A NEW APPROACH FOR THE STRONG UNIQUE CONTINUATION OF ELECTROMAGNETIC SCHröDINGER OPERATOR WITH COMPLEX-VALUED COEFFICIENT

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Abstract. This paper mainly addresses the strong unique continuation property for the electromagnetic Schrödinger operator with complex-valued coefficients. Appropriate multipliers with physical backgrounds have been introduced to prove a priori estimates. Moreover, its application in an exact controllability problem has been shown, in which case, the boundary value determines the interior value completely.

Résumé. Dans cet article, on considère essentiellement la propriété fortement unique pour l’opérateur électromagnétique de Schrödinger avec les coefficients de complexe. La méthode de multiplicateur a été introduite pour démontrer les estimations à priori. En plus, cette théorie est appliquée dans le problème de contrôlabilité exacte, où la valeur sur la frontière déterminera la valeur intérieure complètement.

1. Introduction

Nowadays, quantum studies, especially multiphoton entanglement and interferometry, are attracting many scientists’ attention, either theoretically or practically. A few world-famous high-tech companies, such as Apple, Microsoft, etc. are developing new generation of high-performance computers based on the quantum mechanics phenomena.

In our paper, we discuss an important complex-valued operator in this research field. Let \( \mathbf{A}(x) \) be the vector potential of the magnetic field \( \mathbf{B} \), that is, \( \mathbf{B} = \nabla \times \mathbf{A} \). Clearly, \( \nabla \cdot \mathbf{B} = \text{div} \, \text{rot} \, \mathbf{A} = 0 \). From one of Maxwell’s equations (\( \mu \) is magnetic permeability) \( \nabla \times \mathbf{E} = -\mu \partial \mathbf{B}/\partial t = 0 \), we deduce that \( \mathbf{E} = -\nabla \phi \), where the scalar \( \phi \) represents the electric potential. We choose an appropriate Lagrangian for the non-relativistic charged particle in the electromagnetic field (\( q \) is the electric charge of the particle, and \( \mathbf{v} \) is its velocity, \( m \) is mass), \( \mathcal{L} = m \mathbf{v}^2/2 - q\phi + q\mathbf{v} \cdot \mathbf{A} \). Particularly, the canonical
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momentum is specified by the vector \( \mathbf{p} = \nabla v \mathcal{L} = m \mathbf{v} + q \mathbf{A}. \) Next, we define the classical Hamiltonian by Legendre transform, \( H \triangleq \mathbf{p} \cdot \mathbf{v} - \mathcal{L} = (\mathbf{p} - q \mathbf{A})^2/(2m) + q \phi. \) In quantum mechanics, when \( \mathbf{p} \) is replaced by \( -i\hbar \nabla_i; \) \( \hbar \) is the Planck constant, we have the following operator

\[
P \triangleq (i\hbar \nabla + q \mathbf{A})^2/(2m) + q \phi : \mathcal{H} \to \mathcal{H}^*,
\]

where \( \mathcal{H} \) and \( \mathcal{H}^* \) are corresponding function spaces. Lots of literature is devoted to the research of this kind of operator\([6, 7, 11, 18]\).

Let \( \Omega \subset \mathbb{R}^N \) be an open, connected and bounded domain. From the structure of operator \( P \), we define the corresponding simplified operators

\[
\mathcal{H}^A \triangleq i
\nabla + \mathbf{A}(x) \quad : \quad L^2(\Omega) \to (L^2(\Omega))^N,
\]

\[
\mathcal{H}_2^A \triangleq (i\nabla + \mathbf{A}(x))^2 \quad : \quad L^2(\Omega) \to L^2(\Omega),
\]

where \( \mathbf{A} \in C^1(\Omega) \) is a real-valued potential vector. The corresponding derivative of the magnetic potential \( \mathbf{A} \) is as follows,

\[
D \mathbf{A} = \begin{pmatrix}
nabla_1 a_1 & \nabla_1 a_2 & \cdots & \nabla_1 a_N \\
nabla_2 a_1 & \nabla_2 a_2 & \cdots & \nabla_2 a_N \\
\vdots & \vdots & \cdots & \vdots \\
nabla_N a_1 & \nabla_N a_2 & \cdots & \nabla_N a_N
\end{pmatrix},
\]

where

\[
\nabla_i a_j \triangleq \partial a_j/\partial x_i, \quad i, j = 1, \cdots, N.
\]

In addition, one defines the following \( N \times N \) anti-symmetric matrix \( \Xi_A \) given by

\[
\Xi_A \triangleq (D \mathbf{A})^T - D \mathbf{A}^T = \begin{pmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1N} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2N} \\
\vdots & \vdots & \cdots & \vdots \\
\xi_{N1} & \xi_{N2} & \cdots & \xi_{NN}
\end{pmatrix}
\]

with

\[
\xi_{jk} \triangleq \nabla_j a_k - \nabla_k a_j, \quad k, j = 1, \cdots, N.
\]

In quantum mechanics, \( \Xi_A \equiv 0 \) stands for the case without magnetic field, i.e. \( \mathbf{B} = \text{rot} \mathbf{A} = 0. \)

