HILBERT SERIES AND OBSTRUCTIONS TO ASYMMETRIC SEMISTABILITY

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Abstract. Given a polarized manifold there are obstructions for asymptotic Chow semistability described as integral invariants which can be regarded as characters of the Lie algebra of holomorphic vector fields. In this paper we show that, on toric Fano manifolds, the linear span of those Lie algebra characters coincides with the derivatives of the Laurent series of the Hilbert series.

1. Introduction

Let $M$ be a compact complex manifold of dimension $m$. A polarization $L \rightarrow M$ is an ample line bundle over $M$, i.e. a holomorphic line bundle such that the first Chern class $c_1(L)$ is represented by a positive $(1,1)$-form. Then $c_1(L)$ can be considered as a Kähler class. In [9] Donaldson proved that if a polarized manifold $(M,L)$ admits a constant scalar curvature Kähler metric (cscK metric for short) in $c_1(L)$ and if the automorphism group $\text{Aut}(M)$ of $M$ is discrete then $(M,L)$ is asymptotically Chow stable. This result was extended by Mabuchi [26] when $\text{Aut}(M)$ is not discrete. Namely, Mabuchi proved that if the obstruction introduced in [25] vanishes and $(M,L)$ admits a cscK metric in $c_1(L)$ then $(M,L)$ is asymptotically Chow polystable. The obstruction introduced in [25] is an obstruction for $(M,L)$ to be asymptotically Chow semistable. We will explain this obstruction and the definitions of relevant stability conditions in section 2. We warn the reader that our terminology is slightly different from Mabuchi’s.

Mabuchi’s obstruction was reformulated by the first author in [17] to be the vanishing of a collection of integral invariants. One of these integral invariants coincides with an obstruction to the existence of cscK metric (see [14], [15], [7]). This last obstruction to the existence of cscK metrics is defined as a Lie algebra character on the complex Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector fields on $M$, which we denote by $f : \mathfrak{h}(M) \rightarrow \mathbb{C}$. To explain the collection of integral invariants which obstruct the asymptotic semistability let $\mathfrak{h}_0(M)$ the subalgebra consisting of all holomorphic vector fields $X \in \mathfrak{h}(M)$ which have non-empty zero set. Choose any $X \in \mathfrak{h}_0(M)$. For any Kähler form $\omega$ representing $c_1(L)$ there exists a complex valued smooth function $u_X$ determined up to a constant such that

$$i(X)\omega = -\nabla u_X.$$
When \( u_X \) is a real function the real part of \( X \) is a Hamiltonian vector field, and even if \( u_X \) is not real we call \( u_X \) the Hamiltonian function for \( X \) by the abuse of terminology. The existence of \( u_X \) for \( X \in \mathfrak{h}_0(M) \) is a classically known, see e.g. [23]; a comprehensive proof can be found in [24]. We assume the normalization of \( u_X \) is so chosen that

\[
(1) \quad \int_M u_X \omega^m = 0.
\]

Thus, the Lie subalgebra \( \mathfrak{h}_0(M) \) consists of all holomorphic vector fields \( X \) in \( \mathfrak{h}(M) \) such that \( X \) is written in the form

\[
(2) \quad X = ig^{ij} \frac{\partial u_X}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}
\]

where the Kähler form \( \omega \) is given by

\[
\omega = ig^{ij} dz^i \wedge d\bar{z}^j.
\]

To give another interpretation of \( \mathfrak{h}_0(M) \), let \( \text{Aut}(M) \) be the group of all automorphisms of \( M \). Let \( \text{Aut}(L) \) be the group of all bundle automorphisms of \( L \). Then \( \text{Aut}(L) \) contains \( \mathbb{C}^* \) as a subgroup which acts as fiber multiplications. We put \( \text{Aut}(M, L) := \text{Aut}(L)/\mathbb{C}^* \). Then any element of \( \text{Aut}(M, L) \) induces an automorphism of \( M \), and \( \text{Aut}(M, L) \) is considered as a Lie subgroup of \( \text{Aut}(M) \). The Lie subalgebra in \( \mathfrak{h}(M) \) corresponding to \( \text{Aut}(M, L) \) is exactly \( \mathfrak{h}_0(M) \). This last fact follows from the general fact that giving a moment map \( M \to \mathfrak{h}_0(M)^* \) corresponds to giving a lifting of infinitesimal action of \( \mathfrak{h}_0(M) \) on \( M \) to that on \( L \). Good references for this general fact are [11], section 6.5, and [18], but the reference [18] is more precise in that the ambiguity of Hamiltonian functions up to constant is more carefully treated. The functions \( u_X \) define a moment map, and thus define a lifting of the infinitesimal action of \( \mathfrak{h}_0(M) \) on \( M \) to an infinitesimal action on \( L \). Therefore \( \mathfrak{h}_0(M) \) corresponds to the Lie subgroup \( \text{Aut}(M, L) \) in \( \text{Aut}(M) \). However we encounter the trouble coming from the ambiguity of constants of Hamiltonian functions. Different constants give different liftings. When we discuss stability we have to have a subgroup of special linear group. This point can be overcome by using S. Zhang’s result (Theorem 2.4 in section 2).

Let \( \nabla \) be a type \((1,0)\) connection of the holomorphic tangent bundle, that is a linear connection whose connection form \( \theta \) is expressed as a type \((1,0)\)-form with respect to local holomorphic frames. This last condition is of course equivalent to saying that the \((0,1)\)-part of \( \nabla \) is equal to \( \bar{\partial} \). Denote by \( \Theta = \bar{\partial} \theta \) its curvature form. For a holomorphic vector field \( X \) we also put

\[
L(X) := \nabla_X - L_X
\]

where \( L_X \) and \( \nabla_X \) respectively denote the Lie derivative and covariant derivative by \( X \). It is easy to see that \( L(X) \) defines a smooth section of the endomorphism bundle of the holomorphic tangent bundle. Let \( \phi \) be a \( GL(m, \mathbb{C}) \)-invariant polynomial of degree \( p \) on \( \mathfrak{gl}(m, \mathbb{C}) \). We define \( \mathcal{F}_\phi : \mathfrak{h}_0(M) \to \mathbb{C} \) by

\[
(3) \quad \mathcal{F}_\phi(X) = (m - p + 1) \int_M \phi(\Theta) \wedge u_X \omega^{m-p}
+ \int_M \phi(L(X) + \Theta) \wedge \omega^{m-p+1}.
\]
It can be shown that $\mathcal{F}_\phi(X)$ is independent of the choices of $\omega$ and $\nabla$, see [17] and [20] for the detailed account on $\mathcal{F}_\phi$. Let $\text{Td}^p$ be the $p$-th Todd polynomial which is a $GL(m, \mathbb{C})$-invariant polynomial of degree $p$ on $gl(m, \mathbb{C})$. The reformulation by the first author [17] of Mabuchi’s obstruction [25] to asymptotic Chow semistability of $(M, L)$ is the vanishing of $\mathcal{F}_{\text{Td}^p}$ for all $p = 1, \ldots, m$. Moreover it can be shown that $\mathcal{F}_{\text{Td}^1}$ coincides with the obstruction $f|_{\text{Hb}(M)}$ to the existence of cscK metric up to the multiplication of a non-zero constant.

Choosing a Kähler form $\omega$ in $c_1(L)$, we have the Levi-Civita connection $\theta = g^{-1}\partial g$ and its curvature form by $\Theta = \partial \theta$ where $g$ denotes the Kähler metric of $\omega$. In the definition of $\mathcal{F}_\phi(X)$ above we could have used $\nabla$ to be the Levi-Civita connection with the connection form $\theta$. In this case, because of the torsion-freeness of the Levi-Civita connection, $L(X)$ can be expressed by

$$L(X) = \nabla X = \nabla_j X^i \, dz^j \otimes \frac{\partial}{\partial z^i}$$

regarded as a smooth section of $\text{End}(T'M)$.

The paper [17] was also motivated by the work of Mabuchi and Nakagawa [27] in which they claimed that the obstruction $f$ to the existence of cscK metric is an obstruction to semistability though their proof contained incomplete arguments, see the Erratum in [27]. But in light of their paper it is an interesting question whether or not the other integral invariants $\mathcal{F}_{\text{Td}^2}, \ldots, \mathcal{F}_{\text{Td}^m}$ are linearly dependent on $\mathcal{F}_{\text{Td}^1}$.

In this paper we related these integral invariants $\mathcal{F}_{\text{Td}^p}$ to the Hilbert series for toric Fano manifolds, which is the index character considered by Martelli, Sparks and Yau [29]. The main result of this paper is Theorem 3.2 which claims that on toric Fano manifolds, the linear span of $\mathcal{F}_{\text{Td}^2}, \ldots, \mathcal{F}_{\text{Td}^m}$ restricted to the Lie algebra of the algebraic torus coincides with the linear span of the derivatives of the Laurent series of the Hilbert series. As an application we see that there are 3-dimensional toric Fano manifolds such that those integral invariants span 2-dimension. Thus $\mathcal{F}_{\text{Td}^2}, \ldots, \mathcal{F}_{\text{Td}^m}$ are not in general linearly dependent on $\mathcal{F}_{\text{Td}^1}$.

A question we can not answer in this paper is whether or not there is a polarized manifold $(M, L)$ on which a cscK metric exists in $c_1(L)$ so that $\mathcal{F}_{\text{Td}^1} = 0$ but on which $\mathcal{F}_{\text{Td}^p} \neq 0$ for some $p = 2, \ldots, m$. If the answer is no the assumption on the obstruction in Mabuchi’s result [26] can be omitted. Our computations show that the last question is closely related to a question raised by Batyrev and Selivanova [3]: Is a toric Fano manifold with vanishing $f$ for the anticanonical class necessarily symmetric? If the answer is yes then any toric Fano Kähler-Einstein manifold has vanishing $\mathcal{F}_{\text{Td}^p}$ for $p = 1, \ldots, m$. Recall that a toric Fano manifold $M$ is said to be symmetric if the trivial character is the only fixed point of the action of the Weyl group on the space of all algebraic characters of the maximal torus in $\text{Aut}(M)$. Note that if a toric Fano manifold $M$ is symmetric then the character $f$ for the anticanonical class vanishes. Recall also that Batyrev and Selivanova [3] proved that a toric Fano manifold $M$ admits a Kähler-Einstein metric if $M$ is symmetric, and that Wang and Zhu [37] improved the result of Batyrev and Selivanova to the effect that a toric Fano manifold $M$ admits a Kähler-Einstein metric if the invariant $f$ vanishes for the anticanonical class.

\footnote{After posting the first version of this paper on the arXiv the paper [31] appeared and a seven dimensional example of non-symmetric toric Kähler-Einstein Fano manifold was presented. It is further shown in [32] that, for this example, $\mathcal{F}_{\text{Td}^1} = 0$ and $\mathcal{F}_{\text{Td}^p} \neq 0$ for $p = 2, \ldots, 7$.}
This paper is organized as follows. In section 2, we review the definition of asymptotic Chow semistability, and then we give the result, Theorem 2.1 proved in [17]. In section 3, we prove the main theorem of this paper, Theorem 3.2. In section 4, we give a combinatorial formula (23) for computing the Hilbert series $\text{Hilb}(\text{Chow stability})$. In section 5, we observe that there are toric Fano 3-folds such that $\mathcal{F}_{T_d}$ and $\mathcal{F}_{T_d^1}$ are linearly independent of $\mathcal{F}_{T_d^1}$ using Theorem 3.2 and computation of Hilbert series. In particular, we do computer calculation in such a way as follows:

- Input the combinatorial data of the moment polytope to the formula (23).
- Reduce the fractions to a common denominator.
- Substitute $e^{-tb} = (e^{-b_1 t}, e^{-b_2 t}, e^{-b_3 t}, e^{-4t})$ for $x = (x_1, x_2, x_3, x_4)$.
- Differentiate $C(e^{-tb}, C^*)$ by $b_1, b_2$ and $b_3$ at $(b_1, b_2, b_3) = (0, 0, 0)$.

