A simple and unifying method to show the perfect error-correcting condition is provided based on the quantum mutual information. The one-to-one parameterization of quantum operations and the properties of the quantum relative entropy are used effectively in this paper, where the equivalence between the subspace transmission and the entanglement transmission is clearly presented. We also revisit a variant of the no-cloning and no-deleting theorem based on an information-theoretical tradeoff between two parties for the reversibility of quantum operations, and demonstrate that the no-cloning and no-deleting theorem leads to the perfect error-correcting condition on Kraus operators.

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I. INTRODUCTION

In the past decade, much progress has been made in the theory of the quantum error-correcting codes [1–6] along with information-theoretical developments [7–14]. In particular, the perfect quantum error-correcting condition by Schumacher and Nielsen [8] is of much importance, providing insights on the role of the coherent information [8, 15]. Another approach to the perfect error-correcting condition was also given independently by Knill and Laflamme [10] and by Bennett et al. [9] to establish the algebraic condition on Kraus operators of quantum operations. The above mentioned results are already widely known and, for example, one may find them in the textbook by Nielsen and Chuang [16].

On the other hand, Cerf and Adami [17, 18] introduced a quantum counterpart of the classical mutual information, namely the quantum mutual information, and tried to develop the quantum information theory through it [17–20], while the results by Cerf and Cleve [19, 20] concerning the perfect error-correcting condition were dependent on those of Schumacher and Nielsen [8] and the role of the quantum mutual information remained relatively unclear. Later the quantum mutual information appeared in the formula for the entanglement-assisted capacity [21–23] of quantum channels. Note that the Holevo information [24] and the coherent information [8, 15] are also regarded as quantum counterparts of the classical mutual information, with definite meanings as the capacities of quantum channels for transmitting classical information [25, 26] and quantum information [27, 28], respectively. In this paper, however, we use the term quantum mutual information as one introduced by Cerf and Adami.

The aim of this paper is to provide a simple and unifying method to show the perfect error-correcting condition based on the quantum mutual information. Our approach does not depend on the results of Schumacher and Nielsen [8], but the one-to-one parameterization [29–32] of quantum operations and the properties [33–35] of the quantum relative entropy are used effectively. The arguments in this paper will refine the earlier works [19, 20], and shed light on the role of the quantum mutual information. As an application of our method, a variant [36] of the no-cloning theorem [37–42] and the no-deleting theorem [43, 44] is recaptured based on an information-theoretical tradeoff [20] between two parties for the reversibility of quantum operations. It is also demonstrated that our approach immediately yields the perfect error-correcting condition on Kraus operators [9, 10]. The results themselves in this paper are not always new and should be regarded as a refinement or a recast of the earlier works [8–10, 19, 20, 35, 36]. The methods used here, however, give us clear insights on the role of the quantum mutual information related to the reversibility of the quantum operations.

II. DEFINITIONS AND THE NO-CLONING AND NO-DELETING THEOREM

Let \( \mathcal{H}_A, \mathcal{H}_B, \) and \( \mathcal{H}_C \) be finite dimensional Hilbert spaces, and let \( \mathcal{L}(\mathcal{H}) \) denote the totality of linear operators on a Hilbert space \( \mathcal{H} \). The totality of density operators is denoted by

\[
\mathcal{S}(\mathcal{H}) := \{ \rho \in \mathcal{L}(\mathcal{H}) \mid \rho^* = \rho \geq 0, \text{Tr}[\rho] = 1 \}.
\]

The notion of the reversibility and the vanishing property is defined for quantum operations as follows, related to the quantum error-correcting schemes. A quantum operation \( \mathcal{E} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \), which is a trace preserving and completely positive linear superoperator, is called reversible with respect to (w.r.t.) a subset \( \mathcal{S} \subseteq \mathcal{S}(\mathcal{H}_A) \) if there exists a quantum operation \( \mathcal{R} : \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_A) \) such that \( \forall \rho \in \mathcal{S}, \mathcal{R}\mathcal{E}(\rho) = \rho \). A quantum operation \( \mathcal{E} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) is called vanishing w.r.t. \( \mathcal{S} \subseteq \mathcal{S}(\mathcal{H}_A) \) if there exists a density operator \( \rho_0 \in \mathcal{S}(\mathcal{H}_B) \) such that \( \forall \rho \in \mathcal{S}, \mathcal{E}(\rho) = \rho_0 \).
In the quantum error-correcting schemes, a subspace $K_A \subseteq \mathcal{H}_A$ is chosen as a codebook to be protected from a quantum operation $\mathcal{E}$ so that $\mathcal{E}$ is reversible w.r.t. the set

$$S_1(K_A) := \{|\psi \rangle \langle \psi| \in S(\mathcal{H}_A) \mid |\psi| \in K_A\} \quad (2)$$

of pure states with their eigenvectors included by $K_A$. In this case, we may say that $\mathcal{E}$ is reversible w.r.t. the subspace $K_A$ for simplicity. It should be noted that the reversibility and the vanishing property of $K_A$ are, respectively, equivalent to those of the convex hull of $S_1(K_A)$, say $S(K_A)$, which is the set of density operators with their supports included by $K_A$.