Once the magnetic field exists, then \( \Xi_A \neq 0. \) Consequently, \( \Xi_A \) serves as a test matrix for the magnetic field. Interested readers can refer to \([5, 14, 15]\) for more details concerned with the vector operator \( \mathcal{H}^A \) and self-adjoint operator \( \mathcal{H}_2^A \). In such a manner, (1) is simplified as

\[
\mathcal{H}_2^A - \phi(x) \quad : \quad L^2(\Omega) \to L^2(\Omega),
\]

where the complex-valued function \( \phi \in L^\infty(\Omega). \) In this paper, we focus on the strong unique continuation property(SUCP) for the electromagnetic Schrödinger operator (4). In the following, we introduce a few important definitions.
Definition 1.1. A function \( u \in L^2_{\text{loc}}(\Omega) \) is said to vanish of infinite order at \( x_0 \in \Omega \) if for any sufficiently small \( R > 0 \), one has
\[
\int_{|x-x_0|<R} |u|^2 \, dx = O(R^M), \quad \text{for every } M \in \mathbb{N}^+.
\]

Definition 1.2. We say that the operator (4) has SUCP if every solution \( \omega \) of the equation
\[
\mathcal{H}^2_A \omega = \phi \omega,
\]
which vanishes of infinite order at \( x_0 \) is identically zero in a neighborhood of \( x_0 \).

So far, the strong unique continuation problem for second order elliptic operators is well-understood. In the case of \( \Omega = \mathbb{R}^2 \), Carleman proved the SUCP of the elliptic equation with bounded coefficients and \( V \in L^\infty_{\text{loc}}(\mathbb{R}^2) \)
\[
- \Delta u = W \cdot \nabla u + Vu
\]
by introducing a weighted \( L^2 \)-estimate, the so-called Carleman estimate [4]. For the space dimension \( N \geq 3 \) with bounded coefficients, N. Aronszajn, A. Krzywicki and J. Szarski proved the SUCP by means of Carleman type inequalities, namely, observability inequalities. Afterwards, D. Jerison, C. E. Kenig, C. D. Sogge treated the equation (6) with singular potentials \( V \in L^{N/2}_{\text{loc}}(\mathbb{R}^N) \) and \( W \in L^\infty(\mathbb{R}^N) \), \( N \geq 3 \), by the approach of \( L^p - L^q \) Carleman estimate involving sharp exponents [10, 11, 17]. Afterwards, N. Garofalo and F. H. Lin gave a new proof for the SUCP of the elliptic operator \(-\Delta u = Vu\) with bounded potential by applying a variational method in [9].

There is a large body of work on SUCP for (6) with real-valued coefficients. In this paper, we investigate the complex-valued case. As a matter of fact, the operator \( \mathcal{H}^2_A \) can be decomposed into
\[
\mathcal{H}^2_A \omega = -\Delta \omega + iA \cdot \nabla \omega + i\nabla \cdot (A \omega) + AA^T \omega.
\]
In [12, 13], K. Kurata proved the SUCP for (4) with \( AA^T \in \mathcal{K}^{\text{loc}}_{N}(\Omega) \), where \( \mathcal{K}^{\text{loc}}_{N}(\Omega) \) denotes the Kato class. When the potential \( A \in (L^\infty(\Omega))^N \), in effect, it does not belong to the Kato class. As a result, we can not deduce corresponding results directly from K. Kurata’s work. In this manuscript, we intend to provide a new approach of SUCP for (4) with complex-valued coefficients by developing new multipliers. At the moment one is ready to state the main results.

Theorem 1.3. For \( N \geq 2 \), let the complex-valued \( \omega \in H^2(\mathbb{B}_1) \) be a solution of the problem
\[
-\Delta \omega + iA \cdot \nabla \omega + i\nabla \cdot (A \omega) + AA^T \omega = \phi(x) \omega \quad \text{in } \mathbb{B}_1,
\]
where \( \mathbb{B}_1 \) is a unit ball \( \mathbb{B}_1 \subset \overline{\Omega} \), \( A \in C^1(\overline{\Omega}) \) is a real-valued potential vector and the complex-valued function \( \phi \in L^\infty(\mathbb{R}^N) \). If \( \omega \) vanishes of infinite order at \( x_0 \in \mathbb{B}_1 \), then \( \omega \equiv 0 \) in \( \mathbb{B}_1 \).

By virtue of Theorem 1.3, one is able to prove the following statement for a mixed boundary value problem which is of great importance in the discussion of exact controllability through boundary control [13].
Corollary 1.4. Assume that $\Omega$ is a bounded, open and connected domain in $\mathbb{R}^N$ with the boundary $\Gamma \in C^2$, $A \in C^1(\overline{\Omega})$ is a real-valued potential vector and the complex-valued function $\phi \in L^\infty(\mathbb{R}^N)$. Let $\omega \in H^2(\Omega)$ be the solution of the mixed boundary problem

$$-\Delta \omega + iA \cdot \nabla \omega + i\nabla \cdot (A\omega) + AA^T \omega = \phi(x)\omega \quad \text{in} \ \Omega,$$

$$\omega = \partial \omega / \partial \nu = 0 \quad \text{on} \ \Gamma.$$

Then $\omega$ is identically 0 in $\Omega$.

Remark 1.5. Theorem 1.3 demonstrates, the asymptotic behavior of the solution $\omega$ at an interior point $x_0$ determines the interior value of $\omega$ in $B_1$. In contrast with Theorem 1.3, Corollary 1.4 indicates, the behavior of solution $\omega$ on the boundary determines the interior value of $\omega$ in $\Omega$.

The rest of the paper is organized as follows. First and foremost, in Section 2, we introduce some useful quantities and their particular properties. Next, we give an important comparison lemma and a frequency function. By carefully estimating the derivative of the frequency function, we reach the conclusion in the final analysis. In Section 3, as an important application in exact controllability, we prove Corollary 1.4 in detail.

