Cohomological Classification of Ann-categories

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Abstract

An Ann-category is a categorification of rings. Regular Ann-categories were classified by Shukla cohomology of algebras. In this paper, we state and prove the precise theorem on classification of Ann-categories in the general case based on Mac Lane cohomology of rings.

Keywords: Ann-category, Ann-functor, categorical ring, Mac Lane cohomology, Shukla cohomology

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1 Introduction

Categories with monoidal structures $\oplus$, $\otimes$ (or categories with distributivity constraints) were originally considered by Laplaza in [4]. Kapranov and Voevodsky [3] omitted conditions related to the commutativity constraint with respect to $\otimes$ in the axioms of Laplaza and called these categories ring categories.

In an alternative approach, monoidal categories can be “refined” to become categories with group structure if the objects are all invertible (see [5], [12]). When the underlying category is a groupoid (that is, every morphism is an isomorphism), we obtain the notion of monoidal category group-like [1], or Gr-category [14]. These categories can be classified by the cohomology group $H^3(\Pi, A)$ of groups.

In 1987, Quang [8] introduced the notion of Ann-category which is a categorification of rings. Ann-categories are symmetric Gr-categories (or Picard categories) equipped with a monoidal structure $\otimes$. Since all objects are invertible and all morphisms are isomorphisms, the axioms of an Ann-category are much fewer than those of a ring category (see [7]). The first two invariants of an Ann-category $\mathcal{A}$ are the ring $R = \pi_0 \mathcal{A}$ of isomorphism classes of the objects in $\mathcal{A}$ and the $R$-bimodule $M = \pi_1 \mathcal{A} = \text{Aut}_\mathcal{A}(0)$. Via the structure transport, we can construct an Ann-category of type $(R, M)$ which is Ann-equivalent to $\mathcal{A}$. A family of constraints of $\mathcal{A}$ induces a 5-tuple of functions $(\xi, \alpha, \lambda, \rho : R^3 \to M, \eta : R^2 \to M)$ satisfying certain relations.
This 5-tuple is called a \textit{structure} of an Ann-category of type \((R, M)\). Our purpose is to classify these categories by an appropriate cohomology group.

First we deal with the manifold of \textit{regular} Ann-categories (they satisfy \(c_{A,A} = id\) for all objects \(A\)) which arises from the ring extension problem. In [9], these categories were classified by the cohomology group \(H^3_{Sh}\) of the ring \(R\) (regarded as an \(\mathbb{Z}\)-algebra) in the sense of Shukla [13] (that we misleadingly call Mac Lane-Shukla). This result shows the relation between the notion of regular Ann-category and the theory of Shukla cohomology. Note that the structure \((\xi, \eta, \alpha, \lambda, \rho)\) of a regular Ann-category has an extra condition \(\eta(x, x) = 0\) for the symmetry constraint. This condition is similar to the requirement \(f\left(\begin{array}{cc} 0 & x \\ x & 0 \end{array}\right) = 0\) so that a 3-cocycle \(f\) of Mac Lane cohomology has a realization [6].

In 2007, Jibladze and Pirashvili [2] introduced the notion of \textit{categorical ring} as a slightly modified version of the notion of Ann-category and classified categorical rings by the cohomology group \(H^3_{MacL}(R, M)\). The condition \((Ann - 1)\) and the compatibility of \(\otimes\) with associativity and commutativity constraints with respect to \(\oplus\) are replaced by the compatibility of \(\otimes\) with the “associativity - commutativity” constraint. We prove in [10] that the manifold of all Ann-categories is a subset of the manifold of all categorical rings. We also show that there exists a serious gap in the proof of Proposition 2.3 [2]. The authors of [2] did not prove the existence of the isomorphisms

\[
A \otimes 0 \rightarrow 0, \quad 0 \otimes A \rightarrow 0,
\]

so that the distributivity constraints induce the \(\otimes\)-functors which are compatible with the unit constraints. Thus, it can not be deduced the \(\pi_0A\)-bimodule structure of the abelian group \(\pi_1A\) from axioms of a categorical ring, and therefore results on cohomological classification of categorical rings can not be stated precisely. In the appendix, we give an example of a \textit{categorical ring} which is not an Ann-category, and prove that the classification theorem in [2] is wrong.

A main result of this paper is the cohomological classification theorem for Ann-categories (Theorem 12) in the general case. It is not only a continuation of the results in [9] and in [11], but it also gives a new interpretation of low-dimensional Mac Lane cohomology groups.

After this introductory Section 1, Section 2 is devoted to recalling some well-known results: i) the construction of an Ann-category of type \((R, M)\) which is the reduced Ann-category of an arbitrary one and the determination of a \textit{structure} on such an Ann-category of type \((R, M)\); ii) the Mac Lane cohomology and the obstruction theory of Ann-functors. In section 3 we prove that there is a bijection

\[
\text{Struct}[R, M] \leftrightarrow H^3_{MacL}(R, M)
\]

2
between the set of cohomology classes of structures on $(R, M)$ and the Mac Lane cohomology group of the ring $R$ with coefficients in the $R$-bimodule $M$, and therefore we obtain the precise theorem on classification of Ann-categories and Ann-functors.

For short, we sometimes write $AB$ or $A.B$ instead of $A \otimes B$.

2 Ann-categories of type $(R, M)$

Let us recall some necessary concepts and facts in this section from [8, 9].

A monoidal category is called a Gr-category (or a categorical group) if every object is invertible and the background category is a groupoid. A Picard category (or a symmetric categorical group) is a Gr-category equipped with a symmetry constraint which is compatible with associativity constraint.

2.1 Ann-categories and Ann-functors

**Definition.** An Ann-category consists of
i) a category $\mathcal{A}$ together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$;
ii) a fixed object $0 \in \mathcal{A}$ together with natural isomorphisms $a_+, c, g, d$ such that $(\mathcal{A}, \oplus, a_+, c, (0, g, d))$ is a Picard category;
iii) a fixed object $1 \in \mathcal{A}$ together with natural isomorphisms $a, l, r$ such that $(\mathcal{A}, \otimes, a, (1, l, r))$ is a monoidal category;
iv) natural isomorphisms $L, R$ given by

\[
L_{A,X,Y} : A \otimes (X \oplus Y) \rightarrow (A \otimes X) \oplus (A \otimes Y),
\]

\[
R_{X,Y,A} : (X \oplus Y) \otimes A \rightarrow (X \otimes A) \oplus (Y \otimes A)
\]

such that the following conditions hold:

(Ann - 1) for $A \in \mathcal{A}$, the pairs $(L^A, \bar{L}^A), (R^A, \bar{R}^A)$ defined by

\[
L^A = A \otimes - \quad R^A = - \otimes A
\]

\[
\bar{L}^A_{X,Y} = L_{A,X,Y} \quad \bar{R}^A_{X,Y} = R_{X,Y,A}
\]

are $\oplus$-functors which are compatible with $a_+$ and $c$;