In section 6, we calculate $\mathcal{F}_{T_d}$ of toric Fano 3-folds by using localization formula. We can also confirm the result obtained in section 5 by this way. In section 7, we remark that a part of our construction makes sense in general compact Sasaki manifolds.

2. Obstructions to Asymptotic Chow semistability

In this section we review the obstructions to asymptotic Chow semistability. Though the full account of these obstructions has already appeared in [17] we will reproduce the arguments of [17] for the reader’s convenience. The result in [17] we want to use in this paper is stated as follows.

**Theorem 2.1** ([17]). Suppose that a polarized manifold $(M, L)$ is asymptotically Chow semistable. Then $\mathcal{F}_{T_d}$ vanishes for all $p = 1, \cdots, m$. Suppose further that $H^q(M, \mathcal{O}(L))$ vanishes for all $q > 0$ and let $X$ be a holomorphic vector field in $h_0(M)$ which generates an $S^1$-action on $M$. There is then a lifting of the infinitesimal action of $X$ to $L$ such that the induced $S^1$-action on $L^k$ induces an action of a subgroup in $\text{SL}(H^0(M, \mathcal{O}(L)))$ for all $k \geq 1$ at once.

Note that in the above theorems the $S^1$-action on $L$ might be a finite covering of the $S^1$-action on $M$. Before reproducing the proof of Theorem 2.1 we collect basic definitions and well-known facts used in the proof. First of all we recall the definitions of Chow (semi)stability and asymptotic Chow (semi)stability.

**Definition 2.2** (Chow stability). Let $\Lambda \to M$ be a very ample line bundle over a compact complex manifold $M$. Let $\Phi_{\Lambda|}: M \to \mathbb{P}(V)$ be the Kodaira embedding defined by using the sections of $\Lambda$ where $V = H^0(M, \mathcal{O}(\Lambda))^*$. Let $d$ be the degree of $\Phi_{\Lambda|}(M)$ in $\mathbb{P}(V)$. An element of the product $\mathbb{P}(V^*) \times \cdots \times \mathbb{P}(V^*)$ of $m+1$ copies of $\mathbb{P}(V^*)$ defines $m+1$ hyperplanes $H_1, \cdots, H_{m+1}$ in $\mathbb{P}(V)$. The set of all $m+1$ hyperplanes such that $H_1 \cap \cdots \cap H_{m+1} \cap \Phi_{\Lambda|}(M)$ is non-empty defines a divisor in $\mathbb{P}(V^*) \times \cdots \times \mathbb{P}(V^*)$. Since the degree of $\Phi_{\Lambda|}(M)$ is $d$ this divisor is defined by some $\tilde{M}_\Lambda \in (\text{Sym}^d(V))^\oplus (m+1)$. Of course $\tilde{M}_\Lambda$ is determined up to constant. The point $[\tilde{M}_\Lambda] \in \mathbb{P}((\text{Sym}^d(V))^\oplus (m+1))$ is called the Chow point. $M$ is said to be Chow polystable with respect to $\Lambda$ if the orbit of $\tilde{M}_\Lambda$ in $(\text{Sym}^d(V))^\oplus (m+1)$ under the action of $\text{SL}(V)$ is closed. $M$ is said to be Chow stable with respect to $\Lambda$ if $M$ is polystable and the stabilizer at $\tilde{M}_\Lambda$ of the action of $\text{SL}(V)$ is finite. $M$ is said to be Chow semistable with respect to $\Lambda$ if the closure of the orbit of $\tilde{M}_\Lambda$ in $(\text{Sym}^d(V))^\oplus (m+1)$ under the action of $\text{SL}(V)$ does not contain
A.4. Hilbert-Mumford criterion says that, to check Chow stability, it is sufficient to check the stability condition for all one parameter subgroups \((\cong \mathbb{C}^*)\) in \(SL(H^0(M, \mathcal{O}(\Lambda)))\).

**Definition 2.3** (Asymptotic Chow stability). Let \(L \to M\) be an ample line bundle. For a large positive integer \(k\), \(L^k\) is very ample. We apply Definition 2.2 by taking \(\Lambda\) to be \(L^k\). Put \(V_k := H^0(M, \mathcal{O}(L^k))^*\), and let \(\Phi_{|L^k|} : M \to \mathbb{P}(V_k)\) be the Kodaira embedding defined by using the sections of \(L^k\). Let \(d_k\) be the degree of \(\Phi_{|L^k|}(M)\) in \(\mathbb{P}(V_k)\). An element of the product \(\mathbb{P}(V_k^*) \times \cdots \times \mathbb{P}(V_k^*)\) of \(m + 1\) copies of \(\mathbb{P}(V_k^*)\) defines \(m + 1\) hyperplanes \(H_1, \cdots, H_{m+1}\) in \(\mathbb{P}(V_k)\). The set of all \(m + 1\) hyperplanes such that \(H_1 \cap \cdots \cap H_{m+1} \cap \Phi_{|L^k|}(M)\) is non-empty defines a divisor in \(\mathbb{P}(V_k^*) \times \cdots \times \mathbb{P}(V_k^*)\). Since the degree of \(\Phi_{|L^k|}(M)\) is \(d_k\) this divisor is defined by some \(\tilde{M}_k \in (\text{Sym}^d_k(V_k))^\otimes(m+1)\), which is determined up to constant. The point \([\tilde{M}_k] \in \mathbb{P}((\text{Sym}^d_k(V_k))^\otimes(m+1))\) is called the Chow point. \(M\) is said to be Chow polystable with respect to \(L^k\) if the orbit of \(\tilde{M}_k\) in \((\text{Sym}^d_k(V_k))^\otimes(m+1)\) under the action of \(SL(V_k)\) is closed. \(M\) is said to be Chow stable with respect to \(L^k\) if \(M\) is polystable and the stabilizer at \(\tilde{M}_k\) of the action of \(SL(V_k)\) is finite. \(M\) is said to be Chow semistable with respect to \(L^k\) if the closure of the orbit of \(\tilde{M}_k\) in \((\text{Sym}^d_k(V_k))^\otimes(m+1)\) under the action of \(SL(V_k)\) does not contain \(\mathbf{o} \in (\text{Sym}^d_k(V_k))^\otimes(m+1)\). \(M\) is said to be asymptotically Chow polystable (resp. stable or semistable) with respect to \(L\) if there exists a \(k_0 > 0\) such that \(M\) is polystable (resp. stable or semistable) for all \(k \geq k_0\).}

Next recall a theorem of Zhang:

**Theorem 2.4** (S. Zhang, Theorem 3.4 in [38]). Let \(\Lambda \to M\) be a very ample line bundle over a compact complex manifold \(M\), and let \(\Phi_{|\Lambda|} : M \to \mathbb{P}(V)\) be the Kodaira embedding defined by using the sections of \(\Lambda\) where \(V = H^0(M, \mathcal{O}(\Lambda))^*\) as in Definition 2.2. We identify \(V\) with \(\mathbb{C}^{N+1}\) endowed with the standard Hermitian metric. Then there is a norm, called Chow norm and denoted by \(\| \cdot \|_C\), on \((\text{Sym}^d(V))^\otimes(m+1)\) such that, for any one parameter subgroup \(\sigma_t\) of \(SL(V)\), we have

\[
\frac{d}{dt} \log \|\sigma_t(\tilde{M}_\Lambda)\|_C = \int_M \hat{\varphi}_t \sigma_t^* \omega_{FS}^m,
\]

where \(\varphi_t = \log \|\sigma_t(z)\|/\|z\|\) for \(z \in V - \{\mathbf{o}\}\) with \(|z| \in M \subset \mathbb{P}(V)\), and \(\omega_{FS}\) denotes the Fubini-Study Kähler form restricted to \(M\).

Zhang proved this theorem using Deligne pairing, but direct proofs are also given by Phong and Sturm [33] and the third author [35]. Zhang’s theorem lays a bridge between Chow stability and lifting an infinitesimal action of \(h_0(M)\) to an ample line bundle \(\Lambda\) in such a way that it induces a subgroup of \(SL(H^0(M, \mathcal{O}(\Lambda)))\) for the following reasons. Suppose that \(\sigma_t\) in Theorem 2.4 preserves \(M\) and induces an action on \(M\) generated by a holomorphic vector field \(X\) on \(M\). Then \(\hat{\varphi}_t\) is a Hamiltonian function for \(X\) with respect to the Kähler form \(\sigma_t^* \omega_{FS}\). For a polarized manifold \((M, \Lambda)\) with very ample line bundle \(\Lambda\), let \(V = H^0(M, \mathcal{O}(\Lambda))^*\) and \(M \to \mathbb{P}(V)\) be the Kodaira embedding. If \((M, \Lambda)\) is Chow semistable then the left hand side of (4) has to be zero, for otherwise the Chow norm tends to zero along the orbit of the one parameter subgroup \(\sigma_t\) and the closure of the orbit is the straight line joining \(\tilde{M}_\Lambda\) and the origin \(\mathbf{o}\), and of course contains \(\mathbf{o}\). It follows therefore that if \((M, \Lambda)\) is Chow semistable then the Hamiltonian function \(\hat{\varphi}_t = u_X\) giving a lifting to \(\Lambda\)
Remark 2.6. Recall that two different liftings of \( SL(C) \) is Chow semistable. Suppose also that we have a fiber multiplications by \( C \) of an additive constant of Hamiltonian functions. These constants belongs to \( X \) with respect to the Kähler form \( \sigma \) of the infinitesimal action of \( X \) in section 1, that the infinitesimal action of \( H \) inducing a subgroup of \( SL(H^0(M, \mathcal{O}(\Lambda))) \). Summarizing the arguments given in this paragraph we get the following.

Proposition 2.5. Let \((M, \Lambda)\) be a very ample line bundle and suppose that \((M, \Lambda)\) is Chow semistable. Suppose also that we have a \( C^* \)-action on \( M \) generated by \( X \in \mathfrak{h}_0(M) \). Then the additive constant of Hamiltonian function \( \omega_X \) for \( X \) inducing a lifting of infinitesimal action of \( X \) on \( \Lambda \) such that it generates a subgroup of \( SL(H^0(M, \mathcal{O}(\Lambda))) \) is determined by the normalization \( I \).

The detail of the following remarks the example can be found in [18].

Remark 2.7. Let \( \Lambda \to M \) be an ample line bundle. Let \( \omega \) be a Kähler form representing \( c_1(\Lambda) \), and let \( \theta \) be the connection form form of \( \Lambda \) such that the curvature \( \omega \). For an element \( X \) in the lattice of the Lie algebra of the maximal torus in \( \text{Aut}(M, L) \), a natural choice of a Hamiltonian function for \( X \) with respect to \( \omega \) is \((i/2\pi)\theta(X)\). Then the normalization \( \int_M (i/2\pi)\theta(X)\omega^m \) is a rational number. This number is an invariant related to equivariant cohomology with respect to the lifted action. With this choice of Hamiltonian functions the moment map image is an integral polytope, i.e. a polytope with integer vertices. In particular, \( \omega_X \) satisfying the normalization \( I \) gives a rational polytope as its moment map image.

