We derive a class of solutions to the string sigma-model equations for the closed bosonic string. The tachyon field is taken to form a constant condensate and the beta-function equations at one-loop level are solved for the evolution of the metric and the dilaton. The solutions represent critical string theories in arbitrary dimensions. The spectrum of the subclass of models with a linearly rising asymptotic dilaton is found using the Feigin-Fuks method. Certain approximate solutions arising in string field theory are used to illustrate the results explicitly. An argument based on conformal invariance leads to the conjecture that stringy corrections to at least some singular spacetimes in general relativity result in non-singular metrics. We use the singularities of the big-bang/crunch type appearing in our models to examine this conjecture.
1. Introduction

Many open issues remain in string theory. One is the development of solutions to critical string theories that have non-trivial backgrounds and another is the description of models in non-critical dimensions. Part of the motivation for addressing these issues is the stringy resolution of puzzles in classical and quantum general relativity, such as the role and nature of spacetime singularities. In this paper, we seek solutions to critical string theories that describe models in arbitrary dimensions.

Several methods can be used to find string theories in non-trivial backgrounds. An important one is the sigma-model approach \[1, 2, 3, 4\]. The idea is to impose conformal invariance on an effective theory for selected particle fields contained in the string, by requiring the vanishing of the corresponding beta functions. The resulting equations can be viewed as arising from a spacetime action written in terms of the particle fields. In explicit calculations, the beta functions are usually determined only at tree-level in the string perturbation series and at no more than a few orders in an expansion in powers of the string scale. Nonetheless, solutions of the resulting expressions are presumably perturbative approximations to exact solutions of the full string theory and so may thereby provide insight.

The lowest-lying fields of the closed string are the tachyon, the dilaton, the graviton, and the Kalb-Ramond field. Some authors have addressed the issue of the presence of the tachyon in the beta-function equations. The proof that the weak-field expansion provides a link between a suitable tachyon term in the sigma model and string tachyon scattering amplitudes is given in \[5\], while the beta functions incorporating the tachyon are discussed in \[6, 7, 8\]. However, much of the work on solutions to the beta-function equations has concentrated on the role of the dilaton and graviton \[9, 10, 11, 12, 13\].

In generating solutions to the beta function equations involving some non-zero background values, it is important to realize that it is mathematically consistent to neglect the tachyon $T$ only if the tachyon potential $V(T)$ satisfies $V(T) = V'(T) = 0$. Even if these conditions are satisfied, any resulting solutions are destabilized if $V'' < 0$. In this paper, we seek to construct stable string models in background
fields corresponding to arbitrary dimensions. We consider in particular the case of a constant tachyon expectation, $T = T_0$. This means that we are seeking simple closed-string analogs of the usual de Sitter, Minkowski, and anti-de Sitter solutions arising in a particle field theory when a scalar forms a Higgs condensate. These solutions are dynamical, i.e., they represent evolving universes, in which both the metric and the dilaton fields change with time.

When the action is written in a form as close to the Einstein action as possible, the solutions we find have singularities of the big bang and/or big crunch type. They provide a means of exploring the second open issue mentioned above, namely, whether and how strings permit answers to the issue of the physical meaning of the global structure of spacetime in the presence of spacetime singularities. Here, we argue that string theory cures at least some of the spacetime singularities that arise in general relativity. The argument is based on the possibility of performing conformal transformations that smooth away the singularities.

Provided the asymptotic solution has a linear dilaton background, information about the spectrum in the sigma model can be found by an application of the Feigin-Fuks method \[14\]. Note that a large class of exact solutions at the string tree level can be found in terms of WZW models \[15\], and various models corresponding to non-zero backgrounds have been constructed \[16, 17\]. However, our models are not known to fit in this class \[18\] and so these methods are not of direct use here.

In principle, an alternative approach to string theories in non-trivial backgrounds is via string field theory. For example, one can seek to construct an effective potential for chosen particle fields and then look for non-trivial extrema \[19\]. For the field theory of the closed bosonic string in its critical dimension \[20, 21, 22\], an analysis of this sort \[23\] suggests the existence of stringy solutions of the type we present. This approximate method of solution of the string-field equations of motion indicate the existence of an asymptotic and perturbatively stable vacuum with a non-zero tachyon expectation value. We use these approximate solutions to illustrate explicitly our results obtained from the sigma-model approach and the Feigin-Fuks method.

Here is an outline of the contents of this paper. In section 2, we set up some formalism and present to leading level the beta functions for the tachyon and all
the massless fields in the closed bosonic string. We then describe the basis for the conjecture that singular spacetimes in general relativity may be non-singular in the string extension. In section 3, we derive a class of solutions of the beta function equations, valid for any dimension $d$. They describe cosmologically evolving universes in which nonzero expectation values appear for all the particles in the theory except the Kalb-Ramond field. In section 4, we use the Feigin-Fuks technique to get information about the spectrum of the string solutions at asymptotic times for a subclass of these models. Approximate results derived from closed-string field theory are presented in section 5 and used to provide explicit examples. We summarize and discuss the content of the paper in section 6.

2. String $\beta$-functions

The spectrum of the closed bosonic string in its critical dimension propagating in flat spacetime with metric $\eta_{\mu\nu}$ is well understood. At the lowest level, there is a tachyon $T$ with mass $m$, $m^2 = -4/\alpha'$. The first-excited states are massless and consist of a graviton $h_{\mu\nu} = h_{\nu\mu}$, a dilaton $\phi$ and a Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$. Above these are an infinite number of massive fields. For non-critical string theory in spacetime dimension $d$, the tachyon mass is given by $m^2 = -(d - 2)/6\alpha'$. The first-excited state remains massless but the graviton, dilaton and Kalb-Ramond fields appear as discrete states. Note that $d \leq 26$ seems to be demanded by the no-ghost theorem; see, for example, ref. [24].

