

CONNECTIVITY OF THE GROMOV BOUNDARY OF THE FREE FACTOR COMPLEX

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Abstract. We show that in large enough rank, the Gromov boundary of the free factor complex is path connected and locally path connected.

1. Introduction

A prevailing theme in geometric group theory is to study groups using actions on suitable Gromov hyperbolic spaces. One of the most successful examples of this philosophy is the action of the mapping class group of a surface on the curve graph, which is hyperbolic by the seminal work of Masur and Minsky [MM99], and which has been used to give a hierarchical description of the geometry of mapping class groups.

One core tool when studying Gromov hyperbolic spaces is that they admit natural boundaries at infinity. In the case of the curve graph, this boundary can be explicitly described in terms of topological objects. Namely, Klarreich [Kla] proved that the boundary is the space of ending laminations. That is, the boundary can be obtained from the sphere of projective measured laminations by removing all non-minimal laminations, and then identifying laminations with the same support. Although the Gromov boundaries of curve graphs are fairly complicated topological spaces (in the case of punctured spheres, they are Nöbeling spaces [Gab14]), the connection to laminations can be used to effectively study them. Maybe most relevant for our current work, Gabai [Gab09, Gab14] used this connection to show that the boundary is path-connected and locally path-connected.

In the setting of outer automorphism groups of free groups, there are several possible analogs of the curve graph. In this article, we focus on the free factor complex $\mathcal{FF}_n$, which is hyperbolic by a result of Bestvina-Feighn [BF14]. Similar to Klarreich’s theorem, the Gromov boundary has been identified as the space of arational trees modulo a suitable equivalence [BR15]. The role of the sphere of projective measured laminations is played by the boundary of Culler-Vogtmann’s Outer space (which has much more complicated local topology than a sphere).

In this article, we nevertheless begin a study of connectivity properties of the boundary at infinity of $\mathcal{FF}_n$. More specifically, we show

Theorem 1.1. The Gromov boundary $\partial \mathcal{FF}_n$ of the free factor complex is path connected and locally path connected for $n \geq 18$.

As an immediate consequence we obtain the following coarse geometric property:

Corollary 1.2. The free factor complex $\mathcal{FF}_n$ is one-ended for $n \geq 18$.

From this, one-endedness of various other combinatorial complexes (the free splitting complex, the cyclic splitting complex, and the maximal cyclic splitting complex) can also
be concluded (Corollary 6.3). We also emphasise that we do not claim that the constant 18 is optimal, and in fact expect that the result holds for much lower $n$ (see below).

1.1. Outline of proof. Our strategy is motivated by the case of $\mathcal{PML}$ and ending lamination space, though it requires new ideas. To link the case of $\mathcal{PML}$ to $\partial CV_n$, there is a dense connected set of copies of $\mathcal{PML}$ in $\partial CV_n$ coming from identifying the fundamental group of a surface with one boundary component with $F_n$. For simplicity, lets start ‘upstairs’ in $\partial CV_n$ and assuming that two arational trees, $T, T'$ lie on two different $\mathcal{PML}$s.

We have a chain of copies of $\mathcal{PML}$ that connects the $\mathcal{PML}$ containing $T$ to the one containing $T'$ and where each consecutive pair in the chain intersects in the $\mathcal{PML}$ of a subsurface. Using work from surface case, in particular [CH], which builds on [LS09], we build a path, $p_0$ across this chain of $\mathcal{PML}$s so that every tree in it is arational except at the intersection of the consecutive $\mathcal{PML}$s and at these it is the stable lamination of a (partial) Pseudo-Anosov supported on the subsurface. We wish to have a path entirely of arational trees, so we find a countable sequence of compact sets $K_1, K_2, ...$ so that the arational trees are the complement of $\bigcup K_j$. We iteratively improve our path $p_j$ which by inductive hypothesis

- avoids $K_1, ..., K_j$ entirely
- has that every foliation on it is minimal, except for a finite number of points
- these points are $\lambda_\psi$, the stable laminations of (partial) Pseudo-Anosovs, $\psi$, supported on a subsurface.

to a path $p_{j+1}$ which avoids $K_1, ..., K_j, K_{j+1}$ entirely and has the same structure as above. Most of the work of this paper is in avoiding $K_{j+1}$. (The fact that $p_{j+1}$ avoids $K_1, ..., K_j$ is a consequence of the fact that we can make $p_{j+1}$ as close as we want to $p_j$.)

We now discuss avoiding the $K_{j+1}$. First off, what are the $K_{j+1}$? They are the unions of the trees in $\partial CV_n$ where a fixed proper free factor $E_{j+1}$ is not both free and discrete. To rule this out, it suffices to show that our stable laminations of the partial Pseudo-Anosov is supported on a subsurface whose fundamental group is not contained in $E_{j+1}$ and that $E_{j+1}$ trivially intersects the fundamental group of the complement of the support of partial Pseudo-Anosov. See Lemma 2.21 (It is automatic that the rest of our path avoids $K_j$ because these points are already arational.) In Section 4 we describe how if one of our stable laminations $\lambda_\psi$ is in $K_{j+1}$ we can find a chain of $\mathcal{PML}$s that build a detour around it and moreover, the image of this path under a large power of $\psi$ also avoids $K_{j+1}$. This involves explicitly constructing relations in $\text{Out}(F_n)$, and is technically the most involved part of the paper.

In this way we can improve our path $p_j$ on a small segment around $\lambda_\psi$ to obtain $p_{j+1}$, which now avoids $K_{j+1}$ and where the contraction properties of $\psi$ guarantee that it still avoids $K_1, ..., K_j$. Our sequence of paths $p_0, p_1, ...$ is a Cauchy sequence and so we obtain in the limit $p_\infty$ which is in $(\bigcup K_j)\circ$. This describes an argument that the projection of the set of arational surface type trees to the $\partial FF_n$ is path connected. To upgrade this to showing that $\partial FF_n$ is path connected, we show (basically via Proposition 5.3 which uses folding paths) that we can choose sequences of surface type arational trees converging to our fixed (not necessarily surface type) tree so that the paths stay in a $\delta$ neighborhood of our arational tree. In the outline we ignored some subtleties, most notably how to construct the required relations in $\text{Out}(F_n)$, and that our “paths” in $\partial CV_n$ are not necessarily continuous at the end points, though the projection of these “paths” to $\partial FF_n$ will be. This last point exploits contraction dynamics on the boundary of the hyperbolic space $\partial FF_n$. 

We now briefly state the structure of the paper. Section 2 collects known results about $\mathcal{PML}$ and $\text{Out}(F_n)$ and modifies them for our purposes. Section 3 relates $\partial CV_n$ and $\text{Out}(F_n)$. Section 4 is the technical heart of our paper, describing how to locally avoid a fixed $K_j$. Section 5 proves the main theorem. Section 6 uses the main theorem to establish one-endedness of some other combinatorial complexes. There are three appendices that treat issues related to non-orientable surfaces, which we use to address $\partial FF_n$ when $n$ is odd.

1.2. Previous work. Our work is a direct analogue of Gabai’s work [Gab09] on the connectivity of the ending lamination space $\mathcal{EL}$, which is the Gromov boundary of the curve complex. He obtains optimal path connectivity results and establishes higher connectivity, where appropriate. He goes further and in [Gab14] identifies $\mathcal{EL}$ of punctured spheres with so called Nöbling spaces. This had previously been done in the case of the 5-times punctured sphere by Hensel and Przytycki [?].

However, there is one crucial difference in approaches: throughout his arguments, Gabai homotopes paths in $\mathcal{PML}$, which is a sphere. It is unclear how to do this in our setting, because the topology of $\partial CV_n$ is still poorly understood, and it is not even know if it is locally connected. In particular, our successive improvements of paths do not proceed by homotopy.

Perhaps a better analogy for our work is Leininger and Schleimer’s proof that $\mathcal{EL}$ is connected [LS09]. They do this by using “point pushing” to find a dense path connected set of arational laminations upstairs in $\mathcal{PML}$. Building on this, the second and third named authors use contraction properties of the mapping class group on the curve complex to show that the subsets of uniquely ergodic and of cobounded foliations in $\mathcal{PML}$ are both path connected [CH]. This motivates our approach and especially Proposition 5.3. However, there is again an important difference between the paths built in [LS09] or [CH], and the ones we construct here: in the former sources, the paths are often obtained from lower complexity surfaces by lifting along (branched) covers. Here, we do not have this option, and instead need to construct the paths directly.

Finally, in our setting, this previous work is of little help in getting between adjacent $\mathcal{PML}$s, where the new ideas of this article are needed.

1.3. Questions. We end this introduction with a short list of further questions which this work suggests.

1. Is the boundary $\partial FF_N$ already path-connected for $N \geq 3$? The bound 18 used here certainly carries no special significance, and is an artifact of the proof.
2. Do the boundaries $\partial FF_N$ satisfy (for large enough rank $N$) higher connectivity properties?
3. Are there topological models for the boundaries $\partial FF_N$? Most likely, this would involve showing that they satisfy other universal properties (dimension, locally finite $n$–disk properties)?
4. Is the set of arational trees in $\partial CV_n$ path-connected? We remark that the paths constructed between boundary points of the free factor complex do not yield paths in $\partial CV_n$, as continuity at the endpoints cannot be guaranteed. The corresponding question for $\mathcal{PML}$ is an open question of Gabai.
5. Is the set of points in $\partial FF_n$ which are not of surface type path-connected? The paths we construct contain subpaths each point of which is of surface type. Avoiding
this seems to require new ideas. We were made aware of this question by Camille Horbez.

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2. Out($F_n$) preliminaries

This section collects the necessary facts about (compactified) Culler-Vogtmann Outer space, related spaces where Out($F_n$) acts, and geodesic laminations on surfaces.

2.1. Outer Space. Throughout this article, any tree is understood to be a tree together with an isometric action of $F_n$. Recall that a tree is minimal if it does not contain a proper invariant subtree, and it is nontrivial if it does not have a global fixed point. Unless stated otherwise, all trees will be minimal and nontrivial. When $T$ is an $F_n$-tree and $a$ is an element or a conjugacy class in $F_n$, we write $(T,a)$ for the translation length of $a$ in $T$.

We denote by $cv_n$ (unprojectivized) Outer space in rank $n$, and we denote by $CV_n = cv_n/(0,\infty)$ projectivised Outer space. Points in these spaces correspond to (projectivised) free, simplicial, minimal $F_n$–trees. Compare [CV86, Vog08] for details. One can think of the space $CV_n$ as a free group analog of Teichmüller space. The function $T \mapsto (a \mapsto (T,a))$ defines an embedding of $cv_n$ into the space of length functions $[0,\infty)^{F_n}$. We denote by $cv_n^\infty$ the closure of $cv_n$ in this space and by $CV_n^\infty$ its projectivization. A point in $cv_n^\infty$ determines a tree, unique up to equivariant isometry. Both $cv_n^\infty$ and $CV_n^\infty$ are metrizable ($cv_n^\infty$ is a subspace of the metrizable space $[0,\infty)^{F_n}$ and $CV_n^\infty$ embeds in $cv_n^\infty$; see below). Moreover, $CV_n^\infty$ is compact and $\partial CV_n = CV_n \setminus CV_n^\infty$ plays the role of $\mathcal{PML}$. For our later arguments we choose a distance $\text{dist}$ on $CV_n^\infty$. As usual, this distance defines a (Hausdorff) distance on the set of compact subsets, and we keep the same notation for this distance. An element of $CV_n^\infty$ is represented by a projective class of trees, but we follow the custom and talk about trees as points in $CV_n^\infty$.

Recall the following characterisation of this compactification.

**Definition 2.1.** A nontrivial minimal tree $T$ is very small, if arc stabilizers are trivial or maximal cyclic subgroups, and the fixed set of a nontrivial element does not contain a tripod.

**Proposition 2.2 ([BF, Hor17]).** A nontrivial minimal tree $T$ is contained in $CV_n^\infty$ if and only if it is very small.

2.2. Arational Trees. To describe the boundary of the free factor complex, we need the following notion.

**Definition 2.3.** A tree $T \in \partial CV_n$ is arational if for every proper free factor $A < F_n$ the induced action of $A$ on $T$ is free and discrete.
Two arational trees are *topologically equivalent* if there is an equivariant homeomorphism in the observers’ topology between them (we elaborate on observers’ topology below). Equivalence classes of arational trees in $\text{CV}_n$ can be naturally identified with simplices, analogously to simplices of projectivized transverse measures on geodesic laminations. Denote by $\mathcal{AT} \subset \partial \text{CV}_n$ the space of arational trees, and by $\mathcal{AT}/\sim$ the quotient space obtained by collapsing each equivalence class to a point.

**Theorem 2.4** ([BR15] [Ham]). *The Gromov boundary $\partial FF_n$ of the free factor complex is homeomorphic to $\mathcal{AT}/\sim$.***

We will need a more precise version of this theorem. There is a function $\Phi : \text{CV}_n \rightarrow FF_n$ (in [BR15] this is the map $\pi \cup \partial \pi$) with the following properties:

- The restriction of $\Phi$ to the space $\mathcal{AT}$ of arational trees maps it continuously onto $\partial FF_n$ ([BR15] Proposition 7.5), it is a closed map ([BR15] Lemma 8.6), and the point inverses are exactly the simplices of equivalence classes of arational trees ([BR15] Proposition 8.4).
- The restriction of $\Phi$ to the complement of $\mathcal{AT}$ has $FF_n$ as its range and it is defined coarsely; it maps $T \in \partial \text{CV}_n \setminus \mathcal{AT}$ to a free factor $A$ such that the $A$-minimal subtree of $T$ has dense orbits (or $A$ is elliptic), and it maps $T \in \text{CV}_n$ to a free factor $A$ realized as a subgraph of $T/F_n$. See [BR15] Lemma 5.1 and Corollary 5.3).
- $\Phi$ is coarsely continuous: if $T_i$ is a sequence in $\text{CV}_n$ and $\Delta$ is an equivalence class of arational trees, then $\Phi(T_i) \rightarrow \Phi(\Delta)$ if and only if the accumulation set of $T_i$ is contained in $\Delta$. See [BR15] Proposition 8.3 and Proposition 8.5).

We will have to regularly construct arational trees, and this subsection collects some tools to do so.

We let $\text{cv}_n^+$ be the union of $\text{cv}_n$ together with a point $0$ representing the trivial action. The space $\text{cv}_n^+$ is naturally a cone with $[0, \infty)$ acting by rescaling. In a similar way we define the space $\text{cv}_n^+$ of very small trees, together with a point representing the trivial action.

There are many ways of realizing $\text{CV}_n$ as a section of the cone $\text{cv}_n^+$ (by which we mean a section of $\text{cv}_n^+ \setminus \{0\} \rightarrow \text{CV}_n$). We choose the following: by Serre’s lemma [Ser80], an action of $F_n$ on a tree so that each element of length $\leq 2$ acts elliptically, in fact has a global fixed point. Thus, the sum of translation lengths of all elements of length $\leq 2$ is positive in each tree of $\text{cv}_n$. We identify $\text{CV}_n$ with the subset of $\text{cv}_n$ where the sum of translation lengths for all elements of length $\leq 2$ is equal to 1.

**Lemma 2.5.** *For every proper free factor $A < F_n$, restricting the action of $F_n$ on a tree to a minimal subtree for the $A$–action yields a continuous map $r_A : \text{cv}_n^+ \rightarrow \text{cv}(A)^+$.***

**Proof.** First, observe that if $A$ acts elliptically on $T$, then $r_A(T)$ is the cone point in $\text{cv}(A)^+$. Otherwise, we claim that the restriction is very small. Namely, consider any arc $a \subset T$. If its stabiliser is trivial, the same is obviously true for the restricted action. If the stabiliser is a maximal cyclic subgroup, then the same is true for the restricted action: since $A$ is a free factor, if $1 \neq g \in A$ then the maximal cyclic subgroup in $F_n$ containing $g$ is contained
in $A$. Finally, suppose that $g$ acts nontrivially on the restricted tree. If it would fix a tripod, the same would be true in $T$, violating that $T$ is very small.

Continuity is clear, since translation lengths for the restriction are the same as translation lengths in $T$. □

We define $\rho_A: \partial CV_n \to cv(A)^+$ as the restriction of the maps $r_A$ to the subset $\partial CV_n \subset cv_n^+$ via the above normalization.

We now make the following definition, which will be crucial for our construction:

**Definition 2.6.** Let $F$ be the countable set of conjugacy classes of proper nontrivial free factors of $F_n$. For any $A \in F$ we define

$$K_A = \partial CV_n \setminus \rho_A^{-1}(cv(A))$$

Thus $K_A$ is the set of trees in $\partial CV_n$ where $A$ does not act freely and simplicially. The following is now clear from the above:

**Proposition 2.7.** The collection $\{K_A, A \in F\}$ is a countable collection of closed subsets whose complement is the set of arational trees:

$$AT = \partial CV_n \setminus \left( \bigcup_{A \in F} K_A \right)$$

### 2.3. Trees, currents, and the action of $Out(F_n)$.

Any $\phi \in Out(F_n)$ acts naturally (on the left) on the set of conjugacy classes. To be consistent with our later constructions, we also define a left action of $Out(F_n)$ on the set of trees defined by

$$(\phi T, a) = (T, \phi^{-1}(a))$$

The length pairing can be extended from the set of conjugacy classes to the space $MC_n$ of *measured geodesic currents* that contains positive multiples of conjugacy classes [Mar95, Kap06] (see below for the definition). It admits an action of $(0, \infty)$ by scaling and a left action of $Out(F_n)$ that commutes with scaling and extends the action on conjugacy classes. The length pairing extends to a continuous function $cv_n \times MC_n \to [0, \infty)$ that commutes with scaling in each coordinate [KL09].

### 2.4. Dual laminations and currents.

For more details on this section see [CHL07, CHL08a, CHL08b, CHL08c]. Denote by $\partial F_n$ the Cantor set of ends of $F_n$. A *lamination* $L$ is a closed subset of $\partial^2 F_n := \partial F_n \times \partial F_n \setminus \Delta$ invariant under $(x, y) \mapsto (y, x)$ and under the left action of $F_n$, where $\Delta$ is the diagonal. To every $T \in cv_n$ one associates the *dual lamination* $L(T)$, defined as

$$L(T) = \cap_{\epsilon > 0} L_\epsilon(T)$$

where $L_\epsilon(T)$ is the closure of set of pairs $(x, y) \in \partial^2 F_n$ which are endpoints in the Cayley graph of axes of elements with translation length $< \epsilon$ in $T$. It turns out that $L(T)$ is always *diagonally closed* i.e. $(a, b), (b, c) \in L(T)$ implies $(a, c) \in L(T)$ (if $a \neq c$), so it determines an equivalence relation on $\partial F_n$, and the equivalence classes form an upper semi-continuous decomposition of $\partial F_n$.

The precise definition of a measured geodesic current is that it is an $F_n$-invariant and $(x, y) \mapsto (y, x)$ invariant Radon measure on $\partial^2 F_n$ (i.e. a Borel measure which is finite on
compact sets). For example, a conjugacy class in $F_n$ determines a counting measure on $\partial^2 F_n$ and can be viewed as a current. The topology on the space $\mathcal{MC}_n$ of all currents is the weak* topology. The support $\text{Supp} (\mu)$ of a current is the smallest closed set such that $\mu$ is 0 in the complement; the support of a current is always a lamination. An important theorem relating currents and laminations is the following.

**Theorem 2.8** ([KL10]). Let $T \in \overline{cv}_n$ and $\mu \in \mathcal{MC}_n$. Then $\langle T, \mu \rangle = 0$ if and only if $\text{Supp}(\mu) \subseteq L(T)$.

2.5. **Observers’ topology.** Let $T \in \overline{cv}_n$. Then $T$ is a metric space, but it also admits a weaker topology, called observers’ topology. A subbasis for this topology consists of complementary components of individual points in $T$. One can also form the metric completion of $T$ and add the space of ends (i.e. the Gromov boundary) to form $\hat{T}$. The space $\hat{T}$ also has observers’ topology, defined in the same way. In this topology $\hat{T}$ is always compact, and in fact it is a dendrite (uniquely arcwise connected Peano continuum). The following is a theorem of Coulbois-Hilion-Lustig. There is an alternative description of $L(T)$ in terms of the $Q$-map $Q : \partial F_n \rightarrow \hat{T}$ (see Levitt-Lustig [LL03]): $(a, b) \in L(T)$ if and only if $Q(a) = Q(b)$.

**Theorem 2.9.** [CHL07] Suppose $T$ has dense orbits. Then $\hat{T}$ with observers’ topology is equivariantly homeomorphic to $\partial F_n / L(T)$.

In fact, the $Q$-map realizes this homeomorphism.

