WHEN SIZE MATTERS: SUBSHIFTS AND THEIR RELATED TILING SPACES

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ABSTRACT. We investigate the dynamics of substitution subshifts and their associated tiling spaces. For a given subshift, the associated tiling spaces are all homeomorphic, but their dynamical properties may differ. We give criteria for such a tiling space to be weakly mixing, and for the dynamics of two such spaces to be topologically conjugate.

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1. Introduction

We consider the dynamics of 1-dimensional minimal substitutions subshifts (with a natural \( \mathbb{Z} \) action) and the associated 1-dimensional tiling spaces (with natural \( \mathbb{R} \) actions). Given an alphabet \( \mathcal{A} \) of \( n \) symbols \( \{a_1, \ldots, a_n\} \), a substitution \( \sigma \) on \( \mathcal{A} \) is a function \( \sigma \) from \( \mathcal{A} \) into the non-empty, finite words of \( \mathcal{A} \). Associated with such a substitution is the \( n \times n \) matrix \( M \) which has as its \((i,j)\) entry the number of occurrences of \( a_j \) in \( \sigma(a_i) \). A substitution is primitive if some positive power of \( M \) has strictly positive entries. The substitution \( \sigma \) induces the map \( \sigma : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \) given by \( \langle \ldots u_{-1}, u_0, u_1, \ldots \rangle \xrightarrow{\sigma} \langle \ldots \sigma(u_{-1}), \sigma(u_0), \sigma(u_1), \ldots \rangle \). For any primitive substitution \( \sigma \) there is at least one point \( u \in \mathcal{A}^\mathbb{Z} \) which is periodic under \( \sigma \), and the closure of the orbit of any such \( u \) under the left shift map \( s \) of \( \mathcal{A}^\mathbb{Z} \) forms a minimal subshift \( S \). This subshift \( S \) is uniquely determined by \( \sigma \). To avoid trivialities, we shall only consider primitive, aperiodic substitutions \( \sigma \), i.e., those for which the subshift \( S \) is not periodic.

Given a collection of intervals \( \mathcal{I} = \{I_1, \ldots, I_n\} \), a tiling \( T \) of \( \mathbb{R} \) by \( \mathcal{I} \) is a collection of closed intervals \( \{T_i\}_{i \in \mathbb{Z}} \) satisfying

1. \( \bigcup_{i \in \mathbb{Z}} T_i = \mathbb{R} \),
2. For each \( i \in \mathbb{Z} \), \( T_i \) is the translate of some \( I_{\tau(i)} \in \mathcal{I} \), and
3. \( T_i \cap T_{i+1} \) is a singleton for each \( i \in \mathbb{Z} \).

If the function \( \tau : \mathbb{Z} \to \{1, \ldots, n\} \) is an element of a minimal substitution subshift \( S \) of \( \{1, \ldots, n\}^\mathbb{Z} \), then \( T \) is called a substitution tiling. There is a natural topology on the space \( \mathcal{S} \) of tilings of \( \mathbb{R} \) by \( \mathcal{I} \) that is induced by a metric which measures as close any two tilings \( T \) and \( T' \) that agree on a large neighborhood of 0 up to an \( \varepsilon \) translation. There is then the continuous translation action \( T \) of \( \mathbb{R} \) on \( \mathcal{S} \) : for \( t \in \mathbb{R} \) and the tiling \( T = \{T_i\}_{i \in \mathbb{Z}}, T : (T, t) \mapsto T_t T = \{T_i - t\}_{i \in \mathbb{Z}} \). (A positive element of \( \mathbb{R} \) moves the origin to the right, or equivalently moves tiles to the left). The closure of the translation orbit of any substitution tiling \( T \) in \( \mathcal{S} \) is then a minimal set of the action, the (substitution) tiling space \( \mathcal{T} \) of the tiling \( T \). As the tiling space for any iterate of a substitution \( \sigma \) is the same as the tiling space of \( \sigma \) and since any substitution has a point \( u \) whose right half \( \langle u_0, u_1, \ldots \rangle \) is periodic under substitution, for ease of discussion we shall only consider substitutions with a point \( u \) whose right half \( \langle u_0, u_1, \ldots \rangle \) is fixed under substitution.

Flows under a function provide an alternative description of tiling spaces convenient for our purposes. Given a minimal subshift \( (\mathcal{S}, s) \) of the shift on \( \mathcal{A}^\mathbb{Z} \) and \( f : \mathcal{A} \to (0, \infty) \), there is the flow under \( f \) given by the natural \( \mathbb{R} \) action on \( \mathcal{T}_f = \mathcal{S} \times \mathbb{R} / \sim \), where \( (u, f(u_0)) \sim (s(u), 0) \). If we then associate with each \( a_i \in \mathcal{A} \) a closed interval \( I_i \) of length \( f(a_i) \) and form the tiling on \( T \) of \( \mathbb{R} \) by \( \{I_1, \ldots, I_n\} \) with associated function \( \tau = u \in \mathcal{A}^\mathbb{Z} \), which has the left endpoint of the interval corresponding to \( u_0 \) at \( 0 \in \mathbb{R} \), then the function sending \( T \) to the class of \( (u, 0) \) in \( \mathcal{T}_f \) extends uniquely to a homeomorphism \( \mathcal{T} \to \mathcal{T}_f \) which conjugates the respective \( \mathbb{R} \) actions.

The primary focus of this paper is the extent to which the dynamical systems \( \mathcal{T}_f \) depend on the function \( f \). If all we care about is the topological space, they don’t:

**Theorem 1.1.** Let \( \mathcal{S} \) be a subshift, and let \( f, g \) be two positive functions on the alphabet of \( \mathcal{S} \). Then \( \mathcal{T}_f \) and \( \mathcal{T}_g \) are homeomorphic.
Proof. Every class \( x \in T_f \) has a (unique) representative of the form \((u,t)\), with \(0 \leq t < f(u_0)\). Let \(h(x)\) be the class of \((u,g(u_0)t/f(u_0))\) in \(T_g\). As the function on quotient spaces induced by a continuous function \(S \times \mathbb{R} \to S \times \mathbb{R}\), \(h\) is continuous, and it has a well-defined continuous inverse. \( \Box \)

But what about dynamics? Dekking and Keane [DK] showed that, for the substitution \(a \to abab, \ b \to bbba\), the associated tiling space with \(f(a) = 2, f(b) = 1\) has pointspectrum only at multiples of \(2\pi\). (That is, the associated discrete dynamical system is weakly mixing.) Further, Berend and Radin [BR] showed that a tiling space based on this substitution has trivial point spectrum if \(f(a)/f(b)\) is irrational. Radin and Sadun [RS] obtained very different results for the Fibonacci substitution \(a \to b, \ b \to ab\). For the Fibonacci substitution, \(T_f\) is topologically conjugate to \(T_g\) whenever \(f(a) + \tau f(b) = g(a) + \tau g(b)\), where \(\tau = (1 + \sqrt{5})/2\) is the golden mean. (The conjugacy is not the map \(h\), described above, but is homotopic to \(h\). A generalization of this construction appears below in Section 3.) In particular, all Fibonacci tiling spaces have pure point spectrum, regardless of the sizes of the tiles.

