Abstract

Given a hypergraph $H = (V,E)$ and an integer parameter $k$, a coloring of $V$ is said to be $k$-conflict-free ($k$-CF in short) if for every hyperedge $S \in E$, there exists a color with multiplicity at most $k$ in $S$. A $k$-CF coloring of a graph is a $k$-CF coloring of the hypergraph induced by the (closed or punctured) neighborhoods of its vertices. The special case of 1-CF coloring of general graphs and hypergraphs has been studied extensively.

In this paper we study $k$-CF coloring of graphs and hypergraphs. First, we study the non-geometric case and prove that any hypergraph with $n$ vertices and $m$ hyperedges can be $k$-CF colored with $\tilde{O}(m^{1/k+1})$ colors. This bound, which extends theorems of Cheilaris and of Pach and Tardos (2009), is tight, up to a logarithmic factor.

Next, we study string graphs. We consider several families of string graphs on $n$ vertices for which the 1-CF chromatic number w.r.t. punctured neighborhoods is $\Omega(\sqrt{n})$ (which is the maximum possible for any graph with $n$ vertices), and show that they admit a $k$-CF coloring with only $O(\log n)$ colors for very small constant parameters $k > 1$.

We then focus on $k = 1$ and prove that any string graph on $n$ strings with chromatic number bounded by $t$ admits a 1-CF coloring with $O(t^2 \log n)$ colors w.r.t. punctured neighborhoods; this bound is asymptotically sharp in $n$. Finally, we show that for a large class of string graphs, which includes intersection graphs of frames and of boundaries of pseudo-discs, the 1-CF chromatic number w.r.t. closed neighborhoods can be bounded in terms of the graph’s packing number.

1 Introduction

Let us start with the definition of a $k$-CF-coloring of a hypergraph, the main notion of this paper:

\begin{definition}
Let $H = (V,E)$ be a hypergraph and let $C$ be a coloring $C: V \to \mathbb{N}$. We say that $C$ is a $k$-conflict-free coloring ($k$-CF-coloring in short) if for every hyperedge $S \in E$ there exists a color $i \in \mathbb{N}$ such that $1 \leq |S \cap C^{-1}(i)| \leq k$. That is, for every hyperedge $S \in E$ there is some color assigned to at least one and at most $k$ vertices of $S$. For $k = 1$ we simply call $C$ a CF-coloring.

The minimum integer $\ell$ such that $H$ admits a $k$-CF-coloring with $\ell$ colors is denoted by $\chi_{k-CF}(H)$.
\end{definition}

Conflict-free coloring of hypergraphs. CF-coloring of hypergraphs was introduced in FOCS’2002 by Even et al. [13] and in the Ph.D. of Smorodinsky [22]. The original motivation to study CF-coloring came from spectrum allocation in radio networks. Since then, the notion of CF-coloring...
attracted significant attention from researchers. Many different aspects and variants of CF-coloring were studied in dozens of papers. See, e.g., [2, 5, 7, 8, 19, 23, 34] for a sample of various aspects of CF-coloring. We refer the reader to the survey [44] and the references therein for more on CF-colorings and its applied and theoretical motivation.

\textbf{k-conflict-free coloring.} In 2003, Har-Peled and Smorodinsky [22, 42] introduced the notion of \( k \)-CF coloring. Like in the first papers on CF-coloring, the motivation came from allocation of frequencies in cellular networks, since in reality, the interference between conflicting antennas is a function of the number of such antennas. In particular, they proved that if \( H \) is a hypergraph on \( n \) vertices with VC-dimension \( d \), then for any \( k \geq d \log n \), we have \( \chi_{k\text{-cf}}(H) = O(\log n) \), and that if \( H' \) is a hypergraph whose vertex set \( V \) consists of \( n \) balls in \( \mathbb{R}^3 \) and whose hyperedges are all subsets of \( V \) that cover a given point, then \( \chi_{k\text{-cf}}(H) = O(n^{1/k}) \), for all \( k \geq 2 \).

\textbf{Conflict-free coloring of graphs.} For a simple undirected graph \( G = (V, E) \) and a vertex \( v \in V \), the \emph{punctured neighborhood} of \( v \) in \( G \) is \( N'_G(v) = \{ u \in V : (v, u) \in E \} \). The \emph{closed neighborhood} of \( v \) is \( N_G(v) = N'_G(v) \cup \{ v \} \). A coloring \( C \) of \( V \) is called \emph{k-conflict-free} with respect to closed (respectively, punctured) neighborhoods, or in short, \emph{closed k-CF-coloring} (respectively, \emph{punctured k-CF-coloring}) if for any \( v \in G \), there exists a color \( i \) that appears in the set \( N_G[v] \) (respectively, \( N'_G(v) \)) between 1 and \( k \) times, i.e., \( 1 \leq |N'_G(v) \cap C^{-1}(i)| \leq k \) (or the same for \( N_G(v) \)).

The minimum number of colors in a closed (respectively, punctured) \( k \)-CF-coloring of \( G \) is denoted by \( \chi_{k\text{-cf}}^c(G) \) (respectively, \( \chi_{k\text{-cf}}^p(G) \)). Note that a closed (respectively, punctured) \( k \)-CF-coloring of a graph \( G = (V, E) \) is a \( k \)-CF-coloring of the hypergraph on the same vertex set whose hyperedges are all sets of the form \( N_G[v] \) (respectively, \( N'_G(v) \)) for all \( v \in V \), which we call the \emph{neighborhood hypergraph} of \( G \). As above, when we consider \( k = 1 \), the ‘\( k \)’ is omitted.

CF-coloring of graphs was introduced in 2009 by Cheilaris [6] and Pach and Tardos [34] and was studied extensively since then. In particular, for general graphs on \( n \) vertices, Cheilaris [6] and Pach and Tardos [34] obtained the tight bound \( \chi_{\text{cf}}^p(G) = O(\sqrt{n}) \) for punctured CF-coloring. This bound can be attained as a lower bound even with bi-partite graphs (as we discuss later). The situation for closed CF-coloring is different. Note that for any graph \( G \) it trivially holds that \( \chi_{\text{cf}}(G) \leq \chi(G) \) where \( \chi(G) \) denotes the “classic” chromatic number of \( G \). Pach and Tardos [34] showed that \( \chi_{\text{cf}}^c(G) = O(\log^2 n) \), and Glebov, Szabó, and Tardos [21] proved that this is asymptotically tight. Gargano and Rescigno [19] showed that the problem of computing an optimal closed CF-coloring is NP-complete and gave non-approximability results for punctured CF-coloring, and Abel et al. [1] showed that a slightly modified variant of the classical Hadwiger conjecture holds for closed CF-colorings.

