GEOMETRY OF THE EIGENCURVE AT CM POINTS AND TRIVIAL ZEROS OF KATZ $p$-ADIC $L$-FUNCTIONS

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Abstract. The primary goal of this paper is to study the geometry of the $p$-adic eigencurve at a point $f$ corresponding to a weight one theta series $\theta_\psi$ irregular at $p$. We show that $f$ belongs to exactly three or four irreducible components and study their mutual congruences. In particular, we show that the congruence ideal of one of the CM components has a simple zero at $f$ if, and only if, a certain $L$-invariant $L(\psi)$ does not vanish. Further, using Roy’s Strong Six Exponential Theorem we show that at least one amongst $L(\psi)$ and $L(\psi^{-1})$ is non-zero. Combined with a divisibility proved by Hida and Tilouine, we deduce that the anti-cyclotomic Katz $p$-adic $L$-function of $\psi$ has a simple (trivial) zero at $s = 0$, if $L(\psi)$ is non-zero, which can be seen as an anti-cyclotomic analogue of a result of Ferrero and Greenberg. Finally, we propose a formula for the linear term of the two-variable Katz $p$-adic $L$-function of $\psi$ at $s = 0$ thus extending a conjecture of Gross.

Introduction

Our investigations lie at the crossroad between Iwasawa theory and Hida theory. Iwasawa theory and its subsequent generalizations study $p$-adic $L$-functions of automorphic forms or motives. The Iwasawa Main conjecture postulated that those generate the characteristic ideal of an appropriately defined Greenberg-Selmer group. In the few cases where a proof is available, comparing the orders of the trivial (or extra) zeros is quite involved and requires additional arguments. On the other hand, Hida has extended the construction from his seminal trilogy [21, 22, 23] to congruence modules for $p$-adic families of automorphic forms, and has conjectured that the corresponding characteristic power series is generated by the $p$-adic adjoint $L$-function of the family, and that it does not vanish at classical points of weight at least two. Hida theory has allowed to seek understanding of the Iwasawa Main Conjecture through the study of the geometry of $p$-adic eigenvarieties, and to detect even finer local and global geometric phenomena such as intersection numbers and ramification indices over the weight space.

Let $K$ be an imaginary quadratic field in which the prime number $p$ splits as $(p) = \mathfrak{p}\mathfrak{p}$. Let $\psi$ be a non-trivial, finite order Hecke character of $K$ of conductor relatively prime to $p$. We assume that $\psi(p) = \psi(\mathfrak{p})$, so that the weight 1 theta series attached to $\psi$ has a unique $p$-stabilization, denoted by $f$, and we study the conjectural relation between the congruence power series of a Hida family containing $f$ and its $p$-adic adjoint $L$-function. In the case of a
CM family $\Theta_\psi$, the latter is essentially given by the Katz anti-cyclotomic $p$-adic $L$-function $L^\psi_p(\psi, s)$ of $\psi = \psi/\psi^\tau$, where $\tau$ denotes a complex conjugation, and Katz’s $p$-adic Kronecker Second Limit Formula implies that $L^\psi_p(\psi, s)$ has a trivial zero at $s = 0$. While the Iwasawa Main Conjecture relates $L^\psi_p(\psi, s)$ to the characteristic series of a certain Iwasawa module over the anti-cyclotomic tower of $K$, neither the original formulation, nor its proof by Rubin [26, Thm.I] provide a formula for the orders of their common trivial zeros.

A main theme of this paper is to describe as precisely as possible the local geometry of the eigencurve $\mathcal{E}$ at $f$ and draw some arithmetic consequences, such as a criterion for $L^\psi_p(\psi, s)$ to have a simple trivial zero. Our investigations pushed us to go further and propose the following expression for the leading term of the Katz 2-variable $p$-adic $L$-function (see [23, 4.3]):

\[ L_p(\psi, s, s') = \frac{1}{2} \left( \log_p(p) \cdot s + (\mathcal{L}(\psi) + \log_p(p)) \cdot s' \right) \cdot L^\psi_p(\psi, 0) + \text{higher order terms}, \]

where $\mathcal{L}(\psi)$ is the anti-cyclotomic $\mathcal{L}$-invariant introduced in Definition [1.9] while $L^\psi_p(\psi, 0)$ is related via the $p$-adic Kronecker Second Limit Formula to a $p$-adic logarithm of an elliptic unit, which is non-zero by the Baker-Brumer Theorem.

The above formula falls outside the scope of Gross’ famous conjecture on the leading term at $s = 0$ of $p$-adic $L$-functions of Artin Hecke characters. Indeed, the fixed field of $\psi$ is not a CM field, unless $\psi$ is quadratic, in which case we prove (1) up to a scalar in $\mathbb{Q}^\times$ using Gross’ factorization formula. In the general case, we provide the following evidence:

**Theorem A.** If $\mathcal{L}(\psi) \neq 0$, then $L^\psi_p(\psi, s) = L_p(\psi, s, -s)$ has a simple trivial zero at $s = 0$. Moreover, at least one amongst $\mathcal{L}(\psi)$ and $\mathcal{L}(\psi^\tau)$ is non-zero.

The second part is proved in Proposition [1.10] using the Strong Six Exponentials Theorem, while the Strong Four Exponentials Conjecture would imply that $\mathcal{L}(\psi) \cdot \mathcal{L}(\psi^\tau) \neq 0$.

An important observation made in Theorem [3.3] is that in addition to the two Hida families $\Theta_\psi$ and $\Theta_{\psi^\tau}$ having CM by $K$, $f$ also belongs to at least one Hida family without CM by $K$, hence lies in the closed analytic subspace $\mathcal{E}^\perp$ of $\mathcal{E}$, union of irreducible components having no CM by $K$. Denote by $T$, resp. by $T^\perp$, the completed local ring at $f$ of $\mathcal{E}$, resp. of $\mathcal{E}^\perp$. Both algebras $T$ and $T^\perp$ are finite and flat over the completed local ring $\Lambda = \mathbb{Q}_p[X]$ of the weight space at the point $X = 0$ corresponding to the weight of $f$. Consider the congruence ideal $C^0_\psi = \pi_\psi(\text{Ann}_T(\ker(\pi_\psi)))$ attached to the $\Theta_\psi$-projection $\pi_\psi : T \twoheadrightarrow \Lambda$. Hida and Tilouine showed in [26, Thm.1] that

\[ C^0_\psi \subset (\zeta_\psi) \subset (X), \]

where $\zeta_\psi$ is the Katz anti-cyclotomic $p$-adic $L$-function $L^\psi_p(\psi, s)$, seen as an element of $\Lambda$, and have thus provided an upper bound for the order of the trivial zero, in terms of the local geometry of $\mathcal{E}$ at $f$.

To prove the first part of Theorem A it then suffices to show that $C^0_\psi = (X)$ if, and only if, $\mathcal{L}(\psi) \neq 0$, which is a consequence of our next result.
Theorem B (Thm 4.8). The $\Lambda$-algebra $T$ is isomorphic to $\Lambda \times \mathfrak{q}_p (T^1 \times \mathfrak{q}_p[X]/(X^{r-1})\Lambda)$, where

$$T^1 = \begin{cases} \Lambda[Z]/(Z^2 - X^{r+1}) & \text{if } \mathcal{L}_- (\psi^r) \cdot \mathcal{L}_- (\psi_\tau) = 0, \text{ for some } r \geq 3, \\ \mathfrak{q}_p \times \mathfrak{q}_p \mathbb{Z}[X^{1/e}] & \text{if } \mathcal{L}_- (\psi^r) + \mathcal{L}_- (\psi_\tau) = 0, \text{ for some } e \geq 2, \\ \Lambda \times \mathfrak{q}_p \Lambda & \text{otherwise}. \end{cases}$$

In the last case $r = 2$ and $T$ is isomorphic to $\Lambda \times \mathfrak{q}_p \Lambda \times \mathfrak{q}_p \Lambda \times \mathfrak{q}_p \Lambda$ as $\Lambda$-algebra.

As a consequence we establish, in the case of a weight one cusform with CM, a Conjecture that Darmon, Lauder and Rotger made in [12] and which was previously only known when $\psi_\tau$ is quadratic by the work [30] of Lee (see Corollary 5.5).

Let us now explain some of the ideas and techniques that go in the proof of Theorem B.

Since the 2-dimensional $\Gamma_K$-representation $\rho_f = \text{Ind}_K^\mathbb{Q} \psi$ is locally scalar at $p$, neither its Mazur’s $p$-ordinary deformation functor is representable, nor its deformation to $T$ has a $p$-ordinary filtration. In contrast with the $p$-regular setting considered in [3], where $T$ is isomorphic to the $p$-ordinary deformation ring of $\rho_f$, we need to resort to new methods. First, we divide the difficulties and study separately the components having no CM by $K$. Since the restriction to $\Gamma_K$ of the $T^1$-values pseudo-character is generically irreducible and lifts $\psi + \psi^r$, and since $\text{Ext}_{\Gamma_K}^1 (\psi^r, \psi)$ and $\text{Ext}_{\Gamma_K}^1 (\psi, \psi^r)$ are both 1-dimensional generated by $\rho$ and $\rho'$, respectively, a Ribet style argument shows the existence of pair $(\rho\tau^1, \rho'_\tau)$ of $p$-ordinary $\Gamma_K$-deformations lifting $(\rho, \rho')$. This leads naturally to a universal deformation $\Lambda$-algebra $R^1$ classifying pairs of $p$-ordinary deformations of $(\rho, \rho')$ sharing the same $\tau$-invariant traces and rank one $p$-unramified quotients. We show that $R^1 \simeq \mathfrak{q}_p [X^{1/e}]$ is a discrete valuation ring endowed with flat morphism $R^1 \to T^1$ of $\Lambda$-algebras (see Thm 3.7). Furthermore, the reducibility ideal of the 2-dimensional pseudo-character carried by $R^1$ is generated by an element of valuation $r \geq 2$, with $r = 2$ if, and only if, $\mathcal{L}_- (\psi^r) \cdot \mathcal{L}_- (\psi_\tau) \neq 0$. In § 4.4 we determine all possible extensions to $\Gamma_K$ of the pseudo-character carried by $R^1$ and deduce that $R^1[Z]/(Z^2 - X^{r/e}) \simeq T^1$ as $\Lambda$-algebras. With all the components of $\mathcal{E}$ containing $f$ determined in § 4.4, § 4.1.1 is devoted to the study of the CM ideal, measuring the congruences between a component having CM by $K$ and a component with no CM by $K$. Finally in § 4.2 we determine the ideal $\mathcal{C}_\psi$ governing the congruences between $\Theta_\psi$ and all other families.

While the deformation to $T$ is not $p$-ordinary, its localization at any generic point is, motivating the use in § 5 of a universal “generically” $p$-ordinary deformation ring $R^\triangle$ of $\rho_f$, for which we establish the following modularity result.

Theorem C. There exists an isomorphism of non-Gorenstein local $\Lambda$-algebras $R^\triangle_{\text{red}} \simeq T$.

Let us specify the above results in the simplest case of a quadratic character $\psi_\tau$ defining a biquadratic extension $H$ of $\mathbb{Q}$ in which $p$ splits completely. Denote by $K'$ (resp. $F$) the other imaginary quadratic field (resp. the unique real quadratic field) contained in $H$. In that case
we have $T \simeq \Lambda \times \overline{\mathbb{Q}}_p \Lambda \times \overline{\mathbb{Q}}_p \Lambda$ and, in addition to the families $\Theta_\psi$ and $\Theta_\psi^\tau = \Theta_\psi \otimes \epsilon_F$ having CM by $K$, $f$ is contained in families $\Theta_\psi'$ and $\Theta_\psi'^\tau = \Theta_\psi' \otimes \epsilon_F$ having CM by $K'$. Both $p$-adic $L$-functions $L_p(\psi, s)$ and $L_p(\psi', s)$ then have simple trivial zeros at $s = 0$.

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1. **Slopes and $L$-invariants**

We denote by $\Gamma_L = \text{Gal}(\overline{L}/L)$ the absolute Galois group of a perfect field $L$. We consider the field of algebraic numbers $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ which determines a complex conjugation $\tau \in \Gamma_{\overline{\mathbb{Q}}}$. We let $\epsilon_L$ denote the Dirichlet character of a quadratic extension $L/\mathbb{Q}$. 

Denote by $H$ the splitting field of the anti-cyclotomic character $\psi_\ast = \psi/\psi^\tau$, where $\psi^\tau = \psi(\tau^{-1} \cdot \tau)$ is the Galois conjugate of $\psi$. Recall the assumption that $\psi_\ast(p) = 1$, i.e., $p$ splits completely in $H$. For a prime number $p$, we fix an embedding $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ which determines an embedding $\Gamma_{\mathbb{Q}_p} \to \Gamma_{\mathbb{Q}}$, and a place $v_0$ of $H$ above $p$.

1.1. The slope of $\psi_\ast$. The dihedral group $G = \Gamma_{H/\mathbb{Q}}$ is a semi-direct product of the cyclic group $C = \Gamma_{H/K}$ with $\{1, \tau\}$. The product is direct if and only if the character $\psi_\ast$ is quadratic, in which case $G$ is the Klein group.

Given any global unit $u \in \mathcal{O}_H^\ast$ we consider the element

$$u_{\psi_\ast} = \sum_{g \in C} g^{-1}(u) \otimes \psi_\ast(g) \in (\mathcal{O}_H^\ast \otimes \overline{\mathbb{Q}})[\psi_\ast],$$

where $[\psi_\ast]$ denotes the $\psi_\ast$-isotypic part for the natural left action of $C$.

Minkowski’s proof of Dirichlet’s Unit Theorem shows that the left $G$-module $\mathcal{O}_H^\ast \otimes \mathbb{Q}$ is isomorphic to $(\text{Ind}_K^G(1, \psi_\ast))/\mathbb{Q}$ (see [3, (7)]), hence the 2-dimensional odd representation $\text{Ind}_K^G(\psi_\ast)$ occurs with multiplicity one in it. By the Frobenius reciprocity theorem $(\mathcal{O}_H^\ast \otimes \overline{\mathbb{Q}})[\psi_\ast]$ is a line, and similarly $(\mathcal{O}_H^\ast \otimes \overline{\mathbb{Q}})[\psi_\ast^\tau]$ is a line containing $\tau(u_{\psi_\ast}) = \sum_{g \in C} \tau g^{-1}(u) \otimes \psi_\ast(g)$.

Minkowski’s Theorem guarantees the existence of $u \in \mathcal{O}_H^\ast$ such that $\{g(u) | g \in C \setminus \{1\}\}$ is a basis of $\mathcal{O}_H^\ast \otimes \mathbb{Q}$, called a Minkowski unit, to which one can additionally impose to be fixed by $\tau$. Since $\mathcal{O}_K^\ast$ is finite, the only non-trivial relation between $\{g(u) | g \in C\}$ is given by the kernel of the relative norm $N_{H/K}$, hence for any Minkowski unit $u$ one has $u_{\psi_\ast} \neq 0$. We record those useful facts as a lemma.

**Lemma 1.1.** There exists $u \in \mathcal{O}_H^\ast$, such that

$$(\mathcal{O}_H^\ast \otimes \overline{\mathbb{Q}})[\psi_\ast] = \overline{\mathbb{Q}} \cdot u_{\psi_\ast}, \text{ and } (\mathcal{O}_H^\ast \otimes \overline{\mathbb{Q}})[\psi_\ast^\tau] = \overline{\mathbb{Q}} \cdot \tau(u_{\psi_\ast}).$$

The Baker-Brumer Theorem [7] demonstrates the injectivity of the $\overline{\mathbb{Q}}$-linear homomorphism

$$(2) \quad \log_p : \mathcal{O}_H^\ast \otimes \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \colon u \otimes x \mapsto \log_p(u) \cdot \iota_p(x),$$

allowing to define for $u$ as in Lemma [1.1] the following $p$-adic regulator, that we call the slope

$$(3) \quad \mathcal{S}(\psi_\ast) = \frac{-\log_p(u_{\psi_\ast})}{\log_p(\tau(u_{\psi_\ast}))}.$$

**Lemma 1.2.** The slope $\mathcal{S}(\psi_\ast)$ does not depend on the choice of $u$ as in Lemma [1.1]. Moreover $\mathcal{S}(\psi_\ast) \notin \overline{\mathbb{Q}}$, unless $\psi_\ast$ is quadratic, in which case $\mathcal{S}(\psi_\ast) = -1$. Finally $\mathcal{S}(\psi_\ast^\tau) = \mathcal{S}(\psi_\ast)^{-1}$.

**Proof.** The value of $\mathcal{S}(\psi_\ast)$ remains clearly unchanged if we replace $u$ by $g(u)$ for any $g \in C$. The independence then follows from the fact that any unit lies in the $\mathbb{Q}$-linear span of these. If $\psi_\ast^\tau \neq \psi_\ast$, then $\mathcal{S}(\psi_\ast) \notin \overline{\mathbb{Q}}$ by (2). Finally, taking $u$ to be a Minkowski unit fixed by $\tau$,
one finds that $\tau(u_\psi) = u_{\psi^*}$ which, in view of the definition \[3\] simultaneously shows that $\mathcal{S}(\psi^*) = \mathcal{S}(\psi)^{-1}$ and further, when $\psi$ is quadratic, that $\mathcal{S}(\psi) = -1$.

\[4\]

1.2. A Galois theoretic interpretation of the slope. Let $\eta_p$ be the unique element of $\ker(H^1(K, \overline{\mathbb{Q}_p}) \to H^1(I_p, \overline{\mathbb{Q}_p}))$ whose image in $\im(H^1(K_p, \overline{\mathbb{Q}_p}) \to H^1(I_p, \overline{\mathbb{Q}_p}) \cong \text{Hom}(\mathcal{O}^*_K, \overline{\mathbb{Q}_p})$ is given by $\log_p$. Note that $\eta_p$ and $\eta_p^*$ form a basis of $H^1(K, \overline{\mathbb{Q}_p})$.

Proposition 1.3. The image of the natural restriction homomorphism:

$H^1(K, \psi_\cdot) \sim \im(H^1(K_p, \overline{\mathbb{Q}_p}) \times H^1(K, \overline{\mathbb{Q}_p}) \to H^1(I_p, \overline{\mathbb{Q}_p}) \times H^1(I_p, \overline{\mathbb{Q}_p})) \cong \text{Hom}(\mathcal{O}^*_K \times \mathcal{O}^*_K, \overline{\mathbb{Q}_p})$

is a line generated by $(\log_p, \mathcal{S}(\psi_\cdot) \log_p)$. In particular both maps $H^1(K, \psi_\cdot) \to H^1(I_p, \overline{\mathbb{Q}_p})$ and $H^1(K, \psi_\cdot) \to H^1(I_p, \overline{\mathbb{Q}_p})$ are injective.

Proof. The inflation-restriction exact sequence (see [3, (8)]) yields an isomorphism:

\[\tag{4} H^1(K, \psi_\cdot) \sim H^1(H, \overline{\mathbb{Q}_p})[\psi_\cdot^*] = (\text{Hom}(\Gamma_H, \overline{\mathbb{Q}_p}) \otimes \psi_\cdot)^G.\]

Using Shaprio’s Lemma the vector spaces in [4] are further isomorphic to

\[\tag{5} H^1(Q, \text{Ind}_K^Q \psi_\cdot) \sim H^1(K, \psi_\cdot \oplus \psi_\cdot^*) = (\text{Hom}(\Gamma_H, \overline{\mathbb{Q}_p}) \otimes \text{Ind}_K^Q \psi_\cdot)^G.\]

Since Leopoldt’s Conjecture holds for cyclic extensions of imaginary quadratic fields, the dimension of the latter equals $\dim(\text{Ind}_K^Q \psi_\cdot)^{G-1} = 1$.

Taking the $\psi_\cdot^*$-isotypic part of [5] for the left $C$-action yields an exact sequence

$0 \to H^1(H, \overline{\mathbb{Q}_p})[\psi_\cdot^*] \xrightarrow{\text{res}_p} \text{Hom}(\mathcal{O}^*_H, \overline{\mathbb{Q}_p})[\psi_\cdot^*] \times \text{Hom}(\mathcal{O}^*_H, \overline{\mathbb{Q}_p})[\psi_\cdot^*] \to \text{Hom}(\mathcal{O}^*_H, \overline{\mathbb{Q}_p})$.

