Some examples of equivalent rearrangement-invariant quasi-norms defined via \( f^* \) or \( f^{**} \)

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Abstract
We consider Lorentz–Karamata spaces, small and grand Lorentz–Karamata spaces, and the so-called \( L, R, L^c, L^r, R^c, R^r \) spaces. The quasi-norms for a function \( f \) in each of these spaces can be defined via the nonincreasing rearrangement \( f^* \) or via the maximal function \( f^{**} \). We investigate when these quasi-norms are equivalent. Most of the proofs are based on Hardy-type inequalities. As an application, we demonstrate how our general results can be used to establish interpolation formulae for grand and small Lorentz–Karamata spaces.

KEYWORDS
\( K \)-functional, real interpolation, rearrangement invariant spaces, slowly varying functions

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1 | INTRODUCTION

Let \((\Omega, \mu)\) be a totally \( \sigma \)-finite measure space with a nonatomic measure \( \mu \) and let \( M(\Omega, \mu) \) be the set of all \( \mu \)-measurable functions on \( \Omega \). For each function \( f \in M(\Omega, \mu) \), we consider \( f^* \), the nonincreasing rearrangement of \( f \), given by

\[
    f^*(t) := \inf \left\{ \lambda > 0 : \mu(\{x \in \Omega : |f(x)| > \lambda\}) \leq t \right\}, \quad t > 0,
\]

and

\[
    f^{**}(t) := \frac{1}{t} \int_0^t f^*(u)du, \quad t > 0,
\]

the maximal function of \( f^* \) (see, e.g., [6]). One of the possibilities to define rearrangement-invariant quasi-Banach function spaces on \( \Omega \) is to use expressions for the quasi-norms based on \( f^* \) or \( f^{**} \). The machinery goes back to Lorentz [33] and is used in most papers and books dealing with rearrangement-invariant quasi-Banach function spaces. See, for example, [5–7, 31, 41] and references therein. The same expression written with \( f^* \) and with \( f^{**} \) can produce two quasi-norms that define, a priori, different quasi-Banach function spaces. Thus, a natural question to ask is how the resulting spaces are related. Embeddings in one direction are simple, thanks to the fact \( f^* \leq f^{**} \). Hardy-type inequalities are the classical tool to establish embeddings in the other direction.
For example, given $0 < p, q \leq \infty$, the Lorentz spaces $L_{p,q}$ and $L_{(p,q)}$ are defined by

$$L_{p,q} = \{ f \in \mathcal{M}(\Omega, \mu) : \| f \|_{p,q} = \| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \|_{q,(0,\infty)} < \infty \}$$

and

$$L_{(p,q)} = \{ f \in \mathcal{M}(\Omega, \mu) : \| f \|_{(p,q)} = \| t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \|_{q,(0,\infty)} < \infty \}$$

(see [6, Definitions 4.1 and 4.4], [9, 33]). Using the classical Hardy-type inequality, it can be proved that the quasi-norm $\| \cdot \|_{p,q}$ is equivalent to the norm $\| \cdot \|_{(p,q)}$ when $1 < p \leq \infty$ and $1 \leq q \leq \infty$ (see, e.g., [6, chapter 4, Lemma 4.5]). When $0 < q < 1$, the equivalence between the quasi-norms can be established using another Hardy-type inequality [32, Theorem 2.2] (see, e.g., [35, Theorem 3.8(i)]).

A more general context is the one of weighted Lorentz spaces $\Lambda_p(w) = \{ f \in \mathcal{M}(\Omega, \mu) : \| f \|_{\Lambda_p(w)} = \| w^{p}(t)f^*(t) \|_{q,(0,\infty)} < \infty \}$ and $\Gamma_p(w) = \{ f \in \mathcal{M}(\Omega, \mu) : \| f \|_{\Gamma_p(w)} = \| w^{p}(t)f^{**}(t) \|_{q,(0,\infty)} < \infty \}$, introduced by Lorentz [33] and Calderón [9], respectively, and later studied by many authors. See, for example, [10] and the references therein. If $1 < p < \infty$, Sawyer [39, Theorem 4] proved that $\Lambda_p(w) = \Gamma_p(w)$ if and only if the weight $w$ belongs to the class $B_p$ defined by Ariño and Muckenhoupt in [4]. The well-known classical result of Ariño and Muckenhoupt [4, Theorem 1.7] asserts that the weight $w$ belongs to $B_p$, $1 \leq p < \infty$, if and only if the weighted Hardy operator restricted to nonnegative nonincreasing functions is bounded.

In the particular case $w(t) = t^{\frac{1}{p}-\frac{1}{q}} b(t)$, where $0 < p, q \leq \infty$ and $b$ is a slowly varying function on $(0, \infty)$ (see Definition 2.1 below), the weighted Lorentz spaces are usually called Lorentz–Karamata spaces and denoted by $L_{p,q;b}$ and $L_{(p,q);b}$, respectively. In [28, 34, 35] and [36, Theorem 3.15], it was proved, using weighted Hardy inequalities, that

$$L_{p,q;b} = L_{(p,q);b}$$

if $1 < p \leq \infty$ and $0 < q \leq \infty$. (1.1)

The previous equality means that the spaces are equal as linear spaces and their quasi-norms are equivalent.

Other cases studied in the literature are the Lorentz–Zygmund spaces $L^{p,q}(\log L)^{\alpha}$, investigated by Bennett and Rudnick [5], generalized Lorentz–Zygmund spaces (see [35]), ultrasymmetric spaces introduced by Pustylnik [37], grand and small Lebesgue spaces (see [25]), among other examples.

An important feature of identity (1.1) is that it can be seen from the point of view of interpolation theory. Recall that J. Peetre’s well-known formula for the $K$-functional of a function $f$ in $L_1 + L_\infty$ asserts that

$$K(t,f;L_1,L_\infty) = \int_0^t f^*(\tau) d\tau = t f^{**}(t)$$

(see [6, Theorem V.1.6]) and the interpolation spaces $(L_1,L_\infty)_{\theta,q;b}$ are defined as the set of all $f \in L_1 + L_\infty$ such that

$$\| f \|_{\theta,q;b} := \| t^{\frac{1}{\theta}-\frac{1}{q}} b(t)K(t,f;L_1,L_\infty) \|_{q,(0,\infty)} < \infty,$$

where $0 \leq \theta \leq 1$, $0 < q \leq \infty$ and $b$ is a slowly varying function. Then,

$$(L_1,L_\infty)_{1-\frac{1}{p},q;b} = L_{p,q;b}.$$
Due to (1.1), we have that
\[ (L_1, L_\infty)_{\frac{1}{p} + \frac{1}{q} = L_{(p,q)b} = L_{p,q;b}, \quad \text{if } 1 < p < \infty, 0 < q \leq \infty. \]

The aim of this paper is to prove similar identities for the limiting and extremal constructions \( R, L, L\ell, L\ell R, R\ell, \) and \( R R \) (see precise definitions in Section 3). Specifically, these limiting and extremal methods applied to the couple \((L_1, L_\infty)\) give spaces with quasi-norms expressed in terms of \( f^{**} \). Our goal is to prove that the spaces defined via \( f^* \) or \( f^{**} \) are equal and the quasi-norms are equivalent. See Section 6 for the precise statements. In particular, our results can be applied to grand and small Lebesgue spaces [25], as well as grand and small Lorentz–Karamata spaces, see Theorems 6.3 and 6.4.

We illustrate the utility of our results by including an application to the interpolation of the couples \((L_{p_0}, L_{q_0}, r_0, b_0), L_\infty)\) (see Corollary 7.2) and \((L_{p_0, q_0, r_0, b_0}, L_\infty)\) (see Corollary 7.4). Other examples can be seen in [13–16, 22–24].

The organization of the paper is the following: Section 2 contains the basic tools: the properties of slowly varying functions and new Hardy-type inequalities. In Section 3, we describe the interpolation methods we shall work with, namely, \((A_0, A_1)_{\ell,q; b}, \) the \( \ell \) and \( \ell \) limiting interpolation spaces, and the \( \ell\ell, \ell R, R\ell, \) and \( R R \) extremal interpolation spaces. We also define the function spaces under consideration through quasi-norms expressed in terms of \( f^* \) or \( f^{**} \). In Section 4, we prove the main lemmas. Section 5 is devoted to the interpolation formulae for the couples \((L_m, L_\infty)\) and \((L_{m,\infty}, L_\infty)\) and in Section 6 we deduce the particular case \( m = 1 \). Finally, Section 7 contains the applications.

2 | PRELIMINARIES

Throughout the paper, we write \( X \hookrightarrow Y \) for two (quasi-)normed spaces \( X \) and \( Y \) to indicate that \( X \) is continuously embedded in \( Y \). We write \( X = Y \) if \( X \hookrightarrow Y \) and \( Y \hookrightarrow X \). In this case, \( X \) and \( Y \) are equal as sets and as linear spaces and their quasi-norms are equivalent, and we say that the spaces \( X \) and \( Y \) are identical or equal.

For \( f \) and \( g \) being positive functions, we write \( f \lesssim g \) if \( f \leq C g \), where the constant \( C \) is independent of all significant quantities. Two positive functions \( f \) and \( g \) are considered equivalent \( f \approx g \) if \( f \lesssim g \) and \( g \lesssim f \). We adopt the conventions \( 1/\infty = 0 \) and \( 1/0 = \infty \). The abbreviation LHS(*) and RHS(*) will be used for the left- and right-hand side of the relation (*), respectively. By \( \chi_{(a,b)} \), we denote the characteristic function on an interval \( (a,b) \). We write \( f \uparrow (f \downarrow) \) if the positive function \( f \) is nondecreasing (nonincreasing). Moreover, \( \| * \|_{q,(a,b)} \) is the usual (quasi-)norm in the Lebesgue space \( L_q \) on the interval \( (a,b) \) \((0 < q \leq \infty, 0 \leq a < b \leq \infty)\).

2.1 | Slowly varying functions

In this subsection, we summarize some of the properties of slowly varying functions, which will be required later. For more details, we refer to, for example, [19, 21, 28, 36].

Definition 2.1. A positive Lebesgue measurable function \( b, 0 \neq b \neq \infty \), is said to be slowly varying on \((0, \infty)\) (notation \( b \in \mathcal{S}V \)) if, for each \( \varepsilon > 0 \), the function \( t \mapsto t^{-\varepsilon} b(t) \) is nondecreasing. Moreover, \( b \in \mathcal{S}V \) if \( b \) is slowly varying on \((0, \infty)\).

Examples of \( SV \)-functions include powers of logarithms,
\[ \ell^\alpha(t) = (1 + |\log t|)^\alpha(t), \quad t > 0, \quad \alpha \in \mathbb{R}, \]
“broken” logarithmic functions of [18], reiterated logarithms \((\ell \circ \ldots \circ \ell)(t), t > 0, \alpha \in \mathbb{R}, \) and also the family of functions \( \exp(|\log t|^\alpha), t > 0, \) for \( \alpha \in (0, 1) \).

Lemma 2.2. Let \( b, b_1, b_2 \in \mathcal{S}V, \lambda \in \mathbb{R}, \alpha > 0, 0 < q \leq \infty, \) and \( t \in (0, \infty) \).

