PHASE TRANSITIONS OF THE SIR RUMOR SPREADING MODEL WITH A VARIABLE TRUST RATE

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Abstract. We study a threshold phenomenon of rumor outbreak on the SIR rumor spreading model with a variable trust rate depending on the populations of ignorants and spreaders. Rumor outbreak in the SIR rumor spreading model is defined as a persistence of the final rumor size in the large population limit or thermodynamics limit \( (n \to \infty) \), where \( 1/n \) is the initial population of spreaders. We present a rigorous proof for the existence of threshold on the final size of the rumor with respect to the basic reproduction number \( R_0 \). Moreover, we prove that a phase transition phenomenon occurs for the final size of the rumor (as an order parameter) with respect to the basic reproduction number and provide a criterion to determine whether the phase transition is of first or second order. Precisely, we prove that there is a critical number \( R_1 \) such that if \( R_1 > 1 \), then the phase transition is of the first order, i.e., the limit of the final size is not a continuous function with respect to \( R_0 \). The discontinuity is a jump-type discontinuity and it occurs only at \( R_0 = 1 \). If \( R_1 < 1 \), then the phase transition is second order, i.e., the limit of the final size is continuous with respect to \( R_0 \) and its derivative exists, except at \( R_0 = 1 \), and the derivative is not continuous at \( R_0 = 1 \). We also present numerical simulations to demonstrate our analytical results for the threshold phenomena and phase transition order criterion.

1. Introduction. With the development of information and communication technology (ICT), the speed of rumor spreading has greatly increased and rumors are being without restrictions in space. For example, a widespread rumor held that vaccines cause autism and this rumor led people to the anti-vaccine movement interactions over the Internet [7]. As a result, the number of measles cases is increasing again [11]. In addition to ICT, as society develops, the factors that affect rumors are becoming more diverse, such that it has become very difficult to control rumors that have a negative impact on society. Therefore, to reduce the damage caused by rumors in our society, it is necessary to understand them. Many researchers have

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studied the structure of rumor transfer from the macroscopic perspective using the SIR model.

We focus on why some rumors spread wildly and some rumors spread gradually. The difference between these phenomena is closely related to the phase transition order. We study the threshold phenomena of the thermodynamic limit for the SIR rumor spreading equation and its phase transition order. To understand the dynamics of rumors, we consider a simple model \[2, 3\] based on the SIR equation that conventionally consists of three population densities of ignorants \(I\), spreaders \(S\), and recovered \(R\). Herein, the recovered population in the original epidemic model, \(R\), is comprised of stiflers, because they suppress the spread of the rumor.

We assume the following mechanisms of rumor spreading:

(i) When ignorant \((I)\) meets a spreader \((S)\), he or she believes the rumor with a trust rate \(\lambda\), and becomes spreader.

(ii) A number of spreaders lose their interest in the rumor and they become stiflers \((R)\) with a constant rate \(\delta\).

(iii) When a spreader meets other spreader or stifler, the spreader becomes the stifler because they consider the rumor to be outdated.

The corresponding mathematical model of rumor spreading is given by

\[
\begin{align*}
\dot{I}(t) &= -k\lambda(I(t), S(t))S(t)I(t), \\
\dot{S}(t) &= k\lambda(I(t), S(t))S(t)I(t) - kS(t)(\sigma_S S(t) + \sigma_R R(t)) - \delta S(t), \\
\dot{R}(t) &= kS(t)(\sigma_S S(t) + \sigma_R R(t)) + \delta S(t),
\end{align*}
\]

subject to the initial data

\[
I(0) = I_0, \quad S(0) = S_0, \quad R(0) = R_0,
\]

where \(k\) is the average degree of the network, \(\lambda(x, y)\) is the trust rate of the system, \(\sigma_S\) is the transition rate between spreaders, \(\sigma_R\) is the transition rate between spreaders and stiflers, and \(\delta\) denotes the decay rate of spreaders to stiflers. Note that if we assume \(\sigma_S = \sigma_R = 0\), then the system in (1.1) becomes the classical SIR model, describing epidemic phenomena with incidence rate \(\lambda\).

We assume that all the coefficients \(\lambda(I, S)\), \(\sigma_S\), \(\sigma_R\), \(\delta\), and the initial data \(I^0, S^0, R^0\) are non-negative. Moreover, for simplicity, we assume that \(k = 1\) and

\[
I^0 + S^0 + R^0 = 1.
\]

We notice that the non-negativity assumptions regarding the coefficients and initial conditions are natural. In real situations, the trust rate could change due to the densities of ignorants and spreaders. For example, the conformity effect is one of the important factors in determining the dynamics of rumor spreading \[14\], so the population structure, such as the network topology, plays an important role in rumor acceptance. See also \[12, 16\] for the relation between rumor spreading and psychological effects. To reflect this phenomenon in our model, we assume that \(\lambda\) is a function of the two variables \(I\) and \(S\):

\[
\lambda = \lambda(I, S).
\]

See \[1, 5, 17\] for similar models with general incidence (trust) rate. Throughout this paper, for technical reasons, we assume that \(\lambda \in C^1\) so there is a uniform upper bound \(\Lambda\) such that

\[
0 \leq ||\lambda(\cdot, \cdot)||_{C^1} \leq \Lambda.
\]

(1.3)
For a solution \((I(t), S(t), R(t))\) to the system (1.1) with non-negative initial data, we define the rumor size \(\phi(t)\) and final rumor size \(\phi^\infty\) in the following way:

\[
\phi(t) = \int_0^t S(\tau)d\tau, \quad \phi^\infty(I^0, S^0) := \lim_{t \to \infty} \phi(t).
\]  

(1.4)

At the initial stage, we assume that there are few people who know the rumor. Thus, we consider the following initial data.

\[
I^0_n = 1 - \frac{1}{n}, \quad S^0_n = \frac{1}{n}, \quad R^0_n = 0, \quad n \in \mathbb{N}.
\]  

(1.5)

We specially denote the solution to (1.1) subject to the initial condition (1.5) by \((I_n, S_n, R_n)\) and also define rumor size \(\phi_n(t)\), and \(\phi^\infty_n\) as

\[
\phi_n(t) = \int_0^t S_n(\tau)d\tau, \quad \phi^\infty_n = \phi^\infty_n(I^0_n, S^0_n) = \lim_{t \to \infty} \phi_n.
\]  

(1.6)

We now define rumor outbreak as follows:

**Definition 1.1.** [15, 19, 20] Let \((I_n, S_n, R_n)\) be the solution to (1.1) subject to (1.5) and \(\phi^\infty_n\) be the corresponding final rumor size of \((I_n, S_n, R_n)\) defined in (1.6). We say that a rumor outbreak occurs if the following limit exists,

\[
\phi^e := \lim_{n \to \infty} \phi^\infty_n
\]

and its value is strictly positive.

**Remark 1.** If we choose initial data \((1, 0, 0) = \lim_{n \to \infty} (I^0_n, S^0_n, R^0_n)\) for the solution \((I, S, R)\) to (1.1), then the solution satisfies \((I(t), S(t), R(t)) \equiv (1, 0, 0)\). Therefore, in general, for the solutions \((I_n, S_n, R_n)\) to (1.1) subject to the initial data \((I^0_n, S^0_n, R^0_n)\), we have \((I_n, S_n, R_n) \to (I, S, R)\) as \(n \to \infty\), where \((I, S, R)\) is the solution to (1.1) subject to the initial data \((1, 0, 0)\). Therefore, to obtain the persistence of a rumor in the large population limit, considering \(\phi^e\) in Definition 1.1 is natural. See [13] for a general reference.

We now define the basic reproduction numbers \(R_0\) and \(R_1\) by

\[
R_0 = \frac{\lambda^0}{\delta}, \quad R_1 = -\frac{\lambda^0_I + \sigma_R}{\lambda^0},
\]

where

\[
\lambda^0 = \lambda(1, 0), \quad \lambda^0_I = \frac{\partial \lambda}{\partial I}(1, 0), \quad \lambda^0_S = \frac{\partial \lambda}{\partial S}(1, 0).
\]

Similar to the classical results, we will show that at \(R_0 = 1\), threshold phenomena exist for rumor spreading. Moreover, at the critical case \(R_1 = 1\), the value \(R_1\) determines the positivity of \(\phi^e\).

We define an autonomous auxiliary system as follows.

\[
F'_I(x) = -\lambda(I(x), S(x))F_I(x),
\]
\[
F'_S(x) = \lambda(I(x), S(x))F_I(x) - \sigma_S F_S(x) - \sigma_R F_R(x) - \delta,
\]
\[
F'_R(x) = \sigma_S F_S(x) + \sigma_R F_R(x) + \delta, \quad x > 0,
\]

subject to

\[
F_I(0) = I^0, \quad F_S(0) = S^0, \quad F_R(0) = R^0.
\]  

(1.7)

(1.8)
**Theorem 1.** For a given $\lambda(\cdot, \cdot)$ satisfying (1.3), let $(I_n, S_n, R_n)$ be a series of solutions to system (1.1) subject to (1.5) and let $(F_I, F_S, F_R)$ be a solution with the initial condition

$$(F_I(0), F_S(0), F_R(0)) = (1, 0, 0).$$

We additionally assume that for the smallest positive root $x_\infty$ of $F_S(x) = 0$,

$$F'_S(x_\infty) < 0$$  \hspace{1cm} (1.9)

when $R_0 > 1$ or $R_0 = 1, R_1 > 1$.