Each of the fields $T, h_{\mu\nu}, B_{\mu\nu}$ and $\phi$ are long range and thus may reasonably be expected to form a condensate in spacetime. The other fields all have large masses, and so their direct influence on physics at large scales should be minimal. We can construct the action $I$ for a string propagating in a background where the long-range fields do not vanish. Let the string world sheet $\Sigma$ have a metric $\gamma_{ab}$ and coordinates $\xi^a, a = 1, 2$. Let the location of $\Sigma$ in spacetime be given by $X^\mu(\xi^a)$. The action $I$ is then

$$I = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi \sqrt{\gamma} \left( \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + \frac{1}{2} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + \frac{1}{2} \alpha' T - \frac{1}{4} \alpha' R^{(2)}(\phi) \right).$$

(1)
Here, $g_{\mu\nu}$ is the background metric of a curved spacetime, coming from the combination of $h_{\mu\nu}$ and $\eta_{\mu\nu}$. The symbol $\epsilon^{ab}$ represents the alternating tensor on $\Sigma$ and $R^{(2)}$ is the Ricci curvature scalar formed from the metric $\gamma^{ab}$. Note that $\xi^a = (\tau, \sigma)$, $g_{\mu\nu}$, $B_{\mu\nu}$, the dilaton $\phi$, and the tachyon $T$ are dimensionless, while $X^\mu$ has length dimension one. Various terms involving fields for higher-mass excitations of the string could be added to $I$, but they would lead to apparently unrenormalizable interactions.

The physical significance of $\phi$ can be understood from the action $I$. If $\phi$ is constant over $\Sigma$, the contribution to the action from the dilaton term becomes

$$I_\phi = \langle \phi \rangle \frac{1}{8\pi} \int_\Sigma d^2 \xi \gamma^{1/2} R^{(2)} = \frac{1}{2} \langle \phi \rangle \chi,$$

since the integral over the string world sheet is just the Gauss-Bonnet invariant, which is related to the Euler character $\chi$ of the world sheet. A world sheet of genus $g$ with $h$ ends has an Euler character of $\chi = 2 - 2g - h$. Thus, the effect of adding an extra end to the world sheet is to give an additional contribution to the path integral of $\exp(\frac{1}{2} \langle \phi \rangle)$. This means that the string coupling constant is

$$\hat{g} = \exp(\frac{1}{2} \langle \phi \rangle).$$

Note that nothing unphysical happens if $\langle \phi \rangle \to -\infty$. The string coupling constant merely goes to zero, so free string theory yields the only physical contributions to the path integral. Conversely, if $\langle \phi \rangle \to \infty$ then strong string coupling occurs, and our picture of physics based on string perturbation theory breaks down.

The basic symmetry of string physics is world-sheet conformal invariance. Classically, this symmetry is violated in $I$ by the tachyon and dilaton fields. Nevertheless, it can be restored quantum mechanically. As is apparent from the form of $I$, the background spacetime fields $g_{\mu\nu}, B_{\mu\nu}$, and $\phi$ behave like coupling constants and the field $T$ behaves like a world-sheet cosmological constant. The quantum-mechanical statement of conformal invariance is that the $\beta$ functions for all these fields vanish. Evaluating the $\beta$ functions to one loop gives

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \phi + \nabla_\mu T \nabla_\nu T + \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma}$$

for the metric,

$$\nabla_\lambda H^{\lambda\mu\nu} + (\nabla_\lambda \phi) H^{\lambda\mu\nu} = 0$$
for the Kalb-Ramond field, and

$$\Box T + (\nabla_\mu \phi)(\nabla_\nu T) = V'(T)$$  \hspace{1cm} (6)

for the tachyon. In these equations, $H_{\lambda \mu \nu} = 3 \partial_\lambda B_{\mu \nu}$ is the 3-form field strength found from the exterior derivative of $B_{\mu \nu}$. Also, $V'(T) = \partial V/\partial T$ is the derivative of the tachyon potential $V(T)$, the explicit form of which is discussed in section 5. The beta function for the dilaton can also be calculated. Requiring its vanishing to the same order as Eqs. (4) - (6) gives

$$\hat{c} = R - (\nabla \phi)^2 - 2\Box \phi - (\nabla T)^2 - 2V(T) - \frac{1}{12} H_{\lambda \mu \nu} H^{\lambda \mu \nu} ,$$  \hspace{1cm} (7)

where $\hat{c} = 2(d - 26)/3\alpha'$. In the absence of background fields, $\hat{c}$ would be the central charge $c$ divided by $3\alpha'/2$; see ref. [25].

All these conditions can be derived from the spacetime action

$$I_S = -\frac{1}{2\kappa^2} \int d^d x \sqrt{g} e^\phi \left( \hat{c} - R - (\nabla \phi)^2 + (\nabla T)^2 + 2V(T) + \frac{1}{12} H_{\lambda \mu \nu} H^{\lambda \mu \nu} \right) ,$$  \hspace{1cm} (8)

where $2\kappa^2 = 16\pi G_N$ when $d = 4$. Thus, we can derive the stringy version of the Einstein equation and the relevant wave equations from a symmetry principle.

