BIFURCATION FROM A HOMOCLINIC ORBIT IN PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider a family of partial functional differential equations which has a homoclinic orbit asymptotic to an isolated equilibrium point at a critical value of the parameter. Under some technical assumptions, we show that a unique stable periodic orbit bifurcates from the homoclinic orbit. Our approach follows the ideas of Šil’nikov for ordinary differential equations and of Chow and Deng for semilinear parabolic equations and retarded functional differential equations.

1. Introduction. For an ordinary differential equation

$$\dot{x} = g(x, \epsilon), \quad (1.1)$$

where $x \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}$ is a parameter and $g$ is a smooth function, it is known that if $x = 0$ is a hyperbolic equilibrium for $\epsilon = 0$ and the Jacobian matrix $D_x f(0, 0) = A$ has a unique eigenvalue $\lambda > 0$ which is simple and the real parts of all other eigenvalues are strictly less than $-\lambda$, then under certain additional transversality conditions, a unique stable periodic orbit bifurcates from the homoclinic orbit as the parameter $\epsilon$ changes. See, for example, Andronov et al. [AL73], Chow and Hale [CH86] and Kuznetsov [Ku95]. One of the approaches to the above bifurcation problem, originated in the work of Neimark and Šil’nikov [NS65] and Šil’nikov [Si68] for ordinary differential equations in $\mathbb{R}^n$ with $n \geq 3$, is to reduce the bifurcation problem to a problem of the continuation of fixed points for a one-parameter map in a small neighborhood of the hyperbolic equilibrium. This map resembles the well-known Poincaré map but the points on the stable manifold do not return. In

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what follows we shall call this map the Šil’nikov map and we refer to Kuznetsov [Ku95] for a detailed description of Šil’nikov’s results and techniques.

The above result has been generalized to other kinds of equations, while several other methods have also been developed. These include the work of Blázquez [Bl86] for semilinear parabolic equations, and of Walther [Wa90] for retarded functional differential equations. We should particularly mention the work of Chow and Deng [CD89] for some infinite dimensional dynamical systems including semilinear parabolic differential equations and retarded functional differential equations, where they obtained some subtle estimates related to linear variational equations along semi-orbits of the nonlinear equations and established the smoothness and the existence of a fixed point of the Šil’nikov map.

In this paper, we consider the following one-parameter family of partial functional differential equations:

\[
\dot{u}(t) = Au(t) + L(u_t) + g(u_t, \epsilon),
\]

(1.2)

where \( A \) is the generator of an analytic semigroup, \( L \) is a linear operator and \( g \) is a smooth nonlinear functional. \( g \) depends on not only the current but also the historic status of \( u \). More specific descriptions will be given in next section. This kind of equations is motivated by reaction-diffusion equations where the reaction terms may involve time delay and have been studied by many researchers, see, for example, Faria [Fa99, Fa01], Faria et al. [FHW02], Hale [Ha86], Hale and Ladeira [HL93], He [He90], Martin and Smith [MS90], Memory [Me91], Travis and Webb [TW74, TW78], etc. For an introduction of the fundamental theory of such equations and some related references, we refer to the monograph by Wu [Wu96].

The purpose of this paper is to generalize Šil’nikov’s theorem and Chow and Deng’s techniques to the above partial functional differential equations. In section 2, we introduce the notations and present the main results. The differentiability of solutions of equation (1.2) with respect to the initial values and parameters and the smoothness of the stable and unstable manifolds are proved in section 3. The local analysis of equation (1.2) near the equilibrium is given in section 4. In section 5, we construct the Šil’nikov map and discuss some of its properties. The proof of the main theorem is presented in section 6.

2. The Main Results. Let \( X \) denote a Banach space over \( R = (\infty, \infty) \) and \( B(X, X) \) the Banach space of bounded linear operators from \( X \) to \( X \) equipped with the operator norm. Let \( r > 0 \) be a given constant and \( C = C([-r, 0]; X) \) the Banach space of continuous \( X \)-valued functions on \([-r, 0]\) with the supremum norm \(|\cdot|\). For any real numbers \( a \leq b \), \( t \in [a, b] \) and any continuous mapping \( u : [a - r, b] \rightarrow X \), \( u_t \) denotes the element of \( C \) given by \( u_t(\theta) = u(t + \theta) \) for \( \theta \in [-r, 0] \).

Consider the following family of partial functional differential equations

\[
\dot{u}(t) = Au(t) + L(u_t) + g(u_t, \epsilon),
\]

(2.1)

where \( \epsilon \in (-\epsilon_0, \epsilon_0) \) is a parameter, \( \epsilon_0 \) is a given positive constant, \( A, L \) and \( g \) satisfy the following assumptions:

**(H1)** \( A \) is the infinitesimal generator of an analytic compact semigroup \( \{S(t)\}_{t \geq 0} \) on \( X \).

**(H2)** \( L : C \rightarrow X \) is given by

\[
L\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta), \quad \phi \in C
\]
for a function \( \eta: [-r, 0] \to B(X, X) \) of bounded variation.

\((H3)\) \( g \in C^3(\mathcal{C} \times (-\epsilon_0, \epsilon_0); X) \) and \( g(0, \epsilon) = 0, \) \( D_\epsilon g(0, \epsilon) = 0 \) for \( \epsilon \in (-\epsilon_0, \epsilon_0). \)

The associated linear equation is given by

\[
\dot{u}(t) = Au(t) + L(u_t).
\]

(2.2)

For each complex number \( \lambda \), define the linear operator \( \Delta(\lambda) : \text{Dom}(A) \to X \) by

\[
\Delta(\lambda)u = Au - \lambda u + L(e^{\lambda}u), \quad u \in \text{Dom}(A),
\]

where \( e^{\lambda}u \in \mathcal{C} \) is defined by

\[
(e^{\lambda}u)(\theta) = e^{\lambda \theta}u, \quad \theta \in [-r, 0].
\]

\( \lambda \) is called a characteristic value of equation (2.2) if there exists \( u \in \text{Dom}(A) \setminus \{0\} \) solving the characteristic equation

\[
\Delta(\lambda)u = 0.
\]

A characteristic value \( \lambda \) is simple if \( \dim(\ker(\Delta(\lambda))) = 1 \) for all positive integer \( n \).

We further assume that

\((H4)\) Equation (2.2) has a unique positive characteristic value \( \lambda > 0 \) which is simple and the real parts of all other characteristic values of (2.2) are smaller than \( -\lambda \).

It is known that for each \( \phi \in \mathcal{C} \), the initial value problem

\[
\begin{align*}
\dot{u}(t) &= S(t)\phi(0) + \int_0^t S(t - \alpha)L(u_\alpha)d\alpha, \quad t \geq 0, \\
u_0 &= \phi
\end{align*}
\]

has a unique solution defined for \( t \geq -r \). Denote this solution by

\[
T(t)\phi = u_t(\phi),
\]

then \( \{T(t)\}_{t \geq 0} \) is a strongly continuous semigroup of bounded linear operators on \( \mathcal{C} \) with the generator denoted by \( A_T \). Also, for each \( \phi \in \mathcal{C} \) and \( \epsilon \in (-\epsilon_0, \epsilon_0) \), there exists \( \tau(\phi, \epsilon) > 0 \) and a unique continuous map \( u = u(\phi, \epsilon): [-r, \tau(\phi, \epsilon)) \to X \) such that

\[
u(\phi, \epsilon)(t) = S(t)\phi(0) + \int_0^t S(t - \alpha)[L(u_\alpha(\phi, \epsilon)) + g(u_\alpha(\phi, \epsilon), \epsilon)]d\alpha
\]

for \( t \in [0, \tau(\phi, \epsilon)) \). Using the mapping \( X_0: [-r, 0] \to B(X, X) \) defined by

\[
X_0(\theta) = \begin{cases} 
0, & -r \leq \theta < 0, \\
I, & \theta = 0,
\end{cases}
\]

we have the following variation of constants formula (see He [He90], Memory [Me91] and Wu [Wu96])

\[
u(t) = T(t)\phi + \int_0^t T(t - \alpha)X_0g(u_\alpha, \epsilon)d\alpha,
\]

(2.3)
for $u(\phi, \epsilon)$ on $[0, \tau(\phi, \epsilon))$. By assumption (H4), $C$ can be decomposed as $C = C^s \oplus C^u$, where $C^u$ is the one-dimensional eigenspace of $A_T$ associated with $\{\lambda\}$ and $C^s$ is the generalized eigenspace associated with the remaining spectrum. Let $\phi_\lambda = \phi_\lambda(0)e^{\lambda\cdot}$ be the eigenvector of $A_0$ associated with $\lambda$ and $\phi^*_\lambda$ be the eigenvector corresponding to $\{\lambda\}$ of the formal adjoint operator associated with the bilinear pairing (see Travis and Webb [Tw74])

$$
\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_0^1 \int_0^\theta \psi(\xi - \theta)[d\eta(\theta)]\phi(\xi)d\xi,
$$

where $\psi \in C^* = C([0, r]; X^*)$, $X^*$ is the dual space of $X$ and $\phi \in C$. Then

$$
C^u = \{ \phi; \phi \in C, \phi = a\phi_\lambda \text{ for some } a \in R \},
C^s = \{ \phi; \phi \in C, \langle \phi^*_\lambda, \phi \rangle = 0 \}.
$$

For every $\phi^s + \phi^u \in C^s \oplus C^u$, we have

$$
\phi^u = \langle \phi^*, \phi \rangle \phi_\lambda, \quad \phi^s = \phi - \phi^u.
$$

Note that for $\phi \in C$,

$$
T(t)\phi(\theta) = \begin{cases} 
\phi(t + \theta), & -\tau \leq t + \theta \leq 0, \\
T(t + \theta)\phi(0), & t + \theta \geq 0 
\end{cases} \quad (2.4)
$$

and for $\phi^u \in C^u$,

$$
T(t)\phi^u = \phi^u e^{\lambda t}, \quad t \in R,
\phi^u(\theta) = \phi^u(0) e^{\lambda \theta}, \quad \theta \in [-\tau, 0]. \quad (2.5)
$$

Let $P^s$ and $P^u$ be the projections of $C$ onto $C^s$ and $C^u$, respectively, i.e. $C^s = P^sC, C^u = P^uC$. It is shown that $P^s$ and $P^u$ can be applied to the elements $X_0w$ with $w \in X$. Define $X^s_0$ and $X^u_0$ by

$$
X^u_0w = P^uX_0w, \quad X^s_0w = P^sX_0w, \quad w \in X. \quad (2.6)
$$

Note that if $w \in X$, then $T(t)X^s_0w \in C^u$ for all $t \in R$ and $T(t)X^u_0w \in C^s$ for $t \geq \tau$. Moreover, there exist constants $K_1$ and $\mu > \lambda > 0$ such that

$$
\begin{align*}
|T(t)\phi^s| & \leq K_1 e^{-\mu t} |\phi^s|, \quad t \geq 0, \quad \phi^s \in C^s; \\
|T(t)X^s_0| & \leq K_1 e^{-\mu t}, \quad t \geq 0.
\end{align*} \quad (2.7)
$$

Decompose $u_1(\phi, \epsilon)$ as

$$
\begin{align*}
u_1(\phi, \epsilon) & = u^s_1(\phi, \epsilon) + u^u_1(\phi, \epsilon)
\end{align*}
$$

with $u^s_1(\phi, \epsilon) \in C^s$ and $u^u_1(\phi, \epsilon) \in C^u$. Then we have the following variation of constants formula (see He [He90], Memory [Me91] or Wu [Wu96]):

$$
\begin{align*}
u^s_1(\phi, \epsilon) & = T(t)\phi^s + \int_0^t T(t - \alpha)X^s_0g(u_0(\phi, \epsilon), \epsilon)d\alpha, \\
u^u_1(\phi, \epsilon) & = e^{\lambda t}\phi^u + \int_0^t e^{\lambda(t - \alpha)}X^u_0g(u_0(\phi, \epsilon), \epsilon)d\alpha \quad (2.8)
\end{align*}
$$
for \( t \in [0, \tau(\phi, \epsilon)) \).

Since \( g \) is \( C^3 \)-smooth, by the differentiability of the solution with respect to initial values and parameters (see Theorem 3.1 in section 3), \( u_t(\phi, \epsilon) \) is \( C^3 \)-smooth in \( (\phi, \epsilon) \) for all \( t \geq 0 \) in the maximal interval of existence. Set

\[
v_t^s(\phi, \epsilon) = D_\phi u_t^s(\phi, \epsilon), \quad v_t^u(\phi, \epsilon) = D_\phi u_t^u(\phi, \epsilon). \tag{2.9}
\]

We have

\[
\begin{align*}
v_t^s(\phi, \epsilon) &= T(t)P^s + \int_0^t T(t - \alpha)X_t^s D_\phi g(u_\alpha(\phi, \epsilon), \epsilon)v_\alpha(\phi, \epsilon)d\alpha, \\
v_t^u(\phi, \epsilon) &= T(t)P^u + \int_0^t T(t - \alpha)X_t^u D_\phi g(u_\alpha(\phi, \epsilon), \epsilon)v_\alpha(\phi, \epsilon)d\alpha
\end{align*} \tag{2.10}
\]

with \( T(t)P^s \phi = e^{M_s} \phi, \quad \phi \in C, \quad t \in \mathbb{R} \). By assumptions (H3) and (H4) there exist \( \delta_1 > 0 \) and \( \epsilon_1 \in (0, \epsilon_0) \) such that the local stable and unstable manifolds \( W^s_{\text{loc}}(\epsilon) \) and \( W^u_{\text{loc}}(\epsilon) \) exist and are subsets of \( B(\delta_1) \) for \( \epsilon \in [-\epsilon_1, \epsilon_1] \), where \( B(\delta_1) = \{ \phi \in C; |\phi'| < \delta_1, |\phi''| < \delta_1 \} \) and \( W^s_{\text{loc}}(\epsilon) \) and \( W^u_{\text{loc}}(\epsilon) \) are given by

\[
\begin{align*}
W^s_{\text{loc}}(\epsilon) &= \{ \phi = \phi^s + \phi^u; \phi^u = h_s(\phi^s, \epsilon), |\phi^s| < \delta_1 \}, \\
W^u_{\text{loc}}(\epsilon) &= \{ \phi = \phi^s + \phi^u; \phi^s = h_u(\phi^u, \epsilon), |\phi^u| < \delta_1 \}, \tag{2.11}
\end{align*}
\]

where \( h_s \) and \( h_u \) are \( C^3 \)-smooth (see Theorem 3.2 in section 3) and \( h_u \) is defined by

\[
h_u(\phi^u, \epsilon) = \int_{-\infty}^0 T(-\alpha)X_0^u D_\phi g(u_\alpha(\phi^u, \epsilon), \epsilon)d\alpha, \quad |\phi^u| < \delta_1, \quad \epsilon \in [-\epsilon_1, \epsilon_1] \tag{2.12}
\]

(see Memory [Me91]), and \( u_t^s(\phi^u, \epsilon) \) is the unique bounded solution of (2.1) on \((-\infty, 0]\) with

\[
|u_t^s(\phi^u, \epsilon)| \leq K_2 e^{t\mu_1}|\phi^u|, \quad t \leq 0 \tag{2.13}
\]

for some positive constant \( K_2 \) independent of \( (\phi^u, \epsilon) \).

In order to state the main theorem, we need one additional assumption:

(H5) When \( \epsilon = 0 \), equation (2.1) has a homoclinic orbit \( \Gamma_0 \) asymptotic to the equilibrium 0.

For a fixed \( \epsilon \in (-\epsilon_0, \epsilon_0) \), let \( W^u_\epsilon(\epsilon) \) be the orbit of equation (2.1) through a given \( \phi_0 \in W^u_{\text{loc}}(\epsilon) \) with \( \langle \phi_0^s, \phi_0^u - h_0(\phi_0^s, \epsilon) \rangle > 0 \). Without loss of generality, we assume that the homoclinic orbit \( \Gamma_0 = W^u_0(0) \). Here, a homoclinic orbit \( \Gamma_0 \) asymptotic to 0 is a continuous mapping \( u : R \rightarrow X \) satisfying

\[
u(t) = S(t - s)u(s) + \int_s^t S(t - \alpha)[L(u_\alpha) + g(u_\alpha, 0)]d\alpha
\]

for \( t, s \in R \) with \( t \geq s \), and \( \lim_{t \to \pm \infty} u(t) = 0 \). Now we can state our main theorem on homoclinic bifurcation of (2.1), which is a generalization of the results of Šil’nikov [Si68] and Chow and Deng [CD89] to abstract semilinear functional differential equations.

