Momentum of superconducting electrons and the explanation of the Meissner effect

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Momentum and energy conservation are fundamental tenets of physics, that valid physical theories have to satisfy. In the reversible transformation between superconducting and normal phases in the presence of a magnetic field, the mechanical momentum of the supercurrent has to be transferred to the body as a whole and vice versa, the kinetic energy of the supercurrent stays in the electronic degrees of freedom, and no energy is dissipated in the process. We argue on general grounds that to explain these processes it is necessary that the electromagnetic field mediates the transfer of momentum between electrons and the body as a whole, and this requires that when the phase boundary between normal and superconducting phases is displaced, a flow and counterflow of charge occurs in direction perpendicular to the phase boundary. This flow and counterflow does not occur according to the conventional BCS-London theory of superconductivity, therefore we argue that within BCS-London theory the Meissner transition is a ‘forbidden transition’. Furthermore, to explain the phase transformation in a way that is consistent with the experimental observations requires that (i) the wavefunction and charge distribution of superconducting electrons near the phase boundary extend into the normal phase, and (ii) that the charge carriers in the normal state have hole-like character. The conventional theory of superconductivity does not have these physical elements, the theory of hole superconductivity does.

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I. INTRODUCTION

Experiments [1, 2] and theory [3] show that under ideal conditions the superconductor to normal transition in the presence of a magnetic field is a reversible phase transformation between equilibrium states of matter that occurs without energy dissipation and without increase in the entropy of the universe. In this paper we argue that the conventional BCS-London theory of superconductivity [4, 5] cannot explain how mechanical momentum is conserved in this transition, and for this reason BCS-London theory as it stands is not a viable theory of superconductivity for any superconductor. In other words, BCS theory predicts that the Meissner transition is a ‘forbidden transition’ [6], in contradiction with experiment [1, 2]. Instead, we point out that the alternative theory of hole superconductivity [7] explains how the transition occurs conserving mechanical momentum.

We restrict ourselves to non-relativistic electrons, which is sufficient for most solids. In the absence of electric current the average mechanical momentum of electrons at any point in space is zero. In the presence of an electric current, the mechanical momentum density of electrons at position $\mathbf{r}$ is [8, 9]

$$\mathbf{P}(\mathbf{r}) = \frac{m_e}{e} \mathbf{J}(\mathbf{r})$$

(1)

where $\mathbf{J}(\mathbf{r})$ is the current density at position $\mathbf{r}$, $m_e$ the bare electron mass and $e$ ($<0$) the electron charge. Consider a cylinder of radius $R$ and height $h$ in a uniform magnetic field $H$ parallel to its axis pointing in the $\hat{z}$ direction, hanging from a thread of negligible torsion coefficient. Assume the material is a type I superconductor with thermodynamic critical field $H_c$ and London penetration depth $\lambda_L$, initially in the normal state, and the body is at rest. When it is cooled into the superconducting state the magnetic field is expelled from the interior [1] (assuming $H < H_c$) through the development of a surface current

$$I = \frac{c}{4\pi} hH.$$  

(2)

$I$ flows within a London penetration depth of the surface, so the current density is

$$\mathbf{J} = -\frac{c}{4\pi\lambda_L} H \hat{\mathbf{z}}.$$  

(3)

as follows from Ampere’s law $\nabla \times \mathbf{H} = (4\pi/c)\mathbf{J}$ and the requirement that $\mathbf{B} = 0$ inside the superconductor. Therefore the electrons acquired a non-zero momentum. The momentum density of the supercurrent is, from Eqs. (1) and (3)

$$\mathbf{P} = -\frac{m_e c}{4\pi\lambda_L e} H \hat{\mathbf{z}}.$$  

(4)

in a volume $2\pi R\lambda_L h$, hence the total angular momentum of the supercurrent is

$$\mathbf{L}_e = -\frac{m_e c}{2e} hR^2 H \hat{z}.$$  

(5)

Note that $L_e$ is proportional to the bare electron mass $m_e$.

$L_e$ is a macroscopic angular momentum carried by the supercurrent. For example, for $R = 1cm$, $h = 5cm$ and $H = 200G$, $L_e = 2.84mg-mm^2/s$. From angular momentum conservation we conclude that the body as a whole must rotate carrying equal and opposite angular momentum, since the total angular momentum before the system was cooled was zero and no angular momentum was
FIG. 1: Process I: a magnetic field is applied to a superconductor at rest. The body acquires angular momentum \( \mathbf{L}_i \) antiparallel to the applied magnetic field. The supercurrent acquires angular momentum \( \mathbf{L}_e = -\mathbf{L}_i \) parallel to the magnetic field. \( E_F \) is the Faraday electric field that exists during the process, which is clockwise as seen from the top.

imparted to the system as a whole upon cooling. Measurement of the body's angular momentum was never done this way, but instead by applying a magnetic field to an already superconducting body, which will develop a screening current and angular momentum equal to those given by Eqs. (3) and (5). Indeed the body is found to rotate with angular momentum given by Eq. (5) \([10, 11]\). This is called the 'gyromagnetic effect'.

Thus we cannot doubt that supercurrents carry mechanical momentum and angular momentum. Momentum conservation is a universal law of physics, hence when the state of the system changes so that the supercurrent changes, the change in the angular momentum of the supercurrent must be accounted for. Quantitatively this momentum is certainly non-negligible. Consider that a typical current density in superconductors is of order \( 10^8 \text{A/cm}^2 \), much higher than current densities in normal metals. It is well known that for normal state current densities of order \( 10^6 \text{A/cm}^2 \) one begins to see significant electromigration effects \([12]\), where the momentum of the conduction electrons is transferred to individual ions causing actual displacement of the ions. Such effects are not seen in superconductors, Moreover, no Joule heat is dissipated in the superconductor to normal transition in a magnetic field \([2, 13–15]\). Therefore, superconductors need to have a way to transfer the large momentum of the supercurrent to the body as a whole that is different from the way normal electrons do it. A theory of superconductivity that cannot describe this momentum transfer process cannot account for momentum conservation and hence cannot be a valid theory of superconductivity.

II. WHAT NEEDS TO BE EXPLAINED

Consider three different processes in which a superconducting cylinder hanging from a thread will acquire angular momentum, shown in Figs. 1, 2 and 3. We assume, consistent with the conventional theory of superconductivity and with experiment, that all processes are reversible.

**Process I:** The body is at rest at temperature \( T < T_c \), and a magnetic field \( H < H_c(T) \) is applied (Fig. 1). A clockwise supercurrent develops to prevent the magnetic field from penetrating its interior, and the body starts to rotate in the clockwise direction (as seen from the direction where the magnetic field is pointing).

**Process II:** The body is at rest in a magnetic field \( H \) and has a clockwise supercurrent preventing the magnetic field from entering its interior. Electrons in the
superconducting current are moving counterclockwise. The temperature is raised to slightly above \( T_c(H) \), the body enters the normal state, the supercurrent stops and the body starts rotating in counterclockwise direction (Fig. 2).

