To what extent do the Classical Equations of Motion Determine the Quantization Scheme?

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Abstract

A simple example of one particle moving in a (1+1) space-time is considered. As an example we take the harmonic oscillator. We confirm the statement that the classical Equations of Motion do not determine at all the quantization scheme. To this aim we use two inequivalent Lagrange functions, yielding Euler-Lagrange Equations, having the same set of solutions. We present in detail the calculations of both cases to emphasize the differences occurring between them.

1 Introduction

In classical physics the content of dynamical process is mainly characterized by the Equations of Motion of the physical system. It may happen that these equations can be derived from a certain Lagrange function as the Euler-Lagrange-Equations. If this is the case we may expect that there can be many (even infinitely many!) nonequivalent Lagrange functions linked to these equations of motion, yielding the same set of solutions – so called s-equivalent equations. These fact is well known [1,2].

Two nonequivalent but s-equivalent Lagrange functions lead to two distinct Hamilton functions and distinct canonical momentum variables. If we
take as the starting point for our consideration these two Hamilton functions as well as these two sets of canonical momenta and try to quantize them in a standard way we get, in general, two distinct quantization schemes, differing essentially from each other, although both having common roots coming from the same equations of motion. This fact is not new and well known to some physicists [1,2], but – strange enough – not much attention was paid by them to this problem.

From what was said so far we may infer that the answer to the question posed in the title is that the equations of motion do not determine the quantization scheme.

Below we shall present an elucidating example in favour of the statement made above.

One remark is here in order. If we choose two inequivalent, but s-equivalent Lagrange functions, say, \( L \), and \( L' \), we get two sets of canonical variables

\[
(x, p \equiv \frac{\partial L}{\partial \dot{x}}) \quad \text{and} \quad (x, p' \equiv \frac{\partial L'}{\partial \dot{x}}).
\]

Notice that those canonical variables are not connected to each other by a canonical transformation, viz.

\[
(x, p) \rightarrow (X(x, p), P(x, p)).
\]

Should they be linked to each other by a point transformation

\[
(x, p) \rightarrow (x, p'(x, p))
\]

it would follow from the canonical Poisson brackets that

\[
p' = p + f(x), \quad f(x) - \text{arbitrary function},
\]

which is not the case considered by us in this note.

2 The case of one classical particle in a \((1+1)\) space-time. The Master Equation for \( H'(x, p') \)

To buttress the above observations and make plain the goal of this note, we shall investigate a very simple problem of classical mechanics. So let
us restrict ourselves to the case of one classical particle, moving in a (1+1) space-time. Let us take sufficiently smooth equation of motion

\[ \ddot{x} = f(x, \dot{x}, t) \]  

(2.1)

where \(x(t)\) denotes the location of the particle and \(\dot{x} \equiv dx/dt\) and \(\ddot{x} \equiv d^2x/dt^2\) denote its velocity and acceleration resp., \(t\) being the independent time variable. To every such equation belongs a Lagrange function \([2,3]\), \(L(x, \dot{x}, t)\).

Let us further restrict ourselves for simplicity reason to autonomous Lagrange functions. It is known \([2,4]\) that the most general expression for Lagrange function, \(L'(x, \dot{x})\), s-equivalent to \(L(x, \dot{x})\), is

\[
L' = \dot{x} \int_c^x d\Sigma(H) \frac{\partial^2 L}{\partial \dot{x}^2} |_{x=u} du - \Sigma(H)
\]

(2.2)

where \(\Sigma(H)\) is an arbitrary differentiable function of \(H\), different from zero a.e., and

\[
H \equiv \dot{x} \frac{\partial L}{\partial \dot{x}} - L,
\]

(2.3)

the Hamilton function. The constant \(c\) is so choosen that the integral on the r.h.s. of (2.2) does not diverge. It is easy to see that we have

\[
\frac{\partial L'}{\partial x} - \dot{x} \frac{\partial^2 L'}{\partial \dot{x} \partial x} - \ddot{x} \frac{\partial^2 L'}{\partial \dot{x}^2} = \frac{d\Sigma(H)}{dH} \left( \frac{\partial L}{\partial x} - \dot{x} \frac{\partial^2 L}{\partial \dot{x} \partial x} - \ddot{x} \frac{\partial^2 L}{\partial \dot{x}^2} \right).
\]

(2.4)

Thus \(L'\) is, indeed, s-equivalent to \(L\). We have

\[
H' \equiv x \frac{\partial L'}{\partial x} - L' = \Sigma(H).
\]

(2.5)

We get also

\[
p' \equiv \frac{\partial L'}{\partial \dot{x}}.
\]

(2.6)

