Probabilistic teleportation via a non-maximally entangled GHZ state

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We present a scheme for probabilistic teleportation via a non-maximally entangled GHZ state. Quantum teleportation will succeed with a certain probability if the sender makes a generalized Bell state measurement, the cooperator performs a generalized X basis measurement, and the receiver introduces an auxiliary particle, performs a collective unitary transformation and makes a measurement on the auxiliary particle. The success probability of the teleportation is given. We also obtain the maximum of the success probability of the teleportation.

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Quantum teleportation, which allows transportation of an unknown state from a sender Alice to a spatially distant receiver Bob with the aid of the previously shared entanglement and classical communication, is regarded as one of the most striking results of quantum information theory [1]. It plays an important role in the development of quantum computation and quantum communication [2, 3, 4, 5, 6, 7, 8].

The original protocol of Bennett et al. [2] involves teleportation of an arbitrary state of a qubit via an Einstein-Podolsky-Rosen (EPR) pair and by transmitting two bits of classical information from Alice to Bob. Here Alice knows neither the state to be teleported nor the location of the intended receiver, Bob. They also presented a protocol for teleporting an unknown state of a qubit via a maximally entangled state in $d \times d$ dimensional Hilbert space and by sending $2\log_d d$ bits of classical information.

Since then, quantum teleportation has got great development [9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and has been demonstrated experimentally by several groups [19, 20, 21, 22]. It was generalized to a more general situation, where two parties may not start with a set of pure entangled states, but with a noisy quantum channel. In order to arrive at their goal of transmitting unknown quantum state over this noisy quantum channel, they could use an error correcting code [23], or alternatively they can share the entanglement through this noisy channel first and then use teleportation [24].

Quantum teleportation is also possible for infinite dimensional Hilbert space, for example in position-momentum space with continuous variable states, which is called continuous variable quantum teleportation [9, 10, 11].

Karlsson and Bourennane put forward the controlled quantum teleportation protocol [13, 14, 15, 16]. In the protocol, one can perfectly transport an unknown state from one place to another place via previously shared Greenberger-Horne-Zeilinger (GHZ) state by means of local operations and classical communications under the control of the third party. The signal state can not be transmitted unless the third party gives a permission. The controlled quantum teleportation is useful in networked quantum information processing, and has other interesting applications, such as in opening account authorized by the managers in a network.

If the quantum channel is not in a maximally entangled state then one cannot transport a qubit with unit fidelity and unit probability. However, it was shown that using a non-maximally entangled state one can have unit fidelity teleportation but with a probability less than unit-called probabilistic quantum teleportation [17, 18]. Using a non-maximally entangled basis as a measurement basis this was shown to be possible. Subsequently, this probabilistic scheme has been generalized to teleport $N$ qubits [12] and controlled teleportation [16].

In this paper, we will investigate the probabilistic teleportation via a quantum channel of a non-maximally entangled GHZ state.

First we consider three-partite probabilistic teleportation protocol.

Suppose that Alice has a qubit in state

$$|\phi\rangle_{A_1} = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

and she wants to transport this state to the receiver Bob. Of course, the identity of $|\phi\rangle_{A_1}$ is unknown to Alice.

Suppose that a quantum channel shared by Alice, Bob and the collaborator Charlie is a non-maximally entangled GHZ state

$$|\text{GHZ}_{A_2CB}\rangle = N(|000\rangle + n|111\rangle), N = \frac{1}{\sqrt{1 + |n|^2}}.$$  (2)

Here we assume that Charlie is cooperative and loyal. The particles $A_1$ and $A_2$ are in Alice’s possession, particle $B$ is in Bob’s possession and particle $C$ belongs to Charlie.

