Structure of Ann-categories

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Abstract
Each Ann-category \( A \) is equivalent to an Ann-category of the type \( (R, M) \), where \( M \) is an \( R \)-bimodule. The family of constraints of \( A \) induces a structure on \( (R, M) \). The main result of the paper is:

1. There exists a bijection between the set of structures on \( (R, M) \) and the group of Mac Lane 3-cocycles \( Z^3_{\text{Mac}}(R, M) \).
2. There exists a bijection between \( C(R, M) \) of congruence classes of Ann-categories whose pre-stick is of the type \( (R, M) \) and the Mac Lane cohomology group \( H^3_{\text{Mac}}(R, M) \).

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1 Introduction and Preliminary

The definition of Ann-categories is a natural development of the definitions of monoidal categories, Gr-categories. The axiomatics of Ann-categories was presented in [7]. The first two invariants of an Ann-category is the ring \( R = \Pi_0(A) \) of isomorphic classes of objects of \( A \) and \( R \)-bimodule \( M = \Pi_1(A) = \text{Aut}_A(0) \). With the structure conversion, we can build an Ann-category of the type \( (R, M) \), which is Ann-equivalent to \( A \). The family of constraints on \( A \) induces the one of 5 functions \( (\xi, \alpha, \lambda, \rho : R^3 \to M, \eta : R^2 \to M) \) satisfying certain relations. It is called a structure of an Ann-category of the type \( (R, M) \). In [9], the Classification Theorem was proved for Ann-categories:

(A) There exists a bijection between the set of congruence classes \( C(R, M) \) of pre-sticked Ann-categories of the type \( (R, M) \) and \( S(R, M) \) of cohomology classes of structures on Ann-categories of the type \( (R, M) \).

Ann-categories which satisfy \( c_{A,A} = id \) for all \( A \) are called regular ones. These categories are wide enough to study the problem of ring extension. As in [9], each obstruction of a regular homomorphism corresponds to a structure of a certain regular Ann-category. In [11], the Classification Theorem (A) leads to the Cohomological Classification Theorem for these categories thanks to the cohomology group \( H^3_S \) of \( R \), as a \( \mathbb{Z} \)-algebra, due to Shukla [16]. The difference between the general and the regular cases is that the structure \( (\xi, \eta, \alpha, \lambda, \rho) \) of a regular Ann-category has the bonus property \( \eta(x, x) = 0 \) for the commutativity constraint.

The main result in this paper is the Cohomological Classification Theorem for Ann-categories (Theorem 7.6) in the general case based on Theorem (A) and Mac Lane cohomology.
In 2006, T.Jibladze and M. Pirashvili [1] presented the definition of categorical rings as a slightly modified version of the definition of Ann-categories and classified them with the cohomology group $H^3_{\text{Ann}}(R, M)$. In [15], we have shown that categorical rings contain Ann-categories. “Do they coincide?” is still an open question. We have shown that each categorical ring added the condition (U) is an Ann-category. (U) For each $A \in R$, the pairs $(L^A, \tilde{L}^A), (R^A, \tilde{R}^A)$ defined by

$$
L^A = A \otimes - \quad R^A = - \otimes A \\
\tilde{L}^A_{X,Y} = L_{A,X,Y}^{A,X} \quad \tilde{R}^A_{X,Y} = R_{X,Y,A}^{X,Y}
$$

are $\oplus$-functors which are compatible with the unitivity constraint $(0, g, d)$ of the operation $\oplus$. That is, there exist isomorphisms $\hat{L}^A : A \otimes 0 \to 0$, $\hat{R}^A : 0 \otimes A \to 0$ which make the diagrams in Lemma 1 commute.

Thanks to the condition (U), we may build the structure $\Pi_0(A)$-bimodule for the abelian group $M = \Pi_0(A)$ to the Ann-category $A$, and therefore may build the bimonoidal structure on the category of the type $(R, M)$, (see [9], [11]). The cohomological classification is based on this structure. In [1], authors used these results in [9] without interpreting how to build it without the condition (U). A question is raised: “May we deduce the condition (U) from the definition of categorical rings?” Categorical rings and Ann-categories will not coincide until this question is answered, and the structure $\Pi_0(R)$-bimodule of categorical ring $R$ should be built explicitly.

2 Elementary concepts

We start with some elementary concepts of monoidal categories.

A monoidal category $(C, \otimes, I, a, l, r)$ is a category $C$ which is equipped with a tensor product $\otimes : C \times C \to C$; with an object $I$, called the unit of the category; together with the following natural isomorphisms

$$
a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \\
l_A : I \otimes A \to A \\
r_A : A \otimes I \to A
$$

which are, respectively, called the associativity constraint, the left unicity constraint and the right unicity constraint. These constraints must satisfy the Pentagon Axiom

$$(a_{A,B,C} \otimes \text{id}_D) a_{A,B \otimes C,D} (\text{id}_A \otimes a_{B,C,D}) = a_{A \otimes B, C,D} a_{A,B,C \otimes D},$$

and the Triangle Axiom

$$\text{id}_A \otimes l_B = (r_A \otimes \text{id}_B)a_{A,I,B}.$$
\[ \tilde{F}_{A,B} : FA \otimes FB \to F(A \otimes B) \]

satisfying the following diagrams

\[
\begin{array}{ccc}
FA \otimes (FB \otimes FC) & \xrightarrow{id \otimes \tilde{F}} & FA \otimes F(B \otimes C) \\
\downarrow a' & & \downarrow F(a) \\
(FA \otimes FB) \otimes FC & \xrightarrow{\tilde{F} \otimes id} & FA \otimes (B \otimes FC) \\
\end{array}
\]

\[
\begin{array}{ccc}
FA \otimes FI & \xrightarrow{\tilde{F}} & F(A \otimes I) \\
\downarrow id \otimes F & & \downarrow F(I) \\
FA \otimes I' & \xrightarrow{\alpha} & FA \\
\end{array}
\]

\[
\begin{array}{ccc}
FI \otimes FA & \xrightarrow{\tilde{F}} & F(I \otimes A) \\
\downarrow F \otimes id & & \downarrow F(r) \\
I' \otimes FA & \xrightarrow{\alpha} & FA \\
\end{array}
\]

A natural monoidal transformation \( \alpha : (F, \tilde{F}, \hat{F}) \to (G, \tilde{G}, \hat{G}) \) between monoidal functors, from \( A \) to \( A' \), is a natural transformation \( \alpha : F \to G \) such that the following diagrams commute, for all pairs \((X, Y)\) of objects in \( A \):

\[
\begin{array}{ccc}
FX \otimes FY & \xrightarrow{\alpha_X \otimes \alpha_Y} & GX \otimes GY \\
\downarrow \tilde{F} & & \downarrow G \\
F(X \otimes Y) & \xrightarrow{\alpha} & G(X \otimes Y) \\
\end{array}
\]

\[
\begin{array}{ccc}
FI & \xrightarrow{\alpha_I} & GI \\
\downarrow \tilde{F} & & \downarrow G \\
I' \otimes FA & \xrightarrow{\alpha} & FA \\
\end{array}
\]

A monoidal equivalence between monoidal categories is a monoidal functor \( F : A \to A' \) such that there exists a monoidal functor \( G : A' \to A \) and natural isomorphism monoidal functors \( \alpha : GF \to id_A \) and \( \beta : FG \to id_{A'} \).

\( A \) and \( A' \) are monoidal equivalent if there exists a monoidal equivalence between them.

From Theorem 2 [13], \( F : A \to A' \) is a monoidal equivalence iff \( F \) is a categorical equivalence.

A monoidal category is a Gr-category (or categorical group) if its objects are all invertible and the background category is a groupoid. A Picard category (or Pic-category) is a Gr-category together with a commutativity constraint which is compatible with associativity constraint.

**Definition 2.1.** An Ann-category consists of:

(i) A category \( A \) together with two bifunctors \( \oplus, \otimes : A \times A \to A \);

(ii) A fixed object \( 0 \in A \) with natural isomorphisms \( a_+, c, g, d \) such that \( (A, \oplus, a_+, c, (0, g, d)) \) is a Pic-category;

(iii) A fixed object \( 1 \in A \) with natural isomorphisms \( a, l, r \) such that \( (A, \otimes, a, (1, l, r)) \) is a monoidal category;

(iv) Natural isomorphisms \( \mathcal{L}, \mathcal{R} \)

\[
\mathcal{L}_{A,X,Y} : A \otimes (X \oplus Y) \longrightarrow (A \otimes X) \oplus (A \otimes Y) \\
\mathcal{R}_{X,Y,A} : (X \oplus Y) \otimes A \longrightarrow (X \otimes A) \oplus (Y \otimes A)
\]

satisfy the following conditions:
(Ann - 1) For each object \( A \in \mathcal{A} \), the pairs \((L^A, \hat{L}^A), (R^A, \hat{R}^A)\) defined by the relations:

\[
L^A = A \otimes - \quad R^A = - \otimes A
\]

\[
\hat{L}^A_{X,Y} = \mathcal{L}_{A,X,Y} \quad \hat{R}^A_{X,Y} = \mathcal{R}_{X,Y,A}
\]

are \( \oplus \)-functors which are compatible with \( a_+ \) and \( c \).

(Ann - 2) For any \( A, B, X, Y \in \mathcal{A} \) the following diagrams

\[
\begin{align*}
(AB)(X \oplus Y) & \xrightarrow{\hat{a}_{A,B,X \oplus Y}} A(B(X \oplus Y)) \xrightarrow{id_A \oplus L^B} A(BX \oplus BY) \\
(AB)X \oplus (AB)Y & \xrightarrow{a_{A,B,X \oplus A,B,Y}} A(BX) \oplus A(BY) \\
(X \oplus Y)(BA) & \xrightarrow{a_{X \oplus Y,B,A}} ((X \oplus Y)B)A \xrightarrow{R^B \otimes id_A} (XB \oplus YB)A \\
X(BA) \oplus Y(BA) & \xrightarrow{a_{X,B,A \oplus Y,B,A}} (XB)A \oplus (YB)A \\
(A(X \oplus Y)B) & \xrightarrow{a_{A,X \oplus Y,B}} A((X \oplus Y)B) \xrightarrow{id_A \oplus \hat{R}^B} A(XB \oplus YB) \\
(AX \oplus (AY)B) & \xrightarrow{\hat{R}^B} (AX)B \oplus (AY)B \xrightarrow{a_{X,B} \oplus Y} A(XB) \oplus A(YB) \\
(A \oplus B)(X \oplus (AY)B) & \xrightarrow{L^A \oplus id_B} (A \oplus B)(X \oplus Y) \xrightarrow{\hat{R}^X \oplus Y} A(X \oplus Y) \oplus B(X \oplus Y) \\
(AX \oplus BX) \oplus (AY \oplus BY) & \xrightarrow{\hat{L}^Y} (AX \oplus BY) \oplus (BX \oplus BY)
\end{align*}
\]

commute, where \( v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \rightarrow (U \oplus Z) \oplus (V \oplus T) \) is the unique morphism constructed from \( \oplus, a_+, c, id \) in the symmetric monoidal category \((\mathcal{A}, \oplus)\).

(Ann - 3) For the unit \( 1 \in \mathcal{A} \) of the operation \( \otimes \), the following diagrams

\[
\begin{align*}
1(X \oplus Y) & \xrightarrow{L^1} 1X \oplus 1Y \xrightarrow{L^1} X \oplus Y \\
L^1 & \xrightarrow{L^1} 1X \oplus 1Y \\
X \oplus Y & \xrightarrow{i_{X \oplus Y}} X \oplus Y \\
X \oplus Y & \xrightarrow{i_{X \oplus Y}} X \oplus Y \\
X \oplus Y & \xrightarrow{r_{X \oplus Y}} X \oplus Y \\
X \oplus Y & \xrightarrow{r_{X \oplus Y}} X \oplus Y \\
(X \oplus Y)1 & \xrightarrow{R^1} X1 \oplus Y1 \\
X1 \oplus Y1 & \xrightarrow{R^1} X1 \oplus Y1 \\
1X \oplus 1Y & \xrightarrow{R^1} X1 \oplus Y1 \\
\end{align*}
\]

commute.

