A twisted inclusion between tensor products of operator spaces, with an application to 2-cocycles

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Abstract

Given operator spaces $V$ and $W$, let $\tilde{W}$ denote the opposite operator space structure on the same underlying Banach space. Although the identity map $W \to \tilde{W}$ is in general not completely bounded, we show that the identity map on $V \otimes W$ extends to a contractive linear map $V \otimes \tilde{W} \to V \otimes_{\min} W$, where $\otimes$ and $\otimes_{\min}$ denote the projective and injective tensor products of operator spaces. We then sketch how this aids us in constructing anti-symmetric 2-cocycles on certain Fourier algebras.

Dedicated to John Rainwater, with thanks for his varied contributions and generous spirit.

1 Introduction

We start by emphasising a convention that will be adhered to throughout this note: all operator spaces are assumed, as part of the definition, to be complete.

A core result in the theory of operator spaces is the following observation: the operation on $K(\ell_2)$ given by matrix transpose, $a \mapsto a^\top$, fails to be completely bounded, even though it is an isometric involution of Banach spaces. This fact serves to explain certain phenomena in non-commutative harmonic analysis, and can be exploited to prove structural results about Fourier and Fourier-Stieltjes algebras of locally compact groups: see, for instance, [4, 12].

More generally: given an operator space $W$ — by which we mean a complex vector space $W$, equipped with a sequence of complete norms $\|\cdot\|_{(n)}$ on $M_n(W) = M_n \otimes W$ that satisfy Ruan’s axioms — one may define a new sequence of norms as follows:

$$\|\sum_i a_i \otimes w_i\|_{(n),\text{opp}} := \|\sum_i a_i^\top \otimes w_i\|_{(n)} \quad (a_i \in M_n, w_i \in W).$$

These matrix norms also define an operator space structure on $W$, which we denote by $\tilde{W}$ and call\footnote{A more natural name might be the transposed operator space structure, but the terminology here appears to be the standard one.} the opposite operator space of $W$, or the opposite operator space structure on the underlying Banach space of $W$. While $W$ and $\tilde{W}$ have the same underlying Banach space, in general they are not isomorphic as operator spaces.
All this is well known. However, the following result appears to be new, or at least not recorded in the literature. We write $\widehat{\otimes}$ for the projective tensor product of operator spaces and $\otimes_{\text{min}}$ for the injective tensor product (also known as the minimal tensor product).

**Theorem 1.1 (Main theorem).** Let $V$ and $W$ be operator spaces. The identity map on their algebraic tensor product $V \otimes W$ extends to a linear contraction

$$\Psi_{V,W} : V \widehat{\otimes} \tilde{W} \to V \otimes_{\text{min}} W.$$ 

The proof is based on an interpolation argument suggested to the author by John Rainwater (personal communication) and will be explained in Section 3. Note that in general, $\Psi_{V,W}$ cannot be completely bounded (just take $V = \mathbb{C}$), and so it seems hard to obtain a more direct proof by considering operator-space tensor norms on $V \otimes W$. We also note that in this theorem, we cannot replace the projective tensor product by the Haagerup tensor product, even if we weaken “linear contraction” to “bounded linear map” (Proposition 3.3).

Theorem 1.1 was originally motivated by a technical issue that arose in studying the Hochschild cohomology of Fourier algebras, specifically the problem of higher-dimensional weak amenability as defined in [7]. The main issue is that when $A$ and $B$ are cb-versions of Banach algebras, and $V$ and $W$ are cb-modules over $A$ and $B$ respectively, then $V \widehat{\otimes} \tilde{W}$ might not be a Banach module over $A \otimes B$; however, $V \otimes_{\text{min}} W$ is such a module, and then Theorem 1.1 allows us to replace the bad space $V \widehat{\otimes} \tilde{W}$ with the better space $V \otimes_{\text{min}} W$.

More precise statements can be found in Section 4, whose main result – on existence of certain 2-cocycles – is stated as Theorem 4.8. The presentation in Section 4 is more of a sketch than a comprehensive account: in future work we intend to investigate these cohomology problems in much more detail, and develop the appropriate theoretical framework in a more complete way.

**Remarks on notation.** What we have written here as $\tilde{W}$ is often denoted in the literature by $W^\text{op}$. We have chosen different notation because in some of the intended applications, one is dealing with an operator space $A$ which is also an algebra; and hence there is a potential conflict with the usage of $A^\text{op}$ to denote the “opposite algebra”, i.e. the algebra with the same underlying vector space but with reversed product. In longer expressions, when considering the opposite operator space, we use the notation $(\ldots)^\sim$; for instance $B(H)^\sim$ denotes $B(H)$ equipped with the opposite of its usual operator space structure.

### 2 Conventions and technical preliminaries

We assume the reader is familiar with the basic definitions of operator spaces and completely bounded maps, as presented in [3] or [11]. We remind her that, given operator spaces $E$ and $F$, the space $CB(E,F)$ is itself an operator space in a natural way. If $V$ is an operator space, then the dual Banach space $V^*$ becomes an operator space under the identification $V^* = CB(V, \mathbb{C})$, while $CB(\mathbb{C}, V)$ is completely isometrically isomorphic to $V$.

