Numerical results for influence the flow of MHD nanofluids on heat and mass transfer past a stretched surface

https://doi.org/10.1515/nleng-2021-0003
Received Jul 28, 2020; accepted Jan 28, 2021.

Abstract: Due to its significant applications in physics, chemistry, and engineering, some interest has been given in recent years to research the boundary layer flow of magnetohydrodynamic nanofluids. The numerical results were analyzed for temperature profile, concentration profile, reduced number of Nusselt and reduced number of Sherwood. It has also been shown that the magnetic field, the Eckert number, and the thermophoresis parameter boost the temperature field and raise the thermal boundary layer thickness while the Prandtl number reduces the temperature field at high values and lowers the thermal boundary layer thickness. However, if Lewis number is higher than the unit and the Eckert number increases, the concentration profiles decrease as well. Ultimately, the concentration profiles are reduced for the variance of the Brownian motion parameter and the Eckert number, where the thickness of the boundary layer for the mass friction feature is reduced.

Keywords: nanofluids, stretching sheet, magnetic field, viscous dissipation and Joule heating, Chebyshev pseudospectral technique

Chinese Library Classification: O302

1 Introduction

One of the most important emerging developments of the 21st century is nanotechnology. It has been commonly used in manufacturing and nanometersized materials have special physical and chemical properties. With its increasingly significant and complex effect on a wide range of industries, including biotechnology, oil, electronics and consumer goods, it promises to change our lives in this decade. In many engineering processes with applications in industries such as extrusion, melt-spinning, heat rolling, wire drawing, glass-fiber processing, plastic and rubber sheet manufacturing, cooling of a large metal plate in a bath that may be an electrolyte, etc., flow over a stretching surface is an important issue. The authors in [1] investigate the elastic deformation effects on the boundary layer flow of an incompressible second grade two phase nanofluid model over a stretching surface in the presence of suction and partial slip boundary condition.

In industry, polymer sheets and filaments are manufactured by continuous extrusion of the polymer from a die to a windup roller, which is located at a finite distance away [2]. There are many applications in engineering and industry for boundary layer flow behavior over a stretching surface [3]. Having a low heat transfer in a fluid would cause limited heat transfer and can lead to limited heat transfer efficiency. Due to the high thermal conductivity of metal particles, adding them to a fluid would increase the thermal conductivity and also heat transfer of the resultant mixture fluid. Choi’s [4] initial analysis of the term "nanofluid" identified a liquid suspension containing ultra-fine particles. Nanoparticles (e.g., Copper (Cu), Silver (Ag), Alumina (Al₂O₃), Titanium (TiO₂)) range from 1 to 100 nm in diameter [5]. The base liquid’s thermal conductivity is improved by (10% – 50%) if it is suspended by a low volumetric fraction (less than 5%) of nanoparticles [6–8]. In Ref. [9], the authors examined improvements in the thermal conductivity of fluids (such as oil, water, and ethylene glycol mixture) which are poor heat transfer by suspending nano/micro or large particle materials in these fluids. Kuznetsov and Nield [10] investigated the effect of nanoparticles on the natural convection boundary-layer flow through a vertical plate, using a model in which Brownian motion and thermophoresis are represented.

During the recent decades, convective heat transfer of nanofluids is a hot topic of academic and industrial research due to its various applications in industrial processes such as thermal heating, power generation and chemical processing [11–22].
In many applications in the polymer and metal-
lurgy industries, hydro-magnetic techniques are used [23].
Hence, the influence of the magnetic field has attracted
significant interest in recent years due to its high appli-
cations in physics, chemistry, and engineering [24]. MHD
free convection flow of Sodium Alginate nanofluid on a
solid sphere with prescribed wall temperature is investi-
gated by [25]. The authors examine the free convection
flow of Casson nanofluid in the presence of magnetic field.
The relevant partial differential equations are first con-
verted into non-dimensional equations by using appro-
priate transformation and then computed by utilizing the
Keller box method. MHD peristaltic transport of copper-
water nanofluid in an artery with mild stenosis for differ-
ent shapes of nanoparticles is studied in [26]. The charac-
teristics of MHD, heat sink/source, and convective bound-
ary conditions in chemically reactive radia-
tive Powell-
Eyring nanofluid flow via Darcy channel using a nonlinear
early stretched sheet/surface are used in Rasool and Shafiq [27]. In ref. [28], the authors concern with the
examination of heat transfer rate, mass and motile micro-
organisms for convective second grade nanofluid flow.

