Nonlocal boundary value problem for generalized Hilfer implicit fractional differential equations

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1 | INTRODUCTION

Fractional differential operators (FDOs) permit capturing the memory effects due to its nonlocal nature; consequently, modeling of real-world dynamics having memory effect is more suitable with FDOs than the classical integer-order derivatives. Many FDOs can be found in the literature with the aim that different kinds of real-world physical phenomena can be modeled suitably. Further, the defined FDO must preserve the classical properties of integer-order derivatives. Among the existing classical FDOs, most widely used FDOs are Caputo derivative and Riemann–Liouville derivative. But in many cases, FDOs with the singular kernel cannot describe the nonlocality of many real-world dynamics. Therefore, new FDOs with nonsingular kernel have been introduced, namely, Caputo–Fabrizio (C-F) fractional derivative (with nonsingular exponential function as its kernel) and Atangana–Baleanu–Caputo (ABC) fractional derivative (with nonsingular Mittag–Leffler function as its kernel). For detailed studies on theoretical analysis, approximations, distinct methods of existence and uniqueness solutions, development of various kinds of models, and recent applications of fractional differential equations (FDEs) involving the C-F FDO, we refer the interested reader to previous works. Jajarmi et al. presented and examined a new approach for the mathematical modeling involving ABC fractional derivative for a dengue fever outbreak, a tumor-immune surveillance mechanism, free motion of a coupled oscillator, and the impact of diabetes mellitus on the epidemiology of tuberculosis.

Boundary value problems (BVPs) for FDEs and inclusions with various kinds of boundary conditions is a topic of high interest. Numerous specialists have developed different methods and techniques for analyzing various classes of nonlinear fractional BVPs. Existence and uniqueness results for nonlocal fractional BVPs involving Caputo FDO has been examined by Benchohra et al. and Ahmad et al. through various types of fixed-point theorems. Utilizing the methodology of Ahmad and Nieto, the study has been extended to nonlocal BVPs for nonlinear integrodifferential equations. Baleanu et al. investigated the existence of solutions for a three-step crisis fractional integrodifferential equations. By utilizing the equivalent fractional integral equation to the non-linear implicit fractional differential equations involving $\Psi$-Hilfer fractional derivative subject to nonlocal fractional integral boundary conditions. The existence of a solution, Ulam–Hyers, and Ulam–Hyers–Rassias stability have been acquired by means of an equivalent fractional integral equation.

In this paper, we derive the equivalent fractional integral equation to the non-linear implicit fractional differential equations involving $\Psi$-Hilfer fractional derivative subject to nonlocal fractional integral boundary conditions. The existence of a solution, Ulam–Hyers, and Ulam–Hyers–Rassias stability have been acquired by means of an equivalent fractional integral equation. Our investigations depend on the fixed-point theorem due to Krasnoselskii and the Gronwall inequality involving $\Psi$-Riemann–Liouville fractional integral. Finally, examples are provided to show the utilization of primary outcomes.
the shifted Legendre and Chebyshev polynomials, Kojabad and Rezapour\textsuperscript{22} have given a numerical method for finding solutions for a sum-type Caputo fractional integrodifferential equations. Zhang\textsuperscript{23} obtained results relating to the existence and multiplicity of positive solutions of BVP for nonlinear Caputo FDEs. Jiang and Wang\textsuperscript{24} investigated results concerning the existence of solutions of multipoint BVP for Riemann–Liouville FDEs. Further, significant recent work pertaining to BVPs for fractional differential equations and inclusions can be found in previous works.\textsuperscript{25–28}

Study about Caputo implicit FDEs with initial condition of the form

\[ cD^\alpha x(t) = f(t, x(t), cD^\alpha x(t)), \ x(0) = x_0, \]

have been initiated by Nieto et al.\textsuperscript{29} The major investigation relating the existence and uniqueness of the solution and Ulam–Hyers stabilities for implicit FDEs with different kinds of initial and boundary conditions can be found in the work of Benchohra et al.\textsuperscript{30–36} Kucche et al.\textsuperscript{37} investigated the existence, the interval of existence, and uniqueness of solutions along with various qualitative properties of solution for the implicit FDEs.

On the other hand, Hilfer\textsuperscript{38} defined a two-parameter fractional derivative called Hilfer fractional derivative that incorporates Riemann–Liouville FDO and Caputo FDO. The fundamental work on the theory of initial value problem for FDEs involving Hilfer derivative can be found in Furati and Kassim.\textsuperscript{39} Asawasamrit et al.\textsuperscript{40} initiated the study of BVPs for FDEs involving Hilfer fractional derivatives subject to nonlocal integral boundary conditions. The BVP for fractional integro-differential equations with Hilfer derivative has been researched in Thabet et al.\textsuperscript{41} for the existence and data dependence of solutions. The implicit FDEs with a nonlocal condition involving Hilfer fractional derivative were investigated in previous works\textsuperscript{42,43} for the existence of a solution and Ulam type stabilities.

The Hilfer version of the fractional derivative with another function called \( \Psi \)-Hilfer FDO has been presented by Sousa et al.\textsuperscript{44} The basic study about existence and uniqueness of the solution of a nonlinear \( \Psi \)-Hilfer FDEs with different kinds of initial conditions and the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of its solutions have been explored in previous studies.\textsuperscript{45–52} The implicit FDEs involving \( \Psi \)-Hilfer derivative has been investigated in Sousa and Oliveira\textsuperscript{53} for the existence and uniqueness of the solution and the Ulam–Hyers–Rassias stability.

Thinking about available literature, it is seen that the study of BVPs for nonlinear implicit FDEs involving generalized fractional derivative is still in the underlying stages and numerous parts of this theory should be investigated. Inspired by the work of the papers mentioned above and the work of Benchohra et al.\textsuperscript{17} and Asawasamrit et al.\textsuperscript{40} the primary goal of the present work is to establish the existence of the solutions and to examine the Ulam–Hyers stabilities of the following implicit FDEs involving \( \Psi \)-Hilfer fractional derivative subject to nonlocal integral boundary conditions,

\[
^H D_{a+}^{\mu, \nu} y(t) = f(t, y(t), ^H D_{a+}^{\mu, \nu} y(t)), \ t \in J', \ 1 < \mu < 2, \ 0 \leq \nu \leq 1,
\]

\[
y(a) = 0,
\]

\[
y(b) = \sum_{i=1}^{m} \lambda_i \ I_{a+}^{\delta_i} \Psi y(t_i),
\]

where \( J' = (a, b], ^H D_{a+}^{\mu, \nu} : \Psi (\cdot) \) is the \( \Psi \)-Hilfer fractional derivative of order \( \mu \) and type \( \nu \), \( I_{a+}^{\delta_i} : \Psi (\cdot) \) is the \( \Psi \)-Riemann–Liouville fractional integral of order \( \delta_i > 0 \), \( \lambda_i \in \mathbb{R} \), \( i = 1, 2, \ldots, m \), \( 0 \leq a \leq t_1 < t_2 < t_3 < \ldots < t_m \leq b \) and \( f \in C(J' \times \mathbb{R}, \mathbb{R}) \).

The \( \Psi \)-Hilfer implicit FDEs with nonlocal integral boundary conditions considered in the present paper is the more broad class of BVPs that incorporates the BVP for implicit FDEs including most widely used Riemann–Liouville FDO (for \( \nu = 0 \), \( \Psi(t) = t \)) and Caputo FDO (for \( \nu = 1 \), \( \Psi(t) = t \)). Apart from this, the BVP (1)–(3) for different values of \( \nu \) and \( \Psi \) includes the study of implicit FDEs involving the FDOs: Hilfer, Hadamard, Katugampola, Chen, Jumarie, Prabhakar, Erdélyi-Kober, Riesz, Feller, Weyl, Cassar, and many other FDOs listed in Sousa and Oliveira.\textsuperscript{44} Further, results obtained in the present paper includes the results of Ahmad and Nieto\textsuperscript{18} and Asawasamrit et al.\textsuperscript{40} in particular,

- for \( \Psi(t) = t \), the outcomes acquired in the present paper incorporates the results of Asawasamrit et al.\textsuperscript{40} for nonimplicit Hilfer FDEs with nonlocal BVP.

- \( \nu = 1 \), \( \Psi(t) = t, \delta_i = 0, a = 0, b = 1 \), and \( t_1 = t_2 = \ldots = t_{m-1} = 0 \) incorporates the results of Ahmad and Nieto\textsuperscript{18} for nonimplicit Caputo FDEs with nonlocal BVP.
The main contribution of the present work is to acquire equivalent fractional integral equation and to derive the existence of a solution, Ulam–Hyers, and Ulam–Hyers–Rassias stability for broad class of nonlocal implicit BVP (1)–(3) with minimal conditions.

The paper is structured in five sections as follows: In Sections 2, we present the definitions and the results that are utilized in the paper. Section 3 provides an equivalent fractional integral equation to the nonlocal implicit BVP (1)–(3). Section 4 deals with existence of solution for nonlocal BVP (1)–(3). Ulam–Hyers stability and Ulam–Hyers–Rassias stability have been examined in Section 5. In Section 6, we have provided examples to justify our main results.

2 | PRELIMINARIES

Let $\xi = \mu + \nu (2 - \mu), 1 < \mu < 2, 0 \leq \nu \leq 1$. Then $1 < \xi \leq 2$. Let $\Psi \in C^1(J, \mathbb{R})$ be an increasing function with $\Psi'(t) \neq 0$, for all $t \in J = [a, b]$. Consider the space

$$C_{2-\xi} \Psi(J, \mathbb{R}) = \{ \phi : (a, b) \to \mathbb{R} \mid (\Psi(t) - \Psi(a))^{2-\xi} \phi(t) \in C(J, \mathbb{R}) \} ,$$

with the norm

$$\| \phi \|_{C_{2-\xi} \Psi(t, \mathbb{R})} = \max_{t \in J} \left| (\Psi(t) - \Psi(a))^{2-\xi} \phi(t) \right| . \tag{4}$$

**Definition 2.1** (Kilbas et al.) Let $\mu > 0 (\mu \in \mathbb{R})$, $\phi \in L_1(J, \mathbb{R})$. Then, the $\Psi$-Riemann–Liouville fractional integral of a function $\phi$ with respect to $\Psi$ is defined by

$$I^\mu_{a+} \Psi \phi(t) = \frac{1}{\Gamma(\mu)} \int_a^t \Psi'(s)(\Psi(t) - \Psi(s))^\mu-1 \phi(s) \, ds. \tag{5}$$

**Definition 2.2** (Sousa and Oliveira). Let $n - 1 < \mu < n \in \mathbb{N}$ and $h \in C^n(J, \mathbb{R})$. Then, the $\Psi$-Hilfer fractional derivative $^H D^\mu_{a+} : \Psi \phi(t)$ of a function $\phi$ of order $\mu$ and type $0 \leq \nu \leq 1$ is defined by

$$^H D^\mu_{a+} : \Psi \phi(t) = I^{\mu(n-\nu)}_{a+} \Psi \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n I^{(1-\nu)(n-\mu)}_{a+} : \Psi \phi(t). \tag{6}$$

**Lemma 2.1** (Kilbas et al. and Sousa and Oliveira). Let $\mu, \chi > 0$ and $\delta > 0$. Then

a. $I^\mu_{a+} : \Psi \Psi I^\mu_{a+} : \Psi \phi(t) = I^{\mu+\chi}_{a+} : \Psi \phi(t)$

b. $I^\mu_{a+} : \Psi (\Psi(t) - \Psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mu+\delta)} (\Psi(t) - \Psi(a))^{\mu+\delta-1}$.

c. $^H D^\mu_{a+} : \Psi (\Psi(t) - \Psi(a))^{\delta-1} = 0$.