It is trivial to find \(H'\) and \(p'\) for given \(\Sigma\) and \(L\) as functions of \(x\) and \(\dot{x}\). It is, however, not so simple to get \(H'\) as a function of \(x\) and \(p'\). To get that let as observe that from (2.6) and (2.2) follows

\[
1 = \frac{d\Sigma \frac{\partial^2 L}{\partial \dot{x} \partial p'(x, p')}}{dH \frac{\partial^2 \dot{x}(x, p')}{\partial p'}}.
\]

(2.7)

\(^1\) The presence of \(d\Sigma(H)/dH\) in (2.4) causes that the set of solutions of equation on the l.h.s. of (2.4) and the one on the r.h.s. differ by a set of measure zero.
Taking into account the relation
\[ \frac{\partial H''(x, p')}{\partial p'} = \dot{x}(x, p') \] (2.8)
we get from (2.7) and (2.8)
\[ \frac{\partial^2 H'(x, p')}{\partial p'^2} = \frac{\partial \dot{x}(x, p')}{\partial p'} = \left( \frac{dH'}{dH} \frac{\partial^2 L}{\partial \dot{x}^2} |_{\dot{x}=\partial H'/\partial p'} \right)^{-1}. \] (2.9)

This is the Master Equation for \( H' \) as a function of \( p' \) and \( x \); it is, in general, nonlinear.

The solutions of this Master Equation have to satisfy a physically justified requirement that \( p' \) has to tend to zero as \( \dot{x} \) tends to zero and vice versa, or in other words
\[ \frac{\partial H'}{\partial p'} |_{p'=0} = 0. \] (2.10)

3 Application of the Master Equation

To make use of this Master Equation one has, of course, to specify what \( L \) and \( \Sigma \) are. This will be done now. We choose
\[ L = \frac{1}{2} \dot{x}^2 - V(x) \] (3.1)
where \( V(x) \) is a nonnegative function of \( x \) and
\[ \Sigma(H) = H' = \sqrt{2H} \] (3.2)
Square root means nonnegative root. For this choice of \( L \) and \( \Sigma \) equation (2.9) reduces to
\[ \frac{\partial^2 H'}{\partial p'^2} = H'. \] (3.3)

\(^2\) In case \( V(x) \) is just bounded from below we may make it nonnegative for each \( x \) by adding to it a suitably chosen positive constant.

\(^3\) The equality (3.2) should be understood as follows
\[ H' = \alpha \sqrt{2H}, \quad \alpha - \text{constant} \]
As \( H' \) as well \( H \) should have the same dimensions it follows that the constant \( \alpha \) has to have the dimension
\[ [\alpha] = \frac{g^{1/2} \text{cm/sec}}{} \]
The solution of (3.3) reads

\[ H' = \alpha(x) \sinh p' + \beta(x) \cosh p'. \quad (3.4) \]

This solution has to satisfy the requirement (2.10) and therefore

\[ \alpha(x) = 0. \quad (3.5) \]

From (3.1) and the Hamilton Equation, we obtain

\[ \frac{d}{dt} \frac{\partial H'}{\partial p'} = \ddot{x} = -\frac{\partial V}{\partial x}. \quad (3.6) \]

If we insert in (3.6)

\[ \dot{x} = \beta(x) \sinh p' \quad (3.7) \]

\[ \dot{p}' = -\frac{\partial H'}{\partial x} = -\frac{\partial \beta}{\partial x} \cosh p', \quad (3.8) \]

following from the Hamilton Equations, we get

\[ \frac{1}{2} \frac{\partial}{\partial x} \beta^2 = \frac{\partial V}{\partial x} \]

or

\[ \beta = \pm \sqrt{2(V + C)} \quad (3.9) \]

C being a constant. Hence

\[ H' = \pm \sqrt{2(V(x) + C)} \cosh p'. \quad (3.10) \]

The equation (3.3) reads

\[ \frac{\partial^2 H'}{\partial p'^2} = \frac{1}{\alpha^2 m} H' \]

as \( L \) in (3.1) becomes

\[ L = \frac{1}{2} m \dot{x}^2 - V(x) \]

where \( m \) denotes the mass of the particle. In this note we put

\[ \alpha = m = 1. \]

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According to our choice (3.2) taking into account (3.9) we should have
\[ \pm \sqrt{2(V + C)} \cosh p' = \sqrt{2(\dot{x}^2/2 + V)} \] (3.11)
To keep both sides of the relation (3.11) compatible with each other we have to choose the (+) sign on the l.h.s. of (3.11). Since for \( p' \) tending to zero \( \dot{x} \) should also tend to zero we conclude that \( C = 0 \). Thus eventually we have
\[ H' = \sqrt{2V(x)} \cosh p'. \] (3.12)
Notice that we could as well choose in the definition on the r.h.s. of (3.2) the (–) sign in front of the root or use both signs suitable for certain nonoverlapping intervals of the variable \( x \). This would cause a change of our model. In each case, mentioned above, the Hamilton Equations are s-equivalent to original equations of motions and the Hamilton functions are constants of motion.

