Rigorous multi-asset optimal execution with Bayesian learning of the drift

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Abstract
Liquidity issues have been increasingly addressed recently, especially with regards to optimal execution of large orders. In practice, agents facing these issues are uncertain about the future dynamics of the assets, and face the risk of model misspecification, which could make the execution strategies non optimal. In this paper, we address the problem of uncertainty faced by an agent wishing to execute large orders on multiple assets. The agent only has knowledge about the distribution of the future drift of the assets composing her portfolio. We build on the work in [13] who proposed a model coupling Bayesian learning and dynamic programming techniques. More precisely, in this article, we provide a rigorous solution to the problem of portfolio optimal execution where prices have drifted Bachelier dynamics with an unknown drift. The agent uses Bayesian learning to update her estimation of the drift, while she maximizes the expected exponential utility of her final wealth. We consider the specific case where the prior is a non-degenerate multivariate Gaussian, and the costs are quadratic. We use stochastic optimal control tools to show how the problem of optimal execution simplifies into a system of ordinary differential equations (ODEs) which involves a matrix Riccati ODE with time-dependent coefficients for which classical existence theorems do not apply. However, using a method similar to the one in [10], we provide a rigorous solution to the problem by using a priori estimates obtained thanks to the original control problem.

Key words: Bayesian learning, Optimal execution, Stochastic optimal control, Riccati equations.

1 Introduction
The optimal split of an order through time (scheduling) and space (order routing) is of high importance for market operators. During the last two decades, driven by the increasing complexity of the market microstructure and the recent evolution of regulation worldwide, the optimal execution of large orders has been the center of an extensive literature. Starting from the seminal work of Bertsimas and Lo in [11] and Almgren and Chriss in [6] and [7], this literature addresses the important problem faced by operators who need to execute large orders. During the execution, an agent needs to control her overall costs while also controlling her inventory risk. The costs her trading incurs take the form of the market impact of orders and other trading fees, which are minimal when the trading is slow. However, a balance must be struck between trading slowly to mitigate the effects of market impact, and trading rapidly to mitigate the risk of adverse price moves.

In this article, we address optimal execution in a model à la Almgren-Chriss. The original model is a discrete-time model where the agent posts market orders to maximize a mean-variance objective function. Since then, numerous extensions to this seminal model have been proposed in the literature, providing extensions for every aspect of the model: the dynamics and model parameters, the optimization objectives, the order types, and access to different liquidity pools.

Regarding the general framework parameters and in particular the specifications of execution costs and market impact, the case of random execution costs is addressed in [3] and the case of stochastic liquidity is studied in [4]. Furthermore, the advantages of different market impact and limit order book (LOB) models have also been studied. For instance,
the authors in [31] investigate the influence of the presence of a transient price impact with exponential decay. In [32], later generalized in [1], the authors propose a single-asset market impact model where price dynamics are derived from a dynamic LOB model with resilience. In [2], the authors derive explicit optimal execution strategies in a discrete-time LOB model with general shape functions and an exponentially decaying price impact. We also mention the work in [20] where the authors obtain explicit optimal strategies with a transient market impact in an expected cost minimization setup. As for the optimization objectives, [18] addresses the use of quadratic variation rather than variance in the objective function. [33] uses stochastic optimal control tools to characterize and find optimal strategies for a Von Neumann–Morgenstern investor, and [21] provides results for optimal liquidation within a Von Neumann-Morgenstern expected utility framework with general market impact functions and derives subsequent results for block trade pricing.

Another stream of the literature deals with the integration of practices into the framework such as order types, strategy types, and access to different liquidity pools. Originally, the Almgren-Chriss framework focuses on orders of the Implementation Shortfall (IS) type with market orders only. Volume-Weighted Average Price (VWAP) orders are addressed in [19] [26] [27], and Target Close (TC) and Percentage of Volume (POV) orders are studied in [22]. The incorporation of limit orders has also been proposed and further studied in [9] [24] [23] [16]. Finally, the problem of optimal splitting of orders across different liquidity pools is addressed in [28] [10], and more recently in (Baldacci and Manzini, [8]). Besides, different specifications for the dynamics of prices have also been proposed in the literature. [31] incorporates general dynamics for the drift in the underlying price process accounting for the price impact of other agents, [5] addresses the incorporation of stochastic volatility for the prices, [15] provides a closed-form strategy incorporating order flows from all agents, and [10] [14] addresses multi-asset optimal execution and statistical arbitrage with Ornstein-Uhlenbeck dynamics for the prices.

In the majority of models in the literature, the parameters governing the dynamics of assets are estimated and fixed to obtain the optimal execution schedules. Consequently, in practice, operators first estimate the parameters of the dynamics before the start of the execution. There are two immediate downsides to this common approach. First, if the a priori estimates are correct but the dynamics changes during the execution process, or if the estimates are simply wrong, the strategy derived becomes non-optimal. Second, if the operator is not confident about the parameters, then deriving an optimal strategy is impossible when using the classical frameworks. The uncertainty about the model parameters in these situations have been addressed in the literature as it is commonly faced by market participants in practice. For instance, [30] discusses how to take into account statistical aspects of the variability of estimators of the main exogenous variables such as volumes or volatilities in the optimization phase. Others have addressed this issue by using filtering and learning techniques in a framework with partial information, and mixing online learning with optimal control. For instance, the authors in [29] propose a model in which an agent optimizes her strategy with limit orders while simultaneously learning the parameters of the jump process governing the execution of her orders. [17] also uses a similar approach for deriving optimal strategies with learning of posterior latent state distribution upon which the evolution of prices depend. More recently and similar to the problem addressed in this paper, [12] proposes a general approach for adaptive control which is robust to model misspecification, where the agent continuously learns the drift and her uncertainty follows a jump-diffusion. We also note that combining learning with optimization in mathematical finance is more explored in the context of the optimal portfolio choice literature.

In the Bayesian learning paradigm explored in this paper, the goal is to encode beliefs about the parameters of a given model, and later infer their values by observing data. In our case, an agent makes (reasonable) assumptions about the future distribution of the drifts of the assets she is in charge of, for instance based on historical behavior of the same parameters, or on assumptions based on exogenous variables or predictors. By using such a prior, the agent states that the values of the drifts will fall in some range according to some probability distribution. She then observes the prices of the assets and progressively and continuously updates the latest estimates of the drifts. At any one point of time during the execution process, her latest estimate of the true value of the drifts will then be the expectation of the drifts, given her prior, and conditionally to the filtration generated by the price process up until time \( t \).

Combining online or Bayesian learning with stochastic optimal control to obtain optimal strategies in a partial information framework is appealing. It is in fact interesting to combine online learning which is forward in time, with dynamic programming which classically uses backward induction. The approach is powerful in two ways, first it profits from the online learning approach to continuously learn and adapt to the rapidly changing market trends, and second the optimal strategies obtained not only incorporates the latest information in the observed prices, but also incorporates the uncertainty on the future value of the drift vector, and the fact that the agent will go on learning its posterior
distribution. In this paper, we wish to provide a rigorous solution to the problem of multi-asset optimal execution, where the prices follow drifted Bachelier dynamics, and the drifts are unknown. The model has first been addressed in \[13\] along with other portfolio problems, but the authors fail to provide a rigorous solution to the problem. More precisely, we will consider an agent in charge of a portfolio of assets, who does not know the drift vector of her assets but has a prior Gaussian distribution for them. The agent combines stochastic optimal control for obtaining an optimal trading schedule, with online learning and updating of her estimation of the drift vector based on the information gathered from the prices.

The remainder of this paper is organized as follows. In Section 2 we present the Bayesian learning framework that we will use later for the derivation of our optimal execution strategies. We briefly recall the results obtained by the authors in \[13\] and present some additional results that will be helpful for carrying out computations in the remainder of the paper. In section 3 we present the optimal execution problem faced by an agent observing prices but who does not know the true value of the drift vector of her assets. We then study in depth the case where she has a Gaussian prior on this vector. The optimal execution problem takes the form of a stochastic optimal control problem, and we briefly recall the results obtained by the authors in \[13\] along with other portfolio problems, but the authors fail to provide a rigorous solution to the problem. More precisely, we will consider an agent in charge of a portfolio of assets, who does not know the drift vector of her assets but has a prior Gaussian distribution for them. The agent combines stochastic optimal control for obtaining an optimal trading schedule, with online learning and updating of her estimation of the drift vector based on the information gathered from the prices.

2 Bayesian learning

2.1 Price dynamics and Bayesian learning of the drift

In this section, we introduce the Bayesian learning framework, along with some results proved and discussed in detail in \[13\]. We consider a filtered probability space \((\Omega, \mathcal{F}, P; \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]})\) satisfying the usual conditions, which we consider large enough to support the processes we introduce.

We consider a market with \(d\) assets for some \(d \in \mathbb{N}^* \). We first introduce a \(d\)-dimensional Brownian motion \((W_t)_{t \in [0,T]} = (W^1_t, \ldots, W^d_t)^\top\) adapted to the filtration \((\mathcal{F}_t)_{t \in [0,T]} \). We suppose that the fundamental prices of the assets are modelled as a \(d\)-dimensional drifted Bachelier process \((S_t)_{t \in [0,T]} = (S^1_t, \ldots, S^d_t)^\top\):

\[
dS_t = \mu dt + \sigma \odot dW_t,
\]

where the operator \(\odot\) denotes the element-wise multiplication of vectors. The dispersion vector \(\sigma = (\sigma^1, \ldots, \sigma^d)^\top\) satisfies \(\forall i \in \{1, \ldots, d\}, \sigma^i > 0\), and the drift vector \(\mu = (\mu^1, \ldots, \mu^d)^\top\) is unknown. Equivalently, we can write for all \(i \in \{1, \ldots, d\}\),

\[
dS^i_t = \mu^i dt + \sigma^i dW^i_t.
\]

In what follows, we denote by \(\rho = (\rho^{ij})_{1 \leq i,j \leq d}\) the correlation matrix and by \(\Sigma = (\sigma^{ij})_{1 \leq i,j \leq d}\) the covariance matrix associated with the dynamics of prices.

The drifts of the \(d\) assets \(\mu\) are unknown to the agent but she assumes a sub-Gaussian prior distribution that we denote by \(m_\mu\). It shall be noted that both the drift and the Brownians \((W_t)_{t \in [0,T]}\) are not observed directly by the agent. However, the price of every asset is continuously revealed to the agent, and their evolution reveals information about the true values of the drifts. The classical Bayesian framework for updating her estimation appears to be a natural choice.

1 We denote by \(\mathbb{N}^*\) the set \(\mathbb{N}^* := \mathbb{N}\setminus\{0\}\) of positive integers.

2 The superscript \(\top\) designates the transpose operator. It transforms here a line vector into a column vector.

3 A prior distribution is called sub-Gaussian if it satisfies the following property:

\[
\exists \xi > 0, \mathbb{E}\left[ e^{\xi ||x||^2} \right] = \int_{x \in \mathbb{R}^d} e^{\xi ||x||^2} m_\mu (dx) < +\infty
\]
estimation at every time step of the execution process. In this section, we discuss the results obtained in [13] before presenting how to incorporate this approach into our optimal execution framework à-la Almgren-Chriss.

