Sequential Analysis of a finite number of Coherent states

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We investigate an advantage for information processing of ordering a set of states over making a global quantum processing with a fixed number of copies of coherent states. Suppose Alice has $N$ copies of one of two quantum states $\rho_0$ or $\rho_1$ and she gives these states to Bob. Using the optimal sequential test, the SPRT, we ask if processing the states in batches of size $l$ is advantageous to optimally distinguish the two hypotheses. We find that for the symmetric case $\{|\gamma\rangle,|\omega\rangle\}$ there is no advantage of taking any batch size $l$. We give an expression for the optimal batch size $l_{\text{opt}}$ in the asymmetric case. We give bounds $l_{\text{min}}$ and $l_{\text{max}}$ for when $P_S \approx 1$.

I. INTRODUCTION

The efficient detection of quantum phenomena is a matter of fundamental and practical importance, useful to test a fundamental theory or create a precise detector for a technological application, for example. This kind of problem can be framed within the study of hypothesis testing [1]. This topic can be generalized into quantum hypothesis testing [2,3]. If the information is stored in one of several quantum states the problem is usually called quantum state discrimination [4] as the task concerns with differentiating these quantum states. Optimal systems are very relevant to quantum technologies [5,6] and therefore the discrimination of optical quantum states is an important topic of study [7].

Efficient detection implies the best use of the available resources for the discovery of an event in a signal. A usual approach to analyze the efficiency of a protocol is to fix a number $N$ of resources and find the apparatus that minimizes the errors [8]. However, in practice, it is useful to consider online, on-the-fly detectors such as change-point detection [9-12].

In its simplest form, quantum state discrimination consists in being given a state $\rho$ with the promise of being one of two possible states: $\rho = \rho_0$ or $\rho = \rho_1$ (hypothesis 0 and 1 respectively) and construct a quantum measurement that distinguishes them with the lowest possible average error [13,14]. We call Type-I error for guessing hypothesis 1 as true while it is false and Type-II for guessing hypothesis 0 as true while being false. Such measurement is described by a Positive Operator Valued Measure (POVM). One can consider $N$ copies of states and form tensor states $\rho^\otimes N$. The probability of success will be higher with more copies, as more resources are available [15]. Then, the problem changes to distinguish between two hypotheses with the lowest possible average error using the given number of copies. However, the POVM might imply highly entangled operators which can be hard to build. It is relevant and not trivial to know how well different strategies behave with respect to the total number of resources $N$.

Sequential analysis is a statistical framework that addresses the issue of optimal resource handling [16]. In this framework, the desired error bounds on the Type-I and Type-II errors are fixed beforehand and the number of average samples needed to decide within these bounds. The protocol that minimizes this average number of resources is called the Sequential Probability Ratio Test (SPRT) [17].

Recently the framework of Sequential Analysis has been introduced to quantum theory [18]. It considers the problem of having access to quantum measurements of states. The bounds given in Ref. [18] give us the minimum number of resources needed using quantum measurements. It was found afterward in [19] that the general bounds are attainable with adaptive measurements. The present work can be regarded as an extension of the sequential analysis program when considering coherent states, which imply an infinite dimensional Hilbert space. However, here we consider a fixed (or non-adaptive) protocol. The problem we treat here uses the SPRT and asks for the probability that the protocol stops with $N$ copies or less with the probabilities of Type-I and Type-II errors being less than or equal to given probabilities $\alpha$ and $\beta$ respectively.

Three different strategies are relevant to this work. First, we have the general case when all the $N$ copies are available at once. This case includes possibly entangled operators for measurement. Then when the states are available one by one, we have the online scenario, which implies that the protocol ignores if there is a horizon in the number of copies and therefore is optimal at each step of the process (this protocol is well described in [20]). Finally, there is the sequential scenario, that uses the SPRT and is closely related to the online one. A relevant difference is that the SPRT is a test that minimizes the average number of resources needed.