(i) Then, \( b^\lambda \in \mathcal{S}V, b(1/t) \in \mathcal{S}V, b(t^\lambda b_1(t)) \in \mathcal{S}V, \) and \( b_1 b_2 \in \mathcal{S}V. \)

(ii) If \( f \approx g, \) then \( b o f \approx b o g. \)
(iii) \[ \| u^{-\frac{1}{\beta}} b(u) \|_{q,(0,t)} \approx t^\alpha b(t) \] and \[ \| u^{-\frac{1}{\beta}} b(u) \|_{q,(t,\infty)} \approx t^{-\alpha} b(t). \]

(iv) The functions \[ \| u^{-\frac{1}{\beta}} b(u) \|_{q,(0,t)} \] and \[ \| u^{-\frac{1}{\beta}} b(u) \|_{q,(t,\infty)} \] (if finite) belong to \( S^V \). Moreover, \[ b(t) \lesssim \| u^{-\frac{1}{\beta}} b(u) \|_{q,(0,t)} \] and \[ b(t) \lesssim \| u^{-\frac{1}{\beta}} b(u) \|_{q,(t,\infty)}. \]

(v) \[ \| u^{\lambda-\frac{1}{\beta}} b(u) \|_{q,(t/2,t)} \approx \| u^{\lambda-\frac{1}{\beta}} b(u) \|_{q,(t,2t)} \approx t^\lambda b(t). \]

Along the paper, we will often use these properties without explicit reference every time. The following simple lemma extends the above inequalities.

**Lemma 2.3.** Let \( \lambda \in \mathbb{R} \), \( 0 < q \leq \infty \), and \( b \in S^V \). Then, for all nonnegative nonincreasing Lebesgue measurable functions \( f \) on \((0, \infty)\) and for all \( t > 0 \),

\[ t^\lambda b(t) f(t) \lesssim \| u^{\lambda-\frac{1}{\beta}} b(u) f(u) \|_{q,(0,t)}, \]

\[ t^\lambda b(t) f(t) \lesssim \| u^{\lambda-\frac{1}{\beta}} b(u) f(u) \|_{q,(t,\infty)} \]

and

\[ t^\lambda b(t) f(2t) \lesssim \| u^{\lambda-\frac{1}{\beta}} b(u) f(u) \|_{q,(t,2t)}. \]

**Proof.** Since \( f \) is nonincreasing, by Lemma 2.2(v), we have

\[ t^\lambda b(t) f(t) \approx \| u^{\lambda-\frac{1}{\beta}} b(u) f(u) \|_{q,(0,t)}, \]

\[ t^\lambda b(t) f(t) \lesssim \| u^{\lambda-\frac{1}{\beta}} b(u) f(u) \|_{q,(t,\infty)} \]

and

\[ t^\lambda b(t) f(2t) \lesssim \| u^{\lambda-\frac{1}{\beta}} b(u) f(u) \|_{q,(t,2t)}. \]

The other two estimates can be proved similarly. \( \Box \)

### 2.2 Hardy-type inequalities

The following Hardy-type inequalities and their corollaries will be essential parts of our later arguments.

**Lemma 2.4** ([28, Lemma 2.7]). Let \( 1 \leq q \leq \infty \) and \( b \in S^V \). The inequality

\[ \| u^{\alpha-\frac{1}{\beta}} b(u) \int_0^u f(\tau) d\tau \|_{q,(0,\infty)} \lesssim \| u^{\alpha+1-\frac{1}{\beta}} b(u) f(u) \|_{q,(0,\infty)} \]

holds for all nonnegative Lebesgue measurable functions \( f \) on \((0, \infty)\) if and only if \( \alpha < 0 \).

Next, lemma recovers Corollary 2.9 from [28], and additionally completes it with the cases \( 0 < T < S < \infty \) and \( \alpha \leq -1 \).

**Lemma 2.5.** Let \( b \in S^V \) and \( \alpha < 0 \). Then,

i) If \( 0 < q < 1 \), the inequality

\[ \| u^{\alpha-\frac{1}{\beta}} b(u) \int_0^u f(\tau) d\tau \|_{q,(T,S)} \lesssim T^\alpha b(T) \int_0^T f(s) ds + \| u^{\alpha+1-\frac{1}{\beta}} b(u) f(u) \|_{q,(T,S)} \] \hspace{1cm} (2.1)

holds for all \( 0 \leq T < S \leq \infty \) and all nonnegative nonincreasing Lebesgue measurable functions \( f \) on \((0, \infty)\).

ii) If \( 1 \leq q \leq \infty \), (2.1) holds for all \( 0 \leq T < S \leq \infty \) and all nonnegative Lebesgue measurable functions \( f \) on \((0, \infty)\).
Proof. We can assume that \( f \neq 0 \).

Case \( 0 < q < 1 \). Denote \( G(u) = \int_0^u f(\tau) d\tau \). It is easy to see that

\[
G^q(u) = q \int_0^u G^{q-1}(\tau) f(\tau) d\tau, \quad u > 0,
\]

and hence

\[
\left\| u^{\frac{1}{q} - 1} b(u) \int_0^u f(\tau) d\tau \right\|_{q,(T, S)}^q \approx \int_T^S u^{\frac{1}{q}} b^q(u) G^q(u) \frac{du}{u} = q \int_T^S u^{\frac{1}{q}} b^q(u) \int_0^u G^{q-1}(\tau) f(\tau) d\tau \frac{du}{u}.
\]

Thus, by Fubini’s theorem, (2.2), and Lemma 2.2, we deduce

\[
\left\| u^{\frac{1}{q} - 1} b(u) \int_0^u f(\tau) d\tau \right\|_{q,(T, S)}^q \approx G^q(T) \int_T^S u^{\frac{1}{q}} b^q(u) \frac{du}{u} + \int_T^S \left( \int_0^\infty u^{\frac{1}{q}} b^q(u) \frac{du}{u} \right) G^{q-1}(\tau) f(\tau) d\tau
\]

\[
\leq G^q(T) \int_T^\infty u^{\frac{1}{q}} b^q(u) \frac{du}{u} + \int_T^S \left( \int_0^\infty u^{\frac{1}{q}} b^q(u) \frac{du}{u} \right) G^{q-1}(\tau) f(\tau) d\tau
\]

\[
\approx T^{\frac{1}{q}} b^q(T) G^q(T) + \int_T^S \tau^{\frac{1}{q}} b^q(\tau) G^{q-1}(\tau) f(\tau) d\tau.
\]

Furthermore, the hypothesis \( f \downarrow \) implies \( G(u) \geq f(u) \int_0^u d\tau = uf(u) \) and, since \( q - 1 < 0 \), we have \( G^{q-1}(u) \leq (uf(u))^{q-1}, \ u > 0 \). Multiplying by \( f(u) \), we obtain that the inequality

\[
G^{q-1}(u) f(u) \leq u^{q-1} f^q(u)
\]

holds for all \( u > 0 \). Consequently,

\[
\left\| u^{\frac{1}{q} - 1} b(u) \int_0^u f(\tau) d\tau \right\|_{q,(T, S)}^q \approx T^{\frac{1}{q}} b^q(T) f^q(T) + \int_T^S \tau^{\frac{1}{q}} b^q(\tau) f^q(\tau) d\tau.
\]

Case \( 1 \leq q \leq \infty \). This time, we have

\[
\left\| u^{\frac{1}{q} - 1} b(u) \int_0^u f(\tau) d\tau \right\|_{q,(T, S)}^q \approx \left\| u^{\frac{1}{q} - 1} b(u) \right\|_{q,(T, S)} \int_T^T f(\tau) d\tau + \left\| u^{\frac{1}{q} - 1} b(u) \right\|_{q,(T, S)} \int_T^u f(\tau) d\tau \right\|_{q,(T, S)}
\]

\[
\leq T^{\frac{1}{q}} b(T) \int_T^T f(\tau) d\tau + \left\| u^{\frac{1}{q} - 1} b(u) \right\|_{q,(T, S)} \int_T^u f(\tau) d\tau \right\|_{q,(T, S)}
\]

Moreover, if we apply the Hardy-type inequality of Lemma 2.4, we obtain that the second term of the last expression is bounded by \( \left\| u^{\frac{1}{q} - \frac{1}{q}} b(u) f(u) \right\|_{q,(T, S)}^q \). Indeed,

\[
\left\| u^{\frac{1}{q} - \frac{1}{q}} b(u) \int_T^u f(\tau) d\tau \right\|_{q,(T, S)} \leq \left\| u^{\frac{1}{q} - \frac{1}{q}} b(u) \right\|_{q,(T, S)} \int_T^u \left\| f(\tau) \chi(T, S)(\tau) \right\|_{q,(T, S)} d\tau
\]

\[
\leq \left\| u^{\frac{1}{q} - \frac{1}{q}} b(u) f(u) \chi(T, S)(u) \right\|_{q,(T, S)} \leq \left\| u^{\frac{1}{q} - \frac{1}{q}} b(u) f(u) \right\|_{q,(T, S)}.
\]

This completes the proof.
Corollary 2.6. Let $\alpha < 0$, $0 < q, r \leq \infty$, and $a, b \in SV$. Then, for all nonnegative nonincreasing Lebesgue measurable functions $f$ on $(0, \infty)$

$$
\left\| t^{-1} b(t) \right\|_{q,(t,\infty)} \left\| u^{-\frac{1}{q}} a(u) \right\|_{r,(u,\infty)} \approx \left\| t^{-\frac{1}{r}} b(t) \right\|_{q,(t,\infty)} \left\| u^{1+\frac{1}{q}} a(u) f(u) \right\|_{r,(0,\infty)}. 
$$

(2.3)

Proof. By hypothesis $f \downarrow$, then $\int_{0}^{t} f(\tau) d\tau \geq u f(u)$ and the estimate "$\geq$" holds. Next, we prove the reverse estimate. Lemma 2.5 yields

$$
\text{LHS}(2.3) \leq \left\| t^{-\frac{1}{r}} b(t) a(t) \right\|_{r,(0,\infty)} + \text{RHS}(2.3).
$$

Moreover, by the Hardy-type inequality of Lemma 2.4, Lemma 2.3, and a change of variables, we obtain that the first term of the previous sum is bounded by the second one. That is,

$$
\left\| t^{-\frac{1}{r}} b(t) a(t) \right\|_{r,(0,\infty)} \approx \left\| t^{-\frac{1}{r}} b(t) \right\|_{r,(0,\infty)} \left\| u^{-\frac{1}{r}} a(u) f(u) \right\|_{r,(0,\infty)} \approx \text{RHS}(2.3).
$$

This completes the proof. □

Corollary 2.7. Let $\alpha < 0$, $0 < q, r, s \leq \infty$, and $a, b, c \in SV$. Then, for all nonnegative nonincreasing Lebesgue measurable functions $f$ on $(0, \infty)$

$$
\left\| t^{-\frac{1}{s}} c(t) \right\|_{s,(0,\infty)} \left\| u^{-\frac{1}{s}} b(u) \right\|_{s,(u,\infty)} \left\| u^{-\frac{1}{s}} a(v) \right\|_{s,(v,\infty)} \approx \left\| t^{-\frac{1}{s}} c(t) \right\|_{s,(0,\infty)} \left\| u^{-\frac{1}{s}} b(u) \right\|_{s,(u,\infty)} \left\| v^{1+\frac{1}{s}} a(v) f(v) \right\|_{s,(v,\infty)} \left\| q,(u,\infty) \right\|_{r,(0,\infty)} \left\| s,(0,\infty) \right\|_{r,(0,\infty)} \approx \text{RHS}(2.4).
$$

(2.4)

Proof. Since $f \downarrow$, the estimate "$\geq$" holds. Let us prove the reverse estimate "$\leq$." By Lemma 2.5, we have that LHS(2.4) is bounded by

$$
\left\| t^{-\frac{1}{s}} c(t) \right\|_{s,(0,\infty)} \left\| u^{-\frac{1}{s}} b(u) \right\|_{s,(u,\infty)} \left\| u^{-\frac{1}{s}} a(v) \right\|_{s,(v,\infty)} \approx \left\| t^{-\frac{1}{s}} c(t) \right\|_{s,(0,\infty)} \left\| u^{-\frac{1}{s}} b(u) \right\|_{s,(u,\infty)} \left\| v^{1+\frac{1}{s}} a(v) f(v) \right\|_{s,(v,\infty)} \left\| q,(u,\infty) \right\|_{r,(0,\infty)} \left\| s,(0,\infty) \right\|_{r,(0,\infty)} \approx \text{RHS}(2.4).
$$

Then, to finish the proof, it suffices to show that

$$
I := \left\| t^{-\frac{1}{s}} c(t) \right\|_{s,(0,\infty)} \left\| u^{-\frac{1}{s}} b(u) \right\|_{s,(u,\infty)} \left\| q,(u,\infty) \right\|_{r,(0,\infty)} \left\| s,(0,\infty) \right\|_{r,(0,\infty)} \leq \text{RHS}(2.4).
$$

Using Corollary 2.6, Lemma 2.3, and the simple changes of variables $y = u/2$ and $x = t/2$, we conclude that

$$
I \approx \left\| t^{-\frac{1}{s}} c(t) \right\|_{s,(0,\infty)} \left\| u^{1+\frac{1}{s}} b(u) \right\|_{s,(u,\infty)} \left\| q,(u,\infty) \right\|_{r,(0,\infty)} \left\| s,(0,\infty) \right\|_{r,(0,\infty)} \approx \text{RHS}(2.4).
$$

□
Remark 2.8. Note that in Lemma 2.5 and in Corollaries 2.6 and 2.7, the assumption “nonincreasing” is necessary only if $0 < q < 1$.

