NEW PATH EQUATIONS IN ABSOLUTE PARALLELISM GEOMETRY

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Abstract. The Bazanski approach, for deriving the geodesic equations in Riemannian geometry, is generalized in the absolute parallelism geometry. As a consequence of this generalization three path equations are obtained. A striking feature in the derived equations is the appearance of a torsion term with a numerical coefficients that jumps by a step of one half from equation to another. This is tempting to speculate that the paths in absolute parallelism geometry might admit a quantum feature.

1. Introduction

It is well known that Riemannian geometry possesses two types of paths. The first is the geodesic path which is given by the second order differential equation,

$$\frac{dU^\mu}{dS} + \{^\mu_{\alpha \beta}\} U^\alpha U^\beta = 0,$$

where \{^\mu_{\alpha \beta}\} is Christoffel symbol, S is an evolution parameter varies along the path and \(U^\alpha\) is a unit vector tangent to the path. The second is the null-geodesic path given by the second order differential equation,

$$\frac{dN^\mu}{d\lambda} + \{^\mu_{\alpha \beta}\} N^\alpha N^\beta = 0,$$

where \(N^\alpha\) is a null vector tangent to the path, and \(\lambda\) is the evolution parameter.

In constructing his theory of general relativity (GR), Einstein has used Riemannian geometry in which the first path taken to represent the trajectory of a massive test particle, while the second taken to represent the trajectory of massless particle moving in a gravitational field.

Bazanski (1977, 1989) has established a new approach to derive the equations of geodesic and geodesic deviation simultaneously by carrying out the variation on the following Lagrangian:

$$L_B = g_{\mu \nu} U^\mu \frac{D \Psi^\nu}{DS}.$$
where \( U^\mu = \frac{dx^\mu}{dS} \), \( g_{\mu\nu} \) is the metric tensor, \( \Psi^\mu \) is the deviation vector and \( \frac{D}{DS} \) is the covariant differential operator using Christoffel Symbol.

In the last thirty years some problems appeared in GR, as a result of its applications, especially in the domain of cosmology. Some authors (cf. Mikhail and Wanas (1977), Møller (1978) and Hayashi and Shirafuji (1978)) strongly believe that those defects may be removed using a more general geometry than the Riemannian. A possible candidate geometry for this purpose is the absolute parallelism (AP) geometry (cf. Einstein (1929), Robertson (1932) and McCrea and Mikhail (1956)).

The aim of the present work is to find the possible paths in the AP-geometry that can be considered as generalization of the paths in the Riemannian geometry.

2. Path equations in AP-Geometry

In the AP-geometry, one can define four different affine connexions, Christoffel symbols \( \{ \Gamma^{\alpha}_{\mu\nu} \} \), the non-symmetric connexion \( \Gamma^{\alpha}_{\mu\nu} \) defined as a consequence of the AP-condition, the dual connexion \( \tilde{\Gamma}^{\alpha}_{\mu\nu} \) and the symmetric part of the non-symmetric connexion \( \Gamma^{\alpha}_{(\mu\nu)} \). Using these connexions one can define the following derivatives:

\[
A^{\mu}_{,\alpha} = A^{\mu \beta} \{ \beta_{\alpha} \} A^{\beta}, \quad (4)
\]

\[
A^{\mu \alpha}_{|\alpha} = A^{\mu \beta}_{,\alpha} + \Gamma^{\beta}_{\alpha\beta \alpha} A^{\beta}, \quad (5)
\]

\[
A^{\mu}_{|\alpha} = A^{\mu \beta}_{,\alpha} + \Gamma^{\mu}_{\beta \alpha} A^{\beta}, \quad (6)
\]

\[
A^{\mu}_{-|\alpha} = A^{\mu \beta}_{,\alpha} + \tilde{\Gamma}^{\mu}_{\beta \alpha} A^{\beta}, \quad (7)
\]

where \( A^\mu \) is an arbitrary contravariant vector, and \( (,) \) denotes ordinary partial differentiation. The coordinate derivatives (4), (5), (6) and (7) are related to the parameter derivatives by the following relations:

\[
\frac{DA^\mu}{DS} = A^{\mu \alpha} U^\alpha, \quad (8)
\]

\[
\frac{DA^\mu}{DS^+} = A^{\mu \alpha}_{|\alpha} V^\alpha, \quad (9)
\]

\[
\frac{DA^\mu}{DS^0} = A^{\mu \alpha}_{|\alpha} W^\alpha, \quad (10)
\]
\[
\frac{DA^\mu}{DS^-} = A^-_{\alpha} J^\alpha, \tag{11}
\]

where \( S, S^+, S^0 \) and \( S^- \) are parameters varying along the curves whose tangents are, respectively, \( U^\alpha, V^\alpha, W^\alpha \) and \( J^\alpha \) defined in the usual manner.

