Families of orthogonal Schrödinger cat-like-states

Ludmila Praxmeyer

Institute of Photonics Technologies, National Tsing-Hua University, No. 101, Section 2, Kuang-Fu Road, Hsinchu 30013, Taiwan, Republic of China

E-mail: lpraxm@gmail.com

Received 25 November 2015
Accepted for publication 25 April 2016
Published 13 May 2016

Abstract
We analyze the condition of orthogonality between optical Schrödinger cat-like-states constructed as a superposition of two coherent states. We show that the orthogonality condition leads to the quantization of values of a naturally emerging symplectic form, while values of the corresponding metric form are continuous. A complete analytical solution of the problem is presented.

Keywords: coherent states, Schrödinger cat-like-states, phase-space representation

1. Introduction
A set of nonorthogonal wave functions that naturally appear in the description of a quantum harmonic oscillator was discovered from the beginnings of quantum theory. It was first mentioned by Schrödinger [1] in 1926 and analyzed later in von Neumann’s Mathematische Grundlagen der Quantenmechanik [2]. The name ‘coherent states’ was proposed by Glauber in the context of the description of coherent laser beams in 1963 [3]. Since then, the formalism of coherent states often serves as a language of quantum optics—especially in its phase space representation, and not without reason, as the ‘most classical from quantum states’ coherent states match classical intuitions, when at the same time the superposition of the coherent states are nonclassical enough to reveal purely quantum effects. A typical example usually serves a superposition of two coherent states, a so-called Schrödinger-cat-like state.

Superpositions of coherent states have been studied in the contexts of quantum error correction [4], quantum teleportation [5], and quantum computation [6, 7]. The majority of these applications exploit the fact that a cat-like state split on a beamsplitter produces an entangled state. There are many theoretical proposals of the generation of such superpositions and a number of experimental realizations [8–10]. I believe that many of the proposals mentioned above could benefit from the fact that, unlike single coherent states, their superpositions form families of orthogonal states. This paper presents an analysis showing that the scalar product between the superpositions of coherent states can be exactly zero—not just reach ‘close to zero’ values, as is the case when single coherent states are used, e.g. as logical qubits.

Coherent states minimize the uncertainty relation; they are connected to the eigenvectors of a quantum harmonic oscillator, $|m\rangle_{\text{osc}}$, by the formula

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle_{\text{osc}},$$

and they form an over-complete (but not orthogonal) basis. The average number of photons\(^1\) in (1) is equal to $|\alpha|^2$. Although two coherent states are never orthogonal to each other, the scalar product between them vanishes exponentially with distance $|\langle \beta | \alpha \rangle|^2 = \exp(-|\alpha - \beta|^2)$, which for large enough values of $|\alpha|$ allows us to treat a sum

$$K_{\alpha}(\beta) := |\alpha\rangle + e^{i\beta} | -\alpha \rangle,$$

as a close optical analogue of the superposition of macroscopically distinguishable states from the Schrödinger’s gedanken experiment [11]. For small values of $|\alpha|$ (and a significant overlap between the states) such superpositions are known as Schrödinger’s kitten. Because of an omitted normalization

\(^1\)Because we consider coherent states in the context of quantum optics we talk about the average number of photons rather than the average number of excitations.
factor, we shall refer to \( K_c(\alpha) \) as a vector rather than a state. For \( \varphi = 0 \) and \( \varphi = \pi \), vector \( K_c(\alpha) \) corresponds respectively to even and odd coherent states introduced in [12]. Note, also, that the phase in (2) is not equivalent to the one studied in [13] where superpositions of the form \( |\alpha e^{i\theta}| + |\alpha e^{-i\theta}| \) were considered.

The Wigner function [14] of a cat-like state is often used to illustrate how decoherence affects quantum superpositions [15, 16]. In this phase space representation it is clearly seen that the addition of a noise to the system destroys the interference terms, while Gaussian peaks corresponding to \( |\alpha\rangle \) or \( |-\alpha\rangle \) remain unaffected. It was shown that the coherent states are especially robust to decoherence, and thus can serve as the ‘pointer’ states [16]. More information about the mathematical properties of coherent states and their generalizations can be found in [17, 18]. A short list of symbols used in this paper is presented in appendix A.

