EQUILIBRIUM STATES, PHASE TRANSITIONS AND DYNAMICS IN QUANTUM ANHARMONIC CRYSTALS

YURI KOZITSKY

Abstract. The basic elements of the mathematical theory of states of thermal equilibrium of infinite systems of quantum anharmonic oscillators (quantum crystals) are outlined. The main concept of this theory is to describe the states of finite portions of the whole system (local states) in terms of stochastically positive KMS systems and path measures. The global states are constructed as Gibbs path measures satisfying the corresponding DLR equation. The multiplicity of such measures is then treated as the existence of phase transitions. This effect can be established by analyzing the properties of the Matsubara functions corresponding to the global states. The equilibrium dynamics of finite subsystems can also be described by means of these functions. Then three basic results of this theory are presented and discussed: (a) a sufficient condition for a phase transition to occur at some temperature; (b) a sufficient condition for the suppression of phase transitions at all temperatures (quantum stabilization); (c) a statement showing how the phase transition can affect the local equilibrium dynamics.

1. Generalities

In recent years, remarkable progress has been made in the experimental testing of the fundamentals of quantum physics, as well as in developing quantum information theory and basics of quantum computing, see [7]. Due to these advances the elaboration of the mathematical background of quantum theory returned to the circle of actual tasks of applied mathematics. Developing the statistical description of infinite systems of interacting quantum particles is one of them. Essential results in this direction were obtained by means of methods developed in stochastic analysis, mostly in the approach in which states of such systems are constructed as probability measures on infinite dimensional path spaces. A substantial part of these results appeared due to Michael Röckner’s research activity, see, e.g., [1, 2, 3, 4, 5, 12]. The aim of this work is to outline the main aspects of the theory of equilibrium states of quantum anharmonic crystals obtained in [3] in the approach based on path measures.

1.1. The Anharmonic Crystal. An anharmonic oscillator is a mathematical model of a point particle moving in a potential field with multiple minima and sufficient growth at infinity. In the simplest case, the motion is one-dimensional and the potential has two minima (wells) separated by a potential barrier. If the motion is governed by the laws of classical mechanics, the oscillator’s states are characterized by a couple $(q, p) \in \mathbb{R}^2$, where $q$ is the displacement of the oscillator from a certain point and $p$ is its momentum (amount of motion). In the states with sufficiently small fixed $|p|$, the particle is confined to one of the wells. This produces a degeneracy – the multiplicity of states $(q, p)$ with the same $p$ and energy $E$, that is, the

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multiplicity of $q$ solving the equation

$$H(p,q) := \frac{1}{2m}p^2 + \frac{a}{2}q^2 + V(q) = E. \tag{1.1}$$

Here $H(p,q)$ is the particle’s Hamiltonian in which $m > 0$ is the mass of the particle and the second and third terms constitute the potential energy. If $V \equiv 0$, the oscillator is harmonic (of rigidity $a > 0$), i.e., the third term can be considered as an anharmonic correction to the potential energy. In the quantum case, the particle’s states are vectors of unit norm belonging to the complex Hilbert state $L^2(\mathbb{R})$. The displacement and momentum are then unbounded operators defined on $L^2(\mathbb{R})$, satisfying (on a common domain) the following commutation relation

$$[p,q] := pq - qp = -i, \quad i := \sqrt{-1}. \tag{1.2}$$

In (1.2), we use the physical units in which the Planck constant is set $\hbar = 1$. Assume now that an infinite system of such particles is arranged into a crystal. That is, each particle is attached to its own crystal site $\ell \in \mathbb{Z}^d$, $d \geq 1$, and performs oscillations in its own copy of $\mathbb{R}$. For a finite $\Lambda \subset \mathbb{Z}^d$, the state space of the particles attached to the sites in $\Lambda$ is the tensor product of the single-particle spaces, i.e., $H_\Lambda = L^2(\mathbb{R}^\Lambda)$. The Hamiltonian of this portion of particles is

$$H_\Lambda = \sum_{\ell \in \Lambda} H_\ell + J \sum_{\ell \sim \ell', \Lambda} q_\ell q_{\ell'},$$