(Ann - 2) for all $A, B, X, Y \in \mathcal{A}$, the following diagrams commute

\[
\begin{array}{ccc}
(AB)(X \oplus Y) & \xrightarrow{a_{A,B,X \oplus Y}^{A,B,X \oplus Y}} & A(B(X \oplus Y)) \\
\downarrow{L^{AB}} & & \downarrow{A(BX \oplus AY)} \\
(AB)X \oplus (AB)Y & \xrightarrow{a_{A,B,X \oplus A,B,Y}^{A,B,X \oplus A,B,Y}} & A(BX) \oplus A(BY)
\end{array}
\]

\[
\begin{array}{ccc}
(X \oplus Y)(BA) & \xrightarrow{a_{X,Y,B,A}^{X,Y,B,A}} & ((X \oplus Y)B)A \\
\downarrow{\bar{R}^{BA}} & & \downarrow{\bar{R}^{BA} \otimes id_A} \\
X(BA) \oplus Y(BA) & \xrightarrow{a_{X,B,A \oplus Y,B,A}^{X,B,A \oplus Y,B,A}} & (XB \oplus YB)A
\end{array}
\]
where $v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \to (U \oplus Z) \oplus (V \oplus T)$ is a unique morphism constructed from $\oplus, a_+, c, id$ of the symmetric monoidal category $(A, \oplus)$;

(Ann - 3) for the unit $1 \in A$ of the operation $\otimes$, the following diagrams commute

$$
\begin{array}{ccc}
1(X \oplus Y) & \xrightarrow{l^1} & 1X \oplus 1Y \\
\downarrow 1_X \otimes 1_Y & & \downarrow 1_X \otimes 4_Y \\
X \oplus Y & & X \oplus Y.
\end{array}
$$

\(\hat{L}^A \otimes \hat{R}^A\)

Since each of pairs $(L^A, \hat{L}^A)$, $(R^A, \hat{R}^A)$ is an $\oplus$-functor which is compatible with the associativity constraint in the Picard category $A$, it is also compatible with the unit constraint $(0, g, d)$, so we obtain the following result.

**Lemma 1.** In an Ann-category $A$ there exist uniquely isomorphisms

$$
\hat{L}^A : A \otimes 0 \to 0, \hat{R}^A : 0 \otimes A \to 0
$$

such that the following diagrams commute

$$
\begin{array}{ccc}
AX & \xrightarrow{\hat{L}^A(g)} & A(0 \oplus X) \\
\downarrow 0 \oplus AX & & \downarrow 0 \oplus A \oplus AX \\
0 \oplus AX & \xrightarrow{\hat{L}^A \oplus id} & A0 \oplus AX.
\end{array}
$$

$$
\begin{array}{ccc}
AX & \xrightarrow{\hat{L}^A(d)} & A(X \oplus 0) \\
\downarrow 0 \oplus 0 \oplus AX & & \downarrow 0 \oplus AX \oplus A0 \\
0 \oplus AX & \xrightarrow{id \oplus \hat{L}^A} & AX \oplus A0.
\end{array}
$$

$$
\begin{array}{ccc}
XA & \xrightarrow{R^A(g)} & (0 \oplus X)A \\
\downarrow 0 \oplusXA & & \downarrow 0 \oplus 0 \oplusXA \\
0 \oplusXA & \xrightarrow{R^A \oplus id} & 0A \oplusXA.
\end{array}
$$

$$
\begin{array}{ccc}
XA & \xrightarrow{R^A(d)} & (X \oplus 0)A \\
\downarrow 0 \oplus 0 \oplusXA & & \downarrow 0 \oplus AX \oplus A0 \\
0 \oplusXA & \xrightarrow{id \oplus R^A} & AX \oplus A0.
\end{array}
$$
It is easy to see that if \((F, \hat{F}, \tilde{F}) : (\mathcal{A}, \oplus) \to (\mathcal{A}', \oplus)\) is a monoidal functor between two Gr-categories, then the canonical isomorphism \(\hat{F} : F0 \to 0'\) can be deduced from the others. Thus, we state the following definition.

**Definition.** Let \(\mathcal{A}\) and \(\mathcal{A}'\) be Ann-categories. An Ann-functor \((F, \hat{F}, \tilde{F}, F_\ast) : \mathcal{A} \to \mathcal{A}'\) consists of a functor \(F : \mathcal{A} \to \mathcal{A}'\), natural isomorphisms

\[
\tilde{F}_{X,Y} : F(X \oplus Y) \to F(X) \oplus F(Y), \quad \hat{F}_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y),
\]

and an isomorphism \(F_\ast : F(1) \to 1'\) such that \((F, \hat{F})\) is a symmetric monoidal functor with respect to the operation \(\oplus\), \((F, \hat{F}, \tilde{F}, F_\ast)\) is a monoidal functor with respect to the operation \(\otimes\), and \((F, \hat{F}, \tilde{F})\) satisfies two following commutative diagrams

\[
\begin{align*}
F(X(Y \oplus Z)) & \xrightarrow{\hat{F}} FX.F(Y \oplus Z) & \xrightarrow{id \oplus \hat{F}} FX(FY \oplus FZ) \\
F(XY \oplus XZ) & \xrightarrow{\hat{F}} F(XY) \oplus F(XZ) & \xrightarrow{\hat{F} \oplus \hat{F}} FX.FY \oplus FX.FZ \\
F((X \oplus Y)Z) & \xrightarrow{\hat{F}} F(X \oplus Y).FZ & \xrightarrow{F \oplus id} (FX \oplus FY).FZ \\
F(XZ \oplus YZ) & \xrightarrow{\hat{F}} F(XZ) \oplus F(YZ) & \xrightarrow{\hat{F} \oplus \hat{F}} FX.FZ \oplus FY.FZ.
\end{align*}
\]

These diagrams are called the **compatibility** of the functor \(F\) with the distributivity constraints.

An Ann-morphism (or a homotopy)

\[
\theta : (F, \hat{F}, \tilde{F}, F_\ast) \to (F', \hat{F}', \tilde{F}', F'_\ast)
\]

between Ann-functors is an \(\oplus\)-morphism, as well as an \(\otimes\)-morphism.

If there exists an Ann-functor \((F', \hat{F}', \tilde{F}', F'_\ast) : \mathcal{A}' \to \mathcal{A}\) and Ann-morphisms \(F'F \sim id_{\mathcal{A}}\), \(FF' \sim id_{\mathcal{A'}}\), we say that \((F, \hat{F}, \tilde{F}, F_\ast)\) is an Ann-equivalence, and \(\mathcal{A}, \mathcal{A}'\) are Ann-equivalent.

It can be proved that each Ann-functor is an Ann-equivalence if and only if \(F\) is a categorical equivalence.

**Lemma 2.** Any Ann-functor \(F = (F, \hat{F}, \tilde{F}, F_\ast) : \mathcal{A} \to \mathcal{A}'\) is homotopic to an Ann-functor \(F' = (F', \hat{F}', \tilde{F}', F'_\ast)\), where \(F'0 = 0', \hat{F}'0 = id_{\mathcal{A'}}\), and \(F'1 = 1', F'_\ast = id_{\mathcal{A'}}\).

