AFFINE PAVINGS AND THE ENHANCED NILPOTENT CONE

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Abstract. We construct affine pavings of Springer-type fibers over the enhanced nilpotent cone. This resolves a question of Achar-Henderson and implies the existence of perverse parity sheaves on the enhanced nilpotent cone.

1. Notation and Results

Let \( F \) be an algebraically closed field of arbitrary characteristic and let \( V \) be an \( n \)-dimensional \( F \)-vector space. Let \( G = \text{GL}(V) \) and \( \frak{g} = \text{Lie}(G) \) be its Lie algebra with nilpotent cone \( \mathcal{N} \subset \frak{g} \). The \( G \)-variety \( V \times \mathcal{N} \) is known as the enhanced nilpotent cone. As shown independently by \[\text{[AH08]}\] and \[\text{[Tra09]}\], the \( G \)-orbits in \( V \times \mathcal{N} \) are in bijection with the set \( Q_n \) of bipartitions of \( n \) (meaning ordered pairs of partitions \((\mu; \nu)\) such that \(|\mu| + |\nu| = n\)). The closure of each orbit \( O_{\mu; \nu} \) has a semismall resolution of singularities \( \pi_{\mu; \nu} : \tilde{F}_{\mu; \nu} \to V \times \mathcal{N} \) (whose construction we recall below). The aim of this paper is to construct affine pavings of the fibers of these resolutions. This claim appeared in \[\text{[AH08]}\], but was then retracted in \[\text{[AH11]}\], where it is posed as an open problem. Our construction is a variant of the method introduced in \[\text{[DCLPS88]}\] to construct affine pavings of Springer fibers for classical groups.

To describe the resolutions \( \pi_{\mu; \nu} \), recall that \[\text{[AH08]}\] associate a ‘back-to-back union’ diagram to \((\mu; \nu)\), whose \( i \)-th row contains \( \mu_i + \nu_i \) boxes and \((\mu_1 - i)\)-th column has \( \mu_{i+1} \) boxes for \( i \geq 0 \) and \((\mu_1 + i)\)-th column has \( \nu_i \) boxes for \( i > 0 \). For example, the diagram associated to \(((3,1,1); (3,2)) \in Q_{10}\) is represented as:

Let \( F_{\mu; \nu} \) be the variety of partial flags

\[
0 = W_0 \subset W_1 \subset \cdots \subset W_{\mu_1 + \nu_1} = V,
\]

where \( W_i \) has dimension equal to the number of boxes in or to the left of the \( i \)-th column in the diagram of \((\mu; \nu)\). So in the example above, \( \mu_1 + \nu_1 = 6 \) and the dimensions of the subspaces are: 1, 2, 5, 7, 9 and 10.

We will consider, more generally, for any sequence \( \rho \), \( 0 = r_0 < r_1 < \cdots < r_m = n \), the variety \( F_{\rho} \) of partial flags

\[
0 = W_0 \subset W_1 \subset \cdots \subset W_m = V,
\]

where the dimension of \( W_i \) is \( r_i \).

Recall the resolution of \( O_{\mu; \nu} \) defined in \[\text{[AH08]}\] via the space

\[
\tilde{F}_{\mu; \nu} := \{(v, x, (W_i)) \in V \times \mathcal{N} \times F_{\mu; \nu}, \forall v \in W_{\mu_1}, x(W_i) \subset W_{i-1}\},
\]
and the projection $\pi_{\mu,\nu} : \widetilde{F}_{\mu,\nu} \to V \times \mathcal{N}$ to the first two coordinates. By [AH08, Thm. 4.5], $\pi_{\mu,\nu}$ is a semismall resolution of $O_{\mu,\nu}$.

More generally, for any $j \in \mathbb{Z}$ such that $0 \leq j \leq m$, let $\widetilde{F}_{\rho,j}$ be defined as

$$\widetilde{F}_{\rho,j} := \{(v, x, (W_i)) \in V \times \mathcal{N} \times F_{\rho} | v \in W_j, x(W_i) \subset W_i - 1\},$$

and let $\pi_{\rho,j} : \widetilde{F}_{\rho,j} \to V \times \mathcal{N}$ denote the projection.

