ACCESSIBILITY AND ERGODICITY FOR COLLAPSED ANOSOV FLOWS

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Abstract. We consider a class of partially hyperbolic diffeomorphisms introduced in [BFP] which is open and closed and contains all known examples. If in addition the diffeomorphism is non-wandering, then we show it is accessible unless it contains a su-torus. This implies that these systems are ergodic when they preserve volume, confirming a conjecture by Hertz-Hertz-Ures [CHHU, Conjecture 2.11] for this class of systems.

Keywords: Partial hyperbolicity, ergodicity, accessibility, Anosov flows, foliations. 3-manifold topology, foliations.

1. Introduction

In this paper we study the ergodicity problem for 3 dimensional partially hyperbolic diffeomorphisms. It has been shown in [HHU] that ergodicity (in fact the K-property) is abundant, establishing a conjecture of Pugh and Shub in this setting [PS]. This lead to the belief that non-ergodic partially hyperbolic diffeomorphisms in dimension 3 can be described [HHU2] (see also [CHHU]).

Many recent works including our own, have tried to show that in certain manifolds or isotopy classes ergodicity holds for all volume preserving partially hyperbolic diffeomorphisms (see [HHU2, HU, GS, FP, HRHU]). Thanks to the reductions of [GPS, BuW, HHU] the problem boils down to the study of accessibility, and this has become a problem on its own (see [HP, H, HS, FP] for general results about accessibility).

Here, instead of fixing a manifold or isotopy class, we work on a specific class of partially hyperbolic diffeomorphisms that were introduced in [BFP] motivated by [BFFP, BFFP2, FP2]. They are called collapsed Anosov flows. Very roughly the dynamics of the partially hyperbolic diffeomorphism is semiconjugated to a self orbit equivalence of an Anosov flow, and the semiconjugacy sends flow lines to curves tangent to the center bundle. See section 2 for a precise definition. The class of partially hyperbolic diffeomorphisms which are collapsed Anosov flows is an open and closed subset of all partially hyperbolic diffeomorphisms, and in addition it includes all known examples in manifolds with non-solvable fundamental group. We note that this class is strictly larger than the ones previously known to satisfy [CHHU, Conjecture 2.11] as it contains the partially hyperbolic diffeomorphisms in the connected component of the examples constructed in [BGP, BGHP].

Our main result is the following:

Theorem A. Let $f : M \to M$ be a collapsed Anosov flow of a closed 3-manifold $M$ whose fundamental group is not virtually solvable. Assume that the non-wandering set of $f$ is all of $M$. Then, $f$ is accessible. In particular, if $f$ is $C^{1+}$ and volume preserving it is a K-system (and consequently ergodic and mixing).

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Again, precise definitions are given in § 2. It is worth noting that in [BGP, BGHP] the argument to get ergodic examples is perturbative, so not even the concrete examples constructed there were known to be ergodic (but it was known that an open and dense subset of a volume preserving neighborhood of the examples were). The ergodicity for specific examples follows from the results of this paper.

We note that the assumption that $M$ does not have solvable fundamental group is necessary since the time one map of a suspension of a toral automorphism is not accessible (and in fact contains a $su$-torus, that is, an embedded torus tangent to $E^s \oplus E^u$). We refer the reader to [H] for a complete treatment of accessibility in the class of 3-manifolds with virtually solvable fundamental group.

Theorem A is complementary to what was done in [FP] but some results have intersection (in [FP] it is proved that a more restrictive class, that of discretized Anosov flows, are always accessible without the non-wandering assumption). In fact, the proof of Theorem A uses [FP] at some point. However, a slightly weaker version of Theorem A, dealing with strong collapsed Anosov flows rather than collapsed Anosov flows can be proven independently of [FP] (see § 6.2 for precise definitions). In § 6.3 we explain precisely the dependence of Theorem A on [FP] and provide the unconditional result we show here (c.f. Theorem 6.2).

In [FP] accessibility and ergodicity are established unconditionally in certain manifolds or isotopy classes of diffeomorphisms. On the other hand, in [FP$_2$] we showed that every partially hyperbolic diffeomorphism in a hyperbolic 3-manifold is a (strong) collapsed Anosov flow, so this paper gives a different proof of some of the results of [FP]. The argument of [FP] achieves accessibility in hyperbolic 3-manifolds in a shorter way as it uses much less of the classification: to use the proof we present here to deduce ergodicity in hyperbolic 3-manifolds one should use the full classification of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds given in [FP$_2$] which is way longer. Note that in [FP$_3$] we show that every volume preserving partially hyperbolic diffeomorphism in a unit tangent bundle is a collapsed Anosov flow, so this paper will imply that [CHHU, Conjecture 2.11] also holds for unit tangent manifolds extending [HRHU, FP] where this was shown for certain isotopy classes (but also works in general Seifert manifolds).

The results of this article emphasize what [BFP] proposes: it is valuable to understand the dynamics of collapsed Anosov flows and separate it from the classification of general partially hyperbolic diffeomorphisms. The proof uses some fine properties of (topological) Anosov flows that had not been previously used in the study of partially hyperbolic dynamics (see § 2.1).

2. BACKGROUND

In this paper $M$ will always denote a closed 3-manifold. In this section we briefly recall some basic properties of Anosov flows, partially hyperbolic diffeomorphisms and collapsed Anosov flows. Except in § 2.1 we only present definitions and quote results from elsewhere. In § 2.1 we prove Proposition 2.5 which is an observation about self orbit equivalences of topological Anosov flows that we will use in the proof of Theorem A. We refer the reader to [BFP, FP] for more detailed information.

In § 2.4 we will show that the results of Theorem A still hold if shown for iterates of a finite lift. Thus, we will assume without mention that all objects are orientable and foliations are also transversally orientable and orientations are preserved.

2.1. Topological Anosov flows. A flow $\phi_t : M \to M$ generated by a continuous vector field $X$ is said to be a topological Anosov flow if it is expansive and preserves a topological foliation (see [BFP, §5] for other equivalent definitions). A topological Anosov flow $\phi_t$ preserves two transverse foliations $\mathcal{F}^{s}$ and $\mathcal{F}^{u}$ with the property that orbits in $\mathcal{F}^{s}(x)$ approach the orbit of $x$ in the future while orbits in $\mathcal{F}^{u}(x)$ approach the orbit of $x$ in the past.
approach the orbit of $x$ in the past. Leaves of $\mathcal{F}^{us}$ and $\mathcal{F}^{wu}$ are planes or cylinders (Möbius bands are excluded by our orientability assumptions) and all cylinder leaves contain a unique periodic orbit. We refer the reader to [Bar] for generalities on Anosov flows in dimension 3.

We denote by $\tilde{\phi}_t : \tilde{M} \to \tilde{M}$ the lift of the flow to the universal cover. A fundamental property for us is [Fen1, Ba] that the quotient $\mathcal{O}_\phi$ of $\tilde{M}$ by the orbits of $\tilde{\phi}_t$ is homeomorphic to $\mathbb{R}^2$. In other words, the flow $\tilde{\phi}_t$ in $\tilde{M}$ is topologically a product flow. However, geometrically this is rarely the case, and this will be a key fact (see Theorem 2.2 below for a precise statement) that we will exploit to show accessibility.

