RIGIDITY OF GRAPHS OF GERMS AND HOMOMORPHISMS BETWEEN FULL GROUPS

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ABSTRACT. We study topological full groups of étale groupoids and show that they satisfy new rigidity phenomena of topological dynamical nature. If $G$ is a minimal groupoid of germs on the Cantor set, actions of the (alternating) full group of $G$ on compact spaces satisfy the following dichotomy: either there is a point such that no element has trivial germ at that point, or the action is induced from an action of a (reduced) power of the groupoid $G$. This dichotomy is a simultaneous generalisation of the fact that isomorphisms of full groups are implemented by isomorphisms of the underlying groupoids, and of the simplicity of the alternating full group. Using this result we obtain that, for a vast class of groupoids (defined in terms of the geometry of their Cayley graphs), not only isomorphisms but all embeddings between the full groups are induced from the groupoids in a suitable sense. We also show that various quantitative invariants of étale groupoids, such as the orbital growth and the complexity function, can be used to produce obstructions to the existence of embeddings. A key tool in the proofs is a characterisation of the subgroups of the full group whose conjugacy class does not accumulate on the trivial subgroup in the Chabauty topology. As another application, we provide the first examples of finitely generated groups that do not admit infinite Schreier graphs that grow uniformly subexponentially, but do admit co-amenable subgroups of infinite index.


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1. Introduction

Let \( \mathcal{G} \) be an \( \text{étale groupoid} \) whose unit space, denoted \( X \), is homeomorphic to the Cantor set. A fundamental example is given by the \( \text{groupoid of germs} \) associated to an action \( G \ltimes X \) of a countable group. We are interested in the \( \text{topological full group} \) of \( \mathcal{G} \), denoted \( F(\mathcal{G}) \) or \( F(\mathcal{G}; X) \) in the case of the groupoid of germs of a group action on a compact space. When \( \mathcal{G} \) is a \( \text{groupoid of germs} \) (i.e. when it is an \( \text{effective} \) \( \text{étale groupoid} \)), the group \( F(\mathcal{G}) \) is defined as the group of all homeomorphisms of \( X \) whose germs belong to \( \mathcal{G} \). See Definition 2.11 for a more general and more detailed definition, and Subsection 2.3 for preliminaries on \( \text{étale groupoids} \).

The notion of full group was introduced by Dye in the context of orbit equivalence of measure preserving actions [Dye59]. Topological full groups were introduced by Giordano, Putnam and Skau [GPS99] for minimal \( Z \)-actions on the Cantor set, and by Matui [Mat12, Mat15] for a wider class of \( \text{étale groupoids} \). Matui [Mat06, Mat15] has shown that for some classes of groupoids \( \mathcal{G} \) the derived subgroup \( D(\mathcal{G}) \) of \( F(\mathcal{G}) \) is simple, and in some cases finitely generated. For more general groupoids, Nekrashevych [Nek15a] defines a subgroup \( A(\mathcal{G}) \leq F(\mathcal{G}) \), that we will call the \( \text{alternating full group} \), which will play an important role in this paper (see Definition 2.13). It is shown in [Nek15a] that whenever \( \mathcal{G} \) is a \( \text{minimal} \) groupoid of germs on the Cantor set (i.e. every orbit is dense), the group \( A(\mathcal{G}) \) is simple; see also [MB16, Th. 1.2] for an equivalent result with a different definition of \( A(\mathcal{G}) \). Moreover the group \( A(\mathcal{G}) \) is finitely generated whenever \( \mathcal{G} \) is \textit{expansive} [Nek15a] (for instance, if \( \mathcal{G} \)
the groupoid of germs an expansive action of a finitely generated group). Topological full
groups have been considered in recent year to produce various new interesting examples of groups [Nek16a, JM13, JNdlS16, Mat15].

A fundamental result on topological full groups is Matui’s isomorphism theorem: if $G, H$ are minimal groupoids of germs with Cantor set unit space, and if there exists an isomorphism $\rho: F(G) \rightarrow F(H)$, then there exists an isomorphism of the étale groupoids $G$ and $H$ that induces $\rho$. This result was proven in [GPS99], for $\mathbb{Z}$-actions, and in [Mat15] in general. The same result holds for isomorphisms between alternating full groups [Nek15a]. This result, analogous to the work of Dye on measurable full groups [Dye59], is an example of a reconstruction result. It can also be deduced from a more general reconstruction theory developed by Rubin (see [Rub89]), which provides widely applicable tools to study isomorphisms between “large” groups of homeomorphisms.

The main goal of this paper is to show that actions of topological full groups on compact spaces enjoy an interesting rigidity property that encompasses the isomorphism theorem, and to discuss various consequences of it. Our approach is different from the above-mentioned reconstruction methods, and involves the study the dynamics of the conjugation action on the space of subgroups endowed with the Chabauty topology, combined with the language provided by the setting of étale groupoids.

1.1. A dichotomy for actions on compact spaces. In order to state our results we need to fix some terminology regarding induction of actions of groupoids to actions of their topological full groups.

In analogy with the case of groups, an étale groupoid $G$ can act. The natural objects on which it can act are pairs $(Y, q)$ consisting of a space $Y$ together with a surjective continuous map $q: Y \rightarrow X$; such a pair will be called a fibre space over $X$ (see Subsection 2.4). Every continuous action $G \curvearrowright (Y, q)$ on a compact fibre space over $X$ can be induced to an action of the full group $F(G)$ on $Y$ (as we shall explain in Section 6.1). The action $F(G) \curvearrowright Y$ obtained in this way factors onto the natural action $F(G) \curvearrowright X$ via the map $q$.

We need to consider a slightly more general way to induce actions of $F(G)$, as follows. Let $X^{(r)}$ be the compact space consisting of non-empty finite subsets of $X$ with cardinality at most $r$. The group $F(G)$ acts on $X^{(r)}$ and the germs of this action belong to a natural groupoid $G^{*r}$, that we call the $r$-th reduced power of $G$, that can be thought of as a relative of the usual cartesian power $G^r$ (we refer to Definition 6.5 for the detailed definition of $G^{*r}$). Hence, the group $F(G)$ is naturally a subgroup of $F(G^{*r})$. It follows that every continuous action of the reduced power $G^{*r}$ also induces a continuous action of the group $F(G)$.

We will say for short that a continuous action $F(G) \curvearrowright Y$ is induced from $G^{*r}$ if there exists a continuous surjective map $q: Y \rightarrow X^{(r)}$ and a continuous groupoid action $G^{*r} \curvearrowright (Y, q)$ which induces $F(G) \curvearrowright Y$.

Our first result provides a necessary and sufficient condition ensuring that an action of the alternating full group $\mathbb{A}(G)$ is induced from $G^{*r}$ for some $r \geq 1$. If $G \curvearrowright Y$ is a group action on a compact space, for every point $y \in Y$ we denote by $S_G^0(y)$ the germ-stabiliser, i.e. the subgroup consisting of elements that fix point-wise a neighbourhood of $y$. The following is a consequence of Corollary 6.13.
Theorem 1.1 (Germ-stabiliser rigidity for actions on compact spaces). Let $\mathcal{G}$ be a minimal groupoid of germs with Cantor set unit space $X$. Let $\mathbb{A}(\mathcal{G}) \curvearrowright Y$ be a non-trivial action by homeomorphisms on a compact space. Then:

(i) either there exists a point $y \in Y$ such that $S^0_{\mathbb{A}(\mathcal{G})}(y) = \{1\}$; or
(ii) up to removing a clopen subset of $Y$ consisting of global fixed points (possibly empty), the action is induced from $\mathcal{G}^{\ast r}$ for some $r \geq 1$; in particular, it factors onto the natural action $\mathbb{A}(\mathcal{G}) \curvearrowright X^{(r)}$.

For minimal actions, this statement simplifies as follows:

Corollary 1.2 (Case of minimal actions). A non-trivial minimal action of the group $\mathbb{A}(\mathcal{G})$ on a compact space is either topologically free or it is induced from $\mathcal{G}$.

Recall that an action of a countable group on a compact space is said to be topologically free if a dense set of points have trivial stabiliser. The first special case of Corollary 1.2 was established in [LBMB16, Th. 1.8] for the Thompson’s group $V$ (and for the Thompson’s group $T$).

Remark 1.3. (1) To simplify the discussion in the introduction, we have stated Theorem 1.1 for only actions of the group $\mathbb{A}(\mathcal{G})$. See Corollary 6.13 for a statement which analyses also actions of the group $\mathbb{F}(\mathcal{G})$, or of any subgroup of it that contains $\mathbb{A}(\mathcal{G})$ (with a slightly weaker conclusion).

(2) An action $\mathbb{A}(\mathcal{G}) \curvearrowright Y$ on a compact space can be equivalently thought of as a homomorphism $\mathbb{A}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$, where $\mathcal{H}$ is a groupoid of germs (namely the groupoid of germs of the action). Taking this point of view on Theorem 1.1, it turns out that the assumption that $\mathcal{H}$ is a groupoid of germs is irrelevant: we shall in fact prove a more general version of Theorem 1.1 where we allow $\mathcal{H}$ to be an arbitrary étale groupoid, see Theorem 6.9.

Formally, Theorem 1.1 is a generalisation of the simplicity of the group $\mathbb{A}(\mathcal{G})$. To see the connection, consider a proper normal subgroup $N \trianglelefteq \mathbb{A}(\mathcal{G})$, and apply Theorem 1.1 to the action $\mathbb{A}(\mathcal{G}) \curvearrowright Y = (\mathbb{A}(\mathcal{G})/N) \cup \{\infty\}$ on the one-point compactification of the quotient. It is easy to see that $S^0_{\mathbb{A}(\mathcal{G})}(y) = N$ for every $y \in Y$. Since $Y$ is countable, no subset of it can admit a surjective map to $X^{(r)}$, hence we are in case [i] and $N = \{1\}$. In fact, the proof of Theorem 1.1 is based on a generalisation of the simplicity of the group $\mathbb{A}(\mathcal{G})$. See also Example 6.12.

In addition, Theorem 1.1 recovers the isomorphism theorem. To explain the connection, let $\mathcal{G}_i, i = 1, 2$ be a minimal groupoid of germs on unit spaces $X_i$ homeomorphic to the Cantor set. If there is an isomorphism $p: \mathbb{A}(\mathcal{G}_1) \to \mathbb{A}(\mathcal{G}_2)$, we can apply Corollary 1.2 to the induced actions $\mathbb{A}(\mathcal{G}_1) \curvearrowright X_2$ and $\mathbb{A}(\mathcal{G}_2) \curvearrowright X_1$. These actions are not topologically free and the isomorphism theorem readily follows (see Examples 6.15 for details and for the case of isomorphisms between groups $\mathbb{F}(\mathcal{G}_i)$).

Therefore Theorem 1.1 unifies to a single more general phenomenon these two facts. In addition, it has various new consequences that we outline in the rest of the introduction.
1.2. **Embeddings between full groups.** It is natural to ask whether some sort of generalisation of Matui’s isomorphism theorem holds for homomorphisms between topological full groups. In other words, is it true that all homomorphisms $F(G) \to F(H)$ have to come from the underlying dynamics, in some sense to be made precise?

A moment of reflection shows that one cannot hope for a result of this type to hold in full generality: every action of the group $F(G)$ on the Cantor set gives rise to a homomorphism $\rho : F(G) \to F(H)$ where $H$ is any étale groupoid that contains the groupoid of germs of the action. As every countable group the group $F(G)$ admits a wild multitude of actions on the Cantor set, not all of which are related to the groupoid $G$ in a clear way. However the question becomes interesting if we require some restriction on the étale groupoids we are interested into. For example, when $G, H$ are both groupoids of germs of minimal $\mathbb{Z}$-actions, this problem is raised by Cornulier in [Cor14, Question (2f)].

Theorem 1.1 (along with the more general Theorem 6.9) provides a convenient tool to study homomorphisms. Roughly speaking, the reason is that if we are given a homomorphism $A(G) \to F(H)$, case [1] in Theorem 1.1 can be often be ruled out using geometric information on the groupoid $H$. The possibility of studying homomorphisms beyond the case of isomorphisms is an interesting difference with the respect to the reconstruction methods previously used in the context of full groups or in similar contexts.

We need again to fix some terminology in order to state this result. Let $H$ be an étale groupoid with compact unit space $Y$, that we assume to be *compactly generated* in the sense of Haefliger [Hae02] and Nekrashevych [Nek15b] (see Definition 2.8). This assumption allows to endow every fibre $s^{-1}(y) \subset H$ of the source map of $H$ with the structure of a graph $\tilde{\Gamma}_y(H, S)$, called the *Cayley graph* of $H$ based at $y$. This graph depends on the choice of a generating set $S$ of $G$, but its bi-Lipschitz equivalence type is independent of it (but may depend on the base-point $y \in Y$). A tightly related graph is the *orbital graph* $\Gamma_y(H, S)$, whose vertex set is the $H$-orbit of $y$. We refer to Subsection 2.9 for more details, but for now the reader can have in mind the following example: if $H$ is the groupoid of germs of an action $H \curvearrowright Y$ of a finitely generated group, then it is compactly generated, and for every $y \in Y$ the Cayley graph $\tilde{\Gamma}_y(H, S)$ is bi-Lipschitz equivalent to the Schreier graph of $H$ with respect to the subgroup $S_H^0(y)$ (also called the *graph of germs*, while the orbital graph is bi-Lipschitz equivalent to the Schreier graph with respect to the usual stabiliser $S_H(y)$). In particular, if the action is *free*, both graphs are quasi-isometric the Cayley graph of of the group $H$.

The following definition makes use of the notion of asymptotic dimension, a quasi-isometry invariant of a metric space introduced by Gromov [Gro93] (its definition is recalled in Section 7).

**Definition 1.4 (The class $\mathcal{E}$).** Let $H$ be a compactly generated étale groupoid with compact unit space $Y$. We say that $H$ belongs to the class $\mathcal{E}$ if at least one of the following conditions is satisfied for some finite generating set $S$ of bisections of $H$.

- For every $y \in Y$ the graph $\tilde{\Gamma}_y(H, S)$ has finite asymptotic dimension.
- For every $y \in Y$ the graph $\tilde{\Gamma}_y(H, S)$ has polynomially bounded growth.

**Examples 1.5.** The following classes of groupoids belong to the class $\mathcal{E}$.
(i) The groupoid of germs of actions of \( \mathbb{Z} \) and of \( \mathbb{Z}^d \), and more generally of any topologically free action of a finitely generated group with finite asymptotic dimension. A wide class of groups are known to have finite asymptotic dimension. See [BD08] for a survey.

(ii) The groupoids associated to one-sided shifts of finite type, and products of them. Their topological full groups have been studied by Matui in [Mat15, Mat16] (see Subsection 7.2). They include the family of Higman-Thompson’s groups and their higher-dimensional versions.

(iii) The groupoid of germs associated to the fragmentations of dihedral group actions, introduced by Nekrashevych, whose topological full group are torsion and have intermediate growth [Nek16a].

(iv) Groupoids associated to tilings and to quasi-crystals in euclidean space, see [BJS10, Kel97]. Their topological full group is considered in [Nek15a Section 6.3].

To state the next result, recall that every continuous action \( G \rtimes (Y, q) \) of an étale groupoid on a compact space gives rise to another étale groupoid, called the semi-direct product groupoid \( G \rtimes Y \), whose unit space is \( Y \) (its definition is recalled in Subsection 2.4). We say that a cocycle (i.e. a groupoid homomorphism) between étale groupoids \( b: G \to H \) is a spatial inclusion if it is injective in restriction to the unit space of \( G \).

The following theorem is a consequence of Theorem 7.5.

**Theorem 1.6 (Rigidity of embeddings).** Retain the same assumptions on \( G \) as above and further assume that it is compactly generated. Let \( H \) be an étale groupoid in the class \( \mathcal{E} \). The following are equivalent.

(i) There exists a non-trivial homomorphism \( \mathbb{A}(G) \to \mathbb{F}(H) \).

(ii) There exists an embedding \( \mathbb{F}(G) \to \mathbb{F}(H) \).

(iii) There exists an integer \( r \geq 1 \), an action \( G^{\times r} \rtimes (Z, q) \) on a compact fibre space over \( X^{(r)} \), and a continuous open spatial inclusion

\[
b: G^{\times r} \rtimes Z \to H.
\]

Moreover, for every non-trivial homomorphism \( \rho: \mathbb{A}(G) \to \mathbb{F}(H) \) there exists a spatial inclusion as in (iii) which naturally induces \( \rho \).

For a description of the homomorphisms “naturally induced by” a spatial inclusion \( b: G^{\times r} \rtimes Z \to H \), see Subsection 6.1.

See Corollary 7.8 for a more precise statement in the special case of topological full groups of minimal \( \mathbb{Z} \)-actions on the Cantor set, which gives an answer to [Cor14, Question (2f)]. See also Corollary 7.12 for a statement in the special case of groupoids associated to one-sided SFT’s, which applies in particular to homomorphisms between groups in the family of Higman-Thompson groups \( V_{n,r} \).

### 1.3. Quantitative invariants of groupoids.

Étale groupoids bear rich structure of geometric and dynamical nature, and various quantitative invariants can be associated to them. It is natural to study how these invariants are related to the behaviour of the topological full group.
The isomorphism theorem implies that every groupoid invariant can be used to distinguish topological full groups up to isomorphism. As a motivating application of Theorems 1.1 and 1.6, we strengthen this by showing that various quantitative invariants associated to étale groupoids produce obstructions to embeddings between the corresponding topological full groups. This confirms the natural intuition that a “more complicated” dynamical system produces a “larger” topological full group.

The first invariant that we consider is the orbital growth function, which measures the maximal growth of the orbital graphs $\Gamma_x(G, S)$. Given a graph $\Gamma$ and a vertex $v \in \Gamma$, we denote $b_\Gamma(n, v)$ the ball of radius $n$ around $\Gamma$, and $\overline{b}_\Gamma(n) = \sup_{v \in \Gamma} b(v, n)$. Let $G$ be a compactly generated étale groupoid over $X$ with finite generating set of bisections $S$. The orbital growth of $G$ is defined as the function $\beta_G(n, S) = \sup_{x \in X} b_{\Gamma_x(G, S)}(n, x)$. Given two functions $f, g: \mathbb{N} \to \mathbb{R}$ we write $f \preceq g$ if $f(n) \leq g(Cn)$ for some $C > 0$, and $f \sim g$ if $f \preceq g \preceq f$. The growth rate of $\beta_G$ does not depend on the choice of $S$ up to the equivalence $\sim$, and is denoted $\beta_G(n)$.

Another invariant that we consider is the complexity function. Let $G$ be a finitely generated group and let $G \curvearrowright X \subset A^G$ be a subshift over a finite alphabet, i.e. the translation action on a closed invariant subset of the shift. Let $S$ be a finite generating set of $G$, and let $B_{G,S}(n)$ be the ball of radius $n$ in the Cayley graph of $G$. The complexity function $p_X(n, S)$ is defined as the number of configurations $f: B_{G,S}(n) \to A$ that appear as restrictions $x|_{B_{G,S}(n)}$ of elements $x \in X$. The complexity function of subshifts over the group $\mathbb{Z}$ is a well-studied invariant in symbolic dynamics, see [CN10]. In this case, it is related to the topological entropy of the subshift via the formula $h_{\text{top}}(\mathbb{Z}, X) = \lim \frac{1}{n} \log p_X(n, \{ \pm 1 \})$.

The complexity function of subshift admits a very natural generalisation $\pi_G(n, S)$ that can be defined for every expansive étale groupoid $G$ on the Cantor set endowed with an expansive generating set $S$. A definition was given by Nekrashevych in [Nek16b]; we propose a slightly different one given in Section 2.10. Its growth type $\pi_G(n)$ doesn’t depend on $S$ (up to $\sim$), and it agrees with the previous definition if $G$ is the groupoid of germs of a subshift $G \curvearrowright X$ over a finitely generated groups (these facts are checked in Section 2.10).

A result relating the complexity to the topological full group was proven in [MB14]: if the complexity function of a subshift $\mathbb{Z} \curvearrowright X$ grows slower than $n^\alpha$ for some $\alpha < 2$, then the simple random walk on every finitely generated subgroups of the topological full group has trivial Poisson boundary. In [Nek15a], dynamical systems with linear growth of the repetitivity (which implies linear growth of the complexity) are used to construct étale groupoids whose topological full group has intermediate growth. These results motivate the question whether for more general dynamical systems, the growth and the complexity (or the positivity/ vanishing of entropy) constraints the behaviour of the full group, especially in relation to its subgroup structure, see [Nek16b].

The following theorem is proven in Section 8.

**Theorem 1.7** (Obstructions to embeddings). Retain the same assumptions on $G$ and further assume that it is compactly generated. Let $\mathcal{H}$ be a compactly generated étale groupoid. Then the group $\mathbb{F}(G)$ cannot embed into the group $\mathbb{F}(\mathcal{H})$ provided any of the following holds.
(i) The groupoid $\mathcal{H}$ is a groupoid of germs, and the orbital growth functions of $\mathcal{G}$ and $\mathcal{H}$ satisfy $\beta_\mathcal{G} \not\preceq \beta_\mathcal{H}$.

(ii) The groupoid $\mathcal{H}$ belongs to the class $\mathcal{E}$, both $\mathcal{G}, \mathcal{H}$ are expansive with Cantor set unit space, and their complexity functions satisfy $\pi_\mathcal{G} \not\preceq \pi_\mathcal{H}$.

(iii) The asymptotic dimension of every Cayley graph of $\mathcal{G}$ is strictly larger than the asymptotic dimension of every Cayley graph of $\mathcal{H}$.

It is worth pointing out the following special case of the non-embedding criterion (iii).

**Corollary 1.8.** Let $\mathbb{Z} \curvearrowright X$ and $\mathbb{Z} \curvearrowright Y$ be minimal subshifts whose topological entropies satisfy $h_{\text{top}}(\mathbb{Z}, X) > 0$ and $h_{\text{top}}(\mathbb{Z}, Y) = 0$. Then every homomorphism $\mathbb{F}(\mathbb{Z}, X) \to \mathbb{F}(\mathbb{Z}, Y)$ has abelian image.

Perhaps the simplest illustration of criteria (i) and (iii) is the following: assume that $\mathbb{Z}^d \curvearrowright X, \mathbb{Z}^\ell \curvearrowright Y$ are free actions on the Cantor set, with $\mathbb{Z}^d \curvearrowright X$ minimal. If $d > \ell$, then the group $\mathbb{F}(\mathbb{Z}^d; X)$ cannot embed into $\mathbb{F}(\mathbb{Z}^\ell; Y)$.

For an application of the criterion (iii) to the family of higher-dimensional Higman–Thompson groups introduced by Brin [Bri04], see Corollary 8.7.

### 1.4. Actions by bounded displacement

Let us now describe a more combinatorial rigidity property of the group $\mathbb{A}(\mathcal{G})$. Let $\Gamma$ be a graph of bounded degree. Its group of permutations of bounded displacement, or *wobbling group* $W(\Gamma)$ is the group of permutations $\sigma$ of the set of vertices of $\Gamma$, having the property that

$$\sup_{v \in \Gamma} d_\Gamma(v, \sigma v) < \infty.$$  

For example, let $G$ be a finitely generated group, and $H \leq G$ be a subgroup, and let $\Gamma$ be its *Schreier graph* with respect to a finite symmetric generating set of $G$. Then the action $G \curvearrowright G/H$ defines a homomorphism $G \to W(\Gamma)$ (studying actions by bounded displacement of finitely generated groups is essentially equivalent to the study of their Schreier graphs, but the wobbling group point of view has the advantage to make sense also for non-finitely generated groups).

It is easy to see that the natural action of the group $\mathbb{F}(\mathcal{G})$ on every $\mathcal{G}$-orbit defines a homomorphism $\mathbb{F}(\mathcal{G}) \to W(\Gamma_\mathcal{G}(\mathcal{G}, \mathcal{S}))$, which is injective if $\mathcal{G}$ is minimal. In many interesting cases the graphs $\Gamma_\mathcal{G}(\mathcal{G}, \mathcal{S})$ are very explicit and this allows to take a rather concrete point of view on the group $\mathbb{F}(\mathcal{G})$ (this point of view is used e.g. in [JM13, Nek16a]).

The following theorem says that the group $\mathbb{A}(\mathcal{G})$ cannot act by permutation of bounded displacements on any graph for which the growth function $b_\Delta$ grows slower than $\beta_\mathcal{G}(n)$. Recall that the group $\mathbb{A}(\mathcal{G})$ is finitely generated if and only if $\mathcal{G}$ is expansive [Nek15a].

**Theorem 1.9.** Let $\mathcal{G}$ be a compactly generated minimal groupoid of germs with Cantor set unit space. Let $\Delta$ be a bounded degree graph such that $b_\Delta(n) \not\preceq \beta_\mathcal{G}(n)$. Then there is no non-trivial homomorphism $\mathbb{A}(\mathcal{G}) \to W(\Delta)$.

When $\mathcal{G}$ is expansive, (so that $\mathbb{A}(\mathcal{G})$ is finitely generated), it follows that every non-trivial Schreier graph $\Delta$ of $\mathbb{A}(\mathcal{G})$ (with respect to any finite generating set of $\mathbb{A}(\mathcal{G})$) satisfies $b_\Delta(n) \succeq \beta_\mathcal{G}(n)$.
Similar properties for a given group $G$ were previously known only as a consequence of stronger properties of analytic nature. It is a well-known observation that a group $G$ with property $(T)$ does not admit any Schreier graph of subexponential growth \cite{Gro93} Rem. 0.5.F; see also \cite{Leb14} for a similar statement in terms of wobbling groups. More generally, the same conclusion holds as soon as $G$ does not admit co-amenable subgroups of infinite index, a property which is in some sense a discrete weakening of property $(T)$ (see \cite{Cor15, GM07} for an account of the examples of groups that are known to have this property but do not have $(T)$, and \cite{Cor15} for a similar statement in terms of wobbling groups). Note also that many groups admit Schreier graphs whose growth functions is a slow as possible, i.e. linear in $n$ (for instance any group that virtually has infinite abelianization; another famous example of group with this property is given by the Grigorchuk group \cite{BG00}).

