LP-SASAKIAN MANIFOLDS EQUIPPED WITH ZAMKOVOY CONNECTION AND CONHARMONIC CURVATURE TENSOR

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Abstract. The paper concerns with some results on conharmonically flat, quasi-conharmonically flat and \(\phi\)-conharmonically flat LP-Sasakian manifolds with respect to Zamkovoy connection. Also, it contains study of generalized conharmonic \(\phi\)-recurrent LP-Sasakian manifolds with respect to Zamkovoy connection. Moreover, the paper deals with LP-Sasakian manifolds satisfying \(K^* (\xi, U^*) R^* = 0\), where \(K^*\) denotes conharmonic curvature tensor and \(R^*\) denotes Riemannian curvature tensor with respect to Zamkovoy connection, respectively.

Key words and Phrases: LP-Sasakian manifold, Zamkovoy connection, Conharmonic curvature tensor

1. Introduction

In 1989, K. Matsumoto \([13]\) first introduced the notion of Lorentzian para-Sasakian manifolds (briefly, LP-Sasakian manifolds). Also, in 1992, I. Mihai and R. Rosca \([14]\) introduced independently the notion of Lorentzian para-Sasakian manifolds in classical analysis. The generalized recurrent manifolds was introduced by Dubey \([8]\) and it was studied by De and Guha et al. \([6]\). In this context, \(\phi\)-recurrent LP-Sasakian manifold was first studied by A. A. Shaikh, D. G. Prakasha and Helaluddin Ahmad \([15]\). On the other hand, \(\phi\)-conharmonically flat LP-Sasakian manifold was introduced by A. Taleshian \([16]\). Apart from these, the properties of LP-Sasakian manifolds were studied by several authors, namely U. C. De \([7]\), C. Ozgur \([17]\) and many others.

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In 2008, a new non-metric canonical connection on para contact manifold was introduced by S. Zamkovoy [18]. This connection named as Zamkovoy connection was further studied in Sasakian manifolds, LP-Sasakian manifolds and para-Kenmotsu manifolds by several researcher et al. ([3], [1], [2], [10], [11], [12], [5]). Zamkovoy connection $\nabla^*$ for an $n$-dimensional almost contact metric manifold $M$ equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a $1$-form $\eta$ and a Riemannian metric $g$ is given by

$$\nabla^*_XY = \nabla_XY + (\nabla_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi + \eta(X)\phi Y,$$

for all $X, Y \in \chi(M)$, where $\nabla$ is the Levi-Civita connection and $\chi(M)$ is the set of all vector fields on $M$.

In 1957, Y. Ishii [9] first studied the notion of a conharmonic curvature tensor. A rank three tensor $K$, that remains invariant under conharmonic transformation for an $n$-dimensional Riemannian manifold $M$ is given by

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y]$$

$$- \frac{1}{n-2} [g(Y,Z)QX - g(X,Z)QY],$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all vector fields of the manifold $M$ and $R$ denotes the Riemannian curvature tensor of type $(1,3)$, $S$ denotes the Ricci tensor of type $(0,2)$, $Q$ is the Ricci operator.

The conharmonic curvature tensor $(K^*)$ with respect to Zamkovoy connection is given by

$$K^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{n-2} [S^*(Y,Z)X - S^*(X,Z)Y]$$

$$- \frac{1}{n-2} [g(Y,Z)Q^*X - g(X,Z)Q^*Y],$$

for all $X, Y, Z \in \chi(M)$, where $R^*, S^*$ and $Q^*$ are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection, respectively.

**Definition 1.1.** An $n$-dimensional LP-Sasakian manifold $M$ is said to be generalized $\eta$–Einstein manifold if the Ricci tensor of type $(0,2)$ is of the form

$$S(Y,Z) = k_1g(Y,Z) + k_2\eta(Y)\eta(Z) + k_3\omega(Y,Z),$$

for all $Y, Z \in \chi(M)$, where $k_1, k_2$ and $k_3$ are scalars and $\omega$ is a 2–form.