2. Proof of the main theorem: A new multiplier method

First, we introduce several quantities which will serve as useful tools for our purposes. For every $r \in (0, 1)$, we define the following two quantities

(9) $\Phi(r) \Delta \int_{\partial B_r} |\omega|^2 dS_x,$

where $B_r$ is centered at the origin with radius $r$, $\partial B_r$ denotes its sphere, $dS_x$ stands for the $(N-1)$-dimensional Hausdorff measure on the sphere $\partial B_r$.

(10) $\Psi(r) \Delta \int_{B_r} (|H_A \omega|^2 - \phi^R|\omega|^2) dV_x,$

where $\phi^R$ denotes the real part of $\phi$. Actually, we have

Lemma 2.1. By virtue of divergence theorem, the following identity holds,

(11) $- \text{Re} \int_{\partial B_r} \left( \nabla|\omega|^2 - iA|\omega|^2 \right) \cdot x/rdS_x = \int_{B_r} \left( -2|H_A \omega|^2 + 2\phi^R|\omega|^2 \right) dV_x.$

Proof. On the one hand,

$$\int_{B_r} \frac{H_A^2}{4} |\omega|^2 dV_x = -\int_{\partial B_r} \left( \nabla|\omega|^2 - iA \cdot |\omega|^2 \right) \cdot x/rdS_x + \int_{B_r} A \cdot H_A |\omega|^2 dV_x$$

(12)

$$= -\int_{\partial B_r} \left( \nabla|\omega|^2 - iA|\omega|^2 \right) \cdot x/rdS_x$$

$$+ \int_{B_r} A \cdot (i\omega \nabla \omega + i\omega \nabla \omega + A|\omega|^2) dV_x.$$
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On the other hand,

\[ \int_{B_r} \mathcal{H}_A^2 |\omega|^2 dV_x = \int_{B_r} \left( -\overline{\omega} \Delta \omega - \omega \Delta \overline{\omega} - 2|\nabla \omega|^2 + i \nabla \cdot A |\omega|^2 \right) dV_x \]

(13)

\[ + \int_{B_r} \left( i \omega A \cdot \nabla \omega + i \omega A \cdot \nabla \overline{\omega} + AA^T |\omega|^2 \right) dV_x. \]

Since

\[ \nabla \cdot A |\omega|^2 = \overline{\omega} \nabla \cdot A \omega + \omega A \cdot \nabla \overline{\omega}, \]

then by combining (12) and (13), we have

\[-\text{Re} \int_{\partial B_r} \left( \nabla |\omega|^2 - i A |\omega|^2 \right) \cdot x/rdS_x \]

\[ = \int_{B_r} \left( -\overline{\omega} \Delta \omega - \omega \Delta \overline{\omega} - 2|\nabla \omega|^2 \right) dV_x \]

\[ = \int_{B_r} \left( -2|\nabla \omega|^2 + 2i \omega A \cdot \nabla \overline{\omega} - 2i \overline{\omega} A \cdot \nabla \omega - 2AA^T |\omega|^2 \right) dV_x \]

\[ + \int_{B_r} \left( -i \overline{\omega} \Delta \omega + i \overline{\omega} A \cdot \nabla \omega + i \nabla \overline{\omega} \cdot A \omega + AA^T |\omega|^2 \right) dV_x \]

\[ + \int_{B_r} \left( -i \omega \Delta \overline{\omega} - i \omega A \cdot \nabla \overline{\omega} - i \omega \nabla \cdot A \omega + AA^T |\omega|^2 \right) dV_x \]

\[ = \int_{B_r} \left( -2|\mathcal{H}_A \omega|^2 + 2\phi R |\omega|^2 \right) dV_x. \]

□

Next we calculate the derivatives of \( \Phi(r) \) and \( \Psi(r) \) with respect to \( r \).

**Lemma 2.2.** The derivatives of \( \Phi(r) \) and \( \Psi(r) \) with respect to \( r \) are presented as follows,

(14) \[ \Phi'(r) = (N - 1)\Phi(r)/r + 2\Psi(r). \]

(15) \[ \Psi'(r) = (N - 2)\Psi(r)/r + (N - 2)/r \int_{B_r} \phi R |\omega|^2 dV_x + 2/r \text{Re} \int_{B_r} A \omega \cdot \mathcal{H}_A \omega dV_x \]

\[ + 2/r \text{Re} \int_{B_r} (x \cdot \nabla \omega) \cdot \overline{\phi} \omega dV_x + 2/r \text{Re} \int_{B_r} \omega x (DA)^T \mathcal{H}_A \omega^T dV_x \]

\[ + 2 \int_{\partial B_r} \nu \cdot (i \nabla \omega + A \omega)^2 dS_x - 2 \int_{\partial B_r} (A \omega \cdot \nu) (\mathcal{H}_A \omega \cdot \nu) dS_x \]

\[ - \int_{\partial B_r} \phi R |\omega|^2 dS_x. \]

**Remark 2.3.** (14) shows that \( \Phi(r) \) and \( \Psi(r) \) are closed related with each other. This relation is very important for our discussion.
Proof. First we consider the derivative of $\Phi(r)$ with respect to $r$. Indeed, we have

$$
\Phi'(r) = \int_{\partial B_{1}}(|\omega(ry)|^2 r^{N-1})' dS_y \\
= \int_{\partial B_{1}} \left( \nabla\omega \cdot \nabla \omega + \omega (\nabla \omega \cdot y) r^{N-1} + |\omega|^2 (N - 1) r^{N-2} \right) dS_y \\
= \int_{\partial B_{r}} \left( \nabla\omega \cdot x/r \omega + \omega (\nabla \omega \cdot x/r) \right) dS_x + (N - 1)/r \int_{\partial B_{r}} |\omega|^2 dS_x \\
= (N - 1)/r \Phi(r) + \Re \int_{\partial B_{r}} \left( \nabla |\omega|^2 - i A |\omega|^2 \right) \cdot x/rdS_x \\
= (N - 1)\Phi(r)/r + 2\Psi(r).
$$