Theorem 1.9 has the following corollaries; the first can be compared with a question asked by Cornulier in \cite[Question 1.19 (3)]{Cor15}.

**Corollary 1.10.** There exists a finitely generated group $G$ with the following properties.

(i) the group $G$ has a co-amenable subgroup of infinite index;

(ii) the group $G$ does not admit any infinite Schreier graph $\Gamma$ which has uniformly subexponential growth, i.e. \( \lim_{n \to \infty} \frac{1}{n} \log b_{\Gamma}(n) = 0 \).

**Corollary 1.11.** For every function $f$ which is the orbital growth function of a minimal expansive groupoid of germs, there exists a finitely group $G$ with the following properties.

1. For every non-trivial Schreier graph $\Gamma$ of $G$ we have $b_{\Gamma} \geq f$.
2. There exists a Schreier graph $\Gamma$ of $G$ such that $b_{\Gamma} \sim f$.

Note that the function $f$ in Corollary 1.11 can take a wide range of behaviours, and can be arbitrarily close to exponential but still be subexponential. It is a natural question to characterise exactly the growth types of functions that are realisable as growth functions of expansive groupoids. It seems plausible that rather arbitrary subexponential functions are realisable.

### 1.5. Chabauty space and confined subgroups.

Given a countable group $G$, its space of subgroup $\text{Sub}(G)$ is naturally a compact space, endowed with the Chabauty topology. We say that a subgroup $H \leq G$ is **confined** if the closure of its conjugacy class $\{ gHg^{-1} : g \in G \}$ in the space $\text{Sub}(G)$ does not contain the trivial subgroup $\{1\}$. A homonymous notion of confined subgroups was introduced by Hartley and Zalesskii \cite{HZ97} in the special case of simple locally finite groups, and has been further studied in \cite{LP03, LP02}. The original definition is different, but it is equivalent for locally finite groups (as recently pointed out by Thomas \cite{Tho17}). Therefore we chose to keep the same terminology.

The following result provides a characterisation of all confined subgroups of the group $\mathbb{F}(\mathcal{G})$ and, is a key tool used throughout the paper.

**Theorem 1.12** (Characterisation of the confined subgroups). Let $\mathcal{G}$ be a minimal groupoid of germs with unit space $X$ homeomorphic to the Cantor set. Let $G$ be any subgroup of $\mathbb{F}(\mathcal{G})$ such that $\mathbb{A}(\mathcal{G}) \leq G \leq \mathbb{F}(\mathcal{G})$. A subgroup $H \leq G$ is confined if and only if there exists a unique finite subset $Q \subset X$ (possibly empty) such that $S_{\mathbb{A}(\mathcal{G})}(Q) \leq H \leq S_G(\mathcal{G})(Q)$. 
Here $S_{\mathcal{A}(G)}^0(Q)$ denotes the subgroup of $\mathcal{A}(G)$ consisting of elements that fix point-wise a neighbourhood of $Q$, and $S_G(Q)$ denotes the set-wise stabiliser of $Q$ in $G$.

When $\mathcal{G}$ is an $AF$-groupoid (see Subsection 4.1), Theorem 1.12 provides a classification of the confined subgroup of a class of simple locally finite groups, namely block-diagonal limits of products of finite alternating groups (or LDA-groups), completing previous work of Leinen-Puglisi [LP03] in the case of diagonal limits of product of alternating groups.

A classification of the confined subgroups of a group provides a more precise information than a classification of the uniformly recurrent subgroups (URS’s) in the sense of Glasner and Weiss [GW15]. Recall that a URS of a countable group $G$ is a closed minimal conjugation-invariant subset $Z \subset \text{Sub}(G)$. A URS can be associated to every minimal action of $G \curvearrowright X$ on a compact space, and is called the stabiliser URS of the action, see [GW15, Proposition 1.2] (conversely, every URS arises in this way [MBT17, Ele17]). In particular, Theorem 1.12 implies that the group $\mathcal{A}(\mathcal{G})$ admits a unique non-trivial URS, namely the stabiliser URS of its action $\mathcal{A}(\mathcal{G}) \curvearrowright X$ (see Corollary 4.6). In the case of the LDA groups, the classification of URS’s was recently obtained by Thomas [Tho17].

The first result of uniqueness of the URS’s of a group given by a minimal action on a compact space was provided in [LBMB16], and applies to the Thompson’s groups $T$ and $V$ and to some related groups. The groups $T$ and $V$ are examples of topological full groups of minimal groupoids of germs on the circle and the Cantor set, respectively. Two main novelties arise in Theorem 1.12 with respect to this result. First, it is valid for a much wider class of topological full groups (with Cantor set unit space). In particular, its proof no longer relies on the extreme proximality condition of the natural actions of $T$ and $V$ used in [LBMB16]. As a consequence, it can be applied to groupoids whose Cayley graphs are amenable or have subexponential growth, which is important for Theorem 1.9 and its corollaries. Second, we no longer classify only URS’s but rather all the confined subgroups. This is important to study non-minimal actions, and opens the way to the applications to homomorphisms between topological full groups (given a homomorphism $\mathbb{F}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$, we are certainly not allowed to assume that the image acts minimally on the unit space of $\mathcal{H}$).

One of the purposes of this paper is to illustrate that Chabauty methods and confined subgroups can be used as tools to study embeddings between groups. In a different direction, A. Le Boudec [LB18] shows that, for a class of discrete groups, URS’s can be used as a tool to study lattice embeddings into locally compact groups.

The reader may have noticed a similarity between Theorem 1.1 and a theorem of Nekrashevych [Nek10], stating that if the free group $F_2$ acts faithfully on a rooted tree $T$, then there is a non-cyclic subgroup $G < F_2$ and a point $y \in \partial T$ such that $S_G^0(y) = \{1\}$. Although Theorem 1.1 is not directly related to this result, its formulation draws inspiration from it.

**Structure of the paper.** Section 2 fixes the notations used throughout the paper and contains elementary preliminaries. In Section 3 we prove a refinement of a result from [LBMB16], which provides a tool to study confined subgroups of groups given by a proximal minimal action on a Hausdorff space, and is perhaps of independent interest. In Section 4 we characterise the confined subgroups and the uniformly recurrent subgroups of the
groups $\mathbb{F}(G)$ and $\mathbb{A}(G)$. In Section 5 we use this characterisation to establish Theorem 1.9 and its application to growth and amenability of Schreier graphs. In Section 6 we state and prove the main dichotomy theorem in a more general form (Theorem 6.9) and discuss its first consequences. In Section 7 we show Theorem 1.6. We discuss in more detail how it specialises to the case of topological full groups of Cantor minimal systems and of one-sided shifts of finite type. Section 8 is devoted to obstruction to embeddings; we prove Theorem 1.7 and discuss applications of it in some special cases.

Acknowledgements. The derivation of Theorem 1.6 from Theorem 6.9 uses a property of asymptotic dimension whose proof was explained to me by Alessandro Sisto (Proposition 7.3); I thank him for this suggestion that allowed to improve the statement of Theorem 1.6. I am grateful to Adrien Le Boudec and to Volodia Nekrashevych for many conversations related to the topics of this paper. I also thank Laurent Bartholdi, Yves Cornulier and Adrien Le Boudec for useful remarks on a first draft.

2. Preliminaries

2.1. Notations on stabilisers and graphs of group actions. Throughout the paper, we use the following notations. If $X$ is a set and $A \subset X$ is a subset, we denote $A^c$ its complement. Assume that $G$ is a countable group acting on $X$.

- the stabiliser of a point $x \in X$ will be denoted $S_G(x)$, and the set-wise stabiliser of $A$ will be denoted $S_G(A)$,
- the point-wise stabiliser of $A \subset X$ will be denoted $F_G(A)$,
- we call the subgroup $F_G(A^c)$ the rigid stabiliser of $A$ and denote it $R_G(A)$.

If, moreover, $X$ has a topology we denote $S^0_G(x)$ and $S^0_G(A)$ the subgroups consisting of elements that fix point-wise a neighbourhood of $x$ (respectively, $A$), and call it the germ-stabiliser of $x$ (respectively, $A$). Thus $S^0_G(A) \leq F_G(A) \leq S_G(A)

Note that $S^0_G(x) \leq S_G(x)$. Assume that $X$ is a compact space. Then a simple Baire argument shows that the set of points $x \in X$ such that $S^0_G(x) = S_G(x)$ is a dense $G_δ$-subset of $X$. A point $x \in X$ satisfying this condition will be said to be a regular point.

Let $G$ be a finitely generated group with finite symmetric generating set $S$. Assume that $G \simeq X$ is an action of $G$ on a set. For every point $x \in X$, the orbital graph $\Gamma_x(G, S)$ is the graph whose vertex set is the orbit of $x$ and it has $x$ as a distinguished vertex, where $z, y$ are connected by an oriented edge labelled by $s \in S$ if $sz = y$. It coincides with the Schreier graph of the stabiliser $S_G(x)$, where the Schreier graph of a subgroup $H \leq G$ is the orbital graph $\Gamma_H(G, S)$ for the action of $G$ on the coset space $G/H$. If $X$ has a topology, the Schreier graph of the germ stabiliser $S^0_G(x)$ is called the graph of germs at $x$ and will be denoted $\tilde{\Gamma}_x(G, S)$. We omit $G, S$ when they are clear from the context.

2.2. The Chabauty topology and confined subgroups. Let $G$ be a countable discrete group. The set of subgroups $\text{Sub}(G)$ is a compact space endowed with the topology induced by the product topology on the set of all subsets $\{0, 1\}^G$ of $G$, which is called the Chabauty topology. The conjugation action of $G$ on $\text{Sub}(G)$ is by homeomorphisms.
Given a finite subset \( P \subset G \setminus \{1\} \), we denote
\[
\mathcal{U}_P = \{ H \in \text{Sub}(G) : H \cap P = \emptyset \}, \quad \mathcal{V}_P = \{ H \in \text{Sub}(G) : P \subset H \}.
\]
These sets are open in \( \text{Sub}(G) \), and form a pre-basis of the topology as \( P \) varies over finite subsets of \( G \setminus \{1\} \). In particular, the sets of the form \( \mathcal{U}_P \) form a fundamental system of neighbourhoods of the trivial subgroup \( \{1\} \).

If \( G \) is finitely generated, an equivalent description of the Chabauty topology can be given in terms of the space of marked graphs. Recall that given an integer \( d \geq 0 \) and a finite set \( S \), the space of oriented graphs \( (\Gamma, v) \) with a distinguished base-point \( v \in \Gamma \), degree bounded by \( d \) and edges labelled by \( S \) is naturally a compact metrisable space, where a sequence \( (\Gamma_n, v_n) \) converges to \( (\Gamma, v) \) if for every \( R > 0 \) the ball \( B_{\Gamma_n}(v_n, R) \) is eventually isomorphic to \( B_{\Gamma}(v, R) \) as a rooted labelled graph. The correspondence \( H \mapsto \Gamma_H(G, S) \) embeds homeomorphically \( \text{Sub}(G) \) into the space of marked graphs with degree bounded by \( 2|S| \) and edges labelled by \( S \).

**Definition 2.1.** A subgroup \( H \leq G \) is said to be defined by a characterisation of these properties is given by the following lemma, which readily follows from the definition of the Chabauty topology.

This property can be thought of as a weak notion of normality. Note that being confined is equivalent to the existence of a finite set \( P \subset G \setminus \{1\} \) such that the conjugacy class of \( H \) avoids \( \mathcal{U}_P \), i.e., \( P \cap gHg^{-1} \neq \emptyset \).

The terminology is due to Hartley and Zalesskii [HZ97], who introduced an equivalent property for simple locally finite groups \( ^1 \).

Another concept that will play an important role in this paper is the lower and upper semicontinuity of various maps taking values in \( \text{Sub}(G) \). Given a (Hausdorff) space \( X \), a map \( u : X \to \text{Sub}(G) \) is said to be **upper** (respectively **lower**) semicontinuous if for every net \( (x_\nu) \) in \( X \) converging to \( x \), every cluster point \( K \) of \( u(x_\nu) \) in \( \text{Sub}(G) \) verifies \( K \leq u(x) \) (respectively \( u(x) \leq K \)). A characterisation of these properties is given by the following lemma, which readily follows from the definition of the Chabauty topology.

**Lemma 2.2.** Let \( u : X \to \text{Sub}(G) \) be a map from a Hausdorff space to \( \text{Sub}(G) \).

(i) The map \( u \) is upper semicontinuous if and only if for every \( g \in G \), the set \( \{ x \in X : g \in u(x) \} \) is closed.

(ii) The map \( u \) is lower semicontinuous if and only if for every \( g \in G \), the set \( \{ x \in X : g \notin u(x) \} \) is open.

In particular, \( u \) is continuous if and only if both conditions hold.

Two basic example of upper and lower semicontinuous maps are the following.

**Example 2.3.** Let \( G \) act by homeomorphisms on a compact space \( X \). Then it follows from the lemma above that:

\(^1\)In the original definition, a subgroup of \( H < G \) of a locally finite group is said to be confined if there exists a non-trivial finite subgroup \( P < G \) such that \( gHg^{-1} \cap P \neq \{1\} \), which is clearly equivalent to the above since \( G \) is locally finite. This connection was recently pointed out by Thomas, see [Tho17].
Lemma 2.4. Assume that \( G \) acts continuously on a compact space \( X \). Let \( u: X \to \text{Sub}(G) \) be a lower semicontinuous map such that \( u(x) \neq \{1\} \) for all \( x \in X \). Then the closure of the image of \( u \) does not contain \( \{1\} \). In particular \( u(x) \) is confined for every \( x \in X \).

Proof. Let \( H \) be in the closure of the image of \( u \), and let \( (x_\nu) \subset X \) be a net such that \( u(x_\nu) \) tends to \( H \). Upon extracting, \( x_\nu \) tends to a limit \( x \in X \) and by lower semicontinuity we have \( u(x) \leq H \). It follows that \( H \neq \{1\} \). \( \square \)

Recall also that a uniformly recurrent subgroup, or URS, is a closed minimal invariant subset \( Z \subset \text{Sub}(G) \) \cite{GW15}. Whenever \( G \acts X \) is a minimal action on a compact space, the closure of \( \{S_G(x): x \in X\} \) contains a unique URS, called the stabiliser URS of the action, see \cite{GW15} (this notion will not play an essential role in this paper).

2.3. Étale groupoids. We briefly recall the basic definitions regarding groupoids and étale groupoids. A groupoid is a small category in which every morphism is an isomorphism. In other words it consists of a set of objects \( X \) (called the unit space), a set of morphisms \( G \), two maps \( r, s: G \to X \) called the source and the range that indicate the initial and terminal object of a morphism. The product of two elements \( \gamma, \delta \in G \) is defined if and only if \( s(\gamma) = r(\delta) \) and in this case \( s(\gamma \delta) = s(\delta), r(\gamma \delta) = r(\gamma) \). For every \( x \in X \) there is a unique element \( 1_x \in G \) (called the unit at \( x \)) which satisfies \( r(1_x) = s(1_x) = 1 \) and \( r(1_x) = 1 \) for every \( \gamma \in G \). There is an inversion \( \gamma \mapsto \gamma^{-1} \) so that \( \gamma \gamma^{-1} = 1 \) and \( \gamma^{-1} \gamma = 1 \).

The map \( x \mapsto 1_x \) allows to identify the unit space \( X \) with a subset of \( G \). We will systematically use this identification, and write \( x \) instead of \( 1_x \).

It is a common use to denote the unit space of a groupoid \( G \) by \( G^{(0)} \); we will deviate from this convention to avoid conflicts with other notations used in this paper \((X^{(0)}, G^{\ast}, \ldots)\).

Given a subset \( A \subset X \) of the unit space, the restriction of \( G \) to \( A \) is the subgroupoid \( G|A = \{\gamma \in G: s(\gamma), r(\gamma) \in A\} \).

A cocycle is a groupoid homomorphism, i.e. a map \( c: G_1 \to G_2 \) between groupoids such that \( c \circ r_1 = r_2 \circ c, c \circ s_1 = s_2 \circ c \) and \( c(\gamma \delta) = c(\gamma) c(\delta) \) for every \( \gamma, \delta \in G \) such that \( s_1(\gamma) = r_1(\delta) \), were \( s_i, r_i \) denote the source and range maps of \( G_i, i = 1, 2 \).

A topological groupoid is a groupoid endowed with a topology such that the source and range maps are continuous. Here \( G^{\ast} = \{(\gamma, \delta): s(\gamma) = r(\delta)\} \) is the set of composable pairs, endowed with the topology induced from the product topology on \( G^2 \). The unit space \( X \) is endowed with the topology induced by the inclusion \( X \subset G \).

An étale groupoid is a locally compact topological groupoid \( G \) such that the source and range maps are open and are local homeomorphisms. We do not require the topology on \( G \) to be Hausdorff (as many interesting examples are not). However, we do require the unit space \( X \) to be Hausdorff.
A bisection of an étale groupoid is an open subset \( T \subset \mathcal{G} \) such that \( s: T \to s(T) \) and \( r: T \to r(T) \) are homeomorphisms onto their image. By definition of an étale groupoid, bisections form a basis of the topology. Bisections are multiplied and inverted by the rules

\[
TS = \{ \gamma \delta : \gamma \in T, \delta \in S, s(\gamma) = r(\delta) \}, \quad T^{-1} = \{ \gamma^{-1} : \gamma \in T \}.
\]

Every open bisection \( T \) defines a homeomorphism between \( s(T) \) and \( r(T) \), given by \( s(T) \ni x \mapsto r(Tx) \in r(T) \). Note that \( Tx \) is the unique element \( \gamma \in T \) such that \( s(\gamma) = x \). An étale groupoid is said to be effective, or a groupoid of germs, if every bisection \( T \) is uniquely determined by the associated homeomorphism, or equivalently if the associated homeomorphism is non-trivial unless \( T \subset X \).

Given \( x \in X \), the orbit of \( x \) is the set of \( y \in X \) such that there exists \( \gamma \in \mathcal{G} \) with \( s(\gamma) = x \) and \( r(\gamma) = y \). The groupoid \( \mathcal{G} \) is said to be minimal if every orbit is dense in \( X \).

For every \( x \in X \) the set \( \gamma \in \mathcal{G} \) such that \( r(\gamma) = s(\gamma) = x \) forms a group, called the isotropy group at \( x \). The groupoid is said to be principal of \( \mathcal{G}_x \) is trivial for every \( x \in X \). It is said to be essentially principal if the set of points with trivial isotropy group is dense in \( X \).

**Examples 2.5.** (i) Every countable group \( G \) can be seen as an étale groupoid with one-point unit space \( X = \{1_G\} \) and with the discrete topology. A groupoid of this form is never effective (unless \( G \) is trivial).

(ii) Let \( G \curvearrowright X \) be a countable group acting on a compact space. The associated action groupoid is \( G \times X \). Its unit space is \( X \approx \{1_G\} \times X \) (with the obvious identification), and source and range map are given by \( s((g, x)) = x \) and \( r((g, x)) = gx \). The product and inversion are defined by the rules \( (g_1, g_2)(g_2, x) = (g_1g_2, x) \) and \( (g, x)^{-1} = (g^{-1}, gx) \). It is an étale groupoid if \( G \times X \) is endowed with the product topology, where \( G \) has the discrete topology. A groupoid of this form is always Hausdorff. It is effective if and only if the action \( G \curvearrowright X \) is topologically free, i.e. the set of points with trivial stabiliser is dense.

(iii) Let again \( G \curvearrowright X \) be a countable group action on a compact space. For every \( (g, x) \in G \times X \) denote by \( [g, x] \) the germ of \( g \) at \( x \), i.e. the equivalence class of the pair \( (g, x) \) under the equivalence relation that identifies \( (g_1, x_1) \) with \( (g_2, x_2) \) if \( x_1 = x_2 \) and \( g_1, g_2 \) coincide in restriction to a neighbourhood of \( x_1 \). Then the set of germs is naturally a groupoid \( \mathcal{G} = \mathcal{G}(G \curvearrowright X) \), called the groupoid of germs of the action, with unit space \( X \) identified with the set of germs \( \{[1_G, x] : x \in X\} \). Source, range, composition and inversion are given exactly as in the case of the action groupoid by replacing \( (g, x) \) with \( [g, x] \). The groupoid \( \mathcal{G} \) has a natural topology for which a basis of open sets is given by sets of the form \( \mathcal{U}_{g, U} = \{[g, x] : x \in U\} \) where \( g \in G \) and \( U \subset X \) is open, and is étale with this topology. The groupoid of germs of a group action is always effective. It is not necessarily Hausdorff.

(iv) Let \( G \) be a countable group and let \( X \subset \text{Sub}(G) \) be a closed \( G \)-invariant subset in the Chabauty space. The associated coset étale groupoid is the disjoint union \( \mathcal{G} = \bigsqcup_{H \subset X} G/H \) of all coset spaces of subgroups in \( X \). Its unit space is \( X \), and source and range maps are given by \( s(gH) = H, r(gH) = gHg^{-1} \). Composition and inversion are given by \( g_1(g_2Hg_2^{-1})g_2H = g_1g_2H, (gH)^{-1} = g^{-1}(gHg^{-1}) \). A basis of open sets of its topology is given by sets of the form \( \mathcal{U}_{g, U} = \{gH : H \subset U\} \), where \( g \in G \) and \( U \subset X \) is open. This
topology makes it an étale groupoid. It is effective if and only if there is a dense subset of \( H \in X \) which has trivial normaliser in \( G \).

2.4. Actions of groupoids. Let \((Z, q)\) be a set \( Z \) together with a surjective map \( q: Z \to X \). We will call for short the pair \((Z, q)\) a fibre space over \( X \). It is said to be (continuous if and only if there is a dense subset of \( H \in X \) which has trivial normaliser in \( G \).

A continuous action \( G \curvearrowright (Z, q) \) is a continuous map

\[
G \times_{s, q} Z \to Z, \quad (\gamma, z) \mapsto \gamma z
\]

where \( G \times_{s, q} Z = \{(\gamma, z) \in G \times Z: s(\gamma) = q(z)\} \) denotes the fibre product of \((G, s)\) and \((Z, q)\) over \( X \), with the properties that \( q(\gamma z) = r(\gamma) \) and \( (s \gamma, z) = (\delta, \gamma z) \) for every \( \delta, \gamma, z \) such that \( s(\delta) = r(\gamma) \) and \( s(\gamma) = q(z) \). Note that it follows that every \( \gamma \in G \) defines a homeomorphism between \( q^{-1}(s(\gamma)) \) and \( q^{-1}(r(\gamma)) \).

Example 2.6. Let \( G \curvearrowright X \) be a continuous action on a compact space and let \( G = G \times X \) be its action groupoid. Then it is not difficult to see that actions \( G \curvearrowright (Z, q) \) on compact fibre spaces are in one-to-one correspondence with extensions of the action \( G \curvearrowright X \), namely actions \( G \curvearrowright Z \) on compact spaces such that there exists a continuous, surjective, \( G \)-equivariant map \( q: Z \to X \). If the action \( G \curvearrowright X \) is topologically free, the same holds for the groupoid of germs \( G \) of \( G \curvearrowright X \). (However, this is not true if the action is not topologically free.)

If \( \alpha: G \curvearrowright (Z, q) \) is a continuous \( G \)-action, the set \( G \times_{s, q} Z \) is naturally an étale groupoid, called the action groupoid and denoted \( G \curvearrowright_{\alpha} Z \). Its unit space is \( Z \), identified with the subset \( \{(q(z), z): z \in Z\} \subset G \times_{s, q} Z \). Range and source map are given by \( s(\gamma, z) = z \) and \( r(\gamma, z) = \gamma z \). The product is defined by \( (\delta, \gamma z)(\gamma, z) = (\delta \gamma, z) \).

2.5. Standing assumptions on unit spaces of groupoids. We will be mostly interested in studying topological full groups of étale groupoids whose unit space is homeomorphic to the Cantor set. However unit spaces that are locally homeomorphic to the Cantor set, but non-compact appear naturally in the proofs, for example when considering the restriction \( G|U \) of \( G \) to an open set \( U \subset X \). Moreover we will also consider groupoids with a more general unit space, that will play the role of the “target” groupoid in Theorem 1.6. Therefore, we fix the following convention.

Convention 2.7. For the rest of the paper, unless differently specified, \( G \) will denote a second countable étale groupoid whose unit space \( X \) is isomorphic either to the Cantor set, or to the locally compact non-compact Cantor set. We denote \( H \) an étale groupoid whose unit space \( Y \) is arbitrary.

Recall that the locally compact non-compact Cantor set is described as the unique space up to homeomorphism which is locally compact, metrisable, second countable, totally disconnected and without isolated points. Any open, non-closed subset of the Cantor set is homeomorphic to it.
2.6. **Compactly generated and expansive groupoids.** We recall the definition of a compactly generated étale groupoid, due to Haefliger [Hae02] (in the language of pseudo-groups), see also [Nek15b, Sec. 2.3]. We only recall it in the special case of groupoids with a compact unit space. Note that the definition for groupoids with non-compact unit space is different, see [Hae02].

**Definition 2.8.** Let $\mathcal{H}$ be an étale groupoid with compact unit space $Y$. It is said to be **compactly generated** if there exists a finite set $\mathcal{T} = \{T_1, \ldots, T_r\}$ of open bisections such that every element of $\mathcal{H}$ is a product of elements of $\bigcup_{i=1}^{r} T_i$ and their inverses, and there exists open bisections $\tilde{T}_1, \ldots, \tilde{T}_r$ such that $T_i \subset \tilde{T}_i$ and the closure of $s(T_i)$ in $Y$ is contained in $s(\tilde{T}_i)$. The set $\mathcal{T}$ is called a **finite generating set** (of bisections) of $\mathcal{H}$.

