Nullity and Bounds to the Nullity of Dendrimer Graphs

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ABSTRACT

In this paper, a high zero-sum weighting is applied to evaluate the nullity of a dendrimer graph for some special graphs such as cycles, paths, complete graphs, complete bipartite graphs and star graphs.

Finally, we introduce and prove a sharp lower and a sharp upper bound for the nullity of the coalescence graph of two graphs.

Keywords: Graph spectra, Nullity of graphs

1. Introduction

The characteristic polynomial of the adjacency matrix $A(G)$ is said to be the characteristic polynomial of the graph $G$, denoted by $\varphi (G;x)$. The eigenvalues of $A(G)$ are said to be the eiegenvalues of the graph $G$, the occurrence of zero as an eigenvalue in the spectrum of the graph $G$ is called the “nullity” of $G$ denoted by $\eta (G)$. Brown and others [4] proved that a graph $G$ is singular if, and only if, $G$ possesses a non-trivial zero-sum weighting, and asked, what causes a graph to be singular and what are the effects of this on its properties. Rashid [11] proved that a high zero-sum weighting $M_v(G)$ of a graph $G$, that is (the maximum number of non zero independent variables used in a high zero-sum weighting for a graph $G$, is equal to the nullity of $G$). It is known that $0 \leq \eta (G) \leq p-2$ if $G$ is a non empty graph with $p$ vertices. Cheng and Liu [5] proved that if $G$ has $p$ vertices with no isolated vertices, then $\eta (G) = p-2$ if, and only if, $G$ is isomorphic to a complete bipartite graph $K_{m,n}$, and $\eta (G) = p - 3$ if, and only if, $G$ is isomorphic to a complete 3 partite graph $K_{a,b,c}$. Omidi [10] found some lower bounds for the nullity of graphs and proved that among bipartite graphs with $p$ vertices, $q$ edges and maximum degree $\Delta$ which do not have any cycle of length a multiple of 4 as a subgraph, the greatest nullity is $p - 2 \left\lfloor q/\Delta \right\rfloor$. 
In this paper, we continue the research along the same lines. We derive formulas to determine the nullity of dendrimer graphs.

2 Definition and Preliminary Results

Definition 2.1: [5, p.16] and [8] A vertex weighting of a graph G is a function \( f: V(G) \rightarrow \mathbb{R} \) where \( \mathbb{R} \) is the set of real numbers, which assigns a real number (weight) to each vertex. The weighting of G is said to be non-trivial if there is, at least, one vertex \( v \in V(G) \) for which \( f(v) \neq 0 \).

Definition 2.2: [5, p.16] A non-trivial vertex weighing of a graph G is called a zero-sum weighting provided that for each \( v \in V(G) \), \( \sum f(w) = 0 \), where the summation is taken over all \( w \in NG(v) \).

Clearly, the following weighting for G is a non-trivial zero-sum weighting where \( x_1, x_2, x_3, x_4, \) and \( x_5 \) are weights and provided that \((x_1, x_2, x_3, x_4, x_5) \neq (0, 0, 0, 0, 0)\) as indicated in Figure 2.1.

![Figure 2.1. A non-trivial zero-sum weighting for a graph G.](image)

Theorem 2.3: [4] A graph g is singular if, and only if, there is a non-trivial zero-sum weighting for g.

Hence, the graph G depicted in Figure 2.1 is singular. Out of all zero-sum weightings of a graph G, a high zero-sum weighting of G is one that uses maximum number of non-zero independent variables.

Proposition 2.4: [6, p.35] and [8] in any graph g, the maximum number \( mv(g) \) of non-zero independent variables in a high zero-sum weighting equals the number of zeros as an eigenvalues of the adjacency matrix of g, (i.e. \( mv(g) = \eta(g) \)).

In Figure 2.1, the weighting for the graph G is a high zero-sum weighting that uses 5 independent variables, hence, \( \eta(G) = 5 \).

The complement of the disjoint union of m edges is called a cocktail graph and is denoted by \( CP(m) = (mK_2)c = K_{2,2,...,2} = K_{2m} \).

Proposition 2.5: [6, p.20] The spectrum of the cocktail graph \( CP(m) \) is:

\[
S_p(CP(m)) = \begin{pmatrix}
2m-2 & 0 & -2 \\
1 & m & m-1
\end{pmatrix}
\]

thus \( \eta(CP(m)) = \begin{cases} 
2, & \text{if } m=1, \\
m, & \text{if } m>1.
\end{cases} \)

Proposition 2.6: [2] The adjacency matrix of the wheel graph \( W_p \), \( A(W_p) \), has eigenvalues \( 1+\sqrt{p}, 1-\sqrt{p} \) and \( 2\cos \frac{2\pi r}{p} \), \( r = 0, 1, ..., p-2 \). Hence, \( \eta(W_p) = 2 \) if \( p=1(\text{mod}4) \) and \( \eta(W_p) = 0 \) otherwise.

Proposition 2.7: [4, p.72] i) The eigenvalues of the cycle \( C_p \) are of the form \( 2\cos \frac{2\pi r}{p} \), \( r = 0, 1, ..., p-1 \). According to this, \( \eta(CP) = 2 \) if \( p=0(\text{mod}4) \) and 0 otherwise.

ii) The eigenvalues of the path \( P_p \) are of the form \( 2\cos \frac{\pi r}{p+1} \), \( r = 1,2, ..., p \). And thus, \( \eta(P_P) = 1 \) if \( p \) is odd and 0 otherwise.

iii) The spectrum of the complete graph \( K_p \), consists of \( p-1 \) and -1 with multiplicity \( p-1 \).
iv) The spectrum of the complete bipartite graph $K_{m,n}$, consists of $\sqrt{mn}$, $-\sqrt{mn}$ and zero $m+n-2$ times.

