QUASILINEAR DRIVEN TRANSPORT IN A SHEARED FLOW FIELD

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Abstract

The evolution of a passive scalar field is considered for a slowly varying stratified medium, which is convected in an incompressible sheared flow with many overlapping static flux islands. Within the quasilinear/random phase approximation, a multiple scale expansion is made. Due to the rapid spatial variation of the temperature, the “ensemble” averaged/ slowly varying part of the solution is not described by the arithmetic average of the oscillatory evolution equation. The standard Markovian and continuum approximations are shown to be invalid. For times of order $N$, where there are $O(N^2)$ excited modes, most of the time dependent perturbation phase mixes away and the fluid reaches a new saturated state with small time oscillations about the temperature. This saturated state has smaller resonance layers, (corresponding to magnetic islands) than those that occur in the isolated resonant perturbation case. Thus the quasilinear response to the resonant interactions reduces the effective size of the perturbations. The temperature gradient of the saturated state vanishes at all the excited resonance surfaces but has a nonzero average. Thus either the quasilinear approximation ceases to be valid on long time scales, or the fluid remains essentially in this modified equilibrium and does not evolve diffusively. Thus collisionless, driftless fast particles will not be lost rapidly in equilibria with many small islands.
I. INTRODUCTION

The passive scalar evolution equation is widely studied as a model problem for turbulent systems. One class of research concentrates on self-consistent approximate solutions of the fully nonlinear convective nonlinearity under the simplifying assumptions of homogeneity and isotropicity\(^1\). These assumptions eliminate much of the geometry of the underlying stationary state. We consider systems where a weak ensemble of turbulent modes\(^2,3\) is superimposed on a strongly sheared equilibrium flow. The response of the passive scalar field is resonant at the points where the turbulent wavevector, \(\vec{k}\), is perpendicular to the equilibrium flow velocity.

When the fluctuations are small, three wave interactions are a higher order effect and may be neglected in the evolution of the turbulent modes. Instead the turbulent modes interact only by modifying the equilibrium gradients. Thus the turbulent modes evolve linearly on a slowly varying equilibrium state. The turbulent modes only affect the evolution of the background state. In the turbulence literature, this is known as the “weak coupling approximation”, the “quasilinear approximation”, or the “random phase approximation.” In the standard weak turbulence theory of Galeev and Sagdeev\(^3\), the weak coupling truncation is the first of four formal steps. The last three formal steps, the Markovian and continuum “approximations” and the spatial averaging of the coefficients, will be examined using a multiple scale expansion. Since our analysis is based exclusively on the quasilinear/random phase approximation, our conclusions take the form: “If the truncated system of equations remains valid for long times, then ....”

We consider the driven passive scalar transport problem in a sheared torus. We note that the passive scalar evolution problem is isomorphic to the Liouville equation for the diffusion of trajectories in a nearly integrable Hamiltonian system with one and a half degrees of freedom\(^2\). The problem also is equivalent to field line diffusion in magnetohydrodynamic equilibria with magnetic islands. Past analysis of the transport of particles in stochastic field lines has been based on the quasilinear diffusion
theory for field line diffusion\textsuperscript{4,5}. Our work is motivated by the pioneering analysis of Rosenbluth, Sagdeev, Taylor and Zaslavskii\textsuperscript{2}.

We consider the effective diffusion of the passive convective scalar equation in a torus:

\[
\frac{\partial T(\vec{x}, t)}{\partial t} = -\vec{v}(\vec{x}, t) \cdot \nabla T(\vec{x}, t)
\]

where \(\vec{v}(\vec{x}, t)\) is a given velocity field. We use coordinates \(x, \theta, z\), where \(x\) is the stratified direction, \(\theta\) is the poloidal direction, and \(z\) is the toroidal direction. The torus has a toroidal radius of size \(L\) and a poloidal radius of size \(a\). We assume the aspect ratio, \(\frac{L}{a}\), is order one.

Both the driven velocity field, \(\vec{v}(\vec{x}, t)\), and the passive scalar, \(T(\vec{x}, t)\), are \(2\pi L\) and \(2\pi a\) periodic in the toroidal and poloidal planes. Toroidal periodicity is the natural boundary condition for the breakup of integrable surfaces in Hamiltonian systems and for toroidal magnetic confinement systems. Toroidal periodicity quantizes the toroidal mode spectrum. In contrast, fluid flow in an infinite pipe has a continuous spectrum of resonant modes.

We assume that the equilibrium velocity field is sheared: \(\vec{v}_0(x) = \nabla [v_z(x)\hat{z} - \frac{a}{L} \mu(x)\hat{\theta}]\) and the equilibrium temperature field is stratified in the \(x\) direction: \(T_0(\vec{x}, t) = T_0(x, t)\).

Since the temperature is purely convected, the natural initial boundary value problem is that the temperature is given both on the boundary and in the interior at \(t = 0\) and the boundary values are convected as well. If the normal velocity at the boundary vanishes, our fixed boundary problem is well posed. If in addition, we assume the temperature at the boundary initially depends only on the \(x\) variable, the temperature will remain constant on the boundary. If the normal velocity at the boundary does not vanish, then the problem is wellposed as a free boundary problem.

When the normal velocity at the boundary vanishes, the conservation formulation guarantees that there is no net flux of temperature out of the domain and that temperature is conserved pointwise in Lagrangian coordinates. As the number and strength
of the helical perturbations increases, the trajectories of neighboring temperature surfaces will become increasingly distorted. The $\theta - z$ surface average of these distorted, oscillating temperature surfaces may then tend to flatten and approximate diffusion. In this article, we examine whether this often proposed mixing/Hamiltonian diffusion occurs in our model toroidal problem.

In the $x - \theta$ plane, the characteristic length is $a$, and in the toroidal direction, the characteristic length is $L$. The characteristic velocity of the equilibrium flow is $\mathbf{v}$. Our time is measured in units of $L/v$.

We consider the evolution of the stratified equilibrium in an ensemble of many small, helical perturbations. The velocity field is incompressible, $\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0$. We decompose the perturbing velocity field, $\tilde{\mathbf{v}}$, into a sum of discrete Fourier harmonics:

$$\tilde{\mathbf{v}}(\mathbf{x}, t) = \epsilon_0 \mathbf{v} \sum_{m,n \neq (0,0)} \tilde{v}_{m,n}(x) e^{i(m\theta + n z - \omega_{m,n} L \tau)}$$

where $\tilde{v}_{-m,-n} = \tilde{v}_{m,n}$ and $\omega_{-m,-n} = \omega_{m,n}$. The $\omega_{m,n}$ are the eigenfrequencies of the excited modes and are assumed to be real. If we were interested in infinitely long pipes, the sum over $n$ would be replaced by an integral $dn$. We shall show that the often made substitution of $\sum_n \rightarrow \int dn$ is not valid in our system. Thus the quantization of the mode spectrum induced by the boundary conditions is an essential aspect of our problem.

The mode amplitudes, $\tilde{v}_{m,n}$, may be treated as fixed, or as random variables. Since the $\tilde{v}_{m,n}$ do not have any time dependence, the correlation time for the mode amplitudes $|\tilde{v}_{m,n}|^2$ is infinite. We will show that static or strictly oscillatory perturbations are pathological and do not generate diffusion to second order in the standard amplitude expansion. We note that the finite autocorrelation case may behave diffusively and that our results are limited to velocity perturbations with an infinite mode autocorrelation time. In Ref. 5, Krommes refers to this same problem as the short/zero autocorrelation case. Although the autocorrelation function, $\langle \tilde{v}, \tilde{v} \rangle$, has an infinite autocorrelation time, the effective $\langle \tilde{v}, \tilde{T} \rangle$ correlation decays initially with velocity
shear time, $1/\pi \partial_x \mu(x)$, in both the Eulerian and Lagrangian coordinates. However for long times, the Eulerian autocorrelation function of $\langle \tilde{v}, \tilde{v} \rangle$ ceases to decay and therefore has an infinite autocorrelation function and Kubo number. In Appendix B, we show that the effective Lagrangian autocorrelation function decays on the shear time scale until it saturates at a small constant level. Furthermore, as pointed out in Ref. 5, the neglect of the fully nonlinear terms in this analysis is equivalent to assuming the Kolmogorov exponentiation time or nonlinear decorrelation time is effectively infinite with respect to autocorrelation time.

We suppress the dependence of $\tilde{v}_{m,n}$ on the random variable, $\omega$, and in Appendix F, consider the additional smoothness due to random Doppler shifts. Our results include the case where the eigenfrequencies, $\omega_{m,n}$, are also random but time independent.

We scale the fluctuation amplitudes with the small parameter, $\epsilon_v$. We scale the mode spectrum with a second small parameter, $\epsilon_L$. If we did not scale the mode spectrum, then for small enough $\epsilon_v$, closed flux surfaces would exist and there would be no net diffusion. Thus as $\epsilon_v$ decreases, we must increase the number of excited modes to ensure all flux surfaces are broken.

Our scaling of the mode spectrum must not only guarantee that the Chirikov overlap criteria is satisfied everywhere, but also that the quasilinear truncation is formally valid. Within these restrictions, we believe our multiple scale expansion of the long time quasilinear evolution will be valid for a large class of mode spectra. (See Appendix A.) We assume a strict cutoff in mode amplitudes at the values $N_m$ and $N_n : |\tilde{v}_{mn}| \equiv 0$ if $|m| > N_m$ or $|n| > N_n$. We scale the mode cutoffs as $N_n \sim 1/\epsilon_L$ and $N_m \sim 1/\epsilon_L^\alpha$ where $0 \leq \alpha \leq 1$. $\epsilon_L$ may also be thought of as the characteristic scalelength for nonresonant phenomena. For fixed poloidal mode number, $m$, the distance between adjacent resonance surfaces is proportional to $\epsilon_L$. On the short wavelength scale, $y \equiv \epsilon_L x$, the number of excited rational frequencies is proportional to $N_m$. For convenience, we assume that all of the excited modes have roughly similar amplitudes.
The fluctuating temperature perturbation is decomposed into Fourier harmonics, $\tilde{T}_{m,n}$, which evolve temporally as

$$\frac{\partial \tilde{T}_{m,n}(x,t)}{\partial t} + i \frac{\nu}{L} (nv_z(x) - m\mu(x) - \omega_{m,n}) \tilde{T}_{m,n}(x,t) =$$

$$- \frac{\epsilon_v}{a} \tilde{v}_{m,n}(x) \frac{\partial T_0}{\partial x}(x,t) - \epsilon_v \left( \tilde{v}(\vec{x},t) \cdot \nabla \tilde{T}(\vec{x},t) \right)_{m,n}, \quad (3)$$

where we have used the nondimensional $x$ variable, $x \equiv x_{dim}/a$. The temperature perturbations, $\tilde{T}_{m,n}$, are spatially localized in small neighborhoods where $(nv_z(x) - m\mu(x) - \omega_{m,n})$ is not large and therefore $\tilde{T}_{m,n}$ have a spatial localization of size $\epsilon_L$. Due to the spatial localization of $\tilde{T}_{m,n}$, the nonlinear convective term, $\left( \tilde{v}(\vec{x},t) \cdot \nabla \tilde{T}(\vec{x},t) \right)_{m,n}$, is the sum of $N_n$ terms of roughly order $\epsilon_v^2/\epsilon_L$. When the orientation of the phases is random with respect to each other, the sum scales as $N_n^{1/2}$ times the size of an individual element. The random phase approximation postulates that the sum of $N_n$ terms probabilistically has a zero mean value and thus only its variance determines the magnitude of the nonlinear term. Thus, in the random phase approximation, the nonlinear convective term, $\left( \tilde{v}(\vec{x},t) \cdot \nabla \tilde{T}(\vec{x},t) \right)_{m,n}$, is order $N_n^{1/2} \epsilon_v/\epsilon_L$ smaller than the $\tilde{v}_{m,n}(x) \frac{\partial T_0(x,t)}{\partial x}$. For the quasilinear mode truncation to be valid on the ideal timescale, we assume that $\epsilon_v << \epsilon_L N_n^{1/2}$, in addition to the random phase ansatz. Thus the eddy turnover time, $\epsilon_L/\epsilon_v$, is assumed to be large.

