HOMOGENIZATION OF THE LANDAU-LIFSHITZ-GILBERT EQUATION WITH NATURAL BOUNDARY CONDITION

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Abstract

The full Landau-Lifshitz-Gilbert equation with periodic material coefficients and natural boundary condition is employed to model the magnetization dynamics in composite ferromagnets. In this work, we establish the convergence between the homogenized solution and the original solution via a Lax equivalence theorem kind of argument. There are a few technical difficulties, including: 1) it is proven the classic choice of corrector to homogenization cannot provide the convergence result in the $H^1$ norm; 2) a boundary layer is induced due to the natural boundary condition; 3) the presence of stray field give rise to a multiscale potential problem. To keep the convergence rates near the boundary, we introduce the Neumann corrector with a high-order modification. Estimates on singular integral for disturbed functions and boundary layer are deduced, to conduct consistency analysis of stray field. Furthermore, inspired by length conservation of magnetization, we choose proper correctors in specific geometric space. These, together with a uniform $W^{1,6}$ estimate on original solution, provide the convergence rates in the $H^1$ sense.

1. Introduction

The intrinsic magnetic order of a rigid single-crystal ferromagnet over a region $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$ is described by the magnetization $M$ satisfying

$$M = M_s(T)m, \quad \text{a.e. in } \Omega,$$

where the saturation magnetization $M_s$ depends on the material and the temperature $T$. Below Curie temperature, $M_s$ is modeled as a constant.

A stable structure of a ferromagnet is mathematically characterized as the local minimizers of the Landau-Lifshitz energy functional [7]

$$G_L[m] := \int_{\Omega} a(x)|\nabla m|^2 \, dx + \int_{\Omega} K(x) (m \cdot u)^2 (m) \, dx$$

$$- \mu_0 \int_{\Omega} h_d[M_s m] \cdot M_s m \, dx - \int_{\Omega} h_a \cdot M_s m \, dx$$

$$= \mathcal{E}(m) + A(m) + W(m) + Z(m).$$

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\( \mathcal{E}(\mathbf{m}) \) is the exchange energy, which penalizes the spatial variation of \( \mathbf{m} \). The matrix \( a = (a_{ij})_{1 \leq i,j \leq 3} \) is symmetric, uniformly coercive and bounded, i.e.,

\[
\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \eta_i \eta_j &\geq a_{\min} |\eta|^2 \quad \text{for any } \mathbf{x} \in \mathbb{R}^n, \eta \in \mathbb{R}^n, \\
\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \eta_i \xi_j &\leq a_{\max} |\eta||\xi| \quad \text{for any } \mathbf{x} \in \mathbb{R}^n, \eta, \xi \in \mathbb{R}^n.
\end{aligned}
\]

(1)

In the anisotropy energy \( \mathcal{A}(\mathbf{m}) \), \( \mathbf{u} \) is the easy-axis direction which depends on the crystallographic structure of the material. The anisotropy energy density is assumed to be a non-negatively even and globally Lipschitz continuous function that vanishes only on a finite set of unit vectors (the easy axis). The third term \( \mathcal{W}(\mathbf{m}) \) is the magnetostatic self-energy due to the dipolar magnetic field, also known as the stray field \( \mathbf{h}_d[\mathbf{m}] \). For an open bounded domain \( \Omega \) with a Lipschitz boundary, the magnetization \( \mathbf{m} \in L^p(\Omega, \mathbb{R}^3) \) generates a stray field satisfying

\[
\mathbf{h}_d[\mathbf{m}] = \nabla U_m,
\]

where the potential \( U_m \) solves

\[
\Delta U_m = -\operatorname{div}(\mathbf{m}\chi_{\Omega}), \quad \text{in } D'(\mathbb{R}^3)
\]

with \( \mathbf{m}\chi_{\Omega} \) the extension of \( \mathbf{m} \) to \( \mathbb{R}^3 \) that vanishes outside \( \Omega \). The existence and uniqueness of \( U_m \) follows from the Lax-Milgram Theorem and \( U_m \) satisfies the estimate \([10]\)

\[
\|\mathbf{h}_d[\mathbf{m}]\|_{L^p(\Omega)} \leq \|\mathbf{m}\|_{L^p(\Omega)} \quad 1 < p < \infty.
\]

The last term \( \mathcal{Z}(\mathbf{m}) \) is the Zeeman energy that models the interaction between \( \mathbf{m} \) and the externally applied magnetic field \( \mathbf{h}_a \).

For a composite ferromagnet with periodic microstructures, the material constants are modeled with periodic material coefficients with period \( \varepsilon \), i.e., \( a^\varepsilon = a(\mathbf{x}/\varepsilon) \), \( K^\varepsilon = K(\mathbf{x}/\varepsilon) \), \( M^\varepsilon = M_s(\mathbf{x}/\varepsilon) \), with functions \( a, K, M_s \) periodic over \( \mathbb{Y} = [0, 1]^n \). The associated energy reads as

\[
\mathcal{G}_\varepsilon^\varepsilon[\mathbf{m}] := \int_{\Omega} a^\varepsilon(\mathbf{x}) |\nabla \mathbf{m}|^2 d\mathbf{x} + \int_{\Omega} K^\varepsilon(\mathbf{x}) (\mathbf{m} \cdot \mathbf{u})^2 d\mathbf{x}
\]

\[
- \mu_0 \int_{\Omega} \mathbf{h}_d[M^\varepsilon \mathbf{m}] \cdot M^\varepsilon \mathbf{m} d\mathbf{x} - \int_{\Omega} \mathbf{h}_a \cdot M^\varepsilon \mathbf{m} d\mathbf{x}.
\]

(5)

It is proved in \([2]\) that \( \mathcal{G}_\varepsilon^\varepsilon[\mathbf{m}] \) is equi-mild coercive in the metric space \( (H^1(\Omega, \mathbb{S}^2), d_{L^2(\mathbb{Y}, \mathbb{S}^2)}) \) and \( \varepsilon \)-converges to the functional \( \mathcal{G}_{\text{hom}} \) defined as

\[
\mathcal{G}_{\text{hom}}[\mathbf{m}] = \int_{\Omega} a^0 |\nabla \mathbf{m}|^2 d\mathbf{x} + \int_{\Omega} K^0 (\mathbf{m} \cdot \mathbf{u})^2 d\mathbf{x} - \mu_0 (M^0)^2 \int_{\Omega} \mathbf{h}_d[\mathbf{m}] \cdot \mathbf{m} d\mathbf{x}
\]

\[
- \mu_0 \int_{\Omega \times \mathbb{Y}} |\mathbf{m} \cdot \mathbf{H}_d[M_s(\mathbf{y})](\mathbf{y})|^2 d\mathbf{x} d\mathbf{y} - M^0 \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} d\mathbf{x},
\]

(6)
where \( a^0 \) is the homogenized tensor
\[
a^0_{ij} = \int_Y \left( a_{ij} + \sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) \, dy,
\]
the constants \( M^0 \) and \( K^0 \) are calculated by
\[
M^0 = \int_Y M_s(y) \, dy, \quad K^0 = \int_Y K(y) \, dy,
\]
and the symmetric matrix-valued function \( \mathbf{H}_d[M_s(y)](y) = \nabla_y \mathbf{U}(y) \) with potential function given by
\[
\int_Y M_s(y) \nabla_y \varphi(y) \, dy = -\int_Y \nabla_y \mathbf{U}(y) \cdot \nabla_y \varphi(y) \, dy,
\]
(7)
\[
\mathbf{U}(y) \text{ is } Y\text{-periodic, } \int_Y \mathbf{U}(y) \, dy = 0,
\]
for any periodic function \( \varphi \in H^1_{\text{per}}(Y) \).

In the current work, we are interested in the convergence of the dynamic problem driven by the Landau-Lifshitz energy (5) to the dynamics problem driven by the homogenized energy (6) as \( \varepsilon \) goes to 0. It is well known that the time evolution of the magnetization over \( \Omega_T = \Omega \times [0, T] \) follows the Landau-Lifshitz-Gilbert (LLG) equation [7, 6]
\[
\begin{aligned}
\partial_t \mathbf{m}^\varepsilon - \alpha \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon &= -(1 + \alpha^2) \mathbf{m}^\varepsilon \times \mathcal{H}^\varepsilon_e(\mathbf{m}^\varepsilon) \quad \text{a.e. in } \Omega_T, \\
\mathbf{m}^\varepsilon(0, x) &= \mathbf{m}^\varepsilon_{\text{init}}(x), \quad |\mathbf{m}^\varepsilon_{\text{init}}(x)| = 1 \quad \text{a.e. in } \Omega,
\end{aligned}
\]
(8)
\[
\mathbf{m}^\varepsilon(0, x) = \mathbf{m}^\varepsilon_{\text{init}}(x), \quad |\mathbf{m}^\varepsilon_{\text{init}}(x)| = 1 \quad \text{a.e. in } \Omega,
\]
where \( \alpha > 0 \) is the damping constant, and the effective field \( \mathcal{H}^\varepsilon_e(\mathbf{m}^\varepsilon) = -\frac{\delta G^\varepsilon_e}{\delta \mathbf{m}} \) associated to the Landau-Lifshitz energy (5) is given by
\[
\mathcal{H}^\varepsilon_e(\mathbf{m}^\varepsilon) = \text{div} \left( a^\varepsilon \nabla \mathbf{m}^\varepsilon \right) - K^\varepsilon(\mathbf{m}^\varepsilon \cdot \mathbf{u}) \mathbf{u} + \mu_0 M^\varepsilon \mathbf{h}_d[M^\varepsilon \mathbf{m}^\varepsilon] + M^\varepsilon \mathbf{h}_a.
\]
Meanwhile, the LLG equation associated to the homogenized energy (6) reads as
\[
\begin{aligned}
\partial_t \mathbf{m}_0 - \alpha \mathbf{m}_0 \times \partial_t \mathbf{m}_0 &= -(1 + \alpha^2) \mathbf{m}_0 \times \mathcal{H}^0_e(\mathbf{m}_0) \\
\nu \cdot a^0 \nabla \mathbf{m}_0 &= 0, \quad \text{a.e. on } \partial \Omega \times [0, T] \\
\mathbf{m}_0(0, x) &= \mathbf{m}^\varepsilon_{\text{init}}(x), \quad |\mathbf{m}^\varepsilon_{\text{init}}(x)| = 1 \quad \text{a.e. in } \Omega
\end{aligned}
\]
(10)
\[
\mathbf{m}_0(0, x) = \mathbf{m}^\varepsilon_{\text{init}}(x), \quad |\mathbf{m}^\varepsilon_{\text{init}}(x)| = 1 \quad \text{a.e. in } \Omega
\]
with homogenized effective field \( \mathcal{H}^0_e(\mathbf{m}_0) = -\frac{\delta G^0_{\text{hom}}}{\delta \mathbf{m}} \) calculated by
\[
\mathcal{H}^0_e(\mathbf{m}_0) = \text{div} \left( a^0 \nabla \mathbf{m}_0 \right) - K^0(\mathbf{m}_0 \cdot \mathbf{u}) \mathbf{u} + \mu_0 (M^0)^2 \mathbf{h}_d[\mathbf{m}_0] + \mu_0 \mathbf{H}_d^0 \cdot \mathbf{m}_0 + M^0 \mathbf{h}_a,
\]
(11)
where the matrix \( \mathbf{H}_d^0 = \int_Y M_s(y) \mathbf{H}_d[M_s(y)](y) \, dy \).

Works related to the homogenization of the LLG equation in the literature include \[11, 5, 1, 8, 9, 4\]. As for the convergence rate, most relevantly, the LLG equation (8) with only the exchange term and with the periodic
boundary condition is studied in [8]. Convergence rates between \(m^\varepsilon\) and \(m_0\) in time interval \([0, \varepsilon \sigma T]\) are obtained under the assumption

\[
\|\nabla m^\varepsilon\|_{L^\infty(\Omega)} \leq C, \quad \text{for any } t \in [0, \varepsilon \sigma T],
\]

where \(C\) is a constant independent of \(\varepsilon\) and \(\sigma \in [0, 2)\). As a special case, when \(\sigma = 0\) in assumption (12), i.e., \(\|\nabla m^\varepsilon\|_{L^\infty(\Omega)}\) is uniformly bounded over a time interval independent of \(\varepsilon\), it is proven that \(\|m^\varepsilon - m_0\|_{L^\infty(0,T;L^2(\Omega))} = O(\varepsilon)\) while \(\|m^\varepsilon - m_0\|_{L^\infty(0,T;H^1(\Omega))}\) is only uniformly bounded without strong convergence rate.

In this work, we consider the full LLG model (8) equipped with the Neumann boundary condition, which is the original model derived by Landau and Lifshitz [7]. We prove the convergence rates between \(m^\varepsilon\) and \(m_0\) in the \(L^\infty(0,T;H^1(\Omega))\) sense without the strong assumption (12). It is worth mentioning that, the trick to improve the convergence result into \(H^1\) sense is to find proper correctors \(m_1, m_2\), such that they satisfy geometric properties

\[
m_0 \cdot m_1 = 0, \quad \text{and} \quad m_0 \cdot m_2 = -|m_1|^2,
\]

which are motivated by the length-preserving property of magnetization and asymptotic expansion. A familiar definition of classic first-order homogenization corrector \(m_1\) in (32) would naturally satisfies first property in (13); see [8]. In this article, the suitable corrector \(m_2\) in (13) is obtained. By the usage of these properties, we are able to derive the estimate of consistency error, which is induced by an equivalent form of LLG equation, given in (22), and a sharper estimate than [8] in \(L^\infty(0,T;H^1(\Omega))\) sense is finally obtained.

Instead of the assumption (12), we prove a weak result that \(\|\nabla m^\varepsilon\|_{L^6(\Omega)}\) is uniformly bounded over a time interval independent of \(\varepsilon\). Such a uniform estimate is nontrivial for the LLG equation, since the standard energy estimate usually transforms the degenerate (damping) term into the diffusion term and thus the upper bound becomes \(\varepsilon\)-dependent. To overcome this difficulty, we introduce the interpolation inequality when \(n \leq 3\)

\[
\|\text{div} (a^\varepsilon \nabla m)\|_{L^3(\Omega)}^3 \leq C + C\|\text{div} (a^\varepsilon \nabla m)\|_{L^2(\Omega)}^6 \\
+ C\|m \times \nabla \{\text{div} (a^\varepsilon \nabla m)\}\|_{L^2(\Omega)}^2,
\]

for the \(S^2\)-value function \(m\) satisfying homogeneous Neumann boundary condition. This inequality can help us derive a structure-preserving energy estimate, in which the degenerate term is kept in the energy.

