L-SPACE SURGERIES ON LINKS

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Abstract. An L-space link is a link in $S^3$ on which all large surgeries are L-spaces. In this paper, we initiate a general study of the definitions, properties, and examples of L-space links. In particular, we find many hyperbolic L-space links, including some chain links and two-bridge links; from them, we obtain many hyperbolic L-spaces by integral surgeries, including the Weeks manifold. We give bounds on the ranks of the link Floer homology of L-space links and on the coefficients in the multi-variable Alexander polynomials. We also describe the Floer homology of surgeries on any L-space link using the link surgery formula of Ozsváth and Manolescu. As applications, we compute the graded Heegaard Floer homology of surgeries on 2-component L-space links in terms of only the Alexander polynomial and the surgery framing, and give a fast algorithm to classify L-space surgeries among them.

1. Introduction

1.1. Background on L-spaces. Heegaard Floer homology is a package of invariants for 3-manifolds and links introduced by Ozsváth and Szabó in [32]. It has many applications to topological questions. See [29, 35, 23, 27, 37, 40, 24]. An L-space, which was introduced by Ozsváth and Szabó, is a rational homology sphere with the simplest Heegaard Floer homology. In this paper, for simplicity, we work in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, and then we use the following definition:

Definition 1.1 (Z/2Z-L-space). A 3-manifold $M$ is called an L-space, if it is a rational homology sphere and $\dim_\mathbb{F}(\check{H}_F(M)) = |H_1(M)|$.

There are many examples of L-spaces, such as all 3-manifolds with elliptic geometry and double branched covers over quasi-alternating links. There is a conjecture of Boyer-Gordon-Watson from [1] relating L-spaces with left-orderable of the fundamental group.

Another important property of L-spaces is that they do not admit any taut foliations; see Theorem 1.4 from [29]. Note that in the proof of Theorem 1.4 of [29], it is also pointed out that any $\mathbb{Z}/p\mathbb{Z}$-L-space does not admit a taut foliation for all prime numbers $p$. Thus, any hyperbolic L-space will provide an example of hyperbolic manifold admitting no taut foliations. Closed hyperbolic manifolds admitting no taut foliations were first found in [38] and [3] by considering their fundamental groups.

In [33], L-space knots are introduced by Oszváth and Szabó, in order to study the Berge conjecture on lens space surgeries on knots in $S^3$. There are related results towards the Berge conjecture; see [9, 10].

Definition 1.2 (L-space knot). A knot $K \subset S^3$ is called an L-space knot, if there is a positive integer $n$, such that the $n$-surgery on $K$ is an L-space.

Since every 3-manifold is a surgery on a link in $S^3$, we study what links produce L-spaces by surgeries. In this paper, we focus on a class of these links, which are called L-space links. These links are also natural generalizations of L-space knots. The terminology of L-space links was introduced by Gorsky and Némethi in [8] to study algebraic links. Actually, Oszváth, Stipsicz and Szabó have shown that all plumbing trees are L-space links in [26]. In this paper, we give many examples of hyperbolic L-space links including some families of two-bridge links and chain links. In turn, these hyperbolic L-space links provide many examples of hyperbolic L-spaces, including the Weeks manifold.
manifold; see Section 3. All of these hyperbolic L-spaces are derived from elliptic L-spaces, by using the surgery exact triangle of Floer homology. It turns out that L-space links are rich in geometry and simple in algebra.

Here, all the links are oriented links in $S^3$, and all Floer complexes are of the completed version, meaning over the completion $\mathbb{F}[[U]]$.

1.2. L-space knots. Examples and properties of L-space knots have been extensively studied in the literature. We list some of them here.

Example 1.3. Examples of L-space knots include lens space knots such as Berge knots (up to mirror), algebraic knots (which are torus knots and their cables), and $(-2, 3, q)$ pretzel knots with $q > 1$ odd (which are hyperbolic). See [33, 11, 15, 13].

Fact 1.4. In [37], it is shown that $K$ is an L-space knot if and only if all large surgeries on $K$ are L-spaces (if and only if there is a positive rational L-space surgery on $K$).

Fact 1.5 ([23]). If $K$ is an alternating L-space knot, then $K$ is a $T(2, 2n + 1)$ torus knot.

Fact 1.6 ([23]). An L-space knot is a fibered knot.

Fact 1.7 ([23]). Let $K$ be an L-space knot. The knot Floer homology $\widehat{HFK}(K)$ is determined by the Alexander polynomial of $K$, and $\text{rank}(\widehat{HFK}(K, s)) \leq 1, \forall s \in \mathbb{Z}$.

In fact, it turns out that none of the above properties extends to L-space links immediately.

1.3. L-space links. In [8], Gorsky and Némethi define L-space links in terms of large surgeries.

Definition 1.8 (L-space link). An $l$-component link $L \subset S^3$ is called an L-space link, if all of its positive large surgeries are L-spaces, that is, there exist integers $p_1, \ldots, p_l$, such that $S^3_{n_1, \ldots, n_l}(L)$ is an L-space for all $n_1, \ldots, n_l$ with $n_i > p_i, \forall 1 \leq i \leq l$. Note that whether $L$ is an L-space link does not depend on the orientation of $L$. A link $L$ is called a non-L-space link, if neither $L$ nor its mirror is an L-space link.

The large surgeries on the link $L$ are governed by the generalized Floer complexes $\mathfrak{A}_s^-(L)$’s with $s \in \mathbb{H}(L)$, which were introduced by Manolescu and Ozsváth in [20]. Here, $\mathbb{H}(L)$ is defined below. Also, see Definition 2.1 for the generalized Floer complexes.

Definition 1.9 ($\mathbb{H}(L)$). For an oriented link $L$ with $l$ components, we define $\mathbb{H}(L)$ to be the affine lattice over $\mathbb{Z}^l$,

$$
\mathbb{H}(L) = \bigoplus_{i=1}^l \mathbb{H}(L)_i, \quad \mathbb{H}(L)_i = \mathbb{Z} + \frac{\text{lk}(L_i, L - L_i)}{2}.
$$

Based on the knowledge of $\mathfrak{A}_s^-(L)$, we have the following necessary condition on L-space links.

Lemma 1.10. If $L$ is an L-space link, then all sublinks of $L$ are L-space links.

We also formulate L-space links in three other equivalent ways, which are easy to use. To this end, we study the relation between L-space surgeries and large surgeries on links. Using the L-space surgery induction lemma (Lemma 2.5) and the generalized Floer complexes, we give the following result.

Proposition 1.11. The following conditions are equivalent:

(i) $L$ is an L-space link;

(ii) there exists a surgery framing $\Lambda(p_1, \ldots, p_l)$, such that for all sublink $L' \subseteq L$, $\det(\Lambda(p_1, \ldots, p_l)|_{L'}) > 0$ and $S^3_{\Lambda(p_1, \ldots, p_l)}(L')$ is an L-space;

(iii) $H_s(\mathfrak{A}_s^-(L)) = \mathbb{F}[[U]], \forall s \in \mathbb{H}(L)$;
(iv) \( H_s(\hat{\mathfrak{A}}_s(L)) = \mathbb{F}, \forall s \in \mathbb{H}(L). \)

Using grid diagrams as in \[21\], one can compute \( \hat{\mathfrak{A}}_s \) combinatorially and check condition (iii) or (iv). On the other hand, for special class of links, it is more convenient to use condition (ii). For instance, it follows immediately that an algebraically split link is an \( L \)-space link if and only if it admits a positive surgery \( \Lambda \) such that the surgeries restricted to sublinks are all \( L \)-spaces. Note that if we work with \( \mathbb{Z} \) coefficients, conditions (i) and (ii) are also equivalent.

In contrast to Fact 1.4, a single \( L \)-space surgery (with positive surgery coefficients) on \( L \) fails to imply that all the large surgeries on \( L \) are \( L \)-spaces. See Example 2.3. It leads us to define \textit{weak} \( L \)-space links.

**Definition 1.12 (Weak \( L \)-space link).** A link \( L \) is called a \textit{weak} \( L \)-space link, if there exists an \( L \)-space surgery on \( L \).

There are generalizations of \( L \)-space links, called \textit{generalized} \((\pm \cdots \pm)\)-\( L \)-space links, by considering the corresponding types of generalized large surgeries. There are also parallel theories of \( \hat{\mathfrak{A}}_s \) for generalized large surgeries and the link surgery formula. See Section 2. An \( L \)-space link is literally a \( \text{(+· · · +)}\)-\( L \)-space link. Note that there are generalized \( (-\cdots-\text{)}\)-\( L \)-space links that are \textit{non-}\( L \)-space links.

**Example 1.13.** We have the following examples of \( L \)-space links and generalized \( L \)-space links.

(A) Split disjoint unions of \( L \)-space knots are \( L \)-space links.
(B) Two-bridge links \( b(rq - 1, -q) \) with \( r, q \) being positive odd integers are all \( L \)-space links, which include \( T(2, 2n) \) torus links. See Theorem 3.10. Note that except for \( T(2, 2n) \), they are all hyperbolic links.
(C) A 2-component \( L \)-space link: \( L7n1 \) in the Thistlethwaite link table. See Example 3.19.
(D) Some 3-component \( L \)-space links: Borromean rings, \( L6a5 \), \( L6n1 \), \( L7a7 \) and a link in Example 3.19.
(E) A hyperbolic 4-chain \( L \)-space link: See Example 3.14.
(F) A hyperbolic 5-chain generalized \( (+\cdots+)\)-\( L \)-space link: See Example 3.15.
(G) Two families of hyperbolic \( L \)-space chain links: See Example 3.16 and Example 3.17.
(H) A sequence of plumbing graphs that are generalized \( L \)-space links: See Example 3.18.
(I) All plumbing trees of unknots are \( L \)-space links. This was proved by Ozsváth and Szabó in \[23\]. See Example 3.12.
(J) All algebraic links are \( L \)-space links. This was proved by Gorsky and Némethi in \[8\].
(K) See Table 3.2 for the list of which links with crossing number \( \leq 7 \) are \( L \)-space links.

In contrast to Fact 1.5, there are alternating hyperbolic \( L \)-space links. For example, all two-bridge links \( b(rq - 1, -q) \) with \( r, q > 1 \) being positive odd integers. Surgeries on these hyperbolic \( L \)-space links can give examples of hyperbolic \( L \)-spaces which are neither surgery nor double branched cover over any knot. See Example 3.1. In fact, surgeries on these \( L \)-space two-bridge links are always double branched covers over some links. It is not clear to us whether those links are quasi-alternating or not.

In relation to Example 1.13 (B), we make the following conjecture:

**Conjecture 1.14.** The set of all \( L \)-space two-bridge links is

\[ \{b(rq - 1, -q) : r, q \text{ are positive odd integers}\}. \]

Using the algorithm from \[18\] for computing \( \hat{\mathfrak{A}}_s(L) \) for two-bridge links, we verify that Conjecture 1.14 is true for all two-bridge links \( b(p, q) \) with \( p \leq 100 \).

Compared with Fact 1.7, we study the Alexander polynomials of \( L \)-space links using \( \hat{\mathfrak{A}}_s(L) \).
Theorem 1.15. Suppose $L$ is an $l$-component $L$-space link with $l \geq 2$, and has the multi-variable Alexander polynomial as follows

$$\Delta_L(x_1, \ldots, x_l) = \sum_{i_1, \ldots, i_l} a_{i_1, \ldots, i_l} \cdot x_1^{i_1} \cdots x_l^{i_l}.$$ 

Then,

\begin{align*}
\text{(1.1)} & \quad \text{rank}_F(HFL^-(L,s)) \leq 2^{l-1}, \forall s \in H(L), \\
\text{(1.2)} & \quad -2^{l-2} \leq a_{i_1, \ldots, i_l} \leq 2^{l-2}, \forall i_1, \ldots, i_l.
\end{align*}

In particular, for a 2-component $L$-space link, the multi-variable Alexander polynomial has non-zero coefficients $\pm 1$. Moreover, fixing $i_1$, the signs of non-zero $a_{i_1, i_2}$'s are alternating; similarly fixing $i_2$, the signs of non-zero $a_{i_1, i_2}$'s are alternating.

Remark 1.16. Inequality (1.1) is sharp for $l = 2$. For example, for the Whitehead link $Wh$, $HFL^-(Wh, 0, 1)$ equals to $F[[U]]/U \oplus F[[U]]/U$ as an $F[[U]]$-module. Inequality (1.1) can also be implied by a spectral sequence of Gorsky and Némethi from [8].

Inequality (1.2) is sharp for $l = 3$. The mirror of $L7a7$ is an $L$-space link with Alexander polynomial

$$\Delta_{L7a7}(u, v, w) = u vw - uw - uw + 2u - 2vw + v + w - 1 \sqrt{uvw}.$$ 

In contrast to knots, the Alexander polynomial condition does not give strong constraints for alternating links. In [33], it is shown that if $K$ is an alternating knot with Alexander polynomial satisfying the condition in Fact 1.7 then $K$ is a $T(2, 2n + 1)$ torus knot; see Proposition 4.2 and Theorem 4.3. On the other hand, we find infinitely many hyperbolic alternating links with multi-variable Alexander polynomial satisfying Inequality (1.2), including $L$-space links and non-$L$-space links. See Section 4.2.

Theorem 1.15 also implies that a Floer homologically thin $L$-space 2-component link $L$ has fibered link exterior.

In contrast to Fact 1.6, there are non-fibered $L$-space links. For example, the split disjoint union of two $L$-space knots is a non-fibered $L$-space link, since the complement is not irreducible any more. In fact, there are also many non-fibered $L$-space links among hyperbolic two-bridge links. See Example 3.11.

Actually, there are additional constraints on the Alexander polynomials of an $L$-space link; see Theorem 5.10 and Theorem 5.12 below for the precise statements. As a consequence, either of these theorems implies that $L7a7$ is not an $L$-space link, while Theorem 1.15 fails to do so.

1.4. Surgeries on $L$-space links. In [33], it is shown that for an $L$-space knot $K$, $\widehat{HFK}(K)$ is determined by its Alexander polynomial. In [12], Hom pointed out that the result from [33] further implies that the whole package of Heegaard Floer homology of surgeries on $K$ is determined by the Alexander polynomial of $K$ and the surgery coefficients.

In this paper, we study the Heegaard Floer invariants of integral surgeries on an $L$-space link $L$, including the completed Heegaard Floer homology $\text{HF}^-$, absolute gradings, and the cobordism maps, using the link surgery formula of Manolescu-Ozsváth from [20]. The M-O surgery complex is an object in the category of chain complexes of $F[[U]]$-modules, while it can also be considered as an object in the homotopy category of chain complexes of $F[[U]]$-modules. In [18], this is called a perturbed surgery complex. Since here the complexes are only $\mathbb{Z}/2\mathbb{Z}$-graded considered as $F[[U]]$-modules, standard results in homological algebra regarding bounded below chain complexes do not directly apply. Some algebraic rigidity results are established in [18], which imply that $\text{HFL}^-_s(L)$ is chain homotopic to $F[[U]]$ by a $F[[U]]$-linear chain map preserving the $\mathbb{Z}$-grading.
Thus, for an $L$-space link $L$, the perturbed surgery complex turns out to be largely simplified. When $L$ has 1 or 2 components, all the information needed in the perturbed surgery complex is completely determined by the Alexander polynomial and the surgery framing matrix.

**Theorem 1.17.** For a 2-component $L$-space link $\overline{L} = L_1 \cup L_2$, all Heegaard Floer homology $HF^-(S^3_\Lambda(L))$ together with the absolute gradings on them are determined by the following set of data:

- the multi-variable Alexander polynomial $\Delta_L(x,y)$,
- the Alexander polynomials $\Delta_{L_1}(t)$ and $\Delta_{L_2}(t)$,
- the framing matrix $\Lambda$.

**Remark 1.18.** For $L$-space links with more components, besides the Alexander polynomials more information are needed to determine whether the higher diagonal maps vanish or not.

Furthermore, we explicitly describe $HF^-$ of surgeries on an $L$-space link $L = L_1 \cup L_2$ by a series of formulas in terms of the Alexander polynomials and the surgery framing matrix. These formulas give a fast algorithm computing $HF^-$ of these surgeries. We also give a fast algorithm for classifying $L$-space surgeries. As an application, we study the classification of $L$-space surgeries on two-bridge links, and compute some examples explicitly: $(1,1)$-surgery on a family of $L$-space links with linking number zero, $L_n = b(4n^2 + 4n, -2n - 1)$.

Instead of classifying $L$-space links with more than 2 components, we contend to show the prevalence of surgeries on $L$-space links among 3-manifolds:

**Question 1.19.** Is every 3-manifold a surgery on a (generalized) $L$-space link?

If Question [1.19] had a positive answer, one could hope to compute Heegaard Floer homology by $L$-space links. As a matter of fact, every 3-manifold $M$ can be realized by a surgery on an algebraically split link after connect sum with several lens spaces; see Corollary 2.5 from [25]. It is also interesting to ask whether this algebraically split link can be chosen to be a generalized $L$-space link.

Regarding $L$-space surgeries, there is a more basic question:

**Question 1.20.** Is every $L$-space a surgery on a (generalized) $L$-space link?

1.5. **Organization and conventions.** This paper is organized as follows. In Section 2, we discuss the properties of $L$-space links and generalized $L$-space links. In Section 3, we present examples of $L$-space links and contrast them with $L$-space knots. Section 4 consists of the proof of Theorem [1.15] and related discussions on fiberedness of $L$-space links. In Section 5, we prove Theorem [1.17] in Section 6, we give the algorithm for computing $HF^-$ of surgeries on 2-component $L$-space links and compute some examples.

Since $L$-space links are sensitive to mirrors and the generalized Floer complexes are defined for oriented links, we describe our conventions about oriented two-bridge links $b(p,q)$ and oriented torus links $T(2,2n)$ in Section 3. In addition, the Floer complex $CF^-(S^3)$ is absolutely $\mathbb{Z}$-graded, and the top grading is 0. This convention is needed to compute the $d$-invariants from link surgery formula using minus version Floer complexes.

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2. L-space links

In this section, let us study the relation between L-space surgeries on a link L and its large surgeries. Then, we introduce various notions of L-space links.

2.1. L-space links. Let us recall the definition of generalized Floer complexes of a link L in $S^3$ in [20] Section 4, which govern the large surgeries on L. For simplicity, we only consider generic admissible multi-pointed Heegaard diagrams with each component $L_i$ having only two basepoints $w_i, z_i$.