**Proof.** Consider a family of isomorphisms in \(\mathcal{A}'\):

\[
\theta_X = \begin{cases} 
  id_{F,X} & \text{if } X \neq 0, X \neq 1, \\
  \hat{F} & \text{if } X = 0, \\
  F_\ast & \text{if } X = 1,
\end{cases}
\]

5
for $X \in \mathcal{A}$. Then, the Ann-functor $F'$ can be constructed in a unique way such that $\theta : F \to F'$ becomes a homotopy. Namely,

$$F'X = \begin{cases} 
FX & \text{if } X \neq 0, X \neq 1, \\
0' & \text{if } X = 0, \\
1' & \text{if } X = 1,
\end{cases}$$

$$F'(f : X \to Y) = \theta_Y F(f)(\theta_X)^{-1} : F'X \to F'Y,$$

$$\bar{F}_{X,Y} = (\theta_X \oplus \theta_Y)\bar{F}_{X,Y}\theta_X^{-1}$$,

$$\hat{F} = \hat{F}\theta_1^{-1} = \id_{0'}, F_*' = F_*\theta_1^{-1} = \id_{1'}.$$  

Based on Lemma 2, we refer to $(F, \bar{F}, \hat{F})$ as an Ann-functor.

### 2.2 Reduced Ann-categories

For an Ann-category $\mathcal{A}$, the set $R = \pi_0 \mathcal{A}$ of isomorphism classes of the objects in $\mathcal{A}$ is a ring where the operations $+$, $\times$ are induced by $\oplus$, $\otimes$ on $\mathcal{A}$, and $M = \pi_1 \mathcal{A} = \Aut(0)$ is an abelian group where the operation, denoted by $+$, is just the composition. Moreover, $M = \pi_1 \mathcal{A}$ is an $R$-bimodule with the actions

$$sa = \lambda_X(a), \quad as = \rho_X(a),$$

where $X \in s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$ and $\lambda_X, \rho_X$ satisfy the commutative diagrams

$$\begin{array}{c}
\begin{array}{ccc}
X.0 & \xrightarrow{\bar{i}_X} & 0 \\
0 \xleftarrow{id \oplus a} & & \xleftarrow{\lambda_X(a)} \\
X.0 & \xleftarrow{L_X} & 0,
\end{array} & \begin{array}{ccc}
0.X & \xrightarrow{\hat{\bar{F}}^X} & 0 \\
0 \xleftarrow{a \oplus \id} & & \xleftarrow{\rho_X(a)} \\
0.X & \xleftarrow{\hat{\bar{F}}^X} & 0,
\end{array}
\end{array}$$

We recall briefly some main facts of the construction of the reduced Ann-category $S_\mathcal{A}$ of $\mathcal{A}$ via the structure transport (for details, see [9]). The objects of $S_\mathcal{A}$ are the elements of the ring $\pi_0 \mathcal{A}$. A morphism is an automorphism $(s, a) : s \to s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$. The composition of morphisms is given by

$$(s, a) \circ (s, b) = (s, a + b).$$

For each $s \in \pi_0 \mathcal{A}$, choose an object $X_s \in \mathcal{A}$ such that $X_0 = 0, X_1 = 1$, and choose an isomorphism $i_X : X \to X_s$ such that $i_X^* = id_{X_s}$. We obtain two functors

$$\begin{cases}
G : \mathcal{A} \to S_\mathcal{A} \\
G(X) = [X] = s \\
G(X \xrightarrow{f} Y) = (s, \gamma_{X_s}^{-1}(i_Y f i_X^{-1})),
\end{cases} \quad \begin{cases}
H : S_\mathcal{A} \to \mathcal{A} \\
H(s) = X_s \\
H(s, u) = \gamma_{X_s}(u).
\end{cases} \quad (1)$$
where \( X, Y \in s \) and \( f : X \to Y \), and \( \gamma_X \) is a map defined by the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma_X(a)} & X \\
\downarrow{g_X} & & \downarrow{g_X} \\
0 \oplus X & \xrightarrow{a \oplus id} & 0 \oplus X
\end{array}
\]

Diagram 1

The operations on \( S_A \) are defined by

\[
\begin{align*}
(s \oplus t) &= G(H(s) \oplus H(t)) = s + t, \\
(s, a) \oplus (t, b) &= G(H(s, a) \oplus H(t, b)) = (s + t, a + b), \\
(s \otimes t) &= G(H(s) \otimes H(t)) = st, \\
(s, a) \otimes (t, b) &= G(H(s, a) \otimes H(t, b)) = (st, sb + at),
\end{align*}
\]

where \( s, t \in \pi_0 A \), \( a, b \in \pi_1 A \). Obviously, these operations do not depend on the choice of the set of representatives \((X_s, i_X)\).

The constraints in \( S_A \) are defined by those in \( A \) by means of the notion of stick. A stick in \( A \) is a set of representatives \((X_s, i_X)\) such that

\[
\begin{align*}
i_{0 \oplus X_t} &= g_{X_t}, & i_{X_s \oplus 0} &= d_{X_s}, \\
i_{1 \oplus X_t} &= 1_{X_t}, & i_{X_s \otimes 1} &= r_{X_s}, & i_{0 \otimes X_t} &= \hat{R}_{X_t}, & i_{X_s \otimes 0} &= \hat{L}_{X_s}.
\end{align*}
\]

The unit constraints for two operations \( \oplus, \otimes \) in \( S_A \) are \((0, id, id)\) and \((1, id, id)\), respectively. The functor \( H \) and isomorphisms

\[
\begin{align*}
\tilde{H} &= i_{X_s \otimes X_t}^{-1}, & \tilde{\tilde{H}} &= i_{X_s \oplus X_t}^{-1}.
\end{align*}
\]

transport the constraints \( \alpha, \beta, \gamma, \delta, \epsilon \) of \( A \) to those \( \xi, \eta, \alpha, \lambda, \rho \) of \( S_A \). Then, the category

\[
(S_A, \xi, \eta, (0, id, id), \alpha, (1, id, id), \lambda, \rho)
\]

is an Ann-category which is equivalent to \( A \) by the Ann-equivalence \((H, \tilde{H}, \tilde{\tilde{H}}) : S_A \to A \). Besides, the functor \( G : A \to S_A \) together with isomorphisms

\[
G_{X,Y} = G(i_X \oplus i_Y), \quad \tilde{G}_{X,Y} = G(i_X \otimes i_Y)
\]

is also an Ann-equivalence. We refer to \( S_A \) as an Ann-category of type \((R, M)\), called a reduction of \( A \). We also call \((H, \tilde{H}, \tilde{\tilde{H}}) \) and \((G, \tilde{G}, \tilde{\tilde{G}})\) canonical Ann-equivalences, the family of constraints \( h = (\xi, \eta, \alpha, \lambda, \rho) \) of \( S_A \) a structure of the Ann-category of type \((R, M)\), or simply a structure on \((R, M)\).

