Quantum collapse dynamics with attractive densities

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Abstract

We discuss a model of spontaneous collapse of the quantum state that does not require adding any stochastic processes to the standard dynamics. The additional ingredient with respect to the wave function is a position in the configuration space, which drives the collapse in a completely deterministic way. This new variable is equivalent to a set of positions of all the particles, i.e. a set of Bohmian positions, which obey the usual guiding equation of Bohmian theory. Any superposition of quantum states of a macroscopic object occupying different regions of space is projected by a localization process onto the region occupied by the positions. Since the Bohmian positions are well defined in a single realization of the experiment, a space localization into one region is produced. The mechanism is based on the correlations between these positions arising from the cohesive forces inside macroscopic objects.

The model introduces two collapse parameters, which play a very similar role to those of the GRW and CSL theories. With appropriate values of these parameters, we check that the corresponding dynamics rapidly projects superpositions of macroscopic states localized in different regions of space into a single region, while it keeps a negligible effect in all situations where the predictions of standard quantum dynamics are known to be correct. The possible relations with gravity are briefly speculated. We then study the evolution of the density operator and a mean-field approximation of the dynamical equations of this model, as well as the change of the evolution of the momentum introduced by the localization process. Possible theoretical interpretations are finally discussed. Generally speaking, this model introduces a sharper border between the quantum and classical world than the GRW and CSL theories, and leaves a broader range of acceptable values for the parameters.

Contents

1 Dynamic equation with a localization term
  1.1 Densities .........................................................
  1.2 Attractive dynamics ........................................
  1.3 Modified Schrödinger equation ..............................
  1.4 Coupled evolutions ...........................................

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The standard linear Schrödinger equation predicts the possible occurrence of quantum superpositions of macroscopically distinguishable states (QSMDS). This leads for instance to the famous Schrödinger cat paradox [1,2], to the so called measurement problem, etc. Nevertheless, such QSMDS are apparently never observed, even in experiments involving “macroscopic quantum phenomena” [3] such as superfluidity or superconductivity. The problem arising from this apparent contradiction has given rise to a huge literature [4,5]. Since Bohr, numerous authors have proposed various interpretations of quantum mechanics to deal with this problem [6]. Another possible approach, nevertheless, is to forbid the occurrence of superpositions of QSMDS by modifying the dynamics of quantum mechanics, for instance by adding a small non-linear and stochastic term to the Schrödinger equation. This is the basic idea of “spontaneous collapse theories”, such as the GRW [7] and CSL [8,9] theories; for reviews, see for instance [10] or [11]. It has also been proposed to introduce collapse mechanisms that are driven by gravity [12–15]. A common feature of these theories is the introduction of a stochastic term in the Schrödinger dynamics.

Here we propose a model of spontaneous collapse where the dynamics is deterministic – as we will see, it is actually more a class of models than a specific model since, for instance, the localization function can be chosen in several ways. Instead of adding stochastic processes or functions to the standard quantum description by a wave function, in the configuration space we add a position that drives the collapse mechanism. Adding positions to standard quantum mechanics is of course the basic idea of the de Broglie-Bohm (dBB) interpretation [16–21]. Our model can therefore be seen as nothing but a combination of the dBB theory with spontaneous collapse theories.

A basic remark is that, in most cases, the number density of the Bohmian positions in ordinary space coincides almost perfectly with the quantum single-particle-density obtained from the many-body state...
vector; this is a consequence of the so called “quantum equilibrium” condition, which in turn results from a dynamic that forces the Bohmian positions to follow the wave function. Nevertheless, in Schrödinger-cat like situations, or after a quantum measurement has been performed, this is no longer the case: on the one hand, because the quantum state splits into several macroscopically distinct components, the quantum density divides into two (or more) disconnected regions of space, corresponding for instance to different positions of the pointer of the measurement apparatus; on the other hand, the individual Bohmian positions of the particles must remain all clustered together in only one of these regions. This clustering is a consequence of the internal cohesive forces inside solid objects: the quantum Hamiltonian allows quantum superpositions of states where all particles move together in one, or another, region of space, but forbids states where some of the particles are in one region and others in another region (the cohesive forces inside the pointer of the measurement apparatus forbid states where the pointer is broken into two parts). Because the set of all the Bohmian positions must define a point in configuration space where the wave function does not vanish, they must remain grouped together.

