Weakly periodic Gibbs measures of the Ising model on the Cayley tree of order five and six

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1. Introduction

A Gibbs measure is a mathematical idealization of an equilibrium state of a physical system which consists of a very large number of interacting components. In the language of Probability Theory, a Gibbs measure is simply the distribution of a stochastic process which, instead of being indexed by the time, is parametrized by the sites of a spatial lattice, and has the special feature of admitting prescribed versions of the conditional distributions with respect to the configurations outside finite regions. The physical phenomenon of phase transition is reflected by the non-uniqueness of the Gibbs measures for considered model. The Ising model is realistic enough to exhibit this non-uniqueness of Gibbs measures in which a phase transition is predicted by physics. This fact is one of the main reasons for the physical interest in Gibbs measures. The problem of non-uniqueness, and also the converse problem of uniqueness, are central themes of the theory of Gibbs measures.

Let \( M(H) \) be the set of all Gibbs measures defined by Hamiltonian \( H \). Note that this set contain translation-invariant Gibbs measures, periodic Gibbs measures and non-periodic Gibbs measures and one can consider the problem of phase transition in the classes translation-invariant Gibbs measures, periodic Gibbs measures and non-periodic Gibbs measures respectively. In [1-5] have been studied the translational invariant Gibbs measures of the Ising model and some its generalization on the Cayley tree. The papers [6-8] devoted to study periodic Gibbs measures with period 2 for models with finite radius of interaction. In [9-12] the authors introduced new class of Gibbs measures, so-called weakly periodic Gibbs measures and proved the existence of such measures for the Ising model on the Cayley tree. In the papers [1, 12, 13] have been constructed continuum sets of non periodic Gibbs measures for the Ising model on the Cayley tree. In [9, 10, 14] the authors considered non-periodic weakly periodic Gibbs measures for the Ising model on the Cayley tree of order \( k < 5 \). The present paper is continuation of investigations in [14] and in this paper we study weakly periodic Gibbs measures on the Cayley tree of order five and six.

2. Basic definitions and formulation of the problem

Let \( \Gamma^k = (V, L), k \geq 1, \) be the Cayley tree of order \( k \), i.e. an infinity graph every vertex of which is incident to exactly \( k + 1 \) edges. Here \( V \) is the set of all vertices, \( L \) is the set of all edges of the tree \( \Gamma^k \). It is known that \( \Gamma^k \) can be represented as a non-commutative group \( G_k \), which is the free product of \( k + 1 \) cyclic groups of the second order [13].
For an arbitrary point $x^0 \in V$ we set $W_n = \{x \in V | d(x^0, x) = n\}$, $V_n = \bigcup_{m=0}^{n} W_m$, $L_n = \{< x, y > \in L | x, y \in V_n\}$, where $d(x, y)$ is the distance between the vertices $x$ and $y$ in the Cayley tree, i.e. the number of edges in the shortest path joining the vertices $x$ and $y$. We write $x \prec y$ if the path from $x^0$ to $y$ goes through $x$. We call the vertex $y$ a direct successor of $x$, if $y \succ x$ and $x, y$ are nearest neighbours. The set of the direct successors of $x$ is denoted by $S(x)$, i.e., if $x \in W_n$, then

$$S(x) = \{y_i \in W_{n+1} | d(x, y_i) = 1, i = 1, 2, \cdots, k\}.$$  

Let $\Phi = \{-1, 1\}$ and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e. $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. For subset $A \subset V$ we denote by $\Omega_A$ the space of all configurations defined on the set $A$ and taking values in $\Phi$.

We consider the Hamiltonian of the Ising model:

$$H(\sigma) = -J \sum_{<x, y> \in L} \sigma(x)\sigma(y),$$  

(2.1)

where $J \in R$, $\sigma(x) \in \Phi$ and $< x, y >$ are nearest neighbors.