The following result follows from the axioms of an Ann-category.
We now investigate the effect of different choices of the stick \((X, i_X)\). Proposition 4. Let \(S\) and \(S'\) be reduced Ann-categories of \(A\) corresponding to the sticks \((X, i_X)\) and \((X', i'_X)\), respectively. Then the structures
Theorem 1. Two structures \( \xi, \eta, \alpha, \lambda, \rho \) of \( S \) and \( \xi', \eta', \alpha', \lambda', \rho' \) of \( S' \) satisfy the following relations:

\[ \begin{align*}
A_{13}. & \quad \xi(x, y, z) - \xi'(x, y, z) = \tau(y, z) - \tau(x + y, z) + \tau(x, y + z) - \tau(x, y), \\
A_{14}. & \quad \eta(x, y) - \eta'(y, x) = \tau(x, y) - \tau(y, x), \\
A_{15}. & \quad \alpha(x, y, z) - \alpha'(x, y, z) = xy\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z, \\
A_{16}. & \quad \lambda(x, y, z) - \lambda'(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\tau(y, z) - \tau(xy, xz), \\
A_{17}. & \quad \rho(x, y, z) - \rho'(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \nu(x, y)z - \tau(xz, yz),
\end{align*} \]

where \( \tau, \nu : (\pi_0 A)^2 \to \pi_1 A \) are the functions satisfying the normalization conditions \( \tau(0, y) = \tau(x, 0) = 0 \) and \( \nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0 \).

Two structures \( \xi, \eta, \alpha, \lambda, \rho \) and \( \xi', \eta', \alpha', \lambda', \rho' \) of Ann-categories of type \((R, M)\) are cohomologous if and only if they satisfy the relations \( A_{13} - A_{17} \) in Proposition 4.

Note that two unit constraints of \( \oplus \) and \( \otimes \) in an Ann-category of type \((R, \underline{M})\) are both strict. It is easy to prove the following lemma.

**Lemma 5.** Two structures \( h \) and \( h' \) are cohomologous if and only if there exists an Ann-functor \( (F, \tilde{F}, \tilde{F}) : (R, M, h) \to (R, M, h') \), where \( F = id_{(R, \underline{M})} \).

### 2.3 Mac Lane cohomology groups of rings and obstruction theory

Let \( R \) be a ring and \( M \) be an \( R \)-bimodule. From the definition of Mac Lane cohomology of rings \( \underline{\mathcal{M}_3} \), we obtain the description of elements in the cohomology group \( H_3^{\mathcal{M}_3}(R, M) \).

The group \( Z_3^{\mathcal{M}_3}(R, M) \) of 3-cocycles of \( R \) with coefficients in \( M \) consists of the quadruples \( (\sigma, \alpha, \lambda, \rho) \) of the maps:

\[ \sigma : R^3 \to M; \quad \alpha, \lambda, \rho : R^3 \to M \]

satisfying the following conditions:

\[ \begin{align*}
M_1. & \quad x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0, \\
M_2. & \quad -\alpha(x, z, t) + \alpha(y, z, t) + \alpha(x + y, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \\
& \quad \rho(x, y, z)t = 0, \\
M_3. & \quad -\alpha(x, y, t) - \alpha(x, z, t) + \alpha(x, yz, t) + \alpha(xy, xz, t) - \alpha(xy, xz, t) - \lambda(x, y, zt) + \\
& \quad \lambda(x, y, z)t = 0, \\
M_4. & \quad \alpha(x, y, z) + \alpha(x, y, t) - \alpha(x, yz, t) + x\lambda(y, z, t) - \lambda(xy, z, t) + \lambda(x, yz, yt) = 0, \\
M_5. & \quad -\lambda(x, z, t) - \lambda(y, z, t) + \lambda(x + y, z, t) + \rho(x, y, z) + \rho(x, y, z) - \\
& \quad \rho(x, y, z + t) + \sigma(xz, xt, yz, yt) = 0,
\end{align*} \]
\[M_6. \lambda(r, x, y) + \lambda(r, z, t) - \lambda(r, x + z, y + t) - \lambda(r, x, z) - \lambda(r, y, t) + \lambda(r, x + y, z + t) - \sigma(x, y, z, t) + \sigma(ry, rz, rt) = 0,
\]
\[M_7. - \rho(x, y, r) - \rho(z, t, r) + \rho(x + z, y + t, r) + \rho(x, z, r) + \rho(y, t, r)
- \rho(x + y, z + t, r) - \sigma(xr, yr, zr, tr) + \sigma(x, y, z, t)r = 0,
\]
\[M_8. - \sigma(r, s, u, v) - \sigma(x, y, z, t) + \sigma(r + x, s + y, u + z, v + t)
+ \sigma(r, s, x, y) + \sigma(u, v, z, t) - \sigma(r + u, s + v, x + z, y + t)
- \sigma(r, u, x, z) - \sigma(s, v, y, t) + \sigma(r + s, u + v, x + y, z + t) = 0.
\]
These functions satisfy normalization conditions:

\[
\begin{align*}
\alpha(0, y, z) & = \alpha(x, 0, z) = \alpha(x, y, 0) = 0, \\
\lambda(0, y, z) & = \lambda(x, 0, z) = \lambda(x, y, 0) = 0, \\
\rho(0, y, z) & = \rho(x, 0, z) = \rho(x, y, 0) = 0, \\
\sigma(r, s, 0, 0) & = \sigma(0, 0, u, v) = \sigma(r, 0, u, 0) = \sigma(0, s, 0, v) = \sigma(r, 0, 0, v) = 0.
\end{align*}
\]

The 3-cocycle \(h = (\sigma, \alpha, \lambda, \rho)\) belongs to the group \(B^3_{MaL}(R, M)\) if and only if there exist the functions \(\tau\nu: R^2 \to M\) satisfying:

\[M_9. \sigma(x, y, z, t) = \tau(x, y) + \tau(z, t) - \tau(x + z, y + t) - \tau(x, z) - \tau(y, t)
+ \tau(x + y, z + t),
\]
\[M_{10}. \alpha(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z,
\]
\[M_{11}. \lambda(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\tau(y, z) - \tau(xy, xz),
\]
\[M_{12}. \rho(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \tau(x, y)z - \tau(x, yz),
\]
where \(\tau, \nu\) satisfy the normalization conditions: \(\tau(0, y) = \tau(x, 0) = 0\) and \(\nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0\).

The group \(Z^2_{MaL}(R, M)\) consists of 2-cochains \(g = (\tau, \nu)\) of the ring \(R\) with coefficients in the \(R\)-bimodule \(M\) satisfying

\[\partial g = 0.
\]

The subgroup \(B^2_{MaL}(R, M) \subset Z^2_{MaL}(R, M)\) of 2-coboundaries consists of the pairs \((\tau, \nu)\) such that there exist the maps \(t: R \to M\) satisfying \((\tau, \nu) = \partial_{MaL}t\), that is,

\[M_{13}. \tau(x, y) = t(y) - t(x + y) + t(x),
\]
\[M_{14}. \nu(x, y) = xt(y) - t(xy) + t(x)y,
\]
where \(t\) satisfies the normalization condition, \(t(0) = t(1) = 0\).

The group \(Z^1_{MaL}(R, M)\) consists of 1-cochains \(t\) of the ring \(R\) with coefficients in the \(R\)-bimodule \(M\) satisfying

\[\partial t = 0.
\]

The subgroup of 1-coboundaries, \(B^1_{MaL}(R, M) \subset Z^1_{MaL}(R, M)\), consists of the functions \(t\) such that there exists \(a \in R\) satisfying \(t(x) = ax - xa\).
The quotient group
\[ H^i_{\text{Mac}}(R, M) = \frac{Z^i_{\text{Mac}}(R, M)}{B^i_{\text{Mac}}(R, M)}, \quad i = 1, 2, 3, \]
is called the \(i\)th \textit{Mac Lane cohomology group} of the ring \(R\) with coefficients in the \(R\)-bimodule \(M\).