In Section 2 we examine the general ergodic properties of the flows \(T_f\), showing that these flows are never mixing. We also provide general criteria for the existence of point spectrum and constraints on the form of that spectrum, in terms of the eigenvalues of the substitution matrix and the ratios of lengths of the tiles. The Dekking-Keane and Berend-Radin results are then easily understood.

In Section 3 we provide criteria for \(T_f\) and \(T_g\) to be topologically conjugate. In particular, if only one eigenvalue has magnitude 1 or greater, we show that there exists a constant \(c\) such that \(T_f\) and \(T_cg\) are topologically conjugate. In addition to Pisot substitutions, this result applies to substitutions such as Thue-Morse, whose matrices have zero as an eigenvalue.

Given a substitution, there are two natural choices of tile length. One choice is the standard suspension, where all the tiles have length one. The other choice is to pick lengths according to the left Perron-Frobenius eigenvector of the substitution matrix. This yields tilings that are self-similar in the sense of Solomyak [S].

For Pisot substitutions, there are long-standing conjectures that both the self-similar tiling space and the suspended subshift have pure point spectrum. Partial results have been obtained for both cases, using the “balanced pair” algorithm [BD2; HS] for the suspended subshift and the “overlap algorithm” for the self-similar tiling space [S]. Various relations between the two problems have been derived [SV]. Our results imply that the two problems are in fact equivalent.

The techniques of Section 3 draw heavily on the work of Mossé [M1; M2]. We generalize her computation of the number of times a word can be repeated to allow for fractional repetitions, and show that the set of period lengths yields a conjugacy invariant. The proof of Theorem 3.3 in particular is an extension of ideas in [M1]. Moreover, we are implicitly using Mossé’s bilateral recognizability results whenever we refer to the location of supertiles within a tiling.

Finally, Sadun and Williams [SW] have shown that every finite dimensional tiling space meeting mild conditions (consisting of a finite number of polygonal tiles, appearing in a finite number of orientations, and meeting edge-to-edge) can be deformed into a tiling.
space that is topologically conjugate to the \( d \)-fold suspension of a \( \mathbb{Z}^d \) subshift. This deformation changes the shapes and sizes of the tiles, but not the combinatorics of which tiles touch which others. The difference between the dynamics of general tiling spaces and of subshifts is therefore not combinatorial in nature, but depends rather on the geometry of the individual tiles. This paper is an exploration of this phenomenon in one dimension, where the geometry of a tile is simply its length. We believe that the results of this paper will provide essential tools for exploring the phenomenon in higher dimensions as well.

2. Ergodic Properties

For a substitution tiling space \( \mathcal{T}_f \), let \( L = (f(a_1), \ldots, f(a_n)) \) be the (row) vector that gives the lengths of the various tiles. Let \( L_0 \) be the normalized positive left Perron-Frobenius eigenvector with eigenvalue \( \lambda_{PF} \) of the substitution matrix \( M \), and let \( \mathcal{T}_0 \) be the tiling space associated with the length vector \( L_0 \). Every such tiling space \( \mathcal{T}_0 \) is known to admit a self-homeomorphism derived from \( \sigma \) (also denoted \( \sigma \)) which inflates all the tiles by a factor of \( \lambda_{PF} \) and which maps the class of a point \((x, 0)\) to the class of \((\sigma(x), 0)\), see, e.g., \([BD1]\). The more general substitution tiling space \( \mathcal{T}_f \) then admits a substitution homeomorphism, also denoted \( \sigma \), conjugate to the substitution homeomorphism of \( \mathcal{T}_0 \) via the homeomorphism \( h \) of Theorem 1.1, but this substitution homeomorphism is not generally affine. Solomyak \([S]\) has investigated the ergodic properties of the self-affine \( \mathcal{T}_0 \) in the case that the matrix \( M \) is diagonalizable. Now we proceed to investigate the ergodic properties of the more general tiling spaces \( \mathcal{T}_f \), requiring only that the substitution be primitive and aperiodic. As most proofs in this section are similar to those for analogous known results for substitutions, the proofs shall be abbreviated.

It should be noted that most of the literature is devoted to the ergodic properties of one-sided substitutions, while we are dealing with two-sided substitutions. However, as there are finitely many \( s \)-orbits of words in a one-sided substitution that admit more than one left extension \([Q]\), (see \([BDH]\) for precise bounds), measure theoretic considerations are not altered. For example, by placing all two-sided words with the same right halves into the same partition element, partitions of one-sided substitutions lead to partitions of two-sided substitutions with corresponding partition elements having the same measure.

**Definitions.** For a word \( w = w_0 \cdots w_k \) from the alphabet \( \mathcal{A} \) of the substitution subshift \( \mathcal{S} \) derived from the substitution \( \sigma \),

\[
[w] \overset{\text{def}}{=} \{ u \in \mathcal{S} \mid u_0 \cdots u_k = w \}.
\]

A **cylinder set** \([w] \times I \) in \( \mathcal{T}_f \) for a word \( w = w_0 \cdots w_k \) and interval \( I \subseteq [0, f(w_0)) \) is the set of all \( x \sim (u, t) \in \mathcal{T}_f \) with \( u \in [w] \) and \( t \in I \).

The **population vector** \( v = (v_1, \ldots, v_n)^T \) of the finite word \( w \) gives the number of occurrences \( v_i \) of the letter \( a_i \) in \( w \). Note that \( v \) is a column vector, not a row, and that the substitution matrix acts from the left.

For \( u \in \mathcal{A}^\mathbb{Z} \), a **recurrence word** is a finite word \( w \) in \( u \), \( w = u_r u_{r+1} \cdots u_s \) satisfying the condition that \( u_{s+1} = u_r \), and a **recurrence vector** is the population vector for a recurrence word. For example, for \( \ldots ababcab \ldots \) in \( \{a, b, c\}^\mathbb{Z} \), \( ababc \) is a recurrence word.
for which \((1,3,1)^T\) is the recurrence vector. The vectors \((0,1,0)^T\), \((0,1,1)^T\), and \((0,2,1)^T\) are recurrence vectors for the successive \(b_i\)'s.

A recurrence vector \(v\) is called full if the vectors \(\{M^kv\}\), with \(k\) ranging from 0 to \(n-1\), are linearly independent. If \(M\) is diagonalizable, this is equivalent to \(v\) pairing nontrivially with every left-eigenvector of \(M\). Note that every primitive substitution on two letters admits a full recurrence vector, namely \((0,1)^T\) or \((1,0)^T\).