For long times, the truncated equations may cease to reflect the actual physical system. Nevertheless, the truncated equations are interesting and widely used. Furthermore, the truncated system is analytically tractable using a multiple scale expansion. We concentrate on a detailed examination of the properties of the quasilinear equations. The quasilinear or “weak coupling” equations for the Fourier harmonics are:

$$\frac{\partial \tilde{T}_{m,n}(x,t)}{\partial t} + i \frac{\nu}{L} \mu_{m,n}(x) \tilde{T}_{m,n}(x,t) = - \frac{\epsilon_v}{a} \tilde{v}_{m,n}(x) \frac{\partial T_0}{\partial x}(x,t), \quad (4)$$

where we have defined $\mu_{m,n}(x)$ to be the resonance denominator, $\mu_{m,n}(x) \equiv m\mu(x) - nv_z(x) - \omega_{m,n}$. We denote by $x_{m,n}$, the location of the spatial resonance where $\mu_{m,n}(x)$
vanishes. This first order differential equation in time may be integrated to yield

\[ \tilde{T}_{m,n}(x,t) = -\frac{\epsilon_v}{a} \tilde{v}_{m,n}(x) \int_{s=0}^{t} e^{i\mu_{m,n}(x)(s-t)} \frac{\partial T_o(x,s)}{\partial x} ds. \]  

(5)

In contrast to dissipative systems, this ideal system possesses infinite memory as indicated by the lack of a temporal decay constant in the kernel of Eq.(5). At the resonance surface, \( x_{m,n}, \tilde{T}_{m,n} \) grows linearly in time unless \( \partial_x T_o \) vanishes at the resonance surface. We rescale time, \( \tilde{t} \equiv \delta t \), where the small parameter \( \delta \) is a yet unspecified function of \( \epsilon_v \) and \( \epsilon_L \). If \( T_o(x,t) \) were to vary only on the slow, \( 1/\delta \) timescale, the magnitude of \( \tilde{T}_{m,n} \) can be estimated by replacing \( T_o(x,s) \) by \( T_o(x,t) \) in the integrand. We find that \( \tilde{T}_{m,n} \sim O(\epsilon_v A_{m,n}(x) \frac{\sin(\mu_{m,n}(x) \tilde{t})}{\mu_{m,n}(x)}) \). Within \( O(\epsilon_L \delta) \) of the resonance surface, \( x_{m,n}, \tilde{T}_{m,n} \) would be order \( \frac{\epsilon_v}{\delta} \). This linear in time growth of \( \tilde{T}_{m,n} \) at the resonance surface could easily violate the quasilinear ordering. However, we will show that the growth of the \( \tilde{T}_{m,n} \) saturates due to the flattening of \( \partial_x T_o \) at the resonance surfaces.

The quasilinear evolution equation for the \( m = 0, n = 0 \) mode is

\[ \frac{\partial T_o(x,\tilde{t})}{\partial \tilde{t}} = \frac{\epsilon_v L}{\delta a} \frac{\partial}{\partial x} \sum_{m,n\neq(0,0)} \tilde{v}_{-m,-n}(x,\tilde{t}) \tilde{T}_{m,n}(x,\tilde{t}) = \]

\[ \frac{\epsilon_v^2}{\delta^2} \frac{\partial}{\partial x} \sum_{m,n\neq(0,0)} A_{m,n}(x) \int_{s=0}^{\tilde{t}} \cos(\mu_{m,n}(x) \frac{(s-\tilde{t})}{\delta}) \frac{\partial T_o(x,s)}{\partial x} ds, \]

(6)

where \( A_{m,n}(x) \equiv (L/a)^2 |\tilde{v}_{m,n}(x)|^2 \) and the nondimensional slow time, \( \tilde{t} \equiv \delta v t_{dim}/L \) has been used. We concentrate on analyzing the mathematical properties of Eq. (6) in the limit of many small, short wavelength perturbations.

In the next section, we summarize the pioneering analyses of Rosenbluth, Sagdeev, Taylor and Zaslavskii\(^2\) and the related work by Krommes, et al.\(^5\). We derive a number of mathematical results for Eq. (6) and its Laplace transform, Eq. (12) in Sec. III. We show that Eq. (6) has purely continuous spectrum, except for an eigenmode at zero frequency.
Sec. IV reviews the homogenization theory of rapidly varying dielectric functions. In Sec. V and Appendix E, we make multiple scale expansions to determine asymptotic behavior of $\hat{T}(x,q)$. Sec. V addresses the overlapping resonance case and Appendix E addresses the isolated resonance layer case. In Sec. VI, we determine the time evolution of $T(x,t)$ by inverting the Laplace transform. We show that the truncated system phase mixes to a new Cantor-set like gradient on a time scale of order $N_m$. We conclude by discussing our results for the quasilinear system and their relevance for the actual physical system.

In Appendix A, we discuss the physical basis of our rescaling of the mode spectrum. In Appendix B, we make an equivalent second order truncation in Lagrangian coordinates and show that an equivalent time evolution occurs. In Appendix C, we examine the behavior of the coefficients for small values of the Laplace parameter, $q$, and show that Eq. (12) is analytic in a neighborhood of order $1/N_m$ about the isolated pole at $q = 0$. In Appendix D, we examine the behavior of $\hat{T}(x,q)$ at the real and removable singularities. In Appendix F, we discuss the modifications which result if the eigenmode frequencies, $\omega_{m,n}$ are random, but time independent.
II. SUMMARY OF PREVIOUS FORMAL ANALYSES

This same quasilinear equation, Eq. (6), was originally derived and analyzed by Rosenbluth et al. in Ref. 2. Rosenbluth et al. found that for very small perturbations, a new modified equilibrium state is reached with the new equilibrium temperature gradient vanishing at each excited rational surface. As the perturbations increase, the resonance layers overlap. In Ref. 2, an effective diffusion equation was derived for this “overlapping resonance” regime. We rigorously rederive and systematize the results of Ref. 2 for the “isolated resonance” regime. In contrast, in the “overlapping resonance” regime, we continue to find nearby saturated temperature states which do not decay on a slow diffusive timescale.

We briefly review the standard derivation\(^2\) of an effective diffusion equation for Eq. (6), emphasizing the differences between the traditional approach and a careful multiple scale analysis. We note that no part of the following arguments of Ref. 2 uses the Chirikov criteria of resonant layer overlap.

Since the evolution of the slowly varying part of \(T_o(x, t)\) is of primary interest, previous calculations ignored the rapidly varying part of \(T_o(x, t)\), replaced \(T_o(x, s)\) by \(T_o(x, t)\), and performed the temporal integration. This formal step is called the “Markovian approximation” in the turbulence literature and is one of the principal objects of our investigation. The resulting heat equation has a time dependent heat conductivity:

\[
\kappa(x, t) = \sum_{m,n \neq (0,0)} \frac{c_T^2 L^2}{\delta a^2} A_{m,n}(x) \frac{\sin(\mu_{m,n}(x) \frac{t}{L})}{\mu_{m,n}(x)}. \tag{7}
\]

In general, this approximation is invalid because the rapidly varying kernel will induce rapid variations in \(T_o\). Although these variations in \(T_o\) are small, the variations in the gradients of \(T_o\) are order one and therefore influence the time evolution of \(T_o(x, t)\).

The second step in the traditional analysis is to approximate the sum over toroidal wave number by an integral over \(n\) space. This “continuum” approximation is only valid when the integrand is a slowly varying function of \(n\). Unfortunately, the inte-
grand is a rapidly varying function of \( n \).

Naively, one hopes that the “Markovian approximation” and the “continuum approximation” would become valid in the limit of large shear. If the shear were to be scaled such that it tended to infinity as the modes strengths tended to zero, the density of mode rational surfaces would become large and the kernel term, \( \cos(\mu_{m,n}(x)\tilde{t}/\delta) \) would be rapidly varying away from the resonance surfaces, \( x_{mn} \). However the “Markovian approximation” and the “continuum approximation” are questionable even in the large shear limit, because the selfconsistent solution, \( T_o(x,t) \), will also vary rapidly as the shear tends to infinity.

The third step in the standard approach is to replace \( \kappa(x,t) \) by its limiting form in the long time and spatially averaged limit. In Ref. 2, this step is divided into two parts. First, Rosenbluth et al. state that \( \sin(\alpha t)/\alpha \to \pi \delta(\alpha) \) as \( t \to \infty \). Implicitly, Rosenbluth et al. are assuming that the spectrum, \( A_{m,n} \) depends only weakly on \( n \). We do not understand this third formal step of Ref. 2 if the toroidal harmonics are different. Therefore we restrict our to \( A_{m,n} \equiv A_m + o(1) \) and \( v_z(x) \equiv 1 \) in the rest of this paragraph. Thus Rosenbluth et al. replace Eq. (7) with

\[
\kappa_{lim} = \frac{\pi \varepsilon_v^2}{\delta} \left( \frac{L}{a} \right)^2 \sum_{m \neq 0} \frac{A_{mn}(x_{mn})}{\mu'} \delta(x - x_{mn}) . \tag{8a}
\]

Finally, Rosenbluth et al. spatially average their expression for \( \kappa(x) \):

\[
\kappa_{lim} = \frac{\pi \varepsilon_v^2}{\delta \mu'} \left( \frac{L}{a} \right)^2 \sum_{m \neq 0} A_m(x_{mn}) . \tag{8b}
\]

We believe that this complicated third step is trying to replace the rapidly varying \( \kappa(x,t) \) by a smooth limiting function. However, \( \kappa(x,t) \) tends only weakly, not pointwise, to \( \kappa_{lim} \). The mathematical theory of homogenization\(^6^{--}\)\(^8\) has shown that the limiting solutions to rapidly varying heat equations do not, in general, converge to the solution corresponding to the limiting heat conductivity \( \kappa_{lim} \).

Krommes et al.\(^5\) have reanalyzed a similar problem: the motion of particles on chaotic field lines with a small random walk superimposed. We note that the renor-
malization procedure proposed by Krommes has never been performed in any detailed or explicit fashion and thus unexpected difficulties may occur. Furthermore, the renormalization theory is purely formal and makes many of the same formal manipulations as the quasilinear theory. Thus the difficulties of the quasilinear theory, which we examine in this article, need to be addressed in renormalization theories as well.

Krommes replaces the continuum approximation for toroidal mode numbers with a continuum approximation for $k_\parallel$, because he considers that the kernel varies more slowly with respect to $k_\parallel$ than $k_z$. In reality, the kernel has order one dependencies on $k_\parallel$ for finite time and fast variation with respect to $k_\parallel$ for long times. The $k_\parallel$ continuum approximation has the other difficulties that $k_\parallel$ is a spatially varying function and that the Fourier modes are unlikely to be averagable in the $k_\parallel - k_\theta$ plane. In other words the density of modes is nonuniform in the $k_\parallel - k_\theta$ plane and there is no symmetry such as the “equivalence of harmonics”.$^9$–$^{12}$

A number of articles study the 1 1/2 degree of freedom analog of the passive scalar equation and report to derive quasilinear diffusion coefficients.$^{13}$–$^{14}$ These authors consider a single set of resonances in the limit that the individual mode amplitudes tend to infinity ($\epsilon_v \to \infty$). Our limit, many small amplitude modes interacting with each other by modifying the underlying equilibrium seems to not only be the more physically relevant limit, but also the limit which is more appropriately termed quasilinear. This $\epsilon_v \to \infty$ limit corresponds to scaling the velocity shear to infinity as $\vec{v}_o(x) \sim 1/\epsilon_v^2$! In the ‘derivation’ of Rechester and White for this infinitely large shear limit, they replace rapidly oscillating coefficients with their weak limit. As in Ref. 2, the solution of the limiting equation need not be the limit of the solutions to the actual equations.
III. PROPERTIES OF THE QUASILINEAR EQUATION

In this section, we derive some fundamental mathematical properties of Eq. (6); in particular, that the spectrum of Eq. (6) is purely continuous and oscillatory except for the steady state eigenmode. Equation (6) is a convolution differential equation of the form:

\[ \frac{\partial T}{\partial t}(x, t) = \frac{\partial}{\partial x} \int_0^t K(x, \frac{t-s}{\delta}) \frac{\partial T}{\partial x}(x, s) ds, \]  

(9)

where we have suppressed the subscript, \( o \). The time evolution of the mean squared profile satisfies

\[ \frac{\partial}{\partial t} \int_{-\infty}^{\infty} T^2(x, t) = -\int_0^t \int_{-\infty}^{\infty} \frac{\partial T}{\partial x}(x, \bar{t}) K(x, \frac{\bar{t}-s}{\delta}) \frac{\partial T}{\partial x}(x, s) ds. \]  

(10)

Since \( K(x, \frac{\bar{t}-s}{\delta}) \) is positive at the resonance surfaces and is rapidly oscillating elsewhere, Eq. (10) suggests that \( T(x, \bar{t}) \) might phase mix and decay. However, Eq. (10) also allows predominantly static solutions where the gradient of \( T \) vanishes at all resonances. Furthermore, this simple analysis neglects the rapidly oscillating part of \( T(x, \bar{t}) \).