Let us now recall from [11] some results on Ann-functors. Each Ann-functor \((F, \hat{F}, \tilde{F}) : A \to A'\) induces one \(S_F\) between their reduced Ann-categories. Throughout this section, let \(S\) and \(S'\) be Ann-categories of types \((R, M, h)\) and \((R', M', h')\), respectively.

A functor \(F : S \to S'\) is called a functor of \textit{type} \((p, q)\) if
\[ F(x) = p(x), \quad F(x, a) = (p(x), q(a)), \]
where \(p : R \to R'\) is a ring homomorphism and \(q : M \to M'\) is a group homomorphism such that
\[ q(xa) = p(x)q(a), \quad x \in R, a \in M. \]
The group \(M'\) can be regarded as an \(R\)-module with the action \(sa' = p(s)a'\), so \(q\) is an \(R\)-bimodule homomorphism. In this case, we say that \((p, q)\) is a \textit{pair of homomorphisms} and that the function
\[ k = q, h - p^* h' \tag{4} \]
is an \textit{obstruction} of \(F\), where \(p^*, q_*\) are canonical homomorphisms,
\[ Z^3_{\text{Mac}}(R, M) \xrightarrow{q_*} Z^3_{\text{Mac}}(R, M') \xleftarrow{p^*} Z^3_{\text{Mac}}(R', M'). \]

\textbf{Proposition 6} (Proposition 4.3 [11]). Every Ann-functor \(F : S \to S'\) is a functor of \textit{type} \((p, q)\).

Keeping in mind that \(\gamma\) is the map defined by Diagram 1, we state the following proposition.

\textbf{Proposition 7} (Proposition 4.1 [11]). Let \(A\) and \(A'\) be Ann-categories. Then every Ann-functor \((F, \hat{F}, \tilde{F}) : A \to A'\) induces an Ann-functor \(S_F : S_A \to S_{A'}\) of \textit{type} \((p, q)\), where
\[ p = F_0 : \pi_0 A \to \pi_0 A', \quad [X] \mapsto [FX], \]
\[ q = F_1 : \pi_1 A \to \pi_1 A', \quad u \mapsto \gamma_{F_0}^{-1}(Fu). \]
Further,
\begin{itemize}
  \item[i)] \(F\) is an equivalence if and only if \(F_0, F_1\) are isomorphisms,
\end{itemize}
ii) the Ann-functor $S_F$ satisfies the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow{H} & & \downarrow{G'} \\
S_A & \xrightarrow{S_F} & S_{A'}
\end{array}
$$

where $H, G'$ are canonical Ann-equivalences defined by \([1, 2, 3]\).

Since $\tilde{F}_{x,y} = (\bullet, \tau(x, y))$, and $\tilde{F}'_{x,y} = (\bullet, \nu(x, y))$, we call $g_F = (\tau, \nu)$ a pair of functions associated to $(\tilde{F}, \tilde{F}')$, and hence an Ann-functor $F : S \to S'$ can be regarded as a triple $(p, q, g_F)$. It follows from the compatibility of $F$ with the constraints that

$$
q_* h - p_* h' = \partial(g_F),
$$

(5)

Moreover, Ann-functors $(F, g_F)$ and $(F', g_{F'})$ are homotopic if and only if $F' = F$, that is, they are of the same type $(p, q)$, and there is a function $t : R \to M'$ such that $g_{F'} = g_F + \partial t$.

We write $\text{Hom}^{\text{Ann}}_{(p,q)}[S, S']$ for the set of homotopy classes of Ann-functors of type $(p, q)$ from $S$ to $S'$.

**Theorem 8** (Theorem 4.4, 4.5 [11]). The functor $F : S \to S'$ of type $(p, q)$ is an Ann-functor if and only if the obstruction $[k]$ vanishes in $H_3^{\text{MacL}}(R, M')$. Then, there exists a bijection

$$
\text{Hom}^{\text{Ann}}_{(p,q)}[S, S'] \leftrightarrow H^2_3^{\text{MacL}}(R, M').
$$

(6)

3 Classification of Ann-categories

In order to prove the main result (Theorem [13]) of the paper, we first prove that the set of cohomology classes of structures on $(R, M)$ and the group $H^3_3^{\text{MacL}}(R, M)$ are coincident.

**Lemma 9.** Each structure of an Ann-category of type $(R, M)$ induces a 3-cocycle in $Z^3_3^{\text{MacL}}(R, M)$.

**Proof.** Let $h = (\xi, \eta, \alpha, \lambda, \rho)$ be a structure of an Ann-category $S$ of type $(R, M)$. We define a function $\sigma : R^4 \to M$ by

$$
\sigma(x, y, z, t) = \xi(x + y, z, t) - \xi(x, y, z) + \eta(y, z) + \xi(x, z, y) - \xi(x + z, y, t)
$$

(7)

This equation shows that $\sigma$ is just the morphism $\nu : (x + y) + (z + t) \to (x + z) + (y + t)$
in an Ann-category of type \((R, M)\).

First, the normalized property of \(\sigma\) follows from the ones of \(\xi\) and \(\eta\)

\[
\sigma(0, 0, z, t) = \sigma(x, y, 0, 0) = \sigma(0, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0.
\]

We now show that the quadruple \(\hat{h} = (\sigma, \alpha, \lambda, \rho)\) satisfies the relations \(M_1 - M_8\), and \(\hat{h}\) is therefore a 3-cocycle. The relation \(M_1\) is just the relation \(A_{12}\). The relations \(M_2, M_3, M_4, M_5\) are just \(A_{11}, A_{10}, A_9, A_8\), respectively.

According to the coherence theorem in an Ann-category of type \((R, M)\) the following Diagrams 2, 3 commute

\[
\begin{array}{ccc}
(r(x + y) + (z + t)) & \xrightarrow{\xi} & (r(x + z) + (y + t)) \\
\downarrow \xi & & \downarrow \xi \\
(r(x + y) + r(z + t)) & \xrightarrow{\xi \otimes \xi} & (r(x + z) + r(y + t)) \\
\downarrow \xi \otimes \xi & & \downarrow \xi \otimes \xi \\
(rx + ry) + (rz + rt) & \xrightarrow{\text{id} \otimes v} & (rx + rz) + (ry + rt)
\end{array}
\]

Diagram 2

\[
\begin{array}{ccc}
[(r + s) + (u + v)] + [(x + y) + (z + t)] & \xrightarrow{\nu} & [(r + s) + (x + y)] + [(u + v) + (z + t)] \\
\downarrow \nu & & \downarrow \nu \\
[(r + u) + (s + v)] + [(x + z) + (y + t)] & \xrightarrow{\nu} & [(r + x) + (s + y)] + [(u + z) + (v + t)] \\
\downarrow \nu & & \downarrow \nu \\
[(r + u) + (x + z)] + [(s + v) + (y + t)] & \xrightarrow{\nu + \nu} & [(r + x) + (u + z)] + [(s + y) + (v + t)]
\end{array}
\]

Diagram 3

These commutative diagrams imply the relations \(M_6, M_8\). The relation \(M_7\) follows from a commutative diagram which is analogous to the Diagram 2, where \(r\) is tensored on the right side.