Then, the following holds:

1. The equation $R_0 = 1$ is a rumor spreading threshold. In other words, there exists $\phi^c$ which is positive if $R_0 > 1$, and $\phi^c = 0$ if $R_0 < 1$.

2. When $R_0 = 1, R_1 = 1$ also gives a threshold for the rumor spreading. In particular, $\phi^c > 0$ when $R_1 > 1$, and $\phi^c = 0$ when $R_1 < 1$.

**Remark 2.**

1. We note that by the conformity effect, $\lambda_S^0 \geq 0$ is a natural assumption. For the linear case $\lambda(I, S) = \lambda^0 - \lambda_S^0(1 - I) + \lambda_S^0 I$ under the assumptions $\sigma_R \geq \sigma_S$ and $\lambda_S^0 \geq 0$, by using phase-plane analysis, we easily check that (1.9) in Theorem 1 holds:

$$F'_S(x_\infty) < 0$$

for $R_0 \geq 1$. For the proof of this statement, see the Appendix.

2. With higher regularity assumptions on $\lambda$, we inductively define the third- and higher-order threshold equations

$$R_2 = 1, \quad R_3 = 1, \quad R_4 = 1, \ldots$$

for properly defined values $R_2, R_3, \ldots$ which comes from the Taylor expansion of $\lambda$.

3. The auxiliary equations in (1.7) allow us to simplify the proof of the main theorem. Moreover, they play a crucial role in the proof for the existence of the limit, $\lim_{n \to \infty} \phi_n^c$ when $R_0 > 1$ or $R_0 = 1, R_1 > 1$. However, we cannot use (1.7) for other cases, $R_0 < 1$ or $R_0 = 1, R_1 < 1$. For these cases and the continuity result of $\phi^c$, we need more detailed comparison between $S$ and $1 - I$. See Section 3 for a more detailed argument.

**Corollary 3.** Usually, the first-order phase transition exhibits that the free energy is discontinuous with respect to a thermodynamic variable at the phase transition point and the second-order phase transition exhibits that the free energy is continuous with respect to a thermodynamic variable but its derivative is discontinuous at the phase transition point. In this paper, we prove that if $R_1 > 1$, then the phase transition is of first order in the following sense: the final size $\lim_{n \to \infty} \int_0^\infty S(t) dt$ is not a continuous function with respect to $R_0$, as seen in Figure 5(A). If $R_1 < 1$, then the phase transition is of second order: the final size $\lim_{n \to \infty} \int_0^\infty S(t) dt$ is continuous with respect to $R_0$ but its derivative is not continuous, as seen in Figure 5(B). In the both cases, the discontinuities occur at $R_0 = 1$. For the references of epidemic outbreaks and the large population limit, see [8, 13].

Throughout this paper, for fixed coefficients of $\lambda(\cdot, \cdot), \sigma_S, \sigma_R \geq 0$, we consider $\delta$ as a tuning parameter for the phase transition. Then, the reproduction constant $R_1$ is independent of $\delta$. Moreover, $R_0$ is completely determined by $\delta$. As mentioned above, we present the complete criterion for the first- and second-order phase transition phenomena with respect to $\delta$.

The following is the rest of the main result in this paper.
Theorem 2. Under the same conditions on Theorem 1, \( \phi^e = \phi^e(\delta) \) is a differentiable function on each interval \((0, \lambda_0]\) and \([\lambda_0, \infty)\). Moreover, if \( R_1 > 1 \), then \( \phi^e \) has a jump discontinuity at \( \delta = \lambda_0^0 \), or if \( R_1 < 1 \), then \( \phi^e \) is continuous at \( \delta = \lambda_0^0 \) but \( d\phi^e/d\delta \) is discontinuous at \( \delta = \lambda_0^0 \).

The physical meaning of Theorem 2 is as follows. As a well-known result, the basic reproduction number plays an important role in this epidemiological dynamic, as well as in Theorem 1. Here, we focus on the originality of the rumor dynamics. \( \sigma_S \) expresses how spreaders interact with each other. \( \sigma_R \) expresses the interaction between spreaders and stiflers. At least, in this thermodynamical limit sense, \( \sigma_R \) has a greater impact on the spreading dynamics than \( \sigma_S \). For the case of \( R_1 > 1 \), i.e., \(-\lambda_0^0 > \delta + \sigma_R \), when the basic reproduction number increases and it crosses the threshold, the rumor will spread explosively. However, For the case of \( R_1 < 1 \), although the basic reproduction number increases and it crosses the threshold, the rumor will slowly increase and spread mildly. For example, if a rumor is logically convincing, then stiflers will not disturb the spreaders and \( \sigma_R \) is small. Contrarily, if a rumor is not logically convincing or a rumor is definitely not true, stiflers will disturb the spreaders and \( \sigma_R \) is relatively large. Rationally, the former case would lead to rapid spread if the rumor begins to spread, and the latter case would have a relatively weak effect due to the resistance, even though the rumor will spread.

We close this section with a brief review of the previous literature concerning the thermodynamic limit problem and SIR-type rumor spreading models. The thermodynamic limit problem on the epidemic SIR model is studied in [8]. In this paper, the authors considered the basic SIR model and provide a formal proof. See also [9, 10]. The authors in [15] propose a SIR-type rumor spreading model with a trust mechanism in inhomogeneous networks as well as in homogeneous networks. By using steady-state analysis in formal computation, the authors investigate the thresholds which allow one to determine the rumor spreading. Their consideration holds for \( \sigma_S = \sigma_R \). Hong et al. [6] studied the SIR rumor model of several types of ignorant with different constant trust rates. They rigorously proved the existence of the rumor outbreak threshold with the same condition \( \sigma_S = \sigma_R \). We refer to [18] and [19] for a counterattack mechanism and hibernation mechanism in the rumor spreading dynamics, respectively.

The rest of this paper is organized as follows. In Section 2, we derive (1.7) from (1.1) by introducing the functional of the final rumor size and consider the final rumor size as a variable. Using this auxiliary functional, we analytically study the dynamic behavior of the SIR rumor spreading model for \( R_0 > 1 \) and \( R_0 = 1 \), \( R_1 > 1 \). In Section 3, we provide a rigorous proof of the main theorems. In Section 4, we provide some numerical simulations to demonstrate our results obtained in the previous sections.

2. A system of differential equations for rumor size: Steady-state analysis. In this section, by considering a variable rumor size \( \phi \), we derive the auxiliary system of equations

\[
\begin{align*}
F'_I(x) & = -\lambda(F_I(x), F_S(x))F_I(x), \\
F'_S(x) & = \lambda(F_I(x), F_S(x))F_I(x) - \sigma_S F_S(x) - \sigma_R F_R(x) - \delta, \\
F'_R(x) & = \sigma_S F_S(x) + \sigma_R F_R(x) + \delta,
\end{align*}
\]

subject to

\[
F_I(0) = I^0, \quad F_S(0) = S^0, \quad F_R(0) = R^0.
\]
We demonstrate that the composition of \( \phi \) and the three nonnegative functions \( F_I \), \( F_S \) and \( F_R \) are exactly the same as the density of ignorants \( I \), spreaders \( S \), and stiflers \( R \), respectively.

Similar to \((I_n, S_n, R_n)\), we denote \((F^n_I(x), F^n_S(x), F^n_R(x))\) as solutions corresponding to the initial data

\[
F^n_I(0) = 1 - \frac{1}{n}, \quad F^n_S(0) = \frac{1}{n}, \quad F^n_R(0) = 0, \quad n \in \mathbb{N}. \tag{2.3}
\]

We also prove that \( F^n_S(x) \) converges uniformly on \( x \in [0, 1/\delta] \) as \( n \to \infty \). From this convergence result, we will obtain that the final size of the rumor \( \phi_n \) converges as \( n \to \infty \) in the next section.

**Lemma 2.1.** Let \((I(t), S(t), R(t))\) be a solution to system \((1.1)\) with nonnegative initial data \((1.2)\) and \((F_I(x), F_S(x), F_R(x))\) be a solution to \((2.1)\) with a boundary data \((2.2)\). Then we have

\[
I(t) = F_I(\phi(t)), \quad S(t) = F_S(\phi(t)), \quad R(t) = F_R(\phi(t)),
\]

where \( \phi(t) \) is the rumor size defined in \((1.4)\).