In this article, we explore the freedom of using collective quantum strategies on subsets of copies of coherent states. The collective strategy involves an accumulation of information into one mode [21]. We have a setting as in Fig. 1, Alice gives a state $\rho^\otimes N$ to Bob and we investigate if slicing this set into $N/l$ batches of the state $\rho^\otimes l$ and measuring them in an ordering given by a function

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$f$ is beneficial for Bob in terms of distinguishing which state he was given: $\sigma_0^{\otimes N}$ or $\sigma_1^{\otimes N}$. The function $f$ only represents the fact that we are using a statistical method: the SPRT.

Intuition indicates that there should be a trade-off, as measurements with more copies yield less error, however, if we make the batches too large we will run out of copies for the SPRT, as Alice handles a finite number of copies $N$. Therefore, given $N$ there must be an optimal batch size $l$ in terms of the probability of successfully identifying the given state. We find that this is not always the case, as there are relevant instances where all values of $l$ are equivalent.

We first revisit the pure qubit case with unambiguous from Ref. [18] in Sec. [I]. Afterward, we treat the problem with coherent states. In Sec. [II A] we introduce basic notions of the SPRT. In section [III B] we explore the problem of testing Gaussian distributions and calculate the probabilities for the SPRT to stopping with $N_0$ copies or less depending on which hypothesis is true, given bounds on the Type-I and Type-II errors. Then we introduce in Secs. [IV A] and [IV B] the problem of measuring coherent states and the quantum strategy of accumulating the information of several copies into one mode. This leads to the results of Sec. [V C] where we explore the optimality of $l$ in several cases. We end the article in Sec. [V] with the conclusions.

II. SEQUENTIAL UNAMBIGUOUS POVM

In some cases, nonorthogonal states can be exactly distinguished if we allow the possibility of outcomes that don’t give information. Such discrimination protocols are called unambiguous [4]. Here we study an unambiguous protocol for distinguishing two pure finite-dimensional states. Let us denote without loss of generality, the two possible states as $|\psi_0\rangle$ and $|\psi_1\rangle$ as [22]

$$|\psi_a\rangle = \cos \theta |x\rangle + (1 - \cos \theta) \sin \theta |y\rangle ,$$

where we have written them in terms of an orthonormal basis $|x\rangle$ and $|y\rangle$ of a two-dimensional Hilbert space and an angle $\theta \in [0, \pi/4]$ between them. Let us denote the overlap between them as $\langle \psi_0 | \psi_1 \rangle = \cos 2\theta =: c$. We use a three-outcome POVM because the protocol considered here is unambiguous [4]. Following [18] we have the sequential probability of success for unambiguously discriminating 2 hypotheses when $N$ copies are available goes as $P_{UA}^{1/A} = 1 - c^N$. Remarkably, this is a result that applies to a global strategy as well as for online strategies. This equivalence implies that all batch sizes are equivalent. To see this last statement imagine that Bob makes batches of size $l$ from the original set of $N$ states. We would therefore have the states $|\Psi_a\rangle = |\psi_a\rangle^{\otimes l}$. The effective overlap between the redefined copies is $C = \langle \Psi_0 | \Psi_1 \rangle = \langle \psi_0 | \psi_1 \rangle^l = c^l$. We would therefore have $N/l$ batches. As we have batches of size $l$ then we can see this fact as a redefinition of a copy. We have therefore the probability of success for unambiguous discrimination of these batches as

$$P_{UA}^{1/A} = 1 - C^{N/l} = 1 - c^N .$$

The reason for the simple substitution on Eq. (2) is that the global performance of the unambiguous protocol is achieved by an online strategy [18]. The online strategy is to apply an unambiguous POVM for each available copy. Being an unambiguous measurement then the probability of success with $N$ copies coincide with the probability of stopping at step $N$ because this measurement yields a zero error answer. Only if we get an inconclusive outcome we would have to keep on measuring. However, we can wait to have all the $N$ copies and make a global unambiguous measurement and have a result with the same success probability, therefore there is no gain in the ordering strategy by Bob in the unambiguous protocol.

