Efficient construction of 2-D cluster states with probabilistic quantum gates

Qing Chen1,2†, Jianhua Cheng2, Ke-Lin Wang2, and Jiangfeng Du1,2,3,4

1 Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei 230026, PR China
2 Department of Modern Physics, University of Science and Technology of China, Hefei 230026, PR China
3 Hefei National Laboratory for Physical Sciences at Microscale,
University of Science and Technology of China, Hefei, Anhui 230026, PR China
4 Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542

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We propose an efficient scheme for constructing arbitrary 2-D cluster states using probabilistic entangling quantum gates. In our scheme, the 2-D cluster state is constructed with star-like basic units generated from 1-D cluster chains. By applying parallel operations, the process of generating 2-D (or higher dimensional) cluster states is significantly accelerated, which provides an efficient way to implement realistic one way quantum computers.

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Quantum computation offers a potentially exponential computational speed-up over classical computers in certain tasks, which makes it one of the most promising ways for the future’s computations. Nowadays conceptual quantum computer with few qubits has been demonstrated experimentally in many systems, however, how to extend these systems to a large number of qubits, known as scalability still remains a question. While in most of the current experiments, quantum logic gates are implemented with sequences of highly controlled interactions between selected particles, Robert Raussendorf and Hans J. Briegel recently proposed a different model of a scalable quantum computer, namely the one way quantum computer, which constructs quantum logic gates by performing single qubit measurements on cluster states. In this model, the total resource needed for the quantum computation is a 2-D (two dimensional) cluster state prepared at numerous qubits (e.g. a lattice-like cluster), which serves as a “substrate” for the computation. After the preparation of this universal “substrate”, the remaining work is to perform single qubit measurements and, the final results are read out from those qubits that were not measured in the whole process.

Optical approaches to quantum computation are attractive because of the long decoherence time of photons and near perfect single qubit operations. However, due to the lack of interactions between photons, the main challenge within these approaches is the realization of two-qubit logic gates. A breakthrough is the Knill, Laflamme and Milburn’s (KLM) scheme, in which an entangling gate was proposed based on linear optics. Nevertheless, to implement a near-deterministic gate within these approaches is the realization of two-qubit operations, which makes the present scalable quantum computation inefficient. Moreover, with the restriction of involving only static linear optical systems, there are limitations for the success probability of such kind of entangling gates. In recent works, the one way quantum computer scheme, carried out in a cluster state, was introduced to implement efficient scalable quantum computations with probabilistic entangling gates. Specifically, Ref. [11, 13] show efficient schemes to construct 1-D cluster states and suggest that 2-D cluster states be built from “T-pieces”. Later, L.-M. Duan et al. [12] presented an efficient scheme to construct 2-D lattice-like cluster states. To achieve this goal, they made use of a “cross” (“+” shape) cluster state as the basic unit, which is built from two long 1-D chain cluster states. In their proposal, the temporal overheads increase logarithmically with the qubit number $N$ and polynomially with $1/p$. This result opens up the promising prospect to realize efficient quantum computation with probabilistic entangling gates.

In this paper, we report that an arbitrary 2-D cluster state can be directly constructed from star-like cluster states (bottom right in Fig.1), which can be obtained by performing single qubit measurements on a chain-like cluster state (top in Fig.1). Moreover, in this scheme, the time consuming of generating a 2-D cluster state is nearly equal to that of generating a 1-D chain cluster state.

Specifically, following Duan’s assumptions, one can reliably perform two-qubit controlled phase flip (CPF) gates with a small success probability $p$. Also, all single qubit operations are regarded as perfect, which are well justified in linear optical systems. And in the calculation of temporal overheads, compared with the operation time of the CPF gate, we neglect the time consuming of all single-qubit operations including the single-qubit measurements.

Before stating our proposal, we list some important properties of cluster states for future use. Since cluster state is relevant to the graph theory, we start with graph language. For a given simple undirected graph $G = (V, E)$, where $V$ denotes the vertex set and $E(A,B) = \{(a,b) \in E : a \in A, b \in B, a \neq b\}$ denotes the set of edges between the vertices, one can define a set of commutable operators $K = \sigma_x \prod_{i \in N} \sigma_z$ regard-
ing every vertex of the graph, here \( i \) stands for an arbitrary vertex of the graph and \( N_i \) refers to the set of vertex \( i \)'s neighboring vertices, \( \sigma_{x,y,z}^i \) denote the Pauli matrices corresponding to the \( i \)-th vertex. The cluster state is then defined as the co-eigensates of the group of operators. The local Pauli measurements on cluster states induce some interesting results. Specifically, \( \sigma_{x}^i \) measurement deletes all edges incident to vertex \( i \), while \( \sigma_{y}^i \) measurement simply replaces the subgraph \( G[N_i] \) by its complement \( G[N_i]' \). In the case of the \( \sigma_{z} \) measurement, the resulting state is a cluster state associated with the graph

\[
G' = G \cup E(N_j, N_i) \cup E(N_j \cap N_i, N_j \cap N_i) \cup E\{j\}, N_i - \{j\},
\]

where \( E \cup F = (E \cup F) - (E \cap F) \), and \( j \in N_i \).

