On Difference Between Two Types of $\gamma$-divergence for Regression

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Abstract

The $\gamma$-divergence is well-known for having strong robustness against heavy contamination. By virtue of this property, many applications via the $\gamma$-divergence have been proposed. There are two types of $\gamma$-divergence for regression problem, in which the treatments of base measure are different. In this paper, we compare them and pointed out a distinct difference between these two divergences under heterogeneous contamination where the outlier ratio depends on the explanatory variable. One divergence has the strong robustness under heterogeneous contamination. The other does not have in general, but has when the parametric model of the response variable belongs to a location-scale family in which the scale does not depend on the explanatory variables or under homogeneous contamination where the outlier ratio does not depend on the explanatory variable. [Hung et al. (2017)] discussed the strong robustness in a logistic regression model with an additional assumption that the tuning parameter $\gamma$ is sufficiently large. The results obtained in this paper hold for any parametric model without such an additional assumption.
1 Introduction

The maximum likelihood estimation and least square method have been widely used. However, they are not robust against outliers. To overcome this problem, many robust methods have been proposed mainly using M-estimation (Hampel et al. 2005; Maronna et al. 2006; Huber and Ronchetti 2009). The maximum likelihood estimation can be regarded as the minimization of the empirical estimator of the Kullback-Leibler divergence. As an extension of this idea, some robust estimators were proposed as the minimization of the empirical estimators of the modified divergences (Basu et al. 1998; Scott 2001; Fujisawa and Eguchi 2008).

Recently, some robust regression methods based on divergences have been proposed using $L_2$-divergence (Chi and Scott 2014; Lozano et al. 2016), density power divergence (Ghosh and Basu 2016) and $\gamma$-divergence (Hung et al. 2017; Kawashima and Fujisawa 2017). The robust properties are often investigated under the contaminated model. The difference between i.i.d. and regression problems is whether the outlier ratio in the contaminated model can depend on the explanatory variable or not. They are called the heterogeneous and homogeneous contaminations, respectively. To our knowledge, the robust properties under heterogeneous contamination have not been investigated well. Recently, Hung et al. (2017) pointed out that a logistic regression model with mislabel can be regarded as a logistic regression model with heterogeneous contamination and then applied the $\gamma$-divergence to a usual logistic regression model, which enables us to estimate the parameter of the logistic regression model without estimating a mislabel model even if mislabels exist. They discussed the strong robustness that the latent bias can be sufficiently small against heavy contamination, but assumed that the tuning parameter $\gamma$ is sufficiently large.

There are two types of $\gamma$-divergence for regression problem in which the treatments of base measure are different (Fujisawa and Eguchi 2008; Kawashima and Fujisawa 2017). In this paper, we compare them in detail and prove that one divergence can show the strong robustness for any parametric model under heterogeneous contamination. The other can not in general, but can when the parametric model of the response variable belongs to a location-scale family in which the scale does not depend on the explanatory variables or under homogeneous contamination. This difference will be illustrated in numerical experiments. Comparing them with Hung et al. (2017), the results obtained here holds for any parametric model, including a logistic regression model, and do not assume that $\gamma$ is sufficiently large.

This paper is organized as follows. In Section 2 we review two types of $\gamma$-divergence for regression problem. In Section 3 we elucidate a distinct difference between two types of $\gamma$-divergences from the viewpoint of robustness. In Section 4 numerical experiments are illustrated to verify the difference stated in...
Section 3. The R language code, which was run in our experiments, is available at https://sites.google.com/site/takayukikawashimaspagen/software

2 Regression based on $\gamma$-divergence

The $\gamma$-divergence for regression was first proposed by Fujisawa and Eguchi (2008). It measures the difference between two conditional probability density functions. The other type of the $\gamma$-divergence for regression was proposed by Kawashima and Fujisawa (2017), in which the treatment of the base measure on the explanatory variable was changed. In this section, we briefly review both types of $\gamma$-divergences for regression and present the corresponding parameter estimation.

