ON THE PARAMETRIZATION OF NONLINEAR IMPULSIVE FRACTIONAL INTEGRO–DIFFERENTIAL SYSTEM WITH NON–SEPARATED INTEGRAL COUPLED BOUNDARY CONDITIONS

Ava Sh. Rafeeq

Dept. of Mathematic, Faculty of Science, University of Zakho, Zakho, Kurdistan Region, Iraq (ava.rafeeq@uoz.edu.krd)

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ABSTRACT:

We give a new investigation of periodic solutions of nonlinear impulsive fractional integro-differential system with different orders of fractional derivatives with non-separating integral coupled boundary conditions. Uniformly Converging of the series of solutions according to the main idea of the Numerical-analytic technique, from creating a sequence of functions. An example of impulsive fractional system is also presented to illustrate the theory.

KEYWORDS: Caputo fractional derivative, fractional integro-differential, integral coupled boundary conditions, Periodic solutions, successive approximation method.

1. INTRODUCTION

Fractional differential equations have been developed in the last decade as good tools to describe the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, economics, control theory and image processing, etc. (Kilbas et al., 2006, Sabatier et al., 2007, Lakshmikantham etal., 2009). On the other hand, we observe periodic motions in every field of science and everywhere in real life (Farkas, 1994). The applications and theory of the fractional differential equations have recently been collected by several researchers for different problems, we refer the reader to (Ahmad and Nieto, 2014, Henderson et al., 2015). The theory of impulsive differential equations is a new and important field of differential equation theory, many authors have collected (Agarwal et al., 2010, Wang et al., 2012, Feckan et al., 2014), which has been an object of intensive investigation in recent years, and applications to different areas have been considered. However, the concept of solutions for impulsive fractional differential equations (Zhou and Chu, 2012, Bai et al., 2016, Bai and Zhang, 2016, Dong et al., 2017) has been argued extensively.

Also an extended this method by (Feckan and Marynets, 2017) to nonlinear system of integro-differential equation (Butris et al., 2017, Butris, and Taher, 2019) and Caputo type fractional equations with periodic boundary value problems (Mahmudov et al., 2019). Furthermore, the investigate of a coupled system of fractional order is also very significant because this type of system can often occur in applications. The reader is referred to read (Su, 2009, Wang et al., 2010), and the references cited therein.

Caputo-type fractional integro-differential system with nonlinear coupled integral boundary conditions has been considered in this paper. We apply picard approximations technique proposed by (Ronto and Samolenko, 2000, Ronto et al., 2015, Marynets et al., 2016, Feckan and Marynets, 2018) for investigation the existence, uniqueness and approximation of periodic solutions of nonlinear fractional integro-differential system with nonlinear coupled integral boundary conditions. Motivated by the works mentioned and many known results, we use the numerical analytic method to investigate the existence and uniqueness of periodic solutions and define our problem

\[ \frac{D^\alpha}{D_t^\alpha} u(t) = f(t, \beta(t, \alpha), u(t)) \int_0^t K(t, s)(u(s) - z(s))ds, \quad t \neq t_i \]

\[ \frac{D^\gamma}{D_t^\gamma} z(t) = g(t, \beta(t, \gamma), z(t)) \int_0^t H(t, s)(u(s) - z(s))ds, \quad t \neq t_i \]

\[ \Delta u_{i+1} = i(\beta(t, \alpha), u(t)), \quad \Delta z_{i+1} = j(\beta(t, \gamma), z(t)) \]

with non-separated integral coupled boundary conditions

\[ Au(0) + Bu(T) = \int_0^T h_1(z(s))ds, \quad \text{with} \quad \det(B) \neq 0 \]

\[ Cz(0) + Dz(T) = \int_0^T h_2(u(s))ds, \quad \text{with} \quad \det(D) \neq 0 \]

for all \( t \in [0, T] \), \( A, B, C \) and \( D \) are \( n \times n \) matrices where \( ^cD^\alpha_+ \), \( ^cD^\gamma_+ \), denote the Caputo fractional derivatives, \( 0 < \alpha, \gamma \leq 1 \), also \( \beta(t, \alpha) \) and \( \beta(t, \gamma) \) are said to be special functions provided that (Beta function), the function \( f, g \in C([0, T] \times \Omega \times D_1, D_2, R), \Omega = [0, T] \times \{0, 1\}, D_1 \) and \( D_2 \) are compact subset of, also \( a, b, h_1 \) and \( h_2 \) are continuous functions on \([0, T]\)

2. BACKGROUND MATERIAL

In this section, some definitions of fraction calculus and lemmas are presented which are used for the statement of the problem (1.1) and (1.2).

