CAUCHY’S RESIDUE THEOREM FOR A CLASS OF REAL VALUED FUNCTIONS

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Abstract. Let \([a, b]\) be an interval in \(\mathbb{R}\) and let \(F\) be a real valued function defined at the endpoints of \([a, b]\) and with a certain number of discontinuities within \([a, b]\). Having assumed \(F\) to be differentiable on a set \([a, b] \setminus E\) to the derivative \(f\), where \(E\) is a subset of \([a, b]\) at whose points \(F\) can take values \(\pm \infty\) or not be defined at all, we adopt the convention that \(F\) and \(f\) are equal to 0 at all points of \(E\) and show that

\[
\mathcal{K}H - \text{vt } \int_a^b f = F(b) - F(a),
\]

where \(\mathcal{K}H - \text{vt}\) denotes the total value of the Kurzweil-Henstock integral. The paper ends with a few examples that illustrate the theory.

1. Introduction

Let \([a, b]\) be some compact interval in \(\mathbb{R}\). It is an old result that for an ACGδ function \(F : [a, b] \mapsto \mathbb{R}\) on \([a, b]\), which is differentiable almost everywhere on \([a, b]\), its derivative \(f\) is integrable (in the Kurzweil-Henstock sense) on \([a, b]\) and

\[
\mathcal{K}H - \text{vt } \int_a^b f = F(b) - F(a), \quad [3, \text{Theorem } 9.17].
\]

The aim of this note is to define a new definite integral named the total Kurzweil-Henstock integral that can be used to extend the above mentioned result to any real valued function \(F\) defined and differentiable on \([a, b] \setminus E\), where \(E\) is a certain subset of \([a, b]\) at whose points \(F\) can take values \(\pm \infty\) or not be defined at all. Unless otherwise stated in what follows, we assume that the endpoints of \([a, b]\) do not belong to \(E\). Now, define point functions \(F_{ex} : [a, b] \mapsto \mathbb{R}\) and \(D_{ex} F : [a, b] \mapsto \mathbb{R}\) by extending \(F\) and its derivative \(f\) from \([a, b] \setminus E\) to \(E\) by \(F_{ex}(x) = 0\) and \(D_{ex} F(x) = 0\) for \(x \in E\), so that

\[
F_{ex}(x) = \begin{cases} 
F(x), & \text{if } x \in [a, b] \setminus E \\
0, & \text{if } x \in E
\end{cases}
\]

and

\[
D_{ex} F(x) = \begin{cases} 
f(x), & \text{if } x \in [a, b] \setminus E \\
0, & \text{if } x \in E
\end{cases}.
\]

2. Preliminaries

A partition \(P[a, b]\) of \([a, b] \in \mathbb{R}\) is a finite set (collection) of interval-point pairs \(\{(a_i, b_i), x_i \mid i = 1, \ldots, \nu\}\), such that the subintervals \([a_i, b_i]\) are non-overlapping, \(\bigcup_{i \leq \nu} [a_i, b_i] = [a, b]\) and \(x_i \in [a_i, b_i]\). The points \(\{x_i\}_{i \leq \nu}\) are the tags of \(P[a, b]\), [2]. It is evident that a given partition of \([a, b]\) can be tagged in infinitely many ways by choosing different points as tags. If \(E\) is a subset of \([a, b]\), then the restriction of \(P[a, b]\) to \(E\) is a finite collection of \(\{(a_i, b_i), x_i \mid x_i \in P[a, b]\}\) such that each \(x_i \in E\).

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In symbols, \( P[a, b] |_E = \{(a_i, b_i), x_i \mid x_i \in E, i = 1, \ldots, \nu_E\} \). Let \( \mathcal{P}[a, b] \) be the family of all partitions \( P[a, b] \) of \([a, b]\). Given \( \delta : [a, b] \rightarrow \mathbb{R}_+ \), named a gauge, a partition \( P[a, b] \in \mathcal{P}[a, b] \) is called \( \delta \)-fine if \([a_i, b_i] \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))\). By Cousin’s lemma the set of \( \delta \)-fine partitions of \([a, b]\) is nonempty. [3]

The collection \( \mathcal{I}([a, b]) \) is the family of compact subintervals \( I \) of \([a, b]\). The Lebesgue measure of the interval \( I \) is denoted by \( |I|\). Any real valued function defined on \( \mathcal{I}([a, b]) \) is an interval function. For a function \( f : [a, b] \rightarrow \mathbb{R} \), the associated interval function of \( f \) is an interval function \( F : \mathcal{I}([a, b]) \rightarrow \mathbb{R} \), again denoted by \( f \). [3] If \( f \equiv 0 \) on \([a, b]\) then its associated interval function is trivial.