Remark 2.8. Let \( \Lambda \to M \) be an ample line bundle. Consider an action of \( S^1 \) on \( M \) generated by a holomorphic vector field \( X \in \mathfrak{h}_0(M) \) with period 1. Suppose that a Kähler form \( \omega \in c_1(\Lambda) \) is given. Then the lift of the infinitesimal action of \( X \) to \( \Lambda \) is given locally by

\[
X \mapsto X^z = -2\pi i u_X \frac{\partial}{\partial z} + X^h
\]

where \( z \) is the fiber coordinate and \( X^h \) is the horizontal lift with respect to the connection whose curvature is the given Kähler form \( \omega \). From Remark 2.7 one can see that, for \( u_X \) satisfying the normalization \( I \), \( X^z \) generates an \( S^1 \)-action of integer period. See the next example.

Example 2.9. Let \( M = \mathbb{C}P^1 \) and \( \Lambda = \mathcal{O}(1) \). Then the moment map image with respect to the Hamiltonian using a connection form is \( [n, n + 1] \) for some integer \( n \). If we normalize the Hamiltonian function by \( I \) then the moment map image becomes \([-1/2, 1/2], \) and the period in this case is 2.

Now we are in a position to give proofs of Theorem 2.1. To put it simply, applying Proposition 2.5 by taking \( \Lambda \) to be \( L^k \) for all large \( k \) shows that asymptotic stability assures that the normalization \( I \) for \( L \) gives the normalization \( I \) for \( L^k \) all at once. We give some more detail about this in what follows.
Proof of Theorem 2.7. We take an $X \in h_0(M)$ such that the real part $\text{Re} X$ of $X$ generates an $S^1$-action $M$ with period 1. Let $\omega$ be a Kähler form representing $c_1(L)$, and $u_X$ be the Hamiltonian function with the normalization (1). Then by Remark 2.8, $u_X$ defines a lifting of $X$ to an infinitesimal action of $X^\sharp$ which generates an $S^1$-action with integer period. This induces a $\mathbb{C}^*$-action on $L$ and also on $L^k$ for all $k$ naturally. Fixing $k$, the lifted $\mathbb{C}^*$-action on $L^k$ defines a subgroup of $GL(H^0(M, O(L^k)))$-action but not necessarily $SL(H^0(M, O(L^k)))$. We therefore divide the action by $\det^{1/N_k}$ where $N_k = \dim H^0(M, O(L^k))$ and get a subgroup of $SL(H^0(M, O(L^k)))$. This action induces action on $V_k := H^0(M, O(L^k))^*$, and thus on $L^k$. This last lifting the original $\mathbb{C}^*$-action on $M$ to $L^k$ must correspond to a choice of Hamiltonian function of $X$ for the Kähler form $k\omega \in c_1(L^k)$ by the general principle. This Hamiltonian function must be of the form

$$u_{X,k} = ku_X + c_k$$

where $c_k$ is a constant. Then

$$u_{X,k} = u_X + \frac{c_k}{k}$$

defines a lifting of the infinitesimal action of $X$ on $M$ to an infinitesimal action $X^\sharp_k$ on $L$. By the construction, this $X^\sharp_k$ generates a $\mathbb{C}^*$-action on $L$ inducing a subgroup of $SL(H^0(M, O(L^k)))$.

Now recall that we assume that the polarized manifold $(M, L)$ is asymptotically Chow semistable. Therefore there is a positive integer $k_0$ such that for all $k \geq k_0$, $(M, L^k)$ is Chow semistable. But by Proposition 2.5, $u_{X,k}$ satisfies the normalization (1). Since we also chose $u_X$ to satisfy this normalization we must have $c_k = 0$. This arguments apply for all $k \geq k_0$. Hence the choice of $u_X$ gives a lifting of $\mathbb{C}^*$-action to $L$ in such a way that the natural induced actions on $H^0(M, O(L^k))$, which we shall denote by $\rho_k$, defines a subgroup of $SL(H^0(M, O(L^k)))$ for all $k \geq k_0$.

Now we apply the equivariant Riemann-Roch theorem. Recall that we put $N_k = \dim H^0(M, O(L^k))$. Then the weight $w_k$ of the action $\rho_k$ on $\wedge^{N_k} H^0(M, O(L^k))$ is 0 for $k \geq k_0$ since $\rho_k$ gives an $SL$-action on $H^0(M, O(L^k))$. By the equivariant Riemann-Roch theorem this weight is given by the coefficient of $t$ of the following (c.f. [10]):

$$e^{k(\omega + tu_X)}\text{Td}(tL(X) + \Theta) = \sum_{p=0}^{\infty} \frac{k^p}{p!}(\omega + tu_X)^p \sum_{q=0}^{\infty} \text{Td}^{(q)}(tL(X) + \Theta).$$

By writing the coefficient of $t$ explicitly we have

$$0 = \sum_{p=0}^{m+1} \frac{k^p}{p!} \int_M (\omega^p \wedge \text{Td}^{(m-p+1)}(L(X) + \Theta) + p\omega^{p-1} \wedge u_X \text{Td}^{(m-p+1)}(\Theta))$$

for all $k \geq k_0$. But from a result in [19] (see also Theorem 5.3.10 in [16])

$$\int_M \text{Td}^{(m+1)}(L(X) + \Theta) = 0$$

which implies that the term $p = 0$ in (7) vanishes. The term $p = m+1$ also vanishes because of our normalization (1). Thus the vanishing of the terms for $p = 1, \ldots, m$ in (7) gives the desired result since the terms for $p = 1, \ldots, m$ in (7) coincide with $\mathcal{F}_{\text{td}^p}$ for $p = 1, \ldots, m$. 

7
Conversely suppose that $F_{T^d}$ vanishes for all $p = 1, \cdots, m$ and that $H^q(M, \mathcal{O}(L))$ vanishes for all $q > 0$, then the right hand side of (7) is zero for any $k$. This implies that $\rho_k$ induces SL-action not only for all $k \geq k_0$ but also for all $k \geq 1$. This completes the proof of Theorem 2.1. □

**Remark 2.10.** Since $T^{d_1} = \frac{1}{2}c_1$ and $c_1$ is the trace

\begin{equation}
F_{T^{d_1}}(X) = \frac{m}{2} \int_M S u_X \omega^m
\end{equation}

where $S$ denotes the scalar curvature of the Kähler form $\omega$. This is an obstruction to the existence of cscK metrics since if $S$ is constant then $F_{T^{d_1}}(X) = 0$ because of the normalization (9). In fact the right hand side of (9) is equal to $(mi/2)f(X)$. To define $f$ let $F$ be a smooth function such that

$$S - \int_M S \omega^m / \int_M \omega^m = \Delta F.$$

Then $f$ is defined by

$$f(X) = \int_M X F \omega^m.$$

This $f(X)$ is independent of the choice of $\omega$ and obstructs the existence of a cscK metric in a given Kähler class (15). Then using (2) we have

$$\int_M S u_X \omega^m = \int_M \Delta F u_X \omega^m = - \int_M (\text{grad} u_X) F \omega^m = i \int_M X F \omega^m = if(X).$$

Therefore we get $F_{T^{d_1}} = (mi/2)f(X)$.

**Remark 2.11.** If $M$ is a Fano and $L = K^{-1}_M$ it is more convenient to choose $F$ to be

$$\rho_\omega - \omega = \frac{i}{2\pi} \partial \bar{\partial} F$$

where $\omega$ represents $c_1(M) = c_1(K^{-1}_M)$, the metric $g$ is given as

$$\omega = \frac{i}{2\pi} g_{\overline{z}^i z^j} dz^i \wedge d\overline{z}^j$$

and the Ricci form $\rho_\omega$ is given as

$$\rho_\omega = - \frac{i}{2\pi} \partial \bar{\partial} \log \det(g_{\overline{z}^i z^j}).$$

Then $f$ is defined as

$$f(X) = \int_M X F \omega^m.$$

Consider the second order elliptic differential operator

$$\Delta_F = - g^{\overline{z}^i} \frac{\partial^2}{\partial z^i \partial \overline{z}^j} - g^{\overline{z}^i} \frac{\partial F}{\partial z^i} \frac{\partial}{\partial \overline{z}^j}.$$ 

If a complex valued smooth function $\tilde{v}_X$ satisfies

$$\Delta_F \tilde{v}_X = \tilde{v}_X$$
and put

\[ X := g^j \partial \bar{v}_X \partial z^j. \]

then we have

\[ i(X)\omega = i\bar{v}_X \]

and

\[ f(X) = \frac{1}{(m+1)} \bar{v}_X^m = \int_M \Delta \bar{v}_X \rho_X^m = \int_M \text{div} \bar{v}_X \rho_X^m. \]

The proof of this result can be found in (5.2.1) in [16].

Remark 2.12. Mabuchi [25] states the obstruction to asymptotic Chow semistability by

\[ \rho_k = \rho_{k_0} \]

for all \( k \geq k_0. \)

3. The Hilbert Series

Let \( M \) be a toric Fano manifold of complex dimension \( m \) and \( K_M \) its canonical line bundle so that the real torus \( T^m \) acts on \( M \) and this lifts to an action on \( K_M \) by the pull-back of differential forms. This \( T^m \)-action together with the \( S^1 \)-action by multiplication on the fiber gives a \( T^{m+1} \)-action on \( K_M \) so that \( K_M \) is also toric. It is a standard fact that \( K_M^{-1} \) is very ample and \( H^q(M, \mathcal{O}(K_M^{-1})) \) vanishes for all \( q > 0 \), see [32] and [13]. We wish to consider the formal sum

\[ L(g) = \sum_{k=0}^{\infty} \text{Tr}(g|_{H^0(M, \mathcal{O}(K_M^{-k}))}) \]

where \( \text{Tr}(g|_{H^0(M, \mathcal{O}(K_M^{-k}))}) \) denotes the trace of the induced action of \( g \in T^{m+1} \) on \( H^0(M, \mathcal{O}(K_M^{-k})) \), and regard \( L(g) \) as a function of \( g \). We call \( L(g) \) the index character (c.f. [29]). We may analytically continue \( L(x) \) to \( x \in T^{m+1}_\mathbb{C} \), the algebraic torus.

Let \( S \) be the total space of the associated \( U(1) \)-bundle of \( K_M \). Then \( S \) is a \((2m+1)\)-dimensional Sasaki manifold. Recall that an odd dimensional Riemannian manifold \((S,g)\) is a Sasaki manifold if its Riemannian cone \((C(S),\bar{g})\) with \( C(S) = S \times \mathbb{R}_+ \) and \( \bar{g} = dr^2 + r^2 g \) is a Kähler manifold. Here \( r \) denotes the standard coordinate on \( \mathbb{R}_+ \). In the present case \( C(S) \) is biholomorphic to \( K_M - \{ \text{zero section} \} \), and \( S \) is an \( S^1 \)-bundle over the Fano manifold \( M \). In such a case we say that \( S \) is a regular Sasaki manifold.