It follows from the definition of Ann-categories that

**Lemma 2.2.** In Ann-category \( \mathcal{A} \) there exists uniquely the homomorphisms:

\[
\hat{L}^A : A \otimes 0 \rightarrow 0, \quad \hat{R}^A : 0 \otimes A \rightarrow 0
\]

such that the following diagrams

\[
\begin{align*}
AX & \xrightarrow{L^A_{(0 \otimes X)}} A(0 \otimes X) \xrightarrow{L^A_{(0 \otimes X)}} A0 \otimes AX \\
AX & \xrightarrow{L^A_{(0 \otimes X)}} A(0 \otimes X) \xrightarrow{L^A_{(0 \otimes X)}} A0 \otimes AX \\
AX & \xrightarrow{L^A_{(0 \otimes X)}} A(0 \otimes X) \xrightarrow{L^A_{(0 \otimes X)}} A0 \otimes AX \\
AX & \xrightarrow{L^A_{(0 \otimes X)}} A(0 \otimes X) \xrightarrow{L^A_{(0 \otimes X)}} A0 \otimes AX
\end{align*}
\]
commute. That is, $L^A$ and $R^A$ are $\oplus$-functors which are compatible with the unitivity constraint.

**Proof.** Since the pair $(L^A, \tilde{L}^A)$ is $\oplus$-functors which are compatible with the associativity constraint $\alpha$ of the Picard category $(A, \oplus)$, it is compatible with the unitivity constraint $(0, g, d)$. This means that there exists uniquely isomorphism $\hat{L}^A$ satisfying the first diagrams of the lemma. The proof of $\hat{R}^A$ is completely analogous. □

**Lemma 2.3.** In any Ann-category $A$, the isomorphisms $\hat{L}^A, \hat{R}^A$ have the following properties

(i) The family $\hat{L} = \tilde{L}$ (resp. the family $\hat{R} = \tilde{R}$) is a $\oplus$-morphism from the functor $(R^0, \tilde{R}^0)$ (resp. $(L^0, \tilde{L}^0)$) to the functor $(\theta : A \to 0, \tilde{\theta} = g_0^{-1})$, i.e., the following diagrams

\[
\begin{array}{ccc}
A0 & \xrightarrow{\tilde{L}^A} & B0 \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}
\quad
\begin{array}{ccc}
(X \oplus Y)0 & \xrightarrow{\tilde{R}^0} & X0 \oplus Y0 \\
\downarrow & & \downarrow \\
0 & = & 0 \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}
\]

commute.

(resp. $\hat{R}^B(id \otimes f) = \hat{R}^A$ and $\hat{R}^{X \oplus Y} = g_0(\hat{R}^X \oplus \hat{R}^Y)\tilde{L}^0$).

(ii) For any $A, B \in A$, the following diagrams

\[
\begin{array}{ccc}
X(0Y) & \xrightarrow{a} & (X0)Y \\
\downarrow & & \downarrow \\
X0 & \xrightarrow{\hat{L}^X} & 0
\end{array}
\quad
\begin{array}{ccc}
X(Y0) & \xrightarrow{id \otimes \tilde{L}^Y} & X0 \\
\downarrow & & \downarrow \\
X0 & \xrightarrow{\hat{L}^X \otimes id} & 0
\end{array}
\quad
\begin{array}{ccc}
(X0)Y & \xrightarrow{a} & (X0)Y \\
\downarrow & & \downarrow \\
(XY)0 & \xrightarrow{\hat{L}^{XY}} & 0
\end{array}
\quad
\begin{array}{ccc}
X0 & \xrightarrow{\hat{L}^X} & 0 \\
\downarrow & & \downarrow \\
(XY)0 & \xrightarrow{\tilde{L}^{XY}} & 0
\end{array}
\]

commute and $\hat{R}^{XY} = \tilde{R}^Y(\tilde{R}^X \otimes id)a_{0,X,Y}$.

(iii) $L^1 = l_0, R^1 = r_0$.

The properties of the isomorphisms $\hat{L}^A, \hat{R}^A$ are presented in [7], Proposition 3.2. They are necessary for the proofs of the section 5.
3 Ann-functors and the structure conversion

**Definition 3.1.** Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories. An Ann-functor from $\mathcal{A}$ to $\mathcal{A}'$ is a triple $(F, \tilde{F}, \bar{F})$, where $(F, \tilde{F})$ is a Pic-functor for the operation $\oplus$, $(F, \bar{F})$ is a functor which is compatible with the associativity constraint of the operation $\otimes$ and satisfies two following commutative diagrams:

\[
\begin{array}{ccc}
F(X(Y \oplus Z)) & \xrightarrow{\bar{F}} & FX.F(Y \oplus Z) \\
\downarrow F(\alpha) & & \downarrow F' \\
F(XY \oplus XZ) & \xrightarrow{\bar{F}} & F(XY) \oplus F(XZ) \\
\downarrow F(\beta) & & \downarrow F' \\
F((X \oplus Y)Z) & \xrightarrow{\bar{F}} & F(X \oplus Y).FZ \\
\downarrow F(\gamma) & & \downarrow F' \\
F(XZ \oplus YZ) & \xrightarrow{\bar{F}} & F(XZ) \oplus F(YZ) \\
\end{array}
\]

These commutative diagrams are also called the compatibility of the functor $F$ with the distributivity constraints.

Note that, like ring homomorphism, an Ann-functor $F$ does not necessary satisfy $F(1_\mathcal{A}) \cong 1_{\mathcal{A}'}$. However, since $(\mathcal{A}, \oplus)$ and $(\mathcal{A}', \oplus)$ are Pic-categories, the pair $(F, \tilde{F})$ is compatible with the unitivity constraints of the operation $\oplus$, i.e., there exists the bijection $\tilde{F} : F(0_\mathcal{A}) \to 0_{\mathcal{A}'}$ satisfying certain diagrams. We call $\varphi : F \to G$ an Ann-morphism between two Ann-functors $(F, \tilde{F}, \bar{F})$ and $(G, \tilde{G}, \bar{G})$ if it is both an $\oplus$-morphism and an $\otimes$-morphism.

Ann-functor $(F, \tilde{F}, \bar{F}) : \mathcal{A} \to \mathcal{A}'$ is called an Ann-equivalence if there exists an Ann-functor $(G, \tilde{G}, \bar{G}) : \mathcal{A}' \to \mathcal{A}$ and Ann-natural isomorphisms $\alpha : G \circ F \to id$, $\beta : F \circ G \to id$.

By the Theorem 8 [13] $(F, \tilde{F}, \bar{F})$ is an Ann-equivalence iff $F$ is a categorical equivalence.

Now, we consider the problem of the structure conversion of Ann-categories by an Ann-equivalence.

**Lemma 3.2.** Let $F : \mathcal{A} \to \mathcal{A}'$ be an equivalence of categories and $G : \mathcal{A}' \to \mathcal{A}$ be a quasi-inverse of $F$. Then, we can choose natural isomorphisms $\alpha : G \circ F \simeq id_\mathcal{A}$ and $\beta : F \circ G \simeq id'_{\mathcal{A}}$ such that

\[
F(\alpha_A) = \beta_{F,A}; \quad G(\beta_B) = \alpha_{GB},
\]

for all objects $A \in \mathcal{A}$, $B \in \mathcal{A}'$.

**Lemma 3.3.** Let $(F, \tilde{F}) : \mathcal{A} \to \mathcal{A}'$ be a $\otimes$-functor of $\otimes$-categories and $(F, G, \alpha, \beta)$ be a quadruple satisfying the above relations. Then, there exists the natural isomorphism

\[
\bar{G} = \bar{G}_{XY} : GXGY \to G(XY)
\]

such that $(G, \bar{G})$ is a $\otimes$-functor and $\alpha$, $\beta$ are $\otimes$-morphisms.

For the quadruple $(F, G, \alpha, \beta)$, we can establish the structure conversion from $\mathcal{A}'$ to $\mathcal{A}$. $\otimes$ is the tensor product in $\mathcal{A}'$, then $\mathcal{A}$ could be equipped with a tensor product defined by

\[
U \otimes V = G(FU \otimes FV) \\
u \otimes v = G(Fu \otimes Fv)
\]
for all pairs \((U, V)\) of objects and all pairs \((u, v)\) of morphisms in \(\mathcal{A}\).

For this tensor product, \(F, G\) become tensor functors with natural tensor transformations \(\widetilde{F}, \widetilde{G}\) defined as follows

\[
\widetilde{F}_{U,V} = \beta^{-1}_{F_U \otimes F_V}, \quad \widetilde{G}_{A,B} = G(\beta_A \otimes \beta_B) \quad (3.4)
\]

for all \(U, V \in \mathcal{C}, A, B \in \mathcal{D}\).

If \(\mathcal{A}'\) has the unitivity constraint \((I', l', r')\), we can define an unitivity constraint \((I, l, r)\) of \(\mathcal{C}\) by the following commutative diagrams, where \(I = GI'\)

\[
\begin{align*}
&I \otimes X @>i_X>> X \downarrow \alpha_X \\
\downarrow \text{id} \circ \alpha_X @>\alpha_X \circ \text{id}>> X \downarrow \alpha_X \\
GI' \otimes GFX @>G(\alpha_X)\circ G(I' \otimes FX)>> GFX \downarrow \alpha_X \\
GFX \otimes GI' @>G(FX \otimes I')\circ G(I' \otimes FX)>> GFX \\
\end{align*}
\]

(3.5a)

\[
\begin{align*}
&X \otimes I @>r_X>> X \downarrow \alpha_X \\
\downarrow \alpha_X \circ \text{id} @>\alpha_X \circ \text{id}>> X \downarrow \alpha_X \\
GFX \otimes GI' @>G(FX \otimes I')\circ G(I' \otimes FX)>> GFX \downarrow \alpha_X \\
GFX \otimes GI' @>G(FX \otimes I')\circ G(I' \otimes FX)>> GFX \\
\end{align*}
\]

(3.5b)

Then, \((F, \widetilde{F})\) and \((G, \widetilde{G})\) are \(\otimes\)-functors which are compatible with unitivity constraints of \(\mathcal{A}\) and \(\mathcal{A}'\).

Being different from the unitivity constraint, the associativity constraint \(a\) of \(\mathcal{A}\) induced by the associativity constraint \(a'\) of \(\mathcal{A}'\) only depends on \(\otimes\)-functor \((F, \widetilde{F})\) due to the commutative diagram (1.1).

First, we obtain the result of the structure conversion of monoidal categories.

**Theorem 3.4.** Let \(F : \mathcal{A} \to \mathcal{A}'\) be a categorical equivalence and \((F, G, \alpha, \beta)\) be the quadruple satisfying the relations (3.2). Assume that there is a monoidal structure \((\otimes, a', I', l', r')\) on \(\mathcal{A}'\). In addition, the induced monoidal structure of \(\mathcal{A}\) has the unitivity constraint \((I, l, r)\) and \(\widetilde{F}\) is the natural isomorphism such that \((F, \widetilde{F})\) is compatible with the unitivity constraints of \(\mathcal{A}\) and \(\mathcal{A}'\). Then \(\mathcal{A}\) is a monoidal category. In particular, we can define the natural isomorphism \(\widetilde{F}\) by (3.4) and the unitivity constraint \((I, l, r)\) by (3.5a) and (3.5b).

*Proof.* See Theorem 3 [14].

