Abstract. A g-element for a graded $R$-module is a one-form with properties similar to a Lefschetz class in the cohomology ring of a compact complex projective manifold, except that the induced multiplication maps are injections instead of bijections. We show that if $k(\Delta)$ is the face ring of the independence complex of a matroid and the characteristic of $k$ is zero, then there is a non-empty Zariski open subset of pairs $(\Theta, \omega)$ such that $\Theta$ is a linear set of parameters for $k(\Delta)$ and $\omega$ is a $g$-element for $k(\Delta)/<\Theta>$. This leads to an inequality on the first half of the $h$-vector of the complex similar to the $g$-theorem for simplicial polytopes.

1. Introduction

The combinatorics of the independence complex of a matroid can be approached from several different directions. The $f$-vector directly encodes the number of independent sets of every cardinality, while the $h$-vector contains the same information encoded in a way which is more appropriate for reliability problems [4]. In either case the fundamental question is the same. What vectors are possible?

Let $(h_0, h_1, \ldots, h_r)$ be the $h$-vector of the independence complex of a rank $r$ matroid without coloops. Using a PS-ear decomposition of the complex Chari [3] proved that for all $i \leq r/2$, $h_i \leq h_{r-i}$ and $h_{i-1} \leq h_i$. By showing that the $h$-vector was the Hilbert function of $k(\Delta)/<\Theta>$, where $k(\Delta)$ is the face ring of the complex and $\Theta$ is a linear system of parameters for $k(\Delta)$, Stanley [8] proved that $h_{i+1} \leq h_i^{<i>}$ (see section 3 for a definition of the $<i>$-operator). By combining these two methods we show in Theorem 4.3 that if we define $g_i = h_i - h_{i-1}$, then $g_{i+1} \leq g_i^{<i>}$ for all $i < r/2$. All of these inequalities are immediate consequences of the existence of pairs $(\Theta, \omega)$ such that $\Theta$ is a linear set of parameters for $k(\Delta)$ and $\omega$ is a $g$-element for $k(\Delta)/<\Theta>$. Using a different approach, toric hyperkähler varieties, Hausel and Sturmfels proved the existence of $g$-elements for $k(\Delta)/<\Theta>$ when the matroid is representable over the rationals [7]. A $g$-element is a one-form which acts like a Lefschetz class of a compact complex projective manifold except that it induces injections instead of bijections (Definition 4.1).

The broken circuit complex of a matroid is subcomplex of the independence complex and directly encodes the coefficients of the characteristic polynomial of the matroid. Every broken circuit complex is a cone, and if we remove the cone point we obtain a reduced broken circuit complex. Any independence complex is also a reduced broken circuit complex. Since the $h$-vector is unchanged by the removal of a cone point, the set of $h$-vectors of independence complexes is a (strict) subset of the set of $h$-vectors of broken circuit complexes. A natural question is whether or not Theorem 4.2 holds for broken circuit complexes. In Section 5 we...
show that even if broken circuit complexes satisfy the corresponding combinatorial inequalities, there may be no set of linear parameters for the face ring such that there exist $g$-elements for the quotient ring.

Matroid terminology and notation closely follows \[8\]. The main exception to this is that we use $M - A$ for the deletion of a subset instead of $M \setminus A$. The ground set of the matroid $M$ is always $E$.

2. Complexes

Let $\Delta$ be a finite abstract simplicial complex with vertices $V = \{v_1, \ldots, v_n\}$. The $f$-vector of $\Delta$ is the sequence $(f_0(\Delta), \ldots, f_s(\Delta))$, where $f_i(\Delta)$ is the number of simplices of cardinality $i$ and $s - 1$ is the dimension of $\Delta$. The $h$-vector of $\Delta$ is the sequence $(h_0(\Delta), \ldots, h_s(\Delta))$ defined by,

$$h_i(\Delta) = \sum_{k=0}^{i} (-1)^{i+k} f_k(\Delta) \binom{s-k}{i-k}.$$  

Equivalently, if we let $f_0(t) = 1$, $f_1(t) = \Delta(1+t)$, then $h_0(t) = 1$, $h_1(t) = \Delta(1+t) - \Delta(1)$, and $h_s(t) = \Delta(1+t) - \sum_{i=0}^{s-1} h_i(t) \Delta(i+1)$. The independence complex $\Delta(M)$ of $M$ is the simplicial complex whose vertices are the non-loop elements of $E$ and whose simplices are the independent subsets of $E$. We let $\Delta(M)$ represent the independence complex of $M$.

