Conjugacy Growth and Conjugacy Width of Certain Branch Groups

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Abstract

The conjugacy growth function counts the number of distinct conjugacy classes in a ball of radius \( n \). We give a lower bound for the conjugacy growth of certain branch groups, among them the Grigorchuk group. This bound is a function of intermediate growth. We further prove that certain branch groups have the property that every element can be expressed as a product of uniformly boundedly many conjugates of the generators. We call this property bounded conjugacy width. We also show how bounded conjugacy width relates to other algebraic properties of groups and apply these results to study the palindromic width of some branch groups.

1 Introduction

The conjugacy growth function of a group was first introduced by I. K. Babenko in [Bab88] to study geodesic growth of Riemannian manifolds. It counts the number of distinct conjugacy classes in a ball of radius \( n \). This function had already intensively been studied for manifolds, among others by G. Margulis ([Mar69]) who obtained results in the case of negatively curved manifolds. These results have been generalized by T. Roblin ([Rob02]) to any quotient of a CAT(-1) metric space and further by I. Gekhtman ([Gek13]) to elements of mapping class groups. It was shown by E. Breuillard and Y. Cornulier in [BCT13] that the conjugacy growth function of a solvable group is either polynomially bounded or exponential. Recently, M. Hull and D. Osin proved in [HO13] that for any 'sensible' function \( f(n) \), there exists a finitely generated group such that it has conjugacy growth exactly \( f(n) \). The paper in [GST10] gives a summary of examples and conjectures concerning conjugacy growth.

In this paper we study the conjugacy growth of a wide class of branch groups, among them the Grigorchuk group. The following theorem states that for many classes of branch group this conjugacy growth is bounded from below by an intermediate function.

**Theorem 1** (Theorem 4.2). Let \( G \) be a finitely generated regular branch group acting on a \( d \)-regular rooted tree. Then the conjugacy growth function \( f(n) \) of \( G \) satisfies

\[
\frac{1}{p(n)} \cdot 2^n \succeq f(n),
\]

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where \( p(n) \) is a polynomial of degree at most \( 2(d - 1) \) and \( 0 < \sigma < 1 \), which can be made explicit depending on the group.

It would be desirable to obtain upper bounds for the conjugacy growth as well. In particular for the Grigorchuk group, whose word growth is still only known to be bounded from below by \( e^{n^{0.521}} \) as shown by J. Brieussel in his thesis [Bri08] and from above by \( e^{n^{0.767}} \) as shown by L. Bartholdi in [Bar98]. However, since there exist branch groups of exponential word growth, one would need to restrict the groups under investigation to obtain interesting results.

In the second part of this paper we show that certain branch groups have the property that every element can be written as a product of uniformly boundedly many generators. Properties like these have been studied under the name bi-invariant metrics for various groups (see for example [BGKM13] and [BL08]). The same property has been studied under the name reflection length in Coxeter groups in [Dus12] and [MT11]. It will be shown that bounded conjugacy width implies a number of other algebraic properties. We obtain the following result about branch groups:

**Theorem 2** (Theorem 5.8). Let \( G \) be a just infinite branch group that contains a rooted element and that has finite commutator width. Then \( G \) has bounded conjugacy width.

We will show in Subsection 5.1 that the conditions to be just infinite and to have finite commutator width are necessary. In particular, we will give examples of groups which have finite commutator width but which have unbounded conjugacy width.

We can apply our results to the study of the palindromic width of the Grigorchuk group. A palindrome is a word that reads the same right-to-left as left-to-right. It has been studied by various authors over the last decade, whether for a group \( G \) there exists a uniform bound \( N \), such that every element of \( G \) can be expressed as a product of at most \( N \) palindromes (see for example [BG14c], [BG14a], [Fin14b], [FT14] or [RS14]). It has been shown in [FT14] that if a group is just infinite, then it has finite palindromic width with respect to some finite generating sets. Here we complete this picture for the Grigorchuk group and prove that it has finite palindromic width with respect to all generating sets.

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# 2 Branch Groups

In this section we will recall some of the notation and definitions for branch groups from [Bv01] and [Seg01].

## 2.1 Trees

A tree is a connected graph which has no non-trivial cycles. If \( T \) has a distinguished root vertex \( r \) it is called a rooted tree. The distance of a vertex \( v \) from the root is given by the length of the path from \( r \) to \( v \) and called the norm of \( v \). The number

\[
|\{e \in E(T) : e = (v_1, v_2), v = v_1 \text{ or } v = v_2\}|
\]