Primitive aperiodic substitution tilings are recurrent, meaning that any finite patch of one tiling appears somewhere in every other tiling. As a result, the set of recurrence vectors is the same for every tiling in the space, and so we can speak of the recurrence vectors of a tiling space.

The following lemma and its implication for mixing are proved in much the same way as for substitutions [DK] Theorem 2).

**Lemma 2.1.** Let \(v\) be the recurrence vector of a recurrence word \(r\) and let \(t_m \overset{\text{def}}{=} LM^m v\). For any cylinder set \([w] \times I\) in \(T_f\) and for any \(m\), let \(S_m \overset{\text{def}}{=} ([w] \times I) \cap (T_{-t_m} ([w] \times I))\). Then

\[
\liminf_{m \to \infty} \mu(S_m) \geq \gamma \cdot \mu([w] \times I),
\]

where \(\gamma\) is a positive constant independent of \(w\) and \(I\), depending only on the recurrence word \(r\) of \(v\).

**Proof.** With \(t'_m \overset{\text{def}}{=} (1, \ldots, 1) M^m v\), it follows from [DK] that the measures \(\mu'\) in the substitution subshift of the sets \(S'_m \overset{\text{def}}{=} [w] \cap s^{-t'_m} [w]\) satisfy

\[
\liminf_{m \to \infty} \mu'(S'_m) \geq \gamma' \cdot \mu'([w]),
\]

for some positive constant \(\gamma'\) independent of \(w\). Observe that \(\mu'([w])\) is the limit as \(k \to \infty\) of the average number of iterates \(\{s^i(u) | i = 0, \ldots, k-1\}\) in \([w]\), for any given \(u \in A^2\). And the measure \(\mu([w] \times I)\) in \(T_f\) is the limit as \(T \to \infty\) of the average amount of time that \(\{T^i(x) | i \in [0,T]\}\) spends in \(\mu([w] \times I)\) for any given \(x \in T_f\).

Hence, \(\mu([w] \times I) = \mu'( [w] ) \times \frac{\text{length}(I)}{\mu'( [a_1] ) f(a_1) + \cdots + \mu'( [a_n] ) f(a_n)}\). Now, if \(u \in [w]\) and \(t \in I\), then \(s^{t_m}(u) \in [w]\) if and only if \(T_{t_m}((u,t)) \in [w] \times \{t\}\), and the result follows.

By choosing the cylinder set smaller in measure than a \(\gamma\) for a fixed recurrence word, this leads to the following conclusion.

**Theorem 2.2.** None of the substitution tiling space flows on \(T_f\) are (strongly) mixing.

**Theorem 2.3.** The number \(k\) is in the point spectrum of \(T_f\) if and only if, for every recurrence vector \(v\),

\[
\frac{k}{2\pi} LM^m v \to 0 (\text{mod } 1) \text{ as } m \to \infty.
\]
Moreover, when this condition is met there is a constant $C$ and a $\rho < 1$ satisfying
\begin{equation}
|\exp(ikLM^m v) - 1| < C\rho^m
\end{equation}
for any given $v$.

**Proof.**
Essentially the same arguments as found in [FMN] using Rokhlin stacks and columns apply to show that the condition is both necessary and sufficient for return words, finite words $w$ in $u$, $w = u_ru_{r+1}\cdots u_s$ satisfying for sufficiently large $m$ the conditions $\sigma^m u_r = \sigma^m u_{s+1}$ and $\sigma^m u_r \neq \sigma^m u_j$ for $j = r + 1, \ldots, s$. Since the eigenfunction constructed as in [FMN] using the geometric convergence and the condition for return words is continuous, one need only consider continuous eigenfunctions.

We now show that for continuous eigenfunctions the condition for the general recurrence word is necessary, which is all that need be shown since the condition with return words is already sufficient. Consider then a continuous eigenfunction $f$ with eigenvalue $k$. Let $x \sim (u,0) \in T_f$, where $u \in \mathcal{A}^\mathbb{Z}$ is a point of the substitution $\sigma$ with fixed right half $\langle u_0u_1\ldots \rangle$, and let $v$ be a recurrence vector for a recurrence word between $u_0$ and some $u_s$, and let $t_m \overset{\text{def}}{=} LM^m v$. Note that $T_{t_0}x$ agrees with $x$ on the interval covered by the single tile of $x$ represented by $u_0$. Then $T_{t_m}x = \sigma^m(T_{t_0}x)$ agrees with $x$ on forward time intervals of increasing unbounded length, and so $\exp(ikt_m)$ must converge to 1, and $kt_m/(2\pi)$ must converge to 0 (mod 1). Now let $v_1$ and $v_2$ be two such recurrence vectors. Since $kLM^m v_1/(2\pi)$ and $kLM^m v_2/(2\pi)$ both converge to zero (mod 1), so does $kLM^m(v_1 - v_2)/(2\pi)$. In other words, we can use any recurrence vector for the type of letter (say, $a$) that occurs at $u_0$. Finally, suppose that $v_3$ is a recurrence vector for letter $b$. Since $\sigma$ is primitive, suppose $a$ appears somewhere in the $\ell$-th substitution of $b$, for some integer $\ell$. Then $v_4 = M^\ell v_3$ is a recurrence vector for $a$, so $kLM^m v_3/(2\pi) = kLM^{m-\ell} v_4/(2\pi)$ converges to 0(mod 1). To obtain the geometric convergence as in the statement of the theorem, one applies the argument of [H, Lemme 1] with the lattice $f(a_1)\mathbb{Z} \oplus \cdots \oplus f(a_n)\mathbb{Z}$ in place of the lattice $\mathbb{Z}^n$.

The application of this criterion depends on the eigenvalues of $M$ and on the possible forms of the recurrence vectors.

**Theorem 2.4.** Suppose that all the eigenvalues of $M$ are of magnitude 1 or greater, and that there exists a full recurrence vector. If the ratio of any two tile lengths is irrational, then there is trivial point spectrum. If the ratios of tile lengths are all rational, then the point spectrum is contained in $2\pi \mathbb{Q}/L_1$.

