The fundamental solution of a class of ultra-hyperbolic operators on Pseudo $H$-type groups

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Abstract

Pseudo $H$-type Lie groups $G_{r,s}$ of signature $(r,s)$ are defined via a module action of the Clifford algebra $C\ell_{r,s}$ on a vector space $V \cong \mathbb{R}^{2n}$. They form a subclass of all 2-step nilpotent Lie groups and based on their algebraic structure they can be equipped with a left-invariant pseudo-Riemannian metric. Let $\mathcal{N}_{r,s}$ denote the Lie algebra corresponding to $G_{r,s}$. A choice of left-invariant vector fields $[X_1, \ldots, X_{2n}]$ which generate a complement of the center of $\mathcal{N}_{r,s}$ gives rise to a second order operator

$$\Delta_{r,s} := (X_1^2 + \ldots + X_r^2) - (X_{r+1}^2 + \ldots + X_{2n}^2),$$

which we call ultra-hyperbolic. In terms of classical special functions we present families of fundamental solutions of $\Delta_{r,s}$ in the case $r = 0, s > 0$ and study their properties. In the case of $r > 0$ we prove that $\Delta_{r,s}$ admits no fundamental solution in the space of tempered distributions. Finally we discuss the local solvability of $\Delta_{r,s}$ and the existence of a fundamental solution in the space of Schwartz distributions.

**keywords:** left invariant homogeneous differential operator, distributions, local solvability, Bessel functions

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1 Introduction

The study of existence and explicit representations of a fundamental solution to various geometrically induced differential operators has stimulated some research during the last decades (cf. [6, 8, 14, 18, 21, 22, 24, 26]). In this paper we are concerned with such a problem in case of a second order homogeneous differential operator $\Delta_{r,s}$ which is induced by a pseudo $H$-type Lie group $G_{r,s}$ (see Section 2 for the precise definitions). We write $\mathcal{N}_{r,s}$ for the Lie algebra of $G_{r,s}$ and in the standard way we identify $G_{r,s}$ and $\mathcal{N}_{r,s}$ through the exponential map. The Lie algebra $\mathcal{N}_{r,s}$ is nilpotent of step two and it admits a decomposition $\mathcal{N}_{r,s} = V \oplus Z$ into its center $Z$ and a complement. The subspace $V$ is even dimensional and generates a left invariant distribution on $G_{r,s}$ spanned by $\{X_1, \ldots, X_{2n}\}$. Moreover, $G_{r,s}$ is equipped with a pseudo-Riemannian left-invariant metric which has signature $(n, n)$ when it is restricted to the distribution. The differential operator $\Delta_{r,s}$ induced by this setup has the explicit form

$$\Delta_{r,s} := (X_1^2 + \ldots + X_n^2) - (X_{n+1}^2 + \ldots + X_{2n}^2).$$

Due to its similarity with the classical ultra-hyperbolic operator

$$\mathcal{L} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial x_{n+j}^2}$$

on $\mathbb{R}^{2n}$, we call $\Delta_{r,s}$ an ultra-hyperbolic operator on the Lie group $G_{r,s}$. We note that $\Delta_{r,s}$ degenerates on the center $Z$ and different from $\mathcal{L}$ it has non-constant coefficients such that the theorem of Malgrange-Ehrenpreis on the existence of a fundamental solution is not at disposal. In the special case where $G_{r,s}$ is the Heisenberg group $G_{0,1}$ an explicit representation of a fundamental solution of (1) has been previously obtained in [22, 26].

The global solutions to the equation $\Delta_{r,s}u = 0$ and some of its generalizations on the Heisenberg group has been studied in [16, 17]. Therein the Heisenberg group was realized as a flag manifold $G/Q$ of some classical group $G$ factored by a parabolic subgroup $Q$ and the differential operators are acting on the sections of certain line bundles. It has been observed in [2, 13] that some of the pseudo $H$-type groups can also be interpreted as flag manifolds. In these cases the action of the ultra-hyperbolic operator may be considered on the corresponding bundle.

The goals of this paper are as follows:

(a) Characterize the pairs $(r, s)$ for which the ultra-hyperbolic operator $\Delta_{r,s}$ admits a fundamental solution within the space of tempered or Schwartz distributions.

(b) Explicitly derive a class of fundamental solutions in the space of tempered distributions in all cases in which the existence is guaranteed.

(c) Characterize the local solvability of the ultra-hyperbolic operator $\Delta_{r,s}$.

Our strategy of deriving the fundamental solution of (1) in the most general setting is based on the following observation by J. Tie in [26]. In the case of signature $(r, s) = (0, 1)$
it can be verified that a suitable change from real to complex variables in (1) transforms
the ultra-hyperbolic operator \( \Delta_{0,1} \) into a sub-Laplace operator \( \Delta_{\text{sub}} \) on \( G_{0,1} \). Based on
the sub-ellipticity of \( \Delta_{\text{sub}} \) the existence of the heat kernel and a fundamental solution is
guaranteed. Performing the same transformation to an explicit form of the fundamental
solution of \( \Delta_{\text{sub}} \), produces - after a regularization procedure - a fundamental solution
of the ultra-hyperbolic operator \( \Delta_{0,1} \). In case of an arbitrary pseudo \( H \)-type group
with signature \((r,s)\) a similar change of variables in \( \Delta_{r,s} \) formally gives the sub-Laplace
operator \( \Delta_{\text{sub}} \) corresponding to a certain 2-step nilpotent Lie group. The heat kernel
of \( \Delta_{\text{sub}} \) is known explicitly (c.f. [5, 7, 11]) and can be used to calculate a fundamental
solution of \( \Delta_{\text{sub}} \) via an integration over the "time variable". In the same way as before
the corresponding change of variables in the fundamental solution of \( \Delta_{\text{sub}} \) produces a
formal expression of a fundamental solution of \( \Delta_{r,s} \). However, a regularization process
which rigorously defines a tempered distribution seems only possible in the case \( r = 0 \).
As it turns out this is not an artifact. In the last part of the paper we will show that
for \( r > 0 \) a fundamental solution of \( \Delta_{r,s} \) in the space of tempered distributions or even
within the Schwartz distributions does not exist.

The non-uniqueness of a fundamental solution for a class of second order differential
operators on a Lie group which contains \( \Delta_{r,s} \) in a very special case was observed in
[21,22,26]. In our general setting we will present an uncountable family of fundamental
solutions of \( \Delta_{0,s} \), \( s \geq 1 \) in (1) and relate them to classical special functions (e.g. Bessel
functions of the first and second kind). One of the fundamental solutions \( K_{0,1} \) of the
ultra-hyperbolic operator on the Heisenberg group obtained in [26] coincides with an
iterated integral expression which previously and by different methods was presented by
D. Müller and F. Ricci in [22]. This fact was already noticed by J. Tie in [26] and since
our approach generalizes the expression presented there, it is not surprising that we can
detect \( K_{0,1} \) among the above mentioned family of distributions. In a second step we
generalize D. Müller and F. Ricci’s formula to the case \( s > 1 \) and \( r = 0 \).

Finally we consider the ultra-hyperbolic operator \( \Delta_{r,s} \) in the case \( r > 0 \). It is known
that a left-invariant differential operator on a Lie group \( G \) in general does not possesses
a global fundamental solution among the tempered distributions \( S'(G) \). Additional
assumptions on the group and the operator have to be made to guarantee the existence
(see [23] and the references therein). In Theorem 9.11 we prove that in the case \( r > 0 \)
the ultra-hyperbolic operator \( \Delta_{r,s} \) does not admit a fundamental solution in \( S'(G_{r,s}) \).
One may pose the question whether one can invert \( \Delta_{r,s} \) in the larger space \( D'(G_{r,s}) \) of
Schwartz distributions. On the homogeneous Lie group \( G_{r,s} \) this question is known to
be related to the local solvability of the operator (see [3,20]). Based on a criterion by
D. Müller in [20] Theorem 10.2 we show that \( \Delta_{r,s} \) is not locally solvable if and only if
\( r > 0 \). Using this fact we can answer the above question in a negative sense and prove
non-existence of a fundamental solution of \( \Delta_{r,s}, r > 0 \) in \( D'(G_{r,s}) \).

The paper is organized as follows.

In Section 2 we recall the notion of a pseudo \( H \)-type group attached to a \( CL_{r,s} \)-Clifford module and we present two low dimensional examples of such groups. In suitable
coordinates we explicitly represent the left-invariant ultra-hyperbolic operator $\Delta_{r,s}$ as a difference of two sum-of-squares operators.

Section 3 contains more details on the ultra-hyperbolic operator together with a few technical calculations that play a role in the subsequent analysis. In particular, we consider $\Delta_{r,s}$ after a partial Fourier transform as an operator on the Schwartz space.

Via a suitable change from real to complex coordinates we relate the ultra-hyperbolic operator $\Delta_{r,s}$ to a (hypo-elliptic) sub-Laplace operator $\Delta_{sub}$ on a 2-step nilpotent Lie group in Section 4.

In Section 5 we rigorously show that the distribution derived in Section 4 in fact defines a fundamental solution $K_{0,s}$ of $\Delta_{0,s}$, i.e., in the case where $r = 0$ and $s \geq 1$.

We note that $K_{0,s}$ is not the unique fundamental solution and we relate it to classical special function (Bessel functions of the first and second kind) in Section 6. As a result we present an uncountable family of fundamental solutions.

In Section 7 we present a second form of a (specific) fundamental solution $K_{0,s}$ of $\Delta_{0,s}$ which was obtained in the previous chapter. In the case of $G_{0,1}$ this form coincides with the expression obtained in [21, 26]. However, since the condition $s = 1$ is not required in our setup we have obtained a generalization of the distributions in [21, 26] to Lie groups $G_{0,s}$ with center dimension $s > 1$.

Section 8 discusses the singularities of a fundamental solution in the case $r = 0$. In particular, we determine a cone in $\mathbb{R}^{2n+1}$ containing its singular support.

The remaining two chapters are concerned with the invertibility of $\Delta_{r,s}$ in the case $r > 0$. In Section 9 we prove that in this case no fundamental solution among the tempered distributions exists. In the final Section 10 we relate our results to the problem of local solvability of $\Delta_{r,s}$ for $r > 0$. Based on a theorem by D. M"{u}ller in [20] we show non-local solvability and non-existence of a fundamental solution in the Schwartz distributions $\mathcal{D}'(G_{r,s})$.

In the appendix we relate a family of classical distributions associated to a non-degenerate quadratic form on $\mathbb{R}^{2n}$ in [14] to the representation of the fundamental solution of $\Delta_{0,s}$ which was obtained in Chapter 7.

2 Pseudo $H$-type groups

Let $r, s \in \mathbb{N}_0$ and consider $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$ with the non-degenerate bilinear form (scalar product)

$$(x, y)_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}$$

and the corresponding quadratic form $q_{r,s}(x) := (x, x)_{r,s}$. We denote by $\mathcal{C}\ell_{r,s}$ the Clifford algebra generated by $(\mathbb{R}^{r,s}, q_{r,s})$. Let $V$ be a $\mathcal{C}\ell_{r,s}$-Clifford module, i.e., $V$ is a real vector space with module action

$$J : \mathcal{C}\ell_{r,s} \times V \to V.$$

For $z \in \mathbb{R}^{r,s}$ we use the notation $J_z := J(z, \cdot) : V \to V$. Moreover, we write $\mathbb{R}(n)$ for the set of all $n \times n$ matrices with entries in $\mathbb{R}$. 

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Definition 1. We call the module $V$ of the Clifford algebra $\text{C}ℓ_{r,s}$ admissible if it carries a non-degenerate symmetric bilinear form $\langle \cdot , \cdot \rangle_V$ which satisfies the following properties:

\begin{align*}
\langle J_z X, J_z Y \rangle_V &= \langle z, z \rangle_{r,s} \langle X, Y \rangle_V, \\
\langle J_z X, Y \rangle_V &= -\langle X, J_z Y \rangle_V, \\
J_z^2 &= -\langle z, z \rangle_{r,s} I,
\end{align*}

where $I$ denotes the identity operator. Note that conditions (3) and (4) are equivalent if property (5) holds. The existence of admissible modules for the Clifford algebra $\text{C}ℓ_{r,s}$ was shown in [9, Theorem 2.1]. The following is known, see [9, Proposition 2.2]:

Lemma 2.1. If $s > 0$, then $(V, \langle \cdot , \cdot \rangle_V)$ has positive definite and negative definite subspaces of the same dimension. In particular, we have $\dim V = 2n$ for some $n \in \mathbb{N}$.

In what follows we consider the case $s > 0$ only. We may define a Lie bracket $[\cdot , \cdot ] : V \times V \to \mathbb{R}^{r,s}$ through the following equation

$$\langle J_z X, Y \rangle_V = \langle z, [X,Y] \rangle_{r,s}, \quad z \in \mathbb{R}^{r,s}, \ X, Y \in V.$$ 

Definition 2. Let $V$ be an admissible $\text{C}ℓ_{r,s}$-module. With the above bracket relation and centre $\mathbb{R}^{r,s}$ the space

$$\mathcal{N}_{r,s} := V \oplus \mathbb{R}^{r,s}$$

defines a 2-step nilpotent Lie algebra which we call pseudo H-type algebra. Further information on the algebraic properties of such algebras and an analysis of associated second order differential operators can be found in [4, 9, 12].

Example 2.2. (The Heisenberg algebra $\mathcal{N}_{0,1}$) We represent the Heisenberg Lie algebra as a pseudo H-type algebra via an admissible $\text{C}ℓ_{0,1}$-module. Let $z \in \mathbb{R}^{0,1}$ such that $\langle z, z \rangle_{0,1} = -1$ and consider $V := \mathbb{R}^n \times \mathbb{R}^n$ with basis $(v_1, \ldots, v_n, w_1, \ldots, w_n)$. Define

$$\langle v_i, v_j \rangle_V = \delta_{ij}, \quad \langle w_i, w_j \rangle_V = -\delta_{ij}, \quad \langle v_i, w_j \rangle = 0.$$ 

We put $J_z v_i = w_i$, $J_z w_i = v_i$
and extend $J_z$ by linearity. One obtains:

$$\langle [v_i, w_i], z \rangle_{0,1} = \langle J_z v_i, w_i \rangle_V = \langle J_z w_i, J_z v_i \rangle_V = \langle z, z \rangle_{0,1} \langle v_i, v_i \rangle_V = -1.$$ 

Thus, $[v_i, w_i] = z$, whereas the other commutators vanish.

Example 2.3. (The algebra $\mathcal{N}_{1,1}$) We choose a basis $\{z_1, z_2\}$ of $\mathbb{R}^{1,1}$ with

$$\langle z_1, z_1 \rangle_{1,1} = 1, \quad \langle z_2, z_2 \rangle_{1,1} = -1, \quad \langle z_1, z_2 \rangle_{1,1} = 0.$$ 

Then $\text{C}ℓ_{1,1} \cong \mathbb{R}(2)$ and an admissible module $V$ of minimal dimension is 4-dimensional. In this case we may choose $v \in V$ with $\langle v, v \rangle_V = 1$ and put

$$X_1 = v, \quad X_2 = J_{z_1} v, \quad X_3 = J_{z_1} J_{z_2} v, \quad X_4 = J_{z_2} v.$$ 

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We obtain the following table of commutation relations of $\mathcal{N}_{1,1}$:

| \( [\text{row, column}] \) | \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) |
|----------------|----------|----------|----------|----------|
| \( X_1 \)     | 0        | \( z_1 \) | 0        | \( z_2 \) |
| \( X_2 \)     | \( -z_1 \) | 0        | \( -z_2 \) | 0        |
| \( X_3 \)     | 0        | \( z_2 \) | 0        | \( z_1 \) |
| \( X_4 \)     | \( -z_2 \) | 0        | \( -z_1 \) | 0        |

Moreover, we may show that

\[ \langle X_2, X_2 \rangle_V = 1, \quad \langle X_3, X_3 \rangle_V = -1, \quad \langle X_4, X_4 \rangle_V = -1. \]

For a 2-step nilpotent Lie algebra \( \mathfrak{g} = T_e G \) induced by a connected and simply connected Lie group \( G \), the exponential map \( \exp : \mathfrak{g} \to G \) becomes a diffeomorphism and therefore allows us to identify \( \mathfrak{g} \) and \( G \). From the Baker-Campbell Hausdorff formula, which states for a 2-step nilpotent Lie algebra that

\[ \exp(X) \ast \exp(Y) = \exp \left( X + Y + \frac{1}{2} [X,Y] \right), \]

we may reconstruct the group structure. In what follows we will write \( G_{r,s} \) for the connected, simply connected nilpotent Lie group with Lie algebra \( \mathcal{N}_{r,s} \) and call it \textit{pseudo H-type group}. Note that with the above identification and as a vector space we have

\[ G_{r,s} \cong \mathcal{N}_{r,s} \cong \mathbb{R}^{2n+r+s}. \]

On the pseudo H-type algebra \( \mathcal{N}_{r,s} \) we may define a scalar product by

\[ \langle x + z, x' + z' \rangle_{\mathcal{N}_{r,s}} = \langle x, x' \rangle_V + \langle z, z' \rangle_{r,s} \quad \text{where} \quad x + z, x' + z' \in V \oplus \mathbb{R}^{r+s} \]

and extend it to a left-invariant pseudo-Riemannian metric on \( G_{r,s} \).