Using the subspace $K_A$ as the codebook, an arbitrary quantum state on a Hilbert space $\mathcal{H}_X$ with $\dim \mathcal{H}_X = \dim K_A$ is transmitted over the quantum channel $\mathcal{E}$. The encoding operation for this purpose is given by the isometry encoding

$$C : \rho_X \in S(\mathcal{H}_X) \mapsto \rho_A = V \rho_X V^* \in S(\mathcal{H}_A), \quad (3)$$

where $V : \mathcal{H}_X \to \mathcal{H}_A$ is an isometry satisfying $\text{Im} V = K_A$. Note that $S(K_A)$ defined above is written specifically using the isometry encoding as

$$S(K_A) = \{\rho_A \in S(\mathcal{H}_A) \mid \rho_A = V \rho_X V^*, \rho_X \in S(\mathcal{H}_X)\}.$$

The point here is that we have the one-to-one correspondence (3) between $S(K_A)$ and $S(H_X)$ and that any $\rho_A \in S(K_A)$ may be identified with some $\rho_X \in S(H_X)$.

A further definition is needed to state the following theorem. A quantum operation $\mathcal{E} : S(H_A) \to S(H_B)$ is called a pure state channel w.r.t. a subspace $K_A \subseteq \mathcal{H}_A$ if the output $\mathcal{E}(|\psi \rangle \langle \psi|)$ of any pure state $|\psi \rangle \langle \psi|$ in $S_1(K_A)$ results in a pure state. The following theorem represents the tradeoff between two parties for the reversibility of quantum operations and is regarded as a variant of the no-cloning theorem [37–42] and the no-deleting theorem [43, 44]. Originally the following theorem was shown by Cleve et al. [36] in the case $K_A = H_A$ with explicit arguments for the proof using the perfect error-correcting condition on Kraus operators [9, 10]. One may find a rigorous description [45] of the proof according to the original arguments [36].

**Theorem 1.** Given a quantum operation $\mathcal{E}_{BC} : S(\mathcal{H}_A) \to S(\mathcal{H}_B \otimes \mathcal{H}_C)$, let $\mathcal{E}_B := \text{Tr}_C \mathcal{E}_{BC}$ and $\mathcal{E}_C := \text{Tr}_B \mathcal{E}_{BC}$ be composite maps of quantum operations. Then, concerning the following conditions for a subspace $K_A \subseteq \mathcal{H}_A$,

(i) $\mathcal{E}_B$ is reversible w.r.t. $S(K_A)$,

(ii) $\mathcal{E}_C$ is vanishing w.r.t. $S(K_A)$,

it holds that

(a) (i) $\Rightarrow$ (ii),

(b) (i) $\Leftrightarrow$ (ii) if $\mathcal{E}_{BC}$ is a pure state channel w.r.t. $K_A$ and reversible w.r.t. $S(K_A)$.

In the later sections, we provide a simple proof of the above theorem using an information-theoretical tradeoff [20] of the quantum mutual information between two parties. The process to show the proof will expose the role of the quantum mutual information related to the reversibility of quantum operations.

**III. ONE-TO-ONE PARAMETERIZATION OF QUANTUM OPERATIONS**

The aim of this section is, for readers’ convenience, to summarize the one-to-one parameterization of quantum operations given by Fujiwara and Algoet [29, 30] (see also [31]) based on the work of Choi [32]. The parameterization establishes an one-to-one affine correspondence between the totality of quantum operations,

$$\mathcal{QO} := \{\mathcal{E} : S(\mathcal{H}_A) \to S(\mathcal{H}_B) \mid \mathcal{E} : \text{quantum operation}\},$$

and a set of nonnegative definite operators which is defined below.

Let $d := \dim \mathcal{H}_A$ and

$$|\Phi \rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i \rangle \otimes |i \rangle \quad (4)$$

be the standard maximally entangled state on a bipartite system $\mathcal{H}_R \otimes \mathcal{H}_A$ with a reference system $\mathcal{H}_R$ satisfying $\dim \mathcal{H}_R = \dim \mathcal{H}_A$, where $\{|i\rangle\}_{i=1}^{d}$ is a complete orthonormal basis on $H_A$, and we use the same index for that on $H_R$ for simplicity. Let us consider the output of the maximally entangled state by the extended quantum operation $I_R \otimes \mathcal{E}$ with $I_R$ denoting the identity superoperator, i.e.,

$$M(\mathcal{E}) := (I_R \otimes \mathcal{E})(|\Phi \rangle \langle \Phi|). \quad (5)$$

On the other hand, define the set of nonnegative definite operators on the extended Hilbert space by

$$\mathcal{M} := \left\{ M \in L(\mathcal{H}_R \otimes \mathcal{H}_B) \mid M \geq 0, \text{Tr}_B[M] = \frac{1}{d} I_R \right\}.$$

Then, the map $\mathcal{E} \in \mathcal{QO} \mapsto M(\mathcal{E}) \in \mathcal{M}$ establishes the one-to-one affine correspondence between $\mathcal{QO}$ and $\mathcal{M}$.

