On the gravitational entropy of accelerating black holes

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Abstract

In this paper we have examined the validity of a proposed definition of gravitational entropy in the context of accelerating black hole solutions of the Einstein field equations, which represent the realistic black hole solutions. We have adopted a phenomenological approach proposed in [20] and expanded in [21], in which the Weyl curvature hypothesis is tested against the expressions for the gravitational entropy. Considering the $C$-metric for the accelerating black holes, we have evaluated the gravitational entropy and the corresponding entropy density for four different types of black holes, namely, non-rotating black hole, non-rotating charged black hole, rotating black hole and rotating charged black hole. We end up by discussing the merits of such an analysis and the possible reason of failure in the particular case of rotating charged black hole and comment on the possible resolution of the problem.

KEYWORDS: Gravitational entropy, Accelerating Black holes

I. INTRODUCTION

The $C$-metric was independently discovered by Levi-Civita [1] and Weyl [2] in 1917. Ehlers and Kundt [3] while working on the classification of the degenerated static vacuum fields, constructed a table in which this metric was placed in the slot “$C$”, leading to the name ‘$C$-metric’. Kinnersley and Walker [4] pointed out that this metric is an exact solution of Einstein’s equations which describes the combined electromagnetic and gravitational field of a uniformly accelerating object having mass $m$ and charge $e$, and is an example of “almost everything”. It is for this reason that the $C$-metric is the focus of our attention in this paper.

Dray and Walker [5] showed that this spacetime represents the gravitational field of a pair of uniformly accelerating black holes. Letelier and Oliveira [6], studied the static and stationary $C$-metric and sought its interpretation in details, in particular those cases charaterized by two event horizons, one for the black hole and another for the acceleration. For spacetimes with vanishing or positive cosmological constant, the $C$-metric represents two accelerated black holes in asymptotically flat or de Sitter (dS) spacetime, and for a negative $\Lambda$ term, depending on the magnitude of acceleration [7], it may represent a single accelerated black hole or a pair of causally separated black holes which accelerate away from each other [8]. The acceleration $A$ is due to forces represented by conical singularities arising out of a strut between the two black holes or because of two semi-infinite strings connecting them to infinity [9, 10].

The second law of thermodynamics is one of the most fundamental laws of physics. We know that for an ensemble of ideal gas molecules confined to a closed chamber, the gas spreads out to fill the entire space once the chamber is opened, thereby reaching a state of maximum entropy. However, in the case of the universe with its matter content modelled as a fluid (or gas), this is not exactly true. The universe was born from a very homogeneous state and later on, small density fluctuations appeared due to the effect of gravity, that ultimately led to the formation of structures in the universe. This evolution is contrary to our expectations from the thermodynamic point of view, since the “gas” condenses into clumps of matter, instead of spreading out. Moreover in the past, the universe was much hotter and at some point of time, matter and radiation were in thermal equilibrium, and the entropy was maximum. So, how can the entropy increase if it was maximum in the past? It appears that if the evolution of the universe is dominated solely by gravity, then we may encounter a violation of the second law of thermodynamics, if we are considering the contribution of the thermodynamic entropy only.

To resolve this problem and to provide a proper sequence to the occurrence of gravitational processes, Penrose [11] proposed that we must assign an entropy function to the gravitational field itself. He suggested that the Weyl curvature tensor could be used as a measure of the gravitational entropy. The Weyl tensor $C_{\alpha\beta\gamma\delta}$ in $n$ dimensions is expressed as [12]

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{(n-2)}(g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\delta} R_{\alpha\gamma} - g_{\alpha\delta} R_{\beta\gamma} + g_{\alpha\gamma} R_{\beta\delta}) + \frac{1}{(n-1)(n-2)} R(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}),$$

where $R_{\alpha\beta\gamma\delta}$ is the covariant Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor and $R$ is the Ricciscalar.