**Corollary 2.8:** [4, p.234] If $G$ is a bipartite graph with an end vertex, and if $H$ is an induced subgraph of $G$ obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$. ■

**Corollary 2.9:** [4, p.235] Let $G_1$ and $G_2$ be two bipartite graphs in which $\eta(G_1) = 0$. If the graph $G$ is obtained by joining an arbitrary vertex of $G_1$ by an edge to an arbitrary vertex of $G_2$, then $\eta(G) = \eta(G_2)$. ■

**Coalescence Graphs**

To **identify** nonadjacent vertices $u$ and $v$ of a graph $G$ is to replace the two vertices by a single vertex incident to all the edges which are incident in $G$ to either $u$ or $v$. Denote the resulting graph by $G/\{u, v\}$. To **contract** an edge $e$ of a graph $G$ is to delete the edge and then (if the edge is a link) identify its ends. The resulting graph is denoted by $G/e$.

**Definition 2.10:** [7] Let $(G_1, u)$ and $(G_2, v)$ be two graphs rooted at vertices $u$ and $v$, respectively. We attach $G_1$ to $G_2$ (or $G_2$ to $G_1$) by identifying the vertex $u$ of $G_1$ with the vertex $v$ of $G_2$. Vertices $u$ and $v$ are called **vertices of attachment**. The vertex formed by their identification is called the **coalescence vertex**. The resulting graph $G_1 \circ G_2$ is called the **coalescence (vertex identification)** of $G_1$ and $G_2$.

**Definition 2.11:** [7] Let $\{(G_1, v_1), (G_2, v_2), \ldots, (G_t, v_t)\}$ be a family of not necessary distinct connected graphs with roots $v_1$, $v_2$, $\ldots$, $v_t$, respectively. A connected graph $G= G_1 \circ G_2 \circ \ldots \circ G_t$ is called the **multiple coalescence** of $G_1, G_2, \ldots, G_t$ provided that the vertices $v_1, v_2, \ldots, v_t$ are identified to reform the coalescence vertex $v$. The $t$-**tuple coalescence graph** is denoted by $G$ is the multiple coalescence of $t$ isomorphic copies of a graph $G$. In the same ways $G_1 \circ G_2 \cdots \circ G_t$ is the multiple coalescence of $G_1$ and $t$ copies of $G_2$.

**Remark 2.12:** [7] All coalesced graphs have $v$ as a common cut vertex. Some graphs and their operations will, herein, be illustrated in Figure 2.2.

![Figure 2.2](image)

**Definition 2.13:** [7] Let $G$ be a graph consisting of $n$ vertices and $L = \{H_1, H_2, \ldots, H_n\}$ be a family of rooted graphs. Then, the graph formed by attaching $H_k$ to the $k$-th ($1 \leq k \leq n$) vertex of $G$ is called the **generalized rooted product** and is denoted by $G(L)$; $G$ itself is called the **core** of $G(L)$. If each member of $L$ is isomorphic to the rooted graph $H$, then the graph $G(L)$ is denoted by $G(H)$. Recall $G_1$, $G_2$ and $G_3$ from Figure 2.2. Then, we have
Definition 2.14: [7] The generalization of the rooted product graphs is called the F-graphs, which are consecutively iterated rooted products defined as: \( F^0 = K_1, F^1 = G = H, F^i = G(H), \ldots, F^s = F(H), s \geq 1. \)

Definition 2.15: [7] A family of dendrimers \( D^k \) \((k \geq 0)\) is just a rooted product graph which is defined as follows:
\[
D^0 = K_1, \quad D^1 = G = H, \quad D^{k+1} = D^k(G) = D^k(H), k \geq 1.
\]

In general, \( D^k \) \((k \geq 1)\) is constructed from \( D \), and the number of copies of \( H \) attached to \( D \) obeys some fixed generation law. Hence, \( D^k \) is \( D \) with \( G \) attached to each vertex of \( D \) which is not in \( D^k \), that is to each \( u \in V(D) \setminus V(D^k) \), \( k \geq 1. \)

3 Nullity of Dendrimer Graphs

In this section, we determine the nullity of dendrimer graphs \( D^k \), \( k \geq 0 \), where \( D^1 = G \) of some known graphs such as \( C_p, P_p, K_p \) and \( K_{m,n} \). In each case, we consider that the nullity of the dendrimer graph \( D^0 \) is defined to be, \( \eta(D^0) = \eta(K_1) = 1. \)

The dendrimer \( C_p^k \) for the cycle \( C_p \) is a connected graph with order
\[
P(C_p^k) = p + p(p - 1)^2 + \cdots + p(p - 1)^{k - 1} = \sum_{i=1}^{k} P(p - 1)^{i - 1}. \quad \text{And size}
\]
\[
q(C_p^k) = q + pq + p(p - 1)q + \cdots + p(p - 1)^{k - 2}q
\]
\[
= p + p^2 + p^2(p - 1) + \cdots + p^2(p - 1)^{k - 2} = p + p^2 \sum_{i=2}^{k} (p - 1)^{i - 2}. \]

Moreover, the diameter of \( C_p^k \) is \((2k - 1), diam(C_p)\). Also for \( k > 1 \), the degrees of each vertex of \( C_p^k \) is either 2 or 4.
**Proposition 3.1:** For a dendrimer graph $C^k_p$, $k \geq 1$, we have:

i) If $p = 4n$, $n = 1, 2, \ldots$, then $\eta(C^1_{4n}) = 2$.

And for all $k$, $k \geq 2$, $\eta(C^k_{4n}) = \eta(C^{k-1}_{4n}) + 4n(4n-1)^{k-2}$.

ii) If $p = 4n + 2$, $n = 1, 2, \ldots$, then $\eta(C^1_{4n+2}) = 0$, for all $k$, $k \geq 1$.

iii) If $p = 4n - 1$, $n = 1, 2, \ldots$, then $\eta(C^1_{4n-1}) = 0$, $\eta(C^2_{4n-1}) = 1$.

And for all $k$, $k \geq 3$, $\eta(C^k_{4n-1}) = 0$.

iv) If $p = 4n + 1$, $n = 1, 2, \ldots$, then $\eta(C^1_{4n+1}) = 0$. for all $k$, $k \geq 1$.