**Example 2.9.** If $G \curvearrowright Y$ is a finitely generated group action on a compact space, its groupoid of germs is compactly generated, a finite generating set being given by (the sets of germs of) a finite generating set of $G$.

For groupoids with Cantor set unit space the existence of the bisections $\tilde{T}_i$ in the definition can be replaced by the requirement that $T_1, \ldots, T_r$ are compact open bisections (the resulting definition is equivalent). In this case, we will always consider generating sets consisting of compact open bisections.

Recall that a group action $G \curvearrowright X$ on the Cantor set is said to be **expansive** if there exists a finite partition $\mathcal{P}$ of $X$ into clopen sets whose $G$-translates separate points, equivalently if it is conjugate to a subshift over a finite alphabet. The following definition is due to Nekrashevych [Nek15a].

**Definition 2.10.** Let $\mathcal{G}$ be an étale groupoid with Cantor set unit space. It is said to be **expansive** if it is compactly generated and there exists a generating set $\mathcal{T} = \{T_1, \ldots, T_r\}$ of compact open bisections such that $\bigcup_{r=0}^{\infty} (\mathcal{T} \cup \mathcal{T}^{-1})^r$ is a basis of the topology of $\mathcal{G}$. Such a set is called an **expansive generating set**.

Expansivity of an action of a finitely generated group on the Cantor set is equivalent to the expansivity of its groupoid of germs and to the expansivity of its action groupoid, see [Nek15a, Prop. 5.5].

2.7. **The topological full group.**

**Definition 2.11.** Let $\mathcal{H}$ be an étale groupoid with unit space $Y$. Its **topological full group** is the group of all continuous maps $g : Y \to \mathcal{H}$, $y \mapsto g|_y$, with the following properties:

(i) for every $y \in Y$ we have $s(g|_y) = y$;
(ii) the set $T_g = \{g|_y : y \in Y\}$ is a bisection of $\mathcal{H}$ such that $s(T_g) = r(T_g) = Y$;
(iii) the set $T_g \setminus Y$ is compact, or equivalently $g|_y = y$ for $y$ outside a compact set.

It is a group with the composition defined by $(gh)|_y = g_{r(h|_y)} h|_y$.

There is natural action $\mathbb{F}(\mathcal{H}) \curvearrowright Y$, $(g, y) \mapsto gy$ via the homeomorphism associated to bisections $T_g$, or equivalently given by $gy = r(g|_y)$ for every $y \in Y$ and $g \in \mathbb{F}(\mathcal{H})$. This action is faithful if $\mathcal{H}$ is a groupoid of germs, but not necessarily otherwise. The map

$$\mathbb{F}(\mathcal{H}) \times Y \to \mathcal{H}, (g, y) \mapsto g|_y$$
is a continuous open cocycle from the action groupoid $\mathcal{F}(\mathcal{H}) \times Y$ to $\mathcal{H}$.

The correspondence $g \mapsto T_g$ defines a bijection between $\mathcal{F}(\mathcal{H})$ and the set of bisections of $\mathcal{H}$ verifying [iii] and [iii], and we have $T_{gh} = T_g T_h$ where bisections are are multiplied according to (1). Therefore, one can simply define the group $\mathcal{F}(\mathcal{G})$ as the group of all bisections with these properties, as in [Nek15a]. However, we will keep the distinction between $g$ and the associated bisection $T_g$, to avoid conflict between well-established notations regarding group actions and groupoids. In particular, if $U \subset Y$, the notation $gU$ will refer to the image of $U$ under the natural action $\mathcal{F}(\mathcal{H}) \curvearrowright Y$, while $T_g U$ denotes the set-wise product of $T_g$ and $U$ in $\mathcal{H}$, i.e. $T_g U = \{ \gamma \in T_g : s(\gamma) \in U \}$.

2.8. The alternating full group. In this subsection, let $\mathcal{G}$ be as in Convention [2.7]. We recall the definition of multisections and of the group $\mathcal{A}(\mathcal{G})$, following the point of view of Nekrashevych [Nek15a] (with some minor differences in the notations and the terminology).

**Definition 2.12.** A multisection of degree $d$ of $\mathcal{G}$ is a map $\mathfrak{F}$ from $\{1, \ldots, d\}^2$ to the set of compact open bisections of $\mathcal{G}$, such that the following conditions are satisfied:

(i) we have $\mathfrak{F}(i, i) \subset X$ for every $i = 1, \ldots, d$, and $\mathfrak{F}(i, i) \cap \mathfrak{F}(j, j) = \emptyset$ if $i \neq j$;
(ii) for every $i, j, \ell = 1, \ldots, d$ we have $\mathfrak{F}(i, j) \mathfrak{F}(j, \ell) = \mathfrak{F}(i, \ell)$.

The set $\bigcup_{i=1}^d \mathfrak{F}(i, i)$ is called the domain of the multisection and the subsets $\mathfrak{F}(i, i)$ are called the components of the domain.

There is a natural action of the group $\mathcal{F}(\mathcal{G})$ on the set of multisections given by $(g \mathfrak{F})(i, j) = T_g \mathfrak{F}(i, j) T_g^{-1}$.

A multisection $\mathfrak{F}$ of degree $d$ induces an embedding from the symmetric group $\text{Sym}(d)$ into $\mathcal{F}(\mathcal{G})$, still denoted $\mathfrak{F}$: $\text{Sym}(d) \to \mathcal{F}(\mathcal{G})$. The bisection associated to $\mathfrak{F}(\sigma)$ is given by

$$T_{\mathfrak{F}(\sigma)} = \left( \bigcup_{i=1}^d \mathfrak{F}(\sigma(i), i) \right) \cup (X \setminus \bigcup_{i=1}^d \mathfrak{F}(i, i)).$$

Informally, the associated homeomorphism permutes the components $\mathfrak{F}(i, i)$ in a way prescribed by $\sigma$, and acts within each component according to the bisection $\mathfrak{F}(\sigma(i), i)$. We denote $A(\mathfrak{F})$ the image of the alternating group $\text{Alt}(d)$ under $\mathfrak{F}$.

**Definition 2.13.** The alternating full group of $\mathcal{G}$ is the subgroup $\mathcal{A}(\mathcal{G})$ of $\mathcal{F}(\mathcal{G})$ generated by $\bigcup \mathfrak{F}(\mathfrak{F}(\sigma))$, where $\mathfrak{F}$ varies over multisections of degree $d \geq 3$.

**Remark 2.14.** It follows from the definitions that when the unit space $X$ is the locally compact non compact Cantor set, the group $\mathcal{A}(\mathcal{G})$ is the direct limit of the groups $\mathcal{A}(\mathcal{G}|U)$, where $U \subset X$ varies among compact open subsets of $X$ (and hence $U$ is isomorphic to the Cantor set).

The following theorems are proven in [Nek15a]. Note that it is stated there only for Cantor set space of unit, but the case of the locally compact Cantor set reduces to this by Remark 2.14.
Theorem 2.15 ([Nek15a]). Assume that \( \mathcal{G} \) is a minimal groupoid of germs. Then every non-trivial subgroup of \( \mathcal{F}(\mathcal{G}) \) normalised by \( \mathcal{A}(\mathcal{G}) \) contains \( \mathcal{A}(\mathcal{G}) \). In particular, \( \mathcal{A}(\mathcal{G}) \) is simple and contained in every non-trivial normal subgroup of \( \mathcal{F}(\mathcal{G}) \).

Theorem 2.16 ([Nek15a]). Assume that \( \mathcal{G} \) has Cantor set unit space and is expansive, and that every \( \mathcal{G} \)-orbit contains at least 5 points. Then the alternating full group \( \mathcal{A}(\mathcal{G}) \) is finitely generated.

We now list some useful general facts about multisections and the alternating group, that will be used in the sequel. It is convenient to introduce the following terminology.

Definition 2.17. A multielement of degree \( d \) of \( \mathcal{G} \) is an embedding of the complete equivalence relation on \( \{1, \ldots, d\} \) into \( \mathcal{G} \). Explicitly, an injective map \( g: \{1, \ldots, d\}^2 \to \mathcal{G} \) such that \( g(i, i) \in X \) for every \( i = 1, \ldots, d \) and for every \( i, j, \ell = 1, \ldots, d \) we have \( g(i, j)g(\ell, \ell) = g(i, \ell) \).

Note that it follows that \( s(g(i, j)) = g(j, i) \) and \( r(g(i, j)) = g(i, \ell) \), and that \( g(i, i), i = 1, \ldots, d \) all lie in the same orbit. The group \( \mathcal{F}(\mathcal{G}) \) acts on the set of multielements by \( (g|_{\{i, j\}})g(i, j)g(j, j)^{-1} \).

If \( g \) is a multielement and \( \mathfrak{F} \) a multisection of the same degree, we will write \( g \in \mathfrak{F} \) to mean that \( g(i, j) \in \mathfrak{F}(i, j) \) for every \( i, j \).

Lemma 2.18. Let \( g \) be a degree \( d \) multielement.

(i) Assume \( x_{d+1}, \ldots, x_{d'} \) are units in the same orbit of \( g(1, 1) \) such that the points \( g(1, 1), \ldots, g(d, d), x_{d+1}, \ldots, x_{d'} \) are pairwise distinct. Then there exists a degree \( d' \) multielement \( g' \) such that \( g'|_{\{1, \ldots, d\}^2} = g \) and \( g'(i, i) = x_i \) for \( i = d+1, \ldots, d' \).

(ii) Assume that \( T_{i,j} \) is a neighbourhood of \( g(i, j) \) for every \( i, j = 1 \ldots d \). Then there exists a degree \( d \) multisection \( \mathfrak{F} \) such that \( g \in \mathfrak{F} \) and \( \mathfrak{F}(i, j) \subset T_{i,j} \) for every \( i, j \).

Part [ii] allows us to talk about “sufficiently small” multisections containing \( g \), and we will sometimes do so without mention.

Proof. To prove (i), set \( g'(i, j) = g(i, j) \) for \( 1 \leq i, j \leq d \). Choose arbitrarily for every \( d + 1 \leq j \leq d' \) an element \( g'(j, 1) \in \mathcal{G} \) such that \( s(g'(j, 1)) = g(1, 1) \) and \( r(g'(j, 1)) = x_j \), and set \( g'(1, j) = g'(j, 1) \). In the remaining cases, let \( g'(i, j) = g'(i, 1)g'(1, j) \).

Let us prove (ii). First, for every \( i, j \) we can assume that \( T_{i,j} = T^{-1}_{i,j} \), upon replacing all the \( T_{i,j} \) by \( T_{i,j} \) by \( T_{i,j} \cap T^{-1}_{i,j} \). Choose a decreasing basis \( (W_n) \) of compact open neighbourhoods of \( g(1, 1) \) in \( X \). If \( n \) is large enough, we can assume \( W_n \subset T_{1,1} \) , that \( W_n \subset s(T_{j,1}) \) for every \( j = 2, \ldots, d \), and that \( W_n, r(T_{2,1}W_n), \ldots, r(T_{d,1}W_n) \) are pairwise disjoint. Define a sequence of multisections \( \mathfrak{F}_n \) by setting \( \mathfrak{F}_n(1, 1) = T_{1,1}W_n \) and \( \mathfrak{F}_n(1, j) = \mathfrak{F}_n(j, 1) = \mathfrak{F}_n(j, 1)^{-1} \) for \( j = 1, \ldots, d \) and \( \mathfrak{F}_n(i, j) = \mathfrak{F}_n(i, 1)\mathfrak{F}_n(1, j) \) in the remaining cases. Then for every \( i, j \) the sequence of bisections \( \mathfrak{F}_n(i, j) \) is a basis of neighbourhoods of \( g(i, j) \), and it follows that \( \mathfrak{F}_n(i, j) \subset T_{i,j} \) for all \( i, j \) if \( n \) is large enough. \( \square \)

Lemma 2.19. Assume that every \( \mathcal{G} \) contains at least \( d \) points for some \( d \geq 2 \). Then for every bisection \( T \subset \mathcal{G} \) such that \( s(T) \cap r(T) = \emptyset \), there exists finitely many degree \( d \) multisections \( \mathfrak{F}_1, \ldots, \mathfrak{F}_n \) such that \( T = \bigcup_{i=1}^n \mathfrak{F}_i(1, 2) \).
Proof. Using Lemma \[\text{[2.18]}\] and the fact that every \( G \) orbit has at least \( d \) points, or every \( \gamma \in T \), we can find a degree \( d \) multielement \( g \) such that \( g(1, 2) = \gamma \), and a degree \( d \) multisection \( \mathfrak{s} \) such that \( \mathfrak{s}(1, 2) \subset T \). The conclusion follows by compactness. \( \square \)

The next proposition is \([\text{Nek15a}, \text{Proposition 3.2.}]\). If \( \mathfrak{s}, \mathfrak{S} \) are multisections of the same degree, we write \( \mathfrak{s} \preceq \mathfrak{S} \) if \( \mathfrak{s}(i, j) \subset \mathfrak{S}(i, j) \) for every \( i, j \).

**Proposition 2.20.** Let \( \mathfrak{s} \) be a degree \( d \) multisection. Let \( \mathfrak{s}_\ell \subset \mathfrak{s} \) for \( \ell = 1, \ldots, n \) be multisections such that \( \mathfrak{s}(i, j) = \bigcup_{\ell=1}^n \mathfrak{s}_\ell(i, j) \) for every \( i, j \in \{1, \ldots, d\} \). Then \( A(\mathfrak{s}) \) is contained in the subgroup generated by \( A(\mathfrak{s}_1) \cup \cdots \cup A(\mathfrak{s}_n) \).

**2.9. Graphs and growth functions of étale groupoids.** Let \( \mathcal{H} \) be an étale groupoid with compact unit space \( Y \). Assume that \( \mathcal{H} \) is compactly generated and let \( \mathcal{T} = \{T_1, \ldots, T_r\} \) be a generating set. We assume \( \mathcal{T} \) to be symmetric, i.e. \( T_i \in \mathcal{T} \Rightarrow T_i^{-1} \in \mathcal{T} \).

For every \( y \in Y \), the **orbital graph** \( \Gamma_y(\mathcal{H}, \mathcal{T}) \) is the \( S \)-labelled graph whose vertex set is the \( \mathcal{H} \)-orbit of \( y \), and where \( y_1, y_2 \) are connected by an oriented edge for every \( i = 1, \ldots, r \) such that \( y_2 = r(T_i y_1) \). Such an edge is labelled by \( T_i \). The **Cayley graph** \( \tilde{\Gamma}_y(\mathcal{H}, \mathcal{T}) \) based at the unit \( y \in Y \) is the graph whose vertex set is \( s^{-1}(y) \) and where \( \gamma_1, \gamma_2 \) are connected by an oriented edge labelled by \( T_i \in \mathcal{T} \) if \( \gamma_2 = T_i \gamma_1 \).

Apart from labels and multiple edges, both these graphs depend only on \( T_1 \cup \cdots \cup T_r \).

If \( \mathcal{T}' \) is another generating set, for every base-point \( y \in Y \) the graph \( \Gamma_y(\mathcal{H}, \mathcal{T}') \) (respectively \( \tilde{\Gamma}_y(\mathcal{H}, \mathcal{T}') \)) is bi-Lipschitz equivalent to \( \Gamma_y(\mathcal{H}, \mathcal{T}) \) (respectively \( \tilde{\Gamma}_y(\mathcal{H}, \mathcal{T}) \)), and the bi-Lipschitz equivalence constant only depends on \( \mathcal{T}, \mathcal{T}' \) and not on \( y \) (see e.g. \([\text{Nek15b, Corollary 2.3.4}]\)). However the bi-Lipschitz equivalence type of each graph may depend on the choice of \( y \). We will simply write \( \tilde{\Gamma}_y, \Gamma_y \) when \( \mathcal{H}, \mathcal{T} \) are are clear from the context, or when we are only interested in the graphs up to bi-Lipschitz equivalence.

**Example 2.21.** Assume that \( \mathcal{H} \) is the groupoid of germs of a continuous action \( G \curvearrowright Y \) of a finitely generated group. Then, for a suitable choice of the generating sets, the orbital graphs and the Cayley graphs of \( \mathcal{H} \) at a point \( y \) coincide with the orbital graph and the graph of germs of the action \( G \curvearrowright Y \) as defined in Subsection 2.1.

There is a natural map \( \tilde{\Gamma}_y(\mathcal{H}, \mathcal{T}) \to \Gamma_y(\mathcal{H}, \mathcal{T}) \) given on the set of vertices by \( \gamma \mapsto r(\gamma) \). This map is a Galois covering map, with deck transformation group isomorphic to the isotropy group \( G_y \) (this covering map played an important role in \([\text{Nek16a}]\)). In particular, if \( G_y \) is trivial, the Cayley graph and the orbital graph coincide. If \( \mathcal{H} \) is essentially principal, it follows that \( \tilde{\Gamma}_y(\mathcal{H}, \mathcal{T}) \) and \( \Gamma_y(\mathcal{H}, \mathcal{T}) \) coincide for \( y \) in a dense \( G_y \)-subset.

Given a graph \( \Gamma \) and a vertex \( v \in \Gamma \) we denote \( B_\Gamma(n, v) \) the size of the ball of radius \( n \) around \( v \), and \( \overline{B}_\Gamma(n) = \sup_{v \in \Gamma} |B_\Gamma(n, v)| \).

Given two functions \( f, g: \mathbb{N} \to \mathbb{N} \), we write \( f \preceq g \) if there exists a constant \( C > 0 \) such that \( f(n) \leq g(Cn) \) for every \( n \geq 1 \), and \( f \sim g \) if \( f \preceq g \preceq f \).

**Definition 2.22.** The **growth** of a compactly generated groupoid \( \mathcal{H} \) with generating set \( \mathcal{T} \) is the function of \( n \in \mathbb{N} \) given by

\[
\beta_\mathcal{H}(n, \mathcal{T}) = \max_{y \in Y} |B_{\tilde{\Gamma}_y(\mathcal{H}, \mathcal{T})}(n, y)| = \max_{y \in Y} \overline{B}_{\Gamma_y(\mathcal{H}, \mathcal{T})}(n).
\]
The orbital growth $\beta_\mathcal{H}(n, T)$ is defined similarly by replacing the Cayley graphs with the orbital graphs $\Gamma_y(\mathcal{H}, T)$.

Since changing generating set results in a bi-Lipschitz equivalence of the Cayley and orbital graphs, the growth types of these functions according to $\sim$ does not depend on the choice of $T$ and will be denoted simply $\beta_\mathcal{H}(n)$ and $\tilde{\beta}_\mathcal{H}(n)$.

It is clear that we have $\beta_\mathcal{H}(n) \leq \tilde{\beta}_\mathcal{H}(n)$, because the Cayley graphs cover the orbital graphs. The following proposition shows that in some relevant cases these two functions coincide.

**Proposition 2.23.** if $\mathcal{H}$ is essentially principal and Hausdorff, we have $\beta_\mathcal{H}(n, T) = \tilde{\beta}_\mathcal{H}(n, T)$ for every symmetric generating set $T$ and every $n \geq 1$.

**Proof.** Fix a symmetric finite set of bisection $T$ generating $\mathcal{H}$. It is clear that $\beta_\mathcal{H}(n, T) \geq \tilde{\beta}_\mathcal{H}(n, T)$ and we only have to prove the converse inequality. Let $y \in Y$ that realises the maximum in the definition of $\tilde{\beta}_\mathcal{H}(n, T)$. For every $\gamma \in B_{\tilde{\Gamma}_y}(n, y)$ let $F_\gamma \ni \gamma$ be the intersection of all bisections in $\bigcup_{n=0}^\infty T^n$ to which $\gamma$ belongs, and note that $F_\gamma$ is an open bisection containing $\gamma$. Since $\mathcal{H}$ is Hausdorff, we can choose smaller open bisections $F'_\gamma \subset F_\gamma$ such that $F'_\gamma \cap F'_\delta = \emptyset$ if $\gamma \neq \delta$. Set $U = \bigcap_{\gamma} s(F'_\gamma)$, which is a neighbourhood of $y$ in $Y$. Now let $\gamma \neq \delta$ be in $B_{\tilde{\Gamma}_y}(n, T)(n, y)$ and write $\gamma = T_{i_k} \cdots T_{i_1} y, \delta = T_{j_k} \cdots T_{j_1} y$ with $k, \ell \leq n$. For every $z \in U$, we have that $T_{i_k} \cdots T_{i_1} z \in F'_\gamma$ and $T_{j_k} \cdots T_{j_1} z \in F'_\delta$ and it follows that we also have $\gamma' \neq \delta'$. This shows that $|B_{\tilde{\Gamma}_y}(n, T)(n, z)| \geq |B_{\tilde{\Gamma}_y}(n, T)(n, x)|$. Since $\mathcal{H}$ is essentially principal, we can choose $z \in U$ such that $\mathcal{H}_z$ is trivial, so that the Cayley graph and the orbital graph based at $z$ coincide and we have $|B_{\Gamma_z}(n, T)(n, z)| \geq |B_{\tilde{\Gamma}_y}(n, T)(n, y)|$, proving that $\beta_\mathcal{H}(n, T) \geq \tilde{\beta}_\mathcal{H}(n, T)$.

**Proposition 2.24.** If $\mathcal{H}$ is minimal, then the orbital graph of every point $y \in Y$ satisfies $\tilde{\beta}_{\Gamma_y}(n) \sim \beta_\mathcal{H}(n)$.

**Proof.** The inequality $\tilde{\beta}_{\Gamma_y}(n) \leq \beta_\mathcal{H}(n)$ is obvious. Let us show the converse. Fix $n$, let $z \in Y$ realise the maximum in the definition of $\beta_\mathcal{H}(n)$, and let $z_1 = z, \ldots, z_r$ be the distinct vertices in in the ball of radius $r$ around $z$, where $r = \beta_\mathcal{H}(n)$. This implies that there exists bisections $T_1, \ldots, T_r \in \bigcup_{n=0}^\infty T^n$ such that $z_i = t(T_i y)$ for $i = 1, \ldots, r$. It follows that if $U$ is a sufficiently small neighbourhood of $z$ then $t(T_1 U), \ldots, t(T_r U)$ are pairwise disjoint. Hence for every $w \in U$ the points $t(T_1 w), \ldots, t(T_r w)$ are pairwise distinct. Since $\mathcal{H}$ is minimal, we can choose $w \in U$ which is in the same orbit of $y$, and since the ball of radius $n$ in $\Gamma_y$ centred at $w$ has at least $r$ different points this shows that $\tilde{\beta}_{\Gamma_y}(n) \geq r$.

Let now $\mathcal{G}$ be a compactly generated groupoid with Cantor set unit space, and fix a generating set $T$ consisting of compact open bisections. We will use the following easy Lemma.

**Lemma 2.25.** Assume that $\mathcal{G}$ has no orbit of size less than 3. Then there exists a finitely generated subgroup $G \leq \mathcal{A}(\mathcal{G})$ such that for every $x \in X$ the orbital graph $\Gamma_x(\mathcal{G}, T)$, is bi-Lipschitz equivalent to the the orbital graph $\Gamma_x(G, S)$ of the action $G \curvearrowright X$, where $S$ is any
finite symmetric generating set of $G$, and and the constants in the bi-Lipschitz equivalence are independent of $x \in X$.

Note that if $\mathcal{G}$ is expansive, one can simply choose $G = \Delta(\mathcal{G})$.

**Proof.** We can assume that the generating set $\mathcal{T} = \{T_1, \ldots, T_d\}$ is such that $r(T_i) \cap s(T_i) = \emptyset$ for every $i = 1, \ldots, d$, see [Nek15a, Lemma 5.1]. Using Lemma 2.18 choose for every $i = 1, \ldots, d$ degree 3 multisections $\mathcal{F}_{i,1}, \ldots, \mathcal{F}_{i,k_i}$ such that the sets $\mathcal{F}_{i,j}(1,2), j = 1 \ldots k_i$ cover $T_i$. Let $G$ be the group generated by $S = \bigcup_{i=1}^{d} \bigcup_{j=1}^{k_i} A(\mathcal{F}_{i,j})$. By construction every $y, z$ in the $\mathcal{G}$-orbit of $x$ that are neighbours in $\Gamma_x(\mathcal{G}, \mathcal{T})$ are also neighbours in $\Gamma_x(G,S)$. Conversely it is easy to see that the identity map on vertices $\Gamma_x(G,S) \to \Gamma_x(\mathcal{G}, \mathcal{T})$ is Lipschitz (for any finitely generated subgroup $G < \Delta(\mathcal{G})$ and any finite generating set $S$). \hfill \square

For a proof of the following fact, see [Nek15b, Cor. 2.3.4].

**Proposition 2.26.** Assume that $\mathcal{G}$ is minimal. Let $U \subset X$ be a clopen set. Then $\mathcal{G}|U$ is compactly generated, and for every $x \in U$ the Cayley graph of $\mathcal{G}$ and $\mathcal{G}|U$ based at $x$ are quasi-isometric. The same holds for the orbital graphs.

### 2.10. Complexity of expansive groupoids.

Throughout the subsection let $\mathcal{G}$ be an étale groupoid with Cantor set unit space. Let $\mathcal{T} = \{T_1, \ldots, T_n\}$ be a finite symmetric set of compact open bisections of $\mathcal{G}$ (not necessarily a generating set). We denote by $Q_{\mathcal{T}}(n)$ the finite partition of $X$ generated by the sets $s(T), T \in \bigcup_{r \leq n} \mathcal{T}_r$, i.e. $x, y \in X$ are in the same element of the partition if and only if for every $T \in \bigcup_{r \leq n} \mathcal{T}_r$ we have $x \in s(T) \Leftrightarrow y \in s(T)$.

