{ij}(t), \tilde{Y}_{ij}(t) \in B_r\) for all \(i,j = 1, \ldots, N\) and \(t \in [0, T]\). Then
\[\dot{Y}_{ij} - \dot{\tilde{Y}}_{ij} = -\kappa [d\phi_c(Y_{ij} - \tilde{Y}_{ij}) + o_{G,\phi}(\|Y - \tilde{Y}\|_{\infty})] + O(\|Y - \tilde{Y}\|_{\infty} \cdot \|H\|_{\infty}),\]
and thus
\[\frac{d}{dt} \|Y - \tilde{Y}\|_{\infty} \leq -\kappa [\lambda \|Y - \tilde{Y}\|_{\infty} + o_{G,\phi}(\|Y - \tilde{Y}\|_{\infty})] + O(\|Y - \tilde{Y}\|_{\infty} \cdot \|H\|_{\infty}),\]
where the little o notation is with respect to \(\|Y\|, \|\tilde{Y}\| \rightarrow 0\).

Remark 3.2. Observe that (3.13) is not the difference of (3.10) for \(Y\) and \(\tilde{Y}\). One can accurately guess the leading term \(-\kappa d\phi_c(Y_{ij} - \tilde{Y}_{ij})\) in this manner, but cannot get the correct error terms. It is crucial that the error terms are controlled in terms of \(\|Y - \tilde{Y}\|\), as opposed to simply \(\|Y\|\) and \(\|\tilde{Y}\|\), in order to use (3.14) to fruition; this fact is the core mathematical content of the proof of Theorem 3.1 (2).
Proof of Lemma 3.14. We expand

\[
\dot{Y}_{ij} - \dot{\bar{Y}}_{ij} = \frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} H_i - \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \left[ \frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} \phi \circ \exp(Y_{ki}) - \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} \phi \circ \exp(Y_{kj}) \right] 
\]

\[
- \frac{\text{ad}_{\bar{Y}_{ij}}}{e^{\text{ad}_{\bar{Y}_{ij}}} - 1} H_i + \frac{\text{ad}_{\bar{Y}_{ij}}}{1 - e^{-\text{ad}_{\bar{Y}_{ij}}}} H_j - \frac{\kappa}{N} \sum_{k=1}^{N} \left[ \frac{\text{ad}_{\bar{Y}_{ij}}}{e^{\text{ad}_{\bar{Y}_{ij}}} - 1} \phi \circ \exp(\bar{Y}_{ki}) - \frac{\text{ad}_{\bar{Y}_{ij}}}{1 - e^{-\text{ad}_{\bar{Y}_{ij}}}} \phi \circ \exp(\bar{Y}_{kj}) \right] 
\]

\[
= \left( \frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} - \frac{\text{ad}_{\bar{Y}_{ij}}}{e^{\text{ad}_{\bar{Y}_{ij}}} - 1} \right) (H_i + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \circ \exp(\bar{Y}_{ki})) 
\]

\[
- \left( \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} - \frac{\text{ad}_{\bar{Y}_{ij}}}{1 - e^{-\text{ad}_{\bar{Y}_{ij}}}} \right) (H_j + \frac{\kappa}{N} \sum_{k=1}^{N} \phi \circ \exp(\bar{Y}_{kj})) 
\]

\[
+ \frac{\kappa}{N} \sum_{k=1}^{N} \left[ \frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} \left( \phi \circ \exp(Y_{ki}) - \phi \circ \exp(\bar{Y}_{ki}) \right) - \frac{\text{ad}_{Y_{ij}}}{1 - e^{-\text{ad}_{Y_{ij}}}} \phi \circ \exp(Y_{kj}) - \phi \circ \exp(\bar{Y}_{kj}) \right] 
\]

By functional calculus [2.8],

\[
\frac{\text{ad}_{Y_{ij}}}{e^{\text{ad}_{Y_{ij}}} - 1} - \frac{\text{ad}_{\bar{Y}_{ij}}}{e^{\text{ad}_{\bar{Y}_{ij}}} - 1} = O(|Y_{ij} - \bar{Y}_{ij}|), 
\]

and

\[
\phi \circ \exp(Y_{ij}) - \phi \circ \exp(\bar{Y}_{ij}) = d\phi_{\exp(Y_{ij})}(Y_{ij} - \bar{Y}_{ij}) + o_{G,\phi}(|Y_{ij} - \bar{Y}_{ij}|) 
\]

\[
= d\phi_e(Y_{ij} - \bar{Y}_{ij}) + o_{G,\phi}(|Y_{ij} - \bar{Y}_{ij}|). 
\]