Conformal invariance on the string world sheet induces conformal covariance in spacetime. One can always redefine the metric by conformal transformations involving the dilaton. Thus, under conformal transformations the mapping

$$g_{\mu \nu} \mapsto f(\phi) g_{\mu \nu} , \quad B_{\mu \nu} \mapsto B_{\mu \nu} , \quad T \mapsto T , \quad \phi \mapsto \phi$$  \hspace{1cm} (9)

is an invariance of the system. Nonetheless, a conformal transformation of the form (9) induces non-trivial changes in both the $\beta$ functions and the action because both the connection coefficients and the curvature transform in a non-trivial way.

The action (8) is expressed in the ‘string’ frame, so named because this form arises naturally in the $\sigma$-model approach to string theory. However, to discuss spacetime physics it is useful to write the action in a form as close to the Einstein action as possible. To arrange this, choose

$$f(\phi) = e^{-2\phi/(d-2)} ,$$  \hspace{1cm} (10)
which has the effect of removing the factor $e^\phi$ multiplying $R$ in Eq. (8). The action in the ‘Einstein’ frame is then

$$I_E = -\frac{1}{2\kappa^2} \int_M d^d x \sqrt{g}[ - R + \hat{c} e^{-2\phi/(d-2)} + \frac{1}{d-2} (\nabla \phi)^2 + (\nabla T)^2 + 2 e^{-2\phi/(d-2)} V(T) + \frac{1}{12} e^{4\phi/(d-2)} H_{\lambda\mu\nu} H^{\lambda\mu\nu} ] . \quad (11)$$

Note that the graviton is not directly coupled to the dilaton in this frame.

The question arises as to whether one frame is more ‘natural’ than another. Evidently, if one wishes to compare string predictions with general relativity, it is useful to use the Einstein frame. In contrast, from the sigma-model viewpoint the ‘string’ frame appears more natural. We believe neither frame should be taken as fundamental in the real sense of the word. Conformal invariance ensures that the choice of frame is a matter of convenience in the description.

It is nonetheless true that the arbitrariness of $f(\phi)$ in this formulation affects our view of the physics near singularities. Consider first the situation in general relativity, where spacetime singularities are viewed as the boundary of spacetime. They are usually related to singularities of the metric tensor. However, the converse is not true: a metric singularity may be a coordinate singularity or it may be a spacetime singularity. The way to discern the difference is to seek a diffeomorphism to a new, nonsingular metric. If one exists, the singularity was a coordinate artifact. We view the situation concerning conformal transformations in string theory to be analogous. Locally, rewriting the $\beta$ function in a new conformal frame merely maps the solutions of the equations into new solutions. Globally, the situation is radically different. It might be the case that a particular spacetime appears singular in some coordinate frame. If we can find a $f(\phi)$ removing the singularity, then a non-singular conformal frame exists. We regard combinations of the dilaton field and the metric where this can be done as non-singular. This situation therefore allows us to view as non-singular many spacetimes that would be singular in general relativity. In this way, string theory may resolve one of the perplexing issues of general relativity. This might seem strange from the point of view of spacetime physics, where at a singularity it is impossible to solve differential equations. However, the spacetime is determined
by the string behavior, which may be well-defined even though the spacetime picture suffers from pathologies.

To remove a spacetime singularity via such a transformation, $f(\phi)$ itself must be singular. If this happens because $\phi$ is tending to $-\infty$, then nothing odd happens to string physics because this merely indicates that the string coupling is approaching zero, cf. Eq. (3). However, if $\phi$ is tending to $+\infty$ then perturbation theory will break down, and a more sophisticated treatment of the string may be needed.

3. Solutions of the $\beta$-function Equations

In this section, we address the issue of some solutions to the $\beta$-function equations other than Minkowski spacetime in the critical dimension. One well-known solution for $d > 26$ is the linear dilaton background in which spacetime is flat, $T = 0$, and the dilaton rises linearly with time \[12\]. Examination of the dilaton beta function \[7\] shows that

$$\phi(t) = \mu t \quad \text{,} \quad \mu = \sqrt{\frac{2(d - 26)}{3\alpha'}}. \quad (12)$$

This solution is still plagued by a tachyon, as can be seen by the following argument. Consider the linear part of the tachyon beta function:

$$\square T + (\nabla \chi \phi)(\nabla T) = -\frac{4}{\alpha'} T. \quad (13)$$

The tachyon is minimally coupled to the dilaton. The tachyon mass can be determined from this equation by eliminating the term in $\nabla T$ using \[12\]. Thus, Eq. \[13\] becomes

$$[-(\partial_t + \frac{1}{2}\mu)^2 + \nabla^2]T = -\left(\frac{4}{\alpha'} + \frac{\mu^2}{4}\right)T \quad (14)$$

and the physical mass of the tachyon field is

$$m^2 = \frac{-d + 2}{6\alpha'}. \quad (15)$$

This calculation agrees with the standard calculation based either on the Casimir energy of the ground state or on the anomaly in the Virasoro algebra.

The idea we explore here is to suppose that the tachyon potential has extrema other than the one at $T = 0$. At $T = 0$, $V(T)$ is a maximum and so the excitations
of the field $T$ about the origin are interpreted as tachyons. In general, the mass $m$ of the field that represents small fluctuations about some fixed value $T_0$ of $T$ is given by $m^2 = V''(T_0)$, where $V'(T_0) = 0$. Thus, for $T_0 = 0$ we find $m^2 = -4/\alpha'$. In this section, we assume that $V(T)$ has an extremum at $T = T_0 \neq 0$, with $V''(T_0) \geq 0$ to avoid problems with stability and causality.

