THE DUFFIN-SCHAEFFER TYPE CONJECTURES IN VARIOUS LOCAL FIELDS

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ABSTRACT. This paper discovers a new phenomenon about the Duffin-Schaeffer conjecture, which claims that
\[ \lambda(\bigcap_{m=1}^{\infty} \cup_{n=m}^{\infty} E_n) = 1 \] if and only if \( \sum_{n} \lambda(E_n) = \infty \),
where \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}/\mathbb{Z} \),
\[ E_n = E_n(\psi) = \bigcup_{(m,n)=1}^{\infty} \left( \frac{m-\psi(n)}{n}, \frac{m+\psi(n)}{n} \right), \]
\( \psi \) is any non-negative arithmetical function. Instead of studying \( \bigcap_{m=1}^{\infty} \cup_{n=m}^{\infty} E_n \),
we introduce an even fundamental object \( \cup_{n=1}^{\infty} E_n \) and conjecture there exists a
universal constant \( C > 0 \) such that
\[ \lambda(\bigcup_{n=1}^{\infty} E_n) \geq C \min\{ \sum_{n=1}^{\infty} \lambda(E_n), 1 \}. \]
It is shown that this conjecture is equivalent to the Duffin-Schaeffer conjecture.
Similar phenomena are found in the fields of \( p \)-adic numbers and formal Laurent series. As a byproduct, we answer conditionally a question of Haynes by showing
that one can always use the quasi-independence on average method to deduce
\( \lambda(\bigcap_{m=1}^{\infty} \cup_{n=m}^{\infty} E_n) = 1 \) as long as the Duffin-Schaeffer conjecture is true. We
also show among several others that two conjectures of Haynes, Pollington and Velani are equivalent to the Duffin-Schaeffer conjecture, and introduce for the first
time a weighted version of the second Borel-Cantelli lemma to the study of the
Duffin-Schaeffer conjecture.

1. INTRODUCTION TO THE DUFFIN-SCHAEFFER CONJECTURE

Throughout the paper we use the following notations:
- \( p \) denotes a prime number,
- \( n, h \) denote positive integers,
- \( \varphi(n) \) denotes the Euler phi function,
- \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}/\mathbb{Z} \),
- \( f(x) \uparrow A \) (\( \downarrow A \)) means \( f(x) \) tends increasingly (decreasingly) to \( A \),
- \( f(x) \ll g(x) \) or \( g(x) \gg f(x) \) means \( |f(x)| \leq C|g(x)| \) for some universal
constant \( C > 0 \), \( f(x) \asymp g(x) \) means both \( f(x) \ll g(x) \) and \( g(x) \ll f(x) \),
- \( B(x, r) \) (\( \overline{B}(x, r) \)) denotes the open (closed) ball with center \( x \) and radius \( r \)
in a given metric space.
In this section we will introduce case by case the Duffin-Schaeffer type conjectures in the fields $\mathbb{R}$ of real numbers, $\mathbb{Q}_p$ of $p$-adic numbers, and $\mathbb{F}((X^{-1}))$ of formal Laurent series.

1.1. **Duffin-Schaeffer conjecture.** For any non-negative function $\psi : \mathbb{N} \to \mathbb{R}$ and any positive integer $n$, we define $E_n(\psi) \subseteq \mathbb{R}/\mathbb{Z}$ by

$$E_n = E_n(\psi) = \bigcup_{m=1}^{n} \left( \frac{m - \psi(n)}{n}, \frac{m + \psi(n)}{n} \right).$$

Let $W(\psi)$ denote the collection of points $x \in \mathbb{R}/\mathbb{Z}$ which fall in infinitely many of the sequence of the sets $\{E_n\}_{n} \in \mathbb{N}$, that is,

$$W(\psi) = \limsup_{n \to \infty} E_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n.$$

The Lebesgue measure of $E_n$ is obviously bounded above by $2\psi(n)^2\varphi(n)$. According to the first Borel-Cantelli lemma, we see that if the series

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n} \varphi(n)$$

is convergent, then $W(\psi)$ is of zero Lebesgue measure. In 1942 Duffin and Schaeffer ([9]) proposed the following conjecture in metric number theory:

**Conjecture 1.1.** If (1.3) is divergent, then $\lambda(W(\psi)) = 1$.

Although various partial results are known (see [19] for details and references before 2000 and [1, 4, 22, 25] for recent progresses), the full conjecture represents one of the most fundamental unsolved problems in metric number theory.

The main purpose of this paper is to establish various equivalent forms for the original Duffin-Schaeffer conjecture. We begin with a classical estimate of Pollington and Vaughan ([27, formula (3)]) claiming that if $\psi(n) \leq \frac{n}{2\varphi(n)}$, then $\lambda(E_n) \geq \frac{\psi(n)\varphi(n)}{n}$. This implies unconditionally that

$$\min\left\{ \frac{\psi(n)\varphi(n)}{n}, \frac{1}{2} \right\} \leq \lambda(E_n) \leq \min\{2\frac{\psi(n)\varphi(n)}{n}, 1\}.$$

Instead of studying $W(\psi)$ we introduce an even fundamental object

$$Z(\psi) = \bigcup_{n=1}^{\infty} E_n,$$

whose Lebesgue measure is bounded above by $\min\{2\sum_{n=1}^{\infty} \frac{\psi(n)\varphi(n)}{n}, 1\}$. Similar to (1.4) we conjecture this trivial upper bound is essentially a non-trivial lower bound despite some loss of constant:
Conjecture 1.2. There exists a universal constant $C > 0$ such that for any non-negative function $\psi$,

\begin{equation}
\lambda(Z(\psi)) \geq C \min\left\{ \sum_{n=1}^{\infty} \frac{\psi(n) \varphi(n)}{n}, 1 \right\}.
\end{equation}

At this stage (see also the end of Section 4) it is hard to convince the readers why the above conjecture is possibly true, but we can show

Theorem 1.3. Conjecture 1.2 and the Duffin-Schaeffer conjecture are equivalent.

Similar phenomena will be confirmed in the fields of $p$-adic numbers and formal Laurent series. At the moment we concentrate on the classical case.

At the end of the paper [21] Haynes asked whether there exists a non-negative function $\psi$ for which one cannot use the quasi-independence on average method (see the last part of this section for an introduction) to deduce $\lambda(W(\psi)) = 1$. As a byproduct of Theorem 1.3, we can give a conditional but almost best possible answer: If the Duffin-Schaeffer conjecture is true, then the answer is NO.

Next let us recall three related conjectures due to Sanju Velani and his coauthors. Letting $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a dimension function, and letting $\mathcal{H}^f$ be the corresponding Hausdorff $f$-measure on $\mathbb{R}/\mathbb{Z}$, Beresnevich and Velani ([6, Conjecture 2]) proposed the following Hausdorff measure version of the Duffin-Schaeffer conjecture:

Conjecture 1.4. Let $f$ be a dimension function such that $r^{-1} f(r)$ is monotonic. If

\begin{equation}
\sum_{n=1}^{\infty} f\left(\frac{\psi(n)}{n}\right) \varphi(n)
\end{equation}

is divergent, then $\mathcal{H}^f(W(\psi)) = \mathcal{H}^f(\mathbb{R}/\mathbb{Z})$.

The original statement of Conjecture 1.4 is in fact a $k$-dimensional analogue, which has already been confirmed for $k \geq 2$ ([6, Corollary 1]). As a consequence of their Mass Transference Principle ([6, Theorem 2]), Beresnevich and Velani showed ([6, Theorem 1]) that Conjecture 1.4 is equivalent to the Duffin-Schaeffer conjecture.

Recently, Haynes, Pollington and Velani ([22, Conjectures 1 & 2]) proposed the following “weakening” versions of the Duffin-Schaeffer conjecture by assuming extra divergence:

Conjecture 1.5. Let $f$ be a dimension function such that $r^{-1} f(r) \not\to \infty$ as $r \to 0$. If (1.3) is divergent, then $\mathcal{H}^f(W(\psi)) = \infty$.

Conjecture 1.6. Let $f : [0, \infty) \to \mathbb{R}$ be an increasing non-negative function such that $r^{-1} f(r) \to 0$ as $r \to 0$. If (1.7) is divergent, then $\lambda(W(\psi)) = 1$.

The proposers also believe that the latter two conjectures are in principle easier to establish than the Duffin-Schaeffer conjecture, but we can show
Theorem 1.7. Conjecture 1.5, Conjecture 1.6 and the Duffin-Schaeffer conjecture are all equivalent.

Nevertheless, the interested readers may still study Conjectures 1.5 & 1.6 for some particularly chosen functions to get some partial results (see [22, Problems 1 & 2]). The main tool for proving Theorem 1.7 is the well-known fact that there is no fastest converging or slowest diverging series (see e.g. [28]).

To summarize, we have the equivalence between the Duffin-Schaeffer conjecture for \( W(\psi) \), the newest Lebesgue measure version Conjecture 1.2 for \( Z(\psi) \) as well as the Hausdorff measure version Conjecture 1.4 for \( W(\psi) \). Conjecture 1.2 might be more easier to attack than Conjecture 1.4 as it is a standard Lebesgue measure statement rather than a Hausdorff measure one.

### 1.2. \( p \)-adic approximation.

For any prime \( p \), let \( \mathbb{Q}_p \) denote the field of \( p \)-adic numbers with absolute value \( | \cdot |_p \), and let \( \mathbb{Z}_p \) denote the ring of integers
\[
\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}.
\]

Since \( \mathbb{Q}_p \) is a locally compact topological group under addition, there exists a unique Haar measure \( \mu_p \) on \( \mathbb{Q}_p \) such that \( \mu_p(\mathbb{Z}_p) = 1 \). For any non-negative function \( \psi : \mathbb{N} \to \mathbb{R} \) and any positive integer \( n \), we define
\[
\mathcal{K}_n(\psi) = \bigcup_{a \equiv -n \pmod{n}} \left\{ x \in \mathbb{Z}_p : |x - \frac{a}{n}|_p \leq \frac{\psi(n)}{n} \right\},
\]
and set
\[
W_p(\psi) = \limsup_{n \to \infty} \mathcal{K}_n(\psi) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{K}_n(\psi).
\]

Recently, Haynes ([21]) studied in detail the metric Diophantine approximation in \( \mathbb{Q}_p \) by attacking the next Duffin-Schaeffer type conjecture ([21, Conjecture 1]):

**Conjecture 1.8.** \( \mu_p(W_p(\psi)) = 1 \) if and only if \( \sum_{n \in \mathbb{N}} \mu_p(\mathcal{K}_n(\psi)) = \infty \).

Similar to Conjecture 1.2 we propose

**Conjecture 1.9.** There exists a universal constant \( C > 0 \) depending only on \( p \) such that for any non-negative function \( \psi : \mathbb{N} \to \mathbb{R} \),
\[
\mu_p \left( \bigcup_{n=1}^{\infty} \mathcal{K}_n(\psi) \right) \geq C \min \left\{ \sum_{n=1}^{\infty} \mu_p(\mathcal{K}_n(\psi)), 1 \right\}.
\]

Similar to Theorem 1.3 we can show

**Theorem 1.10.** Conjecture 1.8 and Conjecture 1.9 are equivalent.
The proof of Theorem 1.10 is in essence similar to that of Theorem 1.3, but still needs to be treated independently.

We will also show that one can always use the quasi-independence on average method to deduce \( \mu_p(W_p(\psi)) = 1 \) as long as the \( p \)-adic version of the Duffin-Schaeffer conjecture is true.

Several recent progresses on the classical Duffin-Schaeffer conjecture ([1, 4, 25]) can be naturally transferred into new results on the \( p \)-adic version of the Duffin-Schaeffer conjecture via a lemma of Haynes ([21, Lemma 3]), but we will not pursue this direction in the paper.

1.3. Formal Laurent series. Let \( \mathbb{F} \) be a finite field of \( q \) elements. Throughout we will use the following notations which play the roles of integers, rational numbers, real numbers, \( \mathbb{R}/\mathbb{Z} \), and the absolute value, respectively:

- \( \mathbb{F}[X] \) denotes the set of polynomials with \( \mathbb{F} \)-coefficients,
- \( \mathbb{F}(X) \) denotes the fraction field of \( \mathbb{F}[X] \),
- \( \mathbb{F}((X^{-1})) \) denotes the set of formal Laurent series,
- \( \mathbb{L} \) denotes the set of elements of \( \mathbb{F}((X^{-1})) \) with degrees less than zero,
- \( |f| \triangleq q^{\partial f} \), where \( \partial f \) denotes the degree of \( f \in \mathbb{F}((X^{-1})) \).