Definition 2.1 (Kilbas et al., 2006) For a function \( g \) given on the interval \([a, b]\), the Caputo fractional order derivative of \( g \) is defined by

\[ ^cD^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \left( t - s \right)^{n-\alpha-1} g^{(n)}(s) ds \]

where \( n = [\alpha] + 1 \) and \([\alpha]\) denotes the integer part of \( \alpha \), and \( \Gamma(\cdot) \) denotes the Gamma function.
Definition 2.2 (Kilbas et al., 2006) Let \( g \) be a function which is defined almost everywhere (a.e.) on \([a, b]\), for \( \alpha > 0 \), we define

\[
{\frac{b}{a}}D^{-\alpha}f = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (b - t)^{\alpha - 1} g(t) \, dt \quad (2.2)
\]

provided that the integral (Lebesgue) exists.

Lemma 2.3 Let \( g(t) \) be a continuous function for \( t \in [0, T] \), then the following estimate holds

\[
\left| \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} g(s) \, ds - \left( \frac{1}{\Gamma(\alpha)} \int_{0}^{T} (T - s)^{\alpha - 1} g(s) \, ds \right) \right| ^{2} \leq \omega(t) \max\left\{ \frac{1}{\Gamma(\alpha + 1)}, \frac{1}{(\Gamma(\alpha))^2} \right\}
\]

where \( \omega(t) = \frac{2t^\alpha}{\Gamma(\alpha + 1)} \left( 1 - \frac{t}{T} \right) \).

Proof. It is obvious that

\[
\left| \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} g(s) \, ds - \left( \frac{1}{\Gamma(\alpha)} \int_{0}^{T} (T - s)^{\alpha - 1} g(s) \, ds \right) \right| ^{2} \leq \left( \frac{1}{\Gamma(\alpha)} \right)^{2} \int_{0}^{T} \left( (t - s)^{\alpha - 1} - \left( \frac{1}{\Gamma(\alpha)} \right) (T - s)^{\alpha - 1} \right) g(s) \, ds
\]

Form (Feckan and Marynets, 2017), the terms becomes

\[
(t - s)^{\alpha - 1} - \frac{1}{\Gamma(\alpha)} (T - s)^{\alpha - 1} = (t - s)^{\alpha - 1} (1 - \frac{1}{\Gamma(\alpha)} (T - s) - \frac{1}{\Gamma(\alpha + 1)} (T - s)^{1 - \alpha}) \geq (t - s)^{\alpha - 1} (1 - \frac{1}{\Gamma(\alpha)} (T - s) - \frac{1}{\Gamma(\alpha + 1)} (T - s)^{1 - \alpha}) = (t - s)^{\alpha - 1} (\frac{T - t}{T}) \geq 0
\]

For any \( s \in [0, t] \), we obtain that

\[
\left| \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} g(s) \, ds - \left( \frac{1}{\Gamma(\alpha)} \int_{0}^{T} (T - s)^{\alpha - 1} g(s) \, ds \right) \right| ^{2} \leq \frac{2t^\alpha}{\Gamma(\alpha + 1)} \left( 1 - \frac{t}{T} \right) \max\left\{ \frac{1}{\Gamma(\alpha + 1)}, \frac{1}{(\Gamma(\alpha))^2} \right\} \leq \omega(t) \max\left\{ \frac{1}{\Gamma(\alpha + 1)}, \frac{1}{(\Gamma(\alpha))^2} \right\}
\]

The proof of lemma is complete.

Definition 2.4 The solutions of the system of fractional integro-differential equations (1.1) and integral boundary conditions (1.2) are defining the following integral equations:

\[
u(t, x_{0}, y_{0}) = u_{0} + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} f(s, \beta(t, x, u(s, w, z)), u(s, w, z)) \, ds
\]

\[
\int_{0}^{t} \xi(s, t) w(s, w, z) \, ds
\]

\[
\int_{0}^{t} \xi(s, t) w(s, w, z) \, ds
\]

3. CONDITIONS FOR CONVERGENCE OF SUCCESSIVE APPROXIMATION

For investigate of the successive approximation for periodic solution of the problem (1.1) and (2.2), we need the some conditions, suppose that the vector functions \( f, g \in C([0, T] \times \Omega x D_1 x D_2, R), l_1, l_2 \in C([0, T] x D_1, R) \), \( \Omega = [0, T] \times (0, 1], D_1 \) and \( D_2 \) are compact subset of \( R^2 \), also \( \varphi, h_1 \) and \( h_2 \) are continuous functions on \([0, T]\), and satisfies the following hypothesis.