2. Superposition of coherent states

2.1. A coherent state is orthogonal to ...

We have emphasized that coherent states are not orthogonal to each other. It does not also mean that superpositions of coherent states always have a non-zero scalar product. The most obvious example of a vanishing overlap between Schrödinger-cat-like states is obtained via a change of the relative phase: vector \( K_c(\alpha) \) is orthogonal to \( K_\theta(\alpha) \). This fact is easily proved, one just has to realize the difference in the parity of the corresponding states. Substituting (1) into \( K_\theta(\alpha) \) and \( K_c(\alpha) \) one sees that the former consists only of even number states, while the latter only of odd number states [12]. The same argument proves that \( K_c(\alpha) \) and \( K_c(\beta) \) are always orthogonal and that the vacuum state \( |0\rangle \) is orthogonal to \( K_c(\alpha) \) for any \( \alpha, \beta \neq 0 \). Another example is a class of vectors orthogonal to \( K_\theta(\alpha), d \in \mathbb{R}^c \), defined as

\[ J(d, \delta_k) := (d + i\delta_k) + \gamma d + i\delta_k, \]

where \( \delta_k = \pi(2k + 1)/(2d) \) and \( k \in \mathbb{Z} \). Each value of \( \delta_k \) corresponds to a shift in momentum that makes \( K_\theta(\alpha) \) and \( J(d, \delta_k) \) orthogonal to each other [19].

One might wonder if a superposition \( K_c(\alpha) \), which is orthogonal to a single coherent state different from \( |0\rangle \). The answer is yes: for example, an overlap of vector \( |\alpha\rangle \) for any set but nonzero real \( \alpha \), and \( K_c(\beta_n) \) for \( \beta_n = i\pi(n + 1/2)/\alpha \), vanishes for all natural \( n \), i.e.

\[ \langle \alpha|K_\theta(\alpha)|\beta_n\rangle = 0 \quad \text{for} \quad \alpha \in \mathbb{R}^c, \quad n \in \mathbb{N}. \]

(3)

For a set value of \( n \), the smaller \( |\alpha| \) the further apart are \( |\beta_n\rangle \) and \( |-\beta_n\rangle \), for a set \( \alpha \), the amplitude \( |\beta_n| \) increases with \( n \).

To illustrate the examples used, we will either plot the corresponding Husimi functions [20] or just represent a coherent state \( |\alpha\rangle \) as a circle of radii \( 1/\sqrt{2} \) centered at point (Re(\( \alpha \)), Im(\( \alpha \))). For a given density matrix \( \hat{\rho} \), the Husimi function is defined as

\[ Q_\theta(\gamma) = \text{Tr}[\hat{\rho}|\gamma\rangle\langle\gamma|]. \]

(4)

Figure 1 shows the Husimi functions corresponding to cat-like superpositions orthogonal to the coherent state [5]. The first example, depicted in figure 1(a), corresponds to \( n = 4 \) and vector \( K_\theta(\alpha, 0.9) \). The second, figure 1(b), was calculated for \( n = 15 \) and corresponds to vector \( K_\theta(\alpha, 3.1) \). In both figures, the Husimi function of state [5] is plotted in red, and the Husimi functions of the orthogonal vectors \( K_\theta(\beta_n) \) plotted in blue. Dashed circles of radii 5 and \( |\beta_n| \) denote phase-space trajectories of the free evolution of coherent states [5], \( |\beta_n\rangle \). Note that with respect to the typical notation position (\( \sqrt{2}\text{Re}(\gamma) \)) and momentum (\( \sqrt{2}\text{Im}(\gamma) \)), the axes in these plots are exchanged.

2.2. Even and odd coherent states

2.2.1. Orthogonality between even cats. Consider a superposition of two coherent states of form \( K_\theta(\alpha) \), equation (2). It can be proved that:

**Fact 1** For any \( \alpha, \beta \in \mathbb{C}^c \), the following conditions are equivalent:

1. \( \langle K_\theta(\beta)|K_\theta(\alpha)\rangle = 0 \).

2. There exists \( n \in \mathbb{N} \), such that

\[ K_\theta(\beta) = K_\theta(\beta_n) := -\frac{i\pi(2n + 1)}{2\alpha^*} + \frac{i\pi(2n + 1)}{2\alpha^*}. \]

(5)

It is seen that for a given non-zero \( \alpha \) there exists a whole class of solutions of \( \Gamma \) parametrized by a natural number \( n \), each rotated in phase space by \( \pi/2 \) with respect to \( K_\theta(\alpha) \). Figure 2 shows examples of Husimi functions calculated for pairs of orthogonal vectors \( K_\theta(\alpha) \) (plotted in green) and \( K_\theta(\beta_n) \) (plotted in blue), for parameters \( \alpha = 4 + i8 \) and \( \beta_n \) corresponding to \( n = 4 \) and \( n = 16 \), respectively. Comparison between figures 2(a) and (b) reveals how a change of \( n \) changes the distance between states forming a superposition (5). In general, separation \( 2|\beta_n| \) is different from \( 2|\alpha| \) for every value of parameter \( n \). Below we consider a very special case when the distances between states forming orthogonal superpositions \( K_\theta(\alpha) \) and \( K_\theta(\beta_n) \) are equal.

