COMMUTATOR COVERINGS OF SIEGEL THREEFOLDS

V. GRITSENKO\(^1\) AND K. HULEK\(^1\)

**Abstract.** We investigate the existence and non-existence of modular forms of low weight with a character with respect to the paramodular group \(\Gamma_t\) and discuss the resulting geometric consequences. Using an advanced version of Maass lifting one can construct many examples of such modular forms and in particular examples of weight 3 cusp forms. Consequently we find many abelian coverings of low degree of the moduli space \(A_t\) of \((1, t)\)-polarized abelian surfaces which are not unirational. We also determine the commutator subgroups of the paramodular group \(\Gamma_t\) and its degree 2 extension \(\Gamma^*_t\). This has applications for the Picard group of the moduli stack \(\breve{A}_t\). Finally we prove non-existence theorems for low weight modular forms. As one of our main results we obtain the theorem that the maximal abelian cover \(A_{t \text{com}}\) of \(A_t\) has geometric genus 0 if and only if \(t = 1, 2, 4, 5\). We also prove that \(A_{t \text{com}}\) has geometric genus 1 for \(t = 3, 7\).

**Introduction**

The main theme of this paper are cusp forms of small weight with respect to the paramodular group \(\Gamma_t\) and Siegel modular threefolds. Since cusp forms of weight 3 define canonical differential forms on the moduli space \(A_t\) of \((1, t)\)-polarized abelian surfaces, the existence or non-existence of such forms has important geometric consequences.

For \(t \geq 1\) the paramodular group \(\Gamma_t\) is not a maximal discrete group. One can add to it a number of exterior involutions to obtain a maximal normal extension \(\Gamma^*_t\) such that \(\Gamma^*_t / \Gamma_t\) is a product of \(\mathbb{Z}_2\) components. We then have the following tower of Siegel threefolds

\[
\begin{array}{ccccccc}
A_{t \text{com}} & \longrightarrow & A^{(l)} & \longrightarrow & A_t & \longrightarrow & A^{(r)} & \longrightarrow & A^*_t \\
\| & & \| & & \| & & \| & & \\
\Gamma'_t \setminus \mathbb{H}_2 & \longrightarrow & \Gamma^{(l)} \setminus \mathbb{H}_2 & \longrightarrow & \Gamma_t \setminus \mathbb{H}_2 & \longrightarrow & \Gamma^{(r)} \setminus \mathbb{H}_2 & \longrightarrow & \Gamma^*_t \setminus \mathbb{H}_2 \\
\end{array}
\] (0.1)

where \(\Gamma'_t \subset \Gamma^{(l)} \subset \Gamma_t \subset \Gamma^{(r)} \subset \Gamma^*_t\) and \(\Gamma'_t\) is the commutator subgroup of \(\Gamma_t\). We call these varieties *commutative neighbours* of \(A_t\). They have very interesting properties. Neighbours \(A^{(r)}\) of the spaces \(A_t\) to the right, i.e. finite quotients, were studied in [GH2]. The coverings in (0.1) to the left of \(A_t\) are also galois with a finite abelian Galois group. This paper is devoted to neighbours \(A^{(l)}\) to the left of \(A_t\).

Recall that the geometric genus of \(A_t\) can be zero only for 20 exceptional polarizations \(t = 1, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36\) (see [G1]). Hypothetically, for all of them \(A_t\) is rational or unirational (see [GP] for an announcement of some results in this direction).

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and [G4]). Hence no weight 3 cusp forms should exist for these values of \( t \) (for an easy proof of this fact for \( t \leq 8 \) see Corollary 3.3). On the other hand we shall construct many examples of cusp forms of small weight \( (k \leq 3) \) with respect to \( \Gamma_t \) with a character. It follows that \( \mathcal{A}_t \) usually has a double modular covering with positive geometric genus. One of the main results of this paper is

**Theorem 0.1.** Let \( \tilde{\mathcal{A}}_t^{\text{com}} \) be a smooth projective model of the maximal abelian covering \( \tilde{\mathcal{A}}_t^{\text{com}} = \Gamma_t' \setminus \mathbb{H}_2 \) of \( \mathcal{A}_t \). Then

1. \( \tilde{\mathcal{A}}_t^{\text{com}} \) has geometric genus 0 if and only if \( t = 1, 2, 4 \) or 5;
2. \( \tilde{\mathcal{A}}_3^{\text{com}} \) and \( \tilde{\mathcal{A}}_7^{\text{com}} \) have geometric genus 1.

Therefore \( \tilde{\mathcal{A}}_3^{\text{com}} \) and \( \tilde{\mathcal{A}}_7^{\text{com}} \) are candidates for Calabi–Yau varieties. Some other examples of Siegel threefolds with genus 1 can also be found in section 3 (see Theorem 3.1 and Corollary 3.5).

The paper is organized as follows. In section 1 we collect examples of \( \Gamma_t \)-cusp forms with a character of weight one, two or three. The main ingredient for the construction of these forms is an advanced version of the Maaß lifting of Jacobi forms to modular forms. This lifting result was proved in [GN] and goes back to [G1]. As a consequence of this construction the modular forms which we obtain are in fact forms with respect to a bigger group \( \Gamma_t^+ \) which is an extension of \( \Gamma_t \) of order 2. Characters of \( \Gamma_t \) define coverings of \( \mathcal{A}_t \) (see definition 1.5). As a consequence we obtain many covering spaces of \( \mathcal{A}_t \) for which we can prove that they are not unirational, resp. in some cases we can also show that they have Kodaira dimension \( \geq 1 \). This result is contained in Corollary 1.6. Note that these spaces are covering spaces of rational or unirational moduli spaces \( \mathcal{A}_t \) of degree 2, 3 or 4.

In section 2 we determine, partly using the existence of characters from section 1, the commutator subgroups of \( \Gamma_t \) and \( \Gamma_t^+ \) (Theorem 2.1). This result could be classical (in the case of \( Sp_4(\mathbb{Z}) \) see [R] and [Ma]), but to our knowledge it does not exist in the literature. We also note some consequences of this result for the Picard group of the moduli stack \( \mathcal{A}_t \) of \( (1, t) \)–polarized abelian surfaces. In particular the quotient of \( \Gamma_t \) by its commutator subgroup gives the torsion part of the Picard group (see Proposition 2.3).

In section 3 we study the maximal abelian covering \( \tilde{\mathcal{A}}_t^{\text{com}} \) of \( \mathcal{A}_t \). The crucial technical ingredient in section 3 is an estimate on the vanishing order of Jacobi cusp forms (Proposition 3.2). This result seems interesting in its own right. As an easy corollary one obtains that there exist no weight 3 cusp forms for \( \Gamma_t \) for \( t \leq 8 \) (Corollary 3.3). Among the exceptional polarizations the polarization \( (1, 3) \) looks very interesting. There are twelve commutative neighbours of the threefold \( \mathcal{A}_3 \) to the left. Six of them have geometric genus 1 and the others have genus 0. We also found an example of a Siegel modular threefold, namely a 3:1-covering of \( \mathcal{A}_6 \), which has geometric genus 2 (see Theorem 3.1).

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\section{1. Roots of order 2, 3, 4, 6 and 12 of Siegel modular forms}

In this section we construct roots of Siegel modular forms with respect to the paramodu-
lar group $\Gamma_t$ ($t \in \mathbb{N}$). They will be again modular forms with respect to the full paramodular group (no level structure involved) but with a character $\chi : \Gamma_t \to \mathbb{C}^*$. We will use two realizations of the paramodular group. Let us fix a free $\mathbb{Z}$-module $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$. We define a skew-symmetric form $W_t$ on $L$ as follows

$$W_t(x, y)e_1 \wedge e_2 \wedge e_3 \wedge e_4 = -x \wedge y \wedge w_t, \quad w_t = te_1 \wedge e_3 + e_2 \wedge e_4.$$ 

The group

$$\tilde{\Gamma}_t = Sp(W_t, \mathbb{Z}) = \{g : L \to L \mid W_t(gx, gy) = W_t(x, y)\}$$

is called the integral paramodular group of level $t$. It is easy to check that

if $g = (g_{ij}) \in \tilde{\Gamma}_t$, then $g_{12} \equiv g_{14} \equiv g_{32} \equiv g_{34} \equiv 0 \mod t$.

The paramodular group $\tilde{\Gamma}_t$ is conjugate to a subgroup of the rational symplectic group $Sp_4(\mathbb{Q})$:

$$\Gamma_t := I_t\tilde{\Gamma}_t I_t^{-1} = \left\{ \begin{pmatrix} * & t^* & * & * \\ * & * & t^{-1}\ast & * \\ * & t^* & * & * \\ t^* & t^* & t^* & \ast \end{pmatrix} \in Sp_4(\mathbb{Q}) \mid \text{all } \ast \text{ are integral} \right\}$$

where $I_t = \text{diag}(1, 1, 1, t)$. This is the second realization of the paramodular group. The quotient

$$\mathcal{A}_t = \Gamma_t \setminus \mathbb{H}_2,$$

where $\mathbb{H}_2$ is the Siegel upper half-plane of genus 2, is isomorphic to the moduli space of abelian surfaces with a polarization of type $(1, t)$.

A modular form of weight $k$ with respect to $\Gamma_t$ with a character $\chi : \Gamma_t \to \mathbb{C}^*$ is a holomorphic function on $\mathbb{H}_2$ which for arbitrary $M \in \Gamma_t$ satisfies the functional equation

$$F(M < Z >) = \chi(M)\text{det}(CZ + D)^k F(Z) \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t.$$ 

We denote the space of such modular (cusp) forms by $\mathcal{M}_k(\Gamma_t, \chi)$ ($\mathcal{M}_k(\Gamma_t, \chi)$ respectively). Here we admit a character of the paramodular group $\Gamma_t$ in order to construct roots of certain orders of modular forms with respect to $\Gamma_t$ with trivial character. We remark that no level structure is involved in our considerations.

The group $\Gamma_t$ is not a maximal discrete group acting on $\mathbb{H}_2$ if $t \neq 1$. It has a normal extension $\Gamma_t^*$ such that $\Gamma_t^*/\Gamma_t \cong (\mathbb{Z}/2)^{\nu(t)}$ where $\nu(t)$ is the number of distinct prime divisors of $t$. The modular forms which we construct in this paper are forms with respect to a double extension of $\Gamma_t$:

$$\Gamma_t^+ = \langle \Gamma_t, V_t \rangle \quad \text{where} \quad V_t = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

The double quotient

$$\mathcal{A}_t \xrightarrow{2:1} \mathcal{A}_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2$$
of $\mathcal{A}_t$ can be interpreted as a moduli space of lattice-polarized K3 surfaces for arbitrary $t$ or as the moduli space of Kummer surfaces of $(1,p)$-polarized abelian surfaces for a prime $t = p$ (see [GH2]).

The main tool of the construction of modular forms of type $\sqrt{F(Z)}$, where $F(Z) \in M_k(\Gamma_t)$ is a new variant of the lifting of Jacobi forms to Siegel modular forms proposed in [GN, Theorem 1.12]. The datum for this lifting is a Jacobi form $\phi_{k,R}(\tau, z)\big| \phi_{k,R}(\tau, z)\big| \in \mathbb{M}_k(\Gamma_t)$ together with a character of the full Jacobi group. We define Jacobi forms as modular forms with respect to the maximal parabolic subgroup $\Gamma_{1,\infty}$, i.e.

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \right\}. $$

Let $k$ and $R$ be integral or half-integral. We choose one of the holomorphic square roots $\sqrt{\det(Z)}$ by the condition $\sqrt{\det(Z/i)} > 0$ for $Z = i1$.

We call a holomorphic function $\phi(\tau, z)$ on $\mathbb{H}_1 \times \mathbb{C}$ a Jacobi form of weight $k$ and index $R$ with a multiplier system (or a character) $v : \Gamma_{\infty} \to \mathbb{C}^*$ if the function

$$\tilde{\phi}(Z) := \phi(\tau, z) \exp(2\pi i R\omega), \quad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2,$$

satisfies the functional equation

$$\tilde{\phi}(M < Z >) = v(M)\det(CZ + D)^k\tilde{\phi}(Z) \quad \text{for any } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\infty$$

and $\phi$ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, l} f(n, l) \exp(2\pi i (n\tau + lz))$$

where the summation is taken over $n$ and $l$ from some free $\mathbb{Z}$-modules depending on $v$. The condition $f(n, l) = 0$ unless $4Rn - l^2 \geq 0$ is equivalent to the holomorphicity of $\phi$ at infinity. The form $\phi(\tau, z)$ is called a Jacobi cusp form if $f(n, l) = 0$ unless $4Rn - l^2 > 0$. We call the number $4Rn - l^2$ the norm of the index of the Fourier coefficient $f(n, l)$. We denote the finite dimensional space of such Jacobi forms (cusp forms) by $J_{k, R}(v)$ ($J_{c, k, R}(v)$ respectively).