Next we will show that once the star-like cluster state (depicted in bottom right of Fig. 1) is prepared, one can construct an arbitrary 2-D cluster state directly, and the time consuming of this process is the single CPF gate operation time \( t_a \). For example, an \( N \) qubits lattice cluster state can be generated with \( N \) such basic units, each center qubit corresponding to one site of the lattice. Now we connect these basic units to a 2-D lattice-like cluster state via probabilistic CPF gates. Suppose each unit has \( n_l \) “arms”, there will be \( n_l/4 \) connections between each adjacent pair of sites, and all these operations are to be done at the same time (Note here parallel operations are introduced in this process). At least one successful connection is needed between each pair of sites. After the connections are completed, we keep one successful connection and remove the others by \( \sigma_{x} \) measurements on the qubits adjacent to the center qubits, no matter they succeed or not. Next, we trim the cluster state by applying \( \sigma_{y} \) measurements on the four redundant qubits of the only one kept successful connection, thus the required connections between center qubits are established and a 2-D lattice-like cluster is successfully constructed. Now we should find a proper \( n_l \) so that the lattice cluster state can be created at a satisfying probability. Let \( p_c \) be the probability of at least one connection succeeds between a pair of sites. And one can easily get \( p_c = 1 - (1 - p)^{n_l/4} \). Consequently, the total success probability of the lattice cluster state is \( P_s = p_c^{2N} \). Naturally we order \( P_s > 1 - \epsilon \), where \( \epsilon \) is a small positive number standing for the overall failure probability. If \( N \) is sufficiently large and \( p \) is small, we get \( n_l \sim (4/p) \ln(2N/\epsilon) \).

By now we showed that it is convenient to generate a 2-D cluster state with such basic units. Here we give a method to create these basic units by performing \( \sigma_{x} \) and \( \sigma_{z} \) measurements on chosen qubits of a certain cluster state (see top in Fig. 1). Given a 2\( n_l \) qubits (main chain length) cluster state with half of them having a two-qubit “arm”, by performing \( \sigma_{x} \) measurements on those qubits which do not have an “arm”, one achieves the expected star-like basic unit with \( n_l \) “arms” (In fact, there exists \( n_l \) one-qubit “arms”, which can be easily removed by performing \( \sigma_{z} \) measurements on them). Note that all operations involved in the process are single qubit measurements and local unitary transformations.

In a slightly different way from Duan et al’s proposal [12], the 1-D chain cluster state with “arms” can be generated efficiently. The detailed process is shown below.

Suppose we have some \( n_0 \) qubits cluster chains with half of the qubits having a two-qubit “arm” (\( n_0 \) stands for the number of qubits in the main chain and is supposed to be large enough), we connect two of them together with a probabilistic CPF gate. If this attempt fails, by applying a \( \sigma_{z} \) measurement, two qubits at the end of each cluster’s main chain are removed. And in the next round we connect the left two \( (n_0 - 2) \) cluster chains. The expected length of the connected cluster chain can be calculated as \( n_1 = \sum_{i=0}^{\lfloor n_0/2 \rfloor} 2(n_0 - 2i)p(1 - p)^i \approx 2n_0 - 4(1 - p)/p \). So if we want to get an expected longer cluster chain after a successful connection, \( n_0 \) should be larger than a critical length \( n_c = 4(1 - p)/p \). Iterating this process, one can get an arbitrary length cluster chain. After \( r \) rounds successful connection, the total chain length \( n_r \), the time consuming \( T_r \), and the total number of attempt \( M_r \) follow the recursion rules:
are:

\[ n_r = 2n_{r-1} - n_c, \]
\[ T_r = T_{r-1} + t_a/p, \]
\[ M_r = 2M_{r-1} + 1/p. \] (1)

In writing the recursion rule for \( T_r \), one should keep in mind that two cluster chains for each connection are assumed to be prepared in parallel. From the above three recursion rules, one gets \( n_r = (n_0 - n_c)2^r + n_c, T_r = T_0 + rt_a/p, \) and \( M_r = (M_0 + 1/p)2^r - 1/p. \) According to the first formula, \( r \) can be expressed with the main chain length \( n (n_c) \) after \( r \)th rounds successful connections, which can be written as \( r = \log_2[(n - n_c)/(n_0 - n_c)]. \)