We call $\mathcal{O}_\phi$ the orbit space of $\phi$. The lifts $\mathcal{F}^{us}$ and $\mathcal{F}^{wu}$ are flow invariant and therefore induce transverse one dimensional foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ in the orbit space $\mathcal{O}_\phi$. The fundamental group $\pi_1(M)$ preserves the collection of flow lines, hence induces an action by homeomorphisms on $\mathcal{O}_\phi$ which preserves the foliations $\mathcal{G}^s$ and $\mathcal{G}^u$. Given an orbit $o \in \mathcal{O}_\phi$ we call half leaf of $\mathcal{G}^s(o)$ (or $\mathcal{G}^u(o)$) a connected component of $\mathcal{G}^s(o)\setminus\{o\}$ (or $\mathcal{G}^u(o)\setminus\{o\}$).

Given an orbit $o$ of $\tilde{\phi}_t$ we view it as both an element or point in $\mathcal{O}_\phi$ and as a subset of $\tilde{M}$ consisting of all points in the orbit.

Note that if for some $o \in \mathcal{O}_\phi$ there is some non trivial $\gamma \in \pi_1(M)$ such that $\gamma o = o$ then this means that $o$ corresponds to a periodic orbit $\alpha = \pi(o)$ of $\phi_t$. In fact this is an if and only if property. We say that $\gamma$ acts increasingly (resp. decreasingly) on $o$ if $\gamma x = \phi_t(x)$ for some $t > 0$ (resp. $t < 0$) and some $x \in o$ (since $\gamma$ does not have fixed points in $\tilde{M}$ it is easy to see that this is independent on $x \in o$). Our orientability assumptions imply that if a deck transformation fixes some orbit, then it also fixes all the half leaves of $\mathcal{G}^s(o)$ and $\mathcal{G}^u(o)$, i.e. the connected components of $\mathcal{G}^s(o)\setminus\{o\}$, $\mathcal{G}^u(o)\setminus\{o\}$.

Let $\alpha_1, \alpha_2$ be periodic orbits of $\phi_t$. We say that they are freely homotopic if the unoriented curves $\alpha_1, \alpha_2$ are freely homotopic. With our orientation conditions this is equivalent to saying that there is a non trivial deck transformation $\gamma \in \pi_1(M)$ and lifts $o_1, o_2$ of $\alpha_1, \alpha_2$ respectively such that $\gamma o_1 = o_1$ and $\gamma o_2 = o_2$ (note that we do not require $\gamma$ to act increasingly on both, or any of them). There are many subtleties with defining freely homotopic orbits, having to do with orientation on the orbits, taking powers of orbits; but we will not enter into them here. We refer the reader to [Fen3] for more details.

A key object for the main result here will be lozenges, see figure 1.

**Definition 2.1.** An open subset $\mathcal{L}$ of the orbit space $\mathcal{O}_\phi$ is said to be a lozenge if there are two orbits $o_1, o_2$ called the corners of the lozenge which verify that they have half leaves $A^s_1$ of $\mathcal{G}^s(o_1)$ and $A^u_1$ of $\mathcal{G}^u(o_1)$ which are disjoint from half leaves $A^s_2$ of $\mathcal{G}^s(o_2)$ and $A^u_2$ of $\mathcal{G}^u(o_2)$ and satisfy the following properties:

- A leaf of $\mathcal{G}^s$ intersects $A^s_1$ if and only if it intersects $A^s_2$.
- A leaf of $\mathcal{G}^u$ intersects $A^u_1$ if and only if it intersects $A^u_2$.
- every point $p \in \mathcal{L}$ satisfies that $\mathcal{G}^s(p)$ intersects $A^s_1$ and separates $o_1$ from $o_2$.

In addition $\mathcal{G}^u(p)$ intersects $A^u_1$ and separates $o_1$ from $o_2$.

The half leaves $A^s_1, A^s_2, A^u_1, A^u_2$ are called the sides of the lozenge.

A chain of lozenges is a sequence $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of lozenges such that if $\overline{\mathcal{L}_i}$ denotes the closure of $\mathcal{L}_i$ in $\mathcal{O}_\phi$ then for every $1 \leq i \leq k - 1$ we have that $\overline{\mathcal{L}_i} \cap \overline{\mathcal{L}_{i+1}}$ is either a side or a corner. We allow infinite chain of lozenges if $k = \infty$ and bi-infinite chain of lozenges of the form $\ldots, \mathcal{L}_{-n}, \mathcal{L}_{-n+1}, \ldots, \mathcal{L}_k, \mathcal{L}_{k+1}, \ldots$.

A chain is minimal if a given lozenge occurs at most once in the chain, i.e. no backtracking.
Figure 1. A lozenge. This figure is in the orbit space. So topologically the region of $\widetilde{M}$ which projects into the lozenge is this lozenge times the reals.

We will need the following result, the first claim of which is [Fen$_2$, Theorem 3.3] and the second is [Fen$_3$, Corollary E]. For the last claim see [Fen$_3$] or [BaF, Theorem C].

**Theorem 2.2.** Let $\phi_t : M \to M$ be a topological Anosov flow. Then,

(i) Let $o_1, o_2$ orbits in $O_\phi$ which are fixed by a non trivial $\gamma \in \pi_1(M)$. Then $o_1$ and $o_2$ are connected by a finite chain of lozenges whose corners are fixed by $\gamma$.

(ii) If $\phi_t$ is not a suspension then there are freely homotopic periodic orbits of $\phi_t$.

Moreover, every $\mathbb{Z}^2$ subgroup of the fundamental group is associated to at most two bi-infinite minimal chains of lozenges. The subgroup fixes each such chain of lozenges. There are infinitely many non trivial elements in the $\mathbb{Z}^2$ subgroup which fix every lozenge in these chains.

To get the last statement one uses that the $\mathbb{Z}^2$ subgroup acts on the linearly ordered set of lozenges in the bi-infinite chain of lozenges, which is order isomorphic to $\mathbb{Z}$.

As a consequence one deduces that every topological Anosov flow $\phi_t : M \to M$ which is not a suspension contains a lozenge whose corners are lifts of periodic orbits of $\phi_t$ to $\widetilde{M}$. This is the crucial place where we will use that the fundamental group is not solvable since suspensions are very different from the rest of Anosov flows in this respect.

**Corollary 2.3.** If $\phi_t : M \to M$ is a topological Anosov flow which is not a suspension, then there exists a lozenge $L$ in $O_\phi$ which is fixed by some non trivial $\gamma \in \pi_1(M)$.
Note that a topological Anosov flow is a suspension if and only if the manifold has (virtually) solvable fundamental group [Bru].

An important property of lozenges that we will use is the following (see figures 2 and 3):

**Proposition 2.4.** Let $L \subset O_\phi$ be a lozenge in $O_\phi$ fixed by $\gamma \in \pi_1(M)$. Denote by $o_1, o_2$ the corners of $L$ which correspond to periodic orbits of $\phi_t$. Then, if $\gamma$ acts increasingly on $o_1$ then it acts decreasingly on $o_2$ and vice versa.

**Proof.** This is well known (see for instance [Fen1, Bar]) but we give a detailed proof for the convenience of the reader not familiar with lozenges. Note that if $\gamma$ acts increasingly on $o_1$ then $\gamma$ expands points in $G^s(o_1)$ and contracts points in $G^u(o_1)$.