Although we will not use any category theory, it is convenient occasionally to refer to the category of operator spaces and completely bounded maps, which we denote by $\text{OpSp}$. 
We shall abbreviate the phrase “operator space structure” to o.s.s. Whenever $H$ is a Hilbert space and we refer to $B(H)$ as an operator space, we assume (unless explicitly stated otherwise) that it is equipped with its usual, canonical o.s.s.; note that if we do this, then there is a natural and completely isometric identification of $B(H)$ with $CB(COL_H)$, where $COL_H$ denotes $H$ equipped with the column o.s.s.

**Opposite operator spaces.** The opposite o.s.s. was already defined in the introduction; see also [11 §2.10]. Let us collect some basic properties that do not seem to be mentioned in [3] or [11]. It is easily checked that if $f : X \to Y$ is completely bounded, then so is $f : \tilde{X} \to Y$, with the same cb-norm. For sake of clarity, and to emphasise the functorial behaviour, we write this as $\tilde{f} : \tilde{X} \to Y$. The same calculation gives, with some book-keeping, a more precise result: we omit the details.

**Lemma 2.1.** Given operator spaces $E$ and $F$, we have a completely isometric isomorphism $CB(E,F)\sim \cong_1 CB(\tilde{E},\tilde{F})$. In particular, we can identify $(E^*)\sim$ with $(\tilde{E})^*$.

If $K$ is a Hilbert space, the transpose operator $\top : B(K) \to B(K)$, $b \mapsto b^\top$, is defined by

$$(b^\top,\xi,\eta) := (b\eta,\xi) \quad (\xi,\eta \in K).$$

More explicitly: if we fix a basis for $K$, then the matrix of $b^\top$ with respect to this basis is the transpose of the matrix representing $b$ (with respect to the same basis). It is clear that $\top$ provides a complete isometry from $B(K)$ onto $(B(K))^\sim$, and vice versa.

**Tensor products and tensor norms.** For clarity, we repeat some notation. The algebraic tensor product of two complex vector spaces $E$ and $F$ is denoted by $E \otimes F$. The projective and injective tensor products in the category $\OpSp$ are denoted by $\otimes$ and $\otimes_{\min}$ respectively; this follows the notation of [11], rather than that of [3]. If $f \in CB(E,X)$ and $g \in CB(F,Y)$ then by tensoring we obtain completely bounded maps $E \otimes F \to X \otimes Y$ and $E \otimes_{\min} F \to X \otimes_{\min} Y$; for extra emphasis, these maps will be denoted by $f \otimes g$ and $f \otimes_{\min} g$ respectively. We have $\|f \otimes g\|_{cb} = \|f\|_{cb}\|g\|_{cb}$.

In proving Theorem [11], we exploit the fact that the injective tensor norm on $B(H) \otimes B(K)$ can be calculated in terms of the action of this algebra on $S_2(K,H)$, the space of Hilbert-Schmidt operators $K \to H$.

**Lemma 2.2.**

(i) The linear map $\theta_0 : B(H) \otimes B(K) \to B(S_2(K,H))$ that is defined by $\theta_0(a \otimes b)(c) := acb^\top$ extends to a complete isometry $\theta : B(H) \otimes_{\min} B(K) \to B(S_2(K,H))$.

(ii) There is a complete isometry $\Lambda : B(H) \otimes_{\min} B(K)^\sim \to B(S_2(K,H))$, which satisfies $\Lambda(a \otimes b)(c) = acb$ for all $a \in B(H)$, $b \in B(K)$ and $c \in S_2(K,H)$.

**Sketch of the proof.** Part (i) can be checked by considering the expression $\langle (a \otimes b^\top)\xi,\eta \rangle$ for $\xi,\eta \in H \otimes_2 K$, expanding out $\xi$ and $\eta$ as linear combinations of elementary tensors, and making direct calculations; see also Proposition 2.9.1 in [11], or the calculations in Section 3.5 of [3]. Part (ii) follows from part (i) by composing $\theta$ with the completely isometric isomorphism $\iota \otimes \top : B(H) \otimes_{\min} B(K)^\sim \to B(H) \otimes_{\min} B(K)$.
Instead of the “concrete” definition of the matrix norms on the projective tensor product of operator spaces, we prefer to use its characteristic universal property: it linearizes those bilinear maps $h : E \times F \rightarrow G$ which are “completely bounded” in the following sense

$$\sup_{m,n \geq 1} \{ \| h(x_{ij}, y_{pq}) \|_{M_{mn}(E \otimes F)} : [x_{ij}] \in \text{ball}_1 M_m(E), [y_{pq}] \in \text{ball}_1 M_n(F) \} < \infty.$$  

(Here, our terminology is that of [3]: the reader should beware that often such maps are instead called “jointly completely bounded”, and that the term “completely bounded” is then used for what [3] call “multiplicatively bounded”.) Denoting the space of such maps by $\text{CB}_{\text{bil}}(E \times F; G)$, there are natural and completely isometric identifications

$$\text{CB}(E \hat{\otimes} F, G) \cong_1 \text{CB}_{\text{bil}}(E \times F; G) \cong_1 \text{CB}(E, \text{CB}(F, G)).$$

(See [3, Proposition 7.1.2].) We note, for future reference, that there is a natural and completely isometric identification of $\hat{E} \otimes \hat{F}$ with $(E \otimes F)^\sim$.

