PRISMATIC $F$-CRYSTALS AND CRYSTALLINE LOCAL SYSTEMS

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Abstract. We show that the category of integral crystalline local systems on the generic fiber of a $p$-adic smooth formal scheme $X$ is equivalent to the category of prismatic $F$-crystals over the analytic locus of the prismatic site of $X$. This extends a recent result of Bhatt–Scholze [BS21] to higher dimensions.

1. Introduction

Let $K$ be a complete discretely valued $p$-adic field of characteristic 0 with ring of integers $\mathcal{O}_K$ and perfect residue field $k$. Let $G_K$ be the absolute Galois group of $K$. One of the central goals of $p$-adic Hodge theory is the study of $p$-adic Galois representations, that is, continuous representations of $G_K$ valued in finite dimensional $\mathbb{Q}_p$-vector spaces. The key example is the étale cohomology of a smooth proper algebraic variety over $K$ with coefficients in $\mathbb{Q}_p$.

Let $X$ be a $p$-adic smooth formal scheme over $\mathcal{O}_K$. Motivated by the de Rham and the Hodge theorem in complex geometry, Grothendieck asked if there is a “mysterious functor” relating $\mathbb{Q}_p$-étale cohomology of the generic fiber $X_\eta$ over $K$ and the crystalline cohomology of the special fiber $X_s$ over $k$. This question was answered by work of many mathematicians, including Fontaine [Fon82] and Faltings [Fal89], who found a natural isomorphism between the aforementioned cohomology theories after tensoring with the $p$-adic period ring $B_{\text{crys}}$. Fontaine’s work led to the abstract notion of crystalline Galois representations, which are roughly $\mathbb{Q}_p$-representations that come from a filtered Frobenius module with the trivial Galois action after a base extension to $B_{\text{crys}}$. A natural question then is: can crystalline Galois representations be understood in a more geometric manner?

One possible answer to this question has recently been put forward in work of Bhatt–Scholze [BS19, BS21]. In [BS19], they attached to each $p$-adic smooth formal scheme $X$ a novel Grothendieck site $X_\Delta$, called the prismatic site of $X$. The objects of $X_\Delta$ are given by maps $\text{Spf}(\mathcal{A}/I) \to X$ from prisms $(\mathcal{A}, I)$, that is, $\mathbb{Z}_p$-algebras $\mathcal{A}$ which are (derived) $(p, I)$-complete for a certain ideal $I \subset \mathcal{A}$ and are equipped with an endomorphism on $\mathcal{A}$ lifting Frobenius on $\mathcal{A}/p$; see [BS19, §3]. The cohomology of the associated structure sheaf $\mathcal{O}_\Delta$ specializes to many other important $p$-adic cohomology theories, including $\mathbb{Z}_p$-étale cohomology of the generic fiber $X_\eta$ and crystalline cohomology of the special fiber $X_s$ ([BS19, Thm. 1.8] and [BL22, §1.3] for the absolute version). This suggests that sheaves on the prismatic site can be used to build a bridge between crystalline Galois representations and their associated filtered Frobenius modules. In [BS21] Bhatt–Scholze prove such a connection in the special case $X = \text{Spf}(\mathcal{O}_K)$. More precisely, their result is as follows.

Theorem 1.1. [BS21, Thm. 1.2] Let $X = \text{Spf}(\mathcal{O}_K)$. There is a natural equivalence

$$\text{Vect}^\phi(X_\Delta, \mathcal{O}_\Delta) \longrightarrow \text{Rep}_{\mathbb{Z}_p}^{\text{crys}}(G_K)$$

between the category of prismatic $F$-crystals on $X$ and the category of continuous $G_K$-representations over $\mathbb{Z}_p$ that are crystalline after inverting $p$.

Note that continuous $G_K$-representations over $\mathbb{Z}_p$ can naturally be identified with étale $\mathbb{Z}_p$-local systems over the generic fiber $X_\eta = \text{Spec}(K)$. We can thus rephrase Theorem 1.1 more geometrically as an equivalence between the category of prismatic $F$-crystals on $X$ and the category of crystalline...
étale $\mathbb{Z}_p$-local systems on $X_\eta$:

$$\text{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta) \simeq \text{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X_\eta).$$

Since both categories admit higher-dimensional generalizations, it is natural to ask if the aforementioned equivalence can be extended to general $p$-adic smooth formal schemes over $\mathcal{O}_K$. Our main result in this article gives a positive answer to this question.

**Theorem A.** Let $X$ be a $p$-adic smooth formal scheme over $\mathcal{O}_K$. There is a natural equivalence

$$T : \text{Vect}^{\text{an}, \varphi}(X_\Delta) \longrightarrow \text{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X_\eta)$$

between the category of analytic prismatic $F$-crystals on $X$ and the category of crystalline $\mathbb{Z}_p$-local systems on $X_\eta$.

We now explain in more detail the various notions in the statements above.

**Dramatis personae.** A **prismatic $F$-crystal** (of vector bundles) $(\mathcal{E}, \varphi_{\mathcal{E}})$ consists of a vector bundle $\mathcal{E}$ over the ringed site $(X_\Delta, \mathcal{O}_\Delta)$, together with an $\mathcal{O}_\Delta$-linear isomorphism $\varphi_{\mathcal{E}} : (\varphi^* \mathcal{E})[1/I_\Delta] \rightarrow \mathcal{E}[1/I_\Delta]$; cf. [BS21, Def. 4.1]. Here, $\varphi^* \mathcal{E}$ is the pullback of $\mathcal{E}$ along the Frobenius endomorphism of the structure sheaf $\mathcal{O}_\Delta$ and $I_\Delta$ is the ideal sheaf sending a prism $(A, I)$ to the ideal $I$. Analogously, an **analytic prismatic $F$-crystal** (of vector bundles) is a similar pair $(\mathcal{E}, \varphi_{\mathcal{E}})$, with the difference that over a prism $(A, I)$, the vector bundle $\mathcal{E}$ is only defined on the open subset $\text{Spec}(A) \setminus V(p, I)$ instead of all of $\text{Spec}(A)$; cf. Definition 3.8. We denote the category of analytic prismatic $F$-crystals over $X$ by $\text{Vect}^{\text{an}, \varphi}(X_\Delta)$.

**Example 1.2.** A **prismatic Dieudonné crystal** is a prismatic $F$-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ such that $\varphi_{\mathcal{E}}$ sends the submodule $\mathcal{E}$ into itself, with the cokernel killed by $I_\Delta$. By prismatic Dieudonné theory ([ALB19, Thm. 1.4.4, Prop. 5.2.3]), there is an equivalence of categories between $p$-divisible groups over $X$ and prismatic Dieudonné crystals, where the latter is a full subcategory of prismatic $F$-crystals.

**Remark 1.3.** The category of prismatic $F$-crystals embeds as a full subcategory into the category of analytic prismatic $F$-crystal by restricting to the open subsets (Proposition 3.10). In the special case $X = \text{Spf}(\mathcal{O}_K)$, this inclusion of categories is in fact an equivalence (Proposition 3.11). However, in general not every analytic prismatic $F$-crystal can be extended to a prismatic $F$-crystal. See [DLMS22, Ex. 3.35] for an example.

**Remark 1.4.** The name “analytic prismatic $F$-crystal” comes from the following observation: Let $C$ be a fixed completed algebraic closure of $K$. When $A$ is the perfect prism associated with a $p$-torsionfree perfectoid $\mathcal{O}_C$-algebra (which covers $X$ locally in the prismatic site), [Ked20] shows that the category of vector bundles over $\text{Spec}(A) \setminus (p, I)$ is equivalent to the category of vector bundles over the analytic locus of the adic spectrum $\text{Spa}(A, A)$.

The inclusion $\text{Spec}(A[1/I]) \simeq \text{Spec}(A) \setminus V(I) \hookrightarrow \text{Spec}(A) \setminus V(p, I)$ gives rise to a base change functor

$$\text{Vect}^{\text{an}, \varphi}(X_\Delta) \longrightarrow \text{Vect}^{\varphi}(X_\Delta, \mathcal{O}_\Delta[1/I_\Delta],)$$

where $(-)^\wedge_p$ denotes the derived $p$-adic completion. Thanks to [BS21, Cor. 3.8] (see [MW21] for a different approach), we can naturally identify the target category with étale $\mathbb{Z}_p$-local systems on the generic fiber:

$$\text{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p) \simeq \text{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X_\eta).$$

A combination of the two functors yields the **étale realization functor**

$$T : \text{Vect}^{\text{an}, \varphi}(X_\Delta) \longrightarrow \text{Vect}^{\varphi}(X_\Delta, \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p) \simeq \text{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X_\eta);$$

see Construction 3.12.

On the other hand, the category of étale $\mathbb{Z}_p$-local systems on the generic fiber $X_\eta$ contains the full subcategory $\text{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X_\eta)$ of objects which are **crystalline** in the sense of Faltings ([Fal89,
p. 67]) after inverting \( p \). Under the equivalence of \( \mathbb{Z}_p \)-local systems over a point and continuous \( G_K \)-representations over \( \mathbb{Z}_p \), crystalline local systems in this sense correspond to the crystalline representations appearing in Theorem 1.1. By Theorem 3.14 below, the essential image of \( T \) is contained in \( \text{Loc}^{\text{crys}}(X_\eta) \), and Theorem A says that it is in fact equal.

Example 1.5. Let \( f : Y \to X = \text{Spf}(\mathcal{O}_K) \) be a proper smooth map of \( p \)-adic formal schemes, and let \( i \in \mathbb{Z}_{\geq 0} \). Assume \( H^i(Y_{\mathbb{A}, \text{ét}}, \mathbb{Z}_p) \) is \( p \)-torsionfree. Using the étale and the crystalline comparison theorems in [BS19, Thm. 1.8], one can show that the \( i \)-th prismatic cohomology group \( H^i(Y_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}) \) gives an analytic prismatic \( F \)-crystal over \( X \) after restricting to the analytic locus. Its étale realization corresponds to the crystalline \( G_K \)-representation \( H^i(Y_{\mathbb{A}, \text{ét}}, \mathbb{Z}_p) \).

Remark 1.6. The composition of our étale realization functor with the fully faithful embedding

\[
\text{Vect}^\varphi(X_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}) \to \text{Vect}^{\text{an}, \varphi}(X_{\mathbb{A}}).
\]

from Remark 1.3 recovers the étale realization functor from [BS21, Constr. 4.8]. In particular, by the equivalence between prismatic \( F \)-crystals and analytic prismatic \( F \)-crystals over \( X = \text{Spf}(\mathcal{O}_K) \) mentioned above, our Theorem A indeed extends the result of Bhatt–Scholze in Theorem 1.1 to general \( X \).

Strategy of the proof. We now briefly discuss the idea of the proof of Theorem A. For simplicity, we temporarily assume that \( X = \text{Spf}(R) \) is an affine smooth formal scheme over \( \mathcal{O}_K \) and that there exists a \( p \)-torsionfree perfectoid \( \mathcal{O}_C \)-algebra \( S \) that covers \( R \) in the quasi-syntomic site.

For the full faithfulness, essentially the same proof as in [BS21, §5] carries over to our higher-dimensional situation. Namely, we consider an analog of Fargues’s functor which sends an analytic prismatic \( F \)-crystal over \( \text{Spf}(S) \) to a pair \( (T, \Xi) \), consisting of a \( \mathbb{Z}_p \)-local system \( T \) over \( \text{Spf}(S)_{\eta} \) together with a \( \mathbb{B}_{\text{dR}}^+ \)-lattice \( \Xi \) of the pro-étale sheaf \( T \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{dR}} \) (Construction 4.2). Then the question of the full faithfulness of Theorem A can be reduced to that of the full faithfulness of Fargues’s functor, using a Čech complex argument. See Section 4.1 for details.

For essential surjectivity, let \( T \) be a crystalline \( \mathbb{Z}_p \)-local system on the generic fiber \( X_\eta \). The general strategy of our proof follows that of [BS21, §6]: first construct an analytic prismatic \( F \)-crystal over \( S \) and then descend it to \( X \). Starting with the filtered \( F \)-isocrystal \( (\mathcal{E}, \varphi_\mathcal{E}, \text{Fil}^*(\mathcal{E})) \) over the special fiber \( X_\eta \) associated with the crystalline local system \( T \) (cf. Definition 2.25), one can naturally construct an \( F \)-crystal over the locus \( \{I_{\mathbb{A}} \leq p \neq 0\} \) inside \( \text{Spa}(\Delta_S) \) (Construction 4.8). However, the situation for higher-dimensional formal schemes is different from that of a point as in [BS21, §6.4], in that we cannot use Kedlaya’s slope filtration results [Ked04] for weakly admissible filtered \( \varphi \)-modules to extend the \( F \)-crystal further across the entire locus \( \{p \neq 0\} \). In fact, it is not even clear how to formulate the notion of weak admissibility for filtered \( F \)-isocrystals over higher-dimensional \( X \). Instead, our approach is to construct a canonical analytic \( F \)-crystal \( \mathcal{M}_S \) over \( \text{Spec}(\Delta_S) \setminus V(p, I) \) for every perfectoid algebra \( S \) over \( R \) that is large enough and satisfies a crystal-like pullback compatibility among perfectoid \( S \) (Theorem 4.13). After this construction, we use (as in [BS21]) the Beilinson fiber square developed in [AMMN22] to study the structure of the prism \( \Delta_{S'} \) attached to the \( p \)-complete fiber product \( S' = S \otimes_R S \). Using this structural result, we are then able to extend the descent data on \( \mathcal{M} \) from \( \{I_{\mathbb{A}} \leq p \neq 0\} \) to \( \{p \neq 0\} \) (Proposition 4.18), and eventually to the whole analytic locus (Theorem 4.5).

Remark 1.7. In the final stages of our project, we learned that Du–Liu–Moon–Shimizu independently proved Theorem A via a different approach by studying higher-dimensional Breuil–Kisin prisms and their descent data; cf. [DLMS22, Thm. 1.3]. They show that the subcategory of effective analytic prismatic \( F \)-crystals\(^1\) is equivalent to crystalline \( \mathbb{Z}_p \)-local systems with nonnegative

\(^1\)In [DLMS22, Def. 1.1] such objects are called completed prismatic \( F \)-crystals. To see that their notion is compatible with ours, note that by [DLMS22, Prop. 4.13] and Beauville–Laszlo gluing, the restriction of a completed
Hodge–Tate weights. The equivalence for all analytic prismatic crystals without the assumption that the Frobenius endomorphism is effective (that is, defined on a lattice of the analytic prismatic crystal), can then be obtained by taking Breuil–Kisin and Tate twists; see [DLMS22, Rem. 1.5].

Outline. We briefly explain the structure of the article. After recalling the definitions of various period sheaves (Section 2.1, Section 2.3) and filtered $F$-isocrystals (Section 2.2), we introduce in Section 2 the notion of crystalline $\mathbb{Z}_p$-local systems in Definition 2.25, following Faltings [Fal89]. In Section 3.1, we introduce and study analytic prismatic $F$-crystals. Then we define the étale and the crystalline realization functors in Section 3.2. Moreover, we show that the étale realization sends an analytic prismatic $F$-crystal to a crystalline $\mathbb{Z}_p$-local system. Finally, we prove our Theorem A in Section 4, with Section 4.1 addressing full faithfulness and Section 4.2 essential surjectivity.

Notation and conventions. Let $A$ be a commutative ring and $I$ be an ideal of $A$. We denote the derived $I$-adic completion of $A$ by $A_I^\wedge$. We write $A(I/p)$ for the $p$-completion of the $A$-algebra $A[I/p]$. When $A$ is a a $\delta$-ring, $B$ is an $A$-algebra, and $(f_i)$ a regular sequence in $B$, we let $B\{f_i\}$ be the $p$-completion of the $B$-algebra $B \otimes_{A[x_i]} A[x_i]$, where $A\{x_i\}$ is the free $\delta$-$A$-algebra on generators $x_i$ and the map $A[x_i] \to B$ sends $x_i$ to $f_i$.

We will assume and follow the foundations of the prismatic theory as in [BS19], and we refer the reader to loc. cit. for a detailed exposition. As a convention, our prismatic site is equipped with the $(p,I)$-completely flat topology.

Throughout the article, $K$ will denote a complete discretely valued $p$-adic field of characteristic 0 with ring of integers $\mathcal{O}_K$ and perfect residue field $k$. We fix a completed algebraic closure $C$ of $K$ with ring of integers $\mathcal{O}_C$.

To lighten notation, we will in some places not distinguish notationally between a coherent sheaf on an affine scheme and its global sections.

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2. Local systems on rigid spaces

We fix a complete discretely valued $p$-adic field $K$ of characteristic 0 with ring of integers $\mathcal{O}_K$ and perfect residue field $k$. Let $V_0 := W(k) \subseteq \mathcal{O}_K$ resp. $K_0 := V_0[p^{-1}] \subseteq K$ be the maximal unramified extension of $\mathbb{Z}_p$ resp. $\mathbb{Q}_p$ in $\mathcal{O}_K$ resp. $K$. Moreover, we fix a locally noetherian adic space $X_\eta$ over $\text{Spa}(K, \mathcal{O}_K)$. Following [Sch13, Def. 3.9], we equip $X_\eta$ with the proät topology and consider the associated site $X_\eta,\text{pro\acute{e}t}$. 

2.1. Infinitesimal and de Rham period sheaves. We begin with a reminder of various period sheaves on $X_\eta,\text{pro\acute{e}t}$, following [Sch13]. To this end, recall the following statement, which reduces us to defining these sheaves over affinoid perfectoid rings.

**Proposition 2.1** ([Sch13, Lem. 4.6, Prop. 4.8]). The affinoid perfectoid objects of $X_\eta,\text{pro\acute{e}t}$ form a basis for the topology. In other words, if $\text{Perfd}/X_\eta,\text{pro\acute{e}t}$ denotes the site associated with the full prismatic $F$-crystal to the open subset $\text{Spec}(\mathfrak{S}) \setminus V(p,E)$ of a Breuil–Kisin prism $(\mathfrak{S}, E)$ is a vector bundle. Moreover, the global sections of a vector bundle over $\text{Spec}(A) \setminus V(p, I)$ are a finitely presented module over $A$ (cf. Lemma 3.4).
subcategory of objects $U \in X_{\eta, \text{pro} \acute{e}t}$ for which $\hat{U} := \lim_i U_i$ is affinoid perfectoid, then restriction along the natural inclusion $\text{Perfd}/X_{\eta, \text{pro} \acute{e}t} \rightarrow X_{\eta, \text{pro} \acute{e}t}$ induces an equivalence of topoi

$$\text{Shv}(X_{\eta, \text{pro} \acute{e}t}) \rightarrow \text{Shv}(\text{Perfd}/X_{\eta, \text{pro} \acute{e}t})$$

**Definition 2.2** ([Sch13, Def. 6.1, Def. 6.8], [Sch16]). The sheaf $A_{\inf}$ resp. $B_{\inf}$ resp. $\mathcal{O}B_{\inf}$ in $\text{Shv}(X_{\eta, \text{pro} \acute{e}t})$ is given on an element $U \in \text{Perfd}/X_{\eta, \text{pro} \acute{e}t}$ with $\hat{U} = \text{Spa}(R, R^+)$ by

- $A_{\inf}(R, R^+) := W(R^{+\flat})$ resp.
- $B_{\inf}(R, R^+) := A_{\inf}(R, R^+)[1/p] = W(R^{+\flat})[1/p]$ resp.
- $\mathcal{O}B_{\inf}(R, R^+) := R \otimes_{V_0} B_{\inf}(R, R^+) \cong (R^+ \otimes_{V_0} W(R^{+\flat}))[1/p]$.

Here $R^{+\flat} = \lim_{x \rightarrow x^p} R^+/p$ is the tilt of $R^+$.

Next, we recall the definition of the sheaf versions of Fontaine’s de Rham period rings.