As for the derivative of $\Psi(r)$ with respect to $r$, via the divergence theorem, we have

$$
\Psi'(r) = \int_{\partial B_{r}} |A \omega|^2 dS_x - \int_{\partial B_{r}} \phi^R |\omega|^2 dS_x \\
= 1/r \int_{\partial B_{r}} |A \omega|^2 x \cdot rdS_x - \int_{\partial B_{r}} \phi^R |\omega|^2 dS_x \\
= 1/r \int_{B_{r}} \text{div}(|A \omega|^2 x) dV_x - \int_{\partial B_{r}} \phi^R |\omega|^2 dS_x \\
= N/r \int_{B_{r}} |A \omega|^2 dV_x + 1/r \int_{\partial B_{r}} x \cdot \nabla |A \omega|^2 dV_x - \int_{\partial B_{r}} \phi^R |\omega|^2 dS_x. \\
$$

(16)

As for the derivative of $\Psi(r)$ with respect to $r$, via the divergence theorem, we have
Now we treat the term (I) carefully.

\[
(I) = \sum_{j,k} \frac{1}{r} \int_{B_r} x_j \nabla_j \left( (i \nabla_k + a_k \omega)(i \nabla_k + a_k \omega) \right) dV_x \\
= \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} x_j \left( \nabla_j (i \nabla_k + a_k \omega)(i \nabla_k + a_k \omega) \right) dV_x \\
= \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} i x_j \nabla_j \nabla_k (i \nabla_k + a_k \omega) dV_x + \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} x_j \nabla_j (a_k \omega)(i \nabla_k + a_k \omega) dV_x \\
= \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} i x_j \nabla_j \nabla_k (i \nabla_k + a_k \omega) dV_x - \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} i \nabla_k x_j \nabla_j (i \nabla_k + a_k \omega) dV_x \\
- \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} i x_j \nabla_j \nabla_k (i \nabla_k + a_k \omega) dV_x + \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} x_j \nabla_j a_k (i \nabla_k + a_k \omega) dV_x \\
+ \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} a_k x_j \nabla_j (i \nabla_k + a_k \omega) dV_x \\
= \sum_{j,k} \frac{2}{r} \text{Re} \int_{\partial B_r} x_j (i \nabla_j + a_j \omega)(i \nabla_k + a_k \omega) \nu_k dS_x \\
- \sum_{j,k} \frac{2}{r} \text{Re} \int_{\partial B_r} x_j a_j (i \nabla_k + a_k \omega) \nu_k dS_x - \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} i \nabla_k x_j \nabla_j (i \nabla_k + a_k \omega) dV_x \\
- \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} i x_j \nabla_j \nabla_k (i \nabla_k + a_k \omega) dV_x + \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} x_j \nabla_j a_k (i \nabla_k + a_k \omega) dV_x \\
+ \sum_{j,k} \frac{2}{r} \text{Re} \int_{B_r} a_k x_j \nabla_j (i \nabla_k + a_k \omega) dV_x \\
= 2 \int_{\partial B_r} |\nu \cdot \mathcal{H}_A \omega|^2 dS_x - 2 \text{Re} \int_{\partial B_r} (A \omega \cdot \nu)(\overline{\mathcal{H}_A \omega} \cdot \nu) dS_x \\
- 2/r \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x + 2/r \int_{B_r} A \omega \cdot \overline{\mathcal{H}_A \omega} dV_x + 2/r \int_{B_r} \nu \cdot \partial_x (DA)^T \overline{\mathcal{H}_A \omega} dV_x \\
+ 2/r \int_{B_r} (x \cdot \nabla) \cdot \mathcal{H}_A \omega dV_x \\
= 2 \int_{\partial B_r} |\nu \cdot \mathcal{H}_A \omega|^2 dS_x - 2 \text{Re} \int_{\partial B_r} (A \omega \cdot \nu)(\overline{\mathcal{H}_A \omega} \cdot \nu) dS_x \\
- 2/r \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x + 2/r \int_{B_r} A \omega \cdot \overline{\mathcal{H}_A \omega} dV_x + 2/r \int_{B_r} (x \cdot \nabla \omega) \cdot \overline{\mathcal{H}_A \omega} dV_x \\
+ 2/r \int_{B_r} (x \cdot \nu \cdot A \omega)^T \overline{\mathcal{H}_A \omega} dV_x.
Finally, keeping in mind the definition of $\Psi(r)$, we reach the conclusion immediately. \hfill \Box

Next we show an important comparison lemma.

