Path Integral Quantization of Cosmological Perturbations

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Abstract

We derive the first order canonical formulation of cosmological perturbation theory in a Universe filled by a few scalar fields. This theory is quantized via well-defined Hamiltonian path integral. The propagator which describes the evolution of the initial (for instance, vacuum) state, is calculated.

1 Introduction

The quantization of linearized matter and metric perturbations in the Friedmann Universe has become an especially interesting topic in connection with inflationary scenarios of the Universe evolution.

Actually, if the Universe went through an inflationary stage, then the initial quantum fluctuations can explain the observable large-scale structure of it (see, for instance, [1]).

One of the simplest models for inflation is the chaotic inflation when the Universe is filled by homogeneous scalar fields [2]. In this model, small inhomogeneities in the distribution of scalar fields create the metric perturbations. In addition there can be fluctuations of the metric which are not due to inhomogeneities of the matter (gravitational waves).

The purpose of this paper is to present a selfconsistent quantum theory of these perturbations starting from first principles.

It was shown in [3] that the Lagrangean for the scalar cosmological perturbations derived via expansion of the Einstein-Hilbert action can be ex-
pressed entirely in terms of one gauge-invariant variable which describes the collective degree of freedom of the metric and matter perturbations. This reduced Lagrangean was then the starting point for the quantization.

There are several disadvantages in the above mentioned semi-quantum approach. First, it does not permit us to reveal the connection between fundamental quantum theory of gravity and quantization of small metric perturbations. In particular, the role of gauge group (diffeomorphisms) is not clear there. Second, to reduce the action to the function of only one variable, we need to use some of the Einstein equations.

It is much more easy to clarify the role of these equations in the canonical quantum theory, where they are just algebraic constraints. Further, the calculations in Lagrangean theory are not very straightforward and it is a tricky point to generalize them to include in the consideration several scalar fields [4, 5].

And finally, the Lagrangean theory is not very suited to relate the functional approach to quantization of perturbations with "fundamental" canonical or path integral quantum gravity where the question about "boundary conditions" for a quantized Universe is well posed [6, 7].

Our method in this paper is to expand the canonical ADM action for gravity and $N$ scalar fields up to second order in the perturbation variables. Thus the first order formulation of the classical theory of cosmological perturbations is derived. Then we quantize this theory via path integral in the Hamiltonian formalism where the measure is well defined. As a result the integration over the (infinite) volume of group (diffeomorphisms) is factorized explicitly and the problem is reduced to the quantization of a set of gauge invariant fields with time-dependent masses. The propagators and evolution of initial vacuum state are calculated. The details of calculations will be presented in a forcoming paper [8].

We use the units in which $c = \hbar = 16\pi G = 1$ and adopt MTW conventions [9].

2 Action

If we write the metric in the ADM form [10],

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + \gamma_{ij} dx^i dx^j,$$

(1)
then the first order action for gravity and $N$-scalar fields $\varphi_A$ ($A = 1, \cdots, N$) with potentials $V^A(\varphi)$ is \cite{6,11}:

$$S = \int d^3x dt (\pi^{ij} \dot{\gamma}_{ij} + \sum_A \pi_A^A \dot{\varphi}_A - N^\alpha \mathcal{H}_\alpha), \quad (2)$$

Here, the dot means the derivative with respect to coordinate time $t$, $\pi^{ij}$ and $\pi_A^A$ are the momenta conjugated to $\gamma_{ij}$ and $\varphi_A$. The lapse $N(x,t)$ and the shift $N^i(x,t)$ functions play the role of Lagrangean multipliers.

Correspondingly, the four constraints are

$$\mathcal{H}_0 = G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\gamma} (3) R + \sum_A \left( \frac{\left(\pi_A^A\right)^2}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{2} \gamma^{ij} \dot{\varphi}_i^A \dot{\varphi}_j^A + \sqrt{\gamma} V^A(\varphi) \right) \tag{3}$$

$$\mathcal{H}_i = -2\gamma_{ij} \pi_j^k \dot{\gamma}_{ij} + \sum_A \pi_A^A \dot{\varphi}_i^A \tag{4}$$

where

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}} (\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl}) \quad \gamma = \text{det}(\gamma_{ij}) \tag{5}$$

and the bar $|$ denotes covariant derivatives with respect to the three-metric $\gamma_{ij}$.