**Definition 2.3** ([Sch13, Def. 6.1, Def. 6.8]). Let $U = \lim_i \text{Spa}(R_i, R^{+}_i) \in \text{Perfd}/X_{\eta, \text{pro} \acute{e}t}$ with $\hat{U} = \text{Spa}(R, R^+)$ and let $\bar{\xi} \in A_{\inf}(R, R^+)$ be a generator of the de Rham specialization map $\bar{\theta} : A_{\inf}(R, R^+) \rightarrow R^+$. The filtered sheaf $B_{\text{dr}}^+$ resp. $B_{\text{dr}}$ resp. $\mathcal{O}B_{\text{dr}}^+$ resp. $\mathcal{O}B_{\text{dr}}$ in $\text{Shv}(X_{\eta, \text{pro} \acute{e}t})$ is given on $U$ by

- $B_{\text{dr}}^+(R, R^+) := B_{\inf}(R, R^+)\bar{\xi}$ with the $\bar{\xi}$-adic filtration resp.
- $B_{\text{dr}}(R, R^+) := B_{\text{dr}}^+(R, R^+)[1/\bar{\xi}] = B_{\inf}(R, R^+)[1/\bar{\xi}]$ with the $\bar{\xi}$-adic filtration resp.
- $\mathcal{O}B_{\text{dr}}^+(R, R^+) := \text{colim}((R^+ \otimes_{V_0} A_{\inf}(R, R^+))[1/p])_{\ker(\bar{\theta})}$ with the $\ker(\bar{\theta})$-adic filtration, where the morphism $\bar{\theta} : (R^+ \otimes_{V_0} A_{\inf}(R, R^+))[1/p] \rightarrow R$ is induced from $A_{\inf}(R^+)[p^{-1}] \rightarrow R$ by extension of scalars resp.
- $\mathcal{O}B_{\text{dr}}(R, R^+) := \mathcal{O}B_{\text{dr}}^+(R, R^+)[1/\bar{\xi}]$.

**Warning 2.4.** The definitions of our de Rham period sheaves are different from those of [Sch13], as we complete at $\{\bar{\xi} = 0\}$ instead of $\{\xi = 0\}$, where $\bar{\xi} = \varphi(\xi)$. This is simply to be consistent with various constructions in the prismatic setting later on.

Moreover, $\mathcal{O}B_{\text{dr}}^+$ is equipped with a $B_{\text{dr}}^+$-linear connection $\nabla : \mathcal{O}B_{\text{dr}}^+ \rightarrow \mathcal{O}B_{\text{dr}}^+ \otimes_{\mathcal{O}X_\eta} \Omega^1_{X_\eta}$: if $X_\eta$ is smooth, it satisfies Griffiths transversality, that is, $\nabla(\text{Fil}^i(\mathcal{O}B_{\text{dr}}^+)) \subseteq \text{Fil}^{i-1}(\mathcal{O}B_{\text{dr}}^+) \otimes_{\mathcal{O}X_\eta} \Omega^1_{X_\eta}$ ([Sch13, Cor. 6.13]). It extends to a connection on $\mathcal{O}B_{\text{dr}}$ with the same properties.

With these period sheaves in hand, we come to the definition of de Rham local systems. Let $\mathbf{Z}_p = \lim_n \mathbf{Z}/p^n$ be the lisse local system of $p$-adic integers on $X_{\eta, \text{pro} \acute{e}t}$, in the sense of [Sch13, §8].

**Definition 2.5** ([Sch13, Def. 8.3]). Assume that $X_\eta$ is smooth over $\text{Spa}(K, O_K)$. A sheaf of $\mathbf{Z}_p$-modules $L$ is de Rham if

(i) $L$ is lisse, that is, locally on $X_{\eta, \text{pro} \acute{e}t}$ of the form $\mathbf{Z}_p \otimes_{\mathbf{Z}_p} M$ for some finitely generated $\mathbf{Z}_p$-module $M$, and

(ii) there exists a locally free $\mathcal{O}_{X_\eta}$-module $E$, a separated, exhaustive, locally split, decreasing filtration $\text{Fil}^i E$ on $E$, and an integrable connection $\nabla : E \rightarrow E \otimes_{\mathcal{O}_{X_\eta}} \Omega^1_{X_\eta}$ with $\nabla(\text{Fil}^i E) \subseteq \text{Fil}^{i-1} E \otimes_{\mathcal{O}_{X_\eta}} \Omega^1_{X_\eta}$ for all $i \in \mathbf{Z}$ such that

$$L \otimes_{\mathbf{Z}_p} \mathcal{B}_{\text{dr}}^+ \cong (\text{Fil}^0(E \otimes_{\mathcal{O}_{X_\eta}} \mathcal{O}B_{\text{dr}}))^{\nabla = 0}.$$ 

It is called a de Rham local system if $L$ is furthermore a $\mathbf{Z}_p$-local system (equivalently, torsionfree).

### 2.2. Filtered $F$-isocrystals.

Next, we review the notion of filtered $F$-isocrystals.

**Definition 2.6.** Let $X_s$ be a $k$-scheme. Equip $V_0 = W(k)$ with the standard pd-structure and let $\text{CRIS}(X_s/V_0)$ be the big crystalline site of $X_s$. 
(1) A crystal (in coherent sheaves) on $X_s$ is a sheaf $\mathcal{E}$ of $\mathcal{O}_{X_s/V_0}$-modules on $\text{CRIS}(X_s/V_0)$ such that
   
   (i) for each pd-thickening $(U, T, \gamma)$ in $\text{CRIS}(X_s/V_0)$, the restriction $\mathcal{E}|_T$ of $\mathcal{E}$ to the Zariski site of $T$ is a coherent $\mathcal{O}_T$-module and
   (ii) for each $\alpha: (U, T, \gamma) \to (U', T', \gamma')$, the induced map $\alpha^*(\mathcal{E}|_T) \to \mathcal{E}|_{T'}$ is an $\mathcal{O}_T$-linear isomorphism.

(2) The category $\text{Isoc}(X_s/V_0)$ of isocrystals on $X_s$ has as objects the crystals on $X_s$ and as morphisms

$$\text{Hom}_{\text{Isoc}(X_s/V_0)}(\mathcal{E}, \mathcal{E}') := \text{Hom}_{\mathcal{O}_{X_s/V_0}}(\mathcal{E}, \mathcal{E}') \otimes \mathbb{Z} \mathbb{Q}$$

for any crystals $\mathcal{E}$ and $\mathcal{E}'$ on $X_s$.

The absolute Frobenius on $X_s$ and the Witt vector Frobenius on $V_0$ are compatible and thus induce a continuous and cocontinuous morphism of sites

$$F: \text{CRIS}(X_s/V_0) \to \text{CRIS}(X_s/V_0);$$

to lighten notation, we will use $F$ for both the morphism of sites and the induced morphism of topoi and suppress any mention of $X_s$ or $V_0$ as this is unlikely to cause any confusion.

**Definition 2.7.** An $F$-isocrystal on $X_s$ is a pair $(\mathcal{E}, \varphi)$ consisting of an isocrystal $\mathcal{E}$ on $X_s$ together with an isomorphism $\varphi: F^* \mathcal{E} \to \mathcal{E}$. A morphism between two $F$-isocrystals $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ is a morphism of isocrystals $\alpha: \mathcal{E} \to \mathcal{E}'$ such that the diagram

$$
\begin{array}{ccc}
F^* \mathcal{E} & \xrightarrow{\varphi} & \mathcal{E} \\
\downarrow \alpha & & \downarrow \alpha \\
F^* \mathcal{E}' & \xrightarrow{\varphi'} & \mathcal{E}'
\end{array}
$$

commutes. We denote the resulting category of $F$-isocrystals on $X_s$ by $F\text{-Isoc}(X_s/V_0)$.

We now want to recall a well-known concrete description of $F$-isocrystals when $X_s$ admits a smooth formal deformation to $\text{Spf}(V_0)$. In our eventual application, $X_s$ will arise as the special fiber of a smooth formal scheme $X \to \text{Spf} \mathcal{O}_K$. For motivation, we therefore first remind the reader that such an $X$ locally admits a smooth formal model over $\text{Spf} V_0$.

**Definition 2.8.** Let $X$ be a smooth affine formal scheme over $\text{Spf} \mathcal{O}_K$. A framing of $X$ is a finite étale morphism $X \to \text{Spf} \mathcal{O}_K[x_1^\pm, \ldots, x_d^\pm]$ over $\text{Spf} \mathcal{O}_K$ for some $d \in \mathbb{Z}_{\geq 0}$.

**Lemma 2.9.** Let $X$ be a smooth affine formal scheme over $\mathcal{O}_K$ and $\square: X \to \text{Spf} \mathcal{O}_K[x_1^\pm, \ldots, x_d^\pm]$ be a framing. Then there exists a unique smooth affine formal scheme $\tilde{X}$ over $V_0$ with a framing of $V_0$-formal schemes $\square: \tilde{X} \to \text{Spf} V_0[x_1^\pm, \ldots, x_d^\pm]$ whose base change to $\text{Spf} \mathcal{O}_K$ is $\square$.

**Proof.** Let $R := \Gamma(X, \mathcal{O}_X)$. For any $n \in \mathbb{N}$, the framing $\square$ induces a finite étale morphism $\square_n: \mathcal{O}_K/p^n[x_1^\pm, \ldots, x_d^\pm] \to R/p^n$. By the topological invariance of the étale site (see e.g. [Sp22, Tag 039R]), there is a unique finite étale morphism $\bar{\square}_n: V_0/p^n[x_1^\pm, \ldots, x_d^\pm] \to \bar{R}_n$ whose base change to $\mathcal{O}_K/p^n$ is $\square_n$. The uniqueness shows that the reduction of $\bar{\square}_m$ mod $p^n$ is $\square_n$ for all $m \geq n$. Taking the limit of the resulting inverse system over $n$, we obtain the desired affine formal scheme $\tilde{X}$ and framing $\square$. \hfill $\square$

**Lemma 2.10** (Kedlaya). A smooth formal scheme $X \to \text{Spf} \mathcal{O}_K$ has a basis of affine open formal subschemes that admit a framing.

**Proof.** Cf. e.g. the proof of [Bha18, Lem. 4.9]. \hfill $\square$
We return to the local description of $F$-isocrystals. Let $\tilde{X} = \text{Spf } R$ be an affine smooth $p$-adic formal scheme over $\text{Spf } V_0$ with special fiber $\tilde{X}_s := \tilde{X} \otimes_{V_0} k$ and rigid generic fiber $\tilde{X}_\eta := \tilde{X} \times_{\text{Spf } V_0} \text{Spa}(K_0, V_0)$. The absolute Frobenius $F$ on the smooth affine $\tilde{X}_s$ is flat and hence can be lifted (nonuniquely) to a Frobenius morphism on $\tilde{X}$ because the obstruction space for deformations of $F$ is given by $\text{Ext}^1_{\mathcal{O}_{\tilde{X}_s}}(LF^*L\tilde{X}_s/k, \mathcal{O}_{\tilde{X}_s}) \simeq \text{Ext}^1(F\mathcal{O}_{\tilde{X}_s}^{\mathcal{O}_{\tilde{X}_s}/k}, \mathcal{O}_{\tilde{X}_s}) \simeq 0$.

**Construction 2.11.** Let $\mathcal{E}$ be a crystal on $\tilde{X}_s$. Then $\mathcal{E}(\tilde{X}) := \lim_{\mathcal{E}(\tilde{X}_s, \tilde{X} \otimes_{V_0} V_0/p^r, \gamma)$, where $\gamma$ is the canonical pd-structure on $pR/p^r R$, is a finitely presented $R$-module equipped with a canonical connection. This induces an equivalence between the category of crystals on $\tilde{X}$ and the category of finitely presented $R$-modules $M$ together with a connection $\nabla : M \rightarrow M \otimes_\mathbb{R} \Omega^1_R$ that is integrable and topologically quasi-nilpotent. Here, the topological quasi-nilpotence of a connection is defined as follows: locally on $\tilde{X}$ take a framing $\tilde{X} \rightarrow \text{Spf}(V_0[\tilde{x}^{\pm 1}_1, \ldots, \tilde{x}^{\pm 1}_d])$, which induces a basis $dx_i$ of $\Omega^1_R$. Then for $1 \leq i \leq d$ and $m \in M$, we have $\theta^m_i(m) \in pM$ for $n \gg 0$, where $\nabla = \sum_{i=1}^d \theta_i \otimes dx_i$ for some $\theta_i : M \rightarrow M$. See for example [SP22, Tag 07J7] for details.

If $\mathcal{E}$ is an isocrystal on $\tilde{X}_s$, Construction 2.11 still gives a canonically defined finitely presented projective module over the affinoid algebra $R[1/p]$, or in other words, an $\mathcal{O}_{\tilde{X}_\eta}$-vector bundle $E$ on the rigid generic fiber $\tilde{X}_\eta$, which is still equipped with an integrable connection $\nabla : E \rightarrow E \otimes \mathcal{O}_{\tilde{X}_\eta} \Omega^1_{\tilde{X}_\eta/K_0}$.

Since $E$ is module-finite over $R[1/p]$, it also carries a complete norm $\|\cdot\|$, which is unique up to equivalence [BGR84, Prop. 3.7.3.3]. When there exists a framing $\tilde{X} \rightarrow \text{Spf}(V_0[\tilde{x}^{\pm 1}_1, \ldots, \tilde{x}^{\pm 1}_d])$ as before, the topological quasi-nilpotence of $\nabla$ implies that $\lim_{|\alpha| \rightarrow \infty} \|\theta^\alpha(e)\| = 0$ for all $e \in \Gamma(\tilde{X}_\eta, E)$.\(^2\)

Since $\lim_{n \rightarrow \infty} [p^n/n!] = 0$, this implies that $\lim_{|\alpha| \rightarrow \infty} \left\| \frac{1}{\alpha!} \theta^\alpha(e) \right\| \cdot |p|^{\eta|\alpha|} = 0$.

Lastly, assume that $(\mathcal{E}, \varphi)$ is an $F$-isocrystal. In that case, one can check that the Frobenius structure forces a stronger convergence. This motivates the following notion.

**Definition 2.12** ([Ber96, Cor. 2.2.14]). Let $\tilde{X} = \text{Spf } R$ be an affine smooth $p$-adic formal scheme over $\text{Spf } V_0$ and $E$ be a coherent $\mathcal{O}_{\tilde{X}}$-module, corresponding to a finitely presented $R[1/p]$-module with a complete norm $\|\cdot\|$. An integrable connection $\nabla : E \rightarrow E \otimes \mathcal{O}_{\tilde{X}_\eta} \Omega^1_{\tilde{X}_\eta/K_0}$ is called convergent if we have for any $e \in \Gamma(\tilde{X}_\eta, E)$ and any $0 \leq \eta < 1$

$$\lim_{|\alpha| \rightarrow \infty} \left\| \frac{1}{\alpha!} \theta^\alpha(e) \right\| \cdot \eta^{\eta|\alpha|} = 0$$

in local coordinates on $\tilde{X}_\eta$.

**Remark 2.13.** Using the usual correspondence between connections and isomorphisms of pullbacks to a first-order thickening of the diagonal in $E \times E$, one can rephrase this definition in a way that shows independence of the choice of local coordinates; see [Ber96, Def. 2.2.5].

Conversely, any vector bundle on $E$ with a convergent integrable connection arises from an $F$-isocrystal via this construction.

**Theorem 2.14** ([Ber96, Thm. 2.4.2]). Let $\tilde{X} = \text{Spf } R$ be an affine smooth $p$-adic formal scheme over $\text{Spf } V_0$ with special fiber $\tilde{X}_s := \tilde{X} \otimes_{V_0} k$ and rigid generic fiber $\tilde{X}_\eta := \tilde{X} \times_{\text{Spf } V_0} \text{Spa}(K_0, V_0)$. Fix a lift of absolute Frobenius to $\tilde{X}$, which we denote again by $F$ abusing notation. Then the functor which associates with an $F$-isocrystal $(\mathcal{E}, \varphi)$ on $\tilde{X}_s$ the vector bundle with connection on $\tilde{X}_\eta$ corresponding to $\mathcal{E}(\tilde{X}) := \lim_{\mathcal{E}(\tilde{X}_s, \tilde{X} \otimes_{V_0} V_0/p^r, \gamma)[1/p}$ induces an equivalence between

---

\(^2\)For any multiindex $\alpha \in \mathbb{Z}_{\geq 0}^d$, we use the customary notation $\theta^\alpha := \theta_1^{\alpha_1} \cdots \theta_d^{\alpha_d}, |\alpha| := \alpha_1 + \cdots + \alpha_d$, and $\alpha! := \alpha_1! \cdots \alpha_d!$.

\(^3\)Here we use that $V_0$ is unramified; if the ramification index of $V_0$ was greater than $p - 1$, the analogous limit $\lim_{n \rightarrow \infty} [\pi^n/n!] = \infty$ for a uniformizer $\pi \in V_0$. 

the category \( \mathcal{F}-\text{Isoc}(\tilde{X}_s/V_0) \) of \( \mathcal{F} \)-isocrystals on \( \tilde{X}_s \) and
the category of \( \mathcal{O}_{\tilde{X}_s} \)-vector bundles \( E \) on \( \tilde{X}_s \) together with an integrable and convergent connection \( \nabla : E \to E \otimes \mathcal{O}_{\tilde{X}_s} \Omega^1_{\tilde{X}_s} \) and an isomorphism \( \varphi : F^*E \to E \) which is compatible with connections.

**Remark 2.20.** In [Ber96], Berthelot defines a more general notion of **convergent isocrystals** without any Frobenius structure. Similarly, Ogus introduces in [Ogu84, Ogu90] the convergent site of a \( p \)-adic formal scheme \( \tilde{X} \) over Spf \( V_0 \), which is reminiscent of the crystalline site. Crystals of \( \mathcal{O}_X[1/p] \)-modules on the convergent site can be identified with convergent isocrystals on \( \tilde{X}_s \) in the sense of Berthelot and any \( F \)-isocrystal on \( \tilde{X}_s \) directly gives rise to a convergent isocrystal [Ogu84, Ex. 2.7.3]. In Ogus’s language, the additional convergence provided by the Frobenius structure is obtained via an application of Dwork’s Frobenius trick; see [Ogu84, Prop. 2.18].

**Remark 2.16.** As in [TT19, Lem. 3.5, Cor. 3.7], one can give yet another equivalent definition of \( F \)-isocrystals locally for framed \( \tilde{X} \) with Frobenius lift, as vector bundles \( E \) over the rigid generic fiber \( \tilde{X}_s \) together with an integrable convergent connection \( \nabla \) and a Frobenius morphism \( F^*E \to E \).

In general, \( X \) does not have a global smooth formal model over Spf \( V_0 \). Nevertheless, we can still define an underlying vector bundle with connection over the whole \( X_0 \).

**Proposition 2.17** ([Ogu84, Rem. 2.8.1, Thm. 2.15]). Let \( X \) be a smooth formal scheme over Spf \( \mathcal{O}_K \) with reduced special fiber \( X_s := X_{\text{red}} \) and rigid generic fiber \( X_\eta \), and let \( (\mathcal{E}, \varphi) \) be an \( F \)-isocrystal on \( X_s \). Then there is a vector bundle with integrable connection \( (E, \nabla) \) over \( X_\eta \) such that for any \( V_0 \)-model \( U \) of an affine open subset \( U \subset X \), the restriction \( (E, \nabla)|_{U_\eta} \) is isomorphic to \( (\mathcal{E}(U), \nabla_U) \otimes_{\mathcal{O}_K} K \), where \( (\mathcal{E}(U), \nabla_U) \) is the associated vector bundle with connection from Theorem 2.14.

We call \( (E, \nabla) \) the **underlying vector bundle** (with connection) of the \( F \)-isocrystal \( (\mathcal{E}, \varphi) \).

**Definition 2.18.** Let \( X \) be a smooth formal scheme over Spf \( \mathcal{O}_K \) with special fiber \( X_s := X_{\text{red}} \) and rigid generic fiber \( X_\eta := X \times_{\text{Spf } \mathcal{O}_K} \text{Spa}(K, \mathcal{O}_K) \). A **filtered \( F \)-isocrystal** on \( X \) is a triple \( (\mathcal{E}, \varphi, \text{Fil}^*)(E) \) such that \( (\mathcal{E}, \varphi) \) is an \( F \)-isocrystal on \( X_s \) and \( \text{Fil}^*(E) \) is a separated, exhaustive, locally split, decreasing filtration of the underlying vector bundle \( (E, \nabla) \) of \( (\mathcal{E}, \varphi) \) which satisfies the Griffiths transversality condition

\[
\nabla(\text{Fil}^*(E)) \subseteq \text{Fil}^{*-1}(E) \otimes \mathcal{O}_{X_\eta} \Omega^1_{X_\eta}.
\]

### 2.3. **Crystalline period sheaves and crystalline local systems**

Finally, we treat the sheaf versions of Fontaine’s crystalline period ring.

