Conic Parameterization in $PG(2, 25)$

Emad B. Al-Zangana*, Elaf A.S. Shehab

Department of Mathematics, College of Science, Mustansiriyah University, IRAQ

*Correspondent author email: e.b.abdulkareem@uomustansiriyah.edu.iq

**Abstract**

The main aim of this paper is to parameterize the conics form through the inequivalent 5-arcs in $PG(2, 25)$ using one-one correspondence property between line and conic. The inequivalent 6-arcs in $PG(2, 25)$, also have been computed with some examples.

**Keywords:** Projective space, arc, conic.

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**Introduction**

In $PG(2, q)$, the projective plane of order $q$, there have been many characterizations of the classical curves given by the zeros of quadratic forms called conics. For example, Al-Zangana studied the group effect on the conic in $PG(2, q), q=19, 29, 31$ [1] [2]. Also, Al-Zangana started to parameterized the conics through the inequivalent 5-arc in $PG(2, 19), PG(2, 23)$ [1][3]. It is worth mentioning that, the projective plane $PG(2, 25)$ has been studied by calculated the complete arcs only as in [4] [5].

The purpose of the research is to compute the 5-arc and then parameterized the conics through these 5-arc in $PG(2, 15)$. Also, in this paper, the inequivalent 6-arcs have been computed and then show that, there is a unique 6-arc with ten $B$-points but does not form a 10-arc.

**Preliminary**

**Definition 1**[6]. A k-arc, $K$ in projective plane $PG(2, q)$ is a set of $k$ points no three of them are collinear, but there is some two collinear. A $k$-set, $S$ in projective line $PG(1, q)$ is a set of $k$ distinct points.

**Definition 2**[6]. A line $\ell$ of $PG(2, q)$ is an i-secant of a k-arc $K$ if $|\ell \cap K| = i$. A 2-secant is called a bisecant, a 1-secant is a unisecant and a 0-secant is an external line. The number of bisecants through a point $Q$ out of $K$ is called the index of $Q$ with respect to $K$.

**Definition 3**[6]. Let $K$ be an arc and $c_i$ be the number of points of $PG(2, q) \setminus K$ with index exactly $i$. A point of index three is called a Brianchon point or $B$-point for short.

During this research, write $ij \cdot kl \cdot mn = P_1P_2\cap P_3P_4\cap P_5P_6$ for $B$-point, where $P_iP_j$ represent the line through the points $P_i$ and $P_j$.

**Definition 4**[6]. The zero set of the form $F$ of degree two

$$V(F) = V(aX_0^2 + bX_1^2 + cX_2^2 + dX_0X_1 + eX_0X_2 + fX_1X_2)$$

is called plane quadric. A non-singular plane quadric is called conic.

For details about groups that appear in this paper like, $Z_n \cong Z_m = \text{semi direct product group}$. $S_n = \text{symmetric group of degree n}$, $V_4 = \text{Klein 4-group}$ and $A_n = \text{alternating group of degree n}$, see [7].

To start with this research, the points and lines of $PG(2,25)$ are needed to construct.

