One loop light-cone QCD, effective action for reggeized gluons and QCD RFT calculus

S. Bondarenko(1), L. Lipatov(2), S. Pozdnyakov(1), A. Prygarin(1)

(1) Physics Department, Ariel University, Ariel 40700, Israel
(2) St. Petersburg State University, St. Petersburg 199034 and Petersburg Nuclear Physics Institute, Gatchina 188300, Russia

Abstract

The effective action for reggeized gluons is based on the gluodynamic Yang-Mills Lagrangian with external current for longitudinal gluons added, see [1]. On the base of classical solutions, obtained in [2], the one-loop corrections to this effective action in light-cone gauge are calculated. The RFT calculus for reggeized gluons similarly to the RFT introduced in [3] is proposed and discussed. The correctness of the results is verified by calculation of the propagator of $A^+$ and $A^-$ reggeized gluon fields and application of the obtained results is discussed as well.

1 Introduction

The action for interaction of reggeized gluons was introduced in the series of papers [1] and describes multi-Regge processes at high-energies, see [4]. There are the following important applications of this action: it can be used for the calculation of production amplitudes in different scattering processes and calculation of sub-leading, unitarizing corrections to the amplitudes and production vertices, see [1][4][7]. The last task can be considered as a construction of the RFT (Regge Field Theory) based on the interaction of the fields of reggeized gluons, where different vertices of the interactions are introduced and calculated. The phenomenological RFT based on the Pomeron degrees of freedom was introduced in [3]. From this point of view we consider the effective action for reggeized gluons as RFT calculus based on the degrees of freedom expressed through colored reggeon fields (reggeized gluons).

The construction of RFT based on QCD Lagrangian requires the knowledge of solutions of classical equations of motion in terms of reggeon fields $A^+$ and $A^-$. Inserting these solutions again in the Lagrangian we can develop field theory fully in terms of reggeon fields, with loops corrections to the action determined in terms of these fields as well. Subsequent expansion of the action in terms of the reggeon fields will produce all possible vertices of interactions of the fields, with precision determined by the precision of calculations in the framework of the QFT. From the QFT point of view, therefore, the problem of interest is the calculation of the one-loop effective action for gluon QCD Lagrangian with added external current by use of the the non-trivial classical solutions expressed in terms of new degrees of freedom, see [2]. These calculations of the one-loop corrections to the effective action we perform in light-cone gauge using classical solutions from [2]. The correctness of the obtained results can be checked by calculations of functions which are well-known in small-x BFKL approach, [8]. The basic function there is the gluon Regge trajectory, which determines the form of the propagator of reggeized gluon fields $A^+$ and $A^-$. This propagator in the proposed framework can be considered as operator inverse to the effective vertex of interaction of reggeon fields, this check is performed in the paper. There are also other possibilities to verify the self-consistency of the approach. For example,
it can be calculation of the BFKL kernel, which is an effective vertex of interactions of four reggeons, or calculation of the triple Pomeron vertex, see [9], which is interaction vertex of six reggeon fields. These calculations will be considered further in separate publications.

Thus, below, we calculate one-loop effective action for reggeized gluons and calculate propagator for \( A^+ \) and \( A^- \) reggeon fields. Respectively, in the Section 2, we remind main results obtained for the classical solution of the effective action for reggeized gluons. In Section 3 we consider the expansion of the Lagrangin in terms of fluctuations around these classical solutions, whereas in Section 4 we calculate the one-loop correction to the classical action obtaining effective one loop action in terms of reggeon fields. In Section 5 we discuss RFT calculus based on the effective action and in Section 6 we verify the correctness of the result calculating the propagator of the reggeized gluons fields. Section 7 is the conclusion of the paper, there are also Appendixes A, B, C where the main calculations related to the result are presented.

2 Effective action for reggeized gluons and classical equations of motion

The effective action, see [1], is a non-linear gauge invariant action which correctly reproduces the production of the particles in direct channels at a quasi-multi-Regge kinematics. It is written for the local in rapidity interactions of physical gluons in direct channels inside of some rapidity interval \((y - \eta/2, y + \eta/2)\). The interaction between the different clusters of gluons at different though very close rapidities can be described with the help of reggeized gluon fields \( A^- \) and \( A^+ \) interacting in crossing channels. Those interaction are non-local in rapidity space. This non-local term is not included in the action, the term of interaction between the reggeon fields in the action is local in rapidity and can be considered a kind of renormalization term in the Lagrangian. The action is gauge invariant and written in the covariant form in terms of gluon field \( v \) as

\[
S_{\text{eff}} = -\int d^4x \left( \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a + tr \left[ v_+ J^+(v_+) - A_+ j^+_\text{reg} + v_- J^-(v_-) - A_- j^-\text{reg} \right] \right),
\]

where

\[
J^\pm(v_\pm) = O(x_\pm, v_\pm) j^\pm_{\text{reg}},
\]

with \(O(x_\pm, v_\pm)\) as some operators, see [1], Appendix A and

\[
j^\pm_{\text{reg}} = \frac{1}{C(R)} \partial^2_\perp A^\pm_a,
\]

is a reggeon current, where \(C(R)\) is the eigenvalue of Casimir operator in the representation \(R\) with \(C(R) = N\) in the case of adjoint representation used in the paper. Further in the calculations we will use the form of the reggeon current Eq. (3) borrowed from the CGC (Color Glass Condensate) approach, see [10][12], where this current is written in terms of some color density function defined as

\[
\partial_\perp \partial^- \rho_a = -\frac{1}{N} \partial^2_\perp A^+_a,
\]

or

\[
\rho_a = \frac{1}{N} \partial^{-1} (\partial^i A^a_i),
\]

see [2] for details. There are additional kinematical constraints for the reggeon fields

\[
\partial^- A_+ = \partial^+ A_- = 0,
\]

\(1\)We use the Kogut-Soper convention for the light-cone for the light-cone definitions with \(x_\pm = (x_0 \pm x_3) / \sqrt{2}\) and \(x_\pm = x^\mp\).
corresponding to the strong-ordering of the Sudakov components in the multi-Regge kinematics, see [1]. Everywhere, as usual, \( \partial_i \) denotes the derivative on transverse coordinates. Under variation on the gluon fields these currents reproduce the Lipatov’s induced currents

\[
\delta \left( v_\pm J^\pm(v_\pm) \right) = (\delta v_\pm) J^{\text{ind}}_\pm(v_\pm) = (\delta v_\pm) j^\pm(v_\pm),
\]

with shortness notation \( j^{\text{ind}}_\pm = j^\pm \) introduced. This current possesses a covariant conservation property:

\[
\left( D_\pm j^{\text{ind}}_\pm(v_\pm) \right)^a = \left( D_\pm j^\pm(v_\pm) \right)^a = 0. \tag{8}
\]

Here and further we denote the induced current in the component form in the adjoint representation as

\[
j^{\text{adj}}_\pm \equiv j^{\text{adj}}_\pm(v_\pm) = v_\pm \delta(v_\pm), \tag{9}
\]

while the integration on fluctuations around the classical solutions provides loop corrections to the "net" contribution which is based on the classical solutions only.

3 Expansion of the Lagrangian around the classical solution

In this section we consider the first step in construction of the effective action of the approach, expanding the Lagrangian Eq. (1) in terms of the fluctuations and classical fields. Inserting Eq. (11) in the Lagrangian the only corrections to \( g^2 \) and \( \epsilon^2 \) orders will be preserved. This precision provides a one-loop correction to the "net" effective action, contributions from higher order loops will be considered in a separate publication.

The Lagrangian in light-cone gauge has the following form:

\[
L = -\frac{1}{4} F_{ij}^{\alpha} F_{ij}^{\alpha} + F_{i+}^{\alpha} F_{i-}^{\alpha} + \frac{1}{2} F_{+-}^{\alpha} F_{+-}^{\alpha}, \tag{12}
\]

where \( F_{+-}^{\alpha} F_{+-}^{\alpha} \) term does not consists transverse fluctuations and \( F_{ij}^{\alpha} F_{ij}^{\alpha} \) term does not consist longitudinal fluctuations.

3.1 The \( F_{i+}^{\alpha} F_{i-}^{\alpha} \) term

Inserting expression Eq. (11) in this term we obtain:

\[
F_{i+}^{\alpha} \to F_{i+}^{\alpha}(v_i^{cl}, v_i^{cl}) + \left( D_+(v_i^{cl}) \varepsilon_+ \right)^a - \left( D_+(v_i^{cl}) \varepsilon_+ \right)^a + g f_{abc} \varepsilon_i^{a} \varepsilon_i^{b} \varepsilon_i^{c}, \tag{13}
\]

and

\[
F_{i-}^{\alpha} \to F_{i-}^{\alpha}(v_i^{cl}) - \partial_- \varepsilon_i^{a}. \tag{14}
\]

\[\text{We use } (T_a)_{b c} = -i f_{a b c} \text{ definition of the matrices and write only ”external” indexes of the } f_{a b c} = (f_a)_{b c} \text{ matrix in the trace.} \]
Therefore, accounting contributions which are only quadratic to the fluctuations, we obtain:

\[ F_{i+}^a F_{-i}^a = (F_{i+}^a F_{-i}^a)_{cl} + F_{i-}^a (v_i^{cl}) \left( \left( D_i (v_i^{cl}) \xi_i^a \right)^a - (D_+ (v_+^{cl}) \xi_+^a) + g f_{abc} \left( \partial_- v_i^{cl} \right) \xi_+^b \xi_+^c \right) - \left( \partial_- \xi_i^a \right) F_{i+}^a (v_i^{cl}) - (D_+ (v_+^{cl}) \xi_+^a) - g f_{abc} \left( \partial_- \xi_i^a \right) \xi_+^b \xi_+^c. \]  

In this expression we do not account term which is cubic in respect to fluctuations, the linear to fluctuations terms are canceled because of equations of motion. The terms quadratic to transverse fluctuations contribute to the corresponding propagator in the Lagrangian and term which is quadratic in respect to combination of transverse and longitudinal fluctuations we write as

\[ - g f_{abc} \left( \partial_- v_i^{cl} \right) \xi_+^b \xi_+^c - (D_+ (v_+^{cl}) \xi_+^a) = J_i^a \xi_+^i. \]  

Here the current

\[ J_i^a = \left( \xi_+^i \partial_- \partial_i + g f_{abc} \xi_+^e \left( \partial_- v_i^{cl} - v_i^{cl} \partial_- \right) \right) \]  

is some effective current in the Lagrangian.