**Definition 2.27.** Let $\mathcal{G}, \mathcal{T}$ be as above. For $n \geq 0$ we set

$$\pi_{\mathcal{G}}(n, \mathcal{T}) = |Q_{\mathcal{T}}(n + 1)|$$

and call this function the complexity of the set of bisections $\mathcal{T}$. If $\mathcal{G}$ is expansive and $\mathcal{T}$ is an expansive generating set of $\mathcal{G}$, then the growth type of the function $\pi_{\mathcal{G}}(n, \mathcal{T})$ according to $\sim$ is called the complexity of $\mathcal{G}$ and is denoted $\pi_{\mathcal{G}}(n)$.

The fact the the complexity of an expansive groupoid is indeed a well-defined invariant is proven in the following lemma.

**Lemma 2.28.** Assume that $\mathcal{G}$ is expansive. If $\mathcal{T}_1$ is an expansive generating set of $\mathcal{G}$ and $\mathcal{T}_2$ is any finite set of compact open bisections, we have $\pi_{\mathcal{G}}(n, \mathcal{T}_2) \leq \pi_{\mathcal{G}}(n, \mathcal{T}_1)$. In particular, if $\mathcal{T}_2$ is any expansive generating set, we have $\pi_{\mathcal{G}}(n, \mathcal{T}_2) \sim \pi_{\mathcal{G}}(n, \mathcal{T}_1)$.

**Proof.** By the definition of expansivity, and by compactness, there exists $M > 0$ be such that every element of $\mathcal{T}_2$ is a union of elements of $\bigcup_{r=0}^{M} \mathcal{T}_r$. Let $x, y \in X$ be $Q_{\mathcal{T}_1}(Mn)$-equivalent. Let $T_2 \in \bigcup_{r \leq n} \mathcal{T}_2^r$ be such that $x \in s(T_2)$. Then there exists $T_1 \in \bigcup_{r \geq n} \mathcal{T}_1^{Mr}$ such that $T_1 \subset T_2$ and $x \in s(T_1)$. It follows that also $y \in s(T_1) \subset s(T_2)$. Since $T_2$ was arbitrary, this shows that $x$ and $y$ are $Q_{\mathcal{T}_1}(n)$-equivalent, i.e. the partition $Q_{\mathcal{T}_1}(Mn)$ is finer than the partition $Q_{\mathcal{T}_2}(n)$. This yields the conclusion. \hfill \square
Remark 2.29. Assume that $\mathcal{G}$ is expansive and $\mathcal{T}$ is an expansive generating set. Then two points $x, y$ are in the same element of $Q_{\mathcal{T}}(n + 1)$ if and only if the ball of radius $n$ in the universal cover of the Cayley graphs $\hat{\Gamma}_x(\mathcal{G}, \mathcal{T}), \hat{\Gamma}_y(\mathcal{G}, \mathcal{T})$ around any preimage of $x$ and $y$ are isomorphic as rooted labelled graphs. Nekrashevych considers a different definition of complexity of étale groupoids in [Nek16b], defined as the number $\tilde{\pi}_G(n, \mathcal{T})$ of elements of the partition of $\mathcal{X}$ induced by isomorphism classes of the balls of radius $n$ in the Cayley graphs $\hat{\Gamma}_x(\mathcal{G}, \mathcal{T})$ as rooted, labelled graphs (without passing to the universal cover). We changed this definition because the function $\pi_G(n, \mathcal{T})$ seems to be better behaved for the purposes of this paper, and because expansivity of groupoids is better captured by the universal covers of Cayley graphs then by the Cayley graphs themselves, as shown in [Nek15a]. We have that $\pi_G(n, \mathcal{T}) \leq \tilde{\pi}_G(n, \mathcal{T})$, and the two functions coincide if $\mathcal{G}$ is the action groupoid of an expansive action of a finitely generated group. It would be interesting to study the relation between these two notions of complexity further.

We now want to clarify the relation of the complexity of an expansive groupoid with the classical notion of complexity in symbolic dynamics. Let $G \curvearrowright X$ be an action of a finitely generated group on the Cantor set. Fix a finite generating set $S$ be of $G$ and let $\mathcal{P}$ be a clopen partition of $X$. Given a family of partitions $\{Q_i : i \in I\}$ we denote $\bigvee \{Q_i : i \in I\}$, its join, i.e. the smallest partition which is finer than every partition in the family. The complexity of the partition $\mathcal{P}$ with respect to $S$ is the function of the natural number $n \geq 0$ given by

$$p_G(n, S, \mathcal{P}) = |\bigvee \{g\mathcal{P} : g \in \bigcup_{r=0}^{n} S^n\}|.$$

A change of the generating set $S$ results in a function with the same growth type in the sense of the relation $\sim$. We simply denote $p_G(n, \mathcal{P})$ this growth type.

Recall that an action $G \curvearrowright X$ on the Cantor set is expansive if and only if there exists a generating partition, i.e. a finite partition $\mathcal{P}$ into clopen sets whose $G$-translates separate points.

Example 2.30. Assume that $G \curvearrowright X \subset A^G$ is a subshift over a finite alphabet $A$. Let $\mathcal{P} = \{P_a, a \in A\}$ be the partition of $X$ into cylinders subsets $P_a = \{x \in X : x(1) = a\}$. Then $\mathcal{P}$ is generating and the function $p_G(nS, \mathcal{P})$ coincides with the complexity of subshifts as defined in the introduction. Reciprocally, every generating partition of an expansive action allows to conjugate the action to a subshift over a finite alphabet such that $\mathcal{P}$ has this form.

Proposition 2.31. Let $G$ be a finitely generated group, and let $G \curvearrowright X$ be an expansive action of $G$ on the cantor set and $\mathcal{P}$ be a generating partition of the action. Let $\mathcal{G}$ be either the action groupoid, or the groupoid of germs of the action. Then we have

$$p_G(n, \mathcal{P}) \sim \pi_G(n).$$

In particular the growth type of $\pi_G(n, \mathcal{P})$ according to $\sim$ is independent of the generating partition and is an invariant of the groupoid $\mathcal{G}$.

Proof. We first prove the following Lemma.
Lemma 2.32. Let $\mathcal{G}$ be a groupoid with Cantor set unit space $X$. Let $G \leq \mathbb{F}(\mathcal{G})$ be a subgroup of the full group generated by a finite symmetric set $S$, and let $\mathcal{P}$ be a clopen partition of $X$. Consider the finite set of bisections $\mathcal{T} = \{T_sA: s \in S, A \in \mathcal{P}\}$. Then for every $n \geq 0$ we have

$$\bigvee \{g\mathcal{P} : g \in \bigcup_{r=0}^{n} S^r\} = \mathcal{Q}_T(n+1).$$

Proof. Let us check that the partition $\mathcal{Q}_T(n+1)$ is finer than the partition $\bigvee \{g\mathcal{P} : g \in \bigcup_{r=0}^{n} S^r\}$. Let $x$ and $y$ be in the same atom of $\mathcal{Q}_T(n+1)$. If they are not in the same atom of $\bigvee \{g\mathcal{P} : g \in \bigcup_{r=0}^{n} S^r\}$, there exists $s_1, \ldots, s_k \in S$, with $k \leq n$, such that $s_k \cdots s_1 x$ and $s_k \cdots s_1 y$ lie in different elements of $\mathcal{P}$. We may assume $k$ to be minimal with this property.

Let $x_0 = x, y_0 = y$ and $x_i = s_i \cdots s_1 x, y_i = s_i \cdots s_1 y, i = 1, \ldots, r$. It follows that if $A_0, \ldots, A_k$ are the elements of $\mathcal{P}$ in which $x_0, \ldots, x_k$ lie, then we have $y_0 \in A_0, \ldots, y_{k-1} \in A_{k-1}$ but $y_k \notin A_k$. Choose arbitrarily $s_{k+1} \in S$. Then $T = (T_{s_{k+1} A_k}) (T_{s_k A_{k-1}}) \cdots (T_{s_1 A_0})$ is an element of $\bigcup_{j \geq n} \mathcal{T}^j$ such that $x \in s(T)$ and $y \notin s(T)$, contradicting that $x$ and $y$ are in the same atom of $\mathcal{Q}_T(n+1)$. The proof that the partition $\bigvee \{g\mathcal{P} : g \in \bigcup_{r=0}^{n} S^r\}$ is finer than the partition $\mathcal{Q}_T(n+1)$ is completely analogous. $\square$

Let $S$ be a finite symmetric generating set of $G$. Let $\bar{T}$ be the image of $G$ into $\mathbb{F}(\mathcal{G})$ and let $\mathcal{T}$ be the set of bisections obtained from $(\bar{T}, \mathcal{P})$ as in the statement of Lemma 2.32, which is a generating set of $\mathcal{G}$. Since the partition $\mathcal{P}$ is generating, it follows from Lemma 2.32 that $\bigvee_{n\geq1} \mathcal{Q}_T(n)$ is the point-partition of $X$. By [Nek15a, Prop. 5.3], this implies that $\mathcal{T}$ is an expansive generating set of $\mathcal{G}$, therefore using again Lemma 2.32 we have $p_G(n, \mathcal{P}) = \pi_G(n, \mathcal{T}) \sim \pi_G(n)$. $\square$

3. Confined subgroups and proximal actions

Let $G$ be a group acting faithfully by homeomorphisms on a Hausdorff space $X$. Given an open subset $U \subset X$, we let $R_G(U)$ be its rigid stabiliser, i.e. the point-wise fixator of its complement. In [LBMB16, Sec. 3], a method was developed to relate confined and uniformly recurrent subgroups of $G$ to the rigid stabilisers of such an action. In this section, we recall some preliminaries from there and prove a refinement of one of these results (Theorem 3.3 below), that will be used later.

We first recall a well-known lemma in the case of normal subgroups.

Lemma 3.1. Let $G$ be a countable group acting faithfully by homeomorphisms on a Hausdorff space $X$. Let $A \leq G$ and let $R \leq G$ be a non-trivial subgroup normalised by $A$. Then there exists a non-empty open subset $U \subset X$ such that $R$ contains $[R_A(U), R_A(U)]$.

This is essentially [Nek13, Lemma 3.1]; we repeat a short proof since it is stated there for $A = G$. We use the commutator notation $[g, h] = ghg^{-1}h^{-1}$.

Proof. Let $g \in R$ be non-trivial, and let $U \subset X$ be a non-empty open subset such that $g(U) \cap U = \emptyset$. Let $a, b \in R_A(U)$, observe that $gag^{-1}$ is supported in $g(U)$ and therefore
commutes with $a$ and $b$. Using this and the fact that $A$ normalises $R$, we have $[a, b] = [[a, g], b] \in R$. Hence $[R_A(U), R_A(U)] \leq R$ since $a, b$ were arbitrary.

The following proposition is a variant of [LBMB16] Proposition 3.8.

**Proposition 3.2.** Let $G$ be a group of homeomorphisms of a Hausdorff space $X$, and $A \subseteq G$ be a subgroup. Let $H \subseteq \text{Sub}(G)$ be a subgroup confined by $A$. Let $P = \{g_1, \ldots, g_r\} \subset G \setminus \{1\}$ be a finite subset such that the $A$-orbit of $H$ avoids $U_P$, and assume that $U_1, \ldots, U_r \subseteq X$ are open subsets such that $U_1, \ldots, U_r, g_1(U_1), \ldots, g_r(U_r)$ are pairwise disjoint. Then there exists $\ell = 1, \ldots, r$, a finite index subgroup $\Gamma \leq R_A(U_\ell)$ and a subgroup $K \leq H$ such that the following hold:

(i) the group $K$ leaves $U_\ell$ invariant;

(ii) for every $\gamma \in \Gamma$, there exists $k \in K$ such that $k|U_\ell = \gamma$;

(iii) every $k \in K$ is supported in $U_1 \cup \cdots \cup U_r \cup g_1^{-1}(U_1) \cup \cdots \cup g_r^{-1}(U_r)$.

**Proof.** We only sketch the proof, as this it is exactly the same proof as [LBMB16] Proposition 3.8. We will use a lemma of B.H. Neumann, stating that an infinite group cannot be written as the union of finitely many cosets of infinite index subgroups [Neu54]. Let $L = R_A(U_1) \times \cdots \times R_A(U_r)$ be the subgroup of $A$ generated by the (pairwise commuting) rigid stabilisers $R_A(U_1), \ldots, R_A(U_r)$. Since every element of $G$ conjugates at least an element of $P$ inside $H$, we can write $L = \bigcup_{i=1}^r Y_i$ where $Y_i = \{\gamma \in L : \gamma g_i \gamma^{-1} \in H\}$. For every $i = 1, \ldots, r$ the set $Y_i$ is contained in a coset of the subgroup $L_i = \langle \gamma \delta^{-1} : \gamma, \delta \in Y_i \rangle$, and it follows that there exists $\ell = 1, \ldots, r$ such that $L_\ell$ has finite index in $L$. Let $\Gamma \leq R_A(U_\ell)$ be the image of $L_\ell$ under the natural projection of $L$ onto $R_A(U_\ell)$, and observe that it has finite index and that it is generated by the restriction to $U_\ell$ of all elements of the form $\gamma \delta^{-1}$ where $\gamma, \delta \in Y_\ell$. For $\gamma, \delta \in Y_\ell$ let $a_{\gamma, \delta} = (\gamma g_\ell^{-1} \gamma^{-1})(\delta g_\ell \delta^{-1}) \in H$ and let $K \leq H$ be the subgroup $K = \langle a_{\gamma, \delta} : \gamma, \delta \in Y_\ell \rangle$. Writing $a_{\gamma, \delta} = \gamma (g_\ell^{-1} \gamma^{-1} \delta g_\ell \delta^{-1})$ we see that each element $a_{\gamma, \delta}$ for $\gamma, \delta \in Y_\ell$ is supported in $U_1 \cup \cdots \cup U_r \cup g_1^{-1}(U_1) \cup \cdots \cup g_r^{-1}(U_r)$, leaves $U_\ell$ invariant, and coincides with $\gamma \delta^{-1}$ in restriction to $U_\ell$. It follows that the group $K$ generated by them verifies the desired conclusions.

Recall the the action of $G$ on $X$ is said to be **proximal** if for every pair of points $x, y \in X$, there exists a net $(g_i)$ of elements of $G$ such that $(g_i x)$ and $(g_i y)$ both converge to the same limit.

**Theorem 3.3.** Let $G$ be a countable group acting faithfully by homeomorphisms of a Hausdorff space $X$, and assume $A \subseteq G$ is a subgroup whose action on $X$ is minimal and proximal. Let $H \subseteq \text{Sub}(G)$ be confined by $A$. Then there exists a non-empty open subset $U \subseteq X$ and a finite index subgroup $\Gamma \leq R_A(U)$ such that $H$ contains the derived subgroup $[\Gamma, \Gamma]$.

**Remark 3.4.** This result was shown in [LBMB16] Theorem 3.10] under the stronger assumption that the action of $G$ on $X$ is extremely proximal, i.e. for every proper closed subset $C \subseteq X$ there exists a net of elements $(g_i)$ of $G$ and a point $z \in X$ such that $g_i C$ converges to $z$. It would be interesting to know if the conclusion remains true without any assumption on the action.
Proof. We can and we shall assume that for every non-empty open set \( U \subset X \) the group \( R_A(U) \) is infinite, otherwise the statement is obvious. In particular, we can assume that \( X \) has no isolated points, as these have a trivial rigid stabiliser.

Let \( P = \{g_1, \ldots, g_r\} \subset G \setminus \{1\} \) be such that the \( A \)-orbit of \( H \) avoids \( U_P \). Since \( X \) has no isolated point, every element \( g_i \), \( i = 1, \ldots, r \) moves infinitely many points of \( X \). In particular, we can find \( x_1, \ldots, x_r \) such that \( x_1, \ldots, x_r, g_1(x_1), \ldots, g_r(x_r) \) are pairwise distinct points. Let us denote \( Q = \{x_1, \ldots, x_r\} \cup (\bigcup_{i=1}^r g_i^{-1}\{(x_1, \ldots, x_r)\}) \subset X \).

For every \( i = 1, \ldots, r \), choose a neighbourhood \( U_i \) of \( x_i \), small enough so that the sets \( U_1, \ldots, U_r, g_1(U_1), \ldots, g_r(U_r) \) are pairwise disjoint. We can therefore apply Proposition 3.2. Let \( \ell \in \{1, \ldots, r\} \), \( \Gamma < A_{U_\ell} \) and \( K < H \) be given by the lemma.

Using minimality and proximality of the action of \( A \) on \( X \), we can find \( a \in A \) such that \( a(Q) \subset U_\ell \). For every \( i = 1, \ldots, r \) let \( U'_i \subset U_i \) be a smaller neighbourhood of \( x_i \) to be determined shortly. Set \( W = U'_1 \cup \cdots \cup U'_r \cup (\bigcup_{i=1}^r g_i^{-1}(U'_1 \cup \cdots \cup U'_r)) \), and observe that it is a neighbourhood of \( Q \) that shrinks to \( Q \) when all the \( U_i' \)'s shrink to \( x_i \). We choose \( U'_i \) small enough so that \( a(W) \subset U'_i \). We now apply again Proposition 3.2 this time to the group \( H' = a^{-1}Ha \) (whose \( A \)-orbit, being the same as the \( A \)-orbit of \( H \), it still doesn't intersect \( U_P \)) and to the open sets \( U'_1, \ldots, U'_r \) (that still verify the disjointness assumption in the proposition, since \( U'_i \subset U_i \)). Let \( \ell', \Gamma' < R_A(U'_P) \) and \( K' < H' \) be given by the proposition. Since we are assuming that all rigid stabilisers are infinite, the group \( \Gamma' \) is non-trivial, and therefore so is \( K' \) by part (iii) of the proposition. By part (iii) \( K' \) is supported in \( W \). It follows that \( aK'a^{-1} \) is a non-trivial subgroup of \( H \) supported in \( U_\ell \).

From this point on, the proof proceeds exactly as the last part of the proof of [LBMB16, Theorem 3.10] (with the minor difference that the proof is written there for \( A = G \)). We repeat the argument for the convenience of the reader. Let \( b \in aK'a^{-1} \) be a non-trivial element. For every \( \gamma \in \Gamma \), choose \( k_\gamma \in K \) which coincides with \( \gamma \) in restriction to \( U_\ell \). Observe that, since the support of \( x \) is contained in \( U_\ell \) we have \( k_\gamma b k_\gamma^{-1} = \gamma x \gamma^{-1} \in H \). It follows that the subgroup \( R = \langle k_\gamma b k_\gamma^{-1} \rangle : \gamma \in \Gamma \rangle \leq H \) is normalised by \( \Gamma \). By Lemma 3.1 there exists a non-empty open subset \( V \subset U_\ell \) such that \( [R_\Gamma(V), R_\Gamma(V)] \leq R \leq H \). The conclusion follows by observing that \( R_\Gamma(V) = \Gamma \cap R_A(V) \) has finite index in \( R_A(V) \), as \( \Gamma \) is a finite index subgroup of \( R_A(U_\ell) \leq R_A(V) \). \( \square \)

4. Confined subgroups of topological full groups

Let \( \mathcal{G} \) be an étale groupoid as in in Notation \([2,4]\). In this section we characterise the confined subgroups of \( \mathbb{F}(\mathcal{G}) \) when \( \mathcal{G} \) is a minimal groupoid of germs. This will be an important technical tool used throughout the paper.

Theorem 4.1. Assume that \( \mathcal{G} \) is a minimal groupoid of germs. The following statements on a subgroup \( H \in \text{Sub}(\mathbb{F}(\mathcal{G})) \) are equivalent:

(i) the subgroup \( H \) is confined;
(ii) the subgroup \( H \) is confined by \( k(\mathcal{G}) \);
(iii) there exists a unique finite subset \( Q \subset X \), possibly empty, such that we have

\[ S^0_{k(\mathcal{G})}(Q) \leq H \leq S^0_{\mathbb{F}(\mathcal{G})}(Q). \]
Moreover let $H$ satisfy one of these equivalent conditions, and let $P \subset G \setminus \{1\}$ be a non empty finite subset such that the $\Lambda(G)$-orbit of $H$ does not intersect $\mathcal{U}_P$. Then the set $Q$ in (iii) verifies $|Q| \leq |P| - 1$.

Let us first prove some lemmas in preparation for the proof.

**Lemma 4.2.** Assume that $\mathcal{G}$ is minimal. Then the action of $\Lambda(G)$ on $X$ is proximal.

**Proof.** Let $x, y \in X$ and $U$ be an open subset. Using Lemma 2.13 we can find multielements $\mathfrak{g}, \mathfrak{d}$ of degree $3t$ such that $\mathfrak{g}(1, 1) = x, \mathfrak{d}(1, 1) = y$ and $\mathfrak{g}(2, 2), \mathfrak{d}(2, 2) \in U$ and the points $\mathfrak{g}(i, i), \mathfrak{d}(j, j), i, j = 1, \ldots, 3$ are pairwise disjoint. For any sufficiently small multisecti ons $\mathfrak{f}' \supseteq \mathfrak{g}$ and $\mathfrak{f} \supseteq \mathfrak{d}$ the element $g = \mathfrak{f}((123))\mathfrak{f}((123)) \in \Lambda(G)$ is such that $g\{x, y\} \in U$. □

**Lemma 4.3.** Assume that $\mathcal{G}$ is a minimal groupoid of germs. Let $Q \subset X$ be a finite set. Then we have $\Lambda(G|Q^c) = S^0_{\Lambda(G)}(Q)$.

**Proof.** The inclusion $\Lambda(G|Q^c) \leq S^0_{\Lambda(G)}(Q)$ is clear. Let $g \in S^0_{\Lambda(G)}(Q)$ and let us show that $g \in \Lambda(G|Q^c)$. Since $g \in \Lambda(G)$, there exist degree $3$ multisecti ons $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ of $\mathcal{G}$ and elements $s_i \in \Lambda(\mathfrak{f}_i)$ such that $g = s_n \cdots s_1$. Note that we can assume that $s_i$ is of the form $s_i = \mathfrak{f}_i((123))$ for every $i = 1, \ldots, n$. Let $V$ be a neighbourhood of $Q$ such that $g$ fixes $V$ point-wise.

If none of the domains of the multisecti ons $\mathfrak{f}_i$ intersects $Q$, we have $\Lambda(\mathfrak{f}_i) \leq \Lambda(G|Q^c)$ for every $i = 1, \ldots, n$ and the conclusions follows. Otherwise there are two cases to consider. Assume at first that the union of the domains of the multisecti ons $\mathfrak{f}_i$ does not cover the whole unit space $X$. Let $U \subset X$ be a clopen set which does not intersect the domain of any of the multisecti ons $\mathfrak{f}_i$, and note that $g$ fixes $U$ point-wise. Using Lemma 2.13 and minimality of $\mathcal{G}$, we can find an element $h \in \Lambda(G)$ such that $h(Q) \subset U$ and such that $h$ fixes point-wise the complement of $V \cup U$. In particular, $h$ commutes with $g$ and hence $g = h^{-1}gh = (h^{-1}s_nh) \cdots (h^{-1}s_1h) \in \langle \Lambda(h^{-1}\mathfrak{f}_1), \ldots, \Lambda(h^{-1}\mathfrak{f}_n) \rangle$. Moreover $Q \subset h^{-1}(U)$ and the set $h^{-1}U$ avoids the domain of all the multisecti ons $h^{-1}\mathfrak{f}_i, i = 1, \ldots, n$. Hence we are reduced to the previous case.

Assume now that the union of the domains of the multisecti ons $\mathfrak{f}_i$ covers $X$. Define inductively sets $Q_0, \ldots, Q_n$ by $Q_0 = Q$ and $Q_{i+1} = Q_i \cup s_{i+1}(Q_i) \cup s_{i+1}^2(Q_i)$. Let $W_0$ be a clopen neighbourhood of $Q$ contained in $V$, and define inductively sets $W_i, i = 1, \ldots, n$ by $W_{i+1} = W_i \cup s_{i+1}(W_i) \cup s_{i+1}^2(W_i)$. If $W_0$ shrinks to $Q$ we have that $W_n$ shrinks to $Q_n$, and therefore we may assume that $W_n$ is a proper subset of $X$. For every $i = 1, \ldots, n$ let $\mathfrak{f}_i' \subset \mathfrak{f}_i$ be the multisecti on consisting of all multigerms $\mathfrak{g} \in \mathfrak{f}_i$ with $\mathfrak{g}(1, 1) \in W_i$, let $\mathfrak{f}_i'' \subset \mathfrak{f}_i$ be its complement, and set $s_i' = \mathfrak{f}_i'((123)), s_i'' = \mathfrak{f}_i''((123))$ (if $\mathfrak{f}_i'$ or $\mathfrak{f}_i''$ are empty, set $s_i' = 1$ or $s_i'' = 1$ accordingly). Observe that $s_i'$ and $s_i''$ are commuting elements such that $s_i = s_i's_i''$. Set $g' = s_n' \cdots s_1'$ and $g'' = s_n'' \cdots s_1''$. Since the domain of $\mathfrak{f}_i'$ avoids $W_i \supset Q$ we have $g'' \in \Lambda(G|Q^c)$. Moreover for $i < j$ we have that each $s_i'$ is supported in $W_i \subset W_j$ and $s_j''$ is supported in the complement of $W_j$, and therefore $s_i'$ and $s_j''$ commute. We deduce that $g = g'g''$. Since for every $i$ the domain of $\mathfrak{f}_i'$ is contained in $W_i \subset W_n$ which is a proper subset of $X$, we have $g' \in \Lambda(G|Q^c)$ by the previous case, and hence $g \in \Lambda(G|Q^c)$. □
Lemma 4.4. Let $d \geq 1$. Then the alternating group $\text{Alt}(3d)$ is generated by $\text{Alt}(\{d + 1, \ldots, 3d\}) \cup \left( \bigcup_{i=1}^{d} \text{Alt}(\{i, d + i, 2d + i\}) \right)$.