We now write up the construction more explicitly. Given a pseudo H-type algebra \( \mathcal{N}_{r,s} = V \oplus \mathbb{R}^{r+s} \) with \( V = \text{span} \{ X_j : j = 1, \ldots, 2n \} \) and \( \mathbb{R}^{r,s} = \text{span} \{ Z_k : k = 1, \ldots, r+s \} \) we may identify the generators \( X_j \) of a complement of the center and \( Z_k \) of the center, respectively, with left-invariant vector fields on \( G_{r,s} \cong \mathbb{R}^{2n+r+s} \) as follows:

\[ X_j := \frac{\partial}{\partial x_j} + \sum_{m=1}^{2n} \sum_{k=1}^{r+s} a_{mj} x_m \frac{\partial}{\partial z_k}, \quad j = 1, \ldots, 2n, \quad Z_k := \frac{\partial}{\partial z_k}, \quad k = 1, \ldots, r+s. \]

We assume that

\[ \langle X_i, X_j \rangle_V = \delta_{ij}, \quad i, j \in \{1, \ldots, n\}, \]
\[ \langle X_i, X_j \rangle_V = -\delta_{ij}, \quad i, j \in \{n+1, \ldots, 2n\} \]
\[ \langle X_i, X_j \rangle_V = 0 \quad i \in \{1, \ldots, n\}, \quad j \in \{n+1, \ldots, 2n\}, \]

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The structure constants satisfy $a^k_{ij} = -a^k_{ji}$ and are defined through the commutation relations, i.e., we have

$$[X_i, X_j] = 2 \sum_{k=1}^{r+s} a^k_{ij} Z_k, \quad i, j \in \{1, \ldots, 2n\}.$$ 

In what follows we form the matrices $\Omega_k \in \mathbb{R}(2n)$ with entries $(\Omega_k)_{ij} := a^k_{ij}$. We also denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product on $\mathbb{R}^{2n}$. Then the corresponding Lie group action on $G_{r,s} \cong \mathbb{R}^{2n+r+s}$ is given by

$$(x, z) \ast (y, w) = (x + y, z + w + \sum_{k=1}^{r+s} \langle \Omega_k^T x, y \rangle e_k), \quad (8)$$

where $e_k \in \mathbb{R}^{r+s}$ is the $k$-th canonical unit vector. In what follows we put

$$\Omega(\eta) := \eta_1 \Omega_1 + \ldots + \eta_{r+s} \Omega_{r+s}, \quad \eta \in \mathbb{R}^{r+s}.$$ 

Note that $\Omega(\eta)^T = -\Omega(\eta)$. With the obvious notation the vector fields $X_j$ can be expressed in the form:

$$X_j = \frac{\partial}{\partial x_j} \left( \Omega \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{r+s}} \right) x \right), \quad (9)$$

In what follows we want to consider the second order differential operator $\Delta_{r,s}$ defined in (11) and associated with the pseudo $H$-type group $G_{r,s}$. We call $\Delta_{r,s}$ an ultra-hyperbolic operator, due to its similarity with the classical ultra-hyperbolic operator $L$ in (2) acting on $\mathbb{R}^{2n}$ (see [13, 24]). Our aim is to calculate a fundamental solution for this operator, i.e., we look for a distribution $K_{r,s} \in S'((\mathbb{R}^{2n+r+s})$, which satisfies

$$\Delta_{r,s} K_{r,s} = \delta_0,$$

where $\delta_0$ is the Dirac distribution centered at $0 \in \mathbb{R}^{2n+r+s}$. Note that different from $L$ the operator $\Delta_{r,s}$ does not have constant coefficients and therefore even the existence of such a fundamental solution is not guaranteed. The differential operator $\Delta_{r,s}$ is built from left-invariant vector fields and therefore left-invariant itself. Let $g \in G_{r,s}$ be arbitrary and $K$ a fundamental solution of $\Delta_{r,s}$ as above. With the left-translation:

$$L_g : S((\mathbb{R}^{2n+r+s}) \to S((\mathbb{R}^{2n+r+s}) : \varphi \mapsto L_g(\varphi) = \varphi(g \ast \cdot)$$

consider the distribution $K_g := K \circ L_g$. Then by the last remark $K_g$ solves the equation

$$\Delta_{r,s} K_g = \delta_g,$$

where $\delta_g(\varphi) = \varphi(g)$ denotes the evaluation in $g$. In the formulas below we will observe a close relation between $\Delta_{r,s}$ and $L$ similar to the relations between the sub-Laplacian $\Delta_{\text{sub}} := -\sum_{j=1}^{2n} X_j^2$ on 2-step nilpotent Lie groups and the Laplacian on $\mathbb{R}^{2n}$. 7
3 The ultra-hyperbolic operator

Our approach is based on a formal observation by J. Tie in [26] where a fundamental solution of the ultra-hyperbolic operator was constructed in the special case of the Heisenberg Lie algebra $N_{0,1} \cong \mathbb{R}^{2n} \oplus \mathbb{R}^{0,1}$ (cf. Example 2.2). Here the vector fields in (7) take the form:

$$X_j = \frac{\partial}{\partial x_j} - \frac{x_{j+n}}{2} \frac{\partial}{\partial z}, \quad \text{and} \quad X_{j+n} = \frac{\partial}{\partial x_{j+n}} + \frac{x_j}{2} \frac{\partial}{\partial z}, \quad j = 1, \ldots, n.$$ 

More precisely, it is shown that

$$K_{r,s}(x,z) = \frac{i^{n+1} \Gamma(n)}{8\pi^{n+1}} \int_\mathbb{R} \frac{1}{\left\{ \frac{n}{4} (\sum_{j=1}^n x_j^2 - x_j^{2+n}) \coth \left( \frac{\eta}{4} \right) \right\}^n} \left( \frac{\eta}{4 \sinh \left( \frac{\eta}{4} \right)} \right)^n d\eta$$

is a fundamental solution for (11). Note that $K_{r,s}(x,z)$ needs to be regularized, however a suitable regularization is shown to converge in the space of tempered distributions. Moreover, the structure of the singular support was discussed in [26] and the construction of $K_{r,s}(x,z)$ was reduced to determine the inverse symbol of $\Delta_{r,s}$ in the framework of a pseudo-differential calculus. However, Formula (10) may also be deduced by a formal change of coordinates. Putting

$$\begin{cases}
y_j = -ix_j, & j \in \{1, \ldots, n\}, \\
y_{j+n} = x_{j+n}, & j \in \{1, \ldots, n\}, \\
w = -iz,
\end{cases}$$

we obtain for $j \in \{1, \ldots, n\}$ that

$$\frac{\partial}{\partial x_j} = \frac{\partial y_j}{\partial x_j} = -i \frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial x_{j+n}} = \frac{\partial}{\partial y_{j+n}}, \quad \frac{\partial}{\partial z} = -i \frac{\partial}{\partial w}.$$ 

Then the operator $\Delta_{r,s}$ transforms formally to the corresponding hypoelliptic sub-Laplacian

$$\Delta_{\text{sub}} := \sum_{j=1}^n -Y_j^2 - Y_{j+n}^2,$$

where we have

$$Y_j = \frac{\partial}{\partial y_j} - \frac{y_{j+n}}{2} \frac{\partial}{\partial w}, \quad Y_{j+n} = \frac{\partial}{\partial y_{j+n}} + \frac{y_j}{2} \frac{\partial}{\partial w}, \quad j \in \{1, \ldots, n\}.$$ 

As is well known (e.g. [6]) a fundamental solution of the sub-Laplacian $\Delta_{\text{sub}}$ is given by

$$K_{\text{sub}}(y,w) := \frac{\Gamma(n)}{8\pi^{n+1}} \int_\mathbb{R} \frac{1}{\left\{ \frac{n}{4} \sum_{j=1}^n y_j^2 + y_{j+n}^2 \right\} \coth \left( \frac{\eta}{4} \right) + i\omega \eta}^n \left( \frac{\eta}{4 \sinh \left( \frac{\eta}{4} \right)} \right)^n d\eta.$$
Applying again (11) and transforming $\eta \mapsto -\eta$ we obtain (10) up to the factor $i^{n+1}$, which formally may be interpreted as a "complex Jacobian".

In the setting of a general pseudo $H$-type group we aim to use a similar approach to obtain a fundamental solution for the ultra-hyperbolic operator. For ease of notation we work with the full symbol of the operator, which according to (9) is given by:

$$\sigma(\Delta_{r,s})(x, z, \xi, \eta) = \sum_{j=1}^{n} -\left\{ \xi_j - (\Omega(\eta)x)_j \right\}^2 + \left\{ \xi_{j+n} - (\Omega(\eta)x)_{j+n} \right\}^2.$$ 

As a first step we start by listing some properties of the matrix $\Omega(\eta)$ that reflect the pseudo $H$-type structure. To this end we consider for $\eta \in \mathbb{R}^{r,s}$ the matrix $\rho(\eta) \in \mathbb{R}^{(2n)}$ describing the Clifford module action, i.e., we have

$$J_{Z(\eta)}X_j = \sum_{l=1}^{2n} \rho(\eta)_{lj}X_l,$$ 

where $Z(\eta) = \eta_1 Z_1 + \ldots + \eta_r Z_r + \eta_{r+s} Z_{r+s}$.

Definition 1 (3) implies that:

$$\rho(\eta)^2 = -\langle \eta, \eta \rangle_{r,s} I.$$ (12)

We put $V_+ := \text{span}\{X_1, \ldots, X_n\}$ and $V_- := \text{span}\{X_{n+1}, \ldots, X_{2n}\}$, where $\langle \cdot, \cdot \rangle_V$ is positive definite on $V_+$ and negative definite on $V_-$. In what follows we call the elements of $V_+$ positive and of $V_-$ negative. Notice that $\rho = \rho(\eta)$ is linear with respect to $\eta$ by the definition of $J_{Z(\eta)}$. With the identity element $I \in \mathbb{R}^n$ we define the matrix

$$\tau := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \in \mathbb{R}^{(2n)}.$$ 

Let us express elements $x, y \in V = V_+ \oplus V_-$ and linear maps $A$ on $V$ with respect to the coordinates $X_j$, $j = 1, \ldots, 2n$. Then we obtain with the euclidean inner product $\langle \cdot, \cdot \rangle_V$:

$$\langle Ax, y \rangle_V = \langle Ax, \tau y \rangle = \langle x, A^T \tau y \rangle = \langle x, \tau A^T \tau y \rangle_V.$$ 

Hence we observe that the transpose $A^*$ of $A$ with respect to the non-degenerate bilinear form $\langle \cdot, \cdot \rangle_V$ is given by

$$A^* = \tau A^T \tau.$$ (13)

The properties of the Clifford module action $J_z$ in Definition 1 together with the identity (13) imply that $\rho(\eta) = -\rho(\eta)^*$ for all $\eta \in \mathbb{R}^{r,s}$ and therefore:

**Lemma 3.1.** For $\eta \in \mathbb{R}^{r,s} \cong \mathbb{R}^r \times \mathbb{R}^s$ we have

$$\tau \rho(\eta) = -\tau (\rho(\eta))^T = -\rho(\eta)^T \tau.$$ 

**Remark 3.2.** Let $Z$ be chosen such that $\langle Z, Z \rangle_{r,s} > 0$. According to Definition 1 (1) it follows that $J_Z$ maps positive elements to positive elements and negative elements to negative elements, i.e., $J_Z$ leaves $V_+$ and $V_-$ invariant. Moreover, if $Z$ satisfies $\langle Z, Z \rangle_{r,s} <$
0 then $J_Z$ maps $V_+$ to $V_-$ and vice versa. Together with the previous lemma this implies that for $\eta = (\eta_+, \eta_-) \in \mathbb{R}^r \times \mathbb{R}^s$ the matrix $\rho(\eta)$ may be written as

$$\rho(\eta) = \begin{pmatrix} A(\eta_+) & B(\eta_-) \\ B(\eta_-)^T & D(\eta_+) \end{pmatrix},$$

such that the following assertions hold true:

1. $A(\eta_+)$ and $D(\eta_-)$ are skew-adjoint,
2. $A(\eta_+)^2 = -|\eta_+|^2I = D(\eta_-)^2$ and $B(\eta_-)^TB(\eta_-) = |\eta_-|^2I = B(\eta_-)B(\eta_-)^T$,
3. $A(\eta_+)B(\eta_-) + B(\eta_-)D(\eta_+) = 0$ and $B(\eta_-)^TA(\eta_+) + D(\eta_+)B(\eta_-)^T = 0$.

Here $|\cdot|^2$ denotes the euclidean square norm of a vector.

For the coefficients $a_{ij}^k = \Omega(Z_k)_{ij}$ we have

$$2a_{ij}^k(Z_k, Z_k)_{r,s} = \langle X_i, X_j, Z_k \rangle_{r,s} = \langle JZ_kX_i, X_j \rangle_V = \langle \rho(Z_k) \rangle_{j\ell}(X_j, X_j)_{V}$$

and thus, we obtain for $\eta \in \mathbb{R}^{r,s}$ that

$$\begin{cases} \langle \Omega(\eta_+) \rangle_{ij} = \frac{1}{2}(\rho(\eta_+)^T)_{ij} & \text{for } j = 1, \ldots, n, \\ \langle \Omega(\eta_-) \rangle_{ij} = \frac{1}{2}(\rho(\eta_-)^T)_{ij} & \text{for } j = n+1, \ldots, 2n, \\ \langle \Omega(\eta_-) \rangle_{ij} = -\frac{1}{2}(\rho(\eta_-)^T)_{ij} & \text{for } j = 1, \ldots, n, \\ \langle \Omega(\eta_+) \rangle_{ij} = -\frac{1}{2}(\rho(\eta_+)^T)_{ij} & \text{for } j = n+1, \ldots, 2n. \end{cases}$$

As a consequence the relation between the matrices $\Omega(\eta)$ and $\rho(\eta)$ is given by:

$$\Omega(\eta) = \frac{1}{2} [\tau \rho(\eta)^T], \quad \text{where} \quad \eta \in \mathbb{R}^{r,s}. \quad (15)$$

We collect some properties of the matrix $\Omega(\eta)$:

**Lemma 3.3.** Let $\eta \in \mathbb{R}^{r+s} \cong \mathbb{R}^r \times \mathbb{R}^s$. Then

1. The matrix $\Omega(\eta)$ fulfills the relations:

$$[\tau \Omega(\eta)]^2 = [\Omega(\eta)\tau]^2 = -\frac{1}{4} \langle \eta, \eta \rangle_{r,s} I.$$

   In particular, since $\Omega(\eta)$ is skew-symmetric we have:

$$\Omega(\eta)^T \tau \Omega(\eta) = \Omega(\eta) \tau \Omega(\eta)^T = -\frac{1}{4} \langle \eta, \eta \rangle_{r,s} I.$$

2. If $\langle \eta, \eta \rangle_{r,s} > 0$, then the matrices $\tau \Omega(\eta)$ and $\Omega(\eta)\tau$ only have purely imaginary eigenvalues

$$\lambda_{\eta, \pm} = \pm i \frac{\sqrt{\langle \eta, \eta \rangle_{r,s}}}{2}.$$

