TAMENESS, POWERFUL IMAGES, AND LARGE CARDINALS

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Abstract. We provide comprehensive, level-by-level characterizations of large cardinals, in the range from weakly compact to strongly compact, by closure properties of powerful images of accessible functors. In the process, we show that these properties are also equivalent to various forms of tameness for abstract elementary classes. This systematizes and extends results of [BU17], [BTR16], [Lie18], and [LR16].

1. Introduction

Recent years have seen the rapid development of a literature surrounding equivalences of large cardinal principles, locality properties of Galois types in abstract elementary classes (AECs), and properties of (powerful) images of accessible functors. This synthesis of set theory, abstract model theory, and category theory has its roots in [Bon14] and [LR16]. The centerpiece result of the former is that if $\kappa$ is a strongly compact cardinal, Galois types in any AEC essentially below $\kappa$ are $<\kappa$-tame; that is, types are completely determined by their restrictions to $<\kappa$-sized submodels of their domains. The latter subsequently derived the same result, but by different means—one can characterize equivalence of Galois types via the powerful image of an accessible functor, in which case $<\kappa$-tameness corresponds precisely to $\kappa$-accessibility (and accessible embeddability) of this powerful image. By an older result of [MP89], this $\kappa$-accessibility, too, follows from strong compactness of $\kappa$.

This chain of implications can be tightened to form an equivalence. In [BTR16], a careful reworking of the argument of [MP89] reveals that accessibility of the powerful image of a $\lambda$-accessible functor $F : K \to L$ that preserves $\mu_L$-presentable objects ($\mu_L$ a cardinal computed from $\lambda$ and $L$, which we recall below) follows from the weaker assumption of $\mu_L$-strong compactness of $\kappa$, or, if you prefer, almost strong compactness of $\kappa$, $\mu_L < \kappa$. The loop is closed in [BU17] (building on [She]), where the authors give a combinatorial construction that allows one to infer, subject to certain technical conditions, almost strong compactness of a cardinal $\kappa$ from the fact that AECs below $\kappa$ are $<\kappa$-tame. So we are left with an equivalence:

**Theorem 1.1** ([BU17, Corollary 4.14]). Let $\kappa$ be an infinite cardinal with $\mu^\omega < \kappa$ for all $\mu < \kappa$. The following are equivalent:

1. $\kappa$ is almost strongly compact.
2. The powerful image of any $\lambda$-accessible functor $F : K \to L$, $\mu_L < \kappa$, is $\kappa$-accessible and $\kappa$-accessibly embedded in $L$.
3. Every AEC $K$ with $LS(K) < \kappa$ is $<\kappa$-tame.
There is a great deal more to be said, however. In conjunction with \cite{Bon14}, \cite{BU17} gives a much broader and subtler array of equivalences between gradations of strong compactness (respectively, measurability) and gradations of tameness (respectively, locality). One can also fine-tune the argument of \cite{BT-R16} to match these finer gradations. As noted in \cite{Lie18}, for any reasonably large cardinal $\kappa$, the powerful image of an accessible functor $F : K \to L$ with $\mu_L < \kappa$ will always be $\kappa$-preaccessible—what changes, depending what precise kind of large cardinal $\kappa$ is, are the kinds of colimits under which the powerful image of $F$ is closed in $L$. While \cite{Lie18} is concerned with almost measurable cardinals $\kappa$, closure of the powerful image under $\kappa$-chains, and $\kappa$-locality of Galois types, we refocus on compactness.

After a brief review of the terminology involved, Section \ref{framework-sec} provides a general framework for the arguments that connect large cardinals, Galois type locality, and accessibility of powerful images. In Section \ref{awcsec} we apply this framework to show that a cardinal $\kappa$ is $\delta$-weakly compact just in case accessible functors below $\kappa$ have powerful images closed under $\kappa^+$-small $\kappa$-directed colimits of $\kappa$-presentable objects, and just in case AECs below $\kappa$ are $(<\kappa, \kappa)$-tame (Theorem \ref{awc-thm}). In Section \ref{ascsec}, we prove a similar, level-by-level equivalence for $(\delta, \theta)$-strongly compact cardinals (Theorem \ref{nearlystrongcptequiv}).

2. Preliminaries

We consider large cardinals around the notions of weak compactness and strong compactness. Many of the definitions that are equivalent globally become separate when viewed on a level-by-level basis. We add the adjective ‘logically’ to denote that we are picking out the definitions in terms of compactness of logics.

**Definition 2.1.** Let $\kappa$ be an uncountable cardinal.

(1) (a) $\kappa$ is logically $\delta$-weakly compact if every $< \kappa$-satisfiable theory in $\mathbb{L}_{\delta, \delta}$ theory of size $\kappa$ is satisfiable.
(b) $\kappa$ is almost logically weakly compact if it is logically $\delta$-weakly compact for all $\delta < \kappa$.
(c) $\kappa$ is logically weakly compact if it is logically $\kappa$-weakly compact.

(2) (a) $\kappa$ is logically $(\delta, \lambda)$-strongly compact for $\delta \leq \kappa \leq \lambda$ if every $< \kappa$-satisfiable theory in $\mathbb{L}_{\delta, \delta}$ in a language of size $\lambda$ is satisfiable.
(b) $\kappa$ is logically $(\delta, \infty)$-strongly compact if it is logically $(\delta, \lambda)$-strongly compact for all $\lambda \geq \kappa$.
(c) $\kappa$ is logically $\lambda$-strongly compact if it is logically $(\kappa, \lambda)$-strongly compact.
(d) $\kappa$ is almost logically strongly compact if it is logically $(\delta, \infty)$-strongly compact for all $\delta < \kappa$.
(e) $\kappa$ is logically strongly compact if it is logically $(\kappa, \infty)$-strongly compact.