Our main result is the following:

**Theorem 1.1.** For any $(v, x) \in V \times \mathcal{N}$, the fiber $\pi_{\rho,j}^{-1}(v, x)$ has an affine paving. In particular, the fiber $\pi_{\mu,\nu}^{-1}(v, x)$ admits an affine paving.

As a simple corollary, we observe that this implies the existence of perverse parity sheaves on the enhanced nilpotent cone. For simplicity, we assume for the rest of the introduction that $\mathbb{F} = \mathbb{C}$ the field of complex numbers. Let $k$ be a complete local principal ideal domain. Let $D_G(V \times \mathcal{N}; k)$ denote the $G$-equivariant constructible derived category of $k$-sheaves.

**Corollary 1.2.** For each $G$-orbit $O_{\mu,\nu}$, there exists up to isomorphism one parity sheaf $E_{\mu,\nu} \in D_G(V \times \mathcal{N}; k)$ with support $O_{\mu,\nu}$, and it is perverse.

**Proof.** First note that there are finitely many $G$-orbits in $V \times \mathcal{N}$ and for any $(v, x) \in V \times \mathcal{N}$ the stabilizer is connected [AH08, Prop. 2.8(7)] and has reductive quotient isomorphic to a product of general linear groups [Sun11, Thm. 2.12]. It follows that the orbits are equivariantly simply connected and have equivariant cohomology concentrated in even degrees. Thus, as a $G$-variety, the enhanced nilpotent cone satisfies the parity conditions of [IMW14], which implies the uniqueness statement.

For the existence of $E_{\mu,\nu}$, note that the resolution $\pi_{\mu,\nu}$ is semismall, so the push-forward sheaf $(\pi_{\mu,\nu})_*\mathbb{L}_{\widetilde{F}_{\mu,\nu}}[\dim O_{\mu,\nu}]$ is perverse and Theorem 1.1 implies that it is also a parity complex. It follows that the push-forward sheaf, which has support $O_{\mu,\nu}$, has a perverse indecomposable parity complex $E_{\mu,\nu}$ with support $O_{\mu,\nu}$ as a direct summand. □

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2. **Construction of Affine Paving**

2.1. As defined and shown in [AH08], we may pick a normal basis for $(v, x)$ in this basis, each basis vector of $V$ corresponds to a box of the back-to-back union diagram for $(\alpha; \beta)$. We denote by $v_{i,j}$ the basis vector corresponding to the $j$-th box in the $i$-th row. In this basis the action of $x$ is given by $x v_{i,j} = v_{i,j-1}$ (or $0$ if $j = 1$), and the vector $v$ expressed as $v = \sum_{i=1}^{\alpha_i^f} v_{i,\alpha_i}$. For example, for $((3, 1, 1); (3, 2)) \in \mathcal{Q}_{10}$ we have basis vectors:

$$v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{21}, v_{22}, v_{23}, v_{31}$$

1We consider here only the constant pariversity.
and $v = v_{13} + v_{21} + v_{31}$.

We grade $V$ by giving the basis vector $v_{i,j}$ grading $\alpha_i - j$. Let $V(i)$ denote the $i$-th graded part. This induces a grading on $\mathfrak{g} = \text{Hom}(V, V)$. Let $\mathfrak{g}(i)$ denote the $i$-th graded part of $\mathfrak{g}$ (i.e., $\oplus_j \text{Hom}(V(j), V(j+i))$). Let $V_{\geq 0} = \oplus_{i \geq 0} V(i)$ be the non-negatively graded part of $V$. (In the notation of [AH08], $V_{\geq 0} = E^x v$. See Proposition 2.8(5) of loc. cit.)

Note that $v \in V(0)$ and $x \in \mathfrak{g}(1)$.

Consider the parabolic subalgebra $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}(i)$, its Levi subalgebra $\mathfrak{g}(0) = \oplus_i \text{End}(V(i))$ and unipotent radical $\mathfrak{u}_P = \oplus_{i > 0} \mathfrak{g}(i)$. Let $G_0 = \prod_i \text{GL}(V(i))$ and $P$ be the corresponding Levi and parabolic subgroups of $G$. (In [AH08], following Thm. 4.1, $P$ is denoted $P^{x,v}$.)

Let $\lambda : \mathbb{G}_m \to G$ denote a cocharacter inducing this Levi decomposition.