As a consequence, in one region of space (occupied for instance by the pointer indicating a definite result), the single particle Bohmian density is much larger than the density predicted by the quantum superposition; in another region of space it is smaller, since the Bohmian density vanishes while the quantum density does not. The basic idea of our model is to introduce a dynamics where the quantum state vector is attracted to the first region, and repelled from the others. The quantum dynamics obtained in this way is completely deterministic: in a given realization of an experiment, the only random element is the initial Bohmian position of the configuration space of the physical system; once these positions are determined, no random process takes place (as is also the case in dBB theory).

Our purpose is not, of course, to claim that the dynamics we propose is highly plausible. The main conceptual interest of such models is their very existence, which proves that such approaches are neither impossible nor contradictory with known experimental results. This is similar to the existence of the dBB theory, which shows that some theorems concerning the impossibility of additional quantum variables are irrelevant. In a previous article [22], we have already proposed a dynamics that also includes an attraction of the state vector towards regions of high Bohmian densities. Here we generalize and improve that model by introducing a spatial localization term that introduces even smaller perturbations (except, of course, situations involving QSMDS), because the added differential term in the dynamics remains almost zero in most cases. As a consequence, a larger flexibility is obtained for the values of the parameters of the dynamics; a relation with the Newton constant of gravity then becomes possible.

1 Dynamic equation with a localization term

We consider a system of \( N \) identical spinless particles associated with a quantum field operator \( \Psi (r) \), which is defined at each point \( r \) of ordinary 3D space.

1.1 Densities

When the system is in state \( |\Phi\rangle \), the local (number) density \( D_\Phi (r) \) of particles at \( r \) is:

\[
D_\Phi (r) = \frac{\langle \Phi | \Psi^\dagger (r) \Psi (r) |\Phi\rangle}{\langle \Phi | \Phi\rangle}
\]  

(1)
In dBB theory, the local density $D_B(r)$ of Bohmian positions is a sum of delta functions:

$$D_B(r,t) = \sum_{n=1}^{N} \delta(r - q_n)$$ (2)

where the sum runs over all $N$ particles with Bohmian position $q_n(t)$.

We wish to introduce a dynamics that favors evolutions where $D_\Phi(r)$ is attracted towards regions where $D_B(r)$ takes higher values than $D_\Phi(r)$, and repelled from regions where the opposite is true. Nevertheless, since $D_B(r)$ is singular, it is useful to introduce a space average. For this purpose, we choose a distance $a_L$ and a function $A(r)$ that is localized around the origin of space within a distance $a_L$, for instance:

$$A_L(r) = e^{-\frac{(r - r')^2}{(a_L)^2}}$$ (3)

We then introduce the following integrals of $D_\Phi$ and $D_B$:

$$N_\Phi (r, t) = \int d^3 r' A_L(r - r') D_\Phi (r', t)$$ (4)

and:

$$N_B (r, t) = \int d^3 r' A_L(r - r') D_B (r', t) = \sum_{n=1}^{N} A_L(r - q_n)$$ (5)

The Gaussian form (3) is one possibility, but we could also have made different choices, for instance:

$$A_L(r) = \frac{(a_L)^s}{(a_L)^s + |r|^s}$$ (6)

where $s$ is any integer number that is larger than 2. All the discussion of this article is actually independent of a particular choice of the localization function.