Remark 2.3. If $H$ is a Hilbert space equipped with column o.s.s., the natural left action of $B(H)$ on $H$ defines a completely contractive bilinear map $B(H) \times H \rightarrow H$. This can be verified directly, but it is more illuminating to see it as a special case of the following general fact: given operator spaces $E, F$ and $G$, the composition operation

$$\text{CB}(F, G) \times \text{CB}(E, F) \rightarrow \text{CB}(E, G)$$

is completely contractive as a bilinear map. (To deduce the original statement, take $E = \mathbb{C}$ and $F = G = \text{COL}_H$.) Similarly, if we denote the Banach-space adjoint of an operator $b : H \rightarrow H$ by $b^* : H^* \rightarrow H^*$, then the adjoint action of $B(H)$ on $H^*$ defines a completely contractive bilinear map $H^* \times B(H) \rightarrow H^*$. (Take $E = F = \text{COL}_H$ and $G = \mathbb{C}$.)

We also need a standard lemma on “interchanging tensor products”: see [3, Theorem 8.1.10] for a proof.

Lemma 2.4. Let $E$, $F$ and $G$ be operator spaces. There are complete contractions

$$E \hat{\otimes} (F \otimes_{\text{min}} G) \rightarrow (E \hat{\otimes} F) \otimes_{\text{min}} G, \quad (E \otimes_{\text{min}} F) \hat{\otimes} G \rightarrow E \otimes_{\text{min}} (F \hat{\otimes} G),$$

both of which are the identity map when restricted to elementary tensors.

3 The main technical result

We will deduce the main result (Theorem 1.1) from the special case $V = B(H), W = B(K)$. It is convenient to reformulate this special case slightly, using Lemma 2.2.

Theorem 3.1. Let $H$ and $K$ be Hilbert spaces. Then the linear map $\Phi_0 : B(H) \otimes B(K) \rightarrow B(S_2(K, H))$ that is defined by $\Phi_0(a \otimes b)(c) = acb$ extends to a contractive linear map $\Phi_2 : B(H) \hat{\otimes} B(K) \rightarrow B(S_2(K, H))$.

The following proof is based on suggestions of John Rainwater (see the Acknowledgements for further details) and we thank him for his consent to include the proof here.
Proof (Rainwater). Write $S_0(K, H)$ for the space of finite-rank operators $K \to H$, $S_1(K, H)$ for the space of nuclear operators $K \to H$, and $S_\infty(K, H)$ for the space of all compact operators $K \to H$. Then we have three linear maps

$$
\Psi_j : B(H) \otimes S_j(K, H) \otimes B(K) \to S_j(K, H) \quad (j = 0, 1, \infty),
$$

each of which is defined on elementary tensors by $a \otimes c \otimes b \mapsto acb$. The key observation is that, if we equip $S_1(K, H)$ and $S_\infty(K, H)$ with appropriate o.s. structures, then we can extend both $\Psi_1$ and $\Psi_\infty$ to completely contractive linear maps on the threefold projective tensor product.

The details are as follows. Equip $H$ and $K$ with column o.s.s., and consider the two operator spaces $H \hat{\otimes} K^*$ and $H \otimes_{\min} K^*$. The usual identification of $H \otimes K^*$ with $S(K, H)$ extends to give two isometric isomorphisms of Banach spaces $H \hat{\otimes} K^* \cong_1 S_1(K, H)$ and $H \otimes_{\min} K^* \cong_1 S_\infty(K, H)$. (See e.g. [3, Proposition 8.2.1] So we can identify $\Psi_0$, $\Psi_1$ and $\Psi_\infty$ with the following linear maps:

$$
\begin{align*}
\Psi_0 : & \quad B(H) \otimes [H \otimes K^*] \otimes B(K) \to H \otimes K^* \\
\Psi_1 : & \quad B(H) \otimes [H \hat{\otimes} K^*] \otimes B(K) \to H \hat{\otimes} K^* \\
\Psi_\infty : & \quad B(H) \otimes [H \otimes_{\min} K^*] \otimes B(K) \to H \otimes_{\min} K^*
\end{align*}
$$

On elementary tensors, these maps satisfy

$$
\Psi_j(a \otimes (v \otimes \phi) \otimes b) = av \otimes b^\# \phi \quad (a \in B(H), v \in H, \phi \in K^*, b \in B(K))
$$

for $j = 0, 1, \infty$. (Recall that $b^\# : K^* \to K^*$ is the adjoint in the Banach space sense.)

As noted in Remark [2,3] there are completely contractive maps $\lambda : B(H) \hat{\otimes} H \to H$ and $\rho : K^* \hat{\otimes} B(K) \to K$, satisfying $\lambda(a \otimes v) = av$ $(a \in B(H), v \in H)$ and $\rho(\phi \otimes b) = b^\# \phi$ $(b \in B(K), \phi \in K^*)$. Hence, by associativity of $\hat{\otimes}$, we see that $\lambda \otimes \rho$ defines a complete contraction

$$
B(H) \otimes H \hat{\otimes} K^* \otimes B(K) \to H \hat{\otimes} K^* \tag{1}
$$

which extends $\Psi_1$. Furthermore, by using associativity of $\hat{\otimes}$, and using Lemma [2,4] twice, we have complete contractions

$$
B(H) \hat{\otimes} [H \otimes_{\min} K^*] \otimes B(K) \cong_1 (B(H) \hat{\otimes} [H \otimes_{\min} K^*]) \hat{\otimes} B(K) \\
\to ([B(H) \hat{\otimes} H] \otimes_{\min} K^*) \hat{\otimes} B(K) \\
\to [B(H) \hat{\otimes} H] \otimes_{\min} [K^* \hat{\otimes} B(K)].
$$

Composing these maps with $\lambda \otimes_{\min} \rho$, we obtain a complete contraction

$$
B(H) \hat{\otimes} [H \otimes_{\min} K^*] \hat{\otimes} B(K) \to H \otimes_{\min} K^* \tag{2}
$$

which extends $\Psi_\infty$.