The metric \( \rho \) on \( \mathbb{F}((X^{-1})) \) is naturally defined as \( \rho(f, g) = |f - g| \). Since \( \mathbb{F}((X^{-1})) \) is a locally compact topological group under addition, there exists a unique Haar measure \( \nu \) on \( \mathbb{F}((X^{-1})) \) such that \( \nu(\mathbb{L}) = 1 \). The \( d \)-fold (\( d \in \mathbb{N} \)) product of measure \( \nu \) on \( \mathbb{F}((X^{-1}))^d \) is denoted by \( \nu_d \).

For any non-negative function \( \Psi : \mathbb{F}[X] \to \mathbb{R} \) and any monic \( Q \in \mathbb{F}[X] \), we define

\[
E_Q(\Psi) = \bigcup_{P \in \mathbb{F}[X]} \left\{ f \in \mathbb{L} : |f - \frac{P}{Q}| < \frac{\Psi(Q)}{|Q|} \right\},
\]

and put for any \( d \in \mathbb{N} \),

\[
W^{(d)}(\Psi) = \bigcap_{n=1}^{\infty} \bigcup_{\partial Q \geq n} E_Q(\Psi)^d.
\]

A few years ago Inoue and Nakada ([23, 24]) first studied the metric simultaneous Diophantine approximation in \( \mathbb{F}((X^{-1}))^d \) by attacking the following Duffin-Schaeffer type conjecture ([23, Conjecture]):

**Conjecture 1.11.** \( \nu_d(W^{(d)}(\Psi)) = 1 \) if and only if

\[
\sum_{Q \text{ is monic}} \nu_d(E_Q(\Psi)^d) = \infty.
\]

Similar to Conjectures 1.2 & 1.9 we propose
Conjecture 1.12. There exists a universal constant $C > 0$ depending only on $d$ and the size of $F$ such that for any non-negative function $\Psi : F[X] \to \mathbb{R}$,

$$\nu_d \left( \bigcup_{Q \text{ is monic}} E_Q(\Psi)^d \right) \geq C \min \left\{ \sum_{Q \text{ is monic}} \left( \frac{\Psi(Q)\Phi(Q)}{|Q|} \right)^d, 1 \right\}. \tag{1.14}$$

Similar to Theorems 1.3 & 1.10 we can show

**Theorem 1.13.** Conjecture 1.11 and Conjecture 1.12 are equivalent.

The proof of Theorem 1.13 is fully identical to that of Theorem 1.10.

In contrast to the classical and $p$-adic cases ([21, 27]), progresses on Conjecture 1.11 are rather incomplete. For example, the Sprindžuk type conjecture ([21, 27, 30]) over formal Laurent series hasn’t been established yet. Maybe the best currently known result is due to Inoue, Nakada ([24, Thm. 1]) and Fuchs ([14, Thm. 1]) whose theorems confirmed Conjecture 1.11 under the additional assumption that $\Psi(Q)$ depends only on the degree of $Q$.

Without any extra assumptions on $\Psi$, we will also study a variant of $W^{(d)}(\Psi)$ by establishing a Gallagher type theorem ([16, Thm. 1]). For any non-negative function $\Psi : F[X] \to \mathbb{R}$, any monic $Q \in F[X]$ and any $d \in \mathbb{N}$, we first define

$$H^{(d)}_Q(\Psi) = \mathbb{L}^d \cap \left( \bigcup_{P_i \in F[X]} \prod_{i=1}^d B \left( \frac{P_i}{Q}, \frac{\Psi(Q)}{|Q|} \right) \right) \tag{1.15}$$

then set

$$H^{(d)}(\Psi) = \bigcap_{n=1}^\infty \bigcup_{\partial Q \geq n} H^{(d)}_Q(\Psi). \tag{1.16}$$

Note $H^{(1)}(\Psi) = W^{(1)}(\Psi)$. In the higher-dimensions we will show

**Theorem 1.14.** Let $d \geq 2$. Then $\nu_d(H^{(d)}(\Psi)) = 1$ if and only if

$$\sum_{Q \text{ is monic}} \nu_d(H^{(d)}_Q(\Psi)) = \infty. \tag{1.17}$$

1.4. **Quasi-independence on average method.** In this paper a weighted version of the second Borel-Cantelli lemma will be introduced for the first time to the study of the Duffin-Schaeffer conjecture. We begin with a beautiful result of Gallagher ([15]) called “zero-one law” claiming that $\lambda(W(\psi))$ can be either 0 or 1 for any non-negative function $\psi$. This means if $\lambda(W(\psi)) > 0$, then we must have $\lambda(W(\psi)) = 1$. A useful tool for proving $\lambda(W(\psi)) > 0$ is the following second Borel-Cantelli lemma due to Erdős and Rényi ([11]):
Lemma 1.15. Let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence of events in a probability space \((\Omega, \mathbb{P})\) such that \( \sum_n \mathbb{P}(A_n) = \infty \). Then

\[
\mathbb{P}(\limsup_{n \to \infty} A_n) \geq \limsup_{n \to \infty} \frac{(\sum_{k=1}^n \mathbb{P}(A_k))^2}{\sum_{i=1}^n \sum_{j=1}^n \mathbb{P}(A_i \cap A_j)}.
\] (1.18)

To the author’s knowledge, the proofs of all known results towards the Duffin-Schaeffer conjecture regard Lemma 1.15 as an indispensable tool (see [21]), showing under various additional conditions the quasi-independence on average property for \( \{E_n(\psi)\} \), that is, proving

\[
\sum_{m=1}^N \sum_{n=1}^N \lambda(E_m \cap E_n) \ll (\sum_{n=1}^N \lambda(E_n))^2
\] (1.19)

for infinitely many \( N \in \mathbb{N} \). Recently, Feng, Shen and the author ([13]) generalized the above lemma to

Lemma 1.16. Let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence of events in a probability space \((\Omega, \mathbb{P})\) and let \( \{\omega_n\}_{n \in \mathbb{N}} \) be a sequence of real numbers such that \( \sum_n \omega_n \mathbb{P}(A_n) = \infty \). Then

\[
\lambda(\limsup_{n \to \infty} A_n) \geq \limsup_{n \to \infty} \frac{(\sum_{k=1}^n \omega_k \mathbb{P}(A_k))^2}{\sum_{i=1}^n \sum_{j=1}^n \omega_i \omega_j \mathbb{P}(A_i \cap A_j)}.
\] (1.20)

The novelty of Lemma 1.16 in our study of the Duffin-Schaeffer conjecture is that we can choose \( \omega_n \) to be some particular fraction numbers to obtain some good effects. For example we can show

Theorem 1.17. Let \( \psi : \mathbb{N} \to \mathbb{R} \) be a non-negative function. Then \( \lambda(W(\psi)) = 1 \) if

\[
\sum_{h:S_h \geq 3} \frac{\log S_h}{h \cdot \log \log S_h} = \infty,
\] (1.21)

where

\[
S_h = \sum_{n=2^h+1}^{2^{h+1}} \frac{\psi(n)\varphi(n)}{n}.
\] (1.22)

This generalizes a recent result by Beresnevich et al. ([4, Thm. 2]) who showed \( \lambda(W(\psi)) = 1 \) if there exists a constant \( c > 0 \) such that

\[
\sum_{n=16}^{\infty} \frac{\varphi(n)\psi(n)}{n \exp(c(\log \log n)(\log \log \log n))} = \infty.
\] (1.23)
In metric number theory the second Borel-Cantelli lemma is so frequently used in the study of many other problems ([20]), for which one may naturally expect that Lemma 1.16 could bring new insight as well as new results.

This paper is mainly arranged as follows:

- Sections 3∼5 are devoted to establishing various equivalent forms for the classical Duffin-Schaeffer conjecture. In particular, a general principle will be introduced after the proof of Theorem 1.3.
- Sections 6, 7 are devoted to studying the Duffin-Schaeffer type conjectures in the fields of $p$-adic numbers and formal Laurent series, respectively.
- Section 8 is devoted to the proof of Theorem 1.17.

2. Preliminaries

2.1. Hausdorff measure. A dimension function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing function such that $f(r) \to 0$ as $r \to 0$. A $\rho$-cover of a subset $A$ of $\mathbb{R}$ is a countable collection $\{B_i\}$ of open intervals in $\mathbb{R}$ with radii $r_i \leq \rho$ for each $i$ such that $A \subset \bigcup_i B_i$. Define

$$H^f_{\rho}(A) = \inf \sum_i f(r_i),$$

where the infimum is taken over all $\rho$-covers of $A$. The Hausdorff $f$-measure of $A$ is defined as

$$H_{\rho}(A) = \lim_{\rho \to 0} H^f_{\rho}(A).$$

For detailed discussions of the Hausdorff measure theory, we refer the readers to the classical book [12] by Falconer.

2.2. Series. It is well-known that there is no fastest converging or slowest diverging series, that is, for any convergent non-negative series $\sum_n a_n$, there exists a sequence of (increasing) real numbers $\{x_n\}$ with $x_n \to \infty$ as $n \to \infty$, such that $\sum_n a_n x_n$ is also convergent; and for any divergent non-negative series $\sum_n b_n$, there exists a sequence of (decreasing) real numbers $\{y_n\}$ with $y_n \to 0$ as $n \to \infty$, such that $\sum_n b_n y_n$ is also divergent. In fact, one can choose ([28, Exer. 11 & 12, Chap. 3])

$$x_n = \frac{1}{\sqrt{\sum_{k=n}^{\infty} (a_k + \frac{1}{k^2})}}; \quad y_n = \frac{1}{\sum_{k=1}^{n} (b_k + \frac{1}{k^2})}.$$  

We may further require that $x_{n+1} - x_n \leq 1$ for all $n$ as if not then we can replace $\{x_n\}$ with $\{z_n\}$ defined recursively by $z_1 = x_1,

$$z_{n+1} = z_n + \min\{1, x_{n+1} - x_n\} \quad (n \in \mathbb{N}).$$

The reason is very simple and left as an exercise to the interested readers.
2.3. Upper bounds. In the following we prepare some upper bounds for $\lambda(\mathcal{E}_m \cap \mathcal{E}_n)$, and always assume $m \neq n$, $m, n \geq 2$. First, Duffin and Schaeffer ([9, Lemma II], see also [4, formula (5)]) proved that

(2.1) $\lambda(\mathcal{E}_m \cap \mathcal{E}_n) \leq 8\psi(m)\psi(n)$.

Second, we have

(2.2) $\lambda(\mathcal{E}_m \cap \mathcal{E}_n) \ll \lambda(\mathcal{E}_m)\lambda(\mathcal{E}_n)P(m, n)$,

where

$$P(m, n) = \prod_{p|B(m, n), p > D(m, n)} (1 - \frac{1}{p})^{-1},$$

with $B(m, n) \triangleq \frac{mn}{\text{gcd}(m, n)^2}$, $D(m, n) \triangleq \frac{1}{(m, n)^2} \cdot \max\{n\psi(m), m\psi(n)\}$. If $\{p : p|B(m, n), p > D(m, n)\} = \emptyset$, we understand that $P(m, n) \equiv 1$. This estimate was first stated by Strauch ([32]), but was also given independently by Pollington and Vaughan ([27]). By one of Merten’s theorems ([17, Theorem 328]) we see that if $P(m, n) > 1$ and $D(m, n) \geq \frac{1}{2}$ then

(2.3) $P(m, n) \ll \exp\left(\sum_{D(m, n) < p < \log B(m, n)} \frac{1}{p}\right) \ll \log \log B(m, n)\frac{2}{2 + \log D(m, n)}$.

Finally, we can find in [27] that if $D(m, n) < 0.5$, then $\mathcal{E}_m \cap \mathcal{E}_n$ is an empty set. This fact is also easily implied by formula (10) in [4].

Formula (2.3) is very useful and we will explain in more detail. Throughout the paper for any $h \in \mathbb{N}$ and any non-negative function $\psi$, we denote

- $\Delta_h = \mathbb{N} \cap [2^h + 1, 2^{h+1})$,
- $S_h = S_h(\psi) = \sum_{n \in \Delta_h} \frac{\psi(n)}{n}$,
- $B_h = B_h(\psi) = \lambda(\bigcup_{n \in \Delta_h} \mathcal{E}_n)$,
- $Q_h = Q_h(\psi) = \sum_{(m, n) \in \Delta_h \times \Delta_h, m \neq n} \lambda(\mathcal{E}_m \cap \mathcal{E}_n)$,
- $R_h = R_h(\psi) = Q_h/S_h^2$.