**Proof.** Let $k$ be in the point spectrum, and consider the sequence of real numbers $t_m = kLM^m v/(2\pi)$, where $v$ is a fixed full recurrence vector. Let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ be the characteristic polynomial of $M$. Note that the $a_i$’s are all integers, since $M$ is an
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integer matrix. Since \( p(M) = 0 \), the \( t_m \)'s satisfy a recurrence relation:

\[
t_{m+n} = -\sum_{k=0}^{n-1} a_k t_{m+k}.
\]

By Theorem 2.3, the \( t_m \)'s converge to zero \( \pmod{1} \). That is, we can write

\[
t_m = i_m + \epsilon_m
\]

where the \( i_m \)'s are integers, and the \( \epsilon_m \)'s converge to zero as real numbers. By substituting the division (7) into the recursion (6), we see that both the \( i \)'s and the \( \epsilon \)'s must separately satisfy the recursion (6), once \( m \) is sufficiently large (e.g., large enough that the \( \epsilon \)'s are bounded by \( 1/\sum |a_i| \)). However, any solution to this recursion relation is a linear combination of powers of the eigenvalues of \( M \) (or polynomials in \( m \) times eigenvalues to the \( m \)-th power, if \( M \) is not diagonalizable). Since the eigenvalues are all of magnitude one or greater, such a linear combination converges to zero only if it is identically zero. Therefore \( \epsilon_m \) must be identically zero for all sufficiently large values of \( m \).

Each \( t_m \) is an integer linear combination of the elements of the vector \( kL/(2\pi) \). From a sequence of \( n \) consecutive \( t \)'s, one can recover all the elements of \( kL/(2\pi) \) by inverting an integer matrix whose columns are powers of \( M \) times \( v \). (The invertibility of this matrix depends on the fact that \( v \) is full and that zero is not an eigenvalue.) By Cramer’s rule, this inversion involves integer multiplication and addition, followed by division by the determinant of this matrix. Since the \( t_m \)'s are integers (for \( m \) large enough), the components of \( kL/(2\pi) \) must then all be rational. If \( L_1/L_2 \) is irrational, this implies that \( k \) lies in \( 2\pi\mathbb{Q}/L_1 \).

\[ \square \]

Theorem 2.4 gives partial spectral information. When the alphabet has only two letters and the substitution has constant length, we can say considerably more:

\textbf{Theorem 2.5.} Suppose that we have a primitive substitution on two letters of constant length \( n \), where \( \sigma(a) \) contains \( n_a \) a’s and \( n-n_a \) b’s, while \( \sigma(b) \) contains \( n_b \) a’s and \( n-n_b \) b’s. Suppose further that \( 1 \leq n_a, n_b \leq n-1 \) and \( n_a \neq n_b \). Let \( z \) be the greatest common factor of \( n \) and \( n_a - n_b \). Then the point spectrum \( \sigma_{pp} \) depends as follows on the ratio \( L_1/L_2 \):

1. If \( L_1 = L_2 \), then there is a positive integer \( N \) such that

\[
N\mathbb{Z}[1/n] \subset NL_1\sigma_{pp}/2\pi \subset \mathbb{Z}[1/n].
\]

2. If \( L_1/L_2 \in \mathbb{Q} - \{1\} \), then there exist positive integers \( N_1 \) and \( N_2 \) such that

\[
N_1\mathbb{Z}[1/z] \subset N_2L_1\sigma_{pp}/2\pi \subset \mathbb{Z}[1/z].
\]

3. If \( L_1/L_2 \notin \mathbb{Q} \), then \( \sigma_{pp} = \{0\} \).

\textbf{Proof.} The \( L_1 = L_2 \) case is not new – Coven and Keane [CK] proved an even stronger result over 30 years ago. We prove all three cases together, since the proof neatly illustrates the significance of the ratio \( L_1/L_2 \).
The substitution matrix

$$M = \begin{pmatrix} n_a & n_b \\ n - n_a & n - n_b \end{pmatrix}$$

has eigenvalues \(n\) and \(n_a - n_b\), with left-eigenvectors \((1, 1)\) and \((n - n_a, -n_b)\) and right-eigenvectors \((n_b, n - n_a)^T\) and \((1, -1)^T\). Since \(n_a \neq n_b\), both eigenvalues have magnitude one or greater.

Since the system is aperiodic and there are only two letters, either \((1, 0)^T\) or \((0, 1)^T\) is a (full) recurrence vector. Suppose that \((1, 0)^T\) is (the other case is similar). Then we take

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{n + n_b - n_a} \begin{pmatrix} n_b \\ n - n_a \end{pmatrix} + \frac{n - n_a}{n + n_b - n_a} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$  

As in the proof of Theorem 2.4, if \(k \in \sigma_{pp}\) then the numbers \(t_m = kLM^m v/2\pi\) must eventually be integers. Note that

$$2\pi t_m = kLM^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$2\pi t_{m+1} = kLM^{m+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = kLM^m \begin{pmatrix} n_a \\ n - n_a \end{pmatrix},$$

so

$$2\pi(t_m, t_{m+1}) = kLM^m \begin{pmatrix} 1 \\ 0 \\ n - n_a \end{pmatrix},$$

and

$$kLM^m = 2\pi(t_m, t_{m+1}) \begin{pmatrix} 1 \\ 0 \\ n - n_a \end{pmatrix}^{-1} = \frac{2\pi}{n - n_a} ((n - n_a)t_m, t_{m+1} - n_a t_m).$$

Multiplying on the right by the right-eigenvectors of \(M\) gives two scalar equations for \(k\):

$$k(L_1 - L_2)(n_a - n_b)^m = \frac{2\pi}{n - n_a} (nt_m - t_{m+1})$$

$$k(n_b L_1 + (n - n_a)L_2)n^m = 2\pi((n_b - n_a)t_m + t_{m+1}).$$

If \(L_1 = L_2\), the first equation reads \(0 = 0\), but the second equation says that \(NkL_1 n^m/2\pi\) is an integer for \(m\) sufficiently large, where \(N = n + n_b - n_a\). Thus \(NL_1\sigma_{pp}/2\pi \subset \mathbb{Z}[1/n]\).

If \(L_1/L_2\) is rational but not one, then the first equation says that \(\sigma_{pp}L_1/2\pi\) is contained in a fixed rational multiple of \(\mathbb{Z}[1/(n_a - n_b)]\), while the second says that \(\sigma_{pp}L_2/2\pi\) is contained in a fixed rational multiple of \(\mathbb{Z}[1/n]\). Thus an integer multiple of \(\sigma_{pp}L_1/2\pi\) lies in \(\mathbb{Z}[1/n] \cap \mathbb{Z}[1/(n_a - n_b)] = \mathbb{Z}[1/z]\).

If \(L_1/L_2\) is irrational, then the only simultaneous solution to both equations is \(k = 0\). For if \(k \neq 0\) is a solution, then the ratio of the two equations would imply that \((L_1 - L_2)/(n_b L_1 + (n - n_a)L_2)\) would be rational, which is inconsistent with the irrationality of \(L_1/L_2\). Thus \(\sigma_{pp} \subset \{0\}\) and so \(\sigma_{pp} = \{0\}\). All that remains is to show that \(L_1\sigma_{pp}/2\pi\)
contains $\mathbb{Z}[1/n]$ if $L_1 = L_2$, and that $L_1\sigma_{pp}/2\pi$ contains a rational multiple of $\mathbb{Z}[1/z]$ if $L_1/L_2$ is rational but not one.

