Simplified solution to the problem plane-parallel motion partial control of an autonomous rigid body in incompressible stratified viscous fluid

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Abstract. The report presents a mathematical model of the motion control of an autonomous solid body moving in incompressible stratified viscous fluid and analytical and numerical analysis of this model. It is assumed that the body does not have its own propulsion system, but is equipped with controlled rudders - wings of finite span. The body is moved by the influence of the buoyancy force and wings lift. Partial movement control is achieved by the angle of attack of the wing change (for ensuring access to the given point by this solid body). This body motion is considered to be plane-parallel motion.

1. Introduction

The effectiveness of observations and measurements obtained in the study of the underwater world via underwater vehicles, in particular, unmanned, depends on minimizing the impact of these submersible crafts to surrounding underwater environment. First of all, it refers to a moving apparatus, which movement is carried out by various power plants (screw propeller or other propulsion). Therefore, the reduction or elimination of such effects is an important application. The ideal situation would obviously be the complete lack of engine. This means that movement control of the body can be carried out only by natural hydrodynamic forces, for instance, the Archimedes buoyancy or an wing lift effect (the body can be equipped with wing). Basic terminology and classical results for the body’s motion in continuum can be found in the books [1], [2].

In this paper motion control mathematic modeling direct problem is solved. The authors consider plane-parallel motion of submersible craft case which is moved in stratified fluid with shear flows by the influence of the buoyancy force and wings lift.

A generalization of this problem to the case of three dimensions and the solution of the inverse problem will be presented in the following publications of these authors. It is usually understood that an inverse problem is a problem of control synthesis which leads to achievement of the defined preselected value. In this case the inverse problem is a problem for reaching to a neighborhood of the given point in the required time range by this solid body.
2. The mathematical model solid body plane-parallel motion control

The classical finite-span slender-wing theory by Zhukovsky-Prandtl ([1] – [3]) is the basis for constructing the mathematical model.

As an autonomous rigid body, the authors propose to consider a research submersible – a uniform sphere-shaped rigid body with two similar symmetrically located around the ball centre wings (figure 1). The wings are considered to be an additional "independent source" of external volume acting on the ball forces. Modifications of mutual bracing of the sphere-shaped body and wings necessitate special experimental research of additional hydrodynamic effects that occur in various versions of the technical construction integration. The authors are considered that modifications choice is a separate engineering task and does not consider them. In any modification of mutual bracing of the sphere-shaped body and wings, the proposed mathematical model can be taken as a basis for whole these alternatives. It should be noted that the trajectory of the "ball with wings" system is understood as the trajectory of the inertia center of this system – the trajectory of the center of the sphere.

![Figure 1. Schematic submersible craft image.](image1)

![Figure 2. Double-layer continuum figure.](image2)

The motion of submersible craft is assumed to happen in a limitless borehole bottom reservoir with an ideal incompressible non-conducting stratified liquid with viscosity effect. The viscosity is taken into account as a Stokes' drag force. It is also assumed that each layer has own density, which is known. Furthermore, liquid in each layer has a constant density and can move rectilinearly and uniformly with known velocity along the horizontal axis, which is perpendicular to a wingspread. Motion velocities and directions of the fluid layers are assumed to be known.

In this paper the authors consider plane-parallel motion of submersible craft case. At the initial time this body is located in stationary state at a predetermined depth (figure 2). It is necessary to define the obtaining solution algorithm in a double-layer liquid for building a similar solution in stratified liquid.

In this paper the following simplifications are accepted:

- The object in question has thin high-aspect-ratio wings.
- The plane section hypothesis is applied.
- The "fluid wing" hypothesis is applied. Furthermore, the submersible craft wing form is selected in such a way as to minimize the induced drag force.

The theory of a finite span wing is based on the following hypotheses ([11], [2]):

1) Thin wing hypothesis. This indicates that an airfoil-shaped profile is thin and an airfoil chord and the direction of speed form a small angle.

2) High-aspect-ratio wing. Wing aspect-ratio is the ratio of the square of the wingspan to its area. It is generally considered that aspect-ratio value should be greater than 4, than means the wing is long and narrow.
3) The plane section hypothesis. It allows forming in the \( z = \text{const} \) plane velocities and pressures in the same manner as in the case of an infinite span wing.