**Lemma 2.4.** There exists an $r_0 \in (0, 1)$ such that for every $r \in (0, r_0)$, we have

\begin{equation}
\int_{B_r} |\omega|^2 dV_x \leq r \int_{\partial B_r} |\omega|^2 dS_x.
\end{equation}

**Proof.** On the one hand,

\[
\int_{B_r} \mathcal{H}_A^2 |\omega|^2 \cdot (r^2 - |x|^2) dV_x
\]

\[
= \int_{B_r} |\omega|^2 \cdot \mathcal{H}_A^2 (r^2 - |x|^2) dV_x + \int_{\partial B_r} |\omega|^2 \cdot \partial (r^2 - |x|^2) / \partial \nu |\mathcal{H}_A| dS_x
\]

\[
= \int_{B_r} |\omega|^2 \left( 2N + 2iA \cdot x - i \nabla \cdot A (r^2 - |x|^2) + AA^T (r^2 - |x|^2) \right) dV_x
\]

\[-2r \int_{\partial B_r} |\omega|^2 dS_x.
\]

On the other hand,

\[
\int_{B_r} \mathcal{H}_A^2 |\omega|^2 \cdot (r^2 - |x|^2) dV_x
\]

\[
= \int_{B_r} \left( -\mathcal{H}_A \Delta \omega - \omega \Delta \mathcal{H}_A - 2 |\nabla \omega|^2 \right) (r^2 - |x|^2) dV_x + \int_{B_r} AA^T |\omega|^2 (r^2 - |x|^2) dV_x
\]

\[
+ \int_{B_r} \left( i \nabla \cdot A |\omega|^2 + iA \cdot \nabla \omega + i\omega A \cdot \nabla \omega \right) (r^2 - |x|^2) dV_x
\]

\[
= \int_{B_r} \left( -2 |\mathcal{H}_A \omega|^2 + 2 |\mathcal{H}_A \omega|^2 (r^2 - |x|^2) \right) dV_x + \int_{B_r} AA^T |\omega|^2 (r^2 - |x|^2) dV_x
\]

\[
+ \int_{B_r} \left( i \nabla \cdot A |\omega|^2 + iA \cdot \nabla \omega + i\omega A \cdot \nabla \omega \right) (r^2 - |x|^2) dV_x.
\]

As a result,

\[
\int_{B_r} \left( 2N |\omega|^2 + 2 |\mathcal{H}_A \omega|^2 (r^2 - |x|^2) \right) dV_x = 2r \int_{\partial B_r} |\omega|^2 dS_x.
\]

When $\|\phi^R\|_{L^\infty} > 0$, then we choose $r_0 \in (0, 1/2)$ such that

\[
r_0^2 \leq (N - 1)/\|\phi^R\|_{L^\infty}.
\]

It follows immediately that

\[
\int_{B_r} |\omega|^2 dV_x \leq \int_{B_r} \left( N |\omega|^2 - \phi^R |\omega|^2 (r^2 - |x|^2) \right) dV_x \leq r \int_{\partial B_r} |\omega|^2 dS_x.
\]

When $\|\phi^R\|_{L^\infty} = 0$, then it is evident

\[
\int_{B_r} |\omega|^2 dV_x \leq r/N \int_{\partial B_r} |\omega|^2 dS_x.
\]
Assume that there exists a small $r_1 \in (0, 1)$ such that

\[(18) \quad \Phi(r) \neq 0 \quad \text{for} \quad \forall \quad r \in (0, r_1).\]

Define the frequency function

\[(19) \quad F(r) \triangleq r\Psi(r)/\Phi(r), \quad r \in (0, r_1).\]

Let $r^* \triangleq \min\{r_0, r_1\}$, and we set

\[(20) \quad \mathfrak{B}_{r^*} \triangleq \{r \in (0, r^*) : F(r) > 1\}.\]

With the above definitions, we have the following inequality for the frequency function.

**Lemma 2.5.** Under the assumptions (18)-(20), there exists a positive constant $\tau = \tau(N, \phi)$ which is independent of $r$ such that $F'(r)$ is estimated in a uniform fashion,

\[F'(r) \geq -F(r)\tau.\]

**Proof.** Actually, from (17)-(20), we have

\[
\int_{B_r} |\mathcal{H}_A\omega|^2 dV_x > (1/r^2 - \|\phi^R\|_{L^\infty}) \int_{B_r} |\omega|^2 dV_x.
\]

Indeed,

\[
\begin{align*}
\int_{B_r} |\mathcal{H}_A\omega|^2 dV_x &= \int_{B_r} (|\mathcal{H}_A\omega|^2 - \phi^R|\omega|^2) dV_x + \int_{B_r} \phi^R|\omega|^2 dV_x \\
&> 1/r \int_{\partial B_r} |\omega|^2 dS_x + \int_{B_r} \phi^R|\omega|^2 dV_x \\
&\geq 1/r^2 \int_{B_r} |\omega|^2 dV_x + \int_{B_r} \phi^R|\omega|^2 dV_x \\
&\geq (1/r^2 - \|\phi^R\|_{L^\infty}) \int_{B_r} |\omega|^2 dV_x.
\end{align*}
\]
This indicates the integral \( \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x \) is the dominating part in \( \Psi(r) \). By calculating \( F'(r) \) with respect to \( r \), we have the following identity,

\[
F'(r) = F(r) \left( \Psi'(r)/\Psi(r) + 1/r - \Phi'(r)/\Phi(r) \right)
\]

\[
= F(r) \left( 2 \int_{\partial B_r} \left| \nu \cdot (i \nabla \omega + A \omega) \right|^2 dS_x / \text{Re} \int_{\partial B_r} (x/r \cdot \nabla \omega) \, \omega dS_x \right)
\]

\[
- 2 \text{Re} \int_{\partial B_r} (x/r \cdot \nabla \omega) \overline{\omega} dS_x / \int_{\partial B_r} |\omega|^2 dS_x
\]

\[
+ F(r) \left\{ \frac{(N-2)}{r} \int_{B_r} \phi^R |\omega|^2 dV_x + 2/r \text{Re} \int_{B_r} A \omega \cdot \overline{\mathcal{H}_A \omega} dV_x \right\}
\]

