UNTYING KNOTS IN 4D AND WEDDERBURN’S THEOREM

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Abstract. It is proved that the Wedderburn Theorem on finite division rings implies that all knots and links in the smooth 4-dimensional manifolds are trivial.

1. Introduction

Our brief note contains an algebraic proof of the otherwise known topological fact, that all knots and links in the smooth 4-dimensional manifolds can be untied, i.e. are trivial. The novelty is a surprising rôle of the Wedderburn Theorem [Maclagan-Wedderburn 1905] in the 4-dimensional topology.

Recall that arithmetic topology studies a functor, $F$, between the 3-dimensional manifolds and the fields of algebraic numbers [Morishita 2012]. Such a functor maps 3-dimensional manifolds $M^3$ to the algebraic number fields $K$, so that the knots (links, resp.) in $M^3$ correspond to the prime ideals (ideals, resp.) in the ring of integers $O_K$.

The map $F$ extends to the smooth 4-dimensional manifolds $M^4$ and the fields of hyper-algebraic numbers $K$, i.e. fields with a non-commutative multiplication.

To formulate our result, denote by $O_K$ the ring of integers of the field $K$. A ring $R$ is called a domain, if $R$ has no zero divisors. The $R$ is called simple, if it has only trivial two-sided ideals. Our main result is the following theorem.

Theorem 1.1. $O_K$ is a simple domain.

Remark 1.2. Theorem 1.1 is false for the algebraic integers, since the domain $O_K$ is never simple.

Corollary 1.3. Any knot or link in $M^4$ is trivial.

Proof. If $\mathcal{K} \subset M^4$ ($\mathcal{L} \subset M^4$, resp.) is a non-trivial knot (link, resp.), then $F(\mathcal{K})$ ($F(\mathcal{L})$, resp.) is a non-trivial two-sided prime ideal (two-sided ideal, resp.) in $O_K$. The latter contradicts 1.1, since $O_K$ is a simple ring. □

The paper is organized as follows. Section 2 contains a brief review of the preliminary results. Theorem 1.1 is proved in Section 3.

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2. Preliminaries

2.1. Arithmetic topology. The arithmetic topology studies an interplay between 3-dimensional manifolds and number fields [Morishita 2012] [3]. Let $\mathcal{M}^3$ be a category of closed 3-dimensional manifolds, such that the arrows of $\mathcal{M}^3$ are homeomorphisms between the manifolds. Likewise, let $\mathcal{K}$ be a category of the algebraic number fields, where the arrows of $\mathcal{K}$ are isomorphisms between such fields. Let $\mathcal{M}^3 \in \mathcal{M}^3$ be a 3-manifold, let $\mathcal{S}^3 \in \mathcal{M}^3$ be the 3-sphere and let $O_K$ be the ring of integers of $K \in \mathcal{K}$. An exact relation between 3-manifolds and number fields can be described as follows.

**Theorem 2.1.** The exists a covariant functor $F : \mathcal{M}^3 \to \mathcal{K}$, such that:

(i) $F(S^3) = \mathbb{Z}$;

(ii) each ideal $I \subseteq O_K = F(\mathcal{M}^3)$ corresponds to a link $\mathcal{L} \subset \mathcal{M}^3$;

(iii) each prime ideal $I \subseteq O_K = F(\mathcal{M}^3)$ corresponds to a knot $\mathcal{K} \subset \mathcal{M}^3$.

Denote by $\mathcal{M}^4$ a category of all smooth 4-dimensional manifolds $\mathcal{M}^4$, such that the arrows of $\mathcal{M}^4$ are homeomorphisms between the manifolds. Denote by $\mathcal{K}$ a category of the hyper-algebraic number fields $\mathcal{K}$, such that the arrows of $\mathcal{K}$ are isomorphisms between the fields. Theorem 2.1 extends to 4-manifolds as follows.

**Theorem 2.2.** ([4, Theorem 1.1]) The exists a covariant functor $F : \mathcal{M}^4 \to \mathcal{K}$, such that the 4-manifolds $\mathcal{M}_1^4, \mathcal{M}_2^4 \in \mathcal{M}^4$ are homeomorphic if and only if the hyper-algebraic number fields $F(\mathcal{M}_1^4), F(\mathcal{M}_2^4) \in \mathcal{K}$ are isomorphic.

2.2. Wedderburn Theorem. Roughly speaking, Wedderburn’s Theorem says that finite non-commutative fields cannot exist [Maclagan-Wedderburn 1905] [2]. Namely, denote by $\mathcal{D}$ a division ring. Let $\mathbb{F}_q$ be a finite field for some $q = p^r$, where $p$ is a prime and $r \geq 1$ is an integer number.

