Renormalization-group study of gate charge effects in Josephson-junction chains

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We study the quantum phase transition in a chain of superconducting grains, coupled by Josephson junctions, with emphasis on the effects of gate charges induced on the grains. At zero temperature the system is mapped onto a two-dimensional classical Coulomb gas, where the gate charge plays the role of an imaginary electric field. Such a field is found relevant in the renormalization-group approximation and to change the nature of the superconductor-insulator transition present in the system, tending to suppress quantum fluctuations and helping establish superconductivity. On the basis of this observation, we propose the zero-temperature phase diagram on the plane of the gate charge and the energy ratio.

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In arrays of ultra-small superconducting grains, the charging energy can be dominant over the Josephson coupling energy, giving rise to crucial effects of quantum fluctuations. In particular recent advances in fabrication techniques make it possible to control the physical parameters of the arrays, providing a convenient model system for the study of quantum phase transition between superconducting and insulating phases. In such an array of ultra-small junctions, frustration can be introduced not only by applying magnetic fields but also by inducing gate charges; these control the number of vortices and charges (Cooper pairs), leading to interesting phase transitions and dynamic responses in two dimensions. In one dimension, where quantum fluctuations should be more important, the quantum phase transition has been studied in the absence of gate charges. In such a one-dimensional (1D) array, i.e., a chain of Josephson junctions, gate charges on grains, breaking the particle-hole symmetry, can generate the persistent voltage in the appropriate regime and affect the quantum phase transition substantially, which has been studied analytically via the mean-field approximation and via the perturbation expansion as well as numerically via quantum Monte Carlo simulations. In the analytical approaches, however, the second gives accurate results only in the weak-coupling limit, i.e., for sufficiently small values of the Josephson coupling energy, whereas the first is not expected to be reliable in one dimension, where fluctuations are too strong to be neglected. Both approaches fail to capture the essential physics of the Kosterlitz-Thouless-Berezinskii (KTB) transition present in the system without gate charges. They not only disallow one to explore the detailed nature of the transition but also yield phase boundaries near the particle-hole symmetry line in a rather large discrepancy with those from simulations. It is thus desirable to study the gate charge effects in an analytical way beyond the above approaches, which discloses the nature of the transition.

This paper presents the renormalization-group (RG) study of the quantum phase transition in a chain of Josephson junctions with gate charges. At zero temperature, the system is mapped onto a two-dimensional (2D) classical Coulomb gas, where the gate charge plays the role of an imaginary electric field. This in general leads to the superconductor-insulator transition, driven by vortices, as the ratio of the Josephson coupling energy to the charging energy is varied. Here the field associated with the gate charge is found to grow under the RG transformation and tend to suppress quantum fluctuations, helping to establish superconductivity in the system. On the basis of this observation, the zero-temperature phase diagram is proposed on the plane of the gate charge and the energy ratio.

We consider a 1D array of \( N \) superconducting grains, each coupled to its two nearest-neighbor grains via Josephson junctions of strength \( E_j \). Each grain is characterized by the self-capacitance \( C \), leading to the charging energy scale \( E_C \equiv 4e^2/C \), and a gate charge \( Q \) can be induced on it externally, e.g., by applying gate voltage \( V \) with respect to the ground, giving \( Q = CV \). Such a Josephson-junction chain is described by the Hamiltonian

\[
H = \frac{1}{2K} \sum_{x=1}^{N} (n_x + q)^2 - K \sum_{x=1}^{N} \cos(\phi_x - \phi_{x+1}),
\]

where the lattice constant has been set equal to unity, the energy has been rescaled in units of the Josephson plasma frequency \( \hbar \omega_p \equiv \sqrt{E_C E_J} \), the square root of the energy ratio \( K \equiv \sqrt{E_J/E_C} \) corresponds to the effective coupling, and \( q \equiv Q/2e \) is the uniform gate charge in units of the Cooper pair charge \( 2e \). The number \( n_x \) of Cooper pairs and the phase \( \phi_x \) of the superconducting order parameter at site \( x \) are quantum-mechanically conjugate variables: \( [n_x, \phi_x^*] = i \hbar \). Due to the periodicity in \( q \), we need to consider only the range \( 0 \leq q \leq 1/2 \).