If the function $\phi$ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n \geq 0, l} f(n, l) \exp(2\pi i (n\tau + lz)),$$

then we call it a weak Jacobi form. The space $J^w_{k, R}(v)$ of all such forms is again finite dimensional.
We remark that the group $\Gamma_\infty/\{\pm1_2\} \cong SL_2(\mathbb{Z}) \rtimes H(\mathbb{Z})$, where $H(\mathbb{Z})$ is the integer Heisenberg group, is usually called the Jacobi group. We use the following embeddings of $SL_2(\mathbb{Z})$ and $H(\mathbb{Z})$ into $\Gamma_\infty$

$$i_\infty\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \left(\begin{array}{cccc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad H(\mathbb{Z}) \cong \left\{ \lambda, \mu, \kappa \in \mathbb{Z} \mid [\lambda, \mu; \kappa] = \left(\begin{array}{cccc} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{array}\right) \right\}. \quad (1.3)$$

Here we admit Jacobi forms of half-integral indices. This is the only difference between the definition of the Jacobi forms given above and the definition of $[EZ]$.

The best known examples of forms with half-integral indices are the Jacobi triple product and the quintuple product. The Jacobi theta-series, which was, maybe, the first example of a Jacobi modular form, is defined as

$$\vartheta(\tau, z) = \sum_{n \equiv 1 \text{mod} 2} (-1)^{n-1} \exp\left(\frac{\pi in^2}{4}\tau + \pi inz\right) = \sum_{m \in \mathbb{Z}} \left(\frac{-4}{m}\right) q^{m^2/8} r^{m/2}, \quad (1.4)$$

where $q = \exp(2\pi i\tau)$, $r = \exp(2\pi iz)$ and

$$\left(\frac{-4}{m}\right) = \begin{cases} 
+1 & \text{if } m \equiv \pm1 \text{ mod } 4 \\
0 & \text{if } m \equiv 0 \text{ mod } 2.
\end{cases}$$

This is a Jacobi form of weight $1/2$ and index $1/2$, i.e. an element of $J_{1/2, 1/2}(v_\eta \times v_H)$. The multiplier system $v_\eta^3 \times v_H$ is induced by the $SL_2(\mathbb{Z})$-multiplier system $v_\eta$ of the Dedekind $\eta$-function and the following character of the integral Heisenberg group

$$v_H([\lambda, \mu; \kappa]) := (-1)^{\lambda+\mu+\lambda\mu+\kappa}. \quad (1.5)$$

We recall the famous Jacobi triple product formula

$$\vartheta(\tau, z) = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n). \quad (1.6)$$

The quintuple product is a Jacobi form of weight $1/2$ and index $3/2$

$$\vartheta_{3/2}(\tau, z) = \sum_{n \in \mathbb{Z}} \left(\frac{12}{n}\right) q^{n^2/24} r^{n/2} \in J_{3/2, 3/2}(v_\eta \times v_H) \quad (1.7)$$

where

$$\left(\frac{12}{n}\right) = \begin{cases} 
1 & \text{if } n \equiv \pm1 \text{ mod } 12 \\
-1 & \text{if } n \equiv \pm5 \text{ mod } 12 \\
0 & \text{if } (n, 12) \neq 1.
\end{cases}$$

It is related to the Jacobi theta-series by the formula

$$\vartheta_{3/2}(\tau, z) = \frac{\eta(\tau) \vartheta(\tau, 2z)}{\vartheta(\tau, z)}. $$
Many examples of Jacobi forms of half-integral indices were constructed in [GN]. One can, for instance, define Hecke operators of different types on Jacobi forms. The first type of Hecke operator is the map

$$\phi(\tau, z) \mapsto \phi(\tau, az) \quad (a \in \mathbb{N}).$$

For the Jacobi theta-series we have

$$\vartheta_a := \vartheta(\tau, az) \in J_{1/2,1/2}a(v_\eta^3 \times v_H^a).$$

The second type of Hecke operator which we shall use here are the operators $T^{(Q)}_-(m)$ acting on Jacobi forms in $J_{k,R}(v_\eta^{24/Q} \times v_H^\varepsilon)$, where $Q = 1, 2, 3, 4, 6$ or $12$, $\varepsilon \equiv 2R \mod 2$, the weight $k$ is an integer and $m \equiv 1 \mod Q$. We assume for simplicity that $QR \in \mathbb{N}$ as well. By definition

$$\tilde{\phi}_{kT^{(Q)}_-(m)}(Z) = m^{2k-3} \sum_{\sigma_a \in SL_2(\mathbb{Z})} d^{-k} v_\eta^{24/Q}(\sigma_a) \phi\left(\frac{a\tau + bQ}{d}, az\right) \exp(2\pi imR\omega) \quad (1.8)$$

where $\sigma_a \in SL_2(\mathbb{Z})$ such that $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \mod Q$. One can show that

$$\tilde{\phi}_{kT^{(Q)}_-(m)}(m) \in J_{k,mR}(v_\eta^{24/Q} \times v_H^\varepsilon)$$

(see [GN, Lemma 1.7]). The lifting we shall use in this paper is a new variant of the lifting construction of [G1] proposed in [GN] (see also [G3] for the case of orthogonal groups). All these liftings are advanced versions of the Maass lifting. Let us consider a Jacobi cusp form $\phi \in J_{k,R}(v_\eta^{24/Q} \times v_H^\varepsilon)$, where $Q$, $R$, and $\varepsilon$ are as above. Then the function

$$\text{Lift}(\phi)(Z) = \sum_{m \equiv 1 \mod Q, m > 0} m^{2-k} (\tilde{\phi}_{kT^{(Q)}_-(m)}(Z)) \in \mathfrak{M}_k(\Gamma_t, \chi_Q) \quad (1.9)$$

is a non-zero cusp form with respect to the paramodular group $\Gamma_t$ with $t = QR \in \mathbb{N}$, where

$$\chi_Q : \Gamma_t \rightarrow \{ \sqrt{t} \}$$

is a character of order $Q$ of $\Gamma_t$ induced by $v_\eta^{24/Q} \times v_H^\varepsilon$ and $\varepsilon \equiv \frac{2t}{Q} \mod 2$. I.e. we have

$$\chi_Q|_{SL_2(\mathbb{Z})} = v_\eta^{24/Q}, \quad \chi_Q|_{H(\mathbb{Z})} = v_H^{2t/Q}. \quad (1.10a)$$

Here $SL_2(\mathbb{Z})$ and $H(\mathbb{Z})$ are realized as subgroups of $\Gamma_\infty \subset \Gamma_t$ (see (1.3)) and

$$\chi_Q([0, 0; \frac{K}{t}]) = \exp(2\pi i \frac{K}{Q}) \quad (1.10b)$$
where \( \{0, 0; \kappa/t \mid \kappa \in \mathbb{Z}\} \) is the center of the maximal parabolic subgroup \( \Gamma_{t, \infty} = \Gamma_\infty(\mathbb{Q}) \cap \Gamma_t \) of the group \( \Gamma_t \) ([GN, Theorem 1.12]). Moreover \( \text{Lift}(\phi) \) satisfies the equation

\[
\text{Lift}(\phi)(V_t < Z>) = \text{Lift}(\phi)(Z)
\]

for the involution (1.1). Thus \( \text{Lift}(\phi) \) is in fact a cusp form with respect to the double extension \( \Gamma_t^+ \) of \( \Gamma_t \), i.e.

\[
\text{Lift}(\phi) \in \mathfrak{M}_k(\Gamma_t^+, \chi_{Q, (-1)^k}),
\]

(1.11)

Here \( \chi_{Q, \pm} \) denotes a character of \( \Gamma_t^+ \) such that

\[
\chi_{Q, \pm}|\Gamma_t = \chi_Q, \quad \chi_{Q, \pm}(V_t) = \pm 1.
\]

We shall see in §2 that the last relation and (1.10a–b) determine the character \( \chi_{Q, \pm} \) uniquely. In the sequel we shall sometimes add an additional index \( \chi_{Q, \pm}^{(t)} \) to indicate for which paramodular group the character is defined.

We can consider the modular form \( \text{Lift}(\phi_{k,R}) \) as a root of order \( Q \) of a modular form with respect to \( \Gamma_t \) (\( t = QR \)) with a trivial character. This construction is specially interesting in the case of small weights. For example, there are no modular forms of weight 1 with respect to \( \text{Sp}_4(\mathbb{Z}) \) with trivial character. It follows from a well known result of Skoruppa, that there are also no Jacobi forms of weight one with respect to the full Jacobi group (see [Sk]). On the other hand we shall construct infinitely many \( \Gamma_t \)-forms, which are indeed cusp forms, of weight 1 with characters of order 4, 6 and 12. We will also construct several series of \( \Gamma_t \)-cusp forms of weight 2 and 3 for all \( t \neq 1, 4, 5 \). We remark that the first cusp form of weight 3 with respect to \( \Gamma_t \) with trivial character exists for \( t = 13 \) (see [G1]).

In §2 we shall determine the commutator of the groups \( \Gamma_t \) and \( \Gamma_t^+ \). For this we need the existence of certain characters of \( \Gamma_t^+ \) which we can obtain from the following lemma.

The non-trivial binary character of the modular group \( \Gamma_1 = \text{Sp}_4(\mathbb{Z}) \) was constructed in [R] and [Ma], where it was proved that the commutator subgroup of \( \text{Sp}_4(\mathbb{Z}) \) has index 2.

**Lemma 1.1.** There exist characters \( \chi_{Q, \pm}^{(t)} \) of order \( Q \) of the group \( \Gamma_t^+ \) in the following cases

\[
\chi_{2, \pm}^{(t)} \text{ for arbitrary } t, \quad \chi_{3, \pm}^{(t)} \text{ for } t \equiv 0 \text{ mod } 3, \quad \chi_{4, \pm}^{(t)} \text{ for } t \equiv 0 \text{ mod } 2.
\]

These characters are induced by the character \( \nu_{24/Q} \times \nu_{2t/Q} \).

**Proof.** Since \( \Gamma_t \) is a normal subgroup of \( \Gamma_t^+ \) of index 2 there exists a character of \( \Gamma_t^+ \) which is trivial on \( \Gamma_t \) and has value \(-1\) on \( V_t \). Hence it is enough to construct one character for \( \Gamma_t^+ \) of type \( \chi_{Q, \pm} \).

We consider the Jacobi-Eisenstein series

\[
e_{4,1}(\tau, z) = 1 + (r^2 + 56r + 126 + 56r^{-1} + r^{-2})q + \cdots \in J_{4,1}
\]

of weight 4 and index 1 (see [EZ, §2]). We set

\[
e_{4,m} = e_{4,1}|_{4T^-}(m) \in J_{4,m}
\]
where \( T_-(m) = T^{(1)}_-(m) \) is the Hecke operator (1.8). Using the lifting construction (1.9) we obtain

\[
\text{Lift}(e_4, m \eta^6 \vartheta^2) \in \mathcal{N}_8(\Gamma_{2m+2}, \chi_{2,+}), \quad \text{Lift}(e_4, m \eta^3 \vartheta^3) \in \mathcal{N}_7(\Gamma_{2m+3}, \chi_{2,-})
\]

where we put \( e_{4,0} = 1 \) for \( m = 0 \). For \( t = 1 \) the non-trivial character of \( \Gamma_1 \) is the character of the cusp form \( \text{Lift}(\eta^9 \vartheta) \). To obtain further characters we consider

\[
\begin{align*}
\text{Lift}(e_4, m \eta^3 \vartheta_2) & \in \mathcal{N}_6(\Gamma_{4m+8}, \chi_{4,+}) , \\
\text{Lift}(e_4, m \eta^3 \vartheta) & \in \mathcal{N}_6(\Gamma_{4m+2}, \chi_{4,+}) , \\
\text{Lift}(e_4, m \eta^2 \vartheta^2) & \in \mathcal{N}_6(\Gamma_{3m+3}, \chi_{3,+}) .
\end{align*}
\]

To complete the list of characters we add the character \( \chi_{4,+} \) of the group \( \Gamma_4 \). This is the character of the lifting \( \text{Lift}(\phi_{10,m} \vartheta^2) \), where \( \phi_{10,m} = \phi_{10,1}|_{10} T_-(m) \in J_{10,m} \). The Jacobi form \( \phi_{10,1} = \eta^{18} \vartheta^2 \) is the first Jacobi cusp form of index 1.

\[\square\]

Our next aim is to construct \( \Gamma_t \)-cusp forms of small weights. We shall explain later in this section why this is of geometric interest. To construct such forms we first define Jacobi cusp forms of weights between 1 and 2 using the Jacobi theta-series. The next lemma is an extended version of Lemma 1.18 of [GN].

