Poset Resolutions and Lattice-Linear Monomial Ideals

with an appendix by

Alexandre Tchernev

Timothy B.P. Clark

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Abstract

We introduce the class of lattice-linear monomial ideals and use the LCM-lattice to give an explicit construction for their minimal free resolution. The class of lattice-linear ideals includes (among others) the class of monomial ideals with linear free resolution and the class of Scarf monomial ideals. Our main tool is a new construction by Tchernev that produces from a map of posets \( \eta : P \rightarrow \mathbb{N}^n \) a sequence of multigraded modules and maps.

1 Introduction

Let \( R = \mathbb{k}[x_1, \ldots, x_n] \) be a polynomial ring where \( \mathbb{k} \) is a field, considered with its standard \( \mathbb{Z}^n \)-grading (multigrading) and let \( N \) be an ideal in \( R \) generated by monomials.

In [7], Gasharov, Peeva and Welker express the Betti numbers of \( R/N \) using the homology groups of of the LCM-lattice \( L_N \) of \( N \). They further show that the isomorphism class of \( L_N \) determines the structure of the minimal free resolution of \( R/N \). Motivated by these results, we introduce the class of \emph{lattice-linear} monomial ideals. A lattice-linear ideal has the mapping structure of its minimal free resolution encoded in the covering relations of
its LCM-lattice. In our main result, Theorem 5.3, we construct explicitly the minimal free resolution of any lattice-linear ideal from its LCM-lattice.

The class of lattice-linear ideals contains extensively studied subclasses, including the class of monomial ideals with a linear free resolution [8, 9, 10, 11, 12] and the class of Scarf ideals [1]. For each of these two subclasses, minimal free resolutions have been constructed using different techniques which are also distinct from the one described in this paper.

The key tool used to produce the lattice-linear resolutions is a new construction due to Tchernev which takes a partially ordered set (poset) $P$ as its input and produces a sequence of $k$-vector spaces and $k$-linear maps $C_\bullet(P)$. Two variations of this construction are discussed, each based upon calculating the homology of the poset $P$. Under sufficiently desirable conditions (for example when the atoms of $P$ form a crosscut), the two variations are canonically isomorphic. In general, the sequence $C_\bullet$ need not be a complex, nor need it be exact.

When there exists a map of posets $\eta : P \to \mathbb{N}^n$, we homogenize the sequence $C_\bullet(P)$ to produce a sequence of multigraded modules and multigraded morphisms $F(\eta)$ which "approximates" a free resolution of the monomial ideal whose generators have their degrees given by the images of the atoms. The poset map utilized in our main result to construct the minimal free resolution of a lattice-linear ideal is the degree map, $\text{deg} : L_N \to \mathbb{N}^n$, which sends a monomial to its multidegree.

2 Poset Combinatorics

Let $(P, \leq)$ be a finite poset with least element $0$. A totally ordered subset $\sigma \subseteq P$ which has the form $\alpha_0 < \cdots < \alpha_k$ is called a chain of length $k$, and for $\alpha \in P$, the rank of an element $\alpha$ is $\text{rk}(\alpha) = \sup \{l : \alpha_0 < \cdots < \alpha_l = \alpha\}$. A subset of $P$ comprised entirely of elements which are pairwise incomparable is called an anti-chain. An element $\beta \in P$ is covered by $\alpha$ (which we write $\beta \triangleleft \alpha$) when $\beta < \alpha$ and there exists no $\gamma \in P$ such that $\beta \triangleleft \gamma \triangleleft \alpha$. An open interval in $P$ is denoted $(\beta, \alpha) = \{\sigma : \beta \triangleleft \alpha_i \triangleleft \alpha \text{ for all } \alpha_i \in \sigma\}$, with closed and half-open intervals denoted similarly. Recall that the set of atoms of $P$ is

$$S = \{a \in P : 0 \triangleleft a\},$$

and setting $\text{rk}(0) = 0$, we have $\text{rk}(a) = 1$ for every $a \in S$. The poset $P$ is said to be ranked when $\text{rk}(\alpha) = \text{rk}(\beta) + 1$ for every $\beta \triangleleft \alpha \in P$. When
they exist, the meet (greatest lower bound) and join (least upper bound) of a subset \( A \subseteq P \) are denoted as \( \wedge A \) and \( \vee A \), respectively.

For the purpose of topological analysis of \( P \), recall that the order complex of a poset is the simplicial complex \( \Delta(P) \) whose \( k \)-dimensional faces are in one-to-one correspondence with the length \( k \) chains of \( P \). As is standard, whenever discussing topological properties of \( P \), we are implicitly referring to the topological properties of \( \Delta(P) \). Another topological object of interest is the the crosscut simplicial complex \( \Gamma(P,C) \) associated to \( P \). Recall that a set \( C \subset P \) is called a crosscut if it satisfies the following three properties:

1. \( C \) is an anti-chain,
2. For every finite chain \( \sigma \) in \( P \) there exists some element in \( C \) which is comparable to each element in \( \sigma \),
3. If \( A \subseteq C \) is bounded in \( P \), then either the join \( \vee A \) or the meet \( \wedge A \) exists in \( P \).

For a crosscut \( C \), the crosscut simplicial complex \( \Gamma(P,C) \) is defined as the collection of all subsets of \( C \) which are bounded in \( P \). We introduce a family of simplicial complexes indexed by the elements of \( P \) as follows.

**Definition 2.1.**

1. For \( \lambda \in P \), set \( G_\lambda \) to be the full simplex on the set
   \[ S_\lambda = \{ a \in S : a \leq \lambda \} \]
   and for \( \alpha \in P \) set
   \[ \Gamma_\alpha = \bigcup_{\lambda \prec \alpha} G_\lambda. \]

2. For each \( \lambda \neq \beta \prec \alpha \) we have \( G_\lambda \cap G_\beta \subset \Gamma_\lambda \) and decompose \( \Gamma_\alpha \) as
   \[ \Gamma_\alpha = G_\lambda \cup \left( \bigcup_{\lambda \neq \beta \prec \alpha} G_\beta \right). \]

For a fixed \( \lambda \prec \alpha \) set
   \[ \Gamma_{\alpha,\lambda} = G_\lambda \cap \left( \bigcup_{\lambda \neq \beta \prec \alpha} G_\beta \right) \subset \Gamma_\lambda. \]
The order complex of an open interval \( \Delta_\alpha := \Delta(\hat{0}, \alpha) \) may be realized in a similar way.

**Definition 2.2.**

1. For \( \alpha \in P \), set \( D_\alpha = \Delta(\hat{0}, \alpha) \], so that

\[
\Delta_\alpha = \bigcup_{\lambda \leq \alpha} D_\lambda.
\]

2. For each \( \lambda \neq \beta < \alpha \) we have \( D_\lambda \cap D_\beta \subset \Delta_\lambda \) and decompose \( \Delta_\alpha \) as

\[
\Delta_\alpha = D_\lambda \cup \left( \bigcup_{\beta < \alpha \atop \lambda \neq \beta} D_\beta \right),
\]

and set

\[
\Delta_{\alpha, \lambda} = D_\lambda \cap \left( \bigcup_{\beta < \alpha \atop \lambda \neq \beta} D_\beta \right) \subset \Delta_\lambda.
\]

It is advantageous to consider the families of simplicial complexes \( \{\Delta_\alpha : \alpha \in P\} \) and \( \{\Gamma_\alpha : \alpha \in P\} \) when analyzing the topology of \( P \), as one family may have structural advantages over the other.

**Remark 2.3.** If the set of atoms \( S \) forms a crosscut in \( P \), then the sets \( S_\lambda \) for \( \lambda \leq \alpha \) are the maximal subsets of \( S \) which are bounded in the open interval \((\hat{0}, \alpha)\) of \( P \). Each simplicial complex \( \Gamma_\alpha \) is therefore identical to the crosscut complex \( \Gamma((\hat{0}, \alpha), S_\alpha) \). Invoking the Crosscut Theorem ([2, Theorem 10.8]), then \( \Delta_\alpha \) and \( \Gamma_\alpha \) are homotopic for every \( \hat{0} \neq \alpha \in P \).

Turning our attention to the family of simplicial complexes

\[
\{\Delta_\alpha : \alpha \in P\}
\]

we describe a sequence of vector spaces and vector space maps

\[
C_\bullet(P) : \cdots \to C_i \xrightarrow{\varphi_i} C_{i-1} \to \cdots \to C_1 \xrightarrow{\varphi_1} C_0
\]

whose structure is determined by the simplicial complexes \( \Delta_\alpha \).
Definition 2.4.

