The Electromagnetic Field in Gravitational Wave Interferometers

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Abstract

We analyse the response of laser interferometric gravitational wave detectors using the full Maxwell equations in curved spacetime in the presence of weak gravitational waves. Existence and uniqueness of solutions is ensured by setting up a suitable boundary value problem. This puts on solid ground previous approximate calculations. We find consistency with previous results obtained from eikonal expansions at the level of accuracy accessible to current gravitational wave detectors.

Contents

1 Introduction 2
2 A Boundary Value Problem for Maxwell’s Equations 4
3 Coordinate-Independence 5
4 Justification of Eikonal Expansions 6
5 The Setup 7
6 Scalar Waves 8
6.1 Emission From a Laser ........................................... 8
6.1.1 General Expression for Emitted Waves .................. 10
6.1.2 Emitted Scalar Wave ........................................ 14
6.2 Reflection At a Mirror .......................................... 15
6.2.1 General Expression for Reflected Waves ............... 16
6.2.2 Reflected Scalar Wave ...................................... 18
7 Maxwell’s Equations .................................................. 20
7.1 The Unperturbed Field ........................................... 21
7.2 Boundary Values .................................................. 22
7.3 The Emitted Wave ................................................ 26
7.4 Reflection at a Mirror .......................................... 28
7.5 The Reflected Wave .............................................. 31

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1 Introduction

Interferometric gravitational wave detectors are the only instruments so far that have been able to directly detect gravitational waves (GWs). While the signals observed are certainly derivable by solving Maxwell’s equations in a GW metric, much to our surprise we have not found satisfactory calculations of this kind in the literature. Instead, most approaches to the problem use various approximations which can be ordered in sequence of increasing detail as

Geodesic Deviation → Geometric Optics → Eikonal Expansion → Maxwell’s Equations.

The perhaps simplest method consists in using an approximate solution of Jacobi’s equation for the geodesic deviation of nearby world lines, which model the trajectories of the mirrors at the ends of the two interferometer arms [1, 2]. One key assumption here is that the arm lengths are sufficiently short, so that the geodesic deviation vector is a reasonable approximation of the arm lengths. Further, this implicitly assumes that laser interferometry directly measures such deviation vectors. As shown in [3], this can be understood as a limiting case of geometric optics methods.

The geometric optics approximation consists in computing the difference in round-trip times of light rays in the interferometer arms, which is then related to the phase shift [3–12]. While the results of this method are generally believed to be correct, they provide only very limited information about the precise form of the electromagnetic field producing the detected signal.

Indeed, the geometric optics equations arise at first order in eikonal expansions, where one explicitly introduces a frequency parameter $\omega$ and an eikonal function $\psi$, seeking solutions possessing a series expansion of the amplitude in inverse powers of $\omega$ [13]. The validity of such high-frequency approximations in the considered context has been questioned in [14], and some previous attempts to obtain formulae applicable for all incidence angles
of the gravitational wave have led to expressions which are undefined if the GW propagates parallelly to one of the interferometer arms [15, 16].

Finally, previous important and significant analyses starting with the full Maxwell equations in the context of GW detection have used approximation schemes which raise concerns about well-posedness of the resulting equations, in particular about uniqueness of solutions [14–19]. This significant question of uniqueness of the proposed solutions likewise plagues all the methods mentioned so far.

We resolve the issues arising in these schemes by setting up a boundary value problem for Maxwell’s equations, where light sent into the interferometer arms is described by suitably prescribed values of the electromagnetic field on infinite planes, which we refer to as emission surfaces. The key fact is that this is the only setup known to us which models reasonably the physical problem and which guarantees both existence and uniqueness of solutions in terms of boundary data. Here we use the word “data” in the PDE sense, namely fields that we prescribe in the problem at hand.

The question then arises, what are the physically correct boundary data at the emission surface. There is freedom which ultimately needs to be tied to the experiment considered.

It turns out that there are several issues here. First, one needs to model the apparatus in the absence of gravitational waves. One can then imagine plane waves, or Bessel waves, or Gauss waves, or else, streaming from the emission surface. Since the latter are a superposition of the former, we chose here to consider the simplest model of a plane wave emitted orthogonally to the emission surface, leaving the remaining important cases for future work. This is thus our model for the unperturbed wave.

Next, suppose that the gravitational wave is of order $\epsilon \ll 1$. In Section 2 we show existence of solutions of our model with a convergent expansion in powers of $\epsilon$. The fact that the expansion converges implies in particular that the first non-trivial coefficient in the expansion can be determined by solving a truncated equation. The resulting solutions of Maxwell’s equations are uniquely determined by certain boundary data, which can be prescribed as a convergent power series in $\epsilon$. A careful choice of boundary terms of first order in $\epsilon$ is needed to eliminate the already-mentioned meaningless solutions when gravitational waves are parallel to the electromagnetic wave.

Some freedom still remains in the boundary data, and we calculate the first non-trivial term in an $\epsilon$-expansion of the associated solution for a specific choice of boundary data in Section 7.3. We show that there exist $\epsilon$-accurate solutions expressible in a form compatible with eikonal expansions, and that their amplitude prefactors are analytic in $\omega_g/\omega$, where $\omega_g$ and $\omega$ are the frequencies of the gravitational and electromagnetic waves, respectively. We stress that the Maxwell field as a whole does not have an expansion in terms of $\omega_g/\omega$, only the amplitude prefactor does; this clarifies the difficulties pointed out in [14, 19].

In Section 4 we show quite generally that the freedom in the boundary data does not affect the leading order, both in $\epsilon$ and in $\omega_g/\omega$, of the phase shift of the solutions, leading thus to an unambiguous interference pattern at this level of accuracy. Hence, we confirm that a calculation of the interference pattern based on eikonal expansions, and thus on geometric optics, is consistent with Maxwell equations within the model analysed here, and provides the key information needed for current experiments.

While our calculations are carried out in TT coordinates, which seem to be most convenient for the problem at hand, we show in Section 3 that the leading-order interference pattern is independent of the coordinates in which the analysis is performed.

Our analysis applies not only to vacuum but also to linear dielectrics with refractive index $n \geq 1$, so that the results can be compared with those obtained for optical fibres [20], see Section 8.6. Another motivation for this is to investigate whether new phenomena arise
from different propagation speeds of the gravitational and electromagnetic waves. While there are explicit changes arising both in the phase and in the amplitude of the field, their net result in the interferometer response turn out to be a simple rescaling of the output signal, see Section 8.5.

Our key findings are summarised in Section 8.1, where explicit formulae are given for the electromagnetic field after its round-trip in one of the interferometer arms. The resulting interferometer response is computed in Section 8.4, and the low-frequency limit is analysed in Section 8.5.

2 A Boundary Value Problem for Maxwell’s Equations

Consider Maxwell’s equations or the scalar wave equation in a metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}(\epsilon, x^\alpha)$$

(2.1)

with $\epsilon \ll 1$, where $\eta_{\mu\nu}$ is the Minkowski metric and the $h_{\mu\nu}$’s are bounded real analytic functions of the coordinates and of $\epsilon$. For example, the reader can think of the metric (5.3) below where the error terms are ignored:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon A_{\mu\nu} \cos[\omega_g(x - t) + \chi],$$

(2.2)

where the $A_{\mu\nu}$’s, $\omega_g$ and $\chi$ are constants; it should be clear from the arguments below that this suffices for the calculations relevant for nowadays interferometric GW detectors. The metric (2.2) is indeed real analytic, and depends analytically upon $\epsilon$ and $\omega_g$.

In the notation of (B.5), Appendix B, Maxwell’s equations in the absence of sources are

$$\nabla_\mu F^{\mu\nu} = 0.\quad (2.3)$$

As explained in more detail shortly, we will seek solutions of (2.3) with real-analytic data prescribed on a real-analytic timelike hypersurface $\Sigma$. Detailed calculations are carried out for the case where $\Sigma$ is a timelike coordinate hyperplane $\Sigma = \{m_i x^i = 0\}$ in TT coordinates.

As well-known, one can extract from (2.3) a system of equations which propagates the data in directions transverse to $\Sigma$, so that $\Sigma$ is not characteristic for (2.3). The remaining equation is then a (complex) constraint equation on $\Sigma$ which, again by a standard argument which we reproduce in Appendix C for the convenience of the reader, propagates away from $\Sigma$ if satisfied there.

Assuming that the free data on $\Sigma$ are analytic, the Cauchy-Kovalevskaya theorem shows existence of a neighbourhood of $\Sigma$ on which a solution exists. By Holmgren’s uniqueness theorem, the solution is unique in the class of smooth solutions. Further standard results show that the solution is real analytic in $\epsilon$ and any further parameters (such as $\omega_g$, $m_i$, etc.) if the boundary data on $\Sigma$ are. Hence, there exists a convergent series

$$F_{\mu\nu} = \sum_{k=0}^{\infty} F_{\mu\nu}^{(k)} \epsilon^k,$$

(2.4)

where the (uniquely determined) expansion coefficients $F_{\mu\nu}^{(k)}$ are independent of $\epsilon$. Given the accuracy of the instruments that exist or are currently planned, for applications in gravitational wave detection it suffices to compute $F_{\mu\nu}^{(1)}$. 

4
Solving the equations explicitly (for suitable boundary data), we verify that the truncated series $\mathcal{F}^{(0)} + \epsilon \mathcal{F}^{(1)}$ can be decomposed into amplitude and phase contributions as commonly assumed. Moreover, we show that, within our model, the amplitude is analytic in $\omega_g/\omega$ so that the eikonal expansion of the amplitude in inverse powers of $\omega$ is justified at this level of approximation: we have not found an abstract argument for this last step of the analysis, we had to check the structure of $\mathcal{F}^{(1)}_{\mu\nu}$. Analyticity in $\omega_g/\omega$ implies that our result coincides with the one obtained by a suitably-truncated (cf. [21]) eikonal approximation of our model. In other words, we have provided a justification that eikonal expansions give the correct result at the expected level of accuracy.

While we have concentrated on the Maxwell equations so far, identical arguments apply to the massless scalar wave equation, which is described in Section 6 below as a preparation for the Maxwell case.

It should be recognised that the assumption of real-analyticity of $g_{\mu\nu}$ plays a key role in the justification-part of our analysis. While this is a common assumption in the setting considered, one does not expect “real life metrics” to be real-analytic, in which case our argument will break down. Indeed, the boundary-value problem considered here is well-posed in the analytic case, but is known to be ill-posed in general. Whatever their shortcomings, our arguments justify rigorously some results in the literature concerning the problem at hand, and allow us to pinpoint problems with some existing treatments.

3 Coordinate-Independence

The question has been raised, to what extent the predictions for the interferometric phase shifts are coordinate invariant. Our approach to this question, as a boundary-value problem for Maxwell equations, allows to give a clear answer.

Indeed, let $\Sigma$ be a smooth timelike hypersurface, describing the emission surface, on which smooth data are given as described in Section 2, with a corresponding solution of the Maxwell equations which we denote by $\mathcal{F}^{\mu\nu}$. (As already hinted-to above, there are no general existence theorems in this setting except in the case of analytic data and of an analytic metric; this is however irrelevant for the problem of coordinate-independence, for if a solution does not exist, there is nothing to prove.) By Holmgren’s uniqueness theorem (which does not require analyticity of solutions, smoothness suffices) the field $\mathcal{F}^{\mu\nu}$ is unique, in particular it is independent of the coordinate system which is used to calculate the tensor field $\mathcal{F}^{\mu\nu}$.

Let further $\Sigma'$ be an analytic timelike hypersurface, describing the mirror. Our proposal for the boundary conditions at the mirror can be found in Section 7.4 below. In this proposal the boundary data at $\Sigma'$ are read off the field $\mathcal{F}^{\mu\nu}$ and lead to a boundary value problem for the returning Maxwell field, say $\tilde{\mathcal{F}}^{\mu\nu}$, essentially identical to the one in Section 2. Thus, the resulting electromagnetic field between the mirrors, $\mathcal{F}^{\mu\nu} + \tilde{\mathcal{F}}^{\mu\nu}$ is defined uniquely in our setting. This field, together with the corresponding field from the second arm, determines the interference pattern at the detector. If there exists a corresponding solution of the Maxwell equations (which is the case in the model considered here), then this solution is unique and independent of the coordinates by an identical argument.

The above makes it clear that both our solutions and the resulting signal at the detector are unique, and hence coordinate-independent.
4 Justification of Eikonal Expansions

As shown in detail below, both the scalar field equation and the wave equations implied by Maxwell’s equations are of the general form

\[
\left( \delta J^\mu J_\mu + \epsilon \omega g \sigma J^\mu J_\mu + \epsilon \omega^2 b J^J \right) \phi_J = 0 ,
\]

(4.1)

for some field \(\phi\) with an index \(J\). Here the \(a_J^\mu\)'s and the \(b_J^J\)'s are linear combinations of \(\sin(u)\) and \(\cos(u)\), where \(u\) is the phase of the gravitational wave

\[
u = \kappa \mu x^\mu + \chi ,
\]

(4.2)

and \(\gamma\) is the optical metric (see (5.8) below)

\[
\gamma^{\mu\nu} = \gamma^{(0)} + \epsilon A^{\mu\nu} \cos(u) + O(\epsilon^2) .
\]

(4.3)

Writing the field \(\phi\) as a perturbation of a plane wave

\[
\phi_I = \varphi^{(0)}_I e^{ik_\mu x^\mu} + \epsilon \phi^{(1)}_I + O(\epsilon^2) ,
\]

(4.4)

where the \(\varphi^{(0)}_I\)'s are constants and \(\phi^{(1)}_I\) does not depend upon \(\epsilon\), one obtains, within our model for light emission from a surface \(\Sigma\) (described in detail below), a boundary value problem of the form

\[
\Box^{(0)} \phi^{(1)}_I = \omega^2 f_{1I} (\Omega, u) e^{i k_\mu x^\mu} ,
\]

(4.5)

\[
\phi^{(1)}_I \big|_{\Sigma} = f_{2I} (\Omega, u) e^{-i\omega t} ,
\]

(4.6)

\[
m(\phi^{(1)}_I) \big|_{\Sigma} = \im \omega f_{3I} (\Omega, u) e^{-i\omega t} ,
\]

(4.7)

where the \(f_{1I}\)'s are linear combinations of \(\sin(u)\) and \(\cos(u)\), with coefficients which are analytic in \(\Omega = \omega_g / \omega\), and \(m\) is the unperturbed unit normal to \(\Sigma\). In particular the coefficients \(f_{1I}\) in (4.5) take the form

\[
f_{1I} = -n^2 A(m, m) \varphi^{(0)}_I \cos(u) + O(\Omega) .
\]

(4.8)

By explicit calculations we show that if the \(f_{1I}\)'s satisfy a certain emission condition (which we verify for our model), then the solutions are of the form

\[
\phi^{(1)}_I = \left[ i \varphi^{(0)}_I \psi^{(1)} + \varphi^{(1)}_I (\Omega, u, \omega_g m, x) \right] e^{ik_\mu x^\mu} .
\]

(4.9)

Here, the real-valued function \(\psi^{(1)}\) is of order \(1/\Omega\) and satisfies

\[
2\gamma^{(0)\mu\nu} k_\mu \partial_\nu \psi^{(1)} - A^{\mu\nu} k_\mu k_\nu \cos(u) = 0 ,
\]

(4.10)

and the \(\varphi^{(1)}_I\)'s are analytic functions of \(\Omega\), which are of order unity. Consequently, the overall field can be written as

\[
\phi_I = \left[ \varphi^{(0)}_I + i \epsilon \varphi^{(0)}_I \psi^{(1)} + \epsilon \varphi^{(1)}_I \right] e^{ik_\mu x^\mu} + O(\epsilon^2)
\]

\[
= \left[ \varphi^{(0)}_I + \epsilon \varphi^{(1)}_I \right] \left[ 1 + i \epsilon \psi^{(1)} \right] e^{ik_\mu x^\mu} + O(\epsilon^2)
\]

\[
= \left[ \varphi^{(0)}_I + \epsilon \varphi^{(1)}_I \right] e^{ik_\mu x^\mu + \epsilon \psi^{(1)}} + O(\epsilon^2) = \varphi_I e^{i\psi} + O(\epsilon^2) ,
\]

(4.11)
where the function $\psi$ is

$$\psi := k_\mu x^\mu + \epsilon \psi^{(1)}. \quad (4.12)$$

Due to (4.10), $\psi$ satisfies the eikonal equation to first order in $\epsilon$:

$$\gamma^{\mu\nu} (\partial_\mu \psi) (\partial_\nu \psi) = O(\epsilon^2). \quad (4.13)$$

Moreover, the amplitude $\varphi_I = \varphi^{(0)}_I + \epsilon \varphi^{(1)}_I$ is analytic in the frequency ratio $\Omega$ and thus analytic in the inverse frequency $1/\omega$. This justifies the use of eikonal expansions, at least at first order in $\epsilon$.

Our formulae show that the phase perturbation $\psi^{(1)}$ is independent of $f_2$, $f_3$, and also of the terms not written explicitly in (4.8). Instead, the details of the boundary conditions and the lower order terms in the wave equation only affect the amplitude perturbation $\varphi^{(1)}_I$. In this sense, the phase does not depend on the precise details of the boundary data. In particular as $\psi^{(1)}$ is of order $1/\Omega$, it constitutes the dominant contribution to $\phi^{(1)}$. This shows that previous analyses using geometric optics correctly determine the leading-order formula for $\phi^{(1)}$ in an expansion in both parameters $\epsilon$ and $\Omega$.

### 5 The Setup

We consider a linearised gravitational wave of amplitude $\epsilon$ in transverse-traceless gauge. Thus, neglecting non-linearities, the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad (5.1)$$

where $\epsilon$ is a small constant and $h_{\mu\nu}$ is spatial, transverse, traceless, and satisfies the Minkowskian wave equation:

$$h_{\mu0} = 0, \quad \eta^{\mu\nu} \partial_\mu h_{\nu\rho} = 0, \quad \eta^{\mu\nu} h_{\mu\nu} = 0, \quad \eta^{\mu\nu} \partial_\mu \partial_\nu h_{\rho\sigma} = 0. \quad (5.2)$$

Here $\epsilon$ should be small enough so that the existence arguments above apply. A sharp estimate for this number is beyond the existing mathematical techniques, but we expect that values typical for current GW detectors are compatible with our analysis.

As explained in Section 2, we are concerned with solutions of the scalar wave equation and of the Maxwell equations which are convergent power series in $\epsilon$. Since only the first-order term is of experimental relevance, it suffices to solve a linear equation for the first coefficient in this expansion. By linearity, it suffices to consider the case where $h_{\mu\nu}$ has a single frequency $\omega_g$ and propagates in a fixed direction. We are then led to consider a metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon A_{\mu\nu} \cos(\kappa.x + \chi) + O(\epsilon^2), \quad (5.3)$$

where $\chi$ is a constant phase offset, $\kappa$ is the GW wave-covector

$$\kappa = \omega_g (\hat{\kappa}_i dx^i - dt), \quad (5.4)$$

$\hat{\kappa}_i$ being a constant unit vector in the background geometry, and where for any covector $k$ we write

$$k.x \equiv k_\mu x^\mu, \quad (5.5)$$
so that $\kappa x = \omega_j (\hat{\kappa} x^j - t)$. The signature used here is

$$(\eta_{\mu\nu}) = \text{diag}(-1, +1, +1, +1).$$ \hfill (5.6)

The gauge conditions now imply

$$A_{\mu 0} = 0, \quad \delta^{ij} A_{ij} = 0, \quad A_{\mu \nu} \kappa^\nu = 0,$$ \hfill (5.7)

i.e. $A$ is purely spatial, traceless, and orthogonal to the GW wave-vector $\kappa$ (whose index is raised with the Minkowski metric).

We consider a region filled by an isotropic linear dielectric of constant refractive index $n$. The dielectric is at rest in the coordinate system above, so that its four-velocity is $u = \partial/\partial x^0$. This is a very crude approximation which ignores the elastic properties of the dielectric medium; we plan to return to this in future work. The associated optical metric is given by

$$\gamma^{\mu\nu} = g^{\mu\nu} + (1 - n^2) u^\mu u^\nu.$$ \hfill (5.8)

Here, the inverse metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon A^{\mu\nu} \cos(\kappa x + \chi) + O(\epsilon^2),$$ \hfill (5.9)

where $A^{\mu\nu} \equiv \eta^{\mu\rho} \eta^{\nu\sigma} A_{\rho\sigma}$ numerically coincides with $A_{\mu\nu}$ (in the used coordinate system). Hence, the optical metric at first order in $\epsilon$ is

$$\gamma^{(0)\mu\nu} = \text{diag}(-n^2, 1, 1, 1), \quad \gamma^{(1)\mu\nu} = -\eta^{\mu\nu}.$$ \hfill (5.10)

## 6 Scalar Waves

Before considering Maxwell’s equations and wave solutions thereof in Section 7, we first study a simplified model based on the scalar wave equation. It is folklore knowledge, and confirmed by our analysis (cf. Sections 7.3 and 7.4 below), that this suffices to determine the phase shift in interferometric experiments.

Since Maxwell’s equations imply a wave equation for each component of the electromagnetic field, we consider a scalar field $\phi$ satisfying the scalar wave equation

$$\Box_{\gamma} \phi \equiv \gamma^{\mu\nu} \partial_\mu \partial_\nu \phi = 0,$$ \hfill (6.1)

where $\Box_{\gamma}$ is the wave operator associated to the optical metric (5.8). The first equality here holds because we are working with harmonic coordinates.

### 6.1 Emission From a Laser

To model the emission of light by a laser, we impose the boundary conditions

$$\phi|_{\Sigma} = e^{-i\omega t}, \quad \nu(\phi)|_{\Sigma} = i n \omega e^{-i\omega t},$$ \hfill (6.1.1)

where $\Sigma$ is the timelike hypersurface from which the wave is emitted, $\nu$ denotes its unit normal, and $\nu(\phi) = \nu^\mu \partial_\mu \phi$ is the usual notation for the action of a vector field on a function.

One motivation for these boundary conditions comes from geometric optics with which we want to compare. There, one writes the field as $A e^{i\psi}$, where the gradient of the eikonal $\psi$ is required to be null. Seeking solutions which resemble plane waves in flat space as
closely as possible, we assume the field to take the same value on the emission surface as one would have for a plane wave in Minkowski space. In the language of geometric optics, this corresponds to a constant amplitude $a'$ equal to 1 on $\Sigma$, and the eikonal $\psi$ equal to $-\omega t$ there. Having prescribed $\psi|_{\Sigma} = -\omega t$, it follows that $d\psi = -\omega dt + \alpha \nu$, where $\nu$ is the geometric normal to $\Sigma$ (normalized with respect to the full metric $g$). The null condition then entails $\alpha^2 = n^2 \omega^2$, and the sign of $\alpha$ determines the direction into which light is emitted. Choosing a positive sign of $\alpha$ (emitting along $\nu$), and neglecting the derivative of $A$ in the normal direction, yields the boundary values in (6.1.1).