**Lemma 2.1.** The $P$-orbit of $(v, x)$ in $V^+ \times \mathfrak{u}_P$ is dense.

**Proof.** This is Lemma 4.2 of [AH08]. In loc. cit., $\mathbb{F}$ is assumed to be the field of complex numbers, but the same proof applies more generally. \hfill \square

Now consider the fiber $\pi_{\rho,j}^{-1}(v, x) \subset \mathcal{F}$. Recall that $\mathcal{F}$ can be identified with a conjugacy class of parabolic subalgebras of $\mathfrak{g}$, by associating to a partial flag $\{W_i\}$ its stabilizer subalgebra in $\mathfrak{g}$.

**Proposition 2.2.** The intersection of $\pi_{\rho,j}^{-1}(v, x)$ with any $P$-orbit on $\mathcal{F}$ is smooth.

This statement is a minor generalization of Lemma 4.3 in [AH08] (where only the fibers of $\pi_{\rho,x}$ are considered). As in loc. cit., we follow the strategy of [DCLP88] Prop 3.2.

**Proof.** Let $\{W_i\} \in \pi_{\rho,j}^{-1}(v, x)$ be a partial flag corresponding to a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$. Let $\mathcal{O}$ be the $P$-orbit in $\mathcal{F}$ of $\{W_i\}$ (or equivalently $\mathfrak{q}$). Let $Q$ be the parabolic subgroup of $G$ with Lie algebra $\mathfrak{q}$. Then the stabilizer of $\mathfrak{q}$ in $P$ is the intersection $H = P \cap Q$ and $\mathcal{O} = P \cdot \mathfrak{q} \cong P/H$. For $p \in P$, $pq$ is in the fiber $\pi_{\rho,j}^{-1}(v, x)$ if and only if $(p^{-1}, \text{Ad}(p^{-1})x) \in W_j \times \mathfrak{u}_Q$.

Thus the intersection $\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O}$ is a subvariety of $P/H$ of the type in [DCLP88] Sect. 2.1 relative to the prehomogeneous space $P \cdot (v, x) = V^+ \times \mathfrak{u}_P$ for $P$ and the $H$-stable subspace $U = (W_j \times \mathfrak{u}_Q) \cap (V^+ \times \mathfrak{u}_P)$. We conclude that it is smooth. \hfill \square

Recall that a finite partition of a variety $X$ into subsets is called an $\alpha$-partition if the subsets can be ordered $X_1, X_2, \ldots, X_t$ such that $X_1 \cup X_2 \cup \ldots \cup X_k$ is closed in $X$ for all $k = 1, \ldots, t$. As the Bialynicki-Birula decomposition of $\mathcal{F}$, with respect to $\lambda$ is an $\alpha$-partition, it follows that the intersections $\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O}$, as $\mathcal{O}$ runs over the $P$-orbits in $F$, form an $\alpha$-partition of $\pi_{\rho,j}^{-1}(v, x)$.

2.2. We will now observe that it suffices to construct an affine paving of the fixed point sets $(\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O})^\lambda$. First note that we may regard $\mathcal{O}$ as a vector bundle over $\mathcal{O}^\lambda$ where $\lambda$ acts linearly on the fibers with strictly positive weights and $\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O} \subset \mathcal{O}$ is a $\mathbb{G}_m$-stable smooth closed subvariety.

Suppose, more generally, that $\rho : E \to Y$ is a vector bundle over a smooth variety $Y$, with a fiber preserving $\mathbb{G}_m$-action on $E$ with strictly positive weights and that $Z \subset E$ is a $\mathbb{G}_m$-stable smooth closed subvariety.
As noted in [DCLP88, 1.5], if $\mathbb{F} = \mathbb{C}$, one can conclude that $\pi(Z) = Z^{G_m}$ is smooth and $Z$ is a subbundle of $E$ restricted to $Z^{G_m}$. Thus the preimage of an affine paving of $Z^{G_m}$ is an affine paving of $Z$.

For arbitrary characteristic, it is not clear that $Z$ must be a subbundle of $E$ over $Z^{G_m}$. Nonetheless, the following result can be gleaned from [Jan04, Sect. 11]:

**Theorem 2.3.** Let $\rho : E \to Y$ and $Z \subset E$ be as above. Then:

1. the fixed point variety $Z^{G_m}$ is smooth, and
2. if $Z^{G_m}$ admits an affine paving, then so does $Z$.