**Lemma 1.2.** Let \( a, b, c, d \in \mathbb{N} \). Then

\[
\begin{align*}
\partial_a \partial_b & \in J_{1, \frac{1}{2}(a^2+b^2)}(v_\eta \times v_{H}^{a+b}) \quad \text{if } \frac{ab}{(a,b)} \neq 1 \\
(\partial_{3/2})_a (\partial_{3/2})_b & \in J_{1, \frac{1}{4}(a^2+b^2)}(v_\eta \times v_{H}^{a+b}) \quad \text{if } \frac{(ab, 6)}{ab} \neq 1 \\
\partial_a \partial_b \partial_c & \in J_{\frac{3}{2}, \frac{1}{2}(a^2+b^2+c^2)}(v_\eta \times v_{H}^{a+b+c}) \quad \text{if } \frac{abc}{(a,b,c)} \neq 1 \\
(\partial_{3/2})_a (\partial_{3/2})_b (\partial_{3/2})_c & \in J_{\frac{5}{2}, \frac{1}{4}(a^2+b^2+c^2)}(v_\eta \times v_{H}^{a+b+c}) \quad \text{if } \frac{(abc, 6)}{abc} \neq 1 \\
\partial_a \partial_b \partial_c \partial_d & \in J_{2, \frac{1}{2}(a^2+b^2+c^2+d^2)}(v_\eta \times v_{H}^{a+b+c+d}) \quad \text{if } \frac{abcd}{(a,b,c,d)} \neq 1 \\
(\partial_{3/2})_a (\partial_{3/2})_b (\partial_{3/2})_c (\partial_{3/2})_d & \in J_{2, \frac{5}{4}(a^2+b^2+c^2+d^2)}(v_\eta \times v_{H}^{a+b+c+d}) \quad \text{if } \frac{abcd}{(a,b,c,d)} \neq 1 .
\end{align*}
\]

Moreover

\[
\partial_a (\partial_{3/2})_b \in J_{1, \frac{1}{4}(a^2+3b^2)}(v_\eta \times v_{H}^{a+b}) \quad \text{if } \frac{a}{(a,b)} = 1 \text{ or } \frac{a}{(a,b)} = 2 \text{ or } \frac{b}{(a,b)} , 6 \neq 1 .
\]

**Proof.** The essential point is to prove that we have cusp forms. For this we consider the norm of the indices of the non-zero Fourier coefficients of the Jacobi forms written above. For example, for the third function we have

\[
\partial(\tau, az) \partial(\tau, bz) \partial(\tau, cz) = \sum_{n, l, m \in \mathbb{Z}} \left(\frac{-4}{n}\right) \left(\frac{-4}{m}\right) \left(\frac{-4}{l}\right) q^{\frac{1}{2}(n^2+m^2+l^2)} r^{\frac{1}{2}(an+bm+cl)} .
\]
Thus the norm of the index of \( f(N, L) \) is given by

\[
2(a^2 + b^2 + c^2)N - L^2 = \frac{1}{4} ((bn - am)^2 + (cn - al)^2 + (cm - bl)^2).
\]

For \( a, b, c \) satisfying the condition of the lemma the last sum can only be zero if at least one of the three indices \( n, m \) or \( l \) is even. For such \( (n, m, l) \) the Fourier coefficient is 0.

We have the same formula for the norm of the indices of the fourth function. In that case the norm is zero only if one of \( n, m \) or \( l \) has a common divisor with 6. Similarly, in the first two cases the norm is a sum of two squares, for the fifth and sixth Jacobi form it is a sum of four squares. For the last function we have two different Kronecker symbols in the product and the result follows by similar arguments.

\[
\square
\]

Using Jacobi forms of weight 1 from Lemma 1.2 and the lifting construction gives us five series of \( \Gamma_1 \)-cusp forms of weight 1.

**Lemma 1.3.** The following cusp forms of weight 1 exist:

\[
\text{Lift}(\eta \vartheta_a) \in \mathcal{M}_1(\Gamma^+_{a_2}, \chi_{6,-}), \quad \text{Lift}(\eta(\vartheta_{3/2}a)) \in \mathcal{M}_1(\Gamma^+_{18a_2}, \chi_{12,-})
\]

\[
\text{Lift}(\vartheta_a \vartheta_b) \in \mathcal{M}_1(\Gamma^+_{2(a^2+b^2)}, \chi_{4,-}), \quad \frac{ab}{(a, b)^2} \text{ is even}
\]

\[
\text{Lift}((\vartheta_{3/2}a)(\vartheta_{3/2}b)) \in \mathcal{M}_1(\Gamma^+_{18(a^2+b^2)}, \chi_{12,-}), \quad \left(\frac{ab}{(a, b)^2}, 6\right) \neq 1
\]

\[
\text{Lift}(\vartheta_a(\vartheta_{3/2}b)) \in \mathcal{M}_1(\Gamma^+_{3(a^2+b^2)}, \chi_{6,-}), \quad \left(\frac{a}{(a, b)}, 3\right) = 1 \lor \left(\frac{b}{(a, b)}, 2\right) = 2 \lor \left(\frac{b}{(a, b)}, 6\right) \neq 1.
\]

As we shall see later the cubes of the cusp forms of Lemma 1.3 define canonical differential forms on the corresponding modular threefolds. It is preferable to have cusp forms of weight 3 with a character of the smallest possible order. For example, the cubes of the first and the fourth forms from Lemma (1.3) have a character of order 2. To find differential forms on Siegel modular threefolds is one of the main aims of this section.

Another way to construct new Jacobi forms of half-integral index is to use the differential operators of Eichler-Zagier type. For example if

\[
\phi_1 \in J_{k_1, m_1}(v^d_{\eta} \times v^e_H), \quad \phi_2 \in J_{k_2, m_2}(v^d_{\eta} \times v^e_H)
\]

are two Jacobi forms of integral or half-integral indices, where \( \varepsilon_i = 0 \) or 1, then one can define the Jacobi form

\[
[\phi_1, \phi_2] = \frac{1}{2\pi i} (m_2 \frac{\partial \phi_1}{\partial z} \phi_2 - m_1 \phi_1 \frac{\partial \phi_2}{\partial z}) \in J^c_{k_1+k_2+1, m_1+m_2}(v^d_{\eta} + v^d_{\eta} \times v^{e_1+e_2}).
\]  

(1.12)

(See [EZ, Theorem 9.5] and [GN, Lemma 1.23]). For arbitrary integers \( a \) and \( b \) we obtain in this way the following Jacobi cusp forms

\[
\phi_{2, \frac{1}{2}(a^2+b^2)}(\tau, z) = \frac{4}{ab} [\vartheta_a, \vartheta_b] = 
\]

\[
\sum_{m, n \in \mathbb{Z}} (bn - am) \left(\frac{-4}{m}\right) \left(\frac{-4}{n}\right) q^{\frac{1}{2}(m^2+n^2)} r^\frac{1}{2}(am+bn) \in J^c_{2, \frac{1}{2}(a^2+b^2)}(v^6_{\eta} \times v^{a+b}),(1.13)
\]

\[
\phi_{3, \frac{1}{2}(a^2+b^2+ c^2 + d^2)}(\tau, z) = [\vartheta_a \vartheta_b, \vartheta_c \vartheta_d] \in J^c_{3, \frac{1}{2}(a^2+b^2+c^2+d^2)}(v^{12}_{\eta} \times v^\Sigma_H)
\]
where in the last formula $\Sigma := a + b + c + d$. These Jacobi forms have integral Fourier coefficients.

The Jacobi form $[\vartheta_a \vartheta_b, \vartheta_c \vartheta_d]$ is a cusp form of weight 3 with non-trivial binary character of the full Jacobi group. We can define three different types of such Jacobi cusp forms which give us altogether six series. The first three series are

\[ \eta^3 \vartheta_a \vartheta_b \vartheta_c \text{ of index } \frac{1}{2}(a^2 + b^2 + c^2), \quad (\text{I}) \]
\[ [\vartheta_a, \vartheta_b] \vartheta_c \vartheta_d \text{ of index } \frac{1}{2}(a^2 + b^2 + c^2 + d^2), \quad (a \neq b) \quad (\text{II-a}) \]
\[ [\vartheta_a \vartheta_b, \vartheta_c \vartheta_d] \text{ of index } \frac{1}{2}(a^2 + b^2 + c^2 + d^2). \quad (\text{II-b}) \]

In the III-series we use the quintuple product:

\[ \eta^2 (\vartheta_{3/2})_a \vartheta_b \vartheta_c \vartheta_d \text{ of index } \frac{1}{2}(3a^2 + b^2 + c^2 + d^2), \quad (\text{III-a}) \]
\[ \eta (\vartheta_{3/2})_{a_1} (\vartheta_{3/2})_{a_2} \vartheta_{b_1} \vartheta_{b_2} \vartheta_{b_3} \text{ of index } \frac{1}{2}(3(a_1^2 + a_2^2) + b_1^2 + b_2^2 + b_3^2), \quad (\text{III-b}) \]
\[ (\vartheta_{3/2})_{a_1} (\vartheta_{3/2})_{a_2} (\vartheta_{3/2})_{a_3} \vartheta_{b_1} \vartheta_{b_2} \vartheta_{b_3} \text{ of index } \frac{1}{2}(3(a_1^2 + a_2^2 + a_3^2) + b_1^2 + b_2^2 + b_3^2). \quad (\text{III-c}) \]

The III-series gives us new Jacobi forms if $a \neq b, c, d$ or $a_i \neq b_j$. The character of all Jacobi cusp forms of weight 3 constructed in this way is $v_\eta^{12} \times v_H^\Sigma$, where $\Sigma$ is the sum of the corresponding indices $a, b, c, d$.

In [G1] the first author constructed cusp forms of weight 3 with respect to the paramodular group $\Gamma_t$ with trivial character for all $t$ except

\[ t = 1, 2, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36. \quad (1.14) \]

We call these $t$ exceptional. As a corollary it follows that the geometric genus of any smooth model $\tilde{\mathcal{A}}_t$ of $\mathcal{A}_t$ is positive if $t$ is not exceptional (see below). Here we shall show the existence of many weight 3 cusp forms with a character for the exceptional values of $t$. Again we shall discuss the geometric relevance of this below.

**Theorem 1.4.**

1. *Let $t$ be one of the exceptional polarizations and assume $t \neq 1, 2, 4, 5, 8, 16$. Then there exists a cusp form of weight 3 with respect to $\Gamma_t$ with the character $\chi_2$ of order 2.*

2. *For $t = 8, 16$ there exists a cusp form of weight 3 with respect to $\Gamma_t$ with the character $\chi_4$ of order 4.*

3. *Let $t \equiv 0 \mod 3$ be exceptional and $t \neq 3, 9$. Then there exists a cusp form of weight 3 with respect to $\Gamma_t$ with the character $\chi_3$ of order 3.*

**Proof.** Let $\phi \in J_{3, \frac{1}{2}}^t(v_\eta^{12} \times v_H^t)$, where $t \in \mathbb{N}$. Then we can construct the lifting (1.9)

\[ \text{Lift}(\phi) \in \mathfrak{N}_3(\Gamma_t^+, \chi_{2,-}). \]
Below we give a list of Jacobi forms of weight 3 from the series I–III for the exceptional polarizations (1.14). This will prove the theorem.

We start the list of canonical cusp forms for the exceptional polarizations with the cubes of cusp forms of weight 1:

\[
(\text{Lift}(\eta \theta))^3 = \Delta_3^3 \in \mathcal{N}_3(\Gamma_3^+, \chi_{2,-})
\]

and

\[
(\text{Lift}(\eta \theta_2)(Z))^3 = F^{(12)}(Z) = \Delta_1((\tauz^{-2z} 4w)^3) \in \mathcal{N}_3(\Gamma_{12}^+, \chi_{2,-}).
\]

For \( t = 12 \) there exists another cusp form, namely

\[
\text{Lift}(\vartheta^2[\vartheta, \vartheta_3]) \in \mathcal{N}_3(\Gamma_{12}^+, \chi_{2,-})(t=12)
\]

(see the series II-a above). To see that this is a different form one can calculate the Fourier coefficients of the Jacobi forms we lift.