The order of magnitude of $N_\Phi (r, t)$ is the (quantum) average number of particles within a volume $(a_L)^3$ around point $r$; similarly, the order of magnitude of $N_B (r, t)$ is the number of Bohmian positions inside the same volume. We have $0 \leq N_\Phi (r, t), N_B (r, t)$. Since:

$$\int d^3 r N_\Phi (r, t) = \int d^3 r N_B (r, t) = N \int d^3 r A_L(r)$$ (7)

both these numbers have an upper bound that is $N$ times the space integral of $A_L(r)$.

### 1.2 Attractive dynamics

We then define the (dimensionless) localization operator $L(t)$ by:

$$L(t) = \int d^3 r \Delta(r, t) \Psi^\dagger(r) \Psi(r)$$ (8)

where $\Delta(r, t)$ is defined as the difference:

$$\Delta(r, t) = N_B(r, t) - N_\Phi(r, t)$$ (9)
This allows us to introduce a dynamics that favors evolutions where $D_\Phi (r)$ is attracted towards regions where $\Delta (r, t) > 0$. For this purpose, we add to the usual Hamiltonian $H (t)$ a localization term that is proportional to $L (t)$, and write the modified Schrödinger equation:

$$i\hbar \frac{d}{dt} |\Phi (t)\rangle = \left[ H (t) + i\hbar \gamma_L L (t) \right] |\Phi (t)\rangle$$

(10)

where $\gamma_L$ is a constant localization rate. The new term in the Hamiltonian increases the modulus of the wave function in regions where $\Delta (r, t)$ is positive, reduces it in regions where the opposite is true.

Relation (7) implies that the space integral of $\Delta (r, t)$ vanishes:

$$\int d^3r \Delta (r, t) = 0$$

(11a)

But, as in [22], we could also have defined $\Delta (r, t)$ as:

$$\Delta (r, t) = N_B (r, t)$$

(11b)

The right hand side of relation (11a) would then be given by 5.

Since the operator acting in the right-hand side of (10) is not Hermitian, this equation of evolution does not conserve the norm of $|\Phi\rangle$. Nevertheless, if desired, one can easily obtain a normalized state vector $|\overline{\Phi}\rangle$:

$$|\overline{\Phi}\rangle = |\Phi (t)\rangle = \frac{1}{\sqrt{\langle \Phi (t) | \Phi (t) \rangle}} |\Phi (t)\rangle$$

(12)

which obeys the following equation of evolution:

$$\frac{d}{dt} |\overline{\Phi} (t)\rangle = \left[ H (t) + i\hbar \gamma_L L (t) \right] |\overline{\Phi} (t)\rangle - \gamma_L \frac{1}{3/2} \left( \frac{d}{dt} \langle \Phi (t) | \Phi (t) \rangle \right) |\Phi (t)\rangle$$

(13)

We then obtain:

$$i\hbar \frac{d}{dt} |\overline{\Phi} (t)\rangle = \left[ H (t) + i\hbar \gamma_L \overline{\mathcal{L}} (t) \right] |\overline{\Phi} (t)\rangle$$

(14)

with:

$$\overline{\mathcal{L}} (t) = \int d^3r \left[ \Psi^\dagger (r) \Psi (r) - D_\Phi (r) \right] \Delta (r, t)$$

(15)

If $\Delta (r, t)$ is a constant in space, we remark that the effect of $\overline{\mathcal{L}} (t)$ on any ket with a fixed number of particles vanishes. This is of course the case if $a_L = 0$ ($N_B$, $N_\Phi$ and $\Delta$ then vanish), but also if $a_L = \infty$ (then $N_B = N_\Phi$ and $\Delta (r, t)$ vanishes again). The localization term is effective only if $a_L$ takes an intermediate, finite, value for which $\Delta (r, t)$ varies in space.
1.3 Modified Schrödinger equation