3 \ INTERPOLATION METHODS AND FUNCTION SPACES

We refer to the monographs [6–8, 31, 41] for basic concepts on interpolation theory and function spaces.

3.1 \ Interpolation methods

Next, we collect the definitions and basic properties of the real interpolation methods defined with slowly varying functions. This should give a sufficient background to follow the paper.

In what follows, $\overline{A} = (A_0, A_1)$ will be a compatible quasi-Banach couple such that $A_0 \cap A_1 \neq \{0\}$. For $t > 0$, the Peetre $K$-functional is given by

$$K(t, f; A_0, A_1) \equiv K(t, f) = \inf \left\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1, \; f_i \in A_i, \; i = 0, 1 \right\}.$$

In recent years, the following scale of interpolation spaces defined with slowly varying functions has been intensively studied.

Definition 3.1 ([28, Definition 2.4]). Let $0 \leq \theta \leq 1$, $0 < q \leq \infty$, and $a \in SV$. The real interpolation space $\overline{A}_{\theta,q,a} \equiv (A_0, A_1)_{\theta,q,a}$ consists of all $f$ in $A_0 + A_1$ that satisfy

$$\|f\|_{\overline{A}_{\theta,q,a}} := \left\|t^{\theta - \frac{1}{q}} a(t)K(t, f) \right\|_{q,(0,\infty)} < \infty.$$

Lemma 3.2 ([28, Proposition 2.5]). Let $0 \leq \theta \leq 1$, $0 < q \leq \infty$, and $a \in SV$. The space $\overline{A}_{\theta,q,a}$ is a quasi-Banach space. Moreover, $A_0 \cap A_1 \hookrightarrow \overline{A}_{\theta,q,a} \hookrightarrow A_0 + A_1$ if and only if one of the following conditions is satisfied:

(i) $0 < \theta < 1$,
(ii) $\theta = 0$ and $\|t^{-\frac{1}{q}}a(t)\|_{q,(1,\infty)} < \infty$,
(iii) $\theta = 1$, $\|t^{-\frac{1}{q}}a(t)\|_{q,(0,1)} < \infty$.

If none of these conditions holds, then $\overline{A}_{\theta,q,a} = \{0\}$.

Definition 3.3 ([28, (3.7), (4.2)]). Let $0 \leq \theta \leq 1$, $0 < r, q \leq \infty$, and $a, b \in SV$. The space $\overline{A}^L_{\theta,r,b,q,a} \equiv (A_0, A_1)^L_{\theta,r,b,q,a}$ consists of all $f \in A_0 + A_1$ for which

$$\|f\|_{L;\theta,r,b,q,a} := \left\|t^{-\frac{1}{r}}b(t)\|u^{-\theta - \frac{1}{q}} a(u)K(u, f) \right\|_{q,(0,\infty)} < \infty.$$

Similarly, the space $\overline{A}^R_{\theta,r,b,q,a} \equiv (A_0, A_1)^R_{\theta,r,b,q,a}$ consists of all $f \in A_0 + A_1$ for which

$$\|f\|_{R;\theta,r,b,q,a} := \left\|t^{-\frac{1}{r}}b(t)\|u^{-\theta - \frac{1}{q}} a(u)K(u, f) \right\|_{q,(t,\infty)} < \infty.$$

These spaces naturally arise in reiteration formulae for the limiting cases $\theta = 0$ or $\theta = 1$. In literature (see, e.g., [1, 11, 13, 15, 19, 21]), similar definitions are given alongside with properties of these spaces. As proposed in [19], we refer to these spaces as $L$ and $R$ limiting interpolation spaces. The next two lemmas can be proved in the usual way.
Lemma 3.4 (Cf. [15]). Let $0 \leq \theta \leq 1$, $0 < r, q \leq \infty$, and $a, b \in SV$. The space $A^C_{\theta,r,b,q,a}$ is a quasi-Banach space. Moreover, $A_0 \cap A_1 \leftrightarrow \overline{A^C_{\theta,r,b,q,a}} \leftrightarrow A_0 + A_1$ if and only if one of the following conditions is satisfied:

(i) $0 < \theta < 1$ and $\|t^{-\frac{1}{\theta}}b(t)\|_{r,(1,\infty)} < \infty$,
(ii) $\theta = 0$ and $\|t^{-\frac{1}{\theta}}b(t)\|u^{-\frac{1}{q}}a(u)\|_{q,(1,\infty)} < \infty$,
(iii) $\theta = 1$, $\|t^{-\frac{1}{\theta}}b(t)\|_{r,(1,\infty)} < \infty$, and $\|t^{-\frac{1}{\theta}}b(t)\|u^{-\frac{1}{q}}a(u)\|_{q,(0,1)} < \infty$.

If none of these conditions holds, then $\overline{A^C_{\theta,r,b,q,a}}$ is the trivial space.

Lemma 3.5 (Cf. [15]). Let $0 \leq \theta \leq 1$, $0 < r, q \leq \infty$, and $a, b \in SV$. The space $A^R_{\theta,r,b,q,a}$ is a quasi-Banach space. Moreover, $A_0 \cap A_1 \leftrightarrow \overline{A^R_{\theta,r,b,q,a}} \leftrightarrow A_0 + A_1$ one of the following conditions is satisfied:

(i) $0 < \theta < 1$ and $\|t^{-\frac{1}{\theta}}b(t)\|_{r,(0,1)} < \infty$,
(ii) $\theta = 0$, $\|t^{-\frac{1}{\theta}}b(t)\|_{r,(0,1)} < \infty$, and $\|t^{-\frac{1}{\theta}}b(t)\|u^{-\frac{1}{q}}a(u)\|_{q,(1,\infty)} < \infty$,
(iii) $\theta = 1$ and $\|t^{-\frac{1}{\theta}}b(t)\|u^{-\frac{1}{q}}a(u)\|_{r,(0,1)} < \infty$.

Otherwise, $\overline{A^R_{\theta,r,b,q,a}} = \{0\}$.

In the next definition, we introduce four more interpolation spaces. We follow [14, 15, 22–24], where it has been shown that they appear in relation with the extreme reiteration results.

Definition 3.6. Let $0 \leq \theta \leq 1$, $0 < s, r, q \leq \infty$, and $a, b, c \in SV$. The space $A^C_{\theta,s,r,c,b,q,a}$ is the set of all $f \in A_0 + A_1$ for which the quasi-norm

$$\|f\|_{L^C_{\theta,s,r,c,b,q,a}} := \|t^{-\frac{1}{\theta}}c(t)\|u^{-\frac{1}{r}}b(u)\|v^{-\frac{1}{q}}a(v)K(v,f)\|_{q,(0,u)}\|_{r,(0,t)}\|_{s,(0,\infty)}$$

is finite. The spaces $A^C_{\theta,s,c,r,b,q,a} \equiv (A_0, A_1)^C_{\theta,s,c,r,b,q,a}$, $A^R_{\theta,s,c,r,b,q,a} \equiv (A_0, A_1)^R_{\theta,s,c,r,b,q,a}$, and $A^R_{\theta,s,c,r,b,q,a} \equiv (A_0, A_1)^R_{\theta,s,c,r,b,q,a}$ are defined via the quasi-norms

$$\|f\|_{L^C_{\theta,s,c,r,b,q,a}} := \|t^{-\frac{1}{\theta}}c(t)\|u^{-\frac{1}{r}}b(u)\|v^{-\frac{1}{q}}a(v)K(v,f)\|_{q,(0,u)}\|_{r,(0,t)}\|_{s,(0,\infty)},$$
$$\|f\|_{R^C_{\theta,s,c,r,b,q,a}} := \|t^{-\frac{1}{\theta}}c(t)\|u^{-\frac{1}{r}}b(u)\|v^{-\frac{1}{q}}a(v)K(v,f)\|_{q,(0,u)}\|_{r,(0,t)}\|_{s,(0,\infty)},$$
and
$$\|f\|_{R^C_{\theta,s,c,r,b,q,a}} := \|t^{-\frac{1}{\theta}}c(t)\|u^{-\frac{1}{r}}b(u)\|v^{-\frac{1}{q}}a(v)K(v,f)\|_{q,(0,u)}\|_{r,(0,t)}\|_{s,(0,\infty)},$$

respectively.

For these spaces, it is possible to formulate conditions under which they are trivial. For example, if $\|u^{-\frac{1}{r}}b(u)\|_{r,(1,\infty)} = \infty$, then $\overline{A^C_{\theta,s,c,r,b,q,a}} = \{0\}$. We leave this to the reader. For other combinations of parameters, they are nontrivial interpolation spaces between $A_0$ and $A_1$. In what follows, we will assume that no spaces under consideration are trivial. We refer to these spaces as $L^C, L^R, R^C$, and $R^R$ extremal interpolation spaces.
3.2 | Function spaces

Next, we define the function spaces under consideration whose quasi-norms are based on \( f^* \) and on \( f^{**} \) pairwise. (The latter ones will be denoted as \( L_{(p,q,a)} \) and \((L)\)-spaces). Recall that the principal aim of this paper is to investigate when these spaces are equal (with equivalent quasi-norms).

**Definition 3.7** ([36, Definition 3.5]). Let \( 0 < p, q \leq \infty \), and \( a \in SV \). The Lorentz–Karamata-type spaces \( L_{p,q,a} \) and \( (L)_{p,q,a} \) are the sets of all \( f \in \mathcal{M}(\Omega, \mu) \) such that

\[
\|f\|_{p,q,a} := \left\| \frac{1}{t^{p-1}} a(t)f^*(t) \right\|_{p,(0,\infty)} < \infty
\]

and

\[
\|f\|_{(p,q,a)} := \left\| \frac{1}{t^{p-1}} a(t)f^{**}(t) \right\|_{p,(0,\infty)} < \infty,
\]

respectively.

The Lorentz–Karamata spaces comprise important scales of spaces. It contains, for example, the Lebesgue spaces \( L_q \), Lorentz spaces \( L_{p,q} \), Lorentz–Zygmund, and the generalized Lorentz–Zygmund spaces. We refer to [20, 28, 34, 36] for further information about Lorentz–Karamata spaces and to [1, 3, 6, 18, 20, 28, 35, 36] for important applications in analysis.