Replacing \( r \) in the left two formulae with this equation, one finds that the total time consuming \( T(n) \) and the number of attempts \( M(n) \) scales with the main chain length \( n \) as

\[ T(n) = T_0 + (t_a/p) \log_2 [(n - n_c)/(n_0 - n_c)], \] (2)
\[ M(n) = (M_0 + 1/p) (n - n_c)/(n_0 - n_c) - 1/p, \] (3)

respectively, where \( T_0 \) and \( M_0 \) stand for the time and attempts needed for a \( n_0 \) qubits chain cluster state.

We now figure out the preparation of an \( n < n_c \) qubits cluster chain with the required “arms”. Started from single qubits, we first create four-qubit cluster chains, then connect two of them in the way illustrated in Fig.2. As a result, a four-qubit cluster chain with two two-qubit “arms” is constructed. By iterating this process, one can reach an arbitrary length such cluster chain. Generally speaking, given two \( 2^i (i > 1) \) qubits cluster chains, by applying a probabilistic CPF gate on them successfully, we can get a \( 2^{i+1} \) qubits cluster chain. If this attempt fails, we go back to the single qubit and try the whole process again. Let \( T_i \) and \( M_i \) be the preparation time and the number of attempts respectively, they obey the following recursion rules \( T_i = (1/p)(T_{i-1} + t_a) \) and \( M_i = (1/p)(2M_{i-1} + 1). \) With \( T_1 = t_a(1/p^2 + 1/p) \) and \( M_1 = 2/p^2 + 1/p \) (for preparing a four-qubit chain cluster), one gets the scaling rules \( T(n) \approx t_a(1/p)^{\log_2 n + 1} \) and \( M(n) \approx (2/p)^{\log_2 n + 1/2}. \)

To create an \( n > n_c \) cluster chain, one needs to combine the above two protocols. Let \( n_0 = n_c + 1, \) then \( T_0 = t_a(1/p)^{\log_2 (n_c+1)+1} \) and \( M_0 = (2/p)^{\log_2 (n_c+1)+1/2}. \)

Replacing these equations to Eq. (4) and (5), the overall time consuming \( T(n) \) and the number of attempts \( M(n) \) are:

\[ T(n) \approx t_a(1/p)^{\log_2 (n_c+1)+1} + (t_a/p) \log_2 (n - n_c), \] (4)
\[ M(n) \approx (2/p)^{\log_2 (n_c+1)+1} (n - n_c)/2. \] (5)

![Figure 3](image-url)

**FIG. 3**: \( T - p, T - \ln(2N/\epsilon) \) relations. \( T_1 \) (solid line) stands for time overheads in our scheme, while \( T_2 \) (dash line) as in Duan et al’s scheme [12, 17]. In Fig. 3a, we set \( p = 0.25. \) In Fig. 3b, parameter \( \ln(2N/\epsilon) \) is chosen as 30. The inserted figure in Fig. 3b plots dependence of \( T_1/T_2 \) on \( p. \) [19]

Taking the construction of a 2-D \( N \) qubits lattice-like cluster state as an example, we now calculate the time overheads and the number of attempt needed for constructing a 2-D cluster state. As stated above, if we want to ensure the total success probability of the construction to be larger than \( 1 - \epsilon, \) the basic unit should have \( n_1 = (4/p) \ln(2N/\epsilon) \) “arms”. To construct a square lattice cluster state of \( N \) qubits, we need \( N \) such basic units in total, and each basic unit is made from a 1-D \( 2n_l \) qubits chain cluster with half of the qubits in the chain having a two-qubit “arm”. Consequently, we need to prepare \( N \) such chain clusters (this can be done in parallel), which requires \( N M(2n_l) \) CPF attempts within a time period of \( T(2n_l) \) (refer to Eqs. (4) and (5) for expressions of \( M(n) \) and \( T(n) \)). The final step is to connect the basic units to form an \( N \) qubits square lattice cluster state. Note that all the connections can be done in parallel in this process, so the whole connection takes on average \((N/2)n_l\) CPF attempts in a single CPF operation time \( t_a. \) Summarizing these results, one gets the the overall time consuming and the total number of attempts.
From Fig. 3(b), we see that $T(N)$ varies from 0 to 0.2 as the detector efficiency goes from 0 to 1 (See Fig. [3]). Note that in BK scheme, the success probability $p$ varies from 0.1 to 0.5 as the detector efficiency goes from 0.5 to 1 (See Fig. [4]), which makes BK scheme an excellent physical implementation of our proposal. Also, we notice that the recently proposed “repeat-until-success” scheme can eventually lead to deterministic gates [22, 23], which sheds light on direct implementation of any scalable quantum logic circuit. However, in realistic systems, where photon loss cannot be ignored, this scheme can not produce entangling gates with unitary success probability. In this case, the one way computer scheme appears to be an effective alternative to implement scalable quantum computation, where our scheme proves to be useful.