**Definition 2.19** ([TT19, §2A]). Let \( U \in \text{Perfd }/X_{\eta, \text{proét}} \) with \( \tilde{U} = \text{Spa}(R, R^+) \) a perfectoid \( C \)-algebra and let \( \xi \in A_{\text{inf}}(R^+) \) be a generator of the de Rham specialization map \( \theta : A_{\text{inf}}(R^+) \to R^+ \). The filtered sheaf \( A_{\text{crys}} \) resp. \( B^+_{\text{crys}} \) resp. \( B_{\text{crys}} \) in \( \text{Shv}(X_{\eta, \text{proét}}) \) is given on \( U \) as follows:

- \( A_{\text{crys}}(R, R^+) \) is the \( p \)-completion of the pd-envelope of \( (\xi) \) in \( A_{\text{inf}}(R, R^+) \) with the filtration given by the \( p \)-completions of the divided power ideals \((\xi^i)^\wedge \) for \( i \in \mathbb{Z}_{\geq 0} \) resp.
- \( B^+_{\text{crys}}(R, R^+) := A_{\text{crys}}(R, R^+)[1/p] \) with the induced filtration resp.
- \( B_{\text{crys}}(R, R^+) := B^+_{\text{crys}}(R, R^+)\xi^{-1} \) with the filtration

\[
\text{Fil}^+ B_{\text{crys}}(R, R^+) = \sum_{i \in \mathbb{Z}} \xi^{-i} \cdot \text{Fil}^{i++} B^+_{\text{crys}}(R, R^+).
\]

**Remark 2.20.** Under our definition of de Rham period sheaves (cf. Warning 2.4), there is a natural filtered map of period sheaves \( \varphi^*B_{\text{crys}} \to B_{\text{dR}} \).
Unfortunately, the definitions of the versions “with connections” $\mathcal{O}_{\text{crys}}$ resp. $\mathcal{O}_{\text{crys}}^+$ resp. $\mathcal{O}_{\text{crys}}$ need the assumption that the ground field $K$ is absolutely unramified. To treat the ramified case, we use Faltings’s definition of crystalline local systems [Fal89], which we recall now.

We assume again that $X_0$ arises as the rigid generic fiber of a smooth formal scheme $X$ over $\text{Spf} \mathcal{O}_K$ with special fiber $X^\text{red} := X_0$.

**Definition 2.21.** Let $\mathcal{E} \in \text{Isoc}(X_0/V_0)$ be an isocrystal on $X_0$. Then we define sheaves $\mathcal{A}_{\text{crys}}(\mathcal{E})$ and $\mathcal{B}_{\text{crys}}(\mathcal{E})$ in $\text{Shv}(X_0/\text{proét})$ as follows: given an element $U \in \text{Perfd}/X_0/\text{proét}$ with $\tilde{U} = \text{Spa}(S, S^+)$, as the morphism $\mathcal{A}_{\text{crys}}(S, S^+) \to S^+/p$ is a pro-pd-thickening in $\text{CRIS}(X_0/V_0)$, we can set

- $\mathcal{A}_{\text{crys}}(\mathcal{E})(U) := \mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)) = (\lim_{\alpha} \mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)/p, \mathcal{A}_{\text{crys}}(S, S^+)/p^\gamma), \gamma)$, where $\gamma$ is the canonical pd-structure, and
- $\mathcal{B}_{\text{crys}}(\mathcal{E})(U) := \mathcal{A}_{\text{crys}}(\mathcal{E})(U)[\xi^{-1}]$.

**Lemma 2.22.** The functor $\mathcal{B}_{\text{crys}}(\mathcal{E})$ from Definition 2.21 is a sheaf.

**Proof.** It suffices to prove the statement for $\mathcal{A}_{\text{crys}}(\mathcal{E})$. Given a prétale cover $\{f_\alpha : U_\alpha \to U\}$ in $\text{Perfd}/X_0/\text{proét}$ with $\tilde{U} = \text{Spa}(S_\alpha, S_\alpha^+)$, we need to show that

\[
\mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)) \to \prod_{\alpha} \mathcal{E}(\mathcal{A}_{\text{crys}}(S_\alpha, S_\alpha^+)) \to \prod_{\alpha, \beta} \mathcal{E}(\mathcal{A}_{\text{crys}}(S_\alpha \otimes SS_\beta, (S_\alpha \otimes SS_\beta)^+))
\]

is an equalizer diagram. Since $\mathcal{A}_{\text{crys}}$ is a sheaf on $\text{Perfd}/X_0/\text{proét}$, we know that

\[
\mathcal{A}_{\text{crys}}(S, S^+) \to \prod_{\alpha} \mathcal{A}_{\text{crys}}(S_\alpha, S_\alpha^+) \to \prod_{\alpha, \beta} \mathcal{A}_{\text{crys}}(S_\alpha \otimes SS_\beta, (S_\alpha \otimes SS_\beta)^+)
\]

is an equalizer diagram. On the other hand, since $\mathcal{E}$ is an isocrystal, we have natural isomorphisms $\mathcal{E}(\mathcal{A}_{\text{crys}}(S_\alpha, S_\alpha^+)) \simeq \mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)) \otimes \mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)) \mathcal{A}_{\text{crys}}(S, S^+)$, and likewise for $S_\alpha \otimes SS_\beta$. Thus, Equation (1) can be obtained by tensoring Equation (2) over $\mathcal{A}_{\text{crys}}(S, S^+)$ with $\mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+))$.

Since $\mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+))[1/p]$ is a finite projective $\mathcal{A}_{\text{crys}}(S, S^+)[1/p]$-module, the tensor product is exact after inverting $p$, so $\mathcal{B}_{\text{crys}}(\mathcal{E})$ is a sheaf as desired. \qed

**Remark 2.23.** Note that $\mathcal{B}_{\text{crys}}(\mathcal{E})$ is a module over the sheaf of rings $\mathcal{B}_{\text{crys}}$, and $\mathcal{B}_{\text{crys}}$ is equipped with a Frobenius endomorphism $F: \mathcal{B}_{\text{crys}} \to \mathcal{B}_{\text{crys}}$. Pullback along $F$ is compatible with Frobenius pullback in the special fiber in the sense that there is a canonical isomorphism $F^* \mathcal{B}_{\text{crys}}(\mathcal{E}) \simeq \mathcal{B}_{\text{crys}}(F^* \mathcal{E})$. If $(\mathcal{E}, \varphi) \in F^{-1}\text{Isoc}(X_0/V_0)$, we therefore obtain an induced Frobenius isomorphism

$$\varphi: F^* \mathcal{B}_{\text{crys}}(\mathcal{E}) \simeq \mathcal{B}_{\text{crys}}(F^* \mathcal{E}) \subseteq \mathcal{B}_{\text{crys}}(\mathcal{E}).$$

**Remark 2.24.** If $\text{Fil}^*(E)$ is a filtration on the underlying vector bundle $E$ on $X$ from Definition 2.18, we can define a filtration on $\mathcal{B}_{\text{crys}}(\mathcal{E}) \otimes_{K_0} K$ as follows: Assume first that $X$ is affine framed with a model $\tilde{X} = \text{Spf} R^+$ over $V_0$ and a framing $V_0(x_1^{\pm 1}, \ldots, x_d^{\pm 1}) \to R^+$. Choose a compatible system $x_i^p$ of $p$-th power roots of the $x_i$ in $S^+$. By the infinitesimal lifting criterion of formal étaleness, we can then extend the map

$$V_0(x_1^{\pm 1}, \ldots, x_d^{\pm 1}) \to \mathcal{A}_{\text{crys}}(S, S^+), \quad x_i \to [x_i^p]$$

to a map $\alpha: R^+ \to \mathcal{A}_{\text{crys}}(S, S^+)$ because $\ker(\mathcal{A}_{\text{crys}}(S, S^+)/p^n \to S^+/p^n)$ is a nil ideal for all $n$.

The crystal property of $\mathcal{E}$ applied to the resulting morphism of pd-thickenings $(R^+ \to R^+/p) \to (\mathcal{A}_{\text{crys}}(S, S^+))$ furnishes an isomorphism

$$\mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)) \otimes_{K_0} K \simeq E(X) \otimes_{\mathcal{O}(X)} (\mathcal{B}_{\text{crys}}(S, S^+) \otimes_{K_0} K).$$

We can equip $\mathcal{A}_{\text{crys}}(S, S^+) \otimes_{V_0} \mathcal{O}_K$ with the $J$-adic filtration, where $J$ is the kernel of the natural map $\mathcal{A}_{\text{crys}}(S, S^+) \otimes_{V_0} \mathcal{O}_K \to S^+$, and then $\mathcal{B}_{\text{crys}}(S, S^+) \otimes_{K_0} K = (\mathcal{A}_{\text{crys}}(S, S^+) \otimes_{V_0} \mathcal{O}_K)[\xi^{-1}]$ with the induced filtration given by $\text{Fil}^*(\mathcal{B}_{\text{crys}}(S, S^+) \otimes_{K_0} K) = \sum_{s \in \mathbb{Z}} \xi^{-s} \text{Fil}^*(\mathcal{A}_{\text{crys}}(S, S^+) \otimes_{V_0} \mathcal{O}_K)$. The filtration on $\mathcal{B}_{\text{crys}}(\mathcal{E})(S, S^+) \otimes_{K_0} K$ is the tensor product filtration of $\text{Fil}^*(E)$ and this filtration...
on $\mathbb{B}_{crys}(S, S^+)\otimes_{K_0} K$, using the formula in the equation above. Moreover, [TT19, Rem. 3.20] shows that it is independent of the choice of the morphism $\alpha$ as long as the composition with the natural map $\mathcal{A}_{crys}(S, S^+) \to S^+$ is identified with the inclusion. Therefore, we can glue the filtrations coming from different affine framed subschemes of $X$ to a filtration on $\mathbb{B}_{crys}(E) \otimes_{K_0} K$.

**Definition 2.25** ([Fal89, p. 67]). Let $X \to \text{Spf} \mathcal{O}_K$ be a smooth formal scheme, let $X_s$ be its reduced special fiber, and let $X_\eta$ be its generic fiber, considered as an adic space over $\text{Spa}(K, \mathcal{O}_K)$. A sheaf of $\mathbb{Z}_p$-modules $L$ on $X_{\eta, \text{proet}}$ is **crystalline** if

(i) $L$ is **lisse**, that is, locally on $X_{\eta, \text{proet}}$ of the form $\mathbb{Z}_p \otimes \mathbb{Z}_p M$ for some finitely generated $\mathbb{Z}_p$-module $M$, and

(ii) there exists a filtered convergent $F$-isocrystal $(\mathcal{E}, \varphi, \text{Fil}^i(\mathcal{E}))$ on $X$ in the sense of Definition 2.18 and an isomorphism

$$\vartheta: \mathbb{B}_{crys}(\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{crys} \otimes_{\mathbb{Z}_p} L$$

such that $\vartheta$ commutes with the Frobenius isomorphisms on both sides (cf. Remark 2.23) and $\vartheta \otimes_{K_0} K: \mathbb{B}_{crys}(\mathcal{E}) \otimes_{K_0} K \xrightarrow{\sim} (\mathbb{B}_{crys} \otimes_{K_0} K) \otimes_{\mathbb{Z}_p} L$ sends the filtration described in Remark 2.24 to the filtration coming from the first factor.

It is called a **crystalline local system** if $L$ is furthermore a $\mathbb{Z}_p$-local system (equivalently, torsionfree).

**Remark 2.26.** In Definition 2.25.(ii), one can give an equivalent description of the filtration condition in terms of a $\mathbb{B}_{dR}$-linear filtered isomorphism

$$\mathbb{B}_{crys}(\mathcal{E}) \otimes_{\mathbb{B}_{crys}} \mathbb{B}_{dR} \simeq \mathbb{B}_{dR} \otimes_{\mathbb{Z}_p} L,$$

where the filtration on $\mathbb{B}_{crys}(\mathcal{E}) \otimes_{\mathbb{B}_{crys}} \mathbb{B}_{dR}$ is defined locally as the product filtration on $E(X) \otimes_{\mathcal{O}(X)} \mathbb{B}_{dR}(S, S^+)$ as in Remark 2.24.

To see this, we first notice that by [Fal89, pp. 29–30], the filtration on $\mathbb{B}_{crys} \otimes_{K_0} K$ is separated and the map $\mathbb{B}_{crys} \otimes_{K_0} K \to \mathbb{B}_{dR}$ induces an isomorphism on each graded piece. Hence, this map is an inclusion and we have

$$\text{Fil}^i(\mathbb{B}_{crys} \otimes_{K_0} K) = \text{Fil}^i \mathbb{B}_{dR} \cap (\mathbb{B}_{crys} \otimes_{K_0} K).$$

Now assume the same setup as in Definition 2.25, only with the filtered isomorphism $\vartheta \otimes_{K_0} K$ of (ii) replaced by Equation (3). Since $p$ is invertible in $\mathbb{B}_{crys}$ and $\mathbb{B}_{dR}$, we can write

$$L \otimes (\text{Fil}^i \mathbb{B}_{crys} \otimes_{K_0} K) = (L \otimes \text{Fil}^i \mathbb{B}_{dR}) \cap (L \otimes \mathbb{B}_{crys} \otimes_{K_0} K)$$

$$\simeq (\text{Fil}^i(\mathbb{B}_{crys}(\mathcal{E}) \otimes_{K_0} K \otimes \mathbb{B}_{dR})) \cap (\mathbb{B}_{crys}(\mathcal{E}) \otimes_{K_0} K).$$

On the other hand, by Remark 2.24, locally for a $V_0$-model $R^+$ of $X$ that maps to $\mathcal{A}_{crys}(S, S^+)$ for a perfectoid $\mathcal{O}_{C}$-algebra $(S, S^+)$, the filtrations on $\mathbb{B}_{crys}(\mathcal{E})(S, S^+) \otimes_{K_0} K$ and $(\mathbb{B}_{crys}(\mathcal{E})(S, S^+) \otimes_{K_0} K) \otimes \mathbb{B}_{dR}$ are given by the product filtrations on

$$E(X) \otimes_{\mathcal{O}(X)} (\mathbb{B}_{crys}(S, S^+) \otimes_{K_0} K) \quad \text{and} \quad E(X) \otimes_{\mathcal{O}(X)} \mathbb{B}_{dR}(S, S^+).$$

In particular, as the filtration on the vector bundle $E$ locally splits, we obtain after localizing $(S, S^+)$ further if necessary

$$\text{Fil}^i(\mathbb{B}_{crys}(\mathcal{E})(S, S^+) \otimes_{K_0} K \otimes \mathbb{B}_{dR}(S, S^+)) \cap (\mathbb{B}_{crys}(\mathcal{E})(S, S^+) \otimes_{K_0} K)$$

$$\simeq \text{Fil}^i(E(X) \otimes_{\mathcal{O}(X)} \mathbb{B}_{dR}(S, S^+)) \cap (E(X) \otimes_{\mathcal{O}(X)} (\mathbb{B}_{crys}(S, S^+) \otimes_{K_0} K))$$

$$\simeq \text{Fil}^i(E(X) \otimes_{\mathcal{O}(X)} (\mathbb{B}_{crys}(S, S^+) \otimes_{K_0} K))$$

$$= \text{Fil}^i(\mathbb{B}_{crys}(\mathcal{E})(S, S^+) \otimes_{K_0} K).$$
Since these isomorphisms are functorial in \((S,S^+)\), we can sheafify and get \(\text{Fil}^i(B_{\text{crys}}(E) \otimes K) \to \text{Fil}^i(B_{\text{crys}}(E) \otimes K)\). Combining this with the equations on \(L \otimes (\text{Fil}^i B_{\text{crys}} \otimes K)\) as above, we get the filtered isomorphism as in Definition 2.25.(ii).

**Remark 2.27.** By Lemma 2.9, any framed open \(U \subset X\) has a model \(U = \text{Spf}(R^+)\) over \(\text{Spf} V_0\). The \(p\)-completion of the normalization of \(R^+\) in the maximal étale extension of \(\tilde{R} := R^+[1/p]\) defines a perfectoid cover \((\tilde{R}, R^+) \to (S, S^+)\) of the rigid fiber \(\tilde{U}_\eta\) of this model. Let \(G_{\tilde{U}_\eta}\) be the continuous Galois group of the cover. It is enough to give compatible (between the \(U\)) isomorphisms

\[
\mathcal{E}(B_{\text{crys}}(S, S^+)) \xrightarrow{\sim} B_{\text{crys}}(S, S^+) \otimes_{\mathbb{Z}_p} L(\text{Spa}(S, S^+))
\]

that preserve the \(G_{\tilde{U}_\eta}\)-action, Frobenius, and filtrations after \(\otimes K_0 K\).

Adapting the proof of [TT19, Prop. 3.21], we can show the following compatibility.

**Proposition 2.28.** Let \(X \to \text{Spf} O_K\) be a smooth formal scheme and \(X_s, X_\eta\) as above. Let \(L\) be a \(\mathbb{Z}_p\)-lisse sheaf on \(X_\eta\). If \(L\) is crystalline, then \(L\) is de Rham in the sense of Definition 2.5.

**Proof.** As the question whether \(L\) is de Rham is local on \(X\), we may assume that \(X\) is affine and framed. Let \(\tilde{X} = \text{Spf}(R^+)\) be a \(V_0\)-model of \(X\) with generic fiber \(\tilde{X}_\eta\) over \(K_0\). Let \((E, \varphi, \text{Fil}^\dR(E))\) be a filtered \(F\)-isocrystal on \(X\) as in Definition 2.25. We divide the proof into several parts.

Step 1 We first construct an isomorphism \(L \otimes \mathbb{Z}_p \mathcal{O}_{\mathcal{B}_{\text{dR}}} \simeq E \otimes_{O_{X_\eta}} \mathcal{O}_{\mathcal{B}_{\text{dR}}}\). Let \(U \in \text{Perfd}_{/X_\eta, \text{pro} \acute{e}t}\) with \(\bar{U} = \text{Spa}(S, S^+)\). We have a commutative diagram of pro-pd-thickenings

\[
\begin{array}{c}
\begin{array}{ccc}
R^+ & \rightarrow & R^+/p \\
\downarrow & & \downarrow \\
O_{\text{crys}}(S, S^+) & \rightarrow & S^+/p \\
\Uparrow & & \Uparrow \\
\mathcal{A}_{\text{crys}}(S, S^+) & \rightarrow & S^+/p
\end{array}
\end{array}
\]

where we use the crystalline period sheaf with connection \(O_{\mathcal{A}_{\text{crys}}}\) from [TT19, Def. 2.9] for the \(V_0\)-model \(\tilde{X}\). Therefore, the crystal property of \(\mathcal{E}\) applied to this diagram yields as in [TT19, (3B.5)] a natural isomorphism

\[
(4) \quad \mathcal{E}(\mathcal{A}_{\text{crys}}(S, S^+)) \otimes_{\mathcal{A}_{\text{crys}}(S, S^+)} O_{\mathcal{A}_{\text{crys}}}(S, S^+) \simeq \mathcal{E}(O_{\mathcal{A}_{\text{crys}}}(S, S^+)) \simeq \mathcal{E}(R^+) \otimes R^+ O_{\mathcal{A}_{\text{crys}}}(S, S^+)
\]

isomorphisms that are functorial in \((S, S^+)\). Inverting \(\xi\) and tensoring along the map \(\varphi^\ast O_{\mathcal{B}_{\text{crys}}} \rightarrow O_{\mathcal{B}_{\text{dR}}}\) to the de Rham period sheaf with connection on \(X\), we obtain a natural isomorphism

\[
\eta: (B_{\text{crys}}(E) \otimes K) \otimes (B_{\text{crys}} \otimes K_0 K) \otimes B_{\text{dR}} \simeq E \otimes O_{\mathcal{A}_{\text{crys}}} O_{\mathcal{B}_{\text{dR}}}
\]

of sheaves on \(X\). The composition with the isomorphism \(\vartheta^{-1} \otimes B_{\text{dR}}\) from Definition 2.25 yields the desired isomorphism \(L \otimes B_{\text{dR}} \simeq E \otimes_{O_{X_\eta}} O_{\mathcal{B}_{\text{dR}}}\).

Step 2 Next, we prove that the above \(O_{\mathcal{B}_{\text{dR}}}\)-linear isomorphism is compatible with connections.

As \(B_{\mathcal{B}_{\text{dR}}}(E) \xrightarrow{\sim} B_{\mathcal{B}_{\text{dR}}} \otimes \mathbb{Z}_p L\) are the flat sections of \(L \otimes \mathbb{Z}_p O_{\mathcal{B}_{\text{dR}}}\), it suffices to show that \(B_{\mathcal{B}_{\text{dR}}}(E)\) are the flat sections of the product connection on \(E \otimes O_{X_\eta} O_{\mathcal{B}_{\text{dR}}}\).