Both eigenvalues $\lambda_{\eta,+}$ and $\lambda_{\eta,-}$ have multiplicity $n.$
Proof. From (12) and the above relation between $\Omega(\eta)$ and $\rho(\eta)$ we have:

$$\left[\tau\Omega(\eta)\right]^2 = \frac{1}{4}\left[\rho(\eta)^T\right]^2 = -\frac{\langle\eta,\eta\rangle_{r,s}}{4}I$$

and likewise we obtain the other assertions in (1). By (1) the minimal polynomial of $\Omega(\eta)$ and $\tau\Omega(\eta)$ has the form

$$p_\eta(\lambda) = \lambda^2 + \frac{\langle\eta,\eta\rangle_{r,s}}{4} = \left(\lambda + i\frac{\sqrt{\langle\eta,\eta\rangle_{r,s}}}{2}\right)\left(\lambda - i\frac{\sqrt{\langle\eta,\eta\rangle_{r,s}}}{2}\right)$$

showing the assertion in (2). \hfill \Box

In our analysis we will make use of the Fourier transform with respect to different groups of variables. First we determine an operator $\Delta_{r,s}(\eta)$ such that

$$F_z \mapsto \eta \Delta_{r,s}(\eta) \varphi = \Delta_{r,s}(\eta) F_z \mapsto \eta \varphi, \quad \varphi \in S(\mathbb{R}^{2n+r+s})$$

holds, where $F_{z\mapsto \eta}$ denotes the Fourier transform with respect to the $z$-variables. Let us express $\Delta_{r,s}$ in the above coordinates in (7):

$$\Delta_{r,s} = \sum_{j=1}^{n} \left\{ \frac{\partial}{\partial x_j} + \frac{2n}{r+s} \sum_{m=1}^{r+s} \sum_{k=1}^{2n} a_{m,j+k} x_m \frac{\partial}{\partial z_k} \right\}^2 - \left\{ \frac{\partial}{\partial x_{j+n}} + \frac{2n}{r+s} \sum_{m=1}^{r+s} \sum_{k=1}^{2n} a_{m,j+n+k} x_m \frac{\partial}{\partial z_k} \right\}^2.$$

Thus, we have

$$\Delta_{r,s}(\eta) = \sum_{j=1}^{n} \left\{ \frac{\partial}{\partial x_j} + i \sum_{m=1}^{r+s} \sum_{k=1}^{2n} a_{m,j+k} x_m \right\}^2 - \left\{ \frac{\partial}{\partial x_{j+n}} + i \sum_{m=1}^{r+s} \sum_{k=1}^{2n} a_{m,j+n+k} x_m \right\}^2$$

$$= L - \sum_{j=1}^{n} \left\{ \sum_{k=1}^{r+s} \sum_{m=1}^{2n} a_{m,j+k} x_m \right\}^2 + \sum_{j=1}^{n} \left\{ \sum_{k=1}^{r+s} \sum_{m=1}^{2n} a_{m,j+n+k} x_m \right\}^2$$

$$+ 2i \sum_{j=1}^{n} \left\{ \sum_{k=1}^{r+s} \sum_{m=1}^{2n} a_{m,j+k} x_m \frac{\partial}{\partial x_j} - \sum_{k=1}^{r+s} \sum_{m=1}^{2n} a_{m,j+n+k} x_m \frac{\partial}{\partial x_{j+n}} \right\},$$

where $L$ denotes the classical ultra-hyperbolic operator (2). Note that we have used that $a_{jj}^k = 0$ due to the skew-symmetry of $\Omega_k$. In order to simplify the expression we use that

$$\sum_{k=1}^{r+s} \sum_{m=1}^{2n} a_{m,j+k} x_m = (\Omega(\eta)^T x)_j,$$

which implies

$$\Delta_{r,s}(\eta) = L - \sum_{j=1}^{n} \left\{ (\Omega(\eta)^T x)^2_j - (\Omega(\eta)^T x)_{j+n}^2 \right\} + 2i (\tau\Omega(\eta)^T x) \cdot \nabla_x.$$
Applying Lemma 3.3 we can calculate the sum on the right hand side:

\[
\sum_{j=1}^{n} \left\{ (\Omega(\eta)^T x)^2_j - (\Omega(\eta)^T x^2)_{j+n} \right\} = (\tau \Omega(\eta)^T x) \cdot (\Omega(\eta)^T x) = (\Omega(\eta)^T \tau \Omega(\eta)^T x) \cdot x = \frac{1}{4} \langle \eta, \eta \rangle_{r,s} P(x),
\]

where we define:

\[
P(x) := \langle \eta, \eta \rangle = \sum_{j=1}^{n} x_j^2 - x_{j+n}^2.
\]

Finally, we note that \( \tau \Omega(\eta)^T = -\frac{1}{2} \rho(\eta)^T \), which proves:

**Lemma 3.4.** With the above notation we have:

\[
\Delta_{r,s}(\eta) = L - \frac{\langle \eta, \eta \rangle_{r,s}}{4} P(x) + i x^T \rho(\eta) \nabla_x.
\]

Let now \( F \) denote the Fourier transform on the full space \( \mathbb{R}^{2n+r+s} \). Consider the differential operator \( G_{r,s} \) on \( \mathcal{S}(\mathbb{R}^{2n+r+s}) \) defined through the equation:

\[
F \circ \Delta_{r,s} = G_{r,s} \circ F.
\]

**Lemma 3.4** allows us to easily calculate \( G_{r,s} \) in an explicit form:

**Corollary 3.5.** The operator \( G_{r,s} \) with (17) acts on \( \varphi = \varphi(\xi, \eta_+, \eta_-) \in \mathcal{S}(\mathbb{R}^{2n+r+s}) \) as:

\[
G_{r,s} \varphi = -P(\xi) \varphi + \frac{\langle \eta, \eta \rangle_{r,s}}{4} L \varphi + i \xi^T \rho(\eta)^T \nabla_\xi \varphi,
\]

where \( L \) is the ultra-hyperbolic operator in \( \mathbb{R}^{2n} \) with respect to the variables \( \xi \in \mathbb{R}^{2n} \). Both operators \( G_{r,s} = F \circ \Delta_{r,s} \circ F^{-1} \) and \( \Delta_{r,s} \) are formally selfadjoint on \( L^2(\mathbb{R}^{2n+r+s}) \).

**Proof.** The last statement follows from the observation that \( \Delta_{r,s} \) is formed by \( 2n \) squares of skew-symmetric vector fields. \( \square \)

## 4 From the Sub-Laplacian to the ultra-hyperbolic operator

From the above considerations we can express the full symbol of the ultra-hyperbolic operator \( \Delta_{r,s} \) as follows:

\[
\sigma(\Delta_{r,s})(x, z, \xi, \eta) = -P(\xi) - \frac{\langle \eta, \eta \rangle_{r,s}}{4} P(x) + x^T \rho(\eta) \xi.
\]

Throughout the paper we use the decomposition \( x = (x_+, x_-) \in \mathbb{R}^n \times \mathbb{R}^n \) and \( z = (z_+, z_-) \in \mathbb{R}^r \times \mathbb{R}^s \). As in the case of the Heisenberg algebra we formally transform variables and put

\[
\begin{align*}
  y_+ &= -ix_+,
  y_- &= x_-,
  w_+ &= z_+,
  w_- &= -iz_-. 
\end{align*}
\]
By \((\zeta_+, \zeta_-, \vartheta_+, \vartheta_-) = (i\xi_+, \xi_-, \eta_+, i\eta_-)\) we denote the variables dual to \((y_+, y_- w_+, w_-)\). According to the matrix representation in (14) we can be interpreted \(\rho(\eta)\) as a \(C\)-linear family of linear maps on \(C^{2n}\). Writing the symbol \(\sigma(\Delta_{r,s})\) in new coordinates, we obtain

\[
\sigma(\Delta_{r,s})(y, w, \zeta, \vartheta) = |\zeta|^2 + \frac{|y|^2}{4} + (iy_+, y_-) \rho(\vartheta_+, -i\vartheta_-) \left( -i\zeta_+ \right) + \frac{|y|^2}{4} + y^T \Xi(\vartheta) \zeta,
\]

where as before \(| \cdot |\) denotes the Euclidean norm and

\[
\Xi(\vartheta) := \begin{pmatrix} A(\vartheta_+) & B(\vartheta_-) \\ -B(\vartheta_-)^T & D(\vartheta_+) \end{pmatrix}, \quad \vartheta = (\vartheta_+, \vartheta_-).
\]

Using the properties of the matrices \(A, B, D\) in Remark 3.2 we obtain that \(\tilde{\Xi}(\vartheta) := \frac{1}{2} \Xi(\vartheta)\) is skew-symmetric and \(\tilde{\Xi}(\vartheta)^2 = -\frac{1}{4} |\vartheta|^2 \mathbf{I}\). As above we have

\[
\sigma(\Delta_{r,s})(y, w, \zeta, \vartheta) = \sum_{j=1}^{2n} \left\{ \zeta_j + \left( \tilde{\Xi}(\vartheta)^T y \right)_j \right\}^2,
\]

which is exactly the full symbol of the corresponding sub-Laplacian \(\Delta_{\text{sub}}\) of a two-step nilpotent Lie group with structure matrix \(\tilde{\Xi}(\vartheta)\). In the next step we calculate a fundamental solution of the sub-Laplacian \(\Delta_{\text{sub}}\) from the well known expression of its heat kernel (see [5, 7]).

**Remark 4.1.** Let \(\partial_t + \frac{1}{2} \Delta_{\text{sub}}\) be the sub-elliptic heat operator. Then the heat kernel \(k : (0, \infty) \times \mathbb{R}^{2n+r+s} \rightarrow \mathbb{R}\) uniquely is defined by the conditions

1. \((\partial_t + \frac{1}{2} \Delta_{\text{sub}})k = 0\) and

2. \(\lim_{t \downarrow 0} k(t, \cdot) = \delta_0\), in the sense of distributions.

In what follows we will write \(k_t(\cdot)\) instead of \(k(t, \cdot)\). For a 2-step nilpotent Lie group it is well known (see e.g. [5] or [7, Theorem 10.2.7]) that the heat kernel explicitly can be written explicitly as an integral:

\[
k_t(y, w) = \frac{1}{(2\pi t)^{n+r+s}} \int_{\mathbb{R}^{2n+r+s}} e^{-\frac{T(y, w, \vartheta)}{4t}} W(|\vartheta|) \, d\vartheta,
\]

where

\[
W(|\vartheta|) = \sqrt{\det \frac{\tilde{\Xi}(i\vartheta)}{\sinh(\tilde{\Xi}(i\vartheta))}} = \left( \frac{|\vartheta|^2}{\sinh \left( \frac{|\vartheta|^2}{2} \right)} \right)^n,
\]
is the so-called *volume element* and

\[ f(y, w, \vartheta) = i\langle \vartheta, w \rangle + \frac{|\vartheta|}{4} \coth \left( \frac{|\vartheta|}{2} \right) \sum_{j=1}^{2n} y_j^2 \]

is the *action function*. If we put \( p := n + r + s - 1 \), then the integral

\[ K_{\text{sub}}(y, w) = \int_0^\infty k_{2t}(y, w) \, dt \]

\[ = \frac{1}{(4\pi)^{p+1}} \int_0^\infty \int_{R^{r+s}} \frac{1}{t^{p+1}} \exp \left\{ -\frac{1}{2t} \left[ i\langle \vartheta, w \rangle + \frac{|\vartheta|}{4} \coth \left( \frac{|\vartheta|}{2} \right) \sum_{j=1}^{2n} y_j^2 \right] \right\} W(|\vartheta|) \, d\vartheta \, dt \]

gives formally a fundamental solution of the sub-Laplacian. Indeed, we have

\[ \Delta_{\text{sub}} K_{\text{sub}} = \int_0^\infty \Delta_{\text{sub}} k_{2t} \, dt = -\int_0^\infty \frac{\partial}{\partial t} k_{2t} \, dt = -\left[ k_{2t} \right]_0^\infty = \delta_0. \]

In the following steps we will perform a series of formal calculations changing from real to complex variables in the above integral representation of \( K_{\text{sub}}(y, w) \). Convergence of the integral expressions is not guaranteed in each step. However, this formal consideration produces a distribution which in Section 5 will rigorously be shown to be a fundamental solution of the ultra-hyperbolic operator when \( r = 0 \) and \( s > 0 \). If we use the change of variables (18) in \( K_{\text{sub}}(y, w) \), then we obtain the new kernel:

\[ K_{r,s}(x, z) = \]
\[ = \frac{i^{n+r+s}}{(4\pi)^{p+1}} \int_0^\infty \int_{R^{r+s}} \frac{1}{t^{p+1}} \exp \left\{ -\frac{1}{2t} \left[ i\langle \vartheta, z+ \rangle + \langle \vartheta, z- \rangle - \kappa(|\vartheta|)P(x) \right] \right\} W(|\vartheta|) \, d\vartheta \, dt, \]

where

\[ P(x) := \sum_{j=1}^n x_j^2 - x_{n+j}^2 \quad \text{and} \quad \kappa(|\vartheta|) := \frac{|\vartheta|}{4} \coth \left( \frac{|\vartheta|}{2} \right). \quad (19) \]

The factor \( i^{n+r+s} \) in front of the integral should be interpreted as a complex Jacobian. Note that \( \kappa(\rho) = \frac{\rho}{4} \coth \left( \frac{\rho}{2} \right) \) continuously extends to \( \rho = 0 \) if we define: \( \kappa(0) := \frac{1}{2} \).

In what follows we restrict ourself to the case \( r = 0 \). The existence of a fundamental solution of \( \Delta_{r,s} \) for \( r > 0 \) will be discussed in Section 5. We then have \( \vartheta = \vartheta_-, z = z_- \in \mathbb{R}^n \) and

\[ K_{0,s}(x, z) = \frac{i^{n+s}}{(4\pi)^{n+s}} \int_0^\infty \int_{R^n} \frac{1}{t^{n+s}} \exp \left\{ -\frac{1}{2t} \left[ \langle \vartheta, z \rangle - \kappa(|\vartheta|)P(x) \right] \right\} W(|\vartheta|) \, d\vartheta \, dt. \]

Take a function \( \varphi \in S(\mathbb{R}^{2n+s}) \) and consider a formal integral:

\[
\int_{\mathbb{R}^{n+s}} \varphi(x, z) K_{0,s}(x, z) \, dx \, dz = \]
\[ = \frac{i^{n+s}}{(4\pi)^{n+s}} \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{t^{n+s}} e^{-\frac{1}{2t} \left[ \langle \vartheta, z \rangle - \kappa(|\vartheta|)P(x) \right]} W(|\vartheta|) \varphi(x, z) \, dx \, dz \, d\vartheta \, dt = (*). \]
We change variables $\vartheta \to -\vartheta \in \mathbb{R}^s$ and formally we replace the integration over $t \in \mathbb{R}$ by an integration over $it$ where $t \in \mathbb{R}$:

\[
(*) = \frac{i}{(4\pi)^{n+s}} \int_0^\infty \int_{\mathbb{R}^{2n+s}} \frac{1}{t^{n+s}} e^{\frac{it}{2}} \left[ (-\vartheta, x) - \kappa(|\vartheta|) P(x) \right] W(|\vartheta|) \varphi(x, z) \, dz \, d\vartheta \, dt.
\]

Now, we perform a Fourier transform $F_{x \to \xi}$ in the last integral with respect to $x \in \mathbb{R}^{2n}$. Lemma 4.2 below provides a useful identity, which we apply in the calculation:

**Lemma 4.2.** Let $\beta_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \ldots, 2n$, with $\text{Re}(\beta_j) \geq 0$. Then

\[
F_{x \to \xi}^{-1} \left[ \exp \left\{- \frac{1}{2} \sum_{j=1}^{2n} x_j^2 \beta_j \right\} \right] (\xi) = \left( \prod_{j=1}^{2n} \frac{1}{\beta_j} \right) \exp \left\{- \frac{1}{2} \sum_{j=1}^{2n} \xi_j^2 \beta_j \right\}.
\]

In this formula the holomorphic branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ is chosen such that $\sqrt{\beta_j} > 0$ for $\beta_j > 0$.

