On new structure of $N$-topology

M. Lellis Thivagar*, V. Ramesh and M. Arockia Dasan

Abstract: In this paper, we propose a new formula to get $N$-topologies in a non empty set $X$. Further, we establish its own open sets. We, in addition to it, study its characterizations. Apart from this, we introduce continuous functions on such topological spaces and establish their basic properties and prove the Pasting Lemma.

Keywords: $N$-topology; $N_r$-open sets; $N_r$-closed sets; relative topology ($N_r^*$); $N^*$-continuous functions

1. Introduction

The intrinsic nature and beauty of Mathematics is this: One must be in "love" with Mathematics. As a result, the nature of inquisitiveness in a person gets, needless to mention, always enkindled and triggered by the new theorems or axioms or any new findings, even if it is a small in its nature or incredibly big.

Indeed, the bitopological space propounded and introduced by Kelly in the year 1963, kept haunting our Mathematical mind. He introduced the bitopological space which is a non empty set $X$ equipped with two arbitrary topologies $T_1$ and $T_2$. In this space, the open sets are called pairwise open sets. In this paper, we establish bitopological space with bitopological axioms and prove the structure of a non empty set $X$ equipped with more than two topologies.

Here, we have defined the structure of $N$-topology, that is, a non empty set $X$ equipped with $N$-arbitrary topologies, and which has its own open sets. Further, we introduce continuous functions on $N$-topological space which in turn has its own impact on the Pasting Lemma.
empty set $X$ equipped with more than two topologies. Recently many researchers defined various forms of open sets in this space such as $\tau_1, \tau_2$ (Lellis Thivagar, 1991), $\tau_{1,2}$ (Lellis Thivagar, Ekici, & Ravi, 2008), etc. In addition to our fervent efforts, herein, we have also tried to prove the structure of $N$-topology, that is, a non empty set $X$ equipped with $N$-arbitrary topologies $\tau_1, \tau_2, \ldots, \tau_N$ and also established its own open sets. Further, we study its characterizations. Also, we introduce continuous functions on such topological spaces and establish their basic properties and proved the Pasting Lemma.

2. Preliminaries

Definition 2.1 (Doitchinov, 1988) A quasi-pseudo-metric on a non empty set $X$ is a function $d_1: X \times X \to \mathbb{R}^+ \cup \{0\}$ such that

(i) $d_1(x,x) = 0$ for all $x \in X$

(ii) $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$ for all $x, y, z \in X$.

where $\mathbb{R}^+$ is the set of all positive real numbers.

Definition 2.2 (Grabiec, Cho & Saadati, 2007) Let $d_1$ a quasi-pseudo-metric on $X$, and let a function $d_2: X \times X \to \mathbb{R}^+ \cup \{0\}$ be defined by $d_2(x,y) = d_1(y,x)$ for all $x, y \in X$. Trivially $d_2$ is a quasi-pseudo-metric defined on $X$ and we say that $d_1$ and $d_2$ are conjugate one another.

If $d_1$ is a quasi-pseudo-metric on $X$, then $B_{d_1}(x, k_1) = \{y: d_1(x, y) < k_1\}$, the open $d_1$-sphere with centre $x$ and radius $k_1 > 0$. Classically, the collection of all open $d_1$-spheres forms a base for a topology, the obtained topology is denoted by $\tau_1$ and called the quasi-pseudo-metric topology of $d_1$. Similarly we get a topology $\tau_2$ for $X$, due to the quasi-pseudo-metric $d_2$.

Definition 2.3 (Kelly, 1963) A non empty set $X$ equipped with two arbitrary topologies $\tau_1$ and $\tau_2$ is called a bitopological space and is denoted by $(X, \tau_1, \tau_2)$.

3. $N$-topological spaces

In this section, we introduce the notion of $N$-topological spaces and its own open sets. We derive its basic properties. We also define and discuss the relative topology in $N$-topological spaces.

Definition 3.1 Let $d_1$ and $d_2$ be conjugate, quasi-pseudo-metrics on $X$ and define a function $d_3: X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$d_3(x, y) = \frac{2d_1(x, y) + d_2(y, x)}{3}$$

for all $x, y \in X$.

Then

(i) $d_3(x, x) = \frac{2d_1(x, x) + d_2(x, x)}{3} = 0$ for all $x \in X$.

(ii) $d_3(x, z) = \frac{2d_1(x, z) + d_2(z, x)}{3} \leq \frac{2(d_1(x, y) + d_2(y, x) + d_1(z, y) + d_2(y, z))}{3} = d_3(x, y) + d_3(y, z)$ for all $x, y, z \in X$.

Therefore, $d_3$ is a quasi-pseudo-metric on $X$ and which is called a Mean Conjugate (simply write M.C) of $d_1$, $d_2$, and $d_3$. For each $i = 1, 2, 3$, the quasi-pseudo metric $d_i$ gives a topology $\tau_i$ whose base is $\{B_{d_i}(x, k_i)\}$, where $B_{d_i}(x, k_i) = \{y: d_i(x, y) < k_i\}$. Thus we define a non empty set $X$ equipped with three arbitrary topologies $\tau_1, \tau_2$ and $\tau_3$ is called a tritopological space and is denoted by $(X, 3\tau)$ or $(X, \tau_1, \tau_2, \tau_3)$.

