KAZHDAN–LUSZTIG POLYNOMIALS FOR MAXIMALLY-CLUSTERED HEXAGON-AVOIDING PERMUTATIONS

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ABSTRACT. We provide a non-recursive description for the bounded admissible sets of masks used by Deodhar’s algorithm [Deo90] to calculate the Kazhdan–Lusztig polynomials $P_{x,w}(q)$ of type $A$, in the case when $w$ is hexagon avoiding [BW01] and maximally clustered [Los06]. This yields a combinatorial description of the Kazhdan–Lusztig basis elements of the Hecke algebra associated to such permutations $w$. The maximally-clustered hexagon-avoiding elements are characterized by avoiding the seven classical permutation patterns \{3421, 4312, 4321, 46718235, 46781235, 56718234, 56781234\}. We also briefly discuss the application of heaps to permutation pattern characterization.

1. INTRODUCTION

The Kazhdan–Lusztig polynomials introduced in [KL79] have interpretations for finite Weyl groups as Poincaré polynomials for intersection cohomology of Schubert varieties [KL80] and as a $q$-analogue of the multiplicities for Verma modules [BB81, BK81]. From these interpretations, it is known that the Kazhdan–Lusztig polynomials have nonnegative integer coefficients. However, their combinatorial structure remains obscure and no simple all positive formula for the coefficients is known in general. For an introduction to these polynomials, consult [Hum90, Deo94, Bre04, BB05].

Deodhar [Deo90] has proposed a framework for determining the Kazhdan–Lusztig polynomials which can be described for an arbitrary Coxeter group. The framework gives the Kazhdan–Lusztig polynomials in the form of a combinatorial generating function, but generally involves summation over a certain recursively defined set.

In this paper, we show that when $w$ is a permutation that is hexagon avoiding and maximally clustered, Deodhar’s algorithm [Deo90] gives a simple combinatorial formula for the Kazhdan–Lusztig polynomials associated to such permutations. This also yields a combinatorial description of the corresponding Kazhdan–Lusztig basis elements of the Hecke algebra. The maximally clustered permutations introduced in [Los06] are a generalization of the freely braided permutations developed in [GL02] and [GL04], and these in turn include the fully commutative permutations studied in [Ste96] as a subset. Section 1 describes Deodhar’s algorithm. In Section 2 we state our main result which is a formula for the Kazhdan–Lusztig polynomials associated to maximally-clustered hexagon-avoiding permutations. Section 3 is devoted to the proof of the formula. Section 4 gives a pattern comparison result showing that the hexagon-avoiding property studied in this paper can be characterized by avoiding four 1-line patterns.

1.1. Background. We view the symmetric group $S_n$ as a rank $n-1$ Coxeter group of type $A$ with the set of generators $S = \{s_1, \ldots, s_{n-1}\}$ and relations of the form $(s_is_{i\pm1})^3 = 1$ together with $(s_is_j)^2 = 1$ for $|i - j| \geq 2$. The Coxeter graph for $S_n$ is the graph on the generating set $S$ with edges connecting $s_i$ and $s_j$ whenever $s_i$ does not commute with $s_j$. We may also refer to elements in the symmetric group by the 1-line notation $w = [w_1w_2\ldots w_n]$ where $w$ is the bijection mapping $i$ to $w_i$. Then the

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generators $s_i$ are the adjacent transpositions interchanging the entries $i$ and $i + 1$ in the 1-line notation. Suppose $w = [w_1 \ldots w_n]$ and $p = [p_1 \ldots p_k]$ is another permutation in $S_k$ for $k \leq n$. We say $w$ contains the permutation pattern $p$ or $w$ contains $p$ as a 1-line pattern whenever there exists a subsequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that

$$w_{i_a} < w_{i_b} \text{ if and only if } p_a < p_b$$

for all $1 \leq a < b \leq k$. We call $(i_1, i_2, \ldots, i_k)$ the pattern instance. For example, $[3241]$ contains the pattern $[321]$ in several ways, including the underlined subsequence. If $w$ does not contain the pattern $p$, we say that $w$ avoids $p$.

An expression is any product of generators from $S$ and the length $l(w)$ is the minimum length of any expression for the permutation $w$. Such a minimum length expression is called reduced. Each permutation $w \in S_n$ can have several different reduced expressions representing it. For example, one reduced expression for $[3412]$ is $s_2 s_3 s_1 s_2$. Given $w \in S_n$, we represent reduced expressions for $w$ in sans serif font, say $w = w_1 w_2 \cdots w_p$ where each $w_i \in S$. We call any expression of the form $s_i s_{i \pm 1} s_i$ a short-braid after A. Zelevinski (see [Fan98]). We caution the reader that some authors have used the term short-braid to refer to a commutation move between two entries $s_i$ and $s_j$ where $|i - j| \geq 2$. We say that $x < w$ in Bruhat order if a reduced expression for $x$ appears as a subexpression that is not necessarily consecutive, of a reduced expression for $w$. If $s_i$ appears as the last factor in any reduced expression for $w$, then we say that $s_i$ is a descent for $w$; otherwise, $s_i$ is an ascent for $w$. Let the support of a permutation $w$, denoted $\text{supp}(w)$, be the set of all generators appearing in any reduced expression for $w$, which is well-defined by Tits’ theorem [Tit69]. We say that the element $w$ is connected if $\text{supp}(w)$ is connected in the Coxeter graph of $W$.

We define an equivalence relation on the set of reduced expressions for a permutation by saying that two reduced expressions are in the same commutativity class if one can be obtained from the other by a sequence of commuting moves of the form $s_i s_j \leftrightarrow s_j s_i$ where $|i - j| \geq 2$. If the reduced expressions for an element $w$ form a single commutativity class, then we say $w$ is fully-commutative.

1.2. Heaps. If $w = w_1 \cdots w_k$ is a reduced expression, then following [Ste96] we define a partial ordering on the indices $\{1, \ldots, k\}$ by the transitive closure of the relation $i < j$ if $i < j$ and $w_i$ does not commute with $w_j$. We label each element $i$ of the poset by the corresponding generator $w_i$. It follows quickly from the definition that if $w$ and $w'$ are two reduced expressions for a permutation $w$ that are in the same commutativity class then the labeled posets of $w$ and $w'$ are isomorphic. This isomorphism class of labeled posets is called the heap of $w$, where $w$ is a reduced expression representative for a commutativity class of $w$. In particular, if $w$ is fully-commutative then it has a single commutativity class, and so there is a unique heap of $w$.

As in [BW01], we will represent a heap as a set of lattice points embedded in $\mathbb{N}^2$. To do so, we assign coordinates $(x, y) \in \mathbb{N}^2$ to each entry of the labeled Hasse diagram for the heap of $w$ in such a way that:

1. If an entry represented by $(x, y)$ is labeled $s_j$ in the heap, then $x = i$, and
2. If an entry represented by $(x, y)$ is greater than an entry represented by $(x', y')$ in the heap, then $y > y'$.

Since the Coxeter graph of type $A$ is a path, it follows from the definition that $(x, y)$ covers $(x', y')$ in the heap if and only if $x = x' \pm 1$, $y > y'$, and there are no entries $(x'', y'')$ such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. Hence, we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. This representation enables us to make arguments “by picture” that would otherwise be difficult to formulate.

Although there are many coordinate assignments for any particular heap, the $x$ coordinates of each entry are fixed for all of them, and the coordinate assignments of any two entries only differ in the amount of vertical space between them. In the case that $w$ is fully-commutative, a canonical choice can be made by “coalescing” the entries as in [BW01]. We will adhere to this standard when we illustrate
specific heaps, but our arguments should always be viewed as referring to the underlying heap poset. In particular, when we consider the heaps of general permutations we will only allude to the relative vertical positions of the entries, and never their absolute coordinates.