**Definition 1.2.** An $n$-dimensional LP-Sasakian manifold $M$ is said to be conharmonically flat with respect to Zamkovoy connection if $K^*(X,Y)Z = 0$, for all $X, Y, Z \in \chi(M)$.

**Definition 1.3.** An $n$-dimensional LP-Sasakian manifold $M$ is said to be $\xi$–conharmonically flat with respect to Zamkovoy connection if $K(X,Y)\xi = 0$, for all $X, Y, Z \in \chi(M)$. 
**Definition 1.4.** An $n$-dimensional LP-Sasakian manifold $M$ is said to be generalized conharmonic $\phi$-$\text{recurrent}$ with respect to Zamkovoy connection if

$$\phi^2 (\nabla^*_W K^*) (X,Y) Z = A(W) K(X,Y) Z + B(W) [g(Y,Z) X - g(X,Z) Y],$$

for all $X,Y,Z,W \in \chi(M)$, where $A$ and $B$ are 1-forms and $B$ is non vanishing such that $A(W) = g(W,\rho_1), B(W) = g(W,\rho_2)$ and $\rho_1, \rho_2$ are vector fields associated with 1-forms $A$ and $B$, respectively.

This paper is structured as follows:

After introduction, a short description of LP-Sasakian manifold has been given in section (2). In section (3), we have obtained Riemannian curvature tensor $R^*$, Ricci tensor $S^*$, scalar curvature $r^*$ with respect to Zamkovoy connection in LP-Sasakian manifold. Section (4) contains conharmonically flat and $\xi$-conharmonically flat LP-Sasakian manifolds with respect to Zamkovoy connection. In section (5), we have discussed quasi-conharmonically flat LP-Sasakian manifold with respect to Zamkovoy connection. Section (6) contains $\phi$-conharmonically flat LP-Sasakian manifold with respect to $\nabla^*$. Section (7) concerns with a generalized conharmonic $\phi$-$\text{recurrent}$ LP-Sasakian manifold with respect to $\nabla^*$. In section (8), we have discussed an LP-Sasakian manifold satisfying $K^* (\xi, U). R^* = 0$.

2. Preliminaries

An $n$-dimensional differentiable manifold is called an LP-Sasakian manifold if it admits a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Lorentzian metric $g$ which satisfies:

$$\phi^2 Y = Y + \eta(Y) \xi, \eta(\xi) = -1, \eta(\phi X) = 0, \phi \xi = 0,$$  \hspace{1cm} (5)

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$  \hspace{1cm} (6)

$$g(X, \phi Y) = g(\phi X, Y), \eta(Y) = g(Y, \xi),$$  \hspace{1cm} (7)

$$\nabla_X \xi = \phi X, \quad g(X, \xi) = \eta(X),$$  \hspace{1cm} (8)

$$\nabla_X (\phi Y) = g(X,Y) \xi + \eta(Y) X + 2\eta(X) \eta(Y) \xi,$$  \hspace{1cm} (9)

for all $X, Y \in \chi(M)$, where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

Let us introduced a symmetric $(0, 2)$ tensor field $\omega$ such that

$$\omega(X, Y) = g(X, \phi Y).$$  \hspace{1cm} (10)

Also, since the vector field $\eta$ is closed in LP-Sasakian manifold $M$, we have

$$(\nabla_X \eta) Y = \omega(X,Y), \omega(X, \xi) = 0,$$  \hspace{1cm} (11)

for all $X, Y \in \chi(M)$.
In LP-Sasakian manifold the following relations also hold:

\begin{align}
\eta(R(X,Y)Z) &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \\
R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, \\
R(\xi,Y)Z &= g(Y,Z)\xi - \eta(Z)Y, \\
R(\xi,Y)\xi &= \eta(Y)\xi + Y, \\
S(X,\xi) &= (n-1)\eta(X), \\
S(\phi X,\phi Y) &= S(X,Y) + (n-1)\eta(X)\eta(Y),
\end{align}

\[ Q\xi = (n-1)\xi, Q\phi = \phi Q, S(X,Y) = g(QX,Y), S^2(X,Y) = S(QX,Y). \tag{18} \]