**Fact 2** For \( \alpha, \beta \in \mathbb{C}^c \) such that \( \langle K_\theta(\alpha)|K_\theta(\beta)\rangle = 0 \), the following conditions are equivalent:

1. \( |\alpha| = |\beta| \).

2. There exists \( n \in \mathbb{N} \) such that \( |\alpha| = \sqrt{(n + 1/2)}\pi \).

(6)

An example of two even cat superpositions with an equal average number of photons is presented in figure 5(a). Dashed circles show the only possible values of \( \alpha \) satisfying condition 2 from fact 2. The areas of the bands between the subsequent circles are equal to \( \pi^2 \).

2.2.2. Orthogonality between odd cats. Similarly, one might consider two coherent states forming an ‘odd’ cat-like superposition \( K_c(\alpha') \) and prove the facts listed below:

**Fact 3** For any \( \alpha', \beta' \in \mathbb{C}^c \), the following conditions are equivalent:

1. \( \langle K_c(\beta')|K_c(\alpha')\rangle = 0 \).

2. There exists \( n \in \mathbb{N}^c \), such that
\[ \langle K_\alpha(\beta) | K_\beta'(\beta') \rangle := \frac{\text{Im} \pi}{\alpha^*} - \frac{\text{Re} \pi}{\alpha'}. \]

(Note that, for \( n = 0 \), the orthogonality condition holds, although it is reduced to a trivial case.)

**Fact 4** For \( \alpha', \beta' \in \mathbb{C} \) such that \( \langle K_\alpha(\alpha') | K_\beta'(\beta') \rangle = 0 \) the following conditions are equivalent:

1° \( |\alpha'| = |\beta'| \).

2° There exists \( n \in \mathbb{N}^\times \) such that \( |\alpha'| = \sqrt{n \pi} \).

Figure 3 shows a phase space representation of a pair of orthogonal vectors \( K_\varphi(\alpha), K_\varphi(\beta) \) fulfilling condition \( |\alpha'| = |\beta'| \). Black dashed lines denote other circles of radii of form \( r_n = \sqrt{n \pi} \). As before, the areas of the bands between the closest-neighbors’ circles are equal to \( \pi^2 \). There is a factor of \( \sqrt{\pi} \) difference between these radii and those separating the subsequent Planck–Bohr–Sommerfeld bands [13] corresponding to number states in the semiclassical limit.

### 3. Orthogonality between cat-like states—arbitrary relative phase

Let us consider a more general superposition obtained for non-zero \( \alpha, \beta \) and arbitrary relative phases \( \varphi_1, \varphi_2 \in [0, 2\pi] \). It can be shown that condition

\[ \langle K_{\varphi_1}(\beta) | K_{\varphi_2}(\alpha) \rangle = 0 \]  

is equivalent to

\[ e^{2\text{Re}(\alpha \beta')} \cos(\frac{\varphi_2 - \varphi_1}{2}) + e^{-2\text{Re}(\alpha \beta')} \cos^2\left( \frac{\varphi_2 + \varphi_1}{2} \right) = -2 \cos\left( \frac{\varphi_2 + \varphi_1}{2} \right) \cos\left( \frac{\varphi_2 - \varphi_1}{2} \right) \cos[2 \text{Im}(\alpha \beta')]. \]  

(7)

In the paragraphs that follow, the analysis of the solutions of (7) is presented and the dependence between parameters \( \alpha, \beta, \varphi_1, \varphi_2 \) is examined.

#### 3.1. Case when \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \cos(\frac{\varphi_2 - \varphi_1}{2}) = 0 \)

Note that (7) is an identity when both \( \cos(\frac{\varphi_2 + \varphi_1}{2}) = 0 \) and \( \cos(\frac{\varphi_2 - \varphi_1}{2}) = 0 \). As a result, the scalar product (6) vanishes for any \( \alpha \) and \( \beta \). On a map presented in figure 4(a), phases \( (\varphi_1, \varphi_2) \in \{0, \pi, (\pi, 0)\} \) corresponding to this case are illustrated as red circles. It is an example of the orthogonality between odd and even cats that occurs regardless of \( \alpha, \beta \), which was already described in the beginning of section 2.1.

