A COMPANION OF OSTROWSKI LIKE INEQUALITY FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $L^\infty$ SPACES AND APPLICATIONS

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Abstract. A companion of Ostrowski like inequality for mappings whose second derivatives belong to $L^\infty$ spaces is established. Applications to composite quadrature rules, and to probability density functions are also given.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality (see [15]) for differentiable mappings with bounded derivatives:

**Theorem 1.1.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ whose derivative is bounded on $(a, b)$ and denote $\|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have

$$
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \cdot (b-a) \|f'\|_\infty.
$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

Motivated by [12], Dragomir [8] proved some companions of Ostrowski’s inequality, as follows:

**Theorem 1.2.** Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then the following inequalities

$$
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} 
\left[ \frac{1}{\pi} + \frac{2 \frac{x-a}{b-a}}{\left( \frac{x-a}{b-a} \right)^{q+1}} \right] (b-a) \|f'\|_\infty, & f' \in L^\infty[a, b], \\
\left[ \frac{1}{\pi} + \frac{2 \frac{x-a}{b-a}}{\left( \frac{x-a}{b-a} \right)^{q+1}} \right] (b-a)^{1/q} \|f'\|_p, & f' \in L^p[a, b], \\
\left[ \frac{1}{\pi} + \frac{2 \frac{x-a}{b-a}}{\left( \frac{x-a}{b-a} \right)^{q+1}} \right] \|f'\|_1, & f' \in L^1[a, b]
\end{cases}
$$

hold for all $x \in [a, \frac{a+b}{2}]$.

Recently, Alomari [1] introduced a companion of Dragomir’s generalization of Ostrowski’s inequality for absolutely continuous mappings whose first derivatives are belong to $L^\infty([a, b])$.

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Theorem 1.3. Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous mappings on \((a, b)\) whose derivative is bounded on \([a, b]\). Then the inequality

\[
\left\| \left( (1 - \lambda) \frac{f(x) + f(a + b - x)}{2} + \lambda \frac{f(a) + f(b)}{2} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right\|
\]

(1.3)

\[
\leq \frac{1}{8} (2\lambda^2 + (1 - \lambda)^2) + 2 \left( \frac{x - \left( \frac{3\lambda a + (1 - \lambda)b}{4} \right)}{(b - a)^2} \right)^2 (b - a) \left\| f' \right\|_{\infty}
\]

holds for all \( \lambda \in [0, 1] \) and \( x \in [a + \frac{b - a}{2}, \frac{a + b}{2}] \).

In (1.3), choose \( \lambda = \frac{1}{8} \), one can get

\[
\leq \frac{1}{32} + 2 \left( \frac{x - \frac{5a + 3b}{8}}{(b - a)^2} \right)^2 (b - a) \left\| f' \right\|_{\infty}.
\]

(1.4)

And if choose \( x = \frac{a + b}{2} \), then one has

\[
\leq \frac{1}{8} (b - a) \left\| f' \right\|_{\infty}.
\]

It’s shown in [1] that the constant \( \frac{1}{8} \) is the best possible.

In related work, Dragomir and Sofo [10] developed the following Ostrowski like integral inequality for twice differentiable mapping.

Theorem 1.4. Let \( f : [a, b] \to \mathbb{R} \) be a mapping whose first derivative is absolutely continuous on \([a, b]\) and assume that the second derivative \( f'' \in L^\infty([a, b]) \). Then we have the inequality

\[
\left\| \frac{1}{2} \left[ f \left( \frac{a + b}{2} \right) + \frac{f(a) + f(b)}{2} \right] - \frac{1}{b - a} \int_a^b f(t) \, dt \right\|
\]

(1.6)

\[
\leq \frac{1}{48} + \frac{1}{3} \left| \frac{x - \frac{a + b}{2}}{(b - a)^3} \right| (b - a)^2 \left\| f'' \right\|_{\infty},
\]

for all \( x \in [a, b] \).

In (1.6), the authors pointed out that the midpoint \( x = \frac{a + b}{2} \) gives the best estimator, i.e.,

\[
\leq \frac{1}{48} (b - a)^2 \left\| f'' \right\|_{\infty}.
\]

(1.7)

In fact, we can choose \( f(t) = (t - a)^2 \) in (1.7) to prove that the constant \( \frac{1}{48} \) in inequality (1.7) is sharp.

