The use of the multi–cumulant tensor analysis for the algorithmic search for safe investment portfolios.

Krzysztof Domino† kdomino@iitis.pl

† Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, Gliwice, Poland

May 31, 2016

Abstract

The cumulant analysis plays an important role in non Gaussian distributed data analysis. The shares’ prices returns are good example of such data. The purpose of this research is to develop the cumulant based algorithm and use it to determine eigenvectors that represent “respectively safe” investment portfolios with low variability. Such algorithm is based on the Alternating Least Square method and involves the simultaneous minimisation \(2^{nd} – 6^{th}\) cumulants of the multidimensional random variable (percentage shares’ returns of many companies). Then the algorithm was examined for daily shares’ returns of companies traded on the Warsaw Stock Exchange. It was shown that the algorithm gives the investment portfolios that are on average better than portfolios achieved by other methods, as well as than the proposed benchmark. Remark that the algorithm of is based on cumulant tensors up to the 6’tth order, what is the novel idea. It can be expected that the algorithm would be useful in the financial data analysis on the world wide scale as well as in the analysis of other types of non Gaussian distributed data.

Keywords cumulant tensors, ALS–class algorithm, Singular Value Decomposition, variability minimisation, financial data analysis, stock exchange.

1 Introduction

Let us consider the multidimensional frequency distribution of shares’ prices’ percentage returns. The optimization (minimization) of higher moments of this distribution is used to determine the “respectively safe” investment portfolio. The procedure implies the investigation of distribution’s cumulant as proposed in [1]. Remind, the \(n^{th}\) cumulant of the multidimensional random variable is represented by the \(n–dimensional\ tensor [1, 2]. In this work, I introduce the generalisation of the classical Value At Risk (VAR) method [3]. In the classical VAR the left Eigenvector Decomposition (EVD) of the second cumulant (the covariance) matrix is performed, in order to achieve factor matrix that gives portfolios with the given variance. The portfolio with minimal variance corresponds to the last eigenvector. However, the classical EVD method fails to anticipate the risk of investment portfolios since the second cumulant fails to represent the extreme events, where drops of shares’ prices values are high and cross–correlated. This happens mainly due to the break down of the central limit theorem resulting from long range auto–correlations of shares’ returns [4, 5, 7, 8, 9]. Such extreme events were often observed on financial markets and modelled by “heavily tailed” copula functions [10, 11, 12].
The author believes that high cumulants analysis will anticipate extreme events, improving the search for portfolios with low variability. There are some works implying the use of 2’nd, 3’rd and 4’th cumulant of multivariate shares’ returns [13, 14]. The author proposes to use the 5’th and the 6’th cumulant as well, what is a new approach for multivariate shares’ returns. In general the proposed algorithm is based on the High Order Singular Value Decomposition (HOSVD) and Alternating Least Square (ALS) procedure [2]. To compare the proposed method with others (such as EVD), the author, for each method, creates the family of investment portfolios which are supposed to be “respectively safe”. Then portfolios are compared using the following result functions:

1. an average and median percentage change of portfolios’ values,
2. a maximal loss – the result of the “worst portfolio”.

2 The standard VAR method – the covariance matrix EVD

Let us take the \( M \)-dimensional random variable of size \( T \), \( \mathbf{X} \in \mathbb{R}^{(T \times M)} \), being the percentage returns of \( M \) shares. Its marginal variables are \( X_i \), and values are \( x_{t,i} \):

\[
\mathbf{X} = [X_1, \ldots, X_i, \ldots, X_M] = \begin{bmatrix} x_{t=1,1} & \cdots & x_{t=1,M} \\
\vdots & \ddots & \vdots \\
x_{t=T,1} & \cdots & x_{t=T,M} \end{bmatrix}.
\]  

(1)

The variance of the \( i \)’th marginal random variable \( (X_i) \) is:

\[
\sigma_i^2 = \frac{1}{T} \sum_{t=1}^{T} (x_{t,i} - \bar{X}_i)^2,
\]  

(2)

and the covariance between \( (X_i) \) and \( (X_j) \) is:

\[
\text{cov}_{i,j} = \frac{1}{T} \sum_{t=1}^{T} (x_{t,i} - \bar{X}_i)(x_{t,j} - \bar{X}_j).
\]  

(3)

The variance and the covariance can be represented by the \( M \times M \) symmetric covariance matrix, called also the second cumulant matrix – \( C_2 \) (notice \( \sigma_i^2 = \text{cov}_{i,i} \)):

\[
C_2 = \begin{bmatrix}
\sigma_1^2 & \text{cov}_{1,2} & \cdots & \text{cov}_{1,L} \\
\text{cov}_{2,1} & \sigma_2^2 & \cdots & \text{cov}_{2,L} \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}_{L,1} & \text{cov}_{L,2} & \cdots & \sigma_L^2
\end{bmatrix}.
\]  

(4)

