Q-balls and the chiral vortical effect
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Abstract
In this paper we try to show the possibility of existence of an axial current directing along the axis of rotation in the presence of a non-topological soliton (the Q-ball). For that purpose we’ll analyse the case when the Q-ball will be carrying an axial charge. And this condition will permit us to write terms in the effective action, which will be responsible for the existence of an axial current and even for the chiral magnetic effect.

1 Introduction
The chiral vortical and magnetic effects have attracted a lot of attention recently. For example, they were investigated in [1][2][3]. As an example we may consider a medium consisting of chiral fermions, where conserved chiral charge $Q_5$ exists, and one can introduces chemical potential $\mu_5$. If anyone mentions the chiral magnetic effect, he understands the phenomenon of emergence of electric current $\vec{j}_{el}$ directed along external magnetic field $\vec{B}$ applying to a chiral medium with a non-vanishing $\mu_5$

$$\vec{j}_{el} = \frac{q^2}{2\pi^2} \mu_5 \vec{B},$$ (1)

where $\vec{B}$-magnetic field, $q$ is the charge of the constituent fermion.

And by the chiral vortical effect one understands the phenomenon of emergence of the chiral current in the presence of nontrivial vorticity $\omega_\mu = \frac{i}{2} \epsilon_{\mu\nu\lambda\rho} u^\nu \partial^\lambda u^\rho$, where $u^\nu$ is 4-velocity of the medium.

$$\vec{j}_5 = \frac{1}{2\pi^2} \mu_5^2 \vec{\omega},$$ (2)

and the axial current directed along the axis of rotation.

The most interesting thing to note is that coefficients in front of the vorticity also as in front of the magnetic field are connected with the chiral anomaly. There is the standpoint that, if we deal with a strongly interacting system, the coefficient in (1) in front of the magnetic field is not influenced by the renormalization and can be calculated explicitly [2]. In Son and Surowka’s work [1] it was proved the possibility to come to the conclusion about existence of chiral magnetic and chiral vortical effects only by consideration of hydrodynamic approximation. Moreover, in the work [1] it was not supposed that to obtain these results we need some special knowledge about the properties of the medium, but only thermodynamic parameters, currents and the energy-momentum tensor. It seems interesting to find a model in which it would be possible to check the results described above. In this paper we show that one of such possibilities is given by Q-balls [4], and we try to find the configuration of the field, which describes the state with the minimum energy for the given charge and an analog of the chemical potential. In this case it is possible to construct an effective action for Q-balls in an external magnetic field and for a rotating Q-ball. From this action we may calculate chiral and electric currents, and we can see that the coefficients in front of the rotational quantum number.

2 Description of introduction of the chemical potential
The chiral vortical effect is connected with the chiral chemical potential, so, we need to understand how it may be introduced into the theory. Chemical potentials are introduced to simulate the effects associated with the presence of the medium. They are conjugate to corresponding conserved charges.

$$\delta H = \sum_i \mu_i Q_i^i$$ (3)

From the thermodynamic point of view chemical potential is the energy required to add a single particle carring quantum numbers of corresponding charge into the medium without doing any work. With this in mind the full Hamiltonian is read as

$$H = H + H_{\text{int}} + \delta H$$ (4)

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2We mean that it plays the role similar to the vorticity, because, in case of quantum system, the rotaion is described by rotational quantum number and by direction of the axis of rotation
If we have the energy - charge equation, we may ask a question: "What is the ground state of the system for a given charge?" The answer to this question will lead us to the concept of the Q-ball. The most surprising fact is that we are able to obtain such a ground state, even in the case of a strongly interacting system.

3 Q-ball soliton and chiral currents

In this section we discuss the construction of Q-balls in the field theory. We need to add some remarks connected with our work. First of all, if we decide to introduce axial charge, we must require conservation of the axial current. That means that we must put quark’s masses to zero because of equation

$$\partial_{\mu} j^{5\mu} = 2i m \bar{\psi} \gamma^5 \psi.$$ (5)

The model described in the work [5] allows us to avoid the problem associated with the mass of the scalar particle, we are talking about Nonlinear sigma model with vanishing mass parameter.