According to Penrose, initially after the ‘big bang’, when the universe started evolving, the Weyl tensor component was much smaller than the Ricci tensor component of the spacetime curvature. This hypothesis sounds credible because
the Weyl tensor is independent of the local energy-momentum tensor. Moreover, the universe was in a nearly homogeneous state before structure formation began, and the FRW models successfully describe this homogeneous phase of the evolution. Further, the Weyl curvature is zero in the FRW models. However, the Weyl is large in the Schwarzschild spacetime. Thus we need a description of gravitational entropy, which should increase throughout the history of the universe on account of formation of more and more structures leading to the growth of inhomogeneity [13, 14], and thus preserve the second law of thermodynamics. But there is still doubt regarding the definition of gravitational entropy in a way analogous to the thermodynamic entropy, which would be applicable to all gravitational systems [15]. The definition of gravitational entropy as the ratio of the Weyl curvature and the Ricci curvature faces problems with radiation [16]. Once Senovilla showed that the Bel-Robinson tensor is suitable for constructing a measure of the “energy” of the gravitational field [17], several attempts were made to define the gravitational entropy based on the Bel-Robinson tensor and also in terms of the Riemann tensor and its covariant derivatives [18, 19].

Many efforts has been made to explain the entropy of black holes using the quantized theories of gravity, such as the string theory and loop quantum gravity. However, in this paper we will handle the problem from a phenomenological approach proposed in [20] and expanded in [21], in which the Weyl curvature hypothesis is tested against the expressions for the entropy of cosmological models and black holes. They considered a measure of gravitational entropy in terms of a scalar derived from the contraction of the Weyl tensor and the Ricci tensor, and matched it with the Bekenstein-Hawking entropy [22, 23]. In our current work we will consider the accelerating black holes only, which represent more realistic black holes. We will investigate whether the calculations for gravitational entropy proposed in [20] and [21] can be applied in this context. The structure of our paper is as follows: Sec. II deals with the definition of gravitational entropy and Sec. III enlists the metrics of accelerating black holes considered by us. Sec. IV provides the main analysis of our paper where we evaluate the gravitational entropy and the corresponding entropy density for these black holes. We discuss our results in Sec. V and present the conclusions in Sec. VI.

II. GRAVITATIONAL ENTROPY

The entropy of a black hole can be described by the surface integral [20]

\[ S_\sigma = k_s \int_\sigma \Psi \cdot d\sigma, \]  

(2)

where \( \sigma \) is the surface of the horizon of the black hole and the vector field \( \Psi \) is given by

\[ \Psi = P e_r, \]  

(3)

with \( e_r \) as a unit radial vector. The scalar \( P \) is defined in terms of the Weyl scalar \( (W) \) and the Krestchmann scalar \( (K) \) in the form

\[ P^2 = \frac{W}{K} = \frac{C_{abcd}C^{abcd}}{R_{abcd}R^{abcd}}. \]  

(4)

In order to find the gravitational entropy, we need to do our computations in a 3-space. Therefore, we consider the spatial metric which is defined as

\[ h_{ij} = g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}}, \]  

(5)

where \( g_{\mu\nu} \) is the concerned 4-dimensional space-time metric and the Latin indices denote spatial components, \( i, j = 1, 2, 3 \). So the infinitesimal surface element is given by

\[ d\sigma = \sqrt{h} d\theta d\phi. \]  

(6)

Using Gauss’s divergence theorem, we can easily find out the entropy density [20] as

\[ s = k_s |\nabla \cdot \Psi|. \]  

(7)
III. ACCELERATING BLACK HOLES

A. Non-rotating black hole

The C-metric in spherical type coordinates is given by

\[
 ds^2 = \frac{1}{(1 - \alpha \cos \theta)^2} \left( -Q dt^2 + \frac{dr^2}{Q} + \frac{r^2 d\theta^2}{P} + Pr^2 \sin^2 \theta d\phi^2 \right),
\]

(8)

where \( P = (1 - 2\alpha m \cos \theta) \), and \( Q = \left( 1 - \frac{2m}{r} \right) (1 - \alpha^2 r^2) \). This metric represents an accelerating massive black hole.

It is quite clear from above equation that this metric has two coordinate singularities: one is at \( r_a = \frac{1}{\alpha} \), and the other is at \( r_h = 2m \). The \( r_h = 2m \) singularity stands for the familiar event horizon, but the \( r_a = \frac{1}{\alpha} \) singularity is the acceleration horizon formed due to the acceleration of the black hole [24].