We first define the filtration generated by \((S_t)_{t \in [0,T]}\) which we denote \(\mathbb{F}^S = (\mathcal{F}^S_t)_{t \in [0,T]}\). It is straightforward to notice that \((W_t)_{t \in [0,T]}\) is not a \(\mathbb{F}^S\)-Brownian motion since it is not \(\mathbb{F}^S\)-adapted. Our objective is to find the dynamics of the unknown drifts within this appropriate filtration, which represents the information revealed to the agent. For that purpose, let us introduce the process \(\beta_t\) defined by

\[ \forall t \in [0,T], \quad \beta_t = \mathbb{E}[\mu | \mathcal{F}^S_t]. \]

The process \(\beta_t\) represents the latest estimated value for the drift \(\mu\) given the information in the prices up to time \(t\). We state the next result proved in [13], which gives a formula for the expectation of the drift, given the information in the prices.

**Theorem 1.** Let us define

\[ F : (t, S) \in [0, T] \times \mathbb{R}^d \mapsto \int_{\mathbb{R}^d} \exp \left( z^T \Sigma^{-1} \left[ S - S_0 - t \frac{1}{2} z \right] \right) m_\mu(dz). \]

\(F\) is a well-defined finite-valued \(C^\infty([0, T] \times \mathbb{R}^d)\) function.

We have

\[ \forall t \in [0, T], \quad \beta_t = \Sigma G(t, S_t), \]

where

\[ G = \frac{\nabla_S F}{F}. \]

We define the process \((\hat{W}_t)_{t \in [0,T]}\) by

\[ \forall i \in \{1, \ldots, d\}, \quad \forall t \in [0,T], \quad \hat{W}_t^i = W_t^i + \int_0^t \frac{\mu_i - \beta_i}{\sigma_i} ds. \]

Next, the authors in [13] prove the following result for \((\hat{W}_t)_{t \in [0,T]}\):

**Proposition 1.** \((\hat{W}_t)_{t \in [0,T]}\) is a Brownian motion adapted to \((\mathcal{F}_t^S)_{t \in [0,T]}\), with the same correlation structure as \((W_t)_{t \in [0,T]}\):

\[ \forall i, j \in \{1, \ldots, d\}, \quad d\langle \hat{W}_t^i, \hat{W}_t^j \rangle_t = d\langle W_t^i, W_t^j \rangle_t. \]

We next use the Brownian motion \((\hat{W}_t)_{t \in [0,T]}\) to re-write the dynamics (1) of \(S\) in the appropriate filtration for the purposes of this work. The equation (1) now writes

\[ dS_t = \beta_t dt + \sigma \odot d\hat{W}_t. \]

\[ = \Sigma G(t, S_t) dt + \sigma \odot d\hat{W}_t. \]

(2)

**2.2 The case of a Gaussian prior**

In this section we consider the specific case of a Gaussian prior for the drift, as it will be the prior used in our optimal execution framework. Let us consider a non-degenerate multivariate Gaussian prior that we denote \(m_\mu\), i.e.,

\[ m_\mu(dz) = \frac{1}{(2\pi)^{d/2} |\Gamma_0|^{1/2}} \exp \left( -\frac{1}{2} (z - \beta_0)^T \Gamma_0^{-1} (z - \beta_0) \right) dz, \]

where \(\beta_0 \in \mathbb{R}^d\) and \(\Gamma_0 \in S_d^{++}(\mathbb{R})\). By using Theorem 1 we obtain the following result:
Then we can re-write the dynamics of the prices in (2) as
\[ G(t, S) = \Sigma^{-1} \Gamma(t) \left( \Sigma^{-1} (S - S_0) + \Gamma^{-1}_0 \beta_0 \right), \]
where \( \Gamma(t) = (\Gamma^{-1}_0 + t \Sigma^{-1})^{-1}. \)

Given the form of \( G, \) we can now state the following useful result, for which a proof can be found in [13].

**Proposition 3.** The first order partial derivatives of \( G \) are given by:
\[
\begin{align*}
\forall t \in [0, T], \forall S \in \mathbb{R}^d, \\
D_S G(t, S) &= \Sigma^{-1} \Gamma(t) \Sigma^{-1}, \\
\partial_t G(t, S) &= -\Sigma^{-1} \Gamma(t) G(t, S).
\end{align*}
\]

Then we can re-write the dynamics (6) as
\[ dS_t = -\Gamma(t) \Sigma^{-1} (S_0 - \Sigma \Gamma^{-1}_0 \beta_0 - S_t) dt + \sigma \odot d\hat{W}_t. \]  

At this stage, it is noteworthy to observe that the dynamics of \( S \) in (7) resembles that of an Ornstein-Uhlenbeck process with a time dependent transition matrix. In fact if we define
\[ R(t) = -\Gamma(t) \Sigma^{-1}, \]
then we can re-write the dynamics (7) as
\[ dS_t = R(t) (S - S_t) dt + \sigma \odot d\hat{W}_t. \]

The matrix \( R_t \) steers the deterministic part of the process, and the vector \( S \) acts as an unconditional long-term expectation. This is interesting fact that will make it possible in the next section to provide a rigorous solution to the problem of optimal execution with Bayesian learning of the drift, by using an approach similar to the one in [10] where the authors provide a rigorous solution to the problem of optimal execution under multivariate Ornstein-Uhlenbeck dynamics for the fundamental prices.

For that purpose, we state the following result which gives a solution to the dynamics of the price process \( (S_t)_{t \in [0, T]} \), which will be important in providing a rigorous solution for the optimal execution problem.

**Proposition 4.** Let us define \( R : [0, T] \mapsto \mathcal{M}_d(\mathbb{R}) \) and \( S \) as in (7). Then the solution for the dynamics in (2) is normally distributed and writes
\[ S_t = \exp \left( -\int_0^t R(s) ds \right) S_0 + \left( I - \exp \left( -\int_0^t R(s) ds \right) \right) \overline{S} + \int_0^t \exp \left( -\int_u^t R(s) ds \right) \sigma \odot d\hat{W}_u, \]
\[ Y_t = \exp \left( \int_0^t R(u) du \right) (S_t - \overline{S}). \]

**Proof.** Starting with the dynamics as written in (9), we introduce the following process:
\[ Y_t = \exp \left( \int_0^t R(u) du \right) (S_t - \overline{S}). \]

Then using Ito’s lemma we obtain
\[ dY_t = \left( R(t) \exp \left( \int_0^t R(u) du \right) \right) (S_t - \overline{S}) dt + \exp \left( \int_0^t R(u) du \right) \sigma \odot d\hat{W}_t. \]

But since the matrices \( R(t) \) and \( \int_0^t R(u) du \) commute for all \( t \in [0, T], \) because
\[ R(t) \int_0^t R(u) du = \int_0^t \left( (\Gamma^{-1}_0 \Sigma + u I) (\Gamma^{-1}_0 \Sigma + t I) \right)^{-1} du = \left( \int_0^t R(u) du \right) R(t), \]
then \( R(t) \) and \( \exp \left( \int_0^t R(u) du \right) \) commute as well, and we can write
\[ dY_t = \exp \left( \int_0^t R(u) du \right) \sigma \odot d\hat{W}_t. \]

Integrating both sides in the dynamics of \( Y, \) one obtains the solution in (9), concluding the proof. \( \square \)
3 The optimal liquidation problem with Bayesian learning

3.1 Modelling framework and notations

We consider in this section the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F}^S = (\mathcal{F}^S_t)_{t \in [0,T]})\). Similar to the previous section, we consider the same market with \(d \in \mathbb{N}^+\) assets. Now the agent is in charge of a portfolio consisting of these assets, and she wishes to liquidate her positions within a given period of time \([0, T]\), with \(T > 0\). The following framework has first been introduced by Almgren and Chriss in [6, 7] and later generalized and made rigorous in [21, 22].

The agent has a \(d\)-dimensional inventory process \((q_t)_{t \in [0,T]} = (q^1_t, \ldots, q^d_t)_{t \in [0,T]}\) evolving as

\[
dq_t = \nu_t dt,
\]

with the initial inventory \(q_0 \in \mathbb{R}^d\) given, and where \((\nu_t)_{t \in [0,T]} = (\nu^1_t, \ldots, \nu^d_t)_{t \in [0,T]}\) models the trading speed of the trader for each asset.

As shown in the previous section, the fundamental prices of the \(d\) assets, \((S_t)_{t \in [0,T]} = (S^1_t, \ldots, S^d_t)_{t \in [0,T]}\) have the dynamics

\[
dS_t = \Sigma G(t, S_t)dt + \sigma \odot dW_t,
\]

with \(S_0 \in \mathbb{R}^d\) given.

Let us now define the dynamics for the amount on the cash account as

\[
dx_t = -v^T_t S_t dt - L(\nu_t) dt,
\]

with \(X_0\) given, and where \(L : \mathbb{R}^d \rightarrow \mathbb{R}_+\) is a function representing the temporary market impact of trades along with eventual execution costs incurred by the trader. In what follows, we only consider the case where the function \(L\) has a positive-definite quadratic form, i.e.

\[
L(v) = v^\top \eta v \quad \text{with} \quad \eta \in \mathcal{S}_d^{++}(\mathbb{R}).
\]

In our framework, the agent aims at maximizing the expected exponential utility of her final wealth at the end of the trading window \([0, T]\). The final wealth is valued as the sum of the final amount \(X_T\) on the cash account and the remaining inventory which we evaluate at \(q^T_T S_T - \ell(q_T)\). \(\ell(q_T)\) is a discount term applied to the terminal Mark-to-Market (MtM) value of the assets, and hence penalizes any non-zero final position. In what follows, we only consider the case where \(\ell\) is a positive-definite quadratic form, i.e. \(\ell(q) = q^\top A q\) with \(A \in \mathcal{S}_d^{++}(\mathbb{R})\).

Prior to defining the set of admissible controls, we first introduce a notion of “linear growth” relevant in our context.

Definition 1. Let \(t \in [0, T]\). An \(\mathbb{R}^d\)-valued, \(\mathbb{F}\)-adapted process \((\zeta_s)_{s \in [t,T]}\) is said to satisfy a linear growth condition on \([t, T]\) with respect to \((S_s)_{s \in [t,T]}\) if there exists a constant \(C_{t,T} > 0\) such that for all \(s \in [t, T]\),

\[
\|\zeta_s\| \leq C_{t,T} \left(1 + \sup_{\tau \in [t,s]} \|S_{\tau}\|\right)
\]

almost surely.

What is called an execution strategy of the agent is described by the stochastic process \((v_s)_{s \in [t,T]} \in \mathcal{A} = \mathcal{A}_0\), where for \(t \in [0, T]\)

\[
\mathcal{A}_t = \left\{(v_s)_{s \in [t,T]}, \text{\(\mathbb{R}^d\)-valued \(\mathbb{F}^S\)-adapted process, satisfying the linear growth condition with respect to \((S_s)_{s \in [t,T]}\)}\right\}.
\]

We do not consider the permanent impact of the agent on the prices for convenience since it does not change the result of the optimization problem, as justified in more detail in [10]. The results of this section can easily be generalized to the case of a linear permanent impact.

See [22] for an introduction to this type of models.

The subset of positive-definite and positive semi-definite matrices of \(\mathcal{S}_d(\mathbb{R})\) are respectively denoted by \(\mathcal{S}_d^{++}(\mathbb{R})\) and \(\mathcal{S}_d^{+}(\mathbb{R})\).

This penalization term serves primarily to relax the classical constraint of a zero inventory \(q_T = 0\) found in some liquidation problems. This relaxation enables us to use classical tools of optimal control while the problem with hard constraint is difficult to address mathematically.