Considering these facts, and motivated by above dis-
cussed papers in the field of nanofluids, there is enhance-
ment in the momentum, thermal energy and nanoparticles equations be-
comes:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_f} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma^* B_0^2}{\rho_f} u, \quad (2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho_f} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \tau \left\{ \frac{D_f}{\rho_c f_p} \left[ \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} \right] \right\}
\frac{D_f}{\rho_c f_p} \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right]
+ \frac{\mu_f}{\rho_c f_p} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\sigma^* B_0^2}{\rho_c f_p} u^2, \quad (4)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_f \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{D_f}{\rho_c f_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \quad (5)$$

subject to the boundary conditions:

$$u = u_w(x) = ax, \quad v = s, \quad T = T_w, \quad C = C_w \text{ at } y = 0, \quad (6)$$

$$u = v = 0, \quad T = T_{\infty}, \quad C = C_{\infty} \quad \text{as } y \rightarrow \infty.$$ 

Where, $u$ and $v$ are the velocity components along the axes $x$ and $y$, respectively, $\rho_f$ is the density of the base fluid, $\nu$ is the kinematic viscosity, $\sigma^*$ is the electrical conductivity, $p$ is the fluid pressure, $\alpha = \frac{k_l}{\rho_c f_p}$ is the thermal diffusivity, $D_f$ is the Brownian diffusion coefficient, $D_T$ is the ther-
mophoretic diffusion coefficient, $\tau = \frac{\nu_c f_p}{\rho_c f_p}$ is the ratio be-
tween the effective heat capacity of the nanoparticle mate-
rial and heat capacity of the fluid with $\nu$ being the density,
$c$ is the volumetric volume expansion coefficient and $\mu_f$ is

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Physical diagram of the flow geometry}
\end{figure}

2 Analysis

A uniform magnetic field of force $B_0$ is imposed in the
$y$–direction according to Ref. [8] (see Fig. 1). For the cur-
rent study, the basic steady conservation of mass, mo-
the density of the particles, \( \rho_p \) is the fluid specific heat at constant pressure, \( \mu_f \) is the viscosity of the base fluid and \( s \) is suction (or injection) parameter, respectively. \( T \) is the temperature of the fluid, \( C \) is the fraction of the volume of nanoparticles, \( T_w \) is the temperature of the stretching surface, \( C_w \) is the fraction of the volume of nanoparticles on the stretching surface, \( T_\infty \) is the ambient temperature and \( C_\infty \) is the fraction of the volume of ambient nanoparticles. Under the related work [10].

\[
\eta = \frac{\psi}{\alpha Ra^\frac{1}{2}}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad f(\eta) = \frac{C - C_\infty}{C_w - C_\infty}.
\]

(7)

Where \( \psi \) is a stream function provided by

\[
u = \frac{\partial \psi}{\partial y}, \quad \theta = \frac{\partial \psi}{\partial x}.
\]

(8)

Well then, Eq. (1) identically satisfied. For converting Eqs. (1) – (5) with the boundary conditions (6) in the following nonlinear ordinary differential equations a similarity solution in Ref. [10] was implemented.

\[
f''' + \left( \frac{1}{4 \Pr} \right) \left[ 3f'' - 2(f')^2 \right] - M f' = 0,
\]

(9)

\[
\theta'' + \frac{3}{4} f' \theta' + Nb \phi' \theta' + Nt (\theta')^2 + Ec (f'')^2 + M Ec (f')^2 = 0,
\]

(10)

\[
\phi'' + \frac{3}{4} Le f' \phi' + \frac{Nt}{Nb} \phi'' = 0,
\]

(11)

with the boundary conditions:

\[
f(0) = s, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \phi(0) = 1,
\]

\[
f'(\infty) = 0, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0.
\]

(12)

Where primes indicate differentiation for \( \eta \) and \( \text{Pr}, \ Nb, \ Nt, \ Ec, \ M, \text{and} \ Le \) are Prandtl number, Brownian motion parameter, thermophoresis parameter, Eckert number, magnetic parameter, and Lewis number, respectively. The physical parameters below are described by:

\[
\eta = \frac{Y Ra^\frac{1}{2}}{\alpha}, \quad \text{Pr} = \frac{\nu}{\alpha}, \quad Le = \frac{\alpha}{D_B},
\]

\[
Nb = \frac{(p\rho_c)^{\frac{1}{2}}}{\nu} (\phi_w - \phi_\infty), \quad Nt = \frac{(p\rho_c)^{\frac{1}{2}}}{\nu} (\phi_w - \phi_\infty),
\]

\[
M = \frac{\sigma^* B_0^2 L^2}{\mu_f}, \quad L = \left[ \frac{1}{(1 - \phi_\infty)^2 \beta g(T_w - T_\infty)} \right]^{\frac{1}{2}},
\]

\[
Ec = \sqrt{\left[ g \beta (1 - \phi_\infty)^2 \left( \frac{\nu \alpha Ra^\frac{1}{2}}{T_w - T_\infty} \right) \right]},
\]

\[
Ra_x = \frac{\left( 1 - \phi_\infty \right) \beta g(T_w - T_\infty) x^3}{\nu \alpha}.
\]

(13)