is called the degree of \( v \in V(T) \). The tree is called spherically homogeneous if vertices of the same norm have the same degree. Let \( \Omega(n) \) denote the set of vertices of distance \( n \) from the root. This
set is called the \( n \)-th level of \( T \). A spherically homogeneous tree \( T \) is determined by a finite or infinite sequence \( \ell = \{ l_n \}_{n=0}^{\infty} \) where \( l_n + 1 \) is the degree of the vertices on level \( n \) for \( n \geq 1 \). The root has degree \( l_0 \). Hence each level \( \Omega(n) \) has \( \prod_{i=0}^{n-1} l_i \) vertices. Let us denote this number by \( m_n = |\Omega(n)| \). We denote such a tree by \( T_l \). A tree is called regular if \( l_i = l_{i+1} \) for all \( i \in \mathbb{N}_0 \). Given a spherically homogeneous tree \( T \) we denote by \( T[n] \) the finite tree where all vertices have norm less or equal to \( n \) and write \( T_v \) for the subtree of \( T \) with root \( v \). For all vertices \( v, u \in \Omega(n) \) we have that \( T_u \cong T_v \). Denote a tree isomorphic to \( T_v \) for \( v \in \Omega(n) \) by \( T_n \). This will be the tree with defining sequence \( \{ l_n, l_{n+1}, \ldots \} \). To each sequence \( \ell \) we associate a sequence \( \{ X_n \}_{n \in \mathbb{N}_0} \) of alphabets where \( X_n = \{ v_1^{(n)}, \ldots, v_{l_n}^{(n)} \} \) is an \( l_n \)-tuple so that \( |X_n| = l_n \). A path beginning at the root of length \( n \) in \( T_{\ell} \) is identified with the sequence \( x_1, \ldots, x_i, \ldots, x_n \) where \( x_i \in X_i \) and infinite paths are identified in a natural way with infinite sequences. Vertices will be identified with finite strings in the alphabets \( X_i \). Vertices on level \( n \) can be written as elements of \( Y_n = X_0 \times \cdots \times X_{n-1} \). Alphabets induce the lexicographic order on the paths of a tree and therefore on the vertices.

### 2.2 Automorphisms

An automorphism of a rooted tree \( T \) is a bijection from \( V(T) \) to \( V(T) \) that preserves edge incidence and the distinguished root vertex \( r \). The set of all such bijections is denoted by \( \text{Aut}(T) \). This group acts as an imprimitive permutation group on the set \( \Omega(n) \) of vertices on level \( n \) for each \( n \geq 2 \). Consider an element \( g \in \text{Aut}(T) \). Let \( y \) be a letter from \( Y_n \), hence a vertex of \( T[n] \) and \( z \) a vertex of \( T_n \). Then \( g(y) \) induces a vertex permutation of \( Y_n \). If we denote the image of \( z \) under \( g \) by \( g_y(z) \) then

\[
g(yz) = g(y)g_y(z).
\]

With any group \( G \leq \text{Aut}(T) \) we associate the subgroups

\[
\text{St}_G(u) = \{ g \in G : g(u) = u \},
\]

the stabilizer of a vertex \( u \). Then the subgroup

\[
\text{St}_G(n) = \bigcap_{u \in \Omega(n)} \text{St}_G(u)
\]

is called the \( n \)-th level stabilizer and it fixes all vertices on the \( n \)-th level. Another important class of subgroups associated with \( G \leq \text{Aut}(T) \) consists of the rigid vertex stabilizers

\[
\text{rst}_G(u) = \{ g \in G : \forall v \in V(T) \setminus V(T_u) : g(v) = v \}
\]

Informally speaking, \( \text{rst}_G(u) \) fixes everything outside the subtree \( T_u \) with root \( u \). The subgroup

\[
\text{rst}_G(n) = \prod_{u \in \Omega(n)} \text{rst}_G(u)
\]

is called the \( n \)-th level rigid stabilizer. Obviously \( \text{rst}_G(n) \leq \text{St}_G(n) \).

**Definition 2.1.** Let \( G \) be a subgroup of \( \text{Aut}(T) \) where \( T \) is a spherically homogeneous rooted tree. We say that \( G \) acts on \( T \) as a branch group if it acts transitively on the vertices of each level of \( T \) and \( \text{rst}_G(n) \) has finite index for all \( n \in \mathbb{N} \).
The definition implies that branch groups are infinite and residually finite groups. We can specify an automorphism $g$ of $T$ that fixes all vertices of level $n$ by writing $g = (g_1, g_2, \ldots, g_m)_n$ with $g_i \in \text{Aut}(T_n)$ where the subscript $n$ of the brackets indicates that we are on level $n$. Each automorphism can be written as $g = (g_1, g_2, \ldots, g_m)_n \cdot \alpha$ with $g_i \in \text{Aut}(T_n)$ and $\alpha$ an element of $\text{Sym}(l_{n-1}) \cdot \cdots \cdot \text{Sym}(l_0)$. Automorphisms acting only on level 1 by permutation are called rooted automorphisms. We can identify those with elements of $\text{Sym}(l_0)$.

**Definition 2.2.** An algebraic description of a branch group is given by the existence of a sequence of branching subgroups. In particular, we say $G$ is a branch group if there exist two decreasing sequences of subgroups $(L_i)_{i \in \mathbb{N}_0}$ and $(H_i)_{i \in \mathbb{N}_0}$ and a sequence of integers $(k_i)_{i \in \mathbb{N}_0}$ such that $L_0 = H_0 = G$, $k_0 = 1$, $\bigcap_{i \in \mathbb{N}_0} H_i = 1$ and for each $i$

1. $H_i$ is a normal subgroup of $G$ of finite index,
2. $H_i$ is a direct product of $k_i$ copies of the subgroup $L_i$, in other words there are subgroups $L_i^{(1)}, \ldots, L_i^{(k_i)}$ of $G$ such that $H_i = L_i^{(1)} \times \cdots \times L_i^{(k_i)}$ and each of the factors is isomorphic to $L_i$,
3. $k_i$ properly divides $k_{i+1}$, i.e. $m_{i+1} = k_{i+1}/k_i \geq 2$, and the product decomposition of $H_{i+1}$ refines the product decomposition of $H_i$ in the sense that each factor $L_i(l)$ of $H_i$ contains $m_{i+1}$ of the factors of $H_{i+1}$, namely the factors $L_i^{(l)}$ for $l = (j-1)m_{i+1} + 1, \ldots, jm_{i+1}$,
4. conjugations by the elements in $G$ transitively permute the factors in the product decomposition.

