A Lagrangian, small-scale investigation of turbulent entrainment in an axisymmetric jet.

M. Wolf¹, B. Lüthi¹, M. Holzner², A. Liberzon³, D. Krug¹ and A. Tsinober³

1 Institute of Environmental Engineering, ETH Zurich, 8093 Zurich, Switzerland
2 Max Planck Institute for Dynamics and Self-Organisation, 37073 Goettingen, Germany
3 School of Mechanical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel

E-mail: wolf@ifu.baug.ethz.ch

Abstract. Particle tracking velocimetry (PTV) was applied to study turbulent entrainment in an axisymmetric jet at Re = 5000. Several single-point flow statistics are used to characterize the general flow field of our newly designed jet facility, proving that a self-preserving axisymmetric jet could be established. An analysis of the Lagrangian evolution of small scale quantities, such as vorticity and strain, along trajectories passing the entrainment interface is performed. We find that a particle needs on the order of one Kolmogorov time scale to cross the entrainment interface, which is similar to results of grid turbulence without mean shear. Finally, we perform a conditional investigation of invariants of $\partial u_i/\partial x_j$ at the entrainment interface, analyzing joint probability density functions (joint PDFs) evaluated at different times along trajectories crossing the interfacial region.

1. Introduction

Partly turbulent flows exist in many different configurations in nature and technology, for example as smoke plumes from chimneys, effluent jets from waste-water outlets, atmospheric boundary layers, and wakes of aircrafts. One important feature of those flows is the phenomenon of turbulent entrainment (TE), where fluid particles of surrounding, non-rotational, regions are entrained into the turbulent flow. TE has a direct impact on dynamics and mixing in all partly turbulent flows, e.g. on transport rates of active quantities (momentum, energy, vorticity) and passive quantities (heat, mass) across the entrainment interface. Although the mechanism of the entraining process has been studied for a long time, see e.g. Townsend (1976) and Corrsin & Kistler (1954), numerical and experimental advances have only recently allowed to access the small scale features of the entrainment process related to the gradients of the velocity field. It could be shown for example by Bisset et al. (2002) and Westerweel et al. (2009) that small scale ‘nibbling’ processes play an essential role in turbulent entrainment along with the large scale ‘engulfment’. This illustrates the need for a more detailed analysis of small scale quantities in the study of TE in order to improve the understanding of the process itself. The turbulent/non-turbulent interface (TNTI) is a non-material surface that evolves with the flow and is defined by a threshold of vorticity (Corrsin & Kistler, 1954). The entrainment process is essentially
a Lagrangian phenomenon and therefore requires an experimental approach in the Lagrangian frame of reference. In a previous study of TE and TNTI properties, three-dimensional particle tracking velocimetry (3D-PTV) was performed in a turbulent flow with zero mean shear, e.g. Holzner et al. (2008). We now go one step further and investigate, in a Lagrangian frame of reference, small scale features of TE in a flow with significant mean shear.

The focus of our study lies in the Lagrangian behavior of particles crossing the entrainment interface. Thereby, we perform 3D-PTV measurements to study TE at the interface of a turbulent jet, i.e., to obtain velocity and velocity gradients of flow tracers that cross the TNTI. The questions we want to analyze are: How do these quantities change along trajectories crossing the interfacial region? How much time do particles need to pass this particular area? For this purpose, we consider the evolution of small scale quantities, such as vorticity, \( \omega = \nabla \times \mathbf{u} \), and strain, \( s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \), as well as their production terms, \( \omega_j \omega_i s_{ij} \) and \( -s_{ij} s_{jk} s_{ki} \).

In the following section we will give a brief description of the experimental setup and the measurement configurations. The results shown in this paper are divided into three subsections. First, one point flow statistics for the characterization of the flow field are presented. This is followed by an analysis of the Lagrangian evolution of strain and vorticity along trajectories passing the entrainment interface. Finally, joint probability density functions (joint PDFs) of the second and third invariants of \( \frac{\partial u_i}{\partial x_j} \) are used to classify the flow topology.