The proof is elementary and we omit it.

Proof of Theorem 4.1. To see that $\text{Alt}(3)$, it is enough to check that for every finite set $Q \subset X$, the subgroup $S_0^{\mathcal{A}(G)}(Q)$ is confined. Let $r = |Q|$ and choose $r + 1$ degree 3 multisected with disjoint domains $\mathfrak{f}_1, \ldots, \mathfrak{f}_{r+1}$ (for example using Lemma 2.18). Then for every $g \in \mathbb{F}(G)$, there exists $i$ such that the set $g(Q)$ does not intersect the domains of $\mathfrak{f}_i$, and therefore $A(\mathfrak{f}_i) \leq gS_0^{\mathcal{A}(G)}(Q)g^{-1}$. For every $i = 1, \ldots, r + 1$, choose a non-trivial element $g_i \in A(\mathfrak{f}_i)$ and let $P = \{g_1, \ldots, g_{r+1}\}$. Then the $\mathcal{F}(G)$-orbit of $S_0^{\mathcal{A}(G)}(Q)$ does not intersect the open neighbourhood of the identity $\mathcal{U}P$.

The implication $[\text{iii}] \Rightarrow [\text{ii}]$ is obvious.

It remains to be shown that $[\text{ii}] \Rightarrow [\text{iii}]$ which is the main part of the statement. To this end, let $H \leq \mathbb{F}(G)$ be as in [\text{ii}]. Fix also a finite subset a set $P \subset G \setminus \{1\}$ such that $\mathcal{A}(G)$-orbit of $H$ avoids $\mathcal{U}P$, and write $r = |P|$. We divide the proof into a sequence of steps.

Step 1. Let $D_1 \ldots D_r \subset X$ be disjoint finite subsets, and let $V \subset X$ be an open set. Then there exists $h \in H$ such that $h(D_j) \subset V$.

Proof. Let $g_1, \ldots, g_r$ be the elements of $P$. Since $aP \setminus H$ is not empty for every $a \in \mathcal{A}(G)$, it is enough to observe that there exists an element $a \in \mathcal{A}(G)$ such that $aD_i \subset V$ for every $i = 1, \ldots, r$. We now explain how to construct such an element. Let $U_1, \ldots, U_r \subset X$ be open sets such that $U_1, \ldots, U_r, g_1(U_1), \ldots, g_r(U_r)$ are pairwise disjoint (such sets exist for any finite family of non-trivial homeomorphisms of $X$, see e.g. Lemma 3.1). Set $D = D_1 \cup \cdots \cup D_r$. Since $\mathcal{G}$ is minimal, for every $k = 1, \ldots, r$ and every $x \in D_k$ we can find an element $\gamma_x$ such that $x = s(\gamma_x) \neq r(\gamma_x) \in U_k \setminus (D \cup g_1^{-1}(D) \cup \cdots \cup g_r^{-1}(D))$. Moreover, we can find such elements in such a way that the points $r(\gamma_x), x \in D_k, k = 1, \ldots, r$ are pairwise distinct (for instance, this can be done by ordering all points of $D$ in a list and choosing each $\gamma_x$ at a time in this order; note that at each step only finitely many new points are forbidden for $r(\gamma_x)$). For every $k = 1, \ldots, r$, let $D'_k = \{r(\gamma_x): x \in D_k\} \subset U_k$ and set $D' = D_1 \cup \cdots \cup D'_k$. Let also $D'' = g_k(D'_k) \subset g_k(U_k)$ and $D'' = D'_1 \cup \cdots \cup D''_r$. Note that, by the choices made, the sets $D, D', D''$ are pairwise disjoint. Proceeding in a similar way, choose for every $x \in D''$ an element $\gamma_x \in \mathcal{G}$ such that $x = s(\gamma_x) \neq r(\gamma_x) \in V \setminus (D \cup D' \cup D'')$, and the points $r(\gamma_x)$ for $x \in D''$ are pairwise distinct. For $k = 1, \ldots, r$, let $D''_k = \{r(\gamma_x), x \in D''_k\}$ and $D'' = D'_1 \cup \cdots \cup D''_r$. Finally using Lemma 2.18 find for every $x \in D''$ a degree 3 multisection $g_x$ such that $g_x(1, 2) = \gamma_x$ for every $x \in D''$ and the units $g_x(3, 3)$ for $x \in D''$ are pairwise disjoint and do not belong to $D \cup D' \cup D'' \cup D'''$. Using Lemma 2.18 again, we can find for every $x \in D''$ sufficiently small multisections $\mathfrak{f}_x \supset g_x$ that have pairwise disjoint domains. Let $\sigma = (123) \in \text{Alt}(3)$ and let $a = (\prod_{x \in D} \mathfrak{f}_x(\sigma^{-1})) (\prod_{x \in D'} \mathfrak{f}_x(\sigma))$. For every $k = 1 \ldots r$ we have $a^{-1}(D_k) = D'_k$ and $a(D''_k) = D''_k$ and therefore $a = a^{-1}(D_k) = D''_k \subset V$. This concludes the proof of the Step 1.

A special case of the previous step yields the following one.
Step 2. Among closed $H$-invariant proper subsets of $X$ there is a unique maximal one with respect to inclusion (possibly empty), denoted $Q$. The set $Q$ is finite and verifies $|Q| \leq r - 1$. This will be the set in the statement of the theorem.

Proof. Let $Z \subset X$ be a proper closed $H$-invariant subset. Assume by contradiction that $x_1, \ldots, x_r$ are distinct points in $Z$. By Step 1 applied to the sets $D_i = \{x_i\}, i = 1, \ldots, r$, and to the open set $V = X \setminus Z$, there exists $j$ and $h \in H$ such that $h(x_j) \notin Z$, contradicting the invariance of $Z$. It follows that every closed $H$-invariant proper subset $Z \subset X$ is finite and satisfies $|Z| \leq r - 1$. Let $(Z_n)_{n \geq 0}$ be a sequence of closed $H$-invariant proper subsets. Since for every $m \geq 0$ the union $\bigcup_{n=0}^{m} Z_n$ is also a proper closed invariant subset, it has cardinality at most $r - 1$, and it follows that it has to stabilise for $m$ large enough and therefore the sequence $Z_n$ takes only finitely many values. Therefore there are only finitely many closed $H$-invariant proper subsets and their union $Q$ verifies the conclusion. □

Step 3. There exists a clopen subset $U \subset X$ such that $H$ contains $\mathcal{A}(\mathcal{G}|U)$.

Proof. Since the action of $\mathcal{A}(\mathcal{G})$ on $X$ is proximal, we can apply Theorem 3.3. We obtain an open set $U \subset X$ and a finite index subgroup $\Gamma < R_{\mathcal{A}(\mathcal{G})}(U)$ such that $H$ contains $[\Gamma, \Gamma]$. Since $\mathcal{A}(\mathcal{G}|U) < R_{\mathcal{A}(\mathcal{G})}(U)$ is infinite and simple, we have $\mathcal{A}(\mathcal{G}|U) < [\Gamma, \Gamma]$. □

Step 4. Let $d \geq 3$ and let $g$ be a degree $d$ multielement of $\mathcal{G}|Q^c$. Then for every multisection $\mathcal{T} \ni g$ there exists a multisection $\mathcal{F}$ with $g \in \mathcal{F} \subset \mathcal{T}$ and $A(\mathcal{F}) \leq H$.

Proof. First observe that if there exists an element $h \in H$ such that $hg \in \mathcal{G}|U$, then for every sufficiently small bisection $\mathcal{F} \ni g$ we have that $h\mathcal{F}$ is a multisection of $\mathcal{G}|U$ and $hA(\mathcal{F})h^{-1} = A(h\mathcal{F}) \leq H$ by Step 3 whenever the conclusion. In general, since for every $i = 1, \ldots, d$ the unit $g(i, i)$ does not belong to $Q$, it has a dense orbit by the definition of $Q$. Therefore there exist elements $h_1, \ldots, h_d$ such that $h_ig(i, i) \in U$ for every $i = 1, \ldots, d$. For every $i = 1, \ldots, d$ let $V_i$ be a neighbourhood of $g(i, i)$ such that $h_i(V_i) \subset U$. By minimality of $\mathcal{G}$ we can choose disjoint sets $D_1, \ldots, D_r$ that consist of points that lie in the same orbit of $g(1, 1), \ldots, g(d, d)$ and are disjoint from these and from $Q$, and such that $|D_j \cap V_i| = 2$ for every $j = 1, \ldots, r$ and every $i = 1, \ldots, d$. In particular $|D_j| = 2d$ for every $j = 1, \ldots, r$. By Step 1 there exists $k \in H$ and $\ell = 1, \ldots, r$ such that $kD_\ell \subset U$. Fix such $\ell$ and $h$ and set $D_\ell = \{x_{d+1}, \ldots, x_{3d}\}$ where the numbering is chosen in such a way that $x_{d+i}, x_{d+2i} \in V_i$ for every $i = 1, \ldots, d$. By Lemma 2.18 we can find a degree $3d$ multielement $g'$ such that $g'[1, 3d] = g$ and $g'(j, j) = x_j$ for $j = d + 1, \ldots, 3d$. Observe that we have $h_ig'[1, 3d] \in \mathcal{G}|U$ and $h_ig'[i, d+i, 2d+i] \in \mathcal{G}|U$ for every $i = 1, \ldots, d$. It follows that if $\mathcal{F}'$ is any sufficiently small multisection containing $g'$, we have $A(\mathcal{F}'|i, d+i, 3d] \leq H$ and $A(\mathcal{F}'|i, d+i, 2d+i]) \leq H$ for every $i = 1, \ldots, d$. By Lemma 1.4 we deduce that $A(\mathcal{F}') \leq H$. Setting $\mathcal{F} = \mathcal{F}'|1, d] \subset \mathcal{F}$ we therefore have $A(\mathcal{F}) \leq H$. By Lemma 2.18 we can choose $\mathcal{F}' \subset \mathcal{T}$ sufficiently small so that $\mathcal{F} \subset \mathcal{T}$ as desired. □

Step 5. $H$ contains $S^0_{\mathcal{A}(\mathcal{G})}(Q)$.

Proof. Let $\mathcal{T}$ be a degree $d \geq 3$ a multisection of $\mathcal{G}|Q^c$. By Step 1 and by compactness of $\mathcal{T}$ we can cover $\mathcal{T}$ with finitely many degree $d$ multisections $\mathcal{F}$ such that $A(\mathcal{F}) \leq H$. By
Proposition 2.20, we have \( A(\mathcal{T}) \leq H \). It follows that \( \mathcal{A}(\mathcal{G}|Q^c) \leq H \) since \( \mathcal{T} \) was arbitrary. The conclusion follows from Lemma 4.3.

**Step 6.** The set \( Q \) satisfies \([\text{iii}]\) in the statement of the theorem.

**Proof.** The inclusion \( H \leq S_{\mathcal{F}(\mathcal{G})}(Q) \) follows by the definition of \( Q \) in Step 2 and the inclusion \( S_{\mathcal{A}(\mathcal{G})}^0(Q) \leq H \) was proven in Step 5. Only uniqueness is left to prove. Assume that \( Q_1, Q_2 \) satisfy the condition. In particular we have \( S_{\mathcal{A}(\mathcal{G})}^0(Q_1) \leq H \leq S_{\mathcal{F}(\mathcal{G})}(Q_2) \). Note that every \( x \notin Q_1 \) has an infinite dense orbit under the action of \( S_{\mathcal{A}(\mathcal{G})}^0 \) and therefore we deduce that \( Q_2 \subset Q_1 \). Exchanging the roles, \( Q_1 = Q_2 \).

Finally note that the set \( Q \) verifies \( |Q| \leq |P| - 1 \) by its definition in Step 2 (recall that \( r = |P| \)). This concludes the proof of the theorem.

In practice, the working consequence of Theorem 4.1 will be the following.

**Corollary 4.5.** Let \( G \) be a group such that \( \mathcal{A}(\mathcal{G}) \leq G \leq \mathcal{F}(\mathcal{G}) \). A subgroup \( H \in \text{Sub}(G) \) is confined if and only if there exists a finite subset \( Q \subset X \) such that \( S_{\mathcal{A}(\mathcal{G})}^0(Q) \leq H \leq S_G(Q) \). Moreover we have \( |Q| \leq |P| - 1 \), where \( P \subset G \setminus \{1\} \) is any subset such that the conjugacy class of \( H \) avoids \( U_P \).

**Proof.** If \( H \in \text{Sub}(G) \) is confined, then it is confined by \( \mathcal{A}(\mathcal{G}) \) and hence the theorem applies.

Moreover, we have the following characterisation of the uniformly recurrent subgroups of the group \( \mathcal{A}(\mathcal{G}) \).

**Corollary 4.6.** Assume that \( Y \subset \text{Sub}(\mathcal{A}(\mathcal{G})) \) is a uniformly recurrent subgroup of \( \mathcal{A}(\mathcal{G}) \) such that \( Y \neq \{\{1\}\}, \{\mathcal{A}(\mathcal{G})\} \). Then \( X \) is compact, and \( Y \) is the stabiliser URS associated to the action \( \mathcal{A}(\mathcal{G}) \times X \).

**Proof.** Let \( H \in Y \). Since \( H \) is confined, there exists \( Q \) such that \( S_{\mathcal{A}(\mathcal{G})}^0(Q) \leq H \leq S_G(Q) \). Let \( P_r(X) \) be the compact space of finite subsets of \( X \) of cardinality at most \( r \). If \( X \) is not compact, we can find a sequence \( (g_n) \in \mathcal{A}(\mathcal{G}) \) such that \( g_n(Q) \to \emptyset \) in \( P_r(X) \). Let \( K \in Y \) be any cluster point of \( g_nHg_n^{-1} \). By the lower semicontinuity of \( Q \mapsto S_{\mathcal{A}(\mathcal{G})}^0(Q) \) (Lemma 2.2), we obtain that \( K \) contains \( S_{\mathcal{A}(\mathcal{G})}^0(\emptyset) = \mathcal{A}(\mathcal{G}) \) and hence \( Y = \{K\} = \{\mathcal{A}(\mathcal{G})\} \), a contradiction. Otherwise, by proximality and minimality we can find for every \( x \in X \), a sequence \( (g_n) \) as above such that \( g_n(Q) \to \{x\} \). Using again semi-continuity we get that every cluster point \( K \) of \( g_nHg_n^{-1} \) satisfies \( S_{\mathcal{A}(\mathcal{G})}^0(x) \leq K \leq S_{\mathcal{A}(\mathcal{G})}(x) \). Choosing \( x \) to be a regular point, we get \( K = S_{\mathcal{A}(\mathcal{G})}(x) \in Y \), and therefore by minimality \( Y \) must coincide with the stabiliser URS associated to \( \mathcal{A}(\mathcal{G}) \times X \).

4.1. Example: AF-groupoids and confined subgroups of the LDA-groups. An important class of étale groupoids are the AF-groupoids associated to a Bratteli diagram.

**Definition 4.7.** A Bratteli diagram is a graph \( B = (V, E) \), possibly with multiple edges, together with a map \( d \geq V \to \mathbb{N} \), called the labelling, such that the following hold.
(1) there are partitions $V = \sqcup_{i \geq 1} V_i$ and $E = \sqcup_{i \geq 1} E_i$ of the set of vertices and of edges into non-empty finite subsets, and we have $|E_i| \geq 2$ for every $i$;

(2) for every $i \geq 0$ there are maps $o: E_i \rightarrow V_i$ and $t: E_i \rightarrow V_{i+1}$, such that every edge $e \in E_i$ has endpoints $o(e) \in V_i$ and $t(e) \in V_{i+1}$;

(3) for every $v \in V_i$, $i \geq 0$ there exists at least an edge $e \in E_i$ with $o(e) = v$;

(4) for every $v \in V_i$, we have $k(v) := d(v) - \sum_{e: t(e) = v} (d(s(e)) \geq 0$;

(5) we have $d(v) \geq 1$ if $v \in V_1$.

The Bratteli diagram $B$ is said to be simple if for every $i$ there exists $j > i$ such that every vertex of $V_i$ is connected to every vertex of $V_j$. It is said to be unital if $k(v) = 0$ for all but finitely many vertices $v \in V$.

For every vertex $v \in V$, choose a finite alphabet $A^v = \{a_1^v, a_2^v, \ldots, a_k^v\}$ such that $A^v \cap A^w = \emptyset$ if $v \neq w$ (if $k(v) = 0$, set $A^v = \emptyset$). Define the path space $X_B$ of the Bratteli diagram $B$ to be the set of infinite formal words of the form $a_j^v e_i e_{i+1} e_{i+2} \ldots$ where $v \in V$, $1 \leq j \leq k(v)$, $o(e_i) = v$ and $t(e_i) = o(e_{i+1})$ for every $\ell \geq i$. The space $X_B$ is endowed with the natural product topology, and it is compact if and only if $B$ is unital. A finite path is a finite word $\gamma$ which is a prefix of an element of $X_B$. If $\gamma$ is a finite path of the form $\gamma = a_j^v e_i e_{i+1} \cdots e_k$, its terminal vertex is the vertex $t(\gamma) = t(e_k) \in V_k$, and if $\gamma$ is of the form $a_j^v$ we set $t(\gamma) = v$. Let $\Delta_v$ be the set of finite paths that have $v$ as terminal point. Given a finite path $\gamma$, we denote by $U_{\gamma} \subset X_B$ the set of paths that have $\gamma$ as a prefix. These sets form a basis of clopen sets for the topology on $X_B$.

Fix a vertex $v$. For every $\gamma_1, \gamma_2 \in \Delta_v$, we have a natural homeomorphism $\tau_{\gamma_1, \gamma_2}: U_{\gamma_1} \rightarrow U_{\gamma_2}$ given by $\gamma_1 e_k e_{k+1} \cdots \mapsto \gamma_2 e_k e_{k+1} \cdots$. These maps generate a pseudogroup of partial homeomorphisms of $X_B$. The corresponding groupoid of germs, with its natural étale topology, will be denoted $\mathcal{G}_B$. The groupoid $\mathcal{G}_B$ is minimal if and only if the diagram $B$ is simple.

**Definition 4.8.** An étale groupoid is called an **AF-groupoid** if it is of the form $\mathcal{G}_B$ for some simple Bratteli diagram $B$.

If $\mathcal{G}_B$ be is AF-groupoid, the group $\mathbb{A}(\mathcal{G}_B)$ is a simple locally finite group, that can be expressed as the ascending union $\bigcup_{i \geq 1} \prod_{v \in V_i} \text{Alt}(\Delta_v)$ of finite alternating groups, where each alternating group $\text{Alt}(\Delta_v)$ acts on $U_v = \sqcup_{\gamma \in \Delta_v} U_\gamma$ by permuting initial finite prefixes, and acting trivially outside $U_v$. In fact, in this case the group $\mathbb{A}(\mathcal{G})$ coincides with the derived subgroup of $\mathbb{D}(\mathcal{G})$ of the topological full group. For more details on topological full groups of AF-groupoids, see [Mat06 Section 3].

By a result of Lavrenyuk–Nekrashevych [LN07], the class of simple locally finite groups arising in this way are precisely the **strongly diagonal limits** of products of alternating groups, or **LDA-groups** (with the single exception of the infinite finitary alternating group $\text{Alt}_f(\mathbb{N})$, which is an LDA-group but does not arise in this way).

Confined subgroups of simple locally finite groups have been extensively studied [HZ97, LP03, LP02]. Confined subgroups of the LDA-groups were studied by Leinen–Puglisi in [LP03], who provided a characterisation of the confined subgroups in the special case of
diagonal limits of alternating groups (the case of $\text{Alt}_f(\mathbb{N})$ had been previously treated in [SZ93]). Theorem 4.1 extends this characterisation to all the LDA-groups.

**Corollary 4.9.** Let $G$ be an LDA-group, with $G \neq \text{Alt}_f(\mathbb{N})$, and let $B$ be a simple Bratteli diagram such that $G \simeq \Lambda(B)$. A subgroup $H < G$ is confined if and only if $H$ contains the point-wise fixator of a finite set $Q \subset X_B$ as a subgroup of finite index.

**Proof.** By Corollary 4.5, $H$ is confined if and only if there exists a unique set $Q$ such that $S_{\Lambda(B)}^0(Q) \leq H \leq S_{\Lambda(B)}(Q)$. But an AF-groupoid is principal, hence $S_{\Lambda(B)}^0(Q) = F_{\Lambda(B)}(Q)$ and the latter has finite index in $S_{\Lambda(B)}(Q)$. □

Corollary 4.6 recovers a recent result of Thomas [Tho17, Theorem 2.8].

**Corollary 4.10.** Let $B$ be a simple Bratteli diagram, and assume that $Y \subset \text{Sub}(\Lambda(B))$ is a non-trivial uniformly recurrent subgroup. Then $B$ is unital, and $Y = \{S_{\Lambda_B}(x) : x \in X_B\}$.

5. **Actions by bounded displacement and growth of Schreier graphs**

Let $\Delta$ be a bounded degree graph. Recall that we denote $B_\Delta(n, v)$ the the ball of radius $n$ around $v \in \Delta$, and denote $\overline{B}_\Delta(n) = \sup_{v \in \Delta} |B_\Delta(n, v)|$.

The following lemma is straightforward from the definition of the Chabauty topology in terms of the space of marked graphs.

**Lemma 5.1.** Let $G$ be a finitely generated group with finite symmetric generating set $S$. Then for every function $f : \mathbb{N} \to \mathbb{N}$, the set of subgroups $H \leq G$ such that $\overline{B}_{\text{Ch}(G, S)}(n) \leq f(n)$ for every $n$ is closed in $\text{Sub}(G)$.

The group of permutations of bounded displacement, or **wobbling group** $W(\Gamma)$ is the group of all permutations $\sigma$ of the set of vertices of $\Gamma$ such that $\sup_{v \in \Gamma} d(v, \sigma(v)) < \infty$.

**Theorem 5.2.** Let $\mathcal{G}$ be a compactly generated minimal effective groupoid with Cantor set unit space, and let $\Delta$ be a bounded degree graph such that $\beta_\mathcal{G}(n) \not\leq \overline{B}_\Delta(n)$. Then there is no non-trivial homomorphism $\Lambda(\mathcal{G}) \to W(\Delta)$.

**Proof.** Let $K \leq \Lambda(\mathcal{G})$ be a finitely generated subgroup as in Lemma 2.25 and let $S$ be a finite generating set of $K$. Let $\Delta$ be a graph and consider a homomorphism $\Lambda(\mathcal{G}) \to W(\Delta)$. For every vertex $v \in \Delta$ the orbital Schreier graph $\Gamma_v(K, S)$ is Lipschitz-embedded into $\Delta$, and in particular we have $\overline{b}_{\Gamma_v(K, S)}(n) \leq \overline{B}_\Delta(n)$, with constant independent of the choice of $v$. There are two cases to consider. First, assume that there exists a sequence of vertices $(v_n)$ such that $S_{\Lambda(\mathcal{G})}(v_n)$ tends to $\{1\}$ in $\text{Sub}(\Lambda(\mathcal{G}))$. It follows that $S_K(v_n) = S_{\Lambda(\mathcal{G})}(v_n) \cap K$ tends to $\{1\}$ in $\text{Sub}(K)$ and therefore the corresponding Schreier graphs tend to the Cayley graph of $K$ in the space of marked graphs. In this case we deduce that $b_K(n, S) \leq \overline{b}_\Delta(n)$ by Lemma 5.1 where $b_K(n, S)$ denotes the growth function of $K$ with respect to $S$. The conclusion follows, since Lemma 2.25 and Lemma 5.1 together imply that $\beta_{\mathcal{G}}(n) \sim \overline{b}_{\Gamma_v(K, S)}(n) \leq b_K(n)$. Second, assume that there is no such sequence of vertices. In particular, for every vertex $v \in \Delta$ the group $S_{\Lambda(\mathcal{G})}(v)$ is confined, and therefore we can apply Theorem 4.1 and obtain the existence of a finite set $Q \subset X$ such that $S_{\Lambda(\mathcal{G})}^0(Q) \leq S_{\Lambda(\mathcal{G})}(v) \leq S_{\Lambda(\mathcal{G})}^0(Q)$. If
Theorem 5.2 is equivalent to the following.

**Corollary 5.3.** Assume that \( \mathcal{G} \) is expansive, and let \( S \) be any finite generating set of \( \mathcal{A}(\mathcal{G}) \). Then for every subgroup \( H \leq \mathcal{A}(\mathcal{G}) \), the Schreier graph \( \Gamma_H(\mathcal{A}(\mathcal{G})), S \) satisfies
\[
\frac{\beta_{\Gamma_H(\mathcal{A}(\mathcal{G}), S)}}{(n)} \geq \beta_G(n).
\]

**Proof.** This follows from the theorem, because the natural action of a finitely generated group on the set of vertices of each of its Schreier graphs is by permutation by bounded displacement.

**Corollary 5.4.** There exists a finitely generated group \( G \) with the following properties.

1. For every graph \( \Delta \) which grows uniformly subexponentially, i.e. \( \lim \frac{1}{n} \log \bar{b}_\Delta(n) \), every non-trivial homomorphism \( \mathcal{G} \to W(\Delta) \) is trivial. In particular the group \( G \) has no infinite Schreier graph whose growth is uniformly subexponential.

2. The group \( G \) has a co-amenable subgroup of infinite index.

**Proof.** Let \( H \) be a finitely generated amenable group of exponential growth (e.g. take \( H = \mathbb{Z}/2 \mathbb{Z} \)). Let \( H \curvearrowright X \) be a minimal, free expansive action on the Cantor set, which exists by [Eilenberg]. Let \( \mathcal{G} \) be the groupoid of germs of the action and consider the group \( G = \mathcal{A}(\mathcal{G}) \). Then \( G \) is finitely generated by Theorem 2.16. The orbital graphs of its action on \( X \) are quasi-isometric to \( H \) by 2.25 and hence \( \beta_G(n) \) is exponential. Therefore the first part follows from Theorem 5.2 and point-stabilisers of the action \( \mathcal{A}(\mathcal{G}) \curvearrowright X \) are co-amenable.