For other related results, the reader may be refer to [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] and the references therein.

Motivated by previous works [1] [6] [8] [10], we investigate in this paper a companion of the above mentioned Ostrowski like integral inequality for twice differentiable mappings. Our result gives a smaller estimator than (1.7) (see (2.9) below). Some applications to composite quadrature rules, and to probability density functions are also given.
The following companion of Ostrowski like inequality holds:

**Theorem 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be a mapping whose first derivative is absolutely continuous on \([a, b]\) and assume that the second derivative \( f'' \in L^\infty([a, b]) \). Then we have the inequality

\[
\left| \frac{1}{b-a} \int_a^b K(t) f''(t) dt \right| \leq \frac{1}{3} \left\{ \frac{1}{(b-a)^2} (a+b-x)^2 + \frac{1}{3} \frac{(a+b-x)^3}{(b-a)^3} \right\} (b-a)^2 \| f'' \|_\infty
\]

for all \( x \in [a, b] \). The first constant \( \frac{1}{3} \) in the right hand side of (2.1) is sharp in the sense that it can not be replaced by a smaller one provided that \( x \neq \frac{a+b}{2} \) and \( x \neq a \).

**Proof.** Define the kernel \( K(t) : [a, b] \to \mathbb{R} \) by

\[
K(t) := \begin{cases} 
  t - a, & t \in [a, x], \\
  t - \frac{a+b}{2}, & t \in (x, a+b-x], \\
  t - b, & t \in (a+b-x, b],
\end{cases}
\]

for all \( x \in [a, \frac{a+b}{2}] \). Integrating by parts, we obtain (see [8])

\[
\frac{1}{b-a} \int_a^b K(t) g'(t) dt = \frac{g(x) + g(a+b-x)}{2} - \frac{1}{b-a} \int_a^b g(t) dt.
\]

Now choose in (2.3), \( g(x) = (x - \frac{a+b}{2}) f'(x) \), to get

\[
\frac{1}{b-a} \int_a^b K(t) \left[ f'(t) + \left( t - \frac{a+b}{2} \right) f''(t) \right] dt = \frac{1}{2} \left( x - \frac{a+b}{2} \right) [f'(x) - f'(a+b-x)] - \frac{1}{b-a} \int_a^b \left( t - \frac{a+b}{2} \right) f'(t) dt.
\]

Integrating by parts, we have

\[
\frac{1}{b-a} \int_a^b \left( t - \frac{a+b}{2} \right) f'(t) dt = \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt.
\]

Also upon using (2.3), we get

\[
\frac{1}{b-a} \int_a^b K(t) \left[ f'(t) + \left( t - \frac{a+b}{2} \right) f''(t) \right] dt = \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b K(t) \left( t - \frac{a+b}{2} \right) f''(t) dt.
\]
It follows from (2.4)–(2.6) that
\[
\frac{1}{2(b-a)} \int_a^b K(t) \left( t - \frac{a+b}{2} \right) f''(t) dt \\
= \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} \left[ \frac{f(x) + f(a+b-x)}{2} + \frac{f(a) + f(b)}{2} \right] \\
+ \frac{1}{2} \left( x - \frac{a+b}{2} \right) f'(x) - f'(a+b-x).
\]
(2.7)

Now using (2.4) we obtain
\[
\frac{1}{2} \left[ \frac{f(x) + f(a+b-x)}{2} + \frac{f(a) + f(b)}{2} \right] \\
- \frac{1}{2} \left( x - \frac{a+b}{2} \right) f'(x) - f'(a+b-x) - \frac{1}{b-a} \int_a^b f(t) dt
\]
\[\leq \frac{\|f''\|_{\infty}}{2(b-a)} \int_a^b |K(t)| \left| t - \frac{a+b}{2} \right| dt.
\]
(2.8)