**Definition 3.8** (Cf. [28, (5.21), (5.33)]). Let \( 0 < p, q, r \leq \infty \) and \( a, b \in SV \). The spaces \( L_{p,r,b,q,a} \), \((L)_{p,r,b,q,a} \), \( L_{p,r,b,q,a} \), and \((L)_{p,r,b,q,a} \) are the sets of all \( f \in \mathcal{M}(\Omega, \mu) \) such that

\[
\|f\|_{L_{p,r,b,q,a}} := \left\| \frac{1}{t^{r-1} b(t)} \right\|_{r,(0,\infty)} < \infty,
\]

\[
\|f\|_{(L)_{p,r,b,q,a}} := \left\| \frac{1}{t^{r-1} b(t)} \right\|_{r,(0,\infty)} < \infty,
\]

\[
\|f\|_{L_{p,r,b,q,a}} := \left\| \frac{1}{t^{r-1} b(t)} \right\|_{r,(0,\infty)} < \infty,
\]

and

\[
\|f\|_{(L)_{p,r,b,q,a}} := \left\| \frac{1}{t^{r-1} b(t)} \right\|_{r,(0,\infty)} < \infty,
\]

respectively.

We will require that \( \|t^{-1} b(t)\|_{r,(0,1)} < \infty \) for \( L_{p,r,b,q,a} \) and \((L)_{p,r,b,q,a} \) spaces, and that \( \|t^{-1} b(t)\|_{r,(0,1)} < \infty \) for \( L_{p,r,b,q,a} \) and \((L)_{p,r,b,q,a} \) spaces. Otherwise the corresponding spaces consist only of the null element. Similar definitions can be found in [12, 14, 18, 21, 23]. We refer to these spaces as \( L \) and \( R \) Lorentz–Karamata spaces, respectively. Note that the \( L \) spaces are special cases of generalized gamma spaces with double weights [27, Definition 1.2].

In order to compare our results with those from [3, 21, 24, 27], we introduce the grand and small Lorentz–Karamata spaces.

**Definition 3.9**. Let \( 0 < p, q, r \leq \infty \), and \( b \in SV \). The small Lorentz–Karamata spaces \( L_{(p,q,r)} \) and \((L)_{(p,q,r)} \) and the grand Lorentz–Karamata spaces \( L_{(p)_{p,q,r}} \) and \((L)_{(p)_{p,q,r}} \) are the sets of all \( f \in \mathcal{M}(\Omega, \mu) \) such that

\[
\|f\|_{(p,q,r)} := \left\| t^{-\frac{1}{r}} b(t) \right\|_{r,(0,\infty)} < \infty,
\]

\[
\|f\|_{(L)(p,q,r)} := \left\| t^{-\frac{1}{r}} b(t) \right\|_{r,(0,\infty)} < \infty,
\]

\[
\|f\|_{(p)_{p,q,r}} := \left\| t^{-\frac{1}{r}} b(t) \right\|_{r,(0,\infty)} < \infty,
\]

and

\[
\|f\|_{(L)_{(p)_{p,q,r}}} := \left\| t^{-\frac{1}{r}} b(t) \right\|_{r,(0,\infty)} < \infty,
\]

respectively.
and

\[ \|f\|_{L_b^{p,q,r}} = \left\| t^{-\frac{1}{p}} b(t) u^{-\frac{1}{q}} a(t) f^*(v) \right\|_{L^{pr,qr}(0,\infty)} < \infty, \]

respectively.

Remark 3.10. It is clear that \( L_b^{p,q,r} = L_{p,r,b,q,1} \) and \( L_b^{p,q,r} = L_{p,r,b,q,1} \).

Grand and small Lebesgue and Lorentz spaces find many important applications and they have been intensively studied by different authors. See \[3, 21–24, 27\] and the references therein. These spaces are often defined on a bounded domain \( \Omega \) in \( \mathbb{R}^n \) with \( \mu(\Omega) = 1 \); sometimes, see \[3\], they are also restricted to real valued functions. Here, we do not require that \( \mu(\Omega) = 1 \) and neither that the functions are real valued.

Definition 3.11. Let \( 0 < p, q, r, s \leq \infty \), and \( a, b, c \in SV \). The spaces \( L^{p,q,r}_{p,s,c,r,b,q,a} \), \( L^{p,q,r}_{p,s,c,r,b,q,a} \), \( L^{p,q,r}_{p,s,c,r,b,q,a} \), and \( L^{p,q,r}_{p,s,c,r,b,q,a} \) are the set of all \( f \in \mathcal{M}(\Omega, \mu) \) such that

\[ \|f\|_{L^{p,q,r}_{p,s,c,r,b,q,a}} = \left\| t^{-\frac{1}{p}} c(t) u^{-\frac{1}{q}} b(u) v^{-\frac{1}{r}} a(v) f^*(v) \right\|_{L^{pr,qr}(0,\infty)} < \infty, \]
\[ \|f\|_{L^{p,q,r}_{p,s,c,r,b,q,a}} = \left\| t^{-\frac{1}{p}} c(t) u^{-\frac{1}{q}} b(u) v^{-\frac{1}{r}} a(v) f^*(v) \right\|_{L^{pr,qr}(0,\infty)} < \infty, \]
\[ \|f\|_{L^{p,q,r}_{p,s,c,r,b,q,a}} = \left\| t^{-\frac{1}{p}} c(t) u^{-\frac{1}{q}} b(u) v^{-\frac{1}{r}} a(v) f^*(v) \right\|_{L^{pr,qr}(0,\infty)} < \infty, \]
and
\[ \|f\|_{L^{p,q,r}_{p,s,c,r,b,q,a}} = \left\| t^{-\frac{1}{p}} c(t) u^{-\frac{1}{q}} b(u) v^{-\frac{1}{r}} a(v) f^*(v) \right\|_{L^{pr,qr}(0,\infty)} < \infty, \]

respectively.

The spaces \( L^{p,q,r}_{p,s,c,r,b,q,a} \), \( L^{p,q,r}_{p,s,c,r,b,q,a} \), \( L^{p,q,r}_{p,s,c,r,b,q,a} \), and \( L^{p,q,r}_{p,s,c,r,b,q,a} \) are defined as the set of all \( f \in \mathcal{M}(\Omega, \mu) \) such that (3.1)–(3.4) are satisfied after replacing of \( f^* \) by \( f^{**} \).

We refer to these spaces as \( L^{p,q,r}_{p,s,c,r,b,q,a} \), \( L^{p,q,r}_{p,s,c,r,b,q,a} \), \( L^{p,q,r}_{p,s,c,r,b,q,a} \), and \( L^{p,q,r}_{p,s,c,r,b,q,a} \) Lorentz–Karamata spaces.

The next lemma follows from Lemma 2.3. The proof is left to the reader.

Lemma 3.12. Let \( 0 < p, q, r, s \leq \infty \) and \( a, b, c \in SV \). Then,

\[ L^{p,q,r}_{p,s,c,r,b,q,a} \subset L^{p,q,r}_{p,s,c,r,b,q,a} \subset L^{p,q,r}_{p,s,c,r,b,q,a}, \]
and

\[ L^{p,q,r}_{p,s,c,r,b,q,a} \subset L^{p,q,r}_{p,s,c,r,b,q,a}. \]

In particular,

\[ L^{p,q,r}_{b} \subset L^{p,q,r}_{p,r,b}. \]

Analogous inclusions hold if the \( L \)-spaces are replaced by \( (L) \)-spaces.
3.3 Spaces which quasi-norms are defined via $f^{**}$

Peetre’s formula (1.2) allows us to characterize all $(L)$-spaces as interpolation spaces for the couple $(L_1, L_{\infty})$ through the appropriate interpolation method. Indeed,

**Lemma 3.13.** Let $1 \leq p \leq \infty$, $0 < q \leq \infty$, $a, b \in SV$, and $\theta = 1 - \frac{1}{p}$. Then,

$$L_{(p,q,a)} = (L_1, L_{\infty})_{\theta,q,a}.$$

**Lemma 3.14** (See [28, Lemmas 5.4, 5.9]). Let $1 \leq p \leq \infty$, $0 < q, r \leq \infty$, $a, b \in SV$, and $\theta = 1 - \frac{1}{p}$. Then,

$$\begin{align*}
(L)^c_{p,r,b,q,a} &= (L_1, L_{\infty})_{\theta,r,b,q,a}^c \\
(L)^R_{p,r,b,q,a} &= (L_1, L_{\infty})_{\theta,r,b,q,a}^R.
\end{align*}$$

In particular,

$$\begin{align*}
(L)^c_{b} &= (L_1, L_{\infty})_{\theta,b,q,1}^c \\
(L)^R_{b} &= (L_1, L_{\infty})_{\theta,b,q,1}^R.
\end{align*}$$

**Lemma 3.15.** Let $1 \leq p \leq \infty$, $0 < q, r, s \leq \infty$, $a, b, c \in SV$, and $\theta = 1 - \frac{1}{p}$. Then,

$$\begin{align*}
(L)^{\mathcal{C}, \mathcal{C}}_{p,(s,c,r,b,q,a)} &= (L_1, L_{\infty})^{\mathcal{C}, \mathcal{C}}_{s,c,r,b,q,a} \\
(L)^{\mathcal{C}, \mathcal{R}}_{p,(s,c,r,b,q,a)} &= (L_1, L_{\infty})^{\mathcal{C}, \mathcal{R}}_{s,c,r,b,q,a} \\
(L)^{\mathcal{R}, \mathcal{C}}_{p,(s,c,r,b,q,a)} &= (L_1, L_{\infty})^{\mathcal{R}, \mathcal{C}}_{s,c,r,b,q,a} \\
(L)^{\mathcal{R}, \mathcal{R}}_{p,(s,c,r,b,q,a)} &= (L_1, L_{\infty})^{\mathcal{R}, \mathcal{R}}_{s,c,r,b,q,a}.
\end{align*}$$

4 MAIN LEMMAS

Recall that $(\Omega, \mu)$ denotes a totally $\sigma$-finite measure space with a nonatomic measure $\mu$ and that $\mathcal{M}(\Omega, \mu)$ is the set of all $\mu$-measurable functions on $\Omega$.

Let $0 < m < \infty$. Peetre’s formula (1.2) was generalized by P. Krée [30] in the following sense. For $f \in L_m + L_{\infty}$,

$$K(t, f; L_m, L_{\infty}) \approx \left( \int_0^t f^*(\tau)^m d\tau \right)^{1/m}, \quad t > 0. \quad (4.1)$$

If $f \in L_{m,\infty} + L_{\infty}$, then [29]

$$K(t, f; L_{m,\infty}, L_{\infty}) \approx \sup_{0 < \tau < t} \frac{1}{m} f^*(\tau), \quad t > 0.$$

A general formula for the $K$-functional of the couple $(E, L_{\infty})$ with $E$ an r.i. is given in [17, p. 84]. Moreover, the formula for $K(t, f; \Lambda_m(\omega), L_{\infty})$ (with a simpler proof) can be found in [2, Lemma 3.1].

Let (cf. [38, 40])

$$f^{**}_{(m)}(t) := \frac{1}{t} \left( \int_0^t f^*(\tau)^m d\tau \right)^{1/m}, \quad t > 0. \quad (4.2)$$

Obviously $f^{**}_{(1)} = f^{**}$. Furthermore, $f^{**}_{(m)}$ satisfies the following properties:

**Lemma 4.1** (Cf. [40, (R8)]).

(i) The function $f^{**}_{(m)}$ (if exists) is nonincreasing.
(ii) $f^{**}_{(m)}(t) \geq f^*(t^{**})$, for all $t > 0$.
(iii) $(f + g)^{**}_{(m)}(t) \leq f^{**}_{(m)}(t) + g^{**}_{(m)}(t)$, for all $t > 0$. 