This is counterintuitive so we explain for the action on $G^s(o_1)$. Let $L$ be stable leaf of $r_\phi$ which projects to $G^s(o_1)$ in $O_\phi$. Then $\gamma$ acts as an isometry in $L$ and sends a point $x$ in $o_1$ to $r_\phi^t x$ where $t > 0$. Let $y$ be in $L$ in a nearby orbit at distance $d_0$ from $x$. Then $\gamma(y)$ is at distance $d_0$ from $\phi_t(x)$ since $\gamma$ acts as an isometry on $L$.

But the orbit through $y$ is closer to $r_\phi^t x$ than $y$ is to $x$ because orbits in a stable leaf converge together. So the orbit through $\gamma(y)$ is farther from $o_1$ than the orbit through $y$ is, that is $\gamma$ acts as an expansion on the set of orbits in $L$.

Let $A^s_1$ be the half leaves of $G^s(o_1)$ which are in the boundary of the lozenge, and $A^u_1$ the half leaves of $G^u(o_1)$ which are in the boundary of the lozenge. We proved that $\gamma$ acts in an expanding manner in $G^s(o_1)$. Let $q$ in $G^s(o_1)$. Then $G^u(q)$ intersects $A^s_2$ and separates $o_1$ from $o_2$ by the lozenge property. This implies that $\gamma$ contracts points in $G^s(o_2)$.

Similarly $\gamma$ contracts points in $G^u(o_1)$ and expands points in $G^u(o_2)$.

\[\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure2.png}
\end{array}\]

**Figure 2.** A periodic lozenge fixed by a non trivial deck transformation $\gamma$ which also fixes the corners $o_1, o_2$ of the lozenge. The arrows indicate the action of $\gamma$ on the stable and unstable leaves of $o_i$. Notice again that this figure depicts the situation in the orbit space which is two dimensional.

The figure depicts the situation that $\gamma(x) = \Phi_t(x)$ with $t < 0$ for $x$ in $o_1$.

We now clarify what we mean by the flow not being a geometric product. Consider the setup as in Proposition 2.4. Let $\alpha_i = \pi(o_i)$ be the corresponding periodic orbits of $\phi_t$ in $M$ ($\pi : \tilde{M} \to M$ is the universal covering map). Proposition 2.4 implies that $\alpha_1, \alpha_2$ are freely homotopic to the inverses of each other where free homotopies are considered with orientation. In particular this implies that a positive ray of $o_1$, that
is, the forward flow of some point in $o_1$ is a bounded Hausdorff distance in $\tilde{M}$ from a negative ray of $o_2$ (i.e. the backward flow of a point in $o_2$) and vice versa. Also a positive ray of $o_1$ is not a bounded distance from a positive ray of $o_2$ (see for instance [Fen$_1$, Bar] for proofs of these facts). Hence the flow $\Phi$ in $\tilde{M}$ is very far from being a geometric product. The prototype flow for this behavior is the geodesic flow on a closed, orientable hyperbolic surface. Here every periodic orbit is freely homotopic to the inverse of another periodic orbit (the same geodesic traversed in the opposite direction).

![Figure 3. Lifts $o_1, o_2$ of periodic orbits $\alpha_1, \alpha_2$ which are freely homotopic to the inverses of each other. The arrows in the flow lines indicate the positive flow direction in each orbit. The action of $\gamma$ in each orbit is also indicated. This figure is supposed to be 3-dimensional in $\tilde{M}$ and geometrically correct in that positive rays of $o_1$ are a bounded distance from negative rays of $o_2$ and so on. Hence geometrically the flow is very far from being a product.](image)

A self orbit equivalence of an Anosov flow $\phi_t$ is a homeomorphism $\beta : M \to M$ sending orbits of $\phi_t$ to orbits of $\phi_t$ and preserving their orientation. See [BaG, BFP] for more information on them. A self orbit equivalence is trivial if it leaves invariant every orbit.

We show the following proposition about self orbit equivalences that will be an important technical ingredient to prove our main result. The proof and statement will assume some familiarity with the theory of Anosov flows (we note that very similar arguments can be found in [BaG]; we include the proof since this is not stated explicitly).

The proposition refers to $\mathbb{R}$-covered Anosov flows. This means that the leaf space of the stable foliation of the flow lifted to $\tilde{M}$ is homeomorphic to the reals. In case $\Phi$ is $\mathbb{R}$-covered and not conjugate to a suspension it has what is called skewed type. The skewed type has a well defined structure, in particular every orbit (periodic or not) is one corner of a bi-infinite minimal chain of lozenges. If the orbit is periodic then there is a $\mathbb{Z}$ subgroup leaving this bi-infinite chain invariant. Any such flow has what is called a one step up map: it is self orbit equivalence of the flow $\Phi$ which is homotopic to the identity and has a lift to $\tilde{M}$ which induces a shift by one in any of these bi-infinite chains. We refer the reader to [Fen$_1$] and [Bar] for details on $\mathbb{R}$-covered Anosov flows, including the one step up map.

**Proposition 2.5.** Let $\beta : M \to M$ be a self orbit equivalence of a topological Anosov flow $\phi_t : M \to M$ on a manifold with non-virtually solvable fundamental group. Then, one of the following three possibilities happens:
(i) there is a lozenge $L$ fixed by some non trivial deck transformation $\gamma$ and there is a lift $\hat{\beta}$ of an iterate of $\beta$ such that $\hat{\beta}(L) = L$

(ii) $M$ is hyperbolic, $\phi_t$ is $\mathbb{R}$-covered and $\beta$ is a non trivial power of an one step up map, or

(iii) $M$ is Seifert and $\beta$ acts in the base as a pseudo-Anosov.

Proof. If a 3-manifold $M$ admits an Anosov flow then it is irreducible and the universal cover is homeomorphic to $\mathbb{R}^3$. It follows that the manifold is either hyperbolic, Seifert fibered or it has a non-trivial JSJ decomposition (see e.g. [Bar]).

If $M$ has a non-trivial JSJ decomposition, then there is some iterate of $\beta$ that fixes all tori of the decomposition up to homotopy. Each torus in the torus decomposition is associated with a $\mathbb{Z}^2$ subgroup of $\pi_1(M)$ which leaves invariant a minimal chain of lozenges $C$ (cf. Theorem 2.2). Let $L$ be a lozenge in $C$, which has to be fixed by infinitely many $\gamma$ in the $\mathbb{Z}^2$ subgroup associated with the torus. Since an iterate of $\beta$ leaves the torus invariant up to homotopy, this implies that some lift $\hat{\beta}_0$ of an iterate of $\beta$ leaves invariant $C$. If $\beta_0$ does not fix a lozenge in $C$ then it acts as a translation on the set of lozenges in $C$. A $\mathbb{Z}^2$ subgroup leaving $C$ invariant has a non trivial element $\gamma_1$ acting as translation on $C$, hence some iterate $\hat{\beta}_0^{i}\gamma_1$ with $i$ not zero fixes $L$, and this is a lift $\hat{\beta}$ of a non trivial iterate of $\beta$.

This shows that either the first possibility holds or the manifold is hyperbolic or Seifert fibered.