For every $n$, we define a measure $\mu_n$ on $\Omega_{V_n}$ setting

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\{ -\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x) \},$$  

(2.2)

where $h_x \in R, x \in V$, $\beta = \frac{1}{T}$ ($T$ is temperature, $T > 0$), $\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$, $Z_n^{-1}$ is the normalizing factor, and

$$H(\sigma_n) = -J \sum_{<x, y> \in L_n} \sigma(x)\sigma(y).$$

The compatibility condition for the measures $\mu_n(\sigma_n), n \geq 1$, is

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}),$$  

(2.3)

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $\mu_n, n \geq 1$ be a sequence of measures on the sets $\Omega_{V_n}$ that satisfy compatibility condition (2.3). By the Kolmogorov theorem, we then have a unique limit measure $\mu$ on $\Omega_V = \Omega$ (called the limit Gibbs measure) such that

$$\mu(\sigma_n) = \mu_n(\sigma_n)$$

for every $n = 1, 2, \cdots$. It is known that measures (2.2) satisfies the condition (2.3) if and only if the set $h = \{h_x, x \in G_k\}$ of quantities satisfies the condition

$$h_x = \sum_{y \in S(x)} f(h_y, \theta),$$  

(2.4)

where $S(x)$ is the set of direct successors of the vertex $x \in V$ (see [1], [2], [3]). Here, $f(x, \theta) = \arctanh(\tanh(\theta \tanh x))$, $\theta = \tanh(J \beta)$, $\beta = \frac{1}{T}$.

Let $G_k/\hat{G}_k = \{H_1, \cdots, H_r\}$ be a factor group, where $\hat{G}_k$ is a normal subgroup of index $r \geq 1$. 
Definition 1. A set $h = \{h_x, x \in G_k\}$ of quantities is called $\hat{G}_k$-periodic if $h_{xy} = h_x$, for all $x \in G_k$ and $y \in \hat{G}_k$.

For $x \in G_k$ we denote by $x_1$ the unique point of the set $\{y \in G_k : \langle x, y \rangle \}\setminus S(x)$.

Definition 2. A set of quantities $h = \{h_x, x \in G_k\}$ is called $\hat{G}_k$-weakly periodic, if $h_x = h_{ij}$, for any $x \in H_i, x_1 \in H_j$.

We note that the weakly periodic $h$ coincides with an ordinary periodic one (see Definition 1) if the quantity $h_x$ is independent of $x_1$.

Definition 3. A Gibbs measure $\mu$ is said to be $\hat{G}_k$-(weakly) periodic if it corresponds to the $\hat{G}_k$-(weakly) periodic $h$. We call a $G_k$-periodic measure a translation-invariant measure.

In this paper, we study weakly periodic Gibbs measures and demonstrate that such measures exist for the Ising model on a Cayley tree of order five and six.

3. WEAKLY PERIODIC MEASURES

The level of difficulty in the describing of weakly periodic Gibbs measures is related to the structure and index of the normal subgroup relative to which the periodicity condition is imposed. It is known (see Chapter 1 of [1]) that in the group $G_k$, there is no normal subgroup of odd index different from one. Therefore, we consider normal subgroups of even indices. Here, we restrict ourself to the case of index two.

We describe $\hat{G}_k$-weakly periodic Gibbs measures for any normal subgroup $\hat{G}_k$ of index two. We note (see Chapter 1 of [1]) that any normal subgroup of index two of the group $G_k$ has the form

$$H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i) - \text{even} \right\},$$

where $\emptyset \neq A \subseteq N_k = \{1, 2, \ldots, k + 1\}$, and $\omega_x(a_i)$ is the number of letters $a_i$ in a word $x \in G_k$.

Let $A \subseteq N_k$ and $H_A$ be the corresponding normal subgroup of index two. We note that in the case $|A| = k + 1$, i.e., in the case $A = N_k$, weak periodicity coincides with ordinary periodicity. Therefore, we consider $A \subset N_k$ such that $A \neq N_k$. Then, in view of (2.4), the $H_A$-weakly periodic set of $h$ has the form

$$h_x = \begin{cases} h_1, & x \in H_A, x_1 \in H_A, \\ h_2, & x \in H_A, x_1 \in G_k \setminus H_A, \\ h_3, & x \in G_k \setminus H_A, x_1 \in H_A, \\ h_4, & x \in G_k \setminus H_A, x_1 \in G_k \setminus H_A, \end{cases} \quad (3.1)$$

where $h_i, i = 1, 2, 3, 4$, satisfy the following equations:

$$\begin{cases} h_1 = |A| f(h_3, \theta) + (k - |A|) f(h_1, \theta), \\ h_2 = (|A| - 1) f(h_3, \theta) + (k + 1 - |A|) f(h_1, \theta), \\ h_3 = (|A| - 1) f(h_2, \theta) + (k + 1 - |A|) f(h_4, \theta), \\ h_4 = |A| f(h_2, \theta) + (k - |A|) f(h_4, \theta). \end{cases} \quad (3.2)$$
Consider operator $W : R^4 \to R^4$, defined by

$$
\begin{align*}
 h'_1 &= |A| f(h_3, \theta) + (k - |A|) f(h_1, \theta) \\
 h'_2 &= (|A| - 1) f(h_3, \theta) + (k + 1 - |A|) f(h_1, \theta) \\
 h'_3 &= (|A| - 1) f(h_2, \theta) + (k + 1 - |A|) f(h_4, \theta) \\
 h'_4 &= |A| f(h_2, \theta) + (k - |A|) f(h_4, \theta).
\end{align*}
$$

(3.3)

Note that the system of equations (3.2) describes fixed points of the operator $W$, i.e.

$h = W(h)$.

It is obvious that the following sets are invariant with respect to operator $W$:

$I_1 = \{ h \in R^4 : h_1 = h_2 = h_3 = h_4 \}$, $I_2 = \{ h \in R^4 : h_1 = h_4; h_2 = h_3 \}$, $I_3 = \{ h \in R^4 : h_1 = -h_4; h_2 = -h_3 \}$.

In [9] it was proved that the system of equation (3.2), on the invariant set $I_2$ has the solutions which belong to $I_1$. The system equation (3.2) on the invariant set $I_1$ reduced to the following equation

$$h = kf(h, \theta).$$

(3.4)

The solutions of (3.4) correspond to translation-invariant Gibbs measures. In this paper we will study weakly periodic (non-periodic, in particular non translation-invariant) Gibbs measures, i.e. we will investigate the fixed points of operator $W$ in the invariant set $I_3$.

Let $\alpha = \frac{1 - \theta}{1 + \theta}$. In [11] was proven the following statement.

**Theorem 1.** Let $|A| = k, \alpha > 1$.

1) For $k \leq 3$ all $H_A$-weakly periodic Gibbs measures on $I_3$ are translational invariant.

2) For $k = 4$ there exists a critical value $\alpha_{cr}(\approx 6.3716)$ such that for $\alpha < \alpha_{cr}$ on $I_3$ there exists one $H_A$-weakly periodic Gibbs measure; for $\alpha = \alpha_{cr}$ on $I_3$ there exist three $H_A$-weakly periodic Gibbs measures; for $\alpha > \alpha_{cr}$ on $I_3$ there exist five $H_A$-weakly periodic Gibbs measures.

**Remark 1.** Note that one of the measures described in item 2) of Theorem 1 is translation-invariant, but the other measures are $H_A$-weakly periodic (non-periodic) and differ from measures considered in [9, 10].

In Theorem 1 have been considered the cases with $k \leq 4$ (see [11]). In this paper we consider the cases with $k \geq 5$.

Using the fact that

$$f(h, \theta) = \arctanh(\theta \tanh h) = \frac{1}{2} \ln \frac{(1 + \theta)e^{2h} + (1 - \theta)}{(1 - \theta)e^{2h} + (1 + \theta)},$$

and introducing the variables $z_i = e^{2h_i}$ $i = 1, 2, 3, 4$ one can transform the system of equations (3.2) to the following:

$$
\begin{align*}
 z_1 &= \left( \frac{21 + \alpha}{\alpha z_1 + 1} \right)^{|A|} \cdot \left( \frac{21 + \alpha}{\alpha z_1 + 1} \right)^{(k - |A|)} \\
 z_2 &= \left( \frac{21 + \alpha}{\alpha z_2 + 1} \right)^{|A| - 1} \cdot \left( \frac{21 + \alpha}{\alpha z_2 + 1} \right)^{(k + 1 - |A|)} \\
 z_3 &= \left( \frac{21 + \alpha}{\alpha z_3 + 1} \right)^{|A| - 1} \cdot \left( \frac{21 + \alpha}{\alpha z_3 + 1} \right)^{(k + 1 - |A|)} \\
 z_4 &= \left( \frac{21 + \alpha}{\alpha z_4 + 1} \right)^{|A|} \cdot \left( \frac{21 + \alpha}{\alpha z_4 + 1} \right)^{(k - |A|)}.
\end{align*}
$$

(3.5)
Theorem 2. Let $|A| = k$. Then for arbitrary $k$ the number of $H_A$-weakly periodic (non-periodic) Gibbs measures which correspond to fixed points of operator $W$ on the invariant set $I_3$ does not exceed four.