For \( t = 6, 7, 9, 10, 11, 14 \) and 24 we obtain

\[
\begin{align*}
\text{Lift}(\eta^3 \vartheta^2 \vartheta_2) & \in \mathcal{N}_3(\Gamma_6^+, \chi_{2,-}), \quad (t=6) \\
\text{Lift}(\vartheta^2[\vartheta, \vartheta_2]) & \in \mathcal{N}_3(\Gamma_7^+, \chi_{2,-}), \quad (t=7) \\
\text{Lift}(\eta^3 \vartheta \vartheta_2^2) & \in \mathcal{N}_3(\Gamma_9^+, \chi_{2,-}), \quad (t=9) \\
\text{Lift}(\vartheta \vartheta_2[\vartheta, \vartheta_2]) & \in \mathcal{N}_3(\Gamma_{10}^+, \chi_{2,-}), \quad (t=10) \\
\text{Lift}(\eta^3 \vartheta^2 \vartheta_3) & \in \mathcal{N}_3(\Gamma_{11}^+, \chi_{2,-}), \quad (t=11) \\
\text{Lift}(\eta^3 \vartheta \vartheta_2 \vartheta_3) & \in \mathcal{N}_3(\Gamma_{14}^+, \chi_{2,-}), \quad (t=14) \\
\text{Lift}(\eta^3 \vartheta \vartheta_2^2) & \in \mathcal{N}_3(\Gamma_{24}^+, \chi_{2,-}). \quad (t=24)
\end{align*}
\]

For the remaining values \( t = 12, 15, 18, 20, 30, 36 \) the Jacobi forms of the series I–III provide us with several \( \Gamma_t \)-cusp forms of weight 3. For \( t = 15 \) we have five cusp forms which are liftings of

\[
\vartheta \vartheta_3[\vartheta, \vartheta_2], \ \vartheta \vartheta_2[\vartheta, \vartheta_3], \ \vartheta^2[\vartheta_2, \vartheta_3], \ \eta^2 \vartheta \vartheta_3/2 \vartheta_2^3, \ \eta^2(\vartheta_3/2)2\vartheta^3. \quad (t=15)
\]

The liftings of these Jacobi forms generate a 4-dimensional subspace in \( \mathcal{N}_3(\Gamma_{15}^+, \chi_{2,-}) \). We remark that the first three Jacobi forms are obtained as “permutations” of theta-constants.

For \( t = 18 \) the series I–III give us again five Jacobi cusp forms

\[
\eta^3 \vartheta \vartheta_4, \ \eta(\vartheta_3/2)^2 \vartheta_2^3, \ \vartheta_2 \vartheta_3[\vartheta, \vartheta_2] \quad (t=18)
\]

(one can add two “permutations” in the third Jacobi form). Finally we get nine Jacobi cusp forms for \( t = 30 \) with character \( v_{12}^2 \times 1_H \)

\[
\begin{align*}
\eta^3 \vartheta \vartheta_2 \vartheta_5, \ \eta^2(\vartheta_3/2)^2 \vartheta^2 \vartheta_4, \ \eta^2(\vartheta_3/2)^3 \vartheta_3^3, \ \eta(\vartheta_3/2)^2 \vartheta_2^2 \vartheta_4, \ \vartheta_3 \vartheta_4[\vartheta, \vartheta_2] & \quad (t=30)
\end{align*}
\]
(one can add four “permutations” of the fifth Jacobi form) and six Jacobi forms for \( t = 36 \) with the same character \( v_1^{12} \times 1_H \)

\[
\eta^3 \vartheta_2 \vartheta_4^2, \quad \eta^2 (\vartheta_{3/2})_3 \vartheta_2^2, \quad \eta^2 (\vartheta_{3/2})_3 \vartheta_3^2, \quad \vartheta \vartheta_5 [\vartheta, \vartheta_3] \quad (t=36)
\]

(plus two “permutations” in the last Jacobi form).

A cusp form of weight 3 for \( \Gamma_8^+ \) with the character \( \chi_1^{(8)} \) of order 4 can be obtained, for example, by the lifting of a Jacobi form of type II-a

\[
\text{Lift}(\eta^2 [\vartheta, \vartheta_{3/2}]) \in \mathcal{H}_3(\Gamma_8^+, \chi_1^{(8)}).
\]

To construct a similar cusp form for \( \Gamma_{16}^+ \) we consider the Jacobi form

\[
\eta(\tau)^6 \phi_{0,4}(\tau, z) = \eta(\tau)^6 \frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)}.
\]

The function \( \phi_{0,4}(\tau, z) \) is a weak Jacobi form of weight 0 and index 4. One can prove that \( \eta^2 \phi_{0,4}(\tau, z) \) is a cusp form (see [GN, Lemma 1.21]). Thus

\[
\text{Lift}(\eta^6 \frac{\vartheta_3}{\vartheta}) \in \mathcal{H}_3(\Gamma_{16}^+, \chi_1^{(8)}).
\]

To prove the third claim of Theorem 1.4 we give a list of Jacobi cusp forms of weight 3 with the character \( v_8^8 \times 1_H \):

\[
\begin{align*}
\eta^5 \vartheta_2 \quad (t = 6) & \quad \eta [\vartheta, \vartheta_2] \vartheta_{3/2} \quad (t = 12) & \quad \eta^3 \vartheta_2 \vartheta_{3/2}^2 \quad (t = 15) \\
\eta^5 \vartheta_4 \quad (t = 24) & \quad \eta^4 \vartheta_3 \vartheta_{3/2} \quad (t = 18) & \quad \eta^2 [\vartheta_2, \vartheta_4] \quad (t = 30) \\
& \quad \eta^5 \vartheta_3 \vartheta_4 / \vartheta \quad (t = 36).
\end{align*}
\]

The lifted forms are cusp forms of weight 3 with the character \( \chi_3^{(8)} \).

At this point it is natural to discuss the geometric implications of our results. Weight 3 cusp forms are closely related to canonical differential forms on smooth models of the corresponding modular variety. If \( F \) is a cusp form of weight 3 with respect to a group \( \Gamma \), then \( \omega_F = F(Z) dZ \) is a holomorphic 3-form on the space \( \mathcal{A}_\Gamma^o = (\Gamma \setminus \mathbb{H}_2)^o \), where \(^o\) means that we consider the threefold outside the branch locus of the natural projection from \( \mathbb{H}_2 \) to \( \mathcal{A}_\Gamma \). A very useful extension theorem due to E. Freitag implies that such a form can be extended to any smooth model of \( \mathcal{A}_\Gamma \). To be more precise, let \( \Gamma \) be an arbitrary subgroup of \( Sp_4(\mathbb{R}) \), which contains a principal congruence subgroup \( \Gamma_1(q) \subset Sp_4(\mathbb{Z}) \) of some level \( q \). We then have the following

**Criterion.** (Freitag) An element \( \omega_F = F(Z) dZ \in H^0(\mathcal{A}_\Gamma^o, \Omega_3(\mathcal{A}_\Gamma^o)) \) can be extended to a canonical differential form on a non-singular model \( \overline{\mathcal{A}}_\Gamma \) of a compactification of \( \mathcal{A}_\Gamma \) if and only if the differential form \( \omega_F \) is square integrable.

**Proof.** See [F], Hilfsatz 3.2.1.
It is well known that a $\Gamma$-invariant differential form $\omega_F = F(Z)\,dZ$ is square-integrable if and only if $F$ is a cusp form of weight 3 with respect to the group $\Gamma$. Thus we have the following identity for the geometric genus of the variety $\tilde{A}_\Gamma$:

$$p_g(\tilde{A}_\Gamma) = h^{3,0}(\tilde{A}_\Gamma) = \dim_\mathbb{C} \mathfrak{M}_3(\Gamma). \quad (1.15)$$

We also remark at this point that $\Gamma_t$-cusp forms of weight 2 can be very useful when one wants to prove that some modular threefolds are of general type (see [GH1], [GS] and [S]). All these facts explain our interest in cusp forms of small weight $k$ ($k \leq 3$). To reformulate our above results in geometric terms we introduce the following

**Definition 1.5.** Let $\Gamma$ be a subgroup of $Sp_4(\mathbb{R})$ which contains a principal congruence-subgroup. For any character $\chi_\Gamma : \Gamma \to \mathbb{C}^*$ we define the threefold

$$A(\chi_\Gamma) = \ker(\chi_\Gamma) \setminus \mathbb{H}_2.$$ 

The covering $A(\chi_\Gamma) \to A_\Gamma = \Gamma \setminus \mathbb{H}_2$ is galois with a finite abelian Galois group.

**Corollary 1.6.** Let $t$ be one of the exceptional polarizations (see (1.14)).

1. If $t \neq 1, 2, 4, 5, 8, 16$, then the modular double covering

$$A(\chi_2) \to A_t$$

of the moduli space $A_t$ of abelian surfaces with polarization of type $(1, t)$ has positive geometric genus, and in particular the Kodaira dimension of $A(\chi_2)$ is not negative. Moreover, for $t = 12, 15, 18, 20, 30$ and 36 the Kodaira dimension of $A_t$ is positive.

2. If $t = 6, 12, 15, 18, 24, 30, 36$, then the threefold $A_t(\chi_3) \to A_t$ has positive geometric genus.

3. If $t = 8$ or 16, then the covering $A_t(\chi_4) \to A_t$ has positive geometric genus. In particular all these modular varieties are not unirational.

**Remark 1.7.** If one lifts a Jacobi form which contains a factor $\vartheta$ or $\vartheta_{3/2}$, then one has some information about the divisor of the lifted form (see [GN, Lemma 1.16]). Assume, e.g. that $\vartheta(\tau, az)$ is a factor of a Jacobi form $\phi$, then

$$\text{Div}_{A_t^+}(\text{Lift}(\phi)) \supset \sum_{d|a} H_{d^2}(d).$$

Here we denote by

$$H_D(b) = \pi_t^+(\{\tau Z = (\begin{smallmatrix} \tau & \bar{z} \\ z & \omega \end{smallmatrix}) \in \mathbb{H}_2 | a\tau + bz + t\omega = 0\})$$

the Humbert modular surface of discriminant $D = b^2 - 4at$ in $A_t^+$ where

$$\pi_t^+: \mathbb{H}_2 \to A_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2.$$
is the natural projection (see [GH2]). For example, we obtain
\[ \text{Div}_{A_{15}^+} (\text{Lift}(\eta^2 \vartheta_3/2 \vartheta_2^2)) \supset 3H_1 + 4H_4(2), \quad \text{Div}_{A_{15}^+} (\text{Lift}(\vartheta_3 \vartheta_2)) \supset 3H_1 + H_9(3). \]
The other cusp forms of weight 3 constructed above can be treated similarly.

The modular form \( \Delta_1 = \text{Lift}(\eta \vartheta) \) was studied in [GN], where it was proved that its divisor in \( A_3^+ \) is exactly equal to the Humbert modular surface with discriminant 1. Thus
\[ \text{Div}_{A_3^+} (\text{Lift}(\eta \vartheta^3)) = 3H_1, \quad \text{Div}_{A_{12}^+} (\text{Lift}(\eta \vartheta_2^3)) = 3H_1 + 3H_4(2). \]

On the other hand Brasch [B] has determined the branch locus of the map
\[ \pi_t : \mathbb{H}_2 \to A_t = \Gamma_t \setminus \mathbb{H}_2 \]
as well as the singularities of a toroidal compactification of \( A_t \) (at least in the prime number case). From his work it is possible to obtain the same information about coverings of \( A_t \). Altogether this makes it possible to write down an explicit effective canonical divisor on a suitable smooth model. This information is very useful if one wants to study the geometry of these modular varieties from the point of view of Mori theory. We also hope to find examples of modular varieties which are Calabi–Yau varieties. We hope to return to this in the future.

\section{2. Commutator subgroups}

The aim of this section is to determine the commutator subgroup of \( \Gamma_t \) and \( \Gamma_t^+ \). In the case of \( \Gamma_1 \) this is a classical result due to Reiner [R] and Maaß [Ma]. We shall also discuss some consequences of our result.

\begin{theorem}
For any integer \( t \geq 1 \) let \( t_1 = (t, 12) \) and \( t_2 = (2t, 12) \). If \( \Gamma_t' \) is the commutator subgroup of \( \Gamma_t \), resp. \( (\Gamma_t^+)' \) is the commutator subgroup of \( \Gamma_t^+ \), then the following holds:
(1) \( \Gamma_t / \Gamma_t' \cong \mathbb{Z}/t_1 \times \mathbb{Z}/t_2 \)
(2) \( \Gamma_t^+ / (\Gamma_t^+)' \cong \mathbb{Z}/2 \times \mathbb{Z}/t_2 \).
\end{theorem}

\textbf{Proof.} We shall proceed in several steps.

\textbf{Step 1:} Recall that we have the following two embeddings of \( SL_2(\mathbb{Z}) \) into the paramodular group \( \Gamma_t \):
\[ i_\infty \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cccc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad j_\infty \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & a' & 0 & b't^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & c't & 0 & d' \end{array} \right). \]

We consider the matrices
\[ A = i_\infty \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad B = j_\infty \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \]
It is well known [Ma, p. 130] that the commutator $SL_2(\mathbb{Z})'$ of $SL_2(\mathbb{Z})$ has index 12 in $SL_2(\mathbb{Z})$ and that
\[
SL_2(\mathbb{Z}) = \bigcup_{i=0}^{11} SL_2(\mathbb{Z})' T^i
\]
where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Applying this to the two subgroups in $\Gamma_t$ isomorphic to $SL_2(\mathbb{Z})$ via $i_\infty$, resp. $j_\infty$ it follows that in particular $A_{12}^{12}, B_{12}^{12} \in \Gamma'_t$.

