THREE CONJECTURES ON LAGRANGIAN TORI IN THE PROJECTIVE PLANE

Nikolay A. Tyurin

Abstract. In this paper we extend the discussion on Homological Mirror Symmetry for Fano toric varieties presented in [HV] to more general case of monotone symplectic manifolds with real polarizations. We claim that the Hori–Vafa prediction, proven in [CO] for toric Fano varieties, can be checked in much more wider context. Then the notion of Bohr–Sommerfeld with respect to the canonical class lagrangian submanifold appears and plays an important role. The discussion presents a bridge between Geometric Quantization and Homological Mirror Symmetry programmes both applied to the projective plane in terms of its lagrangian geometry. Due to this relation one could exploit some standard facts known in GQ to produce results in HMS.

Introduction

Lagrangian geometry of compact symplectic manifolds remains to be a subject where not too much is known. Even in the simplest case of two dimensional compact symplectic manifolds (= Riemann surfaces) where the lagrangian condition degenerates and any 1-dimensional submanifold is lagrangian, the classification problems (up to hamiltonian isotopy or up to symplectomorphism) are solved just for certain special cases including the case of the projective line. In dimension 4 one doesn’t know which 2-dimensional manifolds appear as lagrangian submanifolds, and the discussion on the existence of a lagrangian Klein bottle in Kahler surfaces was not finished yet. One believes that the projective plane \( \mathbb{CP}^2 \) admits only lagrangian tori as orientable lagrangian submanifolds and real projective planes as non orientable ones plus some artificial types produced by hands using lagrangian surgery near the intersections of lagrangian tori and real projective planes which gives new topological types of lagrangian submanifolds such as \( T^2 \# \mathbb{RP}^2 \) etc. On the other hand, the classification of lagrangian tori even for \( \mathbb{CP}^2 \) is not completed: one knows two types of lagrangian tori (the Clifford type and the Chekanov type) and it seems that these types belong to different classes of the classification up to hamiltonian isotopy (one refers here to a paper of Chekanov and Schlenk which is coming but not published yet). And of course it doesn’t mean that the set of equivalence classes is exhausted by these Clifford and Chekanov types.

At the same time lagrangian geometry is highly desired for certain approaches to Mirror Symmetry conjecture. According to Homological Mirror Symmetry programme, proposed by M. Kontsevich, [1], the symmetry is an equivalence between

---

1This work was partially supported by RFBR, grants 05 - 01 - 01086, 05 - 01 - 00455 and 07 - 01 - 92211.
the bounded derived category of coherent sheaves over a given algebraic manifold $M$ and certain Fukaya - Floer category of lagrangian submanifolds of its mirror partner $\text{mir}(M)$ where the last one is a symplectic manifold. The objects of the Fukaya - Floer category are presented by lagrangian submanifolds up to hamiltonian isotopy, and the morphisms are given by the Floer cohomology of pairs of objects. Despite of the fact that a rigorous definition of the Floer cohomology doesn’t exist in general, one understands that the full category of lagrangian submanifolds up to hamiltonian isotopy is too big anyway and this implies certain restrictions on the type of the lagrangian submanifolds taken in the specified constructions. Concerning the 4-dimensional case one adopted a variant of the generic Fukaya – Floer theory where an additional data is exploited namely the structure of the Lefschetz pencil on a given symplectic manifold. Then the Fukaya – Seidel category plays the role of the counterpartner of the bounded derived category of coherent sheaves; in the Fukaya – Seidel category one takes not all lagrangian submanifold but vanishing cycles only.

Another type of restriction is proposed for the case of toric Fano varieties: for such an $X$ one takes a toric fibration and considers the fibers $\{S_\alpha\}$ (which are lagrangian submanifolds in $X$) with non trivial Floer cohomology $FH^*(S_\alpha, S_\alpha; \mathbb{C}) \neq 0$. Then the desired category is constructed over the set of the fibers, satisfy this non triviality condition. For this case the prediction of K. Hori and C. Vafa says that the number of such fibers is finite; it should be the euler characteristic of the mirror partner, see [2]. This prediction was proven in [3] modulo certain assumptions:instead of the Floer cohomology one computes the Bott – Morse version; instead of a generic almost complex structure one takes the complex structure of the toric variety or its small hamiltonian deformations; the answer looks very familiar in the framework of Geometric Quantization of toric varieties – the desired fibers are distinguished by certain integrality condition, see formula (10.6) from [3]. And this integrality condition in the toric framework means that one deals with the Bohr – Sommerfeld fibers.

But the main idea of Mirror Symmetry is to relate the algebraic geometry of a given variety to the symplectic geometry of its mirror partner so the answer on the right hand side must be independent on the choice of the compatible complex structure. One exploits certain sufficiently generic almost complex structure to construct the objects like the Gromov invariants or the Floer cohomology, and we know that an integrable complex structure is too far to be generic in this setup (many examples of the answers which must be corrected are known in the gauge theories etc.); moreover the complex structure of a toric variety is even more special (any toric variety is rigid in the class of toric varieties but of course can be deformed to a non toric algebraic variety). Thus one needs to extend the setup of [3] in the way which would be more independent on the complex structure choice. On the other hand, passing in this way one can see that the results of [3] can be understood as more general facts adopted to the specific situation of the toric Fano varieties.

What would be a possible ”more general setup”? Instead of Fano varieties we consider monotone simply connected compact symplectic manifold. Instead of toric varieties we consider symplectic manifolds with real polarizations. And after this translations we reach the situation which is very well known in Geometric Quantization, see f.e. [4], [5], [6].
Geometric Quantization is a set of recepies attaching to a given symplectic manifold certain Hilbert spaces together with homomorphisms of the Lie algebra of smooth functions on this given manifold to the spaces of self adjoint operators acting on the Hilbert spaces (or more generally, it attaches to a given symplectic manifold certain algebraic variety, [7]). One recepie from the set can be applied in the case when our given symplectic manifold admits an additional structure — a real polarization, which is a lagrangian fibration of our given symplectic manifold. In this case the Hilbert spaces are spanned by the fibers which satisfy some specific condition — so - called Bohr - Sommerfeld condition of different levels. And the crucial fact here is the following: in the compact case the number of Bohr - Sommerfeld fibers is finite if the lagrangian fibration is sufficiently good. The toric Fano case is included by this class of ”sufficiently good” lagrangian fibration and the construction from [3] ensures that for this case there is the coincidence between Bohr – Sommerfeld fibers and the fibers with non trivial adopted Floer – Bott – Morse cohomology. To avoid the ambiguity with different versions of the Floer cohomology we will universalize the story having in mind the following general distribution: if a lagrangian submanifold is displaceable then it seems that for any version of the Floer cohomology theory it should have trivial Floer cohomology; and if a lagrangian submanifold is monotone then it seems that for any definition it should have non trivial cohomology. Recall that S is displaceable if it can be moved by some hamiltonian isotopy $\phi_H$ such that the intersection $S \cap \psi_H(S) = \emptyset$. Of course, the distribution is not complete in general, but at least for the basic example of montone simply connected symplectic manifold with real polarization — the projective plane — it seems to be exhaustive. The present text contains results about the fibers of a real polarization with regular degeneration: for a generic monotone symplectic manifold the number of monotone fibers is always finite (Theorem 2); for any real polarization with regular degeneration of the projective plane any Bohr – Sommerfeld with respect to the canonical class fiber is monotone and any other fiber is displaceable (Theorem 3). Let us emphazize that it is true for any real polarization with regular degeneration, not just for toric one. We do not use the toric structure in the constructions below, and in parallel present several (naive) conjectures which extend the statement of Theorem 3 to the case of any lagrangian torus in $\mathbb{CP}^2$, not just for the fibers. One could not expect today that these ones are true but the work in this direction is continued.