Proof. Let $|A| = k$. Then the system of equations (3.5) has the form
\[
\begin{align*}
    z_1 &= (f(z_3))^k \\
    z_2 &= (f(z_3))^{k-1} \cdot (f(z_1)) \\
    z_3 &= (f(z_2))^{k-1} \cdot (f(z_4)) \\
    z_4 &= (f(z_2))^k,
\end{align*}
\]  
(3.6)

where $f(x) = \frac{x\alpha}{\alpha^2 + 1}$. The system of equations (3.6) on the invariant set $I_3$ has the following form:
\[
\begin{align*}
    z_1 &= \left( f\left( \frac{1}{z_2} \right) \right)^k \\
    z_2 &= \left( f\left( \frac{1}{z_2} \right) \right)^{k-1} \cdot (f(z_1))
\end{align*}
\]  
(3.7)

and it can be transformed to the following equation
\[
z_2 = \left( \frac{1 + \alpha z_2}{\alpha + z_2} \right)^{k-1} \frac{\alpha(\alpha + z_2)^k + (1 + \alpha z_2)^k}{(\alpha + z_2)^k + \alpha(1 + \alpha z_2)^k}.
\]  
(3.8)

Assuming $u = f(z_2)$ we reduce the equation (3.8) to the equation
\[
u^{2k} - \alpha u^{2k-1} + \alpha^2 u^{k+1} - \alpha^2 u^{k-1} + \alpha u - 1 = 0. 
\]  
(3.9)

According Descartes’ rule of signs (see for example [15]) the number of positive roots of the polynomial (3.9) is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Therefore, the equation (3.9) has at most five positive solutions. It is easy to verify that this equation (3.9) is factorized as follows:
\[
(u^2 - 1)P_{2k-2}(u) = 0,
\]  
\[
(3.10)
\]

where $P_{2k-2}(u)$ is a polynomial of degree $2k-2$. Since one of the roots of (3.9) is $u = 1$ which corresponds to translational-invariant Gibbs measure, the number of $H_A$-weakly periodic (non-periodic) Gibbs measures does not exceed of four.

\[\square\]

Remark 2. In general, the total number of $H_A$-weakly periodic (non-periodic) Gibbs measures (considered everywhere, not only on the invariant set $I_3$) may be greater than four.

Recall that a polynomial $P = \sum_{i=0}^{n} a_i x^i$ of degree $n$, is called palindromic (antipalindromic) if $a_i = a_{n-i}$ (respectively $a_i = -a_{n-i}$) for $i = 0, 1, \cdots, n$. Note that the polynomial (13) is antipalindromic. It is known that if antipalindromic polynomial of even degree is a multiple of $x^2 - 1$ (it has -1 and 1 as a roots) then its quotient by $x^2 - 1$ is palindromic (see for example [15]).

Theorem 3. Let $|A| = k, k = 5$.

For the weakly periodic Gibbs measures corresponding to the set of quantities from $I_3$ there exists a critical value $\alpha_{cr} (\approx 2.65)$ such that there is not any $H_A$- weakly periodic (nonperiodic) Gibbs measures for $0 < \alpha < \alpha_{cr}$, there are two $H_A$- weakly periodic (nonperiodic)
**Gibbs measures for** \( \alpha = \alpha_{cr} \), **and there are four** \( H_A \)– weakly periodic (nonperiodic) **Gibbs measures for** \( \alpha_{cr} < \alpha \).