Recently, three papers considered CF-coloring of intersection graphs of geometric objects, that is, graphs whose vertices are geometric objects, where two objects are adjacent if their intersection is non-empty. Fekete and Keldenich [14] showed (along with other results) that the intersection graph \( G \) of unit discs satisfies \( \chi_{\text{cf}}^c(G) \leq 6 \) and that the intersection graph \( G' \) of unit squares satisfies \( \chi_{\text{cf}}^p(G') \leq 4 \). Keller and Smorodinsky [24] proved (among other results) that the intersection graph \( G'' \) of pseudo-discs (i.e., simple Jordan regions such that the boundaries of any two of them intersect in at most two points) satisfies \( \chi_{\text{cf}}^p(G'') = O(\log n) \), and that this bound is asymptotically sharp. A few weeks ago, the latter result was generalized by Keszegh [27] who showed that the intersection graph of any family of \( n \) regions with a linear union complexity with respect to a family of pseudo-discs admits a CF-coloring with \( O(\log n) \) colors.
**String graphs.** A string graph is an intersection graph of curves (strings) in the plane. String graphs were introduced by Benzer [4] in 1959, and were studied in numerous papers (see, e.g., [12, 28, 36, 41]), both for practical applications and theoretical interest. Although string graphs seem rather general, in some cases they exhibit very distinct behavior than general abstract graphs. For example, Fox and Pach [15] showed that any string graph with \( m \) edges has a separator of size \( O\left(\frac{m^{3/4}}{\sqrt{\log m}}\right) \). This bound was improved later by Matoušek [31] to \( O\left(\sqrt{m \log m}\right) \). In [17] it was shown that any string graph with \( n \) vertices and \( \Omega(n^2) \) edges contains a subgraph of size \( \Omega(n^2) \) which is the incomparability graph of some partially ordered set. The chromatic number of string graphs was also studied extensively. For example, it was shown that triangle free string-graphs with arbitrarily large chromatic number [38], while outer-string graphs are \( \chi \)-bounded (that is, their chromatic number is bounded by a function of their clique number, see, e.g., [39]). In [10] Chudnovsky et al. considered string graphs which are triangle free but with an arbitrarily large chromatic number, and studied their induced subgraphs. Bounds on the chromatic numbers of some specific classes of string graphs, e.g., intersection graphs of frames and L-shapes, were studied in [29] and [32], respectively.

2 Results

In this paper we study CF-coloring of graphs, focusing on several classes of string graphs. Before presenting our results on string graphs we present a purely combinatorial result that holds for arbitrary hypergraphs.

**Sharp upper bound on the** \( k \)-**CF-chromatic number of general hypergraphs.** In his Ph.D. thesis, Cheilaris [6] showed that for any \( n \)-vertex graph \( G \), we have \( \chi_{p^n}(G) \leq 2\sqrt{n} \). Pach and Tardos [34] noted that the constant can be improved to \( \sqrt{2} \), and that in fact, one can show that any hypergraph with less than \( \binom{m}{2} \) hyperedges can be CF-colored with less than \( m \) colors. This result is sharp, as a 1-subdivision of the complete graph on \( m \) vertices (i.e., a graph \( G \) obtained from \( K_m \) by adding a vertex in the ‘middle’ of each edge, dividing each edge into two edges) has \( n = m + \binom{m}{2} \) vertices, and satisfies \( \chi_{p^n}(G) \geq m \).

We obtain an almost tight generalization of this result to \( k \)-CF-coloring:

**Theorem 2.1.** Let \( H = (V, E) \) be a hypergraph with \( n \) vertices and \( m \) hyperedges. Let \( k > 1 \) be a fixed integer. Then

\[
\chi_{k-cf}(H) = O(m^{\frac{1}{k+1}} \log^{\frac{k}{k+1}} n).
\]

In particular, as the neighborhood hypergraph of a graph on \( n \) vertices has \( n \) hyperedges, this implies that any graph \( G \) on \( n \) vertices satisfies \( \chi_{p^n}(G) = O(n^{\frac{1}{k+1}} \log^{\frac{k}{k+1}} n) \), and the same holds for \( \chi_{k-cf}(G) \).

Our proof technique is different from that of [6, 34]. We make use of the Lovász Local Lemma to find a so-called \((k+1)\)-weak coloring of \( H \) (i.e., a coloring of \( H \) in which no hyperedge of size \( \geq k+1 \) is monochromatic), and then leverage it to a \( k \)-CF coloring of \( H \) using an algorithm presented in [22].

We prove that Theorem 2.1 is tight up to logarithmic factors by presenting string graphs \( G \) on \( n \) vertices that satisfy \( \chi_{k-cf}(G) = \Omega(n^{\frac{1}{k+1}}) \).

Next, we present several results, focusing on string graphs, which we summarize below:
Upper bounds on the \( k \)-CF chromatic number of string graphs. String graphs exhibit a different behavior than intersection graphs of two dimensional Jordan regions. For example, for any intersection graph \( G \) of \( n \) discs it was shown recently in [24] that \( \chi_{\text{cf}}^{\text{pn}}(G) = O(\log n) \). while (as we show here) for any \( n \), there exist \( n \) circles so that their intersection graph \( G' \) satisfy \( \chi_{\text{cf}}^{\text{pn}}(G') = \Omega(\sqrt{n}) \). The bound \( \Omega(\sqrt{n}) \) is the worse possible for any graph with \( n \) vertices. We note that this lower bound can be achieved also with intersection graphs of very special families of strings such as segments, circles, frames (i.e., boundaries of axis-parallel rectangles) etc. This motivates us to exploit the geometry of string graphs by turning to study \( k \)-CF coloring for \( k > 1 \).

We prove an upper bound of \( \chi_{\text{cf}}^{\text{pn}}(G) = O(\log n) \) for several classes of string graphs. In particular, we show that if \( G \) is the intersection graph of \( L \)-shapes then \( \chi_{\text{cf}}^{\text{pn}}(G) = O(\log n) \), and that if \( G' \) is the intersection graph of \( L \)-shapes then \( \chi_{\text{cf}}^{\text{pn}}(G') = O(\log n) \). We note again that for both frames and \( L \)-shapes, a similar result could not be achieved for the punctured \( CF \)-chromatic number; indeed, there exist graphs \( G \) on \( n \) frames (or \( L \)-shapes) with \( \chi_{\text{cf}}^{\text{pn}}(G) = \Omega(\sqrt{n}) \), which matches the upper bound for general graphs mentioned above.

A key step in our proofs is constructing a \((k+1)\)-weak coloring of the punctured neighborhood hypergraph \( H = H(G) \). It is established by constructing an auxiliary graph \( G' \) such that each hyperedge of \( H \) of size \( \geq k + 1 \) contains an edge of \( G' \), and then showing that \( G' \) can be represented as a union of a small number of planar graphs, and thus, has a low chromatic number.

Bounding the punctured \( CF \)-chromatic number of string graphs in terms of their chromatic number. It is clear, as mentioned already that while the closed \( CF \)-chromatic number of a graph is bounded by its chromatic number (as any proper coloring of the graph is a closed \( CF \)-coloring), the punctured \( CF \)-chromatic number can be much higher. For example, the aforementioned \( 1 \)-subdivision of \( K_m \) is bipartite and thus 2-colorable, while its punctured \( CF \)-chromatic number is \( m \). We show that for string graphs, the situation is different.

**Theorem 2.2.** Let \( G = (V, E) \) be a string graph on a set \( S \) of \( n \) strings such that \( \chi(G) \leq t \). Then \( \chi_{\text{cf}}^{\text{pn}}(G) = O(t^2 \log n) \).

We also show that our upper bound is asymptotically sharp in the number of strings \( n \).

Theorem 2.2 implies, for example, that if \( G \) is the intersection graph of \( x \)-monotone curves with a bounded clique number, then \( \chi_{\text{cf}}^{\text{pn}}(G) = O(\log^3 n) \), where the constant in the big ‘O’ notation depends on the clique number. Indeed, this follows from Theorem 2.2 combined with the result of [40] where it was shown that any such graph is \( O(\log n) \)-colorable. Similarly, by [16, 18] and Theorem 2.2 any string graph with a bounded clique number admits a punctured \( CF \)-coloring with a polylogarithmic number of colors.