**Lemma 2.1.** The relation between Zamkovoy connection and Levi-Civita connection in an LP-Sasakian manifold is given by

\[ \nabla_X^* Y = \nabla_X Y + g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y, \tag{19} \]

where the torsion tensor of Zamkovoy connection is

\[ T^*(X,Y) = 2[\eta(X)\phi Y - \eta(Y)\phi X]. \tag{20} \]

**Proof.** In view of (1) and (11), we have

\[ (\nabla_X^* g)(Y,Z) = -2g(Y,\phi Z)\eta(X). \tag{21} \]

Suppose that the Zamkovoy connection \( \nabla^* \) defined on an \( n \)-dimensional LP-Sasakian manifold \( M \) is connected with the Levi-Civita connection \( \nabla \) by the relation

\[ \nabla_X^* Y = \nabla_X Y + P(X,Y), \tag{22} \]

where \( P(X,Y) \) is a tensor field of type \((1,1)\). Then by definition of torsion tensor, we have

\[ T^*(X,Y) = P(X,Y) - P(Y,X). \tag{23} \]

Zamkovoy connection is a non-metric connection and hence from (22), we get

\begin{align}
g(P(X,Y),Z) + g(P(X,Z),Y) &= 2g(Y,\phi Z)\eta(X), \\
g(P(Y,X),Z) + g(P(Y,Z),X) &= 2g(X,\phi Z)\eta(Y), \\
g(P(Z,X),Y) + g(P(Z,Y),X) &= 2g(X,\phi Y)\eta(Z). \tag{26} \end{align}

In view of (24), (25), (26) and (23), we have

\begin{align}
g(T^*(X,Y),Z) + g(T^*(Z,X),Y) + g(T^*(Z,Y),X) &= g(P(X,Y),Z) - g(P(Y,X),Z) - g(P(X,Z),Y) \\
&- g(P(Y,Z),X) - g(P(Z,X),Y) - g(P(Z,Y),X) \\
&= 2g(P(X,Y),Z) - 2g(Y,\phi Z)\eta(X) \\
&- 2g(X,\phi Z)\eta(Y) + 2g(X,\phi Y)\eta(Z). \tag{27} \end{align}

Setting

\begin{align}
g(T^*(Z,X),Y) &= g(\overline{T}(X,Y),Z), \tag{28} \\
g(T^*(Z,Y),X) &= g(\overline{T}(Y,X),Z), \tag{29} \end{align}
in (27), we have
\[ g(T^*(X,Y),Z) + g(T(X,Y),Z) + g(T(Y,X),Z) = 2g(P(X,Y),Z) - 2g(Y,\phi Z)\eta(X) \\
-2g(X,\phi Z)\eta(Y) + 2g(X,\phi Y)\eta(Z), \tag{30} \]
which implies that
\[ P(X,Y) = \frac{1}{2}[T^*(X,Y) + T(X,Y) + T(Y,X)] \\
+\eta(X)\phi Y + \eta(Y)\phi X - g(X,\phi Y)\xi. \tag{31} \]
In reference to (20), (28) and (29), we have
\[ T(X,Y) = 2g(X,\phi Y)\xi - 2\eta(X)\phi Y, \tag{32} \]
\[ T(Y,X) = 2g(X,\phi Y)\xi - 2\eta(Y)\phi X. \tag{33} \]
Using (20), (32) and (33) in (31), we obtain
\[ P(X,Y) = g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{34} \]
In reference to (22) and (34), we can easily bring out the equation (19).

From the equation (19), it is obvious that
\[ \nabla_X\xi = 2\phi X. \tag{35} \]

**Proposition 2.2.** The Zamkovoy connection on an \( n \)-dimensional LP-Sasakian manifold is a non-metric linear connection with torsion tensor given by equation (20).