The discussion below follows the idea that the Bohr - Sommerfeld condition and the non triviality condition for the Floer cohomology are somehow related, and if it is indeed the case we would get a way how to proceed in HMS using known facts and constructions in GQ. For the author the main reason to study the question is that it would be a realization of the ideology, proposed by Andrey Tyurin, which claims that Mirror Symmetry and Geometric Quantization are relatives, [8].

I would like to thank the Max Planck Institute (Bonn) for the hospitality, excellent working condition and the possibility to communicate with high level mathematicians. My gratitude goes to F. Hirzebruch, A. Gorodentsev, P. Pushkar, A. Kokotov, P. Moree and many others. I have to thank D. Auroux and D. Orlov for constant help and remarks. The last but not least thanks go to the staff of the Max - Planck - Institute fur Matematik.
§1. **Bohr - Sommerfeld conditions**

Consider \((M, \omega)\) — a compact simply connected symplectic manifold of real dimension \(2n\) and suppose that the cohomology class \([\omega] \in H^2(M, \mathbb{R})\) is integer. It means that there exists a complex line bundle \(L \to M\) with the first Chern class \(c_1(L) = [\omega]\). Choosing a hermitian structure on \(L\), the space of hermitian connection \(\mathcal{A}_h(L)\) is defined. There exists unique up to gauge transformations hermitian connection \(a \in \mathcal{A}_h(L)\) such that its curvature form \(F_a\) is proportional to \(\omega\):

\[
F_a = 2\pi i \omega.
\]

The pair \((L, a)\) is usually called the *prequantization data*. Consider any integer number \(k \in \mathbb{Z}\) and the corresponding power \(L^k\). The space \(\mathcal{A}_h(L^k)\) contains unique up to gauge transformations hermitian connection \(a_k\) such that its curvature form is proportional to \(\omega\):

\[
F_{a_k} = 2k\pi i \omega.
\]

The pair \((L^k, a_k)\) is called the prequantization data of level \(k\).

Let \(S \subset M\) be a lagrangian submanifold. This means that \(S\) has dimension \(n\) and the restriction \(\omega|_S\) vanishes identically. Then restricting the pair \((L^k, a_k)\) to \(S\) one gets a trivial line bundle with a flat connection since the curvature form vanishes being proportional to the symplectic form. Therefore lagrangian submanifolds can be distinguished using the data of flat connections.

We say that a lagrangian submanifold \(S \subset M\) is **Bohr - Sommerfeld of level** \(k\) if the restricted connection \(a_k|_S\) admits covariantly constant sections.

One could ask whether or not this definition depends on the choices of hermitian structure on \(L\) and a connection \(a\) from the equivalence class of hermitian connections described by the condition \(F_a = 2\pi i \omega\). The point is that the definition is *absolutely universal*: it can be reformulated in our case as follows. Consider \(H_1(S, \mathbb{Z})\) and for each primitive element \(b \in H_1(S, \mathbb{Z})\) consider some representative \(\gamma_b \subset S\). Since \(\pi_1(M)\) is trivial one can find a disc \(D \subset M\) with the boundary \(\partial D = \gamma_b\). It is not hard to see that \(S\) is Bohr - Sommerfeld of level \(k\) if and only if for any \(b \in H_1(S, \mathbb{Z})\) the symplectic area of \(D\) multiplied by \(k\) is integer:

\[
k \cdot \int_D \omega \in \mathbb{Z}.
\]

Note, that we consider the case when \([\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})\) and the last integrality condition doesn’t depend on the choice of particular \(D\). At the same time the last description of the Bohr - Sommerfeld condition doesn’t involve any bundles or connections and therefore the notion is universal. It’s natural to call the numbers

\[
p_k(b) = k \cdot \int_D \omega \mod \mathbb{Z}
\]

the *periods* of our given lagrangian submanifold.

On the other hand the last description relates the local deformations of a given symplectic manifold to the variations of periods of the deformed submanifolds. According to the Darboux - Weinstein theorem, see [9], there exists certain small
tubular neighborhood of a given $S$ such that this neighborhood is symplectomorphic to a neighborhood of the zero section in $T^*S$, endowed with the standard symplectic form. Then local lagrangian deformations of $S$ are presented by graphs of closed 1-forms on $S$. Taking in mind the period description one can see that for a deformation $\psi$ of a given $S$ with the class $[\psi] \in H^1(S, \mathbb{R})$ the periods change as follows

$$p_k(\psi_*(b)) = p_k(b) + k \cdot \psi(b).$$

(1)

Indeed, if we have a loop $\gamma_b$ on the given lagrangian submanifold and then deform it to $\psi(\gamma_b)$ the symplectic area of the tube with boundary $\gamma_b - \psi(\gamma_b)$ is exactly $\psi(b)$ and it implies the formula above.

As a corollary one gets the existence of Bohr - Sommerfeld lagrangian submanifolds of (perhaps, sufficiently big) level $k$ in the case when a single lagrangian submanifold exists.

A variant of the basic definition appears in the case of monotone symplectic manifolds. A symplectic manifold $(M, \omega)$ is called monotone if its canonical class $K_\omega \in H^2(M, \mathbb{Z})$ is proportional to $[\omega]$:

$$K_\omega = k \cdot [\omega].$$

Number $k$ is called the coefficient of monotonicity. For this case for any hermitian structure on $K_\omega$ there exists unique up to gauge transformations hermitian connection $a_{\text{can}}$ with the curvature form proportional to the symplectic form. Then we just repeat the basic definition with respect to $(K_\omega, a_{\text{can}})$. If the restriction $(K_\omega, a_{\text{can}})|_S$ admits covariantly constant sections we say that $S$ is Bohr - Sommerfeld with respect to the canonical class. This specification is reasonable — we will discuss it in the Section 3.