Here, gravitational acceleration, volumetric expansion coefficient of the fluid, nanoparticle volume fraction at the surface, ambient nanoparticle volume fraction attained as \( y \) tends to be infinite, and local Rayleigh number, respectively, are also the symbols \( g, \beta, \phi_w, \phi_\infty, \text{and} Ra_x \). In addition, \( f, \theta, \phi \) are the dimensionless of the stream function, temperature, and volume of nanoparticles respectively. It distinguishes the local skin friction \( C_f \), the reduced Nusselt number \( Nur \) and the reduced Sherwood number \( Shr \).

\[
\frac{1}{2} \left( \frac{Pr Re^2}{Ra^2} \right) C_f = f''(0), \quad Nur = -\theta'(0), \quad Shr = -\phi'(0).
\]

(14)

where, \( Re_x \) is the local Reynolds number based on the stretching velocity \( u_w(x) \).

### 2.1 Chebyshev pseudospectral differentiation matrix technique

A numerical solution based on Chebyshev collocation approximations can be considered as a suitable choice for many practical problems (as described in the literature review and for example Canuto et al. [29] and Peyret [30]). Accordingly, Chebyshev collocation method will be applied for the presented model. The derivatives of the function \( f(x) \) at the Gauss-Lobatto points, \( x_k = \cos \left( \frac{k\pi}{L} \right) \), which are the linear combination of the values of the function \( f(x) \) [31],

\[
f^{(n)} = D^{(n)} f,
\]

where

\[
f = [f(x_0), f(x_1), ..., f(x_L)]^T,
\]

and

\[
f^{(n)} = [f^{(n)}(x_0), f^{(n)}(x_1), ..., f^{(n)}(x_L)]^T,
\]

where

\[
D^{(n)} = [d_{k,j}^{(n)}],
\]

or

\[
f^{(n)}(x_k) = \sum_{j=0}^{L} d_{k,j}^{(n)} f(x_j),
\]

where

\[
d_{k,j}^{(n)} = \frac{2}{L} \sum_{l=0}^{l=n} \sum_{m=0}^{m=\text{even}} \gamma_l^* a_{m,l}^{n} (-1)^{\frac{L+1}{2}} s_{\text{even}} \left[ \frac{m}{l} \right] \left[ \frac{m}{l} \right]^{\frac{1}{2}} \left[ \frac{n}{l} \right]^{\frac{1}{2}} \left[ \frac{n}{l} \right]^{\frac{1}{2}},
\]

where,

\[
a_{m,l}^{n} = \frac{2^n}{(n-1)!c_m} (s - m + n - 1)! (s + n - 1)!.
\]
such that $2s = l + m - n$ and $c_0 = 2$, $c_i = 1$, $i \geq 1$, where $k, j = 0, 1, 2, \ldots, L^* - 1$ and $\gamma_0 = \gamma_1 = \frac{1}{2}$, $\gamma_{j} = 1$ for $j = 1, 2, 3, \ldots, L - 1$. The round off errors (see Appendix A) incurred during computing differentiation matrices $D^{(n)}$ are investigated in [31].

### 2.2 Description of the numerical method

The grid points $(x_i, x_j)$ in this situation are given as $x_i = \cos \left( \frac{i \pi}{L_x^*} \right)$, $x_j = \cos \left( \frac{j \pi}{L_y^*} \right)$ for $i = 1, \ldots, L_x^* - 1$ and $j = 1, \ldots, L_y^* - 1$. The domain in the $x$–direction is $[0, x_{\text{max}}]$ where $x_{\text{max}}$ is the length of the dimensionless axial coordinate and the domain in the $\eta$–direction is $[0, \eta_{\text{max}}]$ where $\eta_{\text{max}}$ corresponds to $\eta_{\text{oo}}$. The domain $[0, x_{\text{max}}] \times [0, \eta_{\text{max}}]$ is mapped into the computational domain $[0, x_{\text{max}}] \times [-1, 1]$. The application of this method to differential equation leads to system of algebraic equations. The first rows and last rows of the coefficients matrix of the algebraic system are replaced by a suitable formulation of the boundary conditions. The Chebyshev collocation method is more accurate in comparison with other techniques for solving this kind of problems, especially the finite difference and the finite elements methods. The finite difference methods replace the derivatives of a function at any point with finite difference approximation formulas in terms of its values on a grid of mesh points that span the domain of interest. The numbers of these mesh points are two and three for the central finite difference of second order of the first and second derivatives, respectively. While in Chebyshev collocation method, the derivatives of a non-singular function at any point from the Chebyshev points are expanded as a linear combination from the values of the function at all of these points. i.e. (the approximations of the derivatives are defined over the whole domain).

