An ILB- Manifold Structure on the Set of Riemannian Metrics on a Noncompact Manifold

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Abstract

In this paper, using the structures of cone and bicone fields on vector bundles, the author introduces a ILB (inverse limit of Banach)-manifold structure on the space of Riemannian metrics on a non-compact manifold $M$. In the last section, it is proven that, this way, on the open submanifold $M_{\text{finite}}$ of finite volume metrics, the canonical Riemannian metric is defined.

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1 Preliminaries

First, let $M$ be a topological manifold, paracompact, with $\partial M = \emptyset$, which need not to be compact. Let $(E, p, M)$ be a topological vector bundle over $M$.

Definition 1.1. [PA-cones]

A cone field on the vector bundle $(E, p, M)$ is a map $K : M \to \mathcal{P}(E)$, $x \mapsto K(x) \subset E_x$ which satisfies the following two conditions:

(K1) $(\forall)x \in M, K(x)$ is a convex cone, closed in $E_x$, pointed, solid;

(K2) $\cup_{x \in M} \text{Int}(K(x))$ and $\cup_{x \in M}(E_x \setminus K(x))$ are open in $E$.

In the definition above, a convex cone is, following [KRA], a set $K$ which satisfies $K + K \subset K$ and $(\forall) \lambda \geq 0, \lambda K \subset K$. A cone $K$ which satisfies $K \cap -K = \{0\}$ will be called pointed cone, and a solid cone is a cone which has interior points in the topology of $E_x$.

The structure consisting by a vector bundle $(E, p, M)$ and a cone field $K$ on it is denoted by $[(E, p, M); K]$.
Example 1.2. [PA-metrics] Let us consider now the bundle \((S^2T^*M, p, M)\) of 2 times covariant symmetric tensors on a given manifold \(M\). We put 

\[
K_k(x) := \{t_x \in S^2T^*M_x | r(t_x) = i_p(t_x)\},
\]

where \(r\) denotes the rank and \(i_p\) denotes the positive inertia index. Then \(x \mapsto K(x) := \bigcup_{k=1}^n K_k(x)\) defines a cone field on the bundle \((S^2T^*M, p, M)\).

There are local and global properties of this structure, exposed in [PA-cones].

Consider now \(\Gamma^0(E)\), the space of continuous sections of the bundle \((E, p, M)\).

Definition 1.3. [PA-cones]

We call a positive section of the structure \([(E, p, M); K]\), a section \(\sigma \in \Gamma^0(E)\) for which \(\sigma(x) \in K(x), (\forall) x \in M\).

The set of positive sections is denoted by \(K^0\); if on the space \(\Gamma^0(E)\) is considered the graph topology \(W^0\) then we have:

Proposition 1.4. [PA-cones]

The set \(K^0\) is a convex cone, pointed, solid, in \(\Gamma^0(E)\). Moreover, \(K^0\) is a generating cone in \(\Gamma^0(E)\), i.e. \(\forall \sigma \in \Gamma^0(E) (\exists) \zeta_1, \zeta_2 \in K^0 \) such that \(\sigma = \zeta_1 - \zeta_2\).

The cone \(K^0\) defines a partial order relation on \(\Gamma^0(E)\) by

\[
\sigma_1, \sigma_2 \in \Gamma^0(E), \ \sigma_1 \leq \sigma_2 \iff \sigma_2 - \sigma_1 \in K^0.
\]

Proposition 1.5. Papuc [PA-cones]

The pair \((\Gamma^0(E), \leq)\) is an ordered vector space, directed on both sides. Endowed with the \(W^0\) topology is a topological vector space iff \(M\) is a compact manifold.

Given a fixed \(\zeta \in Int(K^0)\), we denote by \(\Gamma^0_{\zeta}\) the set of \(\zeta\)-measurable elements of \(\Gamma^0(E)\):

\[
\Gamma^0_{\zeta} := \{\sigma \in \Gamma^0(E) | (\exists) \lambda \in \mathbb{R}_+: -\lambda \zeta \leq \sigma \leq \lambda \zeta\},
\]

