Exponential growth of solution with $L^p$-norm for class of non-linear viscoelastic wave equation with distributed delay term for large initial data

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Abstract: In this work, we are concerned with a problem for a viscoelastic wave equation with strong damping, nonlinear source and distributed delay terms. We show the exponential growth of solution with $L^p$-norm, i.e., $\lim_{t \to \infty} \|u\|_p^{p} \to \infty$.

Keywords: Strong damping, viscoelasticity, nonlinear source, exponential growth, distributed delay.

MSC: 35L05, 35L20, 58G16, 93D20.

1. Introduction

The well known “Growth” phenomenon is one of the most important phenomena of asymptotic behavior, where many researches omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view. Although the interest of the scientific community for the study of delayed problems is fairly recent, multiple techniques have already been explored in depth.

In this direction, we are concerned with the delayed damped system

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_{0}^{t} \varpi(t-s) \Delta u(s) ds \\
+ \mu_1 u_t + \int_{0}^{\tau_2} |\mu_2(s)| u_t(x, t-s) ds = b|u|^{p-2}u, x \in \Omega, t > 0, \\
u(x, t) = 0, x \in \partial \Omega, \\
u_t(x, -t) = f_0(x, t), (x, t) \in \Omega \times (0, \tau_2), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \end{cases}$$

(1)

where $\omega, b, \mu_1$ are positive constants, $p > 2$ and $\tau_1, \tau_2$ are the time delay with $0 \leq \tau_1 < \tau_2$ and $\mu_2$ is bounded function and $\varpi$ is a differentiable function.

It is well known that viscous materials are the opposite of elastic materials which have the capacity to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of other applied sciences. Many searchers have paid attention to this problem.

In the absence of the strong damping $\Delta u_t$, that is for $w = 0$ and in absence of the distributed delay term. Our problem (1) has been investigated by many authors and results on the local/global existence, and stability have been established. See for example [1–4]. In [5], the authors looked into the following system

$$u_{tt} - \Delta u + \int_{0}^{t} \omega(t-s) \Delta u(s) ds + a(x) u_t + |u|^{q}u = 0,$$

(2)
where decay result of an exponential rate was showed.

In [6], Song and Xue considered with the following viscoelastic equation with strong damping:

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}u, \quad x \in \Omega, t > 0, \\
  u(x,0) = u_0(x), u_t(x,0) = u_1(x).
\end{cases}
\]

(3)

The authors showed, under suitable conditions on \(g\), that there were solutions of (3) with arbitrarily high initial energy that blow up in a finite time. For the same Problem (3), in [7], Song and Zhong showed that there were solutions of (3) with positive initial energy that blew up in finite time. In [8], Zennir considered with the following viscoelastic equation with strong damping:

\[
\begin{cases}
  u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = |u|^{p-2}u, \quad x \in \Omega, t > 0, \\
  u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega \\
  u(x,t) = 0, \quad x \in \partial\Omega.
\end{cases}
\]

(4)

They proved the exponential growth result under suitable assumptions.

In [9] the authors considered the following problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds - \Delta u_t + \varepsilon_1 \Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \quad x \in \Omega, t > 0, \\
  u(x,0) = 0, \quad x \in \partial\Omega, t > 0 \\
  u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega.
\end{cases}
\]

(5)

They proved a blow up result if \(p > m\) and established the global existence.

In this article, we investigated Problem (1), in which all the damping mechanism have been considered in the same time, these assumptions make our problem different from those studied in the literature, specially the Exponential Growth of solutions. We will prove that if the initial energy \(E(0)\) of our solutions is negative (this means that our initial data are large enough), then our local solutions in bounded and

\[
\|u\|^p_p \to \infty,
\]

as \(t\) tends to \(\infty\), used idea in [10–13].