**Proof:** i) For $k = 1$ it is clear that $\eta(C^1_{4n}) = 2$, $n = 1, 2, \ldots$, by Proposition 2.7 (i). For $k = 2$, $C^1_{4n}$ ($C^2_{4n}$), is a rooted product of $C^1_{4n}$ and $C^2_{4n}$. So we need to prove that $\eta(C^2_{4n}) = 2 + 4n$. Let $x_{i,j}$, $i, j = 1, 2, \ldots, 4n$ be a weighting for the vertex $v_{i,j}$ in $C^2_{4n}$, $n = 1, 2, \ldots$, as indicated in Figure 3.1.

![Figure 3.1. A weighting of $C^2_{4n}$, $n = 1, 2, \ldots$](image)

From the condition that $\sum_{w \in N_G(v)} w = 0$, for all $v$ in $C^2_{4n}$, $n = 1, 2, \ldots$, we have, for the cycles identified with the vertices $v_{i,1}$.

For $j = 1, 3, \ldots, 4n - 3$.

\[
x_{i,j} + x_{i,j+2} = 0 \quad \Rightarrow \quad x_{i,j} = -x_{i,j+2}
\]

\[
x_{i,j+1} + x_{i,j+3} = 0 \quad \Rightarrow \quad x_{i,j+1} = -x_{i,j+3}
\]

\[
\vdots
\]

\[
x_{i,4n,j} + x_{i,4n,j+2} = 0 \quad \Rightarrow \quad x_{i,4n,j} = -x_{i,4n,j+2}
\]

And, for $j = 2, 4, \ldots, 4n - 2$.

\[
x_{i,j} + x_{i,j+2} = 0 \quad \Rightarrow \quad x_{i,j} = -x_{i,j+2}
\]

\[
x_{i,j+1} + x_{i,j+3} = 0 \quad \Rightarrow \quad x_{i,j+1} = -x_{i,j+3}
\]

\[
\vdots
\]

\[
x_{i,4n,j} + x_{i,4n,j+2} = 0 \quad \Rightarrow \quad x_{i,4n,j} = -x_{i,4n,j+2}
\]

Also, from the condition that $\sum_{w \in N_G(v)} w = 0$, for all $v$ in the central cycle $C^1_{4n}$, we have,

For $i = 1, 3, \ldots, 4n - 3$.

\[
x_{i,1} + x_{i+2,1} = 0 \quad \Rightarrow \quad x_{i,1} = -x_{i+2,1}
\]

And, for $i = 2, 4, \ldots, 4n - 2$.

\[
x_{i,1} + x_{i+2,1} = 0 \quad \Rightarrow \quad x_{i,1} = -x_{i+2,1}
\]
Therefore, for each i in the Equations (3.1), (3.2) and (3.4) we have used exactly two non-zero independent variables, one of which in the weight of $x_{i,1}$, where i is odd and the other in the weight of $x_{i,1}$, where i is even. And from Equation (3.2) we have used $4n$ non-zero independent variables.

Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of $C^2_{4n}, n = 1, 2, ..., = 2 + 4n$.

On the other hand, we have $\eta(C_{4n}) = 2$, $n = 1, 2, ..., , \text{ by Lemma 2.7 (i). But } C^2_{4n} = C_{4n}(C_{4n})$, so each identification of a copy of $C_{4n}$ with a vertex of $C_{4n}$ adds (increases) one to the nullity of a dendrimer graph. Since $C_{4n}$ has $4n$ vertices; thus, $4n$ copies of a cycle $C_{4n}$ are identified to $C_{4n}$.

Therefore, $\eta(C_{4n}(C_{4n})) = \eta(C_{4n}) = 2 + 4n$.

For $k \geq 3$, we use the iteration $C^2_{4n}(C_{4n})$. This graph is a rooted product of $C^2_{4n}$ and $C_{4n}$. Since $C^2_{4n}$ is a dendrimer graph having $4n$ cycles and each cycle has $4n - 1$ vertices to be identified with new vertices, hence we attach a copy of $C_{4n}$ to $4n(4n - 1)$ vertices. Also, each copy of $C_{4n}$ adds (increases) one to the nullity of a dendrimer graph. Therefore, $\eta(C^2_{4n}(C_{4n})) = \eta(C^2_{4n}) + 4n(4n - 1) = 2 + 4n + 4n(4n - 1)$.

Similarly, we have, $\eta(C^k_{4n}(C_{4n})) = \eta(C^k_{4n}) + 4n(4n - 1)^k - 2$ where $k \geq 3$.

i) For each $k, k \geq 1$, there exists no non-trivial zero-sum weighting for $C^k_{4n+2}$, $n = 1, 2, ...$. Thus, by Theorem 2.3, $C^k_{4n+2}$ is non-singular.

ii) For $k = 1$, there exists no non-trivial zero-sum weighting for $C^k_{4n-1}$, $n = 1, 2, ...$. Thus, by Theorem 2.3, $C^k_{4n-1}$ is non-singular. For $k = 2$, $C^k_{4n-1}(C_{4n-1})$, is a rooted product of $C_{4n-1}$ and $C_{4n-1}$. To prove that $\eta(C^k_{4n-1}(C_{4n-1})) = 1$. Let $x_{i,j}, i, j = 1, 2, ..., 4n - 1$ be a weighting for

![Figure 3.2. A weighting of $C^2_{4n-1}, n = 1, 2, ...$](image)

vertex $v_{i,j}$ in $C^2_{4n-1}, n = 1, 2, ...$, as indicated in Figure 3.2.
Then, from the condition that \( \sum_{w \in \mathcal{N}_G(v)} f(w) = 0 \), for all \( v \) in \( C^2_{4n-1}, n = 1, 2, \ldots \), we have:

For \( i = 1, 2, \ldots, 4n-1 \), and \( j = 1, 3, \ldots, 4n-3 \).