**Definition 2.1** (Generalized Floer complexes). Let L be a link in $S^3$ and choose a Heegaard diagram $\mathcal{H}$ as above. For $s \in \mathbb{H}(L)$, the generalized Floer complex $\mathfrak{A}^-(\mathcal{H}, s)$ is the free module over $\mathcal{R} = \mathbb{F}[\{U_1, \ldots, U_l\}]$ generated by $\mathcal{T}_\alpha \cap \mathcal{T}_\beta \in \text{Sym}^{g+k-1}(\Sigma)$, and equipped with the differential:

\[
\partial_s \mathfrak{x} = \sum_{y \in \mathcal{T}(\alpha) \cap \mathcal{T}(\beta)} \sum_{\phi \in \pi_2(x, y)} \frac{\#(M(\phi) / \mathcal{R})}{\mu(\phi) = 1} \cdot U_1^{E^1_1(\phi)} \cdots U_l^{E^l_1(\phi)} \cdot U_{l+1}^{n_{w_1}(\phi)} \cdots U_{k}^{n_{w_k}(\phi)} \cdot y,
\]

where $E^i_1(\phi)$ is defined by

\[
E^i_1(\phi) = \max\{s - A_i(x), 0\} - \max\{s - A_i(y), 0\} + n_{z_i}(\phi)
\]

\[
E^i_1(\phi) = \max\{A_i(x) - s, 0\} - \max\{A_i(y) - s, 0\} + n_{w_i}(\phi).
\]

The stable quasi-isomorphism type of $\mathfrak{A}^-(\mathcal{H}, s)$ is an invariant of L. For simplicity, we also write $\mathfrak{A}^-(L, s), \mathfrak{A}^-(L)$, or $\mathfrak{A}^-(L)$, when the context is clear.

**Notation 2.2.** Let L be an l-component link in $S^3$. In order to simplify the notation, we denote the $(p_1, \ldots, p_l)$-surgery on L by $S^3_{p_1, \ldots, p_l}(L)$ and the surgery framing matrix by $\Lambda(p_1, \ldots, p_l)$, where $p_1, \ldots, p_l$ are surgery coefficients on the link; i.e. $\Lambda(p_1, \ldots, p_l)$ is the matrix with $p_1, \ldots, p_l$ on the diagonal and linking numbers off the diagonal.

Recall that Lemma 1.10 says that every sublink of an L-space link is an L-space link.

**Proof of Lemma 1.10.** By Theorem 10.1 in [20], L is an L-space link if and only if $\mathfrak{A}^-(L)$ has the homology $\mathbb{F}[\{U\}]$ for all $s \in \mathbb{H}(L)$. When the ith component of s, say $s_i$, equals to $+\infty$, there is a destabilization map between $\mathfrak{A}^-(L, s)$ and $\mathfrak{A}^-(L - L_i, s_i + L_i(s))$, which is a quasi-isomorphism. See Example 7.2 in [20]. Roughly, this is because the generalized Floer complexes of $L - L_i$ can be computed from the Heegaard diagram of L by deleting the basepoint $z_i$, which is the same as putting $s_i = +\infty$ in $\mathfrak{A}^-(L, s)$. Thus, $\mathfrak{A}^-(L - L_i, s')$ has homology $\mathbb{F}[\{U\}]$ for all $s' \in \mathbb{H}(L - L_i)$. So $L - L_i$ is an L-space link for $L_i \subset L$. An induction will show that all sublinks are L-space links.

In contrast to knots, a weak L-space link L might be a non-L-space links.

**Example 2.3.** Let $L = L_1 \cup L_2$ be the link consisting of a Figure-8 knot $L_1$ and an unknot $L_2$ as in Figure 2.1. Then by blowing down the unknot, the Figure-8 knot is then unknotted, and thus the surgery $S^3_{-1,1}(L)$ is the lens space $L(n - 4, 1)$, when $n \neq 4$. However, the Figure-8 knot is not an L-space knot. Thus, by Lemma 1.10 L is a weak L-space link but not an L-space link. Similarly, the mirror of L is not a L-space link either.

2.2. L-space induction and generalized large surgeries. In this subsection, we study how to characterize L-space links, by exploiting induction in light of surgery exact triangles.

**Lemma 2.4.** Let L be a 2-component link and $p_1, p_2$ are integers. Suppose $S^3_{p_3}(L_2)$ and $S^3_{p_1, p_2}(L)$ are both L-spaces. Then,

**Case I:** If $(p_1 p_2 - lk^2) \cdot p_2 > 0$, then $S^3_{p_1, p_2}(L)$ is an L-space for all $k \geq 0$. 


Case II: If \((p_1 p_2 - \text{lk}^2) \cdot p_2 < 0\), then \(S^3_{p_1, k, p_2} (L)\) is an L-space for all \(k \leq 0\).

Proof. For the case \((p_1 p_2 - \text{lk}^2) \cdot p_2 > 0\), consider the following exact triangle of surgeries:

\[
\begin{array}{c}
\overset{\text{Case II:}}{\text{ }}
\end{array}
\]

Thus, from the following two conditions

- \(\tilde{HF}(S^3_{p_1, p_2} (L)) = \mathbb{F}^{\pm p_1 - \text{lk}^2}\) and \(\tilde{HF}(S^3_{p_1} (L_1)) = \mathbb{F}^{\pm p_1}\),
- \(|(p_1 + 1)p_2 - \text{lk}^2| = |p_1 p_2 - \text{lk}^2| + |p_2|\),

it follows that \(S^3_{p_1, p_2} (L)\) is an L-space.

Moreover, since \(|(p_1 + k)p_2 - \text{lk}^2| \cdot p_2 = (p_1 p_2 - \text{lk}^2) \cdot p_2 + k \cdot p_2^2 > 0\) for all \(k > 0\), an induction will show that \(S^3_{p_1, k, p_2} (L)\) is an L-space for all \(k \geq 0\). By using the general exact triangle for a triad of 3-manifolds, the case where \((p_1 p_2 - \text{lk}^2) \cdot p_2 < 0\) is similar.

Following the same lines, we can directly generalize this lemma to links with more components.

Lemma 2.5 (L-space surgery induction). Let \(L = L_1 \cup ... \cup L_n\) be a link with \(n\) components, and \(L' = L - L_1\). Let \(\Lambda\) be the framing matrix of \(L\) for the surgery \(S^3_{p_1, ..., p_n} (L)\), and denote by \(\Lambda'\) the restriction of \(\Lambda\) on \(L'\). Suppose \(S^3_{p_1, ..., p_n} (L)\) and \(S^3_{p_1, ..., p_n} (L')\) are both L-spaces. Then,

Case I: if \(\text{det}(\Lambda) \cdot \text{det}(\Lambda') > 0\), then for all \(k > 0\), \(S^3_{p_1 + k, k, p_2, ..., p_n} (L)\) is an L-space;

Case II: if \(\text{det}(\Lambda) \cdot \text{det}(\Lambda') < 0\), then for all \(k > 0\), \(S^3_{p_1 - k, k, p_2, ..., p_n} (L)\) is an L-space.

Proof. Let \(\Lambda_k\) be the framing matrix of the surgery \(S^3_{p_1 + k, k, p_2, ..., p_n} (L)\). Notice that \(\text{det}(\Lambda_k) = \text{det}(\Lambda) + k \text{det}(\Lambda')\). Similar arguments as in Lemma 2.4 apply to the links with more components.

Lemma 2.6 (Positive L-space surgery criterion). An \(l\)-component link \(L\) is an L-space link, if and only if, there exists a surgery framing \(\Lambda(p_1, ..., p_l)\), such that for all sublink \(L' \subseteq L\), \(\text{det}(\Lambda(p_1, ..., p_l)|_{L'}) > 0\) and \(S^3_{\Lambda|_{L'}} (L')\) is an L-space.

In particular, if the surgery framing \(\Lambda(p_1, ..., p_l)\) satisfies the above condition, then for any surgery framing \(\Lambda = \Lambda(n_1, ..., n_l)\) with \(n_i \geq p_i\) for all \(i\), the surgery \(S^3_{\Lambda} (L)\) is an L-space.

Proof. If \(L\) is an L-space link, then every sublink \(L'\) is an L-space link, by Lemma 1.10. Thus, there is a large \((p_1, ..., p_l)\)-surgery on \(L\) such that for all \(L' \subseteq L\), \(\text{det}(\Lambda(p_1, ..., p_l)) > 0\) and \(S^3_{\Lambda|_{L'}} (L')\) is an L-space.
Conversely, let $\Lambda(p_1, \ldots, p_l)$ be the surgery framing satisfying the condition in the proposition. Let $\Lambda' = \Lambda(p_1, \ldots, p_l + 1, \ldots, p_l)$. By the $L$-space surgery induction lemma, we have that for all $L' \subseteq L$, $\mathcal{S}^3_{\Lambda'|L'}(L')$ is an $L$-space and $\det(\Lambda'|L') = \det(\Lambda|L' - L_i) + \varepsilon \det(\Lambda|L' - L_i)$, where $\varepsilon = 1$ if $L_i \subseteq L'$ and $\varepsilon = 0$ if $L_i \not\subseteq L'$. Thus, by induction, we can show that for any surgery framing $\Lambda'' = \Lambda(n_1, \ldots, n_l)$ with $n_i \geq p_i$, the surgery $\mathcal{S}^3_{\Lambda'|L'}(L')$ is an $L$-space for all sublinks $L' \subseteq L$. Particularly, $\mathcal{S}^3_{\Lambda''}(L)$ is an $L$-space, and this finishes the proof.

\textbf{Definition 2.7.} A link is called algebraically split, if all the pairwise linking numbers are 0.

\textbf{Corollary 2.8.} Let $L = L_1 \cup \ldots \cup L_l$ be an algebraically split link. Then $L$ is an $L$-space link if and only if there exists $i$, such that for all positive integers $k_1, k_2 > 0$, $\mathcal{S}^3_{p_1k_1p_2k_2}(L)$ is an $L$-space. Similarly, we define an $l$-component generalized $(\pm \cdots \pm)$-space link.

\textbf{Example 2.10.} The split disjoint union of the left-handed trefoil and the right-handed trefoil is a generalized $(+ -)$-space link. However, it is not an $L$-space link, and neither is its mirror.

Let us look at some examples of $2$-component generalized $L$-space links.

\textbf{Proposition 2.11.} Suppose $L$ is a $2$-component link $L = L_1 \cup L_2$ with $L_1, L_2$ both being the unknots, and $\mathcal{S}^3_{p_1,p_2}(L)$ is an $L$-space. Then,

1. if $p_1p_2 > k_1^2, p_1 > 0, p_2 > 0$, then $\mathcal{S}^3_{p_1k_1p_2k_2}(L)$ are $L$-spaces for all $k_1, k_2 \in \mathbb{N}$;
2. if $p_1p_2 > k_1^2, p_1 < 0, p_2 < 0$, then $\mathcal{S}^3_{p_1k_1, p_2k_2}(L)$ are $L$-spaces for all $k_1, k_2 \in \mathbb{N}$;
3. if $p_1 > 0, p_2 < 0$, then $\mathcal{S}^3_{p_1k_1, p_2k_2}(L)$ are $L$-spaces for all $k_1, k_2 \in \mathbb{N}$;
4. if $p_1p_2 < k_1^2, p_1 > 0, p_2 > 0$, then the surgeries $\mathcal{S}^3_{p_1k_1, p_2k_2}(L)$ with $k_1 \geq 0, k_2 \geq 0$ and $\mathcal{S}^3_{p_1', p_2'}(L)$ with $0 \leq p_1' \leq p_1, 0 \leq p_2' \leq p_2$ are $L$-spaces;
5. if $p_1p_2 < k_1^2, p_1 < 0, p_2 < 0$, then the surgeries $\mathcal{S}^3_{p_1k_1, p_2k_2}(L)$ with $k_1 > 0, k_2 > 0$ and $\mathcal{S}^3_{p_1', p_2'}(L)$ with $0 \leq p_1' \leq p_1, 0 < p_2' \leq p_2$ are $L$-spaces.

\textbf{Proof.} The cases (1), (2), and (3) are proved by induction using the long exact sequences for the surgery triad $(\mathcal{S}^3_{p,q}(L), \mathcal{S}^3_{p+1,q}(L), \mathcal{S}^3_{q}(L_2))$.

For the case (4), first by \textbf{Lemma 2.5}, we have that $\mathcal{S}^3_{p_1,q-1}(L)$, $\mathcal{S}^3_{-1,p_2}(L)$ are both $L$-spaces. From (3), it follows that $\mathcal{S}^3_{p_1k_1, p_2k_2}(L), \mathcal{S}^3_{-1k_1, p_2k_2}(L)$ are all $L$-spaces for all non-negative integers $k_1, k_2$. Second, we can do induction to prove that $\mathcal{S}^3_{p_1', p_2'}(L)$ with $0 < p_1' \leq p_1, 0 < p_2' \leq p_2$ are all $L$-spaces. The case (5) is similar to the case (4).

Proposition 2.11 says that if $L$ is a $2$-component link with unknotted components, then $L$ is a weak $L$-space link if and only if $L$ is a generalized $L$-space link. The following proposition gives another example of generalized $L$-space links.

\textbf{Proposition 2.12.} Let $L$ be an algebraically split link. If there exists a surgery framing $\Lambda(p_1, \ldots, p_l)$ on $L$, such that for all sublink $L' \subseteq L$, $\mathcal{S}^3_{\Lambda|L'}(L')$ is an $L$-space, then $L$ is a generalized $L$-space link of $\epsilon_1 \cdots \epsilon_l$-type, where $\epsilon_i$ is the sign of $p_i$. 
2.4. **Subcomplexes of \( \text{CFL}^\infty(L) \) governing generalized large surgeries.** In this section, we demonstrate that parallel theory of \( \mathfrak{A}_s^- \) can be done by considering generalized large surgeries of different types. Here, we illustrate the idea by using 2-component links.

The generalized Floer complexes \( \mathfrak{A}_s^- (L) \) governs the positive large surgeries on \( L \). In fact, there are also subcomplexes of \( \text{CFL}^\infty(L) \) governing the other types of large surgeries on \( L \) respectively.

For any basic Heegaard diagram of an \( l \)-component link \( L \), there is an Alexander grading on the intersection points \( T_\alpha \cap T_\beta \)

\[
A : T_\alpha \cap T_\beta \to \mathbb{H}(L),
\]

which is characterized by

\[
A_i(x) - A_i(y) = n_{z_i}(\phi) - n_{w_i}(\phi), \quad \forall \phi \in \pi_2(x, y)
\]

and a normalization condition of the Alexander polynomial.

**Definition 2.13 (\( \text{CFL}^\infty \)).** Let \( L \) be an \( l \)-component link and \( \mathcal{H} \) be a basic Heegaard diagram of \( L \). Then, \( \text{CFL}^\infty(\mathcal{H}) \) is a chain complex of \( \mathbb{F}[[U_1, ..., U_l, U_1^{-1}, ..., U_l^{-1}]] \)-modules freely generated by \( x \in T_\alpha \cap T_\beta \) with the differential

\[
\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y) \mid \mu(\phi) = 1} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{n_{z_1}(\phi) - n_{w_1}(\phi)} \cdots U_l^{n_{z_l}(\phi) - n_{w_l}(\phi)} \cdot y,
\]

where the \( U_i \)'s lower the \( \mathbb{Z} \)-grading by 2. There is an Alexander filtration on \( \text{CFL}^\infty(\mathcal{H}) \) extended from \( T_\alpha \cap T_\beta \) by \( A_i(U_k \cdot x) = A_i(x) - \delta_{ij}k \). The filtered chain homotopy type of the filtered complex \( (\text{CFL}^\infty(\mathcal{H}), A) \) is an invariant of \( L \), where \( A \) denotes the Alexander filtration. We denote this filtered homotopy type by \( \text{CFL}^\infty(L) \). By abuse of notation, we also use \( \text{CFL}^\infty(L) \) to denote a chain complex in this class.

To obtain the subcomplexes of \( \text{CFL}^\infty(L) \) governing the large surgeries, let us first recall some facts of knots. Let \( K \) be a knot in \( S^3 \). Following [30], we can also regard \( \text{CFL}^\infty(K) \) as a chain complex \( C \) of \( \mathbb{F} \) vector spaces generated by triples

\[
[x, i, j], x \in T_\alpha \cap T_\beta, i, j \in \mathbb{Z}, \quad \text{with} \quad A(x) = j - i.
\]

The triple \([x, i, j] \) is corresponding to \( U^{-i}x \). Then, Heegaard Floer homology of the positive large surgeries \( HF^-(S^3_p(K)) \) with \( p \gg 0 \) can be computed from the subcomplexes \( C\{\max(i, j - m) \leq s\}'s \), whereas, Heegaard Floer homology of the negative large surgeries \( S^3_p(K) \) with \( p \ll 0 \) can be computed from the subcomplexes \( C\{\min(i, j - m) \leq s\}'s \). See Theorem 4.1 and Theorem 4.4 in [30] or Section 2.2 in [27]. Thus, \( C\{\min(i, j - s) \leq 0\} \) is corresponded to the subcomplex of \( \text{CFL}^\infty(K) \):

\[
\mathfrak{A}^-_s(K) = \{U^kx \mid \min(-k, A(x) - k - s) \leq 0\}.
\]

One can also formulate this complex more explicitly by using a similar approach in [20]. For simplicity, we consider basic Heegaard diagram without free basepoints, i.e. a doubly-pointed Heegaard diagram of a knot \( K \).

**Definition 2.14 (\( \mathfrak{A}^-_s(K) \)).** Let \( \mathcal{H} \) be a doubly-pointed Heegaard diagram of \( K \). For any \( s \in \mathbb{Z} \), the complex \( \mathfrak{A}^-_s(\mathcal{H}, s) \) is the free module over \( \mathcal{R} = \mathbb{F}[[U]] \) generated by \( T_\alpha \cap T_\beta \), and equipped with the differential:

\[
\partial^-_s x = \sum_{y \in T(\alpha) \cap T(\beta)} \sum_{\phi \in \pi_2(x, y)} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^E_s(\phi) \cdot y,
\]

where \( E_s(\phi) \) is defined by

\[
E_s(\phi) = \max(0, s - A(y)) - \max(0, s - A(x)) + n_w(\phi), \quad \forall \phi \in \pi_2(x, y).
\]

The chain homotopy type of \( \mathfrak{A}^-_s(\mathcal{H}, s) \) is an invariant of \( K \) and \( s \), and we denote it by \( \mathfrak{A}^-_s(K) \).
Now we can pass from knots to links.

