A Causal Approach to Massive Yang-Mills Theories

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Summary - We study quantized Yang-Mills theory with massive vector fields in the framework of causal perturbation theory. The most general form of the interaction which is invariant under operator gauge transformations is pointed out. The generator of these transformations generally fails to be nilpotent. This defect, however, is easily cured by including scalar fields in the gauge transformations. Due to gauge invariance these scalar gauge fields couple to the Yang-Mills fields with predicted strength. We also show that invariance under ghost charge conjugation fixes the form of the interaction completely. The coupling of the Yang-Mills fields and the scalar gauge fields to matter is investigated. It is proven that gauge invariance implies unitarity of the physical $S$-matrix. We always work in the Fock space of free quantum fields in which all expressions are mathematically well defined.

1. Introduction

Recently, massless Yang-Mills theory has been successfully studied in the framework of causal perturbation theory [1-4]. The central object in this approach is the causal $S$-matrix

$$ S[g] = 1 + \sum_{n=1}^{\infty} \int d^4 x_1 \cdots d^4 x_n T^{(n)}(x_1, \ldots, x_n) $$

$T^{(1)}$ specifies the theory. For massless Yang Mills theories it is given by

$$ T^{(1)}(x) \overset{\text{def}}{=} -ie f_{abc} \left\{ \frac{1}{2} : A_{\mu a} A_{\nu b} F_{\mu \nu}^{\mu} : - : A_{\mu a} u_\nu : \partial^\mu \tilde{u}_c : \right\}(x) $$

$e$ is the coupling constant and $f_{abc}$ are the structure constants of a non-abelian semi-simple compact gauge group $G$. $A_{\mu a}$ are the free gauge fields, defined by

$$ \partial \cdot \partial A_{\mu a}^\nu(x) = 0, \quad [A_{\mu a}^\nu(x), A_{\nu b}(y)]_+ = i \delta_{ab} g_{\mu \nu} D_0(x - y) $$

where $D_0$ is the Pauli - Jordan commutation function for $m = 0$. $F_{\mu \nu}^{\mu}$ are the free field strengths:

$$ F_{\mu a}^{\mu} \overset{\text{def}}{=} \partial^\mu A_{\mu a}^\nu - \partial^\nu A_{\mu a}^\mu $$

and $u_a$ and $\tilde{u}_a$ are the free ghost fields:

$$ \partial \cdot u_a(x) = \partial \cdot \tilde{u}_a(x) = 0, \quad \{u_a(x), \tilde{u}_b(y)\}_+ = -i \delta_{ab} D_0(x - y) $$

$$ \{u_a(x), u_b(y)\}_+ = \{\tilde{u}_a(x), \tilde{u}_b(y)\}_+ = 0 $$

A detailed discussion of the algebraic properties of these ghost fields can be found in [5].
Differentiating (1.3) we get
\[ [\partial_\mu A_\mu^a(x), \partial_\nu A_\nu^b(y)]_- = 0 \quad (1.7) \]
\[ [\partial_\mu A_\mu^a(x), F_\kappa^\lambda(y)]_- = 0 \quad (1.8) \]

Despite their simplicity, these equations have important consequences. For, let us consider the operator
\[ Q \overset{\text{def}}{=} \int_{x_0=\text{const.}} d^3\vec{x} \partial_\mu A_\mu^a(x) \partial_0 u_a(x) \quad (1.9) \]

Using the Leibnitz rule for graded algebras gives
\[ Q^2 = \frac{1}{2} \{Q, Q\}_+ = \frac{1}{2} \int_{x_0=\text{const.}} d^3\vec{x} \int_{y_0=\text{const.}} d^3\vec{y} \left\{ [\partial_\mu A_\mu^a(x), \partial_\nu A_\nu^b(y)]_- \partial_\mu \partial_\nu \partial_\mu \partial_\nu (u_a(x)u_b(y)) + \right. \]
\[ \left. + (\partial_\mu A_\mu^a(y)\partial_\nu A_\nu^b(x)) \partial_\mu \partial_\nu \partial_\mu \partial_\nu (u_a(x), u_b(y))_+ \right\} = 0 \quad (1.10) \]

Thus eqs. (1.6, 1.7) make \( Q \) a differential operator in the sense of homological algebra. This allows for standard homological notions \[6,7\]: Let \( F = \{F\} \) be the field algebra consisting of the polynomials in the (smeared) gauge and ghost fields and their Wick powers. Consider the ghost charge operator \[5\]
\[ Q_g \overset{\text{def}}{=} \int_{x_0=\text{const.}} d^3\vec{x} \partial_\mu A_\mu^a(x) \partial_0 u_a(x) \quad (1.11) \]

and the corresponding derivation \( \delta_g \) in \( F \)
\[ \delta_g F \overset{\text{def}}{=} [Q_g, F]_- \quad (1.12) \]

We say an operator \( F \) has ghost charge \( z \) if
\[ \delta_g F = zF \quad (1.13) \]

Since \( Q_g \) has integer spectrum \[5\] we have \( z \in \mathbb{Z} \). The operators \( F_z \) with ghost charge \( z \) form the subspace \( \mathcal{F}_z \), and we obviously have
\[ \mathcal{F} = \bigoplus_{z \in \mathbb{Z}} \mathcal{F}_z \quad (1.14) \]

which makes \( \mathcal{F} \) a \( \mathbb{Z} \)-graded algebra. Consider the unitary operator \[5\]
\[ E \overset{\text{def}}{=} (-1)^Q, \quad E^2 = 1 \quad (1.15) \]

It induces the canonical involution \( \omega \) in \( \mathcal{F} \) by
\[ \omega F \overset{\text{def}}{=} EFE, \quad \omega^2 = 1 \quad (1.16) \]

We define the bosonic part \( F_b \) and the fermionic part \( F_f \) of an operator \( F \) by
\[ F_{b(f)} \overset{\text{def}}{=} \frac{1}{2} (1 \overset{+}{\omega}) F, \quad F_{b(f)} \overset{+}{\omega} F_{b(f)} = \overset{+}{\omega} F_{b(f)} \quad (1.17) \]

and the graded bracket of two operators \( F \) and \( G \) by
\[ [F, G] = [F_b + F_f, G_b + G_f] \overset{\text{def}}{=} [F_b, G_b]_- + [F_b, G_f]_- + [F_f, G_b]_- + [F_f, G_f]_+ \quad (1.18) \]

We also define on \( \mathcal{F} \) the operator \( d_Q \) by
\[ d_Q F \overset{\text{def}}{=} [Q, F] = QF - (\omega F)Q \quad (1.19) \]
This is a differential operator:
\[ d_Q^2 = 0, \iff \{Q, \{Q, F_0\}_-\} = \{Q, \{Q, F_f\}_-\} = 0 \]  
(1.20)
and an antiderivation with respect to \( \omega \):
\[ d_Q(FG) = (d_Q F)G + (\omega F)(d_Q G) \]  
(1.21)
The commutator relation
\[ [Q, Q] = -Q \]  
(1.22)
implies
\[ [\delta_g, d_Q] = -d_Q, \quad \Rightarrow d_Q \mathcal{F}_z \subseteq \mathcal{F}_{z-1} \]  
(1.23)
i.e. \( d_Q \) is a homogeneous homomorphism of degree (-1) over \( \mathcal{F} \). This implies in particular that it anticommutes with the canonical involution:
\[ \{d_Q, \omega\}_+ = 0 \]  
(1.24)
We conclude that the quadruplet \( \{\mathcal{F}, \delta_g, \omega, d_Q\} \) fits well into the definition of a graded differential algebra [6].
Let us study the action of \( d_Q \) on \( \mathcal{F} \) more explicitly. We find
\[ d_Q A_\alpha^\mu(x) = i \partial^\mu u_\alpha(x) \]  
(1.25)
\[ d_Q u_\alpha(x) = 0, \quad d_Q \tilde{u}_\alpha(x) = -i \partial_\mu A_\alpha^\mu(x) \]  
(1.26)
Eqs. (1.7, 1.8) immediately give two gauge invariants:
\[ d_Q \partial_\mu A_\alpha^\mu(x) = d_Q F_\alpha^{\mu\nu}(x) = 0 \]  
(1.27)
The above actions of \( d_Q \) on \( \mathcal{F} \) may be called free or asymptotic BRS variations since the (formally defined) full BRS variations of interacting fields [7] reduce to them in the absence of interaction. It is exactly these free variations we are interested in when applying causal perturbation theory, since there we are looking for symmetries of the \( S \)-matrix which is defined in the Hilbert - Fock space \( \mathcal{H} \) of free asymptotic fields. The algebra and homology of free BRS operators is well studied in [7,8]. The variations induced by \( d_Q \) are also called operator valued gauge transformations in [1] since they emerge from the usual asymptotic gauge variations in QED [12] by replacing the gauge function \( \chi \) with the ghost operator \( u_\alpha \). We will often simply call this asymptotic BRS variations gauge variations and their invariants gauge invariants.

The interaction (1.2) is gauge invariant, i.e. we have
\[ d_Q T^{(1)}(x) = \partial_\mu T^{(1)\mu}(x), \quad T^{(1)\mu}(x) \equiv \varepsilon_{abc} : u_a \{ A_{\alpha} F_\alpha^{\mu\nu} + \frac{1}{2} \partial_\mu u_\alpha \partial^\nu u_\alpha \} : (x) \]  
(1.28)
The quintessence of causal perturbation theory is that all higher terms \( T^{(n)}, n \geq 2 \) in (1.1) are determined from \( T^{(1)} \) by Poincaré invariance and causality [9-12]. This determination is unique up to some (finite!) normalization constants, which can be determined by the requirement of symmetries and (finitely many) normalization conditions. \( T^{(n)} \) is given symbolically by
\[ T^{(n)}(x_1, \cdots, x_n) = \Theta[T^{(1)}(x_1) \cdots T^{(n)}(x_n)] \]  
(1.29)
where \( \Theta \) means the time ordered product. This, however, cannot be constructed by multiplying with step functions, since this would lead to the well known UV-divergences[9,10]. Instead one has to use the method of distribution splitting, developed by Epstein and Glaser [11] and applied to QED, for example, by Scharf [12]. Using exactly this construction Duetsch et al. [1-4] have shown that the Yang-Mills theory specified by (1.2) is gauge invariant in all orders, i.e. the following equations hold true:
\[ d_Q T^{(n)}(x_1, \cdots, x_n) = \sum_{l=1}^{n} \partial_\mu T^{(n)\mu}_{l}(x_1, \cdots, x_n), \]  
(1.30)
Thus the gauge variation of the $T^{(n)}$ are total divergences. One would like to conclude from this the more conventional form of gauge invariance:

$$\lim_{g \to 1} d_Q S[g] = 0 \quad (1.32)$$

While this adiabatic limit is well controlled in massive theories [13] the situation is far more difficult in massless theories, where it generally fails to exist in the $S$-matrix elements [12,14]. The strength of eq. (1.30) is to give a formulation of gauge invariance which is completely independent of the infrared problems encountered when passing to the adiabatic limit.