It suffices to see this on the \(\mathcal{A}_{\text{crys}}\)-linear level, and we use the crystal property to describe the product connection as follows: Denote by \(O_{X_\eta}^\dR\) the pullback of \(O_{X_\eta}\) along \(X_\eta\)-proét \(X_\eta\). Define \(O_{\mathcal{A}_{\text{crys}}}\) to be the \(p\)-completed pd-envelope of \(O_{\mathcal{A}_{\text{inf}}} := O_{X_\eta}^\dR \otimes_{V_0} O_{X_\eta}^\dR \otimes V_0\) with respect to the kernel of the natural map \(O_{\mathcal{A}_{\text{inf}}} \rightarrow O_{X_\eta}^\dR\).
Denote the two natural inclusions by $p_1, p_2: \mathcal{O}_{\mathcal{H}^{\text{crys}}_S} \to \mathcal{O}_{\mathcal{H}^{\text{crys}}_S}$; the crystal condition then defines a natural isomorphism

$$c: \mathcal{E}(\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)) \otimes_{\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)}} \mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)} \simeq \mathcal{E}(\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)) \otimes_{\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)}} \mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)},$$

and the difference with the map $\hat{p}_2$ gives

$$\nabla: \mathcal{E}(\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)) \to \mathcal{E}(\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)) \otimes_{\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)}} \Omega^{1, \text{ur}}_{X^\alpha}, s \mapsto p_2^s - c(p_1^s).$$

In particular, using a local coordinate calculation, the above description implies that the flat sections of this connection are $\mathcal{H}^{\text{crys}}(\mathcal{E})$.

**Step 3** Lastly, we show that $\eta$ (and thus $(\Theta^{-1} \otimes \mathcal{B}_{dR}) \circ \eta$) is compatible with filtrations. In Remark 2.24, we chose a map $\beta_0: R^+ \to \mathcal{H}^{\text{crys}}(S, S^+)$ and used it to define a filtration on $\mathcal{B}_{\text{crys}}(\mathcal{E}) \otimes K_0$ (independent of $\beta_0$) via the isomorphism $\mathcal{E}(\mathcal{B}_{\text{crys}}(S(S^+)) \otimes K_0 K \simeq E(X_\eta) \otimes (\mathcal{B}_{\text{crys}}(S(S^+)) \otimes K_0 K)$ coming from the crystal property of $\mathcal{E}$. Let $\beta_1$ be the composition of $\beta_0$ with the natural map $\mathcal{H}^{\text{crys}}(S, S^+) \to \mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)$, let $\beta_2: R^+ \to \mathcal{H}^{\text{crys}}(S, S^+)$ be the natural inclusion, and let $\beta := (\beta_1, \beta_2): R^+ \otimes V_0 R^+ \to \mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)$. There is a diagram of pro-pd-thickenings

$$\begin{array}{ccc}
\mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)} & \xrightarrow{\rho} & S^+/p \\
\beta \downarrow & & \uparrow \\
D(1) & \xrightarrow{p_1} & R^+/p \\
p_1 & \downarrow & \\
R^+ & \xrightarrow{p_2} & R^+/p,
\end{array}$$

where $D(1)$ is the $p$-completed pro-pd-envelope of $R^+ \otimes R^+$ with respect to the multiplication map $R^+ \otimes R^+ \to R^+/p$, and $p_1$ resp. $p_2$ denotes the map induced by the inclusion $r \mapsto r \otimes 1$ resp. $r \mapsto 1 \otimes r$ into the first resp. second factor.

The crystal property of $\mathcal{E}$ then furnishes an isomorphism $(\beta \circ p_2)^* \mathcal{E} \simeq (\beta \circ p_1)^* \mathcal{E}$. Via the usual correspondence of crystals and vector bundles with connections, it is given by the formula

$$1 \otimes s \mapsto \sum_{\alpha \in \mathbb{Z}^{d}_{\geq 0}} \theta^\alpha(s) \otimes \beta(1 \otimes x - x \otimes 1)^{[\alpha]},$$

where $\beta(1 \otimes x - x \otimes 1)^{[\alpha]} := (1 \otimes \beta_2(x_1) - \beta_1(x_1) \otimes 1)^{[\alpha_1]} \cdots (1 \otimes \beta_2(x_d) - \beta_1(x_d) \otimes 1)^{[\alpha_d]}$ for coordinates $x_1, \ldots, x_d$ coming from a framing $V_0(x_1, \ldots, x_d) \to R^+$ and $\nabla = \sum_{i=1}^d \theta_i \otimes dx_i$; cf. [Ber96, (2.2.4)]. After tensoring with $K$, we obtain an isomorphism

$$\begin{align*}
(\mathcal{O}_{\mathcal{B}_{\text{crys}}(S, S^+) \otimes K_0} K) \otimes_{\mathcal{O}(X)} E(X) & \xrightarrow{\sim} E(X) \otimes_{\mathcal{O}(X)} \mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)) \otimes K_0 K}
\end{align*}$$

given by a similar formula. For any $s \in \text{Fil}^i E(X)$ and any $\alpha \in \mathbb{Z}^{d}_{\geq 0}$, we have $\theta^\alpha(s) \in \text{Fil}^{i-\alpha} E(X)$ by Griffiths transversality. On the other hand, $(1 \otimes \beta_2(x_i) - \beta_1(x_i) \otimes 1)$ lies in $\ker \rho$, so $\beta(1 \otimes x - x \otimes 1)^{[\alpha]} \in \text{Fil}^{[\alpha]} \mathcal{O}_{\mathcal{H}^{\text{crys}}_S(S, S^+)$ for all $\alpha \in \mathbb{Z}^{d}_{\geq 0}$. Thus, the isomorphism in Equation (5) is compatible with the product filtrations on both sides, and so is $\eta$ after extending scalars to $\mathcal{O}_{dR}$.

**3. Prismatic $F$-crystal over the analytic locus**

Let $X$ be a quasi-syntomic $p$-adic formal scheme over $\mathcal{O}_K$. Recall from [BS19] and [BS21] that we can attach to $X$ the following two ringed sites:
• (Absolute prismatic site) The opposite category $X_\Delta$ of bounded prisms $(A, I)$ over $X$ with the $(p, I)$-complete flat topology, equipped with the sheaf of rings

$$\mathcal{O}_\Delta: (A, I) \mapsto A.$$ 

The prismatic structure sheaf $\mathcal{O}_\Delta$ admits a natural Frobenius action which lifts that of the reduction mod $p$. 

• (Quasi-syntomic site) The opposite category $X_{\text{qrsp}}$ of quasiregular semiperfectoid algebras over $X$ with the quasi-syntomic topology, equipped with the sheaf of rings

$$\Delta_\bullet: S \mapsto \Delta_S,$$

sending $S \in X_{\text{qrsp}}$ to the initial prism of $\text{Spf}(S)_\Delta$ as in [BS19, Prop. 7.2]. The Frobenius endomorphism of prisms induces a natural endomorphism $\varphi$ of $\Delta_\bullet$.

By a slight abuse of notation, we use $\mathcal{I}_S$ to denote the ideal sheaf over either of the above sites which sends a prism $(A, I)$ to the ideal $I$.

A typical example that plays a major part in next two sections is the ring $\Delta_S$ for a perfectoid $\mathcal{O}_C$-algebra $S$. This leads to the $A_{\text{inf}}$-prism, which we briefly recall below.

**Example 3.1.** Choose a compatible system $\zeta^{p^n}$ of $p^n$-th roots of unity in $\mathcal{O}_C$, and let $\epsilon = (1, \zeta_p, \ldots)$ be the corresponding element of $\mathcal{O}_C^\times = \lim_{\rightarrow \to} \mathcal{O}_C/p$. The initial prism $\Delta_{\mathcal{O}_C}$ of $\mathcal{O}_C$ is given by the ring $A_{\text{inf}}(\mathcal{O}_C) := W(\mathcal{O}_C)$. Its associated ideal $I$ is generated by $\tilde{\xi} := [p]_q = \frac{q^p - 1}{q - 1}$ for $q = \lceil \epsilon \rceil$. We also denote by $\mu$ the element $q - 1$, which satisfies the relation $\varphi(\mu) = \mu \cdot \tilde{\xi}$. One can then further associate various localizations to $\Delta_{\mathcal{O}_C}$ as below

$$\Delta_{\mathcal{O}_C} = A_{\text{inf}} \longrightarrow \Delta_{\mathcal{O}_C}\{\tilde{\xi}/p\} \longrightarrow A_{\text{crys}} [1/\mu] = B_{\text{crys}} \longrightarrow \Delta_{\mathcal{O}_C}[1/\tilde{\xi}]^\wedge_p = B_{\text{dR}},$$

where the third map is the composition of Frobenius endomorphism and the natural inclusion (cf. Warning 2.4), and the diagram is compatible with our setting in Section 2.

3.1. **Analytic prismatic $F$-crystals.** As described in [BS21, Def. 4.1, Constr. 6.2], the ringed sites $X_\Delta$ and $X_{\text{qrsp}}$ naturally come with a notion of vector bundles and $F$-crystals on them. Thanks to [BS19, Prop. 7.11], they are related by the equivalences

$$\text{Vect}(X_\Delta, \mathcal{O}_\Delta) \simeq \text{Vect}(X_{\text{qrsp}}, \Delta_\bullet) \quad \text{and}$$

$$\text{Vect}^p(X_\Delta, \mathcal{O}_\Delta) \simeq \text{Vect}^p(X_{\text{qrsp}}, \Delta_\bullet);$$

see [BS21, Prop. 2.14, Remark 6.3]. Moreover, following [BS21, Prop. 2.7, Prop. 2.13] we can describe the above categories using limit presentations:

$$\text{Vect}(X_\Delta, \mathcal{O}_\Delta) = \lim_{(A, I) \in X_\Delta} \text{Vect}(A) \quad \text{and}$$

$$\text{Vect}(X_{\text{qrsp}}, \Delta_\bullet) = \lim_{S \in X_{\text{qrsp}}} \text{Vect}(\Delta_S).$$

Similarly, one can obtain the derived analogs for perfect complexes. We will freely use the above equivalent definitions of prismatic crystals and prismatic $F$-crystals throughout the article.

Analogously to the above constructions, we introduce a notion of prismatic ($F$-)crystals on $X_\Delta$ and $X_{\text{qrsp}}$ that are only defined over the analytic locus $\text{Spec}(A) \setminus V(p, I)$ for each $(A, I) \in X_\Delta$. We start with vector bundles.

**Definition 3.2.** The category of **analytic prismatic vector bundles** over $X$ is defined as

$$\text{Vect}^{an}(X_\Delta) := \lim_{(A, I) \in X_\Delta} \text{Vect}(\text{Spec}(A) \setminus V(p, I)).$$
Remark 3.3. By a result of Drinfeld–Mathew [Mat22, Thm. 7.8], the analytic vector bundle functor
\[ A \mapsto \text{Vect}(\text{Spec}(A_{(p, I)}) \setminus V(p, I)) \]
is a sheaf for the \((p, I)\)-completely flat topology on \(A\) (cf. [Mat22, Def. 7.1]). In particular, analytic vector bundles satisfy \((p, I)\)-completely flat descent among prisms.

In the next observation, we reinterpret analytic prismatic vector bundles in terms of finitely presented modules over \(\mathcal{O}_{\Delta}\), when restricted to a prism of small size.

Lemma 3.4. Let \((A, I)\) be a prism in \(X_{\Delta}\), such that \(A\) is noetherian and \(A/I\) has no \(p\)-torsion. Then for a vector bundle \(E\) over \(U = \text{Spec}(A) \setminus V(p, I)\), its global sections \(H^0(U, E)\) are a finitely presented \(A\)-module.

Proof. Let \(j: U \to \text{Spec}(A)\) be the open immersion. It suffices to show that \(j_*E\) is coherent. As \(E\) is a vector bundle over \(U\), its only associated points are the generic points of the irreducible components of \(U\). On the other hand, since \(U\) is the complement of the closed subset \(V(p, I)\), which is of codimension 2 in \(\text{Spec}(A)\) by assumption, we have that \(j_*E\) is a coherent sheaf over \(\text{Spec}(A)\) by [SP22, Tags 0BK1, 0BJZ]. As a consequence, the global sections \(H^0(U, E) = H^0(\text{Spec}(A), j_*E)\) are a finitely presented \(A\)-module. \(\square\)

Example 3.5. When \(X\) is \(p\)-adic smooth formal scheme over \(\mathcal{O}_K\), we can always find prisms as in Lemma 3.4 which cover the final object of \(\text{Shv}(X_{\Delta})\) locally. For example, when \(\mathcal{O}_K = W(k)\) is absolutely unramified, we may take a framed open affine subset \(\text{Spf}(R) \to \text{Spf}(W(k)\langle T_i^{\pm 1}\rangle)\) and define a \(\delta\)-structure on the ring \(R[u]\) using the Frobenius of \(W(k)\) and the \(p\)-power map on coordinates. For general base \(\mathcal{O}_K\), one can consider the higher-dimensional analog of the Breuil–Kisin prism as in [DLMS22].

One can also give a presentation of analytic prismatic crystals in terms of the quasi-syntomic site.

Lemma 3.6. Define the category \(\text{Vect}^\text{an}(X_{\text{qsp}})\) to be \(\lim_{S \in X_{\text{qsp}}} \text{Vect}(\text{Spec}(\Delta_S) \setminus V(p, I))\). Then there is a natural equivalence of categories
\[ \text{Vect}^\text{an}(X_{\Delta}) \simeq \text{Vect}^\text{an}(X_{\text{qsp}}) \]
and a similar equivalence for the derived (\(\infty\))-categories of perfect complexes.

Proof. It follows from [BS19, Prop. 7.11] that given a quasi-syntomic map (resp. cover) \(S \to S'\) in \(X_{\text{qsp}}\), the associated map of initial prisms \(\Delta_S \to \Delta_{S'}\) is \((p, I)\)-completely flat (resp. faithfully flat).\(^4\) In particular, by [Mat22, Thm. 7.8] the functor
\[ S \mapsto \text{Vect}(\text{Spec}(\Delta_S) \setminus V(p, I)) \]
is a sheaf over \(X_{\text{qsp}}\). Note that one can find a quasi-syntomic hypercovering \(\text{Spf}(S)\) of \(X\) such that each \(S_n\) is quasiregular semiperfectoid. Moreover, the opposite of \(\Delta_{S_n}\) covers the final object of the prismatic topos \(\text{Shv}(X_{\Delta})\), so by Remark 3.3 the category \(\text{Vect}^\text{an}(X_{\Delta})\) is computed by vector bundles over \(\Delta_{S_n}\). Thus, the claim follows from the equivalences
\[ \text{Vect}^\text{an}(X_{\Delta}) \simeq \lim_{[n] \in \Delta} \text{Vect}(\text{Spec}(\Delta_{S_n} \setminus V(p, I))) \simeq \text{Vect}^\text{an}(X_{\text{qsp}}). \] \(\square\)

To define the notion of analytic prismatic \(F\)-crystals, we need the following small preparation.

\(^4\)To see this, first note that since the ring \(\bar{\mathbb{Z}}_S \otimes_\mathbb{Z} S'\) is relatively semiperfect over \(\bar{\mathbb{Z}}_S\) mod \(p\), the proof of [BS19, Prop. 7.11] shows that \(B := \Delta^\wedge_{\bar{\mathbb{Z}}_S \otimes_\mathbb{Z} S'}\) gives a \((p, I)\)-completely flat prism \((B, IB)\) over \((\Delta_S, I)\). As the derived functor \(\Delta^\wedge\) is symmetric monoidal and \(S \to S'\) is \(p\)-completely flat, \(\Delta^\wedge_{\bar{\mathbb{Z}}_S \otimes_\mathbb{Z} S'} \simeq \Delta^\wedge_{\bar{\mathbb{Z}}_S} \otimes_{\Delta^\wedge_S} \Delta_{S'} \simeq \Delta_S \otimes_{\Delta_S} \Delta_{S'} \simeq \Delta_{S'}\); here, we use that \(\Delta^\wedge_{\bar{\mathbb{Z}}_S}\) satisfies the universal property of the initial prism \(\Delta_S\).
**Lemma 3.7.** The Frobenius map \( \varphi \) of a prism \( (A, I) \) induces a natural endomorphism of the closed subscheme \( V(p, I) \) and its complement in \( \text{Spec}(A) \).

**Proof.** It suffices to show that the ideal \( (p, \varphi(I)) \) is contained in \( (p, I) \). This follows from the formula
\[
\varphi(d) = d^p + pd\delta(d) \in (p, d) \quad \forall d \in I. \tag*{\qed}
\]

**Definition 3.8.** (i) Let \( (A, I) \) be a prism. The category \( \text{Vect}^\varphi(\text{Spec}(A) \setminus V(p, I)) \) has
- objects given by pairs \( (M, \varphi_M) \), where \( M \) is a vector bundle over \( \text{Spec}(A) \setminus V(p, I) \) and \( \varphi_M \) is an \( A \)-linear isomorphism \( \varphi_M^* M[1/I] \to M[1/I] \);
- morphisms given by maps of vector bundles that are compatible with \( \varphi_M \).

(ii) Let \( X \) be a \( p \)-adic formal scheme over \( \mathcal{O}_K \). The category of analytic prismatic \( F \)-crystals over \( X \) is defined as
\[
\text{Vect}^{\an, \varphi}(X_\Delta) := \lim_{(A, I) \in X_\Delta} \text{Vect}^\varphi(\text{Spec}(A) \setminus V(p, I)).
\]

Similarly to Lemma 3.6, analytic prismatic \( F \)-crystals can be computed over the quasi-syntomic site.

**Lemma 3.9.** Define the category \( \text{Vect}^{\an, \varphi}(X_{\text{qsp}}) \) to be \( \lim_{S \in X_{\text{qsp}}} \text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I)). \) There is a natural equivalence of categories
\[
\text{Vect}^{\an, \varphi}(X_\Delta) \simeq \text{Vect}^{\an, \varphi}(X_{\text{qsp}}).
\]

Ditto for the derived (\( \infty \)-)categories of perfect complexes.

By taking restrictions, one obtains an analytic prismatic \( F \)-crystal from a prismatic \( F \)-crystal. Next, we show that this functor is in fact a fully faithful embedding of categories.

**Proposition 3.10.** Let \( X \) be a smooth \( p \)-adic formal scheme over \( \mathcal{O}_K \). The restriction functor induces a fully faithful embedding of categories
\[
\text{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta) \to \text{Vect}^{\an, \varphi}(X_\Delta).
\]

**Proof.** Thanks to Lemma 3.9, it suffices to show that the restriction functor \( \text{Vect}^\varphi(X_{\text{qsp}}, \Delta_S) \to \text{Vect}^{\an, \varphi}(X_{\text{qsp}}) \) is fully faithful. Because of the definition of both categories as 2-limits, it suffices to prove that the restriction \( \text{Vect}^\varphi(\text{Spec}(\Delta_S)) \to \text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I)) \) is fully faithful for each \( S \in X_{\text{qsp}} \). By taking internal Homs, we are reduced to showing that the global sections of each \( \mathcal{E} \in \text{Vect}(\text{Spec}(\Delta_S)) \) coincide with those of its restriction. Since \( \Delta_{\mathcal{O}_C} \to \Delta_S \) is \( (p, \xi) \)-completely faithfully flat [BS19, Prop. 7.10], \( (p, \xi) \) is a regular sequence on \( \Delta_S \) of length 2. Thus, the statement follows from [SP22, Tag 0G7P]. \( \tag*{\qed} \)

In the special case when \( X = \text{Spf}(\mathcal{O}_K) \) is a point, the above functor is in fact an equivalence. That is, the notion of analytic prismatic \( F \)-crystal is the same as that of prismatic \( F \)-crystal over \( \mathcal{O}_\Delta \) from [BS21] for \( X = \text{Spf}(\mathcal{O}_K) \).

**Proposition 3.11.** The functor induced by restriction to open subsets
\[
\text{Vect}^\varphi(\text{Spf}(\mathcal{O}_K)_\Delta, \mathcal{O}_\Delta) \to \text{Vect}^{\an, \varphi}(\text{Spf}(\mathcal{O}_K)_\Delta)
\]
induces an equivalence of categories.

**Proof.** By Proposition 3.10, it remains to prove that the functor is essentially surjective. Set \( \mathcal{A}_{\text{inf}} := \mathcal{A}_{\text{inf}}(\mathcal{O}_C) \) and \( S := \mathcal{O}_C \otimes_{\mathcal{O}_K} \mathcal{O}_C \). Let \( (\mathcal{E}, \varphi_\mathcal{E}) \) be an analytic prismatic \( F \)-crystal over \( \text{Spf}(\mathcal{O}_K) \). We want to show that \( (\mathcal{E}, \varphi_\mathcal{E}) \) extends to a prismatic \( F \)-crystal over \( \text{Spf}(\mathcal{O}_K) \), or equivalently, an object in \( \text{Vect}^\varphi(\mathcal{A}_{\text{inf}}) \) together with descent data at \( \Delta_S \).

