On a Lorentz-Invariant Interpretation

of Noncommutative Space-Time

and Its Implications on Noncommutative QFT

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Abstract By invoking the concept of twisted Poincaré symmetry of the algebra of functions on a Minkowski space-time, we demonstrate that the noncommutative space-time with the commutation relations $[x_\mu, x_\nu] = i\theta_{\mu\nu}$, where $\theta_{\mu\nu}$ is a constant real antisymmetric matrix, can be interpreted in a Lorentz-invariant way. The implications of the twisted Poincaré symmetry on QFT on such a space-time is briefly discussed. The presence of the twisted symmetry gives justification to all the previous treatments within NC QFT using Lorentz invariant quantities and the representations of the usual Poincaré symmetry.

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1 Introduction

Quantum field theories on noncommutative space-time have been lately thoroughly investigated, especially after it has been shown [1] that they can be obtained as low-energy limits of open string theory in an antisymmetric constant background field (for reviews, see [2], [3]). However, the issue of the lack of Lorentz symmetry has remained a challenge to this moment, since the field theories defined on a space-time with the commutation relation of the coordinate operators

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \]  

(1.1)

where \( \theta_{\mu\nu} \) is a constant antisymmetric matrix, are obviously not Lorentz-invariant.

In spite of this well-recognized problem, all fundamental issues, like the unitarity [4], causality [5], UV/IR divergences [6], have been discussed in a formally Lorentz invariant approach, using the representations of the usual Poincaré algebra. These results have been achieved using the Weyl-Moyal correspondence, which assigns to every field operator \( \phi(\hat{x}) \) its Weyl symbol \( \phi(x) \) defined on the commutative counterpart of the noncommutative space-time. At the same time, this correspondence requires that products of operators are replaced by Moyal \( \star \)-products of their Weyl symbols:

\[ \phi(\hat{x})\psi(\hat{x}) \to \phi(x) \star \psi(x), \]  

(1.2)

where the Moyal \( \star \)-product is defined as

\[ \phi(x) \star \psi(x) = \phi(x)e^{\frac{i\theta_{\mu\nu}}{2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} \psi(y)|_{x=y}. \]  

(1.3)

Consequently, the commutators of operators are replaced by Moyal brackets and the equiv-
alent of (1.1) is

\[ [x_\mu, x_\nu]_\star \equiv x_\mu \star x_\nu - x_\nu \star x_\mu = i \theta_{\mu\nu}, \quad (1.4) \]

In fact, admitting that noncommutativity should be relevant only at very short distances, the noncommutativity has been often treated as a perturbation and only the corrections to first order in \( \theta \) were computed. As a result, the NC QFT was practically considered Lorentz invariant in zeroth order in \( \theta_{\mu\nu} \), with the first order corrections coming only from the \( \star \)-product.

Later the fact that QFT on 4-dimensional NC space-time is invariant under the \( SO(1, 1) \times SO(2) \) subgroup of the Lorentz group was used [7] (for several applications, see [8], [9], [10], [11]). However, a serious problem arises from the fact that the representation content of the \( SO(1, 1) \times SO(2) \) subgroup is very different from the representation content of the Lorentz group: both \( SO(1, 1) \) and \( SO(2) \) being abelian groups, they have only one-dimensional unitary irreducible representations and thus no spinor, vector etc. representations. In this respect, one encounters a contradiction with previous calculations, in which the representation content for the NC QFT was assumed to be the one of the Poincaré group.

In this letter we shall show that indeed the transformation properties of the NC space-time coordinates \( x_\mu \) can still be regarded as the transformations under the usual Poincaré algebra, with their representation content identical to the one of the commutative case. At the same time, the commutation relation (1.4) appears as the consequence of the noncommutativity of the coproduct (called noncocommutativity) of the twist-deformed (Hopf) Poincaré algebra when acting on the products of the space-time coordinates \( x_\mu x_\nu \). As a consequence, the QFT constructed with \( \star \)-product on such a NC space-time, though it explicitly violates the
Lorentz invariance, possesses the symmetry under the proper twist-Poincaré algebra.

2 Twist deformation of the Poincaré algebra

The usual Poincaré algebra $\mathcal{P}$ with the generators $M_{\mu\nu}$ and $P_\alpha$ has abelian subalgebra of infinitesimal translations. Using this subalgebra it is easy to construct a twist element of the quantum group theory [12] (for detailed explanations, see the monographs [13], [14]), which permits to deform the universal enveloping of the Poincaré algebra $\mathcal{U}(\mathcal{P})^*$.