**Proof.** Let \( k = 5 \)

In this case equation (3.9) has the form 

\[
u^{10} - \alpha u^9 + \alpha^2 u^6 - \alpha^2 u^4 + \alpha u - 1 = 0. \tag{3.11}
\]

Now equation (3.10) has the form 

\[
(u^2 - 1) (u^8 - \alpha u^7 + u^6 - \alpha u^5 + (\alpha^2 + 1) u^4 - \alpha u^3 + u^2 - \alpha u + 1) = 0. \tag{3.12}
\]

From (3.12) we have \( u^2 - 1 = 0 \) or 

\[
u^8 - \alpha u^7 + u^6 - \alpha u^5 + (\alpha^2 + 1) u^4 - \alpha u^3 + u^2 - \alpha u + 1 = 0. \tag{3.13}
\]

Since \( u > 0 \), we have that \( u = 1 \) is the solution of equation (3.12). We assume that \( u \neq 1 \). Setting \( \xi = u + \frac{1}{u} > 2 \), from (3.13) we obtain the equation 

\[
\xi^4 - \alpha \xi^3 - 3 \xi^2 + 2 \alpha \xi + \alpha^2 + 1 = 0. \tag{3.14}
\]

The equation (3.14) has at most two positive solutions. From equation (3.14) we find the parameter \( \alpha \):

\[
\alpha_1 = \frac{\xi^3 - 2 \xi + \sqrt{\xi^6 - 8 \xi^4 + 16 \xi^2 - 4}}{2} := \gamma_1(\xi), \tag{3.15}
\]

\[
\alpha_2 = \frac{\xi^3 - 2 \xi - \sqrt{\xi^6 - 8 \xi^4 + 16 \xi^2 - 4}}{2} := \gamma_2(\xi). \tag{3.16}
\]

Assume \( v = \xi^2 \) and \( \varphi(v) = v^3 - 8 v^2 + 16 v - 4 \). We consider \( \varphi'(v) = 3 v^2 - 16 v + 16 \). It is clear that \( \varphi'(v) > 0 \) for \( v > 4 \). On the other hand \( \varphi(4) < 0 \), \( \varphi(+\infty) > 0 \). It follows that \( \varphi(v) = 0 \) has a unique solution \( v_0 \) for \( v > 4 \). Therefore, the system of inequalities 

\[
\begin{cases}
\xi^6 - 8 \xi^4 + 16 \xi^2 - 4 \geq 0 \\
\xi > 2,
\end{cases}
\]

valid for \( \xi \in [\xi_0, +\infty) \), where \( \xi_0 = \sqrt{v_0} \approx 2.214 \).

Note that \( \gamma_1(\xi_0) = \gamma_2(\xi_0) \).

One can check that 

\[
\lim_{\xi \to +\infty} \gamma_i(\xi) = +\infty, i = 1, 2. \tag{3.17}
\]

It is clear that the function \( \gamma_1(\xi) \) is increasing on the \( [\xi_0, +\infty) \). Then we get following: for \( \alpha \in (0, \gamma_1(\xi_0)) \) there is not \( \xi > 2 \) satisfying the equation (3.14); if \( \alpha \in [\gamma_1(\xi_0), +\infty) \) then there exists a unique \( \xi > 2 \) which satisfying the equation (3.14).

Note that if \( \xi \in [\xi_0, +\infty) \) then the equation \( \gamma_2'(\xi) = 0 \) has a unique solution, which is \( \xi_1 \approx 2.3841 \), and also we get \( \gamma_2(\xi_0) \approx 3.21, \gamma_2(\xi_1) \approx 2.65 \). Denote \( \alpha_{cr} = \gamma_2(\xi_1) \).

Hence it is evident that the function \( \gamma_2(\xi) \) reaches its minimum in \( [\xi_0, +\infty) \) at \( \xi_1 \). Consequently, for \( \alpha \in (0, \gamma_2(\xi_1)) \) there is not \( \xi > 2 \) satisfying the equation (3.14), for \( \alpha \in \{ \gamma_2(\xi_1) \} \cup (\gamma_2(\xi_0); +\infty) \) there exists a unique \( \xi > 2 \) satisfying the equation (3.14), if \( \alpha \in (\gamma_2(\xi_1), \gamma_2(\xi_0)) \) then there exist two \( \xi > 2 \) satisfying the equation (3.14).
Let \( n_\alpha \) be the number of solutions of the equation (3.14). Then \( n_\alpha \) has the following form

\[
n_\alpha = \begin{cases} 
0, & \text{if } \alpha \in (0, \alpha_{cr}) \\
1, & \text{if } \alpha = \alpha_{cr} \\
2, & \text{if } \alpha \in (\alpha_{cr}, +\infty) .
\end{cases}
\]

For \( \alpha \in (0, \alpha_{cr}) \) from \( u + \frac{1}{u} = \xi \) we get four solutions of the equation (3.13). In this case the equation (3.12) has five solutions. For \( \alpha = \alpha_{cr} \) from \( u + \frac{1}{u} = \xi \) we get that the equation (3.13) has two solutions. Consequently equation (3.12) has three solutions. In the case \( \alpha > \alpha_{cr} \) the equation (3.12) has a unique solution \( u = 1 \).