\[
+ 2/r \text{Re} \int_{B_r} (x \cdot \nabla \omega) \cdot \overline{\phi \omega} dV_x + 2/r \text{Re} \int_{B_r} \omega x(DA)^T \overline{\mathcal{H}_A \omega} dV_x
\]

\[
- 2 \text{Re} \int_{\partial B_r} (A \omega \cdot \nu)(\overline{\mathcal{H}_A \omega} \cdot \nu) dS_x
\]

\[
- \int_{\partial B_r} \phi^R |\omega|^2 dS_x \right\} / \left\{ 1/2 \int_{\partial B_r} x/r \cdot \nabla |\omega|^2 dS_x \right\},
\]

\[
(VIII)
\]
We estimate each term respectively. For (II)-(II'), we apply Hölder’s inequality and obtain

\[
(II) - (II') \\
= 2 \int_{\partial B_r} |\nu \cdot (i \nabla \omega + A \omega)|^2 dS_x / \text{Re} \int_{\partial B_r} (x/r \cdot \nabla \omega) \overline{\omega} dS_x \\
- 2 \text{Re} \int_{\partial B_r} (x/r \cdot \nabla \omega) \overline{\omega} dS_x / |\omega|^2 dS_x \\
= 2 \int_{\partial B_r} |\nu \cdot (i \nabla + A \omega)|^2 dS_x / \text{Re} \int_{\partial B_r} (x/r \cdot (\nabla \omega - i A \omega)) \overline{\omega} dS_x \\
- 2 \text{Re} \int_{\partial B_r} (x/r \cdot (\nabla \omega - i A \omega)) \overline{\omega} dS_x / |\omega|^2 dS_x \\
\geq 2 \int_{\partial B_r} |\nu \cdot (i \nabla + A \omega)|^2 dS_x / \left( \sqrt{\int_{\partial B_r} |(x/r \cdot (\nabla \omega - i A \omega))|^2 dS_x} \sqrt{\int_{\partial B_r} |\omega|^2 dS_x} \right) \\
- 2 \sqrt{\int_{\partial B_r} |(x/r \cdot (\nabla \omega - i A \omega))|^2 dS_x} \sqrt{\int_{\partial B_r} |\omega|^2 dS_x} / \int_{\partial B_r} |\omega|^2 dS_x \\
\geq 0.
\]

In addition, we have

**Lemma 2.6.** There exists a constant $C^*(\phi)$ independent of $r$ such that

\[
\int_{\partial B_r} |\nu \cdot (i \nabla + A \omega)|^2 dS_x / \Psi(r) \leq C^*(\phi) / r.
\]

**Proof.** Indeed, by multiplying $H(x) \cdot H^T \omega$ to $H^T \omega = \phi \omega$ and integrating by parts, we have the following identity,

\[
-1/2 \int_{\partial B_r} \left| \partial \omega / \partial \nu_i H^T \right|^2 \left( H(x) \cdot \nu \right) dS_x \\
= 1/2 \int_{B_r} \left( \nabla \cdot H(x) \right) |H^T \omega|^2 dV_x - \text{Im} \int_{B_r} \phi \omega \cdot \left( H(x) \cdot H^T \omega \right) dV_x \\
- \text{Re} \int_{B_r} H^T \omega (D H)^T H^T \omega dV_x - \text{Re} \int_{B_r} \overline{\omega} \overline{\omega} \omega \omega \overline{H} dV_x.
\]

Since

\[
\int_{B_r} |H^T \omega|^2 dV_x \geq (1/r^2 - \| \phi \|^2_{L^\infty}) \int_{B_r} |\omega|^2 dV_x,
\]

by choosing

\[
H(x) \triangleq x/r,
\]
we have the following estimate,

\[
\frac{1}{2} \int_{\partial B_r} \left| \frac{\partial \omega}{\partial \nu} \right| dS_x \\
\leq \frac{N}{2r} \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x + \frac{1}{2} \| \phi \|_{L^\infty} \int_{B_r} (|\omega|^2 + |\mathcal{H}_A \omega|^2) dV_x \\
+ \frac{1}{r} \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x + \frac{1}{2} \max \| \Xi_A \|_{F} \int_{B_r} (|\omega|^2 + |\mathcal{H}_A \omega|^2) dV_x \\
= \left( \frac{(N+2)}{2r} + \frac{1}{2} \left( \| \phi \|_{L^\infty} + \max \| \Xi_A \|_{F} \right) \right) \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x \\
+ \frac{1}{2} \left( \| \phi \|_{L^\infty} + \max \| \Xi_A \|_{F} \right) \int_{B_r} |\omega|^2 dV_x \\
\leq \left( \alpha + \frac{\beta r^2}{1 - r^2 \| \phi^R \|_{L^\infty}} \right) \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x.
\]

where \( \| \Xi_A \|_{F} \) denotes the Frobenius norm of the test matrix \( \Xi_A \). Since

\[
\Psi(r) = \int_{B_r} (|\mathcal{H}_A \omega|^2 - \phi^R |\omega|^2) dV_x \geq \left( 1 - 2r^2 \| \phi^R \|_{L^\infty} \right) / \left( 1 - r^2 \| \phi^R \|_{L^\infty} \right) \int_{B_r} |\mathcal{H}_A \omega|^2 dV_x,
\]

therefore,

\[
\int_{\partial B_r} |\nu \cdot (i \nabla \omega + A \omega)|^2 dS_x / \Psi(r) \\
\leq \left( 1 - r^2 \| \phi^R \|_{L^\infty} \right) / \left( 1 - 2r^2 \| \phi^R \|_{L^\infty} \right) \left( 2\alpha + \frac{2r^2}{1 - r^2 \| \phi^R \|_{L^\infty}} \beta \right).
\]