Provided that \( \frac{\partial T}{\partial t}(x, \bar{t}) \) is exponentially bounded, this convolution equation may be simplified using the Laplace transform,

\[ \hat{T}(x, q) = i \int_0^\infty e^{-iq\bar{t}} T(x, \bar{t}) d\bar{t}, \quad T(x, \bar{t}) = \frac{1}{2\pi i} \int_{-ic-i\infty}^{-ic+i\infty} e^{iq\bar{t}} \hat{T}(x, q) dq. \]  

(11)

For convenience, we have rotated the Laplace variable plane by making the substitution, \( p \equiv iq \), and taken \( \hat{T}(x, q) \equiv i\hat{T}(x, p) \), where \( p \) is the normal Laplace transform variable. This rotates the contour of integration to an integration below the real \( q \) axis. Noting that \( \frac{\partial}{\partial p} \) is transformed to \( \frac{\partial}{\partial q} \), the equation for \( \hat{T}(x, q) \) becomes:

\[ \frac{\partial}{\partial x} \hat{D}(x, q) \frac{\partial \hat{T}}{\partial x}(x, q) - \hat{T}(x, q) = -T(x, t = 0)/q, \]  

(12)

where \( \hat{D}(x, q) = \frac{\epsilon^2 L^2}{a^2} \sum_{m,n \neq (0,0)} \frac{A_{m,n}(x)}{(\mu_{m,n}(x))^2 - \delta^2 q^2} \),

(13)
and $\hat{K}(x,q) = q\hat{D}(x,q)$. The kernel, $K(x,t)$, of Eq. (6) has the special property that $\hat{K}(x,q)$ vanishes everywhere at $q = 0$. Therefore the substitution, $\hat{K}(x,q) = q\hat{D}(x,q)$, is justified. An immediate consequence is that $\hat{T}(x,q)$ has a pole at $q = 0$ for all values of $x$. Since we will later show that this is an isolated singularity, the pole at $q = 0$ generates the time independent part of the solution.

Equation (12) possesses the following symmetries: $\hat{T}(x,-q) = -\hat{T}(x,q)$ and $\hat{T}(x,q^*) = \hat{T}^*(x,q)$. The second symmetry guarantees the reality of $T(x,t)$. We define the Doppler shifted resonances, $x_{m,n}^\pm(q)$, by $\mu(x_{m,n}^\pm(q)) = \pm\delta q$. Note that $x_{m,n}^\pm(q = 0) \equiv x_{m,n}$.

We examine wellposedness and solution properties of the quasilinear Eq. (12). Representations of the solutions are given in Sec. V and Appendix E. We begin by studying the solution for large values of the Laplace parameter $q$. Expanding Eq. (12) in powers of $1/q$, we find that $q\hat{T}(x,q) - T(x,t = 0) = O(1/q^2)$ and therefore $\partial T/\partial t$ exists and is exponentially bounded.

The poles of $\hat{D}(x,q)$ generate removable singularities of Eq. (12). To show this, we make the substitution, $\hat{Y}(x,q) \equiv \hat{D}(x,q)\hat{T}(x,q)$ which yields

$$\frac{\partial^2 \hat{Y}}{\partial x^2}(x,q) - \hat{D}(x,q)^{-1}\hat{Y}(x,q) = \frac{1}{q}\frac{\partial T}{\partial x}(x,t = 0).$$

(14)

Thus $\partial^2 \hat{Y}/\partial x^2(x,q)$ vanishes at the poles of $\hat{D}(x,q)$. Away from the zeros of $\hat{K}(x,q)$, Eq. (12) is clearly wellposed and depends analytically on the Laplace parameter $q$. At the double poles at $q = 0$, $\hat{T}(x,q)$ remains analytic in $q$ and both $\partial_x \hat{T}(x,q)$ and $\partial_x^2 \hat{T}(x,q)$ vanish at $x_{m,n}(q = 0)$. Thus $\hat{D}(x,q)^{-1}$ plays the role of a quantum mechanical potential, $V(x,q) \equiv \hat{D}(x,q)^{-1}$.

Both Eq. (12) and Eq. (14) are formally selfadjoint equations of the form $(a(x)u')' - a_2(x)u = f(x)$. Therefore we use $u(x)$ to denote either $\hat{T}(x,q)$ or $\hat{Y}(x,q)$. To construct the Greens function, we let $u_L(x,q)$ be a solution of $(a(x)u')' - a_2(x)u = 0$, which satisfies the left boundary condition, and $u_R(x,q)$ satisfy the right boundary condition. The Greens function representation is:
\begin{equation}
A(q)u(x, q) = u_R(x, q) \int_a^x u_L(\xi, q)f(\xi, q)d\xi + u_L(x, q)\int_x^b u_R(\xi, q)f(\xi, q)d\xi,
\end{equation}
where \( A(q) \equiv a_o(x, q)[u'_L(x, q)u_R(x, q) - u_L(x, q)u'_R(x, q)] \) is independent of \( x \).

To examine the existence of normal modes, we multiply Eqs. (12) and (14) by \( \hat{T}^*(x, q) \) and \( \hat{Y}^*(x, q) \) respectively and integrate by parts. The imaginary part of \( \hat{D}(x, q) \) never vanishes except on the real and imaginary \( q \) axes. Thus the imaginary part of the variational principle prevents the existence of normal modes. Clearly no normal modes occur when \( Re(q) = 0 \) and \( q \neq 0 \). Thus normal modes can occur only on the real \( q \) axis.

For \( Im(q) = 0 \), the energy in the variational principle is unbounded unless the normal mode is analytic at each resonance surface. For two or more resonance surfaces, this behavior is nongeneric. The same arguments apply to the normal modes of Eq. (14).

The contour of integration in the inverse Laplace transform can clearly be deformed up to the real \( q \) axis and be replaced by a contour around the continuous spectrum on the real \( q \) axis. The symmetries of \( \hat{T}(x, q) \) imply that Eq. (11) can be rewritten as

\begin{equation}
T(x, \bar{t}) = \frac{4}{2\pi} lim_{\epsilon \rightarrow 0} \int_0^\infty Imag(\hat{T}(x, q_R - i\epsilon)cosq_R\bar{t})dq_R.
\end{equation}

IV. HOMOGENIZATION OF RAPIDLY VARYING EQUATIONS

Homogenization theory\(^6\text{-}^8\) considers an analogous problem of estimating the effective heat conductivity of a composite material with a rapidly varying thermal conductivity. Due to the excitation of rapidly varying, order one gradients, the homogenized thermal conductivity is the harmonic and not the arithmetic mean. A more precise statement of the basic theorem of homogenization follows. Consider the sequence of elliptic problems,

\begin{equation}
\frac{\partial}{\partial x} \kappa(\frac{x}{\epsilon}) \frac{\partial T_\epsilon}{\partial x} - T_\epsilon = f(x),
\end{equation}
where $\kappa(x) \varepsilon$ is a quasiperiodic function of $x$ with $\alpha_1 > \kappa(x) > \alpha_0 > 0$. $T_\varepsilon$ tends weakly in $H_1$ to $T_h$, where $T_h$ solves the homogenized equation, i.e. $T_h$ is the solution of Eq. (17) with $\kappa(x)$ replaced by its harmonic mean: $\kappa_h^{-1} \equiv 1/\kappa(x)$. Convergence in $H_1$ means that $\partial T_\varepsilon / \partial x$ converges weakly to $\partial T_h / \partial x$ and this implies $T_\varepsilon$ converges pointwise to $T_h$.

More generally, we consider a family of conductivities, $\kappa_\varepsilon(x)$. A general theory of the convergence of subsequences of the solutions to the elliptic problem, Eq. (17), as $\varepsilon$ tends to zero, has been developed, and is termed ‘H’ convergence. Since we consider sequences of oscillating solutions, $T_\varepsilon(x)$, weak convergence of the gradient of $T_\varepsilon(x)$ is to be expected. From linear operator theory, weak convergence of all sequences $T_\varepsilon(x)$, where $L_\varepsilon T_\varepsilon(x) = f(x)$ and $f(x)$ is arbitrary, in the Hilbert space $H_1$ is equivalent to the convergence of the operators, $L_\varepsilon^{-1}$, in the weak * operator topology. For one dimensional elliptic problems, the main result of ‘H’ convergence implies that if $\kappa_\varepsilon(x)^{-1}$ converges weakly to its average, then $T_\varepsilon(x)$ converges weakly in $H_1$ to the solution of Eq. (17) with the harmonic mean of $\kappa_\varepsilon(x)$, i.e. $\partial_x \kappa_h \partial_x T_h(x) = f(x)$.

For our problem, we are interested in the weak convergence of solutions of Eq. (12) as $\hat{D}_\varepsilon(x, q)$ is rescaled. We wish to derive a similar effective equation for Eq. (12). To do this, we must specify a family of possible rescalings of $\hat{D}_\varepsilon(x, q)$. The interesting rescalings of $\hat{D}_\varepsilon(x, q)$ are those rescalings where more and more resonant islands appear and the domain has no remaining closed flux surfaces. In the remainder of this article, we show that the physical rescalings of the mode amplitudes, $\{A_{mn}\}$, result in equations where the solutions of $\hat{D}_\varepsilon(x, q)$ converge weakly to an effective equation with zero diffusivity.

We would like to homogenize $\hat{T}_\varepsilon(x, q)$ uniformly in $q$, and then invert the Laplace transform to find an effective time evolution equation for Eq. (6). However, Eq. (12) differs from the standard case of Eq. (17), since $\hat{D}(x, q)$ has zeros and poles and is formally small. Direct homogenization of Eq. (12) should still be possible when $\hat{D}(x, q)^{-1}$ is integrable.
For extremely weak perturbations, the quasilinear diffusion term is small away from the Doppler shifted resonances, i.e. $\hat{D} \sim O(\varepsilon^2 N_m)$. At each Doppler shifted resonance, $x_{m,n}(q)$, a small resonance layer of width $\delta x$ forms in which $\partial_x \hat{D} \partial_x$ becomes order one. The multiple scale analysis of Eq. (12) separates into two cases: the “isolated resonance” regime and the “overlapping resonance” regime.

V. MULTIPLE SCALE ANALYSIS OF OVERLAPPING RESONANCES

The overlapping resonance case is easier to analyze and more physically relevant. Thus we defer the isolated resonance case to Appendix E. In the “isolated resonance” regime, away from the Doppler shifted resonances, $\partial_x \hat{D} \partial_x \hat{T}(x, q)$ is a small correction to the solution, $\hat{T}(x, q) \sim T(x, t = 0)/q$. Near $x_{m,n}(q)$, this ordering breaks down due to the resonance denominator in $\hat{D}$. As shown in Appendix D, the resonance layer width scales as $\delta x \sim (\varepsilon_v/\varepsilon_L)^{2/3}/N_n$ for $q \sim O(1)$, and as $\delta x \sim N^{1/3}_m (\varepsilon_v/\varepsilon_L)^{2/3}/N_n$ for $q \sim O(1/N_m)$, and as $\delta x \sim (\varepsilon_v/\varepsilon_L)^{1/2}/N_n$ for $q = 0$.