**Theorem 2.1.** Suppose (H1) – (H5) hold. Then there exist a neighborhood \( \mathcal{N}(\Gamma_0) \) of \( \Gamma_0 \cap \{0\} \) in \( C \) and \( \epsilon_0 \in (0, \epsilon_0) \) such that \( W^u_\epsilon(\epsilon) \cap W^u_{\text{loc}}(\epsilon) = \emptyset \) if and only if there exists a periodic orbit in \( \mathcal{N}(\Gamma_0) \) for given \( \epsilon \in [-\epsilon_0, \epsilon_0] \). Furthermore, for the given \( \epsilon \in [-\epsilon_0, \epsilon_0] \) this periodic orbit is unique and exponentially asymptotically stable.
Corollary 2.2. Under the same assumptions of Theorem 2.1, there exists a neighborhood $\mathcal{N}(\Gamma_0)$ of $\Gamma_0 \cup \{0\}$ in $\mathcal{C}$ and $0 < \varepsilon_0 < \varepsilon_0$ such that if there is a homoclinic orbit in $\mathcal{N}(\Gamma_0)$ for equation (2.1) at $\epsilon$ with $|\epsilon| \leq \varepsilon_0$, then there exist no periodic orbits of equation (2.1) lying entirely in $\mathcal{N}(\Gamma_0)$ at $\epsilon$.

3. Preliminaries. In this section, we prove the differentiability of solutions of equation (2.1) with respect to initial values and parameters and the smoothness of the stable and unstable manifolds.

3.1. Differentiability with Respect to Initial Values and Parameters. Let $V$ be a neighborhood of 0 in $\mathcal{C}$, $(a, b)$ be an open interval in $R$ and $F \in C^k(V \times (a, b); X)$. Consider

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)F(u_s, \alpha)ds, \quad u_0 = \phi.$$  \hfill (3.1)

Theorem 3.1. The solution $u(\phi, \alpha)$ is $C^k$-smooth with respect to $(\phi, \alpha)$ for $t$ in any compact set of of the domain of definition of $u(\phi, \alpha)$. Moreover, for each $\psi \in \mathcal{C}$, $D\phi u(\phi, \alpha)\psi(t)$ satisfies the linear variational equation

$$v(t) = T(t)\psi(0) + \int_0^t T(t-s)D\phi F(u_s, \alpha, \alpha)v_s ds, \quad v_0 = \psi.$$  \hfill (3.2)

In the proof, we shall use Lemma 4.2 and the argument for Theorem 4.1 in Hale and Verduyn Lunel [HV93].

Proof. Fix $\xi \in V$ and $\alpha_0 \in (a, b)$. There exist constants $M > 0$, $\delta > 0$ and $N > 0$ such that

$$\begin{cases}
|T(t)| \leq M \quad \text{for } 0 \leq t \leq 1, \\
B_\delta(\xi) \subseteq V \text{ with } B_\delta(\xi) = \{\psi \in \mathcal{C}; \|\psi - \xi\| < \delta\}, \\
[a_0 - \delta, a_0 + \delta] \subseteq (a, b), \\
|F(\psi, \alpha)| \leq N, \quad |D\phi F(\psi, \alpha)| \leq N \quad \text{for } (\psi, \alpha) \in B_\delta(\xi) \times [a_0 - \delta, a_0 + \delta].
\end{cases}$$

Now choose $\eta \in (0, 1)$ and $\nu \in (0, 1)$ so that

$$\begin{cases}
\nu < \frac{\delta}{2}, \quad \eta < \frac{\nu}{MN}, \\
\sup_{\theta, \theta' \in [-r, 0]} |\xi(\theta') - \xi(\theta)| < \frac{\delta}{8}, \\
\sup_{\tau \in [0, \eta]} \|T(t)\xi(0) - \xi(0)\| < \frac{\delta}{4}.
\end{cases}$$

Let

$$K(\eta, \nu) = \{w \in C([-r, \eta]; X); w_0 = 0, \|w_t\| \leq \nu \text{ for } t \in [0, \eta]\}.$$  

Clearly $K(\eta, \nu)$ is a closed subset of the Banach space

$$C_0([-r, \eta]) = \{\phi \in C([-r, \eta]; X); \phi(\theta) = 0 \text{ for } \theta \in [-r, 0]\}.$$
equipped with the super-norm.

For each $\phi \in C$, define $\hat{\phi} : [-r, \infty) \to X$ by $\tilde{\phi}_0 = \phi$ and $\hat{\phi}(t) = T(t)\phi(0)$ for $t \geq 0$. Now for fixed $\phi \in B_{\frac{s}{\mu+\tau industries} (\xi)$ and $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$, define $A(\phi, \alpha)$ on $K(\eta, \nu)$ by

$$ A(\phi, \alpha)w(t) = \begin{cases} \int_0^t T(t-s)F(w_s + \hat{\phi}_s, \alpha)ds, & w \in K(\eta, \nu), \quad t \in [0, \eta], \\ 0 & t \in [-r, 0]. \end{cases} $$

It follows that $A(\phi, \alpha)w \in C([-r, \eta] ; X)$. Moreover, since for $s \in [0, \eta]$, $\|w_s\| \leq \nu$ and

$$ \|\hat{\phi}_s - \xi\| \leq \|\tilde{\phi}_s - \xi_s\| + \|\xi_s - \xi\| $$

$$ \leq \|\phi - \xi\| + \sup_{s \in [0, \eta]} \|T(s)||\phi(0) - \xi(0)| + \sup_{\theta \in [-r, 0]} \max_{\phi \in [0, \eta]} \|\xi(s + \theta) - \xi(\theta)\| $$

$$ + \sup_{\theta \in [-r, 0]} \max_{s \in [0, \eta]} \|\xi(s + \theta) - \xi(\theta)\| $$

$$ \leq (1 + M)\|\phi - \xi\| + \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{8} < \frac{\delta}{2}, $$

we have $\|w_s + \hat{\phi}_s - \xi\| < \nu + \frac{\delta}{2} < \delta$ and hence

$$ |F(w_s + \hat{\phi}_s, \alpha)| \leq N $$

for $s \in [0, \eta], \alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$. This shows that

$$ |A(\phi, \alpha)w(t)| \leq MN\eta < \nu \quad \text{for} \quad t \in [0, \eta]. $$

Thus, $A(\phi, \alpha)w \in K(\eta, \nu)$ and $A(\phi, \alpha)K(\eta, \nu) \subseteq K(\eta, \nu)$. Moreover, using $|D_{\phi}F(\phi, \alpha)| \leq N$ for $(\phi, \alpha) \in \hat{B}_3(\xi) \times [\alpha_0 - \delta, \alpha_0 + \delta]$, we have for $w, \tilde{w} \in K(\eta, \nu)$ that

$$ |A(\phi, \alpha)w(t) - A(\phi, \alpha)\tilde{w}(t)| \leq \int_0^t T(t-s)|F(w_s + \hat{\phi}_s, \alpha) - F(\tilde{w}_s + \hat{\phi}_s, \alpha)|ds $$

$$ \leq MN\eta \sup_{s \in [0, t]} \|w_s - \tilde{w}_s\| $$

$$ \leq MN\eta \sup_{s \in [-r, \eta]} \|w(s) - \tilde{w}(s)\| $$

$$ < \nu \sup_{s \in [-r, \eta]} \|w(s) - \tilde{w}(s)\|, $$

Since $\nu < 1$, we conclude that

$$ A(\cdot) : \hat{B}_{\frac{s}{\mu+\tau industries} (\xi) \times [\alpha_0 - \delta, \alpha_0 + \delta] \to K(\eta, \nu) $$

is a uniform contraction. By Lemma 4.2 of Hale and Verduyn Lunel [HV93], for each fixed $(\phi, \alpha) \in \hat{B}_{\frac{s}{\mu+\tau industries} (\xi) \times [\alpha_0 - \delta, \alpha_0 + \delta]$, $A(\cdot)$ has a unique fixed point $w(\phi, \alpha) \in K(\eta, \nu)$ which is continuous in $(\phi, \nu)$.

Note that $\hat{B}_{\frac{s}{\mu+\tau industries} (\xi) \times [\alpha_0 - \delta, \alpha_0 + \delta] \times (\alpha_0 - \delta, \alpha_0 + \delta)$ and $A(\phi, \alpha)w$ has continuous $k$-th derivative with respect to $(\phi, \alpha, w) \in \hat{B}_{\frac{s}{\mu+\tau industries} (\xi) \times [\alpha_0 - \delta, \alpha_0 + \delta] \times K^0(\eta, \nu)$, where

$$ K^0(\eta, \nu) = \{ w \in K(\eta, \nu); \|w_t\| < \nu \quad \text{for} \quad t \in [0, \alpha] \} $$
with $K^0(\eta, \nu)$ being open in $C_0([-r, \eta])$ and $K(\eta, \nu) = \overline{K^0(\eta, \nu)}$. Therefore, by Lemma 4.2, $u(\phi, \alpha)$ is $C^k$-smooth with respect to $(\phi, \alpha) \in B_{\frac{1}{\xi(\xi + 1)}}(\xi) \times (\alpha_0 - \delta, \alpha_0 + \delta)$. Hence, $u(\phi, \alpha) = \tilde{\phi} + w(\phi, \alpha)$ is $C^k$-smooth with respect to $(\phi, \alpha) \in B_{\frac{1}{\xi(\xi + 1)}}(\xi) \times (\alpha_0 - \delta, \alpha_0 + \delta)$ for $t \in [0, \eta]$. Standard continuation argument then leads to the $C^k$-smoothness of $u(\phi, \alpha)$ with respect to $(\phi, \alpha)$ for $t$ in any compact subset of the domain of the definition of $u(\phi, \alpha)$. 

3.2. Smoothness of the Stable and Unstable Manifolds. In this subsection, we study the $C^k$-smoothness of the stable and unstable manifolds of equation (2.1) (Chow and Lu [CL88a, CL88b]). First, we modify assumption (H3) as follows:

(H3') $g \in C^k(\mathcal{C} \times (-\epsilon_0, \epsilon_0); X)$ and $g(0, \epsilon) = 0$, $D_\phi g(0, \epsilon) = 0$ for $\epsilon \in (-\epsilon_0, \epsilon_0)$.

For a given $\epsilon \in (-\epsilon_0, \epsilon_0)$, a (mild) solution of equation (2.1) subject to the initial condition $u_0 = \phi \in \mathcal{C}$ on $[-r, \tau(\phi, \epsilon))$, $\tau(\phi, \epsilon) > 0$, is a continuous mapping $u = u(\phi, \epsilon) : [-r, \tau(\phi, \epsilon)) \rightarrow X$ such that

$$u(\phi, \epsilon)(t) = S(t)\phi(0) + \int_0^t S(t - \alpha)[L(u_\alpha(\phi, \epsilon)) + g(u_\alpha(\phi, \epsilon), \epsilon)]d\alpha$$

for $t \in [0, \tau(\phi, \epsilon))$. Note that $u(0, \epsilon)(t) = 0$ for all $t \geq 0$ is always a solution of (2.1) with $u_0 = 0$. Therefore, for a fixed $\tau > r$, by the basic theory of (2.1) (see Wu [Wu96]) it follows that there exists an open neighborhood $\mathcal{N}_0$ of 0 in $\mathcal{C}$ and $\epsilon_1 \in (-\epsilon_0, \epsilon_0)$ such that for each $\epsilon \in (-\epsilon_1, \epsilon_1)$ and for every $\phi \in \mathcal{N}_0$, equation (2.1) has a solution $u(\phi, \epsilon)$ defined at least on $[-r, \tau_1]$. Let $\tilde{f} : \mathcal{N}_0 \times (-\epsilon_1, \epsilon_1) \rightarrow \mathcal{C}$ be given by

$$\tilde{f}(\phi, \epsilon) = u_{\tau_1}(\phi, \epsilon).$$

Then $\tilde{f}$ is completely continuous, $C^k$-smooth and $D_\phi \tilde{f}(0, 0) = T(\tau_1)$ (defined in section 2). Let

$$\Sigma_s = \{\lambda \in \mathcal{C}; \lambda \text{ is a characteristic value of (2.2) with Re} \lambda < 0\},$$
$$\Sigma_u = \{\lambda \in \mathcal{C}; \lambda \text{ is a characteristic value of (2.2) with Re} \lambda = 0\},$$
$$\Sigma_u = \{\lambda \in \mathcal{C}; \lambda \text{ is a characteristic value of (2.2) with Re} \lambda > 0\}$$

and assume that

$$\Sigma_u \neq \emptyset, \quad \Sigma_u \neq \emptyset.$$

For each $\lambda \in \Sigma_s \cup \Sigma_u$, let $M_\lambda$ be the realized generalized eigenspace of $A_T$ associated with $\lambda$ and denote

$$C^s = \bigoplus_{\lambda \in \Sigma_s} M_\lambda, \quad C^u = \bigoplus_{\lambda \in \Sigma_u} M_\lambda.$$

Then we know that $\dim C^u < \infty$, $C^u$ and $C^s$ are closed subspaces of $\mathcal{C}$ such that $\mathcal{C} = C^s \oplus C^u$ and $T(\tau_1)C^s \subseteq C^s$, $T(\tau_1)C^u \subseteq C^u$.

Theorem 3.2. We have the following results on the smoothness of the stable and unstable manifolds.

(i) There exist a constant $\epsilon_\delta \in (0, \epsilon_0)$, convex open bounded neighborhoods $\mathcal{N}_s$ of 0 in $C^s$ and $\mathcal{N}_u$ of 0 in $C^u$, and a $C^k$-smooth mapping $h_s : \mathcal{N}_s \times (-\epsilon_\delta, \epsilon_\delta) \rightarrow C^u$ with $h_s(0, 0) = 0$, $D_\phi h_s(0, 0) = 0$, $h_s(\mathcal{N}_s \times (-\epsilon_\delta, \epsilon_\delta)) \subseteq \mathcal{N}_u$ such that for $\phi \in \mathcal{C}$ and $\epsilon \in (-\epsilon_\delta, \epsilon_\delta)$, if equation (2.1) has a solution $u(\phi, \epsilon)$ on
and $\infty$

(Theorem II.1 ensures

By Theorem II.1 of Krisztin, Walther and Wu [KWW99], there exist open neighbor-

$\tilde{C}$
yields

Hence,

$\sigma(L(E_\beta)) = \{1\}$,

$\sigma(L(E_\alpha)) = \{z \in \mathbb{C}; |z| \geq 1\}$.}

By Theorem II.1 of Krisztin, Walther and Wu [KWW99], there exist open neighborhoods $\tilde{N}_{sc}$ of 0 in $E_s \oplus E_c$, $\tilde{N}_u$ of 0 in $E_u$ and a $C^k$-smooth mapping $\tilde{h} : \tilde{N}_{sc} \to \tilde{N}_u$ (Theorem II.1 ensures $C^1$-smoothness if $f$ is $C^1$-smooth, the same argument there yields $C^k$-smoothness of $\tilde{h}$ if $f$ is $C^k$-smooth) with

$\tilde{h}(0, 0) = 0, \quad D_{\phi, \epsilon}\tilde{h}(0, 0) = 0, \quad \tilde{h}(\tilde{N}_{sc}) \subseteq \tilde{N}_u$

and

$\bigcap_{n=0}^{\infty} f^{-n}(\tilde{N}_{sc} \cup \tilde{N}_u) \subseteq \tilde{W}$,
where
\[
\hat{W} = \{ (\phi^s, \epsilon) + \tilde{h}(\phi^s, \epsilon); (\phi^s, \epsilon) \in \tilde{N}_{sc} \}.
\]

Let \( \pi_u : E_u \rightarrow C^u \) be the natural projection. Find open neighborhoods \( N_s \) of 0 in \( C^s \), \( \bar{N}_u \) of \( 0 \) in \( C^u \) and \( \epsilon_s \in (0, \epsilon_0) \) so that
\[
N_s \times (-\epsilon_s, \epsilon_s) \subseteq \tilde{h}(\tilde{N}_{sc}), \quad N_u \times \{0\} = \bar{N}_u.
\]