**Remark 2.2.** As can be seen in standard texts (e.g., \cite{Kan03}), the logical global versions (‘logically weak compact,’ ‘logically strong compact,’ etc.) agree with the standard global versions (‘weak compact,’ ‘strong compact,’ etc.). On a level-by-level basis (where the relevant definitions can be found in \cite{BU17}, Definition 2.1], things separate a bit more. Hayut \cite{Hay18} studies this behavior when $\delta = \kappa$, and the same arguments work in $\delta < \kappa$ case. Consider the following three notions:

\texttt{cto-set} \quad \texttt{cto-set} \quad \texttt{cto-set}

\texttt{((*)}_\delta, \kappa, \lambda \texttt{)} \quad \kappa \text{ is logically } (\delta, \lambda)-\text{strongly compact}

\texttt{((**)}_\delta, \kappa, \lambda \texttt{)} \quad \text{any } \kappa \text{-complete filter on } \lambda \text{ can be extended to a } \delta \text{-complete ultrafilter}

\texttt{((***)}_\delta, \kappa, \lambda \texttt{)} \quad \text{there is a } \delta \text{-complete, fine ultrafilter on } \mathcal{P}_\kappa, \lambda
Then, if $\delta < \kappa \leq \lambda = \lambda^{<\kappa}$, we have\footnote{See especially Theorem 3 there; Hayut uses ‘$\mathcal{L}_{\kappa, \kappa}$-compactness for languages of size $2^{\lambda^*}$ to refer to $(**)_\kappa$}.\[
(**)_\delta, \kappa, 2^\lambda \implies (**)_{\delta, \kappa, \lambda} \iff (*)_\delta, \kappa, 2^\lambda \implies (**)_{\delta, \kappa, \lambda}
\]

We make use of the logical characterizations of these cardinals, so prefer this definition.

We recall the following terminology related to accessible categories, and refer readers to \cite{Makkai-Pare} and \cite{Adamek-Rosicky} for further details:

**Definition 2.3.** Let $\lambda$ be a regular cardinal.

1. The colimit of a diagram $D : I \to K$ in a category $K$ is $\lambda$-directed if the underlying poset $I$ is $\lambda$-directed, i.e. any $J \subset I$ with $|J| < \lambda$ has upper bound in $I$.
2. An object $M$ in a category $K$ is $\lambda$-presentable if the associated hom-functor
   \[
   \text{Hom}_K(M, -) : K \to \text{Set}
   \]
   preserves $\lambda$-directed colimits.
3. A category $K$ is $\lambda$-preaccessible if
   - it contains, up to isomorphism, a set of $\lambda$-presentable objects, and
   - any object in $K$ is a $\lambda$-directed colimit of $\lambda$-presentable objects.
4. A category $K$ is $\lambda$-accessible if it is $\lambda$-preaccessible and closed under $\lambda$-directed colimits. We say $K$ is accessible if it is $\lambda$-accessible for some $\lambda$.

**Remark 2.4.** In an accessible category $K$, every object $M$ is $\lambda$-presentable for some $\lambda$. We define the presentability rank of $M$ to be the least such $\lambda$.

**Remark 2.5.** Recall that accessibility (and preaccessibility) do not pass upward as nicely as one might hope: in general, a $\lambda$-accessible category $K$ is $\mu$-accessible, $\mu > \lambda$ regular, if and only if $\mu$ is sharply greater than $\lambda$, denoted $\mu \gg \lambda$. The difficulty is related to, and commensurate with, that of ensuring that a subset of cardinality $\mu$ in a $\lambda$-directed poset can be completed to a $\lambda$-directed set also of cardinality $\mu$. This sharp inequality relation, defined in \cite[2.5.1]{Adamek-Rosicky}, can be reduced to the following (see \cite[2.5]{Makkai-Pare}) if $\mu$ is $\lambda$-closed, i.e. $\theta^{<\lambda} < \mu$ for all $\theta < \mu$, then $\mu \gg \lambda$. When $\mu > 2^{<\lambda}$, the converse holds.

**Remark 2.6.** We recall an important parameter, $\mu_K$, associated with each accessible category $K$ \cite{Adamek-Rosicky}. Fix $\lambda$ such that $K$ is $\lambda$-accessible—in fact, $\mu_K$ will depend on $\lambda$, but we do not include it explicitly in the notation: we trust that no confusion will result. Let $\text{Pres}_\lambda(K)$ denote a full subcategory of $K$ containing exactly one representative of each isomorphism class of $\lambda$-presentable objects, $\beta = |\text{Pres}_\lambda(K)|$. Let $\gamma_K$ be the smallest cardinal with $\gamma_K \geq \beta$ and $\gamma_K \geq \lambda$. Define
   \[
   \mu_K = (\gamma_K^{<\gamma_K})^+.
   \]
   By design, $\mu_K \geq \gamma_K \geq \lambda$. As we will see, this parameter gives a kind of measure of $K$, giving an upper bound on the linguistic resources needed to describe $K$ as a well-behaved class of structures.

**Definition 2.7.** We say that an accessible category $K$ is below $\kappa$, $\kappa$ a cardinal, if $\mu_K < \kappa$. We say $K$ is sharply below $\kappa$ if $\mu_K < \kappa$.