The treatment of the poles as isolated singularities requires that the average distance between resonances exceed this resonance layer width. Since the average distance between resonances is $1/N_m N_n$, the resonances decouple if $N^{3/2}_m \varepsilon_v/\varepsilon_L << 1$ for $q \sim O(1)$. For $q \sim O(1/N_m)$, the isolated resonance criteria is $N^2_m \varepsilon_v/\varepsilon_L << 1$ and at $q = 0$, the criteria is again $N^2_m \varepsilon_v/\varepsilon_L << 1$.

We now restrict our consideration to the overlapping resonance case, where $1/N_m N_m$ exceeds the isolated resonance layer width. There are approximately $N_R \equiv N_m N_n \delta x$ resonances in the layer width. In this case, the scalelength of rapid variation in $\hat{D}(x, q)$ and $\hat{T}$ is $\varepsilon_m \equiv 1/N_n N_m$. The appropriate maximal ordering is $\varepsilon_m \sim \varepsilon^2_v N_m$.

We begin our multiple scale expansion by defining $y = \varepsilon_m x$ and expanding $\hat{T}_\varepsilon$ as $T_o + \varepsilon_m T_1 + \varepsilon^2_m T_2$. Spatial differentiation has the expansion, $1/\varepsilon_m \partial_y + \partial_x$. We expand the differential operator as $\mathbf{L} = \mathbf{L}_0 + \varepsilon_m \mathbf{L}_1 + \varepsilon^2_m \mathbf{L}_2$ where $\mathbf{L}_0 T \equiv \partial_y \hat{D} \partial_y T$ and $\mathbf{L}_1 T \equiv \partial_x \hat{D} \partial_x T$ and $\mathbf{L}_2 \equiv \partial_x \hat{D} \partial_x$.

The zeroth order equation shows that $T_o$ is only a function of $x$. The first order
equation is \( L_0 T_1 + L_1 T_o(x) = 0 \). The averaged first order equation yields \( T_o = T(x, t = 0)/q \).

From the rapidly varying part, we have \( T_1 = T_1(x) + \chi \partial_x T_o \), where \( \chi \) is the solution of \( L_0 \chi = -\partial_y \hat{D} \). This equation may be integrated to yield \( T_o = T(x, t = 0)/q \).

From the rapidly varying part, we have \( T_1 = T_1(x) + \chi \partial_x T_o \), where \( \chi \) is the solution of \( L_0 \chi = -\partial_y \hat{D} \). This equation may be integrated to yield \( \partial_y \chi = \frac{1}{\hat{D}} \). For \( \chi \) to vary only on the fast spatial scale, the constant must be the harmonic mean, \( D_h \), of \( \hat{D} \), i.e. \( \overline{D_h^{-1}}(q) \equiv \frac{1}{\hat{D}(x, q)} \). Thus \( \partial_y \chi = \frac{D_h^{-1}}{\hat{D}} - 1 \), and \( \chi = f'(\overline{D_h^{-1}}/\hat{D} - 1) \).

To have \( T_{\epsilon_\nu, \epsilon_L}(x, q) \) converge to a limiting temperature, \( T_{lim}(x, q) \), pointwise, \( \chi_{\epsilon_\nu, \epsilon_L}(x, q) \) must tend to zero. Since \( \partial_y \chi_{\epsilon_\nu, \epsilon_L}(y, q) \) is \( O(1) \), \( \chi_{\epsilon_\nu, \epsilon_L}(x, q) \) will tend to zero pointwise if and only if

\[
\int_a^x dy \left( \frac{\overline{D_h^{-1}}(q)}{\partial_y \hat{D}(y, q)} - 1 \right) \to 0, \tag{18}
\]

i.e. weak convergence of \( \hat{D}(x, q)^{-1} \) to \( \overline{D_h}(q)^{-1} \). Weaker hypotheses will result in weak convergence in the sense of distributional limits. The separation of scales should be sufficient to ensure the existence of a weak limit. Since we are primarily interested in the limiting \( T(x, t) \), a distributional limit with respect to the Laplace parameter is probably sufficient for our needs.

To the first order, the temperature gradient vanishes at all resonance surfaces:

\[
\partial_x \hat{T}(x, q) = \frac{\overline{D_h^{-1}}(q) \partial_x T(x, t = 0)}{\hat{D}(x, q) q}. \tag{19OB}
\]

The average of the second order equation is \( \langle T_1(x) \rangle = \partial_x \langle \hat{D} [1 + \partial_y \chi] \rangle > \partial_x T_o \), which reduces to \( \langle T_1(x, q) \rangle = \partial_x \overline{D_h(q)} \partial_x T_o \).

The harmonic mean has logarithmic discontinuity across the real \( q \) axis whenever \( \hat{D}(x, q) \) vanishes. When \( \hat{D}(x, q) \) vanishes, our ordering assumption for the overlapping resonance regime is not strictly correct. In this case, the correct ordering is locally identical with the isolated resonance regime. Since the zeros of \( \hat{D} \) generate integrable singularities, the preceding asymptotic expressions should be valid. Our multiple scale expansion correctly captures the behavior of \( \hat{T}(x, q) \) at the zeros and poles of \( \hat{D}(x, q) \).
At the double zeros, $\hat{D}$ vanishes too strongly to ignore the $-\chi$ term. In fact, the expression is meaningless since $\overline{D}_h(q)$ is undefined. The presence of the second term in Eq. (12) removes the nonintegrable singularities of $\hat{D}$. At the double zeros, when $\partial^2_x \hat{D} \gg 1$, the initial value basis solutions, $T_L$ and $T_R$, have singularities of the form $1/(x-z)$. From the Green’s function representation, we see that $\hat{T}$ has logarithmic singularities away from the double zero.

When the two zeros are only slightly displaced, we expect that the solution will approximate $x^\lambda$ by superimposing two logarithmic singularities. The characteristic width for this strong interaction is $z_{m,n}(q) - z_{\overline{m},\overline{n}}(q) = \delta x$. Surprisingly, this never occurs, since the resonance layer decreases proportionally to $\partial_x \hat{D} \sim z_{m,n}(q) - z_{\overline{m},\overline{n}}(q)$. Thus this stronger $x^\lambda$ singularity manifests itself only at the discrete frequencies where pairs of zeros coincide.

The preceding multiple scale expansion was based on the ordering, $\hat{D}/\epsilon_m^2 \gg 1$. For large values of the Laplace parameter, $q$, and spatially localized modes, $A_{m,n}(x/n)$, the preceding expansion must be modified and the correct expansion resembles that of Appendix E. This may indicate extremely complex behavior for short times.

**VI. SINGULARITIES AND LONG TIME ASYMPTOTICS OF THE OVERLAPPING RESONANCE REGIME**

To determine the long time behavior of $T(x, t)$, we analyze the singularities of $\hat{T}(x, q)$. We consider the homogeneous version of Eq. (12) and let $\hat{T}_L(x, q)$ be a solution which satisfies the left boundary condition and let $\hat{T}_R(x, q)$ satisfy the right boundary condition. From the Green’s function representation of $\hat{T}(x, q)$, Eq. (15), $\hat{T}(x, q)$ has a Mittag-Leffler type expansion of the form:

$$\hat{T}(x, q) = \frac{T_f(x)}{q} + \sum_{m,n,\pm} T_{pv}(z_{m,n}^\pm(q), q) ln(x - z_{m,n}^\pm(q)) + \hat{T}_S(x, q) ,$$

where the subscript, $pv$, denotes the principal value and $\hat{T}_S(x, q)$ has at worst $x ln(x)$ singularities. (See Appendix D.) The multiple scale expansion of Sec. V are of this form. At the isolated points where double zeros of $\hat{D}(x, q)$ occur, the logarithmic
singularities in the second term should be replaced by \((x - z_{mn}^\pm(q))^\lambda\) where \(\lambda\) is calculated in Appendix D. For simplicity, we begin our analysis of the time evolution of \(T(x, t)\) by temporarily omitting the terms arising from the degenerate resonances at the double zeros of \(\hat{D}(x, q)\).

Since we are interested in the slow evolution, we remove the rapid time oscillations by filtering. We define \(T_W(x, t) \equiv T(x, t - s) * W(s)\) where \(W(\cdot)\) is a filter with bandwidth \(c\). The standard asymptotic expansion for long times is

\[
T_W(x, t) \equiv \frac{1}{2\pi i} \oint e^{iqt} \hat{T}(x, q) \hat{W}(q) dq = T_f(x) + \frac{-1}{2\pi i} \oint e^{iqt} \hat{T}(x, q) \hat{W}(q) dq .
\]

(21)

As shown in Appendix C, \(\hat{T}(x, q)\) is analytic for \(q^2 < \min_{mn} A_{mn}/C_{mn} \sim 1/N_m\), except for the isolated pole at \(q = 0\). Thus the second contour integral does not encircle these values of \(q\).

If the bandwidth of the filter is \(O(1/N_m)\) or smaller, the contour integral, representing the time dependent part of the solution, is exponentially small and the time averaged state is effectively in the new saturated state.

To examine the time dependent behavior, we need to use filter bandwidths greater than \(O(1/N_m)\). To determine the leading order time dependent behavior, we integrate Eq. (21) by parts:

\[
T_W(x, t) = T_f(x) + \frac{-1}{2\pi i} \oint \frac{e^{iqt} \partial(\hat{T}\hat{W})}{i\hat{T}} (x, q) dq .
\]

(22)

The \(q\) scalelength for the variation of \(\hat{T}(x, q)\) is \(1/N_m\). If the bandwidth of the filter is larger, then \(\hat{W}(q)\) may be removed from the \(q\) derivative. Thus the leading order long time behavior of \(T_W(x, t)\) is

\[
T_W(x, t) \sim T_f(x) + \frac{1}{T} \sum_{m,n,\pm} T_{mn}(x, q_{mn}^\pm(x)) \hat{W}(q_{mn}^\pm(x)) \exp(i\hat{T}q_{mn}^\pm(x)) + O(1/T) .
\]

(23)

In the overlapping resonance case, the leading order term is
\[ T_{pw}(x, q_{mn}^\pm(x)) \sim <D^{-1}(x, q_{mn}^\pm(x)) >^{-1} \frac{T'(x, t = 0)}{q_{mn}^\pm(x)} \]  

(24)

In general, the remainder terms in the integrand satisfy \( \partial_q \hat{T}(x, q) \sim O(N_m \hat{T}(x, q)) \). Thus the size of the time dependent terms relative to the \( T_f(x) - T(x, t = 0) \) scales as \( (N_m/t) \). This corresponds to temperature oscillations persisting until a time of order \( O(N_m) \) before phase mixing. *This time scale is also the time in which the argument of the exponential, \( \exp(i\mu_{mn}t) \), becomes oscillatory on the length scale of \( 1/N_m N_n \), the distance between harmonics.* We cannot expect phase mixing to a Cantor set-like gradient to dominate the time evolution on time scales which cannot distinguish between these neighboring resonances.

The strength of the oscillations for times much longer than the ideal time and approaching \( t \sim N_m \) depends on the behavior of \( \hat{T}(x, q) \) for small \( q \). It is possible that the amplitude of the oscillations decays considerably in the time range \( 1 \ll t < N_m \). Note that our estimate of the time to reach the new steady state is much shorter than the naive diffusion time of \( N_n/\epsilon^2 \). On the longer diffusive timescale, \( \bar{v}t/L > N_m \), these rapidly decaying terms are only of slight interest.

Our estimate of the size and timescale of the decay of the time dependent perturbation is unoptimized, since we have not used the more detailed asymptotic expansions of Sec. V and Appendix E in our bound on the integrand \( |\partial_q \hat{T}| \). In a future publication, we hope to apply our asymptotic representations of \( \hat{T}(x, q) \) to more precisely analyze the time evolution of \( T(x, t) \) to \( T_f(x) \).

