Jordan types for graded Artinian algebras in height two

Nasrin Altafi

Department of Mathematics, KTH Royal Institute of Technology, S-100 44 Stockholm, Sweden.

Anthony Iarrobino

Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

Leila Khatami

Union College, Schenectady, New York, 12308, USA.

Joachim Yaméogo

Université Côte d’Azur, CNRS, LJAD, FRANCE.

June 21, 2020

Abstract

We let $A = R/I$ be a standard graded Artinian algebra quotient of $R = k[x, y]$, the polynomial ring in two variables over a field $k$ by an ideal $I$, and let $n$ be its vector space dimension. The Jordan type $P_\ell$ of a linear form $\ell \in A_1$ is the partition of $n$ determining the Jordan block decomposition of the multiplication on $A$ by $\ell$ – which is nilpotent. The first three authors previously determined which partitions of $n = \dim_k A$ may occur as the Jordan type for some linear form $\ell$ on a graded complete intersection Artinian quotient $A = R/(f,g)$ of $R$, and they counted the number of such partitions for each complete intersection Hilbert function $T$ [AIK].

We here consider the family $G_T$ of graded Artinian quotients $A = R/I$ of $R = k[x, y]$, having arbitrary Hilbert function $H(A) = T$. The cell $\mathcal{V}(E_P)$ corresponding to a partition $P$ having diagonal lengths $T$ is comprised of all ideals $I$ in $R$ whose initial ideal is the monomial ideal $E_P$ determined by $P$. These cells give a decomposition of the variety $G_T$ into affine spaces. We determine the generic number $\kappa(P)$ of generators for the ideals in each cell $\mathcal{V}(E_P)$, generalizing a result

*Keywords: Artinian algebra, Hilbert function, hook code, Jordan type, partition, cellular decomposition, graded ideal. 2010 Mathematics Subject Classification: Primary: 13E10; Secondary: 05A17, 05E40, 13D40, 14C05. Email addresses: nasrinar@kth.se, a.iarrobino@northeastern.edu, khatamil@union.edu, joachim.yameogo@unice.fr
of [AIK]. In particular, we determine those partitions for which \( \kappa(P) = \kappa(T) \), the generic number of generators for an ideal defining an algebra \( A \) in \( G_T \). We also count the number of partitions \( P \) of diagonal lengths \( T \) having a given \( \kappa(P) \).

A main tool is a combinatorial and geometric result allowing us to split \( T \) and any partition \( P \) of diagonal lengths \( T \) into simpler \( T_i \) and partitions \( P_i \), such that \( V(E_P) \) is the product of the cells \( V(E_{P_i}) \), and \( T_i \) is single-block: \( G_{T_i} \) is a Grassmannian.

**Contents**

1 Introduction. 

2 Cells of the variety \( G_T \) and their hook codes.  
   2.1 The variety \( G_T \) and the cells \( V(E_P) \).  
   2.2 Hook code of \( P \).  
   2.3 The cell \( V(E_P) \), and the product of small Grassmannians.  

3 Number of generators for a single-block partition  
   3.1 Lower bound \( \kappa(T) \) on the number of generators of an ideal \( I \) in \( G_T \).  
   3.2 Single block partitions \( P \), and \( \kappa(P) \).  

4 Partitions in \( \mathcal{P}(T) \) having a given number of generators, for single-block \( T \).  

5 Number of generators for multiblock partitions.  

6 Number of cells of special multiblock partitions.  

7 Correspondence between the hook code and the branch label of a partition in \( \mathcal{P}(T) \).  

8 Betti strata of \( V(E_P) \).  

9 Problems.
$H(A)^\vee$ to the Hilbert function $H(A)$ ([HW Proposition 3.64]); and $(A, \ell)$ is weak Lefschetz if the number of parts of $P_{\ell, A}$ is equal to the maximum value of the Hilbert function of $A$, called the Sperner number of $A$ ([HW Proposition 3.5]). We say $A$ is strong Lefschetz, or weak Lefschetz, if the pair $(A, \ell)$ is such for a generic $\ell \in A_1$.

We consider standard graded Artinian algebra quotients $A = R/I$ where $I$ is an ideal of $R = k[x, y]$, the polynomial ring $R$ in two variables over an arbitrary field $k$. That is, $n = \dim_k A$ is finite, and $A$ is generated by $A_1$. We will assume that $I_1 = 0$, so the codimension $\dim_k A_1$ of $A$ is two. The order of the graded ideal $I$ is the lowest degree of a (non-zero) element.

The Hilbert function $T = H(A)$ of such a graded Artinian algebra $A$ in codimension two is a sequence of the following form

$$T = (1, 2, \ldots, d, t_d, t_{d+1}, \ldots, t_j, 0) \text{ where } d \geq t_d \geq t_{d+1} \geq \cdots \geq t_j > 0,$$

where $t_i = \dim_k A_i$, where $j$ is the socle degree of $T$, $d$ is the order of $T$ – the order of any ideal $I$, so $H(R/I) = T$ – and $n = |T| = \sum t_i = \dim_k A$.

The first three authors in [AIK] determined all possible Jordan types $\ell$ for complete intersection (CI) graded Artinian algebras of height two: they assumed that the field $k$ has characteristic zero or is infinite of characteristic $p > j$, the socle degree of the algebra. We make here the same assumption on characteristic, because we use a standard-basis result due to J. Briançon and A. Galligo in showing Lemma 3.6 that requires the restriction.

The CI Hilbert functions – for graded Artinian algebras $A = R/I, I = (f, g)$ – are symmetric about $j/2$ and satisfy

$$T = (1, 2, \ldots, d - 1, d, \ldots, d - 1, \ldots, 2, 1),$$

with Sperner number (order) $d$. In [AIK] it was shown that in characteristic zero the CI Jordan types compatible with the CI Hilbert function $T$ of Equation (1.2) correspond 1-1 to the subsets of the $d$ or $d - 1$ Hessians that can vanish, so there are $2^d$ or $2^{d-1}$ such CI Jordan types of diagonal lengths $T$, depending on whether or not the Sperner number $d$ is repeated in $T$. We give a different proof of the count of CI Jordan types of diagonal lengths $T$ – for a field of characteristic zero, or an infinite field of characteristic greater than the socle degree of $T$ – in Corollary 6.3 below. Our main work here is to generalize the Jordan type results of [AIK] to all height two Hilbert functions – those satisfying Equation (1.1).

To state our results requires some notation. We associate to a partition $P$ a monomial ideal $E_P$, determined by the Ferrers graph of $P$; the diagonal lengths $T = T_P$ is just the Hilbert function $T_P = H(R/E_P)$ (Definition 2.1). The set $\mathcal{P}(T)$ of partitions having diagonal lengths $T$ has a natural structure of lattice arising from the isomorphism $q : \mathcal{P}(T) \to Q(T)$, the set of hook codes, which is a lattice under inclusion of component Ferrers diagrams (Definition 2.4). The affine-space cell $\mathcal{V}(E_P)$ parametrizes the ideals of $R$ having initial ideal $E_P$ (Definition 2.1). We denote by $\kappa(P)$ the minimum number of generators for the ideal $I$ defining a generic element $A = R/I$ in the cell $\mathcal{V}(E_P)$. Likewise, we denote by $\kappa(T)$ the minimum number of generators of a generic ideal $I$ satisfying $H(R/I) = T$. This latter integer $\kappa(T)$ was already well-known (Lemma 3.3).

By a generic element of an irreducible algebraic variety $X$ we mean an element
belonging to a certain non-empty Zariski-dense open subset \( U \subset X \). Our main goal is to determine \( \kappa(P) \) for each partition \( P \in \mathcal{P}(T) \).

**Background.**

The family \( \mathcal{G}_T \) of graded Artinian quotients \( A = R/I \), where \( I \) is an ideal of \( R = k[x,y] \), for which the Hilbert function \( H(A) = T \) is a smooth projective variety \( \mathcal{G}_T \), that is locally an affine space of known dimension (Proposition 2.5). The variety \( \mathcal{G}_T \) has a cellular decomposition into the cells \( \mathcal{V}(E_P) \) where \( P \) runs through the set \( \mathcal{P}(T) \) of partitions \( P \) having diagonal lengths \( T \) and \( E_P \) is a monomial ideal determined by \( P \) (Definition 2.1 and Theorem 2.6).

The set \( \mathcal{P}(T) \) of partitions having diagonal lengths \( T \) – or, equivalently – the set of monomial ideals \( E_P \) such that \( H(R/E_P) = T \), has been studied by the second and last author in [IY1], as well as by others, including [Ev, Con]. That the cells \( \mathcal{V}(E_P) \) for \( P \in \mathcal{P}(T) \) form a cellular decomposition of \( \mathcal{G}_T \) was shown by G. Ellingsrud and S.A. Strømme in [ES1, ES2]; they were as well studied by G. Gotzmann [Gm], and L. Göttsche [Gö1]. By the result of A. Bialynicki-Birula [B-B] and as stated in [Y2, IY1, Gö1] the homology classes of the cells \( \mathcal{V}(E_P) \) form a basis of the homology \( H^*(\mathcal{G}_T) \) over the complexes; this result can be extended to algebraically closed fields \( k \) of characteristic \( p \).

The dimension of the cell \( \mathcal{V}(E_P) \) is given by the number of difference-one hooks in the Ferrers graph of \( P \) [IY1, Theorems 3.12, 3.27]. These results – the cellular decomposition, and the dimension of the cells – lead to concise formulas for the Betti numbers – the homology – of \( \mathcal{G}_T \) [IY1, Theorem 3.29, Gö1, Gm] (Theorem 2.10 below). But the cohomology ring structure for \( \mathcal{G}_T \) is in general still quite open (see [IY1, §4]). The cells \( \mathcal{V}(E_P) \) and their connection with generators and relations of ideals have also been studied by A. Conca and G. Valla in [CoVa] (see Section 8 and Theorem 8.3 below). Generators and relations for height two graded ideals have been studied by many, as [Br, Br-Ga, MR, Con]; J.O. Kleppe studies a scheme analogue of \( \mathcal{G}_T \) [K], L. Evain discusses an equivariant Hilbert scheme, involving weights of the variables [Ev]; some combinatorics of cells in [IY1] are seen in a larger context [LW].

**Results.**

We will answer the questions we consider first in the single-block case where the Hilbert function sequence \( T \) of Equation (1.1) satisfies \( d = j \), so

\[
T = (1, 2, \ldots, d, t_d = t, 0),
\]

that is \( t_i = i + 1 \) for \( 0 \leq i \leq d - 1 \) and \( t_d = t \). An ideal \( I \subset R \) defining an algebra \( A = R/I \) of Hilbert function \( T \) of Equation (1.3) satisfies, letting \( V = I_d \subset R_d \),

\[
I = V \oplus m^{d+1},
\]

where \( m \) is the maximal ideal of \( R \) and \( \dim_k V = d + 1 - t_d \). Thus, the projective variety \( \mathcal{G}_T \) in the single-block case is isomorphic to the Grassmannian \( \text{Grass}(s, R_d) \), \( s = d + 1 - t_d \) parametrizing \( s \)-dimensional subspaces \( V \subset R_d \).

\footnote{The definition of the cells and results need no change when \( \text{char} \ k = 0 \) or \( \text{char} \ k = p > j \); they will need modification when \( p \leq j \), the socle degree (recall the weak cells in [Ia1] used to show the irreducibility and smoothness of \( \mathcal{G}_T \) in any characteristic).}
Component Theorem.

A main tool is a new combinatorial and geometric result allowing us to associate to $T$ and to any partition $P$ of diagonal lengths $T$ their components, a set of simpler, single-block sequences $T_i$ and partitions $P_i$ of diagonal lengths $T_i$. In particular the affine cell $V(E_P)$ of $G_T$ is in a natural way the product of the cells $V(E_{P_i})$ of $G_{T_i}$. This decomposition is closely related to the hook codes studied in [IY1].

A general Hilbert function $T$ satisfying Equation (1.1) is equivalent to a unique sequence $(T_d, \ldots, T_j)$ of single-block Hilbert functions. We set $\delta_i(T) = t_{i-1} - t_i$ for $d(T) \leq i \leq j(T) + 1$, and we let $T_i$ be the single-block Hilbert function

$$T_i = (1, \cdots, d_i - 1, d_i, t_{d_i}, 0)$$

where $d_i = \delta_i(T) + \delta_{i+1}(T)$

and $t_{d_i} = \delta_{i+1}(T)$ for $i \in [d(T), j(T)]$. (1.5)

Each partition $P$ of diagonal lengths $T$ (so $P \in \mathcal{P}(T)$) is equivalent to a sequence of partitions $(P_d, \ldots, P_j)$ where $P_i \in \mathcal{P}(T_i)$ (Definition 2.16, Lemma 2.18). There is a morphism $\pi : V(E_P) \to \prod_{i=d}^{j} V(E_{P_i})$ (Lemma 2.25). We show (Theorem 2.26)

**Theorem 1.** The map $\pi$ is an isomorphism from $V(E_P)$ onto its image $\prod_{i=d}^{j} V(E_{P_i})$.

This key result allows us to reduce our questions about algebras of Hilbert function $T$ to the case of single-block Hilbert functions.

In our subsequent main results we

- determine the number of generators of a generic ideal in the cell $V(E_P)$ first for single-block partitions (Theorem 3.11);
- determine $\kappa(P)$ for arbitrary partitions $P \in \mathcal{P}(T)$ in terms of the $\kappa(P_i)$ for their components (Theorem 5.1);
- show that $P$ is special ($\kappa(P) = \kappa(T)$) if and only if some component $P_i$ is special (Theorem 5.19); and
- determine $\kappa(P)$ for arbitrary Jordan types $P$ (Theorem 6.1).

**Hook code.** We explain in Section 2.2 the hook code for partitions of diagonal lengths $T$. The hook code of $P$ is a sequence $\Omega(P) = (\eta_d(P), \ldots, \eta_j(P))$ of partitions-in-a-box $\mathfrak{B}_i(T), d \leq i \leq j$, where the box $\mathfrak{B}_i(T) = (\delta_{i+1}) \times (1 + \delta_i)$: that is, the Ferrers diagram of each $\eta_i(P)$ has at most $\delta_{i+1}$ rows and $(1 + \delta_i)$ columns (Definition 2.7). The partitions of diagonal lengths $T$ are completely determined by their hook code $\Omega(P)$. Also, the partition $P_i$ of diagonal lengths $T_i$ has as its hook code the degree-$i$ component $\eta_i(P)$ of $\Omega(P)$ (Lemma 2.18).

**Generators.** As mentioned, the minimal number of generators $\kappa(T)$ needed for a generic graded ideal $I$ where $H(R/I) = T$ was known (Lemma 3.3). In Theorem 3.11 we provide a formula giving $\kappa(P)$ for a partition $P \in \mathcal{P}(T)$ when $T$ is single-block, in terms of its difference-one hook code $\Omega(P)$: that is, we assume $T$ satisfies $d = j$ (Equation (1.3)). In Section 4 we study the subset of $\mathcal{P}(T)$ for single-block $T$ where $\kappa(P)$ is specified: in Theorem 4.2 we count the number of single block partitions $P \in \mathcal{P}(T)$ such that $\kappa(P)$ is bigger than or equal to a fixed integer in the possible range depending on $T$.

In Section 5 we give the analogous results for a multiblock Hilbert function $T$. These rely on connecting $\kappa(P)$ with the component $\kappa(P_i)$ (Theorem 5.1). We
also specify the generic number of generators for $I \in \mathcal{V}(E_P)$ in each degree, in terms of the hook code (Theorem 5.13). In Theorem 6.1 we count the number of partitions in $\mathcal{P}(T)$ having $\kappa(P) = k$ in the multiblock case; in Corollary 6.3 we count the number of special partitions $P \in \mathcal{P}(T)$ for which $\kappa(P) > \kappa(T)$.

The proofs involve a careful study of standard generators and relations for the ideals $I$ defining algebras in the cell $\mathcal{V}(E_P)$, using in particular the hook code of a partition $P$. We then compare these invariants to those for the partition $P : x$ corresponding to the ideals $I : x$. This allows us to compare $\kappa(P)$ with $\kappa(P : x)$, and we thus determine how to compute $\kappa(P)$ from the hook code $\Omega(P)$. Our proofs use the hook code $\Omega(P)$ as it is convenient for describing the component partitions $P_i$ given $P$.

**Branch label and ramification.** Another useful way of describing a partition $P$ of diagonal lengths $T$ is the *branch label* $b(P)$, that is used in [AIK]. We give this description in Section 7. In Theorem 7.2 we compare the hook code and branch label for partitions $P \in \mathcal{P}(T)$ where $T$ is single-block. In Theorem 7.3 we give the correspondence for multiblock $T$. The ramification of an ideal in a cell $\mathcal{V}(E_P)$ (Definition 7.5) is important in understanding the Zariski closure of cells (Lemma 7.6). We include a table of branch label, hook code, and ramification for cells of $G_T$, $T = (1, 2, 3, 4, 3, 2, 0)$ (Example 7.7 and Table 7.1).

In Section 8 we explore the Betti strata of cells $\mathcal{V}(E_P)$ and of their Zariski closures $\mathcal{V}(E_P)$. Previous work shows that the Betti sequences (the degrees of generators of $I$) define strata $G_\delta(T)$ of $G_T$ that have particularly good properties (Theorem 8.2). We have earlier determined the generic Betti stratum of each cell $\mathcal{V}(E_P), P \in \mathcal{P}(T)$. A. Conca and G. Valla have shown there is a decomposition of each Betti stratum of the cell $\mathcal{V}(E_P)$ into its degree-$i$ pieces (Theorem 8.3). In Proposition 8.5 we show that every intermediate generator sequence in between the minimum (generic) one for $I \in \mathcal{V}(E_P)$ and the maximum (for the monomial ideal $E_P$) occurs in the cell $\mathcal{V}(E_P)$. We pose Questions 8.6(i)-(vi), concerning the Betti strata and closures of the cells $\mathcal{V}(E_P)$, and we answer some parts: other parts remain open. In Section 9 we state further problems.

## 2 Cells of the variety $G_T$ and their hook codes.

### 2.1 The variety $G_T$ and the cells $\mathcal{V}(E_P)$.

We need some basic notions from [LY1, LY2] (see also [AIK] §4.1).

Recall that we consider graded Artinian quotients $A = R/I$, where $I$ is an ideal of $R = k[x, y]$ the polynomial ring over an arbitrary field $k$. The Hilbert function of $A$ is the sequence $H(A) = (1, t_1, \ldots, t_j)$ where $t_i = \dim_k A_i$, and $j$ is the socle degree of $A$ that is $A_j \neq 0, A_{j+1} = 0$. The family of all such quotients having Hilbert function $H(A) = T$ is denoted by $G_T$, which has a natural structure of subvariety $G_T \subset \Pi_{d \leq i \leq j} \text{Grass}(t_i, R_i)$, where $\text{Grass}(t_i, R_i)$ parametrizes quotients $A_i = R_i/I_i$ of vector space dimension $t_i$. Thus we have

$$t : G_T \to \Pi_{d \leq i \leq j} \text{Grass}(t_i, R_i) : A = R/I \to (I_d, I_{d+1}, \ldots, I_j).$$

We now explain the affine cell decomposition $G_T = \bigcup_{P \in \mathcal{P}(T)} \mathcal{V}(E_P)$ where $P$ runs through the set $\mathcal{P}(T)$ of partitions having diagonal lengths $T$. (Theorem 2.6).
Definition 2.1 (The monomial ideal $E_P$ and diagonal lengths of $P$). Given a partition $P = (p_1, p_2, \ldots, p_t)$ of $n = \sum p_i$ where $p_1 \geq p_2 \geq \cdots \geq p_t$, of diagonal lengths $T$ we let $C_P$ be the set of $n$ monomials that fill the Ferrers diagram $F_P$ of $P$ as follows: for $i \in [1, t]$ the, $i$-th row counting from the the top of $F_P$ is filled by the monomials $y^{i-1}, y^{i-1}x, \ldots, y^{i-1}x^{n-1}$. We let $E_P$ be the complementary set of monomials to $C_P$ and denote by $(E_P)$ the monomial ideal generated by $E_P$. The diagonal lengths $T_P$ of $P$ is the Hilbert function $T = H(R/E_P)$.

In a Ferrers diagram of monomials associated to a partition $P$ of $n$, the $x$-degrees of monomials increase as we go from left to right and the $y$-degrees increase as we go from top to bottom. We count the columns from left to right and the rows from top to bottom. See Figure 1 for the Ferrers diagram of the partition $P = (5, 3, 1)$ of diagonal lengths $T = (1, 2, 3, 2, 1)$; and Example 2.24 and Figure 15 for that of $P = (10^2, 4, 3, 2^5)$. Recall that the basic triangle of $P$ is the largest triangle $\Delta_d$ of shape $(d, d - 1, \ldots, 2, 1)$ comprised of monomials $(1, x, \ldots, x^{d-1}; y, yx, \ldots, yx^{d-2}; \ldots; y^{d-1})$ that fits in the Ferrers diagram; as an example, for $P = (5, 3, 1)$ this is $\Delta_3$, shown in red in Figure 1. For $P = (5, 3, 1)$ the diagonal lengths $T_P = (1, 2, 3, 2, 1)$ (see Figure 1).

Definition 2.2. A hook of a partition $P$ is a subset of the Ferrers diagram $F_P$ consisting of a corner monomial $c$, an arm $(c, xc, \ldots, \nu = x^{\nu-1}c)$ and a leg $(c, yc, \ldots, \mu = y^{\mu-1}c)$, such that $x\nu \in E_P$ and $y\mu \in E_P$ (Figure 2). The arm length is $u$ and the leg length is $v$; the hook has arm-leg difference $u - v$. We term the monomial $\nu$ the hand, and the monomial $\mu$ the foot of the hook.

Example 2.3. Let $P = (4, 3, 1)$. The hook with corner $x$ in the Ferrers diagram $C_P$ has arm length 3, foot length 2, hand $x^3$, foot $yx$, so has (arm − leg) difference one (Figure 3). Here $T_P = (1, 2, 3, 2)$, the basic triangle is $\Delta(P) = \Delta_3$, and the degree-3-diagonal of $\Delta(P)$ has the two monomials $x^3$ and $yx^2$ of $C_P$ and the monomials $y^2x$ and $y^3$ of $E_P$.

Definition 2.4 (Initial ideal of $I$, and the cell $V(E_P)$). The initial monomial $\mu(f) = \text{in}(f)$ of a form $f = \sum_k a_k y^k x^{i-k}, a_k \in k$ in the $y$-direction is the monomial $\mu(f)$. Figure 2: Difference-one hook with hand $h$, foot $f$, corner $c$, $P = (4, 1, 1)$.
\[ \mu(f) = y^s x^{i-s} \text{ of highest } y\text{-degree } s \text{ among those with non-zero coefficients } a_k. \]

Given an ideal \( I \subset R = k[x,y] \), defining the Artinian quotient \( A = R/I \) we denote by \( \text{in}(I) \) the ideal
\[ \text{in}(I) = \{ \{ \text{in}(f), f \in I \} \} \]
generated by the initial monomials of all elements of \( I \). We may identify \( \text{in}(I) \) with an ideal \( E_P \) for a partition \( P = P(I) \) of diagonal lengths \( T = H(A) = H(R/E_P) \).

We denote by \( \forall(E_P) \) the (affine variety) parametrizing all ideals \( I \subset R \) having initial ideal \( E_P \).

For the next two results see [IY1, §3-B, Theorem 3.12, §3-F]; the proof of the second by the last author relies on methods of J. Briançon [Br]. Recall that we denote by \( P(T) \) the set of all partitions of \( n = |T| \) having diagonal lengths \( T \).

We denote by \( \delta_i(T) \) the difference \( \delta_i(T) = t_{i-1} - t_i \), for \( i \geq d(T) \).

**Proposition 2.5** (The smooth projective variety \( G_T \)). [Ia1, Thm. 3.13]. The variety \( G_T \) parametrizing all ideals \( I \) of \( R = k[x,y] \) satisfying \( H(R/I) = T \) is a smooth irreducible projective variety, that is locally an affine space of dimension \( \sum_{i \geq d(T)}(\delta_i + 1)(\delta_{i+1}) \): it has a connected cover by opens in the same affine space.

**Theorem 2.6** (Cellular decomposition of \( G_T \)). [Y3, Y4]. The cell \( \forall(E_P) \) is an affine space of dimension the total number of difference one hooks in \( C_P \), viewed as the Ferrers diagram of the partition \( P \) (Definition 2.1).

The variety \( G_T \) has a finite decomposition into affine cells,
\[ G_T = \bigcup_{P \in P(T)} \forall(E_P). \quad (2.1) \]

### 2.2 Hook code of \( P \).

We review the hook code, using results from [IY1, IY2]. First, given \( T \) satisfying Equation (1.1) we define a sequence \( \mathcal{B}(T) \) of rectangular partitions or boxes. We let \( \delta_i(T) = t_{i-1} - t_i \) for \( i \geq d(T) \).

\[ \mathcal{B}(T) = (\mathcal{B}_d(T), \mathcal{B}_{d+1}(T), \ldots, \mathcal{B}_{i}(T)), \text{ where} \]
\[ \mathcal{B}_i(T) = (\delta_{i+1}) \times (1 + \delta_i), \text{ a rectangular box,} \quad (2.2) \]

with height \( \delta_{i+1} \) and base \( 1 + \delta_i \). We order the monomials of degree \( i \) by \( x^i < x^{i-1}y < \cdots < y^i \) (lex order); certain of these monomials are hands of degree-\( i \) hooks of \( P \), that is end elements of rows of \( C_P \) from Definition 2.1 and we order these correspondingly.
Figure 4: Hook code for $P = (6, 3, 3, 3) : \Omega(P) = (14, (2, 1)_5)$.