In fact, the above identity follows easily for $\beta_j > 0$. By analytic continuation and continuity we observe that the equality holds true for $\text{Re}(\beta_j) \geq 0$. In our case we set

\[
\beta_j := \begin{cases}
\kappa(|\vartheta|), & \text{if } j = 1, \ldots, n, \\
-\kappa(|\vartheta|), & \text{if } j = n + 1, \ldots, 2n.
\end{cases}
\]

Note that $\beta_{j+n} = \bar{\beta}_j$ for $j = 1, \ldots, n$ and thus,

\[
\prod_{j=1}^{2n} \frac{1}{\sqrt{\beta_j}} = \prod_{j=1}^{n} |\beta_j|^{-1} = |\beta_1|^{-n} = \frac{t^n}{\kappa(|\vartheta|)^n} \frac{(4t)^n}{|\vartheta|^n} \tanh^n \left( \frac{|\vartheta|}{2} \right).
\]

Lemma 4.2 shows for all $t > 0$:

\[
F_{x \to \xi}^{-1} \left[ \exp \left\{- \frac{i\kappa(|\vartheta|) P(x)}{2t} \right\} \right] (\xi) = \frac{t^n}{\kappa(|\vartheta|)^n} \exp \left\{ \frac{it}{2} \frac{P(\xi)}{\kappa(|\vartheta|)} \right\}.
\]

We use this formula on the right hand side of $(*)$ to obtain:

\[
(*) = \frac{i(2\pi)^{\frac{n}{2}}}{(4\pi)^{n+s}} \int_0^\infty \int_{\mathbb{R}^{2n+s}} \frac{1}{t^{n+s}} \frac{W(|\vartheta|)}{\kappa(|\vartheta|)^n} \exp \left\{ \frac{it}{2} \frac{P(\xi)}{\kappa(|\vartheta|)} \right\} [F\varphi] \left( \xi, \frac{\vartheta}{2t} \right) \, d\xi \, dz \, d\vartheta \, dt,
\]

where $F$ denotes the Fourier transform on $\mathbb{R}^{2n+s}$. Recall that for $t > 0$:

\[
W(t) = \left( \frac{t}{2 \sinh \left( \frac{t}{2} \right)} \right)^n \frac{4^n}{t^n} \frac{\tanh \left( \frac{t}{2} \right)^n}{\cosh \left( \frac{t}{2} \right)}.
\]

We insert the last relation and change variables $\vartheta \to 2t \vartheta$ in the integration over $\mathbb{R}^s$ to obtain:

\[
(*) = \frac{i}{(2\pi)^{n+s}} \int_0^\infty \int_{\mathbb{R}^{2n+s}} \frac{[F\varphi](\xi, \vartheta)}{\cosh \left( \frac{t}{2} |\vartheta| \right)} \exp \left\{ \frac{\tanh(t|\vartheta|)}{|\vartheta|} \frac{P(\xi)}{|\vartheta|} \right\} \, d\xi \, d\vartheta \, dt.
\]
Finally, consider the change of variables \( t \mapsto \frac{1}{|\vartheta|} \) in the integration over \( \mathbb{R}_+ \). We obtain the expression:

\[
\int_{\mathbb{R}^{2n+s}} \varphi(x, z) K_{0,s}(x, z) \, dx \, dz =
\]

\[
= \frac{i}{(2\pi)^n \frac{n+s}{2}} \int_{\mathbb{R}^{2n+s}} \int_0^\infty \frac{1}{|\vartheta| \cosh^n t} \exp \left\{ \frac{i \tanh t}{|\vartheta|} P(\xi) \right\} \, dt \mathcal{F}[\varphi](\xi, \vartheta) \, d\xi \, d\vartheta.
\]

Hence let us define the kernel \( q(\xi, \vartheta) \) for \( \vartheta \neq 0 \) by:

\[
q(\xi, \vartheta) := \frac{i}{(2\pi)^n \frac{n+s}{2}} \int_0^\infty \frac{1}{|\vartheta| \cosh^n t} \exp \left\{ \frac{i \tanh t}{|\vartheta|} P(\xi) \right\} \, dt.
\]  \hfill (20)

5 The fundamental solution of \( \Delta_{0,s} \)

Assuming \( s > 1 \) we observe that the function \( q \) in (20) satisfies:

\[
q \in L^1(\mathbb{R}^{2n+s}; (1 + |\xi|^2 + |\vartheta|^2)^{-N} \, d(\xi, \vartheta))
\]

for sufficiently large \( N > 0 \). In particular, it defines a tempered distribution on \( \mathbb{R}^{2n+s} \) and we may define \( K_{0,s} \in \mathcal{S}'(\mathbb{R}^{2n+s}) \). Let \( \mathcal{F} \) denote the Fourier transform on \( \mathbb{R}^{2n+s} \). Given a rapidly decreasing function \( \varphi \in \mathcal{S}(\mathbb{R}^{2n+s}) \) we put

\[
K_{0,s}(\varphi) := \int_{\mathbb{R}^{2n+s}} q(\xi, \vartheta) [\mathcal{F}[\Delta_{0,s}\varphi]](\xi, \vartheta) \, d\xi \, d\vartheta.
\]  \hfill (21)

**Theorem 5.1.** The distribution \( K_{0,s} \) defines a fundamental solution of \( \Delta_{0,s} \), i.e., we have \( \Delta_{0,s} K_{0,s} = \delta_0 \) or equivalently

\[
K_{0,s}(\Delta_{0,s}\varphi) = \varphi(0), \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^{2n+s}).
\]

Since our derivation of Theorem 5.1 in Section 4 was purely formal, we have to provide a rigorous proof. To this end we have to show that:

\[
\int_{\mathbb{R}^{2n+s}} q(\xi, \vartheta) [\mathcal{F}\Delta_{0,s}\varphi](\xi, \vartheta) \, d\vartheta \, d\xi = \varphi(0) = \frac{1}{(2\pi)^{n+s}/2} \int_{\mathbb{R}^{2n+s}} [\mathcal{F}\varphi](\xi, \vartheta) \, d\xi \, d\vartheta.
\]

Let \( \mathcal{G}_{0,s} \) be the differential operator on \( \mathcal{S}(\mathbb{R}^{2n+s}) \) defined through the equation (17). As was shown in Corollary 3.5 the operator \( \mathcal{G}_{0,s} \) has the explicit form:

\[
\mathcal{G}_{0,s} = -P(\xi) - \frac{|\vartheta|^2}{4} \mathcal{L} + i \zeta^T \rho(\vartheta)^T \nabla \xi.
\]

Put

\[
\overline{\mathcal{G}}_{0,s} := -P(\xi) - \frac{|\vartheta|^2}{4} \mathcal{L} - i \zeta^T \rho(\vartheta)^T \nabla \xi.
\]
As $q \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^s \setminus \{0\})$ we have for each $\varphi \in \mathcal{S}(\mathbb{R}^{2n+} \times \mathbb{R}^s)$ by partial integration:

$$
\int_{\mathbb{R}^{2n+}} q(\xi, \vartheta)[\mathcal{F} \circ \Delta_{0,s} \varphi](\xi, \vartheta) \, d\vartheta \, d\xi =
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \int_{|\vartheta| > \varepsilon} q(\xi, \vartheta)[\mathcal{G}_{0,s} \circ \mathcal{F} \varphi](\xi, \vartheta) \, d\vartheta \, d\xi =
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \int_{|\vartheta| > \varepsilon} [\mathcal{G}_{0,s} q](\xi, \vartheta)[\mathcal{F} \varphi](\xi, \vartheta) \, d\vartheta \, d\xi.
$$

Thus, the proof of Theorem 5.1 follows from the next lemma.

**Lemma 5.2.** With the above notation we have $[\mathcal{G}_{0,s} q](\xi, \vartheta) = (2\pi)^{-n+s/2}$.

**Proof.** We note that $q$ is of the form $q(\xi, \vartheta) = a(P(\xi), \vartheta)$ where $a = a(v, \vartheta): \mathbb{R} \times \mathbb{R}^s \to \mathbb{C}$. Then we obtain by a straightforward calculation:

$$
[\mathcal{L} q](\xi, \vartheta) = 4P(\xi)\partial_\vartheta^2 a(P(\xi), \vartheta) + 4n\partial_v a(P(\xi), \vartheta)
$$

Moreover, one has:

$$
\xi^T \rho(\vartheta)^T (\nabla_\xi q)(\xi, \vartheta) = 2\langle \tau \rho(\vartheta)\xi, \xi \rangle \partial_v a(P(\xi), \vartheta) = 0,
$$

as $\tau \rho(\vartheta) = -2\tau \Omega(\vartheta)\tau$ is skew-adjoint. Thus, we have

$$
[\mathcal{G}_{0,s} q](\xi, \vartheta) = -P(\xi)a(P(\xi), \vartheta) - n|\vartheta|^2 \partial_v a(P(\xi), \vartheta) - |\vartheta|^2 P(\xi)\partial_\vartheta^2 a(P(\xi), \vartheta).
$$

Using the expression of the function $a(v, \vartheta)$ gives:

$$
[\mathcal{G}_{0,s} q](\xi, \vartheta) = \frac{i}{(2\pi)^{n+s/2}} \int_0^\infty \left\{ -in \tanh t + P(\xi) \frac{\tanh^2 t - 1}{|\vartheta|} \right\} \times
\frac{1}{\cosh^n t} \exp \left\{ \frac{i \tanh t}{|\vartheta|} P(\xi) \right\} \frac{1}{|\vartheta|^2} \cosh^n t dt
$$

and the assertion follows.

### 6 A family of fundamental solutions

Motivated by the previous section we want to discuss now possible restrictions on further fundamental solutions of the ultra-hyperbolic operator $\Delta_{0,s}$. To this end we define
Let\( A := \{(\xi, \vartheta) \in \mathbb{R}^{2n+s} : P(\xi) \neq 0 \text{ and } |\vartheta| \neq 0\} \) and we assume that \( \tilde{K}_{0,s} \) is a fundamental solution of \( \Delta_{0,s} \), which satisfies

\[
\tilde{K}_{0,s}(\varphi) = \int_A \tilde{q}(\xi, \vartheta) [\mathcal{F}\varphi](\xi, \vartheta) \, d\xi \, d\vartheta
\]

for all functions \( \varphi \in S(\mathbb{R}^{2n+s}) \) such that \( \text{supp}(\mathcal{F}\varphi) \subseteq A \). We additionally assume that \( \tilde{q} \in C^\infty(A) \) and

\[
\tilde{q}(\xi, \vartheta) = a(P(\xi), \vartheta), \quad \text{for some } a = a(v, \vartheta) \in C^\infty(\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^s \setminus \{0\}).
\]

As \( \tilde{K}_{0,s} \) was assumed to be a fundamental solution, we have \( \mathbb{K}_{0,s} \tilde{q} = (2\pi)^{-n+\frac{1}{2}} \) on \( A \), where \( \mathbb{K}_{0,s} \) is given as above. As before this implies

\[
(2\pi)^{-\frac{2n+1}{2}} = -P(\xi) a(P(\xi), \vartheta) - n|\vartheta|^2 \partial_\vartheta a(P(\xi), \vartheta) - |\vartheta|^2 P(\xi) \partial_\xi^2 a(P(\xi), \vartheta),
\]

and thus, \( a(\cdot, \vartheta) \) is a solution of the differential equation of second-order

\[
\gamma = -vf(v) - naf'(v) - \alpha vf''(v),
\]

where \( \gamma := (2\pi)^{-\frac{2n+1}{2}} \) and \( \alpha := |\vartheta|^2 \). The general solution may be calculated explicitly. To this end we use the ansatz \( f(v) = g(v/\sqrt{\alpha}) \), which leads to

\[
\frac{\gamma}{\sqrt{\alpha}} = -vg(v) - ng'(v) - vg''(v).
\]

Thus, for \( g(v) = v^{\frac{1-n}{2}} h(v) \) we obtain

\[
\frac{\gamma}{\sqrt{\alpha}} = \left\{-v^{\frac{4n-1}{2}+1} - \left(\frac{1-n}{2}\right) \left(n - \frac{n+1}{2}\right) v^{\frac{3n-2}{2}+1}\right\} h(v) - v^{\frac{2n}{2}+1} h'(v) - v^{\frac{4n}{2}+1} h''(v),
\]

and finally we have

\[
-\gamma \frac{v^{\frac{n+1}{2}-1}}{2} = \left\{v^2 - \left(\frac{n-1}{2}\right)^2\right\} h(v) + v^2 h'(v) + v^2 h''(v).
\]

The general solution of this differential equation is well-known and may be written as

\[
h(v) = \frac{i\gamma}{\sqrt{\alpha}} 2^{\frac{n-1}{2}-1} \sqrt{\pi} \Gamma\left(\frac{1}{2}\right) \left\{c_1 J_{n-1/2}(v) + c_2 Y_{n-1/2}(v) + iH_{n+1/2}(v)\right\},
\]

for coefficients \( c_1, c_2 \in \mathbb{C} \). Here \( J_{n-1/2} \) and \( Y_{n-1/2} \) are the Bessel functions of the first and second kind and \( H_{n+1/2} \) is the Struve function, which is defined via the integral representation below. Recall that \( H_{n+1/2} \) solves the inhomogeneous Bessel equation

\[
\frac{d}{dv} \left(v^{n+\frac{1}{2}} h(v)\right) + \frac{n+\frac{1}{2}}{v} h(v) = v^2 - \left(\frac{n-1}{2}\right)^2 h(v) + v^2 h'(v) + v^2 h''(v),
\]

where

\[
A := \{(\xi, \vartheta) \in \mathbb{R}^{2n+s} : P(\xi) \neq 0 \text{ and } |\vartheta| \neq 0\} \]
whereas $J_{n-\frac{1}{2}}$ and $Y_{n-\frac{1}{2}}$ solve the corresponding homogenous Bessel equation. This implies

$$\tilde{q}(\xi, \vartheta) = \frac{i}{2(2\pi)^{n+s/2} |\vartheta|} \left( \frac{2|\vartheta|}{P(\xi)} \right)^{n-\frac{1}{2}} \left\{ c_1(\vartheta) J_{n-\frac{1}{2}} \left( \frac{P(\xi)}{|\vartheta|} \right) + c_2(\vartheta) Y_{n-\frac{1}{2}} \left( \frac{P(\xi)}{|\vartheta|} \right) \right. + i H_{n-\frac{1}{2}} \left( \frac{P(\xi)}{|\vartheta|} \right) \right\}$$

for measurable functions $c_1, c_2 : \mathbb{R}^s \to \mathbb{C}$.

To recover the fundamental solution from the previous section we use for $\nu \geq \frac{1}{2}$ the well-known integral representation for the Bessel function and the Struve function (cf. formula 12.1.6 of Chapter 12 in [1]):

$$J_\nu(v) = \frac{2 \left( \frac{v}{2} \right)^\nu}{\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right)} \int_0^1 (1 - \rho^2)^{\nu-\frac{1}{2}} \cos(v \rho) \, d\rho,$$

$$H_\nu(v) = \frac{2 \left( \frac{v}{2} \right)^\nu}{\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right)} \int_0^1 (1 - \rho^2)^{\nu-\frac{1}{2}} \sin(v \rho) \, d\rho.$$  

The transformation $\rho = \tanh t$ gives us $\frac{d\rho}{dt} = 1 - \rho^2$, $\cosh^2 t = \frac{1}{1-\rho^2}$, and thus, we obtain

$$J_{n-\frac{1}{2}}(v) + i H_{n-\frac{1}{2}}(v) = \frac{2 \left( \frac{v}{2} \right)^\nu}{\sqrt{\pi} \Gamma \left( \frac{\nu}{2} + \frac{1}{2} \right)} \int_0^\infty \exp \left\{ iv \tanh t \right\} \cosh^n t \, dt.$$  

Thus, the case $c_1 \equiv 1$ and $c_2 \equiv 0$ will give the fundamental solution $q = \tilde{q}$ in (20) which was obtained in the previous section.

Conversely, we may provide functions $c_1, c_2 : \mathbb{R}^s \to \mathbb{C}$ and try to obtain further fundamental solutions. To this end we assume that $c_1$ is (for the sake of simplicity) bounded and that $c_2 \equiv 0$. Define $\lambda := \frac{c_1}{2}, \mu := \frac{1-c_1}{2}$ and let

$$q_{0,s}^{\lambda,\mu}(\xi, \vartheta) := \frac{i}{2(2\pi)^{n+s/2} |\vartheta|} \left( \frac{2|\vartheta|}{P(\xi)} \right)^{n-\frac{1}{2}} \left\{ c_1(\vartheta) J_{n-\frac{1}{2}} \left( \frac{P(\xi)}{|\vartheta|} \right) + i H_{n-\frac{1}{2}} \left( \frac{P(\xi)}{|\vartheta|} \right) \right\}$$

$$= \frac{i}{(2\pi)^{n+s/2} |\vartheta|} \int_0^1 (1 - \rho^2)^{n-\frac{1}{2}} \{ \lambda(\vartheta) e^{\frac{P(\xi)}{|\vartheta|} \rho} - \mu(\vartheta) e^{-\frac{P(\xi)}{|\vartheta|} \rho} \} \, d\rho.$$  

Then

$$K_{0,s}^{\lambda,\mu}(\varphi) := \int_{\mathbb{R}^{2n+s}} q_{0,s}^{\lambda,\mu}(\xi, \vartheta) [\mathcal{F} \varphi](\xi, \vartheta) \, d(\xi, \vartheta)$$

is well-defined for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^{2n+s})$ and it gives a tempered distribution $K_{0,s}^{\lambda,\mu} \in \mathcal{S}'(\mathbb{R}^{2n+s})$. As in the previous chapter we obtain:

**Theorem 6.1.** For any pair of bounded functions $\lambda, \mu : \mathbb{R}^s \to \mathbb{C}$ which satisfies $\lambda + \mu \equiv 1$ we have that $K_{0,s}^{\lambda,\mu}$ defined by (23) is a fundamental solution of $\Delta_{0,s}$. 