Since \( M \) is toric so is \( C(S) \). If the convex polytope (i.e. the moment map image) of \( M \) is given by

\[ P^* := \{ w \in \mathbb{R}^m \mid v_j \cdot w \geq -1 \} \]

where \( v_j \in \mathbb{Z}^m \) generates a 1-dimensional face of the fan then the convex polytope of \( C(S) \) is given by

\[ C^* := \{ y \in \mathbb{R}^{m+1} \mid \lambda_j \cdot y \geq 0 \} \]

where \( \lambda_j = (v_j,1) \in \mathbb{Z}^{m+1} \). We denote by \( P \) the dual polytope of \( P^* \), that is, \( P \) is a convex polytope with vertices \( v_j \in \mathbb{Z}^m \). The integral points in \( C^* \) correspond to the sections of \( K_M^{-k} \) for some \( k \geq 1 \). In fact a point \((u,k) \in \mathbb{Z}^{m+1} \cap C^* \) with
$u \in \mathbb{Z}^m$ and $k \in \mathbb{Z}$ of height $k$ corresponds to an element $H^0(M, \mathcal{O}(K^{-k}_M))$. For $a \in C^* \cap \mathbb{Z}^{m+1}$ and $x \in T^*_C$ we put
\[ x^a := x_1^{a_1} \cdots x_{m+1}^{a_{m+1}}. \]

If $a = (u, k)$ and $\sigma_a$ denotes the holomorphic section of $H^0(M, \mathcal{O}(K^{-k}_M))$ corresponding to $a$ then the action of $x \in T^*_C$ is given by $\sigma_a \mapsto x^a \sigma_a$. We write $C(x, C^*)$ for the index character $L(x)$ for the toric Fano manifold corresponding to the cone $C^*$. Thus we have obtained
\[ C(x, C^*) = \sum_{a \in C^* \cap \mathbb{Z}^{m+1}} x^a. \]

The right hand side is also called the Hilbert series. It is known that the Hilbert series $C(x, C^*)$ of a rational cone $C^*$ can be written as a rational function of $x$, see the books [4] or [30] for this subject.

For $b \in \mathbb{R}^{m+1}$ we write
\[ e^{-tb} = (e^{-b_1 t}, \ldots, e^{-b_{m+1} t}) \]
and consider
\[ C(e^{-tb}, C^*) = \sum_{a \in C^* \cap \mathbb{Z}^{m+1}} e^{-\langle a, b \rangle}. \]

Then $C(e^{-tb}, C^*)$ is a meromorphic function of $t$.

We choose $b \in \mathbb{R}^{m+1}$ from the subset
\[ C_R := \{ b \in \mathbb{R}^{m+1} \mid b = (b_1, \ldots, b_m, m+1), (b_1, \ldots, b_m) \in (m+1)P \}. \]

The intrinsic meaning of the subset $C_R$ can be explained in the context of toric Sasaki geometry as follows (c.f. [28], [29], and also [21], [8]).

We start with a general Sasaki manifold $S$ so that its Riemannian cone $C(S)$ described above is a Kähler manifold. When $C(S)$ is a toric Kähler manifold we say that $S$ is a toric Sasaki manifold. $S$ is identified with $\{ r = 1 \} \subset C(S)$. The Reeb vector field is a vector field $\xi = J(\partial/\partial r)$ on $S \cong \{ r = 1 \}$ where $J$ denotes the complex structure on $C(S)$. It extends to a vector field on $C(S)$ given by $J(\partial/\partial r)$, which we also call the Reeb vector field. The Reeb vector field is a Killing vector field both on $S$ and $C(S)$, and can be regarded as an element $\xi$ of the Lie algebra $t^{m+1}$ of the torus $T^{m+1}$. When the cone $C(S)$ is $\mathbb{Q}$-Gorenstein as a toric variety it can be shown that there is an element $\gamma \in t^{m+1\ast}$ such that the Reeb vector field satisfies
\[ \langle \lambda_j, \gamma \rangle = -1 \]
and
\[ \langle \gamma, \xi \rangle = -m - 1 \]
where $\lambda_j$‘s in $t^{m+1}$ determine the moment cone $C^*$ of $C(S)$ by
\[ C^* := \{ y \in t^{m+1\ast} \mid \langle \lambda_j, y \rangle \geq 0 \}. \]

The smoothness of $C(S)$ implies that $\lambda_j$‘s form a basis over $\mathbb{Z}$ along each 1-dimensional face of $C^*$. Thus by (11), $\gamma$ is uniquely determined from the toric data of $C(S)$. If we vary the Sasaki structure by changing the Reeb vector field keeping the toric structure of $C(S)$, then, since $\gamma$ is not varied, the Reeb vector field $\xi$ has to obey the condition (12). Thus the deformation space of Sasaki structures with fixed toric structure of the cone is given by
For a regular Sasaki manifold we can take a basis of the lattice $\mathbb{Z}_m^{m+1} = \text{Ker}\{\exp : t^{m+1} \to T^{m+1}\}$ such that $\gamma$ and the Reeb vector field $\xi_0$ are denoted as $\gamma = (0, \cdots, 0, -1)$ and $\xi_0 = (0, \cdots, 0, m + 1)$.

Then the deformation space (13) of Sasaki structures with fixed toric structure in this case coincides with the space (10). The tangent space $T_\xi C_R$ of the deformation space $C_R$ at $\xi$ is isomorphic to

$$\{ X \in t^{m+1} \mid \langle \gamma, X \rangle = 0 \}.$$ 

The subspace given by (13) has another intrinsic meaning. Recall that the cone $C(S)$ for the regular Sasaki manifold $S$ is $K_M$ minus the zero section for a Fano manifold $M$. In the toric case $M$ admits an action of the $m$-dimensional torus $T^m$, and together with the circle action of the fiber of $S \to M$, $S$ admits an action of the $(m + 1)$-dimensional torus $T^{m+1}$. This $(m + 1)$-dimensional torus action also gives the toric structure of $C(S)$. Let us consider the liftings of the action of $T^m$ on $M$ to $K_M$. A natural choice is given by the pull-back of differential forms since $K_M$ is the bundle of $(m, 0)$-forms. Any other choice differs from the natural choice by the action along the fibers of $S \to M$. The different choices of the liftings of the $T^m$-action on $M$ to $K_M$ can be described in two ways.

First of all, $C(S)$, which is isomorphic to $K_M$ minus the zero section, admits $T^{m+1}$-action. Any lifting of $T^{m+1}$-action on $M$ to $K_M$ is given by a subgroup of $T^{m+1}$. At the Lie algebra level, this subgroup corresponds to a sub-lattice of rank $m$ in $\mathbb{Z}_m^{m+1} \subset t^{m+1}$. It spans a hyperplane in $t^{m+1}$. In this manner we can regard the hyperplane (14) as a lifting of $T^{m+1}$-action on $M$ to $K_M$ or $C(S)$.

Secondly, the difference of the liftings are described in terms of the normalization of Hamiltonian functions as follows. Let $X$ be a holomorphic vector field on $M$ such that $X$ is the infinitesimal generator of the action of an $S^1$ in $T^m$ and that $\exp(X) = 1$. Choose a lifting of the $S^1$-action on $M$ to $K_M^{-1}$ and let $\tilde{X}$ be its infinitesimal generator. Then any other lift of the $S^1$-action is given by an infinitesimal generator of the form $2\ell \pi i z \partial / \partial z + \tilde{X}$ for some integer $\ell$ where $z$ denotes the coordinate of the fiber of $K_M^{-1} \to M$. Then if $\theta$ is a connection form on the principal $\mathbb{C}^*$-bundle associated with $K_M^{-1}$ then

$$\frac{1}{2\pi i} \tilde{\theta}(2\ell \pi i z \partial / \partial z + \tilde{X}) = \frac{1}{2\pi i} \tilde{\theta}(\tilde{X}) + \ell.$$ 

Let $\rho$ be a $T^m$-invariant Kähler form representing $c_1(M) = -c_1(K_M)$. Then by the Calabi-Yau theorem there is another $T^m$-invariant Kähler form $\omega$ representing $c_1(M) = -c_1(K_M)$ such that the Ricci form $\rho_\omega$ is equal to $\rho$. Let $\tilde{\theta}$ be the connection form on the principal $\mathbb{C}^*$-bundle associated with $K_M^{-1}$ of the Hermitian connection $\nabla$ induced from the Levi-Civita connection of $\omega$. Since $(i/2\pi) \tilde{\theta}(\tilde{X}) + \text{constant}$ is a Hamiltonian function of $X$ for the Ricci form $\rho_\omega$ considered as a symplectic form and since the liftings of $T^m$-action to $K_M^{-1}$ and $K_M$ have the natural correspondence...
the above arguments explain that the difference of the liftings are described in terms of normalizations of Hamiltonian functions. To make this correspondence definitive we need to decide the Hamiltonian functions for the natural lifting by the pull-back of \((m,0)\)-forms.

In the next section we shall consider the derivative of \(C(e^{-t\theta}, C^*)\) at \(b = \xi_0\) along a vector in the tangent space \(T_{\xi_0}C_R\) described as (13). For that purpose we claim the following.

**Proposition 3.1.** Let \(M\) be a Fano manifold and take \(c_1(M)\) as a Kähler class. The following three liftings of \(T^m\)-action on \(M\) to \(K_M\) coincide. Here the lifted action to \(K_M\) naturally induces a lifted action to \(K^{-1}_M\) and vice versa, and they are identified.

(a) The action on \(K_M\) defined by the pull-back of \((m,0)\)-forms.
(b) The lifted action defined by the subspace (14).
(c) The lifted action to \(K^{-1}_M\) defined by the normalization of the Hamiltonian function \(v_X\) for \(X \in t^m \otimes \mathbb{C}\) by

\[
\int_M v_X \omega^m = \frac{i}{2\pi} f(X).
\]

Here \(v_X\) is a Hamiltonian function of \(X\) in the sense that \(i(X)\omega = -\partial v_X\), and \(f(X)\) is the one given in Remark 2.11.

**Proof.** First we see that (a) and (c) coincide. As above let \(\rho\) be a \(T^m\)-invariant Kähler form representing \(c_1(M) = -c_1(K_M)\). Then by the Calabi-Yau theorem (36) there is another \(T^m\)-invariant Kähler form \(\omega\) representing \(c_1(M) = -c_1(K_M)\) such that the Ricci form \(\rho_{\omega}\) is equal to \(\rho\). Express the Kähler form \(\omega = i\theta_{1/2}dz^i \wedge d\bar{z}^j\) as in section 1, and consider its Levi-Civita connection on the tangent bundle and the induced connection on \(K^{-1}_M\) and \(K_M\). The pull back action of \(T^m\) on \(K_M\) is identified with the usual push forward action on \(K^{-1}_M\). Let \(X\) be a holomorphic vector field whose real part belongs to the Lie algebra of \(T^m\), and let \(\tilde{X}\) be its lift to \(K^{-1}_M\) induced by the push forward action. It is easy to compute that for the connection form \(\tilde{\theta}\) on \(K^{-1}_M\) we have

\[
\tilde{\theta}(\tilde{X}) = \text{div} X = \sum_{i=1}^m \nabla_i X^i.
\]

From \(\rho = \frac{1}{2\pi} \tilde{\theta}\) we see that

\[
i(X)\rho = -\frac{i}{2\pi} \tilde{\theta}(\tilde{X})
\]

and \(v_X = \frac{i}{2\pi} \tilde{\theta}(\tilde{X}) = \frac{1}{2\pi} \text{div} X\) is the Hamiltonian function. That (a) and (c) coincide follows from this.