**Definition 2.7.** Suppose that the partition $P$ has diagonal lengths $T$. The (difference-one) *hook code* of $P$ is the sequence

$$\Omega(P) = (h_d(P), \ldots, h_j(P))$$

(2.3)

where $h_i(P)$ is a partition that enumerates the difference-one hooks of hand-degree $i$, according to their $\delta_{i+1}$ degree-$i$ hands. That is, the $k$-th part of $h_i(P)$ is the number of difference-one hooks having the $k$-th possible degree-$i$ hand. It is not hard to see that the number of difference-one hooks per hand is in the interval $[0, \delta_i + 1]$, and that is non-increasing; so $h_i(P)$ is a partition, and $h_i(P) \subset B_i(T)$: the degree-$i$ hook partition fits into the box $B_i(T)$.

Thus, the code is determined by arranging the difference-one hooks of $P$ first, according to their hand-degree $i$, then according to their “hand monomial,” determining for each degree $i \in [d, j]$ a partition $h_i(P)$.

We denote by $Q(T)$ the set of all $(j+1-d)$-tuples of partitions $(h_d, \ldots, h_j)$ satisfying, $h_i \subset B_i(T)$. Here $Q(T)$ is a lattice under the product structure given by inclusion for each component $h_i$: that is, $Q \leq Q'$ if each $h_i \subset h_i'$, in the sense that the Ferrers diagram for $h_i$ fits inside that of $h_i'$.

**Example 2.8** (Hook code for $P = (6, 3, 3, 3)$). Let $T = (1, 2, 3, 4, 3, 2, 0)$ where $d = 4$; we have $\delta_4 = 4 - 3 = 1, \delta_5 = 3 - 2 = 1, \delta_6 = 2 - 0 = 2$. Then $B(T) = (B_4, B_5) = ((1 \times 2)_4, (2 \times 2)_5)$. The partition $P = T^\vee = (6, 3, 1)$ has the maximum hook code $\Omega(P) = B(T)$. But $P = (6, 3^3)$ has the hook code $\Omega(P) = (14, (2, 1)_5)$: the degree four hand monomial is $y^2x^2$ with a single difference-one hook, with corner $y^2$; the degree-5 hand monomials are $x^5$ with two hooks with corners $x, x^4$, and $y^2x^3$ with one hook, corner $y^3x$. See Figure 4 where we visualize the hooks by showing their corners, blue for degree 4 and red for degree 5. Table 7.1 with Example 7.7 gives the hook code and branch labels for all partitions of diagonal lengths $T = (1, 2, 3, 4, 3, 2, 0)$.

See also Example 7.3 and Figure 10 which shows the hook code for $P = (13, 11, 10, 7^2, 5^2, 4^3, 2)$.

The following is stated as part of [LY1] Theorem 3.27], and shown in [LY2] Theorem 1.17. Recall that $P(T)$ is the set of partitions having diagonal lengths $T$. We denote by $q : P(T) \rightarrow Q(T)$ the hook code map taking $P$ to $\Omega(P)$. For a partition $h_i \subset B_i$ we denote by $h_i^c$ the complement of $h_i$ in $B_i$. For an element $h = (h_d, \ldots, h_j) \in Q(T)$, we denote by $h^c = (h_d^c, \ldots, h_j^c)$ the complement in $B(T)$. Recall that, given $P \in P(T)$ we denote by $P^\vee$ the conjugate partition (switch rows and columns in the Ferrers graph of $P$); evidently $P^\vee \in P(T)$.

**Theorem 2.9.** Let $T$ satisfy Equation (1.1), and let $P \in P(T)$. Then in the hook code map $q$, the $i$-th hook code partition $h_i(P) \subset B_i(T)$. Furthermore, the map $q : P(T) \rightarrow Q(T)$ is an isomorphism of sets satisfying $q(P^\vee) = (q(P))^c$.
We endow $\mathcal{P}(T)$ with the structure of a lattice via the isomorphism $q$ to $\mathcal{Q}(T)$ (see Definition 2.7).  

The second and last author showed that the dimension of the cell $\mathcal{V}(E)$ is the total number of difference-one hooks in the partition $P_E$ determined by $E$ (Theorem 2.6, [Y1, Theorem 3.27]): this is just the height of $\mathcal{Q}(P)$ in the lattice $\mathcal{Q}(T)$. It follows from the A. Bialynicki-Birula result [B-B] that over the complexes, the Betti numbers of $G_T$ may be deduced from the cellular decomposition [Y1, Theorem 3.28, Theorem 3.29].

We recall first the generating function $B(a,b) = \sum B(a,b,n)q^n$ for the number of partitions of $n$ whose Ferrers diagram fit in an $a \times b$ box: that is, $B(a,b,n)$ is the number of partitions of $n$ that have less or equal $a$ parts, each no greater than $b$. Denote by $(q)_a = (q^a - 1) \cdot (q^{a-1} - 1) \cdots (q - 1)$. Then $B(a,b)$ is given by the $q$-binomial coefficient,

$$B(a,b)_q = \frac{q_{a+b}}{q_a \cdot q_b}. \quad (2.4)$$

The Poincaré polynomial $B(X) = B(X,q)$ of a variety $X$ is $B^*(X) = \sum h^i(X)q^i$ where $h^i(X) = \dim_k H^i(X,\mathbb{C})$, the dimension of the $i$-th homology group. We have

$$B^*(\text{Grass}(a,a+b)) = B(a,b)_q. \quad (2.5)$$

Now Theorem 2.6 implies

**Theorem 2.10.** [Y1, Theorem 3.29] Let $T$ satisfy (1.1), with first differences $\delta_i = t_{i-1} - t_i$. The Poincaré polynomial $B(G_T)$ for $G_T$ satisfies

$$B(G_T) = \Pi_{v(T) \leq i < j(T)} B^{\text{Grass}}(\delta_{i+1}, 1 + \delta_i + \delta_{i+1}) \quad (2.6)$$

$$= \Pi_{v(T) \leq i < j(T)} B^{\delta_{i+1}, \delta_i + 1}_q. \quad (2.7)$$

The following result concerns single-block Hilbert functions $T = (1,2,\ldots,d,t_d,0)$ [Y1, Section 2C]. We will say that a partition $P$ in a $t \times s$ box is greater or equal to $P'$ also in the box, if the Ferrers graph of $P$ includes that of $P'$.

**Theorem 2.11** (Closure in the Grassmannian). Let $T$ satisfy (1.3), and let $P$ have diagonal lengths $T$. Then the closure $\overline{\mathcal{V}(E_P)}$ of the cell $\mathcal{V}(E_P)$ satisfies

$$\overline{\mathcal{V}(E_P)} = \bigcup_{P' \leq P} \mathcal{V}(E_{P'}). \quad (2.8)$$

*Proof.* This follows from interpreting the closures $\overline{\mathcal{V}(E_P)}$ as Schubert varieties in the Grassmannian Grass$(t,R_d)$. \hfill \Box

**Warning 2.12.** Equation 2.8 describes a frontier property: the closure of a cell is the union of cells. The analog of Theorem 2.11 is not true in general for $G_T$, when $T$ is multiblock. The last author has shown in [Y2] that even for $T = (1,2,3,2,1)$ the closure of the cell $\mathcal{V}(E_P)$ corresponding to the partition $P = (5,2,1,1)$ is not

---

2There is an alternative poset structure $\mathcal{P}_{alt}(T)$ on $\mathcal{P}(T)$, related to the sequences of degree-$i$ monomials in $C_P$. The inverse $Q(T) \cong P(T) \to \mathcal{P}_{alt}(T)$ is an inclusion of posets, not an isomorphism of lattices as stated incorrectly in [Y1, Theorem 3.27]. See the discussion in [Y2].
the union of cells (see also [LY1, Example 3.28]). The closure of the cells are still not well understood, although the last author has shown the following:

**Theorem.** Let $U_c = \mathcal{V}(E_P)$ be a cell in $G_T$ of dimension $c$. Then there are cells $U_+$ and $U_-$ of dimensions $(c + 1)$ and $(c - 1)$, respectively, such that $U_+ \supset U \supset U_{c-1}$.

2.3 The cell $\mathcal{V}(E_P)$, and the product of small Grassmannians.

Throughout this section $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ will be a Hilbert function satisfying Equation (1.1) and $P$ will be a partition of diagonal lengths $T$. We denote by $E_P$ the monomial ideal associated to $P$ and let $\mathcal{V}(E_P)$ be the cell of $G_T$ associated to $P$.

**The component partitions of $P$.**

Let $P$ be a partition with Hilbert function $T = (1, \ldots, d, t_d, \ldots, t_j, 0)$ and difference-one hook code $\Omega(P) = (\delta_d, \delta_{d+1}, \ldots, \delta_j)$ (Definition 2.7). Set $t_{d-1} := d$ and $t_{j+1} := 0$ and for $i = d, \ldots, j$; recall the sequence $T_i$ from Equation (1.5)

$$T_i = (1, 2, \ldots, t_{i-1} - t_{i+1}, t_i - t_{i+1}, 0).$$

(There is a shift in degrees, $T_i$ parametrizes ideals of order (initial degree) $d_i = t_{i-1} - t_{i+1} = \delta_i(T) + \delta_{i+1}(T)$).

We will next define the $i$-th block of $P$, denoted by $P_i$ (Definition 2.16), and we will show that it is the partition with diagonal lengths $T_i$ and hook code $\Omega(P_i) = (\delta_i) = (\delta_i(P))$ (Lemma 2.18). A reader on a first look may just use this Lemma as a definition, but we wish to be precise.

Our purpose here is to give a description of $\mathcal{V}(E_P)$ as a product of cells of “small” Grassmannians (Theorem 2.20). Recall that for a Hilbert function $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$, we set $\delta_i = t_{i-1} - t_i$ ($d \leq i \leq j + 1$).

Given $P = (p_1, \ldots, p_s)$ a partition of diagonal lengths $T$, for each $i$ ($d \leq i \leq j$) we construct a vector space $V_i$ of dimension $n_i = \delta_i + \delta_{i+1} + 1$ such that to each element $I$ of the cell $\mathcal{V}(E_P)$ we can associate a subvector space $I_{V_i}$ of $V_i$ with $\dim_k (I_{V_i}) = \delta_i + 1$. So the vector space $I_{V_i}$ is an element of Grass($\delta_i + 1$, $V_i$) and belongs to a cell described by the partition $h_i(P)$, the degree-$i$ block of the difference-one hook code of $P$.

We first define (as in [LY2] but in a more strict way) the horizontal-border monomials and vertical-border monomials of $E_P$.

**Definition 2.13.** Let $P$ be a partition of diagonal lengths $T$. Denote by $E_P$ the monomial ideal of $k[x, y]$ associated to $P$.

1. We say a monomial $x^a y^b \in E_P$ is a horizontal-border monomial of $E_P$ if ($b > 0$ and $x^{a} y^{b-1} \notin E_P$).
2. We say a monomial $x^a y^b \in E_P$ is a vertical-border monomial of $E_P$ if $a > 0$ and $x^{a-1} y^b \notin E_P$.
Denote by $A(P)_i$ the set of degree-$i$ horizontal-border monomials of $E_P$, $B(P)_i$ the set of degree-$i$ vertical-border monomials of $E_P$ and $(C_P)_i$ the set of degree-$i$ monomials that are not in $E_P$.

**Claim 2.14.**

1. $|A(P)_i| = \begin{cases} t_{i-1} - t_i + 1 = \delta_i + 1, & \text{if } x^i \in (C_P)_i \\ t_{i-1} - t_i = \delta_i, & \text{if } x^i \notin (C_P)_i \end{cases}$

2. $|B(P)_{i+1}| = \begin{cases} t_i - t_{i+1} + 1 = \delta_{i+1} + 1, & \text{if } y^{i+1} \in (C_P)_{i+1} \\ t_i - t_{i+1} = \delta_{i+1}, & \text{if } y^{i+1} \notin (C_P)_{i+1} \end{cases}$

**Proof of claim.** (Note: these formulas have been established in [Y2] to define difference-a hook partitions.)

One can consider the following maps

\[ \varphi_i : (C_P)_{i-1} \rightarrow (C_P)_i \cup A(P)_i, \quad \psi_i : (C_P)_i \rightarrow (C_P)_{i+1} \cup B(P)_{i+1}. \]

The maps $\varphi_i$ and $\psi_i$ are injective. Also note that if $x^i \in (C_P)_i$, then $x^i$ is the only element of $(C_P)_i \cup A(P)_i$ that is not in the image of $\varphi_i$, so we have $|(C_P)_{i-1}| = |(C_P)_i| + |A(P)_i| - 1$. If $x^i \notin (C_P)_i$, then $\varphi_i$ is a bijection, thus $|(C_P)_{i-1}| = |(C_P)_i| + |A(P)_i|$. The formula for $|A(P)_i|$ follows from the fact that for any integer $t$, $|(C_P)_t| = t_i$. Using the same arguments one can verify the formula for $|B(P)_{i+1}|$. \qed

**Remark 2.15.**

1. Each monomial $x^a y^b$ in $A(P)_i$ is just below a degree-$(i-1)$ foot monomial $(x^a y^{b-1})$ of $P$, thus $|A(P)_i|$ counts the number of degree-$(i-1)$ foot monomials in the Ferrers diagram of $P$.

2. The elements of $B(P)_{i+1}$ are each just right to a degree-$i$ hand monomial, so $|B(P)_{i+1}|$ counts the number of degree-$i$ hand monomials in the Ferrers diagram of $P$. In Definition 2.16 we will consider the first (numbering from top to bottom– lex order) $\delta_{i+1}$ degree-$i$ hand monomials of $P$.

3. Recall that $P = (p_1, \ldots, p_s)$, with $p_1 \geq p_2 \geq \cdots \geq p_s > 0$.

   If $x^i \notin (C_P)_i$, then $p_1 \leq i$.

**Definition 2.16** (The partition $P_i$ determined by $P$). For any positive integer $n \in \mathbb{N}$, denote by $\text{Mon}(R_n)$ the set of degree $n$ monomials of $R = k[x, y]$ and consider the lex order on $\text{Mon}(R_n): x^n < x^{n-1} y < \cdots < x y^{n-1} < y^n$. Let $P \in \mathcal{P}(T)$. If $x^i \in (C_P)_i$ we let $V_{i1} = A(P)_i$.

If $x^i \notin (C_P)_i$, we number the $\delta_i$ elements of $A(P)_i$ from top to bottom: $A(P)_i = \{M_{i1}, \ldots, M_{i\delta_i}\}$. Let $x^a y^b$ be the last (lex order) degree-$i$ vertical-border monomial above $M_{i1}$. We then let $V_{i1} = A(P)_i \cup \{x^a y^b\}$ (if $x^i \notin (C_P)_i$).

Thus, $\dim V_{i1} = \delta_i + 1$.

We now consider the set $V_{i2}$ consisting of the first (lex order) $\delta_{i+1}$ hand monomials in $(C_P)_i$ and we let $V_i$ be the vector space spanned by $V_{i1} \cup V_{i2}$.

By definition, $V_i$ has dimension $d_i + 1$ where $d_i = \delta_i + \delta_{i+1}$. So the set $V_{i1} \cup V_{i2}$ is lex ordered and we can consider the one to one correspondence

\[ s_i : V_{i1} \cup V_{i2} \rightarrow \text{Mon}(R_{d_i}) \quad (2.10) \]
that respects the lex ordering (the \( k \)-th element of \( V_{i1} \cup V_{i2} \) is associated to the \( k \)-th element of \( \text{Mon}(R_{d_i}) )\).

The vector space \( s_i(V_{i1}) \) has dimension \( \delta_i + 1 \), so \( \langle s_i(V_{i1}) \rangle + \mathfrak{m}^{d_i+1} \) is the monomial ideal \( (E_{P_i}) \) for a unique partition \( P_i \) of diagonal lengths the single-block Hilbert function \( T_i = (1, \ldots, d_i - 1, d_i, t_{di}, 0) \) where \( d_i = \delta_i + \delta_{i+1} \) and \( t_{di} = \delta_{i+1} \).

**Remark 2.17.** For \( i \in [d,j] \), it may happen that \( t_i = t_{i+1} \), so \( \delta_{i+1} = 0 \). In that case we have:

1. The rectangular box \( \mathfrak{B}_i(T) = (\delta_{i+1}) \times (1 + \delta_i) \) of Equation 2.2 is empty and so the degree-\( i \) partition \( \mathfrak{h}_i(P) \) in Equation 2.3 is empty.
2. \( V_{i2} \) is empty and so \( s_i(V_{i1}) = \text{Mon}(R_{d_i}) \).
3. The partition \( P_i \) associated to the monomial ideal \( \langle s_i(V_{i1}) \rangle + \mathfrak{m}^{d_i+1} \) is just the basic triangle \( \Delta_{d_i} = \Delta_{\delta_i} = (\delta_i, \delta_i - 1, \ldots, 1) \). Of course, if \( \delta_i = 0 \), then \( \Delta_{d_i} = \emptyset \) and \( \langle s_i(V_{i1}) \rangle + \mathfrak{m}^{d_i+1} = \mathfrak{m}_{d_i} \).

The following Lemma follows directly from Definitions 2.7 and 2.16:

**Lemma 2.18.** The difference-one hook code of \( P_i \) is exactly that of the \( i \)-th component of \( \mathfrak{Q}(P) \) (in the difference-one hook code of \( P \)).

Note, however, the shift in degree: the difference-one hook code of \( P_i \) occurs in the degree \( d_i = t_{i-1} - t_{i+1} = \delta_i(T) + \delta_{i+1}(T) \).

In connection with Lemma 5.7 where we will be counting the degree \( i+1 \) relations and corner-monomials of \( E_P \) it is interesting to note that:

**Lemma 2.19.** The bijection \( s_i : V_{i1} \cup V_{i2} \rightarrow \text{Mon}(R_{d_i}) \) induces

1. a one to one correspondence between the degree \( i + 1 \) relations of \( E_P \) and the degree \( d_i + 1 = \delta_i + \delta_{i+1} + 1 \) relations of \( E_{P_i} \).
2. a one to one correspondence between the first (numbering from top to bottom-lex order) \( \delta_{i+1} \) degree \( i + 1 \) vertical-border monomials of the ideal \( E_P \) and the degree \( d_i + 1 \) vertical-border monomials of \( E_{P_i} \).
3. a one to one correspondence between the degree \( i + 1 \) corner-monomials of \( E_P \) and the degree \( d_i + 1 \) corner-monomials of \( E_{P_i} \).

**Proof.** 1. Suppose that the monomial \( x^\alpha y^{i+1-\alpha} \) (\( 0 < \alpha < i+1 \)) corresponds to a degree \( i + 1 \) relation. Then \( x^{\alpha-1} y^{i+1-\alpha} \) is a horizontal-border monomial of \( E_P \) and \( x^\alpha y^{i-\alpha} \) is a vertical-border monomial of \( E_P \). Now, consider the set \( \langle E_P \rangle_{i,\alpha} \) of degree \( i \) horizontal-border monomials of \( E_P \) that are above \( x^\alpha y^{i-\alpha} \). If this set is empty, then \( x^i \notin \langle C_P \rangle_i \), so \( x^\alpha y^{i-\alpha} \in V_{i1} \) and the degree \( i + 1 \) relation of \( E_P \) corresponding to \( x^\alpha y^{i+1-\alpha} \) is sent to a degree \( d_i + 1 \) relation of \( E_{P_i} \). If the set \( \langle E_P \rangle_{i,\alpha} \) is not empty, let \( x^\alpha y^{i-\alpha} \) be the first element of \( \langle E_P \rangle_{i,\alpha} \) just above \( x^\alpha y^{i-\alpha} \). By definition, \( s_i \) sends \( x^{\alpha-1} y^{i+1-\alpha} \) and \( x^\alpha y^{i-\alpha} \) to two consecutive horizontal-border monomials of \( E_{P_i} \), resulting to a degree \( d_i + 1 \) relation of \( E_{P_i} \).
2. By Definition 2.13, we know that any degree $i+1$ vertical-border monomial is just to the right of a unique degree $i$ hand monomial. The second statement of the Lemma is then just a remark based on the fact that any element of $V_2$ is a degree $i$ hand monomial of $P$ and $s_i$ sends the elements of $V_2$ to the degree $d_i$ hand monomials of $E_{P_i}$.

3. Suppose $x^\alpha y^{i+1-\alpha}$ is a degree $i + 1$ corner monomial of $E_P$.
   a) If $\alpha = 0$, then one can easily see that $y^{d_i} \notin E_{P_i}$, so $y^{d_i+1}$ is a corner-mononial of $E_{P_i}$.
   b) If $\alpha > 0$ then the corner-monomial $x^\alpha y^{i+1-\alpha}$ is also a vertical-border monomial of $E_P$ that will correspond via $s_i$ to a degree $d_i + 1$ corner-monomial of $E_{P_i}$.

\[\square\]

Proposition 2.20. Using the notation of Definition 2.16, let
\[V_{i1} \cup V_{i2} = \{x^{\alpha_0}y^{\beta_0}, \ldots, x^{\alpha_{d_i}}y^{\beta_{d_i}}\},\]
where $0 < \alpha_1 < \cdots < \alpha_{d_i}$ (so $\beta_0 > \beta_1 > \cdots > \beta_{d_i}$).

Let $P'$ be the partition obtained from $P$ by removing any column of $P$ whose index does not belong to the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{d_i}\}$ and any row of $P$ whose index does not belong to the set $\{\beta_0, \beta_1, \ldots, \beta_{d_i}\}$. Then $P' = P_i$.

Proof. In constructing the $i$-th component of $\Omega(P)$ (in the difference-one hook code of $P$), we only need the elements of $V_{i2}$ (degree $i$ hand monomials) and the elements of $V_{i1}$ (related to degree $i$ horizontal-border monomials). The purpose of the bijection $s_i : V_{i1} \cup V_{i2} \rightarrow \text{Mon}(R_{d_i})$ is to let us focus on these monomials. So by definition of the bijection $s_i$, the partition $P_i$ is obtained from $P$ by ignoring any column of $P$ whose index does not belong to the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{d_i}\}$ and any row of $P$ whose index does not belong to the set $\{\beta_0, \beta_1, \ldots, \beta_{d_i}\}$. Deleting these unnecessary rows and columns will result in showing only the relevant degree $i$ hands and degree $i - 1$ feet of $P$.

\[\square\]

Remark 2.21. Looking at the Ferrers diagram of a partition $P$, we can visualize the set $V_i = V_{i1} \cup V_{i2}$, defined in Definition 2.16, as follows. Consider the diagonal corresponding to degree-$i$ monomials of $k[x,y]$, see the grey bubbles in Figure 5. Then the set of degree-$i$ horizontal border monomials of $P$, $A(P)_i$ can correspond to the bubbles outside of the Ferrers diagram that are right below the horizontal edges of $P$, see the blue monomials in Figure 5. If the largest part of $P$ is greater than $i$ (i.e., $x^i \in (C_P)_i$), then $V_{i1}$ is the same as $(A(P))_i$. If the largest part of $P$ is at most $i$ (i.e., $x^i \notin (C_P)_i$) then $V_{i1}$ also includes the first degree-$i$ vertical border monomial of $P$ that is above all monomials in $(A(P))_i$. This monomial corresponds to a bubble outside of the Ferrers diagram that is immediately to the right of a horizontal edge of $P$, see the red monomial in Figure 5. Finally, monomials in $V_{i2}$ consists of the first $d_i + 1$ hand monomials of $P$. These correspond to bubbles inside the Ferrers diagram that are at the end of a row of $P$, see the blue monomials in Figure 5.

To visualize Proposition 2.20 we fill out the degree-$i$ bubbles that correspond to $V_i$ by their monomials. We then remove all rows and columns of $P$ that do not include a filled degree-$i$ bubble, see Figure 5.
Figure 5: Decomposition of partition \( P = (6^2, 3, 2^2) \) with diagonal lengths \( T = (1, 2, 3, 4, 5, 3, 1) \) and hook code \( \Omega(P) = ((3, 1)^5, (2)_6) \) into components \( P_5 = (5, 3, 2^2) \) and \( P_6 = (3^2, 1) \) of diagonal lengths \( T_5 = (1, 2, 3, 4, 2) \) and \( T_6 = (1, 2, 3, 1) \), respectively. On the left, the grey bubbles represent degree 5 monomials, filled with monomials in \( V_5 \). The monomials in black represent \( V_{5,2} \) and the monomials in blue represent \( V_{5,1} = (A(P))_5 \). On the right, the gray bubbles represent degree 6 monomials, filled with monomials in \( V_6 \). The monomials in black represent \( V_{6,2} \), the monomials in blue represent \( (A(P))_6 \) and the monomial in red represents the additional vertical border monomial in \( V_{6,1} \). See Remark \[ \text{[2.21]} \].
**Example 2.22.** Consider the three-block partition $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$ with diagonal lengths $T = (1, 2, \ldots, 13, 10_{13}, 6_{14}, 3_{15}, 0)$ and hook code

$$\Omega(P) = ((3, 1^2, 0)_{13}, (5, 4, 1)_{14}, (2^2, 1)_{15}).$$

In Figure [4] we illustrate the process of decomposing $P$ into its single-block components, $P_{13} = (7^2, 5, 4^2, 3, 1^2)$, $P_{14} = (8, 6^2, 4, 3, 2^2)$, and $P_{15} = (6, 5^2, 4, 2^2)$ of diagonal lengths, respectively, $T_{13} = (1, \ldots, 7, 4, 0)$, $T_{14} = (1, \ldots, 7, 3, 0)$, and $T_{15} = (1, \ldots, 6, 3, 0)$.

In each part of Figure [4] the bubbles correspond to degree-$i$ monomials. The hand monomials of $V_{i2}$ are black and the horizontal border monomials, which all belong to $V_{i1}$ are blue. For $i = 15$, since $x^{15} \not\in (C_P)_{15}$, in addition of the horizontal border monomials, the set $V_{15,2}$ also includes a vertical border monomial that is illustrated in red. The rest of the bubbles which are in light grey determine the rows and columns of $P$ that need to be removed, according to Proposition 2.20, in order to obtain the corresponding single block component.

The projection map $\pi$ from $\mathbb{V}(E_P)$ to a product of cells of “small” Grassmannians.