The spatially averaged or slowly varying temperature, \( T_S \), satisfies

\[ \partial_x \hat{T}(q_{mn}(x)) - \hat{T}(q_{mn}(x)) = -T(x, t = 0)/q , \]  

(25)

which corresponds to the evolution equation:

\[ \partial_t T_S(x, t) = \partial_x \int_{s=0}^t K_h(t - s) \partial_x T_S(x, s) ds . \]  

(26)

20
The averaged equation also approaches a steady state as $t \to \infty$. Unfortunately, we are unable to simplify this spatially averaged convolution equation further due to the complicated form of $K_h(t)$. Eqs. (25)-(26) neglect the effect of the double zeros of $\hat{D}(x,q)$, and they may modify the form of Eqs. (25)-(26).

We now present our understanding of the effect of the degenerate resonances at the double zeros of $\hat{D}$ on the time evolution of $T(x,t)$. In the case of overlapping resonances ($|\partial_x^2 \hat{D}(x,q)| \gg 1$), $\hat{T}(x,q)$ has a singularity of order $1/(x - z_{mn})$. So we expect the oscillations at the double zero locations to damp extremely slowly, if they damp at all. In the isolated resonance case ($|\partial_x^2 \hat{D}(x,q)| \ll 1$), the solutions of the homogeneous equation have $x^{-1/2 + \text{ic}}$, where $|c| \gg 1$. Thus we expect the oscillations of $T(x,t)$ at these points to phase mix to zero at least as fast as $t^{-1/2}$.

However, the precise effect of the large complex exponent in the singularity requires further study. The conclusion that the time averaged state, $T_W$, reaches a new Cantor set-like equilibrium is independent of the behavior at the double zeros, since we can filter out this behavior.

In our simple equation, we are able to average spatially before analyzing the time dependent behavior. In general, this type of averaged equation will only occur when the spatial variation is much stronger than the temporal variation.

VII. DISCUSSION

In conclusion, we have considered the evolution of a passive scalar field for a slowly varying stratified medium which is convected in a turbulent sheared flow within the quasilinear or weak coupling approximation. On a nearly ideal time scale, $\tilde{t} \bar{v}/L \sim N_m$, the fluid reaches a new saturated state. The fluid remains in this modified equilibrium and does not evolve diffusively. Our multiple scale expansion shows the final saturated state is
\[ T(x, t = \infty) \sim T(x, t = 0) + \left[ \int_a^x \left( \frac{\hat{D}_h(q = 0)}{\hat{D}(x, 0)} - 1.0 \right) dx \right] \partial_x T_o + \partial_x \hat{D}_h(q = 0) \partial_x T_o. \]  

(27)

Near the resonances, \( \delta T \) is order \( \delta x_q = o \), and \( \partial_x T_o \) vanishes at each rational surface. Since the growth of the harmonic, \( \tilde{T}_{m,n} \), is proportional to \( \partial_x T_o \), \( \tilde{T}_{m,n} \) saturates at each rational surface instead of growing linearly in time.

The resulting saturated state has a “Cantor set” structure in the gradient of the temperature. Such quasilinear Cantor set gradient saturation has been found previously in self-consistent pressure gradient driven turbulence in tokamaks\(^9\)\(^{-12}\). However the quasilinear ballooning mode calculations had an infinite number of poloidal harmonics, but only one\(^9,10\) or a small number of toroidal harmonics\(^11,12\). In the present analysis, we have considered situations where the density of toroidal harmonics tends to infinity as well.

The long time solution of Eq. (6) is determined by balancing the transfer of energy between the saturated state, \( T_f(x) \) and the oscillations, \( \tilde{T}(x, t) \equiv T(x, t) - T_f(x) \).

The rapidly varying part of the kernel, \( \tilde{K}(x, t) \), continuously excites temperature oscillations from the equilibrium gradient. Initially, this transfer is large due to the net coupling at the resonance surfaces. However, the gradients at the rational surfaces rapidly flatten to zero and thereby reduce the excitation to a small manageable (balancable) level.

The standard weak turbulence theory of Galeev and Sagdeev\(^3\) consists of four successive formal steps. First, in the weak coupling or random phase approximation, the nonlinear interaction is truncated. Second, the Markovian approximation replaces time history integrals with the local time. Third, the continuum approximation replaces the discrete sum over mode numbers, \( \sum_k \) with an integral over \( k \) space, \( \int dk \).

Fourth, the rapidly varying diffusion coefficient, \( \kappa(x, t) \), is replaced by its weak limit, i.e. its arithmetic average. In this article, \( \textit{we have shown that our model problem violates the last three formal steps, given the truncated equations of the first formal} \)
In most turbulence formalisms, simplifications of the dynamics are justified by hypothesizing statistical properties of the systems. If the hypothesized system properties are not good approximations, the resulting analysis will fail. Furthermore, these theories do not address the domain of validity of these properties (such as the Markovian and continuum approximations). In contrast, our multiple scale expansion yields self-consistent estimates of the validity of our “Cantor set” gradient saturated state.

Two other well known calculations, the Rechester-Rosenbluth theory\textsuperscript{4,5} of particle diffusion and the estimate of the Kolmogorov exponentiation length\textsuperscript{15} $L_K$, both apply similar analysis techniques as Ref. 2. Thus the results of these calculations could easily be inconsistent with a multiple scale expansion. In particular, nearly collisionless particles follow the field lines almost perfectly, and therefore will diffuse extremely slowly, in contrast to the collisionless Rechester-Rosenbluth diffusion. The Kolmogorov exponentiation length calculation may still be valid, since $L_K$ is a local property and the Markovian approximation failure is related to the toroidal periodicity.

Our ordering implies that the fully nonlinear term is small, and therefore may be neglected on the ideal timescale. For long times of order $N_m$, the nonlinear term may significantly modify the asymptotic behavior of the system. First, the fully nonlinear term will induce an intrinsically nonlinear scattering and diffusion. Secondly and probably more importantly, the nonlinear term will decorrelate the quasilinear trajectories as they repeatedly encircle the torus.

Much of the nonstandard behavior of our solution arises from the infinite memory and oscillatory kernel of Eq. (6). For long wavelength modes, the kernel is initially of order $O(c_w^2 N_m N_n)$. However, on the velocity shear timescale, the nonresonant contribution will decay away, and only a resonant part of the kernel, of order $O(c_v^2 N_m)$, remains time asymptotically. We note that the Markovian approximation corresponds
to replacing the oscillatory cosine kernel, \( \cos(\mu_{m,n}(x)\frac{(s-t)}{\delta}) \), by an exponentially decaying kernel such as \( \exp(\mu_{m,n}(x)\frac{(s-t)}{\delta}) \). Thus the Markovian approximation cuts off the kernel after the first oscillation and ignores the further oscillations. Therefore it ignores the resonant contribution to the kernel.

The nonlinear term should decorrelate this infinite memory and replace it with an exponentially decaying kernel. If we make the common assumption that the nonlinear decorrelation may be modeled by a simple decorrelation time, \( \tau_K \), the model equation becomes:

\[
\frac{\partial T_o}{\partial t} = \epsilon^2 \frac{\partial}{\partial x} \sum_{m,n} A_{m,n}(x) \int_{s=0}^{\tau} \cos(\mu_{m,n}(x)\frac{(s-t)}{\delta}) \exp(\frac{(s-t)}{\tau_K(\epsilon_v, \epsilon_L)}) \frac{\partial T_o}{\partial x}(x,s) ds .
\] (28)

The model of Eq. (28) is not equivalent to assuming the random velocity field has a slowly decaying autocorrelation function, \( <\tilde{v}(x,t), \tilde{v}(x,s)> \sim A_{mn}(x) \exp((t-s)/\tau_k) \). Equation (28) corresponds to the ensemble average of the equations for various realizations of the probabilistic velocity field with the given autocorrelation function. However, the solution to the ensemble averaged equation is not necessarily the ensemble averaged solution. Furthermore, we are interested in the ordering where the nonlinear decorrelation time is long with respect to the toroidal circulation time and Eq. (28) is probably only physically reasonable in the opposite limit where \( \tau_K << L/\nu \).

For large values of \( \tau_K \), the model nonlinear decorrelation of Eq. (28) allows the oscillations to slowly decay in time. Although it is possible, we strongly doubtful that this modified equation will have a \( \tau_K \) independent limit which diffuses according to the standard weak turbulence theory.

In Ref. 15, a positive Kolmogorov length is calculated and this normally implies chaotic behavior. Since we also assume a separation of scales, diffusive behavior is to be expected. Clearly, we expect that the actual physical system will experience diffusive evolution of some general type. However, the second order truncation eliminates the nonlinear decorrelation of the trajectories. The infinite memory of the kernel allows the trajectory to circulate around the torus many times and average over the
phase of perturbing field. Thus the effect of the modes phase mixes away except
at the nearly resonant surfaces and no diffusion results. Higher order calculations
are necessary to determine the strength of the nonlinear scattering and the resulting
transport.

In conclusion, the passive scalar problem differs from self-consistent turbulence in
two significant ways. First, in the Eulerian frame, the autocorrelation time of the
forcing is infinite. *Time dependent random perturbations will tend to mix the fluid
randomly and eliminate infinite memory effects.* Thus the Markovian approximation
is much more likely to be true with time dependent random perturbations or self-
consistent fluctuations.

Second, the presence of a dissipation length would prevent the quasilinear flattening
of the gradients at the smallest wavelengths. If dissipation were to be included,
we still expect that the quasilinear flattening would remain an important effect up to
the dissipation scalelength.

Experimental observations indicate that fast particles have better and not worse
confinement than thermal particles\textsuperscript{16}. On the basis of the Rechester-Rosenbluth cal-
culations, the fusion community has inferred that magnetic islands are not a signific-
ant source of transport in tokamaks. Our results show that the collisionless limit of
Rechester and Rosenbluth is incorrect. The actual dependence of particle losses as a
function of parallel velocity is unknown. Therefore no inference on the presence or
absence of islands is possible.

A similar argument has been advanced in Ref. 16. In this calculation, the net
displacement of fast particles is gyrophase averaged to zero due to the displacement
of the particle trajectories from the magnetic field line. The actual calculation in
Ref. 16 is based on the extremely questionable weak turbulence formalism of Ref.
2. We believe that a multiple scale analysis of the particle trajectories also will
show reduced displacement due to gyrophase averaging. The strength of this effect is
presently unknown.
Both our multiple scale analysis and the formal arguments of Ref. 16 find greatly reduced diffusion due to phase averaging. Our mechanism requires the trajectory to make many circuits about the torus before the orbit is decorrelated. The relative strength of toroidal phase averaging to gyrophase averaging will depend on the mode spectrum and the decorrelation mechanisms and deserves future study.

We now address the numerical estimates of diffusion in overlapping island structures. Most of the calculations report to be in either crude or good agreement with the existing weak turbulence theory. We note that both our analysis and the diffusive theory of Rosenbluth et al. have an infinite number of free parameters, i.e. the spectrum $\tilde{v}$. Often the numerical simulations fix the relative amplitudes of the modes and adjust only the total amplitude, thereby testing only a small part of the total “spectrum” of problems (pun intended).

Furthermore, the quasilinear calculations are only valid in the limit of many small overlapping modes. Naturally, this limit requires much greater numerical resolution than a small number of large perturbations. Thus it is quite possible that the majority of numerical simulations have examined cases where quasilinear theory should not be valid.

The initial decay of the autocorrelation function is on the velocity shear timescale; and this decay time is the basis for the quasilinear diffusion coefficient. Inaccurate numerical schemes will observe this initial decay, but may fail to resolve the saturation of the autocorrelation function at times of order $N_m$. At this numerical resolution, losses should correspond to the incorrect weak turbulence expression.

We note that the recent simulation by Duchs and Montvai$^{17}$ shows that the mean squared displacement saturates or evolves much more slowly than predicted by the standard quasilinear diffusion coefficient. This particular calculation is in strong qualitative agreement with our multiple scale analysis.

