Random Time Dynamical Systems I: General Structures

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Abstract

In this paper we introduce the concept of random time changes in dynamical systems. The subordination principle may be applied to study the long time behavior of the random time systems. We show, under certain assumptions on the class of random time, that the subordinated system exhibits a slower time decay which is determined by the random time characteristics. In the path asymptotic a random time change is reflected in the new velocity of the resulting dynamics.

Keywords: Dynamical systems; random time change; inverse subordinator; long time behavior.

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1 Introduction

The idea to consider stochastic processes with general random times is known at least from the classical book by Gikhman and Skorokhod [GS74]. In the case of Markov processes time changes by subordinators was considered already by Bochner [Boc62]. In this case it gives again a Markov process, so-called Bochner subordinated Markov process. A more interesting situation appears in the case of inverse subordinators. After the time change we do not obtain anymore a Markov process and the study of such kind of processes becomes more challenging. At this point we would like to point out the work of Montroll and Weiss, 1965 [MW04] which considered the physically motivated case of random walks in the random time. This work created a wide research area related to the study of Markov processes with inverse stable subordinators as random time changes, see the book [MS12] for a detailed review and historical comments.

For processes with random time change which are not subordinators or inverse subordinators the situation is much less investigated. On the one hand additional assumptions on the stable subordinator turns the time change process very restrictive in applications. On the other hand, we find technical difficulties in handling general inverse subordinators. These difficulties may be overcome for certain sub-classes of inverse subordinators, see, e.g., [KKdS20b, KKdS20a]. The random time change is essential in modelling several physical systems (ecological and biological models), see, e.g., [MS15] and references therein for other applications. A very particular choice of a random time process has not a special motivation.

There is a natural question concerning the use of a random time change not only in stochastic dynamics but in a more wide class of dynamical problems. In this paper we concentrate on the analysis of this question in the case of dynamical systems in $\mathbb{R}^d$.

Let $X(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$ be a dynamical system in $\mathbb{R}^d$ starting from $x$, that is, $X(0, x) = x$. This system is also a deterministic Markov process. Given $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we define

$$u(t, x) := f(X(t, x)).$$

Then we have a version of the Kolmogorov equation, called the Liouville equation in the theory of dynamical systems, namely

$$\frac{\partial}{\partial t} u(t, x) = Lu(t, x),$$
where $L$ is the generator of a semigroup. This semigroup is given by the solution of the Liouville equation, see for example [EN00, RS75, Yos80] for more details.

If $E(t)$ is an inverse subordinator process (see Section 2 below for details and examples) then we may consider the time changed random dynamical systems

$$Y(t) := X(E(t)).$$

Our aim is to analyze the properties of $Y(t)$ depending on those of the initial dynamical systems $X(t)$. In particular, we can define

$$v(t, x) := \mathbb{E}[f(Y(t, x))]$$

and try to compare the behavior $u(t, x)$ and $v(t, x)$ for a certain class of functions $f$.

In the present paper we would like to present the main problems which appear naturally in the study of random time changes in dynamical systems. We illustrate the solutions to these problems with the simplest examples in Section 3. We postpone to a forthcoming paper the study of particular classes of dynamical systems and random times for which more detailed information may be obtained. In a certain sense, in this paper foundations are presented to study random time changes in dynamical systems. We hope that this program may be interesting and attractive for mathematicians working and studying the interplay between stochastic and dynamical systems theories. Having this in mind, we will not elaborate detailed conditions on the dynamical systems and random times in our considerations. The latter shall be (and may be) done for each particular model we would like to study in detail.