2.6. **Surfaces, Laminations and $\mathcal{CV}_n$.** Suppose that $\Sigma$ is a compact oriented surface with one boundary component. Then the fundamental group $\pi_1(\Sigma)$ is the free group $F_{2g}$. If $\Sigma$ is a nonorientable surface with a single boundary component, the fundamental group $\pi_1(\Sigma)$ is also free, of even or odd rank (depending on the parity of the Euler characteristic).

A lamination $\lambda$ on $\Sigma$ will always mean a measured geodesic lamination. For any surface $\Sigma$, we denote by $\mathcal{PML}(\Sigma)$ the sphere of projective measured laminations. See [FLP12] for a thorough treatment. We will also need measured laminations of non-orientable surfaces, but the only specific result we rely on is that for a nonorientable $\Sigma$, the lifting map

$$\mathcal{PML}(\Sigma) \rightarrow \mathcal{PML}(\Sigma')$$

is a topological embedding, where $\Sigma'$ denotes the orientation double cover.

In our construction we will need paths in $\mathcal{PML}$ consisting only of minimal laminations. In the orientable case these will be provided by the following theorem.

**Theorem 2.10** (Chaika and Hensel [CH]). For a surface of genus at least 5 (with any number of marked points) the set of uniquely ergodic ergodic foliations is path-connected and locally path-connected in $\mathcal{PML}$. Furthermore, given any finite set $B$ of laminations, the complement of $B$ in $\mathcal{PML}$ is still path-connected.

For non-orientable surfaces, we require a similar result. As we do not need the full strength of Theorem 2.10 for our argument, we will prove the following in the appendix (which follows quickly from methods developed in [LS09]):

**Theorem 2.11.** Suppose that $\Sigma$ is a nonorientable surface with a single marked point $p$.

Let $\mathcal{P} \subset \mathcal{PML}(\Sigma)$ be the set of minimal foliations which either

1. do not have an angle–$\pi$ singularity at $p$, or
2. are stable foliations of point-pushing pseudo-Anosovs.
Then $\mathcal{P}$ is path-connected, and invariant under the mapping class group of $\Sigma$. In addition, if $F$ is any finite set of laminations, the set $\mathcal{P} \setminus F$ is still path-connected.

Given a lamination on $\Sigma$, we can lift it to a lamination $\tilde{\lambda}$ of the universal cover $\tilde{\Sigma}$. Since $\partial_\infty \tilde{\Sigma} = \partial_\infty \pi_1(\Sigma) = \partial_\infty F_n$, this allows us to interpret $\lambda$ as a lamination on the free group. It is dual (in the sense above) to the dual tree of the lamination $\tilde{\lambda}$ (in the geometric sense).

The following theorem, due to Skora, describes exactly which trees appear in this way.

**Theorem 2.12.** [Sko96] Suppose a closed surface group acts on a minimal $\mathbb{R}$-tree $T$ with cyclic arc stabilizers. Then $T$ is dual to a measured geodesic lamination on the surface. The same holds for surfaces with boundary if the fundamental group of each boundary component acts elliptically in $T$.

For our purposes, we will need to be a bit more careful about how we identify surface laminations and free group laminations. Namely, for any identification $\sigma : \pi_1(\Sigma) \to F_n$ we obtain the corresponding map

$$\iota_\sigma : \mathcal{PML}(\Sigma) \to \overline{CV_n}$$

mapping a lamination to its dual tree. We denote by

$$\mathcal{PML}_\sigma = \iota_\sigma(\mathcal{PML}(\Sigma))$$

the image of $\iota_\sigma$. In other words, $\mathcal{PML}_\sigma$ consists of those trees which are realisable as duals to geodesic laminations on $\Sigma$, given the identification of $F_n = \pi_1(\Sigma)$ via $\sigma$.

If $\phi$ is any outer automorphism, then the images of $\iota_\sigma$ and $\iota_{\sigma \phi^{-1}}$ differ by applying the outer automorphism $\phi$ (acting on $\overline{CV_n}$). Core to our argument will be to use the intersection of “adjacent” such copies of $\mathcal{PML}$; see Section 3. A typical lamination contained in the intersection is one that fills a suitable subsurface of $\Sigma$.

### 2.7. Density of surface type arational trees.

Arational trees come in two flavors: ones dual to a filling measured lamination on a compact surface with one boundary component (we will call them arational trees of surface type), and the “others” – these are free as $F_n$-trees (see [Rey]). Recall that every arational tree $T$ belongs to a canonical simplex $\Delta_T \subset \partial CV_n$ consisting of arational trees with the same dual lamination. We will need the following lemma.

**Lemma 2.13.** Let $T$ be an arational tree and let $U$ be a neighborhood of the simplex $\Delta_T$ in $\partial CV_n$. Then $U$ contains an arational tree $S$ of surface type.

**Proof.** All arational trees in rank 2 are of surface type (and all associated simplices are points) so we may assume $n \geq 3$. Vincent Guirardel showed in [Gui00a] that for $n \geq 3$ the boundary $\partial CV_n$ contains a unique minimal $\text{Out}(F_n)$-invariant closed set $\mathcal{M}_n$. In particular, $\text{Out}(F_n)$ acts on $\mathcal{M}_n$ with dense orbits. He further showed that any arational tree (or indeed any tree with dense orbits) with ergodic Lebesgue measure belongs to $\mathcal{M}_n$. In our situation this means that the vertices of $\Delta_T$ belong to $\mathcal{M}_n$ and they can be approximated by points in the orbit of a fixed surface type arational tree that also belongs to $\mathcal{M}_n$. $\square$

### 2.8. Dynamics of partial pseudo-Anosovs.

In this section we will assemble the dynamical properties of partial pseudo-Anosov homeomorphism as they act on Outer space. The proofs are standard but we couldn’t find the statements we need in the literature.
The basic theorem is that of Levitt and Lustig. An outer automorphism of $F_n$ is fully irreducible if all of its nontrivial powers are irreducible, i.e. don’t fix any proper free factors up to conjugation.

**Theorem 2.14** ([LL03]). Let $f \in \text{Out}(F_n)$ be a fully irreducible automorphism. Then $f$ acts with north-south dynamics on $\overline{CV_n}$.

We start with a pseudo-Anosov homeomorphism $f : S \to S$ of a compact surface $S$ (with one or more boundary components) with $\pi_1(S) = F_n$. By $\lambda > 1$ denote the dilatation and by $\Lambda^\pm$ the stable and the unstable measured geodesic laminations, so $f(\Lambda^\pm) = \frac{1}{\lambda^\pm}\Lambda^\pm$. Let $T^\pm$ be the trees dual to $\Lambda^\pm$, and let $\mu^\pm$ be the measured currents corresponding to $\Lambda^\pm$. Thus

$$f_*T^\pm = \lambda^\pm T^\pm$$

and

$$f_*(\mu^\pm) = \lambda \mu^\pm$$

This implies $\langle T^\pm, \mu^\pm \rangle = 0$ and $\langle T^\pm, \mu^\mp \rangle > 0$.

**Proposition 2.15.** With notation as above, let $K \subset \overline{CV_n}$ be a compact set of trees such that $\langle T, \mu^+ \rangle \neq 0$ for every $T \in K$. Then $f_*^m K$ converges to $T^+$ as $m \to \infty$.

When $S$ has more than one boundary component $f_*$ is not fully irreducible. It is irreducible if it permutes the boundary components cyclically. The Levitt-Lustig argument can be used to prove the north-south dynamics for irreducible automorphisms, and this would suffice for our purposes since we could arrange that pseudo-Anosov homeomorphisms we use in our construction later have roots that cyclically permute the boundary components. However, we will give a direct argument.

**Proof.** We will view $K \subset \overline{CV_n}$ as a compact set of unprojectivized trees as in Section 2.2.

Let $Y$ be the accumulation set of the scaled forward iterates $f_*^m K/\lambda^m$ of $K$. Note that if $T \in K$ then $(\frac{1}{\lambda^m}f_*^m T, \mu^+) = (T, \frac{1}{\lambda^m}f_*^{-m}(\mu^+)) = (T, \mu^+)$, so the length of $\mu^+$, being bounded below by some $\epsilon > 0$ over $K$, is also bounded below by $\epsilon$ on $Y$. Similarly, $(\frac{1}{\lambda^m}f_*^m T, \mu^-) = (T, \frac{1}{\lambda^m}f_*^{-m}(\mu^-)) = (T, \frac{1}{\lambda^m}\mu^-) \to 0$, so the length of $\mu^-$ is 0 in all trees in $Y$. If $a$ is any conjugacy class other than a power of a boundary component, then we have by surface theory $\frac{1}{\lambda^m}f_*^m(a) \to C_\alpha a$ for some $C_\alpha > 0$, and a similar argument as above shows that $\langle T, a \rangle = C_\alpha \langle T, \mu^+ \rangle > 0$ for every $T \in Y$. However, when $a$ represents a boundary component then $\langle T, a \rangle = 0$ since then $f_*^m(a)$ is a boundary component for all $m$, and thus $\frac{1}{\lambda^m}f_*^m(a) \to 0$. In particular, all iterates $f_*^m K$ are contained in a compact subset of $\overline{CV_n}$ and so the accumulation set in $\overline{CV_n}$ is the projectivization of $Y$. It follows from Skora’s theorem that $Y$ consists of trees dual to measured geodesic laminations on $S$. The only lamination where $\mu^-$ has length 0 in the dual tree is $\Lambda^+$ and hence $Y = \{T^+\}$ (projectively). $\square$

The next result gives a criterion to check the condition required in the previous proposition.

**Proposition 2.16.** With notation as in the paragraph before Proposition 2.15 suppose that $R$ is a compact surface with marking induced by a homotopy equivalence $\phi : S \to R$ (which may not send boundary to boundary). Let $\Lambda$ be a measured geodesic lamination on $R$ and let $T$ be the dual tree. If $(T, \mu^+) = 0$ then $T = T^-$ projectively.
Proof. The support $\text{Supp}(\mu^+)$ of $\mu^+$ is $\Lambda^-$. The complementary component of $\Lambda^-$ in $S$ that contains a boundary component is a “crown region” and adding diagonals amounts to adding infinitely many (non-embedded) lines that start and end in a cusp of the crown region and wind around the boundary component any number of times. These accumulate on the boundary, so each boundary component is in the diagonal closure of $\text{Supp}(\mu^+)$. Any other line will intersect the leaves of $\Lambda^-$ transversally, so it follows that the lamination $L(T^-)$ dual to the tree $T^-$ coincides with the diagonal closure of $\text{Supp}(\mu^+)$ (cf. [BR15, Proposition 4.2(ii)]). Similarly, the lamination $L(T)$ dual to $T$ consists of leaves of $\Lambda$ together with all lines not crossing $\Lambda$ transversally. Since $(T, \mu^+) = 0$ we have $\text{Supp}(\mu^+) \subseteq L(T)$ by the Kapovich-Lustig Theorem [2,8] and hence $L(T^-) \subseteq L(T)$. We now argue that $\Lambda$ must be a filling lamination. First, $\Lambda$ cannot contain closed leaves, for otherwise $L(T)$ would be carried by an infinite index finitely generated subgroup, contradicting the fact that $\Lambda^-$ isn’t. Now suppose that $\Lambda$ contains a minimal component $\Lambda_0$ carried by a proper subsurface $R_0 \subset R$. If there is a leaf of $L(T^-)$ asymptotic to a leaf of $\Lambda_0$, then $L(T^-)$ contains $\Lambda_0$ as well as boundary components of $R_0$ and the diagonal leaves within $R_0$. It then follows that $L(T^-)$ doesn’t contain any other leaves since it equals the diagonal closure of any non-closed leaf, and hence again it is carried by an infinite index finitely generated subgroup. Thus $\Lambda$ must be filling and $L(T^-) = L(T)$. It now follows from the Coulbois-Hilion-Lustig Theorem [2,9] that $T$ and $T^-$ are equivariantly homeomorphic in observers’ topology. But since $T^-$ is uniquely ergodic, projectively $T = T^-$.

Remark 2.17. In fact, $(T, \mu^+) = 0$ forces $T = T^-$ even without assuming that $T$ is dual to a lamination on a surface. This can be proved by noting that $L(T^-) \subseteq L(T)$ (which is proved above for all $T$ with $(T, \mu^+) = 0$) forces $L(T) = L(T^-)$ by [BR15 Proposition 3.2] since $T^-$ is an indecomposable tree. It follows that $f_*$ satisfies the north-south dynamics on all of $\overline{CV_n}$, not just on compact sets dual to surface laminations.

We now generalize this to partial pseudo-Anosov homeomorphisms.

Proposition 2.18. Let $f$ be a homeomorphism of $\Sigma$, which restricts to a pseudo-Anosov on a $\pi_1$-injective subsurface $S \subset \Sigma$, and to the identity in the complement. Suppose that $K \subset \overline{CV_n}$ is compact such that every $T \in K$ satisfies $(T, \mu^+) \neq 0$, where $\mu^+$ is a current supported in $\pi_1(S)$ corresponding to the stable lamination $\Lambda^+$ of $f$.

Then the sequence $f^i K$, $i \to \infty$, converges to the tree $T_f \in \partial CV_n$ dual to $\Lambda^+$ on $\Sigma$.

Note that the condition implies that $\pi_1(S)$ is not elliptic in any $T \in K$. To begin the proof of Proposition 2.18 denote by $Y \subset \overline{CV_n}$ the set of accumulation points of $f^i K$ as $i \to \infty$. We need to show that $Y = \{T_f\}$.

Lemma 2.19. Let $T \in Y$. Every conjugacy class represented by a (not necessarily simple) curve in $\Sigma \setminus S$ is elliptic in $T$. The minimal $\pi_1(S)$-subtree of $T$ is dual to $\Lambda^+$.

Proof. The set $K$ is defined as a subset of $\overline{CV_n}$, but we can lift it to unprojectivized space $\overline{CV_n}$ as in Section 2.2. Now let $\gamma$ be a curve in the complement of $S$. Since $K$ is compact, the set $\{f^i(\gamma) \mid R \in Y\}$ is bounded. Since $f(\gamma) = \gamma$ we deduce that the set $\{\langle R, f^{-i}(\gamma) \rangle \mid R \in Y, i \in \mathbb{Z}\}$ is bounded as well. But $\langle f_* R, \gamma \rangle = \langle R, f_*^{-i}(\gamma) \rangle$, so the length of $\gamma$ in $f_*^i(X)$ stays uniformly bounded for all $i$.

On the other hand, let $\delta$ be a curve in $S$. Applying Proposition 2.15 to the trees obtained from the trees in $K$ by restricting to $\pi_1(S)$ we see that these restrictions converge to the
tree dual to $\Lambda^+$. Moreover, the length of $\delta$ along $f_i^*K$ goes to infinity as $i \to \infty$ (in fact, the length grows like $(\text{const})\lambda^i$ where $\lambda$ is the dilatation). Thus projectively, the length of $\gamma$ will go to 0 as $i \to \infty$. $\square$

**Proof of Proposition 2.18.** Let $T \in Y$. By Lemma 2.19 conjugacy classes of boundary components of $\Sigma$ are all elliptic, so by Skora’s theorem we deduce that $T$ is dual to a measured geodesic lamination on $\Sigma$. Again by Lemma 2.19 this lamination is $\Lambda^+$ on $S$ and possibly curves in $\partial S \setminus \partial \Sigma$ with nonzero weight. Thus the accumulation set is contained in the simplex in $\mathcal{PML}(\Sigma)$ whose vertices are $\Lambda^+$ and these curves, it is compact, and disjoint from the face opposite the vertex $\Lambda^+$. The only $f$-invariant compact set with this property is $\{\Lambda^+\}$ since $f$ acts by attracting towards $\Lambda^+$ all compact sets disjoint from the opposite face. $\square$

**Corollary 2.20.** In the setting of Proposition 2.18, let $A < F_n$ be a free factor such that the action of $A$ on $T^+$ is free and discrete. Then for large $k > 0$ the action of $f_k^*(A)$ on every $T \in K$ is free and discrete.

**Proof.** The subgroup $A$ will act freely and discretely in a neighborhood of $T^+$, and this includes $Kf_k^*$ for large $k$. This is equivalent to the statement. $\square$

There is a simple criterion for deciding if the action of $A$ on $T^+$ is free and discrete.

**Lemma 2.21.** Let $A < F_n$ be a free factor and $T^+$ the tree dual to the stable lamination $\Lambda_f$ of a partial pseudo-Anosov homeomorphism $f$ supported on a subsurface $S \subset \Sigma$. The action of $A$ on $T^+$ is free and discrete if and only if $\pi_1(S)$ is not conjugate into $A$ and no nontrivial conjugacy class in $A$ is represented by an immersed curve in $\Sigma \setminus \Lambda^+$. 

**Proof.** It is clear that these conditions are necessary. Assuming they hold, equip $\Sigma$ with a complete hyperbolic metric of finite area and let $\tilde{\Sigma} \to \Sigma$ be the covering space with $\pi_1(\tilde{\Sigma}) = A$. Lift the hyperbolic metric to $\Sigma$ and let $\tilde{\Sigma}_C \subset \Sigma$ be the convex core. Then $\tilde{\Sigma}_C$ is compact, since any cusp would represent a boundary component of $\Sigma$ whose conjugacy class is in $A$. Lift the lamination $\Lambda^+$ to $\tilde{\Lambda} \subset \Sigma$. Each leaf of $\tilde{\Lambda}$ intersects $\tilde{\Sigma}_C$ in an arc (or not at all) for otherwise $A$ would carry $\Lambda_f$ and would have to contain a finite index subgroup of $\pi_1(S)$. But $A$ is root-closed, so it would contain $\pi_1(S)$, which we excluded. So the intersection of $\tilde{\Lambda}$ with $\tilde{\Sigma}_C$ consists of finitely many isotopy classes of arcs. These arcs are filling, for otherwise we would have a loop in the complement that would represent a nontrivial element of $A$ whose image in $\Sigma$ is disjoint from $\Lambda^+$. The minimal $A$-subtree of $T_f$ is dual to this collection of arcs, and so is free and discrete. $\square$

Finally, we need the following, which describes the dynamics of partial pseudo-Anosovs on free factors. We use the terminology of good and bad subsurfaces, which will be motivated and introduced in Section 3.

**Proposition 2.22.** Identify $\pi_1(\Sigma) = F_n$ and assume that the rank $n$ of the free group is at least 18. Suppose that $\psi$ is a partial pseudo-Anosov, supported on a “good” subsurface $S^g$ of $\Sigma$, and so that the “bad subgroup” $\pi_1(\Sigma - S^g)$ has rank at most 5.

Let $E$ be any free factor, and $B'$ a subgroup which does not contain $\pi_1(S^g)$ up to conjugacy. Then one of the following holds:

1. $\pi_1(S^g) \subset E$, or
Proof. Assume that (1) fails, i.e. \( \pi_1(S^g) \) is not conjugate into \( E \). Choose a hyperbolic metric on \( \Sigma \). From now on, we assume that all curves and laminations are geodesic. By Scott’s theorem, there is a finite cover \( X \to \Sigma \) so that \( E = \pi_1(X_E) \) where \( X_E \subset X \) is a subsurface. We can choose a power \( k \) so that \( \psi^k \) lifts to a partial pseudo-Anosov map \( \hat{\psi} \) of \( X \).

Let \( \hat{\lambda}^\pm \) be the lifts of the stable/unstable lamination of \( \psi \) to \( X \); in other words, these are the stable/unstable laminations of \( \psi \). Since both of these laminations fill \( \pi_1(S^g) \), and we assume that \( \pi_1(S^g) \) is not conjugate into \( E \), no leaf of these laminations is completely contained in \( X_E \).

The intersection \( \hat{\lambda}^{\pm} \cap X_E \) is supported in a subsurface \( Y \subset X_E \), and we let \( Y' = X_E - Y \) be its complement. We emphasise that the complement could be empty. Also observe that any curve in \( Y' \) maps (under the covering map) into \( \Sigma - S^g \).

By compactness and the fact that no leaf of \( \hat{\lambda}^- \) is supported in \( X_E \) there is a number \( \alpha \) with the following property: any geodesic starting in a point \( p \in \hat{\lambda}^- \) and making angle \( < \alpha \) with \( \hat{\lambda}^- \) leaves \( X_E \). In particular we conclude that any closed geodesic in \( X_E \) which intersects \( Y \) (and hence \( \hat{\lambda}^- \)) makes angle \( \geq \alpha \) with \( \hat{\lambda}^- \).

As a consequence, we have the following: for any \( L \) and \( \epsilon \) there is a number \( N(L, \epsilon) \) with the following property. If \( \gamma \subset X_E \) is any geodesic intersecting \( Y \), then \( \hat{\psi}^n \gamma \) contains a segment of length \( \geq L \) which \( \epsilon \)-fellow-travels a leaf of \( \hat{\lambda}^+ \), for all \( n > N(L, \epsilon) \).