**Definition 2.23** (The map $\pi$ of $\mathbb{V}(E_P)$ to the product of small Grassmannians).

Suppose $P \in \mathcal{P}(T)$ and let $I \in \mathbb{V}(E_P)$. Denote by $W_i$ the vector space generated by $V_{i1} \cup (C_P)_i$ for $i \in [d, j]$. Note that $\dim_k W_i = \delta_i + 1 + t_i$. It is straightforward to see that the vector space $I_{W_i} = I \cap W_i$ has dimension $(\delta_i + 1)$. The leading monomial of any non zero element of $I_{W_i}$ belongs to $V_{i1}$ and conversely, given an element $M$ of $V_{i1}$, there is an element of $I_{W_i}$ whose leading monomial is $M$.

Let $K_i$ be the vector space generated by $V_{3} = (C_P)_i \setminus V_{i2}$. We write $W_i$ as a direct sum $W_i = V_i \oplus K_i$ and consider the projection on the first factor $pr_1 : W_i \to V_i$ and let $I_{V_i} = pr_1(I_{W_i})$. Then $I_{V_i} \in \text{Grass}(\delta_i + 1, V_i)$.

Thus, we have constructed a morphism $\mathbb{V}(E_P) \xrightarrow{\pi} \prod_{i=d}^{j} \text{Grass}(\delta_i + 1, V_i)$, and, taking the product $\pi = (\pi_d, \ldots, \pi_j)$ we have a morphism

$$\mathbb{V}(E_P) \xrightarrow{\pi} \prod_{i=d}^{j} \text{Grass}(\delta_i + 1, V_i). \quad (2.11)$$

**Remark 2.24.** Note that by construction and the definition of the difference-one hook code, the image of $\pi_i$ is a Schubert cell $\mathbb{V}(E_P)$ in $\text{Grass}(\delta_i + 1, V_i)$, whose dimension is $|Q(P)_i|$, the length of the $i$-th block of the hook code of $P$.

Also, when $\delta_{i+1} = 0$ (that is $t_i = t_{i+1}$) we have $\dim(V_i) = \delta_i + 1$, $\text{Grass}(\delta_i + 1, V_i)$ is just one point; the $i$-th block of the hook code of $P$ in this case is empty, so has length zero.

---

3 We use the term “small” Grassmannian to distinguish these from the (large) Grassmannians determined by the projection $G_T \to \prod_{d \leq i \leq j} \text{Grass}(i + 1 - t_i, i + 1)$ given by $I \to (I_d, \ldots, I_j)$. 

16
Obtaining $P_{13} = (7^2, 5, 4^2, 3, 1^2)$ from $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$

Obtaining $P_{14} = (8, 6^2, 4, 3, 2^2)$ from $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$

Obtaining $P_{15} = (6, 5^2, 4, 2^2)$ from $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$

Figure 6: Finding single-block components for partition $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$ of Example 2.22 using Proposition 2.20. In each part of the Figure, the black bubbles represent elements of $V_i$, the blue bubbles represent horizontal border monomials in $V_i$ and the red bubble represents the vertical border monomial in $V_{i1}$ (when $V_{i1}$ includes one such monomial).
Lemma 2.25 (Morphism $\pi$ to the component small Grassmannians). Let $P \in \mathcal{P}(T)$. The morphism $\pi$ of (2.11) determines a morphism

$$\pi : \mathbb{V}(E_P) \xrightarrow{\pi} \prod_{i=d}^{i=j} \mathbb{V}(E_{P_i}).$$

Proof. The bijection $s_i : V_i \to \text{Mon}(R_{d_i})$ of Equation (2.10) induces a linear isomorphism $\tilde{s}_i : V_i \to R_{d_i}$. We thus have an isomorphism $\tilde{s}_i : \text{Grass}(\delta_i + 1, V_i) \to \text{Grass}(\delta_i + 1, R_{d_i})$, taking $I_{V_i} = pr_1(I_{V_i})$ to the subspace $\tilde{s}_i(I_{V_i}) \subset R_{d_i}$. Then by Definition 2.16, Equation 2.11 determines the morphism $\pi$ of the Lemma. \qed

Theorem 2.26. The morphism $\pi$ of Lemma 2.25 is an isomorphism from $\mathbb{V}(E_P)$ onto its image $\prod_{i=d}^{i=j} \mathbb{V}(E_{P_i})$.

Proof. Each $\mathbb{V}(E_{P_i})$ is an affine space; from Theorem 2.26 and Lemma 2.18 the dimension of $\mathbb{V}(E_P)$ is the sum $\sum \dim \mathbb{V}(E_{P_i})$. We know that the difference-one hook code of $P_i$ is the $i$-th component $(h_i(P))$.

Using the notation of Definition 2.16 consider the set $V_{i1}$, which contains all the degree-$i$ horizontal border monomials of $E_P$.

Let $V_{i1} = \{M_{i,1}, \ldots, M_{i,\delta_i}, M_{i,\delta_i+1}\}$ (numbered from top to bottom). Denote by $b_{i,l}$ the number of degree-$i$ hand monomials above $M_{i,l}$. Then we have $b_{i,\delta_i+1} \geq b_{i,\delta_i} \geq \cdots \geq b_{i,1}$, and the affine space $\mathbb{V}(E_{P_i})$ has dimension $\sum_{l=1}^{\delta_i+1} b_{i,l} = |(h_i(P))|$

We may think of $\mathbb{V}(E_{P_i})$ as $\prod_{l=1}^{\delta_i+1} k^{b_{i,l}}$ ($b_{i,l}$ free parameters for each $M_{i,l}$).

We now let $\text{hMon}(E_P)$ be the set of horizontal border monomials of $E_P$. Suppose $\text{hMon}(E_P) = \{y^{\beta_0}, xy^{\beta_1}, \ldots, x^m y^{\beta_m}\}$, with $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_m$.

For $0 \leq l \leq m$, let $b_l$ be the number of degree-$(l + \beta_l)$ hand monomials above $x^l y^{\beta_l}$.

Let $P'$ be the partition obtained by deleting the first column of $P$. The morphism $\mathbb{V}(E_P) \to \mathbb{V}(E_{P'}) : I \mapsto (I : x)$ is a trivial fibration whose fiber has dimension $b_0$ and we have $\mathbb{V}(E_{P'}) \cong k^{b_0} \times \mathbb{V}(E_{P'})$ (see for example [Y1], Prop. 1.7). We then obtain the proof of the theorem by induction. \qed

Examples of the projection map $\pi$.

Example 2.27. Let $T = (1, 2, 2, 1)$. We have $\delta_2 = t_1 - t_2 = 0, \delta_3 = t_2 - t_3 = 1, \delta_4 = t_3 - t_4 = 1$, and $\mathcal{B}(T) = (\mathcal{B}_2, \mathcal{B}_3) = ((1 \times 1)_2, (1 \times 2)_3)$. Here the product of “small” Grassmannians is

$$G = \text{Grass}(\delta_2 + 1, \delta_2 + 1 + \delta_3) \times \text{Grass}(\delta_3 + 1, \delta_3 + 1 + \delta_4) = \text{Grass}(1, 2) \times \text{Grass}(2, 3).$$

Consider the partition $P = (3, 3)$.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
\hline
0 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
2 & 3 & 4 \\
\hline
\end{tabular}
\end{center}
By Definition 2.23, we have $V_2 = \langle y^2, x^2 \rangle$, $W_2 = \langle y^2, xy, x^2 \rangle$, $V_3 = \langle xy^2, x^2y, x^3 \rangle$ and $W_3 = V_3$.

Any element $I$ in the cell $\mathcal{V}(E_P)$ is of the form $I = (y^2 + a_1 xy + a_2 x^2, x^3)$, with $(a_1, a_2) \in k^2$. So $\mathcal{V}(E_P)$ is a two dimensional affine space.

For $I = (y^2 + a_1 xy + a_2 x^2, x^3) \in \mathcal{V}(E_P)$ we get $I \cap W_2 = (y^2 + a_1 xy + a_2 x^2)$ and the projection of $I \cap W_2$ on $V_2$ is $I_{V_2} = (y^2 + a_2 x^2) \in \text{Grass}(1, V_2)$. Also $I \cap W_3 = \langle xy^2 + a_1 x^2 y, x^3 \rangle = I_{V_3} \in \text{Grass}(2, V_3)$. We view $\mathcal{V}(E_P)$ as a product of two cells, one in $\text{Grass}(1, 2)$ corresponding to single block $T_2 = (1, 1)$ and another one in $\text{Grass}(2, 3)$ corresponding to $T_3 = (1, 2, 1)$.

The difference-one hook code of $P = (3, 3)$ is

$$\Omega(P) = ((1)_2, (1)_3) \subset \mathcal{B}(T) = ((1 \times 1)_2, (1 \times 2)_3).$$

The code $(1)_2$ corresponds to the vector space $V_2$ and the small cell $\mathcal{V}(E_{P_2}) = \{ \langle y^2 + a_2 x^2 \rangle, a_2 \in k \} \subset \text{Grass}(1, 2) \cong \text{Grass}(1, V_2)$:

$$\begin{array}{c}
| & | \\
\hline
\end{array}$$

The code $(1)_3$ corresponds to the vector space $V_3$ and the small cell $\mathcal{V}(E_{P_3}) = \{ \langle xy^2 + a_1 x^2 y, x^3 \rangle, a_1 \in k \} \subset \text{Grass}(2, 3) \cong \text{Grass}(2, V_3)$ (note, these are labelled by degree: $V_i \subset R_i$):

$$\begin{array}{c|c}
| & | \\
\hline
\hline
\end{array}$$

So we get $\mathcal{V}(E_P)$ as a product of two affine lines: $\mathcal{V}(E_P) = \mathcal{V}(E_{P_2}) \times \mathcal{V}(E_{P_3})$.

**Example 2.28.** Let $T = (1, 2, 3, 4, 5, 4, 2)$ and consider $P = (5^3, 3, 1^3)$. (See Figure 7) We have $(\delta_5, \delta_6, \delta_7) = (1, 2, 2)$. The two single-block Hilbert functions associated to $T$ are $T_5 = (1, 2, 3, 2)$ and $T_6 = (1, 2, 3, 4, 2)$ (see Definition 2.16). We will view the cell $\mathcal{V}(E_P)$ as a product of cells in $\text{Grass}(2, 4) \times \text{Grass}(3, 5)$.

The difference-one hook code of $P$ is $\Omega(P) = ((1)_5, (2)_6)$. The partition $P_5 = (3^2, 2)$ of diagonal lengths $T_5$ has hook code $\Omega(P_5) = ((1)_5)$ (the degree-5 block of $\Omega(P)$). The partition $P_6 = (4^2, 2, 1^2)$ of diagonal lengths $T_6$ has hook code $\Omega(P_6) = ((2, 0))$ (the degree-6 block of $\Omega(P)$).

We now display the isomorphism $\pi : \mathcal{V}(E_P) \rightarrow \mathcal{V}(E_{P_5}) \times \mathcal{V}(E_{P_6})$.

By Definition 2.23 we have

$$V_5 = \langle xy^4, x^2 y^3, x^4 y, x^5 \rangle, W_5 = R_5, V_6 = \langle y^6, x^2 y^4, x^3 y^3, x^4 y^2, x^5 y \rangle = W_6.$$  

The projection of $I_5$ onto $V_5$ is a 2-dimensional vector space

$$I_{V_5} = \langle xy^4 + \alpha_1 x^2 y^3 + \alpha_2 x^4 y, x^5 \rangle, (\alpha_1, \alpha_2) \in k^2.$$  

The projection of $I_6$ onto $V_6$ is a 3-dimensional vector space

$$I_{V_6} = \langle x^2 y^4 + a_1 x^4 y^2, x^3 y^3 + a_2 x^4 y^2, x^5 y \rangle, (a_1, a_2) \in k^2.$$  

19
The bijection $(L_5, L_6) \in \mathcal{V}(E_{P_5}) \times \mathcal{V}(E_{P_6})$ with $L_5 = \langle xy^4 + \alpha_1 x^2 y^3 + \alpha_2 x^4 y, x^5 \rangle$, $(\alpha_1, \alpha_2) \in k^2$ and $L_6 = (x^2 y^3 + a_1 x^4 y^2, x^3 y^3 + a_2 x^4 y^2, x^5 y)$, $(a_1, a_2) \in k^2$. Then, using standard basis techniques, one can see that there is a unique ideal $I \in \mathcal{V}(E_P)$ such that $I_{V_5} = L_5$ and $I_{V_6} = L_6$:

$$I = \langle x^5, x^3 y^3 + a_2 x^4 y^2, x(y^4 + a_1 x^2 y^2) + \alpha_1 (x^3 y + a_2 x^2 y^2) + \alpha_2 x^3 y), y^7 \rangle.$$

**Example 2.29.** Let $T = (1, 2, 3, 4, 5, 6, 7, 5, 2, 0)$ and consider the partition $P = (9, 7^2, 4^2, 2, 1^2)$. (See Figure 8). We have $(\delta_7, \delta_8, \delta_9) = (2, 3, 2)$, $T_7 = (1, 2, 3, 4, 5, 3)$ and $T_8 = (1, 2, 3, 4, 5, 2)$.

The product of small Grassmannians here is

$$G = \text{Grass}(\delta_7 + 1, \delta_7 + 1 + \delta_8) \times \text{Grass}(\delta_8 + 1, \delta_8 + 1 + \delta_9) = \text{Grass}(3, 6) \times \text{Grass}(4, 6).$$

The difference-one hook code of $P$ is $\Omega(P) = ((3, 2, 0)_7, (4, 3)_8)$.

(a) In degree 7 we have $V_{7,1} = \{x^4 y^3, x^2 y^5, x y^6\}$, $V_{7,2} = \{x^6 y, x^3 y^4, y^7\}$, so $V_7$ has basis $(x^6 y, x^4 y^3, x^3 y^4, x^2 y^5, x y^6, y^7)$ in the lex order. We consider the bijection $s_7 : V_{7,1} \cup V_{7,2} \rightarrow \text{Mon}(R_5)$ given by:

$$s_7(x^6 y) = x^5, \quad s_7(x^4 y^3) = x^4 y, \quad s_7(x^3 y^4) = x^3 y^2$$

$$s_7(x^2 y^5) = x^2 y^3, \quad s_7(x y^6) = x y^4, \quad s_7(y^7) = y^5.$$

We then get a one block partition $P_7$ (see Figure 9)

(b) In degree 8, $V_{8,1} = \{x^7 y, x^5 y^3, x^3 y^5, y^8\}$, $V_{8,2} = \{x^8, x^6 y^2\}$. $V_8$ has basis $(x^8, x^7 y, x^5 y^3, x^3 y^5, y^8)$ in the lex order.

The bijection $s_8 : V_{8,1} \cup V_{8,2} \rightarrow \text{Mon}(R_5)$ given by:

$$s_8(x^8) = x^5, \quad s_8(x^7 y) = x^4 y, \quad s_8(x^6 y^2) = x^3 y^2$$

$$s_8(x^5 y^3) = x^2 y^3, \quad s_8(x^3 y^5) = x y^4, \quad s_8(y^8) = y^5.$$

This gives us a one block partition $P_8$ (see Figure 10)
3 Number of generators for a single-block partition

We first state the known bounds for the number of generators of a graded ideal \( I \) of Hilbert function \( H(R/I) = T \) for arbitrary \( T \). In Theorem 3.11 we determine the number of generators \( \kappa(P) \) for generic ideals in the cell \( \mathcal{V}(E_P) \) where \( P \) has diagonal lengths \( T \) satisfying the single-block Equation (3.3).
3.1 Lower bound $\kappa(T)$ on the number of generators of an ideal $I$ in $G_T$.

We recall Equation (1.1) for an arbitrary codimension two Hilbert function $T$:

$$T = (1, 2, \ldots, d-1, d, t_d, \ldots, t_j, 0)$$

where $d \geq t_d \geq t_{d+1} \geq \cdots \geq t_j > 0$.

Here $j$ is the (highest) socle degree of $A = R/I$. Recall from Section 2.1 that $G_T$ is the irreducible projective variety parametrizing the graded ideals $I$ in $R = k[x, y]$ such that $A = R/I$ has Hilbert function $T$.

**Definition 3.1 (Order of a Hilbert function $T$).** Let $T$ be a sequence satisfying Equation (1.1). Set $\nu(T) = d$, usually called the order of $T$: that is $\nu(T)$ is the order of graded ideals $I \in G_T$ — that define an Artinian algebra $A = R/I$ of Hilbert function $T$.

**Definition 3.2.** We let $\kappa(T)$ be the minimum number of generators for the ideal $I$ corresponding to a generic element of $G_T$. If $P$ is a partition of diagonal lengths $T$, we denote by $E_P$ the monomial ideal associated to $P$ and set $\kappa(P)$ to be the minimum number of generators for a generic element $I$ in the cell $V(E_P)$.

Given a sequence $T$ satisfying Equation (1.1), recall that we denote by $\delta_i$ the first difference function of $T$:

$$\delta_i = t_{i-1} - t_i$$

for $i \in [\nu(T), j + 1]$. 

The following result (i)-(ii) is shown in [Ia1, Theorem 4.3, Lemma 4.5], but a separate proof will also result from our work here (see Remark 5.14). A different proof of (ii) is given by M. Mandal and M.E. Rossi in [MR, Theorem 2.1]. The statement (iii) is obvious (see discussion after Lemma 8.1). We denote by $[k]^+ = \max\{k, 0\}$.

**Lemma 3.3.** Let $T$ satisfy Equation (1.1), and let $I$ be a homogeneous ideal such that $A = R/I$ has Hilbert function $T$. Then

i. $I$ has at least $[\delta_i - \delta_{i-1}]^+$ generators of each degree $i \geq \nu(T)$.
ii. A generic graded ideal \( I \in G_T \) has
\[
\kappa(T) = 1 + \delta_{\nu(T)} + \sum_{i > \nu(T)} [\delta_i - \delta_{i+1}]^+ \tag{3.2}
\]
generators, exactly \([\delta_i - \delta_{i-1}]^+\) in each degree \(i \geq \nu(T)\).

iii. The ideal \( E_{P_0} \), \( P_0 = T \) (listed as a partition) has \( \nu(T) + 1 = 1 + \sum_{i \geq \nu(T)} \delta_i \) generators, and \( \kappa_{E_{P_0}} = \delta_i \) for \( i > \nu(T) \) and \( 1 + \delta_{\nu(T)} \) for \( i = \nu(T) \). This is the termwise maximum \( \kappa_I(z) \) that occurs for any ideal \( I \in G_T \): that is \( \kappa_{I,i} \leq \delta_i \) for \( i > \nu(T) \) and \( \kappa_{I,\nu(T)} \leq 1 + \delta_{\nu(T)} \).

**Definition 3.4.** If \( P \) is a partition of diagonal lengths \( T \) such that \( \kappa(P) \neq \kappa(T) \), then we say \( P \) is special. If \( \kappa(P) = \kappa(T) \) we say \( P \) is non-special.

**Example 3.5.** Let \( T = (1,2,3,2,1) \). We have \( \kappa(T) = 2 \), as the generic ideal in \( G_T \) is a complete intersection of generator degrees (3.3). For \( P = (5,3,1) \) we also have \( \kappa(P) = 2 \): here \( \mathcal{V}(E_P) \) is open dense in \( G_T \), so \( P = (5,3,1) \) is non-special. But for \( P = (3,3,1^3) \) we have \( \kappa(P) = 3 \) since an \( R \)-relation between the generators \( y^5, y^2 x + \cdots \) cannot yield the generator \( x^3 \): so \( P = (3,3,1^3) \) is special.

### 3.2 Single block partitions \( P \), and \( \kappa(P) \).

Henceforth in this section we let \( T \) be a Hilbert function that satisfies
\[
T = (1,2,\ldots,d-1,d,t_d,0). \tag{3.3}
\]
where \( d \geq t_d \) and we let \( s = d+1-t_d \). We term this a single-block Hilbert function.

In this case, \( G_T \) is isomorphic to the Grassmannian variety \( \text{Grass}(s,R_d) \) where \( R_d \) is the vector space of the degree \( d \) homogeneous forms of \( R = k[x,y] \):
\[
\Phi: G_T \rightarrow \text{Grass}(s,R_d) \quad I \mapsto I_d
\]

Also, by Equation (3.2) we have for a single-block Hilbert function
\[
\kappa(T) = s + \delta, \text{ where } \delta = \max\{t_d + 1 - s, 0\}. \tag{3.4}
\]

Let \( P \) be a partition of diagonal lengths \( T \). The corners of the Ferrers diagram of \( P \) correspond to monomials \( x^\alpha y^\beta \) that belong to a minimal set of generators for the monomial ideal \( E_P \). We may call such monomials, corner-monomials of \( P \). Let \( I \in G_T \) be an element of the cell \( \mathcal{V}(E_P) \). Then the corner-monomials of \( P \) are leading terms of a system of generators \( \mathfrak{B}(I) \) of \( I \). The system \( \mathfrak{B}(I) \) may not be minimal. By definition of \( T \), a minimal set of generators of \( E_P \) must contain \( s \) degree \( d = \nu(T) \) corner-monomials. These degree \( d \) corner-monomials are leading monomials for the degree \( d \) elements of the system of generators \( \mathfrak{B}(I) \). Since we are looking for a minimal set of generators for \( I \), we want a criterion to decide that a degree \( d + 1 \) element of \( \mathfrak{B}(I) \) can be obtained using a relation involving degree \( d \) elements of \( \mathfrak{B}(I) \). That is where corner “kick-off” comes into play.
Let $a$ be integer such that $0 \leq a < d = \nu(T)$ and set $d' = d - a$. Suppose $m$ is an integer such that $1 \leq m \leq d'$. For any integer $i$ such that $0 \leq i \leq m$, set $K_i = x^{\alpha_i} \left( x^{d-i} y^{d-m+i} \right)$. The $K_i$’s form a set of $m + 1$ consecutive degree $d$ monomials in two variables:

$$x^{a+m} y^{d'-m}, x^{a+m-1} y^{d'-m+1}, \ldots, x^{a+1} y^{d'-1}, x^{a} y^{d'}.$$

From these $m + 1$ consecutive monomials we have $m$ relations: $yK_i - xK_{i+1} = 0$, $(0 \leq i < m)$. Suppose the $K_i$’s are leading monomials of some elements of $\mathfrak{B}(I)$. In order to produce $m$ degree $d + 1$ elements of $\mathfrak{B}(I)$ with these relations, we have to assume that $2(m + 1) \leq d' + 2 = \dim_k(R_{d+1})$, that is, $2m \leq d'$.

Note that by definition, if $f_0, \ldots, f_m$ are degree-$i$ forms such that $f_i$ has leading monomial $K_i$, then $x^a$ divides any element of the ideal $I = (f_0, \ldots, f_m)$. Therefore, for simplicity we now assume $a = 0$, so $K_i = x^{m-i} y^{d-m+i}, 2m \leq d$, and we let $N_1, \ldots, N_m$ be the $m$ degree $d + 1$ monomials given by

$$N_i = x^{\alpha_i} y^{\beta_i}, \quad \alpha_i + \beta_i = d + 1,$$

$$0 \leq \beta_1 < \beta_2 < \cdots < \beta_m < d - m,$$

$$m + 1 < \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 \leq d + 1.$$

Concerning the next Lemma, although J. Briançon and A. Galligo state their standard basis result that we use in characteristic zero, it is valid also for characteristic bigger than the degree $d$. This is the key step in the paper where we need to restrict the characteristic of $k$.

**Lemma 3.6** (How to kick off corners). With the above notation, there exist $m + 1$ degree $d$ forms $f_0, \ldots, f_m$ such that $f_i$ has leading monomial $K_i$ and $N_i$ is a leading monomial of a degree $d + 1$ element of the ideal $I = (f_0, \ldots, f_m)$.

**Proof.** Using a technique of standard basis calculations developed by J. Briançon and A. Galligo in [Br-Ga], we can inductively construct $f_0, \ldots, f_m$ such that $N_i \in I = (f_0, \ldots, f_m)$. Let

$$f_0 = x^m y^{d-m}, \quad f_1 = x^{m-1} y^{d-m+1} + \lambda_1 x^{\alpha_1-1} y^{\beta_1}.$$

One can see that $xf_1 - yf_0 = \lambda_1 x^{\alpha_1} y^{\beta_1}$, so if we set $\lambda_1 \neq 0$, we have

$$N_1 = x^{\alpha_1} y^{\beta_1} \in (f_0, \ldots, f_m).$$

In general, for $0 \leq i < m$, suppose that we have $f_i = x^{m-i} u_i + \lambda_i x^{\alpha_i-1} y^{\beta_i}$ where $u_i$ is a degree $d - m + i$ form such that $u_i(0, y) = y^{d-m+i}$. Then we set

$$f_{i+1} = x^{m-i-1} y \left( u_i + \lambda_i x^{\alpha_i-1-m+i} y^{\beta_i} \right) + \lambda_{i+1} x^{\alpha_{i+1}-1} y^{\beta_{i+1}}.$$

So, $xf_{i+1} - yf_i = \lambda_{i+1} x^{\alpha_{i+1}+1} y^{\beta_{i+1}}$ and for $\lambda_{i+1} \neq 0$, we have $N_{i+1} \in (f_0, \ldots, f_m)$. Note that for $i = 0$, $u_0 = y^{d-m}, \lambda_0 = 0$; for $i = 1$, $u_1 = y^{d-m+1}$; thus, inductively, we have constructed $f_0, \ldots, f_m$ such that $N_i \in I = (f_0, \ldots, f_m)$. \qed
Remark 3.7 (Choosing which corner should be kicked off). Given \( r \) indices \( i_1, \ldots, i_r \) such that \( 1 \leq i_1 < \ldots < i_r \leq m \), in the inductive construction of \((f_0, \ldots, f_m)\) of Lemma 3.6 if we let \( \lambda_{i_l} = 0 \) \((1 \leq l \leq r)\), then none of the monomials \( N_{i_l} \) will be kicked off. So, if \( \lambda_{i_l} = 0 \) for \( 1 \leq l \leq r \), then \( N_{i_l} / \notin (f_0, \ldots, f_m) \).