**Proof.** We define

\[
I_F(t) := F_I(\phi(t)), \quad S_F(t) := F_S(\phi(t)), \quad R_F(t) := F_R(\phi(t)).
\]

Then, by \((2.1)\),

\[
\dot{I}_F(t) = F'_I(\phi(t))\dot{\phi}(t) = -\lambda(F_I(\phi(t)), F_S(\phi(t)))F_I(\phi(t))S(t).
\]

By the definition of \( I_F, S_F \) and \( R_F \), we have

\[
\dot{I}_F(t) = -\lambda(I_F(t), S_F(t))I_F(t)S(t). \tag{2.4}
\]

Similarly, we have

\[
\dot{S}_F(t) = F'_S(S_F(t))\dot{\phi}(t) = \lambda(I_F(t), S_F(t))I_F(t)S(t) - S(t)[\sigma_S S_F(t) + \sigma_R R_F(t)] - \delta S(t), \tag{2.5}
\]

and

\[
\dot{R}_F(t) = F'_R(S_F(t))\dot{\phi}(t) = S(t)[\sigma_S S_F(t) + \sigma_R R_F(t)] + \delta S(t). \tag{2.6}
\]

From the initial data \((2.2)\) and \( \phi(0) = 0 \), it follows that

\[
I_F(0) = I^0, \quad S_F(0) = S^0, \quad R_F(0) = R^0.
\]

By subtracting \((1.1)\) from \((2.4)-(2.6)\), we obtain

\[
\begin{align*}
\dot{I}_F(t) - \dot{I}(t) & = -[\lambda(I_F(t), S_F(t))I_F(t) - \lambda(I(t), S(t))I(t)]S(t), \\
\dot{S}_F(t) - \dot{S}(t) & = [\lambda(I_F(t), S_F(t))I_F(t) - \lambda(I(t), S(t))I(t)]S(t) \\
& \quad - S(t)[\sigma_S(S_F(t) - S(t)) + \sigma_R(R_F(t) - R(t))], \\
\dot{R}_F(t) - \dot{R}(t) & = S(t)[\sigma_S(S_F(t) - S(t)) + \sigma_R(R_F(t) - R(t))].
\end{align*} \tag{2.7}
\]

We multiply the first equation in \((2.7)\) by \( I_F(t) - I(t) \) to obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & |I_F(t) - I(t)|^2 \\
& = -[\lambda(I_F(t), S_F(t))I_F(t) - \lambda(I(t), S(t))I(t)]S(t)(I_F(t) - I(t)).
\end{align*}
\]
Proposition 2.2. Let \((F^n_t, F^n_S, F^n_R)\) be the solution to (2.1) subject to the initial data (2.3) and let \((F_I, F_S, F_R)\) be the solution to (2.1) subject to the initial data \(F_I(0) = 1\), and \(F_S(0) = F_R(0) = 0\). Assume that \(\lambda(\cdot, \cdot)\) is in \(C^1\). Then, 
\[(F^n_t, F^n_S, F^n_R) \to (F_I, F_S, F_R) \quad \text{in} \quad C^1(0, 1/\delta)\] 
as \(n \to \infty\).

Proof. By subtracting two equations for \(F^n_t(x)\) and \(F_I(x)\), we obtain that
\[
(F^n_t)' - (F_I)' = -\lambda(F^n_I, F^n_S)F_t^n + \lambda(F_I, F_S)F_I \\
= -\lambda(F^n_I, F^n_S)(F^n_t - F_I) - (\lambda(F^n_I, F^n_S) - \lambda(F_I, F_S))F_I \\
= -\lambda(F^n_I, F^n_S)(F^n_t - F_I) - (\lambda(F^n_I, F^n_S) - \lambda(F_I, F_S))F_I \\
- (\lambda(F^n_I, F^n_S) - \lambda(F_I, F_S))F_I.
\]

We multiply the both sides of the above equation by \((F^n_t(x) - F_I(x))\) to obtain
\[
\frac{1}{2} \frac{d}{dx} |F^n_t - F_I|^2 = -\lambda(F^n_I, F^n_S)|F^n_t - F_I|^2 \\
- |\lambda(F^n_I, F^n_S) - \lambda(F_I, F_S)|(F^n_t - F_I)F_I \\
- |\lambda(F^n_I, F^n_S) - \lambda(F_I, F_S)|(F^n_t - F_I)F_I.
\]

Since 0 \(\leq F^n_t(x), F_I(x) \leq 1\) for any \(x \geq 0\),
\[
\frac{1}{2} \frac{d}{dx} |F^n_t - F_I|^2 \leq \Lambda |F^n_t - F_I|^2 + \Lambda |F^n_S - F_S| |F^n_t - F_I| \\
+ \Lambda |F^n_I - F_I|^2 \\
\leq C(\Lambda) (|F^n_t - F_I|^2 + |F^n_S - F_S|^2).
\]
Similarly, it follows from the equations for \( F^n_S(x) \) and \( F_S(x) \) that
\[
(F^n_S)' - (F_S)' = \lambda(F^n_I, F^n_S)F^n_I - \lambda(F_I, F_S)F_I \\
- \sigma_S(F^n_S - F_S) - \sigma_R(F^n_R - F_R) \\
= \lambda(F^n_I, F^n_S)(F^n_I - F_I) + (\lambda(F^n_I, F^n_S) - \lambda(F^n_I, F_S))F_I \\
+ (\lambda(F^n_I, F_S) - \lambda(F_I, F_S))F_I \\
- \sigma_S(F^n_S - F_S) - \sigma_R(F^n_R - F_R)
\]
so we can obtain that
\[
1 \frac{d}{dx}|F^n_S - F_S|^2 \leq \Lambda|F^n_I - F_I||F^n_S - F_S| + \Lambda|F^n_S - F_S|^2 \\
+ \Lambda|F^n_I - F_I||F^n_S - F_S| + \sigma_S|F^n_S - F_S|^2 \\
+ \sigma_R|F^n_R - F_R||F^n_S - F_S| \\
\leq C(\Lambda, \sigma_S, \sigma_R)(|F^n_I - F_I|^2 + |F^n_S - F_S|^2).
\]
By (2.8) and (2.9),
\[
\frac{d}{dx}(|F^n_I - F_I|^2 + |F^n_S - F_S|^2) \leq C(|F^n_I - F_I|^2 + |F^n_S - F_S|^2),
\]
where \( C \) is a positive constant depending on \( \Lambda, \sigma_S, \) and \( \sigma_R. \)
Hence, for \( 0 \leq x \leq 1/\delta \),
\[
|F^n_I - F_I|^2 + |F^n_S - F_S|^2 \leq e^{Cx}(|F^n_I(0) - F_I(0)|^2 + |F^n_S(0) - F_S(0)|^2) \\
\leq C/n^2.
\]
This implies that
\[
F^n_I(x) \rightarrow F_I(x), \quad F^n_S(x) \rightarrow F_S(x),
\]
uniformly on \( x \in [0, 1/\delta] \) as \( n \rightarrow \infty \). Moreover, since \( F^n_I + F^n_S + F^n_R = 1 \),
\[
F^n_R(x) \rightarrow F_R(x), \quad \text{uniformly on } x \in [0, 1/\delta] \text{ as } n \rightarrow \infty.
\]
Let \( G^n_I = (F^n_I)', G^n_S = (F^n_S)', G^n_R = (F^n_R)' \) and
\[
G_I = (F_I)', \quad G_S = (F_S)', \quad G_R = (F_R)'.
\]
Then \((G^n_I, G^n_S, G^n_R)\) satisfies
\[
(G^n_I)' = -\frac{\partial \lambda}{\partial I}(F^n_I, F^n_S)F^n_I G^n_I - \frac{\partial \lambda}{\partial S}(F^n_I, F^n_S)F^n_I G^n_S \\
- \lambda(F^n_I, F^n_S)G^n_I \\
+ \lambda(F^n_I, F^n_S)G^n_I + \frac{\partial \lambda}{\partial S}(F^n_I, F^n_S)F^n_I G^n_S \
+ \lambda(F^n_I, F^n_S)G^n_I - \sigma_S G^n_S - \sigma_R G^n_R,
\]
with the initial data
\[
G^n_I(0) = -\lambda \left(1 - \frac{1}{n}, \frac{1}{n} \right) \left(1 - \frac{1}{n} \right), \\
G^n_S(0) = \lambda \left(1 - \frac{1}{n}, \frac{1}{n} \right) \left(1 - \frac{1}{n} \right) - \frac{\sigma_S}{n} - \delta, \\
G^n_R(0) = \frac{\sigma_S}{n} + \delta.
\]
Also, \((G_I(x), G_S(x), G_R(x))\) satisfies (2.11) with the initial data (2.12) when \( n = \infty. \)
Remark 3. As we mentioned in Remark 1, we have

Then there exists a positive number 

Lemma 2.3. Following convergence exists between them.

Thus, Gronwall’s inequality implies that on 

Proof. We first suppose that there is no positive real root 

We can easily check that 

Since 

From the equation for 

Thus, 

Hence, we have 

uniformly on 

as 

uniformly on 

as 

Remark 3. As we mentioned in Remark 1, we have 

However, the result in Proposition 2.2 implies that 

as 

By using this difference, we can obtain the convergence result of 

defined in Definition 1.1.

For given 

and 

there are positive zeros 

and 

respectively, and the following convergence exists between them.

Lemma 2.3. Let 

be the solution to (2.1) subject to initial data (2.3). Then there exists a positive number 

such that 

\[ F^n_S(x_n) = 0, \text{ and } F^n_S(x) > 0 \text{ for } x \in (0, x_n). \]

Proof. We first suppose that there is no positive real root 

of 

on 

Since 

is continuous, we have 

From the equation for 

it follows that 

Thus, 

This implies that 

By (2.13) and the positivity of 

we have 

Since 

are continuous, we get a contradiction. □
Proposition 2.4. Let \((F^n_I, F^n_S, F^n_R)\) be the solution to (2.1) subject to initial data (2.3) and \((F_I, F_S, F_R)\) be the solution to (2.1) subject to initial data

\[ F_I(0) = 1, \quad F_S(0) = F_R(0) = 0. \]

Assume that \(\lambda(\cdot, \cdot)\) is in \(C^1\) and that one of the followings holds:

1. \(R_0 > 1\),
2. \(R_0 = 1\) and \(R_1 > 1\).