**Example 1.1.** One lattice point representation of the heap of \( w = s_2 s_3 s_1 s_2 s_4 \) is shown below, together with the labeled Hasse diagram for the unique heap poset of \( w \).

Suppose \( x \) and \( y \) are a pair of entries in the heap of \( w \) that correspond to the same generator \( s_i \), so they lie in the same column \( i \) of the heap. Assume that \( x \) and \( y \) are a *minimal pair* in the sense that there is no other entry between them in column \( i \). Then, for \( w \) to be reduced, there must exist at least one non-commuting generator between \( x \) and \( y \), and if \( w \) is short-braid avoiding, there must actually be two non-commuting labeled heap entries that lie strictly between \( x \) and \( y \) in the heap. We call these two non-commuting labeled heap entries a *resolution* of the pair \( x, y \). If the generators lie in distinct columns, we call the resolution a *distinct resolution*. The Lateral Convexity Lemma of [BW01] characterizes fully-commutative permutations \( w \) as those for which every minimal pair in the heap of \( w \) has a distinct resolution.

We now describe a notion of containment for heaps. Recall from [BJ06] that an *orientation preserving Coxeter embedding* \( f : \{s_1, \ldots, s_{k-1}\} \to \{s_1, \ldots, s_{n-1}\} \) is an injective map of Coxeter generators such that for each \( m \in \{2, 3\} \), we have

\[
(s_i s_j)^m = 1 \text{ if and only if } (f(s_i) f(s_j))^m = 1
\]

and the subscript of \( f(s_i) \) is less than the subscript of \( f(s_j) \) whenever \( i < j \). We can view this as a map of permutations which we also denote \( f : S_k \to S_n \) by extending it to a word homomorphism which is then applied to any reduced expression in \( S_k \).

Recall that a subposet \( Q \) of \( P \) is called *convex* if \( y \in Q \) whenever \( x < y < z \) in \( P \) and \( x, z \in Q \). Suppose that \( w \) and \( h \) are permutations. We say that \( w \) *heap-contains* \( h \) if there exist commutativity classes represented by \( w \) and \( h \), together with an orientation preserving Coxeter embedding \( f \) such that the heap of \( f(h) \) is contained as a convex labeled subposet of the heap of \( w \). If \( w \) does not heap-contain \( h \), we say that \( w \) *heap-avoids* \( h \). To illustrate, \( w = s_2 s_3 s_1 s_2 s_4 \) from Example 1.1 heap-contains \( s_1 s_2 s_3 \) under the Coxeter embedding that sends \( s_i \mapsto s_{i+1} \), but \( w \) heap-avoids \( s_1 s_2 s_8 \).

In type A, the heap construction can be combined with another combinatorial model for permutations in which the entries from the 1-line notation are represented by strings. The points at which two strings cross can be viewed as adjacent transpositions of the 1-line notation. Hence, we may overlay strings on top of a heap diagram to recover the 1-line notation for the element, by drawing the strings from bottom to top so that they cross at each entry in the heap where they meet and bounce at each lattice point not in the heap. Conversely, each permutation string diagram corresponds with a heap by taking all of the points where the strings cross as the entries of the heap.

For example, we can overlay strings on the two heaps of [3214]. Note that the labels in the picture below refer to the strings, not the generators.
For a more leisurely introduction to heaps and string diagrams, as well as generalizations to Coxeter types $B$ and $D$, see [BJ06]. Cartier and Foata [CF69] were among the first to study heaps of dimers, which were generalized to other settings by Viennot [Vie89]. Stembridge has studied enumerative aspects of heaps [Ste96, Ste98] in the context of fully commutative elements. Green has also considered heaps of pieces with applications to Coxeter groups in [Gre03, Gre04a, Gre04b].

1.3. **Deodhar’s Theorem.** Given any Coxeter group $W$, we can form the Hecke algebra $\mathcal{H}$ over the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis $\{T_w : w \in W\}$, and relations:

$$T_s T_w = T_{sw} \text{ for } l(sw) > l(w)$$

$$(T_s)^2 = (q - 1)T_s + qT_1$$

where $T_1$ corresponds to the identity element. Kazhdan and Lusztig [KL79] described another basis for $\mathcal{H}$ that is invariant under the involution on the Hecke algebra defined by $\overline{T}_s = q^{-1}T_s(q^{-1})$, where we denote the involution with an overline. This basis, denoted $\{C'_{w} : w \in W\}$, has important applications in representation theory and algebraic geometry [KL80]. The Kazhdan–Lusztig polynomials $P_{x,w}(q)$ describe how to change between these bases of $\mathcal{H}$:

$$C'_{w} = q^{-\frac{1}{2}(w)} \sum_{x \leq w} P_{x,w}(q)T_x.$$  

The $C'_{w}$ are defined uniquely to be the Hecke algebra elements that are invariant under the involution and have expansion coefficients as above, where $P_{x,w}$ is a polynomial in $q$ with

$$\deg P_{x,w}(q) \leq \frac{l(w) - l(x) - 1}{2}$$

for $x < w$ in Bruhat order and $P_{w,w}(q) = 1$. We use the notation $C'_{w}$ to be consistent with the literature because there is already a related basis denoted $C_{w}$.

Fix a reduced expression $w = w_1w_2 \cdots w_k$. Define a *mask* $\sigma$ associated to the reduced expression $w$ to be any binary vector $(\sigma_1, \ldots, \sigma_k)$ of length $k = l(w)$. Every mask corresponds with a subexpression of $w$ defined by $w^\sigma = w_1^{\sigma_1} \cdots w_k^{\sigma_k}$ where

$$w_j^{\sigma_j} = \begin{cases} w_j & \text{if } \sigma_j = 1 \\ \text{id} & \text{if } \sigma_j = 0. \end{cases}$$

Each $w^\sigma$ is a product of generators so it determines an element of $W$. For $1 \leq j \leq k$, we also consider initial sequences of masks denoted $\sigma[j] = (\sigma_1, \ldots, \sigma_j)$, and the corresponding initial subexpressions $w^{\sigma[j]} = w_1^{\sigma_1} \cdots w_j^{\sigma_j}$. In particular, we have $w^{\sigma[k]} = w^\sigma$. The mask $\sigma$ is *proper* if it does not consist of all 1 entries, since $w^{(1, \ldots, 1)} = w$ which is the fixed reduced expression for $w$.

We say that a position $j$ (for $2 \leq j \leq k$) of the fixed reduced expression $w$ is a *defect* with respect to the mask $\sigma$ if

$$l(w^{\sigma[j-1]_w}) < l(w^{\sigma[j-1]}).$$

Note that the defect status of position $j$ does not depend on the value of $\sigma_j$. Let $d_w(\sigma)$ denote the number of defects of $w$ for a mask $\sigma$. We will use the notation $d(\sigma) = d_w(\sigma)$ if the reduced word $w$ is fixed.
Deodhar’s framework gives a combinatorial interpretation for the Kazhdan–Lusztig polynomial $P_{x,w}(q)$ as the generating function for masks $\sigma$ on a reduced expression $w$ with respect to the defect statistic $d(\sigma)$. We begin by considering subsets of

$$S = \{ \text{all possible masks } \sigma \text{ on } w \}.$$  

For $E \subset S$, we define a prototype for $P_{x,w}(q)$:

$$P_x(E) = \sum_{\sigma \in E} q^{d(\sigma)}$$

and a corresponding prototype for the Kazhdan–Lusztig basis element $C'_w$:

$$h(E) = q^{-\frac{l(w)}{2}} \sum_{\sigma \in E} q^{d(\sigma)} T_{w^\sigma}.$$  

**Definition 1.2.** [Deo90] Fix $w = w_1 w_2 \ldots w_k$. We say that $E \subset S$ is admissible on $w$ if:

1. $E$ contains $\sigma = (1,1,\ldots,1)$.
2. $E = \tilde{E}$ where $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_{k-1}, 1 - \sigma_k)$.
3. $h(E) = h(\tilde{E})$ is invariant under the involution on the Hecke algebra.