Generally, let $d_1, d_2, \ldots, d_{n-1}$ be quasi-pseudo-metrics on $X$, $d_n$ and $d_1$ be conjugate and $d_{n-1}$ be M.C of $d_n, d_2, d_3, \ldots, d_{n-2}$; $d_{n-1}$ be M.C of $d_1, d_2, d_3, \ldots, d_{n-2}$ and $d_n$, respectively. Define a function $d_n: X \times X \to \mathbb{R}^+ \cup \{0\}$ by
We can easily verify that \( d_n \) is a quasi-pseudo-metric on \( X \). Also we note that for each \( N \), 
\[
d_n(x, y) = \frac{\left| d_1(y, x) + \sum_{i=2}^{N-1} d_i(y, x) \right|}{N}
\]
for all \( x, y \in X \).

Definition 3.2 Let \( X \) be a non empty set, \( \tau_1 \) and \( \tau_2 \) be two arbitrary topologies defined on \( X \) and the collection \( 2\tau \) be defined by
\[
2\tau = \{ S \subseteq X : (A_1 \cup A_2) \cup (B_1 \cap B_2), A_1, B_1 \in \tau_1 \text{ and } A_2, B_2 \in \tau_2 \}
\]
satisfying the following axioms:

(i) \( X, \emptyset \in 2\tau \)
(ii) \( \bigcup_{i=1}^n S_i \in 2\tau \) for all \( S_i \in 2\tau \)
(iii) \( \bigcap_{i=1}^n S_i \in 2\tau \) for all \( S_i \in 2\tau \).

Then the pair \( (X, 2\tau) \) is called a bitopological space on \( X \) and the elements of the collection \( 2\tau \) are known as \( 2\tau \)-open sets on \( X \).

We can generalize the above definition as given below: let \( X \) be a non empty set, \( \tau_1, \tau_2, \ldots, \tau_n \) be \( N \)-arbitrary topologies defined on \( X \) and let the collection \( N\tau \) be defined by
\[
N\tau = \{ S \subseteq X : S = \left( \bigcup_{i=1}^n A_i \right) \cup \left( \bigcap_{i=1}^n B_i \right), A_i, B_i \in \tau_i \},
\]
satisfying the following axioms:

(i) \( X, \emptyset \in N\tau \)
(ii) \( \bigcup_{i=1}^n S_i \in N\tau \) for all \( S_i \in N\tau \)
(iii) \( \bigcap_{i=1}^n S_i \in N\tau \) for all \( S_i \in N\tau \).

Then the pair \( (X, N\tau) \) is called a \( N \)-topological space on \( X \) and the elements of the collection \( N\tau \) are known as \( N\tau \)-open sets on \( X \). A subset \( A \) of \( X \) is said to be \( N\tau \)-closed on \( X \) if the complement of \( A \) is \( N\tau \)-open on \( X \). The set of all \( N\tau \)-open sets on \( X \) and the set of all \( N\tau \)-closed sets on \( X \) are, respectively, denoted by \( N\tau\text{-O}(X) \) and \( N\tau\text{-C}(X) \).

Example 3.3 Let \( X = \{ a, b, c, d \} \). For \( N = 2 \), and assume \( \tau_1 \text{-O}(X) = \{ X, \emptyset, \{ a, b \} \} \) and \( \tau_2 \text{-O}(X) = \{ X, \emptyset, \{ b, c \} \} \), then \( 2\tau \text{-O}(X) = \{ X, \emptyset, \{ b, c \}, \{ a, b, c \} \} \) and \( 2\tau \text{-C}(X) = \{ X, \emptyset, \{ d \}, \{ a, d \}, \{ c, d \}, \{ a, c, d \} \} \). Therefore, \( (X, 2\tau) \) is a bitopological space on \( X \). For \( N = 3 \), and assume \( \tau_1 \text{-O}(X) = \{ X, \emptyset, \{ a \} \} \), \( \tau_2 \text{-O}(X) = \{ X, \emptyset, \{ b, d \} \} \) and \( \tau_3 \text{-O}(X) = \{ X, \emptyset, \{ c, d \} \} \), then \( 3\tau \text{-O}(X) = \{ X, \emptyset, \{ a, d \}, \{ a, c, d \}, \{ b, d \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \} \} \) and \( 3\tau \text{-C}(X) = \{ X, \emptyset, \{ b, c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}, \{ b, c, d \} \} \). Therefore, \( (X, 3\tau) \) is a tritopological space on \( X \).

Remark 3.4

(i) If \( N = 1 \), then \( N\tau = \tau_1 = \tau \).
(ii) Intersection of two \( 2\tau \) is also a \( 2\tau \).

Intersection of two \( 3\tau \) is also a \( 3\tau \).

In general, intersection of two \( N \)-topology is again a \( N \)-topology.
Proof

(i) Proof is trivial.