The original Hamilton function reads
\[ H = \frac{1}{2}p^2 + V(x). \] (3.13)

4 Example of the harmonic oscillator
For the case of the harmonic oscillator
\[ V(x) = \frac{1}{2}x^2 \] (4.1)
and we choose the model \[ H' = x \cosh p'. \] (4.2)

\( H' \) is not bounded from below. As \( H' \) is a conserved quantity the singularity of (4.2) appears at \( x = 0 \). It can be easily removed by taking as the potential
\[ V(x) = \frac{1}{2}x^2 + a, \] (4.3)

\[ ^4 \text{Relation (4.2) has to be understood as follows} \]
\[ H' = \sqrt{m_\omega \alpha x} \cosh \left( \frac{p'}{\sqrt{m_\alpha}} \right) \]
as we have \( L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m_\omega^2 x^2, \quad [\omega] = 1/\text{sec}. \) We put \( m = \alpha = \omega = 1. \)
an arbitrary positive constant. We are not going to use this procedure as it would complicate essentially our further calculations. For the case \((4.3)\) the classical trajectories would be given by

\[ x \cosh p' = b, \quad b - \text{a constant}. \] (4.4)

For \(b > 0\) the phase trajectory lies in the strip \(0 < x \leq b\) for \(b < 0\), in \(b \leq x < 0\), while \(-\infty < p' < +\infty\). If \(x \to b\) then \(dp'/dx \to \infty\).

It is easy to see that \((4.2)\) gives rise to the equation

\[ \ddot{x} + x = 0. \] (4.5)

Indeed, we have

\[ \frac{\partial H'}{\partial x} = \cosh p' = -\dot{p}', \quad \frac{\partial H'}{\partial p'} = x \sinh p' = \dot{x}. \] (4.6)

Then

\[ \ddot{x} = \dot{x} \sinh p' + \dot{p'} x \cosh p' = x (\sinh p')^2 - x (\cosh p')^2 = -x. \] (4.7)

## 5 Quantization

Let us now try to quantize \(H\) and \(H'\). We assume that the operators \((x, \hat{p})\) and \((x, \hat{p}')\) satisfy the standard canonical commutation relations. Then \(\hat{p}\) as well as \(\hat{p}'\) can be replaced by

\[ -i \partial_x \] (5.1)

in formulae \((3.13)\) and \((3.12)\) resp. We get the following differential expressions, viz.

\[ H_Q = -\frac{1}{2} (\partial_x)^2 + V(x) \] (5.2)

\[ H'_Q = \sqrt{\frac{1}{2} V(x) \cos(\partial_x) + \cos(\partial_x) \sqrt{\frac{1}{2} V(x)}} \] (5.3)

This expressions applied to \(C^\infty_0(\mathbb{R})\) define symmetric operators in \(L^2(\mathbb{R})\) \footnote{As it is well known \(H_Q\) can be extended to a self-adjoint operator}.

Notice that the operator \((5.3)\) is not local, viz.

\[ \cos(\partial_x) \Psi(x) = \frac{1}{2} (\Psi(x + i) + \Psi(x - i)) \] (5.4)
and therefore (we denote the operator also by $H'_Q$)

$$H'_Q \Psi(x) = \frac{1}{\sqrt{2}} \left( \sqrt{V(x)} + \sqrt{V(x+i)} \right) \Psi(x+i) +$$

$$+ \frac{1}{\sqrt{2}} \left( \sqrt{V(x)} + \sqrt{V(x-i)} \right) \Psi(x-i) \quad (5.5)$$

### 6 Harmonic oscillator

The potential is given by (4.1). As it is well known the eigenvalues for $H_Q$ are

$$n + \frac{1}{2}, \quad n - \text{natural number or 0} \quad (6.1)$$

and the eigenfunctions are the Hermitean functions

$$\Psi_n = e^{x^2/2} H_n(x) \quad (6.2)$$

where $H_n$ are the Hermitean polynomials [5].

For the case $H'_Q$ we have (see [5.3])

$$H'_Q = \frac{1}{2} (x \cos(\partial_x) + \cos(\partial_x)x). \quad (6.3)$$

The case of nonlocal $H'_Q$ will be investigated in Section 7.

It seems more convenient to start the discussion by using different canonically conjugate variables, namely (hereafter we shall use the letter $p$ instead of $p'$)

$$-i\partial_x \rightarrow p \quad \text{and} \quad x \rightarrow i\partial_p. \quad (6.4)$$

Notice that the two systems of variables are linked by a Fourier transformation.