**Definition 2.15** ($\pm \mathfrak{A}_{s_1,s_2}$). Let $L$ be a 2-component link, and $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ be a basic Heegaard diagram of $L$. Then we define the following subcomplexes of the Alexander filtered complex $\text{CFL}^\infty(L)$

- $\pm \mathfrak{A}_{s_1,s_2} := \{ (1^k U_2 x) \in \text{CFL}^\infty(L) : x \in T_{\alpha} \cap T_{\beta}, \max(0, -k_i, A_i(x) - k_i - s_i) \leq 0, \forall i = 1, 2 \}$;
- $\pm \mathfrak{A}_{s_1,s_2} := \{ (1^k U_2 x) \in \text{CFL}^\infty(L) : x \in T_{\alpha} \cap T_{\beta}, \max(0, -k_i, A_i(x) - k_i - s_i) \leq 0, \min(-k_i, A_i(x) - k_i - s_i) \leq 0, \forall i = 1, 2 \}$;
- $\pm \mathfrak{A}_{s_1,s_2} := \{ (1^k U_2 x) \in \text{CFL}^\infty(L) : x \in T_{\alpha} \cap T_{\beta}, \min(-k_i, A_i(x) - k_i - s_i) \leq 0, \max(-k_i, A_i(x) - k_i - s_i) \leq 0, \forall i = 1, 2 \}$.

Thus, $\pm \mathfrak{A}_s(L)$ is isomorphic to $\mathfrak{A}_s(L)$ defined in [20]. One can also formulate these complexes by using $E_s'(\phi)$'s and $E_s''(\phi)$'s. In fact, these four sets of complexes are equivalent data of $L$. The advantage of considering $\mathfrak{A}_s(L)$'s is that they can be identified with the subcomplexes of $\text{CFL}^- (L)$ which form the Alexander filtration of $\text{CFL}^- (L)$.

**Remark 2.16.** The original $\text{CFK}^\infty(K)$ defined in [30] is slightly different from the formulation of $\text{CFL}^\infty(K)$ here. We adopt the formulation of $\text{CFK}^\infty(K)$ in [19], and it is the same as $\text{CFL}^\infty(K)$ here.

3. **Examples of $L$-space links and generalized $L$-space links**

In this section, we use the lemmas and propositions in Section 2 to show some examples of $L$-space links and generalized $L$-space links.

**Example 3.1** (Two hyperbolic links: the Whitehead link and the Borromean rings). The Whitehead link and the Borromean rings are two well-known hyperbolic links. In fact, they are both $L$-space links.

The (1,1)-surgery on the Whitehead link is the Poincaré sphere. See Example 8 on Page 263 in [39]. The (1,1)-surgery on the Borromean rings is also the Poincaré sphere. See Exercise 4 on Page 269 in [39]. By Corollary 2.8 they are both $L$-space links.

**Remark 3.2.** There are no alternating hyperbolic $L$-space knots. See Theorem 1.3 below cited from [33]. However, Example 3.1 shows that there are $L$-space alternating hyperbolic links. In fact, there are many, see Theorem 3.10.

Moreover, these hyperbolic links provide many examples of hyperbolic $L$-spaces which are neither surgery over knots nor double branched cover over knots. For example, surgeries on the Whitehead link $S^3_{n,2n}(\text{Wh})$ with $n > 0$ are all $L$-spaces but not surgeries nor double branched cover on a knot. The reason is the first homology of these surgeries is neither cyclic nor of odd order.

**Example 3.3** (An $L$-space link providing the Weeks manifold). Consider the link $L = L_1 \cup L_2 \cup L_3$ in Figure 3.2 where $L_1 \cup L_2$ is the Whitehead link (using the convention in [39]) and $L_3$ is the meridian of $L_3$. The (1,2,1)-surgery is the Poincaré sphere, and it satisfies the positive $L$-space surgery criterion. Thus, it is an $L$-space link.

By Lemma 2.6, we have that for any $n_1 \geq 1$, $n_2 \geq 2$, $n_3 \geq 1$, the surgery $(n_1, n_2, n_3)$-surgery on $L$ is an $L$-space. Thus, the (5,3,2)-surgery is an $L$-space, which is the (5,5/2)-surgery on the Whitehead link. This surgery is the Weeks manifold; see [3]. The Weeks manifold has the smallest hyperbolic volume among closed hyperbolic 3-manifolds; see [6]. Thus, we confirm that the Weeks manifold does not admit a taut foliation.

The fact that the Weeks manifold is an $L$-space was already known by experts such as [14] and [4].
Figure 3.1. **The Borromean ring.** The (1,1,1)-surgery on the Borromean link is the Poincaré sphere.

Figure 3.2. **An L-space link giving the Weeks manifold.**

**Example 3.4** \((T(2, 2n)\) torus links). The oriented torus links \(T(2, 2n)\) are \(L\)-space links as Corollary 3.6 below shows. We need to distinguish them from their mirrors, so see Figure 3.3 for the precise definitions of \(T(2, 2n)\).

**Lemma 3.5.** For the torus links \(T(2, 2n)\), we have the following identifications of surgeries

\[
S_{n+1,n-1}(T(2, 2n)) = S^3, \quad S_{n+1,n+1}(T(2, 2n)) = L(2n + 1, 2), \quad S_{n,n+1}(T(2, 2n)) = L(n, 1).
\]

**Proof.** First, for the \((n + 1, n - 1)\)-surgery on \(T(2, 2n)\), we consider a surgery on the upper-left link \(L\) in Figure 3.4, where \(L\) is a plumbing of unknots. After two different blowing-down procedures, we get the identification of \(S_{n+1,n-1}(T(2, 2n))\) with \(S^3\).

Second, for the \((n + 1, n + 1)\)-surgery on \(T(2, 2n)\), we similarly consider a different surgery on \(L\), which is drawn in Figure 3.4. After two different processes of doing Rolfsen twists, we can obtain the identification of \(S_{n+1,n+1}(T(2, 2n))\) with \(L(2n + 1, 2)\). See Figure 3.4. As is similar to the \((n + 1, n + 1)\)-surgery, the \((n, n + 1)\)-surgery is \(L(n, 1)\). \(\square\)

**Corollary 3.6.** The following surgeries on the torus link \(T(2, 2n)\) are all \(L\)-spaces:

\[
\begin{align*}
&\quad S_{n+1+k_1,n+1+k_2}(T(2, 2n)), \quad \forall k_1 \geq 0, \forall k_2 \geq 0, \\
&\quad S_{n+1-k_1,n-1}(T(2, 2n)), \quad \forall k_1 \geq 0, \\
&\quad S_{n-k_1,n-1+k_2}(T(2, 2n)), \quad \forall k_1 \geq 0, \forall k_2 \geq 0, \\
&\quad S_{n,q}(T(2, 2n)) \quad \text{with} \quad q \neq n.
\end{align*}
\]
Figure 3.3. The \((n+1, n-1)\)-surgery on \(T(2, 2n)\). Consider the surgery on the upper-left link \(L\), which is a plumbing of unknots. By blowing down the horizontal unknots \(H_i\)'s, we get the surgery on the lower-left link \(T(2, 2n)\). While blowing down the black unknots \(V_j\)'s, we can get the surgery on the lower-right link, which is \(S^3\).

**Proof.** We combine Proposition 2.11 and Lemma 3.5.

From \(S^3_{n+1,n+1}(T(2, 2n)) = L(2n+1, 2)\), it follows that \(S^3_{n+1+k_1,n+1+k_2}(T(2, 2n))\) are all \(L\)-spaces for \(k_1, k_2 \geq 0\).

From \(S^3_{n+1,n-1}(T(2, 2n)) = S^3\), it follows that \(S^3_{n+1-k_1,n-1}(T(2, 2n))\) are all \(L\)-spaces by Lemma 2.5. Thus, \((-1, n-1)\)-surgery is an \(L\)-space, and so is any \(S^3_{n-k_1,n-1+k_2}(T(2, 2n))\) with \(k_1, k_2 \geq 0\).

From \(S^3_{n+1,n-1}(T(2, 2n)) = S^3\), it follows that \((n, n-1)\)-surgery is an \(L\)-space and thus all \((n, q)\)-surgeries with \(q \leq n-1\) are \(L\)-spaces.

From \(S^3_{n,n+1}(T(2, 2n)) = L(n, 1)\), it follows that all \((n, q)\)-surgery with \(q \geq n+1\) are \(L\)-spaces. \(\square\)

**Example 3.7** (Algebraic links). Gorsky and Némethi showed in [8] that every algebraic link is an \(L\)-space link. An algebraic knot can be obtained by iterated cabling of the unknot. In [11], Hedden proved that algebraic knots are \(L\)-space knots. Note that algebraic knots include all torus knots.

**Proposition 3.8** (A sequence of hyperbolic two-bridge \(L\)-space links). The family of two-bridge links \(b(6n + 2, -3)\) drawn in Figure 3.5 are all \(L\)-space links.
The $(n+1, n+1)$-surgery on the $T(2, 2n)$ torus link. Consider the surgery on upper-middle link $L$, which is a plumbing of unknots. After blowing down the horizontal (blue) unknots $H_i$’s, we get the $(n+1, n+1)$-surgery on the upper-left link $T(2, 2n)$. While after doing Rolfsen twists on the black unknots $V_j$’s, we can get a rational surgery on the lower-middle link $M$, which is a lens space by blowing-down the blue unknots using Rolfsen twists again.

The two-bridge link $b(6n + 2, -3)$. 
Figure 3.6. The 4-plat presentations of two-bridge links. For any continued fraction \([a_1,...,a_m] = q/p\), there is a 4-plat projection of the two-bridge link \(b(p,q)\). When \(m\) is odd, we use (a) to close the 4-braid \(B\) in the box; when \(m\) is even, we use (b) to close the 4-braid \(B\).

Before proving this proposition, let us clarify some conventions. First, the notation \(b(p,q)\) denotes an oriented two-bridge link of slope \(q/p\). For any continued fraction of \(q/p\):

\[
\frac{q}{p} = [a_1, a_2,...,a_m] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_{m-1} + \frac{1}{a_m}}}}}.
\]

a 4-plat projection of \(b(p,q)\) can be obtained in the following ways:

**Case I:** If \(m\) is odd, then the 4-plat is obtained by closing the 4-braid

\[
B = \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_2^{-a_m}
\]

in the way shown in Figure 3.6(a).

**Case II:** If \(m\) is even, then the 4-plat is obtained by closing the 4-braid

\[
B = \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_1^{-a_m}
\]

in the way shown in Figure 3.6(b).

Here, we follow [2] Chapter 12B. We prescribe an orientation on \(b(p,q)\) shown in Figure 3.6. Note that this orientation convention is different from [2].

**Lemma 3.9.** For any integer \(n \geq 1\), the \((n+2,n+2)\)-surgery on the two-bridge link \(b(6n+2,-3)\) is an \(L\)-space.

**Proof.** Consider the 3-component link \(L = L_1 \cup L_2 \cup L_3\) drawn in Figure 3.7. We see that \(L_1 \cup L_2\) is the \(T(2,2n)\) torus link with the linking number \(n\).

Consider the \((n+1,n+1,1)\)-surgery on \(L\), \(S^3_{n+1,n+1,1}(L)\). By blowing down the (+1)-framed component \(L_3\), we get the \((n,n)\)-surgery on the \(T(2,2n+2)\) torus link, \(S^3_{n,n}(T(2,2n+2))\), which is an \(L\)-space by Corollary 3.6. While the \((n+1,n+1)\)-surgery on the \(T(2,2n)\) torus link \(L_1 \cup L_2\),
Figure 3.7. A 3-component link used to study the surgeries on \( b(6n + 2, -3) \). The left link \( L \) is used to study the surgeries on \( b(6n + 2, -3) \). After blowing down the \((-1)\)-framed \( L_3 \), we can get the two-bridge link \( b(6n + 2, -3) \). While if we consider the \((n + 1, n + 1, -1)\)-surgery on \( L \), after blowing down the \((+1)\)-framed component \( L_3 \), we get the \((n, n)\)-surgery on \( T(2, 2n + 2) \), which is an \( L \)-space.

Figure 3.8. The \((n + 2, n + 2)\)-surgery on the two-bridge link \( b(6n + 2, -3) \). Consider the \((n + 1, n + 1, -1)\)-surgery on the left 3-component link \( L \). After blowing down the \((-1)\)-framed component \( L_3 \), we get the \((n + 2, n + 2)\)-surgery on the two-bridge link \( b(6n + 2, -3) \).

\[
S^3_{n+1,n+1}(L_1 \cup L_2), \text{ is also an } L\text{-space. In addition, since}
\[
\begin{align*}
\det \begin{pmatrix}
n + 1 & n & 1 \\
n & n + 1 & -1 \\
1 & -1 & 1 
\end{pmatrix} &= -1 - 2n, \\
\det \begin{pmatrix}
n + 1 & n \\
n & n + 1 
\end{pmatrix} &= 2n + 1,
\end{align*}
\]

from Lemma 2.5 it follows that the surgeries \( S^3_{n+1,n+1,0}(L), S^3_{n+1,n+1,-1}(L) \) are both \( L \)-spaces. By blowing down the \((-1)\)-framed component \( L_3 \) on the \((n + 1, n + 1, -1)\)-surgery on \( L \), we get the \((n + 2, n + 2)\)-surgery on the two-bridge link \( b(6n + 2, -3) \). See Figure 3.8. \( \Box \)
Figure 3.9. The \((n + 1 + k, n + 1 + k)\)-surgery on the two-bridge link \(b(rq - 1, -q)\) with \(r = 2n + 1, q = 2k + 1\). Consider the \((n + 1, n + 1, -\frac{1}{k})\)-surgery on the left 3-component link \(L\). After doing the Rolfsen twists on the \((-1)\)-framed component \(L_3\), we get the \((n + 1 + k, n + 1 + k)\)-surgery on the two-bridge link \(b(rq - 1, -q)\).

Proof of Proposition 3.8. Since \(\det \begin{pmatrix} n + 2 & n - 1 \\ n - 1 & n + 2 \end{pmatrix} > 0\) and \(n + 2 > 0\), Proposition 3.8 directly follows from Lemma 2.5 and Lemma 3.9.

More generally, we have the following theorem.

Theorem 3.10. For all positive odd integers \(r, q\), the two-bridge link \(b(rq - 1, -q)\) is an \(L\)-space link.

Proof. Let \(r = 2n + 1\) and \(q = 2k + 1\). Consider rational surgeries on the 3-component link \(L\) in Figure 3.7 with a rational coefficient on \(L_3\). Then we have an exact triangle for the triad \((S^3_{n+1,n+1,0}(L), S^3_{n+1,n+1,1/k}(L), S^3_{n+1,n+1,1/(k+1)}(L))\):

\[
\widetilde{HF} \left( S^3_{n+1,n+1,0}(L) \right) \longrightarrow \widetilde{HF} \left( S^3_{n+1,n+1,-\frac{1}{k}}(L) \right) \longrightarrow \widetilde{HF} \left( S^3_{n+1,n+1,-\frac{1}{k+1}}(L) \right).
\]

We claim that \(S^3_{n+1,n+1,-\frac{1}{k+1}}(L)\) is an \(L\)-space for all positive integers \(k\). In the proof of Lemma 3.9, we have shown \(S^3_{n+1,n+1,0}(L), S^3_{n+1,n+1,1}(L)\) are both \(L\)-spaces. In addition, since

\[
\left| H_1(S^3_{n+1,n+1,-\frac{1}{k}}(L)) \right| = \det \begin{pmatrix} n + 1 & n & 1 \\ n & n + 1 & -1 \\ k & -k & -1 \end{pmatrix} = -1 - 2n - 2k - 4kn,
\]

we have that

\[
\left| H_1(S^3_{n+1,n+1,-\frac{1}{k+1}}(L)) \right| = \left| H_1(S^3_{n+1,n+1,0}(L)) \right| + \left| H_1(S^3_{n+1,n+1,1}(L)) \right|.
\]
Table 3.1. Alexander polynomials of non-fibered hyperbolic $L$-space links.

| $b(24, -5)$ | $L_1 \cup L_2$ | $\Delta_{L_1 \cup L_2}(t) = \frac{1}{t^2}(2t^6 - 3t^5 + 2t^4 - 3t^3 + 2t^2 - 3t + 2)$ |
| $b(34, -5)$ | $2t^2 - 3t + 2 - \frac{3}{t} + \frac{2}{t^2}$ |
| $b(44, -5)$ | $3t^2 - 4t + 3 - \frac{4}{t} + \frac{3}{t^2}$ |
| $b(54, -5)$ | $4t^2 - 5t + 4 - \frac{5}{t} + \frac{4}{t^2}$ |
| $b(64, -5)$ | $5t^2 - 6t + 5 - \frac{6}{t} + \frac{5}{t^2}$ |
| $b(24, -5)$ | $6t^2 - 7t + 6 - \frac{7}{t} + \frac{6}{t^2}$ |

Hence, from the above exact triangle it follows that $S^3_{n+1,n+1,-\frac{1}{n+1}}(L)$ is an $L$-space by an induction on $k$.

Now by doing Rolfsen twists on $L_3$, we get a $(n + 1 + k, n + 1 + k)$-surgery on the two-bridge link $b(pq - 1, -q) = b(4kn + 2k + 2n, -2k - 1)$. See Figure 3.9. Since the linking number of $b(4kn + 2k + 2n, -2k - 1)$ is $\pm(n - k)$, the determinant $\det \begin{pmatrix} n + 1 + k & \pm(n - k) \\ \pm(n - k) & n + 1 + k \end{pmatrix}$ is positive. Thus, by Lemma 2.6 we get $b(rq - 1, -q)$ is an $L$-space link for all positive odd integers $r, q$. □

**Example 3.11** (Non-fibered hyperbolic $L$-space links). The two-bridge links $b(10n + 4, -5)$ with $n \in \mathbb{N}$ are $L$-space links, by Theorem 3.10. At least for $2 \leq n \leq 6$, they are not fibered links, i.e., there does not exist any Seifert surface $F$ such that the link complement fibers over circle with fiber $F$. The fiberedness of links is detected by the knot Floer homology. See Corollary 1.2 in [23]: An oriented link $\overline{L}$ in $S^3$ is fibered if and only if the knot Floer homology $\widehat{HFK}(\overline{L})$ has a single copy of $\mathbb{Z}$ at the top Alexander grading. Thus, for a homologically thin link $L$, the link $L$ is fibered if and only if its single-variable Alexander polynomial has leading coefficient $\pm 1$. Note two-bridge links are homologically thin. We compute the multi-variable polynomials $\Delta_L(x, y)$ using the algorithm in [18], and plug $t$ or $t^{-1}$ for $x, y$ so as to get the single-variable Alexander polynomials. It turns out that $b(10n + 4, -5)$ is not fibered with any orientation, when $2 \leq n \leq 6$. See Table 3.1. In fact, the fibered two-bridge knots and links are also classified by using continued fractions due to Gabai. See [5]. One should be able to generalize this to all $n \geq 2$ using number theoretic arguments.