The overall state of the whole system reads

$$|\psi\rangle_{A_1A_2CB} = |\phi\rangle_{A_1} \otimes |\text{GHZ}_{A_2CB}\rangle = (\alpha|0\rangle + \beta|1\rangle)N(|000\rangle + n|111\rangle).$$  (3)
We define the generalized Bell states [12]

\[ |\phi^+_m\rangle = M(|00\rangle + m|11\rangle), \]
\[ |\phi^-_m\rangle = M(m^*|00\rangle - |11\rangle), \]
\[ |\psi^+_m\rangle = M(|01\rangle + m|10\rangle), \]
\[ |\psi^-_m\rangle = M(m^*|01\rangle - |10\rangle) \] (4)

and the generalized X basis

\[ |+\rangle = a(|0\rangle + |b\rangle|1\rangle), \]
\[ |-\rangle = a(b^*|0\rangle - |1\rangle). \] (5)

Here

\[ M = \frac{1}{\sqrt{1 + |m|^2}} \]

and

\[ a = \frac{1}{\sqrt{1 + |b|^2}}. \]

Evidently, in general the generalized Bell states are not maximally entangled.

A simple algebraic rearrangement of Eq.(3) in terms of the generalized Bell states Eq.(4) and the generalized X basis Eq.(5) yields

\[ |\psi\rangle_{A_1A_2CB} = \phi^+_A \otimes |\text{GHZ}\rangle_{A_2CB} \]
\[ = N M a \{ |\phi^+_m\rangle_{A_1} |+\rangle_C (|a\rangle|0\rangle + m^*|b\rangle|1\rangle) + |-\rangle_C (|a\rangle|0\rangle - m^*|b\rangle|1\rangle) \}
\[ + |\phi^-_m\rangle_{A_1} |+\rangle_C (m|a\rangle|0\rangle - n|b\rangle|1\rangle) + |-\rangle_C (m|a\rangle|0\rangle + n|b\rangle|1\rangle) \]
\[ + |\psi^+_m\rangle_{A_1} |+\rangle_C (m|a\rangle|0\rangle + n|b\rangle|1\rangle) + |-\rangle_C (m|a\rangle|0\rangle - n|b\rangle|1\rangle) \]
\[ + |\psi^-_m\rangle_{A_1} |+\rangle_C (m|a\rangle|0\rangle + n|b\rangle|1\rangle) + |-\rangle_C (m|a\rangle|0\rangle - n|b\rangle|1\rangle) \}
\[ = N M \left\{ \sqrt{|a|^2 + |mnab|^2} |\phi^+_m\rangle_{A_1} |+\rangle_C \left( \frac{|a\rangle|0\rangle + m^*|b\rangle|1\rangle}{\sqrt{|a|^2 + |mnab|^2}} \right) \right\} \]
\[ + \frac{\sqrt{|a|^2 + |mnab|^2}}{\sqrt{|a|^2 + |mnab|^2}} \left( \frac{|a\rangle|0\rangle - m^*|b\rangle|1\rangle}{\sqrt{|a|^2 + |mnab|^2}} \right) \] (6)

In virtue of Eq.(6), the probabilistic teleportation can be accomplished by the following steps. Firstly, Alice performs a generalized Bell state measurement on qubits \( A_1, A_2 \) shown in Eq.(4). Evidently, \( |\phi^+_m\rangle, |\phi^-_m\rangle, |\psi^+_m\rangle \), and \( |\psi^-_m\rangle \) will occur with probabilities \( N^2M^2(|a|^2 + |mnab|^2), N^2M^2(|ma|^2 + |\beta|^2), N^2M^2(|na|^2 + |\beta|^2) \), and \( N^2M^2(|na|^2 + |\beta|^2) \), respectively. Then Alice communicates to the collaborator Charlie and Bob the outcome of the generalized Bell state measurement and the value of \( m \). Later on Charlie performs a generalized X basis measurement stated in Eq.(5) on his qubit \( C \). After that Charlie tells Bob his measurement outcome and the value of \( b \). The resulting states of Bob’s qubit will be
respectively

\[ \frac{(\alpha|0) + m^*n\beta|1)}{\sqrt{|\alpha|^2 + |mnb\beta|^2}} = I \frac{(\alpha|0) + m^*n\beta|1)}{\sqrt{|\alpha|^2 + |mnb\beta|^2}}. \quad (7) \]

\[ \frac{(ab|0) - m^*n\beta|1)}{\sqrt{|ab|^2 + |mn\beta|^2}} = \sigma_z \frac{(ab|0) - m^*n\beta|1)}{\sqrt{|ab|^2 + |mn\beta|^2}}. \quad (8) \]