These two definitions are in general not equivalent as stated in [BGv03]. However, if a group is a branch group by the geometric definition, then one can easily recover the branching structure that is required in the algebraic definition. For more details, please see [BGv03].

In many cases the structure of branch groups becomes more accessible if we impose another condition.

**Definition 2.3.** A branch group $G$ acting on a rooted regular tree is called regular branch over its normal subgroup $H$ if $H$ has finite index in $G$, $(H, \ldots, H)_1 \leq H$ and if moreover the last inclusion is of finite index.

In particular, this last definition allows us to study branch groups via so-called self-similarity arguments.

## 3 The Grigorchuk Group

The Grigorchuk group, which was introduced in [Gri84] by R. Grigorchuk, is defined via its action on a rooted binary tree. It is generated by four automorphisms. The first one, $a$, swaps the two
top subtrees. The other three are defined recursively as
\[ b = (a, c)_1, \quad c = (a, d)_1, \quad d = (1, b)_1. \] (1)

It is helpful to picture them via their actions on the binary tree (see Figure 1).

![Figure 1: Actions of the generators of the Grigorchuk group depicted on the binary tree.](image)

The Grigorchuk group has become well known due to its property of having intermediate word growth. It was further also the first example of an amenable group that is not elementary amenable.

The following proposition lists some of the many interesting properties of this group.

**Proposition 3.1.** The Grigorchuk group has the following properties:

1. It is a just infinite branch group.
2. It is generated by three elements.
3. It is infinitely presented, but in particular we have the following relations:
   \[ a^2 = b^2 = c^2 = d^2 = 1, \quad [b, c] = [b, d] = [c, d] = 1. \]
4. It has intermediate word growth.

The relations in 3.1(3) help to understand the structure of words in the Grigorchuk group. In particular, we can deduce some normal form:

**Lemma 3.2.** Any element \( g \in G \) can be written as
\[ a^\epsilon * a * a * ... * a^\delta, \] (2)

where * stands for either \( b, c \) or \( d \) and \( \epsilon, \delta \in \{0, 1\} \).

**Proof.** This follows immediately from Proposition 3.1(3).
4 Conjugacy Growth

In this section we study the conjugacy growth function of branch groups acting on a regular rooted tree and then treat the special case of the Grigorchuk group. We will consider this group separately in Subsection 4.2 to get a better estimate than the one which would follow from the general result about branch groups. Our approach for branch groups in general uses its finite index branching subgroup. It emerges from the work of M. Hull and D. Osin in [HO13] that there exists a group with exponential conjugacy growth, but it has an index 2 subgroup which has only 2 conjugacy classes. We emphasize, that in our approach we consider the lengths of the words in the branching subgroup in the word metric coming from the group itself and conjugation will also be considered in the whole group.

Let \( G \) be any finitely generated group. We will for the rest of this paper denote the conjugate of an element \( g \in G \) by another element \( x \in G \) as \( g^x = x^{-1}gx \) and commutators by \([x, y] = x^{-1}y^{-1}xy, x, y \in G\). Further, we denote by \( l_X(g) \) the word length of an element \( g \) in the generators of the group. This word length depends on the chosen generating set. However, one can easily see that a change of the generating set does not change the equivalence class of the conjugacy growth function. It is clear that this function is bounded from above by the word growth \( \gamma_G(n) \) of a group \( G \).

4.1 Regular branch groups

We show that if a branch group \( G \) acts on a regular rooted tree, then its conjugacy growth function is bounded from below by a function equivalent to \( \frac{1}{p(n)} \cdot 2^{n^\sigma} \) for some polynomial \( p(n) \) and \( 0 < \sigma < 1 \).

Theorem 4.2. Let \( G \) be a finitely generated regular branch group acting on a \( d \)-regular rooted tree. Then the conjugacy growth function \( f(n) \) of \( G \) satisfies

\[
\frac{1}{p(n)} \cdot 2^{n^\sigma} \lesssim f(n),
\]

where \( p(n) \) is a polynomial of degree at most \( 2(d - 1) \) and \( 0 < \sigma < 1 \), which can be made explicit depending on the group.
4.1 Regular branch groups

Proof. The group $G$ is by hypothesis branch, so we have the regular branching structure from Definition 2.3. The branching subgroup $H$ has finite index in $G$ hence $H$ and $(H, \ldots, H)_1$ are finitely generated as well. Say $(H, \ldots, H)_1$ is finitely generated by elements $\{y_1, \ldots, y_l\}$. If $X$ is the finite generating set of $G$, then we denote by $M$ the maximum over all word lengths of the $y_i$,

$$M = \max_{j=1, \ldots, l} \{l_X(y_j)\}.$$ 

We have of course that $d \mid l$. In words, it takes at most a word of length $M$ in the letters $X$ to write each generator of each copy of $H$ on each of the $d$ subtrees of level 1. Figure 2 depicts this idea of finding multiple copies of $H$ on the first level which are contained in $H$.