\( H_1 \): There exist positive constants \( M_{\beta \alpha}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta} \), and \( M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta} \) such that

\[
\| f(t, \beta, u, x) \| \leq M_{\beta \alpha}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta} \quad \text{... (3.1)}
\]

\[
\| g(t, \beta, y, z) \| \leq M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta} \quad \text{... (3.2)}
\]

\[
\| l_1(\beta, \eta, y, z) \| \leq M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta} \quad \text{... (3.3)}
\]

\[
\| l_2(\beta, \eta, y, z) \| \leq M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta}, M_{\beta \eta} \quad \text{... (3.4)}
\]

where

\[
M_{\beta \alpha} = \max_{\tau \in [0,T]} \frac{1}{\beta(\tau)}, M_{\beta \eta} = \max_{\tau \in [0,T]} \frac{1}{\eta(\tau)}, x_i = \int_{0}^{t} K(t, s) (u(s) - z(s)) \, ds, \quad y_i = \int_{0}^{t} R(t, s) (u(s) - z(s)) \, ds
\]
for all \( t \in [0, T] \), \( \tau \in (0, T) \), \( u, u_2, z_1, z_2 \in D_1 \) and \( x_1, y_1 \in D_2 \), \( i = 1, 2 \).

H2: There exist positive constants \( p_0, p_1, q_0 \) and \( q_1 \) such that

\[
\begin{align*}
\|h_1(z_1)\| &\leq p_0 \quad \|h_2(u_1)\| \leq q_0 \quad \text{... (3.5)} \\
\|h_1(z_2) - h_1(z_1)\| &\leq p_1 \|z_1 - z_2\| \quad \text{... (3.6)} \\
\|h_2(u_2) - h_2(u_1)\| &\leq q_1 \|u_1 - u_2\| \quad \text{... (3.7)}
\end{align*}
\]

for all \( t \in [0, T] \), \( u, u_2, z_1, z_2 \in D_1 \).

H3: The kernels \( K(t, s) \) and \( R(t, s) \) satisfy the following conditions, when there exist positive constants \( KS \) and \( RS \) such that

\[
\int_0^T \|K(t, s)\| ds \leq KS \quad \text{and} \quad \int_0^T \|R(t, s)\| ds \leq RS 
\]

for all \( s, t \in [0, T] \) ... (3.8)

Let \( \omega (t) = \frac{2\pi^2}{T + 1} \left( 1 - \frac{1}{2} \right)^2 \) and \( \phi(t) = \frac{2\pi^2}{T + 1} \left( 1 - \frac{1}{2} \right)^2 \), then \( \omega(t) \) and \( \phi(t) \) take these maximum value at \( t = \frac{T}{2} \), and \( \|\omega\| = \frac{2\pi^2}{T + 1} \left( 1 - \frac{1}{2} \right)^2 \), \( \|\phi\| = \frac{2\pi^2}{T + 1} \left( 1 - \frac{1}{2} \right)^2 \), with \( \|\cdot\|_\infty = \max_{t \in [0, T]} \|\cdot\| \).

Define the non-empty set

\[
\begin{align*}
D_1 &= D_1 - \frac{T^a}{2^{a+1}T^a} M_{b_0} M + M_1 \\
D_2 &= D_2 - \frac{T^a}{2^{a+1}T^a} M_{b_0} L + M_1
\end{align*}
\]

... (3.9)

where

\[
\begin{align*}
M_1 &= M_2 + 2S_1 M_1 \\
M_2 &= M_3 + 2S_2 L_1 \\
M_3 &= ||B^{-1}||T_0 + ||B^{-1}||A + 1 ||u_0|| \\
and M_4 &= ||D^{-1}||T_0 + ||D^{-1}||C + 1 ||z_0||
\end{align*}
\]

Furthermore, we suppose that the largest eigen-value of the matrix

\[
\Lambda = \begin{pmatrix}
H_1 & H_2 \\
H_3 & H_4
\end{pmatrix}
\]