Following the standard procedure, we introduce the
imaginary time $\tau$ running on the interval $[0, \beta]$, and write the partition function in the imaginary-time path-integral representation:

$$Z = \prod \sum_{x, \tau} \int_0^{2\pi} \frac{d\phi_{x,\tau}}{2\pi} \exp \left\{ \beta^{-1} \sum_{x, \tau} A[n, \phi] \right\}$$

(2)

with the action

$$A[n, \phi] = -i \sum_x^n n_x, \tau \partial_{\tau} \phi_{x, \tau} - \frac{1}{2K} \sum_x^n (n_x, \tau + q)^2$$

$$+ K \sum_x^n \cos \partial_x \phi_{x, \tau},$$

(3)

where the inverse temperature has been rescaled according to $\hbar \omega_c \beta \rightarrow \beta$, and the (imaginary) time slice $\delta\tau$ has been chosen to be unity (in units of $\hbar \omega_c$). Here $\partial_x$ and $\partial_{\tau}$ denote the difference operator with respect to the position $x$ and to the imaginary time $\tau$, respectively:

$$\partial_x \phi_{x, \tau} \equiv \phi_{x+1, \tau} - \phi_{x, \tau}$$

and

$$\partial_{\tau} \phi_{x, \tau} \equiv \phi_{x, \tau+1} - \phi_{x, \tau},$$

and the zero-temperature limit $\beta \rightarrow \infty$ as well as the thermodynamic limit $N \rightarrow \infty$ is to be taken. We then apply the Villain approximation to integrate out the field variable in such a way that $\partial_x \phi_{x, \tau}$ into a real-valued field $\theta_{R \tau}$ and perform the duality transformation [12] to write the partition function in the form

$$Z = \sum_{s_{R \tau}} \exp (-A[s])$$

(4)

with the vorticity neutrality condition $\sum_R m_R = 0$, with $G' (R - R') \equiv 2\pi [G(0) - G(R - R')]$ describes the lattice Coulomb Green’s function with the singular diagonal piece $G(0)$ subtracted and $q$ corresponds to the vortex fugacity. Note that the gate charge $q$ plays the role of an imaginary electric field acting on vortices. Accordingly, the effects of the gate charge on the quantum phase transition in the Josephson-junction chain reduces to those of the (imaginary) electric field in the 2D Coulomb gas.

To investigate the quantum phase transition displayed by the action in Eq. (3), we now consider the correlation function, with attention to the effects of the gate charge. To the lowest order in the fugacity $y$, the vortex correlation function reads

$$\langle m_0 m_R \rangle = -2y^2 R^{-2\pi K} \cos 2\pi q X,$$

(7)

which, like the Hamiltonian in Eq. (4), depends periodically on $q$. We also have the average vorticity

$$\langle m_0 \rangle = -2iy^2 \sum_R R^{-2\pi K} \sin 2\pi q X = 0,$$

(8)

upon summing over the orientation; the gate charge does not alter the average vorticity. Renormalization of the phase correlation function by such vortices then leads to the scaling equations describing the critical behavior. The phase correlation function takes the form

$$g(r-r') \equiv \langle e^{i\phi_R - \phi_{R'}} \rangle = g_{sw}(r-r') \delta_{v}(r-r')$$

(9)

with the spin wave part

$$g_{sw}(r-r') = \exp \left[ -\frac{1}{2\pi K} G'(r-r') \right]$$

(10)

and the vortex contribution given by

$$\ln g_v(r-r') = -\frac{1}{4} y^2 \sum_{R, R_0} r_0^{-2\pi K} \cos 2\pi q x_0 |(r_0 \cdot \nabla_{R_0}) u_{R_0}|^2.$$

(11)

Here Eqs. (7) and (8) have been used, and the function $u_{R \tau}$ defined on the dual lattice satisfies

$$\begin{align*}
\partial_X u_{R \tau} &= -\partial_Y [G'(r-R) - G'(r'-R)] \\
\partial_Y u_{R \tau} &= \partial_X [G'(r-R) - G'(r'-R)].
\end{align*}$$

(12)

Replacing the lattice summation over $r_0$ by the integration in the polar coordinate $(r_0, \theta_0)$ and noting $|(r_0 \cdot \nabla_{R_0}) u_{R_0}|^2 = r_0^2 \cos^2 \theta_0 |\partial_X u_{R_0}|^2 + \sin^2 \theta_0 |\partial_Y u_{R_0}|^2 + 2 \cos \theta_0 \sin \theta_0 (\partial_X u_{R_0}) (\partial_Y u_{R_0})$, we obtain

$$\ln g_v(r-r') = -\frac{\pi y^2}{4} \sum_{R \tau} \int_{-1}^1 dr_0 r_0^{-3-2\pi K} \left[ \left\{ [J_0(2\pi q r_0) + J_2(2\pi q r_0)] |\partial_X u_{R_0}|^2 + [J_0(2\pi q r_0) - J_2(2\pi q r_0)] |\partial_Y u_{R_0}|^2 \right\} \right]$$