Therefore, Eqs. (9) - (11) with the boundary conditions (12) have been solved numerically [18, 19]. Thus by applying the Chebyshev collocation approximation to equations (9) - (11). The following Chebyshev collocation equations can be obtained:

\[
\left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,i,i}^{(3)} f_i \right) + \left( \frac{3}{4 \Pr} \right) f_j \left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,j,i}^{(2)} f_i \right) \]

\[
- \left( \frac{2}{4 \Pr} \right) \left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,i,j}^{(1)} f_i \right) \]

\[
- M \left( \frac{2}{\eta_{\text{max}}} \right) \left( \sum_{i=0}^{L^*} d_{i,j}^{(1)} f_i \right) = 0, \quad (15)
\]

\[
\left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,i,i}^{(2)} f_i \right) + \left( \frac{3}{4 \Pr} \right) f_j \left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,j,i}^{(1)} f_i \right) \]

\[
+ N \left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,i,j}^{(1)} f_i \right) = 0, \quad (16)
\]

\[
\left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,i,i}^{(3)} f_i \right) + \left( \frac{3}{4 \Pr} \right) f_j \left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,j,i}^{(2)} f_i \right) \]

\[
+ N \left( \frac{2}{\eta_{\text{max}}} \right)^2 \left( \sum_{i=0}^{L^*} d_{i,i,j}^{(1)} f_i \right) = 0. \quad (17)
\]

The computer program of the numerical method and the numerical computations have been done by the symbolic computation software Mathematica $\alpha^TM$. Also, the solution of the above equations (15) - (17) for the unknowns $f_i$, $\theta_j$ and $\phi_j$ with boundary conditions (12) where $j = 1(1)L^*$ (take $L^* = 32$) and $\eta_{\text{max}}$ corresponds to $\eta_{\text{oo}}$ are obtained using the Newton-Raphson iteration technique.

### 3 Results and Discussion

#### 3.1 Validation of the numerical solution

The results for the local skin friction $f''(0)$ are compared with those obtained by Abd Elazem [19] for different values of $Pr$, $s$, and $\eta_{\text{max}}$ in Table 1. It is noticed that the comparison shows excellent agreement for each values of $Pr$, $s$, and $\eta_{\text{max}}$. Also, dimensionless similarity functions $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ are matching with the previously published [19] as shown in Fig. 2. Therefore, it is confident that the present results are very accurate.

#### 3.2 Results for temperature and solid volume fraction profiles

Equations (9) – (11) have been numerically solved with the boundary conditions (12) using the Chebyshev pseudospectral technique. It is found that as the distance
increases from the solid boundaries, both the temperature and the concentration profiles start at unity near the wall and reach to vanish. In the case of $Nt = Nb = 0.1$, $Pr = Le = 1$, $s = 0.01$ and $M = 1$, Figure 3 serves to highlight the present numerical results for $Ec = -0.05, -0.01, 0.0, 0.01$ on temperature profiles. It is demonstrated that with an increase in $Ec$, the temperature profiles and thermal boundary layer thickness are enhanced. Physically, the ohmic heating effect due to the effects on electromagnetic work is found to be produced an increase in the fluid temperature, and thus a decrease in the surface temperature gradient. Further, it is found that the effect of viscous heating leads to an increase in the temperature; this effect is more pronounced in the presence of the magnetic field. It is acknowledged that an increase in $Nt$, the temperature profile accelerates also, the temperature profile raises the elevation of the Eckert number as shown in Fig. 4 (positive $Ec$ values correspond to wall heating, while the opposite is true for $Ec$ negative values).

The temperature decreases as $Pr$ increases. This is in agreement with the physical fact that the thermal boundary layer thickness decreases with increasing $Pr$. Further, according to Fig. 5, the numerical results for the profiles of $\theta(y)$ for (a) $Ec = -0.01$ and (b) $Ec = 0.01$ when $Pr = 0.7, 1, 10, 10^2$ at $Nt = Nb = 0.1$, $Le = 1$, $s = 0.01$, and $M = 1$. Physically the thermal boundary-layer thickness, as predicted, is less than the boundary-layer thickness of the momentum when $Pr \gg 1$. Also, the temperature function decreases (see Fig. 5). At high values of the Prandtl number $Pr$ (values of $Pr \gg 1$, decrease conductivity, and increase pure convection). Besides the typical matching of temperature profiles [19] at values ($10 < Pr < 10^5$). Notwithstanding this, it should also be noticed that with the rise in $Ec$ from $-0.01$ to $0.01$, there is a slightly greater difference in $Ec = 0.01$ if $Pr = 0.7, 1$ (see Fig. 5).

By contrast, it is clearly shown that an increase in $M$ and $Ec$ increases the thermal boundary layer thickness and the temperature profiles. Physically, application of a transverse magnetic field to an electrically conducting fluid creates a resistive-type force called the Lorentz force. This conclusion meets the logic that the magnetic field exerts a retarding force on the free-convection flow in the boundary layer and increase its temperature (see Figure 6).