Lemma 3.8 (Counting the corner-monomials of \( P \)). Let \( T = (1, 2, \ldots, d, t_d = t_0, t > 0 \) and set \( s = d + 1 - t \). Suppose that \( P \) is a partition of diagonal lengths \( T \) and difference-one hook code \( \Omega(P) = (h_1^1, \ldots, h_n^0) \) \((where s \geq h_1 > h_2 > \cdots > h_n \geq 0)\). Then the minimum number of generators \( b_1(E_P) \) of the monomial ideal \( E_P \) is given by the following formula.

\[
\begin{align*}
b_1(E_P) &= s + t - n, \quad \text{if } h_1 < s \text{ and } h_n > 0 \\
b_1(E_P) &= s + t - n + 1, \quad \text{if } h_1 = s \text{ and } h_n > 0, \text{ or } h_1 < s \text{ and } h_n = 0 \\
b_1(E_P) &= s + t - n + 2, \quad \text{if } h_1 = s \text{ and } h_n = 0.
\end{align*}
\]

Proof. This is an easy count that we obtain by looking at the Ferrers diagram of \( P \). \(\square\)

Note that \( b_1(E_P) - \kappa(P) \) counts the number of degree \( d + 1 \) corner-monomials we have been able to kick-off.

Example 3.9. Suppose \( T = (1, 2, 3, 4, 5, 6, 7, 8, 4) \). Then \( d = 8, t_d = 4 \) and \( s = 5 \). Let \( P \) be the partition of diagonal lengths \( T \) defined by \( P = (9, 7^2, 6, 4^2, 2, 1) \) (See Figure 11). We have \( \Omega(P) = (5, 4^2, 3) \). The monomial ideal \( E_P \) associated to \( P \) is generated by \((y^8, xy^7, x^2y^6, x^4y^4, x^7y, x^6y^3, x^9)\). Using Lemma 3.6 we see that the degree 9 corners of \( P \) associated to \( x^6y^3 \) and \( x^9 \) can be kicked-off using the degree 8 corners associated to the consecutive monomials \( y^8, xy^7, x^2y^6 \).

Figure 11: Kicking off corners of the partition \( P = (9, 7^2, 6, 4^2, 2, 1) \) (Example 3.9).

Let \( P \) be a partition of diagonal lengths \( T \). Suppose \( P = (p_1, \ldots, p_m) \), with \( p_1 \geq p_2 \geq \cdots \geq p_m \). Let \( P' = (p'_1, \ldots, p'_m) \) with \( p'_i = p_i - 1 \). Let \( T' \) be the
Hilbert function associated to $P'$. If $I$ is an element of the cell $\mathcal{V}(E_P)$ of $G_T$, then $(I : x)$ is an element of the cell $\mathcal{V}(E_{P'})$ of $G_{T'}$. In fact we have a morphism $\varphi : \mathcal{V}(E_P) \to \mathcal{V}(E_{P'})$ defined by $I \mapsto (I : x)$ whose fiber is an affine space of dimension the number of difference-one hooks having their feet at $y^{n-1}$ (Theorem 3.1 and Lemma 3.3).

Lemma 3.10. Assume that $T = (1, \ldots, d, t_d, 0)$, and that $P$ is a partition having diagonal lengths $T$ and difference-one hook code $\Omega(P) = (h_1^t, \ldots, h_n^t)$. Set $s = d + 1 - t_d$. Suppose that $I$ is a generic ideal in the cell $\mathcal{V}(E_P)$ and let $\bar{I} = (I : x)$.

(a) If $h_n = 0$ then $\bar{I} \in \mathcal{V}(E_P)$ where $\bar{P}$ is the partition of diagonal lengths $\bar{T} = (1, \ldots, d-1, t-1)$ and hook code $\Omega(\bar{P}) = (h_1^t, \ldots, h_n^{t-1})$. Furthermore, in this case $\kappa(P) = \kappa(\bar{P}) + 1$.

(b) If $h_n > 0$ then $\bar{I} \in \mathcal{V}(E_P)$ where $\bar{P}$ is the partition of diagonal lengths $\bar{T} = (1, \ldots, d-1, t)$ and hook code $\Omega(\bar{P}) = ((h_1 - 1)^t, \ldots, (h_n - 1)^{t_n})$. Furthermore, in this case

$$\kappa(P) = \begin{cases} \kappa(\bar{P}), & \text{if } \kappa(\bar{P}) \geq s \\ s, & \text{if } \kappa(\bar{P}) = s - 1. \end{cases}$$

Proof. We note that the Ferrers diagram of $\bar{P}$ is always obtained from the Ferrers diagram of $P$ by removing the first column. Let $\mathfrak{B} = \{f_1, \ldots, f_\kappa\}$ be a minimal set of generators for $I$ (the $f_i$'s are ordered according to their leading monomials, from top to bottom in the Ferrers diagram).

Part (a) If $h_n = 0$, then $\mathfrak{B} = \{xf_1, \ldots, xf_\kappa, y^{d+1}\}$ is a minimal set of generators for $I$. Thus the equality in part (a) holds.

Part (b). If $h_n > 0$ then by definition $\bar{s} = s - 1$. This in particular implies that in this case $\kappa(\bar{P}) \geq s - 1$.

Assume that $h_n = 1$. Then the leading term of $f_\kappa$ is $y^d$ (we can even set $f_\kappa = y^d$ here). Let $g$ be a generic polynomial with leading term $y^d$. Then $\mathfrak{B} = \{xf_1, \ldots, xf_{\kappa-1}, g\}$ is a minimal set of generators for $I$. Thus in this case $\kappa(P) = \kappa(\bar{P})$. We also note that since $\kappa(P) \geq s$ (Definition 3.1 and Lemma 3.3) the equality $\kappa(P) = \kappa(\bar{P})$ in particular implies that when $h_n = 1$, we have $\kappa(\bar{P}) \geq s$.

Next, assume that $h_n > 1$. Suppose $\kappa(\bar{P}) = s - 1$ (this is the minimum value possible for $\kappa(\bar{P})$). In this case, all the degree $d$ corner-monomials of $P'$ have been kicked-off. After multiplication by $x$, these degree $d$ corner-monomials of $P'$ become degree $d + 1$ corner-monomials of $P$, so are kicked-off by $(xf_1, \ldots, xf_\kappa)$ and therefore $\mathfrak{B} = \{xf_1, \ldots, xf_\kappa, g\}$, where $g$ is a generic polynomial with leading term $y^d$, is a minimal set of generators of the generic element of $\mathcal{V}(E_P)$, so $\kappa(P) = \kappa(\bar{P}) + 1$.

Now, suppose $\kappa(\bar{P}) > s - 1$. This means that there is at least one degree-$d$ form in any minimal set of generators of a generic element of $\mathcal{V}(E_{P'})$. So we have $\mathfrak{B} = \{f_1, \ldots, f_{\kappa-s+1}, f_{\kappa-s+2}, \ldots, f_\kappa\}$, $\deg(f_{\kappa-s+1}) = d$, $\deg(f_{\kappa-s+2}) = \cdots = \deg(f_\kappa) = d - 1$. Note that $f_\kappa$ has leading monomial $y^{d-1}$. Now, let $g = yf_\kappa + \lambda f_{\kappa-s+1}$ ($\lambda \neq 0$). Then $xy - y(xf_\kappa) = \lambda xf_{\kappa-s+1}$. Since $\lambda \neq 0$, this means that $xf_{\kappa-s+1}$ can be kicked off. If $\mathfrak{B} = \{f_1, \ldots, f_{\kappa-s+1}, f_{\kappa-s+2}, \ldots, f_\kappa\}$ is a minimal
Theorem 3.11 \(T\). Theorem 3.11 case.\(\kappa\) note that as discussed at the beginning of this section, if \(T\) is a minimal set of generators of \(\mathcal{I}\), then a minimal system of generators for \(\mathcal{I}\) is a minimal set of generators of \(\mathcal{I}\). Thus \(\kappa(P) = \kappa(P)\). □

Recall that for a partition \(P\) of diagonal lengths \(T\), we denote the minimum number of generators for a generic element \(I\) in the cell \(\mathcal{V}(E_P)\) by \(\kappa(P)\). Also note that as discussed at the beginning of this section, if \(T = (1, \ldots, d, t_d, 0)\) is a single-block Hilbert function, then a minimal system of generators for \(I\) consists of \(s = d + 1 - t_d\) generators of degree \(d\) and \((\kappa(P) - s)\) generators of degree \(d + 1\). The following theorem provides an explicit formula for \(\kappa(P)\) in the single block case.

**Theorem 3.11 (The invariant \(\kappa(P)\) for a single block \(T\)).** Assume that \(T = (1, \ldots, d, t_d = t, 0)\) and let \(P\) be a partition of diagonal lengths \(T\) and difference-one hook code \(\Omega(P) = (h_1^l, \ldots, h_n^l)\). For \(k = 1, \ldots, n\), let \(\tau_k = \sum_{i=k}^n l_i - h_k\). Then

\[
\kappa(P) = s + \max \{t + 1 - s, 0, \tau_k\}_{k=1, \ldots, n}.
\]

**Proof.** We prove the theorem by induction on \(d\).

First assume that \(d = 2\).

If \(t = 2\), then \(s = 1\). In this case there are three partitions of diagonal lengths \(T\), namely

- (i) Partition \(P = (3, 2)\) with hook code \(\Omega(P) = (1^2)\), \(\kappa(P) = 3\) and
  \[
  s + \max \{t + 1 - s, 0, \tau_1\} = 1 + \max \{2, 0, 2 - 1\} = 3;
  \]

- (ii) Partition \(P = (3, 1^2)\) with hook code \(\Omega(P) = (1, 0)\), \(\kappa(P) = 3\) and
  \[
  s + \max \{t + 1 - s, 0, \tau_1, \tau_2\} = 1 + \max \{2, 0, 2 - 1, 1 - 0\} = 3;
  \]

- (iii) Partition \(P = (2^2, 1)\) with hook code \(\Omega(P) = (0^2)\), \(\kappa(P) = 3\) and
  \[
  s + \max \{t + 1 - s, 0, \tau_1\} = 1 + \max \{2, 0, 2 - 0\} = 3.
  \]

On the other hand, if \(t = 1\), then \(s = 2\). In this case, there are three partitions of diagonal lengths \(T\), namely

- (i) Partition \(P = (3, 1)\) with hook code \(\Omega(P) = (2)\), \(\kappa(P) = 2\) and
  \[
  s + \max \{t + 1 - s, s, s + \tau_1\} = 2 + \max \{0, 0, 1 - 2\} = 2;
  \]

- (ii) Partition \(P = (2^2)\) with hook code \(\Omega(P) = (1)\), \(\kappa(P) = 3\) and
  \[
  s + \max \{t + 1 - s, 0, \tau_1\} = 2 + \max \{0, 0, 1 - 1\} = 2;
  \]

- (iii) Partition \(P = (2, 1^2)\) with hook code \(\Omega(P) = (0)\), \(\kappa(P) = 3\) and
  \[
  s + \max \{t + 1 - s, 0, \tau_1\} = 2 + \max \{0, 0, 1 - 0\} = 3.
  \]
This shows that the desired equality holds when $d = 2$.

Now assume that $d > 2$ and that Equation (3.5) holds for any partition of diagonal lengths $(1, \ldots, d', t, 0)$ with $d' < d$.

Suppose that $P$ is a partition of diagonal lengths $T = (1, \ldots, d, t)$ and hook code $Q(P) = (h_1^1, \ldots, h_n^1)$. Let $\bar{P}$ be the partition associated to $P$ defined in Lemma 3.10. Then by the inductive hypothesis Equation (3.5) holds for $\kappa(\bar{P})$.

**Case 1.** Assume that $h_n = 0$. Then $\bar{t} = t - 1$, $\bar{s} = s$, and for $k = 1, \ldots, n$, we have $\bar{\tau}_k = \tau_k - 1$. Thus

\[
\kappa(\bar{P}) = \bar{s} + \max\{\bar{t} + 1 - \bar{s}, 0, \bar{\tau}_k\}_{k=1,\ldots,n} \\
= s + \max\{t - s, 0, \tau_k - 1\}_{k=1,\ldots,n}.
\]

Since $\tau_n = l_n - h_n = l_n \geq 1$, we have

\[
\max\{t - s, 0, \tau_k - 1\}_{k=1,\ldots,n} = \max\{t - s, \tau_k - 1\}_{k=1,\ldots,n} \\
= \max\{t + 1 - s, \tau_k\}_{k=1,\ldots,n} - 1.
\]

Thus using part (a) of Lemma 3.10 we have

\[
\kappa(P) = \kappa(\bar{P}) + 1 \\
s + \max\{t + 1 - s, \tau_k\}_{k=1,\ldots,n} \\
s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,\ldots,n}.
\]

**Case 2.** Assume that $h_n > 0$. Then $\bar{t} = t$, $\bar{s} = s - 1$, and for $k = 1, \ldots, n$, we have $\bar{\tau}_k = \tau_k + 1$. By the inductive hypothesis

\[
\kappa(\bar{P}) = \bar{s} + \max\{\bar{t} + 1 - \bar{s}, 0, \bar{\tau}_k\}_{k=1,\ldots,n} \\
= s - 1 + \max\{t + 1 - s + 1, 0, \tau_k + 1\}_{k=1,\ldots,n}.
\]

If $h_n = 1$, then $\tau_n = l_n - 1 \geq 0$. Furthermore, if $\kappa(\bar{P}) \geq s$ then $t + 1 \geq s$ or $\tau_k \geq 0$ for some $k$. In either of these cases, we have

\[
\max\{t + 1 - s + 1, 0, \tau_k + 1\}_{k=1,\ldots,n} = \max\{t + 1 - s + 1, \tau_k + 1\}_{k=1,\ldots,n}.
\]

Thus, using Lemma 3.10 we have

\[
\kappa(P) = \kappa(\bar{P}) \\
= s - 1 + \max\{t + 1 - s + 1, \tau_k + 1\}_{k=1,\ldots,n} \\
= s + \max\{t + 1 - s, \tau_k\}_{k=1,\ldots,n} \\
= s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,\ldots,n}.
\]

Finally, if $\kappa(\bar{P}) = s - 1$, then $t + 1 \leq s - 1$ and $\tau_k + 1 \leq 0$, for all $k = 1, \ldots, n$. This in particular implies that in this case $s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,\ldots,n} = s$.

By Lemma 3.10 we also have

\[
\kappa(P) = \kappa(\bar{P}) + 1 \\
= s - 1 + 1 \\
= s
\]
**Corollary 3.12** (Special partitions). Assume that $P$ is a single block partition. Then $P$ is special if and only if some $\tau_k$ from Theorem 3.11 satisfies $\tau_k > \delta$ where $\delta = \max\{t_d + 1 - s, 0\}$.

*Proof.* This follows from Equation 3.12 and Theorem 3.11. $\square$

**Remark 3.13.** We note that if at least one entry in the hook code of $P$ is zero, then $\tau_n = l_n > 0$. Thus, in this case $\kappa(P) \geq s + 1$. This in particular implies that in part (b) of Lemma 3.10 if $h_n = 1$ then the hook code of $P$ has a zero entry and therefore $\kappa(P) \geq s + 1 = s$.

### 4 Partitions in $P(T)$ having a given number of generators, for single-block $T$.

We begin with a result counting the total number of partitions having diagonal lengths a single-block Hilbert function $T$. We then in Theorem 4.2 count those associated to a given generic number of generators $\kappa(P)$. Throughout the section $T$ will be a single-block Hilbert function $T = (1, 2, \ldots, t)$.

Lemma 4.1. The number of partitions having the single-block diagonal lengths $T = (1, 2, \ldots, t)$ satisfies

$$
\#P(T) = \binom{s + t}{t}.
$$

*Proof.* By Theorem 2.9, $\#P(T)$ counts the total number of partitions whose Ferrers diagram can be placed in a $t \times s$ box $B_d(T)$ or, equivalently, the number of lattice paths from $(0, 0)$ to $(s, t)$, which satisfies (4.1). Or, $\#P(T)$ by Equation 2.4 is $\sum_{n=0}^{s-t} B(s, t, n) = \sum_i h^i(\text{Grass}(s, s + t)) = B(s, t)_{q=1} = \binom{s+t}{s}$. $\square$

By Theorem 3.11, the number of generators for a generic ideal in the cell $\mathcal{V}(E_P)$ is $\kappa(P) = s + \max\{\delta, \tau_k\}_{k=1, \ldots, n}$. In particular, for all partitions $P$ of diagonal lengths $T$, we have

$$
\kappa(T) = s + \delta \leq \kappa(P) \leq s + t.
$$

**Theorem 4.2** (Number of special partitions of diagonal lengths $T$). Let $T = (1, \ldots, d, t)$, $s = d + 1 - t$, and $\delta = \max\{t + 1 - s, 0\}$. Assume that $k$ is an integer such that $s + \delta < k \leq s + t$. Then the number of partitions $P$ of diagonal lengths $T$ and $\kappa(P) \geq k$ is

$$
\binom{s + t}{k}.
$$

In particular the number of special partitions of diagonal lengths $T$ is

$$
\binom{s + t}{s + \delta + 1} = \binom{s + t}{\min\{s - 1, t\}}.
$$

And the number of non-special partitions of diagonal lengths $T$ is

$$
\binom{s + t}{s} - \binom{s + t}{s + \delta + 1}.
$$
Proof. We use induction on \(d\). For \(d = 2\) the statement holds by the discussion of this case in the proof of Theorem 3.11.

Now assume that \(d > 2\). For a partition \(P\) with hook code \(\mathcal{Q}(P) = (h_1^l, \ldots, h_n^l)\), we define \(\bar{P}\) as in Lemma 3.10.

**Case 1.** Assume that \(P\) is such that \(h_n = 0\).

Then by Lemma 3.10 we have \(\kappa(P) = \kappa(\bar{P}) + 1\). Thus \(\kappa(P) \geq k\) if and only if \(\kappa(\bar{P}) \geq k - 1\).

Moreover, in this case \(\bar{s} = s\) and \(\bar{t} = t - 1\). Hence in this case,

\[
\bar{\delta} = \max\{\bar{s} - \bar{t}, 0\} = \begin{cases} \bar{s} - \bar{t}, & \text{if } \bar{\delta} \geq 1 \\ 0, & \text{if } \bar{\delta} = 0 \end{cases}
\]

If \(\delta \geq 1\), then \(\bar{\delta} = \delta - 1\) and therefore \(\bar{s} + \bar{\delta} = \bar{s} + \delta - 1 < k - 1\). On the other hand, if \(\delta = 0\) and \(s + 1 < k\), we still have the desired inequalities \(\bar{s} < k - 1 \leq \bar{s} + \bar{\delta}\). Thus by the inductive hypothesis, in these cases, the number of partitions \(P\) with \(\kappa(P) \geq k\) satisfies

\[
\binom{\bar{s} + \bar{t}}{k - 1} = \binom{s + t - 1}{k - 1}.
\]

Finally, if \(\delta = 0\) and \(k = \bar{s} + 1\), then all partitions \(\bar{P}\) have at least \(k - 1 = s = \bar{s}\) generators. The number of such partitions satisfies

\[
\binom{\bar{s} + \bar{t}}{k - 1} = \binom{s + t - 1}{t - 1} = \binom{s + t - 1}{s} = \binom{s + t - 1}{k - 1}.
\]

Thus there are \(\binom{s + t - 1}{k - 1}\) partitions \(P\) with \(\kappa(P) \geq k\) and a zero entry in the hook code.

**Case 2.** Assume that \(h_n > 0\). If \(s + \delta < k \leq s + t\) and \(\kappa(P) \geq k\), then in particular \(\kappa(P) > s\). Thus by part (b) of Lemma 3.10 we have \(\kappa(\bar{P}) \geq s\), and \(\kappa(P) = \kappa(\bar{P})\).

We also have \(\bar{s} = s - 1\) and \(\bar{t} = t\). Thus \(\bar{\delta} = \max\{t + 1 - s, 1, 0\}\). In other words, \(\bar{\delta} = \delta + 1\) if \(s \leq t + 1\), and \(\bar{\delta} = \delta = 0\), otherwise. This shows that if \(s + \delta < k\) then \(\bar{s} + \bar{\delta} < \bar{s}\) as well. We also note that if \(k < t + s\), then \(k \leq \bar{s} + \bar{t}\), while for \(k = s + t\), there is no partition \(\bar{P}\) with \(\kappa(P) \geq k\).

By the inductive hypothesis, the number of partitions \(P\) with \(\kappa(P) \geq k\) is

\[
\binom{\bar{s} + \bar{t}}{k} = \binom{s + t - 1}{k}.
\]

Combining the count for the two possible cases, we see that the total number of partitions \(P\) with \(\kappa(P) \geq k\) is

\[
\binom{s + t - 1}{k - 1} + \binom{s + t - 1}{k} = \binom{s + t}{k}.
\]

The first equality of Equation (4.3) is the special case \(k = s + \delta + 1\), the second follows from \(s + t - (s + \delta + 1) = \min\{s - 1, t\}\).

The count of non-special partitions follows from Lemma 4.4 giving the total number \(\binom{s + t}{t}\) of partitions of diagonal lengths \(T\) and the count of special partitions. \(\square\)
Corollary 4.3. Let \( T = (1, \ldots, d, t_d = t, 0) \), \( s = d + 1 - t \), and \( \delta = \max \{t + 1 - s, 0\} \). For a positive integer \( k \), we define \( \mu(T, k) \) to be the number of partitions \( P \) with diagonal lengths \( T \) and \( \kappa(P) = k \). Then
\[
\mu(T, k) = \begin{cases} 
(s+t) - (s+\delta+1), & \text{if } k = s + \delta \text{ (non-special } P), \\
(s+t) - (k+1), & \text{if } s + \delta < k \leq s + t \\
0, & \text{otherwise}.
\end{cases}
\]

Remark 4.4. Obviously, by its definition, for all \( k \), the invariant \( \mu(T, k) \) is non-negative. We also note that the inequalities
\[
s + t \leq 2 \max \{s, t\} \leq 2 \max \{s, t + 1\} = 2(s + \delta)
\]

imply that \( s + \delta \leq \frac{s+t}{2} \). Thus, if \( s + \delta < k \leq s + t \), then \( \binom{s+t}{k} > \binom{s+t}{k+1} \). As for \( k = s + \delta \), if \( s > t \) then \( s > \frac{s+t}{2} \) and therefore
\[
\binom{s+t}{s} > \binom{s+t}{s+1} = \binom{s+t}{s+\delta+1}.
\]

On the other hand, if \( s \leq t \), then \( s \leq \frac{s+t}{2} \). Therefore, \( \binom{s+t}{s-2} < \binom{s+t}{s} \). Moreover, we have \( \delta = (t + 1) - s \), and therefore,
\[
\binom{s+t}{s+\delta+1} = \binom{s+t}{t+2} = \binom{s+t}{s-2} < \binom{s+t}{s}.
\]

We also note that for \( k = s + \delta \), the number \( \mu(T, k) \) is the coefficient of the degree \( k = s + \delta \) term in \((1+z)^{s+t}(z^\delta - \frac{1}{z})\) while for \( s + \delta < k < s + t \), the number \( \mu(T, k) \) is the same as the coefficient of the degree \( k \) term in \((1+z)^{s+t}(1-\frac{1}{z})\).

Theorem 2.11 implies that for any single-block Hilbert function \( T \), there will be a unique minimal finite set of special partitions of diagonal lengths \( T \), such that any special partition is in the closure of the minimal set. The next example shows that the special cells do not form an irreducible subfamily of \( G_T \).

Example 4.5 (Single block table). Let \( T = (1, 2, 3, 4, 2, 0) \), then \( t = 2, s = 3 \) and \( \mathfrak{B}(T) = (\mathfrak{B}_4(T)) = (2 \times 3) \), and there are \( \binom{5}{2} = 10 \) partitions of diagonal lengths \( T \). We give Table 4.1 for these, specifying the branch label, hook code, and \( \kappa(P) \) for each. The branch label is from the Section 7 Definition 7.1 and Theorem 7.2. Here \( \delta(T) = \max \{0, s + 1 - t\} = 0 \) and \( \kappa(T) = 3 \). We have placed conjugate partitions in symmetric positions from the center line; the two middle partitions of hook codes \((3, 0)\) and \((2, 1)\) are self-conjugate. Note also that the conjugate partition \( P^\vee \) has the reverse branch label, and the complementary hook code in \( \mathfrak{B}_3(T) \).

Figure 4.12 gives the specialization diagram for \( P(T) \), corresponding to inclusion of the Ferrers diagrams for the hook codes \( h_4(P) \) (on the left).

We see from the table that the cells in \( \kappa(P) \geq 4 \) are the union of the closures of cells having hook codes \((1, 1)\) and \((3, 0)\): \( \kappa(P) = 4 \) includes also the cells with hook codes \((2, 0)\) and \((1, 0)\), while the cell with hook code \((0, 0)\) is the unique with \( \kappa(P) = 5 \) (these cells are colored red/blue on the left of Figure 4.12). Thus, the subvariety of cells corresponding to special partitions is here the union of two irreducible components, of dimensions three (closure of \((3, 0)\) and two (closure of \((1, 1)\)), respectively.
Table 4.1: Table of $\mathcal{P}(T)$, $T = (1, 2, 3, 4, 2, 0)$. See Example 4.5.

| $P$       | $b_4$    | $d_4$ | $\tau_1$ | $\tau_2$ | $\kappa(P)$ |
|-----------|----------|-------|-----------|-----------|-------------|
| $(5, 4, 2, 1)$ | $(1^2, 0^3)$ | $(3, 3)$ | $-1$   |           | $3$         |
| $(5, 3, 3, 1)$ | $(1, 0, 1, 0^2)$ | $(3, 2)$ | $-1$   | $-1$     | $3$         |
| $(5, 3, 2, 2)$ | $(1, 0^2, 1, 0)$ | $(3, 1)$ | $0$    | $-1$     | $3$         |
| $(4, 4, 3, 1)$ | $(0^1, 1^2, 0^2)$ | $(2, 2)$ | $0$    |           | $3$         |
| $(5, 3, 2, 1, 1)$ | $(1, 0^3, 1)$ | $(3, 0)$ | $-2$   | $1$      | $4$         |
| $(4, 4, 2, 2)$ | $(0, 1, 0, 1, 0)$ | $(2, 1)$ | $0$    | $0$      | $3$         |
| $(4, 3, 3, 2)$ | $(0^2, 1^2, 0)$ | $(1, 1)$ | $1$    | $0$      | $4$         |
| $(4, 4, 2, 1, 1)$ | $(0, 1, 0^2, 1)$ | $(2, 0)$ | $0$    | $1$      | $4$         |
| $(4, 3, 3, 1, 1)$ | $(0^2, 1, 0, 1)$ | $(1, 0)$ | $1$    | $1$      | $4$         |
| $(4, 3, 2, 2, 1)$ | $(0^3, 1^2)$ | $(0, 0)$ | $2$    |           | $5$         |

Figure 12: Specialization diagram for $\mathcal{P}(T)$, $T = (1, 2, 3, 4, 2, 0)$. See Example 4.5.
5 Number of generators for multiblock partitions.

