Abstract—The solution of a causal fractionary wave equation in an infinite potential well was obtained. First, the so-called “free particle” case was solved, giving as normalizable solutions a superposition of damped oscillations similar to a wave packet. From this results, the infinite potential well case was then solved. The damping coefficient of the equation obtained was matched with the exponent appearing in the Yukawa potential or “screened” Coulomb potential. When this matching was forced, the particle acquires an offset energy of $E = mc^2/2$ which then can be increased by each energy level. The exponential damping of the wave solutions in the box was found to be closely related with the radius of the proton when the particle has a mass equal to the mass of the proton. Lastly the fractionary wave equation was expressed in spherical coordinates and remains to be solved through analytical or numerical methods.

I. INTRODUCTION

Fractional calculus is as old as its classical integer counterpart, but refers to integration and differentiation in non-integer orders. In the same way Euler’s gamma function generalizes the concept of a factorial to any real or complex number, it allows the generalization of any integral or derivative to non-integer order [1], [2], [3]. This branch of calculus has proven to model interesting phenomena such as electron collision in metals [4], quark confinement [2], has been related to fractals [5], cosmological expansion [6], and more importantly non-conservative systems [7], [8], [9], [10].

In a past article [11], a set of equations derived from a non-conservative fractional lagrangian were obtained to model dissipative systems. The object of study was the damped oscillator, modelled by a lagrangian with causal and retrocausal fractional derivatives. They were shown to describe a damped oscillator, which satisfies a homogenous differential equation of the form:

$$\ddot{q} + 2\xi \dot{q} + \omega^2 q = 0$$  \hspace{1cm} (2)

When the coefficient $\xi$ is positive, this type of differential equation has three types of solutions: under-damped, critically-damped and over-damped solutions [14], [15], [16].

After studying how fractional lagrangians provide the differential equations of dissipative systems modelled by equation 1 [11], its quantum counter-part was explored and a set of causal and retrocausal fractionary wave equations were obtained using the principle of least action. The causal wave equation obtained, analogous to the damped oscillator of equation 2 was found to be of the form:

$$\nabla^2 \psi_+ + 2\xi \nabla^{(1/2)^2} \psi_+ + \frac{2m}{\hbar^2} (E - V(r)) \psi_+ = 0$$  \hspace{1cm} (3)

Here, the factor $\xi = m^2 c/\hbar B$ was taken to simplify notation. The constant $c$ is the speed of light in the vacuum. Also note that the fractionary operator $\nabla^{(1/2)^2}$ when squared is equivalent to the scalar operator:

$$\nabla^{(1/2)^2} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$  \hspace{1cm} (4)

Equation 3 can be considered a modified version of the “original” Schrödinger wave equation [17], [18], [19]:

$$\nabla^2 \psi + \frac{2m}{\hbar^2} (E - V(r)) \psi = 0$$  \hspace{1cm} (5)

Schrödinger’s wave equation is known to be generated by a hamiltonian operator, which contains a kinetic energy and a potential energy operator:

$$H = \frac{\mathbf{p}^2}{2m} + V$$  \hspace{1cm} (6)

$$\mathbf{p}_i \overset{\text{Position}}{\rightarrow} i\hbar \frac{\partial}{\partial x_i}$$  \hspace{1cm} (7)

Even though the procedure shown in [11] yielded equation 3 without the use of any momentum operators, it can also be
obtained from a fractional Hamiltonian, containing an additional fractionary momentum operator squared:

$$H_f = \frac{p^2}{2m} + \frac{p^{(1/2)^2}}{2\hbar} + V \tag{8}$$

Here, the fractional momentum $p^{(1/2)}$ is the conjugate variable to the fractionary derivative of order 1/2 for the position vector, obtained through a Legendre transform just the same way as the regular momentum vector operator $p$. This fractionary momentum vector operator transforms to the position basis as follows:

$$P^{(1/2)} \xrightarrow{\text{Position}} i(hmc)^{1/2} \frac{d^{1/2}}{dx^{1/2}} \tag{9}$$

The factor $(hmc)^{1/2}$ ensures the correct momentum units \cite{2} for the fractional derivative of order 1/2 next to it.

As equation 8 shows, the use of the vector operator $p^{(1/2)}$ adds a “fractionary” kinetic energy to the particle, which varies with B, the damping coefficient. In this way, if the fractional hamiltonian is applied in the time-independent eigenvalue problem of the form $H_f \psi = E \psi$, equation 3 is obtained.

Now we proceed to solve equation 3 for the free particle case in 1 dimension.