To attack the Duffin-Schaeffer conjecture we need first assume $\sum_h S_h = \infty$. Note there exists an integer $i \in \{0, 1, 2\}$ such that $\sum_h S_{3h+i} = \infty$. By appealing to the Erdős-Vaaler theorem ([10, 33]), to prove $\lambda(W(\psi)) = 1$ we may assume without loss of generality that $\psi(n) \geq \frac{1}{n}$ whenever $\psi(n) \neq 0$. Now for any two distinct positive integers $h_1 < h_2$ and for any $m \in \Delta_{3h_1+i}$, $n \in \Delta_{3h_2+i}$, we have $\log \log B(m, n) \ll h_2$ and $D(m, n) \geq \frac{n}{m^2} \geq \sqrt{n}$, which in turn gives

(2.4) $P(m, n) \ll \max\{\frac{\log \log B(m, n)}{2 + \log D(m, n)}, 1\} \ll \max\{\frac{h_2}{2h_2}, 1\} \ll 1$.

We should remark that the above kind of arguments was first observed by Haynes, Pollington and Velani ([22], see also [1, 4]). Hence to study the Duffin-Schaeffer conjecture it brings no harm for us to assume $\sum_h S_h = \infty$ together with:

(2.5) $P(m, n) \ll 1$ for any $m, n$ in any corresponding distinct blocks $\Delta_{h_1}, \Delta_{h_2}$. 

3. Equivalence of the Duffin-Schaeffer conjecture (1)

The section is mainly devoted to the proof of Theorem 1.3. A general principle and a higher-dimensional analogue will also be established.

3.1. Proof of Theorem 1.3. For any non-negative function $\psi : \mathbb{N} \to \mathbb{R}$, we denote

$$S(\psi) = \sum_{n=1}^{\infty} \frac{\psi(n)\varphi(n)}{n}.$$  

(3.1)

For any $N \in \mathbb{N}$, we set

$$A_N = \inf \{ \lambda(Z(\psi)) : \text{supp}(\psi) \subset [N, \infty) \text{ is bounded}, S(\psi) \geq 1 \}.$$  

(3.2)

Obviously, $0 \leq A_1 \leq A_2 \leq A_3 \leq \cdots \leq 1$. So we can define

$$A_\infty = \lim_{N \to \infty} A_N.$$  

(3.3)

**Claim 1:** The Duffin-Schaeffer conjecture is true if and only if $A_\infty > 0$.

**Proof of Claim 1:** “$\Leftarrow$” Suppose $A_\infty > 0$. Let $\psi$ be any non-negative function such that $S(\psi) = \infty$. We need to show that $\lambda(W(\psi)) = 1$. Let $N_1 < N_2 < N_3 < \cdots$ be any fixed sequence of positive integers such that

$$\sum_{n=N_k+1}^{N_{k+1}} \frac{\psi(n)\varphi(n)}{n} \geq 1.$$  

By definition if $k$ is large enough, then

$$\lambda(\bigcup_{n=N_k+1}^{N_{k+1}} E_n) \geq \frac{A_\infty}{2}.$$  

By the continuity of the Lebesgue measure we have

$$\lambda(W(\psi)) = \lim_{N \to \infty} \lambda(\bigcup_{n=N_k+1}^{N_{k+1}} E_n) \geq \frac{A_\infty}{2},$$  

which gives $\lambda(W(\psi)) = 1$ by applying Gallagher’s zero-one law. This proves the Duffin-Schaeffer conjecture under the assumption $A_\infty > 0$.

“$\Rightarrow$” Suppose the Duffin-Schaeffer conjecture is true. We argue by contradiction and suppose $A_\infty = 0$. Thus $A_N = 0$ for any $N \in \mathbb{N}$. Obviously, be the definitions of $A_N$ we can find a sequence of non-negative functions $\{\psi_k\}$ and a sequence of positive integers $N_1 < N_2 < N_3 < \cdots$ such that $\text{supp}(\psi_k) \subset [N_k + 1, N_{k+1}]$,

$$\sum_{n=N_k+1}^{N_{k+1}} \frac{\psi_k(n)\varphi(n)}{n} \geq 1,$$

and

$$\lambda(\bigcup_{n=N_k+1}^{N_{k+1}} E_n(\psi_k)) \leq \frac{1}{k^2}.$$  

Gluing this sequence of disjointly supported functions \( \{ \psi_k \} \) into a new function \( \psi \), it is easy to deduce from the first Borel-Cantelli lemma that \( \lambda(W(\psi)) = 0 \). Note \( S(\psi) = \infty \). So we get a contradiction to the assumed truth of Duffin-Schaeffer conjecture. This finishes the whole proof of Claim 1.

**Claim 2:** \( A_1 > 0 \iff A_\infty > 0 \).

**Proof of Claim 2:** Obviously, \( A_1 > 0 \) implies \( A_\infty > 0 \). So we need only to show that \( A_\infty > 0 \) implies \( A_1 > 0 \). By the continuity of the Lebesgue measure it suffices to give a universal lower bound for \( \lambda(Z(\psi)) \), where \( \psi \) is any non-negative function with bounded support and \( S(\psi) \geq 1 \). To this aim we first choose an \( N \in \mathbb{N} \) such that \( A_N \geq \frac{A_\infty}{2} \), then decompose \( \psi \) as the sum of two unique non-negative functions \( \psi_1, \psi_2 \) with corresponding bounded supports in \([1, N-1]\) and \([N, \infty)\). Now we have two cases to consider.

**Case 1:** Suppose \( S(\psi_2) \geq 1 - \frac{A_\infty}{8} \). Let \( \psi_3 \) be any non-negative function with bounded support in \((\max \supp(\psi_2), \infty)\) such that \( S(\psi_3) = \frac{A_\infty}{8} \). Note

\[
\frac{A_\infty}{2} \leq \lambda(Z(\psi_2 + \psi_3)) \leq \lambda(Z(\psi_2)) + 2S(\psi_3) \leq \lambda(Z(\psi_2)) + \frac{A_\infty}{4},
\]

from which we deduce \( \lambda(Z(\psi)) \geq \lambda(Z(\psi_2)) \geq \frac{A_\infty}{4} \).

**Case 2:** Suppose \( S(\psi_2) < 1 - \frac{A_\infty}{8} \). Choose an \( n < N \) such that \( \frac{\psi(n)\varphi(n)}{n} \geq \frac{A_\infty}{8(N-1)} \).

Applying the left hand side of (1.4) gives

\[
\lambda(Z(\psi)) \geq \lambda(\mathcal{E}_n) \geq \min\{\frac{A_\infty}{8(N-1)}, \frac{1}{2}\}.
\]

This finishes the whole proof of Claim 2.

**Claim 3:** For any \( t \geq 1 \) and any non-negative function \( \psi \), \( \lambda(Z(t\psi)) \leq t\lambda(Z(\psi)) \).

**Proof of Claim 3:** By the continuity of the Lebesgue measure, we may assume without loss of generality that \( \psi \) is of bounded support, and suppose this is the case. Since \( \mathcal{E}_n \) is an open set in \( \mathbb{R}/\mathbb{Z} \) for any \( n \in \mathbb{N} \), we see that \( Z(\psi) \) is the union of finitely many pairwise disjointly supported open subsets of \( \mathbb{R}/\mathbb{Z} \), say for example,

\[
Z(\psi) = \bigcup_{k=1}^{M} \{e^{2\pi iy} : y \in I_k\},
\]

where \( I_k = (x_k - r_k, x_k + r_k) \), \( k = 1, 2, \ldots, M \), are pairwise disjointly supported open intervals in \((-1, 2)\). For the sake of simplicity we identify \( \{e^{2\pi iy} : y \in I_k\} \) with \( I_k \).

Now suppose \( x \in Z(t\psi) \). This means one can find a coprime pair \((m, n)\) such that \( |x - \frac{m}{n}| < \frac{\psi(n)}{n} \). According to the decomposition (3.4), there exists an \( I_k \) such that

\[
\left( \frac{m}{n} - \frac{\psi(n)}{n}, \frac{m}{n} + \frac{\psi(n)}{n} \right) \subset I_k.
\]
Comparing the lengths of the above two intervals, we get \( \frac{\psi(n)}{n} \leq r_k \). Consequently,
\[
x \in \left( \frac{m}{n} - \frac{t\psi(n)}{n}, \frac{m}{n} + \frac{t\psi(n)}{n} \right)
\]
\[
= \left( \frac{m}{n} - \psi(n) - \frac{(t-1)\psi(n)}{n}, \frac{m}{n} + \psi(n) + \frac{(t-1)\psi(n)}{n} \right)
\]
\[
\subset (x_k - r_k - (t-1)r_k, x_k + r_k + (t-1)r_k)
\]
\[
= (x_k - tr_k, x_k + tr_k),
\]
which naturally implies that
\[
Z(t\psi) \subset \bigcup_{k=1}^{M} (x_k - tr_k, x_k + tr_k).
\]
So we have \( \lambda(Z(t\psi)) \leq \sum_{k=1}^{M} 2tr_k = t\lambda(Z(\psi)) \). This finishes the proof of Claim 3.

**Proof of Theorem 1.3:** We first note that if Conjecture 1.2 is true, then \( A_1 > 0 \), or equivalently by Claims 1 & 2, the Duffin-Schaeffer conjecture is true. Next, we assume the Duffin-Schaeffer conjecture is true and are going to show that Conjecture 1.2 is also true. To this aim it suffices to establish for any non-negative function \( \psi \) that
\[
\lambda(Z(\psi)) \geq A_1 \min\{S(\psi), 1\},
\]
where \( A_1 > 0 \) follows from Claims 1 & 2. By the continuity of the Lebesgue measure, we may further assume without loss of generality that \( \psi \) is of non-empty bounded support. Now we have two cases to consider.

Case 1: Suppose \( S(\psi) \geq 1 \). By the definition of \( A_1 \) we have \( \lambda(Z(\psi)) \geq A_1 \).

Case 2: Suppose \( S(\psi) < 1 \). Let
\[
t \triangleq \frac{1}{S(\psi)} > 1,
\]
which means \( S(t\psi) = 1 \). By the definition of \( A_1 \), \( \lambda(Z(t\psi)) \geq A_1 \). By Claim 3,
\[
\lambda(Z(\psi)) \geq \frac{\lambda(Z(t\psi))}{t} \geq A_1 S(\psi).
\]
This finishes the whole proof of Theorem 1.3.

**3.2. A general principle.** Let \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) be a probability space and let \( \mathcal{F}_n \) \( (n \in \mathbb{N}) \) be a fixed subset of \( \mathcal{F} \). \( \{\mathcal{F}_n\}_n \) is said to have the Duffin-Schaeffer property if
\[
\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty \Rightarrow \mathbb{P}(\text{lim sup}_{n \to \infty} E_n) = 1
\]
for any sequence of events \( \{E_n\}_n \) with \( E_n \in \mathcal{F}_n \) \( (n \in \mathbb{N}) \). Similarly, \( \{\mathcal{F}_n\}_n \) is said to have the zero-one property if
\[
\mathbb{P}(\text{lim sup}_{n \to \infty} E_n) \in \{0, 1\}
\]
for any sequence of events \( \{ E_n \}_{n} \) with \( E_n \in \mathcal{F}_n \) \( (n \in \mathbb{N}) \). A general principle implied in the proofs of Claims 1 & 2 is the following theorem whose proof is left as a simple exercise to the interested readers.

**Theorem 3.1.** Suppose \( \{ \mathcal{F}_n \}_{n} \) is of the zero-one property. Then \( \{ \mathcal{F}_n \}_{n} \) is of the Duffin-Schaeffer property if and only if

\[
\inf \{ \mathbb{P} \left( \bigcup_{n=1}^{\infty} E_n \right) : \sum_{n=1}^{\infty} \mathbb{P}(E_n) \geq 1, E_n \in \mathcal{F}_n \} > 0.
\]

Note in the simultaneous and multiplicative Diophantine approximation the zero-one property is in general not a big problem (see e.g. [5, 7, 26]) as we have the cross fibering principle due to Beresnevich, Haynes and Velani ([5]). Many conjectures in metric number theory were formulated to prove a particularly chosen \( \{ \mathcal{F}_n \}_{n} \) having the Duffin-Schaeffer property (see e.g. [2, 5, 26]). Hence Theorem 3.1 provides a new way to look at such kind of conjectures.