Where

\[
\begin{align*}
H_1 &= \frac{T^a}{2^{a+1}T^a} M_{b_0} (K_1 + K_2) + 2M_{b_0} S_1 \xi
H_2 &= \frac{T^a}{2^{a+1}T^a} M_{b_0} K_2 + 2M_{b_0} S_1 \xi
H_3 &= \frac{T^a}{2^{a+1}T^a} M_{b_0} L_2 + 2M_{b_0} S_1 \xi
H_4 &= \frac{T^a}{2^{a+1}T^a} M_{b_0} L_3 + 2M_{b_0} S_1 \xi
\end{align*}
\]

Theorem 4.1: If the system (1.1) with boundary conditions (1.2) satisfy the conditions \( H_1, H_2, H_3, H_4 \) and (4.10), then sequences of functions (4.1) and (4.2), which are periodic in \( t \) of period \( T \), converges uniformly as \( m \to \infty \) on the domain:

\[
(t, \tau) \in [0, T] \times D_1 \times D_2 \quad \text{... (4.3)}
\]

with \( u(t, \tau, z_0) = u_0, \tau(t, \tau, z_0) = z_0, m = 0, 1, 2, ... \)

Our main results separate to the following four parts:

4. MAIN RESULTS

4.1 Approximation of Periodic Solution of (1.1) and (1.2)

In this section, we study the periodic approximation solutions of the system of nonlinear fractional integro-differential equations (1.1) with coupled integral boundary conditions (1.2) will be introduced by the following theorem.

In the beginning, we define the following sequence of functions \( \{u_m\}_{m=0}^\infty \) and \( \{z_m\}_{m=0}^\infty \) given by the iterative formulas

\[
\begin{align*}
u_{m+1}(t, u_0, z_0) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \\
&\quad \times f(s, \beta(t, \alpha), u_m(s, u_0, z_0), z_m(s, u_0, z_0)) \mathrm{d}\tau \\
&\quad - \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma-1} \\
&\quad \times g(s, \beta(t, \gamma), z_m(s, u_0, z_0)) \mathrm{d}\tau
\end{align*}
\]

... (4.1)


\[ f(s, \beta(t, \alpha), u(s, u_0, z_0)) = \int_0^{\alpha(s)} K(s, \xi)(u(\xi, u_0, z_0) - z(\xi, u_0, z_0))d\xi \]
\[ - \frac{1}{f(0)} \left( \frac{T}{f} \right)^a \int_0^T (T - s)^{-a} \]
\[ f(s, \beta(t, \alpha), u(s, u_0, z_0)) = \int_0^{\alpha(s)} K(s, \xi)(u(\xi, u_0, z_0) - z(\xi, u_0, z_0))d\xi \]
\[ - \frac{(t)}{\left( \frac{T}{f} \right)^a} \sum_{i=1}^{S_1} \int (\beta(t, \alpha), u(t, u_0, z_0)) \]
\[ + \frac{(t)}{\left( \frac{T}{f} \right)^a} \left[ T - 1 \right] \int_0^T h_1(z(s, u_0, z_0))ds + (B^{-1}A + \lambda)u_0 \]
\[ + \sum_{0 < c_t < t} \int (\beta(t, \alpha), u(t, u_0, z_0)) \]
\[ \text{on the domain (4.3), provided that} \]
\[ \|u(t, u_0, z_0) - z(t, u_0, z_0)\| \leq \Lambda_m(E - A)^{-1} \]
\[ \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) \]
\[ \|z(t, u_0, z_0) - z_0\| \leq \frac{T^a}{22a-1} \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) \]
\[ \text{for all } t \in [0, T], u_0 \in D_{f} \text{ we get } u_1(t, u_0, z_0) \in D_1, \text{ similarly} \]
\[ \|z_1(t, x_0, y_0) - z_0\| \leq \frac{T^a}{22a-1} \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \text{for all } t \in [0, T], z_0 \in D_{f} \text{ we obtain that } z_1(t, u_0, z_0) \in D_2 \]

Thus by the mathematical induction, we find that
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \]
\[ \leq \omega(t) \left\| f(t, \beta(t, \alpha), u_m(t, u_0, z_0), \int_0^{\alpha(t)} K(t, s)(u(s, u_0, z_0) - z_m(s, u_0, z_0))ds \right\| \]
\[ + \sum_{0 < c_t < t} \|\int (\beta(t, \alpha), u_m(t, u_0, z_0)) \]
\[ \|f(t, \beta(t, \alpha), u_{m-1}(t, u_0, z_0), \int_0^{\alpha(t)} K(t, s)(u_{m-1}(s, u_0, z_0) - z_{m-1}(s, u_0, z_0))ds \| \]
\[ + \sum_{0 < c_t < t} \|\int (\beta(t, \alpha), u_{m-1}(t, u_0, z_0)) \]
\[ \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \text{for all } t \in [0, T], u_0 \in D_f, z_0 \in D_2, \text{ we obtain that } z_1(t, u_0, z_0) \in D_2 \]