Throughout this section, \( T = (1, \ldots, d, t_d, \ldots, t_j, 0) \), and \( P \) is a partition lengths \( T \) and difference-one hook code \( \Omega(P) = (\mathfrak{h}_d, \ldots, \mathfrak{h}_j) \).

Recall from Equation (2.9) that for \( i = d, \ldots, j \), we set \( \delta_i = t_{i-1} - t_i \) and \( T_i = (1, \ldots, \delta_i + \delta_{i+1}, \delta_{i+1}, 0) \).

As we saw in Definition 2.16 and Lemma 2.18, a partition \( P \in \mathcal{P}(T) \) can be decomposed into single-block “component” partitions \( P_i \). For \( i = d, \ldots, j \), the partition \( P_i \) has diagonal lengths \( T_i \) and difference-one hook code \( \mathfrak{h}_i \). We note that, although by construction the hook code for \( P_i \) is \( \mathfrak{h}_i \), the corresponding hook degree in \( P_i \) is \( \delta_i + \delta_{i+1} \) and not \( i \). We showed in Theorem 2.26 that the cells \( \mathcal{V}(E_P) \) are naturally the product of the corresponding cells \( \mathcal{V}(E_{P_i}) \).

In this section we count the minimum number of generators for ideals in the cell associated to an arbitrary partition. The main results of this section are the following Theorem relating to the components, and Theorem 5.13, which specifies the number \( \beta_{i,0}(P) \) of degree-\( i \) generators of an ideal \( I \) defining a generic element of \( \mathcal{V}(E_P) \).

**Theorem 5.1** (Decomposition of \( \kappa(P) \) into components). Let \( P \) be a partition lengths \( T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0) \). Then

\[
\kappa(P) = \sum_{i=d}^{j} \kappa(P_i) - (t_d - t_j) - (j - d). \tag{5.1}
\]

Moreover, a minimal set of generators for a generic ideal in the cell \( \mathcal{V}(E_P) \) includes \( d + 1 - t_d \) generators of degree \( d \) and \( \kappa(P_i) - (\delta_i + 1) \) generators of degree \( i + 1 \), for \( i \in [d, j] \).

The value \( \kappa(P_i) \) for a single block partition was determined in Theorem 3.11. We establish the Theorem by proving the series of statements that follow, culminating in Corollary 5.16, which is an avatar of the Theorem.

The \( i \)-th block \( \mathfrak{h}_i(P) \) of the hook code \( \Omega(P) \) is a key element of our statements. So we first give a formula for \( \kappa(P_i) \) when \( \mathfrak{h}_i(P) \) is empty. Recall that the case \( \mathfrak{h}_i = \emptyset \) occurs when \( t_i = t_{i+1} \) and in this case, we can think of \( P_i \) as the basic triangle \( \Delta_{\delta_i} \) (Remark 2.17).

**Proposition 5.2.** If the block \( \mathfrak{h}_i(P) \) is empty, then \( \kappa(P_i) = \delta_i + 1 \).

**Proof.** Recall that the case \( \mathfrak{h}_i = \emptyset \) occurs when \( t_i = t_{i+1} \) and in this case, we can think of \( P_i \) as the basic triangle \( \Delta_{\delta_i} \) (see Remark 2.17). It is then clear that the monomial ideal whose cobasis is the basic triangle \( \Delta_{\delta_i} \) has exactly \( \delta_i + 1 \) generators, the degree \( \delta_i \) monomials. Note that the basic triangle \( \Delta_{\delta_i} \) is empty when \( \delta_i = 0 \).
Example 5.3. Consider the partition $P = (8, 7, 4, 2, 1, 1)$ of diagonal lengths $T = (1, 2, 3, 4, 5, 4, 2, 2, 0)$. The degree 6 block $h_6(P)$ is empty ($t_6 = t_7 = 2$). $\kappa(P_6) = \delta_6 + 1 = 3$ and this number corresponds to the three border monomials $(x^4y^2, x^3y^3, y^6)$ of $E_P$.

From now on, we will assume that the $i$-th block $h_i(P)$ of the hook code $Q(P)$ is not empty. For $i = d, \ldots, j$, the degree $i$ block $h_i$ in the hook code of $P$ can be written as

$$h_i = \left( h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}} \right)$$

where $\sum_{k=1}^{n_i} l_{i,k} = \delta_i + 1$ and $\delta_i + 1 \geq h_{i,1} > h_{i,2} > \cdots > h_{i,n_i} \geq 0$.

As in the single-block case, we construct two sequences of integers $R(P)_i$ and $G(P)_i$, that allow us to count degree $i+1$ relations and degree $i+1$ generators of the monomial ideal $E_P$ associated to $P$.

Definition 5.4 (Sequences of degree $i+1$ relations and generators). Denote by $R_{i,k}$ the set of degree $i+1$ relations below the $(l_{i,1} + \cdots + l_{i,k})$-th degree $i$ hand of $P$. Then we let $R(P)_i = (r_{i,1}, \ldots, r_{i,n_i})$ where $r_{i,k}$ counts the number of elements of $R_{i,k}$.

Also, let $G_{i,1}$ be the set of of degree $i+1$ corner-monomials of $E_P$ and for $2 \leq k \leq n_i$, denote by $G_{i,k}$ the set of degree $i+1$ corner-monomials of $E_P$ below the $(l_{i,1} + \cdots + l_{i,k-1} + 1)$-th degree $i$ hand of $P$. Then $G(P)_i = (g_{i,1}, \ldots, g_{i,n_i})$ where $g_{i,k}$ counts the number of elements of $G_{i,k}$.

Remark 5.5 (chains of degree $i+1$ relations and generators). Note that using the above notation, we have the following sequences of inclusions.

$$R_{i,n_i} \subset R_{i,n_i-1} \subset \cdots \subset R_{i,k+1} \subset R_{i,k} \subset \cdots \subset R_{i,1}$$
$$G_{i,n_i} \subset G_{i,n_i-1} \subset \cdots \subset G_{i,k+1} \subset G_{i,k} \subset \cdots \subset G_{i,1}.$$

So by definition, $R(P)_i = (r_{i,1}, \ldots, r_{i,n_i})$ and $G(P)_i = (g_{i,1}, \ldots, g_{i,n_i})$ are non-increasing sequences.

We now use the hook code $h_i = \left( h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}} \right)$ to give formulas for the integers $r_{i,k}$ and $g_{i,k}$ of Definition 5.4.

Observation 5.6. By definition of the difference-one hook code, for any integer $1 \leq k \leq n_i$, there are $(h_{i,k} - h_{i,k+1})$ horizontal-border monomials below the $(l_{i,1} + \cdots + l_{i,k})$-th degree $i$ hand and above the $(l_{i,1} + \cdots + l_{i,k} + 1)$-th degree $i$ hand of $P$. Let $(M_{i,1}, \ldots, M_{i,h_{i,k} - h_{i,k+1}})$ be the list of these monomials, ordered from top to bottom. Then there is exactly one degree $(i+1)$ relation between any two consecutive monomials $M_{i,m}$ and $M_{i,m+1}$ ($1 \leq m \leq h_{i,k} - h_{i,k+1}$). So, we have exactly $(h_{i,k} - h_{i,k+1} - 1)$ degree
Lemma 5.7 (Counting degree $i + 1$ relations and corner monomials). Let $\mathfrak{h}_i = (h_{i,1}, \ldots, h_{i,k}, \ldots, h_{i,n_i})$ be the $i$-th block of the hook code $\mathcal{Q}(P)$. Then the sequences $R(P)_i = (r_{i,1}, \ldots, r_{i,n_i})$ and $G(P)_i = (g_{i,1}, \ldots, g_{i,n_i})$ of Definition 5.4 are given by the following numbers.

(a) If $h_{i,n_i} > 0$, then

- $g_{i,1} = \begin{cases} 
\delta_{i+1} - (n_i - 1) & \text{if } h_{i,1} = \delta_i + 1 \\
\delta_{i+1} - n_i & \text{if } h_{i,1} < \delta_i + 1 
\end{cases}$

- $g_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - (n_i + 1 - k)$, for $2 \leq k \leq n_i$

- $r_{i,k} = h_{i,k} - (n_i + 1 - k)$, for $1 \leq k \leq n_i$.

(b) If $h_{i,n_i} = 0$, then

- $g_{i,1} = \begin{cases} 
\delta_{i+1} - (n_i - 2) & \text{if } h_{i,1} = \delta_i + 1 \\
\delta_{i+1} - (n_i - 1) & \text{if } h_{i,1} < \delta_i + 1 
\end{cases}$

- $g_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - (n_i - k)$, for $2 \leq k \leq n_i$

- $r_{i,k} = h_{i,k} - (n_i - k)$ for $1 \leq k \leq n_i$.

Proof. Note that by definition of the difference-one hook code, we have $\delta_i + 1 \geq h_{i,1} > \cdots > h_{i,n_i} \geq 0$ and $\sum_{k=1}^{n_i} l_{i,k} = \delta_{i+1}$.

The numbers given in the Lemma come directly from Observation 5.6.

(a) If $h_{i,n_i} > 0$, then we have

- $g_{i,1} = c_{i,1} + (l_{i,2} - 1) + \cdots + (l_{i,n_i} - 1) = c_{i,1} + \sum_{k=2}^{n_i} l_{i,k} - (n_i - 1)$ where

  $c_{i,1} = \begin{cases} 
l_{i,1} & \text{if } h_{i,1} = \delta_i + 1 \\
l_{i,1} - 1 & \text{if } h_{i,1} < \delta_i + 1 \end{cases}$, and for $2 \leq k \leq n_i$, we get

  $g_{i,k} = (l_{i,k} - 1) + \cdots + (l_{i,n_i} - 1) = \sum_{j=k}^{n_i} l_{i,j} - (n_i + 1 - k)$.

- $r_{i,k} = (h_{i,k} - h_{i,k+1} - 1) + \cdots + (h_{i,n_i} - h_{i,n_i} - 1) + (h_{i,n_i} - 1) = h_{i,k} - (n_i - (k - 1))$.

(b) If $h_{i,n_i} = 0$, then we have
Remark 5.8. Using the formulas for $r_{i,k}$ and $g_{i,k}$, if we let $\theta_{i,k} = g_{i,k} - r_{i,k}$ ($1 \leq k \leq n_i$), we obtain

\begin{itemize}
  \item $\theta_{i,1} = \left\{ \begin{array}{ll}
    \delta_{i+1} + 1 - h_{i,1} & \text{if } h_{i,1} = \delta_i + 1 \\
    \delta_{i+1} - h_{i,1} & \text{if } h_{i,1} < \delta_i + 1
  \end{array} \right.$

  \item $\theta_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k}$ for $2 \leq k \leq n_i$.
\end{itemize}

Remark 5.9. For $1 \leq k \leq n_i$, the formulas in Lemma 5.7 can be rewritten in a compact way using the invariants of the hook code Equation (5.2)

\begin{itemize}
  \item $g_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - (n_i + 1 - k) + \max\{1 - h_{i,n_i}, 0\} + \max\{h_{i,k} - \delta_i, 0\}$;

  \item $r_{i,k} = h_{i,k} - (n_i + 1 - k) + \max\{1 - h_{i,n_i}, 0\}$.
\end{itemize}

So, $\theta_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k} + \max\{h_{i,k} - \delta_i, 0\}$ for $1 \leq k \leq n_i$. These are the ingredients for the formula Equation (5.3) for $\beta_{i+1,0}(P) = N_{i+1} - N_i$ in Theorem 5.13 for the number of generators of degree-$i$ for an ideal $I$ defining a generic element $A$ of $\mathcal{V}(E_P)$: we will show there that $\beta_{i+1,0}(P)$ is just the integer max $\{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{1 \leq k \leq n_i}$ of the next Lemma.

Lemma 5.10. With the above notation, for $1 \leq k \leq n_i$, let $\tau_{i,k} = \sum_{j=k}^{n_i} l_{i,j} - h_{i,k}$.

Then max $\{0, \theta_{i,k}\}_{1 \leq k \leq n_i} = \max\{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{1 \leq k \leq n_i}$.

Proof. If $h_{i,1} = \delta_i + 1$, then $\theta_{i,1} = \delta_{i+1} - \delta_i = \tau_{i,1} + 1$.

We then have $\{0, \theta_{i,k}\}_{1 \leq k \leq n_i} = \{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{2 \leq k \leq n_i}$ because $\theta_{i,k} = \tau_{i,k}$ for $2 \leq k \leq n_i$. Thus max $\{0, \theta_{i,k}\}_{1 \leq k \leq n_i} = \max\{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{1 \leq k \leq n_i}$.

If $h_{i,1} < \delta_i + 1$, we have $\theta_{i,k} = \tau_{i,k}$ for $1 \leq k \leq n_i$ and $\delta_{i+1} - \delta_i \leq \tau_{i,1}$.

So again max $\{0, \theta_{i,k}\}_{1 \leq k \leq n_i} = \max\{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{1 \leq k \leq n_i}$.

\[\square\]
Remark 5.11. Let $h_i = \left( h_{i,1}^{l_i,1}, \ldots , h_{i,k}^{l_i,k}, \ldots , h_{i,n_i}^{l_i,n_i} \right)$ be the $i$-th block of the hook code $\Omega(P)$.

Moving along the $i$-th diagonal of $P$ from top to bottom, we see that $\sum_{j=k}^{j=n_i} l_{i,j}$ is, by definition, the number of degree-$i$ hand monomials of $P$ below the $(l_{i,1} + \ldots + l_{i,k-1})$-th degree-$i$ hand monomial of $P$. Also, by definition, $h_{i,k}$ is the number of degree $i$ horizontal-border monomials of $P$ below the $(l_{i,1} + \ldots + l_{i,k-1} + 1)$-th degree-$i$ hand monomial of $P$.

We may visualize the key integer $\tau_{i,k}$ related to $\beta_{i+1,0}(P)$ in the Ferrers diagram of $P$ by coloring the corresponding $\sum_{j=k}^{j=n_i} l_{i,j}$ hand monomials in red and the $h_{i,k}$ horizontal-border monomials in blue, in the next example (Figure 13).

Example 5.12. Let $T = (1,2,\ldots ,12,13,12,6)$ and consider the partition $P$ of diagonal lengths $T$ given by $P = (14^2, 12, 11^2, 10, 7^2, 5^3, 4, 3, 1)$. $T$ is a two-block Hilbert function. The hook code of $P$ is $\Omega(P) = (h_{13}, h_{14})$. We have $\delta_{13} = 1, \delta_{14} = 6, \delta_{15} = 6, h_{13}(P) = (2^3, 1^1, 0^2)$ and $h_{14}(P) = (6^1, 4^2, 2^3)$.

Note that $h_{13,1} = 2 = \delta_{13} + 1, h_{14,1} = 6 < \delta_{14} + 1$ and $\delta_{14} - \delta_{13} = 5$.

- $\tau_{13,1}, \tau_{13,2}$ and $\tau_{13,3}$ are computed using the hook code block $h_{13} = (2^3, 1^1, 0^2)$.

\[ \tau_{13,1} = \text{sum of red integers minus the blue integer} \]
\[ \tau_{i,k} = \sum_{j=k}^{j=n_i} l_{i,j} - h_{i,k} \]

- $\tau_{13,2} = 1 + 2 - 1 = 2$. $\tau_{13,2}$ can be visualised by coloring the subsequence $(1^1, 0^2)$ of $(2^3, 1^1, 0^2)$: $(2^3, 1^1, 0^2) = (2^3, 1^1, 0^2)$, so $\tau_{13,2} = \text{sum of red integers minus the blue integer}$.

- $\tau_{13,3} = 2 - 0 = 2$, computed using $(2^3, 1^1, 0^2) = (2^3, 1^1, 0^2)$; $\tau_{13,3} = \text{sum of red integers minus the blue integer}$.

We then find that $\beta_{14,0}(P) = \max \{ \delta_{14} - \delta_{13}, 0, \tau_{13,1}, \tau_{13,2}, \tau_{13,3} \} = 5$.

- to compute $\tau_{14,1}, \tau_{14,2}$ and $\tau_{14,3}$ we can use the same coloring method on the hook code block $h_{14} = (6^1, 4^2, 2^3)$.

\[ \tau_{14,1} = 1 + 2 + 3 - 6 = 0; \text{ this is } \tau_{14,1} = \text{sum of red integers minus the blue integer}, \text{ using the coloring } (6^1, 4^2, 2^3) \]

\[ \tau_{14,2} = 2 + 3 - 4 = 1 \text{ and } \tau_{14,3} = 3 - 2 = 1 \text{ using the colorings } (6^1, 4^2, 2^3), (6^1, 4^2, 2^3) \]

We then find that $\beta_{15,0}(P) = \max \{ \delta_{15} - \delta_{14}, 0, \tau_{14,1}, \tau_{14,2}, \tau_{14,3} \} = 1$. 

37
We illustrate $\tau_{14,2}$ in Figure 13 by coloring the degree-$i$ hand monomials and the degree-$i$ horizontal-border monomials as suggested in Remark 5.11.

Figure 13: $P = (14^2, 12, 11^2, 10, 7^2, 5^3, 4, 3, 1)$: $\tau_{14,2} = 2 + 3 - 4 = 1$
Theorem 5.13. Let \( P \in \mathcal{P}(T), T = (1, \ldots, d, t_d, \ldots, t_j, 0) \) have hook code \( \Omega(P) = (h_1, \ldots, h_i, \ldots, h_j) \). Suppose \( h_i = \left( h_{i,1}^{l_1}, \ldots, h_{i,k}^{l_k}, \ldots, h_{i,n_i}^{l_{n_i}} \right) \) is the \( i \)-th block of the hook code \( \Omega(P) \) of \( P \). Let \( I \) be a generic element of the cell \( \mathcal{V}(E_P) \) and \( \mathcal{G}(I) \) a minimal set of generators of \( I \), and \( \beta_{i,0}(P) \) the number of degree-\( i \) generators. We denote by \( \kappa(P) = \beta_0(P) = (\beta_{d,0}(P), \ldots, \beta_{j+1,0}(P)) \).

For \( d \leq m \leq j + 1 \), let

\[
N_m = \# \{ f \in \mathcal{G}(I), \text{ such that degree}(f) \leq m \}.
\]

Then \( N_d = d + 1 - t_d \), and using the previously defined numbers \( r_{i,k} \) and \( g_{i,k} \), for \( i \in [d,j] \), we have for \( \beta_{i+1,0}(P) = N_{i+1} - N_i \),

\[
N_{i+1} - N_i = \max \left\{ 0, g_{i,k} - r_{i,k} \right\}_{1 \leq k \leq n_i} = \max \left\{ \delta_{i+1} - \delta_i, 0, \tau_{i,k} \right\}_{1 \leq k \leq n_i}.
\]

(5.3)

Proof. By standard basis construction techniques (see Theorem I.1.9 of [Br], Proposition 2 and Proposition 3 of [Br-Ga]), one can first see that if \( r_{i,n_i} \geq g_{i,n_i} \), then there are enough degree \( i + 1 \) relations to kick out all of the \( g_{i,n_i} \) generators that are just above these relations. Also, if \( r_{i,n_i} < g_{i,n_i} \), then we need \( g_{i,n_i} - r_{i,n_i} \) extra generators whose leading terms are corner monomials of \( E_P \) below the \((l_{i,1} + \cdots + l_{i,n_i-1} + 1)\)-th degree \( i \) hand of \( P \).

It is clear that if for all \( k \) \((1 \leq k \leq n_i)\) we have \( r_{i,k} \geq g_{i,k} \), then we can kick out all degree \( i + 1 \) generators whose leading terms are degree \( i + 1 \) corner-monomials of \( E_P \).

Now, suppose there exists an integer \( k \) such that \( r_{i,k} < g_{i,k} \). Then we can inductively consider the following sets and numbers.

\[
S_0 = \left\{ k \in \mathbb{N}, \quad r_{i,k} < g_{i,k} \right\}, \quad s_0 = \max(S_0);
\]

\[
S_1 = \left\{ k \in \mathbb{N}, \quad k < s_0, \quad r_{i,k} - r_{i,s_0} < g_{i,k} - g_{i,s_0} \right\}, \quad s_1 = \max(S_1);
\]

\[
\vdots
\]

\[
S_q = \left\{ k \in \mathbb{N}, \quad k < s_{q-1}, \quad r_{i,k} - r_{i,s_{q-1}} < g_{i,k} - g_{i,s_{q-1}} \right\}, \quad s_q = \max(S_q);
\]

\[
S_{q+1} = \emptyset.
\]

The meaning of the sets \( S_0, \ldots, S_q \) and the numbers \( s_0, \ldots, s_q \) is the following:

- First, we have \( r_{i,s_0+1} \geq g_{i,s_0+1} \ldots r_{i,n_i} \geq g_{i,n_i} \) and \( r_{i,s_0} < g_{i,s_0} \). So we need \( g_{i,s_0} - r_{i,s_0} \) generators in the \((n_i - s_0 + 1)\)-th part of the chain \( G_{i,n} \subset G_{i,n-1} \subset \cdots \subset G_{i,k+1} \subset G_{i,k} \subset \cdots \subset G_{i,1} \). This means we have used all the relations in the \((n_i - s_0 + 1)\)-th part of the chain \( R_{i,n} \subset R_{i,n-1} \subset \cdots \subset R_{i,k+1} \subset R_{i,k} \subset \cdots \subset R_{i,1} \).

- Since we have used all the relations in \( R_{i,n_i-s_0+1} \), if we are looking for more extra generators, the next step is to consider the chains of inclusions

\[
(R_{i,n_i-s_0} - R_{i,n_i-s_0+1}) \subset \cdots \subset (R_{i,1} - R_{i,n_i-s_0+1})
\]

\[
(G_{i,n_i-s_0} - G_{i,n_i-s_0+1}) \subset \cdots \subset (G_{i,1} - G_{i,n_i-s_0+1}).
\]
Corollary 5.15. Let $S_1 = \{ k \in \mathbb{N}, \quad k < s_0, \quad r_{i,k} - r_{i,s_0} < g_{i,k} - g_{i,s_0} \}$ be not empty, then we set $s_1 = \max(S_1)$ and continue looking for extra generators until $S_{q+1} = \emptyset$ and $S_q \neq \emptyset$ for some index $q$.

By construction, the number of degree $i + 1$ extra generators needed is

$$g_{i,s_0} - r_{i,s_0} + \sum_{j=1}^{j=q} ((g_{i,s_j} - g_{i,s_{j-1}}) - (r_{i,s_j} - r_{i,s_{j-1}})) = g_{i,s_q} - r_{i,s_q}.$$

It is then clear that

$$N_{i+1} - N_i = g_{i,s_q} - r_{i,s_q} = \max \{0, g_{i,k} - r_{i,k}\}_{1 \leq k \leq n_i}.$$

Remark 5.14. Using the notation of Lemma 3.3 and Theorem 5.13 one has

1. $[\delta_{i+1} - \delta_i]^+ \leq \max \{\delta_{i+1} - \delta_i, 0, \tau_{i,k}\}_{1 \leq k \leq n_i} = N_{i+1} - N_i$, so the number of degree $i + 1$ generators of $I \in \mathcal{V}(E_P)$ is at least $[\delta_{i+1} - \delta_i]^+$ (this is statement i. of Lemma 3.3).

2. Suppose $P$ is the partition associated to the generic cell of $G_T$. Then for the $i$-th block $\mathfrak{h}_i = (h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}})$ of the hook code $\mathcal{Q}(P)$ of $P$, we have $n_i = 1$, $h_{i,1} = \delta_i + 1$ and $l_{i,1} = \delta_{i+1}$, that is $\mathfrak{h}_i = (\delta_i + 1)^{h_{i,1}}$. In this case, $g_{i,1} = \delta_{i+1}, r_{i,1} = \delta_i$ and formula 5.3 of Theorem 5.13 gives $N_{i+1} - N_i = [\delta_{i+1} - \delta_i]^+$ (this is statement ii. of Lemma 3.3).

Corollary 5.15. Let $T_i = (1, \ldots, d-1, d, t_d, 0)$ be a single-block Hilbert function where $d = \delta_i + \delta_{i+1}$ and $t_d = \delta_{i+1}$. Suppose $P_i$ is a partition of diagonal lengths $T_i$ and difference-one hook code $\mathcal{Q}(P_i) = \mathfrak{h}_i$ with

$$\mathfrak{h}_i = (h_{i,1}^{l_{i,1}}, \ldots, h_{i,k}^{l_{i,k}}, \ldots, h_{i,n_i}^{l_{i,n_i}});$$

$$\delta_i + 1 \geq h_{i,1} > \cdots > h_{i,n_i} \geq 0 \quad \text{and} \quad \sum_{k=1}^{k=n_i} l_{i,k} = \delta_{i+1}.$$

Then $\kappa(P_i) = \delta_i + 1 + \max \{0, g_{i,k} - r_{i,k}\}_{1 \leq k \leq n_i}$.

Corollary 5.16. Let $T = (1, 2, \ldots, d-1, t_d, \ldots, t_j, 0)$ be a Hilbert function and $P$ a partition of diagonal lengths $T$. Let $I$ be a generic element of the cell $\mathcal{V}(E_P)$ and $\mathcal{G}(I)$ a minimal set of generators of $I$. For $d \leq i \leq j + 1$, let $N_i = \# \{ f \in \mathcal{G}(I), \text{degree}(f) \leq i \}$. Then

(a) for any $i \in [d, j]$, $N_{i+1} - N_i = \kappa(P_i) - (\delta_i + 1)$.