3. Some properties of LP-Sasakian manifold with respect to Zamkovoy connection

Let \( R^* \) be the Riemannian curvature tensor with respect to Zamkovoy connection and it be defined as
\[ R^*(X,Y)Z = \nabla_X\nabla_Y^*Z - \nabla_Y\nabla_X^*Z - \nabla^*_{[X,Y]}Z. \tag{36} \]
Using (5), (8), (9) and (19) in (36), we get the Riemannian curvature \( R^* \) with respect to Zamkovoy connection as
\[ R^*(X,Y)Z = R(X,Y)Z + 3g(X,Z)\eta(Y)\xi \\
-3g(Y,Z)\eta(X)\xi + 3g(Y,\phi Z)\phi X - 3g(X,\phi Z)\phi Y \\
-\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X. \tag{37} \]
Consequently, one can easily bring out the followings:
\[ S^*(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) + 3\psi g(Y,\phi Z), \tag{38} \]
\[ S^*(\xi,Z) = S^*(Z,\xi) = 0, \tag{39} \]
\[ Q^*Y = QY + (n-1)\eta(Y)\xi + 3\psi\phi Y, \tag{40} \]
\[ Q^*\xi = 0, \tag{41} \]
\[ r^* = r - n + 1 + 3\psi^2, \tag{42} \]
\[ R^* (X, Y) \xi = 0, \quad (43) \]
\[ R^* (\xi, Y) Z = 4g (\phi Y, \phi Z) \xi, \quad (44) \]
\[ R^* (X, \xi) Z = -4g (\phi X, \phi Z) \xi, \quad (45) \]

for all \( X, Y, Z \in \chi (M) \), where \( \psi = \text{trace} (\phi) \).

**Proposition 3.1.** Let \( M \) be an \( n \)-dimensional LP-Sasakian manifold admitting Zamkovoy connection \( \nabla^* \), then

(i) The curvature tensor \( R^* \) of \( \nabla^* \) is given by (37),

(ii) The Ricci tensor \( S^* \) of \( \nabla^* \) is given by (38),

(iii) The scalar curvature \( r^* \) of \( \nabla^* \) is given by (42),

(iv) The Ricci tensor \( S^* \) of \( \nabla^* \) is symmetric,

(v) \( R^* \) satisfies:

\[ R^* (X,Y) Z + R^* (Y,Z) X + R^* (Z,X) Y = 0. \quad (46) \]

4. **Conharmonically flat and \( \xi \)-conharmonically flat LP-Sasakian manifolds with respect to Zamkovoy connection**

**Theorem 4.1.** If an \( n \)-dimensional LP-Sasakian manifold \( M (n > 2) \) is conharmonically flat with respect to Zamkovoy connection, then the scalar curvature is given by \( r = n - 1 - 3\psi^2 \).

**Proof.** In view of (2) and (3), we have

\[ K^* (X, Y) Z = K (X, Y) Z + 3g (X, Z) \eta (Y) \xi - 3g (Y, Z) \eta (X) \xi \]
\[ + 3g (Y, \phi Z) \phi X - 3g (X, \phi Z) \phi Y - \eta (X) \eta (Y) (Z) Y + \eta (Y) \eta (Z) X \]
\[ - \frac{n - 1}{n - 2} [g (Y, Z) \eta (X) \xi - g (X, Z) \eta (Y) \xi] \]
\[ - \frac{3\psi}{n - 2} [g (Y, Z) \phi X - g (X, Z) \phi Y] \]
\[ - \frac{n - 1}{n - 2} [\eta (Y) X - \eta (X) Y] \eta (Z) \]
\[ - \frac{3\psi}{n - 2} [g (Y, \phi Z) X - 3\psi g (X, \phi Z) Y]. \quad (46) \]

Let us consider an LP-Sasakian manifold \( M \) which is conharmonically flat with respect to Zamkovoy connection, then from (3), we have

\[ R^* (X, Y) Z = \frac{1}{n - 2} [S^* (Y, Z) X - S^* (X, Z) Y] \]
\[ + \frac{1}{n - 2} [g (Y, Z) Q^* X + g (X, Z) Q^* Y]. \quad (47) \]
Taking inner product of (47) with a vector field $V$, we get

\[ g\left( R^* (X, Y) Z, V \right) = \frac{1}{n-2} \left[ S^* (Y, Z) g(X, V) - S^* (X, Z) g(Y, V) \right] + \frac{1}{n-2} \left[ g(Y, Z) S^* (X, V) - g(X, Z) S^* (Y, V) \right]. \quad (48) \]

Taking an orthonormal frame field of $M$ and contracting (48) over $X$ and $V$, we obtain

\[ r = n - 1 - 3\psi^2. \]

This gives the theorem. \[ \square \]

**Corollary 4.2.** If an $n$-dimensional LP-Sasakian manifold is conharmonically flat with respect to Zamkovoy connection, then its scalar curvature is constant, provided that $\text{trace} (\phi) = 0$.