**Process III:** The body is at rest in a uniform magnetic field and initially at temperature \( T > T_c(H) \). The temperature is lowered, the body enters the superconducting state and expels the magnetic field from its interior and starts rotating in clockwise direction (Fig. 3), while electrons in the generated supercurrent move in counterclockwise direction.

All these processes conserve total mechanical angular momentum (of the electrons in the supercurrent plus the ions in the body). Since there are no electric fields in the initial and final states, there is no momentum in the electromagnetic field. We also assume that the processes are slow enough that no momentum is carried away by electromagnetic radiation. We will argue that only process I can be explained by the conventional theory of superconductivity. Note that only in process I is the direction of the Faraday electric field that develops in the process, \( E_F \), parallel to the motion of the ions, in processes II and III it is antiparallel. In the following we discuss these three processes.

### A. Process I

As the magnetic field is applied, an azimuthal Faraday electric field develops in the region within \( \lambda_L \) of the surface of the cylinder in clockwise direction, given by

\[
E_F = \frac{\lambda_L}{c} \frac{\partial H}{\partial t}
\]  

(6)

assuming the magnetic field penetrates a distance \( \lambda_L \). We have for the velocity of a Bloch electron

\[
\vec{v}_k = \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial \vec{k}}
\]

(7)

where \( \epsilon_k \) is the band energy. Within semiclassical transport theory the equation of motion is

\[
\frac{d\vec{v}_k}{dt} = \frac{1}{\hbar^2} \frac{\partial^2 \epsilon_k}{\partial \vec{k} \partial \vec{k}} \frac{d(h\vec{k})}{dt} = \frac{1}{\hbar^2} \frac{\partial^2 \epsilon_k}{\partial \vec{k} \partial \vec{k}} (e\vec{E}_F)
\]

(8)

Assuming an isotropic band and defining

\[
\frac{1}{m_k^*} = \frac{1}{\hbar^2} \frac{\partial^2 \epsilon_k}{\partial \vec{k} \partial \vec{k}}
\]

(9)

the equation of motion is

\[
\frac{d\vec{v}_k}{dt} = \frac{1}{m_k^*} (e\vec{E}_F) = \frac{1}{m_k^*} \frac{e\lambda_L}{c} \frac{\partial H}{\partial t}
\]

(10)

and the change in velocity of the electron when the magnetic field increases from 0 to \( H \) is

\[
\Delta v_k = \frac{1}{m_k^*} \frac{e\lambda_L}{c} H
\]

(11)

The change in electronic momentum is \( m_v \Delta v_k \), so the total change in electronic momentum is

\[
P_e \equiv \sum_{k \, \text{occ}} m_v \Delta v_k = \sum_{k \, \text{occ}} \frac{m_v}{m_k^*} \frac{e\lambda_L}{c} H
\]

(12)

where the sum over \( k \) here and in what follows is over the *occupied* states in the band. The current density that develops is

\[
J = \frac{1}{V} \sum_{k \, \text{occ}} e\Delta v_k = \frac{1}{V} \sum_{k \, \text{occ}} \frac{1}{m_k^*} \frac{e^2\lambda_L}{c} H
\]

(13)

and using eq. (3) yields for the penetration depth

\[
\frac{1}{\lambda_L^2} = \frac{4\pi e^2}{c^2} \left( \frac{1}{V} \sum_{k \, \text{occ}} \frac{1}{m_k^*} \right)
\]

(14)

Using the expression for the current density Eq. (13) to replace the sum over \( k \) in Eq. (12) and the expression for the current density Eq. (3) yields for Eq. (12)

\[
P_e = V \frac{m_v}{4\pi \lambda_L c} H
\]

(15)

with \( V = 2\pi R \lambda_L h \), in agreement with Eq. (4), and yields Eq. (5) for the total angular momentum acquired by the electrons.

Now the equation of motion for a Bloch electron is

\[
m_v \frac{d\vec{v}}{dt} = e\vec{E}_F + \vec{F}_\text{latt}
\]

(16)

where \( \vec{F}_\text{latt} \) is the force exerted by the ionic lattice on the electron of wavevector \( k \). Using Eqs. (8) and (9) we obtain

\[
\vec{F}_\text{latt} = \frac{m_v}{m_k^*} - 1)\vec{E}_F
\]

(17)

and the total force exerted by the lattice on the electrons is

\[
\vec{F}_\text{latt} = \sum_{k \, \text{occ}} \left( \frac{m_v}{m_k^*} - 1 \right) e\vec{E}_F
\]

(18)

By Newton’s third law, the total force exerted by the electrons on the lattice is then

\[
\vec{F}_{\text{on-latt}} = -\vec{F}_\text{latt} = - \sum_{k \, \text{occ}} \left( \frac{m_v}{m_k^*} - 1 \right) e\vec{E}_F
\]

(19)

The Faraday electric field also exerts a force on the positive ions. Assuming charge neutrality we have the same number of positive ions (charges) as negative electrons in the band, and the total force exerted on the ions (labeled by \( i \)) is

\[
\sum_i \vec{F}_i = \sum_i \frac{m_i}{m_i} \frac{d\vec{v}_i}{dt} = \sum_i |e|\vec{E} + \vec{F}_{\text{on-latt}}
\]

(20)
and using Eq. (19)
\[ \sum_i \vec{F}_i = \sum_i m_i \frac{d\vec{v}_i}{dt} = -\sum_{k \text{ occ}} m_e \frac{e_0 e}{m_e} \vec{E}_F \]
yielding for the total change in ionic momentum
\[ P_i = \sum_i m_i \Delta v_i = -\sum_{k \text{ occ}} m_e \frac{e_0 e}{m_e} \frac{e_0 e}{c} H = -P_e \]
hence the total angular momentum acquired by the ions is
\[ L_i = \frac{m_e c}{2|e|} \hbar R^2 H = -L_e. \]

Thus, for a charge neutral system the electrons and ions acquire equal and opposite momenta and angular momenta, as one would expect. The way the ions acquire momentum and angular momentum is partly from the Faraday field itself and partly from the force exerted by the electrons on the ions, as seen from Eq. (20). Irrespective of this, the total angular momentum acquired by the body (and the electrons) is independent of \( m^* \) and hence of the interactions between electrons and ions, as seen from Eq. (23). The interactions between electrons and ions only enter in determining the magnitude of the London penetration depth as seen from Eq. (14).

B. Process II

In process II (Fig. 2), the angular momentum of the supercurrent \( L_e \) given by Eq. (5) in direction parallel to the applied field has to be transferred in its entirety to the body as a whole when the system goes normal. In other words, the angular momentum of the electrons has to go from \( L_e \) to zero and that of the ions from 0 to \( L_e \).