**Definition 2.8.** Let $\lambda$ be a regular cardinal.

1. We say that a functor $F : K \to L$ is $\lambda$-accessible if $K$ and $L$ are $\lambda$-accessible, and $F$ preserves $\lambda$-directed colimits.
Definition 2.11. Let $F : \mathcal{K} \rightarrow \mathcal{L}$ be a functor.

1. The full image of $F$ is the full subcategory of $\mathcal{L}$ on objects $FA$, $A \in \mathcal{K}$.
2. We denote by $S(F)$ the maximal sieve on the image of $F$, i.e. the full subcategory of $\mathcal{L}$ on objects $B \in \mathcal{L}$ that admit $\mathcal{L}$-morphisms $B \rightarrow FA$, $A \in \mathcal{K}$.
3. The powerful image of $F$, denoted $\mathcal{P}(F)$, is the closure of the full image of $F$ under $\mathcal{L}$-subobjects: as in \[\text{AR94, 2.27}\] in the obvious way.
4. The $\lambda$-pure powerful image of $F$, denoted $\mathcal{P}_\lambda(F)$, is the closure of the full image of $F$ under $\lambda$-pure subobjects (see \[\text{ADAM86}\]): as in \[\text{AR94, 2.27}\], with $\lambda$-pure monomorphisms in place of general morphisms.

We also review a few of the ideas we need in connection with abstract elementary classes (AECs). First defined in \[\text{SH87}\], AECs are a semantic (or, if you like, category-theoretic) abstraction of the classes of models and embeddings arising in well-behaved nonelementary classes, such as those axiomatizable in $L_{\lambda, \omega}$ or, indeed, in the elementary classes from finitary first-order logic. Crucially, AECs retain, in the structure of their class of designated embeddings, certain essential properties of these more elementary cousins. For our purposes, it is enough, perhaps, to recall that an AEC $\mathcal{K}$ has arbitrary directed colimits of $\mathcal{K}$-embeddings, and that it has an associated Löwenheim-Skolem number $LS(\mathcal{K})$ such that any object in $\mathcal{K}$ is a $LS(\mathcal{K})^+$-directed colimit of $LS(\mathcal{K})^+$-presentable objects. Here the presentability rank of an object $M \in \mathcal{K}$ is precisely $|M|^+$, i.e. the successor of the cardinality of the underlying set of $M$.

Notation 2.12. Here (and in general) we do not distinguish notationally between an object $M \in \mathcal{K}$ and its underlying set. We denote by $\mathcal{K}_\lambda$ the set of all $M \in \mathcal{K}$ with $|M| = \lambda$, and define $\mathcal{K}_{<\lambda}$, $\mathcal{K}_{\leq\lambda}$ in the obvious way.

Remark 2.13. Notice that for any AEC $\mathcal{K}$ and $\lambda > LS(\mathcal{K})$, the full subcategory of $\mathcal{K}$ on $\mathcal{K}_{<\lambda}$ is equivalent to $\text{Pres}_\lambda(\mathcal{K})$.

In AECs, the syntactic types familiar from first-order model theory are replaced by Galois (or orbital) types. Classically, a Galois pretype over $M \in \mathcal{K}$ consists of a triple $(M, a, N)$, with $N \supseteq M$ and $a \in N$; equivalently, we may define pretypes to be pairs $(f, a)$, with $f : M \rightarrow N$ a $\mathcal{K}$-embedding.

\footnote{This handy piece of terminology has not seen much use: see e.g. \[\text{CHIR} 2.7\] (in the class-accessible context).}
and \(a \in N\). A Galois type over \(M\) is an equivalence class of pretypes, under the transitive closure of the following relation of atomic equivalence: \((M, a_1, N_1) \equiv_{\text{AT}} (M, a_2, N_2)\) if and only if there are \(\mathcal{K}\)-embeddings \(g_i : N_i \to N\), \(i = 1, 2\), with \(g_1 \upharpoonright M = g_2 \upharpoonright M\) and \(g_1(a_1) = g_2(a_2)\). If we adopt the more category-theoretic formulation, pretypes \((f_1, a_1)\) and \((f_2, a_2)\) are equivalent if the pointed span

\[
\begin{array}{c}
N_1 \\
\downarrow^{g_1} \\
N \\
\uparrow_{g_2} \\
N_2
\end{array}
\]

\(\begin{array}{c}
f_1 \\
\downarrow \\
M \\
\uparrow^{f_2} \\
N_2
\end{array}\)

\(a_1 \in N_1 \xrightarrow{f_1} M \xrightarrow{f_2} N_2 \ni a_2\)

can be extended to a commutative square

\[
\begin{array}{c}
N_1 \\
\downarrow^{g_1} \\
N \\
\uparrow_{g_2} \\
N_2
\end{array}
\]

with \(g_1(a_1) = g_2(a_2)\). We toggle between the two viewpoints, but largely employ the standard model-theoretic formulation. Note that if \(\mathcal{K}\) has the amalgamation property, \(\equiv_{\text{AT}}\) is already transitive, so the equivalence notions coincide.

The notion of tameness of Galois types in an AEC \(\mathcal{K}\)—essentially the requirement that equivalence of pretypes over any \(M\) is determined by restrictions to \(\mathcal{K}\)-substructures of \(M\) of some fixed small size—was first isolated in \([\text{GV06}]\), and has come to play an important role in the development of the classification theory of AECs. We consider several parameterizations of this notion.