6. Dichotomy for actions and for homomorphisms

The purpose of this section is to show the main theorem of this paper (Theorem 6.9 below) from which Theorem 1.1 will follow.

Theorem 1.1 is about actions on compact spaces. However, we will take a more general point of view, as follows. An action of a group \( G \curvearrowright Y \) on a compact space can be equivalently thought as a homomorphism \( G \to \mathbb{F}(\mathcal{H}) \), where \( \mathcal{H} \) is a groupoid of germs. Instead of working directly with actions, it is convenient for us to view them as a special case of homomorphisms \( G \to \mathbb{F}(\mathcal{H}) \) to the topological full groups of an étale groupoid,
and to allow the target groupoid $H$ to be an arbitrary étale groupoid (not necessarily a groupoid of germs). Such a homomorphism $\rho: G \to \mathbb{F}(H)$ gives rise to an action $G \acts Y$ on the unit space of $Y$, obtained by composing $\rho$ with the natural action of $\mathbb{F}(H)$ on $Y$. But the homomorphism $\rho$ also carries additional information encoded by the cocycle:

$$G \times Y \to H; \quad (g, y) \mapsto \rho(g)|_y.$$  

When $H$ is a groupoid of germs, then $\rho(g)|_y$ simply identifies with the germ of $\rho(g)$ at $y$. In different situations it can carry a different information. In some sense, studying homomorphisms of a group $G$ to the topological full groups of étale groupoids is a generalisation of studying its actions on compact spaces.

6.1. **Inducing homomorphisms from groupoids to the full groups.** Our goal is to state a theorem characterising those homomorphisms between topological full groups that “arise from” the groupoids in a sense that we will make precise.

With this goal in mind, we begin by describing three basic constructions on étale groupoids that give rise to homomorphisms between topological full groups, that will play an important role in the statement of our main theorem. These are: homomorphisms arising from spatial inclusions between groupoids, blow up homomorphisms associated to actions of groupoids, and reduced diagonal homomorphisms associated to the reduced power of a groupoid.

**Definition 6.1** (Homomorphisms from spatial inclusions). Let $H_1, H_2$ be étale groupoids with unit spaces $Y_1, Y_2$. We say that a cocycle $b: H_1 \to H_2$ is a spatial inclusion if it is continuous, open, and in restriction to $Y_1$ it is an injective embedding $Y_1 \hookrightarrow Y_2$ with clopen image. (Whenever we are given such a spatial inclusion, we will simply identify $Y_1$ with a subset of $Y_2$). A spatial inclusion $b: H_1 \to H_2$ induces a homomorphisms between the corresponding topological full groups

$$b_*: \mathbb{F}(H_1) \to \mathbb{F}(H_2),$$  

where the image of every $g \in \mathbb{F}(H_1)$ is defined by

$$b_*(g)|_y = \begin{cases} b(g|_y) & \text{if } y \in Y_1 \subset Y_2 \\ y & \text{if } y \in Y_2 \setminus Y_1 \end{cases}$$  

It is easy to check that $b_*(g)$ is a well-defined element of $\mathbb{F}(H_2)$ and that the map $b_*$ is a homomorphisms.

**Example 6.2.** Assume that $H_1$ is the groupoid of germs of an action $G \acts Y_1$ of a countable group, and let $H_2$ be another étale groupoid. Then the data of a spatial inclusion $b: H_1 \to H_2$ is equivalent to the data of an embedding $Y_1 \hookrightarrow Y_2$ with clopen image, together with a group homomorphism $b: G \hookrightarrow \mathbb{F}(H_2)$ (denoted with the same letter $b$ by abuse of notation), such that the image of $G$ is supported in $Y_1$ and the corresponding actions of $G$ agree. In this situation, we have an obvious homomorphisms $b_*: \mathbb{F}(H_1) \to \mathbb{F}(H_2)$ (in fact an embedding).
Next, we define blow-up homomorphisms associated to actions of groupoids. (The definition of groupoid actions and of the action groupoids associated to them are recalled in Subsection 2.4.)

Definition 6.3 (Blow-up from groupoid actions). Let \( \mathcal{H} \) be an étale groupoid with compact unit space \( Y \). Let \( (Z, q) \) be a compact fibre space over \( Y \) and, and \( \alpha : \mathcal{H} \curvearrowright (Z, q) \) be a continuous groupoid action. Then \( \alpha \) gives rise to a homomorphism
\[
\alpha_* : \mathbb{F}(\mathcal{H}) \to \mathbb{F}(\mathcal{H} \triangleright \alpha Z)
\]
defined by \( \alpha_*(g)|_z = (g|_{q(z)}, z) \), for every \( g \in \mathbb{F}(\mathcal{H}) \) and every \( z \in Z \). We call \( \alpha_* \) the blow-up homomorphism associated to \( \alpha \).

We will also say that an action \( \mathbb{F}(\mathcal{H}) \curvearrowright Z \) is induced from an action \( \alpha : \mathcal{H} \curvearrowright (Z, q) \) if it coincides with the action associated to \( \alpha_* \).

It is again elementary to verify that \( \alpha_* \) is a well-defined homomorphisms. Note that it follows from the definition that an action \( \mathbb{F}(\mathcal{H}) \curvearrowright Z \) induced from a groupoid action \( \mathcal{H} \curvearrowright (Z, q) \) factors onto the natural action \( \mathbb{F}(\mathcal{H}) \curvearrowright Y \) through the same map \( q \).

Example 6.4. Assume that \( \mathcal{H} \) is the groupoid of germs of a free action of a finitely generated group \( G \curvearrowright Y \). Then it is not difficult to see that an action \( \alpha : \mathcal{H} \curvearrowright (Z, q) \) corresponds to a unique group action \( \alpha : G \curvearrowright Z \) (that we still denote \( \alpha \) ) that factors onto \( G \curvearrowright Y \) through the map \( q \). The homomorphism \( \alpha_* : \mathbb{F}(G; Y) \to \mathbb{F}(G; Z) \) can be more concretely described as follows. An element \( g \in \mathbb{F}(G; Y) \) can be defined by a locally constant function \( k_\gamma : Y \to G \), called the orbit cocycle where for every \( y \in Y \), there element \( k_\gamma(y) \in G \) is uniquely defined by the condition \( g = k_\gamma(y)g \). Then the element \( \alpha_*(g) \) is defined by the orbit cocycle \( k_\gamma \circ p \), i.e. \( \alpha_*(g)z = k_\gamma(p(z))z \) for every \( z \in Z \).

Given a compact space \( Y \) and an integer \( r \geq 1 \), let \( Y^{(r)} \) be the compact space of finite subsets of \( Y \) of cardinality at most \( r \). It is endowed with the quotient of the product topology on \( Y^r \) via the surjection \( Y^r \to Y^{(r)}, (y_1, \ldots, y_r) \mapsto \{y_1, \ldots, y_r\} \).

Definition 6.5 (Reduced power). Let \( \mathcal{H} \) be an étale groupoid with unit space \( Y \), and \( r \geq 1 \). The reduced power \( \mathcal{H}^{*r} \) of \( \mathcal{H} \) is the groupoid with unit space \( Y^{(r)} \), consisting of all non-empty finite subsets \( Q \subseteq \mathcal{H} \) of cardinality at most \( r \) such that every two distinct elements \( \gamma, \delta \in Q \) have different source and different range.

By convention, we let \( \mathcal{H}^{*0} \) be the trivial groupoid, and let \( Y^{(0)} \) be its unit space (reduced to one point).

The source and range maps \( s, r : \mathcal{H}^{*r} \to Y^{(r)} \) are given by the images of the source and range maps of \( \mathcal{H} \). If \( Q_1, Q_2 \in \mathcal{H}^{*r} \) are such that \( s(Q_1) = r(Q_2) \), then their product is defined as the set-wise product in \( \mathcal{H} \). The topology on \( \mathcal{H}^{*r} \) is induced from the set of all finite subsets of \( \mathcal{H} \) of cardinality at most \( r \) (note that \( \mathcal{H}^{*r} \) is open in the latter).

More explicitly, a basis of open neighbourhoods of every \( Q \in \mathcal{H}^{*r} \) can be described as follows. Let \( \gamma_1, \ldots, \gamma_s \) be the pairwise distinct elements of \( Q \), with \( s \leq r \). Then for every collection of open bisections \( T_1 \ni \gamma_1, \ldots, T_s \ni \gamma_s \) of \( \mathcal{H} \), the set of \( Q' \in \mathcal{H}^{*r} \) such that \( Q' \subset T_1 \cup \cdots \cup T_s \) and \( Q' \cap T_i \neq \emptyset \) for all \( i = 1, \ldots, s \) is an open neighbourhood of \( Q \), and sets of this form are a basis of neighbourhoods of \( Q \).
Definition 6.6 (Reduced diagonal homomorphism). For every $r \geq 0$ we have a homomorphism

$$\delta_r : \mathbb{F}(\mathcal{H}) \to \mathbb{F}(\mathcal{H}^r)$$

defined for $r \geq 1$ by the condition $\delta(g)|_Q = \{g|_y : y \in Q\}$, for every $g \in \mathbb{F}(\mathcal{H})$ and $Q \in Y^r$.

We let $\delta_0$ the trivial homomorphism to the trivial group $\mathbb{F}(\mathcal{H}^0)$.

Example 6.7 (Diagonal homomorphism). For every $r \geq 1$, there is also a homomorphism $\hat{\delta}_r : \mathbb{F}(\mathcal{H}) \to \mathbb{F}(\mathcal{H}^r)$, where $\mathcal{H}^r$ is the usual direct power of $\mathcal{H}$, given by $\hat{\delta}_r(g)|_{(y_1, \ldots, y_r)} = (g|_{y_1}, \ldots, g|_{y_r})$. This homomorphism can be obtained as the composition of homomorphisms of the types above. Let us explain why. Denoting $q : Y^r \to Y^{(r)}$ the natural surjection, we can define an action $\alpha : \mathcal{H}^* \curvearrowright (Y^{(r)}, q)$ by $(Q, (y_1, \ldots, y_r)) \mapsto \langle r(\gamma_{i_1}), \ldots, r(\gamma_{i_r})\rangle$, where $\gamma_{i_j}$ is the unique element of $Q$ with $s(\gamma_{i_j}) = y_{i_j}$. The groupoid $\mathcal{H}^* \ltimes_{\alpha} Y^{(r)}$, is naturally identified with the subgroupoid of $\mathcal{H}^r$ consisting of all $r$-tuples $(\gamma_1, \ldots, \gamma_r)$ with the property that whenever $\gamma_{i_j} \neq \gamma_{i_j}$ we have $s(\gamma_{i_j}) \neq s(\gamma_{i_j})$ and $r(\gamma_{i_j}) \neq r(\gamma_{i_j})$. Explicitly this embedding $b : \mathcal{H}^* \ltimes_{\alpha} Y^r \to \mathcal{H}^r$ is given by $b(Q, (y_1, \ldots, y_r)) = (\gamma_{i_1}, \ldots, \gamma_{i_r})$. With these notations, the diagonal homomorphism $\hat{\delta}_r : \mathbb{F}(\mathcal{H}) \to \mathbb{F}(\mathcal{H}^r)$ coincides with the composition $b_{\ast} \circ \alpha_{\ast} \circ \delta_r$.

Example 6.8. Assume that $\mathcal{H}$ is the groupoid of germs of an action $G \curvearrowright Y$. Then the direct power $\mathcal{H}^r$ as in the previous example is the groupoid of germs of the natural product action $G^r \curvearrowright Y^r$. The reduced power $\mathcal{H}^\ast$, although tightly related, cannot be defined as the groupoid of germs of an action of $G$ or $G^r$ (note that the action $G \curvearrowright Y^{(r)}$ has a groupoid of germs contained in $\mathcal{H}^\ast$ but a-priori much smaller, and that the action $G^r \curvearrowright Y^r$ does not pass to the quotient to an action $G^r \curvearrowright Y^{(r)}$).

6.2. Tame and wild homomorphisms and the dichotomy theorem. In the previous subsection we described three natural dynamical constructions at the level of the groupoids, that give rise to homomorphisms between topological full groups. By composing homomorphisms of this type we obtain more. In particular, let $\mathcal{G}, \mathcal{H}$ be étale groupoids with unit spaces $X, Y$, and assume that there exists an integer $r \geq 0$, an action $\alpha : \mathcal{G}^r \curvearrowright (Z, q)$ on some compact fibre space over $X$, and a continuous open spatial inclusion

$$b : \mathcal{G}^r \ltimes_{\alpha} Z \to \mathcal{H}.$$

Then we have a natural homomorphism $\rho : \mathbb{F}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$ obtained as the composition of the maps

$$\mathbb{F}(\mathcal{G}) \xrightarrow{\delta_r} \mathbb{F}(\mathcal{G}^r) \xrightarrow{\alpha_{\ast}} \mathbb{F}(\mathcal{G}^r \ltimes_{\alpha} Z) \xrightarrow{b_{\ast}} \mathbb{F}(\mathcal{H}).$$

Conversely, it is natural to wonder whether every homomorphism $\rho : \mathbb{F}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$ is of this form. This is far from true in general, but the main theorem of this paper explains exactly why, when $\mathcal{G}$ is a minimal groupoid of germs and has Cantor set unit space.

Theorem 6.9. Let $\mathcal{G}$ be a minimal groupoid of germs with unit space $X$ homeomorphic to the Cantor set, and $G$ be a subgroup of $\mathbb{F}(\mathcal{G})$ such that $\hat{\mathcal{G}}(G) \leq G \leq \mathbb{F}(\mathcal{G})$. Let $\mathcal{H}$ be another étale groupoid with compact unit space $Y$. Then every homomorphism $\rho : G \to \mathbb{F}(\mathcal{H})$ verifies one of the following possibilities.
(i) There exists a point \( y \in Y \) such that the map \( G \to H, g \mapsto \rho(g)|_y \) is injective. The set of such points is closed and \( G \)-invariant.

(ii) There exists a clopen \( \rho(G) \)-invariant subset \( Z \subset Y \), an integer \( r \geq 0 \), continuous surjective \( G \)-equivariant map \( q: Z \to X^{(r)} \). Moreover there exists an action \( \alpha: G^*r \acts (Z, q) \), and a spatial inclusion

\[
b: G^*r \acts_\alpha Z \to H
\]

such that the restriction of \( \rho \) to \( \mathbb{A}(G) \) coincides with the restriction of the homomorphism canonically associated to \( b \) and \( \alpha \) as explained above.

In view of the previous discussion, we a homomorphisms falls in case (ii) if it is close to arise from a set of natural operations at the level of the groupoids, while case (i) gives a characterisation of those homomorphisms that lack a dynamical interpretation. For this reason, we will use the following terminology.

**Definition 6.10.** In the situation of Theorem 6.9, we will say that \( \rho \) is wild if it verifies (i), and that it is tame if it verifies (ii) and call the integer \( r \) its rank.

**Remark 6.11.**

(i) A homomorphism is tame of rank 0 if and only if it is trivial in restriction to \( \mathbb{A}(G) \). In particular the trivial homomorphisms is itself tame of rank 0.

(ii) It is not true that a tame homomorphisms \( \rho \) coincides on the whole of \( G \) with the restriction of the homomorphism induced by \( b \) and \( \alpha \): counterexamples will be given in Examples 7.9. However it still satisfy the weaker condition that the action \( G \acts Z \) induced by \( \rho \) factors onto \( X^{(r)} \). The reason for this somewhat unpleasant subtlety is that it is often possible to “twist” a homomorphism \( \rho \) by multiplying it with homomorphisms defined on the quotient \( F(G)/\mathbb{A}(G) \) whose image commutes with the image of \( \rho \). In many cases, \( F(G)/\mathbb{A}(G) \) surjects onto \( Z \), and hence it is often easy to define such homomorphisms.

Before giving the proof of Theorem 6.9 let us illustrate its statement in some relevant special cases. The simplest example is the following.

**Example 6.12 (Simplicity of \( \mathbb{A}(G) \)).** Let \( H \) be a countable group viewed as an \( \text{étale} \) groupoid with one point unit space \( Y = \{1_H\} \) (Example 2.5 (i)). Let \( \rho: G \to F(H) \simeq H \) be a homomorphism, and apply Theorem 6.9. Then case (i) is equivalent to the fact that \( \rho \) is injective. Furthermore, all tame homomorphisms \( F(G) \to F(H) \) have rank 0 (since otherwise the unit space of \( H \) would be uncountable). Therefore case (ii) is equivalent to the fact that \( \rho(\mathbb{A}(G)) = \{1_H\} \). In other words, in this case the theorem reduces to the fact that every homomorphism from \( G \) to any countable group is either injective or its kernel contains \( \mathbb{A}(G) \). This is equivalent to the fact that \( \mathbb{A}(G) \) is simple (for \( G = \mathbb{A}(G) \)) and contained in every non-trivial normal subgroup of \( F(G) \) (for \( G = F(G) \)), i.e. to Theorem 2.15.

The following corollary is equivalent to Theorem 6.9 in the special case where \( \mathcal{H} \) is assumed to be a groupoid of germs, and implies Theorem 1.1.
Corollary 6.13. Let $G$ be as in Theorem 6.12, and let $G \curvearrowright Y$ be a non-trivial continuous action on a compact space. One of the following holds.

(i) There exists a point $y \in Y$ such that $S^0_G(y) = \{1\}$.
(ii) The (open, $G$-invariant) set $Z \subset Y$ of points that are moved by some element of $A(G)$ is clopen, there exists $r \geq 1$ and a continuous surjective $G$-equivariant map $q: Z \to X^{(r)}$. Moreover the action $A(G) \curvearrowright Z$ is induced from an action $G'^{\ast} \curvearrowright (Z, q)$.
(iii) The action is trivial in restriction to the group $A(G)$.

Proof. Let $H$ be the groupoid of germs of the action $G \curvearrowright Y$, and apply the theorem to $G \to F(H)$. If this homomorphisms is wild, we deduce that there exists a point $y \in Y$ such that every element of $g$ is uniquely determined by its germ at $y$, which is equivalent to say that $S^0_G(y) = \{1\}$. If it is tame of rank $r \geq 1$, we deduce that (ii) holds (the fact that $Z$ is $G$-invariant follows from the fact that $A(G)$ is normal in $G$), and if it is tame of rank 0 we deduce that it is trivial in restriction to $A(G)$ and therefore (iii) holds.

If the action is minimal, the statement can be simplified as follows. Recall that an action of a countable group on a compact space is said to be topologically free if the set of points with trivial stabiliser is dense.

Corollary 6.14. Under the same assumptions, assume that $G \curvearrowright Y$ is minimal. Then one of the following possibilities holds.

(i) the action $G \curvearrowright Y$ is topologically free;
(ii) there exists a continuous surjective $G$-equivariant map $q: Y \to X$, and the action $A(G) \curvearrowright Y$ is induced from an action $G \curvearrowright (Y, q)$.
(iii) The group $A(G)$ acts trivially, in particular the action is not faithful.

Proof. Apply Corollary 6.13. Assume that (i) holds. The set of points $y \in Y$ satisfying the conclusion there is invariant and it easily seen to be closed. Therefore by minimality we have $S^0_{A(G)}(y) = \{1\}$ for every $y \in Y$. Since we have $S^0_{A(G)} = S_{A(G)}(y)$ for a dense $G_\delta$ set of points, it follows that the action is topologically free. If (ii) holds, by minimality we must have $Z = Y$. Moreover, since the map $q: Y \to X^{(r)}$ is surjective, we deduce that the action $G \curvearrowright X^{(r)}$ is also minimal, and hence $r = 1$ (otherwise $X \subset X^{(r)}$ is a closed invariant subset).

Let us explain how this recovers the characterisation of isomorphisms from Mat15.

Example 6.15 (Case of isomorphisms). Let $G_1, G_2$ be both minimal groupoids of germs with Cantor set unit space. Assume that there exists isomorphic groups $G_i$ such that $A(G_i) \leq G_i \leq F(G_i)$, for $i = 1, 2$, and let $\rho: G_1 \to G_2$ be an isomorphism. We apply Corollary 6.14 to the induced action $G_1 \curvearrowright X_2$. For every point $y \in X_2$, there exists non-trivial elements $g \in G_2$ that fix $y$ (for example by Lemma 2.18), and since $\rho$ is an isomorphism this rules out Case (i). Case (iii) also cannot hold. Hence we are in Case (ii) and we obtain that there exists a $G_1$-equivariant map $q: X_2 \to X_1$. By the same reasoning applied to $\rho^{-1}$ there exists a $G_2$-equivariant map $p: X_2 \to X_1$. Now $p \circ q: X_1 \to X_1$ is...
there exists a non-empty open subset $V$ of $X$. Let $A$ denote the normal closure of the conjugation of the corresponding actions, and $G$ and $H$ are isomorphic as étale groupoids.

6.3. **Proof of the dichotomy theorem.** We now turn to the proof of Theorem 6.9. Let $\rho: G \to \mathcal{F}(\mathcal{H})$ be as in the statement. We let $G \curvearrowright Y$ be the action associated to $\rho$, and for every $y \in Y$, we denote $S_G(y)$ the stabilisers with respect to this action. We further set $S^\rho_G(y) = \{ g \in G : \rho(g)|_y = y \}$, and note that it is a subgroup of $G$, and that $\ker(\rho) = \bigcap_{y \in Y} S^\rho_G(y)$. For every $y \in Y$ the subgroup $S^\rho_G(y)$ is a normal subgroup of the stabiliser $S_G(y)$. In fact, in restriction to $S_G(y)$ the map $g \mapsto \rho(g)|_y$ is a homomorphism taking values in the isotropy group $H_y$ and the group $S^\rho_G(y)$ coincides with the kernel of this homomorphism. Case (1) in Theorem 6.9 is equivalent to say that there exists a point $y \in Y$ such that $S^\rho_G(y) = \{ 1 \}$.

**Lemma 6.16.** The map $Y \to \text{Sub}(G)$, $y \mapsto S^\rho_G(y)$ is lower semicontinuous. In particular the set $\{ y \in Y : S^\rho_G(y) = \{ 1 \} \}$ is closed.

**Proof.** We apply Lemma 2.2. For every $g \in G$ we have $\{ y \in Y : g \in S^\rho_G(y) \} = T_{\rho(g)} \cap Y$ which is open in $Y$. The last sentence immediately follows from lower semicontinuity. \hfill \Box

**Lemma 6.17.** Let $U \subset X$ be a dense open set. Then every non-trivial subgroup of $\mathcal{F}(\mathcal{G})$ normalised by $\mathbb{A}(\mathcal{G}|U)$ contains $\mathbb{A}(\mathcal{G}|U)$.

**Proof.** Let $N \leq \mathcal{F}(\mathcal{G})$ be a non-trivial subgroup normalised by $\mathbb{A}(\mathcal{G}|U)$. By Lemma 3.1 there exists a non-empty open subset $V \subset X$ such that $N$ contains $[R_{\mathbb{A}(\mathcal{G}|U)}(V), R_{\mathbb{A}(\mathcal{G}|U)}(V)]$. Since $U$ is dense, the intersection $U \cap V$ is non-empty, and $R_{\mathbb{A}(\mathcal{G}|U)}(V)$ contains $\mathbb{A}(\mathcal{G}|U \cap V)$. The latter is a perfect group, and therefore $N$ contains $\mathbb{A}(\mathcal{G}|U \cap V)$, and thus it contains the normal closure of $\mathbb{A}(\mathcal{G}|U \cap V)$ in $\mathbb{A}(\mathcal{G}|U)$. Since the group $\mathbb{A}(\mathcal{G}|U)$ is simple by Theorem 2.15 we get the conclusion. \hfill \Box

In the proof, we will denote by $\mathcal{P}_r(X) = X^{(r)} \cup \{ \emptyset \}$ the set of all finite subsets of $X$ with cardinality at most $r$, where we denote $\emptyset \subset X$ the empty subset, to stress its role as a point in $\mathcal{P}_r(X)$. The point $\emptyset \subset X$ is the unique isolated point of $\mathcal{P}_r(X)$.

**Lemma 6.18.** Let $r \geq 1$. Then the unique closed $\mathbb{A}(\mathcal{G})$-invariant subsets of $\mathcal{P}_r(X)$ are $\mathcal{P}_s(X)$ for $0 \leq s \leq r$ and $X^{(s)}$ for $1 \leq s \leq r$.

**Proof.** Let $D \in \mathcal{P}_r(X)$ be non-empty, and let $D = \{ x_1, \ldots, x_s \}$, where $s = |D|$. Given any non-empty open sets $U_1, \ldots, U_s$, it is easy to construct (e.g. as in the proof of Lemma 4.2) an element $g \in \mathbb{A}(\mathcal{G})$ such that $g x_i \in U_i$ for every $i = 1, \ldots, s$. This shows that the orbit of $D$ is dense in $X^{(s)}$. The conclusion readily follows. \hfill \Box

The core of the argument in the proof of Theorem 6.9 is contained in the following proposition.
Proposition 6.19. In the situation of Theorem 6.4 assume that for every \( y \in Y \) we have \( S^0_G(y) \neq \{1\} \). Then there exists \( r \geq 0 \) and a continuous surjective \( G \)-equivariant map \( \tilde{q} : Y \to \mathcal{P}_r(X) \) whose image avoids at most the point \( \emptyset_X \), and which is uniquely determined by the condition that \( S^0_{A(Q)}(\tilde{q}(y)) \leq S^0_G(y) \leq S_G(\tilde{q}(y)) \) for every \( y \in Y \).