We remark that unconditionally one cannot expect

\[
\mathbb{P} \left( \bigcup_{n=1}^{\infty} E_n \right) \asymp \min \{ \sum_{n=1}^{\infty} \mathbb{P}(E_n), 1 \}
\]

as we have the following example in \( \mathbb{R}/\mathbb{Z} \): define

\[
\mathcal{F}_n = \begin{cases} 
\{0, [0, 1) \} & (n \text{ is odd}) \\
\{0, [0, \frac{1}{n}) \} & (n \text{ is even}),
\end{cases}
\]

\[
E_n = \begin{cases} 
\emptyset & (n \text{ is odd or } n \text{ is even with } n \leq 2N) \\
[0, \frac{1}{n}) & (n \text{ is even with } n > 2N),
\end{cases}
\]

where \( N \) is large enough. Even though, one may still expect the equivalence between (3.8) and the corresponding Duffin-Schaeffer-type conjecture for some particularly chosen \( \{ \mathcal{F}_n \}_{n} \) (see e.g. Theorems 1.3, 1.13, 1.10 & 3.2).

### 3.3. Higher-dimensional case.

We are going to generalize Theorem 1.3 to the higher dimensions.

**Theorem 3.2.** Let \( d \geq 2 \). Then

\[
\lambda_d \left( \bigcup_{n=1}^{\infty} \mathcal{E}_n^d \right) \geq C_d \min \{ \sum_{n=1}^{\infty} \frac{\psi(n) \varphi(n)}{n}^d, 1 \},
\]

where \( \lambda_d \) denotes the \( d \)-dimensional Lebesgue measure on \( (\mathbb{R}/\mathbb{Z})^d \), \( \mathcal{E}_n^d \) denotes the \( d \)-fold product of \( \mathcal{E}_n \), \( C_d > 0 \) depends only on \( d \).

The key to the proof of Theorem 3.2 is the following generalization of Claim 3 in the higher-dimensions as Claims 1 & 2 have just been generalized by Theorem 3.1, and the Sprindžuk conjecture ([30], also known as the higher-dimensional Duffin-Schaeffer conjecture) was confirmed by Pollington and Vaughan ([27], see also [18]).
Lemma 3.3. Let $d \geq 1$ and $t \geq 1$. Then there exists a constant $C_d > 0$ depending only on $d$ such that

$$\lambda_d\left(\bigcup_{n=1}^{\infty} E_n(t\psi)^d\right) \leq C_d t^d \lambda_d\left(\bigcup_{n=1}^{\infty} E_n(\psi)^d\right).$$  

Proof. We only outline the proof in four steps and the interested readers can easily provide all the details as the proof of Lemma 3.3 is similar to that of Claim 3. First, by the continuity of the Lebesgue measure we may assume $\psi$ is of bounded support, thus $\bigcup_{n=1}^{\infty} E_n(\psi)^d$ is the union of finitely many $d$-dimensional cubes $\{C(x_i, r_i)\}$; next, we apply the classical Vitali covering lemma (see e.g. [31]) to choose a subcollection of disjointly supported cubes $\{C(y_j, s_j)\}$ such that

$$\bigcup_i C(x_i, r_i) \subset \bigcup_j C(y_j, M_d s_j),$$

where $M_d > 0$ is a universal constant depending only on $d$; after that, via elementary geometric observation it is easy to show that

$$\bigcup_{n=1}^{\infty} E_n(t\psi)^d = \bigcup_i C(x_i, tr_i) \subset \bigcup_j C(y_j, M_d^{(2)} ts_j),$$

where $M_d^{(2)} > 0$ depends also only on $d$; finally, comparing the $d$-dimensional measures of the left and right hand sides of the last formula gives the desired result. \(\square\)

4. EQUIVALENCE OF THE DUFFIN-SCHAEFFER CONJECTURE (2)

This section is devoted to providing more equivalent forms for the classical Duffin-Schaeffer conjecture. As a byproduct, the question of Haynes introduced in the first section will be given a conditional but almost best possible answer.

First we observe from Claims 1 & 2 in the proof of Theorem 1.3 (see also Theorem 3.1) that the Duffin-Schaeffer conjecture implies the following

**Conjecture 4.1.** There exists a universal constant $C > 0$ independent of $h \in \mathbb{N}$ and non-negative function $\psi$ such that if $S_h(\psi) = 1$, then $B_h(\psi) \geq C$.

By Claim 3 in the proof of Theorem 1.3, this conjecture is in turn equivalent to

**Conjecture 4.2.** There exists a universal constant $C > 0$ independent of $h \in \mathbb{N}$ and non-negative function $\psi$ such that if $S_h(\psi) \leq 1$, then $B_h(\psi) \geq C S_h(\psi)$.

Both conjectures are not weak at all as we have

**Theorem 4.3.** Conjecture 4.1, Conjecture 4.2 and the Duffin-Schaeffer conjecture are all equivalent.

Obviously, to prove this theorem it suffices to show that Conjecture 4.2 implies the Duffin-Schaeffer conjecture. To this aim, let $\psi$ be any non-negative function
such that $\sum_h S_h(\psi) = \infty$. Without loss of generality we may assume $S_h(\psi) \leq 1$ for all $h \in \mathbb{N}$ as if not then we can study another function $\psi_2 \leq \psi$ defined by

$$\psi_2(n) = \begin{cases} \psi(n) & (n \in \Delta_h, S_h(\psi) \leq 1) \\ \frac{\psi(n)}{S_h(\psi)} & (n \in \Delta_h, S_h(\psi) > 1) \end{cases}$$

to deduce first $\lambda(W(\psi_2)) = 1$ then $\lambda(W(\psi)) = 1$. According to the assumed truth of Conjecture 4.2 and the following proposition (see also the proof of [1, Thm. 1]) we have $\lambda(W(\psi)) = 1$, which proves the Duffin-Schaeffer conjecture.

**Proposition 4.4.** Let $\psi$ be any non-negative function such that $\sum_h S_h(\psi) = \infty$. Then $\lambda(W(\psi)) = 1$ if there exists a constant $C > 0$ independent of $h \in \mathbb{N}$ such that $B_h(\psi) \geq C S_h(\psi)$ for any $h \in \mathbb{N}$.

**Proof.** Thanks to (2.5) we may assume without loss of generality that $P(m, n) \ll 1$ for any $m, n$ in any corresponding distinct blocks $\Delta_{h_1}, \Delta_{h_2}$. As $W(\psi)$ is also of the form $\limsup_{h \to \infty} \bigcup_{n \in \Delta_h} E_n(\psi)$ and $\sum_h \lambda\left( \bigcup_{n \in \Delta_h} E_n(\psi) \right) = \sum_h B_h(\psi) = \infty$, we can apply Lemma 1.15 to deduce

$$\lambda(W(\psi)) \geq \limsup_{N \to \infty} \frac{\left( \sum_{h=1}^{N} \lambda\left( \bigcup_{n \in \Delta_h} E_n(\psi) \right) \right)^2}{\sum_{h_1=1}^{N} \sum_{h_2=1}^{N} \lambda\left( \left( \bigcup_{m \in \Delta_{h_1}} E_m(\psi) \right) \cap \left( \bigcup_{n \in \Delta_{h_2}} E_n(\psi) \right) \right) + \left( \sum_{h=1}^{N} B_h(\psi) \right)^2}$$

$$\geq \limsup_{N \to \infty} \sum_{h=1}^{N} B_h(\psi) + 2 \sum_{1 \leq h_1 < h_2 \leq N} \sum_{m \in \Delta_{h_1}} \sum_{n \in \Delta_{h_2}} \lambda(E_m(\psi) \cap E_n(\psi))$$

$$\gg \limsup_{N \to \infty} \frac{\left( \sum_{h=1}^{N} S_h(\psi) \right)^2}{\sum_{h=1}^{N} S_h(\psi) + \left( \sum_{h=1}^{N} S_h(\psi) \right)^2} > 0,$$

which gives $\lambda(W(\psi)) = 1$ by applying Gallagher’s zero-one law. We are done. \[\square\]

As introduced in the first section Haynes asked whether there exists a non-negative function $\psi$ for which one cannot use the quasi-independence on average method to prove $\lambda(W(\psi)) = 1$. Based on the equivalence between the classical Duffin-Schaeffer conjecture and Conjecture 4.2, Proposition 4.4 and its quasi-independence on average proof, a conditional answer is: If the Duffin-Schaeffer conjecture is true, then the answer is NO. This answer is almost best possible as to give an absolutely complete answer one need first assume that the Duffin-Schaeffer conjecture is false.
Haynes ([21]) also commented that an answer to the above question would bring us closer to the heart of the Duffin-Schaeffer conjecture. Based on all previous discussions, we believe the heart of the Duffin-Schaeffer conjecture is to establish $B_h \asymp S_h$ whenever $S_h \leq 1$

- If anyone can confirm or disprove $B_h \asymp S_h$ when $S_h \leq 1$, then the Duffin-Schaeffer conjecture is true or false, respectively.
- If anyone can confirm $B_h \asymp S_h$ when $S_h \leq 1$ under additional assumptions, then we can use Proposition 4.4 to obtain some partial results.

For example, any of the following assumptions can give $B_h \asymp S_h$:

- (A1) $\sum_{n \in \Delta_h} \frac{\psi(n)\varphi(n)}{n} \leq \frac{1}{h}$,
- (A2) $\sum_{n \in \Delta_h} \psi(n) \leq \frac{1}{\sqrt{h}}$,
- (A3) $\sum_{n \in \Delta_h} \frac{\psi(n)n}{\varphi(n)} \leq 1$.

To this aim we first note from the Cauchy-Schwarz inequality that

\begin{equation}
B_h \geq \frac{S_h^2}{S_h + Q_h}.
\end{equation}

Thus to get $B_h \asymp S_h$ it suffices to establish $Q_h \ll S_h$. Since we not only have the powerful (2.3) but also have the Duffin-Schaeffer estimate (2.1), it is rather easy to deduce $Q_h \ll S_h$ from any of the above three assumptions. Just for one example, suppose $\sum_{n \in \Delta_h} \psi(n)n \leq 1$. We then have

\begin{equation}
Q_h \ll \left( \sum_{n \in \Delta_h} \psi(n) \right)^2 \leq S_h \cdot \sum_{n \in \Delta_h} \frac{\psi(n)n}{\varphi(n)} \leq S_h.
\end{equation}

We remark that the assumption (A1) was first observed by Aistleitner ([1]).

Conjectures 4.1 & 4.2 suggest two directions in the study of the Duffin-Schaeffer conjecture, one is proving $B_h \asymp 1$ under the assumption $S_h \geq f(h)$ for some slowly-increasing function $f$, the other is showing $B_h \asymp S_h$ under the assumption $S_h \leq g(h)$ for some slowly-decreasing function $g$. For example, what Beresnevich et al. have actually proved on their [4, Thm. 2] (one may also deduce it from the proof of Theorem 1.17) is in principle the following estimate:

**Proposition 4.5.** There exists a universal constant $C_\alpha > 0$ depending only on $\alpha > 0$ such that if $S_h \geq \exp(\alpha h \log h)$, then $B_h \geq C_\alpha$.

Finally we explain how has Conjecture 1.2 been proposed. Let $\psi$ be a non-negative function such that $\sum_h S_h(\psi)$ diverges extremely slow to infinity. If $\frac{B_h(\psi)}{S_h(\psi)} \to 0$ as $h \to \infty$, then it is highly possible that $\sum_h B_h(\psi) < \infty$, which implies the Duffin-Schaeffer conjecture is false. On the other hand, if $B_h(\psi) \asymp S_h(\psi)$, then we can use Proposition 4.4 to deduce $\lambda(W(\psi)) = 1$. Motivated by this observation we believe in general $B_h(\psi) \asymp S_h(\psi)$ whenever $S_h(\psi) \leq 1$. 
5. EQUIVALENCE OF THE DUFFIN-SCHAEFFER CONJECTURE (3)

This section is devoted to the proof of Theorem 1.7 which is the combination of two smaller ones.