The angular momentum of electrons and ions will change due to (i) electromagnetic forces, and (ii) interaction between electrons and ions. Let us examine them in turn:

1. Electromagnetic forces

As the magnetic field lines enters the body, a Faraday electric field pointing clockwise is generated throughout the interior of the cylinder (Fig. 2), that tries to prevent the magnetic field from entering (Lenz’s law). This electric field imparts momentum to electrons in counterclockwise direction and to ions in clockwise direction. Thus this momentum transfer is in direction exactly opposite to what is needed to reach the final state, where the counterclockwise electron current has stopped and the ions rotate in counterclockwise direction.

Can there be a magnetic Lorentz force in the azimuthal direction? It can result from radial motion of charge. Since the ionic charge cannot undergo radial motion, a magnetic force cannot be the source of angular momentum for the ions. For the electrons there could in principle be radial motion, however there is no such motion within the conventional theory of superconductivity.

2. Electron-ion forces

The Coulomb interaction between electrons and ions can transfer momentum between the two subsystems. Initially the momentum of the supercurrent is carried by electrons bound in Cooper pairs. As the system becomes normal, Cooper pairs unbind and become normal quasiparticles, and the supercurrent stops. Within the conventional theory this process has been discussed by Eilenberger [16] using in the time-dependent Ginzburg-Landau (TDGL) formalism. A term in the current density describes the current carried by normal electrons stemming from the momentum transferred to the normal electron fluid when the superfluid electron density decreases. Eilenberger states that ’this momentum then decays with the transport relaxation time \( \tau' \). However, such decay would necessarily lead to Joule heat dissipation and hence irreversibility, therefore this approach cannot be correct. More generally, any approach that assumes that the momentum of the Cooper pair is transferred to normal quasiparticles cannot be correct since in normal metallic transport, decay of electric current is necessarily associated with thermodynamic irreversibility and even electromigration for high current densities.

As already recognized by Keesom [14], “it is essential that the persistent currents have been annihilated before the material gets resistance, so that no Joule-heat is developed.” The annihilation of the supercurrent has to be accompanied by transfer of the supercurrent momentum to the body as a whole in order to satisfy momentum conservation, with no energy transfer and no energy dissipation. In its more than 50 years of existence, the conventional theory of superconductivity has offered no clue as to how this happens.

One may speculate that transfer of momentum from the supercurrent to the body may occur through phonon emission or scattering by impurities. However, these are not reversible processes: in the reverse transformation from normal to superconducting the body would not be able to transfer its momentum to the supercurrent by reversing the time arrow in these processes.

C. Process III

Process III, shown in Fig. 3, is even more puzzling than process II. Here one has to explain not only how ions acquire momentum opposite to the direction of the force exerted by the Faraday electric field on ions, but also how electrons acquire momentum in direction opposite to the direction of the force exerted by the Faraday electric field on electrons, all without energy dissipation. Within
the conventional theory the Eilenberger formalism can be applied to describe how electrons acquire their momentum, but no mechanism exists for the body to acquire a compensating momentum in the opposite direction.

In summary, we have pointed out that no valid explanation exists in the literature of conventional superconductivity for how momentum is conserved in the processes shown in Figs. 2 and 3. We argue that any explanation of the momentum transfer between electrons and the body as a whole that involves collisions between electrons and ions, or impurities, or defects, or phonons, is necessarily a source of irreversibility, which is not observed [2], hence is invalid [17, 18]. The only other way we know to transfer the momentum between the supercurrent and the body is mediated through the electromagnetic field, as discussed in the next section.

III. MOMENTUM TRANSFER MEDIATED BY THE ELECTROMAGNETIC FIELD

A key aspect of process II is that the momentum of the supercurrent gets transferred to the body, but its kinetic energy is not: the kinetic energy of the supercurrent remains in the electronic degrees of freedom, where it is used to pay the price of the condensation energy in rendering the superconducting electrons normal [17, 19].

In most physical interactions, momentum transfer is accompanied by energy transfer. An exception is when magnetic fields are involved. A charge moving in a magnetic field will change its momentum but not its kinetic energy: the magnetic field doesn’t do work on moving charges since the magnetic Lorentz force $\vec{V} \times \vec{H}$ is perpendicular to the particle’s velocity $\vec{v}$. The momentum change of the particle is compensated by momentum change of the electromagnetic field. The momentum density of the electromagnetic field is given by

$$\tilde{\rho}_\text{em}(\vec{r}) = \frac{1}{4\pi c} \vec{E} \times \vec{H} \tag{24}$$

with $\vec{E}, \vec{H}$ electric and magnetic fields.

Thus we argue that the process of transfer of momentum of the supercurrent to the body without energy transfer must involve the electromagnetic field in an essential way. It is natural to conclude that the transfer has to happen in 2 steps: the first step would transfer the momentum of the supercurrent to the electromagnetic field, and the second step would transfer the momentum from the electromagnetic field to the body as a whole. Even though both processes may occur concurrently, it is useful to think of them as separate processes.

At first sight Eq. (24) does not appear to help, since the electric and magnetic fields at play in Figs. 2 and 3 are azimuthal and in the z direction respectively, resulting in an electromagnetic momentum $\tilde{\rho}_\text{em}$ in the radial direction. However the momentum of the supercurrent and the body are in the azimuthal direction. This then suggests that in processes II and III an electric field in the radial direction exists. An electric field in the radial direction and a magnetic field in the $z$ direction will give an azimuthal $\tilde{\rho}_\text{em}$.

Consider process II, the annihilation of the supercurrent when the system goes normal. The momentum of the supercurrent, carried by negative electrons, is in counterclockwise direction (Fig. 2). Assume that in the process of the supercurrent disappearing an electric field $\vec{E}$ pointing radially inward is created, thus giving a counterclockwise electromagnetic field momentum according to Eq. (24). This could be a “storage box” for the momentum of the supercurrent. In a subsequent step, this momentum of the electromagnetic field would be transferred to the body in a separate process.

What we have just described would occur if the process of inward motion of the N-S phase boundary would also involve inward motion of negative charge, creating a transitory inward-pointing electric field, followed by inward motion of positive charge to retrieve the momentum stored in the field and pass it on to the body. We show the steps in this process in Fig. 4, where the counterclockwise and inward directions in Fig. 2 corresponds to the leftward and downward directions in Fig. 4 respectively.

In the process shown in Fig. 4, the momentum initially carried by the negative charge is transferred to the positive charge, without any energy transfer. There is also no assumption on the masses of negative and positive charges, they could be the same or different. This situation is unlike what would occur if collisions between the charges would occur, in which case there would be both momentum and energy transfer between the charges and the amounts transferred would depend on the values of

![FIG. 4: Illustration of momentum transfer from a negative to a positive charge through the electromagnetic field, corresponding to process II, stopping of a supercurrent. Left and down directions correspond to counterclockwise and radially inward directions in Fig. 2. Magnetic field $H$ points out of the paper. Initially, the negative charge has momentum $P_i$ pointing to the left, the positive charge is at rest. In step 1 the negative charge moves down, the Lorentz force $F_H$ imparts momentum to the right cancelling $P_i$. After step 1, the two charges are at rest, and an electric field exists giving rise to momentum of the electromagnetic field $P_{em} = P_e$. The mechanical momentum of the negative charge resides now in the electromagnetic field. In step 2 the positive charge moves down and the Lorentz force $F_H$ imparts momentum to the left, which is being transferred out of $P_{em}$. After step 2 the positive charge carries the momentum $P_i = P_e$, originally carried by the negative charge, and the momentum of the electromagnetic field is zero again.](image)
the masses.