In field theory, a tachyon can appear as a consequence of selecting an unstable vacuum state, but disappears after spontaneous symmetry breaking has occurred. In this case, a non-zero value of $V(T_0)$ gives rise to an effective cosmological constant. The vacuum state of the resulting theory is de Sitter space if $V(T_0) > 0$, Minkowski space if $V(T_0) = 0$, or anti-de Sitter space if $V(T_0) < 0$. In contrast, in string theory the vacuum state cannot be so simple because $V(T)$ is coupled to the gravitational field through the dilaton. As is apparent from the dilaton beta function, any curvature causes the dilaton to vary over spacetime. This means the effective cosmological constant can depend on the location in spacetime. We therefore expect to find string solutions that are more complicated than de Sitter, Minkowski, or anti-de Sitter spaces.

A plausible ansatz is to suppose that spatial sections of the spacetime metric are flat $R^{d-1}$ coordinatized by $(x_1, x_2, \ldots, x_{d-1})$. The assumption of flatness is reasonable because the false-vacuum configuration corresponding to $T_0 = 0$ can evolve into a new configuration without any topology change. We therefore seek a spacetime described by a $k = 0$ Friedman-Robertson-Walker-Lemaître universe with line element

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + \ldots dx_{d-1}^2) .$$

(16)

Here, $t$ is called the cosmic time coordinate because it is the same as proper time for geodesic observers at constant $x_i$. If $a(t) = e^{Ht}$ the spacetime is de Sitter, if $a(t)$ is constant the spacetime is Minkowski, and if $a(t) = \cos Ht$ the spacetime is anti-de Sitter. Whatever the functional form of $a(t)$, these spacetimes are conformally flat. This can be seen by defining a conformal time coordinate $\eta$ by

$$\eta = \int \frac{dt}{a(t)} ,$$

(17)

so that

$$ds^2 = a^2(\eta)(-d\eta^2 + dx_1^2 + dx_2^2 + \ldots dx_{d-1}^2) .$$

(18)
In what follows, for simplicity we also assume that the dilaton $\phi$ and the tachyon fields are functions only of $t$, and we take $H_{abc} = 0$.

The $\beta$-function equations can now be written out explicitly. In the string frame, the 00 and $j j$ components of the beta function (4) for the metric are

$$
(d - 1) \frac{\ddot{a}}{a} + \dot{\phi} + \ddot{T}^2 = 0
$$

(19)

and

$$a \ddot{a} + (d - 2) \dot{a}^2 + a \dot{a} \dot{\phi} = 0 \ ,
$$

(20)

respectively. The 0$j$ components identically vanish. Equation (3) for the tachyon becomes

$$
\ddot{T} + (d - 1) \frac{\dot{a}}{a} \dot{T} + \dot{\phi} \dot{T} + V'(T) = 0 
$$

(21)

Finally, Eq. (7) for the dilaton is

$$
\dot{\phi}^2 + \ddot{\phi} + (d - 1) \frac{\dot{a}}{a} \dot{\phi} - 2V(T) = \hat{\alpha} \ .
$$

(22)

In all these equations, a dot over a field represents the $t$ derivative and a prime represents the $T$ derivative.

We can integrate these equations directly. If $V'(T) = 0$, then $T = T_0$ is clearly a solution to Eq. (21). If we now put $h = \ln (\sqrt{\alpha'} \frac{\dot{a}}{a})$ and eliminate $\phi$ from Eqs. (19) and (20), we find

$$
\alpha' \ddot{h} = (d - 1) e^{2h} \ .
$$

(23)

The first integral of this equation is

$$
\dot{h}^2 = \frac{(d - 1)}{\alpha'} e^{2h} + k \ .
$$

(24)

Here, $k$ is an integration constant with dimension $1/\alpha'$. Solutions to this equation lie in three distinct classes depending on whether the integration constant $k$ is positive, negative or zero. For each case, we can then determine $\phi$ and examine Eq. (22) to obtain the value of $c$ to which the solution corresponds. Note that applying the Bianchi identities to (13) - (21) implies that (22) is an identity modulo the determination of $\hat{c}$. 

9
Suppose initially $k = 0$. Two solutions for $a(t)$ arise:

$$a(t) = a_0 t^{1/\lambda}, \quad \phi(t) = \phi_0 + (1 - \lambda) \ln(t/\sqrt{\alpha'})$$

(25)

and

$$a(t) = a_0 t^{-1/\lambda}, \quad \phi(t) = \phi_0 + (1 + \lambda) \ln(t/\sqrt{\alpha'})$$

(26)

Here, $\lambda = (d - 1)^{1/2}$, and $a_0$ and $\phi_0$ are arbitrary constants of integration.

Similarly, for $k > 0$ we find two solutions:

$$a(t) = a_0(tanh 1/2 \lambda t \sqrt{k})^{1/\lambda}$$

$$\phi(t) = \phi_0 + (1 + \lambda) \ln \cosh 1/2 \lambda \sqrt{k} t + (1 - \lambda) \ln \sinh 1/2 \lambda \sqrt{k} t$$

(27)

and

$$a(t) = a_0(tanh 1/2 \lambda t \sqrt{-k})^{-1/\lambda}$$

$$\phi(t) = \phi_0 + (1 - \lambda) \ln \cosh 1/2 \lambda \sqrt{-k} t + (1 + \lambda) \ln \sinh 1/2 \lambda \sqrt{-k} t$$

(28)

For small $t$, these solutions are the same as those with $k = 0$. As $t$ becomes large compared to $1/\lambda \sqrt{k}$, we see that $a(t) \to a_0$ and $\phi(t) \to \lambda \sqrt{k} t$. Thus, for large $t$ these solutions tend asymptotically to flat space with a dilaton field that is rising as a linear function of $t$.