We say that $E$ is bounded on $w$ if $P_x(E)$ has degree $\leq \frac{1}{2}(l(w) - l(x) - 1)$ for all $x < w$ in Bruhat order.

**Theorem 1.3.** [Deo90] Let $x, w$ be elements in any Coxeter group $W$, and fix a reduced expression $w$ for $w$. If $E \subset S$ is bounded and admissible on $w$, then

$$P_{x,w}(q) = P_x(E) = \sum_{\sigma \in E \atop w^\sigma = x} q^{d(\sigma)}$$

and hence

$$C'_w = h(E) = q^{-\frac{l(w)}{2}} \sum_{\sigma \in E} q^{d(\sigma)} T_{w^\sigma}.$$  

Billey and Warrington say that $w \in S_n$ is hexagon-avoiding if it heap-avoids $[46718235] = s_5 s_6 s_7 s_8 s_4 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3$.

When $w$ is fully-commutative, this condition is equivalent to avoiding $[46718235]$, $[46781235]$, $[56718234]$ and $[56781234]$ as permutation patterns. We will show in Section 4 that this permutation pattern characterization remains true in more general settings.

**Theorem 1.4.** [BW01] The set $S$ is bounded and admissible on a reduced expression $w$ if and only if the corresponding permutation $w$ is $[321]$-avoiding and hexagon-avoiding.

More generally, let $W$ be any Coxeter group and $E \subset S$ be a set of masks on some reduced expression $w \in W$. By Lemma 2 of [BW01], we have that $E$ is bounded if and only if for every proper mask $\sigma \in E \setminus \{(1,1,\ldots,1)\}$, we have

$$\# \text{ of zero-defects of } \sigma < \# \text{ of plain-zeros of } \sigma,$$

which we refer to as the **Deodhar bound**. Here, a position in $w$ is a **zero-defect** if it has mask-value 0 and it is also a defect. A position in $w$ is a **plain-zero** if it has mask-value 0 and it is not a defect. We say that an element represented by $w$ is **Deodhar** if $S$ is bounded on $w$.  


1.4. Maximally clustered elements. In [Los06], Losonczy introduced the maximally clustered elements of simply laced Coxeter groups. We will define a set of masks for the maximally-clustered hexagon-avoiding permutations that generalizes Theorem 1.4.

Definition 1.5. [Los06] A braid cluster is an expression of the form

\[ s_{i_1}s_{i_2} \ldots s_{i_k}s_{i_k+1}s_{i_k} \ldots s_{i_2}s_{i_1} \]

where each \( s_{i_p} \) for \( 1 \leq p \leq k \) has a unique \( s_{i_q} \) with \( p < q \leq k + 1 \) such that \( |i_p - i_q| = 1 \).

Let \( w \) be a permutation and let \( N(w) \) denote the number of \([321]\) pattern instances in \( w \). We say \( w \) is maximally-clustered if there is a reduced expression for \( w \) of the form

\[ a_0c_1a_1c_2a_2 \ldots c_Ma_M \]

where each \( a_i \) is a reduced expression, each \( c_i \) is a braid cluster with length \( 2n_i + 1 \) and \( N(w) = \sum_{i=1}^{M} n_i \).

Such an expression is called contracted. In particular, \( w \) is freely-braided if there is a reduced expression for \( w \) with \( N(w) \) disjoint short-braids.

Note that this is not the original definition for the maximally clustered elements; however it is equivalent. The remarks in Section 5 of [GL02] show that the number of \([321]\) pattern instances in \( w \) equals the number of contractible triples of roots in the inversion set of \( w \). Corollary 4.3.3 (ii) and Corollary 4.3.5 of [Los06] prove that \( w \) is a contracted reduced expression for a maximally-clustered element if and only if it has the form given in Definition 1.5. Observe that a maximally-clustered permutation \( w \) is fully-commutative if and only if \( N(w) = 0 \) by [BJS93].

In type \( A \), there exists a standard form for the braid clusters.

Lemma 1.6. Suppose \( x = s_{i_1}s_{i_2} \ldots s_{i_k}s_{i_k+1}s_{i_k} \ldots s_{i_2}s_{i_1} \) is a braid cluster of length \( 2k + 1 \) in type \( A \). Then, \( x = s_{m+1}s_{m+2} \ldots s_{m+k}s_{m+k+1}s_{m+k} \ldots s_{m+2s_m+1} \) for some \( m \).

Proof. By Lemma 4.1.3 of [Los06], there exists a sequence of moves that result in a braid cluster for the same element \( x \) such that the largest generator \( s_{m+k+1} \) appearing in any reduced expression for \( x \) appears in the middle position.

The set of generators appearing in reduced expressions for \( x \) must consist of a connected path in the Coxeter graph, otherwise the original expression for \( x \) is not reduced. By the uniqueness statement in Definition 1.5, the entry next to \( s_{m+k+1} \) must be \( s_{m+k-i} \) for each \( i = 0, \ldots, k - 1 \). \( \square \)

For our work, we will implicitly assume that any braid cluster has the canonical form of Lemma 1.6. Also, we refer to \( s_{m+k}s_{m+k+1}s_{m+k} \) as the central braid of the braid cluster

\[ s_{m+1}s_{m+2} \ldots s_{m+k}s_{m+k+1}s_{m+k} \ldots s_{m+2s_m+1} \]

Recall that the maximally-clustered permutations are characterized by avoiding the permutation patterns

\[ [3421], [4312], \text{ and } [4321] \]

as a result of Proposition 3.2.1 in [Los06], while the freely-braided permutations are characterized by avoiding

\[ [4231], [3421], [4312], \text{ and } [4321] \]

as permutation patterns by Proposition 5.1.1 in [GL02].
2. Main Result

Given a contracted expression \( w \) for a maximally-clustered hexagon-avoiding permutation, our main result is that we can identify a set of masks on \( w \) that turn out to be bounded and admissible. Moreover, this set has a simple non-recursive description.

**Definition 2.1.** Let \( w \) be a contracted expression for a maximally-clustered hexagon-avoiding permutation, where each braid cluster has the form given in Lemma 1.6. We say that a mask \( \sigma \) on \( w \) has a 10*-instance if it has the values
\[
\ldots s_i \ s_{i+1} \ s_i \ldots
\]
on any central braid instance \( s_i s_{i+1} s_i \) of any braid cluster in \( w \), where * denotes an arbitrary mask value. If \( \sigma \) never has the values 1 and 0 (respectively) on the first two entries in any central braid of \( w \), then we say that \( \sigma \) is a 10*-avoiding mask for \( w \).