In exactly the same (reversed) fashion process III can then be explained, as shown in Fig. 5, namely: if the outward motion of the phase boundary is associated with outward motion of negative charge, this would create a transitory radially outgoing electric field and hence a clockwise electromagnetic field momentum that would compensate the counterclockwise mechanical momentum acquired by the outward-moving electron due to the Lorentz force. In a subsequent step, positive charge would move outward and the clockwise momentum of the electromagnetic field would be transferred to the positive charge through the Lorentz force. The end result is negative and positive charges moving with the same momentum in opposite directions, as shown in the rightmost panel of Fig. 5. If the mass of the positive charge is much larger than that of the negative charge, its speed and kinetic energy will be much smaller.

A problem with these explanations is of course that in a solid, positive ions cannot move radially inward nor outward. We will show in the next sections how superconducting positive ions cannot move radially inward nor outward, and kinetic energy will be much smaller.

Next we need to understand how this momentum gets around this problem, through the remarkable Meissner effect. We will show in the next sections how superconducting positive ions cannot move radially inward nor outward, and kinetic energy will be much smaller.

IV. HOW SUPERCURRENT CARRIERS ACQUIRE AND LOSE THEIR MOMENTUM WITHOUT ENERGY DISSIPATION

The radial motion of negative carriers hypothesized in Figs. 4 and 5 is in the same direction as the motion of the phase boundary in the respective situations, as shown in Fig. 6. The momentum parallel to the phase boundary acquired by a negative charge moving a distance $\Delta x$ in direction $\hat{n}$ perpendicular to the phase boundary with radial speed $v_r$ due to the Lorentz force imparted by the magnetic field is

$$\Delta \tilde{p}_n = \int \frac{e}{c} \vec{\nabla} \times \vec{H} dt = \frac{e}{c} \vec{H} \times \vec{n} \times H.$$  \hspace{1cm} (25)

We assume that the momentum imparted by the Faraday electric field (in the opposite direction) is much smaller and hence can be ignored. We will justify this assumption in a later section.

The speed of electrons at the normal-superconductor phase boundary in an applied magnetic field $H$ is, according to Eq. (11)

$$v_s = \frac{e\lambda}{m_e c} H$$  \hspace{1cm} (26)

provided we can replace $m_e^* r$ by the bare electron mass $m_e$ in Eq. (11). We have recently argued [8] that BCS theory itself is inconsistent unless the dynamics of electrons in the supercurrent is governed by the bare mass $m_e$ rather than the effective mass, and we will assume hereafter that this is the case. The momentum of the electron is $m_e v_s$, hence Eqs. (25) and (26) indicate that electrons making the transition from normal to superconducting or from superconducting to normal advance in the direction of the phase boundary motion a distance $\lambda_e$. For process II, this motion stops the velocity of the electron in the supercurrent to zero and stores its momentum in the electromagnetic field as shown in Fig. 4. For process III, this motion gives to the electron the momentum needed to carry the supercurrent and stores momentum of opposite sign in the electromagnetic field as shown in Fig. 5. Thus, this accounts for the transfer of momentum from electrons to the electromagnetic field without energy dissipation, “step 1” in Figs. 4 and 5.

Next we need to understand how this momentum gets transferred back to the body, i.e. the processes denoted “step 2” in Figs. 4 and 5, through a ‘backflow process’ which is necessary to preserve local charge neutrality.
Hall coefficients

FIG. 7: Hall effect in metal bars with negative and positive Hall coefficients $R_H$. The Amperian force on the body is the same independent of the sign of the Hall coefficient. However, the physical interpretation is very different, as discussed in the text.

V. HOW MOMENTUM IS TRANSFERRED TO THE BODY WITHOUT ENERGY DISSIPATION

After step 1 in Figs. 4 and 5 the momentum is stored in the electromagnetic field and needs to be retrieved and transferred to the body in step 2. This is achieved through the motion of normal holes in direction perpendicular to the phase boundary.

Consider the two Hall bars shown in Figure 7. They are identical except one has negative and the other positive Hall coefficient $R_H$. The Amperian force on the bar is given by

$$\mathbf{F}_{\text{Amp}} = \frac{I}{c} \mathbf{L} \times \mathbf{H}$$

(27)

where $L = |\mathbf{L}|$ is the length of the sample and the vector $\mathbf{L}$ points in the direction of the flow of current $I$. The Amperian force is of course independent of the sign of the Hall coefficient.

For the bar in Fig. 7 (a) the Hall coefficient $R_H$ is negative, the carriers are electrons. The current density is given by

$$\mathbf{j}_e = -nev \hat{x} = J_x \hat{x}$$

(28)

flowing in the positive $\hat{x}$ direction ($e < 0$), where $v$ is the magnitude of the drift velocity and $n$ is the density of electron carriers. An electric field pointing to the right (negative $\hat{y}$ direction) exists, given by

$$\mathbf{E}_y = -\frac{v}{c} H \hat{y} = -E_y \hat{y}$$

(29)

that equals the magnetic Lorentz force pointing to the left, so that the forces on electrons in the $\hat{y}$ direction are balanced. Assuming the system is charge neutral, for every conduction electron there is a positive charge $|e|$ belonging to an ion that is not moving. The force on this ionic charge is

$$\mathbf{F}_{\text{ion}} = |e|\mathbf{E}_y = eE_y \hat{y}$$

(30)

pointing to the right. The total force on all the ions is

$$\mathbf{F}_{\text{ion, tot}} = nA|e|\mathbf{E}_y = nAe \frac{v}{c} H \hat{y} = -\frac{J_x A}{c} LH \hat{y}$$

(31)

where $A$ is the cross-sectional area of the sample, so the total current is $I = J_x A$. Hence in this case

$$\mathbf{F}_{\text{ion, tot}} = \mathbf{F}_{\text{Amp}}$$

(32)

For the electrons, electric and magnetic forces are balanced and the electrons move along the $\hat{x}$ direction, hence no other force in the $\hat{x}$ direction is acting on electrons. The Amperian force results from the action of the Hall electric field $E_y$ on the ions.