Lemma 10. Each Mac Lane 3-cocycle \((\sigma, \alpha, \lambda, \rho)\) is induced by a structure \((\xi, \eta, \alpha, \lambda, \rho)\) of an Ann-category of type \((R, M)\).

Proof. Let \((\sigma, \alpha, \lambda, \rho)\) be an element in \(Z^3_{\text{MaL}}(R, M)\). Set

\[
\xi(x, y, z) = -\sigma(x, y, 0, z), \eta(x, y) = \sigma(0, x, y, 0),
\]

we obtain a 5-tuple of functions \(h = (\xi, \eta, \alpha, \lambda, \rho)\). The normalized properties of \(\xi, \eta\) follow from that of \(\sigma\).
We now show that \( h \) is a structure of an Ann-category of type \((R, M)\). First, the relations \( A_{12} - A_9 \) are just \( M_1 - M_4 \). The relation \( A_1 \) follows from \( M_8 \) when \( u = 0 = x = y = z \). The relation \( A_3 \) follows from \( M_8 \) when \( r = s = v = 0 = x = z = t \). The relations \( A_4 \) and \( A_5 \) follow from \( M_6 \) and \( M_7 \), respectively, when \( x = t = 0 \). The relations \( A_6 \) and \( A_7 \) follow from \( M_6 \) and \( M_7 \), respectively, when \( z = 0 \).

To prove the relation \( A_2 \), take \( s = u = 0 = x = z = t \) in \( M_8 \) we obtain

\[
- \xi(r, y, v) + \xi(r, v, y) - \eta(v, y) + \sigma(r, v, y, 0) = 0 \tag{8}
\]

Now, take \( r = u = 0 = y = z = t \) in \( M_8 \) we obtain

\[
- \xi(x, s, v) + \eta(s, x) - \eta(s + v, x) + \sigma(s, v, x, 0) = 0.
\]

In other words,

\[
- \xi(y, r, v) + \eta(r, y) - \eta(r + v, y) + \sigma(r, v, y, 0) = 0 \tag{9}
\]

Subtracting (11) from (8), we obtain the relation \( A_2 \).

Finally, to prove the relation \( A_8 \), note that \( \sigma \) can be presented by \( \xi, \eta \) as in (7). Indeed, take \( v = 0 = x = y = z \) in \( M_8 \) we obtain

\[
\sigma(r, s, u, t) + \xi(r + u, s, t) - \xi(r + s, u, t) - \sigma(r, s, u, 0) = 0. \tag{10}
\]

Now, take \( v = s = y = u \) in (10) we obtain

\[
\xi(r, u, s) - \xi(r, s, u) - \eta(s, u) + \sigma(r, s, u, 0) = 0. \tag{11}
\]

Adding (10) to (11) and doing some appropriate calculations, we get (7).

Because of (7), \( M_8 \) becomes \( A_8 \). This means the 5-tuple of functions \( h = (\xi, \eta, \alpha, \lambda, \rho) \) is a structure of an Ann-category of type \((R, M)\). Further, this structure induces the 3-cocycle \( \widehat{h} = (\sigma, \alpha, \lambda, \rho) \). \( \square \)

**Lemma 11.** The structures \( h \) and \( h' \) of the Ann-category of type \((R, M)\) are cohomologous if and only if the corresponding 3-cocycles \( \widehat{h}, \widehat{h}' \) are cohomologous.

**Proof.** By Lemma 10 the structures \( h \) and \( h' \) induce elements \( \widehat{h} \) and \( \widehat{h}' \) in \( Z^3_{Mal}(R, M) \), respectively. By Lemma 5 the functions \( \alpha - \alpha', \lambda - \lambda', \rho - \rho' \) satisfy the relations \( M_{10} - M_{12} \), where \( \tilde{F} = \tau, \tilde{F} = \nu \). Besides, the following diagram commutes because of the coherence of a symmetric monoidal functor

\[
\begin{array}{ccc}
F((x + y) + (z + t)) & \xrightarrow{\tilde{F}} & F(x + y) + F(z + t) \\
\downarrow F & & \downarrow \tilde{F} \\
F((x + z) + (y + t)) & \xrightarrow{\tilde{F} + \tilde{F}} & (F(x) + F(y)) + (F(z) + F(t)) \\
\downarrow \tilde{F} & & \downarrow \tilde{F}' \\
F(x + z) + F(y + t) & \xrightarrow{\tilde{F} + \tilde{F}'} & (F(x) + F(z)) + (F(y) + F(t)).
\end{array}
\]

14
Note that $F = id$ and $\tilde{F} = \tau$, so the above commutative diagram implies
\[
\sigma(x, y, z, t) - \sigma'(x, y, z, t) = \tau(x + y, z + t) + \tau(x, y) + \tau(z, t) - \tau(x + z, y + t) - \tau(x, z) - \tau(y, t).
\]
That means $\sigma - \sigma'$ satisfies $M_9$. Thus, $\hat{h}$ and $\hat{h}'$ belong to the same cohomology class of $H^3_{MacL}(R, M)$.

Now, assume that $\hat{h} - \hat{h}' \in B^3_{MacL}(R, M)$. Then $\alpha - \alpha'$, $\lambda - \lambda'$, $\rho - \rho'$ satisfy $M_{10} - M_{12}$ which are just the relations $A_{15} - A_{17}$. By (7), the definition of $\sigma$ and the normalized property of $\xi, \eta,$ we have
\[
\xi(x, y, z) = -\sigma(x, 0, y, z), \quad \xi'(x, y, z) = -\sigma'(x, 0, y, z),
\]
\[
\eta(x, y) = \sigma(0, x, y, 0), \quad \eta'(x, y) = \sigma'(0, x, y, 0).
\]
Therefore, $A_{13}, A_{14}$ are obtained from $M_9$, and thus $h, h'$ are cohomologous structures.

Let $\text{Struct}[R, M]$ denote the set of cohomology classes of structures on $(R, M)$. Then, Lemmas [9][10][11] lead to the following result.

**Proposition 12.** There exists a bijection
\[
\text{Struct}[R, M] \to H^3_{MacL}(R, M)
\]
\[ [h = (\xi, \eta, \alpha, \lambda, \rho)] \mapsto [\hat{h} = (\sigma, \alpha, \lambda, \rho)]\]

By the above lemma, we regard each cohomology class $[h] = [(\xi, \eta, \alpha, \lambda, \rho)]$ as an element of the group $H^3_{MacL}(R, M)$.

Let $\textbf{Ann}$ refer to the category whose objects are $\text{Ann}$-categories, and whose morphisms are their $\text{Ann}$-functors.