For $k < 0$, again there are two solutions:

$$a(t) = a_0(tan 1/2 \lambda \sqrt{-k} t)^{1/\lambda}$$

$$\phi(t) = \phi_0 + (1 - \lambda) \ln \sin 1/2 \lambda \sqrt{-k} t + (1 + \lambda) \ln \cos 1/2 \lambda \sqrt{-k} t$$

(29)

and

$$a(t) = a_0(tan 1/2 \lambda \sqrt{-k} t)^{-1/\lambda}$$

$$\phi(t) = \phi_0 + (1 + \lambda) \ln \sin 1/2 \lambda \sqrt{-k} t + (1 - \lambda) \ln \cos 1/2 \lambda \sqrt{-k} t$$

(30)

Again, for small $t$ these have the same behavior as the $k = 0$ case. However, we see that $a(t)$ diverges as $t \to \pi / \lambda \sqrt{-k}$ in (29). Thus, in this solution the universe tends to infinite size in a finite amount of proper time, which presumably is unphysical. In
contrast, in (30) the universe starts out with infinite size and then collapses, with the big crunch at \( t = \pi/\lambda\sqrt{-k} \). This case appears dynamically unstable.

Substitution of all these solutions into (22) yields

\[
\dot{c} + 2V(T_0) = (d - 1)k.
\]  

(31)

For positive, zero, and negative values of \( k \), the solutions are analogs of the de Sitter, Minkowski, and anti-de Sitter solutions in field theory, respectively.

In all these solutions, the spacetime is singular whenever \( a(t) \rightarrow 0 \). The universe then has zero size, corresponding either to the big bang or to the big crunch. However, this singularity occurs in the string frame. While local physics can be translated between one frame and another in a predictable way, the presence or absence of singularities involving the global structure is not straightforward to translate. Since we can pass from one frame to another via conformal transformations of the form \( e^{\alpha \phi}, \alpha \in \mathbb{R} \), the spacetime may well be non-singular in one frame but singular in another. As an example, consider the solution (25). Here, it is best to transform to conformal time. From (17) we find \( \eta \sim t^{1-1/\lambda} \) so that \( a(\eta) \sim \eta^{1/\lambda} \). Since \( \phi(t) = \phi_0 + (1 - \lambda) \ln(t/\sqrt{\alpha'}) \), we find \( e^\phi \sim \eta^{-\lambda} \) and hence \( a(\eta) \sim e^{-\phi} \eta^{1/\lambda} \). Thus, if we conformally rescale the metric for (25) by a factor \( e^{2\phi/\lambda(\lambda-1)} \), the spacetime becomes flat and hence non-singular.

It is natural to ask what kinds of singularities are allowed in general. One possible criterion, suggested by analogy with field theory, is the integrability of the action density at \( t = 0 \). In the present simple example, \( e^\phi \sim t^{1-\lambda} \) and \( g^{1/2} \sim a(t)^{d-1} \sim t^{d-1/\lambda} \) so \( g^{1/2}e^\phi \sim t^{1-\lambda + \frac{d-1}{\lambda}} \). Substituting (22) into the action in the string frame gives

\[
I_S = \frac{1}{\kappa^2} \int d^d x \ g^{1/2} e^\phi \left( \Box \phi + (\nabla \phi)^2 \right).
\]  

(32)

Using the functional form of \( \phi \) and the metric then shows there is no singular contribution to \( I_S \).

This suggests that the singularities found in string metrics are harmless, unlike similar-seeming ones found in general relativity representing the boundary of spacetime. In our example, there is apparently no obstacle to passing through the singularity to some other region of spacetime. It is attractive to conjecture that this
picture holds generically, in which case the issue of singularities in general relativity is resolved by string theory.

4. Feigin-Fuks Treatment for \( k > 0 \)

The consistency of our calculations can be checked for the specific case where \( k > 0 \). For large times \( t \gg 1/\lambda \sqrt{k} \), spacetime is flat and the dilaton is growing linearly in time as \( \phi \sim \lambda \sqrt{k}t \) in both of the cases (27) and (28). It is possible to construct the first-quantized string exactly in such a background by using the Feigin-Fuks method \([14]\). Starting from the action (1) in this background and assuming that the tachyon condensate causes the tachyon to have a constant expectation value \( T_0 \), we discover that the energy-momentum tensor \( \Theta_{ab} \) of the string is given by

\[
\Theta_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu \nu} - \frac{1}{2} \gamma_{ab} \partial_c X^\mu \partial_d X^\nu \eta_{\mu \nu} \gamma^{cd} - \frac{1}{2} \alpha' T_0 \gamma_{ab} + \frac{1}{2} \alpha' \lambda \sqrt{k} (\nabla_a \nabla_b X^\mu - \gamma_{ab} \Box X^\mu) \delta^\mu_0 ,
\]

where \( \eta_{\mu \nu} \) is the target-space metric with signature \((-+++\ldots)\). Note that this tensor is not trace free: the contributions of \( T \) and \( \phi \) to the action are not conformally invariant. The present calculation is semiclassical. Only at the quantum level (discussed below) is full conformal invariance restored.

The equations of motion of the string are

\[
2 \gamma^{ab} \nabla_a \nabla_b X^\mu = -\frac{1}{2} \alpha' \lambda \sqrt{k} R^{(2)} X^0 \delta^\mu_0
\]

where \( R^{(2)} \) is the Ricci scalar of the world-sheet metric. From this it follows that the quantization of the \( X^\mu \) other than \( X^0 \) is unaffected by the time-dependent dilaton. Note that the tachyon condensate \( T_0 \) behaves like a world-sheet cosmological constant.