A drawback of using an unambiguous protocol is that despite that it yields a no-error answer, the whole protocol has, in general, a lower probability of success than a two-outcome POVM. The reason for this is that is a very restrictive protocol. Also, for mixed states, unambiguous discrimination is possible only in very restrictive cases.

III. PROBABILITY THAT THE SPRT STOPS WITH $N_0$ SAMPLES OR LESS

A. Classical SPRT

We first review some basic notions of the SPRT theory by Wald [16]. Consider that we have $N_0 < \infty$ independent and identically distributed (i.i.d.) samples $x_i$ of a random variable $X$ that follows the probability distributions $p(x|0)$ or $p(x|1)$. The $\{0, 1\}$ index denotes the hypotheses 0 or 1 respectively. Observe that $N_0 \neq N,$ $N$ will return afterward. We can define a useful variable

$$z(x) = \log \frac{p(x|0)}{p(x|1)}$$

where log denotes the natural logarithm. Thus, with a set of outputs $\{x_1, \ldots, x_n\},$ we have a set of values...
\{z(x_1), \ldots, z(x_n)\} \) that we will denote as \{z_1, \ldots, z_n\} for simplicity. At step \( n \) we define
\[
Z_n := \sum_{i=1}^{n} z_i.
\] (4)

\( Z_n \) is an example of what is known in the literature as Martingale [23], which is a stochastic process whose mean value for step \( n + 1 \) is the value of step \( n \). The SPRT consists in observing the value of \( Z_n \) when a new sample is available. If \( Z_n \geq C_0 \) we will accept hypothesis 0 as true. If \( Z_n \leq C_1 \) we will accept hypothesis 1 as true. If \( C_1 < Z_n < C_0 \) continue sampling. It can be shown [16] that the bounds \( C_0 \) and \( C_1 \) can be chosen such that the type I error probability is \( \leq \alpha \) and analogously, such that the type II error probability is \( \leq \beta \) for given \( \alpha, \beta \in [0,1] \).

Defining
\[
A := \frac{1-\beta}{\alpha} \quad \text{and} \quad B := \frac{\beta}{1-\alpha},
\] (5)

we have that in a very good approximation [16],
\[
C_0 \approx \log A \quad \text{and} \quad C_1 \approx \log B.
\] (6)

The SPRT is the sequential test that requires fewer samples on average [17].

We are given \( N_0 \) samples and we restrict to the SPRT. The relevant probabilities to calculate correspond
\[
P_0(Z_{N_0} \geq \log A) \quad \text{and} \quad P_1(Z_{N_0} \leq \log B)\] (7)

where \( P_i \) correspond to the probability when hypothesis \( i \) is true. Let us suppose that we are given the hypothesis 0 and 1 with equal priors therefore the total probability of success is
\[
P_S = \frac{1}{2} P_0(Z_{N_0} \geq \log A) + \frac{1}{2} P_1(Z_{N_0} \leq \log B).\] (8)

### B. Testing Gaussians

Suppose now that \( X \) is normally distributed so that the probability distribution when the hypothesis \( i \) is true corresponds to
\[
p(x|i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\theta_i)^2}{2\sigma^2}}.
\] (9)

It is straightforward to show that
\[
z(x) = \frac{1}{2\sigma^2} (2(\theta_0 - \theta_1)x + \theta_1^2 - \theta_0^2).
\] (10)

Recalling Eq. (4) we have that
\[
\sum_{i=1}^{N_0} x_i = \frac{Z_{N_0} 2\sigma^2 - (\theta_1^2 - \theta_0^2)}{2(\theta_0 - \theta_1)}.
\] (11)

Suppose that each \( x_i \) has mean \( \theta \) and variance \( \sigma^2 \). Observe that \( \sum_{i=1}^{N_0} x_i \) is a sum of normally distributed random variables, therefore it is a normally distributed variable with mean \( N_0 \theta \) and variance \( N_0 \sigma^2 \) [24].