In order to define the broken circuit complex for $M$, we first choose a linear order $n$ on the elements of the matroid. Given such an order, a broken circuit is a circuit with its least element removed. The broken circuit complex is the simplicial complex whose simplices are the subsets of $E$ which do not contain a broken circuit. We denote the broken circuit complex of $M$ and $n$ by $\Delta_{BC}(M, n)$. Different orderings may lead to different complexes, see \[1\] Example 7.4.4. Conversely, distinct matroids can have the same broken circuit complex. For instance, let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and let $n$ be the obvious order. Let $M_1$ be the matroid on $E$ whose bases are all triples except $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ and let $M_2$ be the matroid on $E$ whose bases are all triples except $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$. Then $M_1$ and $M_2$ are non-isomorphic matroids but their broken circuit complexes are identical. Both $h_{\Delta(M)}(t)$ and $h_{\Delta_{BC}(M)}(t)$ satisfy similar contraction-deletion formulas.

Proposition 2.1. \[1, 3\]

1. If $e$ is a loop of $M$, then $h_{\Delta(M)}(t) = h_{\Delta(\Delta-e)}(t)$, and $h_{\Delta_{BC}(M)}(t) = 1$.
2. If $e$ is a coloop of $M$, then $h_{\Delta(M)}(t) = h_{\Delta(\Delta-e)}(t)$, and $h_{\Delta_{BC}(M)}(t) = h_{\Delta_{BC}(\Delta-e)}(t)$.
3. If $e$ is neither a loop nor a coloop of $M$, then $h_{\Delta(M)}(t) = h_{\Delta(\Delta-e)}(t)$ and $h_{\Delta_{BC}(M)}(t) = h_{\Delta_{BC}(\Delta-e)}(t)$.
4. If $M = M_1 \oplus M_2$, then $h_{\Delta(M)}(t) = h_{\Delta(M_1)}(t) \cdot h_{\Delta(M_2)}(t)$.
5. If $S$ is a series class of $M$ which is not a circuit, then $h_{\Delta(M)}(t) = h_{\Delta(M/S)}(t) + h_{\Delta(M-S)}(1+t + \cdots + t^{|S|-1})$.

3. Face rings

Let $k$ be a field and let $R = k[x_1, \ldots, x_n]$.
Definition 3.1. The face ring of $\Delta$ is the graded $k$-algebra
$$k[\Delta] = R/I_{\Delta},$$
where $I_{\Delta}$ is the ideal generated by all monomials $x_{i_1} \cdots x_{i_t}$ such that $\{v_{i_1}, \ldots, v_{i_t}\}$ is not a face of $\Delta$.

Let $s - 1$ be the dimension of $\Delta$. Let $\Theta = \{\theta_1, \ldots, \theta_s\}$ be a set of one-forms in $R$. Write each $\theta_i = k_{i1}x_1 + \cdots + k_{is}x_s$ and let $K = (k_{ij})$. To each simplex in $\Delta$ there is a corresponding set of column vectors in $K$. If for every simplex of $\Delta$ the corresponding set of column vectors is independent, then $\Theta$ is a linear set of parameters (l.s.o.p.) for $k(\Delta)$. If $k$ is infinite, then it is always possible to choose $\Theta$ such that every set of $s$ columns of $K$ is independent.

Given a l.s.o.p. $\Theta$ for $k(\Delta)$ let $R(\Delta, \Theta) = k(\Delta)/<\Theta>$. If $\Theta$ is unambiguous, then we just use $R(\Delta)$. Since $\Theta$ is homogeneous $R(\Delta)$ is a graded $k$-algebra.

Theorem 3.2. Let $\Theta$ be a l.s.o.p. for $\Delta(M)$ and let $R(\Delta(M))_i$ be the $i^{th}$ graded component of $R(\Delta(M))$. Then $h_i(\Delta(M)) = \dim_k R(\Delta(M))_i$. Similarly, if $\Theta$ is a l.s.o.p. for $\Delta^{BC}(M)$, then $h_i(\Delta^{BC}(M)) = \dim_k R(\Delta^{BC}(M))_i$.