As in [Ma] we can construct elements in $\Gamma'_t$ in the following way
\[
\begin{pmatrix} V & 0 \\ 0 & tV^{-1} \end{pmatrix} \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} V^{-1} & 0 \\ 0 & tV \end{pmatrix} \begin{pmatrix} 1_2 & -S \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} 1_2 & VS^tV - S \\ 0 & 1_2 \end{pmatrix}
\]
where $V = \begin{pmatrix} a & tb \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ and $S = \begin{pmatrix} m & n \\ n & t^{-1}k \end{pmatrix}$. In particular, we obtain for $W = VS^tV - S$ the following matrices.

1. $W = \begin{pmatrix} 2tn + tk & k \\ k & 0 \end{pmatrix}$ for $V = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
2. $W = \begin{pmatrix} 0 & -2n \\ -2n & 0 \end{pmatrix}$ for $V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3. $W = \begin{pmatrix} 0 & m \\ m & m + 2n \end{pmatrix}$ for $V = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

Together with $A_{12}^{12}, B_{12}^{12} \in \Gamma'_t$ we find that $A_{t^2}^{12}, B_{t^2}^{12} \in \Gamma'_t$.

**Step 2:** We consider the following subgroup of $\Gamma_t$:
\[
G = \bigcup_{l,m=0}^{t_2-1} \Gamma'_t A^l B^m.
\]

**Claim.** $G = \Gamma_t$.

To prove this claim we consider the matrix
\[
C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Our first aim is to show that $C$ is contained in $G$. Indeed, consider
\[
L = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A^{-1} B^t \in G, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma'_t A^{-1} \subset G
\]
(the latter follows from the corresponding statement in $SL_2(\mathbb{Z})$ via the inclusion $i_\infty$).

\[
X = \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_t.
\]
By a straightforward calculation

\[ C = LAXA^{-1}X^{-1}M \in G. \]

Recall from [G1, Lemma 2.2] that \( \Gamma_{t,\infty} = \Gamma_t \cap \Gamma_\infty(\mathbb{Q}) \) and the element

\[
J_t = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & t^{-1} \\
-1 & 0 & 0 & 0 \\
0 & -t & 0 & 0
\end{pmatrix}
\]

generate \( \Gamma_t \). Hence it is enough to show that \( \Gamma_{t,\infty} \) and \( J_t \) are contained in \( G \). The assertion about \( J_t \) follows since both copies of \( SL_2(\mathbb{Z}) \) which are contained in \( \Gamma_t \) are already contained in \( G \). Now consider \( g \in \Gamma_{t,\infty} \). Again using the two copies of \( SL_2(\mathbb{Z}) \) we can assume that

\[
g = \begin{pmatrix}
1 & 0 & 0 & k \\
r & 1 & k & 0 \\
0 & 0 & 1 & -r \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

But then

\[ gC^{-r}B^{-rkt}A^{kt} = \begin{pmatrix}
1 & 0 & k & k \\
0 & 1 & k & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in \Gamma_t' \]

by step 1 and this gives the claim.

**Step 3:** We shall now prove the theorem for \( \Gamma_t^+ \). Recall from Lemma 1.1 that there exist characters

\[ \chi_{t_2,\pm}: \Gamma_t^+ \to \{ \sqrt{1} \} \]

of order \( t_2 \) with \( \chi_{t_2,\pm}|SL_2(\mathbb{Z}) = v_H^{24/t_2} \), \( \chi_{t_2,\pm}|H(\mathbb{Z}) = v_H^\varepsilon \) with \( \varepsilon = 0 \) or \( 1 \) depending on \( t \) and \( \chi_{t_2,\pm}(V_t) = \pm 1 \). Since \( \Gamma_t \) is a normal subgroup of \( \Gamma_t^+ \) of index 2, there also exists a character of order two:

\[ \chi'_2: \Gamma_t^+ \to \{ \pm 1 \} \]

with \( \chi'_2|\Gamma_t = 1 \). Clearly \( \chi_{t_2,-} = \chi_{t_2,+} + \chi'_2 \). Together \( \chi_{t_2,+} \) and \( \chi'_2 \) define a surjective map

\[ \bar{\chi} = (\chi'_2, \chi_{t_2,+}): \Gamma_t^+ \to \mathbb{Z}/2 \times \mathbb{Z}/t_2. \]

On the other hand \( B = V_tAV_t^{-1} \), i.e. \( \chi(A) = \chi(B) \) for every character \( \chi \) of \( \Gamma_t^+ \). Together with step 2 this gives the claim.

**Step 4:** It remains to prove the theorem for \( \Gamma_t \). In view of step 2 this will be a consequence of the following facts:

1. There exists a character \( \chi_{t_2} \) of \( \Gamma_t \) of order \( t_2 \) with \( \chi_{t_2}(A) = \chi_{t_2}(B) = e^{2\pi i/t_2} \).
2. There exist characters \( \chi_{t_1} \) and \( \chi'_{t_1} \) of \( \Gamma_t \) with \( \chi_{t_1}(A) = e^{2\pi i/t_1} \), \( \chi_{t_1}(B) = 1 \) and \( \chi'_{t_1}(A) = 1, \chi'_{t_1}(B) = e^{2\pi i/t_1} \).
3. For every character \( \chi \) of \( \Gamma_t \) the equality \( \chi(A^{t_1}) = \chi(B^{t_1}) \) holds.
Let \( t_{\pm} \) be \( \gamma \) and \( \chi \) be the character of \( \Gamma \). Then the definition of \( \chi \) is given by Notation 2.2. The proof of Theorem 2.1 shows that a character \( \chi \) defined by \( \chi \) is symplectic it follows that \( a, \ldots, d \) are integer entries (because of a bad line break). Since the matrix \( g \) is symplectic it follows that \( ad - bc \equiv 1 \pmod{t} \), i.e.

\[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} \mod t \in SL_2(\mathbb{Z}/t).
\]

On the other hand \( v_\eta \) defines a character \( v_\eta^{24/t_1} : SL_2(\mathbb{Z}/t) \rightarrow \{ k \sqrt{1} \} \)

\[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} \mod t \mapsto v_\eta^{24/t_1}(\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix}).
\]

For every element \( g \in \Gamma \) we set \( \chi_t(g) := v_\eta^{24/t_1}(\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} \mod t) \).

It follows easily from (2.2) that this is indeed a character. The character \( \chi_t \) can then be defined by \( \chi_t(g) = \chi_t(V_t g V_t^{-1}) \). To prove (3) we recall from step 1 (take \( n = 0, k = -m = 1 \)) that \( a, a, \pm \) is determined by its values on \( \Gamma \)'s. We denote by \( A^t B^{-t} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma \).

But then also \( A^t B^{-t} \in \Gamma \) and hence \( \chi(A^t) = \chi(B^t) = 1 \) for every character \( \chi \) of \( \Gamma \).

\[ \square \]

**Notation 2.2.** The proof of Theorem 2.1 shows that a character \( \chi : \Gamma \rightarrow \mathbb{C}^* \) is determined by its values on \( A \) and \( B \). These values can be any root of order \( t_2 \) provided \( \chi(A^{t_2}) = \chi(B^{t_2}) \). We denote by \( \chi_{a, b} : \Gamma \rightarrow \mathbb{C}^* \) (\( 1 \leq a, b \leq t_2 \) and \( (a - b) \equiv t_2/t_1 \)) the unique character of \( \Gamma \) given by \( \chi_{a, b}(A) = e^{2\pi i a/t_2} \) and \( \chi_{a, b}(B) = e^{2\pi i b/t_2} \). Comparing this with the definition of \( \chi_Q \) from section 1 shows that \( \chi_Q = \chi_{a, a} \) if and only if \( aQ = t_2 \).

Similarly a character \( \chi : \Gamma^+ \rightarrow \mathbb{C}^* \) is determined by its values on \( A \) and \( V_t \). We denote by \( \chi_{a, a, \pm} \) the unique character of \( \Gamma^+ \) such that \( \chi_{a, a, \pm}(A) = e^{2\pi i a/t_2} \) and \( \chi_{a, a, \pm}(V_t) = \pm 1 \). Then \( \chi_{a, a, \pm} \mid \Gamma = \chi_{a, a} \).

The classical construction of the character \( \chi_2 \) of order 2 for \( Sp(4) \) can be easily generalized to \( \Gamma \) for \( t \) odd. Again it is more convenient to work with the group \( \Gamma_t = Sp(W_t, \mathbb{Z}) \). Let

\[ Sp^{(2)}(W_t, \mathbb{Z}) = \{ g \in Sp(W_t, \mathbb{Z}), g \equiv 1 \pmod{2} \}. \]
Then one can show as in [Ig1, Lemma V.2.5] that there is an exact sequence

$$1 \to Sp^{(2)}(W_t, \mathbb{Z}) \to Sp(W_t, \mathbb{Z}) \to Sp_4(\mathbb{Z}/2) \to 1.$$  

The group $Sp_4(\mathbb{Z}/2)$ is isomorphic to the symmetric group $S_6$ and hence has a character of order 2. This defines a character of order 2 for $Sp(W_t, \mathbb{Z})$ and hence also for $\Gamma_t$. The moduli space defined by $Sp^{(2)}(W_t, \mathbb{Z})$ is the moduli space of $(1, t)$-polarized abelian surfaces with a level-2 structure.

$$A_t^{(2)} = \left\{ (A, H, \alpha); \ H \text{ is a } (1, t)\text{-polarization, } \alpha \text{ is a level-2 structure} \right\}.$$

The group $Sp_4(\mathbb{Z}/2) \cong S_6$ acts transitively on the set of level-2 structures of a fixed abelian surface. The alternating group $A_6$ has two orbits. We call these orbits classes of level-2 structures. So if $\ker(\chi_2) \subset \Gamma_t$ is the kernel of the character of order 2 constructed above, then

$$A(\chi_2) = \ker(\chi_2) \backslash H_2 = \left\{ (A, H, [\alpha]); \ H \text{ is a } (1, t)\text{-polarization, } [\alpha] \text{ is a class of level-2 structures} \right\}.$$

We want to conclude this section with an application of Theorem 2.1 to the Picard group of the moduli stack of $(1, t)$-polarized abelian surfaces. In his beautiful paper [Mu1] Mumford computes the Picard group of the moduli stack $M_{1,1}$ of elliptic curves. His result is that $\text{Pic}(M_{1,1}) = \mathbb{Z}/12$ which is the quotient of $SL_2(\mathbb{Z})$ by its commutator subgroup. Similarly the computation of the commutator group of $\Gamma_t$ has an interpretation in terms of the Picard group of the moduli stack $A_t$ of $(1, t)$-polarized abelian surfaces.

**Proposition 2.3.** The Picard group $\text{Pic}(A_t)$ is finitely generated of rank $\geq 1$. Moreover

(i) $\text{Pic}(A_t) \cong \mathbb{Z} \times \mathbb{Z}/2$

(ii) $\text{Tor}(\text{Pic}(A_t)) \cong \mathbb{Z}/t_1 \times \mathbb{Z}/t_2$ for $t > 1$.

**Proof.** Let $L$ be the $\mathbb{Q}$-line bundle associated to modular forms of weight 1. We first remark that $L$ is a non-zero element of $\text{Pic}(A_t) \otimes \mathbb{Q}$. In fact $L$ is non-zero whenever the genus $g \geq 2$. Assume that $L \otimes k = \mathcal{O}$ for some $k$. Then $L \otimes k_n = \mathcal{O}$ for all $n \geq 1$. But a trivializing section for $L^{\otimes k}$ would imply the existence of a non-constant modular form without zeroes, contradicting Koecher’s principle for $g \geq 2$. In the principally polarized case it is known (see e.g. [Mu2, Part III]) that $L$ generates $\text{Pic}(A_t) \otimes \mathbb{Q}$ and hence the same is true for $\text{Pic}(A_t) \otimes \mathbb{Q}$. (This follows exactly as in [C, Lemme 1.1].) For the rest of the proof we can now argue very much as in [Mu1, §7] and hence we omit full details. First of all we have an exact sequence

$$0 \to H^0(A_t, \mathbb{Z}/n) \to H^0(A_t, \mathcal{O}^*) \to H^0(A_t, \mathcal{O}^*)$$

$$\to H^1(A_t, \mathbb{Z}/n) \to \text{Pic}(A_t) \to H^1(A_t, \mathbb{Z}/n).$$

Moreover

$$H^1(A_t, \mathbb{Z}/n) = H^1(\Gamma_t, \mathbb{Z}/n) = \text{Hom}(\Gamma_t/\Gamma'_t, \mathbb{Z}/n).$$
By our computation of $\Gamma'_t$ it follows that $H^1(\mathcal{A}_t, \mathbb{Z}/p) = 0$ for every prime $p$ with $(p, 2t) = 1$. As in [Mu1, p.77] this implies that $H^0(\mathcal{A}_t, \mathcal{O}^*) = \mathbb{C}^*$ and that $\text{Pic}(\mathcal{A}_t)$ is finitely generated. But now we have an exact sequence

$$0 \to H^1(\mathcal{A}_t, \mathbb{Z}/n) \to \text{Pic}(\mathcal{A}_t) \xrightarrow{n} \text{Pic}(\mathcal{A}_t)$$

and since the $n$-torsion of $\text{Pic}(\mathcal{A}_t)$ is just the kernel of the last map in this sequence, the result follows from Theorem 2.1.