The projective plane of order twenty five, $PG(2,25)$, has 651 points and lines, 26 points
on each line and 26 lines passing through each point.
Let \( (X) = X^3 - \beta^{16} X - \beta \in F_{25}[X] \), where \( \beta \) is the primitive element of \( F_{25} \). Then \( f \) is primitive polynomial over \( F_{25} \) since
\[
f(0) = \beta^{13}, \quad f(1) = \beta^2, \quad f(\beta) = \beta^{21}, \quad f(\beta^2) = \beta^{10}, \quad f(\beta^3) = \beta^5, \quad f(\beta^4) = \beta, \quad f(\beta^5) = \beta^{16}, \quad f(\beta^6) = \beta^{11}, \quad f(\beta^7) = \beta^5, \quad f(\beta^8) = \beta^{13}, \quad f(\beta^{11}) = \beta^{23}, \quad f(\beta^{10}) = \beta^{20}, \quad f(\beta^{12}) = \beta^{16}, \quad f(\beta^{25}) = \beta, \quad f(\beta^{14}) = \beta^5, \quad f(\beta^{15}) = \beta^{10}, \quad f(\beta^{16}) = \beta^7, \quad f(\beta^{17}) = \beta^6, \quad f(\beta^{18}) = \beta^{18}, \quad f(\beta^{19}) = \beta^{10}, \quad f(\beta^{20}) = \beta^{13}, \quad f(\beta^{21}) = \beta^{15}, \quad f(\beta^{22}) = 1, \quad f(\beta^{23}) = \beta^6.
\]
That is, \( f \) irreducible over \( F_{25} \), but \( f \) has three zeros \( \gamma, \gamma^{25}, \gamma^{625} \) in \( F_{25^3} \), where \( \gamma \) is the primitive element of \( F_{25^3} \). Therefore, the companion matrix of \( f \)
\[
C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta & \beta^{16} & 0 \end{pmatrix}
\]
cycle is projectivity, and then the points of \( PG(2,25) \) are
\[
P(i) = (1,0,0)C(f)^i.
\]
Dually, the lines of \( PG(2,25) \) are
\[
P(i) = \ell \ell_1 C(f)^i,
\]
where \( i = 0,1,\ldots,650 \) and \( \ell \ell_1 = V(X_2) \). The line \( \ell \ell_1 \) in numeral form is
\[
1, 2, 4, 44, 65, 74, 93, 162, 170, 176, 215, 252, 269, 310, 397, 422, 454, 472, 501,
\]
\[
506, 516, 528, 532, 539, 552, 587.
\]
For a comprehensive bibliography and more theoretical details about the lines and points structure see [6], and about field theory see [8].

**Inequivalent 5-Arcs**
From the fundamental theorem of projective geometry, there is projectively a unique 4-arc called frame. The stabilizer group of any 4-arc is \( S_4 \). Let \( \Gamma_{25} = \{U_0, U_1, U_2, U\} \) be the representative 4-arc (standard frame) where
\[
U_0 = [1,0,0] = P(0), \quad U_1 = [0,1,0] = P(1), \quad U_2 = [0,0,1] = P(2), \quad U = [1,1,1] = P(603).
\]
The 5-arcs are formed by adding points of index zero and the inequivalent one are computed using mathematical program language Gap as summarized in the following theorem.

**Theorem 5.** In \( PG(2,25) \), there are eight inequivalent 5-arcs through \( \Gamma_{25} \). The values of the constants \( c_i \) for any 5-arc are \( c_0 = 421; \ c_1 = 210; \ c_2 = 15 \). These arcs with their stabilizer group types are given in Table 1.

| \( \mathcal{A}_i \) | The 5-arc | SG- type |
|---|---|---|
| \( \mathcal{A}_1 \) | \( \Gamma_{25}UP(\beta^{16}, \beta^6, 1) \) | \( Z_2 \) |
| \( \mathcal{A}_2 \) | \( \Gamma_{25}UP(\beta^{18}, \beta^{15}, 1) \) | \( I \) |
| \( \mathcal{A}_3 \) | \( \Gamma_{25}UP(\beta^7, \beta^{10}, 1) \) | \( Z_2 \) |
| \( \mathcal{A}_4 \) | \( \Gamma_{25}UP(\beta^{10}, \beta^9, 1) \) | \( Z_2 \) |
| \( \mathcal{A}_5 \) | \( \Gamma_{25}UP(\beta^{18}, \beta^{15}, 1) \) | \( Z_5 \times Z_4 \) |
| \( \mathcal{A}_6 \) | \( \Gamma_{25}UP(\beta^{22}, \beta^{23}, 1) \) | \( Z_2 \) |
| \( \mathcal{A}_7 \) | \( \Gamma_{25}UP(\beta^{14}, \beta^{18}, 1) \) | \( I \) |
| \( \mathcal{A}_8 \) | \( \Gamma_{25}UP(\beta^{20}, \beta, 1) \) | \( S_3 \) |

**Conic Representation through 5- Arc**
It is well known that, through any 5-arc there is a unique conic and the rational points \( X \) of the conic \( C^* = V(X_1 - X_0X_2) \) parameterized as \((t^2, t, 1) \) [6]. So, There is a unique conic through each 5-arc, \( \mathcal{A}_i \), and since each of this arcs passes through \( \Gamma_{25} \), therefore, each conic \( C_{\mathcal{A}_i} \), take the form
\[
C_{\mathcal{A}_i} = V(F_{\mathcal{A}_i}) = X_0X_1 + aX_0X_2 - (a + 1)X_1X_2.
\]
After substituted the fifth point of the arcs \( \mathcal{A}_i \) into \( F_{\mathcal{A}_i} \) the following are deduced.
Lemma 6 [9].
On $PG(1,25)$, there are precisely eight distinct pentads given with their stabilizer groups in Table 2 and Table 3.