Now generalizing the Bazanski’s Lagrangian (3) using (9), (10) and (11) we get the following Lagragians:

\[
L^+ = \lambda_i^\alpha \lambda_i^\beta \frac{\xi^\beta}{DS^+}, \tag{12}
\]

\[
L^0 = \lambda_i^\alpha \lambda_i^\beta \frac{W^\alpha}{DS^0}, \tag{13}
\]

\[
L^- = \lambda_i^\alpha \lambda_i^\beta \frac{J^\alpha}{DS^-}, \tag{14}
\]

where \( \lambda_i^\alpha \) are the tetrad vectors giving the structure of the AP-space; \( \xi^\beta, \zeta^\beta \) and \( \eta^\beta \) are the vectors giving the deviation from the curves characterized by the evolution parameters \( S^+, S^0 \) and \( S^- \) respectively.

Carrying out the variation formalism on (12), (13) and (14), noting that raising and lowering indices does not commute with the differential operators used in (13), (14), we get respectively

\[
\frac{dV}{dS^+} + \{\alpha_\beta\}^{\mu} V^\alpha V^\beta = -\Lambda_{(\alpha_\beta)}^{\mu} V^\alpha V^\beta, \tag{15}
\]

\[
\frac{dW}{dS^0} + \{\alpha_\beta\}^{\mu} W^\alpha W^\beta = -\frac{1}{2} \Lambda_{(\alpha_\beta)}^{\mu} W^\alpha W^\beta, \tag{16}
\]

\[
\frac{dJ}{dS^-} + \{\alpha_\beta\}^{\mu} J^\alpha J^\beta = 0, \tag{17}
\]

where \( \Lambda_{\mu}^{\alpha} \) is the torsion of space-time defined by

\[
\Lambda_{\mu}^{\alpha} = \Gamma_{\mu}^{\alpha} - \Gamma_{\mu}^{\alpha}, \tag{18}
\]

It can be shown that the 1st integrals of the equations (15), (16) and (17) are given, respectively, by

\[
g_{\alpha_\beta} V^\alpha V^\beta = V^2, \tag{19}
\]

\[
g_{\alpha_\beta} W^\alpha W^\beta = W^2, \tag{20}
\]

\[
g_{\alpha_\beta} J^\alpha J^\beta = J^2; \tag{21}
\]

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where V, W and J are constants along the corresponding paths respectively. But since V, W and J are scalars, one can conclude that these quantities are constants in general (everywhere).

3. Discussion

In generalizing the Bazanski’s approach in the AP-geometry, three equations of paths are obtained. These equations can be considered as generalization of the geodesic equations in Riemannian geometry. Moreover, equation (17) and its first integral (21) give rise to the geodesic, (and null-geodesic upon reparameterization), equation of Riemannian geometry. In this case the vector $J^\mu$ will be reduced to a unit vector (in the case of the geodesic) or a null-vector (in the case of the null-geodesic). Thus, in generalizing the Bazanski’s approach, in the AP-geometry, we get in addition to the geodesic and null-geodesic equations, two more paths (15) and (16) contain torsion terms.

One can look at the three equations (15), (16) and (17) as representing three path equations containing torsion terms with different coefficients. The striking feature is that the coefficients of the torsion terms are $1$, $\frac{1}{2}$ and $0$ in the equations (15), (16) and (17), respectively. It is clear from these equations that there is a jump equal to $\frac{1}{2}$, from one path to another.

It is tempting to speculate that paths in the AP-geometry possess some quantum features. The question, now, is: What are the physical trajectories, if any, that these paths represent? This question might be answered in a forthcoming article.

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