The equivalence between (b) and (c) follows from the arguments given in the proof of Proposition 8.10 in [24]. To explain these arguments we recall basic terminologies in Sasaki Geometry. The Reeb vector field \(\xi\) defines a flow which has a transverse Kähler structure. This means that the local orbit spaces are open Kähler manifolds and that they are patched together isometrically on their overlaps. On these local orbit spaces we have Kähler forms which can be lifted to \(S\) and form a global two form \(\omega^T\) called the transverse Kähler form. The Ricci forms on local orbit spaces also lifted to \(S\) to form a global two form \(\rho^T\) called the transverse Ricci
form. On local orbit spaces of the Reeb flow we have a $\bar{\partial}$ and $\partial$ operators, denoted by $\bar{\partial}_B$ and $\partial_B$. When the Sasaki manifold $S$ has a $Q$-Gorenstein cone $C(S)$ there exists a smooth function $h$ such that

$$\rho^T - (2m + 2)\omega^T = i\partial_B\bar{\partial}_B h. \tag{16}$$

This function $h$ is “basic” in the sense that locally it is obtained by lifting a function on the local orbit space. Note that the coefficient $(2m + 2)$ comes from the normalization of the Sasaki metric so that the length of Reeb vector field to be 1.

With these terminologies in mind, it is proved in the proof of Proposition 8.10 in [21] that, on the toric Sasaki manifold $S$ with $Q$-Gorenstein cone $C(S)$, the tangent space to $C_R$ is equal to

$$\{ X \in t^{m+1} | \Delta^h_B \bar{v}_X = (2m + 2)\bar{v}_X \}, \tag{17}$$

where

$$\bar{v}_X = i((\bar{\partial} - \partial) \log r)(X) = -iX \log r,$$

$r$ being the coordinate on $\mathbb{R}_+$ in $C(S) = S \times \mathbb{R}_+$ and where

$$\Delta^h = -g_B^{ij} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} - g_B^{ij} \frac{\partial h}{\partial z^i} \frac{\partial}{\partial \bar{z}^j},$$

g$_B$ being the transverse Kähler metric. In the case of the regular Sasaki manifold $S$ over a Fano manifold $M$, the Reeb vector field is induced by $\xi_0 = (0, \ldots, 0, m + 1)$.

The hyperplane given by (14) is equal to (17). But in this situation $(2m + 2)\omega^T = \omega$ and $h = ((2m + 2)/2\pi)F$. Then the equation $\Delta^h_B \bar{v} = 2(m + 1)\bar{v}$ is equivalent to $\Delta_F \bar{v} = \bar{v}$ where

$$\Delta_F = -g_B^{ij} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} - g_B^{ij} \frac{\partial F}{\partial z^i} \frac{\partial}{\partial \bar{z}^j}. \tag{18}$$

Here, as in Remark 2.11 we take $\omega = (i/2\pi)g_B dz^i \wedge d\bar{z}^j$. Thus (17) implies $\Delta_F \bar{v}_X = \bar{v}_X$, and we have

$$\int_M \bar{v}_X \omega^m = -\int_M (\bar{v}_X)^i F_i \omega^m = -\int_M X \omega^m = -f(X).$$

But we see that $v_X = \frac{i}{2\pi} \bar{v}_X$ is the Hamiltonian function in the sense of (c) and it satisfies

$$\int_M v_X \omega^m = \frac{i}{2\pi} f(X).$$

This proves that (b) and (c) define the same lifting. \qed

Consider the derivatives of the coefficients of the Laurent series in $t$ of the meromorphic function $C(e^{-tb}, C^*)$ at $b = \xi_0$ in the directions of vectors in the tangent space $T_{\xi_0} C_R$ described as (14). Then those derivatives are characters of $\mathfrak{k} \otimes \mathbb{C}$.

**Theorem 3.2.** The linear span of those derivatives described as above coincides with the linear span of $F_{T_{\xi_0}}$, $\cdots$, $F_{T_{\xi_0}}$ restricted to $\mathfrak{k} \otimes \mathbb{C}$.

**Proof.** First of all, for a square matrix $A$ we have as a general formula in linear algebra

$$\frac{d}{ds}|_{s=0} \text{tr} e^{sA} = \frac{d}{ds}|_{s=0} \det e^{sA} = \text{tr} A \tag{19}$$
where $\text{tr}$ denotes the trace. For a tangent vector $c \in T_{S_0} C_R$ we consider the action of $e^{-t(\xi_0 + \epsilon)}$ on $H^0(M, K_M^{-k})$ and take the derivative with respect to $s$ at $s = 0$. Since
\[
e^{-t\xi_0} = (1, \ldots, 1, e^{-(m+1)t}),
\]
and
\[
(e^{-t\xi_0})^a = e^{-(m+1)t},
\]
$e^{-t\xi_0}$ acts on $H^0(M, K_M^{-k})$ as a scalar multiplication by $e^{-(m+1)t}$, and we see from the general formula (19) that the derivative of $C(e^{-t(\xi_0 + \epsilon)}, C)$ with respect to $s$ at $s = 0$ is the sum $\sum_{k=1}^\infty e^{-k(m+1)}\tilde{w}_k$ where $\tilde{w}_k$ is the weight of the lifted action described in Proposition 3.1. By the equivariant index theorem each $\tilde{w}_k$ is given by
\[
\tilde{w}_k = \sum_{p=0}^{m+1} \frac{k^p}{p!} \int_M (\omega^p \wedge \text{td}^{(m-p+1)}(L(X) + \Theta) + p \omega^{p-1} \wedge \nu_X \text{td}^{(m-p+1)}(\Theta))
\]
where $\nu_X$ satisfies the the normalization (15). Recall that the Hamiltonian function $u_X$ used in the definition of $F_{T^d}(X)$ satisfies the normalization
\[
\int_M u_X \omega^m = 0.
\]
Thus, $v_X = u_X - if(X)/2\pi \text{Vol}(M)$. Inserting this into the right hand side of (20) one sees that our $\tilde{w}_k$ differs from $\sum_{p=1}^m (k^p/p!) F_{T^d}(X)$ by a multiple of $f(X)$. But since
\[
f(X) = -\frac{2i}{m} F_{T^d}(X)
\]
we are done. \hfill \Box

4. The formula of Martelli-Sparks-Yau

The Hilbert series $C(x, C^*)$ of a toric diagram $C^*$, which is the image of the moment map of a toric Calabi-Yau manifold, is getting into the limelight in String theory, especially AdS/CFT correspondence, for example see [5, 12, 29]. Let
\[
C^* = \{ x \in \mathbb{R}^{m+1}, \lambda_i : x \geq 0, i = 1, \ldots, d \}
\]
be an $(m + 1)$-dimensional toric diagram of height 1. Here $\lambda_i = (\nu_i^1, \ldots, \nu_i^m, 1) \in \mathbb{Z}^{m+1}$ for each $i$. For a fixed $b \in C_R$, the Laurent expansion of $C(e^{-t\epsilon b}, C^*)$ at $t = 0$ is written as
\[
C(e^{-t\epsilon b}, C^*) = \frac{C_{-m-1}(b)}{t^{m+1}} + \frac{C_m(b)}{t^m} + \frac{C_{-m+1}(b)}{t^{m-1}} + \cdots.
\]
In [29], Martelli, Sparks and Yau showed that the coefficient of the leading order term $C_{-m-1}(b)$ is a constant multiple of the volume of a Sasaki manifold whose Reeb vector field is generated by $b$. Moreover they proved that if we think of $b$ as variables then the first variation of $C_{-m-1}(b)$ is equal to the Sasaki-Futaki invariant\footnote{Strictly speaking, they proved this in the case when $b$ is a rational vector. The general case was verified by the first two authors and G. Wang in [21].}. Hence it is natural to ask what are the other coefficient $C_i(b)$ and its first variation for each $i$. One of our motivations to write this article is that we want to know the answer to this question. As we saw in the previous section, when $C^*$ corresponds to the canonical bundle of a toric Fano manifold, the first variations...
of $C_i$ at $b = (0, \ldots, 0, m + 1)$ are the linear combinations of the integral invariants $\mathcal{F}_{r,d}$.

**Example 4.1.** Let

$$C^* = \{ v + x^1v_1 + \cdots + x^n v_n; x^1, \ldots, x^n \geq 0 \} \subset \mathbb{R}^n$$

be a rational simplicial cone, i.e. $v_1, \ldots, v_n \in \mathbb{Z}^n$ and these are linearly independent in $\mathbb{R}^n$. Then by Theorem 3.5 of [3], the Hilbert series $C(x, C^*)$ of $C^*$ is

$$C(x, C^*) = \frac{\sigma_\Pi(x)}{(1 - x^v_1) \cdots (1 - x^v_n)} \tag{22}$$

where $\Pi$ is the half-open parallelepiped

$$\Pi = \{ v + x^1v_1 + \cdots + x^n v_n; 0 \leq x^1, \ldots, x^n < 1 \}$$

and

$$\sigma_\Pi(x) = \sum_{a \in \Pi \cap \mathbb{Z}^n} x^a.$$ 

For example let

$$C^* := \{ a(1, 1) + b(-1, 1); a, b \geq 0 \} \subset \mathbb{R}^2.$$

Then,

$$C((x, y); C^*) = \frac{1 + y}{(1 - xy)(1 - x^{-1}y)}.$$ 

In the case when $C^*$ is the toric diagram of height 1 corresponding to the canonical bundle of a toric Fano manifold, Martelli, Sparks and Yau [29] gave the formula to compute $C(x, C^*)$ combinatorially. For example, applying the formula to the Hilbert series in Example 4.1 we easily see that

$$\frac{1 + y}{(1 - xy)(1 - x^{-1}y)} = \frac{1}{(1 - xy)(1 - x^{-1})} + \frac{1}{(1 - x^{-1}y)(1 - x)} = C((x, y), C_1) + C((x, y), C_2),$$

where $C_1 = \{ a(1, 1) + b(-1, 0); a, b \geq 0 \}$, $C_2 = \{ a(-1, 1) + b(1, 0); a, b \geq 0 \}$. To prove the formula, they formally applied the Lefschetz fixed point formula to noncompact manifold $K_M$, the total space of the canonical bundle of a toric Fano manifold $M$. But we can verify the same formula using only combinatorial argument as follows. Let $v_1, \ldots, v_d \in \mathbb{Z}^m$ be the vertices of a Fano polytope $P_M \subset \mathbb{R}^m$. Equivalently, $v_1, \ldots, v_d$ are the generators of 1-dimensional cones of the fan of an $m$-dimensional toric Fano manifold $M$. If we set $\lambda_j = (v_j, 1) \in \mathbb{Z}^{m+1}$, then we see that the cone

$$C_M^* = \{ x \in \mathbb{R}^{m+1}; \lambda_j \cdot x \geq 0, j = 1, \ldots, d \}$$

is a toric diagram of height 1 corresponding to the canonical bundle $K_M$ of $M$. We can also describe this cone $C^*$ as

$$C_M^* = \left\{ \sum_{j=1}^k a^j \mu_j; a^1, \ldots, a^k \geq 0, \right\}$$

where $\mu_j = (w_j, 1) \in \mathbb{Z}^{m+1}$, $w_1, \ldots, w_k$ is the vertices of the polar polytope

$$P_M^o = \{ y \in \mathbb{R}^m; v_j \cdot y \geq -1, j = 1, \ldots, d \}.$$
Let \( e_{j,1}, \ldots, e_{j,m} \in \mathbb{Z}^m \) denote the generators of the edges emanating from a vertex \( w_j \). Note here that \( e_{j,1}, \ldots, e_{j,m} \) is a basis of \( \mathbb{Z}^m \) for each \( j \) since \( P^*_M \) is a Delzant polytope. Hence the Hilbert series of the cone

\[
C_{j,t} = \{ t w_j + x^1 e_{j,1} + \cdots + x^m e_{j,m}; x^1, \ldots, x^m \geq 0 \}
\]

in \( \mathbb{R}^m \) is

\[
C(\bar{x}, C_{j,t}) = \frac{\bar{x}^\text{w}_j}{(1 - \bar{x}^{e_{j,1}}) \cdots (1 - \bar{x}^{e_{j,m}})}
\]

by (22). Here \( \bar{x} = (x_1, \ldots, x_m) \). Then, by Brion’s formula [6], see also Theorem 9.7 of [4] or Theorem 12.13 of [30], we see that

\[
\sum_{\bar{a} \in \text{IP}^*_M} \bar{x}^{\bar{a}} = \sum_{j=1}^k C(\bar{x}, C_{j,t}) = \sum_{j=1}^k \bar{x}^{\text{w}_j} \prod_{b=1}^m \frac{1}{(1 - \bar{x}^{e_{j,b}})}.
\]