We now study the modified Schrödinger equation (14) in the position representation. The localization operator (15) contains a first term in $\Psi^\dagger(r) \Psi(r)$ that has the form of a (symmetric) potential operator, diagonal in the position representation. This operator can also be written as a summation over all particles:

$$\int d^3r \left[ \Psi^\dagger(r) \Psi(r) \right] \Delta(r,t) = \sum_{n=1}^{N} \Delta(R_n,t)$$  \hspace{1cm} (16)

where $R_n$ is the position operator associated with the position of particle $n$. We therefore have:

$$\langle 1 : r_1 ; 2 : r_2 ; \ldots ; N : r_N | \int d^3r \Psi^\dagger(r) \Psi(r) \Delta(r,t) | \Phi(t) \rangle = \sum_{n=1}^{N} \Delta(r_n,t) \Phi(r_1, r_2, \ldots, r_n, \ldots, r_N, t)$$  \hspace{1cm} (17)

where $\Phi(r_1, r_2, \ldots, r_n, r_N, t)$ is the wave function representing the $N$ particle system in configuration space (for the sake of simplicity, we assume that the particles have no spin).

As for the second term in the right hand side of (15), it is just a c-number, proportional to the constant $<\Delta>$ defined by:

$$<\Delta> = \frac{1}{N} \int d^3r \frac{D}{N} \Phi(r) \Delta(r,t)$$  \hspace{1cm} (18)

Since $D\Phi(r)/N$ is a distribution over space that is normalized to unity, $<\Delta>$ is the average of $\Delta(r,t)$ over the one-body density of the wave function.

The evolution of the wave function due to the localization term is therefore:

$$\frac{d}{dt} |_\text{loc} \Phi(r_1, r_2, \ldots, r_n, r_N, t) = \gamma_L \left[ \sum_{n=1}^{N} \Delta(r_n,t) - N <\Delta> \right] \Phi(r_1, r_2, \ldots, r_n, \ldots, r_N, t)$$  \hspace{1cm} (19)

The wave function tends to increase in regions where many $\Delta(r_n,t)$ are larger than their space average value $<\Delta>$, and tends to decrease in regions of space where the opposite is true.

1.4 Coupled evolutions

We assume that the Bohmian positions $q_n$ evolve according to the usual Bohmian equation of motion:

$$\frac{dq_n(t)}{dt} = \frac{\hbar}{m} \nabla_n \xi$$  \hspace{1cm} (20)

where $\xi(r_1, r_2, \ldots, r_N)$ is the phase of the wave function $\Phi(r_1, r_2, \ldots, r_N)$, and $\nabla_n$ the gradient taken with respect to $q_n$. In standard dBB theory, this relation ensures that the condition of “quantum equilibrium” is satisfied at any time, if it is satisfied at the initial time. Nevertheless, this assumes that the equation of evolution of the wave function is the standard Schrödinger equation, which is no longer the case in our model. In [22], we argued that this was not a serious problem, since the localization term is very small, while Towler, Russell and Valentini [24,25] have shown that a fast relaxation process drives the system quickly back to quantum equilibrium. Assuming quantum equilibrium should therefore still be an excellent approximation in most cases; we come back to this point in more detail in §2.2.
2 Collapse dynamics and time constants, examples

We now explore the predictions of the model for different choices of the constants. We will see that the model is very robust: even with a large variation of its constants, it remains compatible with experimental observations. To illustrate this point, we choose either of the following pair of values:

\begin{align}
\gamma_L &= 10^{-24} \text{ s}^{-1} \\
a_L &= 1 \mu\text{m} = 10^{-6} \text{ m}
\end{align}

(21a)

or the same values as those of GRW and CSL:

\begin{align}
\gamma_L &= 10^{-16} \text{ s}^{-1} \\
a_L &= 10 \mu\text{m} = 10^{-5} \text{ m}
\end{align}

(22a)

We now examine a few situations where the consequences of these choices can be evaluated, depending on whether a superposition of macroscopically distinct states is involved or not.