More significantly, as follows from our analysis below, these boundary values can also be motivated from Maxwell’s equations. There, we prescribe the values of the fields on $\Sigma$, and Maxwell’s equations then, in turn, determine the normal derivatives. It turns out that the boundary conditions arising there differ from those here only by polarisation corrections, so that the toy model discussed here is sufficient to determine the phase.

Summarizing: the first condition encodes that every point of the emission surface $\Sigma$ emits light of the same proper frequency $\omega$ ($t$ measures proper time of clocks at rest), with the same phase, and the second equation means that the emission is normal to $\Sigma$. While there are certainly more accurate models for the light emission of lasers (e.g. using Gaussian waves), we only consider this simple case.

For the emission surface, we take the coordinate-plane

$$\Sigma = \{ m_i x^i = 0 \}, \tag{6.1.2}$$

for some (constant) coefficients $m_i$, normalized to $m.m = 1$. Here and in the following, for two spatial vectors $v,w$, we denote by $v.w$ their Euclidean scalar product

$$v.w \equiv \delta^{ij} v_i w_j. \tag{6.1.3}$$

This does not apply to $\kappa, x^\mu$, which are not spatial vectors: instead, $\kappa.x$ denotes the contraction $\kappa_{\mu} x^{\mu}$.

Choosing the orientation of $\nu$ to be the same as the one for the conormal

$$m := m_i dx^i, \tag{6.1.4}$$

one has

$$\nu_i = m_i \left( 1 + \frac{1}{2} \epsilon A(m,m) \cos(\kappa.x + \chi) \right), \tag{6.1.5}$$

where we have used the notation

$$A(m,m) = A^{ij} m_i m_j. \tag{6.1.6}$$

The associated normal vector field $\nu^\alpha = g^{\alpha \beta} \nu_\beta$ is thus

$$\nu^i = \delta^{ij} m_j \left( 1 + \frac{1}{2} \epsilon A(m,m) \cos(\kappa.x + \chi) \right) - \epsilon A^{ij} m_j \cos(\kappa.x + \chi). \tag{6.1.7}$$

Analogously to the discussion of Maxwell fields in Section 2, our boundary conditions (6.1.1) fit well into both the Cauchy-Kovalevskaya theorem, which provides a solution in our setting, and the Holmgren uniqueness theorem, which asserts that solutions of such
boundary value problems are unique within the class of smooth solutions. In any case this
provides a toy model for the problem at hand, and we provide the unique solution below
up to, experimentally irrelevant, error terms $O(\epsilon^2)$.

For the perturbative setting, consider first $\epsilon = 0$ (unperturbed problem) where the
solution is evidently given by
$$\phi^{(0)} = e^{ik\cdot x},$$  \hspace{1cm} (6.1.8)
where
$$k_\mu = \omega(-1, nm).$$  \hspace{1cm} (6.1.9)
In the perturbed case, we thus write
$$\phi = \phi^{(0)} + \epsilon \phi^{(1)} + O(\epsilon^2).$$  \hspace{1cm} (6.1.10)

Inserting this into the scalar wave equation (6.1) and the boundary conditions (6.1.1) leads,
at first order in $\epsilon$, to the boundary value problem
$$\Box\gamma^{(0)} \phi^{(1)} = -n^2 \omega^2 A(m, m) \cos(\kappa \cdot x + \chi) e^{ik\cdot x},$$
$$\phi^{(1)}|_{\Sigma} = 0,$$  \hspace{1cm} (6.1.11)
$$m (\phi^{(1)})|_{\Sigma} = \frac{1}{2} n \omega A(m, m) \cos(\kappa \cdot x + \chi) e^{-i\omega t}.$$

Of course, the right-hand sides are also evaluated at $\Sigma$ whenever the left-hand sides are.

### 6.1.1 General Expression for Emitted Waves

Since we will later consider light emission modelled by Maxwell fields, we will now solve a
slightly more general problem (suppressing the superscript “(1)” momentarily):
$$\Box\gamma^{(0)} \phi = -n^2 \omega^2 A(m, m) \cos(\kappa \cdot x + \chi) e^{ik\cdot x},$$  \hspace{1cm} (6.1.12)
$$\phi|_{\Sigma} = f_2 e^{-i\omega t},$$  \hspace{1cm} (6.1.13)
$$m(\phi)|_{\Sigma} = in \omega f_3 e^{-i\omega t},$$  \hspace{1cm} (6.1.14)
where $f_1$ is a trigonometric function (by which we mean a linear combination of sin and
cos of $\kappa \cdot x + \chi$, and $f_2, f_3$ are linear combinations of such functions of $\kappa \cdot x|_{\Sigma} + \chi$, $\chi - \omega g t$, and possibly involve additive constants. This form covers all the boundary values problems,
arising in the description of light emission, considered in this work.

We find it useful to expand the trigonometric functions in terms of complex exponentials
as follows:
$$f_1 = \alpha^+ e^{+i(\kappa \cdot x + \chi)} + \alpha^- e^{-i(\kappa \cdot x + \chi)},$$
$$f_2 = \beta_1^+ e^{+i(\kappa \cdot x + \chi)} + \beta_1^- e^{-i(\kappa \cdot x + \chi)} + \gamma_1^+ e^{+i(\chi - \omega g t)} + \gamma_1^- e^{-i(\chi - \omega g t)} + 2\delta_1,$$  \hspace{1cm} (6.1.15)
$$f_3 = \beta_2^+ e^{+i(\kappa \cdot x + \chi)} + \beta_2^- e^{-i(\kappa \cdot x + \chi)} + \gamma_2^+ e^{+i(\chi - \omega g t)} + \gamma_2^- e^{-i(\chi - \omega g t)} + 2\delta_2,$$  \hspace{1cm} (6.1.16)
with the right-hand sides of (6.1.16) and (6.1.17) implicitly restricted to $\Sigma$. The factor 2
in front of $\delta_1$ and $\delta_2$ is introduced for notational simplicity in the calculations which follow.

The solution to this problem can then be written as
$$\phi = \phi^+ + \phi^-,$$  \hspace{1cm} (6.1.18)
where $\phi^+ \equiv \phi^+(\alpha^+, \beta_1^+, \beta_2^+, \gamma_1^+, \gamma_2^+)$ is the solution to the problem
$$\Box\gamma^{(0)} \phi^+ = -\alpha^+ \omega^2 e^{i(k \cdot \kappa) \cdot x + i\chi},$$  \hspace{1cm} (6.1.19)
\[ \phi^+|_\Sigma = (\beta_1^+ e^{i(\kappa \cdot x + \chi)} + \gamma_1^+ e^{i(\chi - \omega_g t)} + \delta_1) e^{-i\omega t}, \]  
\[ m(\phi^+)|_\Sigma = i\nu(\beta_2 e^{i(\kappa \cdot x + \chi)} + \gamma_2^+ e^{i(\chi - \omega_g t)} + \delta_2) e^{-i\omega t}, \]

which we will construct shortly, and \( \phi^- \) is obtained from \( \phi^+ \) by replacing the various coefficients \( \alpha^+, \beta_1^+, \ldots \) by the corresponding quantities \( \alpha^-, \beta_1^-, \ldots \), and reversing the sign of both \( \omega_g \) and \( \chi \).

In fact, it is not necessary to write down the phase offset \( \chi \) explicitly. Indeed, given a solution for \( \chi = 0 \), the general case is obtained by the substitution

\[ (\alpha^+, \beta_{1,2}^+, \gamma_{1,2}^+) \rightarrow e^{\pm i\chi}(\alpha^+, \beta_{1,2}^+, \gamma_{1,2}^+), \]

while keeping \( \delta_{1,2} \) unchanged.

Hence, we consider the boundary value problem

\[ \Box_{(0)} \phi = -\omega^2 \alpha e^{i(\kappa \cdot x)}, \]  
\[ \phi|_\Sigma = (\beta_1 e^{i\kappa \cdot x} + \gamma_1 e^{-i\omega_g t} + \delta_1) e^{-i\omega t}, \]  
\[ m(\phi)|_\Sigma = i\nu(\beta_2 e^{i\kappa \cdot x} + \gamma_2 e^{-i\omega_g t} + \delta_2) e^{-i\omega t}, \]

for generic parameters \( \alpha, \beta_{1,2}, \gamma_{1,2}, \delta_{1,2} \). The wave-vectors occurring here are

\[ k_\mu = \omega(-1, nm), \quad \kappa_\mu = \omega_g(-1, \hat{\kappa}). \]

We constrict explicitly a solution \( \phi = \phi(\alpha^+, \beta^+, \gamma^+, \delta) \), which is the unique solution by Holmgren’s theorem.

**Construction of the Solution** A particular solution to the inhomogeneous wave equation (6.1.23) is

\[ \phi_{\text{part.}} = -\frac{\alpha}{\Omega \sigma} e^{i(\kappa + \kappa) \cdot x}, \]

where

\[ \sigma = -\frac{\gamma^{(0)\mu\nu}(k + \kappa) \mu (k + \kappa) \nu}{\omega^2} = 2n(n - m \hat{\kappa}) + \Omega(n^2 - 1), \]

provided that \( \Omega \) and \( \sigma \) do not vanish. We will assume this in all calculations that follow, and consider the low frequency limit \( \Omega \rightarrow 0 \) and the “collinear vacuum limit” (where \( n = 1 \) and \( m \hat{\kappa} \rightarrow 1 \)) so that \( \sigma \rightarrow 0 \) at the end of our calculations.

To satisfy the boundary conditions, we add plane waves which restrict to the desired behaviour on \( \Sigma \). The functions \( \exp(i(k + \kappa - \omega \zeta m) \cdot x) \) and \( \exp(i(k - \omega \xi m) \cdot x - i\omega_g t) \) satisfy the homogeneous wave equation for the following values of \( \zeta \) and \( \xi \):

\[ \zeta_{1,2} = n + \Omega m \hat{\kappa} \mp \sqrt{(n + \Omega m \hat{\kappa})^2 + \Omega \sigma}, \]  
\[ \xi_{1,2} = n \mp n(1 + \Omega), \]

where the first (second) subscript index refers to the upper (lower) sign. Defining also the reflected wave vector

\[ \hat{k}_\mu = \omega(-1, -nm), \]

we make the ansatz

\[ \phi = \phi_S + \phi_R, \]
with
\[
\phi_S = -\frac{\alpha}{\Omega \sigma} e^{i(k+\kappa)x} \left(1 - e^{-i\omega_1 m.x}\right),
\]
\[
\phi_R = \left(\lambda_1 e^{-i\omega_1 m.x} + \lambda_2 e^{-i\omega_2 m.x}\right) e^{i(k+\kappa)x} + \left(\mu_1 e^{-i\omega_1 m.x} + \mu_2 e^{-i\omega_2 m.x}\right) e^{i(k.x-\omega t)} + \nu_1 e^{ik.x} + \nu_2 e^{ik.x}.
\]

Here we have split the function \(\phi\) into a “seemingly singular part” \(\phi_S\) which is undefined for \(\Omega \sigma = 0\), and a remainder \(\phi_R\). As we will see, both \(\phi_S\) and \(\phi_R\) have well-behaved low-frequency and collinear vacuum limits. This ansatz satisfies the boundary conditions (6.1.24) and (6.1.25) if
\[
\lambda_1 + \lambda_2 = \beta_1, \quad \zeta_1 \lambda_1 + \zeta_2 \lambda_2 = \beta_1 (n + \Omega m.\hat{k}) - n\beta_2 - \frac{\alpha \zeta_1}{\Omega \sigma},
\]
\[
\mu_1 + \mu_2 = \gamma_1, \quad \xi_1 \mu_1 + \xi_2 \mu_2 = n(\gamma_1 - \gamma_2),
\]
\[
\nu_1 + \nu_2 = \delta_1, \quad \nu_1 - \nu_2 = \delta_2,
\]
with the solution
\[
\lambda_{1,2} = \frac{1}{2} \left( \beta_1 \pm \frac{n\beta_2 + \frac{\alpha \zeta_1}{\Omega \sigma}}{\sqrt{(n + \Omega m.\hat{k})^2 + \Omega \sigma}} \right),
\]
\[
\mu_{1,2} = \frac{1}{2} \left( \gamma_1 \pm \frac{\gamma_2}{1 + \Omega} \right),
\]
\[
\nu_{1,2} = \frac{1}{2} (\delta_1 \pm \delta_2).
\]

Recall that for computational reasons \(\Omega\) is allowed to be negative, but since we are interested in the low-frequency regime \(|\Omega| < 1\), the case \(1 + \Omega = 0\) is irrelevant for our purposes and, in fact, we will assume \(1 + \Omega > 0\).

**Emission Condition** Note that \(\zeta_1\) and \(\xi_2\) are of order \(\Omega\). The plane waves multiplied by \(\lambda_1, \mu_1\) and \(\nu_1\) thus have wave vectors which are close to the unperturbed wave vector \(k\). In contrast, since \(\zeta_2\) and \(\xi_2\) have an expansion of the form \(2n + O(\Omega)\), the plane waves multiplied by \(\lambda_2, \mu_2\) and \(\nu_2\) correspond to “counter-propagating waves” whose wave vectors are close to the reflected wave vector \(\hat{k} = k - 2n\omega_m\).

The condition for absence of such counter-propagating components is \(\lambda_2 = \mu_2 = \nu_2 = 0\), which translates to the following conditions on the boundary data:
\[
\beta_1 \sqrt{(n + \Omega m.\hat{k})^2 + \Omega \sigma} - n\beta_2 - \frac{\alpha \zeta_1}{\Omega \sigma} = 0, \quad (1 + \Omega)\gamma_1 - \gamma_2 = 0, \quad \delta_1 - \delta_2 = 0.
\]

These conditions could have been imposed from the beginning, but the analysis without assuming (6.1.41) makes it clear that only a limited subclass of solutions are of the eikonal form. If these conditions are met, the solution simplifies to
\[
\phi/e^{ik.x} = -\frac{\alpha}{\Omega \sigma} \left(e^{in.x} - e^{i(k.x-\omega t)}\right) + \beta_1 e^{i(k.x-\omega_1 m.x)} + \gamma_1 e^{-i(\omega t + \omega_1 m.x)} + \delta_1.
\]

We now consider the following two limiting cases which have been excluded so far: (i) the collinear vacuum limit where \(n = 1\) and \(m.\hat{k} \to 1\), so that \(\sigma \to 0\) according to (6.1.28), and (ii) the low frequency limit \(\Omega \to 0\). Both singularities turn out to be removable.
Collinear Vacuum Limit  Consider first the collinear vacuum limit of (6.1.42), where \( n = 1 \) and \( m.\hat{\kappa} \rightarrow 1 \). The last two terms in (6.1.42) remain unchanged in this limit, and the second to last term tends to \( \beta_1 e^{i(\kappa.x)} \) since \( \zeta_1 \rightarrow 0 \) in this limit. Finally, for the remaining term, one has

\[
\lim_{m.\hat{\kappa} \rightarrow 1} \frac{1}{n\sigma} \left( e^{-i\omega_1 mx} - 1 \right) = \frac{i\omega mx}{2(1 + \Omega)},
\]

so all terms in (6.1.42) have a finite collinear vacuum limit. This shows that the singularity at \( m.\hat{\kappa} = 1 \) and \( n = 1 \) is removable.

Low-Frequency Limit  Next, consider the limit \( \Omega \rightarrow 0 \) of (6.1.42). There, one has

\[
\lim_{\Omega \rightarrow 0} \frac{1}{n\sigma} \left( e^{-i\omega_1 mx} - 1 \right) = \frac{i\omega mx}{2n}.
\]

Hence, the limit \( \Omega \rightarrow 0 \) is also well-behaved and the singularity at \( \Omega = 0 \) is removable.

Eikonal Expansion  Using the explicit result (6.1.42), one can now prove the claims made in Section 4. The formula just obtained is already reminiscent of (4.9), but the exact details require further analysis.

Since \( \sigma \) is analytic near \( \Omega = 0 \) with the expansion \( \sigma = 2n(n - m.\hat{\kappa}) + O(\Omega) \), one has

\[
\frac{1}{n\sigma} = \frac{1}{2n\Omega(n - m.\hat{\kappa})} + f(\Omega);
\]

here and in what follows \( f(\Omega) \) denotes a generic function which may change from line to line and which is analytic in \( \Omega \) in a neighbourhood of zero. Similarly, \( \zeta_1 \) is analytic in \( \Omega \) in a neighbourhood of zero and has the expansion \( \zeta_1 = \Omega(m.\hat{\kappa} - n) + O(\Omega^2) \). Consequently, \( \kappa.x - \omega_1 mx = \kappa_0.x + f(\Omega)\omega_0 m.x \), where

\[
\kappa_0.x := \kappa.x + (n - m.\hat{\kappa})\omega_0 m.x.
\]

It then follows that

\[
\frac{\alpha}{n\sigma} \left( e^{i\kappa.x} - e^{i(\kappa.x - \omega_1 mx)} \right) = -\frac{\alpha}{2n\Omega} \frac{e^{i\kappa.x} - e^{i\kappa_0.x}}{n - m.\hat{\kappa}} + f(\Omega, \kappa.x, \omega_0 m.x).
\]

Similarly, the remaining terms in (6.1.42) are of the form \( f(\Omega, \kappa.x, \omega_0 t, \omega_0 m.x) \): the \( \beta_1 \)-term is of the form just considered, and the \( \gamma_1 \)-term can be written as \( \gamma_1 e^{i\omega_0(t - nm.x)} \). Putting all this together, one arrives at the result

\[
\phi/e^{i\kappa.x} = -\frac{\alpha}{2n\Omega} \frac{e^{i\kappa.x} - e^{i\kappa_0.x}}{n - m.\hat{\kappa}} e^{i\kappa.x} + f(\Omega, \kappa.x, \omega_0 t, \omega_0 m.x).
\]

We are primarily concerned with perturbations of plane waves solutions, with the plane wave taking the form \( \phi^{(0)} = \varphi^{(0)} e^{i\kappa.x} \) for some constant \( \varphi^{(0)} \). For \( \varphi^{(0)} \neq 0 \) it suffices to consider the case \( \varphi^{(0)} = 1 \) by linearity. In this case, the structure of the wave equation (4.1) implies that \( \alpha = \frac{i}{2n^2} A(m, m) + O(\Omega) \) so that the overall correction is found to be

\[
\phi^{(1)} = i\psi^{(1)} + f(\Omega, \kappa.x, \omega_0 t, \omega_0 m.x),
\]

where

\[
\psi^{(1)} = -\frac{1}{2n} A(m, m) \frac{\sin(\kappa.x + \chi) - \sin(\kappa_0.x + \chi)}{\Omega(n - m.\hat{\kappa})}.
\]
By a direct calculation one readily verifies that this function satisfies

\[ \gamma^{(0)\mu\nu}k_{\mu}\partial_{\nu}\psi^{(1)} = \frac{1}{2}n^2\omega^2h(m,m), \]  

(6.1.51)

which is the perturbed eikonal equation (4.10).

Alternatively, if the unperturbed field has \( \varphi^{(0)} = 0 \) (this occurs, for example, for polarized EM plane waves, where two components of each of the unperturbed electric and magnetic fields are zero), \( \alpha \) is of order \( \Omega \) and one obtains instead a formula of the kind

\[ \phi^{(1)} = f(\Omega, \kappa, \omega_{\phi}, \omega_{\theta} m, x) \]  

without a leading \( 1/\Omega \) term.

All in all, this proves the claims made in Section 4.

**Next-To-Leading Order Expansion** For later applications, we give an approximate form of (6.1.42) which takes into account both the perturbation of the phase, and also the first correction of the amplitude. Since phase perturbations arise at order \( \Omega^{-1} \) and the first amplitude corrections arise at order \( \Omega^0 \), the desired approximation is obtained by neglecting terms of order \( \Omega \) and higher.

At this level of accuracy, the emission conditions (6.1.41) simplify to yield

\[ \frac{\alpha}{2n^2} + \beta_1 - \beta_2 = O(\Omega), \quad \gamma_1 - \gamma_2 = O(\Omega), \quad \delta_1 - \delta_2 = O(\Omega). \]  

(6.1.52)

If these conditions are satisfied, the field perturbation is given by (6.1.42), in which case one may approximate the phases as

\[ k.x - \omega_0 m.x = \kappa_0 x - \frac{1}{2n} \omega_0 m.x + O(\Omega), \quad -\omega_{\phi} t - \omega_0 m.x = \Omega k.x, \]  

(6.1.53)

(6.1.54)

where \( \kappa_0 x \) is defined in (6.1.46). Expanding the exponentials in this way and using

\[ \frac{1}{\Omega^0} = \frac{1}{2n(n - m.\kappa)} - \frac{n^2 - 1}{4n^2(n - m.\kappa)^2} + O(\Omega), \]  

(6.1.55)

one arrives at

\[ \phi^\pm/e^{ik.x} = \mp \alpha^\pm \frac{e^{\pm ik.x} - e^{\pm i\kappa_0 x}}{2\alpha n(n - m.\kappa)} + \alpha^\pm(n^2 - 1)\left( e^{\pm ik.x} - e^{\pm i\kappa_0 x} \right) \]  

\[ + \left( \beta_1^\pm + i\alpha^\pm \frac{1}{4\alpha^2(n - m.\kappa)} e^{i\kappa_0 x}\right) e^{\pm i\omega_{\phi} m.x} + \delta_1 + O(\Omega), \]  

(6.1.56)

where the first error term grows with the distance \( m.x \), but the second is uniform in the distance.