3 Background

In an isotropic, flat Universe, the coordinate system can be choosen in such a manner that

$$\varphi_A = \varphi_A(t), \quad \pi_\varphi = \pi_\varphi^A(t), \quad \gamma_{ij} = a^2(t) \delta_{ij}, \quad \pi_{\gamma}^{ij} = \delta_{ij} \pi_a / 6a \quad (6)$$

Then the action \cite{3} reduces to

$$S = \int dt d^3x \left( \pi_a \dot{a} + \sum_A \pi_A^A \dot{\varphi}_A - N^0 \mathcal{H}_0 \right), \quad (7)$$

\footnote{The results can easily be generalized for closed and open Universes.}
where now the superhamiltonian
\[ H_0 = -\frac{(\pi_a)^2}{24a} + \sum_A \left( \frac{(\pi^A_\varphi)^2}{2a^3} + a^3V^A(\varphi) \right), \] (8)
is constrained to vanish. Using Hamilton eqs. we can express the momenta \( \pi_a \) and \( \pi^A_\varphi \) in terms of the "velocities" \( \dot{a} \) and \( \dot{\varphi}^A \):
\[ \pi_a = -12a\dot{a}/N, \quad \pi^A_\varphi = a^3\dot{\varphi}^A. \] (9)

From the vanishing Hamiltonian (8) and other Hamilton eqs. with taking into account (9), one obtains the following eqs. for the background Friedmann Universe:
\[ \ddot{\varphi}^A + (3H - \dot{N}/N)\dot{\varphi}^A + N^2V^A = 0, \] (10)
\[ 6H^2 = \sum_A \left( \frac{1}{2}(\dot{\varphi}^A)^2 + N^2V^A(\varphi) \right), \] (11)
where \( H \equiv \dot{a}/a \) coincides with the Hubble parameter only if we choose the gauge \( N(t) = 1 \). Later, these eqs. will heavily be used to simplify the action for cosmological perturbations.

4 Perturbations

To study the behaviour of perturbations in a flat Universe we consider small deviations from the homogeneous solutions (6), such that for the 3-scalars one has
\[ \varphi^A(x,t) = \varphi^A(t) + \delta\varphi^A(x,t) \]
\[ \pi^A_\varphi(x,t) = \pi^A_\varphi(t) + \delta\pi^A_\varphi(x,t) \]
\[ N(x,t) = N(t) + \delta N(x,t) \] (12)

It is convenient to write the perturbations in 3-vector \( \delta N^i \) and 3-tensor \( \delta\gamma_{ij} \) and \( \delta\pi^{ij} \) as a sum of scalar (s), vector (v) and tensor (t) pieces which don’t interfere in the linear approximation:
\[ \delta N^i = (s)B^i + (v)D^i \]
\[ \delta\gamma_{ij} = a^2[-2(s)\psi\delta_{ij} + 2(v)E_{ij} + (s)V_{i,j} + (v)V_{j,i} + (t)h_{ij}] \]
\[ \delta\pi^{ij} = \frac{1}{a^2}[(s)1\pi^{ij} + (s)2\pi^{i,j} + (v)\pi^{i,j} + (v)\pi^{j,i} + (t)\pi^{i,j}] \] (13)
where the comma means usual derivative with respect to corresponding spatial coordinates. The vectors $D^i, V_i, \pi^i$ and tensors $^{(t)}h_{ij}, ^{(t)}\pi^{ij}$ satisfy the following conditions:

$$
D^i, i = 0, \quad V_i, i = 0, \quad ^{(t)}h^i_i = 0 = ^{(t)}h^i_{k,i},
$$
$$
^{(t)}\pi^i_i = 0, \quad ^{(t)}\pi^i_{j;i} = 0 = ^{(t)}\pi^i_{k,ij}.
$$

We will raise and lower indices by the unit tensor $\delta_{ij}$.

Expanding the action $(2)$ in perturbations, we find that the first order terms vanish on the eqs. of motion for the background model whereas the second order terms give the action for the perturbations.