Proof. The first assertion can be proved in the usual way. See, for example, [6, Proposition 3.2]. In order to prove (ii), we use that $f^*$ is nonincreasing to obtain

$$f^{**}_{(m)}(t) \geq \frac{1}{t} f^*(t^m) \left( \int_0^{t^m} d\tau \right)^{1/m} = f^*(t^m).$$

(iii) The equivalence (4.1) and the (quasi)-subadditivity of the $K$-functional imply that

$$(f + g)^*_{(m)}(t) \approx \frac{1}{t} K(t, f + g; L_m, L_{\infty}) \leq \frac{1}{t} K(t, f; L_m, L_{\infty}) + \frac{1}{t} K(t, g; L_m, L_{\infty})$$

$$\approx f^*_{(m)}(t) + g^*_{(m)}(t).$$

This completes the proof of the lemma. □

The following result easily follows from [28, Lemma 5.2]. In what follows, we shall denote $\tilde{b}(u) = b(u^{1/m}), u > 0, 0 < m < \infty$.

**Lemma 4.2.** Let $\theta \in (0, 1], 0 < m < \infty, 0 < q \leq \infty$, and $b \in SV$. Then, for all $f \in L_m + L_{\infty}$ and all $t > 0$

$$\left\| u^{-\theta - \frac{1}{q}} b(u) K(u, f; L_m, L_{\infty}) \right\|_{q,(0,t)} \approx \left\| u^{-\frac{1}{q}} b(u) f^{**}_{(m)}(u) \right\|_{q,(0,t)} \approx \left\| v^{-\frac{1}{m} - \frac{1}{q}} \tilde{b}(v) f^*(v) \right\|_{q,(0,t^m)}.$$

and for all $f \in L_{m,\infty} + L_{\infty}$ and all $t > 0$

$$\left\| u^{-\frac{1}{q}} b(u) K(u, f; L_{m,\infty}, L_{\infty}) \right\|_{q,(0,t)} \approx \left\| v^{-\frac{1}{m} - \frac{1}{q}} \tilde{b}(v) f^*(v) \right\|_{q,(0,t^m)}.$$

In the next two lemmas, we generalize, in some sense, Lemma 4.2. Indeed, we estimate, from above and below, the quasi-norm in $L_q(T, S)$ of the functions

$$u^{-\theta - 1/q} b(u) K(u, f; L_m, L_{\infty}) \quad \text{and} \quad u^{-\theta - 1/q} b(u) K(u, f; L_{m,\infty}, L_{\infty}),$$

for all $0 \leq T < S \leq \infty$.

**Lemma 4.3.** Let $\theta \in (0, 1], 0 < m < \infty, 0 < q \leq \infty$, and $b \in SV$. Then, for all $f \in L_m + L_{\infty}$ and all $0 \leq T < S \leq \infty$,

$$\left\| u^{-\theta - \frac{1}{q}} b(u) K(u, f; L_m, L_{\infty}) \right\|_{q,(T,S)} \geq \left\| v^{-\frac{1}{m} - \frac{1}{q}} \tilde{b}(v) f^*(v) \right\|_{q,(T^m,S^m)},$$

and for all $f \in L_{m,\infty} + L_{\infty}$ and all $0 \leq T < S \leq \infty$,

$$\left\| u^{-\theta - \frac{1}{q}} b(u) K(u, f; L_{m,\infty}, L_{\infty}) \right\|_{q,(T,S)} \geq \left\| v^{-\frac{1}{m} - \frac{1}{q}} \tilde{b}(v) f^*(v) \right\|_{q,(T^m,S^m)}.$$ (4.3)

Proof. Let $f \in L_m + L_{\infty}$. By (4.1) and Lemma 4.1(ii), we have that $K(u, f; L_m, L_{\infty}) \approx u f^{**}_{(m)}(u) \geq u f^*(u^m), u > 0$. Hence, using the change of variables $u = u^m$, it follows

$$\left\| u^{-\theta - \frac{1}{q}} b(u) K(u, f; L_m, L_{\infty}) \right\|_{q,(T,S)} \geq \left\| u^{-\frac{1}{q}} b(u) f^*(u^m) \right\|_{q,(T,S)} \approx \left\| v^{-\frac{1}{m} - \frac{1}{q}} \tilde{b}(v) f^*(v) \right\|_{q,(T^m,S^m)}.$$

Estimate (4.3) can be proved similarly using that

$$K(u, f; L_{m,\infty}, L_{\infty}) \approx \sup_{0 < \tau < u^m} \frac{1}{\tau} f^*(\tau) \geq u f^*(u^m).$$

The proof is completed. □
Lemma 4.4. Let $\theta \in (0, 1]$, $0 < m < \infty$, $0 < q \leq \infty$, and $b \in SV$. Then, for all $f \in L_m + L_\infty$ and all $0 \leq T < S \leq \infty$,
\[
\| u^{-\frac{1}{q} \theta} b(u) K(u, f; L_m, L_\infty) \|_{q(T, S)} \lesssim T^{1-\theta} b(T) f^{**}(m)(T) + \| u^{-\frac{1}{q} \theta} b(v) f^*(v) \|_{q(T, S)}.
\]
and for all $f \in L_{m, \infty} + L_\infty$ and all $0 \leq T < S \leq \infty$,
\[
\| u^{-\frac{1}{q} \theta} b(u) K(u, f; L_{m, \infty}, L_\infty) \|_{q(T, S)} \lesssim T^{1-\theta} b(T) f^{**}(m)(T) + \| u^{-\frac{1}{q} \theta} b(v) f^*(v) \|_{q(T, S)}.
\]

Proof. Let $f \in L_m + L_\infty$. By (4.1), it holds
\[
\| u^{-\frac{1}{q} \theta} b(u) K(u, f; L_m, L_\infty) \|_{q(T, S)} \approx \left\| u^{-\frac{1}{q} \theta} b(u) \left( \int_0^u f^*(\tau)^m d\tau \right)^{1/m} \right\|_{q(T, S)}.
\]
For the last term, the suitable change of variables, Lemma 2.5 and (4.2) yield that
\[
\left\| u^{-\frac{1}{q} \theta} b(u) \left( \int_0^u f^*(\tau)^m d\tau \right)^{1/m} \right\|_{q(T, S)} \approx \left\| v^{-\frac{1}{q} \theta} b(v)^m \left( \int_0^v f^*(\tau)^m d\tau \right)^{1/m} \right\|_{q(T, S)}.
\]
This concludes the first part of the proof.

Let $f \in L_{m, \infty} + L_\infty$ and $u > 0$. Since
\[
K(u, f; L_{m, \infty}, L_\infty) \approx \sup_{0 < z < u^m} z^{1/m} f^*(z) \approx \sup_{0 < z < u^m} f^*(z) \left( \int_0^z d\tau \right)^{1/m} \leq \sup_{0 < z < u^m} \left( \int_0^z f^*(\tau)^m d\tau \right)^{1/m} = \left( \int_0^u f^*(\tau)^m d\tau \right)^{1/m},
\]
we have that
\[
\left\| u^{-\frac{1}{q} \theta} b(u) K(u, f; L_{m, \infty}, L_\infty) \right\|_{q(T, S)} \lesssim \left\| u^{-\frac{1}{q} \theta} b(u) \left( \int_0^u f^*(\tau)^m d\tau \right)^{1/m} \right\|_{q(T, S)}.
\]
Now, using (4.4) we obtain the result.

\[
\]

5  |  INTERPOLATION FORMULAE FOR THE COUPLES $(L_m, L_\infty)$ AND $(L_{m, \infty}, L_\infty)$

As above, we will denote $\bar{b}(t) = b(t^{1/m})$ for all $t > 0$, $0 < m < \infty$. The following theorem extends Corollary 5.3 from [28] and can be proved analogously using Lemma 4.2.
Theorem 5.1. Let $0 < \theta \leq 1$, $0 < m < \infty$, $p = \frac{m}{1-\theta}$, $0 < q \leq \infty$, and $b \in SV$. Then,

\[
(L_m, L_\infty)_{\theta, q, b} = (L_{m, \infty}, L_\infty)_{\theta, q, b} = L_{p, q, b}.
\]

In the following subsections, we study similar identities for the limiting and the extremal constructions.

5.1 | $L$ and $L\ell$ spaces

The next theorem extends [28, Lemma 5.4].

Theorem 5.2. Let $0 < \theta \leq 1$, $0 < m < \infty$, $p = \frac{m}{1-\theta}$, $0 < q, r \leq \infty$, and $a, b \in SV$. Then,

\[
(L_m, L_\infty)_{\theta, r, b, q, a} = (L_{m, \infty}, L_\infty)_{\theta, r, b, q, a} = L_{p, r, b, q, a}.
\]

In particular,

\[
(L_m, L_\infty)_{\theta, r, b, q} = (L_{m, \infty}, L_\infty)_{\theta, r, b, q} = L_{p, r, b, q}.
\]

Theorem 5.3. Let $0 < \theta \leq 1$, $0 < m < \infty$, $p = \frac{m}{1-\theta}$, $0 < q, r, s \leq \infty$, and $a, b, c \in SV$. Then,

\[
(L_m, L_\infty)_{\theta, s, c, r, b, q, a} = (L_{m, \infty}, L_\infty)_{\theta, s, c, r, b, q, a} = L_{\ell, \ell, p, s, c, r, b, q, a}.
\]

Proof. Put $X = (A_0, L_\infty)_{\theta, s, c, r, b, q, a}$, where $A_0 = L_m$ or $A_0 = L_{m, \infty}$. By Lemma 4.2 and the changes of variables $y = u^m$, $x = t^m$, we obtain

\[
\|f\|_X \approx \left\| t^{-1} c(t) \|u^{-\frac{1}{r} b(u)\| u^{\frac{1}{p} - \frac{1}{q} \bar{a}(u) f^*(u)\| q,(0,u^m)\| r,(0,0)\| s,(0,\infty)\| \right. \|
\]

This completes the proof.

5.2 | $R$ and $RR$ spaces

Theorem 5.4. Let $0 < \theta \leq 1$, $0 < m < \infty$, $p = \frac{m}{1-\theta}$, $0 < q, r \leq \infty$, and $a, b \in SV$. Then,

\[
(L_m, L_\infty)_{\theta, r, b, q, a} = (L_{m, \infty}, L_\infty)_{\theta, r, b, q, a} = L_{p, r, b, q, a}.
\]

In particular,

\[
(L_m, L_\infty)_{\theta, r, b, q} = (L_{m, \infty}, L_\infty)_{\theta, r, b, q} = L_{p, r, b, q}.
\]

Proof. Put $X = (A_0, L_\infty)_{\theta, r, b, q, a}$, where $A_0 = L_m$ or $A_0 = L_{m, \infty}$. Lemma 4.3 and the change of variables $x = t^m$ imply that

\[
\|f\|_X \approx \left\| t^{-1} b(t) \|u^{-\frac{1}{r} \bar{a}(u) f^*(u)\| q,(0,0)\| r,(0,0)\| s,(0,\infty)\| \right. \|
\]

This completes the proof.
On the other hand, by (4.1), (4.5), and suitable changes of variables, it follows that

\[ \|f\|_X \lesssim \left\| t^{-\frac{1}{\tau}} b(t) \right\| u^{-\frac{1}{q} - \frac{1}{\tau}} a(u) \left( \int_0^{\mu} f^*(\tau)^m d\tau \right)^{1/m} \|q,(t,\infty)\|_{r,(0,\infty)} \]

\[ = \left\| t^{-\frac{1}{\tau}} b(t) \right\| u^{-\frac{\theta - m}{q}} a(u)^m \left( \int_0^{\mu} f^*(\tau)^m d\tau \right)^{1/m} \|q,(m,t,\infty)\|_{r,(0,\infty)} \]

\[ \approx \left\| t^{-\frac{1}{\tau}} b(t) \right\| y^{-\frac{\theta - m}{q}} a(y)^m \left( \int_0^y f^*(\tau)^m d\tau \right)^{1/m} \|q,(m,t,\infty)\|_{r,(0,\infty)} \]

\[ \approx \left\| x^{-\frac{1}{\tau}} b(x) \right\| y^{-\frac{\theta - m}{q}} a(y)^m \left( \int_0^y f^*(\tau)^m d\tau \right)^{1/m} \|q,(m,x,\infty)\|_{r,(0,\infty)} \]

\[ = \left\| x^{-\frac{m}{\tau}} b(x)^m \right\| y^{-\frac{\theta - m}{q}} a(y)^m \left( \int_0^y f^*(\tau)^m d\tau \right)^{1/m} \|q,(m,x,\infty)\|_{r,(0,\infty)} \].