The case \( c_2 \neq 0 \) is more involved since the Bessel function of the second kind has the following asymptotic behavior

\[
\left( \frac{v}{2} \right)^{-n-1} Y_{-n+1}(v) = -\frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi} \left( \frac{v}{2} \right)^{-(n-1)} + \mathcal{O}(v^{-(n-1)+1}), \quad v \to 0, \quad n > 1.
\]

Thus, a priori it is not clear how to define \( \tilde{K}_{0,s}(\varphi) \) for arbitrary \( \varphi \in \mathcal{S}(\mathbb{R}^{2n+s}) \), and once it has been defined this distribution not necessarily gives a fundamental solution.

**Example 6.2.** We consider the case \( c_1 \equiv 0, \ c_2 \equiv -i, \ n = 2 \) and we use the formula

\[
H_{\nu}(v) - Y_{\nu}(v) = 2 \left( \frac{v}{2} \right)^{\nu} \Gamma\left(\nu+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \int_0^\infty e^{-v\rho}(1+\rho^2)^{\nu-\frac{1}{2}} d\rho,
\]

(cf. formula 12.1.8, Chapter 12 of [1]). Note that \( \nu = \frac{n-1}{2} = \frac{1}{2} \) is a half-integer and therefore \( Y_{\frac{1}{2}}(v) \) is not defined for negative values of \( z \). In particular, the function (24) has a pole at \( z = 0 \). In the case where \( P(\xi) > 0 \), we have \( v = \frac{P(\xi)}{|\vartheta|} > 0 \) and using (24) in the expression of \( \tilde{q}(\xi, \vartheta) \) above gives:

\[
\tilde{q}(\xi, \vartheta) = \frac{-1}{(2\pi)^{2+s/2}} \frac{1}{P(\xi)} \frac{1}{P(\xi)} \int_{\mathbb{R}^s} \mathcal{F}\varphi(\cdot, \vartheta) d\vartheta,
\]

A possible candidate for a fundamental solution of \( \tilde{K}_{0,s} \) would be

\[
\tilde{K}_{0,s}(\varphi) = \frac{-1}{(2\pi)^{2+s/2}} \int_{\mathbb{R}^s} \frac{1}{P(\xi)} \mathcal{F}\varphi(\cdot, \vartheta) d\vartheta,
\]

where \( \frac{1}{P} \) is interpreted as a distribution on \( \mathbb{R}^4 \).

We finally discuss a possible choice of the distribution that appears in the last example. According to [14] we can interpret \( \frac{1}{P(\xi)} \) acting on suitable test functions \( \varphi \) on \( \mathbb{R}^{2n} \) in different ways. With \( \lambda \in \mathbb{C} \) such that \( \text{Re}(\lambda) > 0 \) and using the notation of [14] we consider

\[
(P^\pm_\lambda, \varphi) := \int_{\pm P(\xi) > 0} (\pm P)^\lambda \varphi d\xi \quad \text{and} \quad (P \pm i0)^\lambda := P^\lambda_+ + e^{\pm \pi \lambda} P^\lambda_-.
\]

The following results can be found in Chapter 12 of [14]:

**Proposition 6.3.** The maps \( \lambda \mapsto (P \pm i0)^\lambda, \varphi \) admit a meromorphic extension to the complex plane with only simple poles at most at \( -n, -n-1, -n-2, \ldots \).

We now fix the variable \( \vartheta \in \mathbb{R}^s \) and we consider \( \Delta_{0,s} \) as an operator with respect to \( \xi \in \mathbb{R}^{2n} \). Assuming that the real part of \( \lambda \) is sufficiently large so that all boundary
integrals that appear via partial integration vanish we calculate:

\[
(P^\lambda_{\pm}, \mathcal{F}[\Delta_{0,s}\varphi(\cdot, \vartheta)]) = \int_{\pm P(\xi) > 0} (\pm P(\xi))^\lambda G_{0,s}[\mathcal{F}\varphi](\xi, \vartheta) \, d\xi
\]

\[
= -\int_{\pm P(\xi) > 0} (\pm P(\xi))^\lambda \left\{ P(\xi) + \frac{\left| \vartheta \right|^2}{4} \right\} [\mathcal{F}\varphi](\xi, \vartheta) \, d\xi.
\]

Clearly, the same formula holds true if we replace the distributions \( P^\lambda_{\pm} \) by \( (P \pm i0)^\lambda \) above. Now, we assume again that \( n = 2 \) and we rewrite the last equation in the form:

\[
(P - i0)^\lambda, \mathcal{F}[\Delta_{0,s}\varphi(\cdot, \vartheta)]\big|_{\lambda = -1} = -\int_{\mathbb{R}^4} [\mathcal{F}\varphi](\xi, \vartheta) \, d\xi - \frac{\left| \vartheta \right|^2}{4} \left( (P - i0)^\lambda, \mathcal{L}[\mathcal{F}\varphi](\cdot, \vartheta) \right)\big|_{\lambda = -1}.
\]

The following relations are well-known (cf. [14], p. 258 and formula (4) on p. 277 with \( k = 0 \)) and can be applied to the right hand side of the equation:

\[
(P^\lambda_{\pm -1}, \varphi) = \frac{\pm 1}{4\lambda(\lambda + 1)} (P^\lambda_{\pm}, \mathcal{L}\varphi) \quad \text{and} \quad \text{res}_{\lambda = -2} (P + i0)^\lambda = -\pi^2 \delta_0,
\]

where \( \delta_0 \) means the point evaluation at zero. Therefore we have:

\[
(P - i0)^\lambda, \mathcal{L}\varphi\big|_{\lambda = -1} = -4\text{res}_{\lambda = -2} ((P + i0)^\lambda, \varphi) = (2\pi)^2 \varphi(0).
\]

If we would interpret the action of the distribution \( \frac{1}{P} \) in (25) as \( (P - i0)^\lambda \big|_{\lambda = -1} \), then:

\[
\tilde{K}_{0,s}(\Delta_{0,s}\varphi) = -\frac{1}{(2\pi)^{2+s}} \int_{\mathbb{R}^s} \left( (P - i0)^\lambda, \mathcal{F}[\Delta_{0,s}\varphi(\cdot, \vartheta)] \right)_{\lambda = -1} \, d\vartheta
\]

\[
= \frac{1}{(2\pi)^{2+s}} \int_{\mathbb{R}^{s+s}} \mathcal{F}\varphi(\xi, \vartheta) \, d(\xi, \vartheta) + \frac{1}{(2\pi)^{2+s}} \int_{\mathbb{R}^s} \frac{\left| \vartheta \right|^2}{4} \mathcal{F}\varphi(0, \vartheta) \, d\vartheta
\]

\[
= \varphi(0) + \frac{1}{4} (\Delta_{\vartheta}\varphi)(0).
\]

Here we write \( \Delta_{\vartheta} \) for the Laplace operator with respect to \( \vartheta \in \mathbb{R}^s \). So we have seen that \( \tilde{K}_{s,0} \) fails to be a fundamental solution of \( \Delta_{0,s} \) but rather solves the equation:

\[
\Delta_{0,s} \tilde{K}_{0,s} = \delta_0 + \frac{1}{4} (\delta_{x=0} \mathcal{F}) \otimes (\Delta_{\vartheta}\delta_{z=0}).
\]
7 A second form of fundamental solutions

In the present section we represent the fundamental solutions \( K_{0,s}^{\lambda,\mu} \) of the ultra-hyperbolic operator \( \Delta_{0,s} \) in Theorem \( 6.1 \) in a different form. The formulas which we obtain generalize the expressions derived in \( [22, 26] \) in the special case of the Heisenberg Lie algebra, i.e., \( s = 1 \) and for a specific choice of \( \lambda \) and \( \mu \). We will use the notation in Sections \( 4 \) and \( 5 \). For simplicity we first consider the fundamental solution \( q_{0,1}^{1,0} \) which arises for \( (\lambda, \mu) = (1, 0) \) and therefore will be denoted by \( K_{0, s}^{1,0} := K_{0, s} \). We take \( \varphi(x, z) \in S(\mathbb{R}^{2n+s}) \) and interchange the order of integration in \( [21] \):

\[
K_{0, s}(\varphi) := \frac{1}{(2\pi)^{n+s/2}} \int_0^\infty \int_{\mathbb{R}^{2n+s}} G_0(\xi, \vartheta, t) \left[ \mathcal{F}_\varphi \right](\xi, \vartheta) \, d(\xi, \vartheta) \, dt \tag{27}
\]

where with \( \varepsilon \geq 0 \) we define

\[
G_\varepsilon(\xi, \vartheta, t) := \frac{i}{|\vartheta| \cosh^n t} \exp \left \{ i \frac{\tanh t}{|\vartheta|} P(\xi) - \frac{\varepsilon |\vartheta|}{4} \coth t \right \}.
\]

According to Lemma \( 4.2 \) and with the notation in \( 4 \) we have:

\[
\int_{\mathbb{R}^{2n+s}} G_\varepsilon(\xi, \vartheta, t) \left[ \mathcal{F}_\varphi \right](\xi, \vartheta) \, d(\xi, \vartheta) = \tag{28}
\]

\[
\int_{\mathbb{R}^{2n+s}} [\mathcal{F}_{\xi \rightarrow x} G_\varepsilon](x, \vartheta, t) \left[ \mathcal{F}_{z \rightarrow \vartheta} \varphi \right](x, \vartheta) \, d(x, \vartheta)
\]

\[
= W(2t) \int_{\mathbb{R}^{2n+s}} |\vartheta|^{n-1} \exp \left \{ -\frac{i}{4} |\vartheta| \left[ P(x) - i\varepsilon \right] \cosh t \right \} \left[ \mathcal{F}_{z \rightarrow \vartheta} \varphi \right](x, \vartheta) \, d(x, \vartheta) = (*).
\]

The distribution \( K_{0, s}^{1,0} \) corresponding to the kernel \( q_{0, s}^{1,0} \) in \( [22] \) can be expressed in the same way by replacing \( G_\varepsilon \) with its complex conjugate \( \overline{G_\varepsilon} \). More generally, from these we can derive integral expression for the general case \( q_{0, s}^{\lambda,\mu} \) where \( \lambda + \mu \equiv 1 : \mathbb{R}^s \rightarrow \mathbb{C} \):

\[
K_{0, s}^{\lambda,\mu}(\varphi) = \frac{1}{(2\pi)^{n+s/2}} \int_0^\infty \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n+s}} \left[ \mathcal{F}_{z \rightarrow \vartheta} \varphi \right](x, \vartheta) \times
\]

\[
\times \left( \lambda [\mathcal{F}_{\xi \rightarrow x} G_\varepsilon] + \mu [\mathcal{F}_{\xi \rightarrow x} G_{\varepsilon}] \right)(x, \vartheta, t) \, d(x, \vartheta) \, dt \tag{29}
\]

**Example 7.1.** In case of the Heisenberg group \( G_{0,1} \) of dimension \( 2n + 1 \) in Example \( 2.2 \) and via a specific choice of \( \lambda \) and \( \mu \) we recover from the last formula an expression of a fundamental solution of the ultra-hyperbolic operator \( \Delta_{0,1} \) which previously has been presented in the work by D. Müller and F. Ricci in \( [21] \), (see also \( [26] \), p. 1297).

Let \( s = 1 \) and choose \( \lambda : \mathbb{R} \to [0, 1] \) to be the characteristic function of the non-negative half-line:

\[
\lambda(\vartheta) := \begin{cases} 1, & \text{if } \vartheta \geq 0, \\ 0, & \text{if } \vartheta < 0. \end{cases} \quad \text{and put } \mu(\vartheta) := 1 - \lambda(\vartheta).
\]
Then for all $\vartheta \in \mathbb{R} \setminus \{0\}$ we have:

$$\lambda G_\epsilon + \mu G_\epsilon = \frac{i}{\vartheta \cosh^2t} \exp \left\{ \frac{i \tanh t}{\vartheta} P(\xi) - \frac{\varepsilon |\vartheta|}{4} \coth t \right\}.$$ 

Applying Lemma 4.2 gives:

$$\lambda [F_{\xi \to x} G_\epsilon] + \mu [F_{\xi \to x} G_\epsilon] = \frac{i |\vartheta|^n}{\vartheta \sinh^n t} \exp \left\{ - \frac{i}{4} \vartheta \left[ P(\xi) - i\varepsilon \text{sgn}(\vartheta) \right] \coth t \right\}.$$ 

As usual, $\text{sgn}(\vartheta)$ denotes the sign-function. Therefore, in this example we find:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n+1}} [F_{z \to \vartheta} \varphi] (x, \vartheta) \left( \lambda [F_{\xi \to x} G_\epsilon] + \mu [F_{\xi \to x} G_\epsilon] \right) (x, \vartheta, t) \, dx(\vartheta) =$$

$$= \frac{i}{2^n \sinh^n t} \int_{\mathbb{R}^{2n+1}} (\text{sgn}(\vartheta))^n \vartheta^{n-1} [F_{z \to \vartheta} \varphi] (x, \vartheta) \exp \left\{ - \frac{i}{4} \vartheta P(x) \coth t \right\} \, dx(\vartheta) = (+).$$

If $n$ is even we can calculate the integral on the right more explicitly by the inversion of the Fourier transform:

$$(+) = \frac{\sqrt{2\pi}}{(2i \sinh t)^n} \int_{\mathbb{R}^{2n}} \frac{\partial^{n-1} \varphi}{\partial z^{n-1}} \left( x, -\frac{P(x)}{4} \coth t \right) \, dx.$$ 

Inserting the last formula into (29) gives a fundamental solution in form of an iterated integral:

$$K^{\lambda,\mu}_{0,1}(\varphi) = \frac{1}{(4\pi i)^n} \int_0^\infty \frac{1}{\sinh^n t} \frac{\partial^{n-1} \varphi}{\partial z^{n-1}} \left( x, -\frac{P(x)}{4} \coth t \right) \, dx \, dt. \quad (30)$$

Up to a sign this is the fundamental solution derived in (62) of [26]. Since $\coth t > 1$ on $(0, \infty)$ it follows that the distribution $(30)$ vanishes in $\{(x, z) \in \mathbb{R}^{2n+1} : 4|z| < |P(x)|\}$.