4) The "fluid wing" hypotheses. It assumes the possibility of selecting a vortex system that can replace the solid wing action on the flow and the same movement of the liquid that would be caused by the wing action.

These hypotheses conclusion is the possibility to consider a two-dimensional flow around the airfoil instead of three-dimensional flow near the wing for each \( z = \text{const} \) plane. The velocity of two-dimensional flow around the airfoil depends on \( z, R \leq |z| \leq l + R \), where \( R \) – the sphere radius, \( l \) – the wing length (2\( l \) – the wing span).

A problem of flow about finite-span wing is divided into two parts:

- The problem of forward flow about airfoil.
- The changes determination of circulation around wing span.

The forces acting on the submersible craft are shown in figure 3: \( F_{\text{arch}} \) – the buoyancy force, \( F_g \) – gravity force, \( F_{\text{drag}} \) – head resistance force, \( F_{\text{lift}} \) – wing lift, \( F_i \) – induced drag force.

![Figure 3. Vectors of system of hydrodynamic forces acting on the submersible craft (on the system sphere + wings).](image)

The inertia center motion equations of the submersible craft on the X and Y axes have the following form:

\[
\begin{align*}
\left( m + \frac{2}{3} \rho \pi R^3 \right) \frac{d^2 x}{dt^2} &= F_{\text{arch}} - 2 F_i \cos \delta - \left( F_{\text{drag}}^{(1)} + F_{\text{drag}}^{(2)} \right) \cos \delta - 2 F_{\text{lift}} \sin \delta - F_g, \\
\left( m + \frac{2}{3} \rho \pi R^3 \right) \frac{d^2 y}{dt^2} &= -2 F_i \sin \delta - \left( F_{\text{drag}}^{(1)} + F_{\text{drag}}^{(2)} \right) \sin \delta - 2 F_{\text{lift}} \cos \delta.
\end{align*}
\]

where \( F_{\text{drag}}^{(j)} = C_x^{(j)}(\text{Re}) \cdot S^{(j)} \cdot \frac{\rho v^2}{2} \) (\( j=1 \) for sphere and \( j=2 \) for wings), \( F_i = \frac{\rho}{2} V^2 S \frac{H_i}{2k} \left( \frac{2k \alpha}{1 + \mu_0} \right)^2 \), \( S \) – the wing planform area, \( \alpha = \alpha_0 - \delta \) – the current angle of attack, \( \alpha_0 \) – the initial angle of attack.

3. The mathematical model analysis

The mathematical model of the submersible craft plane-parallel motion is constructed (1). It allows controlling the body through wings angle of attack modifications. The corresponding numerical solution can be obtained in application programs (for instance MATLAB).

Using the change of variables
the mathematical model is transformed to a nonlinear differential equation system

\[
\begin{aligned}
\dot{x} &= z_1, \\
\dot{y} &= z_3, \\
\dot{x} &= z_2, \\
\dot{y} &= z_3 = z_4,
\end{aligned}
\]

Fourth-order of accuracy Runge-Kutta method can be applied for numerical solution of the differential equation system (2).

Motion of the submersible craft in stratified (double-layer) ideal (with viscosity effect) incompressible fluid with shear flow in the line of horizontal axis is considered. At the initial time this body is located at a predetermined depth in stationary state. The system is solved for typical values of hydrodynamic parameters. Authors examine three cases of the variation law of attack angle:

1) \( \alpha = 0 \),

2) \( \alpha = \begin{cases} 0.5 \cdot t, & 0 \leq t \leq 30 \leq \bar{t}, \\ 15, & 30 \leq t \leq \bar{t}, \\ 15 - 0.5 \cdot (t - \bar{t}), & \bar{t} \leq t \leq \bar{t} + 60, \\ -15, & \bar{t} + 60 \leq t, \end{cases} \)

3) \( \alpha = \begin{cases} -0.5 \cdot t, & 0 \leq t \leq 30 \leq \bar{t}, \\ -15, & 30 \leq t \leq \bar{t}, \\ -15 + 0.5 \cdot (t - \bar{t}), & \bar{t} \leq t \leq \bar{t} + 60, \\ 15, & \bar{t} + 60 \leq t. \end{cases} \)

Here \( \bar{t} \) – is an ascent time of the submarine craft in bottom layer.