**Definition 2.1.** The Eigenvalue Decomposition – EVD. Consider the covariance (second cumulant) symmetric matrix. The matrix can be diagonalized in the following way:

\[
C_2 = \mathbf{V} \Sigma \mathbf{V}^\top,
\]  

(5)

where \( \Sigma \) is the diagonal matrix (values of \( \sigma_i^2 \) are sorted in the decreasing order), and \( \mathbf{V} \) is the factors matrix of size \( M \times M \):

\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & 0 & \cdots & 0 \\
0 & \sigma_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_M^2
\end{bmatrix}.
\]  

(6)
The $i$'th column of $V$ is the eigenvector that corresponds with the eigenvalue $\sigma_i^2$. Rows in the $i$'th column of $V$ are factors that give the linear combination of marginal random variables with the combination’s variance $\sigma_i^2$. The last eigenvector would give the linear combination of marginal random variables with the smallest combination’s variance — $\sigma_M^2$.

The standard VAR method has been often used in the portfolio risk determination. However, it requires the multidimensional Gaussian distribution of shares’ returns, where all information about the variability of the frequency distribution is stored in the covariance matrix. As mentioned before the financial data (shares’ returns) are not Gaussian distributed and the standard VAR method has often failed in the investment portfolio’s risk determination [10]. It is why the author proposes to extend the standard VAR model by taking into consideration also cumulants of order higher than 2 – the higher cumulants.

## 3 Cumulants

Let us consider the $M$ dimensional random variable $X = [X_1, \ldots, X_M]$ [1] and the cumulant generation function [15, 16]:

$$K(\tau) = \log \left( E \left( \exp(\tau \cdot X^\top) \right) \right) = \log \left( \frac{\sum_{i=1}^T \left( \sum_{i=1}^M \exp(\tau_i x_{i,t}) \right)}{T} \right),$$

where $\tau$ is the argument vector $\tau = [\tau_1, \ldots, \tau_i, \ldots, \tau_M]$, and $E()$ is the expecting value.

**Definition 3.1.** Cumulants [15, 16] of multidimensional random variable. The $n$’th cumulant $C_n$ of multidimensional random variable $X$ is the $n$–mode tensor [2], its element $\kappa_{\alpha_1,\ldots,\alpha_n}(X)$ at indices $\alpha_1,\ldots,\alpha_n$ are defined by the $n$’th partial derivative of (7) at $\tau = 0$ in the following manner:

$$\kappa_{\alpha_1,\ldots,\alpha_n}(X) = \frac{\partial^n}{\partial \tau_{\alpha_1} \partial \tau_{\alpha_2} \cdots \partial \tau_{\alpha_n}} \log \left( E \left( \exp(\tau \cdot X^\top) \right) \right) \bigg|_{\tau=0}.$$  

(8)

**Example 3.1.** The first cumulant is the vector with elements:

$$\kappa_i(X) = \frac{\partial}{\partial \tau_i} \log \left( E \left( \exp(\tau \cdot X^\top) \right) \right) \bigg|_{\tau=0},$$

(9)

taking the derivative of the log function, and expanding the $\exp()$ into the series:

$$\kappa_i(X) = \frac{\partial}{\partial \tau_i} E \left( 1 + \tau \cdot X^\top + \cdots \right) \bigg|_{\tau=0},$$

(10)

differentiating $E \left( \exp(\tau \cdot X^\top) \right)$ and applying $\tau = 0$, finally we have:

$$\kappa_i(X) = E(X_i).$$

(11)

The first cumulant is simply the mean vector of multivariate data. Analogically it can be shown that the second cumulant $C_2$ is the covariance matrix [4].

**Remark.** The 3’rd and 4’th cumulants tensor elements can be calculated in the following manner [15, 16]. The 3’rd cumulant $C_3$ tensor elements are:

$$\kappa_{i,j,k}(X) = E\left( (X_i - E(X_i))(X_j - E(X_j))(X_k - E(X_k)) \right),$$

(12)
and the 4’th cumulant $C_4$ elements are:

\[
\kappa_{i,j,k,l}(X) = E\left((X_i - E(X_i))(X_j - E(X_j))(X_k - E(X_k))(X_l - E(X_l))\right) \\
- E\left((X_i - E(X_i))(X_j - E(X_j))\right)E\left((X_k - E(X_k))(X_l - E(X_l))\right) \\
- E\left((X_i - E(X_i))(X_k - E(X_k))\right)E\left((X_j - E(X_j))(X_l - E(X_l))\right) \\
- E\left((X_i - E(X_i))(X_l - E(X_l))\right)E\left((X_j - E(X_j))(X_k - E(X_k))\right).
\]

**Remark.** Any cumulant can also be calculated by the direct use of (8). Here analysed data are substituted for the random variable $X \in \mathbb{R}^{(T \times M)}$, and computer differentiations of $K(\tau) - (7)$ are performed at point $\tau = [\tau_1, \ldots, \tau_M] = 0$.