\[ \frac{(ma|0) - n\beta|1)}{\sqrt{|ma|^2 + |n\beta|^2}} = \sigma_z \frac{(ma|0) + n\beta|1)}{\sqrt{|ma|^2 + |n\beta|^2}}. \quad (9) \]

\[ \frac{(mab|0) + n\beta|1)}{\sqrt{|mab|^2 + |n\beta|^2}} = I \frac{(mab|0) + n\beta|1)}{\sqrt{|mab|^2 + |n\beta|^2}}. \quad (10) \]

\[ \frac{(mab^*|1) + m^*n\beta|0)}{\sqrt{|mab|^2 + |m\beta|^2}} = \sigma_z \frac{(mab^*|0) + m^*n\beta|1)}{\sqrt{|mab|^2 + |m\beta|^2}}. \quad (11) \]

\[ \frac{(m\beta|0) - na|1)}{\sqrt{|m\beta|^2 + |na|^2}} = i\sigma_y \frac{(m\beta|0) + na|1)}{\sqrt{|m\beta|^2 + |na|^2}}, \quad (12) \]

\[ \frac{(mnab^*|1) - \beta|0)}{\sqrt{|mnab|^2 + |\beta|^2}} = -i\sigma_y \frac{(mnab^*|0) - \beta|1)}{\sqrt{|mnab|^2 + |\beta|^2}}, \quad (13) \]

\[ \frac{(mnab|1) + \beta|0)}{\sqrt{|mnab|^2 + |\beta|^2}} = \sigma_z \frac{(mnab|0) + \beta|1)}{\sqrt{|mnab|^2 + |\beta|^2}}. \quad (14) \]

Here \( I \) is the two-dimensional identity, and \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices. Depending on Alice’s and Charlie’s measurement results Bob will be able to apply the corresponding unitary operator to his particle \( B \), transforming it to one of the states

\[ \frac{(\alpha|0) + m^*n\beta|1)}{\sqrt{|\alpha|^2 + |mnb\beta|^2}} \]

\[ \frac{(ab|0) - m^*n\beta|1)}{\sqrt{|ab|^2 + |mn\beta|^2}} \]

\[ \frac{(ma|0) + n\beta|1)}{\sqrt{|ma|^2 + |n\beta|^2}} \]

\[ \frac{(mab|0) + n\beta|1)}{\sqrt{|mab|^2 + |n\beta|^2}} \]

\[ \frac{(mab^*|1) + m^*n\beta|0)}{\sqrt{|mab|^2 + |m\beta|^2}} \]

\[ \frac{(m\beta|0) + na|1)}{\sqrt{|m\beta|^2 + |na|^2}} \]

\[ \frac{(mnab^*|1) - \beta|0)}{\sqrt{|mnab|^2 + |\beta|^2}} \]

\[ \frac{(mnab|1) + \beta|0)}{\sqrt{|mnab|^2 + |\beta|^2}} \]

For the state \( \frac{(\alpha|0) + \beta|1)}{\sqrt{|\alpha|^2 + |\beta|^2}} \), in which \( c \) and \( d \) are known, but \( \alpha \) and \( \beta \) are unknown to us, it is possible to obtain the state \( \alpha|0\rangle + \beta|1\rangle \) with a certain probability by the following method.

In order to achieve the above purpose, Bob needs to perform a unitary transformation

\[ U = \begin{pmatrix} \exp(-i \arg c) & 0 \\ \exp(-i \arg d) & 0 \end{pmatrix} \quad (15) \]

on particle \( B \) under the basis \( \{ |0\rangle, |1\rangle \} \), then particle \( B \) is in the state

\[ \frac{(\alpha|c|0\rangle + \beta|d|1\rangle)_B}{\sqrt{|\alpha|^2 + |\beta|^2}}. \quad (16) \]