**Example 3.12** (Plumbing trees). Any plumbing tree $L$ of unknots is an $L$-space link. In fact, any sufficiently negative surgery on $L$ is a negative definite graph without bad vertices, and thus is an $L$-space, by [28] Lemma 2.6. Since the plumbing tree is amphichiral, the sufficiently positive surgeries are also $L$-spaces. Note that if $M$ is an $L$-space, then so is $-M$.

The surgeries on a plumbing tree are generally Seifert manifolds. Actually, we also have examples of plumbing graph of unknots to be generalized $L$-space links, which give rise to hyperbolic manifolds. But one should be very careful about the types of generalized $L$-space links. Also note that for the same graph there are many different plumbings.
Example 3.13 \((L7a7\text{ in the Thistlethwaite Link Table})\). The link \(L\) drawn in Figure \(3.10\) is an \(L\)-space link. It is actually the mirror of \(L7a7\) drawn in the Thistlethwaite Link Table on Knot Atlas. Consider the \((n,n,1)\)-surgery on \(L\). After blowing down the \(1\)-framed component \(L_3\), we get the \((n−1,n−1)\)-surgery on the Whitehead link \(Wh\).

Example 3.14. The plumbing of unknots \(L\) in Figure \(3.11\) is a hyperbolic \(L\)-space link. In fact, consider the \((3,1,3,1)\)-surgery on \(L\), which is \(S^3\). By Lemma \(2.6\) \(L\) is an \(L\)-space link. In fact, this link is derived by resolving the Whitehead link. Thus, all the surgeries on the Whitehead link are surgeries on this link.

Example 3.15. The plumbing shown in Figure \(3.12\) is a generalized \((+++-)\)-\(L\)-space link. The \((1,1,1,1)\)-surgery is the Poincaré sphere. See [39] page 309. In fact, every proper sublink is an \(L\)-space link, since the surgeries on them are lens spaces. Thus, by Lemma \(2.5\) the \((p_1,1,1,1,1)\)-surgery is an \(L\)-space for all \(p_1 \geq 1\), since \(\det(\Lambda(1,1,1,1,1)) = -1\) and \(\det(\Lambda(1,1,1,1,1)|_{L-L_1}) = -1\). Next, from that \(S^3_{p_1,1,1,1}(L-L_2) = L(p_1,1)\) and \(\det(\Lambda(p_1,1,1,1,1)) = \det(\Lambda(p_1,1,1,1,1)|_{L-L_2}) = \)
Example 3.16 (A family of $L$-space chain links). The family of $l$-component chain links in Figure 3.14 are all $L$-space links, and when $l \geq 5$ they are hyperbolic links. In fact, the $(1, 2, \ldots, 2, l-2)$-surgery satisfies the positive $L$-space surgery criterion. First, if we blow down $L_1, L_2, \ldots, L_{l-2}$ successively, then we get the $(1, 1)$-surgery on the Whitehead link, the Poincaré manifold. Moreover, every proper sublink is a union of linear plumbings of unknots, and their surgeries are all connected sum of lens spaces. Thus, we only need to check the positive determinant condition.

Since a handle slide does not change the determinants of the surgery framing matrices, blowing down a $+1$ framed unknot does not change the determinant of the surgery framing matrices. Thus, after successively blowing down $L_1, \ldots, L_{l-2}$ from $L$, we see that $\det(\Lambda(1, 2, \ldots, 2, l-1)) = 1$. For the proper sublinks, we only need to consider a linear plumbing $L' \subset L$. Since the determinant of the surgery framing matrix does not depend on the orientations, we can always orient $L'$ such that all the linking numbers of adjacent components are $-1$. Let $M(k, n)$ denote the following $k \times k$ matrix

$$M(k, n) = \begin{pmatrix} n & -1 & \cdots & -1 \\ -1 & 2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 2 \end{pmatrix},$$

$$-p_1$, it follows that $(p_1, p_2, p_3, 1, 1)$-surgery on $L$ is an $L$-space for all $p_1 \geq 1, p_2 \geq 1$. Similarly, we can get the $(p_1, p_2, p_3, 1, 1)$-surgery is an $L$-space for all $p_1 \geq 1, p_2 \geq 1, p_3 \geq 1$. This is because $S^{3}_{p_1, p_2, 1, 1}(L-L_3) = L(p_1, 1)$, and $\det(\Lambda(p_1, p_2, 1, 1)) = -p_1 p_2$, $\det(\Lambda(p_1, p_2, 1, 1)|_{L-L_3}) = -p_1$.

Now, we can get that the $(p_1, p_2, p_3, p_4, 1)$-surgery on $L$ is an $L$-space, for all $p_1 \geq 3, p_2 \geq 3, p_3 \geq 2, p_4 \leq 1$, since $\det(\Lambda(p_1, p_2, p_3, 1, 1)) = p_2 - p_1 p_2 - p_2 p_3 < 0$, $\det(\Lambda(p_1, p_2, p_3, 1, 1)|_{L-L_4}) = 1 - p_1 - p_3 - p_1 p_2 + p_1 p_2 p_3 > 0$. Finally, we can obtain that the $(p_1, p_2, p_3, p_4, p_5)$-surgery on $L$ is an $L$-space for all $p_1 \gg 0, p_2 \gg 0, p_3 \gg 0, p_4 \ll 0, p_5 \geq 1$, due to

$$\det(\Lambda(p_1, p_2, p_3, p_4, 1)) = p_1 p_2 p_3 p_4 + \text{lower terms} < 0,$$

$$\det(\Lambda(p_1, p_2, p_3, p_4, 1)|_{L-L_5}) = p_1 p_2 p_3 p_4 + \text{lower terms} < 0.$$
which is the surgery framing matrix of the linear plumbing in Figure 3.13. There are four cases for computing $\det(\Lambda|_{L'})$:

- if $L_1 \not\in L', L_l \not\in L'$, then $\Lambda|_{L'} = M(k, 2)$ with $k$ being the number of components in $L'$;
- if $L_1 \not\in L', L_l \in L'$, then $\Lambda|_{L'} = M(k, l - 1)$;
- if $L_1 \in L', L_l \not\in L'$, then after successively blowing down $L_1, L_2, ..., L_l - 2$, we can see $\det(\Lambda|_{L'}) = 1$;
- if $L_1 \in L', L_l \in L'$, then after successively blowing down $L_1, L_2, ..., L_l - 2$ inside $L'$, we can see $\det(\Lambda|_{L'})$ equals to $\det(M(k, n))$ with $k \leq l - 2, n \geq 1$.

It is not hard to see $\det(M(k, 2)) = k + 1$ by induction, and thus $\det(M(k, n)) = nk - k + 1$. Therefore, all determinants are positive.

**Example 3.17** (Another sequence of $L$-space chain links). Similarly, the family of $l$-component chain links in Figure 3.16 are also all $L$-space links for $l \geq 3$. In fact, when $n_1, n_2$ are large enough, the $(1, 2, ..., 2, n_2, n_1)$-surgery satisfies the positive $L$-space surgery criterion. This is because after blowing down $L_1, ..., L_{l-2}$, we have an $(n_1 - l + 2, n_2 - 1)$ framed $T(2, 4)$ torus link. Thus, when $n_1, n_2$ are both large, this surgery is an $L$-space, since $T(2, 4)$ is an $L$-space link. As is similar in Example 3.16, we only need to show when $n_1, n_2$ are large enough, $\det(\Lambda(1, 2, ..., 2, n_2, n_1)|_{L'})$ is positive for any sublink $L'$. For any sublink $L'$, we can blow down the circles on the side of $L_1$, and then obtain a linear plumbing as in Figure 3.15. The surgery matrix is a $k \times k$ matrix in the form...
Figure 3.15. A linear plumbing.

Figure 3.16. Another family of hyperbolic $L$-space chain links. The surgery labelled above satisfies the positive $L$-space surgery criterion, when $n_1, n_2$ are large enough.

of

$$
\begin{pmatrix}
  n_1 - c & -1 &  &  &  \\
  -1 & n_2 & -1 &  &  \\
  & -1 & 2 &  &  \\
  &  &  & \ddots &  \\
  &  &  & 2 & -1 \\
  &  &  & -1 & 2
\end{pmatrix},
$$

where $c$ is the number of times for blowing down $+1$-framed unknots. The determinant of the above matrix is a polynomial of $n_1, n_2$, and the leading term is $\det(M(k-2,2))n_1n_2 = (k-1)n_1n_2$. Thus, for $n_1, n_2$ large enough, all the determinants are positive.

Note that the link in Example 3.15 is the same as the link here for $l = 5$.

Example 3.18. The link $L^{(n)} = V_1 \cup V_2 \cup H_1 \cup \ldots \cup H_n$ shown in Figure 3.17 is a generalized $L$-space link of "++-\cdots-" type, for any $n \geq 1$. One can do similar induction as in Example 3.15 to show the following claim.

Claim: For any $0 \leq k \leq n$ and all integers $p_1 \gg 0, p_2 \gg 0, q_1 \ll 0, \ldots, q_k \ll 0$, the $(p_1, p_2, q_1, \ldots, q_k, -1, \ldots, -1)$-surgery on $L^{(n)}$ is an $L$-space. Notice that the determinant of framing matrix

$$
\det(\Lambda((p_1, p_2, q_1, \ldots, q_k, -1, \ldots, -1))) = (-1)^{n-k}p_1p_2q_1\cdots q_k + \text{lower terms}.
$$

The claim will follow from two induction on $n$ and on $k$. 

Figure 3.17. **Another sequence of generalized L-space link.** Consider the link $L^{(n)}$ used in the proof of Lemma 3.5. It is in fact a generalized L-space link.

Notice that surgeries on $L^{(n)}$ are mostly graph manifolds.

**Example 3.19** (Thistlethwaite Link Table with crossing number $\leq 7$). We examine the links in the Thistlethwaite Link Table with crossing number $\leq 7$ and list the results in Table 3.2.

Using the conditions of Alexander polynomials in Theorem 1.15, we conclude that $L6a1$, $L7a1$, $L7a2$, $L7a4$, and $L7a5$ are all non-L-space links.

The link $L6a2$ is the two-bridge link $b(10, 7)$. Conjecture 1.14 has been verified for all two-bridge links $b(p, q)$ with $p \leq 100$ using the algorithm from [18]. So $L6a2$ is a non-L-space link.

The link $L6a5$ is the mirror of the left link in Figure 3.7 with $n=1$, on which the $(2,2,1)$-surgery is an L-space. Then, it quickly follows from the positive L-space surgery criterion that the mirror of $L6a5$ is an L-space link.

For the link $L6n1$, after blowing down a +1-framed component from it (all three components are symmetric), we get the unlink. So the $(10, 10, 1)$-surgery on $L6n1$ satisfies the positive surgery criterion, and thus showing that $L6n1$ is an L-space link.

The mirror of $L7a3$ consists of two components $L_1$ and $L_2$, where $L_1$ is the right-handed trefoil and $L_2$ is the unknot. Consider the $(n,1)$-surgery on the mirror of $L7a3$ with $n$ large. After blowing down the unknot, we have a large surgery on the right-handed torus knot $T(2,5)$. This is an L-space, since the right-handed torus knot $T(2,5)$ is an L-space knot. Then it follows from the positive surgery criterion that the mirror of $L7a3$ is an L-space link.

The link $L7a6$ is the two-bridge link $b(14, -9)$, and it is not L-space link by direct computation.

The link $L7n1$ has two components $L_1$ and $L_2$, where $L_1$ is the right-handed trefoil knot and $L_2$ is the unknot. Consider the $(10,1)$-surgery on this link. After blowing down the unknot, the trefoil is unknotted and we obtain a lens space surgery. Since the right-handed trefoil is an L-space knot, from the positive surgery criterion it follows that $L7n1$ is an L-space link.

The link $L7n2$ is not an L-space link; see Proposition 5.13 for the proof. Its mirror is not an L-space link neither, since the left-handed trefoil is not an L-space knot. However, $L7n2$ is a generalized $(+ -) L$-space link. The link $L7n2$ consists of two components $L_1$ and $L_2$, with $L_1$ being the right-handed trefoil and $L_2$ being the unknot. Consider the $(n,-1)$-surgery on $L7n2$ with $n$ large. After blowing down the unknot, we get the unknot, thus getting a lens space surgery. Then, since the right-handed trefoil is an L-space knot, $(n, -k)$-surgery is an L-space for all $k > 0$ and large $n$ by Lemma 2.5.
| Links   | L-space link | Alexander polynomial | Comments                                      |
|--------|--------------|----------------------|------------------------------------------------|
| L2a1   | Yes          | Yes                  | The Hopf link                                  |
| L4a1   | Yes          | Yes                  | The $T(2, 4)$ torus link                       |
| L5a1   | Yes          | Yes                  | Mirror of the $L$-space Whitehead link          |
| L6a1   | No           | No                   |                                                |
| L6a2   | No           | Yes                  |                                                |
| L6a3   | Yes          | Yes                  | The $T(2, 6)$ torus link                       |
| L6a4   | Yes          | Yes                  | The Borromean link                             |
| L6a5   | Yes          | Yes                  | The mirror is an $L$-space link                 |
| L6n1   | Yes          | Yes                  |                                                |
| L7a1   | No           | No                   |                                                |
| L7a2   | No           | No                   |                                                |
| L7a3   | Yes          | Yes                  | The mirror is an $L$-space link                 |
| L7a4   | No           | No                   |                                                |
| L7a5   | No           | No                   |                                                |
| L7a6   | No           | Yes                  | The two-bridge link $b(14, -9)$                 |
| L7a7   | Yes          | Yes                  | The mirror is an $L$-space link                 |
| L7n1   | Yes          | Yes                  |                                                |
| L7n2   | No           | Yes                  | Generalized $(+-)$-$L$-space link               |

**Table 3.2. Thistlethwaite Link Table with crossing number $\leq 7$.** Here, by 'Yes' in the column 'L-space link', it means either the link or its mirror is an $L$-space link; by 'Yes' in the column 'Alexander polynomial', it means the conditions on the multi-variable Alexander polynomial in Theorem 1.15 are satisfied.

4. Floer homology and Alexander polynomials of $L$-space links

In this section, we study the link Floer homology and the multi-variable Alexander polynomials of $L$-space links with $l \geq 2$ components. The Alexander polynomial of $L$ is determined by the Euler characteristics of the link Floer homology $HFL^-(L, s)$, due to Equation (2) in [34]

$$\Delta_L(x_1, ..., x_l) = \sum_{(s_1, ..., s_l) \in \mathbb{H}(L)} \chi(HFL^-(L, s_1, ..., s_l)) \cdot x_1^{s_1} \cdots x_l^{s_l},$$

where $f \cong g$ denotes that $f$ and $g$ differ by multiplication by units. Here, we use $CFL^-(L)$ rather than $\hat{CFL}(L)$ as in [33]. Note that $CFL^-(L, s_1, s_2)$ is a finite dimensional $\mathbb{F}$-vector space, and thus $\chi(CFL^-(L, s_1, s_2)) = \chi(HFL^-(L, s_1, s_2))$.

Now we are ready to prove Theorem 1.15 from the introduction.

4.1. Proof of Theorem 1.15

*Proof.* First, we consider the case of $l = 2$. We can identify $\mathfrak{A}_{s_1, s_2}$ with a subcomplex of $CFL^-(L) = CF^-(S^3)$,

$$\mathfrak{A}_{s_1, s_2} = \{ x \in CF^-(S^3) | A_i(x) \leq s_i, \forall i = 1, 2 \}.$$

Consider the short exact sequence of chain complexes

$$0 \to \mathfrak{A}_{s_1 - 1, s_2} \xrightarrow{i_{s_1 - 1, s_2}} \mathfrak{A}_{s_1, s_2} \xrightarrow{j_{s_1, s_2}} \mathfrak{A}_{s_1, s_2} / \mathfrak{A}_{s_1 - 1, s_2} \to 0,$$
where the map $i_{s_1-1,s_2}^+:\mathcal{A}_{s_1,s_2}^+ / \mathcal{A}_{s_1-1,s_2}^-$ is the inclusion map. It induces another short exact sequence of homologies

$$0 \to \text{coker}(i_{s_1-1,s_2}^+) \to \text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-) \to \ker(i_{s_1-1,s_2}^+) \to 0.$$  

The map $(i_{s_1-1,s_2}^+) : \mathbb{F}[U] \to \mathbb{F}[U]$ is a multiplication of $U^k$ for some integer $k \geq 0$.

Since $U_1$ acts on the chain complex $\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-$ as 0, it also acts as 0 on homology. Thus, $\text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-)$ is either 0 or $\mathbb{F}[U]/U$. Note here $\mathbb{F}[U]$ denotes $\mathbb{F}[[U_1,U_2]]/(U_1-U_2)$ as an $\mathbb{F}[[U_1,U_2]]$-module and the $U$-action denotes both actions of $U_1$ and $U_2$.

Furthermore, $\chi(\text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-))$ is either 0 or 1. In fact, if $\text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-) = 0$, then the grading of $1 \in \text{H}_s(\mathcal{A}_{s_1,s_2}^-)$ equals to the grading of $1 \in \text{H}_s(\mathcal{A}_{s_1-1,s_2}^-)$; while if $\text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-) = \mathbb{F}[U]/U$, then the grading of $1 \in \text{H}_s(\mathcal{A}_{s_1,s_2}^-)$ equals to the grading of $1 \in \text{H}_s(\mathcal{A}_{s_1-1,s_2}^-)$ plus 2, and the grading of $1 \in \text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-)$ equals to the grading of $1 \in \text{H}_s(\mathcal{A}_{s_1,s_2}^-)$. Moreover, the complex $\mathcal{A}_{+,+,+}$ is just $\text{CF}^-(S^3)$ and the absolute gradings of elements in $\text{H}_s(\mathcal{A}_{+,+,+})$ are all even integers. An induction will show that all the absolute gradings of elements in $\text{H}_s(\mathcal{A}_{s_1,s_2}^-)$ and $\text{H}_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^-)$ are even integers for all $s_1,s_2 \in \mathbb{H}(L)$.