### 6.1.2 Emitted Scalar Wave

Let us now specialize the general solution just constructed to the concrete problem (6.1.11). Then the coefficients in (6.1.15)—(6.1.17) read

\[ \alpha^\pm = \frac{1}{2}n^2A(m,m), \quad \beta_1^\pm = 0, \quad \beta_2^\pm = \frac{1}{4}A(m,m), \]  

(6.1.57)
and the other parameters $\gamma^\pm_{1,2}$, $\delta_{1,2}$ vanish. These parameters do satisfy the conditions (6.1.52), so that one may use (6.1.56) to obtain

$$
\phi^\pm = -\frac{1}{2} A(m,m)e^{ik.x}\left[\pm \frac{n}{2\Omega} \frac{e^{\pm in.x} - e^{\pm i\kappa_0.x}}{n - m.\kappa} - (n^2 - 1)\frac{e^{\pm in.x} - e^{\pm i\kappa_0.x}}{4(n - m.\kappa)^2}\right.
\pm i\frac{1 - (m.\kappa)^2}{4(n - m.\kappa)}\omega_g m.x e^{\pm i\kappa_0.x} + O(\Omega \omega_g m.x) + O(\Omega),
$$

(6.1.58)

Computing $\phi^{(1)} = \phi^+ + \phi^-$ and restoring the gravitational wave phase shift $\chi$, the first order perturbation is found to be

$$
\phi^{(1)} = -\frac{1}{2} A(m,m)e^{ik.x}\left[in \sin(k.x + \chi) - \sin(k_0.x + \chi)\right] \frac{\Omega}{n - m.\kappa}
- (n^2 - 1)\cos(k.x + \chi) - \cos(k_0.x + \chi)\right]
\left[\frac{\Omega}{2(n - m.\kappa)^2}\omega_g m.x \sin(k_0.x + \chi)\right] + O(\Omega \omega_g m.x) + O(\Omega).
$$

(6.1.59)

Since derivatives of the first term in brackets are of order $\omega$, while those of the remaining terms are of order $\omega_g$, we interpret the purely imaginary first term as a phase correction (recall that in geometric optics, the eikonal is assumed to be rapidly varying, while the amplitude is slowly varying), and write the resulting overall field as

$$
\phi = \mathcal{A} e^{i\psi} + O(\epsilon^2) + O(\epsilon\Omega) + O(\epsilon\omega_g m.x),
$$

(6.1.60)

where

$$
\mathcal{A} = 1 + \frac{1}{4} \epsilon A(m,m)\left[1 - \frac{(m.\kappa)^2}{n - m.\kappa} \sin(k_0.x + \chi)\omega_g m.x
+ (n^2 - 1)\frac{\cos(k.x + \chi) - \cos(k_0.x + \chi)}{(n - m.\kappa)^2}\right],
$$

(6.1.61)

$$
\psi = k.x - \frac{1}{2} \epsilon n A(m,m)\frac{\sin(k.x + \chi) - \sin(k_0.x + \chi)}{\Omega(n - m.\kappa)}.
$$

(6.1.62)

### 6.2 Reflection At a Mirror

We model the mirror by a timelike hypersurface $\Sigma'$ with the following reflection properties. An impinging wave $\phi$ gives rise to a reflected wave $\check{\phi}$ such that

(a) the overall field $\phi + \check{\phi}$ vanishes on $\Sigma'$, and

(b) the normal derivative of $\check{\phi}$ is the same as that of $\phi$.

This is made precise by the equations

$$
(\check{\phi} + \phi)|_{\Sigma'} = 0, \quad \nu(\check{\phi} - \phi)|_{\Sigma'} = 0.
$$

(6.2.1)

To illustrate this in flat space-time, consider a plane wave $\phi = \exp(ik.x)$ with $k_\mu = (-\omega, k_1, k_2, k_3)$ impinging on a mirror $\Sigma'$, which is described by $x = 0$. Choosing the normal $\nu$ to be $\partial/\partial x^1$ one has $\nu(\phi)|_{\Sigma'} = i k_1 \phi|_{\Sigma'}$. The reflected wave $\check{\phi} = -\exp(ik.x)$ with $\check{k}_\mu = (-\omega, -k_1, k_2, k_3)$ then restricts to $-\phi$ on $\Sigma'$ and its normal derivative is

$$
\nu(\check{\phi})|_{\Sigma'} = i k_1 \check{\phi}|_{\Sigma'} = -i k_1 \phi|_{\Sigma'} = i k_1 \phi|_{\Sigma'} = \nu(\phi)|_{\Sigma'}.
$$

(6.2.2)
Since generic waves can be decomposed into plane waves and this argument applies to every Fourier component individually, these conditions are not restricted to plane waves.

In the case of normal incidence in flat spacetime we have \( k_\mu = (-\omega, n\omega, 0, 0) \) and \( \tilde{k}_\mu = (-\omega, -n\omega, 0, 0) \). However, we shall work in coordinates which are not necessarily adapted to the reflection surface, with both \( \Sigma \) and \( \Sigma' \) given by \( m.x = \text{const} \).

Here it should be kept in mind that different components of the electromagnetic field behave differently at mirrors. We will use the current model for scalar waves to get a first insight, before considering the full problem using Maxwell’s equations below.

A simple model for the mirror, we consider the hypersurface

\[
\Sigma' = \{ m.x^i = \ell \}, \tag{6.2.3}
\]

where the coefficients \( m_i \) are the same as for the emission surface \( \Sigma \). The emission and reflection surfaces are thus parallel as coordinate-planes and separated by a coordinate distance \( \ell \).

### 6.2.1 General Expression for Reflected Waves

Before computing the reflected field from the incident field \((6.1.60)\), we consider a more general problem (much like Section 6.1.1) and specialize to the concrete problem later. This will be particularly useful for the discussion of Maxwell fields in Section 7.5.

Proceeding as in \((6.1.18)\) and following, we thus consider the boundary value problem

\[
\Box_{\tilde{\omega}} \tilde{\phi} = -\tilde{\alpha}_\omega^2 e^{i(k+\kappa).x + i\omega t}, \quad m(\tilde{\phi})|_{\Sigma'} = -i\omega \left( \tilde{\beta}_2 e^{i\kappa.x} + \tilde{\gamma}_2 e^{-i\kappa.x} + \tilde{\delta}_2 \right) e^{-i\omega t}, \tag{6.2.4}
\]

where the relevant wave vectors are

\[
k_\mu = \omega(-1, nm), \quad \tilde{k}_\mu = \omega(-1, -nm), \quad \kappa_\mu = \omega g(-1, \tilde{k}). \tag{6.2.7}
\]

Here the phase of the source term in the right-hand side was chosen for convenience \((k.x + n\omega t) \) restricts to \(-\omega t \) on \( \Sigma' \). This is no restriction, as additional phase shifts can always be absorbed in the coefficients \( \tilde{\alpha}, \tilde{\beta}_1, \tilde{\gamma}_2 \) etc.

As in the previous discussion, we work with a vanishing gravitational phase offset \( \chi \). The general case of non-zero \( \chi \) is obtained by a substitution similar to \((6.1.22)\).

The analysis proceeds in almost the same way as in Section 6.1.1. In fact, the only modification is the sign change of \( m \) and the addition of phases of the form \( e^{in\omega t} \). In a way similar to \((6.1.32)\), we write

\[
\tilde{\phi} = \tilde{\phi}_S + \tilde{\phi}_R, \tag{6.2.8}
\]

where

\[
\tilde{\phi}_S = -\tilde{\alpha}_{\Omega\sigma} e^{i(k+\kappa).x + i\omega t} \left( 1 - e^{i\omega\tilde{\chi}_1(m.x-\ell)} \right), \tag{6.2.9}
\]

\[
\tilde{\phi}_R = \left( \tilde{\Lambda}_1 e^{i\omega\tilde{\chi}_1(m.x-\ell)} + \tilde{\Lambda}_2 e^{i\omega\tilde{\chi}_2(m.x-\ell)} \right) e^{i(k+\kappa).x + i\omega t} + \left( \tilde{\beta}_1 e^{i\omega\tilde{\chi}_1(m.x-\ell)} + \tilde{\beta}_2 e^{i\omega\tilde{\chi}_2(m.x-\ell)} \right) e^{i(k.x-\omega t + i\omega t) + \tilde{\delta}_2 e^{i(k.x-n\omega t)}}. \tag{6.2.10}
\]
Here the coefficients $\check{\sigma}, \check{\zeta}_{1,2}$ and $\check{\xi}_{1,2}$ are

\begin{align}
\check{\sigma} &= 2n(n + m.\check{k}) + \Omega(n^2 - 1), \\
\check{\zeta}_{1,2} &= n - \Omega m.\check{k} \mp \sqrt{(n - \Omega m.\check{k})^2 + \Omega \check{\sigma}}, \\
\check{\xi}_{1,2} &= n \mp n(1 + \Omega),
\end{align}

which differ from the previously defined quantities $\sigma, \zeta_{1,2}$ and $\xi_{1,2}$ merely by change of sign of $m$ (in particular, $\check{\xi}_{1,2} = \xi_{1,2}$).

To implement the boundary conditions, we now impose

\begin{align}
\check{\lambda}_1 + \check{\lambda}_2 &= \check{\beta}_1, \\
\check{\zeta}_1 \check{\lambda}_1 + \check{\zeta}_2 \check{\lambda}_2 &= \check{\beta}_1(n - \Omega m.\check{k}) - n\check{\beta}_2 - \frac{\check{\alpha}\check{\zeta}_1}{\Omega \check{\sigma}}, \\
\check{\mu}_1 + \check{\mu}_2 &= \check{\gamma}_1, \\
\check{\zeta}_1 \check{\mu}_1 + \check{\zeta}_2 \check{\mu}_2 &= n(\check{\gamma}_1 - \check{\gamma}_2), \\
\check{\nu}_1 + \check{\nu}_2 &= \check{\delta}_1, \\
\check{\nu}_2 - \check{\nu}_2 &= \check{\delta}_2,
\end{align}

with the solution

\begin{align}
\check{\lambda}_{1,2} &= \frac{1}{2} \left( \check{\beta}_1 \pm \frac{n\check{\beta}_2 + \frac{\check{\alpha}\check{\zeta}_1}{\Omega \check{\sigma}}}{\sqrt{(n - \Omega m.\check{k})^2 + \Omega \check{\sigma}}} \right), \\
\check{\mu}_{1,2} &= \frac{1}{2} \left( \check{\gamma}_1 \pm \frac{\check{\gamma}_2}{1 + \Omega} \right), \\
\check{\nu}_{1,2} &= \frac{1}{2} (\check{\delta}_1 \pm \check{\delta}_2),
\end{align}

cf. (6.1.35) — (6.1.40).

**Reflection Condition** As for the emitted wave, we are primarily interested in the case where no counter-propagating terms arise, which means $\check{\lambda}_2 = \check{\mu}_2 = \check{\nu}_2 = 0$, or equivalently

$$
\sqrt{(n - \Omega m.\check{k})^2 + \Omega \check{\sigma}} \check{\beta}_1 - n\check{\beta}_2 - \frac{\check{\alpha}\check{\zeta}_1}{\Omega \check{\sigma}} = 0, \quad (1 + \Omega)\check{\gamma}_1 - \check{\gamma}_2 = 0, \quad \check{\delta}_1 - \check{\delta}_2 = 0.
$$

(6.2.20)

If these conditions are met, the field simplifies to

$$
\dot{\phi} e^{i(k.x + n.\omega t)} = -\frac{\check{\alpha}}{\Omega \check{\sigma}} (e^{i\kappa.x} - e^{i\kappa.x + \omega\zeta_1(m.x - \ell)}) + \check{\beta}_1 e^{i(n.x + \omega\zeta_1(m.x - \ell))} + \check{\gamma}_1 e^{-i\omega(l + n(m.x - \ell))} + \check{\delta}_1.
$$

(6.2.21)

**Next-To-Leading Order Expansion** Expanding (6.2.20) in powers of $\Omega$, and considering that the boundary values for the reflected wave may contain $1/\Omega$ terms, the condition for the absence of counter-propagating terms at first order in $\Omega$ is found to be

$$
\frac{\check{\alpha}}{2n^2} + \check{\beta}_1 - \check{\beta}_2(1 - \Omega) = O(\Omega), \quad \check{\gamma}_1 - \check{\gamma}_2(1 - \Omega) = O(\Omega), \quad \check{\delta}_1 - \check{\delta}_2 = O(\Omega).
$$

(6.2.22)

Expanding then the first coefficient in (6.2.21) as

$$
\frac{1}{\Omega \check{\sigma}} = \frac{1}{2\Omega n(n + m.\check{k})} - \frac{n^2 - 1}{4n^2(n + m.\check{k})^2} + O(\Omega),
$$

(6.2.23)
cf. (6.1.55), and expanding the phase as
\[
\kappa_1 x + \hat{\zeta}_1 (m.x - \ell) = \kappa_1 x - \frac{1}{2n} (m.\hat{\kappa})^2 \Omega g (\ell - m.x) + O(\Omega^2 \omega_g (\ell - m.x)),
\]
where
\[
\kappa_1 x = \kappa . x + (n + m.\hat{\kappa}) \omega_g (\ell - m.x),
\]
(cf. (6.1.53), one arrives at
\[
\tilde{\varphi}^\pm / e^{i(k.x + n \omega \ell)} = \mp \frac{\tilde{\alpha}^\pm e^{\pm i \kappa_1 x} - e^{\pm i \kappa x}}{2 \Omega n (n + m.\hat{\kappa})} + \alpha^\pm (n^2 - 1) e^{\pm i \kappa x} - e^{\pm i \kappa_1 x} \frac{\Omega \tilde{\beta}^\pm}{2 n (n + m.\hat{\kappa})} \omega_g (\ell - m.x) e^{\pm i \kappa_1 x} + \tilde{\alpha}^\pm (n^2 - 1) e^{\pm i \kappa x} - e^{\pm i \kappa_1 x} \frac{\Omega \tilde{\beta}^\pm}{2 n (n + m.\hat{\kappa})} \omega_g (\ell - m.x) e^{\pm i \kappa_1 x} + O(\Omega^2 \omega_g (\ell - m.x) + O(\Omega).
\]
As in (6.1.56), the first error term grows with the distance from \(\Sigma'\), while the second one is uniform in the distance.

### 6.2.2 Reflected Scalar Wave

In the unperturbed problem \(\epsilon = 0\), the reflected wave is given by
\[
\tilde{\varphi}(0) = -e^{i(k.x + 2i \omega \ell)},
\]
where
\[
\tilde{k} \mu = \omega (-1, -nm).
\]
In the perturbed case, we thus write
\[
\phi = \tilde{\varphi}(0) + \epsilon \tilde{\varphi}(1) + O(\epsilon^2).
\]
From the perturbed wave equation (6.1) one then finds that \(\tilde{\varphi}(1)\) satisfies the inhomogeneous wave equation
\[
\Box \gamma(0) \tilde{\varphi}(1) = -n^2 \omega^2 A(m, m) \cos(\kappa . x + \chi) \tilde{\varphi}(0),
\]
(cf. (6.1.11). According to (6.2.1), the boundary conditions for \(\phi(1)\) are
\[
\tilde{\varphi}(1)|_{\Sigma'} = -\phi(1)|_{\Sigma'}, \quad m(\tilde{\varphi}(1))|_{\Sigma'} = m(\phi(1))|_{\Sigma'} + \nu(\phi(0) - \tilde{\varphi}(0))|_{\Sigma'}.
\]
Using the explicit formula (6.1.7), one finds that the terms arising from the perturbation of the normal \(\nu\) cancel, so that it suffices to implement the conditions
\[
\tilde{\varphi}(1)|_{\Sigma'} = -\phi(1)|_{\Sigma'}, \quad m(\tilde{\varphi}(1))|_{\Sigma'} = m(\phi(1))|_{\Sigma'}.
\]
Using the previously derived expression (6.1.59) for the emitted wave and setting
\[
\varpi = (n - m.\hat{\kappa}) \omega_g \ell
\]
such that \(\kappa_0 . x = \kappa . x + \varpi\) on \(\Sigma'\), one finds the coefficients \(\tilde{\alpha}^\pm\) and \(\tilde{\beta}^\pm_{1,2}\) in (6.2.4) — (6.2.6) to be
\[
\tilde{\alpha}^\pm = -\frac{1}{2} n^2 A(m, m) e^{i \omega \ell \pm i \kappa}.
\]
\[ \beta_1^\pm = \frac{1}{4} A(m, m) e^{i\omega \ell \pm i\chi} \left[ \pm \frac{n}{\Omega} \frac{1 - e^{\pm i\omega}}{n - m.\hat{k}} - \frac{1}{2(n - m.\hat{k})} \frac{1 - e^{\pm i\omega}}{2(n - m.\hat{k})^2} \right] \left( \pm \frac{\omega_g \ell}{2} \right) + O(\Omega) + O(\Omega \omega_g \ell), \]  
(6.2.35)

\[ \beta_2^\pm = \beta_1^\pm (1 \pm \Omega m.\hat{k}/n) - \frac{1}{4} A(m, m) e^{i\omega \ell \pm i\chi} + O(\Omega) + O(\Omega \omega_g \ell), \]  
(6.2.36)

and the coefficients \( \gamma_{1,2}^\pm, \delta_{1,2} \) vanish. These parameters satisfy (6.2.22), so that no counterpropagating terms arise (at the considered level of accuracy) and hence (6.2.26) applies. This yields the following result for the reflected wave:

\[ \phi^\pm = \frac{1}{4} A(m, m) \phi^{(0)} e^{\pm i\chi} \left[ \pm \frac{n}{\Omega} \left( \frac{e^{\pm i\kappa.x} - e^{\pm i\kappa_1.x}}{n + m.\hat{k}} + \frac{e^{\pm i\kappa_1.x} - e^{\pm i(\kappa_1.x + \chi)}}{n - m.\hat{k}} \right) \right] \left( \pm \frac{\omega_g \ell}{2} \right) + O(\Omega) + O(\epsilon \Omega \omega_g \ell), \]  
(6.2.37)

Computing the overall field as \( \phi = \phi^{(0)} + \epsilon (\phi^+ + \phi^-) \), we find

\[ \phi = -s A e^{i\varphi} + O(\epsilon^2) + O(\epsilon \Omega) + O(\epsilon \Omega \omega_g \ell), \]  
(6.2.38)

where

\[ s = 1 - \frac{1}{4} \epsilon A(m, m) \left[ \frac{1}{n^2} - \frac{(m.\hat{k})^2}{2m.\hat{k} \omega_g (\ell - m.x) \sin(\kappa.x + \chi)} \right. \]
\[ \left. - \frac{(n + m.\hat{k}) \omega_g (2\ell - m.x) \sin(\kappa.x + \chi + \omega)}{2(n + m.\hat{k})^2} \right] \left( \cos(k.x + \chi) - \cos(\kappa_1.x + \chi) \right) \left( \frac{\cos(k.x + \chi)}{n + m.\hat{k}} \right)^2 \left( \cos(\kappa_1.x + \chi + \omega) \right) \left( \frac{\cos(\kappa_1.x + \chi) - \cos(\kappa_1.x + \chi + \omega)}{n - m.\hat{k}} \right)^2 \]  
(6.2.39)

\[ \varphi = \kappa.x + 2n.\ell - \frac{1}{4} \epsilon n A(m, m) \sin(u_2 - u_1) \frac{\sin(k.x + \chi) - \sin(\kappa_1.x + \chi + \omega)}{\Omega(n + m.\hat{k})} + \sin(\kappa_1.x + \chi + \omega) - \sin(\kappa_1.x + \chi + \omega) \]  
(6.2.40)

To express the returning field in a concise notation and to compare with the final results of Ref. [21] (see Section 8), we define, for any wave-vector \( \hat{k} \)

\[ H(\tilde{k}, u, u_1) = \frac{\tilde{k}_x \tilde{k}_y}{2\gamma^{(0)}(\tilde{k}, \kappa)} \int_{u_1}^{u_2} h^{\mu\nu}(u) \left( \cos(k.x + \chi) - \sin(k.x + \chi) \right) \frac{\Omega(n + m.\hat{k})}{\Omega(n - m.\hat{k})} \]  
(6.2.41)

so that for the considered metric perturbation \( h^{\mu\nu}(u) = A^{\mu\nu} \cos(u) \) one has

\[ H(k, u, u_1) = -\frac{1}{4} n A(m, m) \sin u_2 - \sin u_1 \frac{\sin(k.x + \chi) - \sin(k.x + \chi + \omega)}{\Omega(n + m.\hat{k})}. \]  
(6.2.42)

The analogous expression with \( k \) replaced by \( \tilde{k} \) is obtained by reversing the sign of \( m \). With this notation, the eikonal perturbation can be written concisely as

\[ \psi^{(1)} = H(\tilde{k}, k.x + \chi, k.x + \chi) + H(k, k.x + \chi, k.x + \chi + \omega). \]  
(6.2.43)
Similarly, we set
\[ \tilde{H}(\tilde{k}, u_2, u_1) = \frac{\tilde{k}_\mu \tilde{k}_\nu}{2\gamma^{0}(\tilde{k}, \kappa)} [h^{\mu\nu}(u_2) - h^{\mu\nu}(u_1)] , \tag{6.2.44} \]
so that the amplitude perturbation can be written as
\[ \mathcal{A}^{(1)} = -\frac{1}{4} A(m, m) \left[ \frac{1 - (m.\kappa)^2}{n^2 - (m.\kappa)^2} \left( 2m.\kappa \omega_g(\ell - m.x) \sin(\kappa_1.x + \chi) 
- (n + m.\kappa) \omega_g(2\ell - m.x) \sin(\kappa_1.x + \chi + \varpi) \right) \right] 
+ \frac{1}{2} (n^2 - 1) \omega_g^2 \left( \frac{\dot{H}(\tilde{k}, \kappa.x + \chi, \kappa_1.x + \chi)}{\gamma^{0}(\kappa, \tilde{k})} + \frac{\dot{H}(k, \kappa_1.x + \chi, \kappa_1.x + \chi + \varpi)}{\gamma^{0}(\kappa, k)} \right) . \tag{6.2.45} \]

### 7 Maxwell’s Equations

Consider the source-free Maxwell equations in the form
\[ \div \bar{F} = 0, \quad \diff F = 0, \tag{7.1} \]
where the field strength \( F \) (a two-form) comprises the electric and magnetic fields \( E \) and \( B \), and the excitation tensor \( \bar{F} \) (a bivector) comprises the fields \( D, H \); see (7.5) below. Here “\( \div \)” denotes the exterior derivative and “\( \diff \)” denotes the divergence with respect to the spacetime metric \( g \). In this work, we consider linear isotropic dielectrics only, for which the relationship between \( F \) and \( \bar{F} \) takes the form
\[ \mu \bar{F}^\alpha_\beta = \gamma^\alpha_\rho \gamma^\beta_\sigma F_{\rho\sigma} , \tag{7.2} \]
where \( \mu \) is the permeability, and \( \gamma \) is the optical metric as defined in (5.8), cf. Ref. [22].