**Lemma 2.2** (Sousa and Oliveira). If $\mu > 0$, and $0 \leq \alpha < 1$, then $I^\mu_{a+} : \Psi$ is bounded from $C_{\alpha} \Psi[a, b]$ to $C_{\alpha} \Psi[a, b]$. In addition, if $\alpha \leq \mu$, then $I^\mu_{a+} : \Psi$ is bounded from $C_{\alpha} \Psi[a, b]$ to $C[a, b]$.

**Lemma 2.3** (Sousa and Oliveira). If $\phi \in C^n[a, b]$, $n - 1 < \mu < n$ and $0 \leq \nu \leq 1$, then

1. $I^\mu_{a+} : ^H D^\mu_{a+} : \Psi \phi(t) = \phi(t) - \sum_{k=1}^{n} (\Psi(t) - \Psi(a))^{k-\nu} C(\xi-k+1) I^\mu_{a+} : \Psi \phi(a)$ where

$$I^\mu_{a+} : \Psi \phi(t) = \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \phi(t).$$

2. $^H D^\mu_{a+} : I^\mu_{a+} : \Psi \phi(t) = \phi(t)$.

**Theorem 1** (Sousa and Oliveira). Let $u, v$ be two integrable functions and $g$ be continuous with domain $[a, b]$. Let $\Psi \in C^1[a, b]$ be an increasing function such that $\Psi'(t) \neq 0, \forall t \in [a, b]$. Assume that

1. $u$ and $v$ are nonnegative;
2. $g$ is nonnegative and nondecreasing.
In this section, we derive equivalent fractional integral equation to the nonlocal BVP (1)–(3).

Let
\[ u(t) \leq v(t) + \frac{g(t)}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} u(s) \, ds, \]
then
\[ u(t) \leq v(t) + \frac{\sum_{k=1}^n [g(t)\Gamma(a)]^k}{\Gamma(k\alpha)} \int_0^t [\Psi(t) - \Psi(s)]^{k\mu-1} v(s) \, ds, \quad t \in [a, b]. \tag{7} \]

Further, if \( v \) is a nondecreasing function on \([a, b] \) then
\[ u(t) \leq v(t) E_{\mu} \left( (g(t)\Gamma(\mu)\Psi(t) - \Psi(a))^\mu \right), \]
where \( E_{\mu}(\cdot) \) is the Mittag–Leffler function of one parameter, defined as
\[ E_{\mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\mu + 1)}. \]

**Theorem 2.5** (Zhou et al., Krasnoselskii). Let \( \mathcal{M} \) be a closed, convex, and nonempty subset of a Banach space \( \mathcal{X} \), and \( P, Q \) are the operators such that
1. \( Px + Qy \in \mathcal{M} \) whenever \( x, y \in \mathcal{M} \);
2. \( P \) is a contraction mapping;
3. \( Q \) is compact and continuous.

Then, there exists \( y^* \in \mathcal{M} \) such that \( y^* = Py^* + Qy^* \).

## 3 | EQUIVALENT FRACTIONAL INTEGRAL EQUATION

In this section, we derive equivalent fractional integral equation to the nonlocal BVP (1)–(3).

**Theorem 3.1.** Let \( 1 < \mu < 2, 0 \leq \nu \leq 1 \) and \( h : J' \to \mathbb{R} \) be a continuous function. Then the nonlocal BVP for \( \Psi \)-Hilfer FDEs
\[ {}^\nu D_{a+}^{\mu} \psi y(t) = h(t), \quad t \in J' = (a, b], \tag{8} \]
\[ y(a) = 0, \tag{9} \]
\[ y(b) = \sum_{i=1}^m \lambda_i I_{a+}^{\delta_i : \psi} y(\tau_i) \tag{10} \]
is equivalent to
\[ y(t) = \frac{(\Psi(t) - \Psi(a))^{\nu-1}}{\Lambda(\xi)} \left[ \sum_{i=1}^m \lambda_i I_{a+}^{\delta_i : \psi} h(\tau_i) - I_{a+}^{\mu : \psi} h(b) \right] + I_{a+}^{\mu : \psi} h(t), \quad t \in J, \]
where \( \lambda_i \in \mathbb{R} (i = 1, 2, \ldots, m) \) are the constants such that
\[ \Lambda = \frac{(\Psi(b) - \Psi(a))^{\nu-1}}{\Gamma(\xi)} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\xi + \delta_i)} (\Psi(\tau_i) - \Psi(a))^{\nu - \delta_i - 1} \neq 0. \tag{11} \]

**Proof.** Assume that \( y \) is the solution of the nonlocal BVP for \( \Psi \)-Hilfer FDEs (8)–(10). Operating \( \Psi \)-fractional integral \( I_{a+}^{\mu : \psi} \) on both sides of Equation (8) and using Lemma 2.3, we obtain
\[ y(t) = \frac{2}{\Gamma(\xi - k + 1)} \sum_{k=1}^n (\Psi(t) - \Psi(a))^{\nu - k} [\Psi(t) - \Psi(a)]^{\nu - k - 1} I_{a+}^{(1-k)(2-\nu) : \psi} h(t), \quad t \in J. \]
But \((1 - \nu)(2 - \mu) = 2 - \xi\). Therefore,

\[
y(t) = \frac{(\Psi(t) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} \left( \frac{1}{\Psi'(t)} \frac{d}{dt} I^{2-\xi}_{a+} : \Psi y(t) \right)_{t=a} + \frac{(\Psi(t) - \Psi(a))^{\xi-2}}{\Gamma(\xi - 1)} I^{2-\xi}_{a+} : \Psi y(t)_{t=a} + I^\mu_{a+} : \Psi h(t)
\]

\[
= \frac{(\Psi(t) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} \int_t^b D_{\alpha^+}^{\xi-1} : \Psi y(t)_{t=a} + I^\mu_{a+} : \Psi h(t).
\]

Set

\[
C_1 = \int_t^b D_{\alpha^+}^{\xi-1} : \Psi y(t)_{t=a} \quad \text{and} \quad C_2 = I^{2-\xi}_{a+} : \Psi y(t)_{t=a}, \quad t \in J.
\]

Then,

\[
y(t) = C_1 \frac{(\Psi(t) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} + C_2 \frac{(\Psi(t) - \Psi(a))^{\xi-2}}{\Gamma(\xi - 1)} I^\mu_{a+} : \Psi h(t), \quad t \in J.
\]

Because \(\lim (\Psi(t) - \Psi(a))^{\xi-2} = \infty\), in the view of boundary condition (9), we must have \(C_2 = 0\). In this case, Equation (12) becomes

\[
y(t) = C_1 \frac{(\Psi(t) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} + I^\mu_{a+} : \Psi h(t), \quad t \in J.
\]

Next, to determine the constant \(C_1\), we utilize the boundary condition (10). Operating \(I^{\delta_i}_a : \Psi\) on both sides of Equation (13), we obtain

\[
I^{\delta_i}_a : \Psi y(t) = C_1 \frac{(\Psi(t) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} + I^\mu_{a+} : \Psi h(t).
\]

From Equations (10) and (14), we have

\[
y(b) = \sum_{i=1}^m \lambda_i I^{\delta_i}_a : \Psi y(t_i)
\]

\[
= C_1 \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\xi + \delta_i)} (\Psi(t_i) - \Psi(a))^{\xi+\delta_i-1} + \sum_{i=1}^m \lambda_i I^{\mu+\delta_i}_a : \Psi h(t_i).
\]

But from Equations (13) and (15), we have

\[
C_1 \frac{(\Psi(b) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} + I^\mu_{a+} : \Psi h(b)
\]

\[
= C_1 \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\xi + \delta_i)} (\Psi(t_i) - \Psi(a))^{\xi+\delta_i-1} + \sum_{i=1}^m \lambda_i I^{\mu+\delta_i}_a : \Psi h(t_i).
\]

Because \(\lambda_i \in \mathbb{R}(i = 1, 2, \ldots, m)\) are the constants such that

\[
\Lambda = \frac{(\Psi(b) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\xi + \delta_i)} (\Psi(t_i) - \Psi(a))^{\xi+\delta_i-1} \neq 0,
\]

Equation (16) can be written as

\[
C_1 = \frac{1}{\Lambda} \left[ \sum_{i=1}^m \lambda_i I^{\mu+\delta_i}_a : \Psi h(t_i) - I^\mu_{a+} : \Psi h(b) \right].
\]

Thus, Equation (13) takes the form

\[
y(t) = \frac{(\Psi(t) - \Psi(a))^{\xi-1}}{\Lambda \Gamma(\xi)} \left[ \sum_{i=1}^m \lambda_i I^{\mu+\delta_i}_a : \Psi h(t_i) - I^\mu_{a+} : \Psi h(b) \right] + I^\mu_{a+} : \Psi h(t), \quad t \in J,
\]
which is the desired equivalent fractional integral equation to the problem (8)–(10).