As it was mentioned above, the conception of Q-balls arises as the answer to the question about the ground state of the system carrying conserved charge $Q$. In the case under consideration it is axial charge $Q_5$ connected with $U(1)$ axial symmetry. Let’s study a case of 4 dimensions with action $S_{\text{field}} = \int d^4x (T - V)$, where $T$ indicates free part of the action, and $V$ is connected with interactions. For better understanding we try to illustrate our ideas for particular case of abelian group only with $U(1)_a$ with action $S = \int d^4x \partial_{\mu} \phi^* \partial_{\mu} \phi - V(|\phi|)$. We are looking for solutions of the field equations in the form

$$\phi = \sigma(r) \exp(i \omega t)$$ (6)

For simple $U(1)$ theory, in the case of non-abelian group we will have some complications related with presence of group generators. Even in the simplest model there are several ways to find the soliton with the given charge and with the same value of $\omega$, and which is implemented depending on the type of potential $V$. Herein we describe a method, which, from our viewpoint, allows to identify the most important features for construction of an effective theory.

Let’s construct the current from the action for the $U(1)_a$, it has the form

$$J_{\mu} = -i(\phi^* \partial_{\mu} \phi - \phi \partial_{\mu} \phi^*)$$ (7)

The charge is zero component of this current $Q = \int d^3x J_0$, then we form the following expression:

$$\epsilon = E_f - \lambda Q$$ (8)

Let’s substitute (6) in it, then the expression becomes

$$\epsilon = \int d^3x (\omega^2 \sigma^2 + (\partial_i \sigma)^2 + V(\sigma) - 2\lambda \omega \sigma^2)$$ (9)

And now we need to find its extremum. This expression will allow us to find the ground state of the system. It seems a bit surprising that it’s possible even in the case of strongly interacting systems. Condition $\frac{\partial \epsilon}{\partial \omega}$ gives us $\lambda = \omega$. Also, we can find the explicit form of function $\sigma(r)$. In the simplest case it is constant till some definite radius.

Let’s analyze the obtained solution. We have find the ground state that having charge $Q_5$. One can see that $\omega$ plays the role similar to the chemical potential. If the charge is axial, so it is possible to find an expression for the chiral chemical potential.

4 Rotation

In this section of our work we try to realize an idea of creation of a solution, which describes the rotating Q-ball. Such well-developed solution is a very laborious task [6]. Seeing that qualitative possibilities of existence of axial and electromagnetic currents are interesting for us, for illustrative purposes, we focus on the simpler case of Q-vortex.

Let’s introduce polar coordinates in the plane $x,y$. To take into account the rotation of the field configuration, phase

$$\Phi = \sigma(r) \exp(i \alpha),$$ (10)

is to be represented in form

$$\alpha = \omega t + N \phi,$$ (11)
where $\phi$ is the polar angle. As we see, such substitution does not change the density of the Q-ball charge, due to
the fact, that $\partial_0 \phi = 0$, it’s quiet clear, that the phase of field $\Phi$ must change to a multiple of two pi when traversing
a closed loop around the point $r = 0$. Thus, we see, that $N$ has only integer values. Let’s seek minimum of field
configuration at the given $N$ and charge $Q$.

Let’s write the equation of motion

$$0 = \sigma'' + \frac{1}{r} \sigma' - \frac{N^2}{r^2} \sigma - \frac{dU(\sigma)}{d\sigma} + \omega^2 \sigma$$

(12)

Energy density per unit length is expressed as

$$E = \pi \int \omega^2 \sigma^2 + (\sigma')^2 + \frac{N^2}{r^2} \sigma^2 + U(\sigma)dr^2.$$

(13)

If $N = 0$, it is just the classical expression for the energy of the Q-ball. From the condition of energy finiteness we
find that $\sigma$ must vanish on the vortex core ($r = 0$). Let us consider the asymptotic behavior of this equation:

When $r \approx 0$

$$\sigma = Ar^N + O(r^{N+1})$$

(14)

and when $r - > \infty$

$$\sigma = \frac{B}{\sqrt{r}} \exp(-\sqrt{(U''(0) - \omega^2)r}).$$

(15)

In addition, it can be shown, that the mechanical moment is quantized in units of charge: $J = \int T_{\phi \phi} drd\phi = NQ$.
This predicating remains correct even when the Q-ball is considering [6].