Our aim in the present work is to extend the existing exponential growth results to strong damping for a viscoelastic problem with distributed delay under the following assumptions:

(A1) \(\varphi \in (R_+,R_+)\) is decreasing function so that

\[
\varphi(t) \geq 0 , \quad 1 - \int_0^\infty \varphi(q)\,dq = l > 0.
\]

(7)

(A2) There exists a constant \(\xi > 0\) such that

\[
\varphi'(t) \leq -\xi \varphi(t) , \quad t \geq 0.
\]

(8)

(A3) \(\mu_2 : [\tau_1, \tau_2] \to R\) is bounded function so that

\[
\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(q)|\,dq \leq \mu_1 , \quad \delta > \frac{1}{2}.
\]

(9)

2. Main results

First, as in [14], we introduce the new variable

\[
y(x,\rho, q, t) = u_t(x, t - q\rho),
\]
then we obtain
\[
\begin{align*}
qy_t(x, \rho, q, t) + y_p(x, \rho, q, t) &= 0 \\
y(x, 0, q, t) &= u_1(x, t).
\end{align*}
\] (10)

Let us denote by
\[
(\omega \mu u) = \int_{\Omega} \int_{0}^{t} \omega(t - q)|u(t) - u(q)|^2 dq.
\] (11)

Therefore, Problem (1) takes the form
\[
\begin{align*}
\begin{cases}
2 < p < \frac{2n - 2}{n - 2}, & n \geq 3; \\
p \geq 2, & n = 1, 2.
\end{cases}
\end{align*}
\] (14)

Then for any initial data
\[
(u_0, u_1, f_0) \in \mathcal{H} / \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)),
\]
with compact support, Problem (13) has a unique solution
\[
u \in C([0, T]; \mathcal{H}),
\]
for some $T > 0$.

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time. One can make use of arguments in [16].

**Theorem 2.** Suppose that (7), (8), (9) and (14) hold. If $u_0 \in W, u_1 \in H_0^1(\Omega)$ and
\[
\frac{bc^2}{l} \left( \frac{2p}{(p - 2)l E(0)} \right)^{-\frac{p}{2}} < 1,
\]
where $C_s$ is the best Poincare’s constant. Then the local solution $u(t, x)$ is global in time.
Lemma 1. Assume (7), (8), (9) and (14) hold, let \( u(t) \) be a solution of (12), then \( \mathcal{E}(t) \) is non-increasing, that is
\[
\mathcal{E}(t) = \frac{1}{2} \| u_t \|_2^2 + \frac{1}{2} \left( 1 - \int_0^t (\omega(q)) \| \nabla u \|_2^2 \right) + \frac{1}{2} \int_0^t \int_{\Omega} q |\mu_2(q)| y^2(x, \rho, q, t) dq dx - \frac{b}{p} \| u \|_p^p.
\]
(16)
satisfies
\[
\mathcal{E}(t) \leq -c_1 \left( \| u_t \|_2^2 + \int_\Omega \int_{\tau_1} |\mu_2(q)| y^2(x, 1, q, t) dq dx \right).
\]
(17)

Proof. By multiplying the Equation (12) by \( u_t \) and integrating over \( \Omega \), we get
\[
\frac{d}{dt} \left( \frac{1}{2} \| u_t \|_2^2 + \left( 1 - \int_0^t (\omega(q)) \| \nabla u \|_2^2 \right) + \frac{1}{2} \int_0^t \int_{\Omega} q |\mu_2(q)| y^2(x, \rho, q, t) dq dx \right) - \frac{b}{p} \| u \|_p^p = \frac{1}{2} \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx.
\]
(18)
and, we have
\[
\frac{d}{dt} \frac{1}{2} \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, \rho, q, t) dq dx = \frac{1}{2} \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx - \frac{1}{2} \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx.
\]
(19)
Then, we get
\[
\frac{d}{dt} \mathcal{E}(t) = -\mu_1 \| u_t \|_2^2 - \int_\Omega \int_{\tau_1} q |\mu_2(q)| u_t y(x, 1, q, t) dq dx + \frac{1}{2} \omega' \omega \| u_t \|_2^2 - \frac{1}{2} \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx.
\]
(20)
By (18) and (19), we get (16). Further using Young’s inequality, (7), (8) and (9) in (20), we obtain (17).