![Figure 2: Self-similarity of $H$.](image)

Now we count the number of distinct conjugacy classes in a ball of radius $n$. If an element $g = (g_1, \ldots, g_d)_1$ is in the first level stabilizer, it means that it does not permute any of the first level vertices. Then a conjugate of $g$ by $h = (h_1, \ldots, h_d)_1 \tau$ has the following form:

$$g^h = \left( g_{\tau(1)}^{h_{\tau(1)}}, \ldots, g_{\tau(d)}^{h_{\tau(d)}} \right)_1,$$

where $\tau \in \text{Sym}(d)$ is the permutation of the first level vertices, coming from the element $h$. This allows us to apply a self-similarity argument. We undercount the number of conjugacy classes by assuming we write a word of length $n/d$ on each of the subtrees. Further, we need to consider that all permutations of the subtrees of the first level are possible by conjugation. However, there are at most as many different permutations of the first level vertices in $G$ as there are elements in $G/st_G(1)$. Denote the index of $st_G(1)$ in $G$ by $K$. So we get a recursive formula

$$f \left( d \cdot M \cdot n \right) \geq \frac{1}{K^2} f(n)^d.$$ 

Here the product $d \cdot M \cdot n$ comes from writing $d$ copies of length $n$, but for each word of length one in each of these copies we need at most $M$ letters. Hence $d \cdot M \cdot n$ is an upper bound for writing a word of length $n$ on each of the $d$ subtrees. The factor $1/K^2$ is explained as follows: We only look at the words that stabilize level 1. The subgroup $st_G(1)$ has index $K$, hence we look at one $K$-th of the elements of $B(n)$. The other $1/K$ comes from the fact that the $d$ subtrees can be permuted by conjugation. However, as $st_G(1)$ has index $K$, there are at most $K$ different such permutations. The power $d$ of $f(n)$ indicates that each different combination of conjugacy classes on the subtrees
gives a different element. We repeat this and get
\[ f((dM)^i n) \geq \frac{1}{(K^2)^i} \cdot f(n)^d. \]
Now we set \( n = 1 \) and get
\[ f((dM)^i) \geq \frac{1}{K^{2i}} \cdot 2^d, \]
because there are at least 2 conjugacy classes within the ball of radius 1. We now substitute \((dM)^i = k\) and get
\[ f(k) \geq \frac{1}{k^{2 \log d} \cdot 2^{\log (dM)}}. \]
We notice that the index of \( st_G(1) \) in \( G \) can be at most \( d! \), hence \( K \leq d! \). This allows us to estimate the degree of the polynomial factor as
\[ \frac{2 \log(K)}{\log (dM)} \leq \frac{2 \cdot \sum_{i=1}^{d} \log(i)}{\log(d) + \log(M)} \leq \frac{2(d-1) \log(d)}{\log(d) + \log(M)} \leq \frac{2d - 2}{1 + \frac{\log(M)}{\log(d)}}, \]
where the third estimate of \( d-1 \) many terms in the sum comes from the fact that \( \log(1) = 0 \) and \( \frac{\log(M)}{\log(d)} > 0 \).

As we will see in the next section, this bound is rather general. By knowing more about the structure of a specific branch group, this lower bound can in some cases be improved significantly. A similar approach can also be applied to branch groups which are not regular. However, in such a case, the constants \( M, K \) and \( d \) in the proof above would be different on every level of the tree. In fact, they might not even follow the same recursion. Hence it appears rather difficult to find bounds for the conjugacy growth of such a branch group with the approach that we have taken above. Examples of branch groups which are not regular can be found in [LS03] or [Fin14a].

Of course an upper bound for the conjugacy growth would be very interesting. This depends apriori heavily on the group, as some branch groups have intermediate word growth, others have exponential word growth. We however suspect, that for most branch groups an upper bound for the conjugacy growth will not be very different than the one for the word growth.

### 4.2 Grigorchuk group

We could now apply Theorem 4.2 to the Grigorchuk group. This group is acting on a binary tree, so we immediately get \( K = d = 2 \). For reasons that we do not want to elaborate here, we have \( M = 24 \), hence yielding a lower bound of \( \frac{1}{2} 2^{n^{1.18}} \) for the conjugacy growth function. However, by studying the structure of the Grigorchuk group more carefully, we obtain a better estimate. First we cite an auxiliary lemma that we will need in the proof.

**Lemma 4.3.** We have a recursive formula:
\[ (B(n), B(m))_1 \subseteq B(2(n + m)). \]
In words, with a word \( w \) of length \( 2(n + m) \) we can write any combination of words of lengths \( n \) respectively \( m \) on the two top subtrees.
4.2 Grigorchuk group