Part (1) follows from a general result [Ive72, Prop. 1.3] which states that the fixed point set of a linearly reductive group acting on a smooth variety is smooth. Part (2) is a slight generalization of [Jan04, Lem. 11.16(b)], which refers to the special case when $E$ is a parabolic orbit on the full flag variety, but the proof only uses the conditions above.

We conclude that $(\pi_{\rho,j}^{-1}(v, x) \cap O)^\lambda$ is a smooth variety and also projective (because it is the intersection of the projective varieties $\pi_{\rho,j}^{-1}(v, x)$ and $O^\lambda$) and that if $(\pi_{\rho,j}^{-1}(v, x) \cap O)^\lambda$ admits an affine paving, then so does $\pi_{\rho,j}^{-1}(v, x)$.

**2.3.** By the previous paragraph, it suffices to construct an affine paving of the $\lambda$-fixed point set $(\pi_{\rho,j}^{-1}(v, x) \cap O)^\lambda$. We proceed by induction on the dimension of $V$. Assume the statement is true for any vector space of dimension less than $n$.

Suppose that there is a nontrivial direct sum decomposition $V = V_1 \oplus V_2$ such that

1. $V_1$ and $V_2$ are preserved by the action of $x$,
2. $V_1$ and $V_2$ are preserved by the action of the cocharacter $\lambda$, and
3. $v \in V_1 \subset V_1 \oplus V_2$.

Let $x_1 = x|_{V_1}$ and $x_2 = x|_{V_2}$.

Let $\chi : G_m \to G$ be the cocharacter that acts on $V_1$ by scaling and on $V_2$ by the inverse. Let $L = GL(V_1) \times GL(V_2)$ be the corresponding Levi subgroup and $P$ the corresponding parabolic.

Let $F^\chi_\rho$ be the $\chi$-fixed point set of $F$. Each component of $F^\chi_\rho$ is contained in a unique $P$-orbit $O$ on $F_\rho$ and is in fact equal to $O^\chi$. Fix $q \in O^\chi$ and let $Q \subset G$ be the corresponding parabolic subgroup and $\{W_i\}_{i=1}^m$ the corresponding partial flag. Then there is an isomorphism $O^\chi \cong L/L \cap Q \cong F_{\rho'} \times F_{\rho''}$. Here $F_{\rho'}$ and $F_{\rho''}$ are partial flag varieties for $GL(V_1)$ and $GL(V_2)$ respectively and $\rho'$ and $\rho''$ are sequences $0 = r'_0 < r'_1 < \cdots < r'_m' = \dim V_1$, $0 = r''_0 < r''_1 < \cdots < r''_m'' = \dim V_1$.

The isomorphism $O^\chi \to F_{\rho'} \times F_{\rho''}$ restricts to an isomorphism

$$\pi_{\rho,j}^{-1}(v, x)^\chi \cap O \to \pi_{\rho',j'}^{-1}(v, x_1) \times \pi_{\rho'',0}^{-1}(0, x_2),$$

where $j'$ is defined as the number between 1 and $m'$ such that $r'_j = \dim(W_j \cap V_1)$.

This isomorphism is compatible with the action of $\lambda$, so taking $\lambda$-fixed points we obtain an isomorphism:

$$\pi_{\rho,j}^{-1}(v, x)^{\chi,\lambda} \cap O \to \pi_{\rho',j'}^{-1}(v, x_1)^\lambda \times \pi_{\rho'',0}^{-1}(0, x_2)^{\lambda}. $$

But $\pi_{\rho,j}^{-1}(v, x)^{\chi,\lambda}$ is also the $\chi$-fixed points of $\pi_{\rho,j}^{-1}(v, x)^\lambda$. We have seen that the latter is smooth and projective, thus the Białynicki-Birula decomposition of

\[ Meanin a reductive group whose category of finite-dimensional representations is semisimple (e.g., a torus). ]
Lemma 2.4. Assume $(v_i, x) \in \mathcal{O}_{(\alpha, \beta)} \in V \times \mathcal{N}$ distinguished if for any direct sum $V = V_1 \oplus V_2$ satisfying conditions (1)-(3) of section 2.3, either $V_1$ or $V_2$ is trivial.