We now claim that there are \( L, \epsilon > 0 \) with the following property: no geodesic \( \gamma \) in \( X \), which represents a conjugacy class of \( B' \), contains a geodesic segment of length \( \geq L \) which \( \epsilon \)-fellow-travels a leaf of \( \hat{\lambda}^+ \).

To see this, we argue by contradiction. Namely, if not, then we could find a sequence of geodesics \( \gamma_n \subset X \) which represent conjugacy classes in \( B' \), and which limit to a leaf of \( \hat{\lambda}^+ \). In particular, the endpoints at infinity of a leaf of \( \hat{\lambda}^+ \) would be contained in the boundary at infinity \( \partial_\infty B' \) of the subgroup \( B' \). Since \( \lambda^+ \) is the stable lamination of a partial pseudo-Anosov supported on \( S^g \), this would imply that \( B' \) contains \( \pi_1(S^g) \) up to conjugacy – which contradicts our assumption.

We let \( N = N(L, \epsilon) \) be the corresponding constant. Now, let \( r > 0 \) be a number so that \( \rho^r \) lifts to \( X \) for any loop \( \rho \) in \( \Sigma \) (this exists, since \( X \) is a finite cover). Let \( \rho \) be any element of \( E \), which is not conjugate into \( \pi_1(\Sigma - S^g) \). Since fundamental groups of surfaces are root-closed, the element \( \rho^r \) is then also not conjugate into \( \pi_1(\Sigma - S^g) \). Then, \( \rho^r \) lifts to a geodesic \( \gamma \) in \( X_E \), which intersects \( Y \). Thus, \( \hat{\psi}^n \gamma \) contains a segment with the property of the previous paragraph, showing that \( \psi^{kn} \rho^r \) is not contained in \( B' \) for all \( n > 0 \).

In other words, if \( \psi^{kn} \rho \) (and thus \( \psi^{kn} \rho^r \)) is contained in \( B' \), then \( \rho \) is contained in \( \pi_1(\Sigma - S^g) \), showing (2). \( \square \)

3. Basic Moves, Good and Bad Subsurfaces

In this section, we will study how to relate different identifications of a free group with the fundamental group of a surface. The basic situation will be to relate two identifications which differ by applying (certain) generators of \( \text{Out}(F_n) \).
3.1. **Standard Geometric Bases.** Let $\Sigma = S_{g,1}$ be a compact oriented surface of genus $g$ with one boundary component. We pick a basepoint $p$, contained in the interior of the surface $\Sigma$. A collection of simple closed curves $a_i, \hat{a}_i, 1 \leq i \leq g$ is called a **standard geometric basis** for $S_{g,1}$ if the following hold:

1. The $a_i, \hat{a}_i$ generate $F_{2g} = \pi_1(\Sigma, p)$.
2. The $a_i, \hat{a}_i$ intersect only in $p$.
3. The cyclic order of incoming and outgoing arcs at $p$ is $\hat{a}_1^+, a_1^+, \hat{a}_1^-, a_1^-, \hat{a}_2^+, a_2^+, \hat{a}_2^-, a_2^-, \ldots, a_g$.

![Figure 1](image.png)  

**Figure 1.** A basis for $\Sigma$ of the type we use in this section.

Compare Figure 1 for an example of such a basis. Given a standard geometric basis, we say that $a_i, \hat{a}_i$ are an **intersecting pair** (of that basis). For ease of notation we define $\hat{\hat{a}}_i = a_i$.

Observe that up to the action of the mapping class group of $\Sigma$ there is a unique standard geometric basis.

To deal with free groups of odd rank, we need to also consider certain nonorientable surfaces. Namely, let $\Sigma = N_{2g+1}$ be the surface obtained from $S_{g}$ by attaching a single twisted band, at the two sides of an initial segment of $a_i^+$. As before, we pick a basepoint $p$ in the interior of $\Sigma$. A collection of simple closed curves $n, a_i, \hat{a}_i, 1 \leq i \leq g$ is called a **standard geometric basis** for $N_{2g+1}$ if the following hold:

1. The curve $n$ is one-sided,
2. all $a_i, \hat{a}_i$ are two-sided,
3. the elements $n, a_i, \hat{a}_i$ generate $F_{2g+1} = \pi_1(\Sigma, p)$,
4. The $n, a_i, \hat{a}_i$ intersect only in $p$.
5. The cyclic order of incoming and outgoing arcs at $p$ is $n^+, \hat{a}_1^+, n^-, a_1^+, \hat{a}_1^-, a_1^-, \hat{a}_2^+, a_2^+, \hat{a}_2^-, a_2^- \ldots$

As above, we say that $a_i, \hat{a}_i$ are an **intersecting pair**. In addition, we also say that $n, \hat{a}_1$ and $n, a_1$ are intersecting pairs. We call $a_1, \hat{a}_1$ the **nonorientable-linked letters**.

**Remark 3.1.** The reason for the somewhat asymmetric setup in the nonorientable setting is as follows. For later arguments, we will need to find two-sided curves which intersect the one-sided curve given by the basis letter in a single point. This forces at least one of the two-sided bands to be linked with the one-sided band. However, since we also need to be able to have an odd total number of bands, the described setup emerges.
Defintion 3.2. Suppose that \( x_1, \ldots, x_n \) is a free basis for \( F_n \). For any \( x = x_i, y = x_j \) with \( i \neq j \), we call an outer automorphism defined by the automorphism

\[
\rho_{x,y}(z) = \begin{cases} 
xy & z = x \\
z & z = x_k, k \neq i 
\end{cases}
\]

or

\[
\lambda_{x,y}(z) = \begin{cases} 
yx & z = x \\
z & z = x_k, k \neq i 
\end{cases}
\]
a basic (Nielsen) move. For \( x = x_i \), we call an outer automorphism defined by the automorphism

\[
i_x(z) = \begin{cases} 
x^{-1} & z = x \\
z & z = x_k, k \neq i 
\end{cases}
\]
a basic (invert) move.

Lemma 3.3. Given any standard geometric basis, \( \text{Out}(F_n) \) is generated by the basic invert moves, and Nielsen moves \( \phi_{x,y} \) for \( x, y \) not an intersecting pair.

Proof. It is well-known that for any basis (in particular, standard geometric bases) all basic moves of the form above generate \( \text{Out}(F_n) \) \cite{Nie24}. So, to prove the lemma, we just need to show that a Nielsen move for an intersecting pair can be written in terms of nonintersecting pairs. This is clear, e.g. for an unrelated letter \( z \) we have:

\[
\phi_{a_i, \hat{a}_i}^{-1} = \phi_{z, \hat{a}_i}^{-1} \phi_{a_i, z} \phi_{a_i, \hat{a}_i} \phi_{a_i, z}.
\]

\( \square \)

Defintion 3.4. Suppose we have chosen an identification \( \sigma : \pi_1(\Sigma) \to F_n \), and basic move \( \phi \). We say a subsurface \( S \subset \Sigma \) good for \( \phi \) if there is a pseudo-Anosov \( \psi \) supported on \( S \) which commutes with \( \phi \) on the level of fundamental groups, under the identification \( \sigma \), i.e.

\[
\phi \sigma \psi \sigma^{-1} = \sigma \psi \sigma^{-1} \phi
\]

We call such a \( \psi \) an associated partial pA. We call the complement of a chosen good subsurface a bad subsurface.

We need a bit more flexibility than basic moves, given by the following definition.

Defintion 3.5. Suppose \( \sigma : \pi_1(\Sigma) \to F_n \) is an identification, and \( B \) is a standard geometric basis for \( \Sigma \). We then call a conjugate of a basic move (of \( B \)) by a mapping class of \( \Sigma \) an adjusted move.

We observe that the good and bad subsurfaces of an adjusted move are obtained from the corresponding subsurfaces of the basic move by applying the mapping class.

Observe that, strictly speaking, neither good nor bad subsurfaces are unique, but we will explain below which ones we choose.

The key reason why we are interested in good and bad subsurfaces is that we will try to apply Lemma 2.21 to the partial pseudo-Anosovs guaranteed to exist on the good subsurface. In order to do this, we will need to find relations in \( \text{Out}(F_n) \) avoiding the following two “problems” (corresponding to the two conditions in Lemma 2.21):

Defintion 3.6. Let \( \phi \) be a basic or adjusted move, and \( E < F_n \) a free factor.
(1) We say that $E$ is an overlap problem for $\phi$ (and a choice of good and bad subsurface) if some nontrivial conjugacy class $w \in E$ is contained (up to conjugacy) in the fundamental group of the bad subsurface.

(2) We say that $E$ is a containment problem for $\phi$ (and a choice of good and bad subsurface) if the fundamental group of the good subsurface is contained (up to conjugacy) in $E$.

Finally, recall that an identification $\sigma : \pi_1(\Sigma) \to F_n$ defines a copy $\text{PML}_\sigma$ of $\text{PML}(\Sigma)$ inside $\text{CV}_n$. The following notion is central for our construction.

**Definition 3.7.** Given any identification $\sigma : \pi_1(\Sigma) \to F_n$, and adjusted move $\phi$ with respect to a standard geometric basis of $\Sigma$, we say that the copies $\text{PML}_\sigma$ and $\phi\text{PML}_\sigma = \text{PML}_{\phi \sigma}$ are adjacent.

### 3.2. Good Subsurfaces.

To find good subsurfaces, we use the following two lemmas. The first constructs an “obvious” good subsurface (which is not large enough for our purposes). The second one will construct curves that yield additional commuting Dehn twists, which extend the good subsurface.

Throughout this section, we fix a standard geometric basis $B$, based at a point $p$.

**Lemma 3.8** ("Obvious" good subsurfaces). Let $x, y$ be two basis elements of $B$ which are not an intersecting pair. Then there is a subsurface $S_0$ with the following properties.

1. If $x$ is two-sided and not linked with the one-sided loop, then $x, \hat{x}, y, \hat{y}$ are disjoint from $S_0$. If $x = n$ is one-sided and linked with $a$, or $x = a, \hat{a}$ is linked with the one-sided letter $n$, then $n, a, \hat{a}, y, \hat{y}$ are disjoint from $S_0$.
2. Any other basis loop in $B$ is freely homotopic into $S_0$, and intersects $\partial S_0$ in two points.
3. Any mapping class supported in $S_0$ commutes with $\iota_x, \lambda_{x,y}$ and $\rho_{x,y}$.

We call the letters as in (1) the active letters of the basis, and all others the inactive.

**Proof.** Let $Y_1, Y_2$ be the subsurfaces filled by the elements of the given standard geometric basis $B^0$ which are between $x, \hat{x}$ and $y, \hat{y}$ in the cyclic ordering induced by the orientation of the surface. We homotope $Y_1, Y_2$ slightly off the basepoint so that they are disjoint from $x, \hat{x}, y, \hat{y}$. Depending on the configuration, one of the $Y_i$ may be empty. If both $Y_i$ are nonempty, choose an arc $\alpha$ connecting $Y_1$ to $Y_2$ disjoint from all $a_i, \hat{a}_i$, homotopic to the product $\hat{y}y^{-1}y^{-1}$. Compare Figure 2 for this setup. We let $S_0$ be the subsurface obtained as a band sum of $Y_1, Y_2$ along $\alpha$, homotoped slightly so the basepoint is outside $S_0$.

Now observe that if $F$ is any mapping class supported in $S_0$, then $F$ acts trivially on all of $x, \hat{x}, y, \hat{y}$. Furthermore, by construction, any loop in $S_0$ can be written in the basis $B$ without $x$. Together, these imply that $F_*$ commutes with $\rho_{x,y}, \lambda_{x,y}$ and $\iota_x$.

The argument in the nonorientable case is very similar, with the three letters $n, \hat{a}_1, a_1$ playing the role of $x, \hat{x}$.

**Lemma 3.9.** In the setting of Lemma 3.8, denote by $x, \hat{x}, y, \hat{y}$ the active letters, and let $S_0$ be the subsurface guaranteed by that lemma.

Then there are two-sided curves $\delta^+, \delta^-$ with the following properties:
Figure 2. Standard geometric bases, and “obvious” good subsurfaces

Figure 3. Constructing “extra twists” in Lemma 3.9

1. $\delta^+$ (or $\delta^-$) intersect $x$ in a single point on $x^+$ (or $x^-$).
2. The curves $\delta^+, \delta^-$ do not cross the band corresponding to $x$.
3. $\delta^+, \delta^-$ are disjoint from $y$.
4. $\delta^+, \delta^-$ intersect $S_0$ essentially.

If one of $y, \hat{y}$ is either one-sided or linked with the one-sided letter, then there is additionally a curve $\delta^0$ intersecting $\partial S_0$ essentially, and which satisfies (2) and (3), but is disjoint from $x$.

Proof. We construct the curves case-by-case, beginning with the orientable case.

Here, we simply take $\gamma$ to be an embedded arc in $\Sigma - S_0$ which intersects $x^+$ (or $x^-$) in a single point and is disjoint from $y, \hat{y}$, and does not intersect the interior of the band corresponding to $x$ (compare Figure 2 and 3). The desired curve is then obtained by concatenating $\gamma$ with any nonseparating arc in $S_0$. 
In the nonorientable case, we do exactly the same, making sure that the arc \( \gamma \) (and the arc in \( S_0 \)) are two-sided.

For some of the arguments in the sequel, we will need explicit descriptions of the curves produced by Lemma 3.9 and the fundamental groups of the resulting bad subsurfaces. As this is a somewhat tedious exercise in constructing and analyzing explicit curves (and the proof follows the exact same strategy in all cases), we only discuss the orientable case here, and defer all further cases to Lemma A.1 in Appendix A.

**Lemma 3.10.** Fix a standard geometric basis \( \mathcal{B} \) of an orientable surface \( \Sigma = \Sigma_{g,1} \), and use it to identify \( \pi_1(\Sigma) \) with \( F_{2g} \). We denote by \( \partial \) the word representing the boundary of the surface, i.e.

\[
\partial = \prod_{i=1}^{g} [\hat{a}_i, a_i^{-1}],
\]

and by \( \partial_w \) the cyclic permutation of \( \partial \) starting with the element \( w \).

Let \( x, y \) be two elements of \( \mathcal{B} \) which are not linked.

- If \( x = a_i \), then the bad subsurface for the right multiplication move \( \rho_{x,y} \) has fundamental group
  \[
  \pi_1(\Sigma - S^g) = \langle y, \hat{y}, x^{-1} \hat{x}x, \partial \hat{a}_{i+1} \rangle.
  \]
  The bad subsurface for the left multiplication move \( \lambda_{x,y} \) has fundamental group
  \[
  \pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{x}, \partial a_i \rangle.
  \]
- If \( x = \hat{a}_i \), then the bad subsurface for the right multiplication move \( \rho_{x,y} \) has fundamental group
  \[
  \pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{x}, \partial a_{i-1} \rangle.
  \]
  The bad subsurface for the left multiplication move \( \lambda_{x,y} \) has fundamental group
  \[
  \pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{a}_i \hat{x} \hat{x}^{-1}, \partial a_i \rangle.
  \]

In both cases, every loop corresponding to a basis letter except \( x, \hat{x}, y, \hat{y} \) is freely homotopic into the good subsurface.

**Proof.** We suppose that \( x = a_i \), and begin with the subsurface \( S_0 \) from Lemma 3.8. Part (3) of that lemma shows that it is indeed good, and part (2) shows that it has the property claimed in the last sentence of the lemma.

We now use curves from Lemma 3.9 to find additional Dehn twists which commute with the basic moves. We begin with the case of \( \phi = \rho_{x,y} \). Here, we use the curve \( \delta^+ \) guaranteed by that lemma (shown in dark green in Figure 3). The action of the twist \( T_{\delta^+} \) on \( \mathcal{B} \) depends on the type of letter. However, the relevant properties for us are the following:

1. \( T_{\delta^+}(x) = wx \), where \( w \) is a word not involving \( x \). Namely, by property (1) of Lemma 3.9, the twisted curve \( T_{\delta^+}(x) \) is obtained by following \( x^+ \) until the intersection point, following around \( \delta^+ \), and then continuing along \( x \). By property (2), the curve \( \delta^+ \) does not cross the band corresponding to \( x \), yielding the desired property of \( w \).

2. \( T_{\delta^+}(y) = y \). This follows since by (3) of Lemma 3.9 \( \delta^+ \) and \( y \) are disjoint.

3. For any other basis element \( z \), the image \( T_{\delta^+}(z) \) is a word in \( \mathcal{B} \) not involving \( x \). Again, this follows from Property (2), since the curve \( \delta^+ \) does not cross the band corresponding to \( x \).
These imply that $T_{\delta^+}$ commute with $\rho_{x,y}$:

1. Since $w$ does not involve $x$, we have
   $$\rho_{x,y} T_{\delta^+}(x) = \rho_{x,y}(wx) = wxy.$$
   Since, by (2) above, the twist fixes $y$, we also have:
   $$T_{\delta^+}(\rho_{x,y}(x)) = T_{\delta^+}(xy) = wxy.$$

2. Since both $T_{\delta^+}$ and $\rho_{x,y}$ fix $y$, we have
   $$\rho_{x,y} T_{\delta^+}(y) = y = T_{\delta^+}(\rho_{x,y}(y))$$

3. Finally, since for any other basis element $z$, the image $T_{\delta^+}(z)$ is a word in $\mathcal{B}$ not involving $x$ by (3) above, we have
   $$\rho_{x,y} T_{\delta^+}(z) = T_{\delta^+}(z) = T_{\delta^+}(\rho_{x,y}(z)).$$

Now, let $S^g$ be a regular neighbourhood of $S_0 \cup \delta^+$. By property (4) of Lemma 3.9, this is strictly bigger than $S_0$. Observe that it is filled by $\delta^+$ and curves contained in $S_0$. Since we have shown that such twists commute with $\rho_{x,y}$, and a suitable product of such twists is a pseudo-Anosov map of $S^g$, it is indeed a good subsurface for $\rho_{x,y}$. It remains to compute the fundamental group, which can be read off from Figure 3.

For the basic move $\lambda_{x,y}$ the proof is analogous, using that $\delta^-$ intersects $x$ only in $x^-$, proving that $T_{\delta^-}(x) = xw'$.

The strategy for the case $x = \hat{a}_i$ is analogous.

We also need the analog for invert moves.

**Lemma 3.11.** With notation as in the previous lemma, consider a basic invert move $\phi = \iota_x$. The bad subsurface has fundamental group
$$\pi_1(\Sigma - S^g) = \langle x, \hat{x}, \partial_{\hat{a}_{i+1}} \rangle$$
if $x = a_i$ or $x = \hat{a}_i$.

**Proof.** Here, only the first part of the proof of Lemma 3.10 is necessary; the desired good subsurface is the subsurface $S_0$ constructed in Lemma 3.8. □

### 3.3. BLAS Paths

In this section we begin to discuss the paths we use as the basis for all of our constructions. The picture to have in mind is that we build paths by concatenating arational paths in different copies of $\mathcal{PML}$, joining them at points which fail to be arational in a very controlled way.

More formally, we say a path $p : [0, 1] \to \partial CV_n$ is a BLAS path if there is a finite set $B_p \subset p([0, 1])$ so that

- Every point on $p([0, 1]) \setminus B_p$ is an arational tree.
- If $T \in B_p$, then there is an identification $F_n = \pi_1(\Sigma)$ of the free group with a surface with one boundary component (where $\Sigma = \Sigma_g$ if $n = 2g$ is even, and $\Sigma = N_{2g+1}$ otherwise), and the following holds: $T$ is the dual tree to the stable foliation of a pseudo-Anosov mapping class $\psi_T$ supported on a subsurface $S$.
- For any $T \in B_p$ there is a neighborhood $\mathcal{N}(T)$ of $T$ in $\partial CV_n$, so that $\mathcal{N}(T) \cap p([0, 1]) \setminus T$ has two connected components, $\gamma_1, \gamma_2$.
  - For each $\gamma_i$ there is a path $\xi_i$ so that $\xi_i \cup \psi_T \xi_i$ is a path and $\gamma_i = \cup_{k=0}^{\infty} \psi_T^k \xi_i$. 
Using Theorem 2.10 and Theorem 2.11 we show:

**Proposition 3.12.** Let $T_s, T_e$ in $\partial CV_n$ be two surface type arational trees, dual to uniquely ergodic laminations (or, in the nonorientable case, in the set $\mathcal{P}$ from Theorem 2.11).

Then $T_s, T_e$ can be connected by a BLAS path. Moreover, if one prescribes a chain of adjacent $\mathcal{PML}$s connecting the $\mathcal{PML}$ on which $T_s$ lies with the one on which $T_e$ lies, we may assume the BLAS path travels exactly through that chain of $\mathcal{PML}$s in exactly the same order.