The rest of the paper is organized as follows. In Section 2 we present the class of inverse subordinators and the associated general fractional derivatives. In Section 3 we consider the simplest examples of dynamical systems and present the first results when the random time is associated to the $\alpha$-space subordinator. In Section 4 we consider a dynamical system as a deterministic Markov processes and introduce the notion of potential and Green measure of the dynamical system. Finally, in Section 5 we investigate the path transformation of a simple dynamical system by a random time.
2 Random Times and Fractional Analysis

In this section we introduce the inverse subordinators and the corresponding general fractional derivatives. Associated to these classes of inverse subordinators we define a kernel $k \in L_{\mathrm{loc}}^{1}(\mathbb{R}_{+})$ which is used to define a general fractional derivatives (GFD), see [Koc11] for details and applications to fractional differential equations. These admissible kernels $k$ are characterized in terms of their Laplace transforms $K(\lambda)$ as $\lambda \to 0$, see assumption (H) below and also Lemma 2.10.

Let $S = \{ S(t), \ t \geq 0 \}$ be a subordinator without drift starting at zero, that is, an increasing Lévy process starting at zero, see [Ber96] for more details. The Laplace transform of $S(t), \ t \geq 0$ is expressed in terms of a Bernstein function $\Phi : [0, \infty) \rightarrow [0, \infty)$ (also known as Laplace exponent) by

$$\mathbb{E}(e^{-\lambda S(t)}) = e^{-\lambda \Phi(\lambda)}, \ \lambda \geq 0.$$  

The function $\Phi$ admits the representation

$$\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda \tau}) \, d\sigma(\tau), \quad (2.1)$$

where the measure $\sigma$ (called Lévy measure) has support in $[0, \infty)$ and fulfills

$$\int_{(0,\infty)} (1 \wedge \tau) \, d\sigma(\tau) < \infty. \quad (2.2)$$

In what follows we assume that the Lévy measure $\sigma$ satisfy

$$\sigma((0, \infty)) = \infty. \quad (2.3)$$

Using the Lévy measure $\sigma$ we define the kernel $k$ as follows

$$k : (0, \infty) \rightarrow (0, \infty), \ t \mapsto k(t) := \sigma((t, \infty)). \quad (2.4)$$

Its Laplace transform is denoted by $K$, that is, for any $\lambda \geq 0$ one has

$$K(\lambda) := \int_{0}^{\infty} e^{-\lambda t} k(t) \, dt. \quad (2.5)$$

The relation between the function $K$ and the Laplace exponent $\Phi$ is given by

$$\Phi(\lambda) = \lambda K(\lambda), \ \forall \lambda \geq 0. \quad (2.6)$$

In what follows we make the following assumption on the Laplace exponent $\Phi(\lambda)$ of the subordinator $S$. 

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(H) \( \Phi \) is a complete Bernstein function (that is, the Lévy measure \( \sigma \) is absolutely continuous with respect to the Lebesgue measure) and the functions \( \mathcal{K}, \Phi \) satisfy

\[
\mathcal{K}(\lambda) \to \infty, \text{ as } \lambda \to 0; \quad \mathcal{K}(\lambda) \to 0, \text{ as } \lambda \to \infty; \quad (2.7)
\]

\[
\Phi(\lambda) \to 0, \text{ as } \lambda \to 0; \quad \Phi(\lambda) \to \infty, \text{ as } \lambda \to \infty. \quad (2.8)
\]

**Example 2.1** (\( \alpha \)-stable subordinator). A classical example of a subordinator \( S \) is the so-called \( \alpha \)-stable process with index \( \alpha \in (0, 1) \). Specifically, a subordinator is \( \alpha \)-stable if its Laplace exponent is

\[
\Phi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda \tau})^{1-\alpha} d\tau.
\]

In this case it follows that the Lévy measure is \( d\sigma_\alpha(\tau) = \frac{\alpha}{\Gamma(1-\alpha)} \tau^{-(1+\alpha)} d\tau \). The corresponding kernel \( k_\alpha \) has the form \( k_\alpha(t) = g_{1-\alpha}(t) := \frac{t^{1-\alpha}}{\Gamma(1-\alpha)}, \ t \geq 0 \) and its Laplace transform is \( \mathcal{K}_\alpha(\lambda) = \lambda^{\alpha-1}, \ \lambda \geq 0 \).