Second, we have

$$CFL^-(L,s_1,s_2) = \{x \in \text{CF}^-(S^3) \mid \text{the Alexander grading } A(x) = (s_1,s_2)\}$$

$$= \mathcal{A}_{s_1,s_2}^- / (\mathcal{A}_{s_1-1,s_2}^- + \mathcal{A}_{s_1,s_2-1}^-)$$

$$= \mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^- \bigg/ \mathcal{A}_{s_1-1,s_2}^- / \mathcal{A}_{s_1-2,s_2-1}^-.$$

From the short exact sequence of chain complexes

$$0 \to \mathcal{A}_{s_1-1,s_2}^- / \mathcal{A}_{s_1-1,s_2-1}^- \to \mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1-1,s_2}^- \to CFL^-(L,s_1-1,s_2) \to 0,$$

it follows that

$$(4.2) \quad \chi(H_s(\mathcal{A}_{s_1-1,s_2}^- / \mathcal{A}_{s_1-1,s_2-1}^-)) - \chi(H_s(\mathcal{A}_{s_1-1,s_2}^- / \mathcal{A}_{s_1-1,s_2-2}^-)) + \chi(\text{HFL}^-(L,s_1-1,s_2)) = 0.$$

Thus, we have

$$\chi(\text{HFL}^-(L,s_1,s_2)) = 0, \pm 1.$$

Let us look at the signs of the multi-variable Alexander polynomial. Suppose

- $|\chi(\text{CFL}^-(L,s_1,s_2))| = 1$,
- $|\chi(\text{CFL}^-(L,s_1+k,s_2))| = 1$,
- $\chi(\text{CFL}^-(L,s_1+i,s_2)) = 0, \forall i = 1,2,\ldots,k-1$.

Then, we have

$$(*) \quad \chi(H_s(\mathcal{A}_{s_1+k,s_2}^- / \mathcal{A}_{s_1+k,s_2-1}^-)) - \chi(H_s(\mathcal{A}_{s_1-1,s_2}^- / \mathcal{A}_{s_1-1,s_2-2}^-))$$

$$= \sum_{i=0}^{k} \chi(\text{CFL}^-(L,s_1+i,s_2))$$

$$= \chi(\text{CFL}^-(L,s_1,s_2)) + \chi(\text{CFL}^-(L,s_1+k,s_2)).$$

Since $\chi(H_s(\mathcal{A}_{s_1,s_2}^- / \mathcal{A}_{s_1,s_2-1}^-)) = 0$ or 1, for all $(s_1,s_2) \in \mathbb{H}(L)$, the top row of Equation $(*)$ is 0 or $\pm 1$. Whereas by the assumption, the bottom row of Equation $(*)$ is 0 or $\pm 2$. Thus, we have

$$\chi(\text{CFL}^-(L,s_1,s_2)) + \chi(\text{CFL}^-(L,s_1-k,s_2)) = 0.$$
For the case of \( l = n \), \( CFL^{-}(L, s) \) with \( s = (s_1, ..., s_n) \) is a successive quotient of \( \mathcal{A}^{-}_s \)'s. Fixing \( \{s_i\}_i \), we denote the following successive quotient complexes by

\[
C_k^{(1)} = \{x, A_l(x) \leq s_i, 1 \leq i \leq k, A_j(x) = s_{j}, k + 1 \leq j \leq n \},
\]

\[
C_k^{(2)} = \{x, A_l(x) \leq s_i, 1 \leq i \leq k-1, A_k(x) \leq s_{k-1}, A_j(x) = s_{j}, k + 1 \leq j \leq n \}.
\]

Then, \( C_n^{(1)} = \mathcal{A}^{-}_s, C_0^{(1)} = CFL^{-}(L, s) \), and \( C_k^{(1)} = C_k^{(1)}/C_k^{(2)} \). As is similar in the case of \( l = 2 \), we have

\[
\left| \chi(H_*(C_{n-2}^{(1)})) \right| \leq 1.
\]

Since \( \chi(H_*(C_{k-1}^{(1)})) = \chi(H_*(C_k^{(1)})) - \chi(H_*(C_k^{(2)})) \), we can inductively prove that

\[
\left| \chi(H_*(C_k^{(1)})) \right| \leq 2^{n-k-2}, \forall k = 0, ..., n - 2.
\]

Hence, we prove Inequality (1.2) by letting \( k = 0 \).

Since \( C_{k-1}^{(1)} = C_k^{(1)}/C_k^{(2)} \), we have

\[
\text{rank}_\mathbb{F}(C_k^{(1)}) \leq \text{rank}_\mathbb{F}(C_k^{(1)}) + \text{rank}_\mathbb{F}(C_k^{(2)}).
\]

From \( \text{rank}_\mathbb{F} H_*(C_{n-1}^{(1)}) \leq 1 \), it follows that \( \text{rank}_\mathbb{F} HFL^{-}(L, s) \leq 2^{l-1} \). Thus, Inequality (1.1) holds.

**Corollary 4.1.** A homological thin \( L \)-space 2-component link \( L = L_1 \cup L_2 \) has fibered link exterior.

**Proof.** Let the symmetrized Alexander polynomial be

\[
\Delta_L(x, y) = \sum_{i,j} a_{i,j} \cdot x^i \cdot y^j.
\]

We choose

\[
x_0 = \max\{i | a_{i,j} \neq 0\}, \quad y_0 = \max\{j | a_{x_0, j} \neq 0\}.
\]

Since

\[
\sum_{(s_1, s_2) \in \mathbb{H}(L)} \chi(\overline{HFL}(L, s_1, s_2)) \cdot x^{s_1} \cdot y^{s_2} = \pm \frac{(x-1)(y-1)}{\sqrt{xy}} \Delta_L(x, y),
\]

we have that \( (x_0 + \frac{1}{2}, y_0 + \frac{1}{2}) \) is an extreme point of the polytope for \( \overline{HFL}(L) \), and \( \chi(\overline{HFL}(L, x_0 + \frac{1}{2}, y_0 + \frac{1}{2})) = \pm 1 \). Furthermore, since \( L \) is homological thin, we have that \( \text{rank}\overline{HFL}(L, x_0 + \frac{1}{2}, y_0 + \frac{1}{2}) = 1 \), and thereby the link exterior of \( L \) is fibered.

**4.2. Examples.** Let us use Theorem 4.1 to filter \( L \)-space links among two-bridge links. Notice that in the knot case the Alexander polynomial gives a strong obstruction for an alternating knot to be an \( L \)-space knot. In [33], it is shown that alternating \( L \)-space knots are only \( (2, 2n + 1) \) torus knots.

**Proposition 4.2** (Ozsváth-Szabó, [33], Proposition 4.1). If \( K \) is an alternating knot with the property that all the coefficients \( a_i \) of its Alexander polynomial \( \Delta_K \) have \( |a_i| \leq 1 \), then \( K \) is the \( (2, 2n + 1) \) torus knot.

**Theorem 4.3** (Ozsváth-Szabó, [33], Theorem 1.5). If \( K \subset S^3 \) is an alternating knot with the property that there is some integral surgery along \( K \) is an \( L \)-space, then \( K \) is a \( (2, 2n + 1) \) torus knot for some integer \( n \).
In contrast to the knot case, by computer experiments, we find many hyperbolic two-bridge non-L-space links whose Alexander polynomials satisfy the constraints in Theorem 1.15. We list some interesting phenomena in the two-bridge links $b(p,q)$ below, where $0 < p \leq 100$. Note that all these phenomena should presumably be true for all positive even integers $p$.

| Links | Alexander polynomial condition: | Hyperbolic link: | L-space link: |
|-------|---------------------------------|------------------|--------------|
| $b(2n,-1)$ | Yes | Torus link $T(2,2n)$ | Yes |
| $b(6n+2,-3)$ | Yes | Hyperbolic link | Yes |
| $b(6n+4,-3)$ | Yes | Hyperbolic link | No, when $6n+2 \leq 100$. |
| $b(10n \pm 2, 5)$ | No | Hyperbolic link | No |

4.3. \textit{HFL}^-$ of L-space links. Let $L$ be an L-space link. In general, $CFL^-(L,s)$ is an iterated quotient complex of $\mathbb{A}_s^{-}$. For every subcomplex $C_1 \subset C$, the quotient complex $C/C_1$ is quasi-isomorphic to the mapping cone of the inclusion map $i : C_1 \to C$. Thus, it leads to an iterated mapping cone construction of $CFL^-(L,s)$ by using $\mathbb{A}_s^{-}$. This provides a spectral sequence converging to $HFL^-(L,s)$ considered as $F$-vector spaces, which is stated in [7]. This spectral sequence also implies Inequality (1.1).

5. SURGERIES ON L-SPACE LINKS

Using the knot surgery formula from [36], the graded Heegaard Floer homology of surgeries on L-space knots are determined by the Alexander polynomial and the surgery coefficient. Using M-O link surgery formula from [20] and algebraic rigidity results from [18], we prove Theorem 1.17 and give some explicit formulas in this section.

In fact, when $L$ is an L-space link, $\mathbb{A}_s^{-}(L)$ is chain homotopic to $F[[U_1]]$ preserving the $Z$-grading. This is done by restricting our scalars to $F[[U_1]]$ and applying the algebraic rigidity results Proposition 5.5 and Corollary 5.6 in [18]. Here, the two complexes are both considered as $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes of $F[[U_1]]$-modules together with a $Z$-grading compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading, where $U_1$ lowers the $Z$-grading by 2.

\textbf{Proposition 5.1} (Proposition 5.5, [18]). Let $A_*, B_*$ be $\mathbb{Z}$-graded complexes of $F$-modules with $U$-action dropping by 2 and commuting with the differential. Suppose $A,B$ are both free $F[[U]]$-modules, and $H_*(A) = H_*(B) = F[[U]]$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong F$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U \cdot H_{2k}(A) = H_{2k-2}(A), U \cdot H_{2k}(B) = H_{2k-2}(B)$.

Then, if $F,G : A \to B$ are both quasi-isomorphisms of $F[[U]]$-modules, then $F,G$ are chain homotopic as maps of $F[[U]]$-modules. Moreover, if $H,K$ are both chain homotopies as homomorphisms of $F[[U]]$-modules between any two chain maps $f,g : A \to B$, i.e. $H \partial + \partial H = K \partial + \partial K = f - g$, then $H - K = \partial T + T \partial$, for some $F[[U]]$-module homomorphism $T : A_+ \to B_{s+2}$.

Using these chain homotopy equivalences, we replace $\mathbb{A}_s^{-}(L)$ by $F[[U_1]]$ in the M-O link surgery complex and replace the maps up to homotopies. In [18], we call this new complex the \textit{perturbed surgery formula}. Thus, we only need to determine the map $\Phi_s^{\mathcal{M}}$ in the perturbed surgery formula, where are either 0 or multiplications of $U^k$.

Combining this with conjugation symmetry, we determine the maps $\Phi_s^{L_1}$ by the coefficients in the multi-variable Alexander polynomials of the sublinks in $L$ and the linking numbers. We also show that in the perturbed surgery complex, $\Phi_s^{\pm L_1 \pm L_2} = 0$ for all $s \in \mathbb{H}(L)$. For higher diagonal maps, more information is needed. For 2-component case, we write down explicit formulas.
5.1. Conjugation symmetry of inclusion maps.

**Definition 5.2** \( (n_s^\hat{M}(L)) \). Suppose \( \hat{L} \) is an oriented \( l \)-component \( L \)-space link and \( \hat{M} \subset \hat{L} \) is a sublink which might not have the induced orientation. Choose a Heegaard diagram \( \mathcal{H} \) of \( L \). The inclusion map \( \hat{I}_s^\hat{M} : \mathfrak{A}^- (\mathcal{H}, s) \to \mathfrak{A}^- (\mathcal{H}, p^\hat{M}(s)) \) is a chain map shifting the \( \mathbb{Z} \)-grading by a definite amount.; see Equation (57) in [20]. Thus, the map induced on homologies \( (\hat{I}_s^\hat{M})_* : H_* (\mathfrak{A}^- (\mathcal{H}, s)) \to H_* (\mathfrak{A}^- (\mathcal{H}, p^\hat{M}(s))) \) is a multiplication of a monomial \( U^k : \mathbb{F}[[U]] \to \mathbb{F}[[U]] \). The integer \( k \) does not depend on the choice of \( \mathcal{H} \), and thus we define it to be \( n_s^\hat{M}(L) \). When the context is clear, we simply denote it by \( n_s^\hat{M} \).

**Remark 5.3.** When \( L \) is a \( L \)-space knot \( K \), these \( n_s^\hat{M}(K) \)'s are just the same as \( V_s \)'s and \( H_s \)'s defined for knots in [21].

**Lemma 5.4** (Conjugation symmetry of \( n_s^\hat{M}(L) \)). Suppose \( L \) is an oriented \( n \)-component \( L \)-space link. Then

\[
n_s^\hat{M} = n_{-s}^\hat{M}, \quad \forall s \in \mathcal{H}(L), \forall \hat{M} \subset L.
\]

**Proof.** Choose an admissible basic Heegaard diagram \( \mathcal{H} = (\Sigma, \alpha, \beta, w^H, z^H) \) for \( \hat{L} \). Then, \( \mathcal{H}' = (-\Sigma, \beta, \alpha, w^{H'}, z^{H'}) \) is also a Heegaard diagram for \( \hat{L} \), where \( w^H = z^{H'}, z^H = w^{H'} \).

There is an \( \mathbb{F}[[U_1, \ldots, U_n]] \)-linear isomorphism of chain complexes

\[
h_s : \mathfrak{A}^- (\mathcal{H}, s) \to \mathfrak{A}^- (\mathcal{H}', -s),
\]

which is the chain map shifting the \( \mathcal{H} \)-grading by a definite amount. Actually, for any \( x, y \in T_\alpha \cap T_\beta \) and a class \( \phi \in \pi_2(x, y) \), the moduli space of holomorphic disks \( M(\phi, \mathcal{H}) \) is identical to \( M(\phi, \mathcal{H}') \). See Theorem 2.4 in [31]. Moreover, it is not hard to see that the Alexander gradings are of opposite signs

\[
A(x, \mathcal{H}) = -A(x, \mathcal{H}').
\]

Thus, we just need to show \( h_s \) is a chain map, i.e.

\[
\partial_E^{\mathcal{H}'} (h_s(x)) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#(M(\phi, \mathcal{H}')) \cdot U_1^{E_{x,s_1}^{\mathcal{H}'}(\phi)} \cdots U_n^{E_{x,s_n}^{\mathcal{H}'}(\phi)} \cdot y
\]

\[
h_s(\partial_E^{\mathcal{H}} (x)) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#(M(\phi, \mathcal{H}')) \cdot U_1^{E_{x,1}^{\mathcal{H}}(\phi)} \cdots U_n^{E_{x,n}^{\mathcal{H}}(\phi)} \cdot y.
\]

In fact, by Equation (2.3), \( \forall \phi \in \pi_2(x, y), \forall 1 \leq i \leq n, \)

\[
E_{s_i}^{\mathcal{H}'}(\phi) = \max(-s_i - A_i^{\mathcal{H}'}(x), 0) - \max(-s_i - A_i^{\mathcal{H}'}(y), 0) + n_{s_i}^{\mathcal{H}'}(\phi)
\]

\[
= \max(-s_i + A_i^{\mathcal{H}}(x), 0) - \max(-s_i + A_i^{\mathcal{H}}(y), 0) + n_{s_i}^{\mathcal{H}}(\phi)
\]

\[
= E_i^{\mathcal{H}}(\phi)
\]

Moreover, by direct computation, we have the following commuting diagram

\[
\begin{array}{ccc}
\mathfrak{A}^- (\mathcal{H}, s) & \xrightarrow{h_s} & \mathfrak{A}^- (\mathcal{H}', -s) \\
\hat{I}_s^\hat{M}(\mathcal{H}) & \searrow & \hat{I}_{-s}^\hat{M}(\mathcal{H}') \\
\mathfrak{A}^- (\mathcal{H}, p^\hat{M}(s)) & \xrightarrow{h_{p^\hat{M}(s)}} & \mathfrak{A}^- (\mathcal{H}', -p^\hat{M}(s)).
\end{array}
\]
Thus, it follows that
\[ n_s^M = n_{-s}^M, \quad \forall s \in \mathbb{H}(L), \forall M \subset L. \]

5.2. Perturbed link surgery formula for 2-component L-space links. We review the link surgery formula of Manolescu-Ozsváth for a 2-component link \( L \). See [20] and Section 4 in [18]. We need some notations. Denote the set of orientations on a link \( N \) by \( \Omega(N) \). We define some projection maps by \( p^\pm L_1(s_1, s_2) = (\pm \infty, s_2), p^\pm L_2(s_1, s_2) = (s_1, \pm \infty), \) and \( p^\pm L_1 \cup \pm L_2(s_1, s_2) = (\pm \infty, \pm \infty) \).

Choose an admissible basic Heegaard diagram \( \mathcal{H} \) and denote \( \mathfrak{A}^- (\mathcal{H}, s) \) by \( \mathfrak{A}^-_s \). Then, the M-O surgery complex \( (C^-(\mathcal{H}, \Lambda), D^-(\Lambda)) \) is as follows:

\[
(C^-(\mathcal{H}, \Lambda), D^-(\Lambda)) := \prod_{(s_1, s_2) \in \mathbb{H}(L)} \mathfrak{A}^-_{s_1, s_2} \xrightarrow{D^0_0(\Lambda)} \prod_{(s_1, s_2) \in \mathbb{H}(L)} \mathfrak{A}^-_{+\infty, s_2} \xrightarrow{D^0_1(\Lambda)} \prod_{(s_1, s_2) \in \mathbb{H}(L)} \mathfrak{A}^-_{+\infty, +\infty},
\]

where \( \forall \delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \{0, 1\}, \)

\[
D^\delta_1 \delta_2 \epsilon_1 \epsilon_2 (\Lambda) = \prod_{(s_1, s_2) \in \mathbb{H}(L)} \left( \sum_{M \in \Omega(\delta_1 L_1 \cup \delta_2 L_2)} \Phi^\delta_1 \delta_2 \epsilon_1 \epsilon_2 L_1 \cup \epsilon_2 L_2 (s_1, s_2) \right).
\]

The M-O surgery complex is in the category of complexes of \( \mathbb{F}[[U_1]] \)-modules, \( \mathbf{Ch} \). If we are working in the homotopy category \( \mathbf{K} \) of \( \mathbb{F}[[U_1]] \)-modules, the M-O surgery complex becomes a perturbed surgery formula. One can even morally think of the M-O surgery complex as an object in the derived category of \( \mathbb{F}[[U_1]] \)-modules \( \mathbf{D} \), with \( \mathfrak{A}^-_s (L) \) replaced by its homology thanks to some algebraic results. But morphisms in the derived category are formally defined and usually cannot be realized by chain maps.