In the case of $Le > 1$, the thickness of the boundary-layer for mass friction function is smaller than the thermal boundary-layer thickness (see Fig. 7). Variations in the concentration function increased with $Ec$ growing far from the boundary. Concentration function profiles generally decrease with the rise in Lewis number as in Fig. 7. It can be seen that an increment in Lewis number decreases the solid volume fraction of nanofluid profiles. Physically, This is due to the fact that mass transfer rate of nanofluid increases as Lewis number increases. It also reveals that the concentration gradient at surface of the stretching sheet increases. Moreover, the concentration at the surface of stretching sheet decreases as Lewis number increases. Finally, Figure 8 shows the variation of Concentration profiles with $Nb$ and $Ec$ when $Nt = 0.1$, $Pr = 10$, $Le = 1.5$, $s = 0.01$, and $M = 3$. It’s clear that the thickness of the boundary layer for mass friction function decreased as $Nb$ increased. Besides, the concentration profiles decrease with the increase of the Eckert number. Physically, the Brownian motion parameter helps to measure the strength of the Brownian diffusion of the nanoparticles in the flow field. Due to the Brownian diffusion, the nanoparticles tend to move away from the surface of the sheet and as result a

| Table 1: Comparison test results for local skin friction $f''(0)$ when (a) $Nb = Nt = 0.1$, $Le = 10$, $M = Ec = 0.0$, and $s = 1$ at different values of $Pr$ (b) $Nt = 0.1$, $Pr = Le = 10$, and $M = Ec = 0.0$ at different values of $s$ (c) $Nb = Nt = 0.1$, $Pr = 10$, $Le = 100$, $M = Ec = 0.0$, and $s = 5$ at different values of $\eta_{max}$

| $(a)$ $Pr$ | 1 | 3 | 10 | $10^3$ | $10^5$ |
|---|---|---|---|---|---|
| Abd Elazem [19] | 1.24162 | 0.591478 | 0.291396 | 0.10255 | 0.108086 |
| Present results | 1.24162 | 0.591478 | 0.291396 | 0.10255 | 0.108086 |
| $(b)$ $s$ | | -10.0 | -5.0 | 5.0 | 10.0 |
| Abd Elazem [19] | | -0.0645743 | -0.116751 | -0.270368 | -0.500422 | -0.854846 |
| Present results | | -0.0645743 | -0.116751 | -0.270368 | -0.500422 | -0.854846 |
| $(c)$ $\eta_{max}$ | | 1 | 3 | 5 | 10 | 20 |
| Abd Elazem [19] | | 0.000522797 | 0.0165552 | 0.0200026 | 0.0200026 | 0.049886 |
| Present results | | 0.000522797 | 0.0165552 | 0.0200026 | 0.0200026 | 0.049886 |

| Table 2: (a) Variation of the reduced Nusselt number $Nu_{r} = -\theta'(0)$ with $Nt$ and $Nb$ (b) Variation of the reduced Sherwood number $Sh_{r} = -\phi'(0)$ with $Nt$ and $Nb$ when $Pr = Le = 1$, $M = 1$, $Ec = 0.01$, and $s = 0.01$

| $(a)$ $Nt = 0.1$ | $Nt = 0.3$ | $Nt = 0.5$ |
|---|---|---|
| $Nb$ | $Nu_{r}$ | $Nb$ | $Nu_{r}$ | $Nb$ | $Nu_{r}$ |
| 0.1 | 0.18658 | 0.1 | 0.166257 | 0.1 | 0.147136 |
| 0.3 | 0.150689 | 0.3 | 0.132122 | 0.3 | 0.11468 |
| 0.5 | 0.11767 | 0.5 | 0.100782 | 0.5 | 0.0849405 |
| $(b)$ $Nt = 0.1$ | $Nt = 0.3$ | $Nt = 0.5$ |
| $Nb$ | $Sh_{r}$ | $Nb$ | $Sh_{r}$ | $Nb$ | $Sh_{r}$ |
| 0.1 | 0.322289 | 0.1 | 0.159911 | 0.1 | 0.0592868 |
| 0.3 | 0.403315 | 0.3 | 0.367966 | 0.3 | 0.351226 |
| 0.5 | 0.418986 | 0.5 | 0.408024 | 0.5 | 0.407115 |
Table 3: (a) Variation of the reduced Nusselt number $Nur = -\theta'(0)$ with $Ec$ and $M$ (b) Variation of the reduced Sherwood number $Shr = -\phi'(0)$ with $Ec$ and $M$ when $Pr = Le = 1$, $Nt = Nb = 0.1$, and $s = 0.01$

(a) $Ec = -0.01$  

| $M$ | $Nur$  | $Ec = 0.0$ | $M$ | $Nur$  | $Ec = 0.01$ |
|----|-------|-----------|----|-------|-----------|
| 0  | 0.598195 | 0  | 0.480109 | 0  | 0.361897 |
| 1  | 0.605839  | 1  | 0.396299  | 1  | 0.18658  |
| 10 | 0.80447   | 10 | 0.216987  | 10 | -0.370631 |

(b) $Ec = -0.01$  

| $M$ | $Shr$  | $Ec = 0.0$ | $M$ | $Shr$  | $Ec = 0.01$ |
|----|-------|-----------|----|-------|-----------|
| 0  | 0.0991581 | 0  | 0.202953 | 0  | 0.306863 |
| 1  | -0.0647157 | 1  | 0.128701 | 1  | 0.322289 |
| 10 | -0.50287   | 10 | 0.0754746 | 10 | 0.653956 |

decrease in nanoparticle volume fraction is encountered within the boundary layer region.