$$H_\ell := \frac{1}{2m}p_\ell^2 + \frac{a}{2}q_\ell^2 + V(q_\ell). \tag{1.3}$$

Here $H_\ell$ is the Hamiltonian of an isolated quantum anharmonic oscillator. The sum in the second term of the first line in (1.3) is taken over all pairs of $\ell, \ell' \in \Lambda$ satisfying $|\ell - \ell'| = 1$. It describes the interaction between the neighboring oscillators located in $\Lambda$ with intensity $J > 0$. The anharmonic potential $V$ is assumed to grow at infinity faster than $q^2$. For simplicity, in this article we take it in the form

$$V(q) = -b_1q^2 + b_2q^4, \quad b_1, b_2 > 0. \tag{1.4}$$

The Hamiltonian $H_\Lambda$ in (1.3) with $V$ as in (1.4) can be defined as a lower bounded self-adjoint operator in $H_\Lambda$ such that $\exp(-\beta H_\Lambda)$ is a positive trace-class operator for each $\beta > 0$. Thus, one can set

$$Z_{\beta,\Lambda} = \text{trace } \exp(-\beta H_\Lambda). \tag{1.5}$$

The state of thermal equilibrium of the oscillators attached to the sites in $\Lambda$ (local Gibbs state) is defined as a positive normalized linear functional $\rho_{\beta,\Lambda} : \mathcal{C}_\Lambda \rightarrow \mathbb{C}$ by the following formula

$$\rho_{\beta,\Lambda}(A) = \text{trace } [A \exp(-\beta H_\Lambda)] / Z_{\beta,\Lambda}, \quad A \in \mathcal{C}_\Lambda. \tag{1.6}$$

Here $\beta = 1/k_B T$, $k_B$ and $T$ are Boltzmann’s constant and temperature, respectively, and $\mathcal{C}_\Lambda$ is the algebra of all bounded linear operators $A : H_\Lambda \rightarrow H_\Lambda$, called observables. By Høegh-Krohn’s theorem [3, page 72] $\rho_{\beta,\Lambda}$ is uniquely determined by its values on the linear span of products

$$a_{t_1}^\Lambda(F_1) \cdots a_{t_n}^\Lambda(F_n), \quad n \in \mathbb{N}, \quad F_1, \ldots, F_n \in \mathfrak{A}_\Lambda, \quad t_1, \ldots, t_n \in \mathbb{R},$$

where $\mathfrak{A}_\Lambda$ is a complete family of multiplication operators by bounded measurable functions $F : \mathbb{R}^\Lambda \rightarrow \mathbb{C}$, whereas

$$a_t^\Lambda(A) := \exp(itH_\Lambda) A \exp(-itH_\Lambda), \quad A \in \mathcal{C}_\Lambda.$$
According to [3, Theorem 1.3.6], $\mathcal{F}_\Lambda$ is complete if it satisfies: (a) for $F_1, F_2 \in \mathcal{F}_\Lambda$, the point-wise products $F_1 F_2$ is also in $\mathcal{F}_\Lambda$; (b) the constant function $1$ belongs to $\mathcal{F}_\Lambda$; (c) for each distinct $x_\Lambda, y_\Lambda \in \mathbb{R}^\Lambda$, one finds $F \in \mathcal{F}_\Lambda$ such that $F(x_\Lambda) \neq F(y_\Lambda)$. Since $H_\Lambda$ is self-adjoint, the map $A \mapsto a^*_\Lambda(A)$ is an isometric automorphism of $\mathcal{C}_\Lambda$. At the same time, the map $\mathbb{R} \ni t \mapsto a^t_\Lambda(A) \in \mathcal{C}_\Lambda$ is the (time) evolution of the observable $A$. The group $\{a^t_\Lambda\}_{t \in \mathbb{R}}$ describes the dynamics of the corresponding finite subsystem. The mentioned above Høegh-Krohn theorem implies that $\rho_{t,\Lambda}$ is determined by the Green functions

$$G_{F_1,\ldots,F_n}^{\beta,A}(t_1,\ldots,t_n) := \rho_{\beta,\Lambda}\left[a^t_1(F_1)\cdots a^t_n(F_n)\right],$$

with all choices of $F_1,\ldots,F_n \in \mathcal{F}_\Lambda$. Each Green function admits an analytic continuation to the domain

$$\mathcal{D}_{n,\beta} := \{(\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^n : 0 < \text{Im}(\zeta_1) < \cdots < \text{Im}(\zeta_n) < \beta\}.$$ (1.8)