B. Non-rotating charged black hole

The charged C-metric in spherical type coordinate is [24]

\[
 ds^2 = \frac{1}{(1 - \alpha \cos \theta)^2} \left( -Q dt^2 + \frac{dr^2}{Q} + \frac{r^2 d\theta^2}{P} + Pr^2 \sin^2 \theta d\phi^2 \right),
\]

(9)

where \( P = (1 - 2\alpha m \cos \theta + \alpha^2 e^2 \cos^2 \theta) \), and \( Q = \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right) (1 - \alpha^2 r^2) \).

C. Rotating black hole

The general line element for a rotating black hole is given by

\[
 ds^2 = \frac{1}{\Omega^2} \left( -\frac{Q}{R} (dt - a \sin^2 \theta d\phi)^2 + \frac{R}{Q} dr^2 + \frac{P}{R} d\theta^2 + \frac{P}{R} \sin^2 \theta [adt - (r^2 + a^2) d\phi]^2 \right),
\]

(10)

where \( \Omega = 1 - \alpha \cos \theta \), \( R = r^2 + a^2 \cos^2 \theta \), \( P = (1 - 2\alpha m \cos \theta + \alpha^2 a^2 \cos^2 \theta) \), and \( Q = (a^2 - 2mr + r^2)(1 - \alpha^2 r^2) \). This metric represents the rotating version of the previous metric, and contains three coordinate singularities, namely \( r_{\pm} = m \pm \sqrt{m^2 - a^2} \), representing the outer and inner horizons, and \( r = \frac{1}{\alpha} \) representing the acceleration horizon [24].

D. Rotating charged black hole

This is the charged version of the previous rotating metric. It may be regarded as the most general case among all the black holes considered by us, and is given by

\[
 ds^2 = \frac{1}{\Omega^2} \left( -\frac{Q}{R} (dt - a \sin^2 \theta d\phi)^2 + \frac{R}{Q} dr^2 + \frac{P}{R} d\theta^2 + \frac{P}{R} \sin^2 \theta [adt - (r^2 + a^2) d\phi]^2 \right),
\]

(11)

where \( \Omega = 1 - \alpha \cos \theta \), \( R = r^2 + a^2 \cos^2 \theta \), \( P = (1 - 2\alpha m \cos \theta + \alpha^2 (a^2 + e^2) \cos^2 \theta) \), and \( Q = (a^2 + e^2 - 2mr + r^2)(1 - \alpha^2 r^2) \).

IV. ANALYSIS

A. Non-rotating black hole

We know that the Kretschmann scalar for a given spacetime geometry is defined by the relation

\[
 K = R_{abcd} R^{abcd},
\]

(12)
where the $R_{abcd}$ is the covariant Riemann curvature tensor. For the $C$-metric (8), the Kretschmann scalar turns out to be

$$K_c = \frac{48 m^2 (\alpha r \cos \theta - 1)^6}{r^6}. \quad (13)$$

The Weyl scalar is defined by

$$W = C_{abcd} C^{abcd}, \quad (14)$$

where the $C_{abcd}$ is the Weyl curvature tensor. For the $C$-metric (non-rotating black hole) the Weyl scalar is evaluated as

$$W_c = \frac{48 m^2 (\alpha r \cos \theta - 1)^6}{r^6}. \quad (15)$$

The scalar function $P$ is defined by the relation (4) as

$$P^2 = C_{abcd} C^{abcd} R_{abcd} R^{abcd}. \quad (16)$$

For this $C$-metric, we observe that $P^2 = 1$. Therefore we assume that $P = +1$ for our entropy calculations, since the entropy must be non-negative.

Now the spatial section corresponding to this metric is

$$h_{ij} = \text{diag} \left[ \frac{1}{(1 - \alpha r \cos \theta)^2 (1 - 2m/r)(1 - \alpha^2 r^2)}, \frac{r^2}{((1 - \alpha r \cos \theta)^2 (1 - 2m \alpha \cos \theta))}, \frac{r^2 \sin^2 \theta (1 - 2m \alpha \cos \theta)}{(1 - \alpha r \cos \theta)^2} \right], \quad (17)$$

with the determinant given by

$$h = \frac{\sin^2 \theta r^5}{(\alpha^2 r^2 - 1)(-r + 2m)(\alpha r \cos \theta - 1)^6}. \quad (18)$$