The essence of the one-to-one parameterization can be seen from the Kronecker product representation of $M(\mathcal{E})$,

$$M(\mathcal{E}) = \frac{1}{d} \sum_{i=1}^{d} \sum_{j=1}^{d} |i \rangle \langle j| \otimes \mathcal{E}(|i \rangle \langle j|) \approx \frac{1}{d} \begin{pmatrix} \mathcal{E}(|i \rangle \langle j|) \end{pmatrix}_{ij}, \quad (6)$$

which means the block matrix including $\mathcal{E}(|i \rangle \langle j|)$ in $(i, j)$-block. Here note that the quantum operation defined in...
$S(\mathcal{H}_A)$ is naturally extended to $\mathcal{L}(\mathcal{H}_A)$ by the linearity and the polar identity,

$$|i\rangle\langle j| = \frac{1}{2} \left( |a\rangle\langle a| - |b\rangle\langle b| + \sqrt{-1} (|c\rangle\langle c| - |d\rangle\langle d|) \right),$$

(7)

where $|a\rangle = (|i\rangle + |j\rangle)/\sqrt{2}$, $|b\rangle = (|i\rangle - |j\rangle)/\sqrt{2}$, $|c\rangle = (|i\rangle + \sqrt{-1} |j\rangle)/\sqrt{2}$, and $|d\rangle = (|i\rangle - \sqrt{-1} |j\rangle)/\sqrt{2}$. Then we can see that (6) has the entire information about $\mathcal{E}$, since the output $\mathcal{E}(|i\rangle\langle j|)$ of each of the complete basis $\{|i\rangle\langle j|\}_J$ on the linear space $\mathcal{L}(\mathcal{H}_A)$ appears in the $(i, j)$-block of $M(\mathcal{E})$; see Appendix for details.

In the same way, we can also make another one-to-one affine parameterization from an arbitrary faithful state \cite{30}. Let $\rho_A > 0$ be a faithful state in $S(H_A)$ and

$$\rho_A = \sum_{i=1}^{d} p_i |i\rangle\langle i|$$

(8)

be the Schatten decomposition of $\rho_A$, where $p_i$ is the eigenvalue corresponding to the eigenvector $|i\rangle$. Then, a purification of $\rho_A$ is given by

$$|\Phi_{\rho_A}\rangle := \sum_{i=1}^{d} \sqrt{p_i} |i\rangle \otimes |i\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A.$$  

(9)

Let $\rho_A := |\Phi_{\rho_A}\rangle\langle \Phi_{\rho_A}|$ and $\rho_R := \text{Tr}_A(\rho_{RA})$, and define the output of the state $\rho_{RA}$ by the extended quantum operation as

$$M_{\rho_{RA}}(\mathcal{E}) := (I_R \otimes \mathcal{E})(\rho_{RA}).$$

(10)

On the other hand, let us define a set of nonnegative definite operators by

$$M_{\rho_{RA}} := \{ M \in \mathcal{L}(\mathcal{H}_R \otimes \mathcal{H}_B) | M \geq 0, \text{Tr}_B[M] = \rho_R \}.$$  

Then the map $\mathcal{E} \in \mathcal{QO} \mapsto M_{\rho_{RA}}(\mathcal{E}) \in M_{\rho_{RA}}$, again, establishes the one-to-one affine correspondence between $\mathcal{QO}$ and $M_{\rho_{RA}}$.

\section{IV. THE QUANTUM RELATIVE ENTROPY}

Let us define the quantum relative entropy between two quantum states $\rho, \sigma \in S(H_A)$ by

$$D(\rho||\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)].$$

(12)

Then, for any quantum operation $\mathcal{E} : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_B)$, it holds that

$$D(\rho||\sigma) \geq D(\mathcal{E}(\rho)||\mathcal{E}(\sigma)),$$

(13)

which is called the monotonicity \cite{46, 47} and is one of the most important properties of the quantum relative entropy. It is known that the equality of the monotonicity (13) holds if and only if $\mathcal{E}$ is reversible w.r.t. $\{\rho, \sigma\}$ \cite{33} (see also Refs. \cite{34, 35}). When the equality of the monotonicity holds, there is a canonical reverse operation depending only on $\sigma$, which given by

$$\mathcal{R}_\sigma(\tau) := \sigma^{1/2} \mathcal{E}^* (\mathcal{E}(\sigma)^{-1/2} \tau \mathcal{E}(\sigma)^{-1/2}) \sigma^{1/2}$$

(14)

on the support of $\mathcal{E}(\sigma)$. Here $\mathcal{E}^* : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A)$ is the dual of $\mathcal{E}$ satisfying

$$\forall \rho \in S(H_A), \forall Y \in \mathcal{L}(H_B), \text{Tr}[\mathcal{E}(\rho)Y] = \text{Tr}[\rho \mathcal{E}^*(Y)].$$

The above fact is summarized as the follows.