and we consider the map

\[
|\cdot|_{\zeta} : \Gamma^0_{\zeta} \to \mathbb{R}, \ |\sigma|_{\zeta} := \min_{\lambda \in \mathbb{R}_+} \{-\lambda \zeta \leq \sigma \leq \lambda \zeta\}.
\]
Proposition 1.6. \textbf{[PA-cones]} \\
1. $\Gamma_0^\zeta = \Gamma_0^0(E)$ iff $M$ is a compact manifold; \\
2. The map $\|\cdot\|_0^\zeta$ defined by equation (3) from above is a norm on $\Gamma_0^\zeta$; \\
3. The set of all $\Gamma_0^\zeta$ is a covering of $\Gamma_0^0(E)$; \\
4. If $\Gamma_0^0_c(E)$ denotes the subspace of compact support sections, then $\Gamma_0^0(E) \subset \Gamma_0^0_c, (\forall)\zeta \in \text{Int}(K_0^0)$; \\
5. If $\zeta \in \Gamma_0^\zeta_1$, with $\zeta, \zeta_1 \in \text{Int}(K_0^0)$ then $\Gamma_0^\zeta_1 \subset \Gamma_0^\zeta$.

Theorem 1.7. \textbf{[VA]} If $\zeta \in K_0^0$, then $(\Gamma_0^\zeta, \|\cdot\|_0^\zeta)$ is a Banach space.

We consider now $(E, p, M)$, a $C^k$-differentiable bundle over $M$, a $C^k$-differentiable manifold, $k \geq 1$, which need not to be compact.

Definition 1.8. \textbf{[VA-bicone]} A bicone field on a vector bundle $(E, p, M)$ is the structure consisting of a cone field $K$ on the bundle $(E, p, M)$ and a second cone field $K_{TM}$ on the tangent bundle $(TM, p, M)$.

We will denote by $[(E, p, M); K; K_{TM}]$ the structure consisting of a bicone field on the vector bundle $(E, p, M)$.

The existence of a bicone field on a vector bundle $(E, p, M)$ is equivalent with the existence of a non zero section $\zeta \in \Gamma^0(E)$ and of a nonzero vector field on $M$.

Now, as a consequence of the vector bundle isomorphism

$$J^k E \cong \bigoplus_{i=1}^{k} L_s(TM^i, E)$$

from \textbf{[Pal]} page 90, we have

Proposition 1.9. \textbf{[VA-bicone]} If $[(E, p, M); K; K_{TM}]$ is a $C^p$-differentiable vector bundle endowed with a bicone field then the vector bundle $(J^k E, p, M)$ is endowed in a natural way with a cone field $K^k, (\forall)k \leq p$.

Next, as usually, we will denote by convention $J^0 E := E, j^0 \zeta := \zeta$.

Definition 1.10. \textbf{[VA-bicone]} A section $\zeta \in \Gamma^k(E)$ which satisfies $\zeta(x) \in K(x)$ and $j^i \zeta(x) \in K^i(x), i = 0, k, (\forall)x \in M$ will be called section positive up to order $k$.

The set of positive sections up to order $k$ will be denoted by $K^k_+$. On $\Gamma^k(E)$, the space of $C^k$-differentiable sections we will consider the Whitney $WO^k$-topology, which on a space of sections can be given by a base of neighborhoods $W(\sigma_0, U)$, where $\sigma_0 \in \Gamma^k(E)$ and $U$ is an open neighborhood of $Im(j^k \sigma_0)$ in $J^k(E)$:

$$W(\sigma_0, U) := \{\sigma \in \Gamma^k(E) \mid j^k \sigma(x) \in U, (\forall)x \in M\}.$$
Proposition 1.11. \( K^k \) is a convex cone, closed, pointed and solid in the space \((\Gamma^k(E), W^0k)\).

Corollary 1.12. \([KRA]\) The cone \( K^k \) defines on \( \Gamma^k(E) \) an order relation by \( \sigma_1 \leq \sigma_2 \iff \sigma_2 - \sigma_1 \in K^k \). In particular, this relation is directed on both sides.

Let \( \zeta \in Int(K^k) \) be fixed.

Definition 1.13. \([VA\text{-bicone}]\) A section \( \sigma \in \Gamma^k(E) \) for which exists \( \lambda \in \mathbb{R}^+ \) s.t.

\[-\lambda j^i \zeta \leq j^i \sigma \leq \lambda j^i \zeta, \text{ } i = 1, k\]

will be called \( \zeta\)- measurable up to order \( k \).