II. 1 DIMENSIONAL FREE PARTICLE

The first and simplest case to start studying the wave equation obtained through the fractionary momentum operator is the free particle case in 1 dimension. From the analysis done in \cite{11}, it is only necessary to solve the causal wave equation, because the retrocausal solution corresponds to the complex conjugate of the causal solution. For this reason, the + subscript in $\psi_+$ will be dropped to simplify notation. Taking $V=0$ in equation 3 results in the following 1 dimensional problem:

$$\frac{d^2 \psi}{dx^2} + 2\xi \frac{d\psi}{dx} + \frac{2mE}{\hbar^2} \psi = 0 \tag{10}$$

Note that equation 10 now has the form of the 1 dimensional damped oscillator as shown in equation 2. In other words, a fractionary momentum operator reproduces a quantum version of the classical damped oscillator. Equation 10 has the following characteristical polynomial:

$$\lambda^2 + 2\xi \lambda + \frac{2mE}{\hbar^2} = 0 \tag{11}$$

Thus the determinant which gives the solutions to be considered is:

$$\Delta = 4\xi^2 - \frac{8mE}{\hbar^2} \tag{12}$$

It was found that only the case where the determinant is negative will give solutions with quantized energies, because the positive solutions grow exponentially or linearly with distance and have no quantization conditions associated. The type of solution sought is known as the “under-damped” solution mentioned earlier. This case, where $\Delta < 0$, must satisfy the condition:

$$\xi^2 < \frac{2mE}{\hbar^2} \tag{13}$$

This condition will be reconsidered later and will be shown to have great importance in choosing which energies are valid solutions. On the other hand, the general solution to equation 10 is consequently, the following set of damped oscillations:

$$\psi(x) = e^{-\xi x} \left( A e^{i\sqrt{\Delta} x/2} + B e^{-i\sqrt{\Delta} x/2} \right) \tag{14}$$

Next if we take the wave number $k = \sqrt{\Delta}/2$ we can write the last equation in a more familiar way:

$$\psi(x) = e^{-\xi x} \left( A e^{ikx} + B e^{-ikx} \right) \tag{15}$$

It is evident that this type of solution “blows up” for $x \rightarrow -\infty$, which is an undesired effect. This free particle problem is very similar to the free particle problem for the Schrödinger wave equation, in which sine and cosine functions appear as plane-wave solutions. Because sine and cosine functions are not normalizable for $x \in [-\infty, \infty]$, the plane-waves have to be taken as a superposition to accurately describe a quantum particle as a wave-packet. Following the same line of thought, for the solutions of equation 15 the following analogous superposition of damped waves is proposed:

$$\psi(x) = \int_{-\infty}^{\infty} A(k) e^{i(x + ik)\xi - \omega(k)t} dk \tag{16}$$

Here the temporal dependence of the waves has been introduced. This last equation is not quite a Fourier transform but it does become a Fourier transform when the coefficient $\xi = 0$. The amplitude $A(k)$ contains the coefficients of the linear superposition of the damped oscillations solutions and can be obtained through an inverse transform when $t=0$.

In the case where $\xi$ is not a function of the wave number $k$, the exponential damping can be taken out of the integral:

$$\psi(x) = e^{-\xi x} \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega(k)t)} dk = e^{-\xi x} u(x,t) \tag{17}$$

For example, lets consider a non-dispersive wave-packet moving to the right with velocity $c$:

$$u(x,t) = e^{-(x-ct)^2+i\kappa_0(x-ct)} \tag{18}$$
With this type of wave-packet, equation 17 reduces to:

$$\psi(x) = e^{-\xi x} e^{-(x-ct)^2+i k_0(x-ct)}$$ \hspace{1cm} (19)$$

By completing squares this can be shown to be equivalent to a spatial translation of the gaussian envelope:

$$\psi(x) = A'(\xi, t) \cdot e^{-(x+\xi^2-ct)^2+i k_0(x-ct)}$$ \hspace{1cm} (20)

With $A'$ equal to:

$$A' = e^{-\xi ct + \xi^2/4}$$ \hspace{1cm} (21)

So when the damping is applied on a wave packet, it does not blow up for $x \to -\infty$.

### III. Particle in a Box

Now we set to solve the equation for this damped particle in a box. If the box has a total length $L$ and we take the potential to be:

$$V(x) = \begin{cases} 
0 & \text{if } x \in [-L/2, L/2] \\
\infty & \text{otherwise}
\end{cases} \hspace{1cm} (22)$$

The solution in the region where the potential is zero is given by equation 15. On the other hand, where the potential is infinite, the solution vanishes. The only thing left to do is to find the constants $A$ and $B$. For this, we apply the appropriate boundary conditions $\psi(x = L/2) = 0$ and $\psi(x = -L/2) = 0$:

$$e^{\xi L/2} (A e^{-i\sqrt{|\Delta|} L/4} + Be^{-i\sqrt{|\Delta|} L/4}) = 0 \hspace{1cm} (23)$$

$$e^{-\xi L/2} (A e^{i\sqrt{|\Delta|} L/4} + Be^{i\sqrt{|\Delta|} L/4}) = 0 \hspace{1cm} (24)$$