**Theorem 2.3.** (Wedderburn Theorem) If $|\mathcal{D}| < \infty$ and $\mathcal{D}$ is finite dimensional over a division ring, then $\mathcal{D} \cong \mathbb{F}_q$ for some $q = p^r$.

We shall use 2.3 along with a classification of simple rings due to Artin and Wedderburn. Recall that a ring $R$ is called simple, if $R$ has only trivial two-sided ideals. By $M_n(\mathcal{D})$ we understand the ring of $n$ by $n$ matrices over $\mathcal{D}$.

**Theorem 2.4.** (Artin-Wedderburn) If $R$ is a simple ring, then $R \cong M_n(\mathcal{D})$ for a division ring $\mathcal{D}$ and an integer $n \geq 1$.

**Remark 2.5.** The ring $M_n(\mathcal{D})$ is a domain if and only if $n = 1$. For instance, if $n = 2$, then the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are zero divisors in the ring $M_2(\mathcal{D})$.

3. Proof of theorem 1.1

Theorem 1.1 will be proved by contradiction. Namely, we show that existence of a non-trivial two-sided ideal in $O_K$ contradicts 2.3. To begin, let us prove the following lemma.

**Lemma 3.1.** $O_K$ is a non-commutative Noetherian domain.
Proof. Recall that $O_K$ is generated by the zeroes of a non-commutative polynomial

$$\mathcal{P}(x) := \sum a_i x b_i x e_i \ldots e_i x l_i,$$

where $a_i, b_i, e_i, \ldots, e_i, l_i \in O_L$ and $K$ is a finite dimensional extension of $L$. By the Hilbert Basis Theorem for non-commutative rings [Amitsur 1970] [1], if $O_L$ is Noetherian, i.e. any ascending chain of the two-sided ideals of $O_L$ stabilizes, then the ring $O_K$ is also Noetherian. Repeating the construction, one arrives at a finite dimensional extension $H \subset K$, where $H$ is the field of quaternions. The ring of the Hurwitz quaternions $O_H$ is known to be Noetherian. Thus $O_K$ is a Noetherian ring. Lemma 3.1 is proved. □

Returning to the proof of theorem 1.1, let us assume to the contrary, that $I$ is a non-trivial two-sided ideal of $O_K$. By lemma 3.1, there exists the maximal two-sided ideal $I_{\text{max}}$, such that

$$I \subseteq I_{\text{max}} \subset O_K.$$  \hspace{1cm} (3.1)

**Lemma 3.2.** The ring $R := O_K/I_{\text{max}}$ is a simple domain.

**Proof.** The ring $R$ is simple, since $I_{\text{max}}$ is the maximal two-sided ideal of $O_K$. The ring $R$ is a domain, since $O_K$ is a domain and the homomorphism

$$h : O_K \to R$$  \hspace{1cm} (3.2)

is surjective. □

**Remark 3.3.** It follows from $R \cong O_K/I_{\text{max}}$, that $|R| < \infty$. Indeed, any non-trivial subgroup of the abelian group $(O_K, +)$ has finite index by the Margulis normal subgroup theorem. In particular, the subgroup $(I_{\text{max}}, +)$ has finite index in $(O_K, +)$.

To finish the proof of theorem 1.1, we write

$$R \cong M_n(\mathcal{D}),$$  \hspace{1cm} (3.3)

where $\mathcal{D}$ is a division ring, see Theorem 2.4. Since $R$ is a domain, we conclude that $n = 1$ in formula (3.3), see remark 2.5. Thus

$$R \cong \mathcal{D}.$$  \hspace{1cm} (3.4)

But remark 3.3 says that $|R| < \infty$ and by the Wedderburn Theorem one gets $R \cong F_q$ for some $q = p^r$. In particular, the homomorphism (3.2) implies that the ring $O_K$ is commutative. Indeed, since $R$ is a commutative ring, one gets $h(xy - yx) = h(x)h(y) - h(y)h(x) = h(x)h(y) - h(x)h(y) = 0$, where 0 is the neutral element of $R$. In other words, the element $xy - yx$ belongs to the kernel of $h$, which is a two-sided ideal $I_h \subset O_K$. If $h$ is not injective, then $I_h$ is non-trivial and taking the multiplicative identity $1 \in I_h$ we obtain a contradiction $h(1) = 0$. Thus $h$ is injective and $xy = yx$ for all $x, y \in O_K$, i.e. $O_K$ is a commutative ring. On the other hand, the ring $O_K$ cannot be commutative by an assumption of theorem 1.1. The obtained contradiction completes the proof of theorem 1.1.
References

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