\footnote{The subset of positive-definite and positive semi-definite matrices of \(\mathcal{S}_d(\mathbb{R})\) are respectively denoted by \(\mathcal{S}_d^{++}(\mathbb{R})\) and \(\mathcal{S}_d^{+}(\mathbb{R})\).}

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\footnote{In all this paper, \(\|,\|\) denotes a fixed norm on \(\mathbb{R}^d\) (for instance, the Euclidean norm).}
Mathematically, the agent wants to solve the dynamic optimization problem

$$\sup_{v \in A} \mathbb{E} \left[ -\exp \left( -\gamma \left( X_T + q_T^T S_T - \ell(q_T) \right) \right) \right], \quad (12)$$

where \( \gamma > 0 \) is the absolute risk aversion parameter of the trader.

In order to solve the dynamic optimization problem \((12)\), we use the tools of stochastic optimal control. Let us introduce the value function associated with the above problem \( w : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) as

$$w(t, x, q, S) = \sup_{v \in A_t} \mathbb{E} \left[ -\exp \left( -\gamma \left( X_T + q_T^T S_T - \ell(q_T) \right) \right) \right],$$

where, for \((t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \) and \( v \) in \( A_t \), the processes \((q_{t,s}^v)_{s \in [t,T]}, (S_{t,s}^v)_{s \in [t,T]} \) and \((X_{t,s}^{t,x,v})_{s \in [t,T]} \) have the following dynamics:

\[
dq_{t,s}^v = v_{t,s} ds, \\
dS_{t,s}^v = \Sigma G(s, S_{t,s}^v) ds + \sigma \odot d\hat{W}_s, \\
and \\
dX_{t,s}^{t,x,v} = -v_{t,s}^T S_{t,s}^v ds - L(v_{t,s}) ds,
\]

with \( S_{t,s}^v = S, q_{t,s}^v = q, \) and \( X_{t,s}^{t,x,v} = x \).

### 3.2 Hamilton-Jacobi-Bellman equation

The HJB equation associated with our problem \((12)\), and which we expect the value function to solve, is

\[
\begin{align*}
\partial_t u(t, x, q, S) + G(t, S)^T \Sigma \nabla_S u(t, x, q, S) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} u(t, x, q, S) \right) \\
+ \sup_{v \in \mathbb{R}^d} \left\{ v^T \nabla_q u(t, x, q, S) - (v^T S + L(v)) \right\} \partial_x u(t, x, q, S) = 0, \\
\end{align*}
\]

for all \((t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \) with terminal condition

\[
\forall (x, q, S) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d, \quad u(T, x, q, S) = -\exp(-\gamma (x + q^T S - \ell(q))). \\
(14)
\]

For reducing the dimensionality of the problem, we consider the following ansatz:

\[
\begin{align*}
u(t, x, q, S) = -\exp(-\gamma (x + q^T S - \theta(t, q, S))),
\end{align*}
\]

(15)

The interest in this ansatz is based on the following result:

**Proposition 5.** Let \( \tau < T \). Suppose there exists \( \theta \in C^{1,2,2}([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d) \) satisfying

\[
\begin{align*}
\partial_t \theta(t, q, S) + G(t, S)^T \Sigma (-q + \nabla_S \theta(t, q, S)) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \theta(t, q, S) \right) \\
+ \gamma (v^T \nabla_q \theta(t, q, S) + \sup_{v \in \mathbb{R}^d} (v^T \nabla_q \theta(t, q, S) - L(v)) = 0,
\end{align*}
\]

for all \((t, q, S) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \) with terminal condition

\[
\forall (q, S) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \theta(T, q, S) = \ell(q),
\]

(17)

Then \( u \) defined by \((15)\) is solution to the HJB equation \((13)\) on \([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \) with terminal condition \((14)\).
Proof. Let us consider \( \theta \in C^{1,2,2}([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d) \) as being a solution to the PDE (19) with terminal condition (17). Then, for \( u \) as defined in (15), we have

\[
\begin{align*}
\partial_t u(t, x, q, S) + G(t, S)\Sigma \nabla q u(t, x, q, S) + \frac{1}{2} \text{Tr} (\Sigma \nabla_S^2 u(t, x, q, S)) \\
+ \sup_{v \in \mathbb{R}^d} \{v^\top \nabla_q u(t, x, q, S) - (v^\top S + L(v)) \partial_x u(t, x, q, S)\}
\end{align*}
\]

\[
= \frac{\gamma}{2} (-q + \nabla_S \theta(t, q, S))^\top \Sigma (-q + \nabla_S \theta(t, q, S)) - \frac{1}{2} \text{Tr} (\Sigma \nabla_S^2 \theta(t, q, S))
\]

\[
+ \frac{\gamma}{2} \nabla q \theta(t, q, S)\Sigma (-q + \nabla_S \theta(t, q, S)) - \frac{1}{4} \nabla q \theta(t, q, S)^\top \Sigma^{-1} \nabla q \theta(t, q, S) + \frac{1}{2} \text{Tr} (\Sigma \nabla_S^2 \theta(t, q, S))
\]

\[
= 0.
\]

As it is straightforward to verify that \( u \) satisfies the terminal condition (14), the result is proved. \( \square \)

The above result does not use the assumptions we made for the functions \( L \) and \( \ell \). With these assumptions, the optimal strategy can be explicitly found in a closed-loop formulation. We first notice that the Legendre-Fenchel transform of the function \( L \) writes

\[
H : p \in \mathbb{R}^d \mapsto \sup_{v \in \mathbb{R}^d} v^\top p - L(v) = \sup_{v \in \mathbb{R}^d} v^\top p + v^\top \eta v = \frac{1}{4} p^\top \eta^{-1} p,
\]

using the fact that the supremum in the expression above is easily found to be reached at \( v^* = \frac{1}{2} \eta^{-1} p \). We can then re-write the HJB equation (19) as

\[
\begin{align*}
\partial_t \theta(t, q, S) + G(t, S)^\top \Sigma (-q + \nabla_S \theta(t, q, S)) + \frac{1}{2} \text{Tr} (\Sigma \nabla_S^2 \theta(t, q, S)) \\
+ \frac{\gamma}{2} (-q + \nabla_S \theta(t, q, S))^\top \Sigma (-q + \nabla_S \theta(t, q, S)) - \frac{1}{4} \nabla q \theta(t, q, S)^\top \Sigma^{-1} \nabla q \theta(t, q, S) = 0,
\end{align*}
\]

with terminal condition

\[
\forall (q, S) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \theta(T, q, S) = \ell(q).
\]

The result in Proposition 3 means that solving the HJB equation (13) reduces to solving the simpler three-variable PDE (19). However, solving equation (19) with terminal condition (20) for a large range of sub-Gaussian priors is not the goal here, as we will next focus on the widely used case of a Gaussian prior distribution for the drift vector. We will show that solving the HJB in this case boils down to solving a matrix Riccati equation. Unfortunately, this Riccati equation is not a classical one as the quadratic term has both positive and negative eigenvalues. Similar Riccati equations are sometimes obtained in some Two-Person Zero-Sum Differential Games, or agent versus the market in our case, and are difficult to address, in particular for obtaining closed-form solutions. We will nevertheless prove a global existence and uniqueness and provide a rigorous solution in that sense.

### 3.3 The case of a Gaussian prior and quadratic costs

In what follows, we consider a Gaussian prior on the drifts, such that the dynamics of the prices \( S \) in the appropriate filtration is given by (3) and (9). In order to further study (19), we introduce a second ansatz and now look for a solution \( \theta \) of the form
\[ \theta(t, q, S) = a(t) + \frac{1}{2} G(t, S)^T b(t) G(t, S) + G(t, S)^T c(t) q + \frac{1}{2} q^T d(t) q + G(t, S)^T e(t) + q^T f(t), \]  

(21)

where \( a(t) \in \mathbb{R}, b(t) \in S_d(\mathbb{R}), c(t) \in M_d(\mathbb{R}), d(t) \in S_d(\mathbb{R}), e(t) \in \mathbb{R}^d, \) and \( f(t) \in \mathbb{R}^d \). The interest in this ansatz is justified in the following proposition:

**Proposition 6.** Let \( \tau < T \). Assume there exists \( a \in C^1([\tau, T]), b \in C^1([\tau, T], S_d(\mathbb{R})), c \in C^1([\tau, T], M_d(\mathbb{R})), d \in C^1([\tau, T], S_d(\mathbb{R})), e \in C^1([\tau, T], \mathbb{R}^d), \) and \( f \in C^1([\tau, T], \mathbb{R}^d) \) satisfying the following system of ODEs:

\[
\begin{align*}
\dot{a}(t) &= -\gamma c(t)^T \Sigma^{-1} \Gamma(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) + \frac{1}{2} f(t)^T \eta^{-1} f(t) - \frac{1}{2} \text{Tr} \left( \Gamma(t) \Sigma^{-1} b(t) \Gamma(t) \Sigma^{-1} \right) \\
\dot{b}(t) &= -\gamma b(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} b(t) + \frac{1}{2} c(t)^T \eta^{-1} c(t) \\
\dot{c}(t) &= -\gamma b(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) + \frac{1}{2} c(t)^T \eta^{-1} d(t) \\
\dot{d}(t) &= -\gamma (b(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) + \frac{1}{2} c(t)^T \eta^{-1} d(t) \\
\dot{\gamma}(t) &= -\gamma (b(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) + \frac{1}{2} c(t)^T \eta^{-1} f(t) \\
\dot{f}(t) &= -\gamma (b(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) + \frac{1}{2} c(t)^T \eta^{-1} f(t) 
\end{align*}
\]

(22)

where \( I_d \) denotes the identity matrix in \( M_d(\mathbb{R}) \), with terminal condition

\[
\begin{align*}
{a(T)} &= 0 \\
{b(T)} &= 0 \\
{c(T)} &= 0 \\
{d(T)} &= A \\
{e(T)} &= 0 \\
{f(T)} &= 0. 
\end{align*}
\]

(23)

Then, the function \( \theta \) defined by (21) satisfies (19) with terminal condition (21).