The importance of the gauge invariance (1.30) lies in the fact that it enables one to proof the unitary of the physical $S$-matrix $S_{\text{phys}}$ defined in the physical subspace $H_{\text{phys}}$ of the total Hilbert-Fock $H$. The latter one can be defined as the cohomology space of $Q$ or, equivalently, as $\{ \text{Ker}Q \} \ominus \{ \text{Ran}Q \}$.

The paper at hand aims at the construction of Yang-Mills theories with massive gauge (and ghost) fields. This is usually done via the Higgs mechanism which is known to give a renormalizable, gauge invariant, and unitary perturbation series. While not questioning the validity of this result we here want to develop a different approach.

To clearly enlighten this difference let us briefly summarize the logical steps used to derive the perturbation series for the Higgs model. There the starting point is a classical Yang-Mills theory defined by an action $\Sigma$ which is invariant under the group of local classical gauge transformations $G_{\text{local}}$. The gauge fields are coupled to scalar fields which interact among each other by a Mexican hat potential. Then one considers the classical energy which is a functional on the configuration space of classical fields. Due to the peculiar form of the classical potential the classical ground state, i.e. the points in the classical configuration space which minimize the classical energy (often very misleadingly called the vacuum), is found to be degenerate. Then an arbitrary representative point from this state is chosen. This is called spontaneous symmetry breaking. After that new classical fields are defined as the original fields shifted by this reference point. Then a gauge fixing term is added to the action and also the corresponding Fadeev-Popov ghost term included. The total action is then shown to be BRS-invariant. Then the action is split into two parts: The free part being at most quadratically in the shifted fields and the interaction part being at least trilinear in these fields. It is only then that quantization comes into play: The quadratic part of the action defines the quantum kinematical setup, i.e. the shifted classical fields are quantized as free quantum fields in a Hilbert-Fock space $H$ with unique (!) vacuum. These free quantized fields describe the in- and outgoing particles. The interacting part of the action describes the interaction of these particles and allows for the perturbative calculation of Green functions of the corresponding (only formally defined) interaction fields and, most important, determines the $S$-matrix in $H$. Again a physical subspace $H_{\text{phys}} \subset H$, a formally defined set of physical interacting observables, and the physical $S$-matrix $S_{\text{phys}}$ are defined via the homology of the BRS-transformation. It is then shown that these physical quantities depend neither on the representative point of the ground state of the classical energy chosen above nor on the the gauge fixing term. Eventually $S_{\text{phys}}$ is shown to be unitary. In this step the BRS-invariance is again the key ingredient. The BRS-charge is often expressed in terms of the only formally defined interacting quantum fields. Due to its conservation it can, however, be expressed in terms of the free asymptotic fields as well. Since these are perfectly well defined the latter method is superior. Moreover, it is exactly this asymptotic BRS-invariance which is needed to proof unitarity of $S_{\text{phys}}$.

This suggests our approach to massive Yang-Mills theories: We will not take any reference to the classical theory. So, neither the classical gauge group $G_{\text{local}}$ nor the concept of spontaneous symmetry breaking will enter our reasoning. Instead, we immediately start with the quantum theory, defined by given asymptotic massive gauge and ghost fields and by the generator of the causal $S$-matrix, $T^{(1)}(x)$. This we demand to be invariant under asymptotic BRS-transformations, and we give a classification of all all $T^{(1)}(x)$ with this property. Since we do not employ the notion of spontaneous symmetry breaking we have no reason to include scalar fields in our discussion from the very beginning. Instead, we derive the presence of these fields by a purely algebraic condition.

The paper is organized as follows: The next chapter deals once more with massless Yang-Mills theories showing that the interaction (1.2) admits gauge invariant generalizations and giving a complete list of them. Chapter 3 starts the investigation of massive Yang-Mills theories. We study theories with only gauge and ghost fields and construct their gauge invariant interactions. We will see, however, that the BRS-charge fails
to be a differential operator in this case. Chapter 4 shows how to cure this defect: Scalar fields have to be included in the definition of the BRS-charge to restore its nilpotency. This changes the gauge transformations and we have to determine the gauge invariant interactions once again. It turns out that the scalar fields couple to the gauge fields with predicted strength. Chapter 5 shows how to incorporate matter fields into the theory. It is proven in chapter 6 that gauge invariance implies unitarity of $S_{\text{phys}}$. There the relation between anomalies and unitarity is clarified, too. The critical discussion can be found in the last chapter.

2. Gauge Invariant Interactions of Massless Yang-Mills Fields

Here we will generalize the interaction of massless Yang-Mills and ghost fields given by (1.1). So we have to classify their possible interactions $T := T^{(1)}(x)$. We will restrain the form of these interactions by requiring it to share the following structural properties with the interaction specified in (1.1):

1.) We demand the interaction to be normalizable. Then only normal products of three or four fields can appear in $T$. In the case of three fields one of them may be differentiated once.

2.) The interaction of the ghost and gauge fields shall be of Yang-Mills type, i.e. the coupling of the gauge and ghost fields, which are in the adjoint representation of the global Group $G$, shall be proportional to the structure constants $f_{abc}$.

3.) We constrain the interaction $T$ to be invariant under the global group $G$. Thus the tensor of the structure constants has to be contracted with three coloured fields. This excludes a posteriori the coupling of four fields in $T$. This is quite satisfactory. For, in the next order $T^{(2)}$, causality and gauge invariance will create these quartic couplings without further ado. The seagull graph in scalar QED and the four gluon coupling in Yang-Mills theory, for example, are generated this way.

4.) We require $T$ to have vanishing ghost charge, i.e $\delta Q_g T = 0$. This makes the two ghost fields $u$ and $\tilde{u}$ always appear together and particularly implies that $T$ is a bosonic operator.

5.) We will not give up invariance of $T$ under the proper Poincaré group $P^\uparrow_+$, of course. Thus all Lorentz indices have to be contracted. This implies that one of the three fields coupled in $T$ has to be differentiated, because otherwise the number of Lorentz indices would be odd.

6.) In order to have a pseudo-unitary $S$-matrix $T$ should be anti-pseudo-hermitian.

There exist exactly four linear independent interaction terms $T_i$ fulfilling these conditions:

$$
T_1 \stackrel{\text{def}}{=} -\frac{1}{2}ie f_{abc} A^\mu_a A^\nu_b F^\rho_{\mu\nu} c, \quad T_2 \stackrel{\text{def}}{=} -ie f_{abc} A^\mu_a u_b \partial^\mu \tilde{u}_c,
$$

$$
T_3 \stackrel{\text{def}}{=} -ie f_{abc} A_{\mu\rho} A^\mu_a \partial^\rho \tilde{u}_b u_c, \quad T_4 \stackrel{\text{def}}{=} ie f_{abc} \partial^\mu A^\mu_a u_b \tilde{u}_c
$$

(2.1)

and any real linear combination of these terms fulfils these conditions, too. We therefore set

$$
T = \sum_{i=1}^{4} \alpha_i T_i
$$

(2.2)

with a priori arbitrary real constants $\alpha_i$.

Now we demand, in addition, gauge invariance, i.e.

$$
d_Q T = \partial_\mu T^\mu
$$

(2.3)

Since $T$ is different from the expression (1.2), $T^\mu := T^{(1)}(x)$ will be different from (1.28), too. It shall, however, retain the following structural properties:

1'.) Normalizability: Only normal products of three or for fields may appear in $T^\mu$. In the case of three fields one may be differentiated once.

2'.) $T^\mu$ shall be of Yang-Mills type, i.e. the coupling of the gauge and ghost fields shall be proportional to the structure constants $f_{abc}$. 

5
3'.) $G$-invariance: All colour indices in $T^\mu$ have to be contracted. This, again, excludes the appearence of normal products of four fields in $T^\mu$.

4'.) $T^\mu$ must have ghost charge $-1$. This implies that either one $u$ and no $\tilde u$ or two $u$ and one $\tilde u$ are present in $T^\mu$ and that $T^\mu$ is fermionic.

5'.) $P^\mu$-covariance: All Lorentz indices except $\mu$ have to be contracted.

6'.) $T^\mu$ should be pseudo-hermitean.

These properties follow naturally from the corresponding properties 1.-6.) of $T$ and eq.(2.3). There exist exactly six linear independent terms fulfilling these conditions:

\[
\begin{align*}
T_1^\mu &\overset{\text{def}}{=} :e f_{a b c} u_a A_{\mu} F^{\mu \nu}_c : , \\
T_2^\mu &\overset{\text{def}}{=} -e f_{a b c} u_a A^\mu_{\nu} \partial_\nu A^\nu_c : , \\
T_3^\mu &\overset{\text{def}}{=} -e f_{a b c} u_a A_{\mu} G^{\mu \nu}_c : , \\
T_4^\mu &\overset{\text{def}}{=} \frac{1}{2} e f_{a b c} u_a u_b \partial_\mu \tilde u_c : , \\
T_5^\mu &\overset{\text{def}}{=} e f_{a b c} u_a \partial_\mu u_b \tilde u_c : , \\
T_6^\mu &\overset{\text{def}}{=} e f_{a b c} \partial^\mu u_a A_{\nu} \partial_\nu A^\nu_c : , \\
\end{align*}
\]

Here we have introduced $G^{\mu \nu}_c := \partial^\mu A^\nu_c + \partial^\nu A^\mu_c$. Any real linear combination of the six expressions above fulfills the requirements 1'-6', too. We therefore set

\[
T^\mu = \sum_{j=1}^{6} \beta_j T_j^\mu 
\]

with a priori arbitrary real constants $\beta_j$.

To study (2.3) we need

\[
\begin{align*}
d_Q T_1 &= : e f_{a b c} A_{\mu \alpha} \partial_\nu u_b F^{\mu \nu}_c : , \\
d_Q T_2 &= : e f_{a b c} (A_{\mu \alpha} u_b \partial_\nu A^\nu_c + \partial_\nu u_a u_b \partial_\mu \tilde u_c) : , \\
d_Q T_3 &= : e f_{a b c} (A_{\mu \alpha} \partial_\nu u_b \partial_\nu A^\nu_c + \partial_\nu u_a \partial_\mu u_b \tilde u_c) : , \\
d_Q T_4 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
\partial_\mu T_1^\mu &= : e f_{a b c} (A_{\mu \alpha} \partial_\nu u_b F^{\mu \nu}_c + A_{\mu \alpha} u_b \partial_\nu A^\nu_c) : , \\
\partial_\mu T_2^\mu &= : e f_{a b c} (A_{\mu \alpha} u_b \partial_\nu A^\nu_c + A_{\mu \alpha} \partial_\nu u_b \partial_\mu A^\nu_c) : , \\
\partial_\mu T_3^\mu &= : e f_{a b c} (A_{\mu \alpha} \partial_\nu u_b G^{\mu \nu}_c + A_{\mu \alpha} u_b \partial_\mu A^\nu_c) : , \\
\partial_\mu T_4^\mu &= : e f_{a b c} (A_{\mu \alpha} \partial_\nu u_b \tilde u_c + A_{\mu \alpha} \partial_\mu \tilde u_c) : , \\
\partial_\mu T_5^\mu &= : e f_{a b c} (\partial^\mu u_a \partial_\nu u_b \tilde u_c + \partial^\nu u_a \partial_\mu \tilde u_c) : , \\
\partial_\mu T_6^\mu &= : e f_{a b c} (A_{\mu \alpha} \partial_\nu u_b \partial_\nu A^\nu_c + A_{\mu \alpha} \partial_\nu u_b \partial^\nu A^\nu_c) : \\
\end{align*}
\]

To derive these formulæ we have used that the fields obey the wave equation. By inserting them in (2.3) we get a system of linear homogeneous equations for the coefficients $\alpha_i$ and $\beta_j$. Due to its homogeneity we can certainly choose freely the overall normalization of these coefficients, and we do that in setting $\alpha_1 = 1$.