This action consists of three decoupled terms which correspond to scalar, vector and tensor (gravitational waves) perturbations:

$$
\delta_2 S = ^{(s)}S(\delta \varphi_A, \delta \pi^A, \psi, E), ^{(v)}\pi, ^{(t)}\pi
$$
$$
+ ^{(s)}S(V_i, ^{(v)}\pi_i) + ^{(t)}S(^{(t)}h_{ij}, ^{(t)}\pi^{ij})
$$

The explicit form of the action for scalar perturbations up to total derivatives is

$$
^{(s)}S = \int dtd^3x \left( \pi \dot{\psi} + \pi_F \dot{F} + \sum_A \delta \pi^A \delta \varphi_A - ^{(s)}\lambda^0 \delta C_0 - ^{(s)}\lambda^1 \delta C_1 - ^{(s)}\mathcal{H} \right)
$$

where the two constraints linear in momenta are

$$
^{(s)}C_0 = \frac{H}{N} \dot{\pi}_\psi - \frac{4a^3}{N^2} (H^2 + \dot{H} - H \frac{\dot{N}}{N} (3\psi - F) - 4a \Delta \psi
$$
$$
+ \sum_A \left( \frac{\dot{\varphi}_A}{N} \delta \pi^A + a^3 \dot{V}_i^A \delta \varphi_A \right),
$$
$$
^{(s)}C_1 = \pi_F - \frac{4a^3}{N} (\psi + F) - \sum_A \frac{a^3}{N} \dot{\varphi}_A \delta \varphi_A,
$$

and the Hamiltonian for scalar perturbations $^{(s)}\mathcal{H}$ takes the form

$$
^{(s)}\mathcal{H} = \frac{N}{8a^3} (3\pi_F^2 + 2\pi_F \pi_\psi) - H (\pi_\psi \dot{\psi} + \pi_F F + 4\pi_F F)
$$
$$
+ \frac{4a^3}{N} (H^2 - \dot{H} + H \frac{\dot{N}}{N} (3\psi^2 - 2\psi F + F^2)) + 2Na \Delta \psi
$$

\[2\]Here and in what follows, we omit repeating the superscripts $(s)$ and $(v)$ whenever it is clear which of the decoupled pieces is considered.
\begin{align*}
&+ \sum_A \left( \varphi^A \delta \pi^A - \mathcal{N} a^3 V^A \delta \varphi_A \right) (3\psi - F) + \frac{\mathcal{N}}{2a^3} \left( \delta \pi^A \right)^2 \\
&\quad - \frac{\mathcal{N} a}{2} \Delta \delta \varphi^A \delta \varphi_A + \frac{\mathcal{N} a^3}{2} V^A \delta \varphi_A (\delta \varphi_A)^2 \right) ,
\end{align*}

(18)

and we introduced the new independent variables\( \sum \)

\begin{align*}
F &= \Delta E, \\
\pi_F &= 2 \left( (s^1) \pi + \Delta (s^2) \pi \right), \\
\pi_\psi &= -2 \left( (s^1) \pi + \Delta (s^2) \pi \right), \\
\lambda^0 &= \delta \mathcal{N}, \\
\lambda^1 &= \Delta B.
\end{align*}

(19)

For the vector perturbations one gets:

\begin{align*}
^{(v)} S &= \int dt d^3 x \left( \pi_V^i \dot{V}_i - ^{(v)} \lambda^i (^{(v)} C_i - ^{(v)} \mathcal{H}) \right)
\end{align*}

(20)

where

\begin{align*}
^{(v)} C_i &= \pi_V^i + \frac{4Ha^3}{\mathcal{N}} \Delta V^i \\
^{(v)} \mathcal{H} &= -2a^3 \left( H^2 - \dot{H} + H \frac{\dot{\mathcal{N}}}{\mathcal{N}} \right) V^i \Delta V_i - 2H \pi_V^i V_i ,
\end{align*}

(21)

and we introduced instead of \( ^{(v)} \pi^i \) and \( D^i \) the independent variables

\begin{align*}
\pi_V^i &= -2 \Delta ^{(v)} \pi^i, \\
\lambda^i &= D^i + \frac{\mathcal{N}}{a^3} ^{(v)} \pi^i
\end{align*}

(22)

Note that in this case we also have only two independent constraints, since

\begin{align*}
\lambda^{i,i} = 0 , \\
C^{i,i} = 0.
\end{align*}

(23)

For the tensor perturbations there are no constraints and the action takes the form

\begin{align*}
^{(t)} S &= \int dt d^3 x \left( \pi^i_{h ij} \dot{h}_{ij} - ^{(t)} \mathcal{H} \right),
\end{align*}

