Bi-directional Evolutionary Structural Optimization of Continuum Structures with Multiple Constraints

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Abstract. Most of the structural topology optimization problems involved in practical engineering applications are problems with many constraints. Thus, this paper presented an improved bi-directional evolutionary structural optimization (BESO) method for problems with many constraints. Slack variables were introduced to transform the inequality constraints into equality constraints. Then the Lagrange multiplier method converted the multi-constrained optimization problem to an unconstrained optimization problem and an advanced multiplier calculation was proposed. Elemental sensitivity numbers were derived according to the obtained multipliers. The design variables were updated by the BESO method. One numerical example that aimed to minimizing the structural mean compliance and involved volume, fundamental frequency and displacement constraints was used to validate the proposed method. The new method extends the BESO method to topology optimization that has a number of constraints, such as volume, displacement and frequency constraints.

1. Introduction
The evolutionary structural optimization (ESO) is a topology optimization method proposed by Xie and Steven et al. [1, 2]. ESO gradually removes inefficient materials from the structure and evolves the structure to the optimal material distribution based on certain criteria. ESO is easy to be implemented and computationally inexpensive. However, ESO has problems such as easily trapped in local minima, got a mesh-dependency and checkerboard design [3-6]. To overcome these shortcomings, ESO has been extended to bi-directional evolutionary structural optimization (BESO) by Querin et.al [7-9]. BESO not only removes inefficient material but also recovers the useful material to the solid with its material addition and removal scheme. Compared with conventional topology optimization methods such as homogenization approach and density approach, BESO has no intermediate density elements [10]. Therefore, a black-and-white topology can be obtained. Programs for BESO can be easily developed by the programming languages. Those programs can solve problems in a wide range of disciplines by interfacing the commercial finite element (FE) codes [11-13]. Therefore, BESO has a great value to explore.

Early BESO method can only deal with problems have a single constraint, and most often it is a volume constraint which is easier to control [14]. Huang et al. solved the compliance minimization problem has volume constraint and local displacement constraint using the Lagrange multiplier method [15]. Wang et.al solved the compliance minimization involving volume constraint and stress constraint using a global stress measure based on aggregation function [16]. However, to our best...
knowledge, few studies are focused on problems involving multiple types of constraints though most of the real engineering design problems involving multiple types of constraints besides volume constraints, such as displacement constraints, frequency constraints. In this study, an improved BESO method is proposed based on Huang’s work [17]. Besides the volume constraint, the displacement and natural frequency constraints are also included. To construct the Lagrange function, first, slack variables are introduced to transform the inequality constraints to equality constraints, and then the Lagrange multipliers are introduced to transform the multi-constrained optimization problem into the unconstrained optimization problem. In order to simplify the derivation of Lagrange multipliers, an improved multiplier search scheme is proposed. This method enables the BESO method to handle multi-constrained optimization problems. Finally, the effectiveness of the proposed method is verified by one numerical example with several constraint combinations.

2. Lagrange Function

2.1. Lagrange Relaxation

Lagrange multipliers method is usually used for multi-constrained problems. Not lose generality, a multi-constrained problems can be formulated as:

\[
\begin{align*}
\text{Find} \quad & X \\
\min \quad & f(X) \\
\text{s.t.} \quad & h_j(X) = 0 \quad j = 1, 2, \ldots, l \\
& g_k(X) \leq 0 \quad k = 1, 2, \ldots, m
\end{align*}
\]

(1)

Where \( f(X) \) is the objective function, \( h(X) \) is the equality constraint, \( g(X) \) is the inequality constraint, \( X \) represents the design variable vector and can be expressed by \( X = \{x_1, x_2, \ldots, x_n\}^T \), \( l \) and \( m \) represents the number of equality and inequality constraints, respectively.

To formulate the Lagrange function, a slack variable \( S \) is introduced to transform an inequality constraint into an equality constraint:

\[
\begin{align*}
g_k(X) + S_k = 0
\end{align*}
\]

(2)

Thus, the Lagrange function of a multi-constrained optimization problem is obtained.

\[
\begin{align*}
L(X, \lambda, \mu, S^2) = f(X) + \sum_{j=1}^{l} \lambda_j h_j(X) + \sum_{k=1}^{m} \mu_k (g_k(X) + S_k^2)
\end{align*}
\]

(3)

Where \( \lambda, \mu \) are Lagrange multipliers for the equality and inequality constraints. Thus, the multi-constrained optimization problem formulated in Eq. (1) is transformed into the minimization of \( L(X, \lambda, \mu, S^2) \). The Eq. (3) is equivalent to \( f(X) \) in Eq. (1) if all constraints are satisfied. \( \lambda, \mu \) and \( S^2 \) can be treated as additional variables. Therefore, the derivatives of Lagrange function with respect to these variables are determined as the following additional derivatives.