Then there exists a positive number \(x_\infty\) on \((0, 1/\delta)\) such that

\[ F_S(0) = F_S(x_\infty) = 0, \quad \text{and} \quad F_S(x) > 0 \quad \text{for} \quad x \in (0, x_\infty). \]

Moreover, if \(F_S'(x_\infty) < 0\), then \(x_n\) converges to \(x_\infty\) of \(F_S(x)\) as \(n \to \infty\), where \(x_n\) is the smallest positive root of \(F_S^n(x) = 0\).

Proof. By Lemma 2.3, there is the smallest positive root \(x_n\) of \(F_S^n(x) = 0\).

Due to the assumptions on \(R_0\) and \(R_1\),

\[ F_S^n(0) > 0 \]

or

\[ (F_S^n)'(0) = 0 \quad \text{and} \quad (F_S^n)''(0) > 0. \]

We suppose that there is no positive real root \(x_\infty\) of \(F_S(x) = 0\) on \((0, 1/\delta)\). Then, we have

\[ F_S > 0 \quad \text{on} \quad (0, 1/\delta). \]

By the same argument in the proof of Lemma 2.3, we can get a contradiction. Therefore, we prove the existence of \(x_\infty \in (0, 1/\delta)\).

From Taylor expansion on \(\lambda\), there exists a constant \(\epsilon_\mu > 0\) such that

\[ \epsilon_\mu \to 0 \quad \text{as} \quad \mu \to 0 \]

and for \(|1 \cdot I| + |S| \leq \mu_0\),

\[ \lambda(I, S) \geq \lambda^0 - \lambda^0_0(1 - I) + \lambda^0_S S - \epsilon_\mu \left( |1 - I| + |S| \right). \]

By using the above Taylor expansion and positivity of \(F^n_I\), we have

\[
\begin{align*}
(F_S^n)'(x) &= \lambda(F_I^n, F_S^n, F_R^n)F_I^n - \sigma_S F_S^n - \sigma_R F_R^n - \delta \\
&\geq \left[ \lambda^0 - \lambda^0_0(1 - F_I^n) + \lambda^0_S F_S^n - \epsilon_\mu \left( |1 - F_I^n| + |F_S^n| \right) \right] F_I^n \\
&\quad - \sigma_S F_S^n - \sigma_R F_R^n - \delta \\
&= \left[ \lambda^0 - \lambda^0_0(1 - F_I^n) + \lambda^0_S F_S^n - \epsilon_\mu \left( |1 - F_I^n| + |F_S^n| \right) \right] \\
&\quad - \left[ \lambda^0 - \lambda^0_0(1 - F_I^n) + \lambda^0_S F_S^n - \epsilon_\mu \left( |1 - F_I^n| + |F_S^n| \right) \right] (1 - F_I^n) \\
&\quad - \sigma_S F_S^n - \sigma_R F_R^n - \delta \\
&\geq (\lambda^0 - \delta) + (-\lambda^0 - \lambda^0_0 - \sigma_R)(1 - F_I^n) + (\lambda^0_S - \sigma_S + \sigma_R)F_S^n \\
&\quad - (|1 - F_I^n| + |F_S^n|) (2\Delta \mu_0 + (1 + \mu) \epsilon_\mu)
\end{align*}
\]

for \(|1 - F_I^n| + |F_S^n| \leq \mu_0\). From the equation for \(F_I^n\), we easily derive that

\[ F_I^n(x) \geq F_I^n(0)e^{-\Delta x}. \]

Therefore, there are \(\eta > 0\) and \(N_0 \in \mathbb{N}\) such that if \(0 \leq x \leq \eta\) and \(n \geq N_0\),

\[ 0 \leq 1 - F_I^n(x) \leq \mu_0/2. \]

Note that \(F_S^n(0) = 1 - F_I^n(0)\) and

\[
(F_S^n)'(x) = (1 - F_I^n(x))' - \sigma_S F_S^n(x) - \sigma_R F_R^n(x) - \delta.
\]
If \( F^n_S, F^n_R \geq 0 \), then

\[
0 \leq F^n_S(x) \leq 1 - F^n_I(x).
\]

Therefore, if \( 0 \leq x \leq \eta \) and \( n \geq N_0 \), then

\[
|1 - F^n_S(x)| + |F^n_S(x)| \leq \mu_0.
\]

Consider \( R_0 > 1 \) case, i.e., \( (\lambda^0 - \delta) > 0 \). We may assume that \( \mu, \epsilon, \mu < 1 \). Hence, by (2.14), there exists a constant \( \mu_0 > 0 \) such that if \( |1 - F^n_I| + |F^n_S| \leq \mu_0 \) and \( F_S \geq 0 \), then

\[
(F^n_S)'(x) \geq (\lambda^0 - \delta) - (4\Lambda + \sigma_R + 2)(1 - F^n_I) - (3\Lambda + \sigma_S + 2)F^n_S.
\]

We choose \( \mu_0 > 0 \) such that

\[
\frac{\lambda^0 - \delta}{2} > (8\Lambda + \sigma_R + \sigma_S + 4)\mu_0.
\]

Then, for \( 0 \leq x \leq \eta \), \( (F^n_S)'(x) > 0 \) and this implies that

\[
F^n_S > 0.
\]

For \( R_0 = 1 \) and \( R_1 > 1 \), by (2.14), there exists a constant \( \mu_0 > 0 \) such that

\[
(F^n_S)'(x) \geq (r_1 - \epsilon_\mu)(1 - F^n_I) - (\Lambda + \sigma_S + \sigma_R + 1)\mu_0
\]

\[
\geq (r_1/2)(1 - F^n_I) - (\Lambda + \sigma_S + \sigma_R + 1)\mu_0,
\]

where \( r_1 = -\lambda^0 - \lambda^1 - \sigma_R \). Therefore, for \( 0 \leq x \leq \eta \),

\[
F^n_S > 0.
\]

For the both cases, we can conclude that for every \( n > N_0 \), we obtain \( x_n \geq \eta \) where \( \eta \) is a uniform constant for \( n > N_0 \). Since \( F^n_S(x) > 0 \) on \( x \in [0, \eta] \subset [0, x_\infty) \), the smallest positive root \( x_n \) of \( F^n_S(x) \) satisfies \( x_n > \eta \).

For any small \( \epsilon > 0 \), there is a small number \( \omega > 0 \) such that \( F_S(x) > \omega \) on \( [\eta, x_\infty - \epsilon] \). As \( n \to \infty \), \( F^n_S \) converges \( F_S \) uniformly on \( [0, x_\infty] \) by Proposition 2.2. Therefore, there is \( N_1 \in \mathbb{N} \) such that

\[
n > N_1 \text{ implies } |F_S(x) - F^n_S(x)| < \omega.
\]

These imply that if \( n > N_1 \), then \( x_n > x_\infty - \epsilon \).

Since \( F^n_S(x_\infty) < 0 \) and \( \epsilon \) is sufficiently small, there is \( \alpha > 0 \) such that \( F_S(x_\infty + \epsilon) \leq -\alpha \). Uniform convergence of \( F^n_S \) implies that there is \( N_2 \in \mathbb{N} \) such that

\[
n > N_2 \text{ implies } |F_S(x) - F^n_S(x)| < \alpha.
\]

Thus, if \( n > N_2 \), then \( F^n_S(x_\infty + \epsilon) < 0 \) and by the continuity of \( F^n_S \), the positive root \( x_n \) is located in \( [0, x_\infty + \epsilon] \). We conclude that \( n > \max\{N_1, N_2\} \), then \( |x_n - x_\infty| < \epsilon \).

\[\square\]

**Proposition 2.5.** Let \((F^n_I, F^n_S, F^n_R)\) be the solution to (2.1) subject to initial data (2.3) and \((F_I, F_S, F_R)\) be the solution to (2.1) subject to the initial data

\[
F_I(0) = 1, \quad F_S(0) = F_R(0) = 0.
\]

Assume that \( \lambda(\cdot, \cdot) \) is in \( C^1 \) and \( R_0 < 1 \).

Then \( x_n \) converges to 0 as \( n \to \infty \), where \( x_n \) is the smallest positive root of \( F^n_S(x) = 0 \).
Proof. We use the same argument in Proposition 2.4. By Lemma 2.3, there is the smallest positive root $x_n$ of $F^n_S(x) = 0$.

