This work presents a selective review of results concerning the mathematical interface between the classical and quantum aspects encountered in problems such as the nuclear mean-field dynamics or quantum Brownian motion. It is shown that the main difference between classical and quantum behaviour arises from the coherence properties of the phase-space distributions known as "action waves" and Wigner functions.
1 Introduction

In classical statistical mechanics, the entropy of a many-body system is defined with respect to a partition of the $2^n$-dimensional one-particle phase-space in elementary cells, but the size of these cells is not specified. The occurrence of $h = 6.626 \times 10^{-34}$ J·s in the Planck distribution for thermal radiation was considered as evidence for such a granular structure, with cells ("quantum states") of volume $h^n$. The old quantum mechanics maintained this view, as for integrable systems a cell structure was introduced by the set of stationary orbits (invariant tori) selected by the Bohr-Sommerfeld quantization conditions. In quantum mechanics however, the physical states of microparticles are described by rays in an abstract Hilbert space, and its main elements (wave-functions and operators) are expressed in terms of only $n$ coordinates. This formalism, though quite successful in atomic physics, is not complete, because it requires additional rules for "quantization" (e.g. algebraic (Dirac), geometric [1,2,3,4]) and interpretation (e.g. probabilistic interpretation of the scalar product). These rules provide an interface between the physical observables defined at classical level and the abstract Hilbert space in some important situations of interest (e.g. the measurement process), but are not enough general to join classical and quantum mechanics into a single theory. One of the difficulties in formulating such a theory is that while thermal fluctuations in classical systems can be explained by the action of random forces, the origin of the quantum fluctuations is unknown. The conceptual gap between classical and quantum mechanics is illustrated for instance by the early (1927) Bohr-Einstein debate on the intrinsic statistical character of quantum mechanics, the paradox of the "Schrödinger’s cat experiment", or the Zeno paradox at the continuous measurement process [5]. There was also a constant effort to bridge this gap by non-standard theories such as the thermodynamics of the isolated particle of L. de Broglie, the Bohmian interpretation of quantum mechanics, or the hypothesis of spontaneous wave function collapse [6].

Considering quantum mechanics as fundamental, in the case of a mixed system containing classical and quantum components one can start with a quantum description of the whole system, and then perform partial tracing over the variables known as classical. Along this line, the classical dynamics of a nuclear collective model can be derived as constrained quantum dynamics if in the quantum time-dependent variational principle (TDVP) for the Schrödinger Lagrangian $L_\psi = \langle \psi | i \hbar \partial_t - H | \psi \rangle$, $\psi$ belongs to a suitable manifold of trial wave functions (e.g. coherent states) [7,8]. A possible mechanism to generate such a phase-space as a symplectic submanifold of the quantum Hilbert space is the spontaneous symmetry breaking [9].

Classical degrees of freedom in nuclear, atomic or molecular systems can be introduced using the Born-Oppenheimer approximation. However, attempts to formulate a theory of genuine mixed classical-quantum dynamics appear in 1994, in the form of coupled Hamilton-Schrödinger [10] or Hamilton-Heisenberg [11] equations, derived from a mixed TDVP. The mixed Lagrangian contains a classical part $L_{cl}(x, \dot{x})$, the quantum part $L_\psi$, and an interaction term $L_{int}(x, \psi)$ depending both on the coordinates $x(t)$ of the classical component and the quantum wave function $\psi(q,t)$.
In the case of a quantum particle coupled to a thermal bath of classical harmonic oscillators by a bilinear interaction the Hamilton-Schrödinger equations can be reduced to a non-linear Schrödinger-Langevin equation, containing noise and non-linear friction terms (due to classical backreaction) \cite{10}. Further applications present two variants of this approach:

- for a many-body quantum system, described within some mean-field approximation, $\psi$ is constrained to a symplectic submanifold of the quantum Hilbert space. The example of a superfluid quantum many-fermion system in a thermal environment was considered in \cite{12, 13}.

- for the study of decoherence $L_{\psi}$ can be expressed in terms of the density matrix, such that instead of the Schrödinger-Langevin equation one obtains a quantum Liouville-Langevin equation \cite{14}. Applications to atomic transition rates in thermal radiation field and decoherence time for a two-level system with Ohmic dissipation are presented in \cite{14}. Numerical integration shows that dissipation produced by the non-linear friction term alone (at zero temperature) resembles the spontaneous decay obtained when the classical environment is quantized. The realistic situation of dissipative atomic tunneling in an asymmetric quartic potential at finite temperature is presented in \cite{15}. Analytically it was shown that if the nonlinear friction term is neglected, then by taking the ensemble average, the results are equivalent to the integration of a quantum Fokker-Planck equation for the density matrix \cite{15}.

The extensive literature on these subjects also includes a detailed analytical study of the environmental decoherence during macroscopic quantum tunneling in a cubic potential using a quantum Kramers equation for the reduced Wigner function of the tunneling particle \cite{16}, or a variational principle describing a classical statistical ensemble on the configuration space interacting with a quantum system, applied to couple quantum matter fields and classical metric \cite{17}.

The variants \cite{10, 12, 14} are summarized in \cite{18}, using for the Lagrangian $L_{tot}$ a more compact form, with the term $L_{\psi}$ alone, but the trial wave functions $\psi$ expressed as a product between quantum, quasiclassical (e.g. mean-field) and classical (action phase-factor) components. The quantum transport equation derived in \cite{15} is also improved by the non-linear friction term to get a Fokker-Planck equation, and integrated numerically for the two-level system. The results show the advantage of the non-linear Liouville-Langevin equation, because the presumed non-linear friction term in the quantum Fokker-Planck equation does not ensure thermalization.

Attempting to find a random force which could simulate the quantum fluctuations, I have arrived at the unexpected result that there is no such force, but instead that a certain "granularity" is required \cite{19}, like in the old quantum mechanics. Thus, while the random force destroys the classical coherence expressed by the Hamilton-Jacobi equation, discretization at a certain scale may induce the "quantum" type of coherence.