The full LLG model (8) we considered contains the stray field, where an independent homogenization problem of potential function in the distribution sense arises, and this complicated the problem when we arrive at the consistency analysis. By using results in [10] and Green’s representation formula, the stray field is rewritten as the derivatives of Newtonian potential. Then we are able to obtain the consistency error by deriving detailed estimate of singular integral for disturbed function and boundary layer.
The effect of boundary layer exists when we apply classic homogenization corrector to the Neumann boundary problem, which would cause the approximation deterioration on the boundary. To avoid this, a Neumann corrector is introduced, which is usually used in elliptic homogenization problems (see [12] for example). In this article, we provide a strategy to apply the Neumann corrector to evolutionary LLG equations, by finding a proper higher-order modification. For a big picture, let us write ahead the linear parabolic equation of error

$$\partial_t e^\varepsilon - L^\varepsilon e^\varepsilon + f^\varepsilon = 0,$$

whose detailed derivation can be found in (26). Following the notation of $e_b^\varepsilon = e^\varepsilon - \omega_b$ with boundary corrector $\omega_b$, one can find by above equation that an $L^\infty(0, T; H^1(\Omega))$ norm of $e_b^\varepsilon$ relies on the boundary data and inhomogeneous term induced by $\omega_b$, which read as

$$\|\nu \cdot a^\varepsilon \nabla \{e^\varepsilon + \omega_b\}\|_{B^{-1/2,2}(\partial \Omega)} \text{ and } \|L^\varepsilon \omega_b\|_{L^2(\Omega)}.$$  

In this end, we divide the corrector $\omega_b$ into two parts as $\omega_b = \omega_N - \omega_M$, such that they can control two terms in (15) respectively. Here $\omega_N$ is the Neumann corrector used in elliptic problems (see [12]), and $\omega_M$ is a modification to be determined. We point out the modification $\omega_M$ is necessary since calculation implies some bad terms in $L^\varepsilon \omega_N$ do not converge in $L^2$ sense. Therefore we construct following elliptic problem to determine $\omega_M$:

$$\text{div}(a^\varepsilon \nabla \omega_M) = \left( \text{Bad Terms in } L^\varepsilon \omega_N \right)$$

with proper Neumann boundary condition. Such a solution $\omega_M$ can be proved to have better estimates than $\omega_N$, by the observation that all “Bad Terms in $L^\varepsilon \omega_N$” can be written in the divergence form. At this point, $\omega_M$ can be viewed as a high-order modification.

This paper is organized as follows. In the next Section, we introduce the main result of our article and outline the main steps of the proof. In Section 3, multiscale expansions are used to derive the second-order corrector $m_2$. In Section 4, we deduce that the consistency error $f^\varepsilon$ only relies on the consistency error of the stray field, which can be estimated by calculation of singular integral for disturbed function and boundary layer. In Section 5, we introduce the boundary corrector $\omega_b$, and derive several relevant estimates of it. Section 6 contains the stability analysis in $L^2$ and $H^1$ sense respectively. And we finally give a uniform regularity analysis of $m^\varepsilon$, by deriving a structure-preserving energy estimate in Section 7.

2. Main result

To proceed, we make the following assumption

**Assumption 1.**

1. **Smoothness** We assume $Y$-periodic functions $a(y) = (a_{ij}(y))_{1 \leq i,j \leq 3}$, $K(y)$, $M_s(y)$, and the time-independent external field $h_a(x)$, alone with
boundary $\partial \Omega$, are sufficiently smooth. These together with the definition in (32), (46), (66), (76) leads to the smoothness of $\mathbf{m}_1$, $\mathbf{m}_2$ and $\omega_0$.

2. **Initial data** Assume $\mathbf{m}^0_{\text{init}}(x)$ and $\mathbf{m}^\varepsilon_{\text{init}}(x)$ are smooth enough and satisfy the Neumann compatibility condition:

\[
\nu \cdot a^\varepsilon \nabla m^\varepsilon_{\text{init}}(x) = \nu \cdot a^0 \nabla m^0_{\text{init}}(x) = 0, \quad x \in \partial \Omega.
\]

Furthermore, we might as well set them satisfying periodically disturbed elliptic problem:

\[
\text{div}(a^\varepsilon \nabla m^\varepsilon_{\text{init}}(x)) = \text{div}(a^0 \nabla m^0_{\text{init}}(x)), \quad x \in \Omega.
\]

(16)-(17) imply $m^0_{\text{init}}(x)$ is the homogenization of $m^\varepsilon_{\text{init}}(x)$. Note that the assumption (17) is necessary not only for the convergence analysis in Theorem 4, but also for the uniform estimate of $m^\varepsilon$ in Theorem 8.

Now let us state our main result:

**Theorem 1.** Let $m^\varepsilon \in L^\infty(0,T;H^2(\Omega))$, $m_0 \in L^\infty(0,T;H^6(\Omega))$ be the unique solutions of (8) and (10), respectively. Under Assumption 1, there exists some $T^* \in (0,T]$ independent of $\varepsilon$, such that for any $t \in (0,T^*)$ and for $n = 2, 3$, it holds

\[
\|m^\varepsilon(t) - m_0(t)\|_{L^2(\Omega)} \leq \beta(\varepsilon), \quad \|m^\varepsilon(t) - m_0(t)\|_{H^1(\Omega)} \leq C\varepsilon^{1/2},
\]

where

\[
\beta(\varepsilon) = \begin{cases} 
C\varepsilon[\ln(\varepsilon^{-1} + 1)]^2, & \text{when } n = 2, \\
C\varepsilon^{5/6}, & \text{when } n = 3.
\end{cases}
\]

In the absence of the stray field, i.e., $\mu_0 = 0$, then it holds for any $t \in (0,T^*)$ and for $n = 1, 2, 3$

\[
\|m^\varepsilon(t) - m_0(t)\|_{L^2(\Omega)} \leq C\varepsilon[\ln(\varepsilon^{-1} + 1)]^2,
\]

\[
\|m^\varepsilon(t) - m_0(t) - (\Phi - \varepsilon)\nabla m_0(t)\|_{H^1(\Omega)} \leq C\varepsilon[\ln(\varepsilon^{-1} + 1)]^2,
\]

where $x$ is spatial variable, $\Phi = (\Phi_i)_{1 \leq i \leq n}$ is the corrector defined in (67). Constant $C$ depends on the initial data $m^\varepsilon_{\text{init}}$ and $m^0_{\text{init}}$, but is independent of $\varepsilon$.

**Remark 2.1.** Comparing (18) and (20), one can see that in the $L^2$ norm, the stray field makes little influence when $n = 2$, but causes $1/6$-order loss of rate when $n = 3$. In the $H^1$ norm, however, the stray field leads to $1/2$-order loss of rate in both cases. Such a deterioration of convergence rate is induced since the zero-extension has been applied for stray field (3), which introduces a boundary layer.

**Remark 2.2.** The logarithmic growth $|\ln(\varepsilon^{-1} + 1)|^2$ in (20) is caused by the Neumann corrector $(\Phi - \varepsilon)\nabla m_0$. For problems (8) and (10) with periodic boundary condition over a cube, by replacing the Neumann corrector in (20)
with the classical two-scale corrector, a similar argument in the current work leads to
\[
\| \mathbf{m}^\varepsilon - \mathbf{m}_0 - \chi \nabla \mathbf{m}_0 \|_{H^1(\Omega)} \leq C\varepsilon,
\]
where \( \chi = (\chi_i)_{1 \leq i \leq n} \) is defined in (33).

Note that (21) is consistent with the \( L^2 \) result in [8]. However, only the uniform boundedness in \( H^1 \) has been shown in [8], while our results (20) and (21) imply that it maintains the same convergence rate in \( L^2 \) and \( H^1 \) norm, by choosing the correctors satisfying specific geometric property (13).

2.1. Some notations and Lax equivalence type theorem. Recall that a classical solution to (8) also satisfies an equivalent form of equation, reads
\[
\mathcal{L}_{\text{LLG}}(\mathbf{m}^\varepsilon) := \partial_t \mathbf{m}^\varepsilon - \alpha \mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) + \mathbf{m}^\varepsilon \times \mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) - \alpha \mu^\varepsilon g_i^\varepsilon(\mathbf{m}^\varepsilon) \mathbf{m}^\varepsilon = 0,
\]
where the \( g_i^\varepsilon(\cdot) \) is the energy density calculated by
\[
g_i^\varepsilon(\mathbf{m}^\varepsilon) = a^i |\nabla \mathbf{m}^\varepsilon|^2 + K^\varepsilon(\mathbf{m}^\varepsilon \cdot \mathbf{u}) \mathbf{u} - \mathbf{h}_d[M^\varepsilon \mathbf{m}^\varepsilon] \cdot M^\varepsilon \mathbf{m}^\varepsilon - \mathbf{h}_a \cdot M^\varepsilon \mathbf{m}^\varepsilon.
\]

For convenience, we also define a bilinear operator deduced from (23), which reads
\[
\mathcal{B}^\varepsilon(\mathbf{m}, \mathbf{n}) = a^i \nabla \mathbf{m} \cdot \nabla \mathbf{n} + K^\varepsilon (\mathbf{m} \cdot \mathbf{u}) (\mathbf{n} \cdot \mathbf{u}) - \mu_0 \mathbf{h}_d[M^\varepsilon \mathbf{m}] \cdot M^\varepsilon \mathbf{n}.
\]

Now let us set up the equation of error, in terms of Lax equivalence theorem kind of argument. Define the approximate solution
\[
\tilde{\mathbf{m}}^\varepsilon(x) = \mathbf{m}_0(x) + \varepsilon \mathbf{m}_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 \mathbf{m}_2(x, \frac{x}{\varepsilon}),
\]
where \( \mathbf{m}_0 \) is the homogenized solution to (10), \( \mathbf{m}_1 \) is the first-order corrector defined in (32), and \( \mathbf{m}_2 \) is the second-order corrector determined by Theorem 2. Then replacing \( \mathbf{m}^\varepsilon \) by \( \tilde{\mathbf{m}}^\varepsilon \) in (22) provides the notation of consistence error \( f^\varepsilon \):
\[
\mathcal{L}_{\text{LLG}}(\tilde{\mathbf{m}}^\varepsilon) = f^\varepsilon.
\]

Together (22) and (25), we can obtain the equation of error \( e^\varepsilon = \mathbf{m}^\varepsilon - \tilde{\mathbf{m}}^\varepsilon \), denoted by
\[
\partial_t e^\varepsilon - \mathcal{L}_e^\varepsilon e^\varepsilon + f^\varepsilon = 0,
\]
where \( \mathcal{L}_e^\varepsilon \) is second-order linear elliptic operator depending on \( \mathbf{m}^\varepsilon \) and \( \tilde{\mathbf{m}}^\varepsilon \), that can be characterized as
\[
\mathcal{L}_e^\varepsilon(e^\varepsilon) = \alpha \mathcal{H}_e^\varepsilon(e^\varepsilon) - \mathbf{D}_1(e^\varepsilon) - \mathbf{D}_2(e^\varepsilon).
\]

Here \( \mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) \) is the linear part of \( \mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) \), i.e.,
\[
\mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) := \mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) - M^\varepsilon \mathbf{h}_a,
\]
procession term \( \mathbf{D}_1 \) is calculated by
\[
\mathbf{D}_1(e^\varepsilon) = \mathbf{m}^\varepsilon \times \mathcal{H}_e^\varepsilon(\mathbf{m}^\varepsilon) - \tilde{\mathbf{m}}^\varepsilon \times \mathcal{H}_e^\varepsilon(\tilde{\mathbf{m}}^\varepsilon) = \mathbf{m}^\varepsilon \times \mathcal{H}_e^\varepsilon(e^\varepsilon) + e^\varepsilon \times \mathcal{H}_e^\varepsilon(\tilde{\mathbf{m}}^\varepsilon),
\]
and the degeneracy term $D_2$ reads as

$$D_2(\varepsilon^\varepsilon) = -\alpha q^\varepsilon_1[\tilde{m}^\varepsilon, m^\varepsilon] + \alpha q^\varepsilon_1[\tilde{m}^\varepsilon, \tilde{m}^\varepsilon]$$

$$= -\alpha(B^\varepsilon[e^\varepsilon, m^\varepsilon] + B^\varepsilon[\tilde{m}^\varepsilon, e^\varepsilon] + M^\varepsilon(h_\varepsilon \cdot e^\varepsilon))m^\varepsilon - \alpha q^\varepsilon_1[\tilde{m}^\varepsilon]e^\varepsilon.$$

Moreover, we define a correctional error $e_b^\varepsilon$ as

$$e_b^\varepsilon = e^\varepsilon - \omega_b,$$

where $\omega_b$ is the boundary corrector satisfying $\omega_b = \omega_N - \omega_M$, for $\omega_N$ the Neumann corrector given in (66), and $\omega_M$ the modification determined in (76). Then equation (26) leads to

$$\partial_t e_b^\varepsilon - L^\varepsilon e_b^\varepsilon + (\partial_t \omega_b - L^\varepsilon \omega_b + f^\varepsilon) = 0.$$

By the Lax equivalence theorem kind of argument, the estimate of error $e_b^\varepsilon$ follows from consistency analysis of (25), energy estimate of boundary corrector, and stability analysis of (30).

### 2.2. Proof of Theorem 1.

**Proof.** Following the above notations, for the consistency error $f^\varepsilon$, Theorem 3 says that it can be divided as $f^\varepsilon = f_0 + \tilde{f}$, satisfying $\|\tilde{f}(t)\|_{L^2(\Omega)} \leq C\varepsilon$, and

$$\|f_0(t)\|_{L^2(\Omega)} = 0, \quad \text{when } \mu_0 = 0,$$

$$\|f_0(t)\|_{L^r(\Omega)} \leq C_r \mu_0 (\varepsilon^{1/r} + \varepsilon \ln(\varepsilon^{-1} + 1)), \quad \text{when } \mu_0 > 0, n \neq 1,$$

where constants $C_r$ and $C$ are independent of $\varepsilon$, for any $t \in (0, T)$, and $1 \leq r < +\infty$. Considering the boundary corrector terms in (30), by Theorem 5 there exists $C = C(\|\nabla m^\varepsilon\|_{L^2(\Omega)})$ such that

$$\|\partial_t \omega_b(t)\|_{L^2(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1),$$

$$\|L^\varepsilon \omega_b(t)\|_{L^2(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1)^2 + C\|e_b^\varepsilon(t)\|_{H^1(\Omega)},$$

for any $t \in (0, T)$. As for initial-boundary data of $e_b^\varepsilon$, using Theorem 4 we write with $C = C(\|\nabla m^\varepsilon\|_{L^2(\Omega)})$,

$$\|e_b^\varepsilon(x, 0)\|_{H^1(\Omega)} + \|\frac{\partial}{\partial x^\varepsilon} e_b^\varepsilon\|_{W^{1, \infty}(0, T; 2^{1/2} H(\Omega))} \leq C \varepsilon \ln(\varepsilon^{-1} + 1).$$

Now let us turn to stability analysis of (30). For the $L^\infty(0,T; L^2(\Omega))$ norm, let $\sigma = 1$ when $n = 1, 2$, and $\sigma = 6/5$ when $n = 3$, we can apply Theorem 6 to derive for $n = 1, 2, 3$

$$\|e_b^\varepsilon\|_{L^\infty(0,T; L^2(\Omega))}^2 + \|\nabla e_b^\varepsilon\|_{L^2(0,T; L^2(\Omega))}^2$$

$$\leq C_6 \left( \|e_b^\varepsilon(x, 0)\|_{L^2(\Omega)}^2 + \|\frac{\partial}{\partial x^\varepsilon} e_b^\varepsilon\|_{L^2(0,T; L^2(\Omega))}^2 + \|f_0\|_{L^2(0,T; L^2(\Omega))}^2 + \|\omega_b\|_{L^2(0,T; L^2(\Omega))}^2 + \gamma(\varepsilon) \|f_0\|_{L^2(0,T; L^2(\Omega))}^2 ight)$$

$$+ \delta \|L^\varepsilon \omega_b\|_{L^2(0,T; L^2(\Omega))}^2 + \varepsilon^2 \|\text{A}_\varepsilon e_b^\varepsilon\|_{L^2(0,T; L^2(\Omega))}^2.$$
with
\[
\begin{cases}
\gamma(\varepsilon) = 1, & \text{when } n = 1, 3, \\
\gamma(\varepsilon) = [\ln(\varepsilon^{-1} + 1)]^2, & \text{when } n = 2.
\end{cases}
\]
Constant \( C_\delta = C_\delta(\|\nabla m^\varepsilon\|_{L^4(\Omega)}) \). Now taking \( \delta \) small enough in (31), and using the fact
\[
\|A_\varepsilon e_0^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C \ln(\varepsilon^{-1} + 1)
\]
with \( C = C(\|A_\varepsilon m^\varepsilon\|_{L^2(\Omega)}) \) from Theorem 5, we finally obtain
\[
\|e_b^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \begin{cases}
\beta(\varepsilon), & \text{when } \mu_0 > 0, n = 2, 3, \\
C\varepsilon \ln(\varepsilon^{-1} + 1)^2, & \text{when } \mu_0 = 0, n = 1, 2, 3,
\end{cases}
\]
where \( \beta(\varepsilon) \) satisfies (19). Using the fact \( m^\varepsilon - m_0 = e_0^\varepsilon + \varepsilon m_1 + \varepsilon^2 m_2 + \omega_b \), along with the estimates of \( \varepsilon m_1, \varepsilon^2 m_2, \omega_b \) in Lemma 4-5, we obtain the \( L^2 \) estimates in Theorem 1.