Also let
\[
h_s : \bar{N}_s \times (-\epsilon_s, \epsilon_s) \rightarrow C^u
\]
be given by
\[
h_s(\phi^s, \epsilon) = \pi_u \tilde{h}(\phi^s, \epsilon).
\]

Then \( h_s \) is \( C^k \)-smooth and satisfies
\[
\begin{align*}
h_s(0, 0) &= \pi_u \tilde{h}(0, 0) = 0, \\
D_\phi h_s(0, 0) &= \pi_u D_\phi \tilde{h}(0, 0) = 0, \\
h_s(N_s \times (-\epsilon_s, \epsilon_s)) &= \pi_u \tilde{h}(N_s \times (-\epsilon_s, \epsilon_s)) \\
&\subseteq \pi_u \tilde{h}(\tilde{N}_{sc}) \subseteq \pi_u \bar{N}_u = N_u.
\end{align*}
\]

Assume that \( u(\phi, \epsilon) \) is a solution of equation (2.1) on \([-r, \infty)\) with \( \epsilon \in (-\epsilon_s, \epsilon_s) \) and \( u_t(\phi, \epsilon) \in N_s \times \bar{N}_u \) for \( t \geq 0 \). Fix \( t \geq 0 \). Then for each integer \( n \geq 0 \),
\[
u_{t+n\pi_i}(\phi, \epsilon) = \hat{f}^n(u_t(\phi, \epsilon), \epsilon) \in N_s \times \bar{N}_u.
\]

Therefore,
\[
f^n(u_t(\phi, \epsilon), \epsilon) = (\hat{f}^n(u_t(\phi, \epsilon), \epsilon), \epsilon) \in (N_s \times (-\epsilon_s, \epsilon_s)) \cup (N_u \times \{0\}) \subseteq \tilde{N}_{sc} \cup \bar{N}_u.
\]

Consequently, \( u_t(\phi, \epsilon) \in \hat{W} \). In other words,
\[
u_t(\phi, \epsilon) = (\tilde{\phi}^s, \epsilon) + \tilde{h}(\tilde{\phi}^s, \epsilon) \text{ for some } (\tilde{\phi}^s, \epsilon) \in \tilde{N}_{sc}.
\]

As \( \tilde{\phi}^s = u^s_t(\phi, \epsilon) \in N_s \) and \( \epsilon \in (-\epsilon_s, \epsilon_s) \), we must have
\[
u^s_t(\phi, \epsilon) = \pi_u \tilde{h}(\tilde{\phi}^s, \epsilon) = h_s(\tilde{\phi}^s, \epsilon) = h_s(u^s_t(\phi, \epsilon), \epsilon).
\]

That is, \( u_t(\phi, \epsilon) \in W \). This proves (i).

(ii) Using Theorem III.1 of Krisztin, Walther and Wu [KWW99], the smoothness of the unstable manifold can be proved similarly. \( \square \)

4. Local Analysis. Under hypothesis (H3), we may assume, without loss of generality, that the constant \( K_2 \) defined in (2.13) is positive and that \( \delta_1 > 0 \) is chosen so that for \( |\phi^s| < \delta_1 \), \( |\phi^s| \leq \delta_1 \) and \( \epsilon \in [-\epsilon_1, \epsilon_1] \), we have
\[
|h_s(\phi^s, \epsilon)| \leq K_2 |\phi^s|^2, \quad (4.1)
\]
\[
|D_\phi h_s(\phi^s, \epsilon) \cdot \psi^s| \leq K_2 |\phi^s| |\psi^s|, \quad \psi^s \in C^s, \quad (4.2)
\]
\[
|D^2_{\phi, \epsilon} h_s(\phi^s, \epsilon) \cdot (\psi^s_1, \psi^s_2)| \leq K_2 |\psi^s_1| |\psi^s_2|, \quad \psi^s_i \in C^s, \quad i = 1, 2 \quad (4.3)
\]
and
\[ |h_u(\phi^n, \epsilon)| \leq K_2|\phi^n|^2, \] (4.4)
\[ |D_{\phi^n}h_u(\phi^n, \epsilon) \cdot \psi^n| \leq K_2|\phi^n| |\psi^n|, \ \psi^n \in C^u, \] (4.5)
\[ |D_{\phi^n}^2h_u(\phi^n, \epsilon) \cdot (\psi_1^n, \psi_2^n)| \leq K_2|\psi_1^n| \cdot |\psi_2^n|, \ \psi_i^n \in C^u, \ i = 1, 2. \] (4.6)

Denote \( u^*_n = u^*_n(\phi^n, \epsilon) \) and \( h_u = h_u(\phi^n, \epsilon) \) for \( |\phi^n| < \delta_1 \) and \( |\epsilon| \leq \epsilon_1 \). By (2.4), the definition of \( X_0 \) and the fact that \( X_0^* + X_0^* = X_0 \), we have
\[
T(-\alpha)X_0^*(\theta) = \begin{cases} 
X_0^*(\theta - \alpha) = -X_0^*(\theta - \alpha), & \theta - \alpha \leq 0, \\
X_0^*(0) + \int_0^{\theta-\alpha} L(T(\beta)X_0^*)d\beta, & \theta - \alpha > 0.
\end{cases}
\] (4.7)

Thus, (2.12) can be rewritten as follows
\[
h_u(\theta) = \int_{-\infty}^{\theta} |X_0^*(0) + \int_0^{\theta-\alpha} L(T(\beta)X_0^*)d\beta|g(u^*_n, \epsilon)d\alpha
- \int_{\theta}^{0} X_0^*(\theta - \alpha)g(u^*_n, \epsilon)d\alpha.
\]

By the smoothness of the local unstable manifold \( W^u_{loc}(\epsilon) \), differentiation of \( h_u(\theta) \) with respect to \( \theta \in [-\epsilon, 0] \) leads to
\[
\frac{d}{d\theta}h_u(\theta) = X_0(0)g(u^*_n, \epsilon) + L \left( \int_{-\infty}^{\theta} T(-\alpha)X_0^*g(u^*_{\alpha+\theta}, \epsilon)d\alpha \right)
- \int_{\theta}^{0} \frac{d}{d\theta} X_0^*(\theta - \alpha)g(u^*_n, \epsilon)d\alpha.
\] (4.3)

Similar to the proof of Proposition 3.2 in Chow and Deng [CD89], we have the following lemma.

**Lemma 4.1.** For \( \phi^n \in C^u \) with \( |\phi^n| < \delta_1, \theta \in [-\epsilon, 0] \) and \( \epsilon \in [-\epsilon_1, \epsilon_1] \), we have
\[
\frac{d}{d\theta}(D_{\phi^n}h_u \cdot \phi_1^n)(\theta) = \left( D_{\phi^n} \frac{d}{d\theta}h_u(\theta) \right) \cdot \phi_1^n, \ \phi_1^n \in C^u, \] (4.8)
\[
\frac{d}{d\theta}(D_{\phi^n}^2h_u \cdot (\phi_1^n, \phi_2^n))(\theta) = \left( D_{\phi^n} \frac{d}{d\theta}h_u(\theta) \right) \cdot (\phi_1^n, \phi_2^n), \ \phi_1^n, \phi_2^n \in C^u, \] (4.9)
\[
\frac{d}{d\theta} \left( D_{\phi^n} \frac{d}{d\theta}h_u \cdot \phi_1^n \right)(\theta) = \left( D_{\phi^n} \frac{d^2}{d\theta^2}h_u(\theta) \right) \cdot \phi_1^n, \ \phi_1^n \in C^u. \] (4.10)

Moreover, there exists a constant \( K_3 > 0 \) depending on \( \delta_1, \epsilon_1, K_1 \) and \( K_2 \) such that
\[
\left| \frac{d}{d\theta} \left[ (D_{\phi^n}^2h_u) \cdot (\phi_1^n, \phi_2^n) \right] \right| \leq K_3|\phi_1^n| |\phi_2^n|, \ \phi_1^n, \phi_2^n \in C^u, \] (4.11)
\[
\left| \frac{d}{d\theta} \left[ (D_{\phi^n} \frac{d}{d\theta}h_u) \cdot \phi_1^n \right] \right| \leq K_3|\phi^n| |\phi_1^n|, \ \phi_1^n \in C^u. \] (4.12)

By the smoothness of the stable and unstable manifolds (Theorem 3.2) and following the argument in the proof of Proposition 3.4 in Chow and Deng [CD89], we have the following lemma.
Lemma 4.2. Let $\phi^u \in C^u$ with $|\phi^u| < \delta_1, \theta \in [-r, 0]$ and $\epsilon \in [-\epsilon_1, \epsilon_1]$. Then there exists a constant $K_4 > 0$ depending on $\delta_1, \epsilon_1, K_1, K_2$ and $K_3$ such that

(i) $\left| \frac{d}{dt} T(t)h_u|_{t=0^+} = \frac{d}{dt} h_u - X_\theta g(h_u + \phi^u, \epsilon) \right|$;
(ii) $\left| T(t) \frac{d}{dt} T(t)h_u|_{t=0^+} \right| \leq K_4 e^{-\alpha t} |\phi^u|^2$, $t \geq 0$.

Define $\bar{\phi} = H(\phi, \epsilon), \phi \in B(\delta_1), |\epsilon| \leq \epsilon_1$, by

$$\bar{\phi}^s = \phi^s - h_u(\phi^u, \epsilon), \quad \bar{\phi}^u = \phi^u. \quad (4.13)$$

In terms of the new variable $\bar{\phi}$, we have

$$W^s_{\text{can}}(\epsilon) = \{ \bar{\phi} : \bar{\phi} \in H(B(\delta_1), \epsilon), \quad \bar{\phi}^s = 0 \}.$$

The inverse $H^{-1}$ of $H$ is given by

$$\phi^s = \bar{\phi}^s + h_u(\bar{\phi}^u, \epsilon), \quad \phi^u = \bar{\phi}^u.$$

The variation of constants formula (2.8) becomes

$$u^s_t(\bar{\phi}, \epsilon) = T(t)\bar{\phi}^s + \int_0^t T(t - \alpha)\bar{f}(\bar{u}_\alpha, \epsilon) d\alpha$$

$$u^u_t(\bar{\phi}, \epsilon) = e^{\lambda t} \bar{\phi}^u + \int_0^t e^{\lambda(t - \alpha)} X^u_0 \bar{g}(\bar{u}_\alpha, \epsilon) d\alpha$$

where

$$\bar{u}_t = \bar{u}_t(\bar{\phi}, \epsilon) = H(u_t(\phi, \epsilon), \epsilon), \quad \bar{g}(\bar{\phi}, \epsilon) = g(H^{-1}(\bar{\phi}, \epsilon), \epsilon),$$

and

$$\bar{f}(\bar{\phi}, \epsilon) = X^s_0 \bar{g}(\bar{\phi}, \epsilon) - \frac{d}{dt} T(t)h_u(\bar{\phi}^u, \epsilon)|_{t=0^+} - D_{\bar{\phi}^u} h_u(\bar{\phi}^s, \epsilon) \cdot [\lambda \bar{\phi}^u + X^u_0 \bar{f}(\bar{\phi}, \epsilon)]. \quad (4.15)$$

Lemma 4.3. There exist constants $0 < \delta_3 < \delta_2$ and $0 < \epsilon_3 < \epsilon_2$ and a map $F : B(\delta_2) \times [-\epsilon_3, \epsilon_3] \to L(C^*, L^\infty), L^\infty = L^\infty([-r, 0], X)$, such that $f(\bar{\phi}, \epsilon) = F(\bar{\phi}, \epsilon) \cdot \bar{\phi}^s$ for $\bar{\phi} \in B(\delta_3)$ and $|\epsilon| < \epsilon_3$, where $\bar{\phi}^s = P^s \bar{\phi}$. Furthermore

(i) $F$ is $C^1$;
(ii) if $\bar{\psi} \in C^s$, then $(F(\bar{\phi}, \epsilon) \cdot \bar{\psi}^s)(\theta)$ is $C^1$ in $\theta \in [-r, 0]$;
(iii) there exists a constant $K_5 > 0$ depending on $\delta_3, \epsilon_3$ and $K_i (i = 1, 2, 3, 4)$ such that for every $\phi \in B(\delta_3)$, $\epsilon \in [-\epsilon_3, \epsilon_3]$,

$$|F(\bar{\phi}, \epsilon) \cdot \bar{\psi}^s|_{L^\infty} \leq K_5 |\bar{\phi}| |\bar{\psi}^s|,$$

$$\sup_{-r \leq \theta \leq 0} \left| \frac{d}{d\theta} (F(\bar{\phi}, \epsilon) \cdot \bar{\psi}^s)(\theta) \right| \leq K_5 |\bar{\psi}^s|.$$

Proof. Claim: There exist $0 < \delta_3 < \delta_2$ and $0 < \epsilon_3 < \epsilon_2$ such that if $|\phi| < \delta_3, \phi \in C^u$ and $|\epsilon| \leq \epsilon_3$, then $f(\bar{\phi}, \epsilon)(\theta) = 0$ for $\theta \in [-r, 0]$.

Suppose $|\phi| < \delta_2$ and $|\epsilon| \leq \epsilon_2$. Define

$$t_0 = t_0(\phi, \epsilon) = \sup\{ t \geq 0 : u_t(\phi, \epsilon) \in H^{-1}(B(\delta_2), \epsilon) \}.$$
If $\delta_2 > 0$ is sufficiently small, then $t_0 > 2r$ for all $|\phi| \leq \delta_2$ and $|\epsilon| \leq \epsilon_2$. Since $H$ is near the identity map, there exists $0 < \epsilon_3 < \epsilon_2$ and $0 < \delta_3 < \delta_2$ such that

$$B(\delta_3) \subset H(B(\delta_2), \epsilon), \quad |\epsilon| \leq \epsilon_3.$$  

To prove the claim, we suppose that there exist $\phi^u_0 \in C^u$ with $|\phi^u_0| < \delta_3$, $|\epsilon_0| \leq \epsilon_3$ and $\theta_0 \in [-r, 0]$, such that

$$\tilde{f}(\phi^u_0, \epsilon_0)(\theta_0) \neq 0. \quad (4.16)$$

Let $\phi_0 \in H^{-1}(\phi^u_0, \epsilon)$. Then $\phi_0 \in W^u_{loc}(\epsilon_0)$ and $|\phi_0| \leq \bar{\delta}_2$. Let $\bar{u}_t = H(u_t(\phi_0, \epsilon_0), \epsilon_0)$, $-\theta_0 \leq t \leq t_0 - \theta_0$. Since $W^u_{loc}(\epsilon_0) \subset \{ \phi : \phi^s = P^s \phi = 0 \}$ and $\bar{u}_{t+\theta_0} = 0$ for $-\theta_0 \leq t \leq t_0 - \theta_0$, $\bar{u}_0 = H(\phi_0, \epsilon) = \phi^u_0$. By (4.14)

$$0 = \bar{u}_{t+\theta_0} = \int_{-\theta_0}^t T(t - \sigma)\tilde{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0) d\alpha.$$

Let $t = \theta_0 + \sigma$ and $\theta < \sigma \leq t_0$ with $\sigma < -\theta_0$ if $\theta_0 \neq 0$. By (2.5), we have

$$0 = \int_{-\theta_0}^{-\theta_0 + \sigma} |T(-\theta_0 + \sigma - \alpha)\tilde{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)| d\alpha$$

$$= \begin{cases} \int_{-\theta_0}^{-\theta_0 + \sigma} \tilde{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)(\sigma - \alpha) d\alpha & \text{if } -\theta_0 > \sigma > 0 \\ \int_{0}^{\sigma} T(\sigma - \alpha)|\tilde{f}(\bar{u}_{\alpha}, \epsilon_0)(0)| d\alpha & \text{if } \theta_0 = 0. \end{cases} \quad (4.17)$$

Dividing (4.17) by $\sigma$ and letting $\sigma \to 0^+$, we have

$$0 = \lim_{\sigma \to 0^+} \frac{1}{\sigma} \int_{-\theta_0}^{-\theta_0 + \sigma} |T(-\theta_0 + \sigma - \alpha)\tilde{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)| d\alpha$$

$$= \tilde{f}(\phi^u_0, \epsilon_0)(\theta_0), \quad \theta_0 \in [-r, 0],$$

which contradicts (4.16). This proves the claim.