**Theorem 5.1.** Conjecture 1.5 is equivalent to the Duffin-Schaeffer conjecture.

Let us briefly explain how will we derive a proof of Theorem 5.1. Obviously, it suffices ([22]) to prove that Conjecture 1.5 implies the Duffin-Schaeffer conjecture. To this aim, we need only to get a contradiction by assuming the truth of Conjecture 1.5 and absurdity of the Duffin-Schaeffer conjecture. Thus suppose there exists a non-negative function \( \psi \) with \( S(\psi) = \infty \) such that \( \lambda(W(\psi)) = 0 \). To give a proof of Theorem 5.1 it suffices to construct a dimension function \( f \) satisfying the assumption of Conjecture 1.5 such that \( H^f(W(\psi)) < \infty \). This is indeed possible as we can learn from the next lemma, hence a proof of Theorem 5.1 is obtained.

**Lemma 5.2.** Let \( A \subset \mathbb{R} \) be of zero Lebesgue measure. Then there exists a dimension function \( f \) with \( r^{-1}f(r) \nearrow \infty \) as \( r \to 0 \), such that \( H^f(A) = 0 \).

\[ \sum_{n=1}^{\infty} \frac{|F_n|g(n)}{e^n} < \infty. \]

Obviously, we may further assume \( g(1) = 1 \). The linear interpolation of \( g \) defined on \([1, \infty)\) is denoted still by \( g \). Note for all non-integer points \( x \in [1, \infty) \),

\[ g'(x) \leq 1 < g(x), \]

where \( g'(x) \) means as usual the derivative of \( g \) at \( x \). With these preparations we define a function \( f : (0, \infty) \to (0, \infty) \) by

\[ f(r) = r \cdot g(\max\{1, -1 - \log r\}), \]

and are going to verify step by step that the function \( f \) satisfies the required claim as follows:

1) Obviously, \( f \) is continuous. It follows from (5.2) that \( f \) is increasing and \( g(n) \leq n \). Since

\[ f\left(\frac{1}{e^{n+1}}\right) = \frac{g(n)}{e^{n+1}} \leq \frac{n}{e^{n+1}}, \]

we see that \( f(r) \to 0 \) as \( r \to 0 \). This shows that \( f \) is indeed a dimension function.
2) \( r^{-1}f(r) = g(\max\{1, -1 - \log r\}) \not\to \infty \) as \( r \to 0 \).

3) Note first

\[ A \subset \bigcup_{i=1}^{\infty} I_i^{(n)} \subset \bigcup_{k=n}^{\infty} \bigcup_{i=1}^{\infty} I_i^{(k)}, \]

which means \( \bigcup_{k=n}^{\infty} \bigcup_{i=1}^{\infty} I_i^{(k)} \) is a \( \frac{1}{e^r} \)-cover of \( A \). Hence if \( n \geq 2 \), then

\[ H^f_\psi(A) \leq \sum_{k=n}^{\infty} \sum_{i=1}^{\infty} f(r_i^{(k)}) \leq \sum_{s=n}^{\infty} |F_s|g(s). \]

In view of (5.1) we must have \( H^f(A) = 0 \). This finishes the proof of Lemma 5.2. \square

**Theorem 5.3.** Conjecture 1.6 is equivalent to the Duffin-Schaeffer conjecture.

*Proof. Obviously, it suffices ([22]) to prove that Conjecture 1.6 implies the Duffin-Schaeffer conjecture. Suppose Conjecture 1.6 is true, and let \( \psi : \mathbb{N} \to \mathbb{R} \) be any non-negative function such that \( \sum_{n} \frac{\varphi(n)\psi(n)}{n} = \infty. \) Our purpose below is to show that \( \lambda(W(\psi)) = 1. \) By appealing to the Erdös-Vaaler theorem ([10, 33]) and to a theorem of Pollington and Vaughan ([27, Thm. 2]), we can assume without loss of generality that \( 1/n \leq \psi(n) \leq 1/2 \) whenever \( \psi(n) \neq 0. \) As discussed in the second section, there exists a decreasing positive function \( g : \mathbb{N} \to \mathbb{R} \) with \( g(n) \searrow 0 \) as \( n \to \infty \) such that

\[ \sum_{n} \frac{\varphi(n)\psi(n)g(n)}{n} = \infty. \]

The linear interpolation of \( g \) defined on \([1, \infty)\) is denoted still by \( g. \) Considering \( g(n) \searrow 0 \) as \( n \to \infty \), it is convenient for us to define \( g(\infty) = 0. \) With these preparations we now define a non-negative function \( f : [0, \infty) \to \mathbb{R} \) by

\[ f(r) = r \cdot g\left( \max\{\frac{1}{\sqrt{r}}, 1\}\right), \]

and are going to verify step by step that the function \( f \) satisfies the required claim as follows:

1) \( f \) is an increasing function as it is the product of the increasing function \( r \mapsto r \) and the non-decreasing function \( r \mapsto g(\max\{1/\sqrt{r}, 1\}). \)

2) \( r^{-1}f(r) = g(\max\{1/\sqrt{r}, 1\}) \to 0 \) as \( r \to 0. \)

3) Considering \( g(\infty) = 0 \) and \( 1/n \leq \psi(n) \leq 1/2 \) whenever \( \psi(n) \neq 0, \) we have

\[ \sum_{n \in \mathbb{N}} f\left( \frac{\psi(n)}{n}\right)\varphi(n) = \sum_{n \in \mathbb{N}} \frac{\psi(n)}{n} \varphi(n)g(\sqrt{\frac{n}{\psi(n)}}) \geq \sum_{n \in \mathbb{N}} \frac{\psi(n)}{n} \varphi(n)g(n) = \infty. \]

Hence \( \lambda(W(\psi)) = 1 \) follows from the assumed truth of Conjecture 1.6. This finishes the proof of Theorem 5.3. \square
6. \( p \)-adic Approximation

This section is devoted to the proof of Theorem 1.10. We will also discuss Haynes’ question in the new setting of \( p \)-adic numbers.

6.1. **Proof of Theorem 1.10.** Every non-zero \( p \)-adic number \( \alpha \) has a unique \( p \)-adic expansion

\[
\alpha = \sum_{k=s}^{\infty} \alpha_k p^k \quad (\alpha_k \in \mathbb{Z}, \ 0 \leq \alpha_k \leq p-1, \ \alpha_s > 0),
\]

and we can do arithmetic in \( \mathbb{Q}_p \) in similar fashion to the way it is done in \( \mathbb{R} \) with decimal expansions. With this form \( |\alpha|_p \triangleq p^{-s}, \) and \( \alpha \in \mathbb{Z}_p \) if and only if \( \alpha_k = 0 \) whenever \( k < 0. \) By the translation-invariant property of the Haar measure \( \mu_p \) on \( \mathbb{Q}_p \) and by \( \mu_p(\mathbb{Z}_p) = 1, \) we see that for any \( \beta \in \mathbb{Q}_p \) and any \( z \in \mathbb{Z}, \) \( \mu_p(B(\beta, p^z)) = p^z, \) which implies further for any \( r > 0, \) \( \mu_p(B(\beta, r)) \leq r \) (♠). With these preparations we can generalize Lemma 3.3 to the \( p \)-adic case.

**Lemma 6.1.** Let \( t \geq 1. \) Then for any non-negative function \( \psi : \mathbb{N} \to \mathbb{R}, \)

\[
\mu_p\left( \bigcup_{n=1}^{\infty} K_n(t\psi) \right) \leq t \mu_p\left( \bigcup_{n=1}^{\infty} K_n(\psi) \right).
\]

**Proof of Lemma 6.1.** The proof of Lemma 6.1 is similar to that of Lemma 3.3, and in fact is more simpler. By the inner regular property of the Haar measure \( \mu_p \) (★), we may assume without loss of generality that \( \psi \) is non-empty bounded support. As any two closed balls in \( \mathbb{Q}_p \) can only have either empty intersection or one is contained in the other (♣), we see that \( \bigcup_n K_n(\psi) \) is the union of finitely many pairwise disjointly supported non-empty closed balls of the following form

\[
\bigcup_n K_n(\psi) = \bigcup_{i=1}^{M} \bigcup_{j=1}^{M_i} \left( B\left( \frac{a_{i,j}}{n_i}, \frac{\psi(n_i)}{n_i} \right) \cap \mathbb{Z}_p \right) \quad - n_i \leq a_{i,j} \leq n_i, \ (a_{i,j}, n_i) = 1,
\]

and

\[
\bigcup_n K_n(t\psi) = \bigcup_{i=1}^{M} \bigcup_{j=1}^{M_i} \left( B\left( \frac{a_{i,j}}{n_i}, \frac{t\psi(n_i)}{n_i} \right) \cap \mathbb{Z}_p \right).
\]

If there exists an pair \((i, j)\) such that \( \mathbb{Z}_p \subset B\left( \frac{a_{i,j}}{n_i}, \frac{\psi(n_i)}{n_i} \right), \) then \( \bigcup_n K_n(\psi) = \mathbb{Z}_p \) and we need to do nothing further. Else by the property ♦ we can assume \( B\left( \frac{a_{i,j}}{n_i}, \frac{\psi(n_i)}{n_i} \right) \subset \mathbb{Z}_p \) for all the pairs \((i, j)\). Consequently, by the property ♦ we have

\[
\mu_p\left( \bigcup_{n=1}^{\infty} K_n(t\psi) \right) \leq \sum_{i=1}^{M} \sum_{j=1}^{M_i} \mu_p\left( B\left( \frac{a_{i,j}}{n_i}, \frac{t\psi(n_i)}{n_i} \right) \right) \leq t \sum_{i=1}^{M} \sum_{j=1}^{M_i} \mu_p\left( B\left( \frac{a_{i,j}}{n_i}, \frac{\psi(n_i)}{n_i} \right) \right) = t \mu_p\left( \bigcup_{n=1}^{\infty} K_n(\psi) \right).
\]

This finishes the proof of Lemma 6.1. \( \square \)
we will use the quasi-independence on average method to deduce
\[ \mu_p(\psi) \]
be any non-negative function such that
\[ H_1 \sim \] the overlap estimates obtained in [27].
\[ \text{proof of } H_3 \ (\text{[21, lemma 2]) is trivial, while that of } H_4 \ (\text{[21, lemma 3]) is similar to } \]
\[ 2\)), while \( H_2 \ (\text{[21, Lemma 1]) is a Cassels-Gallagher type zero-one law \ ([8, 15]). The } \]
\[ \text{We remark } H_1 \ (\text{[21, Thm. 3 (i)]) is a Pollington-Vaughan type theorem \ ([27, Thm. } \]
\[ 6.2. \text{Quasi-independence on average method.} \text{ One may ask in the } p\text{-adic case whether there exists a non-negative function } \psi : \mathbb{N} \to \mathbb{R} \text{ for which one cannot use the quasi-independence on average method to prove } \mu_p(W_p(\psi)) = 1. \text{ We will give a conditional but almost best possible answer to this question: If the } p\text{-adic version of the Duffin-Schaeffer conjecture is true, then the answer is NO. To this purpose we first recall several facts proved by Haynes ([21]):} \]
\[ \bullet \ H_1: \sum_{n \in \mathbb{N}} \mu_p(K_n(\psi)) = \infty \text{ implies } \mu_p(W_p(\psi)) = 1. \]
\[ \bullet \ H_2: \mu_p(W_p(t\psi)) \in \{0, 1\} \text{ is independent of } t > 0. \]
\[ \bullet \ H_3: \text{If } p|n, \text{ then } K_n(\psi) = \emptyset \text{ or } \mathbb{Z}_p. \]
\[ \bullet \ H_4: \text{Suppose that } \frac{\psi(n)}{p^n} \text{ takes value in the set } \{0, 1, p^{-1}, p^{-2}, \ldots \} \text{ and } \psi(n) < \frac{1}{4} \text{ for all } n \in \mathbb{N}. \text{ Then for all } m, n \in \mathbb{N} \text{ with } p \nmid m, n \text{ we have that } \]
\[ \lambda(E_m(\frac{\psi}{2}) \cap E_n(\frac{\psi}{2})) \leq \mu_p(K_m(\psi) \cap K_n(\psi)) \leq \frac{3}{2} \cdot \lambda(E_m(2\psi) \cap E_n(2\psi)). \]
\[ \text{We remark } H_1 \ (\text{[21, Thm. 3 (i)]) is a Pollington-Vaughan type theorem \ ([27, Thm. } \]
\[ 2]), while \( H_2 \ (\text{[21, Lemma 1]) is a Cassels-Gallagher type zero-one law \ ([8, 15]). The } \]
\[ \text{proof of } H_3 \ (\text{[21, lemma 2]) is trivial, while that of } H_4 \ (\text{[21, lemma 3]) is similar to the overlap estimates obtained in [27].} \]
\[ \text{Now suppose the } p\text{-adic version of the Duffin-Schaeffer conjecture is true and let } \psi \text{ be any non-negative function such that } \sum_n \mu_p(K_n(\psi)) = \infty. \text{ In the following we will use the quasi-independence on average method to deduce } \mu_p(W_p(\psi)) = 1. \text{ According to } H_1\sim H_3 \text{ we may further assume without loss of generality that } \psi < \frac{1}{4} \text{ and } \psi(n) = 0 \text{ for all } n \in p\mathbb{N}. \text{ We define} \]
\[ \psi_2(n) = \begin{cases} \psi(n) & (n \in \Delta_h, S_h(\psi) \leq 1) \\ \frac{\psi(n)}{S_h(\psi)} & (n \in \Delta_h, S_h(\psi) > 1). \end{cases} \]
\[ \text{Obviously, } \psi_2 < \frac{1}{4}, \psi_2(n) = 0 \text{ for all } n \in p\mathbb{N}, S_h(\psi_2) \leq 1 \text{ for all } h \in \mathbb{N}, \text{ and} \]
\[ \sum_h S_h(\psi_2) = \infty. \text{ Since we are working in } \mathbb{Z}_p, \text{ it does not change anything on the } p\text{-adic side of things if we round down each of the values taken by the function } n \mapsto \frac{\psi_2(n)}{n} \text{ so that the range of the function } n \mapsto \frac{\psi_2(n)}{n} \text{ is contained in the set } \{0, p^{-1}, p^{-2}, \ldots \}. \text{ Hence by } H_3 \text{ and } H_4 \text{ we have for all } m, n \in \mathbb{N} \text{ that} \]
\[ \lambda(E_m(\frac{\psi_2}{2}) \cap E_n(\frac{\psi_2}{2})) \leq \mu_p(K_m(\psi_2) \cap K_n(\psi_2)) \leq \frac{3}{2} \cdot \lambda(E_m(2\psi_2) \cap E_n(2\psi_2)), \]
\[ \text{which combining Theorem 1.10 gives a universal constant } C > 0 \text{ such that} \]
\[ C S_h(\psi_2) \leq \mu_p(\bigcup_{n \in \Delta_h} K_n(\psi_2)) \leq 3 S_h(\psi_2) \ (h \in \mathbb{N}). \]
As \( \sum h S_h(\psi_2) = \infty \), there exists an integer \( i \in \{0, 1, 2\} \) such that
\[
\sum_{h=1}^{\infty} \mu_p \left( \bigcup_{n \in \Delta_{3h+i}} K_n(\psi_2) \right) = \infty.
\]