As for the stability of (30) in \( L^\infty(0,T;H^1(\Omega)) \) norm, we can apply Theorem 7 to obtain for \( n = 1, 2, 3 \)
\[
\|\nabla e_0^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C \left( \|e_0^\varepsilon(x,0)\|_{H^1(\Omega)}^2 + \|\frac{\partial}{\partial \nu} e_0^\varepsilon\|_{H^1(0,T;B^{-1/2,2}(\partial\Omega))}^2 \right) + \|\mathcal{L} \omega_b^\varepsilon\|_{L^2(0,T;L^2(\Omega))}^2 + \|f^\varepsilon\|_{L^2(0,T;L^2(\Omega))}^2 + \|\partial_t \omega_b^\varepsilon\|_{L^2(0,T;L^2(\Omega))}^2,
\]
where constant \( C = C(\|A_\varepsilon m^\varepsilon\|_{L^2(\Omega)}, \|\nabla m^\varepsilon\|_{L^4(\Omega)}) \). Together with above results, and estimate for \( \|\nabla e_0^\varepsilon\|_{L^2(0,T;L^2(\Omega))}^2 \) in (31), we arrive at
\[
\|\nabla e_0^\varepsilon(t)\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq \begin{cases}
C\varepsilon^{1/2}, & \text{when } \mu_0 > 0, n = 2, 3, \\
C\varepsilon \ln(\varepsilon^{-1} + 1)^2, & \text{when } \mu_0 = 0, n = 1, 2, 3,
\end{cases}
\]
by the representation of \( e_0^\varepsilon \) in (80), together with estimate of \( m_2 \) and \( \omega_M \) in Lemma 5, it leads to the \( H^1 \) estimates in Theorem 1.

Notice that all the constants in our estimate depend on the value of \( \|A_\varepsilon m^\varepsilon(t)\|_{L^2(\Omega)} \) and \( \|\nabla m^\varepsilon(t)\|_{L^4(\Omega)} \), which from Theorem 8 are uniformly bounded with respect to \( \varepsilon \) and \( t \) for any \( t \in (0,T^*) \), with some \( T^* \in (0,T] \). This completes the proof. \( \square \)

3. The Asymptotic Expansion

In this section, we derive the second-order corrector using the formal asymptotic expansion. First, let us define the first-order corrector \( m_1 \) by
\[
m_1(x, y) = \sum_{j=1}^n \chi_j(y) \frac{\partial}{\partial x_j} m_0(x),
\]
where \( \chi_j, j = 1, \ldots, n \) are auxiliary functions satisfying cell problem
\[
\begin{cases}
\text{div}(a(y)\nabla \chi_j(y)) = -\sum_{i=1}^n \frac{\partial}{\partial y_i} a_{ij}(y), \\
\chi_j \quad Y\text{-periodic,}
\end{cases}
\]
such that the first geometric property in (13) holds. As for the second-order corrector $m_2$, we assume it as a two-scale function satisfying

$$
\begin{aligned}
\begin{cases}
m_2(x, y) \text{ is defined for } x \in \Omega \text{ and } y \in Y, \\
m_2(\cdot, y) \text{ is } Y\text{-periodic}.
\end{cases}
\end{aligned}
$$

For notational convenience, given a two-scale function in the form of $m(x, \frac{y}{\varepsilon})$, we denote the fast variable $y = \frac{y}{\varepsilon}$ and have the following chain rule

$$
\nabla m(x, \frac{x}{\varepsilon}) = [(\nabla_x + \varepsilon^{-1}\nabla_y)m](x, y).
$$

Moreover, denoting $A_\varepsilon = \text{div}(a\varepsilon\nabla)$, one can rewrite

$$
A_\varepsilon m(x, \frac{x}{\varepsilon}) = [(\varepsilon^{-2}A_0 + \varepsilon^{-1}A_1 + A_2)m](x, y),
$$

where

$$
\begin{aligned}
\begin{cases}
A_0 &= \text{div}_{\varepsilon}(a(y)\nabla_y), \\
A_1 &= \text{div}_x(a(y)\nabla_y) + \text{div}_y(A(y)\nabla_x), \\
A_2 &= \text{div}_x(a(y)\nabla_x).
\end{cases}
\end{aligned}
$$

The procedure to determine $m_2$ is standard. With the notation in (24), assume $m^\varepsilon$ can be written in form of

$$
m^\varepsilon(x) = \tilde{m}^\varepsilon(x) + o(\varepsilon^2).
$$

One can derive $m_2$ by substituting (36) into (8) and comparing like terms of $\varepsilon$. However, it is a bit fussy in the presence of stray field. Let us outline the main steps here. Revisiting the stray field $h_\varepsilon[M^\varepsilon m^\varepsilon(x)] = \nabla U^\varepsilon$ in (2)-(3), one finds that the potential function $U^\varepsilon = U^\varepsilon[M^\varepsilon m^\varepsilon(x)]$ satisfies

$$
\Delta U^\varepsilon = -\text{div}(M^0(\frac{x}{\varepsilon})m^\varepsilon X_\Omega).
$$

Substituting $U^\varepsilon = \sum_{j=0}^2 \varepsilon^j U_j(x, \frac{y}{\varepsilon}) + o(\varepsilon^2)$ and (36) into (37) and combining like terms of $\varepsilon$ leads to

$$
\begin{aligned}
\text{div}_y(\nabla_y U_0(x, y)) &= 0, \\
\text{div}_y(\nabla_y U_1(x, y)) &= -\text{div}_y(M_s(y)m_0(x)X_\Omega(x)), \\
\text{div}_x(\nabla_x U_0(x, y)) + 2\text{div}_y(\nabla_x U_1(x, y)) &= \text{div}_y(\nabla_y U_2(x, y)) \\
&= -M_s(y)\text{div}_x m_0(x)X_\Omega(x) - \text{div}_y(M_s(y)m_1(x, y)X_\Omega(x)).
\end{aligned}
$$

The first equation in (38) implies that $U_0(x, y) = U_0(x)$ since the Lax-Milgram Theorem ensures the uniqueness and existence of solution (up to a constant). Integrating the third equation in (38) with respect to $y$ yields

$$
\Delta U_0(x) = -\text{div}(M^0m_0X_\Omega).
$$

An application of (2)-(3) implies that $U_0$ is actually the potential function of $h_\varepsilon[M^h m_0]$, i.e.,

$$
\nabla U_0(x) = h_\varepsilon[M^h m_0] = M^h h_\varepsilon[m_0].
$$
With notation given in (7), one can deduce from the second equation in (38) that \( \mathbf{m}_0(x) \lambda_2(x) U(y) = U_1(x, y) \) up to a constant in the \( H^1(Y) \) space. Hence it follows that by (7)

\[
\nabla_y U_1(x, y) = \lambda_2(x) \mathbf{m}_0(x) \cdot H_d[M_s(y)](y).
\]

Substituting (39) and (40) into the expansion of \( U^\varepsilon \), one can deduce that, for \( x \in \Omega \),

\[
\mathrm{h}_d[M^\varepsilon \mathbf{m}^\varepsilon] = \nabla U^\varepsilon = \mathrm{h}_d[M^h \mathbf{m}_0] + \mathbf{m}_0(x) \cdot H_d[M_s(y)](x) + O(\varepsilon).
\]

Substituting (36), (32), (35), (41) into (8) and collecting terms of \( O(\varepsilon^0) \), we obtain the following equations

\[
\begin{align*}
\partial_t \mathbf{m}_0 - \alpha \mathbf{m}_0 \times \partial_t \mathbf{m}_0 &= -(1 + \alpha^2) \mathbf{m}_0 \times \{ A_0 \mathbf{m}_2 + \mathcal{H}_e^a \}, \\
\mathbf{m}_2 &= \text{Y-periodic in } y,
\end{align*}
\]

where

\[
\mathcal{H}_e^a = A_1 \mathbf{m}_1 + A_2 \mathbf{m}_0 - K^\varepsilon(\mathbf{m}_0 \cdot \mathbf{u}) \mathbf{u} + \mu_0 M_s \mathrm{h}_d[M^h \mathbf{m}_0] + \mu_0 M_s \mathbf{m}_0 \cdot H_d[M_s(y)] + M_s \mathbf{h}_a.
\]

Substituting (10) into (42) leads to

\[
\begin{align*}
\mathbf{m}_0 \times A_0 \mathbf{m}_2 &= \mathbf{m}_0 \times \{ \mathcal{H}_e^0(\mathbf{m}_0) - \mathcal{H}_e^a \}, \\
\mathbf{m}_2 &= \text{Y-periodic in } y.
\end{align*}
\]

(44) is the degenerate system that determines \( \mathbf{m}_2 \) in terms of \( \mathbf{m}_0 \).

### 3.1. Second-order corrector.

The well-posedness of (44) is nontrivial due to the degeneracy. In the following Theorem, by searching a suitable solution satisfying (13), we give the existence result, and derive an explicit expression of \( \mathbf{m}_2 \) in terms of \( \mathbf{m}_0 \) and some auxiliary functions.

**Theorem 2.** Given \( \mathbf{m}_0 \in L^\infty ([0, T]; H^2(\Omega)) \) the homogenization solution and \( \mathbf{m}_1 \) calculated in (32), define

\[
\mathcal{V} = \left\{ \mathbf{m} \in H^2(Y) \cap H^1_{\text{per}}(Y) : \, \mathbf{m} \cdot \mathbf{m}_0 = -\frac{1}{2} |\mathbf{m}_1|^2 \text{ a.e. in } \Omega \times Y \right\},
\]

then (44) admits a unique solution \( \mathbf{m}_2(x, y) \in \mathcal{V}/T_{\mathbf{m}_0}(S^2) \) with notation \( T_{\mathbf{m}_0}(S^2) \) denoting the tangent space of \( \mathbf{m}_0 \).

**Proof.** Assume \( \mathbf{m}_2(x, y) \in \mathcal{V} \), i.e., \( \mathbf{m}_2 \cdot \mathbf{m}_0 = -\frac{1}{2} |\mathbf{m}_1|^2 \). Applying \( A_0 \) to both sides of it yields

\[
\mathbf{m}_0 \cdot A_0 \mathbf{m}_2 = -a(y) \nabla_y \mathbf{m}_1 \cdot \nabla_y \mathbf{m}_1 - \mathbf{m}_1 \cdot A_0 \mathbf{m}_1.
\]

Taking the cross-product with \( \mathbf{m}_0 \) to (44) and substituting (45) lead to

\[
A_0 \mathbf{m}_2 = -\{ \mathcal{H}_e^a - \mathcal{H}_e^0(\mathbf{m}_0) \} + \{ \mathbf{m}_0 \cdot (\mathcal{H}_e^a - \mathcal{H}_e^0(\mathbf{m}_0)) \} \mathbf{m}_0
\]

\[
- (\mathbf{m}_1 \cdot A_0 \mathbf{m}_1 + a(y) \nabla_y \mathbf{m}_1 \cdot \nabla_y \mathbf{m}_1) \mathbf{m}_0.
\]
Now using the fact
\[
\int_Y \mathcal{H}_0^0(m_0) - \mathcal{H}_e^0 \, dy = 0,
\]
together with (45), one can check equation (46) satisfies the compatibility condition for \( Y \)-periodic function \( m_2 \) in \( y \). Thus by the application of Lax-Milgram Theorem and smoothness assumption, (46) admits a unique regular solution up to a function independent of \( y \), denoted by \( m_2(x, y) + \tilde{m}_2(x) \). Moreover, one can determine \( \tilde{m}_2(x) \) by
\[
m_0 \cdot (m_2 + \tilde{m}_2) = -\frac{1}{2}|m_1|^2,
\]
such that \( m_2 + \tilde{m}_2 \in V \), and therefore is also a solution to (44) by taking the above transformation inversely.

\[\square\]

**Remark 3.1.** One can check that equation (46) has a solution
\[
m_2 = \sum_{i,j=1}^{n} \theta_{ij} \frac{\partial^2 m_0}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} \left[ \theta_{ij} + \frac{1}{2} \chi_i \chi_j \right] \left( \frac{\partial m_0}{\partial x_i} \cdot \frac{\partial m_0}{\partial x_j} \right) m_0 + T_{low} - (m_0 \cdot T_{low}) m_0
\]
with low-order terms \( T_{low} \) calculated by
\[
T_{low} = -\kappa (m_0 \cdot u) u + \mu_0 \rho \mathbf{h}_d [M^h m_0] + \mu_0 m_0 \cdot \Lambda + M_s \mathbf{h}_s,
\]
where \( \theta_{ij} \) and \( \kappa, \rho, \Lambda \) are given by
\[
\begin{align*}
A_0 \theta_{ij} &= a_{ij}^0 - \left( a_{ij} + \sum_{k=1}^{n} a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) - \sum_{k=1}^{n} \frac{\partial (a_{ik} \chi_j)}{\partial y_k}, \\
A_0 \rho &= M_s(y) - M^0, \quad A_0 \kappa = K(y) - K^0, \\
A_0 \Lambda &= M_s(y) H_d [M_s(y)](y) - H_d^0, \\
\theta_{ij}, \kappa, \rho, \Lambda, \text{ are } Y\text{-periodic.}
\end{align*}
\]

Moreover, one can find \( m_2 \) defined above satisfies geometric property (13), therefore is also the solution to equation (44). In the following, we may assume second-order correct \( m_2 \) takes the form in (48).

4. Consistency Estimate

In this section, we aim to estimate the consistence error \( f^\varepsilon \) defined in (25). Following the notation in (34)-(35), by the definition of \( \tilde{m}^\varepsilon \), (25) can be written in terms of
\[
f^\varepsilon = \varepsilon^{-2} f_{-2} + \varepsilon^{-1} f_{-1} + f_0 + \varepsilon f_1 + \varepsilon^2 f_2.
\]
It is easy to check that \( f_{-2} = f_{-1} = 0 \) by the definition of \( m_0, m_1 \) in Section 3. Along the same line, by the Hölder’s inequality, one has
\[
\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)} \leq C,
\]
where $C$ depends on the $L^2(\Omega)$ and $L^\infty(\Omega)$ norms of $\mathbf{m}_i(x, \frac{\pi}{2})$, $\nabla_x \mathbf{m}_i(x, \frac{\pi}{2})$, $\nabla_y \mathbf{m}_i(x, \frac{\pi}{2})$, $i = 0, 1, 2$, and thus is bounded from above by $\|\nabla \mathbf{m}_0\|_{H^s(\Omega)}$ with the help of smoothness assumption and Sobolev inequality.