**Definition 2.14.** Let \(\mathcal{K}\) be an AEC, and \(\kappa \leq \lambda\).

1. We say that \(\mathcal{K}\) is \((\kappa, \lambda)-\text{tame}\) if for every \(M \in \mathcal{K}_\lambda\) and types \(p \neq q\) over \(M\), there is \(M_0 \preceq_K M\) of size less than \(\kappa\) such that \(p \upharpoonright M_0 \neq q \upharpoonright M_0\).
2. We say \(\mathcal{K}\) is \(\kappa\)-tame if it is \((\kappa, \mu)\)-tame for all \(\mu \geq \kappa\).

We also introduce a related notion: atomic tameness, which is the tameness property for atomic equivalence \(\equiv_{\text{AT}}\) of pretypes. As noted above, this will coincide with conventional tameness in AECs with amalgamation, but we wish to avoid that assumption wherever possible. In any case, we will see in Sections \(\text{BGL}+\text{BGL}+\text{BGL}\) that global assertions of atomic tameness (i.e. “all AECs are atomically tame...”) are equivalent to global assertions of tameness.

**Definition 2.15.** Let \(\mathcal{K}\) be an AEC, and \(\kappa \leq \lambda\).

1. We say that \(\mathcal{K}\) is \((\kappa, \lambda)-\text{atomically tame}\) if for every \(M \in \mathcal{K}_\lambda\) and pretypes \((M, a_1, N_1) \not\equiv_{\text{AT}} (M, a_2, N_2)\) over \(M\), there is \(M_0 \preceq_K M\) of size less than \(\kappa\) such that \((M_0, a_1, N_1) \not\equiv_{\text{AT}} (M_0, a_2, N_2)\).
2. We say \(\mathcal{K}\) is \(\kappa\)-atomically tame if it is \((\kappa, \mu)\)-atomically tame for all \(\mu \geq \kappa\).

It is important to note that, in the category-theoretic formulation, everything that we have described here (and, indeed, all the arguments below) will go through in contexts much more general than AECs. In particular, everything generalizes in straightforward fashion to \(\mu\)-AECs \([\text{BGL}+\text{BGL}]+\text{BGL}\), which are, morally speaking, just accessible categories whose morphisms are monomorphisms: A \(\mu\)-AEC \(\mathcal{K}\) with Löwenheim-Skolem-Tarski number \(\text{LS}(\mathcal{K}) = \lambda\) is a \(\lambda^+\)-accessible category with all morphisms monomorphisms, and a \(\lambda\)-accessible category \(\mathcal{K}'\) whose morphisms are monomorphisms is equivalent to a \(\lambda\)-AEC \(\mathcal{K}\) with \(\text{LS}(\mathcal{K}) = \mu\). This is the level of generality involved in, e.g., \([\text{LR10}], \text{LR11}, \text{LR12}\).
3. General criteria

We extract some general points from the work of [MP89, LR10, BT-R, lcatth] to aid in our analysis. There are two main motivations for this. The first is to create a general framework for variations on these results (which we exploit in later sections to obtain, e.g. Theorems 4.1 and 5.3). The second is to emphasize which part of these constructions are tied to the large cardinal properties, and which are true in ZFC. Note that there is no ‘tameness to compactness’ discussion, as this was already done in a modular manner in [BU17].

3.1. Compactness to powerful images. Given a $\lambda$-accessible category $\mathcal{K}$, one can readily (see, e.g. [AR94, 4.18, 5.33]) build a language $\tau^\lambda_{\mathcal{K}}$ and a theory $T^\lambda_{\mathcal{K}} \subseteq L_{\mu,\mu}(\tau^\lambda_{\mathcal{K}})$ such that

- $\Str \tau^\lambda_{\mathcal{K}}$ is essentially the functor category $\Set^{\Pres_{\lambda}(\mathcal{K})^{op}}$;
- there is a full embedding $E^\lambda_{\mathcal{K}} : \mathcal{K} \to \Str \tau^\lambda_{\mathcal{K}}$, given by the canonical embedding of $\mathcal{K}$ into $\Set^{\Pres_{\lambda}(\mathcal{K})^{op}}$; and
- $E^\lambda_{\mathcal{K}}$ induces an equivalence between $\mathcal{K}$ and $\Mod T^\lambda_{\mathcal{K}}$.

Note that we have decorated these notions with the accessibility cardinal that we are concerned with, but one can typically omit this in practice.

In our set-up, we have the following data: $\lambda$-accessible categories $\mathcal{K}$ and $\mathcal{L}$, and a $\lambda$-accessible functor $F : \mathcal{K} \to \mathcal{L}$ that preserves $\mu_{\mathcal{L}}$-presentable objects—note that, since $\mu_{\mathcal{L}} \geq \lambda$, $F$ is strongly $\mu_{\mathcal{L}}$-accessible. From this, we obtain the comma category $\mathcal{C}_F = \Id_{\mathcal{L}} \downarrow F$, whose objects are morphisms $f : L \to FK$ (for $K \in \mathcal{K}$, $L \in \mathcal{L}$) and the arrows between objects $f : L \to FK$ and $f' : L' \to FK'$ are pairs of morphisms $l : L \to L'$ and $k : K \to K'$ with $F(k)f = f'l$. In fact, we will want to require more—namely that the objects are monomorphisms $L \to FK$ (respectively, $\lambda$-pure monomorphisms) to get the powerful image (respectively, $\lambda$-pure powerful image) of $F$. We postpone this discussion for the moment.