Proof. To see this, we use the standard form of a word in (2). We will show that it is possible to write any word of length $m$ on the right subtree without influencing the left subtree too much. Denote the two subwords by $w_0$ and $w_1$. Obviously $w_0$ and $w_1$ again have the form in (2). To write an $a$ as the first letter of $w_1$, we choose $w$ to begin with $aca$. To write any of the other generators, we use the substitution rules from (1). So to get $ab$ in $w_1$ we write $acad$:

$$w = acad \rightarrow (1, ab)_1$$

$$w = acab \rightarrow (1, ac)_1$$

$$w = acac \rightarrow (1, ad)_1 .$$

It is important to choose $aca$ in $w$ if we want to write $a$ in $w_1$. This will then leave us with a word $a^\varepsilon d^\delta$ on the left subtree, with $\varepsilon, \delta \in \{0, 1\}$. Now $ad$ has order 4. So in the worst case we are left with $adad$ on the left subtree:

$$w = \ldots \rightarrow (adad, w_1)_1 .$$

However, we want to choose the word $w_0$ freely. So if $w_0$ does not start with the word coming from writing $w_1$, we simply slightly modify our choice of letters for $w$ to write $w_1$. This can be done such that it does not affect $w_1$, but leaves us with $adac$, $acac$ or $acad$:

$$w = \ldots * aba * aca \rightarrow (acad, w_1)_1$$

$$w = \ldots * ba * aba \rightarrow (acac, w_1)_1$$

$$w = \ldots * aca * aba \rightarrow (adac, w_1)_1$$

If however one of the first occurrences in $w_1$ is the letter $b$, then we put $d$ into $w$ at the beginning:

$$w = d\ldots \rightarrow (b\ldots , w_1)_1 .$$

This will not affect $w_1$. We notice that with this we are already using 2 letters of $w$. However, we are also already gaining at least one letter in $w_0$. So in total, we need $2 \cdot (l(w_0) + l(w_1))$ letters to write any words $w_0, w_1$. 

First another simple lemma which we will need in the proof:

Lemma 4.4. There are at least 4 different conjugacy classes in the ball of radius 4.

Proof. Obviously two elements which are conjugate have the same order. To prove the lemma, we list 4 elements of length less than or equal to 4 that all have different orders and leave the easy verification to the reader:

$$a, ab, ac, ad : \quad o(a) = 2, \quad o(ab) = 16, \quad o(ac) = 8, \quad o(ad) = 4 .$$

We can now provide a better estimate for the bounds of the conjugacy growth function of the Grigorchuk group.
Theorem 4.5. The conjugacy growth function \( f(n) \) of the Grigorchuk group \( G \) satisfies
\[
\frac{1}{n} e^{n^{0.5}} \lesssim f(n) \lesssim \frac{1}{n} e^{n^{0.767}}.
\]

Proof. The upper bound is given by the word growth as computed by L. Bartholdi in [Bar98], taking into account that cyclic permutations are conjugates. For the lower bound, we look at the action of an element \( g \in G \) on the two subtrees of level 1. Let \( g \) be of length \( n \), and let \( h \) be conjugate to \( g \), of length less than or equal to \( n \). Let \( g \) act as \( g_0 \) and \( g_1 \) on the two subtrees \( T_0 \) respectively \( T_1 \) of level 1 and \( h \) as \( h_0 \) respectively \( h_1 \):
\[
g = (g_0, g_1), \quad h = (h_0, h_1).
\]

First assume that \( g \) fixes the first level, in other words, it contains even many times the generator \( a \). Of course \( h \) must then have the same property. In that case, \( h \) can only be conjugate to \( g \) if we have that either \( g_0 \) is conjugate to \( h_0 \) and \( g_1 \) to \( h_1 \), or we have that \( g_0 \) is conjugate to \( h_1 \) and \( g_1 \) to \( h_0 \). However, if \( g \) contains odd many times the generator \( a \), then \( g \) has the following form:
\[
g = x^{-1}gx = (x_1^{-1}, x_2^{-1}) (g_1, g_2) a (x_1, x_2) = (x_2^{-1}g_2x_1, x_1^{-1}g_1x_2) a.
\]

We rewrite this to
\[
(ga)^2 \cdot a = (g_1^{x_1}, g_2^{x_2}) \cdot (x_1^{-1}x_2, x_2^{-1}x_1) a.
\]

By Lemma 4.3 we can, with a word of length \( n \), write at least any combination of words of lengths \( \frac{n}{4} \) on the two top subtrees. So we can produce at least as many different classes of conjugates of words of length \( n \) as we can have different conjugates of words of length \( \frac{n}{4} \) on each subtree, divided by 2 since the two subtrees can be interchanged by conjugation with \( a \). We can express this recursively, where we have another factor \( \frac{1}{2} \) because we only count words which lie in the first level stabilizer:
\[
f(4n) \geq \frac{1}{2} \cdot \frac{1}{2} f(n)^2 \geq \frac{1}{4} f(n)^2.
\]

We repeat this to get
\[
f \left( 4^{i-1} n \right) \geq \frac{1}{4^{i-1}} f(n)^{2^{i-1}}.
\]

Now choosing \( n = 4 \) and a variable substitution \( k = 4^i \) then gives
\[
f(k) \geq \frac{4}{k} f(4)^{k^{0.5}}.
\]

Now \( f(4) \) is at least 4 by Lemma 4.4 and so we hence get
\[
f(k) \geq \frac{1}{k} 4^{k^{0.5}}
\]
and so
\[
f(k) \gtrsim \frac{1}{k} e^{k^{0.5}}.
\]
An interesting question informally asked by M. Sapir is whether there exist groups which have oscillating word growth, but non-oscillating conjugacy growth. In particular, one source of examples of groups with oscillating word growth is given by examples of L. Bartholdi and A. Erschler in \cite{BE11}.