Given any two integers $i, j > 0$ there is a unique way to write
$$j = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_1}{1}, a_i > a_{i-1} > \cdots > a_1 \geq l \geq 1.$$ 

Given this expansion define,
$$j^{<i>} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_1+1}{l+1}, a_i > a_{i-1} > \cdots > a_1 \geq l \geq 1.$$ 

Theorem 3.3. Let $Q = R/I$, where $I$ is a homogeneous ideal. Let $Q_i$ be the forms of degree $i$ in $Q$ and let $h_i = \dim_k Q_i$. Then $h_{i+1} \leq h_i^{<i>}$. 

Corollary 3.4. For any independence or broken circuit circuit $h_{i+1} \leq h_i^{<i>}$. 

4. The ring $R(\Delta(M))$

In order to study the properties of $h_i(\Delta(M))$ we will look for elements with properties slightly weaker than those provided by Lefschetz elements of the cohomology ring of a compact complex projective manifold.

Definition 4.1. Let $N$ be a (non-negatively) graded $R$-module whose dimension over $k$ is finite. Let $r$ be the last non-zero grade of $N$ and let $\omega$ be a one-form in $R$. Then $\omega$ is a $g$-element for $N$ if for all $i, 0 \leq i \leq r/2$, multiplication by $\omega^{r-2i}$ is an injection from $N_i$ to $N_{r-i}$.

If we replace injection with bijection in the above definition, then we obtain the strong Stanley property in [12].

Let $M$ be a rank $r$ matroid without coloops and $k$ a field of characteristic zero. Let $n = |E|$. Write the elements of $k^{n \times (r+1)}$ in the form $(\Theta, \omega)$, where $\Theta$ consists of $r$ elements in $k^n$ and $\omega$ is also in $k^n$. Identify elements of $k^n$ with the one-forms in $R$ in the canonical way. Let $U$ be the set of all pairs $(\Theta, \omega) \in k^{n \times (r+1)}$ such that $\Theta$ is a l.s.o.p. for $k(\Delta(M))$ and $\omega$ is a $g$-element for $R(\Delta(M), \Theta)$.

Theorem 4.2. Let $M, U$ and $k$ be as above. Then, $U$ is a non-empty Zariski open subset of $k^{n \times (r+1)}$. 

Proof. We first note that \( \Theta \) is a l.s.o.p. for \( k(\Delta) \) if and only if the determinants of the appropriate \( r \times r \) minors of \( K \) are non-zero. Secondly, the multiplication maps \( \omega^{r-2i} \) can be encoded as matrices which are polynomial in the coefficients of \( K \) and \( \omega \). Thus, \( U \) is the intersection of two Zariski open subsets of \( k^{n \times (r+1)} \).

To show that \( U \) is not empty we proceed by induction on \( n \). However, we use a slightly different (but equivalent) induction hypothesis. Let \( C(j) \) be the circuit with \( j \) elements. Let \( P \) be a direct sum of circuits, so we can write \( P = C(j_1) \oplus \cdots \oplus C(j_m) \). The rank of \( M \oplus P \) is \( r' = r + j_1 + \cdots + j_m - m \) and its cardinality is \( n' = n + j_1 + \cdots + j_m \). The induction hypothesis is that given any such \( P \), then \( U = \{ (\Theta, \omega) \in k^{n' \times (r'+1)} : \Theta \) is a l.s.o.p. for \( k(\Delta(M \oplus P)) \) and \( \omega \) is a g-element for \( R(\Delta(M \oplus P), \Theta) \} \) is not empty. If \( M \) consists of a single loop, then \( k(\Delta(M \oplus P)) \simeq k(\Delta(P)) \). As a simplicial complex \( \Delta(P) \) is \( \partial(\Delta^{j-1}) \cdots \partial(\Delta^{j-m-1}) \), where \( \Delta^{j} \) is the \( j \)-simplex. Since this is the boundary of a convex rational polytope we can apply the Hard Lefschetz theorem as in [10] to see that \( U \) is not empty when \( k = \mathbb{Q} \), and hence is not empty for any field of characteristic zero.