\textbf{Proposition 1 (Petz \cite{33–35}).} Given a quantum operation $\mathcal{E} : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_B)$ and $\rho, \sigma \in S(H_A)$, let $\mathcal{R}_\sigma$ be the quantum operation defined by (14). Then the following conditions are equivalent.

(a) $D(\rho||\sigma) = D(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$.

(b) $\mathcal{R}_\sigma(\mathcal{E}(\rho)) = \rho$.

(c) $\mathcal{E}$ is reversible w.r.t. $\{\rho, \sigma\}$.

The quantum relative entropy also satisfies other important properties. One of them is the positivity,

$$D(\rho||\sigma) \geq 0, \quad D(\rho||\sigma) = 0 \Leftrightarrow \rho = \sigma.$$  

(15)

Another one is the invariance under the action of unitary transformations or isometries, i.e., it holds for any isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B$ that

$$D(\rho||\sigma) = D(V \rho V^*||V \sigma V^*).$$

(16)
V. THE QUANTUM MUTUAL INFORMATION AND THE REVERSIBILITY

Let us define the quantum mutual information [17, 18] for a bipartite state $\rho_{XY} \in S(\mathcal{H}_X \otimes \mathcal{H}_Y)$ by

$$I_{\rho_{XY}}(X;Y) := H(X) + H(Y) - H(XY), \quad (17)$$

where $H(X)$, $H(Y)$, and $H(XY)$ are the von Neumann entropy, $H(\rho) := -\text{Tr} [\rho \log \rho]$, of the corresponding states $\rho_X = \text{Tr}_Y \rho_{XY}$, $\rho_Y = \text{Tr}_X \rho_{XY}$, and $\rho_{XY}$, respectively. Hereafter, the subscript $\widetilde{\rho}$ is asked if the state is fixed and no confusion is likely to arise. It is widely known that the quantum mutual information is described by an isometry $\rho$:

Note that the freedom in purifications is essentially described by an isometry $V_R$ acting on $\mathcal{H}_R$ so that another purification is given by $\rho_{RA} = (V_R \otimes I_A)\rho_{RA}(V_R \otimes I_A)^*$. Hence, we can see that (21) is independent of a specific realization of purifications, since the quantum relative entropy is kept invariant under the action of isometries as described in (16).

It is also clear that the monotonicity of the quantum relative entropy (13) yields the data processing inequality [18] for the quantum mutual information (21),

$$I(\rho_A, E) \geq I(\rho_A, E) \geq I(\rho_A, F), \quad (22)$$

where $F : \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_C)$ is an arbitrary further quantum operation. From Proposition 1, the first equality of (22) holds iff

$$R_{RB \otimes RA}(I_R \otimes E)(\rho_{RA}) = \rho_{RA}, \quad (23)$$

where $R_{RB \otimes RA}$ is the reverse operation defined in (14). Note that the above condition is also written as

$$R_{RB \otimes RA}(I_R \otimes E)(\rho_{RA}) = \rho_{RA}, \quad (24)$$

which is demonstrated by Hayden et al. [35]. They discussed the equivalence between the first equality of (22) and (24) with an explicit construction of the recovery operation. Now we have the following lemma, which is essential for later discussions.

**Lemma 1.** For a quantum operation $E : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$, the following three conditions on the reversibility and the vanishing property are, respectively, equivalent.

(a) $E$ is reversible (resp. vanishing) w.r.t. $\mathcal{S}(\mathcal{H}_A)$.

(b) $\forall \rho_A \in \mathcal{S}(\mathcal{H}_A)$, $I(\rho_A, E) = I(\rho_A, I_A)$ (resp. = 0).

(c) $\exists \rho_A > 0$, $I(\rho_A, E) = I(\rho_A, I_A)$ (resp. = 0).

**Proof.** The equivalence of the above conditions for the reversibility is shown as follows.

(a) $\Rightarrow$ (b): If $E$ is reversible w.r.t. $\mathcal{S}(\mathcal{H}_A)$, then there exists a quantum operation $R : \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_A)$ such that $RE = I_A$. Then letting $F = R$ in (22), we have (b).

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): For the state $\rho_A > 0$ given in (c), we have

$$I(\rho_A, E) = I(\rho_A, I_A) \quad \Rightarrow (I_R \otimes R_{RA})(I_R \otimes E)(\rho_{RA}) = \rho_{RA} \quad (25)$$

where (25) follows from Proposition 1 and (24), and (26) follows from the one-to-one parameterization (10). Now (26) implies (a).

Next, we turn to the vanishing condition.

(a) $\Rightarrow$ (b): If $E$ is vanishing w.r.t. $\mathcal{S}(\mathcal{H}_A)$, then it is nothing but the composition map of the trace operation on $\mathcal{H}_A$ and the creation of a state $\rho_0 \in \mathcal{S}(\mathcal{H}_B)$. Therefore, we have

$$I(\rho_A, E) = I(\rho_A, I_A) \quad \Longleftrightarrow (I_R \otimes R_{RA})(I_R \otimes E)(\rho_{RA}) = \rho_{RA} \quad (26)$$

which implies (b).