In addition, for any finite set $F$ of arational trees not containing $T_s, T_e$, the path may be chosen to be disjoint from $F$.

**Proof.** It suffices to show the proposition in the case where $T_s, T_e$ lie in adjacent copies of $\mathcal{PML}$, say $\mathcal{PML}(\Sigma), \phi \mathcal{PML}(\Sigma)$. Furthermore, choose a partial pseudo-Anosov $\psi$ on the good subsurface of $\phi$, and observe that its stable lamination $\lambda^+$ is not dual to any tree contained in the finite set $F$ (since $\psi$ is a partial pseudo-Anosov).

Observe that there is a neighbourhood $U$ of $\lambda^+$ in $\mathcal{PML}(\Sigma)$ so that $\psi(U) \subset U$. We may assume further that no lamination in $U$ is dual to any tree in $F$. Pick any uniquely ergodic lamination $\lambda_0$, and a path $\gamma$ of uniquely ergodic laminations joining $\lambda_0$ to $\psi(\lambda_0)$. In particular, for $k \to \infty$, the paths $\psi^k(\gamma)$ converge to $\lambda^+$ (as no point on it is a lamination disjoint from the boundary of the active subsurface of $\psi$). Thus, we can choose an number $K > 0$ so that $\psi^k(\gamma) \subset U$ for all $k \geq K$. In particular, $\psi^K(\gamma) * \psi^{K+1}(\gamma) * \ldots$ can be completed to a path $p$ joining $\psi^K(\lambda_0)$ to $\lambda_+$ contained in $U$. In particular, no point on $p$ is dual to a tree in $F$.

Now, by Theorem 2.10 or Theorem 2.11 there is a path joining $\lambda_0$ to $\psi^K(\lambda_0)$, and so that no point on it is dual to a tree in $F$. The concatenation of these two paths is the desired path. \hfill \Box

4. Avoiding problems

Recall from Section 2.2 that $K_E$ is the set of trees in $\partial CV_n$ where a free factor $E$ does not act freely and simplicially. The main point of this section is to prove the following local result:

**Theorem 4.1.** Suppose that the rank of the free group $F_n$ is at least 18. Let $F \in \mathcal{F}$ be any proper free factor. Assume that

- $\mathcal{PML}_{\sigma_1}, \mathcal{PML}_{\sigma_2}$ are adjacent copies of $\mathcal{PML}$ (i.e. differ by applying an adjusted move $\phi$),
- $\lambda_1 \in \mathcal{PML}_{\sigma_1}, \lambda_2 \in \mathcal{PML}_{\sigma_2}$ are uniquely ergodic (respectively, in the set $\mathcal{P}$ from Theorem 2.11).
- $\psi$ is a partial pseudo-Anosov mapping class on $\Sigma$ supported on the good subsurface $S \subset \Sigma_1$ of the adjusted move $\phi$ as in Section 3.

Then there exists a BLAS path $q : [0, 1] \to \partial CV_N$ joining the dual trees $T_1, T_2$ of $\lambda_1, \lambda_2$, so that

1. There is a number $m > 0$ so that for all $k = nr \geq 0$ we have $\psi^k(q[0,1]) \cap K_E = \emptyset$.
2. $\psi^k(q[0,1])$ converges to the stable lamination of $\psi$ as $k$ goes to infinity.
Also observe that \( \psi^k(q[0,1]) \) still connect \( PML_{\sigma_1}, PML_{\sigma_2} \), since \( \psi \) commutes with the adjusted move.

We begin by describing how to construct a relation in \( \text{Out}(F_n) \) (which will serve as a “combinatorial skeleton” for a BLAS path), so that overlap and containment problems of \( A \) can be avoided at each step. It is important for the strategy that containment problems can be solved first.

Before we begin in earnest, we want to briefly discuss the setup for the rest of this section. We begin by fixing once and for all an identification \( \pi_1(\Sigma) \cong F_n \), a corresponding “basepoint copy” \( PML(\Sigma) \), and a standard geometric basis \( B \) (as in Section 3.1). In building BLAS paths, we will always use \( \text{Out}(F_n) \) to move the current copy of \( PML \) to this basepoint copy, and work there.

Suppose we have a relation

\[
\phi = \phi_1 \circ \cdots \circ \phi_l
\]

in \( \text{Out}(F_n) \). Associated to this we have a sequence of consecutively adjacent copies of \( PML \)

\[
PML(\Sigma), \phi_1 PML(\Sigma), \phi_2 PML(\Sigma), \ldots, \phi_1 \circ \cdots \circ \phi_l PML(\Sigma) = \phi PML(\Sigma).
\]

The \( i \)-th adjacency of this sequence, i.e. between \( \phi_1 \circ \cdots \circ \phi_{i-1} PML(\Sigma) \) and \( \phi_1 \circ \cdots \circ \phi_{i} PML(\Sigma) \) is the image under \( \phi_1 \circ \cdots \circ \phi_{i-1} \) of the adjacency given by the adjusted move \( \phi_i \). This motivates the following

**Definition 4.2.** Let

\[
\phi = \phi_1 \circ \cdots \circ \phi_l
\]

be a relation in \( \text{Out}(F_n) \). We say that \( E \) is a overlap or containment problem at the \( i \)-th step of the relation, if \( (\phi_1 \circ \cdots \circ \phi_{i-1})^{-1}(E) \) is an overlap or containment problem for the adjusted move \( \phi_i \).

Our strategy will be to replace adjusted moves \( \phi \) by such relations, so that a given factor \( E \) is not an overlap or containment problem at any stage of the relation. Recall that later, such relations will guide the construction of BLAS paths in which \( E \) will no longer be an obstruction to arationality at any point.

The details of this approach are involved and so we now state the two main ingredients, eliminating containment problems (Proposition 4.3) and eliminating overlap problems (Proposition 4.4), and prove Theorem 4.1 conditional on these results. (We will prove Proposition 4.3 in the next subsection, prove Proposition 4.4 in the orientable case in the following subsection and prove Proposition 4.4 in the non-orientable case in Appendix B.)

**Proposition 4.3.** Suppose that \( E < F_n \) is a proper free factor, and \( \phi \) is an adjusted move. If \( E \) is a containment problem for \( \phi \), then there is a relation

\[
\phi = \phi_1 \cdots \phi_l
\]

with the property that \( E \) is not a containment problem at any stage of the relation.

**Proposition 4.4.** Suppose that \( \phi \) is an adjusted move, and \( E \) is a free factor which is not a containment problem for \( \phi \). Then there is a relation

\[
\phi = \phi_1 \cdots \phi_l,
\]

where each \( \phi_i \) is an adjusted move, and so that \( E \) is neither a containment nor an overlap problem at any stage of the relation.
If \( \psi \) is a partial pseudo-Anosov supported on the good subsurface of the basic move \( \phi \) then there is an number \( k > 0 \) so that for any \( n \geq 0 \) the conjugated relation

\[
\phi = \psi^{-kn} \phi_1 \cdots \phi_l \psi^{kn}
\]

has the same property.

**Proof of Theorem 4.1 assuming Propositions 4.3 and 4.4.** Let \( \phi \) be the adjusted move by which the adjacent PMLs differ. Using first Proposition 4.3 and then Proposition 4.4 (to each adjusted move appearing in that first relation), we can replace \( \phi \) by a relation

\[
\phi = \phi_1 \cdots \phi_l
\]

so that in each stage \( E \) is neither an overlap nor containment problem.

Now we will use Proposition 3.12 to find a BLAS path \( q \) going through the chain of PMLs defined by the relation, that is

\[
\text{PML}(\Sigma_1), \phi_1 \text{PML}(\Sigma_1), \ldots, \phi_2 \cdots \phi_l \text{PML}(\Sigma_1), \phi \text{PML}(\Sigma_1) = \text{PML}(\Sigma_2)
\]

We claim that this path \( q \) itself is disjoint from \( K_E \). Observe that it suffices to check this at all of the points of the BLAS path which are not minimal, and therefore dual to the stable lamination of a partial pseudo-Anosov. For these finitely many points, Lemma 2.21 applies (exactly because we have guaranteed that \( E \) is not an overlap or containment problem at any stage of the relation by Proposition 4.4), and shows that they are outside \( K_E \) as well.

To prove Property (1), i.e. that \( \psi^k q \) is disjoint from \( K_E \), observe that \( \psi^k q \) can be thought of as a BLAS path guided by the conjugated relation

\[
\phi = \psi^{-k} \phi_1 \cdots \phi_l \psi^{k},
\]

to which (by the last sentence of Proposition 4.4) the same argument applies.

Finally, Property (2) is implied by Proposition 2.18, assuming that the BLAS path is constructed to never intersect the unstable lamination of \( \psi \) – which can be done by avoiding a single lamination in the construction of the BLAS path, and is therefore clearly possible. □

### 4.1. Containment problems: Proof of Proposition 4.3

The proof of this proposition relies on the construction of a curve with certain properties.

**Lemma 4.5.** Suppose we are given elements \( z, w, a \) of our chosen standard geometric basis \( \mathcal{B} \), all of which are two-sided, and no two of which are linked. Then there is a two-sided curve \( \delta \) with the following properties:

i) \( \delta \) intersects \( a \) in a single point,

ii) \( \delta \) does not cross the band corresponding to \( w \) (i.e. in the fundamental group, \( \delta \) can be written without the letter \( w \)),

iii) There is another, unrelated two-sided letter \( e \), so that \( \delta \) does not cross the band corresponding to \( e \) (i.e. \( \delta \) can be written without \( e \)),

iv) \( \delta \) intersects each basis loop of \( \mathcal{B} \) at most two points, one on an initial and one on a terminal segment,

v) in homology we have \([\delta] = \pm[a] \pm [z]\).

Indeed, the desired curve can be found as the concatenation of \( a \) and \( z \) as in Figure 4.
Figure 4. The curve from Lemma 4.5

Proof of Proposition 4.3. We begin by considering the basic move \( \phi = \lambda_{x,y} \) or \( \phi = \rho_{x,y} \), and assume that \( E \) is a containment problem for the basic move. Let \( z \) be a basis letter so that \( [z] \notin H_1(E) \). Such a letter exists, since \( E \) is a proper free factor. Also observe that by Lemma A.1 (since we assume that \( E \) is a containment problem), \( z \) is one of \( x, \hat{x}, y, \hat{y} \) or the one-sided letter \( n \). Next, choose \( w \) an unrelated, two-sided letter, i.e. different from all of \( x, \hat{x}, y, \hat{y} \), and not linked with \( z \); in particular it is good for \( \phi \) (i.e. contained in the good subsurface) by Lemma A.1. Let \( a \) be a two-sided basis element so that \( a, \hat{a} \) are good for \( \phi \) and \( \lambda_{z,w} \), and are distinct from any of the previously chosen letters.

Now, let \( \delta \) be the curve guaranteed by Lemma 4.5. Define an auxiliary adjusted move

\[
\theta = T_\delta \lambda_{w,z} T_\delta^{-1}.
\]

We observe that \( \theta \) has the following properties:

1. \( \theta \) fixes every basis element, except possibly \( w \).
   Namely, by property iv) of the curve \( \delta \), the Dehn twist \( T_\delta \) acts on each basis element by conjugation, left, or right multiplication by a word obtained by tracing \( \delta \) starting at a suitable point. By property ii), none of these words involve the letter \( w \). Thus, the letter \( w \) appears only in the image of \( w \) in \( T_\delta(B) \). Since \( \lambda_{w,z} \) fixes all letters except \( w \), the claim follows.

2. The word \( \theta(w) \) does not involve the letter \( e \) from Lemma 4.5.
   This follows from the description of the action of \( T_\delta \) above, together with property iii) of \( \delta \).

3. \( [\theta(w)] = [w] + [z] \notin H_1(E) \).
   Namely, by property v) of \( \delta \), we have \( [T_\delta(w)] = [w] \) (as the algebraic intersection between \( \delta \) and \( w \) is zero). Thus, \( [\lambda_{w,z} T_\delta(w)] = [w] + [z] \). Since the algebraic intersection number between \( \delta \) and \( z \) is also zero, we therefore have \( [\theta(w)] = [w] + [z] \). Since \( w \) is good for \( \lambda_{x,y} \), and we assume that \( E \) is a containment problem for \( \phi \), we have \( [w] \in H_1(E) \). Thus, since \( [z] \notin H_1(E) \), the claim follows.

4. \( E \) is not a containment problem for \( \theta \).
   Since \( a \) and \( \hat{a} \) are good for \( \phi \), and we assume that \( E \) is a containment problem, it follows that \( [a] + [\hat{a}] \in H_1(E) \). On the other hand, the good subsurface of \( \theta \) can
be obtained from the good subsurface of $\lambda_{w,z}$ by applying $T_3$. Recall that $a$ is good for $\lambda_{w,z}$, and thus $T_3(a)$ is good for $\theta$. In homology we have $[T_3(a)] = [a] + [\hat{a}] + [z]$. Since $[a] + [\hat{a}] \in H_1(E)$ but $[z] \notin H_1(E)$ this implies $[T_3(a)] \notin H_1(E)$.

(5) The automorphism $\phi^{-1}\theta^{-1}\phi$ fixes all basis elements except possibly $w$.

This is an immediate consequence of the fact that $\theta$ fixes all letters except $w$, which is distinct from $x, y$ (which are the only letters involved in $\phi$).

This shows that there is a relation of the form

$$\phi = \theta \phi D_w \theta^{-1},$$

where $D_w$ is a product of basic moves of the form $\phi_{w,q}$ acting on the letter $w$, and no $q$ is equal to $e$. Hence, $D_w$ commutes with $\lambda_{c,z}$, and we obtain a relation

$$\phi = \theta \phi \lambda_{c,z} D_w \lambda_{c,z}^{-1} \theta^{-1}.$$  

Observe that $D_w$ may be identity; in which case we also remove the $\lambda_{c,z}$-terms from this relation.

We now check that this relation has no containment problems, as claimed. Recall that we have to check this left-to-right.

- **The initial $\theta$ move:** This is (4) above.
- **The move $\phi$:** Here, we observe that $w$ is good for the two possibilities $\lambda_{x,y}$ and $\rho_{x,y}$ that $\phi$ can be. On the other hand, we have that $\theta(w) \notin E$ by (3) above. Thus, $w \notin \theta^{-1}E$, and the claim follows.
- **The auxiliary move $\lambda_{c,z}$:** This move has $w$ as a good letter, and $\theta \lambda_{x,y}(w) = \theta(w) \notin E$ just like above.
- **The moves in $D_w$:** These basic moves all have $e$ as a good letter. Now, we have $\theta \phi \lambda_{c,z}(e) = \theta(\epsilon z) = \epsilon z$, and $[\epsilon] + [z] \notin H_1(E)$.
- **Undoing the auxiliary move $\lambda_{c,z}$:** This move has $w$ as a good letter, and $\theta \phi \lambda_{c,z} D_w(w) = \theta \phi D_w(w) = \phi \theta(w)$ by Equation (1). But, $[\phi \theta(w)] = [z] + [w] \notin H_1(E)$. Thus, $\theta \phi \lambda_{c,z} D_w(w) \notin E$.
- **The final $\theta^{-1}$ move:** Here, we use again (as in (4) above) that $T_3(a)$ is good for $\theta$. We have $[T_3(a)] = [a] + [\hat{a}] + [z]$, and thus

$$[\theta \phi \lambda_{c,z} D_w \lambda_{c,z}^{-1} T_3(a)] = [a] + [\hat{a}] + [z]$$

which does not lie in $H_1(E)$. Thus, we do not have a containment problem.

Finally, we need to deal with a basic inversion move $\phi = \iota_x$. Again, assume that $E$ is a containment problem. Thus, either $[\hat{x}] \notin H_1(E)$ or $[\hat{z}] \notin H_1(E)$. In the former case, we start with the relation

$$\iota_x = \lambda_{x,z} \iota_x \rho_{x,z},$$

and in the latter case, with a relation of the type

$$\iota_x = \rho_{\hat{z},z} \iota_x \rho_{\hat{z},z}^{-1}$$

for an unrelated letter $z$. Observe that in either case, $\lambda_{x,z}^{-1} E$ or $\rho_{\hat{z},z}^{-1} E$ is not a containment problem for $\iota_x$.

Now, suppose we are in the former case (the other one is completely analogous). Then, by the first part of the proof, we can find a relation

$$\lambda_{x,z} = \phi_1 \circ \cdots \circ \phi_i,$$
so that $E$ is not containment problem at any step of this relation. Hence, in the relation

$$\iota_x = \phi_1 \circ \cdots \circ \phi_i \iota_x \rho_{x,z},$$

the factor $E$ is now not a containment problem at the first $i+1$ steps. Now, appealing to the first part of the proof again, we can find a relation

$$\rho_{x,z} = \phi'_1 \circ \cdots \circ \phi'_j,$$

so that $\lambda_{x,z}(E)$ is not a containment problem at any step. Then, the relation

$$\iota_x = \phi_1 \circ \cdots \circ \phi_i \iota_x \rho_{x,z} \circ \cdots \circ \phi'_j,$$

has the desired properties.

Finally, we discuss adjusted moves. Suppose $\phi = \varphi^{-1}\varphi_0\varphi$ is the conjugate of a basic move by a mapping class group element $\varphi$ of $\Sigma$. We then apply the Proposition to the basic move $\varphi_0$ and the factor $\varphi^{-1}E$, and conjugate the resulting by $\varphi$. This resulting relation (of adjusted moves) then has the desired property. □

4.2. Overlap Problems: Proof of Proposition 4.4. The proof of Proposition 4.4 is technically very involved, and the details vary depending on the nature of the move $\phi$. First, observe that exactly as in the last paragraph of the proof of Proposition 4.3, the case of adjusted moves can be reduced to the case of basic moves. The rest of this section is therefore only concerned with basic moves.

For basic moves, we will (again, similar to the proof of Proposition 4.3), reduce the case of invert moves to the case of Nielsen moves. For Nielsen moves the relation claimed in the proposition will be constructed using the following two lemmas, which construct a “preliminary relation”, and “short relations”:

**Lemma 4.6** (Preliminary Relation). Under the assumptions of Proposition 4.4, if $\phi$ is not an invert move, there is a relation

$$\phi = \phi_1 \cdots \phi_r,$$

so that if $B_i$ is the bad subgroup for $\phi_i$ and $B$ is the bad subgroup of $\phi$, then (up to conjugation) the intersection

$$\phi_1 \cdots \phi_{i-1}B_i \cap B$$

is trivial or (up to conjugacy) contained in a “problematic” group of the form $\langle \partial \rangle, \langle x_i \rangle$ or $\langle x_i, \partial \rangle$ for some word $x_i$, where $\partial$ is the boundary component of the surface (compare Section 3.8).\footnote{We remark that, in general, the intersection of two subgroups up to conjugacy could be a collection of conjugacy classes of subgroups. Here, the intersection always consists of at most one such conjugacy class.}

**Lemma 4.7** (Short Relations). For all indices $i$ in Lemma 4.6 where the collection of problematic groups is nonempty, there is a relation

$$\phi_i = \rho_i \phi_i \rho_i^{-1},$$

with the properties

1. No conjugacy class of the problematic group $\langle \partial \rangle, \langle x_i \rangle$ or $\langle x_i, \partial \rangle$ is contained in the bad subsurface of $\rho_i$ and in $E$,
2. and also $E \cap \phi_1 \cdots \phi_{i-1} \rho_i B_i$ is trivial.
The proofs of these lemmas construct the desired relations fairly explicitly, and involve lengthy checks. Before we begin with these proofs, we explain how to use the lemmas in the proof of Proposition 4.4. We need two more tools: first, the following immediate consequence of Proposition 2.22 (this corollary is the reason why in our strategy, containment problems need to be solved before overlap problems). Observe that the Proposition 2.22 may be used, since the fundamental group of good subsurfaces are free factors of rank > $18 - 5$, while the fundamental groups of the complements of good subsurfaces have rank at most 5, and so the former can never be contained in the latter up to conjugacy.

Corollary 4.8. Suppose $\phi$ is a basic move with bad subgroup $B = \pi_1(\Sigma - S^p)$, $\psi$ the commuting partial pseudo-Anosov supported on the good subsurface on $\phi$, and $E$ any free factor. Suppose that $E$ is not a containment problem for $\phi$. Then there is a number $k$ with the following property.

If

$$\phi = \phi_1 \cdots \phi_l$$

is a relation, and $M = km$ is large enough, then the conjugated relation

$$\phi = (\psi^M \phi_1 \psi^{-M})(\psi^M \cdots \psi^{-M})(\psi^M \phi_l \psi^{-M}),$$

has the following property: at every step of the relation, an overlap problems with $E$ occurs exactly if $E$ contains elements conjugate into $B \cap B_i$, the intersection of the bad subgroups of the original relation.

Second, we need the following lemma, guaranteeing that within a relation which replaces a move without containment problem, no new containment problems are created.