**Definition 3.2.** The super–symmetric tensor [2]. Let $A$ be the $n$ mode tensor of size $M \times \cdots \times M$, and let $I : |I| = n$ be the multi–index $I = [i_1, \ldots, i_n]$. Tensor $A$ is super–symmetric if it is invariant under any indices permutation. In other words $A$ is super–symmetric if the element $A_I$ is invariant under any permutations within $I$.

**Remark.** $\forall n$ the $n$’th cumulant $C_n$ is the $n$ mode super–symmetric tensor. It can be seen from (8) since partial derivatives are commutative.

**Definition 3.3.** The super–diagonal tensor element [2]. Let $A$ be the $n$ mode tensor of size $M \times \cdots \times M$, and let $I : |I| = n$ be the multi–index $I = [i_1, \ldots, i_n]$ such that $\forall j \in [1,n]$ $i_j \in [1,M]$. The element $A_{I^*}$ is the super–diagonal element if $I^* = [i_1^*, \ldots, i_n^*]$ such that $\forall j,j' \in [1,n]$ $i_j^* = i_j$.

**Remark.** Super–diagonal elements of the 3’rd and the 4’th cumulant have the statistical meaning. Take the marginal random variable $X_i$, then:

- the super–diagonal element of the 3’rd cumulant has such meaning, that $– \kappa_{iii}/\kappa_{ii}^2$ is the skewness of $X_i$,

- the super–diagonal element of the 4’th cumulant has such meaning, that $– \kappa_{iii}/\kappa_{ii}^2$ is the excess kurtosis of $X_i$.

### 3.1 High Order Singular Value Decomposition of cumulant tensors.

Let us consider $C_n$, the $n$’th cumulant that is the super–symmetric $n$ mode tensor. The High Order Singular Value Decomposition (HOSVD) of $C_n$ is the analogy to the Singular Value Decomposition (SVD) of the 2’nd cumulant matrix $C_2$. Since the $C_2$ matrix is symmetric its SVD corresponds to its Eigenvector Decomposition (EVD) (2.1).

**Definition 3.4.** The tensor times matrix multiplication in mode $r$ [2]. Let $A$ be the $n$ mode tensor: $A \in \mathbb{R}^{L_1 \times \cdots \times L_n}$ and let $V$ be the matrix $V \in \mathbb{R}^{M \times L_r}$. Elements of $A$ are $a_{i_1,\ldots,i_n}$, and element of $V$ are $v_{j_1,j_2}$. Let $1 \leq r \leq n$, the tensor times matrix multiplication in mode $r$ is $A \times_r V \in \mathbb{R}^{L_1 \times \cdots \times L_{r-1}M_{r+1} \times \cdots \times L_n}$ with elements:

\[
(A \times_r V)_{i_1,\ldots,i_{r-1},j_1,i_{r+1},\ldots,i_n} = \sum_{i_{r}=1}^{L_{r}} a_{i_1,\ldots,i_{r-1},i_{r},i_{r+1},\ldots,i_n} v_{j_1,i_{r}}.
\]
**Definition 3.5.** The any mode tensor times matrix multiplication $A \times_{1,...,n} V$. Let $A$ be the $n$ mode tensor of size $L_1 \times \cdots \times L_n$, $A \in \mathbb{R}^{L_1 \times \cdots \times L_n}$ and let $V$ be the matrix $V \in \mathbb{R}^{M \times L}$. The symbol $\times_{1,...,n}$ is the tensor times matrix multiplication in modes $1, \cdots, n$. The result is $A \times_{1,...,n} V \in \mathbb{R}^{M \times \cdots \times M}$ with elements

$$
(A \times_{1,...,n} V)_{i_1,...,i_n} = \sum_{l_1=1}^{L_1} \cdots \sum_{l_n=1}^{L_n} a_{l_1,...,l_n} v_{i_1,l_1} \cdots v_{i_n,l_n}. \tag{18}
$$

**Definition 3.6.** The $r$ mode unfold of the tensor [2]. Let $A$ be the $n$ mode tensor: $A \in \mathbb{R}^{L_1 \times \cdots \times L_n}$ with elements $A_{i_1,...,i_r,...,i_n}$, where $r \in [1, n]$. The $r$ mode fibre is defined as the vector constructed from $A$ with fixed every index but the $r$'th [2]. The $r$ mode matricization (unfolding into the matrix) of the tensor $A$ is represented by $A_{(r)}$ and defined as matrix with the $r$ mode fibres as columns. Let $A_I$ be a tensor element indexed by $I = [i_1, \cdots, i_r, \cdots, i_n]$. It is mapped into the matrix in the following way: $A_{i_1,...,i_r,...,i_n} :\rightarrow A_{(r)_{i_r,j}}$, where [2]:

$$
j = 1 + \sum_{k=1,k\neq r}^{n} (i_k - 1) J_k, \quad J_k = \prod_{m=1,m\neq r}^{k-1} i_m. \tag{19}
$$