Conversely, suppose that \( y \) is the solution of the fractional integral Equation (17). Operating fractional derivative \( ^{H}D_{a+}^{\mu, \nu} : \Psi \) on both sides of Equation (17) and using the Lemma 2.1 and Lemma 2.3, we obtain

\[
^{H}D_{a+}^{\mu, \nu} : \Psi y(t) = \frac{1}{\Gamma(\xi)} \left[ \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) - \int_{a+}^{b} \tau^{\mu} : \Psi h(b) \right] + ^{H}D_{a+}^{\mu, \nu} : \Psi (\Psi(t) - \Psi(a))^{\xi-1} - h(t), \quad t \in J.
\]

This proves \( y \) satisfies Equation (8). Next, we prove that \( y \) given by Equation (17) verifies the boundary conditions. From Equation (17), clearly

\[
y(a) = 0.
\]

Now we prove that \( y \) satisfies the boundary condition (10). For each \( i (i = 1, 2, \ldots, m) \), from Equation (17), we have

\[
\int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) = \frac{1}{\Lambda} \left[ \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) - \int_{a+}^{b} \tau^{\mu} : \Psi h(b) \right] \frac{\Gamma(\xi) - \Lambda}{\Gamma(\xi)} + \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t).
\]

Therefore,

\[
\sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) = \frac{1}{\Lambda} \left[ \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) - \int_{a+}^{b} \tau^{\mu} : \Psi h(b) \right] \sum_{i=1}^{m} \lambda_i \left( \frac{\Gamma(\xi)}{\Gamma(\xi + \delta_i)} - \Lambda \right) + \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t). \tag{20}
\]

But from Equation (11), we have

\[
\sum_{i=1}^{m} \lambda_i \left( \frac{\Gamma(\xi)}{\Gamma(\xi + \delta_i)} - \Lambda \right) = \frac{(\Psi(b) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} - \Lambda.
\]

Thus, Equation (20) reduces to

\[
\sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) = \frac{1}{\Lambda} \left[ \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) - \int_{a+}^{b} \tau^{\mu} : \Psi h(b) \right] \left( \frac{(\Psi(b) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} - \Lambda \right) + \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t)
\]

\[
= \frac{1}{\Lambda} \left[ \sum_{i=1}^{m} \lambda_i \int_{a+}^{b} \tau^{\mu+\delta} : \Psi h(t) - \int_{a+}^{b} \tau^{\mu} : \Psi h(b) \right] \frac{(\Psi(b) - \Psi(a))^{\xi-1}}{\Gamma(\xi)} + \int_{a+}^{b} \tau^{\mu} : \Psi h(b). \tag{21}
\]

Now from Equation (17), we have
In this section, we derive existence result for nonlocal implicit BVP (1)–(3).

Proof. Assume that 

\[ y(b) = \sum_{i=1}^{m} \lambda_i \cdot I_{a+}^{\delta_i} \Psi h(\tau_i) + I_{a+}^{\mu} \Psi h(b), \]

(22)

From Equations (21) and (22), we obtain

\[ y(b) = \sum_{i=1}^{m} \lambda_i \cdot I_{a+}^{\delta_i} \Psi y(\tau_i). \]

(23)

From (18), (19), and (23), it follows that the \( y \) defined by Equation (17) satisfies the problem (8)–(10).

**Theorem 3.2.** Let \( f : J' \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that \( f \left( \cdot, y(\cdot), \frac{H D_{a+}^{\mu, \nu}}{\Psi} y(\cdot) \right) \in C_{2-\xi}; \Psi(J, \mathbb{R}) \) for each \( y \in C_{2-\xi}; \Psi(J, \mathbb{R}) \). Then, the nonlocal BVP for \( \Psi \)-Hilfer implicit FDEs (1)-(3) is equivalent to the fractional integral equation

\[ y(t) = (\Psi(t) - \Psi(a))^{\xi-1} \bar{A}_y + I_{a+}^{\mu} \Psi g_y(t), \]

(24)

where \( g_y(\cdot) \in C_{2-\xi}; \Psi(J, \mathbb{R}) \) satisfies the functional equation

\[ g_y(t) = f \left( t, (\Psi(t) - \Psi(a))^{\xi-1} \bar{A}_y + I_{a+}^{\mu} \Psi g_y(t), g_y(t) \right), \quad t \in J, \]

(25)

and

\[ \bar{A}_y = \frac{1}{\Gamma(\xi)} \sum_{i=1}^{m} \lambda_i \cdot I_{a+}^{\delta_i} \Psi y(\tau_i) - I_{a+}^{\mu} \Psi g_y(b). \]

(26)

**Proof.** Assume that \( y \in C_{2-\xi}; \Psi(J, \mathbb{R}) \) is the solution of integral Equation (24). Operating \( \frac{H D_{a+}^{\mu, \nu}}{\Psi} \) on both sides of Equation (24) and using the Lemma 2.1 and Lemma 2.3, we obtain

\[ \frac{H D_{a+}^{\mu, \nu}}{\Psi} y(t) = \bar{A}_y \frac{H D_{a+}^{\mu, \nu}}{\Psi} (\Psi(t) - \Psi(a))^{\xi-1} + \frac{H D_{a+}^{\mu, \nu}}{\Psi} I_{a+}^{\mu} \Psi g_y(t) \]

\[ = g_y(t) \]

\[ = f \left( t, (\Psi(t) - \Psi(a))^{\xi-1} \bar{A}_y + I_{a+}^{\mu} \Psi g_y(t), g_y(t) \right) \]

\[ = f \left( t, y(t), \frac{H D_{a+}^{\mu, \nu}}{\Psi} y(t) \right). \]

Thus, \( y \) satisfies Equation (1). The proof of the function \( y \) given by Equation (24) satisfies the boundary conditions (2) and (3) and can be completed similarly as in the proof of Theorem 3.1 with \( h(t) \) replaced by \( g_y(t) \).

**4 | EXISTENCE RESULT**

In this section, we derive existence result for nonlocal implicit BVP (1)–(3).

**Theorem 4.1.** Assume the following hypothesis hold:

(H1) \( f : J' \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that \( f \left( \cdot, y(\cdot), \frac{H D_{a+}^{\mu, \nu}}{\Psi} y(\cdot) \right) \in C_{2-\xi}; \Psi(J, \mathbb{R}) \) for each \( y \in C_{2-\xi}; \Psi(J, \mathbb{R}) \) that satisfy the following Lipschitz type condition

\[ |f(t, x_1, y_1) - f(t, x_2, y_2)| \leq K |x_1 - x_2| + L |y_1 - y_2|, \quad t \in J, \]

where \( x_1, x_2, y_1, y_2 \in \mathbb{R}, \ K > 0 \) and \( 0 < L < 1 \). Then, nonlocal BVP for \( \Psi \)-Hilfer implicit FDEs (1)-(3) has at least one solution, provided

\[ \sigma = \frac{K \Gamma(\xi - 1)(\Psi(b) - \Psi(a))^\mu}{1 - L} \left\{ \frac{(\Psi(b) - \Psi(a))^{\xi-1}}{\Gamma(\xi)\Lambda} + \frac{1}{\Gamma(\xi + \mu)\Lambda} \right\} < 1, \]

(27)
\[ \Omega = \sum_{i=1}^{m} \frac{\lambda_i (\Psi(b) - \Psi(a))^{\delta_i}}{\Gamma(\xi + \mu + \delta_i - 1)} + \frac{1}{\Gamma(\xi + \mu - 1)} \]  

(28)

and \( A \) is defined in Equation (11).

**Proof.** In the view of Theorem 3.2, the equivalent fractional integral equation to the nonlocal BVP for \( \Psi \)-Hilfer implicit FDEs (1)–(3) can be written as operator equation as follows:

\[ y = Py + Qy, \quad y \in C_{2-\xi} \Psi(J, \mathbb{R}), \]  

(29)

where \( P \) and \( Q \) are the operators defined on \( C_{2-\xi} \Psi(J, \mathbb{R}) \) by

\[ Py(t) = (\Psi(t) - \Psi(a))^{\xi-1} A_\gamma, \quad t \in J \quad \text{and} \]

\[ Qy(t) = I_{a+}^\mu : \Psi g_y(t) = \frac{1}{\Gamma(\mu)} \int_a^t \Psi'(s)(\Psi(t) - \Psi(s))^{\mu-1} g_y(s) ds, \quad t \in J, \]

where \( g_y \in C_{2-\xi} \Psi(J, \mathbb{R}) \) satisfies the functional equation

\[ g_y(t) = f \left( t, (\Psi(t) - \Psi(a))^{\xi-1} A_\gamma + I_{a+}^\mu : \Psi g_y(t), g_y(t) \right), \quad t \in J, \]

and

\[ A_\gamma = \frac{1}{\Lambda(\xi)} \left[ \sum_{i=1}^{m} \lambda_i I_{a+}^{\mu+\delta_i} : \Psi g_y(\tau_i) - I_{a+}^\mu : \Psi g_y(b) \right]. \]

To prove the nonlocal implicit BVP (1)–(3) has a solution in \( C_{2-\xi} \Psi(J, \mathbb{R}) \) we show that the operator Equation (29) has a fixed point. Define

\[ \zeta = \frac{M(\Psi(t) - \Psi(a))^{\mu}}{1 - L} \left( \frac{(\Psi(t) - \Psi(a))^{\xi-1} A_\gamma}{\Gamma(\xi)} \Omega + \frac{1}{\Gamma(\xi + \mu - 1)} \right), \]

where

\[ M = \max_{t \in J} \left| (\Psi(t) - \Psi(a))^{2-\xi} f(t, 0, 0) \right|. \]

Choose \( r \) such that

\[ r \geq \frac{\zeta}{1 - \sigma} \]

and consider the closed ball

\[ B_r = \left\{ y \in C_{2-\xi} \Psi(J, \mathbb{R}) : \| y \|_{C_{2-\xi} \Psi(J, \mathbb{R})} \leq r \right\}. \]

To obtain the fixed point of the operator Equation (29), we prove that the operators \( P \) and \( Q \) satisfies the conditions
of Theorem 2.5 (Krasnoselskii fixed point theorem). We give the proof in the following steps:

**Step 1:** \(P(x + Qy) \in B_r\) for all \(x, y \in B_r\).

Let any \(x, y \in B_r\). Then, \((P(t) - P(a))^{2-\xi}P_x(t) = (P(t) - P(a))\bar{A}_x \in C(J, \mathbb{R})\). This implies, \(P(x) \in C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})\).

Because \(g_s(\cdot) = f(\cdot, y(\cdot), g_s(\cdot)) \in C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})\) for any \(y \in C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})\), \(0 \leq 2 - \xi < 1\) and \(Qy(t) = I_{a+}^{\mu} \cdot g_y(t)\), by Lemma 2.2, we have \(Qy \in C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})\). Using triangle inequality in the space \(C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})\), we have

\[
\|P(x + Qy)\|_{C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})} \leq \|P(x)\|_{C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})} + \|Qy\|_{C_{2-\xi} \cap \mathcal{P}(J, \mathbb{R})}. \tag{30}
\]

Now, by using hypothesis \((H1)\), we have

\[
\begin{align*}
\left| (P(t) - P(a))^{2-\xi}P_x(t) \right| &= \left| (P(t) - P(a)) \bar{A}_x \right| \\
&= \left| (P(t) - P(a)) \sum_{i=1}^{m} \lambda_i I_{a+}^{\mu+\delta_i} : g_s(t_i) - I_{a+}^{\mu} : g_s(b) \right| \\
&\leq \frac{(P(t) - P(a))}{\Lambda(\xi)} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \psi'(s)[\psi(t_i) - \psi(s)]^{\mu+\delta_i-1} |f(s, x(s), g_s(s))| ds \\
&+ \frac{1}{\Gamma(\mu)} \int_{a}^{b} \psi'(s)[\psi(b) - \psi(s)]^{\mu-1} |f(s, x(s), g_s(s))| ds \\
&\leq \frac{(P(t) - P(a))}{\Lambda(\xi)} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \psi'(s)[\psi(t_i) - \psi(s)]^{\mu+\delta_i-1} \times \\
&\quad \{|f(s, x(s), g_s(s)) - f(s, 0, 0)| + |f(s, 0, 0)|\} ds \\
&+ \frac{1}{\Gamma(\mu)} \int_{a}^{b} \psi'(s)[\psi(b) - \psi(s)]^{\mu-1} \{|f(s, x(s), g_s(s)) - f(s, 0, 0)| + |f(s, 0, 0)|\} ds \\
&\leq \frac{(P(t) - P(a))}{\Lambda(\xi)} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \psi'(s)[\psi(t_i) - \psi(s)]^{\mu+\delta_i-1} \{K |x(s)| + L |g_s(s)| + |f(s, 0, 0)|\} ds \\
&+ \frac{1}{\Gamma(\mu)} \int_{a}^{b} \psi'(s)[\psi(b) - \psi(s)]^{\mu-1} \{K |x(s)| + L |g_s(s)| + |f(s, 0, 0)|\} ds. \tag{31}
\end{align*}
\]