We call the result of applying the substitution $\ell$ times to a single tile (or a single letter) a supertile of order $\ell$. If $L_1 = L_2$, then all supertiles of order $m$ have length $n^mL_1$. Thus the coordinates of the endpoints of such supertiles agree modulo $n^mL_1$. For each pair of integers $(j, m)$ we define a function on our tiling space. If $x$ is a tiling, let

$$
\psi_{j, m}(x) = \exp(2\pi ijp/n^mL_1),
$$

where $p$ is the endpoint of any supertile of order $m$ in $x$. This is manifestly an eigenfunction of translation with eigenvalue $2\pi j/n^mL_1$.

If $L_1/L_2$ is rational but not integral, then the length of each supertile of order $m$ is a fixed rational linear combination of $L_1n^m$ and $L_1(n_a - n_b)^m$. Thus there is a constant $c$, a rational multiple of $L_1$, such that every supertile of order $m$ has length divisible by $cz^m$. For each pair $(j, m)$ we then define the eigenfunction

$$
\phi_{j, m}(x) = \exp(2\pi ijp/cz^m),
$$

which shows that some rational multiple of $\mathbb{Z}[1/z]$ is contained in $L_1\sigma_{pp}/2\pi$. □

As a special case, we recover the results of Dekking and Keane, and then of Berend and Radin, since their substitution $\sigma$ has $n = 4, n_a = 2, n_b = 1$, so $z = 1$. When $L_1 = 2$ and $L_2 = 1$, the point spectrum is $2\pi\mathbb{Z}[1/z] = 2\pi\mathbb{Z}$. This also illustrates the sharp contrast between substitution tiling spaces admitting a flow with point spectrum and spaces admitting equicontinuous flows. If $X$ admits an equicontinuous $\mathbb{R}$ action $\phi$ with point spectrum $\sigma_{pp}$ and if $\phi'$ is any $\mathbb{R}$ action on $X$ obtained by a time change of $\phi$, then the point spectrum of $\phi'$ is $c\cdot\sigma_{pp}$ for some possibly 0 constant $c$, see Egawa [E]. In the substitution $\sigma$ currently under consideration, the $(1, 1)$ flow has purely point spectrum $\sigma_{pp}$ but admits a time change represented by $(2, 1)$ which has discrete spectrum which can only be represented as a proper subset of a multiple of $\sigma_{pp}$.

What about the Fibonacci tiling? In that case the smaller eigenvalue is $1 - \tau$, whose magnitude is less than 1, so Theorem 2.4 does not apply directly. However, there is still something that can be said about such cases:

**Theorem 2.6.** Suppose that there exists a full recurrence vector. Let $S$ be the span of the (generalized left-) eigenspaces of $M$ with eigenvalue of magnitude strictly less than 1. If $k$ is in the point spectrum, then $kL/2\pi$ is the sum of a rational vector and an element of $S$.

**Proof.** Let $v$ be a full recurrence vector, and let $t_m = kLM^mv/2\pi$, as in the proof of Theorem 2.4. As before, $t_m$ converges to zero (mod 1), so we can write $t_m = i_m + \epsilon_m$ with $i_m$ integral and $\epsilon_m$ converging to zero, and with both the $i_m$’s and the $\epsilon_m$’s eventually satisfying the recursion relation (9). Since the $\epsilon_m$’s converge to zero, they must eventually be linear combinations of $m$-th powers of the small eigenvalues of $M$ (possibly times polynomials in $m$, if $M$ is not diagonalizable). By adding an element of $S$ to $L$ (and hence to $kL/2\pi$), we can then get all the $\epsilon_m$’s to be identically zero beyond a certain point.
If $M$ is invertible, the resulting value of $kL/2\pi$ must then be rational by the same argument as in the proof of Theorem 2.4. If $M$ is not invertible, then we need an extra step, since the vectors $M^{m}v, M^{m+1}v, \ldots, M^{m+n-\ell-1}v$ are no longer linearly independent. Let $S_S \subset S$ be the generalized left-eigenspace with eigenvalue zero, and suppose that this space has dimension $\ell$. The vectors $M^{m}v, \ldots, M^{m+n-\ell-1}v$ are linearly independent and span the annihilator of $S_S$, which can be identified with the dual space of $\mathbb{R}^{n*}/S_0$. From the integers $t_m, \ldots, t_{m+n-\ell-1}$ we can reconstruct, by rational operations, a representative of $kL/2\pi$ in $\mathbb{R}^{n*}/S_0$. But that implies that $kL/2\pi$ is a rational vector plus an element of $S_0$. □

Another way of stating the same result is to say that $kL/2\pi$, projected onto the span of the large eigenvectors, equals the projection of a rational vector onto this span.

This theorem can be used in two different ways. First, it constrains the set of vectors $L$ for which the system admits point spectrum. Let $d_b$ be the number of large eigenvalues, counted with (algebraic) multiplicity. There are only a countable number of possible values for the projection of $kL/2\pi$ onto the span of the large (generalized) eigenvectors. If the projection of $L$ does not lie in the ray generated by one of these points, there is no point spectrum. In other words, one must tune $d_b - 1$ parameters to a countable number of possible values in order to achieve a nontrivial point spectrum.

A second usage is to constrain the spectrum for fixed $L$. The rational points in $\mathbb{R}^{n}$, projected onto the span of the large eigenvalues, and then intersected with the ray defined by a fixed $L$, form a vector space over $\mathbb{Q}$ of dimension at most $n + 1 - d_b$. As a result, the point spectrum tensored with $\mathbb{Q}$ is a vector space over $\mathbb{Q}$ whose dimension is bounded by one plus the number of small eigenvalues. Below we derive an even stronger result, in which only the small eigenvalues that are conjugate to the Perron-Frobenius eigenvalue contribute to the complexity of the spectrum.

So far we have assumed that there exists a full recurrence vector $v$. This may fail, either because there is something peculiar about the available recurrence vectors, or because the matrix $M$ has repeated eigenvalues. In such cases we can repeat our analysis on a proper subspace of $\mathbb{R}^{n}$. There are necessarily fewer constraints on the vector $L$, but for fixed $L$ the constraints on $k$ are actually stronger than before.

**Theorem 2.7.** Let $b_{PF}$ be the number of large eigenvalues that are algebraically conjugate to the Perron-Frobenius eigenvalue $\lambda_{PF}$ (including $\lambda_{PF}$ itself), and let $s_{PF}$ be the number of small eigenvalues conjugate to $\lambda_{PF}$. For the system to have nontrivial point spectrum, $L$ must lie in a countable union of subspaces of $\mathbb{R}^{n}$, each of codimension $b_{PF} - 1$ or greater. For fixed $L$, the dimension over $\mathbb{Q}$ of the point spectrum tensored with $\mathbb{Q}$ is at most $s_{PF} + 1$.