Hence

$$\tilde{f}(\phi, \epsilon) = \tilde{f}(\phi^s + \phi^u, \epsilon) = \left( \int_0^1 D_{\phi^s} \tilde{f}(\alpha \phi^s + \phi^u, \epsilon) d\alpha \right) \cdot \phi^s.$$ 

Define

$$F(\phi, \epsilon) = \int_0^1 D_{\phi^s} \tilde{f}(\alpha \phi^s + \phi^u, \epsilon) d\alpha. \quad (4.18)$$

By (4.15), we have

$$D_{\phi^s} \tilde{f}(\alpha \phi^s + \phi^u, \epsilon) = X^u_{\alpha}(\tilde{L}_1 + \tilde{g}_1) - (D_{\phi^s} h) \cdot (X^u_{\alpha} \tilde{g}_1) + X_0 \tilde{g}_1,$$ 

$$\quad (4.19)$$

where $\tilde{L}_1 = D_{\phi^s} \tilde{L}(\alpha \phi^s + \phi^u)$ and $\tilde{g}_1 = D_{\phi^s} \tilde{g}(\alpha \phi^s + \phi^u, \epsilon)$. Thus $F(\phi, \epsilon) : B(\delta_3) \times [-\epsilon_3, \epsilon_3] \to L(C^s, L^\infty)$ is $C^1$ and $[F(\phi, \epsilon) \cdot \tilde{\psi}^s](\theta)$ is $C^1$ in $\theta \in [-r, 0]$ for all $\tilde{\psi}^s \in C^s$.

To prove (iii), we notice from (4.18) and (4.19) that $F(0, \epsilon) = 0$ for $\epsilon \in [-\epsilon_3, \epsilon_3]$. Thus, there exists $K_3 > 0$ depending on $\delta_2, \epsilon_3$ and $K_i (i = 1, 2, 3, 4)$ such that for $|\phi| < \delta_3, |\epsilon| < \epsilon_3$ and $\tilde{\psi}^s \in C^s$,

$$|F(\phi, \epsilon) \cdot \tilde{\psi}^s| \leq K_3 |\phi||\tilde{\psi}^s|.$$
Also, by Lemma 4.1, there exists $\tilde{K}_5 > 0$ depending on $\delta_3, \epsilon_3$ and $K_i (i = 1, 2, 3, 4)$ such that for $|\phi| < \delta_3, |\epsilon| < \epsilon_3$, and $\tilde{\psi} \in C^s$,

$$\left| \frac{d}{d\theta} [F(\phi, \epsilon) \cdot \tilde{\psi}^s] (\theta) \right| \leq \tilde{K}_5 |\tilde{\psi}^s|, \quad -r \leq \theta \leq \theta.$$

Choose $K_5 = \max\{\tilde{K}_5, \tilde{K}_5\}$, we prove (iii). \hfill \square

From (4.15), we have

$$T(t)F(\phi, \epsilon) = T(t)X_{\phi} \tilde{g} - T(t) \frac{d}{dt} T(\tau)h_\alpha |_{\tau = \theta} - T(t)D_{\phi} h_\alpha \cdot (\lambda \tilde{\phi}^u + X_0 \tilde{g}), \quad (4.20)$$

where $\tilde{g} = \tilde{g}(\phi, \epsilon)$ and $h_\alpha = h_\alpha(\tilde{\phi}^u, \epsilon)$. Since $C^s$ is closed, we have $T(t)X_{\phi} \tilde{g} + \tilde{g} \in C^s$ for $t > r$ and $(D_{\phi} h_\alpha) \cdot (\lambda \tilde{\phi}^u + X_0 \tilde{g}) \in C^s$. Thus, by (2.7), (4.20) and Lemma 4.2, there exists a constant $K_6 > 0$ depending on $\delta_3, \epsilon_3$ and $K_i (i = 1, \ldots, 5)$ such that for $\phi \in B(\delta_3)$ and $|\epsilon| \leq \epsilon_3$,

$$|T(t)\tilde{f}(\phi, \epsilon)| \leq K_6 e^{-\mu t}|\phi||\tilde{\phi}^s|, \quad t \geq 0. \quad (4.21)$$

By Lemma 4.3, we can rewrite (4.14) as follows

$$\tilde{u}_t^s(\phi, \epsilon) = T(t)\tilde{\phi}^u + \int_0^t T(t - \alpha)F(\alpha, \epsilon, \tilde{u}_s(\phi, \epsilon))d\alpha$$

$$\tilde{u}_t^u(\phi, \epsilon) = e^{\lambda t} \tilde{\phi}^u + \int_0^t e^{\lambda(t - \alpha)}\tilde{g}(\alpha, \epsilon)d\alpha. \quad (4.22)$$

Denote $\tilde{v}_t = D_{\phi} \tilde{u}_t(\phi, \epsilon)$. Denote $\tilde{v}_t = D_{\phi} \tilde{u}_t(\phi, \epsilon)$. Differentiating (4.22) with respect to $\phi$ in $L^\infty$, we can see that $\tilde{v}_t^s = D_{\phi} \tilde{u}_t^s(\phi, \epsilon)$ and $\tilde{v}_t^u = D_{\phi} \tilde{u}_t^u(\phi, \epsilon)$ satisfy the following variational equations:

$$\tilde{v}_t^s = T(t)P^s + \int_0^t T(t - \alpha)[D_{\phi}F(\alpha, \epsilon, \tilde{u}_s(\phi, \epsilon))]d\alpha, \quad 0 \leq t \leq t_0,$$

$$\tilde{v}_t^u = e^{\lambda t} P^u + \int_0^t e^{\lambda(t - \alpha)}D_{\phi} \tilde{g}(\alpha, \epsilon, \tilde{u}_s(\phi, \epsilon))d\alpha, \quad 0 \leq t \leq t_0. \quad (4.23)$$

**Lemma 4.4.** There is a constant $K_7 > 0$ depending on $\delta_3, \epsilon_3$ and $K_i (i = 1, \ldots, 6)$ such that if $\tilde{u}_t(\phi, \epsilon) \in B(\delta_3)$ for all $0 \leq t \leq t_0$ where $t_0 > 0$ is any given constant, then

$$|\tilde{u}_t^s(\phi, \epsilon)| \leq K_7 |\tilde{\phi}^s| e^{-\mu t}, \quad 0 \leq t \leq t_0.$$

**Proof.** From (4.14), we have

$$|\tilde{u}_t^s(\phi, \epsilon)| \leq |T(t)\tilde{\phi}^s| + \int_0^t |T(t - \alpha)\tilde{f}(\alpha, \epsilon, \tilde{u}_s(\phi, \epsilon))|d\alpha.$$

By (2.7) and (4.21) and the assumption that $\tilde{u}_t \in B(\delta_3)$, we obtain

$$|\tilde{u}_t^s(\phi, \epsilon)| \leq K_1 e^{-\mu t}|\tilde{\phi}^s| + \delta_3 K_6 \int_0^t e^{-\mu(t - \alpha)}|\tilde{u}_s(\phi, \epsilon)|d\alpha.$$
Let \( x(t) = e^{\mu t} |u_t^\ast(\bar{\phi}, \epsilon)| \). Then
\[
x(t) \leq K_1 |\bar{\phi}| + \delta_3 K_6 \int_0^t x(\alpha) d\alpha.
\]
The Gronwall’s Inequality implies that
\[
x(t) \leq K_1 |\bar{\phi}| e^{\delta_3 K_6 t} \quad \text{for} \quad 0 \leq t \leq t_0.
\]
Choosing \( K_7 = K_1 e^{\delta_3 K_6 t_0} \), we obtain the desired inequality. \( \square \)

Denote \( \tilde{\mu} = \frac{\mu - \lambda}{2} \), \( \tilde{\lambda} = \frac{\mu + \lambda}{2} \). Then \( \tilde{\mu} < \mu, \tilde{\lambda} > \lambda, \) and \( -\mu + 2\tilde{\mu} + \tilde{\lambda} = 0 \). Let \( \delta_4 > 0 \) be a small constant such that
\[
\delta_4 \leq \min \left\{ \delta_3, \frac{1}{4K_1 K_2 K_7 (1/\tilde{\mu} + 1/\lambda) + K_5 (\mu - \tilde{\mu})} \right\}.
\]

**Lemma 4.5.** Let \( \bar{\lambda}, \tilde{\mu}, \) and \( \delta_4 \) be as above and \( \bar{\lambda} \in (0, \lambda) \). If \( u_t(\bar{\phi}, \epsilon) \in B(\delta_4) \) for \( 0 \leq t \leq t_0 \) where \( t_0 \) is any given constant, then
\[
|D_{\bar{\alpha}} u_t^\ast(\bar{\phi}, \epsilon)| \leq 2 e^{-\tilde{\mu} t}, \quad t \in [0, t_0], \quad \epsilon \in [-\epsilon_3, \epsilon_3],
\]
\[
|D_{\bar{\alpha}} u_t^\ast(\bar{\phi}, \epsilon)| \leq 2 e^{\tilde{\lambda} t}, \quad t \in [0, t_0], \quad \epsilon \in [-\epsilon_3, \epsilon_3],
\]
\[
|D_{\bar{\alpha}} u_t^\ast(\bar{\phi}, \epsilon)| \leq \frac{1}{2} e^{\tilde{\lambda} t}, \quad t \in [0, t_0], \quad \epsilon \in [-\epsilon_3, \epsilon_3].
\]

**Proof.** Let \( \tilde{V} \) be a subset of \( C \) with a metric
\[
d(w_{1t}, w_{2t}) = \max_{0 \leq t \leq t_0} |w_{1t} - w_{2t}|.
\]

Define a subset \( V \) in \( \tilde{V} \) as follows:
\[
V = \{ w_t : w_t = w_t^* + w_t^u \in \tilde{V}, w_0 = w_0^* + w_0^u = \text{the identity map in } C, \}
\]
\[
|w_t^*| \leq 2 e^{-\tilde{\mu} t}, \quad |w_t^u| \leq 2 e^{\tilde{\lambda} t}, \quad 0 \leq t \leq t_0 \}.
\]

Clearly \( V \) is a closed subset of \( \tilde{V} \). Let \( \Phi : V \to \tilde{V}, \ w_t = \Phi(w_t) \), be defined by
\[
\tilde{w}_t^* = T(t) P^s + \int_0^t T(t - \alpha) [D_{\bar{\alpha}} F(\bar{\pi}_t(\bar{\phi}, \epsilon), \epsilon) \cdot (\bar{\pi}_t, \bar{\pi}_t^*(\bar{\phi}, \epsilon)) + F(\bar{\pi}_t(\bar{\phi}, \epsilon), \epsilon) \cdot \bar{\pi}_t^u] d\alpha,
\]
\[
\tilde{w}_t^u = e^{\lambda t} P^u + \int_0^t e^{\lambda (t - \alpha)} X_t^u D_\bar{\alpha} G(\bar{\pi}_t(\bar{\phi}, \epsilon), \epsilon) \cdot \bar{\pi}_t^u d\alpha.
\]

By using (2.7), (4.18), (4.19), (4.2), Lemmas 4.3 and 4.4, we have
\[
|\tilde{w}_t^*| \leq e^{-\tilde{\mu} t} + 2 K_1 K_2 K_7 \delta_4 e^{-\mu t} \int_0^t e^{-\tilde{\mu} \alpha} d\alpha + 2 K_1 K_5 \delta_4 e^{-\mu t} \int_0^t e^{(\mu - \tilde{\mu}) \alpha} d\alpha
\]
\[
+ 2 K_1 K_2 K_7 \delta_4 e^{-\mu t} \int_0^t e^{\lambda \alpha} d\alpha.
\]
Since \(-\mu < -\bar{\mu}, -\mu + \bar{\lambda} = -2\bar{\mu} < -\bar{\mu}\), we have
\[
|\overline{w}_t^s| \leq e^{-\bar{\mu} t} \left( 1 + \frac{2K_1K_2K_7\delta_4}{\bar{\mu}} + \frac{2K_1K_6\delta_4}{\mu - \bar{\mu}} + \frac{2K_1K_2K_7\delta_4}{\bar{\lambda}} \right).
\]
Since \(\delta_4 \leq 1/[4K_1(K_2K_7(1/\bar{\mu} + 1/\bar{\lambda}) + K_4/(\mu - \bar{\mu}))]\), it follows that
\[
|\overline{w}_t^s| \leq 2e^{-\bar{\mu} t}.
\]

Similarly, since \(\bar{\lambda} > \lambda\), we have
\[
|\overline{w}_t^s| \leq e^{\lambda t} + 2K_2\delta_2^2 e^{\lambda t} \int_0^t e^{-(\lambda + \bar{\mu})\alpha} d\alpha + 2K_2\delta_1^2 e^{\lambda t} \int_0^t e^{(\bar{\lambda} - \lambda)\alpha} d\alpha
\leq e^{\lambda t} \left( 1 + \frac{2K_2\delta_2^2}{\lambda + \bar{\mu}} + \frac{2K_2\delta_1^2}{\lambda - \bar{\lambda}} \right)
\leq 2e^{\lambda t}
\]
provided \(\delta_4 \leq 1/2(K_2(1/\lambda + \bar{\mu}) + 1/(\bar{\mu} - \lambda))]^{\frac{1}{2}}\). It follows that \(\Phi\) maps \(V\) into itself. For \(w_{1t}, w_{2t} \in V\), define another metric \(\overline{d}\) as follows
\[
\overline{d}(w_{1t}, w_{2t}) = \max_{0 \leq t \leq T} (e^{\bar{\mu} t}|\overline{w}_t^w - \overline{w}_t^u| + e^{-\bar{\lambda} t}|\overline{w}_t^w - \overline{w}_t^u|).
\]
Then \((V, \overline{d})\) is a complete space. Let \(\overline{w}_{1t} = \Phi(w_{1t}), \overline{w}_{2t} = \Phi(w_{2t}).\) Then
\[
|\overline{w}_{1t}^w - \overline{w}_{2t}^w| \\
\leq \int_0^t ((K_1K_2K_7\delta_4 e^{-\mu t} + K_1K_5\delta_4 e^{-\mu(t - \alpha)}))|\overline{w}_{1\alpha}^w - \overline{w}_{2\alpha}^w| d\alpha
+ K_1K_2K_7\delta_4 e^{-\mu t} \int_0^t |\overline{w}_{1\alpha}^w - \overline{w}_{2\alpha}^w| d\alpha
= \int_0^t ((K_1K_2K_7\delta_4 e^{-\mu t} + K_1K_5\delta_4 e^{-\mu(t - \alpha)})e^{-\bar{\mu} \alpha}(e^{\bar{\mu} \alpha}|\overline{w}_{1\alpha}^w - \overline{w}_{2\alpha}^w|) d\alpha
+ K_1K_2K_7\delta_4 e^{-\mu t} \int_0^t e^{\lambda \alpha} |\overline{w}_{1\alpha}^w - \overline{w}_{2\alpha}^w| d\alpha
\leq (K_1K_2K_7\delta_4 e^{-\mu t} \int_0^t e^{-\bar{\mu} \alpha} d\alpha + K_1K_5\delta_4 e^{-\mu t} \int_0^t e^{(\bar{\mu} - \bar{\mu}) \alpha} d\alpha) \cdot \overline{d}(\overline{w}_{1t}, \overline{w}_{2t})
+ K_1K_2K_7\delta_4 e^{-\mu t} \int_0^t e^{\lambda \alpha} d\alpha \cdot \overline{d}(\overline{w}_{1t}, \overline{w}_{2t})
\leq K_1\delta_4 \left[ K_2K_7 \left( \frac{1}{\bar{\mu}} + \frac{1}{\bar{\lambda}} \right) + \frac{K_5}{\mu - \bar{\mu}} \right] e^{-\bar{\mu} t} \cdot \overline{d}(\overline{w}_{1t}, \overline{w}_{2t}).
\]
Hence
\[
e^{\bar{\mu} t}|\overline{w}_{1t}^w - \overline{w}_{2t}^w| \leq K_1\delta_4 \left[ K_2K_7 \left( \frac{1}{\bar{\mu}} + \frac{1}{\bar{\lambda}} \right) + \frac{K_5}{\mu - \bar{\mu}} \right] \cdot \overline{d}(\overline{w}_{1t}, \overline{w}_{2t}).
\]
Similarly,
\[
e^{-\bar{\lambda} t}|\overline{w}_{1t}^w - \overline{w}_{2t}^w| \leq K_2\delta_2^2 \left( \frac{1}{\lambda + \bar{\mu}} + \frac{1}{\lambda - \bar{\lambda}} \right) \cdot \overline{d}(\overline{w}_{1t}, \overline{w}_{2t}).
\]
Thus, 
\[ d(\bar{w}_{1t}, \bar{w}_{2t}) \]
\[ = \max_{0 \leq t \leq t_0} (e^{\beta t}|\bar{w}_{1t} - \bar{w}_{2t}| + e^{-\lambda t}|\bar{w}_{10} - \bar{w}_{20}|) \]
\[ \leq \left\{ K_1 \delta_4 \left[ K_2 K_7 \left( \frac{1}{\mu} + \frac{1}{\mu} \right) + K_5 \delta_1^2 \left( \frac{1}{\lambda} + \frac{1}{\lambda - \lambda} \right) \right] \right\} \cdot d(\bar{w}_{1t}, \bar{w}_{2t}) \]
\[ \leq \left( \frac{1}{4} + \frac{1}{4} \right) d(\bar{w}_{1t}, \bar{w}_{2t}) \]
\[ = \frac{1}{2} d(\bar{w}_{1t}, \bar{w}_{2t}) \]

which implies that \( \Phi : V \to V \) is a contractive mapping under the new topology \((V, \tilde{d})\). Hence, \( \Phi \) has a unique fixed point, say \( \bar{w}_t \in V \), such that \( \bar{w}_t = \Phi(\bar{w}_t) \). By the uniqueness of the solution of (4.23), we have 
\[ \bar{w}_t = D_{\varphi} \bar{w}_t(\varphi, \epsilon), \quad \bar{w}_0 = D_{\varphi} \bar{w}_0(\varphi, \epsilon). \]