By Taylor expansion on $\lambda$,

$$\lambda(I, S) \leq \lambda^0 - \lambda^0_I (1 - I) + \lambda^0_S S + \epsilon_\mu(|1 - I| + |S|) \quad \text{for} \quad |1 - I| + |S| \leq \mu_0,$$

where $\epsilon_\mu > 0$ is a constant such that $\epsilon_\mu \to 0$ as $\mu \to 0$. From the above Taylor expansion and positivity of $F^n_T$, it follows that if $|1 - F_I| + |F_S| \leq \mu_0$,

$$(F^n_S)'(x) = \lambda(F^n_T, F^n_S) F^n_T - \sigma_S F^n_S - \sigma_R F^n_R - \delta$$

$$\leq [\lambda^0 - \lambda^0_I (1 - F^n_I) + \lambda^0_S F^n_S + \epsilon_\mu(|1 - F^n_I| + |F^n_S|)] F^n_T$$

$$- \sigma_S F^n_S - \sigma_R F^n_R - \delta$$

$$= [\lambda^0 - \lambda^0_I (1 - F^n_I) + \lambda^0_S F^n_S + \epsilon_\mu(|1 - F^n_I| + |F^n_S|)] - [\lambda^0 - \lambda^0_I (1 - F^n_I) + \lambda^0_S F^n_S + \epsilon_\mu(|1 - F^n_I| + |F^n_S|)] (1 - F^n_I)$$

$$- \sigma_S F^n_S - \sigma_R F^n_R - \delta$$

$$\leq (\lambda^0 - \delta) + (-\lambda^0 - \lambda^0_I - \sigma_R) (1 - F^n_I) + (\lambda^0_S - \sigma_S + \sigma_R) F^n_S$$

$$+ (|1 - F^n_I| + |F^n_S|) (2\Lambda \mu + (1 + \mu) \epsilon_\mu).$$

Note that

$$F^n_I(x) \geq F^n_I(0) e^{-\lambda t},$$

and there are $\eta > 0$ and $N_0 \in \mathbb{N}$ such that if $0 \leq x \leq \eta$ and $n \geq N_0$,

$$0 \leq 1 - F^n_I(x) \leq \mu_0/2.$$

Since $F^n_S(0) = 1 - F^n_I(0)$ and

$$(F^n_S)'(x) = (1 - F^n_I(x))' - \sigma_S F^n_S(x) - \sigma_R F^n_R(x) - \delta,$$

if $F_S, F_R \geq 0$, then

$$0 \leq F^n_S(x) \leq 1 - F^n_I(x).$$

Thus, if $0 \leq x \leq \eta$ and $n \geq N_0$,

$$|1 - F^n_I(x)| + |F^n_S(x)| \leq \mu_0.$$

Assume that $R_0 < 1$ case, i.e., $|\lambda^0 - \delta| < 0$. We may assume that $\mu, \epsilon_\mu < 1$. By (2.15), there exists a constant $\mu_0 > 0$ such that if $|1 - F^n_I| + |F^n_S| \leq \mu_0$ and $F^n_S \geq 0$,

$$(F^n_S)'(x) \leq (\lambda^0 - \delta) + (4\Lambda + 2)|1 - F_I| + (3\Lambda + \sigma_S + \sigma_R + 2)|F^n_S|.$$
The following lemma describes the relationship between the final rumor size and the smallest positive zero of $F^n_S$.

**Lemma 2.6.** Let $(F^n_I,F^n_S,F^n_R)$ be the solution to (2.1) subject to the initial data (2.3). Let $\phi_n(t)$ and $\phi_n^\infty$ be functions defined as in (1.6), and let $x_n$ be the smallest positive real root of $F^n_S(x) = 0$.

Then, the limit of the final rumor size exists.

\[ \phi_n^\infty := \lim_{t \to \infty} \phi_n(t) = x_n. \]

**Proof.** Let $n$ be a fixed natural number. By the definitions of $\phi_n$, $F^n_S$ and Lemma 2.1, we have

\[ \phi_n'(t) = S_n(t) = F^n_S(\phi_n(t)). \]

This implies that

\[ \int_0^{\phi_n(t)} \frac{1}{F^n_S(x)} \, dx = t. \]

Note that $F^n_S(x) > 0$ for $0 \leq x < x_n$ and $F^n_S(x_n) = 0$.

Therefore, we have

\[ \phi_n^\infty = \lim_{t \to \infty} \phi_n(t) = x_n. \]

\[ \square \]

3. The proof of the main theorems. In this section, we present the proofs of Theorem 1 and 2. To prove the two main theorems, we use the preceding auxiliary and original equations as appropriate. For the portion consisting of $R_0 > 1$, $R_0 = 1$ and $R_1 > 1$, the property of the auxiliary equation in Section 2 will be used to prove that the limit $\phi^\infty$ exists. Also, for the part of $R_0 < 1$, the existence of $\phi^e$ is proved by using the uniform convergence of the auxiliary equation in Section 2. However, it is difficult to use the method that employs the auxiliary equation for the remaining case. Therefore, we prove the existence of $\phi^e$ through a more precise comparison with $1 - I$ and $S$. The proof of the phase transition also uses a similar framework and the implicit function theorem. We emphasize that we can obtain the continuity of $\phi^e$ by using the precise approximation for $1 - I$ and $S$ for the original equation. The following simple lemma plays an important key role in this proof.

**Lemma 3.1.** Let $(I,S,R)$ be a solution to (1.1) and $J(t) = 1 - I(t)$. Assume that $\lambda(\cdot, \cdot)$ satisfies the condition in (1.3) and $R_0 \geq 1$.

Then, for any $T > 0$ and arbitrary small $\mu > 0$ satisfying

\[ 0 < \mu < \min \left\{ \frac{1}{2}, \frac{\lambda^0}{4\Lambda} \right\}, \]  

we have

\[ S(t) \leq S(0) + \left( c_0 \mu + \frac{\lambda^0 - \delta}{c_1} \right) (J(t) - J(0)), \quad t \in [0, T), \]

if $|1 - I(t)|, |S(t)| < \mu$ on $t \in [0, T)$, where $c_0$ and $c_1$ are positive constants depending only on $\sigma_S$, $\sigma_R$, and $\Lambda$.

**Proof.** Due to the regularity of $\lambda$, we have

\[ \lambda^0 - \Lambda(1 - |I| + |S|) \leq \lambda(I, S) \leq \lambda^0 + \Lambda(1 - |I| + |S|). \]

\[ (3.2) \]
Assume that $|1 - J(t)|, |S(t)| < \mu$ on $t \in [0, T)$. From the equation for $I(t)$ and (3.2), it follows that $\dot{J} = -I = \lambda SI \geq (\lambda^0 - 2\Lambda \mu)(1 - \mu)S$. Therefore,

$$\dot{J}(t) \geq c_1 S(t),$$

(3.3)

where $c_1 = \lambda^0 / 4 > 0$ due to the condition (3.1). Note that

$$0 \leq I(t), S(t), R(t) \leq 1, \quad \text{and} \quad S(t) + I(t) + R(t) = 1.$$

By (3.2), we have

$$\dot{S} = \lambda SI - [\sigma_S S + \sigma_R (1 - I - S)] S - \delta S$$

$$\leq [\lambda^0 + \Lambda (J + S)] IS - (\sigma_S - \sigma_R) S^2 - \sigma_R JS - \delta S$$

$$\leq \Lambda (J + S) IS - (\sigma_S - \sigma_R) S^2 - \sigma_R JS + \lambda^0 IS - \delta S.$$

Since $0 \leq I, J \leq 1$ and $\lambda^0 - \delta \geq 0$, we have

$$\dot{S} \leq S [\Lambda J + \Lambda S - (\sigma_S - \sigma_R) S - \sigma_R J] + (\lambda^0 - \delta) S$$

$$\leq S [\Lambda + \sigma_R] J + (\Lambda + \sigma_S + \sigma_R) S + (\lambda^0 - \delta) S$$

(3.4)

where $c_2 = 2(\Lambda + \sigma_S + \sigma_R)$.

From (3.3) and (3.4), it follows that

$$J(t) - J(0) \geq c_1 \phi(t),$$

and

$$S(t) - S(0) \leq (c_2 \mu + (\lambda^0 - \delta)) \phi(t).$$

By combining the above two estimates, we have

$$S(t) \leq S(0) + \left( c_0 \mu + \frac{\lambda^0 - \delta}{c_1} \right) (J(t) - J(0)), \quad t \in [0, T),$$

where $c_0 = c_2 / c_1$.

When $R_0 \geq 1$ and $R_1 < 1$, we need to obtain an upper bound of $1 - I$ for the proofs of the main theorems. For this, we define

$$r_0 := \lambda^0 - \delta, \quad r_1 := -\lambda^0 - \sigma_R - \lambda^0, \quad r_2 := \lambda^0_S - \sigma_S + \sigma_R.$$

We note that there are $\epsilon_\mu > 0$ such that if $0 \leq 1 - x, y < \mu$, then

$$\lambda(x, y) \leq \lambda^0 - (\lambda^0_I - \epsilon_\mu)(1 - x) + (\lambda^0_S + \epsilon_\mu)y,$$

(3.5)

where $\epsilon_\mu \to 0$ as $\mu \to 0$.

The following proposition describes the asymptotic behavior ($t \to \infty$) of $1 - I(t)$ near $R_0 = 1$.

**Proposition 3.2.** Let $\lambda(\cdot, \cdot)$ be $C^1$ with (1.3) and $(I_n, S_n, R_n)$ be the solution to (1.1) subject to the initial conditions of (1.5). Assume that $R_1 < 1$. Then, for any small $\mu > 0$, there are $N = N(\lambda, \sigma_S, \sigma_R, \mu) \in \mathbb{N}$ and $\eta = \eta(\lambda, \sigma_S, \sigma_R, \mu) > 0$ such that $n > N$ and $0 \leq \lambda^0 - \delta < \eta$ imply that

$$0 \leq 1 - I_n(t) < \mu \quad \text{for all} \ t > 0.$$
Proof. Let $J_n(t) = 1 - I_n(t)$ and let $c_0, c_1,$ and $c_2$ be generic constants defined in Lemma 3.1 and $c_0c_1 = c_2$. Since $R_1 < 1$, we have $r_1 = -\lambda_0^0 - \sigma_R - \lambda_0 < 0$. We choose $\gamma$ such that

$$1 < 1 + \frac{36\Lambda c_0}{(-r_1)} < \gamma,$$  \hfill (3.6)

where and $\Lambda$ is $C^1$-norm of $\lambda(\cdot, \cdot)$. We notice that $\gamma$ is independent of $\delta$.