**Lemma 5.5.** Let \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \) be an L-space link. Then the Heegaard Floer homologies on all the surgeries \( HF^-(S^3_\Lambda(L)) \) and their absolute gradings are determined by \( \{n_s^+ L_1(L)\}_{s \in \mathbb{H}(L)} \) and \( \{n_s^+ L_2(L)\}_{s \in \mathbb{H}(L)} \).

**Proof.** We restrict our scalars to \( \mathbb{F}[[U_1]] \) from now on. Consider the chain complex \( \mathbb{F}[[U_1]] \), which is freely generated by a single element over \( \mathbb{F}[[U_1]] \) with 0 differential. Since \( L \) is an L-space link, i.e. \( H_s(\mathfrak{A}^-_s (L)) = \mathbb{F}[[U]], \forall s \in \mathbb{H}(L), \mathfrak{A}^-_s (L) \) is in fact chain homotopic to \( \mathbb{F}[[U_1]] \) by Corollary 5.6 in [18] as a \( \mathbb{Z} \)-graded \( \mathbb{F}[[U_1]] \)-module with \( U_1 \) lowering grading by 2.

Thus, we can replace every \( \mathfrak{A}^-_s \) by \( \mathfrak{A}^-_s \) which is isomorphic to \( C^u \) and replace the maps correspondingly so as to get a new complex \( (\hat{C}^-(\mathcal{H}, \Lambda), \hat{D}^-(\Lambda)) \). We call it the perturbed surgery complex, and it is chain homotopic to the original one.

More concretely, we first replace the edge maps in the squares in Equation (5.1) \( \Phi^\pm L_1 \) by

\[
\hat{\Phi}^\pm L_1 = U_1^{\pm L_1} : \mathbb{F}[[U_1]] \to \mathbb{F}[[U_1]].
\]

Next, we replace the diagonal maps \( \Phi^\pm L_1 \cup L_2 \) by

\[
\hat{\Phi}^\pm L_1 \cup L_2 = 0.
\]

The reason we replace the diagonal maps by 0 is that, in the link surgery complex, the \( \mathbb{F}[[U_1]] \)-linear diagonal maps always shift the \( \mathbb{Z} \)-gradings by an odd number.
Finally, we get the new perturbed surgery complex $\tilde{C}(\Lambda)$ as follows:

$$
(\tilde{C}^-((\mathcal{H}, \Lambda), \tilde{D}^-((\mathcal{H}, \Lambda))) := \prod_{(s_1, s_2) \in \mathcal{H}(L)} \tilde{A}^-_{s_1, s_2} \xrightarrow{\tilde{D}_{00}^1(\Lambda)} \prod_{(s_1, s_2) \in \mathcal{H}(L)} \tilde{A}^-_{s_1, +\infty, s_2} \quad \prod_{(s_1, s_2) \in \mathcal{H}(L)} \tilde{A}^-_{s_1, +\infty, +\infty},
$$

where

$$
\tilde{D}_{\varepsilon_1 \varepsilon_2}^1(\Lambda) = \prod_{(s_1, s_2) \in \mathcal{H}(L)} \left( \sum_{M \in \Omega(3, 1, \delta)} \tilde{\Delta}_{p+c_1 L_1 + c_2 L_2}(s_1, s_2) \right), \quad \varepsilon_1, \varepsilon_2 \in \{0, 1\}.
$$

The perturbed complex $\tilde{C}(\Lambda)$ is chain homotopy equivalent to the original surgery complex as $F[[U_1]]$-modules. Moreover, this chain homotopy equivalence is preserving the $\mathbb{Z}$-grading on it. For more details, see Section 5.6 in [18].

Hence, we have $H_s(\tilde{C}(\Lambda)) \cong HF^- \left(S_\Lambda^3(L)\right)$ as an $F[[U_1]]$-module. By Link Surgery Theorem in [20], we have $U_i$ actions on the homology of $HF^- \left(S_\Lambda^3(L)\right)$ are all the same, i.e.

$$
HF^- \left(S_\Lambda^3(L)\right) = H_s(\tilde{C}(\Lambda)) \otimes_{F[[U_1]]} F[[U_1, U_2]]/(U_1 - U_2).
$$

All the inputs of $\tilde{C}(\Lambda)$ are $\{n_1^{+L_1}(s)\}_{s \in \mathcal{H}(L)}$ and $\{n_1^{-L_2}(s)\}_{s \in \mathcal{H}(L)}$. Thus, the proof is done by Lemma 5.3. To compute the absolute grading for $HF^-$, we only need to shift the absolute $\mathbb{Z}$-grading by $-\frac{c_1(s) - 3\chi - 3\pi}{4}$ which can be computed from $\Lambda$.

5.3. Redefining knot Floer homology. We redefine the knot Floer homology by using slightly generalized Heegaard diagrams. Here are the steps. For different generalized versions of knot Floer complex and homology, see [19].

1. Heegaard diagram: We choose a Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \{w_1, \ldots, w_k\}, \{z_1\})$.
2. Alexander grading: For any $x \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta$,

$$
A(U_1^{n_1} \cdots U_k^{n_k} x) = A(x) - n_1.
$$

3. Alexander filtration: The complex $CF^- (S^3)$ is freely generated by $x \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta$ over $F[[U_1, U_2, \ldots, U_k]]$ and the differentials are counting holomorphic disks. For $\forall x \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta$, we have $A(\partial x) \leq A(x)$. This is because for a pseudo-holomorphic disk in $\phi \in \pi_2(x, y)$, $n_{z_1}(\phi) \geq 0$ and

$$
A(x) = A(y) + n_{z_1}(\phi) - n_{w_1}(\phi) = A(U_1^{n_{w_1}(\phi)} \cdots U_k^{n_{w_k}(\phi)} y) + n_{z_1}(\phi).
$$

Thus, the Alexander grading induces a filtration on $CF^- (S^3)$. We define the subcomplex $\mathfrak{A}_s^-(K) := \{x \in CF^- (S^3) | A(x) \leq s\}$.

4. The filtered minus knot Floer homology: We define the chain complex $CFK^-(K, s) = \mathfrak{A}_s^- / \mathfrak{A}_{s-1}^-$ and $HFK^-(K, s) = H_s(CFK^-(K, s))$.

5. The total minus knot Floer homology: We define the chain complex $gCFK^-(K)$ to be freely generated by $\mathcal{T}_\alpha \cap \mathcal{T}_\beta$, and $\forall x \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta$

$$
\partial x = \sum_{y \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta} \sum_{\phi \in \pi_2(x, y)} \#(M(\phi)/\mathbb{R}) \cdot U_1^{n_{w_1}(\phi)} \cdots U_k^{n_{w_k}(\phi)} y, \quad \mu(\phi) = 1, n_{z_1}(\phi) = 0
$$

The homology $HFK^-(K)$ is defined to be the homology of $gCFK^-(K)$.
Remark 5.6. Considered only as \( \mathbb{F} \)-vector spaces, \( HFK^{-}(K) = \bigoplus_{s \in \mathbb{Z}} HFK^{-}(K, s) \). However, considered as \( \mathbb{F}[[U_1, \ldots, U_k]] \)-modules, \( HFK^{-}(K, s) \) is the associated graded of a filtration on \( HFK^{-}(K) \). Note that \( HFK^{-}(K, s) \)'s are always torsion modules.

**Proposition 5.7.** Suppose \( K \subset S^3 \) is a knot. For a multi-pointed Heegaard diagram \( \mathcal{H} = (\Sigma, \alpha, \beta, \{w_1, \ldots, w_k\}, \{z_1\}) \) for \( K \), we have the following:

1. The knot Floer homology \( HFK^{-}(K, s) \) is an \( \mathbb{F}[U] := \mathbb{F}[[U_1, \ldots, U_k]]/(U_2, \ldots, U_k) \)-module, and does not depend on \( \mathcal{H} \) considered as an \( \mathbb{F}[[U]] \)-module.
2. We have the following identity

\[
\sum_{s \in \mathbb{Z}} \chi(HFK^{-}(K, s)) \cdot t^s = \frac{1}{t-1} \Delta_K(t).
\]

**Proof.** This is actually a direct corollary of Theorem 4.10 in [20]. There are six types of Heegaard moves according to [20],

(i) 3-manifold isotopy;
(ii) \( \alpha \)-curve isotopy and \( \beta \)-curve isotopy;
(iii) \( \alpha \)-handle slide and \( \beta \)-handle slide;
(iv) index one/two stabilizations;
(v) free index zero/three stabilizations;
(vi) free index zero/three link stabilizations.

By Proposition 4.13 in [20], we only need to check how the knot Floer homology changes under these Heegaard moves and their inverses.

The Heegaard moves of types (i) to (iv) are chain homotopy equivalences preserving the Alexander filtration, and thus do not change the knot Floer homology.

A Heegaard move of type (v) changes the chain complex \( CF^{-}(\mathcal{H}) \) into \( CF^{-}(\mathcal{H}') \), which is the mapping cone \( CF^{-}(\mathcal{H})[[U_{k+1}]] \xrightarrow{U_{k+1} - U_{i_0}} CF^{-}(\mathcal{H})[[U_{k+1}]] \). Notice that \( U_{k+1} \) does not change the Alexander grading. Thus, if \( i_0 \neq 1 \), then \( CFK^{-}(\mathcal{H}', s) \) is the mapping cone

\[
CFK^{-}(\mathcal{H}, s)[[U_{k+1}]] \xrightarrow{U_{k+1} - U_{i_0}} CFK^{-}(\mathcal{H}, s)[[U_{k+1}]].
\]

If \( i_0 = 1 \), then \( CFK^{-}(\mathcal{H}', s) \) is the mapping cone

\[
CFK^{-}(\mathcal{H}, s)[[U_{k+1}]] \xrightarrow{U_{k+1}} CFK^{-}(\mathcal{H}, s)[[U_{k+1}]].
\]

In both cases, we have that the homology of the mapping cone is

\[
HFK^{-}(\mathcal{H}, s) \otimes_{\mathcal{R}} \mathbb{F}[[U_1, \ldots, U_{k+1}]]/(U_2, \ldots, U_{k+1}),
\]

where \( \mathcal{R} = \mathbb{F}[[U_1, \ldots, U_k]] \).

The Heegaard move of type (vi) changes the complex \( CFK^{-}(\mathcal{H}, s) \) by

\[
CFK^{-}(\mathcal{H}, s) \otimes H_*(S^1) \cong CFK^{-}(\mathcal{H}, s) \oplus CFK^{-}(\mathcal{H}, s).
\]

However, if \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are equivalent Heegaard diagrams both with a single pair of basepoints on \( K \), then total number of copies of \( HFK^{-}(\mathcal{H}_1, s) \)'s in \( HFK^{-}(\mathcal{H}_2, s) \) is one. \( \square \)

5.4. **Reduction of Heegaard diagrams.** Let \( \mathcal{H} \) be Heegaard diagram for a link \( L \). Then there are several Heegaard diagrams \( r_{M^*}(\mathcal{H}) \) of the sublinks of \( L \) reduced from \( \mathcal{H} \). See Definition 4.17 in [20].

There is an identification

\[
\mathfrak{A}^{-}(\mathcal{H}, p_{M^*}^1) \cong \mathfrak{A}^{-}(r_{M^*}(\mathcal{H}), \psi_{M^*}(s)).
\]
Lemma 5.8. Let \( \overline{\mathcal{L}} = \overline{L}_1 \cup \overline{L}_2 \) be a link and \( \mathcal{H} = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\}) \) be a Heegaard diagram for \( \overline{\mathcal{L}} \). Denote \( \mathfrak{A}^- (\mathcal{H}, (s_1, s_2)) \) by \( \mathfrak{A}^-_{s_1, s_2} \), for all \((s_1, s_2) \in \mathbb{H}(L)\). Then
\[
H_*(\mathfrak{A}^-_{+, \infty, s_2}/\mathfrak{A}^-_{+, \infty, s_2-1}) = HFK^-(L_2, s_2 - \frac{lk}{2}).
\]
In particular, \( \chi \left( H_*(\mathfrak{A}^-_{+, \infty, s_2}/\mathfrak{A}^-_{+, \infty, s_2-1}) \right) \) is determined by the Alexander polynomial \( \Delta_{L_2}(t) \).

Proof. We have the following commuting diagram
\[
\begin{array}{ccc}
\mathfrak{A}^-_{+, \infty, s_2-1}(L) & \xrightarrow{\sim} & \mathfrak{A}^-_{s_2-1 - \frac{lk}{2}}(L_2) \\
\downarrow_{\iota^+_{+, \infty, s_2-1}} & & \downarrow_{\iota^+_{s_2-1 - \frac{lk}{2}}} \\
\mathfrak{A}^-_{+, \infty, s_2}(L) & \xrightarrow{\sim} & \mathfrak{A}^-_{s_2 - \frac{lk}{2}}(L_2),
\end{array}
\]
where \( \iota^+_{+, \infty, s_2-1} \) and \( \iota^+_{s_2-1 - \frac{lk}{2}} \) are both the inclusions of subcomplex. Thus, we have
\[
\frac{\mathfrak{A}^-_{+, \infty, s_2}}{\mathfrak{A}^-_{+, \infty, s_2-1}} \cong \frac{\mathfrak{A}^-_{s_2 - \frac{lk}{2}}}{\mathfrak{A}^-_{s_2-1 - \frac{lk}{2}}} = CFK^-(L_2, s_2 - \frac{lk}{2}).
\]
Thus, the lemma follows. \( \square \)

5.5. Proof of Theorem 1.17

Proof. Consider the following factorization of inclusion maps of subcomplexes
\[
I^+_{s_1, s_2} : \mathfrak{A}^-_{s_1, s_2} \xrightarrow{\iota^+_{s_1, s_2}} \mathfrak{A}^-_{s_1, s_2+1} \xrightarrow{I^+_{s_1, s_2+1}} \mathfrak{A}^-_{s_1, +\infty}.
\]
It induces a factorization of the maps on homology \( (I^+_{s_1, s_2})_* = (I^+_{s_1, s_2+1})_* \circ (\iota^+_{s_1, s_2})_* \). From the proof of Theorem 2.1, we see \((\iota^+_{s_1, s_2})_*\) is a multiplication of \( U^{k_1} \), where
\[
k_{s_1, s_2}^+ = n_{s_1, s_2}^+ - n_{s_1, s_2+1}^+.
\]
Moreover, \( k = 0 \) if and only if \( H_*(\mathfrak{A}^-_{s_1, s_2+1}/\mathfrak{A}^-_{s_1, s_2}) = 0 \), and \( k = 1 \) if and only if \( H_*(\mathfrak{A}^-_{s_1, s_2+1}/\mathfrak{A}^-_{s_1, s_2}) = \mathbb{F} \) with an even grading. Then, we have
\[
\chi \left( H_*(\mathfrak{A}^-_{s_1, s_2+1}/\mathfrak{A}^-_{s_1, s_2}) \right) = n_{s_1, s_2}^+ - n_{s_1, s_2+1}^+.
\]
Whereas,
\[
\chi \left( H_*(\mathfrak{A}^-_{s_1+k, s_2+1}/\mathfrak{A}^-_{s_1+k, s_2}) \right) = \chi \left( H_*(\mathfrak{A}^-_{s_1, s_2+1}/\mathfrak{A}^-_{s_1, s_2}) \right) + \sum_{i=1}^{k} \chi (HFL^-(L, s_1 + i, s_2 + 1)), \forall k > 0.
\]
Let \( k \to \infty \), and then we have \( \chi \left( H_*(\mathfrak{A}^-_{s_1, s_2+1}/\mathfrak{A}^-_{s_1, s_2}) \right) = \chi \left( H_*(\mathfrak{A}^-_{+, \infty, s_2}/\mathfrak{A}^-_{+, \infty, s_2-1}) \right) \) determined by \( \Delta_{L_2}(t) \), by Lemma 5.8. Thus, all the \( n_{s_1, s_2}^+ \) are determined by the Alexander polynomials. Similar results hold for \( L_1 \). The theorem follows from Lemma 5.5 and Theorem 5.9. \( \square \)

In fact, when the linking number is not 0, the Alexander polynomials of \( L_1 \) and \( L_2 \) are determined by the Alexander polynomial of \( L = L_1 \cup L_2 \) and the linking number:
Theorem 5.9 (Murasugi, Proposition 4.1 in [22]). Let $\Delta_L(x, y)$ and $\Delta_{L_1}(t)$ be the Alexander polynomial of a link $L = L_1 \cup L_2$ and $L_1$ respectively in $S^3$. Then

$$\Delta_L(t, 1) = \frac{1 - tl_{k}}{1 - t} \Delta_{L_1}(t),$$

where $lk$ is the linking number of $L$.

5.6. Formulas for $n_{s}^{±L_i}(L)$'s. Using the Alexander polynomials of $L, L_1, L_2$, we can get formulas for $n_{s}^{±L_i}(L)$'s.

First of all, we fix the overall signs of these Alexander polynomials to get normalization of Equation (5.5) and Equation (4.1):

(5.6) $\sum_{s \in \mathbb{Z}} \chi \left( HFK^{-}(K, s) \right) \cdot t^{s} = \frac{t}{t^{-1}} \Delta_{K}(t),$

(5.7) $\sum_{(s_1, s_2) \in \mathbb{H}(L)} \chi \left( HFL^{-}(L, s_1, s_2) \right) \cdot x_1^{s_1} \cdot x_2^{s_2} = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \Delta_{L}(x_1, x_2).$

For an $L$-space knot $K$, to get Equation (5.6), we require that $\frac{t}{t^{-1}} \Delta_{K}(t)$ has finitely many non-zero positive powers and all the non-zero coefficients of $\frac{t}{t^{-1}} \Delta_{K}(t)$ are 1, which is equivalent to $\Delta_{K}(1) = 1$.