So we can apply Lemma 1.15 together with the estimates (2.4), (6.1), (6.2) to get
\[
\mu_p(W_p(\psi_2)) \geq \limsup_{N \to \infty} \left( \sum_{h=1}^{N} \mu_p \left( \bigcup_{n \in \Delta_{3h+i}} K_n(\psi_2) \right) \right)^2
\]
\[
\gg \limsup_{N \to \infty} \frac{\sum_{h=1}^{N} S_{3h+i}(\psi_2) + 2 \sum_{1 \leq h_1 < h_2 \leq N} \sum_{m \in \Delta_{3h_{1+i}}} \sum_{n \in \Delta_{3h_{2+i}}} \lambda(E_m(2\psi_2) \cap E_n(2\psi_2))}{\left( \sum_{h=1}^{N} S_{3h+i}(\psi_2) \right)^2} > 0.
\]

Consequently, we can use H2 to deduce \( \mu_p(W_p(\psi_2)) = 1 \). As \( \psi \leq \psi_2 \), \( \mu_p(W_p(\psi)) = 1 \).

Based on H1~H3, to study the \( p \)-adic version of the Duffin-Schaeffer conjecture one may always assume \( \psi < \frac{1}{4} \) and \( \psi(n) = 0 \) for all \( n \in p\mathbb{N} \). With these assumptions we believe the heart of the conjecture is to establish
\[
(6.3) \quad \mu_p \left( \bigcup_{n \in \Delta_h} K_n(\psi) \right) \asymp S_h(\psi)
\]
whenever \( S_h(\psi) \leq 1 \).

### 7. Diophantine approximation over formal Laurent series

The study of the \( p \)-adic version of the Duffin-Schaeffer conjecture highly resembles that of the classical Duffin-Schaeffer conjecture for at least both problems deal with non-negative functions from \( \mathbb{N} \) to \( \mathbb{R} \). In the formal Laurent series case we will study non-negative functions from \( \mathbb{F}[X] \) to \( \mathbb{R} \).

Throughout this section \( Q \) will always be regarded as a monic polynomial wherever you meet. Recall that \( q \) stands for the size of \( \mathbb{F} \).
7.1. **Proof of Theorem 1.13.** Ahead of proving Lemma 7.1 let us explain the construction of the Haar measure \( \nu \) on \( \mathbb{F}(\mathbb{I}(X)^{-1}) \) in a much straightforward way. Any bijection \( \tau : \mathbb{F} \mapsto \{0, 1, \ldots, q-1\} \) naturally induces a map \( \tilde{\tau} : \mathbb{F}(\mathbb{I}(X)^{-1}) \mapsto \mathbb{R} \) sending \( \sum_i a_i X^i \) to \( \sum_i \tau(a_i)q^i \). A subset \( A \) of \( \mathbb{F}(\mathbb{I}(X)^{-1}) \) is said to be \( \tau \)-measurable if \( \tilde{\tau}(A) \) is Lebesgue measurable in \( \mathbb{R} \). For any \( \tau \)-measurable subset \( A \) of \( \mathbb{F}(\mathbb{I}(X)^{-1}) \), we define its \( \tau \)-measure by \( \nu_\tau(A) = \lambda(\tilde{\tau}(A)) \). For any permutation \( \gamma \) on \( \{0, 1, \ldots, q-1\} \) it is easy to show that \( \tilde{\gamma} : \sum_i b_i q^i \in \mathbb{R} \mapsto \sum_i \gamma(b_i)q^i \in \mathbb{R} \) is measure-preserving on \( (\mathbb{R}, \lambda) \). This implies that the concepts of \( \tau \)-measurable and \( \tau \)-measure are independent of the choices of \( \tau \). So it brings no confusion to write \( \nu_\tau \) simply as \( \nu \). The interested readers may easily verify that \( \nu \) is nothing but the unique Haar measure on \( \mathbb{F}(\mathbb{I}(X)^{-1}) \) such that \( \nu(\mathbb{I}) = 1 \). With this construction it is easy to see for any \( f \in \mathbb{F}(\mathbb{I}(X)^{-1}) \) and any \( r > 0 \) that \( r \leq \nu_B(f, r) \leq qr \).

**Lemma 7.1.** Let \( t \geq 1 \). Then for any non-negative function \( \Psi : \mathbb{F}[X] \to \mathbb{R} \),

\[
\nu_d\left( \bigcup_Q \mathcal{E}_Q(t\Psi)^d \right) \leq q^d t^d \nu_d\left( \bigcup_Q \mathcal{E}_Q(\Psi)^d \right).
\]

**Proof of Lemma 7.1.** The proof of Lemma 7.1 is fully identical to that of Lemma 6.1, but we still provide the details to help the readers get familiar with the language of formal Laurent series. By the inner regular property of the Haar measure \( \nu_d \), we may assume without loss of generality that the support of \( \Psi \) is a non-empty set of finite elements. As any two balls in \( \mathbb{F}(\mathbb{I}(X)^{-1}) \) can only have either empty intersection or one is contained in the other, it is rather easy to see that \( \bigcup_Q \mathcal{E}_Q(\Psi)^d \) is the union of finitely many pairwise disjointly supported \( d \)-dimensional non-empty open cubes of the following form

\[
\bigcup_Q \mathcal{E}_Q(\Psi)^d = \bigcup_{i=1}^M \bigcup_{j=1}^{M_i} \left( \bigcap_{k=1}^d \left( P_{i,j,k} \cdot \Psi(Q_i) \right) \right) \cap \mathbb{I}^d.
\]

and

\[
\bigcup_Q \mathcal{E}_Q(t\Psi)^d = \bigcup_{i=1}^M \bigcup_{j=1}^{M_i} \left( \bigcap_{k=1}^d \left( P_{i,j,k} \cdot t\Psi(Q_i) \right) \right) \cap \mathbb{I}^d.
\]

If there exists a pair \( (i, j) \) such that \( \mathbb{I}^d \subset \bigcap_{k=1}^d B\left(\frac{P_{i,j,k}}{Q_i}, \frac{\Psi(Q_i)}{|Q_i|}\right) \), then \( \bigcup_Q \mathcal{E}_Q(\Psi)^d = \mathbb{I}^d \) and we need to do nothing further. Else by the property \( \blacklozenge \blacklozenge \) we can assume \( \prod_{k=1}^d B\left(\frac{P_{i,j,k}}{Q_i}, \frac{\Psi(Q_i)}{|Q_i|}\right) \subset \mathbb{I}^d \) for all the pairs \( (i, j) \). Consequently, by the property \( \blacklozenge \blacklozenge \) we have

\[
\nu_d\left( \bigcup_Q \mathcal{E}_Q(t\Psi)^d \right) \leq \sum_{i=1}^M \sum_{j=1}^{M_i} \nu_d\left( \bigcap_{k=1}^d B\left(\frac{P_{i,j,k}}{Q_i}, \frac{t\Psi(Q_i)}{|Q_i|}\right) \right)
\]

\[
\leq q^d t^d \sum_{i=1}^M \sum_{j=1}^{M_i} \nu_d\left( \bigcap_{k=1}^d B\left(\frac{P_{i,j,k}}{Q_i}, \frac{\Psi(Q_i)}{|Q_i|}\right) \right) = q^d t^d \nu_d\left( \bigcup_Q \mathcal{E}_Q(\Psi)^d \right)
\]

This finishes the proof of Lemma 7.1. \( \square \)
In the formal Laurent series case we do have the Gallagher type zero-one law ([23, Thm. 1]). Thus in view of Theorem 3.1 and Lemma 7.1, a proof of Theorem 1.13 can be easily obtained by mimicking the proofs of Theorems 1.3 & 3.2.

7.2. Proof of Theorem 1.14.

Lemma 7.2. \( \nu_d(\mathcal{H}^{(d)}(t\Psi)) \in \{0, 1\} \) is independent of \( t > 0 \).

This lemma is a Cassels-Gallagher type zero-one law. In the classical case the author established various zero-one laws ([26, Theorems 3.1, 3.2, 3.3, 4.1 & 4.3]) via the cross fibering principle ([5, Thm. 3]) of Beresnevich, Haynes, Velani as well as a multi-purpose Cassels-Gallagher type theorem ([26, Lemma 2.1]). In the formal Laurent series case the key to the proof of Lemma 7.2 is the following concept and generalization of [26, Lemma 2.1]:

Definition 7.3. For any monic \( Q \in \mathbb{F}[X] \), let \( \omega(Q) \) be a fixed non-empty subset of divisors of \( Q \). For any non-negative function \( \Psi : \mathbb{F}[X] \rightarrow \mathbb{R} \) we denote by \( \mathcal{H}(\omega, \Psi) \) the set of \( f \in \mathbb{L} \) for which \( |Qf - P| < \Psi(Q) \) holds for infinitely many triples \( (Q, P, R) \in \mathbb{F}[X]^3 \) with \( \partial P < \partial Q \) and \( P \) being coprime to some \( R \in \omega(Q) \).

Lemma 7.4. \( \nu(\mathcal{H}(\omega, tM)) \in \{0, 1\} \) is independent of \( t > 0 \).

The proof of Lemma 7.4 is similar to those of [24, Thm. 4] and [26, Lemma 2.1] with suitable modifications, and we leave the details to the interested readers. Now we can give a proof of Lemma 7.2 and will only deal with the case \( d = 2 \) for the sake of simplicity. All the other cases are left to the readers to check in a similar way.