Since \( x \in [a, a+b] \), we have
\[
I := \int_a^b |K(t)| \left| t - \frac{a+b}{2} \right| dt \\
= \int_a^x (t-a) \left| t - \frac{a+b}{2} \right| dt + \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right)^2 dt + \int_{a+b-x}^{b} (b-t) \left| t - \frac{a+b}{2} \right| dt \\
= \int_a^x (t-a) \left( \frac{a+b}{2} - t \right) dt + \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right)^2 dt + \int_{a+b-x}^{b} (b-t) \left( t - \frac{a+b}{2} \right) dt \\
= \frac{(a+3b-4x)(x-a)^2}{12} + \frac{2}{3} \left( \frac{a+b}{2} - x \right)^3 + \frac{(a+3b-4x)(x-a)^2}{12} \\
= \frac{(a+3b-4x)(x-a)^2}{12} + \frac{2}{3} \left( \frac{a+b}{2} - x \right)^3,
\]
and referring to (2.8), we obtain the result (2.1).

The sharpness of the constant \( \frac{1}{3} \) can be proved in a special case for \( x = \frac{a+b}{2} \) (see the line behind (1.4)). \( \square \)

**Remark 1.** If we take \( x = \frac{a+b}{2} \) in (2.1), we recapture the sharp inequality (1.7). If we take \( x = a \) in (2.1), we obtain the perturbed trapezoid type inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{b-a}{8} [f'(b) - f'(a)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{24} \|f''\|_{\infty},
\]
which has a smaller estimator than the sharp trapezoid inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{8} \|f''\|_{\infty}
\]

stated in [11, Proposition 2.7].
Proof. Consider Remark 2.

\[ F(x) = \left( \frac{a + 3b}{4} - x \right) (x - a)^2 + \left( \frac{a + b}{2} - x \right)^3 \]

for \( x \in [a, \frac{a+b}{2}] \). It's easy to know that \( F(x) \) obtains its minimal value at \( x = \frac{3a+b}{4} \). Therefore, in (2.1), the point \( x = \frac{3a+b}{4} \) gives the best estimator, i.e.,

\[
\left| \frac{1}{2} \left[ f\left( \frac{3a+b}{4} \right) + f\left( \frac{a+b}{2} \right) \right] + \left( \frac{b-a}{8} \right) \left( \frac{f'\left( \frac{3a+b}{4} \right) - f'\left( \frac{a+b}{2} \right)}{2} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right| 
\]

(2.9) \[ \leq \frac{1}{64} (b-a)^2 \| f'' \|_\infty, \]

the right hand side of which is smaller than that of (1.7).

3. Application to Composite Quadrature Rules

In [10], the authors utilized inequality (1.6) to give estimates of composite quadrature rules which was pointed out have a markedly smaller error than that which may be obtained by the classical results. In this section, we apply our previous inequality (2.1) to give us estimates of new composite quadrature rules which have a further smaller error.

**Theorem 3.1.** Let \( I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \) be a partition of the interval \([a, b]\), \( h_i = x_{i+1} - x_i \), \( \nu(h) := \max\{h_i : i = 1, \cdots, n\} \), \( \xi_i \in [x_i, \frac{x_i + x_{i+1}}{2}] \), and

\[
S(f, I_n, \xi) = \frac{1}{4} \sum_{i=0}^{n-1} [f(x_i) + f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f(x_{i+1})] h_i
\]

\[- \frac{1}{4} \sum_{i=0}^{n-1} h_i \left( \frac{\xi_i - x_i + x_{i+1}}{2} \right) [f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)],
\]

then

\[
\int_a^b f(x)dx = S(f, I_n, \xi) + R(f, I_n, \xi)
\]

and the remainder \( R(f, I_n, \xi) \) satisfies the estimate

\[
|R(f, I_n, \xi)| \leq \frac{1}{3} \| f'' \|_\infty \left[ \sum_{i=0}^{n-1} \left( \frac{x_i + 3x_{i+1}}{4} - \xi_i \right) (x_i - \xi_i)^2 + \sum_{i=0}^{n-1} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right)^3 \right].
\]

**Proof.** Inequality (2.1) can be written as

\[
\left| \int_a^b f(t)dt - \frac{1}{4} \left[ f(a) + f(x) + f(a+b-x) + f(b) \right] (b-a) \right.
\]

\[+ \frac{b-a}{4} \left( x - \frac{a+b}{2} \right) \left| f'(x) - f'(a+b-x) \right| \]