Remark. Geometrically the 2-torsion element in $\text{Pic}(\mathcal{A}_1)$ can be realized as follows: Let $H_1 = \pi_1(\tau_0 \omega) \subset \mathcal{A}_1$ be the Humbert surface of discriminant 1. Then $H_1 = \pi_1(\Delta_5 = 0)$ where $\Delta_5(\mathbb{Z})$ is the product of the ten even theta-characteristics. The function $\Delta_5(\mathbb{Z})$ is a modular form of weight 5 with a non-trivial character of order 2. Hence $2H_1 = 10\mathcal{L}$ in $\text{Pic}(\mathcal{A}_1)$, but $H_1$ and $5\mathcal{L}$ differ by a non-zero 2-torsion element.

§3. The geometric genus of the commutator covering of $\mathcal{A}_t$

In this section we consider modular coverings of the moduli space $\mathcal{A}_t$ with commutative covering groups. The commutator subgroup $\Gamma'_t \subset \Gamma_t$ defines the maximal abelian covering

$$\mathcal{A}^\text{com}_t = \Gamma'_t \backslash \mathbb{H}_2 \to \Gamma_t \backslash \mathbb{H}_2 = \mathcal{A}_t. \quad (3.1)$$

We denote by $\mathcal{A}$ an arbitrary smooth compact model of a threefold $\mathcal{A}$.

**Theorem 3.1.**

1. The maximal abelian covering $\mathcal{A}^\text{com}_t$ has geometric genus 0 if and only if $t = 1, 2, 4, 5$.

2. The threefolds $\mathcal{A}^\text{com}_3 \xrightarrow{18:1} \mathcal{A}_3$, $\mathcal{A}^\text{com}_7 \xrightarrow{2:1} \mathcal{A}_7$ and $\mathcal{A}_6(\chi_2) \xrightarrow{2:1} \mathcal{A}_6$ have geometric genus 1: $h^{3,0}(\mathcal{A}^\text{com}_3) = h^{3,0}(\mathcal{A}^\text{com}_7) = h^{3,0}(\mathcal{A}_6(\chi_2)) = 1$.

3. Let $\chi_3 : \Gamma_6 \to \mathbb{C}^*$ be the character (1.10a–b) of order 3. Then $h^{3,0}(\mathcal{A}_6(\chi_3)) = 2$.

We shall prove the theorem using the lifting construction (1.9) together with a statement about the $q$-order of Jacobi forms which we formulate below. Let

$$\phi_{k,m}(\tau, z) = \sum_{n,l \in \mathbb{Z}} a(n,l) q^n r^l \in J^c_{k,m}$$

be a Jacobi cusp form with trivial character. The $q$-order of $\phi_{k,m}(\tau, z)$ is the minimal $q$-power contained in its Fourier expansion

$$\text{ord}_q(\phi_{k,m}) = \min \{ n \in \mathbb{N} \mid a(n,l) \neq 0 \}.$$
Proposition 3.2. Let \( \phi_{2k,m} \in J_{2k,m}^c \) be a non-zero Jacobi cusp form of even weight. Then

\[
\text{ord}_q(\phi_{2k,m}) \leq \min \left( \frac{3k - 3 + m}{9}, \frac{k + m}{6} \right). \tag{3.2}
\]

Moreover equality in (3.2) holds if and only if \( \phi_{2k,m} = c\Delta^N \eta^{-6m} \vartheta^{2m} \) where \( N > \frac{m}{4} \) is the \( q \)-order, \( 2k = 12N - 2m \) is the weight, and \( c \neq 0 \) is a constant.

The estimation of this proposition works very well for applications concerning modular forms with respect to \( \Gamma_t \) for \( t \leq 8 \). For example we easily obtain the following

Corollary 3.3. For \( t \leq 8 \) the geometric genus of the moduli space of abelian surfaces with a polarization of type \( (1, t) \) is zero:

\[ h^{3,0}(\tilde{A}_t) = 0. \]

Proof of the Corollary. According to (1.15) we have to prove that there are no \( \Gamma_t \)-cusp forms of weight 3 for \( t \leq 8 \). Consider the decomposition

\[
\mathfrak{H}_k(\Gamma_t) = \mathfrak{H}_k^+(\Gamma_t) \oplus \mathfrak{H}_k^-(\Gamma_t), \tag{3.3}
\]

where the \( \pm \)-subspaces consist of \( \Gamma_t \)-modular forms, which are invariant or anti-invariant with respect to the involution \( V_t \) (see (1.1))

\[ F(Z) = \pm F(V_t < Z >), \quad V_t < Z > = \left( \frac{\omega/t}{z} \quad \frac{z}{t\tau} \right). \]

Because of the decomposition (3.3) we can restrict ourselves to \( \Gamma_t^+ \)-cusp forms.

Let \( F_3^{(t)} \in \mathfrak{H}_3^+(\Gamma_t) \). We consider its Fourier-Jacobi expansion

\[
F_3^{(t)}(Z) = \sum_{m \geq M > 0} f_{3,mt}(\tau, z)e^{2\pi imt\omega}.
\]

From \( V_t \)-invariance of \( (F_3^{(t)})^2 \) and Proposition 3.2 follows that \( 2M \leq \text{ord}_q(f_{3,mt}^2) < \frac{6 + 2Mt}{9} \).

For \( t \leq 6 \) we have \( \frac{6 + 2Mt}{9} \leq 2M \), thus there are no cusp forms of weight 3 for \( t \leq 6 \). For \( t = 7 \) and \( t = 8 \) the same inequality holds for \( M \geq 2 \) and \( M \geq 3 \) respectively. We note that \( J_{3,7} = \{0\} \) and \( J_{3,8}^c = J_{3,16}^c = \{0\} \) (see below). This finishes the proof.

Recall that dimension formulae for \( J_{k,m} \) were found in [EZ] and [SkZ]. We use these in the following variant

\[
\dim_{\mathbb{C}} J_{k,m}^c = \begin{cases} 
\sum_{j=0}^{m} (k + 2j)_{12} - \lfloor \frac{j^2}{4m} \rfloor & \text{if } k \text{ is even} \\
\sum_{j=1}^{m-1} (k + 2j - 1)_{12} - \lfloor \frac{j^2}{4m} \rfloor & \text{if } k \text{ is odd}
\end{cases} \tag{3.4}
\]

with

\[
\{m\}_{12} = \begin{cases} 
\left\lfloor \frac{m}{12} \right\rfloor & \text{if } m \neq 2 \text{ mod } 12 \\
\left\lfloor \frac{m}{12} \right\rfloor - 1 & \text{if } m \equiv 2 \text{ mod } 12.
\end{cases}
\]
Remark. The fact that dim $\mathfrak{M}^1_3(\Gamma_t) = 0$ for $t = 2, 3, 5, 7$ was proved in [G2, §3.5] by a different method. Gross and Popescu (cf. [GP]) have announced a result which, among other things, implies that the moduli spaces $A_t$ are unirational for $t \leq 12$. Their approach is via equations for abelian surfaces.

Proof of Proposition 3.2. Let $N = \text{ord}_q(\phi_{2k,m})$. If $6N < k$, then the proposition is proved. Assume that $6N \geq k$ and consider the function $\Delta(\tau)^{-N} \phi_{2k,m}(\tau, z)$, where $\Delta(\tau) = \eta(\tau)^{24}$. According to the definition of $N$ the function $\Delta^{-N} \phi_{2k,m}$ is a weak Jacobi form of weight $2(k-6N)$ (see (1.2)). By [EZ, Theorem 9.3] the ring $J_{ev,*}^{\text{red}}$ of all weak Jacobi forms of even weight is a polynomial algebra in two variables over $\mathfrak{M}_*(SL_2(\mathbb{Z})) = \oplus_k \mathfrak{M}_k(SL_2(\mathbb{Z}))$ with the standard generators $\phi_{0,1}$ and $\phi_{-2,1}$:

$$
\phi_{-2,1}(\tau, z) = \left(\frac{\vartheta(\tau, z)}{\eta(\tau)^3}\right)^2 = \left(\frac{r-1/2}{\prod_{n \geq 1} (1-q^n r)(1-q^n r^{-1})(1-q^n)^{-2}}\right)^2
= (r-2+r^{-1}) - 2q(r^2 - 4r + 6 - 4r^{-1} + r^{-2}) + q^2(\ldots), \tag{3.5}
$$

$$
\phi_{0,1}(\tau, z) = \Delta(\tau)^{-1} \phi_{12,1}(\tau, z)
= (r + 10 + r^{-1}) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) + q^2(\ldots),
$$

where $\vartheta$ is the Jacobi theta-series (1.4) and $\phi_{12,1}$ is the unique Jacobi cusp form of weight 12 and index 1 with integral coprime Fourier coefficients and $q = \exp(2\pi i\tau)$, $r = \exp(2\pi iz)$.

We can represent $\Delta^{-N} \phi_{2k,m}$ as a polynomial in $\phi_{-2,1}$ and $\phi_{0,1}$ over $\mathfrak{M}_*(SL_2(\mathbb{Z}))$:

$$
\Delta^{-N} \phi_{2k,m} = \phi_{-2,1}^{6N-k} \phi_{0, m-6N+k} = \phi_{-2,1}^{6N-k} \sum_{0 \leq i \leq m-6N+k} f_{2i} \phi_{-2,1}^{m-6N+k-i} \tag{3.6}
$$

where the $f_{2i}$ are modular forms of weight $2i$ with respect to $SL_2(\mathbb{Z})$. This shows $6N \geq k + m$. If $6N = k + m$, then the Jacobi form

$$
\phi_{2k,m} = \Delta^{N} \phi_{-2,1}^{m} = \eta^{6(4N-m)} q^{2m}
$$

is of the type stated in the theorem.

Let $6N > k + m$. The Fourier expansion of the weak Jacobi form $\phi_{0,m-6N+k}$ has a non-zero $q^0$-term due to the choice of $N = \text{ord}_q(\phi_{2k,m})$. Let us for the moment suppose that the $q^0$-term of $\phi_{0,m-6N+k}$ is not a constant. Due to the relation $\phi_{2k,m}(\tau, -z) = \phi_{2k,m}(\tau, z)$, the $q^0$-term of the weak Jacobi form $\phi_{0,m-6N+k}$ contains a summand of type $a(r^\beta + r^{-\beta})$ with $a \neq 0$ and a positive integer $\beta$ which we assume to be maximal. This implies that the Fourier expansion of the Jacobi cusp $\phi_{2k,m} = \Delta^{N}(\phi_{-2,1})^{6N-k} \phi_{0,m-6N+k}$ has a non-zero Fourier coefficient of type $q^{N+6N-k+\beta}$. It follows that the norm of the index of this coefficient has to be positive, i.e. $4Nm - (6N - k + \beta)^2 > 0$. Therefore, since $\beta \geq 1$, we have

$$
N < \frac{1}{18} \left(3k - 3\beta + m + \sqrt{(3k - 3\beta + m)^2 - 9(k - \beta)^2}\right) \leq \frac{3k - 3 + m}{9}.
$$

Thus the equality in (3.2) can hold only if the minimum equals $\frac{k + m}{6}$. This case was considered above. To finish the proof of Proposition 3.2 it remains to prove the assumption about the $q^0$-part of $\phi_{0,t}$ if $t \neq 0$. 
Lemma 3.4. Let $\phi_{0,t}$ be a weak Jacobi form of weight zero and index $t$ ($t \neq 0$) with $\text{ord}_q(\phi_{0,t}) = 0$. Then the $q^0$-part of the Fourier expansion of $\phi_{0,t}$ is not a constant.

Proof of the lemma. Similar to (3.6) we obtain

$$\phi_{0,t}(\tau, z) = \sum_{0 \leq i \leq t, \ i \neq 1} f_{2i}(\tau) \phi^{j-2,1}_{-i}(\tau, z) \phi^{j-1}_{0,1}(\tau, z)$$

(3.7)

where $f_{2i}(\tau) = a_i E_{2i}(\tau) + \ldots$ ($a_i \in \mathbb{C}$) are modular forms from $\mathcal{M}_{2i}(SL_2(\mathbb{Z}))$ and $E_{2i}(\tau)$ are the Eisenstein series of weight $2i$ with respect to $SL_2(\mathbb{Z})$ with the constant term 1. We can assume that the other summands are cusp forms.