The situation is different for the Hall bar with $R_H > 0$ shown in Fig. 7 (b). Here the Hall electric field is of opposite sign to the previous case,

$$\mathbf{E}_y = \frac{v}{c} H \hat{y} = E_y \hat{y}$$

(33)

pointing in the positive $\hat{y}$ direction. The current flows in the $\hat{x}$ direction, hence the net force in the $\hat{y}$ direction on current carriers has to be zero. The force on the ions from the electric field is now

$$\mathbf{F}_{\text{ion}} = |e|\mathbf{E}_y = |e|E_y \hat{y}$$

(34)

pointing to the left, i.e. in opposite direction to the Amperian force. How does the Amperian force come about?

The answer is, the electrons flowing in the $\hat{x}$ direction exert a force on the ions, given by

$$\mathbf{F}_{e-i} = 2eE_y \hat{y}$$

(35)

so that the total force on the ion is

$$\mathbf{F}_{\text{ion, tot}} = \mathbf{F}_{\text{ion}} + \mathbf{F}_{e-i} = eE_y \hat{y}$$

(36)

just as in Eq. (30). The total force on the ions is again given by Eq. (32), the Amperian force.

The reason the electrons exert a force on the ions is that the ions exert a force on the electrons that are moving carrying the current. According to the semiclassical equations of motion for Bloch electrons the motion of electrons in solids results from the combined action of the external force and the force exerted by the ions on the electrons. A detailed analysis is given in Appendix A.

This then implies that in a Hall bar with positive Hall coefficient where the drift velocity of current carriers (holes) is $\bar{v}_d$ in the direction of current flow, for each hole that moves a distance $d$, it takes a time interval $\Delta t = d/\bar{v}_d$ and the momentum transferred in that time from the electrons to the ions is

$$\Delta \mathbf{P}_{\text{ion}} = -2e \bar{v}_d \times \mathbf{H} \Delta t = 2e \bar{v}_d \times \mathbf{H} t.$$ 

(37)

This momentum transfer from electrons to ions occurs without any irreversible scattering processes, and can only occur when the carriers are holes.

We can now understand how the momentum transfers between the supercurrent and the body shown in Figs. 4 and 5 occur in the context of the superconductor to
The second term in Eq. (39) is much smaller than the first term and we assume it can be neglected, so that Eq. (39) gives the required momentum of the electron in the supercurrent Eq. (26).

(2) After step 1: the outward motion of electrons in step 1 creates a radially outward electric field in a boundary layer of thickness $\lambda_L$ in the normal region, and stores azimuthal momentum in the electromagnetic field, as shown in the second panel in Fig. 5. The radial electric field drives a radial outflow of current in this boundary layer of thickness $\lambda_L$ (Fig. 8, light grey ring) that moves at the same speed as the phase boundary motion.

(3) Step 2: We assume that the normal current is carried by hole carriers. Normal current flows radially outward in the boundary layer and exerts a force on the body. This force is the Amperian force $\vec{F}_{\text{Amp}}$ shown in Fig. 7 (b) and in Fig. 8. It transfers the momentum stored in the electromagnetic field to the body as a whole, without energy dissipation, so the body acquires rotational velocity in the clockwise direction.

(4) After step 2: the momentum acquired by the electron going superconducting in the counterclockwise direction is exactly compensated by the momentum transferred to the body in the clockwise direction.

The fact that the momenta are exactly compensated does not need proof, it follows from the fact that the backflow propagation of the holes is exactly radial because of the balance of forces: the total azimuthal force acting on the outflowing hole with speed $\dot{r}_0$ is the sum of the clockwise magnetic Lorentz force and counterclockwise Faraday force which equals zero:

$$\vec{F}_{\text{hole}} = -|e|\frac{\dot{r}_0}{c}H_e\hat{\theta} + |e|\vec{E}_F = 0,$$

Nevertheless, let us verify that momentum conservation holds. The momentum acquired by an electron going superconducting and thrusting radially outward a distance $\lambda_L$ is

$$\Delta \vec{p}_e = -\frac{e\lambda_L}{c}H_e\hat{\theta},$$

neglecting the second term in Eq. (39) under the assumption that $v_r >> \dot{r}_0$. The 'backflow' normal holes move at speed $\dot{r}_0$ in the $+\hat{r}$ direction and traverse the boundary layer of thickness $\lambda_L$ in time $\Delta t = \lambda_L/\dot{r}_0$. The net force per carrier exerted on the lattice during that time is given by Eq. (36), where $E_y$ is the Faraday field $E_F$ and the $\hat{y}$ direction is the $\hat{\theta}$ direction

$$\vec{F}_{\text{ion, tot}} = e\vec{E}_F = e\frac{\dot{r}_0}{c}H_e\hat{\theta},$$

which is the same as the Amperian force in Fig. 7 (b). Hence the net momentum transferred to the ions per electron going superconducting is

$$\Delta \vec{p}_i = \vec{F}_{\text{ion, tot}}\Delta t = \frac{e\lambda_L}{c}H_e\hat{\theta}$$

Under the condition

$$\dot{r}_0 << v_r,$$

the second term in Eq. (39) is much smaller than the first term and we assume it can be neglected, so that Eq. (39) gives the required momentum of the electron in the supercurrent Eq. (26).

(2) After step 1: the outward motion of electrons in step 1 creates a radially outward electric field in a boundary layer of thickness $\lambda_L$ in the normal region, and stores azimuthal momentum in the electromagnetic field, as shown in the second panel in Fig. 5. The radial electric field drives a radial outflow of current in this boundary layer of thickness $\lambda_L$ (Fig. 8, light grey ring) that moves at the same speed as the phase boundary motion.

(3) Step 2: We assume that the normal current is carried by hole carriers. Normal current flows radially outward in the boundary layer and exerts a force on the body. This force is the Amperian force $\vec{F}_{\text{Amp}}$ shown in Fig. 7 (b) and in Fig. 8. It transfers the momentum stored in the electromagnetic field to the body as a whole, without energy dissipation, so the body acquires rotational velocity in the clockwise direction.

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$$\vec{F}_{\text{ion, tot}} = e\vec{E}_F = e\frac{\dot{r}_0}{c}H_e\hat{\theta},$$

which is the same as the Amperian force in Fig. 7 (b). Hence the net momentum transferred to the ions per electron going superconducting is

$$\Delta \vec{p}_i = \vec{F}_{\text{ion, tot}}\Delta t = \frac{e\lambda_L}{c}H_e\hat{\theta}$$

Under the condition

$$\dot{r}_0 << v_r,$$
equal and opposite to Eq. (42), as expected.

Exactly the same steps in reverse explain how as the superconducting region shrinks the mechanical momentum of an electron in the supercurrent that becomes normal is transferred to the body through a radially inward flow of holes in a boundary layer of thickness $\lambda_L$.

Returning to the case of the expanding superconducting phase in Fig. 8, we also need to consider the effect of the Faraday field in the superconducting region within $\lambda_L$ of the phase boundary, where the supercurrent flows. Its effect on the body is to impart momentum in counterclockwise direction, which partially compensates the momentum transfer Eq. (44), resulting in a net transfer of momentum which generates the body’s rotation, as discussed in the next section.