### 3.2 The \( F_{ij}^a F_{ij}^a \) term

We have for this term:

\[ F_{ij}^a = F_{ij}^a (v_i^{cl}) + (D_j \xi_j)^a - (D_j \xi_i)^a + g f_{abc} \xi_+^b \xi_+^c. \]  

Therefore, accounting contributions which are only quadratic to the fluctuations, we obtain:

\[ - \frac{1}{4} F_{ij}^a F_{ij}^a = - \frac{1}{4} (F_{ij}^a F_{ij}^a)_{cl} - \frac{1}{2} F_{ij}^a (v_i^{cl}) (D_j \xi_j)^a - (D_j \xi_i)^a - \frac{g}{2} f_{abc} (v_i^{cl}) f_{abc} \xi_+^b \xi_+^c - \frac{1}{2} (D_i \xi_j)^a (D_i \xi_j)^a + \frac{1}{2} (D_i \xi_j)^a (D_j \xi_i)^a, \]  

where as usual linear to fluctuations terms do not contribute to the effective action.

### 3.3 The \( F_{i+}^a F_{i+}^a \) term

This term consists contributions from only longitudinal fluctuations. We have here:

\[ \frac{1}{2} F_{i+}^a F_{i+}^a \rightarrow \frac{1}{2} \left( \partial_- v_+^a \right) \left( \partial_- v_+^a \right) = - \frac{1}{2} v_+^{a cl} \left( \partial_- v_+^{a cl} \right) - \left( \partial_- v_+^{a cl} \right) \xi_+^a - \frac{1}{2} \xi_+^a \left( \partial_-^2 \xi_+^a \right). \]  

The linear term in Eq. (20) is canceled due the equation of motion, therefore the only first and third terms are considered further.

### 3.4 The current term

For the effective current term, taking into account that the linear to fluctuations term is canceling due the equations of motion, we obtain with requested precision the following expansion in terms of longitudinal fluctuations:

\[ v_+^a J_a^+(v_+) = v_+^{a cl} J_{a cl}^+(v_i^{cl}) + \frac{1}{2} \left( \delta^2 \left( v_+^a J_a^+ \right) \right)_{v_+=v_+^{cl}} \xi_+^b \xi_+^c + y. \]  

The same current’s term we can write as

\[ v_+^a J_a^+(v_+) = - v_+^{a cl} O^{ab} (v_i^{cl}) \left( \partial_i \partial_- \rho_+^b \right) - \frac{1}{2} \left( \delta U^{ab}(v_+) \right)_{v_+=v_+^{cl}} \left( \partial_i \partial_- \rho_+^a \right) \xi_+^b \xi_+^c + y. \]
In order to calculate this expression we have to know the expansion of the following function:

\[ U^{ab}(v_+) = \text{tr} \left[ f_a O(v_+) f_b O^T(v_+) \right] \]  

(23)

with respect to the Eq. (11) fluctuation

\[ v_+^a \rightarrow v_+^{a, \text{cl}} + \varepsilon_+^a. \]  

(24)

Using Appendix A formulas we have:

\[ U_x^{ab}(v_+) = U_x^{ab}(v_+^0) + g \left( U_1^{ab} \right)_{xy} c + \frac{1}{2} g^2 \left( U_2^{ab} \right)_{xyz} c + \varepsilon_+^a + \varepsilon_+^a + \ldots, \]  

(25)

where the integration on repeating \( y, z \) indices is assumed. The coefficients of the expansion read as

\[ \left( U_1^{ab} \right)_{xy} = \text{tr} \left[ f_a G_{xy}^+ f_c O_y f_b O_x^T \right] + \text{tr} \left[ f_c G_{xy}^+ f_a O_x f_b O_y^T \right] \]  

(26)

and

\[ \left( U_2^{ab} \right)_{xyz} = \text{tr} \left[ f_a G_{xy}^+ f_c G_{xy}^+ f_d O_x f_b O_y^T \right] + \text{tr} \left[ f_a G_{xy}^+ f_c G_{xy}^+ f_d O_x f_b O_y^T \right] + \text{tr} \left[ f_c G_{xy}^+ f_a G_{xy}^+ f_d O_x f_b O_y^T \right] + \text{tr} \left[ f_c G_{xy}^+ f_a G_{xy}^+ f_d O_x f_b O_y^T \right] + \text{tr} \left[ f_c G_{xy}^+ f_a G_{xy}^+ f_d O_x f_b O_y^T \right] + \text{tr} \left[ f_c G_{xy}^+ f_a G_{xy}^+ f_d O_x f_b O_y^T \right], \]  

(27)

see \[2\] and Appendix A for details. Therefore we obtain for the Eq. (21) expression:

\[ v_+^a J^+_a = - v_+^{a, \text{cl}} O^{ab}(v_+^0) \left( \partial_+ \partial_- \rho^a \right) - \frac{1}{2} g \varepsilon_+^a \left( U_1^{ab} \right)_{xy} \left( \partial_+ \partial_- \rho^a \right)_{xy} \varepsilon_+^a. \]  

(28)

### 4 One loop effective action: integration over fluctuations

The computation of the one loop correction to the effective action in light-cone gauge we perform using non-canonical method, integrating out subsequently transverse and longitudinal fluctuations. The reason for use of this non-canonical method of calculation is simple. The Lagrangian Eq. (1) consist new term in comparison to the usual gluon QCD Lagrangian. Consequently, instead canonical equation of motion which relates transverse and longitudinal fields, we have the following equation:

\[ - \left( D_i \left( \partial_- v^i \right) \right) - \partial_+^2 v_{a+} = j^+_a(v_+) \]  

(29)

see \[2\]. This equation is different from the "canonical" one and will lead to the different constraint in the canonical quantization method, see \[13][14\]. Still, it is possible to make a usual substitution in the Lagrangian which relates these fields, see for example \[14\], and define the canonical light-cone Lagrangian in the usual form in the limit \( g \rightarrow 0 \). But that will case the shift in the argument of the effective current term, especially in light of condition Eq. (10), and in turn that will lead to some complicated expression of the induced current in the equations of motion to \( g^2 \) and higher orders of perturbative theory. Therefore, we prefer to use the non-canonical method of introducing of bare propagators in the theory, calculating the final one-loop expressions in terms of these propagators, see Appendix B. As we see there, after the resummation of one-loop terms, the well known light-cone propagators, see for example \[11][15\], are arising in the expressions. So far it is not clear, is it result of the chosen precision of the calculations or that is a future of the effective Lagrangian Eq. (1), we will investigate this question in the separate publication.
4.1 Integration on transverse fluctuations

Collecting quadratic on transverse fluctuations terms and effective current term we obtain:

\[- \frac{1}{2} \varepsilon_i^a \left( \delta_{ij} \mathcal{D} \mathcal{D} + \partial_i \partial_j \right) - 2 g f_{abc} \left( \delta_{ij} \left( v^{bcl}_k \partial_k - v^{bcl}_+ \partial_- \right) - \frac{1}{2} \left( v^{bcl}_- \partial_+ + v^{bcl}_+ \partial_- - F^b_{ij} \right) \right) -

- g^2 f_{abc} f_{c_1b_1c_1} \left( \delta_{ij} v^{bcl}_k v^{b_1c_1}_k - v^{b_1c_1}_i v^{bcl}_j \right) \varepsilon_i^a + J_i^a \varepsilon_i^a = \]

\[= - \frac{1}{2} \varepsilon_i^a \left( (M_0)^{ac}_{ij} + (M_1)^{ac}_{ij} + (M_2)^{ac}_{ij} \right) \varepsilon_j^c + J_i^a \varepsilon_i^a . \]  

(30)

There are the following operators with respect to the transverse fluctuations which we determine and which we will use in the further calculations. The first one one reads as

\[G_{ij}^{ac} = \left[ (M_0)^{ac}_{ij} + (M_1)^{ac}_{ij} + (M_2)^{ac}_{ij} \right]^{-1} , \]  

(32)

the second one

\[G_{ij}^{ac} = \left[ (M_0)^{ac}_{ij} \right]^{-1} , \]  

(33)

and the the third one, which is the bare propagator of transverse fluctuations, is

\[G_0^{ac}_{ij} = \left[ (M_0)^{ac}_{ij} \right]^{-1} . \]  

(34)

The inverse operator expressions of Eq. (32)-Eq. (32) we write in the following perturbative forms:

\[\tilde{G}_{ij}^{ac}(x, y) = G_{0ij}^{ac}(x, y) - \int d^4 z G_{0ij}^{ab}(x, z) \left( (M_1(z))^{bd}_{j\prime j''} + (M_2(z))^{bd}_{j\prime j''} \right) \tilde{G}_{j\prime j''}^{de}(z, y) \]  

(35)

and

\[G_{ij}^{ac}(x, y) = G_{0ij}^{ac}(x, y) - \int d^4 z G_{0ij}^{ab}(x, z) (M_1(z))^{bd}_{j\prime j''} G_{j\prime j''}^{de}(z, y) , \]  

(36)

with the bare propagator defined through

\[(M_0(x))^{ac}_{ij} G_{0ij}^{bd}(x, y) = \delta^{ab} \delta_{il} \delta^4(x - y) , \]  

(37)

where

\[(M_0)^{ac}_{ij} = \delta_{ac} (\delta_{ij} \mathcal{D} \mathcal{D} + \partial_i \partial_j) . \]  

(38)

The solution of Eq. (37) is simple:

\[G_{0ij}^{ac}(x, y) = - \delta^{ab} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x - y)}}{p^2} \left( \delta_{ij} - \frac{p_i p_j}{2 (p - p_+)} \right) = - \delta^{ab} G_{0ij}(x, y) \]  

(39)

and determines the above operators as perturbative series based on expressions Eq. (35)-Eq. (36).