**Theorem 2.2.** Let \( w \) be a contracted expression for a maximally-clustered hexagon-avoiding permutation in \( S_n \), and let \( E_w \) be the set of 10*-avoiding masks on \( w \). Then for any \( x \in S_n \),
\[
P_{x,w}(q) = P_x(E_w) = \sum_{\sigma \in E_w} q^{d(\sigma)} \quad x^\sigma \quad T_{w^\sigma}
\]
and hence
\[
C'_w = h(E_w) = q^{-\frac{1}{2}l(w)} \sum_{\sigma \in E_w} q^{d(\sigma)} T_{w^\sigma}
\]

**Proof.** This follows from Theorem 1.3 since \( E_w \) is bounded and admissible by Propositions 3.2 and 3.3 below.

We begin by showing that the contracted expressions for maximally-clustered permutations have an especially nice form.

**Lemma 2.3.** Let \( w \) be a contracted reduced expression for a maximally clustered permutation, so \( w \) has the form
\[
a_0 c_1 a_1 \ldots c_M a_M
\]
where each \( c_j \) is a braid cluster, and the \( a_j \) are short-braid avoiding. Then, any generator \( s_i \) that appears in any of the braid clusters \( c_j \) does not appear anywhere else in \( w \).

**Proof.** Suppose for the sake of contradiction that there exists a contracted reduced expression of the form
\[
\ldots s_{m+1} s_m \ldots s_m+1 \ldots s_{m+k-1} s_{m+k-1} \ldots s_{m+2} s_m+1 \ldots s_i \ldots
\]
where \( m + 1 \leq i \leq m + k \). Then, we may choose \( s_i \) to be leftmost to obtain a contracted reduced expression of the form
\[
\tilde{w} = s_{m+1} s_m \ldots s_{m+k-1} s_{m+k-1} \ldots s_{m+2} s_m+1 \ldots s_i
\]
in which there are no \( s_j \) generators among the entries between the braid cluster and \( s_i \) for any \( m + 1 \leq j \leq m + k \). Since this is a factor of a contracted expression, \( \tilde{w} \) is maximally-clustered by Definition 1.5.

We examine the 2-line notation that is built up from the identity permutation by multiplying on the right by \( \tilde{w} \). The columns of this notation encode \( \begin{bmatrix} j & w_j \end{bmatrix} \) for each \( j \in \{1, \ldots n\} \). In particular, the lower row is the usual 1-line notation. The initial braid cluster produces a consecutive \( [\ldots (k+1)23 \ldots k1] \)-instance
\[
\begin{bmatrix}
\ldots & m+1 & m+2 & m+3 & \ldots & m+k & m+k+1 & \ldots \\
\ldots & m+k+1 & m+2 & m+3 & \ldots & m+k & m+1 & \ldots
\end{bmatrix}
\]
in positions \( (m+1) \ldots (m+k+1) \).
Since there are no $s_j$ among the generators not explicitly shown for any $m + 1 \leq j \leq m + k$, the entries in positions $m + 2, \ldots, m + k$ remain fixed as we multiply on the right by subsequent entries from $\tilde{w}$. Therefore, if $m + 1 < i < m + k$ then $s_i$ creates a descent among the $m + 2, \ldots, m + k$ entries so the 1-line notation for $\tilde{w}$ contains a $[4321]$ instance, contradicting that $\tilde{w}$ is maximally-clustered.

Next, suppose that $i = m + k$. Since there are no $s_{m+k}$ generators between the braid cluster and $s_i$ in $\tilde{w}$, the entry with value $m + k$ remains strictly left of position $m + k + 1$, and all of the entries except $m + 1$ that lie strictly right of position $m + k + 1$ always have values $> m + k + 1$. Since the 1-line entries in positions $m + k$ and $m + k + 1$ are inverted, we cannot apply $s_{m+k}$ again in a reduced fashion until we first apply an $s_{m+k+1}$.

Let $x$ be the value of the entry in position $m + k + 1$ just after the last $s_{m+k+1}$ occurs, so $x > m + k + 1$ and we have

\[
\begin{bmatrix}
    \ldots & p & \ldots & m + k & m + k + 1 & \ldots & q & \ldots \\
    \ldots & (m + k + 1) & \ldots & m + k & x & \ldots & (m + 1) & \ldots
\end{bmatrix}.
\]

Once we apply the last $s_{m+k}$, we obtain a $[3421]$ pattern instance, contradicting the maximally-clustered hypothesis.

A similar argument shows that if $i = m + 1$ then the 1-line notation for $\tilde{w}$ contains $[4312]$ as a permutation pattern, contradicting that $\tilde{w}$ is maximally-clustered. Hence, the generators of every braid cluster appear uniquely in any contracted reduced expression for a maximally-clustered permutation. □

**Remark 2.4.** Although we have emphasized the type $A$ case, there is a definition of maximally-clustered and freely-braided for all simply-laced Coxeter groups. It is not true, even in type $D$, that these conditions imply uniqueness for the generators in the short-braid instances. For example, the expression $w = s_2 s_3 s_2 s_1 s_2 s_1 s_2$ in $D_4$ is contracted and freely-braided, but it contains an $s_2$ generator beyond the short-braid instance $s_2 s_3 s_2$. Here, we have labeled the generators so that $s_2$ is adjacent to $s_1$, $s_3$ and $s_3$ in the Coxeter graph.

In type $A$, there is an algorithm for producing a contracted expression from the 1-line notation of the permutation, which is useful for creating computer programs. We have included a description of this algorithm in Appendix A. This algorithm shows that the number of braid clusters in a maximally clustered permutation $w$ is precisely the number of $[(m + 2)23 \ldots (m + 1)1]$-instances in the 1-line notation for $w$. Each such pattern instance contributes $m$ to $N(w)$.

## 3. PROOF OF THE MAIN THEOREM

In this section we will prove Theorem 2.2 by showing that the set of 10*-avoiding masks is bounded and admissible. The proof of each of these properties relies on a map of contracted reduced expression/mask pairs. Here we show that the hexagon-avoiding property is preserved under such maps, which will be used to make inductive arguments in the proofs of Propositions 3.2 and 3.3.

**Lemma 3.1.** Let $w$ be a contracted reduced expression for a maximally-clustered hexagon-avoiding permutation, so $w$ has the form

\[ w = a_0 c_1 a_1 \ldots a_{M-1} c_M a_M \]

where each $c_j$ is a braid cluster and the $a_j$ are short-braid avoiding. Let $u$ be any expression obtained from $w$ by removing some of the entries from the last braid cluster $c_M$ in such a way that

\[ u = a_0 c_1 a_1 \ldots a_{M-1} \tilde{c}_M a_M \]

is still reduced and contracted. Then, the corresponding element $u$ is hexagon-avoiding.
Proof. We suppose that \( u \) contains a hexagon pattern and show that \( w \) must also contain a hexagon pattern. If \( u \) contains a hexagon then there is some reduced expression \( \tilde{u} \) for \( u \) such that the heap of \( \tilde{u} \) contains a hexagonal subheap

in columns \( \{1 + i, \ldots, 7 + i\} \) for some \( i \geq 0 \).

Note that although we assume \( u \) is a contracted reduced expression, our notation \( a_0 c_1 a_1 \ldots a_{M-1} \tilde{c}_M a_M \) might not represent the partition of this expression into braid clusters, because the factors in this notation were defined with respect to \( w \). In particular, \( a_{M-1} \tilde{c}_M a_M \) may be fully-commutative. In any case, every other braid cluster \( c_i \) of \( w \) where \( 1 \leq i \leq M - 1 \) remains a braid cluster in \( u \), and we assume these are in the canonical form of Lemma 1.6. Also, since \( u \) is maximally-clustered, the heaps of \( u \) and \( \tilde{u} \) differ only in the choice of commutativity classes for their braid clusters by [Los06, Corollary 4.3.3].