Moreover, with our basic units, one can directly create an arbitrary 2-D cluster state, which needs not to be the lattice like cluster state. Fig. 5 shows a sketch map of a hexagonal cluster state generated with our basic unit. Following the same calculating process, one gets the time overheads $T(N)$ and the number of $M(N)$’s relation with parameters $N$ (qubit number), $p$, and $\epsilon$, as stated below:

\[
T(N) \simeq t_a (\frac{1}{p})^{\log_2 (4/p - 3) + 1} + \frac{\ln}{p} \log_2 \left( \frac{4}{p} [2 \ln (2N/\epsilon) - 1] \right) + t_a,
\]

and

\[
M(N) \simeq (\frac{2}{p})^{2+\log_2 (4/p - 3)} N [2 \ln (2N/\epsilon) - 1] + \frac{2N}{p} \ln (2N/\epsilon).
\]

We now take a close look at what we have achieved. Since $2N/\epsilon$ is not likely larger than $e^{50}$, it is reasonable that we set $\ln (2N/\epsilon)$ between 5 and 50. Fig. 3 compares the overall time overheads between Duan et al’s scheme (T2) [17] and ours (T1) that is needed for constructing a 2-D lattice-like cluster state. Fig. 3(a) shows that the overall time overheads in our scheme scales $\ln (2N/\epsilon)$ from linear to logarithmic compared with Duan et al’s proposal. From Fig. 3(b), we see that $T_1$ is always smaller than $T_2$ as $p$ varies. More interestingly, the inserted $T_1/T_2 - p$ relation in Fig. 3(b) indicates that our scheme have a relatively more significant improvement when $p$ is larger than 0.2. At last, note that to minimize the time overheads, we implement parallel operations during the whole process. As a result, it costs more attempts within our scheme. Nevertheless, the numbers of attempts in the two schemes have the same scaling with qubit number $N$, specifically, $N \ln (2N/\epsilon)$.

The above results can be analyzed qualitatively from the Eq. [6] and [10]. Note that our improvement lies in the process of constructing 2-D cluster states, which is reflected by substituting the third term $\frac{\ln}{p} \ln (2N/\epsilon)$ in Eq. [11] with $t_a$ in Eq. [6]. For example, if $\ln (2N/\epsilon) = 13$, the first term is comparable with $\frac{\ln}{p} \ln (2N/\epsilon)$ when $p$ is around 0.4. However, as $p$ decreases, the first term increases much more rapidly than $\frac{\ln}{p} \ln (2N/\epsilon)$ . And when $p$ is very small, the first term dominates the total time overheads and the other two terms are negligible.
can be generated from a 1-D cluster chain with two-qubit “arms”. The process scales efficiently with the qubit number and the inverse of the success probability. More interestingly, in our proposal, by applying parallel operations in the process of building 2-D cluster states, the time needed for constructing a 2-D and 1-D cluster state have the same order of magnitude. This result is helpful to construct 2-D cluster states with arbitrary size, which is essential to the implementation of experimental one way quantum computers.

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[17] Taking a close look at Duan et al’s calculation, we note that in the process of generating a “cross” with two long enough 1-D cluster chain, one saves time by costing a lot of resources. While such kind of cost is not involved in our scheme. In order to compare the two schemes, we need to make some tiny modifications to Duan et al’s results. Here we rewrite their time overheads as below:

\[ T(N) \approx t_a(1/p) \log_2(4/p-3)+1 + \frac{1}{p} \log_2 \left( \frac{4}{p} \ln \left( \frac{2N}{\epsilon} \right) - 1 \right) \]  

Here we add an additional factor 1/p to the first two terms of the equation, because in the process of building a “cross” from two long chain clusters, we should do 1/p connections in average to ensure one success.

[18] Note that in order to reach the above results, we need to apply an additional local unitary transformation to the resulted state after each measurement.
[19] In deriving all formulae in this paper, we assume that \( p \) is much smaller than 1 for simplicity of the calculations. However, it is worth noting that our scheme works effectively when \( p \) is comparable with 0.5 or even larger.
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