We apply the 3+1 decomposition given in Ref. [23], where we use the expansion
\[ g_{00} = -1 + O(\epsilon^2), \quad g_{0i} = g_{i0} = 0 + O(\epsilon^2), \quad g_{ij} = \delta_{ij} + e h_{ij} + O(\epsilon^2), \quad h_{ij} = A_{ij} \cos(\kappa.x + \chi), \tag{7.3} \]
where \( g \) denotes the spatial metric, i.e. the Riemannian metric induced by \( g \) on slices of constant \( t \equiv x^0 \). For a linearised gravitational field in TT-gauge the error terms above are zero, but our calculations allow for the above. Since we expand both the metric tensor and the electromagnetic field to first order in \( \epsilon \), all subsequent equations are understood to be correct up to \( O(\epsilon^2) \), where the error term is not always written explicitly. Since \( g_{00} = -1 + O(\epsilon^2) \) and \( \det g = 1 + O(\epsilon^2) \), the definitions in Ref. [23] reduce to
\[ D^i = \bar{F}^{0i}, \quad E_i = F_{i0}, \quad B^i = \frac{1}{2} \varepsilon^{ijk} F_{jk}, \quad H_i = \frac{1}{2} \varepsilon^{ijk} \bar{F}^{jk}, \tag{7.5} \]
where \( \varepsilon^{ijk} = \varepsilon^{ijk} \) is the three-dimensional Levi-Civita symbol. Maxwell’s equations then take the form
\[ \partial_0 B^i + \varepsilon^{ijk} \partial_j E_k = 0, \quad \partial_i B^i = 0, \tag{7.6} \]
\[ \partial_0 D^i - \varepsilon^{ijk} \partial_j H_k = 0, \quad \partial_i D^i = 0. \tag{7.7} \]
These look exactly as in Minkowski spacetime, but the dependence upon the gravitational field enters through the constitutive equation (7.2). Indeed, using (7.2) as well as equation (5.8) for the optical metric $\gamma$, one finds

$$D^i = \varepsilon g^{ij} E_j, \quad B^i = \mu g^{ij} H_j,$$

where $g^{ij}$ is the contravariant spatial metric (i.e. $g^{ij}$ is the matrix inverse to $g_{ij}$), $\varepsilon$ the permittivity of the medium and $\mu$ its permeability.

In order to exploit the electric-magnetic symmetry in the absence of external charges and currents, it is useful to define the complex vector field

$$Z^i = \mu D^i + j n B^i,$$

where $n = \sqrt{\varepsilon \mu}$ and $j$ is a second imaginary unit, independent of $i$ (in particular commuting with it), while we reserve the usage of $i$ for the usual complex description of waves. The field equations then reduce to

$$n \partial_0 Z^i + j \varepsilon \varepsilon^{ijk} \partial_j (g_{kl} Z^l) = 0, \quad \partial_i Z^i = 0,$$

provided that both $\varepsilon$ and $\mu$ are constant.

As shown in Appendix A, these equations imply the following wave equation:

$$n^2 \ddot{Z}^i - \Delta(Z^i) + 2\varepsilon R^{(1)}_{ij} Z^j - 2\varepsilon \delta^{jk} \Gamma^{(1)}_{ijl} \partial_k Z^l + j \varepsilon \varepsilon^{ijk} [n \dot{\Gamma}^{(1)}_{kjl} Z^l + n \dot{h}_{kij} \partial_j Z^l] = 0,$$

where $\Delta(\cdot)$ is the scalar Laplacian defined with respect to the perturbed spatial metric $g_{ij}$, $\Gamma^{(1)}_{ijl}$ are the spatial Christoffel symbols, and $R^{(1)}_{ij}$ is the spatial Ricci tensor, both truncated to first order in $\varepsilon$. Explicitly, one has

$$R^{(1)}_{ij} = \frac{1}{2} \omega^2 g_{ij},$$

and

$$\Gamma^{(1)}_{ijk} = -\omega g^{ijk} \sin(\kappa \cdot x + \chi),$$

where

$$\gamma_{ijk} = \frac{1}{2} \left( \hat{k}_j A_{ki} + \hat{k}_k A_{ij} - \hat{k}_i A_{jk} \right).$$

In the following Section 7.1, we review the description of plane EM waves in the absence of GWs using the complex notation presented here. Boundary data describing the emission of plane EM waves in the presence of GWs are constructed in Section 7.2, from which the perturbed emitted field is computed in Section 7.3. The boundary data for reflection at perfect mirrors are then considered in Section 7.4, and the reflected EM wave is computed in Section 7.5.

### 7.1 The Unperturbed Field

In the unperturbed case (flat space), monochromatic plane waves can be written as

$$Z^{(0)l} = \zeta^l e^{i k \cdot x},$$

where the wave vector is

$$k_\mu = \omega (-1, nm),$$
for some unit vector \( m \), and \( \zeta^i \) is a constant \( i \)-real but \( j \)-complex vector (assuming linear polarisation). Maxwell’s equations further imply

\[
\zeta^i - j \varepsilon^{ijk} m_j \zeta_k = 0,
\]

cf. (7.10). Contracting with the \( j \)-complex conjugate \( \zeta^*_i \), \( \zeta^i \), and \( m_i \), one obtains

\[
\zeta^i \zeta^i = 0, \quad m_i \zeta^i = 0, \quad j \varepsilon^{ijk} \zeta^*_i m_j \zeta_k \geq 0, \quad (7.1.4)
\]

which, together, are equivalent to (7.1.3).

Decomposing the complex field \( Z \) into the \( j \)-real fields \( D \) and \( B \) according to (7.9), (7.1.4) one finds that \( (E, B, m) \) forms a right-handed orthogonal system and

\[
E_i D^i = B_i H^i. \quad (7.1.5)
\]

We choose to normalise \( \zeta \) according to

\[
\zeta^*_i \zeta^i = 1. \quad (7.1.6)
\]

This leaves a \( j \)-phase degree of freedom, which corresponds to the freedom of choosing the polarisation of the electromagnetic field.

Note that (7.1.3) is invariant under

\[
m \to -m, \quad \zeta \to \zeta^*, \quad (7.1.7)
\]

so that counter-propagating waves (as arising from normal reflection at a mirror) are given by \( \zeta^* e^{ikx} \), where the reflected wave vector is \( \vec{k}_\mu = \omega(-1, -nm) \).

In the perturbed case, we find it useful to decompose the field as

\[
Z^i = a \zeta^i + b \zeta^*_i + cm^i, \quad (7.1.8)
\]

where, in general, all three functions \( a, b, c \) are non-zero. For later reference, we note the useful identities

\[
j \varepsilon^{ijk} m_j \zeta_k = +\zeta^i, \quad j \varepsilon^{ijk} m_j \zeta^*_k = -\zeta^{*i}, \quad j \varepsilon^{ijk} \zeta_j \zeta^*_k = +m^i, \quad (7.1.9)
\]

where the first equation is due to (7.1.3), the second equation is obtained from the first by complex conjugation, and the third one is obtained by expanding the left-hand side in the basis \( (m, \zeta, \zeta^*) \) and determining the coefficients from suitable contractions. Moreover, we have

\[
\zeta^{*i} \zeta^i + \zeta^i \zeta^{*j} = \delta^{ij} - m^i m^j, \quad (7.1.10)
\]

and since the metric perturbation \( h \) is traceless, one finds

\[
h(\zeta^*, \zeta) + \frac{1}{2} h(m, m) = 0. \quad (7.1.11)
\]

### 7.2 Boundary Values

In this section, we construct boundary data for Maxwell’s equations on the emission surface \( \Sigma = \{m.x = 0\} \), which model the radiation sent out by a laser.

To describe the emission of plane waves with a given frequency \( \omega \), we require the field on \( \Sigma \) to be of the form

\[
Z^i = \hat{Z}^i e^{-i\omega t}, \quad (7.2.1)
\]
where \( \hat{Z}^k \) is \( i \)-real and normalized according to
\[
g(\hat{Z}^*, \hat{Z}) = 1 \quad \text{at the spatial origin}
\] (7.2.2)
where \( \hat{Z}^* \) denotes the \( j \)-complex conjugate of \( \hat{Z} \); as we will see below, the normalisation \( g(\hat{Z}^*, \hat{Z}) = 1 \) cannot be imposed everywhere on \( \Sigma \). Moreover, we demand
\[
\hat{Z}^i \nu_i = 0, \quad g_{ij} \hat{Z}^i \hat{Z}^j = 0, \quad j\epsilon^{ijk} \hat{Z}^i \nu_j \hat{Z}^k > 0
\] (7.2.3)
i.e. the fields \((\nu, D, B)\) form a right-handed orthogonal basis \((D, B)\) are thus tangent to \( \Sigma \) and satisfy \( E^i D^i = B^i H^i \) on \( \Sigma \). As in the unperturbed case, (7.2.3) is equivalent to
\[
\hat{Z}^i - j\epsilon^{ijk} \nu_j g_{kl} \hat{Z}^l = 0.
\] (7.2.4)

A four-dimensional-covariant formulation of this equation is given in Appendix B.

Using the expansion (7.1.8), the requirement that \( Z \) be tangent to the emission surface is seen to be equivalent to \( c = 0 \) on \( \Sigma \), so it remains to prescribe the functions \( a \) and \( b \) there.

The Unperturbed Case  Before considering the general case, let us briefly return to the unperturbed problem where the spatial metric reduces to the flat Euclidean metric and the surface conormal is given by \( \nu_i = m_i \).

Decomposing the electromagnetic field as in (7.1.8) and projecting the field equations (7.10) with \( \epsilon = 0 \) onto the basis \( \zeta, \zeta^*, m \), one obtains Maxwell’s equations in the form
\[
\zeta(a) + \zeta^*(b) + m(c) = 0, \quad \nu_i D^i = B^i H^i \quad \text{on } \Sigma,
\] (7.2.5)
\[
n\partial_0 a + m(a) - \zeta^*(c) = 0, \quad \nu_i D^i = B^i H^i \quad \text{on } \Sigma,
\] (7.2.6)
\[
n\partial_0 b - m(b) + \zeta^*(c) = 0, \quad \nu_i D^i = B^i H^i \quad \text{on } \Sigma,
\] (7.2.7)
\[
n\partial_0 c - \zeta(a) + \zeta^*(b) = 0.
\] (7.2.8)

The last equation here does not contain any \( m \)-derivatives and is thus a “constraint equation” for boundary data prescribed on \( \Sigma \), while the remaining equations determine the normal derivatives of \( a \), \( b \) and \( c \), once their values on \( \Sigma \) are specified.

In flat space, (7.2.4) is equivalent to \( b = c = 0 \) on \( \Sigma \), so that the “constraint equation” (7.2.8) on \( \Sigma \) reduces to \( \zeta(a) = 0 \). Choosing adapted coordinates \( \tilde{x}, \tilde{y} \) on \( \Sigma \) such that \( \sqrt{2}\zeta = \partial_{\tilde{x}} + j\partial_{\tilde{y}} \) (which is always possible by choosing \( \partial_{\tilde{x}}, \partial_{\tilde{y}} \) and \( m \) to be a right-handed orthonormal basis), this reduces to the Cauchy-Riemann equation
\[
\frac{\partial a}{\partial \tilde{x}} + j\frac{\partial a}{\partial \tilde{y}} = 0.
\] (7.2.9)
Requiring \( a \) to be bounded, one finds that it must be spatially constant (by Liouville’s theorem) and thus a function of time \( t \) alone. Hence, one also has \( \zeta^*(a) = 0 \), so that (7.2.5) yields \( m(c) = 0 \).

Thus, assuming (7.2.4) in flat space and requiring the field on \( \Sigma \) to be bounded, one obtains the following boundary values on the emission surface \( \Sigma \)
\[
n\partial_0 a + m(a) = 0, \quad b = m(b) = 0, \quad c = m(c) = 0,
\] (7.2.10)
where \( a \) is an arbitrary function of time \( t \) alone. In particular to describe the emission of monochromatic plane waves of frequency \( \omega \), we shall take \( a = e^{-i\omega t} \), which entails \( m(a) = -i\omega a \).

Note that since all field components satisfy the wave equation, \( b \) and \( c \) vanish identically everywhere and \( a \) is the only remaining degree of freedom.
The Perturbed Case  

Let us now consider the perturbed case, where we still consider the fields on \( \Sigma \) only. Since \( b \) and \( c \) vanish identically in the unperturbed case, we write

\[
Z = a\zeta + \epsilon (b^{(1)}\zeta^* + c^{(1)}m) \equiv a^{(0)}\zeta + \epsilon (a^{(1)}\zeta + b^{(1)}\zeta^* + c^{(1)}m). \tag{7.2.11}
\]

Inserting this into the field equations (7.10), one obtains Maxwell’s equations at first order in \( \epsilon \) in the form

\[
\zeta(a^{(1)}) + \zeta^*(b^{(1)}) + m(c^{(1)}) = 0, \tag{7.2.12}
\]

\[
n\partial_0 a + \nu(a) - \epsilon \zeta^*(c^{(1)}) + \epsilon a^{(0)}j\epsilon^{ijk}\zeta^*(\partial_j h_{kl})\zeta^l = 0, \tag{7.2.13}
\]

\[
n\partial_0 b^{(1)} - m(b^{(1)}) + \zeta(c^{(1)}) + j\epsilon^{ijk}\zeta^*_i\partial_j (a^{(0)}h_{kl})\zeta^l = 0, \tag{7.2.14}
\]

\[
n\partial_0 c^{(1)} - \zeta(a^{(1)}) + \zeta^*(b^{(1)}) + j\epsilon^{ijk}m_i\partial_j (a^{(0)}h_{kl})\zeta^l = 0. \tag{7.2.15}
\]

Equation (7.2.12) is obtained by inserting (7.2.11) into the second part of (7.10), and the equations (7.2.14) and (7.2.15) are obtained by contracting the first part of (7.10) with either \( \zeta_i \) or \( m_i \). To arrive at (7.2.13), one can contract the first equation in (7.10) with \( \zeta_i^* \) to obtain

\[
n\partial_0 a + g^{ij}\theta_i\partial_j a - \epsilon \zeta^*(c^{(1)}) + \epsilon a^{(0)}j\epsilon^{ijk}\zeta^*_i\partial_j (a^{(0)}h_{kl})\zeta^l = 0, \tag{7.2.16}
\]

where \( \theta \) is defined as

\[
\theta_i = -j\epsilon g_{ij}\epsilon^{kl}\zeta^*_k h_{lm}m^m. \tag{7.2.17}
\]

Expanding \( \theta \) in powers of \( \epsilon \), one has \( \theta_i = m_i + \epsilon \theta_i^{(1)} + O(\epsilon^2) \), by virtue of (7.1.9). Using the contractions

\[
\theta_i^{(1)}\zeta^i = 0, \quad \theta_i^{(1)}\zeta^* = h(m, \zeta^*), \quad \theta_i^{(1)}m^i = h(m, m) + h(\zeta^*, \zeta), \tag{7.2.18}
\]

as well as (7.1.11), one finds

\[
\theta_i = \nu_i + \epsilon \zeta_i h(m, \zeta^*) + O(\epsilon^2). \tag{7.2.19}
\]

Here \( \nu \) is the unit conormal to \( \Sigma \) as defined in (6.1.5). Inserting this into (7.2.16) and using \( \zeta(a^{(0)}) = 0 \) then leads to (7.2.13), as claimed.

As in the unperturbed case, (7.2.15) is a constraint equation for boundary data on \( \Sigma \), while the remaining equations determine the normal derivatives of \( a^{(1)} \), \( b^{(1)} \) and \( c^{(1)} \), once their values on \( \Sigma \) are specified.

Let us now implement the assumptions (7.2.1), (7.2.2) and (7.2.4), together with \( \tilde{Z} = \zeta + O(\epsilon) \). The general solution for \( \tilde{Z} \) satisfying these conditions is

\[
\tilde{Z} = \zeta^i (1 + j\epsilon a) - \frac{1}{2} \epsilon [\delta^i - m^i m^j]h_{jk}\zeta^k, \tag{7.2.20}
\]

where \( \tilde{a} \) is an arbitrary function. In components, this is the same as

\[
a = [1 + \epsilon j\tilde{a} - \frac{1}{2}\epsilon h(\zeta^*, \zeta)]e^{-i\omega t}, \quad b^{(1)} = -\frac{1}{2} h(\zeta, \zeta) e^{-i\omega t}, \quad c^{(1)} = 0. \tag{7.2.21}
\]

Inserting this into (7.2.15), the constraint equation reduces to

\[
\zeta(a^{(1)}) + a^{(0)}\Gamma^{(1)}_{ijk}\zeta^i \zeta^j \zeta^k = 0, \tag{7.2.22}
\]

which is equivalent to

\[
j\zeta(\tilde{a}) + \frac{1}{2} \zeta(h(\zeta^*, \zeta)) - \frac{1}{2} \zeta^*(h(\zeta, \zeta)) = 0. \tag{7.2.23}
\]
To understand the implications of this equation, suppose we have two solutions \( \tilde{a}_1 \) and \( \tilde{a}_2 \). Then their difference satisfies the Cauchy-Riemann equation (7.2.9), so that if both functions are bounded, their difference is spatially constant. Hence, the bounded solutions to (7.2.23) are parameterised by an arbitrary complex function of time alone.

A simple choice would be to prescribe \( \tilde{a}(t, x^i = 0) = 0 \). However, to allow for alternative models of laser emission, which might differ in their prescriptions for the time evolution of the emitted polarisation (e.g. one could demand the polarisation vector to be parallel transported in time in the sense of the perturbed metric), we allow \( \tilde{a} \) to satisfy

\[
\tilde{a}\big|_{x^i=0} \equiv \alpha(t) := \alpha_0 + \alpha_c \cos(-\omega_g t + \chi) + \alpha_s \sin(-\omega_g t + \chi),
\]

(7.2.24)

where \( \alpha_0, \alpha_c \) and \( \alpha_s \) (as well as \( \chi \)) are \( j \)-real constants. This means that the emitted polarization vector at the coordinate origin is normalized (corresponding to an energy density at the coordinate origin which remains constant in time), but we allow for perturbations of the plane of polarization which oscillate with the same frequency as the metric perturbation.

As will be shown below, the parameters \( \alpha_0, \alpha_c \) and \( \alpha_s \) have no impact on the signal detected in the interferometer considered and can thus be set to zero for this concrete application. However, as they describe the perturbation of the emitted polarisation, they would be observable in potential experiments sensitive to light polarisation. We note also that the precise form of \( \alpha(t) \) in (7.2.24) fits into our calculations without further due, while more general functions would require further analysis.

Defining

\[
\zeta = \frac{1}{2k}[(\hat{k}, \zeta)A(\zeta^*, \zeta) - (\hat{k}, \zeta^*)A(\zeta, \zeta)],
\]

(7.2.25)

the unique solution to the constraint equation (7.2.23) which remains bounded on \( \Sigma \) is found to be

\[
\tilde{a} = j\zeta [\cos(k.x) - \cos(-\omega_g t + \chi)] + \alpha_0 + \alpha_c \cos(-\omega_g t + \chi) + \alpha_s \sin(-\omega_g t + \chi).
\]

(7.2.26)

The first term here has a non-zero \( j \)-imaginary part and thus describes \( \epsilon \)-oscillations in the emitted amplitude. Such a behaviour is unavoidable, as it is a consequence of the constraint equation (7.2.23).

Having prescribed the field on \( \Sigma \), the normal derivatives of \( a, b^{(1)} \) and \( c^{(1)} \) can now be obtained from the remaining Maxwell equations

\[
n\partial_0 a + g^{ij} \nu_i \partial_j a + \epsilon a^{(0)} j\epsilon^{ijk} \zeta^*_i (\partial_j h_{kl}) \zeta^k = 0,
\]

(7.2.27)

\[
n\partial_0 b^{(1)} - m(b^{(1)}) + j\epsilon^{ijk} \zeta_i (a^{(0)} h_{kl}) \zeta^l = 0,
\]

(7.2.28)

\[
\zeta(a^{(1)}) + \zeta^*(b^{(1)}) + m(c^{(1)}) = 0.
\]

(7.2.29)

Similarly to the scalar wave case, from now on we assume \( \omega_g \ll \omega \).

From (7.2.27) it follows that the normal derivative of \( a \) is

\[
\nu(a)/\omega = ina + O(\Omega),
\]

(7.2.30)

since \( n\partial_0 a = -in\omega a(1 + \epsilon O(\omega_g/\omega)) \) by virtue of (7.2.21).