Lemma 4.9. Assume that $E$ is not a containment problem for $\phi$, and that

$$\phi = \phi'_1 \circ \cdots \circ \phi'_R$$

is a relation. Then, there is a number $k > 0$ so that conjugating the relation by any large power $kN$ of a partial pseudo-Anosov $\psi$ supported on the good subsurface of $\phi$, we can guarantee that $E$ is not a containment problem at any stage of the resulting relation

$$\phi = \psi^{kN} \circ \phi'_1 \circ \cdots \circ \phi'_R \circ \psi^{-kN}.$$

Proof. Let $G_i$ be the fundamental group of the good subsurface at the $i$-th step of the relation, and note that it is a free factor. After conjugating the relation by $\psi^{-n}$, this good free factor becomes $\psi^{-n} G_i$.

Now, recall that the intersection of the free factor $G_i$ with the bad subgroup $B$ of $\phi$ is a free factor of $B$. In particular, since the rank of the bad subgroups is at most 5, but the good free factor $G_i$ has rank strictly larger than 5, it cannot be completely contained in $B$. In other words, there is some element $g_i \in G_i$ which intersects the subsurface in which $\psi$ is supported.

Now, apply Proposition 2.22 for the factor $E$, and $B' = G_i$. Since Conclusion (1) of that proposition is impossible here (as $E$ is not a containment problem for $\phi$), we see that for large $n = kN$, the only classes contained in $\psi^n E$ and $G_i$ are contained in the bad subsurface fundamental group $B$. Since $g_i$ is not contained in $B$, this shows that $E$ is not a containment problem.

We are now ready for the proof of the central result of the section.
Proof of Proposition 4.4. First, we prove the proposition for basic Nielsen moves. We first apply Lemma 4.6 to obtain the preliminary relation, and then Lemma 4.7 to each index it applies to. We then have a relation
\[ \phi = \phi_1 \cdots \phi_{i-1}(\rho_i \phi_i \rho_i^{-1})\phi_{i+1} \cdots \phi_r, \]
which still may have overlap problems (in particular, since \( \rho_i \) may have other, “new” overlap problems, but at least these will be guaranteed to be outside the intersection \( E \cap \phi_1 \cdots \phi_{i-1}B_i \)).

Now, for all \( N > 0 \), Lemma 4.9 shows that for conjugated relation
\[ \phi = \psi^{kN} \circ \phi_1 \cdots \phi_{i-1}(\rho_i \phi_i \rho_i^{-1})\phi_{i+1} \cdots \phi_r \circ \psi^{-kN} \]
the factor \( E \) is not a containment problem at any stage.

Hence, we can apply Corollary 4.8 to the inserted “small relations”, further replacing them by conjugates of suitable powers of the associated pseudo-Anosov of \( \phi_i \), yielding a relation of the form
\[ \phi = \psi^{kN} \circ \phi_1 \cdots \phi_{i-1}(\psi^{-M_i} \rho_i \psi^M_i \phi_i \psi^{-N_i} \rho_i^{-1} \psi^N_i)\phi_{i+1} \cdots \phi_r \circ \psi^{-kN}. \]
Since the preliminary relation had no containment problems at any stage, Lemma 4.9 can again be applied to guarantee that there replacements also do not have containment problems.

Furthermore, Corollary 4.8 implies that for this relation any overlap problems can only occur within the intersection of the bad factor \( B_i \) of \( \phi_i \) and the bad factor of the move \( \rho_i \). Now, by construction, there are no conjugacy classes that both of those factors have in common with \( (\phi_1 \cdots \phi_{i-1})^{-1}E \). Hence, this final relation indeed solves all containment and overlap problems.

If we conjugate this relation by a further power of \( \psi^k \), then Lemma 4.9 shows that in the resulting relation \( E \) is still no containment problem at any stage, and Corollary 4.8 shows the same for overlap problems. This shows the proposition for basic Nielsen moves.

Now, let \( \phi = \iota_x \) be a basic invert move. Since the subgroup generated by basic Nielsen moves is normal, for any product \( \alpha \) of basic Nielsen moves there is a product of basic Nielsen moves \( \beta \), so that
\[ \iota_x = \alpha \iota_x \beta. \]
By choosing \( \alpha \) to be a large power of a pseudo-Anosov mapping class, we may assume that \( \alpha^{-1}E \) is not a containment or overlap problem for \( \iota_x \).

Now (similar to the proof of Proposition 4.3), by applying the current proposition for basic Nielsen moves, we can write
\[ \alpha = \alpha_1 \circ \cdots \circ \alpha_r, \beta = \beta_1 \circ \cdots \circ \beta_s \]
so that \( E \) is not an overlap or containment problem at any stage of the first relation, and so that \( (\alpha \iota_x)^{-1}E \) is not an overlap or containment problem at any stage of the second. The resulting relation
\[ \iota_x = \alpha_1 \circ \cdots \circ \alpha_r \iota_x \beta_1 \circ \cdots \circ \beta_s \]
then has the desired property. \qed

To prove Lemmas 4.6 and 4.7 which construct relations, we need to collect some results on controlling the intersections between finitely generated subgroups of free groups. These results are basically standard (see [Sta83]), but we present them in a form useful for the
connectivity of $\partial F_n$

Throughout, we denote by $R_n$ the rose labelled by the elements of our chosen standard geometric basis $B$. We identify edge-paths in $R_n$ with words in $B$.

Suppose we are given a subgroup
$$A = \langle \alpha_1, \ldots, \alpha_r \rangle$$
where each $\alpha_i$ is a reduced word in our fixed basis $B$. We denote by $R_A$ the subdivided rose labelled by the $\alpha_i$, and by $f: R_A \to R_n$ the graph morphism inducing the inclusion of $A$ as a subgroup of $F_n$ (recall that graph morphisms map vertices to vertices, and edges to edges).

Let $\Gamma_A$ be a graph obtained by folding from $R_A$, so that $f$ factors as $R_A \xrightarrow{p_A} \Gamma_A \xrightarrow{g_A} R_n$ where $g_A$ is an immersion.

**Definition 4.10.**

1. A subword $w$ of one of the $\alpha_i$ is called a certificate in $\alpha_i$, if there is an embedded path $\gamma_w \subset \Gamma_A$, which lifts to a path $\tilde{\gamma}_i$ in $R_A$ contained in the petal corresponding to $\alpha_i$, and representing $w$.
2. We say that a certificate is uncancelable if $\gamma_w$ is disjoint from the images of all other petals $\alpha_j, j \neq i$ of $R_A$ under $p_A$.
3. A reduced word $w$ in $B$ is impossible in $A$, if the corresponding path in $R_n$ does not lift to $\Gamma_A$ (equivalently, there is no path in $\Gamma_A$ labelled by $w$).

**Lemma 4.11 (Dropping Generators – Impossible Certificates).**

Suppose that $A = \langle \alpha_1, \ldots, \alpha_r \rangle$, $B = \langle \beta_1, \ldots, \beta_s \rangle$ are two subgroups (where the $\alpha_i, \beta_j$ are words in a common basis $B$).

Suppose that $\tau$ is an uncancelable certificate in $\alpha_1$, which is impossible in $A$. Then any conjugacy class contained in $A$ and $B$ is also contained in $\langle \alpha_2, \ldots, \alpha_r \rangle$.

**Proof.** Let $x \in A$ be an element which is not conjugate into $\langle \alpha_2, \ldots, \alpha_r \rangle$. Then, let $\gamma \subset \Gamma_A$ be a geodesic representing $x$. Since any loop representing $x$ in $R_A$ has to involve $\alpha_1$, and by definition of uncancelable certificate, $\gamma$ contains a subpath labelled by $\tau$. Thus, the geodesic $g_A(\gamma)$ contains a subpath $g_A(\tau)$ which, as $\tau$ is impossible for $B$, is in the image of no loop $\gamma' \subset \Gamma_B$ under $g_B$. This shows the claim. $\square$

We need a version of the dropping letters lemma which applies when $A$ and $B$ share a generator.

**Lemma 4.12 (Dropping Generators – Impossible Unique Followup).** We are given two subgroups
$$A = \langle \alpha_1, \ldots, \alpha_r, \delta_A \rangle$$
$$B = \langle \beta_1, \ldots, \beta_s, \delta_B \rangle,$$
where the $\alpha_i, \delta_A, \beta_j, \delta_B$ are words in a fixed basis $B$.

Suppose that
1. $\beta_1$ contains an uncancelable certificate $\tau$,
2. the only path $\tau_A$ in $\Gamma_A$ which lifts to $\tau$ is contained within the geodesic representative $\delta_A$ of the image of $\delta_A$ in the immersed graph $\Gamma_A$,
3. there is an uncancelable certificate $\tau'$ in $\delta_A$ whose image immediately follows $\tau_A$,
(4) no path corresponding to a reduced word $\beta_1 b$ (for $b \in B$). lifts to a path starting with $\tau \tau'$. Then any conjugacy class in $A$ and $B$ is also contained in

$$B = \langle \beta_2, \ldots, \beta_s, \delta_B \rangle.$$ 

**Proof.** The proof is very similar to the previous one. Let $x \in B$ be an element which is not conjugate into $\langle \beta_2, \ldots, \beta_s, \delta_B \rangle$. Then, let $\gamma \subset \Gamma_B$ be a geodesic representing $x$. Since any loop representing $x$ in $R_B$ has to involve $\beta_1$, and by definition of uncancellable certificate, $\gamma$ contains a subpath labelled by $\tau$. Now, suppose $\gamma' \subset \Gamma_A$ is a geodesic representing the same conjugacy class $x$. Then, $\gamma'$ contains a subpath labelled by $\tau$, and by (2) this occurs in $\delta_A$ and is followed by $\tau'$. By uncancelability, $\tau'$ also follows $\tau$ in the loop $\gamma$ – which contradicts (4). □

Finally, we need the following well-known fact.

**Lemma 4.13** (Intersecting with factors). Let

$$\partial = \prod_{i=1}^{g}[\hat{a}_i, a_i^{-1}]$$

or

$$\partial = (n^{\hat{b}\hat{b}^{-1}n}) \prod_{i=1}^{g}[\hat{a}_i, a_i^{-1}]$$

be the boundary of the surface. Then $\partial$ is contained in no proper free factor of the free group.

**Proof.** The claim follows immediately from Whitehead’s algorithm [Whi36], since the Whitehead graph for $\partial$ is a single loop in both cases, and therefore has no cut point. □

We are now ready to prove the lemmas.

**Proof of Lemma 4.6.** The construction depends on the nature of the involved letters (one-or two-sided, linked with the one-sided or not; as in Section 3) of the basic move $\phi$.

Here, we discuss the case of $\phi = \rho_{x,y}$ on an orientable surface in detail. The computations for the other cases follow the same general approach; we have collected the details in Appendix B.

For ease of notation in this construction, we assume that the order of loops in the basis is

$$\hat{x}, x, \hat{y}, y, \hat{a}_3, a_3, \ldots$$

Thus, the bad subgroup is $B = \langle y, \hat{y}, x^{-1} \hat{x} x, \partial_{\hat{y}} \rangle$, where $\partial_{\hat{y}}$ denotes the cyclic permutation of the boundary word

$$\partial = [\hat{x}, x^{-1}][\hat{y}, y^{-1}] \prod_{i>2}^{g}[\hat{a}_i, a_i^{-1}]$$

starting at $\hat{y}$.

We use the relation

$$\rho_{x,y} = \rho_{\hat{y},u}^{-1} \rho_{y,z}^{-1} \rho_{x,y} \rho_{x,z} \rho_{y,z} \rho_{y,\hat{y},u},$$

where $u = a_4, z = a_6$ and $g \geq 7$. 
\[ A = \langle u, \hat{u}, y, \partial y^{-1} \rangle, \]

where \( \partial y^{-1} \) is the cyclic permutation of \( \partial \) beginning with \( y^{-1} \). We need to intersect this subgroup with

\[ B = \langle y, \hat{y}, x^{-1} \hat{x}, \partial \hat{y} \rangle \]

We begin by finding graphs which immerse into the rose with petals corresponding to the basis \( B \), and which represent \( A \) and \( B \).

We begin with \( A \). Here, the starting point is a rose with four petals corresponding to the four generators \( u, \hat{u}, y, \partial y^{-1} \). This is not yet immersed, as the petal corresponding to \( \partial y^{-1} \) begins and ends with segments \( y^{-1} \hat{y}^{-1} y \) and \( \hat{y} \) which can be folded over the other petals. The resulting folded petal \( \partial A \) starts with \( \hat{a}_3 \) and ends with \( a_5 \). Hence, this resulting graph immerses (compare the left side of Figure 5).

The immersed graph for \( B \) is similarly obtained by first folding the first and last segment of the petal labeled by \( x^{-1} \hat{x}x \) together, and then folding the initial commutator \([\hat{y}, y^{-1}]\) and last segment \( x^{-1} \hat{x}x \) of \( \partial \hat{y} \) over the rest. We denote by \( \partial B \) the image of this folded petal; note that it is still based at the same point (compare the right side of Figure 5).

To compute the conjugacy classes in the intersection of these groups, we begin by using Lemma 4.12 with \( \tau = u \) as the input path. Observe that it is indeed uncancellable in \( A \), and appears in \( B \) only in the petal \( \partial B \).

Since \( u = a_4 \) and the rank is at least 6, the petal \( \partial B \) will contain a subpath labelled \([\hat{a}_4, a_4^{-1}]\[\hat{a}_5, a_5^{-1}]\). We let \( \tau' \) be the (uncancellable) path \( a_4[\hat{a}_5, a_5^{-1}] \) following \( u \) in this subpath.

Observe that this it is impossible to achieve such a path in \( A \) starting with \( u \), since \( a_5 \) appears only in the interior \( \partial A \). Hence, Lemma 4.12 applies, and any conjugacy class contained in \( A \) and \( B \) is in fact also contained in

\[ A' = \langle \hat{u}, y, \partial y^{-1} \rangle. \]

Hence, we now aim to compute the intersection of \( A' \) and \( B \) using the same method. The immersed graph for \( A' \) is obtained by simply deleting the petal labeled \( u \) from the graph for \( A \). We can then argue exactly as above (with the input path \( \tau = \hat{u} \)) to also drop the generator \( \hat{u} \), and find that any conjugacy class common to \( A \) and \( B \) is also contained in

\[ A'' = \langle y, \partial y^{-1} \rangle. \]
Observe that this rank-2 group is indeed contained in both $A$ and $B$, and so it is the full intersection. Since it has the desired form, we are done with this step.

b) $\rho_{y,z}^{-1}$ has bad subgroup $A_2 = \langle z, \hat{z}, y^{-1} \hat{y} y, \partial_{a_3} \rangle$, which we need to intersect with
\[ \rho_{\hat{y},u}B = \langle y, \hat{y} u, x^{-1} \hat{x} x, \rho_{\hat{y},u} \partial_{\hat{y}} \rangle. \]

For this intersection, we need to take some care of the order of simplifications. We begin by observing that the path $\hat{y} u$, which corresponds to a petal of the immersed graph of $\rho_{\hat{y},u}B$ is impossible in $A_2$ – the only generator which contains $u = a_4$ at all is $\partial_{a_3}$, and there it is never directly adjacent to $y$. Hence, by Lemma 4.11 we may replace $\rho_{\hat{y},u}B$ by
\[ \langle y, x^{-1} \hat{x} x, \rho_{\hat{y},u} \partial_{\hat{y}} \rangle. \]

Now, we can further remove $\rho_{\hat{y},u} \partial_{\hat{y}}$, as it also contains $\hat{y} u$ as a subword (observe that this would have been impossible as the first step, since this subword was folded over the petal $\hat{y} u$ in the original immersed graph). Now, we need to compare
\[ \langle y, x^{-1} \hat{x} x \rangle \quad \text{and} \quad A_2 = \langle z, \hat{z}, y^{-1} \hat{y} y, \partial_{a_3} \rangle. \]

From the latter, we can drop $\partial_{a_3}$, since it clearly contains uncancellable subwords which are impossible in the former (again, using Lemma 4.11). Then, it is easy to see that the remaining groups have no conjugacy classes in common (by drawing immersed graphs representing them, or further applying Lemma 4.11).

c) $\rho_{x,y}$ has bad subgroup $B = \langle y, \hat{y}, x^{-1} \hat{x} x, \partial_{\hat{y}} \rangle$ and we need to intersect with
\[ \rho_{y,z} \rho_{\hat{y},u}B = \langle yz, \hat{y} u, x^{-1} \hat{x} x, \rho_{y,z} \rho_{\hat{y},u} \partial_{\hat{y}} \rangle. \]

The argument is similar to b). We first focus on the generators $yz, \hat{y} u$ of $\rho_{y,z} \rho_{\hat{y},u}B$. Using Lemma 4.11, we can drop these in order to compute the intersection (as these certificates are impossible $B$). After that is done, we can then also further drop $\rho_{y,z} \rho_{\hat{y},u} \partial_{\hat{y}}$ from $\rho_{y,z} \rho_{\hat{y},u}B$ using Lemma 4.11 again, as $yz$ or $\hat{y} u$ are now certificates (after the previous step, these survive in the immersed graph), which are impossible in $B$. Hence, the intersection is $\langle \hat{x} \rangle$.

d) $\rho_{x,z}$ has bad subgroup $A_3 = \langle z, \hat{z}, x^{-1} \hat{x} x, \partial_{\hat{y}} \rangle$, and we need to intersect with
\[ \rho_{x,y}^{-1} \rho_{y,z} \rho_{\hat{y},u}B = \langle yz, \hat{y} u, yx^{-1} \hat{x} x y^{-1}, \rho_{x,y}^{-1} \rho_{y,z} \rho_{\hat{y},u} \partial_{\hat{y}} \rangle. \]

We begin by dropping $\hat{y} u$ from the latter, since it is impossible in $A_3$. Afterwards, we can also drop $\rho_{x,y}^{-1} \rho_{y,z} \rho_{\hat{y},u} \partial_{\hat{y}}$ (since it also contains $\hat{y} u$, and this subpath is now certainly not folded over anymore, as above). After this, we can remove $\partial_{\hat{y}}$ from $A_3$ since it contains (many) subpaths which are impossible in the other group. At this stage, we need to compare
\[ \langle z, \hat{z}, x^{-1} \hat{x} x \rangle \quad \text{and} \quad \langle yz, yx^{-1} \hat{x} x y^{-1} \rangle, \]
whose intersection is clearly $\langle \hat{x} \rangle$.

e) $\rho_{y,z}$ has bad subgroup $A_2 = \langle z, \hat{z}, y^{-1} \hat{y} y, \partial_{a_3} \rangle$ and we need to intersect with
\[ \rho_{x,y}^{-1} \rho_{x,y}^{-1} \rho_{y,z} \rho_{\hat{y},u}B = \langle yz, \hat{y} u, x^{-1} \hat{x} x, \rho_{x,y}^{-1} \rho_{x,y}^{-1} \rho_{y,z} \rho_{\hat{y},u} \partial_{\hat{y}} \rangle. \]

As before, we start by removing $\hat{y} u$ from the latter, then the boundary word from both. The remaining intersection between
\[ \langle z, \hat{z}, y^{-1} \hat{y} y \rangle \quad \text{and} \quad \langle yz, x^{-1} \hat{x} x \rangle, \]
is trivial.
f) Finally, $\rho_{\hat{y},u}$ has bad subgroup $A = \langle u, \hat{u}, y, \partial_{\hat{y}^{-1}} \rangle$, which we intersect with

$$\rho_{y^{-1},u}^{-1}\rho_{x,\hat{z}}^{-1}\rho_{y,z}^{-1}\rho_{y,\hat{u}}B = \langle y, yu, x^{-1}\hat{x}x, \rho_{x,z}^{-1}\rho_{y,z}^{-1}\rho_{y,\hat{u}} \rangle$$

to find in $\langle y \rangle$ (arguing as before).

The relation for $\lambda_{x,y}$ is similar, with $\rho$ changed to $\lambda$. The case where either $x, y$ or both are “hatted letters” is also analogous.

Proof of Lemma 4.7 As in the previous lemma, the details vary depending on the nature of $\phi$, and the construction is explicit. In contrast to the previous lemma, the arguments are straightforward here, and we only give the details for the case discussed in the proof of Lemma 4.6. The letters below indicate the terms in the relation constructed in that proof.

a) We perform $\lambda_{y,w}$ before this move and $\lambda_{y,\hat{w}}^{-1}$ after. Note that these moves indeed commute with $\rho_{\hat{y},u}^{-1}$.

The bad subgroup of $\lambda_{y,w}$ is $\langle \hat{y}, w, \hat{w}, \partial_{\hat{y}} \rangle$. We want to compute the intersection with the rank 2 intersection group from step a) of the previous lemma, i.e. with $\langle y, \partial_{\hat{y}^{-1}} \rangle$.

