On a Hamiltonian form of an elliptic spin
Ruijseenaars-Schneider system

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1 Introduction

An elliptic Ruijsenaars-Schneider (RS) model \cite{1} is a Hamiltonian system of \( N \) interacting particles with a Hamiltonian

\[
H = \sum_{j=1}^{N} e^{p_j} \prod_{s \neq j} \left( \frac{\sigma(x_j - x_s + \eta)\sigma(x_j - x_s - \eta)}{\sigma^2(x_j - x_s)} \right)^{1/2}
\]

and the canonical symplectic form \( \omega = \sum \delta p_i \wedge \delta x_i \), where \( p_i = \dot{x}_i \).

The equations of motions are

\[
\ddot{x}_i = \sum_{s \neq i} \dot{x}_i \dot{x}_s (V(x_s - x_i) - V(x_i - x_s)),
\]

where \( V(x) = \zeta(x + \eta) - \zeta(x) \), and \( \zeta(x) \) is a Weierstrass zeta function.

The limit when one or two periods of the elliptic curve go to infinity yields a trigonometric or rational system. A RS system is a relativistic generalization of the Calogero-Moser model.

A spin generalization of RS system was suggested in \cite{2}. Each particle additionally carries two \( l \)-dimensional vectors \( a_i \) and \( b_i \) that describe the internal degrees of freedom and affect the interaction. Remarkably, the equations of motion remain integrable and are given by the formulas

\[
\begin{align*}
    \dot{f}_{ij} &= \sum_{k \neq j} f_{ik} f_{kj} V(x_j - x_k) - \sum_{k \neq i} f_{ik} f_{kj} V(x_k - x_i) \\
    \dot{x}_i &= f_{ii},
\end{align*}
\]

where \( f_{ij} = b_i^T a_j \).

It was shown in \cite{3} using the universal symplectic form (proposed in \cite{4}) that a spin elliptic RS system is Hamiltonian. An expression of a symplectic form (or Poisson structure) in explicit coordinates is known only in the rational and trigonometric limits (see \cite{7}).

The aim of this paper is to compute \( \omega \) in the original coordinates \( x_i \) and \( f_{ij} \) in the simplest elliptic case of 2 particles, \( N = 2 \). We compare the obtained 2-form with a symplectic form for a system without spin and with a Poisson structure found in \cite{7} in the rational case.
The general procedure developed by Krichever and Phong in \cite{4} allows to construct action-angle variables for an elliptic RS system and its spin generalization. It was done in \cite{3}. The upshot of the procedure is the following.

A Lax representation with a spectral parameter for an elliptic RS system has been found in \cite{2}. A Lax matrix is

\[ L_{ij} = f_i \Phi(x_i - x_j - \eta), \text{ where } \Phi(x, z) = \frac{\sigma(z + x + \eta)}{\sigma(z + \eta) \sigma(x)} \left[ \frac{\sigma(z - \eta)}{\sigma(z + \eta)} \right]^{x/2\eta}. \quad (4) \]

The spectral parameter \( z \) is defined on an elliptic curve \( \Gamma_0 \) with a cut between points \( z = \eta \) and \( z = -\eta \).

The universal symplectic form is given by the formula

\[ \omega = -\frac{1}{2} \sum_{q \in I} \text{res}_q \text{Tr} (\Psi^{-1} L^{-1} \delta L \wedge \delta \Psi - \Psi^{-1} \delta \Psi \wedge K^{-1} \delta K) \, dz, \quad (5) \]

where the sum is taken over the poles of \( L \) and zeroes of \( \det L \). \( \Psi \) is a matrix composed of eigenvectors of \( L \), which has poles \( \tilde{\gamma}_s \) on the spectral curve \( \tilde{\Gamma} \): \( \det (L - kI) = 0 \) due to normalization of eigenvectors. \( k \) is a meromorphic function on \( \tilde{\Gamma} \) and the matrix \( K = \text{diag}(k_1, \ldots, k_N) \) is composed of values of \( k \) on different sheets of \( \tilde{\Gamma} \).