(24)

where \( \pi_{h}^{ij} \equiv ^{(t)} \pi_{h}^{ij} \) and

\begin{align*}
^{(t)} \mathcal{H} &= \frac{\mathcal{N}}{a^3} \pi_{h}^{ij} \pi_{h}^{ij} - 4H \pi_{h}^{ij} h_{ij} \\
&\quad + \frac{a^3}{\mathcal{N}} \left( H^2 - \dot{H} + H \frac{\dot{\mathcal{N}}}{\mathcal{N}} \right) h_{ij} h_{ij} - \frac{N a}{4} \Delta h_{ij} h_{ij}.
\end{align*}

(25)

\(^3\)Here and throughout the paper, \( \Delta \) denotes the flat three-dimensional Laplacian, e.i., \( \Delta E \equiv E_{ii} \).
Let us mention an interesting point. Starting with a superhamiltonian and fixing the background, we end up with a pure Hamiltonian for perturbations which is responsible for their dynamics with respect to background time. There are still four independent constraints (two for scalar perturbations plus two for vector ones), as it should be. These constraints are linear in momenta and generate themselves the gauge transformations which correspond to diffeomorphisms \[12\].

5 Gauge-invariant variables

The constraints \((^s)C_0, (^s)C_1, (^v)C_i\) and the Hamiltonian

\[H(t) = \int d^3x \left( (^s)\mathcal{H} + (^v)\mathcal{H} + (^t)\mathcal{H} \right)\]

form a closed algebra with respect to Poisson brackets \{ ..., \} defined in the standart manner for our canonically conjugated variables \(\delta \varphi^a, \delta \pi^A; \psi, \pi; F, \pi_F; V_i, \pi^i_V; h_{ij}, \pi^i_h\); that is

\[\{ C_\alpha, C_\beta \} = t^\gamma_{\alpha\beta} C_\gamma \quad \text{and} \quad \{ H(t), C_\alpha \} = t^\beta_\alpha C_\beta + \frac{\partial C_\alpha}{\partial t}.\] (26)

For instance, for the scalar constraints \((^s)C_0, (^s)C_1\) we have

\[\begin{align*}
\{ C_0(x,t), C_0(y,t) \} &= 0 = \{ C_1(x,t), C_1(y,t) \}, \\
\{ H(t), C_0(x,t) \} &= \frac{N}{a^2} \Delta x C_1(x,t) + \frac{\partial C_0(x,t)}{\partial t}, \\
\{ H(t), C_1(x,t) \} &= \frac{\partial C_1(x,t)}{\partial t}.
\end{align*}\] (27)

The action \([16]\) is invariant with respect to the transformations generated by the constraints \([17]\), if we simultaneously transform the Lagrangean multipliers \((^s)\lambda^\alpha\):

\[\begin{align*}
\delta_\xi q &= \{ q, \xi^\alpha C_\alpha \}, \\
\delta_\xi \lambda^\alpha &= \xi^\alpha - \xi^\beta \lambda^\gamma t_{\gamma\beta}^\alpha - \xi^\beta t^\alpha_\beta
\end{align*}\] (28)

where \(\xi \equiv \xi^\alpha\) are the parameters of transformation and \(q\) can be any of the canonical variables \(\delta \varphi^a, \delta \pi^A, \psi, \psi\), etc.

*The appearance of time derivatives on the right-hand-side is due to the explicit time dependence of the constraints.*
Since the constraints are linear in momenta, the transformations (28) correspond to diffeomorphisms [12]. From (28) and (16) with taking into account (26), (27), it is easy to get the following transformation laws:

$$\delta \xi (\delta \varphi_A) = \dot{\varphi}_A^0 / N,$$
$$\delta \xi \lambda_1 = \xi^1 - N \Delta \xi^0 / a^2,$$
$$\delta \xi \lambda_0 = \xi^0,$$
$$\delta \xi F = \xi^1,$$
$$\delta \xi \psi = -H \xi^0 / N,$$  

(29)

Using (29) one can easily construct gauge invariant variables $Q$ for which $\delta \xi Q = 0$. For instance, the most important of them are:

$$v_A = a (\delta \varphi_A + \dot{\varphi}_A^0 \psi)$$
$$\Delta \Phi = \frac{1}{a} \Delta \lambda^0 - \frac{1}{a \partial_t} \frac{a^2}{\mathcal{N}} (\lambda^1 - \dot{F})].$$

(31)

As we will see, the first variable naturally appears when we quantize the theory, whereas the second one has clear physical interpretation, namely, $\Phi$ corresponds to the relativistic Newtonian potential [1].