Using e.g. Lemma 4.12 we can see that the intersection of these two is in fact $\langle \partial_{\hat{y}^{-1}} \rangle$.

By Lemma 4.13 $\langle \partial_{\hat{y}^{-1}} \rangle$ intersects $E$ trivially, and so (1) holds as claimed.

Finally, as $yw$ is not bad for $\rho_{\hat{y},u}^{-1}$, claim (2) holds.

b) No need
c) We perform $\rho_{x,w}$ before this move and $\rho_{x,\hat{w}}^{-1}$ after. Note they commute with $\rho_{x,y}$, and that $\hat{x}$ is not bad for $\rho_{x,w}$. As $\hat{x}w$ is not bad for $\rho_{x,y}$, the conclusion holds.

d) No need
e) No need
f) This is analogous to a).

5. Proof of Theorem 1.1

Before proving the main theorem, we establish the main ingredient, that the set of arational surface type elements of $\overline{CV}_n$ (even in different copies of $\mathcal{PML}$) is path connected. Note that the last sentence of Theorem 1.1 is used with Proposition 5.3 to prove Theorem 1.1

Theorem 5.1. If $x_s$ and $x_c$ are dual to uniquely ergodic (or, in the nonorientable case, elements of $\mathcal{P}$) surface type elements of $\overline{CV}_n$ then there exists $p : [0,1] \to \overline{CV}_n$ continuous so that $p(t)$ is arational for all $t \in [0,1]$. Moreover, for any $\epsilon > 0$ and combinatorial chain of $\mathcal{PML}$s from $x_s$ to $x_c$ we may assume that this path is in an $\epsilon$ neighborhood of that chain.

Proposition 5.2. Let $p : [0,1] \to \overline{CV}_n$ be a BLAS path, $K_E$ as in Proposition 2.7 and $\epsilon > 0$ be given. There exists $p' : [0,1] \to \overline{CV}_n$ so that

1. the distance from $p(x)$ to $p'(x)$ is at most $\epsilon$ for all $x \in [0,1]$,
2. $p'([0,1]) \cap K_E = \emptyset$.

Proof. It suffices to prove the proposition in the case where there is exactly one point in $p([0,1])$ which is not arational, call that point $\sigma$. Recall, from the definition of BLAS paths, that in that case $\sigma$ is the dual tree to a stable lamination of a partial pseudo-Anosov $\psi_\sigma$ (for some identification with a surface). Also recall that in a neighbourhood of $\sigma$ the path $p$ has the form $\cup_b \psi_b^x \xi_i$ for $i = 1, 2$. Let $x_1$ be the starting point of $\xi_1$ (which means $\psi_\sigma x_1$...
trees with the same dual lamination as $p$. Let $q$ be the path as in Theorem 4.1 with $x_1, x_2, K_i$ and $\psi_\sigma$. By Theorem 4.1, there exists $k_0$ so that for all $k \geq k_0$, the distance from $\psi_\sigma^k(q([0, 1])$ to $\sigma$ is at most $\frac{\epsilon}{2}$. Let $k_1 \geq k_0$ so that $\psi_\sigma^{k_1}(q([0, 1])) \cap K_i = \emptyset$ and the Hausdorff distance from $\psi_\sigma^{k_1}$ and $\psi_\sigma^{k_2}$ to $\sigma$ is at most $\frac{\epsilon}{2}$. This exists by Theorem 4.1. Let $p' = p$ outside of $\cup_{i=k_1}^{\infty} \gamma_i$ and let $p'(t) = \psi_\sigma^{k_1}q$ on $p \setminus \cup_{i=k_1}^{\infty} \gamma_i$. Condition (1) is clear for the $x$ so that $p(x) = p'(x)$. All other $x$ have that the distance from both $p(x)$ and $p'(x)$ to $\sigma$ is at most $\frac{\epsilon}{2}$. Condition (2) is obvious for the points in $p'$ that are arational. The other points are contained in $\psi_\sigma^{k_1}q$, which was constructed to avoid $K_i$. 

Proof of Theorem 5.1.\] Enumerate the set of proper free factors in some way as $F = \{E_i, i \in \mathbb{N}\}$, and denote by $K_i = K_{E_i}$. By Proposition 3.12 there exists a BLAS path from $x_s$ to $x_e$. Let $\epsilon > 0$ be given. By Proposition 5.2 with $\epsilon_0 := \epsilon = \frac{\epsilon}{4}$ we may assume $p([0, 1]) \cap K_1 = \emptyset$. Since $K_1$ is closed and $p([0, 1])$ is compact, there exists $\epsilon_1 > 0$ so that $\text{dist}(p([0, 1]), K_i) > \epsilon_1$. Inductively we assume that we are given a BLAS path $p_i$ and a $\epsilon_1, \ldots, \epsilon_i > 0$ so that 

$$\text{dist}(p([0, 1], K_j) > \left(1 - \sum_{i=j+1}^i \epsilon_j \right) \epsilon_j > \frac{1}{2} \epsilon_j$$

for all $j \leq i$. By Proposition 5.2 with $\epsilon = \epsilon_{i+1} = \frac{1}{3} \min \{\epsilon_j\}_{j=1}^i$ and $p = p_i$ and $K = K_{i+1}$ there exists a BLAS path, $p_{i+1}$ from $x_s$ to $x_e$ satisfying equation (3) for all $j \leq i + 1$. Let $p_\infty$ be the limit of the $p_i$. By our inductive procedure our sequence of functions $p_i$ converges. By 3, we have $\text{dist}(p_\infty([0, 1]), K_i) \geq \frac{1}{2} \epsilon_i > 0$ for all $i$. Thus by Proposition 2.7 we have a path from $x_s$ to $x_e$ so that every $p_\infty(t)$ is arational for all $t \in [0, 1]$, establishing Theorem 5.1. \]

To complete the proof of Theorem 1.1 we need the following result:

Proposition 5.3. Let $T \in \partial CV_n$ be an arational tree, and $\Delta$ be the simplex of arational trees with the same dual lamination as $T$.

For every neighborhood $U$ of $\Delta$ in $\overline{CV_n}$ there is a smaller neighborhood $V$ with the following property. Suppose $x, y \in V$ are arational and dual to surface laminations (possibly on different surfaces). If the dual lamination to $T$ is supported on a surface $\Sigma$, we additionally assume that neither $x, y$ are dual to laminations on $\Sigma$. Then $x$ and $y$ can be joined by a chain of consecutively adjacent $\mathcal{PMLC}$’s, each of which is contained in $U$.

The proof of this proposition requires a variant of [BR15, Theorem 4.4]. In its statement we denote by $L(T)$ the dual lamination to a tree $T$. Given a lamination $L$ we denote by $L'$ the sublamination formed by all non-isolated leaves of $L$.

Proposition 5.4. Let $T \in \partial CV_n$ be an arational tree. If $\mu$ is a current so that 

$$\langle T, \mu \rangle = 0,$$

and $U \in \partial CV_n$ is another tree with 

$$\langle U, \mu \rangle = 0,$$

then

1. either, the dual laminations of $T, U$ agree: $L(U) = L(T)$, or
2. $T$ is dual to a lamination on a surface $S$, and the support of $\mu$ is a multiple of the boundary current $\mu_{\partial S}$ of that surface.
Proof. By the assumption on $T, \mu$, [KL10] Theorem 1.1 yields

$$\text{Supp}(\mu) \subset L(T).$$

We begin with the case where $T$ is not dual to a surface lamination. In this case, [BR15] Proposition 4.2 (i]) applies, and shows that $L(T)$ is obtained from the minimal lamination $L'(T)$ by adding isolated leaves, each of which is diagonal and not periodic. On the other hand, the support of a current cannot contain non-periodic isolated leaves. Thus, we then have $\text{Supp}(\mu) \subset L'(T)$, hence $\text{Supp}(\mu) = L'(T)$ by minimality.

Applying [KL10] Theorem 1.1 to $U, \mu$ yields

$$L'''(T) \subset L'(T) = \text{Supp}(\mu) \subset L(U).$$

In this case, [BR15] Corollary 4.3] shows that $L(T) = L(U)$, and we are in case (1).

Now suppose that $T$ is dual to a surface lamination. In this case we need to describe the dual lamination of $T$ more precisely (see also the proof of [BR15] Proposition 4.2 (ii)]). Let $S$ be a hyperbolic surface with one boundary component which is totally geodesic and let $\Lambda$ be a minimal filling measured geodesic lamination on $S$, so that $T$ is the $\mathbb{R}$-tree dual to $\Lambda$.

Consider the universal cover $\tilde{S}$ and the preimage $\tilde{\Lambda}$ of $\Lambda$. The complementary components of $\tilde{\Lambda}$ are ideal polygons and regions containing the lifts of the boundary (these are universal covers of hyperbolic crowns and are bounded by a lift of $\partial S$ and a chain of leaves with consecutive leaves cobounding a cusp) and these, along with non-boundary leaves of $\tilde{\Lambda}$, are in 1-1 correspondence with the points of $T$. The lamination $L(T)$ dual to $T$ consists of pairs of distinct ends of $\tilde{S}$ that are joined by geodesics with 0 measure. Thus the leaves of $L(T)$ are as follows:

(i) leaves of $\tilde{\Lambda}$,
(ii) diagonal leaves in the complementary components that are ideal polygons,
(iii) leaves in the crown regions connecting distinct cusps,
(iv) leaves in the crown regions connecting a cusp with an end corresponding to a lift of $\partial S$,
(v) lifts of $\partial S$.

Recall that $\text{supp}(\mu) \subset L(T)$. Since the leaves of type (ii) and (iii) are isolated and accumulate on leaves of type (i), the measure $\mu$ must assign zero measure to them. Thus the support of $\mu$ is contained in the sublamination of $L(T)$ consisting of leaves of type (i), (iv) and (v). In this sublamination, the leaves of type (iv) are isolated and accumulate on the leaves of both types (i) and (v), so $\mu$ is supported on the disjoint union of $\tilde{\Lambda}$ and the lamination $\Delta$ consisting of the lifts of $\partial S$. Thus

$$\mu = \nu_1 + \nu_2,$$

where $\nu_1$ is supported on $\tilde{\Lambda}$ and $\nu_2$ supported on $\Delta$. If $\nu_2$ assigns $\alpha \geq 0$ to a lift of $\partial S$ then

$$\nu_2 = \alpha \mu_{\partial S}.$$  

Now, if $\nu_1 \neq 0$, then since

$$\langle U, \nu_1 \rangle = 0,$$

we can apply [KL10] Theorem 1.1] to $U, \nu_1$ to obtain

$$L'''(T) = \tilde{\Lambda} = \text{Supp}(\nu_1) \subset L(U),$$

and [BR15] Corollary 4.3] again shows that $L(T) = L(U)$, hence we are in case (1).

Otherwise, $\mu = \nu_2$ and we are in case (2).
Lemma 5.5. For every neighborhood $U$ of $\Delta$ in $\overline{CV}_n$, there is a smaller neighborhood $V$ with the following property. If $\PML(\Sigma)$ intersects $V$, and in the case that $T$ is dual to a surface lamination on $\Sigma'$, is distinct from $\PML(\Sigma')$, then it is contained in $U$.

**Proof.** Suppose such $V$ does not exist. Then we have a sequence of pairwise distinct surfaces $\Sigma_i$ and points $x_i, y_i \in \PML(\Sigma_i)$ such that $x_i \to x \in \Delta$ and $y_i \to y \not\in U$. The boundary curve $\gamma_i$ of $\Sigma_i$ is elliptic in both $x_i$ and $y_i$. After a subsequence, $\gamma_i$ projectively converges to a current $\mu$, and by the continuity of the length pairing we have

$$\langle x, \mu \rangle = \langle y, \mu \rangle = 0.$$

Now, apply Proposition 5.4 to $x, y, \mu$. If we are in case (1) of that proposition, then $y$ has the same dual lamination as $x$ (equivalently, $T$), i.e. $y \in \Delta$. This is a contradiction.

In case (2), we instead conclude that the boundary curves $\gamma_i$ of the $\Sigma_i$ converge (as a current) to the boundary $\gamma$ of the surface $\Sigma$ supporting the dual lamination of $T$.

We choose an $i$ large enough (see below), and consider $\delta = \gamma_i$, written as a cyclically reduced word

$$\delta = \prod \gamma^n_i b_j,$$

in a basis where $\gamma$ is written as a shortest possible word. Since the $\gamma_i$ converge to $\gamma$ as currents, for any given $\epsilon$ we may ensure (by choosing $i$ large enough) that

$$\sum l(b_j) \leq \epsilon \sum n_j l(\gamma).$$

Further, since the $\Sigma_i$ are all distinct, the length of $\gamma_i$ diverges, and we may thus assume that $l(\delta)$ is much larger than $l(\gamma)$.

On the other hand, since the $\gamma_i$ are boundary curves of surfaces, and therefore have uniformly small length $L_0 = l(\gamma)$ in a suitable basis, by Whitehead’s theorem, there is a sequence of Whitehead moves $\phi_k$ so that

$$l(\delta) > l(\phi_1 \delta) > l(\phi_2 \phi_1 \delta) > \ldots$$

Let $B$ be a bounded-cancellation constant that works for all Whitehead moves.

Let $N$ be the largest number so that $l(\phi_1 \cdots \phi_k \gamma) = l(\gamma)$ for all $i \leq N$.

We first claim that $N \geq 1$. Namely, suppose for contradiction that $l(\phi_1 \gamma) \geq l(\gamma) + 1$. Intuitively, the increase in length in $\gamma^n_i$ is much larger than the decrease in length in $b_j$.

More formally, we then have

$$l(\phi_1 \delta) \geq \sum n_j (L_0 + 1 - B)$$

$$\geq \frac{L_0 + 1}{L_0} \sum n_j l(\gamma) - B \sum l(b_j)$$

$$\geq \left( \frac{L_0 + 1}{L_0} - B \epsilon \right) \sum n_j l(\gamma)$$

$$\geq \left( \frac{L_0 + 1}{L_0} - B \epsilon \right) \sum (n_j l(\gamma) + l(b_j)) - \sum l(b_j)$$

$$\geq \left( \frac{L_0 + 1}{L_0} - (B + 1) \epsilon \right) l(\delta),$$

which, if $\epsilon$ is chosen small enough, would imply $l(\phi_1 \delta) > l(\delta)$ which is impossible by the above.
There are a finite number of identifications of $F_n$ with a surface with one boundary component so that the length of its boundary word is $l(\gamma)$. Denote those surfaces by $\Sigma = \Sigma^1, \ldots, \Sigma^K$. By the above, for all $i \leq N$, the maps $\phi_i$ can be represented by homeomorphisms between suitable $\Sigma^r, \Sigma^s$, and thus the same is true for the map $\Psi = \phi_N \cdots \phi_1$.

Now, consider (as unreduced words)

$$\Psi \delta = \prod \Psi(\gamma^{m_j}) \Psi(b_j).$$

We can write $b_j$ as a product of at most $l(b_j)$ simple loops on $\Sigma_j = \Sigma^\omega$. Since $\Psi$ is a surface map, $\Psi(b_j)$ is also a product of at most $l(b_j)$ simple nonseparating loops on $\Sigma^s$. Now, a reduced word representing a simple nonseparating loop cannot contain the square of the boundary as a subword (e.g. by considering Whitehead graphs, and observing that any nonseparating simple loop is primitive). Thus, reducing the description of $\Psi \delta$ above removes at most one copy of the boundary word $\Psi(\gamma)$ for each simple component. Hence, there is a reduced description

$$\Psi \delta = \prod_{j=0}^\infty \gamma_j^m c_j$$

which has

$$\sum m_j l(\gamma) \geq \sum n_j l(\gamma) - \sum l(b_j) \geq (1 - \epsilon) \sum n_j l(\gamma),$$

and

$$(1 + \epsilon) \sum n_j l(\gamma) \geq l(\gamma) > l(\Psi \delta) = \sum n_j l(\gamma) + \sum l(c_j),$$

hence

$$\sum l(c_j) \leq \epsilon \sum n_j l(\gamma) \leq \frac{\epsilon}{1 - \epsilon} \sum m_j l(\gamma).$$

But now, if $\epsilon$ was chosen small enough, so that the argument showing $l(\phi_1 \gamma) = L_0$ also applies to $\epsilon' = \frac{\epsilon}{1 - \epsilon}$, then that same argument shows $l(\phi_{N+1} \Psi(\gamma)) = L_0$, contradicting maximality of $N$.

\[\square\]

**Lemma 5.6.** Let $U$ be a neighborhood of $\Delta$ in $CV_n$. There exists a neighborhood $V$ of $\Delta$ in $CV_n$ so that if $x', y' \in CV_n \cap V$ then any folding path from $x'$ to $y'$ is contained in $U$.

**Proof.** Recall from Section 2.2 that there is a coarsely continuous function $\Phi : CV_n \to FF_n$ that restricted to arational trees gives a quotient map to $\partial CV_n$. This map takes folding paths in $CV_n$ to reparametrized quasigeodesics with uniform constants in $FF_n$, and it takes $\Delta$ to a point $[\Delta] \in \partial FF_n$. By the coarse continuity, there is a neighborhood $U'$ of $[\Delta] \in FF_n$ such that $\Phi^{-1}(U') \subset U$. By hyperbolic geometry there is a neighborhood $V' \subset U'$ of $[\Delta]$ such that any quasigeodesic with above constants with endpoints in $V'$ is contained in $U'$. Finally, let $V$ be a neighborhood of $\Delta$ such that $\Phi(V) \subset V'$ ($V$ exists by the coarse continuity).

\[\square\]

Before we can prove Proposition 5.3, we need one more definition. Namely, given an identification $\sigma$ of the free group with $\pi_1(\Sigma)$, we define the extended projective measured lamination sphere $\tilde{PML}_\sigma$ to be the union of $PML_\sigma$ and the subset of $CV_n$ consisting of graphs where the boundary curve of $\Sigma$ crosses every edge exactly twice (alternatively, the graph can be embedded in the surface with the correct marking).

**Proof of Proposition 5.3** Let $U = U_0$ be a given neighborhood of $\Delta$. For a large (for now unspecified) integer $N$ find neighborhoods

$$U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_N$$
of \( \Delta \) so that each pair \((U_i, U_{i+1})\) satisfies Lemmas 5.5 and 5.6. We then set \( V = U_N \). To see that this works, let \( x, y \in V \) be arational and dual to surface laminations. Let \( P_x \) and \( P_y \) be the extended PML’s containing \( x, y \) respectively. Thus \( P_x, P_y \subset U_{N-1} \). Choose roses \( x' \in P_x \cap CV_n \) and \( y' \in P_y \cap CV_n \). After adjusting the lengths of edges of \( x' \) there will be a folding path from \( x' \) to \( y' \) which is then contained in \( U_{N-2} \). We can assume that the folding process folds one edge at a time. We can choose a finite sequence of graphs along the path, starting with \( x' \) and ending with \( y' \), so that the change in topology in consecutive graphs is a simple fold. It follows that the extended PML’s can be chosen so that the surfaces share a subsurface of small cogenus. Further, in each graph we can collapse a maximal tree so we get a rose. Consecutive roses will differ by the composition of boundedly many Whitehead automorphisms and each Whitehead automorphism is a composition of boundedly many basic moves. We can then insert a bounded chain of extended PML’s between any two in our sequence so that in this expanded chain any two consecutive PML’s differ by a basic move. If \( N \) is sufficiently large this new chain will be contained in \( U = U_0 \). \( \square \)

Proof of Theorem 1.1. To prove that \( \partial FF_n \) is path-connected, it suffices to join by a path points \( \Phi(T), \Phi(S) \in \partial FF_n \) where both \( T, S \in \partial CV_n \) are arational trees and \( S \) is dual to a surface lamination \( \lambda \) which is uniquely ergodic (or, in the nonorientable case, an element of \( \mathcal{P} \)). Additionally, if \( T \) is itself dual to a lamination on a surface \( \Sigma \), we assume that \( \lambda \) is not a lamination of that same surface. For brevity, we call such trees good surface trees in this proof.

Let \( \Delta \subset \partial CV_n \) be the simplex of arational trees equivalent to \( T \), and let \( U_1 \supset U_2 \supset \cdots \) be a nested sequence of smaller and smaller neighborhoods of \( \Delta \) so that each pair \((U_i, U_{i+1})\) satisfies Proposition 2.13. Choose \( S_i \in U_i \) to be a good surface tree, for \( i \geq 1 \) (see Lemma 2.13). By Theorem 5.1 there is a path \( p \) from \( S \) to \( S_1 \) in \( \partial CV_n \) consisting of arational trees, and likewise there is such a path \( p_i \) from \( S_i \) to \( S_{i+1} \). By our choice of the \( U_i \) and the last sentence in Theorem 5.1 we can arrange that each \( p_i \) is contained in \( U_{i-1} \) for \( i \geq 2 \).