(ii) Let $(N_{r})_{1}$ and $(N_{r})_{2}$ be two $N$-topology defined on $X$. Clearly, $X$ and $\emptyset$ are in $(N_{r})_{1} \cap (N_{r})_{2}$. Let $\{ C_{i} \}_{i \in I} \subseteq (N_{r})_{1} \cap (N_{r})_{2}$, $\bigcup_{i \in I} C_{i} \in (N_{r})_{1}$ and $\bigcup_{i \in I} C_{i} \in (N_{r})_{2}$ and so in $(N_{r})_{1} \cap (N_{r})_{2}$. Let $\{ C_{i} \}_{i \in I} \subseteq (N_{r})_{1} \cap (N_{r})_{2} \cap \bigcap_{i \in I} C_{i} \in (N_{r})_{1}$ and $\bigcap_{i \in I} C_{i} \in (N_{r})_{2}$ and so in $(N_{r})_{1} \cap (N_{r})_{2}$. Thus $(N_{r})_{1} \cap (N_{r})_{2}$ is an $N$-topology.

Remark 3.5 Union of two $2\tau$ need not be a $2\tau$.
Union of two $3\tau$ need not be a $3\tau$.
In general, union of two $N$-topology need not be a $N$-topology.

Example 3.6 For $N = 3$, $X = \{a, b, c, d\}$ and assume $\tau_{1}O(X) = \{X, \emptyset, \{a\}\}$, $\tau_{2}O(X) = \{X, \emptyset, \{b, d\}\}$ and $\tau_{3}O(X) = \{X, \emptyset, \{c, d\}\}$, then $(3\tau_{1})O(X) = \{X, \emptyset, \{a, b, d\}\}$, $\{a, b, d\}$ and $(3\tau_{2})O(X) = \{X, \emptyset, \{a, c, d\}\}$, $\bigcap_{i \in I} \{\{a, b, d\}, \{a, c, d\}\}$ is not a tritopological space on $X$. Then $(3\tau_{1})_{b} \cup (3\tau_{2})_{b} = \{X, \emptyset, \{a, b, d\}, \{a, d\}, \{a, b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ is not a tritopology, since $\{a, b\} \in (3\tau_{1})_{b} \cup (3\tau_{2})_{b}$ but $\{a, b\} \notin (3\tau_{1})_{b} \cup (3\tau_{2})_{b}$.

Definition 3.7 Let $X$ be a non empty set and $S$ be a subset of $X$. Then

(i) (a) The $2\tau$-interior of $S$, denoted by $2\tau$-int$(S)$, and is defined by

$$2\tau$-int(S) = \{G \subseteq S \text{ and } G \text{ is } 2\tau\text{-open}\}.$$

(b) The $3\tau$-interior of $S$, denoted by $3\tau$-int$(S)$, and is defined by

$$3\tau$-int(S) = \{G \subseteq S \text{ and } G \text{ is } 3\tau\text{-open}\}.$$

(c) Generally, the $N\tau$-interior of $S$, denoted by $N\tau$-int$(S)$, and is defined by

$$N\tau$-int(S) = \{G \subseteq S \text{ and } G \text{ is } N\tau\text{-open}\}.$$

(ii) (a) The $2\tau$-closure of $S$, denoted by $2\tau$-cl$(S)$, and is defined by

$$2\tau$-cl(S) = \cap \{F : S \subseteq F \text{ and } F \text{ is } 2\tau\text{-closed}\}.$$

(b) The $3\tau$-closure of $S$, denoted by $3\tau$-cl$(S)$, and is defined by

$$3\tau$-cl(S) = \cap \{F : S \subseteq F \text{ and } F \text{ is } 3\tau\text{-closed}\}.$$

(c) Generally, the $N\tau$-closure of $S$, denoted by $N\tau$-cl$(S)$, and is defined by

$$N\tau$-cl(S) = \cap \{F : S \subseteq F \text{ and } F \text{ is } N\tau\text{-closed}\}.$$

Theorem 3.8 Let $(X, N\tau)$ be a $N$-topological space on $X$ and let $A, B \subseteq X$. Then

(i) $N\tau$-cl$(A)$ is the smallest $N\tau$-closed set which containing $A$

(ii) $A$ is $N\tau$-closed if and only if $N\tau$-cl$(A) = A$. In particular, $N\tau$-cl$(\emptyset) = \emptyset$ and $N\tau$-cl$(X) = X$

(iii) $A \subseteq B \Rightarrow N\tau$-cl$(A) \subseteq N\tau$-cl$(B)$

(iv) $N\tau$-cl$(A \cup B) = N\tau$-cl$(A) \cup N\tau$-cl$(B)$

(v) $N\tau$-cl$(A \cap B) \subseteq N\tau$-cl$(A) \cap N\tau$-cl$(B)$

(vi) $N\tau$-cl$(N\tau$-cl$(A)) = N\tau$-cl$(A)$.
Proof

(i) Since intersection of any collection of \(N_r\)-closed sets is also \(N_r\)-closed, then \(N_r\)-cl(A) is a \(N_r\)-closed set. Trivially \(A \subseteq N_r\)-cl(A), by the definition of \(N_r\)-closure of \(A\). Now, let \(B\) be any \(N_r\)-closed set which containing \(A\). Then \(N_r\)-cl(A) = \(\cap\{F\mid A \subseteq F \text{ and } F \text{ is } N_r\text{-closed}\}\). Therefore, \(A\) is the smallest \(N_r\)-closed set which containing \(A\).

(ii) Assume \(A\) is \(N_r\)-closed, then \(A\) is the only smallest \(N_r\)-closed set which containing itself and therefore \(N_r\)-cl(A) = \(A\).