As in \([PA\text{-cones}]\), we have that the map

\[|\cdot|^k_\zeta : \Gamma^k_\zeta \to \mathbb{R}^+, |\sigma|^k_\zeta := \min\{\lambda \in \mathbb{R}^+_+: -\lambda j^i \zeta \leq j^i \sigma \leq \lambda j^i \zeta, \text{ } i = 1, k\}\]

is a norm on the vector space \( \Gamma^k_\zeta \) of \( \zeta\)- measurable sections up to order \( k \), and with this norm, \( \Gamma^k_\zeta \) becomes a Banach space (the proof is absolutely similar to the one from \([VA])\). The open ball in the norm \( |\cdot|^k_\zeta \), centered in \( \sigma \), of radius \( \epsilon \), will be denoted by \( B^k_\zeta(\sigma, \epsilon) \) and as in \([PA\text{-cones}]\), coincides with the open centered intervals in the order relation from corollary 1.12.

Let us denote now by \( \tau^k \) the topology on \( \Gamma^k(E) \) obtained by taking the path connected components of the \( W^0k \)- topology.

Theorem 1.14. \([VA\text{-bicone}]\) For all \( k \in \mathbb{N} \), the \( \tau^k \)- topology on \( \Gamma^k(E) \) is the topology for which a basis of neighborhoods is given by

\[\{B^k_\zeta(\sigma, \epsilon) \mid \zeta \in Int(K^k), \sigma \in \Gamma^k_\zeta, \epsilon \geq 0\}\].

2 The ILB- manifold Structure on the Space of Riemannian Metrics

Let now \((E, p, M)\) be a smooth vector bundle, endowed with a bicone field defined by the cone fields \( K, K_{TM} \).

Definition 2.1. \([VA\text{-bicone}]\) A smooth section \( \zeta \in \Gamma(E) \) which satisfies \( \zeta(x) \in K^k \), \( \forall k \in \mathbb{N} \) will be called a indefinitely positive section.

We will denote by \( K^k \) the set of indefinitely positive sections.
Proposition 2.2. [VA-bicone] The set $K_\Gamma$ is a (nonempty) pointed closed convex cone in $(\Gamma(E), WO^\infty)$.

Corollary 2.3. [VA-bicone] On $\Gamma(E)$ there is an order relation defined by $\sigma \leq \sigma' \iff \sigma' - \sigma \in K_\Gamma(E)$.

Let $\zeta \in \cap_k Int_{WO^k} K^k_\Gamma$ (this set is nonempty, see [VA-bicone]).

Definition 2.4. [VA-bicone] A section $\sigma \in \Gamma(E)$ which is $\zeta$- measurable ($\forall k \in \mathbb{N}$) will be called an indefinitely $\zeta$- measurable section.

The set $\Gamma_\zeta(E)$ of indefinitely $\zeta$- measurable sections is nonempty (e.g. $\zeta \in \Gamma_\zeta$) and is a vector space.

Proposition 2.5. [VA-bicone] The space $\Gamma_\zeta(E)$ is the projective limit of the Banach spaces $\Gamma^k_\zeta(E)$.

Corollary 2.6. [VA-bicone] The following assumptions hold:
(i) $\Gamma_\zeta(E)$ is a complete, locally convex space;
(ii) The $\tau^\infty$- topology on $\Gamma(E)$ is the topology for which a base of neighborhoods is given by the set
\[
\{ B^k_\zeta(\sigma, \epsilon) \mid \zeta \in \cap_k Int_{WO^k} K^k_\Gamma, \ k \in \mathbb{N}, \ \epsilon \geq 0 \};
\]
(iii) The set $\{ \Gamma_\zeta(E), \Gamma^k_\zeta(E) \mid k \in N(0) \}$ is a ILB (inverse limit of Banach)-chain, following Omori’s definition [OMORI], page 5.

Since in the infinite dimensional geometry the notion of manifold might vary, we will refer in this paper to the notion from [MI-KRI], page170, for which the differences from the finite dimensional correspondent is that for each chart is allow a different model space, and the chart changing is require to be only smooth instead of smooth diffeomorphism.

Theorem 2.7. $\Gamma(E)$ is a smooth manifold modelled by the ILB-spaces $\Gamma_\zeta(E)$.