Two following corollaries immediately follow from Theorem 3.4

**Corollary 3.5.** If \(\mathcal{A}'\) is the symmetric monoidal category with the commutativity constraint \(c'\), the monoidal category \(\mathcal{A}\) mentioned in Theorem 3.4 is also symmetric with the commutativity constraint \(c\) defined by

\[
F(c) = F^{-1} \circ c'_{FX, FY} \circ \widetilde{F} \quad (3.6)
\]

**Corollary 3.6.** If \((\mathcal{A}', \oplus, \otimes)\) is an Ann-category, \(\mathcal{A}\) will become a Picard category with the operation induced by the operation \(\oplus\), and \(\mathcal{A}\) is also a monoidal category with the operation induced by the operation \(\otimes\).

For the distributivity constraints, we have the following proposition.
Proposition 3.7. Let \( A' \) be an Ann-category with the family of constraints
\[
(a'_+ , c', (0' , g', d'), a'_+, (I', l', r')) , \mathcal{L}', \mathcal{R}'
\]
and \( A \) be a category with two operations \( \oplus , \otimes \) such that \((A, \oplus)\) is a Picard category with constraints \((a_+, c, (0, g, d))\) and \((A, \otimes)\) is a monoidal category with the constraints \((a, (I, l, r))\).

Furthermore, assume that \( F : A \rightarrow A' \) is a categorical equivalence such that \((F, \bar{F})\) is a Pic-functor and \((F, \tilde{F})\) is a monoidal functor. Then, the isomorphisms \( \mathcal{L}, \mathcal{R} \), defined by the commutative diagrams (3.1) are distributivity constraints and \( A \) becomes an Ann-category, and \((F, \tilde{F}, F')\) becomes an Ann-equivalence.

Proof. See Proposition 4.2 [7], Proposition II.1.4 [9].

The above propositions suggest us to state the theorem of the structure conversion of Ann-categories as below

Theorem 3.8. Let \( F : A \rightarrow A' \) be a categorical equivalence together with the quadruple \((F, G, \alpha, \beta)\) satisfying the relation (3.2), and \( A' \) be an Ann-category. Furthermore, assume that two operations on \( A \) induced by \( F, F' \) have the unitivity constraints \((0, g, d), (I, l, r)\), and \( \bar{F}, \tilde{F} \) are two natural isomorphisms such that \((F, \bar{F}), (F, \tilde{F})\) are, respectively, compatible with unitivity constraints in terms of each operation \( \oplus, \otimes \). Then \( A \) becomes an Ann-category with the inducing constraints, determined by the commutative diagrams (2.1), (3.6), (3.1a), (3.1b), and \((F, \bar{F}, \tilde{F})\) is an Ann-equivalence. Particularly, one can choose \( \bar{F}, \tilde{F} \) by (3.4) and \((0, g, d), (I, l, r)\) by (3.5a), (3.5b).

The first corollary is the “stricticizing” of constraints.

Corollary 3.9. Each Ann-category is Ann-equivalent to an Ann-category whose associativity and unitivity constraints of the operation \( \oplus \) are strict.

Proof. Assume that \( A' \) is an Ann-category. Then \((A', \oplus)\) is monoidal equivalent to a Pic-category \((A, \oplus)\) whose associativity and unitivity constraints are strict (see Theorem 14 [13]). According to the Propositions 3.6 and 3.7, by the structure conversion, the Pic-category \((A, \oplus)\) becomes an Ann-category equivalent to \( A' \).

4 The first invariants of an Ann-category

As we know (See [15]), each Gr-category \( P \) is uniquely determined, up to an equivalence, by three characteristic invariants:

1. The group \( \Pi_0(P) \) of the isomorphic classes of objects of \( P \)
2. \( \Pi_0(P) \)-left module \( \Pi_1(P) = \text{Aut}_P(0) \)
3. The element \( \xi \in H^3(\Pi_0(P), \Pi_1(P)) \) (group cohomology).

The action of the group \( \Pi_0(P) \) on the abelian group \( \Pi_1(P) \) is given by:
\[
su = \gamma_X^{-1} \delta_X(u), \quad X \in s \in \Pi_0(P), u \in \Pi_1(P)
\]
where \( \gamma_X, \delta_X \) are isomorphisms determined by the commutative squares:

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma_X(u)} & X \\
\downarrow & & \downarrow \\
0 \oplus X & \xrightarrow{\oplus 0 \oplus} & 0 \oplus X
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\delta_X(u)} & X \\
\downarrow & & \downarrow \\
X \oplus 0 & \xrightarrow{\oplus \delta_X(0)} & X \oplus 0
\end{array}
\]
When $\mathcal{P}$ is a Picard category, $\gamma_X = \delta_X$; hence, the above action is trivial. Thus, for the Ann-category $\mathcal{A}$ we will construct a new action and prove that with this action, $\Pi_1(\mathcal{A})$ is a bimodule on the ring $\Pi_0(\mathcal{A})$.

Let $\mathcal{A}$ be an Ann-category with the family of constraints 

$$(a_+, c, (0, g, d), a, (1, l, r), L, R).$$

Then, the set of the isomorphic classes of objects $\Pi_0(\mathcal{A})$ of $\mathcal{A}$ is a ring with two operations $+, \times$, induced by the ones $\oplus, \otimes$ in $\mathcal{A}$, and $\Pi_1(\mathcal{A}) = \text{Aut}(0)$ is an abelian group with the composition law denoted by $+$. We construct the actions of the ring $\Pi_0(\mathcal{A})$ on the abelian group $\Pi_1(\mathcal{A})$ so that $\Pi_1(\mathcal{A})$ becomes a $\Pi_0(\mathcal{A})$-bimodule.

**Definition 4.1.** The maps $\lambda_X, \rho_X : \text{Aut}(0) \rightarrow \text{Aut}(0)$ are given by the following commutative diagrams, for $X \in \text{ob}(\mathcal{A})$

$$\begin{array}{ccc}
X.0 & \overset{L_X}{\longrightarrow} & 0 \\
\downarrow id \otimes u & & \downarrow id \otimes u \\
X.0 & \overset{L_X}{\longrightarrow} & 0
\end{array} \quad \begin{array}{ccc}
0.X & \overset{\tilde{L}^X}{\longrightarrow} & 0 \\
\downarrow u \otimes id & & \downarrow u \otimes id \\
0.X & \overset{\tilde{L}^X}{\longrightarrow} & 0
\end{array}
$$

$$\begin{array}{ccc}
X.0 & \overset{\lambda_X(u)}{\longrightarrow} & \lambda_X(u) \\
\downarrow id \otimes u & & \downarrow u \otimes id \\
X.0 & \overset{\lambda_X(u)}{\longrightarrow} & \lambda_X(u)
\end{array} \quad \begin{array}{ccc}
0.X & \overset{\lambda_X(u)}{\longrightarrow} & \lambda_X(u) \\
\downarrow u \otimes id & & \downarrow u \otimes id \\
0.X & \overset{\lambda_X(u)}{\longrightarrow} & \lambda_X(u)
\end{array}
$$

(4.1)

**Proposition 4.2.** If $X, Y \in \text{ob}(\mathcal{A})$, $X = Y$, then $\lambda_X = \lambda_Y$ and $\rho_X = \rho_Y$.

**Proof.** Consider the diagram 1

$$\begin{array}{ccc}
X.0 & \overset{L_X}{\longrightarrow} & 0 \\
\downarrow id \otimes u & & \downarrow id \otimes u \\
X.0 & \overset{L_X}{\longrightarrow} & 0
\end{array} \quad \begin{array}{ccc}
0.X & \overset{\tilde{L}^X}{\longrightarrow} & 0 \\
\downarrow u \otimes id & & \downarrow u \otimes id \\
0.X & \overset{\tilde{L}^X}{\longrightarrow} & 0
\end{array}
$$

Diagram 1

In this diagram, the regions (I) and (III) commute by the definition of $\lambda_X, \lambda_X$; the regions (IV) and (V) commute by the Proposition 2.3; obviously, the outside region commutes by the composition of morphisms. It follows that the region (II) commutes, i.e., $\lambda_X(u) = \lambda_Y(u)$. The proof of $\rho_X = \rho_Y$ is completely analogous.

**Theorem 4.3.** The left and right actions of the ring $\Pi_0(\mathcal{A})$ on the abelian group $\Pi_1(\mathcal{A})$ defined, respectively, by the following relations

$$su = \lambda_X(u), \quad us = \rho_X(u)$$

(4.2)

where $X \in s, s \in \Pi_0(\mathcal{A}), u \in \Pi_1(\mathcal{A})$ make $\Pi_1(\mathcal{A})$ become a $\Pi_0(\mathcal{A})$-bimodule.
Proof. We have to prove that the actions $su, us$ satisfy the following equalities:

\begin{align*}
\text{a)} & \quad s(u_1 + u_2) = su_1 + su_2, \\
\text{b)} & \quad (s_1 + s_2)u = s_1u + s_2u, \\
\text{c)} & \quad s_1(s_2u) = (s_1s_2)u, \\
\text{d)} & \quad 1u = u, \\
\text{e)} & \quad (ru)s = r(us) \\
\text{a')} & \quad (u_1 + u_2)s = u_1s + u_2s \\
\text{b')} & \quad u(s_1 + s_2) = us_1 + us_2 \\
\text{c')} & \quad (us_1)s_2 = u(s_1s_2) \\
\text{d')} & \quad u1 = u \\
\text{e')} & \quad (ru)s = r(us)
\end{align*}

1. In order to prove the relation a), we consider the diagram 2

\[
\begin{array}{c|c|c|c}
\text{id} \otimes u_1 & & \text{id} \otimes u_2 \\
\hline
\text{id} \otimes (u_1 + u_2) & & \lambda_X(u_1) & \lambda_X(u_1 + u_2) \\
\hline
\text{X.0} & \text{X.0} & \text{X.0} & \text{X.0} \\
\hline
\end{array}
\]

Diagram 2

where the regions (I) and (II) commute by the definition of $\lambda_X$, the region (III) commutes by the composition of morphisms; the outside region commutes by the definition of $\lambda_X$. It follows that the region (IV) commutes, i.e., $\lambda_X(u_1 + u_2) = \lambda_X(u_1) + \lambda_X(u_2)$ or $s(u_1 + u_2) = su_1 + su_2$.

The equality a') is analogously proved by the definition of $\rho_X$.

2. In order to prove the relation b), first we consider the diagram

\[
\begin{array}{c|c|c|c}
\gamma_0(u) = u & & \delta_0(v) = v \\
\hline
\gamma_0(u) = u & & \delta_0(v) = v \\
\hline
0 \oplus 0 & \oplus 0 & 0 \oplus 0 & 0 \oplus 0 \\
\hline
d_0 = g_0 & & d_0 = g_0 \\
\hline
\end{array}
\]

where the regions (I), (II) commute, respectively, by the definitions of $\gamma_0, \delta_0$; and $\gamma_0(u) = u (\delta_0(v) = v)$ by the functoriality of $d_0 (g_0)$. It follows that the outside region commutes, i.e.,

\[g_0(u \oplus v)g_0^{-1} = u + v\]  

(3.3)

This equality establishes an isomorphism

\[
\omega : \text{Aut}(0 \oplus 0) \to \text{Aut}(0) \\
u \oplus v \mapsto u + v
\]
Now, we consider the following diagram

\[
\begin{array}{c}
\text{Diagram 3} \\
\end{array}
\]

In this diagram, the region (I) commutes by the definitions of \(\lambda_X, \lambda_Y\); the region (V) commutes by the definition of \(\lambda_{X \oplus Y}\); the regions (II) and (IV) commute by the properties of isomorphisms \(L^A, R^A\); the outside region commutes by the functoriality of \(\check{R}^0\). It follows that the region (III) commutes. Also by the relation (4.3), we have \(\lambda_{X \oplus Y}(u) = \lambda_X(u) + \lambda_Y(u)\), or \((s + r)u = su + ru\) for \(X \in s, Y \in r\). The relation b’ is analogously proved.