Suppose that \( M \) consists of a single coloop. Given \( P \) and \( \Theta \) as in the above paragraph, \( \Theta \cup \{ x_w \} \) is a l.s.o.p. for \( k(\Delta(P \oplus M)) \) and \( \omega \), which can be viewed as an element of \( k[x_1, \ldots, x_n] \), is a \( g \)-element for \( R(\Delta(P \oplus M), \Theta \cup \{ x_w \} \). For the induction step, let \( S \) be a series class of \( M \). Reordering \( M \) if necessary, we assume that \( S = \{ e_1, \ldots, e_s \} \) consists of the first \( s \) elements of \( M \). If \( S \) is a circuit, then \( M = (M - S) \oplus C(s) \). Hence, \( M \oplus P = (M - S) \oplus (C(s) \oplus P) \) and the induction hypothesis applies to \( M - S \). So assume that \( S \) is independent. Let \( x^S = x_1 \cdots x_s \). For \( \Theta \) a l.s.o.p. for \( k(\Delta(M \oplus P)) \) consider the following short exact sequence.

\[
\begin{align*}
0 & \to < x^S > R \\
& \to < x^S > \cap (\Theta + I_{\Delta(M \oplus P)}) \\
& \to R(\Delta(M \oplus P), \Theta) \\
& \to < x^S > \\
& \to 0.
\end{align*}
\]

Since \( S \) is a series class, a subset of \( M - S \) is independent if and only if its union with any proper subset of \( S \) is independent. Hence, the right-hand side is just \( R(\Delta((M - S) \oplus C(s) \oplus P), \Theta) \). Therefore, we can apply the induction hypothesis to \( M - S \) to obtain a non-empty Zariski open subset \( U' \) of \( k^{n \times (r+1)} \) consisting of pairs \( (\Theta', \omega') \) such that \( \omega' \) is a \( g \)-element for \( R(\Delta((M - S) \oplus C(s) \oplus P), \Theta') \).

In order to analyze the left-hand side of (1) choose generators \( \{ \theta_1, \ldots, \theta_s, \ldots, \theta_r \} \) for \( < \Theta > \) so that in the corresponding matrix \( K, k_{ij} = \delta_{ij} \) for \( 1 \leq i \leq s \). Now define an \( R \)-module structure on \( R' = k[x_{s+1}, \ldots, x_{n'}] \) by defining \( (x_i) \cdot f = (x_i - \theta_i) \cdot f \) for \( 1 \leq i \leq s \) and \( f \in R' \). Let \( \phi : R' \to (\leq x^S > R)/(\Theta + I_{\Delta(M \oplus P)}) \) be multiplication by \( x^S \). Since \( S \) is independent, every polynomial in \( (\leq x^S > R)/(\Theta + I_{\Delta(M \oplus P)}) \) is equivalent to a polynomial in \( < x^S > R' \). So, \( \phi \) is surjective. The kernel of \( \phi \) contains \( \Theta' = \{ \theta \in \Theta : \theta = k_{s+1}x_{s+1} + \cdots + k_{n'}x_{n'} \} \). In addition, ker \( \phi \) contains all monomials in \( I_{\Delta((M/S) \oplus P)} \). Since \( \Theta' \) is a l.s.o.p. for \( k(\Delta(M/S)) \), we see that \( \phi \) is a degree \( s \) graded surjective \( R \)-module homomorphism from \( R'/(I_{\Delta((M/S) \oplus P)} + \Theta') \) to the left-hand side of (1). Propositions 2 and 3 show that the \( k \)-dimension of \( R'(\Delta(M/S), \Theta') \) and the l.h.s. of (1) are the same. Hence \( \phi \) is an isomorphism. Therefore, by the induction hypothesis applied to \( M/S \), there is a non-empty Zariski open subset \( U'' \) of \( k^{n \times (r+1)} \) consisting of pairs \( (\Theta'', \omega'') \) such that the multiplication map
Since the first possible problem is in degree 2, the smallest possible rank of
\( \{ x \} \) is an injection for 1 \( \leq i \leq (r' - s)/2 \). Now, \( U' \cap U'' \subseteq U \). Since the intersection of
two non-empty Zariski open subsets of \( k^{n \times (r+1)} \) is not empty, \( U \) is also not empty.

**Theorem 4.3.** Let \( M \) be a rank \( r \) matroid without coloops. Let \( h_i = h_i(\Delta(M)) \).
Then,

1. \( h_0 \leq \cdots \leq h_{\lceil r/2 \rceil} \).
2. \( h_i \leq h_{r-i} \) for all \( i \leq r/2 \).
3. Let \( g_i = h_i - h_{i-1} \). Then, for all \( i < r/2 \), \( g_i < g_i' \).