Therefore we have (23)

\[
C(x, C^*_M) = \sum_{l=0}^\infty \left\{ \sum_{\bar{a} \in \text{C}_{M^*,Z}(\text{IP}^*_M, l) \cap \{a_{m+1} = l\}} x^{\bar{a}} \right\} = \sum_{l=0}^\infty \left\{ \sum_{\bar{a} \in \text{IP}^*_M} \bar{x}^{\bar{a}} \right\} x^{l+1}
\]

\[
= \sum_{l=0}^\infty \left\{ \sum_{j=1}^k \bar{x}^{\text{w}_j} \prod_{b=1}^m \frac{1}{(1 - \bar{x}^{e_{j,b}})} \right\} x^{l+1} = \sum_{l=0}^\infty \left\{ \sum_{j=1}^k \bar{x}^{\text{w}_j} \prod_{b=1}^m \frac{1}{(1 - \bar{x}^{e_{j,b}})} \right\} \prod_{b=1}^m \frac{1}{(1 - \bar{x}^{e_{j,b}})}
\]

\[
= \sum_{j=1}^k \frac{1}{1 - x^{\text{w}_j}} \prod_{b=1}^m \frac{1}{(1 - x^{e_{j,b}})}.
\]

Here \( C_{M^*,Z}^* = C^*_M \cap \mathbb{Z}^{m+1} \). This is the formula given in [29]. Note here that

\[
D_j (x, C^*_M) := \frac{1}{1 - x^{\text{w}_j}} \prod_{b=1}^m \frac{1}{(1 - x^{e_{j,b}})}
\]

diverges at \( x = (1, \ldots, 1, e^{-b^{m+1}t}) = e^{-t(0, \ldots, 0, b^{m+1})} \) for each \( j \). However if we reduce the fractions to a common denominator, at least in the case of \( m = 1, 2, \) and 3, we see that there is a Laurent polynomial \( K_{C^*}(x) \) such that

\[
C(x, C^*_M) = K_{C^*}(x) \prod_{b=1}^m \frac{1}{(1 - x^{e_{j,b}})}
\]

and \( K_{C^*}(x) \) converges at \( x = (1, \ldots, 1, e^{-b^{m+1}t}) \) when \( b^{m+1}, t \neq 0 \). We do not know a general proof of this fact, but can check it using a computer in each cases of \( m = 1, 2 \) and 3. For example, let \( C^* \) be the 2-dimensional cone given in Example 4.1. Then \( 1/(1 - xy)(1 - x^{-1}) \) and \( 1/(1 - x^{-1}y)(1 - x) \) diverge at \( x = 1 \). On the other hand

\[
C((1, y), C^*) = \frac{1 + y}{(1 - y)^2}.
\]

To calculate the Hilbert series of the toric diagram \( C^*_M \) associated with a toric Fano manifold \( M \), we will use (24).
5. Examples

In this section we give some combinatorial data and calculations associated with toric Fano threefolds. We used a computer algebra system Maxima\textsuperscript{4} for computing Hilbert series. Of course you can also utilize other systems, for example, Maple, Mathematica and so on. Since the expressions involved in the calculation are long we omit them in this article.

The equivalence classes of toric Fano threefolds (or 3 dimensional Fano polytopes) are classified by Batyrev completely see \cite{1} or \cite{2}: There are 18 equivalence classes

\[ CP^3, B_1, B_2, B_3, B_4 = CP^2 \times CP^1, C_1, C_2, C_3 = CP^1 \times CP^1 \times CP^1, \]
\[ C_4, C_5, D_1, D_2, E_1, E_2, E_3, E_4, F_1, F_2, \]

and for each equivalence class, the vertices of Fano polytope are specified. Here we use the same symbols as in \cite{2} to represent toric Fano threefolds. Hence we can compute the Hilbert series of the toric diagram associated with the canonical bundles of Fano threefolds using the formula in the previous section.

Let \( M \) be a toric Fano threefold and \( W \) the set of fixed point of the action of the Weyl group on the space of all algebraic characters of the maximal torus in \( \text{Aut}(M) \). Then we see that \( \text{dim} \ W = 0, 1, 2 \).

5.1. The case when \( \text{dim} W = 0 \). Let \( M \) be a Fano threefold with \( \text{dim} W = 0 \), that is \( M = CP^3, CP^2 \times CP^1, CP^1 \times CP^1 \times CP^1, C_5, F_1 \). In such case the Futaki invariant vanishes and by the result of Wang and Zhu, \cite{37}, \( M \) admits a Kähler-Einstein metric. Moreover we see the following by calculation.

**Proposition 5.1.** Let \( M \) be a toric Fano threefold with \( \text{dim} W = 0 \). Then

\begin{equation}
\frac{\partial H_M}{\partial a}(0, 0, 0; t) = \frac{\partial H_M}{\partial b}(0, 0, 0; t) = \frac{\partial H_M}{\partial c}(0, 0, 0; t) = 0.
\end{equation}

Here

\[ H_M(a, b, c; t) = C(\{e^{-at}, e^{-bt}, e^{-ct}, e^{-4t}\}, C_M) \]

Therefore we see that the first variation of \( C_i(a, b, c) \) at \( (a, b, c) = (0, 0, 0) \) vanishes for each \( i = -4, -3, -2, \ldots \).

**Example 5.2.** We give the combinatorial data when \( M = CP^2 \times CP^1 \).

- **The vertices of the Fano polytope** \( P_M \):
  \[ (v_1\ v_2\ v_3\ v_4\ v_5) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} \]

- **The vertices of the polar polytope** \( P_M^* \):
  \[ (w_1\ w_2\ w_3\ w_4\ w_5\ w_6) = \begin{pmatrix} 1 & -2 & 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix} \]

- **The edges** \( \{e_{j,1}, e_{j,2}, e_{j,3}\} \) **emanating from** \( w_j \):

\footnote{Maxima is available from \url{http://maxima.sourceforge.net/}}
5.2. The case when $\dim W = 1$. Let $M$ be a toric Fano threefold with $\dim W = 1$. By the classification of toric Fano threefolds, that is $M = B_1, B_2, B_3, C_1, C_4, E_1, E_3, F_2$.

Proposition 5.3. Let $M$ be a toric Fano threefold with $\dim W = 1$. Then

$$
\left( \frac{\partial H_M}{\partial a}(0, 0, 0; t), \frac{\partial H_M}{\partial b}(0, 0, 0; t), \frac{\partial H_M}{\partial c}(0, 0, 0; t) \right) = f(t)p.
$$

Here $p \in \mathbb{R}^3$ is a non-zero constant vector. As a result the first variation of $C_i(a, b, c)$ at $(a, b, c) = (0, 0, 0)$ is a constant multiple of that of $C_{-4}(a, b, c)$ for each $i = -3, -2, \ldots$.

Example 5.4. We give the combinatorial date when $M = E_2$, the blow-up of $\mathbb{C}P^3$ at a point.

- The vertices of the Fano polytope $P_M$:

$$
(v_1 \ v_2 \ v_3 \ v_4 \ v_5) = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}
$$

- The vertices of the polar polytope $P^*_M$:

$$
(w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_6) = \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 3 \\ -1 & -1 & 1 & -1 & 3 & -1 \end{pmatrix}
$$

- The edges $\{e_{j,1}, e_{j,2}, e_{j,3}\}$ emanating from $w_j$:

$$
(e_{1,1} \ e_{1,2} \ e_{1,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (e_{2,1} \ e_{2,2} \ e_{2,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}
$$

$$
(e_{3,1} \ e_{3,2} \ e_{3,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ (e_{4,1} \ e_{4,2} \ e_{4,3}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
(e_{5,1} \ e_{5,2} \ e_{5,3}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \ (e_{6,1} \ e_{6,2} \ e_{6,3}) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}
$$

- The “gradient vector”.

$$
\left( \frac{\partial H_M}{\partial a}(0, 0, 0; t), \frac{\partial H_M}{\partial b}(0, 0, 0; t), \frac{\partial H_M}{\partial c}(0, 0, 0; t) \right) = -\frac{t e^{8t}(e^{4t} + 3)(3e^{4t} + 1)}{(e^{2t} - 1)^5(e^{2t} + 1)^5}(3, 1, 1)
$$

18
5.3. The case when $\dim W = 2$. Let $M$ be a toric Fano threefold with $\dim W = 2$, that is $M = C_2, D_1, D_2, E_2, E_4$. In this case, two different situations arise.

(a) When $M = C_2$, then the “gradient vector” is the same form as Proposition 6.3. Indeed we see that

- The vertices of the Fano polytope $P_M$:

\[
(v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6) = \begin{pmatrix}
0 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
1 & -1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

- The vertices of the polar polytope $P^*_M$:

\[
(w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_6 \ w_7 \ w_8) = \begin{pmatrix}
0 & 0 & 1 & 1 & -2 & -2 & 1 & 1 \\
0 & 1 & -1 & 1 & 0 & 1 & -3 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

- The edges $\{e_{j,1}, e_{j,2}, e_{j,3}\}$ emanating from $w_j$:

\[
\begin{align*}
(e_{1,1} & \ e_{1,2} \ e_{1,3}) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & (e_{2,1} \ e_{2,2} \ e_{2,3}) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(e_{3,1} & \ e_{3,2} \ e_{3,3}) = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & (e_{4,1} \ e_{4,2} \ e_{4,3}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(e_{5,1} & \ e_{5,2} \ e_{5,3}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}, & (e_{6,1} \ e_{6,2} \ e_{6,3}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
(e_{7,1} & \ e_{7,2} \ e_{7,3}) = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, & (e_{8,1} \ e_{8,2} \ e_{8,3}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{align*}
\]

- The “gradient vector”.

\[
\left( \frac{\partial H_M}{\partial a}(0, 0, 0; t), \frac{\partial H_M}{\partial b}(0, 0, 0; t), \frac{\partial H_M}{\partial c}(0, 0, 0; t) \right)
= -\frac{te^{4t}(e^{4t} + 3)(3e^{4t} + 1)}{(e^{2t} - 1)^3(e^{2t} + 1)^3}(1, -2, 3)
\]

(b) When $M = D_1, D_2, E_2, E_4$, the “gradient vector” has components which are linearly independent as functions of $t$-variable. For example, let $M = D_2$. Then we see that

- The vertices of the Fano polytope $P_M$:

\[
(v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6) = \begin{pmatrix}
0 & 1 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 \\
1 & 0 & -1 & 1 & 0 & 0
\end{pmatrix}
\]
The vertices of the polar polytope $P^2_M$:

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -2 & 1 & -2 \\ 1 & -1 & 1 & 1 & -2 & 1 & -2 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The edges $\{e_{j,1}, e_{j,2}, e_{j,3}\}$ emanating from $w_j$:

$$\begin{pmatrix} e_{1,1} & e_{1,2} & e_{1,3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e_{2,1} & e_{2,2} & e_{2,3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e_{4,1} & e_{4,2} & e_{4,3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} e_{5,1} & e_{5,2} & e_{5,3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} e_{6,1} & e_{6,2} & e_{6,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} e_{7,1} & e_{7,2} & e_{7,3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} e_{8,1} & e_{8,2} & e_{8,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The “gradient vector”.

$$\frac{\partial H_M}{\partial a}(0,0,0;t) = -\frac{te^{st}(2e^{st} + 7e^{4t} + 2)}{(e^{4t} - 1)^5(e^{4t} + 1)^5}$$

$$\frac{\partial H_M}{\partial c}(0,0,0;t) = -\frac{te^{st}(4e^{st} + 13e^{4t} + 4)}{(e^{4t} - 1)^5(e^{4t} + 1)^5}$$

The first variation of $C_i(a,b,c)$ at $(a,b,c) = (0,0,0)$ for $i = -4, -2, -1$.

$$\frac{\partial C_{-1}}{\partial a}(0,0,0) = \frac{\partial C_{-4}}{\partial b}(0,0,0) = -\frac{11}{1024}, \quad \frac{\partial C_{-4}}{\partial c}(0,0,0) = -\frac{21}{1024}$$

$$\frac{\partial C_{-2}}{\partial a}(0,0,0) = \frac{\partial C_{-2}}{\partial b}(0,0,0) = -\frac{13}{768}, \quad \frac{\partial C_{-2}}{\partial c}(0,0,0) = -\frac{9}{256}$$

$$\frac{\partial C_{-1}}{\partial a}(0,0,0) = \frac{\partial C_{-1}}{\partial b}(0,0,0) = -\frac{1}{192}, \quad \frac{\partial C_{-1}}{\partial c}(0,0,0) = -\frac{1}{64}$$

6. **Direct Computations of $F_{T^d}$**

In this section, we shall check the results of the previous section by using the localization formula as in [19]. As we saw in the previous section, $\{F_{T^d}\}_{p=1,2,3}$ on toric Fano threefolds span at most dimension 2. We first show that this is generally true for any toric Fano manifolds.

**Theorem 6.1.** There is a universal linear dependence relation among $\{F_{T^d}\}_{p=1,2,3}$ on any anticanonically polarized toric Fano threefold $(M, K_M^{-1})$.

---

5Note here that $2C_{-4}(a,b,c) = C_{-3}(a,b,c)$ holds. This is because $C_{-3}$ corresponds to the total scalar curvature of the Sasaki manifold whose Reeb vector field is generated by $(a,b,c)$. Therefore $C_{-3}$ is a constant multiple of $C_{-4}$. See [20].
Proof. We shall replace \( \{ F_{T_d^p} \}_{p=1,2,3} \) by the invariants \( \{ G_{T_d^p} \}_{p=1,2,3} \) with respect to the lifted action in Proposition 3.1. Namely \( G_{T_d^p} \) is defined by the right hand side of (28) with the normalization (15). Let \( M \) be a Fano threefold and \( \omega \) be a Kähler form in \( c_1(M) \). Let \( \eta \in c_1(M) \) be another Kähler form whose Ricci form \( \rho_\eta \) equals to \( \omega \). As in Proposition 3.1 for \( X \in h_0(M) \) let \( u_X \) be the Hamiltonian function satisfying the normalization (15). Recall that \( u_X \) satisfies
\[
\Delta_F u_X = \Delta_\eta u_X = u_X,
\]
where \( \Delta_\eta \) is the complex Laplacian with respect to \( \eta \). Remark that the sign of \( \Delta_\eta \) is consistent with \( \Delta_F \) and \( \Delta^h \). Then we have
\[
12 G_{T_d^2}(X) = 2 \int_M (c_1^2 + c_2)(\Theta_\eta)(\Delta_\eta u_X) + (\Delta_\eta u_X) \rho_\eta + \int_M (c_1^2 + c_2)(L_\eta(X) + \Theta_\eta) \wedge \rho_\eta^2
\]
\[
= 2 \int_M (c_1^2 + c_2)(\Theta_\eta) + c_1(L_\eta(X)) c_1(\Theta_\eta)
\]
\[
+ \int_M (c_1^2 + c_2)(L_\eta(X) + \Theta_\eta) \wedge c_1^2(\Theta_\eta)
\]
\[
= \int_M (c_1^4 + c_1^2 c_2)(L_\eta(X) + \Theta_\eta)
\]
(26)
\[
= 4 F_{T_d^1}(X) + \int_M (c_1^2 c_2)(L_\eta(X) + \Theta_\eta).
\]
Also we have
\[
24 G_{T_d^3}(X) = \int_M (c_1 c_2)(\Theta_\eta)(\Delta_\eta u_X) + \int_M (c_1 c_2)(L_\eta(X) + \Theta_\eta) \wedge \rho_\eta
\]
\[
= \int_M (c_1 c_2)(\Theta_\eta) c_1(L_\eta(X)) + \int_M (c_1 c_2)(L_\eta(X) + \Theta_\eta) \wedge c_1(\Theta_\eta)
\]
(27)
\[
= \int_M (c_1^3 c_2)(L_\eta(X) + \Theta_\eta).
\]
Since \( G_{T_d^p}(X) - F_{T_d^p}(X) \) equals to a multiple of \( F_{T_d^1}(X) \) for all \( p \) and \( X \) as pointed out in the proof of Theorem 3.2 the linear span of \( \{ F_{T_d^p} \}_{p=1,2,3} \) equals to the one of \( \{ G_{T_d^p} \}_{p=1,2,3} \). From (26) and (27), \( \{ F_{T_d^p} \}_{p=1,2,3} \) satisfies at least one linear dependence relation. \( \Box \)

This shows that the dimension of the span of \( \{ F_{T_d^p} \}_{p=1,2,3} \) is not more than two for any Fano manifold with the polarization \( L = K_M^{-1} \). To determine the dimension of the span of \( \{ F_{T_d^p} \}_{p=1,2,3} \), it is therefore sufficient to investigate the linear independence between \( \int_M (c_1^2 c_2)(L_\eta(X) + \Theta_\eta) \) and \( \int_M (c_1^3)(L_\eta(X) + \Theta_\eta) \). Since both of them are kind of the integral invariants in [19], we can apply the localization formula for them. More precisely, if \( X \) only has isolated zeroes, then
\[
\int_M (c_1^3)(L(X) + \Theta) = \sum_i \frac{(\text{tr}(L(X_{p_i})))^4}{\det L(X_{p_i})},
\]
(28)
\[
\int_M (c_1^2 c_2)(L(X) + \Theta) = \sum_i \frac{(\text{tr}(L(X_{p_i})))^2 \cdot c_2(L(X_{p_i}))}{\det L(X_{p_i})},
\]
(29)
where \( \text{Zero}(X) = \{ p_i \} \subset M \). As for the localization formula, also see [16].

Now we are in position to do calculations on examples. Firstly, let us compute \( D_2 \). We already saw in 5.3, (b) that for \( D_2 \), \( \{ F_{T_d^p} \}_{p=1,2,3} \) span a two dimensional
vector space. \( D_2 \) is described as the blow up of \( \mathbb{CP}^2 \times \mathbb{CP}^1 \) along \( \mathbb{CP}^1 \times \{ \text{a point} \} \). Let \([Z_0 : Z_1 : Z_2]\) be the homogeneous coordinate on \( \mathbb{CP}^2 \) and \([X_0 : X_1]\) be the homogeneous coordinate on \( \mathbb{CP}^1 \). Then let us consider the blow up of \( \mathbb{CP}^2 \times \mathbb{CP}^1 \) along \( \{(0 : Z_1 : Z_2), [1 : 0]\} \mid [Z_1 : Z_2] \in \mathbb{CP}^1 \}. We denote the blow up by \( \pi : D_2 \rightarrow \mathbb{CP}^2 \times \mathbb{CP}^1 \).

Let \( \sigma(t, \alpha, \beta, \gamma) \) be a flow on \( \mathbb{CP}^2 \times \mathbb{CP}^1 = \{(Z_0 : Z_1 : Z_2), [X_0 : X_1]\} \) defined by

\[
\begin{pmatrix}
  e^{\alpha t} & 0 & 0 & 0 \\
  0 & e^{\beta t} & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & e^{t \gamma}
\end{pmatrix}
\]

where \( t \) is a time parameter and \( \alpha, \beta, \gamma \in \mathbb{R} \). Remark that \( \sigma \) transforms \( \{(0 : Z_1 : Z_2), [1 : 0]\} \mid [Z_1 : Z_2] \in \mathbb{CP}^1 \} \) into itself. So \( \{\sigma\} \) can be lifted as a flow on \( D_2 \). We denote it by the same \( \sigma \). Remark that for generic \( \alpha, \beta \) and \( \gamma \) the set of all fixed points under the flow consists of the following eight isolated points.

\[
\begin{align*}
p_1 &= \pi^{-1}(([1 : 0 : 0], [1 : 0])), \\
p_2 &= \pi^{-1}(([1 : 0 : 0], [0 : 1])), \\
p_3 &= \pi^{-1}(([0 : 1 : 0], [0 : 1])), \\
p_4 &= \pi^{-1}(([0 : 0 : 1], [0 : 1])) \\
p_5 &= (([0 : 1 : 0], [1 : 0]), 0), \\
p_6 &= (([0 : 1 : 0], [1 : 0]), \infty), \\
p_7 &= (([0 : 0 : 1], [1 : 0]), 0), \\
p_8 &= (([0 : 0 : 1], [1 : 0]), \infty). \\
\end{align*}
\]

In above, \( D_2 \) is regarded as the submanifold in \( D_2 \times \mathbb{CP}^1 = D_2 \times (\mathbb{C} \cup \{\infty\}) \) with codimension one. Let \( X \) be the holomorphic vector field on \( M \) associated with \( \sigma \). As for \( p_1 \) and \( p_2 \), we have

\[
L(X) = \text{diag}(\beta - \alpha, -\alpha, \mp \gamma).
\]

As for \( p_3 \), we have

\[
L(X) = \text{diag}(\alpha - \beta, -\beta, \gamma).
\]

As for \( p_4 \), we have

\[
L(X) = \text{diag}(\alpha, \beta, \gamma).
\]

As for \( p_5 \), we have

\[
L(X) = \text{diag}(\alpha - \beta, -\alpha + \beta - \gamma).
\]

As for \( p_6 \), we have

\[
L(X) = \text{diag}(\alpha - \beta, -\gamma, \alpha - \beta + \gamma).
\]

As for \( p_7 \), we have

\[
L(X) = \text{diag}(\alpha, \beta, -\alpha - \gamma).
\]

As for \( p_8 \), we have

\[
L(X) = \text{diag}(\beta, -\gamma, \alpha + \gamma).
\]

From \([23]\) we find

\[
\int_M (c_1^X)(L(X) + \Theta) = -22\alpha + 11\beta + 21\gamma.
\]
Also from (29) we find
\[ \int_M (c_1^2 c_2)(L(X) + \Theta) = 2(-2\alpha + \beta + 3\gamma). \]
Since the one is not proportional to the other, their span is two dimensional.