**Theorem 4.** Let \( |A| = k, k = 6 \).

For the weakly periodic Gibbs measures corresponding to the set of quantities from \( I_3 \) there exists a critical value \( \alpha_{c} (\approx 1.89) \) such that there is not any \( H_{A}^{-} \) weakly periodic (nonperiodic) Gibbs measures for \( \alpha \in (0, \alpha_{c}) \), there are two \( H_{A}^{-} \) weakly periodic (nonperiodic) Gibbs measures for \( \alpha \in [2, 3] \cup \{ \alpha_{c} \} \), and there are four \( H_{A}^{-} \) weakly periodic (nonperiodic) Gibbs measures for \( \alpha \in (\alpha_{c}, 2) \cup (3, +\infty) \).

**Proof.** Let \( k = 6 \).

In this case (3.9) has the form

\[
u^{12} - \alpha u^{11} + \alpha^2 u^7 - \alpha^2 u^5 + \alpha u - 1 = 0 .
\]

(3.18)

The function \( y := y(u) = u^{12} - \alpha u^{11} + \alpha^2 u^7 - \alpha^2 u^5 + \alpha u - 1 \) with \( \alpha = 4.1 \) is plotted in Fig. 1.

In this case, the equation (3.10) has the form
\[(u^2 - 1) \left( u^{10} - \alpha u^9 + u^8 - \alpha u^7 + u^6 + (\alpha^2 - \alpha)u^5 + u^4 - \alpha u^3 + u^2 - \alpha u + 1 \right) = 0. \tag{3.19} \]

From (3.19) we have \(u^2 - 1 = 0\) or
\[u^{10} - \alpha u^9 + u^8 - \alpha u^7 + u^6 + (\alpha^2 - \alpha)u^5 + u^4 - \alpha u^3 + u^2 - \alpha u + 1 = 0. \tag{3.20} \]

Since \(u > 0\), we have that \(u = 1\) is the solution of equation (3.19). We assume that \(u \neq 1\). Setting \(u = 1\) in equation (3.21), we find the parameter \(\alpha\):}

\[
\alpha_1 = \frac{\xi^4 - 3\xi^2 + 1 - \sqrt{\xi(\xi^2 - 1)(\xi^2 - 3)(\xi - 2)(\xi^2 + 2\xi + 2) + 1}}{2} := \alpha_1(\xi), \tag{3.22}
\]

\[
\alpha_2 = \frac{\xi^4 - 3\xi^2 + 1 + \sqrt{\xi(\xi^2 - 1)(\xi^2 - 3)(\xi - 2)(\xi^2 + 2\xi + 2) + 1}}{2} := \alpha_2(\xi). \tag{3.23}
\]

One can check that
\[
\lim_{\xi \to +\infty} \alpha_i(\xi) = +\infty, i = 1, 2,
\]
and \(\xi(\xi^2 - 1)(\xi^2 - 3)(\xi - 2)(\xi^2 + 2\xi + 2) + 1\) is positive for all \(\xi \geq 2\).

Note that if \(\xi \in [2, +\infty)\) the the equation \(\alpha_1(\xi) = 0\) has a unique solution which is \(\xi_0 \approx 2,077\), and also we get \(\alpha_1(\xi_0) \approx 1,89\). Hence it is clear that the function \(\alpha_1(\xi)\) reaches its minimum in \([2, +\infty)\) at \(\xi_0\). Consequently, for \(\alpha \in (0, \alpha_1(\xi_0))\) there is not \(\xi \geq 2\) satisfying the equation (3.21), for \(\alpha \in \{\alpha_1(\xi_0)\} \cup \{\alpha_1(2); +\infty\}\) there exist unique \(\xi \geq 2\) satisfying the equation (3.21), \(\alpha \in (\alpha_1(\xi_0), \alpha_1(2))\) there exist two \(\xi \geq 2\) satisfying the equation (3.21).