The next section reminds the framework of statistical mechanics in which the "classical" coherent distributions are defined, Section 3 outlines the transition to Wigner functions, while the Fokker-Planck equation is discussed in Section 4.
2 The Liouville equation

Let \((M_\mu, \omega_\mu)\) be the phase space of a classical elementary system \(\mu\) (molecule) with \(n\) degrees of freedom \([20]\), and \((M_\Gamma, \omega_\Gamma)\) the \(2nN\) - dimensional phase space of the ensemble \(\Gamma\) (gas) consisting of \(N\) identical elementary subsystems,

\[
M_\Gamma = M_\mu^1 \times M_\mu^2 \times ... M_\mu^N, \quad \omega_\Gamma = \sum_{k=1}^{N} \omega_\mu^k .
\] (1)

In particular, the state of a system composed of \(N\) identical point-like particles is described on the \(3N\) dimensional manifold \(M_\Gamma \equiv T^* \mathbb{R}^{3N}\) by a representative ("phase") point \(m\) of coordinates \((\tilde{q}, \tilde{p})\). To obtain a statistical description of the ensemble each manifold \(M_\mu\) is divided in \(K\) infinitesimal cells \(\{b_j : j = 1, K\}\),

\[
M_\mu = \bigcup_{j=1}^{K} b_j , \quad b_i \cap b_j = \emptyset ,
\] (2)
of volume

\[
\delta \Omega^j_\mu = \int_{b_j} \Omega_\mu , \quad \Omega_\mu = \omega^\mu_n .
\] (3)

Therefore we also obtain a partition of the manifold \(M_\Gamma\) in \(nB = K\) cells \(B_j\) of volume \(\delta \Omega^j_\Gamma\), \(j = 1, n_B\). Denoting by \(w_j\) the probability to find the representative point \(m \in M_\Gamma\) at the time \(t\) in the cell \(B_j\), the ratio \(P_j = w_j / \delta \Omega^j_\Gamma\) defines the distribution function of the probability density \(P\), normalized by

\[
\int_{M_\Gamma} \Omega_\Gamma P = 1 , \quad \Omega_\Gamma = \omega^{Nn}_\Gamma \equiv d\tilde{q}d\tilde{p} .
\]

It is important to remark that to address the issues of continuity and unicity of \(P\) it might be necessary to consider instead of a partition \([2]\) an indexed system of open sets \(\{U_i, i \in I\}\) covering \(M_\mu\) and a system of \(q\)-cochains \([21]\), \(q = 0, 1\), associating to each set of \(q + 1\) indices \(i_0, ..., i_q\) from \(I\) a function \(P_q(i_0, ..., i_q) \in \mathbb{R}\) on \(U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_q}\).

As the hamiltonian flow \(F_t\) on \(M_\Gamma\) preserves the volume element \(\Omega_\Gamma\), the probability density behaves as a perfect fluid described by the continuity (Liouville) equation

\[
\partial_t P + L_{\tilde{H}} P = 0 ,
\] (4)

where \(L_{\tilde{H}} P \equiv -\{\tilde{H}, P\}\) is the Lie derivative defined by the Poisson bracket and \(\tilde{H}\) is the total Hamiltonian, including interaction terms.

The dimensionality of \(P\) depends on the dimension of \(M_\Gamma\), \(2nN\). Because in general \(M_\Gamma\) is not a metric space it is convenient to introduce a fundamental unit \(h\) for \(\omega\), such that \(h^{Nn}\) is the fundamental unit for \(\Omega_\Gamma\). The ratio \(\gamma_j = \delta \Omega_j^\Gamma / h^{Nn}\) is the weight of the cell \(B_j^\Gamma\), while \(P = h^{Nn} \tilde{P}\) is dimensionless, normalized by

\[
\int_{M_\Gamma} \frac{\Omega_\Gamma}{h^{Nn}} \tilde{P} = 1 .
\] (5)
The expectation value of a many-body observable \( \tilde{A} \in \mathcal{F}(M_\Gamma) \) (smooth function on \( M_\Gamma \)), defined by

\[
\langle \tilde{A} \rangle := \int_{M_\Gamma} \frac{\Omega_\Gamma}{h^{Nn}} \tilde{P} \tilde{A}
\]
evolves in time according to

\[
\frac{d \langle \tilde{A} \rangle}{dt} = \langle \{ \tilde{A}, \tilde{H} \} \rangle .
\]

The expectation value of \(-k_B \ln \tilde{P}\), where \( k_B \) is the Boltzmann constant, defines the entropy

\[
S = -k_B \int_{M_\Gamma} \frac{\Omega_\Gamma}{h^{Nn}} \tilde{P} \ln \tilde{P}.
\]

The one-particle probability density \( \rho \) (or \( \bar{\rho} = h^n \rho \)) on the phase-space \( M_\mu \) is related to the density \( P \) on \( M_\Gamma \) by the projection given by integration over \( N-1 \) manifolds \( M_\mu \),

\[
\rho(q, p) = \int d^3q_2...d^3q_N d^3p_2...d^3p_N P(q, q_2, ..., q_N, p, p_2, ..., p_N)
\]

This is well defined because the particles are identical, and although the permutations of coordinate indices \( 1, 2, ..., N \rightarrow \{i_1, i_2, ..., i_N\} \) yield different phase points, \( P \) remains invariant. For instance, if \( P \) is a symmetric functional

\[
P(\tilde{q}, \tilde{p}) = \frac{1}{N!} \sum_{\{i_1, ..., i_N\}} \rho_1(q_{i_1}, p_{i_1}) \rho_2(q_{i_2}, p_{i_2})... \rho_N(q_{i_N}, p_{i_N})
\]
of \( L \leq N \) distinct distribution functions \( \rho_i, i = 1, L \) on \( M_\mu \), then

\[
\rho(q, p) = \frac{1}{N} \sum_{i=1}^{N} \bar{N}_i \rho_i(q, p)
\]
where \( \bar{N}_i \) is the number of particles assigned to \( \rho_i \).