Furthermore, see [3, Theorem 1.2.32, page 78], it can further be continuously extended to the closure $\overline{\mathcal{D}}_{n,\beta}$ of (1.3). The set

$$\mathcal{D}_{n,\beta}^{(0)} := \{(\zeta_1,\ldots,\zeta_n) \in \overline{\mathcal{D}}_{n,\beta} : \text{Re}(\zeta_1) = \cdots = \text{Re}(\zeta_n) = 0\}$$ (1.9)

has the following property: each two continuous functions $f_1, f_2 : \overline{\mathcal{D}}_{n,\beta} \to \mathbb{C}$, analytic on $\mathcal{D}_{n,\beta}$ and equal on $\mathcal{D}_{n,\beta}^{(0)}$, are equal as functions. Then $G_{F_1,\ldots,F_n}^{\beta,A}$ is uniquely determined by its restriction to (1.9), that is, by the Matsubara function

$$\Gamma_{F_1,\ldots,F_n}^{\beta,A}(\tau_1,\ldots,\tau_n) = G_{F_1,\ldots,F_n}^{\beta,A}(i\tau_1,\ldots,i\tau_n).$$ (1.10)

1.2. The Path Measures. The main ingredient of the technique developed in [3] is the following representation, see [3, Theorem 1.4.5],

$$\Gamma_{F_1,\ldots,F_n}^{\beta,A}(\tau_1,\ldots,\tau_n) = \int_{\Omega_{\beta,\Lambda}} F_1(\omega_\Lambda(\tau_1))\cdots F_n(\omega_\Lambda(\tau_n))\mu_{\beta,\Lambda}(d\omega_\Lambda).$$ (1.11)

Here $\mu_{\beta,\Lambda}$ is a probability measure on the Banach space $\Omega_{\beta,\Lambda}$ of ‘temperature loops’, which is

$$\Omega_{\beta,\Lambda} = \{\omega_\Lambda = (\omega_\ell)_{\ell \in \Lambda} : \omega_\ell \in \mathcal{C}_\beta, \|\omega_\Lambda\| = \sup_{\ell \in \Lambda} \|\omega_\ell\|_{\mathcal{C}_\beta},$$

$$\mathcal{C}_\beta = \{\phi \in C([0,\beta] \to \mathbb{R}) : \phi(0) = \phi(\beta)\}, \|\phi\|_{\mathcal{C}_\beta} = \sup_{\tau \in [0,\beta]} |\phi(\tau)|.$$ (1.12)

The measure $\mu_{\beta,\Lambda}$ is constructed in the following way. Let

$$H_{\text{har}} = \frac{1}{2m}p^2 + \frac{a}{2}q^2$$ (1.12)

be the Hamiltonian of a single harmonic oscillator, cf. (1.1) and (1.3), which can be defined as an unbounded self-adjoint operator on $L^2(\mathbb{R})$. It has discrete spectrum consisting of nondegenerate eigenvalues

$$E_n^{\text{har}} = (n + 1/2)\Delta_{\text{har}}, \quad \Delta_{\text{har}} = \sqrt{a/m},$$ (1.13)
see [3] Proposition 1.1.37, page 41. Set $Z^\text{har}_\beta = \text{trace} \exp(-\beta H^\text{har})$, cf. [12], and then

$$S_\beta(\tau, \tau') = \text{trace} \left( q e^{-|\tau-\tau'|H^\text{har}} q e^{-(\beta-|\tau-\tau'|)H^\text{har}} \right) / Z^\text{har}_\beta$$

$$= \left( e^{-|\tau-\tau'|\Delta^\text{har}} + e^{-(\beta-|\tau-\tau'|)\Delta^\text{har}} \right) / 2\sqrt{ma} \left( 1 - e^{-\beta\Delta^\text{har}} \right), \quad \tau, \tau' \in [0, \beta].$$