The conclusion follows immediately. \( \square \)

Taking Lemma 2.6 into account and noticing the fact \( F(r) > 1 \), for the term (III), we have

\[
(III) / \Psi(r) \\
= (N-2) \int_{B_r} \phi^R |\omega|^2 dV_x / \left( r \Psi(r) \right) \\
\leq (N-2) \| \phi^R \|_{L^\infty} \Phi(r) / \Psi(r) \\
\leq r(N-2) \| \phi^R \|_{L^\infty}.
\]
For the term (IV),

$$\left| (IV)/\Psi(r) \right| = \left| 2 \text{Re} \int_{B_r} A\omega \cdot H\omega dV_x \right|/(r\Psi(r))$$

$$\leq 2 \int_{B_r} |A\omega| \cdot |\mathcal{H}\omega| dV_x/(r\Psi(r))$$

$$\leq \left( 1/(2\epsilon r) \right) \int_{B_r} |A\omega|^2 dV_x + 2\epsilon/r \int_{B_r} |\mathcal{H}\omega|^2 dV_x \right)/\Psi(r)$$

$$= \left( 1/(2\epsilon r) \int_{B_r} A^T|\omega|^2 dV_x + 2\epsilon/r \int_{B_r} |\mathcal{H}\omega|^2 dV_x \right)/\Psi(r)$$

$$\leq \|A^T\|_{L^\infty} \Phi(r)/(2\epsilon\Psi(r)) + 2\epsilon/r + 2\epsilon\|\phi^R\|_{L^\infty} \Phi(r)/\Psi(r).$$

Let $\epsilon = r/2$, since $F(r) > 1$, then

$$\left| (IV)/\Psi(r) \right| \leq \|A^T\|_{L^\infty} + 1 + r^2\|\phi^R\|_{L^\infty}.$$

For the term (V),

$$\left| (V)/\Psi(r) \right| = \left| 2 \text{Re} \int_{B_r} (x \cdot \nabla\omega) \cdot \overline{\mathcal{H}\omega} dV_x \right|/(r\Psi(r))$$

$$= \left| 2 \text{Re} \int_{B_r} (x \cdot (\nabla\omega - iA\omega)) \cdot \overline{\mathcal{H}\omega} dV_x \right|/(r\Psi(r))$$

$$\leq \left( \|\phi\|_{L^\infty}/(2\epsilon r) \int_{B_r} |\omega|^2 dV_x + 2/r\epsilon \|\phi\|_{L^\infty} \int_{B_r} |x \cdot \mathcal{H}\omega|^2 dV_x \right)/\Psi(r)$$

$$\leq \left( \|\phi\|_{L^\infty}/(2\epsilon r) \int_{B_r} |\omega|^2 dV_x + 2r\epsilon \|\phi\|_{L^\infty} \int_{B_r} |\mathcal{H}\omega|^2 dV_x \right)/\Psi(r)$$

$$= \left( \|\phi\|_{L^\infty}/(2\epsilon r) \int_{B_r} |\omega|^2 dV_x + 2r\epsilon \|\phi\|_{L^\infty} \int_{B_r} (|\mathcal{H}\omega|^2 - \phi^R|\omega|^2) dV_x \right)$$

$$+ 2r\epsilon \|\phi\|_{L^\infty} \int_{B_r} \phi^R|\omega|^2 dV_x \right)/\Psi(r)$$

$$\leq r\|\phi\|_{L^\infty}/(2\epsilon) + 2r\epsilon \|\phi\|_{L^\infty} + 2r^3\epsilon \|\phi\|^2_{L^\infty}.$$

Let $\epsilon = r/2$, then

$$\left| (V)/\Psi(r) \right| \leq \|\phi\|_{L^\infty}(1 + r^2 + r^4\|\phi\|_{L^\infty}).$$
For the term (VI),
\[
| (VI) / \Psi(r) | = | 2 \text{Re} \int_{B_r} \omega x (DA)^T \overline{\mathcal{H} \omega}^T \, dV_x / (r \Psi(r)) |
\leq 2 \int_{B_r} |\omega x|_2 \| (DA)^T \overline{\mathcal{H} \omega}^T \|_2 dV_x / (r \Psi(r))
\leq 2 \max \| (DA)^T \|_F \int_{B_r} |\omega| \| \mathcal{H} \omega \| dV_x / (r \Psi(r))
\leq (\max \| (DA)^T \|_F / (2 \epsilon) \int_{B_r} |\omega|^2 dV_x + 2 \epsilon \max \| (DA)^T \|_F \left\{ \int_{B_r} (|\mathcal{H} \omega|^2 - \phi^R |\omega|^2) dV_x + \int_{B_r} \phi^R |\omega|^2 dV_x \right\} / \Psi(r)
\leq \max \| (DA)^T \|_F (r^2 / (2 \epsilon) + 2 \epsilon + 2 \epsilon r^2 \| \phi^R \|_{L^\infty}).
\]
Let \( \epsilon = r/2 \), then
\[
| (VI) / \Psi(r) | \leq \max \| (DA)^T \|_F (2r + r^3 \| \phi^R \|_{L^\infty}),
\]
where \( \max \| (DA)^T \|_F \) denotes the Frobenius norm of \( (DA)^T \).