The evaluation of \( (\mathcal{E}, \varphi_\mathcal{E}) \) at the prism \( (\mathcal{A}_{\text{inf}}, \hat{\xi}) \) gives an object of \( \text{Vect}^\varphi(\text{Spec}(\mathcal{A}_{\text{inf}}) \setminus V(\hat{p}, \hat{\xi})) \). Notice that by [BMS18, Lem. 4.6], the restriction functor induces an equivalence of categories between vector bundles on \( \text{Spec}(\mathcal{A}_{\text{inf}}) \setminus V(p, \xi) \) and (necessarily trivial) vector bundles on \( \mathcal{A}_{\text{inf}} \).
Thus, we can extend $(\mathcal{E}, \varphi_{\mathcal{E}})$ to an object $(M, \varphi_M)$ in $\text{Vect}^\varphi(\text{Spec}(A_{\text{inf}}))$, and it suffices to show that the pullbacks of $(M, \varphi_M)$ to $\text{Spec}(\Delta_S)$ admit a descent isomorphism. That is, there is a natural isomorphism of $\Delta_S$-modules $\text{pr}_1^* M \simeq \text{pr}_2^* M$ satisfying the cocycle condition, whose restriction to the open subset $\text{Spec}(\Delta_S) \setminus V(p, \xi)$ coincides with the given descent data $\alpha$: $\text{pr}_1^* \mathcal{E}(\mathcal{O}_C) \simeq \text{pr}_2^* \mathcal{E}(\mathcal{O}_C)$.

To proceed, since $M$ is free over $A_{\text{inf}}$ and thus $p$-torsionfree, it suffices to show that the restriction of $\alpha[1/p]: \text{pr}_1^* M[1/p] \simeq \text{pr}_2^* M[1/p]$ induces an isomorphism of the lattices $\text{pr}_1^* M \stackrel{\sim}{\rightarrow} \text{pr}_2^* M$. On the other hand, as the map $A_{\text{inf}} \to \Delta_S$ is $(p, \xi)$-completely flat ([BS19, Prop. 7.10]), $(p, \xi)$ is a regular sequence of length 2 in $\Delta_S$. In particular, similar to the proof of Proposition 3.10, the natural restriction map

$$\text{pr}_1^* M \simeq \Gamma(\text{Spec}(\Delta_S), \text{pr}_1^* M) \to \Gamma(\text{Spec}(\Delta_S) \setminus V(p, \xi), \text{pr}_1^* M)$$

is an isomorphism. From this, the isomorphism $\alpha: \text{pr}_1^* M \simeq \text{pr}_2^* M$ follows from the assumption that the descent data $\alpha$ is defined over the open subset $\text{Spec}(\Delta_S) \setminus V(p, \xi)$. Thus we are done. □

3.2. Realization functors. Pullback along the open immersions $\text{Spec}(A[1/p]) \subset \text{Spec}(A) \setminus V(p, I)$ and $\text{Spec}(A[1/I]) \subset \text{Spec}(A) \setminus V(p, I)$ induces localizations from analytic prismatic $F$-crystals to prismatic $F$-crystals over $O_{\Delta}[1/p]$ and $O_{\Delta}[1/I_{\Delta}]$, respectively. This leads to the notion of realization functors.

**Construction 3.12** (Realization functors). (i) The étale realization functor is the functor

$$T: \text{Vect}^{an, \varphi}(X_{\Delta}) \to \text{Vect}^\varphi(X_{\Delta}, O_{\Delta}[1/I_{\Delta}]_{\mathcal{P}}) \simeq \text{Loc}_{\mathcal{P}}(X_{\eta})$$

$$(\mathcal{E}, \varphi_{\mathcal{E}}) \mapsto \mathcal{E} \otimes_{O_{\Delta}} O_{\Delta}[1/I_{\Delta}]_{\mathcal{P}}$$

induced by the open immersion $\text{Spec}(A[1/I]) \subset \text{Spec}(A)$ and the equivalence from [BS21, Cor. 3.8, Ex. 4.4]. The étale realization functor for $\text{Vect}^\varphi(X_{\Delta}, O_{\Delta})$ from [BS21, Constr. 4.8] naturally factors through $T$.

(ii) We consider the maps of period sheaves on $X_{\text{grsp}}$

$$O_{\Delta} \to O_{\Delta}[I_{\Delta}/p] =: A_{\text{cryst}}(-) \to O_{\Delta}[I_{\Delta}/p][1/p] =: B_{\text{cryst}}^+(-).$$

As $O_{\Delta} \to B_{\text{cryst}}^+(-)$ factors through the localization $O_{\Delta} \to O_{\Delta}[1/p]$, one can naturally define the rational crystalline realization functor

$$D_{\text{cryst}}: \text{Vect}^{an, \varphi}(X_{\Delta}) \to \text{Vect}^\varphi(X_{\Delta}, B_{\text{cryst}}^+(-))$$

sending an object $(\mathcal{E}_A, \varphi_{\mathcal{E}_A}) \in \text{Vect}^\varphi(\text{Spec}(A) \setminus V(p, I))$ to $(\mathcal{E}_A \otimes B_{\text{cryst}}^+(A), \varphi_{\mathcal{E}_A} \otimes \varphi)$ for any prism $(A, I) \in X_{\Delta}$.

**Remark 3.13.** In the special case when $X = \text{Spf}(S^+)$ for an affinoid perfectoid algebra $(S, S^+)$ and thus $\text{Vect}^\varphi(X_{\Delta}, O_{\Delta}[1/I_{\Delta}]_{\mathcal{P}}) \simeq \text{Vect}^\varphi(A_{\text{inf}}(S^+)[1/I_{\mathcal{P}}],)$, the equivalence $\text{Vect}^\varphi(X_{\Delta}, O_{\Delta}[1/I_{\Delta}]_{\mathcal{P}}) \simeq \text{Loc}_{\mathcal{P}}(X_{\eta})$ from [BS21, Cor. 3.8, Ex. 4.4] can be obtained as follows:

- Given a $\mathbb{Z}_{\mathcal{P}}$-local system $T$ on $X_{\eta}$, take an affinoid pro-étale cover $Y$ of $X_{\eta}$ for which $T|_Y$ is trivial. The module $A_{\text{inf}}[1/I_{\mathcal{P}}]^\varphi(Y) \otimes T(Y)$ is equipped with a Frobenius action from the first factor, and its descent along the cover is a projective $A_{\text{inf}}(S^+)[1/I_{\mathcal{P}}]$-module together with Frobenius.

- Given an $F$-crystal $(M, \varphi_M)$ over $A_{\text{inf}}(S^+)[1/I_{\mathcal{P}}]$, the lisse étale sheaf over $X_{\eta}$ is the inverse limit (with respect to $n$) of the following sheaves of $\mathbb{Z}/p^n$-modules:

$$\text{Perfd}/X_{\eta, \text{proét}} \ni Y \mapsto \left(\frac{M}{p^n \otimes A_{\text{inf}}(S^+)[1/I]} A_{\text{inf}}(Y)[1/I]\right)^{\varphi=1}.$$
Theorem 3.14. Let X be a smooth p-adic formal scheme over \( \mathcal{O}_K \). For \( \mathcal{E} \in \text{Vec}^{\text{an},\varphi}(X_\Delta) \), the étale realization \( T(\mathcal{E}) \) is a crystalline local system on \( X_{q,\text{proét}} \) in the sense of Definition 2.25.

Before the proof of Theorem 3.14, we show an important lemma. The étale realization functor from Construction 3.12.(i) produces an étale \( \mathbb{Z}_p \)-local system \( T(\mathcal{E}) \) over \( X_{q,\text{proét}} \), such that

\[
T(\mathcal{E}) \otimes_{\mathbb{Z}_p} A_{\text{inf}}[1/\mathcal{I}_\Delta]_p^\wedge \simeq \mathcal{E} \otimes_{A_{\text{inf}}[1/\mathcal{I}_\Delta]_p^\wedge} A_{\text{inf}}[1/\mathcal{I}_\Delta]_p^\wedge.
\]

Here, by slight abuse of notation, we also use \( \mathcal{E} \) to denote the restriction of the prismatic \( F \)-crystal to \( \text{Perfd}/X_{q,\text{proét}} \). Choose an affinoid perfectoid object \( \text{Spa}(S,S^+) \) over \( \text{Spa}(C,O_C) \) in \( X_{q,\text{proét}} \) such that \( T|_{\text{Spa}(S,S^+)} \) is trivial (such objects form a basis of \( X_{q,\text{proét}} \)). Evaluated at \( \text{Spa}(S,S^+) \), the above isomorphism of sheaves above leads to an isomorphism of \( A_{\text{inf}}(S^+)[1/\xi]^\wedge_p \)-modules. We show that it can be descended to an \( A_{\text{inf}}[1/\mu] \)-linear isomorphism for \( \mu = q - 1 \).

Lemma 3.15. There is a natural isomorphism of \( A_{\text{inf}}(S^+)[1/\mu] \)-modules

\[
T(\mathcal{E}) \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S^+)[1/\mu] \simeq \mathcal{E}(S^+) \otimes_{A_{\text{inf}}(S^+)} A_{\text{inf}}(S^+)[1/\mu],
\]

which is functorial with respect to perfectoid algebras \( (S,S^+) \in X_{q,\text{proét}} \) for which \( T|_{\text{Spa}(S,S^+)} \) is trivial, and is compatible with Equation (6) after base change along \( A_{\text{inf}}(S^+)[1/\mu] \to A_{\text{inf}}(S^+)[1/\xi]^\wedge_p \).

Here \( T(\mathcal{E}) \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S^+) \) denotes the sections of the pro-étale sheaf \( T(\mathcal{E}) \otimes A_{\text{inf}} \) at the affinoid perfectoid space \( \text{Spa}(S,S^+) \).

Proof of Lemma 3.15. We denote the sections of the local system \( T(\mathcal{E}) \) at \( \text{Spa}(S,S^+) \) by \( T \). As \( \mathcal{E}(S^+)[1/\xi] \) is finite projective over \( A_{\text{inf}}(S^+)[1/\xi] \), we can choose a finitely presented \( A_{\text{inf}}(S^+) \)-submodule \( M \) of \( \mathcal{E}(S^+)[1/\xi] \), generating the latter as an \( A_{\text{inf}}(S^+)[1/\xi] \)-module ([SP22, Tag 01PI]). Let \( \{e_1, \ldots, e_m\} \subset M \) be a set of generators of \( M \) over \( A_{\text{inf}}(S^+) \). The inverse Frobenius \( \varphi^{-1}_\mathcal{E} \) defines a \( \varphi^{-1} \)-semilinear map \( M[1/\xi] = \mathcal{E}(S^+)[1/\xi] \to M[1/\xi] \). After multiplying \( \mathcal{E} \) and \( M \) with \( 1/\mu^n \) for \( n \gg 0 \), we may therefore assume that \( \varphi^{-1}_\mathcal{E}(M) \subseteq M \) because \( M \) is finitely generated. We show now that the isomorphism \( T \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S^+)[1/\xi]^\wedge_p \simeq \mathcal{E}(S^+)[1/\xi]^\wedge_p \) induced by Equation (6) maps \( T \) into \( M[1/\mu] \). By contemplating the same statement for the duals, this will finish the proof.

Our strategy is to use \( v \)-descent to reduce the statement to the case of valuation rings. First, we take the tensor products of the following two finite projective \( A_{\text{inf}}(S^+)[1/\mu] \)-modules with the \( v \)-sheaf \( A_{\text{inf}}[1/\mu] \):

\[
T \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S^+)[1/\mu], \quad \mathcal{E}(S^+) \otimes_{A_{\text{inf}}(S^+)} A_{\text{inf}}(S^+)[1/\mu].
\]

This defines \( v \)-sheaves\(^5\)

\[
T \otimes_{\mathbb{Z}_p} A_{\text{inf}}[1/\mu], \quad \mathcal{E}(S^+)[1/\mu] \otimes_{A_{\text{inf}}(S^+)[1/\mu]} A_{\text{inf}}[1/\mu]
\]

over \( \text{Spa}(S,S^+) \), whose global sections are the two \( A_{\text{inf}}(S^+)[1/\mu] \)-modules above.

Similarly, one obtains two larger isomorphic \( v \)-sheaves \( T \otimes_{\mathbb{Z}_p} A_{\text{inf}}[1/\xi]^\wedge_p \) and \( \mathcal{E}[1/\mu] \otimes_{A_{\text{inf}}(S^+)[1/\mu]} A_{\text{inf}}[1/\xi]^\wedge_p \), which contain the previous two as \( A_{\text{inf}}[1/\mu] \)-linear subsheaves. To show the desired inclusion, it suffices to check that the natural isomorphism \( T \otimes_{\mathbb{Z}_p} A_{\text{inf}}[1/\xi]^\wedge_p \simeq \mathcal{E}[1/\mu] \otimes_{A_{\text{inf}}(S^+)[1/\mu]} A_{\text{inf}}[1/\mu] \) maps the \( A_{\text{inf}}[1/\mu] \)-subsheaf \( T \otimes_{\mathbb{Z}_p} A_{\text{inf}}[1/\mu] \) into \( \mathcal{E}[1/\mu] \otimes_{A_{\text{inf}}(S^+)[1/\mu]} A_{\text{inf}}[1/\mu] \). As the \( v \)-site admits a basis by strictly totally disconnected objects, we can choose a map of \( \mathcal{O}_C \)-algebras \( S^+ \to \prod_i V_i \) such that the corresponding map of adic spectra is a \( v \)-cover and each \( V_i \) is a rank 1 perfectoid valuation ring with algebraically closed fraction field ([Sch17, Prop. 7.16, Lem. 7.18]). It then suffices to show that \( T \otimes_{\mathbb{Z}_p} A_{\text{inf}}(\prod_i V_i) \) is contained in \( \mathcal{E}(S^+)[1/\mu] \otimes_{A_{\text{inf}}(S^+)} A_{\text{inf}}(\prod_i V_i) = (M \otimes_{A_{\text{inf}}(S^+)} A_{\text{inf}}(\prod_i V_i))[1/\mu] \).

\(^5\)The sheafiness comes from that of \( A_{\text{inf}} \), which follows from [Sch17, Thm. 8.7] and the sheafiness of the Witt vector construction.
For any $\mathcal{O}_C$-algebra $V_i$, the base change $M_{V_i} := M \otimes_{A_{\text{inf}(S^+)}(V_i)} A_{\text{inf}}(V_i)$ is a Breuil–Kisin–Fargues module over $A_{\text{inf}}(V_i)$ [BMS18, §4]. By the assumption that $\varphi^{-1}_E$ preserves $M$, the base change $M_{V_i}$ is also sent to itself under the map $\varphi^{-1}_V := \varphi_E \otimes_{A_{\text{inf}(S^+)}(V_i)} A_{\text{inf}}(V_i)$. By the proof of [BMS18, Lem. 4.26], the above condition then implies that the Frobenius invariants of $T \otimes_{\mathbb{Z}_p} A_{\text{inf}}(V_i)[1/\xi]^\wedge_p \simeq M_{V_i}[1/\xi]^\wedge_p$ are contained in the image of $M_{V_i}$ inside $M_{V_i}[1/\mu] \subset M_{V_i}[1/\xi]^\wedge_p$. In other words, we have the commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\subset} & T \otimes A_{\text{inf}}(V_i)[1/\mu] \xrightarrow{\subset} T \otimes A_{\text{inf}}(V_i)[1/\xi]^\wedge_p \\
\downarrow{\cong} & & \downarrow{\cong} \\
M_{V_i} & \xrightarrow{\subset} & M_{V_i}[1/\mu] \xrightarrow{\subset} M_{V_i}[1/\xi]^\wedge_p,
\end{array}
$$

where $M_{V_i}$ is the image of $M_{V_i}$ in $M_{V_i}[1/\mu]$. In particular, the image of any given element $t \in T$ in $M_{V_i}[1/\mu]$ can be expressed as a finite sum

$$\sum_{j=1}^m a_{ij} \epsilon_j, \quad a_{ij} \in A_{\text{inf}}(V_i).$$

We return to the $\mathcal{O}_C$-algebra $\prod_i V_i$. Since both $T$ and $E[1/\mu]$ are locally free, we have natural inclusions of modules

$$T \otimes \mathbb{Z}_p A_{\text{inf}}(\prod_i V_i)[1/\mu] \xrightarrow{\subset} \prod_i (T \otimes \mathbb{Z}_p A_{\text{inf}}(V_i)[1/\mu]) \xrightarrow{\subset} M[1/\mu] \otimes_{A_{\text{inf}(S^+)}} A_{\text{inf}}(\prod_i V_i) \xrightarrow{\subset} \prod_i (M_{V_i}[1/\mu]).$$

As a consequence, by combining the expressions above for all possible $i$, we can write $t \in T$ as an element $\sum_{j=1}^m e_j \cdot (a_{ij})_i \in \prod_i (M_{V_i}[1/\mu])$, where $(a_{ij})_i$ is an element of $A_{\text{inf}}(\prod_i V_i) = \prod_i A_{\text{inf}}(V_i)$. In this way, we see that $t$ lives in the subset $M[1/\mu] \otimes_{A_{\text{inf}(S^+)}} A_{\text{inf}}(\prod_i V_i)$. Thus, we get the inclusion $T \subset M[1/\mu] \otimes A_{\text{inf}}(\prod_i V_i)$. 

\begin{proof}[Proof of Theorem 3.14] First, we construct a natural $F$-isocrystal on $X_s := X_{\text{red}}$ out of the analytic prismatic $F$-crystal $(E, \varphi_E)$. To start, let $\text{Spf} R$ be a small open affine subspace of $X$, where $R$ is a $p$-adic smooth formal $\mathcal{O}_K$-algebra that admits a framing (Definition 2.8). By Lemma 2.9, each framing can be written uniquely as the base change along $V_0 \to \mathcal{O}_K$ of a formal étale $V_0(\pm 1)$-algebra $\tilde{R}$. Moreover,

$$\tilde{R} \to R \otimes_{\mathcal{O}_K} k \leftarrow R$$

defines a natural prism in the absolute prismatic site $X_\Delta$ by setting $\delta(x_i) = 0$. By evaluating the analytic prismatic $F$-crystal $(E, \varphi_E)$ at prisms of this form, we get an $F$-isocrystal whose restriction at $\tilde{R}[1/p]$ is $(E(\tilde{R}), \varphi_E)$ (cf. Remark 2.16). Note that as the prism above is crystalline, the Frobenius $\varphi_E$ on $E(\tilde{R})$ is an isomorphism.