Since $\text{Hom}(\mathcal{O}^*_H, \overline{\mathbb{Q}_p})$, resp. $\text{Hom}(\mathcal{O}^*_H, \overline{\mathbb{Q}_p})$, is the regular representation of $C$, its $(\psi_\cdot^*)$-component is 1-dimensional, generated by $\sum_{g \in C} \psi_\cdot(g) \log_p(g^{-1})$, resp. $\sum_{g \in C} \psi_\cdot(g) \log_p(\tau g^{-1})$.

Hence $H^1(K, \psi_\cdot) \xrightarrow{\sim} H^1(H, \overline{\mathbb{Q}_p})[\psi_\cdot]$ has basis whose restriction to $(\mathcal{O}_H \otimes \mathbb{Z}_p)^{x}$ is given by

$\sum_{g \in C} \psi_\cdot(g) \log_p(g^{-1}) + s \cdot \sum_{g \in C} \psi_\cdot(g) \log_p(\tau g^{-1})$,

for some $s \in \overline{\mathbb{Q}_p}$. The triviality on global units (in particular on $u_{\psi_\cdot}$) implies in view of Lemma [1,2] that $s = \mathcal{S}(\psi_\cdot)$.

\[\tag{6} \mathcal{L}(\psi_\cdot^*) = -\sum_{g \in C} \psi_\cdot(g) \log_p(g^{-1}(y_0)) + \mathcal{S}(\psi_\cdot) \sum_{g \in C} \psi_\cdot(g) \log_p(\tau g^{-1}(y_0)) \quad \text{ord}_{v_0}(y_0) \]

is independent of the choice of $y_0$, since by Lemma [1,2] multiplying $y_0$ by an element of $\mathcal{O}^*_H$ does not affect its value. Another description of $\mathcal{L}(\psi_\cdot^*)$ can be given using the exact sequence:

$1 \to \mathcal{O}_H \to \mathcal{O}_H[\frac{1}{p}] \xrightarrow{\text{ord}} \mathbb{Z}[C]$.
One has a generator of $H^1$ and by Proposition 1.3 and its proof, the restriction of $\eta$ one finds $\operatorname{ord}_p \log_p(\eta)$.

The notation is justified by the fact that one can take $u \in H^1(K, \psi)$, considered as element of $H^1$, restricting to $\Gamma$. Using the exact sequence from Global Class Field Theory:

$$0 \to H^1(K, \overline{\Omega}_p) \to \text{Hom}(K^\times_p \times \overline{\Omega}_p, \overline{\Omega}_p) \to \text{Hom}(O_K[\frac{1}{p}]^\times, \overline{\Omega}_p)$$

one finds $\operatorname{ord}_p(u) \cdot \eta(\text{Frob}_p) = \text{res}_p(\eta)(u) = -\text{res}_p(\eta)(u) = -\log_p(u) = \log_p(u)$. \hfill \Box

Since $\eta \in H^1(K, \psi)$ and $\eta \in H^1(K, \overline{\Omega}_p)$ both restrict to the same element of $H^1$, their difference (considered as element of $H^1(K, \overline{\Omega}_p)$) is unramified at $p$, and the following proposition computes its value on $\text{Frob}_p$.

**Proposition 1.5.** One has $(\eta - \eta)(\text{Frob}_p) = \mathcal{L}(\psi^\ast) - \mathcal{L}(1)$.

**Proof.** Restricting to $\Gamma_H$ allows us to see $(\eta - \eta)$ in $H^1(H, \overline{\Omega}_p)$ and one has to compute its value at $\text{Frob}_{y_0}$. Using the exact sequence from Global Class Field Theory:

$$0 \to H^1(H, \overline{\Omega}_p) \to \text{Hom}(H^\times_{x_0} \times \prod_{x \in H} \tau^\times_{H, x}, \overline{\Omega}_p) \to \text{Hom}(O_H[\frac{1}{x_0}]^\times, \overline{\Omega}_p)$$

and by Proposition [1.3] and its proof, the restriction of $\eta$ to $(O_H \otimes \mathbb{Z}_p)^\times$ is given by

$$\sum_{g \in C} \psi_y(g) \log_p(g^{-1}) + \mathcal{J}(\psi) \sum_{g \in C} \psi_y(g) \log_p(\tau g^{-1}).$$

Therefore, for $y_0 \in O_H[\frac{1}{x_0}]^\times$ such that $\operatorname{ord}_{x_0}(y_0) \neq 0$, one has:

$$\operatorname{ord}_{x_0}(y_0) \res_{x_0}(\eta - \eta)(\text{Frob}_p) = \sum_{g \in C} (1 - \psi_y(g)) \log_p(g^{-1}(y_0)) - \mathcal{J}(\psi) \sum_{g \in C} \psi_y(g) \log_p(\tau g^{-1}(y_0)).$$
The desired formula then follows from the Definition (6), using the computation
\[ \sum_{g \in C} \log_p(g^{-1}(y_0)) = \log_p(N_{H/K}(y_0)) = -L(1) \text{ord}_{v_0}(y_0), \]
the last equality following from the fact that \( p \) splits completely in \( H \).

1.5. An interlude on the Six Exponentials Theorem. We first recall some standard results and conjectures in \( p \)-adic transcendence. As in the previous section, \( \log_p : \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_p \) is the standard \( p \)-adic logarithm sending \( p \) to 0, and we use the same notation for its composition with \( t_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \).

Denote by \( L \subset \overline{\mathbb{Q}}_p \) the \( \overline{\mathbb{Q}} \)-vector space generated by 1 and elements of \( \log_p(\overline{\mathbb{Q}}^*) \).

**Theorem 1.6** (Baker-Brumer [4]). Let \( \lambda_1, \ldots, \lambda_n \in L \) are linearly independent over \( \mathbb{Q} \), then they are linearly independent over \( \overline{\mathbb{Q}} \).

**Conjecture 1.7** (Strong Four Exponentials Conjecture). A matrix \( \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \) with entries in \( L \) whose lines and whose columns are linearly independent over \( \mathbb{Q} \) has rank 2.

We will need the following Strong version of the Six Exponentials Theorem, proved by Roy [34] (see also [36]).

**Theorem 1.8** (Strong Six Exponentials Theorem). A matrix \( L = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix} \) with entries in \( L \) whose lines and whose columns are linearly independent over \( \mathbb{Q} \) has rank 2.

1.6. Non-vanishing of \( L \)-invariants. The following \( L \)-invariant will play a prominent role in the paper, as will be intimately related to the local geometry of the eigencurve at \( f \).

**Definition 1.9.** \( L(\psi_-) = L(\psi_-) - 2L(1) \).

**Proposition 1.10.** At least one amongst \( L(\psi_-) \) and \( L(\psi_-^*) \) is non-zero. If either \( \psi_- \) is quadratic, or the Four Exponentials Conjecture [17] holds, then both are non-zero.

**Proof.** Recall that \( C = \text{Gal}(H/K) \) and \( y_0 \in \mathcal{O}_H[\frac{1}{v_0}]^* \) such that \( \text{ord}_{v_0}(y_0) \neq 0 \). One has
\[ \text{ord}_{v_0}(y_0) L(\psi_-^*) = (\eta - \eta_p + \eta_{\gamma})(y_0) = \]
\[ = \sum_{g \in C - \{1\}} (1 - \psi(g)) \log_p(g^{-1}(y_0)) - \mathcal{S}(\psi_-) \sum_{g \in C} \psi(g) \log_p(\tau g^{-1}(y_0)) - \sum_{g \in C} \log_p(\tau g^{-1}(y_0)) = \]
\[ = -\log_p(u_p,\psi_-) - \mathcal{S}(\psi_-) \log_p(\tau u_p,\psi_-) + 2 \log_p(N_{H/K}(y_0)). \]

Suppose first that \( \psi_- \) is quadratic and denote by \( g \) the non-trivial element of \( C \), so that \( G = \text{Gal}(H/\mathbb{Q}) = \{1, g, \tau, g\tau\} \). Then
\[ \text{ord}_{v_0}(y_0) L(\psi_-^*) = 2 \log_p(g(y_0)) - 2 \log_p(g\tau(y_0)) \neq 0, \]
since \( \frac{g(y_0)}{g\tau(y_0)} \) has infinite order (having a non-zero valuation at \( g(v_0) \)), while having trivial valuations at \( v_0 \) (\( \log_p \) being injective on non-torsion elements of \( \mathcal{O}_H[v_0] \)).
Consider next the case \( \psi \neq \psi^T \). A computation similar to the one above shows that:

\[
\text{ord}_{\nu_0}(y_0) = -\log_p(u_{p,\psi^T}) - \mathcal{L}(\psi^T) \log_p(\tau(u_{p,\psi^T})) + 2 \log_p(N_{H/K}(y_0)).
\]

It suffices to show that Roy’s Six Exponentials Theorem \( \text{[18]} \) applies to the matrix:

\[
\begin{pmatrix}
\log_p(u_{p,\psi^T}) & \log_p(u_{p,\psi^T}) & \log_p(u_{p,\psi^T}) \\
\log_p(u_{p,\psi^T}) & \log_p(u_{p,\psi^T}) & \log_p(u_{p,\psi^T}) \\
\log_p(u_{p,\psi^T}) & \log_p(u_{p,\psi^T}) & \log_p(u_{p,\psi^T})
\end{pmatrix}
\]

where \( u_{p,\psi^T} = \tau(u_{p,\psi^T}) \) and \( u_{p,\psi^T} = \tau(u_{p,\psi^T}) \). Lemma \( \text{[12]} \) implies by looking at the first column entries, that the rows are linearly independent over \( \mathbb{Q} \). It remains to show that the columns are linearly independent over \( \mathbb{Q} \) as well, for which we will only look at the first row entries. Using the facts that \( \log_p(N_{H/K}(u_0)) = 0 \) and \( \log_p(N_{H/K}(y_0)) = 0 \), a linear relation over \( \mathbb{Q} \) between those 3 entries would give integers \( a, b \) and \( c \), not all zero, such that

\[
\sum_{1 \leq g \leq C} (\psi(g) - 1)(a \cdot \log_p(g^{-1}(u_0)) + b \cdot \log_p(g^{-1}(y_0))) + \sum_{g \leq C} (c \cdot \psi(g) + b) \log_p(g^{-1}(\tau(y_0))) = 0.
\]

The numbers \( \{\tau(y_0), g(u_0), g(y_0), g(\tau(y_0))|1 < g < C\} \) are (multiplicatively) linearly independent over \( \mathbb{Q} \) and are all sent by \( \iota_p \) to \( \mathcal{O}_{\mathbb{H},v_0} \cong \mathbb{Z}_p \), hence their \( p \)-adic logarithms are linearly independent over \( \mathbb{Q} \). This contradicts Baker-Brumer’s Theorem \( \text{[16]} \).

The last claim follows by applying the Four Exponentials Conjecture \( \text{[17]} \) to the minors obtained by removing the second (or the third) column of the above matrix. \( \square \)

2. Galois deformations

Recall from \( \text{[31,2]} \) the unique cohomology class \( [\eta] \in H^1(K, \psi) \) whose restriction to \( I_p \) is given by the \( p \)-adic logarithm. Since \( \psi - 1 \) is a basis of the coboundaries, fixing, once and for all, \( \gamma_0 \in \Gamma_K \) such that \( \psi(\gamma_0) \neq 1 \), we let \( \eta : \Gamma_K \to \overline{\mathbb{Q}}_p \) be the unique representative of \( [\eta] \) such that \( \eta(\gamma_0) = 0 \).

We let \( \rho = \begin{pmatrix} \psi & \eta \psi \\ 0 & \psi^T \end{pmatrix} : \Gamma_K \to \text{GL}_2(\overline{\mathbb{Q}}_p) \) in the canonical basis \((e_1, e_2)\) of \( V_{\mathbb{Q}_p} = \overline{\mathbb{Q}}_p \). The unique \( \Gamma_{K_p}\)-stable filtration of \( V_{\mathbb{Q}_p} \) with unramified quotient is given by

\[
0 \to V'_{\mathbb{Q}_p} = \overline{\mathbb{Q}}_p \cdot e_1 \to V_{\mathbb{Q}_p} \to V_{\mathbb{Q}_p}^{un} \to 0.
\]

2.1. Ordinary deformations of \( \rho \). Let \( \mathcal{C} \) be the category of complete Noetherian local \( \overline{\mathbb{Q}}_p \)-algebras \( A \) with maximal ideal \( m_A \) and residue field \( \overline{\mathbb{Q}}_p \), where the morphisms are local homomorphisms of \( \overline{\mathbb{Q}}_p \)-algebras.

Consider the functor \( \mathcal{D}_p^{\text{univ}} \) associating to \( A \in \mathcal{C} \) the set of lifts \( \rho_A : \Gamma_K \to \text{GL}_2(A) \) of \( \rho \) modulo strict equivalence and satisfying

\[
\text{Tr} \rho_A(\tau g) = \text{Tr} \rho_A(g) \text{ for any } g \in \Gamma_K.
\]

Since \( \psi \) and \( [\eta] \) are both non-trivial, the centralizer of the image of \( \rho \) consists only of scalar matrices, hence \( \mathcal{D}_p^{\text{univ}} \) is representable by a universal deformation ring \( \mathcal{R}_p^{\text{univ}} \) (see \( \text{[32]} \)).
Let $\mathcal{D}_{\text{Fil}}$ be the functor assigning to $A$ in $\mathcal{C}$ the set of free $A$-submodules $V_A' \subset A^2$ such that $V_A' \otimes_A \overline{Q}_p = V_{\overline{Q}_p}'$ (such $V_A'$ is necessarily a direct rank 1 summand of $V_A$). It is representable by the strict completed local ring $\mathcal{R}_{\text{Fil}} \cong \overline{Q}_p[U]$ of the projective space $\mathbb{P}^1$ at the point corresponding to the line $V_{\overline{Q}_p}' = \overline{Q}_p \cdot e_1 \subset \overline{Q}_p$. The universal submodule $V'_{\mathcal{R}_{\text{Fil}}}$ has basis given by $e_1 + U e_2$.

**Definition 2.1.** The functor $\mathcal{D}_\rho^{\text{ord}}$ assigns to $A \in \mathcal{C}$ the set of tuples $(V_A, V_A')$ such that

(i) $V_A = A^2$ is endowed with continuous $A$-linear $\Gamma_K$-action, denoted $\rho_A : \Gamma_K \to \text{GL}_2(A)$, such that $V_A \otimes_A \overline{Q}_p = V_{\overline{Q}_p}$ as $\Gamma_K$-modules, and $\text{Tr} \rho_A(\tau \gamma) = \text{Tr} \rho_A(g)$ for all $g \in \Gamma_K$.

(ii) $V_A' \subset V_A$ is a direct free factor over $A$ of rank 1 which is $\Gamma_K$-stable, and such that $\Gamma_K$ acts on $V_A' = V_A/V_A'$ by an unramified character, denoted $\chi_A$,

modulo the strict equivalence relation $(\rho_A, V_A') \sim (P \rho_A P^{-1}, P \cdot V_A')$ with $P \in 1 + M_2(\mathcal{M}_A)$.

One has $V_A' \otimes_A \overline{Q}_p = V_{\overline{Q}_p}'$ since the latter is the unique $\Gamma_K$-stable line in $V_{\overline{Q}_p}$, with unramified quotient. Therefore $\mathcal{D}_\rho^{\text{ord}}$ is a sub-functor of $\mathcal{D}_\rho^{\text{univ}} \times \mathcal{D}_{\text{Fil}}$ defined by a closed condition guaranteeing its representability.

**Lemma 2.2.** (i) The functor $\mathcal{D}_\rho^{\text{ord}}$ is representable by a quotient $\mathcal{R}_\rho^{\text{ord}}$ of $\mathcal{R}_\rho^{\text{univ}}$.

(ii) The determinant of $\rho_{\text{ord}} : \Gamma_K \to \text{GL}_2(\mathcal{R}_\rho^{\text{ord}})$ extends, in two different ways to a character $\Gamma_Q \to (\mathcal{R}_\rho^{\text{ord}})^\times$.

**Proof.** (i) The argument is exactly the same as in the proof of [6, Lemma 1.2].

(ii) Since $2 \det(\rho_{\text{ord}}(\gamma)) = \text{tr}(\rho_{\text{ord}}(\gamma))^2 - \text{tr}(\rho_{\text{ord}}(\gamma^2))$, we deduce that $\det(\rho_{\text{ord}})$ is a $\tau$-invariant character of $\Gamma_K$, hence can be extended to a character of $\Gamma_Q$ by sending the order 2 element $\tau$ either to 1 or to $-1$. Alternatively, one can observe that the character $\det(\rho_{\text{ord}}) \det(\rho)^{-1} : \Gamma_K \to 1 + m_{\mathcal{R}_\rho^{\text{ord}}}$ extends uniquely, since $\mathcal{R}_\rho^{\text{ord}}$ is complete and $K$ has a unique $\mathbb{Z}_p$-extension on which $\tau$ acts trivially (this argument is more general is, as it does not use the existence of a section of $\Gamma_Q \to \Gamma_K/Q$ given by the complex conjugation $\tau$).

Since the two extensions in Lemma 2.2(ii) differ by the quadratic character $\epsilon_K$ of $\Gamma_K/Q$, exactly one of them is odd, and we denote it $\det(\rho_{\text{ord}}) : \Gamma_Q \to \mathcal{R}_\rho^{\text{ord}}$. It reduces modulo the maximal ideal to $\det(\text{Ind}_K^Q \psi) = (\psi \circ \text{Ver}) \cdot \epsilon_K : \Gamma_Q \to \overline{Q}_p^\times$, where $\text{Ver} : \Gamma_K^{ab} \to \Gamma_K^{ab}$ denotes the transfer homomorphism. Since $A \in \mathcal{C}$ is the universal deformation ring of that character (see [3, §6]) the natural transformation $\rho_A \mapsto \det(\rho_A)$, endows $\mathcal{R}_\rho^{\text{ord}}$ with a $\Lambda$-algebra structure.

### 2.2. Reducible and CM deformations of $\rho$

**Definition 2.3.**

(i) Let $\mathcal{D}_\rho^{\text{red}}$ be the subfunctor of $\mathcal{D}_\rho^{\text{ord}}$ consisting of $\Gamma_K$-reducible deformations.

(ii) Let $\mathcal{D}_\rho^K$ the subfunctor of $\mathcal{D}_\rho^{\text{ord}}$ consisting of deformations of $\rho$ having a free $\Gamma_K$-quotient of rank 1 which is unramified at $p$ (i.e., are reducible and ordinary in the same basis).
Lemma 2.4.

(i) The functor $\mathcal{D}_\rho^{\text{red}}$ is representable by a quotient $\mathcal{R}_\rho^{\text{red}}$ of $\mathcal{R}_\rho^\text{ord}$.

(ii) The functor $\mathcal{D}_\rho^K$ is representable by a quotient $\mathcal{R}_\rho^K = \mathcal{R}_\rho^\text{ord}/(c(g); g \in \Gamma_K)$, where $\rho_{\text{ord}}(g) = \left(\begin{array}{cc} a(g) & b(g) \\ c(g) & d(g) \end{array}\right)$ in an ordinary basis.

Proof. (i) Since $\rho(\gamma_0)$ has distinct eigenvalues, Hensel’s lemma provides us with a basis $(e_1^+, e_2^+)$ of $V_{\mathcal{R}_\rho^\text{ord}}$ lifting $(e_1, e_2)$, and in which $\rho^\text{univ}(\gamma_0)$ is a diagonal matrix. Now the assertion follows from exactly the same arguments as those in the proof of [6, Lemma 1.4].

(ii) See [6, Lemma 1.6].

Finally for $? = \{\text{ord}, K, \text{red}\}$, let $\mathcal{D}_{\rho,0}^?$ be the sub-functors of $\mathcal{D}_{\rho}^?$ classifying deformations with fixed determinant. By §2.1 $\mathcal{R}_{\rho,0}^? = \mathcal{R}_{\rho}^?/m_\Lambda\mathcal{R}_\rho^?$ represents $\mathcal{D}_{\rho}^?$.