Let \([p, q]\) be the interval of columns that support the braid cluster \( c_M \) in the heap of \( w \). By Lemma 2.3, we have that the heaps of \( w \) and \( u \) agree on the columns outside of \([p, q]\). In particular, if the hexagon instance in \( \tilde{u} \) does not use any entries from columns \([p, q]\), then the hexagon instance appears in the reduced expression for \( w \) that is obtained by choosing the commutativity class of each braid cluster \( c_1, \ldots, c_{M-1} \) to match the commutativity class of each such braid cluster in \( \tilde{u} \). As this contradicts the hypothesis that \( w \) is hexagon-avoiding, we have that the hexagon instance uses some entry of \( \tilde{u} \) from columns \([p, q]\).

Since the hexagon is fully-commutative, every minimal pair of entries in each column has a distinct resolution. However, by the uniqueness in Definition 1.5, no minimal pair of entries in any braid cluster has a distinct resolution. Thus, we find that the hexagon instance in \( \tilde{u} \) either includes an entry from column \( p \) that corresponds to \( s_{i+7} \), or it includes an entry from column \( q \) that corresponds to \( s_{i+1} \). Let us suppose that we have the former case without loss of generality. Since we only remove entries from columns \([p, q]\) as we pass from \( w \) to \( \tilde{u} \), we find that the entries of \( w \) in column \( p - 1 \) that are used in the hexagon instance of \( \tilde{u} \) surround the entries from the braid cluster in column \( p \).

Hence, if we choose the commutativity class for the braid cluster \( c_M \) which has the form

\[ s_q s_{q-1} \cdots s_{p+1} s_p s_{p+1} \cdots s_{q-1} s_q \]

then the corresponding heap contains a single entry in column \( p \), and so we obtain a hexagon instance in this heap for \( w \). This contradicts our hypothesis that \( w \) is hexagon-avoiding. An illustration for the case when the hexagon instance uses \( s_{7+1} \) from \( c_M \) is given below. The \( \star \) points are entries from the braid cluster \( c_M \) in the heap of \( w \).

\[ \begin{array}{c}
\begin{array}{cccccccc}
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{cccccccc}
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\end{array}
\end{array} \]

In the proof of the next result, we decorate the heap diagrams according to mask-value using Table 1.

**Proposition 3.2.** Let \( w \) be a contracted expression for a maximally-clustered hexagon-avoiding permutation \( w \). Then, the set of 10\( \star \)-avoiding masks on \( w \) is bounded.
Table 1. Heap decorations

| Decoration | Mask-value                          |
|------------|-------------------------------------|
| ◊          | zero-defect entry                   |
| ○          | plain-zero entry (not a defect)     |
| •          | mask-value 1 entry                  |

Proof. Suppose that there exists some proper 10*-avoiding mask \( \sigma \) on \( w \) with

\[
\# \text{zero-defects of } \sigma \geq \# \text{plain-zeros of } \sigma
\]

which violates Equation (1.1). In particular, \( \sigma \) contains at least one zero-defect because it is a proper mask. We show how to extend this mask to an element with one fewer braid cluster while maintaining the non-Deodhar bound. Eventually, we derive a contradiction by Theorem 1.4.

We may adorn the heap diagram of the permutation with strings that correspond to entries in the 1-line notation for the permutation. This construction is a standard technique which is given a detailed description in [BJ06]. We can consider a pair of strings emanating from each entry of the decorated heap such that the strings cross at mask-value 1 entries and bounce at mask-value 0 entries. It follows from the definition that a defect entry must have a pair of strings that cross an odd number of times below the defect. Suppose the last braid cluster has length \( 2k + 1 \) with entries occupying columns \( m, m + 1, \ldots, m + k \) of the heap, and it is in the form given by Lemma 1.6. Let \( \mathcal{B}(w) \) denote the reduced expression obtained from \( w \) by removing the last \( k \) entries of this last braid cluster. By Lemma 2.3 and Definition 1.5, we find that \( \mathcal{B}(w) \) is a contracted reduced expression for a maximally clustered permutation with one fewer braid cluster. We describe how to construct a non-Deodhar mask on \( \mathcal{B}(w) \), starting from the restriction of \( \sigma \) to \( \mathcal{B}(w) \).

First, observe that we remove at least as many plain-zeros as zero-defects from columns \( m, m + 1, \ldots, m + k - 1 \) when we apply \( \mathcal{B} \). To see this, suppose there exists a zero-defect at the top of column \( h \) where \( m \leq h \leq m + k - 1 \). Then the strings for the defect must cross below the defect. By Lemma 2.3 and 1.6, the form of the heap in columns \( m, \ldots, m + k \) is determined as shown in Figure 1(a). In particular, there can be no entries in column \( m - 1 \) lying between the two entries in column \( m \) by Definition 1.5.

![Figure 1](image-url)
Since the mask is 10*-avoiding, the right string of the defect must travel southeast from the defect until it hits a zero in column $g$ with $h < g < m + k$, drop straight down until it hits the next entry in the heap which must also be a zero in the same column $g$, and then continue southwest until it crosses the left string of the defect at the bottom entry in column $h$. Hence, both of the entries in column $g$ must have mask-value 0 to facilitate the string crossing for the defect. Neither entry in column $g$ can be a zero-defect, because such a defect must have a mask-value 1 entry directly below it in the same column to facilitate the string crossing. As we already assumed that the mask values of both entries in column $g$ were 0, the lower entry is not a critical generator. Thus, removing the top entries in columns $m, m + 1, \ldots, m + k - 1$ removes a plain-zero for every defect, so the non-Deodhar bound for the mask $\sigma$ restricted to $B(w)$ is preserved.

Next, we must consider whether the removal of the last $k$ entries from the last braid cluster might destroy the defect status of an entry further to the right in the contracted reduced expression. We will argue that it cannot. Any zero-defect whose left string intersects the braid cluster will have the same string dynamics after we apply $B$ as it did originally since we do not change the mask-values of the remaining entries in the braid cluster. This is illustrated in Figure 1(b). Hence, applying $B$ does not destroy the defect status of such a zero-defect.

Suppose the right string of a zero-defect intersects the braid cluster as in Figure 1(c). Then, the path of this string is prescribed as in the argument above. It must intersect a zero in column $g$ with $g < m + k$, drop straight down until it hits the next entry in the heap which must also be a zero in the same column $g$, and then continue southwest until it leaves the braid cluster and eventually crosses the left string of the zero-defect. No other mask configuration will allow the strings of the zero-defect to cross by Lemma 2.3.

Observe that since there are no entries in column $m - 1$ between the two entries in column $m$, the strings cannot cross at $p$. If the mask-value of $p$ is 1, then we change the mask for $B(w)$ by interchanging the mask-values of $p$ and $q$. Then the string dynamics for the defect will remain the same after we apply $B$, and we preserve the non-Deodhar bound of the mask.

After iteratively applying $B$ to remove each braid cluster of $w$, we eventually obtain a short-braid avoiding reduced expression that is still hexagon-avoiding by Lemma 3.1. The argument above shows that we have a proper mask on the resulting expression in which no more defects than plain-zeros have been removed in comparison with the original mask on $w$. But Theorem 1.4 implies that every proper mask on an element that is short-braid avoiding and hexagon-avoiding must satisfy the Deodhar bound from Equation (1.1)

$$\# \text{ zero-defects of } \sigma < \# \text{ plain-zeros of } \sigma.$$  

This contradicts the existence of our original non-Deodhar mask on $w$. □

**Proposition 3.3.** Let $w$ be a contracted expression for a maximally-clustered hexagon-avoiding permutation. Then, the set of 10*-avoiding masks on $w$ is admissible.