Next, let $S^+$ be a perfectoid $\mathcal{O}_C$-algebra in $X_{\text{qss}}$ whose structure map to $X$ factors through $\text{Spf} \mathfrak{i}$ for some $\mathfrak{i} : R \to S^+$. Following Dwork’s Frobenius trick, there is an integer $n \gg 0$ such that the map $\mathfrak{i} \circ \varphi^n : R/p \to S^+/p$ factors through $\tilde{\mathfrak{i}} \circ \varphi^n$ for a uniquely determined $\tilde{\mathfrak{i}} : R \otimes_{\mathcal{O}_K} k \to S^+/p$. On the other hand, the map $S^+/p \to A_{\text{crys}}(S^+)/p$ is a nil morphism. So by the formal smoothness
of $\tilde{R}$, we can find a (noncanonical) map of crystalline prisms in $R_{\Delta}$
\[
\begin{array}{ccc}
R & \longrightarrow & R/p \\
& \downarrow \alpha & \downarrow \alpha/p \\
R \otimes \mathcal{O}_K & \longrightarrow & \mathcal{A}_{\text{crys}}(S^+)/p \\
& \tilde{R} & \longrightarrow & \mathcal{A}_{\text{crys}}(S^+).
\end{array}
\]

Now we evaluate $(\mathcal{E}, \varphi_{\mathcal{E}})$ at the crystalline prism $\mathcal{A}_{\text{crys}}(S^+)$ to get a $B^+_{\text{crys}}(S^+)$ -module $\mathcal{E}(S^+) \otimes_{A_{\text{inf}}(S^+)} B^+_{\text{crys}}(S^+)$ with a Frobenius isomorphism. By the crystal property for $(\mathcal{E}, \varphi_{\mathcal{E}})$, we have the compatibility
\[
\varphi^*: \mathcal{E}(S^+) \otimes_{A_{\text{inf}}(S^+)} B^+_{\text{crys}}(S^+) \simeq \mathcal{E}(\tilde{R}) \otimes_{\tilde{R}[1/p]} B^+_{\text{crys}}(S^+).
\]

An untwist of Frobenius (which is an isomorphism above as $p$ is inverted) yields an isomorphism
\[
\mathcal{E}(S^+) \otimes_{A_{\text{inf}}(S^+)} B^+_{\text{crys}}(S^+) \simeq \mathcal{E}(\tilde{R}) \otimes_{\tilde{R}[1/p]} B^+_{\text{crys}}(S^+).
\]

Moreover, the localization $B_{\text{crys}}(S^+) = B^+_{\text{crys}}(S^+)[1/\xi]$ naturally contains $A_{\text{inf}}(S^+)[1/\mu]$. Composing with the base change of the isomorphism from Lemma 3.15 along the inclusion $A_{\text{inf}}(S^+)[1/\mu] \to B_{\text{crys}}(S^+)$, we therefore get the crystalline comparison
\[
T(\mathcal{E}) \otimes_{\mathbb{Z}_p} B_{\text{crys}}(S^+) \simeq \mathcal{E}(\tilde{R}) \otimes_{\tilde{R}[1/p]} B_{\text{crys}}(S^+).
\]

Finally, note that the $B_{\text{crys}}(S^+)$ -module $\mathcal{E}(S^+) \otimes_{A_{\text{inf}}(S^+)} B_{\text{crys}}(S^+)$ inherits a natural filtration from $T(\mathcal{E}) \otimes_{\mathbb{Z}_p} B_{\text{crys}}(S^+) \otimes \mathcal{O}_K$. This induces in particular a filtration on the vector bundle
\[
E = \mathcal{E}(\tilde{R}) \otimes_{\mathcal{O}_K} K
\]
over $\text{Spf}(R)$. To see this explicitly, we first observe that by the same argument as in Proposition 2.28, the crystal condition of $\mathcal{E}(\tilde{R})$ gives the following natural isomorphisms of pro-étale sheaves on $X_{\eta}$:
\[
T(\mathcal{E}) \otimes \mathcal{O}_\mathcal{B}_{\text{dR}} \simeq \mathcal{E}(E) \otimes_{B_{\text{crys}}} \mathcal{O}_\mathcal{B}_{\text{dR}} \simeq E \otimes_{\mathcal{O}_{X_{\eta}}} \mathcal{O}_\mathcal{B}_{\text{dR}}.
\]

Moreover, as the stalk of $E$ at a point is of the same rank as that of $T(\mathcal{E})[1/p]$, the isomorphisms in Equation (7) guarantee by the criterion in [LZ17, Thm. 3.9] that the local system $T(\mathcal{E})$ is de Rham, and the filtered vector bundle with connection
\[
(E \simeq \nu_*(T(\mathcal{E}) \otimes \mathcal{O}_\mathcal{B}_{\text{dR}}), \nabla, \text{Fil}^n_e(E) := \nu_*(T(\mathcal{E}) \otimes \text{Fil}^n \mathcal{O}_\mathcal{B}_{\text{dR}}))
\]
is associated with $T(\mathcal{E})$ in the sense of [Sch13, §8], where $\nu: X_{\eta, \text{pro\acute{e}t}} \to X_{\eta, \text{\acute{e}t}}$ is the canonical map of sites. As a consequence, by taking the horizontal sections of Equation (7), we get a filtered isomorphism of $\mathcal{B}_{\text{dR}}$ -linear sheaves
\[
T(\mathcal{E}) \otimes \mathcal{B}_{\text{dR}} \simeq \mathcal{E}(\mathcal{B}_{\text{crys}}) \otimes_{B_{\text{crys}}} \mathcal{B}_{\text{dR}},
\]
which is compatible with the filtration on $E$. In this way, by Remark 2.26 and Definition 2.25, we see that the $\mathbb{Z}_p$ -local system $T(\mathcal{E})$ over $X_{\eta, \text{pro\acute{e}t}}$ is a crystalline local system.

\section{Proof of Theorem A}

In this section, we show the equivalence between the category of crystalline $\mathbb{Z}_p$ -local systems and the category of analytic prismatic $F$ -crystals, generalizing [BS21] to arbitrary $p$ -adic smooth formal schemes $X$ over $\mathcal{O}_K$. More precisely, we prove that the étale realization functor from Construction 3.12, which maps an analytic prismatic $F$ -crystal to a crystalline $\mathbb{Z}_p$ -local system (Theorem 3.14), is both fully faithful and essentially surjective. We refer the reader to [BS21, Constr. 6.2] for a complete summary of various prismatic structure sheaves.
4.1. Full faithfulness. First, we show that the étale realization functor is fully faithful, modifying the case of a point from [BS21, §5]. We begin with faithfulness.

**Lemma 4.1.** Let $X$ be a $p$-adic smooth formal scheme over $\mathcal{O}_K$ and let $f: (\mathcal{E}, \varphi_\mathcal{E}) \to (\mathcal{E}', \varphi_{\mathcal{E}'})$ be a morphism of analytic prismatic $F$-crystals on $X$. If the étale realization of $f$
\[ f \otimes \text{id}: \mathcal{E} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p \to \mathcal{E}' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta[1/I_\Delta]^\wedge_p \]
is 0, then so is $f$.

**Proof.** We can check locally in the Zariski topology on $X$ whether $f$ is 0. As $X$ is smooth, we can therefore use framings to reduce to the case where $X = \operatorname{Spf} R$ is affinoid and admits a $p$-completely flat cover $Y = \operatorname{Spf} S \to X$ for some integral perfectoid ring $S$. Since we endow the prismatic site with the flat topology, it suffices to check that the restriction of $f$ to $Y_\Delta$ is 0.

On the other hand, $\operatorname{Vect}^{an, \varphi}(Y_\Delta) \cong \operatorname{Vect}^\varphi(\operatorname{Spec}(\Delta_S) \setminus V(p, I))$ because $(\Delta_S, I)$ is a final object in $Y_\Delta$, and we may identify $\mathcal{E}|_{Y_\Delta}$ and $\mathcal{E}'|_{Y_\Delta}$ with vector bundles $\mathcal{E}_U$ and $\mathcal{E}'_U$ on $U := \operatorname{Spec}(\Delta_S) \setminus V(p, I)$, respectively. We obtain a diagram
\[
\begin{array}{ccc}
H^0(U, \mathcal{E}_U) & \longrightarrow & \mathcal{E}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U[1/I]^\wedge_p \\
& \downarrow & \downarrow \left[f \otimes \text{id}\right]|_{Y_\Delta} \\
H^0(U, \mathcal{E}_U') & \longrightarrow & \mathcal{E}'_U \otimes_{\mathcal{O}_U} \mathcal{O}_\Delta[1/I]^\wedge_p,
\end{array}
\]
in which the right vertical map is 0 by assumption. Moreover, we claim that the horizontal maps are injective. Granting this, the left vertical map must be 0, which implies the statement.

To see the injectivity of the horizontal maps, we first note that by [BS21, Lem 4.11], we have an injection of quasicoherent sheaves over $U$
\[ \mathcal{O}_U \longrightarrow \mathcal{O}_U[1/I]^\wedge_p. \]
Moreover, since $\mathcal{E}_U$ is a vector bundle (thus finitely presented and flat over $\mathcal{O}_U$), its base change along the above map of sheaves gives an injection
\[ \mathcal{E} \longrightarrow \mathcal{E}[1/I]^\wedge_p. \]
Similarly for $\mathcal{E}'$. Thus, by the left exactness of the global sections functor, we get the injectivity in the diagram before.

The above implies in particular that the étale realization functor is faithful. Now we turn our eyes toward fullness. For this, we need to generalize one easy part of Fargues’ classification of shtukas with one leg over $\operatorname{Spa} C^\flat$ to any integral perfectoid ring.

**Construction 4.2.** Let $S$ be a $p$-torsionfree perfectoid ring over $\mathcal{O}_C$ and $Y := \operatorname{Spf} S$. The **lattice realization** is a functor
\[ \Phi: \operatorname{Vect}^\varphi(\operatorname{Spec}(\Delta_S) \setminus V(p, I)) \to \{(M, \Xi) \mid M \in \operatorname{Loc}_{\mathbb{Z}_p}(Y_\eta), \Xi \subset M \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}^+ \text{ is a } \mathbb{B}_{dR}^+\text{-lattice}\} \]
given on an object $(\mathcal{E}, \varphi) \in \operatorname{Vect}^\varphi(\operatorname{Spec}(\Delta_S) \setminus V(p, I))$ as follows:

- $M$ is the étale realization of $\mathcal{E}$ as in Construction 3.12 and Remark 3.13.
- Base changing the natural isomorphism of sheaves $M \otimes_{\mathbb{Z}_p} \mathbb{A}_{\inf}[1/\mu] \simeq \mathcal{E}(S, S^+)[1/\mu]$ from Lemma 3.15, we obtain naturally an isomorphism
\[ (M \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}) \simeq \varphi^* \mathcal{E}(S^+) \otimes_{\mathbb{A}_{\inf}(S^+, \varphi)} \mathbb{B}_{dR}. \]
We set $\Xi(S', S'^+) := \varphi^* \mathcal{E}(S^+) \otimes_{\mathbb{A}_{\inf}(S^+, \varphi)} \mathbb{B}_{dR}^+(S^+) \subset (M \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR})(S', S'^+)$. This defines a sheaf of $\mathbb{B}_{dR}^+$-lattices $\Xi \subset M \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}$ on $Y_{\eta, \text{proét}}$. 

\[ \Xi(S', S'^+) := \varphi^* \mathcal{E}(S^+) \otimes_{\mathbb{A}_{\inf}(S^+, \varphi)} \mathbb{B}_{dR}^+(S^+) \subset (M \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR})(S', S'^+). \]
A morphism of lattice realizations \( g: (M, \Xi) \rightarrow (M', \Xi') \) is a morphism \( g: M \rightarrow M' \) in \( \text{Loc}_{\mathbb{Z}_p}(Y_\eta) \) such that \((g \otimes \text{id}_{\mathbb{B}_{\text{dR}}})(\Xi) \subseteq \Xi'\).

**Lemma 4.3.** The lattice realization functor from Construction 4.2 is fully faithful.

**Proof.** We follow [BMS18, Rem. 4.29] but for general perfectoid algebras that are flat over \( \mathcal{O}_C \). The composition of \( \Phi \) with the projection onto the first factor is the étale realization \( \text{Vect}^\mu(\text{Spec}(\Delta_S) \setminus V(p, I)) \rightarrow \text{Loc}_{\mathbb{Z}_p}(Y_\eta) \), which is faithful by (the proof of) Lemma 4.1. Thus, \( \Phi \) must be faithful as well.

For fullness, let \((\mathcal{E}, \varphi)\) and \((\mathcal{E}', \varphi')\) be in \( U := \text{Vect}^\mu(\text{Spec}(\Delta_S) \setminus V(p, I)) \) with lattice realizations \((M, \Xi)\) and \((M', \Xi')\), respectively. Let \( g: M \rightarrow M' \) be a morphism such that \((g \otimes \text{id}_{\mathbb{B}_{\text{dR}}})(\Xi) \subseteq \Xi'\); we need to show that there exists a morphism of analytic prismatic \( F \)-crystals \( f: (\mathcal{E}, \varphi) \rightarrow (\mathcal{E}', \varphi') \) with \( \Phi(f) = g \).

Let \( \mathcal{O}[1/\mu] \) be the quasicoherent sheaf on \( U \) corresponding to the \( \Delta_S \)-module \( \Delta_S[1/\mu] \). By Lemma 3.15, \( g \) induces a natural \( \varphi \)-equivariant morphism \( \tilde{g}: \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}[1/\mu] \rightarrow \mathcal{E}' \otimes_{\mathcal{O}} \mathcal{O}[1/\mu] \) of quasicoherent sheaves on \( U \). Since \( \mathcal{E}' \otimes_{\mathcal{O}} \mathcal{O}[1/\mu] = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mu^{-n} \mathcal{E}' \) and the \( \Delta_S[1/\mu] \)-module corresponding to \( \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}[1/\mu] \) is finitely presented, there exists an \( n \in \mathbb{Z}_{\geq 0} \) such that \( \tilde{g} \) factors through \( \mu^{-n} \mathcal{E}' \). Choose \( n \) minimal with that property. It remains to prove that \( n = 0 \).

Assume \( n > 0 \). Replacing \( \mathcal{E}' \) by \( \mu^{-n+1} \mathcal{E}' \), we may assume \( n = 1 \). By [BMS18, Lem. 3.23], we have \( \bigcap_{r \in \mathbb{Z}_{\geq 0}} \mu_{\varphi^{-r}(\mu)} \Delta_{\mathcal{O}_C} = \mu \Delta_{\mathcal{O}_C} \). The flatness of \( \Delta_{\mathcal{O}_C} \) over \( \mathcal{O}_C \) and the local freeness of \( \mathcal{E}' \) ensure a similar intersection formula \( \bigcap_{r \in \mathbb{Z}_{\geq 0}} \mu_{\varphi^{-r}(\mu)} \mathcal{E}' = \mathcal{E}' \) of quasicoherent subsheaves of \( \mu^{-1} \mathcal{E}' \). We will show by induction on \( r \in \mathbb{Z}_{\geq 0} \) that \( \tilde{g} \) factors through \( \mu_{\varphi^{-r}(\mu)} \mathcal{E}' \) for all \( r \) and thus through \( \mathcal{E}' \), contradicting the minimality of \( n \). The base case \( r = 0 \) follows by assumption.

For the inductive step, assume that \( \tilde{g} \) factors through \( \mu_{\varphi^{-r+1}(\mu)} \mathcal{E}' \) for some \( r \in \mathbb{Z}_{\geq 0} \). We want to show that the composition

\[
\mathcal{E} \xrightarrow{\tilde{g}} \mathcal{E}' \xrightarrow{\mu_{\varphi^{-r+1}(\mu)}} \mathcal{E}' \xrightarrow{\varphi^{-r}(\tilde{g})} \mathcal{E}'
\]

is 0. Denote the quasicoherent sheaf on \( U \) corresponding to the \( \Delta_S \)-module \( B_{\text{dR}}^+(S) \) by \( B_{\text{dR}}^+ \). Since \( \varphi^* (\tilde{e}) \) is invertible in \( B_{\text{dR}}^+(S) \) for all \( s > 0 \), the \( F \)-crystal structure on \( \mathcal{E} \) induces an isomorphism \( \varphi^{-1}: \varphi^* \mathcal{E} \otimes B_{\text{dR}}^+ \simeq \varphi^* \mathcal{E} \otimes B_{\text{dR}}^+ \), and similarly for \( \mathcal{E}' \). We obtain the commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\tilde{g}} & \mathcal{E}' \\
\downarrow \varphi^* \mathcal{E} & \xrightarrow{\varphi^* \tilde{g}} & \varphi^* \mathcal{E}' \\
\varphi^* \mathcal{E} \otimes B_{\text{dR}}^+ & \xrightarrow{\varphi^* \tilde{g}} & \varphi^* \mathcal{E}' \otimes B_{\text{dR}}^+ \\
\end{array}
\]

in which the vertical maps from the first two rows are given by the natural base extensions of the coefficient rings. Since the rightmost column consists of isomorphisms, it suffices to show that the bottom row is 0. However, by assumption the bottom left horizontal morphism maps \( \Xi = \varphi^* \mathcal{E} \otimes B_{\text{dR}}^+ \) into \( \Xi' = \varphi^* \mathcal{E}' \otimes B_{\text{dR}}^+ \). Therefore, the image of the composition of the two bottom maps lies in \( \varphi(\mu) \varphi^* \mathcal{E}' \otimes B_{\text{dR}}^+ / \tilde{e} \), which is 0 because \( \tilde{e} \) divides \( \varphi(\mu) \) in \( B_{\text{dR}}^+(S) \). \( \square \)
Now we show fullness of $T$, following the proof of [BS21, Thm. 5.6].

**Proposition 4.4.** Let $X$ be a $p$-adic smooth formal scheme over $\mathcal{O}_K$ and $(\mathcal{E}, \varphi_{\mathcal{E}}), (\mathcal{E}', \varphi_{\mathcal{E}'}) \in \text{Vect}^{\text{an}, \varphi}(X)_\Delta$. Then any morphism $g: T(\mathcal{E}) \to T(\mathcal{E}')$ of étale realizations is given by $T(f)$ for some $f: (\mathcal{E}, \varphi_{\mathcal{E}}) \to (\mathcal{E}', \varphi_{\mathcal{E}'})$.

**Proof.** If we can find $f$ Zariski-locally on an affine open cover of $X$, the different choices must coincide on the intersections by faithfulness of $T$ (Lemma 4.1), so we may assume that $X = \text{Spf} R$ is a framed smooth affine formal scheme. Choose a $p$-completely flat cover $Y = \text{Spf} S \to X$ by an integral perfectoid $\mathcal{O}_C$-algebra $S$. If $f_Y: \mathcal{E}|_{Y_\Delta} \to \mathcal{E}'|_{Y_\Delta}$ is a morphism in $\text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I))$ with $T(f_Y) = g|_Y$, as the structure map $Y \times_X Y \to Y$ is a cover, the pullbacks of $f_Y$ under the two projections $Y \times_X Y \to Y$ coincide again by the faithfulness of $T$ and the analogous statement for the two pullbacks of $g|_Y$. Thus, any such $f_Y$ can be descended to $X$ (cf. Remark 3.3) and it suffices to show the existence of $f_Y$.

However, the étale and lattice realizations fit into a commutative diagram

$$
\begin{array}{ccc}
\text{Vect}^{\text{an}, \varphi}(X_\Delta) & \longrightarrow & \text{Loc}_{\mathbb{Z}_p}^{\text{crys}}(X_\eta) \\
\downarrow & & \downarrow \\
\text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I)) & \longrightarrow & \{(M, \Xi)\}
\end{array}
$$

in which the left vertical arrow is given by the evaluation and the right vertical arrow sends a crystalline local system $L$ on $X_\eta$ to the pair $(M, \Xi)$ where

- $M$ is the pullback of $L$ to $Y_\eta$
- $\Xi := \mathbb{B}^+_\text{crys}(\mathcal{E}) \otimes \mathbb{B}^+_\text{dR}$, where $\mathcal{E}$ is the associated $F$-isocrystal of the crystalline local system $L$.

The statement therefore follows from Lemma 4.3.

**Theorem 4.5.** Let $X$ be a $p$-adic smooth formal scheme over $\mathcal{O}_K$, and let $T$ be a $\mathbb{Z}_p$-crystalline local system over $X_\eta$. Then there is a natural analytic prismatic $F$-crystal in $\text{Vect}^{\text{an}, \varphi}(X_{\text{qcr})}$ whose étale realization recovers $T$.

Throughout the subsection, we consider a $p$-adic smooth formal scheme $X$ over $\mathcal{O}_K$, and a $\mathbb{Z}_p$-crystalline local system $T$ over $X_\eta$. We denote by $(\mathcal{E}, \varphi_\mathcal{E}, \text{Fil}^\bullet(\mathcal{E}))$ the filtered $F$-isocrystal associated with $T$; namely, $\mathcal{E}$ is an isocrystal over the crystalline site of the reduced special fiber $X_s$, $\varphi_\mathcal{E}$ is a rational Frobenius action, and $\text{Fil}^\bullet(\mathcal{E})$ is a descending filtration on the underlying vector bundle $E$ on $X_\eta$ (Definition 2.18).

To start, we introduce a class of objects in $X_{\text{qcr}}$ on which $T$ is trivializable.

**Definition 4.6.** The category $X_{\text{qcr}}^w$ is the full subcategory of $X_{\text{qcr}}$ consisting of those quasiregular semiperfectoid rings $S$ for which the structure map $\text{Spf}(S) \to X$ can be factored as

$$
\begin{array}{ccc}
\text{Spf}(S) & \longrightarrow & \text{Spf}(S') \\
\downarrow & & \downarrow \\
& & \text{Spf}(S')_\eta \longrightarrow \text{Spf}(S')_\eta
\end{array}
$$

such that both $f$ and $g$ are quasi-syntomic and $S' \in X_{\text{qcr}}$ is a perfectoid $\mathcal{O}_C$-algebra on which the restriction $T|_{\text{Spf}(S')_\eta}$ of every $\mathbb{Z}_p$-local system $T$ on $X_{\eta, \text{pro\acute{e}t}}$ is trivial.

**Lemma 4.7.** The subcategory $X_{\text{qcr}}^w$ from Definition 4.6 forms a basis of the quasi-syntomic site $X_{\text{qcr}}$. 
Proof. We may assume that $X = \text{Spf}(R)$ is affine and connected. Let $S$ be the $p$-completion of an absolute integral closure of the integral domain $R$; that is, the $p$-completion of the integral closure of $R$ in an algebraic closure $\text{Frac}(R)$ (which is unique up to nonunique isomorphism). We first show that the natural map $R \to S$ is a quasi-syntomic cover which satisfies the conditions in Definition 4.6.