2.1 Two types of $\gamma$-divergences for regression

First we review the $\gamma$-divergence for the i.i.d. problem. Let $g(u)$ and $f(u)$ be two probability density functions. The $\gamma$-cross entropy and $\gamma$-divergence were defined by

$$
d_{\gamma}(g(u), f(u)) = -\frac{1}{\gamma} \log \int g(u)f(u)^{1+\gamma}du + \frac{1}{1+\gamma} \log \int f(u)^{1+\gamma}du, $$

$$D_{\gamma}(g(u), f(u)) = -d_{\gamma}(g(u), g(u)) + d_{\gamma}(g(u), f(u)).$$

This satisfies the following two basic properties of divergence:

(i) $D_{\gamma}(g(u), f(u)) \geq 0$.
(ii) $D_{\gamma}(g(u), f(u)) = 0 \iff g(u) = f(u)$ (a.e.).

Let us consider the $\gamma$-divergence for regression. Suppose that $g(x, y)$, $g(y|x)$, and $g(x)$ are the underlying probability density functions of $(x, y)$, $y$ given $x$, and $x$, respectively. Let $f(y|x)$ be another conditional probability density function of $y$ given $x$. Let $\gamma$ be the positive tuning parameter which controls a trade-off between efficiency and robustness. Fujisawa and Eguchi (2008) proposed the following cross entropy and divergence:

$$d_{\gamma,1}(g(y|x), f(y|x); g(x))$$

$$= -\frac{1}{\gamma} \log \int \exp\{-\gamma d_{\gamma}(g(y|x), f(y|x))\} g(x)dx$$

$$= -\frac{1}{\gamma} \log \int \left\{ \int g(y|x)f(y|x)^{\gamma}dy \left\{ \int f(y|x)^{1+\gamma}dy \right\}^{\frac{1}{1+\gamma}} \right\} g(x)dx$$

$$= -\frac{1}{\gamma} \log \int \int \left\{ f(y|x)^{\gamma} \left\{ \int f(y|x)^{1+\gamma}dy \right\}^{\frac{1}{1+\gamma}} \right\} g(x,y)dxdy. \quad (2.1)$$
\[ D_{\gamma,1}(g(y|x), f(y|x); g(x)) = -d_{\gamma,1}(g(y|x), g(y|x); g(x)) + d_{\gamma,1}(g(y|x), f(y|x); g(x)). \]  

(2.2)

The cross entropy is empirically estimable, as seen in Section 2.2, and the parameter estimation is easily defined. Kawashima and Fujisawa (2017) proposed the following cross entropy and divergence:

\[ d_{\gamma,2}(g(y|x), f(y|x); g(x)) = -\frac{1}{\gamma} \log \int \left( \int g(y|x)f(y|x)^\gamma dy \right) g(x)dx + \frac{1}{1+\gamma} \log \int \left( \int f(y|x)^{1+\gamma} dy \right) g(x)dx. \]

(2.3)

\[ D_{\gamma,2}(g(y|x), f(y|x); g(x)) = -d_{\gamma,2}(g(y|x), g(y|x); g(x)) + d_{\gamma,2}(g(y|x), f(y|x); g(x)). \]

(2.4)

The base measures on the explanatory variable are taken twice on each term of \( \gamma \)-divergence for the i.i.d. problem. This extension from the i.i.d. problem to the regression problem seems to be more natural than (2.1). The cross entropy is also empirically estimable. We call these two type I and type II, respectively. These two divergences satisfy the following two basic properties of divergence:

(i) \( D_{\gamma,j}(g(y|x), f(y|x); g(x)) \geq 0 \).

(ii) \( D_{\gamma,j}(g(y|x), f(y|x); g(x)) = 0 \iff g(y|x) = f(y|x) \) (a.e.).

The equality holds for the conditional probability density function instead of usual probability density function.

Theoretical properties of \( \gamma \)-divergence for the i.i.d. problem were deeply investigated by Fujisawa and Eguchi (2008). Theoretical properties of \( \gamma \)-divergence for regression were studied by Fujisawa and Eguchi (2008), Kanamori and Fujisawa (2015) and Kawashima and Fujisawa (2017), but not well under heterogeneous contamination, which is special in the regression problem and does not appear in the i.i.d. case. Hung et al. (2017) pointed out that a logistic regression model with mislabel can be regarded as a logistic regression model with heterogeneous contamination and then applied the type I to a usual logistic regression model, which enables us to estimate the parameter of the logistic regression model without estimating a mislabel model even if mislabels exist. They also investigated theoretical properties of robustness, but they assumed that \( \gamma \) is sufficiently large.