Theorem 5.10. Suppose $L = L_1 \cup L_2$ is an $L$-space link. Let $\Delta_{L_1}(t), \Delta_{L_2}(t)$, and $\Delta_{L}(x_1, x_2)$ be the symmetrized Alexander polynomials, such that $\Delta_{L_1}(1) = \Delta_{L_2}(1) = 1$. Let

$$\frac{t}{t - 1} \Delta_{L_1}(t) = \sum_{k \in \mathbb{Z}} a_{L_1}^{k} \cdot t^{k},$$
$$\frac{t}{t - 1} \Delta_{L_2}(t) = \sum_{k \in \mathbb{Z}} a_{L_2}^{k} \cdot t^{k},$$
$$\Delta_{L}(x_1, x_2) = \sum_{i, j} a_{i,j}^{L} \cdot x_1^{i} \cdot x_2^{j}.$$

Suppose $(i_0, j_0)$ satisfies that $a_{i_0,j_0}^{L} \neq 0, a_{i,j}^{L} = 0$ for all $i > i_0$, and $a_{i_0,j}^{L} = 0$ for all $j > j_0$. Then,

- $\chi \left( HFL^{-}(L, i_0 + \frac{1}{2}, j_0 + \frac{1}{2}) \right) = 1$ if and only if $a_{i_0+\frac{1}{2},j_0+\frac{1}{2}}^{L} = a_{i_0+\frac{1}{2},j_0+\frac{1}{2}}^{L_2} = 1$;
- $\chi \left( HFL^{-}(L, i_0 + \frac{1}{2}, j_0 - \frac{1}{2}) \right) = -1$ if and only if $a_{i_0+\frac{1}{2},j_0-\frac{1}{2}}^{L} = a_{i_0+\frac{1}{2},j_0-\frac{1}{2}}^{L_2} = 0$.

Proof. Notice that $\chi \left( \mathfrak{A}_{s_1,s_2}^{-} / \mathfrak{A}_{s_1,s_2-1}^{-} \right)$ can only be 0 or 1 for all $(s_1, s_2) \in \mathbb{H}(L)$. By Equation (4.2), we have two possible cases:

(a) $\chi \left( HFL^{-}(L, s_1, s_2) \right) = 1$ if and only if $\chi \left( \mathfrak{A}_{s_1,s_2}^{-} / \mathfrak{A}_{s_1,s_2-1}^{-} \right) = 1$ and $\chi \left( \mathfrak{A}_{s_1-1,s_2}^{-} / \mathfrak{A}_{s_1-1,s_2-1}^{-} \right) = 0$;

(b) $\chi \left( HFL^{-}(L, s_1, s_2) \right) = -1$ if and only if $\chi \left( \mathfrak{A}_{s_1,s_2}^{-} / \mathfrak{A}_{s_1,s_2-1}^{-} \right) = 0$ and $\chi \left( \mathfrak{A}_{s_1-1,s_2}^{-} / \mathfrak{A}_{s_1-1,s_2-1}^{-} \right) = 1$.

In addition, we have

$$\chi \left( \mathfrak{A}_{i_0+\frac{1}{2},j_0+\frac{1}{2}}^{-} / \mathfrak{A}_{i_0+\frac{1}{2},j_0-\frac{1}{2}}^{-} \right) = \chi \left( \mathfrak{A}_{i_0+\frac{1}{2},j_0+\frac{1}{2}}^{-} / \mathfrak{A}_{i_0+\frac{1}{2},j_0-\frac{1}{2}}^{-} \right) = \chi \left( HFK^{-}(L_2, j_0 + \frac{1}{2} - \frac{lk}{2}) \right).$$

So $\chi \left( HFL^{-}(L, s_1, s_2) \right) = 1$ if and only if $a_{L_2}^{L_2} \cdot x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} = 1$. Symmetrically, we have $\chi \left( HFL^{-}(L, s_1, s_2) \right) = 1$ if and only if $a_{L_2}^{L_2} \cdot x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} = 1$. Similar argument applies to the case (b). □

Definition 5.11 (Normalization of Alexander polynomials for $L$-space links). Suppose $L = L_1 \cup L_2$ is an $L$-space link. Let the symmetrized Alexander polynomial of $L$ be

$$\Delta_{L}(x_1, x_2) = \sum_{i,j} a_{i,j}^{L} \cdot x_1^{i} \cdot x_2^{j},$$
where \( x_i \) corresponds to the component \( L_i \) for \( i = 1, 2 \). Let the symmetrized Alexander polynomials of \( L_1, L_2 \) be \( \Delta_{L_1}(t), \Delta_{L_2}(t) \) in the forms of

\[
\frac{t}{t - 1} \Delta_{L_1}(t) = \sum_{k \in \mathbb{Z}} a_k^{L_1} \cdot t^k, \quad \frac{t}{t - 1} \Delta_{L_2}(t) = \sum_{k \in \mathbb{Z}} a_k^{L_2} \cdot t^k.
\]

Let \((i_0, j_0)\) be such that

\[
j_0 = \max\{j \in \mathbb{Z} + \frac{lk - 1}{2} | a_{i,j}^{L_1} \neq 0\} \quad \text{and} \quad i_0 = \max\{i \in \mathbb{Z} + \frac{lk - 1}{2} | a_{i,j}^{L_2} \neq 0\}.
\]

Then, these Alexander polynomials are called normalized, if

1. the leading coefficient of \( \Delta_{L_i}(t) \) is 1 for both \( i = 1, 2 \), which is equivalent to \( \Delta_{L_i}(1) = 1 \);
2. if \( a_{j_0 - \frac{lk}{2} + \frac{1}{2}}^{L_1} = 1 \), then \( a_{i_0,j_0}^{L_1} = 1 \); while if \( a_{j_0 - \frac{lk}{2} + \frac{1}{2}}^{L_2} = 0 \), then \( a_{i_0,j_0}^{L_2} = -1 \).

After normalization, we have \( \chi(HFL^-(L, s_1, s_2)) = a_{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}}^{L_1} \) and \( \chi(HFK^-(L_i, s)) = a_{s}^{L_i} \) for \( i = 1, 2 \). Therefore,

\[
\chi(H_*(\mathbb{Z}_{s_1, s_2}^-/\mathbb{Z}_{s_1, s_2}^-)) = a_{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}}^{L_2} - \sum_{i=1}^{\infty} a_{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}}^{L_1} = 0 \text{ or } 1.
\]

Hence, we have

\[
(5.8) \quad n_{s_1, s_2}^{+L_2} = \sum_{j=1}^{\infty} \left( a_{s_2 + j - \frac{lk}{2}}^{L_2} - \sum_{i=1}^{\infty} a_{s_1 + i - \frac{lk}{2}, s_2 + j - \frac{1}{2}}^{L_1} \right).
\]

Similarly, we have

\[
(5.9) \quad n_{s_1, s_2}^{+L_1} = \sum_{i=1}^{\infty} \left( a_{s_1 + i - \frac{lk}{2}}^{L_1} - \sum_{j=1}^{\infty} a_{s_1 + i - \frac{lk}{2}, s_2 + j - \frac{1}{2}}^{L_2} \right).
\]

**Theorem 5.12.** Suppose \( L = L_1 \cup L_2 \) is an \( L \)-space link. Under the normalization in Definition 5.11 we have that the formulas in Equation (5.8) and Equation (5.9) are non-negative for all \((s_1, s_2) \in \mathbb{H}(L)\).

In fact, both of Theorem 5.10 and Theorem 5.12 give additional constraints for the Alexander polynomials of an \( L \)-space 2-component link.

**Proposition 5.13.** The link \( L7n2 \) is not an \( L \)-space link.

**Proof.** We give two proofs based on Theorem 5.10 and Theorem 5.12 respectively. Suppose \( L = L7n2 \) is an \( L \)-space link with components \( L_1 \) and \( L_2 \), where \( L_1 \) is the unknot and \( L_2 \) is the right-handed trefoil. Then, we get the normalized Alexander polynomials of \( L_1 \) and \( L_2 \):

\[
\frac{t}{t - 1} \Delta_{L_1}(t) = 1 + t^{-1} + t^{-2} + \cdots,
\]

\[
\frac{t}{t - 1} \Delta_{L_2}(t) = t + t^{-1} + t^{-2} + \cdots.
\]

Since \( \Delta_L(x, y) = \frac{(x-1)(y-1)}{\sqrt{xy}} \) and \( \text{lk} = 0 \), by Theorem 5.10 we have \( a_1^{L_1} = a_1^{L_2} \). This is a contradiction to \( a_1^{L_1} = 0 \) and \( a_1^{L_2} = 1 \).

Another proof is as follows. If we used the normalization in Definition 5.11 for \( L7n2 \), then we get \( n_{0,0}^{+L_1} = -1 \) by Equation (5.9). This is a contradiction to Theorem 5.12. \( \square \)
6. Applications

Classifying $L$-space surgeries on an $L$-space link $L$ is usually challenging. Let us look at an example, $L = T(2, 2n)$. The $T(2, 2)$ torus link is the Hopf link and its surgeries are lens spaces, so we assume $n \geq 2$.

When both of $p$ and $q$ are not equal to $n$, the $(p, q)$-surgery on $T(2, 2n)$ is a Seifert manifold with three singular fibers over the base $S^2$. Using the notational convention in [17], we can write $S^3_{p,q}(T(2, 2n)) = -M(0; \frac{1}{n}, \frac{1}{p-n}, \frac{1}{q-n})$. In [17], Lisca and Stipsicz give a characterization of $L$-space Seifert manifolds.

**Theorem 6.1** (Theorem 1.1, [17]). Suppose $M$ is an oriented rational homology sphere which is Seifert fibered over $S^2$. Then, $M$ is an $L$-space if and only if either $M$ or $-M$ carries no positive, transverse contact structures.

**Theorem 6.2** ([16]). An oriented Seifert rational homology sphere $M = M(e_0; r_1, ..., r_k)$ with $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$ admits no positive, transverse contact structure if and only if

- $e_0(M) \geq 0$, or
- $e_0(M) = -1$ and there are no relatively prime integers $m > a$ such that

$$mr_1 < a < m(1 - r_2), \text{ and } mr_i < 1, i = 3, ..., k.$$  

While the $(n, q)$-surgery on $T(2, 2n)$ is usually a graph manifold. The $(n, q)$-surgeries are discussed in Corollary 3.6. Direct computation gives the following result.

**Proposition 6.3** (Classification of $L$-space surgeries on $T(2, 2n)$ with $n \geq 2$). For all $q \neq n$, the $(n, q)$-surgery on $T(2, 2n)$ is an $L$-space.

When $p \neq n, q \neq n$ and $p \geq q$, $S^3_{p,q}(T(2, 2n))$ is an $L$-space with if and only if one of the following conditions holds:

1. $n + 2 \leq p, n + 1 \leq q$;
2. $2n \leq p, n - 2 \geq q$, and there are no relatively prime integers $m > a > 0$ such that

$$m \frac{n - q - 1}{n - q} < a < m(1 - \frac{1}{n})$$  

and

$$m \frac{1}{p - n} < 1;$$  

3. $n + 2 \leq p \leq 2n, q \leq n - 2$, and there are no relatively prime integers $m > a > 0$ such that

$$m \frac{n - q - 1}{n - q} < a < m(1 - \frac{1}{p - n})$$  

and

$$m \frac{1}{n} < 1;$$  

4. $p = n + 1, q \leq n + 1$, and $q \neq n$;
5. $p = n - 1, q \leq n - 1$;
6. $p \leq n - 2, q \leq p$, and there are no relatively prime integers $m > a > 0$ such that

$$m(1 - \frac{1}{n}) < a < m \frac{1}{n - p}$$  

and

$$m \frac{1}{n - q} < 1.$$  

See Figure 6.1 for the example of $T(2, 20)$.

Nevertheless, the links $T(2, 2n)$ are the simplest two-bridge links. In order to generally study $L$-surgeries on $L$, we give an algorithm computing $\tilde{HF}(S^3_\Lambda(L))$ using the Alexander polynomials.

Another example is the Whitehead link. By the results in Section 6 in [18], we have the following proposition. In order to distinguish it with its mirror, we call it the $L$-space Whitehead link.

**Proposition 6.4.** The $(p_1, p_2)$-surgery on the $L$-space Whitehead link is an $L$-space if and only if $p_1 > 0, p_2 > 0$. 


Figure 6.1. The \textbf{L-space surgeries on }$T(2, 20)$. We draw the L-space surgeries of $T(2, 20)$ on the x-y plane within the range $[-40, 40] \times [-40, 40]$. Every dot $(p, q)$ represents an L-space surgery $(p, q)$. The blue points are Seifert L-space surgeries determined by the characterization of Lisca-Stipsicz, while the red points are determined by induction. The six labelled regions correspond to the six conditions (1) to (6) in Proposition 6.3. The drawn hyperbola indicates the positions of the surgeries with $b_1 = 1$.

6.1. \textbf{Truncated perturbed surgery complex}. The link surgery formula is an infinitely generated $\mathbb{F}[[U_1, U_2]]$-module. A truncation procedure is introduced in Section 8.3 in [20] to reduce it to finitely generated $\mathbb{F}[[U_1, U_2]]$-module. It is called \textit{horizontal truncation} in [20], and we just call it \textit{truncation} here. A truncation for the \(\Lambda\)-surgery on a 2-component link \(L\) is described by four finite subsets of \(\mathbb{H}(L)\),

\[S_{00}^{00}(\Lambda), S_{01}^{01}(\Lambda), S_{10}^{10}(\Lambda), S_{11}^{11}(\Lambda).\]

The way of doing truncation is not unique. Later, we will describe an explicit way which depends on \(L\) and \(\Lambda\).

Define

\[\tilde{C}^{\delta_1\delta_2}(\Lambda) = \bigoplus_{s \in S_{\delta_1\delta_2}} \mathcal{A}_{\delta_1\delta_2}^{\delta_1\delta_2}, \quad \delta_1, \delta_2 \in \{0, 1\}.\]

Then, the truncated perturbed complex \(\tilde{C}(\Lambda)\) for an L-space link is defined as follows:

\begin{equation}
(\tilde{C}^-(\mathcal{H}, \Lambda), \tilde{D}^-(\Lambda)) := \begin{array}{c}
\tilde{C}^{00}(\Lambda) \xrightarrow{\tilde{D}^{00}_{00}(\Lambda)} \tilde{C}^{01}(\Lambda) \\
\Downarrow \tilde{D}^{01}_{01}(\Lambda) & \Downarrow \tilde{D}^{10}_{01}(\Lambda) \\
\tilde{C}^{01}(\Lambda) \xrightarrow{\tilde{D}^{10}_{01}(\Lambda)} \tilde{C}^{11}(\Lambda),
\end{array}
\end{equation}

where \(\tilde{D}^{\delta_1\delta_2}_{\epsilon_1\epsilon_2}(\Lambda)\) are the restrictions of \(\tilde{D}^{\delta_1\delta_2}_{\epsilon_1\epsilon_2}(\Lambda)\) on the truncated complexes.
The surgery complex naturally splits as a direct sum corresponding to Spin$^c$ structures. For the $\Lambda$-surgery on $L$, there is an identification Spin$^c(S^3_\Lambda(L)) = \mathbb{H}(L)/H(L,\Lambda)$, where $H(L,\Lambda)$ is the lattice spanned by $\Lambda$. For $u \in \mathbb{H}(L)/H(L,\Lambda)$, choose $s = (s_1, s_2) \in u$. Denote
\[
\tilde{C}^{\delta_1,\delta_2}(\Lambda, u) = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} \tilde{\Lambda}_{s+i\Lambda_1+j\Lambda_2}.
\]

Then, the summand $\tilde{C}(\Lambda, u)$ is as follows:

\[
(\tilde{C}^-(\Lambda, \Lambda, u), \tilde{D}^-(\Lambda, u)) := \tilde{C}^{00}(\Lambda, u) \xrightarrow{\tilde{D}^{00}_{10}(\Lambda, u)} \tilde{C}^{10}(\Lambda, u) \xrightarrow{\tilde{D}^{01}_{10}(\Lambda, u)} \tilde{C}^{01}(\Lambda, u) \xrightarrow{\tilde{D}^{01}_{11}(\Lambda, u)} \tilde{C}^{11}(\Lambda, u).
\]

By putting $U_1 = 0$, we can get the chain complex of $\mathbb{F}$-vector spaces $\tilde{C}^*(\Lambda, u)$, whose homology is isomorphic to $HF(S^3_\Lambda(L), u)$.

**Lemma 6.5.** Suppose $A, B, C, D$ are finite dimensional $\mathbb{F}$-vector spaces and the following diagram commutes
\[
\begin{array}{ccc}
A & \xrightarrow{h_1} & B \\
\downarrow v_1 & & \downarrow v_2 \\
C & \xrightarrow{h_2} & D.
\end{array}
\]

We form a chain complex $(R_*, d_*)$ supported on degrees $0, 1, 2$, $R : A \xrightarrow{d_2} B \oplus C \xrightarrow{d_1} D$ with $d_2 = h_1 + v_1$ and $d_1 = h_2 \oplus v_2$. Then, we have the following conclusions

(1) $\dim H_*(R) = 2 \dim(\ker h_1 \cap \ker v_1) - 2 \dim(\text{im } v_2 + \text{im } h_2) - \dim A + \dim B + \dim C + \dim D$;

(2) $\dim H_*(R) = 1$ iff one of the following is true

a. $\chi = \dim A - \dim B - \dim C + \dim D = 1$, and $\dim(\ker(h_1) \cap \ker(v_1)) + \dim(\coker(v_2 + h_2)) = 1$.

b. $\chi = \dim A - \dim B - \dim C + \dim D = -1$, and $\dim(\ker(h_1) \cap \ker(v_1)) + \dim(\coker(v_2 + h_2)) = 0$.

**Proof.** Part (1) is a straightforward computation. Notice that $H_0 = \ker(h_2 \oplus v_2)$, $H_2 = \ker(h_1 + v_1)$.

For Part (2), there are only three cases when $H_*(R) = \mathbb{F}$ happens,

(1) $H_0(R) = \mathbb{F}$, $H_1(R) = H_2(R) = 0$;

(2) $H_1(R) = \mathbb{F}$, $H_0(R) = H_2(R) = 0$;

(3) $H_2(R) = \mathbb{F}$, $H_0(R) = H_1(R) = 0$.

In cases (1) and (3), we have that $\chi = 1$ and $\dim H_0 + \dim H_2 = 1$; in case (2), we have that $\chi = -1$ and $\dim H_0 + \dim H_1 = 0$. It is not hard to check the converse. \hfill \Box

If $\ker(v_1), \ker(h_1)$ are both known, then computing $\dim(\ker(v_1) \cap \ker(h_1))$ is equivalent to computing $\dim(\ker(v_1) + \ker(h_1))$, which can be done by Gauss Elimination.

While computing $\ker(v_2 + h_2)$ is the dual question for computing $\ker(v_2^*) \cap \ker(h_2^*)$. While the dual maps $v_2^*$ and $h_2^*$ can be obtained by reversing the arrows, since we are working over $\mathbb{F}$.