**Theorem 4.3.** An $n$-dimensional LP-Sasakian manifold ($n > 2$) is $\xi-$conharmonically flat with respect to Zamkovoy connection if and only if it is so with respect to Levi-Civita connection, provided that the vector fields are horizontal vector fields.

**Proof.** Setting $Z = \xi$ in (46), we have

\[ \mathcal{K}^* (X, Y) \xi = \mathcal{K} (X, Y) \xi + \frac{1}{n-2} \left[ \eta(Y) X - \eta(X) Y \right] - \frac{3\psi}{n-2} \left[ \eta(Y) \phi X - \eta(X) \phi Y \right] \]

\[ = \mathcal{K} (X, Y) \xi, \quad \text{if } X, Y \text{ are horizontal vector fields on } M. \quad (49) \]

This gives the theorem. \[ \square \]

**Theorem 4.4.** If an $n$-dimensional LP-Sasakian manifold ($n > 2$) is $\xi-$conharmonically flat with respect to Zamkovoy connection, then its scalar curvature with respect to Zamkovoy connection vanishes.

**Proof.** Setting $Z = \xi$ in (3), we have

\[ \mathcal{K}^* (X, Y) \xi = \frac{1}{n-2} \left[ \eta(Y) Q^* X - \eta(X) Q^* Y \right]. \quad (50) \]

If $M$ is $\xi-$conharmonically flat with respect to Zamkovoy connection, then it follows from (50) that

\[ 0 = \eta(Y) Q^* X - \eta(X) Q^* Y. \quad (51) \]

Taking inner product of (51) with a vector field $V$, we obtain

\[ 0 = \eta(Y) S^* (X, V) - \eta(X) S^* (Y, V). \quad (52) \]

Setting $Z = \xi$ in (52)

\[ S^* (X, V) = 0. \quad (53) \]
Taking an orthonormal frame field of $M$ and contracting (53) over $X$ and $V$, we get
\[ r^* = 0. \]

This gives the theorem. \qed

5. Quasi-conharmonically flat LP-Sasakian manifold with respect to Zamkovoy connection.

**Theorem 5.1.** If an $n$-dimensional LP-Sasakian manifold $M$ $(n > 2)$ is quasi-conharmonically flat with respect to Zamkovoy connection, then its scalar curvature with respect to Zamkovoy connection vanishes.

**Proof.** Let us consider an LP-Sasakian manifold $M$ which is quasi-conharmonically flat with respect to Zamkovoy connection, i.e.,
\[ g(\mathcal{K}^*(\phi X, Y) Z, \phi V) = 0, \quad (54) \]
for all $X, Y, Z, V \in \chi(M)$.

Then, in view of (3), we have
\[
g(R^*(\phi X, Y) Z, \phi V) = \frac{1}{n-2} [S^*(Y, Z) g(\phi X, \phi V) - S^*(\phi X, Z) g(Y, \phi V)] + \frac{1}{n-2} [g(Y, Z) S^*(\phi X, \phi V) - g(\phi X, Z) S^*(Y, \phi V)]. \quad (55)\]

Let $\{e_i\} (1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold $M$. Setting $Y = Z = e_i$ in the equation (55) and taking summation over $i (1 \leq i \leq n)$, we get
\[
\sum_{i=1}^{n} g(R^*(\phi X, e_i) e_i, \phi V) = \frac{1}{n-2} \left[ \sum_{i=1}^{n} S^*(e_i, e_i) g(\phi X, \phi V) - \sum_{i=1}^{n} S^*(\phi X, e_i) g(e_i, \phi V) \right] \\
+ \frac{1}{n-2} \left[ \sum_{i=1}^{n} g(e_i, e_i) S^*(\phi X, \phi V) - \sum_{i=1}^{n} g(\phi X, e_i) S^*(e_i, \phi V) \right]. \quad (56)\]