**The HOSVD procedure.** For the $n$-mode super–symmetric cumulant tensor the HOSVD is used to decompose it into the $n$–mode super–symmetric core–tensor $T_n$ multiplied in any mode (3.5) by the factor matrix $V$ [2],

$$
C_n = T_n \times_{1,...,n} V. \tag{20}
$$

**Definition 3.7.** The non–truncated HOSVD procedure of the super–symmetric $C_n$ tensor [2]. The tensor $C_n$ has to be unfolded into the matrix (matricized) $C_n :\rightarrow (C_n)_{(r)}$ (3.6) in some mode $(r)$ such that $1 \leq r \leq n$. Since $C_n$ is super–symmetric $- \forall r \in [1, n]$, $(C_n)_{(r)} = (C_n)_{(r^c)}$ – it is not important which mode is concerned for the unfold. Next the factor matrix $V$, with columns that are left eigenvectors of $(C_n)_{(r)}$ is calculated in an analogy to (2.1) [2]. Since $VV^T = I^c$, the core–tensor $(T_n)$ can be reached by transforming [2] (20):

$$
T_n = C_n \times_{1,...,n} V^T. \tag{21}
$$

The HOSVD moves information into the upper left corner of the core–tensor. Hence in the core–tensor the information is structured in the sense of the Frobenius Norm [2]. Let us propose the linear transformation of analysed data $X$:

$$
Y = XV, \tag{22}
$$

where $Y = [Y_1, \ldots, Y_j, \ldots, Y_M]$ (Remind, the author took the non–truncated HOSVD [2] where $V$ is the $M \times M$ factor matrix). Here $Y_j$ is the one dimensional random variable (the linear combination of $X_i$ for $i \in [1, M]$). Elements of $Y$ are:

$$
y_{i,j} = \sum_{i=1}^{M} x_{i,i} V_{i,j} \tag{23}
$$

If the marginal distribution $X_i$ represents percentage returns of the $i$'th share, $Y_j$ would represent percentage returns of the $j$'th portfolio.
Let \( A \). To investigate the financial data the author takes the multi–cumulant decomposition. The variability of the investment portfolio.

The ALS algorithm moves information into the upper left corner of the core–tensor, rear columns of the factor matrix \( V \) give \( Y_j \) (portfolios) with low variability of the investment portfolio.

Remark. High cumulants (of order \( \geq 2 \)) reflect high variability of shares prices – so called “tail events” where simultaneous high drops of shares prices are present \([7, 10, 11, 12]\). The idea of the “respectively safe” portfolio determination is to search portfolios with low values of high cumulants.

3.2 The ALS-class algorithm

Definition 3.8. The tensor times matrix multiplication in any mode but \( r \), \( A \times_{1,\ldots, r-1, r+1, \ldots, n} V \).

Let \( A \) be the \( n \) mode tensor such that: \( A \in \mathbb{R}^{L \times \cdots \times L} \) and let \( V \) be the matrix \( V \in \mathbb{R}^{M \times L} \), let \( r \in [1, n] \). The result of tensor times matrix multiplication in any mode but \( r \) is \( A \times_{1,\ldots, r-1, r+1, \ldots, n} V \in \mathbb{R}^{M \times \cdots \times M} \) with elements \((A \times_{1,\ldots, r-1, r+1, \ldots, n} V)_{i_1,\ldots, i_n}\)

\[
= \sum_{l=1}^{L} \cdots \sum_{l_{r-1}=1}^{L} \sum_{l_{r+1}=1}^{L} \cdots \sum_{l_{n-r}=1}^{L} a_{i_1,\ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_n} v_{i_1, l_1} \cdot \cdots \cdot v_{i_r, l_{r-1}} \cdot v_{i_{r+1}, l_{r+1}} \cdots \cdot v_{i_n, l_{n-r}}.
\]

(24)

To analyse financial data the author is going to use the non–truncated Alternating Least Square (ALS) procedure \([17]\) – where the factor matrices \( V \) are squared. Let us discuss the ALS algorithm \([17]\) in some details. It is used for the decomposition of the \( n \) mode tensor \( A \) into the core–tensor and factor matrices \([20]\). The ALS algorithm is the iterative algorithm, at each \((i)\)th iteration the tensor \( A \) is multiplied by the transposed factor matrices \((V^\top)\) in each mode but \( r \) \([3.8]\). Then the result is unfolded in the \( r \)th mode into matrix \([3.6]\), and finally the left Singular Vector Decomposition (SVD) of \((A \times_{1,\ldots, r-1, r+1, \ldots, n} V^\top)_{(r)}\) is computed to get the new factor matrix for the \( r \)th mode. Within each iteration the procedure is repeated for each mode: \( r \in [1, n] \).

For super–symmetric tensors such as cumulant tensors \( C_n \) the procedure can be simplified. Remark. The ALS algorithm moves information into the upper left corner of the core–tensor and order the information in the sense of the Frobenius Norm. The rear columns of the factor matrix would give linear combination of data with the little magnitude – that is with little variability of the investment portfolio.