But, by hypothesis \((H1)\), we have

\[
|g_s(t)| \leq |f(t, x(t), g_s(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\
\leq K |x(t)| + L |g_s(t)| + |f(t, 0, 0)|, \quad t \in J.
\]
Using Equation (32) in Equation (31), we get

\[
\left| (\Psi(t) - \Psi(a))^\alpha \right| \leq \frac{K}{1-L} |x(t)| + \frac{1}{1-L} |f(t,0,0)|, \ t \in J.
\] (32)

This gives

\[
\begin{align*}
\left| (\Psi(t) - \Psi(a))^\alpha \right| & \leq \frac{(\Psi(t) - \Psi(a))}{\alpha \Gamma(\xi)} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \Psi'(s)(\Psi(\tau_i) - \Psi(s))^{\mu + \delta_i - 1} \right. \\
& \quad \times \left\{ K |x(s)| + L \left\{ \frac{K}{1-L} |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right\} + \left| f(s,0,0) \right\} \right] \, ds \\
& + \frac{1}{\Gamma(\mu)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu - 1} \left( K |x(s)| + L \left\{ \frac{K}{1-L} |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right\} + \left| f(s,0,0) \right\} \right] \, ds \\
& \leq \frac{(\Psi(t) - \Psi(a))}{\alpha \Gamma(\xi)} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \Psi'(s)(\Psi(\tau_i) - \Psi(s))^{\mu + \delta_i - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& + \frac{1}{\Gamma(\mu)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& \leq \frac{(\Psi(t) - \Psi(a))}{\alpha \Gamma(\xi)} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu + \delta_i - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& + \frac{1}{\Gamma(\mu)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& \leq \frac{(\Psi(t) - \Psi(a))}{\alpha \Gamma(\xi)} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu + \delta_i - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& + \frac{1}{\Gamma(\mu)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& \leq \frac{(\Psi(t) - \Psi(a))}{\alpha \Gamma(\xi)} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{a}^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu + \delta_i - 1} \left( K |x(s)| + \frac{1}{1-L} |f(s,0,0)| \right) \, ds \\
& + \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} (\Psi(b) - \Psi(a))^{\mu + \delta_i - 2} \right] \\
& + \frac{M}{1-L} \left\{ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i + \xi - 1)} (\Psi(b) - \Psi(a))^{\mu + \delta_i + \xi - 2} + \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} (\Psi(b) - \Psi(a))^{\mu + \xi - 2} \right\}.
\end{align*}
\]
Therefore,

$$\|Px\|_{C_{2-\xi} y(I, \mathbb{R})} = \max_{t \in I} |(\Psi(t) - \Psi(a))^{\mu+\xi-1} P x(t)|$$

$$\leq \frac{(\Psi(b) - \Psi(a))^{\mu+\xi-1} \Gamma(\xi - 1)}{\Lambda \Gamma(\xi)} \left[ \frac{K r}{1 - L} \left\{ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i + \xi - 1)} (\Psi(b) - \Psi(a))^{\delta_i} + \frac{1}{\Gamma(\mu + \xi - 1)} \right\} \right]$$

$$\leq \left( \frac{\Psi(b) - \Psi(a))^{\mu+\xi-1} \Gamma(\xi - 1)}{\Lambda \Gamma(\xi)} \left( \frac{K r + M}{1 - L} \right) \right) \Omega = A.$$  \hfill (33)

Further, by using hypothesis (H1) and inequality (32), we have

$$\left| \frac{\Psi(t) - \Psi(a))^{2-\xi} Q y(t)}{\Gamma(\mu)} \right|$$

$$\leq \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} g_{y}(s) ds$$

$$\leq \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} \left\{ |f(s, y(s), g_{y}(s))| + |f(s, 0, 0)| \right\} ds$$

$$\leq \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} \left\{ |K| y(s) + L |g_{y}(s)| + |f(s, 0, 0)| \right\} ds$$

$$\leq \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} \left\{ \frac{K}{1 - L} |y(s)| + \frac{1}{1 - L} |f(s, 0, 0)| \right\} ds$$

$$\leq \left( \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \right) \left\{ \frac{K}{1 - L} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} \left( \Psi(s) - \Psi(a) \right)^{2-\xi} f(s, 0, 0) ds \right\}$$

$$+ \frac{1}{(1 - L) \Gamma(\mu)} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(a))^{2-\xi} f(s, 0, 0) ds$$

$$\leq \left( \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \right) \left\{ \frac{K}{1 - L} \|y\|_{C_{2-\xi} y(I, \mathbb{R})} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(a))^{2-\xi} ds \right\}$$

$$+ \frac{M}{1 - L} \int_{a}^{t} \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(a))^{2-\xi} ds$$

$$\leq \left( \frac{(\Psi(t) - \Psi(a))^{2-\xi}}{\Gamma(\mu)} \right) \left\{ \frac{K}{1 - L} \|y\|_{C_{2-\xi} y(I, \mathbb{R})} \Gamma(\xi - 1) \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} (\Psi(t) - \Psi(a))^{2-\xi} \right\}$$

$$+ \frac{M}{1 - L} \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} (\Psi(t) - \Psi(a))^{2-\xi}$$

$$\leq \frac{(\Psi(t) - \Psi(a))^{\mu+\xi-2}}{(1 - L) \Gamma(\mu + \xi - 1)} \left\{ K \|y\|_{C_{2-\xi} y(I, \mathbb{R})} + M \right\}.$$
Therefore,

\[
\|Qy\|_{\mathcal{C}_{2-\xi}, \varphi(J, \mathbb{R})} = \max_{t \in J} |(\varPsi(t) - \varPsi(a))^{2-\xi}Qy(t)| \\
\leq \frac{(\varPsi(b) - \varPsi(a))^{\nu} \Gamma(\xi - 1)}{(1 - L) \Gamma(\mu + \xi - 1)} \{Kr + M\} = B.
\] (34)

From inequalities (33) and (34) and the definition of \( r \) given in Equation (30), we obtain

\[
\|Px + Qy\|_{\mathcal{C}_{2-\xi}, \varphi(J, \mathbb{R})} \leq A + B = r\sigma + \zeta \leq r.
\]

This proves, \( Px + Qy \in B_r \) for all \( x, y \in B_r \).

**Step 2:** Operator \( P \) is contraction on \( B_r \).

Let any \( y_1, y_2 \in B_r \) and \( t \in J \). Then

\[
\left| (\varPsi(t) - \varPsi(a))^{2-\xi}(Px_1(t) - Px_2(t)) \right| \\
\leq \frac{(\varPsi(t) - \varPsi(a))}{\Lambda(\xi)} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{t_i}^{t} \varPsi(s)(\varPsi(\tau_i) - \varPsi(s))^{\mu + \delta_i - 1} \left| g_{y_1}(s) - g_{y_2}(s) \right| ds \\
+ \frac{1}{\Gamma(\mu)} \int_{a}^{b} \varPsi(s)(\varPsi(b) - \varPsi(s))^{\mu - 1} \left| \varPsi(s) - \varPsi(a) \right|^{2-\xi}(y_1(s) - y_2(s)) ds \right].
\] (35)

But by hypothesis (H1), we have

\[
\left| g_{y_1}(t) - g_{y_2}(t) \right| = \left| f(t, y_1(t), g_{y_1}(t)) - f(t, y_2(t), g_{y_2}(t)) \right| \\
\leq K \left| y_1(t) - y_2(t) \right| + L \left| g_{y_1}(t) - g_{y_2}(t) \right|, \ t \in J.
\]

Thus,

\[
\left| g_{y_1}(t) - g_{y_2}(t) \right| \leq \frac{K}{1 - L} \left| y_1(t) - y_2(t) \right|, \ t \in J.
\] (36)

Using Equation (36) in Equation (35), we have

\[
\left| (\varPsi(t) - \varPsi(a))^{2-\xi}(Px_1(t) - Px_2(t)) \right| \\
\leq \frac{K}{1 - L} \left[ \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_{t_i}^{t} \varPsi(s)(\varPsi(\tau_i) - \varPsi(s))^{\mu + \delta_i - 1} \left| \varPsi(s) - \varPsi(a) \right|^{2-\xi}(y_1(s) - y_2(s)) ds \\
+ \frac{1}{\Gamma(\mu)} \int_{a}^{b} \varPsi(s)(\varPsi(b) - \varPsi(s))^{\mu - 1} \left| \varPsi(s) - \varPsi(a) \right|^{2-\xi}(y_1(s) - y_2(s)) ds \right].
\]

Therefore,

\[
\|Py_1 - Py_2\|_{\mathcal{C}_{2-\xi}, \varphi(J, \mathbb{R})} = \max_{t \in J} \left| (\varPsi(t) - \varPsi(a))^{2-\xi}(Py_1(t) - Py_2(t)) \right| \\
\leq \|y_1 - y_2\|_{\mathcal{C}_{2-\xi}, \varphi(J, \mathbb{R})} \frac{KT(\xi - 1)}{1 - L} \frac{\varPsi(b) - \varPsi(a)^{\mu + \xi - 1}}{\Lambda(\xi)} + \Omega \\
\leq \sigma \|y_1 - y_2\|_{\mathcal{C}_{2-\xi}, \varphi(J, \mathbb{R})}.
\]
Because \( \sigma < 1 \), \( P \) is contraction on \( B_r \).