Remark 2.5. While $\rho$ is reducible and ordinary for the same filtration [8], this is not necessarily true for deformations in $\mathcal{D}_{\rho}^\text{ord}$. In fact, we show in §2.3 that

$$\dim \mathcal{D}_\rho^K(\overline{\mathbb{Q}}_p[\epsilon]) = 1 < \dim \mathcal{D}_\rho^{\text{red}}(\overline{\mathbb{Q}}_p[\epsilon]) = \dim \mathcal{D}_\rho^\text{ord}(\overline{\mathbb{Q}}_p[\epsilon]) = 2.$$

2.3. Non-CM deformations of $\rho$. We recall the indecomposable representation $\rho = \left(\begin{array}{c} \psi \\ \eta \end{array}\right)$ from §2.1 and that its universal ordinary deformation $\rho_R = \rho_{\text{ord}}^\text{ord}$. We can do a similar construction with $\psi^\tau$ instead of $\psi$. Namely, we let $\eta': \Gamma_K \to \overline{\mathbb{Q}}_p$ be the unique representative of $[\eta'] \in H^1(K, \psi^\tau)$ whose restriction to $I_p$ is given by the $p$-adic logarithm and such that $\eta'(\gamma_0) = 0$. Then we consider the indecomposable representation

$$\rho' = \left(\begin{array}{cc} \psi & \eta \psi^\tau \\ 0 & \eta \psi^\tau \end{array}\right): \Gamma_K \to \text{GL}_2(\overline{\mathbb{Q}}_p)$$

and denote by $\mathcal{D}_{\rho}^{\text{ord}}$ its ordinary deformation functor, representable by a universal deformation ring $\mathcal{R}_{\rho}^{\text{ord}}$. By definition the universal deformation $\rho'_R$ has a Gal($K/\overline{\mathbb{Q}}$)-invariant trace and admits a rank 1 unramified $\Gamma_K$-quotient, denoted $\chi'_R$.

We highlight that since $[\eta^\tau]$ and $[\eta']$ are proportional, in fact $\eta^\tau = \mathcal{S}(\psi, \eta)^{\psi^{-1}(\gamma_0)} - \mathcal{S}(\psi, \eta)^{\psi^{-1}(\gamma_0)}$, one has $\rho' \approx \rho^\tau$. However $\rho'_R$ need not be isomorphic to $\rho^\tau_R$ as the former admits a $\Gamma_K$-unramified quotient, while the latter admits a $\Gamma_{K_p}$-unramified quotient, and not necessarily vice versa.

The following definition (similar to [6, Def.1.7]) is an attempt to describe Galois theoretically the local ring at $f$ of the closed analytic subspace $\mathcal{E}^\perp$ of $\mathcal{E}^{\text{ord}}$ defined by the union of irreducible components without CM by $K$. As we will later show in Theorem 3.3 one indeed has $f \in \mathcal{E}$. \hspace{1cm} 

Definition 2.6. Let $\mathcal{E}^\perp$ be the functor assigning to $A \in \mathcal{E}$ the set of strict equivalence classes of pairs $((\rho_A, \chi_A), (\rho'_A, \chi'_A))$ in $\mathcal{D}_{\rho}^{\text{ord}}(A) \times \mathcal{D}_{\rho}^{\text{ord}}(A)$ such that $\text{tr}(\rho_A) = \text{tr}(\rho'_A)$ and $\chi_A(\text{Frob}_p) = \chi'_A(\text{Frob}_p)$. 

The definition of $D^k$ is meant to exclude $\Lambda$-adic deformations of $\rho$ having CM by $K$. Note that the $\tau$-conjugate of any $p$-ordinary deformation of $\rho$ is a $\tilde{p}$-ordinary deformation of $\rho'$ which is not necessarily $p$-ordinary, so it does not in general define a point of our functor.

The functor $D^k$ is representable by an $\Lambda$-algebra $R^k$, quotient of $\mathcal{R}_\rho \otimes \mathcal{R}'_\rho$ by the ideal:

$$\left(\text{tr}(\rho_R)(g) \otimes 1 - 1 \otimes \text{tr}(\rho'_R)(g), \chi_R(\text{Frob}_p) \otimes 1 - 1 \otimes \chi'_R(\text{Frob}_p); g \in \Gamma_K\right).$$

**Lemma 2.7.** The natural homomorphisms $\mathcal{R}_\rho \otimes \mathcal{R}'_\rho \to R^k$ and $\mathcal{R}'_\rho \to R^k$ are surjective. Moreover $\rho^k$ and $\rho'_k$ are conjugated by an element of $\text{GL}_2(R^k)$, in particular $\rho^k_{\text{univ}}$ is $p$-ordinary.

**Proof.** The first claim follows from similar arguments that have already been used in [6 Lemma 1.8]. In fact, since $\dim H^1(K, \psi_\psi) = \dim H^1(K, \psi_\tau) = 1$ by Proposition 1.3, one can apply [29 Cor.1.1.4(ii)] to prove that the algebras $\mathcal{R}_\rho \otimes \mathcal{R}'_\rho$ (and hence any of their quotients) are generated by the trace of their universal deformations.

As highlighted before, the representations $(\varphi_\psi \tau) \eta_\psi(\tau) (1 - \psi_\psi(\tau))$ and $\rho^k_R$ both reduce to $\rho'$ and share the same trace, hence they correspond to the same point of $\mathcal{D}_\rho^{\text{univ}}(R^k)$ and thus they are conjugated by an element of $1 + M_2(m_{R^k})$. \qed

The local ring $R^k_0 = R^k/m_{\Lambda}R^k$ represents the subfunctor $D^k_0$ of $D^k$ consisting of deformations with fixed determinant.

### 2.4. The tangent space of ordinary deformations

In this subsection we compute the dimension of the tangent spaces of the earlier introduced functors using results in Galois cohomology, Class Field Theory and some results in transcendence.

Let $\mathcal{Q}_p[\epsilon] = \mathcal{Q}_p[[X]]/(X^2)$ be the $\mathcal{Q}_p$-algebra of dual numbers. It is well know that there is a natural injection:

$$t^\text{univ}_\rho = \mathcal{D}^\text{univ}_\rho(\mathcal{Q}_p[\epsilon]) \to H^1(K, \text{ad}(\rho)),$$

inducing also an injection $t^\text{univ}_\rho,0 \to H^1(K, \text{ad}^0(\rho))$, where $\text{ad}(\rho)$ (resp. $\text{ad}^0(\rho)$) is the adjoint representation on $\text{End}_{\mathcal{Q}_p}(V^\rho_{\mathcal{Q}_p})$ (resp. the sub-representation on trace zero elements. We have a natural decomposition of $\Gamma_K$-modules $\text{ad} \simeq \text{ad}^0 \oplus 1$. The tangent spaces

$$t^\text{ord}_\rho = \mathcal{D}^\text{ord}_\rho(\mathcal{Q}_p[\epsilon]), \quad t^\text{red}_\rho = \mathcal{D}^\text{red}_\rho(\mathcal{Q}_p[\epsilon]) \quad \text{and} \quad t^K_\rho = \mathcal{D}^K_\rho(\mathcal{Q}_p[\epsilon]).$$

of the functors defined in §2.1 and §2.2 are thus naturally subspaces of $H^1(K, \text{ad}(\rho))$. Let

$$0 \to W_\rho \to \text{End}_{\mathcal{Q}_p}(V^\rho_{\mathcal{Q}_p}) \to \text{Hom}_{\mathcal{Q}_p}(V^\rho_{\mathcal{Q}_p}, V''^\rho_{\mathcal{Q}_p}) \simeq \psi_\tau \to 0$$

be the short exact sequence of $\mathcal{Q}_p[\Gamma_K]$-modules arising from the fact that $\rho$ is reducible and let $W^0_\rho = W_\rho \cap \text{End}_{\mathcal{Q}_p}(V^\rho_{\mathcal{Q}_p})$. The choice of the basis $(e_1, e_2)$ of $V^\rho_{\mathcal{Q}_p}$ in which $\rho = \begin{pmatrix} \psi & \eta \\ 0 & \psi_\tau \end{pmatrix}$ yields to a natural identification $\text{End}_{\mathcal{Q}_p}(1_{\mathcal{Q}_p}) \simeq M_2(\mathcal{Q}_p)$, under which $W_\rho$ corresponds to the
subspace of matrices of the form \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \). Consider the short exact sequence of \( \mathbb{Q}_p[\Gamma_K] \)-modules

\[
0 \to W_\rho' \to W_\rho \xrightarrow{d} \text{Hom}_{\mathbb{Q}_p}(V''_{\mathbb{Q}_p}, V''_{\mathbb{Q}_p}) \to 0, \quad \text{and}
\]

(11)

\[
0 \to W_{\rho,0}^\prime \approx \psi_\ast \to W_\rho \xrightarrow{(a,d)} \mathbb{Q}_p^2 \to 0.
\]

For \( h \in \text{Hom}(\Gamma_K, \mathbb{Q}_p) \) we shall denote by \( h^\ast \in \text{Hom}(\Gamma_K, \mathbb{Q}_p) \) the element \( g \mapsto h(\tau g \tau) \).

**Lemma 2.8.** Let \( a, d \in \text{Hom}(\Gamma_K, \mathbb{Q}_p) \) and \( s \in \{ \pm 1 \} \). If \( \psi_\ast + s \psi^\ast d = \psi^\ast a^\ast + \psi d^\ast \) as functions on \( \Gamma_K \), then \( d = a^\ast \).

**Proof.** The relation can be rewritten as \( \psi_\ast (a - d^\ast) = s(a^\ast - d) \). Since \( \psi_\ast \) is unramified at \( p \), it follows that the element \( (a - d^\ast) \) and \( s(a^\ast - d) \) of \( \text{Hom}(\Gamma_K, \mathbb{Q}_p) \) coincide on both inertia groups \( I_p \) and \( I_p^\ast \), hence they are equal. However \( \psi_\ast \) is not a constant function hence \( a - d^\ast = 0 \).

**Definition 2.9.** Let \( \text{Hom}(\Gamma_K, \mathbb{Q}_p)^2 \) be the subspace of \( \text{Hom}(\Gamma_K, \mathbb{Q}_p)^2 \) consisting of elements \( (a, d) \) such that \( d = a^\ast \), and let \( \text{Hom}^1(K, W_\rho)^\ast \) be the inverse image of \( \text{Hom}(\Gamma_K, \mathbb{Q}_p)^2 \) under the morphism \( (a_\ast, d_\ast) : \text{Hom}^1(K, W_\rho) \to \text{Hom}(\Gamma_K, \mathbb{Q}_p)^2 \) coming by functoriality from (11).

Put \( \text{Hom}^1(K, W_\rho)^0 = \text{Hom}^1(K, W_\rho)^\ast \cap \text{Hom}^1(K, W_\rho^0) \).

**Proposition 2.10.** We have \( t^\ast_{\rho,0} = t_{\rho,0}^\ast \in \text{Hom}^1(K, W_\rho)^\ast \) and \( t^\ast_{\rho,0} = t_{\rho,0}^\ast \in \text{Hom}^1(K, W_\rho^0)^\ast \).

**Proof.** (i) Recall that \( t_{\rho,0}^\ast \in t_{\rho,0}^\ast \) by Definition 2.3. To prove the other inclusion, we write an infinitesimal ordinary deformation \( \rho_\varepsilon \) of \( \rho \) in an ordinary basis \( (e_{1,\varepsilon}, e_{2,\varepsilon}) \) of \( V_{\mathbb{Q}_p}[\varepsilon] \):

\[
(12) \quad \rho_\varepsilon = (1 + \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \rho, \quad \text{where} \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \text{Hom}^1(K, \text{ad}(\rho)).
\]

Since \( \text{Hom}^0(K, \psi_\varepsilon^\ast) = \{0\} \), (10) yields the following long exact sequence:

\[
0 \to \text{Hom}^1(K, W_\rho) \to \text{Hom}^1(K, \text{ad} \rho) \xrightarrow{\text{c}_\ast} \text{Hom}^1(K, \psi_\varepsilon^\ast) \to \text{Hom}^2(K, W_\rho),
\]

where the map \( \text{ad} \rho \to \psi_\varepsilon^\ast \) is given by \( \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \mapsto [c] \). By definition the image of \( t_{\rho,0}^\ast \in \text{Hom}^1(K, \text{ad} \rho) \) under the morphism \( \text{c}_\ast \) lands in the subspace

\[
\ker \left( \text{Hom}^1(K, \psi_\varepsilon^\ast) \to \text{Hom}^1(K_p, \mathbb{Q}_p) \right),
\]

which is trivial by Lemma 1.3. Thus \( t_{\rho,0}^\ast \in \text{Hom}^1(K, W_\rho) \). By (10), an element of \( \text{Hom}^1(K, W_\rho) \) corresponds to a reducible infinitesimal deformation in a basis \( (e_+^\ast, e_-^\ast) \) of \( V_{\mathbb{Q}_p}[\varepsilon] \)

\[
\rho_\varepsilon = \begin{pmatrix} 1 + \varepsilon \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \end{pmatrix} \rho = \begin{pmatrix} \psi(1 + \varepsilon a) & \psi^\ast (\eta + \varepsilon (\eta + b)) \\ 0 & \psi^\ast (1 + \varepsilon d) \end{pmatrix}.
\]

If this deformation comes from an element of \( t_{\rho,0}^\ast \), since \( \text{Tr} \rho_\varepsilon = \text{Tr} \rho_\varepsilon^\ast \) (see Definition 2.1), Lemma 2.8 implies that \( a = d^\ast \) in this reducible basis, hence \( t_{\rho,0}^\ast \in \text{Hom}^1(K, W_\rho)^\ast \) (see Definition 2.4). Exactly the same argument shows that \( t_{\rho,0}^\ast = t_{\rho,0}^\ast \in \text{Hom}^1(K, W_\rho^0)^\ast \).
It remains to show that the inclusion $t_p^{\text{ord}} \subset H^1(K, W_{\rho})^* \subset H^1(K, W_{\rho})$ is in fact an isomorphism. As above, an element of $H^1(K, W_{\rho})^*$ corresponds to an infinitesimal deformation

$$\rho_e = \left(1 + \varepsilon \begin{pmatrix} d^T & \ast \\ 0 & d \end{pmatrix} \right) \rho = \begin{pmatrix} \psi(1 + \varepsilon d^T) & \eta \psi^T + \varepsilon \cdot \ast \\ \varepsilon \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \psi(1 + \varepsilon d) \end{pmatrix}$$

in a reducible basis $(e^+_\varepsilon, e^-_\varepsilon)$ of $V_{\mathbb{Q}_p}^\dagger$, where $d \in H^1(K, \mathbb{C})$. It is sufficient to show that $\rho_e$ is $p$-ordinary, i.e., find $\alpha \in \mathbb{Q}_p$ such that the line generated by $e^+_\varepsilon + \varepsilon \alpha \cdot e^-_\varepsilon$ in $V_{\mathbb{Q}_p}$ is $\Gamma_K$-stable with an unramified quotient. A direct computation shows that the restriction of $\rho_e$ to $\Gamma_K$ in the desired ordinary basis $(e^+_\varepsilon + \varepsilon \alpha \cdot e^-_\varepsilon)$ is given by:

$$\begin{pmatrix} 1 & 0 \\ -\varepsilon \alpha & 1 \end{pmatrix} \begin{pmatrix} \psi(1 + \varepsilon d^T) & \eta \psi + \varepsilon \cdot \ast \\ 0 & \psi(1 + \varepsilon d) \end{pmatrix} = \begin{pmatrix} \psi(1 + \varepsilon (d^T + \alpha \eta)) & \eta \psi + \varepsilon \cdot \ast \\ \varepsilon \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \psi(1 + \varepsilon (d - \alpha \eta)) \end{pmatrix}.$$ 

Since $\eta_{\mathbb{Q}_p} = \log_p$ generates the image of the restriction map $H^1(K, \overline{\mathbb{Q}}_p) \to H^1(I_p, \overline{\mathbb{Q}}_p)$, there exists a unique $\alpha \in \mathbb{Q}_p$ such that $d_{\mathbb{Q}_p} = \alpha \eta_{\mathbb{Q}_p}$, for which the character $\chi_e = \psi(1 + \varepsilon (d - \alpha \eta))$ will be unramified at $p$. This completes the proof.

\[ \square \]

Let $H^1(K, W_{\rho}^0) \xrightarrow{\partial_v} H^1(K, \mathbb{Q}_p)$ be natural morphism induced by \((\Pi), \text{i.e., } \left( \begin{smallmatrix} -d \mathbf{h} \\ 0 \end{smallmatrix} \right) \mapsto d \).

**Lemma 2.11.**

(i) One has $\dim_{\mathbb{Q}_p} H^2(K, \psi_\ast) = 0$.

(ii) The map $H^1(K, W_{\rho}^0) \xrightarrow{d_\ast} H^1(K, \mathbb{Q}_p)$ is an isomorphism.

**Proof.** (i) By the global Euler characteristic formula the dimension of $H^2(K, \psi_\ast)$ equals:

$$\dim H^1(K, \psi_\ast) - \dim H^0(K, \psi_\ast) + \dim H^1(C, 1) - [K : \mathbb{Q}] \dim(\psi_\ast) = 1 - 0 + 1 - 2 = 0.$$

(ii) Since $H^2(K, \psi_\ast) = 0$ and $H^1(K, \psi_\ast)$ is 1-dimensional, \((\Pi)\) yields a long exact sequence:

$$0 \to H^0(K, \mathbb{Q}_p) \xrightarrow{\partial_v} H^1(K, \mathbb{Q}_p) \to H^1(K, W_{\rho}^0) \xrightarrow{\partial_\ast} H^1(K, \mathbb{Q}_p) \to 0.$$

Hence, $d_\ast$ is necessarily an isomorphism. \[ \square \]

**Proposition 2.12.** One has $\dim t_p^{\text{ord}} = 2$, $\dim t_{p,0}^{\text{ord}} = \dim t_p^K = 1$ and $t_p^K_{\mathbb{Q}_p} = \{0\}$.

**Proof.** Using the direct sum decomposition $W_{\rho} = W_{\rho}^0 \oplus \mathbb{Q}_p$, Lemma 2.11(ii) yields an isomorphism:

$$(a_\ast - d_\ast, a_\ast + d_\ast) : H^1(K, W_{\rho}) \xrightarrow{\sim} H^1(K, \mathbb{Q}_p) \oplus H^1(K, \mathbb{Q}_p)$$

hence an isomorphism $d_\ast : H^1(K, W_{\rho})^\sim \xrightarrow{\sim} H^1(K, \mathbb{Q}_p)^{\sim -1} \oplus H^1(K, \mathbb{Q}_p)^{\sim -1} = H^1(K, \mathbb{Q}_p)$. In view of Proposition 2.10, we deduce $\dim t_p^{\text{ord}} = \dim t_{p,0}^{\text{ord}} = \dim H^1(K, \mathbb{Q}_p) = 2$. Moreover since $d_\ast$ induces an isomorphism between $H^1(K, W_{\rho})^\sim$ and the anti-cyclotomic line $H^1(K, \mathbb{Q}_p)^{\sim -1}$, one deduces $\dim t_{p,0}^{\text{ord}} = \dim t_{p,0}^{\text{ord}} = 1$.

By definition $t_p^K = \ker(t_p^{\text{ord}} \circ H^1(K, \mathbb{Q}_p) \xrightarrow{\text{res}} \text{Hom}(I_p, \mathbb{Q}_p))$, hence $\dim t_p^K = 1$. Finally

$$t_{p,0}^{\sim} = \ker \left( t_{p,0}^{\text{ord}} : H^1(K, \mathbb{Q}_p)^{\sim -1} \xrightarrow{\text{res}} \text{Hom}(I_p, \mathbb{Q}_p) \right) = \{0\}. \quad \square$$
Let \( t^i = D^i(\overline{\mathbb{Q}}_p[\epsilon]) \), resp. \( t^i_0 = D^i_0(\overline{\mathbb{Q}}_p[\epsilon]) \), be the tangent, resp. relative tangent, space to the functor \( D^i \). Recall the \( L \)-invariants

\[
L_x(\psi) = (\eta - \eta_p + \eta_p')(\text{Frob}_p) \quad \text{and} \quad L_z(\psi) = (\eta' - \eta_p + \eta_p')(\text{Frob}_p)
\]

introduced in Definition 1.9.

**Theorem 2.13.** (i) \( \dim t^i = 1 \), i.e., \( R^i \) is a regular local ring.

(ii) \( t^i_0 = \{0\} \) if, and only if, \( L_z(\psi) + L_x(\psi) \neq 0 \). In particular, \( t^i_0 = \{0\} \) if \( \psi \) is quadratic.