1. Set \( C_0 = C_0(P) = \widetilde{H}_{-1}(\emptyset, k) \cong k \).

2. For \( i \geq 1 \), set \( C_{i,\alpha} = C_{i,\alpha}(P) = \widetilde{H}_{i-2}(\Delta_{\alpha}, k) \) and

\[
C_i = C_i(P) = \bigoplus_{\alpha \in P \setminus \{\hat{0}\}} C_{i,\alpha}.
\]

Remark 2.5. When \( i = 1 \) and \( \alpha \in S \), we have \( \Delta_{\alpha} = D_{\hat{0}} = \emptyset \) and thus, \( C_{1,\alpha} = \widetilde{H}_{-1}(\emptyset, k) \cong k \). If \( i = 1 \) and \( \alpha \notin S \) then \( \Delta_{\alpha} = \bigcup_{\lambda \leq \alpha} D_{\lambda} \neq \emptyset \) and hence \( C_{1,\alpha} = \widetilde{H}_{-1}(\Delta_{\alpha}, k) = 0 \). Therefore,

\[
C_1 = \bigoplus_{\alpha \in S} C_{1,\alpha} = \bigoplus_{\alpha \in S} \widetilde{H}_{-1}(\emptyset, k) \cong \bigoplus_{\alpha \in S} k.
\]

Next, given \( \lambda \leq \alpha \), we consider the Mayer-Vietoris sequence in reduced homology for the triple

\[
\left( D_{\lambda}, \bigcup_{\beta \leq \alpha \leq \lambda} D_{\beta}, \Delta_{\alpha} \right).
\]

We write \( \iota : \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, k) \to \widetilde{H}_{i-3}(\Delta_{\lambda}, k) \) for the map induced in homology by the inclusion map and

\[
\delta_{i-2}^{\alpha,\lambda} : \widetilde{H}_{i-2}(\Delta_{\alpha}, k) \to \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, k)
\]

for the connecting homomorphism from the Mayer-Vietoris sequence. Recall this homomorphism takes the class \([c] \in \widetilde{H}_{i-2}(\Delta_{\alpha}, k)\) to the class \([d_{i-2}(c')] \in \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, k)\) where \( c' + c'' = c \in \widetilde{C}_{i-2}(\Delta_{\alpha}, k) \), and \( c' \) and \( c'' \) are any components of \( c \) that are supported by \( D_{\lambda} \) and \( \bigcup_{\lambda \neq \mu < \alpha} D_{\mu} \) respectively. Here, \( d \) is the usual boundary map.

We can now proceed with the definition of the maps \( \varphi_i : C_i \to C_{i-1} \) for the sequence \( C_\bullet(P) \).

Definition 2.6.
1. Define \( \varphi_1 : C_1 \rightarrow C_0 \) componentwise by \( \varphi_1|_{C_1,\alpha} = \text{id}_{\tilde{H}_{-1}(\emptyset, \k)} \).

2. For \( i \geq 2 \) define \( \varphi_i : C_i \rightarrow C_{i-1} \) componentwise by

\[
\varphi_i|_{C_i,\alpha} = \sum_{\lambda \preceq \alpha} \varphi_i^{\alpha,\lambda}
\]

where

\[
\varphi_i^{\alpha,\lambda} : C_i,\alpha \rightarrow C_{i-1,\lambda}
\]

is the composition \( \varphi_i^{\alpha,\lambda} = \iota \circ \delta_i^{\alpha,\lambda} \).

The construction above may be performed using the family of subcomplexes \( \{ \Gamma_\alpha : \alpha \in \mathcal{P} \} \).

A priori, the sequence \( C_\bullet(P, \Gamma) \) obtained in this way is different from the sequence \( C_\bullet(P) \) based on the family of subcomplexes \( \{ \Delta_\alpha : \alpha \in \mathcal{P} \} \).

However, as the meaning will always be clear from the context, we will use the same notation for the components and the same notation for the maps in both sequences. To be specific, we describe the sequence of vector spaces and vector space maps

\[
C_\bullet(P, \Gamma) : \cdots \rightarrow C_i \xrightarrow{\varphi_i} C_{i-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{\varphi_1} C_0
\]

as follows.

**Definition 2.7.**

1. Set \( C_0 = C_0(P, \Gamma) = \tilde{H}_{-1}(\emptyset, \k) \cong \k \).

2. For \( i \geq 1 \), set \( C_{i,\alpha} = C_{i,\alpha}(P, \Gamma) = \tilde{H}_{i-2}(\Gamma_\alpha, \k) \) and

\[
C_i = C_i(P, \Gamma) = \bigoplus_{\alpha \in \mathcal{P} \setminus \{ \emptyset \}} C_{i,\alpha}.
\]

**Remark 2.8.** Again, when \( i = 1 \) and \( \alpha \in S \), we have \( \Gamma_\alpha = G_\emptyset = \emptyset \) and thus, \( C_{1,\alpha} = \tilde{H}_{-1}(\emptyset, \k) \cong \k \). If \( i = 1 \) and \( \alpha \notin S \) then \( \Gamma_\alpha = \bigcup_{\lambda \preceq \alpha} G_\lambda \neq \emptyset \) and hence \( C_{1,\alpha} = \tilde{H}_{-1}(\Gamma_\alpha, \k) = 0 \). Therefore,

\[
C_1 = \bigoplus_{\alpha \in S} C_{1,\alpha} = \bigoplus_{\alpha \in S} \tilde{H}_{-1}(\emptyset, \k) \cong \bigoplus_{\alpha \in S} \k.
\]
As before, given $\lambda \prec \alpha$, we consider the Mayer-Vietoris sequence in reduced homology for the triple

$$\left( G_{\lambda}, \bigcup_{\lambda \neq \beta < \alpha} G_{\beta}, \Gamma_{\alpha} \right).$$

We set $i : \tilde{H}_{i-3}(\Gamma_{\alpha,\lambda}, k) \to \tilde{H}_{i-3}(\Gamma_{\lambda}, k)$ to be the map induced in homology by the inclusion map and

$$\delta_{i-2}^{\alpha,\lambda} : \tilde{H}_{i-2}(\Gamma_{\alpha}, k) \to \tilde{H}_{i-3}(\Gamma_{\alpha,\lambda}, k)$$

to be the connecting homomorphism from the Mayer-Vietoris sequence. This homomorphism takes the class $[c] \in \tilde{H}_{i-2}(\Gamma_{\alpha}, k)$ to the class $[d_{i-2}(c')] \in \tilde{H}_{i-3}(\Gamma_{\alpha,\lambda}, k)$ where $c' + c'' = c \in C_{i-2}(\Gamma_{\alpha}, k)$, and $c'$ and $c''$ are components of $c$ that are supported by $G_{\lambda}$ and by $\bigcup_{\lambda \neq \beta < \alpha} G_{\mu}$ respectively. Again, $d$ is the usual boundary map.

**Definition 2.9.**

1. Define $\varphi_1 : C_1 \to C_0$ componentwise by $\varphi_1|_{C_1,\alpha} = \text{id}_{\tilde{H}_{-1}(\emptyset, k)}$.

2. For $i \geq 2$ define $\varphi_i : C_i \to C_{i-1}$ componentwise by

$$\varphi_i|_{C_i,\alpha} = \sum_{\lambda < \alpha} \varphi_i^{\alpha,\lambda}$$

where

$$\varphi_i^{\alpha,\lambda} : C_i,\alpha \to C_{i-1,\lambda}$$

is the composition $\varphi_i^{\alpha,\lambda} = i \circ \delta_{i-2}^{\alpha,\lambda}$.

### 3 Properties of $C_*(P)$ and $C_*(P, \Gamma)$

The sequences $C_*(P)$ and $C_*(P, \Gamma)$ are not necessarily complexes of vector spaces, and even if one is a complex, it need not be exact. While necessary and sufficient conditions for the construction to produce an exact complex are not known, if the poset is ranked, $C_*(P)$ is a complex of vector spaces (Proposition 7.1 in the appendix). Depending on the structure of each of the collections, it may be more advantageous to use one instead of the other. When the poset has sufficiently good structure, we prove
Proposition 3.1. Suppose that the set of atoms $S$ forms a crosscut in $P$, and let $f : P \to \Gamma(P, S)$ be the homotopy equivalence given in [2]. Then for every $\beta \preceq \alpha \in P$ and every $i \geq 1$ the map $f$ induces canonically a commutative diagram

$$
\begin{array}{ccc}
\tilde{H}_i(\Delta_\alpha, k) & \xrightarrow{f_*} & \tilde{H}_i(\Gamma_\alpha, k) \\
\varphi_i & & \varphi_i \\
\tilde{H}_{i-1}(\Delta_\beta, k) & \xrightarrow{f_*^{-1}} & \tilde{H}_{i-1}(\Gamma_\beta, k)
\end{array}
$$

where $f_*$ is an isomorphism in homology.