Let us next consider a flat-cylindrical world sheet in the orthonormal gauge, chosen so that the time coordinate \( \tau \) is free and the space coordinate \( \sigma \) varies from 0 to \( \pi \). Solving the string equations of motion yields the usual expression

\[
X^\mu = x^\mu + 2 \alpha' p^\mu \tau + i \sqrt{\frac{1}{2} \alpha'} \sum_{n \neq 0} \left[ \frac{\alpha_n^\mu}{n} exp[-2in(\tau - \sigma)] + \frac{\bar{\alpha}_n^\mu}{n} exp[-2in(\tau + \sigma)] \right].
\]

Here, \( x^\mu \) is the string center-of-mass coordinate, \( p^\mu \) is the momentum, and \( \alpha_n \) and \( \bar{\alpha}_n \) are the usual creation and annihilation operators for the right- and left-moving
excitations, respectively. As usual, quantization imposes the commutation rules

\[ [x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_\mu, \alpha_\nu] = [\bar{\alpha}_\mu, \bar{\alpha}_\nu] = m\delta_{m+n,0}\eta^{\mu\nu}. \] (36)

The ghost sector can be treated similarly. In what follows, we concentrate on the differences between our situation and the usual zero-condensate case.

The (dimensionless) Hamiltonian is proportional to the integral of \( \Theta_{00} \) over a spatial section of the string and is classically given by

\[ H = \sum_{n=1}^{\infty} (\alpha_n \alpha_{-n} + \bar{\alpha}_n \bar{\alpha}_{-n}) + \alpha' p^2 + \frac{T_0}{4}. \] (37)

The Virasoro operators are the Fourier components of the energy-momentum tensor, which are modified from their usual form by the dilaton and tachyon condensates. For the right movers, they are

\[ L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{m-n} - i\lambda \sqrt{\frac{\alpha' k}{8}} m\alpha_m^0 + \frac{T_0}{4} \delta_{m,0} + m\delta_{m+n,0} \] (38)

where we have defined \( \alpha_0^0 = \sqrt{\frac{1}{2}} \alpha' p_0^0 \).

At the quantum level, however, we must deal with the normal-ordering problem. We therefore replace the \( L_m \) by their quantum analogs

\[ L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_n \alpha_{m-n} : - i\lambda \sqrt{\frac{\alpha' k}{8}} m\alpha_m^0 + \frac{T_0}{4} \delta_{m,0} - A \delta_{m,0} \] (39)

where the colons indicate normal ordering. The constant \( A \) corresponds to the normal-ordering shift in the vacuum energy, which can be evaluated by zeta-function regularization. Its value is then \( A = d/24 \). An expression similar to (39) holds for the left movers.

It can be checked that the Virasoro algebra is still satisfied, but the central term becomes modified. Explicitly,

\[ [L_m, L_n] = (m - n)L_{m+n} + \left( \frac{d}{12} - \frac{\alpha' k \lambda^2}{8} \right) m^3 \delta_{m+n,0} - \frac{d}{12} m\delta_{m+n,0}. \] (40)

The coefficient of \( m^3 \delta_{m+n,0} \) is the coefficient of \( R^{(2)} \) in the trace anomaly, that is, the dilaton beta function, while the coefficient of \( m\delta_{m+n,0} \) is the coefficient arising from
the Casimir energy. The ghost contributions can now be incorporated in the usual way. The condition for vanishing cubic term in the central charge is therefore

$$\frac{d - 26}{12\alpha'} = \frac{k\lambda^2}{8}.$$  \hspace{1cm} (41)

This expression should be the same as the condition for the vanishing of the dilaton beta function. Note that Eq. (41) is consistent with Eq. (12) if $\mu^2$ is identified with $k\lambda^2$.

However, there are quantum corrections to this formula from contributions of order $(\alpha')^0$. These have been calculated in the dilaton beta function. In effect, the Feigin-Fuks construction has evaluated the contribution to the dilaton beta function by a different route. The advantage of this method is that it also allows us to make statements about the string spectrum. We thus deduce from Eq. (7) that for consistency to lowest order

$$\frac{d - 26}{12\alpha'} - \frac{k\lambda^2}{8} = \frac{V(T_0)}{4}.$$  \hspace{1cm} (42)

Note that this equation is the same as Eq. (31). This means that we have found a consistent solution if $V(T)$ has an extremum with $V(T_0) > 0$, provided $k > 0$.

At this stage, we can address the question of whether there is a tachyon in the spectrum of the string. The world-sheet hamiltonian must vanish. The Casimir energy of the world-sheet particles can be found as follows. The contribution from the $d$ free bosons is $-d/24$, while the world-sheet reparametrization ghosts provide a factor 1/12. Thus, from (37) we find

$$\alpha' p^2 + \frac{T_0}{4} = \frac{d - 2}{6}.$$  \hspace{1cm} (43)

The mass of the ground state is therefore given by

$$m^2 = \frac{4 - 2d + 3T_0}{12\alpha'},$$  \hspace{1cm} (44)

and $m^2 \geq 0$ if $T_0 \geq 2(2 - d)/3$.