The multi–cumulant decomposition. To investigate the financial data the author takes many cumulant tensors \( C_2, \ldots, C_n \), where \( n = 6 \). The author does not take cumulants of order \( n > 6 \) due to limited number of financial data that are analysed. On the one hand, the calculation of cumulants of order \( n > 6 \) might require larger data series (this fact will be examined in future). On the other hand the non–stationary of financial data \([5]\) makes the investigation of long time series less adequate than shorter data series.

To achieve the factor matrix \( V \) the author proposes the following ALS–class algorithm, where the search for the local maximum of the function \( \Phi(V) \) \([25]\) is performed \([17, 18]\). Following the maximisation procedure which can not be solved precisely, the author will find the local
maximum using the iteration procedure [15] and show that the results are meaningful. The generalisation of the ALS procedure proposed in [11, 15] refers to the search for the common factor matrix \( V \) that decomposes simultaneously the second cumulant matrix \( (C_2) \), the third and the forth cumulant tensors \((C_3, C_4)\). Here the iteration search for the local maximum of the following \( \Phi_4(V) \) can be performed:

\[
\Phi_4(V) = \frac{1}{2!} ||V^TC_2V||^2 + \frac{1}{3!} ||C_3 \times_{1,2,3} V^T||^2 + \frac{1}{4!} ||C_4 \times_{1,2,3,4} V^T||^2.
\]

(25)

The author proposes to extend the analysis up to the 6th cumulant which are more sensitive to extreme “tail events”. Hence the author defines the new function \( \Phi_6(V) \) in the following way:

\[
\Phi_6(V) = \frac{1}{2!} ||V^TC_2V||^2 + \frac{1}{3!} ||C_3 \times_{1,2,3} V^T||^2 + \frac{1}{4!} ||C_4 \times_{1,2,3,4} V^T||^2 + \frac{1}{5!} ||C_5 \times_{1,...,5} V^T||^2 + \frac{1}{6!} ||C_6 \times_{1,...,6} V^T||^2.
\]

(26)

To find the common factor matrix \( V \), the ALS–based algorithm described at the end of the section is used (see Algorithm [1]). The idea is based on the algorithm proposed in [17] where the iteration procedure was used for the search for the local maximum of the following function:

\[
\Phi(V) = ||V^TC_2V||^2 + \alpha_n ||C_n \times_{1,...,n} V^T||^2.
\]

(27)

Further, like in the classical ALS algorithm the \( C_n \) tensor is matricised in the \( r \)th mode \( C_n : \rightarrow C_{n(r)} [3, 6] \) and factor matrix \( V \) is computed, such that its columns are left eigenvectors of the horizontal join of matrices \( C_2 \) and \( C_{n(r)} \) (the later is multiplied here by the scalar value \( \alpha_n \)):

\[
[C_2 \alpha_n \cdot C_{n(r)}].
\]

(28)

Due to the super–symmetry of the cumulant tensor the mode of the unfold \( (r) \) can be arbitrary.

**Definition 3.9.** The horizontal join of matrices, the notation \([A B]\). Consider matrices \( A \in \mathbb{R}^{L \times M_1} \) and \( B \in \mathbb{R}^{L \times M_2} \) with elements \( a_{l,i} \) and \( b_{l,j} \), where \( l \in [1, L] \), \( i \in [1, M_1] \) and \( j \in [1, M_2] \). The horizontal join of matrices \( A \) and \( B \) gives simply the matrix where columns of matrix \( B \) are placed after columns of matrix \( A \). Formally, the matrix \( C \in \mathbb{R}^{L \times (M_1 + M_2)} \) is the horizontal join of matrices \( A \) and \( B \), \( C = [A B] \) if its elements are \( c_{l,k} : l \in [1, L] \), \( k \in [1, (M_1 + M_2)] \) and:

\[
c_{l,k} = a_{l,k} \text{ if } k \leq M_1 \quad \text{and} \quad c_{l,k} = b_{l,(k-M_1)} \text{ if } M_1 < k \leq M_2.
\]

(29)

The algorithm. For the local maximisation of \( \Phi_4(V) \) and \( \Phi_6(V) \) the author proposes the following algorithm based on the non–truncated ALS (Algorithm [1]). The algorithm is supposed to work for the general case (any \( n \)), but computations were performed for \( n = 4 \) and \( n = 6 \).

### 4 The investigation of financial data.

The cumulant analysis was performed in “respectively safe” portfolio searching problem. Let us consider the price of a \( i \)th share at time \( t - p_{t,i} \). Its percentage return is

\[
x_{t,i} = \frac{p_{t,i} - p_{(t-1),i}}{p_{(t-1),i}} \cdot 100%.
\]

(30)

In our case \( t \) numerates trading days (the analyse of daily returns was performed) and \( p_{t,i} \) the closing price of \( i \)th share the given trading day numbered by \( t \). Next the multidimensional random variable \( X \) of percentage returns is constructed [1]. To construct investment portfolios [22] we use the factor matrix \( V \). The \( j \)th portfolio returns are one dimensional random variable \( Y_j \) with elements \( y_{t,j} \) [23].