**Step 3:** The operator \( Q \) is compact and continuous.

We have already proved that \( Q \) is self mapping on \( C_{2-\xi}; \psi(J, \mathbb{R}) \). Further, from Step 1, we have

\[
\|Qy\|_{C_{2-\xi}; \psi(t, \mathbb{R})} \leq \frac{(\Psi(b) - \Psi(a))\mu \Gamma(\xi - 1) (Kr + M)}{(1 - L) \Gamma(\mu + \xi - 1)} \{Kr + M\}, \ y \in B_r. 
\]

Therefore, \( Q(B_r) = \{Qy : y \in B_r\} \) is uniformly bounded. Next, we prove that \( Q(B_r) \) is equicontinuous. Let any \( y \in B_r \) and \( t_1, t_2 \in J \) with \( t_1 > t_2 \). Then

\[
|Qy(t_1) - Qy(t_2)| \leq \frac{1}{\Gamma(\mu)} \int_a^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1}g_y(s)ds - \frac{1}{\Gamma(\mu)} \int_a^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1}g_y(s)ds \\
\leq \frac{1}{\Gamma(\mu)} \int_a^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1}(\Psi(s) - \Psi(a))^{\xi-2} \left| (\Psi(s) - \Psi(a))^{\xi-2} f(s, y(s), g_y(s)) \right| ds \\
- \frac{1}{\Gamma(\mu)} \int_a^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1}(\Psi(s) - \Psi(a))^{\xi-2} \left| (\Psi(s) - \Psi(a))^{\xi-2} f(s, y(s), g_y(s)) \right| ds.
\]

Because \( g_y(\cdot) = f(\cdot, y(\cdot), g_y(\cdot)) \in C_{2-\xi}; \psi(J, \mathbb{R}) \) for any \( y \in C_{2-\xi}; \psi(J, \mathbb{R}) \), \( \Psi(\cdot) - \Psi(\cdot) = \Psi(\cdot) - \Psi(\cdot) = \Psi(\cdot) - \Psi(\cdot) \) is continuous on \( J = [a, b] \). Therefore, there exists \( \mathfrak{R} \in \mathbb{R} \) such that

\[
\left| (\Psi(t) - \Psi(a))^{\xi-2} f(t, y(t), g_y(t)) \right| \leq \mathfrak{R}, \text{ for all } t \in J.
\]

Thus,

\[
|Qy(t_1) - Qy(t_2)| \leq \frac{\mathfrak{R}}{\Gamma(\mu)} \int_a^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1}(\Psi(s) - \Psi(a))^{\xi-2} ds \\
- \frac{\mathfrak{R}}{\Gamma(\mu)} \int_a^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1}(\Psi(s) - \Psi(a))^{\xi-2} ds \\
\leq \frac{\mathfrak{R}}{\Gamma(\mu)} \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} (\Psi(t_1) - \Psi(a))^{\mu^{\xi-2}} - \frac{\mathfrak{R}}{\Gamma(\mu)} \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} (\Psi(t_2) - \Psi(a))^{\mu^{\xi-2}}.
\]

Hence,

\[
|Qy(t_1) - Qy(t_2)| \leq \frac{\mathfrak{R}}{\Gamma(\mu)} \frac{\Gamma(\xi - 1)}{\Gamma(\mu + \xi - 1)} \left| (\Psi(t_1) - \Psi(a))^{\mu^{\xi-2}} - (\Psi(t_2) - \Psi(a))^{\mu^{\xi-2}} \right|.
\]

Using continuity of \( \Psi \), we observe that \( |Qy(t_1) - Qy(t_2)| \to 0 \) as \( |t_1 - t_2| \to 0 \). This proves \( Q(B_r) \) is equicontinuous.

Thus, by Arzelà–Ascoli theorem, \( Q(B_r) \) is relatively compact. We proved that operator \( Q \) is compact.

Because \( P \) and \( Q \) satisfy all the conditions of Krasnoselskii fixed-point theorem (Theorem 2.5), the operator equation defined in Equation (29) has at least one fixed point in the space \( C_{2-\xi}; \psi(J, \mathbb{R}) \), which is the solution of the nonlocal BVP for implicit FDEs (1)–(3).

\[\square\]

## 5 Ulam Stability Results

We begin with definitions regarding Ulam stabilities. To discuss these stabilities for the problem (1), we follow the approach of Sousa and Oliveira.\(^{53}\)

**Definition 5.1.** Equation (1) is Ulam–Hyers stable if there exists a real number \( C_f > 0 \), such that for each \( \epsilon > 0 \) and for each solution \( z \in C_{2-\xi}; \psi(J, \mathbb{R}) \) of the inequality

\[
|^{H}D_{a+}^{\mu, \nu} z(t) - f(t, z(t), ^{H}D_{a+}^{\mu, \nu} z(t))| \leq \epsilon, \quad t \in J,
\]

(37)
there exists solution \( y \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of Equation (1) with
\[
\|z - y\|_{C_{2-\xi}, \varphi(t, \mathbb{R})} \leq C_f \varepsilon, \quad t \in J.
\]

**Definition 5.2.** Equation (1) is generalized Ulam–Hyers stable if there exists a function \( \varphi_f \in C(\mathbb{R}_+, \mathbb{R}_+), \varphi_f(0) = 0, \) such that for each solution \( z \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of the inequality (37), there exists solution \( y \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of Equation (1) with
\[
\|z - y\|_{C_{2-\xi}, \varphi(t, \mathbb{R})} \leq \varphi_f(\varepsilon), \quad t \in J.
\]

**Definition 5.3.** Equation (1) is Ulam–Hyers–Rassias stable, with respect to the \( \chi \in C(\mathbb{R}_+, \mathbb{R}_+), \) if there exists a \( C_{f,\chi} > 0 \) such that for each \( \varepsilon > 0 \) and for each solution \( z \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of the inequality
\[
\left| H D_{a+}^{\mu, \nu} \varphi(z(t) - f(t, z(t), H D_{a+}^{\mu, \nu} \varphi(z(t))) \leq \chi(t), \quad t \in J,
\right.
\]
there exists solution \( y \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of Equation (1) with
\[
\left| (\Psi(t) - \Psi(a))^2 \right| \leq C_{f,\chi} \chi(t), \quad t \in J,
\]

**Definition 5.4.** Equation (1) is generalized Ulam–Hyers–Rassias stable, with respect to the \( \chi \in C(\mathbb{R}_+, \mathbb{R}_+), \) if there exists a \( C_{f,\chi} > 0 \) such that for each \( \varepsilon > 0 \) and for each solution \( z \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of the inequality
\[
\left| H D_{a+}^{\mu, \nu} \varphi(z(t) - f(t, z(t), H D_{a+}^{\mu, \nu} \varphi(z(t)) \leq \chi(t), \quad t \in J,
\right.
\]
there exists solution \( y \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) of Equation (1) with
\[
\left| (\Psi(t) - \Psi(a))^2 \right| \leq C_{f,\chi} \chi(t), \quad t \in J.
\]

**Remark 5.1.** A function \( z \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) is solution of the inequality (37) if and only if there exists a function \( w \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) (which depend on \( z \)) such that
1. \( |w(t)| \leq \varepsilon. \)
2. \( H D_{a+}^{\mu, \nu} \varphi(z(t) = f(t, z(t), H D_{a+}^{\mu, \nu} \varphi(z(t)) + w(t). \)

**Theorem 5.2** (Ulam–Hyers stability). Assume that the hypothesis (H1) hold. Then, Equation (1) is Ulam–Hyers stable provided that the condition (27) hold.

**Proof.** Let any \( \varepsilon > 0. \) Let \( z \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) be any solution of the inequality
\[
\left| H D_{a+}^{\mu, \nu} \varphi(z(t) - f(t, z(t), H D_{a+}^{\mu, \nu} \varphi(z(t)) \leq \varepsilon, \quad t \in J.
\]

Then, there exists \( w \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) such that
\[
H D_{a+}^{\mu, \nu} \varphi(z(t) = f(t, z(t), H D_{a+}^{\mu, \nu} \varphi(z(t)) + w(t), \quad t \in J.
\]

and \( |w(t)| \leq \varepsilon, \quad t \in J. \) In the view of Theorem 3.2,\[
z(t) = (\Psi(t) - \Psi(a))^{2-1} A_{a+}^{\mu, \nu} \varphi(z(t) + I_{a+}^{\mu, \nu} \varphi(g_{z}(t) + I_{a+}^{\mu, \nu} \varphi w(t)
\]
is the solution of Equation (41), where \( g_{z}(\cdot) \in C_{2-\xi}, \varphi(J, \mathbb{R}) \) satisfies the functional equation
\[
g_{z}(t) = f \left( t, (\Psi(t) - \Psi(a))^{2-1} A_{a+}^{\mu, \nu} \varphi(z(t) + I_{a+}^{\mu, \nu} \varphi w(t), g_{z}(t) \right), \quad t \in J.
\]
and

\[
\tilde{A}_\zeta = \frac{1}{\Gamma(\xi)} \left[ \sum_{i=1}^{m} \lambda_i I_{\alpha+}^{\mu+\delta_i} \Psi \left( g_{\xi}(\tau_i) - I_{\alpha+}^{\mu} g_{\xi}(b) \right) \right].
\]

From Equation (42), we have

\[
\left| z(t) - (\Psi(t) - \Psi(a))^{\xi-1} \tilde{A}_\zeta - I_{\alpha+}^{\mu} g_{\xi}(t) \right| \leq I_{\alpha+}^{\mu} |w(t)| \leq \gamma (\Psi(t) - \Psi(a))^{\mu-1} \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1)}. (43)
\]

Let \( y \in C_{2-\xi}, \Psi(J, \mathbb{R}) \) be the solution of the problem

\[
\begin{cases}
\frac{\partial}{\partial \tau}^\mu y(t) = f(t, y(t), y(t)), \\
y(a) = z(a), \\
y(b) = z(b),
\end{cases}
\]

where \( y(b) = \sum_{i=1}^{m} \lambda_i I_{\alpha+}^{\delta_i} y(\tau_i) \) and \( z(b) = \sum_{i=1}^{m} \lambda_i I_{\alpha+}^{\delta_i} z(\tau_i) \). By Theorem 3.2, the equivalent fractional integral equation to Equation (44) is

\[
y(t) = (\Psi(t) - \Psi(a))^{\xi-1} \tilde{A}_\zeta + I_{\alpha+}^{\mu} g_{\xi}(t),
\]

where \( g_{\xi} \) satisfies functional equation

\[
g_{\xi}(t) = f \left[ t, (\Psi(t) - \Psi(a))^{\xi-1} \tilde{A}_\zeta + I_{\alpha+}^{\mu} g_{\xi}(t), g_{\xi}(t) \right], \quad t \in J,
\]

and

\[
\hat{A}_\zeta = \frac{1}{\Gamma(\xi)} \left[ \sum_{i=1}^{m} \lambda_i I_{\alpha+}^{\mu+\delta_i} \Psi \left( g_{\xi}(\tau_i) - I_{\alpha+}^{\mu} g_{\xi}(b) \right) \right].
\]