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APPENDIX A: PHYSICAL BASIS OF SPECTRUM RESCALING

We assume the poloidal and toroidal mode numbers, \( m \) and \( n \), are large and inversely proportional to a second small parameter, \( \epsilon_L \). In our ordering, the \( N_mN_n/2 \) modes are of roughly equal magnitude, and the amplitude scaling must satisfy \( 1/N_m^2N_n < \epsilon_v < 1/N_n\sqrt{N_m} \) for our quasilinear overlapping resonance analysis to apply. Our results should be easily extendable to a much larger class of mode spectra. The lower bound, \( 1/N_m^2N_n < \epsilon_v \) generalizes to the overlap criteria \( \sum_{m,n} \left( \frac{\epsilon_v^2|\tilde{v}_{mn}|^2}{\mu_{m,n}(x_{mn})^2} \right)^{1/4} > 1 \). The upper bound, \( \epsilon_v < 1/N_n\sqrt{N_m} \), generalizes to \( \sum_{m,n,|x-x_{mn}|<1/N_n} \epsilon_v^2n^2|\tilde{v}_{mn}|^2 << 1 \). This upper bound is crude since it is a simple estimate of the relative strength of the fully nonlinear term relative to the quasilinear term.

Our analysis does require that the mode amplitudes have explicit cutoffs at \( N_m \) and \( N_n \). Furthermore, our results on the timescale for the decay of the time dependent oscillations is in terms of the mode cutoff. For more general expressions about the temporal evolution, the multiple scale expansions need to be used directly in Eqs. (21)-(23).

Mode spectra arise from two classes of phenomena: externally driven perturbations and slowly evolving turbulent fluctuations with saturated magnetic islands. We now describe the characteristics of each class of spectra. Although the spectra have different radial scalelengths, their effective properties are nearly identical due to the localizing effect of resonant denominators.

Reference 2 considered the case where the perturbing field is externally driven and therefore \( A_{mn} \) were treated as constants. Due to the presence of resonant denominators of the form, \( 1/(nv_z(x) - m\mu(x) - \omega_{m,n}) \), the perturbations are effectively localized radially about the Doppler shifted resonance. The localization width is proportional to \( \epsilon_L \).

In turbulent situations characterized by a nonlinear cascade of energy, we expect all harmonics within a given wavelength range with rational surfaces to be excited. The higher mode numbers usually have decreasing amplitudes. This corresponds to
a sector in the \((m, n)\) plane with the number of excited modes proportional to \(1/\epsilon_L^2\).

Thus the number of excited poloidal mode numbers, \(N_m\), and excited toroidal mode
numbers, \(N_n\), both scale as \(1/\epsilon_L\). There are two cases of interest: short wavelength
and long wavelength perturbations. Short wavelength modes decay radially on the
scalelength of \(\epsilon_L\), and therefore have an effective mode number density of \(N_m\), and
an energy density of excited modes is proportional to \(\epsilon_L^2 N_m\).

Long wavelength modes decay on the macroscopic scalelength and therefore have
an effective mode number density of \(N_m N_n\), and an energy density of excited modes
is proportional to \(\epsilon_L^2 N_m N_n\). However, the underlying velocity shear causes the effect
of the long wavelength modes to phase mix away except in a small resonance layer,
where \(\vec{k} \cdot \vec{v}_o(x)\) is small. The timescale for the nonresonant terms to decouple is
the velocity shear time, \(1/\partial_x \mu(x)\). The quasilinear diffusion coefficient of Ref. 2 is
essentially the Kubo diffusion coefficient, using \(< \tilde{v}, \tilde{T} >\) in place of \(< \tilde{v}, \tilde{v} >\) with
this nonresonant decorrelation time. However the long time behavior is dominated by
the resonant layers and this modified “nonresonant” Kubo formula is not applicable.

Since the long wavelength modes are resonant in only a small region of order \(1/N_n\),
the resonant energy density is smaller, \(O(\epsilon_L^2 N_m)\).

However, we also wish to consider cases where \(N_n \sim 1/\epsilon_L\) and \(N_m\) is a fixed
subsidiary large parameter. In this case, the energy density of the the excited modes
is independent of \(\epsilon_L\) and proportional \(\epsilon_L^2\). Therefore we specifically distinguish \(N_n\)
and \(N_m\) in order to treat both cases simultaneously.

We assume that the velocity shear, \(\mu'(x) \equiv \frac{\partial}{\partial x} v'_\theta\), is order one. Our transport
becomes small when \(cos(\mu_{m,n}(x)t)\) is oscillatory on the scalelength of the mode sepa-
ration, \(1/N_m N_n\). If \(\mu'(x)\), the \(x\) coordinate, and time were rescaled so that \(\mu_{m,n}(x) \times t\)
were small, a different limiting equation would result.

A number of articles in nonlinear dynamics have considered the 1 1/2 degree of
freedom Hamiltonian system analogous to the passive scalar flow field:
\[
\vec{v}_o(x) = \hat{z} + x \hat{\theta}
\]
and
\[
\tilde{v} = \frac{1}{\epsilon_v} \tilde{v} \hat{x} \sum_{m=\infty}^{-\infty} cos(m\theta - cz + \phi_m).
\]
This flow field is a combination of a single
chain of resonant perturbations of large (not small!) amplitude. We rescale this system by changing variables: \( x_{\text{new}} \equiv \epsilon^2 x_{\text{old}} \) and thereby show that these Hamiltonian system correspond to scaling the velocity shear to infinity as \( \partial_x \vec{v}_o(x) \sim 1/\epsilon^2 \).
**APPENDIX B: LAGRANGIAN TRUNCATION**

We now perform an alternative expansion in Lagrangian coordinates. This expansion is motivated by the results of Kraichnain\(^1\) that Lagrangian direct interaction approximations are necessary to recover the famous Kolmogorov inertial range. We let \(\bar{x}(t, \bar{x}_o)\) denote the position of a particle which was located at \(\bar{x}_o\) initially. We continue to denote the stratified direction by the scalars, \(x\) and \(v\) and assume that the equilibrium velocity, \(\bar{v}_o\) and the wave vectors, \(\vec{k} \equiv (0, m/a, n/L)\), lie in the stratified plane. We expand \(\bar{x}(t, \bar{x}_o)\) as \(\bar{x}^0(t, \bar{x}_o) + \epsilon_o \bar{x}^1(t, \bar{x}_o) + \epsilon_o^2 \bar{x}^2(t, \bar{x}_o)\).

The particle trajectory satisfies

\[
\partial_t \bar{x}(t, \bar{x}_o) = \bar{v}_o(x) + \epsilon V \sum_k \bar{v}_k(x)e^{i\vec{k} \cdot \bar{x}}. 
\]

The zeroth order solution is \(\bar{x}^0(t, \bar{x}_o) = \bar{x}_o + \bar{v}(x_o)t\). The next order equation is

\[
\partial_t \bar{x}^1(t, \bar{x}_o) = \frac{\partial \bar{v}_o(x_o)}{\partial x} x^1(t, \bar{x}_o) + \sum_k \bar{v}_k(x_o)e^{i\vec{k} \cdot \bar{x}^0}. 
\]

We integrate the \(x\) component of Eq. \(B2\), \(\partial_t x^1(t, \bar{x}_o) = \sum_k \bar{v}_k(x_o)e^{i\vec{k} \cdot \bar{x}^0}\), and then substitute \(x^1(t, \bar{x}_o)\) into the \(y\) and \(z\) components to yield:

\[
\bar{x}^1(t, \bar{x}_o) = \sum_k e^{i\vec{k} \cdot \bar{x}_o} \left[ \bar{v}_k(x_o) \left( \frac{e^{i\vec{k} \cdot \bar{v}_o(x)}}{i\vec{k} \cdot \bar{v}_o(x)} - 1 \right) + \frac{\partial \bar{v}_o(x_o)}{\partial x} v_{x,k}(x_o) \left( \frac{e^{i\vec{k} \cdot \bar{v}_o(x)}}{i\vec{k} \cdot \bar{v}_o(x)} - 1 - (i\vec{k} \cdot \bar{v}_o(x)) t \right) \right] 
\]

The second order equation is

\[
\partial_t \bar{x}^2(t, \bar{x}_o) = \frac{\partial \bar{v}_o(x_o)}{\partial x} x^2(t, \bar{x}_o) + \frac{\partial^2 \bar{v}_o(x_o)}{\partial x^2} x^1(t, \bar{x}_o)^2 + \sum_k e^{i\vec{k} \cdot \bar{x}^0} \left( \partial_x \bar{v}_k(x_o) x^1 + \bar{v}_k(x_o) i\vec{k} \cdot \bar{x}^1 \right) 
\]

The normal component reduces:

\[
\partial_t x^2(t, \bar{x}_o) = \sum_k \bar{v}_k \left( v_{k,x}(x_o) \partial_x \bar{v}_{k,x}(x_o) + \bar{v}_{k,x}(x_o) i\vec{k} \cdot \bar{v}_{k'}(x_o) \right) \left( \frac{1 - e^{-i\vec{k} \cdot \bar{v}_o(x) t}}{i\vec{k} \cdot \bar{v}_o(x)} \right) 
\]

\[
+ \sum_k e^{i(\vec{k} + \vec{k}')} \cdot (\bar{v}_o(x) t + \bar{x}_o) \frac{\partial \bar{v}_o(x_o)}{\partial x} v_{k,x}(x_o) v_{k',x}(x_o) \left( \frac{1 - (1 + i\vec{k} \cdot \bar{v}_o(x) t) e^{-i\vec{k}' \cdot \bar{v}_o(x) t}}{(i\vec{k'} \cdot \bar{v}_o(x))^2} \right) 
\]

\[\text{(B5)}\]
The time independent part corresponds to $\vec{k}' \equiv -\vec{k}$ and is identically zero due to incompressibility. For a finite number of modes, these expressions imply that the mean squared displacement, $< |x(t, x_o) - x_o|^2 >$, is bounded except at the rational surfaces. This implies that there is no long time diffusion unless the number of resonance surfaces tends to infinity.

If we make the random phase assumption, the crossterms phase average away and we find the autocorrelation time is

$$< |x(t, x_o) - x_o|^2 > = 2 \sum_k \frac{|v_k(x_o)|^2}{|\vec{k} \cdot \vec{\nu}_o(x_o)|^2} \left( 1 - \cos(\vec{k} \cdot \vec{\nu}_o(x) t) \right). \quad (B6)$$

Thus the Lagrangian truncation also predicts zero diffusion away from the rational surfaces. Our saturated state has $\partial_x T(x_mn) = 0$ at all rational surfaces and therefore is compatible with the Lagrangian result. In the opposite limit of small shear time, $\vec{k} \cdot \vec{\nu}_o(x) t << 1$, we find wave-like transport: $< |x(t, x_o) - x_o|^2 > = \left( \sum_k |v_k(x_o)|^2 \right) t^2$. 

32
APPENDIX C: SMALL Q BEHAVIOR

For small values of $\delta q$, corresponding to large times, the two poles, $x_{mn}^\pm(q)$ coalesce near $x_{mn}$. We now examine $\hat{D}(x, q)$ in a small neighborhood of $x_{mn}$. To separate the resonant and nonresonant parts of $\hat{D}$, we define $C_{mn} \equiv \hat{D}(x, q) - A_{mn}(x)/(\mu_{m,n}(x)^2 - q^2)$. The scalelength for variation of the resonant part, $A_{mn}(x)/(\mu_{m,n}(x)^2 - q^2)$, is $2m\mu_\nu\mu_{m,n}(x)/(\mu_{m,n}(x)^2 - q^2)$. This is large near the poles and order $N_n$ away from the resonance surfaces.