§2. Finitness

Suppose now that we have an additional structure on $M$ — a real polarization. This means that $M$ is fibered over a base $B$ and almost all the fibers are smooth lagrangian. Usually it happens for the phase spaces of completely integrable systems so there exists a commutative sub algebra in the Poisson algebra $(C^\infty(M, \mathbb{R}), \{ , \}_\omega)$ spanned by the set of smooth functions $\{f_1, \ldots, f_n\}$ called integrals such that the differentials $(df_1, \ldots, df_n)$ form a basis in the cotangent space almost everywhere on $M$. The conditions dictate several topological restrictions; the most important for us is that a smooth fiber must be isomorphic to torus.

For a real polarization of $M$ given by some map

$$\pi : M \to B,$$

where $B$ is a convex polytop in $\mathbb{R}^n$, we have the so - called Kodaira – Spencer map; for each regular point $b \in B$ the deformation along the base is reflected somehow by the deformation of the fiber $\pi^{-1}(b) \subset M$ and since the local deformations of a lagrangian submanifold are described by closed 1-forms it induces a map

$$m_{KS} : T_bB \to H^1(\pi^{-1}(b), \mathbb{R}).$$
Now we claim that for a smooth fiber $\pi^{-1}(b)$ the Kodaira – Spencer map is an isomorphism. To prove this fact let us first mention that the dimensions of the entire spaces are the same:

$$\dim T_b B = n = \dim H^1(T^n, \mathbb{R}).$$

If one suppose that a vector $v \in T_b B$ goes to zero under the map then it would imply that the corresponding small deformation is isodrastic so it preserves the periods of $S_b = \pi^{-1}(b)$. But according to formula (1) it could happen if and only if the corresponding closed 1-form $\psi$ is exact. This means that there exists a smooth function $f$ on $S_b$ such that $\psi = df$. But each smooth function must have at least two critical points on a compact manifold, maximal and minimal. This means that the graph of $\psi$ in this case must intersect our given $S_b$ at least in two points. But it is impossible since the fibers can’t intersect each other. Therefore the Kodaira - Spencer map doesn’t have a kernel and due to the dimensional reason it is an isomorphism.

In Geometric Quantization the approach with real polarization gives the following recipe to construct the Hilbert spaces, see [5]. For a real polarization one takes the fibers which are Bohr - Sommerfeld of level $k$ which are $S_1, ..., S_l, ...$ and forms the linear span

$$\sum_i C < S_i > = \mathcal{H}_k.$$  

The point is that the set of such fibers is discrete anyway and finite if the real polarization has sufficiently good degenerations. Indeed, the discretness follows just from the fact that the Kodaira - Spencer map is an isomorphism. Now what are these ”sufficiently good degenerations”? They appear for example in the case of toric varieties. This means that the degenerations are regular so if $B$ is a convex polytop in $\mathbb{R}^n$ then the fibers over the inner part are smooth; the picture over a $n-1$ - dimensional face is again a smooth symplectic manifold fibered over this face with smooth lagrangian fibers over the inner part of this face (which are smooth lagrangian $n-1$ - dimensional tori), etc. In this situation we have that the number of smooth Bohr - Sommerfeld lagrangian fibers is finite. Indeed, the discrete set can have a limiting point only on the boundary. Suppose that the limiting point corresponds to a smooth $n - 1$ - dimensional torus placed over the inner part of a $n-1$ - dimensional face. The preimage of this $n-1$ -dimensional face is a symplectic submanifold $M_1 \subset M$. The limiting process implies that for our fixed $k$ (the level of the Bohr - Sommerfeld property) the normal bundle $N_{M_1/M}|_{S_{l,m}}$ contains a serie of shrinking discs bundles each of them consists of discs of constant symplectic area such that this area multiplied by $k$ is integer. This implies that starting with some sufficiently small disc bundle the symplectic area of the fiber discs must be trivial. But it is impossible since the normal bundle $N_{M_1/M}$ is symplectic and each disc must have nontrivial symplectic volume. Thus the limiting point on the inner part of a $n - 1$ - dimensional face can’t exist. Now suppose that the limiting point is more degenerated. In this situation one can use some natural shift of the chain of lagrangian submanifold resulting with a generic limiting point which already lies on certain $n - 1$ - dimensional face.

Therefore we get the following
Theorem 1. Let $X$ be a simply connected symplectic manifold and $\pi : X \to B$ be a real polarization with regular degeneration. Then for any level $k \in \mathbb{Z}$ the set of Bohr–Sommerfeld lagrangian fibers of level $k$ is finite.

§3. Monotonicity

As we’ve already mentioned in Section 1 there is a variant of the Bohr–Sommerfeld condition natural in the setup of monotone symplectic manifolds. In the Floer cohomology theory one of the most important case is when a given lagrangian submanifold is monotone. For the specialization of such a manifold let us remind first what the Maslov index is. Since we are discussing below lagrangian tori we consider the case of orientable lagrangian submanifolds.

Let $S \subset M$ is an orientable lagrangian submanifold of a simply connected symplectic manifold $(M, \omega)$. Choose any almost complex structure compatible with $\omega$ and realize the anticanonical bundle $K^{-1}_\omega$ as the determinant of the hermitian bundle $(TM, I, \omega)$. For any loop $\gamma \subset S$ choose a disc $D \subset M$ with the boundary $\partial D = \gamma$, and consider a trivialization of the anticanonical bundle $K^{-1}_\omega|_D$ restricted to $D$. This trivialization is unique up to gauge transformations and since $D$ is simply connected the degree of these transformations computed on the boundary must be trivial. Due to the realization this trivialization is presented by a non vanishing on $D$ polyvector field $\eta$ of the type $(n, 0)$. On the other hand the boundary of the disc carries a non vanishing real polyvector field $\theta$ which is given by the determinant of $TS$ restricted to $\gamma$. Thus the hermitian pairing of $\eta$ and $\theta$ gives a map

$$\phi_D : \gamma \to \mathbb{C}^*,$$

since it is not hard to see that the lagrangian condition implies that $<\eta, \theta>_h$ never vanishes. The degree of this map

$$\mu(\gamma, D) = <\phi_D^* h; [\gamma]>$$

(where $h$ is the generator of $H^1(\mathbb{C}^*, \mathbb{Z})$ and $[\gamma] \in H_1(\gamma, \mathbb{Z})$ here is the fundamental class) is an integer number which doesn’t depend on the choice of the almost complex structure. Moreover, it doesn’t depend on the particular choice of $D$ in the same class from $\pi_2(M, S)$ with the image at $[\gamma] \in \pi_1(S)$ under the canonical homomorphism. For another disc $D'$ with the same boundary $\gamma$ the value $\mu(\gamma, D')$ can be computed in the following way:

$$\mu(\gamma, D') = \mu(\gamma, D) + <K^{-1}_\omega; [S^2 = D \cup D']>,$$

hence if $D'$ is homotopy equivalent to $D$ then the numbers must be the same. At the same time the number doesn’t depend on the particular choice of $\gamma$ in a given class $[\gamma] \in \pi_1(S)$. Totally it shows that we have a map

$$\mu : \pi_2(M, S) \to \mathbb{Z},$$

which is called the Maslov index. For any simply connected symplectic manifold and any lagrangian submanifold the index exists and moreover it is invariant under any
lagrangian deformations. It easily follows from its definition — it must be invariant under any continuous deformations. In the case when the ambient symplectic manifold has small second cohomology \((\text{Pic} \ M = \mathbb{Z})\) the index can be reduced to a numerical correspondence
\[
\mu : H_1(S, \mathbb{Z}) \to \mathbb{Z}(\text{mod } \deg K^{-1}_\omega)
\]
which is often called the Maslov number.