**Question 1.** Do the groups of oscillating intermediate growth as defined in \cite{BE11} also have oscillating conjugacy growth?

We expect that the quotient \( q(n) = \frac{\gamma(n)}{f(n)} \) of the word growth \( \gamma(n) \) and the conjugacy growth \( f(n) \) grows very slowly for most branch groups. A lower bound of this quotient is of course given by \( q(n) \sim n \).

**Question 2.** What can be said about the quotient \( \frac{\gamma_G(n)}{f(n)} \) for the Grigorchuk group or for branch groups in general?

We emphasize, that even though the construction in \cite{BE11} is using the Grigorchuk group, the resulting groups are no longer branch.

## 5 Conjugacy Width

The aim of this section is to prove that every element \( g \in G \), where \( G \) is from a certain class of branch groups, can be written as a product of uniformly boundedly many conjugates of the generators. We call this property bounded conjugacy width (BCW).

We start with a general discussion about BCW and will see how it relates to other algebraic properties. This will emphasize, why some of the conditions we set for branch groups to have BCW are necessary. At the end, we draw a connection to the palindromic width of a group and deduce that the Grigorchuk group has finite palindromic width for all generating sets.

### 5.1 First results about BCW

In this subsection we discuss groups which have, or do not have, bounded conjugacy width. We first show that bounded conjugacy width implies finite commutator width. The converse however is not true, we will give examples of groups which have finite commutator width but unbounded conjugacy width. In fact, we will establish that no infinite group of polynomial growth can have BCW. We then show that BCW passes on to finite extensions. Further, we will prove that any group with only finitely many conjugacy classes has BCW and that BCW implies that the abelianisation of the group is finite. These are fairly straight-forward observations and we list them and sketch the proofs for completeness.

It is obvious that having BCW is independent of the chosen generating set. The following proposition says that if a group has bounded conjugacy width then it has finite commutator width, where all commutators have a specific, simple form.

**Proposition 5.1.** If a group \( H \), generated by a minimal set of generators \( X = \{x_1, \ldots, x_k\} \), has bounded conjugacy width \( N \), then it has finite commutator width at most \( 3N \).

**Proof.** Assume an element \( g \in H' \) is of the form

\[
h = x_{i_1}^{p_1} \cdots x_{i_n}^{p_n},
\]  

(3)
where the $x_{ij} \in X$ for the generating set $X$ and $\rho_i \in H$, $n \leq N$. If now $h \in H'$ we complete the product with

$$h = \left( \prod_{j=1}^{n} x_{ij}^{-1} x_{ij} \right) h = \left( \prod_{j=1}^{n} x_{ij}^{-1} x_{ij} \right) \prod_{j=1}^{n} x_{ij}^{\rho_j}.$$  

In order to write the expression as a product of commutators, we shift the factors $x_{ij}$ from the left side into the product on the right. We demonstrate this for the first factor $x_{i1}^{-1}$:

$$h = x_{i1} \cdot \left( \prod_{j=2}^{n} x_{ij}^{-1} x_{ij} \right) \cdot \left[ x_{i1}, \prod_{j=2}^{n} x_{ij}^{-1} x_{ij} \right] \cdot \prod_{j=2}^{n} x_{ij}^{\rho_j}.$$  

We are now left to move $n - 1$ factors $x_{i1}^{-1}$ into the product on the far right. One can see that repeating this procedure will result in a term composed of $\prod_{j=1}^{n} x_{ij} \cdot r$, where $r$ is a product of $2n$ commutators. We can now express the first few terms $z = \prod_{j=1}^{n} x_{ij}$ as an element of $H/H' \cdot H'$, hence we will get a product $z = x_{i1}^{\zeta_1} \cdots x_{ik}^{\zeta_k} \cdot f$, where $f$ is a product of at most $n$ commutators and $\zeta_i \in \mathbb{Z}$. By assumption $h$ was in $H'$, hence the first term $\prod_{i=1}^{k} x_{i1}^{\zeta_i}$ is equal to 1. In total we obtain a commutator width of at most $3N$.

We will now prove that no infinite nilpotent group can have bounded conjugacy width. By a result of M. Gromov ([Gro81]), this says that no group of non-constant polynomial growth can have BCW. It is known that all nilpotent groups have finite commutator width from P. Stroud’s thesis (see [Str66] for a reference), hence BCW is a stronger property than finite commutator width. This in particular implies that the converse of Proposition 5.1 is not true.

We first need the following observations, which we will then apply to nilpotent groups.

**Lemma 5.2.**  

1. If a finitely generated group $G$ has BCW, then its abelianisation $G/G'$ is finite.

2. If $G$ is a finitely generated nilpotent group with finite abelianisation $G/G'$, then $G$ is finite.

3. Let $G$ be a finitely generated infinite nilpotent group. Then $G$ does not have bounded conjugacy width.

**Proof.**  

1: Obviously BCW passes to quotients and no infinite abelian group can have BCW. This shows that $G/G'$ must be finite. 2: See [Seg83, p. 13, Corollary 9]. 3: Assume $G$ had BCW. Then by Lemma 5.1 it must have finite abelianisation. However, 2 implies that $G$ is finite, contradicting the assumption that $G$ is a finitely generated infinite nilpotent group.  

As an application of this we can show that BCW is a stronger property than finite commutator width.