If $M$ is hyperbolic then up to an iterate we can assume that $\beta$ is homotopic to the identity. If $\beta$ has a finite iterate which is trivial, then the finite iterate has a lift fixing all orbits in $\tilde{M}$. Since $M$ is hyperbolic then $\phi_t$ is not conjugate to a suspension Anosov flow. Then the first possibility of the proposition is satisfied automatically because of item (ii) of Theorem 2.2. If no power of $\beta$ is trivial, then in [BaG] it is shown that $\phi_t$ must be $\mathbb{R}$-covered and $\beta$ is a power of a one step up map.

If $M$ is Seifert fibered, then the flow is topologically equivalent to a lift of a geodesic flow in a hyperbolic orbifold [Bru], in particular, every periodic orbit corresponds to a closed curve in the base and all periodic orbits lifted to the universal cover are the corner of some lozenge. If the action in the base of $\beta$ is not pseudo-Anosov, then one of these lozenges will be periodic.

This finishes the proof of the proposition. \qed

2.2. Partially hyperbolic diffeomorphisms. A diffeomorphism $f : M^3 \rightarrow M^3$ is said to be partially hyperbolic if there exists a continuous $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ into one dimensional bundles such that there exists $\ell > 0$ so that for every $x \in M$ and $v^s$ unit vectors in $E^s_\sigma$ ($\sigma = s, c, u$) we have:

$$\|Df^\ell v^s\| < \min\{1, \|Df^\ell v^c\| \} \leq \max\{1, \|Df^\ell v^c\| \} < \|Df^\ell v^u\|.$$  

It is well known that $E^s$ and $E^u$ integrate uniquely into $f$-invariant foliations $W^s$ and $W^u$ (see e.g. [HP2]). In general, the center foliation does not integrate into a foliation, though one dimensionality allows to get some structures that we will not use explicitly here [BI].

Definition 2.6. We say that $f$ is accessible if for every $x, y \in M$ there exists a (piecewise smooth) curve tangent to $E^s \cup E^u$ joining $x$ to $y$.

There is a strong link between accessibility and ergodicity suggested by the Pugh-Shub conjectures [PS]. In our setting the following result is true (see [BuW] for stronger results in higher dimensions that also imply the following):

Theorem 2.7 (Burns-Wilkinson). If $f : M \rightarrow M$ is a volume preserving partially hyperbolic diffeomorphism in a closed 3-manifold $M$ which is accessible and of class $C^{1+}$. Then, $f$ is a K-system, in particular it is ergodic and mixing.
Recall that a measure preserving system is a $K$-system if every non-trivial finite partition has positive metric entropy, this implies the system is ergodic (i.e. invariant sets have zero or full measure). See [BuW, CHHU] for more information.

Lack of accessibility in dimension 3 allows to produce a strong structure that is what we will analyse here (see [HHU, HHU2, CHHU]):

**Theorem 2.8** (Hertz-Hertz-Ures). Let $f : M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold with non-solvable fundamental group whose non-wandering set is all of $M$ and such that $f$ is not accessible. Then, there exists a lamination $\Lambda^{su}$ tangent to $E^s \oplus E^u$ such that:

- $\Lambda$ does not have closed leaves,
- the closure of the complementary regions of $\Lambda^{su}$ (if existing) are $I$-bundles where $E^c$ is uniquely integrable and such that center curves form the $I$-bundle structure.

See [FP, §2.3] for more discussion on this result. The case where the fundamental group is solvable is related with the existence of $su$-tori, see [CHHU].

We will use the following easy property of partially hyperbolic diffeomorphisms that follows directly from uniform transversality of the bundles after iteration (see e.g. [HP2, Proposition 4.2]):

**Proposition 2.9.** There are no closed curves tangent to $E^s$ or $E^u$.

A generalization will be given in Proposition 6.4 below.

### 2.3. Collapsed Anosov flows.

We introduce here the notion of collapsed Anosov flows from [BFP]. As mentioned earlier, it corresponds to an open and closed class of partially hyperbolic diffeomorphisms (see [BFP, Theorem C]) that contains all known examples of partially hyperbolic diffeomorphisms in manifolds with non virtually solvable fundamental group (see [BFP, Theorem A]).

**Definition 2.10.** A partially hyperbolic diffeomorphism $f : M \to M$ is a collapsed Anosov flow if there exists a (topological) Anosov flow $\phi_t : M \to M$, a self orbit equivalence $\beta : M \to M$ and a continuous map $h : M \to M$ homotopic to the identity which is $C^1$ along orbits of the flow and such that $\beta h(\phi_t(x))|_{t=0} \in E^c(h(x))\setminus\{0\}$ and such that $f \circ h = h \circ \beta$.

In [BFP, Theorem A] we actually show that all known examples verify a slightly stronger assumption that we call strong collapsed Anosov flow which under our orientability assumptions is also open and closed among partially hyperbolic diffeomorphisms. In §6.2 we introduce this notion since the proof of Theorem A admits a shortcut if one restricts to this class. (Note that it is an open question if being collapsed Anosov flow implies being strong collapsed Anosov flow.)

### 2.4. Orientability assumptions.

Here we comment on the assumption we have made that all bundles are orientable and that $Df$ preserves their orientation. This is no loss of generality as we shall now explain.

Firstly, let us remark that if $f$ is a partially hyperbolic diffeomorphism of a manifold $M$ and $g$ is a lift of an iterate of $f$ to a finite cover $\hat{M}$ of $M$, then, accessibility of $g$ implies accessibility of $f$. To see this, notice that the strong stable and strong unstable manifolds of an iterate of $f$ are the same as those of $f$, so taking iterates does not change the fact that the whole manifold is an accessibility class. Further, an accessibility class in $\hat{M}$ projects to an accessibility class in $M$, thus, accessibility in $\hat{M}$ for $g$ implies accessibility of $f$ as desired.

Now we need to justify why taking a lift of an iterate of a collapsed Anosov flow to a finite cover is still a collapsed Anosov flow which will end the justification that our assumptions are in fact no loss of generality.
First, notice that taking an iterate of a collapsed Anosov flow is still a collapsed Anosov flow: one just needs to keep the same flow \( \phi_t \) and map \( h \) in Definition 2.10 and pick an iterate of \( \beta \). Consider now a collapsed Anosov flow \( f : M \to M \) and let \( \pi : \tilde{M} \to M \) be a finite lift corresponding to the orientation of the bundles (i.e. \( \pi : \tilde{M} \to M \) is a lift so that the bundles \( E^s, E^c, E^u \) are orientable in the cover). In addition we assume that (up to taking some iterate) \( \beta \) lifts to \( \hat{\beta} : \tilde{M} \to \tilde{M} \) and that \( \hat{f} \) preserves all orientations of the bundles (which can be achieved by further iterates). We claim that \( \hat{f} \) is also a collapsed Anosov flow. The flow \( \hat{\phi}_t \) lifts to \( \hat{\phi}_t \), which is a topological Anosov flow in \( \tilde{M} \). In addition since \( h \) is homotopic to the identity the actions on homotopy induced by \( f \) and \( \beta \) on \( \pi_1(M) \) are the same. Therefore \( \beta \) also lifts to \( \hat{\beta} \) in \( \tilde{M} \). Clearly \( \hat{\beta} \) is a self orbit equivalence of \( \hat{\phi}_t \). Finally the homotopy from the identity to \( h \), lifts \( h \) to \( \hat{h} \) in \( \tilde{M} \). By construction \( \hat{f} \circ \hat{h} = \hat{h} \circ \hat{\beta} \).