2. Experimental setup

![Figure 1. Left: Sketch of the experimental setup (not to scale). Right: Streak visualization of the axisymmetric jet. Two recordings are fitted together in one image in order to capture the flow field from the nozzle exit up to 133 \( x/d \) downstream (nozzle diameter \( d = 3 \) mm). The image shows the superposition of 1500 images corresponding to 0.5 s of recording. The red square indicates the position of the 3D interrogation volume of the 3D-PTV measurements.](image)

Measurements are carried out in our newly designed closed-loop water jet facility. A sketch of the facility is shown on the left in figure 1. The setup consists of a vertical jet emerging from the
end of a pipe (diameter $d = 3$ mm, length $l = 240$ mm) into a transparent cylinder of diameter $D = 300$ mm and length $L = 2000$ mm. The ratio of pipe length to pipe diameter is 80. This ratio is considered to be sufficient to obtain a fully developed turbulent velocity profile at the nozzle exit (Zagarola & Smits, 1998). In choosing the diameter of the test section $D = 300$ mm, a trade-off is made between a tank that is wide enough not to confine the jet too much, yet small enough to reach the measurement domain with our optical tools. The jet axial velocity is generated by the hydrostatic pressure difference between two constant-head reservoirs. In our measurements, the mean velocity of the flow at the jet exit was 1.67 m/s, implying a jet Reynolds number ($Re$) of 5000. A valve and a flow meter in front of the jet pipe are used to control the flow rate ($\hat{Q} = 0.71$ l/min). The water circuit is finally completed by an overflow tank and a pump. A rectangular glass box, filled with the same fluid as the test section, is fitted around the cylindrical water tank to avoid image distortion due to the difference in index of refraction between the in- and outside of the circular tank.

2D-PTV measurements were conducted to characterize the general flow field of the experimental facility. The flow was seeded with neutrally buoyant Polystyrene tracer particles with an average diameter of 45 µm. Illumination of the particles was provided by a continuous 15 Watt Argon-Ion laser. The beam was expanded through a cylindrical lens and formed a thin light sheet of about 1 mm thickness passing through the streamwise jet axis. The flow was recorded at a frame rate of 3000 Hz using a Photron SA 5 camera (1024 x 1024 pixels$^2$, object lens of 60 mm) focusing on a field of view of 200 x 200 mm$^2$. Sequential measurements were performed starting from the nozzle exit of the jet pipe up to 400 mm downstream ($\hat{x} = 133 x/d$), tracking on average 1800 particles in the field of view. A streak visualization of the flow is shown on the right in figure 1.

3. Results

3.1. Large scale flow statistics

For the analysis of the mean flow field, we used an ensemble average of 400 statistically independent 2D-PTV measurement sets. Every measurement set consists of 450 images, corresponding to 0.15 s of recording. Hence, in total we have 60 s of recording for the investigation of the mean velocity profiles from the nozzle exit up to 133 diameters downstream. The spatial resolution of the measurements was 2 mm. The rather coarse grained resolution is a result of the large field of view that was investigated (200 x 200 mm$^2$). However, the main goal
of these measurements was to get a large scale picture of the flow field and to check, within the achievable experimental accuracy, if the axisymmetric turbulent jet exhibits self-similarity in the region where we want to measure. Figure 2 shows the decay of the mean centerline velocity, $U_c$, and the development of the half width, $b_{0.5}$, along the stream-wise jet axis, normalized by the mean exit velocity at the nozzle, $U_{\text{nozzle}}$, and the diameter, $d$, of the jet pipe. The results clearly show a $x^{-1}$ decay for the centerline velocity and a linear development for the half width, which is consistent with the literature, e.g. Hinze (1975) and Pope (2000). The measured velocity decay constant $C_1 = 6.1$ as well as the spreading rate $C_2 = 0.089$ lie in the range reported by others, see Wygnanski & Fiedler (1969), Hussein et al. (1994), and Xu & Antonia (2002). Therefore, it can be concluded that a self-preserving turbulent jet is established in the present experiment.

\begin{equation}
\frac{U_c(x)}{U_{\text{nozzle}}} = C_1 \left( \frac{x}{d} - x_0 \right)
\end{equation}

$C_1 = 6.1164$

$x_0 = 3.2365$

\begin{equation}
\frac{b_{0.5}(x)}{d} = C_2 \left( \frac{x}{d} - x_0 \right)
\end{equation}

$C_2 = 0.089912$

$x_0 = 4.3582$

Figure 2. Variation of normalized centerline mean velocity (left) and normalized half width (right) as a function of the distance from the nozzle exit.