Let us denote the new Hamilton operator by $K$. Then we get from (6.3)

$$K = \frac{i}{2} (\partial_p \cosh(p) + \cosh(p)\partial_p). \quad (6.5)$$

The differential expression (6.3) when applied to $C_0^\infty(R)$ defines a symmetric operator in $L^2(R)$, which we shall also denote by $K$. This statement
as well as the following results are discussed *in extenso* in the Appendix. It is shown that for real $\gamma$
\[0 \leq \gamma < 2\] (6.6)
the system of functions
\[\{\Psi_{2n+\gamma}(p)\}_{n \in \mathbb{Z}},\] (6.7)
where
\[\Psi_{2n+\gamma}(p) = \frac{1}{\sqrt{\pi \cosh p}} \exp(-i(2n + \gamma) \arctan \sinh p)\] (6.8)
are the solution of the equations
\[K\Psi_{2n+\gamma}(p) = (2n + \gamma)\Psi_{2n+\gamma}(p),\] (6.9)
is an orthonormal basis for $L^2(\mathbb{R})$. Thus for each fixed $\gamma$ of the interval (6.6) $K$ has a self-adjoint extension $K_\alpha$, $\alpha \equiv \exp(-i\gamma\pi)$.

### 7 Fourier transform of the eigenfunctions

In this section we investigate Fourier transforms of eigenfunctions of the Hamilton operator (6.5). We find an expression for the Fourier transform in the case when eigenvalues are equal to $n + 1/2$, where $n$ is integer. However, we do not see how to solve the problem for other eigenvalues. We show that in the considered case the Fourier transforms are eigenfunctions of non-local Hamilton operator (6.3). These eigenfunctions form two bases in the Hilbert space. Additionally we get a family of orthogonal polynomials with the weight $(\cosh \pi x)^{-1}$.

Let us rewrite the result (6.8). The normalized eigenfunction belonging to the eigenvalue $\lambda$ reads
\[\Psi_\lambda(p) = \frac{1}{\sqrt{\pi \cosh p}} \exp(-i\lambda \arctan \sinh p).\] (7.1)

We start with two identities:
\[\frac{1}{\sqrt{\cosh p}} \exp(-\frac{i}{2} \arctan \sinh p) = \frac{1 + i \exp p}{1 + i \exp p/2},\] (7.2)
\[\exp(-i \arctan \sinh p) = i \frac{1 - i \exp p}{1 + i \exp p}.\] (7.3)
To check these identities it is most simple to compare the moduli and arguments of the complex number on both sides of the equalities.

Multiplying first identity by \( n\)-th power of the second identity we obtain

\[
\Psi_{n+1/2}(p) = \frac{1}{\sqrt{\pi}} \left( \frac{1 - ie^p}{1 + ie^p} \right)^n \frac{1 + i}{1 + ie^{p/2}}.
\] (7.4)

To get the Fourier transform of (7.4)

\[
\Phi_\lambda(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi_\lambda(p)e^{ipx} dp
\] (7.5)

we employ the method of generating function. We define

\[
\Psi(p, t) \equiv \sum_{n=0}^{\infty} \Psi_{n+1/2}(p)t^n = \frac{1}{\sqrt{\pi}} \frac{1 + i}{1 - it + i(1 + it)e^{p/2}}.
\] (7.6)

The computation of (7.6) amounts to summing up the geometrical series convergent for \(|t| < 1\). The Fourier transform of the generating function (7.6) yields the generating function for the Fourier transforms of the eigenfunctions:

\[
\Phi(x, t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(p, t)e^{ipx} dp = \sum_{n=0}^{\infty} \Phi_{n+1/2}(x)t^n.
\] (7.7)

To evaluate the integral (7.7) we shall use the method of complex analysis. Let us consider the function

\[
\frac{1}{\sqrt{2\pi}} \Psi(p, t)e^{ipx}
\] (7.8)

as the function of a complex variable \( p \) and let us compute the integral of function (7.8) along the contour of the rectangle with the vertices located at the points \((-a, 0), (a, 0), (a, 2\pi i), (-a, 2\pi i), a > 0\), running in the counterclockwise direction. In the limit when \( a \) tends to infinity, the integral along the lower side yields \( \Phi(x, t) \). To get the integral along the upper side of the rectangle we exploit the relation

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(p + 2\pi i, t)e^{i(p + 2\pi i)x} dp = -\Phi(x, t)e^{-2\pi x}
\] (7.9)
which follows from the property

$$\Psi(p + 2\pi i, t) = -\Psi(p, t).$$

(7.10)