Therefore, the infinitesimal surface element has the form

$$d\sigma = \frac{\sqrt{h}}{\sqrt{|h|}} \sqrt{h} \, d\theta d\phi = \frac{r^2 \sin \theta}{(\alpha r \cos \theta - 1)^2} d\theta d\phi. \quad (19)$$

We are now in a position to calculate the magnitude of the gravitational entropy on the event horizon $H_0$ at the location $r_h = 2m$ for this metric, which is

$$S_{\text{grav}} = k_s r_h^2 \int_{\theta = 0}^{\pi} \frac{\sin \theta}{\alpha r_h \cos \theta - 1)^2} d\theta \int_{\phi = 0}^{2\pi C} d\phi = k_s \frac{4\pi C r_h^2}{(1 - r_h^2 \alpha^2)(1 + 2\alpha m)} = k_s \frac{4\pi r_h^2}{(1 - r_h^2 \alpha^2)(1 + 2\alpha m)}. \quad (20)$$

From equation (20) it is evident that the gravitational entropy is proportional to the area of the event horizon of the black hole, as in the case of the Bekenstein-Hawking entropy [22, 23]. Here $C = \frac{1}{(1 + 2\alpha m)}$ is the deficiency factor in the limit of $\phi$ as it runs from $0 \to 2\pi$. If the acceleration of the black hole vanishes, i.e., $\alpha = 0$, then the deficiency factor $C$ becomes unity and $\phi$ reduces to the conventional polar coordinate running from $0 \to \pi$. In FIG. 1, we have shown the variation of the total entropy on the horizon with the acceleration parameter $\alpha$.

Similarly we can compute the entropy density as

$$s = k_s \frac{\partial}{\partial r} \left( \sqrt{h} \frac{P}{\sqrt{|h|}} \right) = \frac{2k_s}{r} \sqrt{(1 - \alpha^2 r^2) \left( 1 - \frac{r_h}{r} \right)}. \quad (21)$$

In the above equation (21), inserting $\alpha = 0$, we get the entropy density for the Schwarzschild black hole. In FIG. 2, the dependence of the gravitational entropy density corresponding to this metric on other relevant parameters have been indicated.
FIG. 1: Plot showing the variation of the total gravitational entropy for the accelerating non-rotating BH with respect to the acceleration parameter $\alpha$, where we have taken $m = 1$ and $k_s = 1$.

FIG. 2: (a) Plot showing the variation of the gravitational entropy density for an accelerating non-rotating BH with respect to the acceleration parameter $\alpha$ and the radial coordinate $r$, for $m = 1$ and $k_s = 1$. (b) Plot showing the variation of the gravitational entropy density for the accelerating non-rotating BH with respect to the radial coordinate $r$, where $\alpha = 0.25$, $m = 1$, and $k_s = 1$.

B. Non-rotating charged black hole

The Kretschmann scalar for the non-rotating charged black hole given by the metric (9) is evaluated to be

$$K = \frac{56 (\alpha \cos \theta - 1)^6 \left( \cos^2 \theta \alpha^2 e^4 r^2 + \frac{10}{3} \left( e^2 - \frac{6}{5} m r \right) r e^2 \alpha \cos \theta + e^4 - \frac{12}{3} e^2 m r + \frac{6}{3} m^2 r^2 \right)}{r^8},$$

(22)

and the corresponding Weyl scalar is

$$W = \frac{4}{3} \left( \alpha \cos \theta - 1 \right)^4 \left( 5 \cos^2 \theta \alpha^2 e^2 r^2 - \sin^2 \theta \alpha^2 e^2 r^2 + \alpha^2 e^2 r^2 - 6 \alpha \cos \theta r^2 - 6 e^2 + 6 m r \right)^2.$$ 

(23)
Therefore the quantity \( P \) is given by the expression

\[
P^2 = \frac{6(e^2 \alpha \cos \theta + e^2 - mr)^2}{(7 \cos^2 \theta \alpha^2 e^4 r^2 + 10 \alpha \cos \theta e^4 - 12 \alpha \cos \theta e^2 m + 7 e^4 - 12 e^2 mr + 6 m^2 r^2)}.
\]  

(24)