Integrating out the transverse fluctuation obtaining the following expression for the effective action:

\[\Gamma = \int d^4 x \left( L_{YM} (v^{cl}_i, v^{cl}_+, \varepsilon_+) - v^{a}_{+cl} J^a_{+} (v^{cl}_+) - A^a_{+} (\partial_i^2 A^a_{+}) \right) +

+ \frac{i}{2} \ln \left( 1 + G_0 M_1 \right) + \frac{i}{2} \ln \left( 1 + G M_2 \right) +

+ \frac{1}{2} \int d^4 x \int d^4 y \left( g \varepsilon^a_{+x} \left( U^a_{1y} \right) (\partial_i \partial_- \rho^b_{-})_{x y} \varepsilon^c_{+y} + J_i^a \tilde{G}_{ij}^{ab}(x, y) J_j^b \right) , \]  

(40)

with all reggeon field terms in the Lagrangian are included.
4.2 Integration on longitudinal fluctuations

Collecting only quadratic to longitudinal fluctuations terms we write the corresponding part of the action as

\[
\Gamma_{\epsilon^2} = -\frac{1}{2} \int d^4x \int d^4y \epsilon^a_+ (N_0)^{++}_{ab} + (N_1)^{++}_{ab} + (N_2)^{++}_{ab} + (N_3)^{++}_{ab} \right)_{xy},
\]

(41)

with inverse propagator term

\[
(N_0)^{++}_{ab} = \delta_{xy} \delta^{ab} \partial_x^2 - \left( \partial_{-x} \partial_{-y} \partial_i x \partial_j y G^{ab}_{0ij}(x, y) \right).
\]

(42)

Correspondingly, other terms in the Lagrangian Eq. (41) are determined by the following expressions:

\[
(N_1)^{++}_{ab} = -g \ (U_1^a)^b_{xy} \left( \partial_i \partial_j \rho^i \right)_x,
\]

(43)

the third term

\[
(N_2)^{++}_{ab} = -2g f_{cdb} \left( \partial_{-y} v^{cl}_{j} \right) \left( \partial_{-x} \partial_{+x} \tilde{G}_{ij}^{ac} \right) - v^{cl}_{j} \left( \partial_{-y} \partial_{-x} \partial_{+x} \tilde{G}_{ij}^{ac} \right) - \]

\[
- g^2 f_{cc,a} f_{dd,b} \left( \left( \partial_{-y} \tilde{v}^{ci}_{j} \right) - \tilde{v}^{ci}_{j} \partial_{-y} \right) \left( \left( \partial_{-x} \tilde{v}^{ci}_{j} \right) - \tilde{v}^{ci}_{j} \partial_{-x} \right) \tilde{G}_{ij}^{cd}(x, y)
\]

(44)

and the last one

\[
(N_3)^{++}_{ab} = - \left( \partial_{-x} \partial_{-y} \right) \tilde{G}_{ij}^{ab}(x, y) - G^{ab}_{0ij}(x, y).
\]

(45)

The term Eq. (42) determines the equation for the longitudinal bare propagator:

\[
\int d^4y N_{0ab}^{++}(x, y) G^{bc}_{0++}(y, z) = \delta^{ac} \delta_{xz}
\]

(46)

with solution

\[
G^{ab}_{0++}(x, y) = -2 \delta^{ab} \int \frac{d^4p (e^{-p(x-y)} - \frac{p_+}{p_-})}{(2\pi)^4 \ p^2} = -\delta^{ab} G_{0++}(x, y).
\]

(47)

Therefore, integrating this fluctuation out, it obtained:

\[
\Gamma = \int d^4x \left( LYM (v_{+}^{cl}, v_{+}^{cl}) - v_{+}^{cl} J_{a}^{+} (v_{+}^{cl}) - A_{+}^{a} \left( \partial_{+}^{2} A_{+}^{a} \right) \right) + \frac{i}{2} \ln \left( 1 + G_{0} M_{1} \right) + \]

\[
+ \frac{i}{2} \ln \left( 1 + G_{0} M_{2} \right) + \frac{i}{2} \ln \left( 1 + G_{0}^{ba} \right) \left( (N_1)^{++}_{ab} + (N_2)^{++}_{ab} + (N_3)^{++}_{ab} \right),
\]

(48)

which is functional of the reggeized gluon fields only.

5 RFT calculus based on the effective action

The construction of the RFT calculus based on effective action Eq. (48) requires the knowledge of classical solutions of Eq. (11) in terms of reggeon fields. This task was performed in [2], the found classical solutions are the following:

\[
v_{+}^{cl} = A_{+}^{a} - 2g^{a} \left[ f_{abc} \left( U_{b1}^{b_1}(A_{+}) \rho_{b_1}^{i} \right) \partial_{+} A_{+}^{c} \right] +
\]

\[
+ 4g^{2} \left[ f_{abc} \left( U_{b1}^{b_1}(A_{+}) \rho_{b_1}^{i} \right) \partial_{+} \left( f_{cb_21} \left( U_{b_2}^{b_2}(A_{+}) \rho_{b_2}^{i} \right) \partial_{+} A_{+}^{c} \right) \right] =
\]

\[
= A_{+}^{a} + g \Phi_{+1}(A_{+}) + g^{2} \Phi_{+2}(A_{+})
\]

(49)
\[ v_{i}^{a cl} = v_{i0}^{a} + g v_{i1}^{a} = U^{ab}(v_{+}^{cl}) \rho_{i} (x^{-}, x_{\perp}) - g \left[ \Box^{-1} \left( \partial_j F_{ji}^{a} + 1 \frac{1}{g} \partial_t \left( \left( \partial_j U^{ab} \right) \rho_{j}^{b} \right) \right) + \partial_t \partial^{-1} j_{a1}^{+} \right] = U^{ab}(v_{+}^{cl}) \rho_{i} + g \Lambda_{i1}^{+} (A_{+}), \]

with some complicated \( P_{ji} \) function, see [2], and

\[ j_{a1}^{+} = f_{abc} v_{j0}^{b} \left( \partial_{-} v_{0}^{jcl} \right). \]

On the base of these solutions, the effective action Eq. (48) can be written as functional of reggeized fields only. We have:

\[ (F_{+}^{a} F_{-}^{a})_{cl} = (\partial_{-} v_{i}^{a}) (D_{+} v_{i})^{a} - (\partial_{-} v_{i}^{a}) (\partial_{i} v_{+})^{a} = v_{i}^{a} (D_{i} (\partial_{-} v_{i}))^{a} + (\partial_{-} v_{i}^{a}) (\partial_{+} v_{i})^{a}. \]

Taking into account an identity from equation of motion

\[ (D_{i} (\partial_{-} v_{i}))^{a} = \partial_{-}^{2} v_{i}^{a} + U^{ab}(v_{+}) \left( \partial_{-} \partial_{i} \rho_{i}^{b} \right), \]

it is obtained to \( g^{2} \) accuracy:

\[ (F_{+}^{a} F_{-}^{a})_{cl} = - g^{2} (\partial_{-} \Phi_{+1}^{a}) (\partial_{-} \Phi_{+1}^{a}) + v_{i}^{a cl} U^{ab}(A_{+}) \left( \partial_{-} \partial_{i} \rho_{i}^{b} \right) + (\partial_{-} v_{i}^{a cl}) (\partial_{+} v_{i}^{a cl}). \]

Correspondingly, there are also the following terms of the ”net” effective action:

\[ \frac{1}{2} (F_{+}^{a} F_{-}^{a})_{cl} = \frac{1}{2} g^{2} (\partial_{-} \Phi_{+1}^{a}) (\partial_{-} \Phi_{+1}^{a}) = 2 g^{2} \Box^{-1} \left[ f_{abc} \left( U^{b d}(A_{+}) \left( \partial_{-} \rho_{d}^{b} \right) \right) \left( \partial_{i} A_{+}^{c} \right) \right] \Box^{-1} \left[ f_{abc1} \left( U^{d1}(A_{+}) \left( \partial_{-} \rho_{d1}^{b} \right) \right) \left( \partial_{+} A_{+}^{c1} \right) \right]. \]

and

\[ - \frac{1}{4} (F_{ij}^{a} F_{ij}^{a})_{cl0} = - \frac{1}{4} (F_{ij}^{a})_{cl0} (F_{ij}^{a})_{cl0} \]

with

\[ (F_{ij}^{a})_{cl0} = \rho_{bj} \partial_{i} U^{ab}(v_{+}^{cl}) - \rho_{bi} \partial_{j} U^{ab}(v_{+}^{cl}) + g f_{abc} \left( U^{b d1}(v_{+}^{cl}) \rho_{b1} \right) \left( U^{c c1}(v_{+}^{cl}) \rho_{c1 j} \right) + g \left( \partial_{i} \Lambda_{j1}^{+} - \partial_{j} \Lambda_{i1}^{+} \right) + g^{2} f_{abc} \left( U^{b b1} \rho_{b1 i} \Lambda_{j1}^{+} + \Lambda_{j1}^{+} U^{c c1} \rho_{c1 j} \right), \]

where we notice that the Eq. (56) expression’s minimal order is \( g^{2} \). Now, basing on connection between \( A_{-} \) field and \( \rho_{i} \) operator

\[ \partial_{i} \partial_{-} \rho_{i}^{a} = - \frac{1}{N} \partial_{i}^{2} A_{-}, \]

or

\[ \rho_{i}^{a}(x^{-}, x_{\perp}) = \frac{1}{N} \int d^{4} z G_{xz}^{-0} \left( \partial_{z} A_{+}^{a}(z^{-}, z_{\perp}) \right), \]

see [2] and Appendix A, the effective action Eq. (48) can be expanded in terms of reggeon fields \( A_{-} \) and \( A_{+} \) as

\[ \Gamma = \sum_{n,m=0} \left( A_{+}^{a_{1}} \cdots A_{+}^{a_{n}} R_{b_{1} \cdots b_{m}}^{a_{1} \cdots a_{n}} A_{-}^{b_{1}} \cdots A_{-}^{b_{m}} \right), \]

that determines this expression as functional of reggeon fields and provides effective vertices of the interactions of the reggeized gluons in the RFT calculus.
6 Interaction kernels and propagators of reggeized gluons

Effective action Eq. (60) can be fully determined in terms of the effective vertices of reggeon fields interactions. Calculating these vertices one after another we will reconstruct expression similar to introduced in [3], see also [10, 20], which can be considered as QCD Hamiltonian for reggeized gluon fields. There are the following well known vertices of interactions of reggeon fields: vertex of interaction of $A_+$ and $A_-$ fields and vertex of interaction of two $A_+$ fields and two $A_-$, which are propagator of reggeized gluons and BFKL kernel correspondingly, see [5]; vertex of interaction of six reggeon fields, which can be identified with triple Pomeron vertex, see [9], or with odderon, see [21]. The expression Eq. (68) consists these vertices plus many other with different precision and different color representations. Calculating the QFT corrections to this effective action we as well will calculate the corrections to these vertices and will determine the expressions for other complex vertices of interactions of reggeized fields in RFT. Anyway, a recalculation of the known vertices it is a good test of self-consistency of the effective action. Therefore, in this paper we calculate the propagator of reggeized gluons, which is the basic element of the small-x BFKL approach, in the framework of the effective action for reggeized gluons.