**Proof.** Let \( \tau < T \), and let us define \( a \in C^1([\tau, T]), b \in C^1([\tau, T], S_d(\mathbb{R})), c \in C^1([\tau, T], M_d(\mathbb{R})), d \in C^1([\tau, T], S_d(\mathbb{R})), e \in C^1([\tau, T], \mathbb{R}^d), f \in C^1([\tau, T], \mathbb{R}^d) \) verifying (22) on \([\tau, T]\) with terminal condition (23). We now define \( \theta : [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) as in (21). Then, by using (14) and (19), we have for all \((t, q, S) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d\),

\[
\begin{align*}
\partial_t \theta(t, q, S) + G(t, S)^T \Sigma \left(-q + \nabla_S \theta(t, q, S)\right) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \theta(t, q, S) \right) \\
+ \frac{\gamma}{2} \left(-q + \nabla_S \theta(t, q, S)\right)^T \Sigma \left(-q + \nabla_S \theta(t, q, S)\right) - \frac{1}{4} \nabla_q \theta(t, q, S)^T \eta^{-1} \nabla_q \theta(t, q, S)
\end{align*}
\]


\[
= \dot{a}(t) + \frac{1}{2} G(t, S)^T \dot{b}(t) G(t, S) + G(t, S)^T \dot{c}(t) q + \frac{1}{2} q^T \dot{d}(t) q + G(t, S)^T \dot{\gamma}(t) + q^T \dot{f}(t)
\]

\[
- G(t, S)^T \Gamma(t) \Sigma^{-1} b(t) G(t, S) - G(t, S)^T \Gamma(t) \Sigma^{-1} c(t) q - G(t, S)^T \Gamma(t) \Sigma^{-1} e(t) \\
+ G(t, S)^T \Sigma \left(-q + \Sigma^{-1} \Gamma(t) \Sigma^{-1} b(t) G(t, S) + \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) q + \Sigma^{-1} \Gamma(t) \Sigma^{-1} e(t) \right)
\]

\[
+ \frac{\gamma}{2} \left(-q + \Sigma^{-1} \Gamma(t) \Sigma^{-1} b(t) G(t, S) + \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) q + \Sigma^{-1} \Gamma(t) \Sigma^{-1} e(t) \right)^T \Sigma \left(-q + \Sigma^{-1} \Gamma(t) \Sigma^{-1} b(t) G(t, S) + \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) q + \Sigma^{-1} \Gamma(t) \Sigma^{-1} e(t) \right)
\]

\[
- \frac{1}{4} \left(c(t)^T G(t, S) + d(t)^T q + f(t)^T \eta^{-1} c(t)^T G(t, S) + d(t)^T q + f(t)\right)
\]

\[
+ \frac{1}{2} \text{Tr} \left( \Gamma(t) \Sigma^{-1} b(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} \right)
\]

\[
= G(t, S)^T \left( \dot{b}(t) + \gamma \dot{b}(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} b(t) - \frac{1}{2} c(t)^T \eta^{-1} c(t)^T \right) G(t, S)
\]

\[
+ q^T \left(d(t) + \gamma \left(-I + c(t)^T \Sigma^{-1} \Gamma(t) \Sigma^{-1} \right) \Sigma \left(-I + \Sigma^{-1} \Gamma(t) \Sigma^{-1} c(t) \right) - \frac{1}{2} d(t)^T \eta^{-1} d(t) \right) q
\]

\[9\text{The function d should not be confused with the number d of risky assets.}

9
In order to tackle the global existence for the system of nonlinear ODEs (22), we first show that solving the system boils down to solving a matrix Riccati equation. Indeed, defining \( \tilde{\Sigma} \) is equivalent to the matrix Riccati equation

\[
\dot{\tilde{\Sigma}}(t) = \Sigma^{-1}(t) \Gamma(t) \Sigma^{-1}(t) - \frac{1}{2} c(t) \eta^{-1} f(t) - g^T \left( \dot{f}(t) + \gamma (-I + c(t)^T) \Sigma^{-1}(t) \Sigma^{-1}(t) - \frac{1}{2} d(t) \eta^{-1} f(t) \right) + \tilde{a}(t) + \frac{\gamma}{2} c(t)^T \Sigma^{-1}(t) \Gamma(t) \Sigma^{-1}(t) e(t) - \frac{1}{4} f(t)^T \eta^{-1} f(t) + \frac{1}{2} \text{Tr} \left( \Gamma(t) \Sigma^{-1} b(t) \Sigma^{-1}(t) \Gamma(t) \Sigma^{-1}(t) \right) = 0
\]

As it is straightforward to verify that \( \theta \) satisfies the terminal condition (20), the result is proved. \( \square \)

**Remark 1.** The system of ODEs (22) deserves a few comments:

- This system of ODEs can be decomposed into three separate groups of equations. A first nonlinear system with the three ODEs for \( b, c, \) and \( d \) with their associated terminal conditions can be solved independently from the others. Moreover, if there exists a global solution to this sub-system, then there exists a global solution to the system (22) with its associated terminal conditions (23). In fact, once \( b, c, \) and \( d \) are known, the ODEs in \( e \) and \( f \) become linear and then a can be obtained by a simple integration.

- If the system of ODEs for \( b, c, \) and \( d \) admits a solution, then \( b \) and \( d \) are symmetric matrices. This result can be obtained by noticing that if \((\tilde{b}, \tilde{c}, \tilde{d})\) is the unique local solution, then \((\tilde{b}^T, \tilde{c}, \tilde{d}^T)\) is also a solution with the same terminal condition.

In order to tackle the global existence for the system of nonlinear ODEs (22), we first show that solving the system boils down to solving a matrix Riccati equation. Indeed, defining \( P : [0, T] \rightarrow S_{2d}(\mathbb{R}) \) as

\[
P(t) = \begin{pmatrix} d(t) & c(t)^T \\ c(t) & b(t) \end{pmatrix}, \tag{24}
\]

and defining \( \tilde{\Sigma}(t) = \Sigma^{-1}(t) \Gamma(t) \Sigma^{-1}(t) \Sigma^{-1}(t) \), we see that the system

\[
\begin{aligned}
\dot{d}(t) &= \gamma \left( -\Sigma + c(t)^T \Sigma^{-1}(t) \Gamma(t) \Sigma^{-1}(t) - c(t) \Sigma^{-1}(t) \tilde{\Sigma}(t) c(t) \right) + \frac{1}{2} d(t)^T \eta^{-1} d(t) \\
\dot{b}(t) &= -\gamma b(t) \tilde{\Sigma}(t) b(t) + \frac{1}{2} c(t)^T \eta^{-1} c(t)^T \\
\dot{c}(t) &= \Sigma + \gamma b(t) \tilde{\Sigma}(t) - \gamma b(t) \tilde{\Sigma}(t) c(t) + \frac{1}{2} c(t)^T \eta^{-1} d(t)
\end{aligned} \tag{25}
\]

with terminal conditions

\[
\begin{aligned}
d(T) &= A \\
c(T) &= 0 \\
b(T) &= 0
\end{aligned} \tag{26}
\]

is equivalent to the matrix Riccati equation

\[
\dot{P}(t) = Q + Y(t)^T P(t) + P(t) Y(t) + P(t) U(t) P(t), \tag{27}
\]

with terminal condition

\[
P(T) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in S_{2d}(\mathbb{R}), \tag{28}
\]

where

\[
Q = \begin{pmatrix} -\gamma \Sigma & \Sigma \\ \Sigma & 0 \end{pmatrix} \in S_{2d}(\mathbb{R}), \quad Y(t) = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma^{-1} \Gamma(t) \end{pmatrix} \in M_{2d}(\mathbb{R}), \quad \text{and} \quad U(t) = \begin{pmatrix} \frac{1}{2} \eta^{-1} & 0 \\ 0 & -\gamma \Sigma(t) \end{pmatrix} \in S_{2d}(\mathbb{R}).
\]

When compared to the matrix Riccati ODEs arising in the linear-quadratic optimal control literature, this Riccati equation appears not to be a classical one. In fact, the matrix \( U(t) \) characterizing the quadratic term has both positive
and negative eigenvalues. Therefore, we cannot rely on classical existing results to prove existence. Nevertheless, in the following, we address the existence of a solution by using a priori estimates for the value function, using a method similar to the one used in [10].

Regarding the set of equations \((23)\), there exists a unique local solution by Cauchy-Lipschitz theorem. In the following section, we therefore first state a verification theorem that solves the problem on an interval \([\tau, T]\), and use that very result to address global existence and uniqueness of a solution on \([0, T]\).

### 3.4 Mathematical results

**Theorem 2.** Let \(\tau \in [0, T]\). Assume there exist \(a \in C^1 ([\tau, T]),\) \(b \in C^1 ([\tau, T], S_d(\mathbb{R}))\), \(c \in C^1 ([\tau, T], M_d(\mathbb{R}))\), \(d \in C^1 ([\tau, T], S_d(\mathbb{R}))\), \(e \in C^1 ([\tau, T], \mathbb{R}^d)\), and \(f \in C^1 ([\tau, T], \mathbb{R}^d)\) satisfying the equations \((22)\) for \(t \in [\tau, T]\) with terminal condition \((23)\). Let us consider the function \(\theta\) defined by \((21)\) and the associated function \(u\) defined by \((15)\).

Then, for all \((t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\) and \(v = (v_s)_{s \in [t, T]} \in \mathcal{A}_t\), we have

\[
\mathbb{E} \left[ - \exp \left( - \gamma \left( X^{t, x, S, v}_T + \hat{q}_T^t \right) S^{t, S} - \ell \left( \hat{q}_T^t, S^{t, S}, v \right) \right) \right] \leq u(t, x, q, S). \tag{29}
\]

Moreover, equality in \((29)\) is obtained by taking the optimal control \((v^*_s)_{s \in [t, T]} \in \mathcal{A}_t\) given by the closed-loop feedback formula

\[
\forall s \in [t, T], \quad v^*_s = \frac{1}{2} \eta^{-1} \left( d(s) \hat{q}_s^t + c(t) G(s, S^{t, S}) + f(s) \right). \tag{30}
\]

In particular \(u = w\) on \([\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\).

**Proof.** Let \(t \in [\tau, T]\), we first prove that \((v^*_s)_{s \in [t, T]}\) is well-defined and admissible, or equivalently \((v^*_s)_{s \in [t, T]} \in \mathcal{A}_t\). For that purpose, we consider the Cauchy problem

\[
\frac{d \hat{q}_s}{ds} = \phi(s) \hat{q}_s + \psi(s, S^{t, S}_s), \quad \hat{q}_t = q,
\]

where \(\phi\) and \(\psi\) are defined by

\[
\begin{align*}
\phi : s \in [t, T] &\mapsto \frac{1}{2} \eta^{-1} d(s), \\
\psi : (s, S) \in [t, T] \times \mathbb{R}^d &\mapsto \frac{1}{2} \eta^{-1} \left( c(t) G(s, S^{t, S}) + f(s) \right).
\end{align*}
\]

The unique solution of this problem is given by

\[
\forall s \in [t, T], \quad \hat{q}_s = \exp \left( \int_t^s \phi(\theta)d\theta \right) \left( q + \int_t^s \psi(\theta, S^{\theta, S}_\theta) \exp \left( - \int_t^\theta \phi(\zeta)d\zeta \right) d\theta \right).
\]

Then \(v^*_s\) writes

\[
\forall s \in [t, T], \quad v^*_s = \frac{d \hat{q}_s}{ds} = \phi(s) \exp \left( \int_t^s \phi(\theta)d\theta \right) \left( q + \int_t^s \psi(\theta, S^{\theta, S}_\theta) \exp \left( - \int_t^\theta \phi(\zeta)d\zeta \right) d\theta \right) + \psi(s, S^{t, S}_s).
\]

Given the definition of \(\psi\) and the affine nature of \(G\) with respect to \(S\), \((v^*_s)_{s \in [t, T]}\) satisfies the required linear growth condition to admissible and in \(\mathcal{A}_t\).