Now, the previous estimate and Corollary 2.6 give

\[ \|f\|_X \lesssim \left\| x^{-\frac{m}{\tau}} b(x)^m \right\| y^{1-\frac{\theta - m}{q}} a(y)^m f^*(y)^m \|q,(m,x,\infty)\|_{r,(0,\infty)} \]

\[ = \left\| x^{-\frac{1}{\tau}} b(x) \right\| y^{1-\frac{\theta - m}{q}} a(y)^m f^*(y)^m \|q,(m,x,\infty)\|_{r,(0,\infty)} \]

\[ = \left\| x^{-\frac{1}{\tau}} b(x) \right\| y^{1-\frac{\theta - m}{q}} a(y) f^*(y) \|q,(x,\infty)\|_{r,(0,\infty)} = \|f\|_{L^R_{p,s,r,b,q,a}}. \]

This completes the proof. □

**Theorem 5.5.** Let \(0 < \theta \leq 1\), \(0 < m < \infty\), \(p = \frac{m}{1-\theta}\), \(0 < q, s, r \leq \infty\), and \(a, b, c \in SV\). Then,

\[ (L_m, L_\infty)_{\theta,s,c,r,b,q,a} = (L_m, L_\infty)_{\theta,s,c,r,b,q,a}^{R,R} = L_{R,R}^{p,s,r,b,q,a}. \]

**Proof.** Put \(X = (A_0, L_\infty)_{\theta,s,c,r,b,q,a}^{R,R}\), where \(A_0 = L_m\) or \(A_0 = L_m, \infty\). Using Lemma 4.3 and the changes of variables \(u^m = y\), \(t^m = x\), we obtain the desired lower bound

\[ \|f\|_X \gtrsim \left\| \frac{1}{t^{\theta}} c(t) \right\| u^{-\frac{1}{\tau}} b(u) \left( \int_0^{\mu} f^*(\tau)^m d\tau \right)^{1/m} \|q,(u,\infty)\|_{r,(0,\infty)} \]

\[ \approx \left\| x^{-\frac{1}{\tau}} c(x) \right\| y^{-\frac{1}{\tau}} b(y) \left( \int_0^{y} f^*(\tau)^m d\tau \right)^{1/m} \|q,(y,\infty)\|_{r,(0,\infty)} = \|f\|_{L^R_{p,s,r,b,q,a}}. \]

Now, we proceed with the upper bound. Using (4.1), (4.5), and making the changes of variables \(v^m = z\), \(u^m = y\), and \(t^m = x\), we deduce that

\[ \|f\|_X \lesssim \left\| \frac{1}{t^{\theta}} c(t) \right\| u^{-\frac{1}{\tau}} b(u) \left( \int_0^{\mu} f^*(\tau)^m d\tau \right)^{1/m} \|q,(u,\infty)\|_{r,(0,\infty)} \]

\[ = \left\| \frac{1}{t^{\theta}} c(t)^m \right\| u^{-\frac{m}{\tau}} b(u)^m \left( \int_0^{\mu} f^*(\tau)^m d\tau \right)^{1/m} \|q,(u,\infty)\|_{r,(0,\infty)} \]

\[ \approx \left\| x^{-\frac{1}{\tau}} c(x)^m \right\| y^{-\frac{1}{\tau}} b(y)^m \left( \int_0^{y} f^*(\tau)^m d\tau \right)^{1/m} \|q,(y,\infty)\|_{r,(0,\infty)} \].
Corollary 2.7 yields the estimate
\[
\|f\|_X \lesssim \left\| x^{-\frac{1}{\theta}} \tilde{c}(x)^m \right\|_{L^\infty(Y)} \left\| y^{-\frac{1}{\gamma}} \tilde{b}(y)^n \right\|_{L^\infty(Z)} \left\| z^{\frac{1}{\alpha}} - \frac{m}{\theta} \tilde{a}(z)^m f^\ast(z)^m \right\|_{L^\infty_m(Y,\infty)} \left\| \frac{1}{m} \right\|_{L^\infty_m(0,\infty)} = \|f\|_{L_{p,\theta,\gamma}^{\infty}}.
\]

5.3 | \(L^p\) and \(R^L\) spaces

Theorem 5.6. Let \(0 < \theta \leq 1, 0 < m < \infty, p = \frac{m}{1-\theta}, 0 < q, r, s \leq \infty, \) and \(a, b, c \in \mathcal{S}V\) with \(b\) satisfying \(\|u^{-\frac{1}{\gamma}}b(u)\|_{R(1,\infty)} < \infty\). Then,
\[
(L_m, L_{\infty})_{\theta, s, r, b, q, a} = (L_m, L_{\infty})_{\theta, s, r, b, q, a} = L_{p, \theta, \gamma}^{L, R}(\mathcal{S}V, R, b, q, a).
\]

Proof. First note that the change of variables \(y = u^m\) and \(x = t^m\) shows that
\[
\|f\|_{L_{p,\theta,\gamma}^{L, R}} = \left\| x^{-\frac{1}{\theta}} \tilde{c}(x) \right\|_{L^\infty(Y)} \left\| y^{-\frac{1}{\gamma}} \tilde{b}(y)^{\frac{1}{p}} \right\|_{L^\infty(Z)} \left\| z^{\frac{1}{\alpha}} \tilde{a}(z)^m f^\ast(z)^m \right\|_{L^\infty_m(Y,\infty)} \left\| \frac{1}{m} \right\|_{L^\infty_m(0,\infty)} \approx \|f\|_{L_{p,\theta,\gamma}^{L, R}}.
\]

Put \(X = (A_0, L_{\infty})_{\theta, \gamma}^{L, R}\), where \(A_0 = L_m\) or \(A_0 = L_{\infty}\). By Lemma 4.3 and (5.1), we have that
\[
\|f\|_X \approx \left\| t^{-\frac{1}{\theta}} c(t) \right\|_{L^\infty(Y)} \left\| u^{-\frac{1}{\gamma}} b(u) \right\|_{L^\infty(Z)} \left\| z^{\frac{1}{\alpha}} \tilde{a}(z)^m f^\ast(z)^m \right\|_{L^\infty_m(Y,\infty)} \left\| \frac{1}{m} \right\|_{L^\infty_m(0,\infty)} \approx \|f\|_{L_{p,\theta,\gamma}^{L, R}}.
\]

Next, we proceed with the reverse estimate. Lemma 4.4 implies that
\[
\|f\|_X \approx \left\| t^{-\frac{1}{\theta}} c(t) \right\|_{L^\infty(Y)} \left\| u^{-\frac{1}{\gamma}} b(u) \right\|_{L^\infty(Z)} \left\| z^{\frac{1}{\alpha}} \tilde{a}(z)^m f^\ast(z)^m \right\|_{L^\infty_m(Y,\infty)} \left\| \frac{1}{m} \right\|_{L^\infty_m(0,\infty)} \approx \|f\|_{L_{p,\theta,\gamma}^{L, R}}.
\]

By (5.1), we have that \(S_2 \approx \|f\|_{L_{p,\theta,\gamma}^{L, R}}\), so to finish the proof it suffices to establish that \(S_1 \lesssim S_2\). Indeed, by Lemma 4.2, Lemma 2.3, and the changes of variables \(x = t^m, y = u^m,\) and \(w = x/2,\) we have
\[
S_1 = \left\| t^{-\frac{1}{\theta}} c(t) \right\|_{L^\infty(Y)} \left\| u^{-\frac{1}{\gamma}} b(u) \right\|_{L^\infty(Z)} \left\| z^{\frac{1}{\alpha}} \tilde{a}(z)^m f^\ast(z)^m \right\|_{L^\infty_m(Y,\infty)} \left\| \frac{1}{m} \right\|_{L^\infty_m(0,\infty)} = S_1 + S_2.
\]

By (5.1), we have that \(S_2 \approx \|f\|_{L_{p,\theta,\gamma}^{L, R}}\), so to finish the proof it suffices to establish that \(S_1 \lesssim S_2\). Indeed, by Lemma 4.2, Lemma 2.3, and the changes of variables \(x = t^m, y = u^m,\) and \(w = x/2,\) we have
\[
S_1 = \left\| t^{-\frac{1}{\theta}} c(t) \right\|_{L^\infty(Y)} \left\| u^{-\frac{1}{\gamma}} b(u) \right\|_{L^\infty(Z)} \left\| z^{\frac{1}{\alpha}} \tilde{a}(z)^m f^\ast(z)^m \right\|_{L^\infty_m(Y,\infty)} \left\| \frac{1}{m} \right\|_{L^\infty_m(0,\infty)} = S_1 + S_2.
\]
\[ \begin{align*}
\leq & \left\| x^{-\frac{1}{q}} c(x) \right\| L_{x}^{\frac{1}{q}, y} \left\| y^{-\frac{1}{q}} b(y) \right\| L_{y}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \tilde{a}(z) f^*(z) \right\| L_{z}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \right\| s, (0, \infty) \\
\leq & \left\| x^{-\frac{1}{q}} c(x) \right\| L_{x}^{\frac{1}{q}, y} \left\| y^{-\frac{1}{q}} b(y) \right\| L_{y}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \tilde{a}(z) f^*(z) \right\| L_{z}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \right\| s, (0, \infty) \\
\leq & \left\| x^{-\frac{1}{q}} c(x) \right\| L_{x}^{\frac{1}{q}, y} \left\| y^{-\frac{1}{q}} b(y) \right\| L_{y}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \tilde{a}(z) f^*(z) \right\| L_{z}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \right\| s, (0, \infty) \\
\approx & \left\| u^{-\frac{1}{q}} c(u) \right\| L_{u}^{\frac{1}{q}, y} \left\| y^{-\frac{1}{q}} b(y) \right\| L_{y}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \tilde{a}(z) f^*(z) \right\| L_{z}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \right\| s, (0, \infty) \\
= & \left\| f \right\| L_{u}^{\frac{1}{q}, y} \left\| y^{-\frac{1}{q}} b(y) \right\| L_{y}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \tilde{a}(z) f^*(z) \right\| L_{z}^{\frac{1}{q}, y} \left\| z^{-\frac{1}{q}} \right\| s, (0, \infty).
\end{align*} \]

The proof is completed. \(\square\)

**Theorem 5.7.** Let \(0 < \theta < 1\), \(0 < m < \infty\), \(p = \frac{m}{1-\theta}\), \(0 < q, r, s \leq \infty\), and \(a, b, c \in SV\). Then,

\[
(L_m, L_\infty)^{R, L}_{\theta, s, c, r, b, q, a} = (L_m, L_\infty)^{R, L}_{\theta, s, c, r, b, q, a} = L^{R, L}_{p, q, r, b, q, a}.
\]

**Proof.** In this case, the changes of variables \(y = u^m\) and \(x = t^m\) show that

\[
\left\| f \right\| s, (0, \infty) = \left\| t^{-\frac{1}{q}} c(t) \right\| u^{-\frac{1}{q}} b(u) \left\| v^{-\frac{1}{q}} a(v) K(v, f; A_0, L_\infty) \right\| s, (0, \infty) \approx \left\| f \right\| R, c, p, s, c, r, b, q, a.
\]