The interaction of reggeized gluons $A_+$ and $A_-$ is defined as effective vertex of interactions of reggeon fields in Eq. (60):

$$\left(K_{xy}^{ab}\right)_-^+ = K_{xy}^{ab} = \left(\frac{\delta^2 \Gamma}{\delta A_+^a \delta A_-^b}\right)_{A_+, A_- = 0},$$

we can call this vertex as interaction kernel as well, see Eq. (68)-Eq. (71) below. The contributions to this kernel are provided by the different terms in the action which are linear with respect to different color representations. Calculating the QFT corrections to this effective action we as well will see [21]. The expression Eq. (48) consists these vertices plus many other with different precision and different color representations. Calculating the QFT corrections to this effective action we as well will calculate the corrections to these vertices and will determine the expressions for other complex vertices of interactions of reggeized fields in RFT. Anyway, a recalculation of the known vertices it is a good test of self-consistency of the effective action. Therefore, in this paper we calculate the propagator of reggeized gluons, which is the basic element of the small-x BFKL approach, in the framework of the effective action for reggeized gluons.

Therefore, there are the following contributions in the kernel. The leading order contribution to $K_{xy0}^{ab}$ is given by the second term in the Eq. (54) and reads as:

$$-2i K_{xy}^{ab} = \left(\frac{\delta^2 \ln (1 + G M)}{\delta A_+^a \delta A_-^b}\right)_{A_+, A_- = 0} =$$

$$= \left[\left(\frac{\delta^2 G}{\delta A_+^a \delta A_+^b} M + \frac{\delta G}{\delta A_+^a} \frac{\delta M}{\delta A_-^b} + \frac{\delta G}{\delta A_-^a} \frac{\delta M}{\delta A_+^b} + G \frac{\delta^2 M}{\delta A_+^a \delta A_-^b}\right)(1 + G M)^{-1} - \left(\frac{\delta G}{\delta A_-^a} M + G \frac{\delta M}{\delta A_-^a}\right)(1 + G M)^{-1}\right]_{A_+, A_- = 0}$$

see the expression of the effective action Eq. (48).

The NLO and NNLO contributions are determined by the logarithms in the r.h.s. of Eq. (48). Further the variation of $\rho$ field with respect to $A_-$ field will be used as well, we have from Eq. (59):

$$\frac{\delta \rho^a(x^-, x^\perp)}{\delta A_-^b(y^-, y^\perp)} = \frac{\delta^2 b}{N} \int d^4 z G^2_{xz} \delta(y^- - z^-) \delta^2(y^\perp - z^\perp) \partial_i z^i.$$  

We note also, that there are also different kernel, related to $< A_+ A_+ >$ and $< A_- A_- >$ propagators, in the approach. In leading order the contributions to these kernels are zero:

$$\left(K_{yy}^{ab}\right)_0^{++} = \left(K_{yy}^{ab}\right)_0^{--} = 0,$$

we do not calculate these propagators in the paper, this task will be considered in separate publication.
Taking into account that
\[ G_{xz}^{-0} = \tilde{G}_{xz}^{-0} \delta(x^+ - z^+) \delta^2(x_\perp - z_\perp), \]  
we write Eq. (64) in the following form:
\[
\frac{\delta \rho^2(x^-, x_\perp)}{\delta A^{\mu}_-(y^-, y_\perp)} = \frac{1}{N} \delta^{ab} \delta^2(x_\perp - y_\perp) \tilde{G}_{xe}^{-0} \partial_{iy},
\]
where the regularization of pole of $\tilde{G}_{xe}^{-0}$ Green's function depends on its definition, see Appendix A.

The propagator corresponding to kernel Eq. (61) is defined as a solution of the following equation:
\[
\int d^4z \left( K_{xz}^{ab} \right)^{++} \left( D_{xy}^{bc} \right)^{+-} = \delta^{ac} \delta_{xy}. \tag{67}
\]
For the propagator of the reggeized gluons, $D_{+-}$, the following perturbative series can be defined therefore:
\[
\left( D_{xy}^{ac} \right)^{+-} = \left( D_{xy}^{ac} \right)_{0+} - \int d^4z \int d^4w \left( D_{xz}^{ab} \right)_{0+} \left( \left( K_{zw}^{bd} \right)^{+-} - \left( K_{zw}^{bd} \right)_{0+} \right) \left( D_{wy}^{bc} \right)^{+-}, \tag{68}
\]
or in short notations:
\[
D_{xy}^{ac} = D_{xy}^{ac} - \int d^4z \int d^4w D_{xz}^{ab} \left( K_{zw}^{bd} - K_{zw}^{bd} \right) D_{wy}^{bc}, \tag{69}
\]
where
\[
K_{zw}^{bd} = \sum_{k=0} K_{zw}^{bd} \tag{70}
\]
and
\[
\int d^4z K_{xz}^{ab} D_{zy}^{bc} = \delta^{ac} \delta_{xy}. \tag{71}
\]
The calculation of the NLO kernel $K_{xy}^{ab}$ is performed in Appendix B and Appendix C. Using Eq. (C.9) and Eq. (C.24), the following expression for the kernel is obtained:
\[
-2i K_{xy}^{ab} = \frac{i g^2 N}{4\pi} \delta_{ix} \left( \int \frac{dp_-}{p_-} \int \frac{d^2p_+}{(2\pi)^2} \int \frac{d^2k_+}{(2\pi)^2} \frac{k_+^2}{p_+^2 (p_+ - k_+)^2} e^{-i k_+(x_- - y_0)} \right), \tag{72}
\]
where the only leading contributions to the kernel are present, see Appendix C. Performing Fourier transform we write Eq. (69) as:
\[
\tilde{D}_{xy}^{ab}(p_+, p_-) = \delta^{ab} - \frac{g^2 N}{32\pi^3} \int \frac{dk'_-}{k'_-} \int d^2k_+ \frac{p_+^2}{k_+^2 (p_+ - k_+)^2} \tilde{D}_{xy}^{ab}(p_+, k'_-), \tag{73}
\]
here we used
\[
D_0^{ab}(x, y) = \delta^{ab} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i p(x - y)}}{p_+^4}, \tag{74}
\]
see Eq. (53) and Eq. (11) definitions. Introducing rapidity variable $y = \frac{1}{2} \ln(\Lambda k_-)$ and taking into account the physical cut-off of the rapidity related with particles cluster size $\eta$, we obtain after integration on $k_-$ variable in the limits $y_0 - \eta/2$ and $y_0 + \eta/2$:
\[
\tilde{D}_{xy}^{ab}(p_+ \perp, \eta) = \delta^{ab} - \frac{g^2 N}{16\pi^3} \int_0^\eta d\eta' \int d^2k_+ \frac{p_+^2}{k_+^2 (p_+ - k_+)^2} \tilde{D}_{xy}^{ab}(p_+ \perp, \eta') \tag{75}
\]
with
\[
\epsilon(p_+^2) = -\frac{\alpha_s N}{4\pi^2} \int d^2k_+ \frac{p_+^2}{k_+^2 (p_+ - k_+)^2} \tag{76}
\]
10
as intercept of the propagator of reggeized gluons. Rewriting this equation as the differential one:

\[
\frac{\partial \tilde{D}^{ab}(p_\perp, \eta)}{\partial \eta} = \tilde{D}^{ab}(p_\perp, \eta) \epsilon(p_\perp^2)
\]  

(77)

we obtain the final expression for the propagator:

\[
\tilde{D}^{ab}(p_\perp, Y) = \delta^{ab} \frac{s}{p_\perp^2} \left( \frac{s}{s_0} \right) \epsilon(p_\perp^2)
\]  

(78)

with some rapidity interval \(0 < \eta < Y = \ln(s/s_0)\) of the problem of interest introduced. We note, that obtained propagator is precisely the well-known one, see \[8\], and thereby we demonstrated the self-consistency of the obtained effective action Eq. (48). It is interesting to note also, that the obtained intercept, the expression Eq. (76), is known as well in CGC approach, this is color charge density there, see for example Kovner et al. in \[11\].

7 Conclusion

The main result of this paper is the expression for one loop effective action for reggeized gluons Eq. (48). On one hand, this effective action can be considered as the one loop effective action for gluodynamics with added gauge invariant source of longitudinal gluons, calculated on the base of non-trivial classical solutions for gluon fields. These classical solutions are fully determined in terms of reggeized gluons fields, and, therefore, the same action can be considered as the one loop effective action for reggeized gluons. The expansion of this action in terms of the reggeons, see expression Eq. (60), determines the vertices of interactions of these reggeized fields. There are all possible vertices in there, the only limitation is the precision of the expression Eq. (48). Whereas the \(A_+\) reggeized field is the argument of the ordered exponential in the classical solutions and the number of derivative in respect to this field is not limited, see expression Eq. (49)-Eq. (50) and [2], the \(A_-\) reggeon field is presented in Eq. (48) in combinations which allows only limited number of derivations in respect to this field. Therefore, the calculation of the complex vertices related to large number of \(A_-\) reggeized fields will require or increase of the precision of the calculations, or calculation of the same effective action in a different gauge, where two types of the ordered exponential, with \(A_+\) and \(A_-\) fields in arguments, will be presented in the classical solutions. In both cases, the precision of the computations will be determined by the QFT methods, namely it will be limited by the orders of the classical solutions, order of loops included in calculations and by combinations of the fields in the final expression which will survive after the \(A_+, A_- \rightarrow 0\) limit application.