The category $\mathcal{C}_F$ is $\mu_{\mathcal{L}}$-accessible ([AR94, 2.51]). Consider the restriction of the domain projection functor $D : \mathcal{L} \to \mathcal{L}$ to the subcategory $\mathcal{C}_F$, $H : \mathcal{C}_F \to \mathcal{L}$, that takes each $f : L \to FK$ to $L$. Note:

1. Since $D$ preserves $\lambda$-directed colimits and $\mu_\lambda$-presentable objects, so does $H$.
2. The full image of $H$ is precisely the subcategory $S(F)$ of $\mathcal{L}$ (Definition [AR94, 2.51]).

So, in thinking about $S(F)$, we can think instead about the full image of $H$. In fact, we translate once more, using the syntactic characterization of accessible categories described above to replace $H$ with a reduct functor, $R$, and realize the full image of $H$—and therefore $S(F)$—as a well-behaved projective class of structures, susceptible to analysis via logical compactness.

The $\mu_{\mathcal{L}}$-accessibility of $\mathcal{C}_F$ gives rise to a functor $E_{\mathcal{C}_F}$ into an associated category of structures, as above, on general grounds. However, the key is that this functor (and the corresponding language and theory) can be written in terms of those of $\mathcal{K}$ and $\mathcal{L}$. In particular, given $f : L \to FK$ and $x : L_0 \to L$ for $L_0 \in \mathcal{L}_{\leq \lambda}$, $\lambda$-presentability of $L_0$ means that the arrow $f \circ x : L_0 \to FK$ factors through one of the objects in the canonical decomposition of $K$ as a $\lambda$-directed colimit of $\lambda$-presentables. That means there is an essentially unique $y : K_0 \to K$ and $h : L_0 \to FK_0$ such that $f \circ x = Fy \circ hf$. Thus, we add to $\tau^\lambda_{\mathcal{K}} \cup \tau^\lambda_{\mathcal{L}}$ a relation $R_h \subseteq S^{L_0} \times S^{K_0}$ that holds in exactly this circumstance. Then we add to the theory $T^\lambda_{\mathcal{K}} \cup T^\lambda_{\mathcal{L}}$ the sentences (for each $L_0 \in \mathcal{L}_{\leq \lambda}$ and $K_0 \in \mathcal{K}_{\leq \lambda}$)

\[
\forall x \in S^{L_0} \exists y \in S^{K_0} \bigvee_{h : L_0 \to FK_0} R_h(x, y)
\]

\[\text{(*)}\]

\[\text{We use essentially the same coding as in [MP89] or [BTR10], which agree in most respects.}\]
Call this language $\tau^k_\mu$ and the theory $T^\lambda_\mu$. By definition of $\mu_\mathcal{L}$, this can all be done in $\mathbb{L}_{\mu_\mathcal{L},\nu_\mathcal{L}}$.

Let $R$ be the reduct functor $R : \text{Mod} T^\lambda_\mu \to \text{Mod} T^\lambda_\nu$, which simply forgets the interpretations of the symbols in $\tau^k_\mu \setminus \tau^k_\nu$. This functor is strongly $\mu_\mathcal{L}$-accessible. In model-theoretic terminology, the image of this functor is precisely the pseudo-elementary class $\text{PC}(T^\lambda_\mu, \tau^k_\mu)$. This means that, given $M \models T^\lambda_\mu$, we have that $M \in \text{im} R$ if and only if it has an expansion realizing $T^\lambda_\mu$.

At long last, we connect this discussion to compactness. Recall that, if $N$ is a $\tau$-structure, then $\tau(N) := \tau \cup \{c_n : n \in N\}$ is the language where a constant for each element of $N$ has been added. Given $M \in \text{Str} \tau^k_\lambda$, we set $T_M := CD^+(M)$ to be the $\mathbb{L}_{\omega,\omega} (\tau^k_\lambda(M))$-theory consisting of the positive, quantifier-free sentences true in $M$. Then $N \models CD^+(M)$ if and only if there is a homomorphism from $M$ to $N$, and this homomorphism is induced by taking $m \in M$ to $c^N_m \in N$. Thus, we have the following chain of equivalences:

$$M \in \text{im} R \iff M \text{ has an expansion that models } T^\lambda_\mu \iff T^\lambda_\mu \cup T_M \text{ is satisfiable}$$

This final piece is the syntactical formulation that we will use.

As noted above, we must adjust our definition of $\mathcal{C}_F$ if we wish to obtain the powerful or $\lambda$-pure powerful image of $F$.

1. **(Powerful image, $\mathcal{P}(F)$):** We are interested in monomorphisms $L \to FK$, which also form a $\mu_\mathcal{L}$-accessible category (see [R15, pseudopullback Theorem], [H15, Proposition 3.1]), here denoted $\mathcal{C}_{F,\text{mono}}$. We may proceed more or less as above, again realizing $\mathcal{P}(F)$ as the full image of a suitable restriction of the domain projection, and translating into structures. Note, though, that we must now ensure syntactically that there is a monomorphism from $M$ into a model of $T^\lambda_\mu$. To achieve this, we expand $T_M$ to

$$T_M^{\text{mono}} = T_M \cup \{c_m \neq c_{m'} : m \neq m' \in M\}$$

2. **($\lambda$-pure powerful image, $\mathcal{P}_\lambda(F)$)** The story is similar, with “$\lambda$-pure monomorphism” in place of “monomorphism,” and with the resulting arrow category denoted by $\mathcal{C}_{F,\lambda-\text{pure}}$. This category is $\mu_\mathcal{L}$-accessible, once again, by [H15, p. 8]. Here we replace $T_M$ by the $\lambda$-pure diagram of $M$, which consists of all positive-primitive and negated positive-primitive formulas in $\mathbb{L}_{\lambda,\lambda} (\tau^k_\lambda(M))$ that are true in $M$ (recall that a formula is positive primitive if it is of the form $\exists \bar{x} \psi(\bar{x})$, where $\psi$ is a conjunction of atomic formulas).