| Type | The pentads |
|------|-------------|
| $P_1$ | $\{\infty, 0, 1, \beta^{12}, \beta^6\}$ |
| $P_2$ | $\{\infty, 0, 1, \beta^{12}, \beta^2\}$ |
| $P_3$ | $\{\infty, 0, 1, \beta^{12}, \beta^2\}$ |
| $P_4$ | $\{\infty, 0, 1, \beta^{12}, \beta^3\}$ |
| $P_5$ | $\{\infty, 0, 1, \beta^4, \beta^2\}$ |
| $P_6$ | $\{\infty, 0, 1, \beta^4, \beta^5\}$ |
| $P_7$ | $\{\infty, 0, 1, \beta, \beta^2\} \quad \Rightarrow \quad \{\infty, 0, 1, \beta, \beta^2\}$ |
| $P_8$ | $\{\infty, 0, 1, \beta, \beta^6\}$ |

Using the corresponding properties between $PG(1,25)$ and the conic $C^*$, the eight 5-sets, $P_i$ in Table 2 are transformed by $t \mapsto (t^2, t, 1)$ into 5-arcs, $P_i^*$ in $C^*$ but not through the frame $\Gamma_{25}$, where $C^* = \{1, 3, 19, 42, 47, 111, 149, 157, 174, 210, 217, 273, 288, 303, 325, 348, 357, 416, 430, 466, 509, 549, 597, 603, 623, 631\}$. Each $P_i^*$ is projectively equivalent to 5-arc, $\mathcal{A}_i$ as given below.

- $P_1^* = \{1,3,603,357,210\}$
- $P_2^* = \{1,3,603,357,273\}$
- $P_3^* = \{1,3,603,357,42\}$
- $P_4^* = \{1,3,603,357,228\}$
- $P_5^* = \{1,3,603,111,42\}$

| Type | SG-type |
|------|----------|
| $P_1$ | $Z_5 \times Z_4 = \langle 1/(t + \beta^{12}), (t\beta^{18} + \beta^{12}) \rangle$ |
| $P_2$ | 1 |
| $P_3$ | $Z_2 = ((t + 1)/(t + \beta^{12}))$ |
| $P_4$ | 1 |
| $P_5$ | $Z_2 = (\beta^4/t)$ |
| $P_6$ | $S_3 = ((\beta^8 t + 1), \beta^5 t/(t + \beta^{17}))$ |
| $P_7$ | $Z_2 = (\beta^2/t)$ |
| $P_8$ | $Z_2 = (t/(t + \beta^{12}))$ |
Theorem 7. By uniqueness properties of conics, the parameterization of each conic $C_{\mathcal{A}_i}$ are given below using the matrix transformation between $C^*$ and $C_{\mathcal{A}_i}$. Let $t \in F_{25} \cup \{\infty\}$.

### Inequivalent 6-Arcs

After calculating the orbit of each 5-arc $\mathcal{A}_i$ and adding one point from each orbit to $\mathcal{A}_i$, the 6-arcs are constructed. In the following theorem the details of inequivalents 6-arcs are given.

**Theorem 8:** In $PG(2,25)$, there are 365 inequivalent 6-arcs through the standard frame. These arcs partitioned according to stabilizer group types and the parameters $[c_0, c_1, c_2, c_3]$ as given below.