The stopping condition for the SPRT \( Z_{N_0} \geq \log A \) translates to
\[
\sum_{i=1}^{N_0} x_i \geq \log A 2\sigma^2 - (\theta_1^2 - \theta_0^2) = \frac{2(\theta_0 - \theta_1)}{2(\theta_0 - \theta_1)}.
\] (12)

Observe that the probability that a normally distributed variable \( X \) to take a value less than or equal \( \lambda \) is given by the cumulative probability \( G(\lambda) \). As we want the probability that a variable takes a value less than or equal to some lambda we need \( 1 - G(\lambda) \). In terms of the Error function [25] defined as
\[
\text{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt,
\]

we thus have the probability
\[
P_0(Z_{N_0} \geq \log A) = \frac{1}{2} \left( 1 - \text{Erf} \left( \frac{2\sigma^2 \log A - (\theta_1^2 - \theta_0^2) - 2N_0 \theta_0 (\theta_0 - \theta_1)}{2(\theta_0 - \theta_1) \sqrt{2N_0} \sigma} \right) \right).
\] (14)

Analogously, we can calculate
\[
P_1(Z_{N_0} \leq \log B) = \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{2\sigma^2 \log B - (\theta_1^2 - \theta_0^2) - 2N_0 \theta_1 (\theta_0 - \theta_1)}{2(\theta_0 - \theta_1) \sqrt{2N_0} \sigma} \right) \right).
\] (15)

### IV. COHERENT STATES

#### A. Wigner function

A coherent state \( |\gamma\rangle \) is described by a complex number \( \gamma \). In the phase space, we can write \( \gamma = q_\gamma + ip_\gamma \) with \( q \) and \( p \) denoting quadratures of the electromagnetic field. The Wigner function of such a state is given by a Gaussian [26]
\[
W_{\gamma} = \frac{1}{\pi} \exp \left[ -2(q - q_\gamma)^2 - 2(p - p_\gamma)^2 \right].
\] (16)

To detect a quadrature of the electromagnetic field one normally uses homodyne detection, which allows us to detect intensity discrepancies in an electromagnetic field. Explicitly we can detect [26]
\[
\Delta I = \sqrt{2}\|\gamma\| \left| \frac{e^{-i\xi a^\dagger + e^{i\xi a}}}{\sqrt{2}} \right|
\]
for the angle \( \xi \). Suppose we measure the quadrature \( q \) with the coherent state \( |\gamma\rangle \), which corresponds to \( \xi = 0 \) the probability distribution is a Gaussian with mean \( \theta = q_\gamma \) and variance \( \sigma^2 = 1/4 \).
B. Multiple copies

If multiple copies of coherent states are available we can accumulate the information into one mode [21]. Consider a beam splitter of transmissivity $T$ and reflectivity $R$, if the coherent states $|\gamma\rangle$ and $|\delta\rangle$ incide into the beam splitter it transforms to

$$|\gamma\rangle \otimes |\delta\rangle \rightarrow \left|\sqrt{T}\gamma + \sqrt{R}\delta\right\rangle \otimes \left|\sqrt{R}\gamma + \sqrt{T}\delta\right\rangle.$$ \hspace{1cm} (18)

Therefore, if $\gamma = \delta$ and we have a 50:50 beam splitter we get $|\sqrt{2}\gamma\rangle \otimes |0\rangle$. In general, $l$ copies can be concentrated into one mode. Suppose several beam splitters are put one after another such that they perform the unitary transformation $|\gamma\rangle^{\otimes l} \rightarrow |\sqrt{l}\gamma\rangle \otimes |0\rangle^{l-1}$ [21]. To achieve this, the beam splitters must have transmissivities and reflectivities given by

$$T_j = \frac{j}{j+1}, \quad R_j = \frac{1}{j+1}.$$ \hspace{1cm} (19)