By means of the ‘propagator’ (1.14) we define a Gaussian measure, $\chi_\beta$, on $C_\beta$ by its Fourier transform

$$\int_{C_\beta} \exp \left( i \int_0^\beta f(\tau) \phi(\tau) d\tau \right) \chi_\beta(d\phi)$$

$$= \exp \left( -\frac{1}{2} \int_0^\beta \int_0^\beta S_\beta(\tau, \tau') f(\tau) f(\tau') d\tau d\tau' \right), \quad f \in C_\beta,$$

see [3] pages 99 and 125. Let $\chi_{\beta,\Lambda}$ be the Gaussian measure on $\Omega_{\beta,\Lambda}$ defined as the product of the corresponding copies of $\chi_\beta$. Then the path measure in (1.11) is

$$\mu_{\beta,\Lambda}(d\omega) = \frac{1}{N_{\beta,\Lambda}} \exp \left( -I_{\beta,\Lambda}(\omega_\Lambda) \right) \chi_{\beta,\Lambda}(d\omega_\Lambda),$$

(1.15)

where $N_{\beta,\Lambda}$ is the normalization factor and

$$I_{\beta,\Lambda}(\omega_\Lambda) = -J \sum_{\ell \sim \ell', \Lambda} \int_0^\Lambda \omega_\ell(\tau) \omega_{\ell'}(\tau) d\tau + \sum_{\ell \in \Lambda} \int_0^\Lambda V(\omega_\ell(\tau)) d\tau.$$

Note that by (1.11), (1.10) and then by (1.17) the measure (1.15) uniquely determines the state (1.6). That is, the local states (1.6) can be constructed as Gibbs measures, similarly as in the case of classical anharmonic crystals. Here, however, the classical variable $q_\ell \in \mathbb{R}$ is replaced by a continuous path $\omega_\ell$, which is an element of an infinite dimensional vector space, $C_\beta$. Going further in this direction, one can define global Gibbs states of the quantum crystal as the probability measures on the space of tempered configurations $\Omega^t_\beta$ satisfying the Dobrushin-Lanford-Ruelle (DLR) equation, see [3] Chapter 3. It can be shown, see [3] Theorem 3.3.6 or [11] Theorem 3.1, that the set of all such measures, which we denote by $\mathcal{G}_\beta$, is a nonempty weakly compact simplex with a nonempty extreme boundary $\text{ex}(\mathcal{G}_\beta)$. By virtue of the DLR equation, the set $\mathcal{G}_\beta$ can contain either one or infinitely many elements. Correspondingly, the multiplicity (resp. the uniqueness) of the Gibbs states existing at a given value of the temperature means that $|\text{ex}(\mathcal{G}_\beta)| > 1$ (resp. $|\mathcal{G}_\beta| = 1$). In the physical interpretation, the multiplicity corresponds to a phase transition, cf. [3] Chapter 7.

For a finite $\Lambda \subset \mathbb{Z}^d$, let $\mathfrak{M}_\Lambda$ be the subset of $\mathcal{C}_\Lambda$ consisting of all multiplication operators by $F \in L^\infty(\mathbb{R}^\Lambda)$. Note that $\mathfrak{M}_\Lambda$ is a maximal $C^*$-subalgebra of $\mathcal{C}_\Lambda$. Each such an $F$ can be considered as a function $F : \mathbb{R}^\mathbb{Z}^d \to \mathbb{C}$. Set

$$\mathfrak{M} = \bigcup_{\Lambda} \mathfrak{M}_\Lambda,$$

where the union is taken over all finite $\Lambda \subset \mathbb{Z}^d$. For $F_1, \ldots, F_m \in \mathfrak{M}$ and $\mu \in \mathcal{G}_\beta$, the Matsubara function corresponding to these $F_i$ and $\mu$ is

$$\Gamma_{F_1, \ldots, F_m}^\mu(\tau_1, \ldots, \tau_n) = \int_{\Omega_{\beta}^\Lambda} F_1(\omega(\tau_1)) \cdots F_n(\omega(\tau_n)) \mu(d\omega),$$