(3.2) \[ \leq \frac{1}{3} \left\{ \left( \frac{a+3b}{4} - x \right) (x-a)^2 + \left( \frac{a+b}{2} - x \right)^3 \right\} \| f'' \|_\infty.
\]
Applying (3.2) on \( \xi_i \in [x_i, \frac{x_i + x_{i+1}}{2}] \), we have
\[
\int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4} \left[ f(x_i) + f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f(x_{i+1}) \right] h_i
+ \frac{h_i}{4} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) \left[ f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i) \right]
\leq \frac{1}{3} \left[ \left( \frac{x_i + 3x_{i+1}}{4} - \xi_i \right) (x_i - \xi_i)^2 + \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right) \right] \|f''\|_{\infty}.
\]

Now summing over \( i \) from 0 to \( n - 1 \) and utilizing the triangle inequality, we have
\[
\left| \int_a^b f(t)dt - S(f, I_n, \xi) \right|
= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4} \sum_{i=0}^{n-1} \left[ f(x_i) + f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f(x_{i+1}) \right] h_i
+ \frac{1}{4} \sum_{i=0}^{n-1} h_i \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) \left[ f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i) \right]
\leq \frac{1}{3} \|f''\|_{\infty} \sum_{i=0}^{n-1} \left[ \left( \frac{x_i + 3x_{i+1}}{4} - \xi_i \right) (x_i - \xi_i)^2 + \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right) \right]
\]
and therefore (3.1) holds.

**Corollary 3.1.** If we choose \( \xi_i = \frac{x_i + x_{i+1}}{4} \), then we have
\[
S(f, I_n) = \frac{1}{4} \sum_{i=0}^{n-1} \left[ f(x_i) + f\left( \frac{3x_i + x_{i+1}}{4} \right) + f\left( \frac{x_i + 3x_{i+1}}{4} \right) + f(x_{i+1}) \right] h_i
+ \frac{1}{16} \sum_{i=0}^{n-1} \left[ f'\left( \frac{3x_i + x_{i+1}}{4} \right) - f'\left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i^2
\]
and
\[
|R(f, I_n)| \leq \frac{1}{64} \|f''\|_{\infty} \sum_{i=0}^{n-1} h_i^3.
\]

**Remark 3.** It is obvious that inequality (3.3) is better than (10) inequality (3.1) due to a smaller error.

**4. Application to probability density functions**

Now, let \( X \) be a random variable taking values in the finite interval \([a, b]\), with the probability density function \( f : [a, b] \rightarrow [0, 1] \) and with the cumulative distribution function
\[
F(x) = Pr(X \leq x) = \int_a^x f(t)dt.
\]

The following result holds:
Theorem 4.1. With the above assumptions, we have the inequality
\[
\left| \frac{1}{2} \left[ \frac{F(x) + F(a + b - x)}{2} + \frac{1}{2} \right] - \frac{1}{2} \left( x - \frac{a + b}{2} \right) \frac{f(x) - f(a + b - x)}{2} - \frac{b - E(X)}{b - a} \right| \leq \left| \left( \frac{a + b}{2} - x \right) \frac{(x - a)^2}{3(b - a)^3} + \frac{1}{3} \left( \frac{a + b}{2} - x \right)^3 \right| (b - a)^2 \| f' \|_{\infty}
\]
(4.1)

for all \( x \in [a, \frac{a + b}{2}] \), where \( E(X) \) is the expectation of \( X \).

Proof. Follows by (2.1) on choosing \( f = F \) and taking into account
\[
E(X) = \int_{a}^{b} t dF(t) = b - \int_{a}^{b} F(t) dt,
\]
we obtain (4.1). \( \square \)

In particular, we have:

Corollary 4.1. With the above assumptions, we have the inequality
\[
\left| \frac{1}{2} \left[ \frac{F(a) + F(a + b)}{2} + \frac{1}{2} \right] - \frac{1}{2} \left( a - \frac{a + b}{2} \right) \frac{f(a) - f(a + b)}{2} - \frac{b - E(X)}{b - a} \right| \leq \frac{1}{64} (b - a)^2 \| f' \|_{\infty}.
\]

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