Next, we prove the reverse estimate. Lemma 4.4 yields that

\[
\left\| f \right\| s, (0, \infty) \lesssim \left\| t^{-\frac{1}{q}} c(t) \right\| u^{-\frac{1}{q}} b(u) \left\| v^{-\frac{1}{q}} a(v) K(v, f; A_0, L_\infty) \right\| s, (0, \infty) \approx \left\| f \right\| R, c, p, s, c, r, b, q, a.
\]

By (5.2), we have that \(T_2 \approx \left\| f \right\| L^{R, L}_{p, q, r, b, q, a}\), so to finish the proof it is enough to obtain that \(T_1 \lesssim T_2\). Indeed, by Lemma 4.2 and Lemma 2.3, we have

\[
T_1 := \left\| t^{-\frac{1}{q}} c(t) \right\| u^{-\frac{1}{q}} b(u) \left\| v^{-\frac{1}{q}} a(v) K(v, f; A_0, L_\infty) \right\| s, (0, \infty) \approx \left\| t^{-\frac{1}{q}} c(t) \right\| y^{-\frac{1}{q}} b(y) \left\| y^{-\frac{1}{q}} a(y) f^*(y) \right\| s, (0, \infty)
\]

\[
\approx \left\| t^{-\frac{1}{q}} c(t) \right\| y^{-\frac{1}{q}} b(y) \left\| y^{-\frac{1}{q}} a(y) f^*(y) \right\| s, (0, \infty)
\]

\[
\approx \left\| t^{-\frac{1}{q}} c(t) \right\| y^{-\frac{1}{q}} b(y) \left\| y^{-\frac{1}{q}} a(y) f^*(y) \right\| s, (0, \infty)
\]
This completes the proof. □

6 | INTERPOLATION FORMULAE FOR THE COUPLES \((L_1, L_\infty)\) AND \((L_{1, \infty}, L_\infty)\)

In this section, taking \(m = 1\) in the results of Section 5, we are in a position to formulate the main theorems of the paper. First, as a direct consequence of Lemma 4.2, we obtain the following corollary whose proof is omitted:

**Corollary 6.1.** Let \(0 < \theta \leq 1, p = \frac{1}{1 - \theta}, 0 < q \leq \infty,\) and \(a \in SV.\) Then, for all \(f \in L_1 + L_\infty\) and all \(t > 0,\)

\[
\left\| u^{-\frac{1}{q}} a(u) K(u, f; L_1, L_\infty) \right\|_{q,(0,t)} = \left\| u^{-\frac{1}{q}} a(u) f^{**}(u) \right\|_{q,(0,t)} \approx \left\| v^{-\frac{1}{q}} a(v) f^{*}(v) \right\|_{q,(0,t)}
\]

and for all \(f \in L_{1,\infty} + L_\infty\) and all \(t > 0,\)

\[
\left\| u^{-\frac{1}{q}} a(u) K(u, f; L_1, L_{\infty}) \right\|_{q,(0,t)} \approx \left\| v^{-\frac{1}{q}} a(v) f^{*}(v) \right\|_{q,(0,t)}.
\]

Theorems 5.1 to 5.7, Corollary 6.1, and Lemmas 3.13 and 3.14 imply the following results.

**Theorem 6.2** (Cf. [28, (5.16)], [40, Proposition 3], and [36, Theorem 3.15]). Let \(0 < \theta \leq 1, p = \frac{1}{1 - \theta}, 0 < q \leq \infty\) and \(a \in SV.\) Then,

\[(L_1, L_\infty)_{\theta,q,a} = (L_{1,\infty}, L_\infty)_{\theta,q,a} = L_{(p,q,a)} = L_{p,q,a}.
\]

**Theorem 6.3** (Cf. [28, Lemma 5.4]). Let \(0 < \theta \leq 1, p = \frac{1}{1 - \theta}, 0 < q, r \leq \infty,\) and \(a, b \in SV.\) Then,

\[(L_1, L_{\infty})_{\theta,r,b,q,a}^c = (L_{1,\infty}, L_\infty)_{\theta,r,b,q,a}^c = (L)_{p,r,b,q,a}^c = L_{p,r,b,q,a}^c.
\]

In particular,

\[(L_1, L_{\infty})_{\theta,r,b,q,a}^c = (L_{1,\infty}, L_\infty)_{\theta,r,b,q,a}^c = (L)_{b}^{(p,q,r)} = L_{b}^{(p,q,r)}.
\]

This proves that the expression of the norm in the small Lorentz–Karamata spaces produces the same space regardless of whether we define it with \(f^*\) or \(f^{**}.\) See [25, Corollary 3.3] for a similar result in the context of small Lebesgue spaces.

**Theorem 6.4.** Let \(0 < \theta \leq 1, p = \frac{1}{1 - \theta}, 0 < q, r \leq \infty,\) and \(a, b \in SV.\) Then,

\[(L_1, L_{\infty})_{\theta,r,b,q,a}^R = (L_{1,\infty}, L_\infty)_{\theta,r,b,q,a}^R = (L)_{p,r,b,q,a}^R = L_{p,r,b,q,a}^R.
\]

In particular,

\[(L_1, L_{\infty})_{\theta,r,b,q,a}^R = (L_{1,\infty}, L_\infty)_{\theta,r,b,q,a}^R = (L)_{b}^{(p,q,r)} = L_{b}^{(p,q,r)}.
\]
Thus, the quasi-norm on grand Lorentz–Karamata spaces can be defined through $f^*$ or $f^{**}$ producing the same function space with equivalence of quasi-norms. See [25, Theorem 4.2] for a similar result in the context of grand Lebesgue spaces.

**Theorem 6.5.** Let $0 < \theta \leq 1$, $p = \frac{1}{1-\theta}$, $0 < q, r, s \leq \infty$, and $a, b, c \in SV$. Then,

$$(L_1, L_\infty)^{\mathcal{E}, \mathcal{E}, \mathcal{E}}_{\theta, s, c, r, b, q, a} = (L_1, L_\infty)^{\mathcal{E}, \mathcal{E}, \mathcal{E}}_{\theta, s, c, r, b, q, a} = (L)^{\mathcal{E}, \mathcal{E}}_{p(s, c, r, b, q, a)} = L^{\mathcal{E}, \mathcal{E}}_{p(s, c, r, b, q, a)}$$

and

$$(L_1, L_\infty)^{R, R}_{\theta, s, c, r, b, q, a} = (L_1, L_\infty)^{R, R}_{\theta, s, c, r, b, q, a} = (L)^{R, R}_{p(s, c, r, b, q, a)} = L^{R, R}_{p(s, c, r, b, q, a)}.$$ 

**Theorem 6.6.** Let $0 < \theta \leq 1$, $p = \frac{1}{1-\theta}$, $0 < q, r, s \leq \infty$, and $a, b, c \in SV$. Then,

$$(L_1, L_\infty)^{\mathcal{E}, \mathcal{E}}_{\theta, s, c, r, b, q, a} = (L_1, L_\infty)^{\mathcal{E}, \mathcal{E}}_{\theta, s, c, r, b, q, a} = (L)^{\mathcal{E}, \mathcal{E}}_{p(s, c, r, b, q, a)} = L^{\mathcal{E}, \mathcal{E}}_{p(s, c, r, b, q, a)}$$

and

$$(L_1, L_\infty)^{R, \mathcal{E}}_{\theta, s, c, r, b, q, a} = (L_1, L_\infty)^{R, \mathcal{E}}_{\theta, s, c, r, b, q, a} = (L)^{R, \mathcal{E}}_{p(s, c, r, b, q, a)} = L^{R, \mathcal{E}}_{p(s, c, r, b, q, a)}.$$ 

7 \ APPLICATIONS

Here, we show how our general assertions can be used to establish interpolation results for grand and small Lorentz–Karamata spaces. For the sake of shortness, we present only two examples. Similar examples can be seen in [13–16, 22–24].

We interpolate $\mathcal{E}$ and $R$ Lorentz–Karamata spaces with $L_\infty$ using reiteration techniques.

**Corollary 7.1** (Cf. [22, Theorem 4.6]). Let $0 < p_0 < \infty$, $0 < r, r_0, q_0 \leq \infty$, and $a, a_0, b_0 \in SV$ with $b_0$ satisfying $\|u^{1/r_0}b_0(u)\|_{r_0(0,1)} < \infty$. Let $\chi(t) = a_0(t)\|u^{1/r_0}b_0(u)\|_{r_0(0,1)}$ and $\rho(t) = t^{1/p_0}\chi(t)$, $t > 0$.

(i) If $\|u^{-1/r}a(u)\|_{r,(0,\infty)} < \infty$, then

$$\left[ L^{R}_{\rho_0, r_0, b_0, q_0, a_0}, L_\infty \right]_{0, \rho; a}^{\rho_0, r_0, b_0, q_0, a_0} = L^{R, \mathcal{E}}_{\rho_0, r_0, b_0, q_0, a_0} = L^{R, \mathcal{E}}_{\rho_0, r_0, b_0, q_0, a_0}.$$ 

(ii) If $0 < \theta < 1$, then

$$\left[ L^{R}_{\rho_0, r_0, b_0, q_0, a_0}, L_\infty \right]_{\theta, r; a}^{\rho_0, r_0, b_0, q_0, a_0} = L_{\theta, r; a}.$$

where $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $a^\theta = \chi^{1-\theta}a_0\rho$.

(iii) If $\|u^{-1/r}a(u)\|_{r,(0,1)} < \infty$, then

$$\left[ L^{R}_{\rho_0, r_0, b_0, q_0, a_0}, L_\infty \right]_{1, \rho; a}^{\rho_0, r_0, b_0, q_0, a_0} = L_{\infty, r; a_0}.$$

**Proof.** Choose $m$ such that $0 < m < \min(p_0, q_0)$. Put $\theta_0 = 1 - \frac{m}{p_0}$ and $X = \left[ L^{R}_{\rho_0, r_0, b_0, q_0, a_0}, L_\infty \right]_{\theta_0, r; a}$. By Theorem 5.4, we know that

$$L^{R}_{\rho_0, r_0, b_0, q_0, a_0} = (L_m, L_\infty)^{R}_{\rho_0, r_0, b_0(t^m), q_0, a_0(t^m)}.$$ 

Thus,

$$X = \left( (L_m, L_\infty)^{R}_{\rho_0, r_0, b_0(t^m), q_0, a_0(t^m)}, L_\infty \right)_{\theta_0, r; a}.$$
Case $0 < \theta \leq 1$. By [13, Theorem 23] and Theorem 5.1, we have

$$X = (L_m, L_\infty)^{\eta, \mu, y} = L_{p, \mu}^\chi,$$

where $\eta = (1 - \theta)\theta + \theta \cdot \frac{1 - \eta}{m}$,

$$y(t) = \left( a_0(t^m)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t)} \right)^{1-\theta} a \left( t^{1-\theta_0} a_0(t^m)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t)} \right),$$

and

$$x(t) = y(t^\frac{1}{m}) = \left( a_0(t)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t^\frac{1}{m})} \right)^{1-\theta} a \left( t^{1-\theta_0} a_0(t)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t^{\frac{1}{m}})} \right).$$

Note that $\frac{1-\theta_0}{m} = \frac{1-\eta}{p_0}$, $\frac{1-\eta}{m} = 1 - \theta \frac{1-\eta}{p_0}$ and $a_0(t)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t)} \approx \chi(t)$. Thus, $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $x(t) \approx \chi(t)^{1-\theta} a \left( t^{1/p_0} \chi(t) \right) = a^n(t)$.