We now consider \((t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\) and an admissible control \(v = (v_s)_{s \in [t, T]} \in \mathcal{A}_t\). We next prove that

\[
\mathbb{E} \left[ u \left( T, X^{t, x, S, v}_T, \hat{q}_T^t, S^{t, S}_T \right) \right] \leq u(t, x, q, S).
\]

First, we use the following notations for the rest of the proof:
\[
\forall s \in [t, T], \quad u^{t,x,q,S,v}_s = u\left(s, X^{t,x,S,v}_s, q^{t,q,v}_s, S^{t,S}_s\right), \quad \\
\forall s \in [t, T], \quad \theta^{t,q,S,v}_s = \theta\left(s, q^{t,q,v}_s, S^{t,S}_s\right).
\]

By Itô's formula, we have for all \(s \in [t, T]\)
\[
du^{t,x,q,S,v}_s = \mathcal{L}^v u^{t,x,q,S,v}_s \, ds + \nabla S u^{t,x,q,S,v}_s \left(\sigma \circ d\hat{W}_s\right).
\]

where
\[
\mathcal{L}^v u^{t,x,q,S,v}_s = \partial_s u^{t,x,q,S,v}_s + G(s, S^{t,S}_s)\mathbf{T} \nabla S u^{t,x,q,S,v}_s + \frac{1}{2} \text{Tr}\left(\Sigma \nabla S u^{t,x,q,S,v}_s \mathbf{T}\right)
\]
\[
+ v^1 \nabla_q u^{t,x,q,S,v}_s - \left(v^1 S^{t,S}_s + L(v)\right) \partial_x u^{t,x,q,S,v}_s.
\]

Using the ansatz definitions (15) and (21), we have
\[
\nabla S u^{t,x,q,S,v}_s = -\gamma u^{t,x,q,S,v}_s (q^{t,q,v}_s - \nabla S \theta^{t,q,S,v}_s)
\]
\[
= -\gamma u^{t,x,q,S,v}_s (q^{t,q,v}_s - \Sigma^{-1} \Gamma(s) \Sigma^{-1} b(s) G(s, S^{t,S}_s) - \Sigma^{-1} \Gamma(s) \Sigma^{-1} c(s) q^{t,q,v}_s - \Sigma^{-1} \Gamma(s) \Sigma^{-1} e(t)).
\]

Let us presently define \(\forall s \in [t, T],\)
\[
\kappa^v_s = -\gamma \left(q^{t,q,v}_s - \Sigma^{-1} \Gamma(s) \Sigma^{-1} b(t) G(s, S^{t,S}_s) - \Sigma^{-1} \Gamma(s) \Sigma^{-1} c(s) q^{t,q,v}_s - \Sigma^{-1} \Gamma(s) \Sigma^{-1} e(s)\right),
\]
and
\[
\xi^v_{t,s} = \exp\left(\int^s_t \kappa^v_{\tau} \left(\sigma \circ d\hat{W}_\tau\right) - \frac{1}{2} \int^s_t \kappa^v_{\tau} \Sigma \kappa^v_{\tau} d\tau\right).
\]

Then
\[
d\left(u^{t,x,q,S,v}_s \left(\xi^v_{t,s}\right)^{-1}\right) = \left(\xi^v_{t,s}\right)^{-1} \mathcal{L}^v u^{t,x,q,S,v}_s \, ds.
\]

By definition of \(u,\) \(\mathcal{L}^v u^{t,x,q,S,v}_s \leq 0.\) Moreover, when the control reaches the supremum (18), we have \(\mathcal{L}^v u^{t,x,q,S,v}_s = 0.\) This only happens when \(v\) has the value
\[
v_s = -\frac{1}{2} q^{-1} \nabla_q \theta^{t,q,S,v}_s
\]
\[
= -\frac{1}{2} q^{-1} (c(s) G(s, S^{t,S}_s) + d(s) q^{t,q,v}_s + f(s)),
\]
which corresponds to the case when \((v_s)_{s \in [t,T]} = \left(v^*_s\right)_{s \in [t,T]}\).

As a consequence, \(\left(u^{t,x,q,S,v}_s \left(\xi^v_{t,s}\right)^{-1}\right)_{s \in [t,T]}\) is nonincreasing, and therefore
\[
u(T, X^{t,x,S,v}_T, q^{t,q,v}_T, S^{t,S}_T) \leq u(t, x, q, S) \xi^v_{t,T},
\]
with equality when \((v_s)_{s \in [t,T]} = \left(v^*_s\right)_{s \in [t,T]}\).

We take expectations above and get
\[
\mathbb{E}\left[-\exp\left(-\gamma \left(X^{t,x,S,v}_T + q^{t,q,v}_T S^{t,S}_T - \ell(q^{t,q,v})\right)\right)\right] = \mathbb{E}\left[u(T, X^{t,x,S,v}_T, q^{t,q,v}_T, S^{t,S}_T) \xi^v_{t,T}\right].
\]

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with equality when $(v_s)_{s \in [t,T]} = (v^*_s)_{s \in [t,T]}$.

We now prove that $\mathbb{E} \left[ \xi^v_{t,T} \right] = 1$, by using the fact that first, $\xi^v_{t,t} = 1$. Second, because $v \in \mathcal{A}$, satisfies the linear growth condition with respect to $(S^v_{t,T})_{s \in [t,T]}$, so does $(q^v_{t,q,v})_{s \in [t,T]}$. Therefore, Using a classical trick due to Beneš (see [26], Chapter 5), we see that $(\xi^v_{t,s})_{s \in [t,T]}$ is a martingale. We conclude that $\mathbb{E} \left[ \xi^v_{t,s} \right] = 1$ for all $s \in [t,T]$.

We obtain

$$\mathbb{E} \left[ -\exp \left( -\gamma \left( X^t_{T,T}S^v_{t,T} + q^v_{t,q,v} \right) - \ell \left( q^v_{t,q,v} \right) \right) \right] = \mathbb{E} \left[ u \left( T, X^t_{T,T}, q^v_{t,q,v}, S^v_{t,T} \right) \right] \leq u(t, x, q, S),$$

with equality when $(v_s)_{s \in [t,T]} = (v^*_s)_{s \in [t,T]}$.

We can conclude that

$$w(t, x, q, S) = \sup_{(v_s)_{s \in [t,T]} \in \mathcal{A}} \mathbb{E} \left[ -\exp \left( -\gamma \left( X^t_{T,T}S^v_{t,T} + q^v_{t,q,v} \right) - \ell \left( q^v_{t,q,v} \right) \right) \right]$$

$$= \mathbb{E} \left[ -\exp \left( -\gamma \left( X^t_{T,T}S^v_{t,T} + q^v_{t,q,v} \right) - \ell \left( q^v_{t,q,v} \right) \right) \right]$$

$$= u(t, x, q, S),$$

proving our verification theorem.

We will next proceed to prove existence and uniqueness of a solution to the system of ODEs (22) on $[0, T]$ with terminal conditions (23). This result provides a rigorous solution to the problem of optimal execution with Bayesian learning of the Gaussian drift.

**Theorem 3.** There exists a unique solution $a \in C^1([0, T]), b \in C^1([0, T], S_d(\mathbb{R})), c \in C^1([0, T], M_d(\mathbb{R})), d \in C^1([0, T], S_d(\mathbb{R})), e \in C^1([0, T], \mathbb{R}^d), f \in C^1([0, T], \mathbb{R}^d)$ to the system of ODEs (22) on $[0, T]$ with terminal condition (23).

**Proof.** As explained in Remark 1, proving Theorem 3 boils down to showing existence and uniqueness of a solution $b \in C^1([0, T], S_d(\mathbb{R})), c \in C^1([0, T], M_d(\mathbb{R})), d \in C^1([0, T], S_d(\mathbb{R})), e \in C^1([0, T], \mathbb{R}^d)$ to the system of ODEs (23) with terminal conditions (24). It is also equivalent to show existence and uniqueness on $[0, T]$ of a solution $P \in C^1([0, T], S_{2d}(\mathbb{R}))$ to the matrix Riccati equation (27) with terminal condition (28).

We first solve the problem near $T$. By the Cauchy-Lipschitz theorem, we know that there exists a unique maximal solution $(b, c, d)$ to the system of ODEs (25) with terminal conditions (28) defined on an open interval $(t_{\min}, t_{\max}) \ni T$, and using Theorem 2, we know that the associated function $u$ defined by (15) is in fact the value function of the problem on $[\tau, T]$ for all $\tau \in (t_{\min}, T)$.

In order to prove our result, our strategy is to show that $t_{\min} = -\infty$. For that purpose, we will prove that the matrix $P(t) = \begin{pmatrix} d(t) & c(t) \\ c(t) & b(t) \end{pmatrix}$ cannot blow up in finite time. This is achieved by, first, finding, thanks to the control problem, lower and upper bounds for the function $\theta$ in the form of polynomials of degree at most 2 in $(q, G)$. We then use the fact that we can convert these bounds into bounds for $P(t)$ with respect to the natural order on symmetric matrices $\mathbb{M}$.

By contradiction, let us assume that $t_{\min} \in (-\infty, T)$ and let $\tau \in (t_{\min}, T)$.

**Upper bound for $P$:** Starting from values $(t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, let us consider the suboptimal strategy $v = (0)_{s \in [t,T]} \in \mathcal{A}_{\tau}$ for which $\forall s \in [t, T], \quad q^v_{t,s,v} = q$ and

$$\mathbb{E} \left[ -\exp \left( -\gamma \left( X^t_{T,T}S^v_{t,T} + q^v_{t,q,v} \right) - \ell \left( q^v_{t,q,v} \right) \right) \right] = \mathbb{E} \left[ -\exp \left( -\gamma \left( x + q^T S + q^T \left( S^v_{t,T} - S \right) - \frac{1}{2} q^T A q \right) \right) \right].$$

(31)

\[10\text{For } \mathbb{M}, \mathbb{M}^T \in S_d(\mathbb{R}), \quad \mathbb{M} \leq \mathbb{M}^T \text{ if and only if } \mathbb{M} - \mathbb{M}^T \in S^+_{2d}(\mathbb{R}).\]
Given the dynamics (39) of \((S_{t,s}^{l,s})_{s \in [t,T]}\) and their solution (40) given in Proposition 4
\[
S_t = \exp \left( - \int_0^t R_s ds \right) S_0 + \left( I - \exp \left( - \int_0^t R_s ds \right) \right) \bar{S} + \int_0^t \exp \left( - \int_u^t R_s ds \right) \sigma \odot d\hat{W}_u,
\]
we know that
\[
S_{t,T}^l - S \sim \mathcal{N}(m_t, \Omega_t),
\]
where
\[
m_t = \left( I - \exp \left( - \int_0^t R_s ds \right) \right) (\bar{S} - S),
\]
and
\[
\Omega_t = \int_t^T \exp \left( - \int_u^t R_s ds \right) \sigma \sigma^\top \exp \left( - \int_u^t R_s ds \right)^\top du.
\]
Then for our sub-optimal strategy we can write
\[
\mathbb{E} \left[ - \exp \left( -\gamma \left( X_T^{l,x,S,v} + (q_T^{l,q,v})^\top S_T^l - \ell(t,q,v) \right) \right) \right] = - \exp \left( -\gamma (x + q^\top S) \right) \exp \left( -\gamma \left( q^\top m_t - q^\top Aq - \frac{1}{2} \gamma q^\top \Omega t q \right) \right).
\]
Since the strategy is sub-optimal, if we consider \(\theta\) defined in (13), we have by Theorem 2
\[
- \exp \left( -\gamma (x + q^\top S - \theta(t,q,S)) \right) \geq - \exp \left( -\gamma (x + q^\top S) \right) \exp \left( -\gamma \left( q^\top m_t - q^\top Aq - \frac{1}{2} \gamma q^\top \Omega t q \right) \right).
\]
We conclude that for all \((t,q,S) \in [t,T] \times \mathbb{R}^d \times \mathbb{R}^d,\)
\[
\theta(t,q,S) = \left( \begin{array}{c} q \\ \frac{q}{G(t,S)} \end{array} \right)^\top P(t) \left( \begin{array}{c} q \\ G(t,S) \end{array} \right) + \frac{f(t)}{e(t)} \left( \begin{array}{c} q \\ G(t,S) \end{array} \right) + \alpha(t)
\leq \left( \begin{array}{c} \frac{q}{G(t,S)} \\ G(t,S) \end{array} \right)^\top \left( \begin{array}{cc} \frac{2}{m_{l,t}} + A & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} q \\ G(t,S) \end{array} \right) + \left( \begin{array}{c} q \\ G(t,S) \end{array} \right).
\]
We therefore necessarily have, for the natural order on symmetric matrices,
\[
\forall t \in [t,T], \quad P(t) \leq \left( \begin{array}{cc} \frac{q}{G(t,S)} & 0 \\ 0 & 0 \end{array} \right).
\]