In Section 3 we will see that the type I is superior to type II under heterogeneous contamination in the sense of strong robustness without assuming that \( \gamma \) is sufficiently large. Here we mention that the density power divergence (Basu et al. 1998) is another candidate of divergence which gives robustness, but it does not have strong robustness (Fujisawa and Eguchi 2008, Hung et al. 2017).
2.2 Estimation for $\gamma$-regression

Let $f(y|x; \theta)$ be a conditional probability density function of $y$ given $x$ with parameter $\theta$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the observations randomly drawn from the underlying distribution $g(x, y)$. Using the formulae (2.1) and (2.3), the $\gamma$-cross entropy for regression can be empirically estimated by

$$\bar{d}_{\gamma, 1}(f(y|x; \theta)) = -\frac{1}{\gamma} \log \frac{1}{n} \sum_{i=1}^{n} f(y_i|x_i; \theta)^{\gamma} \left( \int f(y|x_i; \theta)^{1+\gamma} dy \right)^{-\frac{1}{\gamma}}.$$

$$\bar{d}_{\gamma, 2}(f(y|x; \theta)) = -\frac{1}{\gamma} \log \left( \frac{1}{n} \sum_{i=1}^{n} f(y_i|x_i; \theta) \right)^{\gamma} + \frac{1}{1 + \gamma} \log \left( \frac{1}{n} \sum_{i=1}^{n} \int f(y|x_i; \theta)^{1+\gamma} dy \right).$$

The estimator can be defined as the minimizer by

$$\hat{\theta}_{\gamma, j} = \text{argmin}_{\theta} \bar{d}_{\gamma, j}(f(y|x; \theta)) \quad \text{for } j = 1, 2.$$

In a similar way to in Fujisawa and Eguchi (2008), we can show that $\hat{\theta}_{\gamma, j}$ converges to $\theta_{\gamma, j}^*$ for $j = 1, 2$, where

$$\theta_{\gamma, j}^* = \text{argmin}_{\theta} D_{\gamma, j}(g(y|x), f(y|x; \theta); g(x))$$

$$= \text{argmin}_{\theta} d_{\gamma, j}(g(y|x), f(y|x; \theta); g(x)).$$

Suppose that $f(y|x; \theta^*)$ is the target conditional probability density function. The latent bias is expressed as $\theta_{\gamma, j}^* - \theta^*$. This is zero when the underlying model belongs to a parametric model, in other words, $g(y|x) = f(y|x; \theta^*)$, but not always zero when the underlying model is contaminated by outliers. This issue will be discussed in Section 3.

2.3 Case of location-scale family

Here we show that both types of $\gamma$-divergence give the same parameter estimation when the parametric conditional probability density function $f(y|x; \theta)$ belongs to a location-scale family in which the scale does not depend on the explanatory variable, given by

$$f(y|x; \theta) = \frac{1}{\sigma} s \left( \frac{y - q(x; \xi)}{\sigma} \right),$$

where $s(y)$ is a probability density function, $\sigma$ is a scale parameter and $q(x; \xi)$ is a location function with a regression parameter $\xi$, e.g., $q(x; \xi) = x^T \xi$. Then, we can
obtain
\[
\int f(y|x; \theta)^{1+\gamma} dy = \int \frac{1}{\sigma^{1+\gamma}} \left( \frac{y - q(x; \zeta)}{\sigma} \right)^{1+\gamma} dy \\
= \sigma^{-\gamma} \int s(z)^{1+\gamma} dz.
\]

This does not depend on the explanatory variable \(x\). Using this property, we can show that both types of \(\gamma\)-cross entropy are the same as follows:

\[
d_{\gamma,2}(g(y|x), f(y|x; \theta); g(x)) \\
= -\frac{1}{\gamma} \log \left\{ \int \left( \int g(y|x) f(y|x; \theta)^{\gamma} dy \right) \left( \int f(y|x; \theta)^{1+\gamma} dy \right)^{-\gamma} \right\} g(x) dx \\
= -\frac{1}{\gamma} \log \left\{ \int \int g(x,y) f(y|x; \theta)^{\gamma} dxdy \left( \int f(y|x; \theta)^{1+\gamma} dy \right)^{-\gamma} \right\} \\
= -\frac{1}{\gamma} \log \int \int g(x,y) f(y|x; \theta)^{\gamma} dxdy + \frac{1}{1+\gamma} \log \int f(y|x; \theta)^{1+\gamma} dy \\
= -\frac{1}{\gamma} \log \int \int g(x,y) f(y|x; \theta)^{\gamma} dxdy + \frac{1}{1+\gamma} \log \int f(y|x; \theta)^{1+\gamma} dy \int g(x) dx \\
= d_{\gamma,2}(g(y|x), f(y|x; \theta); g(x)).
\]