In the limit the contributions from both remaining sides of the rectangle vanish. Then the integral along the rectangle in the limit is equal to

$$\Phi(x, t) + \Phi(x, t) e^{-2\pi x}. \tag{7.11}$$

Function (7.6) is a meromorphic function and has inside of the rectangle a simple pole at the point

$$\tilde{p} = \frac{i \pi}{2} - 2i \arctan t. \tag{7.12}$$

We can express the integral of the function (7.6) along the rectangle by residuum of the function at the point $\tilde{p}$ which is equal to

$$- \frac{i}{\pi} e^{-\pi x/2} \frac{1}{\sqrt{1 + t^2}} \exp(2x \arctan t). \tag{7.13}$$

The integral (7.11) is the product of $2\pi i$ and the residuum (7.13). Finally, we get

$$\Phi(x, t) = 2 e^{-\pi x/2} \frac{1}{1 + e^{-2\pi x}} \frac{1}{\sqrt{1 + t^2}} \exp(2x \arctan t). \tag{7.14}$$

Let us set

$$W(x, t) \equiv \frac{1}{\sqrt{1 + t^2}} \exp(2x \arctan t) = \sum_{n=0}^{\infty} W_n(x) \frac{t^n}{n!}. \tag{7.15}$$

The formula (7.15) defines the sequence of polynomials $W_n(x)$. Degree of the polynomial $W_n(x)$ equals $n$. The polynomials $W_n(x)$ are even functions for even $n$ and odd functions for odd $n$.

Comparing definitions (7.7) and (7.15) and the formula (7.14) we can write

$$\Phi_{n+1/2}(x) = 2 e^{-\pi x/2} \frac{W_n(x)}{1 + e^{-2\pi x}} = \Phi_{1/2}(x) \frac{W_n(x)}{n!}. \tag{7.16}$$

This formula holds for nonnegative integer $n$. From the definition of the Fourier transformation (7.5) and from the relation

$$\Psi_{-\lambda}(p) = \Psi_{\lambda}(p) \tag{7.17}$$
follows
\[ \Phi_{-\lambda}(x) = \Phi_{\lambda}(-x). \]  
(7.18)

Thus for nonnegative integer \( n \) we have
\[ \Phi_{-n-1/2}(x) = (-1)^n \frac{2e^{\pi x/2}}{1 + e^{2\pi x}} \frac{W_n(x)}{n!} = (-1)^n \Phi_{-1/2}(x) \frac{W_n(x)}{n!} \]  
(7.19)

which supplements relation (7.10).

Let us investigate the polynomials \( W_n(x) \). For this aim we differentiate \( W(x, t) \), given by (7.15), with respect to \( t \). We get
\[ \frac{\partial W(x, t)}{\partial t} = \frac{2x - t}{1 + t^2} W(x, t). \]  
(7.20)

If we multiple both sides of (7.20) by \((1 + t^2)\) and compare the coefficients of the same power of \( t \) on both sides of (7.20) we obtain
\[
\begin{cases}
W_0(x) = 1 \\
W_1(x) = 2x \\
W_2(x) = 4x^2 - 1 \\
W_3(x) = 8x^3 - 10x \\
W_4(x) = 16x^4 - 56x^2 + 9 \\
\end{cases}
\]  
(7.21)
as well as recurrence formula
\[ W_{n+1}(x) + n^2W_{n-1}(x) = 2xW_n(x), \quad n > 0. \]  
(7.22)

Now we are going to show that the Fourier transformed functions \( \Phi_{n+1/2} \) are eigenfunctions of the nonlocal Hamilton operator (6.3):
\[ H'_Q \Phi_{n+1/2}(x) \equiv i \left( \frac{1}{2} - ix \right) \Phi_{n+1/2}(x + i) - i \left( \frac{1}{2} + ix \right) \Phi_{n+1/2}(x - i) \]  
(7.23)
\[ = (n + 1/2)\Phi_{n+1/2}(x). \]

We shall prove (7.23) for nonnegative integer \( n \); for negative ones the proof is very similar to that for nonnegative.

Taking into account the relation
\[ \Phi_{1/2}(x \pm i) = \mp i\Phi_{1/2}(x) \]  
(7.24)
and the formula (7.16) we conclude that (7.23) holds iff
\[ hW_n(x) \equiv \frac{1}{2} \left( \frac{1}{2} - ix \right) W_n(x + i) + \frac{1}{2} \left( \frac{1}{2} + ix \right) W_n(x - i) = (n + 1/2)W_n(x). \] (7.25)

To prove relation (7.25) let us apply the expression \( h \) upon \( W(x, t) \), namely
\[ hW(x, t) = \frac{1}{2} \left( 1 - ix \right) W(x + i, t) + \frac{1}{2} \left( 1 + ix \right) W(x - i, t) \] (7.26)
\[ = \left( \frac{1}{2} + \frac{2x - t}{1 + t^2} \right) W(x, t) = \frac{1}{2} W(x, t) + t \frac{\partial W(x, t)}{\partial t}. \]

The last equality follows from the formula (7.20). If we compare the coefficients of the same power of \( t \) on both sides of (7.26) we get (7.25). This completes the proof.