The solution of the equations turns out to have three additional free parameters which we call $\alpha$, $\beta$, and $\gamma$. The general solution is:

\[
T = T_1 + \left(\frac{1}{2} - \alpha\right) T_2 + \left(-\frac{1}{2} - \alpha\right) T_3 + (\alpha + \beta) T_4 
\]

\[
T^\mu = (1 + \gamma) T_1^\mu + \left(\frac{1}{2} - \alpha - 2\gamma\right) T_2^\mu + \gamma T_3^\mu + T_4^\mu + \left(-\frac{1}{2} - \alpha\right) T_5^\mu - 2\gamma T_6^\mu
\]

Let us now study the structure of these expressions. We first remark that the special choice $\alpha = -\beta = -\frac{1}{2}$ leads us back to the original interaction (1.2). Setting, in addition, $\gamma = 0$ also reproduces (1.28). This choice is distinguished by its minimalitry: Firstly, there are only two terms in $T$ and $T^\mu$. Secondly, only four elementary fields are used: $A$, $F$, $u$, and $\tilde u$. The other four elementary fields $G$, $\partial \cdot A$, $\partial u$, and $\tilde u$ do not appear at all. This shortens lengthy highly order calculations and the very elaborate proof of gauge invariance in all orders [1-4] by a considerable amount.
Another preferred choice is $\alpha = \beta = \gamma = 0$. This gives

$$T = i e f_{abc} : \left( -\frac{1}{2} A_{\mu a} A_{\mu b} F_{c \nu}^{\mu \nu} - \frac{1}{2} A_{\mu a} u_b \partial^\mu \tilde{u}_c \right) : $$ \quad (2.10)

$$T^\mu = e f_{abc} : u_a \left( A_{\nu b} F_{\mu c}^{\nu \mu} + \frac{1}{2} A_{\nu b} \partial_{\nu} A_{\mu c}^{\nu} + \frac{1}{2} u_b \partial^\mu \tilde{u}_c \right) : $$ \quad (2.11)

This $T$ has an additional symmetry: It is invariant under the ghost charge conjugation $C_g$ [5]. This unitary operator reflects the gauge charge:

$$C_g Q_g C_g^{-1} = -Q_g$$ \quad (2.12)

and acts on the ghost fields in the following way:

$$C_g u_\alpha(x) C_g^{-1} = i \tilde{u}_\alpha(x) \, , \, C_g \tilde{u}_\alpha(x) C_g^{-1} = i u_\alpha(x)$$ \quad (2.13)

This implies indeed:

$$C_g T C_g^{-1} = T$$ \quad (2.14)

and the choice $\alpha = \beta = 0$ is the only one making this equation hold true. This $T$ is actually not only invariant under ghost charge conjugation; it is invariant under SU(1, 1) - “rotations” in ghost space, too [5].

The three parameter freedom in $T$, $T^\mu$ has the following interpretation:

I.) The terms in $T$ which are multiplied by $\alpha$: $T_\alpha$, are a pure divergence:

$$T_\alpha = -T_2 - T_3 + T_1 = \partial_{\mu} \left( i e f_{abc} : A_{\mu a}^\mu u_b \tilde{u}_c \right) : = \partial_{\mu} H^\mu$$ \quad (2.15)

Since $Q$ is $x$ - independent, their gauge variation is a pure divergence, too:

$$\partial_Q T_\alpha = d_Q \partial_{\mu} H^\mu = \partial_{\mu} \left( d_Q H^\mu \right) = \partial_{\mu} T_\alpha^\mu$$ \quad (2.16)

where

$$T_\alpha^\mu \overset{\text{def}}{=} -T_2^\mu - T_3^\mu$$ \quad (2.17)

are exactly those terms in $T^\mu$ which are multiplied by $\alpha$. It follows that the couple $(T_\alpha, T_\alpha^\mu)$ fulfils (2.3) separately.

II.) The term in $T$ which is multiplied by $\beta$: $T_\beta$, is a pure gauge, i.e. a $d_Q$-boundary:

$$T_\beta = T_4 = d_Q L \quad , \quad L \overset{\text{def}}{=} -\frac{1}{2} e f_{abc} : u_a \tilde{u}_b \tilde{u}_c :$$ \quad (2.18)

Since $d_Q$ is a differential operator it is also a $d_Q$-cycle, i.e gauge invariant:

$$d_Q T_\beta = 0$$ \quad (2.19)

It can, therefore, freely be added to $T$ without invalidating (2.3).

III.) The terms in $T^\mu$ which are multiplied by $\gamma$: $T_\gamma^\mu$, are a conserved trilinear current:

$$T_\gamma^\mu = T_1^\mu - 2 T_2^\mu + T_3^\mu - 2 T_4^\mu = 2 \partial_{\mu} \left( \left. e f_{abc} u_a A_{\beta b} A_{\gamma c}^{\nu} \right| \right) = K^\mu \, , \, \partial_{\mu} K^\mu = 0$$ \quad (2.20)

It follows that they can freely be added to $T^\mu$ without invalidating (2.3).

The discussion above allows to reformulate (2.8, 2.9) as

$$T = T_1 + \frac{1}{2} \left( T_2 - T_3 \right) + \alpha \partial_{\mu} H^\mu + \beta d_Q L \, ,$$

$$T^\mu = T_1^\mu + T_4^\mu - \frac{1}{2} \left( T_2^\mu + T_3^\mu \right) + \alpha d_Q H^\mu + \gamma K^\mu$$ \quad (2.21)

In the next chapter we will study how these structures change if the gauge fields and ghost fields are massive.
3. A Direct Route to Massive Yang-Mills Fields

To construct a theory of massive Yang-Mills fields we have to use free asymptotic massive gauge and ghost fields:

\[
(\partial \cdot \partial + M^2) A^\mu_a(x) = (\partial \cdot \partial + M^2) u_a(x) = (\partial \cdot \partial + M^2) \tilde{u}_a(x) = 0 \quad (3.1)
\]

\[
[A^\mu_a(x), A^\nu_b(y)]_- = i \delta_{ab} g^{\mu\nu} D_M(x-y) \quad (3.2)
\]

\[
\{ u_a(x), \tilde{u}_b(y) \}_+ = -i \delta_{ab} D_M(x-y), \quad \{ u_a(x), u_b(y) \}_+ = \{ \tilde{u}_a(x), \tilde{u}_b(y) \}_+ = 0 \quad (3.3)
\]

where \( D_M \), the Pauli-Jordan commutation function for mass \( M > 0 \), appears. The free massive field strengths \( F_a^{\mu\nu} \) are defined as in (1.4). We have given all coloured fields the same mass, since we do not discuss breaking of the global group \( G \) here, while the ghost and the gauge fields have the same mass because they transform among each other under gauge transformations.

The nonvanishing of the mass \( M \) has simple but far reaching consequences: While (1.8) remains true,

\[
[\partial_\mu A^\mu_a(x), F^a_{\nu\lambda}(y)]_- = 0 \quad (3.4)
\]

(1.7) is altered to

\[
[\partial_\mu A^\mu_a(x), \partial_\nu A^\nu_b(y)]_- = i M^2 \delta_{ab} D_M(x-y) \quad (3.5)
\]

Let us define the gauge charge \( Q \) by

\[
Q \overset{\text{def}}{=} \int_{x_0=\text{const.}} d^3 \vec{x} (\partial_\mu A^\mu_a(x)) \frac{\leftrightarrow}{\partial_0} u_a(x) \quad (3.6)
\]

While this is the same expression as in the massless case, it is, of course, a different operator, because now the quantized fields in the integral are the massive ones. Its square is given by

\[
Q^2 = \frac{1}{2} \{ Q, Q \}_+ = \frac{1}{2} \int_{x_0=\text{const.}} d^3 \vec{x} \int_{y_0=\text{const.}} d^3 \vec{y} \cdot \left\{ [\partial_\mu A^\mu_a(x), \partial_\nu A^\nu_b(y)]_-- \frac{\leftrightarrow}{\partial_\nu} \frac{\leftrightarrow}{\partial_0} (u_a(x) u_b(y)) + (\partial_\nu A^\nu_b(y) \partial_\mu A^\mu_a(x)) \frac{\leftrightarrow}{\partial_\mu} \frac{\leftrightarrow}{\partial_0} \{ u_a(x), u_b(y) \}_+ \right\} \quad (3.7)
\]

and this is due to the nonvanishing commutator (3.5) unequal to zero, in contrast to the massless case (1.10). Instead it is given by

\[
Q^2 = i M^2 Q_u, \quad Q_u \overset{\text{def}}{=} i \int_{x_0=\text{const.}} d^3 \vec{x} u_a(x) \frac{\leftrightarrow}{\partial_0} u_a(x) \quad (3.8)
\]

The charge \( Q_u \) has been discussed in the framework of the ghost charge algebra in [5]. So \( Q \) fails to be a differential operator and homological notions do not apply. This applies as well to the gauge variation \( d_Q \), which is defined by (1.19) also in the massive case: (1.20) is changed to

\[
d_Q^2 = i M^2 \delta_u \quad (3.9)
\]

where \( \delta_u \) is the derivation induced by \( Q_u \). The algebraic Eqs. (1.11-1.19, 1.21-1.24) remain true in the massive theory, while the gauge variation of the basic fields changes from (1.25)-(1.27) to

\[
d_Q A^\mu_a(x) = i \partial^\mu u_a(x), \quad d_Q F_a^{\mu\nu}(x) = 0, \quad d_Q (\partial_\mu A^\mu_a(x)) = i M^2 u_a(x), \quad (3.10)
\]

\[
d_Q u_a(x) = 0, \quad d_Q \tilde{u}_a(x) = -i \partial_\mu A^\mu_a(x) \quad (3.11)
\]