\[ x_{i,j} + x_{i,j+2} = 0 \quad \Rightarrow \quad x_{i,j} = -x_{i,j+2} \quad \text{(3.5)} \]

And, for \( i = 1, 2, \ldots, 4n-1 \), and \( j = 2, 4, \ldots, 4n-2 \).

\[ x_{i,j} + x_{i,j+2} = 0 \quad \Rightarrow \quad x_{i,j} = -x_{i,j+2} \quad \text{(3.6)} \]

Hence, from Equations (3.5) and (3.6), we get:

\[ x_{1,1} = x_{1,4} = x_{1,5} = x_{1,8} = x_{1,9} = \ldots = x_{1,4n-4} = x_{1,4n-3} = -x_{1,2} \quad \text{(3.7)} \]

And \( x_{1,2} = x_{1,3} = x_{1,6} = x_{1,7} = \ldots = x_{1,4n-2} = x_{1,4n-1} = -x_{1,1} \quad \text{(3.8)} \)

Also, from the condition that \( \sum_{w \in \mathcal{N}_G(v)} f(w) = 0 \), for all \( v \) in \( C^2_{4n-1}, n = 1, 2, \ldots \), we have:

\[ x_{1,2} + x_{1,4n-1} + x_{2,1} + x_{4n-1,1} = 0 \]

Since \( x_{1,2} = x_{1,4n-1} = -x_{1,1} \), therefore, \( x_{2,1} = x_{4n-1,1} = x_{1,1} \quad \text{(3.9)} \)

Hence, from Equations (3.7), (3.8) and (3.9), we get:

For \( i = 1, 2, \ldots, 4n-1 \).

\[ x_{i,1} = x_{i,4} = x_{i,5} = x_{i,8} = x_{i,9} = \ldots = x_{i,4n-4} = x_{i,4n-3} = x_{i,1} \quad \text{(3.10)} \]

And, for \( i = 1, 2, \ldots, 4n-1 \).

\[ x_{i,2} = x_{i,3} = x_{i,6} = x_{i,7} = \ldots = x_{i,4n-2} = x_{i,4n-1} = -x_{1,1} \quad \text{(3.11)} \]

Therefore, each vertex of \( C^2_{4n-1}, n = 1, 2, \ldots \) has a weight \( x_{1,1} \) or \(-x_{1,1}\).

This means that there exists a non-trivial zero-sum weighting for \( C^2_{4n-1} \) used exactly one non-zero independent variable in a high zero-sum weighting of \( C^2_{4n-1} \). Hence, \( \eta(C^2_{4n-1}) = 1 \).

Finally, the proof of \( \eta(C^2_{4n-1}) = 0 \), for \( k \geq 3 \), is similar to that for \( k=2 \).

**iv) The proof is similar to that of part (ii).**

**Corollary 3.2:** For a dendrimer graph \( C^k_{4n}, k \geq 2 \), \( n = 1, 2, \ldots \), we have

\[ \eta(C^k_{4n}) = 2 + 4n \left[ \sum_{i=0}^{k-2} (4n-1)^i \right] \]

**Proof:** From Proposition 3.1 (i), we have:

\[ \eta(C^k_{4n}) = \eta(C^{k-2}_{4n}) + 4n (4n-1)^{k-2}, \text{ for } k \geq 2 \]

\[ \therefore \eta(C^k_{4n}) = \eta(C^{k-2}_{4n}) + 4n (4n-1)^{k-3} + 4n (4n-1)^{k-2} \]

\[ = \eta(C^{k-2}_{4n}) + 4n (4n-1)^{k-4} + 4n (4n-1)^{k-3} + 4n (4n-1)^{k-2} \]

\[ \vdash \]

\[ = \eta(C^2_{4n}) + 4n (4n-1)^{k-3} + \ldots + 4n (4n-1)^{k-3} + 4n (4n-1)^{k-2} \]

\[ = 2 + 4n + 4n (4n-1) + \ldots + 4n (4n-1)^{k-3} + 4n (4n-1)^{k-2} \]

\[ = 2 + 4n [1 + (4n-1) + \ldots + (4n-1)^{k-3} + (4n-1)^{k-2}] \]

\[ = 2 + 4n \left[ \sum_{i=0}^{k-2} (4n-1)^i \right] \]

\[ \therefore \eta(C^k_{4n}) = 2 + 4n \left[ \sum_{i=0}^{k-2} (4n-1)^i \right], \text{ for } k \geq 2.\]
Let $P_p$ be a path with usually labeled vertices $v_1, v_2, ..., v_p$. If $p$ is odd, this graph has a non-trivial zero-sum weighting, say $x, 0, -x, 0, ..., x$, which provides that an odd path is singular. Moreover, the dendrimer $P_p^k$ has order

$$q(P_p^k) = p + p(p-1) + p(p-1)^2 + ... + p(p-1)^{k-1}$$

and size

$$p(P_p^k) = p(P_p^k) - 1.$$

While, the diameter of $D^k$ depends on the choice of the rooted vertex. Also, the maximum degree will be either 3 or 4 for $k \geq 2$, while the minimum degree is 1.

In general, $diam(P_p^k) \leq (2k - 1)(p - 1)$, equality holds if $k=1$ or the rooted vertex is an end vertex of the path.

**Proposition 3.3:** For a dendrimer graph $P_p^k$, $k \geq 1$ we have:

i) If $p = 2n$, $n = 1, 2, ..., $ then $\eta(P_p^k) = 0$ for all $k$, $k \geq 1$.

ii) If $p = 2n + 1$, $n = 1, 2, ..., $ and the rooted vertex has a non-zero weight, then $\eta(P_{2n+1}^k) = 1$ for all $k$, $k \geq 1$.

iii) If $p = 2n + 1$, $n = 1, 2, ..., $ and the rooted vertex has a zero weight, then $\eta(P_{2n+1}^k) = 1$, $\eta(P_{2n+1}^{2k}) = 2n + 1$, and $\eta(P_{2n+1}^{2k-1}) = (2n + 1)(2n)^{k-2} + \eta(P_{2n+1}^{2k-2})$, for all $k$, $k \geq 3$.