Corollary 3.2. If $P$ is a lattice or a geometric semilattice, then $C_\bullet(P)$ and $C_\bullet(P, \Gamma)$ are canonically isomorphic.

Proof. The set of atoms of $P$ forms a crosscut. □

Proof of Proposition 3.1. Recall that the barycentric subdivision of a simplicial complex $\Omega$ may be realized as the simplicial complex $\text{sd}(\Omega) = \Delta(P(\Omega))$. Write $P_{\geq x} = \{ y \in P : y \geq x \}$ and suppose that $\sigma$ is a chain of $P$. An order reversing map of posets $g : P(\Delta(P)) \to P(\Gamma(P, S))$ is defined in [2] as

$$
\sigma \mapsto \{ x \in S : \sigma \in \Delta(P_{\geq x}) \}.
$$

This map induces in the usual way a simplicial map on the corresponding order complexes $h : \text{sd}(\Delta(P)) \to \text{sd}(\Gamma(P, S))$ which gives rise to a chain map $h_* : C_\bullet(\text{sd}(\Delta(P))) \to C_\bullet(\text{sd}(\Gamma(P, S)))$.

The homotopy equivalence $f$ in [2] can be defined on the level of chains as $f_* = \text{unsd}_* \circ h_* \circ \text{sd}_*$ where

$$
\begin{array}{c}
\tilde{C}_\bullet(\Delta(P)) \xrightarrow{\text{sd}_*} \tilde{C}_\bullet(\text{sd}(\Delta(P))) \xrightarrow{h_*} \tilde{C}_\bullet(\text{sd}(\Gamma(P, S))) \xrightarrow{\text{unsd}_*} \tilde{C}_\bullet(\Gamma(P, S))
\end{array}
$$

the simplicial map $\text{unsd}$ is defined by fixing a total ordering on the set of atoms $S$ and sending $S_a = \{ x \in S : x \leq a \}$ to its minimum element $s_a = \min S_a$ under this ordering; and $\text{unsd}_*$ is the induced chain map. Recall that the star of a vertex $v$ in a simplicial complex $K$, denoted $\text{St}(v, K)$, is
the union of the interiors of the simplices in \( K \) that have \( v \) as a vertex. Given any vertex \( S_a \) of \( \text{sd}(\Gamma(P,S)) \) then the vertex \( s_a \) of \( \Gamma(P,S) \) satisfies

\[
\text{St}(S_a, \text{sd}(\Gamma(P,S))) \subset \text{St}(s_a, \Gamma(P,S)).
\]

Invoking [13, Lemma 15.1], it follows that \( \text{unsd} \) is a simplicial approximation to the identity, and therefore via the algebraic subdivision theorem [13, Theorem 17.2] \( \text{unsd}_\sharp \) is a chain map that is a homotopy inverse to the subdivision map \( \Gamma(P,S) \to \text{sd}(\Gamma(P,S)) \).

Next, recall that under barycentric subdivision a face \( \sigma = \{a_0, \ldots, a_k\} \in \Delta(P) \) with \( a_0 < \cdots < a_k \in P \) has image

\[
\text{sd}_\sharp(\sigma) = \sum_{\rho \in \Sigma_{k+1}} \varepsilon_\rho \{\{a_{\rho(k)}\}, \{a_{\rho(k-1)}, a_{\rho(k)}\}, \ldots, \{a_{\rho(0)}, \ldots, a_{\rho(k)}\}\}
\]  

where \( \Sigma_{k+1} \) is the group of permutations on the set \( \{0, 1, \ldots, k\} \) and \( \varepsilon_\rho \) denotes the sign of the permutation \( \rho \). Applying \( h \) to the chain

\[
\{\{a_{\rho(k)}\}, \{a_{\rho(k-1)}, a_{\rho(k)}\}, \ldots, \{a_{\rho(0)}, \ldots, a_{\rho(k)}\}\}
\]

yields the face

\[
\{S_{a_{\rho(k)}}, S_{a_{\rho(k-1)}}, \ldots, S_{a_{\rho(0)}}\}.
\]

Unless \( \rho \) is the identity of \( \Sigma_{k+1} \), this face has dimension less than or equal to \( k - 1 \). It follows that under the chain map \( h_\sharp \), the sum in equation (3.3) has image

\[
h_\sharp(\text{sd}_\sharp(\sigma)) = \{S_{a_k}, S_{a_{k-1}}, \ldots, S_{a_0}\}.
\]  

(3.4)

Considering the simplicial map \( \text{unsd} : \text{sd}(\Gamma(P,C)) \to \Gamma(P,C) \), if \( s_{at} = s_{at-1} \) for some \( 1 \leq t \leq k \), then the face

\[
\{s_{a_k}, \ldots, s_{a_0}\} = \text{unsd}(\{S_{a_k}, S_{a_{k-1}}, \ldots, S_{a_0}\})
\]

has dimension less than or equal to \( k \). Thus,

\[
\text{unsd}_\sharp(\{S_{a_k}, S_{a_{k-1}}, \ldots, S_{a_0}\}) = 0
\]

except when \( s_{a_k} < s_{a_{k-1}} < \cdots < s_{a_0} \). Therefore,

\[
f_\sharp(\sigma) = \text{unsd}_\sharp(h_\sharp(\text{sd}_\sharp(\sigma))) = \begin{cases} 
\{s_{a_k}, \ldots, s_{a_0}\} & \text{when } s_{a_k} < \cdots < s_{a_0} \\
0 & \text{otherwise}
\end{cases}
\]

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It is clear from this description that for each $\alpha$ and each $\sigma \in \Delta(\tilde{0}, \alpha]$ we have $f_\sharp(\sigma) \in \tilde{C}_\bullet(G_\alpha)$ and therefore the horizontal maps in our diagram are well defined and are isomorphisms by Remark 2.3.

We now turn to proving the commutativity of the diagram. Fix $\beta \ll \alpha$, suppose that $[\pi]$ is a homology class in $\tilde{H}_i(\Delta_\alpha)$ and write

$$
\pi = \sum_{\dim(\sigma) = i} c_\sigma \cdot \sigma \quad (3.5)
$$

for a representative of this class where $\sigma = \{a_0^\sigma, \ldots, a_i^\sigma\}$ is oriented by $a_0^\sigma < \cdots < a_i^\sigma$ and $c_\sigma \in k$ is a scalar. Applying the isomorphism $f$, we have

$$
f([\pi]) = [f_\sharp(\pi)]
$$

$$
= \left[ \sum_{\dim(\sigma) = i} c_\sigma \cdot f_\sharp(\sigma) \right]
$$

$$
= \left[ \sum_{a_i^\sigma \leq \beta} c_\sigma \cdot f_\sharp(\sigma) + \sum_{a_i^\sigma \not\leq \beta} c_\sigma \cdot f_\sharp(\sigma) \right].
$$

Since the terms appearing in the first summand are faces in $G_\beta$ and the terms appearing in the second summand are faces in

$$
\bigcup_{\beta \neq \gamma \ll \alpha} G_\gamma,
$$

then applying the map $\varphi_i$ to $[f_\sharp(\pi)]$ yields

$$
\varphi_i([\pi]) = \left[ \sum_{a_i^\sigma \leq \beta} c_\sigma \cdot d_i \circ f_\sharp(\sigma) \right].
$$

Again taking $[\pi] \in \tilde{H}_i(\Delta_\alpha)$, we apply $\varphi_i$ and achieve the homology class

$$
\varphi_i([\pi]) = \left[ \sum_{a_i^\sigma \leq \beta} c_\sigma \cdot d_i(\sigma) \right].
$$
Lastly, we apply the isomorphism $f$ to achieve

$$f(\varphi_i([\pi])) = \left[ \sum_{a^i_1 \leq \beta} c_{\sigma} \cdot f_{z} \circ d_i(\sigma) \right]$$

$$= \left[ \sum_{a^i_1 \leq \beta} c_{\sigma} \cdot d_i \circ f_{z}(\sigma) \right]$$

$$= \varphi_i([f_{z}(\pi)])$$

so that the proposition is proven. \qed

4 Poset Resolutions

Let $R = \mathbb{k}[x_1, \ldots, x_n]$, let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the unique graded maximal ideal of $R$, and $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$. We appeal to the standard $\mathbb{Z}^n$-grading (mutigrading) of $R$ and use the notation of Section 2 for ordering in the partially ordered set $\mathbb{Z}^n$. We use the degree map $x^\alpha \mapsto \alpha$ and identify the monomials in $R$ with the elements of $\mathbb{N}^n \subset \mathbb{Z}^n$.