Let $\mu > 0$ be a small number satisfying that

$$\frac{\mu}{\gamma} \leq \frac{(-r_1)}{9c_0|r_2| + 1}, \quad \epsilon_\mu + \mu|\lambda_0^0| \leq \frac{(-r_1)}{2}, \quad \epsilon_\mu + \mu|\lambda_0^0| < 1,$$  \hfill (3.7)

where $\epsilon_\mu$ is the constant in (3.5). We assume that $\lambda_0^0 - \delta$ is small such that

$$0 \leq \lambda_0^0 - \delta < \min \left\{ \frac{c_1(-r_1)}{9(|r_2| + 1)}, \frac{(-r_1)}{12}, \frac{c_2}{\gamma} \right\},$$  \hfill (3.8)

and $n$ is sufficiently large such that

$$\frac{1}{n} < \min \left\{ \frac{(-r_1)}{9(|r_2| + 1)}, \frac{\mu}{\gamma}, \frac{c_0}{\gamma}, \frac{\mu^2}{\gamma^2} \right\}.$$  \hfill (3.9)

Since we assume (3.9) and $S_n(0) = J_n(0) = 1/n$, there is $t_n^* > 0$ such that

$$J_n(t) < \frac{\mu}{\gamma}, \quad t \in [0, t_n^*).$$

Let

$$T_\gamma = \sup\left\{ t_n^* > 0 : J_n(t) < \mu/\gamma, \quad t \in [0, t_n^*) \right\}.$$  \hfill (3.10)

If $T_\gamma = \infty$, then by $\gamma > 1$,

$$0 \leq J_n(t) < \frac{\mu}{\gamma} < \mu \quad \text{for all } t > 0.$$  

Therefore, it suffices to consider $T_\gamma < \infty$ case only. Then for $0 \leq t < T_\gamma$,

$$0 \leq J_n(t) < \frac{\mu}{\gamma} \quad \text{and} \quad J_n(T_\gamma) = \frac{\mu}{\gamma}.$$  

We additionally define

$$T = \sup\left\{ t_n > 0 : J_n(t) < \mu, \quad t \in [0, t_n) \right\},$$

and we claim that $T = \infty$.

Assume not, i.e., $T$ is a positive real number such that $T > T_\gamma$,

$$0 \leq J_n(t) < \mu \quad \text{on} \quad 0 \leq t < T \quad \text{and} \quad J_n(T) = \mu.$$  

Since $J_n(t)$ is increasing,

$$J_n(t) \geq J_n(T_\gamma) = \frac{\mu}{\gamma}, \quad t \geq T_\gamma.$$  

We apply Lemma 3.1 to the interval $[0, T_\gamma]$ to obtain

$$S_n(T_\gamma) \leq S_n(0) + \left( c_0 \frac{\mu}{\gamma} + \frac{\lambda_0^0 - \delta}{c_1} \right) (J_n(T_\gamma) - J_n(0))$$

$$\leq S_n(0) + \left( c_0 \frac{\mu}{\gamma} + \frac{\lambda_0^0 - \delta}{c_1} \right) \left( \frac{\mu}{\gamma} - J_n(0) \right)$$

$$= \frac{1}{n} + \left( c_0 \frac{\mu}{\gamma} + \frac{\lambda_0^0 - \delta}{c_1} \right) \left( \frac{\mu}{\gamma} - \frac{1}{n} \right).$$
Since we assume that \( r_1 + \epsilon_\mu + \mu|\lambda^0_1| \leq r_1/2, \epsilon_\mu + \mu|\lambda^0_S| < 1, \mu/\gamma > 1/n \) and \( \lambda^0 - \delta \geq 0 \), we have

\[
(r_1 + \epsilon_\mu + \mu|\lambda^0_1|)J_n(T_\gamma) + (r_2 + \epsilon_\mu + \mu|\lambda^0_S|)S_n(T_\gamma)
\leq \frac{r_1}{2} J_n(T_\gamma) + (|r_2| + 1)S_n(T_\gamma)
\leq \frac{r_1}{2} \frac{\mu}{\gamma} + (|r_2| + 1) \left[ \frac{1}{n} + \left( c_0 \frac{\mu}{\gamma} + \frac{\lambda^0 - \delta}{c_1} \right) \left( \frac{\mu}{\gamma} - \frac{1}{n} \right) \right] \tag{3.11}
\]

By (3.7), \( \mu \) is sufficiently small and

\[
\frac{r_1}{9} \frac{\mu}{\gamma} + (|r_2| + 1) c_0 \frac{\mu^2}{\gamma^2} \leq 0. \tag{3.13}
\]

Note that by (3.9), \( n \) is sufficiently large such that

\[
\frac{r_1}{9} \frac{\mu}{\gamma} + \frac{|r_2| + 1}{n} \leq 0. \tag{3.12}
\]

By (3.8), \( \lambda^0 - \delta \) satisfies

\[
\frac{r_1}{9} \frac{\mu}{\gamma} + (|r_2| + 1) \left( \frac{\lambda^0 - \delta}{c_1} \right) \frac{\mu}{\gamma} \leq 0. \tag{3.14}
\]

Then, by (3.11)-(3.14), we have

\[
(r_1 + \epsilon_\mu + \mu|\lambda^0_1|)J_n(T_\gamma) + (r_2 + \epsilon_\mu + \mu|\lambda^0_S|)S_n(T_\gamma) \leq \frac{r_1 \mu}{6\gamma} < 0. \tag{3.15}
\]

By the equation for \( S_n(t) \) and (3.5), if \( |J_n|, |S_n| < \mu \), then

\[
\dot{S}_n \leq \left[ \lambda^0 - \lambda^0 + (1 - I_n) + (\lambda^0_S + \epsilon_\mu)S_n \right] I_n S_n - \sigma_S - \sigma_R S_n - \sigma_R J_n S_n - \delta S_n
\leq \left[ \lambda^0 - \epsilon_\mu \right] I_n - \lambda^0 - \sigma_R \dot{J}_n S_n + [\lambda^0 + \epsilon_\mu] I_n - \lambda^0 - \sigma_R \dot{S}_n
\leq \left[ \lambda^0 - \epsilon_\mu \right] I_n - \lambda^0 - \sigma_R \dot{J}_n S_n \tag{3.16}
\]

This implies that if \( |J_n|, |S_n| < \mu \), then

\[
\dot{S}_n \leq \left[ \left[ \lambda^0 + \mu|\lambda^0_1| + \lambda^0 - \sigma_R \right] J_n
+ \lambda^0 + \mu|\lambda^0_S| + \epsilon_\mu - (\sigma_S - \sigma_R) \right] S_n + (\lambda^0 - \delta) \right) S_n \tag{3.16}
\]

By (3.8), (3.15) and (3.16), we have

\[
\dot{S}_n \leq \frac{r_1 \mu}{6\gamma} S_n \leq \frac{r_1 \mu}{12\gamma} S_n < 0, \quad \text{at } t = T_\gamma.
\]
Therefore, there is $\epsilon > 0$ such that $\dot{S}(t)$ is decreasing on $[T_\gamma, T_\gamma + \epsilon]$.

Let $\epsilon_M$ be the maximum value of $\epsilon > 0$ such that $S_n(t)$ is decreasing on $[T_\gamma, T_\gamma + \epsilon]$. Since $J_n(t)$ is increasing and $S_n(t)$ is decreasing on $[T_\gamma, T_\gamma + \epsilon_M)$ and $r_1 < 0$, on $t \in [T_\gamma, T_\gamma + \epsilon_M)$,

$$
(r_1 + \epsilon \mu + \mu |\lambda^0_1|)J_n(t) + (r_2 + \epsilon \mu + \mu |\lambda^0_0|)S_n(t)
\leq \frac{r_1}{2} J_n(t) + (|r_2| + 1)S_n(t)
\leq \frac{r_1}{2} J_n(T_\gamma) + (|r_2| + 1)S_n(T_\gamma).
$$

(3.17)

If $T_\gamma + \epsilon_M < T$, then by (3.8), (3.16) and (3.17), on $t \in [T_\gamma, T_\gamma + \epsilon_M)$,

$$
\dot{S}_n \leq \left( \frac{r_1 \mu}{\delta \gamma} + (\Lambda^0 - \delta) \right) S_n \leq \frac{r_1 \mu}{12 \gamma} S_n < 0.
$$

This contradicts to the definition of $\epsilon_M$. Therefore, we have $T \leq T_\gamma + \epsilon_M$ and $\dot{S}_n < 0$ on $[T_\gamma, T)$. By this result, (3.8), and (3.16)-(3.17), on $t \in [T_\gamma, T)$,

$$
\dot{S}_n \leq \frac{r_1 \mu}{12 \gamma} S_n.
$$

(3.18)

Thus, (3.18) implies that on $t \in [T_\gamma, T)$,

$$
\dot{J}_n = \lambda J_n S_n \leq \Lambda S_n \leq \Lambda S_n(T_\gamma) e^{\frac{r_1 \mu}{12 \gamma}(t - T_\gamma)},
$$

and we obtain the following estimate: on $[T_\gamma, T)$,

$$
J_n(t) - J_n(T_\gamma) \leq \Lambda S_n(T_\gamma) e^{\frac{r_1 \mu}{12 \gamma}(t - T_\gamma)} - 1 \leq - \frac{\Lambda S_n(T_\gamma)}{r_1 \mu}.
$$

By (3.8), (3.9) and (3.10),

$$
S_n(T_\gamma) \leq \frac{1}{n} + \left( c_0 \frac{\mu}{\gamma} + \frac{\lambda^0 - \delta}{c_1} \right) \left( \frac{\mu}{\gamma} - 1 \right) \leq 3c_0 \frac{\mu^2}{\gamma^2}.
$$

This and (3.6) implies that on $[T_\gamma, T)$,

$$
J_n(t) \leq J_n(T_\gamma) - \frac{3 \Lambda c_0 \frac{\mu^2}{\gamma^2}}{r_1 \mu} \leq \mu - \frac{36 \Lambda c_0 \mu}{r_1 \gamma} = \left( 1 + \frac{36 \Lambda c_0}{r_1} \right) \frac{\mu}{\gamma} < \mu.
$$

This is a contradiction to the definition of $T$. Thus, $T = \infty$ and directly,

$$
0 \leq J_n(t) < \mu \quad \text{for all} \quad t > 0.
$$

$\square$

Now we are ready to prove our main theorems.