**Proof.** (i) By definition

\[
t^i = D^i(\overline{\mathbb{Q}}_p[\epsilon]) = \{(\rho, \rho') \in t^\text{ord}_p \times t^\text{ord}_p \mid \text{tr}(\rho) = \text{tr}(\rho') \text{ and } \chi(\text{Frob}_p) = \chi'(\text{Frob}_p)\}.
\]

By Proposition 2.10 and Lemma 2.11(ii), there exists \( d \in H^1(K, \overline{\mathbb{Q}}_p) \) such that

\[
\rho_\epsilon = \begin{pmatrix}
\psi(1 + \epsilon d^\tau) & \psi^\tau \eta + \epsilon \cdot \star \\
0 & \psi^\tau(1 + \epsilon d)
\end{pmatrix}
\]

in the reducible basis \((e_\tau^+, e_\tau^-)\) of \( V_{\overline{\mathbb{Q}}_p[\epsilon]} \) (see (14)). Similarly we have

\[
\rho'_\epsilon = \begin{pmatrix}
\psi^\tau(1 + \epsilon d'^\tau) & \psi \eta' + \epsilon \cdot \star \\
0 & \psi(1 + \epsilon d')
\end{pmatrix}.
\]

Write \( d = \alpha \eta_p + \beta \eta_p' \) with \( \alpha, \beta \in \overline{\mathbb{Q}}_p \) in the basis \( \{\eta_p, \eta_p'\} \) of \( H^1(K, \overline{\mathbb{Q}}_p) \) (see §1.2). Note that:

\[
\text{tr}(\rho_\epsilon) = \psi + \psi^\tau + \epsilon(\psi d^\tau + \psi^\tau d) \quad \text{and} \quad \det(\rho_\epsilon) = \psi \psi^\tau(1 + \epsilon(d + d^\tau)).
\]

Since \( \text{tr} \rho'_\epsilon = \text{tr} \rho_\epsilon \), Lemma 2.8 implies that \( d' = d^\tau \).

By the end proof of Proposition 2.10 one has \( \chi = \psi(1 + \epsilon(d - \alpha \eta)) \), hence

\[
\chi(\text{Frob}_p) = (1 + \epsilon(d - \alpha \eta)(\text{Frob}_p))\psi(p) = (1 + \epsilon[\alpha(\eta_p - \eta)(\text{Frob}_p) + \beta \eta_p(\text{Frob}_p)])\psi(p).
\]

Analogously, from \( d' = d^\tau = \beta \eta_p + \alpha \eta_p' \), one deduces \( \chi'_\epsilon = \psi(1 + \epsilon(d^\tau - \alpha \eta')) \) hence

\[
\chi'(\text{Frob}_p) = (1 + \epsilon(d^\tau - \beta \eta')(\text{Frob}_p))\psi(p) = (1 + \epsilon[\beta(\eta_p' - \eta)(\text{Frob}_p) + \alpha \eta_p](\text{Frob}_p))\psi(p).
\]

Comparing (18) with (19) shows that

\[
t^i \simeq \{(\alpha, \beta) \in \overline{\mathbb{Q}}_p^2 \mid \beta(\eta_p - \eta')(\text{Frob}_p) + \alpha \eta_p(\text{Frob}_p) = \alpha(\eta_p - \eta)(\text{Frob}_p) + \beta \eta_p(\text{Frob}_p)\}.
\]

Using Lemma 1.4 and Proposition 1.5 the tangent space can be rewritten in terms of \( L \)-invariants:

\[
t^i \simeq \{(\alpha, \beta) \in \overline{\mathbb{Q}}_p^2 \mid \alpha L_x(\psi) = \beta L_z(\psi)\}.
\]

By Proposition 1.11 \( L_z(\psi) \) and \( L_x(\psi) \) cannot both be zero. Hence \( \dim t^i = 1 \).

(ii) It follows from (17) that the tangent space \( t^i_0 \) is parametrized by \( (\alpha, \beta) \in \overline{\mathbb{Q}}_p^2 \) satisfying \( \alpha + \beta = 0 \) in addition to (20). Hence \( t^i_0 = \{0\} \) if, and only if, \( L_z(\psi) + L_x(\psi) \neq 0 \). \( \square \)
Taking \((\alpha, \beta)\) from the above computations as coordinates of the plane \(t^\text{ord}_\rho\), the lines \(t^\perp\) and \(t^\text{ord}_{\rho,0}\) can be drawn as follows:

\[
\begin{align*}
t^\text{ord}_{\rho,0} : & \alpha + \beta = 0 \\
t^\perp : & \alpha.L.(\psi \bar{\tau}) = \beta.L.(\psi) \\
t^\text{K}_\rho : & \alpha = 0 \text{ (CM line)}
\end{align*}
\]

\[\beta\text{-line} \quad \alpha\text{-line} \quad \text{CM line} \]

**Corollary 2.14.** We have \(t^\text{ord}_\rho = t^\perp \oplus t^\text{K}_\rho\) if, and only if, \(L.(\psi \bar{\tau}) \neq 0\).

**Proof.** As in the proof of Theorem 2.13 one can use \((\alpha, \beta) \in \mathbb{Q}_p^2\) as coordinates on \(t^\text{ord}_\rho\). By (20) the equation defining \(t^\perp\) is \(\alpha.L.(\psi \bar{\tau}) = \beta.L.(\psi)\), while by (15) the equation defining \(t^\text{K}_\rho\) is \(\alpha = 0\). \(\square\)

### 3. Components of the Eigencurve Containing \(f\)

The weight 1 theta series \(\theta_\psi\) has level \(N = N_{K/Q}(\epsilon_\psi) \cdot D\), where \(-D\) is the fundamental discriminant of \(K\) and \(\epsilon_\psi\) is the conductor of \(\theta_\psi\).

The \(p\)-adic cuspidal eigencurve \(\mathcal{E}\) of tame level \(N\) is endowed with a flat and locally finite morphism \(\kappa : \mathcal{E} \rightarrow \mathcal{W}\) of reduced rigid analytic spaces over \(\mathbb{Q}_p\), where the weight space \(\mathcal{W}\) represents continuous homomorphisms from \(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times\) to \(\mathbb{G}_m\). Locally \(O_\mathcal{E}\) is generated as \(O_\mathcal{W}\)-algebra by the Hecke operators \(U_p\) and \(T_\ell\) for \(\ell \nmid Np\). There exists a universal 2-dimensional pseudo-character \(\tau_\mathcal{E} : \Gamma_\mathbb{Q} \rightarrow O(\mathcal{E})\) unramified at \(\ell \nmid Np\) and sending \(\text{Frob}_\ell\) on \(T_\ell\) (see [3, (15)]). It is defined by interpolation, so that its specialization at a classical point \(y \in \mathcal{E}(\mathbb{Q}_p)\) equals the trace of the semi-simple \(p\)-adic Galois representation \(\Gamma_\mathbb{Q} \rightarrow \text{GL}_2(\mathbb{Q}_p)\) attached to the corresponding eigencurve.

If \(\psi\) is trivial then the corresponding theta series \(\theta_\psi\) coincides with the Eisenstein series attached to \(\psi\) and \(\psi\epsilon_K\), and the local structure of the eigencurve at such points has been extensively studied in [6].

If \(\psi_\perp(p) \neq 1\), the corresponding weight 1 theta series \(\theta_\psi\) is a cuspform whose \(p\)-stabilization \(\theta_\psi(z) - \psi(\bar{p})\theta_\psi(pz)\), resp. \(\theta_\psi(pz) - \psi(p)\theta_\psi(pz)\), belongs to a unique Hida family \(\Theta_\psi\), resp. \(\Theta_{\psi\tau}\), having CM by \(K\) (see [3, Cor.1.2]).

We henceforth assume that \(\psi\) is non-trivial but \(\psi_\perp(p) = 1\) (cuspidal, \(p\)-irregular case).

#### 3.1. Components with CM by \(K\) containing \(f\)

Since \((p) = \bar{p}\bar{p}\) splits in \(K\), by Class Field Theory, we have a short exact sequence of groups:

\[
1 \rightarrow (O_K^\times/pO_K^\times)^{(p)} \rightarrow \mathcal{E}_{K_p}^{(p)} \rightarrow \mathcal{E}_{K_p}^{(p)} \rightarrow 1,
\]
where the exponent $(\cdot)^{(p)}$ denotes the $p$-primary part of an abelian group. Letting $\mathbb{W}_p$ denote the maximal torsion-free quotient of $\mathcal{E}\ell_K(p^{\infty})$, which corresponds to the unique $p$-ramified $\mathbb{Z}_p$-extension of $K$, for any prime to $p$ ideal $c$ of $K$, there exists an isomorphism

$$\mathrm{Art}: \mathcal{E}\ell_{K^{(p)}}(p^{\infty}c)/_{\mathrm{tor}} \rightarrow \mathbb{W}_p$$

sending $1 + 2p\mathbb{Z}_p \subset \mathcal{O}^\times_{K_2,p}$ to the inertia subgroup $\mathbb{I}_p \subset \mathbb{W}_p$ at $p$. The quotient $\mathbb{W}_p/I_p$ is cyclic of order $p^h$, for some $h \in \mathbb{Z}_{\geq 1}$, and we let $w_p \in \mathbb{W}_p$ be the topological generator such that $w_p^h = \mathrm{Art}(1 + 2p) \in I_p$. A $p$-adic avatar of a Hecke character of infinity-type $(1,0)$ is given by the continuous character (valued in the ring of integers $\mathcal{O}$ of a sufficiently large $p$-adic field)

$$(21) \quad \varepsilon_p: \Gamma_K \rightarrow \mathbb{W}_p \rightarrow \mathcal{O}^\times,$$

where the second map sends $w_p$ to $\varepsilon_p \in 1 + m_{\mathcal{O}}$ such that $\varepsilon_p^h = 1 + 2p$.

The homomorphism $\eta_p: \Gamma_K \rightarrow \overline{\mathbb{Q}}_p$ from $\mathbb{Z}_2$ factors through $\mathbb{W}_p$ and its restriction to $\mathbb{I}_p$ is given by $\log_p \circ \mathrm{Art}^{-1}$. It follows that $\eta_p = \log_p \circ \varepsilon_p$ on $I_p$. The completed strict local ring of $\Lambda_p = \mathcal{O}[[\mathbb{W}_p]]$ at the augmentation ideal $\ker \left( \mathcal{O}[[\mathbb{W}_p]] \rightarrow \mathcal{O} \right)$ is isomorphic to $\Lambda$. In fact, the inclusion $\mathcal{O}[[\mathbb{I}_p]] \rightarrow \mathcal{O}[[\mathbb{W}_p]]$ can be identified to the generically etale map $\mathcal{O}[X] \rightarrow \mathcal{O}[X]$ sending $X \rightarrow (1 + X)^{p^h} - 1$. We fix such an isomorphism in a way so that the corresponding universal character $\chi_p: \Gamma_K \rightarrow \mathbb{W}_p \rightarrow \mathcal{O}[[\mathbb{W}_p]]^\times \rightarrow \Lambda^\times$ is such that

$$(22) \quad \chi_p = 1 + \eta_pX \pmod{X^2}.$$ 

It is easy to check that, up to an automorphism of $\Lambda$, $\chi_p \circ \mathrm{Ver}$ equals the universal cyclotomic character $\chi_p$ (see [33, §2.4] for the definition of $\chi_p$) and $\eta_p \circ \mathrm{Ver} = \eta_1$, where $\mathrm{Ver}: \Gamma_K^{ab} \rightarrow \Gamma_K^{ab}$ is the transfer homomorphism. Analogously one can define $\chi_{p}$ by replacing $p$ by $\overline{p}$.

The local ring $\Lambda_p$ is a quotient of the universal $p$-ordinary Hecke algebra of tame level $N$ whose kernel corresponds to the minimal prime ideal attached the CM Hida family $\Theta_\psi$.

The specialization of $\Theta_\psi$ in weight $k \in \mathbb{Z}_{\geq 1}$ is given by the classical theta series $\Theta_{\psi,k}$ corresponding to the Hecke character $\lambda_{\psi,k} = \varepsilon_{p}^{k-1} \circ \pi_\psi: \mathcal{E}\ell_K(p^{\infty}c_\psi) \rightarrow \Lambda^\times_p \rightarrow \overline{\mathbb{Q}}_p$ of infinity type $(k-1,0)$, where $\varepsilon_{p}^{k-1}: \Lambda_p \rightarrow \overline{\mathbb{Q}}_p$ is the algebra homomorphism attached to the group homomorphism $\varepsilon_{p}^{k-1}: \mathbb{W}_p \rightarrow \overline{\mathbb{Q}}_p$ and $\pi_\psi: \mathcal{E}\ell_K(p^{\infty}c_\psi) \rightarrow \Lambda^\times_p$ is the projection sending $(z, w) \in \mathcal{E}\ell_K(p^{\infty}c_\psi)_{\mathrm{tor}} \times \mathbb{W}_p = \mathcal{E}\ell_K(p^{\infty}c_\psi)$ to $\psi(z, w)[w]$.

**Definition 3.1.** The CM by $K$ part $\mathcal{E}^K$ of $\mathcal{E}$ is the closed locus where $\tau_{\mathcal{E}}(g)$ vanishes for all $g \in \Gamma_{\mathcal{Q}} \backslash \Gamma_K$. We will say that an irreducible component $\mathcal{Z}$ of $\mathcal{E}$ has CM by $K$, if $\mathcal{Z} \in \mathcal{E}^K$.

Since $\rho_f = \mathrm{Ind}_K^{\mathcal{Q}} \psi$ is irreducible, the localization of the pseudo-character $\tau_{\mathcal{E}}$ at $f$ yields

$$(23) \quad \rho_T: \Gamma_{\mathcal{Q}} \rightarrow \mathrm{GL}_2(T)$$

deforming $\rho_f$ and such that $\mathrm{tr}(\rho_T)(\mathrm{Frob}_\ell) = T_\ell$ for all primes $\ell \nmid Np$ (see [33]).
Proposition 3.2. Let $\mathcal{T}^K$ be the completed local ring of $\mathcal{E}^K$ at $f$. There is an isomorphism of local $A$-algebras $\mathcal{T}^K \cong A \times_{q_p} A$. Moreover, the Galois representations of the two components of $\mathcal{E}^K$ containing $f$ are $\text{Ind}^{\mathcal{O}_K}_A(\psi_\chi)$ and $\text{Ind}^{\mathcal{O}_K}_A(\psi^\tau \chi)$.

Proof. Let $Z$ be an irreducible components of $\mathcal{E}$ having CM by $K$ and containing $f$, and denote by $A = \mathcal{O}_Z$ its completed local $A$-algebra. Since $\rho_f$ is irreducible, the pseudo-character $\tau_A : \Gamma_Q \to A$ comes from a representation $\rho_A : \Gamma_Q \to \text{GL}_2(A)$, which in view of Definition 3.1 (and the uniqueness of the lifting) is such that $\rho_A \simeq \rho_A \otimes \epsilon_K$. By slightly adapting the arguments of [13] Lemma 3.2] to local rings in the category $\mathcal{E}$ we obtain that $\rho_A = \text{Ind}^{\mathcal{O}_K}_A \psi_A$, where $\psi_A : \Gamma_K \to A^\times$ is a character lifting, say, $\psi$ (otherwise it would lift $\psi^\tau$ and the argument would be similar). By p-ordinariness the character $\psi_A \psi^{-1} : \Gamma_K \to 1 + m_A$ factors through $\mathbb{W}_p$. Moreover one has $\chi_p = \det(\text{Ind}_K^A(\psi_A \psi^{-1})) \cdot \epsilon_K = (\psi_A \psi^{-1}) \circ \text{Ver}$, hence $\psi_A = \chi_p \psi$ (or $\psi_A = \chi_p \psi^\tau$). So far we have shown that there is an injection

$$\mathcal{T}^K \to A \times_{q_p} A, \; T_\ell \mapsto ((\chi_p \psi + \chi_p \psi^\tau)(\lambda), (\chi_p \psi + \chi_p \psi^\tau)(\lambda)),$$

where $\ell \nmid Np$ is a prime splitting in $K$ as $(\ell) = \mathfrak{l} \mathfrak{L}$. To show that it is an isomorphism, it suffices to see that $\chi_p \psi + \chi_p \psi^\tau$ and $\chi_p \psi + \chi_p \psi^\tau$ are not congruent mod $X^2$, which in view of (22) amounts to show that $\eta_p \psi + \eta_p \psi^\tau \neq \eta_p \psi^\tau + \eta_p \psi$. This is clear as $\psi_p \neq 1$. \hfill $\Box$

3.2. Existence of a component without CM by $K$ containing $f$. The full eigencurve $\mathcal{E}_{\text{full}}$ is locally generated over $\mathcal{E}$ by the Hecke operators $U_q$ for $q \mid N$. The resulting morphism $\mathcal{E}_{\text{full}} \to \mathcal{E}$ is locally finite, surjective, compatible with all the structures, and not an isomorphism when $N > 1$. Hida’s p-ordinary Hecke algebra of tame level $N$ is a canonical integral model for the ordinary locus of $\mathcal{E}_{\text{full}}$ which is the admissible open of $\mathcal{E}_{\text{full}}$ defined by $|U_p|_p = 1$, and whose irreducible components correspond to Hida families of tame level $N$. In this language Proposition 3.2 asserts that $\Theta_\psi$ and $\Theta_\psi^\tau$ are the only Hida families containing $f$ and having CM by $K$.

Let $U$ be an affinoid subdomain of $\mathcal{W}$ containing $\kappa(f)$ and small enough to ensure the existence of the finite type, projective $\mathcal{O}(U)$-module $S^{1,0}_U(N)$ of p-ordinary families of cuspidal overconvergent modular forms of tame level $N$. By construction, there exists an affinoid neighbourhood $V$ of $f$ in $\mathcal{E}_{\text{full}}$ such that $U = \kappa(V)$, and $\mathcal{O}(V)$ is the finite $\mathcal{O}(U)$-subalgebra of $\text{End}_{\mathcal{O}(U)}(S^{1,0}_U(N))$ generated by $T_\ell$ for $\ell \nmid Np$ and $U_q$ for $q \mid Np$ (see [9] §7.1). Taking the first coefficient in $q$-expansions at the cusp $\infty$ yields a natural isomorphism of $\mathcal{O}(U)$-modules (see Hida [24] §2 and Coleman [10] Prop.B.5.6]):

$$S^{1,0}_U(N) \cong \text{Hom}_{\mathcal{O}(U)-\text{mod}}(\mathcal{O}(V), \mathcal{O}(U)), \; \mathcal{G} \mapsto (T \mapsto a_1(T \cdot \mathcal{G})).$$

By [3] Prop.7.1, $\mathcal{E}$ and $\mathcal{E}_{\text{full}}$ are locally isomorphic at $f$. In particular, $U_q \in \mathcal{T}$ for all $q \mid N$. It follows then from (24) (see also [11] Proof of Prop.1.1) that

$$\text{Hom}_{\mathcal{E}_{\text{full}}-\text{mod}}(\mathcal{T}/m_A \cdot \mathcal{T}, \overline{\mathbb{Q}}_p) \simeq S^{1,0}_U(N)[f].$$
where $\mathcal{T}/\mathfrak{m}_A \cdot \mathcal{T}$ is the local ring of the fiber $\kappa^{-1}(\kappa(f))$ at $f$, and $S_1^1(N)[f]$ is the generalized eigenspace attached to $f$ inside the space of weight 1, level $N$, ordinary $p$-adic modular forms.

Let $\mathcal{T}_{\text{split}}$ be the sub-algebra of $\mathcal{T}$ generated over $A$ by $T_\ell$ for $\ell \nmid Np$ split in $K$.

**Theorem 3.3.** (i) The operator $U_p$ does not belong to $\mathcal{T}_{\text{split}}$.

(ii) There exists an irreducible component $\mathcal{Z}$ of $\mathcal{E}^{\text{ord}}$ containing $f$ and without CM by $K$.

**Proof.** (i) A direct computation shows that $(U_p - \psi(p))(\theta_\psi) = \psi(p)f$, and since clearly $\theta_\psi$ and $f$ share the same Hecke eigenvalues away from $p$ (i.e. for the action of $\mathcal{T}_{\text{split}}$), the space spanned by those two forms is contained in $S_1^1(N)[f]$. Moreover by the above computation, the action of $U_p$ on that space is not semi-simple, while $\mathcal{T}_{\text{split}}$ acts on that space semi-simply (diagonally).