(b) $\kappa(P) = \# \mathcal{G}(I) = N_j + 1 = N_d + \sum_{i=d}^{i=j} (\kappa(P_i) - (\delta_i + 1))$. 

40
Remark 5.17. Note that if $t_i = t_{i+1}$, then it is clear that $N_{i+1} - N_i = 0$. Also, we found (Proposition 5.22) that when $t_i = t_{i+1}$, we have $\kappa(P_i) = \delta_i + 1$. So the empty hook blocs contribute to zero in the sum computing $\kappa(P)$ in Corollary 5.16.

Proposition 5.18. Let $P$ be a partition of diagonal lengths $T = (1, 2, \cdots, d-1, t_d, \cdots, t_j, 0)$ and suppose $\mathbf{b}_i = (h_{i,1}^{t_i}, \ldots, h_{i,k}^{t_i}, \ldots, h_{i,n_i}^{t_i})$ is the $i$-th block of the hook code $\Omega(P)$ of $P$. Denote by $b_{i+1}(E_P)$ the number of degree $i + 1$ corner-monomials (generators) of $E_P$. Then we have

$$b_d = d + 1 - t_d,$$

$$b_{i+1}(E_P) = \delta_{i+1} - n_i + \max\{1 - h_{i,n_i}, 0\} + \max\{h_{i,1} - \delta_i, 0\}.$$ 

Proof. The number of degree $d$ corner-monomials of $E_P$ is of course $d+1-t_d$. By Definition 5.4 we have $b_{i+1}(E_P) = g_{i,1}$. The proof of the Proposition then follows directly from Lemma 5.7 and Remark 5.9. 

We note that Theorem 5.1 is in fact a restatement of Corollary 5.16 above.

Recall from Definition 3.4 that a partition $P$ of diagonal lengths $T$ is special if $\kappa(P) \neq \kappa(T)$. In other words, $P$ is special if $\kappa(P)$ does not have the minimum value $\kappa(T)$ possible for partitions of diagonal lengths $T$, from Equation 3.2. The following immediate corollary of Theorem 5.1 gives a necessary and sufficient condition for a partition $P$ to be special. Recall that Corollary 3.12 specifies when a single-block partition is special.

Theorem 5.19 (Component Theorem for $P$ special). Assume that $T$ is a Hilbert function of height $d$ and socle degree $j$, and that the partition $P$ of diagonal lengths $T$ decomposes into single block partitions $P_d, \ldots, P_j$. Then $P$ is special if and only if $P_i$ is special, for some $i \in [d,j]$.

Proof. By Theorem 5.1 the value of $\kappa(P)$ is minimum if and only if $\kappa(P_i)$ is minimum for all $i \in [d,j]$. Thus $P$ is non-special if and only if at least one component $P_i$ is non-special for an integer $i \in [d,j]$. 

Example 5.20. Consider the partition $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$ from Example 2.22. Then $P$ has diagonal lengths $T = (1, 2, \ldots, 13, 10_{13}, 6_{14}, 3_{15}, 0)$ and hook code (see Figure 14)

$$\Omega(P) = ((3, 1^2, 0)_{13}, (5, 4, 1)_{14}, (2^2, 1)_{15}).$$

Then, as we saw in Example 2.22 $P$ can be decomposed into the following three single-block partitions. Partition $P_{13} = (7^2, 5, 4^2, 3, 1^2)$ of diagonal lengths $T_{13} = (1, \ldots, 7, 4, 0)$ and hook code $\Omega(P_{13}) = (3, 1^2, 0)$, partition $P_{14} = (8, 6^2, 4, 3, 2^2)$ of diagonal lengths $T_{14} = (1, \ldots, 7, 3, 0)$ and hook code

41
Figure 14: Ferrers diagram of the partition $P = (15, 12^4, 11, 7, 6^2, 5, 3^4)$ of diagonal lengths $T = (1, 2, \ldots, 13, 10_{13}, 6_{14}, 3_{15}, 0)$ and difference-one hook code $\Omega(P) = ((3, 1^2, 0)_{13}, (5, 4, 1)_{14}, (2^2, 1)_{15})$. Each labeled box represents a hook corner, it is labeled by a \circ if its hand degree is 13, with a \bullet if the hand degree is 14, and with a dark \bullet when the hand degree is 15 (Example 5.20).

$\Omega(P_{14}) = (5, 4, 1)$, and partition $P_{15} = (6, 5^2, 4, 2^2)$ of diagonal lengths $T_{15} = (1, \ldots, 6, 3, 0)$ and hook code $\Omega(P_{15}) = (2^2, 1)$.

By Theorem 3.11 we have

$$\kappa(P_{13}) = 6, \quad \kappa(P_{14}) = 5, \quad \text{and} \quad \kappa(P_{15}) = 5.$$  

Thus by Theorem 5.1 we have

$$\kappa(P) = 6 + 5 + 5 - (10 - 3) - 2 = 7.$$  

Let $I$ is a generic ideal in the cell $\mathcal{V}(E_P)$. Then a minimal set of generators for $I$ consists of seven generators. Of these seven generators, four have degree 13, two have degree 14, and one has degree 15.

**Elementary and non-elementary Hilbert functions.**

We say that a sequence $T$ satisfying Equation (1.1) is **elementary** if there is no integer $i \in [d, j]$ such that $t_i = t_{i+1} < d$ [LY1 §4A1]; then we also say that $G_T$ is elementary. It is well known that when a Hilbert function $T = H(R/I)$ satisfies $t_i = t_{i+1} = s < d$ then there is a form $f \in R_s$ such that

$$I_i = fR_{i-s} \text{ and } I_{i+1} = f \cdot R_{i+1-s}. \quad (5.4)$$

If follows that $f|I_u$ for $u \leq i + 1$. This is usually shown using the properties of $\tau(V) = \dim_k R_1 V - \dim_k V$ for vector subspaces $V \subset R_i$: this integer is the number of generators of an “ancestor ideal” $I = (V) \oplus_{u=1}^i V : R_u$, and $\tau(I_i) = 1$ when $t_i = t_{i+1}$; see [La1] p. 56 or [La2] Lemma 2.2.

We will define implicitly in the next Theorem “elementary factors” $T(i)$ of Hilbert sequences $T$ which have constant subsequences of height $s < d$.  

42
These factors have no relation with the single block components $T_i$ for each $T$, defined in Equations (1.3) and (2.9), and a major topic for us. In fact if $T$ splits into elementary components $T(i)$ they are not usually single-block.

**Lemma 5.21.** [IV, Lemma 4.2] There is a decomposition of $G_T$ as a product

$$G_T = \prod_k G_{T(k)}$$

for $T(k)$ elementary.

**Proof.** Assume there is a single maximal consecutive subsequence $t_i = t_{i+1} = \cdots = t_{i+k} = s$ with $k \geq 1$ and $s < d$. Then consider $T(1) = (1, 2, \ldots, s_{s-1}, s, \ldots, s_{i+k}, t_{i+k+1}, \ldots, t_j)$, and $T(2)$ defined by $T(2)_u = t_{u+s} - s$ for $u \leq i - s$. Let $p_I \in G_T$ be a point parametrizing the graded ideal $I$ such that $A = R/I$ satisfies $H(A) = T$. Then we let $I(1) = (f_s, I)$. We have $I_{u+s} = f_s V_u$ for $0 \leq u \leq i - s$: we define $I(2)_u = V_u$ for $u \in [0, i - s]$. Then the pair $(I(1), I(2))$ determines $I$ and conversely. This proves the Lemma for $k = 2$, it is straightforward to extend it to $k \geq 2$. $\square$

**Remark 5.22.** Let $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ be a Hilbert function as in Equation (1.1) and $P$ a partition of diagonal lengths $T$. Suppose there is an integer $i \in [d, j - 1]$ such that $t_i = t_{i+1} = \cdots = t_{i+k} = s$ with $k \geq 1$ and $s < d$. Let $I$ be a generic element of the cell $V(E_P)$ of $G(I)$ a minimal set of generators of $I$. Let $G(I)_1 = \{ f \in G(I), \text{degree}(f) \geq i + 1 \}$ and $G(I)_2 = \{ f \in G(I), \text{degree}(f) \leq i \}$. Setting $m_1 = |G(I)_1|$ and $m_2 = |G(I)_2|$, we get $\kappa(P) = m_1 + m_2$. We know from Equation (5.4) that there is a degree $s$ form $f_s$ such that $f_s$ divides each of the elements $f_1, \ldots, f_{m_2}$ of $G(I)_2$. Let $I(1) = (f_s, I)$ and $I(2) = (I : f_s)$. Then $I(1)$ is a generic element of a cell $V(E_{P(1)})$ and $I(2)$ is a generic element of a cell $V(E_{P(2)})$. It is clear that $\kappa(P(1)) = m_1 + 1$ and $\kappa(P(2)) = m_2$, so $\kappa(P) = \kappa(P(1)) + \kappa(P(2)) - 1$.

**Proposition 5.23.** Suppose that the variety $G_T$ decomposes as $G_T = \prod_{k=1}^{r} G_{T(k)}$ with each $T(k)$ elementary. Then any cell $V(E_P)$ of $G_T$ decomposes as

$$V(E_P) = \prod_{k=1}^{r} V(E_{P(k)})$$

for $P(k)$ a partition of diagonal lengths $T(k)$.

Also $\kappa(P) = \sum_{k=1}^{r} \kappa(P(k)) - r + 1$.

**Proof.** The Proposition follows from Remark 5.22. $\square$

**Example 5.24.** (See Figure 15) Let $P = (10^2, 4, 3^2, 2^5)$ be the partition of diagonal lengths

$$T = (1, 2, \ldots, 6_5, 5_6, 4_7, 4_8, 4_9, 2_{10}, 0),$$

43
and difference-one hook code

\[ \Omega(P) = ((0)_{6}, (1,0)_{9}, (2,1)_{10}). \]

Let \( I \) be a generic element of \( \mathcal{V}(E_P) \). The elementary components of \( T \), explained in Lemma 5.21 and Remark 5.22 are

\[ T(1) = (1, 2, 3, 4, 4_{5}, \ldots, 4_{9}, 2_{10}, 0), \quad \text{and} \quad T(2) = (1, 2, 1). \]

As it is explained in Remark 5.22 we let \( I(1) = (f_{4}, I) \) be a generic element in the cell \( \mathcal{V}(E_{P(1)}) \) with the Hilbert function \( T(1) \). Also \( I(2) = (I : f_{4}) \) is a generic element in the cell \( \mathcal{V}(E_{P(2)}) \) with the Hilbert function \( T(2) \). We have \( P(1) = (10^{2}, 2^{8}) \) and \( P(2) = (2, 1^{2}) \) which are subpartions of \( P \) in different colors in Figure 15. We easily see that \( \kappa(P(1)) = \kappa(P(2)) = 3 \) and therefore by Proposition 5.23 we get

\[ \kappa(P) = \kappa(P(1)) + \kappa(P(2)) - 1 = 5. \]

We could also compute \( \kappa(P) \) by decomposition of \( P \) and \( T \) into single-block components, see Equation 2.8. Single-block component partitions \( P_{6}, \ldots, P_{10} \) of diagonal lengths \( T_{6}, \ldots, T_{10} \) as follows,

\[ P_{6} = (2, 1, 1), \quad P_{7} = (1), \quad P_{8} = (0), \quad P_{9} = (3, 1, 1) \quad \text{and} \quad P_{10} = (4, 4, 2, 2), \]

\[ T_{6} = (1, 2, 1), \quad T_{7} = (1, 0), \quad T_{8} = (0), \quad T_{9} = (1, 2, 2, 0) \quad \text{and} \quad T_{10} = (1, 2, 3, 4, 2, 0). \]

The hook codes of \( P_{6}, P_{9} \) and \( P_{10} \) are \( h_{6} = (0), h_{9} = (1, 0) \) and \( h_{10} = (2, 1) \) respectively. Using Theorem 3.11 we get that

\[ \kappa(P_{6}) = \kappa(P_{9}) = \kappa(P_{10}) = 3, \]

and for \( P_{7} = \Delta_{1} \) and \( P_{8} = \Delta_{8} \) by Remark 2.17 we conclude that

\[ \kappa(P_{7}) = 2, \quad \kappa(P_{8}) = 1. \]

Therefore, Theorem 5.19 implies that

\[ \kappa(P) = 3 + 2 + 1 + 3 + 3 - (4 - 2) - (10 - 6) = 5. \]

We also note that of these five generators, two generators have degree 6 and one generator has degree 7 (corresponding to generators of \( P_{6} \)), and two have degree 10.

Note that \( \dim G_{T_{10}} = 2(3) = 6, \dim G_{T_{9}} = (2)(1) = 2, \) and \( \dim G_{T} = 8 \), since \( G_{T} \) is fibred over \( \mathbb{P}_{4} \) parametrizing the generator \( f_{4} \) of \( I_{8} \) by a Grassmanian Grass(2, 4) parametrizing \( I_{10}/f_{4}R_{6} \), a two-dimensional subspace of \( R_{10}/f_{4}R_{6} \), which has dimension four.
6 Number of cells of special multiblock partitions.

Using Corollary 4.3 and Theorem 5.1 we are able to count the number of multiblock partitions with a given number of generators.

Theorem 6.1. Assume that $T = (1, \ldots, d, t_d, \ldots, t_j, 0)$ and for $d \leq i \leq j$, let $T_i = (1, \ldots, t_{i-1} - t_{i+1}, t_i - t_{i+1}, 0)$. Then for every positive integer $k$, the number of partitions $P$ of diagonal lengths $T$ and $\kappa(P) = k$, denoted by $\mu(T, k)$, satisfies

$$\mu(T, k) = \sum_{(k_d, \ldots, k_j) \in Q_k} \left( \prod_{i=d}^{j} \mu(T_i, k_i) \right),$$

(6.1)

where $Q_k = \{(k_d, \ldots, k_j) \in \mathbb{Z}^{j+1-d} \mid k_d + \cdots + k_j = k + (t_d - d) - (t_j - j)\}$.

Proof. This is an immediate consequence of Theorem 5.1. Also recall that Corollary 4.3 provides an explicit formula for $\mu(T_i, k_i)$, for every $d \leq i \leq j$. \qed

Remark 6.2. For each $i \in [d, j]$, by Corollary 4.3 $\mu(T_i, k_i)$ is non-zero if and only if $\max\{t_i - t_{i+1} + 1, t_{i-1} - t_i + 1\} + 1 \leq k_i \leq t_{i-1} - t_{i+1} + 1$. Thus in Equation (6.1) we are effectively taking the sum over the points in the hyperplane defined by $k_d + \cdots + k_j = k + (t_d - d) - (t_j - j)$ in the hyper cubes obtained by the product of line segments of the form $[\max\{t_i - t_{i+1} + 1, t_{i-1} - t_i + 1\} + 1, t_{i-1} - t_{i+1} + 1]$ in $\mathbb{Z}^{j+1-d}$.

Recall that $\mathcal{P}(T)$ is the set of all partitions of diagonal lengths $T$. Denote by $A$ the cardinality of $\mathcal{P}(T)$. Also Recall from Definition 3.4 that a partition $P$ of diagonal lengths $T$ is called special if $\kappa(P) > \kappa(T)$ and denote by $S$ the number of special partitions of diagonal lengths $T$.

Using Definition 2.16 we decompose a partition $P$ of diagonal lengths $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ into $j+1-d$ single block partitions, $P_d, \ldots, P_j$, where for each $d \leq i \leq j$, the diagonal lengths of $P_i$ is...
$T_i = (1, \ldots, t_{i-1} - t_{i+1}, t_i - t_{i+1}, 0)$. For each $d \leq i \leq j$, we denote the total number of partitions of diagonal lengths $T_i$ by $A_i$ and the number of special partitions of diagonal lengths $T_i$ by $S_i$. The number of special partitions is equal to $\sum_{k>\kappa(T)} \mu(T, k)$, where $\mu(T, k)$ is described in the above theorem.

In the following, we provide the number of special partitions of diagonal lengths $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$, using the inclusion-exclusion principal.

**Corollary 6.3 (Number of special partitions).** The number of special partitions of diagonal lengths $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ is equal to

$$S = \sum_{i=1}^{j-d+1} (-1)^{i+1} \sum_{\lambda \subseteq \{d, \ldots, j\}, |\lambda| = i} S_\lambda A_{\{d, \ldots, j\} \setminus \lambda}, \quad (6.2)$$

where $S_\lambda = \prod_{i \in \lambda} S_i$ and $A_{\{d, \ldots, j\} \setminus \lambda} = \prod_{i \in \{d, \ldots, j\} \setminus \lambda} A_i$.

**Proof.** Theorem [5.19] implies that $P$ is special if and only if $P_i$ is special for some $i \in [d, j]$.

Note that for each $i \in [d, j]$ the number of partitions of diagonal lengths $T_i$ is equal to

$$A_i = \begin{pmatrix} t_{i-1} - t_{i+1} + 1 \\ t_i - t_{i+1} \end{pmatrix}.$$ 

On the other hand, Theorem [4.2] provides the number of special single block partitions. Using Equation [4.3] for each $d \leq i \leq j$ we obtain the number of special partitions of diagonal lengths $T_i$ as the following

$$S_i = \begin{pmatrix} t_{i-1} - t_{i+1} + 1 \\ t_i - t_{i+1} - \delta_i - 1 \end{pmatrix},$$

where $\delta_i = \max\{2t_i - 2t_{i+1} - t_{i-1} + t_{i+1}, 0\} = \max\{2t_i - t_{i+1} - t_{i-1}, 0\}$.

Now using the inclusion-exclusion principal we get the equality of Equation (6.2). \qed

As a consequence of the above Proposition, we recover a result of [AIK, Theorem 3.7] providing the number of complete intersection Jordan types $P \in \mathcal{P}(T)$. Recall that a complete intersection Jordan type of diagonal lengths $T$ is a partition $P$ of diagonal lengths $T$ such that $\kappa(P) = 2$.

**Corollary 6.4.** (a) The number of complete intersection Jordan types of diagonal lengths

$$T = (1_0, 2_1, \ldots, (d - 1)_{d-2}, d_{d-1}, (d - 1)_d, \ldots, 2_{2d-3}, 1_{2d-2})$$

is equal to $2^{d-1}$.

(b) The number of complete intersection Jordan types with diagonal lengths

$$T = (1_0, 2_1, \ldots, (d - 1)_{d-2}, d_{d-1}, \ldots, d_{d+k-2}, (d - 1)_{d+k-1}, \ldots, 2_{2d-4+k}, 1_{2d-3+k})$$

is equal to $2^d$. 

46
Proof. (a) In this case we have that $j = 2d - 2$ and the number of blocks in 
this case is equal to $d - 1$, we also have $t_{d-1} = d, t_d = d - 1, \ldots, t_j = 1$. 
For each $d \leq i \leq j$ we have that $T_i = (1, 2, 1)$, and clearly $A_i = 3$ 
and $S_i = 1$. So the total number of partitions of diagonal lengths $T$ 
is $A = 3^{d-1}$. On the other hand, using (6.2), we obtain the number of 
special partitions

$$S = \sum_{i=1}^{d-1} (-1)^{i+1} \sum_{\lambda \subseteq \{d, \ldots, 2d-2\}, |\lambda| = i} 1^{i} \cdot 3^{d-1-i}$$

$$= \sum_{i=1}^{d} (-1)^{i+1} \binom{d - 2}{i} 3^{d-1-i}$$

$$= 3^{d-1} - 2^{d-1}.$$

Thus the number of complete intersection Jordan types with the Hilbert 
function in (a) is equal to $A - S = 2^{d-1}$.

(b) In this case we have that $j = 2d - 3 + k$ and $t_{d-1} = \ldots = t_{k+d-2} = d, t_{d+k-1} = d - 1, \ldots, t_{2d+k-3} = 1$. For each $i \in [d, d + k - 3]$ we have 
$T_i = 0$ and clearly $A_i = 1$ and $S_i = 0$. We have $T_{d+k-2} = (1, 1)$, so 
$A_{d+k-2} = 2$ and $S_{d+k-2} = 0$. There are $d - 1$ more components for 
each $i \in [d + k - 1, 2d + k - 3]$ where $T_i = (1, 2, 1)$, $A_i = 3$ and $S_i = 1$, 
similar to the previous case. So the total number of partitions in this 
case is $A = 2 \cdot 3^{d-1}$

Using Equation (6.2) we obtain the number of special partitions

$$S = \sum_{i=1}^{d+k-2} (-1)^{i+1} \sum_{\lambda \subseteq \{d, \ldots, 2d-2\}, |\lambda| = i} 1^{i} \cdot 3^{d-1-i} \cdot 2$$

$$= 2 \sum_{i=1}^{d-1} (-1)^{i+1} \binom{d + k - 3}{i} 3^{d-1-i}$$

$$= 2 \cdot 3^{d-1} - 2^{d}.$$

Therefore the number of complete intersection Jordan types in this 
case is equal to $A - S = 2^{d}$.

\[\square\]

7 Correspondence between the hook code and 
the branch label of a partition in $\mathcal{P}(T)$.

The branch label $b(P)$ of $P$ specifies the lengths of branches of $P$ attached 
to the $d + 1$ attachment points of the basic triangle, listed from the top: 
for $P = (5, 3, 1)$ the branch label is $b(P) = (2, 1, 0, 0)$. There is a subtlety 
about horizontal vs. vertical branches, see Equation (7.2).
Definition 7.1 (Branch label). Let \( P \) be a partition of diagonal lengths \( T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0) \) satisfying Equation (1.1). The Ferrers diagram of a partition \( P \) is obtained by attaching branches of different lengths between 0 and \( j - d + 1 \) to the \( d + 1 \) attachment places of the basic triangle \( \Delta_d \). Let \( e \) be the lowest gap in the Ferrer’s diagram of \( P \). In other words,

\[
e = \max\{i \mid P \text{ has a gap in position corresponding to } x^{d-i} y^i\}.
\] (7.1)

Then in \( P \), we attach horizontal branches to rows 1 through \( e \) of \( \Delta_d \) (counted from the top), and vertical branches to columns 1 through \( d - e \) of \( \Delta_d \) (counted from the left). We write the branch label of \( P \) as

\[
b(P) = (b_1, \ldots, b_e, 0, v_1, \ldots, v_{d-e}),
\] (7.2)

where for \( i = 1, \ldots, e \), the integer \( b_i \) is the the length of the horizontal attachment to row \( i \), and for \( j = 1, \ldots, d - e \), the integer \( v_j \) is the the length of the vertical attachment to column \( j \).

For a single block partition \( P \), the correspondence of the branch label and the hook code associated to \( P \) is simply described using the definition.

Recall that \( T \) is a single-block Hilbert function if it satisfies

\[
T = (1, \ldots, d - 1, d, t_d = t, 0),
\] (7.3)

with \( t_i = i + 1 \) for \( i < d = \nu(T) \) and \( d \geq t_d \). We set \( s = d + 1 - t \). For any partition \( P \) of diagonal lengths \( T \) the Ferrers diagram \( C_P \) for \( P \) is comprised of \( \Delta_d \) augmented by \( t \) branches of length one: these are monomials of degree \( d \). Thus, the branch label, reckoned here from top right to bottom left, has the form

\[
b(P) = \left(\begin{array}{cccc}
0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, \ldots, 1, \ldots, 1, 0, \ldots, 0
\end{array}\right)
\] (7.4)

where \( n \) is the number of maximal 1-sequences in the branch label, and \( \sum_{i=1}^{n} l_i = t \).

Proposition 7.2 (Single block branch labels and hook code). Assume that \( T \) satisfies (7.3). The branch labels of partitions having diagonal lengths \( T \) run through all sequences as in Equation (7.4) satisfying

\[
h_i \leq s = d + 1 - t \text{ for each } i \in [1, n], \text{ and}
\]

\[
\sum_{i=1}^{n} l_i = t.
\] (7.5)

The hook code of a partition \( P \) with the single-block Hilbert function \( T \) of Equation (3.3) satisfies

\[
\Omega(P) = (h_1^{l_1}, h_2^{l_2}, \ldots, h_n^{l_n}),
\]
where \( s \geq h_1 > h_2 > \cdots > h_{n-1} > h_n \geq 0 \) and \( \sum_{i=1}^{n} l_i = t \). Every such sequence occurs.

The hook codes for elements of \( \mathcal{P}(T) \) run through all partitions having at most \( t \) non-zero parts, each less or equal \( s \). That is, the hook codes run through all partitions whose Ferrers graph lies in a \( t \times s \) box.

More generally, let \( P \) be an arbitrary partition of \( n \) of diagonal lengths \( T = (1, \ldots, d, t_d, \ldots, t_j, 0) \) and hook code \( \mathfrak{Q}(P) = (h_d, h_{d+1}, \ldots, h_j) \). Recall the decomposition of \( P \) into a sequence of single block partitions \( P_i \) with Hilbert functions \( T_i = (1, 2, \ldots, t_{i-1} - t_{i+1}, t_i - t_{i+1}, 0) \), for \( i = d, \ldots, j \) as in (2.9). By Lemma 2.18 we have that the hook code of each single block partition \( P_i \) is \( \mathfrak{Q}(P_i) = \mathfrak{h}_i \), which by Equation (5.2) we denote by

\[
\mathfrak{h}_i = (h_{i,1}^{t_i}, \ldots, h_{i,n_i})
\]

where \( \sum_{k=1}^{n_i} l_{i,k} = t_i - t_{i+1} = \delta_{i+1} \).

For each \( i = 1, \ldots, j \) denote by \( b(P_i) \) the branch label of the single block partition \( P_i \). Using Proposition 7.2 for each \( i = 1, \ldots, j \) we have

\[
b(P_i) = \begin{pmatrix}
0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 0, 0, \ldots, 0 \\
\delta_i+1-h_{i,1}, \delta_i+1-h_{i,1}, h_{i,1} - h_{i,2}, \ldots, h_{i,n_i} - h_{i,n_i}
\end{pmatrix}
\]

Directly going from \( b(P_i) \) to \( b(P) \) is complicated, the natural route is via the hook code.