\[
\begin{align*}
\frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial L}{\partial \lambda_j} = 0, \quad \cdots \\
& \vdots \\
\frac{\partial L}{\partial \lambda_j} = h_j(X) = 0 \\
& \vdots \\
\frac{\partial L}{\partial \mu_m} = g_m(X) + S_m^2 = 0 \\
\frac{\partial L}{\partial S_m} = 2\mu_m S_m
\end{align*}
\]

(4)

A solution could be reached based on Eq. (4). However, it is difficult to demonstrate this solution is
the global optimum without the Hessian matrix. It is impractical to calculate the Hessian matrix since large amounts of variables in the design domain.

Nevertheless, a heuristic estimation can be made based on Eq. (4) at least. For problems that minimize the objective functions. If the derivative of Lagrange function with respect to \( \lambda_j \) is greater than zero. \( \lambda_j \) should be reduced so as to reduce the Lagrange function in the next iteration, and vice versa. If the derivative of Lagrange function with respect to \( \lambda_j \) is equal to zero, the \( j \)-th constraint is satisfied and \( \lambda_j \) should remain unchanged.

For the inequality conditions, if the derivative of Lagrange function with respect to \( \mu_k \) is greater than zero. That means \( g_k(X) + S_k^2 \) is positive. The multiplier \( \mu_k \) should be reduced to minimize the Lagrange function. If \( S_k \) is not equal to zero, \( \mu_k \) needs to be zero to satisfy the \( k \)-th inequality constraint.

2.2. Advanced Search Scheme for the Lagrange Multipliers

Note that, the Lagrange multipliers are continuous in the infinite domain. It is infeasible to find a solution of these additional variables. Thus \( \lambda_j \) and \( \mu_k \) are redefined by a scaling function of replacement factors \( \gamma_j \) and \( \phi_k \) that belong to a finite domain \([0, 1)\).

\[
\lambda_j = \frac{\gamma_j}{1 - \gamma_j} \\
\mu_k = \frac{\phi_k}{1 - \phi_k}
\]

(5)

(6)

If the corresponding constraint is satisfied, the Lagrange multiplier needs to be zero which can be realized by making replacement factor equal to zero. The Lagrange multipliers are increased as the replacement factors increased. As the replacement factors approach one, the Lagrange multipliers would be infinity. Therefore, this scaling function makes the determination of Lagrange multipliers easy to be implemented.

Sensitivity numbers are sensitive to the redistribution of material. To avoid a large fluctuation of the sensitivity numbers, the replacement factors are increased or reduced step by step. In addition, the increment of the replacement factor \( \Delta \gamma \) or \( \Delta \varphi \) (for simplicity, \( \Delta \gamma \) and \( \Delta \varphi \) are denoted by \( \Delta \varphi \) in the rest part of this paper) is determined by following steps:

Step 1. If a constraint is inactive, the replacement factor should be reduced, and vice versa. To avoid the numerical problems, the replacement factor should not be greater than 0.999. If the response is less than 0.9 times that of the specific constraint value \( R^* \), the replacement factor subtracts \( \Delta \varphi \). If the response is greater than 1.1 times that of \( R^* \), the replacement factor adds \( \Delta \varphi \). \( \Delta \varphi \) is selected in \([0.01, 0.02]\).

Step 2. If the response is greater than 0.9 times that of \( R^* \) and is less than 1.1 times that of \( R^* \). The increment \( \Delta \varphi \) is determined by following function:

\[
\Delta \varphi = \text{sign}(R - R^*) \cdot 100 \tau_m \left| \frac{R - R^*}{R^*} \right|^2
\]

(7)

Step 3. Particular responses of the constraints could approach or exceed the imposed constraint values as the volume fraction decreased. To make sure these responses returned to the specific limit in the next several iterations, the corresponding replacement factors are initialized as a small value 0.1 when these responses are greater than 0.95 times that of \( R^* \) for the first time. This procedure is illustrated in Figure 1.
3. Topology Optimization Formulation

This study considers compliance minimization with the volume constraint, displacement constraint and natural frequency constraint. Thus, the topology optimization problem for BESO can be formulated as:

$$\min : \, C$$

$$s.t. \, \sum x_i v_i = V^*$$

$$u \leq u_*$$

$$\omega \geq \omega_*$$

Where \(C\) denotes the compliance of the structure, \(x_i\) is the topology design variable assigned with the finite element \(i\). If \(x_i\) equals one, element \(i\) is presence in the FE model for next iteration. If \(x_i\) equals to a small value, e.g. 0.001, element \(i\) is absence in the FE model for next iteration. \(v_i\) represents the volume of element \(i\). \(u\) and \(\omega\) represent the displacement of the node and the natural frequency of the structure. \(V^*, u^*, \omega^*\) represent the prescribed volume, displacement and natural frequency.