Without loss of generality we assume that \( |c| < |d| \). Furthermore, Bob needs to introduce an auxiliary particle \( D \) with the initial state \( |0\rangle_D \) and performs a collective unitary transformation

\[ U_{BD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1 - |\xi|^2} & \sqrt{1 - |\xi|^2} \\ 0 & 0 & -\sqrt{1 - |\xi|^2} & \sqrt{1 - |\xi|^2} \end{pmatrix} \quad (17) \]

on particles \( B \) and \( D \) under the basis \( \{ |0\rangle_BD, |0\rangle_BD | 1\rangle_BD, |1\rangle_BD, |1\rangle_BD \} \). After that the state of the particles \( B \) and \( D \) becomes

\[ U_{BD} \frac{(\alpha|c|0\rangle + \beta|d|1\rangle)_B}{\sqrt{|\alpha|^2 + |\beta|^2}} \quad (18) \]

\[ = \frac{|c|(\alpha|0\rangle + \beta|1\rangle)_B |0\rangle_D - \beta\sqrt{|d|^2 - |c|^2} |1\rangle_B |1\rangle_D}{\sqrt{|\alpha|^2 + |\beta|^2}}. \]

Then a measurement on Bob’s auxiliary particle \( D \) in the basis \( \{ |0\rangle_D, |1\rangle_D \} \) follows. If \( |0\rangle_D \) occurs, we obtain the state \( \alpha|0\rangle + \beta|1\rangle \) with probability \( \frac{|c|^2}{|\alpha|^2 + |\beta|^2} \), i.e. the teleportation is successful. Otherwise the teleportation fails.

Based on the above argument, it is easy to image that the final step of the teleportation protocol is to introduce an auxiliary particle and make a collective unitary on the signal particle and the auxiliary particle. Then Bob performs a measurement on the auxiliary particle.

By Eq. (6), it is easy to show that the success probability of the teleportation is

\[ P = 2N^2M^2a^2(\min\{|1, |mb|^2\} + \min\{|b|^2, |mn|^2\}) \]

\[ + \min\{|m|^2, |nb|^2\} + \min\{|n|^2, |mb|^2\}). \quad (19) \]

Let us define

\[ \xi = N^2, \quad \zeta = M^2, \quad \eta = a^2. \quad (20) \]

Then we obtain

\[ P = 2(\min\{|\xi\zeta\eta, (1 - \xi)(1 - \zeta)(1 - \eta)\}) \]

\[ + \min\{|\xi(1 - \eta), (1 - \xi)(1 - \zeta)(1 - \eta)\}) \]

\[ + \min\{|(1 - \xi)\zeta\eta, (1 - \xi)(1 - \zeta)(1 - \eta)\}) \]. \quad (21) \]

Obviously,

\[ P(\xi, \zeta, \eta) = P(1 - \xi, \zeta, \eta) = P(\xi, 1 - \zeta, \eta) = P(\xi, \zeta, 1 - \eta). \quad (22) \]

Without loss of generality, we suppose

\[ 0 < \xi \leq \frac{1}{2}, \quad 0 < \zeta \leq \frac{1}{2}, \quad 0 < \eta \leq \frac{1}{2}. \quad (23) \]

Therefore

\[ P = 2(\xi \zeta \eta + \min\{|\xi\zeta(1 - \eta), (1 - \xi)(1 - \zeta)(1 - \eta)\}) \]

\[ + \min\{|(1 - \xi)\zeta\eta, (1 - \xi)(1 - \zeta)(1 - \eta)\}) + \min\{|1 - \xi)\zeta\eta, (1 - \xi)(1 - \zeta)(1 - \eta)\}). \quad (24) \]
Next we will find the maximum of $P$ when $\xi$ is fixed. Since
\[ P(\xi, \zeta, \eta) = P(\xi, \eta, \zeta), \]
so the maximum of success probability $P$ must occur in the region
\[ 0 < \zeta \leq \eta \leq \frac{1}{2}. \]
Assume
\[
\begin{align*}
\xi \zeta (1-\eta) &> (1-\xi)(1-\zeta)\eta, \\
(1-\xi)\zeta \eta &> (1-\zeta)(1-\eta),
\end{align*}
\]
we have
\[ \xi^2 > (1-\xi)^2, \]
which does not hold as $\xi \leq \frac{1}{2}$. So the hypothesis Eq.(27) is wrong. Similarly we can prove that
\[
\begin{align*}
(1-\xi)\zeta \eta &> (1-\zeta)(1-\eta), \\
(1-\xi)(1-\zeta)\eta &> (1-\zeta)(1-\eta),
\end{align*}
\]
is not holed also.
Evidently
\[
\begin{align*}
\xi \zeta (1-\eta) &> (1-\xi)(1-\zeta)\eta, \\
\xi (1-\zeta)\eta &< (1-\zeta)(1-\eta),
\end{align*}
\]
is wrong, because if Eq.(30) holds, we must have
\[ (1-\eta)/\eta > (1-\zeta)/\zeta \]
that means $\zeta > \eta$, which contradicts with Eq.(26).
Therefore, the whole region indicated by Eq.(26) can be divided into the following three regions $E, F, G$. We will discuss the maximal success probability in each region.