In either case, we must also change the additional sentence in $T^\lambda_\mu$, namely that in the displayed equation $(\star)$ above. In particular, we simply restrict the disjunction to be over monomorphisms $h : L_0 \to FK_0$ (respectively, $\lambda$-pure morphisms $h : L_0 \to FK_0$), obtaining a new theory $T^\lambda_{F,\text{mono}}$ (respectively, $T^\lambda_{F,\lambda-\text{pure}}$) for use in place of $T^\lambda_\mu$.

We conclude with a useful lemma.

**Lemma 3.1.** Suppose that $K$ is the colimit of a $\delta$-directed diagram $\mathcal{D} : I \to K$ with $|I| + \delta$.

1. if $X \subseteq E^\lambda_K(I) \setminus \delta$, then $i_* \in I$ such that $X \subseteq E^\lambda_K(\mathcal{D}(i_*))$.
2. $|E^\lambda_K(A)| = \delta + \lambda$.

The proof is straightforward.

The following also reduces the work necessary in our later results, and follows from the syntactic characterization and strong $\mu_\mathcal{L}$-accessibility of the reduct functor, $R$: 

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Lemma 3.2. For any \( \lambda \text{-accessible functor } F : \mathcal{K} \to \mathcal{L} \) that preserves \( \mu_\mathcal{L} \)-presentable objects, the inclusion of \( S(F) \) in \( \mathcal{L} \) is strongly \( \kappa \)-accessible for all \( \kappa \geq \mu_\mathcal{L} \). The same holds for \( P(F) \) and \( P_\lambda(F) \).

Crucially, this holds in ZFC. The large cardinal assumptions considered in Sections \( 2 \) and \( 3 \) will guarantee partial closure under colimits of these categories in \( \mathcal{L} \); that is, the large cardinal properties of \( \kappa \) will determine how close these subcategories come to being \( \kappa \)-accessibly embedded (Definition \( 2.8(\text{awcsec}) \)).

3.2. Powerful images to tameness. Fix an AEC \( \mathcal{K} \) with \( \text{LS}(\mathcal{K}) < \lambda \).

In order to properly code Galois types in our framework, we define two auxiliary AECs:

1. \( \mathcal{K}^< \) consists of all Galois pretypes over the same domain (this is \( \mathcal{L}_2 \) in the context of \([LR16, 5.2]\)). Structures in \( \mathcal{K}^< \) consist of \( (M, N_1, N_2, a_1, a_2) \) in \( \tau(\mathcal{K}) \cup \{R, R_1, R_2, c_1, c_2\} \) such that
   a. \( M \prec_M N_\ell \) for \( \ell = 1, 2 \)
   b. \( a_\ell \in N_\ell \)

Then \( (M, N_1, N_2, a_1, a_2) \prec (M^*, N_1^*, N_2^*, a_1^*, a_2^*) \) if and only if
   a. \( M \prec_M M^* \) and \( N_\ell \prec_M N_\ell^* \)
   b. \( a_\ell^* = a_\ell \)

2. \( \mathcal{K}^\square \) consists of all witnesses to atomic equality of pretypes (this is \( \mathcal{L}_1 \) in the context of \([LR16, 5.2]\)). Structures in \( \mathcal{K}^\square \) consist of tuples \( (M, N_1, N_2, N_+, a_1, a_2, f_1, f_2) \) in \( \tau(\mathcal{K}) \cup \{R, R_1, R_2, R_+, c_1, c_2, F_1, F_2\} \) such that
   a. \( (M, N_1, N_2, a_1, a_2) \in \mathcal{K}^< \) and \( N_+ \in \mathcal{K} \)
   b. \( f_\ell : N_\ell \to N_+ \) is a \( \mathcal{K} \)-embedding
   c. \( f_1, f_2 \) agree on \( M \) and \( f_1(a_1) = f_2(a_2) \)

Then the following is straightforward:

Proposition 3.3. Let \( \mathcal{K} \) be an AEC. Then \( \mathcal{K}^< \) and \( \mathcal{K}^\square \) are AECs with \( \text{LS}(\mathcal{K}^<) = \text{LS}(\mathcal{K}^\square) = \text{LS}(\mathcal{K}) \).

Moreover, considering these AECs as categories in the obvious way, \( \mu_{\mathcal{K}^<} = \mu_{\mathcal{K}^\square} = \mu_{\mathcal{K}} \).

We also build \( U_\mathcal{K} : \mathcal{K}^\square \to \mathcal{K}^< \) by forgetting the extra structure. This functor preserves directed colimits, but it is by no means clear that its image is an AEC or even closed under directed colimits. Indeed, write \( E^\mathcal{K}_{AT} \) for the full image of \( U_\mathcal{K} \) in \( \mathcal{K}^< \). This notation is suggestive because \( (M, N_1, N_2, a_1, a_2) \in E^\mathcal{K}_{AT} \) if and only if
   \( (a_1, M, N_1) \equiv_{AT} (a_2, M, N_2) \)

This category is closed under subobjects in \( \mathcal{K}^< \) so is also the powerful image of \( U_\mathcal{K} \). The crucial observation of \( \text{LRclass} \) is that (under amalgamation at least) the closure of \( E^\mathcal{K}_{AT} \) under colimits of certain kinds is intimately connected to the type locality properties (tameness, locality, etc.) of \( \mathcal{K} \).

We revisit this in later sections (building also on \([LR16, 5.2]\)).