C. Optimal $l$

We return to the scenario of Fig. [1]. Suppose that we are given $N_0 = N/l$ batches of copies of coherent states. For each batch of $l$ states, we implement the process of accumulation from section IV B. Therefore, the probability distributions we are comparing are given by

$$p(x|i) = \sqrt{2\pi} e^{-2(x - \sqrt{\theta_i})^2},$$ \hspace{1cm} (20)

where $\theta_i$ is given by the real part of the coherent state $\theta_i = \Re(\gamma_i)$. Notice that $\theta_i \rightarrow \sqrt{l}\theta_i$ with respect to section III B. Therefore, following Eqs. (14) and (15) we have

$$P_0(Z_{N/l} \geq \log A) = \frac{1}{2} \left(1 - \text{Erf} \left(\frac{\frac{1}{2} \log A - l(\theta_0^2 - \theta_0^2) - 2N\theta_0(\theta_0 - \theta_1)}{\sqrt{2}(\theta_0 - \theta_1)}\sqrt{\frac{\theta_0}{\theta_1}}\right)\right).$$ \hspace{1cm} (21)

and

$$P_1(Z_{N/l} \leq \log B) = \frac{1}{2} \left(1 + \text{Erf} \left(\frac{\frac{1}{2} \log B - l(\theta_0^2 - \theta_0^2) - 2N\theta_1(\theta_0 - \theta_1)}{\sqrt{2}(\theta_0 - \theta_1)}\sqrt{\frac{\theta_0}{\theta_1}}\right)\right).$$ \hspace{1cm} (22)

Observe that $P_0(Z_{N/l} \geq \log A)$ and $P_1(Z_{N/l} \leq \log B)$ depend on $l$.

In Fig. 2 the SPRT is illustrated for several values of $l$. The Gaussian distribution is a numerical approximation truncated in $\{-10, 10\}$. In that figure, we observe random realizations, some of which surpass the bound corresponding to $\log A$, which correspond to the success instances. The mean value of the sampling distribution corresponds to $\theta_0$ and thus we see that the martingales tend to go upwards.

Figure 2: Examples of martingales $Z_n$ for different values of $l$ indicated above each plot. We take 1000 samples in each figure. In each figure $N = 100$, $\theta_0 = 0.1$, $\alpha = -0.1$, $\alpha = 0.01$, and $\beta = 0.05$. The mean value of the sampling distribution is $\theta_0$. The dark blue paths correspond to random realizations of $Z_n$. The dark blue is the mean path over the 1000 samples.

The cost function that needs to be optimized is the total probability from Eq. (8). It remains to optimize it over $l$. To this end, we need to investigate the sum of
Error functions. Let us then define
\begin{align}
y_A &:= \frac{1}{2} \log A - l(\theta_1^2 - \theta_0^2) - 2N\theta_0(\theta_0 - \theta_1), \\
y_B &:= \frac{1}{2} \log B - l(\theta_1^2 - \theta_0^2) - 2N\theta_1(\theta_0 - \theta_1).
\end{align}
\tag{23, 24}
Therefore,
\[ P_S = \max_l \{ \frac{1}{4} (2 - \text{Erf}(y_A) + \text{Erf}(y_B)) \}. \tag{25} \]

1. Symmetric case

The frequently used Dolinar receiver \cite{27} normally works with a symmetric pair of coherent states \{\gamma, -\gamma\}. If we are in this symmetric case then we have that \( \theta_0 = -\theta_1 \). This implies
\begin{align}
y_A &:= \frac{1}{2} \log A - 4N\theta_0^2 \\
y_B &:= \frac{1}{2} \log B + 4N\theta_0^2
\end{align}
\tag{26, 27}
We see that there is no dependence on \( l \), therefore any batch size is equally good.

2. Non-symmetric case

Suppose now that \( \theta_0 \neq -\theta_1 \). In general, \( P_S \) can have three behaviors as shown in Fig. (3). We can change the value of \( l \) such that we move in the \( x \) axis of the figures in question. The optimization over \( l \) depends on the case we have at hand.

If we are in case I there is nothing to do, we have that \( P_S \leq 1/2 \). In this case the best guess for the hypothesis at hand is random.