Now, we claim that the sequences of functions (4.1) and (4.2) are uniformly convergent on the domain (4.3).
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \]
\[ \leq \omega(t) \left\| f(t, \beta(t, \alpha), u_m(t, u_0, z_0), \int_0^{\alpha(t)} K(t, s)(u(s, u_0, z_0) - z_m(s, u_0, z_0))ds \right\| \]
\[ + \sum_{0 < c_t < t} \|\int (\beta(t, \alpha), u_m(t, u_0, z_0)) \]
\[ \|f(t, \beta(t, \alpha), u_{m-1}(t, u_0, z_0), \int_0^{\alpha(t)} K(t, s)(u_{m-1}(s, u_0, z_0) - z_{m-1}(s, u_0, z_0))ds \| \]
\[ + \sum_{0 < c_t < t} \|\int (\beta(t, \alpha), u_{m-1}(t, u_0, z_0)) \]
\[ \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \text{for all } t \in [0, T], u_0 \in D_f, z_0 \in D_2, \text{ we obtain that } z_1(t, u_0, z_0) \in D_2 \]

Therefore, we get
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \text{for all } t \in [0, T], u_0 \in D_f, z_0 \in D_2, \text{ we obtain that } z_1(t, u_0, z_0) \in D_2 \]

We rewrite inequalities (4.8) and (4.9) in vector form to gain
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq \omega(t) \left( \frac{T^a}{22a-1} \right) \left( \frac{M_{\beta A} + M_4}{M_{\beta L}} \right) + M_5 \]

By mathematical induction, we obtain that
Proof. Assume that \( \bar{u}(t, u_0, z_0) \) \( \ddot{z}(t, u_0, z_0) \) is another solution for the system (1.1) with boundary conditions (1.2), so that \( \bar{u}(t, u_0, z_0) = u_0 + \frac{1}{l(t)} \int_0^t \left( t - s \right)^{r-1} f(s, \beta(t, s), u(s, u_0, z_0)) ds \) \( \bar{z}(t, u_0, z_0) = z_0 + \frac{1}{l(t)} \int_0^t \left( t - s \right)^{r-1} g(s, \beta(t, s), z(s, u_0, z_0)) ds \)

for all \( t \in \mathcal{D} \). Then, by using the relation (4.12) and proceeding in (4.1) and (4.2), we have

\[
\| u(t, u_0, z_0) - \bar{u}(t, u_0, z_0) \| \leq 2 \| t \| M_{\beta t} (K_1 + K_2 K_3) + 2 M_{\beta t} S_1 K_3 (u(t, u_0, z_0) - \bar{u}(t, u_0, z_0)) + (\omega(t) M_{\beta t} K_2 K_3 + \| B \| T_{P_1} T_{P_2} (u(t, u_0, z_0) - \bar{u}(t, u_0, z_0)) \)
\]

and

\[
\| z(t, u_0, z_0) - \bar{z}(t, u_0, z_0) \| \leq 2 \| t \| M_{\beta t} L_2 R_3 + \| D^{-1} T_{q_1} \| (u(t, u_0, z_0) - \bar{u}(t, u_0, z_0)) + (\varphi(t) M_{\beta t} (L_1 + L_2 R_3) + 2 M_{\beta t} S_2 K_3) (z(t, u_0, z_0) - \bar{z}(t, u_0, z_0)) \) \]

We rewrite inequalities (4.15) and (4.16) in vector form to gain

\[
\| u(t, u_0, z_0) - \bar{u}(t, u_0, z_0) \| \leq \Lambda \| z(t, u_0, z_0) - \bar{z}(t, u_0, z_0) \| \) \]

By mathematical induction, we obtain that

\[
\| u(t, u_0, z_0) - \bar{u}(t, u_0, z_0) \| \leq \Lambda \| z(t, u_0, z_0) - \bar{z}(t, u_0, z_0) \| \)

From the condition (4.10), shows that the solution \( \bar{u}(t, u_0, z_0) \) \( \bar{z}(t, u_0, z_0) \) is unique periodic solution on the domain (4.3).