Suppose that $\eta : P \to \mathbb{N}^n$ is a map of partially ordered sets, and $S$ is the set of atoms of $P$. Let $N$ be the ideal in $R$ generated by the monomials $G(N) = \{ x^{\eta(a)} : a \in S \}$.

The complex of vector spaces $C_\bullet(P)$ constructed in Section 2 and associated to $P$ is homogenized using the map $\eta$ to produce

$$\mathbb{F} = \mathbb{F}(\eta) : \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0,$$

a sequence of free multigraded $R$-modules and multigraded $R$-module homomorphisms which approximates a free resolution of the multigraded module $R/N$. This homogenization is carried out by constructing $F_0 = R \otimes_{\mathbb{k}} C_0$ and grading the result with $\deg(x^\alpha \otimes v) = \alpha$ for each $v \in C_0$. Similarly, for $i \geq 1$, we set

$$F_i = \bigoplus_{\delta \neq \lambda \in P} F_{i,\lambda} = \bigoplus_{\delta \neq \lambda \in P} R \otimes_{\mathbb{k}} C_{i,\lambda}$$

where the grading is defined as $\deg(x^\alpha \otimes v) = \alpha + \eta(\lambda)$ for each $v \in C_{i,\lambda}$. The differential in this sequence of multigraded modules is defined componentwise.
in homological degree 1 as
\[ \partial_1|_{F_1,\lambda} = x^{\eta(\lambda)} \otimes \varphi_1|_{c_1,\lambda} \]
and for \( i \geq 1 \), the map \( \partial_i : F_i \rightarrow F_{i-1} \) is defined as
\[ \partial_i|_{F_i,\alpha} = \sum_{\lambda \preceq \alpha} \partial_i^{\alpha,\lambda} \]
where \( \partial_i^{\alpha,\lambda} : F_{i,\alpha} \rightarrow F_{i-1,\lambda} \) takes the form \( \partial_i^{\alpha,\lambda} = x^{\alpha-\eta(\lambda)} \otimes \varphi_i^{\alpha,\lambda} \) for \( \lambda \prec \alpha \).

We are now in a position to make

**Definition 4.1.** If \( \mathbb{F}(\eta) \) is an acyclic complex of multigraded modules, then we say that it is a poset resolution of the ideal \( N \).

**Example 4.2.**

1. The Taylor resolution \([14]\) can be realized as a poset resolution where \( P = B_r \), the Boolean lattice. The map \( \eta : B_r \rightarrow \mathbb{N}^n \)

is defined on a lattice element \( I \subseteq \{1, \ldots, r\} \) via \( I \mapsto \deg(m_I) \), where \( m_I = \text{lcm}(m_i \in G(N) : i \in I) \).

2. An arbitrary stable monomial ideal is shown to admit a minimal poset resolution in \([4]\) using a poset of Eliahou-Kervaire admissible symbols.

We conclude this section with the following general remark on notation. Any sequence \( P \) of morphisms of free multigraded modules can be decomposed as
\[ P = \bigoplus_{\alpha \in \mathbb{Z}^n} P_\alpha \]
where each \( P_\alpha \) is a sequence of maps of vector spaces, and is called the (multigraded) strand of \( P \) in degree \( \alpha \). We denote by \( (P_\alpha)_i \), the \( i^{th} \) component of the sequence \( P_\alpha \). Using the multigrading, it is clear that
\[ (mP)_\alpha = \sum_{\beta \prec \alpha} x^{\alpha-\beta} P_\beta \subset P_\alpha. \]

Further, we will identify \( x^{\alpha-\beta} P_\beta \) with \( P_\beta \) so that we may write \( P_\beta \subset P_\alpha \) for \( \beta \prec \alpha \). This allows us to consider \( P_\gamma \subset P_\alpha \) for all \( \gamma < \alpha \). In addition, we may now write
\[ (mP)_\alpha = \sum_{\beta \preceq \alpha} P_\beta = \sum_{\gamma \preceq \alpha} P_\gamma \subset P_\alpha. \]
5 Lattice-Linear Monomial Ideals

We now turn to the class of ideals that are the focus of this paper. Recall that the LCM-lattice associated to a monomial ideal $N$ is the set $L_N$ of least common multiples of the subsets of the set of minimal generators of $N$ (where by convention, 1 is considered to be the least common multiple of the empty set) and ordering in $L_N$ is given by divisibility. Recall that we identify monomials with their degree in $\mathbb{N}^n$. In particular, we consider $L_N$ as a sublattice of $\mathbb{N}^n$. As an immediate consequence of [3, Theorem 3.1a], if the $i^{th}$ multigraded Betti number $\beta_{i,\alpha}(R/N) \neq 0$, then $\alpha \in L_N$. In particular, this means that if $B_i$ is any multihomogeneous basis of the free module $F_i$ in the minimal free resolution $F$ of $R/N$ then $\deg(v) \in L_N$ for each $v \in B_i$.

Definition 5.1. Let $F$ be a minimal multigraded free resolution of $R/N$. We say that $N$ is lattice-linear if multigraded bases $B_k$ of $F_k$ can be fixed for all $k$ so that for any $i \geq 1$ and any $v \in B_i$ the differential

$$\partial^F(v) = \sum_{v' \in B_{i-1}} m_{v,v'} \cdot v'$$

has the property that if the coefficient $m_{v,v'}$ is nonzero then $\deg(v') \leq \deg(v) \in L_N$.

Remark 5.2. The notion of lattice-linearity is dependent upon the characteristic of the ground field $\mathbb{k}$. For example, the ideal

$$N = \langle x_1x_2x_3, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5, x_1x_2x_6, x_1x_4x_6, x_2x_5x_6, x_3x_4x_6, x_3x_5x_6 \rangle$$

in $R = \mathbb{k}[x_1, x_2, x_3, x_4, x_5, x_6]$ is lattice-linear if and only if $\text{char}(\mathbb{k}) \neq 2$.

We now state our main result.

Theorem 5.3. Define the map $\deg : L_N \to \mathbb{N}^n$ by sending a monomial $m \in L_N$ to its degree $\deg(m) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Then the monomial ideal $N$ is lattice-linear if and only if $F(\deg)$ is its minimal free resolution.

We postpone the proof of Theorem 5.3 in favor of two examples of lattice-linear ideals.

First, the class of lattice-linear ideals clearly contains those ideals with a linear free resolution, whose minimal free resolutions have been constructed in
using tools from Discrete Morse Theory. Our methods allow us to provide a considerably simpler and more transparent approach to constructing these minimal free resolutions.

For our second example we recall from [1] the Scarf simplicial complex

$$\Delta_N = \{I \subseteq \{1, \ldots, r\} \mid m_I \neq m_J \text{ for all } J \subseteq \{1, \ldots, r\} \text{ other than } I\}.$$ 

When $I \in \Delta_N$, then $I$ is uniquely determined from $m_I$ as the set $I = \{i \mid m_i < m_I\}$. The ideal $N$ is called a Scarf ideal if its minimal free resolution is supported on $\Delta_N$. For instance, the so-called generic [1, 12] ideals are Scarf.

Note that when $N$ is Scarf, the differential in its minimal free resolution takes the unique basis element $e_I$ labeled by the monomial $m_I$ to

$$\sum_{j=1}^{|I|} (-1)^{j+1} \frac{m_{I \setminus \{i_j\}}}{m_I} \cdot e_{I \setminus \{i_j\}}$$

where $I = \{i_1, \ldots, i_{|I|}\}$.

**Proposition 5.4.** Every Scarf ideal is a lattice-linear monomial ideal. In particular, for $\eta : P(\Delta_N) \rightarrow \mathbb{N}^n$ where $I \mapsto \deg(m_I)$, the complex $\mathbb{F}(\eta)$ is the minimal free resolution of $R/N$.