2.1 Various situations

In \([15]\), only the first term in the bracket defining the localization operator \(\mathcal{L}(t)\) is physically effective. This is because the second term in \(D_\Phi(r)\) introduces only a change of the norm of the whole state vector, which does not change its physical content; we will therefore ignore it in this section. Moreover, if \(\Delta(r,t)\) is uniform in space, condition (11a) shows that \(\mathcal{L}(t)\) vanishes: the localization operator is non-zero only if, after averaging over a volume \((a_L)^3\), the Bohmian and quantum densities still have different variations in space. Consider a region of space \(R\) where the difference \(N_B(r,t) - N_\Phi(r,t)\) takes a typical value \(\Delta_R\); a localization process occurs in this region with a rate \(\gamma_L \Delta_R N_R\), where \(N_R\) is the number of particles contained in \(R\). A distortion of the wave function, and therefore a physical modification of the properties of the system, occurs only if two (or more) regions \(R\) and \(R'\) have values for \(\Delta_R\) and \(\Delta_{R'}\) with different values.

For microscopic systems, the effect of the localization term remains extremely slow. Since both \(\Delta_R\) and \(N_R\) have an upper bound equal to the number of particles \(N\), no localization in any region can occur at a rate higher than \(\gamma_L N^2\). If we assume for instance \(N = 10^6\), with both choices (21) and (22) we obtain upper bounds of the localization rate of the order of \(10^{-4} \text{ s}^{-1}\), or even much less. Moreover, these upper bounds can be approached only if the sign of \(\Delta(r,t)\) is opposite in two different regions of space extending over more that \(a_L\): if a microscopic system is localized in space in a region smaller than \(a_L\), its localization rate is therefore even much smaller.

Clearly, to obtain a significative localization effect, it is required to have at the same time a large number of particles involved and spatial separations exceeding \(a_L\). The optimal situation to detect an effect with a microscopic or mesoscopic system would probably be an interference experiment with a very large molecule or cluster \([26]\), assuming that the distance between the slits is larger than \(a_L\). Nevertheless, even with \(10^8\) particles in the cluster, the time of flight along the two different paths should be at last 1 second for a significant localization effect to be obtained if (22a) is selected, more than one year if (21a) is selected. In most cases, mesoscopic systems seem to be unaffected by the localization term.

The situation is radically different if a QSMDS is created, as is indeed the case during a quantum measurement. We assume that the distance between the positions of the pointer of the measurement apparatus indicating different results (or the distance between the positions of macroscopically distinct
states) is larger than $a_L$. For simplicity, we assume that only two positions are possible, corresponding to two results of measurement (the generalization to more results is trivial). In this case, $\Delta(r,t)$ has large but opposite values in two distant regions of space. This is because $N_\phi(r,t)$ and $N_B(r,t)$ have completely different behaviors: on the one hand, the quantum density of particles and $N_\phi(r,t)$ is distributed among two wave packets; on the other hand, in a single realization of the experiment, the Bohmian factor $N_B(r,t)$ vanishes in one of the wave packets (the empty wave packet), while it takes a maximum value in the other. As discussed in the introduction, this results from the cohesive forces inside the pointer, which create strong quantum correlations between the positions of its constituent particles: in quantum mechanics, these particles can be at the same time in two different regions of space, but they have to remain all together in the same region. Therefore, since the Bohmian positions must define a point in configuration space where the N-particle wave function does not vanish, they have to remain grouped: all of them are in the same wave packet, none is in the other (empty wave). In one channel, $\Delta(r,t) \approx N_B(r,t) - N_\phi(r,t) \approx +N_\phi(r,t)$, in the other channel $\Delta(r,t) \approx 0 - N_\phi(r,t) \approx -N_\phi(r,t)$.