The gauge invariant variables for the vector perturbations can be constructed in the same manner. The tensors $h_{ij}$ and $\pi_{ij}^\mu$ are gauge invariant by themselves since there are no constraints which contain these variables. For more details, refer to [8].

6 Path integral

To quantize the perturbations we write the path integral ($h = 1$)

$$K = \int_i^f D\mu e^{i\langle \delta N^\alpha \rangle} D\mu \mathcal{D} \mu$$

which is just the propagator from some initial ($i$) to final ($f$) state. The diffeomorphism invariant measure $D\mu$ can be taken as

$$D\mu = D(\delta \gamma_{ij}) D(\delta \pi_{ij}) D(\delta N^\alpha) = \mathcal{J} D^{(s)} \mu D^{(c)} \mu D^{(t)} \mu$$

(33)

where $\mathcal{J}$ is an irrelevant constant Jacobian which can be absorbed in normalization and

$$D^{(s)} \mu = D(\delta \varphi_A) D(\delta \pi_A^i) D(\delta \psi) D(\delta \pi_{ij}) D F D \pi_{ij} D^{(s)} \lambda^0 D^{(s)} \lambda^1,$$
$$D^{(c)} \mu = DV_{\nu} D(\delta \pi_{ij}) D^{(c)} \lambda^i,$$
$$D^{(t)} \mu = Dh_{ij} D(\delta \pi_{ij})$$

(34)
are the measure, correspondingly, for scalar, vector and tensor perturbations. For $D(\delta \varphi^A)$, $D(\delta \pi^A)$, $D\psi$, $D\pi\psi$ etc, we take the usual Liouville measure. Then the propagator $K$ can be represented as a product of propagators for different types of perturbations. Let us calculate them separately.

For the **scalar perturbations**:

\[
^{(s)}K = \int D^{(s)}\mu e^{(s)S} \tag{35}
\]

where $(^{(s)}S)$ is given by (16) and $D^{(s)}\mu$ is defined in (34).

Let us integrate at first over the Lagrangean multipliers $(^{(c)}\lambda^0, ^{(c)}\lambda^1)$. Then we get the delta functions of constraints $(^{(c)}C_\alpha$, and since they are linear in momenta they permit us to integrate easily over $\pi\psi, \pi F$. Then, changing the variables (in particular, introducing the gauge invariant variables $v_A$, see (30), instead of $\delta\varphi_A$), with help of the eqs. of motion for background (9), (10) and (11) and after integration over the momenta we obtain the following result

\[
^{(s)}K = M \int \prod_A Dv_A e^{^{(s)}S(v_A)} , \quad Dv_A = \prod_i \frac{dv_A^i}{\sqrt{2\pi i \Delta\eta}} \tag{36}
\]

where

\[
M \propto \int D\psi DF \tag{37}
\]

corresponds to the explicitly factorized infinite volume of the gauge group and can be absorbed in normalization. The action $S(v_A)$ depends only on the gauge invariant variable $v_A$, (see (30)):

\[
^{(s)}S(v_A) = \frac{1}{2} \int d^3x d\eta \left( \sum_A (v^2_A - v_i^Av_i^A) + \sum_{AB} \Omega_{AB}v^Av^B \right). \tag{38}
\]

Here the prime means derivative with respect to the new time parameter

\[
\eta = \int \frac{Ndt}{a}, \tag{39}
\]

and the function of time $\Omega_{AB}$ is

\[
\Omega_{AB} = \left[ \left( \frac{z''_A}{z_A} + 2\frac{\tilde{H}'}{H} - \tilde{H} \frac{\tilde{H}'}{H} \right)\frac{z'_A}{z_A} + \frac{\tilde{H}''}{H} \tilde{H} + 2\tilde{H}^2 \right] \delta_{AB} + \frac{1}{2a^2}(\tilde{H} z_A z_B)', \tag{40}
\]

\[
z_A = \frac{a\varphi'}{H}, \quad \tilde{H} = \frac{a'}{a}.
\]

\[^5\text{We also skip from the action a lot of total derivatives.}\]
We see that the metric perturbations $\psi, F$ do not have their own dynamical degrees of freedom. They are entirely due to the perturbations in the matter distribution. In this sense the quantization of scalar cosmological perturbations is just the same as quantization of matter itself with taking into account the gravitational field created by this matter.