3. In order to prove the relation c), we consider the diagram

\[
\begin{array}{c}
\text{Diagram 4} \\
\end{array}
\]

In this diagram, the region (I) commutes by the definition of \(\lambda_{X,Y}\); the region (VI) commutes by the definition of \(\lambda_X\); the region (IV) commutes by the definition of \(\lambda_Y\); the region (III) commutes by the functoriality of \(a\), the regions (II) and (V) commute by the properties of isomorphisms \(L^A, R^A\). It follows that the outside region commutes, i.e., \(\lambda_{X,Y}(u) = \lambda_X(\lambda_Y(u))\), or \((sr)u = s(ru)\) for \(X \in s, Y \in r\). The relation c) is analogously proved.
4. By the relation $\hat{L}^1 = l_0$ and by the nature of $l_0$ we have the commutative diagram

$$
\begin{array}{ccc}
1.0 & \overset{\hat{L}^1 = l_0}{\longrightarrow} & 0 \\
\downarrow & & \downarrow \\
1.0 & \overset{\hat{L}^1 = l_0}{\longrightarrow} & 0 \\
\end{array}
$$

This means that $u = \lambda_1(u)$ by the definition of $\lambda_1$. Thus $1u = u$. Analogously, $u1 = u$.

5. Finally, we prove that $(su)r = s(ur)$. Consider the diagram 5 In this diagram, the regions (I) and (V) commute by the properties of isomorphisms $L^A$, $R^A$, the regions (II) and (VII) commute by the definition of $\rho_Y$; the region (III) commutes by the nature of $a$; the regions (IV) and (VI) commute by the definition of $\lambda_X$. It follows that the outside region commutes, i.e., $\rho_Y(\lambda_X(u)) = \lambda_X(\rho_Y(u))$, or $(su)r = s(ur)$.

**Diagram 5**

**Remark 4.4.** The isomorphisms $\lambda_X, \rho_X$ and the isomorphisms $\gamma_X$ in the diagrams (4.0), (4.1) concern with each other by the relations

$$
\lambda_X(u) = \gamma_X^{-1}(id \otimes u); \quad \rho_X(u) = \gamma_X^{-1}(u \otimes id) \quad (4.3)
$$
Proof. First, we can see that if \((F, \tilde{F}) : \mathcal{A} \to \mathcal{A}\) is a \(\oplus\)-functor which is compatible with the unit \((0, g, d)\), then the following square

\[
\begin{array}{ccc}
F0 & \xrightarrow{\gamma_{F0}} & F0 \\
\downarrow \tilde{F} & & \downarrow \tilde{F} \\
0 & \xrightarrow{u} & 0
\end{array}
\]

commutes.

Indeed, the family \((\gamma_X(u))_X, X \in \text{ob}(\mathcal{A})\) is an automorphism of the identical functor \(I_A\). We also have \(\gamma_0(u) = u\) so the above square commutes. Now, replacing \(F0 = X.0\) and \(u = \gamma_{X,0}^{-1}(id \otimes v)\), we have the commutative diagram

\[
\begin{array}{ccc}
X0 & \xrightarrow{id \otimes v} & X0 \\
\downarrow \hat{L}^X & & \downarrow \hat{L}^X \\
0 & \xrightarrow{\gamma_{X,0}^{-1}(id \otimes v)} & 0
\end{array}
\]

This shows that \(\gamma_{X,0}^{-1}(id \otimes v) = \lambda_X(v)\) by the definition of \(\lambda_X\). The relation of \(\rho_X\) follows analogously.

\begin{remark}
If \(\mathcal{A}\) is an \(\oplus\)-category together with the unitivity constraint \((0, g, d)\) and \((F, \tilde{F}) : \mathcal{A} \to \mathcal{A}\) is an \(\oplus\)-functor which is compatible with the unit, we have the following relation

\[
F(\gamma_X(u)) = \gamma_{FX}(\gamma_{F0}^{-1}(Fu)) : F(\delta_X(u)) = \delta_{FX}(\delta_{F0}^{-1}(Fu)) \tag{4.4}
\]

\end{remark}

Proof. Consider the following diagram 6

\[
\begin{array}{ccc}
0 \oplus FX & \xrightarrow{\gamma_{F0}^{-1}(Fu) \oplus id} & F0 \oplus FX \\
\downarrow \hat{F} \oplus id & & \downarrow \hat{F} \\
F0 \oplus FX & \xrightarrow{F(gx)} & FX
\end{array}
\]

Diagram 6
In this diagram, the regions (I) and (V) commute by the compatibility of the functor \((F, \hat{F})\) with the unitivity constraint \((0, g, d)\); the region (IV) commutes by the definition of \(\gamma_X\) (images via \(F\)); the region (II) commutes by the diagram in the proof of Remark 4.4; the region (III) commutes by the nature of the isomorphism \(\hat{F}\). It follows that the outside region commutes, i.e.,

\[
F(\gamma_X(u)) = \gamma_{FX}(\gamma_{F0}^{-1}(Fu))
\]

The second equality of (4.4) is analogously proved.

To show the invariant of \(\Pi_0(\mathcal{A})\) and \(\Pi_1(\mathcal{A})\) of Ann-category \(\mathcal{A}\) we prove the following two theorems

**Theorem 4.6.** Let \(\mathcal{A}\) and \(\mathcal{A}'\) be two Ann-categories. Then, each Ann-functor \((F, \hat{F}, \tilde{F}) : \mathcal{A} \to \mathcal{A}'\) induces a pair of ring homomorphisms

\[
\begin{align*}
\hat{F} : & \quad \Pi_0(\mathcal{A}) \to \Pi_0(\mathcal{A}') \\
\tilde{F} : & \quad \Pi_1(\mathcal{A}) \to \Pi_1(\mathcal{A}')
\end{align*}
\]

satisfying the relations

\[
\begin{align*}
0^0_{\hat{F}}(su) &= \hat{F}(s)0^0_F(u); \\
0^0_{\tilde{F}}(us) &= 0^0_F(u)\hat{F}(s)
\end{align*}
\]

where \(\Pi_1(\mathcal{A})\) is regarded as a ring with the null multiplication. Furthermore, \(F\) is an equivalence iff \(\hat{F}, \tilde{F}\) are isomorphisms.

**Proof.** First, we prove the following lemma

**Lemma 4.7.** With the assumption of Theorem 4.6 the following diagram

\[
\begin{array}{ccccccc}
F(A \otimes O) & \xrightarrow{F(L^A)} & FO & \xleftarrow{\hat{F}} & O' \\
\downarrow{\hat{F}} & & & & & \\
FA \otimes FO & \xleftarrow{id \otimes \hat{F}} & FA \otimes O' & \xrightarrow{\hat{L}^F} & O'
\end{array}
\]

commutes.

**Proof.** Indeed, consider two \(\oplus\)-functors \((G_1, \hat{G}_1)\) and \((G_2, \hat{G}_2)\) as below

\[
G_1 : X \mapsto F(A \otimes X); \quad G_2 : X \mapsto FA \otimes FX
\]

Clearly, \(G_1 = F \circ L^A\). Since \(L\) is compatible with the unit \((0, g, d)\) of \(\mathcal{A}\) and \(F\) is compatible with the units \((0, g, d), (0', g', d')\) of \(\mathcal{A}\) and \(\mathcal{A}'\), the composition functor \(G_1 = F \circ L^A\) is also compatible with the units \((0, g, d), (0', g', d')\).

For a similar reason, the functor \(G_2 = L^F \circ A\) is compatible with the unitivity constraints \((0, g, d), (0', g', d')\) of \(\mathcal{A}, \mathcal{A}'\). Furthermore, \(\oplus\)-morphism of functors

\[
\varphi_X = \hat{F}_{A,X} : G_1X \to G_2X, \quad X \in \text{ob}\mathcal{A}
\]

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are isomorphisms, the following diagram
\[
\begin{array}{ccc}
G_1 0 & \xrightarrow{\varphi_0} & G_2 0 \\
\downarrow & & \downarrow \\
\hat{G}_1 & & \hat{G}_2 \\
\end{array}
\]
(b)

commutes, where \(\hat{G}_1, \hat{G}_2\) are isomorphisms determined by the following commutative diagrams

This means that the diagram (b) is just the diagram (a).

Now, we prove the proposition by considering the diagram 7

Diagram 7

In this diagram, the region (I) commutes by the property of the isomorphism \(\gamma_X\) (the diagram (*)); the region (II) commutes by the definition of \(\lambda_A(u)\) (images via \(F\)); the region (III) commutes by the nature of the isomorphism \(\hat{F}\); the region (IV) commutes by the same reason as the region (I); the regions (V), (VI) commute by the above lemma. It follows that the outside region commutes, i.e., from the definition of \(\lambda_X\) we have
\[
\gamma_{F_0}^{-1}(F(\lambda_A(u))) = \lambda_{FA}(\gamma_{F_0}^{-1}Fu)
\]
Note that if \( s = clA \), then by (4.2), we have
\[
0 F(su) = F(\lambda_A(u)) = \gamma_{F_0}^{-1}(F(\lambda_A(u)))
\]
and
\[
\lambda_{F,A}(\gamma_{F_0}^{-1}(Fu)) = \lambda_{F,A}(Fu) = \gamma^{-1} F(\lambda_{F,A}(u))
\]
It follows that
\[
0 F(su) = F(s) F(u)
\]
The second relation is analogously proved. After that, we can verify that \( F \) and \( 0 F \) are two ring homomorphisms. Moreover, \( F \) is an equivalence iff \( F \) and \( 0 F \) are isomorphisms. This completes the proof.

5 Reduced Ann-categories

Let \( A \) be an Ann-category. Consider the category \( S \) whose objects are the elements of \( \Pi_0(A) \), and whose morphisms are automorphisms, in the concrete, for \( r \in \Pi_0(A) \)
\[
\text{Aut}_S(r) = \{r\} \times \Pi_1(A)
\]
The composition law of two morphisms is defined by
\[
(r, u). (r, v) = (r, u + v)
\]
We can use the structure conversion to make \( S \) be an Ann-category which is equivalent to \( A \). In \( A \), we choose the representatives \( (X_s), s \in \Pi_0(A) \), and a family of isomorphisms \( i : X \to X_s \) Then, we can determine two functors:
\[
G : A \to S
\]
\[
G(X) = clX = so
\]
\[
G(f) = (s, \gamma_{i_X}^{-1}(if_{i_X}^{-1}))
\]
for \( X, Y \in s \) and \( f : X \to Y \), and \( \gamma_X \) is determined by the diagram (4.0). The functor morphisms \( \alpha = (\gamma_{i_X}^{-1}(if_{i_X}^{-1})) : GH \to id_S \) and \( \beta = (i_X^{-1}) : HG \to id_A \) satisfy the conditions (3.2)
\[
H \alpha = \beta H, \quad G\beta = \alpha G
\]
Thus, we can perform the structure conversion from \( A \) to \( S \). In the concrete, the operations \( \oplus, \otimes \) determined on the reduced category \( S \) are those obtained by the conversion of the laws \( \oplus, \otimes \) on \( A \) by \( (H, G, \alpha, \beta) \), i.e.,
\[
s \oplus t = G(H(s) \oplus H(t)) = s + t, \quad s, t \in \Pi_0(A)
\]
\[
s \otimes t = G(H(s) \otimes H(t)) = st
\]
\[
(s, u) \oplus (t, v) = G(H(s, u) \oplus H(t, v)), \quad u, v \in \text{Aut}(0)
\]
\[
(s, u) \otimes (t, v) = G(H(s, u) \otimes H(t, v))
\]
Proposition 5.1. The operations $\oplus, \otimes$ of the morphisms have the precise form
\[
(s, u) \oplus (t, v) = (s + t, u + v) \\
(s, u) \otimes (t, v) = (st, sv + ut),
\]
so they do not depend on the selection of the representatives $(X_s, i_X)$.