4. Sensitivity Analysis

4.1. Sensitivity of Lagrange Multipliers

On the basis of Eq. (3) and procedures in [17, 18], the total sensitivity of the Lagrange function can be expressed by:

$$\alpha_i = - \frac{\partial L}{\partial x_i} = \alpha_{i,C} + \sum_{j=1}^l \lambda_j \alpha_{i,u} + \sum_{k=1}^m \mu_k \alpha_{i,\omega}$$

4.2. Displacement Sensitivity

It is assumes that the arbitrary node \(j\) in the structure subject to a specific displacement \(u_j\). The derivative of this displacement with respect to the design variable \(x_i\) is expressed as \(du_j/dx_i\):

A virtual unit load \(f_j\) is introduced in which only the corresponding \(j\)-th component is equal to one. The other components are equal to zero. Thus, the specific displacement \(u_j\) can be expressed as:

$$u_j = f_j^T u$$

The equilibrium equation of the structure subjected to this load can be expressed as:

$$f_j = Ku.$$
Where $K$ is the global stiffness matrix of the structure. $\mathbf{u}^*$ is the nodal displacement vector of the structure. The equilibrium equation of the structure subjected to a real load $\mathbf{F}$ is expressed as:

$$F=K\mathbf{u}$$  (12)

The differentiation on both sides of Eq. (12) with respect to arbitrary variable $x_i$ can be expressed as:

$$\frac{\partial K}{\partial x_i} \mathbf{u} + K \frac{\partial \mathbf{u}}{\partial x_i} - \frac{\partial \mathbf{F}}{\partial x_i} = 0$$  (13)

It is assumed that the real load is independent from the design variables, and the Eq. (13) can be simplified as:

$$\frac{\partial \mathbf{u}}{\partial x_i} = -K^{-1} \frac{\partial K}{\partial x_i} \mathbf{u}$$  (14)

The differentiation on both sides of Eq. (10) can be expressed as:

$$\frac{du_j}{dx_i} = f^T_j \frac{\partial \mathbf{u}}{\partial x_i} = u^T_j K^T \frac{\partial \mathbf{u}}{\partial x_i}$$  (15)

Substituting Eq. (14) into Eq. (15), the derivative with respect to $x_i$ can be expressed as:

$$\frac{du_j}{dx_i} = -u^T_j \frac{\partial K}{\partial x_i} \mathbf{u}$$  (16)

To avoid the re-meshing and singularity of the global stiffness matrix, a solid isotropic material with penalization (SIMP) like material interpolation scheme is introduced:

$$\rho_i = x_i \rho_0 \quad , \quad D_i = x_i^p D_0$$  (17)

Where $\rho_i$ represents the material density of the arbitrary element $i$, $\rho_0$ represents the material density of the solid material. $D_i$ and $D_0$ represent the constitutive matrix of arbitrary element $i$ and the solid material. $p$ represents the penalty exponent and equals to three in this study. The stiffness matrix of element $i$ can be expressed as:

$$k_i = \int_{\Omega} \int_{\Omega} B^T_i D_i B_i d\Omega = px_i^{p-1} \int_{\Omega} \int_{\Omega} B^T_i D_0 B_i d\Omega = px_i^{p-1} k_{i,(0)}$$  (18)

Where $k_i$ denotes the elemental stiffness matrix, $k_{i,(0)}$ is the stiffness matrix of solid element. $B_i$ relates the displacement and strain. The derivative of global stiffness matrix with respect to $x_i$ is only determined by the stiffness matrix of element $i$. Therefore, Eq. (16) is simplified as:

$$\frac{du_j}{dx_i} = -u^T_j k_i \mathbf{u} = -px_i^{p-1} u^T_j k_{i,(0)} \mathbf{u}$$  (19)

Thus, the sensitivity for the displacement is expressed as:

$$\alpha_{i,j} = -\frac{du_j}{dx_i} = px_i^{p-1} u^T_j k_{i,(0)} \mathbf{u}$$  (20)

5. Numerical Implementation

BESO is an easily implemented optimization method. This study proposes a program interfaces the commercial FE code ABAQUS. The implementation of BESO can be summed up as follows:

Step 1. Discretize the design domain with a FE model.
Step 2. Define optimization parameters, such as evolutionary ratio $ER$, sensitivity filter radius $r_{\text{min}}$, maximal admission ratio $AR_{\text{max}}$ and the penalty exponent $p$.

Step 3. Perform FE analysis, calculate the sensitivity number based on Eq. (9).