Bounding the closed \( CF \)-chromatic number of certain string graphs in terms of their packing number. The packing number of a graph \( G = (V, E) \), denoted by \( \nu(G) \), is the maximum size of an independent set of vertices in \( G \). For a simple closed Jordan curve \( C \) in the plane, we denote by \( \text{Int}(C) \) the bounded connected component of \( \mathbb{R}^2 \setminus C \). Put \( \bar{C} = C \cup \text{Int}(C) \). We prove:

**Theorem 2.3.** Let \( \mathcal{F} \) be a family of simple closed Jordan curves in the plane, let \( G = G(\mathcal{F}) \) be the intersection graph of \( \mathcal{F} \), and denote \( p = \nu(G) \). Let \( \mathcal{F}' = \{ \bar{C} : C \in \mathcal{F} \} \) be the family of bounded Jordan regions whose boundaries are the curves of \( C \), and let \( G' \) be the intersection graph of \( \mathcal{F}' \). If for any induced subgraph \( G_1' \) of \( G' \) we have \( \chi_{\text{cf}}^{\text{cn}}(G_1') \leq r \), then \( \chi_{\text{cf}}^{\text{cn}}(G) \leq O(\sqrt{r \cdot p \log p}) \).

\(^1\)Recall that by the Jordan curve theorem \( \mathbb{R}^2 \setminus C \) has exactly two connected components exactly one of which is bounded.
Thus, if the packing number of $G$ is small, then we can leverage an upper bound on the closed CF-chromatic number of $G'$ into an upper bound on the closed CF-chromatic number of $G$.

By combining Theorem 2.3 with results of [14, 24], we show that the intersection graphs $G$ of frames and of boundaries of pseudo-discs satisfy $\chi^\text{cf}_\text{cn}(G) = O(\sqrt{p} \log p)$ and that the overlap graph $G'$ of intervals on a line (that is, a graph whose vertices are intervals, and where two intervals are adjacent if exactly one endpoint of each of them is included in the other one) satisfies $\chi^\text{cf}_\text{cn}(G') = O(\sqrt{p} \log p)$, where $p$ is packing number of the graph.

Organization of the paper. In Section 3 we present definitions and previous results that will be used throughout the paper. In Section 4 we study $k$-CF coloring of general hypergraphs and prove Theorem 2.1. Then, in Section 5 we consider several families of string graphs on $n$ vertices for which the punctured CF chromatic number is $\Omega(\sqrt{n})$, and show that they admit a $k$-CF coloring with only $O(\log n)$ colors for very small constant parameters $k > 1$. We then focus on $k = 1$ and prove Theorems 2.2 and 2.3 in Sections 6 and 7, respectively.

3 Preliminaries

In this section we present several standard definitions and previous results that will be used throughout the paper.

Degree. The degree of a vertex $v$ in a hypergraph $H = (V, E)$ is the number of hyperedges that contain $v$. We denote the maximum degree of $H$, i.e., the maximum degree of a vertex $v \in V(H)$, by $\Delta = \Delta(H)$.

Induced hypergraph. An induced sub-hypergraph $H' = (V', E')$ of a hypergraph $H = (V, E)$ is a sub-hypergraph of $H$ in which $V' \subset V$, and the hyperedges in $E'$ are the restriction of the hyperedges in $E$ to $V'$.

Proper coloring and $t$-weak coloring. Let $H = (V, E)$ be a hypergraph. We say that $C$ is a proper coloring of $H$ if for every hyperedge $S \in E$ with $|S| \geq 2$ there exist two vertices $u, v \in S$ such that $C(u) \neq C(v)$. That is, if every hyperedge with at least two vertices is non-monochromatic.

$C$ is called a $t$-weak coloring if the same condition holds for any hyperedge of size $\geq t$. This notion was used implicitly in [22, 43] and then was explicitly defined and studied in the Ph.D. thesis of Keszegh [26, 25]. It is also related to the notion of cover-decomposability and polychromatic colorings (see, e.g., [20, 35, 37]).

$r$-degenerate graphs and planarity. A graph $G$ is called $r$-degenerate if each induced subgraph of $G$ contains a vertex of degree at most $r$.

The following easy claim relates the property of being $r$-degenerate to $(r + 1)$-colorability.

Claim 3.1. Any $r$-degenerate graph is $(r + 1)$-colorable.

The proof is an easy inductive argument. Indeed, one can pick a vertex of degree $\leq r$, use the induction hypothesis to properly color the remaining vertices in $r + 1$ colors, and then color the picked vertex in a color that differs from the colors of its $\leq r$ neighbors.

We shall use Claim 3.1 for graphs that can be represented as a union of planar graphs, via the following claim.

Claim 3.2. Let $G = (V, E)$ be a union of $s$ planar graphs on vertex set $V$ (that is, $G = (V, E_1 \cup E_2 \cup \ldots \cup E_s)$), where for each $i$, the graph $(V, E_i)$ is planar. Then $G$ is $6s$-colorable.
Proof. As each $G_i = (V, E_i)$ is a planar graph, it follows by Euler’s formula that $|E_i| \leq 3n - 6$ for all $i$, where $n = |V|$. Hence, $|E| \leq 3ns - 6s$, and therefore, the average degree of $G$ is at most $6s - \frac{12s}{n}$. This implies that there exists a vertex in $V$ whose degree is at most $6s - 1$. Since planar graphs are closed under vertex removal, the same holds for any induced subgraph of $G$, and thus, $G$ is $(6s - 1)$-degenerate. Therefore, by Claim 3.1, $G$ is 6-colorable, as asserted. \[\square\]

A planarity lemma. We shall use another lemma, whose proof is a standard planarity argument, of a type that appears many times in the context of string graphs (see, e.g., [30, 33, 40]).

Lemma 3.3. Let $S_1$ and $S_2$ be two sets of pairwise disjoint strings (i.e., the strings in $S_1$ are pairwise disjoint, and the same for $S_2$). Assume that each string in $S_2$ intersects exactly two strings of $S_1$. Let $G = (S_1, E)$ be the graph whose edges are all pairs $a, b \in S_1$ that are intersected by the same string $x \in S_2$. Then $G$ is planar.

The proof of the lemma is presented in Appendix A.

4 \( k \)-Conflict-Free coloring for arbitrary hypergraphs

In this section we prove Theorem 2.1 which asserts that for any hypergraph $H$ with $n$ vertices and $m$ hyperedges, and for any $k > 1$, we have $\chi_{k-cf}(H) = O(m^{\frac{1}{k+1}} \log^{\frac{1}{k+1}} n)$, and that this bound is tight, up to logarithmic factors.

We need two lemmas. The first lemma bounds the number of colors needed in a \((k + 1)\)-weak coloring of a hypergraph $H$ in terms of the maximum degree $\Delta(H)$.

Lemma 4.1. Let $H = (V, E)$ be a hypergraph. Let $\Delta$ denote the maximum degree of $H$. For any $k > 1$, there exists a \((k + 1)\)-weak coloring of $V$ with $O(\Delta^{\frac{k}{k+1}})$ colors.

Proof. We first observe that without loss of generality we can assume that all hyperedges have cardinality $k + 1$. Indeed, we can discard hyperedges of cardinality strictly less than $k + 1$ as this will not affect the desired coloring property. Moreover, we can replace any hyperedge with cardinality strictly more than $k + 1$ with one of its subsets of cardinality $k + 1$. Note that if such a subset is not monochromatic for a given coloring, so is the superset. Note also, that in this process we might replace many hyperedges with one single hyperedge. Obviously, the resulting hypergraph has maximum degree at most $\Delta$. Next, we proceed by a probabilistic proof that will make use of the Lovász’ Local Lemma (see, e.g., [33]). We will use $i = A\Delta^{\frac{1}{k+1}}$ colors for some appropriate constant $A$ to be determined later. We color randomly, uniformly and independently each vertex $v \in V$ with a color so that each color is assigned to $v$ with probability $p = \frac{1}{i}$. Notice that a bad event $B_S$ that a given hyperedge $S \in E$ is monochromatic happens with probability $p^k$. Note also that for every two hyperedges $S_1, S_2$ such that $S_1 \cap S_2 = \emptyset$ the corresponding events $B_{S_1}$ and $B_{S_2}$ are independent. Thus for every hyperedge $S$, the event $B_S$ is independent from all but at most $(k + 1)\Delta$ other events as $S$ has a non-empty intersection with at most $(k + 1)\Delta$ hyperedges. We choose $A$ so that $ep^k[(k + 1)\Delta + 1] < 1$, and thus, $\frac{e[(k+1)\Delta+1]}{A^{k+1}} < 1$, which holds for, e.g., $A = 13$. Hence, By the Lovász’ Local Lemma, we have that:

$$Pr[\bigcup_{S \in E} B_S] < 1$$

In particular, this means that there exists a coloring for which none of the bad events happen. This completes the proof of the lemma. \[\square\]
The second lemma bounds the $k$-CF-chromatic number of a hypergraph in terms of its maximum degree.