Quite the opposite is the case when the product \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \cos(\frac{\varphi_2 - \varphi_1}{2}) \) vanishes, but either \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \neq 0 \) or \( \cos(\frac{\varphi_2 - \varphi_1}{2}) \neq 0 \). Then, there are no solutions of (7). In figure 4(a), phases corresponding to this condition are plotted as black open edges of an inscribed green square. These lines divide the map \( (\varphi_1, \varphi_2) \) into two areas: a green square where \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \cos(\frac{\varphi_2 - \varphi_1}{2}) < 0 \) and a remaining yellow area where \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \cos(\frac{\varphi_2 - \varphi_1}{2}) > 0 \). Because of the periodic boundary

---

**Figure 1.** Phase space Husimi representation of coherent state |5⟩ (plotted in red) and orthogonal to it cat-like superposition \( K_\delta(\beta) \) (plotted in blue). Amplitudes of orthogonal states take on discrete values and increase with \( n \). Blue circles in plots (a) and (b) correspond to vectors \( K_\delta(i\pi(0.9)) \) and \( K_\delta(i\pi(3.1)) \) obtained from (3) for \( n = 4 \) or \( n = 15 \), respectively.

---

**Figure 3.** Shows a phase space representation of a pair of orthogonal vectors \( K_\varphi(\alpha), K_\varphi(\beta) \) fulfilling condition \( |\alpha'| = |\beta'| \). Black dashed lines denote other circles of radii of form \( r_n = \sqrt{n \pi} \). As before, the areas of the bands between the closest-neighbors’ circles are equal to \( \pi^2 \). There is a factor of \( \sqrt{\pi} \) difference between these radii and those separating the subsequent Planck–Bohr–Sommerfeld bands [13] corresponding to number states in the semiclassical limit.
From (9) it is convenient to rewrite this from (5), for (a).\[ Z = \text{depicted in green. Plotted in blue are the Husimi } \alpha, \beta \text{ of } \alpha, \beta. \text{ It is also worth noting that the quantization conditions can change continuously. For more details, see } \alpha, \beta \text{ and } \alpha, \beta \text{ or } \alpha, \beta \text{ or } \alpha, \beta. \text{ The initial cat } n = 4 \text{ and } n = 16, \text{ respectively.}\]

conditions, both surfaces have exactly the same topology—as is seen in figure 4(b).

3.2. Case when \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \cos(\frac{\varphi_2 - \varphi_1}{2}) \neq 0 \)

To find solutions of (7) in a nonsingular case of \( \cos(\frac{\varphi_2 + \varphi_1}{2}) \cos(\frac{\varphi_2 - \varphi_1}{2}) \neq 0 \), it is convenient to rewrite this equation as

\[
-2\cos[2\text{Im}(\alpha\beta^*)] = \exp[2\text{Re}(\alpha\beta^*)] \cos\left(\frac{\varphi_1 - \varphi_2}{2}\right) \cos\left(\frac{\varphi_1 + \varphi_2}{2}\right) + \exp[-2\text{Re}(\alpha\beta^*)] \cos\left(\frac{\varphi_1 + \varphi_2}{2}\right) \cos\left(\frac{\varphi_1 - \varphi_2}{2}\right).
\]

Values of the left-hand side of (8) form a closed segment \([-2, 2]\). The right-hand side took the form of function \( f(z) = \frac{1}{z + 1/\epsilon} \), and its values belong to \([2, \infty[ \) or \( ] -\infty, -2] \) for positive and negative \( z \), respectively. It is easy to check that \( f(z) = -2 \) iff \( z = -1 \), and \( f(z) = 2 \) iff \( z = 1 \). A combination of these facts leads to the conclusion that (8) is satisfied only when both its sides are simultaneously equal to plus or minus 2. Thus, for (6) to hold\(^2\) one of the following conditions has to be satisfied:

\[
\begin{align*}
\text{Im}(\alpha^*) &= k\pi & \text{for } k \in \mathbb{Z} \\
\text{Im}(\alpha\beta^*) &= (2k + 1)\pi/2 & \text{for } k \in \mathbb{Z}.
\end{align*}
\]

Note that the requirements imposed on \( \text{Im}(\alpha\beta^*) \) and \( \text{Re}(\alpha\beta^*) \) are separated: phases \( \varphi_1, \varphi_2 \) define \( \text{Re}(\alpha\beta^*) \) unambiguously, but are independent from the quantization conditions imposed on \( \text{Im}(\alpha\beta^*) \). It is also worth noting that the quantization conditions are imposed only on a naturally emerging symplectic form, \( \text{Im}(\alpha\beta^*) \), while values of the corresponding metric form, \( \text{Re}(\alpha\beta^*) \), can change continuously. For more details, see appendix B.