**Proof.** We let $E_w$ denote the set of 10*-avoiding masks on $w$. In order for $E_w$ to be admissible, we must show that it satisfies the three properties in Definition 1.2.

The first property follows since the mask $\sigma = (1, 1, \ldots, 1)$ avoids 10*, and the second property holds because avoiding 10* imposes no restrictions on the last entry of a mask.

In order to show that $h(E_w)$ is invariant under the Hecke algebra involution, we will use that

$$h(E_w) = h(\overline{E_w})$$

if and only if $h(E_w^c) = h(\overline{E_w^c})$.

where $E_w^c$ denotes the set complement $S \setminus E_w$. Indeed, Deodhar observes in [Deo90, Proposition 3.5] that $h(S) = C_{w_1}^l C_{w_2}^l \ldots C_{w_l}^l$, so in particular $h(S) = \overline{h(S)}$. Since $S = E_w \cup E_w^c$, we have $h(E_w) = h(S) - h(\overline{E_w})$, so by linearity any set of masks is invariant under the involution whenever its complement is.
We proceed by induction on the number $N(w)$ of short-braids in the contracted expression $w$. If $w$ is short-braid avoiding, then $\mathcal{E}_w = \mathcal{S}$, and this set is admissible.

Next, suppose that $w$ has $N(w)$ short-braids, and that $h(\mathcal{E}_v) = h(\mathcal{E}_w)$ whenever $v$ is a contracted expression for a maximally-clustered hexagon-avoiding permutation with fewer than $N(w)$ short-braids. We will prove that $h(\mathcal{E}_w)$ is invariant under the involution.

Suppose $B \subset \{1, \ldots, l(w)\}$ is a set of positions in $w$. Let $\mathcal{E}_w^B$ denote the set of masks $\sigma$ on $w$ such that $\sigma$ has a $10^*$ instance starting at position $b$ if and only if $b \in B$. Not every subset can correspond to a set of masks; for example, if $B$ includes positions that do not correspond to central braids in braid clusters of $w$ then $\mathcal{E}_w^B$ will necessarily be the empty set. In such a situation we define $h(\mathcal{E}_w^B) = 0$.

Observe that we can decompose $\mathcal{E}_w^c$ as $\bigsqcup \mathcal{E}_w^B$, where we take the disjoint union over all non-empty sets of positions. Then, we have

$$h(\mathcal{E}_w^c) = \sum_{\text{non-empty position sets } B} h(\mathcal{E}_w^B)$$

so it suffices by linearity to show that

$$h(\mathcal{E}_w^B) = h(\mathcal{E}_w^B)$$

for each non-empty set of positions $B$.

Towards this end, we define a map $\varphi$ on reduced expression/mask pairs $(w, \sigma)$. We denote the image of $w$ under the map by $\varphi(w)$ via an abuse of notation since $\varphi(w)$ depends on the mask $\sigma$. Then,

$$\varphi : \{w\} \times \mathcal{E}_w^B \rightarrow \mathcal{E}_w^{B \setminus \text{max}(B)}$$

is defined by:

$$\left( \begin{array}{c} \cdots \ s_i \ \cdots \ s_{i+k-1} \ s_{i+k} \ \cdots \ s_{i+k-1} \ \cdots \ s_{i+1} \ s_i \ \cdots \\ \cdots \ 1 \ \cdots \ 1 \ \cdots \ 1 \ * \ \cdots \\ b_1-k+1 \ \cdots \ b_1 - b_1+1 \ b_1+2 \ \cdots \ b_1+k \ b_1+k+1 \end{array} \right) \varphi \left( \begin{array}{c} \cdots \ s_i \ \cdots \\ \cdots \ (1-*) \ \cdots \\ \cdots \ b_1-k+1 \ \cdots \end{array} \right)$$

where the notation indicates the mask-value and location of each entry on the second and third rows of the array, respectively. The segment between indices $b_1 - k + 1$ and $b_1 + k + 1$ is chosen to be the maximal segment that has symmetric intervals of mask-value 1 entries about the central braid. In particular, the segment may not be the entire braid-cluster instance in $w$. In the definition, $b_1 = \text{max}(B)$ is the rightmost position in $B$, and $(1-*)$ indicates that we take the mask-value on $s_i$ that is opposite to the mask-value given on the second $s_i$ generator in location $b_1 + k + 1$ of the $10^*$ instance. In the image, the entire segment between indices $b_1 - k + 1$ and $b_1 + k + 1$ is replaced by the single entry at location $b_1 - k + 1$.

Observe that applying this move $\varphi$ to the reduced expression/mask pairs $(w, \sigma) \in \{w\} \times \mathcal{E}_w^B$ has the following effects:

1. The map $\varphi$ gives a bijection between $\mathcal{E}_w^B$ and $\mathcal{E}_w^{B \setminus \text{max}(B)}$ because it is reversible.
2. $\varphi(w)$ is a contracted reduced expression for a maximally-clustered hexagon-avoiding permutation by Lemma 3.1 and Lemma 2.3, with exactly $k$ fewer short-braid instances than $w$, and $l(\varphi(w)) = l(w) - 2k$.
3. $\varphi$ removes exactly $k$ defects from the mask $\sigma$ on $w$, because $w^{\sigma[\sigma[1+b_1+k+1]} = \varphi(w)^{\sigma[\sigma[1+b_1-k+1]}$, so the defect status of subsequent entries remains precisely the same.
4. $w^{\sigma} = \varphi(w)^{\varphi(\sigma)}$ follows from (3).
5. The map $\varphi$ introduces no new $10^*$-instances because the choice of the segment $(b_1 - k + 1, b_1 + k + 1)$ about $b_1$ is maximal.

To verify (3), note that by Lemma 2.3 there are no other $s_j$ generators for $i \leq j \leq i + k$ anywhere else in the contracted expression $w$. Hence, the second $s_{i+k-1}, \ldots, s_i$ generators are always defects, while none of the entries in the first segment are. Since $s_{i+k}$ has mask-value 0, removing this entry has no
effect on the defect status of any subsequent entries in $w$. Furthermore, we can also remove the other entries and update the mask as shown without changing the defect status of any subsequent entry. This follows because if both $s_i$ entries had mask-value 1, then they would cancel after the collapse of the pairs of generators $s_{i+1}, \ldots, s_{i+k-1}$ in any calculation of the defect status of an entry further to the right. If the second $s_i$ had mask-value 0, then it would play no role in the calculation of the defect status of an entry further to the right, so could be removed.

Next, we may calculate

$$h(E_B^w) = q^{-\frac{1}{2}l(w)} \sum_{\sigma \in \langle \ldots s_i \ldots s_{i+k-1} s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i \ldots \rangle} q^{d(\sigma)} T_{w^\sigma}$$

$$= q^{-\frac{1}{2}(l(w)-2k)} \sum_{\sigma \in \langle \ldots s_i \ldots (1-* \ast \ldots) \rangle} q^{d(\sigma)} T_{\varphi(w)^\sigma}$$

$$= q^{-\frac{1}{2}(l(w)-2k)} \sum_{\sigma \in \langle \ldots s_i \ldots (1-* \ast \ldots) \rangle} q^{d(\sigma)} T_{\varphi(w)^\sigma}$$

to see that

$$h(E_B^w) = h(E_{\varphi(w)}^B).$$

If we let $m = |B|$ then the masks in $E_B^w$ all have exactly $m$ 10*-instances. Therefore, after $m$ applications of the map $\varphi$ we will obtain a set $\varphi^m(E_B^w)$ of 10*-avoiding masks on some contracted expression $\varphi^m(w)$ that has at least $m$ fewer short-braids than $w$. This follows because we always remove at least one short-braid instance from $w$ every time we apply $\varphi$. In particular, since $m > 0$, we have that $h(\varphi^m(E_B^w))$ is known to be invariant under the involution by induction.