**The proof of Theorem 1:** Assume that $R_0 > 1$.

By Lemma 2.6, for $n \in \mathbb{N}$, $\phi_n^\infty$ is the least positive zero of $F^S_\gamma(x) = 0$, where $(F^n_F, F^n_S, F^n_R)$ is the corresponding solution to (1.7). Now by applying Proposition 2.4, we obtain that $\lim_{n \to \infty} \phi_n^\infty$ exists and

$$
x_\infty = \lim_{n \to \infty} \phi_n^\infty > 0.
$$

The proof of the case when $R_0 = 1$ and $R_1 > 1$ is completely analogous to the case when $R_0 > 1$ so we omit it.

Consider the case when $R_0 < 1$. By Proposition 2.5 and Lemma 2.6, the following limit exists.

$$
\phi^* = \lim_{n \to \infty} \phi_n^\infty = 0.
$$
For $R_0 = 1$ and $R_1 < 1$, due to Proposition 3.2 and $S_n(t) \leq 1 - I_n(t)$, we have

$$\lim_{n \to \infty} \inf_{t > 0} I_n(t) = 1.$$ 

Since $I_n(t) \leq I_n(0)e^{-\phi_n(t)/\lambda}$, we can conclude that

$$\lim_{n \to \infty} \sup_{t > 0} \phi_n(t) = 0.$$ 

The proof of Theorem 2: Let $R_1 > 1$. We consider the neighborhood of $R_0 = 1$. The rest of the cases are almost the same. Thus, we assume that $R_0 > 1 - \epsilon_0$, i.e., $\delta \in (0, \lambda^0 + \epsilon_1)$ for sufficiently small $\epsilon_0, \epsilon_1 > 0$. Let $\{F_I, F_S, F_R\}$ be the solution to (1.7) subject to (1.8) and $x_\infty$ be the smallest positive root of $F_S(x)$. Considering $x_\infty(\delta)$ as a function of $\delta$, we have

$$F_S = F_S(\delta, x) \quad \text{and} \quad F_S(\delta, x_\infty(\delta)) = 0,$$

for $\delta \in (0, \lambda^0]$. Since $F'_S(x_\infty) < 0$ when $\delta = \lambda^0$, for sufficiently small $\epsilon_1 > 0$, $x_\infty(\delta)$ is defined on $(0, \lambda^0 + \epsilon_1)$ such that

$$F_S = F_S(\delta, x) \quad \text{and} \quad F_S(\delta, x_\infty(\delta)) = 0,$$

and

$$F'_S(x_\infty) < 0, \quad \delta \in (0, \lambda^0 + \epsilon_1).$$

We note that on $\delta \in (0, \lambda^0]$, $x_\infty$ is the smallest positive root of $F_S(x)$. From Lemma 2.6 and Theorem 1, it follows that for $\delta \in (0, \lambda^0]$, 

$$x_\infty = \phi^c := \lim_{n \to \infty} \phi^c_n > 0.$$ 

Clearly, $\partial F_S/\partial x$ is continuous. By the result in [4], the partial derivative $\partial F_S(x)/\partial \delta$ exists and the following holds.

$$\frac{\partial}{\partial \delta} \left( \frac{\partial F_I}{\partial \delta} \right) = \frac{\partial \lambda}{\partial t} (F_I, F_S) \frac{\partial F_I}{\partial \delta} F_I - \frac{\partial \lambda}{\partial S} (F_I, F_S) \frac{\partial F_S}{\partial \delta} F_I - \lambda (F_I, F_S) \frac{\partial F_I}{\partial \delta},$$

$$\frac{\partial}{\partial \delta} \left( \frac{\partial F_S}{\partial \delta} \right) = \frac{\partial \lambda}{\partial t} (F_I, F_S) \frac{\partial F_I}{\partial \delta} F_I + \frac{\partial \lambda}{\partial S} (F_I, F_S) \frac{\partial F_S}{\partial \delta} F_I + \lambda (F_I, F_S) \frac{\partial F_I}{\partial \delta} - \sigma S \frac{\partial F_S}{\partial \delta} - \sigma R \frac{\partial F_R}{\partial \delta} - 1,$$

$$\frac{\partial}{\partial \delta} \left( \frac{\partial F_R}{\partial \delta} \right) = \sigma S \frac{\partial F_S}{\partial \delta} + \sigma R \frac{\partial F_R}{\partial \delta} + 1,$$

for, at least, $x \in (0, 1/\delta)$ and $\delta \in (0, \lambda^0 + \epsilon_1)$. Thus, $\partial F_S(x)/\partial \delta$ is considered the solution of the inhomogeneous linear system of differential equations. Note that the coefficients and inhomogeneous term are continuous. Thus, $\partial F_S(x)/\partial \delta$ is also continuous with respect to $\delta$ and $x$. Therefore, $F_S$ is in $C^1((0, \lambda^0 + \epsilon_1) \times (0, 1/\delta))$.

Since we assume that

$$\frac{\partial F_S}{\partial \delta}(\delta, x_\infty(\delta)) = F'_S(x_\infty) < 0$$

near $\delta = \lambda^0$, the implicit function theorem tells us that $x_\infty(\delta)$ is $C^1$ near $\delta = \lambda^0$. Since $\phi^c(\delta) = x_\infty(\delta)$ on $\delta \in (0, \lambda^0]$, we have $\phi^c$ is $C^1$ near $\delta = \lambda^0$. When $R_0 < 1$, $\phi^c(\delta) = 0$ for $\delta \in (\lambda^0, \infty)$ by Theorem 1 so we obtain that

$$\phi^c \in C^1((0, \lambda^0]) \cap C^1([\lambda^0, \infty)).$$
Additionally, by Theorem 1, $\phi^\varepsilon$ has jump discontinuity at $R_0 = 1$ for $R_1 > 1$ case. We consider the case when $R_0 \leq 1$ and $R_1 < 1$. For this case, by Theorem 1,
$$\lim_{\delta \to \lambda^0} \phi^\varepsilon(\delta) = 0.$$ Similar to the case of $R_1 > 1$, we can prove that $\phi^\varepsilon \in C^1(0, \lambda^0)$. By Proposition 3.2, we can obtain the continuity of $\phi^\varepsilon$.

Therefore, we have
$$\phi^\varepsilon(\delta) \in C^0(0, \infty) \cap [C^1(0, \lambda^0) \cup C^1([\lambda^0, \infty))].$$ Moreover, we can check the result by using implicit differentiation and L’Hospital’s rule,
$$\lim_{\delta \to \lambda^0^-} (\phi^\varepsilon)'(\delta) = -1$$ and $(\phi^\varepsilon)'$ is discontinuous at $\delta = \lambda^0$.

4. **Numerical simulations.** In this section, we carry out some numerical simulations. We use the forth-order Runge-Kutta method with a time step size $\Delta t = 0.01$. We assume that the variable trust rate (or incidence rate) is
$$\lambda(I, S) = \lambda^0 - \lambda^0_I (1 - I) + \lambda^0_S S.$$ In this linear case, as we mentioned in Remark 2, if $\lambda^0_S \geq 0$ and $\sigma_R \geq \sigma_S$, then the following condition holds.
$$F'_S(x_\infty) < 0,$$ where $x_\infty$ is the smallest positive root of $F_S(x)$. For the thermodynamic limit, we take the initial condition such that
$$\left(I^0, S^0, R^0\right) = \left(1 - \frac{1}{n}, \frac{1}{n}, 0\right). \quad (4.1)$$ Throughout this section, we fix
$$k = 1, \quad \sigma_S = 1, \quad \text{and} \quad \lambda^0_S = 1.$$ If we take $\lambda^0 = \lambda^0_I = \sigma_R = 1$ and $\delta = 2$, then $R_0 = 0.5$. The time evolution of solutions to (1.1) depending on $n$ are presented in Figure 1. Since $R_0 < 1$, the total size of a rumor goes to zero as $n$ goes to infinity. On the other hand, if we take $\lambda^0 = \lambda^0_I = \sigma_R = 1$ and $\delta = 0.5$, then $R_0 = 2$. In this case, the time evolution of solutions are given in Figure 2. We can observe that the total size of a rumor is not zero.