Let us assume that \( \alpha = 0 \) and \( \text{Re}(\alpha\beta^*) = \omega \). From (9) or (10) we obtain

\[
\beta = (\omega - i\kappa)\alpha^* \text{ or } \beta = (\omega - i(k + \frac{1}{2})\pi)\alpha^*, \text{ } k \in \mathbb{Z},
\]

respectively. It is seen that for every real \( \omega \) and non-zero \( \alpha \) there exists a parameterized by integer \( k \), family \( \{\beta_k\} \) of values of \( \beta \) satisfying relations (9) or (10). For \( \omega = 0 \) and \( (\varphi_1, \varphi_2) = (0, 0) \), the problem reduces to that analyzed in facts

\(^2\) In the case when \( \cos(\frac{\varphi_1 + \varphi_2}{2}) \cos(\frac{\varphi_1 - \varphi_2}{2}) = 0 \).
For $\omega = 0$ and $(\varphi_1, \varphi_2) = (\pi, \pi)$, it reduces to the case described by facts 3–4.

If we are looking for orthogonal vectors $\mathcal{K}_\varphi(\alpha), \mathcal{K}_\varphi(\beta)$ with the same average number of photons, relation $|\alpha|^2 = |\beta|^2$ has to be satisfied. Depending on the quantization condition, it implies either $|\alpha|^2 = |\beta|^2 = \sqrt{\omega^2 + k^2 \pi^2}$ or

$$|\alpha|^2 = |\beta|^2 = \sqrt{\omega^2 + \left( k + \frac{1}{2} \right)^2 \pi^2}. \quad (12)$$

Examples of orthogonal vectors $\mathcal{K}_\varphi(\alpha), \mathcal{K}_\varphi(\beta)$ fulfilling (10) are presented in figures 5(a) and (b). Both plots were made under the assumption that $\varphi_1 = 0$ and $k = 1$. Vector $\mathcal{K}_0(\alpha)$ is plotted in green, the orthogonal vector $\mathcal{K}_\varphi(\beta)$ in yellow. Dashed circles denote the only possible values of $\alpha$ fulfilling condition (12) for a set parameter $\omega$. Figure 5(a) corresponds to $\omega = 0$ (for details on the effects of the vanishing real part of $\alpha^*$, see the paragraph below); figure 5(b) corresponds to $\omega = 2\pi$. It is seen that non-zero values of $\omega$ modify an angle of phase space rotation between the orthogonal states, making it $k$-dependent. It is also clear that in figure 5(b) areas of bands between the subsequent dashed circles change with $k$, and a simple calculation shows that in the limit of large $|k|$ they approach $\pi^2$. For $\omega = 0$, figure 5(a), areas of the bands are always equal to $\pi^2$ only in the limit of $|k| \to \infty$.

Figure 4. (a) Map of $(\varphi_1, \varphi_2)$ that organizes solutions of (7). Its periodic boundary conditions make it equivalent to a torus (b) showing that the green and yellow areas have the same topology. Red dots correspond to the case of odd and even cats which are orthogonal to each other for all values of $\alpha, \beta$. Dashed white and dashed blue lines correspond to cases when $\text{Re}(\alpha^*) = 0$ and one of the phases is equal to 0 or $\pi$ while the second is arbitrary. Values of $\varphi_1, \varphi_2$ for which $\cos(\varphi_1 - \varphi_2)\cos(\varphi_1 + \varphi_2) < 0$ are plotted as a green inscribed square. Its black edges (open segments) denote the values for which only one of these cosines vanishes, and there are no solutions of (7). The remaining yellow triangles denote these $\varphi_1, \varphi_2$ for which $\cos(\varphi_1 - \varphi_2)\cos(\varphi_1 + \varphi_2) > 0$. For angles forming open green and yellow areas there are always ambiguous solutions of (7).