Now, a lagrangian submanifold \(S \subset M\) is monotone if there exists an integer number \(k\) such that for any loop \(\gamma \subset S\) and any disc \(D \subset M, \partial D = \gamma\) one has
\[
\mu(\gamma, D) = k \cdot \int_D \omega,
\]
where \(\mu\) is the Maslov index of the loop \(\gamma\) with respect to the disc \(D\). The existence of a monotone lagrangian submanifold imposes strong restrictions on the topology of \(M\) itself — it must be monotone itself. And if it is monotone then it is reasonable to exploit the notion of Bohr - Sommerfeld with respect to canonical bundle lagrangian submanifolds. It is not hard to see that a lagrangian submanifold is monotone only if it is Bohr - Sommerfeld with respect to the canonical class. Indeed, the identity (2) is possible only in the case when for each \(\gamma, D\) the symplectic area of \(D\), multiplied by \(k\), is an integer number. But it is exactly our Bohr – Sommerfeld condition with respect to the canonical class.

On the other hand, for a Bohr - Sommerfeld with respect to the canonical bundle lagrangian submanifold in a monotone simply connected symplectic manifold one can define a characteristic class which is called the universal Maslov class, see [10]. Leaving aside its first definition, we define it here as follows: for a given Bohr - Sommerfeld with respect to the canonical class \(S \subset M\) with \(K^{-1}_\omega = k \cdot [\omega]\) for any loop \(\gamma\) and any disc \(D, \partial D = \gamma\) consider the difference:
\[
m_S(\gamma, D) = \mu(\gamma, D) - k \cdot \int_D \omega \in \mathbb{Z}.
\]
Then the value of \(m_S\) doesn’t depend on the choice of \(D\). Moreover, this numerical correspondence is linear and consequently \(m_S\) is a cohomology class from \(H^1(S, \mathbb{Z})\). Let’s remind that this class is correctly defined if and only if our given lagrangian submanifold is Bohr - Sommerfeld with respect to the canonical bundle. And since this property is stable with respect to hamiltonian deformations only the resulting cohomology class is invariant under hamiltonian deformations only, see [10].

From this description we get tautologically that \(S\) is monotone if and only if it is Bohr - Sommerfeld with respect to the canonical class and its universal Maslov class vanishes
\[
m_S = 0.
\]

Resuming the discussion of this Section, we have the following proposition:

**Theorem 2.** Let \(X\) be a simply connected monotone symplectic manifold, and \(\pi : X \to B\) be a real polarization with regular degeneration. Then the number of monotone lagrangian fibers is finite.

The prove is straightforward.
§4. Lagrangian tori in $\mathbb{CP}^2$

The resting part of the paper discusses the case of the basic example of the monotone simply connected symplectic manifold — the projective plane $\mathbb{CP}^2$.

Take the projective plane $\mathbb{CP}^2$ with the standard Fubini — Study Kahler form $\omega$ which we consider as a symplectic form. Thus the cohomology class $[\omega]$ is integer and presents a generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$. As a symplectic manifold, it is monotone, $K = -3[\omega]$. We are interested in lagrangian fibrations of $\mathbb{CP}^2$ to verify the following naive conjecture which can be attached to any simply connected monotone symplectic manifold endowed with a real polarization with regular degeneration: if a smooth lagrangian fiber is displacable it is not Bohr–Sommerfeld with respect to the canonical class, and if this fiber is Bohr–Sommerfeld with respect to the canonical class it is monotone (one could call this conjecture Extremely Naive, or ENC for short). Below we show that this conjecture is true for the projective plane. But let us start with the basic example of lagrangian tori in $\mathbb{CP}^2$.

The first and simplest example of lagrangian fibration of $\mathbb{CP}^2$ comes from the toric geometry; it can be given by the following construction. Choose homogenous coordinates $[z_0 : z_1 : z_2]$ and consider a subset of $\mathbb{CP}^2$ defined by the system of equations

$$z_i = r_i e^{i \phi_i}, \quad i = 0, 1, 2,$$

where $r_i$ are fixed positive real numbers satisfy $r_0 + r_1 + r_2 = 1$ and $\phi_i$ are real parameters. In $\mathbb{C}^3$ it would give a 3-dimensional torus, but after the phase factorization it gives us a smooth 2-torus in $\mathbb{CP}^2$. Varying $r_i$s we get a family of lagrangian tori and hence a lagrangian fibration of $\mathbb{CP}^2$ over a triangle $\Delta \subset \mathbb{R}^2$. Indeed, one can attach to any smooth torus the pair $(r_0, r_1)$ (since the third $r_2$ is defined by $(r_0, r_1)$ uniquely), and the possible values of $(r_0, r_1)$ form the triangle $\Delta$. The degenerations of this lagrangian fibration are regular: over segments $\{r_0 = 0, 0 < r_1 < 1\}, \{r_1 = 0, 0 < r_0 < 1\}, \{r_0 + r_1 = 1, 0 < r_i < 1\}$ one has 1-dimensional torical fibers and the vertex of the triangle $\Delta$ correspond to the maximal degenerations, 0-dimensional tori or just points. Denote the smooth fiber of $(r_0, r_1) \in \Delta$ as $S_{r_0, r_1}$. These are called the Clifford tori and the fibration is called the Clifford fibration of the projective plane.

Since the symplectic form is integer, the question arises about the Bohr–Sommerfeld fibers of this lagrangian fibration. The line bundle $L = \mathcal{O}(1)$ with a hermitian connection $a$ whose curvature form is proportional to the symplectic form distinguish a set of Bohr–Sommerfeld fibers of different level. And the specification is very simple: the fiber $T_{r_0, r_1}$ is Bohr–Sommerfeld of level $k$ if and only if

$$k \cdot r_0, k \cdot r_1 \in \mathbb{Z}.$$ 

Indeed, the periods of the fiber torus $T_{r_0, r_1}$ are given by numbers $r_0$ and $r_1$ for certain generators of $H_1(T_{r_0, r_1}, \mathbb{Z})$ and this implies the statement. This shows that:

— there are no Bohr–Sommerfeld fibers of level 1 and 2;

— there is unique fiber which is Bohr–Sommerfeld of level 3 and therefore which is Bohr–Sommerfeld with respect to the canonical class;

— the number of fibers which are Bohr–Sommerfeld of level $k$ is exactly the same as $\dim H^0(\mathbb{CP}^3, \mathcal{O}(k - 3))$. 