VI. MACROSCOPIC TORQUE

Let us now analyze how the macroscopic rotation of the body comes about. The Amperian force per unit volume exerted on a radially outgoing current $J_r$ in the presence of magnetic field $H$ is

$$
\vec F_{\text{Amp}} = -\frac{H}{c} J_r \hat{\theta}
$$

where $\hat{\theta}$ is positive in counterclockwise direction. The radial hole current is given by

$$
\vec J_r = n_s |e| \hat{r}_0 \hat{r}
$$

and this current occupies a boundary layer of thickness $\lambda_L$, with volume $V = 2\pi r_0 \lambda_L h$, with $h$ the height of the cylinder. Hence the torque exerted by the Amperian force on the boundary layer of thickness $\lambda_L$ flowing outward with speed $\hat{r}_0$ is

$$
\vec \tau_1 = \frac{H c}{2} \pi r_0^2 \lambda_L n_s e \hat{r}_0 \hat{z}
$$

pointing in the $-\hat{z}$ direction, i.e. opposite to the direction of the magnetic field.

There is also a countertorque due to the clockwise force exerted by the Faraday electric field $E_F$ on the ions in the superconducting region within distance $\lambda_L$ of the superconductor-normal phase boundary, where supercurrent flows. The Faraday field in that region is given by [20]

$$
\vec E_F(r) = \frac{H c}{c} \hat{r}_0 e^{(r - r_0) / \lambda_L} \hat{\theta}
$$

and it exerts a torque

$$
\vec \tau_2 = -2\pi n_s e h \int_0^{r_0} E_F(r) r^2 dr \hat{z}
$$
on the body. Doing the integral and assuming $r_0 >> \lambda_L$ yields

$$
\vec \tau_2 = \frac{H c}{2} 2\pi n_s e h \lambda_L [r_0^2 - 2 r_0 \lambda_L + 2\lambda_L^2] \hat{\theta}
$$

so that the net torque on the body is (neglecting the higher order term proportional to $\lambda_L^3$)

$$
\vec \tau = \vec \tau_1 + \vec \tau_2 = \frac{H c}{c} 4\pi \lambda_L^2 n_s e r_0 \hat{r}_0 \hat{z}
$$

By conservation of momentum, the ionic angular momentum is minus the electronic angular momentum Eq. (5)

$$
\vec L_e = -\frac{m_e c}{2 e} h r_0 \hat{r}_0 \hat{z}
$$

and the associated torque is

$$
\vec \tau_i = \frac{d\vec L_i}{dt} = \frac{m_e c}{e} h r_0 \hat{r}_0 \hat{z}
$$

Equating Eq. (53) to the net torque exerted on the body, Eq. (51), we find

$$
\frac{1}{\lambda_L} = \frac{4 \pi n_s e^2}{m_e c^2}
$$

This is the well-known expression for the London penetration depth [4]. On the other hand, we find from our formula for Bloch electrons Eq. (14) for either a band close to empty or close to full

$$
\frac{1}{\lambda_L^2} = \frac{4 \pi n_s e^2}{m^* c^2}
$$

where $n_s$ is

$$
n_s = \frac{1}{V} \sum_{k \text{ occ}} 1
$$

for a band close to empty, or

$$
n_s = \frac{1}{V} \sum_{k \text{ unocc}} 1
$$

for a band close to full, and $m^* = m^*_k$ at the bottom of the band for an almost empty band, or $m^* = -m^*_k$ at the top of the band for an almost full band.

In deriving the expression Eq. (55) for the London penetration depth, we assumed that superconducting carriers respond to the induced Faraday field as if they were Bloch electrons with effective mass $m^*_k$, Eq. (9). However, to satisfy momentum conservation we found here that the London penetration depth has to be given by Eq. (54) with the bare electron mass $m_e$. The implication of this is inescapable: our original assumption Eq. (10) leading to Eq. (55) was incorrect. Unlike normal Bloch electrons, superconducting carriers respond to external field with their bare electron mass, in other words they
are completely ‘undressed’ from the electron-ion interaction. We recently reached this same conclusion through a completely different path, by examining inconsistencies within conventional BCS-London theory [8].

In summary, the macroscopic rotation of the body when the superconducting region expands results from the torque exerted by the radially outgoing hole current Eq. (46) on the body in the clockwise direction exceeding the countertorque Eq. (50) exerted by the Faraday electric field on the ions in the counterclockwise direction in the region where supercurrent flows by the amount given by the net torque Eq. (51). For a shrinking superconducting region all the signs are simply reversed.

VII. NEW PHYSICS OF SUPERCONDUCTIVITY

In the previous sections we have described a plausible way to explain the momentum transfer between the electronic degrees of freedom and the body as a whole in a reversible way in processes II and III. We don’t know any other possible way to do this, and no other way has been proposed in the literature. Next let us consider what is required of a microscopic theory of superconductivity to allow this to occur. We argue that the following are necessary conditions:

(i) The wavefunction and charge distribution of superconducting electrons close to the phase boundary extend into the normal state.

(ii) The charge carriers in the normal state that are condensing to give rise to the supercurrent in the superconducting state are holes.

Requirement (i) follows from the fact that we assumed in the previous sections that when electrons go from normal to superconducting they ‘thrust’ into the normal region a finite distance $\lambda_L$. Within BCS theory it is assumed that the superconducting order parameter does leak into the normal region, leading e.g. to Josephson effects and proximity effects, however BCS theory does not predict that the order parameter has any charge associated with it. This is because charge has to have a sign (negative or positive), and BCS theory is intrinsically electron-hole symmetric, so the order parameter is not associated with either negative or positive charge. Therefore, BCS theory does not satisfy this requirement.

BCS theory also does not satisfy requirement (ii), since within BCS the normal state carriers may be electron-like or hole-like.

Therefore, we conclude that BCS theory does not have the physical elements required to explain the reversible momentum transfer between electrons and the body that takes place in the superconductor-normal and normal-superconductor transitions in a magnetic field.

Instead, the theory of hole superconductivity [7] does have those physical elements, as discussed in earlier papers [17, 20, 21] and recounted briefly in what follows:

(i) Within the theory of hole superconductivity electrons in the condensate reside in mesoscopic orbits of radius $2\lambda_L$ [22], while they reside in microscopic orbits of radius $k_F^{-1}$ in the normal state. Thus, when electrons go from normal to superconducting they expand their orbits to radius $2\lambda_L$, and since they are at the normal-superconductor phase boundary this is associated with negative charge leaking into the normal region. The azimuthal velocity acquired by expanding the orbit to radius $2\lambda_L$ is Eq. (26), the same as in a linear ‘thrust’ over length $\lambda_L$ in direction perpendicular to the phase boundary [23].

(ii) Within the theory of hole superconductivity, as discussed in numerous papers and for numerous reasons [7], the normal state carriers are necessarily holes.