Figure 5. Phase space representation of orthogonal vectors $\mathcal{K}_0(\alpha)$ (green) and $\mathcal{K}_\varphi(\beta)$ (yellow) with an equal mean number of photons $|\alpha|^2 = |\beta|^2 = \sqrt{\omega^2 + \left( k + \frac{1}{2} \right)^2 \pi^2}$, calculated for $k = 1$ and (a) $\omega = 0$; (b) $\omega = 2\pi$. Dashed circles denote the only possible values fulfilling conditions $\langle \mathcal{K}_\varphi(\alpha) \rangle \langle \mathcal{K}_\varphi(\beta) \rangle = 0$ and $|\alpha| = |\beta|$, for a given $\omega$. The width of the bands between subsequent circles strongly depend on $\omega$: in case (a) corresponding to odd cat-like superpositions, described in section 2.2, areas of subsequent bands are always equal to $\pi^2$. In case (b) areas of the bands depend on $k$ and are equal to $\pi^2$ only in the limit of $|k| \to \infty$. 
3.3. When the real part of $\alpha^3$ vanishes

Consider a special case when $\omega = \Re(\alpha^3) = 0$. If $\cos[2\Im(\alpha^3)] = 1$, from (9) it follows that one of the phases $\varphi_1, \varphi_2$ has to be equal to $\pi$, the other can be arbitrary. If $\cos[2\Im(\alpha^3)] = -1$, one of the phases has to be equal to 0 and the other is arbitrary. In figure 4 the former case is denoted by the dashed white lines, the latter by dashed blue lines, i.e. edges of a larger yellow-green square. Note, that the case when $(\varphi_1, \varphi_2) \in [(0, \pi), (\pi, 0)]$ takes us once again to the orthogonality between odd and even cats. It is clear that in this case, for a known $\alpha \neq 0$ and one and only one of the phases equal to 0 or $\pi$, the value of $\beta$ is determined unambiguously, while the second phase is arbitrary.

From now on we will assume that the real part of $\alpha^3$ is nonzero, which means that we will consider only phases depicted in figure 4 by green or yellow open triangles (without edges).

3.3.1. Set values of $\varphi_1, \varphi_2$ and $\alpha$. To avoid repetitions, we assume now that $\cos(\frac{\varphi_1 + \varphi_2}{2}) = 0$ and that $\varphi_1, \varphi_2$ are different than 0 or $\pi$. For given phases $\varphi_1, \varphi_2$, the quantization condition depends on whether point $(\varphi_1, \varphi_2)$ belongs to a green or yellow area in figure 4. In the first case, phases are such that $\cos(\frac{\varphi_1 + \varphi_2}{2})/\cos(\frac{\varphi_1 - \varphi_2}{2}) < 0$, which imposes condition (9) and, consequently, (9). The second case, when $\cos(\frac{\varphi_1 + \varphi_2}{2})/\cos(\frac{\varphi_1 - \varphi_2}{2}) > 0$ leads to conditions (10) and (10). In both cases, for a known $\varphi_1, \varphi_2$ and $\alpha \neq 0$, the value of $\beta$ can be determined from (11). Table 1 summarizes the results obtained so far.

3.3.2. Set values of $\alpha, \beta$ and $\varphi_1$. To analyze conditions (9), (10), for set values of $\alpha, \beta$ and $\varphi_1$ or set $\alpha, \beta$ and $\varphi_2$, it is convenient to introduce new variables: $a = \tan(\frac{\varphi_1}{2}), b = \tan(\frac{\varphi_2}{4})$, that transform (9) into analytically solvable quadratic equations. After some calculations it can be shown, that for a given $\alpha, \beta$ and $\varphi_1$, solutions of (10) for $\varphi_2$ are

$$\varphi_2^+ = \arctan \left[ \frac{\sqrt{W(\varphi_1, \alpha, \beta)^2 + U(\varphi_1, \alpha, \beta)^2} - W(\varphi_1, \alpha, \beta)}{U(\varphi_1, \alpha, \beta)} \right],$$

$$\varphi_2^- = \arctan \left[ \frac{\sqrt{W(\varphi_1, \alpha, \beta)^2 + U(\varphi_1, \alpha, \beta)^2} + W(\varphi_1, \alpha, \beta)}{-U(\varphi_1, \alpha, \beta)} \right],$$

corresponding to $\Re(\alpha^3) > 0$ or $\Re(\alpha^3) < 0$, respectively, and where

$$W(\varphi_1, \alpha, \beta) := 2 \tan \left( \frac{\varphi_1}{4} \right)[1 + \exp(2\Re(\alpha^3))],$$

$$U(\varphi_1, \alpha, \beta) := [1 - \exp(2\Re(\alpha^3))][ \left( \tan \left( \frac{\varphi_1}{4} \right) \right)^2 - 1].$$