In a quantum measurement situation, we can for instance assume that the apparatus contains a pointer that is a solid containing $N_\phi \approx 10^{11}$ particles per cubic micron (this rough order of magnitude seems to be reasonable for the number of atoms; the number of nucleons, or electrons, would be larger). If the total number of particles in the pointer is $N_P$, the differential rate of relaxation between the full wave (that associated with the result of measurement) and the other empty wave(s) is of the order of $2\gamma N_\phi N_P$. If for instance the pointer is a tiny cube with $100\mu$m side only, $N_P \approx 10^{17}$, and we see that a superposition of two (or more) spatially separate states of the pointer disappears in about $10^{-4}$s, with \(11\), or $10^{-12}$ s. with \(12\). This is the time it takes the measurement apparatus to display a definite result. The model therefore ensures a rapid collapse of the wave function, even for tiny apparatuses of measurement. Another remark is that the so called “surrealistic trajectories” should not exist within this model; actually, even within standard dBB theory, they already do not occur with macroscopic bodies \(27\).

We also note that, even if $N_\phi(r,t)$ may strongly differ from $N_B(r,t)$ during a single realization of the experiment, this is only a short transient effect taking place while the measurement is completed; then the additional localization term in the Schrödinger equation rapidly ensures that $N_\phi(r,t)$ relaxes towards $N_B(r,t)$. It modifies the state vector so that $N_\phi(r,t)$ vanishes in all wave packets but one. After the measurement is completed, the dynamical relaxation process studied in \(24,25\) ensures that the difference $N_\phi(r,t) - N_B(r,t)$ tends rapidly to zero. We discuss in § \(22\) why it is possible to assume that the quantum equilibrium condition is restored when a second experiment is started.

One may wonder if the addition of a nonlinear localization term in the dynamics could produce dramatic unexpected effects, despite the extremely small value of the nonlinear coefficient. Indeed, the very purpose of the model is to obtain a dramatic effect during a measurement process: the state vector is suddenly projected onto one of its components, and all empty waves disappear. Does this extend to other situations? Mathematically, a similar question occurs with the Gross-Pitaevskii equation, which describes interacting Bose-Einstein condensates within mean field theory: an ideal gas is only marginally stable, since an infinitesimal attractive nonlinear term is sufficient to produce a collapse of the boson gas. The question then is: in what circumstances is the standard dBB theory only marginally stable with respect to the addition of an arbitrarily small nonlinear perturbation? Within our model, strong localizations effects occur as soon as the Bohmian density differs significantly from the quantum density. In the absence of a QSMDS states, there is no special reason why this difference should be large. In a localized piece of bulk solid for instance, the Bohmian positions of the particles are randomly distributed in the volume occupied by the solid, and the coincidence between the two densities is rather good; the two terms in \(19\) then almost cancel each other (moreover, even without this cancellation, the localization
term would only then to localize the solid inside its own volume, with no dramatic effect). The localization term is efficient mostly in the presence of quantum superpositions where many particles occupy different regions of space, that is basically QSMDS.

This discussion of various possible physical situations shows that a broad class of models is indeed compatible with the experimental observations that are known at present. In fact, we can choose either definition (9) or (11b) of $\Delta (r, t)$, and then select either (21) or (22) for the constants of the model: in all cases we obtain a fast time constant for the appearance of a single result of measurement, without introducing appreciable perturbations of microscopic systems. The basic reason for this flexibility is the quadratic dependence of the product $N_q N_p$ in the density of macroscopic objects, which introduces a fast relaxation rate even with very small values of $\gamma_L$. In other words, for macroscopic objects, the relaxation rate varies proportionally to the square of the Avogadro number, which is an enormous number.

### 2.2 Quantum equilibrium

The quantum equilibrium condition applies in configuration space, and therefore introduces more stringent condition than the equality of densities in ordinary space discussed in the preceding subsection. It also relates to an ensemble of realizations of the same experiment. Assuming that, initially, the Bohmian positions are randomly distributed, and that their distribution coincides with the quantum probability density (the modulus square of the wave function in the configuration space), the usual dBB theory ensures that the coincidence remains exact at all times.