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): For the state $\rho_A > 0$ given in (c), it follows from (15) that

$$I(\rho_A, E) = 0 \quad \Longleftrightarrow (I_R \otimes E)(\rho_{RA}) = \rho_R \otimes \rho_0. \quad (27)$$

Thus, the one-to-one parameterization (10) of $E$ coincides with that of the composition map of the trace operation and the creation of the state $E(\rho_A)$, which implies (a). \[\Box\]

Considering the isometry encoding discussed in (3), Lemma 1 is strengthened as follows.

**Theorem 2.** For a quantum operation $E : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ and a subspace $\mathcal{K}_A \subseteq \mathcal{H}_A$, the following three conditions on the reversibility and the vanishing property are, respectively, equivalent.

(a) $E$ is reversible (resp. vanishing) w.r.t. $\mathcal{S}(\mathcal{K}_A)$.

(b) $\forall \rho_A \in \mathcal{S}(\mathcal{K}_A)$, $I(\rho_A, E) = I(\rho_A, I_A)$ (resp. = 0).

(c) There exists a density operator $\rho_A$ with its support $\mathcal{K}_A$ such that $I(\rho_A, E) = I(\rho_A, I_A)$ (resp. = 0).
Proof. Consider the isometry encoding $C$ discussed in (3) to send quantum states on a Hilbert space $\mathcal{H}_X$ with its dimension $\dim \mathcal{H}_X = \dim K_A$. Then, any $\rho_A \in \mathcal{S}(K_A)$ is identified with a state $\rho_X \in \mathcal{S}(\mathcal{H}_X)$ by the one-to-one correspondence $\rho_A = C(\rho_X)$, and it holds that
\[
I(\rho_X, I_X) = I(\rho_X, C) = I(\rho_A, I_A), \tag{29}
\]
\[
I(\rho_X, E) = I(\rho_A, E), \tag{30}
\]
where the first equality of (29) follows from the reversibility of $C$ and Lemma 1, and the other equalities follow from the one-to-one correspondence $\rho_A = C(\rho_X)$. Applying Lemma 1 with $E$ and $\mathcal{H}_A$ replaced with $E\mathcal{C}$ and $\mathcal{H}_X$, respectively, we obtain the assertion. \qed

As for the reversible conditions, Theorem 2 is just a recast of the famous perfect error-correcting condition by Schumacher and Nielsen [8] using the coherent information,
\[
I_c(\rho_A, E) := H(B) - H(RB). \tag{31}
\]
They proved that
\[
I_c(\rho_A, E) = H(\rho_A) \tag{32}
\]
holds iff there exists a quantum operation $\mathcal{R} : \mathcal{S}(\mathcal{H}_B) \rightarrow \mathcal{S}(\mathcal{H}_A)$ such that
\[
(I_R \otimes \mathcal{R}\mathcal{E})(\rho_{RA}) = \rho_{RA}. \tag{33}
\]
The equivalence of their condition (32) and Theorem 2 is verified by
\[
I(\rho_A, E) = H(R) + I_c(\rho_A, E), \tag{34}
\]
\[
I(\rho_A, I_A) = H(R) + H(\rho_A), \tag{35}
\]
where (35) follows from $H(RA) = 0$ (see Ref. [35]). It is interesting to observe from Theorem 2, (34), and $H(R) = H(A)$ that the vanishing conditions are equivalent to
\[
I_c(\rho_A, E) = -H(\rho_A). \tag{36}
\]
Note that the quantum data processing inequality [8] for the coherent information follows from (22), i.e.,
\[
H(\rho_A) \geq I_c(\rho_A, E) \geq I_c(\rho_A, F\mathcal{E}). \tag{37}
\]
Using relations [7, 10] between the entanglement fidelity and the average fidelity, Schumacher and Nielsen [8] also showed the equivalence between the subspace transmission and the entanglement transmission, that is, the equivalence between the condition (a) in Theorem 2 and the condition (33) for some $\rho_A \in \mathcal{S}(K_A)$. One of the findings in Theorem 2 lies in a simple exposition of the equivalence between them, as well as providing the clear approach to treat the perfect error-correcting condition. In our approach, the meaning of the entanglement fidelity,
\[
F_c(\rho_A, \mathcal{R}\mathcal{E}) := \langle \Phi_{\rho_A} | (I_R \otimes \mathcal{R}\mathcal{E}) | \Phi_{\rho_A} \rangle, \tag{38}
\]
should be translated into the measure of how close the quantum operation $\mathcal{R}\mathcal{C}$ is to the identity operation $I_A$, at least on the support of $\rho_A$, in the sense of the one-to-one affine parameterization (10) [29, 30].

It is remarked that, in Ref. [48], the condition (c) on the vanishing property in Lemma 1 was used as the unauthorized condition for quantum secret sharing schemes [36, 49, 50].

VI. TRADEOFF BETWEEN TWO PARTIES

In this section, we explore the tradeoff between two parties for the reversibility of quantum operations, and provide a simple proof of Theorem 1. The tradeoff for the reversibility can be clearly seen through the following theorem. Originally, the equality (40) in the following theorem was given by Cerf [20] in the case of error-correcting schemes.