We determine the category $H^3_{\text{Ann}}$ whose objects are triples $(R, M, [h])$, where $[h] \in H^3_{MacL}(R, M)$. A morphism $(R, M, [h]) \to (R', M', [h'])$ in $H^3_{\text{Ann}}$ is a pair $(p, q)$ such that there exists a function $g : R^2 \to M'$ so that $(p, q, g) : (R, M, h) \to (R', M', h')$ is an $\text{Ann}$-functor, that is, $[p^*h'] = [q_*h] \in H^3_{MacL}(R, M')$. The composition in $H^3_{\text{Ann}}$ is defined by
\[
(p', q') \circ (p, q) = (p'p, q'q).
\]
Note that, $\text{Ann}$-functors $F, F' : \mathcal{A} \to \mathcal{A}'$ are homotopic if and only if $F_i = F_i', i = 0, 1$ and $[g_F] = [g_{F'}]$ in $H^3_{MacL}(R, M)$. Denote by
\[
\text{Hom}_{\text{Ann}}^{(p, q)}[\mathcal{A}, \mathcal{A}']
\]
the set of homotopy classes of $\text{Ann}$-functors from $\mathcal{A}$ to $\mathcal{A}'$ inducing the same pair $(p, q)$, we prove the following classification result.
Theorem 13 (Classification Theorem). There is a functor
\[ d : \text{Ann} \to H^3_{\text{Ann}} \]
\[ \mathcal{A} \mapsto (\pi_0 \mathcal{A}, \pi_1 \mathcal{A}, [h_{\mathcal{A}}]) \]
which has the following properties:

i) \(dF\) is an isomorphism if and only if \(F\) is an equivalence,

ii) \(d\) is surjective on objects,

iii) \(d\) is full, but not faithful. For \((p, q) : \mathcal{A} \to \mathcal{A}'\), there is a bijection
\[ \tilde{d} : \text{Hom}_{\text{Ann}}^{\mathcal{A}}[\mathcal{A}, \mathcal{A}'] \to H^2_{\text{MacL}}(\pi_0 \mathcal{A}, \pi_1 \mathcal{A}'). \]
\[ (12) \]

Proof. In the Ann-category \( \mathcal{A} \), for each stick \((X_s, i_X)\) one can construct a reduced Ann-category \((\pi_0 \mathcal{A}, \pi_1 \mathcal{A}, h_{\mathcal{A}})\). If the choice of the stick is modified, then the 3-cocycle \(h\) changes to a cohomologous 3-cocycle \(h'\). Therefore, \( \mathcal{A} \) uniquely determines an element \([h] \in H^3(\pi_0 \mathcal{A}, \pi_1 \mathcal{A})\).

For Ann-functors \( \mathcal{A} \overset{F}{\to} \mathcal{A}' \overset{F'}{\to} \mathcal{A}'' \), it can be seen that \( d(F' \circ F) = dF' \circ dF \), and \( d(id_{\mathcal{A}}) = id_{d_{\mathcal{A}}} \). Therefore, \( d \) is a functor.

i) According to Proposition 7.

ii) If \((R, M, [h])\) is an object of \( H^3_{\text{Ann}} \), then \( S = (R, M, h) \) is an Ann-category of type \((R, M)\), and obviously \( dS = (R, M, [h]) \).

iii) If \((p, q)\) is a morphism in \( \text{Hom}_{H^3_{\text{Ann}}}^{\mathcal{A}}(d\mathcal{A}, d\mathcal{A}')\), then there is a function \( g = (\tau, \nu) : (\pi_0 \mathcal{A})^2 \to \pi_1 \mathcal{A}' \) satisfying the relation \((13)\), and therefore
\[ K = (p, q, g) : (\pi_0 \mathcal{A}, \pi_1 \mathcal{A}, [h_{\mathcal{A}}]) \to (\pi_0 \mathcal{A}', \pi_1 \mathcal{A}', [h_{\mathcal{A}'}]) \]
is an Ann-functor. Thus, the composition \( F = H'KG : \mathcal{A} \to \mathcal{A}' \) is an Ann-functor and \( dF = (p, q) \). This shows that \( d \) is full.

In order to obtain the bijection \((12)\), we prove that the correspondence
\[ \Omega : \text{Hom}_{\text{Ann}}^{\mathcal{A}}(\mathcal{A}, \mathcal{A}') \to \text{Hom}_{\text{Ann}}^{\mathcal{A}}(S_{\mathcal{A}}, S_{\mathcal{A}'}) \]
\[ [F] \mapsto [S_F] \]
\[ (13) \]
is a bijection.

Clearly, if \( F, F' : \mathcal{A} \to \mathcal{A}' \) are homotopic then induced Ann-functors \( S_F, S_{F'} \) are homotopic. Conversely, if \( S_F \) and \( S_{F'} \) are homotopic then the compositions \( E = H'(S_F)G \) and \( E' = H'(S_{F'})G \) are homotopic. Ann-functors \( E \) and \( E' \) are homotopic to \( F \) and \( F' \), respectively. So, \( F \) and \( F' \) are homotopic. This shows that \( \Omega \) is an injection.

Now, if \( K = (p, q, g) : S_{\mathcal{A}} \to S_{\mathcal{A}'} \) is an Ann-functor then the composition
\[ F = H'KG : \mathcal{A} \to \mathcal{A}' \]
is an Ann-functor with \( S_F = K \), that is, \( \Omega \) is surjective. Now, the bijection \((12)\) is the composition of \((13)\) and \((6)\). \( \square \)
Based on Theorem 13, Ann-categories having the same first two invariants can be classified up to equivalence.

Let \( R \) be a ring with a unit, \( M \) be an \( R \)-bimodule which is regarded as a ring with null-multiplication. We say that the Ann-category \( A \) has a pre-stick of type \( (R, M) \) if there is a pair of ring isomorphisms \( \epsilon = (p, q) \)

\[
p : R \to \pi_0 A, \quad q : M \to \pi_1 A
\]

which are compatible with the module action,

\[
q(su) = p(s)q(u),
\]

where \( s \in R, u \in M \). The pair \( (p, q) \) is called a pre-stick of type \( (R, M) \) to the Ann-category \( A \).

A morphism between two Ann-categories \( A, A' \) having pre-sticks of type \( (R, M) \) (with their pre-sticks are \( \epsilon = (p, q) \) and \( \epsilon' = (p', q') \), respectively) is an Ann-functor \( (F, \bar{F}, \tilde{F}) : A \to A' \) such that the following diagrams commute

\[
\begin{array}{ccc}
\pi_0 A & \xrightarrow{F_0} & \pi_0 A' \\
\downarrow p & & \downarrow p' \\
R & \xrightarrow{R} & \pi_0 A' \\
\pi_1 A & \xrightarrow{F_1} & \pi_1 A' \\
\downarrow q & & \downarrow q' \\
M & \xrightarrow{M} & \pi_1 A'
\end{array}
\]

where \( (F_0, F_1) \) is a pair of homomorphisms induced by \( (F, \bar{F}, \tilde{F}) \).

Clearly, it follows from the definition of an Ann-functor that \( F_0, F_1 \) are isomorphisms, therefore \( F \) is an equivalence.

Denote by

\[
\text{Ann}[R, M]
\]

the set of equivalence classes of Ann-categories whose pre-sticks are of type \( (R, M) \). One can prove the following result based on Theorem 13.