The last coincidence can be restored to direct equality "number of $k$ - Bohr – Sommerfeld fibers = dimension of holomorphic section space of $O(k)$" if one generalizes the situation and consider singular fibers as well. Then it would be exactly three Bohr – Sommerfeld fibers of level 1 (= three points over the vertices of $\Delta$, 0-dimensional tori), six Bohr – Sommerfeld fibers of level 2 (= three points above + three middle 1-dimensional tori live over edges of the triangle), ten Bohr – Sommerfeld fibers of level 3 (= three points above + two for each edge of $\Delta$ 1-dimensional tori + our regular fiber), etc. This effect is known in Geometric Quantization of toric varieties.

But we are interested here in regular fibers only. Now let us see what would be the result of the Homological Mirror Symmetry approach. To proceed with one takes the fibers which have non trivial Floer cohomology. Leaving aside possible definitions of the Floer cohomology $FH^*(S,S;\mathbb{Z}_2)$ we can exploit here our reminiscent: if a lagrangian submanifold $S$ is displaceable then it has trivial Floer cohomology. And the displaceability means that there exists a hamiltonian isotopy $\psi_t$ such that $\psi_t(S)$ doesn’t intersect $S$ for some $t$:

$$\psi_t(S) \cap S = \emptyset.$$ 

It is not hard to see that if both of $r_0$ and $r_1$ are not equal to 1/3 then $T_{r_0,r_1}$ is displaceable. Indeed, we have for $\mathbb{CP}^2$ the subalgebra of symbols in the Poisson algebra $(C^\infty(\mathbb{CP}^2),\{;\}_\omega)$, see [7], which correspond to self adjoint operators on $\mathbb{C}^3$. The hamiltonian flow which moves $T_{r_0,r_1}$ to $T_{r_1,r_0}$ is generated by the self adjoint operator which interchange $z_0$ and $z_1$ in $\mathbb{C}^3$. And since the fibers don’t intersect each other we get that if $r_0 \neq r_1$ then $T_{r_0,r_1}$ is displaceable, and the same is true if $r_1 \neq 1 - r_0 - r_1$. It remains one absolutely symmetric possibility: when all $r_i = 1/3$. And the point is that this is precisely the Bohr – Sommerfeld with respect to the canonical class lagrangian fiber. To examine whether or not it has non trivial Floer cohomology we use the following argument: the lagrangian torus $T_{1/3,1/3}$ is monotone. Indeed, it is Bohr - Sommerfeld with respect to the canonical class and it is minimal therefore the universal Maslov class is trivial, see [10]. This fact is exploited in [11] to prove that the Floer cohomology of $S$ is isomorphic to the de Rham cohomology of it:

$$FH^*(S,S;\mathbb{C}) = H^*_{dR}(S,\mathbb{C}),$$

and the last one is very well known for a torus.

Thus for the standard toric fibration of $\mathbb{CP}^2$ (and the same is true for any projective space) the Bohr – Sommerfeld with respect to the canonical class condition is equivalent to the monotonicity condition and furthermore to the non displaceability condition (and in particular one could get the results from [11]).

Now there is a natural simple extension of the toric case: a lagrangian torus in $\mathbb{CP}^2$ is called of the Clifford type if there exists a hamiltonian isotopy which moves this torus to a standard fiber of the Clifford fibration. Since the Floer cohomology is invariant under hamiltonian deformations (and it is the main property of it, which even could be taken for its general definition) as well as the following three conditions:
— the Bohr – Sommerfeld condition of any level;
— the monotonicity condition;
— non displacability condition,
are, all what we’ve said is true for any Clifford torus.

Thus we complete the discussion of the Clifford tori in $\mathbb{CP}^2$, resuming that for the standard toric fibration of $\mathbb{CP}^2$ our Extremely Naive Conjecture is true.

The conjectures mentioned in the title of this text look rather naive as well being based mainly on the known examples and the facts that their statements are true if we replace there ”a lagrangian torus” by ”a fiber of a real polarization with regular polarization”, see below, but nevertheless we would like to formulate them in these extended form.

**Conjecture 1.** If $S \subset \mathbb{CP}^2$ is a Bohr – Sommerfeld lagrangian torus of level $k$ then $k$ must be greater of equal to 3.

If this conjecture is true then the class of Bohr – Sommerfeld with respect to the canonical class lagrangian tori is ”pure” in the following sence: *a priori* any lagrangian torus which is Bohr - Sommerfeld of level 1 should be automatically included to the set of Bohr - Sommerfeld with respect to the canonical class lagrangian tori (since 3 is divisible by 1), but symplectically this torus is too far from the set of ”pure” Bohr - Sommerfeld with respect to the canonical class tori.

The next one is

**Conjecture 2.** For any lagrangian torus $S \subset \mathbb{CP}^2$, Bohr – Sommerfeld with respect to the canonical class, its universal Maslov class is trivial:

$$H^1(S, \mathbb{Z}) \ni m_S = 0.$$  

If this conjecture is true then every Bohr – Sommerfeld with respect to the canonical class lagrangian torus should be monotone and thus must have non trivial Floer cohomology. This implies our third suggestion

**Conjecture 3.** A smooth lagrangian torus $S \subset \mathbb{CP}^2$ of the projective plane is non displacable if and only if $S$ is Bohr – Sommerfeld with respect to the canonical class.

Let us note again that all these conjectures are too strong for proving a general version of the Hori – Vafa prediction: it would be sufficient to exploit a weaker statement which looks as follows:

**Conjecture.** Let $X$ be a monotone simply connected symplectic manifold and $\pi : X \rightarrow B$ be a real polarization. Then a smooth fiber $\pi^{-1}(b) = S_b$ is non displacable if it is Bohr - Sommerfeld with respect to the canonical class.

Consider two examples both of non toric type.

**Toy example.** Consider $\mathbb{CP}^1 = S^2$ endowed with the standard symplectic form. Any smooth loop $\gamma \subset \mathbb{CP}^1$ is a lagrangian submanifold, and the topological type of smooth lagrangian submanifold actually is exhausted by $T^1$, 1 - dimensional torus. Then the line bundle $L = \mathcal{O}(1)$ together with the appropiate hermitian connection $a \in \mathcal{A}_b(L)$ defines the Bohr – Sommerfeld condition of level $k$ which reads in this case as follows: a smooth loop $\gamma \subset \mathbb{CP}^2$ is Bohr – Sommerfeld of level $k$ if and only if it divides the surface into two pieces both of the symplectic area from $\frac{k}{2} \mathbb{Z}$. This means that $\gamma \subset \mathbb{CP}^1$ is Bohr – Sommerfeld with respect to the canonical class if and only if it divides the surface into equal pieces. On the other hand, it is only
the case when $\gamma$ is non displaceable. This means that for a smooth loop in $\mathbb{CP}^1$ the Conjecture above is true.