The essential physics of the Meissner effect within the theory of hole superconductivity is orbit expansion driven by lowering of quantum kinetic energy [24, 25]. Instead, in conventional BCS theory, superconductivity is driven by lowering of potential energy. We have argued that no theory that explains superconductivity as driven by potential rather than kinetic energy can explain the Meissner effect [25].

VIII. MOMENTUM IN THE ELECTROMAGNETIC FIELD

Within our theory, the superfluid charge density is slightly inhomogeneous, since the orbit expansion leads to higher negative charge density within a London penetration depth of the surface, as shown schematically in Fig. 9. The excess charge density is given by [26]

$$\rho_\perp = e n_s \frac{\hbar}{4m_e \lambda_L c}$$

which gives rise to an outward pointing electric field in the interior of superconductors, that attains its maximum value $E_m$ near the surface, given by

$$E_m = -\frac{\hbar c}{4e \lambda_L^2},$$

so that $E_m = -4\pi \rho_- \lambda_L$ [26]. Therefore, there is electromagnetic momentum in the region within $\lambda_L$ of the surface where both electric and magnetic fields are present, according to Eq. (24). The total electromagnetic angular momentum in that region, of volume $2\pi r_0 \lambda_L \hbar$, is

$$\hat{L}_{em} = -\frac{1}{4\pi c} E_m H_e r_0 (2\pi r_0 \lambda_L ) \hat{z}. \quad (59)$$

On the other hand, the extra mass density $\rho_-/e$ carries mechanical angular momentum, given by (using Eqs. (57) and (5))

$$\hat{L}_{e extra} = \frac{\hbar}{4m_e \lambda_L c} \hat{L}_e = -\frac{\hbar}{4m_e \lambda_L c} \frac{me c}{2e} \hbar r_0^2 H_e \hat{z} \quad (60)$$
FIG. 9: Schematic view of superconducting region. Excess negative charge density $\rho_-$ resides within a London penetration depth of the phase boundary. A radial electric field exists in the interior. The extra mechanical momentum $L_{e\text{extra}}$ carried by the excess charge density $\rho_-$ is compensated by momentum $L_{\text{em}}$ in the electromagnetic field.

so that

$$L_{\text{em}} = -L_{e\text{extra}}$$

(61)
as required for momentum conservation.

It is interesting to note that for any value of $\rho_-$ Eq. (61) would hold, provided that $E_m = -4\pi \rho_- \lambda_L$ which is the condition for the ‘surface charge density’ $\sigma = \lambda_L \rho_-$ to screen the internal field $E_m$ so that it does not leak out of the superconductor: the angular momentum of the mass density $\rho_- / e$ moving at speed Eq. (26) is exactly compensated by the angular momentum stored in the electromagnetic field.

In summary, the total electronic mechanical angular momentum of superconducting electrons in a magnetic field is compensated by the angular momentum of the body plus a small contribution ($\sim 1/10^6$) of electromagnetic field momentum:

$$L_{e\text{tot}} = (1 + \frac{\rho_-}{e\lambda_L})L_e = -(L_{\text{body}} + L_{\text{em}})$$

(62)
which completely accounts for momentum conservation and the mechanisms responsible for it.

IX. DISCUSSION

In this paper we have argued that the only way that momentum can be transferred between the supercurrent and the body as a whole in a reversible way in processes where the normal-superconductor phase boundary moves is through mediation of the electromagnetic field, which necessitates flow of charge in direction perpendicular to the phase boundary, and necessitates hole carriers in the normal state. Any alternative way to transfer the momentum between electrons and the body, i.e. scattering by impurities or phonons, would be an irreversible process incompatible both with experiment and established principles of superconductivity.

We have furthermore argued that conventional BCS-London theory does not have the necessary physical elements to describe these processes in a reversible fashion. At the very least, it is a fact that no such description exists in the scientific literature.

An important aspect of our explanation is that it only works if it is assumed that the mechanical momentum of an electron in the supercurrent is

$$\vec{p}_e = -\frac{e\lambda_L}{c} \hat{H} \dot{\theta}$$

(63a)
rather than

$$\vec{p}_e = -\frac{m_e e\lambda_L}{m^* c} \hat{H} \dot{\theta}$$

(63b)
as predicted by the conventional theory [8]. The momentum formula Eq. (63b) works to explain momentum conservation in process I, but cannot explain how momentum is conserved in processes II and III. The reason is, the explanation of how momentum is transferred to the body discussed in Sect. V involves momentum transfer perpendicular to the motion, for which $m^*$ does not play a role: the momentum transferred to the ions Eq. (44) does not depend on $m^*$. We have discussed elsewhere [8] other reasons for why the correct momentum expression has to be Eq. (63a) rather than (63b).

BCS advocates argue that because at low temperatures the superconducting state with the magnetic field excluded has lower free energy than the normal state with the magnetic field inside, the system will ‘somehow’ ‘find its way’ to the lower free energy state and expel the magnetic field. They do not feel compelled to explain in the scientific literature how, within the confines of BCS theory or even within time-dependent Ginzburg Landau theory, the process occurs respecting momentum conservation and reversibility. We argue that such a stance is unacceptable. One might say that it is equivalent to saying that because an electron-positron pair has the same energy as a single 1.022 MeV photon, a theory predicting that the former will decay into the latter has a claim to validity. It does not, because such a process with a single photon would violate momentum conservation, two photons are needed. Hence the theory with only one photon cannot be a valid theory, no matter what other valid predictions it makes.

We argue that within BCS theory the Meissner transition is a ‘forbidden’ transition [6], since the transition cannot take place respecting momentum conservation if only the supercurrent changes its momentum. As in other forbidden transitions in physics, it is necessary to ascertain how long it will take the system to get around the selection rules originating in conservation laws by using higher order processes. For example, consider gamma
decay of excited atomic nuclei. Changes in the nuclear angular momentum by more than one unit cannot occur by emission of a single photon because this would violate angular momentum conservation. Changes by more than 1 unit can occur, but each additional unit of spin change inhibits the decay rate by about 5 orders of magnitude. For the highest known spin change of 8 units, the decay rate is suppressed by a factor $10^{35}$ and takes $10^{15}$ years instead of $10^{-12}$ seconds. Similarly we believe any route to explain the Meissner effect within BCS theory satisfying conservation laws is highly ‘forbidden’ and would take time beyond the age of the universe for macroscopic systems. We would like to challenge BCS advocates to show that this is not so, by explaining the mechanism by which momentum is transferred between electrons and the body in a reversible fashion.

In contrast, the theory of hole superconductivity does have the physical elements necessary to explain these processes [7]. In summary, those physical elements, that are not part of BCS theory, are: (i) normal carriers are necessarily holes; (ii) when a system goes superconducting, not only the occupation of Bloch states near the Fermi energy changes as predicted by BCS, keeping the individual Bloch states unaltered; instead, the electronic wavefunction expands, and the highly dressed normal carrier becomes an undressed carrier with an extended wavefunction that does not ‘see’ the short-wavelength ionic potential [27, 28]; (iii) as a consequence of (ii), supercarriers respond to external fields according to the bare electron mass [8] rather than the effective mass as predicted by BCS, and (iv) as electrons become superconducting, negative charge extends beyond the normal-superconductor boundary into the normal region.