**Proof.** Suppose that $N$ is a Scarf ideal, set $L_N$ as the LCM-lattice of $N$ and let $F$ denote the minimal free resolution of $R/N$ with differential $\partial F$. Fix a homological degree $p > 0$ and an $I \subseteq \{1, \ldots, r\}$. For every $J \subset I$, the monomial $m_J < m_I$ in $L_N$. Supposing that $N$ is not lattice-linear, there exists $J = \{a_1, \ldots, \hat{a}_j, \ldots, a_p\}$ so that the coefficient of $e_J$ in the expansion of $\partial F(e_I)$ is nonzero, and yet $m_J$ is not covered by $m_I$ in the lattice $L_N$. Thus, there exists $m \in L_N$ so that $m_J < m < m_I$. Since $N$ is Scarf, $I$ and $J$ are uniquely determined from $m_I$ and $m_J$. By definition $m = \text{lcm}(m_{a_1}, \ldots, m_{a_t})$ for some $\{a_1, \ldots, a_t\}$, and it follows that $J \subset \{a_1, \ldots, a_t\} \subseteq I = J \cup \{a_j\}$, forcing $\{a_1, \ldots, a_t\} = I$. Therefore, $m = m_I$, and $m_J \preceq m_I$. Hence $N$ is lattice-linear.

**6 Proof of Theorem 5.3**

**Proof.** It is clear that if $\mathbb{F}(\eta)$ is the minimal free resolution then $N$ is lattice-linear. It remains to show that lattice linearity implies that $\mathbb{F}(\eta)$ is a resolution of $R/N$. We remark that since $L_N$ is a lattice, its set of atoms $G(N)$
forms a crosscut, so that Corollary 3.2 implies \( C_\bullet(L_N, \Gamma) = C_\bullet(L_N) \). We will use \( C_\bullet(L_N, \Gamma) \) for our computation of \( \mathbb{F}(\eta) \).

Suppose that \( N \) is a lattice-linear monomial ideal with minimal free resolution \( F \) and let \( B_\eta \) be a basis for \( F \) as in Definition 5.1. With this choice of basis, let \( F_{i,\alpha} \) be the free submodule of \( F_i \) spanned by the set

\[
B_{i,\alpha} = \{ v \in B_i : \deg(v) = \alpha \}.
\]

Hence,

\[
F_i = \bigoplus_{\alpha \in L_N} F_{i,\alpha},
\]

and in particular

\[
(F_\alpha)_i = \bigoplus_{\beta \leq \alpha} V_{i,\beta}
\]

where

\[
V_{i,\beta} = \mathbb{k}\langle v : v \in B_{i,\beta} \rangle
\]

and \( x^{\alpha-\beta}V_{i,\beta} \) is identified with \( V_{i,\beta} \).

Making use of \( T \), the Taylor resolution of \( R/N \), consider the exact sequence

\[
0 \longrightarrow \sum_{\beta \in \alpha} T_\beta \longrightarrow T_\alpha \longrightarrow \frac{T_\alpha}{\sum_{\beta \in \alpha} T_\beta} \longrightarrow 0. \tag{6.1}
\]

The exactness of the Taylor resolution implies that \( T_\alpha \) is an exact complex of vector spaces for \( \hat{0} \neq \alpha \in L_N \). Indeed, \( T_\alpha \) is acyclic with \( H_0(T_\alpha) \cong (R/N)_\alpha \) and \( (R/N)_\alpha = 0 \) for \( x^\alpha \in N \).

Passing from (6.1) to the long exact sequence in homology, the connecting homomorphism yields an isomorphism

\[
H_i \left( \frac{T_\alpha}{\sum_{\beta \in \alpha} T_\beta} \right) \xrightarrow{\mu_1} H_{i-1} \left( \sum_{\beta \in \alpha} T_\beta \right),
\]

which takes the class

\[
[\bar{v}] \in H_i \left( \frac{T_\alpha}{\sum_{\beta \in \alpha} T_\beta} \right)
\]

to the class

\[
[\partial^T(v)] \in H_{i-1} \left( \sum_{\beta \in \alpha} T_\beta \right)
\]

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whenever \( \bar{v} \) is a cycle in \( T_\alpha / \sum_{\beta \leq \alpha} T_\beta \) represented by an element \( v \in T_\alpha \).

Since \( F \) is the minimal free resolution of \( R/N \), we make the identifications

\[
H_i \left( \frac{F_\alpha}{(mF)_\alpha} \right) = \left( \frac{F_\alpha}{\sum F_\beta} \right)_i \\
= \left( \frac{F_\alpha}{\sum B_\beta} \right)_i \\
= \left( \frac{(F_\alpha)_i}{\sum (F_\beta)_i} \right) \\
= \bigoplus_{\beta \leq \alpha} V_{i,\beta} / \bigoplus_{\beta < \alpha} V_{i,\beta} \\
= V_{i,\alpha}.
\]

Fixing an embedding of \( F \) as a direct summand of \( T \), we have \( T = F \bigoplus G \) for some split exact complex of multigraded free modules \( G \), and in particular, \( T_\alpha = F_\alpha \bigoplus G_\alpha \) for every \( \alpha \). Since the induced map of complexes

\[
\frac{F_\alpha}{\sum_{\beta \leq \alpha} F_\beta} \longrightarrow \frac{T_\alpha}{\sum_{\beta \leq \alpha} T_\beta}
\]

is split inclusion and is an isomorphism in homology, we consider \( V_{i,\alpha} \) as a subspace of \( Z_i \left( \frac{T_\alpha}{\sum_{\beta \leq \alpha} T_\beta} \right) \) and obtain the canonical identification

\[
Z_i \left( \frac{T_\alpha}{\sum_{\beta \leq \alpha} T_\beta} \right) = V_{i,\alpha} \bigoplus B_i \left( \frac{T_\alpha}{\sum_{\beta \leq \alpha} T_\beta} \right).
\]

Recalling the definitions of \( \Gamma_\alpha \) and \( \Gamma_{\alpha,\gamma} \) we see that

\[
\sum_{\beta \leq \alpha} T_\beta = \tilde{C}_\bullet(\Gamma_\alpha, k)
\]

and

\[
\sum_{\beta \leq \alpha \atop \gamma \neq \beta} T_{\gamma \wedge \beta} = \tilde{C}_\bullet(\Gamma_{\alpha,\gamma}, k).
\]
Using these identifications, we have the following diagram for each $\gamma < \alpha$ and each $i \geq 2$:

\[
\begin{array}{cccccc}
V_{i,\alpha} & \xrightarrow{\partial^F} & \left(\sum_{\beta < \alpha} F_\beta\right)_{i-1} & \xrightarrow{\text{proj}_\gamma} & V_{i-1,\gamma} \\
\downarrow\text{incl} & & & \downarrow\text{incl} & \\
Z_i \left(\mathbb{T}_\alpha / \sum_{\beta < \alpha} \mathbb{T}_\beta\right) & \xrightarrow{\text{proj}} & H_i \left(\mathbb{T}_\alpha / \sum_{\beta < \alpha} \mathbb{T}_\beta\right) & \xrightarrow{\mu_i \cong} & Z_{i-1} \left(\mathbb{T}_\gamma / \sum_{\nu < \gamma} \mathbb{T}_\nu\right) & \xrightarrow{\text{proj}} & H_{i-1} \left(\mathbb{T}_\gamma / \sum_{\nu < \gamma} \mathbb{T}_\nu\right) \\
\downarrow\mu_i \cong & & & & \downarrow\mu_{i-1} & & \\
H_{i-1} \left(\sum_{\beta < \alpha} \mathbb{T}_\beta\right) & \xrightarrow{\delta_{i-1}^{\alpha,\gamma}} & H_{i-2} \left(\sum_{\beta < \alpha, \gamma \neq \beta} \mathbb{T}_{\gamma,\beta}\right) & \xrightarrow{\lambda} & H_{i-2} \left(\sum_{\nu < \gamma} \mathbb{T}_\nu\right) & & \\
\downarrow & & & & \downarrow & & \\
\tilde{H}_{i-2}(\Gamma_\alpha, k) & \xrightarrow{\delta_{i-1}^{\alpha,\gamma}} & \tilde{H}_{i-3}(\Gamma_{\alpha,\gamma}, k) & \xrightarrow{\lambda} & \tilde{H}_{i-3}(\Gamma_\gamma, k) \\
\end{array}
\] (6.2)

We claim this diagram is commutative.