1. The region $E$ is defined by
\[
\begin{align*}
\xi \zeta (1-\eta) &\leq (1-\xi)(1-\zeta)\eta, \\
(1-\xi)\zeta \eta &\leq (1-\zeta)(1-\eta),
\end{align*}
\]
In this region, the success probability
\[ P = 2[(\xi(\eta + \zeta) + (1-2\xi)\zeta\eta]. \]
One can easily deduce that the maximum of $P$ in this region must occur in the boundary of $E$.

2. The region $F$ satisfies
\[
\begin{align*}
\xi \zeta (1-\eta) &\leq (1-\xi)(1-\zeta)\eta, \\
(1-\xi)\zeta \eta &\leq (1-\zeta)(1-\eta), \\
(1-\xi)(1-\zeta)\eta &\geq (1-\zeta)(1-\eta).
\end{align*}
\]
In this region, we have
\[ \xi \leq \zeta, \eta. \]
It is straightforward to obtain the success probability
\[ P = 2\xi. \]
in region $F$.

3. The region $G$ satisfies
\[
\begin{align*}
\xi (1-\eta) &\leq (1-\xi)(1-\zeta)\eta, \\
(1-\xi)\zeta \eta &\leq (1-\xi)(1-\eta), \\
(1-\xi)(1-\zeta)\eta &\geq (1-\xi)(1-\eta).
\end{align*}
\]
In this region, we have
\[ \zeta \leq \xi, \eta. \]
Obviously,
\[ P = 2\zeta < 2\xi \]
in region $G$.

Synthesizing all cases above, we arrive at the conclusion that the maximum of the success probability of the teleportation is
\[ P_{max} = 2\xi \]
and it occurs in the region $F$.

Now we generalize the above teleportation protocol to the case of $L$-parties. Suppose that the state Alice wants to transmit is stated in Eq.(1), and the quantum channel shared between Alice and other $L-1$ parties is
\[ |\text{GHZ}_{12\cdots L} = N(|00\cdots0 + n|11\cdots1)_{12\cdots L}, \]
where $N$ is defined by Eq.(2). We assume that all members of $L$-parties are cooperative and loyal, particles $A$ and $1$ belong to Alice, particle $i$ is in party $i$’s possession, $i = 2, \cdots, L$. The total state of the whole system can be written as
\[ |\psi\rangle_{A12\cdots N} = |\phi\rangle_A \otimes |\text{GHZ}_{12\cdots N}. \]

Without loss of generality, we assume that Alice wants the $L$-th party to receive the state. Firstly Alice performs a generalized Bell basis measurement on qubits $A$ and $1$ shown in Eq.(4) and publishes the outcome of the measurement. After that party $i$ makes the generalized $X$ basis measurement on his qubit $i$ and announces the result of the measurement, $i = 2, \cdots, L - 1$. Finally, according to the results of the measurements, party $L$ introduces an auxiliary particle, implements a collective unitary transformation on the particle $L$ and the auxiliary particle, and performs a measurement on the auxiliary particle. By completing the above steps, an unknown quantum state has been transmitted with unit fidelity to the receiver with a certain probability.

In summary, we have presented a scheme for probabilistic teleportation via a non-maximally entangled GHZ state. Quantum teleportation will succeed with a certain probability if the sender makes a generalized Bell state measurement, the cooperater performs a
generalized $X$ basis measurement, and the receiver introduces an auxiliary particle, performs a collective unitary transformation and makes a measurement on the auxiliary particle. The success probability of the teleportation is given. We also obtain the maximum of the success probability of the teleportation. The teleportation scheme has also been generalized to the more general case of $L$-parties.

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