Using the explicit form of \( b^{(1)} \) in (7.2.21), we have

\[
\frac{1}{\omega} n\partial_0 b^{(1)} = -inb^{(1)} + O(\omega_g/\omega),
\]

(7.2.31)
since \( h \) varies with frequency \( \omega_g \) only. Furthermore, since \( a^{(0)} \) is constant on \( \Sigma \) but varies rapidly in the direction of \( m \) (again by a factor \( \omega/\omega_g \) faster than the metric \( h \)), we have

\[
\frac{1}{2} j \varepsilon^{ijk} \zeta_i \partial_j (a^{(0)} h_{kl}) \zeta^l = j m(a^{(0)}/\omega) e^{ijk} \zeta_i m_j h_{kl} \zeta^l + O(\omega_g/\omega) \\
= -m(a^{(0)}/\omega) h(\zeta, \zeta) + O(\omega_g/\omega) \\
= -i a^{(0)} h(\zeta, \zeta) + O(\omega_g/\omega) \\
= 2i n b^{(1)} + O(\omega_g/\omega),
\]

where we have used (7.1.9), \( m(a^{(0)}) = i n \omega a^{(0)} \), and again the explicit form of \( b^{(1)} \) given in (7.2.21) to arrive at the last line. Plugging all this into (7.2.28), one obtains

\[
m(b^{(1)})/\omega = i n b^{(1)} + O(\Omega).
\]

Finally, a direct calculation shows that

\[
\zeta^* (b^{(1)}) = -a^{(0)} \Gamma^{(1)}_{ijk} \zeta^i \zeta^j \zeta^k,
\]

so that equation (7.2.29) leads to

\[
m(c) = a^{(0)} \left( \Gamma^{(1)}_{ijk} \zeta^i \zeta^j \zeta^k + \Gamma^{(1)}_{ijk} \zeta^i \zeta^j \zeta^k \right).
\]

Inserting (7.1.10) as well as the explicit expression for the spatial Christoffel symbols (7.13), this yields

\[
m(c) = \frac{1}{2} \omega_g a^{(0)} (\hat{\kappa} \cdot \zeta) A(m, m) \sin(\kappa x + \chi).
\]

To summarize, we have obtained the following boundary data for Maxwell equations on the emission surface \( \Sigma = \{ m.x = 0 \} \).

\[
a = e^{-i \omega t} \left[ 1 + \epsilon \partial \tilde{a} - \frac{1}{2} \epsilon h(\zeta^*, \zeta) \right],
\]

\[
b^{(1)} = -\frac{1}{2} e^{-i \omega t} h(\zeta, \zeta),
\]

\[
c^{(1)} = 0,
\]

\[
\nu(a) = i n \omega a + O(\omega_g),
\]

\[
\nu^{(0)} (b^{(1)}) = i n \omega b^{(1)} + O(\omega_g),
\]

\[
\nu^{(0)} (c^{(1)}) = \frac{1}{2} \omega_g a^{(0)} (\hat{\kappa} \cdot \zeta) A(m, m) \sin(\kappa x + \chi).
\]

### 7.3 The Emitted Wave

The wave equation for the perturbation of the emitted field is now obtained by inserting the unperturbed expression

\[
Z^{(0)i} = \zeta^i e^{ik.x},
\]

with \( k_\mu = \omega(-1, nm) \), into the general wave equation (7.11). This leads to

\[
\Delta(Z^i) - n^2 Z^i = -i \epsilon \tilde{a} \left[ 2 \Gamma^{(1)}_{ijk} m^j \zeta^k - n j \zeta^j \right] e^{ik.x} \\
+ \epsilon \left[ 2 R^{(1)}_{ijkl} \zeta^l + n j \zeta^j \right] e^{ik.x},
\]

with \( k_\mu = \omega(-1, nm) \), into the general wave equation (7.11). This leads to

\[
\Delta(Z^i) - n^2 Z^i = -i \epsilon \tilde{a} \left[ 2 \Gamma^{(1)}_{ijk} m^j \zeta^k - n j \zeta^j \right] e^{ik.x} \\
+ \epsilon \left[ 2 R^{(1)}_{ijkl} \zeta^l + n j \zeta^j \right] e^{ik.x},
\]

(7.3.2)
where $\Delta$ is the scalar Laplacian defined with respect to the perturbed spatial metric $\delta_{ij} + \epsilon h_{ij}$. Projecting this equation onto the basis $\zeta, \zeta^*, m$, one finds the wave equations for the components $a, b, c$ to be

\begin{align}
\Box_{\gamma(0)} a^{(1)} &= -n^2 \omega^2 A(m, m) \cos(\kappa x + \chi) e^{ikx} \\
&\quad + in\omega \left[ 2\gamma_{ijk} \zeta^i m^j \zeta^k + nA(\zeta^*, \zeta) \right] \sin(\kappa x + \chi) e^{ikx} \\
&\quad + \omega^2 \left[ A(\zeta^*, \zeta) + nj^i \zeta^i \gamma_{\omega^2 k} \zeta^k \right] \cos(\kappa x + \chi) e^{ikx},
\end{align}

(7.3.3)

\begin{align}
\Box_{\gamma(0)} b^{(1)} &= in\omega \left[ 2\gamma_{ijk} \zeta^i m^j \zeta^k - nA(\zeta, \zeta) \right] \sin(\kappa x + \chi) e^{ikx} \\
&\quad + \omega^2 \left[ A(\zeta, \zeta) + nj^i \zeta^i \gamma_{\omega^2 k} \zeta^k \right] \cos(\kappa x + \chi) e^{ikx},
\end{align}

(7.3.4)

\begin{align}
\Box_{\gamma(0)} c^{(1)} &= 2in\omega \left[ \gamma_{ijk} m^i m^j \zeta^k \right] \sin(\kappa x + \chi) e^{ikx} \\
&\quad + \omega^2 \left[ A(m, \zeta) + nj^i \zeta^i \gamma_{\omega^2 k} \zeta^k \right] \cos(\kappa x + \chi) e^{ikx},
\end{align}

(7.3.5)

The leading order term in (7.3.3) is the same as for the scalar wave equation (6.1.11), so that the sub-leading terms can be interpreted as polarisation terms.

Having determined the wave equations and the boundary values for the functions $a, b$ and $c$, the emitted electromagnetic wave can now be obtained using the formulae derived in Section 6.1.1.

**The $a$ component** For the $a$ component, the value of $\alpha^\pm$ can be read from (7.3.3), $\beta_1^\pm$ is determined by (7.2.37), and $\beta_2^\pm$ is obtained from (7.2.40). These parameters $\alpha^\pm$, $\beta_1^\pm$ and $\beta_2^\pm$ are almost identical to those of the emitted scalar wave, as given in (6.1.57), so that we merely state the correction terms to be added to the values given there. We denote them by $\delta\alpha, \delta\beta$ etc. Finally, due to (7.2.24), the coefficients $\gamma_{1,2}$ and $\delta$ are no longer zero in general:

\begin{align}
\delta\alpha^\pm &= \mp n\Omega \left( \gamma_{ijk} \zeta^i m^j \zeta^k + \frac{1}{2}nA(\zeta^*, \zeta) \right) + O(\Omega^2),
\end{align}

(7.3.6)

\begin{align}
\delta\beta_{1,2} &= -\frac{1}{2} A(\zeta^*, \zeta) - \frac{1}{2} \kappa + O(\Omega),
\end{align}

(7.3.7)

\begin{align}
\gamma_{1,2} &= \frac{1}{2}\zeta + j\frac{1}{2}[\alpha_c + i\alpha_s] + O(\Omega),
\end{align}

(7.3.8)

\begin{align}
\delta_{1,2} &= \frac{1}{2} j\alpha_0 + O(\Omega).
\end{align}

(7.3.9)

**The $b$ component** For the $b$ component, the parameters $\alpha^\pm$ are determined by (7.3.4), where we obtain at leading order, using the explicit form of the Christoffel symbols (7.14),

\begin{align}
2\gamma_{ijk} \zeta^i m^j \zeta^k - nA(\zeta, \zeta) &= (m, \kappa - n)A(\zeta, \zeta),
\end{align}

(7.3.10)

Next, the parameters $\beta_1^\pm$ can be read from (7.2.37), and the values of $\beta_2^\pm$ are determined by (7.2.40). The non-vanishing components are then

\begin{align}
\alpha^\pm &= \pm \frac{1}{2} n\Omega (n - m, \kappa) A(\zeta, \zeta) + O(\Omega^2),
\end{align}

(7.3.11)

\begin{align}
\beta_{1,2} &= -\frac{1}{4} A(\zeta, \zeta) + O(\Omega).
\end{align}

(7.3.12)

**The $c$ component** For the remaining $c$ component, the value of $\alpha^\pm$ are determined from (7.3.5). Using again the explicit form of the Christoffel symbols, one obtains

\begin{align}
2\gamma_{ijk} m^i m^j \zeta^k &= (\kappa, \zeta) A(m, m),
\end{align}

(7.3.13)
which determines $\alpha^\pm$ to leading order in $\Omega$:

$$
\alpha^\pm = \mp \frac{1}{2} n\Omega(\mathbf{k}, \zeta)A(m, m) + O(\Omega^2),
$$

(7.3.14)

while all other parameters vanish to the considered order, as follows from (7.2.39) and (7.2.42).

**Result**  All the above parameter sets satisfy the emission condition (6.1.52), so that the fields are given by the general formula (6.1.56). This yields the following result for the emitted field:

$$
\begin{align*}
a^{(1)} &= \phi^{(1)} + \left(\gamma_{ijk}\kappa^i m^j \zeta^k + \frac{1}{2} nA(\zeta^*, \zeta)\right) \frac{\cos(u) - \cos(u_0)}{n - m.\mathbf{k}} e^{ik.x} \\
&\quad - \frac{1}{2} A(\zeta^*, \zeta) \cos(u_0) e^{ik.x} + j\alpha_0 e^{ik.x} \\
&\quad - \zeta[\cos(u_0) - \cos(\omega_g(nm.x - t) + \chi)] e^{ik.x} \\
&\quad + j[\alpha_c \cos(\omega_g(nm.x - t) + \chi) + \alpha_s \sin(\omega_g(nm.x - t) + \chi)] e^{ik.x} \\
&\quad + O(\Omega) + O(\Omega \omega_g m.x),
\end{align*}
$$

(7.3.15)

$$
\begin{align*}
b^{(1)} &= - \frac{1}{2} h(\zeta, \zeta) e^{ik.x} + O(\Omega) + O(\Omega \omega_g m.x),
\end{align*}
$$

(7.3.16)

$$
\begin{align*}
c^{(1)} &= \frac{1}{2} A(m, m)(\mathbf{k}, \zeta) \frac{\cos(u) - \cos(u_0)}{n - m.\mathbf{k}} e^{ik.x} + O(\Omega) + O(\Omega \omega_g m.x),
\end{align*}
$$

(7.3.17)

where $\phi^{(1)}$ is the perturbation of the emitted scalar field given in (6.1.59), and where we have used the abbreviations

$$
u = \kappa.x + \chi, \quad u_0 = \kappa_0.x + \chi.
$$

(7.3.18)

Note that (7.3.16) entails that

$$
g(Z, Z) = O(\epsilon \Omega) + O(\epsilon \Omega \omega_g m.x),
$$

(7.3.19)

so all scalar invariants of the electromagnetic field vanish to the considered accuracy, cf. (B.16).

### 7.4 Reflection at a Mirror

Having found the electromagnetic wave emitted from a laser, we now consider its reflection off a perfectly reflecting mirror.

Recall that the jumps (denoted by $\Delta D, \Delta B$ etc.) of the fields at an interface with unit normal $\nu$ satisfy

$$
\begin{align*}
\nu \cdot \Delta D &= s, \\
\nu \cdot \Delta B &= 0, \\
\nu \times \Delta E &= 0, \\
\nu \times \Delta H &= j,
\end{align*}
$$

(7.4.1)

where $s$ is the surface charge density and $j$ is the surface current density.

Since all fields vanish inside (or behind) the mirror, the overall field at the reflecting side of the mirror (being a sum of the incident and the reflected fields) satisfies

$$
\begin{align*}
\nu \cdot D &= s, \\
\nu \cdot B &= 0, \\
\nu \times E &= 0, \\
\nu \times H &= j.
\end{align*}
$$

(7.4.2)

Let $Z$ denote the incident Maxwell field, with $Z^*$ its $j$-complex conjugate, and let $\tilde{Z}$ be the reflected field. Using our notation, the standard conditions for reflection on a perfect mirror can be written as

$$
\begin{align*}
\nu \cdot (\tilde{Z} - Z^*) &= 0, \\
\nu \times (\tilde{Z} + Z^*) &= 0.
\end{align*}
$$

(7.4.3)
Indeed, the imaginary part of the first equation is equivalent to \( \nu \cdot B = 0 \), and the real part of the latter is equivalent to \( \nu \times E = 0 \). The remaining parts (i.e. the real part of the first and the imaginary part of the second equation) say that the surface charges and currents have equal contributions from the incident and the reflected waves, cf. equations (56) and (58) in Ref. [24]. This can be understood as follows. The reflected wave is caused by the movement of charge carriers which respond to the impinging wave as to cancel the field inside the conductor. But the true sources must be computed from the total field (the incident \textit{and} the reflected wave) and a linear response of the mirror demands that the contributions of both parts are proportional to each other. Equality then follows from the condition that the amplitudes of the incident and reflected waves coincide.

Equivalently, the equation can be interpreted to say that the normal part of the electric field and the tangential parts of the magnetic field are unchanged by the reflection process, while the tangential part of the electric field and the normal part of the magnetic field change sign.

A covariant formulation of (7.4.3) is given in Appendix E. There, it is also shown that this condition is consistent in the sense that if the incident field \( \mathbf{Z} \) satisfies the constraint equation arising from Maxwell’s equations, then so does the reflected field \( \mathbf{\tilde{Z}} \).

The unique solution to (7.4.3) is given by

\[
\mathbf{\tilde{Z}} = -Z^* + 2g(\nu, Z^*)\nu, \tag{7.4.4}
\]

i.e. \( \mathbf{\tilde{Z}} \) is obtained by reflecting \( Z \) along the normal \( \nu \), taking the \( j \)-complex conjugate and reversing the sign of the field. In other words, equation (7.4.3) is equivalent to (7.4.4).

Inserting the incident field of the form

\[
Z = a\zeta + \epsilon(b(1)\zeta^* + c(1)m), \tag{7.4.5}
\]

a direct calculation shows that the reflected field is given by

\[
\mathbf{\tilde{Z}} = -a^*\zeta^* - \epsilon(b(1)^*\zeta - c(1)^*m). \tag{7.4.6}
\]

Writing (everywhere) the field as

\[
\mathbf{\tilde{Z}} = \mathbf{b}\zeta^* + \epsilon(\mathbf{a}(1)\zeta + \mathbf{c}(1)m) \equiv (\mathbf{\tilde{b}}(0) + \epsilon\mathbf{\tilde{b}}(1))\zeta^* + \epsilon(\mathbf{\tilde{a}}(1)\zeta + \mathbf{\tilde{c}}(1)m), \tag{7.4.7}
\]

the components of the reflected field on the mirror surface are seen to be

\[
\mathbf{\tilde{a}}(1) = -b(1)^*, \quad \mathbf{\tilde{b}} = -a^*, \quad \mathbf{\tilde{c}}(1) = c(1)^*. \tag{7.4.8}
\]

Note that by virtue of equation (7.3.16), we have

\[
\mathbf{\tilde{a}}(1) = -\frac{1}{2}\mathbf{\tilde{b}}(0)h(\zeta^*, \zeta^*). \tag{7.4.9}
\]

Having found the values of the functions \( \mathbf{\tilde{a}}, \mathbf{\tilde{b}} \) and \( \mathbf{\tilde{c}} \) on the mirror surface \( \Sigma' \), we must compute their normal derivatives from Maxwell’s equations. One of the conditions stems from the divergence equation

\[
\nabla_i\mathbf{\tilde{Z}}^i = 0, \tag{7.4.10}
\]

and the other ones are obtained from the evolution equation

\[
n\partial_t\mathbf{\tilde{Z}}^i + j\epsilon^{ijk}\epsilon_{klm}\partial_j\mathbf{\tilde{Z}}^l + j\epsilon^{ijk}(\partial_j\epsilon_{klm})\mathbf{\tilde{Z}}^l = 0. \tag{7.4.11}
\]
We obtain, using the vector \( \theta \) defined in (7.2.17):

\[
\begin{align*}
 n\partial_t \tilde{b} - g^{ij} \theta^i \partial_j \tilde{b} + \epsilon \zeta (\tilde{c}(1)) + j\epsilon \tilde{b}(0) \epsilon^{ijkl} \zeta_l (\partial_j h_{kl}) \zeta^* = 0, \\
n\partial_t \tilde{a}^{(1)} + m(\tilde{a}^{(1)}) - \zeta^*(\tilde{c}(1)) + j\epsilon \tilde{b}(0) \epsilon^{ijkl} \zeta_l (\partial_j h_{kl}) \zeta^* = 0, \\
n\partial_t \tilde{c}^{(1)} - \zeta(\tilde{a}^{(1)}) + \zeta^*(\tilde{b}(1)) + j\epsilon \tilde{b}(0) m_{i} \partial_{j} (\tilde{b}(0) h_{kl}) \zeta^* = 0, \\
\zeta(\tilde{a}^{(1)}) + \zeta^*(\tilde{b}(1)) + m(\tilde{c}^{(1)}) = 0.
\end{align*}
\]

Here we have used the fact that \( \tilde{b}(0) \) coincides with the unperturbed reflected scalar wave, defined in (6.2.27):

\[
\tilde{b}(0) \equiv \tilde{\phi}(0) = -e^{i(k.x+2m\omega t)},
\]

and thus \( \zeta^*(\tilde{b}(0)) = 0 \).

Equation (7.4.14), obtained by contracting the evolution equation with \( m_i \), is the “constraint equation” which, as shown in Appendix E, is automatically satisfied.

Using the explicit expression for \( \theta \) from equation (7.2.19), we have

\[
\theta^* = \nu + \epsilon h(m, \zeta) \zeta^*,
\]

where the second term, proportional to \( \zeta^* \), gives no contribution in the expression \( g^{ij} \theta^i \partial_j \tilde{b} \) because of \( \zeta^*(\tilde{b}(0)) = 0 \), as just shown. The relevant equations to determine the normal derivatives of the fields are thus

\[
\begin{align*}
 n\partial_t \tilde{b} - \nu(\tilde{b}) + \epsilon \zeta (\tilde{c}(1)) + \tilde{b}(0) \epsilon^{ijkl} \zeta_l (\partial_j h_{kl}) \zeta^* = 0, \\
n\partial_t \tilde{a}^{(1)} + m(\tilde{a}^{(1)}) - \zeta^*(\tilde{c}(1)) + j\epsilon \tilde{b}(0) \epsilon^{ijkl} \zeta_l (\partial_j h_{kl}) \zeta^* = 0, \\
\zeta(\tilde{a}^{(1)}) + \zeta^*(\tilde{b}(1)) + m(\tilde{c}^{(1)}) = 0.
\end{align*}
\]

From this, it follows that on the mirror surface \( \Sigma' \) it holds that

\[
\begin{align*}
 \nu(\tilde{b}) &= n\partial_t \tilde{b} + O(\Omega), \\
m(\tilde{a}^{(1)}) &= -i n \omega \tilde{a}^{(1)} + O(\Omega), \\
m(\tilde{c}^{(1)}) &= -i n \omega \tilde{c}^{(1)} + O(\Omega).
\end{align*}
\]

The first equation is immediate, since tangential derivatives of \( \tilde{c}(1) \) and all derivatives of the perturbed metric are of order \( \omega \), as can be seen from (7.4.8) and (7.3.17). For the second equation, we can again neglect the term \( \zeta^*(\tilde{c}(1)) \) but must analyse the last term, where the main contribution comes from the derivative of \( \tilde{b}(0) \) in the direction of \( m \):

\[
\begin{align*}
 j\epsilon^{ijkl} \zeta_l (\partial_j h_{kl}) \zeta^* = j\epsilon^{ijkl} \zeta_l (\partial_j h_{kl}) \zeta^* m_{i} m_{j} h_{kl} \zeta^* m(\tilde{b}(0))/\omega + O(\Omega) \\
= h(\zeta^*, \zeta^*) m(\tilde{b}(0))/\omega + O(\Omega) \\
= -ih(\zeta, \zeta^*) \tilde{b}(0) + O(\Omega) \\
= 2i n \tilde{a}^{(1)} + O(\Omega),
\end{align*}
\]

where we have used (7.1.9), \( m(\tilde{b}(0)) = -i n \omega \tilde{b} \), which is the unperturbed form of (7.4.21), and (7.4.9). Finally, to arrive at the last equation, one can use (7.4.8) obtain

\[
m(\tilde{c}^{(1)}) = \zeta^*(a^{(1)*}) + \zeta(b^{(1)*}).
\]
Using the fact that the incident field satisfies
\[ \zeta(0) + \zeta^*(1) + m(\epsilon(1)) = 0, \] (7.4.26)
as well as the explicit form of \( \epsilon(1) \) in (7.3.17), and also the reflection condition (7.4.8), one obtains
\[ m(\bar{\epsilon}(1))/\omega = -m(\epsilon(1))^* /\omega = -i\epsilon c(1) + O(\Omega) = -i\epsilon c(1) + O(\Omega), \] (7.4.27)
which establishes (7.4.23).

### 7.5 The Reflected Wave

The unperturbed reflected wave is evidently given by
\[ \tilde{Z}(0) = -\zeta^* e^{ik.x+2in\omega t}, \] (7.5.1)
where \( \hat{k}_\mu = \omega (-1, -n m) \). Plugging this into the general wave equation (7.11), we find
\[ \Delta(\tilde{Z}^*) - n^2 \partial_n^2 \tilde{Z}^* = -i\epsilon_n \omega \left[ 2\Gamma^{(1)ij}_{jk} m^j \zeta^{*k} - nj \zeta^{*j} m_j \hat{h}_{kl} \zeta^{*l} \right] e^{ik.x + 2m\omega t} \]
\[ - \epsilon \left[ 2R^{(1)ij}_{jk} \zeta^{*j} + nj \zeta^{*j} \hat{\Gamma}_{jkl} \zeta^{*l} \right] e^{ik.x + 2m\omega t}. \] (7.5.2)

The wave equations for \( \tilde{a} \), \( \tilde{b} \) and \( \tilde{c} \) are now obtained by contracting with the basis vectors \( \zeta^*, \zeta \) and \( m \), respectively.
\[ \Box_{\gamma(0)} \tilde{a}^{(1)} = -in\omega^2 \omega (n + m\hat{n}) A(\zeta^*, \zeta^*) \sin(u) \tilde{b}^{(0)} \]
\[ + \omega^2 \left[ A(\zeta^*, \zeta^*) + nj \zeta^{*j} \Gamma_{jkl} \zeta^{*l} \right] \cos(u) \tilde{b}^{(0)}, \] (7.5.3)
\[ \Box_{\gamma(0)} \tilde{b}^{(1)} = -n^2 \omega^2 A(m, m) \cos(u) \tilde{b}^{(0)} \]
\[ - in\omega^2 \omega \left[ 2\gamma_{ijk} m^j \zeta^{*k} - nA(\zeta^*, \zeta^*) \right] \sin(u) \tilde{b}^{(0)} \]
\[ + \omega^2 \left[ A(\zeta^*, \zeta^*) + nj \zeta^{*j} \zeta_{jkl} \zeta^{*l} \right] \cos(u) \tilde{b}^{(0)}, \] (7.5.4)
\[ \Box_{\gamma(0)} \tilde{c}^{(1)} = -in\omega^2 \omega (\hat{n}, \zeta^*) A(m, m) \sin(u) \tilde{b}^{(0)} \]
\[ + \omega^2 \left[ A(m, \zeta^*) + nj \zeta^{*j} m_j \gamma_{jkl} \zeta^{*l} \right] \cos(u) \tilde{b}^{(0)}. \] (7.5.5)

The components \( \tilde{a}^{(1)} \), \( \tilde{b}^{(1)} \) and \( \tilde{c}^{(1)} \) can now be determined using the general formulae of Section 6.2.1. As done there, we momentarily suppress the GW phase offset \( \chi \) for notational simplicity and reinstate it only in the final result of the computed field.