Next we shall compute $C_2$. As we saw in 5.3, (a) that $C_2$ is an example for which the invariants do span only a one dimensional space although the set of fixed point of the action of the Weyl group on the space of all algebraic characters of the maximal torus in $\text{Aut}(M)$ is two dimensional. There exists only single example among toric Fano threefolds, denoted by $C_2$ in the list of Batyrev, which is $\mathbb{P}_S(O \oplus O(l))$ where $S_1$ is the blow up of $\mathbb{CP}^2$ at a point and $l^2 = 1$ on $S_1$, i.e., the curve $l$ has the self-intersection +1. Let $[Z_0 : Z_1 : Z_2]$ be the homogeneous coordinate on $\mathbb{CP}^2$ and $S_1$ be the blow up of $\mathbb{CP}^2$ at [1 : 0 : 0]. Then, $S_1$ can be regarded as the submanifold of $\mathbb{CP}^2 \times \mathbb{CP}^1$ defined as (30). Since
\[
\begin{pmatrix}
e^{\alpha t} & 0 & 0 \\
0 & e^{\beta t} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]fixes $[1 : 0 : 0]$ in $\mathbb{CP}^2$, so $\sigma$ induces a flow on $C_2$. We denote it by the same $\sigma$ as the previous case. For generic $\alpha, \beta$ and $\gamma$ the set of all fixed points under the action of $\sigma$ consists of the following eight isolated points.

\begin{align*}
p_1 &:= ([1 : 0 : 0], [1 : 0], [1 : 0]), & p_2 &:= ([1 : 0 : 0], [1 : 0], [0 : 1]), \\
p_3 &:= ([1 : 0 : 0], [0 : 1], [1 : 0]), & p_4 &:= ([0 : 1 : 0], [0 : 1], [0 : 1]), \\
p_5 &:= ([0 : 1 : 0], [1 : 0], [1 : 0]), & p_6 &:= ([0 : 1 : 0], [1 : 0], [0 : 1]), \\
p_7 &:= ([0 : 0 : 1], [0 : 1], [1 : 0]), & p_8 &:= ([0 : 0 : 1], [0 : 1], [0 : 1]).
\end{align*}

Let $X$ be the holomorphic vector field on $M$ associated with $\sigma$. As for $p_1$ and $p_2$, we have

\[ L(X) = \text{diag}(\beta - \alpha, -\beta, \mp \gamma). \]

As for $p_3$ and $p_4$, we have

\[ L(X) = \text{diag}(-\alpha, \beta, \mp \gamma). \]

As for $p_5$ and $p_6$, we have

\[ L(X) = \text{diag}(\alpha - \beta, -\beta, \pm(\alpha - \beta - \gamma)). \]

As for $p_7$ and $p_8$, we have

\[ L(X) = \text{diag}(\alpha, \beta, \pm(\alpha - \gamma)). \]

Then we have
\[
\int_M (c_1^2)(L(X) + \Theta) = 4 \int_M (c_1^2 c_2)(L(X) + \Theta) = -16(-4\alpha + 2\beta + 3\gamma).
\]
The above equality implies our desired conclusion.
7. Extension to general Sasaki manifolds

In this section we remark that the invariants defined by (3) extend to compact Sasaki manifolds. As was explained in section 3 the Reeb vector field $\xi$ on a Sasaki manifold $S$ is defined as $J\frac{\partial}{\partial r}$. Let $\mathcal{F}_\xi$ be the Reeb foliation on $S$ generated by $\xi$. It is convenient to extend $\xi$ to a vector field $\tilde{\xi} = J(r\frac{\partial}{\partial r})$ on $C(S)$. It is well known that $\tilde{\xi} - i\tilde{J}\tilde{\xi} = \tilde{\xi} + i\frac{\partial}{\partial r}$ is a holomorphic vector field on $C(S)$, and thus there is an action on $C(S)$ of the holomorphic flow generated by $\tilde{\xi} - i\tilde{J}\tilde{\xi}$. The collection of local orbit spaces of this action defines a transversely holomorphic structure on $S$ of $\mathcal{F}_\xi$.

We also put $\partial$ on $S$ is biholomorphic. We then have $\partial$ and $\bar{\partial}$ operators on each $V_\alpha$. They define well-defined operators, denoted by $\partial_B$ and $\bar{\partial}_B$, on the basic forms on $S$. Here a differential form $\psi$ on $S$ is said to be basic if

$$i(\xi)\psi = 0 \text{ and } \mathcal{L}_\xi \psi = 0.$$ 

We also put $d_B^\psi = \frac{i}{2}(\bar{\partial}_B - \partial_B)$. Let $G$ be a complex Lie group. We say that a principal $G$-bundle $P$ over $S$ is transversely holomorphic if the transition function from $P|_{U_\beta}$ to $P|_{U_\alpha}$ on the overlap $U_\alpha \cap U_\beta$ is a holomorphic $G$-valued function on $\pi_\beta(U_\alpha \cap U_\beta)$ for any $\alpha$ and $\beta$. A connection on $P$ is said to be a type $(1, 0)$ connection if the connection form on $P|_{U_\alpha}$ consists of type $(1, 0)$ components on $V_\alpha$ and $G$. For a type $(1, 0)$ connection on $P$ let $\Theta$ be its curvature 2-form. Then $\Theta$ does not have type $(0, 2)$ components.

A typical such principal bundle is the frame bundle of the normal bundle $\nu(\mathcal{F}_\xi)$ of the Reeb foliation $\mathcal{F}_\xi$ with $G = GL(m, \mathbb{C})$. The Levi-Civita connections given by the transverse Kähler metric on local orbit spaces naturally define a global connection on $\nu(\mathcal{F}_\xi)$. This is a typical example of type $(1, 0)$ connection.

Let $\eta_0$ be the dual 1-form of $\xi$ on $S$. Then $\eta_0$ is a contact 1-form and $\frac{i}{2}d\eta_0$ gives a transverse Kähler form. Any other Sasaki structure compatible with the Reeb vector field $\xi$ is given by the deformation of $\eta_0$ into $\eta = \eta_0 + 2d_B^\psi \phi$ for a basic function on $S$. This transformation induces the usual Kähler deformation in the transverse direction since it deforms $\frac{i}{2}d\eta_0$ into

$$\frac{i}{2}d(\eta_0 + 2d_B^\psi \phi) = \frac{i}{2}d\eta_0 + d_Bd_B^\psi \phi = \frac{i}{2}d\eta_0 + i\partial_B\bar{\partial}_B \phi.$$

Let $E$ be the set of all such contact forms $\eta = \eta_0 + 2d_B^\psi \phi$.

We pick an $\eta \in E$ and fix it for the moment. Let $h_0$ be the Lie algebra of all holomorphic vector fields on $C(S)$ commuting with $\tilde{\xi} - i\tilde{J}\tilde{\xi}$. Then a vector field in $h_0$ defines naturally a vector field on $S$. By the abuse of notation we also denoted by $h_0$ the Lie algebra of all such vector fields on $S$. For such a vector field $X \in h_0$ we put

$$u_X = \eta(X) = \int_S \eta(X) \eta \wedge (d\eta)^m / \int_S \eta \wedge (d\eta)^m.$$
Let $P^i(G)$ denote the set of all $G$-invariant polynomials of degree $p$ on $g$. For any $\phi \in P^i(G)$ we define $\mathcal{F}_\phi : g \to \mathbb{C}$ by

\[
\mathcal{F}_\phi(X) = (m - p + 1) \int_S \phi(\Theta) \wedge u_X (d\eta)^{m-p} \wedge \eta \\
+ \int_S \phi(\theta(X) + \Theta) \wedge (d\eta)^{m-p+1} \wedge \eta.
\]

(31)

Then one can prove the following theorem just as in [17] using Lemma 9.1 and Lemma 9.2 in [21].

**Theorem 7.1.** $\mathcal{F}_\phi(X)$ is independent of the choices of $\eta \in \mathcal{E}$ and type $(1,0)$ connection $\theta$.

Let $L \to M$ be an ample line bundle. Then the total space $S$ of the associated $U(1)$-bundle is a Sasaki manifold. If there is a torus action of the Sasaki structure we can deform the Sasaki structure by deforming the Reeb vector field, and we can consider $\mathcal{F}_\phi$ for irregular Sasaki manifolds.

Let $M$ be a toric Fano manifold and take $L$ to be $K^{-1}_M$. We can consider the integral invariants $\mathcal{F}_\phi$ for irregular Sasaki structures obtained by deforming the Reeb vector field. But it is not clear how the integral invariants $\mathcal{F}_{Td^i}$ and the Hilbert series are related when the Reeb vector field is irregular.

The following example is intriguing because it provides an example of a Sasaki manifold for which $\mathcal{F}_{Td^i}$ vanishes but the first variation of $C_i(b)$ at the volume minimizing Reeb vector field does not vanish for some $i$. Let $M = \mathbb{CP}^2 \# \mathbb{CP}^2$. Then the total space of the associated $U(1)$-bundle has a Reeb field obtained by the volume minimization, and thus there is a Sasaki-Einstein metric. For this Reeb vector field $\mathcal{F}_{Td^i}$ must vanish because $\mathcal{F}_{Td^i}$ is a multiple of the Sasaki-Futaki invariant. But the computation using a computer shows the first variation of $C_{-1}(b)$ at this Reeb vector field does not vanish. We give the combinatorial data and some calculations with respect to our $M$ below.

- The vertices of the Fano polytope $P_M$:

  \[
  (v_1 \ v_2 \ v_3 \ v_4) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}
  \]

- The vertices of the polar polytope $P^o_M$:

  \[
  (w_1 \ w_2 \ w_3 \ w_4) = \begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}
  \]

- The edges $\{e_{j,1}, e_{j,2}\}$ emanating from $w_j$:

  \[
  (e_{1,1} \ e_{1,2}) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (e_{2,1} \ e_{2,2}) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}
  \]

  \[
  (e_{3,1} \ e_{3,2}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (e_{4,1} \ e_{4,2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
  \]

- $C_i(a, b)$ ($i = -3, -2, -1$):
\[ C_{-3}(a, b) = \frac{2(2b - a + 12)}{(b + 3)(b - 2a - 3)(b - a + 3)(b + a - 3)}, \quad C_{-2}(a, b) = \frac{3}{2}C_{-3}(a, b), \]
\[ C_{-1}(a, b) = \frac{-6b^2 + 2a^2b - 6ab - 18b - a^3 + 9a^2 + 9a - 162}{6(b + 3)(b - 2a - 3)(b - a + 3)(b + a - 3)} \]

- The volume minimizing \((a_0, b_0)\): We call \((a_0, b_0)\) volume minimizing if the gradient of \(C_{-3}(a, b)\) vanishes at \((a_0, b_0)\) and \((a_0, b_0)\) is in the interior of \(3P^0_M\). In this case, we see that \((a_0, b_0) = (0, \sqrt{13} - 4)\). On the other hand,
  \[ \frac{\partial C_{-1}}{\partial a}(a_0, b_0) = \frac{4(137\sqrt{13} - 491)}{(\sqrt{13} - 7)^4(\sqrt{13} - 1)^4}, \quad \frac{\partial C_{-1}}{\partial b}(a_0, b_0) = \frac{32(157\sqrt{13} - 568)}{(\sqrt{13} - 7)^4(\sqrt{13} - 1)^4} \]
which is non-zero and hence the variation does not vanish.

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