The following Proposition describes how we obtain the hook code of \( P \), \( \mathfrak{Q}(P) \) from its branch label, \( b(P) \). For each \( i \in \{1, \ldots, j - d + 1\} \), an entry \( i \) in \( b(P) \) corresponds to an attachment of degree \( d + i \) and therefore an entry in \( \mathfrak{h}_{d+i} \).

**Proposition 7.3.** *(Hook code and branch label)* Let \( i \in [1, j - d + 1] \) and \( b_k = i \), for some \( k = 1, \ldots, e \); then the entry of \( \mathfrak{h}_{d+i-1} \) corresponding to it is equal to

\[
\# \{ j \in [k, e] \mid b_j = i \text{, } b_{j+1} < b_j \} + \# \{ j \in [k+1, e] \mid b_j = i-1 \text{, } b_{j+1} \leq b_j \} + \\
\# \{ j \in [1, d - e] \mid v_j = i - 1 \}.
\]  

(7.6)

If \( v_k = i \), for some \( k = 1 \ldots d - e \), then the entry of \( \mathfrak{h}_{d+i-1} \) corresponding to it is

\[
\# \{ j \in [1, k - 1] \mid v_j = i - 1 \}.
\]  

(7.7)

**Proof.** By the Definition 2.7 the number of difference one hooks for an attachment of degree \( d + i - 1 \) in the Ferrers diagram of \( P \) is equal to the number of columns to the left of that attachment with feet of degree \( d + i - 2 \). Consider an attachment of degree \( d + i - 1 \) in the Ferrers diagram of \( P \) which corresponds to an entry \( i \) in \( b(P) \).
First assume that the entry \( i \) is the \( k \)-th entry in the horizontal part of \( b(P) \) for some \( k = 1, \ldots, e \). The number of columns to the left of this attachment with feet of degree \( d + i - 2 \) corresponding to the horizontal attachments of \( b(P) \) is equal to the number of entries of length \( i - 1 \) and after \( b_k \) which is equal to the sum of the first two terms in Equation (7.6). Moreover, the number of difference one hooks in the vertical attachments with hand corresponding to \( b_k \) is equal to the number of entries in the vertical part of \( b(P) \) that are equal to \( i - 1 \). So we obtain the Equation (7.6).

Now assume that the entry \( i \) is the \( k \)-th entry in the vertical part of \( b(P) \) for some \( k = 1, \ldots, d - e \). Then, similar to the previous case, the number of entries \( i - 1 \) in the vertical part of \( b(P) \) and after \( v_k \) is equal to the number of difference one hooks with hand corresponding to \( v_k \), which is equal to the integer in Equation (7.7).

**Example 7.4.** Let \( P \) be the partition with diagonal lengths \( T = (1, 10, 9, 6, 2, 0) \) and the branch label 
\[
b(P) = (3, 2, 2, 0, 1, 0, 1, 2, 2, 3, 1),
\]

namely \( P = (13, 11, 10, 7^2, 5^2, 4^3, 2) \) (see Figure 16). Observe that \( d = 10, e = 5 \) and \( j = 12 \). The hook code of \( P \) has three components and is denoted by 
\[
\mathbf{Q}(P) = (h_{10}, h_{11}, h_{12}).
\]

The hook code in degree 12 has two entries corresponding to the two entries in \( b(P) \) which are equal to 3, namely \( b_1 \) and \( v_4 \). For the entry \( b_1 = 3 \), the sum in Equation (7.6) is equal to 5 and for \( v_4 = 3 \) we get from (7.7) that the corresponding hook code is 2. We conclude that, \( h_{12} = (5, 2) \). Four entries \( b_2, b_3, v_2 \) and \( v_3 \) in \( b(P) \) which are equal to 2 correspond to four entries in \( h_{11} \) and using the formulas in the above theorem we have that \( h_{11} = (4^2, 1^3) \). Similarly, we get that \( h_{10} = (1, 0^2) \).

**Ramification**

We here recall the ramification partitions \( \mathbf{Q}(P) \) of an ideal in \( \mathbb{V}(E_P) \) at \( x \) ([Y1, Definition 2.3 and 3.3] or [Y2, Definition 1.6]). Each degree- \( i \) component \( \mathbf{Q}_i(P) \) is the special case for the projective line \( \mathbb{P}^1 \) of the ramification of a linear system on a curve, at a point of the curve. See [CuEs, GaSc1, GaSc2, Lak] for further discussion and context. We will use Lemma 7.6 in Example 8.8.

**Definition 7.5 (Ramification partition).** Let \( P, P' \in \mathcal{P}(T) \) and consider \( C_P \) from Definition 2.1

i. Writing \( (C_P)_i = \{x^{u_1}y^{i-u_1}, \ldots, x^{u_k}y^{i-u_k}, k = t_i \} \) with \( u_1 \geq u_2 \geq \cdots \) and similarly writing \( (C_{P'})_i = \{x^{u'_1}y^{i-u'_1}, \ldots, \} \) with \( u'_1 \geq \cdots \geq u'_{k}, \) then 
\[
(C_P)_i \geq_{lex} (C_{P'})_i \text{ if } u_t \geq u'_t \text{ for each } t \in [1, t_i]. \tag{7.8}
\]

We say that \( P \geq_{lex} P' \) if \( (C_P)_i \geq_{lex} (C_{P'})_i \) for each \( i \in [d, j] \).
Figure 16: Ferrers diagram for Example 7.4 where $P = (13, 11, 10, 7^2, 5^2, 4^3, 2)$ and $T = (1, \ldots, 10, 9, 6, 2, 0)$. Attachments of degrees 10, 11 and 12 are colored in blue, green and red respectively. Corners of difference-one hooks with hands in different degrees are shown by bullets with corresponding colors.

ii. Given $(C_P)_i$ as in (i) we denote by $Q_i(P) = (u_k, u_{k-1} - 1, \ldots, u_1 - (k - 1))$ the ramification partition in degree-$i$ of $P$.

Given $P \in \mathcal{P}(T)$ the Ferrers diagram of the ramification partition $Q_i(P)$ is included in a $t_i \times (i + 1 - t_i)$ box $B_{i,T}$. That is, $Q_i(P)$ has $t_i$ parts each no greater than $(i + 1 - t_i)$. Some of the parts may be zero.

We say $Q_i(P') \subset Q_i(P)$ if the Ferrers diagram of the first partition is included in that of the second.

Lemma 7.6. [IY2, Lemma 1.7 and Theorem 1.10] Let $T$ satisfy Equation (1.1) and let $P \in \mathcal{P}(T)$. The Zariski closure of the cell $\nabla(E_P)$ in $G_T$ satisfies

$$\nabla(E_P) \subset \bigcup_{P' \leq_{\text{lex}} P} \nabla(E_{P'}). \quad (7.9)$$

Also, $P' \leq_{\text{lex}} P$ if and only if $Q_i(P') \subset Q_i(P)$ for each $i \in [d, j]$.

The table of partitions of diagonal lengths $T = (1, 2, 3, 4, 3, 2, 0)$.

In the following example we summarize the main results of the paper for partitions with diagonal lengths $T = (1, 2, 3, 4, 3, 2, 0)$.

Example 7.7 (Table for $T = (1, 2, 3, 4, 3, 2, 0)$). See Figure 7.1. Each partition of diagonal lengths $T$ has a hook code, subpartitions of the blocks $\mathfrak{B}(T) = ((1 \times 2)_1, (2 \times 2)_5).$ According to Section 7.1, Theorem 7.2 and Definition 7.1 each of these hook partitions corresponds to a component branch label, the first $b_4$ arising from a partition $h_4$ of diagonal lengths $T_1 = (1, 2, 1)$ and the second $b_5$ from a partition of diagonal lengths $T_2 = (1, 2, 3, 2, 0)$. In the table we list the partition, its branch label, the component branch label.
labels, the hook code, and the excess number of generators over the expected as a triple for \( P \) and for the two components; the last column of Table 7.1 is the ramification partition of Definition 7.3. The excess number of generators is studied in an earlier section (Theorem 6.4).

The partitions in the table are arranged symmetrically around the midline of the table: the conjugate partition \( P^\vee \) has the complementary hook code to \( h_P \) in \( \mathfrak{B}(T) \) (see Theorem 2.9), and is placed the same distance from the end as \( P \) is from the start; it also has the complementary ramification partition in \( B_4, B_5 = (2^3, 4^2) \). The two middle partitions, of hook codes \(((1),(2,0)) \) and \(((1),(1,1))\), respectively, are each self conjugate. Convention for branch label: top right to lower left: this behaves well here under conjugation: the branch label for \( P^\vee \) is just that for \( P \) written backwards.\(^4\)

| \( P \) | \( b \) | \( b_4 \) | \( b_5 \) | \( h_4 \) | \( h_5 \) | \( (s; s_4, s_5) \) | \( Q_4, Q_5 \) |
|------|------|------|------|------|------|-----------------|-----------------|
| \((6,5,3,1)\) | \((2,2,1,0,0)\) | \((1,0,0)\) | \((1,1,0,0)\) | \((2)\) | \((2,2)\) | \((0; 0, 0)\) | \((2^3), (4^2)\) |
| \((6,4,4,1)\) | \((2,1,2,0,0)\) | \((1,0,0)\) | \((1,0,1,0)\) | \((2)\) | \((2,1)\) | \((0; 0, 0)\) | \((2^3), (4,3)\) |
| \((6,5,2,2)\) | \((2,2,0,1,0)\) | \((0,1,0)\) | \((1,1,0,0)\) | \((1)\) | \((2,2)\) | \((0; 0, 0)\) | \((2^2, 1), (4^2)\) |
| \((5,5,4,1)\) | \((1,2,2,0,0)\) | \((1,0,0)\) | \((0,1,1,0)\) | \((2)\) | \((1,1)\) | \((0; 0, 0)\) | \((2^3), (3^2)\) |
| \((6,4,2,1^3)\) | \((2,1,0,0,2)\) | \((1,0,0)\) | \((1,0,1,0)\) | \((1)\) | \((2,0)\) | \((0; 0, 0)\) | \((2^2), (4)\) |
| \((6,3^3)\) | \((2,0,1,2,0)\) | \((0,1,0)\) | \((1,0,1,0)\) | \((1)\) | \((2,1)\) | \((0; 0, 0)\) | \((2^2), (4,2)\) |
| \((6,5,2,1^2)\) | \((2,2,0,0,1)\) | \((0,0,1)\) | \((1,1,0,0)\) | \((0)\) | \((2,2)\) | \((1; 1,0)\) | \((2^2), (4^2)\) |
| \((5,5,2,1^2)\) | \((1,2,0,0,2)\) | \((1,0,0)\) | \((0,1,1,0)\) | \((2)\) | \((1,0)\) | \((0; 0, 0)\) | \((2^2), (3)\) |

Table 7.1: Hook and branch components for partitions of diagonal lengths \( T = (1,2,3,4,3,2,0) \), with \( \mathfrak{B}(T) = ((1 \times 2)_4, (2 \times 2)_5) \). Convention- branch label is top right to lower left (Examples 7.7 and 8.8). The last column \((Q_4, Q_5) \subset (B_4, B_5) = (2^3, 4^2) \) are the ramification partitions (Definition 7.3).

\(^4\)This symmetry does not always hold in similar tables for larger \( T \), as there is a non-symmetry in the choices of vertical/horizontal branches.
8 Betti strata of $\mathbb{V}(E_P)$.

Let $T$ be a Hilbert function satisfying Equation (1.1) (possible for an Artinian graded quotient of $R = k[x, y]$). There is a well behaved Betti stratification of the variety $G_T$ parametrizing all graded quotients $A = R/I$; the strata have known codimension and their closures satisfy a frontier property (Theorem 8.2). The stratification of $G_T$ by the affine cells $\mathbb{V}(E_P)$ on the other hand is rather complicated when $T$ is not single-block: for example, the closures of the cells do not satisfy a frontier property (Warning 2.12).

What can we say about the intersection of the two stratifications? We have determined the lowest (so generic) number of generators $\kappa(P)$ for ideals in $\mathbb{V}(E_P)$ (Theorems 3.11, 5.1, and 5.13); the highest number of generators for ideals in the cell occurs for the monomial ideal $E_P$ itself (Proposition 5.18).

We prove below that each Betti sequence between these occurs for some ideal in $\mathbb{V}(E_P)$ (Proposition 8.5). A. Conca and G. Valla have studied the Betti numbers possible for ideals in a cell [CoVa]. We compare our results with theirs and propose additional problems, some of which we solve, and some of which remain open.

For an ideal $I \subset R = k[x, y]$ of height two, we have the exact sequence

$$0 \rightarrow \bigoplus_{i \in \mathbb{N}} R(-i)\beta_{1,i}(I) \rightarrow \bigoplus_{i \in \mathbb{N}} R(-i)\beta_{0,i}(I) \rightarrow R \rightarrow R/I \rightarrow 0. \quad (8.1)$$

Here $\beta_{0,i}(I)$ is the number of generators if $I$ in degree $i$ and $\beta_{1,i}$ the number of relations among these generators occurring in degree $i$. The following result extends Lemma 3.3. Recall $\delta_i = t_{i-1} - t_i$ and $d(T)$ is the order of ideals defining algebras in $G_T$; for the invariant $\tau(I)$ see Equation (5.4)ff).

**Lemma 8.1.** [Ia2, Prop. 2.7] Let $T$ be a sequence satisfying Equation (1.1), and let $I$ be an ideal defining the algebra $A \in G_T$. Let $i \geq d(T)$. Then

i. The minimal number of generators of $I$ having degree $i + 1$ satisfies

$$(\delta_{i+1} - \delta_i)^+ \leq \beta_{0,i+1}(I) \leq \delta_{i+1}. \quad (8.2)$$

ii. The minimal number of relations of $I$ having degree $i + 1$ satisfies

$$(\delta_i - \delta_{i+1})^+ \leq \beta_{1,i+1} \leq \delta_i.$$

iii. The $\tau$ invariant satisfies

$$1 + \min\{\delta_i, \delta_{i+1}\} \geq \tau(I) \geq 1.$$

iv. For generic $I \in G_T$ the left hand side of the inequalities in (i),(ii),(iii) are equalities.

The upper bound in Equation (8.2) is a consequence of Equation (5.4) and is achieved by setting $I_i = R_{i-t_i}f_i$ where $f_i$ has degree $t_i$ and each $f_i$ divides $f_{i-1}$ for $i \in [d+1,j]$ (this is setting $\tau(I_i) = 1$ for all $i \in [d,j]$).
We term the generic values \( \beta_{0,i+1}(T), \beta_{1,i+1}(T), \tau_{i+1}(T) \). Evidently, we have \( \beta_{0,i+1}(T) \cdot \beta_{1,i+1}(T) = 0 \), and for \( I \in G_T \),
\[
\beta_{0,i+1}(I) - \beta_{0,i+1}(T) = \beta_{1,i+1}(I) - \beta_{1,i+1}(T). \tag{8.3}
\]
So, as is well-known, fixing \( T = H(A) \), the integer \( \beta_{0,i}(I) \) determines \( \beta_{1,i}(I) \), as is shown by either Equation (8.1) or (8.3). We will denote by \( \beta_0(I) = (\beta_{0,0}(I), \ldots, \beta_{0,i}(I), \ldots, \beta_{0,j+1}(I)) \) the sequence of generator degrees – we will sometimes write \( \beta_1(I) \) for \( \beta_{0,i}(I) \) and \( \beta(I) \) for \( \beta_0(I) \). We say a sequence \( \beta' \geq \beta \) if each \( \beta'_i \geq \beta_i \). By the Hilbert-Burch theorem the map \( \alpha \) in the exact sequence of Equation (8.1) corresponds to an irreducible variety of codimension \( s \) in \( \beta, \beta_0, \beta_1 \). Evidently, we have the codimension formula extends to Betti strata of the postulation scheme.

Theorem 8.2. \[a2, \text{Theorem 2.18}\] Let \( T \) satisfy Equation (1.1) and
(i). \( G_{\beta,T} \) is an irreducible variety of codimension \( \sum_{i=0}^{j+1} \beta_{0,i} \cdot \beta_{1,i} \) in \( G_T \),
(ii). The closure \( \overline{G_{\beta,T}} = G_{\geq \beta,T} \).
(iii). This closure is Cohen-Macaulay with singular locus \( G_{> \beta,T} \).

Note, the codimension formula extends to Betti strata of the postulation punctual Hilbert scheme \( \text{Hilb}^T(\mathbb{P}^2) \subset \text{Hilb}^n(\mathbb{P}^2) \) – see \[a2, \text{Remark 3.7}\] these codimension formulas for the postulation scheme were shown differently by A. Constantinescu \[\text{Con}\].

Given \( T \) we will denote by \( b = (b_d, b_{d+1}, \ldots, b_j, b_{j+1}) \) a sequence of non-negative integers satisfying the inequalities of Equation (8.2). Given a partition \( P \in \mathcal{P}(T) \) we define \( \bigvee(E_P, b) = \bigvee E_P \cap G_{b,T} \), the variety (possibly reducible) parametrizing those ideals \( I \in \bigvee E_P \) having minimal-generator degrees \( b \); we define analogously \( \bigvee(E_P, \geq b) = \bigvee E_P \cap G_{\geq b,T} \). We have the following result of A. Conca and G. Valla:

Theorem 8.3. \[\text{CoVa}, \text{Lemma 4.1}\] The variety \( \bigvee(E_P, \geq b) \) is the transversal intersections of the determinantal varieties \( \{ \bigvee(E_P, \beta_i = b_i) \} \) for \( i \in [d, j+1] \). In particular, the codimension of \( \bigvee(E_P, \geq b) \) is the sum of the codimensions of the \( \{ \bigvee(E_P, \beta_i = b_i) \} \). Also \( \bigvee(E_P, \geq b) \) is irreducible if \( \bigvee(E_P, \beta_i = b_i) \) is irreducible for every \( i \in [d, j+1] \).
They specify determinantal ideals defining these varieties [CoVa Lemma 4.2].

In the following example we calculate the Betti strata for closures of certain cells: we will use these calculations to give some partial answers to our Question 8.6 below. We let $b(I) = \beta_0(I) = (\beta_{0,0}(I), \ldots, \beta_{0,i+1}(I))$, (notation of [CoVa p.167]) and $\kappa(I) = |b(I)| = \sum_{i=d}^{i+1} \beta_{0,i}$. We have determined $\kappa(P) = \kappa(I)$ for $I$ generic in $\mathbb{V}(E_P)$. A. Conca and G. Valla study each Betti stratum $\mathbb{V}(E_P, b)$.

**Example 8.4.** We determine the Betti strata in the closures of certain cells, for three Hilbert functions.

i. Let $T = (1,2,3,4,5,2,0)$, a single-block Hilbert function for which $\mathcal{B}_4 = (4,4)$ and $|\mathcal{P}(T)| = \binom{6}{2} = 15$. We consider three partitions, first, $P = (6,5,3,2,1) = T^\gamma$, the generic partition for $T$, so $\mathbb{V}(E_P)$ is open dense in $G_T$, and $P$ has hook code $H_4(P) = \mathcal{B}_4$. Then $P_1 = (6,4,3,2,1,1)$ of hook code $H_4(P_1) = (4)$, and $P_2 = (5,4,4,2,2)$ of hook code $H_4(P_2) = (2,1)$. The dimension of $G_T$ is 8, and the codimensions of the three given cells are, respectively, 0, 4 and 5.

The Betti stratum of $G_T$ associated to $b = (b_5, b_6)$ is non-empty if and only if $b_5 = 4$ and $0 \leq b_6 \leq 2$. From Lemma 8.1 and Equation (8.3) we can conclude $\beta_{1,6} = b_6 + 1$; so by Theorem 8.2(i) the Betti strata of $G_T$ have codimension 0, 2, 6, respectively, for $b_6 = 0,1,2$. For each $P_i$ and $b$ we consider the Betti stratum $\mathbb{V}(E_P, b)$. $b = (b_5, b_6)$.

a. The open-dense cell $\mathbb{V}(E_P)$ with $\kappa(P) = \kappa(T) = 4$ has three Betti strata, each of the expected codimension and irreducible by Theorem 8.2, the analogue is true always for the generic cell (see [CoVa Cor. 4.6,4.7] for a second proof).

b. The cell $\mathbb{V}(E_{P_1})$, with $H_4(\mathcal{I}_1) = (4)$, so dimension four, and $\kappa(P_1) = 5$ has generic Betti stratum $b = (4_5, 1_6)$. The unique proper Betti stratum $\mathcal{b} = (4_5, 2_6)$, is irreducible of dimension one, as it consists of ideals $I = (y^6, \ell R_3, x^6)$ where $\ell = y + ax$.

The closure of $\mathbb{V}(E_{P_1})$ in $G_T$ contains four more cells. The first three are comprised of ideals uniquely of Betti sequence $b = (4_5, 1_6)$: they are from the partitions $(5,5,3,2,1,1)$ of hook code $H_4(\mathcal{I}_1) = (3)$, $(5,4,4,2,1,1)$ of code $H_4(\mathcal{I}_2) = (2)$, and $(5,4,3,3,1,1)$ of code $H_4(\mathcal{I}_3) = (1)$. The fourth cell is $P_0 = (5,4,3,2,1,1)$ of code $H_4(\mathcal{I}_4) = (0)$; it has Betti sequence $b = (4_5, 2_6)$ and is comprised of the single monomial ideal $E_{P_0} = (y^6, x^2y^3, x^3y^2, x^4y, x^5)$. This is also the only other ideal in the closure of the proper Betti stratum $\mathcal{b} = (4_5, 1_6)$ of $\mathbb{V}(E_{P_1})$, obtained by letting $a \to \infty$ in $\ell = y + ax$.

c. The cell $\mathbb{V}(E_{P_2})$ with $H_4(\mathcal{I}_2) = (2,1)$, so dimension 3, and $\kappa(P_2) = 4$, has only one Betti stratum, $b = (4,0)$.

The closure of $\mathbb{V}(E_{P_2})$ in $G_T$ also contains four more cells. The first three are, as for $P_1$, comprised of ideals uniquely of Betti sequence $b = (4_5, 1_6)$: they are from the partitions $(5,4,4,2,1,1)$ of code...
\( S_4 = (2, 0), (5, 4, 3, 3, 2) \) of code \( S_4 = (1, 1) \), and \((5, 4, 3, 3, 1, 1)\) of code \( S_4 = (1, 0) \). The fourth, as before, is \( P_0 = 5, 4, 3, 2, 2, 1 \) of code \((0), b = (4, 2)\), comprised of a single monomial ideal.

Of note, in the closure of \( \mathbb{V}(E_{P_0}) \) the Betti stratum \( b = (4_5, 1_6) \) has dimension two and is comprised of three cells; this closure has two irreducible components – determined by the two cells with the incomparable hook codes \( S_4 = (2) \) and \((1, 1)\). (Recall that by Theorem 2.11 the closures of cells in a single-block \( G_T \) are completely determined by the poset of hook codes).

For the same reason the closure \( \overline{\mathbb{V}(E_P)} \supset \mathbb{V}(E_{P_0}) \).

ii. Let \( T = (1, 2, 3, 2, 1) \), a multiblock Hilbert function; \( \mathfrak{B} = (\mathfrak{B}_3, \mathfrak{B}_4) = ((1 \times 2), (1 \times 2)) \). Here \( \mathcal{P}(T) \) has nine elements, each corresponding to a pair \( S(T) \) satisfying \((0_3, 0_4) \leq S(T) \leq (2_3, 2_4) \) (see [AIK] Example 4.6 and Figure 17). We noted in Warning 2.12 that the closure of the 2-dimensional cell corresponding to \( P' = (5, 2, 1, 1) \) is not the union of cells. The Betti stratum of \( G_T \) associated to \( b = (b_3, b_4, b_5) \) is nonempty if and only if \( b_3 = 2 \) and \( b_4, b_5 \in \{0, 1\} \). The generic partition \( P = (5, 3, 1) \) and \( Q = (3, 3, 3) \) in \( \mathcal{P}(T) \) both satisfy \( \kappa(P) = \kappa(Q) = \kappa(T) = 2 \); each of these cells is comprised generically of complete intersection ideals with two generators in degree 3 and a single Koszul relation in degree 6. All four Betti strata for \( G_T \) occur for the generic cell \( \mathbb{V}(E_P) \). But there is a unique Betti stratum for \( Q \): an element of \( \mathbb{V}(E_Q) \) satisfies \( I = (f_3, x^3) \), where \( f_3 = y^3 + ay^2x + byx^2 \), so is a complete intersection with \( b = (2_3, 0, 0) \).

iii. Consider \( T = (1, 2, 1, 1) \), a multiblock Hilbert function with \( \mathfrak{B} = (\mathfrak{B}_3') = (1 \times 1) \). The generic partition is \( P = (4, 1) \). Here \( \kappa(P) = \kappa(T) = 3 \) and each element \( I \in \mathbb{V}(E_P) \) satisfies \( I_a = ((y + ax)R_1, x^3) \) for a constant \( a \in k \), so requires three generators. The other partition in \( \mathcal{P}(T) \) is \( P_0 = (2, 1, 1, 1) \); here \( \mathbb{V}(E_{P_0}) = E_{P_0} = (y^4, xy, x^3) \), also with three generators. There is a single \( \beta \) stratum \( b = (2_2, 0_3, 1_4) \) for \( G_T \).

Given the partition \( P \in \mathcal{P}(T) \), there is a generic value \( \tilde{\kappa}(P) = b_{\min}(\mathbb{V}(E_P)) \) and a maximum value \( \overline{\kappa}(P) = b_{\max}(\mathbb{V}(E_P)) \) for the generator sequence of ideals in the cell \( \mathbb{V}(E_P) \): the minimum occurs for a generic ideal in the cell (Theorem 5.13) and the maximum occurs for the monomial ideal \( E_P \) where \( \beta_{i,j}(E_P) \) is the number of degree-\( i \) corner-monomials of \( E_P \), and is reckoned in terms of the hook code of \( P \) (Proposition 5.18).