The ensemble of identical particles can also be described using the Boltzmann representation of “occupation numbers” in the \( \mu \)-space. Thus, on \( M_\mu = T^*\mathbb{R}^3 \), each particle is represented by a point of coordinates \((r, p)_{i_i}, i = 1, N \). Let \( [2] \) be a partition of \( M_\mu \) in \( K \) elementary cells, and \( N_j \) the average number of such representative points localized in the cell \( b_j \). The ratio \( f_j = N_j/\delta \Omega_\mu \) defines the distribution function of the particle density \( \rho = N \rho \) on \( M_\mu \), normalized by

\[
\int_{M_\mu} \Omega_\mu f = N
\]

If there are no interactions between components, \( f \) satisfies the one-particle Liouville equation

\[
\partial_t f + L_H f = 0
\]
where \( H \in \mathcal{F}(M_\mu) \) is the one-particle Hamiltonian and \( L_H f \equiv -\{ H, f \} \). If \( M_\mu = T^*\mathbb{R}^3 \), then

\[
L_H = (\nabla_p H) \cdot \nabla - (\nabla H) \cdot \nabla_p ,
\]

where \( \nabla_p \equiv \partial_p, \nabla \equiv \partial_q \), and for

\[
H(p, q) = \frac{p^2}{2m} + V(q)
\]

(13)

(11) becomes

\[
\frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla f - \nabla V \cdot \nabla_p f = 0
\]

(14)

2.1 Classical coherent states

To solve (14) it is convenient to use the Fourier transform \( \tilde{f}(q, k, t) \) in momentum,

\[
\tilde{f}(q, k, t) \equiv \int d^3p \; e^{ik \cdot p} f(q, p, t)
\]

(15)

which is a density on the configuration space \( \mathbb{R}^3 \) related to the particle (n) or current (j) densities by

\[
n(q, t) \equiv \int d^3p \; f(q, p, t) = \tilde{f}(q, 0, t)
\]

(16)

\[
\dot{j}(q, t) \equiv \int d^3p \; \frac{p}{m} f(q, p, t) = -\frac{i}{m} \nabla_k \tilde{f}(q, 0, t)
\]

(17)

Thus, if \( f(q, p, t) \) is a solution of (14), then its Fourier transform \( \tilde{f}(q, k, t) \) will satisfy

\[
\frac{\partial \tilde{f}}{\partial t} - \frac{i}{m} \nabla_k \cdot \nabla \tilde{f} + ik \cdot (\nabla V) \tilde{f} = 0
\]

(18)

An important class of solutions for the one-particle Liouville equation (14) is represented by the "action distributions"

\[
f_0(q, p, t) = n(q, t) \delta(p - \nabla S(q, t))
\]

(19)

These are coherent functionals in the sense that remain all the time a product between \( n(q, t) \) and \( \delta(p - \nabla S(q, t)) \). The two real functions of coordinates and time, \( n(q, t) \) and \( S(q, t) \) are related by the hamiltonian flow because for

\[
\tilde{f}_0(q, k, t) = n(q, t)e^{ik \cdot \nabla S(q, t)}
\]

(20)

(18) reduces to the system of equations

\[
\partial_t n = -\nabla j
\]

(21)

\[
n\nabla[\partial_t S + \frac{(\nabla S)^2}{2m} + V] = 0
\]

(22)
where \( \mathbf{j} \equiv n \nabla S/m \) is the current density \([17]\). Thus, presuming the existence of a “momentum potential” \( S(q,t) \) we get both the continuity and Hamilton-Jacobi equations.

In general the solutions of \([22]\) are multi-valued, and \( f_0 \) is a sum

\[
f_0 = \sum_i n_i \delta(p - \nabla S_i)
\]

over different branches. The subspace of solutions \( f_0 \equiv n[S](q,t) \) corresponding to the same function \( S \) satisfy the superposition principle,

\[
(n_1 + n_2)[S] = n_1[S] + n_2[S]
\]

and can be called “action waves”.

If \( n(q) \) is a solution of the system \([21], [22]\), then \(-n(q)\) is also a solution. To obtain only positive solutions it is convenient to search \( n \) of the form

\[
n = |\psi|^2
\]

where \( \psi \) can be a complex function. When

\[
\hat{\psi} = \int d^3q \ (d\psi^* \wedge d\psi)
\]

is the symplectic form induced by the complex structure of the Hilbert space \( H = \{ \psi \in L^2(\mathbb{R}^3) \} \) generated by \( \psi \). The constant \( \sigma \) and \( h \) from \([5]\) have both dimensionality of action, and in the quantum theory \( \sigma = h/2\pi = \hbar \), where \( \hbar \) is the Planck constant.

It is important to remark that while the singularity of \( f_0 \) is necessary for coherence, it yields infinite entropy, and therefore is not realistic. To obtain finite entropy we can replace for instance the delta function \( \delta(p - \nabla S) \) by a Gaussian

\[
g[S](p) = \frac{1}{(\sqrt{\pi b})^3} e^{-(p - \nabla S)^2/b}
\]

where \( b \) is a finite constant, but in general \( ng[S] \) is not coherent. However, in the particular case of the harmonic oscillator potential \( V(q) = \frac{1}{2} m \omega^2 q^2 \) we can find coherent solutions of the form \( \rho_G(q,p) = g_X(q)g_Y(p) \), where

\[
g_X(q) = \frac{1}{\sqrt{\pi a}} e^{-(q - X)^2/a}, \quad g_Y(p) = \frac{1}{\sqrt{\pi b}} e^{-(p - Y)^2/b}
\]

if \( b/a = m^2 \omega^2 \) and \( X, Y \) are time-dependent vectors which satisfy the classical equations of motion, \( \dot{X} = Y/m, \dot{Y} = -m\omega^2 X \). These are solutions of constant entropy, which can also be written in the form

\[
\rho_G(q,p) = \frac{1}{2\pi} \int dk \ e^{-ikp} \ (\psi_G(q + \frac{\sigma k}{2})\psi_G^*(q - \frac{\sigma k}{2}) \), \quad \sigma = \sqrt{ab}
\]

where \( \psi_G(q) = \sqrt{g_X(q)} e^{iY/\sigma} \). If \( \sigma = h \) then \( \psi_G \) are (up to a phase factor) the non-stationary solutions of the TDSE known as Glauber coherent states. An application of such states to describe “preformed” alpha particles in heavy nuclei, with relevance for the Geiger-Nuttall law \([22]\), was presented in \([23]\). At astronomic scale, we may presume that for a suitable constant \( \sigma \) similar considerations might explain the “preformation” of planets along the orbits described by the Titius-Bode law.
3 Discretization, coherence and quantization

The partial derivative \( \mathbf{k} \cdot \nabla S(\mathbf{q}, t) \) in (20) is the limit of

\[
\frac{k}{\ell} [S(\mathbf{q} + \frac{\ell}{2k} \mathbf{k}, t) - S(\mathbf{q} - \frac{\ell}{2k} \mathbf{k}, t)] ,
\]

with \( k = |\mathbf{k}| \neq 0 \), when \( \ell \to 0 \). However, if \( k \to 0 \) too, a more detailed discussion might be necessary.