The vector perturbations do not possess any dynamics. This can easily be seen if we integrate in $(v)^S$, at first over the Lagrangean multiplier $(v)^{\lambda^i}$ and then over the momenta $(v)^{\pi^i}$. As a result we find that the action $(v)^S(V_i)$ vanishes (up to the total derivatives). The integral over $V_i$ corresponds to the integration over the gauge volume, like in (37).

If the Universe would be filled by some matter which has vector degrees of freedom (like, for instance, by a perfect fluid) then the quantization of vector perturbations would become nontrivial.

For tensor perturbations the calculations are very simple. Integrating over the momenta $(v)^{\pi^i}$ one arrives at:

\begin{equation}
(v)^K = \int d\eta d^3x (v)^{ij} e^{(v)} S \int (v)^{ij} e^{(v)} S \epsilon_{ij},
\end{equation}

where

\begin{equation}
(v)^S = \frac{1}{2} \int d\eta d^3x [\epsilon'_{ij} \epsilon'_{ij} - \epsilon_{ij,m} \epsilon_{ij,m} + \frac{a''}{a} \epsilon_{ij} \epsilon_{ij}],
\end{equation}

and $e_{ij} = a h_{ij}/\sqrt{2}$ is the the gauge invariant variable.

Thus the quantization of the cosmological perturbations in a Friedmann Universe has been reduced to the quantization of a set of gauge invariant fields with time-dependent masses in a flat space-time.

7 Propagator

As an example, we calculate the propagator and the evolution of an initial vacuum state for scalar perturbations in the Schrödinger representation. To simplify the consideration we assume that there is only one scalar field in the Universe, that is $v_A \equiv v$ and $\Omega_{AB} \equiv \Omega = z''/z$.

Expanding $v(\vec{x}, \eta)$ in Fourier modes as

\begin{equation}
v(\vec{x}, \eta) = \sqrt{2} \left( \int_{k_3 > 0} \frac{d^3k}{(2\pi)^{3/2}} v_k(\eta) \sin(k\vec{x}) + \int_{k_3 \leq 0} \frac{d^3k}{(2\pi)^{3/2}} v_k(\eta) \cos(k\vec{x}) \right)
\end{equation}

where the Fourier coefficients $v_k$ are real, one gets

\begin{equation}
S(v) = \frac{1}{2} \int d^3k d\eta \left( v_k^2 - k^2 v_k^2 + \Omega(\eta) v_k^2 \right).
\end{equation}
Thus, we can write the propagator from some initial \((i)\) to final \((f)\) field configuration as

\[
K(f|i) \propto \prod_k K_k(v_f, \eta_f|v_i, \eta_i),
\]

(45)

where

\[
K_k(v_f, \eta_f|v_i, \eta_i) = \int_{v_k(\eta_i)=v_i}^{v_k(\eta_f)=v_f} \mathcal{D}v_k \exp\left\{ \frac{i}{2} \int_{\eta_i}^{\eta_f} d\eta (v_k'^2 - k^2 v_k^2 + \Omega(\eta)v_k^2) \right\}
\]

(46)

is the propagator for the harmonic oscillator with the time-dependent frequency

\[
\omega^2_k(\eta) = k^2 - \Omega(\eta).
\]

(47)

The result for the path integral (46) is well known \([13, 14]\) :

\[
K_k(v_f, \eta_f|v_i, \eta_i) = \left( \frac{1}{2\pi i f(\eta_f, \eta_i)} \right)^{1/2} \exp\left\{ \frac{i}{2} (D_f'v_f^2 - C_i'v_i^2 - 2\frac{v_i v_f}{f(\eta_f, \eta_i)}) \right\}
\]

(52)

To simplify the notation we also skip the index \(k\) in some of the formulae whenever it is clear which of functions depend on \(k\).

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6To simplify the notation we also skip the index \(k\) in some of the formulae whenever it is clear which of functions depend on \(k\).
where
\[ f(\eta_f, \eta_i) = \frac{2}{D'_i - C'f} \quad \text{and} \quad C'_i \equiv \frac{\partial}{\partial \eta_i} C(\eta_i) , \text{etc} \quad (53) \]
do not depend of \( v_i, v_f \).