The direct problem solution is obtained from the given control laws of attack angle. Then appropriate motion trajectories of the submarine craft can be calculated by solving the system (2) (figure 4). The differential equation system is sequentially solved for each layer starting with the bottom layer. Its initial conditions are supposed zero conditions. For other layers initial conditions are recalculated depending on coordinates of submarine craft inertia center at the transitional point from layer to layer.
From a practical standpoint, it is presented that the system (1) (and (2) too) is too lengthy for operational applications. The authors show below, that this system can be simplified without the loss of required accuracy in significant from the applied point of view cases.

By analyzing numerical solution of differential equation system (2) it can be noted that

\[
\sqrt{z_2^2 + z_4^2} \text{ can be taken a Taylor series expansion near the point } z_4 / z_2 = 0 \text{ and it can be assumes that } \sqrt{z_2^2 + z_4^2} \approx z_4.
\]

Also a term \(2b_3 \cdot \alpha \cdot z_2 \cdot z_4\) can be excluded from the second equation of (2). In consequence differential equation system is transformed into

\[
\begin{cases}
\dot{z}_1 = z_2, \\
\dot{z}_2 \cdot b_0 = b_1 - (b_2 \cdot \alpha^2 + b_3 + 2b_4) \cdot z_2^2, \\
\dot{z}_3 = z_4, \\
\dot{z}_4 = -(b_2 \cdot \alpha^2 + b_3 + 2b_4) \cdot z_2 \cdot z_4 + 2b_3 \cdot \alpha \cdot z_2^2.
\end{cases}
\]

(3)

If the function of attack angle \(\alpha\) is time-invariant (constant value), the system (3) can be solved [5] with zero initial conditions:

\[
\begin{cases}
z_2 = \left(\frac{k_1}{k_2}\right)^{1/2} + \frac{1}{k_2} \ln \left(\frac{\exp(2\sqrt{k_1k_2} t) + 1}{2}\right), \\
z_2 = \left(\frac{k_1}{k_2}\right)^{1/2} \frac{\exp(2\sqrt{k_1k_2} t) - 1}{\exp(2\sqrt{k_1k_2} t) + 1}, \\
z_3 = z_4, \\
z_4 = \frac{2\exp(2\sqrt{k_1k_2} t)}{\exp(2\sqrt{k_1k_2} t) + 1} \cdot \frac{k_3}{k_2} \int_0^t \left(\frac{\exp(2\sqrt{k_1k_2} t) - 1}{\exp(2\sqrt{k_1k_2} t) + 1}\right)^2 \cosh \sqrt{k_1k_2} t \, dt,
\end{cases}
\]

(4)

where \(k_i = \frac{b_i}{b_0}, k_2 = \frac{b_2 \cdot \alpha^2 + b_3 + 2b_4}{b_0}, k_3 = \frac{2b_3 \cdot \alpha}{b_0} \).

The numerical experiments show that the solutions of the original (2) and simplified (4) systems almost coincide.
For instance the submarine craft movement in homogeneous (one-layer) ideal (with viscosity effect) incompressible liquid with shear flow in the line of horizontal axis is considered. It is assumed that at the initial time this body was located at a predetermined depth in stationary state. The systems (2) and (4) are solved for typical values of hydrodynamic parameters. The wing attack angle is assumed to be 0.3 radian. In such a case the calculation error will be 0.2 %. This value is inappreciable for this problem.

4. Conclusion
In this paper direct problem of submersible craft motion control in ideal incompressible stratified fluid with viscosity effect was solved. The buoyancy force and the body's absence of its own propulsion system were taken into account. Thus, the authors solved the following problems:

- the mathematical model of the inertia center motion of submersible craft is constructed;
- the motion trajectory of the submarine craft calculation algorithm is presented;
- the attack angle control possibility of the submarine craft movement is analysed;
- the transformation to the simplified equation system is justified in some applied cases.

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