The spatial metric for this case is

\[
h_{ij} = \frac{1}{(1 - \alpha \cos \theta)^2} \text{diag} \left[ \frac{1}{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)(-\alpha^2 r^2 + 1)}, \frac{r^2}{\beta}, \frac{r^2}{\beta} \sin^2 \theta \right],
\]  

where

\[
\beta = (1 - 2 \alpha \cos \theta + \alpha^2 e^2 \cos^2 \theta).
\]

(25)

Consequently the determinant of the spatial \( h_{ij} \) metric is given by

\[
h = -\frac{\sin^2 \theta \, e^6}{(\alpha^2 r^2 - 1)(e^2 - 2mr + r^2)(\alpha \cos \theta - 1) \, e^5},
\]  

(26)

and the infinitesimal surface element is

\[
d\sigma = \sqrt{h} \, d\theta d\phi = \frac{r^2 \sin \theta}{(\alpha \cos \theta - 1)^2} d\theta d\phi.
\]  

(27)

Next we calculate the gravitational entropy on the horizon \( H_0 \) at \( r_h = r_\pm = m \pm \sqrt{m^2 - e^2} \), which turns out to be

\[
S_{grav} = \kappa_s r_h^2 \int_0^{\alpha_r} \frac{P(r_h, \theta) \sin \theta}{(\alpha \cos \theta - 1)^2} d\theta \int_{\phi = 0}^{2\pi C} d\phi = \frac{\kappa_s (4\pi r_\pm^2)}{(1 + 2\alpha m)} \int_\theta \frac{P_\pm(\theta) \sin \theta d\theta}{2(\alpha \cos \theta - 1)^2},
\]  

(28)

where the quantity \( P_\pm \) corresponds to the value calculated for \( r_\pm \).

From equation (28) we find that the gravitational entropy is proportional to the area of the event horizon of the black hole, just as in the case of the Bekenstein-Hawking entropy. We can further check the validity of our result by setting \( \alpha = 0 \) in (28), to see whether it leads us to the desired expression for the entropy of the Reissner-Nordstrom (RN) black hole. This exercise yields the result

\[
S_{grav}^{RN} = \kappa_s (4\pi r_{\pm}^2) \int_\theta \frac{P_{\pm}^{RN}(\theta) \sin \theta}{2} d\theta.
\]  

(29)

We can easily see that \( P_\pm^{RN}(\theta) = P_\pm(\alpha = 0) = \frac{6e^4 - 12e^2 mr + 6m^2 r^2}{7e^4 - 12e^2 mr + 6m^2 r^2} \), and therefore the gravitational entropy for the RN black hole is

\[
S_{grav}^{RN} = \kappa_s (4\pi r_{\pm}^2) \sqrt{\frac{6e^4 - 12e^2 mr + 6m^2 r^2}{7e^4 - 12e^2 mr + 6m^2 r^2}} \int_\theta \frac{\sin \theta}{2} d\theta = \kappa_s (4\pi r_{\pm}^2) \sqrt{\frac{6e^4 - 12e^2 mr + 6m^2 r^2}{7e^4 - 12e^2 mr + 6m^2 r^2}}.
\]  

(30)

The entropy density for the non-rotating charged black hole is obtained as

\[
s = \frac{16\sqrt{6} \kappa_s \sqrt{(-\alpha^2 r^2 + 1)(e^2 - 2mr + r^2)} \left(7e^4 \alpha^2 \cos^2 \theta r^2 + 10 \left(e^2 - \frac{6mr}{5}\right) r e^2 \cos \theta + 7e^4 - 12e^2 mr + 6m^2 r^2\right)^{3/2}}{r^2 \left[\cos^3 \theta \alpha^3 e^6 r^3 + \frac{15e^4 \alpha^2 \cos^2 \theta r^2}{8} + \frac{9 e^2 - 13mr}{10} + \frac{3e^2 - 12mr}{6} + m^2 r^2\right]}
\]  

\[
\times \left[7e^6 - \frac{3mr}{4} \left(\frac{13e^4}{4} - 3e^2 mr + m^2 r^2\right)\right].
\]  

(31)

If in this expression we substitute \( e = 0 \), and consider the absolute value of this quantity, then we get back the expression (21) for the entropy density of the accelerating black hole. In the FIG. 3 we have shown the dependence of the gravitational entropy density of the non-rotating charged black hole on different parameters appearing in (31).
FIG. 3: (a) Plot showing the variation of the gravitational entropy density for the accelerating non-rotating charged BH with respect to the acceleration parameter $\alpha$ and the radial coordinate $r$, where $m = 1$, $k_s = 1$, $e = 0.5$, and $\theta = \frac{\pi}{2}$. (b) Plot showing the variation of the gravitational entropy density for the accelerating non-rotating charged BH with respect to the radial coordinate $r$ and the charge $e$, where $\alpha = 0.45$, $m = 1$, $k_s = 1$, and $\theta = \frac{\pi}{2}$.