4. Almost weakly compact

Recall that a colimit is said to be \( \kappa \)-small if its diagram is of cardinality less than \( \kappa \).

Theorem 4.1. Suppose \( \delta = \delta^\omega < \kappa = \kappa^\delta \). The following are equivalent:

1. \( \kappa \) is logically \( \delta \)-weakly compact.
2. If \( F : \mathcal{K} \to \mathcal{L} \) is \( \lambda \)-accessible and preserves \( \mu_\mathcal{L} \)-presentable objects, \( \mu_\mathcal{L} < \delta \), then the inclusion \( \mathcal{P}(F) \to \mathcal{L} \) is strongly \( \kappa \)-accessible and \( \mathcal{P}(F) \) is closed in \( \mathcal{L} \) under \( \kappa \)-small \( \kappa \)-directed colimits of \( \kappa \)-presentable objects.
(3) Every $AEC\ K$ with $\text{LS}(K) < \delta$ is $(< \kappa, \kappa)$-atomically tame.

(4) Every $AEC\ K$ with $\text{LS}(K) < \delta$ is $(< \kappa, \kappa)$-tame.

**Proof:** (3) $\implies$ (1): The proof of [But17, Theorem 4.9.(1)] uses an AEC with amalgamation (Claim 4.7]), so atomic tameness suffices: it is equivalent to tameness in such classes.

(1) $\implies$ (2): Strong $\lambda$-accessibility of the inclusion follows from Lemma [strongacc-lem] Fix $D: \mathcal{I} \to \mathcal{P}(F) \subset \mathcal{L}$ as in the hypothesis. Since $I$ is $\kappa$-directed, it is also $\lambda$-directed, so there is a colimit $A \in \mathcal{L}$. Then $E_L(A)$ is a $\tau_L$-structure and $|E_L(A)| = \kappa$ by Lemma [prelim-lem]

Thus $T_{E_\lambda}(A)$ has size $\kappa$, in which case $T_{E_\lambda}(A) \cup T_{F_{\text{mono}}}^\text{mono}$ is an $L_{\mu_L, \mu_L}(\tau_L(E_L(A)))$-theory of size $\kappa$, so we are in a position to use $\delta$-weak compactness of $\kappa$ to prove its satisfiability. We need to show that every $< \kappa$-sized subset of this theory is satisfiable.

Let $T_0 \subset T_{E_\lambda}(A) \cup T_{F_{\text{mono}}}^\text{mono}$ of size $< \kappa$. Write $T_1$ for $T_0 \cap T_{F_{\text{mono}}}^\text{mono}$. Then there is some $X \subset E_L(A)$ of size $< \kappa$ that contains every $c_m$ in $T_1$. By Lemma [strongacc-lem] there is $i \in I$ so $i_m \leq i$ for all $c_m \in X$.

Pushing back through $E_L$, this means that $T_1 \subset T_{E_L(D(i))}$. But then

$$T_0 \subset T_{E_L(D(i))} \cup T_{F_{\text{mono}}}^\text{mono}$$

which is satisfiable by virtue of the fact that $\mathcal{D}$ maps to $\mathcal{P}(F)$.

Thus, by the $\delta$-weak compactness of $\kappa$, $T_{E_\lambda}(A) \cup T_{F_{\text{mono}}}^\text{mono}$ is satisfiable and $E_L(A) \in \mathcal{P}(T_{F, \tau_L})$. So $A \in \mathcal{P}(F)$.

(2) $\implies$ (3): Fix such an $AEC\ K$. $(a_1, M, N_1), (a_2, M, N_2) \in K^3$ such that

$$(a_1, M_0, N_1) \equiv_{AT} (a_2, M_0, N_2)$$

for every $M_0 \in K_{< \kappa}$ with $M_0 \prec M$. Note that, in $K^<$, $(a_1, a_2, M, N_1, N_2)$ is the $\kappa$-directed colimit of the collection of $(a_1, a_2, M_0, N_1, N_2)$ for $M_0 \in I := \{M_0 \in K_{< \kappa} : M_0 \prec M\}$ (this is its canonical cocone up to isomorphism), and each such $(a_1, a_2, M, N_1, N_2)$ is in the image of $U_K$. So we have a $\mathcal{D}$ as in (2). Thus, $(a_1, a_2, M, N_1, N_2)$ is in the image of $U_K$, as desired.

(4) $\implies$ (1): This is [But17 Theorem 4.9.(3)].

(1) $\implies$ (4): This is [Bone17 Theorem 6.4].

By a similar argument, and the discussion preceding Lemma [prelim-lem] the same holds if we replace the powerful image with $\mathcal{P}_x(F)$, the $\lambda$-pure powerful image of $F$, or with $\mathcal{S}(F)$, the maximal sieve on the image of $F$.

5. LEVEL-BY-LEVEL ALMOST STRONG COMPACT

**Theorem 5.1.** Let $\delta$ be an inaccessible cardinal, and $\theta = \theta^{+\kappa}$. Each of the following statements implies the next:

1. $\kappa$ is logically $(\delta, \theta)$-strong compact.
2. If $F : K \to \mathcal{L}$ is $\lambda$-accessible, preserves $\mu_\mathcal{L}$-presentable objects, and $\mu_\mathcal{L} < \delta$, then the inclusion $\mathcal{P}(F) \to \mathcal{L}$ is strongly $\lambda$-accessible and $\mathcal{P}(F)$ is closed under $\theta^+-$small $\kappa$-directed colimits of $\kappa$-presentables.
3. Every $AEC\ K$ with $\text{LS}(K) < \delta$ is $(< \kappa, \kappa)$-atomically tame.
4. Every $AEC\ K$ with $\text{LS}(K) < \delta$ is $(< \kappa, \theta)$-tame.
Proof. (3) $\implies$ (1): As before, atomic tameness suffices in the proof of [Bu17, Theorem 4.9.(3)] since the AEC under consideration has amalgamation.