Conversely, assume \(N_r\)-cl(A) = \(A\). Then \(A\) is the smallest \(N_r\)-closed set containing itself. Therefore, \(A\) is \(N_r\)-closed. Particularly, since \(\emptyset\) and \(X\) are \(N_r\)-closed sets, then \(N_r\)-cl(\(\emptyset\)) = \(\emptyset\) and \(N_r\)-cl(\(X\)) = \(X\).

(iii) Assume \(A \subseteq B\), and since \(B \subseteq N_r\)-cl(B), then \(A \subseteq N_r\)-cl(B). Since \(N_r\)-cl(A) is the smallest \(N_r\)-closed set which containing \(A\), we have, \(N_r\)-cl(A) \(\subseteq N_r\)-cl(B).

(iv) Since \(A \subseteq A \cup B\) and \(B \subseteq A \cup B\). Then by (iii), we have \(N_r\)-cl(A) \(\cup N_r\)-cl(B) \(\subseteq N_r\)-cl(A \(\cup B\)). On the other hand, by (i), \(A \cup B \subseteq N_r\)-cl(A) \(\cup N_r\)-cl(B). Since \(N_r\)-cl(A \(\cup B\)) is the smallest \(N_r\)-closed set which containing \(A \cup B\). Then \(N_r\)-cl(A \(\cup B\)) \(\subseteq N_r\)-cl(A) \(\cup N_r\)-cl(B). Therefore we have, \(N_r\)-cl(A \(\cup B\)) = \(N_r\)-cl(A) \(\cup N_r\)-cl(B).

(v) Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), then \(N_r\)-cl(A \(\cap B\)) \(\subseteq N_r\)-cl(A) \(\cap N_r\)-cl(B).

(vi) Since \(N_r\)-cl(A) is a \(N_r\)-closed set, then \(N_r\)-cl(\(N_r\)-cl(A)) = \(N_r\)-cl(A).

Example 3.9 Let \(X = \{a,b,c,d\}\). For \(N = 2\), consider \(\tau_O(X) = \{X,\emptyset,\{a\},\{b,d\},\{a,b,d\}\}\) and \(\tau_O(X) = \{X,\emptyset,\{a\},\{b,d\},\{a,b,d\}\}\) and also \(\tau_O(X) = \{X,\emptyset,\{a\},\{b,c\},\{a,c\},\{b,c,d\}\}\).

Let \(A = \{a\}\) and \(B = \{b,c\}\), then \(2r\)clA = \(\{a,\emptyset,\{a\},\{b,d\},\{a,b,d\}\}\) and also \(2r\)cl(B) = \(\{b,c,d\}\) and \(2r\)cl(A \(\cap B\)) = \(\emptyset\). Therefore, \(2r\)cl(A \(\cap B\)) \(\neq 2r\)cl(A) \(\cap 2r\)cl(B). That is, equality does not hold in (v) of theorem 3.8.

Theorem 3.10 Let \((X,N_r)\) be a \(N\)-topological space on \(X\). Then \(N_r\)-closure satisfies Kuratowski closure axioms given below:

(i) \(N_r\)-cl(\(\emptyset\)) = \(\emptyset\)

(ii) \(A \subseteq N_r\)-cl(A) for each \(A \subseteq X\)

(iii) \(N_r\)-cl(A \(\cup B\)) = \(N_r\)-cl(A) \(\cup N_r\)-cl(B) for all \(A,B \subseteq X\)

(iv) \(N_r\)-cl(\(N_r\)-cl(A)) = \(N_r\)-cl(A) for each \(A \subseteq X\).

Proof Proof is follows from (i), (ii), (iv) and (vi) of theorem 3.8.

Theorem 3.11 Let \((X,N_r)\) be a \(N\)-topological space on \(X\) and \(A \subseteq X\). Then \(x \in N_r\)-cl(A) if and only if \(G \cap A \neq \emptyset\) for every \(N_r\)-open set \(G\) containing \(x\).

Proof Assume \(x \in N_r\)-cl(A) and \(G\) is a \(N_r\)-open set containing \(x\), then \(X - G\) is \(N_r\)-closed set and \(x \notin X - G\). Suppose that \(G \cap A = \emptyset\), then \(A \subseteq X - G\). That is, \(X - G\) is a \(N_r\)-closed set containing \(A\). Since \(N_r\)-cl(A) is the smallest \(N_r\)-closed set which containing \(A\), then \(N_r\)-cl(A) \(\subseteq X - G\). Which is contradicting to \(x \notin X - G\). Hence \(G \cap A \neq \emptyset\) for every \(N_r\)-open set \(G\) containing \(x\). Conversely, assume \(G \cap A \neq \emptyset\) for every \(N_r\)-open set \(G\) containing \(x\). Suppose that \(x \notin N_r\)-cl(A), then \(x \in X - N_r\)-cl(A), which is a \(N_r\)-open set. By hypothesis, \((X - N_r\)-cl(A)) \(\cap A \neq \emptyset\). Since \(X - N_r\)-cl(A) \(\subseteq X - A\) implies \((X - N_r\)-cl(A)) \(\cap A \subseteq (X - A) \cap A\), then \((X - A) \cap A \neq \emptyset\), which is a contradiction. Therefore, \(x \in N_r\)-cl(A).