Case $\theta = 0$. By [14, Theorem 28] and Theorem 5.7, we arrive at

$$X = (L_m, L_\infty)^{R, \ell}_{p_0, r, a_0, b_0(r)} = L_{p_0, r, a_0, b_0, q_0, a_0}^{R, \ell},$$

where

$$x(t) = a \left( t^{1-\theta_0} a_0(t^m)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t)} \right),$$

and hence,

$$\bar{x}(t) = a \left( t^{1-\theta} a_0(t)\|u^{-1/r}b_0(u^m)\|_{r_0(0,t^{\frac{1}{m}})} \right) \approx a \left( t^{1/p_0} \chi(t) \right) = (a \circ \rho)(t).$$

This completes the proof. \hfill \qed

In view of Remark 3.10, we obtain the following statement.

**Corollary 7.2** (Cf. [22, Corollary 5.9], [13, Corollary 49], and [14, Corollary 62]). Let $0 < p_0 < \infty$, $0 < r, r_0, q_0 \leq \infty$, and $a, b_0 \in SV$ with $b_0$ satisfying that $\|u^{-1/r}b_0(u)\|_{r_0, (0,1)} < \infty$. Let $\chi(t) = \|u^{-1/r_0}b_0(u)\|_{r_0, (0,t)}$ and $\rho(t) = t^{1/p_0} \chi(t)$, $t > 0$.

(i) If $\|u^{-1/r}a(u)\|_{r, (1, \infty)} < \infty$, then

$$\left( L_{b_0}^{p_0, q_0, r_0}, L_\infty \right)_{\theta, r, a} = L_{p_0, r, a_0, b_0, q_0, a_0}^{R, \ell}.$$

(ii) If $0 < \theta < 1$, then

$$\left( L_{b_0}^{p_0, q_0, r_0}, L_\infty \right)_{\theta, r, a} = L_{p_0, r, a_0, b_0, q_0, a_0}^{R, \ell},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $a^n = \chi^{1-\theta} a \circ \rho$.

(iii) If $\|u^{-1/r}a(u)\|_{r, (0,1)} < \infty$, then

$$\left( L_{b_0}^{p_0, q_0, r_0}, L_\infty \right)_{\theta, r, a} = L_{p_0, r, a_0, b_0, q_0, a_0}^{R, \ell}.$$

Next, two corollaries can be proved similarly by using Theorems 5.1 to 5.3, [13, Theorem 13], and [14, Theorem 25].

**Corollary 7.3** (Cf. [24, Theorem 4.10]). Let $0 < p_0 < \infty$, $0 < r, r_0, q_0 \leq \infty$, and $a, a_0, b_0 \in SV$ with $b_0$ satisfying $\|u^{-1/r_0}b_0(u)\|_{r_0, (1, \infty)} < \infty$. Let $\chi(t) = a_0(t)\|u^{-1/r_0}b_0(u)\|_{r_0, (t, \infty)}$ and $\rho(t) = t^{1/p_0} \chi(t)$, $t > 0$. 

(i) If \( \|u^{-1/r}a(u)\|_{r,(1,\infty)} < \infty \), then
\[
\left( L^r_{p_0,\rho_0,b_0,q_0,a_0}, L^\infty \right)_{0,r,a} = L^r_{p_0,\rho_0,b_0,q_0,a_0} \cap L^r_{p_0,\rho_0,b_0,q_0,a_0},
\]
where \( a^{##}(t) = \|u^{-1/r_0}b_0(u)\|_{r_0,(1,\infty)}a_0 \rho(t), t > 0 \).

(ii) If \( 0 < \theta < 1 \), then
\[
\left( L^r_{p_0,\rho_0,b_0,q_0,a_0}, L^\infty \right)_{\theta,r,a} = L^r_{p_0,\rho_0,a^#},
\]
where \( \frac{1}{p} = \frac{1-\theta}{p_0} \) and \( a^# = \chi^{1-\theta}a_0 \rho \).

(iii) If \( \|u^{-1/r}a(u)\|_{r,(0,1)} < \infty \), then
\[
\left( L^r_{p_0,\rho_0,b_0,q_0,a_0}, L^\infty \right)_{1,r,a} = L^\infty_{r,a_0 \rho}.
\]

The following result extends [13, Corollary 40] and [14, Corollary 60] (Cf. also [22, Corollary 5.12]).

**Corollary 7.4.** Let \( 0 < p_0 < \infty, 0 < r, r_0, q_0 \leq \infty \), and \( a, b_0 \in SV \) with \( b_0 \) satisfying \( \|u^{-1/r_0}b_0(u)\|_{r_0,(1,\infty)} < \infty \). Let \( \chi(t) = \|u^{-1/r_0}b_0(u)\|_{r_0,(t,\infty)} \) and \( \rho(t) = t^{1/p_0}\chi(t), t > 0 \).

(i) If \( \|u^{-1/r}a(u)\|_{r,(1,\infty)} < \infty \), then
\[
\left( L^{(p_0,q_0,r_0)}_{b_0}, L^\infty \right)_{0,r,a} = L^{(p_0,q_0,r_0)}_{b_0} \cap L^r_{p_0,\rho_0,b_0,q_0,a_0},
\]
where \( a^{##} = \chi a_0 \rho \).

(ii) If \( 0 < \theta < 1 \), then
\[
\left( L^{(p_0,q_0,r_0)}_{b_0}, L^\infty \right)_{\theta,r,a} = L^{r}_{p_0,\rho_0,a^#},
\]
where \( \frac{1}{p} = \frac{1-\theta}{p_0} \) and \( a^# = \chi^{1-\theta}a_0 \rho \).

(iii) If \( \|u^{-1/r}a(u)\|_{r,(0,1)} < \infty \), then
\[
\left( L^{(p_0,q_0,r_0)}_{b_0}, L^\infty \right)_{1,r,a} = L^\infty_{r,a_0 \rho}.
\]

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**REFERENCES**

[1] I. Ahmed, D. E. Edmunds, W. D. Evans, and G. E. Karadzhov, *Reiteration theorems for the K-interpolation method in limiting cases*, Math. Nachr. 284 (2011), no. 4, 421–442.

[2] I. Ahmed, A. Fiorenza, and M. R. Formica, *Interpolation of generalized gamma spaces in a critical case*, J. Fourier Anal. Appl. 28 (2022), no. 3, Paper No. 54, 23.

[3] I. Ahmed, A. Fiorenza, and A. Hafeez, *Some interpolation formulae for grand and small Lorentz spaces*, Mediterr. J. Math. 17 (2020), no. 2, Art. 57, 21.

[4] M. A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. 320 (1990), no. 2, 727–735.

[5] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces*, Dissertationes Math. 175 (1980).

[6] C. Bennett and R. Sharpley, *Interpolation of operator*, Academic Press, Boston, MA, 1988.

[7] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer, Berlin-Heidelberg-New York, 1976.
Interpolation of linear operators between $L_p$ and the theorem of Marcinkiewicz, Studia Math. 26 (1966), 273–299.

M. Cwikel, A. Kamińska, L. Maligranda, and L. Pick, Are generalized Lorentz “spaces” really spaces? Proc. Amer. Math. Soc. 132 (2004), no. 12, 3615–3625.

R. Ya. Doktorskii, Reiteration relations of the real interpolation method, Soviet Math. Dokl. 44 (1992), 665–669.

L. Ya. Doktorski, An application of limiting interpolation to Fourier series theory, in The Diversity and Beauty of Applied Operator Theory. Operator Theory: Advances and Applications, A. Böttcher, P. Stollmann, and D. Wenzel, Eds., vol. 268, Birkhäuser, Cham, Basel, 2018.

L. Ya. Doktorski, Reiteration formulae for the real interpolation method including $L$ or $R$ limiting spaces, J. Funct. Spaces (2020), Art. ID 6858993, 15.

L. Ya. Doktorski, Some reiteration theorems for $R$, $L$, $RR$, $RL$, $LR$, and $LL$ limiting interpolation spaces, J. Funct. Spaces (2021), Art. ID 8513304, 31.

L. Ya. Doktorski, P. Fernández-Martínez, and T. Sages, Reiteration Theorem for $R$ and $L$-spaces with the same parameter, J. Math. Anal. Appl. 508 (2022), Paper No. 125846, 32.

L. Ya. Doktorski, P. Fernández-Martínez, and T. Sages, Reiteration formula for the real interpolation method including limiting $L$ and $R$ spaces, J. Math. Anal. Appl. 508 (2022), no. 1, Paper No. 125846, 32.

E. Pustylnik, Ultrasymmetric spaces.

A. Gogatishvili, B. Opic, and W. Trebels, Limit reiteration for real interpolation with slowly varying functions, Quart. J. Math. 63 (2012), no. 1, 133–164.

P. Fernández-Martínez and T. Sages, An application of interpolation theory to renorming of Lorentz-Karamata type spaces, Ann. Acad. Sci. Fenn. Math. 39 (2014), no. 1, 97–107.

P. Fernández-Martínez and T. Sages, Limit cases of reiteration theorems, Math. Nachr. 288 (2015), no. 1, 25–47.

P. Fernández-Martínez and T. Sages, General reiteration theorems for $R$ and $L$ classes: case of left $R$-spaces and right $L$-spaces, J. Math. Anal. Appl. 494 (2021), no. 2, Paper No. 124649, 33.

P. Fernández-Martínez and T. Sages, General reiteration theorems for $R$ and $L$ classes: mixed interpolation of $R$ and $L$-spaces, Positivity 26 (2022), no. 3, Paper No. 47, 45.

P. Fernández-Martínez and T. Sages, General reiteration theorems for $R$ and $L$ classes: case of right $R$-spaces and left $L$-spaces, Mediterr. J. Math. 19 (2022), Paper No. 193, 36.

A. Fiorenza and G. E. Karadzhov, Grand and small Lebesgue spaces and their analogs, Z. Anal. Anwend. 23 (2004), 657–681.

A. Fiorenza, M. R. Formica, and A. Gogatishvili, On grand and small Lebesgue and Sobolev spaces and some applications to PDE’s, Differ. Equ. Appl. 10 (2018), no. 1, 21–46.

A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kojaliiani, and J. M. Rakotoson, Characterization of interpolation between grand, small or classical Lebesgue spaces, Nonlinear Anal. 177 (2018), part B, 422–453.

A. Gogatishvili, B. Opic, and W. Trebels, Limit reiteration for real interpolation with slowly varying functions, Math. Nachr. 278 (2005), no. 1–2, 86–107.

T. Holmstedt, Interpolation of quasi-normed spaces, Math. Scand. 26 (1970), 177–199.

P. Krée, Interpolation d’espaces qui ne sont ni normés, ni complets. Applications, Ann. Inst. Fourier 17 (1967), 137–174.

S. G. Krein, Ju. I. Petunin, and E. M. Semenov, Interpolation of linear operators, Translations of Math. Monographs, vol. 54, Amer. Math. Soc., Providence, RI, 1982.

S. Lai, Weighted norm inequalities for general operators on monotone functions, Trans. Amer. Math. Soc. 340 (1993), no. 2, 811–836.

G. G. Lorentz, On the theory of spaces $L$, Pacific J. Math. 1 (1951), 411–429.

J. S. Neves, Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings, Dissertationes Math. (Rozprawy Mat.) 405 (2002), 46.

B. Opic and L. Pick, On generalized Lorentz-Zygmund spaces, Math. Ineq. Appl. 2 (1999), 391–467.

D. Peña, Lorentz-Karamata spaces. https://arxiv.org/abs/2006.14455v2, 2021.

E. Pustylnik, Ultrasymmetric spaces, J. London Math. Soc. (2) 68 (2003), 165–182.

Y. Sager, Interpolation of $p$-Banach spaces, Stud. Math. 41 (1972), no. 1, 45–70.

E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, Studia Math. 96 (1990), 145–158.

K. Yoshinaga, Lorentz spaces and the Calderón-Zygmund theorem, Bull. Kyushu Inst. Tech. 16 (1969), 1–38.

H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland, Amsterdam, 1978.

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