In the case of the center dimension $s > 1$ and $n > 1$ we may pass to polar coordinates in the $\vartheta$-integration of $(*)$. For the rest of the section we only consider the fundamental solution $K^{\lambda,\mu}_{0,s}$. However, all calculations can be done in a similar way for the general fundamental solutions $K^{\lambda,\mu}_{0,s}$ in (29). With the standard surface measure $\sigma$ on the euclidean sphere $S^{s-1}$ in $\mathbb{R}^s$ and $r > 0$ we put:

$$\varphi(x, r) = \int_{S^{s-1}} [F_{z \to \vartheta} \varphi] (x, r \omega) \, d\sigma(\omega).$$

Then we have

$$(*) = W(2t) \int_{\mathbb{R}^{2n}} \int_0^\infty r^{n+s-2} \exp \left\{ - \frac{ir}{4} \left[ P(x) - i\varepsilon \right] \coth t \right\} \varphi(x, r) \, dr \, dx.$$ 

In the $r$-integral we can perform $(n-1)$ partial integrations without producing boundary terms and using:

$$e^{-\frac{4i}{P(x) - i\varepsilon} \coth t} = \frac{(4i)^{n-1} \tanh^{n-1} t}{[P(x) - i\varepsilon]^{n-1}} \, dr^{n-1} \, e^{-\frac{4i}{P(x) - i\varepsilon} \coth t}.$$
It follows that
\[
(\ast) = \frac{2^{n-2}}{i^{n-1}} \frac{1}{\sinh t \cosh^{n-1} t} \int_{\mathbb{R}^{2n}} \frac{\varphi_{t,\varepsilon}(x)}{[P(x) - i\varepsilon]^{n-1}} \, dx,
\]
where the function \( \varphi_{t,\varepsilon}(x) \) is given by:
\[
\varphi_{t,\varepsilon}(x) = \int_0^\infty \frac{d^{n-1}}{dr^{n-1}} \left[ r^{n+s-2} \tilde{\varphi}(x, r) \right] e^{-ir \frac{t}{4} [P(x) - i\varepsilon] \coth t} \, dr \in S(\mathbb{R}^{2n}).
\]

Summarizing the calculation we have shown:

**Lemma 7.2.** Let \( \varphi = \varphi(x, z) \in S(\mathbb{R}^{2n+s}) \), then:
\[
K_{0,s}^{1,0}(\varphi) = \left( \frac{2}{i} \right)^{n-2} \frac{1}{(2\pi)^{n+2}} \int_0^\infty \frac{1}{\sinh t \cosh^{n-1} t} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \frac{\varphi_{t,\varepsilon}(x)}{[P(x) - i\varepsilon]^{n-1}} \, dx \, dt.
\]

In the next step we wish to calculate the limit in the \( t \)-integrand of (32). We fix a Schwartz function \( \psi \in S(\mathbb{R}^{2n}) \) and study the existence of the limit
\[
D(\psi) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \frac{\psi(x)}{[P(x) - i\varepsilon]^{n-1}} \, dx = i^{n-1} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \frac{\psi(x)}{[iP(x) + \varepsilon]^{n-1}} \, dx.
\]
Let \( \beta \in \mathbb{C} \) with \( \text{Re}(\beta) > 0 \) and recall the well-known integral representation of the Gamma function:
\[
\Gamma(s) = \frac{\beta^s}{\Gamma(s)} = \int_0^\infty t^{s-1} e^{-\beta t} \, dt.
\]

It follows that
\[
D(\psi) = \frac{i^{n-1}}{\Gamma(n-1)} \lim_{\varepsilon \to 0} \int_0^\infty t^{n-2} e^{-\varepsilon t} \int_{\mathbb{R}^{2n}} \psi(x)e^{-iP(x)t} \, dx \, dt.
\]

We decompose the outer integral into the integrals \( \int_0^1 \) and \( \int_1^\infty \). The first \( \varepsilon \)-dependent family of integrals will be denoted \( I_1^\varepsilon(\psi) \) and the second by \( I_2^\varepsilon(\psi) \). Clearly, \( I_1^\varepsilon(\psi) \) converges as \( \varepsilon \to 0 \) to:
\[
I_1(\psi) := \frac{i^{n-1}}{\Gamma(n-1)} \int_0^1 t^{n-2} \int_{\mathbb{R}^{2n}} \psi(x)e^{-iP(x)t} \, dx \, dt.
\]
Moreover, independently of \( \varepsilon > 0 \) and with the \( L^1 \)-norm on \( L^1(\mathbb{R}^{2n}) \) we have the estimate:
\[
|I_1^\varepsilon(\psi)| \leq \frac{1}{(n-1)!} \|\psi\|_{L^1}.
\]

In the next step we consider the integrals over \( [1, \infty) \) and we perform a Fourier transform in the \( x \)-integral. Using Lemma 4.2 we find:
\[
\mathcal{F} \left[ e^{-iP(x)t} \right](\xi) = (2t)^{-n} e^{-\frac{1}{4} P(\xi)}.
\]
Therefore:
\[
\int_{\mathbb{R}^{2n}} \psi(x)e^{-ip(x)t} \, dx = \frac{1}{(2t)^n} \int_{\mathbb{R}^{2n}} \left[ \mathcal{F}^{-1}\psi \right] (\xi)e^{\frac{i}{t}P(\xi)} \, d\xi.
\] (34)

According to the last expression we find:
\[
I_2^\varepsilon(\psi) = \frac{i^{n-1}}{2^n \Gamma(n-1)} \int_1^\infty \frac{e^{-\varepsilon t}}{t^2} \int_{\mathbb{R}^{2n}} \left[ \mathcal{F}^{-1}\psi \right] (\xi)e^{\frac{i}{t}P(\xi)} \, d\xi \, dt.
\]

The iterated integrals exist in the case \( \varepsilon = 0 \) and by the dominated convergence theorem we have
\[
\lim_{\varepsilon \to 0} I_2^\varepsilon(\psi) = \frac{i^{n-1}}{2^n \Gamma(n-1)} \int_1^\infty \frac{1}{t^2} \int_{\mathbb{R}^{2n}} \left[ \mathcal{F}^{-1}\psi \right] (\xi)e^{\frac{i}{t}P(\xi)} \, d\xi \, dt.
\]

Moreover, independent of \( \varepsilon > 0 \) one obtains the estimate:
\[
\left| I_2^\varepsilon(\psi) \right| \leq \frac{1}{2^n \Gamma(n-1)} \left\| \mathcal{F}^{-1}\psi \right\|_{L^1}.
\]

We summarize the above observations:

**Proposition 7.3.** Let \( \psi \in \mathcal{S}(\mathbb{R}^{2n}) \). Then the limit
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \frac{\psi(x)}{P(x) - i\varepsilon} \, dx = \frac{1}{P^{n-1}}[\psi]
\]
exists and there is a constant \( C > 0 \) independent of \( \psi \) and \( \varepsilon \) such that:
\[
\left| \int_{\mathbb{R}^{2n}} \frac{\psi(x)}{P(x) - i\varepsilon} \, dx \right| \leq C \left( \left\| \psi \right\|_{L^1} + \left\| \mathcal{F}^{-1}\psi \right\|_{L^1} \right).
\] (35)

**Remark 7.4.** In [14] various distributions associated to the quadratic form \( P(x) \) have been defined. We remark that \( 1/P^{n-1} \) in Proposition 7.3 coincides with the value of \( (P + i0)^{\lambda} \) in [14, Chapter III, Section 2.4] at \( \lambda = -n + 1 \), cf. Proposition 6.3. More precisely, it holds:
\[
\frac{1}{P^{n-1}}[\psi] = \left( (P + i0)^{-n+1}, \psi \right) \text{ for all } \psi \in \mathcal{S}(\mathbb{R}^{2n}).
\] (36)

Equation (36) is proven in Proposition 11.1 of the Appendix. In the case \( n = 2 \) the left hand side of (36) has been represented in another form in [21, Formula (5.6), p. 331].

In order to modify inequality (35) we perform a standard estimate:
\[
\left\| \mathcal{F}^{-1}\psi \right\|_{L^1} = \frac{1}{(2\pi)^n} \left\{ \int_{|x| \leq 1} + \int_{|x| > 1} \right\} \int_{\mathbb{R}^{2n}} \psi(y)e^{ix \cdot y} \, dy \, dx.
\]

We can choose a constant \( c > 0 \) such that the first integral can be estimated by:
\[
\int_{|x| \leq 1} \int_{\mathbb{R}^{2n}} |\psi(y)| \, dy \, dx \leq c\left\| \psi \right\|_{L^1}.
\]
Let $\Delta = \sum_{j=1}^{2n} \partial^2_{x_j}$ denote the Laplace operator on $\mathbb{R}^{2n}$ and fix $\ell \in \mathbb{N}$. Then we can use the relation

$$F^{-1} \Delta^\ell \psi = (-1)^\ell \cdot 2^\ell F^{-1} \psi$$

and choose $\ell \in \mathbb{N}$ sufficiently large such that $| \cdot |^{-2\ell}$ becomes an integrable function over the exterior domain $\{ x \in \mathbb{R}^{2n} : |x| > 1 \}$ of the unit ball. Then we have:

$$\int_{|x| > 1} |F^{-1} \psi|(x) \, dx = \int_{|x| > 1} |x|^{-2\ell} |F^{-1} \Delta^\ell \psi(x)| \, dx$$

$$\leq \frac{1}{(2\pi)^n} \int_{|x| > 1} \int_{\mathbb{R}^{2n}} |x|^{-2\ell} |(\Delta^\ell \psi)(u)| \, du \, dx \leq C \| \Delta^\ell \psi \|_{L^1}.$$  

Hence, with a suitable constant $d > 0$ we have for all $\psi \in \mathcal{S}(\mathbb{R}^{2n})$:

$$\| F^{-1} \psi \|_{L^1} \leq d (\| \psi \|_{L^1} + \| \Delta^\ell \psi \|_{L^1}).$$

Therefore the following corollary of Proposition 7.3 holds:

**Corollary 7.5.** There is $\ell \in \mathbb{N}$ and a constant $C_\ell > 0$ independent of $\psi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\varepsilon > 0$ such that

$$\left| \int_{\mathbb{R}^{2n}} \frac{\psi(x)}{\| P(x) - i\varepsilon \|^{n-1}} \, dx \right| \leq C_\ell \left( \| \psi \|_{L^1} + \| \Delta^\ell \psi \|_{L^1} \right).$$

With the notation in (21) we estimate

$$\int_{\mathbb{R}^{2n}} \frac{\varphi_{t,\varepsilon}(x)}{\| P(x) - i\varepsilon \|^{n-1}} \, dx = \int_{\mathbb{R}^{2n}} \frac{\varphi_{t,0}(x)}{\| P(x) - i\varepsilon \|^{n-1}} \, dx + \int_{\mathbb{R}^{2n}} \frac{\varphi_{t,\varepsilon}(x) - \varphi_{t,0}(x)}{\| P(x) - i\varepsilon \|^{n-1}} \, dx.$$  

According to Proposition 7.3 the first integral converges for all $t > 0$ as $\varepsilon \to 0$ with limit:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \frac{\varphi_{t,0}(x)}{\| P(x) - i\varepsilon \|^{n-1}} \, dx = \frac{1}{P^{n-1}} [\varphi_{t,0}].$$

Since the $L^1$-norms:

$$\| \varphi_{t,\varepsilon} - \varphi_{t,0} \|_{L^1} \quad \text{and} \quad \| \Delta^\ell (\varphi_{t,\varepsilon} - \varphi_{t,0}) \|_{L^1}$$

tend to zero as $\varepsilon \to 0$, it follows again from Corollary 7.5 that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \frac{\varphi_{t,\varepsilon}(x) - \varphi_{t,0}(x)}{\| P(x) - i\varepsilon \|^{n-1}} \, dx = 0.$$  

Therefore we obtain the following second form of the fundamental solution:

**Theorem 7.6.** Let $\varphi = \varphi(x, z) \in \mathcal{S}(\mathbb{R}^{2n+s})$. With the notation in Lemma 7.2 we have:

$$K_{0,s}^{1,0}(\varphi) = \left( \frac{2}{i} \right)^{n-2} \frac{1}{(2\pi)^{n+1/2}} \int_0^\infty \frac{1}{\sinh t \cosh^{n-1} t} \frac{1}{P^{n-1}} \left[ \varphi_t \right] \, dt,$$

where

$$\varphi_t(x) := \varphi_{t,0}(x) = \int_0^\infty \frac{d^{n-1}}{dr^{n-1}} \left[ r^{n+s-1} \varphi(x, r) \right] e^{-\frac{r}{2}P(x) \coth t} \, dr \in \mathcal{S}(\mathbb{R}^{2n}).$$
8 On the singular support of $K^{1,0}_{0,s}$

In order to obtain some information on the singular support of the distribution $K_{0,s}$ in (27) (or Theorem 5.1) we move the Fourier transform in the integral (28) from the function $\varphi$ to the integral kernel. More precisely, we need to determine the Fourier transforms

$$F_{\theta \to z} \left\{ |\theta|^{n-1} e^{-|\theta|} \right\}(z) \in L^2(\mathbb{R}^s),$$

where $\lambda \in \mathbb{C}$ with Re($\lambda$) > 0. Recall the following formula (cf. [25], p. 219):

$$F_{\theta \to z} \left[ e^{-|\theta|} \right](z) = c_s \frac{\lambda}{(\lambda^2 + |z|^2)^{\frac{s+1}{2}}}, \quad \text{where} \quad c_s = \frac{2^n}{\sqrt{\pi}} \Gamma \left( \frac{s+1}{2} \right).$$

Taking derivatives with respect to $\lambda$ under the integral yields:

$$F_{\theta \to z} \left[ |\theta|^{n-1} e^{-|\theta|} \right](z) = (-1)^{n-1} \frac{d^{n-1}}{d\lambda^{n-1}} F_{\theta \to z} \left[ e^{-|\theta|} \right](z)$$

$$= (-1)^{n-1} c_s \frac{d^{n-1}}{d\lambda^{n-1}} \frac{\lambda}{(\lambda^2 + |z|^2)^{\frac{s+1}{2}}}$$

$$= \sum_{j=0}^{n-1} \frac{Q_j(\lambda)}{(\lambda^2 + |z|^2)^{\frac{s+1}{2}+j}}, \quad (37)$$

where $Q_j$ is a polynomial of the variable $\lambda$ of some degree $\alpha_j := \deg Q_j \in \mathbb{N}_0$. By induction one verifies:

$$2 \left( \frac{s+1}{2} + j \right) - \alpha_j \geq s + n - 1, \quad \text{where} \quad j = 1, \ldots, n - 1. \quad (38)$$

In particular, with $\lambda := \frac{1}{4} [iP(x) + \varepsilon] \coth t$ we find

$$\lambda^2 + |z|^2 = \frac{1}{16} \left[ -P(x)^2 + 2iP(x)\varepsilon + \varepsilon^2 \right] \coth^2 t + |z|^2. \quad (39)$$

By using (28) it follows that:

$$\int_{\mathbb{R}^{2n+s}} G_{\varepsilon}(\zeta, \theta, t) \left[ F\varphi \right](\zeta, \theta) \, d(\zeta, \theta) = \frac{1}{(2 \sinh t)^n} \sum_{j=0}^{n-1} \int_{\text{supp}(\varphi)} \frac{Q_j(\lambda)\varphi(x, z)}{(\lambda^2 + |z|^2)^{\frac{s+1}{2}+j}} \, d(x, z). \quad (40)$$

Suppose that

$$\text{supp}(\varphi) \cap \left\{ (x, z) \in \mathbb{R}^{2n+s} : 4|P(x)| \leq |z| \right\} = \emptyset, \quad (41)$$

and define:

$$S := \left\{ (x, z) \in \mathbb{R}^{2n+s} : P(x) = 0 \right\}.$$
By the condition (41) and because of \( \coth t > 1 \) for all \( t > 0 \) it is clear from (39) that we may put \( \varepsilon = 0 \) on the right hand side of (40) without causing a singularity in the integrand. Since \( \text{supp}(\varphi) \) does not intersect with \( S \) we find from (39) in the case \( \varepsilon = 0 \):

\[
\lambda^2 + |z|^2 = -\frac{P(x)^2}{16} \coth^2 t + |z|^2 \in O(t^2) \quad \text{as} \quad t \to 0.
\]

Hence (37) and (38) imply:

\[
\int_{\text{supp}(\varphi)} Q_j(\lambda) \varphi(x, z) \, d(x, z) \in O(t^{s+n-1}) \quad \text{as} \quad t \downarrow 0.
\]

Choose \( \varepsilon = 0 \) such that \( \lambda = \lambda_0 = \frac{4}{t} P(x) \coth t \) and \( (x, z) \in \mathbb{R}^{2n+s} \). Put

\[
K(x, z) := \int_0^\infty \frac{1}{(2 \sinh t)^n} \sum_{j=0}^{n-1} Q_j(\lambda_0) \left( -\frac{P(x)^2}{4} \coth^2 t + |z|^2 \right)^{\frac{n+1}{2}+j} \, dt.
\]

Assuming (41) for \( \varphi \in \mathcal{S}(\mathbb{R}^{2n+s}) \) we have shown that \( K(x, z) \) defines a smooth function in a neighborhood of \( \text{supp}(\varphi) \). Moreover, (40) and (27) imply that

\[
K_{1,0}^{0,0}(\varphi) = \frac{1}{(2\pi)^{n+s/2}} \int_{\mathbb{R}^{2n+s}} K(x, z) \varphi(x, z) \, d(x, z).
\]

Summarizing these observations we have shown:

**Theorem 8.1.** The singular support of the fundamental solution \( K_{0,s} \) is contained in

\[
\left\{ (x, z) \in \mathbb{R}^{2n+s} : 4|P(x)| \leq |z| \right\}.
\]

**Example 8.2.** In case of the Heisenberg group and the specific choice of the functions \( \lambda \) and \( \mu \) in Example 7.1 we have even seen that, cf. p. 1295 in [26]

\[
\text{supp}(K^{\lambda,\mu}_{0,1}) \subset \left\{ (x, z) \in \mathbb{R}^{2n+s} : 4|P(x)| \leq |z| \right\}.
\]

A close relation between the structure of geodesics tangent to the left invariant distribution spanned by \( \{X_1, \ldots, X_{2n}\} \) and the cone described by \( P(x) = 0 \) was observed in [19].