**Lemma 4.2.** Let $H = (V, E)$ be a hypergraph with $|V| = n$. Let $\Delta$ denote the maximum degree of $H$. Let $k > 1$ be an integer. Then $\chi_{k-\text{cf}}(H) = O(\Delta^k \log n)$.

To prove the lemma, we use an algorithm due to Har-Peled and Smorodinsky ([22]; see also [44]) which allows us to leverage any $(k + 1)$-weak coloring of a hypergraph $H$ into a $k$-CF coloring of $H$.

**Algorithm 1** $k$-CFcolor($H$): $k$-Conflict-Free-color a hypergraph $H = (V, E)$.

1. $i \leftarrow 0$: $i$ denotes an unused color
2. while $V \neq \emptyset$ do
3. Increment: $i \leftarrow i + 1$
4. Auxiliary coloring: find a weak $k + 1$-coloring $\chi$ of $H(V)$ with “few” colors
5. $V' \leftarrow$ Largest color class of $\chi$
6. Color: $f(x) \leftarrow i$, $\forall x \in V'$
7. Prune: $V \leftarrow V \setminus V'$, $H \leftarrow H(V)$
8. end while

**Theorem 4.3 ([22]).** Algorithm 1 outputs a valid $k$-CF-coloring of $H$.

**Proof of Lemma 4.2.** The proof is constructive. We use Algorithm 1 and Lemma 4.1 to produce a valid $k$-CF-coloring. It remains to argue about the total number of colors used. Note that by Lemma 4.1, each time we execute step 4 of Algorithm 1, we use at total of $O(\Delta^k)$ auxiliary colors, so in step 7 we discard at least $\Omega\left(\frac{|V|}{\Delta^k}\right)$ vertices. Thus, if $n$ is the number of vertices of $H$, after the $i$th step there are at most $O(n(1 - \frac{1}{\Delta^k})^i)$ remaining vertices. Hence, for $i = O(\frac{\Delta^k}{\log n})$ there are only a constant number of remaining vertices. Since the number of final colors used by Algorithm 1 equals the number of iterations, we obtain the asserted bound.

We are ready to prove Theorem 2.1. Let us recall its statement.

**Theorem 2.1** Let $H = (V, E)$ be an arbitrary hypergraph with $n$ vertices and $m$ hyperedges. Let $k > 1$ be a fixed integer. Then $\chi_{k-\text{cf}}(H) = O(m^{\frac{k+1}{k+1}} \log^{\frac{1}{k+1}} n)$.

**Proof.** Let $\Delta$ be an integer parameter to be determined later. Similarly to the proof of the upper bound on $\chi_{\text{cf}}(H)$ for general hypergraphs by Cheilaris [6] (presented in [34]), we start by iteratively coloring vertices contained in more than $\Delta$ hyperedges. Each time we find such a vertex we color it with a unique color that we will not use again. We then remove this vertex together with all hyperedges containing this vertex. Note that in each step we remove at least $\Delta$ hyperedges so the total number of colors we use is at most $m^\frac{m}{\Delta}$. Next, we are left with a hypergraph with at most $n$ vertices and at most $m$ hyperedges such that each vertex belongs to at most $\Delta$ hyperedges. By Lemma 4.2 we can use additional $O(\Delta^k \log n)$ colors for $k$-CF coloring this hypergraph. It is easy to verify that in the obtained coloring, for every hyperedge there exists a color with multiplicity at least one but at most $k$. Indeed, if the hyperedge was removed in the first step, it contains a vertex with a unique color. If it was input to the algorithm of the second step then it contains a color with multiplicity at most $k$ by Lemma 4.2. Note that the total number of colors used is at most $O(m^\frac{m}{\Delta} + \Delta^k \log n)$. We choose $\Delta = \frac{m^{\frac{k+1}{k+1}}}{\log^{\frac{1}{k+1}} n}$ and obtain the asserted bound.

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Corollary 4.4. For any graph $G$ with $n$ vertices, $\chi^{\text{pfn}}_{k\text{-cf}}(G) = O(n^{\frac{1}{k+1}} \log^{\frac{1}{k+1}} n)$.

Proof. Recall that a punctured $k$-CF-coloring of $G$ is a CF-coloring of the hypergraph on the same set of vertices, whose hyperedges are the punctured neighborhoods of any vertex in $G$. The number of hyperedges in this hypergraph is $n$, and thus the asserted bound follows from Theorem 2.1.

The following proposition shows that the upper bounds of Theorem 2.1 and of Corollary 4.4 are tight up to logarithmic factors, even for the class of string graphs on which we focus in this paper.

Proposition 4.5. There exist string graphs on $n$ vertices that require $\Omega(n^{\frac{1}{k+1}})$ colors in order to be $k$-CF-colored with respect to punctured neighborhoods.

The proof of the proposition is presented in Appendix B.

5 $k$-Conflict-Free coloring of some geometric objects and its implications on string graphs

Proposition 4.5 shows that in general, the punctured $k$-CF-chromatic number of a string graph may be as large as $\Omega(n^{\frac{1}{k+1}})$. In this section we show that for a large class of string graphs, $\chi^{\text{pfn}}_{k\text{-cf}}(G)$ is much lower, and can be bounded by $O(\log n)$, even for very small values of $k$.

Let $F$ be a family of subsets of $\mathbb{R}^2$, referred to as objects, such that each $\gamma \in F$ is a union of interior disjoint simple curves $\gamma_1, \ldots, \gamma_m$. Assume that for any $\gamma, \gamma' \in F$, the curves $\gamma_i, \gamma'_i$ are disjoint for each $i$. We say that $\{i, j\}$ is an existing intersection pattern in $F$ if $\gamma_i$ intersects $\gamma'_j$ for some $\gamma, \gamma' \in F$. Two intersecting objects $\gamma, \gamma' \in F$ are said to give an intersection pattern $\{i, j\}$ if $\gamma_i$ intersects $\gamma'_j$ or $\gamma_j$ intersects $\gamma'_i$. A family as above with at most $m$ existing intersection patterns is called an $m$-family.

For example, if $F$ is a family of frames (that is, boundaries of axis-parallel rectangles) in general position in the plane, then we can represent each $\gamma \in F$ as $\gamma = \gamma_1 \cup \ldots \cup \gamma_4$, where $\gamma_1$ is the left side, $\gamma_2$ is the bottom side, $\gamma_3$ is the right side, and $\gamma_4$ is the top side. Clearly, $\gamma_i \cap \gamma'_j = \emptyset$ for any $\gamma, \gamma' \in F$. The existing intersection patterns are $\{1, 2\}, \{1, 4\}, \{2, 3\}$, and $\{3, 4\}$, and each pair of intersecting frames gives either 2 or 4 intersection patterns. E.g., if the upper-right corner of one of the frames is included in the interior of the other frame, then the intersection patterns given by these frames are $\{1, 4\}, \{2, 3\}$.