Thus, $h(E_B^w) = h(E_c^w)$ as well. \qed

This completes the proof of Theorem 2.2, and shows that there is a non-recursive algorithm to compute the Kazhdan–Lusztig basis elements associated to maximally-clustered hexagon-avoiding permutations.

4. Patterns of maximally-clustered elements

In this section, we prove that the property of heap-avoiding the hexagon in the maximally-clustered and freely-braided cases can be characterized by avoiding the four 1-line patterns

$$\{[46718235], [46781235], [56718234], [56781234]\}.$$  

This is implicit in [BW01] for the fully-commutative elements. More generally, we describe when it is possible to translate between heap-avoidance and classical permutation pattern avoidance. This also provides a methodology for using heaps to study classical permutation pattern classes.

The definitions and results in this section generalize those in [BJ06, Section 11] to arbitrary subsets of permutations that are characterized by classical pattern avoidance. See [BJ06, Section 11] for results that hold in type $D$ comparing embedded factor avoidance and classical 1-line pattern avoidance. The proofs from that work are very similar to those given here but need to verified in the more general setting, so we reproduce them for convenience.
Let $S^P = \bigcup_{n \geq 1} S^P_n$ denote the permutations characterized by avoiding a set of 1-line patterns $P$. The most important pattern classes for this work are the maximally-clustered permutations and the freely-braided permutations, characterized by avoiding the patterns from (1.2) and (1.3), respectively. Given a permutation $h$, let $S^P(h)$ be the subset of $S^P$ consisting of those permutations that heap-avoid the single pattern $h$. Our goal is to find a set of 1-line patterns $Q$ such that $S^P(h) = S^Q$. Note that this is not always possible, as demonstrated in Example 11.1 of [BJ06].

If $r(h)$ is the rank of the symmetric group containing $h$, then we let $U^P(h)$ denote the set of all elements in $S^P_{r(h)}$ that heap-contain $h$. We will show that when the patterns in $U^P(h)$ satisfy an additional hypothesis called the ideal pattern condition, heap-avoiding $h$ is equivalent to avoiding the permutations of $U^P(h)$ as 1-line patterns. This set is finite since it includes only permutations from a fixed rank. If the support of $h$ is connected in the Coxeter graph, then the only orientation preserving Coxeter embedding $f : S_r(h) \to S_r(h)$ is the identity, so the elements of $U^P(h)$ are precisely the elements of $S^P_{r(h)}$ that contain $h$ as a factor. In this case, $U^P(h)$ is the upper order ideal generated by $h$ in the two-sided weak Bruhat order on $S_{r(h)}$.

**Proposition 4.1.** Let $w \in S^P$ and $h$ be a permutation. If $w$ heap-contains $h$, then $w$ contains an element of $U^P(h)$ as a 1-line pattern.

**Proof.** We begin by choosing a commutativity class of $w$ whose heap contains a collection of lattice points corresponding to the heap of $h$. Highlight a shifted copy of the heap of $h$ as a set of lattice points inside the heap of $w$. Then, we can build the heap of $w$ starting from the shifted copy of the heap of $h$ by sequentially adding lattice points that are maximal or minimal with respect to the intermediate heap poset.

Since the heap of $h$ is a convex subposet of the heap of $w$, any linear extension of the heap of $h$ can be extended to a linear extension of the heap of $w$. To be precise, let $p_1 < p_2 < \cdots < p_k$ be a linear extension of the heap poset of $w$ and suppose that the interval $p_i < p_{i+1} < \cdots < p_j$ of this linear extension consists exactly of the entries from the heap of $h$. If we add lattice points to the heap of $h$ in the order $p_{i-1}, p_{i-2}, \ldots, p_1, p_{j+1}, p_{j+2}, \ldots, p_k$ then at each step we sequentially add lattice points that are maximal or minimal in the heap poset of $w$ restricted to the lattice points that were added in previous steps. If the heap of $w$ contains multiple connected components then we may add an entry at some stage to start the new component and this entry will be unrelated to the points previously added. Eventually, we add all of the lattice points and obtain the heap of $w$.

Next, suppose that the shifted copy of the heap of $h$ occupies columns $s, s+1, \ldots, t$ in the heap of $w$. Then, we begin with the set of strings $S = \{s, s+1, \ldots, t+1\}$ that appear in the shifted copy of the heap of $h$. These strings initially correspond to the 1-line pattern $h$, and we show by induction that $S$ continues to encode a 1-line pattern from $U^P(h)$ as we add minimal or maximal lattice points to the heap. Consider the relative order of the strings in $S$ when we add a maximal lattice point to the heap. Since the point is maximal, the strings being crossed by the point are adjacent. Thus, if the new point crosses a pair of strings that are both in $S$, then the new string configuration on $S$ corresponds to an element in $U^P(h)$. If the new point crosses a pair of strings such that at most one is contained in $S$, then the string configuration on $S$ is unchanged. Similarly, the relative order of the strings in $S$ corresponds to an element in $U^P(h)$ when we add a minimal lattice point, since the strings being crossed at each stage are adjacent.

At the end of this inductive construction, $w$ contains the 1-line pattern encoded by the strings in $S$, and the element corresponding to this 1-line pattern heap-contains $h$. \hfill $\square$

The converse of Proposition 4.1 can fail in general, as demonstrated in Example 11.1 of [BJ06]. However, on the special patterns defined below a converse can be stated.
Definition 4.2. Let \( p \in S^P \). Then, we say that \( p \) is an ideal pattern in \( S^P \) if for every \( q \in S^P_{r(p)+1} \) containing \( p \) as a 1-line pattern, we have that \( q \) heap-contains \( p \).

This finite test extends to permutations of all ranks according to the following result.

Proposition 4.3. If \( h \in S^P \) is an ideal pattern and \( w \in S^P \) contains \( h \) as a 1-line pattern, then \( w \) heap-contains \( h \).

Proof. Consider the case that \( w \in S^P \). Then by Definition 4.2 we have that \( w \) heap-contains \( h \), so we can highlight an instance of the heap of \( h \) inside the heap of \( w \).

By induction, assume the proposition holds for all elements in \( \bigcup_{k=1}^{n} S^P_k \) and let \( w \in S^P_{n+1} \). If \( w \) contains \( h \) as a 1-line pattern then \( w \) contains some \( h' \in S^P_n \) as a 1-line pattern, with the property that \( h' \) heap-contains \( h \), and we want to show that the heap of \( w \) must also contain a copy of the heap of \( h \).

The string diagram imposed on the heap of \( w \) can be obtained from the string diagram on the heap of \( h' \) by adding one additional string. The additional string will add extra points to the heap at each crossing. This string may cut through the highlighted copy \( C \) of the heap of \( h \), but since \( h \) is ideal, the extra points that are added together with \( C \) must heap-contain \( h \) by Definition 4.2. Therefore, \( w \) heap-contains \( h \) as a subheap.

Thus combining Proposition 4.1 and Proposition 4.3, we have shown the following result.

Theorem 4.4. Suppose \( S^P(H) \) is the subset of permutations characterized by avoiding a finite set \( P \) of 1-line patterns and heap-avoiding a finite set \( H \) of permutations. If each of the elements in \( P' = \bigcup_{h \in H} U^P(h) \) is an ideal pattern, then \( S^P(H) = S^{P \cup P'} \), so is characterized by avoiding the permutations in \( P \cup P' \) as 1-line patterns.