**Proof:** i) The proof is similar to that of Proposition 3.1 (ii).

ii) For $k = 1$, it is clear that $\eta(P_{2n+1}) = 1$ by Proposition 2.7 (ii). For $k = 2$, $P_{2n+1}^2 = P_{2n+1}^1(P_{2n+1})$, is a rooted product of $P_{2n+1}^1$ and $P_{2n+1}$. To prove that $\eta(P_{2n+1}^2) = 1$, let $x_{i,j}$, $i, j = 1, 2, ..., 2n + 1$ be a weighting for the vertex $v_{i,j}$ in $P_{2n+1}^2$, $n = 1, 2, ..., $ as indicated in Figure 3.3.

![Figure 3.3](image)

**Figure 3.3.** A weighting of $P_{2n+1}^2$, where the rooted vertex has a non-zero weight.

Then, from the condition that $\sum_{w \in N(v)} f(w) = 0$, for all $v$ in $P_{2n+1}^2$, $n = 1, 2, ..., $ we have:

For all $i$, $i = 1, 2, ..., 2n + 1$.

$$x_{i, 2n} = 0.$$ \hfill (3.12)

Because $x_{i, 2n}$ are the neighbors of the end vertices.

Also, for all $i, j$, for which $i, j = 1, 2, ..., 2n + 1$
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\[ x_{i,j} = -x_{i,j+2} \quad \text{and} \quad x_{i,j} = -x_{i+2,j} \quad \ldots \text{(3.13)} \]

Thus, from Equations (2.13) and (2.14), we get:

For \( i = 1, 2, \ldots, 2n + 1 \) and \( j = 2, 4, \ldots, 2n \).

\[ x_{i,j} = 0. \quad \ldots \text{(3.14)} \]

Hence, from the condition that \( \sum_{w \in V(G)} f(w) = 0 \), for all \( v \in P_{2n+1}^2 \) and Equations (3.13) and (3.14), we get:

\[ x_{1,2} + x_{2,1} = 0 \implies x_{2,1} = 0. \]

While, from Equation (3.13) and for all \( i \) and \( j \), for which \( i = 2, 4, \ldots, 2n \) \( j = 1, 2, \ldots, 2n + 1 \), we have:

\[ x_{i,j} = 0. \]

Therefore, each vertex of \( P_{2n+1}^2 \) has the weight 0 or \( x_{1,2n+1}^2 \) or \( -x_{1,2n+1} \). Thus, any high zero-sum weighting of \( P_{2n+1}^2 \) will use only one non-zero variable, say \( x_{1,2n+1} \). Therefore, \( \eta(P_{2n+1}^2) = 1 \) where the rooted vertex has non-zero weight, and for \( k \geq 3 \), similar steps for the proof hold as in the case where \( k = 2 \). Thus, any high zero-sum weighting of \( P_{2n+1}^2 \), \( k \geq 3 \), will use only one non-zero variable. Hence, \( \eta(P_{2n+1}^k) = 1 \).

iii) For \( k = 1 \), it is clear that \( \eta(P_{2n+1}^1) = 1 \) by Proposition 2.7 (ii). For \( k = 2 \),

\[ P_{2n+1}^2 = P_{2n+1}^2(P_{2n+1}^1), \]

is a rooted product of \( P_{2n+1}^2 \) and \( P_{2n+1}^1 \). To prove that \( \eta(P_{2n+1}^2) = 2n + 1 \), where the rooted vertex has zero weight, let the rooted vertex is neighbor of end vertex in \( P_{2n+1}^2 \), and let \( x_{i,j}, i, j = 1, 2, \ldots, 2n + 1 \) be a weighting for \( P_{2n+1}^2 \), \( n = 1, 2, \ldots \), as indicated in Figure 3.4.

![Figure 3.4](image)

Figure 3.4. A weighting of \( P_{2n+1}^2 \) where the rooted vertex has a zero weight.

Then, from the condition that \( \sum_{w \in V(G)} f(w) = 0 \), for all \( v \in P_{2n+1}^2 \), \( n = 1, 2, \ldots \), we have:

For all \( i, j \), for which \( i = 1, 2, \ldots, 2n + 1 \) and \( j = 2, 4, \ldots, 2n \).

\[ x_{i,j} = 0 \quad \ldots \text{(3.15)} \]

And, for all \( i \) and \( j \), for which \( i = 1, 2, \ldots, 2n + 1 \) and \( j = 1, 3, \ldots, 2n + 1 \).

\[ x_{i,j} = -x_{i,j+2} \quad \ldots \text{(3.16)} \]

Therefore, for each \( i \) we use one variable. Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of \( P_{2n+1}^2 \) is equal to \( 2n + 1 \). Hence, \( \eta(P_{2n+1}^2) = 2n + 1 \).
On the other hand, \( P_{2n+1}^2 = P_{2n+1}(P_{2n+1}) \), since \( P_{2n+1} \) has \( 2n+1 \) vertices to be attachment and each vertex adds (increases) one to the nullity, thus:
\[
\eta(P_{2n+1}^2) = (2n+1)*1 = 2n+1.
\]
For \( k = 3 \), use the iteration \( P_{2n+1}^k = P_{2n+1}(P_{2n+1}) \). Since, \( P_{2n+1}^2 \) is a dendrimer graph having \( 2n+1 \) paths and each path has \( 2n \) vertices to be attachment, thus we attach \( P_{2n+1} \) to \( (2n+1)(2n) \) vertices. But, each copy of \( P_{2n+1} \) adds (increases) one to the nullity of a dendrimer graph, and together the variable used in a high zero-sum weighting of \( P_{2n+1} \). Therefore,
\[
\eta(P_{2n+1}^3) = (2n+1)(2n) + \eta(P_{2n+1}^2)
= (2n+1)(2n) + 1.
\]
Similarly, we have: \( \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2}) \), for each \( k, k \geq 3. \]