**The \( \tilde{a} \) Component** For the component \( \tilde{a}^{(1)} \), the value of \( \tilde{a}^\pm \) can be read from (7.5.3), and \( \tilde{\beta}_2^\pm \) is determined by \(- (\tilde{b}^{(1)})^* \) according to (7.4.8), which is given explicitly in (7.3.16). Similarly, \( \tilde{\beta}_2^\pm \) is determined from (7.4.22). The such obtained non-vanishing parameters are
\[ \tilde{a}^\pm /e^{i\nu} = \mp \frac{1}{4} n\Omega (n + m\hat{n}) A(\zeta^*, \zeta^*) + O(\Omega^2), \] (7.5.6)
\[ \tilde{\beta}_2^\pm /e^{i\nu} = \mp \frac{1}{4} A(\zeta^*, \zeta^*) + O(\Omega) + O(\Omega \omega^3). \] (7.5.7)
The $\mathbf{\hat{b}}$ Component For the component $\hat{b}^{(1)}$, the parameter $\hat{a}^\pm$ is determined from (7.5.4), $\hat{\beta}^\pm_1$ is determined from $-(a^{(1)})^*$ according to (7.4.8), which is given explicitly in (7.3.15), and $\hat{\beta}^\pm_2$ can be obtained from (7.4.21). These coefficients almost coincide with those for the reflected scalar wave given in (6.2.34)-(6.2.36). Here we merely state the deviations from the values given there:

$$\delta \hat{a}^\pm /e^{\text{inmol}} = \mp n\Omega \left( \gamma_{ijk} \hat{\zeta}^i m^j \hat{\zeta}^k - \frac{1}{2} n A(\zeta^*, \zeta) \right) + O(\Omega^2),$$

$$\delta \hat{\beta}^\pm_{1,2} /e^{\text{inmol}} = -\frac{1}{2} \left( \gamma_{ijk} \hat{\zeta}^i m^j \hat{\zeta}^k + \frac{1}{2} n A(\zeta^*, \zeta) \right) \frac{1 - e^{\pm i\omega}}{n - m.\hat{\kappa}}$$

$$+ \frac{1}{2} A(\zeta^*, \zeta) e^{\pm i\omega} + \frac{1}{2} \hat{\kappa} e^{\pm i\omega} + O(\Omega) + O(\Omega\omega_\ell),$$

$$\tilde{\gamma}_{1,2} /e^{\text{inmol}} = -\frac{1}{2} [\zeta^* - j(\alpha_c \mp i\alpha_s)] e^{\pm i\omega t} + O(\Omega) + O(\Omega\omega_\ell),$$

$$\tilde{\delta}_{1,2} /e^{\text{inmol}} = \frac{1}{2} j\alpha_0 + O(\Omega) + O(\Omega\omega_\ell),$$

where $\omega$ is defined in (6.2.33).

The $\mathbf{\hat{c}}$ Component For the last component, $\hat{c}^{(1)}$, the parameter $\hat{a}^\pm$ can be inferred from (7.5.5), $\hat{\beta}^\pm_1$ is determined from the incident field $c^{(1)}$ in accordance with (7.4.8), which is given explicitly in (7.3.17), and the parameter $\hat{\beta}^\pm_2$ is determined from (7.4.23). To the considered level of approximation, the non-vanishing parameters are thus found to be

$$\hat{a}^\pm /e^{\text{inmol}} = \mp \frac{1}{2} n(\hat{\kappa}, \zeta^*) A(m, m) + O(\Omega^2),$$

$$\hat{\beta}^\pm_{1,2} /e^{\text{inmol}} = \mp \frac{1}{4} (\hat{\kappa}, \zeta^*) A(m, m) \frac{1 - e^{\pm i\omega}}{n - m.\hat{\kappa}} + O(\Omega) + O(\Omega\omega_\ell),$$

where $\omega$ is defined in (6.2.33).

Result One checks that the parameters just computed satisfy the conditions (6.2.22) (decorated with the $\pm$ superscripts), so that the components of the reflected electromagnetic field are readily found using (6.2.26):

$$\hat{a}^{(1)} = -\frac{1}{2} A(\zeta^*, \zeta^*) \cos(u) \hat{b}^{(0)} + O(\Omega) + O(\Omega\omega_\ell),$$

$$\hat{b}^{(1)} = \hat{c}^{(1)} = \mp \frac{1}{2} A(\zeta^*, \zeta) \cos(u_1 + \omega) \hat{b}^{(0)}$$

$$+ \left( \gamma_{ijk} \hat{\zeta}^i m^j \hat{\zeta}^k - \frac{1}{2} n A(\zeta^*, \zeta) \right) \frac{\cos(u) - \cos(u_1)}{n + m.\hat{\kappa}} \hat{b}^{(0)}$$

$$+ \left( \gamma_{ijk} \hat{\zeta}^i m^j \hat{\zeta}^k + \frac{1}{2} n A(\zeta^*, \zeta) \right) \frac{\cos(u_1) - \cos(u_1 + \omega)}{n - m.\hat{\kappa}} \hat{b}^{(0)}$$

$$- \zeta^* [\cos(u_1 + \omega) - \cos(-\omega t - \omega_\ell n(m.x - 2\ell) + \chi)]$$

$$- j\alpha_c \cos(-\omega t - \omega_\ell n(m.x - 2\ell) + \chi) \hat{b}^{(0)}$$

$$- j\alpha_s \sin(-\omega t - \omega_\ell n(m.x - 2\ell) + \chi) \hat{b}^{(0)} + j\alpha_0 \hat{b}^{(0)} + O(\Omega) + O(\Omega\omega_\ell),$$

$$\hat{c}^{(1)} = -\frac{1}{2} (\hat{\kappa}, \zeta^*) A(m, m) \left[ \frac{\cos(u) - \cos(u_1)}{n + m.\hat{\kappa}} + \frac{\cos(u_1) - \cos(u_1 + \omega)}{n - m.\hat{\kappa}} \right] \hat{b}^{(0)}$$

$$+ O(\Omega) + O(\Omega\omega_\ell).$$

Returning Field Finally, we evaluate the reflected field at the spatial coordinate origin $x^i = 0$, where the beam splitter is positioned. Let us use the notation

$$u_R := u_1 |\Sigma = u + (n + m.\hat{\kappa})\omega_\ell,$$

$$u_E := (u_1 + \omega)|\Sigma = u + 2n\omega_\ell,$$

$$32.$$
to denote the values of the GW phase at the (retarded) times of light reflection and emission, respectively. In order to make contact with the notation of [21], given a vector \( \tilde{k} \) we set

\[
\Gamma^{i}_{j}(\tilde{k}, u_{2}, u_{1}) = \frac{1}{\gamma^{(0)}(\tilde{k}, \kappa)} \int_{u_{1}}^{u_{2}} \Gamma^{i}_{jl} u \, du ,
\]

so that

\[
\Gamma^{i}_{j}(k, u_{2}, u_{1}) = -\gamma^{i}_{jk} m^{k} \frac{\cos(u_{2}) - \cos(u_{1})}{n - m \tilde{k}} ,
\]

\[
\Gamma^{i}_{j}(\tilde{k}, u_{2}, u_{1}) = +\gamma^{i}_{jk} m^{k} \frac{\cos(u_{2}) - \cos(u_{1})}{n + m \tilde{k}} .
\]

Using the function \( \alpha(t) \) defined in (7.2.24), we have

\[
\tilde{a}^{(1)}|_{0} = -\frac{1}{2} h(\zeta^{*}, \zeta^{*}) \tilde{b}^{(0)} + O(\Omega) + O(\Omega \omega \ell) ,
\]

\[
\tilde{b}^{(1)}|_{0} = \tilde{\phi}^{(1)} - \frac{1}{2} h(\zeta^{*}, \zeta)|_{u=u_{E}} \tilde{b}^{(0)} - j \alpha(t - 2n \ell) \tilde{b}^{(0)}
\]

\[
- \zeta_{i} \Gamma^{i}_{j}(\tilde{k}, u_{R}) \zeta^{*} \tilde{b}^{(0)} - \zeta_{i} \Gamma^{i}_{j}(k, u_{R}, u_{E}) \zeta^{*} \tilde{b}^{(0)}
\]

\[
+ \frac{1}{4} n^{2} \omega_{g} \omega \left( \frac{h(m, m)|^{u}_{u_{R}} + h(m, m)|^{u_{R}}}{\gamma^{(0)}(k, \kappa)} \tilde{b}^{(0)} + O(\Omega) + O(\Omega \omega \ell) ,
\]

\[
\tilde{c}^{(1)}|_{0} = \frac{1}{2} n \omega_{g} \omega (\tilde{\kappa}, \zeta^{*}) \left( \frac{h(m, m)|^{u}_{u_{R}} + h(m, m)|^{u_{R}}}{\gamma^{(0)}(k, \kappa)} \tilde{b}^{(0)} + O(\Omega) + O(\Omega \omega \ell) .
\]

### 8 Michelson Interferometers

We are now in a position to describe the gravitational wave response of a Michelson interferometer, as sketched in Figure 1.
We use the approach of Ref. [21] to describe such interferometers. In Minkowski spacetime, i.e. the unperturbed geometry, let $m_i$ and $m_{ii}$ denote the unit vectors pointing along the two interferometer arms (away from the beam splitter), defining the two emission surfaces $\Sigma_i = \{m_i.x = 0\}$ and $\Sigma_{ii} = \{m_{ii}.x = 0\}$. The laser emits coherent light with wave vector $k = \omega(m_i - dt)$ and polarisation $\zeta$. Part of the ray passes through the beam splitter unchanged except for a reduction in amplitude, and thus defines boundary data on $\Sigma_i$, corresponding to normal emission with frequency $\omega$ and polarisation $\zeta$. Similarly, the deflected part of the beam gives rise to boundary data on $\Sigma_{ii}$, corresponding again to normal emission with frequency $\omega$, but with the polarisation vector $\eta = R_{\text{BS}}^{(0)}\zeta$ where $R_{\text{BS}}^{(0)}$ is the orthogonal matrix interchanging $m_i$ and $m_{ii}$ while leaving their orthogonal complements unchanged (see below for details). There is also a phase shift at the beam splitter, which depends upon the details of the apparatus [25, Section 2.4]. Since overall phases are irrelevant for our analysis, it suffices to include the relative phase shift of $\pi$ in the Michelson interferometer, which we incorporate in the output field below; see (8.3).

This description of the beam splitter can be carried over to the perturbed geometry with minimal modifications. To this end, we prescribe boundary data for Maxwell’s equations on the same surfaces $\Sigma_i$ and $\Sigma_{ii}$ (whose unit normal one-forms $\nu_i$ and $\nu_{ii}$ are proportional to $m_i$ and $m_{ii}$, respectively) as in the Minkowski case. We assume again normal emission from both surfaces $\Sigma_i$ and $\Sigma_{ii}$, and the polarisation on $\Sigma_i$ is assumed to be of the form given in (7.2.20) (as dictated by the assumption of normal emission of plane waves), with the polarisation on $\Sigma_{ii}$ (sent into the second arm) related to the one on $\Sigma_i$ by the orthogonal map $R_{\text{BS}}$ which now interchanges $\nu_i$ and $\nu_{ii}$ while leaving their orthogonal complement untouched.

A simple renaming of the relevant data in the solutions derived above provides then the new solutions, as needed to determine the resulting field at the interferometer output.

As such, there are various readout schemes used in gravitational wave detection. For simplicity, we focus here on the DC readout scheme currently used in the LIGO and Virgo detectors. In this scheme, the arm lengths are chosen to be unequal ($\Delta\ell = \ell_i - \ell_{ii} \neq 0$) such that even in the absence of a gravitational wave there is a constant output signal registered by the photodiode, and the gravitational wave then causes a small fluctuation of this signal, which we determine below.

To describe the output of the interferometer, we collect the results obtained so far to describe the field returning to the beam splitter as a function of the emitted field. We then use our model of the beam splitter to describe how the output of the laser is transferred into the two interferometer arms, and to describe how the returning rays are transferred to the detector.

### 8.1 The Returning Field

For ease of further reference, we summarise our calculations so far. We have solved Maxwell’s equations in the GW metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \equiv \eta_{\mu\nu} + \epsilon A_{\mu\nu} \cos(u),$$

(8.1.1)

where

$$u = \kappa.x + \chi \equiv \omega_y(-t + \kappa.x) + \chi,$$

(8.1.2)

with the electromagnetic field prescribed on the emission surface

$$\Sigma = \{m.x = 0\},$$

(8.1.3)
to be emitted normally and to have unit norm at the coordinate origin, so that

$$Z_{in|\Sigma} = e^{-i\omega t} \left[ \zeta (1 + \epsilon j \tilde{a} - \frac{1}{2} \epsilon h(\zeta^*, \zeta)) - \frac{1}{2} \epsilon \zeta^* h(\zeta, \zeta) \right] ,$$  \hspace{1cm} (8.1.4)

where \( \zeta \) is the unperturbed polarisation vector, free to choose subject to the conditions (7.1.4), the asterisk indicates \( j \)-complex conjugation, and \( \tilde{a} \) is the function

$$\tilde{a} = \alpha(t) + j \zeta [\cos(\kappa. x) - \cos(-\omega_g t + \chi)] ,$$  \hspace{1cm} (8.1.5)

where \( \alpha \) and \( \zeta \) are defined in (7.2.24) and (7.2.25), respectively.

To write the returning field in a concise notation, we use the definitions of \( H \), \( \dot{H} \) and \( \Gamma^i_j \) from (6.2.41), (6.2.44), and (7.5.19), as well as

$$k_\mu = \omega(-1, + nm_i), \quad \dot{k}_\mu = \omega(-1, - nm_i),$$  \hspace{1cm} (8.1.6)

together with the unperturbed optical metric

$$\gamma^{(0)\mu\nu} = \text{diag}(-n^2, 1, 1, 1),$$  \hspace{1cm} (8.1.7)

and set

$$u^E = u + 2 n \omega_g \ell ,$$  \hspace{1cm} (8.1.8)
$$u^R = u + (n + m \hat{\kappa}) \omega_g \ell ,$$  \hspace{1cm} (8.1.9)
$$\Delta h = \frac{h(m, m)}{\gamma^{(0)(\kappa, \hat{k})}} \bigg|_{u^R} \left[ u^{u^R} + \frac{h(m, m)}{\gamma^{(0)(\kappa, \hat{k})}} \bigg|_{u^E} \right].$$  \hspace{1cm} (8.1.10)

Then, the returning field at the coordinate origin, given by (7.5.22)—(7.5.24) and (6.2.43) as well as (6.2.45), can be written as

$$Z_{out} = -sA \hat{Z} e^{i\psi} ,$$  \hspace{1cm} (8.1.11)

with

$$\psi = -\omega t + 2 n \omega_e + \epsilon H(\tilde{k}, u, u^R) + \epsilon H(k, u^R, u^E) ,$$  \hspace{1cm} (8.1.12)
$$sA = 1 - \frac{1}{2} \epsilon \omega_g \ell A(m, m) \frac{1 - (m \hat{\kappa})^2}{n^2 - (m \hat{\kappa})^2} \left[ (m \hat{\kappa}) \sin(u^R) - (n + m \hat{\kappa}) \sin(u^E) \right]$$
$$+ \frac{1}{2} \epsilon (n^2 - 1) \omega_g \left[ \frac{\dot{H}(\tilde{k}, u, u^R)}{\gamma^{(0)(\kappa, \hat{k})}} + \frac{\dot{H}(k, u^R, u^E)}{\gamma^{(0)(\kappa, \hat{k})}} \right] ,$$  \hspace{1cm} (8.1.13)

$$\hat{Z} = \zeta^* \left[ 1 - \epsilon j \alpha(t - 2 n \ell) - \frac{1}{2} \epsilon h(\zeta^*, \zeta) \right]_{u^E}$$
$$- \epsilon \zeta_i [\Gamma^i_j (\tilde{k}, u, u^R) + \Gamma^i_j (k, u^R, u^E)] \zeta^{*j} + \frac{1}{2} \epsilon n^2 \omega_g \omega \Delta h \right]$$
$$- \frac{1}{2} \epsilon \zeta^* h(\zeta^*, \zeta^*) + \frac{1}{2} \epsilon n \omega_g \omega (\zeta^*, \hat{\kappa}) \Delta h .$$  \hspace{1cm} (8.1.14)

Since the returning field at the origin only involves the function \( \tilde{a} \) evaluated again at the spatial coordinate origin (but at a retarded time) the \( \zeta \) terms in (8.1.5), which were necessary to satisfy the constraint equations, are irrelevant for the interferometer output.

This expression serves as the basis for the description of interferometers, as it describes how light emitted from a surface \( \Sigma \) returns after being reflected at a perfect mirror placed a distance \( \ell \) away.
8.2 Non-Polarising Beam Splitters

To relate the light polarisation vectors sent into the two interferometer arms (which are both derived from the incoming laser beam) and to compute the output laser field from the returning light rays, one must model the polarisation transfer at the beam splitter. Following the model put forward in [21], we assume the polarisation of the field emitted by the laser to have the form

\[ \hat{Z}_{\text{in}}|_{x^i=0} = \sqrt{2}\zeta(1 + \epsilon j\alpha - \frac{1}{2}ch(\zeta^*, \zeta)) - \frac{1}{2}eh(\zeta, \zeta)\zeta^*, \tag{8.2.1} \]

as already considered in the previous calculation (up to the scaling factor \( \sqrt{2} \) which is conventional). Assuming a 50:50 beam splitter, the amplitudes of the two rays transferred to the two arms are reduced by a factor \( \frac{1}{\sqrt{2}} \). Concerning the polarisation vectors, we assume that the part of the ray which passes straight through the beam splitter maintains its polarisation, while the polarisation sent into the orthogonal arm is given by applying the transformation

\[ \hat{Z} \mapsto -R_{\text{bs}}\hat{Z}^*, \tag{8.2.2} \]

analogously to eq. (7.4.4), where \( R_{\text{bs}} \) is the orthogonal matrix which interchanges the unit conormals \( \nu_i \propto m_i \) and \( \nu_{\text{ii}} \propto m_{\text{ii}} \) (see Figure 1)

\[ R_{\text{bs}}^i_j = \delta^i_j - [1 - g(\nu_i, \nu_{\text{ii}})]^{-1}g^{i_k}(\nu_{\text{ii}} - \nu_i)(\nu_{\text{ii}} - \nu_i)_j. \tag{8.2.3} \]

As shown in Appendix F, see (F.24), the action of this operator on the polarisation vector (7.2.20) yields

\[ R_{\text{bs}}[\zeta(1 + \epsilon j\alpha - \frac{1}{2}eh(\zeta^*, \zeta)) - \frac{1}{2}eh(\zeta, \zeta)\zeta^*] = \eta(1 + \epsilon j(\alpha + \bar{a}) - \frac{1}{2}h(\eta^*, \eta^*)) - \frac{1}{2}\epsilon\eta^* h(\eta, \eta), \tag{8.2.4} \]

where \( \eta \) is the unperturbed polarisation vector of the second arm:

\[ \eta = \zeta - (m_{\text{ii}} - m_i)(m_{\text{ii}}, \zeta), \tag{8.2.5} \]

and \( \bar{a} \) is the \( j \)-imaginary part of the function \( h(\zeta^*, \zeta - \eta) \):  

\[ \bar{a} = \Im_j[h(\zeta^*, \zeta - \eta)]. \tag{8.2.6} \]

For the returning rays, the behaviour of the fields is similar. The ray which first passed through the beam splitter without deflection is now deflected towards the detector and thus undergoes the transformation (8.2.2) with its amplitude reduced by \( \frac{1}{\sqrt{2}} \). The other beam, which was deflected before completing its round trip, now maintains its polarisation vector when being transferred towards the detector. Similarly to the first ray, its amplitude is reduced by \( \frac{1}{\sqrt{2}} \), but additionally there is the standard phase shift of \( \pi \).

For a more detailed description of the beam splitter transformations, see [21, Sect. 6].

8.3 Output Field

We are now in the position to compute the output field reaching the detector, which is a sum of the fields from the two interferometer arms.
Ray I  Consider the ray which is deflected only after its round-trip in one of the interferometer arms. According to our model of the beam splitter, the corresponding field emanating from $\Sigma_i$ is thus (see (8.1.4)—(8.1.5))

$$Z_{i\text{in}}|_{z^i=0} = +e^{-i\omega t}\[1+i\epsilon\alpha - \frac{1}{2}i\epsilon h(\zeta^*,\zeta)] - \frac{1}{2}i\epsilon h(\zeta,\zeta)\].$$  

(8.3.1)

The returning field is given by (8.1.11)—(8.1.14). To evaluate this at the coordinate origin, we set

$$u^E_i = u + 2n\omega_g\ell_1, \quad u^R_i = u + (n + \kappa m_i)\omega_g\ell_1.$$  

(8.3.2)

Recall that at the coordinate origin we have $u|_{z^i=0} = -\omega_g t + \chi$. Applying the beam splitter transformation (8.2.2), reducing the amplitude by $1/\sqrt{2}$, and taking the $i$-real part, one obtains the contribution of this ray to the field at the detector output

$$Z_i = \frac{1}{\sqrt{2}}a_i\tilde{Z}_i \cos(\psi_i),$$  

(8.3.3)

where

$$\psi_i = -\omega t + 2n\omega_g\ell_1 + \epsilon H(k_i, u, u^R_i) + \epsilon H(k_i, u^E_i, u),$$  

(8.3.4)

$$a_i = 1 - \frac{1}{2}\epsilon\omega_g\ell_1 A(m_i, m) \left[ -\frac{(m_i, \kappa)^2}{n^2 - (m_i, \kappa)^2} \left[ (m_i, \kappa) \sin(u^R_i) - (n + m_i, \kappa) \sin(u^E_i) \right] ight] + \frac{1}{2}\epsilon(n^2 - 1)\omega_g^2 \left[ \frac{H(k_i, u, u^R_i)}{\gamma^{(0)}(k, k_i)} + \frac{H(k_i, u^E_i, u)}{\gamma^{(0)}(k, k_i)} \right],$$  

(8.3.5)

and

$$\tilde{Z}_i = \eta \left[ 1 + \epsilon j(\alpha(t - 2n\ell_i) + \bar{a}(t)) - \frac{1}{2}i\epsilon h(\eta^*, \eta) + \frac{1}{2}i\epsilon h(\zeta^*, \zeta) \right] u^E_i 
- \epsilon\zeta^* \left[ \Gamma_j^i(k_i, u, u^R_i) + \Gamma_j^i(k_i, u^E_i, u) \right] \zeta + \frac{1}{4}\epsilon n^2\omega_g\omega\Delta h_1 
- \frac{1}{2}\epsilon\eta^* \left[ \eta \eta^* \right] + \frac{1}{2}\epsilon m \omega_g \omega(\kappa, \zeta) \Delta h_1,$$  

(8.3.6)

and

$$\Delta h_i := \frac{h(m_i, m)}{\gamma^{(0)}(k, k_i)} \big| \frac{u^R_i}{u^R_i} + \frac{h(m_i, m)}{\gamma^{(0)}(k, k_i)} \big| \frac{u^E_i}{u^E_i}.$$  

(8.3.7)

Ray II  The other ray is deflected at the beam splitter before the round-trip. The corresponding field emanating from $\Sigma_{ii}$ is

$$Z_{i\text{in}}|_{z^i=0} = -e^{-i\omega t} \left[ \gamma(1 - \epsilon j(\alpha + \bar{a}) - \frac{1}{2}i\epsilon h(\eta^*, \eta^*) ) - \frac{1}{2}i\epsilon h(\eta^*, \eta^*) \right].$$  

(8.3.8)

and the returning field is directly obtained from (8.1.11)—(8.1.14) via the substitutions

$$\zeta \rightarrow \eta^*, \quad \zeta^* \rightarrow \eta, \quad \alpha \rightarrow - (\alpha + \bar{a}) \quad m \rightarrow m_{ii} \quad k \rightarrow k_{ii},$$  