(ii) Assume that all the components of $\mathcal{E}$ specializing to $f$ have CM by $K$, i.e., $\mathcal{T} = \mathcal{T}_K$. Then, by Proposition 3.2 $\mathcal{T} = A \times \mathbb{Q}_p$ and $\mathcal{T}/\mathfrak{m}_A \cdot \mathcal{T} \simeq \mathbb{Q}_p[X]/(X^2)$, hence by (23) we have

\[
\dim_{\mathbb{Q}_p} S_1^1(N)[f] = \dim_{\mathbb{Q}_p} (\mathcal{T}/\mathfrak{m}_A \cdot \mathcal{T}) = 2,
\]

in particular $\{f, \theta_\psi\}$ is a basis of $S_1^1(N)[f]$. On the other hand, Proposition 3.2 also implies that the image of $U_p$ in $\mathcal{T}_K$ equals $(\psi_\lambda(p), (\psi^* \lambda_p)(p)) \in \Lambda^c$ as $\psi_\lambda(p) = 1$. Hence $(U_p - \psi(p)) \in \mathfrak{m}_A$ and $U_p$ acts as a scalar on $S_1^1(N)[f]$, yielding a contradiction with the conclusion that it acts non-semi-simply. \qed

**Remark 3.4.** Theorem 3.3(ii) not only answers a question raised in [14, §7.4(1)] about the existence of a Hida family without CM by $K$ whose specializing in weight 1 has CM by $K$, but it further asserts that this is systematically the case: any CM weight 1 cuspform irregular at $p$ belongs to such a family.

**Definition 3.5.** Let $\mathcal{E}^\perp$ be the closed analytic subspace of $\mathcal{E}$, union of irreducible components having no CM by $K$.

Theorem 3.3 implies that $f \in \mathcal{E}^\perp$. As well known from Hida theory, $\mathcal{T}$ is equidimensional of dimension 1, hence so is the completed local ring $\mathcal{T}_f$ of $\mathcal{E}^\perp$ at $f$, which can also be seen as the largest non-CM by $K$ quotient of $\mathcal{T}$. We will use this modular input to prove that the Krull dimension of $\mathcal{R}_f$ is at least 1 (recall that in §2.4 we have already showed that its dimensions it at most 1).

The specialization of (23) by the surjective homomorphism $\mathcal{T} \to \mathcal{T}_f$, yields a deformation $\rho_f^\perp : \Gamma_K \to \text{GL}_2(\mathcal{T}_f)$ of $\rho_f$. Next, we show that after conjugating $\rho_{\mathcal{T}_f}^{\perp} | \Gamma_K$ by a matrix in the total field of fractions $Q(\mathcal{T}_f)$ we can obtain $\mathcal{T}_f$-valued deformations of both $\rho$ and $\rho'$. 

**Proposition 3.6.** Denote $\chi_f^\perp$ the unramified character of $\Gamma_{K_p} \subset \Gamma_K$ sending Frobp to $U_p$.

There exists a deformation $\rho_{\mathcal{T}_f}^{\perp} : \Gamma_K \to \text{GL}_2(\mathcal{T}_f)$ of $\rho$ such that $\rho_{\mathcal{T}_f}^{\perp} | \Gamma_{K_p} = \left( \epsilon^* \chi_f^\perp \right)$, and $\text{tr} \rho_{\mathcal{T}_f}^{\perp} = \text{tr} (\rho_f^\perp | \Gamma_K)$. Similarly, there exists a deformation $\rho_{\mathcal{T}_f}^{\perp} : \Gamma_K \to \text{GL}_2(\mathcal{T}_f)$ of $\rho'$ such
that $\text{tr}(\rho'_{T'}) = \text{tr}(\rho_{T'})$ and $\rho'_{T'|\Gamma_{K_p}} = \left( \begin{array}{cc} * & * \\ 0 & \chi \end{array} \right)$, hence there is a $\Lambda$-algebra homomorphism $\varphi^i : \mathcal{R}^i \to T^i$.

**Proof.** We first observe that for any localization $L$ of $T^i$ at a minimal prime, the resulting representation $\rho_L : \Gamma_Q \to \text{GL}_2(L)$ remains irreducible when restricted to $\Gamma_K$. In fact, if $\text{Hom}_{\Gamma_K}(\rho_L, \psi_L) \neq \{0\}$ for some character $\psi_L : \Gamma_K \to L^*$, the Frobenius reciprocity theorem would imply that $\text{Hom}_{\Gamma_Q}(\rho_L, \text{Ind}_{K}^{Q} \psi_L) \neq \{0\}$, thus contradicting the assumption that the above minimal prime does not correspond to a component having CM by $K$. Proposition 1.3 then allows us to apply [1] Cor.2 to find an adapted basis $(e_1', e_2')$ (resp. $(e_1'', e_2'')$) of $Q(T^i)^2$ such that the representation $\rho_{Q(T^i)} : \Gamma_K \to \text{GL}_2(Q(T^i))$ takes values in $\text{GL}_2(T^i)$ and reduces to $\rho$ (resp. $\rho''$) modulo the maximal ideal of $T^i$. The resulting representations, denoted $\rho_{T^i}$ and $\rho'_{T'}$, share the same trace and determinant.

For the ordinarity statement at $p$, we write an exact sequence of $T^i[\Gamma_{K_p}]$-modules as in [3] (16) and adapt the argument as follows: The fact that $\rho_{\mathfrak{p}_{L}}$ (resp. $\rho''_{\mathfrak{p}_{L}}$) has an infinite image and admits a unique $I_{\mathfrak{p}}$-stable line, shows that the last term of that exact sequence is a monogenic $T^i$-module, hence it is free, since it is generically free of rank 1 and $T^i$ is reduced. □

Since $\mathcal{R}^i$ is generated over $\Lambda$ by the trace of its universal deformation (see [29] Cor.1.4.4(ii)), the Cebotarev Density Theorem and Proposition 3.6 imply that the image of the $\Lambda$-algebra homomorphism $\varphi^i : \mathcal{R}^i \to T^i$ equals the image $T^i_{\text{split}}$ of $T^i_{\text{split}}$ in $T^i$.

**Theorem 3.7.** The $\Lambda$-algebra $\mathcal{R}^i$ is a discrete valuation ring isomorphic to $T^i_{\text{split}}$ and $\rho_{\mathcal{R}}^i$ is irreducible. The structural homomorphism $\Lambda \to \mathcal{R}^i$ etale if, and only if, $L_{\mu}(\psi)+L_{\mu}(\psi') \neq 0$.

**Proof.** Since by Theorem 2.13(i) the Zariski tangent space $t^i$ of $\mathcal{R}^i$ has dimension 1 and since $T^i_{\text{split}}$ flat over $\Lambda$, it follows that the local complete Noetherian $\Lambda$-algebra $\mathcal{R}^i$ is a discrete valuation ring and that $\varphi^i : \mathcal{R}^i \to T^i_{\text{split}}$ is an isomorphism. The irreducibility of $\rho_{\mathcal{R}}^i$ then follows from Proposition 3.6. The last claim follows directly from Theorem 2.13(ii) where one gives a necessary and sufficient condition for the vanishing of the relative tangent space $t^i_0$ of $\mathcal{R}^i$ over $\Lambda$. □

3.3. The reducibility ideal of $\mathcal{R}^i$. Let $\rho_{\mathcal{R}}$ (resp. $\rho'_{\mathcal{R}}$) be the $\mathfrak{p}$-ordinary deformation of $\rho$ (resp. $\rho'$) obtained by functoriality from the natural surjection $\mathcal{R}_{\mathfrak{p}}^\text{ord} \to \mathcal{R}^i$ (resp. $\mathcal{R}_{\mathfrak{p}}' \to \mathcal{R}^i$) from Lemma 2.7. In order to determine $T^i$ we need to study the possible extensions of $\text{tr}(\rho_{\mathcal{R}}) = \text{tr}(\rho'_{\mathcal{R}})$ to $\Gamma_Q$. For this we first observe that the reducibility ideal (see [2] §1.5]) of the pseudo-character $\text{tr}(\rho_{\mathcal{R}}) = \text{tr}(\rho'_{\mathcal{R}})$ is non-zero, as Theorem 3.7 implies that $\rho_{\mathcal{R}}^i$ is irreducible.

**Definition 3.8.**

(i) Let $e \geq 1$ be the ramification index of $\mathcal{R}^i$ over $\Lambda$, so that $\mathcal{R}^i = \mathcal{O}_\mathfrak{p}[Y]$, with $Y^e = X$. 

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(ii) Let \( r \geq 1 \) be the valuation of the reducibility ideal of \( \text{tr}(\rho_{R^1}) \).

We state for later use a lemma whose elementary proof uses the fact that \( \rho \) has non-isomorphic Jordan-Hölder factors, and that \( R^1/(Y^r) \) is a complete local ring, hence Henselian.

**Lemma 3.9.** For \( m \leq r \), \( \text{tr}(\rho_{R^1}) \mod Y^m \) can be uniquely written as sum of two characters.

It follows from Proposition 2.10 that \( r \geq 2 \). Much of what follows will depend on \( r \), so we first give a numerical criterion for its lowest (conjecturally only possible) value.

**Lemma 3.10.** One has \( r = 2 \) if, and only if, \( \mathcal{L}(\psi_\ast) \cdot \mathcal{L}(\psi_\ast') \neq 0 \).

**Proof.** By (16) and (20) there exists a basis \((e^+, e^-)\) of \( V_{\overline{\mathbb{Q}}_p}^r(Y)/\langle Y^3 \rangle \) in which

\[
\rho_{R^1} \equiv \begin{pmatrix}
\psi + \psi(\mathcal{L}(\psi)\eta_p + \mathcal{L}(\psi)\eta_g)Y + Y^2, \\
\psi\eta + Y^2, \\
\psi^r\eta + Y^2 \end{pmatrix} \quad \text{(mod } Y^3) ,
\]

where \( c \in Z^1(\Gamma_K, \psi_\ast') \) is such that \( c(\gamma_0) = 0 \). By Proposition 3.11 one has \( r \geq 3 \) if, and only if \( c = 0 \), which in view of the injectivity of the restriction \( H^1(K, \psi_\ast) \to H^1(K_p, \mathbb{Q}_p) \) (see Proposition 1.3) is also equivalent to \( c|_{\Gamma_K} = 0 \). We will now use the \( p \)-ordinarity of \( \rho_{R^1} \) and show that the vanishing of \( c|_{\Gamma_{K_p}} \) is equivalent to \( \mathcal{L}(\psi_\ast) \cdot \mathcal{L}(\psi_\ast') = 0 \).

By the proof of Proposition 2.10 the \( p \)-ordinary filtration modulo \( Y^3 \) is generated by a vector of the form \( e_\ast + (\mathcal{L}(\psi_\ast) Y + Y^2 \ast e_-) \), and a direct computation using (26) shows that:

\[
c(g) = \mathcal{L}(\psi_\ast) \cdot (\mathcal{L}(\psi_\ast) \cdot (\eta + \eta_p - \eta_g)(g) - \mathcal{L}(\psi_\ast') \cdot (\eta_g - \eta_p)(g)) = \mathcal{L}(\psi_\ast) \cdot \mathcal{L}(\psi_\ast') \cdot \eta'(g)
\]

for all \( g \in \Gamma_{K_p} \), which completes the proof, since \( \eta'|_{\Gamma_{K_p}} \neq 0 \).

Let \( \iota \) be the automorphism sending \( Z \) to \( -Z \) of the local \( \Lambda \)-algebra

\[
\tilde{R}^1 = R^1/Z/(Z^2 - Y^r) = \begin{cases} \Lambda \times_{\Lambda(\mathbb{F}_2^2)} \Lambda, & r = 2, \\ A \times A(\mathbb{F}_2^2), & r = 4 \text{ even}, \\ A[Z]/(Z^2 - X^r), & r = 3 \text{ odd}. \end{cases}
\]

Note that in the latter case \( \tilde{R}^1 \simeq \mathbb{Q}_p[[X]][X^{\frac{1}{2}}] \) is not normal.

We recall the element \( \gamma_0 \in \Gamma_K \) from 22 such that \( \psi_\ast(\gamma_0) \neq 1 \) and \( \eta(\gamma_0) = 0 \) and write \( \rho_{R^1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in a basis \((e_\ast, e_-)\), where \( \rho_{R^1} = (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}) \), and similarly we write \( \rho'_{R^1} = \rho'_{R^1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \) in a basis \((e'_\ast, e'_-)\), where \( \rho'_{R^1} = (\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}) \).

**Proposition 3.11.** The reducibility ideal \( (Y^r) \) equals \((c(g); g \in \Gamma_K) = (c'(g); g \in \Gamma_K)\). Moreover \( \bar{\rho}_K = \begin{pmatrix} Z & 0 \\ 0 & 1 \end{smallmatrix} \rho_{R^1} \begin{pmatrix} Z & 0 \\ 0 & 1 \end{smallmatrix} \) extends to a representation \( \tilde{\rho}_K : \mathbb{Q} \to \text{GL}_2(\tilde{R}^1) \) reducing to \( \rho_f \) modulo the maximal ideal of \( \tilde{R}^1 \). Finally \( \iota(\rho^\prime_K) = (\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}) \rho_K \otimes \epsilon_K (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \).
Proof. Recall that \((Y^r)\) is the smallest ideal modulo which \(\text{tr}(\rho_{R^1}) = \text{tr}(\rho'_{R^1})\) can be written as a sum of two characters. A direct computation shows that \((Y^r) = (c(g)b(g'); g, g' \in \Gamma_K) = (c'(g)b'(g'); g, g' \in \Gamma_K)\) implying the first claim, since \((b(g'); g' \in \Gamma_K) = (b'(g'); g' \in \Gamma_K) = \mathcal{R}^1\).

Since \(\dim \mathbb{H}^1(K, \psi) = \dim \mathbb{H}^1(K, \psi^\tau) = 1\), there exist upper triangular matrices \(P_0\) and \(P_1\) in \(\text{GL}_2(\mathbb{Q}_p)\) such that the pair of \(\mathcal{R}^1\)-valued \(\Gamma_K\)-representations

\[
\left( P_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rho^\tau_R, \left( \begin{pmatrix} 0 & Y^r \\ 1 & 0 \end{pmatrix} P_0^{-1} \right) \rho^\tau_R, \left( \begin{pmatrix} 0 & Y^r \\ 1 & 0 \end{pmatrix} P_1^{-1} \right) \rho^\tau_R \right)
\]

reduces to \(\left( \rho, \rho' \right)\) modulo \(m_{\mathcal{R}^1}\). We will now check that it defines the same \(\mathcal{R}^1\)-point of \(\mathbb{D}^1\) as \(\left( \rho_{R^1}, \rho'_{R^1} \right)\). In fact, all representations are \(\mathcal{R}^1\)-valued and share the same \(\tau\)-invariant trace and determinant, and, by Lemma \ref{lemma-2.7}, they are all \(p\)-ordinary (with same unramified character). Hence the two pairs are strictly conjugated, in particular there exists \(P \in \text{GL}_2(\mathbb{R}^1)\) such that for all \(g \in \Gamma_K\) we have

\[
\left( P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rho_{R^1}, \left( \begin{pmatrix} 0 & Y^r \\ 1 & 0 \end{pmatrix} P_0^{-1} \right) \rho_{R^1}, \left( \begin{pmatrix} 0 & Y^r \\ 1 & 0 \end{pmatrix} P_1^{-1} \right) \rho_{R^1} \right) = P \rho_{R^1}(g) P^{-1} = P \left( \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} \right) \rho_{R^1}(g) P^{-1}.
\]

Since \(d' \not\equiv d \pmod{Y}\), reducing the above equation modulo \(Y^r\) yields \(P \in \left( \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right) \pmod{Y^r}\). Therefore \(\left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \rho_{R^1}(g) \rho_{R^1}(g) P^{-1} = P \rho_{R^1}(g) P^{-1} = \rho_{R^1}(g)\) \(\mathbb{R}^1\)-invariant trace is the unique \(\mathcal{R}^1\)-valued representation. Hence \(\rho_{R^1}\) is an absolutely irreducible representation (since \(\rho_{R^1}\) is), reducing modulo the maximal to \(\psi \oplus \psi^\tau\). Since since \(\mathcal{R}^1 = \mathbb{Q}_p\), Schur’s lemma applied to a generic point of \(\mathcal{R}^1\) implies that \(\mathcal{R}^1\) is scalar. Therefore \(M^2 = \det(M) = 1\) and putting \(\psi_{\mathcal{R}^1}(\tau) = M\) (or \(-M\)) yields the desired extension. Note that \(\rho_f\) is the unique extension of \(\psi \oplus \psi^\tau\) to \(\Gamma_Q\). The last point is a direct computation. 

Denote by \(\rho_{\mathcal{R}^1}\) the push-forward of \(\rho_{\mathcal{R}^1}\) along the projection \(\mathcal{R}^1 \to \mathcal{R}^1\) sending \(Z\) to \(Y^r\), when \(r\) is even, and let \(\rho_{\mathcal{R}^1} = \rho_{\mathcal{R}^1}\), when \(r\) is odd. By the Frobenius reciprocity (applied over a finite extension of the fraction field of the integral domain \(\mathcal{R}^1\), \(\rho_{\mathcal{R}^1}\) and \(\rho_{\mathcal{R}^1} \otimes \epsilon_K\) are the only possible extensions of \(\rho_{\mathcal{R}^1}\) to \(\Gamma_Q\).

**Corollary 3.12.** The pseudo-characters \(\text{tr}(\rho_{\mathcal{R}^1})\) and \(\text{tr}(\rho_{\mathcal{R}^1} \cdot \epsilon_K)\) are the only possible extensions of \(\text{tr}(\rho_{\mathcal{R}^1})\) to \(\Gamma_Q\). Moreover \(\text{tr}(\rho_{\mathcal{R}^1}) \cdot \epsilon_K = \epsilon \circ \text{tr}(\rho_{\mathcal{R}^1})\).

### 3.4. Components without CM by \(K\) containing \(f\)

Theorem \ref{thm-3.3} implies that there exists a Hida family \(F\) without CM by \(K\) and specializing to \(f\) in weight one. Since \(f \otimes \epsilon_K = f\), the twist \(F \otimes \epsilon_K\) of \(F\) would then also be a Hida family specializing to \(f\).

**Proposition 3.13.** The local ring \(\mathcal{R}^1\) is topologically generated over \(A\) by \(\mathcal{R}^1/\mathbb{Z}_{\mathcal{R}^1}(\Gamma_Q)\).

**Proof.** Let \(A\) be the subring of \(\mathcal{R}^1\) topologically generated over \(A\) by \(\mathcal{R}^1_{\mathbb{Q}}(\Gamma_Q)\). Since \(A \supset \mathcal{R}^1\) (see Theorem \ref{thm-3.7}), it suffices to show that \(A\) contains the local parameter \(Z\). By \ref{thm-3.3} Thm.1]
there exists a deformation $\rho_A = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : \Gamma \rightarrow \text{GL}_2(A)$ of $\rho_f$ such that $\text{tr}(\rho_A) = \tau_k^A$ and $\rho_A(\gamma_0)$ is diagonal. Then the reducibility ideal of $\tau_k^A = \tau_k \rho_R$ equals $(Y') = (b(g) \cdot c(g', g, g' \in \Gamma_K)$. We first show that the ideals $(b(g); g \in \Gamma_K)$ and $(c(g); g \in \Gamma_K)$ of $A$ are equal by showing the double inclusion. To show $(b(g); g \in \Gamma_K) \subset (c(g); g \in \Gamma_K)$ one reduces the equality $\rho_A(\tau g \tau) = \rho_A(\tau) \rho_A(g) \rho_A(\tau)$ for $g \in \Gamma_K$ modulo $(c(g); g \in \Gamma_K)$ and uses the fact that $\rho_A(g)$ and $\rho_A(\tau g \tau)$ are upper triangular and that $\rho_A(\gamma_0)$ is diagonal.

Assume first that $r$ is odd. In that case $\tilde{R}^\perp$ is an integral domain, which is finite, flat and ramified over $R^\perp$. It suffices then to show that $A$ contains an element of $Y^\sharp \cdot \mathcal{O}_p[Y]/(Y^\sharp)^s$, and any $b(g_0) \in A$ generating $(b(g); g \in \Gamma_K) = (Y^\sharp)$ would do.