In other words \( \hat{f} \) is also a collapsed Anosov flow. This concludes the proof of our claim.

### 3. Idea of the proof of Theorem A

Here we explain some of the main ideas to prove Theorem A.

Since \( f \) is a collapsed Anosov flow associated with an Anosov flow \( \phi_t \), there is a self orbit equivalence \( \beta \) of the flow \( \phi_t \), and a map \( h \) homotopic to the identity so that \( f \circ h = h \circ \beta \).

Our assumptions give that there are periodic orbits \( \alpha_1, \alpha_2 \) of \( \phi_t \), which are freely homotopic to the inverses of each other and which lift to orbits \( \tilde{\alpha}_1, \tilde{\alpha}_2 \) that are corners of a lozenge \( \mathcal{L} \) invariant by deck transformation \( \gamma \) associated to \( \alpha_1 \) (c.f. Corollary 2.3).

By Proposition 2.5 (and because the other cases where dealt in [FP]) we can choose \( \alpha_1, \alpha_2 \) so that the lozenge \( \mathcal{L} \) is also invariant by a lift of a power of the self orbit equivalence \( \beta \). Let \( \alpha_1 = \tilde{\alpha}_i \).

Roughly, the idea of the proof of Theorem A is that the orbits \( \alpha_1, \alpha_2 \) above and the fact that they are (essentially) transverse to the lamination \( \Lambda^{su} \) force a closed curve in a leaf of \( \Lambda^{su} \) which is invariant by a power of \( f \). It is easy to show that this leads to a contradiction (see Proposition 6.4).

Let us explain how the curve is found: to simplify the explanation let us assume in this section that \( h \) is injective, hence a homeomorphism. Notice that \( h \) sends flow lines of \( \phi_t \) to \( C^1 \) curves tangent to \( E^c \) and hence transverse to the lamination \( \Lambda^{su} \).

Lift the homotopy of \( h \) to the identity to produce \( \tilde{h} \).

We consider the action of \( \gamma \) on \( \tilde{h}(\tilde{\alpha}_1) \) and \( \tilde{h}(\tilde{\alpha}_2) \). By Proposition 2.4, \( \gamma \) acts increasing on \( \tilde{h}(\tilde{\alpha}_1) \) and decreasingly on \( \tilde{h}(\tilde{\alpha}_2) \). This is the crucial fact. In § 5 we show that this implies that for any leaf \( E \) of \( \tilde{\Lambda}^{su} \) then \( E \) cannot intersect both \( \tilde{h}(\tilde{\alpha}_1) \) and \( \tilde{h}(\tilde{\alpha}_2) \) because \( E \) separates \( \tilde{M} \). Hence, starting with \( E \) intersecting \( \alpha_1 = \tilde{\alpha}_1 \), we can produce (see figure 5) a unique leaf \( L \) of \( \tilde{\Lambda}^{su} \) which separates in a specific way \( \tilde{h}(\tilde{\alpha}_1) \) from \( \tilde{h}(\tilde{\alpha}_2) \) (see Proposition 5.1). This leaf \( L \) is fixed by \( \gamma \).

In § 4 we do a careful analysis of how a surface like \( L \) can intersect the image under \( \tilde{h} \) of the stable and unstable leaves of \( \tilde{\alpha}_i \) and Proposition 4.6 produces a curve in the intersection which is invariant by \( \gamma \). By the choice above we can prove this curve is also invariant by a power of a lift of an iterate of \( f \). This is where the semiconjugacy between \( f \) and \( \beta \) by \( h \) is used. This projects to a closed curve in a leaf of \( \Lambda^{su} \) which is invariant by a power of \( f \) which contradicts Proposition 6.4 (which slightly generalizes Proposition 2.9).

In the rest of the article we carefully carry out this strategy.
4. Surfaces transverse to collapsed Anosov flows

The main result in this section is Proposition 4.5 below, stating that in the orbit space of an Anosov flow, the boundary of the orbits that intersect a surface uniformly transverse to the flow is made of entire weak stable and weak unstable leaves. This somewhat extends [Fen4, Proposition 4.3]. The context is slightly different since we need to take care of a different setup, but some of the ideas are very similar.

The goal is to understand the intersection between a surface transverse to the center bundle of a (collapsed) Anosov flow and how it behaves in its boundary in the “collapsed orbit space”. The main result of this section is Proposition 4.6 which produces some closed curves in the intersection of certain surfaces transverse to the center bundle and the image under $h$ of the weak stable or weak unstable foliation of the Anosov flow. This will be used later for certain $su$-surfaces (i.e. surfaces tangent to $E^s \oplus E^u$) to find some contradiction assuming non accessibility.

4.1. Setup. Let $\phi_t: M \to M$ be a (topological) Anosov flow and let $Y$ be a non-singular vector field in $M$ such that there is a map $h: M \to M$ continuous and homotopic to the identity, with $h$ being $C^1$ along orbits of the flow, and in addition such that $Y(h(x))$ belongs to $\mathbb{R}_+ \partial_t h(\phi_t(x))|_{t=0}$. By non singular we mean that $Y$ is continuous and never zero.

We consider in $\tilde{M}$, the universal cover of $M$, the orbit space $\mathcal{O}_\phi$ of the flow $\tilde{\phi}_t$ lifted to $\tilde{M}$. We can take $\tilde{h}$ a lift of $h$ commuting with deck transformations and at bounded distance from the identity (i.e. take a lift of a homotopy to the identity to construct $\tilde{h}$) and we denote by $\tilde{Y}$ the lifted vector field.

For $x \in \tilde{M}$ we denote by $o_x$ the orbit of $\tilde{\phi}_t$ containing it.

Definition 4.1. Given an orbit $o$ of $\tilde{\phi}_t$ we define $c_o$ to be the image by $\tilde{h}$ of $o$ which is a curve tangent to $\tilde{Y}$.
We consider the sets $\widetilde{W}^{cs}(c_o)$ and $\widetilde{W}^{cu}(c_o)$ to be the images by $\tilde{h}$ of the weak stable and weak unstable foliations of $o$. Note that these are a priori only topological objects and not necessarily $C^1$ immersed surfaces. In addition a priori there may be topological crossings, so the collection of these may not form a (topological) branching foliation. Note that in § 6.3 when dealing with strong collapsed Anosov flows these collections will be known to be $C^1$ and tangent to the partially hyperbolic bundles (which is part of the definition of a strong collapsed Anosov flows), and this will make the argument simpler in this setting.

4.2. Transverse surfaces. We consider $S$ a properly embedded surface in $\tilde{M}$ which is uniformly transverse to $\tilde{Y}$. We will assume that for each $o \in O_\phi$ the curve $c_o$ intersects $S$ in at most one point. This is the case, for instance, when $S$ separates $\tilde{M}$ in two connected components, e.g. if $S$ is the lift of a leaf of a Reebless foliation.