Let $v \in V_{i,\alpha}$ so that by the assumption of lattice-linearity,

\[
\partial^F(v) = \sum_{\beta < \alpha} v_\beta
\]

where each $v_\beta \in V_{i-1,\beta}$. Canonically, $\text{incl}(v) = v$. Under projection, the cycle $v$ is sent to its corresponding class in homology, $[v]$. As mentioned above, the connecting map $\mu_i$ is an isomorphism, and

\[
\mu_i([v]) = [\partial^T(v)] = [\partial^F(v)] = \left[\sum_{\beta < \alpha} v_\beta\right].
\]
Applying $\delta_{i-1}^{\alpha,\gamma}$, which is the connecting Mayer-Vietoris map,

$$\delta_{i-1}^{\alpha,\gamma} \left( \sum_{\beta \leq \alpha} v_{\beta} \right) = [\partial^F(v_{\gamma})].$$

Lastly, $\iota$ is the homological inclusion map and thus, $\iota([\partial^F(v_{\gamma})]) = [\partial^F(v_{\gamma})]$.

Again taking $v \in V_{i,\alpha}$, we appeal to the differential of $F$, and obtain

$$(\partial^F \circ \iota)(v) = \partial^F(v) = \sum_{\beta \leq \alpha} v_{\beta}.$$ 

Projecting onto $V_{i-1,\gamma}$, we have

$$\text{proj}_\gamma \left( \sum_{\beta \leq \alpha} v_{\beta} \right) = v_{\gamma}.$$ 

The inclusion map now gives $\text{incl}(v_{\gamma}) = v_{\gamma}$, and passing this cycle to homology yields $[v_{\gamma}]$. Through the isomorphism, $\mu_{i-1}([v_{\gamma}]) = [\partial^F(v_{\gamma})]$, which completes the proof of the commutativity of the diagram.

We complete the proof of the Theorem by establishing the connection between lattice linearity and the poset construction.

For each $i \geq 0$ define the isomorphism of free $R$-modules $\psi_i : F_i \longrightarrow F(\eta)_i$ on a basis element $v \in V_{i,\alpha} \subset B_i$ by applying the left column of (6.2), thus

$$\psi_i(v) = \otimes [\mu_i \circ \text{proj} \circ \text{incl}(v)] = 1 \otimes [\partial^F(v)] \in R \otimes C_{i,\alpha} \subset F(\eta)_i.$$ 

By the commutativity of (6.2), we have a commutative diagram,

$$
\begin{array}{ccc}
(F)_i & \xrightarrow{\partial^F} & (F)_{i-1} \\
\psi_i \downarrow & & \downarrow \psi_{i-1} \\
(F(\eta))_i & \xrightarrow{d_i} & (F(\eta))_{i-1}
\end{array}
$$

for every $i \geq 2$. Furthermore, (6.3) commutes trivially for $i = 1$. It follows that the sequences $F$ and $F(\eta)$ are isomorphic, hence $F(\eta)$ is the minimal free resolution of $R/N$.

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7 Appendix: Properties of $C_\bullet(P)$, by Alexandre Tchernev

ALEXANDRE TCHERNEV, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY AT ALBANY SUNY, ALBANY, NY 12222
E-mail address: tchernev@math.albany.edu

We present with proofs some additional properties of the sequence $C_\bullet(P)$ as constructed in Section 2. The same properties hold for the sequence $C_\bullet(P, \Gamma)$ with similar proofs which we leave to the reader as an exercise.

Proposition 7.1. If $P$ is a ranked poset then $C_\bullet(P)$ is a complex of vector spaces.

We begin by establishing notation for some of the relevant objects. Set $\Delta_\alpha^{(0)} = D_\alpha$, and for $j \geq 1$, write

$$\Delta_\alpha^{(j)} = \bigcup_{\beta \prec \alpha, \text{rk}(\beta) = \text{rk}(\alpha) - j} \Delta_\beta^{(0)}.$$ 

Using the inclusion $\Delta_\alpha^{(1)} \subset \Delta_\alpha^{(0)}$, and the fact that $\Delta_\alpha^{(0)}$ is contractible, we obtain a canonical isomorphism

$$\widetilde{H}_i(\Delta_\alpha^{(0)}/\Delta_\alpha^{(1)}, \mathbb{K}) \xrightarrow{\sim} \widetilde{H}_{i-1}(\Delta_\alpha^{(1)}, \mathbb{K})$$

for every $i$ using the long exact sequence in relative homology. Further, the equality of reduced chain complexes

$$\widetilde{C}_\bullet(\Delta_\alpha^{(j)}/\Delta_\alpha^{(j+1)}) = \bigoplus_{\beta \prec \alpha, \text{rk}(\beta) = \text{rk}(\alpha) - j} \widetilde{C}_\bullet(\Delta_\beta^{(0)}/\Delta_\beta^{(1)})$$

for each $j$ gives rise to an isomorphism on the level of reduced homology,

$$\widetilde{H}_*(\Delta_\alpha^{(j)}/\Delta_\alpha^{(j+1)}) = \bigoplus_{\beta \prec \alpha, \text{rk}(\beta) = \text{rk}(\alpha) - j} \widetilde{H}_*(\Delta_\beta^{(0)}/\Delta_\beta^{(1)})$$

which we refer to as reindexing.
Through the inclusion of simplicial complexes $\Delta_\alpha^{(2)} \subset \Delta_\alpha^{(1)}$, we obtain an exact sequence of reduced chain complexes,

$$0 \longrightarrow \tilde{C}_\bullet(\Delta_\alpha^{(2)}) \longrightarrow \tilde{C}_\bullet(\Delta_\alpha^{(1)}) \longrightarrow \tilde{C}_\bullet(\Delta_\alpha^{(1)}/\Delta_\alpha^{(2)}) \longrightarrow \bigoplus_{\beta \leq \alpha} \tilde{C}_\bullet(\Delta_\beta^{(0)}/\Delta_\beta^{(1)}) \longrightarrow 0$$

which in turn gives rise to a long exact sequence in reduced homology,

$$\cdots \longrightarrow \tilde{H}_i(\Delta_\alpha^{(2)}, \mathbb{k}) \longrightarrow \tilde{H}_i(\Delta_\alpha^{(1)}, \mathbb{k}) \longrightarrow \tilde{H}_i(\Delta_\alpha^{(1)}/\Delta_\alpha^{(2)}, \mathbb{k}) \longrightarrow \bigoplus_{\beta \leq \alpha} \tilde{H}_i(\Delta_\beta^{(0)}/\Delta_\beta^{(1)}, \mathbb{k}) \longrightarrow \cdots$$

The simplicial inclusions $\Delta_\alpha^{(3)} \subset \Delta_\alpha^{(2)} \subset \Delta_\alpha^{(1)}$ also give rise to an exact sequence of relative chain complexes

$$0 \rightarrow \tilde{C}_\bullet(\Delta_\alpha^{(2)}/\Delta_\alpha^{(3)}, \mathbb{k}) \rightarrow \tilde{C}_\bullet(\Delta_\alpha^{(1)}/\Delta_\alpha^{(3)}, \mathbb{k}) \rightarrow \tilde{C}_\bullet(\Delta_\alpha^{(1)}/\Delta_\alpha^{(2)}, \mathbb{k}) \rightarrow 0$$

which produces a long exact sequence in reduced homology

$$\cdots \longrightarrow \tilde{H}_i(\Delta_\alpha^{(1)}/\Delta_\alpha^{(3)}, \mathbb{k}) \overset{\mu}{\longrightarrow} \tilde{H}_i(\Delta_\alpha^{(1)}/\Delta_\alpha^{(2)}, \mathbb{k}) \overset{D}{\longrightarrow} \tilde{H}_{i-1}(\Delta_\alpha^{(2)}/\Delta_\alpha^{(3)}, \mathbb{k}) \longrightarrow \cdots \quad (7.3)$$

Finally, we have the equality

$$\tilde{H}_{i-1}(\Delta_\alpha^{(2)}/\Delta_\alpha^{(3)}, \mathbb{k}) = \bigoplus_{\gamma < \alpha} \tilde{H}_{i-1}(\Delta_\gamma^{(0)}/\Delta_\gamma^{(1)}, \mathbb{k}),$$

which is given by reindexing.

We can now proceed with the proof of Proposition 7.1, which we break into three lemmas.