5. String Field Theory and the Tachyon Potential

The discussion in the preceding sections has been in the context of first quantization. Another way of obtaining information about background structure is to use
string field theory. Given a particle field $f$ in a string field theory, we can find possible condensates of $f$ by constructing an effective potential for $f$ and looking for extrema. The physics is then determined by small oscillations of the particle fields about a particular extremum. For example, in this way we can find, at least in principle, the spectrum of the string theory in a non-trivial background. This procedure is analogous to that in ordinary field theories. For instance, the electroweak model in the naive first-quantized vacuum has massless vectors and a tachyon $h$, called the Higgs field. Interactions in the field theory generate a potential for $h$, with minima away from the origin in which a Higgs condensate is favored. If we look at small oscillations in the background Higgs field, we see that the physics is changed: there is no tachyon, and massive vector bosons appear.

The field-theoretic approach has an advantage over first-quantized methods in that issues about the meaning or existence of vacuum structure are avoided. The point is that string field theory purports to be a consistent off-shell formulation, and as such provides a definite off-shell framework within which to examine the question of a non-trivial background. In practice, however, the presence in string field theory of an infinite number of particle fields and, especially for the closed string, the complexities of the interactions makes an exact treatment along these lines difficult. Instead, approximation methods are needed.

One approximation scheme is presented in refs. [19, 23], which examined the issue of the formation of condensates in string field theories in their critical dimension using the level-truncation method. In particular, ref. [23] obtained an approximation to a stable ground state of the closed bosonic string via the following path. In the particle-field expansion of the closed-string field about a flat background in 26 dimensions, apply the truncation at the first-massive level, thereby keeping the tachyon, dilaton, graviton, and Kalb-Ramond fields but disregarding all higher-mass fields. For simplicity, restrict attention to the cubic term in the nonpolynomial action [20, 21, 22] and choose the Siegel-Feynman gauge [26], which eliminates certain auxiliary fields for convenience. At this stage, one is left with about 50 interaction terms in the lagrangian density. Presumably, an ansatz of the form (16) would lead to solutions including those we presented in section 3. However, most of these interactions are
derivative couplings, which can be disregarded if an approximation with only stable ground states is sought. The few remaining terms form the static potential. The usual 26-dimensional vacuum is an unstable extremum of this potential, in which the background string field is zero. At this level of approximation another extremum exists, in which the tachyon acquires a constant nonzero expectation. Condensates of the dilaton, graviton, and Kalb-Ramond fields do not appear.

The tachyon potential \( \hat{V}(\hat{T}) \) appearing in the lagrangian of the string field theory has mass dimension 26 and takes the form

\[
\hat{V}(\hat{T}) = -\frac{2}{\alpha'} \hat{T}^2 + \frac{\bar{g}}{3!} \hat{T}^3 + \ldots .
\]  

(45)

Here, \( \hat{T} \) represents the tachyon field (with mass dimension 12) appearing in the expansion of the closed-string field, and \( \bar{g} \) is the tree-level three-tachyon coupling defined at zero momentum. The latter is related to the on-shell tree-level three-tachyon coupling \( g \) by \( \bar{g} = \frac{3}{9} g / 2^{12} \). Extremizing the first two terms in \( \hat{V} \) shows that, in addition to the local maximum at \( \hat{T} = 0 \), there is a minimum at \( \hat{T} = \hat{T}_0 = 8/\bar{g} \alpha' \) of depth \( \hat{V}(\hat{T}_0) = -\frac{2}{3} \bar{g}^2 \alpha'^3 \) and curvature \( \hat{V}''(\hat{T}_0) = 4/\alpha' \). For definiteness in what follows, we neglect the quartic and higher terms in this expression. Note, however, that in principle terms involving arbitrary powers of \( \hat{T} \) are of importance in determining the exact tree-level potential [27].

In previous sections, we have worked with a dimensionless tachyon \( T \) and the tachyon potential \( V(T) \) with mass dimension two. The connection between these and \( \hat{T}, \hat{V} \) is given by

\[
T = \kappa \bar{g} \alpha' \hat{T} , \quad V(T) = \kappa^2 \bar{g}^2 \alpha'^2 \hat{V}(\hat{T}) , \quad \kappa = \frac{1}{4} ,
\]  

(46)

so that

\[
V = -\frac{2}{\alpha'} T^2 + \frac{1}{3! \kappa \alpha'} T^3 + \ldots ,
\]  

(47)

with a minimum at \( T = T_0 = 8 \kappa \) of depth \( V(T_0) = -2^7 \kappa^2 / 3 \alpha' \) and curvature \( V''(T_0) = 4/\alpha' \).

We can use these approximate expressions to obtain explicit results from the formulae found in sections 3 and 4. From Eq. (31), we find

\[
k = \frac{2(d - 26 - 128 \kappa^2)}{3(d - 1) \alpha'} .
\]  

(48)
With $\kappa = \frac{1}{4}$, this would suggest the presence of a nonzero tachyon condensate causes the critical dimension to shift from 26 to 34. It is possible that modular invariance could be recovered for this case since 34 is a special case of $8n + 2$ [28]. Note also that for $d = 26$ we have $k = -256\kappa^2/75\alpha' < 0$.

From the Feigin-Fuks analysis of section 4, we find the lowest-lying state in the theory using Eq. (44). Using the minimum at $T_0 = 8$ gives

$$m^2 = \frac{2 + 12\kappa - d}{6\alpha'}.$$

(49)

With $\kappa = \frac{1}{4}$, this suggests the existence of a string theory without massless states, and hence infrared finite, for $d \leq 4$.

6. Discussion

In this paper, we demonstrate that the closed bosonic string in its critical dimension has solutions that describe critical strings in arbitrary dimensions. The change in dimensions is accomplished by balancing the tachyon condensate against the central charge. We derive solutions that satisfy the leading-order terms in the beta-function equations in an expansion in powers of $\alpha'$. As usual, the higher-order corrections to these equations are expected to induce higher-order corrections in the solutions, which are calculable at least in principle.