C. Rotating black hole

For the rotating black hole metric, the Kretschmann scalar is

$$K = 48m^2 \left(\alpha r \cos(\theta) - 1\right)^6 \left((a^4 \alpha + a^3) \cos^3 \theta + 3a^2 r (a \alpha - 1) \cos^2 \theta - 3ar^2 (a \alpha + 1) \cos \theta - r^3 (a \alpha - 1)\right)$$

$$\times \left((a^4 \alpha - a^3) \cos^3 \theta - 3a^2 r (a \alpha + 1) \cos^2 \theta - 3ar^2 (a \alpha - 1) \cos \theta + r^3 (a \alpha + 1)\right)$$

$$\times \frac{1}{(r^2 + a^2 \cos^2 \theta)^6},$$

(32)

and the Weyl scalar is

$$W = 48m^2 \left(\alpha r \cos(\theta) - 1\right)^6 \left((a^4 \alpha + a^3) \cos^3 \theta + 3a^2 r (a \alpha - 1) \cos^2 \theta - 3ar^2 (a \alpha + 1) \cos \theta - r^3 (a \alpha - 1)\right)$$

$$\times \left((a^4 \alpha - a^3) \cos^3 \theta - 3a^2 r (a \alpha + 1) \cos^2 \theta - 3ar^2 (a \alpha - 1) \cos \theta + r^3 (a \alpha + 1)\right)$$

$$\times \frac{1}{(r^2 + a^2 \cos^2 \theta)^6}.$$ (33)

Therefore $P^2 = 1$, i.e. $P = +1$. Hence the total gravitational entropy in this case is given by

$$S_{grav} = k_s \int_{\sigma} \Psi \, d\sigma = k_s \int_{\sigma} \frac{2\pi C}{\sqrt{-g}} \sqrt{g_{\theta \theta}} d\theta d\phi.$$ (34)

The entropy evaluated at $r_\pm$ is obtained as

$$S_\pm = k_s \int_{\sigma} \Psi \, d\sigma = k_s \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi C} \sqrt{g_{\theta \theta}} d\theta d\phi.$$ (35)

If we substitute $a = 0$ in (35), then we get back the expression (20) for the entropy of the non-rotating accelerating black holes.

The entropy density can be calculated by using the full four-dimensional metric determinant $g$ in the expression involving the covariant derivative, and we get

$$s = k_s |\nabla \Psi| = \frac{k_s}{\sqrt{-g}} \left(\frac{\partial}{\partial r} \sqrt{-g} P\right) = 2k_s \frac{2 \cos^3 \theta a^2 \alpha + \cos \theta a r^2 + r}{(1 - \alpha r \cos \theta)(r^2 + a^2 \cos^2 \theta)},$$

(36)

where $g = -\sin^2 \theta \frac{(a^2 \cos^2 \theta + r^2)^2}{(\alpha r \cos \theta - 1)^8}$. 

FIG. 4: Plot showing the variation of the gravitational entropy density for the accelerating rotating BH with respect to the radial coordinate $r$ and the angular coordinate $\theta$, where $\alpha = 0.45$, $m = 1$, and $k_s = 1$. This figure clearly shows that at the ring singularity ($r = 0$, $\theta = \frac{\pi}{2}$) the gravitational entropy density diverges.

Substituting $\alpha = 0$ in the above expression of entropy density, we get the entropy density for the Kerr black hole:

$$s_{kerr} = \frac{2k_s r}{(r^2 + a^2 \cos^2 \theta)}.$$  

(37)

FIG. 4 clearly shows that the measure of entropy density is well behaved everywhere except at the ring singularity. In figures FIG. 5 and FIG. 6, we have shown that for different $\theta$ we can have a diverging or finite entropy density at $r = 0$. In FIG. 6(b), the entropy density diverges for the condition $r = 0$, $a = 0$, because it corresponds to the central singularity of an accelerating non rotating BH.