The concatenation \( q = p * p_1 * p_2 * \cdots \) is a path parametrized by a half-open interval that accumulates on \( \Delta \) since it is eventually contained in \( U_i \) for every \( i \). It may not converge in \( \partial CV_n \) unless \( \Delta \) is a point (that is, \( T \) is uniquely ergometric, see [CHL07]) but \( \Phi(q) \) converges to \( \Phi(T) \), proving path connectivity.

Local path connectivity is similar. The key observation is that, in the construction of paths above, if we choose \( S \in U_{i+1} \) then the path joining \( S \) to the simplex of \( T \) can be chosen to lie in \( U_i \).

Now, recall that a space \( X \) is locally path connected at \( x \in X \) if for every neighborhood \( U \) of \( x \) there is a smaller neighborhood \( V \) of \( x \) so that any two points in \( V \) are connected by a path in \( U \); this implies the ostensibly stronger property that \( x \) has a path connected neighborhood contained in \( U \) (namely, by taking the path component of \( x \) in \( U \)).

Hence, we want to show that for every \( i \) there is \( j > i \) so that if \( T' \in U_j \) is arational, there is a path of arational ns defined on an open interval accumulating to the associated simplices \( \Delta, \Delta' \) on the two ends which is contained in \( U_i \).

We begin by choosing \( j > i \) so that when \( T' \in U_j \) is arational then its simplex \( \Delta' \) is contained in \( U_{i+1} \) (this is possible because \( \Phi(U_{i+1}) \) contains neighbourhoods of \( \Phi(T) \)).

Now pick a \( S \in U_{i+1} \) which is a good surface tree (for both \( T, T' \)) close to \( \Delta' \). The key observation above implies that for any such \( S \) we can find a path from \( S \) accumulating on \( \Delta \) which lies in \( U_i \). If we choose \( S \) close enough to \( \Delta' \), then again by the key observation, we can also find a path from \( S \) accumulating on the simplex of \( \Delta' \), which is contained in \( U_i \).
(by choosing it to lie in the corresponding sequence \( U'_k \) for \( \Delta' \) for a large enough \( k \)). Putting them together gives the desired path. \( \Box \)

6. One-endedness of other combinatorial complexes

In this section, we discuss one-endedness of various combinatorial complexes. To this end, we use the following criterion.

**Proposition 6.1.** Let \( X,Y \) be \( \delta \)-hyperbolic spaces, \( G \) a group acting coboundedly by isometries on \( X \) and \( Y \), and let \( \pi : X \to Y \) be an equivariant Lipschitz map which is alignment preserving. Suppose there is some \( g \in G \) which is loxodromic in \( Y \) (and therefore also in \( X \)).

If \( Y \) is 1-ended, so is \( X \).

Recall [Gui00b, KR14] that a map \( \pi \) is alignment preserving if there is a constant \( C \geq 0 \) such that the image of any geodesic segment is contained in the \( C \)-neighborhood of any geodesic joining the images of the endpoints.

**Remark 6.2.** We want to remark that [KR14] use only the apparently weaker property that \( \pi([x,y]) \) is bounded whenever \( \pi(x), \pi(y) \) are close, rather than alignment preserving. However, they also show that a map between hyperbolic metric spaces with this weaker property is alignment preserving in the stronger sense.

**Proof of Proposition 6.1** Let \( K_X \) be a metric ball in \( X \). We define \( L_X \) to be the Hausdorff \( N \)-neighborhood of \( K_X \), with \( N \) sufficiently large. Let \( x_1, x_2 \in X \setminus L_X \). We will connect \( x_1, x_2 \) by a path in the complement of \( K_X \).

Fix an axis \( \ell \) in \( X \) of \( g \) (i.e. a quasi-geodesic line where \( g \) acts by translation). Since the action of \( G \) on \( X \) is cobounded, there is a translate \( \ell_1 \) of \( \ell \) that passes within a bounded distance from \( x_1 \). Let \( r \) be the ray starting at \( x_1 \), having a bounded initial segment joining \( x_1 \) with \( \ell_1 \), and the rest is one of the two half-lines in \( \ell_1 \). Since \( K_X \) is quasi-convex, there is a choice of a half-line so that if \( N \) is sufficiently large, \( r \) is disjoint from \( K_X \). The image of \( r \) in \( Y \) follows an axis of a conjugate of \( g \), so it goes to infinity in \( Y \). We can thus join \( x_1 \) by a path missing \( K_X \) to a point \( x'_1 \) whose image in \( Y \) misses a bounded set \( P \) such that points in the complement of \( P \) can be joined by paths missing the \( N \)-neighborhood of \( \pi(K_X) \). In the same way we can join \( x_2 \) to a point \( x'_2 \). It now remains to join \( x'_1 \) to \( x'_2 \).

Join \( \pi(x'_1), \pi(x'_2) \) by a path missing the \( N \)-neighborhood of \( \pi(K_X) \). We will now coarsely lift this path to the desired path. Let \( \pi(x_1) = y_1, y_2, \cdots, y_n = \pi(x_2) \) be points along the path at distance \( \leq 1 \). For each \( y_i \) choose a point \( \tilde{y}_i \in X \) whose image in \( Y \) is at a bounded distance from \( y_i \) (this is possible since \( \pi \) is coarsely onto), and so that \( \tilde{y}_1 = x_1 \) and \( \tilde{y}_n = x_2 \). The desired path is the concatenation of geodesic segments joining the consecutive \( \tilde{y}_i \). Since \( \pi \) is alignment preserving, the images in \( Y \) of these geodesic segments are uniformly bounded, so when \( N \) is large they will miss \( \pi(K_X) \), and the path between \( x_1 \) and \( x_2 \) will miss \( K_X \). \( \Box \)

**Corollary 6.3.** For \( n \geq 18 \) the free splitting complex \( FS_n \), the cyclic splitting complex \( FZ_n \), and the maximal cyclic splitting complex \( FZ_n^{max} \) are all 1-ended.

**Proof.** There are natural coarse maps

\[ CV_n \to FS_n \to FZ_n^{max} \to FZ_n \to FF_n \]
and they are equivariant with respect to the action of $Out(F_n)$. Except on $CV_n$, all these spaces are hyperbolic and the $Out(F_n)$ action is cobounded. Proofs of hyperbolicity show that images of folding paths in $CV_n$ are reparametrized quasi-geodesics with uniform constants. This implies that all the maps starting from $FS_n$ are alignment preserving. Fully irreducible automorphisms are loxodromic in all four complexes. □

APPENDIX A. EXPLICIT CONSTRUCTIONS OF CURVES AND SUBSURFACES

In this section we collect the constructions of curves and subsurfaces claimed in Lemma 3.9.

We begin with the construction of curves in Lemma 3.9. In Figure 6, the curves for the right multiplication moves are shown in green; the curves for the left multiplication moves are shown in purple. Dashed lines entering a group of bands are understood to follow around the boundary of the surface, not intersecting the basis loops corresponding to the loops.

Finally, in Figure 7, the additional curves for the last claim of the lemma are shown.

From these explicit descriptions, fundamental groups of the bad subsurfaces can be read off. We collect the results in the following lemma.

**Lemma A.1.** For a move $\phi = \lambda_{x,y}$ or $\rho_{x,y}$, the bad subsurfaces have the following fundamental groups. We denote by $\partial$ the word representing the boundary of the surface, i.e.

$$\partial = \prod_{i=1}^{g} [\hat{a}_i, a_i^{-1}],$$

if the surface is orientable, and

$$\partial = (n\hat{a}_1\hat{a}_2^{-1}na) \prod_{i=2}^{g} [\hat{a}_i, a_i^{-1}],$$

otherwise (here, $\hat{a}_1, a_1$ are linked with the nonorientable letter $n$, and $\hat{a}_2, \ldots$ are the following letters). We denote by $\partial_w$ the cyclic permutation of $\partial$ starting with the letter $w$.

**x two-sided, not linked with one-sided:** Here, three possibilities for $y$ exist.

**y two-sided, not linked with one-sided:** For $x = a_i$ (not a hatted letter), and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \langle y, \hat{y}, x^{-1} \hat{x} x, \partial_{a_{i+1}} \rangle$$

For $x = a_i$ (not a hatted letter), and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{x}, \partial_{a_i} \rangle$$

For $x = \hat{a}_i$ (a hatted letter), and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{x}, \partial_{a_{i+1}} \rangle$$

For $x = \hat{a}_i$ (a hatted letter), and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \langle y, \hat{y}, x \hat{x} x^{-1}, \partial_{a_i} \rangle$$

**y one-sided or linked with one-sided:** Here, we call the one-sided letter $n$, the linked two-sided letters $a, \hat{a}$ (i.e. $y$ is one of these three), and we assume that $x$ is one of the adjacent $b, \hat{b}$ (this is enough due to the previous normalisation).

In this case, the fundamental group of the bad subsurface $\Sigma - S^g$ has rank three,
Figure 6. Constructing “extra twists”
with two generators $g_1, g_2$ depending solely on $y$, and the final one on $x$ and the type of move. Namely, put

$y = a$:

$g_1 = a, \quad g_2 = n\hat{a}\hat{a}^{-1}$

$y = \hat{a}$:

$g_1 = \hat{a}, \quad g_2 = na$

$y = n$:

$g_1 = n, \quad g_2 = (a^{-1}n^{-1}\hat{a})a(a^{-1}n^{-1}\hat{a})^{-1}$

For $x = b$, and the right multiplication move $\rho_{x,y}$ we have

$\pi_1(\Sigma - S^y) = \langle g_1, g_2, b^{-1}\hat{b}, \partial_{b^{-1}} \rangle$
For \( x = b \), and the left multiplication move \( \lambda_{x,y} \) we have
\[
\pi_1(\Sigma - S^g) = \langle g_1, g_2, \hat{b}, \partial_{\hat{b}_-1} \rangle
\]

For \( x = \hat{b} \), and the right multiplication move \( \rho_{x,y} \) we have
\[
\pi_1(\Sigma - S^g) = \langle g_1, g_2, b, \partial_{b_-1} \rangle
\]

For \( x = \hat{b} \), and the left multiplication move \( \lambda_{x,y} \) we have
\[
\pi_1(\Sigma - S^g) = \langle g_1, g_2, \hat{bb}^{-1}, \partial_{\hat{b}} \rangle
\]

1. **two-sided, linked with one-sided:** Again, we call the one-sided letter \( n \) and the linked two-sided letters \( a, \hat{a} \) (of which \( x \) is one), and we assume that \( y \) is one of the adjacent \( b, \hat{b} \) (this is enough due to the previous normalisation).

   - For \( x = a \), and the right multiplication move \( \rho_{x,y} \) we have
     \[
     \pi_1(\Sigma - S^g) = \langle b, \hat{b}, a^{-1}n^{-1} \hat{a}, a^{-1} \hat{a}a, \partial_{\hat{a}} \rangle
     \]

   - For \( x = a \), and the left multiplication move \( \lambda_{x,y} \) we have
     \[
     \pi_1(\Sigma - S^g) = \langle \hat{a}, b, \hat{b}, \hat{a}a^{-1}n, \partial_a \rangle
     \]

   - For \( x = \hat{a} \), and the right multiplication move \( \rho_{x,y} \) we have
     \[
     \pi_1(\Sigma - S^g) = \langle b, \hat{b}, a, \hat{a}^{-1}n \hat{a}, \partial \rangle
     \]

   - For \( x = \hat{a} \), and the left multiplication move \( \lambda_{x,y} \) we have
     \[
     \pi_1(\Sigma - S^g) = \langle b, \hat{b}, n, \hat{a}a^{-1} \hat{a}, \partial_n \rangle
     \]

2. **one-sided:** Again, we call the one-sided letter \( n = x \), the linked two-sided letters \( a, \hat{a} \), and we assume that \( y \) is one of the adjacent \( b, \hat{b} \) (this is enough due to the previous normalisation).

   - For \( x = n \), and the right multiplication move \( \rho_{x,y} \) we have
     \[
     \pi_1(\Sigma - S^g) = \langle b, \hat{b}, \hat{a}, a^{-1} \hat{a}n, \partial_{\hat{a}} \rangle
     \]

   - For \( x = n \), and the left multiplication move \( \lambda_{x,y} \) we have
     \[
     \pi_1(\Sigma - S^g) = \langle b, \hat{b}, a \hat{a}^{-1}n, \partial_n \rangle
     \]

In all cases, any loop corresponding to a basis element except for \( x, \hat{x}, y, \hat{y} \) (and possibly \( n \), if one of \( x, y \) is linked) can be homotoped into the good subsurface.

**Appendix B. Proof of Lemma 4.6**

Throughout, we call the argument in the proof of Proposition 4.4 Case 1.

First, we observe that the argument of Case 1 extends to the case nonorientable surface. The only difference in this case is that the boundary word \( \partial \) has a slightly different form (see Section 3). However, as we may assume that all the auxiliary letters used above are two-sided, and the boundary word of the nonorientable surface also contains commutators of all the two-sided letters which are not linked with the one-sided letter, the argument works completely analogously.

It remains to discuss the remaining cases in the case of a non-orientable surface, where either \( x \) or \( y \) is one-sided or linked with the one-sided. Unjustified claims about intersections
between subgroups are proved using the arguments in the proof of Proposition 4.4. We make use of the following notation and assumptions throughout:

1. We denote by \( \partial x \) the cyclic permutation of the boundary word \( \partial \) starting at (the first occurrence of) the (signed) letter \( x \).
2. All "auxiliary letters" are chosen to be separated by at least one index from all active letters and from each other (so that the subword detection arguments from Case 1 apply).
3. If \( x \) is a chosen, two-sided letter (i.e. \( x = a_i \) or \( \hat{a}_i \)), then we denote by \( x_+ \) the next letter of the same type (i.e. \( x_+ = a_{i+1} \) or \( \hat{a}_{i+1} \)) and \( \hat{x}_+ \) the next letter of opposite type (i.e. \( \hat{x}_+ = \hat{a}_{i+1} \) or \( a_{i+1} \)).

B.1. Case 2. This case concerns two general \( x \) two-sided, linked to one-sided. Let \( n \) be the one-sided letter. The fundamental group of the bad surface is

\[ B = \langle y, \hat{y}, x^{-1}n^{-1}, x, x^{-1}x, \hat{x}, \partial \rangle. \]

The relation we will use is:

\[ \rho_{x,y} = \rho_{\hat{y},u}^{-1} \rho_{y,z}^{-1} \rho_{n,w} \rho_{x,y} \rho_{x,z} \rho_{n,w} \rho_{y,z} \rho_{\hat{y},u}. \]

Check:

a) \( \rho_{y,u}^{-1} \) has bad subgroup \( \langle u, \tilde{u}, y, y \rangle \) which intersects \( B \) in \( \langle y, \partial \rangle \).

b) \( \rho_{y,z}^{-1} \) has bad subgroup \( \langle z, \hat{z}, y^{-1}\hat{y}y, \partial \rangle \) which intersects \( \rho_{y,u}B = \langle y, \hat{y}, x^{-1}n^{-1}, \hat{x}, x^{-1}x, \rho_{y,u} \partial \rangle \) trivially. Indeed, by Lemma 4.11 applied to the path \( \hat{y}y^{-1}y^{-1} \) we may drop \( \partial \) from \( \langle z, \hat{z}, y^{-1}\hat{y}y, \partial \rangle \). Having done this, applying Lemma 4.11 (to a number of paths) we may drop \( \rho_{y,u} \partial \) from \( \rho_{y,u}B \). Having done this we may apply Lemma 4.11 to \( \hat{y}u \) and \( y^{-1}\hat{y}y \) we may drop \( \hat{y}u \) from \( \rho_{y,u} \langle y, \hat{y}, x^{-1}n^{-1}, \hat{x}, x^{-1}x \rangle \) and \( y^{-1}\hat{y}y \) from \( \langle u, \tilde{u}, y^{-1}\hat{y}y \rangle \). The rest of this case is straightforward.

c) \( \lambda_{n,w}^{-1} \) has bad subgroup \( \langle n, \hat{n}, x^{-1}, x, w, \hat{w}, \partial \rangle \) which intersects \( \rho_{y,z} \rho_{y,u}B = \langle y, \hat{y}, x^{-1}n^{-1}, \hat{x}, x^{-1}x, \rho_{y,z} \rho_{y,u} \partial \rangle \) in \( \langle \hat{x} \rangle \) up to conjugation.

Indeed, as in the previous step, by Lemma 4.11 applied to the path \( \hat{y}y^{-1}y^{-1} \) we may drop \( \partial \) from \( \langle n, \hat{n}, x^{-1}, x, w, \hat{w}, \partial \rangle \). Having done this we may drop \( \rho_{y,z} \rho_{y,u} \partial \) and then \( yz \) and \( \hat{y}u \) from \( \rho_{y,z} \rho_{y,u}B \). So it suffices to consider the intersection of \( \langle n, \hat{n}, x^{-1}, x, w, \hat{w} \rangle \) and \( \langle x^{-1}n^{-1}, x, x^{-1}x \rangle \). Since these are free factorise we consider the abelianization of these which are isomorphic to \( \mathbb{Z}^2 \) where the vector \( (a, b, c) \) represents \( n^ax^by^c \). The claim follows from the fact that the subspace spanned by \( \{(1, 1, 0), (0, -1, -1)\} \) intersects the subspace spanned by \( \{(-1, 1, -1), (1, 1, 0)\} \) trivially.

d) \( \rho_{x,y} \) has bad subgroup \( B \) which intersects \( \lambda_{n,w} \rho_{y,z} \rho_{y,u}B = \langle y, \hat{y}, x^{-1}n^{-1}, \hat{x}, x^{-1}x, \lambda_{n,w} \rho_{y,z} \rho_{y,u} \partial \rangle \) in \( \langle \hat{x} \rangle \).

e) \( \rho_{x,z} \) has bad subgroup \( \langle z, \hat{z}, x, x^{-1}, \hat{x}, x^{-1}x, \partial \rangle \) which intersects \( \rho_{x,y} \lambda_{n,w} \rho_{y,z} \rho_{y,u}B = \langle y, \hat{y}, yx^{-1}n^{-1}, \hat{x}, x^{-1}x, \rho_{x,z} \lambda_{n,w} \rho_{y,z} \rho_{y,u} \partial \rangle \) in \( \langle \hat{x} \rangle \) up to conjugation.

f) \( \lambda_{n,w} \) has bad subgroup \( \langle n, \hat{n}, x, x^{-1}, w, \hat{w}, \partial \rangle \) which intersects \( \rho_{x,y} \rho_{x,z} \lambda_{n,w} \rho_{y,z} \rho_{y,u}B = \langle y, \hat{y}, yx^{-1}n^{-1}, \hat{x}, x^{-1}x, \rho_{x,y} \rho_{x,z} \rho_{n,w} \rho_{y,z} \rho_{y,u} \partial \rangle \) trivially. Similarly to in previous cases we apply Lemma 4.11 to first drop \( \partial \) and then \( \rho_{x,y} \rho_{x,z} \rho_{n,w} \rho_{y,z} \rho_{y,u} \partial \) from their respective subgroups. It is now clear that we can restrict our consideration to
We only indicate how the checks above need to be amended in this case. As neither \(\hat{x}\) nor any word containing the subword \(nw\) is contained in the former, the claim follows.

g) \(\rho_{y,z}\) has bad subgroup \(\langle z, z^{-1}y, y, \rho\rangle \) which intersects \(\langle x, z^{-1}w, x, \rho\rangle \) trivially.

h) \(\rho_{y,u}\) has bad subgroup \(\langle z, z^{-1}y, y, \rho\rangle \) which intersects \(\langle x, z^{-1}w, x, \rho\rangle \) trivially.

This completes the checks for the preliminary relation.

We now collect some variants on this case. First is Case 2' of the left multiplication move \(\lambda_{x,y}\). Here, the bad subgroup is \(B = \langle \hat{x}, y, x^{-1}n, \partial \rangle\). We use the relation

\[
\lambda_{x,y} = \rho_{y,\hat{x}}^{-1}\rho_{\hat{y},\hat{x}}^{-1}\lambda_{x,y}\lambda_{x,z}^{\lambda_{n,w}}\rho_{y,\hat{y},\hat{u}}.
\]

We only indicate how the checks above need to be amended in this case.

a), b), c) are similar to case 2.

d) \(\lambda_{x,z}\) which has bad subgroup \(\langle z, z^{-1}x, \rho\rangle \) which intersects \(\langle x, z^{-1}w, x, \rho\rangle \) in \(\langle x, \rho\rangle\).

e) \(\lambda_{x,y}\) which has bad subgroup \(B\) which intersects \(\lambda_{x,z}\) trivially.

f), g) and h) are similar to case 2.