5 Chiral effects and Q-ball

At the first step we consider a non-rotating Q-ball, which has a high charge, so high, that its radius $R \sim Q^{1/3}$ is
much greater than the Compton wavelength $E_{sol}^{-1} \sim Q^{-1}$. Then, in this limit, the Q-ball may be considered simply
as a classical field configuration.

Now let’s study simple $U(1)$ model for the Q-ball. We need to understand how anomalies arise in this task. For
this we consider a system on scales, where fermionic degrees of freedom exist. Then the full Lagrangian may be
expressed as

$$L = \int d^4x \bar{\psi}_i \gamma^\mu \partial_\mu \psi + \int d^4x L_{intfg} + \int d^4x L_{intfq},$$

(16)

where term $L_{intfg}$ is responsible for interacting of fermions with ”‘gluon’” fields, and term $L_{intfq}$ is responsible
for interacting of fermions and the Q-ball. Effective action must be invariant with respect to the axial symmetry
which acts on the fields as follows

$$\phi \rightarrow \exp(i\delta)\phi, \psi \rightarrow \exp(iq\gamma^5)\psi,$$

(17)

where $\delta$ - is corresponding charge of scalar field. To satisfy these requirements we select an interaction term as

$$L_{intfq} = \frac{1}{2} \bar{\psi} \partial_\mu Ln(\frac{\Phi}{\Phi*}) \gamma^\mu \gamma^5 \psi$$

(18)

the most simple case that is valid by symmetries, and we also include the interaction with $U(1)$-gauge field $q \bar{\psi} A_\mu \gamma^\mu \psi$.
To obtain the final expression for the current we need to integrate over the ”‘gluon’” fields. In article [7] they
explain the way of doing that. Then, using the Fujikawa-Vergeles method [8], we find currents under fermionic
field transformations

$$\psi \rightarrow exp(i\gamma^5\alpha + i\beta)\psi$$

(19),

where $\alpha, \beta$ are independent parameters of transformation. As a result we will have two currents. The first one is
an ordinary electromagnetic current:

$$\partial_\mu j^\mu = -\frac{q}{4\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu A_\nu \partial_\lambda \partial_\rho ln(\frac{\Phi}{\Phi*})$$

(20)

3under the term ”‘gluon’” we understand all the other fields that are responsible for the formation of bosonic degrees of freedom
4for simplicity we will set it equal to 1,
On larger scales we have no fermionic degrees of freedom, but the anomaly is still there. Thus we see, there is a macroscopic current

$$ J^\mu = -\frac{q}{4\pi^2} \varepsilon^{\mu \nu \lambda \rho} \partial_\lambda \ln \frac{\Phi}{\Phi^*} \partial_\nu A_\rho, $$

(21)

if we insert the equation for the Q-ball in it, we can see, that the electric current exists there, is

$$ \vec{J} = -\frac{q}{2\pi^2} \omega \vec{B}, $$

(22)

it also has been shown in [5]. One can see that within this theory we may come to the conclusion of the existence of the effect similar to the chiral vortical effect. The fact is, that in the axial current term

$$ J_5^\mu = \frac{1}{16\pi^2} \varepsilon^{\mu \nu \lambda \rho} \partial_\nu \ln \frac{\Phi}{\Phi^*} \partial^\lambda \partial^\rho \ln \frac{\Phi}{\Phi^*}, $$

(23)

arises. As we see, we can give rotation to the Q-ball by replacing $\omega t \rightarrow \omega t + n\phi$, where we introduce the cylindrical coordinates, and angle $\phi$ is the angle in the xy plane. In the expression for the axial current there is the commutator of two derivatives, and for any smooth function this commutator is equal to zero. And still, in case under consideration we may prove that:

$$ \frac{1}{4\pi} \oint \partial_i \ln \frac{\Phi}{\Phi^*} dx^i = n $$

(24)

Using the Stokes’ theorem we see $\varepsilon^{ijk} \partial_j \partial_k \ln \frac{\Phi}{\Phi^*} = 4\pi \delta(x, y)n$, taking into account this relation, we find that the expression for the axial current becomes

$$ J_5^z = \frac{1}{2\pi} \omega n \delta(x, y). $$

(25)

We integrate this current in the plane x,y and find that the axial current that is directed along the axis of rotation and equal to

$$ J_5^z = \frac{1}{2\pi} \omega n, $$

(26)

must be there. Thus, in the simplest case, we can prove existence of the chiral magnetic effect and the effect similar to the chiral vortical effect.