Another important result of the paper is the calculation of the propagator of reggeized gluons Eq. (78) in the framework of the approach. Although this propagator is well known and widely used in all applications of the effective action, see [4] and [8], the computation of the propagator fully in the framework of interest was done first time. This calculation we can consider as the check of the self-consistency of the approach and also as the explanation of the methods of the calculation of small-x BFKL based vertices in the framework. There are other important vertices which can be similarly calculated basing on the expression Eq. (48). These vertices are important ingredients of the unitary corrections to different production and interaction amplitudes of the processes at high energies and they will be considered in separate publications.

As we mentioned above, the expression Eq. (48) describes the interactions of the reggeized gluons inside a cluster of particles in the limited range of rapidity. Therefore, following to [9], this expression we can define as RFT Hamiltonian written in terms of the QCD reggeized gluons. The performed calculations, in turn, we can consider as the construction of QCD RFT in terms of QCD degrees of freedom. This RFT construction is interesting because it allows to consider the field theory in terms of \(A_-\) and \(A_+\) fields only, developing approach to the calculation of reggeon loops, applications of the
Hamiltonian in the integrable systems frameworks and in condensed matter physics, [22], and use of the methods in the effective gravity approach, see [1].

In conclusion we emphasize, that the paper is considered as the additional step to the developing of the effective theory for reggeized gluons which will be useful in a variety of applications in high energy physics and other research fields.
Appendix A: Representation and properties of \(O\) and \(O^T\) operators

For the arbitrary representation of gauge field \(v_+ = i T^a v^a_+\) with \(D_+ = \partial_+ - g v_+\), we can consider the following representation of \(O\) and \(O^T\) operators:

\[
O_x = \delta^{ab} + g \int d^4y G_{xy}^{a \alpha_1} (v_+ (y))_{a_1 b} = 1 + g G_{xy}^+ v_+ \quad (A.1)
\]

and correspondingly

\[
O_x^T = 1 + g v_+ G_{yx}^+ \quad (A.2)
\]

which is redefinition of the operator expansions used in \([1]\) in terms of Green’s function instead integral operators, see \([2]\) for more details. The Green’s function in above equations we understand as Green’s function of the \(D_+\) operator and express it in the perturbative sense as

\[
G_{xy}^+ = G_{xy}^{+0} + g G_{xz}^{+0} v_+ G_{zy}^+ \quad (A.3)
\]

and

\[
G_{yx}^+ = G_{yx}^{+0} + g G_{yz}^{+0} v_+ G_{zx}^+ \quad (A.4)
\]

with the bare propagators defined as (there is no integration on \(x\) variable)

\[
\partial_{+x} G_{xy}^{+0} = \delta_{xy}, \quad \partial_{+x} \partial_{+x} = -\delta_{xy}. \quad (A.5)
\]

There are the following properties of the operators can be derived:

1.

\[
\delta G_{xy}^+ = g G_{xz}^{+0} (\delta v_+ z) G_{zy}^+ + g G_{xz}^{+0} v_+ \delta G_{zy}^+ = g G_{xz}^{+0} (\delta v_+ z) G_{zy}^+ + G_{xz}^{+0} v_+ (\delta G_{zp}^+) D_{zp} G_{pz}^+ = g G_{zp}^+ \delta v_+ p G_{pz}^+. \quad (A.6)
\]

2.

\[
\delta O_x = g G_{xz}^+ (\delta v_+ y) + g (\delta G_{xz}^+) v_+ y = g G_{zp}^+ \delta v_+ p (1 + g G_{pz}^{+0} v_+ y) = g G_{zp}^+ \delta v_+ p O_p; \quad (A.7)
\]

3.

\[
\partial_{+x} \delta O_x = g (\partial_{+x} G_{zp}^+) \delta v_+ p O_p = g (1 + g v_+ x G_{zp}^+) \delta v_+ p O_p = g O_x^T \delta v_+ x O_x; \quad (A.8)
\]

4.

\[
\partial_{+x} O_x = g (\partial_{+x} G_{xy}^+) v_+ y = g v_+ x (1 + g G_{xy}^{+0} v_+ y) = g v_+ O_x; \quad (A.9)
\]

5.

\[
O_x^T \partial_{+x} = g v_+ y (G_{yx}^+ \partial_{+x}) = -g (1 + v_+ y G_{yx}^+) v_+ x = -g O_x^T v_+ x. \quad (A.10)
\]

We see, that the operator \(O\) and \(O^T\) have the properties of ordered exponents. For example, choosing bare propagators as

\[
G_{xy}^{+0} = \theta(x^+ - y^+) \delta_{xy}^3, \quad G_{yx}^{+0} = \theta(y^+ - x^+) \delta_{xy}^3, \quad (A.11)
\]

we immediately reproduce:

\[
O_x = P e^{g \int_{-\infty}^0 dx^+ v_+ (x^+)} , \quad O_x^T = P e^{g \int_{-\infty}^\infty dx^+ v_+ (x^+)} \quad (A.12)
\]

\footnote{Due the light cone gauge we consider here only \(O(x^+)\) operators. The construction of the representation of the \(O(x^-)\) operators can be done similarly.}
The form of the bare propagators which correspond to another possible integral operator will lead to
the more complicated representations of $O$ and $O^T$ operators, see in [1].

Now we consider a variation of the action’s full current:

$$\delta \text{tr} [v_+ x O_x \partial_i^2 A^+] = \frac{1}{g} \delta \text{tr} [(\partial_+ x O_x) \partial_i^2 A^+] = \frac{1}{g} \text{tr} [(\partial_+ x \delta O_x) \partial_i^2 A^+] = \text{tr} [O^T_x \delta v_+ x O_x (\partial_i^2 A^+)],$$

(A.13)

which can be rewritten in the familiar form of the Lipatov’s induced current used in the paper:

$$\delta (v_+ J^+) = \delta \text{tr} [(v_+ x O_x \partial_i^2 A^+)] = \delta v_+^a \text{tr} [T_a O T_b O^T] (\partial_i^2 A_b^+) .$$

(A.14)

We note also, that with the help of Eq. (A.1) representation of the $O$ operator the full action’s current
can we written as

$$\text{tr} [(v_+ x O_x - A_+ \partial_i^2 A^+)] = \text{tr} [(v_+ - A_+ + v_+ x G_{xy}^+ v_{xy}) (\partial_i^2 A^+)].$$

(A.15)
Appendix B: Calculation of $K_{xy}^{ab}$ effective kernel

Below we present all contributions to the vertex of interest and present the general answer for the kernel. We note, that there are new effective propagators which are arising during the calculations, see Eq. (B.22), Eq. (B.30)-Eq. (B.31) below. Introducing these propagators the effective resummation of the different contributions occurs, see expressions Eq. (B.23) and Eq. (B.37). This resummation can be performed directly in the expansion of logarithms in Eq. (48), before taking the derivatives. Is that redefinitions of the propagators valid for the all order contributions, i.e is it can be done on the level of the Lagrangian this is a subject of the separate publication, we do not consider this problem in the paper.

Transverse loop terms contributions

First of all, consider the $M_2$ term in Eq. (48) to $g^2$ accuracy :

$$(M_2)^{ab}_{ij} = - g^2 f_{acc_1} f_{c_2} b \left( \delta_{ij} v_{k0}^c v_{k0}^d - v_{l0}^c v_{l0}^d \right). \quad (B.1)$$

We note that this term is quadratic with respect to $\rho$ field and therefore it does not contribute to the kernel of interest.

For the $M_1$ term in Eq. (48) we have:

$$- 2 \frac{\partial}{\partial A_{-y}} \left( K_{xy}^{ab} \right) = \left[ G_0 \frac{\delta^2}{\delta A_{+x}^c \delta A_{-y}^b} (1 + G_0 M_1)^{-1} - G_0 \frac{\delta}{\delta A_{-y}^b} \frac{\delta (M_1)^{cd}}{\delta A_{+x}^a} (1 + G_0 M_1)^{-1} \right]_{A_+, A_-=0}. \quad (B.2)$$

Taking into account that

$$M_1^{ab}_{ij} = - 2 g f_{abc} \left( \delta_{ij} \left( v_{k0}^c \partial_k - v_{l0}^c \partial_l \right) - \frac{1}{2} \left( v_{j0}^c \partial_i + v_{i0}^c \partial_j - F_{ij}^c \right) \right), \quad (B.3)$$

with requested accuracy reads as

$$M_1^{ab}_{ij} = - 2 g f_{abc} \left( U_{cc}^l \delta_{ij} \rho_i^c \partial_k - \frac{1}{2} U_{cc}^l \left( \rho_j^c \partial_l + \rho_i^c \partial_j \right) \right) - \delta_{ij} A_+^l \partial_-, \quad (B.4)$$

we obtain

$$- 2 \frac{\partial}{\partial A_{-y}} \left( K_{xy}^{ab} \right) = - \int d^4 z d^4 t \left( \frac{\delta}{\delta A_{-y}^b} \left( M_1^{cd}_{ij} \right) z \right) G_0 j_k(z, t) \left( \frac{\delta}{\delta A_{+x}^a} \left( M_1^{de} \right) t \right) G_0 l_i(t, z). \quad (B.5)$$

Longitudinal loop terms contribution

There are a few contributions arose from the last logarithm in the expression Eq. (48). We account only non-zero ones and calculate them one by one.
\[(N_1)_{ab}^{++} \text{ term contribution}\]