Proof. From [VA] and [VA-bicone] we have $\Gamma(E) = \lim_{\zeta \in Int(K_\Gamma)} \Gamma_\zeta$. The topology induced above on $\Gamma(E)$ is the $\tau^\infty$- topology. Then, again by the equation above, $\Gamma(E) = \cup_{\zeta \in Int(K_\Gamma)} \Gamma_\zeta(E)$.

Let $\sigma_0 \in \Gamma(E)$. There exists a positive section $\zeta_0 \in Int(K_\Gamma)$ such that $\sigma_0 \in \Gamma_\zeta(E) = \cap_k \Gamma^k_\zeta(E)$. Obviously, $U_{\zeta_0}(\sigma_0) := \cap_k B^k_{\zeta_0}(\sigma_0)$ is a nonempty open in $\tau^\infty$- topology neighborhood of $\sigma_0$. Let $\phi_{\sigma_0} : U_{\zeta_0}(\sigma_0) \subset \Gamma(E) \rightarrow \Gamma_\zeta$ be the restriction of the identity map $Id_{\Gamma_\zeta(E)}$. The pair $(U_{\zeta_0}(\sigma_0), \phi_{\sigma_0})$ is a chart around $\sigma_0$. 

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The charts changing is smooth. Indeed, Let \((U_{\zeta_1}(\sigma_1), \phi_{\sigma_1}), (U_{\zeta_2}(\sigma_2), \phi_{\sigma_2})\) be two charts with \(U_{\zeta_1}(\sigma_1) \cup U_{\zeta_2}(\sigma_2) \neq \emptyset\). In particular, it follows that \(U_{\zeta_1}(\sigma_1) \cup U_{\zeta_2}(\sigma_2) \subset \Gamma_{\zeta_1} \cap \Gamma_{\zeta_2}\). But from [PA-cones], the set \(\{\Gamma_{\zeta}(E)|\zeta \in Int(K)\}\) is ordered and directed on both sides, by the inclusion. So there exists \(\zeta_0 \in Int(K)\) such that \(U_{\zeta_1}(\sigma_1) \cap U_{\zeta_2}(\sigma_2) \subset \Gamma_{\zeta_1} \cap \Gamma_{\zeta_2} \subset \Gamma_{\zeta_0}(E)\), and so the chart changing \(\phi_{\sigma_2} \circ \phi_{\sigma_1}^{-1}\) is the restriction to an open set of the identity map \(Id_{\Gamma_{\zeta_0}(E)}\), and so is smooth. Q.E.D.

**Remark 2.8.** In virtue of the example [L2] \(\Gamma(\mathcal{S}^2T^*M)\), the space of two times covariant, symmetric tensor fields on the manifold \(M\) has the structure of a ILB-manifold, modelled by the spaces \(\Gamma_{\zeta}(\mathcal{S}^2T^*M)\), with \(g \in Int(K)\).

From [GiM-MICH] \(\mathcal{M} = Int(K) \cap \Gamma(\mathcal{S}^2T^*M)\), the space of all Riemannian metrics on the manifold \(M\) is \(\tau^\infty\)- open in \(\Gamma(\mathcal{S}^2T^*M)\).

**Corollary 2.9.** The space \(\mathcal{M}\) of all Riemannian metrics on \(M\) is an open submanifold of \(\Gamma(\mathcal{S}^2T^*M)\).

## 3 The Riemannian Geometry of the Space of Riemannian Metrics of Finite Volume

We denote by \(\mathcal{M}_{finite}\) the set of all Riemannian metrics of finite volume on \(M\).

**Remark 3.1.** \(\mathcal{M}_{finite}\) is \(\tau^\infty\)- open in \(\mathcal{M}\). Indeed, let \((g_n)_{n \geq 0}\) a sequence of Riemannian metrics that converges in the \(\tau^\infty\)- topology to \(g_0\), a finite volume metric. In particular, it follows that \((\forall)n \geq 0, g_n\) and \(g_0\) differ only on a compact set, so each \(g_n\) is a finite volume metric.