Now we look again for the general gauge invariant interaction, i.e. any couple \( (T, T^\mu) \) fulfilling the general conditions discussed in the preceding chapter and \( d_Q T = \partial_\mu T^\mu \). We can take over the expressions
(2.1) and (2.4) and the Ansätze (2.2) and (2.5). Since, however, the fields in these expressions are now massive (2.6) and (2.7) change to

\[ dQ T_1 =: e f_{abc} A_{\mu a} \partial_\nu u_b F_{\mu \nu}^c, \quad dQ T_2 =: e f_{abc} (A_{\mu a} u_b \partial^\mu \partial_\nu A_{\nu c}^e + \partial_\mu u_a u_b \partial^\mu \tilde{\partial}_c) : , \]

\[ dQ T_3 =: e f_{abc} (A_{\mu a} \partial^\mu u_b \partial_\nu A_{\nu c}^e + \partial_\mu u_a u_b \partial^\mu \tilde{\partial}_c) : , \quad dQ T_4 =: e f_{abc} M^2 u_a u_b \tilde{\partial}_c : \] (3.12)

\[ \partial_\mu T_1^\mu =: e f_{abc} (A_{\mu a} \partial^\mu u_b F_{\mu \nu}^c + A_{\mu a} u_b \partial^\mu \partial_\nu A_{\nu c}^e) : , \quad \partial_\mu T_2^\mu =: e f_{abc} (A_{\mu a} u_b \partial_\nu A_{\nu c}^e + A_{\mu a} u_b \partial^\mu \tilde{\partial}_c) : , \]

\[ \partial_\mu T_3^\mu =: e f_{abc} (A_{\mu a} \partial^\mu u_b \partial_\nu A_{\nu c}^e) : , \quad \partial_\mu T_4^\mu =: e f_{abc} (\partial_\mu u_a u_b \partial^\mu \tilde{\partial}_c - \frac{1}{2} M^2 u_a u_b \tilde{\partial}_c) : , \]

\[ \partial_\mu T_6^\mu =: e f_{abc} (\partial^\mu u_a \partial_\nu \tilde{\partial}_c + \partial^\mu u_a u_b \partial_\nu \tilde{\partial}_c - M^2 u_a u_b \tilde{\partial}_c) : , \]

\[ \partial_\mu T_6^\mu =: e f_{abc} (\partial^\mu u_a \partial_\nu \tilde{\partial}_c) (4.13) \]

This leads to the result

\[ T = T_1 + \frac{1}{2} (T_2 - T_3) + \alpha \partial_\mu H^\mu , \]

\[ T^\mu = T_1^\mu + T_2^\mu - \frac{1}{2} (T_2^\mu + T_3^\mu) + \alpha dQ H^\mu + \gamma K^\mu \] (3.14)

where \( H^\mu \) and \( K^\mu \) are the expressions defined in (2.15) and (2.20).

The result has the same structure and interpretation as (2.21) in the massless theory, but for one difference: There is no free term of the form \( \beta dQ L \) here. The reason is clear: Since \( dQ \) fails to be a differential operator such a term would not be gauge invariant. Since this term is absent, no analogon of the minimal choice exists in the massive theory. A differential operator such a term would not be gauge invariant. Since this term is absent, no analogon of the minimal choice exists in the massive theory. A symmetric interaction \( T \), however, does exist, and is again uniquely given by setting \( \alpha = 0 \). For this case, the expressions \( T \) and \( T^\mu \) are identical to those in the massless case. So one might say that this additional symmetry has stabilized the theory against perturbations by mass terms.

We have succeeded in constructing gauge invariant interactions for massive Yang-Mills fields. This theory, however, has no physical interpretation. For, we will see in chapter six that the not abundandable unitarity of the physical S-matrix is only guaranteed if both equations: \( dQ T = \partial_\mu T^\mu \) and \( Q^2 = 0 \) hold true, but the latter one fails here. These problems have also been studied in the canonical framework [7].

We would like to point out that the theory defined above may none the less be quite useful: It is a gauge invariant infrared regulator for the massless theory, and a properly done comparsion of the two can give important results.

Knowing now exactly where the straightforward approach to massive quantized Yang - Mills fields fails we will not find it to difficult to cure this problem. This is done in the next chapter.

4. The Algebraic Introduction of Scalar Gauge Fields

From (3.7) we learn that the the reason why \( Q \) fails to be a differential operator is simply the nonvanishing commutator (3.5). Consider now \( \dim G \) hermitean free quantized Klein-Gordon fields \( h_a(x) \) which are, like the gauge fields \( A^a_\mu \) and the ghosts fields \( u_a \) and \( \tilde{u}_a \), in the adjoint representation of \( G \) and which have the same mass as these fields. They obey

\[ (\partial \cdot \partial + M^2) h_a(x) = 0 , \quad [h_a(x), h_b(y)]_\mu = -i \delta_{ab} D_M(x - y) \] (4.1)

Their commutator has the opposite sign to (3.5). It follows that the fields \( \partial_\mu A^a_\mu(x) + M h_a(x) \) have vanishing commutators with themselves:

\[ [\partial_\mu A^a_\mu(x) + M h_a(x), \partial_\nu A^e_\nu(y) + M h_b(y)]_\mu = 0 \] (4.2)

This suggests the following new definition for \( Q \):

\[ Q \overset{\text{def}}{=} \int_{x_0 = \text{const.}} d^3 \vec{x} (\partial_\mu A^a_\mu(x) + M h_a(x)) \partial_\mu u_a(x) \] (4.3)
For, this implies

\[
Q^2 = \frac{1}{2} \int_{x_0=\text{const.}} d^3 \vec{x} \int_{y_0=\text{const.}} d^3 \vec{y} \left\{ \left[ \partial_\mu A_\mu^a(x) + M h_a(x), \partial_\nu A_\nu^a(y) + M h_b(y) \right] - \partial_\nu \leftrightarrow \partial_\nu \left( u_a(x) u_b(y) \right) + \left( \partial_\nu A_\nu^b(y) + M h_b(y) \right) \left( \partial_\mu A_\mu^a(x) + M h_a(x) \right) \right\} = 0
\]

(4.4)

So we have managed to recover this important algebraic property which, together with gauge invariance, guarantees unitarity of \(S_{\text{phys}}\)! Note that all eqs. (1.11-1.20) hold true anew.

The gauge variations of the elementary fields are given by

\[
d_Q A_\mu^a(x) = i \partial^\mu u_a(x) , \; d_Q h_a(x) = i M u_a(x) , \; d_Q u_a = 0 , \; d_Q \tilde{u}_a = -i (\partial_\mu A_\mu^a(x) + M h_a(x)) \quad \text{(4.5)}
\]

Consequently,

\[
d_Q \partial_\mu A_\mu^a(x) = -i M^2 u_a(x) , \; d_Q F_\mu^\nu_a(x) = d_Q (\partial_\mu A_\mu^a(x) + M h_a(x)) = 0 \quad \text{(4.6)}
\]

We see that the scalar fields \(h_a(x)\) are affected by the gauge variation \(d_Q\) and that they appear in the gauge variations of other fields. Hence it is appropriate to call them scalar gauge fields. We will see in chapter six that these fields are unphysical, i.e. their projections onto \(H_{\text{phys}}\) vanish.

We now have to determine the possible gauge invariant interactions \((T, T^\mu)\) once more. We will not dispense with the structural conditions discussed in chapter 2. These conditions, however, can now be fulfilled by more expressions of \(T_i\) and \(T_i^\mu\) than in the preceding chapters, since the presence of the scalar fields allows for the constructions of new terms. We give the following complete list:

\[
T_1 = -\frac{1}{2} i e f_{abc} A_\mu^a A_{\nu b} F_c^{\mu \nu} , \; T_2 = -i e f_{abc} A_\mu^a u_b \partial^\mu \tilde{u}_c , \; T_3 = -i e f_{abc} A_\mu^a \partial^\mu u_b \tilde{u}_c , \; T_4 = i e f_{abc} A_{\mu b} u_b \partial^\mu \tilde{u}_c , \; T_5 = i e f_{abc} M h_a u_b \tilde{u}_c , \; T_6 = \frac{1}{2} i e f_{abc} A_\mu^a h_b \partial^\mu h_c
\]

\[
T_7 = e f_{abc} M u_a A_\mu^a h_c , \; T_8 = -\frac{1}{2} e f_{abc} u_b h_b \partial^\mu h_c
\]

(4.7)

Using

\[
d_Q T_1 = e f_{abc} A_\mu^a \partial_\mu u_b F_c^{\mu \nu} , \; d_Q T_2 = e f_{abc} \left( A_\mu^a u_b \partial^\mu \left[ \partial_\nu A_\nu^c + M h_c \right] + \partial_\mu u_a \partial^\mu \tilde{u}_c \right) , \; d_Q T_3 = e f_{abc} \left( A_\mu^a \partial^\mu u_b \left[ \partial_\nu A_\nu^c + M h_c \right] + \partial_\mu u_a \partial^\mu \tilde{u}_c \right) , \; d_Q T_4 = e f_{abc} \left( -M \partial_\mu A_\mu^a u_b h_c + M^2 u_a u_b \tilde{u}_c \right) , \; d_Q T_5 = e f_{abc} \left( M \partial_\mu A_\mu^a u_b h_c - M^2 u_a u_b \tilde{u}_c \right) , \; d_Q T_6 = e f_{abc} \left( -M \partial_\mu u_b \partial^\mu h_c - M A_\mu^a u_b \partial^\mu h_c - M A_\mu^a h_b \partial^\mu u_c \right) , \; d_Q T_7 = e f_{abc} \left( -M \partial_\mu u_b \partial^\mu h_c - M A_\mu^a u_b \partial^\mu h_c - M A_\mu^a h_b \partial^\mu u_c \right) , \; d_Q T_8 = e f_{abc} \left( -M \partial_\mu u_b \partial^\mu h_c - M A_\mu^a u_b \partial^\mu h_c - M A_\mu^a h_b \partial^\mu u_c \right)
\]

(4.9)
we find the general gauge invariant interaction as follows:

\[
T = T_1 + \left(\frac{1}{2} - \alpha\right)T_2 - \left(\frac{1}{2} + \alpha\right)T_3 + (\alpha + \beta)T_4 + \beta T_5 + \frac{1}{2}T_6 = \\
= T_1 + \frac{1}{2}(T_2 - T_3 + T_6) + \alpha \partial_\mu H^\mu + \beta d_Q L
\]

(4.11)

\[
T^\mu = (1 + \gamma)T^\mu_1 - \left(\frac{1}{2} + \alpha + 2\gamma\right)T^\mu_2 + \gamma T^\mu_3 + T^\mu_4 - \left(\frac{1}{2} + \alpha\right)T^\mu_5 - 2\gamma T^\mu_6 - \alpha T^\mu_7 + \frac{1}{2}T^\mu_8 = \\
= T^\mu_1 + T^\mu_3 + \frac{1}{2}(T^\mu_8 - T^\mu_2 - T^\mu_6) + \alpha d_Q H^\mu + \gamma K^\mu
\]

(4.12)

where

\[
H^\mu \overset{\text{def}}{=} :ie f_{abc}A^\mu_a u_b \tilde{u}_c :, \quad L \overset{\text{def}}{=} -\frac{1}{2}e f_{abc} u_a u_b \tilde{u}_c :, \quad K^\mu \overset{\text{def}}{=} 2\partial_\nu \{ :e f_{abc} u_a A^\mu_b A^\nu_c :\}
\]

(4.13)

We notice that the term \(T_6\) enters in \(T\) with a fixed coefficient. This term describes the interaction of the scalar gauge fields \(h_a\) with the Yang-Mills fields \(A^\mu_a\). Thus the strength of this interaction is fixed by the condition of gauge invariance. The free terms multiplied by \(\alpha, \beta, \) and \(\gamma\) have the same interpretation as in the massless case: \(\partial_\mu H^\mu\) is a pure divergence, \(d_Q L\) is pure gauge, and \(K^\mu\) is a conserved current. The condition of \(C_\gamma\)-invariance on \(T\) uniquely fixes \(\alpha = \beta = 0\). Since the current \(K^\mu\) is not related to the gauge structure of the theory, \(\gamma = 0\) is certainly a sensible choice, too. In this case the interaction is given explicitly by

\[
T = ie f_{abc} A^\mu_a u_b \tilde{u}_c \left\{ -\frac{1}{2} A^\mu_{ab} F^{\mu\nu}_c \tilde{u}_c + \frac{1}{2} A^\mu_{ab} u_b \tilde{u}_c + \frac{1}{4} A^\mu_{ab} h_b \partial^\mu h_c \right\}
\]

(4.14)

We have now succeeded in constructing gauge invariant interactions of Yang-Mills fields, ghost fields, and scalar gauge fields. In the next chapter we study how these fields couple to matter fields.