3.3 Results for reduced Nusselt number and reduced Sherwood number

Table 2 calculates the numerical results for the reduced Nusselt number and the reduced Sherwood number when $Nt = Nb = 0.1, 0.3, 0.5$ for different $Nb$ values, where $Pr = Le = 10, M = 1$, and $s = 0.01$. It is reported that the reduced Nusselt number is a decreasing function while an increasing function is the reduced Sherwood number. Physically, it is also found that the impact of Joule heating on electromagnetic operation has resulted in a rise in the fluid temperature and therefore a decrease in the gradient of the surface temperature. Furthermore, the actual impact of viscous heating results in a temperature increase; this effect is more pronounced in the presence of the magnetic field. As shown in Table 3, it is clear that the reduced Nusselt number is a monotonous function (i.e. it is an increasing function at $Ec = -0.01$, whereas a decreasing function at $Ec = 0.0, 0.01$).

4 Conclusion

The effects of various physical parameters on nanofluids that flow past a stretching surface were explored. This study is very important for engineers and researchers in nearly every branch in engineering and science. Nuclear
power plants, gas turbines and the various propulsion devices for aircraft, missiles, satellites and space vehicles are examples of such engineering areas. Numerically a system of nonlinear ordinary differential equations was solved using Chebyshev’s pseudospectral technique at specified physical parameters. From previous results, it can be concluded that:

1. It has been found that the present results show that the reduced number of Nusselt is a decreasing function at a fixed value of $Ec = 0.01$, while the reduced number of Sherwood is an increasing function for variation of $Nt$ with $Nb$, at $Pr = Le = 10$, $M = 1$, and $s = 0.01$.

2. It has been noted that the current results indicate that the reduced number of Nusselt and the reduced number of Sherwood are monotonous functions for variation of $M$ with $Ec$ when $Nb = Nt = 0.1$, $Pr = Le = 1$, and $s = 0.01$.

3. The magnetic field, Joule heating, Eckert number and thermophoresis parameter improves the temperature field and increases the thermal boundary layer thickness while the Prandtl number $Pr$ reduces the temperature field.
and decreases the thermal boundary layer thickness where the Prandtl number Pr does not affect the temperature profile ($10 < Pr \leq 10^5$).

4. Changing the parameter $Nb$ as known at values ($0.1, 0.3, 0.5$) decreases the profile of the concentration. Furthermore, the Eckart number (from $-0.01 \leq Ec \leq 0.01$) has a direct impact on the concentration profile, where the Eckart number reduces the profile of concentration.

**Funding information:** The authors state no funding involved.

**Figure 7:** Effects of $Ec$ and $Le$ on concentration profiles for $Nt = Nb = 0.1, Pr = 1, s = 0.01$, and $M = 1$

**Figure 8:** Effects of $Ec$ and $Nb$ on concentration profiles for $Nt = 0.1, Pr = 10, Le = 1, s = 0.01$, and $M = 1$

**Author contributions:** All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

**Conflict of interest:** The authors state no conflict of interest.
A Chebyshev collocation approximation

A.1 Rounding off error analysis

The round off errors incurred during computing differentiation matrices $D^{(0)}$ investigated by [31]. Elbarbary and El-sayed [31] show that, the elements of the first order differentiation matrix, there would be round off error as in:

$$d_{1,j}^* - d_{1,j}^1 < 4jδ \left( \delta - O\left( \frac{1}{N^2} \right) \left( \frac{N^2}{3} + \frac{1}{6} \right) \right) \left( \frac{N^2}{3} + \frac{1}{6} \right)$$

The element $d_{01}^*$ is the major elements concerning its values. Accordingly, it bears the major error responsibility comparing the other elements. Meanwhile, Baltensperger and Trummer [32] show that, the error in the evaluation of the element $D_{01}$ from the classical matrix $D$ is of order $O(N^6\delta)$ where $\delta$ is the machine precision and $D_{01} - D_{01} = \frac{8N^6}{3} \delta$, whereas in Elbarbary and El-sayed [31] find the error of order $O(N^2\delta)$ where,

$$d_{01}^* - d_{01}^1 \leq \left( \frac{1}{3} - N + \frac{2}{3} N^2 \right) \delta$$