Thus, we have established the first two estimates. To show the third estimate, we differentiate the second equation in (4.22) and obtain 
\[ D_{\varphi} \bar{w}_t(\varphi, \epsilon) = e^{\lambda t} + \int_0^t e^{\lambda(t-s)} X_0 u D_{\varphi} \bar{y}(\bar{u}_t(\varphi, \epsilon), \epsilon) \cdot D_{\varphi} \bar{w}_t(\varphi, \epsilon) da, \]
\[ D_{\varphi} \bar{w}_0(\varphi, \epsilon) = 1. \]

Let \( t_1 = \sup\{ t : 0 \leq t \leq t_0, D_{\varphi} \bar{w}_t(\varphi, \epsilon) \geq 0 \} \). Then \( t_1 > 0 \). We will show that \( t_1 = t_0 \). Suppose \( t_1 < t_0 \). Then \( D_{\varphi} \bar{w}_t(\varphi, \epsilon) = 0 \). We have 
\[ \frac{dD_{\varphi} \bar{w}_t(\varphi, \epsilon)}{dt} = \lambda D_{\varphi} \bar{w}_t(\varphi, \epsilon) + X_0 u D_{\varphi} \bar{y}(\bar{u}_t(\varphi, \epsilon), \epsilon) \cdot D_{\varphi} \bar{w}_t(\varphi, \epsilon) \]
\[ \geq \tilde{\lambda} D_{\varphi} \bar{u}_t(\varphi, \epsilon) - |X_0 u D_{\varphi} \bar{y}(\bar{u}_t(\varphi, \epsilon), \epsilon) \cdot D_{\varphi} \bar{u}_t(\varphi, \epsilon)|. \]

It follows that 
\[ D_{\varphi} \bar{w}_t(\varphi, \epsilon) \geq \frac{1}{2} e^{\lambda t}, \quad 0 \leq t \leq t_1. \]

Hence \( D_{\varphi} \bar{w}_t(\varphi, \epsilon) \geq \frac{1}{2} e^{\lambda t_1} \neq 0 \), a contradiction. So \( t_1 = t_0 \) and this completes the proof. \( \square \)

Note that in terms of the variable \( \bar{\varphi} \), the local stable and unstable manifolds are given by 
\[ W^u_{loc}(\epsilon) = \{ \bar{\varphi} : \bar{\varphi}^0 = \bar{h}_s(\bar{\varphi}^0, \epsilon), |\bar{\varphi}^0| < \delta_4 \}, \]
\[ W^s_{loc}(\epsilon) = \{ \bar{\varphi} : \bar{\varphi}^0 = 0, |\bar{\varphi}^0| < \delta_4 \}, \]
where \( \bar{h}_s \) is \( C^3 \), \( \bar{h}_s(0, \epsilon) = 0, \epsilon \in [-\epsilon_3, \epsilon_3] \) and \( D_{\varphi} \bar{h}_s(0, 0) = 0 \). For every \( \epsilon \in [-\epsilon_3, \epsilon_3] \), we define 
\[ \Omega = \Omega(\delta_4, \rho, \epsilon) = \{ \bar{\varphi} : |\bar{\varphi}^0| < \delta_4/K_7, |(\bar{\varphi}^0, \bar{\varphi}^0 - \bar{h}_s(\bar{\varphi}^0, \epsilon)| < \rho \} \subset B(\delta_4), \]
\[ \Omega^+ = \Omega^+(\delta_4, \rho, \epsilon) = \{ \bar{\varphi} : \bar{\varphi} \in \Omega(\delta_4, \rho, \epsilon), 0 < \langle \bar{\varphi}^0, \bar{\varphi}^0 - \bar{h}_s(\bar{\varphi}^0, \epsilon) \rangle < \rho \}, \]
\[ \Omega^- = \Omega^-(\delta_4, \rho, \epsilon) = \{ \bar{\varphi} : \bar{\varphi} \in \Omega(\delta_4, \rho, \epsilon), -\rho < \langle \bar{\varphi}^0, \bar{\varphi}^0 - \bar{h}_s(\bar{\varphi}^0, \epsilon) \rangle < 0 \}, \]
where \( \langle \cdot ; \cdot \rangle \) is defined in section 2. Note that since \( W^u_{loc}(\epsilon) \) has codimension one and \( \Omega^+ \cap \Omega^- = \emptyset \), \( \Omega = \Omega^+ \cup W^u_{loc}(\epsilon) \cup \Omega^- \), we must have
Lemma 4.6. Consider $\bar{u}_t(\bar{\phi}, \epsilon)$ satisfying (4.22) in $B(\delta_4)$ and $0 < \rho < \delta_4$, $\delta_4 < 1/(2K_2)$. If $\bar{\phi} \in \Omega^+ \cup \Omega^-$, then there exists $\tau = \tau(\bar{\phi}, \epsilon) > 0$ such that

$$
\bar{u}_t^n(\bar{\phi}, \epsilon) - \bar{u}_t^n(\bar{\phi}^s + \bar{h}_s(\bar{s}^s, \epsilon), \epsilon) = \begin{cases} 
\delta_4/2 & \text{if } \bar{\phi} \in \Omega^+, \quad |\epsilon| \leq \varepsilon_3 \\
-\delta_4/2 & \text{if } \bar{\phi} \in \Omega^-, \quad |\epsilon| \leq \varepsilon_3.
\end{cases}
$$

Furthermore, if $\bar{\phi} \in \Omega^+ \cup \Omega^-$ and $|\epsilon| \leq \varepsilon_3$, then

$$
\frac{1}{\lambda} \ln \frac{\delta_4}{4|\phi^s_{\lambda}, \phi^u - \bar{h}_s(\bar{s}^s, \epsilon)|} \leq \tau \leq \frac{1}{\lambda} \ln \frac{\delta_4}{|\phi^s_{\lambda}, \phi^u - \bar{h}_s(\bar{s}^s, \epsilon)|},
$$

$$
|D_\bar{\phi} \tau(\bar{\phi}, \epsilon)| \leq \frac{8}{\lambda \delta_4 - 8K_2 \delta_4} e^{\lambda \tau(\bar{\phi}, \epsilon)}.
$$

Proof. If $\bar{\phi} \in \Omega^+ \cup \Omega^-$ $\subset \Omega \setminus W^s_{\text{loc}}(\epsilon)$, then the solution cannot stay in $B(\delta_4)$ for all $t > 0$. If $|\bar{s}^s| < \frac{\delta_4}{2K_7}$, then Lemma 4.4 implies that

$$
|\bar{u}_t^n(\bar{\phi}, \epsilon)| \leq K_7 \cdot \frac{\delta_4}{2K_7} \cdot e^{-\mu t} \leq \frac{\delta_4}{2} \quad \text{for } 0 \leq t \leq t_0,
$$

where $t_0 > 0$ is such that the solution $\bar{u}_t(\bar{\phi}, \epsilon) \in B(\delta_4)$ for all $0 \leq t \leq t_0$. Hence, $\bar{u}_t(\bar{\phi}, \epsilon)$ has to leave $B(\delta_4)$ through either $\bar{u}_t^n(\bar{\phi}, \epsilon) = \delta_4$ or $\bar{u}_t^n(\bar{\phi}, \epsilon) = -\delta_4$. Thus,

$$
\bar{\tau} = \inf \{t > 0 : |\bar{u}_t^n(\bar{\phi}, \epsilon)| = \delta_4\}
$$

is well defined.

Let

$$
\Delta_t(\bar{\phi}, \epsilon) = (\phi^s_{\lambda}, \bar{u}_t^n(\bar{\phi}, \epsilon) - \bar{u}_t^n(\bar{s}^s + \bar{h}_s(\bar{s}^s, \epsilon), \epsilon)), \quad 0 \leq t \leq \bar{\tau}.
$$

Note that $\Delta_t$ is $C^2$. Since $\bar{\phi} \in \Omega$,

$$
|\Delta_0(\bar{\phi}, \epsilon)| = |(\phi^s_{\lambda}, \bar{u}_t^n(\bar{\phi}, \epsilon) - \bar{h}_s(\bar{s}^s, \epsilon))| < \rho < \delta_4/4.
$$

Note that by (4.1), if $|\bar{s}^s| < \delta_4$, then $|\bar{h}_s(\bar{s}^s, \epsilon)| \leq K_2 \delta_4^2 < \delta_4/2$. Since $\bar{u}_t(\bar{\phi}^s + \bar{h}_s(\bar{s}^s, \epsilon), \epsilon) \in W^s_{\text{loc}}(\epsilon)$, we have

$$
|\Delta_t(\bar{\phi}, \epsilon)| \geq |(\phi^s_{\lambda}, \bar{u}_t^n(\bar{\phi}, \epsilon))| - |(\phi^s_{\lambda}, \bar{u}_t^n(\bar{s}^s + \bar{h}_s(\bar{s}^s, \epsilon), \epsilon))| > \delta_4 - \delta_4/2 = \delta_4/4.
$$

Thus, by the Intermediate Value Theorem,

$$
\tau(\bar{\phi}, \epsilon) = \inf \{t > 0 : \Delta_t(\bar{\phi}, \epsilon) = \delta_4/4\}
$$

is well defined for $\bar{\phi} \in \Omega^+ \cup \Omega^-$. We shall prove that

$$
\Delta_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon) = \begin{cases} 
\delta_4/2 & \text{if } \bar{\phi} \in \Omega^+ \\
-\delta_4/2 & \text{if } \bar{\phi} \in \Omega^-.
\end{cases}
$$

We only prove $\Delta_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon) = \delta_4/2$ for $\bar{\phi} \in \Omega^+$, the other case can be treated similarly.

By the way of contradiction, suppose $\Delta_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon) = -\delta_4/2$. We have

$$
\langle \phi^s_{\lambda}, \bar{u}_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon) \rangle = -\delta_4/2 + \langle \phi^s_{\lambda}, \bar{u}_{\tau(\bar{\phi}, \epsilon)}(\bar{s}^s + \bar{h}_s(\bar{s}^s, \epsilon), \epsilon) \rangle < -\delta_4/4.
$$

(4.25)
Define
\[ d_t(\tilde{\phi}, \epsilon) = \langle \phi_\lambda^s, \tilde{u}_t^u(\tilde{\phi}, \epsilon) - \bar{h}_a(u_t^*(\tilde{\phi}, \epsilon), \epsilon) \rangle. \]

Since \( \tilde{\phi} \in \Omega^+ \), we have
\[ d_0(\tilde{\phi}, \epsilon) = \langle \phi_\lambda^s, \tilde{\phi}^u - \bar{h}_a(\tilde{\phi}^s, \epsilon) \rangle = \Delta_0(\tilde{\phi}, \epsilon) > 0. \]

On the other hand, by (4.25), we have
\[ d_t(\tilde{\phi}, \epsilon) < -\delta_4/4 - \langle \phi_\lambda^s, \bar{h}_a(u_t^*(\tilde{\phi}, \epsilon), \epsilon) \rangle < 0. \]

Thus the Intermediate Value Theorem implies that there exists \( \tau_0, 0 < \tau_0 < \tau(\tilde{\phi}, \epsilon) \), such that
\[ d_{\tau_0}(\tilde{\phi}, \epsilon) = 0 \]
which implies that \( \tilde{u}_{\tau_0}(\tilde{\phi}, \epsilon) \in W^{s}_{\text{loc}}(\epsilon) \). Therefore,
\[ \tilde{u}_t(\tilde{\phi}, \epsilon) \in W^{s}_{\text{loc}}(\epsilon) \]
which contradicts \( \tilde{\phi} \in \Omega \setminus W^s_{\text{loc}}(\epsilon) \). This proves (4.24).

Since \( \Delta_t(\phi, \epsilon) \) is \( C^2 \) and
\[ \frac{\partial}{\partial t} \Delta_t(\phi, \epsilon) = \lambda \Delta_t(\phi, \epsilon) + \bar{g}(\tilde{u}_t(\phi, \epsilon), \epsilon) - \bar{g}(\tilde{u}_t(\phi^s + \bar{h}_a(\phi^s, \epsilon), \epsilon), \epsilon), \]
we have
\[ \frac{\partial}{\partial t} \Delta_t(\phi, \epsilon)|_{t=\tau(\tilde{\phi}, \epsilon)} \geq \frac{\lambda \delta_4}{2} - 4K_2 \delta_4^2 > 0, \quad \tilde{\phi} \in \Omega^+. \] (4.26)

Applying the Implicit Function Theorem to
\[ \Delta_t(\phi, \epsilon) = \delta_4/2, \quad \phi \in \Omega, \quad \epsilon \in [-\epsilon_2, \epsilon_2], \]
it follows that \( \tau(\tilde{\phi}, \epsilon) \) is \( C^2 \). Moreover, by Lemma 4.5, we have
\[ \delta_4/2 = \langle \phi_\lambda^s, \tilde{u}_{\tau(\tilde{\phi}, \epsilon)}(\tilde{\phi}, \epsilon) - \bar{u}_{\tau(\tilde{\phi}, \epsilon)}(\tilde{\phi}^s + \bar{h}_a(\phi^s, \epsilon), \epsilon) \rangle \]
\[ \leq 2e^{\lambda \tau(\tilde{\phi}, \epsilon)} |\langle \phi_\lambda^s, \tilde{\phi}^u - \bar{h}_a(\phi^s, \epsilon) \rangle| \]
Therefore,
\[ \tau(\tilde{\phi}, \epsilon) \geq \frac{1}{\lambda} \ln \frac{\delta_4}{4|\langle \phi_\lambda^s, \phi^u - \bar{h}_a(\phi^s, \epsilon) \rangle|}. \]

Similarly we can show that
\[ \tau(\tilde{\phi}, \epsilon) \leq \frac{1}{\lambda} \ln \frac{\delta_4}{|\langle \phi_\lambda^s, \phi^u - \bar{h}_a(\phi^s, \epsilon) \rangle|}. \]

Let \( \tilde{\phi} \in \Omega^+ \cup \Omega^- \). Differentiating (4.24) and using the Chain Rule, we have
\[ \frac{\partial}{\partial t} \Delta_t(\tilde{\phi}, \epsilon)|_{t=\tau(\tilde{\phi}, \epsilon)} \cdot D_{\tilde{\phi}} \tau(\tilde{\phi}, \epsilon) + D_{\tilde{\phi}} \Delta_t(\tilde{\phi}, \epsilon)|_{t=\tau(\tilde{\phi}, \epsilon)} = 0. \]

Therefore, by (4.26) and Lemma 4.4, we obtain
\[ |D_{\tilde{\phi}} \tau(\tilde{\phi}, \epsilon)| \leq \frac{8}{\lambda \delta_4 - 8K_2 \delta_4^2} e^{\lambda \tau(\tilde{\phi}, \epsilon)}. \]

This completes the proof. \( \square \)
Lemma 4.7. Let $\tilde{\phi} \in \Omega$, $|\epsilon| \leq \epsilon_3$ and $t_0 > 2r$ be as in the proof Lemma 4.3. Then there exists a constant $K_8 > 0$ depending on $\delta_3, \epsilon_3$ and $K_i (i = 1, \ldots, 7)$ such that if a solution $\tilde{u}_t(\tilde{\phi}, \epsilon)$ of (4.22) is in $B(\delta_4)$ for $0 \leq t \leq t_0$, then $\tilde{u}_t(\tilde{\phi}, \epsilon)$ is differentiable in $t \in (r, t_0)$ and satisfies

$$ \left| \frac{d}{dt} \tilde{u}_t^*(\tilde{\phi}, \epsilon) \right| \leq K_8 |\tilde{\phi}| e^{-\mu t}, \quad r < t < t_0,$$

where $\frac{d}{dt}$ is taken in $L^\infty$.