From the early days of differential calculus, it was presumed that in (25) \( \ell \) can be arbitrarily small, but finite. It seems though that for microparticles there is a physical limit \( \ell_0 > 0 \), and \( \ell \to \ell_0 > 0 \). The existence of an elementary length \( \ell_0 > 0 \), proposed by W. Heisenberg (\( \ell_0 \sim 10^{-15} \) m) and M. Planck (\( \ell_0 \sim 10^{-34} \) m), was developed in the framework of general relativity theory, by the model of crystalline lattice of the physical space [24], or in string theory [25]. Independently of these considerations, the assumption of a finite limit \( \ell_0 \), expected for each massive particle near its Compton wavelength (\( \ell_0 \sim 1/m_0 \)), was used in [19, 26, 27] to justify the transition from a classical coherent distribution (20) to a “quantum” distribution of the form (24). Let us presume that in the Fourier transform (42) from \( \mathbf{p} \) to \( \mathbf{k} \), we approximate

\[
\partial_q S \approx \frac{S(\mathbf{q} + \frac{\ell}{2}, t) - S(\mathbf{q} - \frac{\ell}{2}, t)}{\ell} , \quad n(q) \to \sqrt{n(q + \frac{\ell}{2}, t)n(q - \frac{\ell}{2}, t)} ,
\]

where \( \ell = \sigma k \).

Thus, the space derivative \( \partial_q S(\mathbf{q}, t) \) is replaced by the finite differences expression with respect to a minimum length \( \ell \) depending linearly on \( k \), while the integration on \( k \) is limited by the size of the domain in which \( n(\mathbf{q}, t) \neq 0 \). In terms of the new parameter \( \sigma = \ell/k \), if \( k \neq 0 \) then

\[
\tilde{f}_0(\mathbf{q}, \mathbf{k}, t) = \lim_{\sigma \to 0} \tilde{f}_\psi(\mathbf{q}, \mathbf{k}, t)
\]

where

\[
\tilde{f}_\psi(\mathbf{q}, \mathbf{k}, t) \equiv \psi^*(\mathbf{q} - \frac{\sigma \mathbf{k}}{2}, t)\psi(\mathbf{q} + \frac{\sigma \mathbf{k}}{2}, t)
\]

with \( \psi = \sqrt{\pi} \exp(iS/\sigma) \). In the limit \( k \to 0 \)

\[
S(\mathbf{q} \pm \sigma_0 \mathbf{k}, t) = S(\mathbf{q}, t) \pm \sigma_0 \mathbf{k} \cdot \partial_q S(\mathbf{q}, t) + \frac{\sigma_0^2}{8} (\mathbf{k} \cdot \partial_q)^2 S(\mathbf{q}, t) \pm ...
\]

and if the terms containing \( (\sigma_0 k)^m \), \( m \geq 3 \), are neglected then

\[
\mathbf{k} \cdot \partial_q S(\mathbf{q}, t) = \frac{1}{\sigma_0} [S(\mathbf{q} + \frac{\sigma_0}{2} \mathbf{k}, t) - S(\mathbf{q} - \frac{\sigma_0}{2} \mathbf{k}, t)]
\]

for any dimensional constant \( \sigma_0 > 0 \). Therefore, within a suitable domain for \( \mathbf{k} \), we may consider \( \sigma \) from (28), (29) as a finite constant, related eventually to the size of the cells \( b_j \) used in the partition (2). If \( \sigma = h \) then \( f_\psi \) obtained inverting (15),

\[
f_\psi(\mathbf{q}, \mathbf{p}, t) = \frac{1}{(2\pi)^3} \int d^3k \, e^{-i\mathbf{k} \cdot \mathbf{p}} \, \tilde{f}_\psi(\mathbf{q}, \mathbf{k}, t)
\]

8
is the Wigner transform \[28, 29\] of the complex "wave function" \( \psi = \sqrt{n} \exp(iS/\sigma) \). Some properties of this functional are summarized below:

- \( f_\psi \) is not positive definite, and in general it cannot represent particle density. However it is integrable, and the normalization condition (5) takes the form

\[
\int d^3q d^3p \ f_\psi(q, p, t) = \int d^3q \ |\psi(q, t)|^2 \equiv \langle \psi|\psi \rangle = N .
\]  

(31)

This condition is due to the fact that as long as the particles are distributed over the cells \(2\), \( n = |\psi|^2 \) is the density over the configuration space \(3\).

- The "overlap" integral between two distributions \( f_{\psi_1}, f_{\psi_2} \), over the phase-space is \[26\]

\[
(f_{\psi_1}, f_{\psi_2}) \equiv \int d^3q d^3p \ f_{\psi_1} f_{\psi_2} = \frac{N}{\hbar^3} < \hat{\tilde{f}}_{\psi_2} > |\rho_{\psi_2}| = \frac{|\langle \psi_1 | \psi_2 \rangle|^2}{(2\pi\sigma)^3} \]  

(32)

where

\[
\langle \psi_1 | \psi_2 \rangle \equiv \int d^3q \ \psi_1^*(q, t) \psi_2(q, t) \]  

(33)

is the scalar product between \( \psi_1 \) and \( \psi_2 \) as elements of the quantum Hilbert space \( H \). Thus, the overlap (32) is positive and directly related to the statistical interpretation of the scalar product in quantum mechanics, suggesting again the choice \( \sigma = \hbar \). In particular, the overlap between \( f_\psi \) and the Gaussian (24) is positive.