**Theorem 5.3.** Any finitely generated infinite nilpotent group has finite commutator width but has unbounded conjugacy width.
The following lemma applies to the groups constructed by V. Ivanov ([Ols91]) or by D. Osin in [Osi10].

**Lemma 5.4.** Assume that a finitely generated group $G$ has only $n$ conjugacy classes. Then it has bounded conjugacy width.

**Proof.** Take for each conjugacy class a representative of shortest length. Because we only have $n$ conjugacy classes, we can take the maximum over the lengths of these representatives, denoted by $M$. Then it follows that each element of $G$ is a product of at most $M$ conjugates of the generators.

Together with Proposition 5.1 and Lemma 5.2(1) this implies that these groups have finite commutator width and finite abelianisation.

**Theorem 5.5.** If a finitely generated group $H$ has BCW and $G$ is a finite extension of $H$, then $G$ has BCW.

**Proof.** Let $H = \langle h_1, \ldots, h_m \rangle$ be a finite index subgroup of $G$, such that $H$ has BCW. Let $M$ be the maximal length of the generators of $H$ with respect to the finite generating set $X = \langle x_1, \ldots, x_n \rangle$ of $G$. Then by assumption, every element of $H$ can be written as

$$h = \prod_{i=1}^{K} h_i^{t_i} = \prod_{i=1}^{K} \left( \prod_{j=1}^{M} x_{k_{i,j}} \right)^{t_i} = \prod_{i=1}^{K} x_i^{s_i},$$

The latter is a finite product of conjugates of elements from $X$. Every element $g \in G$ can be written as $g = f \cdot h$ for $f \in G/N$, $h \in N$, where $N = \bigcap_{g \in H} H^g$, where again $N$ has finite index in $G$ because $H$ has. If we take for $f$ the coset representative of shortest length, then we can denote the maximum over all lengths of such minimal coset representatives by $T$. It is then clear that every element of $H$ is a product of at most $T + KM$ conjugates of the generators $\langle x_1, \ldots, x_n \rangle$.

This implies that the group of exponential conjugacy growth constructed by M. Hull and D. Osin [HO13] with the index 2 subgroup with 2 conjugacy classes has finite commutator width and finite abelianisation.

**Question 3.** Does there exist a group $G$ that has bounded conjugacy width, but contains a subgroup $H$ of finite index in $G$ such that $H$ does not have bounded conjugacy width?

### 5.2 Certain branch groups

The aim of this subsection is to demonstrate that if a branch group is just infinite, contains a rooted automorphism and has finite commutator width, then every element can be written as a product of uniformly many conjugates of the generators.
5.2 Certain branch groups

Lemma 5.6. Assume that $G$ is a branch group that contains a rooted element and has finite commutator width. Let $H, L$ be its branching structure as defined above. Then every element of the form $\left(\left[\kappa, \sigma\right], 1\right)_1, \kappa, \sigma \in L_1$ is a product of at most $4 \cdot M$ conjugates of the generators of $G$, where $M$ is the length of the shortest rooted element contained in $G$.

Proof. We aim to express a commutator $\left(\left[\kappa_1, \kappa_2\right], 1, \ldots, 1\right)_1 \in (L'_1, 1)_1$ as a product of conjugates of the generators. By assumption there exists a rooted element $x$. Without loss of generality we can assume that $x$ acts in such a way that it moves the leftmost top subtree to the second leftmost top subtree. Choose $\kappa = (\kappa^{-1}_1, 1, \ldots, 1)_1$ with $\kappa_1 \in L_1$. Then

$$t = x^\kappa x^{-1} = (\kappa_1, \kappa^{-1}_1, 1, \ldots, 1)_1.$$ 

We now proceed and conjugate $t$ with $\lambda = (\kappa_2, 1, \ldots, 1)_1$ to get

$$t^\lambda t^{-1} = (\kappa_1^2, \kappa_1, 1, \ldots, 1)_1 = ([\kappa_2, \kappa_1], 1, \ldots, 1)_1.$$ 

We see that this is a product of 4 conjugates of the element $x$, hence of $4 \cdot l(x)$ conjugates of the generators of $G$.

Lemma 5.7. Let $G$ be a just infinite branch group that contains at least one rooted element. Then there exists a number $T$ such that every commutator of the form $[\alpha, \beta]$ for $\alpha, \beta \in G$ is a product of at most $T$ conjugates of the generators of $G$.

The bound $T$ can be explicitly expressed as $4M + S$, where $M$ is the maximal length of a minimal coset representative of the branching subgroup $H_1$ of $G$ and $S$ is the constant coming from Lemma 5.6.