**Proof.** The first two inequalities follow from the injectivity properties of any \( g \)-element \( \omega \) for \( R(\Delta(M), \Theta) \). Since \( g_i = (R(\Delta(M), \Theta)/< \omega >)_{\lceil i \rceil} \), when \( i < r/2 \), the last inequality follows from Theorem 4.3.

The first two inequalities were obtained by Chari using a PS-ear decomposition of \( \Delta(M) \). See [6] for details on PS-ear decompositions. Hausel and Sturmfels used toric hyperkähler varieties to prove the last inequality for matroids representable over the rationals [5]. The proof of Theorem 4.3 is essentially an algebraic version of a PS-ear decomposition [6, Theorem 2]. Indeed, the proof works without change for any simplicial complex with a PS-ear decomposition.

5. THE RING \( R(\Delta^{BC}(M)) \)

As shown in [6] the cone on any independence complex is a broken circuit complex
(for some other matroid). Since the \( h \)-vector of the cone of a simplicial complex
is the same as the \( h \)-vector of the original complex, the \( h \)-vectors of independence
complexes form a (strict) subset of the the \( h \)-vectors of broken-circuit complexes. It
is natural to ask whether or not Theorem 4.3 holds for \( \Delta^{BC}(M) \). The last non-zero
element of the \( h \)-vector of \( \Delta^{BC}(M) \) is \( r-m \), where \( m \) is the number of components
of \( M \). It is not difficult to modify the proof of Theorem 4.3 to produce injections from
\( R(\Delta^{BC}(M))_0 \) to \( R(\Delta^{BC}(M))_{r-m} \) and from \( R(\Delta^{BC}(M))_1 \) to \( R(\Delta^{BC}(M))_{r-m-1} \).
Since the first possible problem is in degree 2, the smallest possible rank of \( M \) for
which Theorem 4.3 does not hold for \( \Delta^{BC}(M) \) is six.

Let \( G(s) \) be the graph obtained by subdividing each edge of the graph consisting
of \( s \) parallel edges into two edges. Let \( M(s) \) be the cycle matroid of \( G(s) \). The rank
of \( M(s) \) is \( s+1 \) and \( M(s) \) has \( 2s \) elements.

**Proposition 5.1.** Let \( \Theta \) be a l.s.o.p. for \( M(s) \), \( n \) a linear order of the elements
of \( M(s) \) and \( \omega \) a linear form in \( k[x_1, \ldots, x_{2s}] \). Then, multiplication by \( \omega \) has a
non-trivial kernel in \( R(\Delta^{BC}(M(s)))_2 \).

**Proof.** Let \( E_l \) consist of the greatest \( l \) elements of \( M(s) \) with respect to \( n \). Let \( \{e_i, e_j\} \) be the first pair of edges to appear in \( E_l \) as \( l \) goes from 1 to \( s+1 \) such that they come from the subdivision of one of the parallel edges used to construct \( G(s) \). Consider the ideal \( x_i x_j \subseteq R(\Delta^{BC}(M(s), n)) \). Using the same reasoning as in the proof of Theorem 4.2, the choice of \( \{e_i, e_j\} \) implies that \( x_i x_j \) is isomorphic as an \( R \)-module to \( R'(\Delta^{BC}(\Delta(M(s)/\{e_i, e_j\}, n'), \Theta') \), where \( R' \) and \( \Theta' \) are defined as in the proof of Theorem 4.2 and \( n' \) is the order on \( M(s)/\{e_i, e_j\} \) induced from
Now, $M(s)/\{e_i, e_j\}$ is the cycle matroid of the $G(s)$ with the two edges $\{e_i, e_j\}$ contracted. For any such pair and any linear order $\Delta^{BC}(M(s)/\{e_i, e_j\})$ is an $s-2$ dimensional simplex. Hence $<x_ix_j> \simeq k$ and will vanish under any multiplication map. □

Repeated application of Proposition 2.1 shows that $h_i(\Delta^{BC}(M(s))) = \binom{s}{i}$ when $i \neq 1$ and $h_1(\Delta^{BC}(M(s))) = s - 1$. When $s \geq 5$, the $h$-vector of the broken circuit complex of $M(s)$ satisfies the combinatorial conditions of Theorem 4.3 but there is no l.s.o.p. for the face ring such that the quotient ring has $g$-elements. As far as we know, whether or not broken circuit complexes satisfy the combinatorial inequalities of Theorem 4.3 remains an open question.

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