Our main result in this section is the following:

Theorem 5.1. Let $F$ be an $\ell s$-family of size $n$, and assume that any two intersecting $\gamma, \gamma' \in F$ give at least $\ell$ intersection patterns. Let $G$ be the intersection graph of $F$. Then $G$ admits a punctured 2$s$-CF-coloring with $O(\ell s \cdot \log n)$ colors.

Proof. Let $K \subset F$. For each existing intersection pattern $i, j$ in $F$, we define $G_{\langle i,j \rangle}(K)$ to be the graph whose vertex set is $\{\gamma_i : \gamma \in K\}$, where there is an edge between two curves $\gamma_i$ and $\gamma'_j$ if they are consecutive along some curve $\gamma''_{ij}$ of $F$. (Namely, there exist two points $x \in \gamma_i \cap \gamma''_{ij}, y \in \gamma'_j \cap \gamma''_{ij}$ such that on $\gamma''_{ij}$ there is no point between $x$ and $y$ that belongs to $\bigcup_{\delta \in K} \delta$.) Note that although we deal with a subfamily $K$ of $F$, the edges in the corresponding graphs $G_{\langle i,j \rangle}$ are defined with respect to the entire initial family $F$. Since the definition of $G_{\langle i,j \rangle}(K)$ is sensitive to the order of the indices $i, j$, the number of graphs $G_{\langle i,j \rangle}(K)$ is twice the number of existing intersection patterns in $F$, and thus, by assumption it is at most $2\ell s$. 


We observe that each graph $G_{ij}(K)$ is planar. Indeed, define the vertex set to be the collection of curves $\gamma_i$, and when $\gamma_i$ and $\gamma_i'$ form an edge in $G_{ij}(K)$, we take the appropriate subcurve of the curve $\gamma_j''$ along which $\gamma_i$ and $\gamma_i'$ are consecutive. By a small perturbation (which is needed only when some $\gamma_i, \gamma_i', \gamma_i''$ are consecutive along some $\gamma_j''$), we are exactly in the setting of Lemma 3.3, whence $G_{ij}(K)$ is, indeed, planar.

Now, let $G(K)$ be the graph whose vertex set is $K$, where there is an edge between $\gamma, \gamma' \in K$ if some $G_{ij}(K)$ contains the edge $\{\gamma_i, \gamma_i'\}$.

**Claim 5.2.** $G(K)$ can be properly colored with at most $12\ell s$ colors.

**Proof.** As each graph $G_{ij}(K)$ is planar, and there are at most $2\ell s$ such graphs, $G(K)$ is a union of at most $2\ell s$ planar graphs. Hence, by Claim 3.2 it is $12\ell s$-colorable, as asserted.

Let $H = (\mathcal{F}, \mathcal{E})$ be the punctured neighborhood hypergraph of $\mathcal{F}$, i.e., $\mathcal{E} = \{N_G(\gamma) : \gamma \in \mathcal{F}\}$. Recall that an $r$-weak coloring of a hypergraph is a coloring of its vertices such that no edge of size $\geq r$ is monochromatic.

**Claim 5.3.** Any induced sub-hypergraph of $H$ admits a $(2s + 1)$-weak coloring with at most $12\ell s$ colors.

**Proof.** We present the proof for $H$; it will be apparent that the same argument holds for any induced sub-hypergraph as well. By Claim 5.2, the graph $G(\mathcal{F})$ admits a proper coloring $C$ with at most $12\ell s$ colors. We claim that $C$ is actually a $(2s + 1)$-weak coloring of $H$.

Indeed, let $e \in \mathcal{E}$ be a hyperedge of size $\geq 2s + 1$. We have to show that $e$ is not monochromatic in the coloring $C$. Let $\gamma \in \mathcal{F}$ be such that $e = N_G(\gamma)$. As each intersecting pair $\gamma, \gamma' \in \mathcal{F}$ gives at least $\ell$ intersection patterns, the total number (including multiplicities) of intersection patterns given by pairs of the form $(\gamma, \gamma')_{\gamma' \in e}$ is $\geq \ell(2s + 1)$. Hence, by the pigeonhole principle, there are two objects $\gamma', \gamma'' \in e$ and indices $i, j$ such that $\gamma_i'$ and $\gamma_i''$ intersect $\gamma_j$, in the points $x, y$ respectively. (Otherwise, any existing intersection pattern $a, b$ appears at most twice - once for $\gamma_a$ and once for $\gamma_b$, and thus we get at most $2\ell s$ intersection patterns, a contradiction.) We can assume w.l.o.g. that $\gamma_i'$ and $\gamma_i''$ are consecutive along $\gamma_j$, in the sense that there does not exist $\gamma''' \in \mathcal{F}$ s.t. an intersection point of $\gamma'''$ and $\gamma_j$ lies between $x$ and $y$. Thus, $(\gamma', \gamma'') \in E(G_{ij}(\mathcal{F}))$, which in turn implies that $(\gamma', \gamma'')$ is an edge of $G(\mathcal{F})$. Therefore, $C(\gamma') \neq C(\gamma'')$, implying that $e$ is not monochromatic, as asserted.

Finally, as Claim 5.2 holds for any $K \subset \mathcal{F}$, the same argument applies for any induced sub-hypergraph of $H$. This completes the proof of the claim.

To complete the proof of the theorem, we apply Algorithm 1 and Theorem 4.3, where we substitute $k = 2s$ in the parameter of Algorithm 1, and the number of colors in Step 4 of each iteration is at most $12\ell s$. In Step 7 of each iteration, we remove at least $\Omega(|V|/12\ell s)$ of the vertices, and thus, the number of iterations – which is equal to the number of colors in the resulting 2s-CF coloring of $H$ – is at most $O\left(\log_{\frac{12\ell s}{n}} n\right) = O(\ell s \cdot \log n)$.

This completes the proof of the theorem.

In order to illustrate how Theorem 5.1 can be applied, we present two easy corollaries, concerning families of frames and $L$-shapes. In both of them, we assume that the elements of $\mathcal{F}$ are in a general position in the plane, meaning that no intersection of two of them contains a segment.
Corollary 5.4. Let $F$ be a family of $n$ frames and let $G$ be the intersection graph of $F$. Then $\chi_{4-\text{cf}}^\text{pn}(G) = O(\log n)$.

Proof. There are at most 4 existing intersection patterns in $F$ and any two intersecting frames provide at least 2 intersection patterns. Hence, the assertion follows by applying Theorem 5.1 to $F$ with $\ell = 2$ and $s = 2$.

Corollary 5.5. Let $F$ be a family of $n$ $L$-shapes and let $G$ be the intersection graph of $F$. Then $\chi_{2-\text{cf}}^\text{pn}(G) = O(\log n)$.

Proof. There is only a single intersection pattern in $F$, and any two intersecting $L$-shapes give a single intersection pattern. Hence, the assertion follows by applying Theorem 5.1 to $F$ with $\ell = 1$ and $s = 1$.

Recall that our main motivation to study $k$-CF colorings (with $k \geq 2$) of intersection graphs of frames (or $L$-shapes), rather than 1-CF colorings, is justified by the following proposition, which shows that in both these cases, punctured 1-CF coloring may require as many as $\Omega(\sqrt{n})$ colors, which is equal to the upper bound on the punctured CF-chromatic number for general families.

Proposition 5.6. There exist families of $n$ $L$-shapes and families of $n$ frames whose intersection graphs satisfy $\chi_{\text{cf}}^\text{pn}(G) = \Omega(\sqrt{n})$.

The proof of the proposition is presented in Appendix C.

6 String graphs with a bounded chromatic number

For general graphs, having a small chromatic number does not imply any bound on the punctured CF-chromatic number. For example, if $G$ is the aforementioned 1-subdivision of $K_m$, then $G$ is bipartite and thus 2-colorable, while $\chi_{\text{cf}}^\text{pn}(G) = \Omega(\sqrt{|V(G)|})$, which matches the upper bound on $\chi_{\text{cf}}^\text{pn}(G)$ for general graphs. In this section we show that in contrast to general graphs, for string graphs, $\chi_{\text{cf}}^\text{pn}(G)$ can be bounded in terms of the chromatic number of $G$.