- The mean value of a classical observable such as \( q, p, p_2L = q \times p \), \( <A> = \int d^3q d^3p \ f_\psi A = \int d^3q \ \psi^*(q, t) \hat{A} \psi(q, t) \equiv \langle \psi|\hat{A}|\psi \rangle \),

is the "expected value" of the usual operator \( \hat{A} \) on \( H \) associated to the observable \( A \):

\[
\hat{q} = q, \ \hat{p} = -i\sigma \nabla, \ \hat{p}^2 = -\sigma^2 \Delta, \ \hat{L} = -i\sigma q \times \nabla .
\]

For the Hamiltonian (13) \( <H>_{\psi} = \langle \hat{H} \rangle_{\psi} \), \( \hat{H} = -\sigma^2 \Delta/2m + V \), and the energy density becomes

\[
w_q = \int d^3p f_\psi H = \frac{\sigma^2}{2m} [||\nabla \psi||^2 - \frac{1}{4} \text{div}(\psi^* \nabla \psi + \psi \nabla \psi^*)] + V \psi^* \psi .
\]  

(34)

- A symplectic diffeomorphism \( \Phi \) on \( (M, \omega) \) which acts by \( (f_\psi)' = \Phi^* f_\psi \) yields, according to (32), a unitary transformation \( \hat{U}_\Phi \) of the state vectors \( \psi \in H \) of the form \( \psi' = \hat{U}_\Phi \psi \), such that

\[
\Phi^* f_\psi = f_{\hat{U}_\Phi^{-1} \psi} .
\]  

(35)

In particular, when \( \Phi \) is the action of a Lie group \( G \), the infinitesimal transformations take the form \( \hat{U}_\epsilon = 1 + i\epsilon \hat{J} \), where \( \hat{J} \) are Hermitian operators associated to the elements of the algebra \( \mathfrak{g} \) of \( G \). Thus, the main difference between the classical and quantum realization of symmetries (e.g. at spontaneous symmetry breaking) is due to the difference

\[3\] For a quantum system the particles are distributed over wave functions, \( \hat{f} \) takes the form \( \hat{f} = \sum_{\psi} N_{\psi} \hat{\rho}_\psi \) where \( N_{\psi} \) is the number of particles in the state \( \psi \), \( \langle \psi|\psi \rangle = 1 \), and \( S \) is independent of \( \ln \hat{\rho}_\psi \).
between the coherence properties of \( f_0 \) and \( f_\psi \).

In general, a functional \( f_{n,S} \) of \( n \) and \( S \), will be called coherent with respect to the classical Liouville equation if during time-evolution it remains the same functional, although \( n \) and \( S \) may change. According to [19], if the potential in \( H \) is a constant, linear, or quadratic polynomial of \( q \), then \( f_\psi \) is an exact solution of the Liouville equation

\[
\partial_t f_\psi = \{ H, f_\psi \}
\]

if \( \psi \) is an exact solution of

\[
i\sigma \partial_t \psi = \hat{H} \psi , \quad \hat{H} = -\frac{\sigma^2}{2m} \Delta + V ,
\]

formally identical to the time dependent Schrödinger equation (TDSE). Thus, \( f_0 \) is coherent for any Hamiltonian, but \( f_\psi \) is coherent only for polynomial potentials of degree at most 2. This restriction was derived before using different arguments both in algebraic and geometric quantization [1, 27].

Beside the stability of the functional form, another aspect of the "coherence" property is that for \( f_0 \) and \( f_\psi \) the two functions \( n \) and \( S \) play also the role of canonically conjugate variables. This aspect is particularly important for real waves quanta such as photons or phonons [30], and it can be shown [19] that the equations of motion for these variables can be derived from a variational principle related to infinite-dimensional Hamiltonian systems of the form

\[
i X_H \hat{\omega} = d\mathcal{H} ,
\]

where \( \hat{\omega} = d\hat{\theta} \),

\[
\hat{\theta} = \int d^3 q \; n \delta S , \quad X_H = \int d^3 q \; (\partial_t n \frac{\delta}{\delta n} + \partial_t \varphi \frac{\delta}{\delta \varphi}) ,
\]

and \( \mathcal{H} = \int d^3 q \; w \) is the classical \((w \equiv w_{cl} = nH(\nabla S, \mathbf{q}))\) or quantum, \((w \equiv w_q)\) energy functional of \( n \) and \( S \).

For an integrable distribution \( f \in L^1(M) \) on the symplectic manifold \( M = T^*Q \), the "coordinates" \((n, S)\) presume a foliation of \( M \) by Lagrangian submanifolds \( \Lambda_S \subset M \) generated by \( S \in C^1(W), W \subset Q \), and the projection

\[
\pi : L^1(M) \mapsto L^1(Q) , \quad \pi(f) = n
\]

defined by integration on \( \Lambda_S \). If \( S \) is the solution of the Hamilton-Jacobi equation, then the asymptotic solution of the Schrödinger equation (36) in the WKB approximation \( \psi \sim \exp(iS/\sigma) \) is related to the subspace of polarized sections \( r \in \Gamma_L(M, \Lambda_S) \) autoparallel on \( \Lambda_S \) \((\nabla_X r = 0, \forall X \in T\Lambda_S)\) in a complex Hermitian line-bundle with connection \((L, \nabla)\) over \( M \) [27] [3] [2]. Thus, the subspace of the coherent functionals \( f_\psi \) defined by
the Wigner transform arises by a peculiar lift of $\Gamma_L(M, \Lambda_S)$ to $\Gamma_L(M)$.