5. The Coupling of Matter Fields

All fields we have studied so far, the Yang-Mills fields \(A^\mu_a\), the scalars \(h_a\), and the ghosts \(u_a\) and \(\tilde{u}_a\) may be called gauge fields, since they transform among each other under the gauge variation \(d_Q\). Now we will study additional fields, which we call matter fields. We will study scalar matter fields: \(\phi_i\) (adjoint fields: \(\phi^i_{\bar{a}}\)) and Dirac matter fields: \(\psi_i\) (Dirac-adjoint fields: \(\bar{\psi}_i\)). \(i\) is an internal index numbering different fields, while the bispinor indices \(\alpha\) for the Dirac fields are always suppressed. Let us assume that the fields \(\phi_i, \psi_i\) transform under a certain irreducible representation \(R\) of \(G\) in which the hermitean generators are given by \(R^T_a\). The matter fields will form bilinear currents in the following way:

1.) The scalar fields constitute the currents

\[
S^\mu_a =: \phi^+_i R^T_a \partial^\mu \phi_j : \quad (5.1)
\]

If we demand strictly \(G\)-invariance, all fields \(\phi_i\) should have the same mass \(m_S\):

\[
(\partial \cdot \partial + m_S^2)\phi_i = 0 \quad (5.2)
\]

In this case the currents are conserved:

\[
\partial_\mu S^\mu_a = 0 \quad (5.3)
\]

Let us, however allow for \(G\)-breaking in the mass sector of the scalar matter fields, i.e. replace (5.2) by

\[
(\partial \cdot \partial \delta^{ij} + (m_S^2)^{ij}) \phi_j = 0 \quad (5.4)
\]
where \((m_S^2)\) is the positive mass-square matrix of the scalar fields in \(R\). Then (5.3) does not hold. Instead we find
\[
\partial_\mu S^\mu_a =: \phi^+_i [i (m_S^2), R_a]_{ij} \phi_j : \tag{5.5}
\]

2.) The Dirac fields form two kinds of currents. The vector currents are defined by
\[
V^\mu_a =: \overline{\psi}_i R^{ij}_a \gamma^\mu \psi_j : \tag{5.6}
\]
while the axial currents are given by
\[
X^\mu_a =: \overline{\psi}_i R^{ij}_a \gamma^\mu \gamma_5 \psi_j : \tag{5.7}
\]
Strict \(G\)-invariance would require that all Dirac fields in \(R\) have the same mass \(m_D\):
\[
(i \gamma^\mu \partial_\mu - m_D) \psi_i = 0 \tag{5.8}
\]
In this case the vector current would be conserved and the divergence of the axial current would be the pseudo-scalar
\[
\partial_\mu V^\mu_a = 0 \ , \ \partial_\mu X^\mu_a = 2 m_D : \overline{\psi}_i R^{ij}_a i \gamma_5 \psi_j : \tag{5.9}
\]
Let us, however allow for \(G\)-breaking in the mass sector of the Dirac matter fields, i.e. replace (5.8) by
\[
(i \gamma^\mu \partial_\mu \delta^{ij} - (m_D)^{ij}) \psi_j = 0 \tag{5.10}
\]
where \((m_D)\) is the hermitean mass matrix of the Dirac fields in \(R\). Then (5.9) does not hold. Instead we find
\[
\partial_\mu V^\mu_a =: \overline{\psi}_i [(m_D), R_a]^{ij}_\mu \psi_j : , \ \partial_\mu X^\mu_a =: \overline{\psi}_i \{ (m_D), R_a \}^{ij}_\mu i \gamma_5 \psi_j : \tag{5.11}
\]
The interaction \(T, T^\mu\) constructed in the last chapter contains only gauge fields. So let us call it \(T_{\text{gauge}}, T^\mu_{\text{gauge}}\) from now on. We now add to it \(T_{\text{matter}}, T^\mu_{\text{matter}}\) to describe the interaction of the matter fields with the ghost fields:
\[
T = T_{\text{gauge}} + T_{\text{matter}} \ , \ T^\mu = T^\mu_{\text{gauge}} + T^\mu_{\text{matter}} \tag{5.12}
\]
\(T_{\text{matter}}\) is constructed by coupling the above currents to the gauge fields: Let us introduce the total current
\[
J^\mu_a \overset{\text{def}}{=} i (e_S S^\mu_a + e_V V^\mu_a + e_X X^\mu_a) \tag{5.13}
\]
and set
\[
T_{\text{matter}} = i J^\mu_a A_\mu_a + \cdots \tag{5.14}
\]
The dots indicate that we will soon add other terms to this expression. For, let us study gauge invariance. Since \(Q\) (4.3) is entirely composed of gauge fields it commutes with the matter fields and their currents:
\[
d_Q \phi_i = d_Q \psi_i = d_Q J^\mu_a = 0 \tag{5.15}
\]
This leads to
\[
d_Q T_{\text{matter}} = -J^\mu_a \partial_\mu u_a + d_Q (\cdots) = \partial_\mu \{ -J^\mu_a u_a \} + (\partial_\mu J^\mu_a) u_a + d_Q (\cdots) \tag{5.16}
\]
The first term on the right hand side of this equation is already a divergence. The second term vanishes if the currents \(J^\mu_a\) are conserved but is not a divergence otherwise. So it has to be compensated by the third term, i.e. we get the condition
\[
d_Q (\cdots) = - (\partial_\mu J^\mu_a) u_a \tag{5.17}
\]
which is easily solved by
\[
\cdots = i M^{-1} (\partial_\mu J^\mu_a) h_a \tag{5.18}
\]
Such we have found the following gauge invariant interaction of the matter fields with the gauge fields:
\[
T_{\text{matter}} = i \{ J^\mu_a A_\mu_a + M^{-1} \partial_\mu J^\mu_a h_a \} \ , \ T^\mu_{\text{matter}} = - J^\mu_a u_a , \ d_Q T_{\text{matter}} = \partial_\mu T^\mu_{\text{matter}} \tag{5.19}
\]
Let us interpret this result. In the case of conserved currents the matter fields couple only to the Yang-Mills fields and this interaction has the same form as the coupling of matter fields to massless Yang-Mills fields [15]. More interesting is the case of nonconserved currents: There the matter fields couple to the scalar gauge fields, too. We conclude that the scalar gauge fields are a very important part of the whole theory: They allow a consistent treatment of massive Yang-Mills fields and of nonconserved currents at the same time. We also notice that the coupling of the nonconserved currents to the scalar gauge fields is proportional to the inverse mass of the gauge fields. That is only possible if this mass does not vanish, which is in agreement with experiment: The conserved strong vector currents couple to massless (though confined) gluons while the nonconserved weak axial currents couple to the massive weak bosons!

Let us remark that the currents $S_\mu$, $V_\mu$, and $X_\mu$ couple with different coupling constants $e_S$, $e_V$, and $e_X$ to the Yang-Mills fields, as described in (5.13, 5.14). We also have always used the same irreducible representation $R$ for the matter fields. This is, of course, not necessary: The scalar fields will generally occur in other representations than the Dirac fields and the vector currents will generally be made out of fermions in representations different from those to which the fermions constituting the axial currents belong. Moreover, the representations $R$ need not to be irreducible. Instead, they can be the direct sum of several irreducible parts. The above formulae remain true if one defines the total current (5.13) as the sum of all currents in the various representations $R$. Then each representation $R$ can have its own coupling constant $e_R$. We remark, however, that this is a typical first order phenomenon. The condition of gauge invariance in second order will certainly give restrictions on the various coupling constants which seem to arbitrary at the moment. This also happens in massless Yang-Mills theory, where the gauge invariance of certain second order tree graphs implies the equality of the Yang-Mills self-coupling constant with the one in the coupling of the Yang-Mills fields to matter [15].

We now have carefully studied the possible interactions of quantized Yang-Mills fields, ghost fields, scalar gauge fields, and scalar and Dirac matter fields. Though only working in first order $T^1$ we have discovered very interesting structures. To complete the theory, we would have to study gauge invariance in all orders, eq.(1.30). Before we take on this Herculean task we like to know what we get if we succeed. It is the unitarity of $S_{\text{phys}}$. This is shown in the next chapter.

6. Gauge Invariance and Unitarity of the Physical $S$-Matrix

Unitarity of the physical $S$-matrix in the case of massless Yang-Mills fields was proven in [4]. Here we treat the massive case, i.e. the interaction constructed in the two preceding chapters.