The second equality holds from (2.6). As a result, both types of \(\gamma\)-divergence give the same parameter estimation, because the estimator is defined by the empirical estimation of cross entropy. However, it should be noted that both types of \(\gamma\)-divergence are not the same, because \(d_{\gamma,1}(g(y|x), g(y|x); g(x)) \neq d_{\gamma,2}(g(y|x), g(y|x); g(x))\).

3 Robust properties

In this section, we show a distinct difference between two types of \(\gamma\)-divergence.

3.1 Contamination model and basic condition

Let \(\delta(y|x)\) be the contamination conditional probability density function related to outliers. Let \(\epsilon(x)\) and \(\epsilon\) denote the outlier ratios which depends on \(x\) and does not, respectively. Suppose that the underlying conditional probability density functions under heterogeneous and homogeneous contaminations are given by

\[
g(y|x) = (1 - \epsilon(x)) f(y|x; \theta^*) + \epsilon(x) \delta(y|x), \\
g(y|x) = (1 - \epsilon) f(y|x; \theta^*) + \epsilon \delta(y|x).
\]
Let

\[ \nu_{f,\gamma}(x) = \left\{ \int \delta(y|x)f(y|x)^\gamma dy \right\}^{\frac{1}{\gamma}}, \quad \nu_{f,\gamma} = \left\{ \int \nu_{f,\gamma}(x)g(x)dx \right\}^{\frac{1}{\gamma}}. \]

Here we assume that \( \nu_{f,\gamma} \approx 0. \)

This is an extended assumption used for the i.i.d. problem \(^{\text{Fujisawa and Eguchi}}\) to the regression problem. This assumption implies that \( \nu_{f,\gamma}(x) \approx 0 \) for any \( x \) (a.e.) and illustrates that the contamination conditional probability density function \( \delta(y|x) \) lies on the tail of the target conditional probability density function \( f(y|x; \theta^*). \) For example, if \( \delta(y|x) \) is the dirac function at the outlier \( y_\gamma(x) \) given \( x \), then we have \( \nu_{f,\gamma}(x) = f(y_\gamma(x)|x; \theta^*) \approx 0, \) which is reasonable because \( y_\gamma(x) \) is an outlier.

Here we also consider the condition \( \nu_{f,\gamma} \approx 0, \) which is used later. This will be true in the neighborhood of \( \theta = \theta^*. \) In addition, even when \( \theta \) is not close to \( \theta^*, \) if \( \delta(y|x) \) lies on the tail of \( f(y|x; \theta), \) we can see \( \nu_{f,\gamma} \approx 0. \)

To make the discussion easier, we prepare the monotone transformation of both types of \( \gamma \)-cross entropies for regression by

\[
\tilde{d}_{r,1}(g(y|x), f(y|x; \theta); g(x)) = -\exp\left\{ -\gamma d_{r,1}(g(y|x), f(y|x; \theta); g(x)) \right\}
= -\int \int \frac{f(y|x; \theta)^\gamma}{\left( \int f(y|x; \theta)^{1+\gamma} dy \right)^{\frac{1}{\gamma}}} g(y|x)g(x)dx dy,
\]

\[
\tilde{d}_{r,2}(g(y|x), f(y|x; \theta); g(x)) = -\exp\left\{ -\gamma d_{r,2}(g(y|x), f(y|x; \theta); g(x)) \right\}
= -\int \left( \int g(y|x)f(y|x; \theta)^\gamma dy \right) g(x)dx \frac{\left( \int f(y|x; \theta)^{1+\gamma} dy \right)^{\frac{1}{\gamma}}}{\left( \int \int f(y|x; \theta)^{1+\gamma} dy \right)^{\frac{1}{\gamma}}}
\]