On \(\mathcal{M}\) there is a canonical Riemannian metric \(G\), invariant under the natural action by pull- back of the group \(Diff(M)\) of diffeomorphisms of \(M\) on \(\mathcal{M}\), described in [EBIN], or [GiM-MICH]:

\[
G_g : T_g \mathcal{M} \times T_g \mathcal{M} \rightarrow \mathbb{R}, \quad G_g(h, k) = \int_M \text{trace}(g^{-1}hg^{-1}k) d\nu_g, \quad (6)
\]

To make clear the notation \(g^{-1}hg^{-1}k\) we can regard the bundle \(\mathcal{S}^2(T^*M)\) as \(\{h \in \mathcal{L}(TM, T^*M)| h^t = h\}\), subbundle of \(\mathcal{L}(TM, T^*M)\), where \(h^t\) is the composition \(TM \xrightarrow{\iota} T^*M \xrightarrow{h^*} T^*M\). On the other side, since \(g \in \mathcal{M}\), as a Riemannian metric is a fiberwise inner product on \(TM\) it induces a fiberwise inner product on any tensor bundle over \(M\), in particular on \(\mathcal{S}^2(T^*M)\). This is, in fact \(\langle \cdot, \cdot \rangle = \text{trace}(g^{-1} \cdot g^{-1})\). For the metric \(G_g\), instead of the notation above, we will use the classical notations from Riemannian geometry.
(the ’♯’ symbol denotes the ’sharp’ isomorphism induced by the metric $g$ so we will omit to put indices as $♯_g$):

$$G_g : T_g\mathcal{M} \times T_g\mathcal{M} \to \mathbb{R}, \quad G_g(h, k) = \int_{\mathcal{M}} \sum_{i=1}^{n} h(k(E_i)^g, E_i) d\nu_g,$$

Where $(E_i)$ denotes a local field of orthonormal frames.

**Theorem 3.2.** The Riemannian metric $G_g$ is defined on the tangent space $T_g\mathcal{M}_{finite} = \Gamma_g$.

**Proof.** Since $h \in \Gamma_g$, we have $h \in \Gamma^0_g$. This means that $(\exists)\lambda \in \mathbb{R}_+$ s.t. $\lambda g \leq h \leq \lambda g$. Because of equation (3), we have that $-|h|_g^0 g \leq h \leq |h|_g^0 g$. Hence, as in $\text{PA-cones}$, in particular,

$$-|h|_g^0 g(k(E_i)^g, E_i) \leq h(k(E_i)^g, E_i) \leq |h|_g^0 g(k(E_i)^g, E_i), \quad i = 1, n;$$

By summation, we have

$$-|h|_g^0 \sum_{i=1}^{n} g(k(E_i)^g, E_i) \leq \sum_{i=1}^{n} h(k(E_i)^g, E_i) \leq |h|_g^0 \sum_{i=1}^{n} g(k(E_i)^g, E_i),$$

and this means

$$-|h|_g^0 \sum_{i=1}^{n} k(E_i, E_i) \leq \sum_{i=1}^{n} h(k(E_i)^g, E_i) \leq |h|_g^0 \sum_{i=1}^{n} k(E_i, E_i). \quad (7)$$

But $k \in \Gamma_g$, so we have $k \in \Gamma^0_g$. This means that $(\exists)\lambda \in \mathbb{R}_+$ s.t. $\lambda g \leq k \leq \lambda g$. As above, $-|k|_g^0 g \leq k \leq |k|_g^0 g$, and in particular

$$-|k|_g^0 g(E_i, E_i) \leq k(E_i, E_i) \leq |k|_g^0 g(E_i, E_i), \quad i = 1, n;$$

By summation

$$-|k|_g^0 \sum_{i=1}^{n} g(E_i, E_i) \leq \sum_{i=1}^{n} k(E_i, E_i) \leq |k|_g^0 \sum_{i=1}^{n} g(E_i, E_i). \quad (8)$$

From equations (7) and (8) follows that

$$-|h|_g^0 |k|_g^0 \sum_{i=1}^{n} g(E_i, E_i) \leq \sum_{i=1}^{n} h(k(E_i)^g, E_i) \leq |h|_g^0 |k|_g^0 \sum_{i=1}^{n} g(E_i, E_i)$$

and so

$$-n|h|_g^0 |k|_g^0 \leq \sum_{i=1}^{n} h(k(E_i)^g, E_i) \leq n|h|_g^0 |k|_g^0.$$

Now, by integrating with respect to the measure $\nu_g$

$$-n|h|_g^0 |k|_g^0 Vol(M, g) \leq G_g(h, k) \leq n|h|_g^0 |k|_g^0 Vol(M, g)$$

Q.E.D.
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