Of course, it says almost nothing for any other case since it is based on the fact that for a smooth loop in $\mathbb{CP}^1$ there is only one symplectic invariant which characterizes the loop uniquely up to symplectomorphism, namely the symplectic area of the disc, bounded by this loop. But for other dimensions it could be no longer true: one claims that there is at least one more type for lagrangian tori in $\mathbb{CP}^2$ which was called the lagrangian tori of the Chekanov type, see [12], [13]. Thus we have another

"Non toric example". This example can be found in [13], where one characterizes it as a non toric fibration of $\mathbb{CP}^2$. It is defined as follows: consider the family of conics \( \{Q_\varepsilon\} \) in $\mathbb{CP}^2$ given by the equation

\[
Q_\varepsilon = \{z_0z_1 = \varepsilon z_2\},
\]

where $\varepsilon \in \hat{\mathbb{C}}$, and $[z_0 : z_1 : z_2]$ is a homogenous coordinate system. For this pencil with based points $[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]$ one has exactly two singular conics:

- when $\varepsilon = 0$ the conic is two intersecting lines;
- when $\varepsilon = \infty$ the conic is the double line $z_2^2 = 0$.

Inside this pencil one could take 1-dimensional real subfamily consists of the conics of the form $Q_{a, e^{it} - \mu}$ where $\mu \in \mathbb{C}^*$, $a \in \mathbb{R}^+$ are fixed numbers and $t$ is real parameter. For each entire of this subfamily, say, $Q_{a, e^{it_0} - \mu}$ one has the natural fibration

\[
\pi : Q_{a, e^{it_0} - \mu} \to (-1, 1) \subset \mathbb{R}
\]

which is given by the hamiltonian action of the symbol which preserves each conics in the pencils (this symbol is essentially unique up to scale, see [14]). The fiber can be labeled by number $\delta \in (-1, 1)$ and this correspondence has certain meaning: the symplectic area of the disc which bounds loop $T_{t_0, \delta}^1$ equals to $\delta \mod \mathbb{Z}$. Fixing a value of $\delta$ we distinguish the corresponding fiber $T_{t_0, \delta}^1$. Now let us vary $t_0$ in the real subfamily $\{Q_{a, e^{it} - \mu}\}$; it gives us the corresponding family $T_{t, \delta}^1$ which forms certain 2-torus $T_{\delta}^2 = T_{a, \mu, \delta}^2 \subset \mathbb{CP}^2$. The point is that this torus is lagrangian, see [13]. All tori, constructed in this way for different $a$, form certain fibration of $\mathbb{CP}^2 \setminus l_2$ where $l_2$ is the line $z_2 = 0$ with only one singular torus $T_{|\mu|, \mu, 0}^2$ with one shrinked loop (therefore strictly speaking it is not a real polarization with regular degeneration).

One claims that for a fixed $\mu \neq 0$ the fibration of $\mathbb{CP}^2$ consists of two types of lagrangian tori:

- if $a > |\mu|$ then the torus $T_{a, \delta}^1$ is of the Clifford type;
- if $a < |\mu|$ then the torus $T_{a, \delta}^1$ is of the Chekanov type,

see [13], and these types are different$^2$.

What it gives for our discussion and conjectures? Let us note that

- every torus $T_{a, \delta}^2$ of the Clifford type has been discussed above so the conjectures are true for them;

$^2$It looks a bit strange since there is the case when $a = |\mu|$ and $\delta \neq 0$. What is the type of this smooth lagrangian torus? It can be deformed to both the Clifford and the Chekanov types so if one takes in mind the fact that every continious family of smooth lagrangian tori (submanifolds) with the same periods consists of hamiltonically equivalent tori, the types should be related.
— every torus of Chekanov type $T^2_{a,\delta}, a < |\mu|$, is \textit{displacable} and thus has trivial Floer cohomology to itself;
— there is no a torus of Chekanov type which is Bohr – Sommerfeld with respect to the canonical class.

Indeed, it is not hard to construct a smooth function on $\mathbb{CP}^2$ whose Hamiltonian flow moves $T^2_{a,\mu,\delta}$ to $T^2_{a,-\mu,\delta}$. This function is not generic, it is a symbol which corresponds to the self adjoint operator $A = \text{diag}(0, 0, 1)$. The Hamiltonian flow then acts as the rotation of the parameter space $\mathbb{C}$ with two fixed points $0$ and $\infty$. Then since a torus of the Chekanov type corresponds to the case when two circles of the same radius $a$ with centers at $\mu$ and $-\mu$ do not intersect each other one gets that it is displacable. On the other hand, direct computations show that there is no Bohr – Sommerfeld with respect to the canonical class lagrangian fiber of the Chekanov type.

However the last example is excluded by our main setup of real polarizations with regular degeneration. Regularity of degenerations took place for a given case of a real polarization imposes a number of natural arguments and facts.

Consider any real polarization $\pi : \mathbb{CP}^2 \rightarrow B \subset \mathbb{R}^2$ where $B$ is a convex polytop. Suppose that it has only regular degenerations. This means that there exists a set of symplectic divisors $D_1, \ldots, D_m \subset \mathbb{CP}^2$ such that $\dim_{\mathbb{R}} D_i = 2$. These divisors lie over the edges of $B$. Then it follows that

— each $D_i$ represents the class $[D] \in H_2(\mathbb{CP}^2, \mathbb{Z})$ Poincaré dual to the cohomology class $[\omega]$;
— the number of the symplectic divisor is $\deg K^{-1} = 3$.

Indeed, the total degree of the boundary components must be the degree of the anticanonical class since the "inner" part of $\mathbb{CP}^2$ modulo $B$ admits non vanishing holomorphic vector 2-field with respect to an almost complex structure, compatible with $\omega$ and $\pi$. Each component from $\pi^{-1}(\partial B)$ must have positive degree with respect to $[\omega] \in H^2(\mathbb{CP}^2, \mathbb{Z})$. On the other hand the number of components equals to the number of edges of our convex polytop $B$ which is must be greater or equal to 3. This shows that for $\mathbb{CP}^2$ regularity dictates the form of $B$ and the type of $\pi^{-1}(\partial B)$.

Furthermore, the analysis of the system would become simpler if it were possible to find integrals of special type. Namely, since the "inner" part of $\mathbb{CP}^2$ is topologically equivalent to the direct product $(B - \partial B) \times T^2$ we can choose a basis in $H_1(\pi^{-1}(b), \mathbb{Z})$ uniformly for all smooth fibers of $\pi$. Moreover it can be done relatively to the boundary components $D_1, D_2, D_3$ if one chooses any two from the set. The point is that for any $D_i$ there exists uniquely determined basic element from $H_1(\pi^{-1}(b), \mathbb{Z})$ which degenerates when passing to a limit fiber in $D_i$. This means that we have distinguished primitive elements $d_i, d_j, i \neq j$, form a basis. Let us choose and fix $d_1, d_2$ as a basis. Then there exists a lift of the period map with respect to the boundary data:

$$p_{d_1, d_2} = (p_{d_1, d_2}^1, p_{d_1, d_2}^2) : B \rightarrow \mathbb{R}^2,$$

such that

— $p_{d_1, d_2}$ is smooth on $B - \partial B$;
— $p_{d_1, d_2}^i |_{\pi(D_i)} = 0$;
— \( p_{d_1,d_2}^i(b) = \int_D \omega \mod \mathbb{Z} \) where \( D \subset \mathbb{CP}^2 \) is a disc with boundary \( \partial D = \gamma_{d_i} \subset \pi^{-1}(b) \) and \( [\gamma_{d_i}] = d_i \in H_1(\pi^{-1}(b), \mathbb{Z}) \).