(8.3.9)

together with a change of the overall sign. This ray passes through the beam splitter with its amplitude reduced by $1/\sqrt{2}$ and the phase shifted by $\pi$, but without modification of the polarisation vector. Finally, taking the $i$-real part to obtain the physical field, the contribution to the output field from this ray is

$$Z_{ii} = \frac{1}{\sqrt{2}}a_{ii}\tilde{Z}_{ii} \cos(\psi_{ii}),$$  

(8.3.10)

37
where

\[ \psi_i = -\omega t + 2n\omega \xi_i + \pi + \epsilon H(k_i, u, R_i) + \epsilon H(k_i, R_i, E), \quad (8.3.11) \]

\[ \mathcal{A}_n = 1 - \frac{1}{2} \epsilon \omega R_i A(m_i, m_i) \frac{1}{n^2} (m_i^2 - (n^2) m_i^2) \left[ (m_i^2) \sin(u_i) - (n + m_i) \sin(u_i) \right] \]

\[ + \frac{1}{2} \epsilon \omega (n^2 - 1) \omega \left[ \frac{H(k_i, u_i) R_i}{\gamma^{(0)}(\xi, R_i)} + \frac{H(k_i, R_i)}{\gamma^{(0)}(\xi, R_i)} \right], \quad (8.3.12) \]

\[ \dot{Z}_{i} = \eta \left[ 1 + \epsilon j (\alpha (t - 2n\xi_i) + \bar{a}(t - 2n\xi_i)) - \frac{1}{2} \epsilon h(\eta, \eta) \right] - \epsilon \eta \left[ \bar{T}^i_j (k_i, u_i R_i) + \bar{T}^i_j (k_i, u_i) \right] \eta^j + \frac{1}{2} \epsilon n^2 \omega \omega \Delta h_i, \quad (8.3.13) \]

with

\[ \Delta h_i := \frac{h(m_i, m_i)}{\gamma^{(0)}(\xi, k_i)} u_i \xi_i + \frac{h(m_i, m_i)}{\gamma^{(0)}(\xi, k_i)} u_i R_i, \quad (8.3.14) \]

and

\[ u_i^E = u + 2n\omega \xi_i, \quad u_i^R = u + (n + \xi_k) m_i \omega \xi_i . \quad (8.3.15) \]

### 8.4 Output Power

Adding the results of the two previous sections, the physical field at the output of the interferometer is found to be

\[ Z_{\text{out}} = Z_1 + Z_{ii} \equiv \frac{1}{\sqrt{2}} \mathcal{A}_n \dot{Z}_{i} \cos(\psi_i) + \frac{1}{\sqrt{2}} \mathcal{A}_n \dot{Z}_{ii} \cos(\psi_i), \quad (8.4.1) \]

whose energy density is proportional to the norm

\[ T_{00} \propto g(Z_1^* + Z_{ii}^*, Z_1 + Z_{ii}), \quad (8.4.2) \]

compare Appendix D. The exact proportionality factor is irrelevant for our purposes, as we are concerned only with the ratio of the output power relative to the input power. Following the standard time-averaging procedure, which we denote by \( \langle \rangle \), we find the observable output power, normalised to the input power, to be

\[ P = \langle g(Z_{\text{out}}^*, Z_{\text{out}}) \rangle / \langle g(Z_{\text{in}}^*, Z_{\text{in}}) \rangle . \quad (8.4.3) \]

As we have normalised the input field (8.2.1) to \( \langle g(Z_{\text{in}}^*, Z_{\text{in}}) \rangle = 1 \), we get

\[ P = \langle g(Z_1^* + Z_{ii}^*, Z_1 + Z_{ii}) \rangle \]

\[ = \frac{1}{2} \mathcal{A}_n^2 g(\dot{Z}_i, \dot{Z}_i) + \frac{1}{2} \mathcal{A}_n^2 g(\dot{Z}_{ii}, \dot{Z}_{ii}) + \frac{1}{2} \mathcal{A}_n [g(\dot{Z}_i, \dot{Z}_{ii}) + g(\dot{Z}_{ii}, \dot{Z}_i)] \cos(\psi_i - \psi_i) . \quad (8.4.4) \]

This is directly measured in the commonly used DC readout scheme of gravitational wave detectors. In the unperturbed case, the normalised output power takes the simple form

\[ P^{(0)} = \frac{1}{2} - \frac{1}{2} \cos(2n\omega \Delta \ell) = \sin^2(n\omega \Delta \ell) . \quad (8.4.5) \]
which vanishes for $\Delta \ell \equiv \ell_{\text{I}} - \ell_{\text{II}} = 0$. In the perturbed case, a direct calculation based on eqs. (8.3.6) and (8.3.13) shows that

\[
\begin{align*}
  g(\hat{Z}_{\text{I}}^*, \hat{Z}_{\text{I}}) &= 1 + \epsilon n^2 \omega_{\phi} \omega \Delta h_1, \\
  g(\hat{Z}_{\text{II}}^*, \hat{Z}_{\text{II}}) &= 1 + \epsilon n^2 \omega_{\phi} \omega \Delta h_{\text{II}}, \\
  2\Re\delta_{ij}g(\hat{Z}_{\text{I}}^*, \hat{Z}_{\text{II}}) &= 2 + \epsilon n^2 \omega_{\phi} \omega [\Delta h_1 + \Delta h_{\text{II}}].
\end{align*}
\]

Using the explicit form of the phase and amplitude perturbations given by eqs. (8.3.4), (8.3.5), (8.3.11) and (8.3.12), one finds that the perturbed output power can be written in the form

\[
P = P^{(0)} + \epsilon \delta_{\psi} P + \epsilon \delta_{\phi} P + \epsilon \delta_n P + \epsilon \delta_K P + O(\epsilon^2),
\]

where

\[
\begin{align*}
  \delta_{\psi} P &= \frac{1}{2} \epsilon \sin(2n\omega \Delta \ell) \\
  &\times \left[ H(\hat{k}_1, u, u^R_1) + H(\hat{k}_1, u, u^E_1) - H(\hat{k}_1, u, u^R_1) - H(\hat{k}_{\text{II}}, u, u^R_{\text{II}}, u^E_{\text{II}}) \right], \\
  \delta_{\phi} P &= -\frac{1}{2} \epsilon \sin^2(n\omega \Delta \ell) \\
  &\times \left[ \omega_{\phi} \ell_1 A(m_1, m_1) - \frac{1 - (m_1, \hat{k})^2}{n^2} \left[ (m_1, \hat{k}) \sin(u^R_1) - (n + m_1, \hat{k}) \sin(u^E_1) \right] \\
  &+ \omega_{\phi} \ell_{\text{II}} A(m_{\text{II}}, m_{\text{II}}) - \frac{1 - (m_{\text{II}}, \hat{k})^2}{n^2} \left[ (m_{\text{II}}, \hat{k}) \sin(u^R_{\text{II}}) - (n + m_{\text{II}}, \hat{k}) \sin(u^E_{\text{II}}) \right] \right], \\
  \delta_n P &= \frac{1}{2} \epsilon (n^2 - 1) \omega_{\phi}^2 \sin^2(n\omega \Delta \ell) \\
  &\times \left[ \frac{\hat{H}(\hat{k}_1, u, u^R_1)}{\gamma^{(0)}(\kappa, \hat{k}_1)} + \frac{\hat{H}(\hat{k}_1, u, u^E_1)}{\gamma^{(0)}(\kappa, \hat{k}_1)} + \frac{\hat{H}(\hat{k}_{\text{II}}, u, u^R_{\text{II}}, u^E_{\text{II}})}{\gamma^{(0)}(\kappa, \hat{k}_{\text{II}})} \right], \\
  \delta_K P &= \frac{1}{2} \epsilon n^2 \omega_{\phi} \sin^2(n\omega \Delta \ell) \\
  &\times \left[ \frac{h(m_1, m_1) u^R_1}{\gamma^{(0)}(\kappa, \hat{k}_1)} + \frac{h(m_1, m_1) u^E_1}{\gamma^{(0)}(\kappa, \hat{k}_1)} + \frac{h(m_{\text{II}}, m_{\text{II}}) u^R_{\text{II}}}{\gamma^{(0)}(\kappa, \hat{k}_{\text{II}})} + \frac{h(m_{\text{II}}, m_{\text{II}}) u^E_{\text{II}}}{\gamma^{(0)}(\kappa, \hat{k}_{\text{II}})} \right],
\end{align*}
\]

This confirms the results of [21] and generalises them beyond the vacuum case $n = 1$: the terms $\delta_{\psi} P$, $\delta_{\phi} P$ and $\delta_K P$ are trivial modifications of the corresponding expressions for vacuum, while the term $\delta_n P$ has no counterpart in the vacuum case.

Notably, both in vacuum and in dielectrics, the perturbation of the polarisation vectors produce no (linear) perturbation of the output intensity. This is because $\epsilon$-perturbations of these vectors produce only $\epsilon^2$ terms in their inner products with the unperturbed vectors.

As in [21], $\delta_{\psi} P$ can be attributed to the GW perturbation of the optical phase, and $\delta_K P$ corresponds to a perturbation of the EM frequency, which rescales the unperturbed signal (as can be seen from the $\sin^2(\omega \Delta \ell)$ term). While in vacuum the perturbation of the amplitude produces only one correction similar to $\delta_{\phi} P$, in the case of $n \neq 1$ there is a further term $\delta_n P$ arising from the difference in propagation speeds of the electromagnetic and gravitational waves.

**8.5 Low Frequency Limit**

For current interferometers, the arm lengths are short compared to the gravitational wavelength, so that it suffices to expand to leading order in $\omega_{\phi} \ell_1$ and $\omega_{\phi} \ell_{\text{II}}$. Moreover,
the difference in the arm lengths is typically very small: \( \Delta \ell/\ell \ll 1 \). Neglecting relative errors of this size, we replace \( \ell \) and \( \ell_i \) by \( \ell \) everywhere except in the multiplicative factors \( \sin^2(n\omega\Delta \ell) \) and \( \sin(n\omega\Delta \ell) \). Using these approximations, we obtain

\[
\delta_{\psi} P \approx \frac{1}{2} n\omega \ell \sin(2n\omega\Delta \ell) \left[ h(m, m) - h(m_i, m_i) \right],
\]

\[
\delta_{\delta} P \approx -\frac{1}{2} n\omega \ell \sin^2(n\omega\Delta \ell) \left[ h'(m, m) \frac{1 - (\kappa.m)^2}{n^2 - (\kappa.m)^2} + h'(m_i, m_i) \frac{1 - (\kappa.m_i)^2}{n^2 - (\kappa.m_i)^2} \right],
\]

\[
\delta_\delta P \approx -\frac{1}{2} n\omega \ell (n^2 - 1) \sin^2(n\omega\Delta \ell) \left( \frac{h'(m, m)}{n^2 - (\kappa.m)^2} + \frac{h'(m_i, m_i)}{n^2 - (\kappa.m_i)^2} \right),
\]

\[
\delta_K P \approx n\omega \ell \sin^2(n\omega\Delta \ell) \left[ h'(m, m) + h'(m_i, m_i) \right],
\]

where

\[
h'(m, m) = \partial_u h(m, m) = -A(m, m) \sin(u).
\]

In the low frequency limit, the perturbations “beyond the eikonal”, \( \delta_{\delta} P \), \( \delta_\delta P \) and \( \delta_K P \), have a similar form and can thus be summarised as

\[
\delta_{\psi\delta} P := \delta_{\delta} P + \delta_\delta P + \delta_K P
\]

\[
\approx \frac{1}{2} n\omega \ell \sin^2(\omega\Delta \ell) \left[ h'(m, m) + h'(m_i, m_i) \right],
\]

Factoring out trigonometric functions depending on the GW phase and the interferometer arm lengths, and splitting the GW polarisation into the two polarisation states as \( A = \alpha_+ A_+ + \alpha_\times A_\times \), where \( \alpha_+ \) and \( \alpha_\times \) denote the amplitudes of the respective polarisation modes, see [21], the detector response in the low frequency limit takes the form

\[
\delta_{\psi} P = n\omega \ell \cos(u) \sin(2n\omega\Delta \ell) (\alpha_+ F_+ + \alpha_\times F_\times),
\]

\[
\delta_{\psi\delta} P = n\omega \ell \sin(u) \sin(n\omega\Delta \ell)(\alpha_+ f_+ + \alpha_\times f_\times),
\]

with the detector response pattern functions

\[
F_\lambda = +\frac{1}{2} [A_\lambda(m, m) - A_\lambda(m_i, m_i)],
\]

\[
f_\lambda = -\frac{1}{2} [A_\lambda(m, m) + A_\lambda(m_i, m_i)],
\]

where \( \lambda \) takes the values + and \( \times \). Parametrising the GW propagation direction and polarisation using Euler angles \( \Phi, \Theta, \Psi \) as in [21] with the polarisation angle \( \Psi \) set to zero as usual, we find

\[
F_+ = \frac{1}{2} (1 + \cos^2 \Theta) \cos 2\Phi, \quad F_\times = -\cos \Theta \sin 2\Phi, \quad F_+ = \frac{1}{2} \sin^2 \Theta \cos 2\Phi, \quad f_\times = 0,
\]

as in [21] (which was concerned with vacuum only). The function \( f_\times \) vanishes identically, and the remaining functions \( F_+ \), \( F_\times \) and \( f_+ \) are plotted (in absolute values) in Figure 2.
In the low-frequency limit, the presence of a dielectric thus results merely in a rescaling of the detector response by the refractive index \( n \), as can be seen from (8.5.7).

### 8.6 Comparison with Fibre Optics

Finally, we compare our results (applying to plane waves in infinitely extended dielectrics) with those of [20] for guided waves in step-index optical fibres. As the final expressions given there are restricted to one-way propagation of light in the low frequency limit, we consider the same setting here.

For one-way light propagation, the phase perturbation in the current setup reads

\[
\delta \psi = -\frac{1}{2} n A(m, m) \frac{\sin(u) - \sin(u + (n - m.\hat{\kappa})\omega_g m.x)}{\Omega(n - m.\hat{\kappa})}
\]

where \( \Omega = \omega_g/\omega \) and \( u = \omega_g(\hat{\kappa}.x^i - t) + \chi \), cf. (6.1.60). Evaluating this at a distance \( \ell \) from the point of emission, at \( x^i = \ell m^i \), the low-frequency limit \( \omega_g \ell \ll 1 \) yields

\[
\delta \psi = \frac{1}{2} n A(m, m) \cos(-\omega_g t + \chi) \omega_\ell + O(\omega_g \ell),
\]

which is to be compared with the phase perturbation in optical fibres [20, Eq. (6.11)]

\[
\delta \psi_{\text{fibre}} = \frac{1}{2} c_1 A(m, m) \cos(-\omega_g t + \chi) + O(\omega_g \ell)
\]

(with the phase offset \( \chi \) added as needed), where we have adapted the overall sign of the phase for consistency with the sign convention used here. The multiplicative coefficient \( c_1 \) in (8.6.3) had to be determined numerically as a function of the core and cladding refractive indices \( n_1, n_2 \), and also of the core radius \( \rho \), but was generally found to be close to the effective refractive index \( c_1 \approx \bar{n} \approx n_1 \) [20, Sect. 8]. Hence, we find good agreement of the phase perturbations acquired by plane waves in infinitely extended dielectrics and Bessel modes of azimuthal mode index \( m = 1 \) in step-index optical fibres.

Concerning perturbations of the amplitude and polarisation, we note that such effects arise in the current setup only at next-to-leading order in the ratio \( \omega_g/\omega \), which was neglected in [20]. Nonetheless, perturbations of this kind were obtained for optical fibres [20, Sect. 7], but they were found to be suppressed relatively to the phase perturbation by a factor \( (1 - n_2/n_1)^2 \) which vanishes in the case considered here. So although the results for the amplitude and polarisation perturbations are not directly comparable, they agree...
to the extent that such effects in media are smaller than the phase perturbation by a factor of \( \omega_g/\omega \). In any case a more accurate comparison of such effects can only be obtained by repeating the calculations here using Bessel waves in lieu of plane waves.

9 Discussion

We have computed the perturbation of monochromatic plane wave solutions of Maxwell’s equations due to a plane gravitational wave and determined the resulting signal in Michelson interferometers.

While the problem is simpler in the commonly used geometric optics approximation, the validity of such an approximation was not clear. Our analysis of the full Maxwell equations confirms its validity, putting the approximation on firm grounds. The unperturbed electromagnetic field, together with its first order perturbation, can indeed be approximated to current experimental accuracy by using a form which separates amplitude and phase, with the amplitude function analytic in the frequency ratio \( \omega_g/\omega \) within the setup considered here.

By computing both phase and amplitude perturbations for arbitrary incidence angles of the gravitational wave, our result generalises previous ones which have either allowed for arbitrary incidence angles but have neglected amplitude perturbations [19] or, conversely, described amplitude or polarisation perturbations only for specific alignments [14, 18, 26].

Moreover, by including a dielectric refractive index, we were able to generalise our previous result [21] (which relied on the geometric optics approximation), and compare with analogous calculations for fibre optics [20]. In both cases, we find agreement of the essential results.

While we admit that the calculations presented here are unlikely to have a direct impact on signal analysis and interpretation of current gravitational wave detectors, we stress that previous models based on geometric optics relied on the tacit assumption that the true solution of the Maxwell equations in this setup admits a convergent eikonal expansion. Our analysis here describes a model where this is indeed the case.
A The Wave Equation for the Electromagnetic Field

Starting from the field equations (7.10) in the form
\begin{align}
n\partial_0 Z^i + j i\varepsilon^{ijk} \nabla_j Z^k = 0, \\
\nabla_i Z^i = 0,
\end{align}
(A.1)
we now derive a wave equation for the electromagnetic field \( Z^i \). Throughout this section indices are raised and lowered with the perturbed spatial metric \( g_{ij} \) by default. To derive an equation of second order, we apply \( n\partial_0 \) to the first equation, use \( \partial_0 \nabla_j Z^k = \partial_j \partial_0 Z^k - \partial_0 \Gamma^l_{jk} Z_l \), and thus obtain
\begin{align}
n^2 \ddot{Z}^i + j i\varepsilon^{ijk} \nabla_j (n\partial_0 Z^k) = 0,
\end{align}
(A.2)
since the connection is torsion-free. Here and elsewhere, an overset dot indicates differentiation with respect to the time coordinate \( t \equiv x^0 \). Next, lowering the index of the first-order evolution equation, one obtains
\begin{align}
n\dot{Z}^k = -j i\varepsilon^{lm} \nabla_l Z^m + n\dot{g}^{kl} Z^l,
\end{align}
(A.3)
so that the second-order equation can be written as
\begin{align}
n^2 \ddot{Z}^i - \Delta Z^i + R^i_j Z^j + j i\varepsilon^{ijk} \nabla_j (n\dot{g}^{kl} Z^l) = 0.
\end{align}
(A.4)

Using \( \nabla_j Z^j = 0 \), the second term can be simplified to yield
\begin{align}
(g^{il} g^{jm} - g^{im} g^{jl}) \nabla_l \nabla_i Z_m = g^{il} \nabla_l \nabla_i Z^j - g^{il} \nabla_l \nabla_j Z^i - \Delta Z^i = -\Delta Z^i + R^i_j Z^j,
\end{align}
(A.5)
where \( \Delta = g^{ij} \nabla_i \nabla_j \) is the spatial Laplacian operator (here acting on a vector field) and \( R^i_j \) is the spatial Ricci tensor. This leads to the equation
\begin{align}
n^2 \ddot{Z}^i - \Delta Z^i + R^i_j Z^j + j i\varepsilon^{ijk} \nabla_j (n\dot{g}^{kl} Z^l) = 0.
\end{align}
(A.6)

Using \( \varepsilon^{ijk} \nabla_j \alpha_k = \varepsilon^{ijk} \partial_j \alpha_k \) and \( \partial_j g_{jk} = \Gamma_{jki} + \Gamma_{kij} \), as well as the symmetry of the Christoffel symbols, one finally obtains
\begin{align}
n^2 \ddot{Z}^i - \Delta Z^i + R^i_j Z^j + j i\varepsilon^{ijk} [n\dot{\Gamma}_{kjl} Z^l + n\dot{g}_{kl} \partial_j Z^i] = 0.
\end{align}
(A.7)

A direct calculation shows that the components of the Laplacian acting on the vector field \( Z \) are related to the Laplacian acting on the component functions \( Z^i \) by
\begin{align}
(\Delta Z)^i = (\Delta(Z^i) + \epsilon \delta^{ij} \partial_j \Gamma_{kli}^1 \nabla_k Z^l + 2 \epsilon \delta^{ij} \Gamma_{kli}^1 \partial_k Z^l - \epsilon \delta^{ij} \Gamma_{kli}^1 \partial_j Z^i + O(\epsilon^2)).
\end{align}
(A.8)

For the particular spatial metric (7.4), the Christoffel symbols take the form
\begin{align}
\Gamma_{ijkl}^1 = -\frac{1}{2} \epsilon [\kappa_j A_{ki} + \kappa_k A_{ij} - \kappa_i A_{jk}] \sin(\kappa x + \chi),
\end{align}
(A.9)
so that the last term in (A.8) is seen to vanish. Consequently, one has
\[ \delta_{jk} \partial_j \Gamma^{(1)}_{ikl} = -\frac{1}{2} \omega_2 g^{(1)}_{il} = -R^{(1)}_{il}, \]  
(A.10)
where \( R_{ij} \) is the spatial Ricci tensor with components
\[ R^{(1)}_{ij} = \frac{1}{2} \omega_2 g^{(1)}_{ij}. \]  
(A.11)
Combining these intermediate results, the wave equation takes the form
\[ n^2 \dddot{Z}_i - \Delta(Z_i) + 2\epsilon R^{(1)}_{ij} Z^j + j\epsilon \varepsilon^{ijk}[n \dot{\Gamma}^{(1)}_{kjl} Z_l + n \dot{g}^{(1)}_{kl} \partial_j Z^l] = 0. \]  
(A.12)

In all terms involving derivatives of the metric (i.e. its time derivative, Christoffel symbols or the Ricci tensor), we may use the unperturbed expression for the electromagnetic field.

**B Covariant Formulation of the Emission Condition**

Here, we discuss a covariant formulation of the emission condition (7.2.4), which states that
\[ Z^i - j\varepsilon^{ijk} \nu_j Z_k = 0, \]  
(B.1)
where \( \nu \) is the conormal of the emission surface \( \Sigma \).