It can be easily seen that
\[
\sum_{i=1}^{n} g(e_i, e_i) = n, \quad (57) \\
\sum_{i=1}^{n} S^*(\phi X, e_i) g(e_i, \phi V) = S^*(\phi X, \phi V), \quad (58) \\
\sum_{i=1}^{n} S^*(e_i, e_i) = r^*. \quad (59)\]
Using (57), (58) and (59) in (56), we get
\[ r^* = 0. \]

This gives the theorem. \(\square\)

6. \(\phi\)--conharmonically flat LP-Sasakian manifold with respect to Zamkovoy connection

**Theorem 6.1.** If an \(n\)--dimensional LP-Sasakian manifold \(M\) \((n > 2)\) is \(\phi\)--conharmonically flat with respect to Zamkovoy connection, then \(M\) is a generalized \(\eta\)--Einstein manifold.

**Proof.** Let us consider an LP-Sasakian manifold \(M\) which is \(\phi\)--conharmonically flat with respect to Zamkovoy connection, i.e.,
\[ g(K^* (\phi X, \phi Y) \phi Z, \phi V) = 0, \quad (60) \]
for all \(X, Y, Z, V \in \chi(M)\).

Then in view of (3), we have
\[
g (R^* (\phi X, \phi Y) \phi Z, \phi V) = \frac{1}{n-2} [S^* (\phi Y, \phi Z) g (\phi X, \phi V) - S^* (\phi X, \phi Z) g (\phi Y, \phi V)] \\
+ \frac{1}{n-2} [g (\phi Y, \phi Z) S^* (\phi X, \phi V) - g (\phi X, \phi Z) S^* (\phi Y, \phi V)]. \quad (61)\]

Let \(\{e_i, \xi\} \ (1 \leq i \leq n-1)\) be a local orthonormal basis of the tangent space at any point of the manifold \(M\). Using the fact that \(\{\phi e_i, \xi\} \ (1 \leq i \leq n-1)\) is also a local orthonormal basis of the tangent space and setting \(Y = Z = e_i\) and taking summation over \(i(1 \leq i \leq n-1)\) it follows from (61) that
\[
\sum_{i=1}^{n-1} R^* (\phi X, \phi e_i, \phi e_i, \phi V) \\
= \frac{1}{n-2} \left[ \sum_{i=1}^{n-1} S^* (\phi e_i, \phi e_i) g (\phi X, \phi V) - \sum_{i=1}^{n-1} S^* (\phi X, \phi e_i) g (\phi e_i, \phi V) \right] \\
+ \frac{1}{n-2} \left[ \sum_{i=1}^{n-1} g (\phi e_i, \phi e_i) S^* (\phi X, \phi V) - \sum_{i=1}^{n-1} g (\phi X, \phi e_i) S^* (\phi e_i, \phi V) \right]. \quad (62)\]
It can be easily seen that
\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \]  
(63)
\[ \sum_{i=1}^{n-1} S^* (\phi X, \phi e_i) g(\phi e_i, \phi V) = S^* (\phi X, \phi V), \]  
(64)
\[ \sum_{i=1}^{n-1} S^* (\phi e_i, \phi e_i) = r^*. \]  
(65)

Using (63), (64) and (65) in (62), we have
\[ S(X, V) = (r - n + 1 + 3\psi^2) g(X, V) \]
\[ + (r - 2n + 2 + 3\psi^2) \eta(X) \eta(V) - 3\psi \omega(X, V), \]  
(66)
where \( \omega(X, V) = g(X, \phi V) \) and \( \psi = \text{trace}(\phi). \)

Therefore \( M \) is a generalized \( \eta \)-Einstein manifold. \( \square \)