(13)

where $J_n(x)$ is the $n$th order Bessel function of the first kind.
where \( J_0 \) and \( J_2 \) are Bessel functions and we have used:

\[
\sum_{\mathbf{R}_0} [(\partial_{\mathbf{X}_0} u_{\mathbf{R}_0})^2 + (\partial_{\mathbf{Y}_0} u_{\mathbf{R}_0})^2] = \sum_{\mathbf{R}_0} (\nabla_{\mathbf{R}_0} u_{\mathbf{R}_0})^2 = 4\pi G'(r-r') \\
\sum_{\mathbf{R}_0} [(\partial_{\mathbf{X}_0} u_{\mathbf{R}_0})^2 - (\partial_{\mathbf{Y}_0} u_{\mathbf{R}_0})^2] = 0. 
\]

Equation (12) thus reduces to

\[
\ln g(r) = \ln g_{svw}(r) + \ln g_{v}(r) \\
= -\frac{1}{2\pi K} G'(r) - 2\pi^2 y^2 G'(r) \int dr_0 r_0^3 - 2\pi K J_0(2\pi q r_0) \\
\equiv -\frac{1}{2\pi K_{eff}} G'(r), \tag{14}
\]

which defines the effective coupling \( K_{eff} \). It is then straightforward to derive the scaling equations

\[
\frac{dK^{-1}}{d\ell} = 4\pi^3 y^2 J_0(2\pi q) \\
\frac{dy}{d\ell} = (2 - \pi K)y \tag{15}
\]

\[
\frac{dq}{d\ell} = q, 
\]

the third of which demonstrates that the gate charge is relevant, growing under the RG transformation: \( q(\ell) = q e^\ell \) with \( q \equiv q(\ell=0) \) being the initial (physical) value of the gate charge. This allows us to write the scaling equations in the form

\[
\frac{dK^{-1}}{d\ell} = 4\pi^3 y^2 J_0(2\pi q e^\ell) \\
\frac{dy}{d\ell} = (2 - \pi K)y, \tag{16}
\]

which reduce, for \( q > 0 \), to the standard scaling equations for the XY model. Accordingly, the system undergoes a KT transition from the insulating phase to the superconducting one as \( K \) is increased.

When the gate charge is present \( (q \neq 0) \), on the other hand, the Bessel function in Eq. (16) first decreases with \( \ell \) and can even become negative as the renormalization proceeds. The effective temperature \( K^{-1} \) then increases less rapidly and may even decrease under RG transformation: in this way the gate charge tends to suppress quantum fluctuations, helping to establish superconductivity in the system. Note also that the periodicity in \( q \) has been lost in Eq. (15), which results from the continuum approximation, i.e., the replacement of the lattice summation over \( \mathbf{R}_0 \) by the integration over \( r_0 \) and \( \theta_0 \). To be more accurate, we may avoid the continuum approximation and employ the Poisson summation formula:

\[
\sum_{r_0} f(r_0) = \sum_{j,k=-\infty}^{\infty} \int dr_0 r_0 \int d\theta_0 f(r_0, \theta_0) \\
\times \exp[2\pi ir_0(j \cos \theta_0 + k \sin \theta_0)], \tag{17}
\]

This leads to

\[
\frac{dK^{-1}}{d\ell} = 2\pi^3 y^2 \sum_{j,k} \left[ J_0(2\pi \sqrt{(j-q)^2 + k^2}e^\ell) + J_0(2\pi \sqrt{(j-q)^2 + k^2}e^\ell) \right], \tag{18}
\]

which indeed manifests the periodicity, in place of the first one in Eq. (15), the second one remains unchanged. For \( q = 0 \), the terms of \( j \neq 0 \) as well as those of \( k \neq 0 \) give only small quantitative corrections, and we may use the approximate scaling equations given by Eq. (13) or (14). On the other hand, for \( q \neq 0 \), the contributions of nonzero \( j \) terms can have crucial effects, and should be taken into account.