The variation of the first term of the last logarithm in Eq. (45) gives:

\[-2t \left( K_{xy}^{ab} \right)_2 = \left[ G_{0}^{++} \frac{\delta^2 (N_1)_{cc}^{++}}{\delta A_{++}^z \delta A_{--}^y} (1 + G_{0}^{++} N_{1}^{++})^{-1} - \right.\]

\[- G_{0}^{++} \frac{\delta (N_1)_{cc}^{++}}{\delta A_{--}^y} (1 + G_{0}^{++} N_{1}^{++})^{-1} G_{0}^{++} \frac{\delta (N_1)_{cc}^{++}}{\delta A_{++}^z} (1 + G_{0}^{++} N_{1}^{++})^{-1} \right]_{A_{++}, A_{--} = 0}, \tag{B.6}\]

here we used \( G_{0}^{++} \rightarrow \delta^{ab} G_{0}^{++} \) definition. The \( N_1 \) term, which is determined by Eq. (43), is quadratic in respect to the reggeon fields, therefore the only first term in Eq. (B.6) gives the non-zero contribution:

\[-2 t \left( K_{xy}^{ab} \right)_2 = \left[ G_{0}^{++} \frac{\delta^2 (N_1)_{cc}^{++}}{\delta A_{++}^z \delta A_{--}^y} \right]_{A_{++}, A_{--} = 0}. \tag{B.7}\]

We have:

\[\frac{\delta^2 (N_1)_{cc}^{++}}{\delta A_{++}^a \delta A_{--}^b} = \frac{g}{N} \frac{\delta}{\delta A_{++}^a} \frac{\delta}{\delta A_{--}^b} = \frac{g^2}{N} \left( U_2^{cb} \right)_{ztw} \frac{\delta}{\delta A_{++}^a} \left( \frac{\delta^2}{\delta g_{y_{+}z_{-}y_{-}z_{-}}^2} \right) \partial_{i z}^2 \delta. \tag{B.8}\]

At requested accuracy we have:

\[\frac{\delta v_{+w}^{a c l}}{\delta A_{++}^a} = \delta_{a c l} \left( \frac{\delta^2}{\delta g_{x_{+}w_{-}}^2} \right) \delta_{y_{-}z_{-}}, \tag{B.9}\]

see Eq. (49) and

\[\left( \left( U_2^{cb} \right)_{ztw} \right)_{A_{++}, A_{--} = 0} = \frac{1}{2} N^2 \delta^{ab} \left[ \left( G_{tw}^{+0} G_{wt}^{+0} + G_{tw}^{+0} G_{zw}^{+0} \right) + \right.\]

\[\left. + 2 \left( G_{zt}^{+0} G_{tw}^{+0} + G_{zt}^{+0} G_{wz}^{+0} + G_{zt}^{+0} G_{z0}^{+0} + G_{zt}^{+0} G_{w0}^{+0} \right) \right]. \tag{B.10}\]

Therefore, we obtain:

\[-2 t \left( K_{xy}^{ab} \right)_2 = \frac{1}{2} g^2 N \delta^{ab} \int d^4 z d^4 t d^4 w \left( G_{zt}^{+0} + \left( \delta_{x_{+}w_{-}}^2 \right) \left( \delta_{y_{-}z_{-}}^2 \right) \right) \partial_{i z}^2 \delta. \tag{B.11}\]

\[(N_2)_{ab}^{++} \text{ term contribution}\]

The only contribution from this term in the vertex of interest comes from the expression

\[-2 t \left( K_{xy}^{ab} \right)_3 = \left[ G_{0}^{++} \frac{\delta^2 (N_2)_{cc}^{++}}{\delta A_{++}^a \delta A_{--}^b} \right]_{A_{++}, A_{--} = 0}, \tag{B.12}\]

which determined by the following derivatives:

\[\frac{\delta^2 (N_2)_{cc}^{++}}{\delta A_{++}^a \delta A_{--}^b} = -2 g f_{cl} \frac{\delta}{\delta A_{--}^b} \left( \left( \partial_{-} v_{j}^{d, cl} \right) - v_{j}^{d, cl} \partial_{-} \right) \frac{\delta}{\delta A_{++}^a} \partial_{-} \partial_{i z} \tilde{G}_{ij}^{c l}(z, t). \tag{B.13}\]

Taking into account that

\[\frac{\delta}{\delta A_{++}^a} \partial_{-} \partial_{i z} \tilde{G}_{ij}^{c l}(z, t) = - \int d^4 w \left( \partial_{-} \partial_{i z} \tilde{G}_{ij}^{c l}(z, w) \right) \frac{\delta (M_1(w))_{ij}^{c l}}{\delta A_{++}^a} \tilde{G}_{ij}^{c l}(w, t), \tag{B.14}\]

\[\begin{array}{c}
16
\end{array}\]
we collect all terms together obtaining:

\[
-2 t \left( K_{x y}^{ab} \right)_{4-a} = 2 g f_{cl \delta} \int d^4 t d^4 z d^4 w G_{0++}(t, z) \left( \partial_{-z} \partial_{i z} G_{0ij}(z, w) \right) \frac{\delta (M_{1j_1j_2}^{c_l})_{w}}{\delta A_{+x}^{a}} \left( \frac{\delta}{\delta A_{-y}^{b}} \left( \left( \partial_{-} \nu_{j}^{d c l} \right) - \nu_{j}^{d c l} \partial_{-} \right) \frac{\partial_{-} \nu_{j}^{d c l} \partial_{-}}{t} G_{0j_2j}(w, t) \right) . \tag{B.15}
\]

Here we have from Eq. (B.15)

\[
\partial_{-} \rho^{a}_{k} = \frac{1}{N} \partial_{i} A_{a}^{\alpha},
\]

that

\[
\frac{\delta}{\delta A_{-y}^{b}} \left( \left( \partial_{-} \nu_{j}^{d c l} \right) - \nu_{j}^{d c l} \partial_{-} \right) \frac{\partial_{-} \nu_{j}^{d c l} \partial_{-}}{t} = - \delta^{ab} \delta_{g}^{a} t \left( \delta_{y}^{-0} \partial_{-} G_{t-y}^{-0} \partial_{-} \right) \partial_{j} t \tag{B.17}
\]

**(N₃)⁺⁺ term contribution**

The \((N₃)_{ab}^{++}\) term with required accuracy reads as:

\[
(N₃)_{ab}^{++} = \int d^4 z \left( \partial_{-z} \partial_{i z} G_{0ij}(x, z) \right) (M_{1}(z))^{a d}_{j_1 j_2} \left( \partial_{-y} \partial_{j_y} G_{j_2 j}(z, y) \right) = \int d^4 z \left( \partial_{-z} \partial_{i z} G_{0ij}(x, z) \right) (M_{1}(z))^{b d}_{j_1 j_2} \left( \partial_{-y} \partial_{j_y} G_{j_2 j}(z, y) \right) - \int d^4 z d^4 t \left( \partial_{-x} \partial_{i x} G_{0ij}(x, x) \right) (M_{1}(z))^{d c}_{j_1 j_2} \left( \partial_{-y} \partial_{j_y} G_{j_2 j}(z, y) \right) \tag{B.18}
\]

here the shorten notations \(G_{0ij}(x, z) \rightarrow G_{0ij}^{z,x} \) in the last expression were used and only \(M_{1}\) term was preserved in comparison with Eq. (B.15), the \(M_{2}\) does not contribute to the vertex of interest. Therefore, the first contribution to the kernel is given by the following expression:

\[
-2 t \left( K_{x y}^{ab} \right)_{4-a} = G_{0++} \left[ \frac{\delta^2 (N₃)_{xx}^{++}}{\delta A_{+x}^{a} \delta A_{-y}^{b}} \right]_{A_{+},A_{-}=0} = - G_{0++} \left[ \frac{\delta^2 (N₃)_{ab}^{++}}{\delta A_{+x}^{a} \delta A_{-y}^{b}} \right]_{A_{+},A_{-}=0} = \int d^4 z d^4 t \left( \partial_{-z} \partial_{i z} G_{0ij}(x, z) \right) (M_{1}(z))^{a d}_{j_1 j_2} \left( \partial_{-y} \partial_{j_y} G_{j_2 j}(z, y) \right) \tag{B.19}
\]

\[
\left( \frac{\delta (M_{1}^{d c}_{j_1 j_2})}{\delta A_{+x}^{a}} \right)_{A_{+},A_{-}=0} = -2 \int d^4 w d^4 s \partial_{-w} \partial_{i w} G_{0ij}(z, y) \frac{\delta (M_{1}^{d c}_{j_1 j_2})}{\delta A_{+x}^{a}} \left( \partial_{-z} \partial_{j_z} G_{j_2 j}(z, y) \right) \tag{B.19}
\]

The second contribution of this term to the kernel is given by:

\[
-2 t \left( K_{x y}^{ab} \right)_{4-b} = - \left[ G_{0++} \frac{\delta (N₃)_{cd}^{++}}{\delta A_{+x}^{a}} G_{0++} \frac{\delta (N₃)_{de}^{++}}{\delta A_{+x}^{d}} \right]_{A_{+},A_{-}=0} . \tag{B.20}
\]

Inserting the leading contribution from Eq. (B.18) in Eq. (B.20) we obtain:

\[
-2 t \left( K_{x y}^{ab} \right)_{4-b} = - \int d^4 w d^4 s d^4 z d^4 t \left( \partial_{-z} \partial_{i z} G_{0ij}(x, x) \right) \left( \partial_{-} \partial_{i} G_{0ij}(x, x) \right) \frac{\delta (M_{1}^{d c}_{j_1 j_2})}{\delta A_{+x}^{a}} \left( \partial_{-t} \partial_{i t} G_{0ij}(x, x) \right) \frac{\delta (M_{1}^{d c}_{j_1 j_2})}{\delta A_{+x}^{a}} \left( \partial_{-m} \partial_{i m} G_{0ij}(x, x) \right) \tag{B.21}
\]
We introduce now an additional operator:

\[ \hat{G}_{ij}(x, y) = G_{0ij}(x, y) + \int d^4z \, d^4t \, (\partial_z - \partial_k \partial_k G_{0ik}(x, z)) \, G_{0++}(z, t) \, (\partial_t - \partial_t \partial_t G_{0ij}(t, y)) = \]

\[ = - \delta_{ij} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i p(x-y)}}{p_j} \]  

(B.22)

and rewrite the following sum

\[ (K_{xy}^{ab})_{1,4} = (K_{xy}^{ab})_{1} + (K_{xy}^{ab})_{4-a} \]  