Proof. Let $t > r$. By (4.22) we have

$$\tilde{u}_t^*(\theta) = T(t + \theta)\tilde{\phi}^*(0) + \int_0^{t+\theta} T(t + \theta - \alpha) [F(\tilde{u}_\alpha, \epsilon) \cdot \tilde{u}_\alpha^* (0)] d\alpha$$

$$+ \int_{t+\theta}^t [F(\tilde{u}_\alpha, \epsilon) \cdot \tilde{u}_\alpha^*](t + \theta - \alpha) d\alpha, \quad \theta \in [-r, 0].$$

By (4.18) and (4.19), we have

$$\int_{t+\theta}^t [F(\tilde{u}_\alpha, \epsilon) \cdot \tilde{u}_\alpha^*](t + \theta - \alpha) d\alpha = \int_{t+\theta}^t \tilde{F}(\alpha, t + \theta - \alpha) \cdot \tilde{u}_\alpha^* d\alpha,$$

where

$$\tilde{F}(\alpha, \theta) \cdot \tilde{\psi}^* = [F(\tilde{u}_\alpha, \epsilon) \cdot \tilde{\psi}^*](\theta) - X_0(\theta) \int_0^1 \tilde{g}_1(\tilde{u}_\alpha, \epsilon, \beta)d\beta \cdot \tilde{\psi}^*.$$

Lemma 4.3 implies that $\tilde{F}(\alpha, \theta)$ is $C^1$ in $\theta \in [-r, 0]$ and

$$\left| \frac{\partial}{\partial \theta} \tilde{F}(\alpha, \theta) \cdot \tilde{\psi}^* \right| \leq K_5 |\tilde{\psi}^*|, \quad |\tilde{F}(\alpha, \theta) \cdot \tilde{\psi}^*| \leq K_5 |\tilde{\psi}^*| \quad (4.27)$$

for $0 \leq \alpha \leq t_0, \theta \in [-r, 0]$ and $\tilde{\psi}^* \in C^*$. Thus

$$\frac{d}{dt} \tilde{u}_t^*(\theta) = L(T(t(\theta)\tilde{\phi}^*) + [F(\tilde{u}_{t+\theta}, \epsilon) \cdot \tilde{u}_{t+\theta}^*](0) + \int_0^{t+\theta} L(T(t + \theta - \alpha)F(\tilde{u}_\alpha, \epsilon) \cdot \tilde{u}_\alpha^* d\alpha$$

$$+ \tilde{F}(t, \theta) \cdot \tilde{u}_t^* - \tilde{F}(t + \theta, 0) \cdot \tilde{u}_{t+\theta}^* - \int_{t+\theta}^t \frac{\partial}{\partial \theta} \tilde{F}(\alpha, t + \theta - \alpha) \cdot \tilde{u}_\alpha^* d\alpha.$$

This, together with (2.4), (4.27), Lemmas 4.1 and 4.4, implies the desired estimate and completes the proof. □

Define a function $\ell : B(\delta_4) \times [-\epsilon_3, \epsilon_3] \to R$ by

$$\ell(\tilde{\phi}_1, \tilde{\phi}_2, \epsilon) = \max_{i=1,2} \{ \langle \phi_i^*, \tilde{\phi}_i^* - \tilde{h}_s(\tilde{\phi}_i^*, \epsilon) \rangle \}.$$
Lemma 4.8. Let $\tilde{\phi} \in \Omega^+(\delta_4, \rho, \epsilon, \rho < \frac{\lambda}{4})$ and $\epsilon \in [-\epsilon_3, \epsilon_3]$. There exist constants $K_0 > 0$ and $a > 0$ depending on $\delta_4, \epsilon_3$ and $K_i (i = 1, \ldots, 8)$ such that if $\bar{\phi}_1, \bar{\phi}_2 \in \Omega^+(\delta_4, \rho, \epsilon) \in [-\epsilon_3, \epsilon_3]$, then

$$|\bar{u}_{\tau(\bar{\phi}_1, \epsilon)}(\bar{\phi}_1, \epsilon) - \bar{u}_{\tau(\bar{\phi}_2, \epsilon)}(\bar{\phi}_2, \epsilon)| \leq K_0 |\ell(\bar{\phi}_1, \bar{\phi}_2, \epsilon)|^a |\bar{\phi}_1 - \bar{\phi}_2|. \quad (4.28)$$

**Proof.** For $\bar{\phi}_i = \bar{\phi}_i^s + \bar{\phi}_i^u \in \Omega^+(\delta_4, \rho, \epsilon)$, $i = 1, 2$, define

$$\tilde{\phi}^s(\theta, \epsilon) = (1 - \theta)\bar{\phi}_1^s + \theta\bar{\phi}_2^s,$$

$$\tilde{\phi}^u(\theta, \epsilon) = \bar{\phi}_1^u - h_s(\bar{\phi}_1^s, \epsilon) + h_s(\tilde{\phi}^s(\theta, \epsilon), \epsilon), \quad 0 \leq \theta \leq 1$$

and

$$\tilde{\phi}^s(\theta, \epsilon) = \bar{\phi}_1^s,$$

$$\tilde{\phi}^u(\theta, \epsilon) = (1 - \theta)\bar{\phi}_1^u(1 + \epsilon) + \theta\bar{\phi}_2^u, \quad 0 \leq \theta \leq 1.$$

Then

$$\tilde{\phi}^s(\theta, \epsilon) + \tilde{\phi}^u(\theta, \epsilon) \in \Omega^+(\delta_4, \rho, \epsilon), \quad \tilde{\phi}^s(\theta, \epsilon) + \tilde{\phi}^u(\theta, \epsilon) \in \Omega^+(\delta_4, \rho, \epsilon), \quad 0 \leq \theta \leq 1.$$

By the Chain Rule, Lemmas 4.5, 4.6, and 4.7, it follows that

$$|D_\phi \bar{u}_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon)| = \left| \frac{d}{d\theta} \bar{u}_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon) \cdot D_\phi \tau(\bar{\phi}, \epsilon) \right| + \left| \frac{d}{d\theta} \bar{u}_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon)_{t=\tau(\bar{\phi}, \epsilon)} \right|$$

$$\leq \left| K_0 |\ell(\bar{\phi})|^a e^{-\lambda \tau(\bar{\phi}, \epsilon)} \right| + \left| \frac{8K_0}{\lambda - 8K_2\delta_4} \cdot \bar{\phi}^s(\bar{\phi}, \epsilon) \right| + 2|\bar{u}_{\tau(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon)| \lambda \tau(\bar{\phi}, \epsilon)$$

$$\leq 8K_0 \left( \frac{\delta_4}{\lambda - 8K_2\delta_4} \right)^{[\mu - \lambda]/\lambda} |\bar{\phi}^s(\bar{\phi}, \epsilon)|^{[\mu - \lambda]/\lambda} \left[ (\bar{\phi}^s(\bar{\phi}, \epsilon))^\mu / \lambda \right].$$

Also, by (4.1) and (4.2), we have

$$\left| \frac{d}{d\theta} (\tilde{\phi}^s(\theta, \epsilon) + \tilde{\phi}^u(\theta, \epsilon)) \right| = \left| \tilde{\phi}^s - \tilde{\phi}^u \right| \leq (1 + K_2\delta_4^2)|\tilde{\phi}^s - \tilde{\phi}^u|$$

and

$$\left| \frac{d}{d\theta} (\tilde{\phi}^s(\theta, \epsilon) + \tilde{\phi}^u(\theta, \epsilon)) \right| = \left| \tilde{\phi}^s - \tilde{\phi}^u(1, \epsilon) \right|$$

$$\leq \left| \tilde{\phi}^s - \tilde{\phi}^u \right| + \left| \bar{h}_s(\bar{\phi}^s, \epsilon) - \bar{h}_s(\phi^s, \epsilon) \right|$$

$$\leq (1 + K_2\delta_4^2)|\tilde{\phi}_1 - \tilde{\phi}_2|.$$
Thus

\[ |\tilde{u}^u_{\tau(\tilde{\phi}_1,\epsilon)}(\tilde{\phi}_1, \epsilon) - \tilde{u}^u_{\tau(\tilde{\phi}_2,\epsilon)}(\tilde{\phi}_2, \epsilon)| \]

\[ = |\tilde{u}^u_{\tau(\tilde{\phi}_1 + \tilde{\phi}_4,\epsilon)}(\tilde{\phi}_1^* + \tilde{\phi}_4^*, \epsilon) - \tilde{u}^u_{\tau(\tilde{\phi}_2 + \tilde{\phi}_4,\epsilon)}(\tilde{\phi}_2^* + \tilde{\phi}_4^*, \epsilon)| \]

\[ + |\tilde{u}^u_{\tau(\tilde{\phi}_2 + \tilde{\phi}_4,\epsilon)}(\tilde{\phi}_2^* + \tilde{\phi}_4^*, \epsilon) - \tilde{u}^u_{\tau(\tilde{\phi}_2 + \tilde{\phi}_4,\epsilon)}(\tilde{\phi}_2^* + \tilde{\phi}_4^*, \epsilon)| \]

\[ = \left| \int_0^1 \frac{d}{d\theta} \tilde{u}^u_{\tau(\tilde{\phi}^*(\theta, \epsilon) + \tilde{\phi}^*(\theta, \epsilon))}(\tilde{\phi}^*(\theta, \epsilon) + \tilde{\phi}^*(\theta, \epsilon), \epsilon) \right| \cdot \frac{d}{d\theta} (\tilde{\phi}^*(\theta, \epsilon) + \tilde{\phi}^*(\theta, \epsilon)) d\theta \]

\[ \leq \bar{K}_9(\delta_4) \int_0^1 |(\tilde{\phi}_1^*(\theta, \epsilon), \tilde{\phi}^*(\theta, \epsilon) - \bar{h}_4(\tilde{\phi}^*(\theta, \epsilon), \epsilon))|^{1\alpha} d\theta \cdot |\tilde{\phi}_1 - \tilde{\phi}_2| \]

Next, by (4.1) and (4.24),

\[ |\tilde{u}^u_{\tau(\tilde{\phi}_1,\epsilon)}(\tilde{\phi}_1, \epsilon) - \tilde{u}^u_{\tau(\tilde{\phi}_2,\epsilon)}(\tilde{\phi}_2, \epsilon)| \]

\[ = |\tilde{\phi}_4/2 + \tilde{u}^u_{\tau(\tilde{\phi}_1,\epsilon)}(\tilde{\phi}_1 + \bar{h}_4(\tilde{\phi}_4, \epsilon), \epsilon) - \tilde{\phi}_4/2 + \tilde{u}^u_{\tau(\tilde{\phi}_2,\epsilon)}(\tilde{\phi}_2 + \bar{h}_4(\tilde{\phi}_4, \epsilon), \epsilon)| \]

\[ = \left| \bar{h}_4(u_{\tau(\tilde{\phi}_1,\epsilon)}(\tilde{\phi}_4^*, \epsilon) + \bar{h}_4(\tilde{\phi}_4^*, \epsilon), \epsilon) - \bar{h}_4(u_{\tau(\tilde{\phi}_2,\epsilon)}(\tilde{\phi}_4^*, \epsilon) + \bar{h}_4(\tilde{\phi}_4^*, \epsilon), \epsilon) \right| \]

\[ \leq K_2 \delta_4^2 |\tilde{u}^u_{\tau(\tilde{\phi}_1,\epsilon)}(\tilde{\phi}_4^*, \epsilon) - \tilde{u}^u_{\tau(\tilde{\phi}_2,\epsilon)}(\tilde{\phi}_4^*, \epsilon)| \]

\[ \leq 2K_2 \delta_4^2 \bar{K}_9(\delta_4) |\bar{l}(\tilde{\phi}_1, \tilde{\phi}_2, \epsilon)|^{1\alpha} |\tilde{\phi}_1 - \tilde{\phi}_2| \]

Let \( K_9 = 2(1 + K_2 \delta_4^2) \bar{K}_9(\delta_4) \). Then we obtain

\[ |\tilde{u}_{\tau(\tilde{\phi}_1,\epsilon)}(\tilde{\phi}_1, \epsilon) - \tilde{u}_{\tau(\tilde{\phi}_2,\epsilon)}(\tilde{\phi}_2, \epsilon)| \leq K_0 |\bar{l}(\tilde{\phi}_1, \tilde{\phi}_2, \epsilon)|^{1\alpha} |\tilde{\phi}_1 - \tilde{\phi}_2| \]

This completes the proof. \( \Box \)

5. The Šil’nikov Map. In this section, we shall define a map in a small neighborhood of the hyperbolic equilibrium which is closely related to the Poincaré map but is somehow different. The idea of construction of the map is due to Šil’nikov [Si68], so we call it a Šil’nikov map. We shall show that the Šil’nikov map is Lipschitzian with a small Lipschitz constant.

5.1. Construction of the map \( \pi^1 \). Let \( \delta_4 \) and \( \phi < \delta_4/4 \) be fixed. Denote

\[ \rho_0 = \min\{2^{1+a} (1 + K_2 \delta_4^2) K_0(\delta_4)\}^{-1/a}, \rho/2 \}, \quad (5.1) \]

\[ B(\rho_0) = \{ \phi : |\tilde{\phi}^*| < \rho_0, |\tilde{\phi}^u| < \rho_0 \}, \quad (5.2) \]

\[ S(\delta_4, \epsilon) = \{ \phi : (\tilde{\phi}^*_4, \tilde{\phi}^u - h_4(\tilde{\phi}^*, \epsilon)) = \delta_4/4, |\tilde{\phi}^*| < \delta_4/2 \}. \quad (5.3) \]
Since $W^s_{\text{loc}}(\epsilon)$ and $\Omega(\delta_4, \rho, \epsilon)$ continuously depend on $\epsilon \in [-\epsilon_3, \epsilon_4]$, there exist a small $0 < \epsilon_4 < \epsilon_3$ such that $B(\rho_0) \cap W^s_{\text{loc}}(\epsilon) \neq \emptyset$ and $B(\rho_0) \subset \Omega(\delta_4, \rho, \epsilon)$ for $\epsilon \in [-\epsilon_4, \epsilon_4]$. Define a map $\tilde{\pi}^1 : B(\rho_0) \times [-\epsilon_4, \epsilon_4] \to C$ by

$$\tilde{\pi}^1(\bar{\phi}, \epsilon) = \begin{cases} \bar{u}_{\sigma(\bar{\phi}, \epsilon)}(\bar{\phi}, \epsilon) / \delta_4 & \text{if } \bar{\phi} \in \Omega^+(\delta_4, \rho, \epsilon) \cap B(\rho_0), \\ \delta_4/2 & \text{if } \bar{\phi} \in \{\Omega - \Omega^+\} \cap B(\rho_0). \end{cases}$$

(5.4)

Notice that for each $\epsilon \in [-\epsilon_4, \epsilon_4]$, $\tilde{\pi}^1$ maps $B(\rho_0)$ into $S(\delta_4, \epsilon)$.