This twist element $F \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$ does not touch the multiplication in $\mathcal{U}(\mathcal{P})$, i.e. preserves the corresponding commutation relations among $M_{\mu\nu}$ and $P_\alpha$,

\[
\begin{align*}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha}) , \\
[M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu) ,
\end{align*}
\]

with the essential physical implication that the representations of the algebra $\mathcal{U}(\mathcal{P})$ are the same. However, the action of $\mathcal{U}(\mathcal{P})$ in the tensor product of representations is defined by the coproduct given, in the standard case, by the symmetric map (primitive coproduct)

\[
\Delta_0 : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})
\]

\[
\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y ,
\]

for all generators $Y \in \mathcal{P}$. The twist element $F$ changes the coproduct of $\mathcal{U}(\mathcal{P})$ [12]

\[
\Delta_0(Y) \mapsto \Delta_t(Y) = F\Delta_0(Y)F^{-1} .
\]

*For a deformed Poincaré group with twisted classical algebra, see [15].
This similarity transformation is consistent with all the properties of $\mathcal{U}(\mathcal{P})$ as a Hopf algebra if $\mathcal{F}$ satisfies the following twist equation$^1$:

$$\mathcal{F}(\Delta_0 \otimes \text{id}) \mathcal{F} = \mathcal{F}(\text{id} \otimes \Delta_0) \mathcal{F}. \quad \text{(2.4)}$$

Taking the twist element in the form of an abelian twist [16],

$$\mathcal{F} = \exp\left(\frac{i}{2} \theta^{\mu\nu} P_\mu \otimes P_\nu\right), \quad \text{(2.5)}$$

one can check that the twist equation (2.4) is valid.

Since the generators of translations $P_\alpha$ are commutative, their coproduct is not deformed ($\Delta_t = \Delta_0$ is primitive)

$$\Delta_t(P_\alpha) = \Delta_0(P_\alpha) = P_\alpha \otimes 1 + 1 \otimes P_\alpha. \quad \text{(2.6)}$$

However, the coproduct of the Lorentz algebra generators is changed:

$$\Delta_t(M_{\mu\nu}) = \text{Ad} e^{\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) = e^{\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) e^{-\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta}. \quad \text{(2.7)}$$

Using the operator formula $\text{Ad} e^B C = e^B C e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[n \underbrace{[B, [B, \ldots [B, C]]}_{n}\right] = \sum_{n=0}^{\infty} \frac{\text{Ad} B^n}{n!} C$

and the commutation relation between $M_{\mu\nu}$ and $P_\alpha$ (last line of (2.1)), we obtain the explicit form of the coproduct$^1$ $\Delta_t(M_{\mu\nu})$:

$$\Delta_t(M_{\mu\nu}) = \text{Ad} e^{\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu})$$

$$= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta}((\eta_{\alpha\mu} P_\nu - \eta_{\alpha\nu} P_\mu) \otimes P_\beta$$

$^1$See more detailed explanations in monographs on quantum groups (e.g. [13], [14]).

$^2$After the submission of the present work to the hep-th Archive, we were informed that the result (2.8) appears also in [17], which is an extended version of the talk given by Julius Wess in the "Balkan Workshop 2003".
\[ + \ P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu) \]  

It is known (cf. [13], [18]) that having a representation of a Hopf algebra \( \mathcal{H} \) in an associative algebra \( \mathcal{A} \) consistent with the coproduct \( \Delta \) of \( \mathcal{H} \) (a Leibniz rule)

\[ h(a \cdot b) = h_1(a) \cdot h_2(b) , \quad \Delta(h) = h_1 \otimes h_2 , \]  

(2.9)

the multiplication in \( \mathcal{A} \) has to be changed after twisting \( \mathcal{H} \). The new product of \( \mathcal{A} \) consistent with the twisted coproduct \( \Delta_t \) is defined as follows: let \( \mathcal{F} = \sum f_1 \otimes f_2 \), then

\[ a \star b = \sum \overline{f}_1(a) \cdot \overline{f}_2(b) , \]  

(2.10)

where \( \overline{\mathcal{F}} = \sum \overline{f}_1 \otimes \overline{f}_2 \) denotes the representation of \( \mathcal{F}^{-1} \) in \( \mathcal{A} \otimes \mathcal{A} \), and the action of elements \( \overline{f} \in \mathcal{H} \) on elements \( a, b \in \mathcal{A} \) is the same as without twisting.