**Proof.** As a first step we diagonalize $M$ over the rationals as far as possible. By rational operations we can always put $M$ in block-diagonal form, where the characteristic polynomial of each block is a power of an irreducible polynomial. Since the Perron eigenvalue $\lambda_{PF}$ has algebraic multiplicity one, every eigenvalue algebraically conjugate to $\lambda_{PF}$ also has multiplicity one. Thus there is a unique block whose characteristic polynomial has $\lambda_{PF}$ for a root. Every recurrence vector is integral, and pairs nontrivially...
with $L_0$, and so pairs nontrivially with every left-eigenvector of this block. We then consider the constraints on the spectrum that can be obtained from this block alone.

We repeat the argument of the previous two theorems. As before, we modify $L$ by adding a linear combination of small eigenvectors so as to make $t_m$ eventually integral. Since the block diagonalization was over the rationals, the vector $\vec{t} = (t_m, \ldots, t_{m+n-1})^T$, expressed in the new basis, is still rational. From this vector we deduce, as before, the projection of $kL/(2\pi)$ on any large eigenvector with which our recurrence vector $v$ pairs nontrivially. This gives at least $b_{PF}$ independent constraints on the pair $(L,k)$, hence $b_{PF} - 1$ constraints on $L$ if the spectrum is to be nontrivial.

Next we consider the projection of $kL/(2\pi)$ onto the large eigenspaces of the Perron-Frobenius block. Since only $b_{PF} + s_{PF}$ components of $\vec{t}$ (expressed in the new basis) contribute, this is the projection of $Q_{b_{PF}+s_{PF}}$ onto $\mathbb{R}^{b_{PF}}$, whose real span is all of $\mathbb{R}^{b_{PF}}$. Intersected with the ray defined by a fixed $L$, this gives a vector space of dimension at most $s_{PF} + 1$ in which $k$ can live. \qed

3. Topological conjugacies

In Section 2 we studied the point spectrum of a tiling space. This is a conjugacy invariant; systems with different discrete spectra cannot be conjugate. However, we found many different values of $L$ that gave exactly the same point spectrum – namely $\{0\}$. In this section we give criteria for determining when two values of $L$ give rise to conjugate tiling spaces. We supply sufficient conditions for conjugacy in terms of the matrix $M$. We also supply necessary conditions; these require knowing something about the actual substitution, and not just its matrix $M$. Finally, we exhibit examples that demonstrate the need for such details. Different substitutions with the same matrix may result in different criteria for conjugacy.

Throughout this section, $S$ is a substitution subshift with substitution map $\sigma$ and substitution matrix $M$ and we consider two possible length vectors $L = (f(a_1), f(a_2), \ldots)$ and $L' = (g(a_1), g(a_2), \ldots)$. The substitution applied $\ell$ times to a single tile (or letter) is called a supertile of order $\ell$.

**Theorem 3.1.** If, for some integer $k$,

$$\lim_{m \to \infty} \left( LM^{m+k} - L'M^m \right) = 0,$$

then $T_f$ is conjugate to $T_g$.

**Corollary 3.2.** If a substitution has only one eigenvalue of magnitude 1 or greater (counted with multiplicity), then for arbitrary $f$ and $g$ there exists a constant $c$ such that $T_f$ is conjugate to $T_{cg}$.

**Proof of corollary.** Recall that $L_0$ is the normalized left Perron-Frobenius eigenvector of $M$. By rescaling $g$ if necessary, we can assume that $L - L'$ is a linear combination of (generalized) left eigenvectors other than $L_0$. However, all the remaining eigenvectors have eigenvalue less than one, so condition (18) is met with $k = 0$. \qed

**Proof of Theorem.** We prove the theorem in two steps. First we prove it in the special case that $k = 0$, and then in the case that $L = L'M$. Combining these results
then gives the theorem for all \( k \leq 0 \). The situation for \( k > 0 \) is the same, only with \( T_f \) and \( T_g \) reversed.

If \([18]\) holds with \( k = 0 \), then the supertiles in the \( T_f \) system asymptotically have the same size as the corresponding supertiles in the \( T_g \) system, and the convergence is exponential. We then adapt the argument that Radin and Sadun \([RS]\) applied to the Fibonacci tiling. If \( x \) is a tiling in \( T_f \) with tiles in a sequence \( u = \ldots u_{-1} u_0 u_1 \ldots \), then we require \( \phi \) to be a tiling with the exact same sequence of tiles. The only question is where to place the origin. Let \( d_\ell \) be the coordinate of the right edge of the order-\( \ell \) supertile of \( x \) that contains the origin, and let \(-e_\ell \) be the coordinate of the left edge. Place the origin in \( \phi(x) \) a fraction \( e_\ell/(d_\ell + e_\ell) \) of the way across the corresponding supertile of order \( \ell \) in the \( T_g \) system. (This is essentially the map \( h \) of Theorem \([11]\) applied to supertiles of order \( \ell \).) Since the sizes of the supertiles converge exponentially, the location of the origin in \( \phi(x) \) converges, and we can define \( \phi(x) = \lim_{\ell \to \infty} \phi_\ell(x) \).

All that remains is to show that \( \phi \) is a conjugacy. The approximate map \( \phi_\ell \) is not a conjugacy; translations in \( x \) that keep the origin in the same supertile of order \( \ell \) get magnified (in \( \phi(x) \)) by the ratio of the sizes of the supertiles of order \( \ell \) containing the origin in \( x \) and \( \phi(x) \). However, this ratio goes to one as \( \ell \to \infty \), while the range of translations to which this ratio applies grows exponentially. In the \( \ell \to \infty \) limit, \( \phi \) commutes with translation by any \( s \in (\epsilon_\infty, d_\infty) \). Typically this is everything. The case where \( x \) contains two infinite-order supertiles is only slightly trickier – once one sees that \( \phi \) preserves the boundary between these infinite-order supertiles, it is clear that \( \phi \) commutes with all translations.

Now suppose that \( L = L'M \). Then the tiles in the \( T_f \) system have exactly the same size as the corresponding supertiles of \( T_g \) system. We define our conjugacy \( \phi \) as follows: If \( x \) is a tiling in \( T_f \) with tiles in a sequence \( u = \ldots u_{-1} u_0 u_1 \ldots \), then \( \phi(x) \) to a tiling in \( T_g \) with sequence \( \sigma(u) \), with each tile in \( x \) aligned with the corresponding order-1 supertile in \( \phi(x) \).