**Lower bound for** \(P\): Now, for \((t,x,q,S) \in [t,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d,\) we have
\[
\sup_{v \in A_t} \mathbb{E} \left[ - \exp \left( -\gamma \left( X_T^{l,x,S,v} + (q_T^{l,q,v})^\top S_T^l - (q_T^{l,q,v})^\top Aq_T^{l,q,v} \right) \right) \right] \quad (34)
\]
\[
= \sup_{v \in A_t} \mathbb{E} \left[ - \exp \left( -\gamma \left( x + q^\top S + \int_t^T (q_s^{l,q,v})^\top ds - (q_T^{l,q,v})^\top A_T^{l,q,v} \right) \right) \right] \quad (35)
\]
If \((v_s)_{s \in [t,T]} \in A_t,\) it is straightforward to see that the process \((q_s^{l,q,v})_{s \in [t,T]}\) is in the space of admissible controls \(A_t^{\text{Merton}},\) defined in (40). In the Appendix we study a Merton problem within the same framework of Bayesian
Bayesian learning of the unknown drift is solved and the optimal strategy is given by the closed-loop feedback control

$$\sup_{\tau \in \mathcal{A}} \mathbb{E} \left[ -\exp \left( -\gamma \left( X_{T}^{x,S,v} + (q^{q,q,v})^{T} S_{T}^{x,S} - (q^{q,q,v})^{T} \Gamma q^{q,q,v} \right) \right) \right]$$

$$\leq \exp \left( -\gamma (x + q^{T}S) \right) \sup_{q \in \mathcal{A}^{Merton}} \mathbb{E} \left[ -\exp \left( -\gamma \left( q_{T}^{q} dS_{e} \right) \right) \right]. \quad (36)$$

As shown in the Appendix A, the inequality (36) becomes

$$-\exp \left( -\gamma (x + q^{T}S - \theta(t,q,S)) \right) \leq -\exp \left( -\gamma \left( x + q^{T}S + \hat{\theta}(t,S) \right) \right),$$

where $\hat{\theta}(t,S) = \frac{1}{\gamma} G(t,S)^{T} \hat{b}(t) G(t,S) + \hat{\dot{e}}(t)^{T} G(t,S) + \hat{\dot{a}}(t)$, with $\hat{\dot{\theta}} \in C^{1}([\tau,T], S_{d}(\mathbb{R}))$, $\hat{\dot{e}} \in C^{1}([\tau,T], \mathbb{R}^{d})$, $\hat{\dot{a}} \in C^{1}([\tau,T], \mathbb{R})$ are defined by

$$\begin{cases}
\hat{\dot{b}}(t) = \frac{1}{\gamma} \left( \Gamma(t)^{-1} \Sigma^{2} - \Gamma(t)^{-2} \Sigma^{2} \Gamma(T) \right),
\hat{\dot{e}}(t) = 0,
\hat{\dot{a}}(t) = \int_{t}^{T} \frac{1}{2 \gamma} \text{tr} \left( \Sigma \Gamma u \Sigma^{-1} \right) du - \frac{1}{2 \gamma} (T - t) \text{tr} \left( \Gamma(T) \Sigma^{-1} \right).
\end{cases}$$

We conclude that for all $(t,q,S) \in [\tau,T] \times \mathbb{R}^{d} \times \mathbb{R}^{d}$,

$$\theta(t,q,S) = \left( \begin{array}{c} q \\ G \end{array} \right)^{T} \left( \begin{array}{c} P(t) \left( \begin{array}{c} q \\ G \end{array} \right) + \left( \begin{array}{c} f(t) \\ e(t) \end{array} \right)^{T} \left( \begin{array}{c} q \\ G \end{array} \right) + a(t) \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 0 \end{array} \right)^{T} \left( \begin{array}{c} q \\ G \end{array} \right) - \hat{a}(t).$$

Therefore, we necessarily have

$$\forall t \in [\tau,T], \quad P(t) \geq \left( \begin{array}{cc} 0 & 0 \\ 0 & -\frac{1}{\gamma} \left( \Gamma(t)^{-1} \Sigma^{2} - \Gamma(t)^{-2} \Sigma^{2} \Gamma(T) \right) \end{array} \right).$$

We have therefore from the two bounds above, $\forall t \in (t_{\text{min}}, T), \forall t \in [\tau,T]$:

$$\left( \begin{array}{cc} 0 & 0 \\ 0 & -\frac{1}{\gamma} \left( \Gamma(t)^{-1} \Sigma^{2} - \Gamma(t)^{-2} \Sigma^{2} \Gamma(T) \right) \end{array} \right) \leq P(t) \leq \left( \begin{array}{cc} \frac{2}{\gamma} \Sigma_{t} + A & 0 \\ 0 & 0 \end{array} \right).$$

As $t_{\text{min}}$ is supposed to be finite, there exists $\underline{M}, \overline{M} \in S_{d}(\mathbb{R})$ with $\underline{M} \leq \overline{M}$ such that $\forall t \in [t_{\text{min}}, T]$, $P(t)$ stays in the compact set $\{ M \in S_{d}(\mathbb{R}) \mid \underline{M} \leq M \leq \overline{M} \}$. This contradicts the maximality of the solution, hence $t_{\text{min}} = -\infty$.

\[ \square \]

Theorem 3 implies that Theorem 2 can be applied with $\tau = -\infty$. In particular, the optimal execution problem with Bayesian learning of the unknown drift is solved and the optimal strategy is given by the closed-loop feedback control (30).
Conclusion

In this paper, we have shown how to rigorously incorporate the continuous Bayesian learning of the drifts of the assets into an optimal execution framework relying on stochastic optimal control tools. This approach is appealing as it gives the agent the ability to rigorously incorporate her uncertainty about the future drifts, and let the data update her estimates as well as their distribution. The obtained strategy is optimal as it simultaneously takes into account (i) the optimal balance between trading costs and inventory risk, (ii) the latest estimate of the drift, and (iii) the fact that there is still uncertainty along with the fact that the agent will continue learning. The advantages of such an approach are numerous, as one can also extend the results for statistical arbitrage strategies, or incorporate exogenous information in the form of other assets or predictive signals into the dynamics of $S$ in order to enhance the execution process.
A Appendix - Merton portfolio optimization problem with Bayesian learning and an exponential utility

A.1 Modelling framework

In this appendix, we address rigorously a Merton problem where the prices \((S_t)_{t \in [0,T]} = (S^1_t, \ldots, S^d_t)^T\) follow a \(d\)-dimensional drifted Bachelier dynamics. The result of this appendix are central to proving the results of theorem 3.

Similar to the framework explored in this paper, we consider a market with \(d\) risky assets with the following dynamics:

\[
dS_t = \mu dt + V \circ dW_t,
\]

where, the volatility vector \(\sigma = (\sigma^1, \ldots, \sigma^d)^T\) satisfies \(\forall i \in \{1, \ldots, d\}, \sigma^i > 0\), and where the drift vector \(\mu = (\mu^1, \ldots, \mu^d)^T\) is unknown. We consider that the agent has a non-degenerate multivariate Gaussian prior \(m_\mu\) for the drift of the form

\[
m_\mu(dz) = \frac{1}{(2\pi)^{d/2} |\Gamma_0|^{1/2}} \exp\left(-\frac{1}{2} (z - \beta_0)^\top \Gamma_0^{-1} (z - \beta_0)\right) dz,
\]

where \(\beta_0 \in \mathbb{R}^d\) and \(\Gamma_0 \in \mathbb{R}^{d \times d}\). As shown in Propositions 1 and 2, one can write the dynamics of the prices \(S\) as

\[
dS_t = \beta_0 dt + \sigma \circ d\tilde{W}_t = \Sigma G(t, S_t) dt + \sigma \circ d\tilde{W}_t,
\]

where \(\Gamma(t) = (\Gamma_0^{-1} + t\Sigma^{-1})^{-1}\), and \((\tilde{W}_t)_{t \in \mathbb{R}_+}\) is a Brownian motion adapted to \((\mathcal{F}^S_t)_{t \in \mathbb{R}_+}\), with the same correlation structure as \((W_t)_{t \in \mathbb{R}_+}\), i.e.

\[
\forall i, j \in \{1, \ldots, d\}, \quad d\langle \tilde{W}^i, \tilde{W}^j \rangle_t = d\langle W^i, W^j \rangle_t.
\]

The Merton problem with Bayesian learning of the drift is closely related to the optimal execution problem presented in this paper. It can be seen as some form of limit case corresponding to no execution costs (i.e. \(L = 0\)) and no terminal penalty (i.e. \(\ell = 0\)). The results obtained in this appendix are used in our proof for the existence of a global solution to the system of ODEs \((25)\) on \([0, T]\) with terminal condition \((26)\) (see Theorem 3).

We consider an agent in charge of a portfolio comprising the \(d\) assets. She wants to optimize her wealth over a period \([0, T]\) by controlling at each time the quantity she is holding in every asset, i.e. she controls a \(d\)-dimensional process \((q_t)_{t \in [0,T]} = (q^1_t, \ldots, q^d_t)^T\), where we define for all \(i \in \{1, \ldots, d\}, q^i_t\) as the number of assets \(i\) in her portfolio at time \(t\)\(^{11}\)

Let us now suppose that the process \((q_t)_{t \in [0,T]}\) lies in the space of admissible controls \(A_0^{Merton}\), where for all \(t \in [0, T]\), the set \(A_t^{Merton}\) is defined as

\[
A_t^{Merton} := \{(q_s)_{s \in [t,T]}, \mathbb{R}^d\text{-valued } \mathcal{F}^S_{s}\text{-adapted process, satisfying the linear growth condition with respect to } (S_s)_{s \in [t,T]}\}.
\]

Similar to the model in this paper, we consider that the agent has a constant absolute risk aversion denoted by \(\gamma > 0\). She aims at maximizing the following objective function:

\[
E \left[ -e^{-\gamma \mathcal{V}_T} \right],
\]

\(^{11}\)The control variable is here the number of assets held at time \(t\), and not the traded volume at time \(t\). It is only when execution costs are taken into account that one has to consider trading rates as the relevant control variable.
over the set of admissible controls \((q_t)_{t \in [0, T]} \in A_{Merton}^0\), where the process \((V_t)_{t \in [0, T]}\) models the MtM value of the agent’s portfolio, i.e.

\[
\forall t \in [0, T], \quad V_t = V_0 + \int_0^t q_s^T dS_s, \quad V_0 \in \mathbb{R} \text{ given.}
\]