Finally, Case 2'' and Case 2''': with \(x\) hatted for both \(\rho\) and \(\lambda\) are similar.

B.2. Case 3. \(x\) general unhatted, \(y\) unhatted two-sided and linked to one-sided. Let \(n\) be the one-sided letter and \(u, z, w, v\) be general.

\[
B = \langle y, n^{-1}y, x^{-1}x, \partial \rangle
\]

Check:

a) \(\rho_{x,\hat{z}}^{-1}\) has bad subgroup \(\langle z, z^{-1}x, \rho\rangle\) which intersects \(B\) at most in \(\langle x, \partial \rangle\). In fact, by considering immersed graphs representing the subgroups, one can show that the intersection is \(\langle \partial \rangle\), but we do not need this fact.

b) \(\lambda_{x,\hat{z}}^{-1}\) has bad subgroup \(\langle n^{-1}y, y^{-1}, u, \partial \rangle\) which intersects \(\rho_{x,\hat{z}}B\) in \(\langle n^{-1}y, y^{-1} \rangle\). Indeed, by applying Lemma \([4.1]\) as above we may drop \(\partial\), \(\rho_{x,\hat{z}}\partial\), and \(u\) in sequence. So it suffices to consider the intersection of \(\langle n^{-1}y, y^{-1} \rangle\) and \(\langle n^{-1}y, y^{-1} \rangle\). As both of these are free factors, the intersection is again a free factor. In particular, either the two factors are equal, or the intersection is of rank at most 1. Since neither is contained in the other (e.g. by considering Abelianisations), the intersection is at most cyclic. As \(\langle n^{-1}y, y^{-1} \rangle\) is contained in both, the claim follows.

c) \(\rho_{y,\hat{z}}^{-1}\) has bad subgroup \(\langle v, \hat{v}, n^{-1}y, y^{-1}, \rho \rangle\) which intersects \(\lambda_{n,\rho_{x,\hat{z}}}B\) trivially.

d) \(\lambda_{x,\hat{z}}^{-1}\) has bad subgroup \(B\) which intersects \(\rho_{y,v}^{\lambda_{n,\rho_{x,\hat{z}}}B}\) trivially.

e) \(\rho_{x,\hat{v}}^{-1}\) has bad subgroup \(\langle v, \hat{v}, n^{-1}y, y^{-1}, \rho \rangle\) which intersects \(\rho_{x,\hat{v}}^{\lambda_{n,\rho_{x,\hat{z}}}B}\) trivially.

f) \(\rho_{x,\hat{v}}^{-1}\) has bad subgroup \(\langle v, \hat{v}, x^{-1}x, \partial \rangle\) which intersects \(\rho_{x,\hat{v}}^{\lambda_{n,\rho_{x,\hat{z}}}B}\) trivially in the conjugacy class \(\langle \hat{x} \rangle\).
This completes the checks for the preliminary relation. We now collect some variants on this case. The case of $\lambda_{x,y}$ is analogous using $\lambda_{x,y} = \rho_{x,z}^{-1} \lambda_{x,y} \rho_{x,z}$. Indeed the bad subgroup is the same except $x^{-1} \hat{a} \hat{x}$ is replaced by $x$, and $\partial_{x^{-1}}$ by $\partial_{x}^{-1}$.

**Case 3’** is the case of $x$ general, $y$ hatted two-sided and linked to one-sided. The bad subgroup now is $B = \langle y, ny, x^{-1} \hat{a} \hat{x}, \partial_{x^{-1}} \rangle$.

We use the relation $\rho_{x,y} = \rho_{x,z}^{-1} \rho_{x,y} \rho_{x,z}$ and the steps are the same except the overlap of the bad factor for $\rho_{x,y}$ and $\rho_{x,u} \rho_{x,z} B$ is $\langle ny \rangle$.

**Case 3”** is $x$ general hatted, $y$ unhatted two sided and linked to one-sided. The bad subgroup is $B = \langle y, ny \hat{a}^{-1}, \hat{x}, \partial_{x^{-1}} \rangle$. This is similar.

Finally, **Case 3’’** is $x$ general hatted, $y$ hatted two-sided and linked to one-sided. The bad subgroup is $B = \langle \hat{a}, na, b, \partial_{x^{-1}} \rangle$. Again, this is similar.

**B.3. Case 4.** $x$ general unhatted, $y$ one-sided, $\rho_{x,y}$ and let $a, \hat{a}$ denote the letters that are linked with $y$.

The bad subgroup is $B = \langle y, (a^{-1} y^{-1} \hat{a}) a (a^{-1} y^{-1} \hat{a})^{-1}, x^{-1} \hat{a} \hat{x}, \partial_{x^{-1}} \rangle$, and we use the relation

$$\rho_{x,y} = \rho_{y,u} \rho_{x,z}^{-1} \rho_{x,y} \rho_{x,u} \rho_{x,z} \rho_{y,u}$$

**Check:**

a) $\rho_{y,u}^{-1}$ has bad subgroup $\langle \hat{a}, a \hat{a}^{-1} y, u, \hat{a}, \partial_{x} \rangle$ which intersects $B$ in $\langle \partial_{x} \rangle$.

Indeed, as in previous cases by Lemma 12, we may drop $u, \hat{u}$ and $x^{-1} \hat{a} \hat{x}$. We now consider $\langle y, (a^{-1} y^{-1} \hat{a}) a (a^{-1} y^{-1} \hat{a})^{-1}, \partial_{x^{-1}} \rangle$ and $\langle \hat{a}, a \hat{a}^{-1} y, \partial_{x} \rangle$. By considering $y^{-1} \hat{a} \hat{a} u \hat{a} y$, a subword of $(a^{-1} y^{-1} \hat{a}) a (a^{-1} y^{-1} \hat{a})^{-1}$ which can not occur in $\langle \hat{a}, a \hat{a}^{-1} y, \partial_{x} \rangle$ we reduce to $\langle \hat{a}, a \hat{a}^{-1} y, \partial_{x^{-1}} \rangle$ and $\langle y, \partial_{x} \rangle$. Similarly we may remove $a \hat{a}^{-1} y$ and then $y$ and $\hat{a}$.

b) $\rho_{x,z}^{-1}$ has bad subgroup $\langle \hat{a}, u, \hat{a}, x^{-1} \hat{a} \hat{x}, \partial_{x^{-1}} \rangle$ which intersects $\rho_{y,u} B = \langle y, (a^{-1} u^{-1} y^{-1} \hat{a}) a (a^{-1} u^{-1} y^{-1} \hat{a})^{-1}, x^{-1} \hat{a} \hat{x}, \rho_{y,u} \partial_{x^{-1}} \rangle$ trivially.

c) $\rho_{x,y}$ has bad subgroup $B$ which intersects $\rho_{x,z} \rho_{y,u} B = \langle y, (a^{-1} u^{-1} y^{-1} \hat{a}) a (a^{-1} u^{-1} y^{-1} \hat{a})^{-1}, x^{-1} \hat{a} \hat{x}, \rho_{y,u} \partial_{x^{-1}} \rangle$ trivially.

d) $\rho_{x,u}$ has bad subgroup $\langle u, \hat{u}, x^{-1} \hat{a} \hat{x}, \partial_{x^{-1}} \rangle$ which intersects $\rho_{y,u} \rho_{x,z} \rho_{y,u} \partial_{x^{-1}} \rangle$ trivially.

e) $\rho_{x,z} \rho_{y,u}$ has bad subgroup $\langle x, \hat{x}, \partial_{x^{-1}} \rangle$ which intersects $\rho_{x,u} \rho_{x,z} \rho_{y,u} \partial_{x^{-1}} \rangle$ trivially.

f) $\rho_{y,u}$ has bad subgroup $\langle \hat{a}, a \hat{a}^{-1} y, \hat{a}, \partial_{x^{-1}} \rangle$ which intersects $\rho_{y,u} \rho_{x,z} \rho_{y,u} \partial_{x^{-1}} \rangle$ trivially.

Namely, as before, we can drop the (modified) boundary words, as well as $y u x^{-1} \hat{x} u x^{-1} \hat{y}^{-1} \hat{a}$. We now need to control the intersection of $\langle u, \hat{u}, a \hat{a}^{-1} y \rangle$ and $\langle y, (a^{-1} u^{-1} y^{-1} \hat{a}) a (a^{-1} u^{-1} y^{-1} \hat{a})^{-1} \rangle$. Since both are free factors, and their Abelianisations do not intersect, the claim follows.
The case of $x$ general hatted and the relevant $\lambda$ cases are similar.

**B.4. Case 5.** $x$ one sided, $y$ general. As before, we denote by $a, \hat{a}$ the linked two-sided letters. Here, we consider $\rho_{x,y}$ which has bad subgroup

$$B = \langle y, \hat{y}, \hat{a}, a\hat{a}^{-1}x, \partial_a \rangle.$$ 

We use the relation

$$\rho_{x,y} = \rho_{y,u}^{-1}\rho_{y,z}^{-1}\rho_{x,y}^{-1}\rho_{x,z}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}.$$

a) $\rho_{y,u}^{-1}$ has bad subgroup $\langle u, \hat{a}, y, \partial_{y^{-1}} \rangle$. This intersects $B$ in $\langle y, \partial_{y^{-1}} \rangle$.

b) $\rho_{y,z}^{-1}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}, \partial_{y^+} \rangle$ and this intersects $\rho_{y,u}B = \langle y, \hat{y}, u, \hat{a}, a\hat{a}^{-1}x, \partial_a \rangle$ trivially.

c) $\lambda_{a,w}^{-1}$ has bad subgroup $\langle a\hat{a}^{-1}, x, w, \hat{w}, \partial_n \rangle$ and this intersects $\rho_{y,z}\rho_{y,u}B = \langle y, \hat{y}, u, \hat{a}, a\hat{a}^{-1}x, \partial_a \rangle$ in $\langle a\hat{a}^{-1}x \rangle$. Namely, after dropping the boundary terms as usual, we can also drop $yz, \hat{y}u, w, \hat{w}$. The resulting rank 2 free factors $\langle a\hat{a}^{-1}, x \rangle$ and $\langle \hat{a}, a\hat{a}^{-1}x \rangle$ have Abelianisations that intersect in a rank 1 submodule. The intersection is therefore at most a rank 1 free factor, hence it is the one claimed.

d) $\rho_{x,y}$ has bad subgroup $B$ which intersects $\lambda_{a,w}\rho_{y,z}\rho_{y,u}B = \langle y, \hat{y}, w, \hat{a}, a\hat{a}^{-1}w^{-1}x, \partial_n \rangle$ trivially.

e) $\rho_{x,z}$ has bad subgroup $\langle \hat{a}, a\hat{a}^{-1}x, z, \hat{z}, \partial_n \rangle$ which intersects $\rho_{x,y}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}B = \langle y, \hat{y}, w, \hat{a}, a\hat{a}^{-1}w^{-1}x, \partial_n \rangle$ trivially.

f) $\lambda_{a,w}^{-1}$ has bad subgroup $\langle a\hat{a}^{-1}, x, w, \hat{w}, \partial_n \rangle$ which we need to intersect with $\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}B = \langle y, \hat{y}, u, \hat{a}, a\hat{a}^{-1}x^{-1}y^{-1}, \rho_{x,y}^{-1}\rho_{x,z}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}\partial_n \rangle$. As usual, we can discard the boundary word terms, and clean up generators to compare $\langle a\hat{a}^{-1}, x, w, \hat{w} \rangle$ and $\langle y, \hat{y}, u, \hat{a}, a\hat{a}^{-1}w^{-1}x \rangle$. We can drop $yz, \hat{y}u$ from the latter, replacing it with $\langle w, a\hat{a}^{-1}w^{-1}x \rangle$. Since the $w\hat{a}$ is not homologous into the former factor, the intersection is at most rank 1. Thus, the intersection is $\langle w, a\hat{a}^{-1}w^{-1}x \rangle$.

g) $\rho_{z,y}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}, \partial_{y^+} \rangle$ which intersects $\lambda_{a,w}^{-1}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}B = \langle y, \hat{y}, u, \hat{a}, a\hat{a}^{-1}x^{-1}y^{-1}, \rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}\partial_n \rangle$ trivially.

h) $\rho_{z,u}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}} \rangle$ which intersects $\rho_{y,z}^{-1}\lambda_{a,w}^{-1}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{a,w}\rho_{y,z}\rho_{y,u}B = \langle y, \hat{y}, u, \hat{a}, a\hat{a}^{-1}x^{-1}y^{-1} \rangle$ trivially.

**Case 5’** This is the analogous left-multiplication move $\lambda_{x,y}$ with notation as in Case 5, and thus the bad subgroup is

$$B = \langle x\hat{a}, a\hat{a}^{-1}, y, \hat{y}, \partial_n \rangle$$

where $a, \hat{a}$ is linked to $x$. Let $z, w, u$ be general. We use the relation

$$\lambda_{x,y} = \rho_{y,u}^{-1}\rho_{y,z}^{-1}\rho_{x,z}^{-1}\lambda_{x,y}\rho_{a,w}\rho_{y,z}\rho_{y,u}.$$ 

The checks here are similar to Case 5. Indeed, Check:

a) $\rho_{y,u}^{-1}$ has bad subgroup $\langle u, \hat{a}, y, \partial_{y^{-1}} \rangle$. This intersects $B$ in $\langle y, \partial_{y^{-1}} \rangle$.

b) $\rho_{y,z}^{-1}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}, \partial_{y^+} \rangle$ and this intersects $\rho_{y,u}B = \langle x\hat{a}, a\hat{a}^{-1}, y, \hat{y}, \partial_{y^+} \rangle$ trivially.

c) $\rho_{a,w}^{-1}$ has bad subgroup $\langle w, \hat{w}, a, a\hat{a}^{-1}x, \partial_n \rangle$ and this intersects $\rho_{y,z}\rho_{y,u}B = \langle x\hat{a}, a\hat{a}^{-1}, yz, \hat{y}, \rho_{a,w}\rho_{y,z}\rho_{y,u}\partial_n \rangle$ in $\langle a\hat{a}^{-1}x \rangle$.

d) $\lambda_{x,z}$ has bad subgroup $\langle x\hat{a}, a\hat{a}^{-1}, z, \hat{z}, \partial_n \rangle$ which intersect $\rho_{a,w}\rho_{y,z}\rho_{y,u}B = \langle x\hat{a}w, aw^{-1}a^{-1}, yz, \hat{y}, \rho_{a,w}\rho_{y,z}\rho_{y,u}\partial_n \rangle$ trivially.

e) $\lambda_{x,y}$ has bad subgroup $B$ which intersects $\lambda_{x,z}^{-1}\rho_{a,w}\rho_{y,z}\rho_{y,u}B = \langle z^{-1}x\hat{a}w, aw^{-1}a^{-1}, yz, \hat{y}, \lambda_{x,z}^{-1}\rho_{a,w}\rho_{y,z}\rho_{y,u}\partial_n \rangle$ trivially.
f) \( \rho_{a,w} \) has bad subgroup \( \langle w, w', a, a^{-1} x \bar{a}, \partial_n \rangle \) which intersects \( \lambda_{x,y,1}^{-1} \lambda_{x,y,1}^{-1} \rho_{\bar{a},w} \rho_{\bar{y},z} \rho_{\bar{y},u} B = \langle z^{-1} y^{-1} \bar{x}w, aw^{-1} a^{-1}, yz, \bar{y}u, \lambda_{x,y,1}^{-1} \lambda_{x,y,1}^{-1} \rho_{\bar{a},w} \rho_{\bar{y},z} \rho_{\bar{y},u} \partial_n \rangle \) in \( \langle aw^{-1} \bar{a}^{-1} \bar{x}a \rangle \).

g) \( \rho_{y,z} \) has bad subgroup \( \langle z, z, y^{-1} \bar{y}u, \partial_{y^+} \rangle \) which intersects \( \rho_{\bar{a},w}^{-1} \lambda_{x,y,1}^{-1} \rho_{\bar{a},w} \rho_{\bar{y},z} \rho_{\bar{y},u} B = \langle z^{-1} y^{-1} \bar{x}a, a \bar{a}^{-1}, yz, \bar{y}u, \rho_{\bar{a},w}^{-1} \lambda_{x,y,1}^{-1} \rho_{\bar{a},w} \rho_{\bar{y},z} \rho_{\bar{y},u} \partial_n \rangle \) trivially.

h) \( \rho_{y,u} \) has bad subgroup \( \langle u, \bar{u}, y, \bar{y}u, \partial_{y^{-1}} \rangle \) which intersects \( \rho_{y,z}^{-1} \rho_{\bar{a},w}^{-1} \lambda_{x,y,1}^{-1} \rho_{\bar{a},w} \rho_{\bar{y},z} \rho_{\bar{y},u} B = \langle y^{-1} \bar{x}a, a \bar{a}^{-1}, y, \bar{y}u, \rho_{\bar{a},w}^{-1} \lambda_{x,y,1}^{-1} \rho_{\bar{a},w} \rho_{\bar{y},z} \rho_{\bar{y},u} \partial_n \rangle \) in \( \langle y \rangle \).

**Appendix C. Minimal foliations for nonorientable surfaces**

The purpose of this appendix is to show the following result, which was stated as Theorem 2.11 above.

**Theorem C.1.** Suppose that \( \Sigma \) is a nonorientable surface with a single boundary component or marked point. Then, there is a path-connected subset

\[ \mathcal{P} \subset \mathcal{M}(\Sigma) \subset \mathcal{PML}(\Sigma) \]

consisting of minimal measured foliations, which is invariant under the mapping class group of \( \Sigma \). In addition, if \( F \) is any finite set of laminations, the set \( \mathcal{P} \setminus F \) is still path-connected.

The proof uses methods established in [LS09]. We consider throughout the case of a surface \( \Sigma = (S, p) \) with a marked point; the other claim is equivalent.

We begin by observing that any foliation on \( S \) defines a foliation on \( \Sigma \), and the resulting foliations of \( \Sigma \) are exactly those which do not have an angle-\( \pi \) singularity at \( p \).

**Lemma C.2.** A foliation \( F \) on \( S \) is minimal (as a foliation on \( S \)) if and only if it is minimal as a foliation of \( \Sigma \).

**Proof.** This follows, since any essential simple closed curve on \( \Sigma \) defines an essential simple closed curve on \( S \) (i.e. after forgetting the marked point).

**Definition C.3.** We define \( \mathcal{P} \subset \mathcal{PML}(\Sigma) \) to be the set of minimal foliations which either

1. do not have an angle-\( \pi \) singularity at \( p \), or
2. are stable foliations of point-pushing pseudo-Anosov.

It is clear from construction that \( \mathcal{P} \) is invariant under the mapping class group of \( \Sigma \). We aim to show that any foliation in \( \mathcal{P} \) of the first type can be connected by a path to any foliation of the second type, which will prove Theorem 2.11 as we have full flexibility which point-pushes to use.

To do so, we need to recall some facts about point-pushing maps; compare [LS09, CH]. Let \( \gamma : [0, 1] \to S \) be an immersed smooth loop based at \( p \). We let

\[ D_\gamma : [0, 1] \to \text{Diff}(S) \]

be a smooth isotopy starting in the identity, so that \( D_\gamma(t)(p) = \gamma(t) \). By definition, the endpoint \( D_\gamma(1) \) is then a representative of the point-pushing mapping class \( \Psi_\gamma \) defined by \( \gamma \).

Suppose that \( F \) is a foliation of \( S \) which is minimal. Then, the same is true for \( D_\gamma(t)F \) (as they are indeed isotopic). When seen as minimal foliations of \( \Sigma \), the assignment

\[ t \mapsto D_\gamma(t)F \]
is a continuous path of minimal foliations joining $\mathcal{F}$ to $\Psi_2 \mathcal{F}$: minimality follows by Lemma C.2, and continuity since $D_\gamma$ is smooth and intersections with $\mathcal{F}$ vary continuously with the curve.

Now, we use the following:

**Lemma C.4.** Let $\Psi$ be a point-pushing pseudo-Anosov of $\Sigma$. Then $\Psi$ acts on $\mathcal{PL}(\Sigma)$ with north-south dynamics, and both fixpoints have an angle-$\pi$–singularity.

**Proof.** Let $X \to \Sigma$ be the orientation double cover. Then $\Psi$ lifts to a pseudo-Anosov of $X$, and the first claim follows. The second claim follows since point-pushes have angle–$\pi$ singularities at the marked point [LS09]. □

By the lemma, the path $\{D_\gamma(t)\mathcal{F}, t \in [0, 1]\}$ is disjoint from the repelling fixed point of $\Psi$, and thus

$$\bigcup_{n \in \mathbb{N}} \Psi^n \{D_\gamma(t)\mathcal{F}, t \in [0, 1]\}$$

is the desired path joining $\mathcal{F}$ to the stable foliation of $\Psi$.

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