Assume next that $r$ is even. In that case $\tilde{R}^\perp \cong R^\perp \times \mathcal{O}_p[Y]/(Y^\sharp) R^\perp$, and $A \subset \tilde{R}^\perp$ surjects on both components $R^\perp$. Hence $A = R^\perp \times \mathcal{O}_p[Y]/(Y^\sharp) R^\perp$ with $s \geq \frac{r}{2}$, and by Proposition 3.11 one has

$$(-1 \ 0 \ 0 \ 1) (\tilde{\rho}_K^A \otimes \epsilon_K) (-1 \ 0 \ 0 \ 1) = \iota(\tilde{\rho}_K^A) \equiv \tilde{\rho}_K^A \pmod{Y^s},$$

where $\iota$ is the involution of $\tilde{R}^\perp$ exchanging its two components. Writing $b(g) = (b_1(g), b_2(g)) \in A$, there exists $g_0 \in \Gamma_K$ such that $b_1(g_0) = -b_2(g_0) \in Y^\sharp \cdot \mathcal{O}_p[Y]/(Y^\sharp)^s$. The above congruence implies that $-b_1(g_0) \equiv b_1(g_0) \pmod{Y^s}$, hence $s = \frac{r}{2}$.

Since $T^\perp$ carries a pseudo-character of $\Gamma_Q$, extending the $R^\perp \cong T^\perp_{\text{split}}$-valued pseudo-character of $\Gamma_K$, Corollary 3.12 implies that given any minimal prime ideal $p$ of $T^\perp$ the resulting pseudo-character $\Gamma_Q \rightarrow T^\perp/p \otimes \epsilon_K$ factors through $\text{tr}(\tilde{\rho}_K^A)$. Together with Proposition 3.13 this implies the existence of a $A$-algebra homomorphism $\varphi^\perp : \tilde{R}^\perp \rightarrow T^\perp$ fitting in the commutative diagram:

$$\xymatrix{ \Gamma_K \ar[d]_{\tau_k^A} & \tilde{R}^\perp \ar[d]_{\tilde{\rho}_K^A} \ar[rr]_{\varphi^\perp} & & T^\perp \ar[dd] \ar[d]_{\text{split}} \\
\Gamma_Q & \tilde{R}^\perp \ar[rr]_{\varphi^\perp} & & T^\perp. }$$

Since $\varphi^\perp \circ \tilde{\rho}_K^A$ is surjective by definition, so is $\varphi^\perp$. Both algebras being reduces, it suffices to check injectivity on generic points, which holds since $\tilde{R}^\perp$ is a free $R^\perp$-module of rank 2, while $T^\perp$ has rank $\geq 2$ over $T^\perp_{\text{split}}$. As a consequence we determine the $A$-algebra structure of $T^\perp$.

**Theorem 3.14.** There is an isomorphism of $A$-algebras $\varphi^\perp : \tilde{R}^\perp \rightarrow T^\perp$ extending $\varphi^\perp : R^\perp \rightarrow T^\perp_{\text{split}}$ and under which $\iota$ is sent to the involution of $T^\perp$ mapping $T_\ell$ to $\epsilon_K(\ell) T_\ell$, for $\ell \nmid Np$.

In summary, we have shown that $f$ belongs to exactly 3 or 4 irreducible components of $\mathcal{E}$ depending on the parity of $r$, corresponding to the Hida families $\Theta_\psi$, $\Theta_\psi^\ast$, $F$ and $F \otimes \epsilon_K$ containing $f$ (the last two being Galois conjugates if $r$ is odd).
4. Geometry of the eigencurve and congruence ideals

4.1. Iwasawa cohomology and the CM ideal. Let $K_\infty^-$ be the anti-cyclotomic $\mathbb{Z}_p$-extension of $K$ and

$$\chi_- = \frac{\chi_p}{\chi_p^*} : \Gamma_K \to \text{Gal}(K_\infty^-/K) \to \Lambda^*$$

be the universal $p$-adic anti-cyclotomic character. It satisfies the congruence

$$(28) \quad \chi_- \equiv 1 + (\eta_p - \eta_p^*) \cdot X \pmod{X^2}.$$ 

Put $\Psi = \chi_-.\psi$ and for $n \in \mathbb{Z}_{\geq 1}$ let $\Psi_n = \Psi \mod X^n$. Finally, let $\Gamma_K^{Np}$ be the Galois group of the maximal extension of $K$ unramified outside $Np$.

**Proposition 4.1.**

(i) The $\Lambda$-module $H^1(\Gamma_K^{Np}, \Psi^n)$ is free of rank 1 and for all $n \in \mathbb{Z}_{\geq 1}$ the following specialization morphism is an isomorphism:

$$H^1(\Gamma_K^{Np}, \Psi^n) \otimes_{\Lambda} A/(X^n) \xrightarrow{\sim} H^1(\Gamma_K^{Np}, \Psi^n/\Lambda).$$

(ii) Assume that $\mathcal{L}(\psi_-) \neq 0$. Then for every $n \geq 1$, the natural restriction map is injective:

$$H^1(\Gamma_K^{Np}, \Psi^{-n}) \rightarrow H^1(\Gamma_K, \Psi^{-n}).$$

**Proof.** (i) The proof is similar to [8] Prop.2.9 since $\dim H^1(K, \psi_-) = \dim H^1(K, \psi_0^*) = 1$ (by Proposition 1.3) and $\dim H^2(K, \psi_-) = \dim H^2(K, \psi_0^*) = 0$ (by Lemma 2.11).

(ii) As in loc. cit. we have to check that $\eta' - \eta_p^* \in \mathcal{X}$ generate distinct lines in $H^1(K, 1)$. The claim follows from the fact that $(\eta' - \eta_p^* + \eta_p^*)(\text{Frob}_b) \neq 0.

Let $A_\psi \simeq \Lambda$ be the completed local ring at $f$ of the irreducible component of $\mathcal{E}$ corresponding to $\Theta_\psi$.

**Corollary 4.2.** There is a natural isomorphism $R^K_\rho \xrightarrow{\sim} A_\psi$.

**Proof.** Proposition 2.12 shows that the structural homomorphism $\Lambda \to R^K_\rho$ is surjective (since it is unramified), while Proposition 4.1 yields a surjection $R^K_\rho \to A_\psi \simeq \Lambda$. 

Let $J$ be the kernel of the $\Lambda$-algebra homomorphism $R^\text{ord}_\rho \to R^K_\rho \simeq A_\psi$. Since $R^\text{ord}_\rho$ is Noetherian we know that $J/J^2$ is a module of finite type over $R^\text{ord}_\rho/J \simeq \Lambda$.

**Proposition 4.3.** If $\mathcal{L}(\psi_-) \neq 0$, then the $\Lambda$-module $J/J^2$ is torsion of length $\leq 1$.

**Proof.** It is enough to show that for all $n \geq 1$ the length of the $\Lambda$-module $\text{Hom}_\Lambda(J/J^2, A/(X^n))$ is at most 1. We use an homological machinery as in [6] Prop.4.11. Let us fix a realization $\rho_{R^\text{ord}_\rho}(g) = \left( \begin{array}{cc} a(g) & b(g) \\ c(g) & d(g) \end{array} \right)$ of the universal representation in an ordinary basis. By Lemma 2.3 the ideal $J$ of $R^\text{ord}_\rho$ is generated by the set of $c(g)$ for $g \in \Gamma_K$. Since

$$\rho_{R^\text{ord}_\rho} \otimes (R^\text{ord}_\rho/J) = \left( \begin{array}{cc} \psi \chi_p & * \\ 0 & \psi^* \chi_p \end{array} \right)$$
a direct computation using that the basis is $\Gamma_{K_p}$-ordinary shows that the function

$$(29) \quad \widetilde{c} : \Gamma_K \to J/J^2, \; g \mapsto (\psi_{K_p})^{-1}(g) \cdot c(g) \mod J^2$$

belongs to $\ker \left( Z^1(G_{K_p}^\nu, \psi^\nu) \otimes_A (J/J^2) \right) \to Z^1(K_{p^2}, \chi^P \psi^P) \otimes A(J/J^2) \bigg)$. Since the image of $\tilde{c}$ contains a set of generators of $J/J^2$ as $R_\rho^{ord}/J \simeq \Lambda$-module, the natural map

$$\text{Hom}_A(J/J^2, A/(X^n)) \to \ker \left( Z^1(G_{K_p}^\nu, \psi^\nu) \to Z^1(K_{p^2}, \psi^\nu) \right), \; h \mapsto h \circ \tilde{c}$$

is injective for all $n \geq 1$. Moreover by Proposition 4.5, one has:

$$\ker \left( Z^1(G_{K_p}^\nu, \psi^\nu) \to Z^1(K_{p^2}, \psi^\nu) \right) = \ker \left( B^1(G_{K_p}^\nu, \psi^\nu) \to B^1(K_{p^2}, \psi^\nu) \right).$$

Since $\Psi_{1,1} = \psi_1 |_{\Gamma_{K_p}} = 1$, while $\Psi_2 |_{\Gamma_{K_p}} \neq 1$, the latter is given by the length 1 $A$-module $(X^n)/(X^n) \simeq \mathbb{Z}_p$, proving the desired assertion. \hfill \Box

We begin the study of the congruences between $\Theta_\psi$ and a family $\mathcal{F}$ without CM by $K$, by determining the exact congruence ideal at split primes (note that the eigenvalues of $\mathcal{F}$ and $\mathcal{F} \otimes \epsilon_k$ coincide as such primes). This amounts to the study of the image $\mathcal{T}_\rho^{split}$ of $\mathcal{T}_\rho$ in $\mathcal{T}_\rho^{split} \times A_\psi$, which we will now relate to the universal deformation rings $R_\rho^{ord}$ from §2.

Recall the functorial homomorphism $R_\rho^{ord} \to R_\rho^{\Lambda} \times \mathbb{Q}_p A_\psi$ which surjects on both components. As observed in the proof of Lemma 2.4, the $A$-algebra $R_\rho^{ord}$ is generated by the trace of the universal ordinary deformation of $\rho$, hence there exists an integer $m \geq 1$ and a commutative diagram of surjective homomorphisms of $A$-algebras:

$$\xymatrix{ R_\rho^{ord} \ar[r] & R_\rho^{\Lambda} \times \mathbb{Q}_p [X]/(X^m) \ar[r] & \Gamma_K^\Lambda \ar[d] \ar[r] & \Gamma_K \ar[d] \ar[r] & R_\rho^{K} \ar[d] \ar[r] & \mathcal{T}_\rho^{split} \ar[r] & \mathcal{T}_\rho^{split} \times \mathbb{Q}_p [X]/(X^m) \ar[r] & A_\psi. }$$

**Proposition 4.4.** If $\mathcal{L}(\psi) \neq 0$, there is an isomorphism of complete local intersection rings

$$R_\rho^{ord} \xrightarrow{\sim} \mathcal{T}_\rho^{split} = \mathcal{R}^{\Lambda} \times \mathbb{Q}_p A_\psi.$$

**Proof.** Corollary 2.14 implies that $m = 1$. The isomorphism is a consequence of a variant of Wiles’ numerical criterion due to Lenstra [31] in view of Proposition 4.3. \hfill \Box

Just as $R_\rho^{K}$ is a quotient of $R_\rho^{ord}$, the CM ideal $(X^m)$ contains the reducibility ideal $(X^r)$ and plays a role analogous to that of the Eisenstein ideal.

**Proposition 4.5.** Assume that $\mathcal{L}(\psi) = 0$. Then $m = r - 1$, and $\mathcal{T}_\rho^{split} = \Lambda \times \mathbb{Q}_p [X]/(X^{r-1}) A_\psi$.

**Proof.** Since $\mathcal{L}(\psi) = 0$, the structural homomorphism $\Lambda \xrightarrow{\sim} \mathcal{R}^{\Lambda}$ is an isomorphism by Theorem 3.10. We recall that $r \geq 3$ by Proposition 3.10 and that

$$\rho_{\mathcal{R}}(g) \equiv \begin{pmatrix} A(g) \\
\psi(g) \eta'(g) \cdot X^r \\
B(g) \\
D(g) \end{pmatrix} \mod X^{r+1},$$
in a reducible basis \((e_+, e_-)\), where \(A, B, D\) are polynomials in \(X\) of degree \(\leq r\). Since 
\(\mathcal{R}^\downarrow = \mathbb{Q}_p[X]\) is a discrete valuation ring, an ordinary filtration of \(\rho_{\mathcal{R}^\downarrow}\) is generated by \(e_++X^s e_-\) for some \(s \geq 1\) and, using the corresponding \(p\)-ordinary basis \((e_++X^s e_-, e_-)\), one has
\[
\rho_{\mathcal{R}^\downarrow}(g) \equiv \begin{pmatrix}
A(g) - B(g) \cdot X^s & B(g) \\
\psi(g) e(g) \cdot X^r + (A(g) - D(g)) \cdot X^s - B(g) \cdot X^{2s} & D(g) + B(g) \cdot X^s
\end{pmatrix} \mod X^{r+1}.
\]

It is crucial to first observe that \(s \leq r - 1\). Let us proceed by absurd, assuming that \(s \geq r\). The lower left entry of \(\rho_{\mathcal{R}^\downarrow}(g) \mod X^{r+1}\) in the ordinary basis \((e_++X^s e_-, e_-)\) equals
\[
\psi(g) \left(\eta'(g) + (\psi^{\tau}(g) - 1) \cdot X^{s-r}\right) \cdot X^r
\]
and it has to be trivial on \(I_p\), contradicting Proposition 1.3 as \(0 \neq [\eta'] \in H^1(K, \psi^{\tau})\).

Therefore \(s < r\), hence \(D \mod X^r\) is \(p\)-ramified. The \(p\)-ordinarity of \(\rho_{\mathcal{R}^\downarrow}\) \mod \(X^r\) yields
\[
A(g) \equiv 1 \mod X^{r-s} \quad \text{and} \quad D(g) \equiv 1 \mod X^s, \quad \forall \ g \in I_p.
\]

Since \(m \geq 2\) (see Cor. 2.14) one has \(\text{tr}(\rho_{\mathcal{R}^\downarrow}) \equiv \psi \chi_p + \psi^\tau \chi_p^r \mod X^2\). Lemma 3.9 implies
\[
A \equiv \psi \chi_p \equiv \psi \left(1 + \eta_p \cdot X\right) \mod X^2, \quad \text{and} \quad D \equiv \psi^\tau \chi_p^r \equiv \psi^\tau \left(1 + \eta_p^r \cdot X\right) \mod X^2
\]
which together with \((31)\) implies that \(s = r - 1\), since \(\eta_p | \eta_p \neq 0\). It then follows from the second relation of \((31)\) that \(D \mod X^{r-1}\) is unramified at \(p\), hence there exists a morphism \(A_\psi \rightarrow \mathcal{R}^\downarrow/(X^{r-1})\) sending \(\psi^\tau \chi_p^r\) on \(D \mod X^{r-1}\). Using the \(\tau\)-invariance of \(\text{tr}(\rho_{\mathcal{R}^\downarrow})\), a second application of Lemma 3.9 yields that \(A \equiv D^\tau \mod X^r\). Hence \(\mathcal{R}^\downarrow/(X^{r-1})\) is topologically generated by \(\{D(g), g \in \Gamma_K\}\) and therefore the morphism \(A_\psi \rightarrow \mathcal{R}^\downarrow/(X^{r-1})\) defined above is in fact a surjection, and it yields that \(m \geq r - 1\).

It remains to show that \(m < r\), i.e. \(\text{tr}(\rho_{\mathcal{R}^\downarrow}) \neq \psi \chi_p + \psi^\tau \chi_p^r \mod X^r\). Since \(A \equiv D^\tau \mod X^r\), it suffices (again by Lemma 3.9) to show that \(D \neq \psi^\tau \chi_p^r \mod X^r\) which follows from the fact that \(D \mod X^r\) is ramified at \(p\) as earlier observed. \(\square\)

4.2. The congruence ideal. By Proposition 4.11 one may attach to \(\Theta_\psi\) (resp. \(\Theta_{\psi^r}\)) a \(\Gamma_K\)-reducible deformation of \(\rho\) (resp. \(\rho\)'s) whose semi-simplification is \(\psi \chi_p \oplus \psi^\tau \chi_p^r\) (resp. \(\psi^r \chi_p^r \oplus \psi^\tau \chi_p^r\)).

We recall that the completed local ring \(A_\psi\) (resp. \(A_{\psi^r}\)) at \(f\) of the component corresponding to \(\Theta_\psi\) (resp. \(\Theta_{\psi^r}\)) is isomorphic to \(A\). We recall that we have a natural surjective morphism
\[
\mathcal{R}_p^{\text{ord}} \rightarrow \mathcal{T}_p^{\text{split}} = \mathcal{R}^\downarrow \times_{\mathbb{Q}_p[X]/(X^{r-1})} A_\psi,
\]
sending \(\chi_R(\text{Frob}_p) \in \mathcal{R}_p^{\text{ord}}\) on \(U_p = (U_p(\mathcal{F}), U_p(\Theta_\psi)) \in \mathcal{T}_p^{\text{split}}\). This implies in particular that
\[
U_p(\mathcal{F}) \equiv U_p(\Theta_\psi) \mod X^{r-1},
\]
and the following Proposition would show that this congruence is optimal.

**Proposition 4.6.** Assume that \(\mathcal{L}_\psi(\nu_\psi) = 0\), then \(U_p(\mathcal{F}) \neq U_p(\Theta_\psi) \mod X^r\).
Assume that Theorem 4.7. arising naturally from (32).

Since \( \{(X^r,0),(X,X)\} \in T^\text{split}_\rho = R^\perp_{\mathfrak{m}_p(X)/(X^r)} A \), then \( \{(X^r,0),(0,X^r)\} \in m_A \cdot T^\text{split}_\rho \).

Thus, \( U_p(F) - U_p(\Theta_\varphi) \equiv 0 \mod X^r \) if and only if, \( U_p - \psi(p) \in m_A \cdot T^\text{split}_\rho \). On the other hand, we have a natural surjection

\[
R^\text{ord}_{\rho,0} \twoheadrightarrow T^\text{split}_{\rho,0} = T^\text{split}_\rho / m_A \cdot T^\text{split}_\rho
\]

arising naturally from (32).

Since the 1-dimensional local ring \( T^\text{split}_\rho \) is ramified over \( A \) (because it is singular), the tangent space of \( T^\text{split}_{\rho,0} \) is non-trivial. Meanwhile, it has been shown in Proposition 2.12 that \( \dim T^\text{ord}_{\rho,0} = 1 \). In particular, the surjection (33) rises to an isomorphism between the tangent spaces

\[
\text{Hom}_{\text{alg}}(T^\text{split}_{\rho,0}, \mathcal{O}_p[\epsilon]) \cong \text{Hom}_{\text{alg}}(R^\text{ord}_{\rho,0}, \mathcal{O}_p[\epsilon]) = \mathcal{O}_p[\epsilon].
\]

In addition, the equation defining the tangent space of \( t^\text{ord}_{\rho,0} \) inside \( t^\text{ord}_{\rho,0} \cong \text{Hom}(\Gamma_K, \mathcal{O}_p) \cong \{\alpha \cdot \eta_b + \beta \cdot \eta_p \mid (\alpha, \beta) \in \mathcal{O}_p \} \) is given by \( \alpha = -\beta = 1 \). Hence, we deduce from (19) (by plugin \( \alpha = -\beta = 1 \)) that \( \chi_e(\text{Frob}_p) \mod m_A = (1 + \epsilon(\eta_p - \eta)(\text{Frob}_p - \text{Frob}_p)) \psi(p) = (1 - \epsilon \mathcal{L}(\psi_\tau^\perp))^\prime(0) \psi(p), \) where \( \mathcal{L}(\psi_\tau^\perp) \neq 0 \) by Proposition 1.10. This contradicts that \( U_p \equiv \psi(p) \mod m_A \cdot T^\text{split}_\rho \) since the image of \( \chi_e(\text{Frob}_p) \) under (32) is \( U_p \). Finally, this shows that \( U_p(F) - U_p(\Theta_\varphi) \not\equiv 0 \mod X^r \).