Now we are ready to state and prove our main result. For this purpose, we define
\[
H(t) = -\mathcal{E}(t)
\]
(21)
\[
= \frac{b}{p} \| u \|_p^p - \frac{1}{2} \| u_t \|_2^2 - \frac{1}{2} \left( 1 - \int_0^t (\omega(q)) \| \nabla u \|_2^2 \right) + \frac{1}{2} \left( 1 - \omega' \omega \| u_t \|_2^2 - \frac{1}{2} \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx.
\]
(22)
Theorem 3. Suppose that (7)-(9) and (14). Assume further that \( E(0) < 0 \) holds. Then the unique local solution of problem (12) grows exponentially.

Proof. From (16), we have
\[
\mathcal{E}(t) \leq E(0) \leq 0.
\]
(22)
Hence
\[
H' = -\mathcal{E}' \geq c_1 \left( \| u_t \|_2^2 + \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx \right)
\]
\[
\geq c_1 \int_\Omega \int_{\tau_1} q |\mu_2(q)| y^2(x, 1, q, t) dq dx \geq 0,
\]
(23)
and
\[
0 \leq H(0) \leq H(t) \leq \frac{b}{p} \| u \|_p^p.
\]
(24)
We set
\[ K(t) = H(t) + \varepsilon \int_{\Omega} u u_i d\mathbf{x} + \frac{\varepsilon H_1}{2} \int_{\Omega} u^2 d\mathbf{x} + \frac{\varepsilon \omega}{2} \int_{\Omega} (\nabla u)^2 d\mathbf{x}, \] (25)
where \( \varepsilon > 0 \) to be specified later.

Multiplying (12) by \( u \) and taking derivative of (25), we obtain
\[
K'(t) = H'(t) + \varepsilon \|u_i\|^2 + \varepsilon \int_{\Omega} \nabla u \int_{0}^{t} \omega(t-q) \nabla u(q)dqdx \\
- \varepsilon \|\nabla u\|^2 + eb \int_{\Omega} |u|^p d\mathbf{x} - \varepsilon \int_{\Omega} \int_{\tau_1}^{t_2} |\mathcal{M}_2| uy(x,1,q,t)dqdx. \] (26)

Using
\[
\varepsilon \int_{\Omega} \int_{\tau_1}^{t_2} |\mathcal{M}_2| uy(x,1,q,t)dqdx \leq \varepsilon \left\{ \delta_1 \left( \int_{\tau_1}^{t_2} |\mathcal{M}_2| dq \right) \|u\|^2 + \frac{1}{4 \delta_1} \int_{\Omega} \int_{\tau_1}^{t_2} |\mathcal{M}_2| y^2(x,1,q,t)dqdx \right\}, \] (27)

and
\[
\varepsilon \int_{0}^{t} \omega(t-q)dq \int_{\Omega} \nabla u \int_{\Omega} \nabla u(q)dqdx = \varepsilon \int_{0}^{t} \omega(t-q)dq \int_{\Omega} \nabla u \nabla u(q) - \nabla u(t)dqdx + \varepsilon \int_{0}^{t} \omega(q)dq \int_{\Omega} \nabla u(q)dqdx \\
= \frac{\varepsilon}{2} \int_{0}^{t} \omega(q)dq \int_{\Omega} \nabla u^2 - \frac{\varepsilon}{2} (\omega \nabla u). \] (28)

We obtain, from (26),
\[
K'(t) \geq H'(t) + \varepsilon \|u_i\|^2 - \varepsilon \left( 1 - \frac{1}{2} \int_{0}^{t} \omega(q)dq \right) \|\nabla u\|^2 + eb \|u\|^p \\
- \varepsilon \delta_1 \left( \int_{\tau_1}^{t_2} |\mathcal{M}_2| dq \right) \|u\|^2 + \frac{1}{4 \delta_1} \int_{\Omega} \int_{\tau_1}^{t_2} |\mathcal{M}_2| y^2(x,1,q,t)dqdx + \frac{\varepsilon}{2} (\omega \nabla u). \] (29)