It will be important to understand better which curves $c_o$ the surface $S$ intersects. Since $Y$ is only continuous, distinct curves $c_o$ may intersect. Hence it is easier and more convenient to understand this set of intersections with $S$ from the point of view of the orbit space of $\tilde{\phi}_t$ as follows. We define:

$$S_0 = \{ o \in O_\phi : c_o \cap S \neq \emptyset \}.$$ 

**Lemma 4.2.** The set $S_0$ is open in $O_\phi$.

**Proof.** This is just transversality of $S$ and $\tilde{Y}$ and that $\tilde{h}$ maps orbits of $\tilde{\Phi}$ to curves tangent to $\tilde{Y}$. \hfill \Box

We will define a function that indicates how the surface approaches $c_o$, particularly when $o$ is a boundary point of $S_0$. This function depends on some choices, but its asymptotic behavior for points in the boundary is well defined. Fix a transversal $D$ to $\tilde{\phi}_t$. We can define $\tau^S : D \to \mathbb{R} \cup \{ \infty \}$ as

$$\tau^S(y) = \infty \quad \text{if} \quad \tilde{h}(o_y) \cap S = \emptyset,$$

$$\tau^S(y) = t_y \in \mathbb{R} \quad \text{if} \quad \tilde{h}(\tilde{\phi}_t(y)) \in S.$$

The function $\tau^S$ depends on the choice of $D$ which is left implicit. Suppose that the orbit $o$ of $\tilde{\phi}_t$ intersects $D$. Then clearly $c_o$ intersects $S$ if and only if $\tau^S(o \cap D)$ is a real number, that is, it is finite. But in fact one deduces that $\tau^S$ is uniformly bounded in a neighborhood of $o \cap D$ when $c_o$ cuts $S$. Intuitively if $\tau^S(y) > 0$ then $S$ is “in front of” (or flow forward of) $\tilde{h}(y)$ and if $\tau^S(y) < 0$, then $S$ is “behind” $\tilde{h}(y)$. By uniform transversality, it is easy to see that:

**Lemma 4.3.** The function $\tau^S$ is continuous at every point on which $\tau^S$ is finite.

4.3. Boundaries. We want to understand how the function $\tau^S$ goes to infinity when one approaches from $S_0$ the points of $O_\phi$ which are in the boundary of $S_0$.

**Lemma 4.4.** If $o \notin S_0$ intersects the domain $D$ of $\tau^S$, then the function $\tau^S$ is uniformly bounded above in $\mathcal{F}^{ws}(o) \cap D$ and uniformly bounded below in $\mathcal{F}^{cu}(o) \cap D$.

More precisely, for every transverse disk $D$ we have that there exists $K > 0$ such that for every $z \in D$ such that $o_z \notin S_0$ then:

- If $y \in \mathcal{F}^{ws}(o_z) \cap D$ we have that $\tau^S(y) \in (-\infty, K) \cup \{ \infty \}$.
- If $y \in \mathcal{F}^{cu}(o_z) \cap D$ then we have that $\tau^S(y) \in (-K, +\infty) \cup \{ \infty \}$.

**Proof.** By uniform transversality between $S$ and the vector field $Y$, we know that if points $x, y \in \tilde{M}$ verify that $d(x, y) < \varepsilon$ and $z \in S$ then every curve tangent to $Y$ through $y$ will intersect $S$. 


We prove the first property as the second is entirely analogous. Since $h$ is continuous there is $\delta > 0$ so that if $d(x, y) < \delta$ then $d(\tilde{h}(x), \tilde{h}(y)) < \varepsilon$. By the contraction property of Anosov flows we know that there exists a uniformly bounded $t_0 > 0$ such that for every $z \in D$ we have that if $y \in \mathcal{T}^{\text{ops}}(o_x) \cap D$ and $t > t_0$ then $d(\tilde{h}_t(y), \tilde{\varphi}_t(z)) < \delta$. It follows that if $o_x \notin S_0$ but $o_y \in S_0$, then $\tau^S(y) < t_0$. The uniform choice of $t_0$ above implies the result. 

Recall that we denote by $\mathcal{G}^s$ and $\mathcal{G}^u$ to the foliations in $\mathcal{O}_\phi$ induced by $\mathcal{T}^{\text{ops}}$ and $\mathcal{F}^{\text{wu}}$. As a consequence, we get:

**Proposition 4.5.** If $o \in \partial S_0$ then only one of $\mathcal{G}^s(o)$ or $\mathcal{G}^u(o)$ is contained in $\partial S_0$.

The first case happens if there is $V$ a neighborhood of $o \cap D$ in $D$ such that $\tau^S$ is bounded below in $V$ and the second case happens if $\tau^S$ is bounded above in a similar neighborhood.

**Proof.** As in the previous setup, pick $x \in o$ and a disk $D$ transverse to $\tilde{\varphi}_t$ through $x$. Consider $x_n \in D$ so that $x_n \in D$ verify that $o_{x_n} \in S_0$, and $x_n$ converges to $x$. In particular $\tau(x_n)$ is a real number for all $n$. First notice that $|\tau^S(x_n)|$ converges to $\infty$. Otherwise up to subsequence it converges to a real number and then by continuity $S$ intersects $\tilde{h}(o_x)$, contradiction.

For definitness we can assume that there is a subsequence where $\tau^S(x_n) \to +\infty$ (the other case is symmetric). We fix $K > 0$ as in Lemma 4.4 for $D$, so for large enough $n$ we get that $\tau^S(x_n) > K$ and thus for every $y \in \mathcal{T}^{\text{ops}}(o_{x_n}) \cap D$ we have that $o_y \in S_0$. Therefore we get that given $z \in \mathcal{T}^{\text{ops}}(o) \cap D$ there is a sequence $z_n \in D$ such that $z_n \to z$ and $o_{z_n} \in S_0$. Moreover, $o_z \notin S_0$ since $\tau^S(z_n) \to +\infty$: this is because if $o_z \in S_0$ then $\tau^S(z)$ is real and uniformly bounded in a neighborhood of $z$ in $D$. This shows that the local weak stable of the manifold of $o$ is contained in $\partial S_0$.

Now we iterate this analysis. For any $y \in \mathcal{G}^s(o)$ we find $D_1 = D, D_2, ..., D_n$ small disks transverse to $\tilde{\varphi}_t$, consecutive ones intersecting a common orbit of $\tilde{\varphi}_t$, and so that the segment in $\mathcal{G}^s(o)$ between $o$ and $y$ is contained in the union of the orbits of $\tilde{\varphi}_t$ through the $D_i$. In the last paragraph we showed that for any $z \in \mathcal{T}^{\text{ops}}(o) \cap D_1$ we find

$$z_n \in D \text{ with } z_n \to z, \quad o_{z_n} \in S_0, \quad \text{and } \tau^S(z_n) \to +\infty.$$ 

Choose $z$ so that $o_z$ also intersects $D_2$ and now apply the result to $D_2$. We get that for every $y \in \mathcal{T}^{\text{ops}}(o) \cap D_2$ then $o_y \in \partial S_0$. Induction proves that $\mathcal{G}^s(o) \subset \partial S_0$.

In addition $S_0$ is connected and hence in this case intersects only one complementary component of $\mathcal{G}^s(o)$ in $\mathcal{O}_\phi$.

The remaining option in the analysis above is that $\tau^S(x_n)$ converges to $-\infty$ as $x_n$ converges to $x$. Then the same type of arguments show that $\mathcal{G}^u(o) \subset \partial S_0$. In addition for any $z \in \mathcal{T}^{\text{wu}}(o)$ and for $z_n$ in $D$ with $o_{z_n} \in S_0$ converging to $z$ then $\tau^S(z_n)$ converges to $-\infty$.

By the complementary condition these two possibilities are incompatible. This proves the proposition. 