Combining the cases, we obtain conjugacy whenever \([18]\) applies. \( \square \)

Theorem \([3.1]\) gives sufficient conditions for two tiling spaces to be conjugate. To derive necessary conditions we must understand the extent to which words in our subshift \( S \) (and its associated tiling spaces) can repeat themselves. We will show that words that repeat themselves \( p > 1 \) times or more can be grouped into a finite number of families, and we construct a conjugacy invariant from the asymptotic lengths of these words. Comparing these invariants for different tile lengths then gives necessary conditions for conjugacy (Theorem \([3.6]\) below).

Mossé shows in \([21]\) that in every primitive aperiodic substitution subshift there exists an integer \( N \) so that no word is ever repeated \( N \) or more times. We will need to refine these results by considering words that are repeated a fractional number of times. Counting fractional degrees depends on the length vector: the word \( ababa \) has its basic period \( ab \) repeated \( 2 + \frac{f(a)}{f(a)+f(b)} \) times.

**Definition.** A recurrence vector \( v \) is a repetition vector of degree \( p \) of a tiling space \( T \) if there are finite words \( w \) (with population vector \( v \)) and \( w' \), such that: 1) \( w' \) contains \( w \), 2) \( w' \) is periodic with period equal to the length of \( w \), 3) the length \( Lw' \) of \( w' \) is at
least \( p \) times the length \( Lw \) of \( w \), and 4) \( w' \) appears in some (and therefore every) tiling in \( T \).

Note that every repetition vector of degree \( p > p' \) is also a repetition vector of degree \( p' \). Note also that the word \( w \) is typically not uniquely defined by \( v \). A cyclic permutation of the tiles in \( w \) typically yields a word that works as well. For instance, if \( w' = ababa \) appears in a tiling, then \( v = (1, 1)^T \) is a repetition vector with degree \( 2 + \frac{f(a)}{f(a)+f(b)} \), and we may take either \( w = ab \) or \( w = ba \).

Recall that \( T_0 \) is the tiling space with length vector \( L_0 \), where \( L_0 \) is the left Perron-Frobenius eigenvector of \( M \). For any word \( w \), we denote by \( |w| \) the length of the word \( w \) in \( T_0 \). That is, \( |w| = L_0 v \), where \( v \) is the population vector of \( w \). If \( v \) is a repetition vector of degree \( p \) for \( T_0 \), representing the word \( w \) sitting inside \( w' \) in \( u \in \mathbb{A}^2 \), then \( Mv \) is a repetition vector of degree \( p \), representing the word \( \sigma(w) \) sitting inside \( \sigma(w') \) in \( \sigma(u) \in \mathbb{A}^2 \). The degree is the same, since in \( T_0 \) the substitution \( \sigma \) stretches each word by exactly the same factor, namely \( \lambda_{PF} \). Thus, every repetition vector \( v \) gives rise to an infinite family of repetition vectors \( M^k v \). The following theorem limits the number of such families.

**Theorem 3.3.** Let \( p > 1 \). There is a finite collection of vectors \( \{v_1, \ldots, v_N\} \) such that every repetition vector of degree \( p \) for \( T_0 \) is of the form \( M^k v_i \) for some pair \((k, i)\).

**Proof.** From Mossé [M1, M2], we know that there is a recognition length \( D_S \) of the subshift \( S \) such that knowing a letter, its \( D_S \) immediate predecessors and its \( D_S \) immediate successors determines the supertile of order 1 containing that letter, and the position within that supertile of the letter. Thus for every substitution tiling space \( T \) there is a recognition length \( D_T \) such that the neighborhood of radius \( D_T \) about a point determines the supertile of order 1 containing that point, and the position of that point within the supertile. (E.g., one can take \( D_T \) to be \( 1 + D_S \) times the size of the largest tile.) Let \( D = D_{T_0} \).

Suppose \( v \) is a repetition vector of degree \( p \) for \( T_0 \), corresponding to a word \( w \) that is repeated \( p \) times in a word \( w' \), whose length is much greater than \( D \). Then there is an interval of size \( |w'| - 2D \) in which the supertiles of order 1 are periodic with period \( |w| \). A cyclic permutation of \( w \) is therefore the union of distinct supertiles of order 1. Thus there exists a population vector \( v_1 \) of a word \( w_1 \), such that \( v = M v_1 \), and such that \( v_1 \) is a repetition vector of degree \((|w'| - 2D)/|w| = p - 2D/\lambda_{PF}|w_1| \). Note that \( w \) itself does not have to equal \( \sigma(w_1) \) — it may happen that \( w \) is a cyclic permutation of \( \sigma(w_1) \).

Repeating the process, we find words \( w_i \) such that \( \sigma(w_i) \) equals \( w_{i-1} \) (up to cyclic permutation of tiles), and such that the population vector of \( w_i \) is a repetition vector of degree

\[
(19) \quad p - \frac{2D}{|w_i|} (\lambda_{PF}^1 + \lambda_{PF}^2 + \cdots + \lambda_{PF}^0) \geq p - \frac{2D}{|w_1|} \sum_{\ell=1}^{\infty} \lambda_{PF}^{-\ell} = p - \frac{2D}{|w_1|} (\lambda_{PF} - 1).
\]

Now pick \( \epsilon < p - 1 \). We have shown that every repetition vector of degree \( p \) is of the form \( M^k v_i \), where \( v_i \) is a repetition vector of degree \( p - \epsilon \), and where \( L_0 v_i \) is bounded by
Lemma 3.4. Let $L$ be decomposed as $L = cL_0 + L_r$, where $L_r$ is a linear combination of (generalized) eigenvectors with eigenvalue less than $\lambda_{PF}$. For each word $w$, let $r_w$ be the ratio of the length of $w$ in $T_f$ to the length of $w$ in $T_0$. For each $\epsilon > 0$ there exists a length $R$ such that all words with $|w| > R$ have $|r_w - c| < \epsilon$.

As defined, the degree of a repetition vector depends on the length vector of the tiling space. This difference between tiling spaces disappears in the limit of long repetition vectors:

Lemma 3.5. Let $\epsilon > 0$, and let $L$ and $L'$ be fixed. There is a length $R$ such that, if $v$ is a repetition vector of degree $p$ in $T_f$, representing a word $w$ of length greater than $R$, then $v$ is a repetition vector of degree $p - \epsilon$ in $T_g$.

For the repetition vector $v$ to have degree $p - \epsilon$ in $T_g$, the ratio of the lengths of $w'$ and $w$ in $T_g$ must be at least $p - \epsilon$. However, by Lemma 3.4, the ratios of lengths in $T_f$ and $T_g$ both agree, up to a small error, with the ratios of lengths in $T_0$. Since the ratio is at least $p$ in $T_f$, by choosing $R$ large enough we can make the ratio at least $p - \epsilon$ in $T_g$. □

Theorem 3.3 showed there there are only a finite number of families of repetition vectors of a given degree. Lemma 3.5 shows that the families are essentially the same for $T_f$ and $T_g$, up to a small error in the degree. We next choose a degree $p$ for which the families are exactly the same for all choices of tile length, and show that the asymptotic length of the repetition vectors of degree $p$ yields a conjugacy invariant.