3.2. Lagrangian evolution of small scale quantities along trajectories

This section deals with the statistical analysis of Lagrangian flow information at the entrainment interface. Figure 3a illustrates the evolution of enstrophy, $\omega^2$, and strain, $s^2$, along trajectories crossing the entrainment interface. All trajectories are centered at the time $\hat{t}$, when a fixed threshold of $\omega^2$ is exceeded, i.e. $\tilde{t} = t - \hat{t}$, and normalized by $\tau_\eta$. The value of the threshold was set to 10% of the mean value of $\omega^2$ in the turbulent region. For the statistical analysis, all trajectories crossing the interface are then ensemble averaged, conditioned on $\tilde{t}$. About $3 \cdot 10^4$ trajectories with lengths ranging from 2 to 10 $\tau_\eta$ were processed in this way. The average length of the trajectories was 5 $\tau_\eta$. Enstrophy and strain values in the plot are normalized by $(U_c/b_{0.5})^2$.

It can be seen that in the non-turbulent region enstrophy is very small in magnitude, lying in the range of the noise level of the measurements. When particles reach the entrainment interface, $\omega^2$ increases quite suddenly at the interface and grows gradually from then on with increasing time in the turbulent region. Strain in comparison is already significant before particles pass the entrainment interface and it is about twice as large as enstrophy at the interface and beyond. Also, the jump at the interface is not as sudden as for $\omega^2$. The results obtained in the present study are compared to findings by Holzner et al. (2008) in a zero-mean-shear flow generated by an oscillating grid. Holzner et al. found that within the first few $\tau_\eta$ after passing the entrainment interface strain and enstrophy do not settle to a similar level, but maintain an offset. This feature is even more distinct for the axisymmetric jet. In the present study, the offset is even intensified such that a constant factor of 2 is kept between the two quantities. In addition, strain and enstrophy also continue to grow after passing the interfacial region and do not settle to a constant value. In fully developed turbulence, enstrophy and strain are "equal partners" (Tsinober, 2009). Therefore, it can be argued that the particles crossing the
entrainment interface do not reach a region where statistically homogeneous conditions apply within the first few \( \tau_\eta \) after passing the interfacial region, i.e., within the time they could be tracked in our measurements. Similar to the results of Holzner et al. (2008), a particle needs a time on the order of \( \tau_\eta \) to cross the region where the sudden increase of \( \omega^2 \) occurs. Hence, it seems that the addition of mean shear does not influence the time of residence of a particle within the interfacial region.

Figure 3b shows the conditionally averaged evolution of the components of the enstrophy production term, \( \omega_i \omega_j s_{ij} \) and a strain production term, \( -4/3 s_{ij} s_{jk} s_{ki} \). The enstrophy production term, \( \omega_i \omega_j s_{ij} \), behaves similarly to \( \omega^2 \). It is small in the non-turbulent region and increases suddenly at the entrainment interface. In contrast, the term \( -4/3 s_{ij} s_{jk} s_{ki} \) already grows significantly before a particle crosses the interfacial region. Similar to the observation made for \( \omega^2 \) and \( 2s^2 \), there remains an offset for the respective production terms.

![Figure 3](image.jpg)

**Figure 3.** (a) Conditionally averaged Lagrangian evolution of \( \omega^2 \) and \( 2s^2 \) normalized by \( (U_c/b_{0.5})^2 \) (b) conditionally averaged Lagrangian evolution of \( \omega_i \omega_j s_{ij} \) and \( -4/3 s_{ij} s_{jk} s_{ki} \) normalized by \( (U_c/b_{0.5})^3 \).