This can be taken into consideration, as itself, as modifying the classical matrix $D$. Due to,

$$d_{01}^1 = \frac{N-1}{N} \sum_{k=1}^{N-1} k^2 x_k - N = -\frac{1}{3} + \frac{(4/\pi^2)N^2}{2} + O\left( \frac{1}{N^2} \right)$$

with error upper bound

$$d_{01}^* - d_{01}^1 \leq \frac{N-1}{N} \sum_{k=1}^{N-1} k^2 \delta_k \leq \left( \frac{1}{3} - N + \frac{2}{3} N^2 \right) \delta$$

Also, Elbarbary and El-sayed [31] show that the error bound for the second order derivatives can be given by

$$d_{2,j}^* - d_{2,j}^1 \leq 4 \gamma_j^* \left( \delta - O\left( \frac{1}{N^2} \right) \left( \frac{N^2}{3} + \frac{1}{6} \right) \right) \left( \frac{N^2}{3} + \frac{1}{6} \right)$$

and the error bound for the third order derivatives is given by Elbarbary and El-sayed [31]

$$d_{3,j}^* - d_{3,j}^1 \leq \gamma_j^* \left( \delta - O\left( \frac{1}{N^2} \right) \right) \left( \frac{2N^6}{105} - \frac{N^4}{15} - \frac{N^2}{15} + \frac{4}{35} \right)$$

Finally, the error bound for the fourth order derivatives is given by Elbarbary and El-sayed [31]

$$d_{4,j}^* - d_{4,j}^1 \leq \gamma_j^* \left( \delta - O\left( \frac{1}{N^2} \right) \right) \left( \frac{2N^8}{945} - \frac{8N^6}{315} + \frac{2N^4}{45} + \frac{124N^2}{945} - \frac{16}{105} \right)$$

Table 4 lists the computed errors in the elements $d_{01}^*$ and $D_{01}$.

A.2 Example 1

Consider the following boundary value problem [33]

$$f''(\eta) + f(\eta)f'(\eta) + (f'(\eta))^2 = 0,$$

$$f'(0) = 1, f(0) = 0, f(\infty) = 0.$$ 

The exact solution is given by

$$f(\eta) = \sqrt{\frac{2}{\eta}} \tanh \left( \frac{\eta}{\sqrt{2}} \right)$$

Table 5 represents the values of $f(\eta)$ for the exact solution.

| $N$ | $d_{01}^* - d_{01}^1$ | Error upper bound $D_{01}^* - D_{01}^1$ |
|-----|---------------------|----------------------------------------|
| 16  | $7.80 \times 10^{-15}$ | $3.44 \times 10^{-14}$ |
| 32  | $4.06 \times 10^{-14}$ | $1.45 \times 10^{-13}$ |
| 64  | $9.07 \times 10^{-14}$ | $1.70 \times 10^{-13}$ |
| 128 | $3.05 \times 10^{-13}$ | $2.40 \times 10^{-12}$ |
| 256 | $5.29 \times 10^{-12}$ | $9.64 \times 10^{-12}$ |
| 512 | $2.15 \times 10^{-12}$ | $3.87 \times 10^{-11}$ |
| 1024| $9.35 \times 10^{-12}$ | $1.55 \times 10^{-10}$ |
| 2048| $3.88 \times 10^{-10}$ | $6.20 \times 10^{-10}$ |
| 4096| $2.39 \times 10^{-9}$  | $2.48 \times 10^{-9}$  |

Table 4: Computed errors in $d_{01}^*$ and $D_{01}$

Table 5: Values of $f(\eta)$ for the exact solution, the shooting method and the present method.

| $\eta$ | Exact solution | Chebyshev solution | Shooting method |
|--------|---------------|--------------------|----------------|
| 0.0    | 0.0           | 0.0                | 3.1102 $\times 10^{-21}$ |
| 0.19261  | 0.19265989904503 | 0.19265989837756 | 0.192659892303245 |
| 0.30448182  | 0.29986273742903 | 0.299862946450528 | 0.299865455103407 |
| 0.674122  | 0.62731370623597 | 0.62729911341429 | 0.627295450854674 |
| 1.7157  | 0.96114135766676 | 0.96107004597040 | 0.961054342666112 |
| 1.7772  | 1.02428480516056 | 1.024364318410284 | 1.024364540992483 |
| 2.4687  | 1.33066586576608 | 1.330574346579722 | 1.3307838606424 |
| 3.6079  | 1.39711413277153 | 1.390685580848372 | 1.39066522682298 |
| 4.3928  | 1.40854970895860 | 1.408367808484652 | 1.40836784924106 |
| 5.1614  | 1.412392434570513 | 1.41208434920526 | 1.41208442224472 |
| 5.8859  | 1.413527132803757 | 1.413276902414456 | 1.413275693434824 |
| 6.5375  | 1.413940523570454 | 1.413565377029086 | 1.413653447875072 |
| 7.0924  | 1.414049909002763 | 1.41377674540998 | 1.41377667862842 |
| 7.5269  | 1.41416423312725 | 1.41382195821038 | 1.41382183070493 |
| 7.92314 | 1.41417507928212 | 1.41382204813232 | 1.41382237952506 |
| $\eta_{max}$ = 8 | 1.4141798443479533 | 1.413822620116837 | 1.413822547075252 |
the shooting method and the present method. The error of Shooting method \((E_{e,\text{Shooting}})\) and (the error \(E_{e,\text{ChC}}\)) of the present method is given in Table 6.