Let us begin with discussing the Krein structure [5,8,16,17] in the Hilbert-Fock space of the gauge fields. The massive Yang-Mills fields are quantized as

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int dk \{ \epsilon_\lambda^\mu(k)a_{\lambda,a}(k)e^{-ikx} + \epsilon_\lambda^\mu(k)a^K_{\lambda,a}(k)e^{ikx} \}$$

$k$ is always on the mass shell $\mathcal{M}$:

$$k \equiv (k_0, \vec{k}), \quad k_0 \equiv +[(\vec{k})^2 + M^2]^{\frac{1}{2}}, \quad dk \equiv \frac{d^3k}{2\pi^2}, \quad \delta(k-k') \equiv 2k_0(2\pi)^3\delta(3)\left(\vec{k}-\vec{k}'\right)$$

$\epsilon_\lambda^\mu$ are four polarisation vectors satisfying

$$\epsilon_0^\mu(k) = \frac{k^\mu}{M}, \quad g_{\mu\nu}\epsilon_\lambda^\mu(k)\epsilon_\lambda^\nu(k) = g_{\kappa\lambda},$$

$$\sum_{\lambda=1}^{3} \epsilon_\lambda^\mu(k)\epsilon_\lambda^\nu(k) = -\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}\right], \quad \sum_{\lambda=0}^{3} g_{\lambda\kappa}\epsilon_{\lambda}^\mu(k)\epsilon_{\kappa}^\nu(k) = g_{\mu\nu},$$

$$a_{\lambda,a}(k)$$ are $4(\text{dim} G)$ standard (distributional) bosonic annihilation operators [12,17,18,19] acting in the Hilbert-Fock space

$$H_A = \bigoplus_{n=0}^{\infty} \left\{ \bigoplus_{\lambda=0}^{\frac{3}{2}} \bigoplus_{a=1}^{\text{dim} G} \left[ L^2(\mathcal{M}, \text{dk}) \right]_{\lambda,a} \right\}$$
which is equipped with the standard positive scalar product \((a, b)_A\). The operator \(O^+\) denotes the adjoint of \(O\) with respect to this scalar product. The Fock space operators fulfil
\[
[a_{\lambda, a}(k), a_{\lambda, b}^+(k')] = \delta_{\lambda, \lambda'} \delta_{ab} \delta(k - k')
\]
(6.5)
The number operators for a given polarization \(\lambda\) are defined by
\[
N_\lambda \overset{\text{def}}{=} \int dk a_{\lambda, a}^+(k) a_{\lambda, a}(k)
\]
(6.7)
The Krein operator \(J_A\) in \(H_A[5,8,16,17]\) is defined by
\[
J_A \overset{\text{def}}{=} (-1)^{N_0}
\]
(6.8)
It defines a pseudo-conjugation \(O^K[4,5,12,16,17]\) of an operator \(O\) by
\[
O^K \overset{\text{def}}{=} J_A O^+ J_A
\]
(6.9)
Sometimes the form \(\langle a, b \rangle_A := (a, J_A b)_A\) is called an indefinite scalar product. We will not follow this terminology here. The word orthogonal (hermitean, unitary) will always mean orthogonal (hermitean, unitary) with respect to the positive inner product. Otherwise we say pseudo-orthogonal (pseudo-hermitean, pseudo-unitary).

The gauge invariant physical Yang-Mills fields \(A_{\text{phys}}\) have only the three transversal polarisations:
\[
(A_{\text{phys}})_a^\mu(x) = \sum_{\lambda=1}^3 \int dk \left\{ \epsilon_\lambda^\mu(k) a_{\lambda, a}(k) e^{ikx} + \epsilon_\lambda^\mu(k) a_{\lambda, a}^+(k) e^{-ikx} \right\}, \quad dQ A_{\text{phys}} = 0
\]
(6.10)
and the commutator
\[
[(A_{\text{phys}})_a^\mu(x), (A_{\text{phys}})_b^\nu(y)] = -\left[ g^{\mu\nu} + \frac{\partial\mu \partial\nu}{M^2} \right] \delta_{ab} (-i) D_m(x - y)
\]
(6.11)
The unphysical Yang Mills \(A_{\text{unphys}}\) fields are given by
\[
(A_{\text{unphys}})_a^\mu(x) = A^\mu_a(x) - (A_{\text{phys}})_a^\nu(x) = \frac{-1}{M^2} \partial^\mu \partial_\nu A^\nu_a(x)
\]
(6.12)
The following conjugation properties are easily checked:
\[
A = A^K, \quad A_{\text{phys}} = A^K_{\text{phys}} = A^+_{\text{phys}}, \quad A_{\text{unphys}} = A^K_{\text{unphys}} = -A^+_{\text{unphys}}
\]
(6.13)
We also note
\[
F_a^\mu = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu = \partial^\mu (A_{\text{phys}})_a^\nu - \partial^\nu (A_{\text{phys}})_a^\mu = (F_a^\mu)^K = (F_a^\mu)^+
\]
(6.14)
\[
\partial^\mu A_a^\mu = \partial^\nu (A_{\text{unphys}})_a^\nu = (\partial^\nu A_a^\nu)^K = - (\partial^\nu A_a^\nu)^+
\]
(6.15)
The representation of the proper Poincaré group \(P^+_+\) in \(H_A\) is defined by
\[
U_A(a, \Lambda) A^\mu_a(x) U_A(a, \Lambda)^{-1} = \Lambda^\mu_\nu A^\nu_a(\Lambda x + a), \quad U_A(a, \Lambda) \Omega_A = \Omega_A
\]
(6.16)
where \(\Omega_A\) is the vacuum in \(H_A\). It is pseudo-unitary:
\[
U_A(a, \Lambda) U_A(a, \Lambda)^K = 1
\]
(6.17)
and, since it commutes with \(J_A\):
\[
J_A U_A(a, \Lambda) J_A = U_A(a, \Lambda)
\]
(6.18)
unitary as well:

\[ U_A(a, \Lambda)U_A(a, \Lambda)^+ = 1 \]  

(6.19)

The last two equations fail in the massless case where \( \epsilon_0^+ (k) \) cannot be chosen covariantly.

Next we come to the hermitean scalar gauge fields which are quantized in the usual way:

\[ h_a(x) = \int dk \left\{ b_a(k)e^{-ikx} + b_a^+(k)e^{ikx} \right\} = h_a^+(x) \]  

(6.20)

as operators in the Hilbert-Fock space

\[ H_h = \bigoplus_{n=0}^{\infty} \left( \bigoplus_{i=\pm}^{\dim G} \left[ L^2(\mathcal{M}, dk) \right]_{i,a} \right) \]  

(6.21)

with standard positive scalar product \((a, b)_h)\). We have

\[ [b_a(k), b_a^+(k')]_+ = \delta_{ab}\delta(k-k') \]  

(6.22)

We do not introduce an additional Krein structure in \( H_h \). This is equivalent to saying that \( J_h = 1 \) and that the two conjugations agree over \( H_h \):

\[ \langle a, b \rangle_h = \langle a, b \rangle_h. \]

The representation of \( P^+_\uparrow \) in \( H_h \): \( U_h(a, \Lambda) \) is unitary. The total number of scalar gauge particles is given by

\[ N_h = \int dk b_a^+(k)b_a(k) \]  

(6.23)

Now we consider the ghost fields. They have been extensively studied in [5]. So we summarize only the most important formulae here. The ghost fields

\[ u_a(x) = \int dk \left\{ c_{1,a}(k)e^{-ikx} + c_{-1,a}(k)e^{ikx} \right\}, \quad \bar{u}_a(x) = \int dk \left\{ -c_{-1,a}(k)e^{-ikx} + c_{1,a}(k)e^{ikx} \right\} \]  

(6.24)

are defined in the Hilbert-Fock space

\[ H_g = \bigoplus_{n=0}^{\infty} \left( \bigotimes_{i=\pm}^{\dim G} \left[ L^2(\mathcal{M}, dk) \right]_{i,a} \right) \]  

(6.25)

with positive inner product \((a, b)_g)\). The index \( i \) distinguishes ghost from antighost particles. We have

\[ \{c_{i,a}(k), c_{j,b}^+(k')\}_+ = \delta_{ij}\delta_{ab}\delta(k-k') \]  

(6.26)

The Krein operator in \( H_g \) is defined by

\[ J_g = i^{N_g-1} \]  

(6.27)

Here \( N_g \) denotes the total number of ghost and antighost particles:

\[ N_g = \int dk c_{i,a}^+(k)c_{i,a}(k) \]  

(6.28)

while \( \Gamma \) is defined by

\[ \Gamma = \int dk \left\{ c_{1,a}^+(k)c_{-1}(k) + c_{-1}^+(k)c_{1}(k) \right\} \]  

(6.29)

Again one considers the indefinite form \( \langle a, b \rangle_g := (a, J_g b) \) and defines \( O^K = J_g O^+ J_g \). The representation of \( P^+_\uparrow \) in \( H_g \): \( U_g(a, \Lambda) \) is unitary and, since it commutes with \( J_g \), pseudo-unitary as well [5].
The scalar and Dirac matter fields are quantized in their own Hilbert-Fock space $H_{\text{matter}}$ in the usual way. The scalar product in this space is again positive and the Krein structure $J_{\text{matter}}$ is the unit operator. The representation $U_{\text{matter}}(a, \Lambda)$ is unitary.

The Hilbert space $H$ of the total system is the tensor-product of the spaces above:

\[ H = H_A \otimes H_h \otimes H_g \otimes H_{\text{matter}} \]  

(6.30)

The Krein operator and the representation of $P^+_\uparrow$ factorize accordingly:

\[ J = J_A \otimes J_h \otimes J_g \otimes J_{\text{matter}} \]

(6.31)

\[ U(a, \Lambda) = U_A(a, \Lambda) \otimes U_h(a, \Lambda) \otimes U_g(a, \Lambda) \otimes U_{\text{matter}}(a, \Lambda) \]

(6.32)

This $U$ is unitary and pseudo-unitary, since it commutes with $J$. The positive scalar product in $H$ is denoted by $(a, b)$ and $\|a\| := (a, a)$, $\langle a, b \rangle := (a, Jb)$, $O_K := JO^+J$.

Our next task is to study more closely the gauge charge $Q$ (4.3). It is expressed in momentum space as

\[ Q = M \int dk \left\{ c_{1,a}(k) [a_{0,a}(k) + ib_a(k)] - [a^+_{0,a}(k) + ib^+_a(k)] c_{1,a}(k) \right\} \]

(6.33)

Its adjoint $Q^+$ is given by:

\[ Q^+ = M \int dk \left\{ c^+_{1,a}(k) [-a_{0,a}(k) + ib_a(k)] + [a^+_{0,a}(k) - ib^+_a(k)] c_{-1,a}(k) \right\} \]

(6.34)

$Q$ and $Q^+$ are both pseudo-hermitean $P^+_\uparrow$ invariant differential operators:

\[ Q^2 = (Q^+)^2 = 0, \quad Q = Q^K, \quad (Q^+)^K = Q^+, \quad U(a, \Lambda)Q^{(+)}U(a, \Lambda)^{-1} = Q^{(+)} \]

(6.35)

We now follow Razumov and Rybkin [8] who showed that the physical Hilbert space of a gauge theory with quadratic BRS charge $Q$ can be defined as

\[ H_{\text{phys}} \overset{\text{def}}{=} \text{kernel } \{Q, Q^+\}_+ \]

(6.36)

Razumov showed the equivalence of this definition with the more conventional one using equivalent classes in semidefinite metric spaces [7]. Razumov’s definition is advantageous because it realizes $H_{\text{phys}}$ as a concrete subspace of the Hilbert space $H$ which has a clear particle interpretation. To work this out we only have to calculate the above anticommutator. We find:

\[ \{Q, Q^+\}_+ = 2[N_0 + N_h + N_g] \overset{\text{def}}{=} 2N \]

(6.37)

i.e. $N$ is the number of longitudinal Yang-Mills fields plus the number of scalar gauge fields plus the number of ghost and antighost particles. Thus all these particles are unphysical. The only physical particles are the transverse quanta of the Yang-Mills fields and the matter particles.