Fix $x \in B(H) \otimes B(K)$ such that $\|x\|_{B(H) \hat{\otimes} B(K)} \leq 1$, and consider the corresponding linear map $\Phi_0(x) : S_0(K, H) \to S_0(K, H)$. In view of [1] and [2], we obtain contractive linear maps

$$
\Phi_1(x) : S_1(K, H) \to S_1(K, H) , \quad \Phi_\infty(x) : S_\infty(K, H) \to S_\infty(K, H),
$$

which both extend the map $\Phi_0(x)$. 

Viewing \((S_1(K, H), S_\infty(K, H))\) as a compatible interpolation couple of Banach spaces, we have \((S_1(K, H), S_\infty(K, H))_{1/2} \cong S_2(K, H)\). By the Riesz–Thorin interpolation theorem, \(\Phi_0(x)\) extends to a contractive linear map \(\Phi_2(x) : S_2(K, H) \to S_2(K, H)\). It is now routine to check that \(\Phi_2\) defines a linear contraction \(B(H) \otimes B(K) \to B(S_2(K, H))\), which completes the proof of Theorem 3.1. \(\square\)

**Proof of Theorem 3.4.** Let \(V\) and \(W\) be operator spaces, and fix two completely isometric embeddings \(j_V : V \to B(H)\) and \(j_W : W \hookrightarrow B(K)\) for some choices of Hilbert spaces \(H\) and \(K\). Consider the diagram

\[
\begin{array}{ccc}
B(H) \otimes B(K) & \xrightarrow{\Psi_{B(H),B(K)}} & B(H) \otimes_{\min} B(K) \\
\downarrow{j_V \otimes j_W} & & \downarrow{j_V \otimes_{\min} j_W} \\
V \otimes \tilde{W} & \xrightarrow{\text{compatible}} & V \otimes_{\min} W
\end{array}
\]

where the top arrow \(\Psi_{B(H),B(K)}\) restricts to the identity map on elementary tensors; note that \(\Psi_{B(H),B(K)}\) is well-defined and contractive by combining Theorem 3.1 with Lemma 2.2. Now observe that the left-hand vertical arrow in the diagram is a (complete) contraction, while the right-hand vertical arrow is a (complete) isometry (using the "injective" property of \(\otimes_{\min}\)). Hence, for any elementary tensor \(x \in V \otimes W\), we have

\[
\|x\|_{V \otimes_{\min} W} = \|(j_V \otimes j_W)(x)\|_{B(H) \otimes_{\min} B(K)} \\
\leq \|\Psi_{B(H),B(K)}\| \|(j_V \otimes j_W)(x)\|_{B(H) \otimes_{\min} B(K)} \leq \|x\|_{V \otimes \tilde{W}},
\]

which completes the proof. \(\square\)

**Remark 3.2.** If \(K\) is infinite-dimensional, \(\Phi_2\) cannot be completely bounded. However, let \(\text{OpSp}_{K,H}\) denote the Hilbert space \(S_2(K, H)\) equipped not with column o.s.s., but with the self-dual o.s.s. of Pisier; then by applying complex interpolation in the category \(\text{OpSp}\), one can show that \(\Phi_2\) is completely contractive as a map \(B(H) \otimes B(K) \to \text{CB}(\text{OpSp}_{K,H})\). We have chosen to omit the details from this note, since this sharper form of Theorem 3.1 does not seem to be helpful for the applications in Section 4.

We finish by briefly justifying the claim, made after the statement of Theorem 3.1, that one cannot replace the projective tensor product with the Haagerup tensor product in that theorem. The result is probably well known but we include a proof for sake of completeness.

**Proposition 3.3.** There exist C*-algebras \(A\) and \(B\) such that the identity map \(A \otimes B \to A \otimes B\) does not extend to any bounded linear map \(A \otimes_{h} B \to A \otimes_{\min} B\).

**Proof.** For convenience we take \(A = B = B(K)\) for an infinite-dimensional Hilbert space \(K\): it will be seen that we actually get separable counterexamples in the end.

Recall that \(b \mapsto b^\top\) is a complete isometry on \(B(K)\). Hence it suffices to show that the map \(\text{id} \otimes \top : B(K) \otimes B(K) \to B(K) \otimes B(K)\) has no bounded extension to a map \(B(K) \otimes_{h} B(K) \to B(K) \otimes_{\min} B(K)\).
Fix an infinite sequence of isometries \(s_1, s_2, \ldots\) in \(B(K)\) with the property that the range projections \(s_j s_j^*\) are pairwise orthogonal. In particular, \(s_j^* s_k = 0\) for \(j \neq k\) and \(P_n := \sum_{j=1}^{n} s_j s_j^*\) is an orthogonal projection for each \(n\).