(1.16)
where \( \tau_1, \ldots, \tau_n \in [0, \beta] \). Then \( \mu \) is said to be \( \tau \)-shift invariant if, for each \( \vartheta \in [0, \beta] \), the following holds
\[
\Gamma_{F_1, \ldots, F_n}^\mu (\tau_1 + \vartheta, \ldots, \tau_n + \vartheta) = \Gamma_{F_1, \ldots, F_n}^\mu (\tau_1, \ldots, \tau_n),
\tag{1.17}
\] where the addition is modulo \( \beta \). Let \( G_\beta^{\text{phase}} \) be the subset of \( \text{ex}(G_\beta) \) consisting of all \( \tau \)-shift invariant measures. Its elements are called thermodynamic phases or states of thermal equilibrium of the quantum crystal. Each \( \mu \in G_\beta^{\text{phase}} \) is defined by its Matsubara functions \((1.10)\) corresponding to all possible choices of \( n \in \mathbb{N} \) and \( F_1, \ldots, F_n \in \mathfrak{M} \), cf. \[6\]. If \( G_\beta \) is a singleton, then clearly \( G_\beta = G_\beta^{\text{phase}} \). A state \( \mu \in G_\beta^{\text{phase}} \) is called translation invariant if its Matsubara functions are invariant with respect to the shifts of the lattice \( \mathbb{Z}^d \).

2. The Results

Now we present three main results concerning the properties of the set \( G_\beta^{\text{phase}} \).

2.1. Phase Transitions and Quantum Stabilization. It can be shown, see [3, Theorem 3.7.4] or [11, Theorem 3.8], that there exist translation invariant \( \mu^\pm \in G_\beta^{\text{phase}} \) such that, for each \( \ell \in \mathbb{Z}^d \) and \( \mu \in G_\beta^{\text{phase}} \), the following holds
\[
M^- \leq M^\mu_\ell \leq M^+, \quad M^- = -M^+,
\tag{2.1}
\]
where
\[
M^\mu_\ell = \int_{\Omega_\beta^d} \omega_\ell(\tau)\mu(d\omega),
\tag{2.2}
\]
and \( M^\pm = M^\mu^\pm \). In view of (1.17), the integral in (2.2) is independent of \( \tau \). By (2.1) we have that \( M^+ = M^- = 0 \) whenever \( G_\beta \) is a singleton and \( M^+ > 0 \) implies that \( |G_\beta^{\text{phase}}| > 1 \). Moreover, \( M^+ = 0 \) is also sufficient for \( |G_\beta| = 1 \), see [11 2]. Assume that the lattice dimension satisfies \( d \geq 3 \). Set
\[
E(p) = \sum_{j=1}^d [1 - \cos p_j], \quad \theta(d) = \frac{d}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{dp}{E(p)}.
\]
It is possible to show that \( \theta(d) > 1 \) for all \( d \geq 3 \) and \( \theta(d) \to 1^+ \) as \( d \to +\infty \). For \( u \in [0, 1) \), set \( t(u) = (\sqrt{1/2})[\log(1 + \sqrt{u}) - \log(1 - \sqrt{u})] \). Then \( t \) is an increasing function and \( \lim_{u \to 1^+} t(u) = +\infty \). Let \( u(t), \, t \in \mathbb{R}_+ \) be its inverse, which is an increasing function such that \( \lim_{t \to +\infty} u(t) = 1 \). For \( b_1, b_2 \) as in (1.3) and \( a \) as in (1.2), set
\[
u = \frac{2b_1 - a}{12b_2}.
\]
Note that \( \nu > 0 \) whenever \( b_1 > a/2 \), and thereby the potential energy in (1.1) has two wells. Recall that \( J > 0 \) is the intensity of the interaction of a given pair of oscillators, see (1.3). Then \( \tilde{J} := 2dJ \) is the intensity of the interaction of a given oscillator with all its neighbors. The next statement, cf. [8, Theorem 3.1] or [3, Theorem 6.3.6], gives a sufficient condition for the existing of phase transitions in our model.