3.2 Type-I of \( \gamma \)-divergence

We see

\[
\tilde{d}_{r,1}(g(y|x), f(y|x; \theta); g(x))
= -\int \frac{\int g(y|x)f(y|x; \theta)^\gamma dy}{\left( \int f(y|x; \theta)^{1+\gamma} dy \right)^{\frac{1}{\gamma}}} g(x)dx
\]

\[
= -\int \frac{\int \left( (1 - \varepsilon(x)) f(y|x; \theta^*) + \varepsilon(x) \delta(y|x) \right)f(y|x; \theta)^\gamma dy}{\left( \int f(y|x; \theta)^{1+\gamma} dy \right)^{\frac{1}{\gamma}}} g(x)dx
\]
where $\tilde{g}(x) = (1 - \varepsilon(x))g(x)$. From this relation, we can easily show the following theorem.

**Theorem 3.1.** Under the condition $\nu_{f_0, \gamma} \approx 0$ and $\int f(y|x; \theta)^{1+\gamma}dy > 0$, we have

$$\tilde{d}_{r,1}(g(y|x), f(y|x; \theta); g(x)) \approx \tilde{d}_{r,1}(f(y|x; \theta'), f(y|x; \theta); \tilde{g}(x)).$$

Using this theorem, we can expect that the latent bias $\theta_{r,1}'' - \theta' \approx 0$.

The last equality holds even when $g(x)$ is replace by $\tilde{g}(x) = (1 - \varepsilon(x))g(x)$.

In addition, we can have the modified Pythagorian relation approximately.

**Theorem 3.2.** Under the condition $\nu_{f_0, \gamma} \approx 0$ and $\int f(y|x; \theta)^{1+\gamma}dy > 0$, the modified Pythagorian relation among $g(y|x)$, $f(y|x; \theta')$, $f(y|x; \theta)$ approximately holds:

$$D_{r,1}(g(y|x), f(y|x; \theta); g(x)) \approx D_{r,1}(g(y|x), f(y|x; \theta'); g(x)) + D_{r,1}(f(y|x; \theta'), f(y|x; \theta); \tilde{g}(x)).$$

The Pythagorian relation implies that the minimizer of $D_{r,1}(g(y|x), f(y|x; \theta); g(x))$ is almost the same as the minimizer of $D_{r,1}(f(y|x; \theta'), f(y|x; \theta); \tilde{g}(x))$, which is $\theta'$. This also implies the strong robustness.

In the theorems, we assume $\nu_{f_0, \gamma} \approx 0$ and $\int f(y|x; \theta)^{1+\gamma}dy > 0$. The former condition was already discussed in Section 3.1. Here we investigate the latter condition. When the parametric conditional probability density function belongs to a location-scale family $\{\tilde{f}, \tilde{g}\}$, this condition will be expected to hold, because

$$\int f(y|x; \theta)^{1+\gamma}dy = \int \frac{1}{\sigma^{1+\gamma}} (\frac{y - q(x; \xi)}{\sigma})^{1+\gamma}dy = \frac{1}{\sigma^\gamma} \int s(z)^{1+\gamma}dz.$$  

We can also verify that this condition holds for a logistic regression model, a Poisson regression model, and so on.

Finally we mention the homogeneous contamination. The modified Pythagorian relation in Theorem 3.2 is changed to the usual Pythagorian relation, because we can easily see $D_{r,1}(f(y|x; \theta'), f(y|x; \theta); \tilde{g}(x)) = D_{r,1}(f(y|x; \theta'), f(y|x; \theta); g(x))$ under homogeneous contamination.
3.3 Type 2 of $\gamma$-divergence

First, we illustrate that the strong robustness does not hold in general under heterogeneous contamination, unlike for type 1. We see

$$
\tilde{d}_{\gamma,2}(g(y|x), f(y|x; \theta); g(x)) = -\frac{\int \left(\int g(y|x)f(y|x; \theta)\gamma dy\right) g(x)dx}{\left\{\int \left(\int f(y|x; \theta)^{1+\gamma} dy\right) g(x)dx\right\}^{\frac{1}{\gamma}}}
$$

$$
= -\frac{\int \left(\int (1 - \epsilon(x))f(y|x; \theta')f(y|x; \theta)\gamma dy + \int \epsilon(x)\delta(y|x)f(y|x; \theta)\gamma dy\right) g(x)dx}{\left\{\int \left(\int f(y|x; \theta)^{1+\gamma} dy\right) g(x)dx\right\}^{\frac{1}{\gamma}}}
$$

$$
= -\frac{\int \int f(y|x; \theta')f(y|x; \theta)\gamma dy(1 - \epsilon(x))g(x)dx}{\left\{\int \left(\int f(y|x; \theta)^{1+\gamma} dy\right) g(x)dx\right\}^{\frac{1}{\gamma}}}
$$