We can directly apply the above lemma for each truncated perturbed complex $\tilde{C}^*(\Lambda, u)$ for each Spin$^c$ structure. Thus, we only need to describe the truncated regions $S^{00}(\Lambda), S^{01}(\Lambda), S^{10}(\Lambda), S^{11}(\Lambda)$ and the kernels of the maps $\tilde{D}^{**}(\Lambda, u)$ and their dual.
**Figure 6.2. The truncation.** The vectors $\Lambda_1$ and $\Lambda_2$ are determined by the surgery framing matrix. The edges of the parallelogram $Q$ are parallel to $\Lambda_1$ and $\Lambda_2$, and they indicate the border lines of various acyclic subcomplexes or quotient complexes. Thus, the parallelogram $Q$ roughly indicates the support of the truncated complex.

**Proposition 6.6.** Suppose $L$ is an L-space link. Fix a surgery framing $\Lambda$ and a Spin$^c$ structure $u$. Then, $HF(\Lambda, u) = \mathbb{F}$ iff in the truncated complex $\tilde{C}^*(\Lambda, u)$, one of the following is true,

(A) $\#S^{00}(\Lambda, u) - \#S^{01}(\Lambda, u) - \#S^{10}(\Lambda, u) + \#S^{11}(\Lambda, u) = 1$, and $\dim(\text{Ker}(\hat{D}_{00}^{10}) \cap \text{Ker}(\hat{D}_{00}^{01})) + \dim \text{Coker}(\hat{D}_{01}^{10} + \hat{D}_{10}^{01}) = 1$.

(B) $\#S^{00}(\Lambda, u) - \#S^{01}(\Lambda, u) - \#S^{10}(\Lambda, u) + \#S^{11}(\Lambda, u) = -1$, and $\dim(\text{Ker}(\hat{D}_{00}^{10}) \cap \text{Ker}(\hat{D}_{00}^{01})) + \dim \text{Coker}(\hat{D}_{01}^{10} + \hat{D}_{10}^{01}) = 0$.

6.2. Truncations. We explicitly describe the truncated regions $S^{00}(\Lambda), S^{01}(\Lambda), S^{10}(\Lambda), S^{11}(\Lambda)$ here. Let us briefly recall the procedure to form these truncated regions for a general two-component link $L$ in Section 8.3 [20].

(1) Choose a number $b \in \mathbb{N}$, such that the inclusion maps $I_{s_i}$'s are quasi-isomorphisms whenever $\pm s_i \geq b$.

(2) Determine a parallelogram $Q$ in the plane, with vertices $P_1, P_2, P_3, P_4$ counterclockwise labelled, satisfying the following condition: The point $P_i$ has the coordinate $(x_i, y_i)$ such that

$$\begin{align*}
    x_1 > b, & \quad y_1 > b, \\
    x_2 < b, & \quad y_2 > b, \\
    x_3 < b, & \quad y_3 < b, \\
    x_4 > b, & \quad y_4 < b.
\end{align*}$$

We also require that every edge is either parallel to the vector $\Lambda_1$ with length greater than $\|\Lambda_1\|$ or parallel to $\Lambda_2$ with length greater than $\|\Lambda_2\|$.

(3) Decide which is the case among the six cases of the surgeries described in Figure 22 in [20].

Then, we can decide the corresponding truncated regions according to Section 8.3 in [20].

The way of doing truncation is not unique. One explicit way to choose the parallelogram $Q$ to be centered at the origin as follows. See Figure 6.2

Let

$$\{P_1, P_2, P_3, P_4\} = \left\{ \frac{i_0A_1 + j_0A_2}{2}, -\frac{i_0A_1 + j_0A_2}{2}, \frac{i_0A_1 - j_0A_2}{2}, -\frac{i_0A_1 - j_0A_2}{2} \right\}.$$
with $i_0, j_0$ being positive integers, such that Equations (6.3) hold. Fix $\Lambda$ and $u \in \mathbb{H}(L)/H(L, \Lambda)$. Suppose

$$s = \theta_1 \Lambda_1 + \theta_2 \Lambda_2 \in u, \quad P_1 = a_1 \Lambda_1 + a_2 \Lambda_2.$$ 

We denote

$$A_1 = [-\theta_1 - |a_1|], \quad A_2 = [-\theta_1 + |a_1|],$$

$$B_1 = [-\theta_2 - |a_2|], \quad B_2 = [-\theta_2 + |a_2|].$$

Then, the truncated regions in the six cases are as follows.

**Case I:**

$$S^{00}(\Lambda, u) = u \cap Q,$$

$$S^{10}(\Lambda, u) = u \cap (Q + \Lambda_1),$$

$$S^{01}(\Lambda, u) = u \cap (Q + \Lambda_2),$$

$$S^{11}(\Lambda, u) = u \cap (Q + \Lambda_1 + \Lambda_2).$$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1 \delta_2}(\Lambda, u) = \{s + i \Lambda_1 + j \Lambda_2 | A_1 + \delta_1 \leq i \leq A_2, B_1 + \delta_2 \leq j \leq B_2 \}.$$

**Case II:**

$$S^{00}(\Lambda, u) = u \cap Q,$$

$$S^{10}(\Lambda, u) = u \cap (Q \cup (Q + \Lambda_1)),$$

$$S^{01}(\Lambda, u) = u \cap (Q \cup (Q + \Lambda_2)),$$

$$S^{11}(\Lambda, u) = u \cap (Q \cup (Q + \Lambda_1) \cup (Q + \Lambda_2) \cup (Q + \Lambda_1 + \Lambda_2)).$$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1 \delta_2}(\Lambda, u) = \{s + i \Lambda_1 + j \Lambda_2 | A_1 - \delta_1 \leq i \leq A_2, B_1 - \delta_2 \leq j \leq B_2 \}.$$

**Case III:**

$$S^{00}(\Lambda, u) = u \cap Q,$$

$$S^{10}(\Lambda, u) = u \cap (Q \cap (Q + \Lambda_1)),$$

$$S^{01}(\Lambda, u) = u \cap (Q \cup (Q + \Lambda_2)),$$

$$S^{11}(\Lambda, u) = u \cap \{Q \cup (Q + \Lambda_2)\} \cap \{Q \cup (Q + \Lambda_2) \cup (Q + \Lambda_1) \}. $$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1 \delta_2}(\Lambda, u) = \{s + i \Lambda_1 + j \Lambda_2 | A_1 + \delta_1 \leq i \leq A_2, B_1 \leq j \leq B_2 + \delta_2 \}.$$

**Case IV:**

$$S^{00}(\Lambda, u) = u \cap Q,$$

$$S^{10}(\Lambda, u) = u \cap (Q \cup (Q + \Lambda_1)), $$

$$S^{01}(\Lambda, u) = u \cap (Q \cap (Q + \Lambda_2)), $$

$$S^{11}(\Lambda, u) = u \cap (Q \cap (Q + \Lambda_2) \cup (Q \cap (Q + \Lambda_2) + \Lambda_1) \}. $$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1 \delta_2}(\Lambda, u) = \{s + i \Lambda_1 + j \Lambda_2 | A_1 \leq i \leq A_2 + \delta_1, B_1 + \delta_2 \leq j \leq B_2 \}.$$
Case V: This case is similar to Case I, but the regions $S^{10}(\Lambda, u), S^{01}(\Lambda, u)$ have two more points at the corners.

\[
\begin{align*}
S^{00}(\Lambda, u) &= u \cap Q, \\
S^{10}(\Lambda, u) &= (u \cap Q \cap (Q + \Lambda_1)) \cup T^{10}, \\
S^{01}(\Lambda, u) &= (u \cap Q \cap (Q + \Lambda_2)) \cup T^{01}, \\
S^{11}(\Lambda, u) &= u \cap Q \cap (Q + \Lambda_1 + \Lambda_2),
\end{align*}
\]

where $T^{10} = \{s + A_2\Lambda_1 + B_1\Lambda_2\}, T^{10} = \{s + A_1\Lambda_1 + B_2\Lambda_2\}$.

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

\[
S^{\delta_1\delta_2}(\Lambda, u) = \{s + i\Lambda_1 + j\Lambda_2|A_1 + \delta_1 \leq i \leq A_2, B_1 + \delta_2 \leq j \leq B_2,\} \cup T^{\delta_1\delta_2},
\]

where $T^{00} = T^{11} = \emptyset$.

Case VI: This is similar to Case V.

\[
\begin{align*}
S^{00}(\Lambda, u) &= u \cap Q \cap (Q - \Lambda_1 - \Lambda_2), \\
S^{10}(\Lambda, u) &= (u \cap Q \cap (Q - \Lambda_1)) \cup T^{10}, \\
S^{01}(\Lambda, u) &= (u \cap Q \cap (Q - \Lambda_2)) \cup T^{01}, \\
S^{11}(\Lambda, u) &= u \cap Q,
\end{align*}
\]

where $T^{10} = s + A_1\Lambda_1 + B_2\Lambda_2, T^{01} = s + B_1\Lambda_2 + A_1\Lambda_1$.

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

\[
S^{\delta_1\delta_2}(\Lambda, u) = \{s + i\Lambda_1 + j\Lambda_2|A_1 - 1 - \delta_1 \leq i \leq A_2, B_1 - 1 - \delta_2 \leq j \leq B_2,\} \cup T^{\delta_1\delta_2},
\]

where $T^{00} = T^{11} = \emptyset$.

Remark 6.7. In all of the above cases, $\#S^{00}(\Lambda, u) - \#S^{01}(\Lambda, u) - \#S^{10}(\Lambda, u) + \#S^{11}(\Lambda, u) = \pm 1$.

6.3. Kernel of $\tilde{D}^{*\ast}(\Lambda, u)$. In fact, all the mapping cones of $\tilde{D}^{*\ast}(\Lambda, u)$ split as a direct sum of mapping cones in a common form. They look like the mapping cones in computing +1-surgery on knots. Since this type of mapping cones looks like zigzags, we just call them "zigzags". We denote the set of integers in $[a, b]$ by $[a; b]$, where we allow $a = b$.

Definition 6.8 (Zigzags). A zigzag mapping cone $C$ is a mapping cone of $\mathbb{F}$-vector spaces:

\[
\bigoplus_{a_1 \leq s \leq a_2} A_s \xrightarrow{f + g} \bigoplus_{b_1 \leq t \leq b_2} B_t,
\]

where

\[
\begin{align*}
A_s &= \mathbb{F}, \forall a_1 \leq s \leq a_2, \\
B_t &= \mathbb{F}, \forall b_1 \leq t \leq b_2, \\
f &= \bigoplus f_s, \quad f_s : A_s \to B_s, \\
g &= \bigoplus g_s, \quad g_s : A_s \to B_{s+1}.
\end{align*}
\]

The code of the zigzag $C$ is a set of data $\{(a_1; a_2), [b_1; b_2], S_1, S_2\}$, where

\[
\begin{align*}
S_1 &= \{s \in \mathbb{Z}|f_s \neq 0\}, \\
S_2 &= \{s \in \mathbb{Z}|g_s \neq 0\}.
\end{align*}
\]

We define $\text{Ker}(C)$ (resp. $\text{Coker}(C)$) to be $\text{Ker}(f + g)$ (resp. $\text{Coker}(f + g)$).

Definition 6.9. For any element $x$ in $\bigoplus_{a_1 \leq s \leq a_2} \mathbb{F}.e_s$, we can represent it uniquely by $x = \sum_{s \in \Gamma} e_s$. We call $\Gamma$ the support of $x$, and denote it by $\text{Supp}(x)$. Similarly, for $X = \{x_1, \ldots, x_n\}$, we denote $\{\text{Supp}(x_1), \ldots, \text{Supp}(x_n)\}$ by $\text{Supp}(X)$.
Proposition 6.10. For a zigzag $C$ with the code $\{[a_1, a_2], [b_1, b_2], S_1, S_2\}$, we represent $S_1 \cap S_2$ by a minimal disjoint unions

$$S_1 \cap S_2 = \prod_{i \in [1:K]} [\alpha_i; \beta_i],$$

with $\beta_i \leq \alpha_{i+1} + 2, \forall i$. Then, $\text{Ker}(C)$ has a basis with the following support

$$\left\{ \{s\} \mid s \in [a_1, a_2] \setminus (S_1 \cup S_2) \right\} \cup \left\{ [\alpha_j - 1, \beta_j + 1] | \alpha_j - 1 \in S_2, \beta_j + 1 \in S_1 \right\}.$$

Proof. Straightforward. \qed

Definition 6.11. Let $L = L_1 \cup L_2$ be an $L$-space link. For all $s_1 \in \mathbb{H}_1(L), s_2 \in \mathbb{H}_2(L)$, we define

$$\nu_{s_1}^{L_2}(L) = \min \{ s_2 \in \mathbb{H}_2(L) | n_{s_1, s_2}^{L_2} \neq 0 \},$$

and choose $s = (s_1, s_2) \in u$. Before truncation, we have

$$\text{cone}(\hat{D}_{00}(L, u)) = \prod_{i \in \mathbb{Z}} \text{cone}(\prod_{j \in \mathbb{Z}} (\hat{\Phi}_{s_1}^{L_2} + \hat{\Phi}_{s_1}^{L_2})).$$

After truncation, $\text{cone}(\hat{D}_{00}(L, u))$ splits into direct sums of zigzags in form of

$$\text{cone}(\prod_{j \in \mathbb{Z}} (\hat{\Phi}_{s_1}^{L_2} + \hat{\Phi}_{s_1}^{L_2})) \cap \hat{C}^{*}(L).$$

Let us figure out the codes of these zigzags. Suppose the code of the above zigzag is

$$\{[a_1; a_2], [b_1; b_2], S_1, S_2\}.$$ 

Then, it is not hard to get the following formulas for the code,

$$\begin{align*}
[a_1; a_2] &= \{ j \in \mathbb{Z} | s + i \Lambda_1 + j \Lambda_2 \in S_{00}(L, u) \}, \\
[b_1; b_2] &= \{ j \in \mathbb{Z} | s + i \Lambda_1 + j \Lambda_2 \in S_{01}(L, u) \}, \\
S_1 &= \{ j \in \mathbb{Z} | s_2 + i \cdot \text{lk} + j \cdot p_2 \geq \nu_{s_1}^{L_2} + j \cdot \text{lk}(L) \}, \\
S_2 &= \{ j \in \mathbb{Z} | s_2 + i \cdot \text{lk} + j \cdot p_2 \leq -\nu_{s_1}^{L_2} - i \cdot p_1 - j \cdot \text{lk}(L) \}. 
\end{align*}$$

6.4. Examples: $L$-space surgeries on two-bridge links. From Proposition 2.11, we see that if a two-bridge link has an $L$-space surgery, then it is a generalized $L$-space link. By taking mirrors, we can reduce these links to two types: $L$-space links and generalized $(\pm \nu)$-space links. We have discussed two-bridge $L$-space links in Section 3. By the method in this section, it is convenient to make computer programs for computing $\hat{H}F$ of their surgeries and give characterizations of $L$-space surgeries. However, to find a general formula of $\hat{H}F$ is not easy.

In fact, finding $L$-space homology spheres is more interesting. Let us try some examples here, by looking at $(1, 1)$-surgeries on $L_n = b(4n^2 + 4n, -2n - 1)$ for all positive integers $n$. This sequence of $L$-space links have linking numbers $0$, and $L_1$ is the Whitehead link.

Proposition 6.12. For all $n \geq 2$, the $(1, 1)$-surgery on $b(4n^2 + 4n, -2n - 1)$ is not an $L$-space.
With the help of a computer program, we get the Alexander polynomials of \( L_n \):
\[
\Delta_{L_n}(x, y) = \sum_{j=-n}^{n-1} \sum_{i=-n}^{-n-i+j+\frac{1}{2}} (-1)^{i+j} x^{i+j} y^{i+j}.
\]

After normalizing \( \Delta_{L_n}(x, y) \) by Definition 5.11, we can get formulas for \( n_{s_1, s_2}^{L_1} \) by Equation (5.9), (5.8). We list the numbers \( \{ n_{s_1, s_2}^{L_2}(L_n) \} -4 \leq s_1 \leq 4, -4 \leq s_2 \leq 4 \) for \( n = 1, 2, 3, 4 \) as follows:

\[
\begin{align*}
\{ n_{s_1, s_2}^{L_2}(L_1) \} : \\
\{ n_{s_1, s_2}^{L_2}(L_3) \} : \\
\{ n_{s_1, s_2}^{L_2}(L_4) \} :
\end{align*}
\]

In particular, we get the following formulas for all \( s_1 \in \mathbb{Z} \),
\[
\nu_{s_1}^{L_2}(L_n) = \begin{cases} n - |s_1|, & |s_1| \leq n, \\ 0, & |s_1| \geq n. \end{cases}
\]

Since \( L_n \) is a two-bridge link, we have the symmetry \( \nu_{s_1}^{L_2} = \nu_{s_1}^{L_1} \), when \( s_1 = s_2 \).

Thus, we can let \( b(L_n) = n \). Then, as described in Section 6.3, the truncation regions are determined by the parallelogram \( Q \), with vertices \( P_1 = (n, n), P_2 = (-n, n), P_3 = (-n, -n), P_4 = (n, -n) \). The surgery framing is in Case I, so we have the truncated regions
\[
S_{\delta_1 \delta_2} = \{(i, j) \in \mathbb{Z}^2 | -n + \delta_1 \leq i \leq n, -n + \delta_2 \leq j \leq n \}.
\]

Now we can see
\[
\hat{s}_{s_1, s_2}^{L_1} = 0, \forall -n - 1 \leq s_1 \leq n - 1, -n + 1 \leq s_2 \leq n - 1, i = 1, 2.
\]
So \( \hat{s}_{s_1, s_2} \in \hat{C}^{00} \) with \(-n < s_1 < n, -n < s_2 < n \) are all in the kernel of \( \hat{D}_{00}^{10} \) and \( \hat{D}_{00}^{01} \). So when \( n \geq 2 \), we have that \( \text{Ker}(\hat{D}_{00}^{01}) \cap \text{Ker}(\hat{D}_{00}^{10}) \) has rank at least \( n^2 + (n-1)^2 > 1 \). Thus, by Proposition 6.6, the \((1, 1)\)-surgeries on \( L_n \) with \( n \geq 2 \) are never \( L \)-spaces. Similar arguments apply to \((\pm 1, \pm 1)\)-surgeries on these links.

**Proposition 6.13.** On the two-bridge \( L \)-space links \( L_n = b(4n^2 + 4n, -2n - 1) \) with \( n \geq 2 \), there are no \( L \)-space homology sphere surgeries.

In fact, direct computations using the zigzags give that \( \hat{H}(S^{3}_{1,1}(L_n)) \) has dimension \((2n - 1)^2\).
L-Space surgeries on links

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