Following a similar procedure to that of [19] when solving Schrödinger’s wave equation in a box, these last equations can be re-written in matrix form as:

$$\begin{bmatrix} 
-e^{-i\sqrt{|\Delta|} L/4} & e^{i\sqrt{|\Delta|} L/4} \\
e^{i\sqrt{|\Delta|} L/4} & -e^{-i\sqrt{|\Delta|} L/4}
\end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \hspace{1cm} (25)$$

Forcing the determinant to be zero for non-trivial solutions in $A$ and $B$, one gets the following condition:

$$e^{-i\sqrt{|\Delta|} L/2} - e^{i\sqrt{|\Delta|} L/2} = 0 \hspace{1cm} (26)$$

Multiplying and dividing on both sides by $2i$, the energy quantization condition is obtained:

$$2i \sin \left( \frac{\sqrt{|\Delta|} L}{2} \right) = 0 \hspace{1cm} (27)$$

And this last condition gives the allowed energy levels:

$$E_n = \frac{\hbar^2}{8m} \left[ 4\xi^2 + \left( \frac{2\pi n}{L} \right)^2 \right] \hspace{1cm} (31)$$

Note that the energies depend on the level $n$, but also depend on the damping coefficient $\xi$. If the energy solutions are substituted for the wave numbers, one obtains:

$$k_n = \frac{2\pi n}{L} \hspace{1cm} (32)$$

Note that the wave numbers obtained are twice the value obtained for the particle in a box case when solving the original wave equation [19]. This happens because the quadratic equation formula gives a 1/2 factor to be accounted for. If any of the two boundary conditions (equations 23 or 24) are reconsidered for the new wave numbers, one obtains:

$$A e^{-i\pi n} + B e^{i\pi n} = 0 \hspace{1cm} (33)$$

$$A + B e^{i2\pi n} = 0 \hspace{1cm} (34)$$

$$A = -e^{i2\pi n} B \hspace{1cm} (35)$$

$$A = -B \hspace{1cm} (36)$$

For this last condition, the solutions for the particle in the box are odd functions:

$$\psi_n(x) = A e^{-\xi x} \left( e^{i k_n x} - e^{-i k_n x} \right) = A' e^{-\xi x} \sin(k_n x) \hspace{1cm} (37)$$

In contrast with the original wave function solution in the potential well, where there are the same number of symmetric and antisymmetric solutions, we obtain only anti-symmetric solutions in the form of sine functions. This effect should be expected by the damping effect introduced, because equation 10 itself is not parity-invariant. The $A'$ factor can be found through the normalization condition:

$$\int_{-L/2}^{L/2} |\psi(x)|^2 dx = 1 \hspace{1cm} (38)$$
Which gives the condition:

\[
A'_0 = \sqrt{2\xi \left( \frac{\xi L}{2\pi^2} \right)^2 + 1} c\cosh(\xi L)
\]  
(39)

Where \(c\cosh(x)\) is the hyperbolic cosecant function.

IV. THE YUKAWA POTENTIAL CLUE

The exponential decreasing factor \(e^{-\xi x}\) in equation 15 resembles a familiar effect that appears in nuclear physics, modelled by the Yukawa potential or “screened” Coulomb potential. This nuclear phenomena has also an exponential decreasing factor, and is given by [20]:

\[
V(x) = \frac{g}{4\pi x} e^{-mcx/\hbar}
\]  
(40)

Now, if this value for \(\xi\) is substituted for the energy solutions of equation 26, one obtains:

\[
E_n = \frac{1}{2} mc^2 + \left( \frac{\hbar \pi n}{mL} \right)^2
\]  
(42)

Note that a \(mc^2/2\) energy appears, which corresponds to half the relativistic rest energy of the particle. Now, reconsidering the condition of equation 13, necessary for having an under-damped oscillation, for our new value of \(\xi\) we should argue that:

\[
\left( \frac{mc}{\hbar} \right)^2 < \frac{2mE}{\hbar^2} \Rightarrow \frac{mc^2}{2} < E
\]  
(43)

Which is consistent with the \(E_n\) energy values for \(n = 1, 2, 3, \ldots\) etc. Also, when comparing the new value of \(\xi\), \(B\) must fulfill that:

\[
\xi = \frac{m^2 c}{\hbar B} = \frac{mc}{\hbar} \Rightarrow B = m
\]  
(44)