**Theorem 3.** Given a quantum $\mathcal{E}_{BC} : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B \otimes \mathcal{H}_C)$, let $\mathcal{E}_B := \text{Tr}_C \mathcal{E}_{BC}$ and $\mathcal{E}_C := \text{Tr}_B \mathcal{E}_{BC}$.

(a) It holds for any $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ that
\[
I(\rho_A, I_A) \geq I(\rho_A, \mathcal{E}_B) + I(\rho_A, \mathcal{E}_C). \tag{39}
\]
(b) If $\mathcal{E}_{BC}$ is a pure state channel w.r.t. a subspace $K_A \subset \mathcal{H}_A$ and reversible w.r.t. $\mathcal{S}(K_A)$, then it holds for any $\rho_A \in \mathcal{S}(K_A)$ that
\[
I(\rho_A, I_A) = I(\rho_A, \mathcal{E}_BC) = I(\rho_A, \mathcal{E}_B) + I(\rho_A, \mathcal{E}_C). \tag{40}
\]

**Proof.** First, we show the assertion (b). The first equality of (40) follows from the reversibility of $\mathcal{E}_{BC}$ and “(a) $\Rightarrow$ (b)” of Theorem 2. If $\mathcal{E}_{BC}$ is a pure state channel w.r.t. $K_A$ and reversible w.r.t. $\mathcal{S}(K_A)$, the output of $\rho_A \in \mathcal{S}(K_A)$ is written as $\mathcal{E}_{BC}(\rho_A) = W \rho_A W^\dagger$ by a partial isometry $W : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_C$ satisfying $(\text{Ker} W)^\perp = K_A$. Therefore, the output of the purification,
\[
\rho_{RBC} := (I_R \otimes \mathcal{E}_{BC}) | \Phi_{\rho_A} \rangle | \Phi_{\rho_A} \rangle^\dagger = (I_R \otimes W) | \Phi_{\rho_A} \rangle | \Phi_{\rho_A} \rangle (I_R \otimes W)^*, \tag{41}
\]
is also a pure state. Then the purity of $\rho_{RBC}$ yields the following equality [20],
\[
2H(R) = I(R; B) + I(R; C), \tag{42}
\]
which is verified by using $H(B) = H(RC)$ and $H(RB) = H(C)$ as
\[
I(R; B) + I(R; C) = H(R) + H(B) - H(RB) + H(R) + H(C) - H(RC) = 2H(R). \tag{43}
\]
On the other hand, we have
\[
I(R; BC) = H(R) + H(BC) - H(RBC) = 2H(R),
\]
which follows from \(H(R) = H(BC)\) and \(H(RBC) = 0\). Now (40) follows from (42) and (44).

The assertion (a) is shown as follows. Let us consider the Stinespring dilation \([51]\) \(\mathcal{E}_{BC}(\rho_A) = \text{Tr}_E[V\rho AV^*]\), where \(V\) is an isometry from \(\mathcal{H}_A\) to the composite system of \(\mathcal{H}_B, \mathcal{H}_C\), and an environment system \(\mathcal{H}_E\). Let \(\mathcal{E}_{BC}(\rho_A) := V\rho AV^*\), then the quantum operation \(\mathcal{E}_{BC}\) is a pure state channel w.r.t. \(\mathcal{H}_A\) and reversible w.r.t. \(\mathcal{S}(\mathcal{H}_A)\). Therefore, applying the above arguments, we have
\[
I(\rho_A, I_A) = I(\rho_A, \mathcal{E}_B) + I(\rho, \mathcal{E}_C) \geq I(\rho_A, \mathcal{E}_B) + I(\rho, \mathcal{E}_C),
\]
where the last inequality follows from the monotonicity (22).

A proof of Theorem 1 immediately follows from Theorem 2 and Theorem 3 as follows. First, we show the assertion (b) of Theorem 1. Let us take an arbitrary state \(\rho_A \in \mathcal{S}(\mathcal{K}_A)\) which has the support \(\mathcal{K}_A\). Suppose that \(\mathcal{E}_{BC}\) is a pure state channel w.r.t. \(\mathcal{K}_A\) and reversible w.r.t. \(\mathcal{S}(\mathcal{K}_A)\). Then we have
\[
\mathcal{E}_B\text{ is reversible w.r.t. } S(K_A) \iff I(\rho_A, \mathcal{E}_B) = I(\rho_A, I_A) \geq I(\rho_A, \mathcal{E}_B) + I(\rho, \mathcal{E}_C),
\]
where (46) and (48) follow from Theorem 2, and (47) follows from (40).

The assertion (a) of Theorem 1 is shown in the same way as
\[
\mathcal{E}_B\text{ is reversible w.r.t. } S(K_A) \iff I(\rho_A, \mathcal{E}_B) = I(\rho_A, I_A) \iff I(\rho_A, \mathcal{E}_C) = 0 \iff \mathcal{E}_C\text{ is reversible w.r.t. } S(K_A),
\]
where (50) follows from (39), and (51) follows from the positivity (15) of the quantum relative entropy. It was also shown in Ref. [48] that (49) implies (51) by a different method.