**Theorem 4.7.** Assume that \( \mathcal{L}(\psi_\tau^\perp) \neq 0 \). Then \( T^\text{split}[U_p] = A_\psi \times \mathcal{O}_p \left( \mathcal{O}_p^\perp \times \mathcal{O}_p[\epsilon]/(X^r) A_\psi^r \right) \).

(i) If moreover \( 0 \neq \mathcal{L}(\psi_\tau^\perp) = -\mathcal{L}(\psi^\perp) \), then \( T^\text{split} = \left\{ (a,b,c) \in A_\psi \times \mathcal{O}_p A_\psi^r \mid (\mathcal{L}(\psi_\tau^\perp) + \mathcal{L}(\psi^\perp)) b'(0) = \mathcal{L}(\psi_\tau^\perp) a'(0) + \mathcal{L}(\psi^\perp) c'(0) \right\} . \)

(ii) If moreover \( \mathcal{L}(\psi_\tau^\perp) = 0 \), then \( e = 1, r \geq 3 \) and there exists \( \xi \in \mathcal{O}_p^\times \) such that \( T^\text{split} = \left\{ (a,b,c) \in A_\psi \times \mathcal{O}_p \left( A \times \mathcal{O}_p[\epsilon]/(X^r) A_\psi^r \right) \mid (b-c)^{(r-1)}(0) = \xi \cdot (a-b)^{(r)}(0) \right\} . \)

(iii) If moreover \( \mathcal{L}(\psi^\perp) = -\mathcal{L}(\psi_\tau^\perp) \), then \( e \geq 2 = r \) and there exists \( \xi \in \mathcal{O}_p^\times \) such that \( T^\text{split} = \left\{ (a,b,c) \in A_\psi \times \mathcal{O}_p \left( R^\perp_{\mathfrak{m}_p(X)/(X^r)} A_\psi^r \right) \mid b'(0) = \xi \cdot (a-c)^{(r)}(0) \right\} . \)

**Proof.** According to Propositions 3.2 and 4.4 and 4.5 one knows that \( T^\text{split} \) is a \( A \)-sub-algebra of the amalgamated product \( A = A_\psi \times \mathcal{O}_p \left( R^\perp_{\mathfrak{m}_p(X)/(X^r)} A_\psi^r \right) \) surjecting to each two amongst the three factors. Hence \( (X^2,0,0) \in T^\text{split} \) and

\[
A_\psi \times \mathcal{O}_p[\epsilon] \left( R^\perp_{\mathfrak{m}_p[X]/(X^r)} A_\psi^r \right) \subseteq T^\text{split}.
\]

Since \( T^\text{split} \not\supseteq T^\text{split}[U_p] \subseteq A \) one deduces that \( m_{T^\text{split}} \) is given by the kernel of a linear form on the amalgamated product \( m_A \) taking valued in \( \mathcal{O}_p \), which we will determine as precisely as possible in each case. Since we know by Theorem 3.3(1) that \( U_p \not\in T^\text{split} \), we deduce that the above linear form is non-zero and that \( T^\text{split}[U_p] = A \), proving the first part of the theorem.

Let use not write an equation for \( m_{T^\text{split}} \) in each case.
(i) One sees exactly as in [4] that $m_{\tau_{\text{split}}} \supset (m_A^2)^3$. Moreover the image of $m_{\tau_{\text{split}}}$ in $(m_A/m_A^2)^3$ is necessarily a plane (as it surjects onto any two amongst the three factors). The precise equation follows from [17] and Proposition 3.2 in view of (20).

(ii) In this case $\mathcal{L}(\psi^\tau) = 0 \neq \mathcal{L}(\psi)$, hence $e = 1$ and $m_{\tau_{\text{split}}}$ contains the elements $(X, X, X), \ (X^2, 0, 0)$ and $(0, X^r, 0)$, generating an ideal which has co-dimension 2 in the $\mathbb{Q}_p$-vector space of $m_A$. Again the surjectivity onto each two amongst the three factors shows that $m_{\tau_{\text{split}}}$ has co-dimension 1 in $m_A$, hence satisfies a linear equation of the desired form.

(iii) We leave its proof, which is very similar to the above case, to the interested reader. □

**Theorem 4.8.** One has $C^0 = (X)$ if, and only if, $\mathcal{L}(\psi) \neq 0$. More precisely

(i) If $\mathcal{L}(\psi) \cdot \mathcal{L}(\psi^\tau) = 0$, then $T = A_\psi \times_{\mathbb{Q}_p^2} \mathbb{R}^k \times_{\mathbb{Q}_p^2} \mathbb{R}^k \times_{\mathbb{Q}_p^2} A_\psi$.

(ii) If $\mathcal{L}(\psi^\tau) = 0$, then $T = A_\psi \times_{\mathbb{Q}_p^2} A[Z] / (Z^2 - X^r) \times_{\mathbb{Q}_p^2} \mathbb{R}^k \times_{\mathbb{Q}_p^2} A_\psi$ where $r \geq 2$ and the projection $A[Z] / (Z^2 - X^r) \to \mathbb{R}_p[X] / (X^{r-1})$ is modulo the non-principal ideal $(Z, X^{r-1})$.

Proof. The claim about the congruence ideal would follow from (i) and (ii). Part (i) follows directly from Theorems 4.7(i)(iii) and 3.14 so it remains to show (ii).

By Theorem 4.7(ii) it suffices to show that $(0, Z, 0) \in T$. Since $\mathcal{L}(\psi) + \mathcal{L}(\psi^\tau) = 0$ in this case, we have $R^k = \Lambda$. Recall the involution $\iota$ of $\tilde{R}^k$ fixing $R^k = \Lambda$ and sending $Z$ to $-Z$.

For all $\gamma \in \Gamma_K$ one has

\[ \iota \circ \text{tr}(\tilde{\rho}_R^\gamma) = \text{tr}(\tilde{\rho}_R^\gamma) = \text{tr}(\tilde{\rho}_R^\gamma) = -\text{tr}(\tilde{\rho}_R^\gamma), \]

hence $\text{tr}(\rho_R^\gamma) \in Z \cdot \Lambda$. Finally, Proposition 3.13 implies that the $\Lambda$-sub-module of $\tilde{R}^k$ generated by $\text{tr}(\tilde{\rho}_R^\gamma)(\gamma \Gamma_K)$ contains $Z$. □

We conclude this subsection by determining the congruences ideal between $\mathcal{F}$ and $\Theta_\psi$.

**Corollary 4.9.** (i) If $\mathcal{L}(\psi) = 0$, then congruence ideal between $\mathcal{F}$ and $\Theta_\psi$ is given by $(Y) \subset \mathcal{F}^k = \mathbb{Q}_p[Y]$, if $r \geq 2$ is even, and by $(Y, Z) \subset \mathcal{F}^k = Z^k / (Z^2 - X^r)$, if $r \geq 3$ is odd.

(ii) Assume that $\mathcal{L}(\psi) = 0$. The congruence ideal between $\mathcal{F}$ and $\Theta_\psi$ is given by $(X^2) \subset \mathcal{F}^k = A$, if $r \geq 4$ is even, and by $(X^{r-1}, Z) \subset \mathcal{F}^k = A[Z] / (Z^2 - X^r)$, if $r \geq 3$ is odd.

4.3. **The Katz $p$-adic $L$-function at $s = 0$.** Given an ideal $\mathfrak{c}$ of $K$ relatively prime to $p$, Katz constructed a measure $\mu_\mathfrak{c}$ on the ray class group $\mathcal{C}_K(p^\infty \mathfrak{c})$ whose value on (the $p$-adic avatar of) a Hecke character $\varphi$ of conductor dividing $\mathfrak{c}$ and infinite type $(k_1, k_2) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\leq 0}$ is given by (see [5, Theorem 3.1] based on [28] and [13]):

\[ (k_1 - 1)! \left( \frac{\Omega_p}{\Omega_\infty} \right)^{k_1 - k_2} \left( \frac{\sqrt{D}}{2\pi} \right)^{k_2} (1 - \varphi^{-1}(\overline{p})) (1 - \varphi(p)^{p-1}) L_\varphi(\varphi^{-1}, 0), \]

where $\Omega_{\infty}$ (resp. $\Omega_p$) is a complex (resp. $p$-adic) period of the CM elliptic curve attached to $\mathfrak{c}^{-1}$. The above interpolation formula uniquely characterizes $\mu_\mathfrak{c}$ as these characters are Zariski
dense in \(\text{Spec} \mathcal{O}[[\mathcal{C}t_K(p^{\infty} \zeta)]]\). It does not apply however to finite order characters as their infinity type is \((0,0)\).

The Katz \(p\)-adic \(L\)-function of a finite order \textit{non-trivial anti-cyclotomic} character \(\varphi\) of conductor dividing \(\zeta\), is defined as the following continuous function on \(\mathbb{Z}_p^2\)

\[
L_p(\varphi, s_p, s_p) = \mu_c(\varphi \varepsilon_p^{s_p} \varepsilon_p^{s_p}),
\]

where \(\varepsilon_p\) is the character defined in \((21)\) and \(\varepsilon_p = \varepsilon_p^\tau\). In other terms, \(L_p(\varphi, s_p, s_p)\) is the \(p\)-adic Mellin transform of the push-forward of \(\mu_c\) by \(\mathcal{C}t_K(p^{\infty} \zeta)_{\text{tor}} \to \mathcal{W}_p \times \mathcal{W}_p\). Since the function \(s \mapsto z_p^s\) is equal to the analytic function \(\exp_p(s \cdot p^{-h} \log_p(1 + 2p))\) on \(p^h\mathbb{Z}_p\), and since \(\mathcal{C}t_K(p^{\infty})_{\text{tor}} \to \mathcal{W}_p \times \mathcal{W}_p \to p^{-h}\mathbb{Z}_p \times p^{-h}\mathbb{Z}_p\) consists of pairs with difference in \(\mathbb{Z}_p\), it follows that \(L_p(\varphi, s_p, s_p)\) is analytic on the subset of \((s_p, s_p) \in \mathbb{Z}_p^2\) such that \(s_p + s_p \in p^h\mathbb{Z}_p\), in particular it is locally analytic at \((0, 0)\).

Furthermore Katz constructed a 1-variable improved \(p\)-adic \(L\)-function \(L_p^*(\varphi, s)\) such that

\[
L_p(\varphi, s, 0) = (1 - (\varphi^{-1} \varepsilon_p^{-s})(\bar{p})) \cdot L_p^*(\varphi, s),
\]

(see \([28] \S 7.2\)) and proved a \(p\)-adic analogue of the Kronecker’s Second Limit Formula

\[
L_p^*(\varphi, 0) = -(1 - \varphi(p)^{-1}) \cdot \log_p(u_\varphi).
\]

(see \([28] \text{Cor.10.2.9, 27 Thm.1.5.1}\)\), where \(u_\varphi \in (\mathcal{O}_H[\zeta^{-1}] \otimes \overline{Q})[\varphi] = (\mathcal{O}_H \otimes \overline{Q})[\varphi]\) is a specific unit, defined using Robert units (see \([27] (1.5.5)\)\).

\textit{Remark 4.10.} Assume the Heegner hypothesis for the ideal \(\zeta\), \textit{i.e.,} \(\mathcal{O}_K/\mathcal{I} \mathcal{O}_K \cong \mathbb{Z}/C\mathbb{Z}\) and that \(D \neq 3, 4\). Let \(Y_1(C)/\mathbb{Q}\) be the open modular curve of level \(C\), \(\zeta\) be a \(C\)-th root of the unity and \(g_\zeta(q) = q^{-\zeta} \prod_{n=0}^{1-q^n}(1-q^n \zeta)(1-q^n \zeta^{-1}) \in \mathcal{O}(Y_1(C))^\times\) be the Siegel unit associated to \(\zeta\). The theory of Complex Multiplication implies that the elliptic curve \((\mathbb{C}/\zeta^{-1}, C^{-1})\) defines a point of \(Y_1(C)\) over the ray class field \(K_\zeta\) of conductor \(\zeta\) of \(K\). The evaluation of the Siegel units \(g_\zeta(q)\) at this point is an elliptic unit \(u_{\zeta, \zeta} \in \mathcal{O}_{K_\zeta}^\times\), which can be used to define \(u_\varphi\).

Suppose that \(\varphi(p) = 1\). The Euler factor \((1 - \varphi(p))\) in \((34)\) vanishes yielding a trivial zero

\[
L_p(\varphi, 0, 0) = 0.
\]

(37)

Note that since \(\log_p(u_\varphi) \neq 0\) by the Baker-Brumer Theorem, a trivial zeros of the Katz \(p\)-adic \(L\)-function at \((0, 0)\) can occur only in this case.

\textit{Remark 4.11.} When \(\varphi\) is trivial, a case considered in \([6]\) which we have excluded from the present paper, then one has \(L_p(1, 0, 0) = \frac{1}{2}(1 - p^{-1}) \log_p(u_p) \neq 0\), where \(u_p\) is a \(\bar{p}\)-unit of \(K\) of minimal valuation. Note that there is no improved \(p\)-adic \(L\)-function is that case.

A first insight into what should be the leading term of \(L_p(\varphi, s_p, s_p)\) at the trivial zero point \(L_p(\varphi, 0, 0) = 0\), is obtained by differentiating \((35)\)

\[
\frac{dL_p(\varphi)}{ds_p}(0, 0) = -\varphi(\bar{p}) \eta_p(\text{Frob}_\bar{p}) \cdot L_p^*(\varphi, 0) = \mathcal{H}(1) \cdot L_p^*(\varphi, 0).
\]

(38)
To gain further intuition about the linear term, we consider the Katz anti-cyclotomic $p$-adic $L$-function $L_p^-(\varphi, s) = L_p(\varphi, s, -s)$.

Since $\varphi$ is anti-cyclotomic of finite order, there exists a finite order Hecke character $\psi$ which can be chosen to have conductor $c_\psi$ relatively prime to $p$, such that $\varphi = \psi_\omega = \psi/\psi^\tau$.

Let $\zeta_{\psi, \omega}^{-} \in \Lambda_p$ be the formal power series corresponding to the the push-forward of $\mu_\epsilon$ by

$$\mathcal{C}_\ell (p^{\infty} \zeta_{\psi}) \xrightarrow{z \mapsto z^\tau} \mathcal{C}_\ell (p^{\infty} \epsilon_{\psi}) \rightarrow \mathcal{C}_\ell (p^{\infty} \epsilon_{\psi}) \xrightarrow{\pi} \Lambda_p^\times,$$

where $\pi$ is defined in 3.1. By definition $(\epsilon_{\psi}^{k_1}, \epsilon_{p}^{l_1}) = (\mu_{c_\psi}(\psi, \epsilon_{p}^{\epsilon_\omega} \epsilon_\omega)) = L_p^- (\psi, s)$ is analytic in $s \in \mathbb{Z}_p$. The vanishing of $L_p^- (\psi, 0)$ implies $\zeta_{\psi, \omega}^{-} \in \ker (\mathcal{O}[\mathcal{W}_p] \xrightarrow{[w_{\psi}] = [1]} \mathcal{O})$, hence the image of $\zeta_{\psi, \omega}^{-}$ under the localization morphism $\mathcal{O}[\mathcal{W}_p] \rightarrow \Lambda$ lands in $(X)$.

**Theorem 4.12 (Hida-Tilouine).** The anti-cyclotomic $p$-adic $L$-function $\zeta_{\psi, \omega}^{-}$ divides the generator of the congruence ideal $\mathcal{C}_\psi^0$ in $\Lambda$.

**Proof.** Recall from 3.1 the one variable Iwasawa algebra $\Lambda_p = \mathcal{O}[\mathcal{W}_p]$ and the Hecke character $\lambda_{\psi, k}$ corresponding to the classical theta series $\theta_{\psi, k}$ which is the specialization in weight $k \in \mathbb{Z}_{\geq 1}$ of the CM Hida family $\Theta_{\psi}$.

Let $H_\psi \in \Lambda_p$ be a generator of the congruence ideal attached to $\Theta_{\psi}$ introduced by Hida and Tilouine in [20] (6.9)]. Given a Hecke finite order character $\phi$ of $K$ having a prime to $p$ conductor $c_\phi$ and taking values in $\mathcal{O}$, Hida constructed in [25] Thm.I an element $\mathcal{O}$ of the fraction field of $\Lambda_p \mathcal{W}_p \Lambda_p$ such that $H_\psi : \mathcal{O} \rightarrow \Lambda_p \mathcal{W}_p \Lambda_p$, satisfying for all $k > l \geq 2$ the interpolation

$$L_p^{-}(\phi, \omega) \approx \frac{D_p (1 + \frac{k_1 l_1}{2}, \theta_{\psi, k}, \theta_{\lambda_{\psi, k}})}{(\theta_{\psi, k}, \theta_{\lambda_{\psi, k}})} \approx \frac{L(0, \lambda_{\phi, \omega} \lambda_{\psi, \omega}^{-1}) L(0, \lambda_{\phi, \omega} \lambda_{\psi, \omega}^{-1})}{L(0, \lambda_{\psi, \omega}^{-1}) L(0, \lambda_{\psi, \omega}^{-1})},$$

up to a factor made explicit in [25] Thm.I, where $D_p$ is the Rankin-Selberg convolution without Euler factors at $p$ and $(\theta_{\psi, k}, \theta_{\lambda_{\psi, k}})$ is the Petersson inner product.

Let $L_1$ (resp. $L_2, L_3$) be the unique element of $\Lambda_p \mathcal{W}_p \Lambda_p$ whose specialization at $(\epsilon_{p}^{k_1 - 1}, \epsilon_{p}^{l_1 - 1})$ is given by the value $\mu_{c_\psi}(\lambda_{\psi, k} \cdot (\lambda_{\phi, \omega}))$ (resp. $\mu_{c_\psi}(\lambda_{\psi, k} \cdot (\lambda_{\phi, \omega}^{-1})), \mu_p(\lambda_{\psi, k} \cdot (\lambda_{\phi, \omega}^{-1}))$) of the Katz $p$-adic measure. Note that the measure used in [26] differs from $\mu_\epsilon$ by the involution $g \mapsto g^{-1}$ of $\mathcal{C}_\ell (p^{\infty} c)$. Note also that by definition $L_\psi^{-}$ and $H_\psi$ are elements of the embedding of $\Lambda_p$ in $\mathcal{W}_p \Lambda_p$ via the first coordinate.

Let $\Psi \in \Lambda_p \mathcal{W}_p \Lambda_p$ be the Euler product defined in [20] p.249 bottom]. By comparing their respective specializations at each $k > l \geq 2$, Hida and Tilouine showed in [26] Thm.8.1 that the two elements $H_\psi : \mathcal{O}$ and $\frac{\Psi L_1 L_2 H_\psi}{L_3}$ differ by a unit, i.e. their quotient belongs to $(\Lambda_p \mathcal{W}_p \Lambda_p)^{1/p^\times}$. The divisibility $L_\psi^{-} \mid H_\psi$ in $(\Lambda_p \mathcal{W}_p \Lambda_p)^{1/p}$ (hence in $\Lambda_p^{1/p}$) follows from [26] Thm.8.2 where it is shown that $L_1, L_2$ and $\Psi$ are all relatively prime to $L_\psi^{-}$.

The ideal $\mathcal{C}_\psi^0$, resp. $(\zeta_{\psi, \omega}^{-})$, of $\Lambda$ is the localization of $(H_\psi)$, resp. $L_\psi^{-}$ at the height one prime ideal $(X)$ of $\Lambda_p$ corresponding to $f$, hence the theorem.
Alternatively, as we are only interested at establishing this divisibility in $\Lambda$ (i.e. after localizing at $(X)$), we can instead argue as follows. Choosing $\phi$ such that $\phi(\overline{p}) \neq 1 + \phi(p)$, the formulas \[(41)\] and \[(42)\] imply the non-vanishing of $L_1$ and of $L_2$ when evaluated at $k = l = 1$, and the same is true for $\Psi$ by the explicit formula defining it.

The description of the congruence ideal in Theorem 4.13 has the following consequence.

**Corollary 4.13.** Assume that $\mathcal{L}(\varphi) \neq 0$. Then $C^0_\psi = (\zeta_{\psi\cdot}) = (X)$, in particular, the order of vanishing of the anti-cyclotomic Katz $p$-adic $L$-functions $L_p^*(\varphi, s)$ at $s = 0$ is 1.