(B.23)

as

\[ -2t \left( K_{xy}^{ab} \right)_{1,4} = - \int d^4z d^4t \, \hat{G}_{ij}^{zt} \left( \frac{\delta}{\delta A_{+x}^a} \left( M_{1ji1}^{cd} \right)_t \right) \hat{G}_{jiz2} \left( \frac{\delta}{\delta A_{-y}^b} \left( M_{1j2i}^{dc} \right)_z \right). \]

(B.24)

The functional derivatives in the above expression are given by:

\[ \frac{\delta}{\delta A_{+x}^a} \left( M_{1ji1}^{cd} \right)_t = 2g \delta_{j,t} \delta_{\parallel x, \perp} \, \delta_{\perp} \, f_{cad} \, \partial_t \]  

(B.25)

and

\[ \frac{\delta}{\delta A_{-y}^b} \left( M_{1j2i}^{dc} \right)_z = 2g f_{abc} G_{z-y}^{0} \, \delta_{\parallel y, \perp} \, \left( \delta_{j2x} (\partial_{\parallel} z)^2 - \delta_{j2z} (\partial_{\parallel} x)^2 \right). \]  

(B.26)

\( (N_1)^{++}_{ab} \) and \( (N_3)^{++}_{ab} \) terms contribution

For the non-diagonal contribution from \( (N_1)^{++}_{ab} \) and \( (N_3)^{++}_{ab} \) terms we have:

\[ -2t \left( K_{xy}^{ab} \right)_{5} = - \left[ G_{0++} \, \delta \left( \frac{N_{1}^{cd}}{\delta A_{+x}^a} \right) + G_{0++} \, \delta \left( \frac{N_{3}^{dc}}{\delta A_{-y}^b} \right) \right]_{A_+, A_- = 0}. \]

(B.27)

Using results from the previous chapters, we obtain:

\[ -2t \left( K_{xy}^{ab} \right)_{5} = - \frac{g}{N} \int d^4w \, d^4s \, d^4z \, d^4t \, d^4w_1 \left( G_{0++}^{zt} \left( \left( U_{1}^{cb} \right)_t \right)_w \right)_{A_+, A_- = 0} \, (\delta_{\parallel x, \perp} \, \delta_{\perp} \, f_{cad} \, \partial_t) \]  

\[ \cdot \, G_{0++}^{wz} \left( \partial_{\perp} \partial_{\parallel} G_{0ij1}^{*} \delta_{\parallel z} \right) \left( \frac{\delta}{\delta A_{+x}^a} \left( M_{1j1j}^{dc} \right)_{w_1} \right) \left( \partial_{\perp} \partial_{\parallel} z \, G_{0j2z}^{0} \right). \]

(B.28)

Here we have:

\[ \left( \left( U_{1}^{cb} \right)_t \right)_w \right)_{A_+, A_- = 0} = \frac{1}{2} N f_{cab} \left( G_{tw}^{0} - G_{wt}^{0} \right), \]

(B.29)

see Eq. (26). Now we introduce new operators:

\[ \hat{G}_{+j}(x, y) = \int d^4z \, G_{0++}(x, z) \, (\partial_{\perp} \partial_{\parallel} z \, G_{0ij}(z, y)) = - \int \frac{d^4p}{(2\pi)^4} \, \frac{e^{-i p(x-y)}}{p_j} \, p_j \]  

(B.30)

and

\[ \hat{G}_{j+}(x, y) = \int d^4z \, (\partial_{\perp} \partial_{\parallel} z \, G_{0ij}(x, z)) \, G_{0++}(z, y) = \hat{G}_{j+}(y, x). \]

(B.31)

Finally, with the help of these operators, we rewrite Eq. (B.28) as:

\[ -2t \left( K_{xy}^{ab} \right)_{5} = - \frac{1}{2} g f_{cab} \int d^4w \, d^4t \, d^4w_1 \left( \left( G_{tw}^{0} - G_{wt}^{0} \right) \, (\delta_{\parallel x, \perp} \, \delta_{\perp} \, f_{cad} \, \partial_t) \right) \]  

\[ \cdot \, \hat{G}_{wz}^{wz} \left( \frac{\delta}{\delta A_{+x}^a} \left( M_{1j1j}^{dc} \right)_{w_1} \right) \hat{G}_{j+1}^{wz}. \]

(B.32)
\((N_2)_{ab}^{++}\) and \((N_3)_{ab}^{++}\) terms contribution

For the non-diagonal contribution from \((N_2)_{ab}^{++}\) and \((N_3)_{ab}^{++}\) terms we have:

\[
- 2t \left( K_{xy}^{ab} \right)_6 = - \left[ G_{0}^{++} \frac{\delta (N_2)_{cd}^{++}}{\delta A_{-y}^{--}} G_{0}^{++} \frac{\delta (N_3)_{de}^{++}}{\delta A_{+x}^{++}} \right]_{A_+, A_- = 0} .
\]  
(B.33)

Using results of previous calculations we write:

\[
\frac{\delta ((N_2)_{cd}^{++})_{tw}}{\delta A_{-y}} = - 2g f_{c_1 d_1} \left( \frac{\delta}{\delta A_{-y}} (\partial_{-y} v_{d_1}d_{c_1}) - v_{d_1}d_{c_1} \partial_{-y} \right) (\partial_{-t} \partial_{t} \hat{G}^{c_1}_{ij}(t, w)) \]  
(B.34)

and

\[
\frac{\delta ((N_3)_{de}^{++})_{wx}}{\delta A_{+x}} = \int d^4w_1 \left( \partial_{-s} \partial_{t} s G^{w_1}_{0 ij_1} \right) \left( \frac{\delta}{\delta A_{+x}} \left( M_{1 j_1 j_2}^{dc} \right) \right) (\partial_{-z} \partial_{z} z G^{w_1 z}_{0 j_2}) .
\]  
(B.35)

Therefore, we have for the contribution of Eq. (B.33):

\[
- 2t \left( K_{xy}^{ab} \right)_6 = 2g f_{c_1 d_1} \int d^4t d^4y d^4w d^4s d^4w_4 \left( G_{0}^{zt} \left( \frac{\delta}{\delta A_{-y}} \left( \partial_{-y} v_{d_1}d_{c_1} \right) - v_{d_1}d_{c_1} \partial_{-y} \right) \right) \\
(\partial_{-t} \partial_{t} G^{w_1}_{0 ij_1}) \left( \frac{\delta}{\delta A_{+x}} \left( M_{1 j_1 j_2}^{dc} \right) \right) (\partial_{-z} \partial_{z} z G^{w_1 z}_{0 j_2}z) =
\]

\[
= 2g f_{c_1 d_1} \int d^4t d^4w d^4w_1 \left( \frac{\delta}{\delta A_{-y}} \left( \partial_{-y} v_{d_1}d_{c_1} \right) - v_{d_1}d_{c_1} \partial_{-y} \right) \hat{G}^{w_1}_{jz_2} (\partial_{-t} \partial_{t} G^{w_1}_{jz_2} - G^{w_1 w}_{0 j_2}) .
\]
(B.36)

Now the following sum

\[
(K_{xy}^{ab})_{6,3} = (K_{xy}^{ab})_6 + (K_{xy}^{ab})_3
\]
(B.37)
can be written as

\[
- 2t \left( K_{xy}^{ab} \right)_{6,3} = 2g f_{c_1 d_1} \int d^4w d^4w_1 \hat{G}^{w_1}_{jz_2} \left( \frac{\delta}{\delta A_{+x}} \left( M_{1 j_1 j_2}^{dc} \right) \right) (\partial_{-z} \partial_{z} z G^{w_1 z}_{0 j_2})
\]

with the derivatives in the expression given by Eq. (B.17) and Eq. (B.23).

Final expression for the kernel

Taking all contributions together we obtain the following expression for the kernel to required order:

\[
- 2t K_{xy}^{ab} = \frac{1}{2} g^2 N \delta^{ab} \int d^4z d^4t d^4w \partial_{tz}^2 \left( G_{0}^{z_0} \left( \frac{\delta}{\delta z_{x+y+w}} \right) \right) \left( \frac{\delta^2}{\delta t_{x+y+w}} \right) .
\]

- \left[ \left( G_{zt}^{w_0} G_{w_0}^{t_0} + G_{zt}^{t_0} G_{w_0}^{w_0} \right) + 2 \left( G_{zt}^{t_0} G_{w_0}^{w_0} + G_{zt}^{w_0} G_{w_0}^{t_0} + G_{zt}^{w_0} G_{w_0}^{t_0} + G_{zt}^{w_0} G_{w_0}^{w_0} \right) \right] -
\]

- \int d^4z d^4t \hat{G}_{ij}^{zt} \left( \frac{\delta}{\partial A_{+x}^{+}} \left( M_{1 j_1 j_2}^{+} \right) \right) \hat{G}_{ij}^{zt} \left( \frac{\delta}{\partial A_{-y}^{+}} \left( M_{1 j_1 j_2}^{+} \right) \right) -
\]

- \int d^4z d^4t \hat{G}_{ij}^{zt} \left( \frac{\delta}{\partial A_{+x}^{+}} \left( M_{1 j_1 j_2}^{+} \right) \right) \hat{G}_{ij}^{zt} \left( \frac{\delta}{\partial A_{-y}^{+}} \left( M_{1 j_1 j_2}^{+} \right) \right) +
\]

- \frac{1}{2} g f_{c_1 d_1} \int d^4w d^4t d^4w_1 \hat{G}_{ij}^{zt} \left( \frac{\delta}{\partial A_{+x}^{+}} \left( M_{1 j_1 j_2}^{+} \right) \right) \hat{G}_{ij}^{zt} \left( \frac{\delta}{\partial A_{-y}^{+}} \left( M_{1 j_1 j_2}^{+} \right) \right) \hat{G}_{ij}^{zt} + \hat{G}_{ij}^{zt},
\]
(B.39)

where the expressions for all functional derivatives are determined above.
Appendix C: Calculation of the final answer for $K_{xy1}^{ab}$ effective kernel

Below we present final answer for the vertex of interest after the calculations of integrals in Eq. (B.39). We note that the second and fourth integrals in Eq. (B.39) are a contribution of the usual gluon field (to one-loop precision) to the propagator of the reggeized gluons, whereas the first and third terms represent new contributions to the propagator leading at high-energy limit.