Analogously, introducing

$$W(\varphi_1, \alpha, \beta) := 2 \tan \left( \frac{\varphi_1}{4} \right)[1 - \exp(2\Re(\alpha^3))],$$

$$U(\varphi_1, \alpha, \beta) := [1 + \exp(2\Re(\alpha^3))][ \left( \tan \left( \frac{\varphi_1}{4} \right) \right)^2 - 1].$$

the solution of (9) can be written as

$$\varphi_2^+ = \arctan \left[ \frac{\sqrt{W(\varphi_1, \alpha, \beta)^2 + U(\varphi_1, \alpha, \beta)^2} - W(\varphi_1, \alpha, \beta)}{U(\varphi_1, \alpha, \beta)} \right],$$

$$\varphi_2^- = \arctan \left[ \frac{\sqrt{W(\varphi_1, \alpha, \beta)^2 + U(\varphi_1, \alpha, \beta)^2} + W(\varphi_1, \alpha, \beta)}{-U(\varphi_1, \alpha, \beta)} \right],$$

for $\Re(\alpha^3) > 0$ and $\Re(\alpha^3) < 0$, respectively.

We have shown unambiguous solutions of (7) in the case of known parameters $\alpha, \beta$ and $\varphi_1$. In the case when $\varphi_1$ is an unknown variable and $\varphi_2$ is a known parameter, values of $\varphi_1$ can be found in the same fashion because all the equations used were symmetric under transformation $\varphi_1 \leftrightarrow \varphi_2$. This ends the analysis of equation (7).

To summarize the results of this section: if $\cos[2\Im(\alpha^3)] = \pm 1$ one can always find phases $\varphi_1, \varphi_2$ such that $|\mathcal{K}_{\varphi_1}(\beta)|$ is orthogonal to $|\mathcal{K}_{\varphi_2}(\alpha)|$. Moreover,

- for $\Re(\alpha^3) > 0$ and any $\varphi_1$ different then 0 or $\pi$ there exists exactly one $\varphi_2$ that makes respective vectors orthogonal. Both $\varphi_1$ and $\varphi_2$ are orthogonal smaller or larger than $\pi$.
- for $\Re(\alpha^3) < 0$ and any $\varphi_1$ different then 0 or $\pi$ there exists exactly one $\varphi_2$ that makes respective vectors orthogonal, and either $\varphi_1$ or $\varphi_2$ is larger than $\pi$.
- For $\Re(\alpha^3) = 0$ and $\cos[2\Im(\alpha^3)] = 1$, to obtain orthogonality one phase has to be equal to $\pi$, the other phase can be arbitrary (white dashed lines in figure 4). A special case when $(\varphi_1, \varphi_2) \in [(0, \pi), (\pi, 0)]$ corresponds to the always orthogonal cat-like superpositions of different parity.
- For $\Re(\alpha^3) = 0$ and $\cos[2\Im(\alpha^3)] = -1$, one of the phases has to be equal to 0—the second can be arbitrary (dashed blue lines in figure 4). A special case, $(\varphi_1, \varphi_2) \in [(0, \pi), (\pi, 0)]$, corresponds to the always orthogonal cat-like superpositions of different parity.
4. Summary and outlook

We have presented a full analytical solution of a problem of finding orthogonal vectors within a set of cat-like superpositions of coherent states, (2). We have shown that in the case of known \( \alpha, \beta, \) and \( \varphi_1 \) the condition \( \langle K_{\varphi_2}(\beta)|K_{\varphi_1}(\alpha) \rangle = 0 \) determines \( \varphi_2 \) unambiguously, thus, measurements of the scalar product can be used to measure the phase. We have also shown that for any given cat-like superposition \( K_{\varphi_1}(\alpha) \) and set \( \varphi_2, \) such that \( \langle \varphi_1, \varphi_2 \rangle \) belongs to either green or yellow areas in figure 4, there exist a whole family of vectors of the form \( K_{\varphi_2}(\beta) \) orthogonal to \( K_{\varphi_1}(\alpha), \) and have presented the explicit solutions. We have proved that the considered orthogonality condition imposes on antisymmetric (simplectic) form \( \Im(\alpha\beta^*) \) a quantization condition permitting only discreet values, and has no such restriction on metric form \( \Re(\alpha\beta^*). \)

The results presented in this paper show, among others, that it is possible to directly and deterministically distinguish cat-like superpositions—in contrast to only probabilistic distinguishability allowed between ‘single’ coherent states. This fact has potentially many applications: from the ability to perfectly distinguish different cat states follows, in principle, the possibility to use precise measurement of a scalar product for a precise measurement of phase, and vice versa. In the context of quantum communication, the existence of infinite sets of orthogonal states allows us to send binary sequences without repetition of code words, and a condition \( \cos[2\Im(\alpha\beta^*)] = \pm 1 \) adds a possibility of additional spin-like encoding.

Acknowledgments

LP thanks Prof Ray-Kuang Lee for his hospitality and stimulating discussions. This work was supported by NTHU project no. 104N1807E1.