Theorem 6.1. Let $G = (V, E)$ be a string graph on a set $S$ of $n$ strings such that $\chi(G) \leq t$. Then $\chi_{\text{cf}}^\text{pn}(G) = O(t^2 \log n)$.

On the other hand, for any $n$ there exists a 2-colorable string graph $G$ such that $\chi_{\text{cf}}^\text{pn}(G) = \Omega(\log n)$.

Of course, Theorem 6.1 includes Theorem 2.2 stated in the Introduction.

In the proof of the theorem we use the following lemma.

Lemma 6.2. Let $S$ be a set of $n$ pairwise disjoint strings and let $S_1, S_2, \ldots, S_t$ be another family of $t$ sets of strings each of which contains pairwise disjoint curves. Let $G = (S, E)$ be a graph on $S$ whose edges are all pairs $x, y \in S$ such that there exists a string in $\bigcup_{i=1}^t S_i$ that intersects only them (among the elements of $S$). Then $\chi(G) \leq 6t$.

Proof. We have $E = \bigcup_{i=1}^t E_i$, where $E_i$ is the set of pairs witnessed by strings in $S_i$ (that is, pairs for which there exists a string in $S_i$ that intersects only them). By Lemma 3.3, each graph $G_i = (S, E_i)$ is planar, and thus, $G$ is a union of $t$ planar graphs. Thus, by Claim 3.2, $G$ is 6$t$-colorable, as asserted.
Proof of Theorem 6.1. Let $H$ be the punctured neighborhood hypergraph of $G$. That is, $H = (S, \mathcal{E})$ where $\mathcal{E} = \{N_G(s) | s \in S\}$. By definition, a CF-coloring of $H$ is a punctured CF-coloring of $G$.

As $\chi(G) \leq t$ by hypothesis, we can partition $S$ into $t$ sets $S_1, \ldots, S_t$ so that each set $S_i$ consists of pairwise disjoint strings. For any $1 \leq i \leq t$ we define an induced subhypergraph $H_i$ of $H$ whose vertex set is $S_i$. Note that $\{V(H_i)\}_{i=1,\ldots,t}$ is a partition of $S$.

Our aim is to CF-color each $H_i$ with $O(t \log n)$ colors, using a distinct palette of colors to each $H_i$. Thus we will obtain a coloring of all $n$ elements of $S$ with $O(t^2 \log n)$ colors, which is a CF-coloring of $H$. Indeed, any hyperedge that contains at least two elements of the same palette contains a uniquely colored element due to the coloring rule of each $H_i$. In any hyperedge that contains at most one element from each $S_i$, any color is unique since we used a distinct palette of colors for each $H_i$.

In order to CF-color each $H_i$ with $O(t \log n)$ colors, we start by proper-coloring it with $6t$ colors, and then we leverage the $6t$-proper coloring into a CF-coloring with $O(t \log n)$ colors using Algorithm 1 (Note that our setting indeed corresponds to the case $k = 1$ of Algorithm 1 as a 2-weak coloring is simply a proper coloring.)

Fix $H_i = (S_i, \mathcal{E}_i)$ and replace each hyperedge $e \in \mathcal{E}_i$ that contains more than two strings from $S_i$ with an edge, namely, a subset of $e$ of cardinality 2. This is done in the following way: Suppose that $e$ is witnessed by a string $s \in \bigcup_{j \neq i} S_j$, namely, $e = N_G(s)$. If $e$ contains more than two strings from $S_i$, we shrink $s$ into a string $s'$ such that $s$ contains $s'$ but $s'$ intersects exactly two members of $S_i$. We replace $e$ with that pair. Notice that there are many ways to shrink $s$ so there might be many ways to replace $e$ with an edge, but this does not matter to us.

Denote by $G_i$ the graph whose vertex set is $S_i$ and whose edges are the resulting pairs after the shrinking process. By a small perturbation, we are in the setting of Lemma 6.2 and thus we have $\chi(G_i) \leq 6t$. By the construction of $G_i$, any proper coloring of $G_i$ is a proper coloring of $H_i$, since any hyperedge $e \in \mathcal{E}_i$ contains an edge of $G_i$, and thus is non-monochromatic. Hence, we proved that for any $1 \leq i \leq t$, $\chi(H_i) \leq \chi(G_i) \leq 6t$. The same holds also for any induced subhypergraph of $H_i$, by the same arguments.

We can leverage the proper coloring of $H_i$ into a CF-coloring using Algorithm 1 and Theorem 4.3 where we substitute $k = 1$ in the parameter of Algorithm 1 and the number of colors in Step 4 of each iteration is $6t$. In Step 7 of each iteration, we remove at least $\Omega(|S_i|/6t)$ vertices, and thus, the number of iterations which is equal to the number of colors in the resulting CF coloring of $H_i$ – is at most $O\left(\log_{6t} \frac{n}{6t} \right) = O(t \log n)$.

The union of the colorings of all the $H_i$’s (whose vertices are a partition of $S$) with distinct palettes of colors is a CF-coloring of $H$, as we pointed out at above. This completes the proof of the upper bound.

The proof of the tightness statement appears in Appendix D.

Theorem 6.1 enables us to obtain upper bounds on the CF-chromatic number of different classes of string graphs whose chromatic number is low. One example is the following corollary:

Corollary 6.3. Let $G$ be an intersection graph of $x$-monotone curves with a bounded clique number. Then $\chi^{\text{pn}}_c(G) = O(\log^3 n)$.

Proof. It was proved in [39] that we have $\chi(G) = O(\log n)$. Hence, by Theorem 6.1 $\chi^{\text{pn}}_c(G) = O(\log^3 n)$.
7 CF-Coloring of string graphs with a low packing number

The packing number of a graph $G = (V, E)$, denoted by $\nu(G)$, is the maximum size of an independent set of vertices in $G$ (namely, a subset of $V$ that does not contain any edge).

In this section we prove Theorem 2.3 which we restate here:

**Theorem 2.3.** Let $\mathcal{F}$ be a family of simple closed Jordan curves in the plane, let $G = G(\mathcal{F})$ be the intersection graph of $\mathcal{F}$, and denote $p = \nu(G)$. Let $\mathcal{F}' = \{C : C \in \mathcal{F}\}$ be the family of bounded Jordan regions whose boundaries are the curves of $\mathcal{F}$, and let $G'$ be the intersection graph of $\mathcal{F}'$. If for any induced subgraph $G'_1$ of $G'$ we have $\chi^c_\text{cf}(G'_1) \leq r$, then $\chi^c_\text{cf}(G) \leq O(\sqrt{r \cdot p \log p})$.

Throughout this section, we let $\mathcal{F}$ be a family of simple closed Jordan curves in general position in the plane, and let $\mathcal{F}'$ be the family of the corresponding closed bounded Jordan regions. In the proof of the theorem we will use the following two simple observations and two well-known easy propositions.

**Observation 7.1.** For any $\bar{C}_1, \bar{C}_2 \in \mathcal{F}'$, exactly one of the following holds:

- $\bar{C}_1 \subset \bar{C}_2$ or $\bar{C}_2 \subset \bar{C}_1$
- $\bar{C}_1 \cap \bar{C}_2 \neq \emptyset$ if and only if $\bar{C}_1 \cap \bar{C}_2 \neq \emptyset$.