**Theorem 2.1.** For \( d \geq 3 \), assume that \( 4mv^2 \tilde{J} > \theta(d) \), and hence the equation
\[
4mv^2 \tilde{J}u(\beta/4mv) = \theta(d)
\tag{2.3}
\]
has a unique solution, \( \beta_* \). Then \( |G_\beta^{\text{phase}}| > 1 \) whenever \( \beta > \beta_* \).
As follows from Theorem 2.1 the absence of phase transitions, i.e., the fact that $|G^\text{phase}_\beta| = 1$ for all $\beta > 0$ implies $4m \nu^2 \tilde{J} \leq \theta(d)$. In order to get the corresponding sufficient condition let us turn to the spectral properties of the Hamiltonian $H_\ell$ given in the second line of (1.3), which can be defined as a self-adjoint lower bounded operator in $L^2(\mathbb{R})$. By [8, Proposition 4.1] or [3, Theorem 1.1.60], the spectrum of $H_\ell$ entirely consists of simple eigenvalues $E_n$, $n \in \mathbb{N}$. The simplicity means that each $E_n$ corresponds to exactly one state, contrary to the classical case where the mentioned degeneracy might occur. By means of the analytic perturbation theory for linear operators it is possible to prove, see [8, Theorem 4.1], that degeneracy might occur. By means of the analytic perturbation theory for linear operators it is possible to prove, see [8, Theorem 4.1], that $\Delta := \inf_n (E_{n+1} - E_n)$ is a continuous function of $m \in (0, +\infty)$ such that $m^{2/3} \Delta \to \Delta_0$ as $m \to 0^+$ for some $\Delta_0 > 0$. Then $R_m := m \Delta^2$ is a continuous function of $m \in (0, +\infty)$ such that $R_m \sim m^{-1/3} \Delta^2$ as $m \to 0^+$. In the harmonic case (1.13), we have $R_m^\text{har} = a$. By analogy, we call $R_m$ quantum effective rigidity, which, however, depends on $m$ as just discussed. The sufficient condition mentioned above is, see [8, Theorem 4.6] or [3, Theorem 7.3.1].

**Theorem 2.2.** Let the parameters introduced above satisfy $\tilde{J} < R_m$. Then $G_\beta$ is a singleton for all $\beta > 0$.

According to Theorem 2.2 quantum stabilization occurs if the interaction intensity is smaller than the effective rigidity, see [2] [3] and Part 2 of [3] for a physical interpretation of this effect. Note that $R_m$ can be made arbitrarily big either by making $m$ small or $\Delta$ big (e.g., by making the wells closer to each other). On the other hand, it satisfies $R_m \leq 1/4m \nu^2$, see [8, Theorem 4.2] or [3, Theorem 7.1.1]. Therefore, $\tilde{J} < R_m$ implies that $4m \nu^2 \tilde{J} < 1$, cf. (2.4).

### 2.2. Local Dynamics

In this subsection, we show that the dynamics of the oscillators indexed by the elements of a finite $\Lambda$ can be influenced by the phase transitions described in Theorem 2.1. To this end, we use the notion of a *stochastically positive KMS system*, see [9]. Such a system is the tuple $(\mathcal{C}, \mathcal{B}, \{a_t\}_{t \in \mathbb{R}}, \varpi)$, where $\mathcal{C}$ is a $C^*$-algebra; $\{a_t\}_{t \in \mathbb{R}}$ is a group of automorphisms of $\mathcal{C}$; $\mathcal{B}$ is a commutative $C^*$-subalgebra of $\mathcal{C}$ such that the algebra generated by $\cup_{t \in \mathbb{R}} a_t(\mathcal{B})$ is $\mathcal{C}$; $\varpi$ is a faithful state on $\mathcal{C}$ which is stochastically positive and satisfies the KMS condition with some fixed $\beta > 0$. The latter means that, for each $A, B \in \mathcal{C}$, there exists a function, $\Phi_{A,B}(z)$, analytic in the strip $\{ z \in \mathbb{C} : \text{Im} z \in (0, \beta) \}$ and continuous on its closure, such that $\varpi(A a_t(B)) = \Phi_{A,B}(t)$ and $\varpi(a_t(B)A) = \Phi_{A,B}(t+i\beta)$, holding for all $t \in \mathbb{R}$. It can be shown, cf. [9, Theorem 2.1], that, for each collection $A_1, \ldots, A_n$ of the elements of $\mathcal{C}$, the Green function

$$G_{A_1, \ldots, A_n}^\mathcal{C}(t_1, \ldots, t_n) := \varpi(a_{t_1}(A_1) \cdots a_{t_n}(A_n)), \quad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