In a small neighborhood of $x_{mn}$, only the resonant denominator variation is important and we treat the nonresonant contribution, $C_{mn}(x, q)$ as a constant, $C_{mn}$. Thus we define $C_{mn} \equiv \hat{D}(x, q) - A_{mn}(x)/(\mu_{m,n}(x)^2 - q^2)$ and suppress the $x$ and $q$ dependencies in $C_{mn}$ and $A_{mn}$. Note that $\hat{D}(x, q)^{-1}$ is approximately

$$\left( C_{mn} + \frac{A_{mn}}{\mu_{m,n}(x)^2 - q^2} \right)^{-1} = \frac{1}{C_{mn} - \frac{A_{mn}}{\mu_{m,n}(x)^2 - q^2}}$$

Thus if $q^2 > A_{mn}/C_{mn}$, $D(x, q)$ has two zeros, $z_{mn}^\pm(q)$, nestled between the poles, $x_{mn}^\pm(q)$. We denote the inverse function by $q_{mn}^\pm(x)$. Note that the jumps in the values of the logarithmic terms at the branch points, $z_{mn}^\pm(q)$, do not cancel. At the zeros, $z_{mn}^\pm(q)$,

$$\partial_x \hat{D}(z_{mn}^\pm, q) = \sum_{m'n'} 2\frac{\mu_{m'n'}(z_{mn}^\pm)\mu_{m'n'}'(z_{mn}^\pm)A_{m'n'}}{(\mu_{m'n'}^2 - q^2)^2} + \frac{A_{m'n'}'(z_{mn})}{(\mu_{m'n'}^2 - q^2)}$$

$$\sim 2\mu_{m,n}(z_{mn}^\pm)\mu_\nu\mu_{m,n}C_{mn}/A_{mn}.$$ 

In our ordering $A_{mn}/C_{mn}$ is order $1/N_n$, which is a subsidiary small parameter, thus $\partial_x \hat{D}(z_{mn}^\pm, q)$ is $O(\delta q N^2/\nu)$. Since the jump in $\partial_x \hat{D}(z_{mn}^\pm, q)$ and there are approximately $2N_n N_m$ zeros, the total contribution to the change in $\hat{T}(x, q)$ from the sum of all the resonances is exactly the same order as the nonresonant contribution.

At $q^2 = A_{mn}/C_{mn}$, the two zeros coalesce and $\hat{D}(x, q)$ is approximately $C_{mn}\mu_{m,n}(x)^2/Q_{mn}$. The solutions of the homogeneous equation, $T_L$ and $T_R$, are highly oscillatory when
\[ |\mu_{m,n}(x)| < q. \]  \( \hat{T}(x,q) \) has singularities of order \((x - x_{m,n})^\lambda \) where \( \lambda = (-1 \pm \sqrt{1 - 4/g})/2 \) where \( g \equiv C_{mn} (m\mu')^2/q^2 \). In our ordering with clearly separated poles, \( g \) is \( O(\epsilon_v^2 N_m^2/\epsilon_L^2) \), which is much less than one.

For \( q^2 < A_{mn}/C_{mn} \), Eq. (12) has no singularities except a simple pole at \( q = 0 \). The pole generates a steady solution, \( T_f(x) \), corresponding to \( \hat{T}(x,q) \sim T_f(x)/q \). Since \( \hat{T}_\epsilon \) is analytic in \( q \) for \( 0 < q^2 < A_{mn}/C_{mn} \), this range of \( q \) does not contribute to the time evolution.

For \( |q|^2 > max_{x,m,n} \mu_{m,n}(x)^2 \), \( \hat{D}(x,q) \) is negative and does not vanish. Thus the continuous spectrum exists only in the range \( 1/\sqrt{N_m} < |q| < N_m \).
APPENDIX D. SINGULARITIES OF $\hat{T}(x, q)$: REAL AND REMOVABLE

The long time asymptotic behavior of $T(x, t)$ depends on the regularity of $\hat{T}(x, q)$ which in turn depends on the behavior of $\hat{D}(x, q)$. Due to the positivity of $A_{m,n}$, $\hat{D}(x, q)$ has zeros and poles only on the real $q$ axis. $\hat{K}(x, q)$ can be decomposed into pairs of poles,

$$\hat{K}(x, q) \equiv \sum_{m,n} A_{m,n}(x) \left( \frac{1}{\mu_{m,n}(x) - q} - \frac{1}{\mu_{m,n}(x) + q} \right).$$

As $q$ approaches the real axis, we can split $\hat{K}(x, q)$ into a real principal part, $\hat{K}_{PV}(x, q)$, plus a sum of delta functions:

$$\lim_{q_I \to +0} \hat{K}(x, q) = \hat{K}_{PV}(x, q) + \pi i \sum_{m,n} \left( \frac{A_{m,n}(x^+_m,n)}{\mu'_{m,n}(x)} \delta(x - x^+_m,n(q)) + \frac{A_{m,n}(x^-_{m,n})}{\mu'_{m,n}(x)} \delta(x - x^-_{m,n}(q)) \right).$$

Furthermore, for small $q$, $\hat{K}_{PV}(x, q)$ tends to zero (the real parts of the denominators cancel). The standard derivation\(^2\) of quasilinear diffusion averages the limiting $\hat{K}(x, q)$ spatially. The flaw in this analysis is that the limit of the solution is not the solution of the limit. In fact, the poles of $\hat{D}(x, q)$ generate removable singularities. The zeros of $\hat{D}(x, q)$ generate actual singularities in $\hat{T}$. The limiting potential, $V_{lim}(x, q_R)$, in Eq. (14) satisfies

$$\lim_{q_I \to +0} V(x, q) = V_{PV}(x, q) + \pi i \sum_{m,n} \left( V(z^+_m,n) \delta(x - z^+_m,n(q)) + V(z^-_{m,n}) \delta(x - z^-_{m,n}(q)) \right),$$

where $V(x, q) \equiv \hat{D}(x, q)^{-1}$, and the poles of $V(x, q_R)$ are the zeros of $\hat{D}(x, q_R)$. When the potentials, $V_\epsilon(x, q)$, are uniformly bounded from above and below, $0 < c_l < V_\epsilon(x, q) < c_u$, the limiting equation for $\hat{Y}(x, q) \equiv \hat{D}(x, q) \partial_x \hat{T}(x, q)$ exists, and the solution of the limit is the limit of the solutions. In our case, the potentials, $V_\epsilon(x, q)$ are not bounded from above and below, but are integrable except at the double zeros of $\hat{D}(x, q)$. Since the multiple scale expansion appears to only require integrability, we believe similar convergence theorems will hold.
As the Laplace parameter, \( q \), is varied, the positive poles, \( x^+_{m,n}(q) \), move to \(+\infty\) with speed, \( \partial_q x^+_{m,n}(q) = 1/\mu'_{m,n}(x) \). On its path, each pole will collide with a large number of other poles. As \( q \) varies, the Doppler shifted poles are displaced and occasionally coincide. For \( q \neq 0 \), \( \hat{D}(x, q) \) remains with a \( 1/x \) singularity. Thus only the strength and not the order of the singularity is increased.

Between each two adjacent poles, there will be one, two or no zeros. If the adjacent poles have the same parity, there will be an odd number of zeros, generically, one. Between adjacent poles of opposite parity, there will be an even number of zeros. When the poles are infinitesimally close together, there are no zeros. However as the poles separate, a double zero may form at a critical value of \( q \). As the distance further increases, the double zero will divide into two separate isolated zeros. In Appendix C, we show that near \( q = 0 \), the zeros of \( \hat{D}(x, q) \) evolve from no zeros to a double zero to two single zeros as \( q \) increases.

The single and double zeros generate spatially localized temperature oscillations which decay algebraically in time. We now examine the local solutions of Eq. (12) at the poles and then at the zeros. We note that \( \hat{D}(x, q) \) has double poles only at \( q = 0 \). Thus the generic case is when \( \hat{D}(x, q) \) is proportional to \( 1/(x - x_o(q)) \) at each resonance surface. From Eq. (15), we see that \( \hat{Y}(x, q) \equiv \hat{D}(x, q) \partial_x \hat{T}(x, q) \) is locally analytic in \( q \) at the poles and double poles of \( \hat{D} \). Thus \( \partial_x \hat{T}(x, q) \) must vanish at \( x_{mn}(q) \). At the double poles at \( q = 0 \), both \( \partial_x \hat{T}(x, q) \) and \( \partial^2_x \hat{T}(x, q) \) vanish. Thus the poles of \( \hat{D}(x, q) \) generate removable singularities.

The zeros of \( \hat{D}(x, q) \) generate singularities of the first kind in \( \hat{T}(x, q) \). We now examine the behavior of the solutions of the homogeneous equations near a zero, \( z_o(q) \). We let \( c_o(q)^{-1} = \partial \hat{D}(z_o(q), q) \). Near the zero, there is a solution, \( \hat{T}_A(x, q) \), which is locally analytic in \( q \) with \( \hat{T}_A(z_o(q), q) = 1 \) and \( \partial_x \hat{T}_A(z_o(q), q) = c_o(q) \). The corresponding locally analytic solution of Eq. (14) is \( \hat{Y}_A(x, q) \) with \( \hat{Y}_A(x, q) \sim x - z_o(q) + c_o(x) (x - z_o(q))^2 \).

The second solution, \( \hat{T}_S(x, q) \), has a logarithmic singularity of the form \( \hat{T}_S(x, q) = (x - z_o(q))^2 \).
\[ \ln(x - z_o(q)) \tilde{T}_A(x, q) + \tilde{T}_C(x, q) \] where \( \tilde{T}_C(z_o(q), q) = 0 \). The corresponding local solution of Eq. (14) is \( \tilde{Y}_S(z_o(q), q) = \ln(x - z_o(q)) \tilde{Y}_A(x, q) + \tilde{Y}_C(x, q) \), where \( \tilde{Y}_C(z_o(q), q) = 1/c_o(q) \) and \( \tilde{Y}_C(x, q) \) is locally analytic. We note that any solution satisfies the jump condition:

\[ \frac{\partial Y}{\partial x} \big|_{z_o(q)-\epsilon}^{z_o(q)+\epsilon} = \pm \pi i c_o(q) \hat{Y}(z_o(q), q). \]

At a double zero of \( \hat{D} \), there are solutions, \( \hat{T}(x)_\pm \sim z^\lambda \) where \( \lambda = -1/2 \pm \sqrt{1/4 - 1/\hat{g}} \) and \( g \equiv \hat{D}''(x) \). Thus if \( 0 < g << 1 \), then the singularities are order \( z^{-1/2} \). For \( \hat{D}''(x) >> 1 \), the singularity approaches \( z^{-1} \). Overlapping resonance layers correspond to the case \( \hat{D}''(x) >> 1 \).

A second aspect of the singularities of \( \hat{T}(x, q) \) is the thickness of the resonance layers about the zeros and poles of \( \hat{D}(x, q) \). The resonance layer is the subdomain where the behavior of \( \hat{T}(x, q) \) is dominated by the singularity. We examine the intrinsic layer width for poles, double poles, zeros and double zeros.

For the case of a single isolated resonance, the resonant layer width scales as 
\[ (n\delta x)^3 \sim (\epsilon_v/\epsilon_L)^2 (n/m\mu') A_{mn} / (\mu_{mn}(x) + q). \]
In our evaluations of the asymptotic size of terms, we need estimates of \( O(\frac{1}{\mu_{mn} + q}) \) near a resonance of \( \mu_{mn} - q \). This naturally depends on our choice of \( q \). For small \( q \), \( q = O(1/N_m) \), the denominator is \( O(N_m) \). However, the inverse Laplace transform involves integrals, \( dq \), and the small \( q \) are downweighted in the integration due to their small measure. We find the \( q \) weighted average scaling of the terms is not significantly altered by the modified scaling at \( q << 1 \).

When \( q \) is \( O(1) \), the layer width scales as \( (n\delta x)^3 \sim (\epsilon_v/\epsilon_L)^2. \) For \( q \sim 1/N_m \), the layer width is \( (n\delta x)^3 \sim (\epsilon_v/\epsilon_L)^2 N_m \). Finally, at \( q = 0 \), all poles are double poles and the resonance layer width scales as \( (n\delta x)^4 \sim (\epsilon_v/\epsilon_L)^2 (n/m\mu')^2 \). The total area occupied by resonance layers scales roughly as \( N_m N_n \delta x \). For \( q \sim O(1) \), \( N_m N_n \delta x \sim (\epsilon_v/\epsilon_L)^2/3 N_m \). For \( q \sim O(1/N_m) \), \( N_m N_n \delta x \sim (\epsilon_v/\epsilon_L)^2/3 N_m^{4/3} \). For \( q = 0 \), \( N_m N_n \delta x \sim (\epsilon_v/\epsilon_L)^{1/2} N_m \).

At the zeros, \( z_{m,n}(q), \) of \( \hat{D} \), the boundary layer thickness is much smaller, \( \delta x \sim \partial_x \hat{D}(z_{m,n}^\pm, q) = 2\mu_{m,n}(z_{m,n})^\pm \mu_{m,n} C_{mn}^2 / A_{mn} \), which is \( O(m\epsilon_v^2 N_m^2 \sqrt{q^2 - A_{mn}/C_{mn}}) \). Note
this is the boundary layer thickness of the homogeneous equation and not the inhomogeneous equation.