**Proposition 8.5.** For each sequence \( b \) in between \( \tilde{\kappa}(P) \) and \( \overline{\kappa}(P) \) the Betti stratum \( \mathbb{V}(E_P, b) \) is nonempty.

**Proof.** In the proof of Theorem 5.13 we used the standard basis technique of J. Briançon ([Br]) and A. Galligo ([Br-Ga]) to kick out as many corner monomials as possible. The same technique can be used to kick out a fixed number of degree-\( i \) corner monomials of \( E_P \), as explained in Remark 3.7. \( \square \)
Question 8.6. (i) Do the strata of Theorem 8.2 intersect the cell $\mathcal{V}(E_P)$ properly?

Answer: No. Sometimes, when $\mathcal{V}(E_P)$ is not the generic cell, the generator sequence $\underline{b} \neq \underline{b}(T)$ but $\mathcal{V}(E_P, \underline{b}) = \mathcal{V}(E_P)$

Consider, for example, $T = (1, 2, 3, 2, 1)$ for which $\dim G_T = 4$ and $\underline{b}(T) = (23, 04, 05)$. The Betti stratum associated to $\underline{b} = (23, 14, 05)$ has codimension one in $G_T$. Let $P = (3, 2^3)$. Then $\mathcal{V}(E_P)$ has codimension three in $G_T$ and $\mathcal{V}(E_P, (23, 04, 15)) = \mathcal{V}(E_P)$.

(ii) Do the Betti strata of Theorem 8.2 intersect the closure of the cell $\mathcal{V}(E_P)$ properly?

Answer: No, in general. Even for the single-block $T = (1, 2, 3, 4, 5, 2, 0)$ of Example 8.4.i.c) the closure of $\mathcal{V}(E_{P_3}) (\beta = (4, 0_6))$ consists of the dimension three cell itself, then a dimension two reducible Betti locus $\underline{b} = (45, 16)$ (so codimension one in the closure) and the cell comprised of a single monomial ideal, of Betti numbers $(4, 2_6)$. Here the codimensions of the Betti strata in $G_T$ are $(0), (2)$ and 6 for $\beta_{0,6} = 0, 1, 2$, respectively. It is not so surprising that some cells $E_P$ are better aligned with the Betti stratum $\underline{b} = (45, 16)$ than the generic cell.

Yes, sometimes, even for a non-generic cell: see Example 8.8 below.

(iii) If we know $P$ and the codimension of a stratum $G_\beta(T)$ in $G_T$ can we determine the codimension in $\mathcal{V}(E_P)$ of a stratum $\underline{b}$ corresponding to $\beta$ in $\mathcal{V}(E_P)$? In other words, what is the dimension of $\mathcal{V}(E_P, \underline{b})$? If the stratum $\mathcal{V}(E_P, \underline{b})$ has several irreducible components, what are their dimensions? (see part (vi)).

Discussion. The answer will not be simple. Consider Example 4.4 of A. Conca-G. Valla ([CoVa], p.169-170]) where the Hilbert function is $T = (1, 2, 3, 4, 5, 6, 3)$ and $P = (7, 5, 4^2, 3, 1)$. There are four Betti strata for this $G_T$, associated to the vectors $\frac{b_0}{b_0} = (4, 6, 0_7)$ (codimension zero), $\frac{b_1}{b_1} = (4, 6, 1_7)$ (codimension one), $\frac{b_2}{b_2} = (4, 6, 2_7)$ (codimension four), and $\frac{b_3}{b_3} = (4, 6, 3_7)$ (codimension nine), each is irreducible by Theorem 8.2 (i). In their example, A. Conca and G. Valla found that $\mathcal{V}(E_P, (4, 1_7))$ has two irreducible components of codimension one each in $\mathcal{V}(E_P)$ (so preserving codimension but not preserving irreducibility), and $\mathcal{V}(E_P, (4, 2_7))$ is irreducible of codimension three in $\mathcal{V}(E_P)$ (not preserving codimension).

(iv) Given a vector $\underline{b} = (b_d, \ldots, b_{i+1})$ of integers, determine the set of partitions $P$ such that $\mathcal{V}(E_P, \underline{b})$ is not empty.

Discussion. The generic value $\vec{\kappa}(P)$ is given by our main results, Theorem 5.4 (Corollary 5.10), and Theorem 8.11 and Proposition 5.18 gives $\underline{b}(E_P)$. Proposition 8.5 states that each intermediate sequence occurs, so question (iv) is entirely combinatorial.

(v) Recall that by Proposition 8.5 if $\underline{b}$ is in between $\vec{\kappa}(P)$ and $b(E_P)$ then the Betti stratum $\mathcal{V}(E_P, \underline{b})$ is not empty.
So, given a partition $P \in \mathcal{P}(T)$, does every Betti sequence for $T$ greater than $b(E_P)$ occur in the closure of the cell $\mathcal{V}(E_P)$?

**Discussion.** Let $\mathcal{V}(E_{P_0})$ be the zero dimensional cell of $G_T$. The Betti sequence of the monomial ideal $E_{P_0}$ is maximal and we know by Theorem 8.2 that $\mathcal{V}(E_{P_0})$ is in the closure of any Betti stratum. Also, by [Y4] Theorem 2.7, $\mathcal{V}(E_{P_0})$ is in the closure of any cell $\mathcal{V}(E_P)$. More precisely, in [Y4] Theorem 2.7 it is shown that given any partition $P$ of diagonal lengths $T$ such that $\dim \mathcal{V}(E_P) > 0$, one can construct a partition $P'$ of diagonal lengths $T$ such that $\dim \mathcal{V}(E_{P'}) = \dim \mathcal{V}(E_P) - 1$ and $\mathcal{V}(E_{P'}) \subset \overline{\mathcal{V}(E_P)}$. Looking at how $P'$ is constructed from $P$ in the proof of [Y4] Theorem 2.7, one can see that $\kappa(P) \leq \kappa(P') \leq \kappa(P) + 1$. Thus if we let $m = \dim \mathcal{V}(E_P)$, then there is a sequence $P = P'_1, \ldots, P'_m = P_0$ of partitions of diagonal lengths $T$ such that $\mathcal{V}(E_{P'_{i+1}}) \subset \overline{\mathcal{V}(E_{P'_i})}$ and $\kappa(P'_i) \leq \kappa(P'_i + 1)$. Note that there may be many different such sequences.

The existence of such sequences of partitions might suggest that given a vector $\underline{b} = (b_d, \ldots, b_d)$ of integers, pairwise greater than $b(E_P)$, there is a chance that there is a partition $P''$ such that $\mathcal{V}(E_{P''}) \subset \overline{\mathcal{V}(E_{P'})}$ and $\underline{b} = (b_d, \ldots, b_d)$ is in between $\kappa(P'_i)$ and $b(E_{P'})$. The problem is open.

(vi) Are there Betti strata $\mathcal{V}(E_P, \underline{b})$ for suitable $(P, \underline{b})$ that have several irreducible components? If so, determine when this occurs.

**Answer:** Yes. See (iii) above, an example from [CoVa].

How about the same question for $P \in \mathcal{P}(T)$ where $T$ is single block? Note that the single-block $P_2 = (5, 4, 4, 2, 2)$ in Example 8.4 (ii, c) shows that there can be two irreducible components of the $\underline{b} = (4_5, 1_6)$ Betti stratum in the closure of that cell. The problem is open.

The following example illustrates Proposition 8.5

**Example 8.7.** Consider $T = (1, 2, 3, 2, 1), P = (5, 3, 1)$ and the cell $\mathcal{V}(E_P)$, which is the generic cell, of dimension four. The generic ideal $I$ in the cell is a complete intersection of generator degrees $(3, 3)$, so $\underline{b}_{\text{min}} = (2_3, 0_4, 0_5)$. The monomial ideal $E_P$ has generator degrees $\underline{b}_{\text{max}} = (2_3, 1_4, 1_5)$, since here $\beta_{0,i}(E_P)$ is just the number of corners in the Ferrers graph of $P$ having degree $i$. An ideal having these maximum generator degrees, must be such that the generator $f_4 = x^3 \ell, \ell = y + ax$, and $f_3 = y^3 + \cdots + g_3 = x h_2, h_2 = y^2 + \cdots$ satisfy $h_2 = \ell', \ell' = y + bx$ and $\langle f_3, g_3 \rangle = R_1 h_2$. Thus, the dimension of the stratum is two, parametrized by $\{a, b\}$.

An ideal $I \in \mathcal{V}(E_P)$ has generator degrees $(2_3, 1_4, 0_5)$ if and only if $\langle f_3, g_3 \rangle = R_1 h_2$ but $\ell$ does not divide $h_2$; so this Betti stratum is determined by $h_2, \ell$ with three parameters. An ideal $I \in \mathcal{V}(E_P)$ has generator degrees $(2_3, 0_4, 1_5)$ if and only if $\ell$ divides both $f_3, g_3$ but they do not have a common degree-2 factor: this also gives three parameters. This example illustrates both Theorem 8.2 and the codimension addition formula of Theorem 8.3.
For this partition $P = (5, 3, 1)$ each intermediate $\beta$ sequence occurs (an illustration of Proposition [8.3] with the expected codimension (a bonus). Also, each Betti stratum is irreducible.

The next example shows a well-behaved closure of a cell, corresponding to a Yes in this case for Question (8.0)(v).

**Example 8.8.** Let $T = (1, 2, 3, 4, 3, 2, 0)$ as in Example [7.7] and Figure [7.1]

Betti strata for $G_T$. Here dim $G_T = 6$, $\mathfrak{B}(T) = ((1 \times 2)_4, \langle 2 \times 2 \rangle_5)$ and the generic Betti stratum is $h = (2_4, 0_5, 1_6)$ by Lemma [3.3] the maximum Betti stratum, that of the monomial ideal $E_{P_0}$ is $b_0 = (2_4, 1_5, 2_6)$. The first special Betti strata are associated to $b_1 = (2_4, 1_5, 1_6)$ and $b_2 = (2_4, 0_5, 2_6)$. We can calculate their codimensions in $G_T$ using Theorem [8.2] but we give a direct analysis below. The stratum $b_1 = (2_4, 1_5, 1_6)$ has codimension one: we need a common degree three factor $b_3$ of the two degree-four generators $f_4, g_4$ – losing 3 parameters – but we also need a new generator $w_5$ of degree five - gaining two parameters, and giving a net codimension one. For $b_2,$ we need a common degree two factor $f_2 = (y^2 + axy + bx^2)$ of the degree-four generators $f_4, g_4,$ so $\langle f_4, g_4 \rangle = f_2V, V \subset R_2$ with dim $k \langle f_2 \rangle = 2$: that gives a total dimension of four so a codimension two condition for $b_2$. By the Theorem [8.2] the $b_3 = (2_4, 1_5, 2_6)$ stratum in $G_T$ has codimension $(1 \cdot 1 + 1 \cdot 2) = 3$; to verify this directly note that here the degree-5 generator $w_5 = x^3f_2,$ while $f_2 \psi$ is a common factor of $\langle f_4, g_4 \rangle = f_2 \psi R_1$ confirming that $b_4$ has dimension three, so codimension three.

Betti strata for the closure of the cell $V(E_P), P = (4, 4, 4, 3)$ of hook code $(\mathfrak{h}_4 = (1), \mathfrak{h}_5 = (1, 0)).$ Here $V(E_P)$ has dimension three, the total number of hooks (see Table [7.1]. We have $b(I) = (2_4, 0_5, 1_6)$ for every ideal $I$ in $V(E_P)$! There are no higher Betti strata in the cell itself. By Lemma [7.6] the closure of $V(E_P)$ is contained in the union of cells having ramification loci that are no greater than that of $P$: by Table [7.1] the stratification of this closure is the same (in this case) as that by the hook codes. So here, the closure of $V(E_P)$ is in the union of cells whose hook sequences are no greater than $((1), (1, 1)).$ The two candidate strata of $G_T$ having dimension two are for $P_1 = (4^3, 1^3)$ of hook code $((\mathfrak{h}_4, \mathfrak{h}_5) = ((1), (1, 0)) and P_2 = (4, 3^3, 2) of hook code ((0)_4, (1, 1)_5). Both cells are wholly contained in the closure of $V(E_P).$ For $I = (xy^3 + ax^2y^2 + bx^3y, x^4, y^6) \in V(E_{P_1})$ we find $I \in \overline{V(E_P)},$ since

$$I = \lim_{t \to \infty} I(t), I(t) = (f_4 = y^4 + ty^3x + atyx^2 + btyx^3, x^4, y^3x^3).$$

(Note that $m^6 \subset J$ for any ideal $J \in G_T.$) However, again, there is a single Betti sequence possible, namely $h = (2_4, 0_5, 1_6)$ for $I \in V(E_{P_1});$ the partition $P_1$ is non-special, $\kappa(P_1) = \kappa(T) = 3.$ Note that $P_1$ and $P$ have the same Betti sequence.

\footnote{Example 2.2 of [LY2] shows this is not generally true, even for $T = (1, 2, 2, 1);$ in general the lattice $\Omega(T)$ determined by the hook codes is a subposet of $\mathcal{P}(T)$ with the $\text{lex}$ order [LY2] Theorem 2.3.}
The cell $V(E_{P_2})$ is comprised of ideals
\[ I = (x^3y, x^4, y^5 + ay^4x + by^3x^2, y^4x^2), \]
whose Betti sequence is always $b_1 = (2, 1, 1)$. This indeed has codimension one in the closure of $V(E_P)$. The two cells of dimension one in the closure correspond to $(4, 3^2, 2^3, 1)$ and $(2, 1, 1)$ of hook code $((0), (1, 0))$ -- all ideals in this cell have $b_1 = (2, 0, 1)$. Finally, the cell $P_0 = (4, 3^2, 2^3, 1)$ the stratum of dimension zero, consisting of a single monomial ideal, has codimension three in $V(E_P)$.

In sum, we have found elements of each Betti stratum of $G_T$ higher than $b(E_P)$ in the closure of the cell $V(E_P)$: each occurring in the same codimension they have in $G_T$. This is in contrast to Example 8.4(i.c) where each higher Betti stratum occurs in the closure of the cell, but sometimes in smaller codimension than for $G_T$.

**Question 8.9.** What more can we say about the intersections of the two stratifications? In particular what is the closure of $V_{CR}(E_P)$ in $G_T$, for an arbitrary $P \in P(T)$? Which Betti strata occur in the closure of the cell $V(E_P)$, and what is the codimension of these strata?

Also, we know that the classes of the cells give a basis for the homology of $G_T$ (whose ring structure is in general unknown, except for the single-block case). What are the homology classes of the closures of the Betti strata? Will an answer here help in answering some of the other questions about $V(E_P, b)$? A notable fact is that the Betti strata are invariant under the action of $PGL_2(k)$ -- that is, under a change of basis for $R_1$. But the cells $V(E_P)$ depend on a choice of basis $(y, x)$ for $R_1$: the cells are invariant under the upper triangular maps $y \rightarrow y + ax, x \rightarrow x$ of determinant 1.

**Remark 8.10.** Concerning the question of finding $\kappa(P)$, Aldo Conca has suggested that it might also be approached from the viewpoint of [CoVa].

i. the smallest number of generators in degree $i$ corresponds to the largest rank of the matrix $M(p)_i$, see the formula (4.5) at p. 168 of [CoVa].

ii. the rank of $M(p)_i$ is the rank of the matrix $M(p)_i + a$ given constant that depend on the ideals $E$ (p. 168 line -3).

iii. the matrix $M(p)_i$ is a matrix of variables with some zeroes in the upper-right corner (example (4.6) of [CoVa]).

iv. So the problem is turned into the combinatorial problem of finding the rank of a matrix as the one in (4.6) where the location of the zeros is determined by the original ideal $E$.

We of course, give the answer to this combinatorial problem, which we believe is subtle, and also show the relation to hooks, and to a decomposition theorem for cells.
9 Problems.

We discuss some further problems.

Low characteristic, and pairs of Jordan types.

We have assumed in this paper that the characteristic of \( k \) is zero, or greater than the socle degree \( j \) of \( T \). D. Cook in [Cook] has studied when the generic Jordan type of a graded Artinian quotient of \( k[x, y] \) is strong Lefschetz. This is a somewhat different focus than ours, because of the emphasis on generic Jordan type, and because of handling also low characteristic.

Question 9.1. How do our results extend to small characteristic?

Pairs \( (P, P') \) that may occur for \( (\ell, \ell') \) in \( A \).

Restrictions, some at first surprising, among the pairs \( (P_\ell, P_{\ell'}) \) that can occur for \( \ell, \ell' \) in a local ring \( R = k\{x, y\}/I \) have been studied by several groups. When we restrict to graded algebras \( A \) of Hilbert function \( T \), this problem of simultaneous Jordan types has been studied in [LY1, §4] were it is shown to be closely related to determining the homology ring structure of \( G_T \), which is open in general. Although the Poincaré series (additive structure of the homology of \( G_T \) is known (Theorem 2.10), the multiplicative structure is known only in a few cases. One such case is the single block case, when it is just the homology ring of a Grassmanian; there, the problem of compatible Jordan types is closely related to Wronskian determinants and to the Schubert calculus [LY1, §2-B].

The homology ring for \( G_{T(d,j)} \) is given in [LY1 Theorem 4.5]: here \( T(d, j) = (1, 2, \ldots, d - 1, d, d, \ldots, d_j, 1) \), and \( G_{T(d,j)} \) is a \( \mathbb{P}^{d-1} \) bundle over the projective space \( \mathbb{P}_d \) parametrizing the generator \( f_d \) of \( I_u \). Another case where the homology class of \( G_T \) in the product of Grass\( (a, R_d) \times \) Grass\( (b, R_{d+1}) \) (“large” Grassmannians) is known, using an idea of G. Ellingsrud, is \( T = (1, 2, \ldots, d - 1, d, a, b, 0) \). The problem of understanding the homology rings of \( G_T \) reduces to that for elementary \( T \) by Lemma 5.21 but the rings for elementary \( T \) other than the special cases mentioned above is very open.

These ideas, are related to “ideals” of linear systems over the projective line \( \mathbb{P}^1 \) [LY1 §3-A]; the ramification loci of linear systems on curves, and the connection with Wronskian determinants has been extensively studied, as by [CuEs, GaSc1, GaSc2, Lak]. It could be of interest to consider analogues of the cells studied here, in connection to ideals of linear systems for other curves than \( \mathbb{P}^1 \).

Ideals related to a vector space of forms.

Let \( V \subset R_i \) be a vector space of degree-\( i \) forms in \( R = k[x, y] \). There are several ideals related to \( V \), that have been studied by several: the ideal
(V), the ancestor ideal $\nabla = (V) \oplus \sum_{i=1}^{n} V : R_i$ discussed briefly here at Equation 5.4. The parameter varieties $\text{Grass}(T)$ parametrizing $V$ such that $H(R/V) = T$ are related to tangent bundles and normal bundles to the linear system on $\mathbb{P}^1$ determined by $V$; they also are very well behaved: their Zariski closures satisfy a frontier property $\text{Grass}(T) = \bigcup_{T' \geq T} \text{Grass}(T')$. These have not been studied with respect to the Jordan type of $A = R/V$, when $\text{dim}_k A = n$ is specified. Suppose a family $V_w, w \in W$ is a family of vector subspaces of $R_i$ having fixed dimension $d$. What can we say about the family of generic Jordan types $J_{A(t), \ell}$ for the algebras $A_w = R/V_w$? What can we say about Jordan types occurring in closures of the family $A_w, w \in W$ (where the lengths may change?).

**Extension to higher dimensions**

The D. Hilbert-L. Burch theorem gives the minimal resolutions of height two graded ideals; the next well behaved case is the D. Eisenbud-D. Buchsbaum Pfaffian structure theorem for height three Gorenstein ideals. What can we say about the Jordan types? Some initial work has been done in higher dimensions by several: in particular the Jordan degree type is symmetric for graded Gorenstein ideals (see [AIK, Lemma 3.22], [H-W, §4.1] [CsGo, Lemma 4.6] and references cited there).

**Acknowledgment.** The first author was supported by the Swedish Research Council grant VR 2013-4545. The authors are grateful to the organizers of the conference “Lefschetz Properties in Algebra, Geometry and Combinatorics” at Centro Internazionale per la Ricerca Matematica (CIRM) at Levico, Italy, in June 2018; and to the successor conference “Lefschetz Properties in Algebra, Geometry and Combinatorics, II” at Centre International de Rencontres Mathématiques (CIRM) at Lumini, France in October, 2019, where they participated in the working group on Jordan type.

**References**

[AIK] N. Altafi, A. Iarrobino, and L. Khatami: Complete intersection Jordan types in height two, J. Algebra 557 (2020), 224–277.

[B-B] A. Białynicki-Birula: Some properties of the decompositions of algebraic varieties determined by actions of a torus, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 no. 9, (1976), 667–674.

[Br] J. Briançon: Description de Hilb$^n \mathbb{C}\{x,y\}$, Inventiones math. 41 (1977), 45-89.

[Br-Ga] J. Briançon and A. Galligo, Déformations distinguées d’un point de $\mathbb{C}^2$ ou $\mathbb{R}^2$, Astérisque 7,8 (1973), 129-138.
[Bu] L. Burch, *On ideals of finite homological dimension in local rings*, Proc. Cambridge Philos. Soc. 64 (1968), 941–948.

[CoVa] A. Conca and G. Valla: *Canonical Hilbert-Burch matrices for ideals of* $k[x,y]$, Special volume in honor of Melvin Hochster. Michigan Math. J. 57 (2008), 157–172.

[Con] A. Constantinescu: *Parametrizations of ideals in* $K[x,y]$ and $K[x,y,z]$, J. Algebra 346 (2011), 1–30.

[Cook] D. Cook: *The strong Lefschetz property in codimension two*, J. Commut. Algebra 6 no. 3 (2014), 323–345.

(CsGo) B. Costa and R. Gondim: *The Jordan type of graded Artinian Gorenstein algebras*, Adv. in Appl. Math. 111 (2019), 101941, 27 pp.

[CuEs] C. Cumino, and E. Esteves: *Limits of special Weierstrass points*, Int. Math. Res. Pap. IMRP 2008, no. 2, Art. ID rpno01, 65 pp.

[Ell] G. Ellingsrud: *Sur le schéma de Hilbert des variétés de codimension 2 dans* $\mathbb{P}^e$ *à cône de Cohen-Macaulay*, Ann. Sci. Ecole Norm. Sup. (4) 8 (1975), 423–432.

[ES1] G. Ellingsrud and S. Strømme: *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. 87 (1987), 343–352.

[ES2] G. Ellingsrud and S. Strømme: *On a cell decomposition of the Hilbert scheme of points in the plane*, Invent. Math. 91 (1988), 365–370.

[Ev] L. Evain: *Irreducible components of the equivariant punctual Hilbert schemes*, Adv. Math. 185 (2004), no. 2, 328–346.

[GaSc1] L. Gatto and I. Scherbak: *On one property of one solution of one equation or Linear ODEs, Wronskians and Schubert calculus*, Moscow Math. J. 12(2) (2012) 275–291. (Also arXiv math.AG/1310.3345).

[GaSc2] L. Gatto and I. Scherbak: *On generalized Wronskians. Contributions to algebraic geometry*, 257–295, EMS Ser. Congr. Rep., Eur. Math. Soc., Zurich, 2012.

[Gö1] L. Gött sche: *Betti numbers for the Hilbert function strata of the punctual Hilbert scheme in two variables*, Manuscr. Math. 66 (1990), 253–259.

[Gm] G. Gotzmann: *Stratifizierungen von Hilbert schemata- Teil 1: Hilbert schemata von Punkten in der affinen Ebene*, Preprint (1991). Rheine, 134 p.

[H-W] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe: *The Lefschetz properties*. Lecture Notes in Mathematics, vol. 2080. Springer, Heidelberg, ISBN: 978-3-642-38205-5 (2013), xx+250 pp.
[Ia1] A. Iarrobino: *Punctual Hilbert Schemes*, Mem. Amer. Math. Soc. vol. 10 (1977), 111p. #188.

[Ia2] A. Iarrobino: *Betti strata of height two ideals*, J. Algebra, 285 no. 2 (2005), 835–855.

[IY1] A. Iarrobino and J. Yaméogo: *The family $G_T$ of graded Artinian quotients of $k[x,y]$ of given Hilbert function*, in: Special issue in honor of Steven L. Kleiman, Comm. Algebra 31 (8), (2003), 3863–3916.

[IY2] A. Iarrobino and J. Yameogo: *Graded Ideals in $k[x,y]$ and partitions of diagonal lengths $T$: the hook code*, preprint (2020), 82p.

[Kl] J. O. Kleppe: *Families of Artinian and one-dimensional algebras*, J. Algebra 311 (2007), no. 2, 665–701.

[Lak] D. Laksov: *Wronskians and Plücker formulas for linear systems on curves*, Ann. Sci. École Norm. Sup. (4) 17 (1984), no. 1, 45–66.

[LW] N. Loehr and G. Warrington: *A continuous family of partition statistics equidistributed with length*, J. Combin. Theory Ser. A 116 (2009), no. 2, 379–403.

[MR] M. Mandal and M.E. Rossi: *The tangent cone of a local ring of codimension 2*, Acta Math. Vietnam. 40 (2015), no. 1, 85–100.

[Y1] J. Yaméogo, Sur l’alignement dans les schémas de Hilbert ponctuels du plan, Math Ann. 285, (1989), 511–525.

[Y2] J. Yaméogo: *Fibrés en droites amples sur des familles d’idéaux homogènes de $C[x,y]$*, in: Algebraic geometry (Catania, 1993/Barcelona, 1994), in: Lecture Notes in Pure and Appl. Math. vol. 200, Dekker, New York, (1998), pp. 229–244.

[Y3] J. Yaméogo: *Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $C[[x,y]]$. Incidence des cellules. I*, Compositio Math. 90 (1) (1994), 81–98.

[Y4] J. Yaméogo: *Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $C[[x,y]]$. Incidence des cellules. II*, J. Reine Angew. Math. 450 (1994), 123–137.