Therefore, using (23) and by setting \( \delta_1 \) so that \( \frac{1}{4 \delta_1} = \kappa \), substituting in (29), we get
\[
K'(t) \geq [1 - \varepsilon \kappa] H'(t) + \varepsilon \|u_i\|^2 - \varepsilon \left[ \left( 1 - \frac{1}{2} \int_{0}^{t} \omega(q)dq \right) \|\nabla u\|^2 + eb \|u\|^p \right] \\
- \varepsilon \delta_1 \left( \int_{\tau_1}^{t_2} |\mathcal{M}_2| dq \right) \|u\|^2 + \frac{\varepsilon}{2} (\omega \nabla u). \] (30)

For \( 0 < a < 1 \), from (21)
\[
eb \|u\|^p = ep(1-a) H(t) + \frac{ep(1-a)}{2} \|u_i\|^2 + eba \|u\|^p \\
+ \frac{ep(1-a)}{2} \left( \int_{0}^{t} \omega(q)dq \right) \|\nabla u\|^2 + \frac{ep(1-a)}{2} \left( p(1-a) \omega \nabla u \right) \\
+ \frac{ep(1-a)}{2} \int_{\Omega} \int_{\tau_1}^{t_2} q|\mathcal{M}_2| y^2(x,\rho,q,t)dqdx. \] (31)

Substituting in (30), we get
\[
K'(t) \geq [1 - \varepsilon \kappa] H'(t) + \varepsilon \left[ \left( \frac{p(1-a)}{2} \right) + 1 \right] \|u_i\|^2 \\
+ \varepsilon \left[ \left( \frac{ep(1-a)}{2} \right) \left( \int_{0}^{t} \omega(q)dq \right) \right] \|\nabla u\|^2 + ep(1-a) H(t) + eba \|u\|^p \\
- \frac{\varepsilon}{4 \delta_1 \kappa} \left( \int_{\tau_1}^{t_2} |\mathcal{M}_2| dq \right) \|u\|^2 + ep(1-a) H(t) + eba \|u\|^p \\
+ \frac{ep(1-a)}{2} \int_{\Omega} \int_{\tau_1}^{t_2} q|\mathcal{M}_2| y^2(x,\rho,q,t)dqdx + \frac{ep(1-a)}{2} \left( p(1-a) + 1 \right) (\omega \nabla u). \] (32)
Using Poincare’s inequality, we obtain
\[ K'(t) \geq [1 - \varepsilon \kappa] H'(t) + \varepsilon \left[ \frac{p(1-a)}{2} + 1 \right] \| u_t \|^2_2 + \frac{\varepsilon}{2} (p(1-a) + 1) (\omega \partial_\nu u) + \varepsilon \left( \frac{p(1-a)}{2} - 1 \right) - \int_0^t \omega(q) dq \left( \frac{p(1-a) - 1}{2} \right) \]
\[ - \frac{c}{4c_1 \kappa} \left( \int_{t_1}^{t_2} |\mu_2(q)| dq \right) \| \nabla u \|^2_2 + \epsilon ab \| \mu_2 \|^p_2 + \epsilon (1-a) H(t) + \epsilon \int_0^t \int_{t_1}^{t_2} q |\mu_2(q)| y^2(x, p_2, q, t) dq dp dx. \]  (33)

At this point, we choose \( a > 0 \) so small that
\[ \alpha_1 = \frac{p(1-a)}{2} - 1 > 0, \]
and assume
\[ \int_0^{a_0} \omega(q) dq < \frac{p(1-a) - 1}{\left( \frac{p(1-a) - 1}{2} \right)} = \frac{2\alpha_1}{2\alpha_1 + 1}, \]  (34)
then we choose \( \kappa \) so large that
\[ \alpha_2 = \left( \frac{p(1-a)}{2} - 1 \right) - \int_0^t \omega(q) dq \left( \frac{p(1-a) - 1}{2} \right) - \frac{c}{4c_1 \kappa} \left( \int_{t_1}^{t_2} |\mu_2(q)| dq \right) > 0. \]