Calculating integrals of Eq. (B.39), we remind, that in the framework of approach we consider the cluster of the particles, where we can limit the integration on $p_-$ variable by $p_+ > 0$ region. It corresponds to the integration over $0 < y < \eta$ limits in the integral over rapidity. In this case we have:

$$\frac{1}{p^2} \rightarrow \frac{1}{2p_- (p_+ - \frac{p^2}{2p_-} + i\varepsilon)} , \quad \varepsilon > 0 , \quad (C.1)$$

that determines the form of the integration contours in the $p_+$ integrals. Correspondingly, in this kinematic region the usual QCD one-loop gluon contribution, i.e. the second and fourth integrals in Eq. (B.39), are zero. In general, an expansion of the value of $p_-$ variable to the $p_- < 0$ region in integrals of Eq. (B.39), will lead to the change to the limits of integration in integrals over rapidity only:

$$\int_\eta^\eta d\eta' \rightarrow \int_0^\eta d\eta' + \int_0^\eta d\eta' = \int_0^\eta d\eta' , \quad (C.2)$$

and will not change the integrals over transverse momenta, these integrals are symmetrical with respect to sign of $p_-$. and the final expressions there do not depend on it. In this case the usual gluon one-loop contribution will be not zero anymore, but it will be sub-leading in the high energy regime, therefore we will not consider it here.

Non-zero contribution: first term of Eq. (B.39)

For the $p_+ > 0$ integration limit, the non-vanishing contributions in this term read as:

$$I_1 = \frac{1}{2} g^2 N \delta^{ab} \int d^4z d^4t d^4w \left( \delta^2_{x_+ w_+} \delta^2_{y_+ w_+} \right) \left( \delta^2_{y_+ z_+} \delta^2_{z_+} \right) \left( G^{+0}_w G^{+0}_w + \right) (C.3)$$

$$+ 2 \left( G^{+0}_w G^{-0}_x + G^{-0}_x G^{+0}_w \right) \quad (C.4)$$

which can be written as following:

$$I_1 = g^2 N \delta^{ab} \partial^2_{x+} \left( \delta^2_{x_+ y_+} \int dt^+ \theta(t^+ - x^+) \int dz^+ \theta(x^+ - z^+) \right) \int dp_- \int dp_+ \int \frac{d^2 p_+}{(2\pi)^2} \frac{p_+}{p_-} e^{-ip_+(t^+ + z^+)} , \quad (C.5)$$

see results of Appendix A, Eq. (17) and Kovner et al. in [11]. After the regularization of the theta functions we have:

$$\int dt^+ \theta(t^+ - x^+) e^{-ip_+ t^+} = - \frac{1}{p_+ - i\varepsilon} e^{-ip_+ x^+} \quad (C.6)$$

and

$$\int dz^+ \theta(x^+ - z^+) e^{ip_+ z^+} = - \frac{1}{p_+ - i\varepsilon} e^{ip_+ x^+} . \quad (C.7)$$

Inserting these expression in Eq. (C.5) we obtain:

$$I_1 = - g^2 N \delta^{ab} \partial^2_{x+} \left( \delta^2_{x_+ y_+} \int \frac{dp_-}{2p_-} \int \frac{d^2 p_+}{(2\pi)^2} \int \frac{dp_+}{p_+} \frac{1}{p_+ - \frac{p^2}{2p_-} + i\varepsilon} \frac{1}{p_+ - i\varepsilon} \right) , \quad (C.8)$$

that after the integration on $p_+$ gives finally:

$$I_1 = g^2 N \delta^{ab} \partial^2_{x+} \left( \int \frac{dp_-}{p_-} \int \frac{d^2 p_+}{p_+^2} \int \frac{d^2 k_\perp}{(2\pi)^2} e^{-ik_\perp (x_+ - y_+)} \right) . \quad (C.9)$$
Non-zero contribution: third term of Eq. (B.39)

In the $p_+ > 0$ integration limit, the non-vanishing contribution reads as:

$$I_2 = \frac{1}{2} g f_{ab} \int d^4 w d^4 t d^4 w_1 G_{w_1}^+ 0 \left( \delta^2_{i \perp, y \perp} \delta_{t_- y_-} \right) \hat{G}_{w_1}^{u w_1} \left( \frac{\delta}{\delta A_{x_+}^w} \left( M_{1 j_1 j_2}^{d c} \right)_{w_1} \right) \left( \partial^2_{t_+} \hat{G}_{j_2}^{w_1 t} \right). \quad (C.10)$$

Here we have:

$$\frac{\delta}{\delta A_{x_+}^w} \left( M_{1 j_1 j_2}^{d c} \right)_{w_1} = 2 g \delta_{j_1 j_2} \delta^2_{w_1 x_+} \delta_{w_1 t_+} f_{d c} \partial_{- w_1}, \quad (C.11)$$

and

$$G_{w_1}^{+ 0} = \theta(w^+ - t^+) \delta_{w_1 x_+} \delta_{w_1 t_-}, \quad (C.12)$$

see Eq. (B.25) and Appendix A. Inserting these expressions in Eq. (C.10) we obtain for the first contribution:

$$I_2 = g^2 N \delta^{a b} \delta^2_{i x} \int dt^+ \int dw_+ \theta(w^+ - t^+) \int dt^- \int dw^- \delta_{w^+ t^-} \int dt^- \delta_{t^- y^-} \int dw^- \delta_{w^- t^-} \int dt^- \delta_{t^- y^-} \int dw^+ \delta_{w^+ t^+} \int dw^- \delta_{w^- t^-} \int dt^- \delta_{t^- y^-} \hat{G}_{j_2}^{w_1 t} \left( \partial_{- w_1} \partial^2_{t_+} \hat{G}_{j_2}^{w_1 t} \right). \quad (C.13)$$

The propagator in Eq. (C.13) is defined above, see Eq. (B.39), it is:

$$\hat{G}_{j_2}^{w_1 t} = - \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i p(w-w_1)}}{p^2} \frac{p_j}{p_-} \quad (C.15)$$

and correspondingly

$$\partial_{- w_1} \partial^2_{t_+} \hat{G}_{j_2}^{w_1 t} = - i \partial^2_{t_+} \int \frac{d^4 k}{(2\pi)^4} e^{i k (w_1 - t)} k_j k^2. \quad (C.16)$$

Now, performing integration on delta-functions, we obtain:

$$I_2 = i g^2 N \delta^{a b} \delta^2_{i x} \left( \int dt^+ \int dw^+ \theta(w^+ - t^+) \int dw^- e^{i w^- (p_- + k_-)} \int dp^- \frac{(2\pi)^4}{p_-} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \frac{p_j}{p_-} e^{-i y^- (p_- + k_-)} e^{-i (x_i - y_i) (p_+ + k_+)} e^{-i p_+ (w^+ - x^+)} e^{i k_+ (x^+ - t^+)} \right). \quad (C.17)$$

The integral on $w^-_1$ variable provides

$$\int dw^-_1 e^{i w^-_1 (p_- + k_-)} = 2 \pi \delta_{p_- - k_-}, \quad (C.19)$$

and integration on $w^+$ variable gives

$$\int dw^+ \theta(w^+ - t^+) e^{-i p_+ w^+} = \int_{t^+}^{\infty} dw^+ e^{-i w^+ (p_+ - i \varepsilon)} = - \frac{1}{p_+ - i \varepsilon} e^{-i p_+ t^+}. \quad (C.20)$$

Therefore, we obtain for Eq. (C.17):

$$I_2 = -2 \pi g^2 N \delta^{a b} \delta^2_{i x} \left( \int dk_+ \int dt^+ e^{-i t^+ (p_+ + k_+)} \int dp_+ \frac{(2\pi)^4}{p_-} \int dp_+ \frac{2 \pi}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{p_- - i \varepsilon} \right) \frac{p_j}{2 p_-} \left( \frac{p_+ - p_+^2}{2 p_-} + i \varepsilon \right) \frac{k_j}{2 p_-} \left( k_+ + \frac{k_+^2}{2 p_-} - i \varepsilon \right) e^{i (p_+ - k_+) (x_i - y_i) e^{i x^+ (k_+ + p_+)}}. \quad (C.21)$$
Integrating now on $t^+$ and $k_+$ variables we have

\[
I_2 = (2\pi)^2 g^2 N \delta^{ab} \partial_{t,x}^2 \left( \int \frac{dp_-}{p_-} \int \frac{d^2p_\perp}{(2\pi)^2} \int \frac{d^2k_\perp}{(2\pi)^2} \right.
\]

\[
\left. (p_j k_j) e^{-i(p_i + k_i)(x_i - y_i)} \int \frac{dp_+}{4p_-^2} \left( \frac{1}{p_+ - \frac{k_+}{2p_-} + i\varepsilon} \right) \left( \frac{1}{p_+ - \frac{p_+^2}{2p_-} + i\varepsilon} \right) \right),
\]

(C.22)

and performing $p_+$ integration we obtain:

\[
I_2 = i g^2 N \delta^{ab} \partial_{t,x}^2 \left( \int \frac{dp_-}{p_-} \int \frac{d^2p_\perp}{(2\pi)^2} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{p_j k_j}{p_+^2 - k_+^2} e^{-i(p_i + k_i)(x_i - y_i)} \right).
\]

(C.23)

Performing the variable change $p_\perp + k_\perp \to k_\perp$ we rewrite the integral in the more familiar form:

\[
I_2 = i \delta^{ab} \frac{g^2}{4\pi} \partial_{t,x}^2 \left( \int \frac{dp_-}{p_-} \int \frac{d^2p_\perp}{(2\pi)^2} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{k_\perp^2}{p_+^2 - k_\perp^2} e^{-i k_i(x_i - y_i)} \right) -
\]

\[
- i \delta^{ab} \frac{g^2}{(2\pi)^3} \partial_{t,x}^2 \left( \int \frac{dp_-}{p_-} \int \frac{d^2p_\perp}{p_\perp^2} \int \frac{d^2k_\perp}{(2\pi)^2} e^{-i k_i(x_i - y_i)} \right).
\]

(C.24)

Other contributions to Eq. (B.39)

Other terms in Eq. (B.39) will give non-zero contributions in the kernel in the $p_- < 0$ integration region only, see discussion above.
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