According to (3.5) the $q^0$-term $\phi^{(0)}_{0,t}(z)$ of $\phi_{0,t}(\tau, z)$ is equal to

$$\phi^{(0)}_{0,t}(z) = \sum_{0 \leq i \leq t, \ i \neq 1} a_i X^i (X + 12)^{t-i} \quad (X = r - 2 + r^{-1}).$$

If $\phi^{(0)}_{0,t}$ is a constant, then we obtain a triangular linear system of $t$ equations with $t$ unknowns $a_i$ ($i \neq 1!$)

$$X^k \sum_{i \leq k} a_i C_{t-i}^{t-k} = 0 \quad (1 \leq k \leq t).$$

This implies that all $a_i = 0$ and hence $\phi^{(0)}_{0,t} = 0$, a contradiction to the assumption about the $q$-order of $\phi_{0,t}$.

Proof of Theorem 3.1. The “only if” part of the first claim of the theorem follows from Corollary 1.6.

Let $t = 1, 2, 4, 5$. According to (1.15) we have to prove that there are no cusp forms of weight 3 with respect to $\Gamma_t'$. For this it is clearly enough to prove that for any character $\chi_{a,b} : \Gamma_t \to \mathbb{C}^*$ (see Notation 2.2) of the paramodular group $\Gamma_t$ for $t = 2, 4, 5$ we have $\dim(\mathfrak{H}_3(\Gamma_t, \chi_{a,b})) = 0$. (For $t = 1$ this follows from Igusa’s result [Ig2] about the graded ring of Siegel modular forms for $Sp_4(\mathbb{Z})$, but it can be easily obtained by the same method which we apply for $t = 5$.) For $t = 3, 6, 7$ we will show that the corresponding spaces of cusp forms of weight 3 contain only one function.

Let $F_{\chi_{a,b}} \in \mathfrak{H}_k(\Gamma_t, \chi_{a,b})$. The restriction of $\chi_{a,b}$ to the centre of the maximal parabolic subgroup $\Gamma_{\infty,t}$, which is generated by the element $B$ (see (2.1)), determines the type of the Fourier-Jacobi expansion of the cusp form $F$. Let $t_2 = (2t, 12)$ as in Theorem 2.1. Then

$$F_{\chi_{a,b}}(Z) = \sum_{m \equiv b \mod t_2} f^{(a)}_{k,tm/t_2}(\tau, z) \exp(2\pi im \frac{t}{t_2} \omega).$$

(3.8a)

The character of the Jacobi forms $f^{(a)}_{k,tm/t_2}$ is equal to $\nu^{24a/t_2}_H \times \nu^2_H$, where $\varepsilon \equiv 2t b/t_2 \mod 2$. We also have $F|_k V_t \in \mathfrak{H}_k(\Gamma_t, \chi_{b,a})$ since $V_t B V_t = A$ and the character $\chi_{b,a}$ is $V_t$-conjugate.
to $\chi_{a,b}$, i.e. $\chi_{b,a}(\gamma) = \chi_{a,b}(V_t \gamma V_t)$ for $\gamma \in \Gamma_t$. Therefore, we can define a $V_t$-invariant modular form $F \cdot F|_k V_t \in \mathcal{M}_{2k}(\Gamma_t, \chi_{a+b,a+b})$. It has the following Fourier-Jacobi expansion

\[
(F \cdot F|_k V_t)(Z) = \sum_{m \equiv a+b \mod t_2} f_{2k,tm/t_2}(\tau, z) \exp(2\pi i m \frac{t}{t_2} \omega). \tag{3.8b}
\]

We divide the proof of Theorem 3.1 into six steps depending on the values of the polarization $t$.

(V). Let $t = 5$. According to Theorem 2.1 there is only one non-trivial character $\chi_2 = \chi_{1,1} : \Gamma_5 \to \{\pm 1\}$. (If the character is trivial our result follows from Corollary 3.3.) Let $F \in \mathcal{M}_3(\Gamma_5, \chi_{1,1})$. Then $G^{(5)} = F \cdot F|_6 V_5 \in \mathcal{M}_9^+(\Gamma_5)$ is a $V_5$-invariant cusp form. According to the dimension formula (3.4) $\dim(J_{6,5}^c) = 1$ and a basis of this space can be given explicitly in terms of Jacobi forms of half-integral indices, namely

$$J_{6,5}^c = \mathbb{C}\varphi_{6,5}, \text{ where } \varphi_{6,5} = \eta^2 \vartheta^6[\vartheta, \vartheta_{3/2}]$$

(see (1.7) and (1.12)). Calculating the Fourier expansion of $[\vartheta, \vartheta_{3/2}]$ one gets

$$\frac{1}{2} \eta^{-4}[\vartheta, \vartheta_{3/2}] = (r + 4 + r^{-1}) + q(\ldots).$$

Thus $\varphi_{6,5}$ is not a square of a Jacobi form of weight 3 and index 5/2. It follows that the first Fourier-Jacobi coefficient $g_{6,5,M}$ of $G^{(5)}$ has $M \geq 2$. The $q$-order of $g_{6,5,M}$ is at least $M$, since the $V_5$-invariant cusp form $G^{(5)}$ has order 5 with respect to the variable $\omega$. We shall use this type of argument repeatedly in the sequel. According to Proposition 3.2 $\text{ord}_q(g_{6,5,m}) \leq \frac{6 + 5m}{9} < m$ if $m \geq 2$. Hence $G^{(5)} = 0$.

(II) and (IV). Let $t = 2$ or $t = 4$. An arbitrary character $\chi_{a,b}$ of $\Gamma_t$ has order 1, 2 or 4. Let us consider cusp forms $F = F_{\chi_{a,b}} \in \mathcal{M}_3(\Gamma_t, \chi_{a,b})$. Then

$$G^{(t)}(Z) = ((F \cdot F|_t V_t)(Z))^4 = \sum_{m \geq 2} g_{24,tm}^{(t)}(\tau, z) \exp(2\pi itm\omega) \in \mathcal{M}_{24}^+(\Gamma_t) \quad (t = 2, 4)$$

and again use Proposition 3.2. The inequality $\text{ord}_q(g_{24,tm}) \leq \frac{12 + tm}{6} < m$ holds if $m \geq 4$ for $t = 2$ or $m \geq 7$ for $t = 4$. Thus to prove that $G^{(2)} = 0$, resp. $G^{(4)} = 0$, it is enough to show that the Fourier-Jacobi expansion of $F \cdot F|_t V_t$ starts with coefficients $f_{6,2}$ or $f_{6,7}$ for $t = 2$ or 4 respectively. To show this we check that for any $b \leq a$ such that $a + b < 4$, resp. $a + b < 7$ (see (3.8b)) the first possible Fourier-Jacobi coefficient $f_{k,6,4}^{(a)}$ of any $F_{\chi_{a,b}} \in \mathcal{M}_3(\Gamma_t, \chi_{a,b})$ is zero. For this we have to consider the character $\chi_{1,1}$ of the group $\Gamma_2$ and the characters $\chi_{a,1}$ ($1 \leq a \leq 4$), $\chi_{a,2}$ ($2 \leq a \leq 4$) and $\chi_{3,3}$ of $\Gamma_4$. We have

$$f_{3,1/2}^{(1)} \cdot \eta^3 \vartheta^5 \in J_{7,3} = \{0\}, \quad f_{3,b}^{(a)} \cdot \vartheta^{8-2a} \in J_{7-a, b+4-a}^c = \{0\}$$

for the values of $a$ and $b$ chosen above (see (3.4)).
III. Let $t = 3$. By Theorem 2.1 the order of any character of $\Gamma_3$ is a divisor of 6. For given $F = F_{\chi_{a,b}} \in \mathfrak{M}_3(\Gamma_3, \chi_{a,b})$ we consider the form

$$G^{(3)}(Z) = (F \cdot F|_3)(Z)^6 = \sum_{m \geq 2} g_{36,3m}(\tau, z) \exp(6\pi im\omega) \in \mathfrak{M}_{36}^+(\Gamma_3).$$

According to Proposition 3.2 $\text{ord}_q(g_{36,3m}) \leq \frac{18 + 3m}{6} < m$ if $m \geq 7$. Thus if we prove that the Fourier-Jacobi coefficients of $F \cdot F|_3$ of indices smaller than $7/2$ are zero, then it follows that $F_{\chi_{a,b}} = 0$.

Recall that $\chi_{a,b}(A)^3 = \chi_{a,b}(B)^3$ (see Theorem 2.1). Therefore, due to (3.8a) and (3.8b) we have to consider only the characters $\chi_{1,1}$, $\chi_{1,3}$, $\chi_{5,1}$, $\chi_{2,2}$, $\chi_{4,2}$ and $\chi_{3,3}$. For the second, the third, the fourth and the fifth character we have that the first Fourier-Jacobi coefficient in the expansion (3.8a) of the corresponding cusp form $F_{\chi_{a,b}}$ is zero:

$$f_{3,1/2}^3 \cdot \eta^3 \vartheta^3 \in J_{8,3}^c = \{0\} \quad (\chi_{3,1}), \quad f_{3,1/2}^5 \cdot \eta \vartheta \in J_{4,1}^c = \{0\} \quad (\chi_{5,1}),$$

$$f_{3,1}^2 \cdot \eta^2 \vartheta^2 \in J_{5,2}^c = \{0\} \quad (\chi_{2,2}), \quad f_{3,1}^3 \cdot \eta^3 \vartheta^3 \in J_{7,3}^c = \{0\} \quad (\chi_{4,2}).$$

Thus the cusp forms $F_{\chi_{a,b}}$ of weight 3 are zero for these characters.

For the first character the situation is more complicated. The first Fourier-Jacobi coefficient of $F_{\chi_{1,1}}$ satisfies the relation

$$f_{3,1/2}^1 \cdot \eta^5 \vartheta^5 \in J_{8,3}^c = \mathbb{C} \cdot \eta^6 \vartheta^4 \phi_{3,1},$$

where $\phi_{3,1}(\tau, z) = \phi_{12,1}(\tau, z)/\eta(\tau)^{18} \in J_{3,1}(v^6_0 \times 1_H)$ and $\phi_{12,1}$ is the unique (up to a constant) Jacobi cusp form of weight 12 and index 1. The Jacobi form $\phi_{3,1}(\tau, z)$ does not vanish for $z = 0$ since $\phi_{12,1}(\tau, 0) = \eta(\tau)^{24}$. Thus $f_{3,1/2}^1 = 0$. As a result we have proved that there are no cusp forms of weight 3 for all but 1 character of the group $\Gamma_3$:

$$\bigoplus_{(a,b) \neq (3,3)} \mathfrak{M}_3(\Gamma_3, \chi_{a,b}) = \{0\}. \quad (3.9)$$

Let us consider the last character $\chi_{3,3} = \chi_2$ (see (1.10a-b)) of order 2. This character can be extended to a character of the group $\Gamma_3^+$, thus we can consider a decomposition of the space $\mathfrak{M}_3(\Gamma_3, \chi_2)$ of type (3.3), namely $\mathfrak{M}_3(\Gamma_3, \chi_2) = \mathfrak{M}_3(\Gamma_3^+, \chi_{3,3,+}) \oplus \mathfrak{M}_3(\Gamma_3^+, \chi_{3,3,-})$. The first Fourier-Jacobi coefficient of any element $F_{\chi_2} \in \mathfrak{M}_3(\Gamma_3^+, \chi_{3,3,+})$ is a Jacobi form of index 1/2

$$f_{3,1/2}^3 \in J_{3,4}^c(v_{12}^1 \times v_H) = \mathbb{C} \cdot \eta^3 \vartheta^3.$$

The last space is one-dimensional since $\left( J_{3,4}^c(v_{12}^1 \times v_H) \right)^2 \subset J_{6,3}^c$ and $\dim J_{6,3}^c = 1$. We can define the lifting of $\eta \vartheta$ (see (1.9))

$$\Delta_1(Z) = \text{Lift}(\eta \vartheta)(Z) \in \mathfrak{M}_1(\Gamma_3, \chi_{1,1}).$$

Thus there exists a constant $c$ such that

$$F_{\chi_2}(Z)^2 - c \Delta_1(Z)^6 = \sum_{m \geq 2} f_{6,3m}(\tau, z)e^{6\pi im\omega} \equiv 0$$
according to Proposition 3.2.

(VII). Let \( t = 7 \). According to Theorem 2.1 there exists only one non-trivial character \( \chi_2 = \chi_{1,1} \) of \( \Gamma_7 \) of order 2. (The case of trivial character was considered in Corollary 3.3.)