Because these issues are basic and fundamental to the understanding of superconductivity, we argue that it is imperative to resolve them, and physicists should stop using conventional BCS-London theory to describe real superconductors unless or until it can be shown that the theory does not violate momentum conservation.

Appendix A: Elementary derivation of the Hall coefficient and Hall force on the lattice

We consider the Hall effect in the geometry of Fig. 7. The Hall coefficient is defined as

$$R_H = \frac{E_y}{J_x H}$$

(A1)

with $\vec{H} = H \hat{z}$ the applied magnetic field, $\vec{J}_x = J_x \hat{x}$ the current density and $E_y = E_y \hat{y}$ the Hall field. The external force on an electron of wavevector $k$ in direction perpendicular to the current ($\hat{y}$ direction) is

$$F_{ext}^k = e E_y = \frac{e}{c} v_k H$$

(A2)

with

$$v_k = \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial k}$$

(A3)

We assume an isotropic band with energy $\epsilon_k$ and omit vector labels on the wavevectors. The total force on an electron of wavevector $k$ in direction perpendicular to the current is

$$F_{tot}^k = m_e \frac{d v_k}{dt} = \frac{m_e}{m_k^*} \frac{d (\hbar k)}{dt} = \frac{m_e}{m_k^*} \left( e E_y - \frac{e}{c} v_k H \right)$$

(A4)

according to the semiclassical equation of motion, with

$$\frac{1}{m_k^*} = \frac{1}{\hbar^2} \frac{\partial^2 \epsilon_k}{\partial k \partial k}$$

(A5)

the effective mass tensor. On the other hand we can write the force on the electron of wavevector $k$ as the sum of the external force and the force exerted by the lattice

$$F_{tot}^k = F_{ext}^k + F_{latt}^k = e E_y - \frac{e}{c} v_k H + F_{latt}^k$$

(A6)

The total force on carriers per unit volume in direction perpendicular to the current is, from integrating Eq. (A4) over the occupied states

$$F_{tot} = \int \frac{d^3 k}{4 \pi^3} F_{tot}^k \rho_{occ} \int \frac{d^3 k}{4 \pi^3} \frac{1}{m_k^*} (E_y - \frac{H}{c} v_k)$$

(A7)

and the total force per unit volume exerted by the lattice on electrons in the transverse direction is, from Eq. (A6)

$$F_{latt} = \int \frac{d^3 k}{4 \pi^3} F_{latt}^k \rho_{occ},$$

(A8)

Next we evaluate Eqs. (A7) and (A8) for the cases of almost empty and almost full bands.

1. Almost empty band

The number of carriers and current are given by

$$n_e = \int \frac{d^3 k}{4 \pi^3}$$

(A9a)

$$J_x = \int \frac{d^3 k}{4 \pi^3} v_k$$

(A9b)

and we assume that for the occupied states near the bottom of the band

$$\frac{1}{m_k^*} \sim 1 > 0$$

(A10)

independent of $k$. Eq. (A7) yields

$$F_{tot} = \frac{m_e}{m^*} (n_e e E_y - \frac{H}{c} J_x)$$

(A11)

and setting $F_{tot} = 0$ yields

$$E_y = \frac{J_x H}{n_e e c}$$

(A12)
and from Eq. (A8)

\[ F_{\text{latt}} = - (n_e e E_y - J_x H / c) = 0 \]  

using Eq. (A12). Therefore, for this case, the Hall coefficient \( R_H \) is negative, the total force exerted by the lattice on the carriers is zero, and conversely the total force exerted by the carriers on the lattice is zero.

2. Almost full band

The number of carriers and current are given by

\[ n_h = \int \frac{d^3k}{4\pi^3} \quad \text{(A15a)} \]

\[ J_x = \int \frac{d^3k}{4\pi^3} v_k = - \int \frac{d^3k}{4\pi^3} v_k \quad \text{(A15b)} \]

and we assume

\[ \frac{1}{m_k^*} \sim - \frac{1}{m^*} < 0 \quad \text{(A16)} \]

independent of \( k \), for the unoccupied states near the top of the band. We have then

\[ \int \frac{d^3k}{4\pi^3} \frac{1}{m_k^*} = - \int \frac{d^3k}{4\pi^3} \frac{1}{m_k^*} = \frac{n_h}{m^*} \quad \text{(A17)} \]

\[ e \int \frac{d^3k}{4\pi^3} v_k = - e \int \frac{d^3k}{4\pi^3} v_k = - \frac{J_x}{m^*} \quad \text{(A18)} \]

dependent on the above and using Eq. (A7)

\[ F_{\text{tot}} = \frac{m_e}{m^*} (n_h e E_y + \frac{H}{c} J_x) \quad \text{(A19)} \]

and setting \( F_{\text{tot}} = 0 \) yields

\[ E_y = \frac{J_x H}{n_h e c} \quad \text{(A20)} \]

\[ R_H = \frac{E_y}{J_x H} = - \frac{1}{n_h e c} \quad \text{(A21)} \]

Therefore, in this case the Hall coefficient \( R_H \) is positive. To find the force exerted by the lattice on electrons from Eq. (A8) we use that

\[ \int \frac{d^3k}{4\pi^3} = \int \frac{d^3k}{4\pi^3} - \int \frac{d^3k}{4\pi^3} = \frac{2}{v} - n_h \quad \text{(A22)} \]

with \( v \) the volume of the unit cell, and use Eq. (A15b), and obtain

\[ F_{\text{latt}} = -(e E_y (\frac{2}{v} - n_h) + \frac{H}{c} J_x) \quad \text{(A23)} \]

and using Eq. (A20)

\[ F_{\text{latt}} = - \frac{2e E_y}{v} \quad \text{(A24)} \]

which unlike Eq. (A14) is not zero. Hence, the total force per unit volume exerted by the carriers on the lattice is

\[ F_{\text{on-latt}} = \frac{2e E_y}{v} \quad \text{(A25)} \]

Now the electric field \( E_y \) also exerts a force on the lattice. The compensating ionic charge density per unit volume is \(|e| (2/v - n_h)\), hence the direct force of the electric field on the ions per unit volume is

\[ F_{\text{on-latt}}^{E_y} = -e E_y (\frac{2}{v} - n_h) \quad \text{(A26)} \]

so that the net force on the lattice per unit volume is

\[ F_{\text{on-latt}}^{\text{net}} = F_{\text{on-latt}} + F_{\text{on-latt}}^{E_y} = n_h e E_y \quad \text{(A27)} \]

or, using Eq. (A20)

\[ \vec{F}_{\text{on-latt}}^{\text{net}} = \frac{H}{c} J_x \hat{y} \quad \text{(A28)} \]

in agreement with Eq. (31).

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