Note that such a lift exists and there are exactly 4 possibilities for the extension \( p_{d_1,d_2} \) since there are exactly 4 possible choices of the signs for \( d_1 \) and \( d_2 \) (compare this fact with the discussion on the choice of spin structures in [3], [11]). Let us fix the signs in such a way that \( p_{d_1,d_2} \) is non negative on \( B \).

Denote as \( a_{ij} \) the intersection points, such that

\[ a_{ij} = D_i \cap D_j. \]

Then it is easy to see that

\[ p_{d_1,d_2}(a_{12}) = (0,0), p_{d_1,d_2}(a_{13}) = (0,1), p_{d_1,d_2}(a_{23}) = (1,0). \]

From this one deduce that

\[ d_3 = d_1 + d_2 \in H_1(\pi^{-1}(b), \mathbb{Z}). \]

Indeed, \( d_3 \) can be represented as \( d_3 = pd_1 + qd_2 \) being a primitive element, where \( p, q \) are coprime integers. But the symplectic area is an additive functional and from this one deduces that \( p = q = 1 \).

Now let us impose the fact, proven in Section 2: the Kodaira – Spencer map is an isomorphism. This implies one very important property of our lifted period function:

**Lemma.** The function \( p_{d_1,d_2} \) is strictly monotone in both arguments.

Indeed, since each component of \( p_{d_1,d_2} \) is monotone on the corresponding boundary side and the fact, that the Kodaira - Spencer map in this situation coincides with the differential of \( p_{d_1,d_2} \) one sees that

— the lifted period map \( p_{d_1,d_2} \) doesn’t have any critical points on \( B - \partial B \);
— for any level line \( L_c = \{ p_{d_1,d_2}^i = c, 0 \leq c < 1 \} \) the restriction \( p_{d_1,d_2}^i |_{L_c} \) is a strictly monotone (increasing) function.

Now examine the statements of Conjectures 1 – 3 for this situation.

**Conjecture 1.** From the monotonicity of \( p_{d_1,d_2} \) it follows that for any regular fiber \( S_b = \pi^{-1}(b), b \in B - \partial B \), one has

\[ 0 < p_{d_1,d_2}^1(b) + p_{d_1,d_2}^2(b) < 1. \]

By the definitions of \( p_{d_1,d_2} \) and of the Bohr – Sommerfeld fiber of level \( k \) we get that the minimal possible non empty level is 3.

**Conjecture 2.** Again from the monotonicity of \( p_{d_1,d_2} \) we get that there exists unique fiber which is Bohr – Sommerfeld of level 3 or with respect to the canonical class. Note that for this fiber \( S_{can} \) one has

\[ p_{d_1,d_2}^1(S_{can}) = p_{d_1,d_2}^2(S_{can}) = \frac{1}{3}. \]

To prove the monotonicity of \( S_{can} \) it is sufficient to find for each generator of \( H_1(S_{can}, \mathbb{Z}) \) a smooth loop \( \gamma \) representing this generator, and a smooth disc \( D \),
bounded by $\gamma$, such that the Maslov index of $[\gamma, D]$ would be three times the symplectic area of $D$ (and it is enough since for any other disc $D'$ with the same boundary $\gamma$ the relation should be the same due to the monotonicity of $\mathbb{CP}^2$). Note that since the set of lagrangian fibers is connected the Maslov index is the same for all lagrangian tori.

Take our distinguished generator $d_1 \in H_1(S_{\text{can}}, \mathbb{Z})$ and choose a smooth loop $\gamma_1 \subset S_{\text{can}}$ such that $[\gamma_1] = d_1$. Take the level set $C_{\frac{1}{3}} = \{p_{d_1,d_2} = \frac{1}{3}\}$ and choose the segment $B_t \subset C_{\frac{1}{3}}, t \in [0; \frac{1}{3}]$, which corresponds to the inequality $p_{d_1,d_2} \leq \frac{1}{3}$. There exists a family of smooth loops $\gamma_1^t, t \in [0; \frac{1}{3}]$ such that

- $\gamma_1^t = \gamma_1 \subset S_{\text{can}}$;
- $\pi(\gamma_1^t) = b(t) \in B_t \subset B$;
- $\gamma_1^t \subset S_{b(t)}$ and $[\gamma_1^t] = d_1 \in H_1(S_{b(t)})$.

The point is that the family $\{\gamma_1^t\}$ shrinks to point $\gamma_1^0$ which lies on the symplectic divisor $D_1$.

It is not hard to see that the family $\{\gamma_1^t\}$ forms a disc

$$\cup_{t \in [0; 1/3]} \gamma_1^t = D \subset \mathbb{CP}^2$$

such that

$$\int_D \omega = \frac{1}{3}.$$ 

On the other hand the maslov index of $[\gamma_1, D]$ is equal to 1. Indeed, since we shrink $\gamma_1$ to a point over the level line of $p_{d_1,d_2}^2$ it follows that the Maslov index of $D$ must be the degree of the normal bundle of $D_1$. Thus we have

$$\mu([\gamma_1, D]) = 1 = 3 \cdot \frac{1}{3} = 3 \cdot \int_D \omega,$$

and since we can repeat the arguments for a smooth loop $\gamma_2 \subset S_{\text{can}}$, which represents the generator $d_2$, and it follows that $S_{\text{can}}$ is monotone.

**Conjecture 3.** To prove the fact that if a fiber $S_b$ is not Bohr – Sommerfeld with respect to the canonical class then it is displaceable it is sufficient to prove that the same happens for the fiber which doesn’t lie over the ”diagonal” $\{p_{d_1,d_2}^1 = p_{d_1,d_2}^2\}$. Indeed, our choice of $d_1, d_2$ was made arbitrary and taking another pair, say, $(d_1, d_3)$ we shall get the same result for the corresponding ”diagonal”, and since the intersection of the ”diagonals” consists of exactly one point which is Bohr – Sommerfeld with respect to the canonical class, it implies that the Conjecture 3 is true for fibers of a real polarization with regular degeneration of $\mathbb{CP}^2$.