**Example 2.2** (Gamma subordinator). The Gamma process \( Y^{(a,b)} \) with parameters \( a, b > 0 \) is another example of a subordinator with Laplace exponent

\[
\Phi_{(a,b)}(\lambda) = a \log \left(1 + \frac{\lambda}{b}\right) = \int_0^\infty (1 - e^{-\lambda \tau}) a \tau^{-1} e^{-b \tau} d\tau,
\]

the second equality is known as the Frullani integral. The Lévy measure is given by \( d\sigma_{(a,b)}(\tau) = a \tau^{-1} e^{-b \tau} d\tau \). The associated kernel \( k_{(a,b)}(t) = a \Gamma(0, bt), \ t > 0 \) (here \( \Gamma(\nu, z) := \int_z^\infty e^{-t} t^{\nu-1} dt \) is the incomplete gamma function, see Section 8.3 in [GR15]) and its Laplace transform is \( \mathcal{K}_{(a,b)}(\lambda) = a \lambda^{-1} \log(1 + \frac{\lambda}{b}), \ \lambda > 0 \).

**Example 2.3** (Truncated \( \alpha \)-stable subordinator). The truncated \( \alpha \)-stable subordinator (see Example 2.1-(ii) in [Che17]) \( S_\delta, \ \delta > 0 \) is a driftless \( \alpha \)-stable subordinator with Lévy measure given by

\[
d\sigma_\delta(\tau) := \frac{\alpha}{\Gamma(1-\alpha)} \tau^{-(1+\alpha)} \mathbb{I}_{[0,\delta]}(\tau) d\tau, \quad \delta > 0.
\]

The corresponding Laplace exponent is

\[
\Phi_\delta(\lambda) = \lambda^\alpha \left(1 - \frac{\Gamma(-\alpha, \delta \lambda)}{\Gamma(-\alpha)}\right) + \frac{\delta^{-\alpha}}{\Gamma(1-\alpha)},
\]

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and the associated kernel $k_\delta$ is given by

$$k_\delta(t) := \sigma_\delta((t, \infty)) = \frac{\Pi_{(0,\delta)}(t)}{\Gamma(1-\delta)}(t^{-\beta} - \delta^{-\beta}), \ t > 0.$$  

**Example 2.4** (Sum of two alpha stable subordinators). Let $0 < \alpha < \beta < 1$ be given and $S_{\alpha,\beta}(t), \ t \geq 0$ the driftless subordinator with Laplace exponent given by

$$\Phi_{\alpha,\beta}(\lambda) = \lambda^\alpha + \lambda^\beta.$$  

It is clear from Example 2.1 that the corresponding Lévy measure $\sigma_{\alpha,\beta}$ is the sum of two Lévy measures, that is,

$$d\sigma_{\alpha,\beta}(\tau) = d\sigma_{\alpha}(\tau) + d\sigma_{\beta}(\tau) = \frac{\alpha}{\Gamma(1-\alpha)}\tau^{-(1+\alpha)}\,d\tau + \frac{\beta}{\Gamma(1-\beta)}\tau^{-(1+\beta)}\,d\tau.$$  

Then the associated kernel $k_{\alpha,\beta}$ is

$$k_{\alpha,\beta}(t) := g_{1-\alpha}(t) + g_{1-\beta}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{t^{-\beta}}{\Gamma(1-\beta)}, \ t > 0$$  

and its Laplace transform is $K_{\alpha,\beta}(\lambda) = K_\alpha(\lambda) + K_\beta(\lambda) = \lambda^{\alpha-1} + \lambda^{\beta-1}, \ \lambda > 0$.