**Lemma 7.4.** We have $\varphi_1 \circ \varphi_2 = 0$. 

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Proof. Suppose \([w] \in \widetilde{H}_0(\Delta_0^{(1)}, k)\), with representative cycle \(w\). We therefore have
\[
w = \sum_{\lambda \in (0, \alpha)} c_\lambda \cdot \{\lambda\}
\]
with \(\sum_{\lambda \in (0, \alpha)} c_\lambda = 0\). Choosing a partition of \((0, \alpha)\) into a disjoint union
\[
(0, \alpha) = \bigcup_{\beta < \alpha} P_\beta
\]
(7.5)
of subsets \(P_\beta\) such that for every \(\lambda \in P_\beta\) one has \(\lambda \leq \beta\), we get
\[
w = \sum_{\beta < \alpha} w_\beta
\]
with
\[
w_\beta = \sum_{\lambda \in P_\beta} c_\lambda \cdot \{\lambda\}.
\]
Therefore,
\[
\varphi_{2, \alpha, \beta}([w]) = [d(w_\beta)] = \left[ \sum_{\lambda \in P_\beta} c_\lambda \cdot \emptyset \right]
\]
where \(d\) is the usual boundary map.

Applying \(\varphi_1\), we now have
\[
\varphi_1 \circ \varphi_2([w]) = \sum_{\beta < \alpha} \varphi_1 \circ \varphi_{2, \alpha, \beta}([w])
\]
\[
= \sum_{\beta < \alpha} \varphi_1 \left( \left[ \sum_{\lambda \in P_\beta} c_\lambda \cdot \emptyset \right] \right)
\]
\[
= \sum_{\beta < \alpha} \left[ \sum_{\lambda \in P_\beta} c_\lambda \cdot \emptyset \right]
\]
\[
= \sum_{\beta < \alpha} \sum_{\lambda \in P_\beta} c_\lambda \cdot \emptyset
\]
\[
= \sum_{\lambda \in (0, \alpha)} c_\lambda \cdot \emptyset
\]
\[
= 0.
\]
Lemma 7.6. For each $i \geq 1$ the diagram

$$\begin{array}{ccc}
\tilde{H}_i(\Delta^{(1)}_a/\Delta^{(2)}_a, k) & \xrightarrow{D} & \tilde{H}_{i-1}(\Delta^{(2)}_a/\Delta^{(3)}_a, k) \\
\bigoplus_{\beta \leq \alpha} \tilde{H}_i(\Delta^{(0)}_\beta/\Delta^{(1)}_\beta, [k]) & \cong & \bigoplus_{\gamma \leq \alpha, \operatorname{rk}(\gamma) = \operatorname{rk}((\alpha)-2)} \tilde{H}_{i-1}(\Delta^{(0)}_\gamma/\Delta^{(1)}_\gamma, [k]) \\
\cong \bigoplus_{\beta \leq \alpha} \tilde{H}_{i-1}(\Delta^{(1)}_\beta, [k]) & \xrightarrow{\varphi + 1} & \bigoplus_{\gamma \leq \alpha, \operatorname{rk}(\gamma) = \operatorname{rk}((\alpha)-2)} \tilde{H}_{i-2}(\Delta^{(1)}_\gamma, [k])
\end{array}$$

is commutative.

Proof. To verify commutativity, it suffices to show that for each $\delta < \alpha$ with $\operatorname{rk}(\delta) = \operatorname{rk}(\alpha) - 2$ the components of $\varphi_i \circ (\bigoplus_{\beta \leq \alpha} \theta_{i, \beta})$ and $(\bigoplus_{\beta \leq \alpha} \theta_{i-1, \gamma}) \circ D$ are the same in $\tilde{H}_{i-2}(\Delta^{(1)}_\delta, [k])$.

Indeed, suppose that $[\bar{w}]$ is a representative for the homology class generated by the image $\bar{w}$ in $\tilde{C}_i\left(\Delta^{(1)}_a/\Delta^{(2)}_a\right)$ of the relative cycle

$$w = \sum_{a^*_\delta \leq \alpha} c_\sigma \cdot \sigma = \sum_{\beta \leq \alpha} w_\beta$$

of $\left(\Delta^{(1)}_a, \Delta^{(2)}_a\right)$, where $c_\sigma \in k$, each face $\sigma = \{a^*_0, \ldots, a^*_i\}$ is oriented by $a^*_0 < \cdots < a^*_i$ and $w_\beta = \sum_{a^*_\delta = \beta} c_\sigma \cdot \sigma$. Since $w$ is a relative cycle, we must have

$$\sum_{a^*_\delta = \beta} c_\sigma \cdot d(\hat{\sigma}) = 0$$

where $\hat{\sigma} = \{a^*_0, \ldots, a^*_{i-1}\}$ and $d$ is the usual boundary map. Therefore, each $w_\beta$ is a relative cycle for $\left(\Delta^{(0)}_\beta, \Delta^{(1)}_\beta\right)$ and $[\bar{w}]$ has

$$[\bar{w}] = \sum_{\beta \leq \alpha} [\bar{w}_\beta]$$

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as its reindexing decomposition. Thus,

$$(\oplus \theta_{i,\beta}([\bar{w}]) = \sum_{\beta < \alpha} \theta_{i,\beta}([\bar{w}]) = \sum_{\beta < \alpha} [d(w_\beta)] = \sum_{\beta < \alpha} [v_\beta]$$

where

$$v_\beta = d(w_\beta) = (-1)^i \sum_{a_i^\sigma = \beta} c_\sigma \cdot \hat{\sigma}$$

is a cycle in $\Delta^{(0)}_\beta$.

Next, choose for each $\beta < \alpha$ a partition

$$(\hat{0}, \beta) = \bigsqcup_{\gamma < \beta} P_{\beta, \gamma}$$

of $(\hat{0}, \beta)$ such that $\lambda \leq \gamma$ for each $\lambda \in P_{\beta, \gamma}$ and write

$$v_\beta = \sum_{\gamma < \beta} w_{\beta, \gamma}$$

where

$$w_{\beta, \gamma} = (-1)^i \sum_{a_i^\sigma = \beta, a_i^{\sigma - 1} \in P_{\beta, \gamma}} c_\sigma \cdot \hat{\sigma}.$$ 

It follows that the component of $\varphi_{i+1}[(\oplus \theta_{i,\beta})([\bar{w}])]$ in $\widetilde{H}_{i-2}(\Delta^{(1)}_\delta, k)$ is given by

$$\sum_{\beta : \delta < \beta < \alpha} \varphi_{i+1}([v_\beta]) = \sum_{\beta : \delta \leq \beta < \alpha} \varphi_{i+1}^{\beta, \delta} \left( \left[ \sum_{\gamma < \beta} w_{\beta, \gamma} \right] \right) = \sum_{\beta : \delta \leq \beta < \alpha} [d(w_{\beta, \delta})].$$

On the other hand, since $v_\beta$ is a cycle, each $w_{\beta, \gamma}$ is a relative cycle for
\((\Delta_2^{(2)}, \Delta_3^{(3)})\). As \(D\) is the connecting map in (7.6), we have
\[
D([\bar{w}]) = \left[d(w)\right] = \sum_{\beta \triangleleft \alpha} d(w_\beta) = \left[\sum_{\beta \triangleleft \alpha} \bar{v}_\beta\right] = \left[\sum_{\beta \triangleleft \alpha} \left(\sum_{\gamma \triangleleft \beta} w_{\beta,\gamma}\right)\right] = \sum_{\gamma \triangleleft \alpha, \text{rk}(\gamma) = \text{rk}(\alpha) - 2} \left[\sum_{\gamma \triangleleft \beta \triangleleft \alpha} \bar{w}_{\beta,\gamma}\right]
\]
as its reindexing decomposition. Therefore, the component of \((\oplus_{\gamma} \theta_{i-1,\gamma}) \circ (D([\bar{w}]))\) in \(\widetilde{H}_{i-2}(\Delta_1^{(1)}, k)\) is equal to
\[
\theta_{i-1,\beta} \left(\sum_{\beta, \delta \triangleleft \beta \triangleleft \alpha} \bar{w}_{\beta,\delta}\right) = \left[d\left(\sum_{\beta, \delta \triangleleft \beta \triangleleft \alpha} w_{\beta,\delta}\right)\right] = \sum_{\beta, \delta \triangleleft \beta \triangleleft \alpha} [d(w_{\beta,\delta})],
\]
which proves commutativity. \(\square\)

**Lemma 7.7.** For each \(i \geq 1\) the diagram
\[
\begin{array}{ccc}
\widetilde{H}_i(\Delta_1^{(1)}/\Delta_3^{(3)}, k) & \xrightarrow{\mu} & \widetilde{H}_i(\Delta_1^{(1)}/\Delta_2^{(2)}, k) \\
\oplus_{\beta \triangleleft \alpha} \widetilde{H}_i(\Delta_0^{(0)}/\Delta_1^{(1)}, k) & & \\
\widetilde{H}_i(\Delta_1^{(1)}, k) & \xrightarrow{\varphi_{i+2}} & \bigoplus_{\beta \triangleleft \alpha} \widetilde{H}_{i-1}(\Delta_1^{(1)}, k),
\end{array}
\]
is commutative.