Although the relationship between $\psi$ and $f_\psi$ is nonlinear, and

$$f_{\psi_1+\psi_2} \neq f_{\psi_1} + f_{\psi_2},$$

we note that if $\psi_1, \psi_2$ are solutions of TDSE, and $f_{\psi_1}, f_{\psi_2}$ satisfy the Liouville equation, then $f_{\psi_1+\psi_2}$ is also a solution of the Liouville equation. Using the Dirac notation $\hat{P}_\psi \equiv |\psi \rangle \langle \psi|$ for the projection operator associated to the state function $\psi$, $\langle \psi | \psi \rangle = 1$, the distribution $f_\psi$ [30] takes the form

$$f_\psi(q, p) = W(\hat{P}_\psi) \equiv \frac{1}{(2\pi)^3} \int d^3k e^{-ikp/2} \hat{U}_{k/2} \hat{P}_\psi \hat{U}_{k/2} |q\rangle$$

with $\hat{U}_{k/2} = e^{i\hat{k} \cdot \hat{p}/2}$. Thus, between $f_\psi$ and $\hat{P}_\psi$ there exists a linear relationship by the transform $W$.

Moreover, if $C$ denotes a complete set of states, then

$$\sum_{\psi \in C} f_\psi(q, p) = \frac{1}{(2\pi)^3}.$$  (39)

In terms of group actions, the configuration space $Q = \mathbb{R}^3$ is homogeneous space for the Lie group $G = \mathbb{R}^3$ of the space translations, $T_q Q \simeq T_e G \equiv g$, such that the momentum space $T^*_q Q$ is parameterized by $p \in g^*$. The eigenfunctions $\psi_\mu$ of $\hat{H}$,

$$\hat{H} \psi_\mu = E_\mu \psi_\mu$$

are stationary solutions of [36], $\psi_\mu(t) = e^{-iE_\mu t/\sigma} \psi_\mu(0)$, and correspond to distributions $f_{\psi_\mu}$ independent of time, of energy $E_\mu$,

$$E = \langle H \rangle = \int d^3q d^3p f_{\psi_\mu} H = \langle \psi_\mu | \hat{H} | \psi_\mu \rangle = E_\mu .$$  (41)

It is important to remark that this equality, which is used in many stationary variational calculations, holds for any Hamiltonian of the form [13] [31].

### 3.1 Relativistic Wigner functions and Schrödinger equation

The problem of relativistic Wigner functions and Schrödinger equation for massive particles was studied in [26] within the extended phase-space $M^* = T^* \mathbb{R}^4$ presented in [32]. Thus, the energy ($E$) and time ($t \equiv q_0/c$) become conjugate variables, evolving with respect to a true parameter $u$, called universal time. A particular class of coherent solutions for the relativistic Liouville equation (RLE) consists of the "action distributions"

$$f_0(q^e, p^e, u) = n^e(q^e, u) \delta(p_0 - \partial_0 S) \delta(p - \nabla S) ,$$  (42)

where $n^e$ is the localization probability density in space-time. Considering $\partial_a S = m_0 c^2$, in the case of a free particle we get the continuity equation

$$m_0 \partial_a n^e = \partial_0 (n^e \partial_0 S) - \nabla \cdot (n^e \nabla S) ,$$  (43)
Figure 1. $m_0c^2/\Gamma$ from experimental data (*) and the interpolation functions $2.1 + C/\Gamma$ (solid) for 32 light unflavored meson resonances ($\omega, \eta, \pi, \rho, a, b, f$) with $\Gamma \geq 8.43$ MeV (A) and 48 baryon resonances ($N, \Delta, \Lambda, \Sigma$) with $\Gamma \geq 15.6$ MeV (B) [26].

For a density $n_e(q_e, u) = \delta(q_0 - cu) n(q, u)$, localized in time, (43) reduces in the nonrelativistic limit to the usual continuity equation $\delta(t - u) [m_0 \partial_0 n + \nabla \cdot (n \nabla S)] = 0$. The nonrelativistic identification of $u$ as time may appear when $t$ is a quasiclassical variable [18], described by a Gaussian wave-packet such that $\langle t \rangle = u$. The width of this wave-packet sets a lower limit for the classical time-intervals, and a minimum space-length $\ell_0$. Evidence for the existence of such an elementary time-interval $\delta t_0 = \ell_0/c = \hbar/m_0 c^2$ was found in the particle data [26]. Thus, the ratio $m_0c^2/\Gamma = \tau_L/\delta t_0$ between the mass (in MeV) and decay width (\Gamma), calculated using the experimental data for meson and baryon resonances is well interpolated by functions of the form $2.1 + C/\Gamma$, where $C$ is $1222$ MeV for mesons and $1487$ MeV for baryons (Figure 1, Appendix), indicating that the lifetime $\tau_L = \hbar/\Gamma$ is limited below by $2\delta t_0$.

A quantum distribution

$$\tilde{f}^e_{\Psi}(q^e, k^e, u) \equiv \Psi(q^\mu + \sigma k^\mu/2, u) \Psi^*(q^\mu - \sigma k^\mu/2, u),$$

is a "static" solution $(\partial_0 \tilde{f} = 0)$ of RLE if $-\sigma^2 \square \Phi = m_0^2 c^2 \Phi, \square \equiv \partial_0^2 - \nabla^2$. When $\sigma = \hbar$ this represents the Klein-Gordon equation.

The extended phase-space is also the suitable framework to describe the electromagnetic field [34]. In vacuum the electric and magnetic fields $E$ and $B$ appear as coefficients of two dual 2-forms $\omega_f, \omega_f^*$ on the space-time manifold $\mathbb{R}^4$,

$$\omega_f = -B \cdot dS + E \cdot dq_0 \wedge dq$$

$$\omega_f^* = E \cdot dS + B \cdot dq_0 \wedge dq$$

where $dS_1 = dq_2 \wedge dq_3, dS_2 = -dq_1 \wedge dq_3, dS_3 = dq_1 \wedge dq_2$. In the presence of the field the canonical symplectic form $\omega_0^\mu$ on $T^*\mathbb{R}^4$ for a relativistic massive particle which carries the electric charge $q_e$ and the magnetic charge $q_m$ becomes [34]

$$\omega^e = \omega_0^\mu + \frac{q_e}{c} \omega_f + \frac{q_m}{c} \omega_f^*.$$
to account for the Lorentz forces \( \mathbf{F}_B = q_e \mathbf{v} \times \mathbf{B} / c \) and \( \mathbf{F}_E = -q_m \mathbf{v} \times \mathbf{E} / c \). By specific integrality conditions these two forms provide electric or magnetic charge quantization, while the exterior derivatives in vacuum \( d\omega_f = d\omega_f^* = 0 \) yield the wave equation \( \Box \mathbf{E} = \Box \mathbf{B} = 0 \). Such an equation has as coherent solutions any vector function of \( \tau = t \pm \mathbf{n} \cdot \mathbf{q} / c \), where \( \mathbf{n} \) is the unit vector along the propagation direction. For instance, \( \mathbf{E} \) can be a harmonic or a Gaussian function of \( \tau \), as we may have a plane wave or a localized pulse. However, \( \tau \) is not Lorentz-invariant, and instead it is convenient to consider coherent functionals of a Lorentz-invariant function \( \varphi(q_0, \mathbf{q}) \sim \tau \).