Let us return to the consideration of the previous section. There we learned that for any fixed \( \lambda \) the functions \( \Psi_{2n+\gamma}, n = 0, \pm 1, \pm 2, \pm 3, ... \), form an orthonormal basis in \( L^2(\mathbb{R}) \). It is known that the Fourier transformation maps an orthonormal basis into a new orthonormal basis.

We have found the Fourier transforms of the eigenfunctions only for \( \gamma = 1/2 \) and \( \gamma = 3/2 \). Further we choose \( \gamma = 1/2 \) and consider orthonormal basis:
\[ \Phi_{2n+1/2}, \quad n = 0, \pm 1, \pm 2, \pm 3, ... \] (7.27)
Taking into account the definition (7.16) and (7.18) and property of polynomials \( W_n(-x) = (-1)^nW_n(x) \) we get for nonnegative integer \( n \) and \( k \)
\[ \int_{-\infty}^{+\infty} \Phi_{2n+1/2}(x)\Phi_{2k+1/2}(x)dx = \int_{-\infty}^{+\infty} \Phi_{2n+1/2}(x)\Phi_{2k+1/2}(x) + \Phi_{2n+1/2}(-x)\Phi_{2k+1/2}(-x) \] dx
\[ = \int_{-\infty}^{+\infty} W_{2n}(x) W_{2k}(x) \frac{dx}{(2n)!} \frac{dx}{(2k)!} \cosh \pi x. \] (7.28)

Similarly computation for positive integer \( n \) and \( k \) gives
\[ \int_{-\infty}^{+\infty} \Phi_{-2n+1/2}(x)\Phi_{-2k+1/2}(x)dx = \int_{-\infty}^{+\infty} W_{-2n-1}(x) W_{-2k-1}(x) \frac{dx}{(2n - 1)!} \frac{dx}{(2k - 1)!} \cosh \pi x. \] (7.29)
Functions on the left sides of (7.28) and (7.29) are orthonormal. Therefore for nonnegative integer \( n \) and \( k \), boths odd or even, we have

\[
\int_{-\infty}^{+\infty} \frac{W_n(x)W_k(x)}{\cosh\pi x} \, dx = (n!)^2 \delta_{n,k}.
\]  

(7.30)

If \( n \) is odd and \( k \) is even then the integrated function is odd and the integral vanish.

We may regard the set of the polynomials \( W_n(x) \) as the system of orthogonal polynomials with respect to the scalar product

\[
< f, g > \equiv \int_{-\infty}^{+\infty} \frac{f(x)g(x)}{\cosh\pi x} \, dx.
\]

(7.31)

We may prove orthogonality relation (7.30) directly, not referring to Fourier transformation. We are going to use the generating function \( W(x, t) \).

Let us calculate in two different ways an integral

\[
\int_{-\infty}^{+\infty} \frac{W(x, s)W(x, t)}{\cosh\pi x} \, dx
\]

(7.32)

On the one hand we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} s^n t^k \int_{-\infty}^{+\infty} \frac{W_n(x)W_k(x)}{\cosh\pi x} \, dx.
\]

(7.33)

On the other hand we have

\[
\int_{-\infty}^{+\infty} \frac{\exp(2x(\arctan s + \arctan t))}{\sqrt{1 + s^2}\sqrt{1 + t^2}\cosh\pi x} \, dx = \frac{1}{1 - st} = \sum_{n=0}^{\infty} s^n t^n.
\]

(7.34)

Comparing the coefficient of the same power \( s \) and \( t \) in the formulae (7.33) and (7.34) we get (7.30). To compute the integral (7.34) we exploit the relations

\[
\int_{-\infty}^{+\infty} \frac{e^{2x\theta}}{\cosh\pi x} \, dx = \frac{1}{\cosh\theta}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}
\]

(7.35)
and
\[ \cos(\arctan s + \arctan t) = \frac{1 - st}{\sqrt{1 + s^2}\sqrt{1 + t^2}}. \] (7.36)

The integral (7.35) we can calculate using complex analysis method in the very similar way as we calculated Fourier transform of the generating function (7.7). We exploit the following property of integrated function \( f(x) \equiv \exp(2x\theta)/\cos\pi x \):
\[ f(x + i) = -f(x)e^{2i\theta}. \] (7.37)