Proof. It is enough to show that $\varphi_{i+2}$ and $(\oplus \theta_{i, \beta}) \circ D \circ \mu \circ \pi$ have the same component in $\tilde{H}_{i-1}(\Delta_{\beta}^{(1)}, k)$.

Suppose that $[v]$ is a homology class in $\tilde{H}_i(\Delta_{\alpha}^{(1)}, k)$ represented by the cycle

$$v = \sum_{a_\ell^\alpha < \alpha} c_{\sigma} \cdot \sigma.$$ 

Under projection

$$\pi([v]) = [\pi(v)] = \left[ \sum_{a_\ell^\alpha < \alpha} c_{\sigma} \cdot \sigma' \right],$$

where $\sigma' = \pi(\sigma)$ is the image of $\sigma$ under the standard projection $\tilde{C}_*(\Delta_{\alpha}^{(1)}) \longrightarrow \tilde{C}_*(\Delta_{\alpha}^{(1)}/\Delta_{\alpha}^{(2)})$. Applying $\mu$ to $\pi([v])$, we have

$$\mu([\pi(v)]) = \left[ \sum_{a_\ell^\alpha < \alpha} c_{\sigma} \cdot \bar{\sigma} \right] = \sum_{\beta < \alpha} [\bar{w}_\beta],$$

where $\bar{\sigma}$ is the image of $\sigma$ under the projection $\tilde{C}_*(\Delta_{\alpha}^{(1)}) \longrightarrow \tilde{C}_*(\Delta_{\alpha}^{(1)}/\Delta_{\alpha}^{(2)})$, the process of reindexing has produced

$$w_\beta = \sum_{a_\ell^\alpha \in P_\beta} c_{\sigma} \cdot \sigma,$$

and $P_\beta$ is as in (7.5). Thus

$$\left(\oplus \theta_{i, \beta}\right) \left(\sum_{\beta < \alpha} [w_\beta]\right) = \sum_{\beta < \alpha} \theta_{i, \beta}([w_\beta]) = \sum_{\beta < \alpha} [d(w_\beta)] = \sum_{\beta < \alpha} [v_\beta]$$

where

$$v_\beta = d(w_\beta) = (-1)^i \left( \sum_{a_\ell^\alpha = \beta} c_{\sigma} \cdot \hat{\sigma} \right) + d \left( \sum_{a_\ell^\alpha \in P_\beta \setminus \{\beta\}} c_{\sigma} \cdot \sigma \right)$$

and $\hat{\sigma} = \sigma \setminus \{a_\ell^\alpha\}$. Clearly, $[v_\beta]$, is the component of $(\oplus \theta_{i, \beta}) \circ D \circ \mu \circ \pi$ in $\tilde{H}_{i-1}(\Delta_{\beta}^{(1)}, k)$.
Next, given $\beta \lessdot \alpha$ we have

$$v = \sum_{\alpha_i < \alpha} c_\sigma \cdot \sigma = \sum_{\beta \lessdot \alpha} \left( \sum_{\alpha_i \in P_\beta} c_\sigma \cdot \sigma \right)$$

and therefore,

$$\varphi_{i+2}([v]) = \sum_{\beta \lessdot \alpha} \varphi_{i+2}^{\alpha,\beta}([v])$$

$$= \sum_{\beta \lessdot \alpha} \left[ d \left( \sum_{\alpha_i \in P_\beta} c_\sigma \cdot \sigma \right) \right]$$

$$= \sum_{\beta \lessdot \alpha} \left[ (-1)^i \left( \sum_{\alpha_i = \beta} c_\sigma \cdot \sigma \right) + d \left( \sum_{\alpha_i \in P_\beta \setminus \{ \beta \}} c_\sigma \cdot \sigma \right) \right]$$

$$= \sum_{\beta \lessdot \alpha} [v_\beta]$$

so that the commutativity has been proved.

\[\square\]

**Proof of Proposition 7.7.** We show $\varphi_{i-1} \circ \varphi_i = 0$. Lemma 7.4 establishes this for $i = 2$, and the commutative diagrams of Lemmas 7.6 and 7.7 may be combined so that for $i \geq 3$ we have the equality $\varphi_{i-1} \circ \varphi_i = (\oplus \theta_{i-1,\gamma}) \circ D \circ \mu \circ \pi$. Since $D$ and $\mu$ are consecutive maps in an exact sequence, $D \circ \mu = 0$. Therefore $\varphi_{i-1} \circ \varphi_i = 0$ for each $i$ which completes the proof that $C_\bullet(P)$ is a complex.

The general case can always be reduced to the case that $P$ is ranked because of the following.

**Lemma 7.9.** There is a rank preserving canonical embedding $P \subset P'$, where $P'$ is a ranked poset, so that for every $i \geq 0$ and $\alpha \in P'$ one has

$$C_{i,\alpha}(P') = \begin{cases} C_{i,\alpha}(P) & \text{if } \alpha \in P, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $C_i(P) = C_i(P')$ for every $i \geq 0$. 

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Proof. For $\beta < \alpha$ with $\text{rk}_P(\alpha) - \text{rk}_P(\beta) \geq 2$, let
\[ P_{\alpha,\beta} = \{ \gamma_{i}^{\alpha,\beta} : 1 \leq i \leq \text{rk}_P(\alpha) - \text{rk}_P(\beta) - 1 \} \]
be a set of symbols, and define $P'$ as the disjoint union
\[ P' = P \bigsqcup \left( \bigsqcup_{\beta \lesssim \alpha \in P} P_{\alpha,\beta} \right). \]
We order the elements of $P'$ by describing the covering relations: the Hasse diagram of $P'$ is obtained from the Hasse diagram of $P$ by breaking up an edge $\beta < \alpha$ into $n$ edges $\beta < \gamma_1^{\alpha,\beta} < \cdots < \gamma_n^{\alpha,\beta} < \alpha$ where $n = \text{rk}_P(\alpha) - \text{rk}_P(\beta)$. It is clear that in this way, $P'$ is canonically determined by $P$ and $\text{rk}_P(\alpha) = \text{rk}_{P'}(\alpha)$ for each $\alpha \in P$. For the remainder of the proof, we transfer this notation to any set (poset, simplicial complex) $X$ associated to $P$ and write $X'$ for the corresponding set (poset, simplicial complex).

Since $P'$ can be obtained iteratively by breaking up one edge in two at a time, to prove the second claim of the Lemma it is enough to assume that only one additional poset element $\gamma$ is added to $P$ and $\beta < \gamma < \alpha$ in $P'$ for some $\beta < \alpha$ in $P$.

We have
\[ \Delta'_{\gamma} = \bigcup_{\rho < \gamma} G'_{\rho} = G'_{\beta} = G_{\beta} \]
since $\gamma$ uniquely covers $\beta$ by construction. Thus, $\Delta'_{\gamma}$ is a cone with apex $\beta$ and hence contractible. Therefore, $C_{i,\gamma}(P') = 0$ for each $i$. Next, let $\delta \in P$. If $\delta \nless \alpha$ then $\Delta'_{\delta} = \Delta_{\delta}$ and so $C_{i,\delta}(P') = C_{i,\delta}(P)$ for each $i$. Thus, it remains to consider the case $\delta \geq \alpha$.

Let
\[ \Omega = \text{St}(\Delta'_{\delta}, \{ \alpha \}) = G'_{\beta} * \Delta ([\alpha, \delta]) = G_{\beta} * \Delta ([\alpha, \delta]). \]

Thus $\Delta'_{\delta} = \Delta_{\delta} \cup \Omega$ and $\Omega \cap \Delta_{\delta} = G_{\beta} * \Delta ([\alpha, \delta])$ is a cone with apex $\beta$, hence contractible. The Mayer-Vietoris sequence in reduced homology on the triple $(\Delta_{\delta}, \Omega, \Delta'_{\delta})$ yields
Since both $\Omega$ and $\Omega \cap \Delta_\delta$ are contractible we get the desired conclusion. □

**Remark 7.10.** Replacing the poset $P$ with the ranked poset $P'$ may result in different sequences $C_\bullet(P)$ and $C_\bullet(P')$. Indeed, although the components are identical by Lemma 7.9, the maps in the sequences are different in general. In particular, the maps present in $C_\bullet(P')$ will in general have more trivial components.

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