A function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be Kurzweil-Henstock integrable to a real number \( A \) on \([a, b]\) if for every \( \varepsilon > 0 \) there exists a gauge \( \delta_\varepsilon : [a, b] \rightarrow \mathbb{R}_+ \) such that \( \left| \sum_{i \leq \nu} |f(x_i)| |a_i, b_i| \right| - A < \varepsilon \), whenever \( P[a, b] \) is a \( \delta_\varepsilon \)-fine partition of \([a, b]\). In symbols, \( A = \mathcal{KH}\int_a^b f \).

### 3. Main Results

In what follows we will use the following notations.

\[
\Xi_f(P[a, b]) = \sum_{i \leq \nu}|f(x_i)| |a_i| \quad \text{and} \quad \Sigma_\Phi(P[a, b]) = \sum_{i \leq \nu}(\Phi(b_i) - \Phi(a_i)).
\]

Now, we are in a position to introduce the total Kurzweil-Henstock integral.

**Definition 1.** For any compact interval \([a, b]\) \( \in \mathbb{R} \) let \( E \) be a non-empty subset of \([a, b]\). A function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be totally Kurzweil-Henstock integrable to a real number \( \mathcal{H} \) on \([a, b]\) if there exists a nontrivial interval function \( \Phi : \mathcal{I}([a, b]) \rightarrow \mathbb{R} \) with the following property: for every \( \varepsilon > 0 \) there exists a gauge \( \delta_{\varepsilon} \) on \([a, b]\) such that \( |\Xi_f(P[a, b]) - \Sigma_\Phi(P[a, b] |_{[a, b]\setminus E})| < \varepsilon \) and \( \Sigma_\Phi(P[a, b]) = \mathcal{H} \), whenever \( P[a, b] \in \mathcal{P}[a, b] \) is a \( \delta_{\varepsilon} \)-fine partition and \( P[a, b] |_{[a, b]\setminus E} \) is its restriction to \([a, b]\) \( \setminus E \). In symbols, \( \mathcal{KH}\int_a^b f = \mathcal{H} \).

**Definition 2.** Let \( E \) be a non-empty subset of \([a, b]\). Then, an interval function \( \Phi : \mathcal{I}([a, b]) \rightarrow \mathbb{R} \) is said to be basically summable (BSI\(\delta_{\varepsilon}\)) to the sum \( \mathcal{R} \) on \( E \) if there exists a real number \( \mathcal{R} \) with the following property: given \( \varepsilon > 0 \) there exists a gauge \( \delta_{\varepsilon} \) on \([a, b]\) such that \( |\Sigma_\Phi(P[a, b] |_E) - \mathcal{R}| < \varepsilon \), whenever \( P[a, b] \in \mathcal{P}[a, b] \) is a \( \delta_{\varepsilon} \)-fine partition and \( P[a, b] |_E \) is its restriction to \( E \). If \( E \) can be written as a countable union of sets on each of which the interval function \( \Phi \) is BSI\(\delta_{\varepsilon}\), then \( \Phi \) is said to be BSG\(\delta_{\varepsilon}\) on \( E \).

Our main result reads as follows.

**Theorem 1.** For any compact interval \([a, b]\) \( \in \mathbb{R} \) let \( E \) be a non-empty subset of \([a, b]\) at whose points a real valued function \( F \) can take values \( \pm \infty \) or not be defined at all. If \( F \) is defined and differentiable on the set \([a, b]\) \( \setminus E \), then \( D_{ex} F \) is totally Kurzweil-Henstock integrable on \([a, b]\) and

\[
\mathcal{KH}\int_a^b D_{ex} F = F(b) - F(a).
\]