Let \( \epsilon = r/2 \). By Schwartz’s inequality, we have the estimate below for the term (VII),
\[
| (VII) / \Psi(r) | = | - 2 \text{Re} \int_{\partial B_r} (A \omega \cdot \nu) \overline{\mathcal{H} \omega \cdot \nu} dS_x / \Psi(r) |
\leq 2 \int_{\partial B_r} |A \omega \cdot \nu| \| \mathcal{H} \omega \cdot \nu \| dS_x / \Psi(r)
\leq \left( 1 / (2 \epsilon) \int_{\partial B_r} |A \omega|^2 dS_x + 2 \epsilon \int_{\partial B_r} |\mathcal{H} \omega \cdot \nu|^2 dS_x \right) / \Psi(r)
\leq \| AA^T \|_{L^\infty} \int_{\partial B_r} |\omega|^2 dS_x / (2 \epsilon \Psi(r)) + 2 \epsilon \int_{\partial B_r} |\mathcal{H} \omega \cdot \nu|^2 dS_x / \Psi(r)
\leq \| AA^T \|_{L^\infty} / (2 \epsilon) + 2 \epsilon \int_{\partial B_r} |\mathcal{H} \omega \cdot \nu|^2 dS_x / \Psi(r)
\leq \| AA^T \|_{L^\infty} + C^*(\phi),
\]
where \( C^*(\phi) \) is from Lemma 2.6.

For the last term (VIII), a simple calculation leads to
\[
| (VIII) / \Psi(r) | = \int_{\partial B_r} \phi^R |\omega|^2 dS_x / \Psi(r) \leq r \| \phi^R \|_{L^\infty},
\]
From the above estimates, we conclude that there exists a positive constant \( \tau = \tau(N, \phi) \) which is independent of \( r \) such that
\[
F'(r) \geq -F(r)\tau.
\]

\[ \square \]

It follows that \( \exp(\tau r)F(r) \) is monotonically increasing on \((0, r^*)\), that is to say,
\[
\exp(\tau r)F(r) \leq \exp(\tau r^*)F(r^*).
\]

Keeping in mind the case \( \phi \leq 1 \), we know that, \( \phi(r) \) is bounded on \((0, r^*)\). Since \( \Phi'(r) = (N-1)/r \Phi(r) + 2\Psi(r) \), then
\[
\left( \log(\Phi(r)/r^{N-1}) \right)' = 2\Psi(r)/\Phi(r) = 2F(r)/r \leq C(\tau)/r.
\]

We integrate from \( \gamma \) to \( 2\gamma \), then
\[
\log(2^{1-N}\Phi(2\gamma)/\Phi(\gamma)) \leq C(\tau) \log 2.
\]

It follows that
\[
\Phi(2\gamma) \leq 2^{C(\tau)+N-1}\Phi(\gamma).
\]

Finally, integrating with respect to \( \gamma \) gives
\[
\int_{\mathbb{B}_2} |\omega|^2 dV_x \leq 2^{C(\tau)+N-1} \int_{\mathbb{B}_1} |\omega|^2 dV_x.
\]

Since \( \mathbb{B}_1 \) is connected, then our theorem follows immediately.

**Remark 2.7.** It is of great interest to explore the strong unique continuation for a variety of Schrödinger operators with singular or nonlinear potentials by the multiplier method. More results will be available in sequential papers.

### 3. Proof of Corollary 1.4

In this section, we show an important application of Theorem 1.3 in [14].

**Proof of Corollary 1.4:** Let \( \mathbb{B} \) be an arbitrarily small open ball such that
\[
\Gamma \cap \mathbb{B} \neq \emptyset.
\]

Set
\[
\Omega^1 \triangleq \Omega \cup \mathbb{B},
\]
and define
\[
\omega^1 \triangleq \left\{ \begin{array}{ll}
\omega & \text{in } \Omega; \\
0 & \text{in } \mathbb{B} \setminus \Omega.
\end{array} \right.
\]

It is sufficient to verify that \( \omega^1 \in H^2 \). Denote by \( \omega^1_j, \omega^1_{jk} \) the extension by zero to \( \Omega^1 \) of the derivatives \( \nabla_j \omega, \nabla_j \nabla_k \omega, j, k = 1, \cdots, N \). Then \( \omega_j, \omega_{jk} \in L^2(\Omega^1) \) and it is necessary to demonstrate that, for \( \forall \zeta \in \mathcal{D}(\Omega^1) \),
\[
\int_{\Omega^1} \omega^1 \nabla_j \zeta dx = -\int_{\Omega^1} \omega^1_j \zeta dx,
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and

\[
\int_{\Omega} \omega_j \nabla_k \zeta \, dx = - \int_{\Omega} \omega_{jk} \zeta \, dx.
\]

Indeed, since \( \omega_j = \omega_{jk} \equiv 0 \) outside of \( \Omega \), \( \zeta \equiv 0 \) on \( \Gamma \setminus (\Gamma \cap B) \) and \( \omega = \partial \omega / \partial \nu \equiv 0 \) on \( \Gamma \cap B \), we have

\[
\int_{\Omega} \omega_j \nabla_k \zeta \, dx = \int_{\Gamma} \omega \nabla_j \zeta \, d\Gamma = \int_{\Gamma} \omega \nabla \zeta \nu_j \, d\Gamma - \int_{\Omega} (\nabla_j \omega) \zeta \, dx
\]

and

\[
\int_{\Omega} \omega_j \nabla_k \zeta \, dx = \int_{\Omega} \nabla_j \omega \nabla_k \zeta \, dx = \int_{\Gamma} \nabla_j \omega \nabla \zeta \nu_k \, d\Gamma - \int_{\Omega} (\nabla_k \nabla_j \omega) \zeta \, dx
\]

Thus, the result is concluded due to the connectedness of \( \Omega \).

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