The naive method of factor matrix determination uses the Eigenvalue Decomposition (EVD) of the covariance matrix, and is proposed in a classical Value At Risk (VAR) [3]. This method
Algorithm 1 the ALS–class algorithm, computing the non–truncated factor matrix.

Require: $C_2$ – symmetric 2’nd cumulant matrix of size $M \times M$, $C_i$ – super–symmetric $i$ mode, $i$’th cumulants tensors of size $M \times \cdots \times M$, where $i \in [3, n]$, $s_{\text{min}}, s_{\text{max}}$ – minimal number of steps, $tol$ – tolerance.

1: procedure ALS($C_2, \cdots, C_n, s_{\text{min}}, s_{\text{max}}, tol$)

2: $V_1 = \text{left eigenvec} \left[ \frac{1}{2!} \cdot C_2 \cdots \frac{1}{n!} \cdot C_{n(1)} \right]^{n-1}$ $\triangleright$ columns of $V_1$ are left eigenvectors of $\left[ \cdots \right]$

3: the matrix that is the horizontal join of matrices $C_2, \cdots, C_{n(1)}$ multiplied by scalars.

4: for $k \leftarrow 1$ to $s_{\text{max}}$ do

5: for $i \leftarrow 3$ to $n$ do

6: $S_i = C_i \times_{2, \cdots, i} V_k^\top$ $\triangleright$ tensor times matrix in any mode but 1 (3.8)

7: $S_i = S_i(1)$ $\triangleright$ unfold into matrix the in first mode $S_i(1) \in \mathbb{R}^{M \times (i \cdot M)}$

8: $S_2 = C_2 V_k$

9: $V_{k+1} = \text{left eigenvec} \left[ \frac{1}{2!} \cdot S_2 \cdots \frac{1}{i!} \cdot S_i(1) \cdots \frac{1}{n!} \cdot S_{n(1)} \right]^{n-1}$ $\triangleright$ left eigenvectors of the

horizontal join of matrices (3.9)

10: for $i \leftarrow 3$ to $n$ do

11: $T_i = C_i \times_{1, \cdots, i} V_k^\top$ $\triangleright i$’th core tensor

12: end for

13: $T_2 = V^\top C_2 V$

14: for $i \leftarrow 2$ to $n$ do

15: $r_i = \frac{||C_i-T_i||_F}{||C_i||_F}$ $\triangleright$ Frobenius norm

16: end for

17: if ($k > s_{\text{min}} \wedge r_2 > tol \wedge \ldots \wedge r_n > tol \wedge \det(V_{k+1}) = 1$) then

18: return $V_{k+1}, T_2, \ldots, T_n$

19: end if

20: end for

21: end procedure
| i | company | contribution to WIG20 % | contribution to benchmark % (BP<sub>i</sub>) |
|---|---------|-------------------------|---------------------------------|
| 1 | PKOBP   | 14.64                   | 18.31                           |
| 2 | PZU     | 14.04                   | 17.55                           |
| 3 | PEKAO   | 11.65                   | 14.57                           |
| 4 | PKNORLEN| 8.45                    | 10.57                           |
| 5 | PGE     | 7.52                    | 9.40                            |
| 6 | KGHM    | 7.14                    | 8.93                            |
| 7 | BZWBK   | 5.21                    | 6.51                            |
| 8 | LPP     | 4.77                    | 5.96                            |
| 9 | PGNIG   | 3.55                    | 4.43                            |
| 10| MBANK   | 3.00                    | 3.75                            |

Table 1: The 10 most liquid companies of the WIG20 index, their value contribution to the WIG20 index (at 20.03.2015) and as their value contribution to proposed benchmark portfolio.

is not fully adequate since shares returns are not Gaussian distributed [4, 5, 7, 11, 12]. To anticipate higher cumulants of shares returns as well, the author proposes to determine the factor matrix \( V \) by searching for the local maximum of the \( \Phi_4(V) \) function (25) as well as \( \Phi_6(V) \) function (26) – using cumulant tensors up to the 6’th order, what is a new approach.

4.1 The data analysis.

The author examined \( M = 10 \) dimensional random variable (see Tab. 1) of length \( T = 1250 \), being daily percentage returns of the shares of 10 most liquid companies from the WIG20 index at the time 12.05.2010 – 13.05.2015 (the WIG20 index includes 20 most liquid companies traded on the Warsaw Stock Exchange). Let us recall that there was a descending trend on the Warsaw Stock Exchange (that started at 14.05.2015) such that the WIG20 index lost approximately 33% of its value during 170 following trading days – this descending trend can be called the speculation break down of the stock exchange market.

Respectively safe portfolios determination – training. Given the training set, the factor matrix is determined using different methods, such as EVD, \( \Phi_4(V) \) and \( \Phi_6(V) \). Here also the Independent Component Analysis (ICA) was used for more general comparison. The \( \Phi_4(V) \) method requires the calculation of 3’rd and 4’th cumulants. For \( \Phi_6(V) \) also 5’th and 6’th cumulant tensors are required, which were calculated by the direct use of (8) – see the Appendix. Given \( \Phi_4(V) \) and \( \Phi_6(V) \) Algorithm 1 was used for the factor matrix \( V \) determination.