So, this means that the viscous coefficient is no longer an arbitrary parameter, but it must equal the mass of the particle to generate the same exponential decay as the Yukawa potential. For this value of \(B\), the fractionary wave equation takes the following form:

\[
\nabla^2 \psi + \frac{2mc}{\hbar} \nabla^{(1/2)} \psi + \frac{2m}{\hbar^2} (E - V) \psi = 0
\]  
(45)

In this case, the Hamiltonian operator of equation 8 takes the form:

\[
H = \frac{P_T^2}{2m} + V
\]  
(46)

Where the total momentum squared \(P_T^2\) is formed by the addition of the classical momentum and the fractionary momentum vectors squared:

\[
P_T^2 = P^2 + P^{(1/2)^2}
\]  
(47)

Furthermore, if one evaluates the penetration distance \(\lambda = 1/\xi\) for the mass of the proton, one obtains that:

\[
\lambda = \frac{1}{\xi} = \frac{\hbar}{m_p c} \approx 2.10 \text{ fm}
\]  
(48)

Now, we compare the radius of the proton \(r_p \approx 0.842 \text{ fm}\) with four penetration distances (44). This may seem rather arbitrary, but \(4\lambda\) and sometimes \(5\lambda\) are parameters used to measure the decay of an exponential curve in control engineering. At \(4\lambda\), the exponential factor has caused the oscillatory solution to decay to \(e^{-4} \times 100 \approx 1.8\%\) of its maximum value. Under this comparison one gets that:

\[
\frac{4\lambda}{r_p} = 0.9995
\]  
(49)

This result shows that taking a fractionary momentum into consideration forces most of the probability for finding a particle with \(m = m_p\) to be confined to a region which has a length roughly equal to that of a proton radius, within an approximate 0.05\% error margin. If one uses the alternative value of \(r_p \approx 0.833 \text{ fm}\), then the error raises to approximately 1.03\%.

V. FRACTIONARY WAVE EQUATION IN SPHERICAL COORDINATES

We wish to express equation 3 in spherical coordinates, even though there is no actual consensus on how to transform a fractionary operator to spherical coordinates [2]. To overcome this, we start with a cartesian representation and then transform it to spherical coordinates. Recall that the fractionary del operator squared acts according to:

\[
\nabla^{(1/2)^2} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = (\hat{x}, \hat{y}, \hat{z}) \cdot \nabla
\]  
(50)

This last dot product can be taken in spherical coordinates using the usual jacobian transformation [21]:

\[
\begin{bmatrix}
\hat{r} \\
\hat{\theta} \\
\hat{\phi}
\end{bmatrix} = \begin{bmatrix}
sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{bmatrix} \begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix}
\]  
(51)
It is necessary to take the del operator and del operator squared in spherical coordinates [22]:

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{\partial}{\partial \phi}$$

Taking $\kappa = 2mc/h$ in equation 45 to simplify notation, equation 3 in spherical coordinates turns out to be:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Equation 54 showed a relationship with spin phenomena. To match the Yukawa potential exponent, the arbitrary damping coefficient $B$ taken in the wave equation must be equal to the mass $m$, redefining the total momentum squared of the particle to contain a fractionary momentum vector squared. Also, it was shown that the probability to find the particle in this case gets confined to a length roughly equal to a proton radius if the mass of the particle is taken to be the mass of the proton.

This raises a new questions. First, ¿is there an underlying relationship between the fractionary operator formalism and the physics of nuclei and their interactions?. The energy $mc^2/2$ fits Einstein's description for half of the rest energy of a particle. This raises a second question, ¿Is there an underlying relationship between general relativity and fractionary operators?. To answer these questions, much work is further needed.

Lastly, a form of the fractionary wave equation was obtained in spherical coordinates in which first order derivate corrections appear marked by the $\kappa$ factor. Solving equation 54 has proven to be a real challenge, even for the simplest cases. Its analytical or numerical solutions must be left out for further works, but it should be said that no rotationally-invariant solutions seem to appear. This makes sense, since fractionary momentum operators are not rotationally invariant [2], and one was used to obtain such equation. Furthermore, fractionary operators have been linked to the spin of particles [2], and it would be expected that equation 54 showed a relationship with spin phenomena.

VI. CONCLUSIONS

The causal fractionary wave equation obtained in [11] was rederived through the use of a fractionary momentum operator formalism, which is independent of the causal-retrocausal formalism. This equation was solved for the free particle case, showing a wave-packet-like solution, and was also solved for the infinite potential well. This particle in a box was shown to have discrete energy levels and anti-symmetric damped waves as solutions. If the exponential decreasing factor of the damped oscillations is taken to match the Yukawa screening factor, a $mc^2/2$ energy appears in the allowed energy values. The energies of the particle in the box then has an offset value of $mc^2/2$ which increases with each level $n$.

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