**Example 2.5** (Kernel with exponential weight). Given $\gamma > 0$ and $0 < \alpha < 1$ consider the subordinator with Laplace exponent

$$\Phi_\gamma(\lambda) := (\lambda + \gamma)^\alpha = \left(\frac{\lambda + \gamma}{\lambda}\right)^{1+\alpha} \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda \tau})\tau^{-1-\alpha}\,d\tau.$$  

It follows that the Lévy measure is given by $d\sigma_\gamma(\tau) = \left(\frac{\lambda + \gamma}{\lambda}\right)^{1+\alpha} \frac{\alpha}{\Gamma(1-\alpha)}\tau^{-(1+\alpha)}\,d\tau$ which yields a kernel $k_\gamma$ with exponential weight, namely

$$k_\gamma(t) = g_{1-\alpha}(t)e^{-\gamma t} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}e^{-\gamma t}.$$  

The corresponding Laplace transform of $k_\gamma$ is given by $K_\gamma(\lambda) = \lambda^{-1}(\lambda + \gamma)^\alpha, \ \lambda > 0$.

**Remark 2.6.** The subordinators from Examples 2.1–2.5 provides different types of kernels $k$ which give rise to different types of fractional derivatives. These fractional derivatives were introduced to study the theory of relaxation and diffusion equations, see [Koc11] and references therein.
Denote by $E$ the inverse process of the subordinator $S$, that is,

$$E(t) := \inf\{s > 0 \mid S(s) > t\} = \sup\{s \geq 0 \mid S(s) \leq t\}. \quad (2.9)$$

For any $t \geq 0$ we denote by $G_t^k(\tau) := G_t(\tau), \quad \tau \geq 0$ the marginal density of $E(t)$ or, equivalently

$$G_t(\tau) d\tau = \frac{\partial}{\partial \tau} P(E(t) \leq \tau) d\tau = \frac{\partial}{\partial \tau} P(S(\tau) \geq t) d\tau = -\frac{\partial}{\partial \tau} P(S(\tau) < t) d\tau. \quad (2.10)$$

As the density $G_t(\tau)$ plays an important role in what follows, we collect the most important properties needed later on.

**Remark 2.7.** If $S$ is the $\alpha$-stable process, $\alpha \in (0, 1)$, then the inverse process $E(t)$ has Laplace transform (cf. Prop. 1(a) in [Bin71]) given by

$$E(e^{-\lambda E(t)}) = \int_0^\infty e^{-\lambda \tau} G_t(\tau) d\tau = \sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(n\alpha + 1)} = E_{\alpha}(-\lambda t^\alpha). \quad (2.11)$$

It follows from the asymptotic behavior of the Mittag-Leffler function $E_{\alpha}$ that $E(e^{-\lambda E(t)}) \sim Ct^{-\alpha}$ as $t \to \infty$. Using the properties of the Mittag-Leffler function $E_{\alpha}$, we can show that the density $G_t(\tau)$ is given in terms of the Wright function $W_{\mu, \nu}$, namely $G_t(\tau) = t^{-\alpha} W_{-\alpha,1-\alpha}(\tau t^{-\alpha})$, see [GLM99] for more details.

For a general subordinator, the following lemma determines the $t$-Laplace transform of $G_t(\tau)$, with $k$ and $K$ given in (2.4) and (2.5), respectively. For the proof see [Koc11] or Lemma 3.1 in [Toa15].

**Lemma 2.8.** The $t$-Laplace transform of the density $G_t(\tau)$ is given by

$$\int_0^\infty e^{-\lambda \tau} G_t(\tau) d\tau = K(\lambda)e^{-\tau K(\lambda)}. \quad (2.11)$$

The double $(\tau, t)$-Laplace transform of $G_t(\tau)$ is

$$\int_0^\infty \int_0^\infty e^{-\lambda \tau} e^{-\lambda t} G_t(\tau) d\tau d\tau = \frac{K(\lambda)}{\lambda K(\lambda) + p}. \quad (2.12)$$