It remains to estimate $f_0$, let us prove that $f_0$ only depends on the consistence error of stray field, by the help of geometric property (13). Denote the consistence error of stray field by

$$
(50) \quad \mathbf{h} = \mu M^T \mathbf{h}_d[(M^T - M^h)\mathbf{m}_0] - \mu M^T \mathbf{H}_d[M_s(\mathbf{y})](\frac{x}{\varepsilon}) \cdot \mathbf{m}_0
$$

with $\mathbf{H}_d$ given in (7). After some algebraic calculations and the usage of (42) and (43), one has

$$
(51) \quad f_0 = \partial_t \mathbf{m}_0 - \alpha \left\{ \mathcal{A}_0 \mathbf{m}_2 + \mathcal{H}_e^a + \mathbf{h} \right\} + \mathbf{m}_0 \times \left\{ \mathcal{A}_0 \mathbf{m}_2 + \mathcal{H}_e^a + \mathbf{h} \right\} - \alpha g_l^0[\mathbf{m}_0] \mathbf{m}_0 - (a^T \nabla y \mathbf{m}_1 \cdot \nabla \mathbf{m}_1) \mathbf{m}_0 - 2(a^T \nabla y \mathbf{m}_1 \cdot \nabla \mathbf{m}_0) \mathbf{m}_0.
$$

Notice that the classical solution $\mathbf{m}_0$ to (10) also satisfies the equivalent form

$$
(52) \quad \partial_t \mathbf{m}_0 - \alpha \mathcal{H}_e^0(\mathbf{m}_0) + \mathbf{m}_0 \times \mathcal{H}_e^0(\mathbf{m}_0) - \alpha g_l^0[\mathbf{m}_0] \mathbf{m}_0 = 0,
$$

where

$$
\begin{align*}
\nonumber & g_l^0[\mathbf{m}] := a^0|\nabla \mathbf{m}|^2 + K^0(\mathbf{m} \cdot \mathbf{n}) \mathbf{n} - \mu_0(M^0)^2 \mathbf{H}_d[\mathbf{m}] \cdot \mathbf{m} \\
& \quad - \mu_0 \mathbf{m} \cdot \mathbf{H}_d^0 \cdot \mathbf{m} - \mathbf{h} \cdot M^0 \mathbf{m}.
\end{align*}
$$

Substituting (52) into (51) and using (46) lead to

$$
(53) \quad f_0 = -\alpha \mathbf{h} + \mathbf{m}_0 \times \mathbf{h} - \alpha \left\{ \mathbf{m}_0 \cdot \left( \mathcal{H}_e^a - \mathcal{H}_e^0(\mathbf{m}_0) \right) \right\} \mathbf{m}_0 \\
\quad + \alpha (\mathbf{m}_1 \cdot \mathcal{A}_0 \mathbf{m}_1 - 2a^T \nabla y \mathbf{m}_1 \cdot \nabla \mathbf{m}_0) \mathbf{m}_0 + \alpha g_l^0[\mathbf{m}_0] \mathbf{m}_0 - \alpha g_l^0[\mathbf{m}_0] \mathbf{m}_0.
$$

Note that $\mathcal{A}_2 \mathbf{m}_0 = \mathcal{A}_1 \mathbf{m}_0 - \varepsilon^{-1} \mathcal{A}_1 \mathbf{m}_0$, one can deduce

$$
\mathcal{H}_e^a = \mathcal{H}_e^0(\mathbf{m}_0) + \mathcal{A}_1 \mathbf{m}_1 + \varepsilon^{-1} \mathcal{A}_0 \mathbf{m}_1 - \mathbf{h}.
$$

Substituting it into (53), and using the fact

$$
\mathbf{m}_0 \cdot \mathcal{H}_e^0(\mathbf{m}_0) = -g_l^0[\mathbf{m}_0], \quad \mathbf{m}_0 \cdot \mathcal{H}_e^a(\mathbf{m}_0) = -g_l^0[\mathbf{m}_0],
$$

one has

$$
(54) \quad f_0 = -\alpha \mathbf{h} + \mathbf{m}_0 \times \mathbf{h} - \alpha \left\{ \mathbf{m}_0 \cdot \left( \mathcal{A}_1 \mathbf{m}_1 + \varepsilon^{-1} \mathcal{A}_0 \mathbf{m}_1 - \mathbf{h} \right) \right\} \mathbf{m}_0 \\
\quad + \alpha (\mathbf{m}_1 \cdot \mathcal{A}_0 \mathbf{m}_1 - 2a^T \nabla y \mathbf{m}_1 \cdot \nabla \mathbf{m}_0) \mathbf{m}_0.
$$

Apply $\mathcal{A}_0$ and $\mathcal{A}_1$ to both sides of $\mathbf{m}_0 \cdot \mathbf{m}_1 = 0$ respectively, and substitute resulting equations into (54). After simplification, we finally obtain

$$
(55) \quad f_0 = -\alpha \mathbf{h} + \mathbf{m}_0 \times \mathbf{h} + \alpha \left( \mathbf{m}_0 \cdot \mathbf{h} \right) \mathbf{m}_0.
$$

(55) implies that the convergence of $f^\varepsilon$ depends on the convergence of stray field field error $\mathbf{h}$. In fact, we have
Lemma 1. For any $1 \leq r < \infty$, and $n = 2, 3$, it holds that
\begin{equation}
\| M^\varepsilon h_a[(M^\varepsilon - M^h) m_0] - M^\varepsilon H_a[M_a(y)](\frac{x}{\varepsilon}) \cdot m_0 \|_{L^r(\Omega)} \leq C \varepsilon^{1/r} + C \varepsilon \ln(\varepsilon^{-1} + 1),
\end{equation}
where $C$ depends on $\| \nabla m_0 \|_{W^{1,\infty}(\Omega)}$, $\| \nabla M_a(y) \|_{H^1(\Gamma)}$ and is independent of $\varepsilon$.

Proof of Lemma 1 will be given in Section 4.2. Lemma 1 directly leads to the consistency error:

Theorem 3. (Consistency) Given $f^\varepsilon$ defined in (25), it can be divided as $f^\varepsilon = f_0 + f$, satisfying $\| f \|_{L^2(\Omega)} \leq C \varepsilon$, and
\begin{equation}
\begin{aligned}
\| f_0 \|_{L^2(\Omega)} &= 0, \quad \text{when } \mu_0 = 0, \\
\| f_0 \|_{L^r(\Omega)} &\leq C_r \mu_0 (\varepsilon^{1/r} + \varepsilon \ln(\varepsilon^{-1} + 1)), \quad \text{when } \mu_0 > 0, n \neq 1,
\end{aligned}
\end{equation}
for any $1 \leq r < +\infty$. Here constant $C$ and $C_r$ depend on $\| \nabla m_0 \|_{H^4(\Omega)}$, $\| \nabla M_a(y) \|_{H^1(\Gamma)}$, and are independent of $\varepsilon$.

4.1. Estimate of some singular integral. The strategy to prove Lemma 1 is to rewrite the stray field into derivatives of Newtonian potential, thus the consistency estimate turns into the estimate of singular integrals. The following Lemmas introduce the estimate of singular integral in terms of distribution function and boundary layer. We will use the cut-off function $\eta^\varepsilon$ within the interior of area away from boundary:
\begin{equation}
\begin{cases}
0 \leq \eta^\varepsilon \leq 1, & |\nabla \eta^\varepsilon| \leq C \varepsilon^{-1}, \\
\eta^\varepsilon(x) = 1 & \text{if } \text{dist}(x, \partial \Omega) \geq \frac{2}{3}\varepsilon, \\
\eta^\varepsilon(x) = 0 & \text{if } \text{dist}(x, \partial \Omega) \leq \frac{1}{3}\varepsilon.
\end{cases}
\end{equation}
where $\text{dist}(x, \partial \Omega)$ denotes the distance between $x$ and $\partial \Omega$, and cut-off function $\phi^\varepsilon$ in a small ball:
\begin{equation}
\begin{cases}
0 \leq \phi^\varepsilon \leq 1, & |\nabla \phi^\varepsilon| \leq C \varepsilon^{-1}, \\
\phi^\varepsilon(x) = 1 & \text{if } |x| \leq \frac{1}{3}\varepsilon, \\
\phi^\varepsilon(x) = 0 & \text{if } |x| \geq \frac{2}{3}\varepsilon.
\end{cases}
\end{equation}
Denote the boundary layer $\Omega^\varepsilon$ as
\[ \Omega^\varepsilon = \{ x \in \Omega, \text{dist}(x, \partial \Omega) \leq \varepsilon \}. \]

Lemma 2. Assume that scalar functions $f(y) \in C^1(R^n)$ is Y-periodic, $g(x) \in C^1(\Omega)$, define for $x \in \Omega$
\[ u(x) = \int_{\Omega} \frac{|f(\frac{z}{\varepsilon}) - f(\frac{x}{\varepsilon})|}{|x - z|^n} \, dz, \quad v(x) = \int_{\Omega^\varepsilon} \frac{|g(x) - g(z)|}{|x - z|^n} \, dz, \]
then $u(x) \in L^\infty(\Omega)$ logarithmically grows with respect to $\varepsilon$, satisfying
\[ \| u \|_{L^\infty(\Omega)} \leq C \ln(\varepsilon^{-1} + 1) \| f(y) \|_{L^\infty(Y)} + C \| \nabla f(y) \|_{L^\infty(Y)}, \]
and \( v(x) \in L^r(\Omega) \) decreases at speed of \( O(\varepsilon^{1/r}) \) for any \( 1 \leq r < \infty \), satisfying
\[
\|v\|_{L^r(\Omega)} \leq C\varepsilon^{1/r} \ln \varepsilon^{-1} + 1 \left( \|g(x)\|_{L^\infty(\Omega)} + \varepsilon \|\nabla g(x)\|_{L^\infty(\Omega)} \right).
\]

Constant \( C \) is independent of \( \varepsilon \).

Proof. Splitting the integral in \( u \) into \( \int_{\Omega-B(x,\varepsilon)} + \int_{B(x,\varepsilon)} \), one can estimate it by
\[
|u(x)| \leq C \int_{\Omega-B(x,\varepsilon)} \frac{\|f(y)\|_{L^\infty(\Omega)}}{|x-z|^n} \, dz + C\varepsilon^{-1} \int_{B(x,\varepsilon)} \frac{\|\nabla f(y)\|_{L^\infty(\Omega)}}{|x-z|^n} \, dz,
\]
therefore the estimate of \( u \) in Lemma follows by simple integral. As for the estimate of \( v \), by application of cut-off function \( \phi^\varepsilon = \phi^\varepsilon(x-z) \), one has
\[
|v(x)| = \int_{\Omega^c} \frac{\phi^\varepsilon(x-g(z))}{|x-z|^n} \, dz + \int_{\Omega^c} \frac{(1-\phi^\varepsilon)|g(x)-g(z)|}{|x-z|^n} \, dz
\leq C\|\nabla g\|_{L^\infty(\Omega)} \int_{\Omega^c} \frac{\phi^\varepsilon}{|x-z|^{n-1}} \, dz + C\|g\|_{L^\infty(\Omega)} \int_{\Omega^c} \frac{1-\phi^\varepsilon}{|x-z|^{n-1}} \, dz
= R_1 + R_2.
\]

For \( R_1 \), one can write by Fubini’s Theorem
\[
\|R_1\|_{L^r(\Omega)} \leq C\|\nabla g\|_{L^\infty(\Omega)} \left( \int_{\Omega^c} \left( \int_{\Omega^c} \frac{\phi^\varepsilon}{|x-z|^{n-1}} \, dz \right)^r \, dx \right)^{1/r}
\leq C\|\nabla g\|_{L^\infty(\Omega)} \sup_{x \in \Omega^c} \left( \int_{\Omega^c} \frac{\phi^\varepsilon}{|x-z|^{n-1}} \, dz \right)^{1-1/r}
\times \sup_{z \in \Omega^c} \left( \int_{\Omega^c} \frac{1-\phi^\varepsilon}{|x-z|^{n-1}} \, dx \right)^{1/r}
\leq C\|\nabla g\|_{L^\infty(\Omega)} \cdot C\varepsilon^{r-1} \cdot C\varepsilon \cdot C\varepsilon.
\]

As for \( R_2 \), applying the same argument leads to
\[
\|R_2\|_{L^r(\Omega)} \leq C\|g\|_{L^\infty(\Omega)} \sup_{x \in \Omega^c} \left( \int_{\Omega^c} \frac{1-\phi^\varepsilon}{|x-z|^{n}} \, dz \right)^{1-1/r}
\times \sup_{z \in \Omega^c} \left( \int_{\Omega^c} \frac{1-\phi^\varepsilon}{|x-z|^{n}} \, dx \right)^{1/r}
\leq C\|g\|_{L^\infty(\Omega)} \cdot C\ln(\varepsilon^{-1} + 1)^{1-1/r} \cdot C\ln(\varepsilon^{-1} + 1) \cdot C\varepsilon.
\]

\[ \square \]

**Lemma 3.** Assume that a scalar function \( f^\varepsilon(x) \in L^\infty(\Omega) \) satisfies \( f^\varepsilon(x) = 0 \) when \( x \in \Omega - \Omega^c \), which means \( f^\varepsilon \) is nonzero only in boundary layer. Let \( w(x) \) be the Newtonian potential of \( f^\varepsilon \) in \( \Omega \), i.e.,
\[
w(x) = \int_{\Omega} \Phi(x-z)f^\varepsilon(z) \, dz, \quad x \in \Omega,
\]
where \( \Phi \) is the fundamental solution of Laplace’s equation. Then \( w(x) \in W^{2,p}(\Omega) \) satisfies for any \( 1 \leq p < +\infty \)
\[
\|\nabla^2 w\|_{L^p(\Omega)} \leq C \left( \varepsilon^{1/p} + \varepsilon \ln(\varepsilon^{-1} + 1) \right) \left( \|f^\varepsilon(x)\|_{L^\infty(\Omega)} + \varepsilon \|\nabla f^\varepsilon(x)\|_{L^\infty(\Omega)} \right).
\]
Constant $C$ is independent of $\varepsilon$.