Given the propagator, we can calculate the evolution of any initial state functional \( \Psi(v_i, \eta_i) \). For instance, if this initial state can be represented in the form
\[ \Psi(v_i(\vec{x}, \eta_i), \eta_i) \propto \prod_k \Psi_k(v_i, \eta_i) \quad (54) \]
then as a result of evolution we find that
\[ \Psi(v_f(\vec{x}, \eta_f), \eta_f) \propto \prod_k \Psi_k(v_f, \eta_f) \quad (55) \]
where
\[ \Psi_k(v_f, \eta_f) = \int_{-\infty}^{+\infty} dv_i K_k(v_f, \eta_f|v_i, \eta_i) \Psi_k(v_i, \eta_i) . \quad (56) \]
Thus, there will be no mixing of the Fourier components with different \( k \) in the linear approximation as it should be.

If we choose, for instance, a normalized gaussian state with dispersion \( \sigma_0 \) (which can depend on \( k \)) as initial wavefunction,
\[ \Psi_k(v_i, \eta_i) = (\pi \sigma_0^2)^{-1/4} \exp\left\{ -\frac{v_i^2}{2\sigma_0^2} \right\} , \quad (57) \]
then the result for the integral (56) is
\[ \Psi_k(v_f, \eta_f) = e^{i\frac{\theta}{2}} (\pi \Sigma^2)^{-1/4} \exp\left\{ -\frac{1}{2} \frac{v_f^2}{\Sigma^2} \left( \frac{1}{\Sigma^2} - i\Lambda \right) \right\} , \quad (58) \]
where \( \Sigma, \Lambda \) and \( \theta \) depend on \( \eta_i, \eta_f, k \) and \( \sigma_0 \):
\[ \Sigma^2 \equiv \frac{4}{\sigma_0^2} \left[ 1 + \sigma_0^2 C'^2_i \right] , \quad \Lambda \equiv D'_f + \frac{\sigma_0^2}{\Sigma^2} C'_i \quad (59) \]
\[ \cot \theta \equiv \sigma_0^2 C'_i . \]
Thus, the wave function at time \( \eta_f \) describes still a gaussian state, but now with a modified complex dispersion.

Since all information about the system is contained in the state function (58), it is standard to compute expectation values \( \langle \hat{v}_k \hat{v}_k \rangle \), etc.
If we want to take as initial state the vacuum one, we need to specify the dispersion \( \sigma_0 \) entering (57). As it is well known there is an ambiguity in the definition of vacuum in the presence of external fields [13]. In our case this ambiguity is due to the time dependence of the frequency \( \omega_k \) in (47). However for any definition of vacuum we should have

\[
\sigma_0^2 \rightarrow \frac{1}{k} \quad \text{for} \quad k^2 \gg \Omega(k) \quad (60)
\]

Later we shall see that this condition is sufficient to get unambiguous predictions for the spectrum of fluctuations in the most interesting range of scales in inflationary model.

The final step is to calculate the two-point correlation function of the gauge invariant relativistic potential \( \Phi(\vec{x}, \eta) \), (see (31)).

### 8 Correlation function and power spectrum

For the purpose of comparing with observation we need the two-point correlation function \( \xi(r) \) of the gauge invariant metric fluctuation \( \Phi(\vec{x}, \eta) \):

\[
\xi(r) = \langle \Psi(v, \eta) | \hat{\Phi}(\vec{x}) \hat{\Phi}(\vec{x} + \vec{r}) | \Psi(v, \eta) \rangle = \int_0^\infty \frac{dk}{k} \frac{\sin(kr)}{kr} |\delta_k|^2 \quad (61)
\]

where \( \Psi(v, \eta) \) is the state functional and the power spectrum \( |\delta_k| \) is the measure of the amplitude of metric fluctuations on a comoving scale \( \sim 1/k \).

The relation between \( \Phi(\vec{x}, \eta) \) and \( v(\vec{x}, \eta) \), and, correspondingly, between the quantum operators \( \hat{\Phi} \) and \( \hat{v} \), follows from Hamilton eqs., (see [1]):

\[
\Delta \Phi = \frac{\varphi'}{4a}(\pi_v - \frac{z'}{z}v) \quad (63)
\]

where \( \pi_v \) is the momentum conjugated to \( v \).