The spectrum of the number operator $N$ are the natural numbers and 0:

\[ N = \sum_{n=0}^{\infty} nP_n \]

(6.38)

where $P_n$ is the orthogonal projector on the subspace with $n$ unphysical particles. (6.36) means that the orthogonal projector on $H_{\text{phys}}$ is given by $P_0$:

\[ H_{\text{phys}} = P_0 H \]

(6.39)
The operator $N$ can be inverted on the orthogonal complement of its kernel:

$$N^{-1} \overset{\text{def}}{=} 0P_0 + \sum_{n=1}^\infty n^{-1}P_n, \quad N^{-1}N = NN^{-1} = (1 - P_{\text{phys}})$$ (6.40)

Since $Q$ and $Q^+$ are $P_+^\dagger$ invariant, so are $N$, $N^{-1}$, and $P_n$:

$$U(a, \Lambda) \{ N; N^{-1}; P_n \} U(a, \Lambda)^{-1} = \{ N; N^{-1}; P_n \}$$ (6.41)

We also note that $N$, $P_n$, and $N^{-1}$ commute with $Q$ and $Q^+$:

$$[Q(\cdot), N]_\pm = [Q(\cdot), P_n]_\pm = [Q(\cdot), N^{-1}]_\pm = 0$$ (6.42)

We now follow again [8] and introduce the following subspaces of $H$:

$$H_K \overset{\text{def}}{=} \text{kernel } Q, \quad H_{K^+} \overset{\text{def}}{=} \text{kernel } Q^+, \quad H_R \overset{\text{def}}{=} \text{range } Q, \quad H_{R^+} \overset{\text{def}}{=} \text{range } Q^+$$ (6.43)

Let us study the relations between these spaces and $H_{\text{phys}}$. Let $a_0 \in H_{\text{phys}}$. By

$$0 = (a_0, \{ Q, Q^+ \}, a_0) = ||Q^+ a_0||^2 + ||Q a_0||^2$$ (6.44)

we find

$$H_{\text{phys}} = H_K \cap H_{K^+}$$ (6.45)

Since $Q^2 = (Q^+)^2 = 0$ we have

$$H_R \subseteq H_K, \quad H_{R^+} \subseteq H_{K^+}$$ (6.46)

Let $a_K \in H_K$, $b_{R^+} \in H_{R^+}$. Since

$$(a_K, b_{R^+}) = (a_K, Q^+ b) = (Q a_K, b) = 0$$ (6.47)

$H_K$ and $H_{R^+}$ are orthogonal to each other, and replacing $Q$ by $Q^+$ shows that the same holds true for $H_R$ and $H_{K^+}$:

$$H_K \perp H_{R^+}, \quad H_R \perp H_{K^+}$$ (6.48)

Combining this with (6.46) gives

$$H_R \perp H_{R^+}$$ (6.49)

while (6.45) now implies

$$H_{\text{phys}} \perp H_R, \quad H_{\text{phys}} \perp H_{R^+}$$ (6.50)

We conclude that the three spaces $H_{\text{phys}}$, $H_R$, and $H_{R^+}$ are all mutually orthogonal. Let now $a \in H$. Then we can write

$$a = P_0 a + (1 - P_0) a = P_0 a + N N^{-1} a = P_0 a + \frac{1}{2} QQ^+ N^{-1} a + \frac{1}{2} Q^+ Q N^{-1} a \overset{\text{def}}{=} a_0 + a_R + a_{R^+}$$ (6.51)

where $a_0 \in H_{\text{phys}}$, $a_R \in H_R$, and $a_{R^+} \in H_{R^+}$. This shows that the Hilbert space $H$ is the direct orthogonal sum of the three spaces $H_{\text{phys}}$, $H_R$, and $H_{R^+}$:

$$H = H_{\text{phys}} \perp H_R \perp H_{R^+}$$ (6.52)

Since the first two of this spaces are subspaces of $H_K$ and the third is orthogonal to it we can also write:

$$H = H_K \perp H_{R^+}, \quad H_K = H_{\text{phys}} \perp H_R$$ (6.53)
i.e the physical Hilbert space is also given by

\[ H_{\text{phys}} = H_K \ominus_{\perp} H_R \quad (6.54) \]

Moreover, since \( H_{\text{phys}} \) and \( H_R^+ \) are subspaces of \( H_K^+ \) and this space is orthogonal to \( H_R \) one can also write

\[ H = H_K^+ \ominus_{\perp} H_R , \quad H_K^+ = H_{\text{phys}} \ominus_{\perp} H_R^+ \quad (6.55) \]

i.e. we get one more characterization of \( H_{\text{phys}} \) as

\[ H_{\text{phys}} = H_K^+ \ominus_{\perp} H_R^+ \quad (6.56) \]

The orthogonal decompositions above were already given in [8], and, in the specific context of massless Yang-Mills theories, in [4]. We denote the orthogonal projections on \( \{ H_{\text{phys}}; H_K; H_R; H_K^+ H_R^+ \} \) by \( \{ P_0; P_K; P_R; P_{K^+}; P_{R^+} \} \) and the vectors in these spaces by \( \{ a_0; a_K; a_R; a_{K^+}; a_{R^+} \} \). From the preceding equations we find:

\[ P_0 P_R = P_0 P_{R^+} = P_R P_{R^+} = 0 , \quad P_0 + P_R + P_{R^+} = 1 , \quad P_0 = P_0^+ , \quad P_R = P_R^+ , \quad P_{R^+} = P_{R^+} \quad (6.57) \]

\[ P_R = \frac{1}{2} Q Q^+ N^{\sim 1} , \quad P_{R^+} = \frac{1}{2} Q^+ Q N^{\sim 1} \quad (6.58) \]

Let us now study the structure of some important operators with respect to the orthogonal decomposition (6.52). The operators \( Q \) and \( Q^+ \) map the complements of their kernels onto their range. This gives:

\[ Q = P_R Q P_{R^+} , \quad Q^+ = P_{R^+} Q^+ P_R \quad (6.59) \]

Then (6.37) implies that the decomposition of \( N \) and \( N^{\sim 1} \) are given by

\[ N^{(\sim 1)} = P_R N^{(\sim 1)} P_R + P_{R^+} N^{(\sim 1)} P_{R^+} \quad (6.60) \]

Let now \( I \) be a gauge invariant operator, i.e. \( d_Q I = 0 \). Then \( H_K \) and \( H_R \) are stable under the action of \( I \), that is

\[ IP_K = P_K IP_K , \quad IP_R = P_R IP_R \quad (6.61) \]

Thus we get

\[ I = P_0 IP_0 + P_0 IP_{R^+} + P_R IP_0 + P_R IP_{R^+} + P_{R^+} IP_R + P_{R^+} IP_{R^+} \quad (6.62) \]

Next we use the pseudo-hermiticity of \( Q \) and \( Q^K \) to get information about the Krein operator \( J \). Let \( a_K \in H_K, \quad a_R = Q a_R \in H_R \). Then we have

\[ (a_K, J a_R) = < a_K, Q a_R > = < Q a_K, a_R > = 0 \quad (6.63) \]

This means:

\[ P_K JP_R = 0 \quad (6.64) \]

Taking the adjoint gives:

\[ P_R JP_K = 0 \quad (6.65) \]

Using \( Q^+ \) instead of \( Q \) in the argument above gives

\[ P_{K^+} JP_{R^+} = P_{R^+} JP_{K^+} = 0 \quad (6.66) \]

A direct inspection of \( J \) in (6.31) gives the additional information that \( J \) agrees on \( H_{\text{phys}} \) with the unit operator:

\[ P_0 JP_0 = P_0 \quad (6.67) \]
The last four equations are summarized in

\[ J = P_0 J P_0 + P_R J P_R^+ + P_R^+ J P_R \]  

(6.68)

The second of eqs. (6.53) means that \( H_K \) can be interpreted as a linear fiber bundle: \( H_{\text{phys}} \) is the base space and the fibers are the elements of \( H_R \). Eqs. (6.64, 6.65) show that the fibers are pseudo-orthogonal to any vector in \( H_K \). Moreover, writing \( a_K = a_0 + a_R \) according to the orthogonal decomposition (6.53) gives

\[ \langle a_K, b_K \rangle = (a_0, b_0) \]  

(6.69)

This shows that the form \( \langle \cdot, \cdot \rangle \) agrees on \( H_{\text{phys}} \) with the positive form \( (\cdot, \cdot) \), that it is positive semidefinite in \( H_K \), and that its kernel as a quadratic form in this space are the fibers:

\[ H_R = \text{kernel} \langle \cdot, \cdot \rangle_K \]  

(6.70)

where \( \langle \cdot, \cdot \rangle_K \) means the restriction of the form \( \langle \cdot, \cdot \rangle \) to \( H_K \). So we get another expression for \( H_{\text{phys}} \):

\[ H_{\text{phys}} = H_K \ominus H_R \]  

(6.72)

The form \( \langle \cdot, \cdot \rangle_K \) is constant along the fibers in both arguments separately:

\[ \langle a_K + a_R, b_K + b_R \rangle_K = \langle a_K, b_K \rangle_K \]  

(6.73)

and the same holds true for the matrix elements of any gauge invariant operator \( I \) with respect to this form:

\[ \langle a_K + a_R, I (b_K + b_R) \rangle_K = \langle a_K, I b_K \rangle_K \]  

(6.73)

This allows to choose any linear cross section \( H_S \) in \( H_K \), i.e. any subspace of \( H \) which is a (pseudo-orthogonal but generally not orthogonal) complement of \( H_R \) (in \( H_K \)) as a realization of the physical Hilbert space. The scalar product in \( H_S \) is the restriction of the form \( \langle \cdot, \cdot \rangle \) to this space, and there it is positive definite. All this spaces are unitarily equivalent, and the matrix elements of gauge invariant operators do not depend on the section chosen. So one might also consider the equivalence class of all this spaces and that is what is usually done in the literature [7]. We prefer to use \( H_{\text{phys}} \) as a concrete realization of the physical Hilbert space since it is the only section which is orthogonal to the fibers and which allows for a simple interpretation of the quanta of the elementary fields as physical or unphysical particles. The projections onto the sections along the fibers are also called gauge transformations. For \( H_{\text{phys}} \), and only for it, they agree with the orthogonal projection.