Theorem 3.12 Let \((X,N_r)\) be a \(N\)-topological space \(X\) and \(A \subseteq X\). Then
(i) $N_r\text{-int}(X - A) = X - N_r\text{-cl}(A)$
(ii) $N_r\text{-cl}(X - A) = X - N_r\text{-int}(A)$.

Proof

(i) Assume $x \in N_r\text{-int}(X - A)$ and suppose $x \notin X - N_r\text{-cl}(A)$, then $x \notin N_r\text{-cl}(A)$ implies $G \cap A \neq \emptyset$ for every $N_r$-open set $G$ containing $x$. Therefore $G \nsubseteq X - A$ for every $N_r$-open set $G$ containing $x$. Then $x \notin N_r\text{-int}(X - A)$, which is a contradiction. Thus $x \in X - N_r\text{-cl}(A)$. On the other hand, let $x \in X - N_r\text{-cl}(A)$, then $x \notin N_r\text{-cl}(A)$. Suppose $x \notin N_r\text{-int}(X - A)$, then $G \not
in X - A$ for every $N_r$-open set $G$ containing $x$. That is, $G \cap A \neq \emptyset$ for every $N_r$-open set $G$ containing $x$. Then $x \notin N_r\text{-cl}(A)$, which is a contradiction. Thus, $x \in N_r\text{-int}(X - A)$. Therefore, $N_r\text{-int}(X - A) = X - N_r\text{-cl}(A)$.

(ii) Let $x \in X - N_r\text{-int}(A)$. Then $x \notin N_r\text{-int}(A)$ implies $G \nsubseteq A$ for every $N_r$-open set $G$ containing $x$. That is, $G \cap (X - A) \neq \emptyset$ for every $N_r$-open set $G$ containing $x$. Then $x \in N_r\text{-cl}(X - A)$. On the other hand, let $x \in N_r\text{-cl}(X - A)$, then $G \cap (X - A) \neq \emptyset$ for every $N_r$-open set $G$ containing $x$. That is, $G \nsubseteq A$ for every $N_r$-open set $G$ containing $x$. Then $x \notin N_r\text{-int}(A)$ implies $x \in X - N_r\text{-int}(A)$. Therefore, $N_r\text{-cl}(X - A) = X - N_r\text{-int}(A)$.

Remark 3.13 If we take complement of either side of (i) and (ii) of previous theorem, we get

(i) $N_r\text{-cl}(A) = X - N_r\text{-int}(X - A)$
(ii) $N_r\text{-int}(A) = X - N_r\text{-cl}(X - A)$.

Theorem 3.14 Let $(X, N_r)$ be a $N$-topological space $X$ and $A, B \subseteq X$. Then

(i) $N_r\text{-int}(A)$ is the largest $N_r$-open set contained in $A$
(ii) $A$ is $N_r$-open set if and only if $N_r\text{-int}(A) = A$. In particular, $N_r\text{-int}(\emptyset) = \emptyset$ and $N_r\text{-int}(X) = X$
(iii) $A \subseteq B$, then $N_r\text{-int}(A) \subseteq N_r\text{-int}(B)$
(iv) $N_r\text{-int}(A \cup B) \supseteq N_r\text{-int}(A) \cup N_r\text{-int}(B)$
(v) $N_r\text{-int}(A \cap B) = N_r\text{-int}(A) \cap N_r\text{-int}(B)$
(vi) $N_r\text{-int}(N_r\text{-int}(A)) = N_r\text{-int}(A)$.

Proof

(i) Since union of any collection of $N_r$-open sets is again a $N_r$-open set, then $N_r\text{-int}(A)$ is $N_r$-open set
and by definition of $N_r$-interior of $A$, $N_r\text{-int}(A) \subseteq A$. Now, let $B$ be any $N_r$-open set which contained in $A$. Then $B \subseteq \bigcup \{G | G \subseteq A \text{ and } G \text{ is } N_r\text{-open}\}$ and $B$ is $N_r$-open $= N_r\text{-int}(A)$ and therefore, $N_r\text{-int}(A)$ is the largest $N_r$-open set which contained in $A$.

(ii) Assume $A$ is $N_r$-open set if and only if $N_r\text{-cl}(X - A) = X - A$ if and only if $x \notin N_r\text{-cl}(X - A)$ if and only if $N_r\text{-int}(A) = A$. In particular, since $\emptyset$ and $X$ are $N_r$-open sets, then $N_r\text{-int}(\emptyset) = \emptyset$ and $N_r\text{-int}(X) = X$.

(iii) Assume $A \subseteq B$, then $X - B \subseteq X - A$ implies $N_r\text{-cl}(X - A) \subseteq N_r\text{-cl}(X - B)$ implies $N_r\text{-int}(A) \subseteq N_r\text{-int}(B)$.

(iv) Assume $x \in N_r\text{-int}(A) \cup N_r\text{-int}(B)$, then $x \notin (N_r\text{-int}(A) \cup N_r\text{-int}(B))$ implies $x \notin (N_r\text{-cl}(X - A) \cap (N_r\text{-cl}(X - B))$, then $x \notin N_r\text{-cl}(X - (A \cup B))$ implies $x \notin N_r\text{-int}(A \cup B)$. Therefore, $N_r\text{-int}(A \cup B) \subseteq N_r\text{-int}(A) \cup N_r\text{-int}(B)$.