The last approximation holds from $\nu_{\nu_0,\gamma}(x) \approx 0$. This can not be expressed using $d_\gamma(f(y|x; \theta'), f(y|x; \theta); h(x))$ with an appropriate base measure $h(x)$, unlike for type 1, because the base measure of the numerator on the explanatory variable is different from that of the denominator. As in numerical experiments, the type 2 presents a significant bias under heterogenous contamination. However, as already mentioned, when the parametric conditional probability density function belongs to a location-scale family (2.5), the cross entropy for type 2 is the same as that for type 1 and then the type 2 has the strong robustness. In addition, under homogeneous contamination, we have $\tilde{d}_{\gamma,2}(g(y|x), f(y|x; \theta); g(x)) \approx (1 - \epsilon)d_{\gamma,2}(f(y|x; \theta'), f(y|x; \theta); g(x))$ and then we expect that the latent bias $\theta_{\nu_0,\gamma} - \theta'$ is sufficiently small.

4 Numerical experiment

In this section, using a simulation model, we compare the type 1 with the type 2. As shown in Section 3, the distinct difference occurs under heterogeneous contamination when the parametric conditional probability density function $f(y|x; \theta)$ does not belong to a location-scale family. Therefore, we used the logistic regression model as the simulation model, given by

$$
Pr(y = 1|x) = \pi(x; \beta), \ Pr(y = 0|x) = 1 - \pi(x; \beta),
$$
where \( \pi(x;\beta) = \{1 + \exp(-\beta_0 - x_1\beta_1 - \cdots - x_p\beta_p)\}^{-1} \). The sample size and the number of explanatory variables were set to be \( n = 1000 \) and \( p = 5 \), respectively. The true coefficients were given by

\[
\beta_0 = 0, \beta_1 = 1, \beta_2 = -1, \beta_3 = 1, \beta_4 = -1, \beta_5 = 0.
\]

The explanatory variables were generated from a normal distribution \( N(0, \Sigma) \) with \( \Sigma = (0.2|i-j|)_{1 \leq i, j \leq p} \). We generated 100 random samples.

Outliers were incorporated into simulations. We investigated four outlier ratios (\( \epsilon = 0.1, 0.2, 0.3 \) and 0.4) and the following outlier pattern: The outliers were generated around the edge part of the explanatory variable, where the explanatory variables were generated from \( N(\mu_{\text{out}}, 0.5^2I) \) where \( \mu_{\text{out}} = (20, 0, 20, 0, 0) \) and the response variable \( y \) is set to 0.

In order to verify the fitness of regression coefficient, we used the mean squared error (MSE) as the performance measure, given by

\[
\text{MSE} = \frac{1}{p + 1} \sum_{j=0}^{p} (\hat{\beta}_j - \beta_j^*)^2,
\]

where \( \beta_j^* \)'s are the true coefficients. The tuning parameter \( \gamma \) in the \( \gamma \)-divergence was set to 0.5 and 1.0.

Table 1 shows the MSE in the case \( \epsilon = 0.1, 0.2, 0.3 \) and 0.4. The type 2 presented smaller MSEs than the type 1. The difference between two types was larger as the outlier ratio \( \epsilon \) was larger.

| Methods | \( \gamma = 0.5 \) | \( \gamma = 1.0 \) |
|---------|----------------|----------------|
| \( \epsilon = 0.1 \) |                  |                |
| Type 1  | 0.00620         | 0.00712        |
| Type 2  | 0.00810         | 0.0276         |
| \( \epsilon = 0.2 \) |                  |                |
| Type 1  | 0.0136          | 0.0149         |
| Type 2  | 0.0215          | 0.110          |
| \( \epsilon = 0.3 \) |                  |                |
| Type 1  | 0.0262          | 0.0282         |
| Type 2  | 0.0472          | 0.282          |
| \( \epsilon = 0.4 \) |                  |                |
| Type 1  | 0.0514          | 0.0547         |
| Type 2  | 0.0998          | 0.648          |
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