Proof. The first relation is proved by H. X. Sinh [13]. Here, we prove the second one. We have
\[
G(H(s, u), H(t, v)) = G(\gamma_X(u) \otimes \gamma_X(v)) = (st, \gamma_X^{-1}(i_X^t(\gamma_X(u) \otimes \gamma_X(v)))i_X^s)
\]
Apply the formula (4.4) for the functor $L_X$ and then apply the formula (4.3), we have
\[
\text{id}_{X_s} \otimes \gamma_X^t(v) = \gamma_X(u) \otimes \text{id}_{X_t} = \gamma_X(u) \otimes \rho_X(v) = \gamma_X(v) = \gamma_X(v)
\]
Similarly, for the functor $R_X$, we have
\[
\gamma_X(u) \otimes \text{id}_{X_t} = \gamma_X(u) \otimes \lambda_X(v) = \gamma_X(v) = \gamma_X(v)
\]
It follows that
\[
\gamma_X(u) \otimes \gamma_X(v) = (id_{X_s} \otimes \gamma_X(v)) \gamma_X(u) \otimes id_{X_t} = \gamma_X(u) \otimes \gamma_X(v)
\]
From this, by the functoriality of $\gamma(u)$, we have
\[
i_{X_sX_t, X}^{-1}(sv + ut) = \gamma_X(sv + ut)
\]
So we have the relation $(s, u) \otimes (t, v) = (st, sv + ut)$.

Proposition 5.2. $\text{Aut}_S(0)$ is a ring with the null multiplication.

Proof. Indeed, it follows from the second relation of Proposition 5.1 that
\[
(0, u) \otimes (0, v) = (0, 0)
\]

In order to perform the Ann-category structure conversion of $A$ to $S$, we construct the isofunctors $\hat{H}, H$ of the functor $H : S \to A$ base on the concept stick defined as follows

Definition 5.3. Let $A$ be an Ann-category. A stick in $A$ consists of a representatives $(X_s), s \in \Pi_0(A)$ such that $X_0 = 0, X_1 = 1$ and the isomorphisms
\[
\varphi_{s,t} : X_s \oplus X_t \to X_{s+t}, \quad \psi_{s,t} : X_sX_t \to X_{st}
\]
for all $s, t \in \Pi_0(A)$ satisfying
\[
\varphi_{0,t} = g_{X_t}, \quad \varphi_{s,0} = d_{X_s} \\
\psi_{1,t} = 1_{X_t}, \quad \psi_{s,1} = r_{X_s}, \quad \psi_{0,t} = R_{X_t}, \quad \psi_{s,0} = L_{X_s}
\]
The conditions (5.2) for the isomorphisms $\varphi_{s,t}$ and $\psi_{s,t}$ are completely satisfied since $g_0 = d_0, l_1 = r_1$ and $\tilde{r} = r_0, \tilde{L} = l_0$ (according to the definition of the isofunctors $g, d, l, r$ and Proposition 3.3).

For two operations $\otimes, \oplus$ on $\mathcal{S}$, the unitality constraints are chosen, respectively, as $0, (0, id, id)$ and $(1, id, id)$. Then, by setting $\varphi = (\varphi_{s,t})$ and $\psi = (\psi_{s,t})$ we have

**Proposition 5.4.** Let $(X_0, \varphi, \psi)$ be a stick in Ann-category $\mathcal{A}$. Then $(\tilde{H}, \tilde{H} = \varphi^{-1}) : \mathcal{A} \rightarrow \mathcal{S}$ is an $\otimes$-functor which is compatible with the constraints $(0, 0, id, id)$ and $(0, g, d)$ and $(\tilde{H}, \tilde{H} = \psi^{-1}) : \mathcal{S} \rightarrow \mathcal{A}$ is an $\otimes$-functor which is compatible with the unitality constraints $(1, 0, id, id)$ and $(1, l, r)$. It follows that $\mathcal{S}$ is an Ann-category with the constraints induced by those in $\mathcal{A}$ by $\tilde{H} = \varphi^{-1}, \tilde{H} = \psi^{-1}$, and $(\tilde{H}, \tilde{H}, \tilde{H})$ is an Ann-equivalence. We call $\mathcal{S}$ a reduced Ann-category of $\mathcal{A}$ and $(\tilde{H}, \tilde{H}, \tilde{H})$ is an canonical Ann-equivalence.

**Proof.** According to the proof of Proposition 5.1, we can prove that the isomorphisms $\tilde{H} = \varphi^{-1}$ and $\tilde{H} = \psi^{-1}$ are natural. Obviously, $(\tilde{H}, \tilde{H})$ is compatible with $(0, 0, id, id)$ and $(0, g, d)$ by the definition of stick (here $\tilde{H} : H(0) \rightarrow 0$ is the identity). We also deduce analogously for $(\tilde{H}, \tilde{H})$. The functor $\tilde{H}$ and the isomorphisms $\tilde{H}, \tilde{H}$ satisfy the conditions of the structure conversion. Thus $\mathcal{S}$ becomes an Ann-category with the structure induced by $\tilde{H}, \tilde{H}, \tilde{H}$, and $(\tilde{H}, \tilde{H}, \tilde{H})$ becomes an Ann-equivalence. □

The precise determination of the induced constraints of the reduced Ann-category $\mathcal{S}$ and their properties are presented, respectively, in the propositions 5.5 - 5.8 as follows

**Proposition 5.5.** Let $a_+, c$ be the associativity and commutativity constraint of the operation $\oplus$ of Ann-category $\mathcal{A}$. Then, the associativity constraint $\xi = H^*(a_+)$ and the commutativity constraint $\eta = H^*(c)$ induced on $\mathcal{S}$, respectively, determined by the commutative diagram

$$
\begin{array}{ccccc}
X_r \oplus (X_r \oplus X_t) & \xrightarrow{\varphi_{r,s,t}} & X_r \oplus X_{r+s+t} & \xrightarrow{\varphi_{r+s,t}} & X_{r+s+t} \\
X_r \oplus X_s & \xrightarrow{\varphi_{s,t}} & X_{r+s} & \xrightarrow{\varphi_{r+s,t}} & X_{r+s+t}
\end{array}
$$

are two functions whose values in $\Pi_1(\mathcal{A})$ and satisfy the equalities:

$$
\begin{align*}
\xi(s, t, u) - \xi(r, t, s) + \xi(r, s + t, u) - \xi(r, s, t + u) + \xi(r, s, t) &= 0 \\
\xi(0, s, t) = \xi(r, 0, t) = \xi(r, s, 0) &= 0 \\
\eta(r, s) + \eta(r, s) &= 0 \\
\xi(r, s, t) - \xi(r, t, s) &+ \xi(t, r, s) + \eta(r + s, t) - \eta(r, t) - \eta(s, t) &= 0
\end{align*}
$$

and hence $\eta(0, s) = \eta(r, 0) = 0$.

**Proof.** By the structure conversion, we have $(\mathcal{S}, \oplus)$ is a symmetric monoidal category. Thus, the axioms of this category give us the above equalities. □
Proposition 5.6. The reduced Ann-category $S$ has the associativity constraint inducing $\alpha = H^*(a)$ to the operation $\otimes$, determined by the commutative diagram

$$
\begin{array}{c}
X_r(X_sX_t) \xrightarrow{id \otimes \psi_{r,t}} X_rX_{st} \xrightarrow{\psi_{r,st}} X_{rst} \\
\downarrow \alpha \\
(X_rX_sX_t) \xrightarrow{\psi_{r,s} \otimes \psi_{r,t}} X_{rs}X_{st} \xrightarrow{\gamma_{X_{rst}}(\alpha(r,s,t))} X_{rst}
\end{array}
$$

is a function $\pi_0(A)^3 \to \Pi_1(A)$ satisfying the relations:

$$
\begin{align*}
ra(s, t, u) - \alpha(rs, t, u) + \alpha(r, st, u) - \alpha(r, s, tu) + \alpha(r, s, t)u &= 0 \\
\alpha(1, s, t) &= \alpha(r, 1, t) = \alpha(r, s, 1) = 0 \\
\alpha(0, s, t) &= \alpha(r, 0, t) = \alpha(r, s, 0) = 0
\end{align*}
$$

Proof. By the structure conversion, $(S, \otimes)$ is a monoidal category with the strict unitivity constraint so from the compatibility of $a$ with $(1, id, id)$ we obtain $\alpha(1, s, t) = \alpha(r, 1, t) = \alpha(r, s, 1) = 0$. The first relation follows from the Pentagon Axiom. Finally, we have to prove

$$
\alpha(0, s, t) = \alpha(r, 0, t) = \alpha(r, s, 0) = 0
$$

- For $r = 0$, we consider the following diagram

$$
\begin{array}{c}
0(X_rX_t) \xrightarrow{id \otimes \psi_{r,t}} 0X_{st} \xrightarrow{\gamma_{0}(\alpha(0,s,t))} 0X_{rst} \\
\downarrow \alpha \\
0(X_rX_t)X_t \xrightarrow{\psi_{0} \otimes \psi_{r,t}} 0X_{t} \xrightarrow{\gamma_{0}(\alpha(0,s,t))} 0X_{rst}
\end{array}
$$

In this diagram, the region (II) commutes by the properties of the isomorphisms $R^A$; the outside region commutes by the determination of $\alpha$. It follows that the region (I) commutes. Since the properties of the isomorphisms $R^A$ (Lemma 3.3), which yields $\gamma_{0}(\alpha(0, s, t)) = id$, i.e., $\alpha(0, s, t) = 0$.

- For $s = 0$ and $t = 0$ we, respectively, consider the following diagrams

$$
\begin{array}{c}
X_r(0X_t) \xrightarrow{id \otimes \psi_{r,t}} X_r0 \xrightarrow{\gamma_{0}(\alpha(r,0,t))} 0X_{rst} \\
\downarrow \alpha \\
(0X_t)X_t \xrightarrow{\psi_{0} \otimes \psi_{r,t}} 0X_{t} \xrightarrow{\gamma_{0}(\alpha(r,0,t))} 0X_{rst}
\end{array}
$$

and also applying the properties of the isomorphisms $L^A, R^A$ (Lemma 3.3), we obtain

$$
\alpha(r, 0, t) = \alpha(r, s, 0) = 0
$$
Proposition 5.7. Let \( L, R \) be the distributivity constraints of the Ann-category \( A \). Then the inducing distributivity constraints \( \lambda = H^*(L), \rho H^*(R) \) defined by the commutative diagrams

\[
\begin{align*}
X_r(X_s \oplus X_t) & \xrightarrow{id \oplus \phi_{s,t}} X_r X_s + X_r X_t & \xrightarrow{\psi_{r,s,t}} & \gamma_{X_r X_s + X_r X_t}^{\lambda(r,s,t)} \\
(X_r X_s) & \oplus (X_r X_t) \xrightarrow{\psi_{r,s,t} \oplus \phi_{r,s}} X_r X_s \oplus X_r X_t & \xrightarrow{\psi_{r,s,t}} & \gamma_{X_r X_s \oplus X_r X_t}^{\lambda(r,s,t)} \\
(X_r \oplus X_s) X_t & \xrightarrow{\psi_{r,s,t} \oplus \phi_{r,s}} X_r X_t \oplus X_s X_t & \xrightarrow{\psi_{r,s,t}} & \gamma_{X_r X_t \oplus X_s X_t}^{\lambda(r,s,t)} \\
(X_r \oplus X_s) & \oplus (X_r \oplus X_s) X_t \xrightarrow{\psi_{r,s,t} \oplus \phi_{r,s}} X_r X_t \oplus X_r X_t & \xrightarrow{\psi_{r,s,t}} & \gamma_{X_r X_t \oplus X_r X_t}^{\lambda(r,s,t)}
\end{align*}
\]

are the functions satisfying the properties

\[
\begin{align*}
\lambda(1, s, t) &= \lambda(0, s, t) = \lambda(0, 0, t) = \lambda(r, s, 0) = 0 \\
\rho(0, s, t) &= \rho(0, r, t) = \rho(r, s, 0) = \rho(r, s, 1) = 0
\end{align*}
\]

Proof. The pair \((\lambda, \rho)\) defined as above is just the distributivity constraints induced by \((H, \tilde{H}, \tilde{H})\) and \(L, R\) by the diagrams (3.1a), (3.1b). Since \(S\) is an Ann-category whose strict univity constraint to the operation \(\otimes\), it follows from the axiom (Ann-3) that \(\lambda(1, s, t) = \rho(0, s, t) = 0\). For \(r = 0\), we consider the diagram

\[
\begin{array}{ccc}
0(X_s \oplus X_t) & \xrightarrow{id \oplus \phi_{s,t}} & 0X_s + 0X_t \\
& \xrightarrow{\psi_{r,s,t}} & \gamma_0(0(s,t)) \\
& \xrightarrow{\phi_{s,t} \oplus \phi_{0,s}} & \gamma_0(0(s,t)) \\
0X_s \oplus 0X_t & \xrightarrow{\psi_{r,s,t} \oplus \phi_{0,s}} & \gamma_0(0(s,t))
\end{array}
\]

where the outside region commutes by the determination of the inducing isomorphism \(\lambda\); the region (I) commutes by the properties of the isomorphisms \(R^A\). It follows that the region (II) commutes. Then, since the properties of \(R^A\), we have \(\gamma_0(\lambda(0, s, t)) = id\). Since \(\gamma_0\) is an isomorphisms, \(\lambda(0, s, t) = 0\).