**Corollary 3.4:** For a dendrimer graph \( P_{2n+1}^k, \ k \geq 2, \ n=1,2,..., \) and the rooted vertex has zero weight, we have:

i) If \( k \) is odd, \( k \geq 3 \), then: \( \eta(P_{2n+1}^k) = 1 + (2n+1) \sum_{i=1}^{k-1} (2n)^{2i-1}. \)

ii) If \( k \) is even, \( k \geq 2 \), then: \( \eta(P_{2n+1}^k) = (2n+1) \sum_{i=0}^{k-2} (2n)^{2i}. \)

**Proof:** i) From Proposition 3.3 (iii), we have:
\[
\eta(P_{2n+1}) = 1, \ \eta(P_{2n+1}^2) = 2n+1, \ \text{and}
\]
\[
\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2}), \ \text{for each} \ k, k \geq 3
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \eta(P_{2n+1}^{k-4})
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)^{k-2} + \eta(P_{2n+1}^2)
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1) + 1
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1) \sum_{i=1}^{k-1} (2n)^{2i-1} + 1.
\]

ii) From Proposition 3.3 (iii), we have:
\[
\eta(P_{2n+1}) = 1, \ \eta(P_{2n+1}^2) = 2n+1, \ \text{and}
\]
\[
\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2}), \ \text{for each} \ k, k \geq 3
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \eta(P_{2n+1}^{k-4})
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)^{k-4} + \eta(P_{2n+1}^2)
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1) + 1
\]
\[
\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)^{k-4} + \ldots + (2n)^{k-2}.
\]
\[
(2n + 1) \sum_{i=0}^{\frac{k-2}{2}} (2n)^{2i}.
\]

\[\eta(P_{2n+1}^k) = (2n + 1) \sum_{i=0}^{\frac{k-2}{2}} (2n)^{2i}, \text{ if } k \text{ is even, } k \geq 2.\]

The nullities of dendrimers of complete graphs are determined in the next proposition.

**Proposition 3.5:** For a dendrimer graph \( K_p^k \), \( k \geq 1 \) we have:

i) If \( p = 3 \), then \( \eta(K_p^k) = 0 \) for all \( k \), \( 2 \leq k \leq 1 \). And \( \eta(K_p^1) = 1 \).

ii) If \( p \geq 4 \), then \( \eta(K_p^k) = 0 \) for all \( k \), \( k \geq 1 \).

**Proof:** The proof is immediate by Proposition 3.1.

Every complete bipartite graph \( K_{m,n} \), \( m,n \geq 2 \) has exactly 3 distinct eigenvalues, while the dendrimer \( K_{m,n}^k \), \( k \geq 2 \), loses this property.

**Proposition 3.6:** For a dendrimer graph \( K_{m,n}^k \), \( k \geq 1 \), \( m,n \geq 2 \), we have:

\[\eta(K_{m,n}^k) = m + n - 2, \text{ and}\]

\[\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m + n)(m + n - 1)^{k-2}(m + n - 3), \text{ for all } k, \ k \geq 2.\]

**Proof:** For \( k = 1 \), it is clear that \( \eta(K_{m,n}^1) = m + n - 2 \) by Prop. 2.7(iv). For \( k = 2 \), \( K_{m,n}^2 = K_{m,n}(K_{m,n}) \) is a rooted product of \( K_{m,n} \) and \( K_{m,n} \). To prove that \( \eta(K_{m,n}^2) = (m + n - 2) + (m + n)(m + n - 3) \), which is the number of independent variables used in a high zero-sum weighting for \( K_{m,n}^2 \). For \( k \geq 3 \), we use the iteration \( K_{m,n}^3 = K_{m,n}^2(K_{m,n}) \), since \( K_{m,n}^2 \) is a dendrimer graph having \( (m + n) \) complete bipartite graphs \( K_{m,n} \), and each graph has \( (m + n - 1) \) vertices to be attached; hence, we attach \( K_{m,n} \) to \( (m + n)(m + n - 1) \) vertices, but each copy of \( K_{m,n} \) adds (increases) \( (m + n - 3) \) to the nullity of the dendrimer graph.

Thus, \( \eta(K_{m,n}^3) = \eta(K_{m,n}^{k-1}) + (m + n)(m + n - 1)^{k-2}(m + n - 3) \), for all \( k, \ k \geq 2.\)

**Corollary 3.7:** For a dendrimer graph \( K_{m,n}^k \), \( k \geq 2 \), \( m,n \geq 2 \),

\[\eta(K_{m,n}^k) = (m + n - 2) + (m + n)(m + n - 3) \frac{(m + n - 1)^{k-1} - 1}{m + n - 2}.
\]

**Proof:** From Proposition 2.15, we have: \( \eta(K_{m,n}) = m + n - 2 \), and

\[\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m + n)(m + n - 1)^{k-2}(m + n - 3), \text{ for all } k, \ k \geq 2.
\]

\[\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m + n)(m + n - 1)^{k-3}(m + n - 3)
\]

\[+ (m + n)(m + n - 1)^{k-2}(m + n - 3)
\]

\[= \eta(K_{m,n}^{k-1}) + (m + n)(m + n - 1)^{k-1}(m + n - 3)
\]

\[+ ... + (m + n)(m + n - 1)^{k-3}(m + n - 3)
\]

\[+ (m + n)(m + n - 1)^{k-2}(m + n - 3)\]
\[(m + n - 2) + (m + n)(m + n - 3) \]
\[+ (m + n)(m + n - 1) + (m + n)(m + n - 1)^k \]
\[+ ... + (m + n)(m + n - 1)^k (m + n - 3) \]
\[+ (m + n)(m + n - 1)^k (m + n - 3) \]
\[= (m + n - 2) + (m + n)(m + n - 3)[1 + (m + n - 1)] + ...(m + n - 1)^k - (m + n - 1)^k - 3 \]
\[= (m + n - 2) + (m + n)(m + n - 3) (m + n - 3), \text{ for all } k, \ k \geq 2. \]

**Star graphs** are special cases of complete bipartite graphs, namely $S_{1,n-1}$ is $K_{1,n-1}$ with a partite set consisting of a single vertex called the central vertex.