(2.4)
can be continued to a function analytic in the domain defined in (1.8) and continuous on its closure. The stochastic positivity of $\varpi$ means that, for each collection of positive elements $F_1, \ldots, F_n$ of $\mathcal{B}$, the function defined in (2.4) satisfies

$$G_{F_1, \ldots, F_n}^\mathcal{B}(\tau_1, \ldots, \tau_n) \geq 0, \quad 0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \beta.$$

For a finite $\Lambda \subset \mathbb{Z}^d$, let us define

$$\mathcal{D}_\Lambda = \{ Q_{u_\Lambda} = \exp \left( i \sum_{\ell \in \Lambda} u_\ell q_\ell \right) : u_\Lambda \in \mathbb{Q}^\Lambda \},$$

where $\mathbb{Q}$ stands for the set of rational numbers. Clearly, $\mathcal{D}_\Lambda \subset \mathfrak{M}_\Lambda$ is countable and complete. The latter follows by the fact that $\mathcal{D}_\Lambda$ is closed with respect to the
point-wise multiplication, contains the unit element and separates the points of $\mathbb{R}^\Lambda$. Let $\mathfrak{M}_\Lambda$ be the closure (in the norm of $\mathfrak{M}_\Lambda$) of the set of all linear combinations of the elements of $\mathfrak{M}_\Lambda$ with rational coefficients. Then $\mathfrak{M}_\Lambda$ is a separable Banach algebra. Note that $\mathfrak{M}_\Lambda$ is a proper subset of $\mathfrak{M}_\Lambda$, dense in $\mathfrak{M}_\Lambda$ in the $\sigma$-weak topology. For the mentioned above states $\mu^\pm \in \mathcal{G}_\beta^{\text{phase}}$, we have, cf. [1.16], the Matsubara functions $\Gamma_{\mu^\pm}^{F_1,\ldots,F_n} : F_1,\ldots,F_n \in \mathfrak{M}_\Lambda$. These functions determine two types of dynamics of the considered portion of oscillators.

**Theorem 2.3.** Let $\mathfrak{M}_\Lambda$ and $\mu^\pm \in \mathcal{G}_\beta^{\text{phase}}$ be as just described. Then there exist stochastically positive KMS systems, $(\mathfrak{C}_\pm, \mathfrak{B}_\pm, \{\alpha_i^\pm\}_{i \in \mathbb{R}, \omega^\pm})$, and injective homomorphisms, $\pi^\pm : \mathfrak{M}_\Lambda \to \mathfrak{B}_\pm$, such that

$$\Gamma_{\mu^\pm}^{F_1,\ldots,F_n} (\tau_1,\ldots,\tau_n) = G_{\pi_\pm(F_1),\ldots,\pi_\pm(F_n)}^{\omega_\pm}(i\tau_1,\ldots,i\tau_n), \quad 0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \beta, \quad (2.5)$$

holding for all choices of $F_1,\ldots,F_n \in \mathfrak{M}_\Lambda$.

The proof of this statement readily follows from [6, Theorem 3.1], see also [10]. Its meaning can be seen from the following fact. For $\ell \in \Lambda$, let $F_\ell(q_\ell)$ be real, odd and strictly positive for $q_\ell > 0$, $i = 1, 2, 3$. Assume also that $|\mathcal{G}_\beta^{\text{phase}}| > 1$, and hence $\mu^+ \neq \mu^-$, see Theorem [2.1]. By the first GKS inequality, see [3, Theorem 3.2.2], it follows that

$$\Gamma_{F_\ell,\ldots,F_\ell}^{\mu^+} (\tau_1,\tau_2,\tau_3) > 0, \quad \text{and} \quad \Gamma_{F_\ell,\ldots,F_\ell}^{\mu^-} (\tau_1,\tau_2,\tau_3) < 0,$$

for some $\tau_1,\tau_2,\tau_3$. The second inequality follows from the first one by changing the signs of all $\omega_i$. Then by (2.5) one obtains that

$$G_{\pi_+(F_1),\pi_+(F_2),\pi_+(F_3)}^{\omega_+} \neq G_{\pi_-(F_1),\pi_-(F_2),\pi_-(F_3)}^{\omega_-},$$

which means that the oscillators in $\Lambda$ distinguish between the wells in this case, which can be experimentally detected.

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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, 20-031 Lublin, Poland
E-mail address: jkozi@hektor.umcs.lublin.pl