6 General case

In the simplest case we succeed in fixing the existence of an effect similar to the chiral vortical effect in the case of existence of the chiral magnetic effect, and the role of the chiral chemical potential $\mu_5 \omega$ plays. But such description is not accurate enough. The reason is, that, if the interaction term has the form $\frac{1}{2} \bar{\psi} \partial_\mu \ln(\frac{\Phi}{\Phi^*}) \gamma^\mu \gamma^5 \psi$, and then, as one can easily find, the existence of the Q-ball is reduced to the appearance of the ”chiral chemical potential” in the current, and it is independent of the configuration of the field $\sigma$. But symmetries allow to write a much more general case of interaction. Now we need to modify the action for the purpose to take it into account. The most general term we can write

$$ +g\bar{\psi} \gamma^\mu \gamma^5 f(...){\partial_\mu} \zeta \psi, $$

(27)

where $\zeta$-describes the phase of the Q-ball, and g is the coupling constant, nd f-scalar function, that is invariant with respect to the axial transformations. Next, we define $f_\mu = g f_\mu \zeta$, where $\zeta$-denotes the phase of the Q-ball. In order to obtain the chiral magnetic effect we also need to consider the interaction of fermions with U(1) gauge field, to do this we add to the Lagrangian term $q \bar{\psi} A_\mu \gamma^\mu \psi$. Now, using the method of Fujikawa-Vergeles [8] we will find 4-divergence of the axial and vector currents

$$ \partial_\mu j^\mu = \frac{1}{2\pi^2} q \bar{\psi} \varepsilon^{\mu \nu \lambda \rho} \partial_\lambda A_\nu \partial_\rho f_\mu $$

(28)

$$ \partial_\mu j_\mu^\nu = -\frac{1}{4\pi^2} \varepsilon^{\mu \nu \lambda \rho} \partial_\rho f_\mu \partial_\lambda f_\nu $$

(29)

out of (28) we can see the existence of a macroscopic electrical current

$$ J^\mu = -\frac{q}{2\pi^2} \varepsilon^{\mu \nu \lambda \rho} f_\rho \partial_\lambda A_\nu $$

(30)
Let’s consider the case of the mentioned above non-rotating Q-ball, when \( \phi = \omega t \), and if we neglect with effects, associated with the boundary of the Q-ball, the current becomes

\[
J = -\frac{1}{2\pi^2} g f \omega \vec{B}
\]  

(31)

where \( B^a = \frac{1}{2} \epsilon^{abc} F_{bc}, a, b, c = 1, 2, 3 \), and we have the results of the preceding paragraph, connectd with the chiral magnetic effect \[7\], where the value \( g f \omega = \mu_5 \) plays the role of the “chiral chemical potential”. Let us now consider how changes in the action affect the existence of the axial current We substitute the explicit dependence of the phase \( \zeta = \omega t + n\phi \), \( \phi \) - the polar angle. Then, the axial current along the axis of rotation (in this case - \( z \)) is given by

\[
J^z = \frac{1}{4\pi^2} f^2 g^2 n |\partial_x \partial_y - \partial_x \partial_y| \phi
\]  

(32)

and thus we again meet the situation that the commutator in (31) vanishes everywhere except at the origin of coordinates, so, the presence of the effect is determined by the behavior of function \( f \). If we take as \( f \), for example, the function \( g\sigma^2 \) with asymptotic behavior (14), it is easy to see the axial current will not exist.

6 Conclusion

In this work we have considered the effective theory describing the origin of the chiral magnetic effect and the existence of a chiral current directed along the axis of rotation in presence of the axial Q-ball. It is shown that the presence of the chiral magnetic effect can be fixed only on the basis of consideration of symmetries. As for chiral current directed along the Q-ball axis of rotation its existence depends on the details of interaction.

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