Proof. Since \( S^0_G(y) \neq \{1\} \) for every \( y \in Y \), we deduce from Lemma 6.16 and Lemma 2.4 that the subgroup \( \{1\} \) does not belong to the closure of its image. Hence, the image of this map avoids a neighbourhood of \( \{1\} \) of the form \( U_P \) for some non-empty finite set \( P \subset G \setminus \{1\} \). Set \( r = |P| - 1 \). For every \( y \in Y \) we can apply Corollary 4.5 to the subgroup \( H = S^0_G(y) \) and we get a map \( \tilde{q} : Y \to \mathcal{P}_r(X) \), where \( \tilde{q}(y) \in \mathcal{P}_r(X) \) is the unique set such that \( S^0_{A(Q)}(\tilde{q}(y)) \leq S^0_G(y) \leq S_G(\tilde{q}(y)) \). The map \( \tilde{q} \) is clearly equivariant with respect to the action \( G \rtimes \mathcal{P}_r(X) \) induced by \( \rho \) and to the action \( G \times \mathcal{P}_r(X) \).

Lemma 6.20. The map \( \tilde{q} \) is continuous.

Proof. The equivariance of \( \tilde{q} \) automatically implies that we have the inclusion \( S_G(y) \leq S_G(\tilde{q}(y)) \) for every \( y \in Y \). We therefore have for every \( y \in Y \) the chain of subgroups:

\[
S^0_{A(Q)}(\tilde{q}(y)) \leq S^0_G(y) \leq S_G(\tilde{q}(y)) \leq S_G(y).
\]

We will deduce the continuity of \( \tilde{q} \) from these inclusions and from the facts that the maps \( \mathcal{P}_r(X) \to \text{Sub}(G), Q \mapsto S^0_{A(Q)}(Q) \) and \( Y \to \text{Sub}(G), y \mapsto S^0_G(y) \) are lower semicontinuous (Example 2.3 and Lemma 6.16), while the maps \( y \mapsto S_G(y) \) and \( Q \mapsto S_G(Q) \) are upper semicontinuous (see Example 2.3). Let \( (y_i) \subset Y \) be a net converging to a limit \( y \). Let \( Q \) be a cluster point of \( (\tilde{q}(y_i)) \) in \( \mathcal{P}_r(X) \), and let us show that \( Q = \tilde{q}(y) \). Up to taking a subnet, we can assume that \( (\tilde{q}(y_i)) \) converges to \( Q \) and that the four nets \( (S^0_{A(Q)}(\tilde{q}(y_i))), (S^0_G(y_i)), (S_G(y_i)), (S_G(\tilde{q}(y_i))) \) all converge in \( \text{Sub}(G) \) to limits denoted \( H_1, \ldots, H_4 \) respectively. First, passing to the limit the inclusions \( S^0_G(y_i) \leq S_G(\tilde{q}(y_i)) \) and using semicontinuity we obtain \( S^0_G(y) \leq H_2 \leq H_4 \leq S_G(Q) \), and therefore \( S^0_G(y) \leq S_G(Q) \). Next, passing to the limit the inclusion \( S^0_{A(Q)}(\tilde{q}(y_i)) \leq S_G(\tilde{q}(y)) \) and using semicontinuity in the same way we obtain \( S^0_{A(Q)}(Q) \leq S_G(y) \). This implies that \( S^0_{A(Q)}(Q) \) normalises \( S^0_G(y) \), since the latter is a normal subgroup of \( S_G(y) \). Since \( S^0_{A(Q)}(Q) = A(Q) \) (Lemma 4.3), the open set \( U = Q^c \) is dense in \( X \), and we are assuming that \( S^0_G(y) \) is non-trivial, we deduce from Lemma 6.17 that \( S^0_{A(Q)}(Q) \leq S^0_G(y) \). We have proven that the set \( Q \) satisfies \( S^0_{A(Q)}(Q) \leq S^0_G(y) \leq S_G(Q) \) and therefore \( \tilde{q}(y) = Q \) by the uniqueness of \( Q \) in Theorem 4.1. This completes the proof of the continuity of \( \tilde{q} \).

Finally, the image \( \tilde{q}(Y) \) is a closed \( G \)-invariant subset of \( \mathcal{P}_r(X) \). By Lemma 6.18 it must be either \( \mathcal{P}_s(X) \) for some \( 0 \leq s \leq r \) or \( X^{(s)} \) for some \( 1 \leq s \leq r \). Up to reducing \( r \), we may assume that \( r = s \). This shows the claim on the image of \( \tilde{q} \) and concludes the proof of the Proposition.

Lemma 6.21. Let \( r \geq 1 \). Then for every \( Q \in \mathcal{G}^{\mathcal{G}} \) there exists \( g \in A(Q) \) such that \( Q \subset T_g \).
Proof. Assume at first that \( s(Q) \) and \( r(Q) \) are disjoint. Let \( \gamma_1, \ldots, \gamma_s \) for \( s \leq r \) be the distinct elements of \( Q \). Using Lemma \ref{3.18} and the fact that every \( G \)-orbit is infinite and dense, we can find degree 3 multielements \( g_1, \ldots, g_d \) such that \( g_i(1,2) = \gamma_i \) for \( i = 1, \ldots, s \) and such that the points \( g_i(3,3), i = 1, \ldots, s \) are all disjoint and disjoint from \( s(Q) \) and \( r(Q) \), and degree 3 multisections \( \tilde{\mathcal{F}}_1 \owns g_1 \ldots \tilde{\mathcal{F}}_s \owns g_s \) with disjoint support. Then the element \( g = \tilde{\mathcal{F}}_1((123)) \cdots \tilde{\mathcal{F}}_s((123)) \) verifies the desired conclusion. If \( s(Q) \) and \( r(Q) \) intersect non-trivially, using again minimality we can write \( Q = Q_1Q_2 \) where \( Q_1, Q_2 \in G^{\ast r} \) fall in the previous case. Hence we can choose \( g_1, g_2 \in \mathbb{A}(G) \) such that \( Q_i \subset T_{g_i} \) for \( i = 1, 2 \), and \( g = g_1g_2 \) verifies the conclusion. \( \square \)

Proof of Theorem \ref{6.19} Let \( r \geq 0 \) and \( \tilde{q} : Y \to P_r(X) \) be given by the proposition.

Assume that the image of \( \tilde{q} \) is \( P_0(X) = \{ \varnothing_X \} \). We then deduce that \( \mathbb{A}(G) = \mathbb{S}^0_{\mathbb{A}(G)}(\varnothing_X) \leq S^0_G(y) \) for every \( y \in Y \), and therefore \( \mathbb{A}(G) \leq \ker(\rho) = \bigcap_{y \in Y} S^0_G(y) \). In particular, if we choose \( Z = Y, r = 0 \) and \( q \) : \( Z \to X^{(0)} = \{ * \} \) the constant map, then \([\text{iii}]\) is trivially satisfied because \( \rho \) coincides with the trivial homomorphism in restriction to \( \mathbb{A}(G) \). Otherwise, the image of \( \tilde{q} \) is either \( P_r(X) \) or \( X^{(r)} \) for some \( r \geq 1 \). In both cases, let \( Z = \tilde{q}^{-1}(X^{(r)}) \), and observe that it is clopen and \( G \)-invariant by continuity and equivariance of \( \tilde{q} \). Rename \( q \) the restriction of \( \tilde{q} \) to \( Z \). Let us define an action \( \alpha : G^{\ast r} \curvearrowright (Z, q) \) and a spatial inclusion \( b : G^{\ast r} \rtimes \alpha Z \to \mathcal{H} \) as follows. For every \( Q \in G^{\ast r} \) and \( z \in Z \) such that \( s(Q) = q(z) \), choose an element \( g \in \mathbb{A}(G) \) such that \( Q \subset T_g \) (such an element exists by Lemma \ref{5.21}), and set \( \gamma y = \rho(g)y \) and \( b(\gamma, y) = \rho(g)|_y \). To see that action and cocycle are well defined let \( g_1, g_2 \in \mathbb{A}(G) \) be such that \( Q \subset T_{g_1} \) and \( Q \subset T_{g_2} \). Then, using the last sentence in Proposition \ref{6.19} we have that \( g_1^{-1}g_2 \in \mathbb{S}^0_{\mathbb{A}(G)}(s(Q)) = \mathbb{S}^0_{\mathbb{A}(G)}(q(y)) \leq S^0_G(y) \leq S^0_G(y) \), and hence \( \rho(g_1)y = \rho(g_2)y \) and \( \rho|_y \). The representative element \( g \in \mathbb{A}(G) \) chosen to define \( b(Q, z) \) can also be chosen as a representative to define \( b(Q', z') \) for every \( (Q', z') \) in a neighbourhood of \( (Q, z) \), and this fact implies that \( b \) is an open map. Moreover \( b \) is the identity in restriction to \( Z \). Let \( \rho_0 \) be the homomorphism \( \rho_0 : F(G) \to F(H) \) obtained as the composition of

\[
\begin{align*}
F(G) \xrightarrow{\delta_\alpha} F(G^{\ast r}) \xrightarrow{\alpha} F(G^{\ast r} \rtimes \alpha Z) \xrightarrow{b_\ast} F(H).
\end{align*}
\]

From the construction of \( \alpha \) and of \( b \), we see that for every \( g \in \mathbb{A}(G) \) and \( z \in Z \) we have \( \rho(g)|_z = b(\{g|_x : x \in q(z)\}, z) \), and \( \rho(g)|_y = y \) if \( Y \subset Y \setminus Z \). By the the definition of \( \delta_\ast, \alpha_\ast \) and \( b_\ast \) in Subsection \ref{5.1} the same is true for \( \rho_0(g) \) for, and therefore \( \rho \) and \( \rho_0 \) coincide in restriction to \( \mathbb{A}(G) \). This concludes the proof of Theorem \ref{6.9} \( \square \)

7. Homomorphisms and Cayley graphs of groupoids

Theorem \ref{6.9} can be used to show that for a certain class of groupoids \( \mathcal{H} \), every homomorphism \( F(G) \to F(H) \) must be tame. The starting observation is the following corollary of Theorem \ref{6.9}. We say that a graph \( \Gamma \) is Lipschitz-embedded in a graph \( \Delta \) if there is an injective map from the vertices of \( \Gamma \) to the vertices of \( \Delta \) which is Lipschitz for the corresponding simplicial distances (we do not require the map to be defined on edges).
Corollary 7.1. Let $G, H$ be as in Theorem 6.9 and assume that $H$ is compactly generated. If there exists a wild homomorphism $\rho: \mathcal{A}(G) \to \mathcal{F}(H)$, then $H$ has a Cayley graph that contains the Cayley graph of every finitely generated subgroup of $\mathcal{A}(G)$ as a Lipschitz-embedded subgraph.

Proof. Let $H$ be a finitely generated subgroup of $\mathcal{A}(G)$ generated by a finite symmetric generating set $S$, and let $y \in Y$ be as in Theorem 6.9(i). Let $T$ be a finite generating set of bisections of $H$. Up to enlarging $T$ we may assume that it contains the bisections $T_{\rho(s)}$ for every $s \in S$. Then the map $h \mapsto \rho(h)|_y$ defines a 1-Lipschitz embedding of the Cayley graph of $(H, S)$ into the Cayley graph $\Gamma_y(H, T)$.

Lemma 7.2. Let $G$ be a compactly generated minimal étale groupoid with Cantor set unit space. Then for every $n \geq 1$ there exists a finitely generated subgroup $K < \mathcal{A}(G)$ whose Cayley graph contains a Lipschitz-embedded copy of $\mathbb{Z}^n$.

Proof. Let $U_1, \ldots, U_n \subset X$ be disjoint clopen subsets. The groupoid $G|U_i$ is also compactly generated (see Proposition 2.26) and minimal, and therefore by Lemma 2.23 we can find an infinite finitely generated subgroup $K_i < A(G|U_i)$ for every $i = 1, \ldots, n$. Let $K = K_1 \times \cdots \times K_n \leq \mathcal{A}(G)$, Since every $K_i$ is infinite, it contains a Lipschitz-embedded copy of $\mathbb{Z}$, and hence $K$ contains a Lipschitz-embedded copy of $\mathbb{Z}^n$.

We will use the notion of asymptotic dimension. Let us recall its definition. Given a metric space $(\Sigma, d)$, we say that $\mathrm{asdim}(\Sigma) \leq n$ if for every $R > 0$ there exists a cover $\mathcal{U}$ of $\Sigma$ into subsets of uniformly bounded diameter such that every ball of radius $R > 0$ in $\Sigma$ intersects at most $n + 1$ elements of $\mathcal{U}$. The asymptotic dimension $\mathrm{asdim}(\Sigma)$ is defined to be the smallest integer $n$ such that $\mathrm{asdim}(\Sigma) \leq n$ if there is such an integer, and $\infty$ otherwise.

The following proposition was explained to the author by A. Sisto.

Proposition 7.3. Let $\Gamma, \Delta$ be graphs of bounded degree, and assume that $\Gamma$ admits a Lipschitz-embedding into $\Delta$. Then $\mathrm{asdim}(\Gamma) \leq \mathrm{asdim}(\Delta)$.

Proof. We identify the vertex set of $\Gamma$ with a subset of the vertex set of $\Delta$ using the Lipschitz embedding. Up to replacing $\Delta$ with a quasi-isometric graph by adding to it edges of bounded length, we may assume that the embedding of $\Gamma$ into $\Delta$ is 1-Lipschitz. If $\mathrm{asdim}(\Delta) = \infty$, there is nothing to prove. Assume that $\mathrm{asdim}(\Delta) = n < \infty$. Let $R > 0$. By definition of asymptotic dimension, there exists a cover $\mathcal{U}$ of $\Delta$ such that every $R$-ball in $\Delta$ intersects at most $n + 1$ elements of $\mathcal{U}$, and the diameter of elements of $\mathcal{U}$ is uniformly bounded. Since $\Delta$ has bounded degree, there exists $N > 0$ such that every element of $\mathcal{U}$ contains at most $N$ points. Let $\mathcal{V}_t$ be the cover of $\Gamma$ induced by $\mathcal{U}$, and let $\mathcal{V}$ be the cover of $\Gamma$ consisting of $2R$-coarsely connected components of elements of $\mathcal{V}_t$. These are defined as follows: given a subset $A \subset \Gamma$ and an integer $K > 0$, two vertices $x, y \in A$ are in the same $K$-coarsely connected component if there exists a finite sequence $x_0 = x, x_1, \ldots, x_r = y$ of vertices in $A$ such that $d_T(x_i, x_{i+1}) \leq K$ for $i = 0, \ldots, r - 1$. We claim that $\mathcal{V}$ satisfies the conditions in the definition of $\mathrm{asdim}(\Gamma) \leq n$. To see that it has uniformly bounded diameter, note that by construction every element of the cover $\mathcal{V}$ has at most $N$ points and is $2R$-coarsely connected, hence it has diameter bounded by $2RN$. Moreover, every $R$-ball
in $\Gamma$ intersects at most $n + 1$ elements of $\mathcal{V}_1$, and for every $V \in \mathcal{V}_1$ it intersect at most one $2R$-coarsely connected component of it. We deduce that it intersects at most $n + 1$ connected components of $\mathcal{V}$. Since $R$ was arbitrary, we deduce that $\text{asdim}(\Gamma) \leq n = \text{asdim}(\Delta)$, as desired. □

**Definition 7.4.** We say that an étale groupoid $\mathcal{H}$ belongs to the class $\mathcal{E}$ if it has compact unit space $Y$, it is compactly generated, and for every finite generating set $S$ of bisections of $\mathcal{H}$ and every $y \in Y$ the Cayley graph $\tilde{\Gamma}_y = \tilde{\Gamma}_y(\mathcal{H}, S)$ verifies one of the following conditions.

(i) The graph $\tilde{\Gamma}_y$ has finite asymptotic dimension.

(ii) The graph $\tilde{\Gamma}_y$ has growth function bounded above by a polynomial (that may depend on $y$).

The following theorem implies Theorem 1.6.

**Theorem 7.5.** Let $\mathcal{G}$ be a minimal groupoid of germs with Cantor set unit space, and assume that $\mathcal{G}$ is compactly generated. Let $G$ be such that $\mathcal{A}(\mathcal{G}) \leq G \leq \mathcal{F}(\mathcal{G})$. Let $\mathcal{H}$ be an étale groupoid in the class $\mathcal{E}$. Then every homomorphism $\rho : G \to \mathcal{F}(\mathcal{H})$ is tame.

**Proof.** If there exists a wild homomorphism $\rho : G \to \mathcal{F}(\mathcal{H})$ by Corollary 7.1 and Lemma 7.2 we conclude that $\mathcal{H}$ has a Cayley graph $\tilde{\Gamma}_y$ that contains a Lipschitz-embedded copy of $\mathbb{Z}^n$ for every $n$. In particular it has infinite asymptotic dimension by Proposition 7.3, and its growth cannot be bounded by a polynomial. □

**Examples 7.6.** The following classes of groupoids belong to the class $\mathcal{E}$.

(i) The groupoid of germs actions of $\mathbb{Z}^d$ for $d \geq 1$, and of every finitely generated group of polynomial growth.

(ii) More generally, the groupoid of germs of any topologically free action $G \curvearrowright Y$, where $G$ is a finitely generated group with finite asymptotic dimension. This is because the Cayley graphs of the corresponding groupoid are quasi-isometric to the Cayley graphs of $G$. Groups with finite asymptotic dimension include discrete subgroups of connected Lie groups, hyperbolic groups and relatively hyperbolic groups whose peripheral subgroups of finite asymptotic dimension, mapping class groups, etc. See [BD08] for a survey.

(iii) The groupoids associated to (products of) one-sided shifts of finite type, whose topological full groups is studied in great detail by Matui in [Mat15]. These include in particular the family of Higman-Thompson’s groups, and their higher-dimensional generalisations. In fact, the Cayley graphs of the underlying groupoids are (products of) trees (see Lemma 7.11), and therefore they have finite asymptotic dimension.

(iv) The groupoid of germs of Nekrashevych’s fragmentations of dihedral group actions, whose topological full group are torsion and have sometimes intermediate growth [Nek16a]. The Cayley graphs of this class of groupoids (described in detail in [Nek16a]) are quasi-isometric to $\mathbb{Z}$ or to the connected sum of finitely many copies of $\mathbb{N}$, hence they have linear growth and asymptotic dimension 1.

(v) Groupoids associated to quasicrystals of the euclidean space (such as the Penrose tiling) see [BJS10, Kel97, KP00]. Their topological full group is considered in [Nek15a].
Section 6.3). The Cayley graphs of this class of groupoids are quasi-isometric to the euclidean space (hence have both polynomial growth and finite asymptotic dimension).

For later use let us prove the following lemma. It provides a criterion to bound the rank of tame homomorphisms in Theorem 7.5 or to show that tame homomorphisms between the topological full groups of certain groupoids cannot exist at all.

**Lemma 7.7.** Let $G$ be a compactly generated groupoid of germs with unit space $X$ homeomorphic to the Cantor set, and let $S$ be a finite generating set of bisections of $G$. Let $H$ be a compactly generated étale groupoid with compact unit space $Y$ and $T$ be a finite generating set of bisections of $H$. Assume that there exists a tame homomorphism $\rho: \mathcal{H}(G) \to \mathcal{F}(H)$ and let $r \geq 1$ be its rank. Then there exist points $y \in Y, x_1, \ldots, x_r \in X$ such that the Cayley graph $\tilde{\Gamma}_y(H, T)$ contains a Lipschitz-embedded subgraph which is quasi-isometric to $\tilde{\Gamma}_{x_1}(G, S) \times \cdots \times \tilde{\Gamma}_{x_r}(G, S)$.

**Proof.** Let $\alpha: G^r \curvearrowright (Z, q)$ be the action and $b: G^r \times_{\alpha} Z \to H$ be the spatial inclusion associated to $\rho$. Choose $Q \in X^{(r)}$ that verifies the following properties, each of which is verified by a dense $G^\delta$ subset of $X^{(r)}$ (for the second, this follows from the fact that a groupoid of germs is essentially principal [Nek15a Prop. 2.1]):

(i) we have $|Q| = r$

(ii) every $x \in Q$ has a trivial isotropy group $G_x$.

Let $z \in q^{-1}(Q)$, and $Q = \{x_1, \ldots, x_r\}$. Let $U_1, \ldots, U_r \subset X$ be disjoint clopen neighbourhoods of the points $x_1, \ldots, x_r$. By Lemma 2.25 for every $i = 1, \ldots, r$ the groupoid $G|U_i$ is compactly generated and has Cayley graphs quasi-isometric to the corresponding ones in $G$. Let $S_i, i = 1, \ldots, r$ be finite generating sets of $G|U_i$. We define a map between the (the set of vertices of the) graphs $\varphi: \tilde{\Gamma}_{x_1}(G|U_1, S_1) \times \cdots \times \tilde{\Gamma}_{x_r}(G|U_r, S_r) \to \tilde{\Gamma}_{b(z)}(H, T)$, by setting $\varphi(\gamma_1, \ldots, \gamma_r) = b(\{\gamma_1, \ldots, \gamma_r\}, z)$. This map is Lipschitz, as follows from the continuity of $b$. Let us show that it is also injective. Assume that $\varphi(\gamma_1, \ldots, \gamma_r) = \varphi(\gamma'_1, \ldots, \gamma'_r) = \delta$, and set $w = r(\delta) \in Z$. Applying the map $q: Z \to X^{(r)}$ we obtain that $\{r(\gamma_1), \ldots, r(\gamma_r)\} = \{r(\gamma'_1), \ldots, r(\gamma'_r)\} = \{q(w)\}$. Both sets $\{r(\gamma_1), \ldots, r(\gamma_r)\}, \{r(\gamma'_1), \ldots, r(\gamma'_r)\}$ have exactly one element in $U_i$ for every $i = 1, \ldots, r$, (since the sets $U_i$ are disjoint), and this implies that actually $(r(\gamma_1), \ldots, r(\gamma_r)) = (r(\gamma'_1), \ldots, r(\gamma'_r))$ as ordered $r$-tuples. We have proven that for every $i = 1, \ldots, r$, the elements $\gamma_i, \gamma'_i$ both have source $x_i$ and they have equal range. We deduce that $\gamma_i = \gamma'_i$, since $x_i$ has trivial isotropy group by the choices made. This proves injectivity of the map $\varphi$. \hfill $\square$

7.1. **Example: Cantor minimal systems.** We would like to illustrate Theorem 7.5 in a concrete special case, by giving a statement that doesn’t make use of groupoid language. A *Cantor minimal system* is a minimal dynamical system $(X, u)$ where $u$ is a homeomorphism of the Cantor set $X$. We denote $\mathbb{F}(X, u)$ its topological full group (of the corresponding action of $\mathbb{Z}$), and $\mathbb{D}(X, u)$ its derived subgroup. An element $g \in \mathbb{F}(X, u)$ is uniquely determined by the associated *orbit cocycle*, i.e. a continuous function $k_g : X \to \mathbb{Z}$.
which is uniquely defined by the condition $gx = u^{k_y(x)}x$.

The following corollary provides a dynamical description of embeddings between this class of topological full groups, giving an answer to a question of Cornulier [Cor14, Question (2f)].

**Corollary 7.8 (Case of Cantor minimal systems).** Let $(X, u)$ be a Cantor minimal system, and $(Y, v)$ be any dynamical system on a compact space $Y$. Then the following are equivalent.

1. The group $F(X, u)$ embeds into $F(Y, v)$.
2. There exists a non-trivial homomorphism $D(X, u) \to F(Y, v)$.
3. There exists an element $w \in F(Y, v)$ supported in a clopen subset $Z \subset Y$, such that the system $(Z, w)$ factors onto $(X, u)$.

Moreover, for every non-trivial homomorphism $\rho : D(X, u) \to F(Y, v)$, there exists a pair $(Z, w)$ and factor map $q : Z \to X$ as in (iii) which determine $\rho$ uniquely by the rule

$$\rho(g)y = \left\{ \begin{array}{ll} w^{k_y(q(g))}y & y \in Z \\ y & y \notin Z. \end{array} \right.$$

**Proof.** Let $G, H$ be the groupoid of germs of the $\mathbb{Z}$-actions associated to $(X, u)$ and $(Y, v)$ respectively. Condition (iii) is equivalent to the fact that there exists an action $\alpha : G \rightarrow (Z, q)$ on some compact fibre space over $X$ and an open spatial inclusion $b : G \times X Z \rightarrow H$ (see Examples 6.2 and 6.4). If this happens, then we have an embedding of $\rho = b_* \circ \alpha_* : F(G) \hookrightarrow F(H)$. Hence (iii) $\Rightarrow$ (i). It is clear that (i) $\Rightarrow$ (ii). Let us show that (ii) $\Rightarrow$ (iii) Let $\rho : D(G) \to F(H)$ be a non-trivial homomorphism. Since all Cayley graphs of $H$ are quasi-isometric to $Z$, $H$ belongs to the class $C'$, and hence $\rho : D(G) \to F(H)$ is tame. Using Lemma 7.7 and the fact that all Cayley graphs of $G$ and $H$ are quasi-isometric to $Z$, we get that $\rho$ is tame of rank 1. Hence (iii) follows. The last sentence follows from the fact that $A(G) = D(G)$ in this case, since $D(G)$ is simple by Matui [Mat06], and from the definition of blow-up homomorphism and of spatial inclusion homomorphisms specialised to this case (see Examples 6.2 and 6.4). \hfill \Box

Examples 7.9. We now would like to give two examples showing that the last sentence of Corollary 7.8 can fail for homomorphisms $\rho : F(X, u) \hookrightarrow F(Y, v)$ defined on the whole topological full group, and is in general only true for the restriction to the derived subgroup. In particular in the main dichotomy theorem 6.9 it is not true that a tame homomorphism is itself the composition of homomorphisms of the three fundamental types discussed in Subsection 6.1 before taking its restriction to $A(G)$.