Let us now consider the commutative algebra \( \mathcal{A} \) of functions, \( f(x), g(x), \ldots \), depending on coordinates \( x_\mu, \mu = 0, 1, 2, 3 \), in the Minkowski space \( M \). In \( \mathcal{A} \) we have the representation of \( \mathcal{U}(\mathcal{P}) \) generated by the standard representation of the Poincaré algebra:

\[ P_\mu f(x) = i \partial_\mu f(x) , \quad M_{\mu\nu} f(x) = i (x_\mu \partial_\nu - x_\nu \partial_\mu) f(x) , \]  

(2.11)

acting on coordinates as follows:

\[ P_\mu x_\rho = i \eta_{\mu\rho} , \quad M_{\mu\nu} x_\rho = i (x_\mu \eta_{\nu\rho} - x_\nu \eta_{\mu\rho}) . \]  

(2.12)

The Poincaré algebra acts on the Minkowski space \( x_\mu, \mu = 0, 1, 2, 3 \) with commutative multiplication:

\[ m(f(x) \otimes g(x)) := f(x)g(x) . \]  

(2.13)
When twisting $U(\mathcal{P})$, one has to redefine the multiplication according to (2.10), while retaining the action of the generators of the Poincaré algebra on the coordinates as in (2.12):

$$m_t(f(x) \otimes g(x)) =: f(x) \star g(x) = m \circ e^{-\frac{i}{2} \eta^{\alpha \beta} P_\alpha \otimes P_\beta} (f(x) \otimes g(x)) = m \circ e^\frac{\frac{i}{2} \eta^{\alpha \beta} \partial_\alpha \otimes \partial_\beta} (f(x) \otimes g(x)).$$

(2.14)

Specifically, one can now easily compute the commutator of coordinates:

$$m_t(x_\mu \otimes x_\nu) = x_\mu \star x_\nu = m \circ e^{-\frac{i}{2} \eta^{\alpha \beta} P_\alpha \otimes P_\beta} (x_\mu \otimes x_\nu)$$

$$= m \circ [x_\mu \otimes x_\nu + \frac{i}{2} \eta^{\alpha \beta} \eta_{\alpha \mu} \otimes \eta_{\beta \nu}]$$

$$= x_\mu x_\nu + \frac{i}{2} \eta^{\alpha \beta} \eta_{\alpha \mu} \eta_{\beta \nu};$$

$$m_t(x_\nu \otimes x_\mu) = x_\nu \star x_\mu = x_\nu x_\mu + \frac{i}{2} \eta^{\alpha \beta} \eta_{\alpha \nu} \eta_{\beta \mu}. \quad (2.15)$$

Hence,

$$[x_\mu, x_\nu]_\star = \frac{i}{2} \eta^{\alpha \beta} (\eta_{\alpha \mu} \eta_{\beta \nu} - \eta_{\alpha \nu} \eta_{\beta \mu}) = i \theta_{\mu \nu},$$

(2.16)

which is indeed the Moyal bracket (1.4).

### 3 QFT on space-time with twisted Poincaré symmetry

Comparing (1.3) and (2.14) (or equivalently (1.4) and (2.16)), it is obvious that building up the noncommutative quantum field theory through Weyl-Moyal correspondence is equivalent to the procedure of redefining the multiplication of functions, so that it is consistent with the twisted coproduct of the Poincaré generators (2.6), (2.8). The QFT so obtained is invariant under the twisted Poincaré algebra. The benefit of reconsidering NC QFT in the latter approach is that it makes transparent the invariance under the twist-deformed Poincaré algebra, while the first approach highlights the violation of the Lorentz group.
To show this invariance, let us take, as an instructive example, the product $f_{\rho \sigma}(x) = x_\rho x_\sigma$. In the standard non-twisted case, the action of the Lorentz generators on this product reads as:

$$M_{\mu \nu} f_{\rho \sigma} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) f_{\rho \sigma} = i(f_{\mu \sigma} \eta_{\nu \rho} - f_{\nu \sigma} \eta_{\mu \rho} + f_{\rho \nu} \eta_{\mu \sigma} - f_{\rho \mu} \eta_{\nu \sigma}) ,$$  

(3.17)

expressing the fact that $f_{\rho \sigma}$ is a rank-two Lorentz tensor. In the twisted case, $f_{\rho \sigma}$ should be replaced, according to (2.14), by the symmetrized expression\(^6\) $f^t_{\rho \sigma} = x_{(\rho} \star x_{\sigma)} = \frac{1}{2} (x_\rho \star x_\sigma + x_\sigma \star x_\rho)$, and correspondingly the action of the Lorentz generator should be applied through the twisted coproduct:

$$M^t_{\mu \nu} f^t_{\rho \sigma} = m_t \circ (\Delta_t(M_{\mu \nu})(x_\rho \otimes x_\sigma)) .$$  