7. Generalized conharmonic \( \phi \)-recurrent LP-Sasakian manifold with respect to Zamkovoy connection

**Theorem 7.1.** If an \( n \)-dimensional LP-Sasakian manifold \( M (n > 2) \) is generalized conharmonic \( \phi \)-recurrent with respect to Zamkovoy connection, then 1-forms \( A \) and \( B \) are related as
\[ B(W) = \left[ \frac{r - n + 1 + 3\psi^2}{n(n-1)} \right] A(W), \]  
where \( W \) is an arbitrary vector field on \( M \) and \( \psi = \text{trace}(\phi). \)

**Proof.** Let \( M \) be a generalized conharmonic \( \phi \)-recurrent LP-Sasakian manifold with respect to Zamkovoy connection, then
\[ \phi^2 (\nabla^*_W K^*) (X, Y) Z \]
\[ = A(W) K^* (X, Y) Z + B(W) [g(Y, Z) X - g(X, Z) Y], \]  
(67)
where the 1-forms are given by \( A(W) = g(W, \rho_1), B(W) = g(W, \rho_2), B(W) \neq 0 \) and \( \rho_1, \rho_2 \) are vector fields associated with 1-forms \( A \) and \( B \), respectively.

Using (5) in (67), we have
\[ (\nabla^*_W K^*) (X, Y) Z \]
\[ = -\eta((\nabla^*_W K^*) (X, Y) Z) \xi A(W) K^* (X, Y) Z \]
\[ + B(W) [g(Y, Z) X - g(X, Z) Y]. \]  
(68)
The inner product of the equation (68) with vector field \( V \) gives
\[ g((\nabla^*_W K^*) (X, Y) Z, V) \]
\[ = -\eta((\nabla^*_W K^*) (X, Y) Z) \eta(V) + A(W) g(K^* (X, Y) Z, V) \]
\[ + B(W) [g(Y, Z) g(X, V) - g(X, Z) g(Y, V)]. \]  
(69)
In view of (3), it is easily seen that

\[
\begin{align*}
g ((\nabla^W K^*) (X, Y) Z, V) \\
= g ((\nabla^W R^*) (X, Y) Z, V) \\
- \frac{1}{n-2} \left[ (\nabla^* S^*) (Y, Z) g (X, V) - (\nabla^* S^*) (X, Z) g (Y, V) \right] \\
- \frac{1}{n-2} \left[ g (Y, Z) (\nabla^* S^*) (X, V) - g (X, Z) (\nabla^* S^*) (Y, V) \right],
\end{align*}
\]

(70)

\[
\begin{align*}
\eta ((\nabla^W K^*) (X, Y) Z) \\
= g ((\nabla^W R^*) (X, Y) Z, \xi) \\
- \frac{1}{n-2} \left[ (\nabla^* S^*) (Y, Z) \eta (X) - (\nabla^* S^*) (X, Z) \eta (Y) \right],
\end{align*}
\]

(71)

\[
\begin{align*}
g (K^* (X, Y) Z, V) \\
= g (R^* (X, Y) Z, V) \\
- \frac{1}{n-2} \left[ S^* (Y, Z) g (X, V) - S^* (X, Z) g (Y, V) \right] \\
- \frac{1}{n-2} \left[ g (Y, Z) S^* (X, V) - g (X, Z) S^* (Y, V) \right].
\end{align*}
\]

(72)

Using (70), (71) and (72) in (69), we get

\[
\begin{align*}
g ((\nabla^W R^*) (X, Y) Z, V) \\
= \frac{1}{n-2} \left[ (\nabla^* S^*) (Y, Z) g (X, V) - (\nabla^* S^*) (X, Z) g (Y, V) \right] \\
+ \frac{1}{n-2} \left[ g (Y, Z) (\nabla^* S^*) (X, V) - g (X, Z) (\nabla^* S^*) (Y, V) \right] \\
+ \frac{1}{n-2} \left[ (\nabla^* S^*) (Y, Z) \eta (X) - (\nabla^* S^*) (X, Z) \eta (Y) \right] \eta (V) \\
+ g (R^* (X, Y) Z, V) A (W) - g ((\nabla^W R^*) (X, Y) Z, \xi) \eta (V) \\
- \frac{1}{n-2} \left[ S^* (Y, Z) g (X, V) - S^* (X, Z) g (Y, V) \right] A (W) \\
- \frac{1}{n-2} \left[ g (Y, Z) S^* (X, V) - g (X, Z) S^* (Y, V) \right] A (W) \\
+ [g (Y, Z) g (X, V) - g (X, Z) g (Y, V)] B (W).
\end{align*}
\]