### 3.3. Analysis of joint PDFs of the second and third invariants of the velocity gradient tensor

In this section we use the second and third invariant of the velocity gradient tensor, \( A_{ij} = \frac{\partial u_i}{\partial x_j} \), to classify the flow topology during the entrainment process. The second invariant, \( Q = \frac{1}{2}(\omega^2 - 2s^2) \), quantifies the relative strength of enstrophy and strain, identifying enstrophy dominated (\( Q > 0 \)) or strain dominated (\( Q < 0 \)) regions (Tsinober, 2009). The third invariant is defined as \( R = -\frac{1}{3}(s_{ij} s_{jk} s_{ki} - \frac{2}{3} \omega_i \omega_j s_{ij}) \) and compares the production terms of \( \omega^2 \) and \( s^2 \). Joint PDFs of \( R \) and \( Q \) have proven to be a valuable tool for the analysis of small-scale flow properties. It was found by a number of investigators that different kinds of turbulent flows all show similar, quasi “universal” features in the R-Q plane (Chacín & Cantwell, 2000). The iso-probability contours in the R-Q joint PDF for example have a characteristic “teardrop” shape, with distinct accumulations of data points in the second (\( Q > 0, R < 0 \)) and fourth (\( Q < 0, R > 0 \)) quadrant, representing the swirling motion of turbulence on one hand and the predominance of the strain production term, \( -s_{ij} s_{jk} s_{ki} \), over \( \omega_i \omega_j s_{ij} \) in strain dominated regions on the other hand. A detailed review of the invariants of the velocity gradient tensor and its physical meaning is given in Cantwell (1992), Soria et al. (1994), Chertkov et al. (1999), and Ooi et al. (1999) among others.

A conditional investigation of the invariants for different distances from the entrainment interface was recently performed by da Silva & Pereira (2008) analyzing a direct numerical simulation (DNS) of a planar turbulent jet as well as by Khashehchi et al. (2010) applying
tomographic particle image velocimetry (Tomo-PIV) to a round turbulent jet. Following their approach, we also employ a conditional investigation of R and Q at the entrainment interface, but in a Lagrangian frame of reference, conditioning the plots for different times along trajectories crossing the interfacial region. Like in the previous section, all trajectories are centered at the time $\hat{t}$, which defines the point in time where the enstrophy threshold is exceeded. Figure 4 shows the joint PDFs of R and Q for three different periods, before, during and after the interface crossing, i.e. at $\tilde{t} = t - \hat{t} \approx -1\tau_\eta$, at $\tilde{t} \approx 1\tau_\eta$, and at $\tilde{t} \approx 4\tau_\eta$. The dashed line defined by $Q^* = -3(R^2_\pi)^{1/3}$ divides the area into two distinct regions. For $Q > Q^*$, the eigenvalues of $A_{ij}$ are real and there is a predominance of vortex stretching for $R < 0$ and a predominance of vortex compression for $R > 0$. While for $Q < Q^*$, the eigenvalues include a pair of complex conjugates, where $R < 0$ signifies a sheetlike structure and $R > 0$ a tubelike structure. A detailed description of the physical meaning related to the different quadrants of the R-Q plane is given in Ooi et al. (1999). For our analysis, the curve $Q^*$ is important for two reasons. First, in the quasi-irrotational flow region, i.e. for $\tilde{t} \approx -1\tau_\eta$, most data points fall below the dashed line because there is little vorticity in this region, hence there is quasi no vortex stretching or compression, as can be seen in figure 4a. Secondly, in the turbulent flow, data points with $Q < 0$ should predominantly be in the proximity of the right branch of $Q^*$, a feature related to the previously mentioned “teardrop” shape. Figures 4b and 4c exhibit the “universal” teardrop shape for times $\tilde{t} \approx 1\tau_\eta$ and $\tilde{t} \approx 4\tau_\eta$. This implies that the distinct feature of the turbulent flow develops already shortly after a particle has crossed the interfacial region. The finding is consistent with the studies of da Silva & Pereira (2008) and Khashechhi et al. (2010) focusing on the position in space instead of a point in time. The result emphasizes again that the residence time of a particle in the interfacial region is of the order of $\tau_\eta$, which fits to explanations by Corrsin & Kistler (1954).

**Figure 4.** Joint PDFs of the R and Q at three conditional points in time: (a) $\tilde{t} \approx -1\tau_\eta$, (b) $\tilde{t} \approx 1\tau_\eta$, and (c) $\tilde{t} \approx 4\tau_\eta$. R and Q are normalized by $\tau_\eta^3$ and $\tau_\eta^2$ respectively.

### 4. Conclusions

2D-PTV and 3D-PTV measurements were conducted on an axisymmetric water jet at $Re = 5000$. The results of the large scale two-dimensional velocity measurements show a $x^{-1}$ decay.
for the centerline velocity and a linear development for the half width, indicating that a self-preserving jet is established in the present experiment. The most interesting finding of this study is that the averaged time of residence of a particle within the entrainment interface is of the order of $\tau_\eta$, which fits to explanations by Corrsin & Kistler (1954). The result was obtained in two ways. First, a Lagrangian evolution of small scale quantities of trajectories crossing the interfacial region showed the finding. And secondly, an analysis of the joint PDFs of $R$ and $Q$ confirmed the assumption. The finding is similar to results obtained in a previous study of a turbulent flow with zero mean shear, e.g. Holzner et al. (2008). Even though this implies that the addition of shear does not influence the time scale for entraining fluid particles, further careful study is needed to elucidate this and other issues related to the presence of mean shear.