8 Final remarks

1. Let us define the expression.
\[ Rf(x) \equiv \frac{i}{2} \left( \frac{1}{2} - ix \right) f(x + i) - \frac{i}{2} \left( \frac{1}{2} + ix \right) f(x - i) + x f(x). \] (8.1)

Then \( R \) behaves as an "creation operator"
\[ [h, R] = R, \] (8.2)

where \( h \) is defined by formula (7.25). Therefore
\[ hf = \lambda f \quad \text{implies} \quad hRf = (\lambda + 1)Rf. \] (8.3)

That confirms the designation "creation operator" for \( R \). More exactly, we have
\[ RW_n = W_{n+1}. \] (8.4)

2. The equation
\[ \frac{i}{2} \left( \frac{1}{2} - ix \right) \Phi(x + i) - \frac{i}{2} \left( \frac{1}{2} + ix \right) \Phi(x - i) = \frac{1}{2} \Phi(x) \] (8.5)
is not only solved by the function
\[ \Phi_{1/2}(x) = \frac{2e^{-\pi x/2}}{1 + e^{-2\pi x}} \] (8.6)

but also by the function
\[ \Phi(x) \equiv \Phi_{1/2}(x + a). \] (8.7)
The Fourier transform of $\Phi$ reads

$$
\Psi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x) e^{-ipx} dx = \Psi_{1/2}(p)e^{-ipa}
$$

(8.8)

where

$$
\Psi_{1/2}(p) = \frac{1+i}{1+ie^{p/2}}.
$$

(8.9)

We have

$$
K\Psi(p) = \left(\frac{1}{2} + a\cosh p\right)\Psi(p).
$$

(8.10)

and for $a \neq 0$ the function $\Psi(p)$ is not an eigenfunction of $K$.

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Appendix

Let us start with (6.5), viz.

$$
K = \frac{i}{2}(\partial_p\cosh(p) + \cosh(p)\partial_p).
$$

(A1)

This differential expression when applied to $C^\infty_0(\mathbb{R})$ defines a symmetric operator in $L^2(\mathbb{R})$ as stated in section 6. We shall denote this operator also by $K$. Indeed, if $\Psi$ and $\Phi$ belong to $C^\infty_0(\mathbb{R})$ then

$$
\int_{-\infty}^{+\infty} \overline{\Psi(p)}K\Phi(p)dp - \int_{-\infty}^{+\infty} \overline{K\Psi(p)}\Phi(p)dp
$$

$$
= i \int_{-\infty}^{+\infty} \frac{d}{dp} \left(\cosh(p)\overline{\Psi(p)}\Phi(p)\right) dp
$$

$$
= i \cosh(p)\overline{\Psi(p)}\Phi(p)|_{-\infty}^{+\infty} = 0
$$

(A2)

Let $\lambda$ belong to $\mathbb{C}$. Then the equation

$$
K\Psi_\lambda(p) = \lambda\Psi_\lambda(p)
$$

(A3)
has the solution
\[ \Psi_\lambda(p) = C_\lambda \frac{1}{\sqrt{\cosh p}} \exp(-i \lambda \arctan \sinh p) \] (A4)
which is unique up to the multiplicative constant \( C_\lambda \). We may use this constant to normalize \( \Psi_\lambda \). Clearly \( \Psi_\lambda \) belongs to \( L^2(\mathbb{R}) \) as
\[ \int_{-\infty}^{+\infty} \overline{\Psi(p)} \Psi(p) dp = |C_\lambda|^2 \int_{-\infty}^{+\infty} \frac{dp}{\cosh p} = |C_\lambda|^2 \pi. \] (A5)

Thus the defect subspaces for the adjoint operator, \( K^+ \), are one dimensional and the defect indices are \( \{-1, 1\} \). Hence \( K \) has the one-parameter family of self-adjoint extensions, which can be defined as follows. Let
\[ \alpha \in \mathbb{C}, \quad |\alpha| = 1 \] (A6)
and define
\[ M_\alpha = \{ \Phi \in C^\infty(\mathbb{R}) : \lim_{p \to -\infty} \left( \sqrt{\cosh p} \, \Phi(p) \right) = \alpha \lim_{p \to +\infty} \left( \sqrt{\cosh p} \, \Phi(p) \right) \}. \] (A7)