**Lemma 5.1.** $\tilde{\pi}^1(\bar{\phi}, \epsilon)$ is continuous in $(\bar{\phi}, \epsilon) \in B(\rho_0) \times [-\epsilon_4, \epsilon_4]$ and is Lipschitzian continuous in $\bar{\phi} \in B(\rho_0)$.

**Proof.** The continuity of $\tilde{\pi}^1$ follows from Lemmas 4.5 and 4.6. To show $\tilde{\pi}^1$ is Lipschitzian, denote

$$\Delta_1 = \langle \phi^*_1, \bar{\phi}^*_1 - h_4(\bar{\phi}^*_1, \epsilon) \rangle > 0, \quad \Delta_2 = \langle \phi^*_1, \bar{\phi}^*_2 - h_4(\bar{\phi}^*_2, \epsilon) \rangle \leq 0$$

and let

$$\bar{\phi}_2(\theta) = \theta[h_4(\bar{\phi}^*_2, \epsilon) + \bar{\phi}^*_2 - h_4(\bar{\phi}^*_1, \epsilon)] + (1 - \theta)\bar{\phi}^*_2, \quad 0 \leq \theta \leq 1.$$

We have

$$|\tilde{\pi}^1(\bar{\phi}_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}_2, \epsilon)| \leq |\tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}_2(1), \epsilon) - \tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}_2(0), \epsilon)| + |\tilde{\pi}^1(\bar{\phi}^*_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}^*_2, \epsilon)|.$$  

(5.5)

Since $\langle \phi^*_1, \bar{\phi}_2(1) - h_4(\bar{\phi}^*_2, \epsilon) \rangle = \langle \phi^*_1, \bar{\phi}^*_1 - h_4(\bar{\phi}^*_1, \epsilon) \rangle = \Delta_1 > 0, \bar{\phi}_2 + \bar{\phi}_2(1) \in \Omega^+(\delta_4, \rho, \epsilon)$. Lemma 4.8 implies that

$$|\tilde{\pi}^1(\bar{\phi}^*_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}^*_2, \epsilon)| \leq K_9(\delta_4)|l(\bar{\phi}^*_1, \bar{\phi}^*_1, \epsilon)| + |\bar{\phi}^*_1 - \bar{\phi}^*_2(1)|.$$  

(5.6)

Note that $\langle \phi^*_1, \bar{\phi}_2(1) - h_4(\bar{\phi}^*_2, \epsilon) \rangle = \Delta_1 > 0$, but $\langle \phi^*_1, \bar{\phi}_2(0) - h_4(\bar{\phi}^*_2, \epsilon) \rangle = \langle \phi^*_1, \bar{\phi}^*_1 - h_4(\bar{\phi}^*_1, \epsilon) \rangle = \Delta_2 \leq 0$, there must exist $0 \leq \theta \leq 1$ such that $\langle \phi^*_1, \bar{\phi}_2(\theta) - h_4(\bar{\phi}^*_2, \epsilon) \rangle = 0$ and $\langle \phi^*_1, \bar{\phi}_2(\theta) - h_4(\bar{\phi}^*_2, \epsilon) \rangle > 0$ for $\bar{\theta} < \theta \leq 1$, which implies that $\bar{\phi}^*_2 + \bar{\phi}_2(\theta) \in \Omega^+(\delta_4, \rho, \epsilon)$ for $\bar{\theta} < \theta \leq 1$. By the definition of $\tilde{\pi}^1$, $\tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}_2(\bar{\theta}), \epsilon) = \tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}^*_2, \epsilon) = \delta_4/2$. Thus, Lemma 4.8 implies that

$$|\tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}_2(1), \epsilon) - \tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}^*_2, \epsilon)|$$

$$= |\tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}_2(1), \epsilon) - \tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}_2(\bar{\theta}), \epsilon)|$$

$$\leq K_9(\delta_4)|l(\bar{\phi}^*_1, \bar{\phi}^*_2, \epsilon)|^n|\bar{\phi}_2(1) - \bar{\phi}_2(\bar{\theta})|$$

$$= K_9(\delta_4)|l(\bar{\phi}^*_1, \bar{\phi}^*_2, \epsilon)|^n[(1 - \bar{\theta})h_4(\bar{\phi}^*_2, \epsilon) - h_4(\bar{\phi}^*_1, \epsilon)] + |\bar{\phi}^*_1 - \bar{\phi}^*_2|$$

$$\leq (1 + K_2 \delta_4)K_9(\delta_4)|l(\bar{\phi}^*_1, \bar{\phi}^*_2, \epsilon)|^n(|\bar{\phi}^*_1 - \bar{\phi}^*_2| + |\bar{\phi}^*_1 - \bar{\phi}^*_2|).$$  

(5.7)

Therefore, by (5.6) and (5.7), we have

$$|\tilde{\pi}^1(\bar{\phi}^*_1 + \bar{\phi}^*_2, \epsilon) - \tilde{\pi}^1(\bar{\phi}^*_2 + \bar{\phi}^*_2, \epsilon)|$$

$$\leq 2(1 + K_2 \delta_4)K_9(\delta_4)|l(\bar{\phi}^*_1, \bar{\phi}^*_2, \epsilon)|^n(|\bar{\phi}^*_1 - \bar{\phi}^*_2| + |\bar{\phi}^*_1 - \bar{\phi}^*_2|).$$

(5.8)
which means that \( \pi^1 \) is Lipschitzian. □

For \( \epsilon \in [-\epsilon_4, \epsilon_4] \), define
\[
W^u_+(\epsilon) = \{ \phi : \text{there exists } t > 0 \text{ and } \psi \in W^u_{\text{loc}}(\epsilon) \text{ with } \\
\langle \phi^*_\lambda, \psi^u - h_u(\psi^u, \epsilon) \rangle > 0 \text{ and } \phi = u_t(\psi, \epsilon) \},
\]
and \( \phi_1(\epsilon) = W^u_+(\epsilon) \cap \Sigma(\delta_4/2, \epsilon) \), \( \epsilon \in [-\epsilon_4, \epsilon_4] \).

where
\[
\Sigma(\delta_4/2, \epsilon) = \{ \phi : \langle \phi^*_\lambda, \tilde{\phi}^u - \tilde{h}_u(\tilde{\phi}^u, \epsilon) \rangle = \delta_4/2, \tilde{\phi} = H(\phi, \epsilon) \}.
\]

Fix \( 0 < \rho_0 < \delta_4/2 \). Since \( H \) is near the identity map, there exists \( 0 < \rho_1 < \rho_0 \) and \( 0 < \epsilon_5 < \epsilon_4 \) such that
\[
B(\rho_1) \subset H^{-1}(\Omega(\delta_4, \rho_0, \epsilon), \epsilon), \quad \epsilon \in [-\epsilon_5, \epsilon_5].
\]

Let \( \phi_0 \in W^s_{\text{loc}}(0) \) with \( |\phi_0| < \rho_1 \) be fixed. By the continuity property of \( W^s_{\text{loc}}(\epsilon) \) in \( \epsilon \), for every \( 0 < \rho < \text{dist}(\partial B(\rho_1), \phi_0) \), there exists \( 0 < \epsilon_6(\rho) < \epsilon_5 \) such that
\[
B(\phi_0, \rho) \cap W^s_{\text{loc}}(\epsilon) \neq \emptyset, \quad \epsilon \in [-\epsilon_6, \epsilon_6].
\]

where \( B(\phi_0, \rho) = \{ \phi \in C : |\phi^s - \phi^1_0| < \rho, |\tilde{\phi}^u - \phi^u_0| < \rho \} \). Define \( H \times I : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to C \times R \) by
\[
(H \times I)(\phi, \epsilon) = (H(\phi, \epsilon), \epsilon)
\]

and \( \pi^1 \times I : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to C \times R \) by
\[
(\pi^1 \times I)(\phi, \epsilon) = (\pi^1(\phi, \epsilon), \epsilon).
\]

Thus \( \pi^1 : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to C \) given by
\[
\pi^1(\phi, \epsilon) = H^{-1}((\pi^1 \times I) \circ (H \times I)(\phi, \epsilon), \epsilon)
\]
is well-defined. Notice that by (4.2) and (4.13),
\[
|D_\phi H(\tilde{\phi}, \epsilon)| \leq 2 + K_2\delta_4^2, \\
|D_\phi H^{-1}(\tilde{\phi}, \epsilon)| \leq 2 + K_2\delta_4^2
\]

for \( (\phi, \epsilon) \in B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \). Thus, by Lemma 4.8, \( \pi^1 \) is also continuous and Lipschitzian in \( \phi \) and \( \phi_1(\epsilon) \) is the unique intersection point of \( W^u_+(\epsilon) \) and \( \Sigma(\delta_4/2, \epsilon) \).

The above discussion can be summarized as a lemma.

**Lemma 5.2.** The map \( \pi^1 : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to C \) defined in (5.6) satisfies:

(a) If \( |\epsilon| < \epsilon_6(\rho), |\phi - \phi_0| < \rho \) and \( \langle \phi^*_\lambda, \phi^s - h_u(\phi^u, \epsilon) \rangle > 0 \), then \( \pi^1(\phi, \epsilon) \) is the intersection point of \( \Sigma(\delta_4/2, \epsilon) \) and the solution orbit of (2.1) with the initial value \( \phi \) and parameter \( \epsilon \). If \( \langle \phi^*_\lambda, \phi^s - h_u(\phi^u, \epsilon) \rangle \leq 0 \), then \( \pi^1(\phi, \epsilon) = \phi_1(\epsilon) \).

(b) \( \pi^1(\phi, \epsilon) \) is continuous in \( (\phi, \epsilon) \) and is Lipschitzian in \( \phi \) for each fixed \( \epsilon \).

**5.2. Construction of the map \( \pi^2 \).** Let \( t_0 > 0 \) be the time such that \( u_{t_0}(\phi_1(0), 0) = \phi_0 \) and \( \bar{\Gamma}_0 = \{ \phi : \phi = u_t(\phi_1(0), 0), 0 \leq t \leq t_0 \} \subset \Gamma_0 \). Define \( \pi^2 : B(\phi_1(0), \rho_1) \times [-\epsilon_6, \epsilon_6] \to B(\rho_1) \) by
\[
\pi^2(\phi, \epsilon) = u_{t_0}(\phi, \epsilon).
\]
By the differentiability of solutions to (2.1) with respect to initial values (see Theorem 3.1), there exist $0 < \rho_2 < \rho_1$ and $0 < \epsilon_7 < \epsilon_6$ such that

$$|D(\phi, \epsilon)\pi^2(\phi_1, \epsilon)| \leq |D(\phi, \epsilon)\pi^2(\phi_1(0), 0)| + 1$$

for $(\phi_1, \epsilon) \in B(\phi_1(0), \rho_2) \times [-\epsilon_7, \epsilon_7]$. Let

$$K_10 = |D(\phi, \epsilon)\pi^2(\phi_1(0), 0)| + 1. \quad (5.15)$$

Then for every $0 < \rho < \rho_2$ and $0 < \epsilon \leq \epsilon_7$, $\pi^2$ is continuous and Lipschitzian with Lipschitz constant $K_{10}$.

5.3. Construction of the map $\pi$. Since $\phi_1(\epsilon)$ is continuous in $\epsilon \in [-\epsilon_6(\rho), \epsilon_6(\rho)]$ for every $0 < \rho < \text{dist } (\phi_0, \overline{\partial B(\rho_1)})$, there exists $0 < \epsilon_8(\rho) < \epsilon_6(\rho)$ such that

$$|\phi_1(\epsilon) - \phi_1(0)| < K_9(\delta_4)\rho^{\gamma + 1}, \quad \epsilon \in [-\epsilon_6(\rho), \epsilon_8(\rho)]. \quad (5.16)$$

Let

$$\rho_3 = \min\left\{\left[\frac{1}{2K_{10}K_9(\delta_4)}\right]^{-\frac{1}{\gamma}}, \left[\frac{\rho_2}{2K_9(\delta_4)}\right]^{-\frac{1}{\gamma}}, \text{dist } (\partial B(\rho_1), \phi_0)\right\}, \quad (5.17)$$

$$\epsilon_9 = \min\{\epsilon_7, \epsilon_6(\rho_2)\}. \quad (5.18)$$

Then $2K_{10}K_9(\delta_4)\rho_3^\gamma < 1/2$ and the composite map

$$\pi(\cdot, \epsilon) = \pi^2(\pi^1(\cdot), \epsilon, \epsilon) \quad (5.19)$$

mapping $B(\phi_0, \rho_3) \times [-\epsilon_8, \epsilon_8]$ into $B(\phi_0, \rho_3/2)$. By Lemma 5.2 and the definition of $\pi^2$, $\pi$ is continuous and is Lipschitzian in $\phi$. The Lipschitz constant is $K_{10}K_9(\delta_4)\rho_3^\gamma < 1/2$ by (5.17) and independent of $\epsilon \in [-\epsilon_8, \epsilon_8]$. Thus, $\pi$ is continuous for every $\epsilon \in [-\epsilon_8, \epsilon_8]$. We now summarize the properties of $\pi$ in the following result:

**Theorem 5.3.** (i) $\pi$ is continuous and $\pi(\cdot, \epsilon) : B(\phi_0, \rho_3) \to B(\phi_0, \rho_3)$ is a contraction with a contraction constant less than $1/2$ uniformly in $\epsilon \in [-\epsilon_8, \epsilon_8]$.

(ii) If $\phi \in B(\phi_0, \rho_3)$ and $\langle \phi^\ast, \phi^\ast - h_s(\phi^\ast, \epsilon) \rangle > 0$, then $\pi(\phi, \epsilon)$ is on the orbit of (2.1) containing $\phi$. If $\langle \phi^\ast, \phi^\ast - h_s(\phi^\ast, \epsilon) \rangle \leq 0$, then $\pi(\phi, \epsilon)$ is a constant map with $\pi(\phi, \epsilon) = \pi^2(\phi_1(\epsilon), \epsilon)$.

(iii) For every $\epsilon \in [-\epsilon_8, \epsilon_8]$,

$$B(\phi_0, \rho_3) \cap \{\phi : \langle \phi^\ast, \phi^\ast - h_s(\phi^\ast, \epsilon) \rangle > 0\} \neq \emptyset,$$

$$B(\phi_0, \rho_3) \cap \{\phi : \langle \phi^\ast, \phi^\ast - h_s(\phi^\ast, \epsilon) \rangle \leq 0\} \neq \emptyset.$$

6. The Proof of the Main Results. To prove Theorem 2.1, we need the following lemmas.

**Lemma 6.1.** There exist $\epsilon_9 > 0$ and neighborhoods $N_1$ of $\{0\}$ and $N_2$ of $\tilde{\Gamma}_0$, respectively, such that

(i) $N(\Gamma_0) = N_1 \cup N_2$ is a neighborhood of the homoclinic orbit $\Gamma_0$;

(ii) if $\gamma$ is an orbit of (2.1) at $\epsilon \in [-\epsilon_9, \epsilon_9]$ satisfying $\gamma \cap N_1 \neq \emptyset$, $\gamma \subseteq N(\Gamma_0)$, then

$$\langle \phi^\ast, \phi^\ast - h_s(\phi^\ast, \epsilon) \rangle > 0$$

for every $\phi \in \gamma \cap N_1$;
(iii) if $\phi \in N_2$ and $\epsilon \in [-\epsilon_9, \epsilon_9]$, then $u_t(\phi, \epsilon) \in B(\phi_0, \rho_3)$ for some $t > 0$.