We remark that

\[
J_{3,7}^{c}(v_{12}^{12} \times v_H) = \mathbb{C} \cdot \vartheta^2[\vartheta, \vartheta_2]
\]

(see (1.12)). This space is spanned by one Jacobi form, since for any \( \phi \in J_{3,7}^{c}(v_{12}^{12} \times v_H) \) we obtain \( \phi(\eta\vartheta)^3 \in J_{6,5}^{c} \), and the last space is one dimensional.

Let

\[
F^{(7)}(Z) = \sum_{m \equiv 1 \text{ mod } 2} f_{3,7m}(\tau, z)e^{7\pi i m \omega} \in \mathcal{M}_{3}(\Gamma_{7}^{+}, \chi_{2,\pm}).
\]

Then there exists a constant \( c \) such that

\[
F^{(7)}(Z)^2 - c\Xi(Z)^2 = \sum_{m \geq 2} f_{6,7m}(\tau, z)e^{14\pi i m \omega}
\]

where \( \Xi = \text{Lift}(\vartheta^2[\vartheta, \vartheta_2]) \) (see \( t=7 \) in the proof of Theorem 1.4) and \( f_{6,7m} \in J_{6,7m}^{c} \).

Due to Proposition 3.2 \( \text{ord}_q(f_{6,7m}) < \frac{6+7m}{9} \leq m \) if \( m \geq 3 \). Thus \( (F^{(7)})^2 = c\Xi^2 \) if we prove that there is no Jacobi cusp form \( f_{6,14} \) of weight 6 and index 14 such that \( \text{ord}_q(f_{6,14}) \geq 2 \).

We shall now show this.

If \( \Delta^{-3}f_{6,14} \) is a weak holomorphic Jacobi form, then using representation (3.7) we obtain \( \Delta^{-3}f_{6,14} = (\phi_{-2,1})^{15}\phi_{0,-1} \), a contradiction. Thus \( f_{6,14} \) would have \( q \)-order 2 and

\[
\Delta^{-2}f_{6,14} = (\phi_{-2,1})^9\phi_{0,5}, \quad \phi_{0,5} \in J_{6,5}^{\text{weak}}.
\]

If \( \phi_{0,5} \) contains \( r^d \) with \( d \geq 2 \), then we get a contradiction as in Lemma 3.4. For \( t = 5 \) all modular forms in the representation (3.7) are Eisenstein series. Thus, there is only one weak Jacobi form \( \phi_{0,5}(\tau, z) \) with \( q^0 \)-term equal to \( r^1 + c + r^{-1} \). It is easy to determine this function using (3.7) or using two weak Jacobi forms found in [GN, Example 4.5]. As a result we obtain

\[
5\phi_{0,5}(\tau, z) = (5r^1 + 2 + 5r^{-1}) + q(-r^5 + \ldots).
\]

The Jacobi form \( \eta^{48}(\phi_{-2,1})^{9}\phi_{0,5} \) is not holomorphic, since its Fourier expansion contains the term \( cq^3r^{14} (c \neq 0) \). Therefore \( f_{6,14} = 0 \). This finishes the proof for \( t = 7 \).

(VI). Let \( t = 6 \). Any cusp form \( F_{\chi_2} \) of weight 3 with character \( \chi_2 = \chi_{6,6} \) of order 2 has a Fourier expansion

\[
F_{\chi_2}(Z) = \sum_{m \equiv 1 \text{ mod } 2} f_{3,3m}(\tau, z)e^{6\pi i m \omega} \in \mathcal{M}_{3}(\Gamma_6, \chi_2).
\]

Similar to (III) (see also \( t=6 \) of the proof of Theorem 1.1)

\[
J_{3,3}^{c}(v_{12}^{12} \times 1_H) = \mathbb{C} \cdot \eta^3\vartheta^2\vartheta_2
\]
is one-dimensional, since \( \dim J_{6,6}^c = 1 \). Thus there exists a constant \( c \) such that
\[
F_{\chi_2}(Z)^2 - c \operatorname{Lift}(\eta^3 \vartheta^2 \vartheta_2)(Z)^2 = \sum_{m \geq 2} f_{6,6m}(\tau, z) e^{12\pi i m \omega} \equiv 0,
\]
since due to Proposition 3.2 \( \operatorname{ord}_q(f_{6,6m}) < \frac{6+6m}{9} \leq m \) for \( m \geq 2 \). Hence \( h^{3,0}(\tilde{A}_6(\chi_2)) = 1 \).

We remark that we have in fact proven that \( \dim(\mathfrak{N}_6^+(\Gamma_6)) = 1 \). The same arguments show us that \( \dim(\mathfrak{N}_6^- (\Gamma_6)) = 0 \) since the generator \( (\eta^3 \vartheta^2 \vartheta_2)^2 \) of the space \( J_{6,6} \) has \( q \)-order one.

Now we can find the geometric genus of \( \tilde{A}_6(\chi_3) \) where \( \chi_3 = \chi_{4,4} : \Gamma_6 \to \mathbb{C}^* \) is the character induced by \( v_6^4 \times 1_H \) (see (1.10a–b) and Notation 2.2). Let \( \chi_{a,b} \) be a different character of \( \Gamma_6 \). It follows from the definition of the characters \( \chi_{a,b} \) that \( \ker(\chi_{a,b}) = \ker(\chi_{4,4}) \) if and only if \( \chi_{a,b} = \chi_{4,4}^2 = \chi_{8,8} \). In Theorem 1.4 we constructed the cusp form \( \operatorname{Lift}(\eta^3 \vartheta_2) \in \mathfrak{N}_3(\Gamma_6, \chi_{4,4}) \). A modular form from \( \mathfrak{N}_3(\Gamma_6, \chi_{4,4}) \) was found in [GN, Example 1.15]. This is the 2-lifting (or \(-1\)-lifting) of the Jacobi form \( \eta^3 \vartheta_2 \):
\[
\operatorname{Lift}_2(\eta^3 \vartheta_2) = \sum_{m \equiv 2 \mod 3} m^{-1}(\eta(\tau)^5 \vartheta(\tau, 2z) \exp(4\pi i \omega)) | (\tau, z) \in \mathfrak{N}_3(\Gamma_6, \chi_{8,8}).
\]
Let \( F_{\chi_{4,4}} \in \mathfrak{N}_3(\Gamma_6, \chi_{4,4}) \) and \( F_{\chi_{8,8}} \in \mathfrak{N}_3(\Gamma_6, \chi_{8,8}) \) be arbitrary cusp forms. Then it is clear that \( F_{\chi_{4,4}} \cdot F_{\chi_{8,8}} \in \mathfrak{N}_6(\Gamma_6) \). This space is one dimensional (see above), thus
\[
h^{3,0}(\tilde{A}_6(\chi_{4,4})) = \dim(\mathfrak{N}_3(\Gamma_6, \chi_{4,4})) + \dim(\mathfrak{N}_3(\Gamma_6, \chi_{8,8})) = 2.
\]
Hence Theorem 3.1 is proved.

We want to conclude this section with a brief discussion of the commutative neighbours of \( A_3 \). This case is of special geometric interest and is also closely related to the Calabi-Yau threefold found by Barth and Nieto [BN].

**Corollary 3.5.** There are exactly 12 Siegel modular varieties between \( A_3^{\text{com}} \) and \( A_3 \). Of these six have geometric genus 1, the others have geometric genus 0.

**Proof.** Recall from Theorem 2.1 that \( \Gamma_3 / \Gamma_3' \cong \mathbb{Z} / 3 \times \mathbb{Z} / 6 \). This group is generated by elements \( A \) and \( B \) with the relations \( A^6 = B^6 = 1 \) and \( A^3 = B^3 \). The Siegel modular varieties between \( A_3^{\text{com}} \) and \( A_3 \) are in 1:1 correspondence with the subgroups of \( G_3 = \Gamma_3 / \Gamma_3' \). The lattice of subgroups of \( G_3 \) is represented in Diagram 1. The proof of Theorem 3.1 shows that for a group \( \Gamma \) between \( \Gamma_3 \) and \( \Gamma_3 \) the corresponding moduli space \( \mathcal{A}(\Gamma) \) has geometric genus 1 if and only if \( \Gamma \) is contained in \( \ker(\chi_{3,3}) \). The latter is generated by \( A^2 \) and \( B^2 \) and isomorphic to \( \mathbb{Z} / 3 \times \mathbb{Z} / 3 \). This, together with the diagram below shows the claim. 

\[\square\]
Diagram 1. Lattice of the subgroups of the group $G_3 = \Gamma_3/\Gamma'_3$.

Remark 3.6. The lattice of the subgroups of $G_3$ corresponds to a diagram of coverings of Siegel modular varieties. All but one of these coverings are of type $A(\chi)$ for some character $\chi$ of the paramodular group $\Gamma_t$. To be more precise we have to label the subgroups $(\mathbb{Z}/3)_i$ and $(\mathbb{Z}/6)_i$. We put

$$(\mathbb{Z}/3)_1 = \langle A^2 \rangle, \quad (\mathbb{Z}/3)_2 = \langle B^2 \rangle, \quad (\mathbb{Z}/3)_3 = \langle A^2B^2 \rangle, \quad (\mathbb{Z}/3)_3 = \langle A^4B^2 \rangle = \langle AB^{-1} \rangle = \langle A^{-1}B \rangle.$$ 

This also determines the subgroups $(\mathbb{Z}/6)_i$. Note that

$$\mathbb{Z}/3 \times \mathbb{Z}/3 = \langle A, B \rangle, \quad \mathbb{Z}/2 = \langle A^3 \rangle = \langle B^3 \rangle.$$ 

Altogether we have 18 characters $\chi_{a,b}; 1 \leq a \leq b, a-b \equiv 0 \pmod{2}$. Each of the subgroups $(\mathbb{Z}/3)_i$, resp. $(\mathbb{Z}/6)_i$ is the kernel of two different characters, the subgroup $\mathbb{Z}/3 \times \mathbb{Z}/3$ is the kernel of $\chi_{3,3}$. The precise relation between subgroups of $G_3$ and characters is given
by the following table

| group     | character | order of character |
|-----------|-----------|--------------------|
| $(\mathbb{Z}/3)_1$ | $\chi_{3,1}$ and $\chi_{3,5}$ | 6 |
| $(\mathbb{Z}/3)_2$ | $\chi_{1,3}$ and $\chi_{5,3}$ | 6 |
| $(\mathbb{Z}/3)_3$ | $\chi_{1,5}$ and $\chi_{5,1}$ | 6 |
| $(\mathbb{Z}/3)_4$ | $\chi_{1,1}$ and $\chi_{5,5}$ | 6 |
| $(\mathbb{Z}/6)_1$ | $\chi_{6,2}$ and $\chi_{6,4}$ | 3 |
| $(\mathbb{Z}/6)_2$ | $\chi_{2,6}$ and $\chi_{4,6}$ | 3 |
| $(\mathbb{Z}/6)_3$ | $\chi_{4,2}$ and $\chi_{2,4}$ | 3 |
| $(\mathbb{Z}/6)_4$ | $\chi_{2,2}$ and $\chi_{4,4}$ | 3 |
| $\mathbb{Z}/3 \times \mathbb{Z}/3$ | $\chi_{3,3}$ | 2 |
| $G_3$ | $\chi_{6,6}$ | 1 |

The subgroup $\mathbb{Z}/2$ is not the kernel of a character, but it can be written as the intersection of the kernel of two characters in several ways. Note that for example $(\mathbb{Z}/6)_1 = < A >$, $(\mathbb{Z}/6)_2 = < B >$ and hence $\mathbb{Z}/2 = \ker (\chi_{6,2}) \cap \ker (\chi_{2,6})$.

The lattice of the subgroups of $G_3$ then corresponds to the following diagram of coverings

Diagram 2. Commutative coverings of $A_3$. 
The modular varieties corresponding to the vertices of the left square, resp. right square have geometrical genus 1, resp. 0.

**Remark 3.7.** There are exactly 20 Siegel modular varieties between $A_{3}^{\text{com}}$ and $A_{3}^{+} = \Gamma_{3}^{+} \backslash \mathbb{H}_{2}$. (Not all of these are abelian covers of $A_{3}^{+}$.) Of these, exactly 6 have geometric genus 1, whereas the others have geometric genus 0 (by (3.9)). The six modular varieties of geometric genus 1 are natural candidates for Calabi–Yau varieties, whereas we expect the other varieties to be unirational.

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**St. Petersburg Department of Steklov Mathematical Institute, Fontanka 27, 191011 St. Petersburg, Russia**

*E-mail address*: gritsenk@kurims.kyoto-u.ac.jp (till 28.02.97); gritsenk@mpim-bonn.mpg.de

**Institut für Mathematik, Universität Hannover, Postfach 6009, D-30060 Hannover, Germany**

*E-mail address*: hulek@math.uni-hannover.de