The photon, as relativistic particle of vanishing rest mass and energy \( \epsilon = c|\mathbf{p}| \), associated with the (real) electromagnetic waves, can be introduced considering the energy-density continuity equation

\[
\partial_t w_f = - \text{div} \mathbf{Y} ,
\]

and the eikonal equation

\[
(\nabla \varphi)^2 = (\partial_0 \varphi)^2 ,
\]

which are similar to (43) and (44) with \( m_0 = 0 \). Here \( w_f = (E^2 + B^2)/2 \) is the energy density of the field and \( \mathbf{Y} = c\mathbf{E} \times \mathbf{B} \) is the Poynting vector. Because photons are free particles there are no "zero-point energy" terms in the Planck distribution, (accurately retrieved in the 2.7 K cosmic microwave background spectrum \([35]\)), or in the vacuum energy density \([36]\).

The case of particles in states of negative energy \( E < 0 \) is peculiar because according to \([26]\), in such states the Lorentz group \( SO(1, 3) \) is replaced by \( SO(4) \), isomorphic to \( SU(2) \times SU(2) \). This means that for \( E < 0 \) the distinction between space and time coordinates disappears, and apparently such particles live in closed, unobservable space-time domains. It is interesting to remark that in general relativity the metric outside a spherical shell of mass \( M \) \([37]\),

\[
d s^2 = (1 - 2\gamma_0 M / c^2) dq_0^2 - (1 - 2\gamma_0 M / c^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]

\( \gamma_0 = 6.67 \cdot 10^{-11} \text{Nm}^2/\text{kg}^2 \), shows no clear distinction between space and time coordinates at \( r < R_g = 2\gamma_0 M / c^2 \), when formally the gravitational binding energy approaches \( -Mc^2 \). Beside this similarity, we may speculate that the effect of the inertial parameter (mass) on the metric described at macroscopic scale by the general relativity may turn at atomic scale into an effect on the constant \( \ell_0 \).

4 Finite temperature effects

At finite temperature \( (T) \) the thermal noise affects the statistical ensemble of the "pure" coherent distributions, which evolve towards the classical equilibrium density \( f_e \),

\[
f_e = N e^{-\beta H} / Z_\mu , \quad Z_\mu = \int_{M_\mu} \Omega_\mu / h^3 \ e^{-\beta H} ,
\]

13
\[ \beta = 1/k_B T, \] according to the Fokker-Planck equation
\[ \partial_t \langle \hat{f} \rangle + \frac{1}{m} \mathbf{p} \cdot \nabla \langle \hat{f} \rangle - \nabla V \cdot \nabla \langle \hat{f} \rangle = \gamma \nabla_p \cdot \left( \frac{\mathbf{p}}{m} + \frac{\nabla \beta}{\beta} \right) \langle \hat{f} \rangle, \tag{50} \]
where \( \gamma \) denotes the friction coefficient. By the Fourier transform in momentum \[ \hat{f}(\mathbf{k}) = \frac{1}{\sigma} \langle \mathbf{q} \mathbf{f}_ab \rangle, \]
becomes
\[ \partial_t \hat{f}(\mathbf{k}) - \frac{i}{\sigma} \nabla_k \cdot \nabla \hat{f}(\mathbf{k}) + \mathbf{k} \cdot (i \nabla V + \frac{\gamma}{m} \nabla \mathbf{k}) \hat{f}(\mathbf{k}) = -\gamma k_B T k^2 \hat{f}(\mathbf{k}). \tag{51} \]
A function \( \tilde{f} \) of the quantum form \( \tilde{f}_\psi(q, k) = \psi(a)\psi(b)^* \), with \( a = q + \hbar k/2 \) and \( b = q - \hbar k/2 \), can be written as a matrix element \( \hat{f}_{ab} = \langle a | \psi \rangle \langle \psi | b \rangle \) of the operator \( |\psi\rangle \langle \psi| \) between the eigenstates \( |a\rangle, |b\rangle \) of the position operator \( \hat{q} \). With this notation we also get \( \partial_t \hat{f}_{ab} = (\partial_t \hat{f})_{ab} \),
\[ k = \frac{1}{\sigma} (a - b), \quad k \hat{f}_{ab} = \frac{1}{\sigma} [\hat{q}, \hat{f}]_{ab}, \quad k^2 \hat{f}_{ab} = \frac{1}{\sigma^2} [\hat{q}, [\hat{q}, \hat{f}]]_{ab}, \]
\[ \nabla_k = \frac{\sigma}{2} (\nabla_a - \nabla_b), \quad \nabla_k \hat{f}_{ab} = \frac{\sigma}{2} \{ \nabla, \hat{f} \}'_{ab}, \quad \nabla_k \cdot \nabla \hat{f}_{ab} = \frac{\sigma}{2} [\Delta, \hat{f}]_{ab}, \]
where \( \{,\} \) denotes the anticommutator. At small \( k \)
\[ \sigma \mathbf{k} \cdot \nabla V \hat{f}_{ab} = (V_a - V_b) \hat{f}_{ab} = [V, \hat{f}]_{ab}, \]
and as indicated in \[ [19], \] for a single microscopic particle when \( \sigma = \hbar, \hat{f} = \hat{\rho}, Tr\hat{\rho} = 1 \), \[ (51) \] takes the form of the quantum Fokker-Planck equation for the density matrix \( \hat{\rho} \),
\[ i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] + \frac{\gamma}{2m} [\hat{q}, \{ \hat{p}, \hat{\rho} \}'] - i\frac{\gamma k_B T}{\hbar} [\hat{q}, [\hat{q}, \hat{\rho}]] \tag{52} \]
This equation is similar to the one proposed in \[ [38], \] and by replacing \( \langle \hat{p} \rangle \) for \( \hat{p} \) in \( \{ \hat{p}, \hat{\rho} \}' \), it takes the form considered in \[ [18]. \] Though, none of them has a satisfactory form, apparently due to the dissipative term. In classical mechanics, the effect of dissipation is not only energy loss, but also decrease in the phase-space volume. Thus, near the volume of the elementary cell one can expect a change in the dissipation mechanism. In fact, when \( \hat{H} = \hat{p}^2/2m \) the quantum equilibrium distributions
\[ \hat{\rho}_\pm = \frac{1}{\zeta e^\pm H + a} \pm 1 \tag{53} \]
(\( \zeta \) is a normalization constant), can be obtained as stationary solutions of the nonlinear equation
\[ i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] + \frac{\gamma}{2m} [\hat{p}, \{ \hat{p}, \hat{\rho} \}'] - \frac{i\gamma k_B T}{\hbar} [\hat{q}, [\hat{q}, \hat{\rho}]] \tag{54} \]
in which the term \( \mp \gamma \{ \hat{p}, \hat{\rho}^2 \}'/2m \) could be assigned to a density-dependent friction force \[ [31]. \] However, before thermalization, while the thermal noise decreases the coherence domain, one can expect a transition from complex (\( \psi \)) “probability waves” to real (\( \mathbf{n} \)) density waves \[ [19]. \]
5 Conclusions