In the Schrödinger representation, working with separate Fourier modes, one has for the appropriate momentum operator \( \hat{\pi}_k = -i\partial/\partial v_k \).

Now using the definitions (61), taking into account (62) and (63) and calculating the corresponding correlation functions

\[
\langle \hat{v}_k \hat{v}_k \rangle, \quad \langle \hat{\pi}_k \hat{\pi}_k \rangle, \quad \langle \hat{v}_k \hat{\pi}_k + \hat{\pi}_k \hat{v}_k \rangle \quad (64)
\]

7 In the case under consideration the correlation function just depends on \( r = |\vec{r}| \).
for the state (58), we get the following result for the power spectrum $|\delta_k|$:  

$$
|\delta_k|^2 = \frac{4\pi}{(2\pi)^3} k^3 \langle \hat{\phi}_k \hat{\phi}_k \rangle = \left( \frac{\varphi'_{,k}}{8\pi a} \right)^2 \left[ \frac{1}{k} \left( \Sigma^2 + \Lambda - \frac{z'}{z} \right)^2 \right]
$$

Thus, given an initial gaussian state at $\eta = \eta_i$, specified by the dispersion $\sigma_0(k)$, we derived the power spectrum $|\delta_k|$ at an arbitrary later moment of time $\eta = \eta_f$. This spectrum can be expressed entirely in terms of the two functions $C(\eta)$ and $D(\eta)$ which satisfy the eq. (49) with the boundary conditions (51).

The functions $\Sigma$ and $\Lambda$, in terms of $C$ and $D$, are given in (59). For an initial “vacuum” state the asymptotic dependence of $\sigma_0(k)$ on $k$ at $k^2 \gg \Omega$ is specified in (60).

As an example, let us consider chaotic inflation in the model with a massive scalar field: $V(\varphi) = \frac{1}{2} m^2 \varphi^2$. In this case from (63) one obtains the following spectrum for the metric fluctuation after the end of inflation in the most interesting range of galactic scales:  

$$
|\delta_k| \simeq \left( \frac{\sqrt{3}}{10\pi} \right) m \ln(\lambda_{ph}/\lambda_\gamma),
$$

where $\lambda_{ph} = a(\eta)/k$ is the physical wavelength of the perturbation and $\lambda_\gamma$ is some characteristic wavelength of the cosmic background radiation. This result coincides with the one obtained by other methods (see [1]). The inflation amplifies the initial vacuum fluctuations enough to explain the large-scale structure of the Universe only if $m \sim 10^{13} GeV$. The other interesting applications are considered in a forthcoming paper [8].

9 Discussion

Starting with the Hamiltonian (ADM) formulation of General Relativity, we deduced the Hamiltonian theory of cosmological perturbations in the Friedmann Universe.

To derive the action for linearized metric and matter perturbations in a first order Hamiltonian formalism, we expanded the ADM action for gravity and the action for the matter (scalar fields) up to second order in perturbations. The concrete calculations have been done for the case when the background Friedmann Universe has zero spatial curvature. This is a good
approximation if, afterwards, we want to quantize only the perturbations assuming that the background is classical.

The generalization of the developed formalism to the closed Universe is straightforward and will be presented in the future publication [8]. This case is especially interesting if we also want to quantize the Universe as a whole. However if the zero mode perturbation which correspond to the Friedmann background is in a quasiclassical region, then only the quantization of inhomogeneities becomes interesting.

We have further studied the diffeomorphism transformations in Hamiltonian formalism and constructed explicitly gauge (diffeomorphism) invariant variables for perturbations.

Then, the first order Hamiltonian theory was quantized via well defined Hamiltonian path integral. The volume of the gauge group (diffeomorphism) was factorized explicitly and the problem was reduced to the quantization of a set of gauge invariant fields and, finally, to the quantization of harmonic oscillators with time dependent frequences.

We calculated the propagator which describes the evolution of the quantum state and then found a closed expression for the power spectrum of fluctuations for initial gaussian, for instance, vacuum state.

All the calculations have been done in the Schrödinger picture which is very convenient to study, for example, the decoherence problem for the cosmological perturbations [16, 17]. The results were finally applied to the concrete model of chaotic inflation.

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