Appendix A. Notation

In this paper a standard notation was used: greek letters \( \alpha, \beta, \gamma, \ldots \) denote complex numbers and \( |\alpha\rangle, |\beta\rangle, |\gamma\rangle \) are the corresponding coherent states. A set of real and complex numbers are denoted by \( \mathbb{R} \) and \( \mathbb{C}, \) respectively, while \( \mathbb{R}^n \) and \( \mathbb{C}^n \) denote sets of real or complex numbers without zero. A set of integer numbers is denoted as \( \mathbb{Z}, \) and a set of natural numbers (with zero) as \( \mathbb{N}, \) whereas \( \mathbb{N}^* := \mathbb{N} \setminus \{0\}. \)

Appendix B

Decomposition of a complex number \( \gamma \in \mathbb{C} \) into real and imaginary parts, \( \Re(\gamma) \) and \( \Im(\gamma), \) defines a canonical isomorphism of real vector spaces

\[
\gamma : \mathbb{C} \ni \gamma \mapsto \tilde{\gamma} := (\Re(\gamma), \Im(\gamma)) \in \mathbb{R}^2.
\]  

It means that the multiplication of complex numbers \( \alpha\beta^* \) defines two bilinear forms on \( \mathbb{R}^2 \)

\[
g : \mathbb{R}^2 \times \mathbb{R}^2 \ni (\alpha, \beta) \mapsto g(\alpha, \beta) := \Re(\alpha\beta^*) \in \mathbb{R} \]  

(2.2)

\[
h : \mathbb{R}^2 \times \mathbb{R}^2 \ni (\alpha, \beta) \mapsto h(\alpha, \beta) := \Im(\alpha\beta^*) \in \mathbb{R}.
\]  

(2.3)

Form \( g \) is symmetric and defines a metric (Riemann) structure on \( \mathbb{R}^2. \) Form \( h \) is antisymmetric and defines a symplectic structure on \( \mathbb{R}^2. \)

Results of section 3.2 show that the orthogonality condition between vectors \( K_{\varphi_1}(\alpha) \) and \( K_{\varphi_2}(\beta) \) impose a quantization condition on possible values of symplectic form \( h(\alpha, \beta) \) on \( \mathbb{R}^2, \) as is clearly seen from (9) and (10). At the same time connected with phases \( \varphi_1, \varphi_2 \) values of metric form \( g(\alpha, \beta) \) can take on arbitrary real values and are continuous (see (9) and (10)).

References

[1] Schrödinger E 1926 Naturwissenschaften 14 664
[2] von Neumann J 1932 Mathematische Grundlagen der Quantenmechanik (Berlin: Springer)
[3] Glauber R 1963 Phys. Rev. 130 2529
[4] Cochrane P T, Milburn G J and Munro W J 1999 Phys. Rev. A 59 2631
[5] van Enk S J and Hirota O 2001 Phys. Rev. A 64 022313
[6] Jeong H and Kim M S 2002 Phys. Rev. A 65 042305
[7] Ralph T C, Gilchrist A, Milburn G J, Munro W J and Glancy S 2003 Phys. Rev. A 68 042319
[8] Ourjoumtsev A, Tualle-Brouri R, Laurat J and Grangier P 2006 Science 312 83
[9] Ourjoumtsev A, Jeong H, Tualle-Brouri R and Grangier P 2007 Nature 448 06054
[10] Takahashi H, Waku K, Suzuki S, Takeoka M, Hayasaka K, Furusawa A and Sasaki M 2008 Phys. Rev. Lett. 101 233605
[11] Schrödinger E 1935 Naturwissenschaften 23 807
[12] Hillery M 1987 Phys. Rev. A 36 3796
[13] Schleich W, Pernigo M and Kien F L 1991 Phys. Rev. A 44 2172
[14] Wigner E 1932 Phys. Rev. 40 749
[15] Buzek V and Knight P 1995 Quantum interference, superposition states of light, and nonclassical effects Prog. in Opt. XXXIV (Amsterdam: Elsevier)
[16] Zurek W H 1991 Phys. Today 44 36–44
[17] Perelomov A 1986 Generalized Coherent States and their Applications (Berlin: Springer)
[18] Gazeau J P 2009 Coherent States in Quantum Physics (Weinheim: Wiley)
[19] Praxmeyer L, Wasylczyk P, Radzewicz C and Wódkiewicz K 2007 Phys. Rev. Lett. 98 063901
[20] Husimi K 1940 Proc. Phys. Math. Soc. Japan 22 264