**Proof.** The case of $1 < p < +\infty$ follows directly by the property of Newtonian potential:

$$\|
abla^2 w\|_{L^p(\Omega)} \leq C\|f^\varepsilon\|_{L^p(\Omega)} \leq C\|f^\varepsilon\|_{L^p(\Omega^\varepsilon)},$$

and the fact

$$\|f^\varepsilon\|_{L^p(\Omega^\varepsilon)} \leq |\Omega^\varepsilon|^{1/p}\|f^\varepsilon\|_{L^\infty(\Omega)}.$$ 

Now let us consider the case of $p = 1$ and write

$$\frac{\partial^2 w}{\partial x_i \partial x_j} = \int_\Omega \frac{\partial^2}{\partial x_i \partial x_j} \{ \Phi(x - z) \} : \{ f^\varepsilon(z) - f^\varepsilon(x) \} \, dz 
+ f^\varepsilon(x) \int_{\partial \Omega} \nu^i \cdot \frac{\partial}{\partial x_j} \{ \Phi(x - z) \} \, dz.
=: S_1 + S_2.$$

For $S_1$, one can apply Lemma 2 to derive

$$\|S_1\|_{L^1(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1)\left(\|f^\varepsilon(x)\|_{L^\infty(\Omega)} + \varepsilon\|\nabla f^\varepsilon(x)\|_{L^\infty(\Omega)}\right).$$

As for $S_2$, we can split the integral into $\int_{\partial \Omega - B(x, \varepsilon)} + \int_{\partial \Omega \cap B(x, \varepsilon)}$, and write

$$\|S_2\|_{L^1(\Omega)} \leq \sup_{x \in \Omega} \int_{\partial \Omega - B(x, \varepsilon)} \nu^i \cdot \frac{\partial}{\partial x_j} \{ \Phi(x - z) \} \, dz \times \int_{\Omega} f^\varepsilon(x) \, dx 
+ \sup_{z \in \partial \Omega \cap B(x, \varepsilon)} \int_{\Omega \cap B(z, \varepsilon)} \nu^i \cdot \frac{\partial}{\partial x_j} \{ \Phi(x - z) \} \cdot f^\varepsilon(x) \, dx \times \int_{\partial \Omega} 1 \, dz 
\leq C \ln(\varepsilon^{-1} + 1) \times \varepsilon\|f^\varepsilon\|_{L^\infty(\Omega)} + C\varepsilon\|f^\varepsilon\|_{L^\infty(\Omega)},$$

here in the second line we have used the Fubini’s theorem. Thus the Lemma is proved. 

**4.2. Consistency error of stray field.** Now we are ready to prove the consistency error of stray field $\tilde{h}$ in Lemma 1. The idea is to use result in [10] and Green’s representation formula, to rewrite $\tilde{h}$ into singular integral that of the types estimated in above Lemmas.

**Proof.** (Proof of Lemma 1) Recall from (2) the stray field in LLG equation can be calculated by

$$h_0[(M^\varepsilon - M^h)\hat{m}_0] = \nabla U,$$

where $U = U[(M^\varepsilon - M^h)\hat{m}_0]$ satisfies

$$\Delta U = -\text{div}[(M^\varepsilon - M^h)\hat{m}_0 \chi_\Omega] \text{ in } D'(R^n).$$

Denotes the $i$th component of $m_0$ by $m_{0,i}$. Using the fact $|m_0| = 1$, one can write [10]

$$U(x) = -\sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i} \Phi(x - z)(M^\varepsilon(z) - M^h) m_{0,i}(z) \, dz.$$
Substituting above representation of $U(x)$ into (61) and making the use of cut-off function $\eta^\varepsilon$ defined in (58), one can derive
\[
\begin{align*}
\mathbf{h}_d[(M^\varepsilon - M^h)m_0] \\
&= -\nabla \left( \sum_{i=1}^{n} \int_{\Omega^\varepsilon} \frac{\partial}{\partial x_i} \Phi(x - z) \eta^\varepsilon(z)(M(z) - M^h)m_{0,i}(z) \, dz \right) \\
&\quad - \nabla \left( \sum_{i=1}^{n} \int_{\partial \Omega^\varepsilon} \Phi(x - z)(1 - \eta^\varepsilon(z))(M(z) - M^h)m_{0,i}(z) \, dz \right) \\
&=: \mathcal{P}^\varepsilon + \mathcal{P}^\varepsilon,
\end{align*}
\]

where $\mathcal{P}^\varepsilon$ is the derivative of Newtonian potential in boundary layer that can be estimated by Lemma 3. Define $\bar{U}(y)$ as the solution of
\[
\Delta \bar{U}(y) = -(M_s(y) - M^h), \quad U(y) \text{ is } Y\text{-periodic in } y,
\]
then one can write from (7) and (63) that
\[
\begin{align*}
\mathbf{H}_d[M_s(y)](x) &\equiv \varepsilon^2 \nabla^2 \bar{U}(x) \\
&\equiv \varepsilon^2 \nabla^2 \{ \eta^\varepsilon(x)\bar{U}(x) \} + \varepsilon^2 \nabla^2 \{(1 - \eta^\varepsilon(x))\bar{U}(x)\}.
\end{align*}
\]

Note that by Green’s representation formula,
\[
\varepsilon^2 \eta^\varepsilon(x)\bar{U}(x) = -\int_{\Omega} \Phi(x - z)\Delta(\varepsilon^2 \eta^\varepsilon(z)\bar{U}(z)) \, dz.
\]
Substituting the above formula into (64) and using the fact of $\bar{U}$
\[
-\Delta(\varepsilon^2 \eta^\varepsilon(z)\bar{U}(z)) = \eta^\varepsilon(M(z) - M^h) - \varepsilon^2 \Delta \eta^\varepsilon(z)\cdot \bar{U}(z) + 2\varepsilon^2 \nabla \eta^\varepsilon(z) \cdot \nabla \bar{U}(z),
\]
we finally obtain
\[
\begin{align*}
m_0 \cdot \mathbf{H}_d[M_s(y)](x) \\
&= m_0 \cdot \nabla^2 \int_{\Omega} \Phi(x - z) \eta^\varepsilon(z)(M(z) - M^h) \, dz + m_0 \cdot \left\{ \varepsilon^2 \nabla^2 \{(1 - \eta^\varepsilon(x))\bar{U}(x)\} \right\} \\
&\quad + \nabla^2 \int_{\Omega} \Phi(x - z) \left\{ \varepsilon^2 \Delta \eta^\varepsilon(z) \cdot \bar{U}(z) + 2\varepsilon^2 \nabla \eta^\varepsilon(z) \cdot \nabla \bar{U}(z) \right\} \, dz \\
&=: \mathcal{Q}^\varepsilon + \mathcal{Q}^\varepsilon,
\end{align*}
\]

where the boundary layer term $\mathcal{Q}^\varepsilon$ can be estimated by Lemma 3 and (60). Now in order to estimate the left-hand side of (56) in the Lemma, it only remains to consider the term $\mathcal{P}^\varepsilon - \mathcal{Q}^\varepsilon$. Notice that one can write
\[
\mathcal{P}^\varepsilon - \mathcal{Q}^\varepsilon = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial}{\partial x_i} \left\{ \nabla_x \Phi(x - z) \right\} \eta^\varepsilon(z) \\
\times (M(z) - M^h)(m_{0,i}(x) - m_{0,i}(z)) \, dz.
\]
With the notation \((63)\), one has
\[
(M(\vec{z}) - M^h) = \nabla z \cdot \{ \nabla z (\varepsilon^2 \tilde{\nabla} \vec{z}) - \nabla \varepsilon^2 \tilde{\nabla} \frac{x}{\varepsilon} \}.
\]
After substituting it into \((65)\) and applying integration by parts, the leading integrals are estimated directly by application of Lemma 2. \(\square\)

5. Boundary Corrector

5.1. Neumann corrector. Let us give the definition of Neumann corrector \(\omega_N\) as
\[
\omega_N = \sum_{i=1}^{n} (\Phi_i - x_i - \varepsilon \chi^\varepsilon_i) \frac{\partial m^0}{\partial x_i}
\]
with the notation \(\chi^\varepsilon_i(x) = \chi_i(\frac{x}{\varepsilon})\), \(x_i\) is the \(i\)th component of spatial variable, and \((\Phi_i)_{1 \leq i \leq n}\) is given by
\[
\begin{align*}
\text{div}(a^\varepsilon \nabla \Phi_i) &= \text{div}(a^0 \nabla x_i) \text{ in } \Omega, \\
\frac{\partial}{\partial \nu^\varepsilon} \Phi_i &= \frac{\partial}{\partial \nu^0} x_i \text{ on } \partial \Omega.
\end{align*}
\]
Here we denote \(\frac{\partial}{\partial \nu^\varepsilon} = \nu \cdot a^\varepsilon \nabla, \frac{\partial}{\partial \nu^0} = \nu \cdot a^0 \nabla\). Thus \(x_i\) is the homogenized solution of \(\Phi_i\) from \((67)\). Since \(\Phi_i\) is unique up to a constant, one may assume \(\Phi_i(\tilde{x}) - \tilde{x} = 0\) for some \(\tilde{x} \in \Omega\). We introduce that \(\Phi_i - x_i - \varepsilon \chi^\varepsilon_i\) has following property.

**Lemma 4.** For \(\Phi_i\) given in \((67)\), under the smoothness assumption on \(A(y)\) and \(\partial \Omega\), it holds that (see [12])
\[
\| \nabla \Phi_i - \nabla x_i - \varepsilon \nabla \chi^\varepsilon_i \|_{L^\infty(\Omega)} \leq C, \quad \| \nabla^2 \Phi_i \|_{L^\infty(\Omega)} \leq C,
\]
and
\[
\| \Phi_i - x_i \|_{L^\infty(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1),
\]
where \(C\) is independent of \(\varepsilon\).

**Proof.** In fact, one has the estimate
\[
| \nabla \Phi_i - \nabla x_i - \varepsilon \nabla \chi^\varepsilon_i | \leq C \max\{1, \varepsilon \text{dist}(x, \partial \Omega) \}^{-1}
\]
from Lemma 7.4.5 in [12]. This, together with the fact \(\Phi_i(\tilde{x}) - \tilde{x} = 0\), yields \((69)\) by following integrals:
\[
|\Phi_i(x) - x_i| = \left| \int_0^1 \frac{d}{ds} \left\{ \Phi_i(\tilde{x} + s(x - \tilde{x})) - (x_i + s(x_i - \tilde{x})) \right\} \, ds \right|
\leq C \int_0^1 \max\{1, \varepsilon (1 - s)^{-1}\} \, ds \leq C \varepsilon \ln(\varepsilon^{-1} + 1),
\]
for any \(x \in \Omega\).
As for the second inequality in (68), we prove by making use of the Neumann function for operator $A_\varepsilon$ from [12] Section 7.4, denoted by $N_\varepsilon(x, z)$, and write from (67) that

\begin{equation}
(70) \quad \Phi_i(x) = -\sum_{k=1}^{n} \int_{\partial \Omega} \nu_k \cdot a_{ki}^0 N_\varepsilon(x, z) \, dz + \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \Phi_i(z) \, dz.
\end{equation}

Let us denote the projection of $\frac{\partial}{\partial x_j}$ along $\frac{\partial}{\partial \nu_\varepsilon}$ by $P_{x_j}$, and define $P_{x_j}^\perp = \frac{\partial}{\partial \nu_\varepsilon} - P_{x_j}$, one can write for $z \in \partial \Omega$

\[ \frac{\partial}{\partial z_j} N_\varepsilon(x, z) = (P_{z_j} + P_{z_j}^\perp) N_\varepsilon(x, z) = P_{z_j}^\perp N_\varepsilon(x, z). \]

Now applying $\frac{\partial^2}{\partial x_l \partial x_j}$ to both sides of (70), using above formula and integration by parts on $\partial \Omega$ lead to

\begin{equation}
(71) \quad \frac{\partial^2}{\partial x_l \partial x_j} \Phi_i(x) = -\sum_{k=1}^{n} \int_{\partial \Omega} P_{z_k}^\perp P_{z_j}^\perp \nu_k(z) \cdot a_{ki}^0 N_\varepsilon(x, z) \, dz,
\end{equation}

here we have used the fact that $P_{z_j}^\perp$ is a tangential derivative on the boundary, and $N_\varepsilon(x, z) = N_\varepsilon(z, x)$ by the symmetry of $A_\varepsilon$. (71) implies the second inequality in (68) by the smoothness assumption of boundary.

---

5.2. A high-order modification. As noted in Section 1, we use $\omega_N$ to control the Neumann boundary data, and use a modification function $\omega_M$ to control the inhomogeneous term that induced by $\omega_N$, written in (15) separately. In order to explain the construction of the modification function, we point out that there are some bad terms appear when we calculate $L^\varepsilon \omega_N$, which have no convergence in $L^2$ norm. Denote the bad terms by $T_{bad}^1$ and $T_{bad}^2$, then they can be written as

\begin{equation}
T_{bad}^1 = 2 \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_k} \left\{ a_{ki}^\varepsilon (\Phi_j - x_j - \varepsilon \chi_j^\varepsilon) \cdot \frac{\partial^2 m_0}{\partial x_i \partial x_j} \right\} - \sum_{i,j,k=1}^{n} \left\{ \frac{\partial}{\partial x_k} a_{ik}^\varepsilon \cdot (\Phi_j - x_j - \varepsilon \chi_j^\varepsilon) \right\} \frac{\partial^2 m_0}{\partial x_i \partial x_j},
\end{equation}

and

\begin{equation}
T_{bad}^2 = \alpha \sum_{i,j=1}^{n} \left( a_{ij}^\varepsilon \frac{\partial \omega_N}{\partial x_i} \cdot \frac{\partial (2\tilde{m}^\varepsilon + \omega_N)}{\partial x_j} \right) (\tilde{m}^\varepsilon + \omega_N).
\end{equation}

Notice that these terms cannot converge for the existence of $\frac{\partial \omega_N}{\partial x_i}$ and $\frac{\partial a_{ik}^\varepsilon}{\partial x_k}$.

Now let us rewrite $T_{bad}^1$ and $T_{bad}^2$ into divergence form up to a small term. For $T_{bad}^1$, notice that $\sum_{k=1}^{n} \frac{\partial a_{ik}^\varepsilon}{\partial x_k} = A_\varepsilon(\varepsilon \chi_i^\varepsilon)$ from (33), substitute it into the
second term on the right-hand side of (72), it leads to

\[ \mathcal{T}_{\text{bad}}^1 = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} (a_{kl}^i G^1_i(x)) \]

\[ + \sum_{i,j=1}^{n} \varepsilon \chi^i \cdot A_{\varepsilon} \left\{ \left( \Phi_j - x_j - \varepsilon \chi^j \right) \cdot \frac{\partial^2 m_0}{\partial x_i \partial x_j} \right\}, \]

where \( G^1_i(x) \) in the divergence term reads

\[ G^1_i(x) = 2 \sum_{j=1}^{n} \left( \Phi_j - x_j - \varepsilon \chi^j \right) \frac{\partial^2 m_0}{\partial x_l \partial x_j} + \sum_{i,j=1}^{n} \left\{ \frac{\partial}{\partial x_i} (\varepsilon \chi^i) \right\} \left( \Phi_j - x_j - \varepsilon \chi^j \right) \frac{\partial^2 m_0}{\partial x_i \partial x_j}. \]

As for \( \mathcal{T}_{\text{bad}}^2 \), a direct calculation implies it can be rewritten as

\[ \mathcal{T}_{\text{bad}}^2 = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} (a_{kl}^i G^2_i(x)) - \alpha \sum_{i,j=1}^{n} \left\{ \omega_{N} \cdot A_{\varepsilon} \left\{ 2 \tilde{m}^\varepsilon + \omega_{N} \right\} \right\} \left( \tilde{m}^\varepsilon + \omega_{N} \right), \]

\[ - \alpha \sum_{i,j=1}^{n} \left( \omega_{N} \cdot A_{\varepsilon} \left\{ 2 \tilde{m}^\varepsilon + \omega_{N} \right\} \right) \left( \tilde{m}^\varepsilon + \omega_{N} \right), \]

where \( G^2_i(x) \) in the divergence term can be calculated by

\[ G^2_i(x) = \alpha \left( \omega_{N} \cdot \frac{\partial \left\{ 2 \tilde{m}^\varepsilon + \omega_{N} \right\}}{\partial x_l} \right) \left( \tilde{m}^\varepsilon + \omega_{N} \right). \]

Moreover, one can apply Lemma 4 to deduce from (73) and (74) that for \( i = 1, 2 \)

\[ \| \mathcal{T}_{\text{bad}}^i - \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} (a_{kl}^i G^i_l(x)) \|_{L^2(\Omega)} \leq C \varepsilon [\ln(\varepsilon^{-1} + 1)]^2, \]

here we have use the fact \( A_{\varepsilon}(\Phi_i - x_i - \varepsilon \chi^i) = 0 \). Constant \( C \) depends on \( \| \nabla m_0 \|_{W^{2,\infty}(\Omega)}, \| A_{\varepsilon} \tilde{m}^\varepsilon \|_{L^\infty(\Omega)} \), but is independent of \( \varepsilon \).