**Observation 7.2.** Let $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_m \in \mathcal{F}'$ be a chain with respect to inclusion, i.e., $\bar{C}_1 \subset \bar{C}_2 \subset \ldots \subset \bar{C}_m$, and consider the set of corresponding curves $S_0 = \{C_1, \ldots, C_m\}$. For any $C \in \mathcal{F}$, the indices of the curves in $S_0$ that intersect $C$ are consecutive, i.e., $\{1 \leq i \leq m : C \cap C_i \neq \emptyset\}$ is a set of consecutive integers.

Both observations are an immediate consequence of the Jordan curve Theorem.

**Proposition 7.3 ([14]).** Let $H$ be the discrete interval hypergraph, whose vertex set is $[n] = \{1, \ldots, n\}$ and whose hyperedges are all the subsets of $[n]$ consisting of consecutive elements of $[n]$. Then $\chi^{\text{cf}}(H) = \lceil \log n \rceil + 1$.

The following is an immediate corollary of Observation 7.2 and Proposition 7.3. We use here the notations of Theorem 2.3.

**Corollary 7.4.** Let $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_m \in \mathcal{F}'$ be a chain with respect to inclusion and let $S_0 = \{C_1, \ldots, C_m\}$ be the set of corresponding curves. Let $G_0$ be the induced subgraph of $G$ whose vertices are $\{C \in \mathcal{F} : 1 \leq i \leq m : C \cap C_i \neq \emptyset\}$ (that is, all curves in $\mathcal{F}$ that intersect some $C_i$). Then $\chi^{\text{cf}}_n(G_0) \leq \lceil \log m \rceil + 2$.

We color the vertices corresponding to $S_0$ with $\lceil \log m \rceil + 1$ colors according to Proposition 7.3, identifying each $C_i$ with the integer $i \in [m]$ in the statement of the proposition. We then color all other vertices with one additional color. It is easy to see that this coloring is valid. Indeed, the coloring provided by Proposition 7.3 supplies each vertex of $\bigcup_{j=1}^m N_G(C_j)$ with a uniquely-colored neighbor in its closed neighborhood, while each $C_i \in S_0$ is itself’s uniquely colored neighbor.

Finally, we use the following proposition, which is the ‘easy direction’ of the classical Dilworth’ theorem [11].

**Proposition 7.5.** Let $S$ be a finite partially ordered set. For any $p \in \mathbb{N}$, either $S$ contains a chain (i.e., a linearly ordered subset) of size $p + 1$, or else $S$ can be partitioned into $p$ anti-chains (i.e., subsets in which any two elements are incomparable).
Now we are ready to present the proof of the theorem.

Proof of Theorem 2.3. We define a partial ordering on $\mathcal{F}'$ by inclusion. That is, $C_1 \preceq C_2$ if $C_1 \subseteq C_2$. We color the vertices of $G = G(\mathcal{F})$ using the following iterative process.

Let $\ell$ be the maximum length of a chain in $\mathcal{F}'$. Note that $\ell \leq p$ since any chain in $\mathcal{F}'$ corresponds to an independent set in $V(G)$, and by assumption, $\nu(G) = p$. Assume first that $\ell \geq t$, for some threshold parameter $t$ to be determined later. Let $C_1 \prec C_2 \prec \ldots \prec C_{ \ell}$ be a maximum-length chain in $\mathcal{F}'$. By Corollary 7.4, we can color $\{C_1, C_2, \ldots, C_{ \ell}\}$ with $O(\log \ell) \leq O(\log p)$ colors, such that each $C \in \cup_{j=1}^{\ell} N_{G[j]}$ will have a uniquely-colored neighbor. Hence, with $O(\log p)$ colors (that will not be used anymore) we supply uniquely-colored neighbors for $\cup_{j=1}^{\ell} N_{G[j]}$. Now, we remove these vertices (i.e., these curves) of $\cup_{j=1}^{\ell} N_{G[j]}$ and repeat the process for the remaining graph (i.e., the remaining sub-family), using new colors. We repeat the process until the maximum length of a chain of the remaining graph drops below the threshold $t$. Note that in each step, the packing number of the remaining graph is reduced by at least $t$ (since the set of vertices that we remove contains an independent set of size $\ell \geq t$ and we also remove the neighborhoods of its members), and thus, we reach the case $\ell < t$ after performing at most $p/t$ steps, and using at most $O(\log p)$ colors. At this step, the remaining sub-family $\tilde{\mathcal{F}}'$ has maximum chain length $\ell < t$. By Proposition 7.5, $\tilde{\mathcal{F}}'$ can be partitioned into at most $t$ anti-chains $S_1, S_2, \ldots, S_t$, where $S_i = \{C_1^i, C_2^i, \ldots\}$. By Observation 7.1, for each $i$, the intersection graph of $S_i = \{C_1^i, C_2^i, \ldots\}$ is the same as the intersection graph of $S_i$. By the assumption of the theorem we have $\chi_{\text{cf}}^\nu(G_{S_i}) \leq r$, and thus, we can color the elements of $S_i$ with $r$ colors such that each of them will have a uniquely-colored neighbor in its closed neighborhood. Using a different set of colors for each $S_i$, we obtain a CF-coloring of $\tilde{G}$ with at most $tr$ colors.

Overall, we obtain a CF-coloring of $G$ with at most $O((p \log p)/t) + tr$ colors. Choosing $t = \sqrt{p \log p}/r$ in order to minimize the upper bound, we obtain a coloring of $G$ with $O(\sqrt{p \log p} \cdot r)$ colors, as asserted.

To demonstrate the generality of Theorem 2.3, let us briefly present three applications. The first concerns the intersection graph of frames – boundaries of rectangles.

Corollary 7.6. Let $\mathcal{F}$ be a family of frames in the plane, and let $G$ be the intersection graph of $\mathcal{F}$. If $p = \nu(G)$, then $\chi_{\text{cf}}^\nu(G) = O(\sqrt{p} \log p)$.

Proof. By Theorem 2.3, the assertion will follow once we show that the intersection graph $G'$ of axis-parallel rectangles in the plane satisfies $\chi_{\text{cf}}^\nu(G') \leq O(\log p)$, where $p$ is its packing number. This indeed follows easily from the proof of [24] Theorem 1.6. (Note that the proof, as written in [24], yields only $\chi_{\text{cf}}^\nu(G') \leq O(\log |V(G')|)$. However, it can be easily modified to yield $O(\log p)$ instead.)

The second application concerns boundaries of pseudo-discs, that is, simple Jordan regions such that the boundaries of any two of them intersect in at most two points. Well-studied families of pseudo-discs are discs, homothets of a convex set, etc.

Corollary 7.7. Let $\mathcal{F}'$ be a family of pseudo-discs in the plane, and let $\mathcal{F}$ be the family of the corresponding boundaries. Denote by $G$ be the intersection graph of $\mathcal{F}$. If $p = \nu(G)$, then $\chi_{\text{cf}}^\nu(G) = O(\sqrt{p} \log p)$.

Proof. Like in Corollary 7.6, the assertion will follow once we show that the intersection graph $G$ of pseudo-discs satisfies $\chi_{\text{cf}}^\nu(G) \leq O(\log p)$, where $p$ is its packing number. This indeed follows from the first step of the proof of [24] Theorem 1.3.
Our third example is not a direct application of Theorem 2.3 but can be proved in exactly the same way.

Claim 7.8. Let $F$ be a family of closed intervals on a line, and let $G_{\text{olap}}$ be the overlap graph of $F$ (i.e., $(I_1, I_2) \in E(G_{\text{olap}})$ if the endpoint of one of them is included in the second). If $p = \nu(G_{\text{olap}})$, then $\chi_{\text{cl}}(G_{\text{olap}}) \leq O(\sqrt{p \log p})$.