Once \( \kappa \) and \( a \) are fixed, we pick \( \epsilon \) so small enough so that
\[ \alpha_4 = 1 - \epsilon \kappa > 0, \]
and
\[ K'(t) \leq \frac{b}{p} \| u \|^p_2. \]  (35)

Thus, for some \( \beta > 0 \), estimate (33) becomes
\[ K'(t) \geq \beta \left\{ H(t) + \| u_t \|^2_2 + \| \nabla u \|^2_2 + (\omega \partial_\nu u) + \| u \|^p_2 + \int_\Omega \int_0^1 \int_{t_1}^{t_2} q |\mu_2(q)| y^2(x, p_2, q, t) dq dp dx \right\}, \]  (36)
and
\[ K(t) \geq K(0) > 0, \quad t > 0. \]  (37)

Next, using Young’s and Poincare’s inequalities, from (25) we have
\[ K(t) = H + \varepsilon \int_\Omega uu_t dx + \frac{\varepsilon H_1}{2} \int_\Omega u^2 dx + \frac{\varepsilon \omega}{2} \int_\Omega \nabla u^2 dx \]
\[ \leq c[H(t) + \| u \|^2_2 + \| \nabla u \|^2_2] \leq c[H(t) + \| \nabla u \|^2_2 + \| u \|^2_2]. \]  (38)

For some \( c > 0 \), Since, \( H(t) > 0 \), we have from (12)
\[ -\frac{1}{2} \| u \|^2_2 - \frac{1}{2} \left( 1 - \int_0^t \omega(q) dq \right) \| \nabla u \|^2_2 - \frac{1}{2} \omega \partial_\nu u \]
\[ -\frac{1}{2} \int_\Omega \int_0^1 \int_{t_1}^{t_2} q |\mu_2(q)| y^2(x, p_2, q, t) dq dp dx + \frac{b}{p} \| u \|^p_2 > 0, \]  (39)
then
\[ \frac{1}{2} \left( 1 - \int_0^t \omega(q) dq \right) \| \nabla u \|^2_2 < \frac{b}{p} \| u \|^p_2 < \frac{b}{p} \| u \|^p_2 + (\omega \partial_\nu u) + \int_\Omega \int_0^1 \int_{t_1}^{t_2} q |\mu_2(q)| y^2(x, p_2, q, t) dq dp dx. \]
In the other hand, using (7), to get
\[
\frac{1}{2} (1 - l) \| \nabla u \|_2^2 < \frac{b}{p} \| u \|_p^p < \frac{b}{p} \| u \|_p^p + (\omega \nabla u) + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x, \rho, q, t)dqdx. \tag{40}
\]
Consequently,
\[
\| \nabla u \|_2^2 < \frac{2b}{p} \| u \|_p^p + 2(\omega \nabla u) + l \| \nabla u \|_2^2 + 2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x, \rho, q, t)dqdx. \tag{41}
\]
Inserting (41) into (38), to see that there exists a positive constant \( k_1 \) such that
\[
\mathcal{K}(t) \leq k_1 [H(t) + \| \nabla u \|_2^2 + \| u_t \|_2^2 + \frac{b}{p} \| u \|_p^p + (\omega \nabla u)(t) + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\mu_2(q)|y^2(x, \rho, q, t)dqdx, \forall t > 0. \tag{42}
\]
From inequalities (36) and (42) we obtain the differential inequality
\[
\mathcal{K}'(t) \geq \lambda \mathcal{K}(t), \tag{43}
\]
where \( \lambda > 0 \), depending only on \( \beta \) and \( k_1 \).