Table 6: The maximum absolute error

| \(E_{e,\text{ChC}}\) | \(E_{e,\text{Shooting}}\) |
|----------------------|-----------------------------|
| 0                    | \(3.1102 \times 10^{-21}\)  |
| 1.97548 \times 10^{-8} | 1.71093 \times 10^{-8}      |
| 3.09104 \times 10^{-6} | 3.15618 \times 10^{-6}      |
| 1.44949 \times 10^{-5} | 1.41610 \times 10^{-5}      |
| 3.41792 \times 10^{-5} | 3.57495 \times 10^{-5}      |
| 6.42833 \times 10^{-5} | 6.40506 \times 10^{-5}      |
| 9.546 \times 10^{-5}   | 9.5079 \times 10^{-5}       |
| 1.45825 \times 10^{-4} | 1.45891 \times 10^{-4}      |
| 1.82163 \times 10^{-4} | 1.82186 \times 10^{-4}      |
| 2.18649 \times 10^{-4} | 2.18682 \times 10^{-4}      |
| 2.53528 \times 10^{-4} | 2.53561 \times 10^{-4}      |
| 2.85146 \times 10^{-4} | 2.85183 \times 10^{-4}      |
| 3.12135 \times 10^{-4} | 3.1217 \times 10^{-4}       |
| 3.33375 \times 10^{-4} | 3.3341 \times 10^{-4}       |
| 3.52671 \times 10^{-4} | 3.52707 \times 10^{-4}      |
| 3.56423 \times 10^{-4} | 3.56459 \times 10^{-4}      |

**Nomenclature**

- \(x, y\): Cartesian coordinates \(m\)
- \(u, v\): Horizontal and vertical velocity components \(m \cdot s^{-1}\)
- \(u_w(x)\): Stretching velocity \(m \cdot s^{-1}\)
- \(\rho_f\): Dynamic viscosity of base fluid \(Pa \cdot s\)
- \(\nu\): Kinematic viscosity \(m^2 \cdot s^{-1}\)
- \(k_f\): Thermal conductivity \(W \cdot m^{-1} \cdot K^{-1}\)
- \(a\): Stretching rate \(s^{-1}\)
- \(\rho_f\): Density \(kg \cdot m^{-3}\)
- \(P\): Fluid pressure
- \(\sigma^*\): Fluid Electric conductivity \((\Omega m)^{-1}\)
- \(B_0\): Applied magnetic field intensity \(A \cdot m^{-1}\)
- \(\alpha\): Thermal diffusivity
- \(\tau\): Heat capacity ratio for fluid and nanoparticles
- \(D_B\): Brownian diffusion
- \(T\): Local temperature of the fluid \(K\)
- \(C\): Concentration distributions \(kg \cdot m^{-3}\)
- \(\rho C\): Fluid’s productive heat capacity \(J \cdot m^{-3}\)
- \(s\): Suction or injection parameter
- \(T_w\): Temperature of the sheet \(K\)
- \(T_{\infty}\): Temperature of the fluid far away from the sheet \(K\)
- \(D_T\): Thermophoresis diffusion coefficient \(m^2 / s\)
- \(C_w\): Solid volume friction of the sheet
- \(Shr\): Reduced Sherwood number
- \(g\): Gravitational acceleration
- \(\beta\): Volumetric expansion coefficient of the fluid
- \(\phi_w\): Nanoparticle volume fraction at the surface
- \(\phi_{\infty}\): Ambient nanoparticle volume fraction attained as \(y\) tends to be infinite
- \(Ra_x\): Local Rayleigh number
- \(C_{\infty}\): Solid volume friction far away from the sheet
- \(Ec\): Eckert number
- \(\phi\): Dimensionless of the concentration
- \(f\): Dimensionless of the stream function
- \(\theta\): Dimensionless of the temperature
- \(\psi\): Stream function
- \(Pr\): Prandtl number
- \(C_f\): Local skin friction coefficient \(Pascal\)
- \(\eta\): Dimensionless space variable
- \(\rho C_p\): Nanoparticles’ productive heat capacity \(J \cdot m^{-3}\)
- \(MHD\): Magnetohydrodynamics
- \(Le\): Lewis number
- \(Nb\): Brownian motion parameter
- \(Nt\): Thermophoresis parameter
- \(M\): Magnetic parameter
- \(Re_x\): Reynolds number
- \(Nur\): Reduced Nusselt number

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