(v) Assume $x \in N_r\text{-int}(A \cap B)$, then $x \notin (N_r\text{-cl}(X - A) \cup (N_r\text{-cl}(X - B))$, then $x \in (X - N_r\text{-cl}(X - A)) \cap (X - N_r\text{-cl}(X - B))$. Then $x \notin N_r\text{-int}(A \cap N_r\text{-int}(B)$. Thus, $N_r\text{-int}(A \cap B) \subseteq N_r\text{-int}(A) \cap N_r\text{-int}(B)$. On the other hand, let $x \in N_r\text{-int}(A \cap N_r\text{-int}(B)$, then
Example 3.15 Let \( X = \{ a, b, c, d, e \} \). For \( N = 3 \), consider \( \tau_0(X) = \{ X, \emptyset, \{ a, b \} \} \), \( \tau_1(X) = \{ X, \emptyset, \{ a, b \}, \{ a, b, c, d \} \} \) and \( \tau_2(X) = \{ X, \emptyset, \{ c \} \} \). Then, we have \( 3\tau_0(X) = \{ X, \emptyset, \{ a, b \}, \{ a, b, c, d \} \} \) and also \( \tau_1(X) = \{ X, \emptyset, \{ a, b, c, d \}, \{ a, b, c, d \} \} \). Let \( A = \{ a, c, d \} \) and \( B = \{ b \} \), then \( 3\tau_0(A) \) is not a \( N \)-open set.

Theorem 3.16 Let \( (X, N) \) be a \( N \)-topological space on \( X \) and \( A \subseteq X \). Then

(i) \( N^r\text{-int}(A) \supseteq \tau_1\text{-int}(A) \cup \tau_2\text{-int}(A) \cup \ldots \cup \tau_N\text{-int}(A) \)

(ii) \( N^r\text{-cl}(A) \subseteq \tau_1\text{-cl}(A) \cap \tau_2\text{-cl}(A) \cap \ldots \cap \tau_N\text{-cl}(A) \).

Definition 3.18

(i) Let \( Y \) be a non-empty subset of a bitopological space \( (X, 2\tau) \). Then the bitopology \( (2\tau)^* = \{ Y \cap \tau : \tau \in 2\tau \} \) is called the relative (simply induced or subspace) topology on \( Y \) for \( 2\tau \).

(ii) Let \( Y \) be a non-empty subset of a \( N \)-topological space \( (X, N) \). Then the \( N \)-topology \( (N\tau)^* = \{ Y \cap \tau : \tau \in N\tau \} \) is called the relative (simply induced or subspace) topology on \( Y \) for \( N\tau \).

Theorem 3.20 Let \( (Y, (N\tau)^*) \) be a subspace of \( (X, 2\tau) \) and \( A \subseteq Y \). Then

(i) \( A \) is \( (N\tau)^* \)-closed in \( Y \) if and only if \( A = Y \cap F \), where \( F \) is \( N\tau \)-closed in \( X \).

(ii) If \( A \) is \( (N\tau)^* \)-closed in \( Y \) and \( Y \) is \( N\tau \)-closed in \( X \). Then \( A \) is \( N\tau \)-closed in \( X \).
Proof

(i) $A$ is $(N_r)^{-}$-closed $\iff Y - A$ is $(N_r)^{-}$-open $\iff Y - A = Y \cap O$ for some $O \in N_r \Rightarrow A = Y \cap F$ where $F = X - O$ is $N_r$-closed.

(ii) Since $A$ is $(N_r)^{-}$-closed in $Y$, $A = Y \cap F$ for some $N_r$-closed set $F$ in $X$. Since $Y$ and $F$ are both $N_r$-closed in $X$, so is $Y \cap F$. \hfill \Box

4. Continuity in $N$-topological spaces

In this section, we introduce continuous functions in $N$-topological spaces and discuss the different properties of it. Also, we prove the Pasting Lemma. Throughout this section, the $N$-topological spaces $(X, N_r)$ and $(Y, N_r)$ are represented by $X$ and $Y$, respectively.

Definition 4.1 Let $X$ and $Y$ be two $N$-topological spaces. A function $f : X \to Y$ is said to be $N^*$-continuous on $X$ if the inverse image of every $N_r$-open set in $Y$ is a $N_r$-open set in $X$.

Example 4.2 For $N = 2$, let $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$. Consider $r_1O(X) = \{X, \emptyset, \{a, b\}\}$, $r_2O(X) = \{X, \emptyset, \{a\}\}$ and $r_2O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2rO(X) = \{X, \emptyset, \{a, b\}\}$, $2rO(Y) = \{Y, \emptyset, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$, $f(c) = z$, $f(d) = z$. Then $f^{-1}(Y) = X$, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{x\}) = \{a\}$.

Theorem 4.3 A function $f : X \to Y$ is $N^*$-continuous on $X$ if and only if the inverse image of every $N_r$-closed set in $Y$ is a $N_r$-closed set in $X$.