4.3 Existence of Periodic Solutions of (1.1) and (1.2)

The problem of the existence of the periodic solution for the system (1.1) with boundary condition (1.2) is uniquely connected with the existence of the zeros of the vector functions.
\[\Delta_1(0, u_0, z_0) = -\frac{\alpha}{T^a} \int_0^T (T - s)^{a-1} \]
\[f(s, \beta(t, \alpha), u(s, u_0, z_0), K(s, \xi)(u(\xi, u_0, z_0)) - z(\xi, u_0, z_0)) d\xi ds \]
\[-\frac{\Gamma(\alpha + 1)}{T^a} \sum_{t=1}^{s_1} l_t(\beta(t, \alpha), u(t, u_0, z_0)) + \frac{\Gamma(\alpha + 1)}{T^a} \left[B^{-1} \int_0^T h_1(z(s, u_0, z_0)) ds -(B^{-1} A + I)u_0\right] \]
and
\[\Delta_2(0, u_0, z_0) = -\frac{\gamma}{T^y} \int_0^T (T - s)^{y-1} \]
\[g(s, \beta(t, \gamma), z(s, u_0, z_0), \varphi(s, \xi)(u_m(\xi, u_0, z_0)) - z(\xi, u_0, z_0)) d\xi ds \]
\[-\frac{\Gamma(\gamma + 1)}{T^y} \sum_{t=1}^{s_2} l_t(\beta(t, \gamma), z(t, u_0, z_0)) + \frac{\Gamma(\gamma + 1)}{T^y} \left[D^{-1} \int_0^T h_2(u(s, u_0, z_0)) ds -(D^{-1} C + I)z_0\right] \]
\[\Delta_1m(0, u_0, z_0) \quad \text{and} \quad \Delta_2m(0, u_0, z_0) \]
\[\Delta_1m(0, u_0, z_0) = -\frac{\alpha}{T^a} \int_0^T (T - s)^{a-1} \]
\[f(s, \beta(t, \alpha), u_m(s, u_0, z_0), K(s, \xi)(u_m(\xi, u_0, z_0)) - z_m(\xi, u_0, z_0)) d\xi ds \]
\[-\frac{\Gamma(\alpha + 1)}{T^a} \sum_{t=1}^{s_1} l_t(\beta(t, \alpha), u_m(t, u_0, z_0)) + \frac{\Gamma(\alpha + 1)}{T^a} \left[B^{-1} \int_0^T h_1(z_m(s, u_0, z_0)) ds -(B^{-1} A + I)u_0\right] \]
and
\[\Delta_2m(0, u_0, z_0) = -\frac{\gamma}{T^y} \int_0^T (T - s)^{y-1} \]
\[g(s, \beta(t, \gamma), z_m(s, u_0, z_0), \varphi(s, \xi)(u_m(\xi, u_0, z_0)) - z_m(\xi, u_0, z_0)) d\xi ds \]
\[-\frac{\Gamma(\gamma + 1)}{T^y} \sum_{t=1}^{s_2} l_t(\beta(t, \gamma), z_m(t, u_0, z_0)) + \frac{\Gamma(\gamma + 1)}{T^y} \left[D^{-1} \int_0^T h_2(u_m(s, u_0, z_0)) ds -(D^{-1} C + I)z_0\right] \]
\[d_1 = \frac{1}{\Gamma(\alpha)} M_{\beta u} K_1 + K_2 + \frac{\Gamma(\alpha + 1)}{T^a} M_{\beta u} K_3 S_1 \]
\[d_2 = \frac{1}{\Gamma(\alpha)} M_{\beta u} K_1 + \frac{\Gamma(\alpha + 1)}{T^a} ||B^{-1}|| T p_1 \]
\[d_3 = \frac{1}{\Gamma(\gamma)} M_{\beta y} L_2 R_2 + \frac{\Gamma(\gamma + 1)}{T^y} ||D^{-1}|| T q_1 \]
\[d_4 = \frac{1}{\Gamma(\gamma)} M_{\beta y} L_2 R_2 + \frac{\Gamma(\gamma + 1)}{T^y} M_{\beta y} L_3 S_2 \]
\[\text{and} \quad \eta = \left(\frac{2^{\alpha - 1}}{\Gamma(\alpha)} M_{\beta u} + M_4 \right) \left(\frac{2^{\gamma - 1}}{\Gamma(\gamma)} M_{\beta y} + M_5 \right) \]