So we may simplify

\[
\dot{Y}_{ij} - \dot{\bar{Y}}_{ij} = O(||Y - \bar{Y}||_{\infty} ||H||_{\infty} + \kappa ||Y||_{\infty} + \kappa ||\bar{Y}||_{\infty}) 
\]

\[
+ \frac{\kappa}{N} \sum_{k=1}^{N} \left[ \left( \phi \circ \exp(Y_{ki}) - \phi \circ \exp(\bar{Y}_{ki}) \right) - \left( \phi \circ \exp(Y_{kj}) - \phi \circ \exp(\bar{Y}_{kj}) \right) \right] 
\]

\[
= O(||Y - \bar{Y}||_{\infty} ||H||_{\infty} + \kappa \cdot o_{G}||Y - \bar{Y}||_{\infty}) 
\]

\[
+ \frac{\kappa}{N} \sum_{k=1}^{N} \left[ d\phi_e(Y_{ki} - \bar{Y}_{ki} - Y_{kj} + \bar{Y}_{kj}) + o_{G,\phi}||Y - \bar{Y}||_{\infty} \right]. 
\]

Again, by the Baker-Campbell-Hausdorff formula [3.7],

\[
Y_{ki} - Y_{kj} - \bar{Y}_{ki} + \bar{Y}_{kj} = Y_{ij} + \bar{Y}_{ij} - a(Y_{jk}, Y_{ki}) + a(\bar{Y}_{jk}, \bar{Y}_{ki}) 
\]

\[
\equiv -Y_{ij} + \bar{Y}_{ij} + O_G(||Y||_{\infty} + ||\bar{Y}||_{\infty})||Y - \bar{Y}||_{\infty}, 
\]

and this last big O term is again $o_G(||Y - \bar{Y}||_{\infty})$. Thus

\[
\dot{Y}_{ij} - \dot{\bar{Y}}_{ij} = -\kappa \left[ d\phi_e(Y_{ij} - \bar{Y}_{ij}) + o_{G,\phi}||Y - \bar{Y}||_{\infty} \right] + O(||Y - \bar{Y}||_{\infty} \cdot ||H||_{\infty}). 
\]
Proof of Theorem $\text{3.2}(2)$. If $c_1$ is small enough so that $\|Y\|, \|\hat{Y}\| < c_1$ implies that $Y, \hat{Y} \in B_r$ and that the $o(\|Y - \hat{Y}\|_\infty)$ term in (3.14) is at most $\frac{\lambda\|Y - \hat{Y}\|_\infty}{3}$, (3.14) becomes

$$d\|rac{\partial Y - \hat{Y}}{\partial t}\|_\infty \leq -\frac{2\kappa\lambda}{3}\|Y - \hat{Y}\|_\infty + C\|Y - \hat{Y}\|_\infty \cdot \|H\|_\infty$$

for some absolute constant $C > 0$. By choosing $\kappa > \frac{3C}{\lambda}$ we have

$$d\|rac{\partial Y - \hat{Y}}{\partial t}\|_\infty \leq -\frac{\kappa\lambda}{3}\|Y - \hat{Y}\|_\infty$$

for times $t$ at which $\|Y(t)\|, \|\hat{Y}(t)\| < c_1$. This hypothesis is satisfied by Theorem 3.2 (1) if $\|Y(0)\|, \|\hat{Y}(0)\| < c_1$, in which case we have

$$\|Y(t) - \hat{Y}(t)\|_\infty \leq \|Y(0) - \hat{Y}(0)\|_\infty \exp\left(-\frac{\kappa\lambda}{3}t\right).$$

Remark 3.3. Question 1.1 may be reformulated as follows. Is it possible that in a semi-local regime with intermediate coupling strength $\kappa$:

$$c_1' < \max_{i,j=1,\ldots,N} d(X_i^0, X_j^0) < c_1, \quad c\|H\|_\infty < \kappa < c'\|H\|_\infty,$$

with different constants $c_1'$ and $c'$, (3.10) holds yet (3.13) fails?

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