Now,

\[
|\hat{A}_\zeta - \tilde{A}_\zeta| = \left\{ \left| \frac{1}{\Gamma(\mu)} \int_a^b \frac{1}{\Gamma(\mu + \delta_i)} \int_a^{\tau_i} \Psi'(s)(\Psi(\tau_i) - \Psi(s))^{\mu+\delta_i-1} g_{\xi}(s) ds \right| \right\}
\]

\[
- \frac{1}{\Gamma(\mu)} \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{\mu-1} g_{\xi}(s) ds \right\}
\]

\[
- \left\{ \left| \frac{1}{\Gamma(\mu)} \int_a^b \frac{1}{\Gamma(\mu + \delta_i)} \int_a^{\tau_i} \Psi'(s)(\Psi(\tau_i) - \Psi(s))^{\mu+\delta_i-1} g_{\xi}(s) ds \right| \right\}
\]

\[
- \frac{1}{\Gamma(\mu)} \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{\mu-1} g_{\xi}(s) ds \right\}
\]

\[
\leq \frac{1}{\Gamma(\mu)} \left\{ \left| \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\mu + \delta_i)} \int_a^{\tau_i} \Psi'(s)(\Psi(b) - \Psi(s))^{\mu-1} g_{\xi}(s) ds \right| \right\}
\]

\[
\leq \frac{1}{\Gamma(\mu)} \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{\mu-1} |g_{\xi}(s) - g_{\xi}(s)| ds \]

By hypothesis (H1), we obtain

\[
\left| g_{\xi}(t) - g_{\xi}(t) \right| = \left| f(t, y(t), g_{\xi}(t)) - f(t, z(t), g_{\xi}(t)) \right| \leq K \left| y(t) - z(t) \right| + L \left| g_{\xi}(t) - g_{\xi}(t) \right|
\]

This gives

\[
\left| g_{\xi}(t) - g_{\xi}(t) \right| \leq \frac{K}{1-L} \left| y(t) - z(t) \right|. (47)
\]
Using Equation (47) in Equation (46), we get
\[
|\ddot{A}_y - \ddot{A}_z|
\]
\[
\leq K \frac{1}{1 - L \Lambda(\xi)} \left\{ \sum_{i=1}^{m} \frac{\lambda_i}{1 + \delta_i} \int_a^b \psi'(s)(\psi(t) - \psi(s))^{\mu+\delta_i} - 1 \ |y(s) - z(s)| \ ds \right. \\
\left. - \frac{1}{1 - \Lambda(\xi)} \int_a^b \psi'(s)(\psi(b) - \psi(s))^{\mu-1} |y(s) - z(s)| \ ds \right\}
\]
\[
\leq K \frac{1}{1 - L \Lambda(\xi)} \left\{ \sum_{i=1}^{m} \lambda_i \Gamma_{a+}^{\mu+\delta_i} \psi |y(t_i) - z(t_i)| + I_a^{\mu+} \psi |y(b) - z(b)| \right\}.
\] (48)

Because \(y(b) = z(b)\), we must have \(y(t_i) = z(t_i)(i = 1, 2, \ldots, m)\). Therefore, from inequality (48), we obtain \(\ddot{A}_y = \ddot{A}_z\). Using Equations (43) and (47), we have
\[
|z(t) - y(t)|
\]
\[
= |z(t) - \left( \psi(t) - \psi(a) \right)^{\mu-1} \dot{A}_y + \frac{1}{1 - \Lambda(\xi)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\mu-1} g_s(s) ds \right| \\
\leq |z(t) - \left( \psi(t) - \psi(a) \right)^{\mu-1} \dot{A}_z - \frac{1}{1 - \Lambda(\xi)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\mu-1} g_s(s) ds | \\
+ \frac{1}{1 - \Lambda(\xi)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\mu-1} g_s(s) ds - \frac{1}{1 - \Lambda(\xi)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\mu-1} g_s(s) ds \\
\leq \epsilon \left( \frac{\psi(b) - \psi(a)}{\Gamma(\mu + 1)} \right) + \frac{1}{1 - \Lambda(\xi)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\mu-1} |g_s(s) - g_s| ds \\
\leq \epsilon \left( \frac{\psi(b) - \psi(a)}{\Gamma(\mu + 1)} \right) + K \frac{1}{1 - \Lambda(\xi)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\mu-1} |z(s) - y(s)| ds.
\]

Applying Gronwall inequality (Theorem 1) with \(u(t) = |z(t) - y(t)|\), \(v(t) = \epsilon \left( \frac{\psi(b) - \psi(a)}{\Gamma(\mu + 1)} \right)\) and \(g(t) = K \frac{1}{1 - \Lambda(\xi)}\), we obtain
\[
|z(t) - y(t)| \leq \epsilon \left( \frac{\psi(b) - \psi(a)}{\Gamma(\mu + 1)} \right) E_\mu \left( \frac{K}{1 - \Lambda(\psi(t) - \psi(a))} \right), \ t \in J.
\]
Therefore,
\[
\|z - y\|_{C_{2-\psi}(J, \mathbb{R})} = \max_{t \in J} |(\psi(t) - \psi(a))^{2-\psi}(z(t) - y(t))| \\
\leq (\psi(t) - \psi(a))^{2-\psi} \epsilon \left( \frac{\psi(b) - \psi(a)}{\Gamma(\mu + 1)} \right) E_\mu \left( \frac{K}{1 - \Lambda(\psi(t) - \psi(a))} \right) \\
= C_f \epsilon.
\] (49)

where \(C_f := \left( \frac{\psi(b) - \psi(a)}{\Gamma(\mu + 1)} \right) E_\mu \left( \frac{K}{1 - \Lambda(\psi(b) - \psi(a))} \right)\). Thus, Equation (1) is Ulam–Hyers stable.

Remark 5.3. Define \(\varphi_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) by \(\varphi_f(\epsilon) = C_f \epsilon\). Then, \(\varphi_f \in C(\mathbb{R}_+, \mathbb{R}_+)\) and \(\varphi_f(0) = 0\). Then Equation (49) can be written as
\[
\|z - y\|_{C_{2-\psi}(J, \mathbb{R})} \leq \varphi_f(\epsilon).
\]

Thus, Equation (1) is generalized Ulam–Hyers stable.

Theorem 5.4 (Ulam–Hyers–Rassias stability). Assume that the hypothesis (H1) hold. Let there exist nonincreasing function \(\chi \in C(J, \mathbb{R}_+)\) and \(K^* > 0\) such that
\[
\frac{1}{1 - \Lambda(\xi)} \int_a^b \psi'(s)(\psi(t) - \psi(s))^{\mu-1} \chi(s) ds \leq K^* \chi(t), \text{ for all } t \in J.
\] (50)
Then, the implicit FDE (1) is Ulam–Hyers–Rassias stable provided the condition (27) hold.

**Proof.** Let any $\varepsilon > 0$ and let $z \in C_{2-\varepsilon}, \varphi(J, \mathbb{R})$ be any solution of the inequality

$$\left| H^\mu D^\nu_{a+} z(t) - f(t, z(t), H^\mu D^\nu_{a+} z(t)) \right| \leq \varepsilon \chi(t), \ t \in J. \quad (51)$$

Then, proceeding as in the proof of Theorem 5.2, we obtain

$$\left| z(t) - (\Psi(t) - \Psi(a)) z_e - I^\mu_{a+} g_e(t) \right| \leq I^\mu_{a+} |w(t)| \leq \varepsilon I^\mu_{a+} \chi(t), \ t \in J. \quad (52)$$

Taking $y \in C_{2-\varepsilon}(J, \mathbb{R})$ as any solution of the problem (44), and following similar steps as in the proof of Theorem 5.2, we obtain

$$|z(t) - y(t)| \leq \delta e K^+ \chi(t) + \frac{K}{1 - L} \int^t_a \Psi'(s)(\Psi(t) - \Psi(s))|z(s) - y(s)| \ ds, \ t \in J.$$

By applying Gronwall inequality (Theorem 1), we get

$$|z(t) - y(t)| \leq \delta e K^+ \chi(t) + \frac{K}{1 - L} \psi(\delta e K^+ \chi(t), t \in J,$$

which gives,

$$|\Psi(t) - \Psi(a)^{2-\varepsilon} (z(t) - y(t))| \leq C_{f, \varepsilon} \chi(t), \ t \in J. \quad (53)$$

where $C_{f, \varepsilon} := K^+(\Psi(b) - \Psi(a))^{2-\varepsilon} E_\mu \left( \frac{K}{1 - L} (\Psi(b) - \Psi(a)) \right)$. This proves, Equation (1) is Ulam–Hyers–Rassias stable.

**Remark 5.5.** The proof of Equation (1) is generalized Ulam–Hyers–Rassias stable, which follows by taking $\varepsilon = 1$, in the inequality (53).

### 6 EXAMPLES

**Example 6.1.** Consider the following nonlocal BVP for implicit FDE involving $\Psi$-Hilfer fractional derivative

$$H^\mu D^\nu_{0+} y(t) = \frac{\cos t}{10 e^{t+1}} \left[ \sin y(t) + H^\mu D^\nu_{0+} y(t) \right], \ t \in (0, 1], \ 1 < \mu < 2, \ 0 \leq \nu \leq 1, \quad (54)$$

$$y(0) = 0, \quad (55)$$

$$y(1) = 10^\mu \int^1_{0+} y \left( \frac{1}{3} \right) + \frac{13}{6} \int^1_{0+} y \left( \frac{1}{2} \right). \quad (56)$$

Define $f : (0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t, u, v) = \frac{\cos t}{10 e^{t+1}} [\sin u + v].$$

Then, for any $u_1, v_1 \in \mathbb{R} (i = 1, 2)$ and $t \in (0, 1]$, we have

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{10 e} |\sin u_1 - \sin u_2| + |v_1 - v_2|. \quad (57)$$

Without loss of generality, we may take $u_1 < u_2$. Then, by mean value theorem, $\exists \gamma \in (u_1, u_2)$ such that $\sin u_1 - \sin u_2 = (u_1 - u_2) \cos \gamma$. This gives $|\sin u_1 - \sin u_2| \leq |u_1 - u_2|$. Thus,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{10 e} \left[ |u_1 - u_2| + |v_1 - v_2| \right].$$
This proves $f$ satisfies the Lipschitz type condition in hypothesis (H1) with $K = L = \frac{1}{10}\epsilon = 0.0368$. Further comparing the problem (54)–(56) to the problem (1)–(3), we have $a = 0$, $b = 1$, $\lambda_1 = \frac{10}{7}$, $\lambda_2 = \frac{12}{5}$, $\delta_1 = \frac{4}{5}$, $\delta_2 = \frac{2}{3}$, $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{2}$. With these constants, Equations (11), (27), and (28) becomes

$$
\sigma = \frac{0.0368 \Gamma(\xi - 1)(\Psi(1) - \Psi(0))}{1 - 0.0368} \left\{ \frac{(\Psi(1) - \Psi(0))^{\xi - 1}}{\Lambda^2(\xi)} \Omega + \frac{1}{\Lambda^2(\xi + \mu - 1)} \right\},
$$

(57)

where

$$
\Lambda = \frac{(\Psi(1) - \Psi(0))^{\xi - 1}}{\Gamma(\xi)} - \left[ \frac{10(\Psi(1) - \Psi(0))^{\xi - 1 + \frac{1}{2}}}{7 \Gamma(\xi + \frac{4}{5})} + \frac{13}{6} \frac{(\Psi(1) - \Psi(0))^{\xi - 1 + \frac{1}{2}}}{\Gamma(\xi + \frac{2}{3})} \right]
$$