**Proof.** Let $\delta_4, \rho$ and $\Omega(\delta_4, \rho, \epsilon)$ be as in Lemma 4.5, and $\epsilon_4$ and $S(\delta_4, \epsilon)$ (see (5.3)) be as in Lemma 5.1. Define

\[
\tilde{N}(\epsilon) = \{ \phi : |\phi'| < \delta_4, -\delta_4/4 < \langle \phi_x', \phi^n - h_x, \epsilon \rangle < \delta_4 \},
\]

\[
\partial \tilde{N}^- = \{ \phi : |\phi'| \leq \delta_4, -\delta_4/4 = \langle \phi_x', \phi^n - h_x(\phi^*, \epsilon) \rangle \}
\]

for $\epsilon \in [-\epsilon_4, \epsilon_4]$. Then

\[
\Omega(\delta_4, \rho, \epsilon) \subset \tilde{N}(\epsilon), \quad S(\delta_4, \epsilon) \subset \tilde{N}(\epsilon), \quad \epsilon \in [-\epsilon_4, \epsilon_4].
\]

By Lemma 5.1, for every $\phi \in \tilde{N}(\epsilon) \cap \Omega^-(\delta_4, \rho, \epsilon)$, there exists $t > 0$ such that $u_t(\phi, \epsilon) \in \partial \tilde{N}^-(\epsilon)$ for $\epsilon \in [-\epsilon_4, \epsilon_4]$. Denote

\[
N_1 = H^{-1}(\tilde{N}(0), 0), \quad \partial N_1^- = H^{-1}(\partial \tilde{N}^-(0), 0).
\]

Since $H$ is continuous in $\epsilon$, there exists $0 < \epsilon_{10} < \epsilon_9$ such that

\[
g(x, \epsilon_{10}) \in N_1 \supset H^{-1}(\Omega(\delta_4, \rho_0, \epsilon_{10}), \epsilon_{10}) \supset H^{-1}(B(\rho_0), \epsilon_{10}) \supset B(\phi_0, \rho_3),
\]

for $\epsilon \in [-\epsilon_{10}, \epsilon_{10}]$, where $B(\rho_0)$, $B(\phi_0, \rho_3)$, $S(\delta_4, \epsilon)$ and $\Sigma(\epsilon)$ are defined in section 5. This implies the following properties:

(A) if $\phi \in N_1 \cap H^{-1}(\Omega^-(\delta_4, \rho_0, \epsilon), \epsilon) \setminus W_{loc}(\epsilon)$ and $\epsilon \in [-\epsilon_{10}, \epsilon_{10}]$, then $u_t(\phi, \epsilon)$ will leave $N_2$ through $\partial N_1^-$;

(B) $\partial N_1^-$ is closed with $\Gamma_0 \cap \partial N_1^- = \emptyset$.

Thus, for $t_0$ and $\tilde{\Gamma}_0$ defined in section 5 and for every $\tilde{\phi} = u_t(\phi_1(0), 0) \in \tilde{\Gamma}_0$ there exists $\bar{\rho} = \bar{\rho}(t) > 0$ and $\bar{\epsilon} = \bar{\epsilon}(t) > 0$ such that

\[
B(\tilde{\phi}, \bar{\rho}) \cap \partial N_1^- = \emptyset
\]

and if $\phi \in B(\tilde{\phi}, \bar{\rho}), \epsilon \in [-\bar{\epsilon}, \bar{\epsilon}]$, then

\[
u_{t_0-\epsilon}(\phi, \epsilon) \in B(\phi_0, \rho_3).
\]

Note that $\bigcup_{0 \leq t \leq t_0} B(\tilde{\phi}, \bar{\rho})$ is an open cover of $\tilde{\Gamma}_0$. By the compactness of $\tilde{\Gamma}_0$, there exists a finite open cover $N_2 = \bigcup_{i=1}^{n} B(\tilde{\phi}_i, \bar{\rho}_i)$, where

\[
\tilde{\phi}_i = u_{t_i}(\phi_1(0), 0), \quad \bar{\rho}_i = \bar{\rho}_i(t), \quad 0 \leq t_1 < t_2 < \cdots < t_n \leq t_0.
\]

Define $\epsilon_{10} = \min_{1 \leq i \leq n} \{ \bar{\epsilon}(t_i) \}$ and $N(\Gamma_0) = N_1 \cup N_2$. Then $N(\Gamma_0)$ is an open neighborhood of $\tilde{\Gamma}_0 \cup \{0\}$, which implies (i). Since $N_2 \cap \partial N_1^- = \emptyset$ and $\partial N_1^- \subset \partial N_1 \subset \partial N(\Gamma_0)$, by property (A), we obtain (ii). (iii) follows from the definition of $B(\phi, \bar{\rho})$. $\square$
Lemma 6.2. There exist $0 < \epsilon_9 < \epsilon_{10}$ and $\hat{\rho} > 0$ such that

(i) if $(\phi, \epsilon) \in B(\phi_0(0), \hat{\rho}) \times [-\epsilon_9, \epsilon_9]$, then $u_t(\phi, \epsilon) \in N_2$ for all $0 \leq t \leq t_0$;

(ii) if $\epsilon \in [-\epsilon_9, \epsilon_9]$, then $|\phi_1(\epsilon) - \phi_0(0)| < \hat{\rho}/4$;

(iii) if $\phi_*(\epsilon)$ is the unique fixed point of $\pi$ and $\epsilon \in [-\epsilon_9, \epsilon_9]$, then

$$
|\langle \phi_0^*, \phi_0^* - h_2(\phi_0^*, \epsilon) \rangle| < \min\left\{ \frac{1}{4}K_{10}\hat{\rho}, \rho_3 \right\},
$$

where $K_{10}$ and $ho_3$ are given in (5.15) and (5.17), respectively.

Proof. By Lemma 6.1, if $\phi \in B(\phi_0(0), \hat{\rho}(0))$ and $\epsilon \in [-\epsilon_9, \epsilon_9]$, then $u_t(\phi, \epsilon)$ is defined for all $0 \leq t \leq t_0$. We claim that there exist $0 < \epsilon_{10} < \epsilon_9$ and $0 < \hat{\rho} < \hat{\rho}(0)$ such that for every $(\phi, \epsilon) \in B(\phi_0(0), \hat{\rho}) \times [-\epsilon_{10}, \epsilon_{10}]$,

$$u_t(\phi, \epsilon) \in N_2, \quad 0 \leq t \leq t_0.
$$

Suppose the contrary. Then there exists a sequence $\{ (\phi_k, \epsilon_k, t_k) \}_{k}$ with $(\phi_k, \epsilon_k) \to (\phi_0(0), 0)$ as $k \to \infty$ and $0 \leq t_k \leq t_0$ for every $k = 1, 2, \ldots$, such that

$$u_{t_k}(\phi_k, \epsilon_k) \in \partial N_2.
$$

Since $[0, t_0]$ is closed, without loss of generality, assume that $t_k \to t_0 \in [0, t_0]$ as $k \to \infty$. Thus, $\lim_{k \to \infty} u_{t_k}(\phi_k, \epsilon_k) = u_{t_0}(\phi_0(0), 0) \in \tilde{\Gamma}_0$, which contradicts $\tilde{\Gamma}_0 \subset N_2$. This proves (i), (ii) and (iii) follow from the continuity of $\phi_*(\epsilon)$ and $\phi_1(\epsilon)$ in $\epsilon$. □

Lemma 6.3. Let $N(\Gamma_0)$ be the neighborhood of the homoclinic orbit $\Gamma_0$ as in Lemma 6.1 and $\bar{\epsilon}$ and $\phi_*(\epsilon)$ be as in Lemma 6.2. If $\gamma$ is a periodic or homoclinic orbit of equation (2.1) at $\epsilon \in [-\epsilon_9, \epsilon_9]$ and $\gamma \subset N(\Gamma_0)$, then $\phi_*(\epsilon) \in \gamma$.

Proof. If $\gamma$ is a homoclinic orbit, then $\phi_1(\epsilon) \in \gamma$ and $\phi_2(\epsilon) = \pi^2(\phi_1(\epsilon), \epsilon) \in B(\phi_0, \rho_3)$, where $\rho_3$ is defined in (5.17) and $\epsilon \in [-\epsilon_9, \epsilon_9]$. Since $\gamma \subset N(\Gamma_0)$, by Lemma 6.1(ii), $\gamma \cap N_1 \cap H^{-1}(\Omega^-(\delta_4, \rho, \epsilon), \epsilon) = \emptyset$. Theorem 5.3(b) then implies that

$$(\pi)^k(\phi_2(\epsilon), \epsilon) \in \gamma, \quad k = 0, 1, 2, \ldots,
$$

where $(\pi)^k(\phi_2(\epsilon), \epsilon) = \pi((\pi)^{k-1}(\phi_2(\epsilon), \epsilon))$ is the $k$th iterate of $\pi$. Since $\gamma \cap W^u_{loc}(\epsilon) \neq \emptyset$, there exists $K \geq 0$ such that $(\pi)^K(\phi_2(\epsilon), \epsilon) \in W^u_{loc}(\epsilon)$. Once again by Theorem 5.3,

$$
\pi((\pi)^K(\phi_2(\epsilon), \epsilon)) = \pi^2(\phi_1(\epsilon), \epsilon) = \phi_2(\epsilon),
$$

which means $\phi_2(\epsilon)$ is a fixed point of $\pi^{K+1}$, thus $\phi_2(\epsilon) = \phi_*(\epsilon)$. The case that $\gamma$ is a periodic orbit can be proved similarly. □

Proof of Theorem 2.1. Necessity. Suppose $W^u_{loc}(\epsilon) \subset N(\Gamma_0)$ for some $\epsilon \in [-\epsilon_9, \epsilon_9]$, then Lemma 4.1(iii) implies that $W^u_{loc}(\epsilon) \cap H^{-1}(\Omega^- (\delta_4, \rho, \epsilon), \epsilon) = \emptyset$. We claim that

$$\phi_*(\epsilon) \notin H^{-1}(\Omega^- (\delta_4, \rho, \epsilon), \epsilon).
$$

If not, by Theorem 5.3(b), $\phi_*(\epsilon) = \pi(\phi_*(\epsilon), \epsilon) = \pi^2(\phi_1(\epsilon), \epsilon) \in W^u_{loc}(\epsilon)$, a contradiction. Thus, $\phi_*(\epsilon, \epsilon) = \phi_*(\epsilon) \in \gamma$, a solution orbit of equation (2.1). It follows that $\gamma$ is a periodic orbit.
Let $\tau = \tau(\bar{\phi}, \varepsilon)$ be as in Lemma 4.6 and

$$\tau_* (\varepsilon) = \tau(H(\phi_\ast (\varepsilon), \varepsilon), \varepsilon).$$

By Lemma 4.6, $u_t (\phi_\ast (\varepsilon), \varepsilon) \in N_1$ for $0 \leq t \leq \tau_* (\varepsilon)$. By (5.17), the continuity of $\pi^1$ and Lemma 6.2, we have

$$|\pi^1(\phi_\ast (\varepsilon), \varepsilon) - \phi_1 (0)| \leq |\pi^1(\phi_\ast (\varepsilon), \varepsilon) - \phi_1 (\varepsilon)| + |\phi_1 (\varepsilon) - \phi_1 (0)| \leq \bar{\rho}/2,$$

where $\bar{\rho}$ is given in Lemma 6.2. Thus, Lemma 6.1 implies that

$$u_{t+\tau_\ast (\varepsilon)}(\phi_\ast (\varepsilon), \varepsilon) \in N_2, \quad 0 \leq t \leq t_0.$$

Therefore, $\gamma \subset N(\Gamma_0)$.

**Sufficiency.** Let $\gamma$ be a periodic orbit. Then Lemma 4.4 implies that $\phi_\ast (\varepsilon) \in \gamma$. Let $\phi_2 (\varepsilon) = \pi^1(\phi_1 (\varepsilon), \varepsilon) \in W^u_\ast (\varepsilon) \cap B(\phi_0, \rho_3)$ be as in the proof of Lemma 6.3 and $\tau = \tau(\bar{\phi}, \varepsilon)$ be as in Lemma 4.6. Define

$$\phi_2^k (\varepsilon) = (\pi)^k (\phi_2 (\varepsilon), \varepsilon),$$

$$\tau_2^k (\varepsilon) = \tau(H(\phi_2^k (\varepsilon), \varepsilon), \varepsilon)$$

for $\varepsilon \in [-\bar{\varepsilon}_0, \bar{\varepsilon}_0]$ and $k = 1, 2, \ldots$

**Claim A.** $\phi_2^k (\varepsilon) \in H^{-1}(\Omega^+ (\delta_4, \rho_3, \varepsilon), \varepsilon), \varepsilon \in [-\bar{\varepsilon}_0, \bar{\varepsilon}_0], \ k = 1, 2, \ldots$.

Suppose not, then there exists $K$ such that

$$\phi_2^K (\varepsilon) \in H^{-1}(\Omega^- (\delta_4, \rho_3, \varepsilon), \varepsilon) \cup W^s_\text{loc} (\varepsilon)$$

and

$$\phi_2^k (\varepsilon) \notin H^{-1}(\Omega^- (\delta_4, \rho_3, \varepsilon), \varepsilon) \cup W^s_\text{loc} (\varepsilon), \quad k = 1, 2, \ldots, K - 1.$$

By Theorem 5.3(b), we have

$$\pi(\phi_2^k (\varepsilon), \varepsilon) = \pi((\pi)^K (\phi_2 (\varepsilon), \varepsilon), \varepsilon) = \phi_2 (\varepsilon),$$

which implies that $\phi_2 (\varepsilon)$ is the fixed point of $\pi^{K+1}$, thus, the fixed point of $\pi$. Therefore $\phi_2 (\varepsilon) = \phi_\ast (\varepsilon) \in \gamma$. This contradiction proves the claim.

**Claim B.** $\bigcup_{0 \leq t \leq t_0} u_t (\phi_1 (\varepsilon), \varepsilon) \subset N(\Gamma_0)$.

By Theorem 5.3(b), $\phi_2^k (\varepsilon) \in W^u_\ast (\varepsilon)$ for every $k = 1, 2, \ldots$. Thus, Lemma 6.3 implies the claim.

**Claim C.** $\bigcup_{0 \leq t \leq \tau_\ast (\varepsilon) + t_0} u_t (\phi_2^k (\varepsilon), \varepsilon) \subset N(\Gamma_0) \text{ for } \varepsilon \in [-\varepsilon_0, \varepsilon_0], k = 1, 2, \ldots$.

Theorem 5.3(a) and Lemma 6.2 imply that

$$|\phi_2^k (\varepsilon) - \phi_\ast (\varepsilon)| \leq \left(\frac{1}{2}\right)^k |\phi_\ast (\varepsilon) - \phi_\ast (\varepsilon)|$$

$$\leq \left(\frac{1}{2}\right)^k \frac{K_1 \bar{\rho}}{4},$$

where $\bar{\phi}_\ast (\varepsilon) = \phi_\ast (\varepsilon) + h_\ast (\phi_\ast (\varepsilon), \varepsilon) \in W^s_\text{loc} (\varepsilon)$. Thus, by the continuity of $\pi^1$, (5.17) and Claim A, we have

$$|\pi^1(\phi_2^k (\varepsilon), \varepsilon) - \pi^1(\phi_\ast (\varepsilon), \varepsilon)| \leq \bar{\rho}/8.$$
Hence
$$|π^1(φ_k^2(ε), ε) − φ_1(0)| < \bar{ρ}/2 + \bar{ρ}/8 < \bar{ρ},$$
is this, together with Lemma 6.2(i), implies that
$$\bigcup_{\bar{τ}_k^2(ε) ≤ τ ≤ \bar{τ}_k^2(ε) + t_0} \{u_t(φ_k^2(ε), ε)\} \subset N_2.$$By the definition of $N_1$ and Lemma 4.6, we have
$$\bigcup_{0 ≤ t ≤ \bar{τ}_k^2(ε)} \{u_t(φ_k^2(ε), ε)\} \subset N_1.$$Therefore, Claim C is proved.

Given a $φ \in W^u_+{ε}$, by Claims A, B, and C, $φ$ is in either
$$\bigcup_{0 ≤ t ≤ t_0} \{u_t(φ_k^2(ε), ε)\} \subset N(Γ_0)$$or
$$\bigcup_{0 ≤ t ≤ \bar{τ}_k^2(ε) + t_0} \{u_t(φ_k^2(ε), ε)\} \subset N(Γ_0)$$for some $k = 1, 2, \ldots$. Therefore, $W^u_+(ε) \subset N(Γ_0)$.

Finally, the exponentially asymptotic stability of $γ$ follows from Theorem 5.3(a).

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