Proof. From equations (4.18) to (4.21), we obtain that
\[\Delta_1(0, u_0, z_0) - \Delta_1m(0, u_0, z_0) \leq \left(\frac{1}{\Gamma(\alpha)} M_{\beta u} + M_4 \right) \left(\frac{2^{\alpha - 1}}{\Gamma(\alpha)} M_{\beta u} + M_5 \right) \]
\[\rightarrow \Delta_1(0, u_0, z_0) - \Delta_1m(0, u_0, z_0) \leq \left(\frac{1}{\Gamma(\alpha)} M_{\beta u} K_1 + K_2 + \frac{\Gamma(\alpha + 1)}{T^a} M_{\beta u} K_3 S_1 \right) \]
\[\text{Proof.} \quad \text{From equations (4.18) to (4.21), we obtain that} \]
\[\quad \Delta_1(0, u_0, z_0) - \Delta_1m(0, u_0, z_0) \leq \left(\frac{1}{\Gamma(\alpha)} M_{\beta u} K_1 + K_2 + \frac{\Gamma(\alpha + 1)}{T^a} M_{\beta u} K_3 S_1 \right) \]
\[\quad \text{and} \quad \eta = \left(\frac{2^{\alpha - 1}}{\Gamma(\alpha)} M_{\beta u} + M_4 \right) \left(\frac{2^{\gamma - 1}}{\Gamma(\gamma)} M_{\beta y} + M_5 \right) \]

Since \[\|u(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq \Lambda^{m}(E - \Lambda)^{-1} \]
\[\text{and} \quad \||z(t, u_0, z_0) - z_m(t, u_0, z_0)\| \leq \Lambda^{m}(E - \Lambda)^{-1} \]

hence rewrite the equations (4.23) and (4.24) as a vector form.

The inequality (4.22) is hold for all m \geq 0.

**Theorem 4.4.** Let the vector functions \(f(t, \beta(t, \alpha), u, x)\) and \(g(t, \beta(t, \gamma), z, x)\) be defined on the intervals \([a_1, b_1]\) and \([a_2, b_2]\) on \(\mathbb{R}^t\) and periodic in t of period T, suppose that for all m \geq 0, then the sequences of the functions \(\Delta_1m(0, u_0, z_0)\) and \(\Delta_2m(0, u_0, z_0)\) which are defined in (4.20)and (4.21) satisfy the inequalities:-

\[\min_{u_m, z_m, \Delta_1m, \Delta_2m} \Delta_1m(0, u_0, z_0) \leq -\left(\frac{d_1}{d_2}, A^{m}(E - \Lambda)^{-1} \eta\right) \]
\[\max_{u_m, z_m, \Delta_1m, \Delta_2m} \Delta_2m(0, u_0, z_0) \leq \left(\frac{d_1}{d_2}, A^{m}(E - \Lambda)^{-1} \eta\right) \]

Then (1.1) and (1.2) has a periodic solution \((u(t, u_0, z_0), x(t, u_0, z_0))\) such that

\[u \in L_1 = \left[a + \frac{T^a}{2^{\gamma-1} \Gamma(\alpha + 1)} M_{\beta u} + M_4, b - \frac{T^a}{2^{\alpha-1} \Gamma(\alpha + 1)} M_{\beta u} + M_4 \right] \]
\[z \in L_2 = \left[c + \frac{T^y}{2^{\gamma-1} \Gamma(\gamma + 1)} M_{\beta y} + M_5, d - \frac{T^y}{2^{\alpha-1} \Gamma(\gamma + 1)} M_{\beta y} + M_5 \right] \]

**Proof.** Let \(u_1, u_2\) and \(z_1, z_2\) be any points belonging to the intervals \(I_1\) and \(I_2\) respectively, such that

\[\Delta_1m(0, u_1, z_1) = \min_{u_m, z_m, \Delta_1m, \Delta_2m} \Delta_1m(0, u_0, z_0) \]
\[\Delta_2m(0, u_2, z_2) = \max_{u_m, z_m, \Delta_1m, \Delta_2m} \Delta_1m(0, u_0, z_0) \]
\[ \Delta m(0, u_1, z_1) = \min_{u_1 \in U, z_1 \in Z} \Delta m(0, u_0, z_0) \]
\[ \Delta m(0, u_2, z_2) = \max_{u_2 \in U, z_2 \in Z} \Delta m(0, u_0, z_0) \]

By using inequalities (4.23) to (4.24), the following are obtained

\[ \Delta_2(0, u_2, z_2) = \Delta m(0, u_2, z_2) + (\Delta(0, u_2, z_2) - \Delta m(0, u_2, z_2)) > 0 \]
\[ \Delta_2(0, u_1, z_1) = \Delta m(0, u_1, z_1) + (\Delta(0, u_1, z_1) - \Delta m(0, u_1, z_1)) < 0 \]
\[ \Delta_2(0, u_0, z_0) = \Delta m(0, u_0, z_0) + (\Delta(0, u_0, z_0) - \Delta m(0, u_0, z_0)) > 0 \]