FIG. 5: (a) Plot showing the variation of the gravitational entropy density for the accelerating rotating BH with respect to the radial coordinate $r$ and the acceleration parameter $\alpha$, where $a = 0.5$, $m = 1$, $k_s = 1$, and $\theta = \frac{\pi}{2}$. This figure clearly shows that at the ring singularity ($r = 0$, $\theta = \frac{\pi}{2}$) the gravitational entropy density diverges. (b) Plot showing the variation of the gravitational entropy density for the accelerating rotating BH with respect to the radial coordinate $r$ and the acceleration parameter $\alpha$, where $a = 0.5$, $m = 1$, $k_s = 1$, and $\theta = \frac{\pi}{6}$. This figure clearly shows that at $r = 0$ and $\theta = \frac{\pi}{6}$, the gravitational entropy density is finite.
D. Rotating charged black hole

For the rotating charged black hole, the Kretschmann scalar $K$ is

$$K = 48 \frac{(\alpha r \cos(\theta) - 1)^6}{(r^2 + a^2 \cos(\theta))^2}$$

$$+ 2 \left( a^2 m + \frac{5}{6} e^2 r \right) (e^2 - 6mr) a^4 \alpha (\cos(\theta))^5$$

$$+ \left( 15a^4 m^2 - 20a^4 \alpha^2 e^2 m^3 - 15a^6 \alpha^2 m^2 r^2 - \frac{17a^2 \alpha^2 e^4 r^4}{3} - 10a^4 e^2 m^2 r + 7/6a^4 e^4 \right) (\cos(\theta))^4$$

$$- 20 \left( -e^2 m r^2 + \left( -2a^2 m^2 + \frac{19e^4}{30} \right) r + a^2 e^2 m \right) e^2 a^2 \alpha (\cos(\theta))^3$$

$$+ \left( 7/6a^2 e^4 r^6 - \frac{17a^2 e^4 r^2}{3} - 15a^2 m^2 r^4 + 10a^2 \alpha^2 e^2 m^3 r^3 + 15a^4 \alpha^2 e^4 m^2 r^2 + 20a^2 e^2 m^3 r \right) (\cos(\theta))^2$$

$$+ 10 \left( e^2 - 6/5mr \right) r^4 \alpha \left( a^2 m + 1/6e^2 r \right) \cos(\theta) + \left( -a^2 \alpha^2 m^2 + m^2 \right) r^6 - 2e^2 m^2 r^5 + 7/6e^4 r^4), \quad (38)$$

and the Weyl scalar $W$ is

$$W = 48 \frac{(\alpha r \cos(\theta) - 1)^6}{(r^2 + a^2 \cos(\theta))^2}$$

$$\times$$

$$\left( (e^2 r^2 + am (aa + 1)) a^2 \cos^3(\theta) + (2ae^2 r^2 \alpha + 3a^2 m (aa - 1) r + a^2 e^2) \cos^2(\theta) \right.$$

$$\left. + (-e^2 \alpha \alpha^3 - 3am (aa + 1) r^2 + 2ae^2 r) \cos(\theta) - r^2 \left( m (aa - 1) r + e^2 \right) \right)$$

$$\left( (e^2 r^2 + am (aa - 1)) a^2 (\cos(\theta))^3 + (-2ae^2 r^2 \alpha - 3a^2 m (aa + 1) r + a^2 e^2) \cos(\theta))^2 \right.$$

$$\left. + (-e^2 \alpha \alpha^3 - 3am (aa - 1) r^2 - 2ae^2 r) \cos(\theta) + r^2 \left( m (aa + 1) r - e^2 \right) \right). \quad (39)$$
From the above scalars we can calculate the ratio $P = \sqrt{\frac{W}{K}}$, defined in [21], which serves as the measure of gravitational entropy, $S_{grav}$ of black holes. The four-dimensional determinant of the metric is

$$g = -\frac{(\sin (\theta))^2 (r^2 + a^2 (\cos (\theta))^2)^2}{(\alpha r \cos (\theta) - 1)^8}.$$  \hfill (40)

The entropy density is calculated by using the metric determinant $g$ in the covariant derivative. We thus have

$$s = k_s |\nabla \Psi| = \frac{k_s}{\sqrt{-g}} \left( \frac{\partial}{\partial r} \sqrt{-g} P \right).$$  \hfill (41)

Here we have intentionally avoided writing the exact expression of entropy density as it is lengthy and too much complicated, but we can easily check the validity of the result. We have checked that if we substitute $e = 0$ in these calculations, then we get back the result for the accelerating rotating black hole.