Proof. If $[\gamma, \xi]$ is a commutator with $\gamma, \xi \in G$, then we can write $\gamma, \xi$ as $\gamma = \sigma \kappa, \xi = \tau \lambda$ with $\kappa, \lambda \in H_1$. Because $H_1$ has finite index in $G$, there are only finitely many minimal choices for $\sigma, \tau$, hence their length is uniformly bounded over all elements. Denote the maximal length of a coset representative by $M$ and denote $\sigma = x_1 \cdots x_n, \tau = y_1 \cdots y_n$ for $n \leq M$, where the $x_i$ and $y_i$ are some generators of $G$. The commutator $[\gamma, \xi]$ can with the help of basic commutator identities be written as

$$[\gamma, \xi] = [\sigma \kappa, \tau \lambda] = [\sigma, \tau \lambda]\kappa[\kappa, \lambda] = [x_1 \cdots x_n, \tau \lambda]\kappa[\kappa, \lambda][\kappa, \lambda] = \left(\prod_{i=1}^{n} [x_i, \tau \lambda]\kappa[i] \right) [\kappa, \lambda] \left(\prod_{i=1}^{n} [\kappa, y_{n-i+1}]\eta[i] \right),$$

where $\zeta = \kappa \cdot \prod_{i=1}^{n} x_i$ and $\eta = \lambda \cdot \prod_{i=1}^{n} y_i$. By assumption we have $\kappa, \lambda \in H_1 = L_1 \times L_1$, so $[\kappa, \lambda] \in (L'_1, 1)_1$ and it actually has the form $(\left[\kappa_0, \lambda_0\right], [\kappa_1, \lambda_1])_1$. By Lemma 5.6 there exists a number $t$ such that the commutator in each component is a product of at most $t$ conjugates of the generators. Each commutator of the form $[x_i, \rho]$ for some $\rho \in G$ is a product of 2 conjugates of $x_i$:

$$[x_i, \rho] = x_i^{-1} \cdot x_i^\rho.$$ 

So we get $2 \cdot 2n$ conjugates for the commutators and $2 \cdot t$ more for the commutator $[\kappa, \lambda]$. In total we hence need $4n + 2t$ conjugates of the generators to express a commutator of $G$.

Theorem 5.8. Let $G$ be a just infinite branch group that contains a rooted element and that has finite commutator width. Then $G$ has bounded conjugacy width.
Proof. Because \( G \) is just infinite the normal subgroup \( G' \) of \( G \) has finite index in \( G \). Hence every element can be written in the form \( \gamma = \xi \cdot \rho \), for some \( \rho \in G' \) and there are only finitely many minimal choices for \( \xi \). Denote by \( M \) the length of the longest minimal coset representative. The fact that \( G \) has finite commutator width gives us that \( \rho \) is a product of at most \( C \) commutators of the form \([x, y], x, y \in G\). By Lemma 5.7 there exists a number \( T \) such that each of them is a product of at most \( T \) conjugates of the generators of \( G \). Hence every element of \( G \) is a product of at most \( M + C \cdot T \) conjugates of the generators of \( G \).

We can see from Proposition 5.1 that the condition of having finite commutator width is necessary. However, proving that a group has BCW would also provide an effective way to prove that it has finite commutator width. To see that the condition to be just infinite is necessary, we need another theorem.

Theorem 5.9. \([\text{Gri00}]\) A branch group \( G \) is just infinite if and only if all \( H'_i \) have finite index in \( H_i \).

In particular, this implies that \( G \) cannot be just infinite if \( H_0 = G \) does not have finite abelianisation. Hence we obtain that the condition to be just infinite cannot be omitted.

Corollary 5.10. There exist branch groups which do not have BCW.

Proof. The groups studied by the author in \([\text{Fin14a}]\) are not just infinite, hence cannot have BCW.

At this moment, the Grigorchuk group is the only known branch group that satisfies all hypotheses of Theorem 5.8. In particular it has been shown to have finite commutator width by I. Lysenok, A. Miasnikov and A. Ushakov (LMU13).

Corollary 5.11. The Grigorchuk group has bounded conjugacy width.

Computer experiments of L. Bartholdi have suggested that for the Grigorchuk group we might in fact have that every element of \( G' \) is a product of only four conjugates of the generator \( a \). This would in particular imply that its commutator width is at most 2 because

\[
a^x a^y = x^{-1} xy^{-1} [xy^{-1}, a] a y = [xy^{-1}, a]^y,
\]

which uses in particular that \( a^2 = 1 \). This leads to the following open question:

Question 4. Does the Grigorchuk group \( G \) have commutator width 2?

Because of the above computer experiments, if there exists an element \( g \in G' \) which is not a product of 2 commutators, then its length in the standard generators must be at least 17.

5.3 Palindromes

A palindrome is a group word which reads the same left-to-right as right-to-left. It has been studied over the last decade by various authors whether a group has the property that every element is a product of uniformly boundedly many palindromes, see \([\text{BG14c}], [\text{BG14a}], [\text{BG14b}], [\text{Fin14b}], [\text{RS14}]\). This notion is not known to be independent of the generating set and many examples depend on a specific generating set. In some cases, the question of bounded conjugacy width and finite palindromic width coincide:
Lemma 5.12. If $G$ has a generating set $X = \langle x_1, \ldots, x_n \rangle$ where every $x_i^2 = 1$, then the palindromes in $G$ with respect to this generating set are exactly the conjugates of the generators $x_i$.

Proof. We note that if every generator has order 2, then taking inverses amounts to writing a word backwards.

It has been shown by A. Thom and the author in [FT14], that if a group is just infinite, then after a possibly slight modification of the generating set, it will have finite palindromic width. This modification in particular rules out that every generator has order 2. Here we prove that the Grigorchuk group has bounded conjugacy width, hence together with the result from [FT14] it follows that

Corollary 5.13. The Grigorchuk group has finite palindromic width with respect to all generating sets.

For a more detailed study of palindromic width we recommend any of the papers mentioned above.

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