We define the agent’s value function \(\hat{w} : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) as:

\[
\hat{w}(t, V, S) = \sup_{(q_s)_{s \in [t, T]} \in A_{Merton}^t} \mathbb{E} \left[ -e^{-\gamma V_T} \right] \quad \forall (t, V, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,
\]

where \((V_s^{t, V, S, q})_{s \in [t, T]}\) denotes the process defined by

\[
dV_s^{t, V, S, q} = q_s^T dS_s, \quad V_t^{t, V, S, q} = V,
\]

where

\[
S_t^{t, S} = \Sigma G(s, S_t^{t, S}) ds + \sigma \circ d\hat{W}_s, \quad S_t^{t, S} = S.
\]

### A.2 HJB equation

The HJB equation associated with the problem (41) is given by

\[
0 = \partial_t \hat{u}(t, V, S) + \nabla S \hat{u}(t, V, S)^T \Sigma G(t, S) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \hat{u}(t, V, S) \right)
+ \sup_{q \in \mathbb{R}^d} \left\{ \partial_v \hat{u}(t, V, S) q^T \Sigma G(t, S) + \frac{1}{2} \partial^2_{vv} \hat{u}(t, V, S) q^T \Sigma q + \partial_v \nabla S \hat{u}(t, V, S)^T \Sigma q \right\},
\]

for all \((t, V, S) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d\), with terminal condition

\[
\hat{u}(T, V, S) = -e^{-\gamma V} \quad \forall (V, S) \in \mathbb{R} \times \mathbb{R}^d.
\]

To solve the HJB equation, we introduce the ansatz

\[
\hat{u}(t, V, S) = -e^{-\gamma (V + \hat{\theta}(t, S))}.
\]

This ansatz is used based on the following result:

**Proposition 7.** If there exists \(\hat{\theta} \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})\) solution to

\[
0 = \partial_t \hat{\theta}(t, S) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \hat{\theta}(t, S) \right) + \frac{1}{2\gamma} G(t, S)^T \Sigma G(t, S)
\]

on \([0, T) \times \mathbb{R}^d\), with terminal condition

\[
\hat{\theta}(T, S) = 0 \quad \forall S \in \mathbb{R}^d,
\]

then the function \(\hat{u} : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) defined by (44) for all \((t, V, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d\) is a solution to (42) on \([0, T) \times \mathbb{R} \times \mathbb{R}^d\) with terminal condition (43).

**Proof.** Let \(\hat{\theta} : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) be a solution to (45) on \([0, T) \times \mathbb{R}^d\) with terminal condition (46), then we have for all
(t, V, S) ∈ [0, T) × \mathbb{R} × \mathbb{R}^d:

\begin{align*}
\partial_t \hat{w}(t, V, S) + \nabla_S \hat{w}(t, V, S)^T \Sigma G(t, S) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \hat{w}(t, V, S) \right) \\
+ \sup_{q \in \mathbb{R}^d} \left\{ \partial_V \hat{w}(t, V, S)q^T \Sigma G(t, S) + \frac{1}{2} \partial^2_{VV} \hat{w}(t, V, S)q^T \Sigma q + \partial_V \nabla_S \hat{w}(t, V, S)^T \Sigma q \right\} \\
= -\gamma \hat{w}(t, V, S)\partial_t \hat{\theta}(t, S) - \gamma \hat{w}(t, V, S)\nabla_S \hat{\theta}(t, S)^T \Sigma G(t, S) - \frac{\gamma}{2} \hat{w}(t, V, S)\text{Tr} \left( \Sigma D^2_{SS} \hat{\theta}(t, S) \right) \\
+ \frac{\gamma^2}{2} \hat{w}(t, V, S)\nabla_S \hat{\theta}(t, S)^T \Sigma \nabla_S \hat{\theta}(t, S) \\
+ \sup_{q \in \mathbb{R}^d} \left\{ -\gamma \hat{w}(t, V, S)q^T \Sigma G(t, S) + \frac{\gamma^2}{2} \hat{w}(t, V, S)q^T \Sigma q + \gamma^2 \hat{w}(t, V, S)\nabla_S \hat{\theta}(t, S)^T \Sigma q \right\}
\end{align*}

The supremum in the above equation is reached for \( q = q^*(t, S) = \frac{1}{\gamma} G(t, S) - \nabla_S \hat{\theta}(t, S) \). Hence we can write:

\begin{align*}
\partial_t \hat{w}(t, V, S) + \nabla_S \hat{w}(t, V, S)^T \Sigma G(t, S) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \hat{w}(t, V, S) \right) \\
+ \sup_{q \in \mathbb{R}^d} \left\{ \partial_V \hat{w}(t, V, S)q^T \Sigma G(t, S) + \frac{1}{2} \partial^2_{VV} \hat{w}(t, V, S)q^T \Sigma q + \partial_V \nabla_S \hat{w}(t, V, S)^T \Sigma q \right\} \\
= -\gamma \hat{w}(t, V, S)\left( \partial_t \hat{\theta}(t, S) + \frac{1}{2} \text{Tr} \left( \Sigma D^2_{SS} \hat{\theta}(t, S) \right) + \frac{1}{2\gamma} G(t, S)^T \Sigma G(t, S) \right) \\
= 0.
\end{align*}

As \( \hat{w} \) satisfies the terminal condition (43), the result is proved.

Similar to the approach for solving the problem in the paper, we use a second ansatz and look for a function \( \hat{\theta} \) solution to (45) on \([0, T) \times \mathbb{R}^d \) with terminal condition (46) with the following form:

\begin{equation}
\hat{\theta}(t, S) = \frac{1}{2} G(t, S)^T \hat{b}(t) G(t, S) + \hat{c}(t)^T G(t, S) + \hat{a}(t).
\end{equation}

We justify the use of this ansatz in the following result:

**Proposition 8.** Assume there exists \( \hat{b} \in C^1 \left( [0, T], \mathcal{S}_d(\mathbb{R}) \right) \), \( \hat{c} \in C^1 \left( [0, T], \mathbb{R}^d \right) \), \( \hat{a} \in C^1 \left( [0, T], \mathbb{R} \right) \) satisfying the system of ODEs

\begin{equation}
\begin{cases}
\dot{\hat{b}}(t) = 2\Gamma(t) \Sigma^{-1} \hat{b}(t) - \frac{1}{2} \Sigma \\
\dot{\hat{c}}(t) = \Gamma(t) \Sigma^{-1} \hat{c}(t) \\
\dot{\hat{a}}(t) = -\frac{1}{2} \text{Tr} \left( \Gamma(t) \Sigma^{-1} \hat{b}(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} \right) 
\end{cases}
\end{equation}

with terminal condition

\begin{equation}
\hat{b}(T) = \hat{c}(T) = \hat{a}(T) = 0.
\end{equation}

Then the function \( \hat{\theta} \) defined by (47) satisfies (45) on \([0, T) \times \mathbb{R}^d \) with terminal condition (46).

**Proof.** Let us consider \( \hat{b} \in C^1 \left( [0, T], \mathcal{S}_d(\mathbb{R}) \right) \), \( \hat{c} \in C^1 \left( [0, T], \mathbb{R}^d \right) \), \( \hat{a} \in C^1 \left( [0, T], \mathbb{R} \right) \) verifying (48) on \([0, T) \) with terminal
condition (19). Let us consider \( \hat{\theta} : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) defined by (17). Then we obtain for all \( (t, S) \in [0, T) \times \mathbb{R}^d \):

\[
\frac{\partial \hat{\theta}}{\partial t}(t, S) + \frac{1}{2} \text{Tr} \left( \Sigma D_{\hat{S}}^2 \hat{\theta}(t, S) \right) + \frac{1}{2\gamma} G(t, S)^T \Sigma G(t, S) 
= G(t, S)^T \left( -\frac{1}{2} \Gamma(t)^T \Sigma^{-1} \hat{b}(t) + \frac{1}{2} \hat{b}(t) \Sigma^{-1} \Gamma(t) + \frac{1}{2\gamma} \Sigma \right) G(t, S) 
+ \left( \hat{\epsilon}(t) - \hat{\epsilon}(t)^T \Sigma^{-1} \Gamma(t) \right) G(t, S) + \hat{a}(t) + \frac{1}{2\gamma} \text{Tr} \left( \Gamma(t) \Sigma^{-1} \hat{b}(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} \right) 
= 0.
\]

As it is straightforward to verify that \( \hat{\theta} \) satisfies the terminal condition (19), the result is proved.

It is straightforward to see that there exists a unique solution \( \hat{b} \in C^1([0, T], \mathcal{S}_d(\mathbb{R})), \hat{\epsilon} \in C^1([0, T], \mathbb{R}^d), \hat{a} \in C^1([0, T], \mathbb{R}) \) to (18) with terminal condition (19), we state the following Proposition to provide a closed-form solution:

**Proposition 9.** Let \( \hat{b} \in C^1([0, T], \mathcal{S}_d(\mathbb{R})), \hat{\epsilon} \in C^1([0, T], \mathbb{R}^d), \hat{a} \in C^1([0, T], \mathbb{R}) \) be the solutions to (18) with terminal condition

\[
\hat{b}(T) = \hat{\epsilon}(T) = \hat{a}(T) = 0,
\]

then, for all \( t \in [0, T] \),

\[
\begin{cases}
\hat{b}(t) = \frac{1}{\gamma} \left( \Gamma(t)^{-1} \Sigma^2 - \Gamma(t)^{-2} \Sigma^2 \Gamma(T) \right), \\
\hat{\epsilon}(t) = 0, \\
\hat{a}(t) = \int_t^T \frac{1}{2\gamma} \text{Tr} \left( \Sigma \Gamma_u \Sigma^{-1} \right) du - \frac{1}{2\gamma} \left( T - t \right) \text{Tr} \left( \Gamma(T) \Sigma^{-1} \right).
\end{cases}
\]

**Proof.** We proceed to prove the solutions for respectively \( \hat{b}, \hat{\epsilon}, \) and \( \hat{a} \).

**Solution for \( \hat{b} \):** The equation for \( \hat{b} \) writes:

\[
\dot{\hat{b}}(t) = 2 \Gamma(t) \Sigma^{-1} \hat{b}(t) - \frac{1}{\gamma} \Sigma = 2 \left( \Gamma_0^{-1} \Sigma + tI \right)^{-1} \hat{b}(t) - \frac{1}{\gamma} \Sigma.
\]

We look for a solution of the form

\[
\hat{b}(t) = \frac{1}{\gamma} \left( \Gamma_0^{-1} \Sigma + tI \right) \Sigma + \left( \Gamma_0^{-1} \Sigma + tI \right)^2 C,
\]

where \( C \) is a constant matrix. By straightforward calculations we prove that, for all \( t \in [0, T] \):

\[
\hat{b}(t) = \frac{1}{\gamma} \left( \Gamma_0^{-1} \Sigma + tI \right) \Sigma - \frac{1}{\gamma} \left( \Gamma_0^{-1} \Sigma + tI \right)^2 \left( \Gamma_0^{-1} + T \Sigma^{-1} \right)^{-1} = \frac{1}{\gamma} \left( \Gamma(t)^{-1} \Sigma^2 - \Gamma(t)^{-2} \Sigma^2 \Gamma(T) \right)
\]

**Solution for \( \hat{\epsilon} \):** The equation for \( \hat{\epsilon} \) writes:

\[
\dot{\hat{\epsilon}}(t) = \Gamma(t) \Sigma^{-1} \hat{\epsilon}(t).
\]

We look for a solution of the form \( \hat{\epsilon}(t) = \left( \Gamma_0^{-1} \Sigma + tI \right) C \), where \( C \) is constant vector. Using the terminal condition for \( \hat{\epsilon} \) and by straightforward calculations, we prove that \( \epsilon(t) = 0 \) for all \( t \in [0, T] \).