**VII. ERROR-CORRECTING CONDITION ON KRAUS OPERATORS**

We shall demonstrate that Theorem 1 immediately yields the quantum error-correcting condition on Kraus operators which is proved independently by Knill and Laflamme [10] and by Bennett et al. [9],

**Proposition 2** ([9, 10]). Let \(\mathcal{E} : S(\mathcal{H}_A) \to S(\mathcal{H}_B)\) be a quantum operation represented by the Kraus representation [52],
\[
\mathcal{E}(\rho) = \sum_k E_k \rho E_k^*,
\]
and \(K_A \subseteq H_A\) be a subspace. Then the following conditions are equivalent, where the projection onto the subspace \(K_A\) is denoted by \(P_{K_A}\).

(a) \(\mathcal{E}\) is reversible w.r.t. \(S(K_A)\).

(b) For each pair of indices \((k,l)\), there exists \(c_{kl} \in \mathbb{C}\) such that \(P_{K_A} E_k^* E_l P_{K_A} = c_{kl} P_{K_A}\).

Proof. Let us consider the Stinespring dilation \(\mathcal{E}_{BE}(\rho_A) = \text{Tr}_E[V\rho AV^*]\), where \(V\) is an isometry from \(\mathcal{H}_A\) to the composite system of \(\mathcal{H}_B\) and an environment system \(\mathcal{H}_E\), i.e.,
\[
V : |i\rangle \in \mathcal{H}_A \mapsto \sum_l E_l |i\rangle \otimes |l\rangle \in \mathcal{H}_B \otimes \mathcal{H}_E.
\]
Let \(\mathcal{E}_{BE}(\rho_A) := V\rho AV^*\) and \(\mathcal{E}_E := \text{Tr}_E \mathcal{E}_{BE}\), then the quantum operation \(\mathcal{E}_{BE}\) is a pure state channel and reversible w.r.t. \(S(\mathcal{H}_A)\), and hence, it follows from Theorem 1 that the condition (a) above is equivalent to the condition,

(c) \(\mathcal{E}_E\) is vanishing w.r.t. \(S(K_A)\).

We shall show that the condition (c) is equivalent to the condition (b) above. Using (54), \(\mathcal{E}_E\) is explicitly written as
\[
\mathcal{E}_E(\rho_A) = \sum_k \sum_l \text{Tr}[\rho_A E_k^* E_l] |l\rangle \langle k|.
\]
Therefore, (c) is equivalent to
\[
\forall (k,l), \exists c_{kl} \in \mathbb{C}, \forall \rho_A \in S(K_A), \text{Tr}[\rho_A E_k^* E_l] = c_{kl}.
\]
Now let \(\{|i\rangle\}_{i=1}^{\dim K_A}\) be a complete orthonormal basis on \(K_A\). Then, by the polar identity (7), we can show that (56) is also equivalent to the existence of \(c_{kl} \in \mathbb{C}\) such that
\[
\langle i| E_k^* E_l |j\rangle = c_{kl} \delta_{ij} \quad (i, j = 1, \ldots, \dim K_A),
\]
which implies that the matrix components of \(P_{K_A} E_k^* E_l P_{K_A}\) are the same as those of \(c_{kl} P_{K_A}\). Hence (57) is equivalent to the condition (b) above.

**VIII. CONCLUDING REMARKS**

We have laid a simple and unifying method to show the perfect error-correcting condition based on the quantum mutual information (Theorem 2). In our approach, the one-to-one affine parameterization of quantum operations played an important role to show the equivalence
between the subspace transmission and the entanglement transmission, as well as the meaning of the entanglement fidelity as the measure of closeness between a quantum operation and the identity operation. We have also revisited the no-cloning and no-deleting theorem (Theorem 1) based on the information-theoretical tradeoff (Theorem 3) between two parties for the reversibility of quantum operations, and demonstrated that the no-cloning and no-deleting theorem leads to the perfect error-correcting condition on Kraus operators.

In this paper, the study of the error-correcting schemes was restricted to those with perfect reconstruction of the encoded states, while it is important to extend our approach to the error-correcting schemes allowing small errors or asymptotically vanishing errors. In the process of extending our results, the following pair of equalities given in Ref. [53],

\[ H(\rho_A) = \frac{1}{2} \{ I(\rho_A, \mathcal{E}) + I(\rho_A, \mathcal{E}_E) \}, \quad (58) \]

\[ I_c(\rho_A, \mathcal{E}) = \frac{1}{2} \{ I(\rho_A, \mathcal{E}) - I(\rho_A, \mathcal{E}_E) \}, \quad (59) \]

will play a crucial role, where (58) is just the equality (42) by Cerf [20] applied to the case in the proof of Proposition 2, and (59) is verified in the same way. Note that (58) is closely related to the reversibility of the quantum operation \( \mathcal{E} \). On the other hand, (59) will establish some relations between the quantum mutual information and the coherent information related to the various capacities [21–23, 27, 28] of quantum operation \( \mathcal{E} \). These developments are given in the subsequent paper by the author.