For any $\alpha \in (0, 1)$ the Caputo-Dzhrbashyan fractional derivative of order $\alpha$ of a function $u$ is defined by (see e.g., [KST06] and references therein)

$$(D_t^\alpha u)(t) = \frac{d}{dt} \int_0^t k_\alpha(t - \tau) u(\tau) d\tau - k_\alpha(t)u(0), \quad t > 0, \quad (2.13)$$
where \( k_\alpha \) is given in Example 2.1, that is, \( k_\alpha(t) = g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \ t > 0 \). In general, starting with a subordinator \( S \) and the kernel \( k \in L^1_{\text{loc}}(\mathbb{R}_+) \) given in (2.4), we may define a differential-convolution operator by

\[
(\mathbb{D}_t^{(k)} u)(t) = \frac{d}{dt} \int_0^t k(t-\tau)u(\tau) \, d\tau - k(t)u(0), \ t > 0.
\] (2.14)

The operator \( \mathbb{D}_t^{(k)} \) is also known as generalized fractional derivative.

**Example 2.9** (Distributed order derivative). Consider the kernel \( k \) defined by

\[
k(t) := \int_0^1 g_\alpha(t) \, d\alpha = \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \, d\alpha, \ t > 0.
\] (2.15)

Then it is easy to see that

\[
\mathcal{K}(\lambda) = \int_0^\infty e^{-\lambda t} k(t) \, dt = \frac{\lambda - 1}{\lambda \log(\lambda)}, \ \lambda > 0.
\]

The corresponding differential-convolution operator \( \mathbb{D}_t^{(k)} \) is called distributed order derivative, see [APZ09, DGB08, Han07, Koc08, GU05, MS06] for more details and applications.

We say that the functions \( f \) and \( g \) are *asymptotically equivalent at infinity*, and denote \( f \sim g \) as \( x \to \infty \), meaning that

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

We say that a function \( f \) is slowly varying (see [Fel71, Sen76]) if

\[
\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = 1, \ \text{for any} \ \lambda > 0.
\]

The following lemma may be extracted from the proof of Theorem 4.2 in [KK17] which is used below.

**Lemma 2.10.** Assume that the subordinator \( S(t) \) and its inverse \( E(t), \ t \geq 0 \) are such that

\[
\mathcal{K}(\lambda) \sim \lambda^{-\gamma} Q \left( \frac{1}{\lambda} \right), \ \lambda \to 0,
\] (2.16)
where $0 \leq \gamma \leq 1$ and $Q$ is a slowly varying function. In addition define

$$A(t, z) := \int_0^\infty e^{-z\tau} G_t(\tau) d\tau, \quad t > 0, \ z > 0.$$ 

Then

$$A(t, z) \sim \frac{1}{z} \frac{t^{\gamma-1}}{\Gamma(\gamma)} Q(t), \quad t \to \infty.$$ 

Remark 2.11. We point out that the condition (2.16) on the Laplace transform of the kernel $k$ is satisfied by all Examples 2.1–2.5 and 2.9 above. The case of Example 2.4 is easily checked as

$$k(\lambda) = \lambda^\alpha + \lambda^\beta = \lambda^{-(1-\alpha)}(1 + \lambda^{-(\alpha-\beta)}) = \lambda^{-\gamma} Q\left(\frac{1}{\lambda}\right),$$

where $\gamma = 1 - \alpha > 0$ and $Q(t) = 1 + t^{\alpha-\beta}$ is a slowly varying function.

### 3 Random Time Dynamics

In this section we study the effect of the subordination by the density $G_t(\tau)$ of the inverse process $E(t), t \geq 0$ of a dynamical system.