Let \(x_n = \sum_{j=1}^{n} s_j \otimes s_j^* \in B(H) \otimes B(K)\): then using the standard formula for the Haagerup tensor norm (see [IIII, Chapter 5]),
\[
\|x_n\|_{B(K) \otimes_h B(K)} \leq \left\| \sum_{j=1}^{n} s_j s_j^* \right\|^{1/2} \left\| \sum_{k=1}^{n} (s_k^*)^* s_k^* \right\|^{1/2} = \|P_n\|^{1/2} \|P_n^\top\|^{1/2} \leq 1 \;
\]

On the other hand, let \(y_n = (\text{id} \otimes \top)(x_n) = \sum_{j=1}^{n} s_j \otimes s_j \in B(K) \otimes_{\text{min}} B(K)\). By the \(C^*\)-identity \(\|y_n\| = \|y_n^* y_n\|^{1/2}\), with both norms taken in \(B(K) \otimes_{\text{min}} B(K)\). But since the \(s_j\) are isometries with pairwise orthogonal ranges,
\[
y_n^* y_n = \sum_{j,k=1}^{n} s_j^* s_k \otimes s_j^* s_k = nI \otimes I .
\]

Hence \(\|(\text{id} \otimes \top)(x_n)\|_{B(K) \otimes_{\text{min}} B(K)} \geq n^{1/2} \|x_n\|_{B(K) \otimes_h B(K)}\), and since \(n\) is arbitrary the result follows. \hfill \Box

**Remark 3.4.** Since \(\otimes_h\) and \(\otimes_{\text{min}}\) are both injective tensor norms, the proof of Proposition \[\text{3.3}\] shows that we could take \(A\) to be the \(C^*\)-algebra generated by the isometries \((s_j)_{j \geq 1}\), i.e. the Cuntz algebra \(\mathcal{O}_\infty\), and \(B = A^{\text{op}}\), the \(C^*\)-algebra obtained from \(A\) by reversing the product. In particular, we get nuclear examples.

### 4 Constructing 2-cocycles on certain Fourier algebras

In this longer section, we give an outline of a problem arising in the study of Fourier algebras, and how Theorem \[\text{1.1}\] supplies a missing piece of the puzzle. This may be of interest to some specialists in abstract harmonic analysis. Since this will not be a systematic or comprehensive exposition of the Hochschild cohomology groups of Fourier algebras, we shall omit many definitions and details. The reader is referred to the articles cited in this section, and their bibliographies, for the relevant definitions and background.

#### 4.1 Obtaining cocycles from derivations

Consider commutative algebras \(A\) and \(B\) (not necessarily unital), a symmetric \(A\)-bimodule \(X\), and a symmetric \(B\)-bimodule \(Y\). Given derivations \(D_A : A \to X\), \(D_B : B \to Y\), we can “wedge” \(D_A \otimes i_B\) and \(i_A \otimes D_B\) to obtain an antisymmetric 2-cocycle \(F_0 \in Z^2(A \otimes B, X \otimes Y)\):
\[
F_0(a_1 \otimes b_1, a_2 \otimes b_2) := [D_A(a_1) \cdot a_2] \otimes [b_1 \cdot D_B(b_2)] - [a_1 \cdot D_A(a_2) \otimes D_B(b_1) \cdot b_2] \quad (3)
\]
for \(a_1, a_2 \in A\) and \(b_1, b_2 \in B\). One can verify the 2-cocycle identity by hand: this makes it clear where we require \(D_A\) and \(D_B\) to be derivations, and also why we want the bimodules \(X\) and \(Y\) to be symmetric.

Furthermore, under various mild conditions (for instance, if \(A\) and \(B\) are unital), \(F_0\) is non-zero provided that \(D_A\) and \(D_B\) are non-zero. Since degree-2 coboundaries are symmetric as functions of two variables, while \(F\) is antisymmetric, it follows that \(F_0\) defines a non-zero element of the Hochschild cohomology group \(H^2(A \otimes B, X \otimes Y)\).

The following example should be kept in mind as a motivation for the general construction.
Example 4.1. Let $A = X = \mathbb{C}[w]$, $B = Y = \mathbb{C}[z]$, $D_A g = dg/dw$ and $D_B h = dh/dz$. Identifying $A \otimes B$ with $\mathbb{C}[z, w]$, the 2-cocycle $F$ is given by 

$$F(f_1, f_2) = \frac{\partial f_1 \partial f_2}{\partial w \partial z} - \frac{\partial f_2 \partial f_1}{\partial w \partial z} \quad (f_1, f_2 \in \mathbb{C}[z, w]).$$

With minor modifications, the same procedure still works in the setting of commutative Banach algebras and symmetric Banach bimodules. Let $\hat{\otimes}_\gamma$ denote the projective tensor product of Banach spaces. Then, given commutative Banach algebras $A$ and $B$, symmetric Banach bimodules $X$ and $Y$ over $A$ and $B$ respectively, and continuous derivations $D_A : A \to X$, $D_B : B \to Y$, the formula (3) defines an antisymmetric, continuous 2-cocycle

$$F : A \hat{\otimes}_\gamma B \times A \hat{\otimes}_\gamma B \to X \hat{\otimes}_\gamma Y. \quad (4)$$

In particular, this gives a method for finding commutative Banach algebras whose degree-2 continuous Hochschild cohomology groups are non-zero. For instance, since $\ell^1(\mathbb{Z}_+^2) \cong_1 \ell^1(\mathbb{Z}_+) \hat{\otimes}_\gamma \ell^1(\mathbb{Z}_+)$, a modified version of Example 4.1 shows that $H^2(\ell^1(\mathbb{Z}_+^2), \mathbb{C}_\varphi) \neq 0$ for suitable choices of a character $\varphi : \ell^1(\mathbb{Z}_+) \to \mathbb{C}$.