We have thus computed numerically the RG flow from the above scaling equations for various (initial) values of \( q \) as well as \( K \), and show the typical RG flow in Fig. 1. The dotted line represents the initial configuration, i.e., the physical value of the fugacity, chosen to be \( y = e^{-(\pi^2/2)K} \). It is obvious that the Gaussian fixed line \( (y = 0) \) is stable for \( K > 2/\pi \) and unstable for \( K < 2/\pi \). When the initial coupling is strong \( (K \gg 1) \), the system thus approaches the Gaussian fixed line with finite (renormalized) values of \( K \), yielding superconductivity; at weak (initial) couplings \( (K \ll 1) \), the system flows to the high (effective) temperature limit \( (K^{-1} \to \infty) \) and becomes insulating. For intermediate couplings, the non-zero gate charge, growing under the RG transformation, can make \( K \) increase and alter the flow qualitatively: The RG flow starting near \( K = 2/\pi \) can eventually turn around, approaching the fixed line, as illustrated in Fig. 1. Note that the projections of the flow trajectories onto the \( K^{-1} - y \) plane are depicted here. In fact the trajectories exist in the three-dimensional \( (K^{-1}, y, q) \) space, and do not cross with each other. The minimum initial value of the coupling, which yields such turning flow, then gives the critical coupling strength \( K_c \), below which the system is driven by quantum fluctuations to the insulating state. Via extensive numerical computation, we have obtained \( K_c \) for various values of the gate charge \( q \) and found that \( K_c \), starting from 0.748 at \( q = 0 \), decreases monotonically with \( q \); in this way the gate charge tends to suppress quantum fluctuations, helping to establish superconductivity in the system. Of particular interest is the case of the maximal gate charge \( (q = 1/2) \), where the presence of degeneracy is believed to restore superconductivity at arbitrarily weak couplings. The RG approach, with the contributions of several nonzero \( j \) terms in Eq. (18) taken into account, indeed indicates that \( K_c \) reaches its minimum at \( q = 1/2 \). Unfortunately, however, it is not allowed to confirm that \( K_c \) actually vanishes in this case: For large gate charges \( (q \sim 1/2) \), the scaling equations yield the RG flow going through the region of large values of the fugacity \( (y \gtrsim 1) \), for which the scaling equations themselves are not reliable.

The critical coupling strength \( K_c \), obtained numerically from the scaling equations as above, leads to the
phase diagram in Fig. 3. Note that due to the employed Villain approximation, the coupling $K$ in the scaling equations cannot be identified directly with $\sqrt{E_J/E_C}$ in the original Josephson-junction chain system. Comparison of the interaction form in the Villain action and the original one shows that the critical coupling strength $K_c \approx 0.748$ for $q = 0$ corresponds to the critical value $(E_J/E_C)_c \approx 1.035$, which sets the overall scale. Accordingly, for each value of $q$, the obtained value of $K_c$ is transformed into the corresponding value of $(E_J/E_C)_c$, which gives the boundary between the superconducting phase and the insulating one in Fig. 3 manifested is the suppression of quantum fluctuations by the gate charge. At $q = 0$ the transition between the two phases is of the standard KTB type, driven by vortices in the 2D spacetime. On the other hand, the nonzero gate charge, which breaks the particle-hole symmetry, makes a relevant perturbation and changes the transition in a significant way. The critical RG flow corresponding to $K_c$ approaches a fixed point on the Gaussian fixed line with an arbitrarily large renormalized value instead of the standard KTB value $2/\pi$, thus suggesting a transition of the Gaussian nature. For large $q$, the RG flow goes through the region of large values of the fugacity, for which the scaling equations are not reliable. The corresponding boundary, which is thus somewhat speculative, is drawn as a dashed line. Note that the perturbation expansion is valid in the limit $E_J/E_C \to 0$, gives accurate results around $q \approx 1/2$. In this sense the RG approach here complements the perturbation expansion. Here it is pleasing that Fig. 3 agrees well with the results of quantum Monte Carlo simulations, particularly near the symmetry line, where the perturbation expansion as well as the mean-field approximation gives large discrepancy.

In summary, we have employed the renormalization-group transformation method to investigate the quantum phase transition in a chain of Josephson junctions, with attention to the effects of gate charges. At zero temperature the system has been mapped onto a two-dimensional classical Coulomb gas, which undergoes a superconductor-insulator transition as the energy ratio is varied. In the absence of the gate charge the transition, driven by vortices, is of the KTB type. Here the gate charge has been found to play the role of an imaginary electric field, which grows under the renormalization-group transformation. In particular, it not only changes the nature of the transition but also tends to suppress quantum fluctuations, helping establish superconductivity in the system. On the basis of this observation, we have obtained the zero-temperature phase diagram on the plane of the gate charge and the energy ratio, which is, unlike the diagram obtained in the previous analytical approaches, fully consistent with the results of quantum Monte Carlo simulations.

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FIG. 1. Typical renormalization-group flow diagram on the $K^{-1} - y$ plane. Here the initial (physical) value of the gate charge is $q = 0.1$ and the dotted line represents the locus of the initial configuration.

FIG. 2. Phase diagram of a Josephson-junction chain, displaying the boundary between the superconducting phase (S) and the insulating phase (I) on the $E_J/E_C - q$ plane. The somewhat speculative boundary is depicted by the dashed line.