We now consider again operators \( A, B, C \cdots \) over \( H \). We define the orthogonal projection of \( A \) on \( H_{\text{phys}} \): \( A_0 \) by

\[ A_0 \overset{\text{def}}{=} P_0 A P_0 \]  

(6.74)

\( A_0 \) is still an operator from \( H \) to \( H \). It is zero on the orthogonal complement of \( H_{\text{phys}} \): \( H_{\text{phys}}^\perp \). Since this zero is certainly not very interesting we define \( A_{\text{phys}} \) to be the restriction of \( A_0 \) to \( H_{\text{phys}} \):

\[ A_{\text{phys}} \overset{\text{def}}{=} (A_0)_{\downarrow H_{\text{phys}}} \]  

(6.75)

The map \( A \rightarrow A_{\text{phys}} \) is certainly linear:

\[ (\alpha A)_{\text{phys}} = \alpha A_{\text{phys}}, \quad (A + B)_{\text{phys}} = A_{\text{phys}} + B_{\text{phys}} \]  

(6.76)

More interesting is the projection of the product of two operators. We calculate:

\[ P_0 A BP_0 = P_0 A P_0 + P_R A P_R P_0 A P_0 + P_R A P_R P_0 A P_0 + P_0 A \frac{1}{2} Q Q^+ N^{-1} B P_0 + P_0 A \frac{1}{2} N^{-1} Q^+ Q B P_0 \]  

(6.77)
Let us concentrate on the second summand: \( X \). Since \( P_0 Q = 0 \) we can replace \( AQ \) by \( \{ A, Q \}_\pm \). We take the anticommutator if \( A \) is fermionic and the commutator if it is bosonic. This gives

\[
X = P_0 \frac{1}{2} \{ A, Q \}_\pm Q^+ N^{-1} B P_0
\]  

(6.78)

Now we use \( P_0 Q^+ = 0 \) to replace that by

\[
X = P_0 \frac{1}{2} \{ \{ A, Q \}_\pm, Q^+ \} \pm N^{-1} B P_0
\]

(6.79)

And finally we use \( P_0 N^{-1} = 0 \) to write

\[
X = P_0 \frac{1}{2} \left[ \{ \{ A, Q \}_\pm, Q^+ \} \pm , N^{-1} \right] _- B P_0
\]

(6.80)

There are always two commutators and one anticommutator in this expression. Thus it can be uniquely written as

\[
X = P_0 (TA) B P_0
\]

(6.81)

where the triple variation \( T \) is defined by

\[
T \equiv \frac{1}{2} \delta_{(N^{-1})} d_{(Q^+)} d_Q
\]

(6.82)

Here \( \delta_{(N^{-1})} \) is the derivation induced by \( N^{-1} \), and \( d_Q \) and \( d_{(Q^+)} \) are the antiderivations induced by \( Q \) and \( Q^+ \), respectively (see chapter 1). Note that the triple variation of gauge invariant operators vanishes. In the same way the third summand in (6.77): \( Y \) is written as

\[
Y = P_0 A(TB) P_0
\]

(6.83)

We thus have found the important projection formula:

\[
A_{\text{phys}} B_{\text{phys}} = (AB)_{\text{phys}} - \{ (TA) B + A(TB) \}_{\text{phys}}
\]

(6.84)

This implies

**Theorem I:** The product of the physical projections of two operators with vanishing triple variation, especially of two gauge invariant operators, is identical to the physical projection of their product. The physical projection of a group (of an algebra) of operators with vanishing triple variation, especially of gauge invariant operators, is a representation of this group (algebra).

Next we consider the physical projection of the pseudo-adjoint \( A^K \) of an operator \( A \). So we have to study

\[
P_0 A^K P_0 = P_0 J A^+ J P_0
\]

(6.85)

Now we use that (6.68) implies

\[
P_0 J = J P_0 = P_0 J P_0 = P_0
\]

(6.86)

to conclude:

\[
P_0 A^K P_0 = P_0 A^+ P_0 = (P_0 A P_0)^+
\]

(6.87)

which means

\[
(A^K)_{\text{phys}} = (A_{\text{phys}})^+
\]

(6.88)

Thus we have found

**Theorem II:** The physical projection of the pseudo-adjoint operator is identical to the adjoint of the physical projection of this operator. The physical projection of a pseudo-hermitean operator is hermitean.
Taking the pseudo-conjugate of this equation gives:

\[ \left( T^{(1)}(x) \right)^K = -T^{(1)}(x) \] (6.89)

This guarantees the pseudo-unitarity of \( S[g] \) [11,12,15]:

\[ S[g]S^K[g] = 1 \] (6.90)

This is [11] equivalent to

\[ \sum_{I \uplus J = N} T(I)T^K(J) = 0, \ \forall N \neq \emptyset \] (6.91)

Here \( T(I) \) means \( T^{(r)}(x_{i_1}, \cdots, x_{i_r}) \), \( T^K(J) \) means \( (T^{(s)}(x_{j_1}, \cdots, x_{j_s}))^K \), where \( r + s = n \), and the sum \( \Sigma \) runs over all direct decompositions of the set \( N = \{1, \cdots, n\} \) into two subsets \( I = \{i_1, \cdots, i_r\} \) and \( J = \{j_1, \cdots, j_s\} \). Let us now assume that gauge invariance holds true in all orders, i.e. we have

\[ d_Q T^{(n)} = \sum_{k=1}^{4n} \partial^k T^{(n)}_k \] (6.92)

Taking the pseudo-conjugate of this equation gives:

\[ d_Q \left( T^{(n)} \right)^K = -\sum_{k=1}^{4n} \partial^k \left( T^{(n)}_k \right)^K \] (6.93)

Then (6.84,6.88) and (6.90-6.92) imply

\[ \sum_{I \uplus J = X} T_{\text{phys}}(I) (T_{\text{phys}}(J))^+ = \sum_{k=1}^{4n} \partial^k W^{(n)}_k(X), \]

\[ W^{(n)}_k(X) \overset{\text{def}}{=} \frac{1}{2} \sum_{I \uplus J = X} \left\{ -\theta_I(k) \left( \delta_{(N-1)}d_{(Q^+)}T_k(I) \right) T^K(J) + \theta_J(k)T(I) \left( \delta_{(N-1)}d_{(Q^+)}T^K(J) \right) \right\}_{\text{phys}} \] (6.94)

where

\[ \theta_I(k) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } k \in I \\ 0, & \text{if } k \notin I \end{cases} \] (6.95)

This is the exact perturbative, pre-adiabatic expression for the unitarity of the physical \( S \)-matrix. If the adiabatic limit:

\[ S_{\text{phys}} = \lim_{g \to 1} (S[g])_{\text{phys}} \] (6.96)

exists and has the same analytic properties as in the scalar theory discussed in [13], and if the boundary terms \( \int \partial^k W^{(n)}_k \) vanish, (6.94) will imply

\[ S_{\text{phys}}S^K_{\text{phys}} = 1 \] (6.97)

We note, however, that this adiabatic limit may have additional subtleties if the heavy gauge particles are coupled to light matter, since then the gauge fields become unstable, i.e. are not really asymptotic fields.

Let us end this long chapter with a short discussion of \( P^+_1 \)-invariance. The remark after (6.32) and eq. (6.41) immediately imply that \( U(a, \Lambda)_{\text{phys}} \) is an unitary representation of \( P^+_1 \), while the \( P^+_1 \)-invariance of \( S[g] \):

\[ U(a, \Lambda)S[g]U(a, \Lambda)^{-1} = S[g, \Lambda], \quad g_{a, \Lambda}(x) \overset{\text{def}}{=} \Lambda^{-1}(x - a) \] (6.98)
and (6.41) lead direct to

\[ U(a, \Lambda)S[g]_{\text{phys}}U(a, \Lambda)^{-1} = S[g, \Lambda]_{\text{phys}} \]  

(6.99)

which is the \( P^+ \)-invariance of the physical \( S \)-matrix.

The situation is different in the massless case where \( J \) is not \( P^+ \)-invariant. However, the three theorems above and the projection formula (6.84) hold true in this case, too. Since \( Q \) is \( P^+ \)-invariant also in the massless case, the theorems show that \( U(a, \Lambda)_{\text{phys}} \) is indeed an unitary representation of \( P^+ \), while the \( P^+ \)-invariance of \( S[g] \) and (6.84,6.88) imply that (6.99) is violated only by boundary terms. The latter should vanish in physical quantities like cross sections, for example.

### 7. Discussion

We first have shown that the interaction of massless Yang-Mills fields studied in [1-4] admits generalizations. Then we have constructed gauge invariant first order interactions between massive Yang-Mills fields, scalar gauge fields, and matter fields. We have shown how invariance under ghost charge conjugation fixes the interaction uniquely. The scalar gauge fields had to be introduced for a pure algebraic reason, i.e. to have \( Q^2 = 0 \). Moreover, we have proven that gauge invariance implies unitarity of the physical \( S \)-matrix and that the latter is Poincaré invariant. However, gauge invariance were only shown to hold true in the first order. This is certainly not enough. Before we can claim to have a completely consistent theory we have to proof gauge invariance in all orders, i.e. the absence of anomalies. We intend to tackle this labourious though important task in future publications.

The analysis of unitarity has shown that all scalar gauge fields are unphysical. This means that our (pure) gauge theory is not a Higgs-model. One could, of course, introduce a physical scalar particle in the matter sector of the theory. But there seems to be no logical reason for doing so (unless one finds out that this would be the only way to avoid anomalies) since, in contrast to the Higgs model, the matter sector and the gauge sector are independent structures in our theory. Our theory has structural similarity to Stueckelberg type gauge theories. There [20] all scalar gauge fields are unphysical, too. We remark, however, that our theory is not identical to the Stueckelberg models discussed in [20], it differs from them in the detailed structure of the propagators and couplings. Stueckelberg type theories are most often discussed under the aspect of classical gauge symmetry and gauge independence while BRS-invariance is only a derived concept. We have followed a completely different route and have made (free) BRS-invariance a first principle. This seems appropriate since it is exactly this invariance which implies unitarity. All Stueckelberg models studied in [20] where either nonrenormalizable or nonunitary. This was shown by direct calculation, but the reason why remained a mystery. Our model is renormalizable by construction, and unitarity follows from gauge invariance. So, should it fail to be consistent, we know at least why: There have to be anomalies. One could then study these anomalies to learn how to avoid them.

To proof unitarity of the physical \( S \)-matrix we have extended Razumovský and Rybkiński investigation of quadratic BRS systems [8] by giving the important projection formula (6.84). This formula shows that the triple variation \( \mathcal{T} \) is more important for the algebraic structure of the theory than the gauge variation is. Indeed, it follows from our calculations in the last chapter that a theory which is not gauge-invariant but is invariant under the triple variation \( \mathcal{T} \) would still have an unitary physical \( S \)-matrix! Thus one might call anomalies of gauge invariance which do not violate triple invariance weak anomalies and anomalies which do violate it strong anomalies. Well, it could happen that all known anomalies are strong ones, but this is far from obvious. A careful algebraic analysis of anomalies with respect to the triple variation is certainly worth doing, and we plan to investigate this interesting structure in the future.

Another thing which one has to do before our theory can make contact with “physics” is the extension to the nonsimple group \( G = U(2) \) and an analysis of global \( G \)-breaking which allows for different masses of the various gauge fields.

Let us summarize: A direct algebraic analysis of massive quantized Yang-Mills theories done in the framework of causal perturbation theory and free of any classical notions as spontaneous symmetry breaking has naturally led to the construction of a Higgs free model of massive gauge fields and has revealed new interesting algebraic structures. Further investigations remain to be done.
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