Proof. We apply the proof of Theorem 2.3 verbatim, with $G_{\text{olap}}$ in place of $G$ and $G'$ being the intersection graph of $F$. The first step of the proof (i.e., the case $\ell \geq t$) works since if $I_1 \subset I_2 \subset \ldots \subset I_m$ is a chain of intervals, then for each interval $I$, the indices $\{1 \leq i \leq m : I \text{ overlaps } I_i\}$ are consecutive, and thus, Corollary 7.4 can be applied. The second step (i.e., the case $\ell < t$) applies with $k = 3$, since by [14], the intersection graph $G'$ of closed intervals on a line satisfies $\chi_{\text{cl}}(G') \leq 3$. This completes the proof. 

\section*{A Proof of Lemma 3.3}

In this appendix we present the proof of Lemma 3.3. Let us recall its formulation.

Lemma 3.3. Let $S_1$ and $S_2$ be two sets of pairwise disjoint strings (i.e., the strings in $S_1$ are pairwise disjoint, and the same for $S_2$). Assume that each string in $S_2$ intersects exactly two strings of $S_1$. Let $G = (S_1, E)$ be a graph whose edges are all pairs $a, b \in S_1$ that are intersected by the same string $x \in S_2$. Then $G$ is planar.

Proof. We present a planar drawing of the graph $G$. For any string $s \in S_1$, we choose a point $x_s \in s$. These points represent the vertices of $G$. For each edge $e = (s_1, s_1') \in E$, we define the planar drawing of $e$ as follows. Since $e \in E$, there exists $s_2 \in S_2$ that intersects $s_1$ at the point $x_1$ and $s_1'$ at the point $x_2$ (see Figure 1).

The planar representation of the edge $(s_1, s_1')$ consists of the union of three curves: The first one connects $x_{s_1}$ to $x_1$ on the curve $s_1$, the second one connects $x_1$ to $x_2$ on $s_2$, and the third one connects $x_2$ to $x_{s_1'}$ on $s_1'$.

In order to prove that $G$ is planar, it is sufficient to show that for any two vertex-disjoint edges $(s_1, s_1'), (t_1, t_1') \in E$, the planar drawing of $(s_1, s_1')$ is disjoint with the planar drawing of $(t_1, t_1')$. (That this condition is sufficient follows, e.g., from the well-known Hanani-Tutte Theorem [9, 15].)

Since the strings in $S_1$ are pairwise disjoint, the first and the third parts of the planar drawing of $(s_1, s_1')$ are disjoint with the first and the third parts of the planar drawing of $(t_1, t_1')$. Similarly, the
second part of the planar drawing of \((s_1, s'_1)\) is disjoint with the second part of the planar drawing of \((t_1, t'_1)\) as the strings in \(S_2\) are pairwise disjoint. Moreover, since we assumed that any string in \(S_2\) intersects exactly two strings in \(S_1\), the second part of the planar drawing of \((s_1, s'_1)\) is disjoint with the first and the third parts of \((t_1, t'_1)\) and vice versa. This completes the proof.

B Proof of Proposition 4.5

In this appendix we present the proof of Proposition 4.5. Let us recall its formulation.

**Proposition 4.5.** There exist string graphs on \(n\) vertices that require \(\Omega(n^{\frac{1}{k+1}})\) colors in order to be \(k\)-CF-colored w.r.t. punctured neighborhoods.

**Proof.** Take a family \(K\) of \(m \geq k + 2\) curves such that any \(k + 1\) of them intersect in a distinct point (one can slightly perturb each curve if one wants a family in a general position). For each subset of \(k + 1\) curves, add to \(K\) a curve close to their intersection point, so that this added curve intersects exactly this subset, and denote the set of added curves by \(S\). Clearly, if one wants to \(k\)-CF-color \(G\), the intersection graph of \(K \cup S\), w.r.t. punctured neighborhoods, no subset of \(k + 1\) curves from \(K\) can have the same color, as otherwise the neighborhood of some curve from \(S\) is monochromatic. Hence, while the total number of curves is \(n = m + \binom{m}{k+1}\), the number of colors required is at least \(m/k\) (by the pigeonhole principle). Note that we have

\[
n = m + \binom{m}{k+1} \leq 2 \binom{m}{k+1} \leq \frac{2m^{k+1}}{(k+1)!} \leq \frac{2e^{k+1}}{e^{k+1}} \leq \left(\frac{em}{k+1}\right)^{k+1},
\]

where the first inequality holds since \(m \geq k + 2\). Therefore,

\[
\chi_{\text{pn}}^{\text{cf}}(G) \geq \frac{m}{k} \geq \frac{1}{e} \frac{em}{n^{\frac{1}{k+1}}} \geq \frac{1}{e} n^{\frac{1}{k+1}} = \Omega\left(n^{\frac{1}{k+1}}\right),
\]

as asserted.

C Proof of Proposition 5.6

In this appendix we prove Proposition 5.6. Let us recall its formulation.

**Proposition 5.6.** There exist families of \(n\) \(L\)-shapes and families of \(n\) frames whose intersection graphs satisfy \(\chi_{\text{cf}}^{\text{pn}}(G) \geq \Omega(\sqrt{n})\).

**Proof.** Let \(G\) be the graph obtained from the clique \(K = K_m\) by adding a set \(V\) of \(\binom{m}{2}\) vertices such that any pair of vertices in \(K\) is adjacent to a distinct vertex of \(V\). (Note that this is not exactly a 1-subdivision of \(K_m\), since the edges of the original \(K_m\) are not removed. In particular, \(\chi(G) = m.\)) It is easy to see that \(\chi_{\text{cf}}^{\text{pn}}(G) \geq m.\) Indeed, if two vertices of \(K\) are colored with the same color, then the vertex in \(V\) adjacent to both of them has a monochromatic punctured neighborhood. Setting \(n = |V(G)|\), we get for \(m \geq 3:\)

\[
n = m + \binom{m}{2} \leq 2 \binom{m}{2} \leq m^2.
\]

Hence, \(\chi_{\text{cf}}^{\text{pn}}(G) \geq m \geq \sqrt{n}.

It remains to show that $G$ is the intersection graph of some family of $n$ $L$-shapes, and also of some family of $n$ frames. We present the argument for $L$-shapes; exactly the same argument works for frames.

Take $m$ pairwise intersecting $L$-shapes such that no three of them intersect at the same point, and for each pair, take an intersection point. Then, for every pair $(\gamma_1, \gamma_2)$ of shapes, add a small shape $\gamma$ close to the selected intersection point of $\gamma_1$ with $\gamma_2$, such that $\gamma$ intersects only $\gamma_1$ and $\gamma_2$. The intersection graph of the resulting family is $G$. \hfill \Box

\section{D Tightness of Theorem 6.1}

In order to prove the tightness of Theorem 6.1, we need Proposition 7.3. For sake of convenience, we restate it here.

**Proposition 7.3.** Let $H$ be the discrete interval hypergraph, whose vertex set is $[n] = \{1, \ldots, n\}$ and whose hyperedges are all the subsets of $[n]$ consisting of consecutive elements of $[n]$. Then $\chi_{cf}(H) = \lfloor \log n \rfloor + 1$.

Consider a family $S$ of $n = \binom{t}{2} + 2t$ segments, consisting of $t$ vertical parallel segments $s_1, \ldots, s_t$ and of $\binom{t}{2} + t$ horizontal parallel segments, such that for any consecutive subset of $s_1, \ldots, s_t$ there exists a horizontal segment that intersects exactly that subset (see Figure 2).

Let $G$ be the intersection graph of $S$ (which is obviously a string graph). Clearly, $\chi(G) = 2$. It is easy to see that the punctured neighborhood hypergraph of $G$ contains an isomorphic copy of the discrete interval hypergraph, and thus, $\chi_{cn}(G) = \Omega(\log n)$. This proves that the assertion of Theorem 6.1 is sharp in $n$.

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