5. Acknowledgments

We gratefully acknowledge the support of this work by the Georg Fischer Fund under grant no. ETH-14 10-2.

References

Bisset, D. K., Hunt, J. C. R. & Rogers, M. M. 2002 The turbulent/non-turbulent interface bounding a far wake. *J. Fluid Mech.* **451**, 383–410.

Cantwell, B.J. 1992 Exact solution of a restricted euler equation for the velocity-gradient tensor. *Physics of Fluids* **4** (4), 782–793.

Chacín, Juan M. & Cantwell, Brian J. 2000 Dynamics of a low Reynolds number turbulent boundary layer. *J. Fluid Mech.* **404**, 87–115.

Chertkov, Misha, Pumir, Alain & Shraiman, Boris I. 1999 Lagrangian tetrad dynamics and the phenomenology of turbulence. *Physics of Fluids* **11** (8), 2394–2410.

Corrsin, S. & Kistler, A. 1954 The free-stream boundaries of turbulent flows. *NACA, TN-3133, TR-1244*, pp. 1033–1064.

Hinze, J. O. 1975 *Turbulence*, 2nd edn. New York: McGraw-Hill.

Holzner, M., Liberzon, A., Nikitin, N., Lüthi, B., Kinzelbach, W. & Tsinober, A. 2008 A Lagrangian investigation of the small-scale features of turbulent entrainment through particle tracking and direct numerical simulation. *J. Fluid Mech.* **598**, 465–475.

Hoyer, K., Holzner, M., Lüthi, B., Guala, M., Liberzon, A. & Kinzelbach, W. 2005 3d scanning particle tracking velocimetry. *Exp. Fluids* **39** (5), 923–934.

Hussein, N. J. C., Capp, S. P. & George, W. K. 1994 Velocity measurements in a high-reynolds-number, momentum-conserving, axisymmetric, turbulent jet. *J. Fluid Mech.* **258**, 31–75.

Khashehchi, M., Elsinga, G., Ooi, A., Soria, J. & Marusic, I. 2010 Studying invariants of the velocity gradient tensor of a round turbulent jet across the turbulent/NOTurbulent interface using tomo-piv. 15th Int Symp on Applicaitons of Laser Techniques to Fluid Mechanics, Lisbon, Portugal, 05-08 July.

Ooi, Andrew, Martín, Jesus, Soria, Julio & Chong, M.S. 1999 A study of the evolution and characteristics of the invariants of the velocity-gradient tensor in isotropic turbulence. *J. Fluid Mech.* **381**, 141–174.

Pope, Stephen B. 2000 *Turbulent Flows*. Cambridge University Press.

Da Silva, C. B. & Pereira, C. F. 2008 Invariants of the velocity-gradient, rate-of-strain, and rate-of-rotation tensors across the turbulent/non-turbulent interface in jets. *Physics of Fluids* **20** (5).
Soria, J., Sondergaard, R., Cantwell, B. J., Chong, M. S. & Perry, A. E. 1994 A study of the fine-scale motions of incompressible time-developing mixing layers. *Physics of Fluids* **6** (2), 871–884.

Townsend, A. A. 1976 *The Structure of Turbulent Shear Flow*, 2nd edn. Cambridge University Press.

Tsinober, A. 2009 *An informal conceptual introduction to turbulence*, 2nd edn. Springer, Berlin, New York.

Westerweel, J., Fukushima, C., Pedersen, J. & Hunt, J. 2009 Momentum and scalar transport at the turbulent/non-turbulent interface of a jet. *J. Fluid Mech.* **631**, 199–230.

Wygnanski, I. & Fiedler, H. 1969 Some measurements in the self-preserving jet. *J. Fluid Mech.* **38** (3), 577–612.

Xu, G. & Antonia, R. A. 2002 Effect of different initial conditions on a turbulent round free jet. *Exp. Fluids* **33**, 677–683.

Zagarola, M. V. & Smits, A. J. 1998 Mean-flow scaling of turbulent pipe flow. *J. Fluid Mech.* **373**, 33–79.