Note that $S$ is a perfectoid $\mathcal{O}_C$-algebra by [Bha20, Lem. 4.20] (and because $\mathcal{O}_C$ is the completion of the integral closure of $\mathcal{O}_K$ in $K$). Furthermore, $S$ is faithfully flat [Bha20, Thm. 5.16], and thus in particular also quasi-syntomic over $R$: indeed, by the transitivity triangle on cotangent complexes for $\mathbb{Z}_p \to R \to S$ and the smoothness of $R$, one can show that $L_{S/R}$ has $p$-complete Tor amplitude in $[-1, 0]$. It remains to check that $T|_{\text{Spf}(S)_n}$ is trivial for every $\hat{\mathbb{Z}}_p$-local system $T$ of $X_{\eta, \text{proet}}$.

By [Sch13, Prop. 8.2], we have $T = \lim_n T_n$ for (pullbacks to the pro-étale site of) étale $\mathbb{Z}/p^n$-local systems $T_n$ on $X_n$. It suffices to show that $T_n|_{\text{Spf}(S)_n}$ can be trivialized for each $n$: using the surjection $(\mathbb{Z}/p^{n+1})^\times \to (\mathbb{Z}/p^n)^\times$, one can then make these trivializations compatible recursively on $n$, so that their inverse limit will give the desired trivialization of $T$.

Since $T_n$ is an étale $\mathbb{Z}/p^n$-local system, we can trivialize $T_n|_{X_n}$ for some finite étale morphism of adic spaces $X_n = \text{Spa}(R_n, R_n^\times) \to \text{Spf}(R)_{\eta}$ (i.e., $R[1/p] \to R_n$ finite étale and $R_n^\times$ is the integral closure of $R$ in $R_n$). We may assume that $X_n$ is connected because $X$ is so. As $R[1/p] \to R_n$ is finite étale, we can find an embedding $\text{Frac}(R_n) \hookrightarrow \text{Frac}(R)$ extending $\text{Frac}(R) \hookrightarrow \text{Frac}(R)$. Moreover, since $S$ is the $p$-completion of the integral closure of $R$ in $\text{Frac}(R)$, the natural map $R \to S$ factors through $R_n^\times$. Consequently, the morphism $\text{Spf}(S)_n \to \text{Spf}(R)_{\eta}$ factors through $\text{Spa}(R_n, R_n^\times)$, so that $T_n|_{\text{Spf}(S)_n}$ must be trivializable as well. This concludes the argument that $R \to S$ is a quasi-syntomic cover which satisfies the conditions in Definition 4.6.

To finish, let $\text{Spf}(R') \to X$ be any cover by a quasiregular semiperfectoid ring. Then the $p$-complete tensor product $S' := S \otimes_R R'$ is again quasiregular semiperfectoid by the proof of [BMS19, Lem. 4.27] and $R' \to S'$ is a quasi-syntomic cover by [BMS19, Lem. 4.16.(2)], hence defines an object of $X_{\text{qsp}}$ covering $R'$. Further, $X_{\text{qsp}}$ is closed under base change and composition, so we are done.

Following [BS21, Constr. 6.5], our first step is to extend filtered $F$-isocrystals to prismatic $F$-crystals over $\Delta_\bullet(\mathcal{I}_\Delta/p)[1/p]$.

Construction 4.8. Let $(\mathcal{E}, \varphi_{\mathcal{E}}, \text{Fil}^*(E))$ be the filtered $F$-isocrystal associated with the $\hat{\mathbb{Z}}_p$-crystalline local system $T$ as above. For each $S \in X_{\text{qsp}}$, the ring $A_{\text{crys}}(S) = \Delta_\bullet(\mathcal{I}_\Delta/p)$ is a pro-$p$-thickening over $S$, so $\mathcal{M}_1(S) := \mathcal{E}(A_{\text{crys}}(S))$ naturally defines a sheaf of finite projective $\Delta_\bullet(\mathcal{I}_\Delta/p)$-modules $\mathcal{M}_1$. The rational endomorphism $\varphi_{\mathcal{E}}$ induces a Frobenius isomorphism of $\mathcal{M}_1$.

Composing the Frobenius $\Delta_\bullet(\mathcal{I}_\Delta/p) \to \Delta_\bullet(\varphi(\mathcal{I}_\Delta)/p)$ with the natural inclusion $\Delta_\bullet(\varphi(\mathcal{I}_\Delta)/p) \hookrightarrow \Delta_\bullet(\mathcal{I}_\Delta/p)$ (cf. [BS19, Cor. 2.38]) and inverting $p$, we obtain a natural map $\tilde{\varphi}: \Delta_\bullet(\mathcal{I}_\Delta/p)[1/p] \to \Delta_\bullet(\mathcal{I}_\Delta/p)[1/p]$. Use $A_{\text{crys}}$-notation in the following? The extension of scalars of $\mathcal{M}_1$ along $\tilde{\varphi}$ is a sheaf of $\Delta_\bullet(\mathcal{I}_\Delta/p)[1/p]$-modules together with a Frobenius action

$$\mathcal{M}_2 := \mathcal{M}_1 \otimes_{\Delta_\bullet(\mathcal{I}_\Delta/p), \tilde{\varphi}} \Delta_\bullet(\mathcal{I}_\Delta/p).$$

Finally, note that by the proof of [BS21, Lem. 6.7], $\Delta_\bullet[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p) \simeq \Delta_\bullet(\mathcal{I}_\Delta/p)[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p) \simeq B_{dR}^+}$ and $\Delta_\bullet[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p) \simeq \Delta_\bullet(\mathcal{I}_\Delta/p)[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p) \simeq B_{dR}^+}$. In exactly the same way as in Remark 2.24, we can equip the $B_{dR}^+$-linear quasi-syntomic sheaf $\mathcal{E}(\Delta_\bullet(\mathcal{I}_\Delta/p)) \otimes_{\Delta_\bullet(\mathcal{I}_\Delta/p), \tilde{\varphi}} \Delta_\bullet[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p) \simeq B_{dR}^+}$ with a filtration; it is compatible with its pro-étale analog from Remark 2.24 when evaluated at perfectoid algebras. Since the filtration on $B_{dR}^+$ is the $\mathcal{I}_\Delta$-adic one, we can then use Beauville–Laszlo gluing to modify the $\Delta_\bullet(\mathcal{I}_\Delta/p)[1/p]$-module $\mathcal{M}_2$ at $\mathcal{V}(\mathcal{I}_\Delta)$ by the $\Delta_\bullet[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p) \simeq B_{dR}^+}$-module

$$\text{Fil}^0(\mathcal{E}(\Delta_\bullet(\mathcal{I}_\Delta/p)) \otimes_{\Delta_\bullet(\mathcal{I}_\Delta/p), \tilde{\varphi}} \Delta_\bullet[1/p]^{\Delta_\bullet(\mathcal{I}_\Delta/p)}).$$
along the gluing isomorphism
\[
\text{Fil}^0(\mathcal{E}(\Delta_{\{I/\mathbb{A}/p}\}) \otimes_{\Delta_{\{I/\mathbb{A}/p\}}(I/p)^{\Delta_{\{I/\mathbb{A}/p\}}[1/p]} \simeq \mathcal{E}(\Delta_{\{I/\mathbb{A}/p\}}) \otimes_{\Delta_{\{I/\mathbb{A}/p\}}(I/p)^{\Delta_{\{I/\mathbb{A}/p\}}[1/p]} \simeq M_2 \otimes_{\Delta_{\{I/\mathbb{A}/p\}}(I/p)^{\Delta_{\{I/\mathbb{A}/p\}}[1/p]}}
\]

We thus obtain a sheaf of finite projective \(\Delta_{\{I/\mathbb{A}/p\}}[1/p]\)-modules \(M_3\) together with a Frobenius action \(\varphi_{M_3}\) after inverting \(\mathcal{I}_{\mathbb{A}}\). When evaluated at a perfectoid algebra \(S \in X_{w,\text{qrsp}}\), its complete localization at \(V(\mathcal{I}_{\mathbb{A}})\) is
\[
\text{Fil}^0(\mathcal{E}(\mathcal{A}_{\text{cris}}(S)) \otimes_{\mathcal{A}_{\text{cris}}(S)}(I/p)^{\Delta_{\{I/\mathbb{A}/p\}}} \simeq \mathcal{E}(\mathcal{A}_{\text{cris}}(S)[1/p]) \otimes_{\mathcal{B}_{\text{cris}}(S[1/p])} \simeq \mathcal{E}(\mathcal{B}_{\text{cris}}(S[1/p]))
\]

Remark 4.9. As in [BS21, Constr. 6.5], one can construct \(M_3\) independently out of a filtered \(F\)-isocrystal \((\mathcal{E}, \varphi_{\mathcal{E}}, \text{Fil}^\bullet(\mathcal{E}))\), without knowing if it is associated with a local system.

Remark 4.10. Let \(S \in X_{w,\text{qrsp}}\) such that \(T|_{\text{Spec}(S)}\) is trivial. Since the crystalline local system \(T\) is associated with the filtered \(F\)-isocrystal \((\mathcal{E}, \varphi_{\mathcal{E}}, \text{Fil}^\bullet(\mathcal{E}))\), one has a natural isomorphism of \(\Delta_{S,\text{perf}}\)-modules
\[
T(S[1/p]) \otimes_{\mathbb{Z}_p} \Delta_{S,\text{perf}}(1/p^\varphi[I]) \simeq \mathcal{E}(\Delta_{S,\text{perf}}[1/p])[1/\varphi[I]].
\]

In particular, a base change along the Frobenius morphism \(\Delta_{\perfn}(1/p) \to \Delta_{\perfn}(1/p) \to \Delta_{\perfn}(1/p)\) as before gives
\[
T(S[1/p]) \otimes_{\mathbb{Z}_p} \Delta_{S,\text{perf}}(1/p^\varphi[I]) \simeq \mathcal{E}(\Delta_{S,\text{perf}}(1/p^\varphi[I]) \otimes_{\varphi_{\mathcal{E}}} \Delta_{S,\text{perf}}(1/p^\varphi[I]).
\]

Remark 4.11. Similarly to [BS21, Rem. 6.6], by taking Frobenius pullbacks and gluing in the Hodge-Tate lattice at \(V(\mathcal{I}_{\mathbb{A}})\), a prismatic \(F\)-crystal \((\mathcal{M}, \varphi_{\mathcal{M}})\) over \(\Delta_{\perfn}(\mathcal{I}_{\mathbb{A}})[1/p]\) extends uniquely to a prismatic \(F\)-crystal over \(\Delta_{\perfn}(\mathcal{I}_{\mathbb{A}})[1/p]\) for all \(n \in \mathbb{N}\). Explicitly, given \(n \in \mathbb{N}\) and \(\mathcal{M} \in \text{Vect}_{w}(X_{w,\text{qrsp}}, \Delta_{\perfn}(\mathcal{I}_{\mathbb{A}})[1/p])\), we apply Beauville–Laszlo gluing over \(\text{Spec}(\Delta_{\perfn}(\varphi^n(\mathcal{I}_{\mathbb{A}}))[1/p])\) as below:

- over the open subset \(\text{Spec}(\Delta_{\perfn}(\varphi^n(\mathcal{I}_{\mathbb{A}}))[1/p]) \setminus V(\mathcal{I}_{\mathbb{A}} \cdots \varphi^n[I_{\mathbb{A}}])\), take the restriction of \((\varphi_{\mathcal{M}}^n, \mathcal{M})\)
- at each \(V(\mathcal{I}_{\mathbb{A}})\) for \(0 \leq i \leq n - 1\), the complete localization is modified by \(\varphi_{i,\ast} \mathcal{M}^2_{\mathcal{I}_{\mathbb{A}}}\).

Remark 4.12. Philosophically, Construction 4.8 and Remark 4.11 can be thought of as follows: The \(\Delta_{\perfn}(\mathcal{I}_{\mathbb{A}})[1/p]\)-module \(M_3\) from Construction 4.8 is a “sheaf on the analytic disk \(\{|I| \leq |p| \neq 0\}\),” which is obtained from the constant sheaf \(M_2\) by modifying at \(V(\xi)\) with the Hodge-Tate lattice. Moreover, the pullback \(\varphi_{\mathcal{M}}\) considered in Remark 4.11 is a “sheaf on the bigger analytic disk \(\{|I| \leq |p| \neq 0\}\); the Frobenius isomorphism \(\varphi_{\mathcal{M}}: \varphi_{\mathcal{M}}(\mathcal{M}[1/\xi]) \simeq (\varphi_{\mathcal{M}}(\mathcal{M}[1/\xi]) \simeq \mathcal{M}[1/\xi]\) guarantees that the restriction of \(\varphi_{\mathcal{M}}\) to the original disk \(\{|I| \leq |p| \neq 0\}\) agrees with \(\mathcal{M}\) except at the locus \(V(\xi)\), which is further modified with the Hodge–Tate lattice. Continuing in this manner, \(M_3\) can be extended to a “sheaf on the entire open analytic disk \(\{|p| \neq 0\}\).” See Figure 4.12 for a pictorial description.

In the remainder of this section, we extend the \(F\)-crystal on the open unit disk as in Remark 4.12 to the entire analytic locus, yielding the desired analytic prismatic \(F\)-crystal. As a first step in this direction, we perform the construction on \(S \in X_{w,\text{qrsp}}\), using the crystalline local system and its associated \(F\)-isocrystal.

**Theorem 4.13.** One can associate with each \(S \in X_{w,\text{qrsp}}\) an analytic prismatic \(F\)-crystal \(M_S\) in \(\text{Vect}_{w}(\text{Spec}(\Delta_{S,\text{perf}}) \setminus V(p, I))\) such that:
Figure 1. A cartoon of Spa $\Delta_S$. The sheaf $\varphi^{n,*}M_3$ is defined on the area between the vertical coordinate axis and the line $V(\varphi^n(I))$. Its complete localizations at the red lines are the de Rham lattice and the complete localizations at the blue lines the Hodge–Tate lattice. Every time the sheaf is pulled back via $\varphi$, everything is shifted down one line. The red line $V(\xi)$ lands on the blue line $V(\tilde{\xi})$, and Beauville–Laszlo gluing happens at $V(\tilde{\xi})$.

(i) The assignment $S \mapsto M_S$ is functorial. That is, for any arrow $f: S_1 \to S_2$ in $X_{wqrsp}$, there is an isomorphism of vector bundles over $\text{Spec}(\Delta_{S_2,\text{perf}}) \setminus V(p,I)$

$$\beta_f: f^*_{\text{Spec}(\Delta_{S_1,\text{perf}}) \setminus V(p,I)} M_{S_1} \to M_{S_2},$$

which satisfies the cocycle condition for any composition of arrows.

(ii) There is a natural equivalence

$$X_{qsp} \ni S \longmapsto \left( M_S(I/p)[1/p] \simeq M'_{S,\text{perf}} \otimes M_S \right)$$

compatible with (i), where $M' := M_3 \in \text{Vect}^e(X_{qsp}, \Delta(I/p)[1/p])$ is the prismatic $F$-crystal from Construction 4.8.

Here $S_{\text{perf}}$ denotes the perfection of the quasiregular semiperfectoid ring $S \in X_{qsp}$ as in [BS19, §8]. As a consequence, the above defines an analytic $F$-crystal over the perfect prismatic site.

Remark 4.14. The construction of Theorem 4.13 makes essential use of the assumption that the restriction of the local system $T |_{S_{\text{perf}}}$ is trivial, which is true for any object in $X_{qsp}^w$. This is one of the main motivations for the introduction of the subcategory $X_{qsp}^w$ inside $X_{qsp}$.

Remark 4.15. Note that when $S$ is itself perfectoid, $M_S$ is by construction an analytic prismatic $F$-crystal over $\text{Spf}(S)$. Moreover, any automorphism of the perfectoid ring $S$ over $X$ will induce an automorphism of $M_S$, thanks to Theorem 4.13.(i). In particular, when the generic fiber $\text{Spf}(S)_\eta \to X_\eta$ is a Galois pro-étale cover of an open subset of $X_\eta$, the Galois group preserves the subring of integral elements and therefore induces a natural action on $M_S$.

Proof. Let $S \in X_{qsp}^w$, and let $\mathcal{Y} = \text{Spa}(\Delta_{S,\text{perf}}) \setminus V(p,I)$ be the analytic pre-adic space over $\text{Spa}(\mathcal{O}_C, \mathcal{O}_C) \setminus V(p,I)$. By the algebraicity of vector bundles on $\mathcal{Y}$ from [Ked20, Thm. 3.8], it suffices to build an $F$-vector bundle on the pre-adic space $\mathcal{Y}$. Following the convention in [SW20, §12.2], we denote by $\mathcal{Y}_{[0,\infty)}$ the open subset of $\mathcal{Y}$ away from the locus $\{(S^p)^{\infty} = 0\}$, where $S^p$ is the perfection of $S/p$ and $(S^p)^{\infty}$ is the set of topologically nilpotent elements in $S^p$. In order to construct the $F$-vector bundle on $\mathcal{Y}$, we will glue the prismatic $F$-crystal $M' := M_3 \in \text{Vect}^e(X_{qsp}, \Delta(I/p)[1/p])$
on \(|I| \leq |p| \neq 0\) from Construction 4.8 with a different \(F\)-vector bundle \(\mathcal{F}_S\) on \(\mathcal{Y}_{[0,\infty)}\) along the open subset \(\{|I| \leq |p| \neq 0\}\) \(\cap \mathcal{Y}_{[0,\infty)}\) of \(\mathcal{Y}\).

For the construction of \(\mathcal{F}_S\), consider the trivial bundle \(T|_{S_{\text{perf}}} \otimes \mathcal{O}_{\mathcal{Y}_{[0,\infty)}}\) over the pre-adic space \(\mathcal{Y}_{[0,\infty)}\). As the union \(\bigcup_{i \geq 0} V(\varphi^{-i}(I))\) is a closed subset of \(\mathcal{Y}_{[0,\infty)}\), we can use Beauville–Laszlo gluing to modify \(T|_{S_{\text{perf}}} \otimes \mathcal{O}_{\mathcal{Y}_{[0,\infty)}}\) at each locus \(V(\varphi^{-i}(I))\) for \(i > 0\) by the \(\hat{\mathcal{O}}_{\mathcal{Y},V(\varphi^{-i}(I))}\)-module

\[
\varphi^*(\mathcal{E}(\Delta_{S_{\text{perf}}} \{\varphi^{-i}(I)/p\})) \otimes \Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i}(I)}.
\]

Here, we identify the ring \(\Delta_{S_{\text{perf}}} \{\varphi^{-i+1}(I)/p\}\) with the universal \(p\)-adic \(\mathcal{O}_{\text{perf}}\)-envelope \(D_{(p,\varphi^{-i}(I))}(\Delta_{S_{\text{perf}}})\) over \(\mathbb{Z}_p\) ([BS19, Cor. 2.38]), which maps to the complete local ring \(\Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i}(I)} = \hat{\mathcal{O}}_{\mathcal{Y}_{[0,\infty)}, V(\varphi^{-i}(I))}\). To obtain the gluing isomorphism, we twist the de Rham comparison isomorphism

\[
\varphi^*(\mathcal{E}(\Delta_{S_{\text{perf}}} \{I/p\})) \otimes \Delta_{S_{\text{perf}}} [1/p]_{I} \simeq T(S_{\text{perf}}[1/p]) \otimes \Delta_{S_{\text{perf}}} [1/p]_{I}^\wedge
\]

along the natural isomorphism \(\varphi^{-i} \colon \Delta_{S_{\text{perf}}} [1/p]_{I} \rightarrow \Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i}(I)}\); the crystal condition for \(\mathcal{E}\) guarantees that

\[
\varphi^{-1,*} \left( \mathcal{E}(\Delta_{S_{\text{perf}}} \{\varphi^{-i}(I)/p\}) \otimes \Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i}(I)} \right)
\]

\[
= \mathcal{E}(\Delta_{S_{\text{perf}}} \{\varphi^{-i}(I)/p\}) \otimes \Delta_{S_{\text{perf}}} \{\varphi^{-i-1}(I)/p\} \Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i-1}(I)}
\]

\[
\simeq \mathcal{E}(\Delta_{S_{\text{perf}}} \{\varphi^{-i-1}(I)/p\}) \otimes \Delta_{S_{\text{perf}}} \{\varphi^{-i-1}(I)/p\} \Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i-1}(I)}.
\]

This produces a vector bundle \(\mathcal{F}_S\) over \(\mathcal{Y}_{[0,\infty)}\).