The Fourier transform in momentum $\tilde{F}$ of the distribution function on the classical phase space is a density on the configuration space $Q$, such that coherent solutions of the Liouville equation, expressed as functionals of only two functions on $Q$, $n$ and $S$, can be found. The action waves $f_0$ [42] are localized in momentum and evolve according to the Hamilton-Jacobi equation at infinite entropy. In the particular case of the harmonic oscillator Hamiltonian can also be found coherent distributions [24] of finite entropy. These are coherent not only as Gaussian distributions on the phase-space, but also as the Wigner transform of Glauber coherent states for the TDSE. In general, a functional $f_0$ takes the form of the Wigner function $f_\psi$ [30] by space discretization. Although $f_\psi$ is not positive definite, it has a positive overlap with [24], and in this sense can be considered as particle (or probability) density. During time evolution $f_\psi$, with $\psi$ an exact solution of TDSE, remains coherent only for polynomial potentials of degree at most 2. This means for instance that in a Coulomb potential either the classical Liouville equation, or TDSE should contain correction terms, which can be calculated and compared to other corrections (e.g. relativistic, QED [39]), or experimental data (e.g. the transition time in single atoms [40]). However, the equality [41] $\langle H \rangle |_{f_\psi} = \langle \hat{H} \rangle_{\psi}$, which can be used in time-independent variational calculations for the dominant part of the (quasi) stationary equilibrium distributions, holds for any potential [31].

Relativistic Wigner functions can be defined similarly, by space-time discretization of the action distributions on the extended phase space. Though, the presumed dependence of the minimum interval of time (the ”present”) on the inertial parameter, or the problem of negative mass, indicate that the suitable framework for discussion is the general relativity.

At finite temperature the Fourier transform [51] of the classical Fokker-Planck equation takes the form of the quantum transport equation [52] simply by considering the density $\tilde{f}_\psi(q,k)$ as a matrix element. However, to obtain the quantum equilibrium distributions [53] as stationary solutions an additional, density-dependent dissipative term, is necessary.

The results summarized above indicate that the functional coherent distributions on the classical phase-space may provide the missing link between classical mechanics and quantum phenomenology. The ”action waves” [42] and the Wigner functions [30] are two examples of coherent distributions $f_{[n,S]}$ related to the classical and quantum behaviour, respectively, but the space of such solutions, its relationship to ”granularity”, and the various aspects of ”decoherence” remain so far unexplored.
6 Appendix

Table 1. Comparison between the experimental value of the mass ($M$) and the estimate $2.1\Gamma+ 1222$ MeV for some meson resonances ($M, \Gamma$ from [33]).

| Resonance | $\Gamma$ (MeV) | $J^{PC}$ | $M$ (MeV) | $2.1\Gamma+1222$ MeV |
|-----------|----------------|----------|-----------|----------------------|
| $f_1$ (1285) | 24.2 ±1.1 | 1++ | 1282.1 ±0.6 | 1272.8 ±2.3 |
| $\eta$ (1295) | 55 ±5 | 0−+ | 1294 ±4 | 1337.5 ±10.5 |
| $f_0$ (1500) | 109 ±7 | 0++ | 1505 ±6 | 1450.9 ±14.7 |
| $\pi$ (1800) | 208 ±12 | 0−+ | 1720 ±6 | 1658.8 ±25.2 |

Table 2. Comparison between the experimental value of the mass ($M$) and the estimate $2.1\Gamma+ 1487$ MeV for some barion resonances ($M, \Gamma$ from [33]).

| Resonance | $\Gamma$ (MeV) | $J^P$ | $M$ (MeV) | $2.1\Gamma+1487$ MeV |
|-----------|----------------|-------|-----------|----------------------|
| $\Lambda(1520)$ | 15.6 ±1 | $\frac{3}{2}^-$ | 1519.5 ±1 | 1519.7 ±2.1 |
| $N(1700)$ | 150 (100-250) | $\frac{3}{2}^-$ | 1700(1650-1750) | 1802(1697-2012) |
| $\Sigma(1940)$ | 220 (150-300) | $\frac{3}{2}^-$ | 1940 (1900-1950) | 1949 (1802-2117) |
| $N(2600)$ | 650 (500-800) | $\frac{1}{2}^+$ | 2600(2550-2750) | 2852 (2537-3167) |

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