In Fig. 1 some cumulant value of the one dimensional random variable, that is the \( j \)’th investment portfolio \( Y_j \) (with elements \( y_{t,i} = \sum_{i=1}^{M} x_{t,i} V_{i,j} \)) are presented for different methods of the factor matrix determination. Generally large cumulants values were “stored” in first two portfolios where \( j = 1, 2 \) and 8 rear portfolios, where \( j \in [3, 10] \) have low variability. In Fig. 2 chosen normalized cumulants values for “respectively safe” portfolios \( (Y_j : j \in [3, 10]) \) are presented. It can be concluded that the \( \Phi_6(V) \) method gives lowest values cumulants for \( j \in [3, 10] \). Finally for algorithm testing, the statistics of returns of all “respectively safe” portfolios \( (Y_j \text{ where } j \in [3, 10]) \) are compared for each method of the factor matrix \( V \) determination.

Testing “respectively safe” portfolios. After the training (the determination of \( V \)) has been completed, the testing of portfolios is performed. The factor matrices \( (V) \) columns contain both positive and negative values, the later corresponds to the negative value of shares in the portfolio – the short sale. To diminish the use of the sort sale, the test portfolios were compared with the benchmark portfolio. Shares values contributions in benchmark portfolio – \( BP_i \) are
Figure 1: Chosen cumulants values for all portfolios.

Figure 2: Normalized cumulants values for “relatively safe” portfolios.
given in Tab. 1. In proposed test portfolios the value contribution of the \(i\)'th share in the \(j\)'th portfolio would be:

\[
TV_{i,j} = \frac{\alpha BP_i + V_{i,j}}{\sum_{i=1}^{10}(\alpha BP_i + V_{i,j})},
\]

the \(\alpha = 7\) was taken, to make cases of the short sale rare. For testing, shares prices of companies (see Tab. 1) since 14.05.2015 were taken. Testing set is represented by: \(p_{t',i}\), where \(t' = 1\) at 14.05.2015, e.i. a day after training was complete. The percentage return of \(j\)'th portfolio after \(L\) trading days is:

\[
Pr_j(L) = \frac{\sum_{i=1}^{10} TV_{i,j} \left( \frac{p_{(t'=L+1,i)}}{p_{(t'=1,i)}} \right) - \sum_{i=1}^{10} TV_{i,j}}{\sum_{i=1}^{10} TV_{i,j}} = \sum_{i=1}^{10} TV_{i,j} \left( \frac{p_{(t'=L+1,i)}}{p_{(t'=1,i)}} \right).
\]

In Fig. 3, returns after 70 trading days – \(Pr_j(L = 70)\) are presented as an example. Remark, in this research transaction costs were not taken into account. The benchmark portfolio contributions can be reproduced by simply substituting \(\forall i,j \ V_{i,j} = 0\) to (31).

Discussion. Analysing Fig. 3, one can see that algorithm \(\Phi_6(V)\) gives 5 portfolios that are better than the benchmark – their losses are smaller that the benchmark’s loss. One can either conclude, that each method of factor matrix determination (\(\Phi_6(V), \Phi_4(V), \text{EVD}, \text{ICA}\)) produces the worst portfolio

\[
\min_{j \in [3,10]} Pr_j(L),
\]

which return is minimal and smaller than benchmark’s return. Those minimum of portfolios’ returns are presented in Fig. 4 top panel. Analysing minimum of portfolios’ returns one can conclude that out of all methods (\(\Phi_6(V), \Phi_4(V), \text{EVD}, \text{ICA}\)) the \(\Phi_6(V)\) method gives smallest loss.

It is worth checking now, which portfolio determination method is best on average. In Fig. 4 mean panel, mean values of portfolios’ returns are presented:

\[
\frac{1}{8} \sum_{j=3}^{10} Pr_j(L).
\]
Figure 4: The statistics of portfolios’ returns.
Remark that all methods but $\Phi_6(V)$ give an average return similar to the benchmark. It has been expected, since it is hard to “beat” the benchmark (based on the stock exchange index) on average. Importantly the $\Phi_6(V)$ method is on average better than the benchmark.

To examine an “average” portfolio the median of portfolios’ returns can be mentioned as well. It was presented in Fig. 4, bottom panel. Here also all methods but $\Phi_6(V)$ give a median return similar to the benchmark. For most lengths of the testing data set – $L$, the method $\Phi_6(V)$ gives results, better than the benchmark.

5 Conclusions

The author has used the multi–cumulant tensor analysis to analyse financial data and determine the “respectively safe” investment portfolio (with low variability). The author has analysed daily returns of shares traded on the Warsaw Stock Exchange to determine the factor matrix that represents “respectively safe” portfolios. Those portfolios were tested during the break down of the stock market – the recent break down of the Warsaw Stock Exchange.