To glue \(\mathcal{F}_S\) with \(\mathcal{M}'\), we need to check the compatibility of \(\mathcal{F}_S\) with \(\mathcal{M}'_{\text{perf}}\) after restriction to the open subset \(\{|I| \leq |p| \neq 0\} \cap \mathcal{Y}_{[0,\infty)}\), where \(\mathcal{M}'_{\text{perf}}\) is the evaluation of \(\mathcal{M}'\) at \(S_{\text{perf}}\). Away from \(V(I)\), the restriction of \(\mathcal{M}'_{\text{perf}}\) is \(\mathcal{E}(\Delta_{S_{\text{perf}}} \{I/p\}) \otimes \Delta_{S_{\text{perf}}} \{I/p\} \Delta_{S_{\text{perf}}} [1/p]_{I}\). In particular, by the crystalline comparison isomorphism from Remark 4.10, this is naturally isomorphic to \(T \otimes \Delta_{S_{\text{perf}}} \{I/p\}[1/I]\), which coincides with \(\mathcal{F}_S\) when restricted to the open subset

\[
\left( \mathcal{Y}_{[0,\infty)} \setminus \bigcup_{i \geq 0} V(\varphi^{-i}(I)) \right) \cap \{|I| \leq |p| \neq 0\}.
\]

Moreover, over each of the loci \(V(\varphi^{-i}(I))\) for \(i \geq 0\), both \(\mathcal{F}_S\) and \(\mathcal{M}'_{\text{perf}}\) have the same completions by their constructions (cf. Construction 4.8, Remark 4.11) and the twisted de Rham comparison

\[
T \otimes \hat{\mathcal{O}}_{\mathcal{Y},V(\varphi^{-i}(I))} \simeq \text{Fil}^0 \left( \mathcal{E}(\Delta_{S_{\text{perf}}} \{\varphi^{-i}(I)/p\}) \otimes \Delta_{S_{\text{perf}}} \{\varphi^{-i}(I)/p\} \Delta_{S_{\text{perf}}} [1/p]_{\varphi^{-i}(I)} \right).
\]

This finishes the gluing process, and in particular proves the compatibility of the underlying vector bundles in (ii).

Note that by the assumption that the local system \(T\) is defined over \(X_\eta\), the crystal property of \(\mathcal{E}\), and the fact that \(\mathcal{M}'\) is a prismatic \(F\)-crystal over \(X_{\text{qsp}}\), one has the natural pullback compatibility

\[
\mathcal{M}_{S_1} \otimes \Delta_{S_{1,\text{perf}}} \Delta_{S_{2,\text{perf}}} \simeq \mathcal{M}_{S_2}
\]

for any arrow \(S_1 \rightarrow S_2\) in \(X_{\text{qsp}}^w\) (Definition 4.6). This finishes part (i).

Lastly, there is a natural Frobenius structure \(\varphi_{\mathcal{M}_S}\) of \(\mathcal{M}_S\) away from \(V(I)\). Over the open subset \(\mathcal{Y}_{[0,\infty)} \setminus \bigcup_{i \geq 0} V(\varphi^{-i}(I))\), it is induced by the pullback of the structure sheaf

\[
\varphi^*(T \otimes \mathcal{O}_{\mathcal{Y}_{[0,\infty)}} \bigcup_{i \geq 0} V(\varphi^{-i}(I))[1/I]) \simeq (T \otimes \mathcal{O}_{\mathcal{Y}_{[0,\infty)}}) \bigcup_{i \geq 0} V(\varphi^{-i}(I))[1/I].
\]

Over the closed complement \(\bigcup_{i \geq 0} V(\varphi^{-i}(I))\), we notice that by the crystal property of \(\mathcal{E}\), similarly to the second paragraph above, one has the natural isomorphism below coming from pullback along
Proof. The proof is identical to that of [BS21, Lem. 6.9], replacing the ring assumptions of [BS21, Prop. 6.8].

On the other hand, as the crystalline comparison in Remark 4.10 is Frobenius equivariant, running again the argument in the last paragraph, we see that the gluing process for $\mathcal{M}_S$ is compatible with the Frobenius. Thus, we get a Frobenius isomorphism of $\mathcal{M}_S$ away from $V(I)$. \qed

Remark 4.16. If we complete the construction above at $V(p)$, then we get for each perfectoid algebra $S \in X^w_{\text{qp}}$ that

$$(\mathcal{M}_S \otimes \mathcal{O}_T \mathcal{O}_T[1/I])^{\wedge}_p = (T \otimes \mathcal{O}_T[1/I])^{\wedge}_p = T \otimes \Delta_S[1/I]^{\wedge}_p.$$ 

In particular, by [BS21, Prop. 3.5] and Theorem 4.13.(i), the étale realization of $\mathcal{M}$ descends to the original crystalline local system $T$.

We now prove the boundedness of the descent data at the boundary of the open unit disk in the Fargues–Fontaine curve. As a preparation, we use the Beilinson fiber square as in [BS21, §6.3] to study the structure of the initial prism for a quasiregular semiperfectoid algebra.

Lemma 4.17. Let $X = \text{Spf}(R)$ be a $p$-adic smooth formal $\mathcal{O}_K$-scheme, let $S$ be a perfectoid $\mathcal{O}_C$-algebra in $X^w_{\text{qp}}$, and let $S'$ be the $p$-complete tensor product $S \otimes_R S$. Write $q_1, q_2 \in \Delta_{S'}$ for the image of $q \in \Delta_{\mathcal{O}_C}$ along the two structure maps $\mathcal{O}_C \to S$.

(i) The structure map

$$\Delta_{S'}(1/(p(q_1 - 1)(q_2 - 1)))^{\wedge} = 1 \to \Delta_{S'}(1/(p(q_1 - 1)(q_2 - 1)))^{\wedge}$$

is an isomorphism on cohomology in degree 0.

(ii) The element $(q_1 - 1)(q_2 - 1)$ is invertible in $\Delta_{S'}(p/(q_1 - 1)^2)[1/p]$.

(iii) The natural maps of rings give rise to a short exact sequence

$$0 \to \Delta_{S'} \to \Delta_{S'}(p/(q_1 - 1)^2/p) \to \Delta_{S'}(p/(q_1 - 1)^2/p) \to 0.$$ 

Proof. The proof is identical to that of [BS21, Lem. 6.9], replacing the ring $R$ there by our ring $S'$. Note that since $R$ is flat over $\mathcal{O}_K$ and $R \to S$ is a quasi-syntomic map, the $p$-complete tensor product $S'$ is a $p$-torsionfree quasiregular semiperfectoid algebra over $\mathcal{O}_C$, and hence satisfies the assumptions of [BS21, Prop. 6.8]. \qed

Using the above lemma, we are able to extend the descent data from the locus $\{|I| \leq |p| \neq 0\}$ to the whole open subset $\{p \neq 0\}$.

Proposition 4.18. Assume $X = \text{Spf}(R)$ is affine over $\mathcal{O}_K$.

(i) Let $S \in X^w_{\text{qp}}$ be a perfectoid $\mathcal{O}_C$-algebra covering $X$. Let $\mathcal{M} \in \text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(I,p))$ be an analytic prismatic $F$-crystal on $S$. Assume that $\mathcal{M}(I/p)[1/p]$ can be descended to $\text{Vect}^\varphi(X_{\text{qp}}, \Delta(T/I/p)[1/p])$. Then the submodule $\mathcal{M}(I/p)[1/p]$ can be descended to $\text{Vect}^\varphi(X_{\text{qp}}, \Delta[1/p])$.

(ii) The descent process in (i) is functorial in the following sense: assume $f: S_1 \to S_2$ is a quasi-syntomic map of perfectoid rings in $X^w_{\text{qp}}$ and $\mathcal{M}_1 \in \text{Vect}^\varphi(\text{Spec}(\Delta_{S_1}) \setminus V(I,p))$ such that

- there is a pullback isomorphism $\beta_f: f^*_{\text{Spec}(\Delta_S) \setminus V(I,p)} \mathcal{M}_1 \to \mathcal{M}_2$;

- both $\mathcal{M}_i(I/p)[1/p]$ descend to the same object of $\text{Vect}^\varphi(X_{\text{qp}}, \Delta(T/I/p)[1/p])$ and the evaluations of this object at the structure maps

$$R \longrightarrow S_1, \quad R \longrightarrow S_1 \longrightarrow S_2$$ 

are compatible with $\beta_f(I/p)[1/p]$.


Then the descent process in (i) for $\mathcal{M}_i[1/p]$ yields the same object, and the evaluation maps of this object are compatible with $\beta_f[1/p]$.

Proof. We follow the proof of [BS21, Prop. 6.10]. First, note that for any quasi-syntomic $p$-adic formal scheme $Y$, a prismatic $F$-crystal on $Y_{\text{q}}$ over $\Delta_s(\mathcal{I}_\wedge/p)[1/p]$ extends uniquely to one over $\Delta_s(\varphi^n(\mathcal{I}_\wedge)/p)[1/p]$ for any $n \in \mathbb{N}$ by Remark 4.11. In particular, a descent datum for $\mathcal{M}(I/p)[1/p]$ uniquely extends to one for $\mathcal{M}(\varphi^n(I)/p)[1/p]$.

Let $S'$ be the $p$-completed tensor product $S \otimes_R S$. It is a quasicrregular semiperfectoid $R$-algebra. For $i = 1, 2$, let $p_i: \text{Spec}(\Delta_{S'}) \to \text{Spec}(\Delta_S)$ be the two projection maps. By assumption, there is an isomorphism of $\Delta_{S'}/(q_1 - 1)\varphi/p)[1/p]$-modules

$$\alpha_{(I/p)}: p_1^*\mathcal{M}(I/p)[1/p] \to p_2^*\mathcal{M}(I/p)[1/p],$$

satisfying the cocycle condition for descent data. As the natural ring map $\Delta_{S'}/[1/p] \to \Delta_{S'/}(I/p)[1/p]$ is injective for $S' \in X_{\text{q}}$, we want to show that $\alpha_{(I/p)}$ induces an isomorphism of the submodules $p_1^*\mathcal{M}[1/p]$ and $p_2^*\mathcal{M}[1/p]$, which means that the descent datum extends to one for $\Delta_S[1/p]$. In fact, by contemplating the inverse, it suffices to prove that $\alpha_{(I/p)}$ sends $p_1^*\mathcal{M}[1/p]$ into $p_2^*\mathcal{M}[1/p]$.

The inclusion $\alpha_{(I/p)}(p_1^*\mathcal{M}[1/p]) \subseteq p_2^*\mathcal{M}[1/p]$ will prove (ii) as well: Indeed, let $S_1 \to S_2$ be a map of perfectoid rings together with $\mathcal{M}_i \in \text{Vect}^w(\text{Spec}(\Delta_S) \setminus V(I, p))$ and a pullback isomorphism $\beta_f$ satisfying the assumption of (ii). Then the pullback of the descent isomorphism $\alpha_{1,(I/p)}$ along $f \otimes_R f$ preserves the submodules $p_1^*\mathcal{M}_2[1/p]$ and is identified with $\alpha_{2,(I/p)}$ via the isomorphism $\beta_f$. Thus, the compatibility of the descent isomorphism $\alpha_{i,(I/p)}$ with the pullback isomorphism $\beta_f$ implies that both $\mathcal{M}_i[1/p]$ descend to the same object in $\text{Vect}^w(X_{\text{q}}, \Delta_{S'}[1/p])$.

The strategy is then to extend the descent data to the ring $\Delta_{S'/}{(q_1 - 1)\varphi/p)[1/p]}$ and the ring $\Delta_{S'/}(p/(q_1 - 1)\varphi/p)[1/p]$ separately, and use the short exact sequence in Lemma 4.17.(iii) to glue to one over $\Delta_{S'/}[1/p]$. We carry out this strategy in the following three steps:

Step 1 To extend the isomorphism $\alpha_{(I/p)}$ to $\alpha_{((q_1 - 1)\varphi/p)}$ over the ring $\Delta_{S'/}(q_1 - 1)\varphi/p)[1/p]$, recall from the first paragraph of the proof that there is a unique extension $\alpha_{\varphi(I/p)}$ of $\alpha_{(I/p)}$. We can thus take the base change of $\alpha_{\varphi(I/p)}$ along the natural map of rings

$$\Delta_{S'/}(q_1 - 1)\varphi/p)[1/p] \to \Delta_{S'/}(q_1 - 1)\varphi/p)[1/p];$$

this map exists since the element $\varphi(q_1 - 1)/p$ is contained in $\Delta_{S'/}(q_1 - 1)\varphi/p)$. 

Step 2 To extend the isomorphism $\alpha_{(I/p)}$ to the ring $\Delta_{S'/}(p/(q_1 - 1)^\varphi/p)[1/p]$, we use the étale realization functor from Construction 3.12.(i). The étale realization $T(\mathcal{M})$ of the analytic prismatic $F$-crystal $\mathcal{M}$ is a local system on $\text{Spa}(S[1/p], S)$. By our assumption of $S \in X_w$, the $\mathbb{Z}_p$-local system $T(\mathcal{M})$ is trivializable and is determined by its global sections, which form a finite free $\mathbb{Z}_p$-module (cf. Definition 4.6). Moreover, by Lemma 3.15 and Remark 3.13, we have the following Frobenius equivariant isomorphism of $\Delta_S[1/(q_1 - 1)]$-modules:

$$T(\mathcal{M}) \otimes \mathbb{Z}_p \Delta_S[1/(q_1 - 1)] \simeq \mathcal{M}[1/(q_1 - 1)].$$

Furthermore, after choosing a basis of $T(\mathcal{M})$, the base change of $\alpha_{(I/p)}$ to $\Delta_{S'/}(I/p)[1/p(q_1 - 1)(q_2 - 1)]$ corresponds to a matrix in $\Delta_{S'/}(I/p)[1/p(q_1 - 1)(q_2 - 1)]$ that is invariant under the Frobenius morphism. Thanks to Lemma 4.17.(i), the matrix has entries in the subring $\Delta_{S'}[1/p(q_1 - 1)(q_2 - 1)]$. In particular, as the element $(q_1 - 1)(q_2 - 1)$ is invertible in $\Delta_{S'}[p/(q_1 - 1)^\varphi/p)[1/p]$ by Lemma 4.17.(ii), a further base change to $\Delta_{S'}[p/(q_1 - 1)^\varphi/p)[1/p]$ extends $\alpha_{(I/p)}$ to an isomorphism

$$\alpha_{(p/(q_1 - 1)p)}: p_1^*\mathcal{M}(p/(q_1 - 1)^\varphi/p)[1/p] \to p_2^*\mathcal{M}(p/(q_1 - 1)^\varphi/p)[1/p].$$

Step 3 Finally, note that both $\alpha_{(q_1 - 1)^\varphi/p}$ and $\alpha_{(p/(q_1 - 1)p)}$ extend $\alpha_{\varphi(I/p)}$ along the natural inclusions of rings. In particular, they agree after base change to $\Delta_{S'}[p/(q_1 - 1)^\varphi/p][1/p]$. As a consequence, by gluing $\alpha_{(q_1 - 1)^\varphi/p}$ and $\alpha_{(p/(q_1 - 1)p)}$ along the short exact sequence from
Lemma 4.17.(iii), we get a unique map $\alpha : p_1^* M[1/p] \to p_2^* M[1/p]$ over $\Delta_S[1/p]$ compatible with all the other descent isomorphisms, which finishes the proof. \hfill \square

An application of Proposition 4.18 to the construction of Theorem 4.13 immediately yields the following:

**Corollary 4.19.** Assume that $X = \text{Spf}(R)$ is affine over $\mathcal{O}_K$. Let $S \in X^{w}_{\text{qsp}}$ be a ring that covers $X$. Then the $p$-inverted prismatic $F$-crystal $M_{S}[1/p] \in \text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I))$ from Theorem 4.13 descends to a prismatic $F$-crystal $\tilde{M}[1/p] \in \text{Vect}^\varphi(X_{\text{qsp}}, \Delta_{\bullet}[1/p])$, and the latter is independent of choice of $S \in X^{w}_{\text{qsp}}$.

We are finally ready to prove the essential surjectivity theorem.

**Proof of Theorem 4.5.** Let us first assume $X = \text{Spf}(R)$ is affine. Let $S \in X^{w}_{\text{qsp}}$ be any perfectoid $\mathcal{O}_C$-algebra that covers $X$ in the quasi-syntomic site and let $S' = S \widehat{\otimes}_R S$ be the $p$-completed tensor product. As $S' \in X_{\text{qsp}}$ and the two structure maps $S \to S'$ are $p$-completely flat covers, the associated maps of initial prisms $\Delta_S \to \Delta_{S'}$ are also $(p, I)$-completely flat covers by [BS19, Prop. 7.11] (see the footnote in Lemma 3.6). In particular, by taking the colimit of $\Delta_S \to \Delta_{S'}$ with respect to Frobenius and reducing mod $I$, we see the induced maps of perfectoid algebras

$$p_i : S \longrightarrow S'_{\text{perf}} = (\text{colim}_{\varphi_{S'}} \Delta_{S'})/I.$$  

are $p$-completely faithful flat; cf. [BS19, §8.1]. As a consequence, the perfectoid ring $S'_{\text{perf}}$ is a quasi-syntomic cover of $\text{Spf}(S)$ and of $X$ under the structure maps. The same also holds for the perfection of $p$-completed tensor products of $S$ over $R$.

Using Corollary 4.19, we obtain a prismatic $F$-crystal $\tilde{M}[1/p] \in \text{Vect}(X_{\text{qsp}}, \Delta_{\bullet}[1/p])$, which is the descent of the two analytic prismatic $F$-crystals $M_S \in \text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I))$ and $M_{S'} \in \text{Vect}^\varphi(\text{Spec}(\Delta_{S'}) \setminus V(p, I))$ constructed in Theorem 4.13 after inverting $p$. So to prove that $M_S$ descends to an analytic prismatic $F$-crystal on $X$, it suffices to show that $M_S[1/I]_{\tilde{p}}$ also descends compatibly with $\tilde{M}[1/p]$. Applying Theorem 4.13.(i) to the two canonical maps $p_i$, we obtain a natural isomorphism of vector bundles over $\text{Spec}(\Delta_{S'}_{\text{perf}}) \setminus V(p, I)$

$$(*) \quad p_{1, \text{Spec}(\Delta_{\bullet_{\text{perf}}})}^* M_S \simeq p_{2, \text{Spec}(\Delta_{\bullet_{\text{perf}}}) \setminus V(p, I)}^* M_{S'},$$

which is compatible with the descent data of $\tilde{M}[1/p]$ by Proposition 4.18.(ii). In particular, applying $[-]_{\tilde{p}}$ to the isomorphism in $(*)$, we get the following descent isomorphism over $\Delta_{S', \text{perf}}[1/I]_{\tilde{p}}$, which is compatible with the descent data of $\tilde{M}[1/p]$:

$$p_1^* M[1/I]_{\tilde{p}} \simeq p_2^* M[1/I]_{\tilde{p}}.$$  

By Beauville–Laszlo gluing, we can combine the two descent data to an isomorphism $p_1^* M_S \simeq p_2^* M_S$, which lets us descent $M_S$ to the desired analytic prismatic $F$-crystal. Its étale realization recovers the original local system $T$ by Remark 4.16. So we are done with the affine case.

To extend everything to the general case, we want to show that the above process is compatible with respect to open immersions from affine subsets. By taking an affine open cover of $X$ and refining the intersections, we reduce to the case where $X = \text{Spf}(R)$ admits two quasi-syntomic covering maps $f_i : \text{Spf}(S) \to \text{Spf}(R)$ by a perfectoid algebra $S$ in $X^{w}_{\text{qsp}}$. By the same argument as in the first paragraph above, the perfection of $S \widehat{\otimes}_{f_i, R} S$ is also a quasi-syntomic cover of $S$ under either of the structure maps. The rest follows by exactly the same strategy as in the previous paragraph: By Corollary 4.19, each of the structure maps $f_i$ induces an object $M_{S,f_i} \in \text{Vect}^\varphi(\text{Spec}(\Delta_S) \setminus V(p, I))$, and their pullbacks to $\text{Vect}^\varphi(\text{Spec}(\Delta_{S'}_{\text{perf}}) \setminus V(p, I))$ are isomorphic to $M_{S'}$. So thanks to Theorem 4.13.(ii), all of them descend to a single object $\tilde{M}[1/p]$ after inverting.
Moreover, at the locus $V(p)$, this follows from the pullback compatibility as in Theorem 4.13.(i). Thus, the descended objects of $\mathcal{M}_{S,f}$ coincide with that of $\mathcal{M}_{(S \otimes_{f_1} f_2)_{\text{perf}}}$, and we are done. □

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