As a basis for both examples, let $(Y, v)$ be Cantor minimal system. Let $X \subset Y$ be a proper clopen subset of $Y$, and let $u : X \to X$ be the first return map to $X$, defined by $x \mapsto v^{\tau_X(x)}(x)$ where $\tau_X(x) = \min\{n \geq 1, v^n(x) \in X\}$. Then $(X, u)$ is again a Cantor minimal system, and the obvious inclusion $\rho_0 : F(X, u) \hookrightarrow F(Y, v)$ is a spatial inclusion homomorphism. We will make use of the facts, shown in [Mat06], that for every Cantor minimal system $(X, u)$ there exists a surjective homomorphism $I : F(X, u) \twoheadrightarrow \mathbb{Z}$, and that the derived subgroup $D(X, u)$ is simple (in particular $A(X, u) = D(X, u)$).
(i) Let us modify the inclusion $F(X, u) \hookrightarrow F(Y, v)$ in the following way. Choose $t \in F(Y, v)$ supported outside $X$ (in particular it commutes with $F(X, u)$). Define a homomorphism $\rho: F(X, u) \rightarrow F(Y, v)$ by

$$\rho(g) = gt^{I(g)}.$$  

It is a well defined homomorphism since $t$ commutes with $F(X, u)$. Since the derived subgroup $D(X, u)$ is contained in the kernel of the map $I: F(X, u) \rightarrow \mathbb{Z}$, the restriction $\rho|_{D(X, u)}$ coincides with the standard inclusion $D(X, u) \hookrightarrow F(Y, v)$, but the homomorphism $\rho$ longer arises from spatial inclusion.

(ii) Choose an element $s \in F(Y, v)$ having order 2 (such elements always exist). Perhaps after changing the choice of the clopen set $X \subset Y$, we can assume $s(X) \cap X = \emptyset$. In particular $F(s(X), sus^{-1}) = sF(X, u)s^{-1}$ is an isomorphic copy of $F(X, u)$ which commutes with it. Let $\rho_0: F(X, u) \rightarrow F(Y, v)$ be the diagonal embedding into their product, given by $\rho_0(g) = g(sgs)$. Note that $s$ commutes with the image of $\rho_0$. Set $Z = X \sqcup s(X)$ and $w = u(sus)$. Then $(Z, w)$ is a Cantor system that factors onto $(X, u)$ via the map $p: Z \rightarrow X$ given by

$$p(x) = \begin{cases} x & x \in X \\ s(x) & x \in s(X) \end{cases},$$

and one can check that $\rho_0$ coincides with the composition $F(X, u) \hookrightarrow F(Z, w) \hookrightarrow F(Y, v)$ of the corresponding blow-up homomorphism and spatial inclusion homomorphism. Now define a new homomorphism $\rho: F(X, u) \rightarrow F(Y, v)$ by

$$\rho(g) = \rho_0(g)s^{I(g)}.$$  

Then $\rho$ coincides with $\rho_0$ in restriction to $D(X, u)$, but it is no longer itself the composition of homomorphisms of that form.

### 7.2. Example: one-sided SFT’s and Higman-Thompson’s groups

To give an illustration of Theorem 7.3 for a class of groupoids that are not given by free actions of finitely generated groups, let us discuss the example of groupoids associated to one sided shifts of finite type. Their full groups were studied in detail by Matui in [Mat15], and can be seen as generalisations of the Higman-Thompson’s groups.

Let $\Sigma = (\mathcal{V}, \mathcal{E})$ be a directed graph, with finite sets of vertices $\mathcal{V}$ and of edges $\mathcal{E}$. We assume that $\Sigma$ is irreducible, i.e. for any vertices $v, w \in \mathcal{V}$ there is a directed path going from $v$ to $w$, and that it is not a cycle (hence there are at least two such paths for any pair of vertices). We let $X_\Sigma$ be the set of all one sided infinite directed paths in $\Sigma$. Note that by a path we mean its set of edges. The set $X_\Sigma$ is endowed with the topology induced from the product topology on $\mathcal{E}^\mathbb{N}$, which makes it homeomorphic to the Cantor set. The map

$$S: X_\Sigma \rightarrow X_\Sigma, \quad e_0e_1e_2 \cdots \mapsto e_1e_2 \cdots,$$

is called an irreducible shift of finite type associated to $\Sigma$.

The map $S$ is not invertible, but it is invertible locally. More precisely, for every edge $e \in \mathcal{E}$ let $U_e \subset X_\Sigma$ be the clopen subset consisting of paths that start with $e$. Then the map

$$S_e: U_e \rightarrow \sigma(U_e)$$
is a homeomorphism. We let $G_{\Sigma}$ be the groupoid generated by the germs of all such homeomorphisms and their inverses. An alternative description of $G_{\Sigma}$ is as follows. Given a finite directed path $e_1 \cdots e_k$ in $\Sigma$, we denote $U_{e_1 \cdots e_k} \subset X_{\Sigma}$ the clopen subset of paths that start with $e_1 \cdots e_k$. Let $d_1 \cdots d_k$ be another path with the same endpoint as $e_1 \cdots e_k$ (but not necessarily with the same starting point, nor with the same length). By composing maps of the form $S_e$ and their inverses we obtain a homeomorphism

$$S_{d_1}^{-1} \cdots S_{d_k}^{-1} S_{e_k} \cdots S_{e_1} : U_{e_1 \cdots e_k} \rightarrow U_{d_1 \cdots d_k}, \quad e_1 \cdots e_k e'_k e_{k+2} \cdots \rightarrow d_1 \cdots d_k e'_k e_{k+2} \cdots.$$

Then $G_{\Sigma}$ is the set of germs of all maps of this form. The set of germs of $S_{d_1}^{-1} \cdots S_{d_k}^{-1} S_{e_k} \cdots S_{e_1}$ is a compact open bisection of $G_{\Sigma}$, and bisections of this form are a basis of the topology of $G_{\Sigma}$. In particular the set $S = \{S_e : e \in \mathcal{E}\}$ is a generating set of $G_{\Sigma}$, which is moreover expansive. (Here and in what follows, we view $S_e$ as a bisection of $G_{\Sigma}$ by identifying $S_e$ with the set of its germs).

**Example 7.10.** Let $\Sigma$ be a graph consisting of one vertex and two loops. The corresponding map $S : X_{\Sigma} \rightarrow X_{\Sigma}$ is conjugate to the one-sided shift on $\{0,1\}^\mathbb{N}$. The group $F(\Sigma)$ is isomorphic to the Higman-Thompson group $V$ acting on the Cantor set. In a similar way, the extended family of Higman-Thompson’s groups $(V_{n,d})$ are topological full groups of one-sided shifts of finite type, see Matui Sec. 6.7.1.

We have learned the following observation from V. Nekrashevych.

**Lemma 7.11.** For every $x \in X_{\Sigma}$ the Cayley graph $\Gamma_x(G_{\Sigma},S)$ is a tree.

**Proof.** Let $x \in X_{\Sigma}$ and assume that $x$ belongs to a non-trivial cycle in $\Gamma_x(G,S)$ without backtracking. Let $S^{e_1} \cdots S^{e_k}$ be the word in read on the labels of this cycle, where $e_i \in \{-1,1\}$. This means that the domain of the corresponding transformation contains $x$ and acts trivially on a neighbourhood $U$ of $x$ contained in the domain. Let $y \in U$ be a point which is not an eventually periodic sequence (note that such sequences are dense). Let $y_0 = y$, and $y_i = S^{e_i} \cdots S^{e_1} y$. Note that $y_{i+1}$ is obtained from $y_i$ by either appending or removing $e_{i+1}$. Since $y_n = y$ and is not eventually periodic, it follows that $S^{e_n} \cdots S^{e_1}$ represents a trivial element in the free group over $S \cup S^{-1}$, and hence the cycle in $\Gamma_x(G,S)$ had to contain some backtracking, a contradiction. \hfill \Box

In particular, the Cayley graphs of $G_{\Sigma}$ have finite asymptotic dimension. Therefore Theorem 7.10 can be applied to this class of groups. Moreover, it can be strengthened to yield the following result. To make the statement shorter, we state it only for homomorphisms defined on the derived subgroup $D(G_{\Sigma})$. Note that the latter is simple by a result of Matui (therefore in this case we have $D(G_{\Sigma}) = \mathbb{A}(G_{\Sigma})$).

**Corollary 7.12.** Let $G_{\Sigma_1}, G_{\Sigma_2}$ be groupoids associated to irreducible one sided shifts of finite type. Let $\rho : D(G_{\Sigma_1}) \rightarrow \mathbb{F}(G_{\Sigma_2})$ be a non-trivial homomorphism. Then its image is supported in a clopen subset $Z \subset X_{\Sigma_2}$ such that the action $D(G_{\Sigma_1}) \acts Z$ is induced from an action $\alpha : G_{\Sigma_1} \acts (Z,q)$ for some continuous surjective map $q : Z \rightarrow X_{\Sigma_1}$. In particular, the action $D(G_{\Sigma_1}) \acts Z$ factors onto the canonical action $D(G_{\Sigma_1}) \acts X_{\Sigma_1}$ through the map $q$. 
This statement appears to be new also for groups in the family of the Higman–Thompson groups $V_{n,d}$.

**Proof.** By Theorem 7.5 and Lemma 7.11 the homomorphism $\rho$ is tame. We only need to show that its rank is equal to 1. This follows from the combination of Lemma 7.7 and of Proposition 7.3 since the product of $r$ trees has asymptotic dimension $r$. □

8. Obstructions to Homomorphisms

In this section we apply our results to show that geometric and dynamical asymptotic invariants of groupoids give rise to obstructions to the existence of embeddings between the corresponding topological full groups. This gives an interpretation of these invariants in terms of the algebraic behaviour of the topological full group, formalising the intuition that they “constraint” their structure.

**Theorem 8.1** (Obstructions to embeddings). Let $\mathcal{G}$ be a compactly generated minimal groupoid of germs with Cantor set unit space. Let $\mathcal{H}$ be another compactly generated étale groupoid. Assume that one of the following conditions holds.

(i) the groupoid $\mathcal{H}$ is a groupoid of germs, and the orbital growth functions satisfy $\beta_\mathcal{H} \not\leq \beta_\mathcal{G}$;

(ii) the groupoid $\mathcal{H}$ belongs to the class $\mathcal{E}$, both $\mathcal{G}$ and $\mathcal{H}$ are expansive (with Cantor set unit space), their complexity functions satisfy $\pi_\mathcal{G} \not\leq \pi_\mathcal{H}$.

(iii) the asymptotic dimension of every Cayley graph of $\mathcal{G}$ is strictly larger than the asymptotic dimension of every Cayley graph of $\mathcal{H}$.

Then there is no non-trivial homomorphism $\mathbb{A}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$. In particular every homomorphism $\mathbb{F}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$ factors through $\mathbb{F}(\mathcal{G})/\mathbb{A}(\mathcal{G})$.

**Proof.** Assume by contradiction that there is a non-trivial homomorphism $\rho: \mathbb{A}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$. Assume that (i) holds. Since $\mathcal{H}$ is a groupoid of germs the group $\mathbb{F}(\mathcal{H})$ acts faithfully on $Y$, and since $\mathbb{A}(\mathcal{G})$ is simple it must act faithfully on every $\mathcal{H}$-orbit by permutations of bounded displacement on the orbital graph. We get a contradiction by Theorem 5.2.

Assume that (ii) holds. By Theorem 7.5 $\rho$ must be tame, and the following lemma provides a contradiction.

**Lemma 8.2.** Let $\mathcal{G}, \mathcal{H}$ be étale groupoids with Cantor set unit space. If there exists a non-trivial tame homomorphism $\rho: \mathbb{F}(\mathcal{G}) \to \mathbb{F}(\mathcal{H})$ then for every set finite set of compact open bisections $T$ of $\mathcal{G}$ there exists a finite set of compact open bisections $S$ of $\mathcal{H}$ whose complexities satisfy $\pi_\mathcal{G}(n,T) \leq \pi_\mathcal{H}(n,S)$ for every $n \geq 1$. In particular, if $\mathcal{G}$ and $\mathcal{H}$ are expansive, we have $\pi_\mathcal{G}(n) \leq \pi_\mathcal{H}(n)$.

**Proof.** Let $r \geq 1, Z \subset Y, \alpha: \mathcal{G}^* \rhd (Z,q)$ and $b: \mathcal{G}^* \ltimes_\alpha Z \to \mathcal{H}$ be as in Theorem 6.9. Given a bisection $T$ of $\mathcal{G}$, we denote $T^* = \{Q \in G^*: Q \subset T\}$, which is a bisection of $\mathcal{G}^*$. Let $T^* = \{T^* : T \in T\}$. Observe that for every $n \geq 1$, the partition $Q_{T^*}(n)$ induces on $X \subset X^{(r)}$ exactly the partition $Q_T(n)$, and therefore we have $\pi_\mathcal{G}(n,T) \leq \pi_{\mathcal{G}^*}(n,T^*)$. Now for every $T \in T$ the set $T^* s \times Z = \{ (Q,z) : Q \in T^*, s(Q) = q(z) \}$ is a bisection of $\mathcal{G}^* \ltimes_\alpha Z$. Let $S = \{ b(T^* s \times Z) : T \in S\}$. It is a finite set if bisection of $\mathcal{H}$. The partition
\( Q_S(n) \) induces on \( Z \) the preimage under \( q \) of the partition \( Q_{T^*}(n) \). We deduce that \( S \) verifies the desired conclusion. The last sentence follows from Lemma 2.28. □

Finally assume that [iii] holds. Then the conclusion follows from the combination of Lemma 7.7 and Proposition 7.3. This concludes the proof of the theorem. □

8.1. Example: case of free actions of finitely generated groups. Let us state in the following corollary how this theorem reads in the case of the topological full group of free actions of a finitely generated group actions.

**Corollary 8.3.** Let \( G \bowtie X; H \bowtie Y \) be topologically free actions of finitely generated groups, where \( G \bowtie X \) is minimal and \( X \) is the Cantor set. Assume that one of the following holds.

(i) The growth functions of \( G \) and \( H \) satisfy \( b_G(n) \preceq b_H(n) \).

(ii) The group \( H \) has finite asymptotic dimension, and the actions \( G \bowtie X \) and \( H \bowtie Y \) are subshifts over a finite alphabet and their complexity functions satisfy \( p_X(n) \preceq p_Y(n) \).

(iii) We have \( \text{asdim}(H) < \text{asdim}(G) \).

Then there is no non-trivial homomorphism \( A(G; X) \rightarrow F(G; Y) \). In particular the group \( F(G; X) \) cannot embed in \( F(G; Y) \).

**Proof.** All Cayley graphs of the corresponding groupoids of germs are quasi-isometric to \( G \) and \( H \) respectively, and [i] and [iii] immediately follow from the theorem. [ii] follows from Proposition 2.31 and from Theorem 8.1. □

An interesting special case of [ii] is the case of Cantor systems \( (X, u) \), i.e. actions of the group \( \mathbb{Z} \) generated by a homeomorphisms \( u: X \rightarrow X \). If \( (X, u) \) is a subshift, its complexity is tightly related to the topological entropy of \( (X, u) \) by \( h_{\text{top}}(X, u) = \lim_{n \rightarrow \infty} \frac{1}{n} \log p_u(n) \) (here we compute the complexity using the generating set \( \{u^\pm 1\} \) of \( \mathbb{Z} \)).

**Corollary 8.4.** Let \( (X, u), (Y, v) \) be subshifts over finite alphabets, with \( (X, u) \) minimal. If there is a non-trivial homomorphism \( D(X, u) \rightarrow F(Y, v) \), then the corresponding complexity functions satisfy \( p_X \preceq p_Y \). In particular if \( h_{\text{top}}(X, u) > 0 \) and \( h_{\text{top}}(Y, v) = 0 \), then every homomorphism \( F(X, u) \rightarrow F(Y, v) \) has abelian image.

**Remark 8.5.** The last sentence fails if one assumes only \( h_{\text{top}}(X, u) > h_{\text{top}}(Y, v) \). For example, for every system \( (X, u) \), the group of \( F(X, u^2) \) is a subgroup of \( F(X, u) \), and the corresponding entropies verify \( h_{\text{top}}(X, u^2) = 2h_{\text{top}}(X, u) \).

8.2. Example: products of SFT’s and Brin’s groups \( nV \). As an example of application to groupoids that are not given by free actions of finitely generated groups, let us consider again the case of groupoids associated to one-sided shifts of finite type (see Subsection 7.2). Matui studies in [Mat16] étale groupoids arising from products of such groupoids, i.e. étale groupoids of the form \( G = \mathcal{G}_{\Sigma_1} \times \cdots \times \mathcal{G}_{\Sigma_n} \), where each \( \mathcal{G}_{\Sigma_n} \) is the groupoid associated to an irreducible one-sided shift of finite type. As an illustration of Theorem 8.1 [iii], we have the following.
Corollary 8.6. Let $G, H$ be respectively the products of $n$ and $m$ étale groupoids arising from irreducible shifts of finite type. If $n > m$, then every homomorphism $\mathbb{F}(G) \to \mathbb{F}(H)$ has abelian image.

Proof. The derived subgroup $D(G)$ is simple by [Mat15, Th. 4.2] since $G$ is purely infinite (see [Mat16, Sec. 5.2]), and therefore we have $D(G) = A(G)$. Moreover it follows from Lemma 7.11 that every Cayley graph of $G$ is a product of $n$ trees and thus it has asymptotic dimension $n$, while every Cayley graph of $H$ has asymptotic dimension $m$. The conclusion follows from Theorem 8.1 (iii). \[\square\]

A special case of topological full groups of products of one sided shifts of finite type is the family of higher dimensional Higman-Thompson’s groups $nV$, introduced by Brin [Bri04]. The group $nV$ coincides with the topological full group of the product $G = G_\Sigma \times \cdots \times G_\Sigma$, where $G_\Sigma$ is as in Example 7.10.

It was shown in [Bri04] that the group $2V$ is not isomorphic to the group $V$ as a consequence of Rubin’s theorem [Rub89]. Later this was extended in [BL10] to show that $nV$ and $mV$ are isomorphic if and only if $n = m$, also using Rubin’s theorem. Theorem 8.6 implies the following stronger fact.

Corollary 8.7. The group $nV$ embeds in $mV$ if and only if $n \leq m$.

References

[BD08] G. Bell and A. Dranishnikov. Asymptotic dimension. Topology Appl., 155(12):1265–1296, 2008.
[BG00] L. Bartholdi and R. I. Grigorchuk. On the spectrum of Hecke type operators related to some fractal groups. Tr. Mat. Inst. Steklova, 231(Din. Sist., Avtom. i Beskon. Gruppy):5–45, 2000.
[BJS10] Jean Bellissard, Antoine Julien, and Jean Savinien. Tiling groupoids and Bratteli diagrams. Ann. Henri Poincaré, 11(1-2):69–99, 2010.
[BL10] Collin Bleak and Daniel Lanoue. A family of non-isomorphism results. Geom. Dedicata, 146:21–26, 2010.
[Bri04] Matthew G. Brin. Higher dimensional Thompson group $s$ $nV$, introduced by Brin [Bri04].
[CN10] Julien Cassaigne and François Nicolas. Factor complexity. In Combinatorics, automata and number theory, volume 135 of Encyclopedia Math. Appl., pages 163–247. Cambridge Univ. Press, Cambridge, 2010.
[Cor14] Yves Cornulier. Groupes pleins-topologiques [d’après Matui, Juschenko, Monod,...]. Astérisque, (361):Exp. No. 1064, 2014. Séminaire Bourbaki. Vol. 2012/2013.
[Cor15] Yves Cornulier. Irreducible lattices, invariant means, and commensurating actions. Math. Z., 279(1-2):1–26, 2015.
[Dye59] H. A. Dye. On groups of measure preserving transformation. I. Amer. J. Math., 81:119–159, 1959.
[Ele17] G. Elek. On uniformly recurrent subgroups of finitely generated groups. arXiv:1702.01631, 2017.
[GJS09] Su Gao, Steve Jackson, and Brandon Seward. A coloring property for countable groups. Math. Proc. Cambridge Philos. Soc., 147(3):579–592, 2009.
[Gla76] Shmuel Glasner. Proximal flows. Lecture Notes in Mathematics, Vol. 517. Springer-Verlag, Berlin-New York, 1976.
[GM07] Y. Glasner and N. Monod. Amenable actions, free products and a fixed point property. Bull. Lond. Math. Soc., 39(1):138–150, 2007.
[GPS99] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Full groups of Cantor minimal systems. Israel J. Math., 111:285–320, 1999.
[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In Geometric group theory, Vol. 2 (Sussex, 1991), volume 182 of London Math. Soc. Lecture Note Ser., pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[GW15] E. Glasner and B. Weiss. Uniformly recurrent subgroups. In Recent trends in ergodic theory and dynamical systems, volume 631 of Contemp. Math., pages 63–75. Amer. Math. Soc., Providence, RI, 2015.

[Hae02] André Haefliger. Foliations and compactly generated pseudogroups. In Foliations: geometry and dynamics (Warsaw, 2000), pages 275–295. World Sci. Publ., River Edge, NJ, 2002.

[HZ97] B. Hartley and A. E. Zalesskii. Confined subgroups of simple locally finite groups and ideals of their group rings. J. London Math. Soc. (2), 55(2):210–230, 1997.

[JdlS15] Kate Juschenko and Mikael de la Salle. Invariant means for the wobbling group. Bull. Belg. Math. Soc. Simon Stevin, 22(2):281–290, 2015.

[JM13] Kate Juschenko and Nicolas Monod. Cantor systems, piecewise translations and simple amenable groups. Ann. of Math. (2), 178(2):775–787, 2013.

[JNdlS16] Kate Juschenko, Volodymyr Nekrashevych, and Mikael de la Salle. Extensions of amenable groups by recurrent groupoids. Invent. Math., 206(3):837–867, 2016.

[Ke97] Johannes Kellendonk. Topological equivalence of tilings. J. Math. Phys., 38(4):1823–1842, 1997. Quantum problems in condensed matter physics.

[KP00] Johannes Kellendonk and Ian F. Putnam. Tilings, $C^*$-algebras, and K-theory. In Directions in mathematical quasicrystals, volume 13 of CRM Monogr. Ser., pages 177–206. Amer. Math. Soc., Providence, RI, 2000.

[LB18] A. Le Boudec. Amenable uniformly recurrent subgroups and lattice embeddings. in preparation, 2018.

[LBMB16] A. Le Boudec and N. Matte Bon. Subgroup dynamics and C*-simplicity of groups of homeomorphisms. Ann. Sci. Ecole Norm. Sup. (to appear), 2016.

[LN07] Y. Lavrenyuk and V. Nekrashevych. On classification of inductive limits of direct products of alternating groups. J. Lond. Math. Soc. (2), 75(1):146–162, 2007.

[LP02] Felix Leinen and Orazio Puglisi. Confined subgroups in periodic simple finitary linear groups. Israel J. Math., 128:285–324, 2002.

[LP03] Felix Leinen and Orazio Puglisi. Diagonal limits of finite alternating groups: confined subgroups, ideals, and positive definite functions. Illinois J. Math., 47(1-2):345–360, 2003. Special issue in honor of Reinhold Baer (1902–1979).

[Mat06] Hiroki Matui. Some remarks on topological full groups of Cantor minimal systems. Internat. J. Math., 17(2):231–251, 2006.

[Mat12] Hiroki Matui. Homology and topological full groups of étale groupoids on totally disconnected spaces. Proc. Lond. Math. Soc. (3), 104(1):27–56, 2012.

[Mat15] Hiroki Matui. Topological full groups of one-sided shifts of finite type. J. Reine Angew. Math., 705:35–84, 2015.

[Mat16] Hiroki Matui. étale groupoids arising from products of shifts of finite type. Adv. Math., 303:502–548, 2016.

[MB14] Nicolás Matte Bon. Subshifts with slow complexity and simple groups with the Liouville property. Geom. Funct. Anal., 24(5):1637–1659, 2014.

[MB16] N. Matte Bon. Full groups of bounded automaton groups. J. Fractal Geometry (to appear), 2016.

[MBT17] N. Matte Bon and T. Tsankov. Realizing uniformly recurrent subgroups. arXiv:1702.07101, 2017.

[Nek10] V. Nekrashevych. Free subgroups in groups acting on rooted trees. Groups Geom. Dyn., 4(4):847–862, 2010.

[Nek13] V. Nekrashevych. Finitely presented groups associated with expanding maps. arXiv:1312.5654v1, 2013.
[Nek15a] V. Nekrashevych. Simple groups of dynamical origin. *Ergodic Theory Dynam. Syst (to appear)*, 2015.

[Nek15b] Volodymyr V. Nekrashevych. Hyperbolic groupoids and duality. *Mem. Amer. Math. Soc.*, 237(1122):v+105, 2015.

[Nek16a] V. Nekrashevych. Palindromic subshifts and simple periodic groups of intermediate growth. *Ann. Math. (2) (to appear), arXiv:1601.01033*, 2016.

[Nek16b] Volodymyr Nekrashevych. Growth of étale groupoids and simple algebras. *Internat. J. Algebra Comput.*, 26(2):375–397, 2016.

[Neu54] B. H. Neumann. Groups covered by permutable subsets. *J. London Math. Soc.*, 29:236–248, 1954.

[Rub89] Matatyahu Rubin. On the reconstruction of topological spaces from their groups of homeomorphisms. *Trans. Amer. Math. Soc.*, 312(2):487–538, 1989.

[_SZ93] S. K. Sehgal and A. E. Zalesskii. Induced modules and some arithmetic invariants of the finitary symmetric groups. *Nova J. Algebra Geom.*, 2(1):89–105, 1993.

[Tho17] S. Thomas. Uniformly recurrent subgroups of simple locally finite groups. *Preprint*, 2017.