(58)

and

$$
\Omega = \frac{10}{7} \frac{(\Psi(1) - \Psi(0))^{\xi}}{\Gamma(\xi - 1 + \mu + \frac{4}{5})} + \frac{13}{6} \frac{(\Psi(1) - \Psi(0))^{\xi}}{\Gamma(\xi - 1 + \mu + \frac{2}{3})} + \frac{1}{\Gamma(\xi + \mu - 1)}.
$$

(59)

Because all the assumptions of Theorem 4.1 are satisfied, the problem (54)–(56) has at least one solution provided $\sigma < 1$. Further, by Theorem 5.2, for any solution $z \in C_{2-\xi}, \Psi([0, 1], \mathbb{R})$ of the inequality

$$
\left| H D_{0+}^\nu \Psi(t) - \frac{\cos t}{10 e^{t+1}} \left[ \sin z(t) + H D_{0+}^\nu \Psi(t) \right] \right| \leq \epsilon, \quad t \in [0, 1],
$$

there exists a unique solution $y \in C_{2-\xi}, \Psi([0, 1], \mathbb{R})$ of the problem (54)–(56) such that

$$
\|z - y\|_{C_{2-\xi}, \Psi([0, 1], \mathbb{R})} \leq C_f \epsilon,
$$

where

$$
C_f = \frac{(\Psi(1) - \Psi(0))^{\xi - 1 + \frac{1}{2}}}{\Gamma(\mu + 1)} E_{\mu} \left( \frac{0.0368}{1 - 0.0368} (\Psi(1) - \Psi(0)) \right).
$$

Define

$$
\chi(t) = E_{\mu} \left( \frac{1}{9} (\Psi(t) - \Psi(0)) \right), \quad t \in [0, 1].
$$

(60)

Then, $\chi : [0, 1] \to \mathbb{R}$ is continuous nondecreasing function such that

$$
I_{\mu+}^\nu : \Psi E_{\mu} \left( \frac{1}{9} (\Psi(t) - \Psi(0)) \right) = \frac{1}{9} \left[ E_{\mu} \left( \frac{1}{9} (\Psi(t) - \Psi(0)) \right) - 1 \right] \leq \frac{1}{9} E_{\mu} \left( \frac{1}{9} (\Psi(t) - \Psi(0)) \right).
$$

This proves $\chi$ satisfies condition (50) with $K^\star = \frac{1}{9}$. Therefore, by Theorem 5.4, for each solution $z \in C_{2-\xi}, \Psi([0, 1], \mathbb{R})$ of the inequality

$$
\left| H D_{0+}^\nu \Psi(t) - \frac{\cos t}{10 e^{t+1}} \left[ \sin z(t) + H D_{0+}^\nu \Psi(t) \right] \right| \leq \epsilon E_{\mu} \left( \frac{1}{9} (\Psi(t) - \Psi(0)) \right), \quad t \in [0, 1],
$$

there exists a unique solution $y \in C_{2-\xi}, \Psi([0, 1], \mathbb{R})$ of the problem (54)–(56) such that

$$
\left| (\Psi(t) - \Psi(0))^{\xi - \nu} \chi(t) \right| \leq \epsilon C_{f, \chi} E_{\mu} \left( \frac{1}{9} (\Psi(t) - \Psi(0)) \right),
$$

where

$$
C_{f, \chi} = \frac{1}{9} (\Psi(1) - \Psi(0))^{\xi - \nu} E_{\mu} \left( \frac{0.0368}{1 - 0.0368} (\Psi(1) - \Psi(0)) \right).
$$

A particular case: For $\mu = \frac{3}{2}$, $\nu = 1$ and $\Psi(t) = t$, the problem (54)–(56) reduces to the following problem

$$
C D_{\frac{3}{2}}^\frac{1}{2} y(t) = \frac{\cos t}{10 e^{t+1}} \left[ \sin y(t) + y(t) \right],
$$

(61)

$$
y(0) = 0,
$$

(62)

$$
y(1) = \frac{10}{7} f_{0+} \left( \frac{1}{3} \right) + \frac{13}{6} f_{0+} \left( \frac{1}{2} \right),
$$

(63)
which is the nonlocal BVP for implicit FDEs involving Caputo fractional derivative. In this case, \( \xi = \mu + \nu(2 - \mu) = 2 \).

Further putting the values of the constants \( \xi, \mu \) with \( \Psi(t) = t \) in Equations (57), (58), and (59) has the following values \( \lambda = 0.87045, \Omega = 1.35464, \) and \( \sigma = 0.0881987 < 1 \). It can be seen that all the assumptions of Theorem 4.1 and Theorem 5.2 are satisfied. Therefore, by Theorem 4.1, the Caputo nonlocal implicit BVP (61)–(63) has at least one solution. Further, for any solution \( z \in C((0, 1], \mathbb{R}) = C \) of the inequality

\[
\left| C D_{0+}^{\frac{3}{2}} z(t) - \cos t 10 e^{t^2} \left[ \sin z(t) + C D_{0+}^{\frac{3}{2}} z(t) \right] \right| \leq \epsilon, \quad t \in [0, 1],
\]

there is a solution \( y \in C((0, 1], \mathbb{R}) = C \) of problem (61)–(63) such that

\[
\| z - y \|_C \leq C_\epsilon \epsilon,
\]

where \( \| \cdot \|_C \) is supremum norm on \( C \) and the problem (61) is Ulam–Hyers stable. Further, for \( \Psi(t) = t, \mu = \frac{3}{2} \) from Equation (60), we have \( \chi(t) = E_{\frac{3}{2}} \left( \frac{1}{9} t^2 \right) \), which satisfy

\[
I_{0+}^{\frac{3}{2}} E_{\frac{3}{2}} \left( \frac{1}{9} t^2 \right) \leq \frac{1}{9} E_{\frac{3}{2}} \left( \frac{1}{9} t^2 \right).
\]

Therefore, by Theorem 5.4, for each solution \( z \in C \) of the inequality

\[
\left| C D_{0+}^{\frac{3}{2}} z(t) - \cos t 10 e^{t^2} \left[ \sin z(t) + C D_{0+}^{\frac{3}{2}} z(t) \right] \right| \leq \epsilon \chi(t), \quad t \in [0, 1],
\]

there is a solution \( y \in C \) of problem (61)–(63) such that

\[
| z(t) - y(t) | \leq \epsilon C_{\chi} \chi(t),
\]

and hence Equation (61) is Ulam–Hyers–Rassias stable.

**Example 6.2.** Consider the following nonlocal BVP for implicit FDE involving \( \Psi \)-Hilfer fractional derivative

\[
^H D_{0+}^{\mu, \nu} \Psi y(t) = \frac{1}{8} \left[ \frac{|y(t)|}{3 + |y(t)|} + \frac{|^H D_{0+}^{\mu, \nu} \Psi y(t)|}{3 + |^H D_{0+}^{\mu, \nu} \Psi y(t)|} \right], \quad t \in (0, 5], \quad 1 < \mu < 2, \quad 0 \leq \nu \leq 1,
\]

(64)

\[
y(0) = 0,
\]

(65)

\[
y(5) = \frac{9}{5} I_{0+}^{\frac{11}{3}} \Psi y(2) + \frac{11}{39} I_{0+}^{\frac{22}{3}} \Psi y(3) + \frac{14}{3} I_{0+}^{\frac{33}{3}} \Psi y(4).
\]

(66)

Define \( f : (0, 5] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
f(t, u, v) = \frac{1}{8} \left[ \frac{|u|}{3 + |u|} + \frac{|v|}{3 + |v|} \right].
\]

Then, for any \( u_i, v_i \in \mathbb{R} \) \( (i = 1, 2) \) and \( t \in (0, 5] \) we have

\[
| f(t, u_1, v_1) - f(t, u_2, v_2) | = \frac{1}{8} \left| \frac{|u_1|}{3 + |u_1|} + \frac{|v_1|}{3 + |v_1|} - \frac{|u_2|}{3 + |u_2|} - \frac{|v_2|}{3 + |v_2|} \right|
\]

\[
\leq \frac{3}{8} \left( |u_1 - u_2| + |v_1 - v_2| \right).
\]

This proves \( f \) satisfies the Lipschitz type condition in hypothesis (H1) with \( K = L = \frac{3}{8} = 0.375 \). Further comparing the problem (64)–(66) to the problem (1)–(3), we have \( a = 0, \ b = 5, \lambda_1 = \frac{9}{5}, \lambda_2 = \frac{11}{39}, \lambda_3 = \frac{14}{3}, \delta_1 = \frac{13}{11}, \delta_2 = \frac{3}{17}, \delta_3 = \frac{12}{17} \).
\[ \sigma = \frac{0.375 \Gamma(\xi - 1)(\Psi(5) - \Psi(0))^{\mu}}{1 - 0.375} \left( \frac{(\Psi(5) - \Psi(0))^{\xi-1}}{\Lambda \Gamma(\xi)} + \frac{1}{\Gamma(\xi + \mu - 1)} \right), \]  

(67)

where

\[ \Lambda = \frac{(\Psi(5) - \Psi(0))^{\xi-1}}{\Gamma(\xi)} - \left[ \frac{9}{5} \frac{(\Psi(2) - \Psi(0))^{\xi+\frac{11}{11}}}{\Gamma(\xi + \frac{11}{11})} + \frac{11}{39} \frac{(\Psi(3) - \Psi(0))^{\xi+\frac{22}{17}}}{\Gamma(\xi + \frac{22}{17})} + \frac{14}{3} \frac{(\Psi(4) - \Psi(0))^{\xi+\frac{33}{5}}}{\Gamma(\xi + \frac{33}{5})} \right], \]  

(68)

and

\[ \Omega = \frac{9}{5} \frac{(\Psi(5) - \Psi(0))^{\xi+\frac{11}{11}}}{\Gamma(\xi + \frac{11}{11})} + \frac{11}{39} \frac{(\Psi(5) - \Psi(0))^{\xi+\frac{22}{17}}}{\Gamma(\xi + \frac{22}{17})} + \frac{14}{3} \frac{(\Psi(5) - \Psi(0))^{\xi+\frac{33}{5}}}{\Gamma(\xi + \frac{33}{5})} + \frac{1}{\Gamma(\xi + \mu - 1)}. \]  

(69)

Because all the assumptions of Theorem 4.1 are satisfied, the problem (64)–(66) has at least one solution provided \( \sigma < 1 \). Further, by Theorem 5.2, for any solution \( z \in C_{2-\xi}; \varphi([0, 5], \mathbb{R}) \) of the inequality

\[ \left\| H D_{0+}^{\mu, \nu} z(t) - \frac{1}{8} \left[ \frac{|z(t)|}{3 + |z(t)|} + \frac{|H D_{0+}^{\mu, \nu} z(t)|}{3 + |H D_{0+}^{\mu, \nu} z(t)|} \right] \right\| \leq \epsilon, \quad t \in [0, 5], \]

there exists a unique solution \( y \in C_{2-\xi}; \varphi([0, 5], \mathbb{R}) \) of the problem (64)–(66) such that

\[ \| z - y \|_{C_{2-\xi}, \varphi([0, 5], \mathbb{R})} \leq C_f \epsilon, \]

where \( C_f = \frac{(\Psi(5) - \Psi(0))^{\mu+1}}{\Gamma(\mu+1)} E_{\mu} \left( 0.375, \frac{(\Psi(5) - \Psi(0))^{\mu}}{1 - 0.375} \right) \).