| $C_{\mathcal{A}_i}$ | Matrix trans. of $C_{\mathcal{A}_i}$ to $C^*$ | Parameterization of $C_{\mathcal{A}_i}$ |
|---------------------|---------------------------------------------|----------------------------------------|
| $C_{\mathcal{A}_1}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{10} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-1}(t^2-\beta^{-1}t),\beta^{-10}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_2}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{10} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-1}(t^2-\beta^{-1}t),\beta^{-10}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_3}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{4} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-11}(t^2-\beta^{-1}t),\beta^{-4}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_4}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{19} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-15}(t^2-\beta^{-1}t),\beta^{-19}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_5}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{14} \\ \beta^{15} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-10}(t^2-\beta^{-1}t),\beta^{-15}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_6}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{5} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-13}(t^2-\beta^{-1}t),\beta^{-5}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_7}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{13} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-2}(t^2-\beta^{-1}t),\beta^{-13}(1-\beta t), \beta^{-13}t)$ $\}$ |
| $C_{\mathcal{A}_8}$ | $\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \beta^{6} \\ \beta^{12} & \beta^{13} & \beta^{14} \end{pmatrix}$ | $\{$ $P(\beta^{-20}(t^2-\beta^{-1}t),\beta^{-6}(1-\beta t), \beta^{-13}t)$ $\}$ |

| SG-type:No. | |
|--------------|----------|
| $l$: 255     |          |
| $Z_7$: 53    |          |
| $Z_3^2$: 29  |          |
| $V_4$: 5, $Z_4$: 4 |   |
| $S_3$: 12    |          |
| $A_4$: 5     |          |
| $G_{36}$: 1  |          |
| $S_5$: 1     |          |

The elements of the group $G_{36}$ have order as follows.
\[ G_{36} \]

| Ord(g):No. |
|------------|
| 2: 9       |
| 3: 8       |
| 4: 18      |

| \([c_0, c_1, c_2, c_3]\) | :No. |
|--------------------------|------|
| [320, 300, 15, 10]       | :1   |
| [324, 288, 27, 6]        | :6   |
| [326, 282, 33, 4]        | :9   |
| [327, 279, 36, 3]        | :32  |
| [328, 276, 39, 2]        | :50  |
| [329, 273, 42, 1]        | :133 |
| [330, 270, 45, 0]        | :134 |

Example 9:
The unique 6-arc with stabilizer group of order 120 and ten \( B \)-points is \( \mathcal{H} = A_5 \cup P(\beta^{12}, \beta^{18}, 1) \).
The arc \( \mathcal{H} \) in numeral form is \( \{1, 2, 3, 603, 17, 430\} \).
The ten \( B \)-points of \( \mathcal{H} \) in numeral form is \( \mathcal{K}_{10} = \{176, 93, 396, 268, 624, 380, 533, 351, 517, 574\} \), where

| \(ij \cdot kl \cdot mn\) | Point in coordinate form | Point in numeral form |
|-------------------------|--------------------------|-----------------------|
| 12 \cdot 34 \cdot 56    | \(P(1,1,0)\)             | 176                   |
| 12 \cdot 35 \cdot 46    | \(P(\beta^{12}, 1,0)\)   | 93                    |
| 13 \cdot 24 \cdot 56    | \(P(1,0,1)\)             | 396                   |
| 13 \cdot 26 \cdot 45    | \(P(\beta^{12}, 0,1)\)   | 268                   |
| 14 \cdot 25 \cdot 36    | \(P(\beta^{18}, 1,1)\)   | 624                   |
| 14 \cdot 26 \cdot 35    | \(P(\beta^{12}, 1,1)\)   | 380                   |
| 15 \cdot 23 \cdot 46    | \(P(0, \beta^6, 1)\)     | 533                   |
| 15 \cdot 24 \cdot 36    | \(P(1, \beta^6, 1)\)     | 351                   |
| 16 \cdot 23 \cdot 45    | \(P(0, \beta^{18}, 1)\)  | 517                   |
| 16 \cdot 25 \cdot 34    | \(P(\beta^{18}, \beta^{18}, 1)\) | 574 |

The set \( \mathcal{K}_{10} \) does not form 10-arc since it has ten 3-secants as given below.

| \(\mathcal{K}_{10} \cap \ell_{93}\) | = 93,268,624 |
| \(\mathcal{K}_{10} \cap \ell_{112}\) | = 176,380,533 |
| \(\mathcal{K}_{10} \cap \ell_{176}\) | = 176,268,351 |
| \(\mathcal{K}_{10} \cap \ell_{265}\) | = 268,533,574 |
| \(\mathcal{K}_{10} \cap \ell_{323}\) | = 93,396,574 |
| \(\mathcal{K}_{10} \cap \ell_{348}\) | = 93,351,517 |

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