4.4. **Additional invariance.** We now assume in addition that there is a nontrivial $\gamma \in \pi_1(M)$ such that:

(i) $\gamma$ fixes a lozenge $L$ of $\tilde{\varphi}_t$ whose corners we denote by $e_1$ and $e_2$ and such that $\gamma$ acts increasing in the orientation of $e_1$ and decreasingly in the orientation of $e_2$,

(ii) $\gamma(S) = S$,

(iii) $S$ does not intersect $c_1 := e_1 - \tilde{h}(e_1)$ nor $c_2 := e_2 - \tilde{h}(e_2)$,
Proof. The set \( S_0 \) is an open connected set which intersects \( \mathcal{L} \) and it is \( \gamma \)-invariant, because \( S, \mathcal{L} \) and the sides are \( \gamma \) invariant.

Assume that there is a point \( o \in \mathcal{L} \cap \partial S_0 \). Using Proposition 4.5 assume without loss of generality that \( \mathcal{G}^u(o) \subset \partial S_0 \). Since \( o \) is in the interior of the lozenge it follows that \( \mathcal{G}^s(o) \cap \mathcal{G}^s(e_i) \) for \( i = 1, 2 \). Since \( S_0 \) is \( \gamma \)-invariant and \( \gamma \) acts as an expansion on \( \mathcal{G}^s(e_1) \) (resp. \( \gamma \) acts as a contraction on \( \mathcal{G}^s(e_2) \)) we get that \( S_0 \) must accumulate in both \( e_1 \) and \( e_2 \) from inside \( \mathcal{L} \).

However, since \( S_0 \) is connected this is forbiden by the fact that \( \mathcal{G}^u(o) \subset \partial S_0 \) and so \( S_0 \) cannot accumulate in both. This contradiction proves the lemma. \( \square \)

We can now prove

Proof of Proposition 4.6. Since \( \mathcal{L} \subset S_0 \) and \( e_1, e_2 \notin S_0 \) by assumption we deduce that \( e_1, e_2 \in \partial S_0 \).

We can then apply Proposition 4.5 and without loss of generality we assume that \( \mathcal{G}^u(e_1) \subset \partial S_0 \). We will show that \( \tilde{W}^{cu}(e_1) \cap S \) is a curve which is invariant under \( \gamma \).

First, we note that since \( e_1 \) is in the boundary of \( S_0 \) and \( \mathcal{G}^s(e_1) \subset \partial S_0 \) it follows that \( \mathcal{G}^u(e_1) \) is not contained in \( \partial S_0 \). Since \( \mathcal{L} \) is contained in \( S_0 \) which is \( \gamma \) invariant, then it follows that there is a half leaf \( A \) of \( \mathcal{G}^u(e_1) \) (i.e. a connected component of \( \mathcal{G}^u(e_1) \{ e_1 \} \)) such that every \( o \in A \) verifies that \( \tilde{h}(o) \) intersects \( S \). In other words \( A \subset S_0 \).

We know that both \( A \) and \( S \) are \( \gamma \)-invariant. Since for each orbit \( o \) we have that the intersection point \( S \cap \tilde{h}(o) \) is unique, we deduce that \( S \cap \tilde{W}^{cu}(e_1) \) is connected. This implies that it is a \( \gamma \)-invariant curve as desired and that if a homeomorphism preserves \( S \) and \( \tilde{W}^{cu}(e_1) \) then it must preserve this curve. \( \square \)

Remark 4.8. The curve may auto-intersect both in \( \tilde{M} \) (due to the lack of injectivity of \( h \)) as well as in its projection in \( M \) (also due to the action of \( \pi_1(M) \)). But the point is that its projection to \( M \) is the image by a continuous map of a circle and this is all we will use. This is because the curve is preserved by the non trivial element \( \gamma \in \pi_1(M) \).

5. Transverse laminations

We continue in the setup of § 4.1.

Let \( \Lambda \) be a lamination transverse to \( Y \) which verifies that:

- \( \Lambda \) does not have compact leaves,
- the closure of the complementary regions of \( \Lambda \) (if existing) are \( I \)-bundles where \( Y \) is uniquely integrable and such that flowlines of \( Y \) form the \( I \)-bundle structure.

The standing assumption will be that the Anosov flow \( \phi_t \) is not a suspension (see Corollary 2.3). Under this assumption we can show that the assumptions in § 4.4 are verified for any surface \( L \) in the lift of the lamination \( \Lambda \) to the universal cover,
as follows. First notice that since $\Lambda$ is transverse to $Y$ and $\Lambda$ is closed, then $\Lambda$ is uniformly transverse to $Y$. In addition, the second condition above implies that $\Lambda$ can be completed to a foliation $\mathcal{F}$ so that it does not have compact leaves and $\mathcal{F}$ is transverse to $Y$ (this is well known, but see explicit proofs and explanations in [FP, Lemma 3.9]). Since $\mathcal{F}$ does not have compact leaves it is Reebless, therefore any curve in $\tilde{M}$ transverse to $\mathcal{F}$ intersects a leaf of $\mathcal{F}$ at most once. In particular any curve tangent to $\tilde{Y}$ intersects a leaf of $\tilde{\Lambda}$ at most once as required in § 4.2.

**Proposition 5.1.** For every lozenge $\mathcal{L}$ fixed by a non trivial deck transformation $\gamma \in \pi_1(\mathcal{L})$ there is a $\gamma$-invariant leaf $L$ of $\tilde{\Lambda}$ so that $L$ intersects the image by $\tilde{h}$ of some orbit in the interior of the lozenge but does not intersect the image of the corner orbits of $\mathcal{L}$.

**Proof.** Let $o_1, o_2$ be the corners of the lozenge $\mathcal{L}$. Take a leaf $E \in \tilde{\Lambda}$ intersecting $c_1 = \tilde{h}(o_1)$. There is always such a leaf because $c_1$ is a properly embedded curve tangent to $Y$ and the closures of complementary regions of $\Lambda$ are $I$-bundles where $Y$ is uniquely integrable.

**Claim 5.2.** The leaf $E$ cannot intersect $c_2 = \tilde{h}(o_2)$.

**Proof.** Assume that $\gamma$ acts increasingly on $o_1$. Proposition 2.4 implies that $\gamma$ acts decreasingly on $o_2$. Since $\tilde{h}$ commutes with deck transformations we get that the same happens in $c_1$ and $c_2$. Now we again use that $\Lambda$ can be completed to a Reebless foliation. This implies that any leaf of $\tilde{\Lambda}$ separates $\tilde{M}$. Since $\gamma$ acts increasingly in $o_1$ and hence also in $c_1$, it follows that $\gamma(E)$ is on the positive side of $E$ with respect to $Y$. If $E$ intersects $c_2$ then since $\gamma$ acts decreasingly on $o_2$, the same argument shows that $\gamma(E)$ is contained in the negative side of $E$ with respect to $Y$. This is a contradiction and proves the claim.

Therefore, we can consider $V$ to be the region in $\tilde{M}$ between $E$ and $\gamma(E)$. We claim that $c_2$ cannot intersect $V$: $c_2$ is $\gamma$-invariant, and if it intersects $V$ then it must intersect $\gamma^{-1}(V)$ and thus intersect $E$ a contradiction.