## 9 On the fundamental solution of \( \Delta_{r,s} \) in the case \( r > 0 \)

In the case \( r > 0 \) is seems difficult to interpret the formal expression of \( K_{r,s}(x, y) \) in Section 4 in a meaningful way as a tempered distribution. In fact, in this last section we
will show that $\Delta_{r,s}$ with $r > 0$ does not even have a fundamental solution in $\mathcal{S}'(\mathbb{R}^{2n+r+s})$. In Corollary 3.5 we calculated the differential operator $G_{r,s}$ with

$$\mathcal{F} \circ \Delta_{r,s} = G_{r,s} \circ \mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathcal{S}(\mathbb{R}^{2n+r+s})$. By using the relation (15) between $\Omega(\eta)$ and $\rho(\eta)$ we can rewrite $G_{r,s}$ as:

$$G_{r,s} \phi = -P(\xi) \phi + \frac{|\eta_+|^2 - |\eta_-|^2}{4} \mathcal{L} \phi - 2i \langle \Omega(\eta) \tau \xi, \nabla \xi \phi \rangle,$$

where $\mathcal{L}$ was the ultra-hyperbolic operator on $\mathbb{R}^{2n}$ defined in (2).

Let us fix $\eta \in \mathbb{R}^{r+s}$ and consider the one-parameter matrix group:

$$\mathbb{R} \ni t \mapsto e^{t \Omega(\eta) \tau} \in \mathbb{R}^{2n \times 2n}.$$

By applying the relations in Lemma 3.3 we can calculate the exponents.

Corollary 9.1. Let $\eta = (\eta_+, \eta_-) \in \mathbb{R}^{r+s}$ with $|\eta_+|^2 - |\eta_-|^2 > 0$. Then $A_\eta$ is not injective as an operator on $\mathcal{S}(\mathbb{R}^{2n})$.

Now we wish to express $B_\eta$ in a geometric form. Let $\eta \in \mathbb{R}^{r+s}$ be fixed and consider the one-parameter matrix group:

$$\mathbb{R} \ni t \mapsto e^{t \Omega(\eta) \tau} \in \mathbb{R}^{2n \times 2n}.$$

By applying the relations in Lemma 3.3 we can calculate the exponentials.
Lemma 9.2. Let $t \in \mathbb{R}$ and let $\eta \in \mathbb{R}^{r+s} \cong \mathbb{R}^{r,s}$ be fixed. With the notation

$$|\eta|_{r,s} := \sqrt{||\eta_+|^2 - |\eta_-|^2}$$

one obtains:

$$e^{t\Omega(\eta)\tau} = \begin{cases} \cos\left(\frac{t|\eta|_{r,s}}{2}\right) I + \frac{2}{|\eta|_{r,s}} \sin\left(\frac{t|\eta|_{r,s}}{2}\right) \Omega(\eta)\tau & \text{if } |\eta_+|^2 - |\eta_-|^2 > 0. \\ \cosh\left(\frac{t|\eta|_{r,s}}{2}\right) I + \frac{2}{|\eta|_{r,s}} \sinh\left(\frac{t|\eta|_{r,s}}{2}\right) \Omega(\eta)\tau & \text{if } |\eta_+|^2 - |\eta_-|^2 < 0. \end{cases} \tag{45}$$

and

$$e^{t\tau\Omega(\eta)} = \begin{cases} \cos\left(\frac{t|\eta|_{r,s}}{2}\right) I + \frac{2}{|\eta|_{r,s}} \sin\left(\frac{t|\eta|_{r,s}}{2}\right) \tau\Omega(\eta) & \text{if } |\eta_+|^2 - |\eta_-|^2 > 0. \\ \cosh\left(\frac{t|\eta|_{r,s}}{2}\right) I + \frac{2}{|\eta|_{r,s}} \sinh\left(\frac{t|\eta|_{r,s}}{2}\right) \tau\Omega(\eta) & \text{if } |\eta_+|^2 - |\eta_-|^2 < 0. \end{cases} \tag{46}$$

In particular, if $|\eta_+|^2 - |\eta_-|^2 > 0$, then $e^{t\Omega(\eta)\tau}$ and $e^{t\tau\Omega(\eta)}$ are periodic in $t$ with period $q_\eta = \frac{4\pi}{|\eta|_{r,s}}$.

Proof. We conclude from Lemma 3.3 (1) that for $|\eta_+|^2 - |\eta_-|^2 > 0$

$$(\Omega(\eta)\tau)^{2k} = \left(-\frac{|\eta_+|^2 - |\eta_-|^2}{4}\right)^k = (-1)^k \left(\frac{|\eta|_{r,s}}{2}\right)^{2k},$$

and therefore the power series expansion of the exponential function gives:

$$e^{t\Omega(\eta)\tau} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (\Omega(\eta)\tau)^{2k} + \Omega(\eta)\tau \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (\Omega(\eta)\tau)^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{t|\eta|_{r,s}}{2}\right)^{2k} + \Omega(\eta)\tau \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{t|\eta|_{r,s}}{2}\right)^{2k+1} 2^{|\eta|_{r,s}}.$$

In the case $|\eta_+|^2 - |\eta_-|^2 < 0$ we use

$$(\Omega(\eta)\tau)^{2k} = (-1)^k \left(\frac{i|\eta|_{r,s}}{2}\right)^{2k}, \quad \cos(\alpha i) = \cosh \alpha, \quad \sin(\alpha i) = i \sinh \alpha, \quad \alpha \in \mathbb{R}$$

and finish the proof of (45). A similar calculation shows (46). \hfill \Box

Lemma 9.3. Let $\eta \in \mathbb{R}^{r+s}$. Then for all $t \in \mathbb{R}$ one has

$$e^{-t\tau\Omega(\eta)} e^{t\Omega(\eta)\tau} = \tau \quad \text{and} \quad e^{-t\Omega(\eta)\tau} e^{t\tau\Omega(\eta)} = \tau. \tag{47}$$

In particular, the level sets of $P(\xi) = \sum_{j=1}^{n} \xi_j^2 - \xi_{j+n}^2$ are invariant under the flows $e^{t\Omega(\eta)\tau}$ and $e^{t\tau\Omega(\eta)}$, i.e., for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$:

$$P \circ e^{t\Omega(\eta)\tau} \xi = P(\xi) \quad \text{and} \quad P \circ e^{t\tau\Omega(\eta)} \xi = P(\xi). \tag{48}$$
Proof. Let \( \eta = (\eta_+, \eta_-) \in \mathbb{R}^{r+s} \) with \(|\eta_+|^2 - |\eta_-|^2 > 0\) as above and \( t \in \mathbb{R} \). By using Lemma 9.2 we calculate the following product:

\[
e^{-t\tau\Omega(\eta)\tau\xi} e^{t\Omega(\eta)\tau\xi} = \left[ \cos \left( t \frac{|\eta|_{r,s}}{2} \right) \tau - \frac{\sin(t \frac{|\eta|_{r,s}}{2} \tau)}{\frac{2}{|\eta|_{r,s}}} \left( \tau\Omega(\eta)\tau \right) \right] \times \left[ \cos \left( t \frac{|\eta|_{r,s}}{2} \right) I + \frac{\sin(t \frac{|\eta|_{r,s}}{2} \tau)}{\frac{2}{|\eta|_{r,s}}} \left( \Omega(\eta)\tau \right) \right] \tag{49}
\]

We know that \( \left( \tau\Omega(\eta)\tau \right)^2 = -\frac{|\eta_+|^2-|\eta_-|^2}{4} I = -\frac{|\eta|^2}{4} I \) from Lemma 3.3 (1). Inserting it to (49) we obtain the necessary equality.

If \(|\eta_+|^2 - |\eta_-|^2 < 0\) then we calculate

\[
e^{-t\tau\Omega(\eta)\tau\xi} e^{t\Omega(\eta)\tau\xi} = \cosh \left( t \frac{|\eta|_{r,s}}{2} \tau \right) - \frac{\sinh^2(t \frac{|\eta|_{r,s}}{2} \tau)}{\frac{2}{|\eta|^2}} \left( \tau\Omega(\eta)\tau \right)^2 \tau \tag{50}
\]

Substituting \( \left( \tau\Omega(\eta)\tau \right)^2 = -\frac{|\eta_+|^2-|\eta_-|^2}{4} I = -\frac{|\eta|^2}{4} I \) into (50), we get the assertion. The second equality in (47) follows analogously.

To show the second statement we write \( P(\xi) = \langle \tau\xi, \xi \rangle \) and calculate

\[
P \circ e^{t\Omega(\eta)\tau\xi} = \langle e^{t\Omega(\eta)\tau\xi}, e^{t\Omega(\eta)\tau\xi} \rangle = \langle e^{-t\tau\Omega(\eta)\tau\xi} e^{t\Omega(\eta)\tau\xi}, \xi \rangle = \langle \tau\xi, \xi \rangle = P(\xi)
\]

for all \( \xi \in \mathbb{R}^{2n} \). \[\square\]

**Lemma 9.4.** Let \( \eta \in \mathbb{R}^{r+s} \cong \mathbb{R}^{r,s} \) be fixed. Then the operator \( B_\eta \) can be expressed in the form

\[
B_\eta \varphi = -2i \left. \frac{d}{dt} \right|_{t=0} \varphi(e^{t\Omega(\eta)\tau\xi}) \quad \text{where} \quad \varphi \in \mathcal{S}(\mathbb{R}^{2n}).
\]

**Proof.** Let \( \varphi \in \mathcal{S}(\mathbb{R}^{2n}) \) and \( \xi \in \mathbb{R}^{2n} \). Then

\[
\left. \frac{d}{dt} \right|_{t=0} \varphi(e^{t\Omega(\eta)\tau\xi}) = [\nabla_\xi \varphi] \cdot \left. \frac{d}{dt} \right|_{t=0} e^{t\Omega(\eta)\tau\xi} = (\nabla_\xi \varphi) \Omega(\eta)\tau\xi = \langle \Omega(\eta)\tau\xi, \nabla_\xi \varphi \rangle = \frac{i}{2} B_\eta(\varphi).
\]

\[\square\]

For each \( t \in \mathbb{R} \), fixed \( \eta \in \mathbb{R}^{r+s} \) and \( \varphi \in \mathcal{S}(\mathbb{R}^{2n}) \) we define two composition operators \( C_{\eta,t}^{(j)}, j = 1, 2 \), on \( \mathcal{S}(\mathbb{R}^{2n}) \) by:

\[
C_{\eta,t}^{(1)} \varphi = \varphi(e^{t\tau\Omega(\eta)\xi}), \quad C_{\eta,t}^{(2)} \varphi = \varphi(e^{t\Omega(\eta)\tau\xi}).
\]

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Lemma 9.5. The operator of multiplication by $P(\xi)$ on $S(\mathbb{R}^n)$ commutes with both composition operators $C_{\eta,t}^{(j)}$, $j = 1, 2$. Moreover, $C_{\eta,t}^{(2)}$ and $B_\eta$ commute.

Proof. We only treat $C_{\eta,t}^{(1)}$ and use the invariance property (48) in Lemma 9.3:

$$PC_{\eta,t}^{(1)}\varphi(\xi) = P(\xi)\varphi\left(e^{t\tau\Omega(\eta)}\xi\right)$$

$$= P\left(e^{t\tau\Omega(\eta)}\xi\right)\varphi\left(e^{t\tau\Omega(\eta)}\xi\right) = C_{\eta,t}^{(1)}(P\varphi)(\xi).$$

The second statement follows by combining Lemma 9.3 and Lemma 9.4.

As a consequence of Lemma 9.5 we also have:

Lemma 9.6. Let $F$ denote the Fourier transform on $S(\mathbb{R}^{2n})$. Then we have for all $t \in \mathbb{R}$ and $\eta \in \mathbb{R}^{r,s}$:

$$C_{\eta,t}^{(2)} \circ F = F \circ C_{\eta,t}^{(1)} \quad \text{and} \quad C_{\eta,t}^{(1)} \circ F = F \circ C_{\eta,t}^{(2)} \quad (51)$$

In particular, the ultra-hyperbolic operator $L$ commutes with $C_{\eta,t}^{(j)}$ for $j = 1, 2$ and by Lemma 9.5

$$[A_\eta, C_{\eta,t}^{(j)}] = 0 \quad j = 1, 2.$$

Proof. First we calculate the commutation relations (51). It is sufficient to prove the first formula. Let $\varphi \in S(\mathbb{R}^{2n})$, then:

$$F \circ C_{\eta,t}^{(1)} \varphi(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \varphi\left(e^{t\tau\Omega(\eta)}x\right) e^{-ix \cdot \xi} \, dx = (*) \quad (51).$$

According to Lemma 3.3 (2) it follows:

$$\left| \det e^{-t\tau\Omega(\eta)} \right| = \left| \det e^{-i\tau\Omega(\eta)\tau} \right| = 1$$

and the transformation rule for the integral implies:

$$(*) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \varphi(x) \exp\left\{-i\left(e^{-t\tau\Omega(\eta)}x\right) \cdot \xi\right\} \, dx$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \varphi(x) \exp\left\{-ix \cdot \left(e^{i\Omega(\eta)\tau} \xi\right)\right\} \, dx = C_{\eta,t}^{(2)} \circ F\varphi(\xi).$$

Using (51) we can prove the second statement and we only treat the case $j = 1$. Considered as operators on $S(\mathbb{R}^{2n})$ we have

$$F \circ P = -F \circ L \quad \text{and} \quad F \circ L = -P \circ F.$$
and from (51) and Lemma 9.5 it follows:
\[ F \circ L \circ C_{\eta,t}^{(1)} = -P \circ F \circ C_{\eta,t}^{(1)} \]
\[ = -P \circ C_{\eta,t}^{(2)} \circ F \]
\[ = -C_{\eta,t}^{(2)} \circ P \circ F \]
\[ = C_{\eta,t}^{(2)} \circ F \circ L = F \circ C_{\eta,t}^{(1)} \circ L. \]

Since \( F \) is bijective on \( \mathcal{S}(\mathbb{R}^{2n}) \) it follows that \( L \circ C_{\eta,t}^{(1)} = C_{\eta,t}^{(1)} \circ L \). The case \( j = 2 \) can be treated similarly.

**Corollary 9.7.** Let \( \eta \in \mathbb{R}^{r,s} \) be fixed. Then \( B_\eta \) commutes with \( P \) and \( L \). In particular, \( B_\eta \) commutes with \( A_\eta \) in (43).

**Proof.** Since \( A_\eta \) is a linear combination of \( P \) and \( L \) it is sufficient to prove the first statement. Lemma 9.4 and Lemma 9.5 give with \( \varphi \in \mathcal{S}(\mathbb{R}^{2n}) \):

\[ B_\eta \circ P \varphi = -2i \frac{d}{dt} \bigg|_{t=0} C_{\eta,t}^{(2)}(\varphi) = -2iP \frac{d}{dt} \bigg|_{t=0} C_{\eta,t}^{(2)}(\varphi) = P \circ B_\eta \varphi. \]

Therefore we have \([B_\eta, P] = 0\). Replacing \( P \) with \( L \) in the above calculation and applying Lemma 9.6 shows that also the commutator \([B_\eta, L]\) vanishes.

Let now \( \varphi \in \mathcal{S}(\mathbb{R}^{2n}) \) and fix \( \eta \in \mathbb{R}^{r,s} \) with \(|\eta_+|^2 - |\eta_-|^2 > 0\). Consider the operator

\[ D_\eta : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^{2n}) : [D_\eta \varphi](\xi) := \int_0^{q_\eta} [C_{\eta,t}^{(2)} \varphi](\xi) \, dt, \]

where \( q_\eta = \frac{4\pi}{|\eta_+|} \) is the period of \( \mathbb{R} \ni t \mapsto C_{\eta,t}^{(2)} \) in Lemma 9.2. Applying the transformation \( t \to \frac{4\pi}{|\eta_+|} \rho \) we can also write

\[ [D_\eta \varphi](\xi) = \frac{4\pi}{|\eta_+|} \int_0^1 \varphi \left( \exp \left( \frac{4\pi \rho}{|\eta_+|} \Omega(\eta(t)) \xi \right) \right) \, d\rho. \]

**Lemma 9.8.** With the notation above \( D_\eta \) maps \( \mathcal{S}(\mathbb{R}^{2n}) \) into the kernel of \( B_\eta \).