(3.18)

In the above equation, $M^t_{\mu \nu}$ denotes the usual Lorentz generator, but with the action of a twisted coproduct. A straightforward calculation gives:

$$M^t_{\mu \nu} f^t_{\rho \sigma} = i(f^t_{\mu \sigma} \eta_{\nu \rho} - f^t_{\nu \sigma} \eta_{\mu \rho} + f^t_{\rho \nu} \eta_{\mu \sigma} - f^t_{\rho \mu} \eta_{\nu \sigma}) ,$$  

(3.19)

which is analogous to (3.17), confirming the (expected) covariance under the twisted Poincaré algebra. This argument extends to any symmetrized tensor formed from the $\star$-products of $x$’s. For example, the invariance of Minkowski length $s^2_t = x_\mu \star x^\mu = x_\mu x^\mu$ is obvious: multiplying (3.19) by $\eta^{\rho \sigma}$, one obtains $M^t_{\mu \nu} s^2_t = 0$.

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\(^6\)We use the symmetrization because, due to the commutation relation $[x_\mu, x_\nu]_\star = i \theta_{\mu \nu}$ (where $\theta_{\mu \nu}$ is twisted-Poincaré invariant, as shown also in the consistency check performed below), every tensorial object of the form $x_\mu \star x_\nu \star \cdots \star x_\sigma$ can be written as a sum of symmetric tensors of lower or equal ranks, so that the basis of the representation algebra $A_t$ is symmetric. This statement is valid in general in the case of the universal enveloping algebras of Lie algebras.
As a consistency check, we shall calculate the action of $M^t_{\mu\nu}$ on the antisymmetric combination $2x_\rho \star x_\sigma = [x_\rho, x_\sigma]_*$:

\[
M^t_{\mu\nu}([x_\rho, x_\sigma]_*) = ([x_\mu, x_\sigma]_* - i\theta_{\mu\rho})\eta_{\nu\rho} - ([x_\nu, x_\sigma]_* - i\theta_{\nu\sigma})\eta_{\mu\rho}
- ([x_\mu, x_\rho]_* - i\theta_{\mu\rho})\eta_{\nu\sigma} + ([x_\nu, x_\rho]_* - i\theta_{\nu\rho})\eta_{\mu\sigma} = 0.
\]

(3.20)

Thus, we have $M^t_{\mu\nu}\theta_{\rho\sigma} = 0$, since $\theta_{\rho\sigma} = -i[x_\rho, x_\sigma]_*$, i.e. the antisymmetric tensor $\theta_{\rho\sigma}$ is twisted-Poincaré invariant.

Therefore, the Lagrangian obtained by replacing all the usual products of fields in the corresponding commutative theory with $\star$-products, though it breaks the Lorentz invariance in the usual sense, it is, however, invariant under the twist-deformed Poincaré algebra.

Another important feature of the QFT with twist-deformed Poincaré symmetry deserves a special highlighting: the representation content of the NC QFT is exactly the same as for its commutative correspondent. It is easy to see that the action of the Pauli-Ljubanski operator, $W_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}M^{\beta\gamma}P^\delta$ is not changed by the twist (due to the commutativity of the translation generators) and $P^2$ and $W^2$ retain their role of Casimir operators. Consequently, the representations of the twisted Poincaré algebra will be, just as in the commutative case, classified according to the eigenvalues of these invariant operators, $m^2$ and $m^2s(s+1)$, respectively. Besides justifying the validity of the results obtained so far in NC QFT using the representations of the Poincaré algebra, this aspect will cast a new light on other closely-related fundamental issues, such as the CPT and the spin-statistics theorems in NC QFT [9, 10, 19].
4 Conclusions

In this letter we have shown that the quantum field theory on NC space-time possesses symmetry under a twist-deformed Poincaré algebra. The twisted Poincaré symmetry exists provided that: (i) we consider \( \star \)-products among functions instead of the usual one and (ii) we take the proper action of generators specified by the twisted coproduct. As a byproduct with major physical implications, the representation content of NC QFT, invariant under the twist-deformed Poincaré algebra, is identical to the one of the corresponding commutative theory with usual Poincaré symmetry. Some of the applications of the present treatment of the symmetry properties of NC QFT will be considered in a forthcoming communication [20].

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