A simple integration of (43), we obtain
\[
\mathcal{K}(t) \geq \mathcal{K}(0)e^{\lambda t}, \forall t > 0. \tag{44}
\]
From (24) and (35), we have
\[
\mathcal{K}(t) \leq H(t) \leq \frac{b}{p} \| u \|_p^p. \tag{45}
\]
By (44) and (45), we have
\[
\| u \|_p^p \geq Ce^{\lambda t}, \forall t > 0.
\]
Therefore, we conclude that the solution in the \( L_p \)-norm grows exponentially. This completes the proof. \( \Box \)

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References
[1] Bouhali, K., & Ellaggoune, F. (2018). Existence and decay of solution to coupled system of viscoelastic wave equations with strong damping in \( \mathbb{R}^N \). Boletim da Sociedade Paranaense de Matematica, doi:10.5269/bspm.41175.
[2] Cavalcanti, M. M., Domingos Cavalcanti, V. N., & Ferreira, J. (2001). Existence and uniform decay for a non-linear viscoelastic equation with strong damping. Mathematical Methods in Applied Sciences, 24(14), 1043-1053.
[3] Piskin, E. (2015). Growth of solutions with positive initial energy to systems of nonlinear wave equations with damping and source terms. Advances in Mathematical Physics, 2015.
[4] Piskin, E., & Ekinci, F. (2019). Blow up, exponential growth of solution for a reaction-diffusion equation with multiple nonlinearities. Tbilisi Mathematical Journal, 12(4), 61-70.
[5] Cavalcanti, M. M., Cavalcanti, V. D., Prates Filho, J. S., & Soriano, J. A. (2001). Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. Differential and Integral Equations, 14(1), 85-116.
[6] Song, H., & Xue, D. (2014). Blow up in a nonlinear viscoelastic wave equation with strong damping. Nonlinear Analysis: Theory, Methods & Applications, 109, 245-251.
[7] Song, H., & Zhong, C. (2010). Blow-up of solutions of a nonlinear viscoelastic wave equation. Nonlinear Analysis: Real World Applications, 11(5), 3877-3883.
[8] Zennir, K. (2013). Exponential growth of solutions with \( L_p \)-norm of a nonlinear viscoelastic hyperbolic equation. Journal of Nonlinear Sciences & Applications (JNSA), 6(4), 252-262.
[9] Guo, L., Yuan, Z., & Lin, G. (2015). Blow up and global existence for a nonlinear viscoelastic wave equation with strong damping and nonlinear damping and source terms. *Applied Mathematics, 6*(5), 806-816.

[10] Braik, A., Miloudi, Y., & Zennir, K. (2018). A finite-time blow-up result for a class of solutions with positive initial energy for coupled system of heat equations with memories. *Mathematical Methods in the Applied Sciences, 41*(4), 1674-1682.

[11] Benaissa, A., Ouchenane, D., & Zennir, K. (2012). Blow up of positive initial-energy solutions to systems of nonlinear wave equations with degenerate damping and source terms. *Nonlinear Studies, 19*(4), 523-535.

[12] Ouchenane, D., Zennir, K., & Bayoud, M. (2013). Global nonexistence of solutions for a system of nonlinear viscoelastic wave equations with degenerate damping and source terms. *Ukrainian Mathematical Journal, 65*(5), 723-739.

[13] Zennir, K. (2014). Growth of solutions with positive initial energy to system of degenerately damped wave equations with memory. *Lobachevskii Journal of Mathematics, 35*(2), 147-156.

[14] Nicaise, S., & Pignotti, C. (2008). Stabilization of the wave equation with boundary or internal distributed delay. *Differential and Integral Equations, 21*(9-10), 935-958.

[15] Georgiev, V., & Todorova, G. (1994). Existence of a solution of the wave equation with nonlinear damping and source terms. *Journal of Differential Equations, 109*(2), 295-308.

[16] Wu, S. T., & Tsai, L. Y. (2006). On global existence and blow-up of solutions for an integro-differential equation with strong damping. *Taiwanese Journal of Mathematics, 10*(4), 979-1014.

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