This condition cannot be directly transposed to our modified dynamics, where different Bohmian positions associated with the same initial wave function lead to different wave functions at later times. The distribution of the positions can then no longer be compared to a single quantum probability density. For instance, just after a measurement has been performed (or, more generally, when a QSMDS has been projected onto one of its macroscopic components), the physical system is described by several different wave functions, depending on the result obtained in the experiment. We therefore have to modify the condition by requesting that, if we consider only the sub-ensemble of realizations having provided a specific result of measurement, the new wave function provides a density in configuration space that matches the distribution of Bohmian positions for this sub-ensemble. Clearly, this condition cannot be exactly fulfilled at all times, in particular during the (very short) projection process of the wave function.

Nevertheless, once the projection process is complete, in the many particle system made of the entangled measured system $S$ and measurement apparatus $M$, one can again rely on the dynamical relaxation process discussed by Valentini et al. \[24,25\] to restore quantum equilibrium. Indeed, these authors have shown that, at least in simple systems, a statistical distribution of the Bohmian position relaxes very quickly towards the modulus square of the wave function. Admittedly, we are making some extrapolation at this stage: as far as we know, there exist no systematic study in configuration space of the Bohmian dynamics of large entangled systems. Valentini has nevertheless shown \[28\] that the approach to quantum equilibrium can be derived from a statistical “subquantum theorem”, based on statistical assumptions that are analogous to those of classical statistical mechanics. The assumption we are making at this stage could probably be tested by more systematic numerical simulations of the Bohmian dynamics.

Within this scenario, assume that a second measurement is performed on the same system $S$ after the first experiment has been completed (its result has been registered). It is then legitimate to assume that the quantum equilibrium is obeyed for the sub-ensemble of experiments that have given a specific result in the first experiment. We then recover the standard rules of dBB mechanics and the Born rule, within a possible relative error of $10^{-16}$ or less if the whole experiment lasts one second; such an error rate is of course totally undetectable.
In other words, within our model, quantum equilibrium is no longer considered as a condition that is exactly met at all times. For instance, equilibrium is not yet reached while the result of an experiment is appearing on the pointer of a measurement apparatus; one has to wait until the result of measurement is fully registered. Quantum equilibrium is then rather seen as an emergent phenomenon \[29\] that reappears after each measurement, so that the initial conditions for the next experiment that are extremely close to this equilibrium.

2.3 Introducing the gravitational constant

Instead of two arbitrary constants as in \eqref{eq:22}, it is possible to introduce only a single constant, for instance \(a_L\), provided the Newton constant \(G\) is taken into account. We may assume that:

\[
\gamma_L = \frac{m^2 G}{\hbar a_L}
\]

where \(m\) is the mass of a nucleon for instance. If we choose value \eqref{eq:21b} for \(a_L\), we obtain:

\[
\gamma_L \simeq \frac{10^{-54}}{10^{-34}} \frac{10^{-11}}{10^{-6}} \simeq 10^{-24} \text{ s}^{-1}
\]

which is indeed compatible with \eqref{eq:21a}. One can then assume that the limit between the macroscopic and microscopic world occurs at a characteristic length \(a_L = 1\mu m\), and then use relation \eqref{eq:23} to “explain” why \(\gamma_L\) has the extremely small value given in \eqref{eq:21a}.

We can for instance consider that the collapsing field originates from the average gravitational attraction of the other identical particles within a range \(\alpha L\). The source of this classical field is the average Bohmian density (not the average quantum density), which localizes the quantum state in space. This is similar to the theory proposed in Ref. \[15\], within a stochastic dynamics. In quantum cosmogenesis, similar ideas have been proposed in Refs. \[30\] \[31\] within the dBB theory, in order to treat the metric of general relativity classically by considering the Bohmian positions as the sources of gravity.