Therefore, the open region

$$R = \bigcup_n \gamma^n(V \cup E)$$

contains $c_1$ and is disjoint from $c_2$. The boundary of $R$ is accumulated by translates of $E$ under $\gamma^n$, $n \to \infty$, therefore is saturated by leaves of $\tilde{\Lambda}$. There must be a single leaf $L \in \tilde{\Lambda}$ in the boundary of $R$ that separates $c_1$ from $c_2$. By construction this leaf does not intersect $c_1$ nor $c_2$ and is invariant by $\gamma$.

We need to show that $L$ intersects the image of some orbit in the lozenge. But this is true because otherwise one could connect $c_1$ and $c_2$ by a path with endpoints one in $c_1$ one in $c_2$ and the interior a path which is the projection of a path in the lozenge $\mathcal{L}$. Hence $c_1, c_2$ would be in the same connected component of the complement of $L$. This completes the proof.

As a consequence of Proposition 5.1 and Proposition 4.6 we deduce:

**Corollary 5.3.** A lamination with the properties stated in the beginning of this section verifies that there is a leaf $L \in \tilde{\Lambda}$ invariant under a non trivial $\gamma \in \pi_1(\mathcal{L})$ and a $\gamma$-invariant orbit $o \in \mathcal{O}_o$ such that $\tilde{\mathcal{W}}^s(c_o) \cap L$ contains a curve which is $\gamma$-invariant.
6. Accessibility and ergodicity of partially hyperbolic diffeomorphisms

6.1. Setup. We let \( f : M \to M \) be a collapsed Anosov flow with respect to an Anosov flow \( \phi_t : M \to M \) which is not a suspension. There is a self orbit equivalence \( \beta : M \to M \) and \( h : M \to M \) a map homotopic to the identity, so that \( h \) sends orbits of \( \phi_t \) injectively onto curves tangent to \( E^c \). In addition \( f \circ h = h \circ \beta \).

In addition Remark [BFP, Remark 2.6] shows that for a collapsed Anosov flow, the center bundle \( E^c \) is orientable, and it shows that \( \partial_t h((\phi_t(x)))_{t=0} \) induces an orientation on the center bundle \( E^c \). Therefore we choose \( Y \) to be the vector field of norm one such that

\[
Y(h(x)) \in \mathbb{R}_+ \partial_t h((\phi_t(x)))_{t=0}.
\]

The vector field \( Y \) is contained in the \( E^c \) bundle. Note that this is the context of §4.1.

To prove Theorem A we will assume that the non-wandering set of \( f \) is all of \( M \). This allows us to apply Theorem 2.8. We will assume by contradiction that \( f \) is not accessible so that Theorem 2.8 implies that there is a lamination \( \Lambda^{su} \) tangent to \( E^s \otimes E^u \) which satisfies the conditions of §5.

As explained before, the proof of Theorem A is not affected if we take finite lifts and iterates, so we will assume for simplicity that all bundles of \( f \) are orientable and their orientation is preserved by \( Df \). We will reach a contradiction that will prove that \( f \) is accessible, then the ergodicity part of Theorem A follows immediately from Theorem 2.7.

6.2. Strong collapsed Anosov flows. We first give a direct proof under the extra assumption that \( f \) is a strong collapsed Anosov flow (see [BFP] for a precise definition and discussions). The definition implies in particular that the map \( h \) maps weak stable and weak unstable leaves of \( \phi_t \) into surfaces tangent to \( E^{cs} = E^s \oplus E^c \) and \( E^{cu} = E^c \oplus E^u \) respectively. The reason we first show this case is that here we will not need to use the dynamics at all, and will get a direct contradiction to Proposition 2.9.

Note that by assumption, in this case \( \mathcal{W}^{cs} \) is a surface tangent to \( E^{cs} \) and \( \Lambda^{su} \) is tangent to \( E^s \oplus E^u \), hence the intersection is tangent to \( E^s \). Applying Corollary 5.3 we obtain a curve in \( \tilde{M} \) tangent to \( E^s \) which is \( \gamma \)-invariant. This contradicts Proposition 2.9.

Remark 6.1. As explained above, in principle, collapsed Anosov flows may always have this stronger property (that is, we currently know no example which is a collapsed Anosov flow, but not a strong collapsed Anosov flow).

6.3. Collapsed Anosov flows. The proof of Theorem A is harder since it will need to appeal to [FP] for some cases. To be clear on the dependence on [FP] let us state an unconditional result:

**Theorem 6.2.** Let \( f : M \to M \) be a collapsed Anosov flow associated to a self orbit equivalence \( \beta : M \to M \) of an Anosov flow \( \phi_t \). Assume moreover that the non-wandering of \( f \) is all of \( M \) and that \((M, \phi_t, \beta)\) is not one of the following:

- \( M \) is hyperbolic, \( \phi_t \) is \( \mathbb{R} \)-covered and \( \beta \) is a non-trivial power of a one step up map, or,
- \( M \) is Seifert and \( \beta \) acts in the base as pseudo-Anosov.

Then, \( f \) is accessible.

To complete the proof of Theorem A it is enough to note that in [FP] both cases where treated (see [FP, Theorem B] for the hyperbolic manifold case and [FP, Theorem D] for the Seifert case with base pseudo-Anosov).
The proof of this theorem is just slightly harder than the case of strong collapsed Anosov flows treated in the previous subsection: instead of using Proposition 2.9 we will use that since \( f \) does not belong to one of those classes (due to Proposition ??) the curve given by Corollary 5.3 is also \( \hat{f} \)-invariant for some lift \( \hat{f} \) of some iterate of \( f \) which will be enough to get a contradiction.

We now come back to the setting of § 6.1 and do not make the assumption on \( h \) we did in § 6.2.

**Lemma 6.3.** If \( f \) is not accessible then there is some iterate of \( \beta \) which has a lift \( \hat{\beta} \) to \( \bar{M} \) which fixes some lozenge \( L \) in \( \Omega_\beta \), and so that \( L \) is invariant by some non trivial \( \gamma \in \pi_1(M) \).

**Proof.** We apply Proposition 2.5. Unless the conclusion of the lemma is verified, we get that one of the excluded cases in Theorem 6.2 happens. This completes the proof. \( \square \)

As a consequence we deduce from Corollary 5.3 that there is a leaf \( L \in \Lambda^{su} \) which is invariant under the deck transformation \( \gamma \) as in the previous lemma, as well as \( L \) is invariant under some lift \( \hat{f} : \bar{M} \to \hat{M} \) of an iterate of \( f \). The proof of Theorem A is therefore completed with the following extension of Proposition 2.9:

**Proposition 6.4.** There are no closed curves (not necessarily injectively embedded) invariant under \( f \) in an \( f \) invariant surface tangent to \( E^s \oplus E^u \).  

**Proof.** If an \( f \)-invariant curve is not tangent to \( E^s \), then iterates of it have segments closer and closer to segments in unstable leaves. Again since the curve is invariant one obtains that the curve contains a segment which is contained in an unstable leaf. But then invariance implies that the entire curve is contained in an unstable leaf. This is impossible by Proposition 2.9. \( \square \)

**Remark 6.5.** Another proof of the previous proposition follows from the fact that one can construct a Lyapunov function for the action of \( f \) in the image of \( \eta \) to show that \( f \) is an expansive homeomorphism in the image of \( \eta \) (see \cite{Pot}, §2). Since the circle does not admit expansive homeomorphisms, that gives a contradiction.

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