Using a timelike, future-pointing unit normal field \( u \), every two-form \( F \) admits the covariant representation
\[ F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu + \epsilon_{\mu\nu\rho} B^\rho, \]  
(B.2)
with a vector \( B \) and a covector \( E \) satisfying \( E_\mu u^\mu = 0 = B^\mu u_\mu \). Therein, the “spatial epsilon tensor” \( \epsilon_{\mu\nu\rho} \) is obtained from the epsilon tensor (with \( \epsilon_{0123} = \sqrt{-\det g_{\mu\nu}} \)) via the contraction \( \epsilon_{\mu\nu\rho} := u^\sigma \epsilon_{\sigma\mu\nu\rho} \). The projections \( E_i := P_i^\mu E_\mu \) and \( B^i := P_i^\mu B^\mu \) with \( P_\mu^\nu := \delta_\mu^\nu + u^\mu u_\nu \) yield the 3-vectors that reproduce the representation (7.5) of the field strength tensor in coordinates with \( u_\mu = (-1/\sqrt{-g^{00}}, 0, 0, 0) \). Analogously, the displacement tensor may be written as
\[ \bar{F}^{\mu\nu} = u_\mu D_\nu - u_\nu D_\mu + \epsilon^{\mu\nu\rho} H_\rho. \]  
(B.3)
For the dual field strength tensor \( *F^{\mu\nu} := \frac{1}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} \) one obtains
\[ *F^{\mu\nu} = -u_\mu B_\nu + u_\nu B_\mu + \epsilon^{\mu\nu\rho} E_\rho. \]  
(B.4)
Assuming now a linear isotropic dielectric with \( D_\mu = \varepsilon g^{\mu\nu} E_\nu, B_\mu = \mu g^{\mu\nu} H_\nu \) and \( n := \sqrt{\varepsilon \mu} \), we consider the \( j \)-complex combination
\[ \mathcal{F}^{\mu\nu} := \mu \bar{F}^{\mu\nu} - jn \varepsilon *F^{\mu\nu} \]  
\[ = u_\mu Z_\nu - u_\nu Z_\mu - \frac{j}{n} \epsilon^{\mu\nu\rho} Z_\rho, \]  
(B.5)
where
\[ Z_\mu := \mu D_\mu + jn B_\mu, \]  
\[ Z_\mu := g_{\mu\nu} Z_\nu = n^2 E_\nu + j\mu n H_\nu. \]  
(B.6)
Note that the electromagnetic field in a general medium can be expressed similarly using two independent complex vectors, cf. [27].
In accordance with geometric optics, cf. e.g. [21], we consider light of frequency $\omega$ (as measured by observers with 4-velocity $u$) emitted orthogonally to the surface $\Sigma$, with normal $\nu$, which we assume orthogonal to $u$. The corresponding wave vector then reads

$$k_\mu = \omega(u_\mu + n\nu_\mu).$$ \hspace{1cm} (B.7)

In such a setting one imposes on $\Sigma$ the conditions

$$\bar{F}^{\mu\nu}k_\nu = 0, \quad *F^{\mu\nu}k_\nu = 0. \hspace{1cm} (B.8)$$

In our complex notation, this can be summarized as

$$\mathcal{F}^{\mu\nu}k_\nu = 0, \hspace{1cm} (B.9)$$

which can be rewritten as

$$Z^\mu + n\nu^\mu Z^\nu - j\epsilon^{\mu\nu\rho}Z^\rho = 0. \hspace{1cm} (B.10)$$

Contracting this equation with $u_\mu$ yields $\nu_\mu Z^\nu = 0$, so that (B.10) simplifies to

$$Z^\mu - j\epsilon^{\mu\nu\rho}Z^\rho = 0, \hspace{1cm} (B.11)$$

as used in the main body of the paper.

To calculate the invariants of the electromagnetic field we recall the definition of the dual of the excitation tensor $\bar{F}^{\alpha\beta}$:

$$*\bar{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\bar{F}^{\alpha\beta} = -u_\mu H_\nu + u_\nu H_\mu + \epsilon_{\mu\nu\rho}D^\rho. \hspace{1cm} (B.12)$$

To relate $*\bar{F}^{\mu\nu}$ to $*F^{\rho\sigma}$, let $\gamma^{\mu\nu}$ denote the inverse to the contravariant optical metric $\gamma^{\mu\nu}$, which evaluates to

$$\gamma^{\mu\nu} = g^{\mu\nu} + (1 - n^{-2})u^\mu u^\nu. \hspace{1cm} (B.13)$$

One then finds

$$\Pi *\bar{F}^{\mu\nu} = \frac{\det(\gamma^{\alpha\beta})}{\det(g^{\alpha\beta})} \gamma^{\mu\rho} \gamma^{\nu\sigma} *F^{\rho\sigma} = n^2 \gamma^{\mu\rho} \gamma^{\nu\sigma} *F^{\rho\sigma}. \hspace{1cm} (B.14)$$

In the last identity we used the fact that $\det(\gamma^{\alpha\beta})/\det(g^{\alpha\beta})$ is a scalar, so that it may be computed in the local inertial system of the medium, where $g^{\alpha\beta} = \eta^{\alpha\beta}$ and $\gamma^{\alpha\beta} = \text{diag}(-n^2, 1, 1, 1)$. Using (7.2), contraction of $F^{\rho\sigma}$ with the optical metric then yields

$$\bar{F}^{\mu\nu} := \gamma^{\mu\rho} \gamma^{\nu\sigma} F^{\rho\sigma} = F^{\mu\nu} - \frac{j}{n} \Pi \bar{F}^{\mu\nu} = n^{-2}(u_\mu Z_\nu - u_\nu Z_\mu - jn\epsilon_{\mu\nu\rho}Z^\rho). \hspace{1cm} (B.15)$$

Thus setting both invariants $F^{\mu\nu} \bar{F}^{\mu\nu}$ and $F^{\mu\nu} * F^{\mu\nu}$ of the electromagnetic field to zero is seen to be equivalent to the complex equation

$$\mathcal{F}^{\mu\nu} \bar{F}^{\mu\nu} \equiv 2\Pi F^{\mu\nu} \bar{F}^{\mu\nu} - 2jnF^{\mu\nu} * F^{\mu\nu} \equiv -\frac{4}{n^2} Z^\nu Z_\nu = 0, \hspace{1cm} (B.16)$$

which is already implied by (B.11).
C The Boundary-Constraint Equation

Maxwell's equations written in terms of the tensor field (B.5) read
\[ \nabla_{\mu} F^{\mu \nu} = 0, \]  
\[ (C.1) \]
or equivalently
\[ \partial_{\mu} (\sqrt{-\det g} F^{\mu \nu}) = 0. \]  
\[ (C.2) \]

Let us choose a coordinate system so that a non-characteristic hypersurface \( \Sigma \) is given by the equation \( \{x^0 = 0\} \); we emphasise that we do not assume that \( x^0 \) is a time coordinate, only that \( g^{00}|_{\Sigma} \) has no zeros. The field
\[ C := \partial_{i} (\sqrt{-\det g} F^{i 0}) \equiv \partial_{\mu} (\sqrt{-\det g} F^{\mu 0}), \]  
\[ (C.3) \]
which involves only derivatives tangential to \( \Sigma \), has to vanish for solutions of (C.1). Hence, the equation \( C = 0 \) has the character of a constraint equation when propagating fields away from \( \Sigma \).

The complementing part of the constraint equation \( C = 0 \) in (C.1) are the propagation equations
\[ \partial_{0} (\sqrt{-\det g} F^{0 j}) = -\partial_{j} (\sqrt{-\det g} F^{0 j}). \]  
\[ (C.4) \]
Here, propagation does not mean propagation in time, but propagation away from \( \Sigma \). Every solution of these propagation equations satisfies
\[ \partial_{0} C = \partial_{0} \partial_{j} (\sqrt{-\det g} F^{0 j}) = \partial_{j} \partial_{0} (\sqrt{-\det g} F^{0 j}) = \partial_{j} \partial_{i} (\sqrt{-\det g} F^{0 j}) = 0, \]  
\[ (C.5) \]
where we have used the antisymmetry of \( F^{\mu \nu} \). We conclude that if \( C \) vanishes on \( \Sigma \) and if \( (C.4) \) holds, then \( C \) vanishes on the domain of definition of the coordinates above.

D The Energy-Momentum Tensor

The field equations can be derived from a variational principle based on the Lagrangian density
\[ L = -\frac{1}{4} \sqrt{-g} \tilde{F}^{\alpha \beta} F_{\alpha \beta}, \]  
\[ (D.1) \]
where \( F \) is regarded as the curvature form of an abelian gauge potential \( F = dA \). The overall scaling depends on the choice of units, which is irrelevant here as we assume no external charges or currents. An energy tensor \( T_{\mu \nu} \) is obtained using the standard formula
\[ \delta L = -\frac{1}{2\pi} \sqrt{-g} T_{\mu \nu} \delta g^{\mu \nu}, \]  
\[ (D.2) \]
where the 4-velocity \( u \) is varied according to \( \delta u^{\mu} = -\frac{1}{2} u^{\mu} u_{\alpha} u_{\beta} \delta g^{\alpha \beta} \) in order to preserve the normalisation \( g_{\mu \nu} u^{\mu} u^{\nu} = -1 \). Using \( \gamma_{\mu \nu} \) as defined in (B.13), this can be written as
\[ 4\pi T_{\mu \nu} = \gamma_{\mu \nu} \tilde{F}^\alpha \beta F_{\nu \beta} - \frac{1}{4} \tilde{F}^\alpha \beta F_{\alpha \beta} g_{\mu \nu} + (n^2 - 1) \gamma_{\rho \sigma} \tilde{F}^\alpha \beta F_{\sigma \beta} u^\rho u^\sigma u^\mu u^\nu, \]  
\[ (D.3) \]
cf. [29]. Using the decomposition of \( F \) and \( \tilde{F} \) from (B.2) and (B.3), and letting \( g_{\mu \nu} = g_{\mu \nu} + u_{\mu} u_{\nu} \) be the “spatial metric” (which can be used to raise and lower indices of spatial vectors), this can be written as
\[ 4\pi T_{\mu \nu} = \frac{1}{2} (E.D + B.H)(u_{\mu} u_{\nu} + g_{\mu \nu}) + 2u_{(\mu} \epsilon_{\nu)}^{\rho \sigma} E_{\rho} H_{\sigma} - D_{\mu} E_{\nu} - B_{\mu} H_{\nu}. \]  
\[ (D.4) \]
This tensor is evidently symmetric and trace-free with respect to the spacetime metric. In terms of the complex field \( Z \) it may be expressed as
\[
8\pi\mu_2 T_{\mu\nu} = Z_\rho Z^\rho (u_\mu u_\nu + g_{\mu\nu}) + 2jn^{-1}u_\mu \epsilon_{\mu\nu}^{\rho\sigma} Z_\rho Z^\sigma - Z_\mu Z^*_{\nu} - Z^*_{\mu} Z_\nu ,
\] (D.5)
so that the energy density is seen to be given by
\[
T_{00} = \frac{1}{8\pi} (E.D + B.H) = \frac{1}{8\pi\mu_2} Z^* Z .
\] (D.6)

E Compatibility of Reflection with Constraints

Consider the reflection conditions
\[
\nu \cdot (\bar{Z} - Z^*) = 0 , \quad \nu \times (\bar{Z} + Z^*) = 0 ,
\] (E.1)
where \( Z \) is the incident field, \( \bar{Z} \) is the reflected field, and \( \nu \) is the normal to the reflecting surface. This can be covariantly rewritten using the complex tensor \( F \) defined in (B.5):
\[
(\bar{F} - F^{\ast})_{\alpha\beta}^\nu \nu_\beta = 0 .
\] (E.2)

Let us verify that this prescription is consistent in the following sense: if the incident field \( Z \) satisfies the constraint equation which arises from Maxwell’s equations, then so does the reflected field \( \bar{Z} \). Recall that the constraint equation is obtained from the time-evolution equation by contraction with the conormal \( \nu_i \). Note that since \( \nu_i \) is proportional to \( m_i \), we may use the latter instead, which has the advantage that all its derivatives vanish. By assumption, the incident field \( Z \) satisfies the (conjugate) constraint equation
\[
n\partial_0 (m_i Z^*_{\nu i}) - j\epsilon^{ijk} \partial_j (m_i Z^*_{k}) = 0 .
\] (E.3)
Using \( m_i Z^*_{\nu i} = m_i \bar{Z}^i \) and \( \epsilon^{ijk} m_i Z^*_{k} = -\epsilon^{ijk} m_i \bar{Z}_k \), we obtain
\[
n\partial_0 (m_i \bar{Z}^i) + j\epsilon^{ijk} \partial_j (m_i \bar{Z}_k) = 0 ,
\] (E.4)
so the reflected field indeed satisfies the constraint equation. The remaining projections of Maxwell’s equations then determine the normal derivatives of the various components of \( Z \), thereby encoding the law of reflection.

In fact, we can formulate a covariant argument which shows that the requirement that partial derivatives of \( m \) vanish is inessential. Since \( \nu \) is hypersurface-orthogonal, it can be locally written as \( \nu = g df \), where \( f \) and \( g \) are smooth functions. By linearity, we may equally formulate the reflection condition with the exact form \( df \) instead of \( \nu \):
\[
(\bar{F} - F^{\ast})_{\alpha\beta}^\nu \nu_\beta = 0 .
\] (E.5)
Taking the divergence, we obtain
\[
\bar{F}_{\alpha\beta} f_{\beta\gamma} - F^{\ast\alpha\beta} f_{\beta\gamma} = 0 .
\] (E.6)
Since the connection used is torsion-free, \( f_{\beta\gamma} \) is symmetric, so the last term vanishes due to the anti-symmetry of the field tensors. Thus, \( \bar{F} \) satisfies the constraint equation \( \bar{F}_{\alpha\beta} \nu_\beta \) if and only if the incident field \( F \) does. Note that the presence of sources on the mirror surface has no influence on this argument since the current four-vector is tangent to the mirror surface and thus annihilated by the normal one-form \( \nu \).
F  Polarization Reflection at a Beam Splitter

We do not attempt to model the precise details of beam splitters in curved spacetime, and use a simplified model which relates the electromagnetic fields at the various surfaces of the beam splitter, pointing towards the laser, detector, and along the two interferometer arms, see Figure 1.

Denote by $\Sigma_t$ and $\Sigma_l$ the surfaces facing the mirrors, as in Figure 1. We describe these surfaces by $\Sigma_t = \{m_t(x = 0)\}$ and $\Sigma_l = \{m_l(x = 0)\}$, where

$$m_t.m_l = \delta^{ij}m_{ti}m_{lj} = 0, \quad (F.1)$$

so that the unperturbed normals are orthogonal in the background metric. The normals in the perturbed metric are then

$$\nu_{ti} = m_{ti} \left( 1 + \frac{1}{2} \varepsilon h(m_t, m_t) \right), \quad \nu_{ti}^* = m_{ti} \left( 1 + \frac{1}{2} \varepsilon h(m_t, m_t) \right) - \varepsilon h^{ij}m_{ij}, \quad (F.2)$$

$$\nu_{li} = m_{li} \left( 1 + \frac{1}{2} \varepsilon h(m_l, m_l) \right), \quad \nu_{li}^* = m_{li} \left( 1 + \frac{1}{2} \varepsilon h(m_l, m_l) \right) - \varepsilon h^{ij}m_{ij}, \quad (F.3)$$

where indices of $m_t$ and $m_l$ are raised with the background metric $\delta_{ij}$. The corresponding reflection operator which interchanges $\nu_i$ and $\nu_{li}$ while leaving their orthogonal complement unaltered is

$$R^e_j = \delta_j^i - [1 - g(\nu_i, \nu_i)]^{-1}(\nu_i - \nu_i^*)(\nu_i - \nu_i)_j, \quad (F.4)$$

By virtue of (F.1), one has $g(\nu_i, \nu_i) = -\varepsilon h(m_t, m_t) + O(\varepsilon^2)$, and thus

$$R^e_j = \delta_j^i - [1 - \varepsilon h(m_t, m_t)](\nu_i - \nu_i^*)(\nu_i - \nu_i)_j + O(\varepsilon^2). \quad (F.5)$$

We wish to compute $R^e_j \hat{Z}^j$, where

$$\hat{Z}^j = a\hat{\zeta}^j - \frac{1}{2} \varepsilon h(\hat{\zeta}, \hat{\zeta})\hat{\zeta}^j, \quad (F.6)$$

with

$$a = 1 + e\hat{\alpha} - \frac{1}{2} \varepsilon h(\hat{\zeta}^*, \hat{\zeta}). \quad (F.7)$$

Defining the unperturbed reflected polarization

$$\eta_i = R^{(0)}_j \hat{\zeta}_j = \zeta_i - (m_t - m_l)_j(\eta_l, \zeta)_j, \quad (F.8)$$

one can expand $R^e_j \hat{Z}^j$ in the basis $\eta$, $\eta^*$, $m_l$. Since $\hat{Z}$ is orthogonal to $m_l$, the reflected vector is orthogonal to $m_l$, so that one can write

$$R^e_j \hat{Z}^j = b\eta^j + c\eta^* \hat{\zeta}^i, \quad (F.9)$$

for some constants $b$, $c$ to be determined. Since $\hat{Z}$ satisfies $g(\hat{Z}, \hat{Z}) = 0$, the reflected vector also has zero norm. As $b = 1 + O(\varepsilon)$ the factor $c$ is determined by

$$h(\eta, \eta) + 2c = 0, \quad (F.10)$$

so that it remains to determine $b$. For this, one may use the expansion (F.9) to find

$$b = \eta^* R^e_j \hat{Z}^j = \zeta^* R^e_j \hat{Z}^j - (m_l, \zeta^*) (m_l - m_t)_j R^e_j \hat{Z}^j, \quad (F.11)$$
where we consider the second term first. There, one has \( m_{ii} R_{j} \hat{Z}^j \propto \nu_{ii} R_{j} \hat{Z}^j = \nu_{ii} \hat{Z}^i = 0 \). For the other term, \( m_{ii} R_{j} \hat{Z}^j \), one may use the explicit form of \( \nu_i, \nu_i \), and \( z \) to obtain
\[
b = \zeta^i R^i_j \hat{Z}^j + a(m_{ii}, \zeta^i)(m_{ii}, \zeta)[1 + \frac{1}{2} \epsilon h(m_{ii}, m_{ii}) - \frac{1}{2} \epsilon h(m_{ii}, m_{ii}) - \frac{1}{2} \epsilon h(\zeta, \zeta)(m_{ii}, \zeta^i)^2]. \quad \text{(F.12)}
\]
Contracting (7.1.10) with \( m_{ii} m_{ij} \) yields \((m_{ii}, \zeta^i)(m_{ii}, \zeta) = \frac{1}{2}\), and using \( a = 1 + O(\epsilon) \), this simplifies to
\[
b = \zeta^i R^i_j \hat{Z}^j + \frac{1}{2} a + \frac{1}{4} \epsilon h(m_{ii}, m_{ii}) - \frac{1}{2} \epsilon h(m_{ii}, m_{ii}) - \frac{1}{2} \epsilon h(\zeta, \zeta)(m_{ii}, \zeta^i)^2. \quad \text{(F.13)}
\]
Consider, now, the first term. Using the explicit expression for the reflection operator, as well as the intermediate results
\[
\zeta^i \hat{Z}^i = a, \quad \zeta^i (\nu_i - \nu_i)^t = [1 + \frac{1}{2} \epsilon h(m_{ii}, m_{ii})](m_{ii}, \zeta^i) - \epsilon h(\zeta, m_{ii} - m_{ii}), \quad \text{(F.14)}
\]
\[
(\nu_i - \nu_i)^t \hat{Z}^i = [1 + \frac{1}{2} \epsilon h(m_{ii}, m_{ii})][a(m_{ii}, \zeta) - \frac{1}{2} \epsilon h(\zeta, \zeta)(m_{ii}, \zeta^i)], \quad \text{(F.15)}
\]
and using again \((m_{ii}, \zeta^i)(m_{ii}, \zeta) = \frac{1}{2}\) as well as \( a = 1 + O(\epsilon) \), one finds
\[
\zeta^i R^i_j \hat{Z}^j = \frac{1}{2} a + \frac{1}{4} \epsilon h(m_{ii}, m_{ii}) - \frac{1}{2} \epsilon h(m_{ii}, m_{ii}) + \epsilon h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta) + \frac{1}{2} \epsilon h(\zeta, \zeta)(m_{ii}, \zeta^i)^2. \quad \text{(F.16)}
\]
Plugging this into (F.13) then yields
\[
b = a + \epsilon h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta) - \frac{1}{4} \epsilon h(m_{ii} - m_{ii}, m_{ii} - m_{ii}), \quad \text{(F.17)}
\]
and inserting the explicit form of \( a \) from (F.7), one arrives at
\[
b = 1 + \epsilon j \alpha - \frac{1}{2} \epsilon h(\zeta^i, \zeta) + \epsilon h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta) - \frac{1}{4} \epsilon h(m_{ii} - m_{ii}, m_{ii} - m_{ii}). \quad \text{(F.18)}
\]
Decomposing \( b \) analogously as \( a \) was decomposed in (F.7) in the form
\[
b = 1 + \epsilon j \beta - \frac{1}{2} \epsilon h(\eta^i, \eta) \]
\[
= 1 + \epsilon j \beta - \frac{1}{2} \epsilon h(\zeta^i, \zeta) + \frac{1}{2} \epsilon h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta) + \frac{1}{2} \epsilon h(\zeta, m_{ii} - m_{ii} - m_{ii} - m_{ii}), \quad \text{(F.19)}
\]
one arrives at
\[
j \beta = j \alpha + \frac{1}{2} \epsilon h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta) - \frac{1}{2} \epsilon h(\zeta, m_{ii} - m_{ii})(m_{ii}, \zeta^i), \quad \text{(F.20)}
\]
or equivalently
\[
\beta = \alpha + \Im j [h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta)], \quad \text{(F.21)}
\]
where \( \Im \) denotes the \( j \)-imaginary part.
Summarizing, given a polarization vector of the form
\[
\hat{Z}^i = \zeta^i [1 + j \epsilon a - \frac{1}{2} \epsilon h(\zeta^i, \zeta)] - \frac{1}{2} \epsilon \zeta^i h(\zeta, \zeta) + O(\epsilon^2), \quad \text{(F.22)}
\]
the vector obtained by applying the reflection operator (F.4) is
\[
R_{j} \hat{Z}^j = \eta^j \left( 1 + j \epsilon a + j \epsilon \Im j [h(\zeta^i, m_{ii} - m_{ii})(m_{ii}, \zeta)] - \frac{1}{2} \epsilon h(\eta^i, \eta) \right) - \frac{1}{2} \epsilon \eta^i h(\eta, \eta) + O(\epsilon^2), \quad \text{(F.23)}
\]
where \( \eta \) is the unperturbed reflected vector, as defined in (F.8).
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