Proof of Lemma 7.2 (\( d = 2 \)): For any \( s, t > 0 \), we denote by \( \mathcal{H}_{s,t}(\Psi) \) the set of \( (f, g) \in \mathbb{L}^2 \) for which

\[
|f - \frac{P_1}{Q}| < \frac{s\Psi(Q)}{|Q|} \quad \text{and} \quad |g - \frac{P_2}{Q}| < \frac{t\Psi(Q)}{|Q|}
\]

for infinitely many triples \( (Q, P_1, P_2) \in \mathbb{F}[X]^3 \) with \( \partial P_1, \partial P_2 < \partial Q \), \( (Q, P_1, P_2) = 1 \). Obviously, \( \nu_2(\mathcal{H}_{s,t}(\Psi)) = \nu_2(\mathcal{H}_{t,s}(\Psi)) \) (\( \star \)). We decompose \( \mathcal{H}_{s,t}(\Psi) \) as disjoint unions

\[
\cup_{g \in \mathbb{L}} \mathcal{H}_g(s\Psi, t\Psi) \times \{g\},
\]

where \( \mathcal{H}_g(s\Psi, t\Psi) \) denotes the set of \( f \in \mathbb{L} \) for which (7.1) holds for infinitely many triples \( (Q, P_1, P_2) \in \mathbb{F}[X]^3 \) with \( \partial P_1, \partial P_2 < \partial Q \), \( (Q, P_1, P_2) = 1 \). For any monic \( Q \in \mathbb{F}[X] \), we denote

\[
\omega_t(Q) = \{Q\} \cup \{(Q, P_2) : P_2 \in \mathbb{F}[X], \partial P_2 < \partial Q, |g - \frac{P_2}{Q}| < \frac{t\Psi(Q)}{|Q|}\}.
\]

By Definition 7.3 it is easy to see that \( \mathcal{H}_g(s\Psi, t\Psi) = \mathcal{H}(\omega_t, s\Psi) \). Hence by Fubini’s theorem, Lemma 7.4 and \( \star \), \( \nu_2(\mathcal{H}_{s,t}(\Psi)) \) is independent of \( s, t > 0 \). On the other hand, we note from Lemma 7.4 that each fiber \( \mathcal{H}_g(s\Psi, t\Psi) \) of \( \mathcal{H}_{s,t}(\Psi) \) with horizontal direction has \( \nu \)-measure either 0 or 1. In a similar way one can show that each fiber of \( \mathcal{H}_{s,t}(\Psi) \) with vertical direction also has \( \nu \)-measure either 0 or 1. Consequently, \( \nu_2(\mathcal{H}_{s,t}(\Psi)) \in \{0, 1\} \) follows the cross fibering principle [5, Thm. 3]. This suffices to finish the proof of Lemma 7.4 as we have \( \mathcal{H}^{(2)}(t\Psi) = \mathcal{H}_{t,t}(\Psi) \).
Lemma 7.5. Suppose $d \geq 2$. Then $\nu_d(\mathcal{H}_Q^{(d)}(\Psi)) \geq \frac{3}{16} \min\{\Psi(Q)^d, 1\}$.

This lemma can be regarded as either a Gallagher type estimate ([16, formula (9)], see also [3]) or a Pollington-Vaughan type estimate (1.4). To give a proof we may assume without loss of generality that $\Psi(Q) < 1$, and suppose this is the case. Thus $\mathcal{H}_Q^{(d)}(\Psi)$ is the union of pairwise disjointly supported $d$-dimensional open cubes of the following form

$$\mathcal{H}_Q^{(d)}(\Psi) = \bigcup_{P_i \in \mathbb{F}[X]} \prod_{i=1}^d B\left(\frac{P_i}{Q}, \frac{\Psi(Q)}{|Q|}\right).$$

By the property $\clubsuit\clubsuit$, we have

$$\nu_d(\mathcal{H}_Q^{(d)}(\Psi)) \geq \left(\frac{\Psi(Q)}{|Q|}\right)^d \cdot \Theta^{(d)}(Q),$$

where $\Theta^{(d)}(Q)$ denotes the size of the set of $P = (P_1, \ldots, P_d) \in \mathbb{F}[X]^d$ such that $\partial P = \max_i \partial P_i < \partial Q$, $(P, Q) = (P_1, P_2, \ldots, P_d, Q) = 1$. With the help of the Möbius function defined by

$$\mu(Q) = \begin{cases} 1, & \partial Q = 0 \\ (-1)^k, & \text{if } Q \text{ is the product of } k \text{ distinct monic irreducible polynomials} \\ 0, & \text{if } Q \text{ is divisible by the square of an irreducible polynomial} \end{cases}$$

and one of its fundamental properties $\sum_{R|Q} \mu(R) = 0$ whenever $\partial Q \geq 1$, where $\sum_{R|Q}$ denotes the sum over all monic divisors $R$ of $Q$, we have

$$\Theta^{(d)}(Q) = \sum_{\partial P < \partial Q} \sum_{R|Q} \mu(R) = \sum_{R|Q} \left(\sum_{P:R|P, \partial P < \partial Q} \mu(R) \cdot \sum_{R|Q} 1\right)$$

$$= \sum_{R|Q} \mu(R) \cdot \left(\frac{|Q|}{|R|}\right)^d = |Q|^d \cdot \sum_{R|Q} \frac{\mu(R)}{|R|^d}.$$ 

Consequently, $\nu_d(\mathcal{H}_Q^{(d)}(\Psi)) \geq \Psi(Q)^d \cdot \sum_{R|Q} \frac{\mu(R)}{|R|^d}$. Now we suppose $d \geq 2$ and have two cases to consider.

Case 1: Suppose $q^{d-1} \geq 3$. In this case we have

$$\sum_{R|Q} \frac{\mu(R)}{|R|^d} \geq 1 - \sum_{k=1}^{\infty} \sum_{R|Q, \partial R = k} \frac{1}{q^k} \geq 1 - \sum_{k=1}^{\infty} \frac{q^k}{q^{kd}} \geq 1 - \sum_{k=1}^{\infty} \frac{q}{q^{kd}} \geq \frac{1}{2}.$$ 

Case 2: Suppose $q = d = 2$. In this case we have

$$\sum_{R|Q} \frac{\mu(R)}{|R|^d} \geq 1 - \frac{2^1}{2^2} - \frac{2^2 - 3}{2^4} - \frac{2^3}{2^6} - \frac{2^4}{2^8} - \cdots = \frac{3}{16},$$

where $2^2 - 3$ comes from the contribution made only by $X^2 + X + 1$ as it is the sole second order element whose Möbius value is negative. This suffices to conclude the proof of Lemma 7.5.
Lemma 7.6. For any \( z_1, z_2 \in \mathbb{Z} \) and any nonzero element \( g \in \mathbb{F}((X^{-1})) \),
\[
\sum_{P \in \mathbb{F}[X]^d \setminus \{0\}} \nu_d(\mathbb{L}_{z_1}^{d} \cap (\mathbb{L}_{z_2}^{d} + gP)) \leq \frac{\nu_d(\mathbb{L}_{z_1}^{d}) \cdot \nu_d(\mathbb{L}_{z_2}^{d})}{|g|^d},
\]
where \( \mathbb{L}_z \) denotes the set of elements of \( \mathbb{F}((X^{-1})) \) with degrees less than \( z \).

Proof. Without loss of generality we may assume that \( z_1 \geq z_2 \). If \( z_1 \leq \partial g \), then we have nothing to prove as the left hand side of the desired inequality is zero. So we can assume \( z_1 > \partial g \). In this case for any \( P \in \mathbb{F}[X]^d \) with \( \mathbb{L}_{z_1}^{d} \cap (\mathbb{L}_{z_2}^{d} + gP) \neq \emptyset \), one must have \( \partial P < z_1 - \partial g \). This implies
\[
\sum_{P \in \mathbb{F}[X]^d \setminus \{0\}} \nu_d(\mathbb{L}_{z_1}^{d} \cap (\mathbb{L}_{z_2}^{d} + gP)) \leq (q^{z_1 - \partial g})^{d} \cdot \nu_d(\mathbb{L}_{z_2}^{d}).
\]
This finishes the proof of Lemma 7.6. \( \square \)

Proof of Theorem 1.14: According to Lemmas 7.2, 7.5 and the first Borel-Cantelli lemma, we may assume without loss of generality that \( \Psi(Q) \in \{q^{-1}, q^{-2}, q^{-3}, \ldots\} \) for all \( Q \in \mathbb{F}[X] \). By Lemma 7.2 and the second Borel-Cantelli lemma it suffices to establish the quasi-independence property for \( \mathcal{H}_Q^{(d)}(\Psi) \) and \( \mathcal{H}_{Q'}^{(d)}(\Psi) \), where \( Q, Q' \) are any two distinct monic elements of \( \mathbb{F}[X] \). To this aim, we first denote \( U(s) = \mathbb{L}_s^{d} \), then rewrite \( \mathcal{H}_Q^{(d)}(\Psi) \) as
\[
\mathcal{H}_Q^{(d)}(\Psi) = \bigcup_{\partial P < \partial Q \atop (P, Q) = 1} (\bigcup(\frac{\Psi(Q)}{|Q|}) + \frac{P}{Q}).
\]
Thus
\[
\nu_d(\mathcal{H}_Q^{(d)}(\Psi) \cap \mathcal{H}_{Q'}^{(d)}(\Psi)) \leq \sum_{\partial P < \partial Q \atop (P, Q) = 1} \sum_{\partial P' < \partial Q'} \nu_d\left(\bigcup\left(\frac{\Psi(Q)}{|Q|}\right) \cap \bigcup\left(\frac{\Psi(Q')}{|Q'|}\right) + \frac{P'}{Q'} - \frac{P}{Q}\right).
\]
It is easy to verify that \( \frac{P'}{Q'} - \frac{P}{Q} \) is always a nonzero element of \( \mathbb{F}((X^{-1}))^d \) whenever \( \partial P < \partial Q, \partial P' < \partial Q', (P, Q) = (P', Q') = 1 \), and if we further have \( \frac{P'}{Q'} - \frac{P}{Q} = \frac{R'}{Q'} - \frac{R}{Q} \), \( \partial R < \partial Q, \partial R' < \partial Q' \), \( (R, Q) = (R', Q') = 1 \), then
\[
\partial(P - R) < \partial Q \quad \text{and} \quad \frac{Q}{(Q, Q')} |(P - R).
\]
These facts imply that every fixed (nonzero) element of the form \( \frac{P'}{Q'} - \frac{P}{Q} \) can be repeated at most \( |(Q, Q')|^d \) times. Consequently, by Lemmas 7.5 and 7.6 we have
\[
\nu_d(\mathcal{H}_Q^{(d)}(\Psi) \cap \mathcal{H}_{Q'}^{(d)}(\Psi)) \leq |(Q, Q')|^d \cdot \frac{\frac{\Psi(Q)}{|Q|}^{d} \cdot \frac{\Psi(Q')}{|Q'|}^{d}}{|Q Q'|^{d}} \cdot \Psi(Q)^d \cdot \Psi(Q')^d \leq \frac{256}{9} \cdot \nu_d(\mathcal{H}_Q^{(d)}(\Psi)) \cdot \nu_d(\mathcal{H}_{Q'}^{(d)}(\Psi)).
\]
This finishes the whole proof of Theorem 1.14.
8. Weighted second Borel-Cantelli lemma

This section is devoted to the proof of Theorem 1.17. First let us explain why Theorem 1.17 is a generalization of the Beresnevich-Harman-Haynes-Velani theorem ([4, Thm. 2]) which claims $\lambda(W(\psi)) = 1$ if for some $c > 0$,

$$\sum_{n=16}^{\infty} \frac{\varphi(n)\psi(n)}{n \exp(c(\log \log n)(\log \log \log n))} = \infty.$$  \hspace{1cm} (8.1)

Suppose (8.1) is true. Obviously, there exists a $c_1 > 0$ such that (see the second section for the meanings of $\Delta_h, S_h, B_h, Q_h, R_h$)

$$\sum_{h \in \mathbb{N}} \frac{S_h}{\exp(c_1 h \log h)} = \infty,$$

from which we can deduce a $c_2 > 0$ and an infinite subset $H$ of $\mathbb{N}$ such that for all $h \in H$, $S_h \geq \exp(c_2 h \log h)$. In fact, we can define $h \in H$ by

$$\frac{S_h}{\exp(c_1 h \log h)} \geq \frac{1}{h^2}.$$

Thus $\lambda(W(\psi)) = 1$ follows easily from Theorem 1.17.