For \(s = 0\), we consider the diagram

\[
\begin{array}{ccc}
X_r(0 \oplus X_t) & \xrightarrow{id \oplus \psi_{r,t}} & X_r X_t \\
& \xrightarrow{\psi_{r,s,t}} & \gamma_{X_r X_t}(\lambda(r,s,t)) \\
& \xrightarrow{\phi_{r,s,t} \oplus \psi_{r,s,t}} & \gamma_{X_r X_t}(\lambda(r,s,t)) \\
X_r 0 \oplus X_r X_t & \xrightarrow{\phi_{r,s,t} \oplus \psi_{r,s,t}} & X_r X_t
\end{array}
\]

where the outside region commutes by the determination of the inducing isomorphism \(\lambda\); the region (I) commutes by the compatibility of the distributivity constraint \(L\) with the univity constraint \((0, g, d)\); the region (II) commutes by the composition law of morphisms.
It follows that the region (III) commutes. But by the functoriality of $g$, this means that $\gamma_{X_{S}}(\lambda(r,0,t)) = \text{id}$. Since $\gamma_{X_{S}}$ is an isomorphism, $\lambda(r,0,t) = 0$.

Similarly, we have $\lambda(r,s,0) = 0$ and $\rho(0,s,t) = \rho(r,0,t) = \rho(r,s,0)$. □

The above results suggest us to state the following proposition

**Proposition 5.8.** In the reduced Ann-category $S = (S, \xi, \eta, (0, id, id), \alpha, (1, id, id), \lambda, \rho)$, the constraints $\xi, \eta, \alpha, \lambda, \rho$ are the functions whose values in $\Pi_{1}(A)$ and which satisfy the following relations for all $x,y,z,t \in \Pi_{0}(A)$.

$$S1. \quad \xi(x, y, z, t) - \xi(x + y, z, t) + \xi(x, y + z, t) - \xi(x, y + z + t) + \xi(x, y, z) = 0$$
$$S2. \quad \xi(x, y, z) - \xi(x, z, y) + \xi(z, x, y) + \eta(x + y, z) - \eta(x, z) = 0$$
$$S3. \quad \eta(x, y) + \eta(y, x) = 0$$
$$S4. \quad x\eta(y, z) - \eta(xy, xz) = \lambda(x, y, z) - \lambda(x, z, y)$$
$$S5. \quad x\eta(y, z) - \eta(xz, yz) = \rho(x, y, z) - \rho(y, x, z)$$
$$S6. \quad x\xi(y, z, t) - \xi(xy, xz, xt) = \lambda(x, y, z) - \lambda(x, y + z, t) + \lambda(x, y + z + t) - \lambda(x, y, z)$$
$$S7. \quad \xi(x, y, z) - \xi(xt, yt, zt) = \rho(y, z, t) - \rho(x + y, z, t) + \rho(x, y + z, t) - \rho(x, y, z)$$
$$S8. \quad \rho(x, y, z + t) - \rho(x, y, z) - \rho(x, y, t) + \lambda(x, y, z) + \lambda(y, z, t) - \lambda(x + y, z, t) = \xi(x, y, z + t) - \xi(xy, xz, yt) + \xi(xz, yt, zt)$$
$$S9. \quad \alpha(x, y, z + t) - \alpha(x, y, z) = -\alpha(x, y, t) = x\lambda(y, z, t) + \lambda(xy, xz, yt) - \lambda(xy, zt)$$
$$S10. \quad \alpha(x, y, z + t) - \alpha(x, y, t) - \alpha(x, z, t) = x\rho(y, z, t) - \rho(xy, xz, t) + \lambda(xy, zt, xt) - \lambda(xy, zt)$$
$$S11. \quad \alpha(x + y, z, t) - \alpha(x, y, t) - \alpha(y, z, t) = -\rho(x, y, z) - \rho(xy, xy, z) - \rho(xy, zt)$$
$$S12. \quad x\alpha(y, z, t) - \alpha(xy, xz, t) + \alpha(x, yz, t) - \alpha(x, z, t) + \alpha(x, y, z) = 0$$

and satisfy "normalization" conditions:

$$\xi(0, y, z) = \xi(x, 0, z) = \xi(x, y, 0) = 0$$
$$\alpha(0, y, z) = \alpha(1, y, z) = \alpha(x, 1, z) = \alpha(x, y, 1) = 0$$
$$\alpha(0, y, z) = \alpha(x, 0, z) = \alpha(x, y, 0) = 0$$
$$\lambda(0, y, z) = \lambda(1, y, z) = \lambda(0, 0, z) = \lambda(x, y, 0) = 0$$
$$\rho(x, 0, 1) = \rho(0, 0, y) = \rho(0, s, t) = \rho(r, 0, t) = \rho(r, s, 0).$$

*Proof.* The relations 1 - 3, 12 and "normalization" conditions were just proved right in the Propositions 5.5 - 5.7. The remaining relations follow directly from the axioms of Ann-categories. (Here, the order of the relations as above differs from their order in the definition but it is convenient for the later use). □

Thus, from a stick $(X_{S}, \varphi, \psi)$ of Ann-category $A$, we establish a reduced Ann-category $S$ which is Ann-equivalent to $A$. We have known that the operations induced on $S$ do not depend on the selection of the stick. We now consider the effect of different choices of sticks in the inducing constraints on $S$. For convenience, we also call $(\xi, \eta, \alpha, \lambda, \rho)$ a family of constraints of Ann-category $S$. 
Proposition 5.9. If S and S' are two reduced Ann-categories of A corresponding to two sticks \((X_s, \varphi, \psi), (X'_s, \varphi', \psi')\) then

1. There exists an Ann-equivalence \((F, \hat{F}, \hat{F}) : S' \rightarrow S\), where \(F = \text{id}\)

2. Two families of the constraints \((\xi, \eta, \alpha, \lambda, \rho)\) and \((\xi', \eta', \alpha', \lambda', \rho')\), respectively, of \(S\) and \(S'\) satisfy the following relations

\[
\begin{align*}
\xi'(x, y, z) - \xi(x, y, z) &= \mu(y, z) - \mu(x + y, z) + \mu(x, y, z) - \mu(x, y) \\
\eta'(x, y) - \eta(y, x) &= \mu(x, y) - \mu(y, x) \\
\alpha'(x, y, z) - \alpha(x, y, z) &= x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z \\
\lambda'(x, y, z) - \lambda(x, y, z) &= \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\nu(y, z) - \mu(xy, xz) \\
\rho'(x, y, z) - \rho(x, y, z) &= \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \mu(x, y)z - \mu(x, yz)
\end{align*}
\]

where \(\mu, \nu : \Pi_0(A)^2 \rightarrow \Pi_1(A)\) are two functions satisfying the conditions \(\mu(0, y) = \mu(x, 0) = 0\) and \(\nu(0, y) = \nu(x, o) = \nu(1, y) = \nu(x, 1) = 0\).

Proof. 1. According to Proposition 5.4, \((H, \varphi^{-1}, \psi^{-1})\) and \((H', \varphi'^{-1}, \psi'^{-1})\) are two canonical Ann-equivalence corresponding to \(S, S'\). Then there exits an Ann-equivalence \(K, \hat{K}, \hat{K} : A \rightarrow S\) which carries each object \(X\) to the class consisting of \(X\). Setting \(F = KH', \hat{F} = KH', \hat{F} : S' \rightarrow S\) is an Ann-equivalence (according to II.1.2.[9]) and \(F = \text{id}\).

2. Setting \(\mu = \hat{K}H'\), \(\nu = \hat{K}H'\). Then, from the compatibility of Ann-functor \((F = \text{id}, \mu, \nu)\) with the corresponding constraints of \(S'\) and \(S\), we obtain the above relations. Moreover, we can verify that \(\mu(o, y) = \nu(x, 1) = 0\) by the “normality” of the functions \(\xi, \eta, \alpha, \lambda, \rho\).

Proposition 5.10. In Proposition 5.9, we can choose the stick \((X'_s, \varphi', \psi')\) instead of the stick \((X_s, \varphi, \psi)\) such that \(\mu, \nu : \Pi_0(A)^2 \rightarrow \Pi_1(A)\) are two arbitrary functions satisfying the conditions \(\mu(x, o) = \mu(o, y) = 0\) and \(\nu(x, 0) = \nu(0, y) = \nu(x, 1) = \nu(1, y) = 0\).

Proof. Suppose that \(\mu, \nu\) are two functions mentioned in the proposition. After choosing the representatives \((X'_s)\) such that \(X'_0 = 0, X'_1 = 1\), we may construct the function \(H' : S' \rightarrow A\). Then \(\hat{H}', \hat{H}'\) are chosen such that \(\hat{K}H' = \mu\) and \(\hat{K}H' = \nu\), where \(K, \hat{K}, \hat{K}\) is mentioned in the proof of Proposition 5.9. It follows that \(\varphi' = (\hat{H}')^{-1}, \psi' = (\hat{H}')^{-1}\).

Note that in an Ann-category of the type \((R, M)\), two unitiy constraints of \(\oplus\) and \(\otimes\) are both strict, we have the following definition

Definition 5.11. Two structures \((\xi, \eta, \alpha, \lambda, \rho)\) and \((\xi', \eta', \alpha', \lambda', \rho')\) of Ann-category of the type \((R, M)\) are cohomologous iff they satisfy the relations in Proposition 5.9. Clearly, we have

Lemma 5.12. Two structures \(f, f'\) are cohomologous iff they are two families of constraints of an Ann-category of the type \((R, M)\) which are compatible with each other to the Ann-functor \((F, \hat{F}, \hat{F}) : (R, M) \rightarrow (R, M)\), where \(F = \text{id}_A\). We now present the third invariant of Ann-categories.
Pre-sticked Ann-categories of the type \((R, M)\)

We may simplify the problem of equivalence classification of Ann-categories by the classification of Ann-categories which have the same (up to an isomorphism) the first two invariants. The results in this section were first presented in [9].