If the associated interval function of \( F_{ex} \) defined by \( (1.7) \) is in addition basically summable (BSI\(\delta_{\varepsilon}\)) to the sum \( \mathcal{R} \) on \( E \), then

\[
F(b) - F(a) = \mathcal{KH}\int_a^b D_{ex} F + \mathcal{R}.
\]
Lemma 1. Let $E$ be a non-empty subset of $[a,b]$. If a function $f : [a,b] \to \mathbb{R}$ is totally Kurzweil-Henstock integrable on $[a,b]$ and $\Phi$ is basically summable (BS$_{a}$) to the sum $\Re$ on $E$, then $f$ is Kurzweil-Henstock integrable on $[a,b]$ and

$$
\mathcal{KH} - \nu \int_{a}^{b} f = \mathcal{KH} - \int_{a}^{b} f + \Re.
$$

Proof. Given $\varepsilon > 0$ we will construct a gauge for $f$ as follows. Since $f$ is totally Kurzweil-Henstock integrable on $[a,b]$ it follows from Definition 1 that there exist a real number $\exists$ and an interval function $\Phi$ with the following property: for every $\varepsilon > 0$ there exists a gauge $\delta^*_\varepsilon$ on $[a,b]$ such that $|\Xi_f (P [a,b]) - \Sigma_f (P [a,b] \vert_{[a,b] \setminus E})| < \varepsilon$ and $\Sigma_f (P [a,b]) = \exists$, whenever $P [a,b] \in P [a,b]$ is a $\delta^*_\varepsilon$-fine partition and $P [a,b] \vert_{[a,b] \setminus E}$ is its restriction to $[a,b] \setminus E$. Choose a gauge $\delta^*_\varepsilon (x)$ as required in Definition 2 above. The function $\delta_x = \min (\delta^*_\varepsilon, \delta^*_\varepsilon)$ is a gauge on $[a,b]$. We now let $P [a,b] = \{([a_i, b_i], x_i) \mid i = 1, \ldots, \nu\}$ be a $\delta_x$-fine partition of $[a,b]$. It is readily seen that

$$
|\Xi_f (P [a,b]) - \exists + \Re| =
= |\Xi_f (P [a,b]) - \exists + \Sigma_f (P [a,b] \vert_{E}) - \Sigma_f (P [a,b] \vert_{[a,b] \setminus E}) - \Re| \leq
\leq |\Xi_f (P [a,b]) - \Sigma_f (P [a,b] \vert_{[a,b] \setminus E})| + |\Sigma_f (P [a,b] \vert_{E}) - \Re| < 2\varepsilon.
$$

Therefore, $f$ is Kurzweil-Henstock integrable on $[a,b]$ and $\mathcal{KH} - \int_{a}^{b} f = \exists - \Re$, that is

$$
\mathcal{KH} - \nu \int_{a}^{b} f = \mathcal{KH} - \int_{a}^{b} f + \Re.
$$

We now turn to the proof of Theorem 1.

Proof. Given $\varepsilon > 0$. By definition of $f$ at the point $x \in [a,b] \setminus E$, given $\varepsilon > 0$ there exists $\delta_x (x) > 0$ such that if $x \in [a, v] \subseteq [x - \delta_x (x), x + \delta_x (x)]$ and $x \in [a,b] \setminus E$, then

$$
|F (v) - F (u) - f (x) (v - u)| < \varepsilon (v - u).
$$

For $F_{ex}$ defined by \Box let $F_{ex} : [a,b] \to \mathbb{R}$ be its associated interval function. We now let $P [a,b] = \{([a_i, b_i], x_i) \mid i = 1, \ldots, \nu\}$ be a $\delta_x$-fine partition of $[a,b]$. Since $F (b) - F (a) = \sum_{i=1}^{\nu} [F_{ex} (b_i) - F_{ex} (a_i)]$ and (remember if $x \in E$, then $D_{ex} F = 0$)

$$
|\Xi_f (P [a,b]) - \Sigma_f (P [a,b] \vert_{[a,b] \setminus E})| =
= |\Xi_f (P [a,b] \vert_{[a,b] \setminus E}) - \Sigma_f (P [a,b] \vert_{[a,b] \setminus E})| < \varepsilon (b - a),
$$

it follows from Definition 1 that $D_{ex} F$ is totally Kurzweil-Henstock integrable on $[a,b]$ and

$$
\mathcal{KH} - \nu \int_{a}^{b} D_{ex} F = F (b) - F (a).
$$

Finally, based on the result of Lemma 1

$$
F (b) - F (a) = \mathcal{KH} - \int_{a}^{b} D_{ex} F + \Re.
$$

Finally, based on the result of Lemma 1

By Definition 2 one can easily see that if $\Re = 0$ then $F$ has negligible variation on $E$, [11, Definition 5.11]. So, we now in position to define a residual function of $F$.  

\[ \Box \]
Definition 3. Let \( F : [a, b] \rightarrow \mathbb{R} \). A function \( R : [a, b] \rightarrow \mathbb{R} \) is said to be a residual function of \( F \) on \([a, b]\) if given \( \varepsilon > 0 \) there exists a gauge \( \delta_\varepsilon \) on \([a, b]\) such that \( |F(a) - F(a_i) - R(x_i)| < \varepsilon \), whenever \( P[a, b] \in \mathcal{P}[a, b] \) is a \( \delta_\varepsilon \)-fine partition.