By using these inequalities and the continuity of the functions \( \Delta_1(0, u_0, z_0) \) and \( \Delta_2(0, u_0, z_0) \), we can find that \( \Delta_1(0, x_0, y_0) = 0 \) and \( \Delta_2(0, x_0, y_0) = 0 \). This means that (1.1) has a periodic solution \( z(t, u_0, z_0) \).

**Remark 4.5.** Theorem 4.3 is provided when \( R^n = R^1 \), i.e. \( u_0 \) and \( z_0 \) are scalar singular points and should be isolated. For more details, see (Samoilenko and Ronto, 1976).

### 4.4 Stability of Periodic Solution of (1.1) and (1.2)

**Theorem 4.6.** Let the vector functions \( \Delta_1(0, u_0, z_0) \) and \( \Delta_2(0, u_0, z_0) \) be defined by the inequalities (4.18) and (4.19), where \( u(t, u_0, z_0) \) is a limit of the sequence of the function (4.1). The function \( z(t, u_0, z_0) \) is the limit of the sequence of the function (4.2), then the following inequalities hold:

\[ \| \Delta_1(0, u_0, z_0) \| \leq \frac{1}{\Gamma(\alpha + 1)} M_{\beta_0} M_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_0} M_2 S_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_0} M_3 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_0} M_4 S_3 \]

\[ \| \Delta_2(0, u_0, z_0) \| \leq \frac{1}{\Gamma(\gamma)} M_{\beta_0} K_1 + \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} M_{\beta_0} K_2 S_1 + \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} M_{\beta_0} K_3 S_2 \]

where

\[ E_1 = \frac{1}{\Gamma(\alpha)} M_{\beta_0} (K_1 + K_2 S_1 + 2M_{\beta_0} S_1 K_3) \]
\[ E_2 = \frac{1}{\Gamma(\gamma)} M_{\beta_0} K_2 S_1 \]
\[ E_3 = \frac{1}{\Gamma(\gamma)} M_{\beta_0} L_2 + \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} M_{\beta_0} L_1 S_2 + \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} M_{\beta_0} L_3 S_3 \]
\[ E_4 = \frac{1}{\Gamma(\gamma)} M_{\beta_0} (L_1 + L_2 + 2M_{\beta_0} S_2 L_3) \]
\[ E_5 = \frac{1}{\Gamma(\gamma)} M_{\beta_0} (L_1 + L_2 + 2M_{\beta_0} S_2 L_3) \]

By rewriting (3.32) and (3.34) by vector form, we obtain (3.31). From inequality (4.18), we get

\[ \| \Delta_1(0, u_0, z_0) \| \leq \frac{1}{\Gamma(\alpha)} M_{\beta_0} (K_1 + K_2 S_1 + 2M_{\beta_0} S_1 K_3) \]
\[ \| \Delta_2(0, u_0, z_0) \| \leq \frac{1}{\Gamma(\gamma)} M_{\beta_0} K_2 S_1 + \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} M_{\beta_0} K_3 S_2 \]

where the functions \( u(t, u_0, z_0) \), \( u(t, u_0, z_0) \), and \( z(t, u_0, z_0) \) are solutions of the equation:

\[ u(t, u_0, z_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, \beta(t, \alpha), w(s, u_0, z_0), K(s, \xi) (u(\xi, u_0, z_0) - z(\xi, u_0, z_0))) \, ds \]

\[ f(s, \beta(t, \alpha), u(s, u_0, z_0), K(s, \xi) (u(\xi, u_0, z_0) - z(\xi, u_0, z_0))) \, ds \]

\[ - \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} \, ds \]

and

\[ z(t, u_0, z_0) = z_0 + \frac{1}{\Gamma(\gamma)} \int_0^T (t - s)^{\gamma - 1} \, ds \]
Consider the following system of fractional integro-differential equation.

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Remark 4.7. (Mitropol’sky and Martynyuk, 1979). Theorem 4.6 confirms the stability of the solution of (1.1) and (1.2), that is when a few change happens in the points \(u_0, z_0\), then a few change will happen in the functions \(\Delta_1(0, u_0, z_0)\) and \(\Delta_2(0, u_0, z_0)\).

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