(73)
Taking an orthonormal frame field of $M$ and contracting (73) over $Y$ and $Z$, we get
\[
\left(\nabla^*_W S^*\right)(X, V) = 1 + \frac{1}{n-2} \left[\nabla^*_W r^* g(X, V) - g(\nabla^*_W S^*)(X, \xi) \eta(V)\right]
\]
\[
+ \frac{1}{n-2} \left[n \left(\nabla^*_W S^*\right)(X, V) - \left(\nabla^*_W S^*\right)(X, \xi) \eta(V)\right]
\]
\[
+ \frac{1}{n-2} \left[S^*(X, V) A(W) - \frac{1}{n-2} \left[r^* g(X, V) - S^*(X, V)\right] A(W)\right]
\]
\[
+ \frac{n-1}{n-2} \left[S^*(X, V) A(W) + (n-1) g(X, V) B(W)\right].
\]

(74)

Setting $V = \xi$ in (74),
\[
B(W) = \left[\frac{r - n + 1 + 3\psi^2}{(n-2)(n-1)}\right] A(W).
\]

(75)

This gives the theorem.

\[\square\]

8. LP-Sasakian manifold satisfying $K^*(\xi, U).R^* = 0$

**Theorem 8.1.** If in an $n$-dimensional $(n > 2)$ LP-Sasakian manifold $M$, the condition $K^*(\xi, U).R^* = 0$ holds, then the equation
\[
S^2(Y, U) + 9\psi^2 g(Y, U) + \left[(n-1)^2 + 9\psi^2\right] g(Y) \eta(U) + 6\psi S(Y, \phi U) = 0,
\]
is satisfied on $M$, where $Y, U \in \chi(M)$ and $\psi = \text{trace}(\phi)$.

**Proof.** Let us consider an LP-Sasakian manifold $M$ satisfying the condition
\[
(K^*(\xi, U).R^*)(X, Y) Z = 0.
\]

(76)

Then, we have
\[
0 = K^*(\xi, U) R^*(X, Y) Z - R^*(K^*(\xi, U) X, Y) Z
\]
\[
- R^*(X, K^*(\xi, U) Y) Z - R^*(X, Y) K^*(\xi, U) Z.
\]

(77)

Replacing $Z$ by $\xi$ in (77), we get
\[
0 = K^*(\xi, U) R^*(X, Y) \xi - R^*(K^*(\xi, U) X, Y) \xi
\]
\[
- R^*(X, K^*(\xi, U) Y) \xi - R^*(X, Y) K^*(\xi, U) \xi.
\]

(78)

In view of (37), (40), (3) and (78), we have
\[
0 = R^*(X, Y) K^*(\xi, U) \xi
\]
\[
= R^*(X, Y) Q^* U
\]
\[
= R^*(X, Y) Q U + 3\psi R^*(X, Y) \phi U.
\]

(79)

The inner product of the equation (79) with vector field $V$ gives
\[
0 = g(R^*(X, Y) Q U, V) + 3\psi g(R^*(X, Y) \phi U, V).
\]

(80)
Let \( \{e_i\} (1 \leq i \leq n) \) be an orthonormal basis of the tangent space at any point of the manifold \( M \). Setting \( X = V = e_i \) and taking summation over \( i (1 \leq i \leq n) \) and using (18) in (80), we get

\[
0 = S^2(Y, U) + 9\psi^2 g(Y, U) + \left[ (n - 1)^2 + 9\psi^2 \right] \eta(Y) \eta(U) + 6\psi S(Y, \phi U).
\]

This gives the theorem. \( \square \)

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