Now we define the modification function \( \omega_M = \omega^1_M + \omega^2_M \), where \( \omega^i_M, i = 1, 2 \) satisfies

\[ \begin{cases} A_{\varepsilon} \omega^i_M = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} (a_{kl}^i G^i_l(x)) \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu} \omega^i_M = \sum_{k,l=1}^{n} \nu^k \cdot a_{kl}^i G^i_l(x) \quad \text{on } \partial \Omega, \end{cases} \]

where \( \nu^k \) is the \( k \)-th component of vector \( \nu \). By the Lax-Milgram theorem, one can obtain the existence and uniqueness of \( \omega^i_M, i = 1, 2 \) up to a constant. Let \( \int_{\partial \Omega} \omega^i_M \, dx = 0 \), then the correctors yield the following estimate.
Lemma 5. For $\omega_M^i$, $i = 1, 2$ defined in (76), under smooth assumption of $m_0$ and $\partial \Omega$, it holds that for $n \leq 3$

$$\left\| \omega_M^i \right\|_{L^\infty(\Omega)} \leq C \varepsilon, \quad \left\| \nabla \omega_M^i \right\|_{L^\infty(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1),$$

where $C$ depends on $\| \nabla m_0 \|_{W^{3,\infty}(\Omega)}$ and is independent of $\varepsilon$.

Proof. Here we use the Neumann function $N^\varepsilon(x, z)$ for operator $A_\varepsilon$, see [12] Section 7.4. (76) implies for $i = 1, 2$

$$\omega_M^i = \sum_{k,l=1}^n \int_{\Omega} a_{kl}^i \frac{\partial}{\partial z_k} \left\{ N^\varepsilon(x, z) \right\} G_l^i(z) \, dz.$$  

Using the fact $\nabla_z N^\varepsilon(x, z) \leq C|x - z|^{1-n}$, see [12] p.159, we can derive

$$\left\| \omega_M^i \right\|_{L^\infty(\Omega)} \leq C \left\| G_i^i \right\|_{L^\infty(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1).$$

As for the second inequality in (77), it follows from [12], Lemma 7.4.7:

$$\left\| \nabla \omega_M^i \right\|_{L^\infty(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1) \left\| G_i^i \right\|_{L^\infty(\Omega)} + C \varepsilon \left\| \nabla G_i^i \right\|_{L^\infty(\Omega)}$$

with the estimate

$$\left\| \nabla G_i^i \right\|_{L^\infty(\Omega)} \leq C,$$

from Lemma 4. Here constant $C$ depends on $\| \nabla m_0 \|_{W^{3,\infty}(\Omega)}$, $\| \nabla (\Phi_j - x_j - \varepsilon \chi_j) \|_{L^\infty(\Omega)}$, but is independent of $\varepsilon$ by Lemma 4. \hfill \Box

5.3. Estimates of initial-boundary data.

Theorem 4. For $e_0^\varepsilon$ given in (29), with $\omega_0 = \omega_N - \omega_M$ given in (66), under the smooth assumption, it holds that initial data of $e_0^\varepsilon$ satisfies

$$\| e_0^\varepsilon(x, 0) \|_{H^1(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1),$$

where $C$ depends on $\| \nabla^2 m_0 \|_{H^1(\Omega)}$ and is independent of $\varepsilon$. And for the boundary data, it holds that

$$\frac{\partial}{\partial \nu^\varepsilon} e_0^\varepsilon \bigg|_{W^{1,\infty}(\Omega)} \leq C \varepsilon \ln(\varepsilon^{-1} + 1),$$

where $C$ depends on $\| \nabla^2 m_0 \|_{W^{1,\infty}(\Omega)}$ and is independent of $\varepsilon$.

Proof. We rewrite $e_0^\varepsilon$ from its definition as

$$e_0^\varepsilon = m^\varepsilon - m_0 - \sum_{i=1}^n (\Phi_i - x_i) \frac{\partial m_0}{\partial x_i} - \varepsilon^2 m_2 + \omega_M.$$  

First, let us prove (78). By the initial condition of $m^\varepsilon$ and $m_0$, along with the smoothness condition, one can check

$$ e_0^\varepsilon(x, 0) = m_0^\varepsilon_{\text{init}} - m_0^0_{\text{init}} - \sum_{i=1}^n (\Phi_i - x_i) \frac{\partial m_0^0_{\text{init}}}{\partial x_i} - \varepsilon^2 m_2_{\text{init}} + \omega_M_{\text{init}} $$

with notation $m_0^0_{\text{init}}, \omega_M_{\text{init}}$ defined the same as $m_0, \omega_M$ except we replace $m_0$ by $m_0^0_{\text{init}}$. From the assumption (16)-(17), $m_0^0_{\text{init}}$ is the homogenized
solution of \( m_{\text{init}}^\varepsilon \), by classical homogenization theorem of elliptic problems in [12], one has
\[
\|m_{\text{init}}^\varepsilon - m_{\text{init}}^0 - (\Phi - x)\nabla m_{\text{init}}^0\|_{H^1(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1).
\]
Also note that by definition of \( m_2 \) and Lemma 5, one has
\[
\|\varepsilon^2 m_{2,\text{init}}\|_{H^1(\Omega)} + \|\omega_{M,\text{init}}\|_{H^1(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1)\|\nabla^2 m_{\text{init}}^0\|_{H^1(\Omega)}.
\]
Therefore the inequality (78) follows from (81) and above estimates.

Notice that by the boundary condition of \( m_0 \) in (10) and \( \Phi_k \) in (67), we have
\[
\sum_{k=1}^n \frac{\partial}{\partial \nu} (\Phi_k - x_k) \cdot \frac{\partial m_0}{\partial x_k} = - \frac{\partial}{\partial \nu} m_0, \quad x \in \partial \Omega,
\]
therefore applying \( \frac{\partial}{\partial \nu} \) to both sides of (80) leads to
(82)
\[
\frac{\partial}{\partial \nu} e_b^\varepsilon = - \sum_{k=1}^n (\Phi_k - x_k) \cdot \frac{\partial}{\partial \nu} (\frac{\partial m_0}{\partial x_k}) - \varepsilon^2 \frac{\partial}{\partial \nu} m_2 + \frac{\partial}{\partial \nu} \omega_M, \quad x \in \partial \Omega.
\]

Under the smoothness assumption of \( m_0 \) and \( a^\varepsilon \), we can also derive the smoothness of \( (\Phi - x) \), \( m_2 \) and \( \omega_M \) over \( \Omega \). Thus by Lemma 4 and Lemma 5, one can directly obtain from (82)
\[
\|\frac{\partial}{\partial \nu} e_b^\varepsilon\|_{B^{-1/2,2}(\partial \Omega)} \leq C\|\Phi - x\|_{L^\infty(\Omega)} + C\varepsilon^2 \|\nabla m_2\|_{L^\infty(\Omega)} + C\|\nabla \omega_M\|_{L^\infty(\Omega)}
\leq C\varepsilon \ln(\varepsilon^{-1} + 1),
\]
where \( C \) depends on \( \|\nabla^2 m_0\|_{B^{-1/2,2}(\partial \Omega)} \) and \( \|\nabla^2 (\partial_t m_0)\|_{B^{-1/2,2}(\partial \Omega)} \). The same argument for \( \|\frac{\partial}{\partial \nu} (\partial_t e_b^\varepsilon)\|_{B^{-1/2,2}(\partial \Omega)} \) leads to (79).

5.4. Estimates of inhomogeneous terms. From the above definition and property, we get the main result of this section.

**Theorem 5.** For \( e_b^\varepsilon \) given in (29), with \( \omega_b = \omega_N - \omega_M \) given in (66) and \( L^\varepsilon \) defined in (27), under the smooth assumption, it holds that
(83)
\[
\|\partial_t \omega_b\|_{L^2(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1),
\]
(84)
\[
\|L^\varepsilon \omega_b\|_{L^2(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1)^2 + C\|m^\varepsilon - \tilde{m}^\varepsilon - \omega_b\|_{H^1(\Omega)},
\]
where \( C \) depends on \( \|m^\varepsilon\|_{H^1(\Omega)}, \|\nabla^2 m_0\|_{W^{2,\infty}(\Omega)}, \|\nabla (\partial_t m_0)\|_{W^{1,\infty}(\Omega)} \) and is independent of \( \varepsilon \). Moreover, one has the estimate
(85)
\[
\|A^\varepsilon e_b^\varepsilon\|_{L^2(\Omega)} \leq C\ln(\varepsilon^{-1} + 1),
\]
where \( C \) depends on \( \|A^\varepsilon m^\varepsilon\|_{L^2(\Omega)}, \|\nabla^2 m_0\|_{W^{2,\infty}(\Omega)}, \|\nabla (\partial_t m_0)\|_{W^{1,\infty}(\Omega)} \) and is independent of \( \varepsilon \).
Proof. In order to estimate left-hand side of (84), we split it as $\|L^\varepsilon \omega_N - L^\varepsilon \omega_M\|_{L^2(\Omega)} \leq R_1 + R_2 + R_3$, with

$$
\begin{align*}
R_1 &= \|L^\varepsilon \omega_N - \{A\varepsilon \omega_N - m^\varepsilon \times A\varepsilon \omega_N - D_2(\omega_N)\}\|_{L^2(\Omega)}, \\
R_2 &= \|\{A\varepsilon \omega^1_M - m^\varepsilon \times A\varepsilon \omega^1_M - A\varepsilon \omega^2_M\} - L^\varepsilon \omega_M\|_{L^2(\Omega)}, \\
R_3 &= \|A\varepsilon (\omega_N - \omega^1_M) - m^\varepsilon \times A\varepsilon (\omega_N - \omega^1_M) - (D_2(\omega_N) - A\varepsilon \omega^2_M)\|_{L^2(\Omega)}.
\end{align*}
$$

One can check by definition of $L^\varepsilon$ that $R_1$ does not have derivative of $\omega_N$, and $R_2$ only contains first-order derivative of $\omega_M$, thus they can be estimated by Lemma 4 and Lemma 5 as

$$R_1 + R_2 \leq C\varepsilon \ln(\varepsilon^{-1} + 1),$$

where $C$ depends on $\|m^\varepsilon\|_{H^1(\Omega)}$, $\|\nabla^2 m_0\|_{W^{2,\infty}(\Omega)}$. As for $R_3$, in the view of (75), it can be bounded from above by

$$\|A\varepsilon \omega_N - \nabla^2(\omega_N)\|_{L^2(\Omega)} + \|m^\varepsilon \times (A\varepsilon \omega_N - \nabla^2(\omega_N))\|_{L^2(\Omega)} + \|D_2(\omega_N) - \nabla^2(\omega_N)\|_{L^2(\Omega)}.$$

In above terms, the first term can be estimated by applying $A\varepsilon(\Phi_i - x_i - \varepsilon x^i_\varepsilon) = 0$ to derive that $\|A\varepsilon \omega_N - \nabla^2(\omega_N)\|_{L^2(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1)$, with $C$ independent of $\varepsilon$. The same result holds for the second term. Now let us estimate the last term. We assert that

$$\|D_2(\omega_N) - \nabla^2(\omega_N)\|_{L^2(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1) + C\|m^\varepsilon - m^\varepsilon - m_0\|_{H^1(\Omega)},$$

where $C$ depends on $\|\nabla^2 m_0\|_{W^{1,\infty}(\Omega)}$ and $\|m^\varepsilon\|_{H^1(\Omega)}$. In fact, we denote the terms in $D_2(\omega_N)$ that contain derivatives of $\omega_N$ by $\tilde{D}_2(\omega_N)$, then it reads

$$\tilde{D}_2(\omega_N) = \alpha \sum_{i,j=1}^n (a_{ij}^\varepsilon \frac{\partial \omega_N}{\partial x_i} \cdot \frac{\partial m^\varepsilon}{\partial x_j} + a_{ij}^\varepsilon \frac{\partial m^\varepsilon}{\partial x_i} \cdot \frac{\partial \omega_N}{\partial x_j}) m^\varepsilon,$$

and one can check the remaining terms satisfy

$$\|D_2(\omega_N) - \tilde{D}_2(\omega_N)\|_{L^2(\Omega)} \leq C(1 + \|\nabla m_0\|_{L^2(\Omega)}) \|\omega_N\|_{L^2(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1).$$

Substituting $m^\varepsilon = (m^\varepsilon - \tilde{m}^\varepsilon - \omega_N) + (\tilde{m}^\varepsilon + \omega_N)$ into $\tilde{D}_2(\omega_N)$, one can write

$$\tilde{D}_2(\omega_N) = \nabla^2(\omega_N) + \alpha \sum_{i,j=1}^n (a_{ij}^\varepsilon \frac{\partial \omega_N}{\partial x_i} \cdot \frac{\partial (m^\varepsilon - \tilde{m}^\varepsilon - \omega_N)}{\partial x_j}) m^\varepsilon + \alpha \sum_{i,j=1}^n (a_{ij}^\varepsilon \frac{\partial \omega_N}{\partial x_i} \cdot \frac{\partial (\tilde{m}^\varepsilon + \omega_N)}{\partial x_j}) (m^\varepsilon - \tilde{m}^\varepsilon - \omega_N).$$

Hence it follows that

$$\|\tilde{D}_2(\omega_N) - \nabla^2(\omega_N)\|_{L^2(\Omega)} \leq C(1 + \|\nabla m_0\|_{L^2(\Omega)}) \|m^\varepsilon - \tilde{m}^\varepsilon - \omega_N\|_{H^1(\Omega)}.$$

The assertion is proved.
As for (83), one can deduce from Lemma 4 and the proof of Lemma 5 to obtain
\[ \|\partial_t \omega_N\|_{L^\infty(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1), \]
\[ \|\partial_t \omega_M\|_{L^\infty(\Omega)} \leq C \|\partial_t G_i\|_{L^\infty(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1), \]
where \( C \) depends on \( \|\nabla(\partial_t m_0)\|_{W^{1,\infty}(\Omega)} \). In order to prove (85), we use Lemma 4, Lemma 5 and definition of \( \tilde{m}_\varepsilon \), to deduce the estimates
\[ \|A_\varepsilon \tilde{m}_\varepsilon\|_{L^2(\Omega)} + \|A_\varepsilon \omega_b\|_{L^2(\Omega)} \leq C \ln(\varepsilon^{-1} + 1), \]
then (85) follows with some constant \( C \) depending on \( \|A_\varepsilon \tilde{m}_\varepsilon\|_{L^2(\Omega)} \). Therefore Theorem is proved. □

6. Stability Analysis

In this section, we will discuss the stability of following initial-boundary problem, which is motivated by equation (26):
\[
\begin{cases}
\partial_t e - L^\varepsilon(e) = F & \text{in } \Omega, \\
\nu \cdot a^\varepsilon \nabla e = g & \text{on } \partial \Omega, \\
e(0, x) = h & \text{in } \Omega.
\end{cases}
\]

The following two inequalities will be used. The first inequality is motivated by \( W^{1,p} \) estimate for oscillatory elliptic problem.