Proof Assume that $f : X \to Y$ is $N^*$-continuous on $X$ and let $A$ be a $N_r$-closed set in $Y$. Then $Y - A$ is a $N_r$-open set in $Y$. Since $f$ is a $N^*$-continuous function on $X$, then $f^{-1}(Y - A)$ is $N_r$-open set in $X$. Then $X - f^{-1}(A)$ is $N_r$-closed set in $X$. Conversely, assume the inverse image of every $N_r$-closed set in $Y$ is $N_r$-closed set in $X$ and let $B$ be a $N_r$-open set in $Y$. Then $Y - B$ is a $N_r$-closed set in $Y$ and $f^{-1}(Y - B)$ is a $N_r$-closed set in $X$. Then $f^{-1}(B)$ is a $N_r$-open set in $X$.

Theorem 4.4 A function $f : X \to Y$ is $N^*$-continuous on $X$ if and only if $f(N_r-cl(A)) \subseteq N_r-cl(f(A))$ for every $A \subseteq X$.

Proof Assume $f : X \to Y$ be a $N^*$-continuous function on $X$ and let $A \subseteq X$. Then $f(A) \subseteq Y$ and $N_r-cl(f(A))$ is $N_r$-closed set in $Y$. Since $f$ is $N^*$-continuous function on $X$, then $f^{-1}(N_r-cl(f(A)))$ is $N_r$-closed set in $X$. Since $f(A) \subseteq N_r-cl(f(A))$, then $A \subseteq f^{-1}(N_r-cl(f(A)))$. Since $N_r-cl(A)$ is the smallest $N_r$-closed set in $X$ containing $A$, then $N_r-cl(A) \subseteq f^{-1}(N_r-cl(f(A)))$. Then $f(N_r-cl(A)) \subseteq N_r-cl(f(A))$ for every $A \subseteq X$. Conversely, assume $f(N_r-cl(A)) \subseteq N_r-cl(f(A))$ for every $A \subseteq X$ and let $F$ be a $N_r$-closed set in $Y$. Since $f^{-1}(F) \subseteq X$, then $f(N_r-cl(f^{-1}(F))) \subseteq N_r-cl(f^{-1}(F)) = N_r-cl(F)$. Then $N_r-cl(f^{-1}(F)) \subseteq f^{-1}(N_r-cl(f(A))) = f^{-1}(F)$. Since $F$ is a $N_r$-closed set in $Y$ and also $f^{-1}(F) \subseteq N_r-cl(f^{-1}(F))$. Then $f^{-1}(F) = N_r-cl(f^{-1}(F))$ and also $f^{-1}(F)$ is $N_r$-closed set in $X$. Therefore, $f$ is $N^*$-continuous function on $X$.

Example 4.5 For $N = 2$. Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$. Consider $r_1O(X) = \{X, \emptyset, \{a, c\}\}$, $r_2O(X) = \{X, \emptyset, \{a\}\}$ and also $r_1O(Y) = \{Y, \emptyset, \{x, z\}\}$. Then $2rO(X) = \{X, \emptyset, \{a, c\}\}$, $2rO(Y) = \{Y, \emptyset, \{y\}\}$. Then $2rO(X) = \{X, \emptyset, \{a, b\}\}$, $2rO(Y) = \{Y, \emptyset, \{y\}\}$ and also $2rO(Y) = \{Y, \emptyset, \{y\}\}$.
2rC(Y) = \{y, \emptyset, \{w\}, \{y, w\}, \{x, z, w\}\}. Define \(f: X \to Y\) by \(f(a) = y, f(b) = x, f(c) = y\) and \(f(d) = x\). Clearly, \(f\) is \(2^r\)-continuous function on \(X\). If \(A = \{a, c\} \subseteq X\). Then \(f(2r-cl(A)) = f(\{a, c\}) = \{y\}\). But, \(2r-cl(f(A)) = 2r-cl(\{y, w\}) = \{y, w\}\). Thus, \(f(2r-cl(A)) \neq 2r-cl(f(A))\), even though \(f\) is \(2^r\)-continuous function on \(X\). That is, equality does not hold in the theorem 4.4, even though \(f\) is \(2^r\)-continuous function on \(X\).

**Theorem 4.6** A function \(f: X \to Y\) is \(N^r\)-continuous on \(X\) if and only if \(Nr-cl(f^{-1}(B)) \subseteq f^{-1}(Nr-cl(B))\) for every \(B \subseteq Y\).

**Proof** Let \(f: X \to Y\) be a \(N^r\)-continuous on \(X\) and let \(B \subseteq Y\). Then \(Nr-cl(B)\) is \(Nr\)-closed set in \(Y\). Since \(f\) is \(N^r\)-continuous function on \(X\), then \(f^{-1}(Nr-cl(B))\) is \(Nr\)-closed in \(X\). That is, \(Nr-cl(f^{-1}(Nr-cl(B)))=f^{-1}(Nr-cl(B))\). Since \(B \subseteq Nr-cl(B)\), then \(f^{-1}(Nr-cl(B)) \subseteq f^{-1}(Nr-cl(B))\). Thus, \(Nr-cl(f^{-1}(B)) \subseteq f^{-1}(Nr-cl(B))\) for every \(B \subseteq Y\). Conversely, assume that \(Nr-cl(f^{-1}(B)) \subseteq f^{-1}(Nr-cl(B))\) for every \(B \subseteq Y\) and let \(B\) be a \(Nr\)-closed set in \(Y\). Then \(Nr-cl(B) = f^{-1}(Nr-cl(B))\) and \(f\) be \(Nr\)-continuous on \(X\). Thus, \(f^{-1}(2r-cl(f^{-1}(A)))\) even though \(f\) is \(2^r\)-continuous function on \(X\). That is, equality does not hold in the theorem 4.6, even though \(f\) is \(2^r\)-continuous function on \(X\).