![Graphs](image-url)
Figure 1. Time evolutions of \((I(t), S(t), R(t))\) with initial data (4.1) when \(R_0 = 0.5\)

Figure 2. Time evolutions of \((I(t), S(t), R(t))\) with initial data (4.1) when \(R_0 = 2\)
Furthermore, with the same parameters of Figure 1, as we proved in Theorem 1, \( \phi^e \) exists and it is zero. See Figure 3(A) and Figure 3(B). In these figures, we display the size of the rumor, that is, 
\[
\int_0^T S(\tau) \, d\tau
\]
with initial data 
\[
\left( 1 - \frac{1}{n}, \frac{1}{n}, 0 \right) \quad \text{and} \quad T = 10^4.
\]
As seen in Figure 1 and 2, the sequence of the solutions \((I_n(t), S_n(t), R_n(t))\) converges to a function rapidly as \(n\) goes to infinity if \(R_0 \neq 1\). Therefore, \(T = 10^4\) is enough to obtain \(\phi^e\) instead of taking \(t\) to \(\infty\). In Figure 3(A) and Figure 3(B), \(n\) changes in [1, 10^3] and [10^{100}, 10^{101}], respectively. Especially, we take a large number \(n \in \mathbb{N}\) in Figure 3(B) to ensure the result for the thermodynamic limit. On the contrary, with the same parameters of Figure 2, Figure 3(C) and Figure 3(D) show that \(\phi^e\) exists and it is positive as we proved in Theorem 1. Therefore, we can numerically observe that rumor outbreak occurs in the large population limit sense if \(R_0 > 1\) and rumor outbreak does not occur in the large population limit sense if \(R_0 < 1\).

![Figure 3](image_url)

**Figure 3.** Numerical simulations for \(\phi^\infty_n\) when \(T = 10^4\)
Now, we investigate the rumor dynamics when $R_0 = 1$. In this case, a larger $T$ is required as $n$ increases. Therefore, we take $T = 10^6$. We fix parameters $\lambda^0 = \delta = 1$ so that $R_0 = 1$. If $\lambda_j^0 = \sigma_R = 1$, then $R_1 = -2$. According to Theorem 1, $\phi^e = 0$. Indeed, we can numerically verify that the same result holds in Figure 4(A) and 4(B), $\phi^e = 0$. On the contrary, if we set $\lambda_j^0 = -3$ and $\sigma_R = 1$, then $R_1 = 2$. In this case, $\phi^e$ is strictly positive. See Figure 4(C) and 4(D).

We next verify the result of Theorem 2. We change $1/\delta$ from 0.5 to 1.5 and all other parameters are fixed as $\lambda^0 = 1$, $\lambda_j^0 = 1$ and $\sigma_R = 1$. Additionally, to consider the limit case, we take the final time $T$ and the size of population $n$ as sufficiently large, such that

$$T = 10^6 \quad \text{and} \quad n = 10^{10}.$$  

Then, $R_0$ varies from 0.5 to 1.5 and $R_1 = -2$. As we proved in Theorem 2, $\phi^e$ is continuous and differentiable except for $R_0 = 1$. If we fix parameters, such as $\lambda^0 = 1$, $\lambda_j^0 = -3$ and $\sigma_R = 1$, then $R_1$ becomes 2. As seen in Figure 5(A), $\phi^e$ has a jump discontinuity at $R_0 = 1$. That is, the phase transition is the first order, as we have seen in Theorem 2. In numeric simulations, the second-order phase transition is also observed. See Figure 5(B).
5. Conclusion. In this paper, we consider a large population limit for the SIR rumor spreading model when the trust rate depends on the densities of ignorants and spreaders. In this limit, a threshold phenomenon appears for the final size of the rumor. We rigorously prove that the final size of the rumor exists and it is positive if $R_0 > 1$. Moreover, if $R_0 < 1$, then the final size of the rumor is zero. Due to the variable trust rate and original rumor dynamics, two kinds of phase transition phenomena occur. We also found a criterion to determine whether the phase transition is of the first or the second order. The critical number $R_1 = -\frac{(\lambda_I^0 + \sigma_R)}{\lambda_0}$ determines whether the phase transition is of first or second order.

When we consider the basic reproduction number or the decay rate $\delta$ as a variable for the phase transition, then the final size of the rumor (or the order parameter) is discontinuous at $R_0 = 1$ if $R_1 > 1$ and it is continuous at $R_0 = 1$ if $R_1 < 1$.

To prove this result, we use an auxiliary system for the $\phi^c > 0$ case and a detailed analysis to compare $S$ and $1 - I$ for the $\phi^c = 0$ case.

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Appendix A. Proof of Remark 2 (1). Let $\lambda(I, S) = \lambda^0 - \lambda_I^0(1 - I) + \lambda_S^0 S$. We assume that $\sigma_R \geq \sigma_S$, and $\lambda_S^0 \geq 0$. Then the solution $(F_I, F_S, F_R)$ to (1.7) subject to $(I^0, S^0, R^0) = (1, 0, 0)$ satisfies

$$F'_I(x) = G_I(F_I, F_S), \quad F'_S(x) = G_S(F_I, F_S), \quad x > 0,$$

subject to $F_I(0) = 1, F_S(0) = 0$, where

$$G_I(F_I, F_S) = -\lambda(F_I, F_S)F_I(x),$$

$$G_S(F_I, F_S) = \lambda(F_I(x), F_S(x))F_I(x) - \sigma_S F_S(x) - \sigma_R (1 - F_I(x) - F_S(x)) - \delta.$$

Since the above system is autonomous, we can use a phase-plane analysis.

Consider the trajectory $(F_I(x), F_S(x))$ starting at $x = 0$ with $(F_I(0), F_S(0)) = (1, 0)$. On the region of $\{G_S(F_I, F_S) > 0\}$, $F_S(x)$ is increasing. Since $F_S(x)$ is increasing at $x = 0$ for $R_0 > 1$ or $R_0 = 1, R_1 > 1, F_I(x)$ is increasing if $(F_I(x), F_S(s))$
is located near \((1,0)\) and satisfies \(F_1(x) > 0\) and \(0 \leq F_S(x) \leq 1\). As mentioned in Proposition 2.4, there exists a positive number \(x_\infty\) on \((0, 1/\delta)\) such that
\[
F_S(0) = F_S(x_\infty) = 0, \quad \text{and} \quad F_S(x) > 0 \quad \text{for} \quad x \in (0, x_\infty).
\]

We consider the two cases: (1) \(\sigma_R > \sigma_S\) or \(\lambda_0^S > 0\), (2) \(\sigma_R = \sigma_S\) and \(\lambda_0^S = 0\).

We first consider \(\sigma_R > \sigma_S\) or \(\lambda_0^S > 0\) and restrict our domain as \((F_1, F_S) \in [0, 1] \times [0, \infty)\). Let \(f(y)\) be a function on \([0, 1]\) such that
\[
G_S(y, f(y)) = 0.
\]
From the linearity of \(\delta\), it follows that
\[
G_S(F_1, F_S) = (\lambda^0 - \lambda_1^0(1 - F_1(x)) + \lambda_0^S F_S(x))F_1(x) - \sigma_S F_S(x) - \sigma_R(1 - F_1(x) - F_S(x)) - \delta,
\]
and
\[
f(y) = \frac{\delta + \sigma_R - \lambda_0^0y + \lambda_0^ Ry - \sigma_Ry - \lambda_0^0y^2}{\lambda_0^0y + \sigma_R - \sigma_S}.
\]
Note that
\[
f'(y) = -\frac{\delta \lambda_0^S + (\sigma_R - \sigma_S)(\lambda^0 - \lambda_1^0 + \sigma_R) + \sigma_R \lambda_0^S + 2\lambda_1^0(\sigma_R - \sigma_S)y + \lambda_0^0 \lambda_0^R y^2}{(\sigma_R - \sigma_S + \lambda_0^S y)^2}.
\]
By elementary calculation, we can easily check that if \(\sigma_R \geq \sigma_S\), and \(\lambda_0^S \geq 0\), there is \(0 < I_c < 1\) satisfying \(f(I_c) = 0\) and the following holds
\[
f'(y) < 0 \quad \text{for} \quad 0 \leq y \leq I_c. \quad (A.1)
\]

Based on the curve \((y, f(y))\), let
\[
S_1 = \{(u, v) : v > f(u)\}, \quad S_2 = \{(u, v) : v < f(u)\}.
\]
Then \(F'_S(x) > 0\) holds on \(S_1\) and \(F'_S(x) < 0\) holds on \(S_2\). Clearly,
\[
F'_1(x) > 0, \quad \text{for} \quad F_1(x) > 0. \quad (A.2)
\]
Therefore, by standard phase-plane analysis and (A.1)-(A.2), we have
\[
F'_S(x_\infty) < 0.
\]

See Figure 6.

Next we consider the rest case of \(\sigma_R = \sigma_S\) and \(\lambda_0^S = 0\). Then we also easily check that \(G_S(F_1, F_S)\) is independent of \(F_S\) and there is \(0 < I_c < 1\) such that
\[
0 = G_S(I_c) = G_S(I_c, F_S). \quad \text{Thus,} \quad F'_S(x) < 0 \quad \text{holds on} \quad \{F_1(x) < I_c\} \quad \text{and} \quad F'_S(x) > 0.
\]
holds on \( \{ F_I(x) > I_c \} \). It also holds that \( F'_I(x) > 0 \) for \( F_I(x) > 0 \). Similarly, by phase-plane analysis, we have \( F'_S(x_\infty) < 0 \).

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