**Theorem 14.** There is a bijection

\[
\Gamma : \text{Ann}[R, M] \to H^3_{\text{MacL}}(R, M)
\]

\[
[A] \mapsto q_*^{-1} p^*[h_A]
\]

**Proof.** By Theorem 13 each Ann-category \( A \) determines a unique element \( [h_A] \in H^3_{\text{MacL}}(\pi_0 A, \pi_1 A) \), and hence an element

\[
\epsilon[h_A] = q_*^{-1} p^*[h_A] \in H^3_{\text{MacL}}(R, M).
\]

Now if \( F : A \to A' \) is a functor between Ann-categories whose pre-sticks are of type \( (p, q) \), then the induced Ann-functor \( S_F = (p, q, g_F) \) satisfies the relation (5), and therefore

\[
p^*[h_{A'}] = q_*[h_A].
\]
One can check that
\[ \epsilon'[h_A'] = \epsilon[h_A]. \]
This means \( \Gamma \) is a map. Moreover, it is an injection. Indeed, if \( \Gamma[A] = \Gamma[A'] \), then
\[ \epsilon(h_A) - \epsilon'(h_A') = \partial g. \]
Thus, there exists an Ann-functor \( J \) of type \((id, id)\) from \( \mathcal{I} = (R, M, \epsilon(h_A)) \) to \( \mathcal{I}' = (R, M, \epsilon'(h_A')) \). The composition
\[ \mathcal{A} \xrightarrow{G} S_A \xrightarrow{\epsilon^{-1}} \mathcal{I} \xrightarrow{J} \mathcal{I}' \xrightarrow{\epsilon'} S_A' \xrightarrow{H'} \mathcal{A}' \]
shows that \([A] = [A']\), and \( \Gamma \) is an injection. Obviously, \( \Gamma \) is surjective. □

In [9], the author proved that each structure of a regular Ann-category of type \((R, M)\) (that is, a structure satisfies the regular condition, \( \eta(x, x) = 0 \)) is an element in the group \( Z^3_{Sh}(R, M) \) of Shukla 3-cocycles. From Classification Theorem 4.4 [9] and Theorem [3] the following result is obtained.

**Corollary 15.** There is an injection

\[ H^3_{Sh}(R, M) \hookrightarrow H^3_{MacL}(R, M). \]

## 4 Appendix: A categorical ring which is not an Ann-category

Below, we construct a categorical ring which is not an Ann-category.

Let \( R \) be a ring with a unit and \( A \) be a \( R \)-bimodule. Then, one constructs a categorical ring \( \mathcal{R} \) as follows. First, \( \mathcal{R} \) is a category defined as in Section 2. The objects of \( \mathcal{R} \) are elements of \( R \), the morphisms in \( \mathcal{R} \) are automorphisms \((r, a) : r \to r, a \in A\). Composition is the addition on \( A \). Operations \( \oplus, \otimes \) on \( \mathcal{R} \) are given by

\[ r \oplus s = r + s, \quad (r, a) \oplus (s, b) = (r + s, a + b), \]
\[ r \otimes s = rs, \quad (r, a) \otimes (s, b) = (rs, rb + as). \]

Suppose that the system \((\mathcal{R}, \oplus, \otimes)\) has a left distributivity constraint

\[ \lambda_{r,s,t} : r(s + t) \to rs + rt \]

given by \( \lambda_{r,s,t} = (\bullet, \lambda(r, s, t)) \), where \( \lambda : R^3 \to A \), and other constraints are strict. Then, the commutative diagrams in the axioms of a categorical ring are equivalent to the equations

\[ R_1. \ r\lambda(s, t, u) - \lambda(rs, t, u) + \lambda(r, st, su) = 0, \]
\[ R_2. \ \lambda(r, s, t)u - \lambda(r, su, tu) = 0, \]
\[ R_3. \ \lambda(1, s, t) = 0, \]
Let $R$ be the ring of dual numbers on $\mathbb{Z}$, $R = \{a + be \mid a, b \in \mathbb{Z}, \epsilon^2 = 0\}$ and $A = \mathbb{Z} \cong R/\langle \epsilon \rangle$. Then, $A$ is a $R$-bimodule with the natural actions

$$(a + be)k = ak = k(a + be).$$

The function $\lambda : R^3 \to A$, defined by

$$\lambda(r, s, t) = b_r(a_s + a_t),$$

is satisfies the equations $R_1 - R_5$, so that $R$ is a categorical ring.

It is clear that if $b_r \neq 0$ and $a_s 
eq 0$, then $\lambda(r, 0) \neq 0$. Thus, by Theorem 3 $R$ is not an Ann-category.

One can deduce that:

1. Since the function $\lambda$ is not normalized, $\hat{h} = (0, \lambda, 0, 0) \notin Z^3_{\text{MacL}}(R, A)$. This means that the classification theorem in [2] is wrong.

2. The condition $(U)$ in the following theorem is necessary.

**Theorem 4** [10]. Each categorical ring $R$ is an Ann-category if and only if it satisfies the following condition.

$(U)$: Each of pairs $(L^A, \hat{L}^A)$, $(R^A, \hat{R}^A)$, $A \in R$, is an $\oplus$-functor which is compatible with the unit constraint $(0, g, d)$ with respect to the operation $\oplus$.

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**References**

[1] A. Fröhlich and C. T. C Wall, *Graded monoidal categories*, Compositio Math. 28 (1974). 229-285.

[2] M. Jibladze and T. Pirashvili, *Third Mac Lane cohomology via categorical rings*, J. Homotopy Relat. Struct. 2 (2007) 187-216.

[3] M. M. Kapranov and V. A. Voevodsky, *2-Categories and Zamolodchikov Tetrahedra Equations*, Proceedings of Symposia in Pure Mathematics, Vol. 56 (1994), Part 2, 177-259

[4] M. L. Laplaza, *Coherence for distributivity*, Lecture Notes in Math, 281 (1972), 29-65.

[5] M. L. Laplaza, *Coherence for Categories with Group Structure: an alternative approach*, J. Algebra, 84 (1983), 305-323.
[6] S. Mac Lane, *Extensions and obstruction for rings*, Illinois J. Mathematics, 2 (1958), 316-345.

[7] C. T. K. Phung, N. T. Quang, N. T. Thuy, *Relation between Ann-categories and ring categories*, Comm. Korean Math. Soc. 25, No 4 (2010), 523-535.

[8] N. T. Quang, *Introduction to Ann-categories*, J. Math. Hanoi, No.15, 4 (1987), 14-24. arXiv:math.CT/0702588v2 21 Feb 2007.

[9] N. T. Quang, *Structure of Ann-categories and Mac Lane-Shukla cohomology*, East-West J. Mathematics, Vol 5, No 1 (2003), 51-66.

[10] N. T. Quang, D. D. Hanh and N. T. Thuy, *On the Axiomatics of Ann-categories*, JP Journal of Algebra, Number Theory and Applications, Vol 11, No 1 (2008), 59-72.

[11] N. T. Quang, D. D. Hanh, *Homological classification of Ann-functors*, East-West J. of Mathematics, Vol 11, No 2 (2009), 195-210.

[12] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag Berlin and New York (1972).

[13] U. Shukla, *Cohomologie des algebras associatives*, Ann. Sci. Ecole Norm. Sup., 78 No 2 (1961), 163-209.

[14] H. X. Sinh, *Gr-catégories*, Université Paris VII, Thése de doctorat (1975).

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