Corollary 4.5. Let \( w \) be any permutation. Then, \( w \) is freely-braided and hexagon-avoiding if and only if \( w \) avoids

\[
\{[3421], [4231], [4312], [4321], [46718235], [46781235], [56718234], [56781234]\}
\]

as 1-line patterns. Also, \( w \) is maximally-clustered and hexagon-avoiding if and only if \( w \) avoids

\[
\{[3421], [4312], [4321], [46718235], [46781235], [56718234], [56781234]\}
\]

as 1-line patterns.

Proof. It is straightforward to verify that

\[
U^P([46718235]) = \{[46718235], [46781235], [56718234], [56781234]\}
\]

for \( P \in \{\) maximally-clustered, freely-braided\} and each of these patterns are ideal. The corollary then follows from Theorem 4.4.

Remark 4.6. Theorem 4.4 can also be applied to study classical permutation pattern classes using heap-avoidance. If \( P \) is an upper order ideal in 2-sided weak Bruhat order consisting of connected ideal patterns, and \( P \) has finitely many minimal elements \( H \), then \( S^P = S(H) \).

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APPENDIX A.

The idea of the following algorithm is that each \([(m+2)23\ldots(m+1)1]\)-instance in the 1-line notation for \(w\) corresponds to a length \(2m+1\) braid cluster in the reduced expression \(w\) that we build. Since each of the \([(m+2)23\ldots(m+1)1]\)-instances contributes exactly \(m\) to the number of [321]-instances \(N(w)\), we have that \(w\) is contracted by Definition 1.5.

Algorithm A.1. (Produce a contracted expression for a maximally clustered permutation.)

1. Given a maximally clustered permutation \(w\) in 1-line notation, choose the unique [321]-instance in positions \((i, j, k)\) such that \(j\) is leftmost. This [321]-instance may be part of a larger \([(m+2)23\ldots(m+1)1]\)-instance corresponding to a braid cluster.
2. Then \(i\) can always be brought to the right by a reduced sequence of length-decreasing moves so that \(i\) is made adjacent to \(j\).
3. At this point, we can mark positions

\[
(\ldots, i, j, \ldots, h, \ldots, k, \ldots)
\]

in the 1-line notation for \(w\) where all entries weakly right of \(h\) and strictly left of \(k\) have value greater than \(i\). Those entries lying strictly left of \(h\) and right of \(j\) have values between those of \(i\) and \(j\), and include no descents.
4. If there exists some other \([(m'+2)23\ldots(m'+1)1]\)-instance that uses \(k\) then it occurs in the segment between \(h\) and \(k\). The instance can be made consecutive, and we apply a braid cluster to bring \(k\) to the left. Otherwise, \(k\) does not participate in any other \([(m'+2)23\ldots(m'+1)1]\)-instance and we simply move \(k\) to the left by adjacent transpositions.
5. Eventually, \(k\) will be moved past \(h\). We apply a braid cluster to undo the consecutive instance in positions \((i, j, \ldots, h)\).
6. Repeat from (1), until there are no more \([(m+2)23\ldots(m+1)1]\)-instances for any \(m\).

Proof. We will show that in a reduced fashion the algorithm makes every \([(m+2)23\ldots(m+1)1]\) pattern instance consecutive and then applies a braid cluster to undo the \([(m+2)23\ldots(m+1)1]\) instance. In particular, we must never move an entry that plays the role of 3 past an entry that plays the role of 2 in some [321] instance, unless we are applying a braid cluster to an instance that has already been made consecutive. Similarly, we cannot move an entry that plays the role of 2 past an entry that plays the role of 1 in some [321] instance unless the move is part of a braid cluster.

First, observe that no entry can play the role of 2 in more than one [321]-instance. To see this, suppose that the 2 in 321 is used in some other instance 3′21′. If 3′ is not distinct from 3, then we have a forbidden pattern when we attempt to place 1′ among the 321 entries:

\[
32(1.5)1 = [321]; \quad 321(1.5) = [312]; \quad 32(0.5)1 = [312]; \quad 321(0.5) = [312]
\]

Here we have indicated the value of 1′ in parentheses, and the corresponding permutation pattern where the relative values have been normalized to form a permutation in \(S_4\). Otherwise, 3′ is distinct from 3 and then we have a forbidden pattern when we attempt to place 3′ among the 321 entries:

\[
(2.5)321 = [321]; \quad 3(2.5)21 = [321]; \quad (3.5)321 = [321]; \quad 3(3.5)21 = [321]
\]

Hence if an entry plays the role of 2 in any [321]-instance then the instance is uniquely determined, proving (1).

A length-decreasing move in (2) is always available since otherwise we have a forbidden pattern 3(3.5)21 = [3421]. By the leftmost choice in (1), we never move \(i\) past another entry that plays the role

2. In any instance with \(i\).

Next, note that if any entry between \(j\) and \(k\) has value less than the value at \(j\), then we have a forbidden [4312] or [4321] instance. Moreover, if an entry located between \(j\) and \(k\) with value greater than the value
at \( i \) occurs to the left of an entry located between \( j \) and \( k \) with value less than the value at \( i \), then we have a forbidden \([3421]\)-instance. Therefore, all entries with value greater than the value at \( i \) occur to the right of all the entries with values between those of \( j \) and \( i \). Also, if there is a descent among the entries with value less than the value at \( i \) then we have a forbidden \([4321]\)-instance so these values are all increasing. Hence, the 1-line notation for \( w \) has marked positions \((\ldots,i,j,k,\ldots)\) as described in (3).

To prove (4) note that \( k \) can always be moved to the left in a length decreasing fashion, or else we obtain a \([4312]\) pattern. It remains to check that we can consecutively undo any other \([m+2]\)\(\ldots\)\((m+1)1]\)-instance in which \( k \) participates. Observe that \( k \) cannot play the role of 3 nor 2 in another pattern, or we obtain a forbidden \([4321]\) pattern.

Hence, we suppose that \( k \) plays the role of 1 in an instance of the form \((m+2)\ldots2'3'\ldots(m+1)'1\) where \((m+2)\)' must occur in a position weakly to the right of \( h \) because there are no descents among the entries with values < \( i \). Also, the entries \( 2',3',\ldots,(m+1)' \) must be consecutive, since any entry with value > \((m+2)'\) among these yields a forbidden \([3421]\) pattern, while any entry with value < 1 yields a forbidden \([4312]\) pattern.

Then the entries of our 1-line notation are of the form

\[32\ldots(m+2)'\ldots2'3'\ldots(m+1)'1\]

and we can assume that \(2'3'\ldots(m+1)'1\) and 32 have been made consecutive by previous steps. A length-decreasing move to make \((m+2)'\) closer to \(2'\) is always available since otherwise we have a \([3421]\) pattern. The only other obstruction is an entry \(2''\) that participates with \((m+2)\)' in another \([321]\] instance. However, if \(2''\) is not part of the \((m+2)'\ldots2'3'\ldots(m+1)\) cluster, then we must have that \(2''\) has value less than that of 1. Hence, we obtain a \([4312]\) pattern from the \(322''1\) entries.

It is evident that (5) and (6) can be accomplished using the previous steps of the algorithm.

Since the permutation is assumed to be finite, and we reduce the length at each step, the algorithm eventually terminates. By construction, the reduced expression we produce from the algorithm has a braid cluster for each \([m23]\ldots(m-1)1]\)-instance. Hence, it is maximally clustered. \(\square\)

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