This relates conjecture of Hida on the $p$-adic adjoint $L$-function of the CM family $\Theta_\psi$ (see [26, Conjecture, p.192]) to the well established Four Exponentials Conjecture in Transcendence Theory, and shows that it holds for more than half of the characters, i.e., for at least one in each couple $\{\psi, \psi^\tau\}$.

Combining Corollary 4.13 with (38) leads us to propose the following formula:

\[(40)\]

\[L_p(\varphi, s_p, s_p^\prime) = \mathcal{L}(1) \cdot s_p + (\mathcal{L}(\varphi) - \mathcal{L}(1)) \cdot s_p^\prime \cdot L_p^*(\varphi, 0) + \text{higher order terms.}\]

or equivalently, using Katz cyclotomic $p$-adic $L$-function $L_p(\varphi, s) = L_p(\varphi, s, s)$, the formula:

\[(41)\]

\[L_p(\varphi, s) \cong \frac{(1 - p^{-1})}{\text{ord}_\psi(\tau(u_{\varphi}))} \left( \frac{\log_p(u_{\varphi})}{\log_p(u_{\varphi}^\tau)} \right) \cdot s + O(s^2).\]

**Remark 4.14.** Benois has formulated (see [1]) a very precise conjecture on trivial zeros in a generality that covers the case of rank 2 Artin motives which are not critical in the sense of Deligne. His definition of a cyclotomic $\mathcal{L}$-invariant depends on a choice of a regular sub-module which, in the case of a Galois representation which is locally scalar at $p$, can be arbitrarily chosen. We expect however that there should exist a choice for which (41) concords with his conjecture. Let us also mention a recent work [5] of Büyükkodu and Sakamoto on the leading term using methods completely different from ours.

For the rest of this section we assume that $\varphi = \psi_e$ is quadratic and we will check the compatibility between (41) and the Greenberg-Ferrero formula [16]. Denote by $F = \mathbb{Q}(\sqrt{d})$ the real quadratic subfield of $H$ and by $K' \neq K$ its other imaginary quadratic subfield. Recall Gross’s factorization of the Katz cyclotomic $p$-adic $L$-function as product of two Kubota-Leopoldt $p$-adic $L$-functions (see [19], and also [18] for an arbitrary conductor)

\[(42)\]

\[L_p(\varphi, s) = L_p(\epsilon_{K'}, \omega_p, s)L_p(\epsilon_F, 1 - s),\]

where $\omega_p$ is the Teichmüller character. Then $L_p(\epsilon_{K'}, \omega_p, s)$ has a trivial zero at $s = 0$ and

\[L_p(\epsilon_{K'}, \omega_p, 0) = -\mathcal{L}(\epsilon_{K'})L(\epsilon_{K'}, 0),\]

where $\mathcal{L}(\epsilon_{K'})$ is the cyclotomic $\mathcal{L}$-invariant of the odd Dirichlet character $\epsilon_{K'}$ (see [3] (15))], which turns out to be equal to $\mathcal{L}(\varphi)$. By the Class Number Formula $L(\epsilon_{K'}, 0) = \frac{h_{K'}}{|\mathcal{O}_{K'}/(1)|}. $
On the other hand, Leopoldt’s formula yields that
\[ L_p(\epsilon_F, 1) = -(1 - p^{-1}) \sum_{a=1}^d \epsilon_F(a) \log_p(1 - \zeta_d^a). \]
Since \( \sum_{a=1}^d \epsilon_F(a) \log_p(1 - \zeta_d^a) \in O_F^\times[\epsilon_F] \subset O_H^\times[\varphi] \) it follows that (41) holds, at least up to a multiplication by an element of \( Q^\times \), in this case.

5. An \( R = T \) modularity lifting theorem for non-Gorenstein local rings

The goal of this section is to introduce a universal ring \( R^\triangle \) representing deformations of \( \rho \) which are “generically ordinary”, and to show that \( R^\triangle_{\text{red}} \cong T \).

5.1. Ordinary framed deformations of a 2-dimensional trivial representation. The deformation functor \( D_{\text{loc}}^\triangle \) associating to \( A \) in \( \mathcal{C} \) the set of framed deformations \( \rho_A : \Gamma_{Q_p} \to \text{GL}_2(A) \) of \( 12 \) is representable by \( R_{\text{loc}}^\triangle \) together with \( \rho^\triangle : \Gamma_{Q_p} \to \text{GL}_2(R_{\text{loc}}^\triangle) \).

Recall that \( \rho_A \in D_{\text{loc}}^\triangle(A) \) is ordinary if and only if, there exists a \( \Gamma_{Q_p} \)-stable direct factor \( V_A' \subset V_A = A^2 \) of rank 1 over \( A \) and such that \( \Gamma_{Q_p} \) acts on \( V_A'' = V_A/V_A' \cong A \) by an unramified character \( \chi_A \). However “being ordinary” is not a deformation condition on \( D_{\text{loc}}^\triangle \), since it does not satisfy the Schlessinger Criterion (see for example [17, §3.1]).

Recall that the for any \( \overline{Q}_p \)-scheme \( S \), an \( S \)-point of \( \mathbb{P}^1 \) is an invertible \( O_S \)-sub-module \( \text{Fil}_S \subset \mathcal{O}_S^2 \) which is locally a direct summand. We introduce a variant, when the residual characteristic is 0, of a functor defined in [17, §3.1].

**Proposition 5.1.** Let \( D_{\text{loc}}^\triangle \) be the subfunctor of \( \mathbb{P}^1 \times_{\overline{Q}_p} R_{\text{loc}}^{\triangle}[U] \) whose \( A \)-points correspond to a filtration \( \text{Fil}_A \in \mathbb{P}^1(A) \) and a morphism \( \pi_A : R_{\text{loc}}^{\triangle}[U] \to A \), such that \( \text{Fil}_A \) is \( \Gamma_{Q_p} \)-stable for the action on \( V_A \) defined via \( R_{\text{loc}}^{\triangle} \to R_{\text{loc}}^{\triangle}[U] \overset{\pi_A}{\longrightarrow} A \) and moreover \( V_A/\text{Fil}_A \) is unramified with \( \text{Frob}_p \) acting by \( \phi_A = \pi_A(1 + U) \in A^\times \).

The functor \( D_{\text{loc}}^\triangle \) is representable by a closed subscheme \( \mathcal{X} \subset \mathbb{P}^1 \times_{\overline{Q}_p} R_{\text{loc}}^{\triangle}[U] \).

**Proof.** See [17, Lemma 3.1.1].

Let \( \text{Spec} R_{\text{loc}}^\triangle \subset \text{Spec} R_{\text{loc}}^{\triangle}[U] \) be the scheme theoretical image of \( \mathcal{X} \) under the second projection \( \text{pr}_2 : \mathbb{P}^1 \times_{\overline{Q}_p} R_{\text{loc}}^{\triangle}[U] \to \text{Spec} R_{\text{loc}}^{\triangle}[U] \) (i.e., \( \text{Spec} R_{\text{loc}}^\triangle \) is the Zariski closure of \( \text{pr}_2(\mathcal{X}) \)). Since \( \mathbb{P}^1 \) is proper (hence universally closed) over \( \overline{Q}_p \) and \( \mathcal{X} \) is a closed subscheme of \( \mathbb{P}^1 \times_{\overline{Q}_p} R_{\text{loc}}^{\triangle}[U] \), the natural morphism \( \text{pr}_2 : \mathcal{X} \to \text{Spec} R_{\text{loc}}^{\triangle} \) is surjective.

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\text{pr}_2} & \mathbb{P}^1 \times_{\overline{Q}_p} R_{\text{loc}}^{\triangle}[U] \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\
\text{Spec}(R_{\text{loc}}^\triangle) & \xrightarrow{\phi} & \text{Spec}(R_{\text{loc}}^{\triangle}[U]).
\end{array}
\]

We denote by \( \phi \in (R_{\text{loc}}^\triangle)^\times \) the image of \((1+U)\) under the canonical surjection \( R_{\text{loc}}^{\triangle}[U] \to R_{\text{loc}}^\triangle \).
5.2. Generically ordinary deformations of $\rho_f$. Let $(e_1, e_2)$ be a basis of $\mathbb{Q}_p^2$ where $\rho_f|_{\Gamma_K} = \begin{pmatrix} \psi & 0 \\ 0 & \psi^{-1} \end{pmatrix}$. Let $\rho_{\text{univ}} : \Gamma_Q \to \text{GL}_2(\mathbb{Q}_p^\text{univ})$ be the universal deformation of $\rho_f$.

Let $\mathcal{R}^{\triangle} = \mathcal{R}^{\triangle}_{\text{loc}} \otimes_{\mathcal{R}^{\text{univ}}_{\text{loc}}} \mathcal{R}^{\text{univ}}_{\rho_f}$ and $\rho_{\mathcal{R}} : \Gamma_Q \to \text{GL}_2(\mathcal{R}^{\triangle})$ be the corresponding universal deformation, where the natural morphism $\mathcal{R}^{\triangle}_{\text{loc}} \to \mathcal{R}^{\text{univ}}_{\rho_f}$ arises from $(\rho_{\text{univ}}|_{\Gamma_{q_p}}) \otimes \psi^{-1}$.

We show in the following lemma that the schematic points of $\text{Spec} \mathcal{R}^{\triangle}$ correspond to ordinary representations.

**Lemma 5.2.** Let $x : \mathcal{R}^{\triangle} \to L$ be a point over a field $L$. Then the corresponding representation $\rho_x : \Gamma_Q \to \text{GL}_2(L)$ has an unramified rank 1 $\Gamma_{q_p}$-quotient on which $\text{Frob}_p$ acts by $x(\phi \otimes 1) \in L^\times$.

**Proof.** By definition of $\mathcal{R}^{\triangle}$, $x$ yields a point $y : \mathcal{R}^{\triangle}_{\text{loc}} \to L$ such that $y(\phi) = x(\phi \otimes 1) \in L^\times$. Since $\text{pr}_2 : \mathfrak{X} \to \text{Spec} \mathcal{R}^{\triangle}_{\text{loc}}$ is surjective, we can lift $y$ to $y' \in \mathfrak{X}(L')$, with $L'$ a field containing $L$. By Proposition 5.1, $(\rho_x \otimes L')|_{\Gamma_{q_p}}$ stabilizes a line $\text{Fil}_{L'} \subset V_{L'}$ and acts on the quotient $V_{L'}/\text{Fil}_{L'}$ by an unramified character sending $\text{Frob}_p$ to $\text{pr}_2(y')(\phi) \in (L')^\times$.

The commutativity of the diagram (44) implies that $\text{pr}_2(y')(\phi) = y(\phi) \in L$, and therefore $\text{Fil}_{L'}$ has an $L$-rational basis, and $\text{Frob}_p$ acts on the quotient by $x(\phi \otimes 1)$ as requested. \qed

5.3. Modularity in the non-Gorenstein case. Recall that $\mathcal{T}$ denotes the completed local $A$-algebra of $\mathfrak{e}$ at $f$. The restriction of the deformation $\rho_{\mathcal{T}} : \Gamma_Q \to \text{GL}_2(\mathcal{T})$ to $\Gamma_{q_p}$ yields a natural morphism $\mathcal{R}^{\triangle}_{\text{loc}}[U] \to \mathcal{T}$ sending $\psi(p)(1+U)$ on $U_p$. Moreover, $\rho_{\mathcal{T}} \otimes Q(\mathcal{T})$ is ordinary at $p$, yielding a $Q(\mathcal{T})$-point of $\mathfrak{X}$, hence, in view of (44), we have a following commutative diagram

$$
\begin{array}{c}
\mathcal{R}^{\triangle}_{\text{loc}}[U] \\
\downarrow \quad \downarrow \\
\mathcal{T} \\
\downarrow \quad \downarrow \\
Q(\mathcal{T})
\end{array}
$$

Since $\mathcal{T}$ is reduced, the above diagram can be commutatively completed with a morphism $\mathcal{R}^{\triangle}_{\text{loc}} \to \mathcal{T}$. Finally, by (23) there exists a homomorphism $\mathcal{R}^{\text{univ}}_{\rho_f} \to \mathcal{T}$ and since $\mathcal{T}$ is topologically generated over $A$ by the image of $\text{tr}(\rho_{\text{univ}})$ and by $U_p$, we obtain a natural surjective $A$-algebra homomorphism

(44)

$$
\mathcal{R}^{\triangle} \to \mathcal{T}.
$$

**Theorem 5.3.** The morphism (44) induces an isomorphism $\mathcal{R}^{\triangle}_{\text{red}} \cong \mathcal{T}$, where $\mathcal{R}^{\triangle}_{\text{red}}$ is the nilreduction of $\mathcal{R}^{\triangle}$. In particular $\mathcal{R}^{\triangle}$ is equidimensional of dimension 1.

**Proof.** It suffices to show that for every minimal prime ideal $q$ of $\mathcal{R}^{\triangle}$ there exists a surjective homomorphism $\mathcal{T} \to \mathcal{R}^{\triangle}/q$ of $\mathcal{R}^{\triangle}$-algebras. In fact, it will then follow that the kernel of $\mathcal{R}^{\triangle} \to \mathcal{T}$ is contained in the nilradical of $\mathcal{R}^{\triangle}$ (which is given by the intersection of its primes). Since $\mathcal{T}$ is reduced, the desired isomorphism $\mathcal{R}^{\triangle}_{\text{red}} \cong \mathcal{T}$ would follow.
To prove the above claim, we let $\mathcal{A} = \mathcal{R}^\wedge / q \neq \overline{\mathbb{Q}}_p$ and consider the push-forward $\rho_{\mathcal{A}} : \Gamma_\mathcal{Q} \to \text{GL}_2(\mathcal{A})$ of $\rho_{\mathcal{R}}^\wedge$ along $\mathcal{R}^\wedge \to \mathcal{A}$. Since $Q(\mathcal{A})$ is a field, Lemma 5.2 implies that $\rho_{\mathcal{A}} \otimes Q(\mathcal{A})$ has an unramified free rank 1 $\Gamma_\mathcal{Q}_p$-quotient on which $\text{Frob}_p$ acts by the image of $\phi \otimes 1$ in $\mathcal{A}^\wedge$. Note that $\rho_{\mathcal{A}}$ need not be $p$-ordinary a priori (a posteriori this will be the case when $\mathcal{A}$ is a discrete valuation ring). One distinguishes two cases:

- $\rho_{\mathcal{A}^{\text{Gal}}_K}$ is reducible. Exactly as in the proof of Proposition 3.2 one shows that $\rho_{\mathcal{A}} \simeq \text{Ind}^G_{K} \psi_{\mathcal{A}}$, where $\psi_{\mathcal{A}} : \Gamma_K \to \mathcal{A}^\wedge$ is a character lifting $\psi$. The $p$-ordinariness of $\rho_{\mathcal{A}} \otimes Q(\mathcal{A})$ implies that $\psi^{-1}{\psi_{\mathcal{A}}}$ is equal either to $\chi_p$ or to $\chi_p^\tau$. In both cases $\mathcal{T} \to \mathcal{A} = \Lambda$.

- $\rho_{\mathcal{A}^{\text{Gal}}_K}$ is irreducible. As in Proposition 3.6 one may construct a point of $\mathcal{D}_1(\mathcal{A})$, i.e. a homomorphism of integral domains $\mathcal{R}^1 \to \mathcal{A}$, which is necessarily injective as $\mathcal{R}^1$ is a discrete valuation ring. It follows from Theorem 5.14 that either $\mathcal{A} \simeq \mathcal{R}^1$ when $r$ is even, or $\mathcal{A} \simeq \hat{\mathcal{R}}^1$ when $r$ is odd, hence in all cases $\mathcal{T} \to \mathcal{T}^1 \to \mathcal{A}$. 

The following lemma in commutative algebra justifies the title of the section.

**Proposition 5.4.** The dimension of the $\overline{\mathbb{Q}}_p$-vector space $\mathcal{T}/(m_\mathcal{A} + m_\mathcal{T}^2)$ is 4.

The eigencurve $\mathcal{E}$ is not Gorenstein at $f$.

**Proof.** The 1-dimensional reduced local ring $\mathcal{T}$ is Gorenstein if, and only if, its quotient $\mathcal{T}/a\mathcal{T}$ by the regular element $a = (X, Y, Y, X, X)$ is Gorenstein, where $\mathcal{R}^1 = \overline{\mathbb{Q}}_p[Y^e]$, with $Y^e = X$. Since $\mathcal{T}/a\mathcal{T}$ is Artinian, this is equivalent to its socle $\mathcal{M} = \text{Hom}_\mathcal{Q}(\overline{\mathbb{Q}}_p, \mathcal{T}/a\mathcal{T})$ being 1-dimensional. Assume first that $\mathcal{L}(\psi_\mathcal{T}) \cdot \mathcal{L}(\psi_\mathcal{T}) \neq 0$. By Theorem 5.8 one has $\mathcal{T} = A \times_{\overline{\mathbb{Q}}_p} \mathcal{R}^1 \times_{\overline{\mathbb{Q}}_p} \mathcal{R}^1 \times_{\overline{\mathbb{Q}}_p} \Lambda$ hence $m_\mathcal{T}^2 = a \cdot m_\mathcal{T}$ and $\mathcal{M} = m_\mathcal{T}/a\mathcal{T}$. It follows that $\dim(\mathcal{M}) = \dim(m_\mathcal{T}/(a\mathcal{T} + m_\mathcal{T}^2)) = \dim(m_\mathcal{T}/m_\mathcal{T}^2) - 1 = \dim(m_\mathcal{T}/(m_\mathcal{A} + m_\mathcal{T}^2)) = 3 > 1$, hence $\mathcal{T}$ is not Gorenstein.

Assume next that $\mathcal{L}(\psi_\mathcal{T}) \cdot \mathcal{L}(\psi_\mathcal{T}) = 0$. Then $\mathcal{T} \simeq A \times_{\overline{\mathbb{Q}}_p} \mathcal{A}$, with $\mathcal{A} = \left( A[Z]/(Z^2 - X^r) \times_{\overline{\mathbb{Q}}_p[X]/(X^{r-1})} \Lambda \right)$. Since $m_\mathcal{A}/m_\mathcal{A}^2$ has $\overline{\mathbb{Q}}_p$-basis $((X, X), (Z, 0), (0, X^{r-1}))$, one deduces that $m_\mathcal{A}^2 = m_\mathcal{A}m_\mathcal{A}$ and $m_\mathcal{T}^2 = m_\mathcal{A}m_\mathcal{T}$. Again $\mathcal{M} = m_\mathcal{T}/m_\mathcal{A}1$ has dimension 3, hence $\mathcal{T}$ is not Gorenstein.

In both cases $m_\mathcal{T}/(m_\mathcal{A} + m_\mathcal{T}^2)$ is 3-dimensional, hence $\mathcal{T}/(m_\mathcal{A} + m_\mathcal{T}^2)$ is 4-dimensional. \qed

We close this paper by establishing Conjecture 4.1 from [12] about the generalized eigenspace $S_1^1(N)[f] = S_1^1(N)[m_\mathcal{T}]$ attached to $f$ inside the space of weight 1, level $N$, ordinary $p$-adic modular forms. To be precise, Darmon, Lauder and Rotger only consider the subspace $S_1^1(N)[m_\mathcal{T}^2]$. The subspace of classical forms $S_1(Np)[m_\mathcal{T}^2]$ is two-dimensional having $\{f, \theta_\psi\}$ as basis, and a supplement of this space is given by the space $S_1^1(N)[m_\mathcal{T}^2]_0$ of normalized generalized eigenforms, i.e. forms having first and $p$-th Fourier coefficients both 0.

**Corollary 5.5.** The $\overline{\mathbb{Q}}_p$-vector space $S_1^1(N)[m_\mathcal{T}^2]$ has dimensions 4. Moreover Conjecture 4.1 from [12] holds in the CM case, i.e. the two-dimensional vector space $S_1^1(N)[m_\mathcal{T}^2]_0$ is canonically isomorphic to $H^1(\mathbb{Q}, \text{ad}^0(\rho_f))$. 


Proof. The first claim is merely a translation of the first claim of Proposition 5.4 in view of the duality \[ (25) \]. It follows that \( \dim S^1_1(N)[m^2_T]_0 = 4 - 2 = 2 \) and the conjecture then follows from [3, Lemma 3.2] asserting that \( \dim H^1(Q, \text{ad}^0(\rho_f)) = 2 \) as well. \( \square \)

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