**Solution for \( \hat{a} \):** The equation for \( \hat{a} \) writes:

\[
\dot{\hat{a}}(t) = -\frac{1}{2} \text{Tr} \left( \Gamma(t) \Sigma^{-1} \hat{b}(t) \Sigma^{-1} \Gamma(t) \Sigma^{-1} \right) 
= \frac{1}{2\gamma} \text{Tr} \left( \Sigma \left( \Gamma_0^{-1} \Sigma + tI \right)^{-1} \right) + \frac{1}{2\gamma} \text{Tr} \left( \Gamma(T) \Sigma^{-1} \right)
\]

The solution is then obtained by a simple integration.
We can now prove the following verification theorem:

**Theorem 4.** We consider the functions \( \hat{b} \in C^1 ([0, T], \mathcal{S}_d(\mathbb{R})) \), \( \hat{\varphi} \in C^1 ([0, T], \mathbb{R}^d) \), \( \hat{\alpha} \in C^1 ([0, T], \mathbb{R}) \) solutions to 

\[
\hat{b}(T) = \hat{\varphi}(T) = \hat{\alpha}(T) = 0,
\]
i.e. for all \( t \in [0, T] \),

\[
\begin{align*}
\hat{b}(t) &= \frac{1}{2} \left( \Gamma (t)^{-1} \Sigma^2 - \Gamma (t)^{-1} \Sigma^2 \Gamma (t) \right), \\
\hat{\varphi}(t) &= 0, \\
\hat{\alpha}(t) &= \int_t^T \frac{1}{2} \operatorname{Tr} \left( \Sigma \Gamma_u \Sigma^{-1} \right) du - \frac{1}{2} \left( T - t \right) \operatorname{Tr} \left( \Gamma (t) \Sigma^{-1} \right).
\end{align*}
\]

We consider the function \( \hat{\theta} \) defined by

\[
\hat{\theta}(t, S) = \frac{1}{2} G(t, S)^T \hat{b}(t) G(t, S) + \hat{\varphi}(t) G(t, S) + \hat{\alpha}(t),
\]
and the associated function \( \hat{u} \) defined by

\[
\hat{w}(t, \mathcal{V}, S) = - e^{-(\gamma + \hat{\theta}(t, S))}.
\]

For all \((t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \) and \( q = (q_s)_{s \in [t, T]} \in \mathcal{A}_t^{Merton} \), we have

\[
\mathbb{E} \left[ - e^{-(\gamma \mathcal{V}^\mathcal{V}, S)} \right] \leq \hat{u}(t, \mathcal{V}, S).
\]

Moreover, equality is obtained in \((50)\) by taking the optimal control \((q^*_s)_{s \in [t, T]} \in \mathcal{A}_t^{Merton} \) given by the closed-loop feedback formula

\[
q^*_s = \frac{1}{\gamma} \left( I_d - \Gamma (s)^{-1} \Sigma^2 + \Gamma (s)^{-1} \Sigma^2 \Gamma (t) \right) G (s, S) \tag{51}
\]

In particular, \( \hat{w} = \hat{u} \).

**Proof.** It is first easy to notice that \((q^*_s)_{s \in [t, T]} \) is well defined and admissible, i.e. \((q^*_s)_{s \in [t, T]} \in \mathcal{A}_t^{Merton} \) since

\[
\exists C_{t, T} > 0, \forall s \in [t, T], \quad \| q^*_s \| \leq C_{t, T} \left( 1 + \sup_{\tau \in [t, s]} \| S_{\tau} \| \right).
\]

We now consider \((t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \) and an admissible strategy \( q = (q_s)_{s \in [t, T]} \in \mathcal{A}_t^{Merton} \). We want to prove that

\[
\mathbb{E} \left[ \hat{u} \left( T, \mathcal{V}_T^{t, S}, S_T^{t, S} \right) \right] \leq \hat{u}(t, \mathcal{V}, S).
\]

We use the following notations for readability

\[
\forall s \in [t, T], \quad \hat{u}_s^{t, \mathcal{V}, S} = \hat{u} \left( s, \mathcal{V}_s^{t, \mathcal{V}, S}, S_s^{t, S} \right), \quad \hat{\theta}_s^{t, S} = \hat{\theta} \left( s, S_s^{t, S} \right).
\]

By Itô’s formula, we have \( \forall s \in [0, T] \)

\[
d \hat{u}_s^{t, \mathcal{V}, S} = \mathcal{L} \hat{u}_s^{t, \mathcal{V}, S} ds + (\partial_{\mathcal{V}} \hat{u}_s^{t, \mathcal{V}, S} q_s + \nabla S \hat{u}_s^{t, \mathcal{V}, S})^T \left( \sigma \otimes d \hat{W}_s \right),
\]
where

\[
\mathcal{L} \hat{u}_s^{t, \mathcal{V}, S} = \partial_{t} \hat{u}_s^{t, \mathcal{V}, S} + \left( \nabla S \hat{u}_s^{t, \mathcal{V}, S} q_s + \nabla S \hat{u}_s^{t, \mathcal{V}, S} \right)^T \Sigma G (s, S_s^{t, S}) + \partial_{\mathcal{V}} \hat{u}_s^{t, \mathcal{V}, S} q_s \Sigma G (s, S_s^{t, S}) + \frac{1}{2} \operatorname{Tr} \left( \Sigma D_{SS}^2 \hat{u}_s^{t, \mathcal{V}, S} q_s \right) + \frac{1}{2} \partial_{\mathcal{V}}^2 \hat{u}_s^{t, \mathcal{V}, S} q_s \Sigma q_s.
\]
Using the definitions in (44) and (47), we know that
\[ \nabla_S \hat{u}^t,V,S,q = -\gamma \hat{u}^t,V,S,q \nabla_S \hat{q}^t \]
\[ = -\gamma \hat{u}^t,V,S,q \left( \Sigma^{-1} \Gamma (s) \Sigma^{-1} \hat{b}(s) G \left( t, S^t_s \right) + \Sigma^{-1} \Gamma (s) \Sigma^{-1} \hat{c}(s) \right) \]
and
\[ \partial_V \hat{u}^t,V,S,q = -\gamma \hat{u}^t,V,S,q. \]

We define for all \( s \in [t, T] \)
\[ \kappa^q_s = -\gamma \left( q_s + \Sigma^{-1} \Gamma (s) \Sigma^{-1} \hat{b}(s) G \left( t, S^t_s \right) + \Sigma^{-1} \Gamma (s) \Sigma^{-1} \hat{c}(s) \right), \]
\[ \xi^q_{t,s} = \exp \left( \int_t^s \kappa^q_{\theta}^T \left( \sigma \otimes d\hat{W}_\theta \right) - \frac{1}{2} \int_t^s \kappa^q_{\theta}^T \Sigma \kappa^q_{\theta} d\theta \right). \]

We then have
\[ d \left( \hat{u}^t,V,S,q \left( \xi^q_{t,s} \right)^{-1} \right) = (\xi^q_{t,s})^{-1} \mathcal{L}^q \hat{u}^t,V,S,q ds. \]

By definition of \( \hat{u} \), \( \mathcal{L}^q \hat{u}^t,V,S,q \leq 0 \). Moreover, equality holds when we take a control for which the supremum is reached in (42). It is straightforward to see that this is the case for the unique value
\[ q_s = \frac{1}{\gamma} G \left( s, S^t_s \right) - \nabla_S \hat{\theta} \left( s, S^t_s \right) \]
\[ = \frac{1}{\gamma} G \left( s, S^t_s \right) - \hat{b}(s) G \left( s, S^t_s \right) - \hat{c}(s) \]
\[ = \frac{1}{\gamma} \left( I_d - \Gamma^{-1} (s) \Sigma^2 + \Gamma^{-2} (s) \Sigma^2 \Gamma (T) \right) G \left( s, S \right) \]
which corresponds to \( (q_s)_{s \in [t, T]} = (q^*_s)_{s \in [t, T]} \).

As a consequence, \( \left( \hat{u}^t,V,S,q \left( \xi^q_{t,s} \right)^{-1} \right)_{s \in [t, T]} \) is nonincreasing and therefore
\[ \hat{u} \left( T, V^T_t,V,S,q, S^t_T \right) \leq \hat{u}(t, V, S) \xi^q_{s,t}, \]
with equality when \( (q_s)_{s \in [t, T]} = (q^*_s)_{s \in [t, T]} \).

We now take the expectations in both sides of the above inequality and get
\[ \mathbb{E} \left[ \hat{u} \left( T, V^T_t,V,S,q, S^t_T \right) \right] \leq \hat{u}(t, V, S) \mathbb{E} \left[ \xi^q_{s,t} \right]. \]

The next step is to prove that \( \mathbb{E} \left[ \xi^q_{s,t} \right] \) is equal to 1. To do so, we use that \( \xi^q_{s,t} = 1 \) and prove that \( (\xi^q_{s,t})_{s \in [t, T]} \) is a martingale under \( (\mathbb{P}; \mathcal{F})_{s \in [t, T]} \). Given the form of \( \kappa \) and that \( (q^*_s)_{s \in [t, T]} \) satisfies a linear growth condition with respect to \( (S^t_s)_{s \in [t, T]} \), one can easily show that there exists a constant \( C \) such that
\[ \sup_{s \in [t, T]} \| \kappa_s^q \|^2 \leq C \left( 1 + \sup_{s \in [t, T]} \| W_s - W_t \|^2 \right). \]

By using the classical properties of the Brownian motion, we prove that
\[ \exists \epsilon > 0, \forall s \in [t, T], \quad \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_s^{(s+\epsilon) \wedge T} \kappa^q_{\theta}^T \Sigma \kappa^q_{\theta} d\theta \right) \right] < +\infty. \]
Finally, using a classical trick due to Beneš (see [26], Chapter 5), we conclude that \((\xi^q_{t,s})_{s \in [t,T]}\) is a martingale under \((\mathbb{P}, \mathbb{F} = (\mathcal{F}_s)_{s \in [t,T]})\).

We then obtain
\[
\mathbb{E} \left[ \tilde{u} \left( T, \mathcal{V}_T^{t,\mathcal{V},\mathcal{S},q}, S_T^{t,S} \right) \right] \leq \hat{u}(t, \mathcal{V}, \mathcal{S}),
\]
with equality when \((q_s)_{s \in [t,T]} = (q^*_s)_{s \in [t,T]}\).

We conclude that
\[
\hat{w}(t, \mathcal{V}, \mathcal{S}) = \sup_{(q_s)_{s \in [t,T]} \in \mathcal{A}_{\text{Merton}}} \mathbb{E} \left[ -\exp \left( -\gamma \mathcal{V}_T^{t,\mathcal{V},\mathcal{S},q} \right) \right]
\]
\[
= \mathbb{E} \left[ -\exp \left( -\gamma \mathcal{V}_T^{t,\mathcal{V},\mathcal{S},q^*} \right) \right]
\]
\[
= \hat{u}(t, \mathcal{V}, \mathcal{S}).
\]

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