In FIG. 7 we find that the gravitational entropy density is not smooth, but contains several singularities. The above analysis clearly shows that the measure of gravitational entropy used above is not adequate to explain the case of the accelerating rotating black holes. Therefore we have to use the measure proposed in [21] for the expression of $P$, which is

$$P = C_{abcd} C^{abcd}.$$  \hfill (42)

Using this definition of $P$ we have calculated the gravitational entropy density. In the FIG. 8 we have shown the variation of the gravitational entropy density with the radial distance and the acceleration parameter. In FIG. 9, we indicate the variation of the gravitational entropy density with radial distance and the angular coordinate, using the modified expression (42).

V. DISCUSSIONS

From the figures in FIG. 9, we find that the entropy density measure diverges not only at the ring singularity but also at $\theta = \pi$, which renders this measure inappropriate for measuring gravitational entropy in these cases. This is a disturbing feature of this method of analysis. For a possible resolution of this problem we want to point out that in the case of rotating black holes, the existence of stationary observer is not well defined because of the effect of
Frame dragging. Nevertheless, we have worked with the chosen definition of gravitational entropy density to get an overall idea of the way things work out. For such cases of axisymmetric space-times, it is not possible to determine the spatial metric $h_{ij}$ because of the presence of the metric coefficient $g_{t\phi}$ in the metric (11) and in metric (10). This is because the object is rotating and the spatial position of each event in the space-time depends on time. Therefore the covariant divergence is calculated from the determinant of the full metric and is given in equations (36) and (41).

We have also discussed the possibility of having an angular component in the vector field $\Psi$ for axisymmetric spacetimes as proposed in [21]. Using this modified definition of $\Psi$ we now calculate the gravitational entropy density for axisymmetric space-times, which yields the expression

$$s = k_s |\nabla . \Psi| = \frac{k_s}{\sqrt{-g}} \left| \frac{\partial}{\partial r} (\sqrt{-g} P) + \frac{\partial}{r \partial \theta} (\sqrt{-g} P) \right|.$$

VI. CONCLUSIONS

In this paper we have adopted a phenomenological approach of determining the gravitational entropy of accelerating black holes as done in [20] and [21]. We find that the gravitational entropy proposal [20] for the accelerating black holes and charged accelerating black holes works pretty well, except for the rotating charged metric where we faced difficulties in this regard. We then considered the alternative definition of $P$ given in [21] to compute the entropy density and showed that the gravitational entropy is well defined in this case. In the end we considered the vector $\Psi$ to have additional angular components for axisymmetric spacetimes, as proposed in [21], to compute the entropy density for accelerating rotating and accelerating charged rotating black holes. From our calculations and the corresponding plots, we can conclude that for the rotating black holes the entropy density will be well-defined if we change our definition of the vector field $\Psi$, be it in the magnitude ($P$) of it, or in the vector directions (having additional angular components).

Acknowledgments

SC is grateful to CSIR, Government of India for providing junior research fellowship. SG gratefully acknowledges IUCAA, India for an associateship and CSIR, Government of India for approving the major research project No.
FIG. 9: (a) Plot showing the variation of the gravitational entropy density for the accelerating rotating charged BH with respect to the radial coordinate $r$ and the angular coordinate $\theta$, using the modified expression given in [21], where $\alpha = 0.25$, $e = 0.2$, $a = 0.45$, $m = 1$, and $k_s = 1$. (b) Plot showing the variation of the gravitational entropy density for the accelerating rotating BH with respect to the radial coordinate $r$ and the angular coordinate $\theta$, using the modified expression given in [21], where $\alpha = 0.25$, $e = 0$, $a = 0.95$, $m = 1$, and $k_s = 1$.

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