Define a random dynamical systems

$$Y(t, x, \omega) := X(E(t, \omega), x, \omega), \quad x \in \mathbb{R}^d.$$ 

For suitable functions $f : \mathbb{R}^d \to \mathbb{R}$ define

$$v(t, x) := \mathbb{E}[f(Y(t, x))].$$

Then $v(t, x)$ is the solution to an evolution equation with the same generator $L$ but with generalized fractional derivative (see (2.14)), namely

$$D_t^{(k)} v(t, x) = L v(t, x). \quad (3.1)$$

As a result the following subordination formula holds:

$$v(t, x) = \int_0^\infty u(\tau, x) G_t(\tau) \ d\tau. \quad (3.2)$$
The problem (as in the Markov case) is to study the change of the behavior of \( u \) after subordination. The formula (3.2) is the main object for the analysis of this problem.

In the following proposition we consider the simplest evolution equation in \( \mathbb{R}^d \)
\[
\frac{d}{dt}X(t) = v \in \mathbb{R}^d, \quad X(0) = x_0 \in \mathbb{R}^d.
\]
The corresponding dynamics is
\[
X(t) = x_0 + vt, \quad t \geq 0.
\]
We assume \( x_0 = 0 \) for simplicity. Take \( f(x) = e^{-\alpha|x|}, \alpha > 0 \). The corresponding solution to the Liouville equation is
\[
u(t, x) = e^{-\alpha|v|}, \quad t \geq 0.
\]

**Proposition 3.1.** Assume that the assumptions of Lemma 2.10 are satisfied. Then
\[
v(t, x) \sim \frac{1}{\alpha|v|\Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.
\]

*Proof.* From the explicit form of the solution \( u(t, x) \) using (3.2) and Lemma 2.10 we obtain
\[
v(t, x) \sim \frac{1}{\alpha|v|\Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.
\]

In particular, for the \( \alpha \)-stable subordinator considered in Example 2.1, we obtain \( v(t, x) \sim Ct^{-\alpha} \), \( C \) is a constant. Therefore, starting with a solution \( u(t, x) \) with exponential decay after subordination we observe a polynomial decay with the order defined by the random time characteristics. \( \square \)

**Example 3.1.** For \( d = 1 \) consider the dynamics
\[
\beta \frac{d}{dt}X(t) = \frac{1}{X^{\beta-1}(t)}, \quad \beta \geq 1.
\]
It is clear that the solution is given by
\[
X(t) = (t + C)^{1/\beta}.
\]
Take the function \( f(x) = \exp(-a|x|^\beta), \quad a > 0 \), then the long time behavior of the subordination \( v(t, x) \) is given by
\[
v(t, x) \sim \frac{e^{-aC}}{a} \frac{t^{\gamma-1}}{\Gamma(\gamma)} Q(t), \quad t \to \infty.
\]
In particular if we choose the density \( G_t(\tau) \) of the inverse subordinator \( E(t) \) corresponding to the Example 2.4, then we obtain

\[
v(t, x) \sim Ct^{-\alpha}(1 + t^{\alpha-\beta}) \sim Ct^{-\alpha}, \quad t \to \infty.
\]

4 Potentials and Green Measures

The notion of potential is classical in the theory of Markov processes, see, e.g., [BG68]. Recently it was proposed the concept of Green measure as a representation of potentials in an integral form, see [KdS20a]. The modification of these concepts for time changed Markov processes was investigated in [KdS20b].

Considering a dynamical system as a deterministic Markov processes, we have the possibility to study the notion of potential and Green measure in this context.