(1) $\implies$ (2): Strong $\lambda$-accessibility of the inclusion follows from Lemma [str-fact] as before. Fix $\mathcal{D}: I \to \mathcal{P}(F) \subset \mathcal{L}$ as in the hypothesis. $I$ is $\kappa$-directed, has colimit $A \in \mathcal{L}$. Then $E_\mathcal{L}(A)$ is a $\tau_\mathcal{L}$-structure and $|E_\mathcal{L}(A)| = \theta$ by Lemma [awc-thm], meaning that $T_{E_\mathcal{L}}^{mono}(A) \cup T_{F}^{mono}$ is in a language of size $\theta$.

Let $T_0 \subset T_{E_\mathcal{L}}^{mono}(A) \cup T_{F}^{mono}$ of size $< \kappa$. Write $T_1$ for $T_0 \cap T_{F}^{mono}$. Then there is some $X \subset E_\mathcal{L}(A)$ of size $< \kappa$ that contains every $c_m$ in $T_1$. By Lemma [awc-thm], there is $i \in I$ so $i_m \leq i$ for all $c_m \in X$.

Pulling back through $E_\mathcal{L}$, this means that $T_1 \subset T_{E_\mathcal{L}(\mathcal{D}(i))}$. But then

$$T_0 \subset T_{E_\mathcal{L}(\mathcal{D}(i))}^{mono} \cup T_{F}^{mono}$$

which is satisfiable by virtue of the fact that $\mathcal{D}$ maps to $\mathcal{P}(F)$.

By the $(\delta, \theta)$-strong compactness of $\kappa$, then $T_{E_\mathcal{L}(\mathcal{D}(i))}^{mono} \cup T_{F}^{mono}$ is satisfiable and $E_\mathcal{L}(A) \in PC(T_F, \tau_\mathcal{L})$. So $A \in \mathcal{P}(F)$.

(2) $\implies$ (3): Fix such an AEC $\mathbb{K}$. Let $(a_1, M, N_1), (a_2, M, N_2) \in \mathbb{K}^3$ such that

$$(a_1, M_0, N_1) \equiv_{AT} (a_2, M_0, N_2)$$

for every $M_0 \in \mathbb{K}_{<\kappa}$ with $M_0 \prec M$. Note that, in $\mathbb{K}<\kappa$, $(a_1, a_2, M, N_1, N_2)$ is the $\kappa$-directed colimit of the collection of $(a_1, a_2, M_0, N_1, N_2)$ for $M_0 \in I := \{M_0 \in \mathbb{K}_{<\kappa} : M_0 \prec M\}$ (this is its canonical cocone, up to isomorphism), and each such $(a_1, a_2, M_0, N_1, N_2)$ is in the image of $U_\mathbb{K}$. So we have a $\mathcal{D}$ as in (2). Thus, $(a_1, a_2, M, N_1, N_2)$ is in the image of $U_\mathbb{K}$, as desired.

$\dagger$

(4) $\implies$ (1): This is [Bu17, Theorem 4.9.(1)].

(1) $\implies$ (4): This is [Bonl4, Theorem 4.5].

As noted after Theorem [awc-thm] we may replicate this argument in the $\lambda$-pure powerful case, as well. Note that we do not have a perfect equivalence in Theorem [awc-thm]. The following would close the loop, but the cardinal arithmetic is just off (since $\delta^{(\theta^{<\kappa})} \geq 2^\theta > \theta$).

Fact 5.2 ([Bu17, Theorem 4.9.(3)]). If every AEC $\mathbb{K}$ with LS$(\mathbb{K}) = \delta$ is $(< \kappa, \delta^{(\theta^{<\kappa})})$-tame, then $\kappa$ is $(\delta^+, \theta)$-strong compact (in the sense that $(**)_\delta^{+, \kappa, \theta}$ holds).

This allows us to give a nice characterization at $\kappa$-closed, strong limit cardinals. Adopting the notational convention that a class is $(\mu, < \chi)$-tame if it is $(\mu, \chi_0)$-tame for all $\mu \leq \chi_0 < \chi$, we have:

Theorem 5.3. Let $\delta$ be an inaccessible cardinal, and $\theta$ be a $\kappa$-closed strong limit cardinal. The following are equivalent:

1. $\kappa$ is logically $(\delta, < \theta)$-strong compact.
2. If $F : \mathbb{K} \to \mathcal{L}$ is $\lambda$-accessible, $\mu_\mathcal{L} < \delta$, and preserves $\mu_\mathcal{L}$-presentable objects, then the inclusion $\mathcal{P}_F \to \mathcal{L}$ is strongly $\kappa$-accessible and $\mathcal{P}(F)$ is closed under $< \theta$-small $\kappa$-directed colimits of $\kappa$-presentables.
3. Any AEC $\mathbb{K}$ with LS$(\mathbb{K}) < \delta$ is $(< \kappa, < \theta)$-tame.

Proof. Combine Theorem [awc-thm] with Fact 5.2. □
One final time, we note that this argument can be adapted, in straightforward fashion, to the case of $\mathcal{P}_\lambda(F)$, the $\lambda$-pure powerful image, or $\mathcal{S}(F)$, the maximal sieve on the image of $F$.

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