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APPENDIX

The use of the one-to-one parameterization [29, 30, 32] (see also [31]) of quantum operations is essential in this paper. In this appendix, for readers’ convenience, we show that the map \( \mathcal{E} \in \mathcal{QO} \mapsto M(\mathcal{E}) \in \mathcal{M} \) defined in (5) actually establishes the one-to-one affine parameterization.

First, the fact that \( M(\mathcal{E}) \in \mathcal{M} \) if \( \mathcal{E} \in \mathcal{QO} \) is verified as follows. The first requirement \( M(\mathcal{E}) \geq 0 \) for \( \mathcal{M} \) follows from the complete positivity of the quantum operation \( \mathcal{E} \), and the second requirement \( \text{Tr}_B[M(\mathcal{E})] = \frac{1}{d} I_R \) is a consequence of the trace preserving condition on \( \mathcal{E} \). Actually, we have

\[ \text{Tr}_B[M(\mathcal{E})] \approx \frac{1}{d} \left( \text{Tr}[\mathcal{E}(\langle i | j \rangle)] \right)_{ij} \approx \frac{1}{d} I_R. \quad (60) \]

Injectivity of the map \( M(\mathcal{E}) \) is clear from the representation (6). For \( \mathcal{E}, \mathcal{F} \in \mathcal{QO} \) and \( t \in [0, 1] \), we have

\[
M(t\mathcal{E} + (1-t)\mathcal{F}) = (tI_R \otimes (t\mathcal{E} + (1-t)\mathcal{F}))(\langle \Phi | \Phi \rangle) = t(I_R \otimes \mathcal{E})(\langle \Phi | \Phi \rangle) + (1-t)(I_R \otimes \mathcal{F})(\langle \Phi | \Phi \rangle) = tM(\mathcal{E}) + (1-t)M(\mathcal{F}),
\]

(61)

which shows that \( M(\mathcal{E}) \) is an affine mapping.

Conversely, by taking the \((i, j)\)-block of \( M \in \mathcal{M} \) as

\[ E_M(\langle i | j \rangle) = d \cdot \text{Tr}_R[(|i \rangle \langle j | \otimes I_B)^* M], \quad (62) \]

we obtain the corresponding quantum operation \( \mathcal{E}_M \) from \( M \in \mathcal{M} \). In fact, we can show that \( \mathcal{E}_M \) actually yields a quantum operation as follows. From the condition \( M \geq 0 \) of \( \mathcal{M} \), we obtain the following decomposition

\[ M = \sum_k |\Psi_k \rangle \langle \Psi_k| \quad (63) \]

by using \( |\Psi_k \rangle \in \mathcal{H}_R \otimes \mathcal{H}_B \), for example, which is given by the spectral decomposition of \( M \). Here let \( \mathcal{L}(\mathcal{H}_A \rightarrow \mathcal{H}_B) \) be the totality of linear operators from \( \mathcal{H}_A \) to \( \mathcal{H}_B \). Then the map

\[ E \in \mathcal{L}(\mathcal{H}_A \rightarrow \mathcal{H}_B) \mapsto (I \otimes E) \langle \Phi | \Phi \rangle \in \mathcal{L}(\mathcal{H}_R \otimes \mathcal{H}_B) \]

(64)

defines an one-to-one linear correspondence between \( \mathcal{L}(\mathcal{H}_A \rightarrow \mathcal{H}_B) \) and \( \mathcal{L}(\mathcal{H}_R \otimes \mathcal{H}_B) \). Therefore, there exists an operator \( E_k \in \mathcal{L}(\mathcal{H}_A \rightarrow \mathcal{H}_B) \) for each \( k \) such that \( |\Psi_k \rangle = (I_R \otimes E_k)|\Phi \rangle \), and hence, (63) is written as follows

\[
M = \sum_k (I_R \otimes E_k) |\Phi \rangle \langle \Phi| (I_R \otimes E_k)^* = \frac{1}{d} \sum_i \sum_j |i \rangle \langle j | \otimes \left( \sum_k E_k |i \rangle \langle j | E_k^* \right). \quad (65)
\]

The above formula implies that \( \mathcal{E}_M \) is represented by the Kraus representation,

\[ \mathcal{E}_M(\rho_A) = \sum_k E_k \rho_A E_k^*, \quad (66) \]

which ensures the complete positivity of \( \mathcal{E}_M \). The trace preserving requirement for \( \mathcal{E}_M \) easily follows from \( \text{Tr}_B[M] = \frac{1}{d} I_R \), i.e.,

\[
\text{Tr}_B \mathcal{E}_M(\langle i | j \rangle) = d \cdot \text{Tr}_B \text{Tr}_R[(|i \rangle \langle j | \otimes I_B)^* M],
\]

\[ = d \cdot \text{Tr}_B(|i \rangle \langle j | \text{Tr}_B[M]) = \delta_{ij}. \quad (67)\]
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