Define

\[ \chi(t) = (\Psi(t) - \Psi(0)), \quad t \in [0, 5]. \]  

(70)

Then, \( \chi : [0, 5] \to \mathbb{R} \) is continuous nondecreasing function such that

\[ I_{0+}^{\mu, \nu} \chi(t) = I_{0+}^{\mu, \nu} (\Psi(t) - \Psi(0)) \]

\[ = \frac{1}{\Gamma(\mu)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0)) \, ds \]

\[ \leq (\Psi(t) - \Psi(0)) \frac{1}{\Gamma(\mu)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\mu-1} \, ds \]

\[ \leq \frac{(\Psi(5) - \Psi(0))^{\mu}}{\Gamma(\mu + 1)} \chi(t), \quad t \in [0, 5]. \]

This proves \( \chi \) satisfies condition (50) with \( K^* = \frac{(\Psi(5) - \Psi(0))^{\mu}}{\Gamma(\mu+1)} \). Therefore, by Theorem 5.4, for each solution \( z \in C_{2-\xi}; \varphi([0, 5], \mathbb{R}) \) of the inequality

\[ \left\| H D_{0+}^{\mu, \nu} z(t) - \frac{1}{8} \left[ \frac{|z(t)|}{3 + |z(t)|} + \frac{|H D_{0+}^{\mu, \nu} z(t)|}{3 + |H D_{0+}^{\mu, \nu} z(t)|} \right] \right\| \leq \epsilon (\Psi(t) - \Psi(0)), \quad t \in [0, 5], \]

there exists a unique solution \( y \in C_{2-\xi}; \varphi([0, 5], \mathbb{R}) \) of the problem (64)–(66) such that

\[ \left| (\Psi(t) - \Psi(a))^{2-\xi} (z(t) - y(t)) \right| \leq \epsilon C_f \chi (\Psi(t) - \Psi(0)), \]

where \( C_f \chi = \frac{(\Psi(5) - \Psi(0))^{\mu+1}}{\Gamma(\mu+1)} E_{\mu} \left( 0.375, \frac{(\Psi(5) - \Psi(0))^{\mu}}{1 - 0.375} \right) \).
A particular case: For $\mu = \frac{5}{4}, \nu = 0$ and $\Psi(t) = t$, the problem (64)–(66) reduces to the following problem

$$R^L_{0+}D^\frac{5}{4} y(t) = \frac{1}{8} \left[ \frac{|y(t)|}{3 + |y(t)|} + \frac{|R^L_{0+}D^\frac{5}{4} y(t)|}{3 + |R^L_{0+}D^\frac{5}{4} y(t)|} \right], \quad t \in (0, 5],$$

(71)

$$y(0) = 0,$$

(72)

$$y(5) = \frac{9}{5} I^\frac{5}{4}_{0+} y(2) + \frac{11}{39} I^\frac{7}{4}_{0+} y(3) + \frac{14}{3} I^\frac{7}{4}_{0+} y(4),$$

(73)

which is the nonlocal BVP for implicit FDEs involving Riemann–Liouville fractional derivative. In this case, $\xi = \mu + \nu(2 - \mu) = \frac{5}{4}$. Further, putting the values of the constants $\xi, \mu$ with $\Psi(t) = t$ in Equations (67), (68), and (69) has the following values $A = -16.0654, \Omega = 19.6428$, and $\sigma = -14.4552 < 1$. It can be seen that all the assumptions of the Theorem 4.1 and Theorem 5.2 are satisfied. Therefore by Theorem 4.1, the Riemann–Liouville nonlocal implicit BVP (71)–(73) has at least one solution. Further, for any solution $z \in C ([0, 5], \mathbb{R}) = C^*$ of the inequality

$$\left| R^L_{0+}D^\frac{5}{4} z(t) - \frac{1}{8} \left[ \frac{|z(t)|}{3 + |z(t)|} + \frac{|R^L_{0+}D^\frac{5}{4} z(t)|}{3 + |R^L_{0+}D^\frac{5}{4} z(t)|} \right] \right| \leq \epsilon, \quad t \in [0, 5],$$

there is a solution $y \in C ([0, 5], \mathbb{R}) = C^*$ of problem (71)–(73) such that

$$\|z - y\|_{C^*} \leq C_f \epsilon,$$

where $\| \cdot \|_{C^*}$ is supremum norm on $C^*$ and the problem (71) is Ulam–Hyers stable. Further, for $\Psi(t) = t, \mu = \frac{5}{4}$ from Equation (70), we have $\chi(t) = t$, which satisfy

$$I^\frac{5}{4}_{0+} t \leq \frac{5}{4} t \cdot \frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}.$$ 

Therefore, by Theorem 5.4, for each solution $z \in C^*$ of the inequality

$$\left| R^L_{0+}D^\frac{5}{4} z(t) - \frac{1}{8} \left[ \frac{|z(t)|}{3 + |z(t)|} + \frac{|R^L_{0+}D^\frac{5}{4} z(t)|}{3 + |R^L_{0+}D^\frac{5}{4} z(t)|} \right] \right| \leq \epsilon \chi(t), \quad t \in [0, 5],$$

there is a solution $y \in C^*$ of problem (71)–(73) such that

$$|z(t) - y(t)| \leq \epsilon C_f \cdot \chi(t),$$

and hence Equation (71) is Ulam–Hyers–Rassias stable.

7 | CONCLUDING REMARKS

In the present manuscript, we have derived equivalent fractional integral equations for the $\Psi$-Hilfer implicit FDEs with nonlocal integral boundary conditions and through which we have investigated the existence of solutions, Ulam–Hyers and Ulam–Hyers–Rassias stabilities. The results obtained in the present manuscript are also true for the implicit FDEs with nonlocal integral boundary conditions involving FDOs listed in Sousa and Oliveira.44
Taking into account the importance and collection of recent applications of Caputo–Fabrizio FDO and Atangana–Baléanu–Caputo FDO, one can investigate the nonlocal implicit BVP (1)–(3) with these FDOs having non-singular kernel.

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CONFLICTS OF INTEREST
This work does not have any conflicts of interest.

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REFERENCES
1. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations, North–Holland Mathematics Studies, vol. 207. Amsterdam: Elsevier; 2006.
2. Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl. 2015;1(2):73-85.
3. Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm Sci. 2016;20(2):763-769.
4. Alsaedi A, Baleanu D, Etemad S, Rezapour S. On coupled systems of time-fractional differential problems by using a new fractional derivative. J Funct Spaces. 2016;2016:4626940. https://doi.org/10.1155/2016/4626940
5. Baleanu D, Mousalou A, Rezapour S. A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative. Adv Differ Equ. 2017;2017:51. https://doi.org/10.1186/s13662-017-1088-3
6. Baleanu D, Mousalou A, Rezapour S. On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations. Bound Value Probl. 2017;2017:145. https://doi.org/10.1186/s13661-017-0867-9
7. Baleanu D, Mousalou A, Rezapour S. The extended fractional Caputo-Fabrizio derivative of order 0 ≤ σ ≤ 1 on C_G[0, 1] and the existence of solutions for two higher-order series-type differential equations. Adv Differ Equ. 2018;2018:255. https://doi.org/10.1186/s13662-018-1696-6
8. Aydogan MS, Baleanu D, Mousalou A, Rezapour S. On approximate solutions for two higher-order Caputo-Fabrizio fractional integro-differential equations. Adv Differ Equ. 2017;2017:221. https://doi.org/10.1186/s13662-017-1258-3
9. Aydogan MS, Baleanu D, Mousalou A, Rezapour S. On high order fractional integro-differential equations including the Caputo-Fabrizio derivative. Bound Value Probl. 2018;2018:90. https://doi.org/10.1186/s13661-018-1008-9
10. Baleanu D, Rezapour S, Saberpour Z. On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation. Bound Value Probl. 2019;2019:79. https://doi.org/10.1186/s13661-019-1194-0
11. Yıldız TA, Jajarmi A, Yıldız B, Baleanu D. New aspects of time fractional optimal control problems within operators with nonsingular kernel. Discr Contin Dyn Syst -S. 2020;13(3):407-428.
12. Baleanu D, Jajarmi A, Mohammadi H, Rezapour S. A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative. Chaos Solitons Fract. 2020;134:109705. https://doi.org/10.1016/j.chaos.2020.109705
13. Jajarmi A, Ghanbari B, Baleanu D. A new and efficient numerical method for the fractional modeling and optimal control of diabetes and tuberculosis co-existence. Chaos: An Interdiscip J Nonlin Sci. 2019;29(9):93111. https://doi.org/10.1063/1.5112177
14. Jajarmi A, Baleanu D, Sajjadi SS, Asad JH. A new feature of the fractional Euler-Lagrange equations for a coupled oscillator using a nonsingular operator approach. Front Phys. 2019;7:196. https://doi.org/103389/fphy.2019.00196
15. Baleanu D, Jajarmi A, Sajjadi SS, Mozyrska D. A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator. Chaos: An Interdiscip J Nonlin Sci. 2019;29(8):83127. https://doi.org/10.1063/1.5096159
16. Jajarmi A, Arshad S, Baleanu D. A new fractional modelling and control strategy for the outbreak of dengue fever. Phys A: Stat Mech Appl. 2019;535:122524. https://doi.org/10.1016/j.physa.2019.122524
17. Benchhra M, Hamani S, Ntouyas SK. Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlin Anal Theory Meth Appl. 2009;71(7-8):2391-2396.
18. Ahmad B, Nieto JJ. Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. Abstr Appl Anal. 2009;2009:494720. https://doi.org/10.1155/2009/494720
19. Ahmad B. On nonlocal boundary value problems for nonlinear integro-differential equations of arbitrary fractional order. Results Math. 2013;1(63):183-194.
54. Sousa JVC, Oliveira EC. A Gronwall inequality and the Cauchy-type problem by means of $\psi$-Hilfer operator. *Diff Equ Appl*. 2019;11(1):87-106.

55. Gorenflo R, Kilbas AA, Mainardi F, Rogosin SV. *Mittag-Leffler Functions, Related Topics and Applications*. Verlag Berlin Heidelberg: Springer; 2014.

56. Zhou Y, Wang J, Zhang L. *Basic Theory of Fractional Differential Equations*. Singapore: World Scientific; 2014.

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