Can we do something similar for Fourier algebras of locally compact groups? We now know many examples of locally compact groups $G$ for which there exist non-zero continuous derivations from $A(G)$ to suitable symmetric bimodules (see [8] for some of the history and a guide to the relevant literature). However, attempting to use (4) runs into a problem:

- the natural map $A(G_1) \hat{\otimes}_\gamma A(G_2) \to A(G_1 \times G_2)$ is surjective if and only if either $G_1$ or $G_2$ has an abelian subgroup of finite index [9];
- if $G$ has an abelian subgroup of finite index, then all continuous derivations from $A(G)$ into symmetric Banach $A(G)$-bimodules must vanish [4, Theorem 3.3].

Since there is a completely isometric algebra isomorphism $A(G_1) \hat{\otimes} A(G_2) \cong_1 A(G_1 \times G_2)$ (see e.g. [3, Theorem 7.2.4]), we might try to mimic (1) in the operator-space category $\text{OpSp}$. However, if we try to do this directly, we run into a new problem: all algebras, module actions and derivations would now live in $\text{OpSp}$; and in this setting, all derivations of the appropriate form are zero (see [13, Theorem 5.2] or [14]).

Nevertheless, not all is lost. For many non-abelian, connected groups $G$, one can actually find derivations that are completely bounded as maps $A(G) \to (A(G)^*)^\sim$. This is not mentioned explicitly in the original articles, although it is lurking implicitly (see e.g. Remark 3.2 in [1] and [8, Theorem 2.1]). The next subsection shows how we may exploit this.

4.2 The needle returns to the start of the song

We use some slightly non-standard terminology. By a $cb$-Banach algebra, we mean an operator space $A$ equipped with a bilinear, completely bounded and associative map $A \times A \to A$. Given such an $A$, we define a $cb$-Banach $A$-bimodule to be an operator space $X$, equipped with an $A$-bimodule structure such that the left action $A \times X \to X$ and the right action $X \times A \to X$ are both completely bounded.

\footnote{This is essentially the same as the example given in [5, §5].}
These notions interact well with the “opposite o.s.s. functor”, but care is needed. If $A$ is a cb-Banach algebra then so is $\tilde{A}$; and if $X$ is a cb-Banach $A$-bimodule, $\tilde{X}$ is a cb-Banach $\tilde{A}$-bimodule. (Note that when passing to the opposite o.s.s. we are not reversing the algebra product or the module actions in any way; we are merely changing the way we put norms on certain maps between matricial spaces.) However, there is no reason to suppose $\tilde{X}$ will be a cb-Banach $A$-bimodule! In particular, although $A$ and $A^*$ are cb-Banach $A$-bimodules, $\tilde{A}$ and $(A^*)^\sim$ need not be.

**Proposition 4.2.** Let $A$ and $B$ be cb-Banach algebras; let $X$ be a cb-Banach $A$-bimodule and $Y$ a cb-Banach $B$-bimodule. Let $T_A \in CB(A, \tilde{X})$ and $T_B \in CB(B, \tilde{Y})$. Then, if we define $F_1, F_2 : (A \otimes B) \times (A \otimes B) \to X \otimes Y$ by

$$F_1(a_1 \otimes b_1, a_2 \otimes b_2) = [T_A(a_1) \cdot a_2] \otimes [b_1 \cdot T_B(b_2)],$$

$$F_2(a_1 \otimes b_1, a_2 \otimes b_2) = [a_1 \cdot T_A(a_2)] \otimes [T_B(b_1) \cdot b_2],$$

both $F_1$ and $F_2$ extend to bounded bilinear maps $(A \hat{\otimes} B) \times (A \hat{\otimes} B) \to X \hat{\otimes} Y$.

**Proof.** We will only give the proof for $F_1$; the proof for $F_2$ is very similar.

Since $T_A : A \to \tilde{X}$ is completely bounded, so is $\tilde{T}_A : \tilde{A} \to X$. Therefore, $S = \tilde{T}_A \hat{\otimes} \iota_{\tilde{B}} \otimes \iota_A \hat{\otimes} T_B$ is a complete contraction from $\tilde{A} \hat{\otimes} \tilde{B} \hat{\otimes} A \hat{\otimes} B$ to $X \hat{\otimes} \tilde{B} \otimes A \hat{\otimes} Y$. Then, since $X$ is a cb-Banach $A$-bimodule, and $\tilde{Y}$ is a cb-Banach $\tilde{B}$-bimodule, we have a complete contraction $R : X \hat{\otimes} \tilde{B} \hat{\otimes} A \hat{\otimes} \tilde{Y} \to X \hat{\otimes} \tilde{Y}$, which satisfies $R(x \otimes b_1 \otimes a_2 \otimes y) = (x \cdot a_2) \otimes (b_1 \otimes y)$.