Next, let us explain how will we make use of the weighted version Lemma 1.16 in the proof of Theorem 1.17. According to the discussions in the last paragraph of the second section, we can assume $P(m, n) \ll 1$ for any $m, n$ lying in any corresponding distinct blocks $\Delta_{h_1}, \Delta_{h_2}$. By appealing to a theorem of Pollington and Vaughan ([27, Theorem 2]), we may also assume without loss of generality that $\psi(n) \leq 1/2$ for all $n \in \mathbb{N}$. This means $\lambda(E_n) = 2^{\psi(n)\varphi(n)}$ for all $n \in \mathbb{N}$. Fix arbitrarily $\omega_{\Delta_h} \in [0, 1]$ such that $\sum_h \omega_{\Delta_h} S_h = \infty$. With $\omega_n \triangleq \omega_{\Delta_h}$ ($n \in \Delta_h$) and $A_n \triangleq E_n$ ($n \geq 5$) we can apply Lemma 1.16 to get

$$\lambda(W(\psi)) \gg \limsup_{N \to \infty} \frac{\left( \sum_{h=1}^{N} \omega_{\Delta_h} S_h \right)^2}{\sum_{h=1}^{N} \omega_{\Delta_h}^2 S_h + \sum_{h=1}^{N} \omega_{\Delta_h}^2 Q_h + \left( \sum_{h=1}^{N} \omega_{\Delta_h} S_h \right)^2}.$$  \hspace{1cm} (8.2)

Thus if one can show that

$$\sum_{h=1}^{N} \omega_{\Delta_h}^2 Q_h \ll \left( \sum_{h=1}^{N} \omega_{\Delta_h} S_h \right)^2$$  \hspace{1cm} (8.3)

for infinitely many $N$, then $\lambda(W(\psi)) = 1$ follows from Gallagher’s zero-one law.

**Proof of Theorem 1.17:** Suppose we have (1.21), which is equivalent to

$$\sum_{h: S_h \geq e^x} \frac{\log S_h}{h \cdot \log \log S_h} = \infty.$$  \hspace{1cm} (8.4)
Our purpose is to show that $\lambda(W(\psi)) = 1$. If there exist infinitely many $h \in \mathbb{N}$ such that $S_h \geq \exp(h \log h)$, then we can apply the aforementioned Beresnevich-Harman-Haynes-Velani theorem to deduce $\lambda(W(\psi)) = 1$. Thus we can assume without loss of generality that

\begin{equation}
S_h \leq \exp(h \log h) \quad (h \in \mathbb{N}).
\end{equation}

For any $h \in \mathbb{N}$ with $S_h \geq e^e$, let $D^{(h)}_j$ $(j \geq -1)$ denote the collection of pairs $(m, n) \in \Delta_h \times \Delta_h$, $m \neq n$, such that $e^e \leq D(m, n) < e^{j+1}$. Denote

$$A^{(h)}_j = A^{(h)}_j(\psi) = \sum_{(m, n) \in D^{(h)}_j} \psi(m)\psi(n)\frac{\varphi(m)}{m} \frac{\varphi(n)}{n}.$$ 

Define two functions $f^{(h)}$, $g^{(h)}$ on the set of integers $\mathbb{Z}$ respectively by

$$f^{(h)}(j) = \begin{cases} A^{(h)}_j & (j \geq -1) \\ 0 & (j < -1), \end{cases}$$

$$g^{(h)}(j) = \begin{cases} \frac{h}{j+2} & (-\frac{\log S_h}{\log \log S_h} \leq j \leq 1) \\ 0 & (j > 1 \text{ or } j < -\frac{\log S_h}{\log \log S_h}). \end{cases}$$

The convolution of $f^{(h)}$ and $g^{(h)}$ is defined usually as

$$(f^{(h)} * g^{(h)})(k) = \sum_{j \in \mathbb{Z}} f^{(h)}(j)g^{(h)}(k-j) \quad (k \in \mathbb{Z}).$$

For simplicity we denote $y_h = \frac{\log S_h}{\log \log S_h}$. Note it is easy to deduce from $e^e \leq S_h \leq \exp(h \log h)$ that $y_h \leq h$. With these preparations we have

$$\sum_{k: 0 \leq k \leq \log S_h} \sum_{j=k-1}^{k+y_h} \frac{h}{j+2} A^{(h)}_j \leq \|f^{(h)} * g^{(h)}\|_{l^1(\mathbb{Z})} \leq \|f^{(h)}\|_{l^1(\mathbb{Z})} \cdot \|g^{(h)}\|_{l^1(\mathbb{Z})} \ll S_h^2 \cdot h \cdot \log \log S_h.$$ 

Thus there exists a non-negative integer

\begin{equation}
k_h \leq \log S_h
\end{equation}

such that

$$\sum_{j=k_h-1}^{k_h+y_h} \frac{h}{j+2} A^{(h)}_j \ll \frac{S_h^2 \cdot h \cdot \log \log S_h}{\log S_h},$$

which is equivalent to

\begin{equation}
\sum_{j=k_h-1}^{k_h+y_h} \frac{h}{j+2} A^{(h)}_j \left(\frac{\psi}{e^{k_h}}\right) \ll S_h \left(\frac{\psi}{e^{k_h}}\right)^2 \cdot \frac{h \cdot \log \log S_h}{\log S_h}. \tag{8.7}
\end{equation}

On the other hand,

\begin{equation}
\sum_{j \geq k_h+y_h+1} \frac{h}{j+2} A^{(h)}_j \left(\frac{\psi}{e^{k_h}}\right) \ll \frac{h}{y_h} \cdot S_h \left(\frac{\psi}{e^{k_h}}\right)^2 = S_h \left(\frac{\psi}{e^{k_h}}\right)^2 \cdot \frac{h \cdot \log \log S_h}{\log S_h}. \tag{8.8}
\end{equation}
Combining (8.7) and (8.8) gives
\[
\sum_{j \geq kh - 1} h \frac{A_j^{(h)}(\psi)}{e^{kh}} \leq S_h\left(\frac{\psi}{e^{kh}}\right)^2 \cdot \frac{h \cdot \log \log S_h}{\log S_h},
\]
which followed by applying (2.2) and (2.3) gives
\[
(8.9) \quad R_h\left(\frac{\psi}{e^{kh}}\right) \ll \frac{h \cdot \log \log S_h}{\log S_h}.
\]
We now define
\[
\psi(n) = \begin{cases} \frac{\psi(n)}{e^{kh}} & (n \in \Delta_h, S_h \geq e^\epsilon) \\ 0 & \text{(otherwise)}, \end{cases}
\]
\[
\omega_{\Delta_h} = \begin{cases} \frac{1}{S_h(\psi)} & (S_h \geq e^\epsilon) \\ 0 & \text{(otherwise)}. \end{cases}
\]
By (8.4)~(8.6) we see that \(\omega_{\Delta_h} \leq 1\) and \(\sum_h \omega_{\Delta_h} S_h(\overline{\psi}) = \infty\). By (8.9) we have
\[
(8.10) \quad \sum_{(N)} \omega_{\Delta_h}^2 Q_h(\overline{\psi}) \ll \sum_{(N)} \frac{\log S_h}{h \cdot \log \log S_h} \ll \left(\sum_{(N)} \frac{\log S_h}{h \cdot \log \log S_h}\right)^2 = \left(\sum_{(N)} \omega_{\Delta_h} S_h(\overline{\psi})\right)^2
\]
for large enough \(N\), where \(\sum_{(N)}\) denotes the sum over all \(h \in [1, N]\) with \(S_h \geq e^\epsilon\). It follows from the definition of \(\omega_{\Delta_h}\) and (8.10) that for large enough \(N\),
\[
\sum_{h=1}^{N} \omega_{\Delta_h}^2 \cdot Q_h(\overline{\psi}) \ll \left(\sum_{h=1}^{N} \omega_{\Delta_h} S_h(\overline{\psi})\right)^2,
\]
which verifies (8.3) for the function \(\overline{\psi}\). So we get \(\lambda(W(\overline{\psi})) = 1\). Since \(\overline{\psi} \leq \psi\), we immediately have \(\lambda(W(\psi)) = 1\). This finishes the whole proof of Theorem 1.17.

**Remark 8.1.** We remark that one can also use the weighted second Borel-Cantelli lemma to reprove a theorem of the author ([25]) claiming \(\lambda(W(\psi)) = 1\) if
\[
\sum_{n=1}^{\infty} \psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty
\]
for some \(\epsilon > 0\), where \(\psi\) is any prescribed bounded non-negative function. Obviously, we may assume \(\epsilon < 0.5\). By [18, Lemma 5] there exists a sequence of distinct integers \(\{n_k\}\) such that \(\psi(n_k)^\epsilon \cdot \lambda(E_{n_k}(\psi)) < 1/k\) for all \(k \in \mathbb{N}\) and
\[
\sum_{k=1}^{\infty} \psi(n_k)^\epsilon \cdot \lambda(E_{n_k}(\psi)) = \infty.
\]
We can first apply the weighted second Borel-Cantelli lemma with \(\omega_k \triangleq \psi(n_k)^\epsilon\) and \(A_k \triangleq E_{n_k}(\psi)\) to give a lower bound for \(\lambda(W(\psi))\), then mimic Harman’s proof of [18, Thm. 1] to deduce the positiveness of this lower bound. Finally by Gallagher’s zero-one law, we are done. The details are left to the interested readers to verify.
9. Further questions

Based on Theorem 1.3, we know that if the classical Duffin-Schaeffer conjecture is true, then there exists a universal constant $C > 0$ such that

$$\lambda(\bigcup_{n=1}^{\infty} E_n(\psi)) \geq C \sum_{n=1}^{\infty} \lambda(E_n(\psi))$$

for any non-negative function $\psi : \mathbb{N} \to \mathbb{R}$ with $\sum_{n=1}^{\infty} \lambda(E_n(\psi)) \leq 1$. This lower bound may not well reflect the true value of $\lambda(\bigcup_{n=1}^{\infty} E_n(\psi))$ when $\sum_{n=1}^{\infty} \lambda(E_n(\psi))$ is sufficiently large, so we are bold enough to propose

**Conjecture 9.1.** There exists a universal constant $M_d > 0$ depending only on $d \in \mathbb{N}$ such that if $\sum_{n=1}^{\infty} \lambda_d(E_n(\psi)^d) \geq M_d$, then $\lambda_d(\bigcup_{n=1}^{\infty} E_n(\psi)^d) = 1$.

Conjecture 9.1 implies the classical Duffin-Schaeffer and Sprindžuk conjectures. Taking for granted that Conjecture 9.1 is true, we can understand the quantitative theory pioneered by Schmidt ([29]) even better. Let $\psi : \mathbb{N} \to \mathbb{R}$ be any non-negative function such that $\sum_{n=1}^{\infty} \lambda_d(E_n(\psi)^d) = \infty$, and let $0 = N_0 < N_1 < N_2 < N_3 < \cdots$ be the unique sequence of integers such that for all non-negative integers $k$,

$$\sum_{n=N_k+1}^{N_{k+1}-1} \lambda_d(E_n(\psi)^d) < M_d \leq \sum_{n=N_k+1}^{N_{k+1}} \lambda_d(E_n(\psi)^d) < M_d + 1.$$

Note for any $N \in \mathbb{N}$, there exists a unique $k \in \mathbb{N} \cup \{0\}$ such that $N_k < N \leq N_{k+1}$, from which we can easily deduce

$$kM_d \leq \sum_{n=1}^{N} \lambda_d(E_n(\psi)^d) \leq (k + 1)(M_d + 1).$$

By the assumed truth of Conjecture 9.1 we have for almost all $\mathbf{x} \in (\mathbb{R}/\mathbb{Z})^d$,

$$M(N, \mathbf{x}) \triangleq \#\{n \in \mathbb{N} : n \leq N_k, \; \mathbf{x} \in E_n(\psi)^d\} \geq M(N_k, \mathbf{x}) \geq k$$

$$\geq \frac{\sum_{n=1}^{N} \lambda_d(E_n(\psi)^d)}{M_d + 1} - 1.$$

Note also

$$\int_{(\mathbb{R}/\mathbb{Z})^d} M(N, \cdot) = \sum_{n=1}^{N} \lambda_d(E_n(\psi)^d).$$

Consequently, the above lower bound is best possible despite some loss of constant.

Conjecture 9.1 might be too strong to hold, so we instead propose a weaker version.

**Conjecture 9.2.** There exists a universal constant $M_{d, \gamma} > 0$ depending only on $d \in \mathbb{N}$ and $\gamma \in (0, 1)$ such that if $\sum_{n=1}^{\infty} \lambda_d(E_n(\psi)^d) \geq M_{d, \gamma}$, then $\lambda_d(\bigcup_{n=1}^{\infty} E_n(\psi)^d) \geq \gamma$. 
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