\( \omega \) doesn’t depend on the gauge transformations \( L \rightarrow gLg^{-1} \) and the normalization of eigenvectors on the leaves where the form \( \delta \ln k \, dz \) is holomorphic. \( \text{Tr} (...) \, dz \) is a meromorphic differential, and the sum of all its residues is zero. Using these facts, one can show that on the leaves

\[ \omega = \sum_s \delta \ln k(\tilde{\gamma}_s) \wedge \delta z(\tilde{\gamma}_s). \quad (6) \]

Computations performed in \cite{3} for Lax matrix \cite{4} show that

\[ \omega = \sum_i \delta \ln f_i \wedge \delta x_i + \sum_{i \neq j} V(x_i - x_j) \delta x_i \wedge \delta x_j, \quad (7) \]

where

\[ f_i = e^{p_i} \prod_{s \neq i}^N \left( \frac{\sigma(x_i - x_s + \eta) \sigma(x_i - x_s - \eta)}{\sigma^2(x_i - x_s)} \right)^{1/2} \]

and the Hamiltonian for system \cite{2} is \( H = \sum_{i=1}^N f_i \).

A Lax representation with a spectral parameter for an elliptic spin RS system \cite{3} has been found in \cite{4}. The Lax matrix is \( L_{ij} = f_{ij} \Phi(x_i - x_j - \eta) \). Formally, equations \cite{3} are Hamiltonian with \( H = \sum_{i=1}^N f_{ii} \) and symplectic form \cite{4} (see \cite{3} for details). The goal of this paper is to compute form \cite{4} in the original coordinates \( x_i \) and \( f_{ij} \).
After the gauge transformation by a diagonal matrix \( g = \text{diag}(\Phi(x_1, z), \lambda \Phi(x_2, z)) \) with an appropriate choice of \( \lambda \), the matrix \( L_{ij} \) becomes

\[
L = \begin{pmatrix}
-f_1 \frac{\sigma(z)}{\sigma(\eta)} & f_2 \frac{\sigma(z - x_1 + x_2)\sigma(z + x_1 + \eta)\sigma(x_2)}{\sigma(x_2 - x_1 - \eta)\sigma(x_2 - x_1 - \eta)\sigma(x_2)} \\
-f_3 \frac{\sigma(z - x_1 + x_2)\sigma(z + x_1 + \eta)\sigma(x_2)}{\sigma(x_2 - x_1 - \eta)\sigma(x_2 - x_1 - \eta)\sigma(x_2)} & 1 \\
\end{pmatrix}
\]

where \( f_1 \equiv f_{11}, f_2 \equiv f_{22} \) and \( f_3 \equiv \sqrt{f_{12}f_{21}} \).

The matrix \( L \) is defined on a curve \( \Gamma \) of genus \( g = 2 \), which is a 2-sheeted cover of the elliptic curve \( \Gamma_0 \) with 2 branch points \( z = \eta \) and \( z = -\eta \).

The spectral curve \( \hat{\Gamma} \) of \( L \) is defined by the equation \( R = \det(L_{ij} - k) = 0 \).

The set \( I \) in (5) is \( I = \{ \gamma_0, 0, \pm z_0 \} \), where \( z_0 \) is defined by the equation \( \det(L_{ij}) = 0 \), or

\[
f_1 f_2 \sigma^2(z_0) - f_3^2 \sigma(z_0 - x_1 + x_2)\sigma(z_0 - x_1 + x_2) = 0.
\]

Notice, that we can use variables \((x_1, x_2, f_1, f_2, z_0)\) instead of \((x_1, x_2, f_1, f_2, f_3)\).

**Theorem 1.** In the case \( N = 2 \) the elliptic spin RS system is Hamiltonian with a symplectic form

\[
\omega = -\delta \ln f_1 \wedge \delta x_1 - \delta \ln f_2 \wedge \delta x_2 + 2 \hat{V}(x_1 - x_2) \delta x_1 \wedge \delta x_2
\]

and Hamiltonian \( H = f_1 + f_2 \), where \( \hat{V}(x) = \zeta(x + z_0) - \zeta(x) \). The spinless case corresponds to \( z_0 = 0 \).

**Proof.** The eigenvector \( \psi \) of \( L \) in any normalization is a meromorphic function on \( \Gamma \) and it has \( \hat{g} + 1 = 6 \) poles \( \hat{\gamma}_s \). The proof of formula (6) in [6] assumes that the situation is in general position, i.e. projections of points \( \hat{\gamma}_i \) don’t coincide with \( \gamma_0 \).

Most appropriate normalization here is \( \psi_1 \equiv 1 \), because it easily allows us to find poles \( \hat{\gamma}_s \) of \( \psi \). Two of them \((s = 1, 2)\) lie above the point \( z = x_1 - x_2 \), and the other are above \( z = -x_1 - \eta \) \((s = 3, 4, 5, 6)\). This is not the case of general position, but it turns out that the same formula (6) still holds.