Since $\tilde{A} \hat{\otimes} \tilde{B} = (A \hat{\otimes} B)^\sim$, the composite map $RS$ defines a completely bounded bilinear map from $(A \hat{\otimes} B)^\sim \times (A \hat{\otimes} B)$ to $X \hat{\otimes} \tilde{Y}$, which agrees with $F_1$ on $(A \hat{\otimes} B) \times (A \hat{\otimes} B)$. In particular, $F_1$ extends to a bounded bilinear map (no longer completely bounded!) from $(A \hat{\otimes} B) \times (A \hat{\otimes} B)$ to $X \hat{\otimes} \tilde{Y}$. □

**Remark 4.3.** The proof of Proposition 4.2 might seem like overkill. Things would be much easier if $T_A \otimes \iota_B \hat{\otimes} \iota_A \hat{\otimes} T_B$ extended to a continuous linear map from $(A \hat{\otimes} B)^\sim \otimes_{\gamma} (A \hat{\otimes} B)$ to $(X \hat{\otimes} A) \hat{\otimes}_{\gamma} (B \hat{\otimes} Y)$; however, we see no reason why this should always hold.

Why does Proposition 4.2 not give us the version of Equation (3) that we need? Ultimately, we want to consider $F_1 - F_2$ as our candidate for a 2-cocycle on the Banach algebra $A \hat{\otimes} B$, and so we need it to take values in a Banach $(A \hat{\otimes} B)$-bimodule. But there seems to be no reason that the action $a \otimes b \otimes x \otimes y \mapsto ax \otimes by$ should extend to a linear map $(A \hat{\otimes} B) \hat{\otimes}_{\gamma} (X \hat{\otimes} Y) \to X \hat{\otimes} \tilde{Y}$; in general $X \hat{\otimes} \tilde{Y}$ and $X \hat{\otimes} \tilde{Y}$ need not have the same underlying Banach space! On the other hand, there is no problem with $X \otimes_{\min} Y$.

**Lemma 4.4.** Let $A$ and $B$ be cb-Banach algebras; let $X$ be a cb-Banach $A$-bimodule and $Y$ a cb-Banach $B$-bimodule. Then $X \otimes_{\min} Y$ is a cb-Banach $A \hat{\otimes} B$-bimodule.

**Proof.** Consider the map $(A \hat{\otimes} B) \hat{\otimes}_{\gamma} (X \otimes_{\min} Y) \cong_1 A \hat{\otimes}_\otimes (X \otimes_{\min} Y) \hat{\otimes}_{\gamma} B \to (A \hat{\otimes} X) \otimes_{\min}(Y \hat{\otimes} B)$; this is well-defined and completely contractive by repeated use of Lemma 2.4. Then the left and right actions of $A$ and $B$ on $X$ and $Y$ give two completely bounded bilinear maps $A \hat{\otimes} X \to X$ and $Y \hat{\otimes} B$; min-tensoring these and composing with the previous map, we obtain the desired cb-actions of $A \hat{\otimes} B$ on $X \otimes_{\min} Y$. □
4.3 The application to Fourier algebras

The next definition is purely for convenience of presentation.

**Definition 4.5**. $G \in S$ if there exists a non-zero bounded derivation $A(G) \to A(G)^*$ that is completely bounded as a map $A(G) \to (A(G)^*)^\sim$.

Note that by Herz’s restriction theorem, if $G$ contains a closed subgroup belonging to $S$, then $G \in S$.

We take a diversion to record some easy calculations, which would otherwise clutter up the proof of Theorem 4.8.

**Lemma 4.6.** Let $R$ be a commutative algebra. Consider the following subsets of $R$:

\[ X_{1,1} = \{ab: a, b \in R\} \quad , \quad X_2 = \{a^2: a \in R\} \]

\[ X_{1,1,1,1} = \{abcd: a, b, c, d \in R\} \quad , \quad X_{2,2} = \{a^2b^2: a, b \in R\} \quad , \quad X_4 = \{a^4: a \in R\}. \]

Then $\text{lin} X_{1,1} = \text{lin} X_2$ and $\text{lin} X_{1,1,1,1} = \text{lin} X_{2,2} = \text{lin} X_4$.

**Idea of the proof.** These follow from the polarization identities $ab = \frac{1}{4} [(a + b)^2 - (a - b)^2]$ and $x^2y^2 = \frac{1}{24}[(x + y)^4 + (x - y)^4 - (x + iy)^4 - (x - iy)^4]$. We leave it to the reader to fill in the details. \(\square\)

**Corollary 4.7.** Let $G$ be a locally compact group. Then $\{a^4: a \in A(G)\}$ and $\{b^2: b \in A(G)\}$ both have dense linear span in $A(G)$.

**Proof.** Let $A_0 = A(G) \cap C_c(G)$. Standard results on Fourier algebras tell us that $A_0$ is dense in $A(G)$; moreover, for each $f \in A_0$ there exists $g \in A_0$ such that $fg = f$. It then follows from Lemma 4.6 that $A_0 = \text{lin}\{a^4: a \in A_0\}$ and $A_0 = \text{lin}\{a^2: a \in A_0\}$. The rest is clear. \(\square\)

Putting everything together, we arrive at our main theorem concerning Fourier algebras.