The solutions we find are presented in Eqs. (25) to (30). They fall into three classes characterized by the sign of a parameter $k$. This situation is reminiscent of the de Sitter, Minkowski, and anti-de Sitter solutions that appear in general relativity when a Higgs condensate forms. The parameter $k$ controls the balance between the central charge and the energy density in the tachyon condensate, cf. Eq. (31), and so the value of $k$ determines the dimensionality of the string theory involved. For $k > 0$, the dilaton is asymptotically rising, which means the Feigin-Fuks method can be applied to determine the spectrum of the theory. We use this to find the value of the tachyon potential at the tachyon expectation value in terms of $k$ (Eq. (42)) and a condition that determines whether the ground state in the background configuration of fields has positive mass (Eq. (44)). The operator and beta-function methods give results in agreement where they can both be used.
An approximate solution can be found to the string field theory of the closed bosonic string that describes a stable background with constant tachyon expectation value. In section 5, we use this approximation to illustrate our results, thereby generating explicit expressions for the parameter $k$ and the spectrum in terms of the dimension $d$.

On the basis of the freedom to apply conformal transformations to the string action, we conjecture that the issue of spacetime singularities in general relativity may be solved by their stringy extensions. An encounter with a spacetime singularity can be avoided in a string theory by an appropriate choice of conformal gauge, which does not change the local physics but which removes the singularity. We test this conjecture in the context of our cosmological solutions in $d$ dimensions, for which there are spacetime singularities associated with the big bang/crunch. It is indeed possible to avoid the usual problems by a choice of conformal frame.

We expect generalizations of our solutions to exist. For example, a solution is likely to exist describing a background with a non-zero condensate of the Kalb-Ramond field. Similarly, extensions of the equations to superstrings and heterotic strings probably also permit consistent solutions of the type discussed here. In this context, it would be particularly interesting to examine the beta functions for the heterotic strings with tachyons, in which tachyon condensates might be expected to form.

6. Acknowledgments

V.A.K. thanks Trinity College, Cambridge, the Theory Division at CERN, and the Aspen Center for Physics for hospitality while part of this work was done. This work was supported in part by the North Atlantic Treaty Organization under grant number CRG 910192 and by the United States Department of Energy under contracts DE-AC02-84ER40125 and DE-FG02-91ER40661.

7. References

1. C.G. Callan, D. Friedan, E.J. Martinec, and M.J. Perry, Nucl. Phys. B 262 (1985) 593.
2. E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B 261 (1985) 11.
3. C. Lovelace, Nucl. Phys. B 273 (1986) 413.
4. C. Callan, I. Klebanov, and M. Perry, Nucl. Phys. B 278 (1986) 78.
5. R. Brustein, D. Nemeschansky, and S. Yankielowicz, Nucl. Phys. B 301 (1988) 224.
6. C.G. Callan and Z. Gan, Nucl. Phys. B 272 (1986) 647.
7. S.R. Das and B. Sathiapalan, Phys. Rev. Lett. 56 (1986) 2664.
8. A.A. Tseytlin, Phys. Lett. B 264 (1991) 311.
9. R.C. Myers, Phys. Lett. B 199 (1987) 371.
10. I. Antoniadis, C. Bachas, J. Ellis and D. Nanopoulos, Phys. Lett. B 211 (1988) 393; Nucl. Phys. B328 (1989) 117; Phys. Lett. B 257 (1991) 278.
11. S.P de Alwis, J. Polchinski, and R. Schimmrigk, Phys. Lett. B 218 (1989) 449.
12. J. Polchinski, Nucl. Phys. B 346 (1990) 253.
13. M. Mueller, Nucl. Phys. B337 (1990) 37.
14. B.L. Feigin and D.B. Fuks, Funct. Anal. Appl. 16 (1982) 114.
15. J. Wess and B. Zumino, Phys. Lett. B 37 (1971) 95; E. Witten, Commun. Math. Phys. 92 (1984) 455.
16. I. Bars and K. Sfetsos, preprint UCS-92/HEP-B1, hep-th/9205037. I. Bars, Phys. Lett. B 293 (1992) 315.
17. C. Nappi and E. Witten, Phys. Lett. B 293 (1992) 309.
18. P. Ginsparg and F. Quevedo, Nucl. Phys. B385 (1992) 527.
19. V.A. Kostelecký and S. Samuel, Phys. Rev. Lett. 63 (1990) 224; Nucl. Phys. B336 (1990) 263.
20. T. Kugo and K. Suehiro, Nucl. Phys. B337 (1990) 434; T. Kugo, H. Kunimoto, and K. Suehiro, Phys. Lett. B 226 (1989) 48.
21. M. Saadi and B. Zweibach, Ann. Phys. 192 (1989) 213; B. Zweibach, preprint IASSNS-HEP-92/41, hep-th/9206084.
22. M. Kaku, Phys. Rev. D 41 (1990) 3734.
23. V.A. Kostelecký and S. Samuel, Phys. Rev. D 42 (1990) 1289.
24. C.B. Thorn, Nucl. Phys. B 248 (1984) 551.
25. D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. 54 (1985) 620.
26. W. Siegel, Phys. Lett. B 142 (1984) 276; 151 (1984) 391, 396.
27. V.A. Kostelecký and S. Samuel, Phys. Rev. D 39 (1989) 683.
28. J. Thierry-Mieg, Phys. Lett. B 171 (1986) 163.