**Lemma 6.** Assume \( u \in H^2(\Omega) \), \( \nu \cdot a^\varepsilon \nabla u = g \) on \( \partial \Omega \), with \( g \in B^{-1/2,2}(\partial \Omega) \), then it holds that for \( n \leq 3 \),
\[ \|\nabla u\|_{L^6(\Omega)} \leq C \|A_\varepsilon u\|_{L^2(\Omega)} + C \|g\|_{B^{-1/2,2}(\partial \Omega)}, \]
moreover, if \( g = 0 \), then one has for \( n \leq 3 \)
\[ \|\nabla u\|_{L^6(\Omega)} \leq C \|A_\varepsilon u\|_{L^2(\Omega)}. \]

Constant \( C \) is independent of \( \varepsilon \).

**Proof.** We refer that Lemma 6 is a direct corollary of Theorem 6.3.2 in [12]. One can find the proof in [12] [Pages 144-152]. □

We also introduce Sobolev inequality with small coefficient when \( n = 2 \).

**Lemma 7.** For any function \( f \in H^2(\Omega) \), one has when \( n = 2 \)
\[ \|f\|_{L^\infty(\Omega)} \leq C\varepsilon \ln(\varepsilon^{-1} + 1) \|f\|_{H^1(\Omega)} + \varepsilon \|A^\varepsilon f\|_{L^2(\Omega)}, \]
where constant \( C \) is independent of \( \varepsilon \).

**Proof.** Using the Neumann function, see [12] Section 7.4, one has
\[ f = -\int_\Omega \nabla_z N^\varepsilon(x,z) \cdot a^\varepsilon \nabla f(z) \, dz + \frac{1}{|\partial \Omega|} \int_{\partial \Omega} f \, dz =: P_1 + P_2. \]
Applying cut-off function \( \phi^\varepsilon = \phi^\varepsilon(x - z) \), the first term yields by integration by parts

\[
P_1 = - \int_{\Omega} (1 - \phi^\varepsilon) \nabla_x N^\varepsilon(x, z) \cdot \phi^\varepsilon \nabla f(z) \, dz + \int_{\Omega} \phi^\varepsilon N^\varepsilon(x, z) \cdot A_\varepsilon f(z) \, dz
\]

\[
+ \int_{\Omega} \nabla_x \phi^\varepsilon(x - z) \cdot N^\varepsilon(x, z) \cdot \phi^\varepsilon \nabla f(z) \, dz
\]

\[
\leq C \varepsilon \| \phi^\varepsilon L^2(\Omega) + C \ln(\varepsilon^{-1} + 1) \| \nabla f \| L^2(\Omega),
\]

here in the last line we have used the fact \( \nabla_x N^\varepsilon(x, z) \leq C |x - y|^{-1} \) and \( N^\varepsilon(x, z) \leq C \{1 + \ln(\|x - z\|^{-1})\} \) for \( n = 2 \), see [12] page 159. As for \( P_2 \), one has by trace inequality that \( P_2 \leq C \| f \| H^1(\Omega) \). The Lemma is proved. \( \square \)

Now let us give the stability of system (87) in terms of \( h, g, F \) in \( L^\infty(0, T; L^2(\Omega)) \) and \( L^\infty(0, T; H^1(\Omega)) \) norm, respectively.

6.1. Stability in \( L^\infty(0, T; L^2(\Omega)) \).

**Theorem 6.** Let \( e \in L^\infty(0, T; H^2(\Omega)) \) be a strong solution to (87). Assume \( h \in L^2(\Omega), g \in L^\infty(0, T; B^{-1/2, 2}(\partial \Omega)), \) and \( F = F_1 + F_2 \) satisfies

(88)

\[
F_1 \in L^2(0, T; L^\sigma(\Omega)), \quad F_2 \in L^2(0, T; L^2(\Omega))
\]

with \( \sigma = 1 \) when \( n = 1, 2 \), and \( \sigma = 6/5 \) when \( n = 3 \), then it holds that, for any \( 0 \leq t \leq T \)

\[
\| e \|_{L^\infty(0, T; L^2(\Omega))} + \| \nabla e \|_{L^2(0, T; L^2(\Omega))} \leq C_\delta \left( \| h \|_{L^2(\Omega)}^2 + \| g \|_{L^2(0, T; B^{-1/2, 2}(\partial \Omega))}^2 + \gamma(\varepsilon) \| F_1 \|_{L^2(0, T; L^\sigma(\Omega))}^2 \right)
\]

\[
+ \delta \| F_2 \|_{L^2(0, T; L^2(\Omega))}^2 + \varepsilon^2 \| A_\varepsilon e \|_{L^2(0, T; L^2(\Omega))}^2,
\]

for any small \( \delta > 0 \), where

\[
\gamma(\varepsilon) = \begin{cases} 1, & \text{when } n = 1, 3, \\ \ln(\varepsilon^{-1} + 1) \varepsilon, & \text{when } n = 2. \end{cases}
\]

\( C_\delta \) is a constant depending on \( \| \nabla m^\varepsilon \|_{L^1(\Omega)}, \| \nabla \tilde{m}^\varepsilon \|_{L^4(\Omega)}, \) but is independent of \( t \) and \( \varepsilon \).

**Proof.** The inner product between (87) and \( e \) in \( L^2(\Omega) \) leads to

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} | e |^2 \, dx - \alpha \int_{\Omega} \tilde{H}_\varepsilon(e) \cdot e \, dx
\]

\[
= - \int_{\Omega} D_1(e) \cdot e \, dx - \int_{\Omega} D_2(e) \cdot e \, dx - \int_{\Omega} F \cdot e \, dx.
\]

Now let us give the estimates to (90) term by term. Integration by parts for the second term on the left-hand side yields

\[
- \int_{\Omega} \tilde{H}_\varepsilon(e) \cdot e \, dx \geq \sum_{i,j=1}^n \int_{\Omega} a_{\varepsilon ij} \frac{\partial e}{\partial x_i} \cdot \frac{\partial e}{\partial x_j} \, dx - \int_{\partial \Omega} g \cdot e \, dx - C \int_{\Omega} | e |^2 \, dx
\]
with the boundary term satisfying
\[
\int_{\partial \Omega} \mathbf{g} \cdot \mathbf{e} \, d\mathbf{x} \leq \|\mathbf{g}\|_{B^{1/2,2}(\partial \Omega)} \|\mathbf{e}\|_{B^{1/2,2}(\partial \Omega)} \leq C \|\mathbf{g}\|_{B^{-1/2,2}(\partial \Omega)} \|\mathbf{e}\|_{H^1(\Omega)}.
\]
By integration by parts and the same argument for the boundary term, the first term on the right-hand side can be estimated as
\[
- \int_{\Omega} \mathbf{D}_1(\mathbf{e}) \cdot \mathbf{e} \, d\mathbf{x} \leq C \int_{\Omega} |\mathbf{e}|^2 \, d\mathbf{x} + \delta C \int_{\Omega} |\nabla \mathbf{e}|^2 \, d\mathbf{x} - C \|\mathbf{g}\|_{B^{-1/2,2}(\partial \Omega)} \|\mathbf{e}\|_{H^1(\Omega)}.
\]
For the second term on the right-hand side of (90), using the estimates
\[
\int_{\Omega} (\mathbf{B}^\varepsilon[\mathbf{e}, \mathbf{m}^\varepsilon]) \mathbf{m}^\varepsilon \cdot \mathbf{e} \, d\mathbf{x} \leq C \|\nabla \mathbf{m}^\varepsilon\|_{L^4(\Omega)} \|\mathbf{e}\|_{L^4(\Omega)} \|\nabla \mathbf{e}\|_{L^2(\Omega)} + C \|\mathbf{m}^\varepsilon\|_{L^4(\Omega)} \|\mathbf{e}\|_{L^4(\Omega)} \|\mathbf{e}\|_{L^2(\Omega)},
\]
\[
\int_{\Omega} q_\varepsilon[\mathbf{m}^\varepsilon] \mathbf{e} \cdot \mathbf{e} \, d\mathbf{x} \leq C \|\nabla \mathbf{m}^\varepsilon\|_{L^4(\Omega)} \|\mathbf{e}\|_{L^4(\Omega)}^2 + C \|\mathbf{m}^\varepsilon\|_{L^4(\Omega)} \|\mathbf{e}\|_{L^4(\Omega)} \|\mathbf{e}\|_{L^4(\Omega)},
\]
and the same argument can be applied to the other terms, we finally obtain by Sobolev inequality
\[
- \int_{\Omega} \mathbf{D}_2(\mathbf{e}) \cdot \mathbf{e} \, d\mathbf{x} \leq C + C \int_{\Omega} |\mathbf{e}|^2 \, d\mathbf{x} + \delta C \int_{\Omega} |\nabla \mathbf{e}|^2 \, d\mathbf{x},
\]
where \( C = C^0\left(1 + \|\nabla \mathbf{m}^\varepsilon\|_{L^4(\Omega)}^2 + \|\nabla \mathbf{m}^\varepsilon\|_{L^4(\Omega)}^2\right). \) For the last term in (90), by the assumption (88), we apply Sobolev inequality for \( n = 1, 3, \) and apply Lemma 7 for \( n = 2, \) it follows that
\[
- \int_{\Omega} \mathbf{F}_1 \cdot \mathbf{e} \, d\mathbf{x} \leq \begin{cases} 
C \|\mathbf{F}_1\|_{L^1(\Omega)}^2 + \delta \|\mathbf{e}\|_{H^1(\Omega)}^2, & n = 1 \\
C[\ln(\varepsilon^{-1} + 1)]^2 \|\mathbf{F}_1\|_{L^1(\Omega)}^2 + \delta \|\mathbf{e}\|_{H^1(\Omega)}^2 + \varepsilon^2 \|\mathbf{A}\\mathbf{e}\|_{L^2}^2, & n = 2 \\
C \|\mathbf{F}_1\|_{L^6/5(\Omega)}^2 + \delta \|\mathbf{e}\|_{H^1(\Omega)}^2, & n = 3
\end{cases}
\]
\[
- \int_{\Omega} \mathbf{F}_2 \cdot \mathbf{e} \, d\mathbf{x} \leq \delta^* \|\mathbf{F}_2\|_{L^2(\Omega)}^2 + C \|\mathbf{e}\|_{L^2(\Omega)},
\]
with any small \( \delta, \delta^* > 0. \) Substituting above estimates, one can derive from (90) that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathbf{e}|^2 \, d\mathbf{x} + (\alpha a_{\min} - 2\delta) \int_{\Omega} |\nabla \mathbf{e}|^2 \, d\mathbf{x} \leq C \int_{\Omega} |\mathbf{e}|^2 \, d\mathbf{x} + C \|\mathbf{g}\|_{B^{-1/2,2}(\partial \Omega)}^2 + C[\ln(\varepsilon^{-1} + 1)]^2 \|\mathbf{F}_1\|_{L^4(\Omega)}^2 + \delta^* \|\mathbf{F}_2\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\mathbf{A}\\mathbf{e}\|_{L^2}^2.
\]
Then (89) follows directly by taking \( \delta \) small enough, and the application of Grönwall’s inequality.
6.2. Stability in $L^\infty(0,T;H^1(\Omega))$.

**Theorem 7.** Let $e \in L^\infty(0,T;H^2(\Omega))$ be a strong solution to (87). Assume $h \in H^1(\Omega)$, $g \in H^1(0,T;B^{-1/2,2}(\partial\Omega))$, and $F \in L^2(0,T;L^2(\Omega))$, it holds

\[
\|\nabla e\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C(\|h\|_{H^1(\Omega)}^2 + \|\nabla g\|_{L^2(0,T;B^{-1/2,2}(\partial\Omega))}^2 + \|F\|_{L^2(0,T;L^2(\Omega))}^2),
\]

where $C$ depends on $\|\nabla m^\varepsilon\|_{L^4(\Omega)}$, $\|\nabla \tilde{m}^\varepsilon\|_{L^4(\Omega)}$ and $\|H_{\varepsilon}(\tilde{m}^\varepsilon)\|_{L^4(\Omega)}$, but is independent of $t$ and $\varepsilon$.

**Proof.** The inner product between (87) and $\tilde{H}_\varepsilon^\varepsilon(e)$ in $L^2(\Omega)$ leads to

\[
-\int_\Omega \partial_t e \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx + \alpha \int_\Omega \tilde{H}_\varepsilon^\varepsilon(e) \cdot e \, dx = \int_\Omega D_1(e) \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx + \int_\Omega D_2(e) \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx + \int_\Omega F \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx.
\]

(92)

In the following we give the estimates to (92) term by term. Note that integration by parts yields

\[
-\int_\Omega \partial_t e \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx = \frac{d}{dt} \int_\Omega G_\varepsilon^\varepsilon(e) - \int_\Omega \partial_t e \cdot g \, dx,
\]

where

\[
\frac{d}{dt} \int_\Omega G_\varepsilon^\varepsilon(e) - \int_\Omega \partial_t e \cdot g \, dx - C\|\partial_t g\|_{B^{-1/2,2}(\partial\Omega)} \|e\|_{H^1(\Omega)}.
\]