In our framework above for a function \( f : \mathbb{R}^d \to \mathbb{R} \) consider the solution to the Cauchy problem

\[
\frac{\partial}{\partial t} u(t, x) = Lu(t, x),
\]

\[
u(0, x) = f(x).
\]

Then we obtain

\[
u(t, x) = (e^{tL}f)(x).
\]

Define the potential for the function \( f \) by

\[
U(f, x) := \int_0^\infty u(t, x) \, dt = \int_0^\infty (e^{tL}f)(x) \, dt = -(L^{-1}f)(x), \quad x \in \mathbb{R}^d.
\]

The existence of \( U(f, x) \) is not clear at all. It depends on the class of functions \( f \) and the Liouville generator \( L \). Assuming the existence of \( U(f, x) \) we would like to obtain an integral representation

\[
U(f, x) = \int_{\mathbb{R}^d} f(y) \, d\mu^x(y)
\]

(4.1)

with a Radom measure \( \mu^x \) on \( \mathbb{R}^d \). This measure we will call the Green measure for our dynamical system. As in the case of Markov processes, the definition of the potential is easy to introduce but difficult to analyze for each particular model.
Moreover, on the base of certain particular examples we may assume that the potentials are well defined for very special classes functions $f$. But the existence of Green measure we can not expect. This point we will discuss in details in a forthcoming paper.

After a random time change we will have the subordinated solution $v(t, x)$ for the fractional equation (see equation (3.1) above). Then we can try again to define the potential

$$V(f, x) := \int_0^\infty v(t, x) \, dt, \quad x \in \mathbb{R}^d.$$ 

But it is not hard to see that for general random times this integral will be divergent. In fact, it follows from the subordination formula (3.2), and the Fubini theorem that

$$V(f, x) = \int_0^\infty \int_0^\infty u(\tau, x) G_t(\tau) \, d\tau \, dt = \int_0^\infty u(\tau, x) \left( \int_0^\infty G_t(\tau) \, dt \right) \, d\tau.$$ 

The inner integral is not convergent due to equality (2.11) and assumption (2.7). In view to overcome this difficulty we may use the notion of renormalized potential. More precisely, inspired by the time change of Markov processes (see [KdS20b] for details), we define the renormalized potential

$$V_r(f, x) := \lim_{t \to \infty} \frac{1}{N(t)} \int_0^t v(s, x) \, ds, \quad t \geq 0, \quad (4.2)$$

where $N(t)$ is defined by $N(t) := \int_0^t k(s) \, ds$. Then assuming the existence of $U(f, x)$ it is not difficult to show that

$$V_r(f, x) = \int_0^\infty u(t, x) \, dt.$$ 

5 Path Transformation

Now we investigate the transformation of the trajectories of dynamical systems under random times. As above we have the Liouville equation for

$$u(t, x) := f(X(t, x)), \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

that is,

$$\frac{\partial}{\partial t} u(t, x) = Lu(t, x), \quad u(0, x) = f(x),$$
where $L$ is the generator of a semigroup. In addition, let $E(t)$, $t \geq 0$ be the inverse subordinator process, then we can consider the time changed random dynamical systems

$$Y(t,x) = X(E(t), x), \quad t \geq 0, \ x \in \mathbb{R}^d.$$ 

Define

$$v(t, x) := E[f(Y(t, x)].$$

The subordination formula gives

$$v(t, x) = \int_0^\infty u(\tau, x)G_t(\tau) \, d\tau.$$

Now we will take the vector-function

$$f(x) = x \in \mathbb{R}^d.$$ 

Then the average trajectories of $Y(t, x)$ is given by

$$\mathbb{E}(Y(t, x)) = \int_0^\infty X(\tau, x)G_t(\tau) \, d\tau.$$

**Example 5.1.** If we consider the dynamical system of Proposition 3.1, that is, $X(t, x) = vt$, then we obtain

$$\mathbb{E}[Y(t, x)] = v \int_0^\infty \tau G_t(\tau) \, d\tau.$$

Therefore, we need to know the first moment of the density $G_t$. Consider the case of the inverse $\alpha$-stable subordinator from Example 2.1. Then

$$\int_0^\infty \tau G_t(\tau) \, d\tau = Ct^\alpha.$$ 

Therefore, the asymptotic of the time changed trajectory will be slower (proportional to $t^\alpha$) instead of initial linear $vt$ motion. In a forthcoming paper we will study in detail these results for other classes of inverse subordinators.

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