By the previous paragraph, we are reduced to studying $\pi^{-1}_{\rho, j}(v, x)^\lambda$ for distinguished pairs $(v, x)$.

We first classify distinguished pairs.

**Lemma 2.4.** If $(v, x) \in \mathcal{O}_{(\alpha, \beta)}$ is distinguished then either (1) $\alpha = \emptyset$ (i.e., $v = 0$) and $\beta = (n)$ (i.e., $x$ is a regular nilpotent) or (2) $\alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k)$ and $\alpha_1 > \alpha_2 > \ldots > \alpha_k > 0$, $\beta_1 > \beta_2 > \ldots > \beta_k$.

**Proof.** Assume $(v, x) \in \mathcal{O}_{(\alpha, \beta)}$ is distinguished. For a partition $\mu$, let $\ell(\mu)$ denote the number of nonzero terms.

Suppose that $\ell(\beta) > \ell(\alpha)$, so $\beta_{\ell(\alpha)+1} > 0$. Let $V_2 \subset V$ be the subspace spanned by the basis vectors $v_{\ell(\alpha)+1,j}$ for all $j$ and $V_1 \subset V$ be the subspace spanned by the complementary set of basis vectors. It is clear that this is a direct sum decomposition and satisfies conditions (1)-(3) of 2.3. As $(v, x)$ is distinguished and $V_2$ is non-trivial by definition, we conclude that $V_1$ is trivial and so $\alpha = \emptyset$ and $\ell(\beta) = 1$.

On the other hand, suppose that $\ell(\beta) \leq \ell(\alpha)$ and let $k = \ell(\alpha)$.

If $\alpha_l = \alpha_{l+1}$ for some $l < k$, we let $V_2 \subset V$ be the subspace spanned by the basis vectors $v_{l,j}$ for all $j$. Let $V_1$ be the span of the basis vectors $v_{i,j}$ for all $i \neq l, l+1$ and the vectors $v_{l,j} + v_{l+1,j}$ for all $j$ such that $1 \leq j \leq \alpha_{l+1} + \beta_{l+1}$. Note that $V = V_1 \oplus V_2$, $V_1$ and $V_2$ are both nontrivial and the conditions (1)-(3) of 2.3 are satisfied. This contradicts the assumption that $(v, x)$ be distinguished.

Similarly, suppose that $\beta_l = \beta_{l+1}$ for some $l < k$. Let $V_1 \subset V$ be the span of the basis vectors $v_{l,j}$ for all $i \neq l, l+1$ and the vectors $x^m(v_{l+1,j} + v_{l,j+1, \alpha_{l+1} + \beta_l})$ for all $m$. Let $V_2 \subset V$ be the span of the basis vectors $v_{l+1,j}$ for all $j$. Again we have $V = V_1 \oplus V_2$. $V_1$ and $V_2$ are both nontrivial, and the conditions (1)-(3) of 2.3 are satisfied. This contradicts the assumption that $(v, x)$ be distinguished. \qed

We can now check that we have an affine paving in both cases.

Case (1): In this case $(\pi^{-1}_{\rho, j}(v, x) \cap \mathcal{O})^\lambda$ is either empty or a single point.

Case (2): As no two parts of $\alpha$ are equal, the kernel of $x$ breaks up under the action of $\lambda$ into a direct sum of 1-dimensional weight spaces with distinct weights.

For any partial flag $\{V_i\}_{i=0}^m \in \pi^{-1}_{\rho, j}(v, x)^\lambda$, $V_1$ must be contained in the kernel of $x$ and also be a direct sum of $\lambda$-weight spaces. Let $A$ denote the finite set of such $r_1$-dimensional subspaces of the kernel of $x$. Consider the forgetful map from $\pi^{-1}_{\rho, j}(v, x)^\lambda$ to $A$. The fiber of this map over a point $W \in A$ is simply $\pi^{-1}_{\rho, j}(v, x)^\lambda$, where $\rho = (0 < r_2 - r_1 < r_3 - r_1 < \cdots < r_m - r_1 = n - r_1, j = \hat{j} - 1)$ (or 0 if $j = 0$), $\hat{v}$ is the image of $v$ in the quotient $V/W$ and $\hat{x}$ is the induced action on $V/W$. Having reduced to the case of a smaller dimensional vector space, we are done.
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