It is clear that the function \(\alpha_2(\xi)\) is increasing on the \([2, +\infty)\). Then we get following: for \(\alpha \in (0, \alpha_2(2))\) there is not \(\xi > 2\) satisfying the equation (3.21); \(\alpha \in [\alpha_2(2), +\infty)\) there exist a unique \(\xi > 2\) which satisfying the equation (3.21).

Denote \(\alpha_1(\xi_0) = \alpha_c\).

Let \(n_\alpha\) be the number of solutions of the equation (3.20). Then \(n_\alpha\) has the following form

\[
n_\alpha = \begin{cases} 
0, & \text{if } \alpha \in (0, \alpha_c) \\
1, & \text{if } \alpha \in (\alpha_1(2), \alpha_2(2)) \cup \{\alpha_c\} \\
2, & \text{if } \alpha \in (\alpha_c, \alpha_1(2)) \cup [\alpha_2(2), +\infty). 
\end{cases}
\]

For \(\alpha \in (0, \alpha_c)\) equation (3.19) has a unique solution \(u = 1\). For \(\alpha \in (2, 3) \cup \{\alpha_c\}\) from \(u + \frac{1}{u} = \xi\) we get two solutions of the equation (3.20). In this case the equation (3.19) has five solutions. Note that \(\alpha_1(2) = 2\) and \(\alpha_2(2) = 3\) and in the case \(\xi = 2\) from \(u + \frac{1}{u} = \xi\) we get \(u = 1\). Consequently, for \(\alpha = 2\) and \(\alpha = 3\) we get two solution of equation (3.20) which different from \(u = 1\). In this case equation (3.19) has three solutions. For \(\alpha \in (\alpha_c, \alpha_1(2)) \cup (\alpha_2(2), +\infty)\) from \(u + \frac{1}{u} = \xi\) we get that the equation (3.20) has two solutions. In this the equation (3.19) has a five solutions.
Let \( N_\alpha \) be the number of solutions of the equation (3.19). Then \( N_\alpha \) has the following form

\[
N_\alpha = \begin{cases} 
1, & \text{if } \alpha \in (0, \alpha_c) \\
3, & \text{if } \alpha \in [2, 3] \cup \{\alpha_c\} \\
5, & \text{if } \alpha \in (\alpha_c, 2) \cup (3, +\infty).
\end{cases}
\]

\[\square\]

4. DISCUSSION

It is well known [16-19] the formulation of Ising model according to which every node \( x \) of a lattice corresponds to the two-valued variable \( \sigma(x) \) with values +1 or −1. If "objects" connected with nodes \( x \) and \( x' \) are in the same state, then \( \sigma(x)\sigma(x') = +1 \) but if they are in different state, then \( \sigma(x)\sigma(x') = -1 \). Clear that in such an interpretation the definition of "object" connected with the node \( x \) can be discussed very widely. For example, it may be a magnetic moment of an ion in a crystal having two directions [20] or it may be atoms of two kinds in a binary alloy [21] (value \( \sigma(x) = +1 \) corresponds to an occupation of \( x \)-th node by an atom of one kind, and \( \sigma(x) = -1 \) when occupation take place by an atom of another kind). Others interpretations of the Ising model are connected with an investigation of adsorption phenomenon on a surface [22], with DNA melting [23], with the theory of latticed gas [24] and with other questions of theory of change of a phase of type order-disorder. For considered model on \( \Gamma^5 \) we find critical value \( \alpha_{cr}(\approx 2, 65) \) such that there are two \( H_A \)-weakly periodic (non-periodic) Gibbs measures for \( \alpha = \alpha_{cr} \), and there are four \( H_A \)-weakly periodic (non-periodic) Gibbs measures for \( \alpha_{cr} < \alpha \).

On \( \Gamma^6 \) we find critical value \( \alpha_c(\approx 1, 89) \) such that there are two \( H_A \)-weakly periodic (non-periodic) Gibbs measures for \( \alpha \in [2, 3] \cup \{\alpha_c\} \), and there are four \( H_A \)-weakly periodic (non-periodic) Gibbs measures for \( \alpha \in (\alpha_c, 2) \cup (3, +\infty) \).

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