The main result of this work is the introduction of the algorithm that uses 2’nd – 6’th cumulant tensors to analyse multivariate data and determine the one dimensional linear combination of those multivariate data that have low variability – the “respectively safe” portfolio. It was shown that the introduced algorithm gives “safest” portfolios, and is better on average than the stock exchange index based benchmark. The author believes that such algorithm can be used to examine stock exchanges worldwide, and determine “safe”, more break down / crisis resistant, investment portfolios. This is an important issue in the financial risk management. On the other hand the algorithm can be used to analyse other (non–financial) data in the data mining approach.

6 Appendix – calculation of high order cumulants

Consider the $M$ dimensional random variable $X \in \mathbb{R}^{(T \times M)}$. Recall the cumulant generation function (7) and the cumulant definition (8). Consider the $n$ mode cumulant tensor with elements $\kappa_{\alpha_1,\alpha_2,\ldots,\alpha_n}$, where $\forall i \in [1,n] \; \alpha_i \in [1,M]$

$$\kappa_{\alpha_1,\alpha_2,\ldots,\alpha_n}(X) = \frac{\partial^n}{\partial \tau_{\alpha_1}, \partial \tau_{\alpha_2}, \ldots, \partial \tau_{\alpha_n}} \log (E(\exp(\tau \cdot X^T))) \bigg|_{\tau=0}. \quad (35)$$

To calculate elements of arbitrary cumulant tensor the author used the ForwardDiff and Dual-Numbers library in Julia programming [19].

Acknowledgements

The research was partially financed by the National Science Centre, Poland - project number 2014/15/B/ST6/05204

References

[1] J. Morton, “Algebraic models for multilinear dependence,” 2009.
[2] T. G. Kolda and B. W. Bader, “Tensor decompositions and applications,” SIAM review, vol. 51, no. 3, pp. 455–500, 2009.
[3] P. Best, Implementing value at risk. John Wiley & Sons, 2000.
[4] B. B. Mandelbrot, The variation of certain speculative prices. Springer, 1997.
[5] D. Grech and G. Pamula, “The local hurst exponent of the financial time series in the vicinity of crashes on the polish stock exchange market,” Physica A: Statistical Mechanics and its Applications, vol. 387, no. 16, pp. 4299–4308, 2008.

[6] Ł. Czarnecki, D. Grech, and G. Pamula, “Comparison study of global and local approaches describing critical phenomena on the polish stock exchange market,” Physica A: Statistical Mechanics and its Applications, vol. 387, no. 27, pp. 6801–6811, 2008.

[7] G. L. Vasconcelos, “A guided walk down wall street: an introduction to econophysics,” Brazilian Journal of Physics, vol. 34, no. 3B, pp. 1039–1065, 2004.

[8] K. Domino, “The use of the hurst exponent to predict changes in trends on the warsaw stock exchange,” Physica A: Statistical Mechanics and its Applications, vol. 390, no. 1, pp. 98–109, 2011.

[9] K. Domino, “The use of the hurst exponent to investigate the global maximum of the warsaw stock exchange wig20 index,” Physica A: Statistical Mechanics and its Applications, vol. 391, no. 1, pp. 156–169, 2012.

[10] U. Cherubini, E. Luciano, and W. Vecchiato, Copula methods in finance. John Wiley & Sons, 2004.

[11] K. Domino and T. Błachowicz, “The use of copula functions for modeling the risk of investment in shares traded on the warsaw stock exchange,” Physica A: Statistical Mechanics and its Applications, vol. 413, pp. 77–85, 2014.

[12] K. Domino and T. Błachowicz, “The use of copula functions for modeling the risk of investment in shares traded on world stock exchanges,” Physica A: Statistical Mechanics and its Applications, vol. 424, pp. 142–151, 2015.

[13] J. C. Arismendi and H. Kimura, “Monte carlo approximate tensor moment simulations,” Available at SSRN 2491639, 2014.

[14] E. Jondeau, E. Jurlczenko, and M. Rockinger, “Moment component analysis: An illustration with international stock markets,” Swiss Finance Institute Research Paper, no. 10-43, 2015.

[15] M. G. Kendall et al., “The advanced theory of statistics,” The advanced theory of statistics, no. 2nd Ed, 1946.

[16] E. Lukacs, “Characteristics functions,” Griffin, London, 1970.

[17] L. De Lathauwer and J. Vandewalle, “Dimensionality reduction in higher-order signal processing and rank-(r 1, r 2, . . . , r n) reduction in multilinear algebra,” Linear Algebra and its Applications, vol. 391, pp. 31–55, 2004.

[18] B. Savas and L.-H. Lim, “Quasi-newton methods on grassmannians and multilinear approximations of tensors,” SIAM Journal on Scientific Computing, vol. 32, no. 6, pp. 3352–3393, 2010.

[19] J. Revels, T. Papamarkou, and M. Lubin, “Forwarddiff.jl,” JuliaDiff/ForwardDiff.jl, 2015.