**Theorem 4.8** (Constructing antisymmetric 2-cocycles on Fourier algebras). Let $H, L \in S$ with corresponding non-zero derivations $D_H: A(H) \to A(H)^*$ and $D_L: A(L) \to A(L)^*$ that are “completely bounded into the opposite o.s.s.” Then the bilinear map

\[ F_0: (A(H) \otimes A(L)) \times (A(H) \otimes A(L)) \to A(H)^* \otimes A(L)^*, \]

defined as in Equation 4.8, extends to a bounded bilinear map $F: A(H \times L) \times A(H \times L) \to A(H \times L)^\sim$. Moreover, $F$ is a non-zero, antisymmetric 2-cocycle.

**Proof.** For this proof, just to ease notation slightly, we denote the Fourier algebras of $H, L$ and $H \times L$ by $A_H, A_L$ and $A_{H \times L}$ respectively.

By Proposition 4.2 $F_0$ extends to a bounded bilinear map from $(A_H \otimes A_L) \times (A_H \otimes A_L)$ to $A_H^* \otimes (A_L^*)^\sim$. Applying Theorem 4.11 with $X = A_H^*$ and $Y = A_L^*$, we obtain a bounded bilinear map

\[ F: (A_H \otimes A_L) \times (A_H \otimes A_L) \to A_H^* \otimes_{\min} A_L^* \]
that extends $F_0$. Clearly $F$ is antisymmetric, by construction. As mentioned after Equation (3), one can check that $F$ satisfies the 2-cocycle identity on the dense subalgebra $A_H \otimes A_L$. A routine continuity argument shows that it satisfies the identity on all of $A_H \hat{\otimes} A_L$.

Recall that $A_H \hat{\otimes} A_L \cong_1 A_{H \times L}$ as algebras, while $A_H^* \otimes_{\min} A_L^* \cong_1 \text{VN}(H) \otimes_{\min} \text{VN}(L)$ embeds isometrically in $\text{VN}(H \times L) = (A_{H \times L})^*$ (and this is an embedding of $A_{H \times L}$ bimodules). So $F$ can be viewed as a bilinear map $A_{H \times L} \times A_{H \times L} \to (A_{H \times L})^*$.

Finally, we must show that $F$ is not identically zero. Since $F_0$ takes values in $\text{VN}(H) \otimes \text{VN}(L)$ and the natural map $\text{VN}(H) \otimes \text{VN}(L) \to \text{VN}(H \times L)$ is injective, it suffices to show that $F_0$ is not identically zero. Observe that if $a \in A_H$ and $b \in A_L$ we have

$$F_0(a^3 \otimes b, a \otimes b) = [D_H(a^3) \cdot a] \otimes [b \cdot D_L(b)] - [a^3 \cdot D_H(a)] \otimes [D_L(b) \cdot b]$$

$$= 2a^3 \cdot D_H(a) \otimes b \cdot D_L(b)$$

$$- \frac{1}{4} D_H(a^4) \otimes D_L(b^2).$$

By Corollary 4.7, elements of the form $a^4$ span a dense subspace of $A_H$, and elements of the form $b^2$ span a dense subspace of $A_L$. Therefore, since $D_H$ is continuous and non-zero, there exists $a \in A_H$ such that $D_H(a^4) \neq 0$; similarly, there exists $b \in A_L$ such that $D_L(b^2) \neq 0$. We conclude that $F_0(a^3 \otimes b, a \otimes b) \neq 0$, as required. 

**Remark 4.9.** The last part of the proof of Theorem 4.8 was somewhat fiddly because we wished to formulate the theorem in a very general setting. For many of the known examples of groups in $S$, one can find derivations where it is “obvious” that the resulting 2-cocycle $F_0$ cannot vanish identically on $(A(H) \otimes A(L)) \times (A(H) \otimes A(L))$.

It is tricky to give precise references for the fact that certain groups belong to the class $S$, since the extra “completely bounded into the opposite o.s.s.” property is merely implicit rather than explicit in the relevant papers [1, 2, 6, 8]. We shall merely indicate some places where specific cases can be analyzed: in all cases, the key point is that the check map on $A(G)$, defined by $\hat{f}(x) = f(x^{-1})$, defines a complete isometry from $A(G)$ onto $A(G)^\sim$.

**Example 4.10.** By examining the calculations in [1 §3] or [8 Proposition 2.4], it is quite easy to check that $\text{SO}(3, \mathbb{R})$ and $\text{SU}(2, \mathbb{C})$ belong to $S$. With a little more work, one sees that the same is true for the affine group of the real line: see Theorem 2.9 in [8]. Therefore, by the remark after Definition 4.5 and structure theory for Lie groups, it follows that every non-abelian connected semisimple Lie group belongs to $S$.

It seems plausible, given the results of [8], that every connected non-abelian Lie group belongs to $S$; we intend to investigate this in future work.

Finally: suppose $H \in S$ and $L \in S$, and let $G$ be any locally compact group that contains a copy of $H \times L$ as a closed subgroup. For instance: in view of Example 4.10 we could take $H = L = \text{SU}(2, \mathbb{C})$ and then $G = \text{SU}(n, \mathbb{C})$ for any $n \geq 4$. Then Herz’s restriction theorem provides a quotient homomorphism from $A(G)$ onto $A(H \times L)$, and so by routine arguments (see e.g. [2]) we can “lift” the non-trivial antisymmetric 2-cocycle on $A(H \times L)$ to get the same kind of 2-cocycle on $A(G)$. In the language of [7], $A(G)$ is not 2-dimensionally weakly amenable; these are the first confirmed examples of Fourier algebras with this property.
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