The proof in [5] and [6] implies that 2-form (5) in the normalization \( \psi_1 \equiv 1 \) equals to \( \omega_0 = \sum_{s=1}^2 \delta \ln k(\hat{\gamma}_s) \wedge \delta z(\hat{\gamma}_s) \).
A change of normalization of $\Psi$ from $\psi_1 \equiv 1$ to $\sum \psi_i \equiv 1$ (the last one is in "general position") corresponds to the transformation $\tilde{\Psi} = \Psi V$, where

$$V = \begin{pmatrix} \frac{L_{12}}{k_1 - L_{11} + L_{12}} & 0 \\ \frac{L_{12}}{k_2 - L_{11} + L_{12}} & \frac{L_{12}}{k_1 - L_{11} + L_{12}} \end{pmatrix}.$$ 

According to the computations in [6],

$$\omega = \omega_0 + \sum_{q \in \mathcal{I}} \text{res}_q \text{Tr} \left( K^{-1} \delta K \wedge \delta VV^{-1} \right) dz.$$ 

Since $\omega$ has to be restricted to the leaves where $\delta \ln kdz$ is holomorphic (which is equivalent to 2 conditions: $\delta \eta = 0$ and $\delta z_0 = 0$), the only non-zero residue in the second term is at the point $z(\gamma_i) = -x_1 - \eta$. After computing the residue, we get that $\omega = \omega_0 + \sum_{s=3}^6 \delta \ln k(\tilde{\gamma}_s) \wedge \delta z(\tilde{\gamma}_s)$, i.e. effectively formula (6) holds in both normalizations.

Substituting $\tilde{\gamma}_s$ in (6), we find that

$$\omega = -\delta \ln f_1 \wedge \delta x_1 - \delta \ln f_2 \wedge \delta x_2 + 2\tilde{V}(x_1 - x_2)\delta x_1 \wedge \delta x_2,$$

where $\tilde{V}(x) = \zeta(x + z_0) - \zeta(x)$.

The Hamiltonian $H = f_1 + f_2$ defines the flow

$$\begin{cases} \dot{f}_1 = -f_1 f_2 (\zeta(x_1 + x_2) - \zeta(x_0 - x_1 + x_2) - 2\zeta(x_1 - x_2)) \\ \dot{f}_2 = f_1 f_2 (\zeta(x_1 + x_2) - \zeta(x_0 - x_1 + x_2) - 2\zeta(x_1 - x_2)) \\ \dot{x}_1 = f_1 \\ \dot{x}_2 = f_2. \end{cases}$$

Using identities for Weierstrass $\sigma$-functions, namely,

$$\sigma(a + c)\sigma(a - c)\sigma(b + d)\sigma(b - d) - \sigma(a + d)\sigma(a - d)\sigma(b + c)\sigma(b - c) = \sigma(a + b)\sigma(a - b)\sigma(c + d)\sigma(c - d),$$

and

$$\zeta(a) + \zeta(b) + \zeta(c) - \zeta(a + b + c) = \frac{\sigma(a + b)\sigma(b + c)\sigma(a + c)}{\sigma(a)\sigma(b)\sigma(c)\sigma(a + b + c)},$$

it follows from the definition of $z_0$ that

$$f_1 f_2 (2\zeta(x_1 - x_2) + \zeta(z_0 - x_1 + x_2) - \zeta(z_0 + x_1 - x_2)) = f_3^2 (2\zeta(x_1 - x_2) + \zeta(\eta - x_1 + x_2) - \zeta(\eta + x_1 - x_2)).$$

With the help of this identity, we can show that the above equations are equivalent to

$$\begin{cases} \dddot{x}_1 = f_3^2 (2\zeta(x_1 - x_2) + \zeta(\eta - x_1 + x_2) - \zeta(\eta + x_1 - x_2)) \\ \dddot{x}_2 = -f_3^2 (2\zeta(x_1 - x_2) + \zeta(\eta - x_1 + x_2) - \zeta(\eta + x_1 - x_2)), \end{cases}$$

which is an RS system.

The spinless case occurs when $f_3^2 = f_1 f_2$ and $z_0 = \eta$ as one can observe from [6].
Remark. A Poisson structure was found in [7] in the rational limit for arbitrary \( N \) (see formula (3.31) in [7]). In the case of 2 particles it is non-degenerate and defined on a 6-dimensional space \((f_{11}, f_{12}, f_{21}, f_{22}, x_1, x_2)\). The corresponding 2-form is defined on the same space and coincides with \( \delta z_0 = 0 \) and after reduction with respect to the action \( f_{12} \rightarrow f_{12}/\lambda, f_{21} \rightarrow f_{21}/\lambda \).

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References

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