**Proposition 3.8:** For a dendrimer graph $S^k_{1,n-1}$, $k \geq 1$, $n \geq 3$, we have:

i) If the rooted vertex of $S_{1,n-1}$ is the central vertex, then

\[\eta(S_{1,n-1}) = n - 2, \ \eta(S^2_{1,n-1}) = n(n - 2), \text{ and} \]
\[\eta(S^k_{1,n-1}) = n(n - 1)^k(n - 2) + \eta(S^{k-1}_{1,n-1}), \text{ for all } k, \ k \geq 3. \]

ii) If the rooted vertex of $S_{1,n-1}$ is a non-central vertex, then

\[\eta(S_{1,n-1}) = n - 2, \text{ and } \eta(S^k_{1,n-1}) = n(n - 1)^k(n - 3) + \eta(S^{k-1}_{1,n-1}), \text{ for all } k, \ k \geq 2. \]

**Proof:** i) For $k = 1$, it is clear that $\eta(S_{1,n-1}) = n - 2$ by Proposition 2.7 (iv). For $k = 2$, $S^2_{1,n-1} = S_{1,n-1}(S_{1,n-1})$, is a rooted product of $S_{1,n-1}$ and $S_{1,n-1}$. To prove that $\eta(S^2_{1,n-1}) = n(n - 2)$; let $x_i, j, i, j = 1, 2, ..., n$ be a weighting for $S^2_{1,n-1}$, as indicated in Figure 3.5.

**Figure 3.5.** A weighting of $S^2_{1,n-1}$, where the rooted vertex of $S_{1,n-1}$ is the central vertex.

Then, from the condition that $\sum_{w \in \mathcal{N}_G(v)} f(w) = 0$, for all $v$ in $S^2_{1,n-1}$, we have:

\[x_{1,n} = x_{2,n} = ... = x_{n,n} = 0 \]
\[...(3.17) \]

And,
Nullity and Bounds to the Nullity of Dendrimer Graphs

\[ x_{1,1} + x_{1,2} + \ldots + x_{1,n-1} = 0 \]
\[ x_{2,1} + x_{2,2} + \ldots + x_{2,n-1} = 0 \]
\[ \vdots \]
\[ x_{n,1} + x_{n,2} + \ldots + x_{n,n-1} = 0 \]

Then,
\[ x_{1,n-1} = -x_{1,1} - x_{1,2} - \ldots - x_{1,n-2} \]
\[ x_{2,n-1} = -x_{2,1} - x_{2,2} - \ldots - x_{2,n-2} \]
\[ \vdots \]
\[ x_{n,n-1} = -x_{n,1} - x_{n,2} - \ldots - x_{n,n-2} \]

Then, from Equation (3.18), the number of independent variables used in a high zero-sum weighting of \( S_{l,n-1}^2 \) is equal to \( n(n-2) \).

Hence, \( \eta(S_{l,n-1}^2) = n(n-2) \).

For \( k = 3 \), use the iteration \( S_{l,n-1}^3 = S_{l,n-1}^2(S_{l,n-1}) \), since \( S_{l,n-1}^2 \) is a dendrimer graph having \( n \) star graphs \( S_{l,n-1} \) and each graph has \( n - 1 \) vertices to be attachment, thus we attach \( S_{l,n-1} \) to \( n(n-1) \) vertices. But also, each copy of \( S_{l,n-1} \) adds (increases) \( n - 3 \) to the nullity of a dendrimer graph, together the variable used in a high zero-sum weighting of \( S_{l,n-1} \).

Therefore,
\[ \eta(S_{l,n-1}^3) = n(n-1)(n-2) + \eta(S_{l,n-1}^2) \]
\[ = n(n-1)(n-2) + (n-2) . \]

Similarly, we have:
\[ \eta(S_{l,n-1}^k) = n(n-1)^{k-2} (n-2) + \eta(S_{l,n-1}^{k-2}) \], for each \( k, k \geq 3 \).

ii) The proof is similar to that of Proposition 3.6■

**Corollary 3.9:** For a dendrimer graph \( S_{l,n-1}^k, k \geq 2, n \geq 3 \), we have:

i) If \( k \) is odd, \( k \geq 3 \), and the rooted vertex of a graph \( S_{l,n-1} \) is its central vertex, then,
\[ \eta(S_{l,n-1}^k) = (n-2) + n(n-2) \sum_{i=0}^{k-1} (n-1)^{2i-1} . \]

ii) If \( k \) is even, \( k \geq 2 \), and the rooted vertex of a graph \( H = S_{l,n-1} \) is its central vertex, then:
\[ \eta(S_{l,n-1}^k) = n(n-2) \sum_{i=0}^{k-2} (n-1)^{2i} . \]

iii) For all \( k, k \geq 2 \), if the rooted vertex of a graph \( H = S_{l,n-1} \) is a non central vertex, then,
\[ \eta(S_{l,n-1}^k) = (n-2) + n(n-3) \frac{(n-1)^{k-1}-1}{n-2} , \text{for all } k, k \geq 2 . \]