Let us further define the operator \( K_\alpha \) as
\[ K_\alpha \Phi(p) = \frac{i}{2} (\sinh(p) + 2 \cosh(p) \frac{d}{dp}) \Phi(p), \quad \Phi \in M_\alpha. \] (A8)
The operator \( K_\alpha \) is essentially self-adjoint. To see this we observe that
\[ \text{Ran}(K_\alpha \pm i \mathbb{1}) \]
is dense in \( L^2(\mathbb{R}) \). Indeed, taking
\[ \Psi = \Psi_{\pm i}, \]

\[ \int_{-\infty}^{+\infty} \overline{\Psi_\lambda(p)} \Psi_{\mu}(p) dp = C_\lambda C_\mu \frac{\sin(\lambda - \mu)\pi/2}{(\lambda - \mu)\pi/2}. \]
we see that (A2) does not hold for all \( \Phi \in M_\alpha \). Thus \( \Psi \) do not belong to the domain \( D(K_\alpha^+) \) of \( K_\alpha^+ \). Consequently if \( \Psi \in D(K_\alpha^+) \) then
\[
K \Psi = K_\alpha^+ \Psi = \pm i \Psi \quad \text{implies} \quad \Psi = 0. \quad (A9)
\]
This implies that if
\[
(\Psi, (K_\alpha \mp i \mathbb{1}) \Phi) = ((K_\alpha^+ \pm i \mathbb{1}) \Psi, \Phi) = 0 \quad (A10)
\]
for all \( \Phi \in M_\alpha \) then \( \Psi = 0 \) and hence
\[
\text{Ran}(K_\alpha \pm i \mathbb{1}) \quad (A11)
\]
is dense and \( K_\alpha \) is essentially self-adjoint. We shall denote the closure of \( K_\alpha \) by \( \overline{K_\alpha} \).

Let us now consider the function (A4). For real \( \lambda \) we have
\[
\lim_{p \to -\infty} \left( \sqrt{\cosh p} \Psi_{2n+\gamma}(p) \right) = \alpha \lim_{q \to +\infty} \left( \sqrt{\cosh p} \Psi_{2n+\gamma}(p) \right) \quad (A12)
\]
where
\[
\alpha = e^{i\pi \gamma}.
\]
We conclude that for any \( 0 \leq \gamma < 2 \) the functions
\[
\{ \Psi_{2n+\gamma} \}_{n \in \mathbb{Z}} \quad (A13)
\]
form a system of eigenvectors for the self-adjoint operator
\[
\overline{K_\alpha}, \quad \alpha = e^{-i\gamma \pi} \quad (A14)
\]
The corresponding eigenvalues are
\[
\{ 2n + \gamma \}_{n \in \mathbb{Z}} \quad (A15)
\]
It follows from (A5) that if \( C_{2n+\gamma} = 1/\sqrt{\pi} \) then \( \Psi_{2n+\gamma} \) are normalized. We show that for any
\[
0 \leq \gamma < 2
\]
the system
\[
\{ \Psi_{2n+\gamma} \}_{n \in \mathbb{Z}}
\]
is an orthonormal basis for $L^2(\mathbb{R})$. It is sufficient to show that for any $f \in L^2(\mathbb{R})$

$$\sum_{n \in \mathbb{Z}} |(f, \Psi_{2n+\gamma})|^2 = ||f||^2 \quad (A16)$$

where $(.,.)$ and $||.||$ denote the inner product and the norm in $L^2$ resp. To prove (A16) let us observe that

$$(f, \Psi_{2n+\gamma}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(p)(\cosh p)^{-1/2} \exp(-i(2n+\gamma) \text{arctan sinh } p) dp \quad (A17)$$

Let us change the variables as follows

$$\sinh p = \tan \frac{\theta}{2}, \quad -\pi \leq \theta \leq \pi, \quad (A18)$$

$$\cosh p = \frac{1}{\cos \frac{\theta}{2}}, \quad dp = \frac{d\theta}{\cos \frac{\theta}{2}}.$$ 

Then (A17) takes the form

$$(f, \Psi_{2n+\gamma}) = \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{+\pi} f(\text{arcsinh tan } \frac{\theta}{2})(\cos \frac{\theta}{2})^{-1/2} \exp(-\gamma \frac{i\theta}{2}) \exp(-in\theta) d\theta \quad (A19)$$

We get further

$$\sum_{n \in \mathbb{Z}} |(f, \Psi_{2n+\gamma})|^2 = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} \tilde{f}(\theta) e^{-in\theta} d\theta \right|^2 =$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} |\tilde{f}(\theta)|^2 d\theta = \int_{-\infty}^{+\infty} |f(q)|^2 dq = ||f||^2 \quad (A20)$$

where

$$\tilde{f}(\theta) \equiv f(\text{arcsinh tan } \frac{\theta}{2})(\cos \frac{\theta}{2})^{-1/2} e^{i\gamma \theta/2} \quad (A21)$$

Formula (A20) is the Parseval equality.
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