Step 4. The procedures proposed in [10] are introduced to stabilize the optimization process.

Step 5. Determine the volume of the structure for next iteration first and update the design variables based on their sensitivity number. A ‘soft-kill’ technique is utilized in this study. It assigns the element with a soft material if the corresponding design variable is equal to $x_{\text{min}}$.

Step 6. Repeated the Steps 3~5 until the convergence criteria is satisfied. The convergence criteria is the same as that proposed in [17].

6. Numerical Example

The numerical example is illustrated in Figure 2. A plate simply supported at the lower corners and a vertical load is applied to the middle point of the top edge. The material Young’s modulus is 1GPa, density is 8000kg/m$^3$ and Poisson’s ratio is 0.3. The design domain is discretized with 200×100 four-node plane stress elements. The study aims at minimizing the mean compliance with the volume, displacement and natural frequency constraints. The final volume fraction is set to 0.3. The $x$-direction displacement of the lower right corner denoted as point $d_A$ is set as displacement constraint. The first-order natural frequency of the structure is set as natural frequency constraint. The evolutionary volume ratio $ER$ is 0.02, the maximal admission ratio $AR_{\text{max}}$ is 0.01, the filter radius $r_{\text{min}}$ is 1.5mm. The increment of replacement factor $\Delta \varphi$ is 0.015, the penalty exponent $p$ is 3.

To demonstrate the effectiveness of the method proposed in this study. Four different optimization cases are design as shown in Table 1.

![Figure 2. Dimensions and load conditions of example 1](image)

**Table 1. Constraint combinations of example 1**

| Cases  | Constraints               |
|--------|---------------------------|
| Case 1 | No constraints on $d_A$ and $\omega_1$ |
| Case 2 | $d_A<1.4mm$, $\omega_1>240Hz$ |
| Case 3 | $d_A<1.3mm$, $\omega_1>240Hz$ |
| Case 4 | $d_A<1.4mm$, $\omega_1<160Hz$ |

The optimal designs of the four cases are shown in Figure 3. The performances of the four cases are listed in Table 2. They are almost the same as those obtained in [17]. The optimal design of Case 1 is shown in Figure 3 (a). The distribution of the material is a symmetric configuration. The optimal designs of Case 2 and Case 3 have similar material distribution for they have the same natural frequency constraint and similar displacement constraint. It is noted that the optimal design of Case 2 is stiffer than that of Case 3 because of the bigger imposed displacement constraint value. The optimal design of Case 4 approximates a symmetric configuration.
Case 2 is solved by the existing method proposed in [17] and the corresponding results are listed in the last row of Table 2. It is found that the errors between results of optimal designs in this study and the imposed constraint values are smaller than that obtained in [17] though it needs more iterations. Compliance of optimal design due to Case 2 in this study is lower than that obtained in [17]. The advanced Lagrange multipliers search scheme proposed in this study is robust than that proposed in [17] for the increment of the replacement factor is smaller when the response approaching the imposed constraint value. This robustness also stabilizes the optimization process.

![Optimal design of Case 1](a) ![Optimal design of Case 2](b) ![Optimal design of Case 3](c) ![Optimal design of Case 4](d)

Figure 3. Optimal designs of example 1

| Cases   | Iteration number | $C$ (Nmm) | $d_A$ (mm) | $\omega_1$ (Hz) |
|---------|------------------|-----------|------------|-----------------|
| Case 1  | 54               | 102.76    | 1.61       | 236.6           |
| Case 2  | 145              | 106.37    | 1.399      | 248.53          |
| Case 3  | 98               | 111.23    | 1.296      | 251.23          |
| Case 4  | 96               | 110.98    | 1.386      | 152.86          |
| Case 2 [17] | 78       | 113.13    | 1.346      | 257.4           |

7. Conclusions
This paper presented an improved topology optimization method for the multi-constrained topology optimization of continuum structures. The general inequality constraints were transformed into equality constraints using the slack variables. A Lagrange function was established by taking all constraints into the objective function. Thus, the multi-constrained optimization problem was converted to an unconstrained optimization problem. An advanced Lagrange multipliers search scheme was proposed to find a solution of the additional Lagrange multipliers. The design variables were updated by the BESO method.

A numerical example that minimizing mean compliance of the structure involving displacement and natural frequency constraints besides the volume constraint was solved by the proposed topology optimization method. The result shows that different constraint combinations result in different material distributions. Compared with the existing method, though the improved multiplier search scheme makes the optimization process converged with more iterative steps, the mean compliance is converged to a smaller value and all of the constraints are satisfied. The responses of optimal designs and the imposed constraint values are closer. The advanced Lagrange multipliers search scheme is robust which stabilizes the optimization process.

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