**Proof.** Let \( \varphi \in \mathcal{S}(\mathbb{R}^{2n}) \). Then we have by Corollary 9.4

\[ B_\eta \circ D_\eta \varphi(\xi) = -2i \frac{d}{d\rho} \bigg|_{\rho=0} \int_0^{q_\eta} C_{\eta,t}^{(2)} \varphi \left( e^{\rho \Omega(\eta(t)) \xi} \right) \, dt \]
\[ = -2i \frac{d}{d\rho} \bigg|_{\rho=0} \int_0^{q_\eta} \varphi \left( e^{(\rho+t)\Omega(\eta(t)) \xi} \right) \, dt. \]

Since we integrate a periodic function on \( \mathbb{R} \) over a full period we have:

\[ \int_0^{q_\eta} \varphi \left( e^{(\rho+t)\Omega(\eta(\xi)) \xi} \right) \, dt = \int_0^{q_\eta} \varphi \left( e^{\Omega(\eta(\xi)) \xi} \right) \, dt \]
and therefore the integrand does not depend on \( \rho \). Hence it follows \( B_\eta \circ D_\eta \varphi \equiv 0 \) as it was claimed.
Lemma 9.9. Let \( \eta \in \mathbb{R}^{r,s} \) be fixed with \( |\eta_+|^2 - |\eta_-|^2 > 0 \) and choose \( \varphi \in \mathcal{S}(\mathbb{R}^{2n}) \) in the kernel of \( A_\eta \). Then

\[
A_\eta \circ D_\eta \varphi = 0.
\]

Moreover, \( D_\eta \) maps \( \ker A_\eta \subset \mathcal{S}(\mathbb{R}^{2n}) \) into \( \ker A_\eta \cap \ker B_\eta \subset \mathcal{S}(\mathbb{R}^{2n}) \).

Proof. According to Lemma 9.8 it is sufficient to prove the first statement. Let \( \varphi \in \ker A_\eta \subset \mathcal{S}(\mathbb{R}^{2n}) \). Since \( A_\eta \) commutes with \( C^{(2)}_{\eta,t} \) according to Lemma 9.6 we have:

\[
A_\eta \circ D_\eta \varphi = \int_0^{q_\eta} [A_\eta \circ C^{(2)}_{\eta,t} \varphi](\xi) \, dt = \int_0^{q_\eta} C^{(2)}_{\eta,t}(A_\eta \varphi)(\xi) \, dt = 0.
\]

This proves the assertion. \( \Box \)

In particular, we may choose the function \( \varphi_\eta \in \ker A_\eta \) in (44):

\[
\varphi_\eta(\xi) = \exp \left( -c_\eta |\xi|^2 \right) \in \mathcal{S}(\mathbb{R}^{2n}) \quad \text{where} \quad c_\eta = \frac{1}{\sqrt{|\eta_+|^2 - |\eta_-|^2}}.
\]

Since \( \varphi_\eta \) only has positive values, the same holds for the function \( D_\eta \varphi_\eta \) and, in particular, it is non-zero.

For the moment we consider the operator \( G_{r,s} = A_\eta + B_\eta \) in Lemma 3.4 for fixed \( \eta \in \mathbb{R}^{r,s} \) with \( |\eta_+|^2 - |\eta_-|^2 > 0 \) as an operator on \( \mathcal{S}(\mathbb{R}^{2n}) \). Lemma 9.9 implies:

\[
0 \neq D_\eta \varphi_\eta \in \ker A_\eta \cap \ker B_\eta \subset \ker (A_\eta + B_\eta) = \ker G_{r,s} \subset \mathcal{S}(\mathbb{R}^{2n}).
\]

Now we consider \( G_{r,s} \) again as an operator on \( \mathcal{S}(\mathbb{R}^{2n+r+s}) \) as in Lemma 3.4. We assume that \( r > 0 \) such that

\[
\mathcal{K} = \left\{ \eta \in \mathbb{R}^{r,s} : |\eta_+|^2 - |\eta_-|^2 > 0 \right\}
\]

is a non-empty open subset in \( \mathbb{R}^{r,s} \). Choose a compactly supported cut-off function \( \omega \in C_0^\infty(\mathbb{R}^{r+s}) \) with

\[
\text{supp} \omega \subset \mathcal{K},
\]

and taking values in the interval \([0,1]\). Consider the function:

\[
\psi(\xi,\eta) := \omega(\eta)[D_\eta \varphi_\eta](\xi) \in \mathcal{S}(\mathbb{R}^{2n+r+s}).
\]

Then by construction \( \psi \) only takes values in \( \mathbb{R}_+ = (0,\infty) \) and

\[
G_{r,s}\psi = \omega(\eta)(A_\eta + B_\eta)D_\eta \varphi_\eta \equiv 0.
\]

Hence we have shown:

Corollary 9.10. Let \( r > 0 \), then there is a non-trivial, non-negative valued function \( \psi \in \mathcal{S}(\mathbb{R}^{2n+r+s}) \) in the kernel of the operator \( G_{r,s} \).

Combining the previous observations, we can now prove the main result:
Theorem 9.11. Let $r > 0$, then the ultra-hyperbolic operator $\Delta_{r,s}$ does not have a fundamental solution in $S'(\mathbb{R}^{2n+r+s})$, i.e., there is no tempered distribution $K_{r,s} \in S'(\mathbb{R}^{2n+r+s})$ such that

$$\Delta_{r,s} K_{r,s} = \delta_0.$$ 

Proof. Let $r > 0$ and assume that the ultra-hyperbolic operator $\Delta_{r,s}$ admits a fundamental solution $K_{r,s} \in S'(\mathbb{R}^{2n+r+s})$. Then

$$\Delta_{r,s} K_{r,s} = K_{r,s} \left( \Delta_{r,s} \psi \right) = \delta_0 \psi = \psi(0) \quad \text{for all} \quad \psi \in S(\mathbb{R}^{2n+r+s}).$$

Choose a non-trivial, non-negative valued function $\psi \in S(\mathbb{R}^{2n+r+s})$ in the kernel of $G_{r,s}$ according to Corollary 9.10. With the Fourier transform $F$ on $S(\mathbb{R}^{2n+r+s})$ we have the relation

$$\Delta_{r,s} \circ F^{-1} = F^{-1} \circ G_{r,s}.$$

Hence it follows:

$$K_{r,s}(F^{-1} \circ G_{r,s} \psi) = K_{r,s}(\Delta_{r,s} \circ F^{-1} \psi) = \left[ F^{-1} \psi \right](0) = \frac{1}{(2\pi)^{n+r+s}} \int_{\mathbb{R}^{2n+r+s}} \psi(\xi,z) \, d\xi \, dz > 0.$$

On the other hand, since $G_{r,s} \psi = 0$ we have $K_{r,s}(F^{-1} \circ G_{r,s} \psi) = 0$ which leads to a contradiction. \hfill $\square$

10 On the local solvability of $\Delta_{r,s}$

Recall that a left-invariant differential operator $L$ on $G_{r,s}$ is called locally solvable at $x_0 \in G_{r,s}$ if one can find an open neighborhood $U$ of $x_0$ such that

$$LC^{\infty}(U) \supset C_0^{\infty}(U).$$

As usual $C_0^{\infty}(U)$ denotes the space of compactly supported smooth functions on $U$. From the left-invariance of $L$ it follows that the local solvability of $L$ at a fixed point $x_0$ is equivalent to the local solvability of $L$ at any point in $G_{r,s}$. Hence we may use the term local solvability without specifying the point.

Lemma 10.1. Let $r > 0$, then there is a function $\varphi \in S(\mathbb{R}^{2n+r+s})$ which lies in the kernel of $\Delta_{r,s}$ and fulfills $\varphi(0) = 1$.

Proof. Let $\psi \in S(\mathbb{R}^{2n+r+s})$ be the function in Corollary 9.10 and put $\varphi_0 = F^{-1}\psi$. Then

$$c := \varphi_0(0) = \frac{1}{(2\pi)^{n+r+s}} \int_{\mathbb{R}^{2n+r+s}} \psi(\xi,z) \, d\xi \, dz > 0.$$

If we put $\varphi = c^{-1}\varphi_0$, then $\varphi(0) = 1$ and

$$\Delta_{r,s} \varphi = c^{-1}\Delta_{r,s} \circ F^{-1} \psi = c^{-1} F^{-1} \circ G_{r,s} \psi = 0$$

since $\psi$ is in the kernel of $G_{r,s}$. \hfill $\square$
Recall that $G_{r,s}$ is a homogeneous Lie group, i.e., there is a family $\{\delta_\rho\}_{\rho > 0}$ of dilations which at the same time are automorphisms. In fact, with respect to the coordinates defined in Section 2 and $\rho > 0$, we put
\[
\delta_\rho : G_{r,s} \cong \mathbb{R}^{2n+r+s} \to G_{r,s} : \delta_\rho(x,z) := (\rho x, \rho^2 z).
\]
Then the product formula in (8) shows that
\[
\delta_\rho \left( (x,z) \ast (y,w) \right) = \delta_\rho(x,z) \ast \delta_\rho(y,w).
\]
From the explicit form of the vector fields $X_j$ in (7) one immediately checks that $\Delta_{r,s}$ is homogeneous of degree 2 with respect to $\delta_\rho$, i.e., $\delta_\rho^* \Delta_{r,s} = \rho^2 \Delta_{r,s}$ for all $\rho > 0$.

In this setting Lemma 10.1 can be used to prove that $\Delta_{r,s}$ is locally solvable if and only if $r = 0$. The arguments are based on the non-injectivity of the ultra-hyperbolic operator $\Delta_{r,s}$ on Schwartz functions in case of $r > 0$, see [10]. We may also use a more refined criterion on local non-solvability of homogeneous left-invariant differential operators which is due to D. Müller and can be found in [20]:

**Theorem 10.2** (D. Müller, [20]). Let $L$ be a left-invariant homogeneous differential operator on a homogeneous, simply connected nilpotent Lie group $G$ with transpose $L^\tau$. Assume there exists a sequence $\{\psi_j\}_{j=1}^\infty$ of Schwartz functions on $G$ with (i) and (ii):

(i) $\psi_j(0) = 1$ for every $j$,

(ii) For every continuous semi-norm $\| \cdot \|_{(N)}$ on the Schwartz space $S(G)$ it holds:
\[
\lim_{j \to \infty} \| \psi_j \|_{(N)} \| L^\tau \psi_j \|_{(N)} = 0.
\]

Then $L$ is not locally solvable.

Now we can prove the following extension of Theorem 9.11.

**Theorem 10.3.** In the case $r > 0$ the ultra-hyperbolic operator $\Delta_{r,s}$ is not locally solvable. In particular, $\Delta_{r,s}$ does not even admit a fundamental solution in the space of Schwartz distributions $\mathcal{D}'(G_{r,s})$. Moreover, $\Delta_{0,s}$ for $s > 0$ is locally solvable and
\[
\Delta_{0,s} C^\infty(\mathbb{R}^{2n+s}) = C^\infty(\mathbb{R}^{2n+s}).
\]

**Proof.** Since $\Delta_{r,s}$ coincides with its transpose the local non-solvability follows from Theorem 10.2 and Lemma 10.1. In fact, we may choose $\{\psi_j\} \subset S(G_{r,s})$ to be the constant sequence $\psi_j = \varphi \in S(G_{r,s})$ where $\varphi$ denotes the function in Lemma 10.1. Then (i) and (ii) above are fulfilled and the first statement follows from Theorem 10.2. It is known that the following properties are equivalent (see 3 [20]):

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(a) $\Delta_{r,s}$ is locally solvable,

(b) $\Delta_{r,s} C^\infty(G_{r,s}) = C^\infty(G_{r,s})$,

(c) $\Delta_{r,s}$ has a fundamental solution in $\mathcal{D}'(G_{r,s})$.

By the equivalences (a) $\iff$ (c) the second statement follows from the first. 

11 Appendix

In this appendix we link the distribution $1/P^{n-1}$ in Proposition 7.3 to the value of $(P + i0)^{\lambda}$ at $\lambda = -n + 1$, cf. Proposition 6.3 and [14]. Assume that $\lambda \in \mathbb{C}$ and let $z \in \mathbb{C}$ be in the upper half plane, i.e. $\Im(z) > 0$. We write:

$$z^\lambda = \exp \left\{ \lambda \log |z| + i \lambda \arg(z) \right\} \quad \text{where} \quad 0 < \arg(z) < \pi$$

and use the notation of Proposition 7.3 and [14, Chapter III, Section 2.4].

Proposition 11.1. Let $\psi \in \mathcal{S}(\mathbb{R}^{2n})$, then one has:

$$\frac{1}{P^{n-1}}[\psi] = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \psi(x)(P(x) + i\varepsilon)^{-n+1} \, dx = \left( (P + i0)^{-n+1}, \psi \right).$$

(52)

Proof. Let $\mathcal{L}$ be the ultra-hyperbolic operator in (2). For each $\varepsilon > 0$ the following identity can be verified by a straightforward calculation:

$$\mathcal{L}[P(x) + i\varepsilon]^\lambda + 1 = 4(\lambda + 1)(n + \lambda)[P(x) + i\varepsilon]^\lambda - i\varepsilon 4\lambda(\lambda + 1)[P(x) + i\varepsilon]^{\lambda - 1}. \quad (53)$$

Assume in addition that $1 > \Re(\lambda) > -n$. Then it holds:

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\mathbb{R}^{2n}} \psi(x)[P(x) + i\varepsilon]^\lambda - 1 \, dx = 0 \quad \text{for all} \quad \psi \in \mathcal{S}(\mathbb{R}^{2n}). \quad (54)$$

In order to prove (54) we can use the integral representation (33) of the Gamma function and the identity (34). By a decomposition of the integral similar to the calculation following Lemma 7.2 one only needs to verify the estimate:

$$0 \leq \int_{1}^{\infty} t^{-\Re(\lambda) - n} e^{-te} \, dt = e^{\Re(\lambda) + n} \int_{1}^{\infty} s^{-\Re(\lambda) - n} e^{-s} \, ds$$

and note that under the above assumption $\Re(\lambda) + n > 0$:

$$0 \leq \varepsilon \int_{1}^{\infty} t^{-\Re(\lambda) - n} e^{-te} \, dt = \varepsilon e^{\Re(\lambda) + n} \int_{\varepsilon}^{\infty} s^{-\Re(\lambda) - n} e^{-s} \, ds$$

$$\leq \left\{ \begin{array}{ll} e^{\Re(\lambda) + n} & \text{if } \Re(\lambda) \neq -n + 1, \\ \varepsilon \log \left( \frac{1}{\varepsilon} \right) + e^{\Re(\lambda) + n} & \text{if } \Re(\lambda) = -n + 1 \rightarrow 0, \quad \text{as } \varepsilon \downarrow 0. \end{array} \right.$$
Obviously, (54) remains valid in the case of \( \Re(\lambda) \geq 1 \). Hence, multiplying both sides of (53) with \( \psi \in \mathcal{S}(\mathbb{R}^{2n}) \) and performing a partial integration over \( \mathbb{R}^{2n} \) shows for all \( \lambda \in \mathbb{C} \) with \( \Re(\lambda) \geq -n \) (provided that the limit exists):

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} [L \psi](x) (P(x) + i\varepsilon)^{\lambda+1} \, dx = 4(\lambda + 1)(n + \lambda) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \psi(x) (P(x) + i\varepsilon)^{\lambda} \, dx.
\]

Note that after a \( k \)-fold (\( k \in \mathbb{N} \)) iteration of the last equation one has:

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} [L^k \psi](x) (P(x) + i\varepsilon)^{\lambda+k} \, dx = \frac{1}{\Lambda(\lambda, k)} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \psi(x) (P(x) + i\varepsilon)^{\lambda} \, dx,
\]

where we define:

\[
\Lambda(\lambda, k) := \frac{1}{4^k \prod_{j=1}^{k} (\lambda + j)(n + \lambda + j - 1)}.
\]

For sufficiently large integer \( k \) and complex exponents \( \lambda \) with \( \Re(\lambda) \geq -n \) we show the existence of the limit above. If \( \mu \in \mathbb{C} \) with \( \Re(\mu) > 0 \) then:

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \psi(x) (P(x) + i\varepsilon)^{\mu} \, dx = \left( P + i0 \right)^{\mu}, \psi \right).
\]

Choose \( k \in \mathbb{N} \) such that \( \Re(\lambda) + k > 0 \). Inserting the above relation gives:

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2n}} \psi(x) (P(x) + i\varepsilon)^{\lambda} \, dx = \Lambda(\lambda, k) \left( \left( P + i0 \right)^{\lambda+k}, \psi \right) = \left( \left( P + i0 \right)^{\lambda}, \psi \right). \]  

(56)

Hence the right hand side of (56) defines a meromorphic extension to the complex plane of the limit on the left with removable singularities at the negative integers \( \lambda = -1, -2, \ldots, -n + 1 \). Moreover, note that the left hand side of (56) is continuous for \( \lambda \in \{ z : \Re(z) \geq -n + 1 \} \). In fact, this can be seen from an integral representation of the limit based on the calculations following Lemma 7.2. By choosing \( \lambda = -n + 1 \) in (56) the equality (52) follows.

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