**Definition 6.1.** Let \(R\) be a ring with the unit, \(M\) be an \(R\)-bimodule and be a ring with the null multiplication. We say that an Ann-category \(A\) has a pre-stick of the type \((R, M)\) if there exists a pair of ring isomorphisms \(\epsilon = (\epsilon_0, \epsilon_1)\)

\[
\begin{align*}
\epsilon_0 &: R \to \Pi_0(A), \\
\epsilon_1 &: M \to \Pi_1(A)
\end{align*}
\]

which is compatible with the module actions, i.e.,

\[
\epsilon_1(su) = \epsilon_0(s).\epsilon_1(u), \quad \epsilon_1(us) = \epsilon_1(u).\epsilon_0(s)
\]

for \(s \in R, u \in M\). The pair \(\epsilon = (\epsilon_0, \epsilon_1)\) is called a pre-stick of the type \((R, M)\) to the Ann-category \(A\).

A morphism between two Ann-categories \(A, A'\), whose pre-sticks are of the type \((R, M)\) (with, respectively, the pre-sticks \(\epsilon, \epsilon'\) ) is an Ann-functor \((F, \tilde{F}, \tilde{F}) : A \to A'\) such that the following triangles

\[
\begin{array}{ccc}
\Pi_0(A) & \xrightarrow{F} & \Pi_0(A') \\
\downarrow{\epsilon_0} & & \downarrow{\epsilon_0'} \\
R & \xrightarrow{\epsilon_1} & M
\end{array} \quad \begin{array}{ccc}
\Pi_1(A) & \xrightarrow{\tilde{F}} & \Pi_1(A') \\
\downarrow{\epsilon_1} & & \downarrow{\epsilon_1'} \\
M & \xrightarrow{\tilde{F}} & R
\end{array} \tag{6.1}
\]

commute, where \(F\) and \(\tilde{F}\) are ring homomorphisms induced by \((F, \tilde{F}, \tilde{F})\). Clearly, \(F\) and \(\tilde{F}\) are ring isomorphisms and thus \(F\) is an Ann-equivalence.

**Theorem 6.2.** Each congruence class of an Ann-category pre-sticked by \((R, M)\) defines uniquely a cohomological class \(k = (\xi, \eta, \alpha, \lambda, \rho)\) of structures of Ann-categories of the type \((R, M)\), and thus \(k\) is the third invariant of \(A\).

Proof. Suppose that \(A\) is an Ann-category whose pre-stick is of the type \((R, M)\). In \(A\), we choose a stick \((X_\epsilon, \varphi, \psi)\). Then, this stick induces a reduced Ann-category \(S\) together with the Ann-equivalences \((G, \tilde{G}, \tilde{G})\) as in Proposition 5.4. Assume that \(\epsilon = (\epsilon_0, \epsilon_1)\) is a pre-stick of the type \((R, M)\) of \(A\). Let \(\mathcal{I}\) denote the category built from the ring \(R\) and \(R\)-bimodule \(M\), which is similar with the reduced Ann-category when we replace \(\Pi_0(A), \Pi_1(A)\) with, respectively, \(R, M\). Then, since \(\epsilon : \mathcal{I} \to S\) is an isomorphism, we can perform the structure conversion from \(S\) to \(\mathcal{I}\) so that \(\mathcal{I}\) becomes an Ann-category where

\[
(\epsilon, id, id) : \mathcal{I}(R, M) \to S
\]

is an Ann-equivalence. Therefore, we obtain a structure \((\xi, \eta, \alpha, \lambda, \rho)\) - a family of constraints of \(\mathcal{I}\) induced by the family of constraints of \(S\).

Now, if we choose in \(A\) an other stick \((X'_\epsilon, \varphi', \psi')\), we will obtain a corresponding structure \((\xi', \eta', \alpha', \lambda', \rho')\) of \(\mathcal{I}\) in the same way. Then, from Proposition 5.9 and the isomorphism property of \(\epsilon\), it is easy to see that the structures \((\xi, \eta, \alpha, \lambda, \rho)\) and \((\xi', \eta', \alpha', \lambda', \rho')\) are cohomologous.
Assume that $\mathcal{A}$ and $\mathcal{A}'$ are pre-sticked Ann-categories of the type $(R, M)$ and equivalent, i.e., there exists an Ann-equivalence $(F, \tilde{F}, \tilde{F}) : \mathcal{A} \to \mathcal{A}'$ such that the diagrams (6.1) commute. In $\mathcal{A}$, we choose an arbitrary stick $(X_s, \varphi, \psi)$ and from that, we obtain the structure $(\xi, \eta, \alpha, \lambda, \rho) = f$ via the canonical Ann-equivalences:

$$
\mathcal{T}(R, M) \xrightarrow{(\epsilon, \text{id}, \text{id})} \mathcal{S} \xrightarrow{(H, H', \tilde{H})} \mathcal{A}
$$

Similarly, in $\mathcal{A}'$ we choose the stick $(X'_s, \varphi', \psi')$ and obtain the structure $(\xi', \eta', \alpha', \lambda', \rho') = f'$ via the canonical Ann-equivalences:

$$
\mathcal{T}(R, M) \xrightarrow{(\epsilon', \text{id}, \text{id})} \mathcal{S}' \xrightarrow{(H', H', \tilde{H}') \mathcal{A}'}
$$

Ann-equivalence $(F, \tilde{F}, \tilde{F})$ induces the pair of ring isomorphisms $(\mathcal{T}, \mathcal{F})^0$; this pair is regarded as an Ann-functor $\mathcal{S} \to \mathcal{S}'$ making the diagram

$$
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{(\mathcal{T}, \mathcal{F})^0} & \mathcal{S}' \\
\mathcal{I} \downarrow & & \downarrow \mathcal{I}' \\
\mathcal{I} & & \mathcal{I}'
\end{array}
$$

commute.

Like in the proof of Proposition 5.4, there exists an Ann-equivalence $(G, \tilde{G}', \tilde{G}') : \mathcal{A}' \to \mathcal{S}'$ such that $G'H' \simeq \text{id}$. Then, the following diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\
\mathcal{S} & \xleftarrow{(\mathcal{T}, \mathcal{F})^0} & \mathcal{S}' \\
\mathcal{I} \uparrow & & \uparrow \mathcal{I}' \\
\mathcal{H} & & \mathcal{G} \\
\mathcal{S} & \xrightarrow{(\mathcal{T}, \mathcal{F})^0} & \mathcal{S}'
\end{array}
$$

commutes.

Set $K = \epsilon'.G'.H.H.\epsilon$, then $(K, \tilde{K}, \tilde{K}) : \mathcal{I} \to \mathcal{I}$ is an Ann-equivalence which is compatible with the constraints $f, f'$. Since $K = \text{id}$, $f$ and $f'$ are cohomologous structures. 

**Theorem 6.3.** (Theorem 3.4[11]) There exists a bijection

$$
\Phi : C(R, M) \to S(R, M)
$$

between the set of congruence classes of the pre-sticked Ann-categories of the type $(R, M)$ and the set of cohomologous classes of the structures of Ann-categories of the type $(R, M)$.

**Proof.** Let $\Phi$ be the function which maps each congruence class of Ann-category $\mathcal{A}$ to the invariant $\tilde{K}$. Let $\mathcal{A}, \mathcal{A}'$ be pre-sticked Ann-categories of the type $(R, M)$ whose corresponding structures $f$ and $f'$ are cohomologous. Then, there exists an Ann-functor

$$(K, \tilde{K}, \tilde{K}) : \mathcal{I}(R, M) \to \mathcal{I}(R, M) \text{ (where $K = \text{id}$)}$$

24
which is compatible with the family of constraints \((\xi, \eta, \alpha, \lambda, \rho), (\xi', \eta', \alpha', \lambda', \rho')\). Then, we consider Ann-functor of composition

\[ F = H' \cdot e', K \cdot e^{-1} \cdot G : \mathcal{A} \rightarrow \mathcal{A}' \]

where \((G, \tilde{G}, \tilde{G}) : \mathcal{A} \rightarrow \mathcal{S}\) is an Ann-equivalence such that \(G \cdot H \simeq \text{id}\). Clearly, \((F, \tilde{F}, \tilde{F})\) is an Ann-equivalence and the pair of induced ring isomorphisms \((\tilde{F}, \tilde{F}) : \mathcal{S} \rightarrow \mathcal{S}'\) makes the diagrams (1) commute. It shows that \((F, \tilde{F}, \tilde{F})\) is a morphism and \(\mathcal{A}, \mathcal{A}'\) belong to the same congruence class, and thus \(\Phi\) is an injection.

Now, if \(\mathcal{K} = (\xi, \eta, \alpha, \lambda, \rho)\) belongs to \(\mathcal{S}(R, M)\), then \(\mathcal{T}(R, M)\) will be an Ann-category with the family of constraints \((\xi, (0, \text{id}, \text{id}), \eta, \alpha, (1, \text{id}, \text{id}), \lambda, \rho)\). This shows that \(\Phi\) is a surjection. 

\[\square\]

7 Classification theorems

Let us recall that an Ann-category \(\mathcal{A}\) is regular if its commutativity constraint satisfies \(c_{X,X} = \text{id}\) for all \(X \in \mathcal{A}\). In [9], [11], authors showed that each structure \((\xi, \eta, \alpha, \lambda, \rho)\) of the regular Ann-category \(\mathcal{A}\) satisfies \(\eta(x, x) = 0\), and therefore becomes a Shukla 3-cocycle of \(\mathbb{Z}\)-algebra \(R\), with coefficients in \(R\)-bimodule \(M\). Therefore, from Theorem 7, we have

**Theorem 7.1.** (Theorem 4.4[11]) There exists a bijection between \(\text{Cr}(R, M)\) - the set of congruence classes of pre-sticked regular Ann-categories of the type \((R, M)\) and the Shukla cohomology group \(H^3_{\text{Sh}}(R, M)\).

We now use the cohomology group \(H^3_{\text{Mal}}(R, M)\) due to MacLane to prove the Classification Theorem in the general case, Theorem 7.6.

From the definition of ring cohomology of MacLane [6], we may obtain the description of the elements of \(H^3_{\text{Mal}}(R, M)\).

The group \(Z^3_{\text{Mal}}(R, M)\) of 3-cocycles of \(R\), with coefficients in \(M\), consists of the quadruples \((\sigma, \alpha, \lambda, \rho)\) of the maps:

\[\sigma : R^4 \rightarrow M; \quad \alpha, \lambda, \rho : R^3 \rightarrow M\]

satisfying the following conditions:

\[\begin{align*}
M1. \quad & xa(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, z)t = 0 \\
M2. \quad & -\alpha(x, z, t) + \alpha(y, z, t) + \alpha(x + y, z, t) + \alpha(xyz, t) - \rho(x, y, zt) + \rho(x, y, z)t = 0 \\
M3. \quad & -\alpha(x, y, t) + \alpha(x, yz, t) + \alpha(x, y + z, t) + \alpha(xyz, t) - \rho(x, y, zt) = 0 \\
M4. \quad & -\alpha(x, y, t) + \alpha(x, yz, t) + \alpha(x, y, z + t) + x\lambda(y, z, t) - \lambda(xy, z, t) + \lambda(xy, z, t) = 0 \\
M5. \quad & \lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) - \rho(x, y, z) - \rho(x, y, t) \\
\text{and } & +\rho(x, y, z + t) + \sigma(xz, xt, yz, yt) = 0 \\
M6. \quad & \lambda(x, a, b) + \lambda(x, c, d) - \lambda(x, a + c, b + d) - \lambda(x, a, c) - \lambda(x, b, d) \\
\text{and } & +\lambda(x, a + b, c + d) - x\sigma(a, b, c, d) + \sigma(xa, xb, xc, xd) = 0 \\
M7. \quad & -\rho(a, b, x) - \rho(c, d, x) + \rho(a + c, b + d, x) + \rho(a, c, x) + \rho(b, d, x) \\
\text{and } & -\rho(a + b, c + d, x) - \sigma(ax, bx, cx, dx) + \sigma(a, b, c, d)x = 0
\end{align*}\]
M8. \(-\sigma(a, b, c, d) - \sigma(x, y, z, t) + \sigma(a + x, b + y, c + z, d + t) + \sigma(a, b, x, y) + \sigma(c, d, z, t) - \sigma(a + c, b + d, x + z, y + t) - \sigma(a, c, x, z) - \sigma(b, d, y, t) + \sigma(a + b, c + d, x + y, z + t) = 0\)

and these 4 functions satisfy canonical conditions:

\[
\begin{align*}
\alpha(0, y, z) &= \alpha(x, 0, z) = \alpha(x, y, 0) = 0 \\
\lambda(0, y, z) &= \lambda(x, 0, z) = \lambda(x, y, 0) = 0 \\
\rho(0, y, z) &= \rho(x, 0, z) = \rho(x, y, 0) = 0 \\
\sigma(a, b, 0, 0) &= \sigma(0, 0, c, d) = \sigma(a, 0, c, 0) = \sigma(0, b, 0, d) = \sigma(a, 0, 0, d) = 0.
\end{align*}
\]