Definition 4. Let \( E \) be a non-empty subset of \([a, b]\) and let \( F : [a, b] \rightarrow \mathbb{R} \) be a function whose associated interval function \( F : I([a, b]) \rightarrow \mathbb{R} \) is \( BS\delta_\varepsilon \) \( (BSG_\delta) \) to the sum \( R \) on \( E \). Then, a residual function \( R : [a, b] \rightarrow \mathbb{R} \) of \( F \) is said to be also \( BS\delta_\varepsilon \) \( (BSG_\delta) \) to the same sum \( R \) on \( E \). In symbols, \( \sum_{x \in E} R(x) = R \).

Clearly, Definition 4 establishes a causal connection between Definitions 2 and 3. If \( E \) is a countable set, the causality is so obvious. However, if \( E \) is an infinite set, then this connection is not necessarily a causal connection. Namely, if \( F : [a, b] \rightarrow \mathbb{R} \) has negligible variation on some subset \( E \) of \([a, b]\), which is a countably infinite set, then its residual function \( R \) vanishes identically on \( E \), so that the sum \( \sum_{x \in E} R(x) \) is reduced to the so-called indeterminate expression \( \infty \cdot 0 \) that have, in this case, the null value. On the contrary, if \( F \) has no negligible variation on \( E \), and its residual function \( R \) also vanishes identically on \( E \), as in the case of the Cantor function, then the sum \( \sum_{x \in E} R(x) \) is reduced to the indeterminate expression \( \infty \cdot 0 \) that actually have, in Cantor’s case, the numerical value of 1. By Definition 4, we may rewrite (3.3) as follows,

\[ (3.5) \quad F(b) - F(a) = \mathcal{K}H - \int_a^b D_\varepsilon x F + \sum_{x \in E} R(x). \]

If \( f \) in addition vanishes identically on \([a, b] \setminus E\), then

\[ (3.6) \quad F(b) - F(a) = \sum_{x \in E} R(x). \]

The previous result is an extension of Cauchy’s residue theorem result in \( \mathbb{R} \).

4. Examples

For an illustration of (3.5) and (3.6) we consider the Heaviside unit function defined by

\[ (4.1) \quad F(x) = \begin{cases} 0, & \text{if } a \leq x \leq 0 \\ 1, & \text{if } 0 < x \leq b \end{cases}. \]

In this case, if \( a < 0 \), then \( \mathcal{K}H - vt \int_a^b D_\varepsilon x F = 1 \), in spite of the fact that \( D_\varepsilon x F \equiv 0 \) on \([a, b]\). Accordingly, it follows from (3.5) and (3.6) that \( R(0) = 1 \), since

\[ (4.2) \quad f(x) = \begin{cases} +\infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}, \]

where \( f \) is the derivative of \( F \), and \( \mathcal{K}H - \int_a^b D_\varepsilon x F = 0 \).

Let \([a, b] \subset \mathbb{R}\) be an arbitrary compact interval within which is the point \( x = 0 \). For an illustration of the result (3.2) of Theorem 1 we consider the real valued function \( F(x) = 1/x \) that is differentiable to \( f(x) = -1/x^2 \) at all but the exceptional set \( \{0\} \) of \([a, b]\). In spite of the fact that \( f \) is not Kurzweil-Henstock integrable on \([a, b]\) it follows from (3.2) that \( \mathcal{K}H-vt \int_a^b D_\varepsilon x F = (a - b) / (ab) \). In this case, \( R(x) \) is not defined at the point \( x = 0 \), that is

\[ (4.3) \quad R(x) = \begin{cases} +\infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}. \]
and $\mathcal{KH}-vt\int_a^b D_{ex}F$ is reduced to the so-called indeterminate expression $\infty - \infty$ (in the sense of the difference of limits) that actually have, in this situation, the real numerical value of $(a - b) / (ab)$.

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