We calim that there exists a Hamiltonian deformation of $\mathbb{CP}^2$ which generates the corresponding Hamiltonian isotopy which interchanges fibers with values $(c_1, c_2)$ and $(c_2, c_1)$ with respect to the function $p_{d_1,d_2}^j$. The desired Hamiltonian deformation is constructed explicitly as follows. Consider the level sets of the sum $p_{d_1,d_2}^1 + p_{d_1,d_2}^2$, lifted to $\mathbb{CP}^2$. The possible values are in $[0; 1]$. There are two exceptional level sets: for $c = 0$ we have the point $D_1 \cap D_2$; for $c = 1$ it is $D_3$. For any other $\alpha \in (0; 1)$ the level set

$$C_\alpha = \pi^{-1}(\{p_{d_1,d_2}^1 + p_{d_1,d_2}^2 = \alpha\}$$
is a smooth 3-sphere. The restriction of the symplectic form \( \omega \) to \( C_\alpha \) defines a fibration
\[
p_\alpha : C_\alpha \to S^2_\alpha
\]
which is topologically the Hopf bundle. Indeed, we take the kernels of \( \omega|_{C_\alpha} \), and the corresponding 1-dimensional distribution is integrable which gives the fibration. Additionally one has

— a symplectic form \( \omega_\alpha \) on \( S^2_\alpha \) which is the result of the reduction applied to \( \omega \);
— a smooth circle \( S^1_\alpha \) which is the result of the phase factorization of the "diagonal" torus with periods \((\alpha/2, \alpha/2)\).

Note that when \( \alpha \) tends to 1 this Hopf bundle \( C_\alpha \to S^2_\alpha \) degenerates to \( D_3 \) with a marked circle \( S^1_1 \subset D_3 \). Moreover, the triple \((D_3, \omega|_{D_3}, S^1_1)\) is the result of the limiting procedure applied to \((S^2_\alpha, \omega_\alpha, S^1_\alpha)\) when \( \alpha \) tends to 1. On the other hand, the other limit \( \alpha \to 0 \) is realized as a conformal shrinking of the triple \((S^2_\alpha, \omega_\alpha, S^1_\alpha)\) to the point \( D_1 \cap D_2 \). Indeed, it is clear that the symplectic volume
\[
\int_{S^2_\alpha} \omega_\alpha = \alpha.
\]

Let us fix for the symplectic 2-sphere \( D_3 \) a smooth function \( f_1 \in C^\infty(D_3, \mathbb{R}) \) such that \( f_1 \) is a height function and it has two non degenerated critical points \( p^N_1, p^S_1 \) both of which lie on the marked circle \( S^1_1 \). One can construct now using inverse limiting process a family of smooth functions \( \{f_\alpha\} \) for the family of 2-spheres \( \{S^2_\alpha\} \) for \( \alpha \in (0; 1] \). We take an appropriate normalization for the functions such that
\[
\int_{S^2_\alpha} f_\alpha \omega_\alpha = \alpha^2,
\]
and then lift each function \( f_\alpha \) to \( C_\alpha \) via the canonical projection:
\[
F_\alpha = f_\alpha \circ p_\alpha : C_\alpha \to \mathbb{R}.
\]

Then we claim that these lifted functions can be combined to a global smooth function \( F \) such that
\[
F|_{C_\alpha} = F_\alpha.
\]

This function has exactly three critical points:
— the intersection point \( D_1 \cap D_2 \),
— two points \( p^N_1, p^S_1 \) which lie on \( D_3 \).

The hamiltonian vector field \( X_F \) generates the flow on \( \mathbb{C}P^2 \) which is a 1-parameter family of symplectomorphisms \( \phi_t \) of \( \mathbb{C}P^2 \) such that \( \phi_{2\pi} = \text{id} \). Indeed, it follows the rotation of the spheres \( S^2_\alpha \) with fixed points \( p^N_\alpha, p^S_\alpha \) which lie on the "diagonal" circle \( S^1_\alpha \). And it is not hard to see that the result of this rotation applied to a fiber of the given real polarization with periods \( (c_1, c_2) \) should be the fiber with periods \( (c_2, c_1) \). It ends the prove of the Conjecture 3 for fibers of real polarization with regular degeneration.

Resuming the discussion we see that the following fact takes place:

**Theorem 3.** For \( \mathbb{C}P^2 \) the Extremely Naive Conjecture is true.
Note that the method of lifting of the period map can be applied to any compact simply connected symplectic manifold, and the main property of the Kodaira – Spencer map can be exploited to establish the strict monotonicity of this lifted period function which was crucial in our construction for $\mathbb{CP}^2$ above. Thus one could expect that the same method will be useful for more general cases, for other monotone symplectic manifolds.

At the same time before studying the Conjectures 1 – 3, which were formulated for any lagrangian tori in $\mathbb{CP}^2$, one could try to find the answer on the following natural question: is there a geometric condition on a lagrangian torus in $\mathbb{CP}^2$ which should detect whether or not this torus can be included to a family of lagrangian fibers of a real polarization with regular degeneration?

References

[1] M. Kontsevich, Homological algebra of mirror symmetry, ICM -1994 Proceedings, Zurich, Birkhauser, 1995.
[2] K. Hori, C. Vafa, Mirror symmetry, hep-th/0002222.
[3] C.-H. Cho, Y.-G. Oh, Floer cohomology and discs instantons of lagrangian torus fibers in Fano toris manifolds, Asian J. Math., vol. 10, No. 4 (2006), 773 - 814.
[4] V. Guillemin, S. Sternberg, Symplectic technique in physics, Cambridge Univ. Press (1990).
[5] J. Sniatycki, Quantization and quantum mechanics, Springer, Berlin (1970).
[6] N. Woodhause, Geometric quantization, Oxford Univ. Press (1980).
[7] N. Tyurin, Geometric quantization and algebraic lagrangian geometry, London Math. Soc. Lecture Notes, 338, 279 - 318.
[8] A. Tyurin, Geometric quantization and mirror symmetry, arXiv: math/9902027.
[9] A. Weinstein, Lagrangian submanifolds and hamiltonian systems, Ann. of Math., 98 (1973), 377 - 410.
[10] N. Tyurin, Universal Maslov class of Bohr - Sommerfeld lagrangian embedding into pseudo Einstein symplectic manifold, Teoret. Mat. Fiz. 150 (2007), No 2, 325 - 337.
[11] C. - H. Cho, Holomorphic discs, spin structures and the Floer cohomology of the Clifford torus, PhD Thesis, Univ. of Wisconsin - Madison, 2003.
[12] Yu. Chekanov, T. Schlenk, Lagrangian tori in projective spaces, in preparation.
[13] D. Auroux, Mirror symmetry and T - duality in the complement of an anticanonical divisor, arXiv: 0706.3207.
[14] S. Belev, Proper non linear quantum subsystems of standard quantum systems, Bachelor diploma Thesis, BLTP JINR (Dubna), 2007.

MPI (Bonn), BLTP JINR (Dubna)
E-mail address: ntyurin@theor.jinr.ru jtyurin@mpim-bonn.mpg.de