Using the fact $m^\varepsilon \times \tilde{H}_\varepsilon^\varepsilon(e) \cdot \tilde{H}_\varepsilon^\varepsilon(e) = 0$, the first term on the right-hand side of (92) can be estimate by Sobolev inequality as

\[
\int_\Omega D_1(e) \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx \leq C\|H^\varepsilon_\varepsilon(e)\|_{L^4(\Omega)} \|e\|_{L^4(\Omega)} \|\tilde{H}_\varepsilon^\varepsilon(e)\|_{L^2(\Omega)}
\]

\[
\leq C\|e\|_{H^1(\Omega)}^2 + \delta C\|\tilde{H}_\varepsilon^\varepsilon(e)\|_{L^2(\Omega)}^2,
\]

where $C = C^0(1 + \|H^\varepsilon_\varepsilon(e)\|_{L^4(\Omega)}^2)$. For the second term on the right-hand side of (92), note that we have the estimate

\[
\int_\Omega (B^\varepsilon(e,m^\varepsilon)) m^\varepsilon \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx \leq C\|\nabla m^\varepsilon\|_{L^4(\Omega)} \|\nabla e\|_{L^4(\Omega)} \|\tilde{H}_\varepsilon^\varepsilon(e)\|_{L^2(\Omega)}
\]

\[
+ C\|m^\varepsilon\|_{L^4(\Omega)} \|e\|_{L^4(\Omega)} \|\tilde{H}_\varepsilon^\varepsilon(e)\|_{L^2(\Omega)},
\]

in which one can deduce

\[
\|\nabla e\|_{L^4(\Omega)}^2 \leq \|\nabla e\|_{L^2(\Omega)}^{1/3} \|\nabla e\|_{L^6(\Omega)}^{2/3}
\]

\[
\leq C\|\nabla e\|_{L^2(\Omega)}^{1/3}(1 + \|\tilde{H}_\varepsilon^\varepsilon(e)\|_{L^2(\Omega)} + \|g\|_{B^{-1/2,2}(\partial\Omega)})^{2/3},
\]

using interpolation inequality and Lemma 6. The other terms can be estimated in the same fashion. After the application of Young’s inequality, one finally obtains

\[
\int_\Omega D_2(e) \cdot \tilde{H}_\varepsilon^\varepsilon(e) \, dx \leq C\|e\|_{H^1(\Omega)}^2 + \delta C\|\tilde{H}_\varepsilon^\varepsilon(e)\|_{L^2(\Omega)}^2 + C\|g\|_{B^{-1/2,2}(\partial\Omega)}^2.
\]
where \( C = C^0(1 + \|\nabla m^\varepsilon\|^2_{L^4(\Omega)} + \|\nabla \tilde{m}^\varepsilon\|^2_{L^4(\Omega)}) \). Substituting above estimates into (92), we arrive at
\[
\frac{d}{dt} G^e_L[\varepsilon] + (\alpha - C\delta) \int_\Omega |\tilde{H}^e(\varepsilon)|^2 \, dx - \partial_t \int_{\partial\Omega} \mathbf{e} \cdot \mathbf{g} \, dx \\
\leq C \left( \|e\|_{H^1(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 + \|\mathbf{g}\|_{B^{-1/2,2}(\partial\Omega)}^2 + \|\partial_t \mathbf{g}\|_{B^{-1/2,2}(\partial\Omega)}^2 \right).
\]
Integrating the above inequality over \([0, t]\) with \(0 < t < T\) and using the facts
\[
\int_{\partial\Omega} \mathbf{e} \cdot \mathbf{g} \, dx \leq \delta \|e\|_{H^1(\Omega)}^2 + C \|\mathbf{g}\|_{B^{-1/2,2}(\partial\Omega)}^2,
\]
\[
G^e_L[\varepsilon] \geq \frac{a_{\min}}{2} \|\nabla e\|_{L^2(\Omega)}^2 - C,
\]
one can finally derive
\[
\left( \frac{a_{\min}}{2} - \delta \right) \|\nabla e(t)\|_{L^2(\Omega)}^2 + (\alpha - C\delta) \int_0^t \|\tilde{H}^e(\varepsilon(t))\|_{L^2(\Omega)}^2 \, dt \\
\leq C \int_0^t \left( \|e\|_{L^2(\Omega)}^2 + \|\nabla e\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{g}\|_{B^{-1/2,2}(\partial\Omega)}^2 \right) \, dt + \mathcal{J}(h),
\]
where \( \mathcal{J}(h) \) yields
\[
\mathcal{J}(h) = G^e_L[h] - \int_{\partial\Omega} h \cdot \frac{\partial}{\partial \nu^e} h \, dx \leq C \|h\|_{H^1(\Omega)}^2 + \|\frac{\partial}{\partial \nu^e} h\|_{B^{-1/2,2}(\partial\Omega)}^2.
\]
(91) is then derived after taking \( \delta \) small enough and the application of Grönwall’s inequality. \( \square \)

7. Regularity

In the estimate of boundary corrector and stability analysis by Theorem 5, Theorem 6, Theorem 7, the constant we deduced rely on the value of \( \|\mathcal{A}e_m^e\|_{L^2(\Omega)} \) and \( \|\nabla m^e\|_{L^6(\Omega)} \). In this section, we introduce the uniform regularity on \( m^e \), over a time interval independent of \( \varepsilon \). For this purpose, we intend to derive a structure-preserving energy inequality, in which the degenerate term are kept in the energy.

First, let us introduce an interpolation inequality of the effective field \( \mathcal{H}^e(m^e) \) for some \( S^2 \)-valued function \( m^e \), which is the generalization of (14). The following estimates will be used:

(93) \[ a_{\max}^{-1} \|m^e \cdot \mathcal{A}e_m^e\|_{L^3(\Omega)}^3 \leq \|\nabla m^e\|_{L^6(\Omega)}^6 \leq a_{\min}^{-1} \|\mathcal{A}e_m^e\|_{L^3(\Omega)}^3, \]
(94) \[ \|\mathcal{A}e_m^e\|_{L^p(\Omega)} - C_p \leq \|\mathcal{H}^e(m^e)\|_{L^p(\Omega)} \leq \|\mathcal{A}e_m^e\|_{L^p(\Omega)} + C_p, \]

with \( 1 < p < +\infty \), here the first line follows from the fact \(-a^e|\nabla m^e|^2 = m^e \cdot \mathcal{A}e_m^e \) by \( |m^e| = 1 \) and assumption of \( a^e \) in (1), and in second line the estimate (4) is used. We also introduce a orthogonal decomposition to any vector \( a \) as
\[
\mathbf{a} = (m^e \cdot \mathbf{a}) m^e - m^e \times (m^e \times \mathbf{a}).
\]
Lemma 8. Given \( m^\varepsilon \in H^3(\Omega) \) that satisfies \( |m^\varepsilon| = 1 \) and Neumann boundary condition \( \nu \cdot a \varepsilon \nabla m^\varepsilon = 0 \), then it holds for \( n \leq 3 \) and any \( 0 < \delta < 1 \)
\[
\| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^3(\Omega)}^3 \leq C_\delta + C_\delta \| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^6 + \delta \| m^\varepsilon \times \nabla \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^2,
\]
where \( C_\delta \) is a constant depending on \( \delta \) but independent of \( \varepsilon \).

Proof. Applying decomposition (95) by taking \( a = \mathcal{H}^\varepsilon_e(m^\varepsilon) \), one can write
\[
\| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^3(\Omega)}^3 \leq \int_\Omega |m^\varepsilon \cdot \mathcal{H}^\varepsilon_e(m^\varepsilon)|^3 \, dx
\]
\[
+ \int_\Omega |m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon)|^3 \, dx
=: I_1 + I_2.
\]
Now let us estimate the right-hand side of (97) separately. For \( I_1 \), we apply (93) and Remark 6 to derive
\[
I_1 \leq C + C \| \nabla m^\varepsilon \|_{L^6(\Omega)}^6 \leq C + C \| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^6.
\]
As for \( I_2 \), we have by Sobolev inequality for \( n \leq 3 \)
\[
I_2 \leq C + C \| m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^6 + \delta^* \| m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{H^1(\Omega)}^2,
\]
here in the last term, we can apply (93)-(94) to derive:
\[
\delta^* \| \nabla m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{H^1(\Omega)}^2 \leq \delta^* \| \nabla m^\varepsilon \|_{L^6(\Omega)}^6 + \delta^* \| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^3(\Omega)}^3
\]
\[
\leq C + C \delta^* \| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^3(\Omega)}^3.
\]
Now let us turn back to (97), we finally obtain
\[
(1 - C\delta^*) \| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^3(\Omega)}^3 \leq C + C \| \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^6 + \delta^* \| m^\varepsilon \times \nabla \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^2.
\]
Let \( \delta^* < \frac{1}{2C} \), one can derive (96) with \( \delta = \delta^*/(1 - C\delta^*) < 1 \).

Now let us recall some energy property of LLG equation, and give the uniform regularity result. Using the formula of vector outer production
\[
a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,
\]
one can rewrite LLG equation (22) into a degenerate form
\[
\partial_t m^\varepsilon + (a m^\varepsilon) \times (m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon)) + m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon) = 0.
\]
Multiplying (99) by \( \mathcal{H}^\varepsilon_e(m^\varepsilon) \) and integrating over \((0, t)\), we derive the energy dissipation identity
\[
\mathcal{G}_L^\varepsilon[m^\varepsilon(t)] + \alpha \int_0^t \| m^\varepsilon \times \mathcal{H}^\varepsilon_e(m^\varepsilon) \|_{L^2(\Omega)}^2 \, d\tau = \mathcal{G}_L^\varepsilon[m^\varepsilon(0)],
\]
Together (99) and (100) leads to the integrable of kinetic energy
\[
\alpha \int_0^t \| \partial_t m^\varepsilon \|_{L^2(\Omega)}^2 \, d\tau \leq \mathcal{G}_L^\varepsilon[m^\varepsilon(0)].
\]
The energy identity (100) implies the uniform regularity of \( \| m^\varepsilon \times A_m m^\varepsilon \|_{L^2(\Omega)}^2 \), however, this is not enough to obtain the regularity of \( \| A_m m^\varepsilon \|_{L^2(\Omega)}^2 \) due to the degeneracy. In this end we introduce that:
Theorem 8. Let $m^\varepsilon \in L^2(0, T; H^3(\Omega))$ be a solution to (8). Assume $n \leq 3$, then there exists $T^* \in (0, T]$ independent of $\varepsilon$, such that for $0 \leq t \leq T^*$,

$$\|A_\varepsilon m^\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \|m^\varepsilon \times \nabla H^\varepsilon_0(m^\varepsilon)(\tau)\|^2_{L^2(\Omega)} d\tau \leq C,$$

and therefore, by the Sobolev-type inequality in Remark 6,

$$\|\nabla m^\varepsilon(\cdot, t)\|^2_{L^2(\Omega)} \leq C,$$

where $C$ is a constant independent of $\varepsilon$ and $t$.

Proof. Applying $\nabla$ to (99) and multiplying by $a^\varepsilon \nabla H^\varepsilon_0(m^\varepsilon)$ lead to

$$- \int_\Omega \nabla(\partial_t m^\varepsilon) \cdot a^\varepsilon \nabla H^\varepsilon_0(m^\varepsilon) \, dx$$

$$= \alpha \int_\Omega \nabla\left( m^\varepsilon \times (m^\varepsilon \times H^\varepsilon_0(m^\varepsilon)) \right) \cdot a^\varepsilon \nabla H^\varepsilon_0(m^\varepsilon) \, dx$$

$$+ \sum_{i,j=1}^n \int_\Omega \frac{\partial}{\partial x_i} m^\varepsilon \times H^\varepsilon_0(m^\varepsilon) \cdot a^\varepsilon_{ij} \frac{\partial}{\partial x_j} H^\varepsilon_0(m^\varepsilon) \, dx =: J_1 + J_2. \tag{102}$$

Denote $\Gamma^\varepsilon(m^\varepsilon) = H^\varepsilon_0(m^\varepsilon) - A_\varepsilon m^\varepsilon$. After integration by parts, the left-hand side of (102) becomes

$$- \int_\Omega \nabla(\partial_t m^\varepsilon) \cdot a^\varepsilon \nabla H^\varepsilon_0(m^\varepsilon) \, dx = \int_\Omega A_\varepsilon(\partial_t m^\varepsilon) \cdot (A_\varepsilon m^\varepsilon + \Gamma^\varepsilon(m^\varepsilon)) \, dx,$$

where the right-hand side can be rewritten as

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |A_\varepsilon m^\varepsilon|^2 \, dx + \frac{d}{dt} \int_\Omega A_\varepsilon m^\varepsilon \cdot \Gamma^\varepsilon(m^\varepsilon) \, dx - \int_\Omega A_\varepsilon(m^\varepsilon) \cdot \Gamma^\varepsilon(\partial_t m^\varepsilon) \, dx.$$

Now let us consider the right-hand side of (102). For $J_1$, one can derive by swapping the order of mixed product

$$J_1 = -\alpha \sum_{i,j=1}^n \int_\Omega \left( m^\varepsilon \times \frac{\partial}{\partial x_i} H^\varepsilon_0(m^\varepsilon) \right) \cdot a^\varepsilon_{ij} \left( m^\varepsilon \times \frac{\partial}{\partial x_j} H^\varepsilon_0(m^\varepsilon) \right) \, dx + F_1,$$

here the first term on right-hand side is sign-preserved due to the uniform coerciveness of $a^\varepsilon$ in (1). As for $J_2$, we apply (95) by taking $a = a^\varepsilon_{ij} \partial_j H^\varepsilon_0(m^\varepsilon)$, it leads to

$$J_2 = \sum_{i,j=1}^n \int_\Omega m^\varepsilon \times \left( \frac{\partial}{\partial x_i} m^\varepsilon \times H^\varepsilon_0(m^\varepsilon) \right) \cdot \left( m^\varepsilon \times a^\varepsilon_{ij} \frac{\partial}{\partial x_j} H^\varepsilon_0(m^\varepsilon) \right) \, dx$$

$$- \sum_{i,j=1}^n \int_\Omega \left( m^\varepsilon \times H^\varepsilon_0(m^\varepsilon) \right) \cdot \frac{\partial}{\partial x_i} m^\varepsilon \cdot a^\varepsilon_{ij} \frac{\partial}{\partial x_j} H^\varepsilon_0(m^\varepsilon) \, dx. \tag{103}$$
Using property of vector outer production (98) for first term, and integration by parts for the second term, (103) becomes

\[ J_2 = 2 \sum_{i,j=1}^{n} \int_{\Omega} (m^\epsilon \cdot \mathcal{H}_\epsilon^x (m^\epsilon)) (m^\epsilon \times a^\epsilon_{ij} \frac{\partial}{\partial x_j} \mathcal{H}_\epsilon^x (m^\epsilon) \cdot \frac{\partial}{\partial x_i} m^\epsilon) \, dx + F_2 \]

\[ \leq C \| \nabla m^\epsilon \|_{L^6}^6 + C \| \mathcal{H}_\epsilon^x (m^\epsilon) \|_{L^3}^3 + \delta \| m^\epsilon \times \nabla \mathcal{H}_\epsilon^x (m^\epsilon) \|_{L^2}^2 + F_2. \]

Here low-order terms \( F_i, i = 1, 2 \) satisfies by (94) and Hölder’s inequality

\[ F_i \leq C + C \| \nabla m^\epsilon \|_{L^6}^6 + C \| \mathcal{H}_\epsilon^x (m^\epsilon) \|_{L^3}^3. \]

Substituting above estimates into (102), applying estimate (93) and Lemma 8, we finally arrive at

\[ \frac{1}{2} \frac{d}{dt} \| A_\epsilon m^\epsilon \|_{L^2}^2 + (\alpha a_{\text{min}} - C \delta) \| m^\epsilon \times \nabla \mathcal{H}_\epsilon^x (m^\epsilon) \|_{L^2}^2 \]

\[ \leq C + C \| A_\epsilon m^\epsilon \|_{L^2}^6 + C \| \partial_t m^\epsilon \|_{L^2}^2 - \frac{d}{dt} \int_{\Omega} A_\epsilon m^\epsilon \cdot \Gamma^\epsilon (m^\epsilon) \, dx, \]

Integrating (104) over \([0,t]\), using the integrability of kinetic energy (101) and the following inequality

\[ \int_{\Omega} A_\epsilon m^\epsilon \cdot \Gamma^\epsilon (m^\epsilon) \, dx \leq C \| \Gamma^\epsilon (m^\epsilon) \|_{L^2}^2 + \frac{1}{4} \| A_\epsilon m^\epsilon \|_{L^2}^2, \]

one has for any \( t \in (0, T] \)

\[ \frac{1}{4} \| A_\epsilon m^\epsilon (t) \|_{L^2}^2 \leq C + C \int_0^t \| A_\epsilon m^\epsilon (\tau) \|_{L^2}^6 \, d\tau, \]

where \( C \) depends on \( \| A_\epsilon m^\epsilon_{\text{init}} \|_{L^2} \), \( G^\epsilon_L [m^\epsilon_{\text{init}}] \) thus is independent of \( \epsilon \) and \( t \) by assumption (16)-(17) and Lemma 6. Denote the right-hand side of (105) by \( F(t) \) and write

\[ \frac{d}{dt} F(t) \leq C F^3(t). \]

By the Cauchy-Lipshitz-Picard Theorem [3] and comparison principle, there exists \( T^* \in (0, T] \) independent of \( \epsilon \), such that \( F(t) \) is uniformly bounded on \([0, T^*]\), thus \( \| A_\epsilon m^\epsilon (t) \|_{L^2}^2 \) is uniformly bounded by (105). The Lemma is proved.

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