**Proof:** i) From Proposition 3.8 (i), we have:
\[ \eta(S_{l,n-1}) = n-2, \ \eta(S_{l,n-1}^2) = n(n-2) , \text{and} \]
\[ \eta(S_{l,n-1}^k) = n(n-1)^{k-2} (n-2) + \eta(S_{l,n-1}^{k-2}) , \text{for all } k, k \geq 3 . \]
\[ \therefore \ \eta(S_{l,n-1}^k) = n(n-1)^{k-2} (n-2) + n(n-1)^{k-4} (n-2) + \eta(S_{l,n-1}^{k-4}) \]
\[ \begin{align*}
&= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)(n-2) + \eta(S^k_{1,n-1}) \\
&= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)(n-2) + (n-2) \\
&= n(n-2)[(n-1)^{k-2} + (n-1)^{k-4} + \ldots + (n-1)] + (n-2) \\
&= n(n-2) \sum_{i=1}^{k} (n-1)^{2i-1} + (n-2). \\
\end{align*} \]

\[ \therefore \eta(S^k_{1,n-1}) = (n-2) + n(n-2) \sum_{i=1}^{k} (n-1)^{2i-1}, \text{ if } k \text{ is odd, } k \geq 3. \]

ii) From Proposition 3.8 (i), we have:

\[ \begin{align*}
&= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)^2(n-2) + \eta(S^k_{1,n-1}) \\
&= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)^2(n-2) + n(n-2) \\
&= n(n-2)[(n-1)^{k-2} + (n-1)^{k-4} + \ldots + (n-1)^2 + 1] \\
&= n(n-2) \sum_{i=0}^{k-2} (n-1)^{2i}. \\
\end{align*} \]

\[ \therefore \eta(S^k_{1,n-1}) = n(n-2) \sum_{i=0}^{k-2} (n-1)^{2i}, \text{ if } k \text{ is even, } k \geq 2. \]

iii) From Proposition 3.8 (ii), we have:

\[ \begin{align*}
&= n(n-1)^{k-2}(n-3) + n(n-1)^{k-3}(n-3) + \eta(D^k_{1,n-1}) \\
&= n(n-1)^{k-2}(n-3) + n(n-1)^{k-3}(n-3) + \ldots + n(n-1)(n-3) + \eta(S^k_{1,n-1}) \\
&= n(n-1)^{k-2}(n-3) + n(n-1)^{k-3}(n-3) + \ldots + n(n-1)(n-3) + (n-3) + (n-2) \\
&= n(n-3)[(n-1)^{k-2} + (n-1)^{k-3} + \ldots + (n-1) + 1] + (n-2) \\
&= (n-2) + n(n-3) \frac{(n-1)^{k-1}-1}{n-2}. \\
\end{align*} \]

\[ \therefore \eta(S^k_{1,n-1}) = (n-2) + n(n-3) \frac{(n-1)^{k-1}-1}{n-2}, \text{ for all } k, k \geq 2. \]

4. Upper Bounds for the Nullity of Coalescence Graphs

In this section, we shall introduce and prove a lower and an upper bound for the nullity of the coalescence graph \( G_1 \circ G_2 \).

**Proposition 4.1:** For any singular graphs \( G_1 \) and \( G_2 \),

\[ \eta(G_1) + \eta(G_2) - 1 \leq \eta(G_1 \circ G_2) \leq \eta(G_1) + \eta(G_2) + 1 \]
**Proof:** Let $G_1$ and $G_2$ be two singular graphs of orders $p_1$ and $p_2$, respectively, thus first we label the vertices of $G_1$ by $u_1, u_2, ..., u_{p_1}$, with a high zero-sum weighting $x_1, x_2, ..., x_{p_1}$ and the vertices of $G_2$ by $v_1, v_2, ..., v_{p_2}$, with a high zero-sum weighting $y_1, y_2, ..., y_{p_2}$.

Assume that $u_i$ and $v_i$ are rooted vertices of $G_1$ and $G_2$ respectively. Then equality holds at the left if either or both rooted vertices are non-zero weighted because there exists a high zero-sum weighting for $G_1 \circ G_2$ which is the enlargement of high zero-sum weightings for $G_1$ and $G_2$ reducing or vanishing one non-zero weight at the identification vertex. See Figure 4.1 where $G_1 = G_2 = P_3$.

Moreover, strictly holds at the left side if both rooted vertices have zero weights in their high zero-sum weightings, because there exists a zero-sum weighting which is the union of both high zero-sum weightings of $G_1$ and $G_2$.

Equality holds at the right side if both rooted vertices are cut vertices with zero weights in their high zero-sum weightings, and each component obtained with a deleting of a rooted cut vertex is singular, because there exists a high zero-sum weighting for $G_1 \circ G_2$ that uses an extra independent variable further than the variables used in high zero-sum weightings of $G_1$ and $G_2$. See Figure 4.2.

Moreover, strictly holds at the right side if one rooted vertices does not satisfy the condition of equality as indicated above. ■

**Note:** Let $w$ be the identification vertex $w = (u = v)$ of $G = G_1 \circ G_2$. Then, by interlacing Theorem [2, p314], $|\eta(G) - \eta(G - w)| \leq 1$ i.e.

$|\eta(G) - \eta(G_1 - u) - \eta(G_2 - v)| \leq 1 \quad \forall \ u \in G_1, v \in G_2$. Hence,

$\eta(G_1 - u) + \eta(G_2 - v) - 1 \leq \eta(G_1 \circ G_2) \leq \eta(G_1 - u) + \eta(G_2 - v) + 1$. 

---

**Figure 4.1.** $G_1 \circ G_2$ where either or both rooted vertices have non-zero weight.

Moreover, strictly holds at the left side if both rooted vertices have zero weights in their high zero-sum weightings, because there exists a zero-sum weighting which is the union of both high zero-sum weightings of $G_1$ and $G_2$.

Equality holds at the right side if both rooted vertices are cut vertices with zero weights in their high zero-sum weightings, and each component obtained with a deleting of a rooted cut vertex is singular, because there exists a high zero-sum weighting for $G_1 \circ G_2$ that uses an extra independent variable further than the variables used in high zero-sum weightings of $G_1$ and $G_2$. See Figure 4.2.

Moreover, strictly holds at the right side if one rooted vertices does not satisfy the condition of equality as indicated above. ■

**Figure 4.2.** $G_1 \circ G_2$ where both rooted vertices are cut vertices with zero weight and each component obtained by the deleting of rooted vertex is singular.
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