If we are in case II observe that there is a point where the sum of Error functions attain a maximum. This maximum can be approximated with the Taylor expansion of the exponential around 0. Using the Eq. \( 13 \) we obtain a Taylor expansion for the Error function around 0
\[ \text{Erf}(y) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{n!(2n+1)}. \tag{28} \]
At order zero, we see that
\[ \text{Erf}(y) \approx \frac{2}{\sqrt{\pi}} y. \tag{29} \]
Using this, by symmetry, we can obtain an approximation to the optimal value of \( l \). Notice that the point where the zero-order approximation in case II in Fig. (3) cross each other marks the optimal value of the sum of Error functions. Therefore, the maximum of \( P_S \) is found when
\[ -y_A \approx y_B. \tag{30} \]

We thus approximate value for the optimal \( l \), we define
\[ l_{\text{opt}} := N + \frac{\log A + \log B}{4(\theta_1^2 - \theta_0^2)}. \tag{31} \]
This value only makes sense when
\[ 0 \leq l_{\text{opt}} \leq N. \tag{32} \]

In Fig. (4) we have a graph of the total probability of success \( P_S \) dependent on \( l \) and see that it attains its maximum at \( l_{\text{opt}} \) given by Eq. \( 31 \).

If we are in case III there are limits for \( x \) were \( P_S \approx 1 \) in Fig. (5) as \( x_{\text{min}} \) and \( x_{\text{max}} \). This implies bounds for \( l \)
These equations give the bounds that are defined as follows
\[
\begin{align*}
-\frac{2}{\sqrt{\pi}} y_A &= 1 \\
\frac{2}{\sqrt{\pi}} y_B &= 1.
\end{align*}
\] (33)

These equations give the limits
\[
\begin{align*}
\ell_{\min} &:= \frac{1}{2(\theta_1 + \theta_0)} \left( \frac{\log B}{\theta_1 - \theta_0} + 4N\theta_1 + \sqrt{2N\pi} \right) \\
\ell_{\max} &:= \frac{1}{2(\theta_1 + \theta_0)} \left( \frac{\log A}{\theta_1 - \theta_0} + 4N\theta_0 - \sqrt{2N\pi} \right). 
\end{align*}
\] (34)

These bounds are only defined for
\[
\begin{align*}
0 \leq \ell_{\min} \leq N, \\
0 \leq \ell_{\max} \leq N.
\end{align*}
\] (35)

V. CONCLUSIONS

We extend the study of sequential analysis protocols for coherent states. Specifically, we study the probability that a specific statistical test, the SPRT accepts one of two possible hypotheses with N, a given number of resources. In so doing we investigate the duality of collective measurements with many copies and the necessity of having to process the measurements optimally with the SPRT.

We find that in the symmetric case, \{ |\gamma\rangle, -|\gamma\rangle \} there is no advantage of taking batches of any size. In contrast, with the adaptive protocol used in the Dolinar receiver [27] the protocol we consider here is non-adaptive. The independence with respect to \ell seems to come from the fact that we are considering optimal sequential processing. For non-symmetric cases, two cases are relevant to us. In the first one, there is a unique \ell that achieves the maximum labeled \ell_{opt}, which is approximated using the Taylor expansion of the Error function. The second relevant case implies a range of values of \ell for which, using the expansion of the Error function we define a lower bound \ell_{\min} and an upper bound \ell_{\max} for the range of values of \ell that attain the optimal \text{P}_S.

Operationally speaking, the SPRT shows an advantage when considering small type-I and type-II error probabilities. However, notice that the protocol we are considering is more general than only making a collective, entangled measurement. The reason for this is that the batch could be of size N always i.e., \ell = N. We show that in general, this is not the case and that there is an advantage when taking into account the statistical process.

The treatment here was with the most simple quantum strategy that involves only pure states and fixed measurements. Perhaps an adaptive strategy in the measurement apparatus gives more insight into when sequential information processing is necessary [19].

The results from this work could be generalized to the mixed-state case for finite-dimension states. It would be necessary a calculation of the probabilities of Eqs. \([14]\) and \([15]\).

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