3-cocycle \( h = (\sigma, \alpha, \lambda, \rho) \) belongs to \( B^3_{\text{Mal}}(R, M) \) iff there exist \( \mu, \nu : R^2 \to M \) satisfying:

M9. \( \sigma(x, y, z, t) = \mu(x, y) + \mu(z, t) - \mu(x + z, y + t) - \mu(x, z) - \mu(y, t) + \mu(x + y, z + t) \)

M10. \( \alpha(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, yz) \)

M11. \( \lambda(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\mu(y, z) - \mu(xy, xz) \)

M12. \( \rho(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \mu(x, y)z - \mu(xz, yz) \)

Finally,
\[
H^3_{\text{Mal}}(R, M) = Z^3_{\text{Mal}}(R, M)/B^3_{\text{Mal}}(R, M)
\]

Now we assume that \( A \) is a pre-sticked Ann-category of the type \((R, M)\) and \((\xi, \eta, \alpha, \lambda, \rho)\) is one of its structures. We define a function \( \sigma : R^4 \to M \) given by

\[ \sigma(x, y, z, t) = \xi(x + y, z, t) - \xi(x, y, z) + \eta(y, z) + \xi(x, z, y) - \xi(x + z, y, t) \]

This equality shows that \( \sigma \) is just the morphism \( v \)
\[
v : (x + y) + (z + t) \to (x + z) + (y + t)
\]
in an Ann-category of the type \((R, M)\).

**Proposition 7.2.** The quadruple \((\sigma, \alpha, \lambda, \rho)\), induced by the structure \((\xi, \eta, \alpha, \lambda, \rho)\), belongs to \( Z^3_{\text{Mal}}(R, M) \).

**Proof.** First, the normalization of \( \sigma \) follows from the normalization of \( \xi, \eta \)

\[ \sigma(0, 0, z, t) = \sigma(x, y, 0, 0) = \sigma(0, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0. \]

We will show that the quadruple \((\sigma, \alpha, \lambda, \rho)\) satisfy the relations M1 - M8. Relation M1 is just the relation S12 in Theorem 5.8. The relations M2, M3, M4, M5 are, respectively, S11, S10, S9, S8 in Theorem 5.8.

According to the Coherence Theorem, in Ann-categories of the type \((R, M)\), the following diagrams 1, 2 commute. Thus the relations M6, M9 follows. M7 is decuced from the commutative diagram which is similar with the diagram 1, where \( s \) tensored on the right.
The remaining relations of a 3-cocycle are deduced directly from the remaining axiomatics of Ann-categories.

\[ s(x + y) + s(z + t) \]  
\[ \xi \]  
\[ s(x + y) + s(z + t) \]  
\[ s(x + z) + s(y + t) \]  
\[ \xi \otimes \xi \]  
\[ (sx + sy) + (sz + st) \]  
\[ [(a + b) + (c + d)] + [(x + z) + (y + t)] \]  
\[ \nu \]  
\[ [(a + b) + (x + y)] + [(c + d) + (z + t)] \]  
\[ v + v \]  
\[ [(a + x) + (b + y)] + [(c + z) + (d + t)] \]  
\[ v + v \]  
\[ [(a + c) + (x + z)] + [(b + d) + (y + t)] \]  
\[ \nu + v \]  
\[ [(a + x) + (c + z)] + [(b + y) + (d + t)] \]  
\[ \nu + v \]

(Diagram 1)

The above relation turns into:

\[ -\xi(y, a, d) + \eta(a, y) - \eta(a + d, y) + \sigma(a, d, y, 0) = 0 \]  
\[ (2) \]

Proposition 7.3. Each Mac Lane 3-cocycle \( h = (\sigma, \alpha, \lambda, \rho) \) is induced by a structure \( f = (\xi, \eta, \alpha, \lambda, \rho) \), of Ann-category of the type \( (R, M) \).

Proof. Let \( h = (\sigma, \alpha, \lambda, \rho) \) be a given element in \( Z^3_{MacL}(R, M) \). Set

\[ \xi(x, y, z) = -\rho(x, y, 0, z), \quad \eta(x, y) = \rho(0, x, y, 0) \]

we obtain a tuple of 5 functions \( f = (\xi, \eta, \alpha, \lambda, \rho) \). The normalization of the functions \( \xi, \eta \) follows from the normalization of \( \sigma \). We now show that \( f \) is a structure of Ann-category of the type \( (R, M) \). First, the relations \( M1 - M4 \) are just \( S12 - S9 \) in the structure of Ann-category of the type \( (R, M) \).

The relation \( S1 \) follows from \( M8 \) when we choose \( c = 0 = x = y = z \).

The relation \( S3 \) follows from \( M8 \) when we choose \( a = b = d = 0 = x = z = t \).

The relations \( S4 \) and \( S5 \), respectively, follow from \( M6 \) and \( M7 \) when we choose \( a = d = 0 \).

The relations \( S6 \) and \( S7 \), respectively, follow from \( M6 \) and \( M7 \) when we choose \( c = 0 \).

We now prove the relation \( S2 \). In \( M8 \), we first choose \( b = c = 0 = x = z = t \), we obtain

\[ -\xi(a, y, d) + \xi(a, d, y) - \eta(d, y) + \sigma(a, d, y, 0) = 0 \]  
\[ (1) \]

Then we choose \( a = c = 0 = y = z = t \), we obtain:

\[ -\xi(x, b, d) + \eta(b, x) - \eta(b + d, x) + \sigma(b, d, x, 0) = 0. \]

The above relation turns into:

\[ -\xi(y, a, d) + \eta(a, y) - \eta(a + d, y) + \sigma(a, d, y, 0) = 0 \]  
\[ (2) \]
Let (1) be subtracted by (2), we obtain:

\[-\xi(a, y, d) + \xi(a, d, y) + \eta(a, a + d, y) - \eta(a, y) - \eta(d, y) = 0.\]

Finally, we prove the relation $S_8$. To do this, we prove that $\sigma$ can be presented via $\xi, \eta$ according to (*). Indeed, in $M_8$, we choose $d = 0 = x = y = z$, we obtain:

$$
\sigma(a, b, c, t) + \xi(a + c, b, t) - \xi(a + b, c, t) - \sigma(a, b, c, 0) = 0 \quad (3).
$$

In (1), set $d = b, y = c$, we obtain:

$$
\xi(a, c, b) - \xi(a, b, c) - \eta(b, c) + \sigma(a, b, c, 0) = 0 \quad (1').
$$

Let (1') be added by (3), and use suitable computation, we obtain (*).

Now, thanks to (*), $M_8$ becomes the relation $S_8$. This means the tuple of 5 functions $f = (\xi, \eta, \alpha, \lambda, \rho)$, is a structure of Ann-category of the type $(R, M)$. Moreover, from (*), this structure induces Mac Lane 3-cocycle $h = (\sigma, \alpha, \lambda, \rho)$. $\square$

**Corollary 7.4.** There exists a bijection from the set of structures of Ann-categories of the type $(R, M)$ to the set of 3-cocycle $H^3_{\text{MaL}}(R, M)$.

**Proof.** The result follows from the proof of Proposition 7.2, 7.3. $\square$

**Proposition 7.5.** There exists a bijection

$$
\Gamma : S(R, M) \rightarrow H^3_{\text{MaL}}(R, M)
$$

from the set of congruence classes of structures of Ann-categories of the type $(R, M)$ to the cohomology group $H^3_{\text{MaL}}(R, M)$.

**Proof.** Indeed, assume that structures $f, f'$ are cohomologous. Then, they are compatible with an Ann-functor $(F = \text{id}, \bar{F}, \tilde{F})$, thus $\alpha - \alpha', \lambda - \lambda', \rho - \rho'$ satisfy M12-M14, where $\bar{F} = \mu, \tilde{F} = \nu$. Besides, the following diagram commutes thanks to coherence property of an symmetric monoidal functor:

$$
\begin{array}{c}
F((a + y) + (b + y)) \\
\downarrow F(a + y) \\
F((a + x) + (b + y)) \\
\downarrow F \\
F(a + x) + F(b + y)
\end{array}
\begin{array}{c}
\bar{F} \\
F(a + y) \\
F(a) + F(b) + (F(x) + F(y)) \\
\tilde{F} \\
(F(a) + F(b)) + (F(x) + F(y))
\end{array}
\begin{array}{c}
\tilde{F} \\
(F(a) + F(x)) + (F(b) + F(y))
\end{array}
$$

(Diagram 3)
Note that $F = id$ and $\tilde{F} = \mu$, the above diagram gives us

$$\sigma'(x, y, z, t) - \sigma(x, y, z, t) = \mu(x + y, z + t) + \mu(x, y) + \mu(z, t) - \mu(x, z) - \mu(y, t)$$

That means $\sigma - \sigma'$ satisfies $M9$. Thus $h, h'$ belong to the same cohomologous class of $H^3_{MaL}(R, M)$, this shows that $\Gamma$ is a function.

Now assume that $\Gamma(\overline{f}) = \overline{h}$, $\Gamma(\overline{f'}) = \overline{h'}$, and $h - h' \in H^3_{MaL}(R, M)$. Then $\alpha - \alpha'$, $\lambda - \lambda'$, $\rho - \rho'$ satisfy $M10 - M12$, and these are just the last 3 relations in Theorem 4.8. Thanks to (*), the definition of $\sigma$ and the normalization property of $\xi, \eta$, we have

$$\xi(x, y, z) = -\sigma(x, 0, y, z) ; \xi'(x, y, z) = -\sigma'(x, 0, y, z)$$
$$\eta(x, y) = \sigma(0, x, y, 0) ; \eta'(x, y) = \sigma'(0, x, y, 0).$$

Therefore, from $M9$, we have the remaining 2 relations in Theorem 4.8, and thus $f, f'$ are cohomologous structures, i.e., $\Gamma$ is an injection.

We now prove that $\Gamma$ is a surjection. According to Proposition 7.3, each element $h = (\sigma, \alpha, \lambda, \rho)$ in $Z^3_{MaL}(R, M)$ is induced by the structure $f = (\xi, \eta, \alpha, \lambda, \rho)$ of the Ann-category of the type $(R, M)$, i.e., $\Gamma(\overline{f}) = \overline{h}$, and therefore $\Gamma$ is a surjection.

**Theorem 7.6.** There exists a bijection

$$C(R, M) \to H^3_{MaL}(R, M)$$

from the set of congruence classes of pre-sticked Ann-categories of the type $(R, M)$ to the cohomology group $H^3_{MaL}(R, M)$ due to Maclane.

**Proof.** It is deduced from Theorem 6.3 and Proposition 7.5. \hfill \Box

**Corollary 7.7.** There exists an injection from $H^3_{Sh}(R, M)$ to $H^3_{MaL}(R, M)$.

**Proof.** It directly follows from Theorems 7.1 and 7.6. \hfill \Box

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