Pick $p_0 > 1$ such that there exist repetition vectors of degree strictly greater than $p_0$ in $T_0$. By Theorem 3.3, there are a finite number of vectors $v_i$ such that every repetition vector of degree $p_0$ is of the form $M^k v_i$. Now, for each $k, i$, let $p_{k,i}$ be the maximal degree of $M^k v_i$, and let $p_i = \lim_{k \to \infty} p_{k,i}$. Note that for fixed $i$, the sequence $p_k$ i is nondecreasing in $k$, since the maximal degree of $\sigma v$ is at least the maximal degree of $v$. Thus the $p_i$’s are a finite set of real numbers, all greater than or equal to $p_0$ and some strictly bigger than $p_0$. There are real numbers $p$ and $\epsilon$ such that $p > p_0 + \epsilon$ and such that each $p_i$ is either greater than $p + \epsilon$ or equal to $p_0$. We restrict our attention to the (nonempty!) indices for which $p_i$ is greater than $p + \epsilon$. Let $L$, $L'$ and $L_0$ be given. By Lemma 3.5, there is a length $R$ such that the set of repetition vectors $v$ of degree $p$ and with $L_0 v > R$ is precisely the same for the three tiling systems $T_f$, $T_g$ and $T_0$. Pick generators $v_1, \ldots, v_N$ of the finite families of these repetition vectors, with $L_0 v_i \leq L_0 v_{i+1}$, and with $L_0 v_N < \lambda_{PF} L_0 v_1$.

Theorem 3.6. Let $p > 1$ be chosen as in the previous paragraph. Suppose that $T_f$ and $T_g$ are conjugate and that $\{v_1, \ldots, v_N\}$ is a collection of repetition vectors of degree $p$ that generates all repetition vectors of degree $p$. Then, given $\delta > 0$, for every $i \in \{1, \ldots, N\}$

$$\frac{2 D \lambda_{PF}}{\epsilon (\lambda_{PF} - 1)}.$$ However, there are only a finite number of words of this length, hence only a finite number of possible repetition vectors $v_i$. □
and for any sufficiently large integer \( m \), there exist \( j, m' \) such that \( |LM^m v_i - L'M^{m'} v_j| < \delta \).

**Proof.** Let \( \phi : T_f \to T_g \) be the conjugacy. Since \( T_f \) and \( T_g \) are compact metric spaces, \( \phi \) is uniformly continuous. Now suppose \( v_i \) is a repetition vector of degree \( p \). Let \( t_{m,i} = LM^m v_i \). Then for every tiling \( x \in T_f \) there is a range of real numbers \( r \) of size roughly \( (p-1)t_{m,i} \), such that \( T_r(x) \) is very close to \( T_{r+m,i}(x) \), where \( T_r \) denotes translation by \( r \). Specifically, as \( m \to \infty \), the distance from \( T_{r+m,i}(x) \) to \( T_r(x) \) can be made arbitrarily small, and the size of the range of \( r \)'s, divided by \( t_{m,i} \), can be made arbitrarily close to \( p-1 \). This implies that \( \phi(T_r(x)) \) and \( \phi(T_{r+m,i}(x)) \) are extremely close. But \( \phi(T_r(x)) = T_r(\phi(x)) \) and \( \phi(T_{r+m,i}(x)) = T_{r+m,i}(\phi(x)) \), so \( T_g \) admits a repetition vector whose pairing with \( L' \) is extremely close to \( t_{m,i} \), and whose degree is extremely close to \( p \). However, for \( m \) large, all such repetition vectors are of the form \( M^{m'} v_j \). \( \square \)

**Examples**

1. The substitution \( a \to aaaaab, b \to babbaa \) has matrix \((\frac{4}{2} \frac{3}{2})\), whose eigenvalues are 6 and 2. It is easy to see that \( v = (1,0)^T \) is a repetition vector of degree 5, and this vector generates the only family of repetition vectors with \( p = 5 \). If \( T_f \) and \( T_g \) are conjugate, for each large \( m \) there must exist \( m' \) such that \( (LM^m - L'M^{m'})v \) is small. Since \( v \) is full, and since both eigenvalues are large, this implies there is an integer \( k \) such that \( L' = LM^k \).

2. The substitution \( a \to aaaaab, b \to bbbbaa \), has exactly the same substitution matrix as the first example, but the repetitions are different. \( v_1 = (1,0)^T \) and \( v_2 = (0,1)^T \) are both repetition vectors with degree 6. The possibility of having \( j \neq i \) in Theorem 3.6 allows for additional conjugacies. Indeed, \( T_f \) and \( T_g \) are conjugate if (and only if) either \( (L'_1, L'_2) = (L_1, L_2)M^k \) or \( (L'_1, L'_2) = (L_2, L_1)M^k \) for some (possibly negative) integer \( k \).

3. The substitution \( a \to b, b \to ac, c \to ab \), has a matrix whose eigenvalues are \( \tau, -1 \) and \(-1/\tau \), where \( \tau \) is the golden mean. However, the eigenvalue \(-1 \) is irrelevant, as its left-eigenvector \((1,1,0)\) is orthogonal to all recurrence vectors, and in particular to all repetition vectors. (Every \( a \) is preceded by a \( b \), and every \( b \) is followed by an \( a \)). This system behaves essentially like the Fibonacci substitution, with all choices of tile lengths giving conjugate dynamics, up to an overall scale.

Notice that in the first example, our necessary condition was the same as the sufficient condition of Theorem 3.1 while in the second and third examples there were conjugacies between length choices that did not meet the condition of Theorem 3.1. The difference between the first two examples also illustrates that one cannot obtain sharp conditions on conjugacy from the substitution matrix alone. One needs some details about the substitution itself. Our final example further illustrates this point:

4. The substitution \( a \to aababbbbaa, b \to bbabbaaabab \) has matrix \( M = (\frac{4}{2} \frac{1}{2}) = (\frac{2}{1} \frac{1}{2})^2 \) and is the square of the substitution \( a \to aab, b \to bba \), so its associated tiling space admits (among others) conjugacies with \( L' = L(\frac{2}{1} \frac{1}{2})^m \), where \( m \) can be any integer. However, the substitution \( a \to aaaaaabb, b \to abbbabaa \), with the same
matrix $M$, has only one family of repetition vectors of degree 8, namely those generated by $(1, 0)^T$. In this case all conjugacies are of the form $L' = LM^k = L (\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix})^{2k}$.

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