For double zeros, the extent of the subdomain where resonant behavior dominates depends on the “exterior” solution and cannot be determined \textit{a priori}. 
APPENDIX E: MULTIPLE SCALE ANALYSIS OF THE ISOLATED RESONANCES REGIME

In this appendix, we find multiple scale solutions to Eq. (12) in the isolated resonance case. The expansions can then be used in Eqs. (21)-(23) to derive the leading order time dependent asymptotics. In practice, we use the isolated resonance expansion only to show that the solutions do not become pathologically large or singular. With some additional work, this expansion could be incorporated into Sec. VI. To solve the zeroth order equation, we used the W.K.B. expansion with turning points.

Due to the smallness of \( \hat{D}(x, q) \), the standard multiple scale ordering\(^{6-8} \) must be slightly modified. For a maximal ordering, we assume \( \epsilon_v^2 N_m \sim \epsilon_L^2 \). This ordering enables us to correctly treat the resonances of \( \hat{D} \). We let \( y = \epsilon_L x \) and expand \( \hat{T}_\epsilon \) as \( T_o + \epsilon T_1 + \epsilon^2 T_2 \). Spatial differentiation has the expansion, \( 1/\epsilon_L \partial_y + \partial_x \). We expand the differential operator as \( \mathbf{L} = \mathbf{L}_o + \epsilon \mathbf{L}_1 + \epsilon^2 \mathbf{L}_2 \) where \( \mathbf{L}_o T \equiv \partial_y \hat{D} \partial_y T - T \) and \( \mathbf{L}_1 \equiv \partial_x \hat{D} \partial_y + \partial_y \hat{D} \partial_x \) and \( \mathbf{L}_2 \equiv \partial_x \hat{D} \partial_x \). \( \mathbf{L}_o \) contains both slowly and rapidly varying parts due to the presence of the identity operator. The slowly varying part of \( \mathbf{L}_o \) satisfies \( < \mathbf{L}_o T > = -< T(x, t = 0) > \).

The zeroth order equation reduces to \( T_o = T(x, t = 0)/q \). To next order, we have \( T_1 = \chi \partial_x T_o \), where \( \chi \) is the rapidly varying solution of \( \mathbf{L}_o \chi = -\partial_y \hat{D} \). \( \chi(x, q) \) will have logarithmic singularities at the zeros of \( \hat{D}(x, q) \) and \( (x - z)^\lambda \) singularities at the double zeros.

Within this resonance layer, the full equation, \( \mathbf{L}_o \chi = -\partial_y \hat{D} \), must be solved. Since \( \partial_x \hat{T}(x_{m, n}(q), q) = 0 \), \( \partial_y \chi(x_{m, n}(q), q) = -1 \) and thus near \( x_{m, n}(q) \), \( \chi \) is order \( \delta x \), independent of the value of \( \epsilon_v \sqrt{N_m}/\epsilon_L \). Within the layer, \( \partial_y \chi(x, q) \sim \partial_y \chi(x, q) \partial_q x_{m, n}(q) \), and therefore \( \partial_y \chi(x, q) \) is order \( 1/mu' \).

Away from the poles, \( x_{m, n}(q) \), and assuming \( \delta q \sim O(1) \), \( \hat{D} \) is \( O(\epsilon_v^2 N_m) \), \( \partial_x \hat{D} \) is \( O(\epsilon_v^2 N_m/\epsilon_L) \), and \( \partial_q \partial_x \hat{D} \) is \( O(\epsilon_v^2 N_m/\epsilon_L) \). If \( \partial_y \ln(\chi) \) is \( O(1) \), \( \partial_y \hat{D} \partial_y \chi \) is order \( (\epsilon_v/\epsilon_L)^2 N_m \) smaller than \( \chi \) away from the the resonances, \( x_{m, n}(q) \).
We would like to expand $\chi(x, q)$ in powers of $(\epsilon_v/\epsilon_L)^2 N_m$ away from the poles. Between the poles of $\hat{D}(x, q)$, a particular solution of the inhomogeneous equation is $\chi_p = \partial_y \hat{D}$, accurate to order $O((\epsilon_v/\epsilon_L)^2 N_m)$. Away from the poles of $\hat{D}(x, q)$, the solution would appear to converge to

$$T(x; q) = \frac{T(x, t = 0) + \partial_x \hat{D}(x, q) \partial_x T(x, t = 0) + \ldots}{q}. \quad (E1)$$

This formal solution clearly fails near the poles of $\hat{D}(x, q)$ where the actual solution has at worst an $(x-x_{m,n}(q))\ln(x-x_{m,n}(q))$ singularity, as shown by the local analysis and the Green’s function representation.

In the Green’s function of Eq. (15), $T_L(x, q)$ is a solution which satisfies the left boundary condition and $T_R(x, q)$ satisfies the right boundary condition. Replacing $T_L, R$ by $\partial_\xi \hat{D} \partial_\xi T_L, R$ in Eq. (15) and integrating by parts yields

$$A(q)T(x; q) = \frac{A(q)T(x, t = 0)}{q} - T_R(x; q) \int_a^x (\hat{D}(\xi, q) \partial_\xi T_L(\xi, q)) \partial_\xi f(\xi, q) d\xi - T_L(x, q) \int_x^b (\hat{D}(\xi, q) \partial_\xi T_R(\xi, q)) \partial_\xi f(\xi, q) d\xi, \quad (E2)$$

where $A(q) \equiv \hat{D}(x; q) W(T_L, T_R, x)$ is independent of $x$ and $f(x, q) = -T(x, t = 0)/q$.

Away from the zeros and poles of $\hat{D}(x, q)$, $T_L(x, q)$ and $T_R(x, q)$ can be represented in terms of the W.K.B. expansion$^{19-21}$:

$$T_{\pm}(x, q) = c_{\pm}(q) \exp(\pm \frac{\Phi(x, q)}{\epsilon_L}) \sum_k \epsilon L \tilde{p}_{k}^{\pm}(x, q), \quad (E3)$$

where $\partial_x \Phi(x, q) \equiv 1/\sqrt{\hat{D}(x, q)}$ and $p_0^{\pm}(x, q) = \hat{D}(x, q)^{-1/4}$. The sign of $q_I$ determines the branch cut of $\Phi$ in the W.K.B. representation of $T_L$ and $T_R$ and this breaks the analyticity of $\chi$ with respect to $q$.

In regions where $\hat{D} > 0$, the solutions grow exponentially, and this exponentially growth localizes the kernel of the Green’s function to a neighborhood of $x$. When $\hat{D} < 0$, the solutions are oscillatory, and the entire subinterval can contribute to the kernel, as well as neighborhoods of the two adjacent transition points.
A further integration by parts yields

\[ A(q)T(x; q) = \frac{A(q)T(x, t = 0)}{q} + B(q) + \]

\[ T_R(x; q) \int_a^x T_L(\xi; q) \partial_\xi \hat{D}(\xi, q) \partial_\xi f(\xi; q) d\xi + T_L(x; q) \int_x^b T_R(\xi; q) \partial_\xi \hat{D}(\xi, q) \partial_\xi f(\xi; q) d\xi , \]

(E4)

where \( B(q) \equiv \hat{D}(x; q)(\partial_x f(x; q))T_L(\xi, q)T_R(\xi, q)|_{\xi=a} \) is a boundary term, independent of \( x \) and exponentially small. This procedure of replacing \( T_L,R \) by \( \partial_\xi \hat{D} \partial_\xi T_L,R \) in Eq. (15) and integrating by parts may be repeated successively to formally rederive the expansion in Eq. (E1). Again the derivation is flawed due to the poles of \( \hat{D} \). Away from the transition points, in regions where \( \hat{D} > 0 \), the effect of the poles of \( \hat{D} \) is exponentially small and may be neglected.

Thus we need to consider only regions where \( \hat{D} \) is negative and regions near a zero or pole of \( \hat{D} \). Equation (16) shows that the time evolution of \( T(x, t) \) is completely determined by \( \text{Imag}(\hat{T}(x, q_R)) \). Since \( \hat{T}_L \) and \( \hat{T}_R \) are analytic at the poles of \( \hat{D} \), the poles of \( \hat{D} \) do not generate imaginary \( \hat{T}(x, q) \). Therefore points, \( (x, q) \), which lie on anti-Stokes lines between two adjacent poles are well represented by Eq. (E1), and do not contribute to the time evolution according to Eq. (22). If the point, \( (x, q) \), lies on anti-Stokes lines between two zeros of \( \hat{D} \), the expansion of Eqs. (E1) and (E4) converges. Thus we need only evaluate the Green’s function when the point of interest, \( x \), lies in an interval of negative \( \hat{D}(x, q) \) which is bordered by one pole and one zero.

Using the method of stationary phases, it can be shown that the only contributions to the integral representation of Eq. (E4) occur at the adjacent transition points and at the point of interest, \( x \). The zero of \( \hat{D}(x, q) \) requires special treatment because \( T_L \) and \( T_R \) are analytic functions except at the zero and the imaginary part of \( T_L \) and \( T_R \) are generated exclusively at the zero. The pole of \( \hat{D}(x, q) \) requires special treatment because the naive expansion fails. To solve these problems, \( T_L \) and \( T_R \) need to be represented near the transition points using comparison equations.
Near the pole of $\hat{D}$, the comparison equation is $\partial_x(d/x)\partial_x T + T = 0$, which transforms to $\partial_y(1/y)\partial_y T - T = 0$ under the transformation $y = -d^{-1/3}x$. We identify the pole, $x_L$, with $x_{mn}(q)$ and $d \equiv A_{mn}(x_{mn}(q))/\mu'_{mn}(x_{mn}(q))$. Since $T_L$ increases exponentially as $x \to x_L$ from the left, in a neighborhood of the origin, we can approximate it by $T_L(x) \sim T_L(x_L)\alpha'(-d^{-1/3}x)/\alpha'(0)$. Using this and similar expressions, a uniformly valid asymptotic expansion may be constructed.
APPENDIX F: RANDOM RESONANCE BROADENING VERSUS THE CONTINUUM APPROXIMATION

The continuum approximation has been justified by resonance broadening. A specific type of resonance broadening occurs when the real eigenfrequencies, $\omega_{m,n}$ are random, i.e. $\omega_{m,n} = \omega_{m,n}^0 + \delta\omega_{m,n}$. When the random piece of the eigenfrequencies, $\delta\omega_{m,n}$, is time independent, the resonances will be sharp for each distinct realization of $\delta\omega_{m,n}$, but the ensemble average will be smoother due to blurring effects. The additional smoothness, which an ensemble averaged solution possesses, will result in a faster decay of the time dependent perturbation of the ensemble averaged solution.

This random eigenfrequency resonance blurring is not equivalent to the continuum approximation. The correct physical problem is to determine the expectation of $T(x,t; \delta A_{m,n}, \delta \omega_{m,n})$. The continuum approximation is essentially averaging or taking the expectation of the the kernel $K(x,t; \delta A_{m,n}, \delta \omega_{m,n})$ with respect to resonance broadening. However the expectation of Eq. (6) with a random kernel is not equal to Eq. (6) with the expectation of the kernel.

A second physical variant of Eq. (6) is the everywhere resonant perturbation. In this case, the eigenfrequencies have a spatial variation such that they are everywhere in resonance, i.e. $\mu_{m,n}(x) \equiv m\mu(x) - nv_z(x) - \omega_{m,n}(x) \equiv 0$. In this everywhere resonant case, Eq. (6) reduces to a random wave equation, $T_{tt}(x,t) = \partial_x(\sum_{m,n} A_{m,n}(x) \partial_x T(x,t))$. This random wave equation may easily be homogenized to yield $T_{tt}(x,t) = \partial_x(A_h \partial_x T(x,t))$, where $A_h$ is the harmonic mean of $\sum_{m,n} A_{m,n}(x)$. The characteristic loss time is the minor radius divided by the effective wave velocity: $a/\sqrt{A_h}$.
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