**Theorem 4.8** A function \(f: X \to Y\) is \(N^r\)-continuous on \(X\) if and only if \(f^{-1}(Nr-int(B)) \subseteq Nr-int(f^{-1}(B))\) for every \(B \subseteq Y\).

**Proof** Let \(f: X \to Y\) be a \(N^r\)-continuous on \(X\) and let \(B \subseteq Y\). Then \(Nr-int(B)\) is \(Nr\)-open set in \(Y\). Since \(f\) is \(N^r\)-continuous function on \(X\), then \(f^{-1}(Nr-int(B))\) is \(Nr\)-open in \(X\). That is, \(Nr-int(f^{-1}(Nr-int(B)))=f^{-1}(Nr-int(B))\). Since \(Nr-int(B) \subseteq B\), then \(f^{-1}(Nr-int(B)) \subseteq f^{-1}(B)\). Thus, \(f^{-1}(Nr-int(B)) \subseteq Nr-int(f^{-1}(B))\) for every \(B \subseteq Y\). Conversely, assume \(f^{-1}(Nr-int(B)) \subseteq Nr-int(f^{-1}(B))\) for every \(B \subseteq Y\) and let \(B\) be \(Nr\)-open set in \(Y\). Then \(Nr-int(B) = f^{-1}(Nr-int(B))\) and \(f\), \(Nr-int(f^{-1}(B)) \subseteq Nr-int(f^{-1}(B))\) which implies \(f^{-1}(Nr-int(B)) \subseteq Nr-int(f^{-1}(B))\). Thus, \(f^{-1}(Nr-int(B)) \subseteq Nr-int(f^{-1}(B))\) for every \(B \subseteq Y\) and \(B\) be a \(Nr\)-open set in \(Y\). Therefore, \(f\) is \(N^r\)-continuous function on \(X\).

**Example 4.9** For \(N = 2\), let \(X = \{a, b, c, d\}\) and \(Y = \{x, y, z, w\}\). Consider \(r_7O(X) = \{X, \emptyset, \{a\}\}\). Define \(f: X \to Y\) by \(f(a) = x, f(b) = y, f(c) = z, f(d) = z\). Clearly, \(f\) is \(2^r\)-continuous function on \(X\). If \(B = \{x, y\} \subseteq Y\). Then \(f^{-1}(2r-int(B)) \neq f^{-1}(\{a\})\). But, \(2r-int(f^{-1}(B)) = 2r-int(\{a, b\}) = \{a, b\}\). Thus, \(f^{-1}(2r-int(B)) \neq 2r-int(f^{-1}(B))\), even though \(f\) is \(2^r\)-continuous. That is, equality does not hold in the theorem 4.8, even when \(f\) is \(2^r\)-continuous.

**Theorem 4.10** (The Pasting Lemma)

Let \(X\) and \(Y\) be two \(N\)-topological spaces with \(X = A \cup B\), where \(A\) and \(B\) are \(Nr\)-closed sets in \(X\). Let \(f: A \to Y\) and \(g: B \to Y\) be \(N^r\)-continuous. If \(f(x) = g(x)\) for every \(x \in A \cap B\), then \(f\) and \(g\) combine to give a \(N^r\)-continuous function \(h: X \to Y\), defined by setting \(h(x) = f(x)\) if \(x \in A\), and \(h(x) = g(x)\) if \(x \in B\).
**Proof** Let $F$ be a $N^r$-closed set in $Y$. Now $h^{-1}(F) = f^{-1}(F) \cap g^{-1}(F)$, by elementary set theory. Since $f$ is $N^u$-continuous, $f^{-1}(F)$ is $N^r$-closed in $A$ and therefore, $N^r$-closed in $X$. Similarly, $g^{-1}(F)$ is $N^r$-closed in $B$ and therefore, $N^r$-closed in $X$. Thus their union $h^{-1}(F)$ is $N^r$-closed in $X$. \[\blacksquare\]

5. Conclusion

In this paper, we introduce a new venture to establish more topologies on a non empty set. Such efforts prompt us to blissfully convey that these concepts are also applicable in other areas of General topology, Fuzzy topology, intuitionistic topology, ideal topology so on and so forth. The course of human history, unmistakably shown and revealed to us that many great leaps of learning, discoveries, and understanding come from a source not so anticipated, and that in any field of sciences or humanities, and in particular in the field of basic researches often bear fruit well within hundred years, so to say. However, the more we come to grapple with and invest our time and energy to comprehend anything that is new, the better will we be, in order to handle and deal with the challenges and queries that keep facing us in the future, and come up with better results and findings.

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**Author details**

M. Lellis Thivagar
E-mail: mlthivagar@yahoo.co.in
ORCID ID: http://orcid.org/0000-0001-5997-5185

V. Ramesh
E-mail: kabilanchelian@gmail.com
ORCID ID: http://orcid.org/0000-0002-1243-6958

M. Arockia Dasan
E-mail: dassfredy@gmail.com
ORCID ID: http://orcid.org/0000-0002-1243-6958

1 School of Mathematics, Madurai Kamaraj University, Madurai 625021, Tamil Nadu, India.

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