Needless to say, in relations \eqref{eq:21}, we can multiply \(\gamma_L\) by any factor \(\lambda\) and \(a_L\) by \(1/\lambda\) without changing this agreement. Actually, relation \eqref{eq:23} only determines a velocity \(c_L\) as:

\[
c_L = \gamma_L a_L = \frac{m^2 G}{\hbar}
\]

One can also generalize \eqref{eq:23} by introducing a universal dimensionless constant \(\alpha_L\) as:

\[
\alpha_L = \frac{m^2 G}{\hbar c_L}
\]

and consider models where this constant takes on any arbitrary dimensionless value, for instance \(2\pi\), \(1/137\), etc.

3 Density operator

We now examine the effect of the localization term on the evolution of the density operator, either in a single realization of the experiment, or by average over many realizations.
3.1 Time evolution (single realization)

If the system is in a normalized pure state \(|\Phi(t)\rangle\), the density operator \(\rho(t)\) is defined as:

\[
\rho(t) = |\Phi(t)\rangle \langle \Phi(t)|
\]  

(27)

The density \(D_\Phi(r)\) is now defined by:

\[
D_\rho(r,t) = \text{Tr}\left\{ \Psi^\dagger(r) \Psi(r) \rho(t) \right\}
\]  

(28)

\(N_\Phi(r,t)\) is replaced by \(N_\rho(r,t)\), obtained by substituting \(D_\rho(r,t)\) to \(D_\Phi(r)\) in (4). The same changes are made in (9) and in the definition (15) of \(L(t)\). Then \(\rho(t)\) evolves according to the equation:

\[
i\hbar \frac{d}{dt}\rho(t) = [H(t), \rho(t)] + i\hbar \gamma_L [L(t), \rho(t)]_+
\]  

(29)

where \([A, B]_+\) is the anticommutator \(AB + BA\) of the two operators \(A\) and \(B\). This equation is nonlinear since \(D_\rho(r,t)\), and therefore \(L(t)\), depends on \(\rho(t)\).

We check that:

\[
i\hbar \frac{d}{dt}\text{Tr}\{\rho(t)\} = 2i\hbar \gamma_L \text{Tr}\left\{ \int d^3r \left[ \Psi^\dagger(r) \Psi(r) - D_\rho(r) \right] \Delta(r,t) \rho(t) \right\}
\]  

\[
= 2i\hbar \gamma_L \text{Tr}\left\{ \int d^3r \left| D_\rho(r) - D_\rho(r) \right| \Delta(r,t) \rho(t) \right\} = 0
\]  

(30)

3.2 Average over many realizations

The evolution of the density operator describing the average of many realizations of an experiment is given by the average of equation (29) over these realizations. If the system contains a single particle, its Bohmian position is different for each realization; during time evolution, it explores various regions of the wave function, as shown in the figures of [24,25]. Therefore, when the average over many realizations is taken, \(N_B(r,t)\) as well as \(\Delta(r,t)\) play the role of random functions. If the system contains \(N\) particles, \(\Delta(r,t)\) is then the sum of \(N\) fluctuating functions. In both cases, (29) becomes similar to a stochastic differential equation. We remark that, if the initial distribution of Bohmian variables coincides with the quantum distribution, the ensemble average of \(\Delta(r,t)\) vanishes. The same is true of the average of the localization operator, which is linear in \(\Delta(r,t)\). Therefore, if we take an average over many realizations of the experiment, and if \(\Delta(r,t)\) and \(\rho(t)\) remain uncorrelated, the average contribution of the localization term in the right-hand side of (29) vanishes. It is non-zero only when \(\rho(t)\) and the fluctuations of the Bohmian positions around their quantum equilibrium positions become correlated.

The situation is therefore similar to a relaxation phenomenon created by an ensemble of \(r\)-dependent fluctuating perturbations \(\Delta(r,t)\). The so called “motional narrowing” condition (see for instance [32,33]) expresses that the perturbations have very little effect during their correlation time. In our case, for a single particle, this condition reads \(\gamma_L \tau_c \ll 1\), which is easily fulfilled with the very small value \(\gamma_L\) of \(\gamma_L\).

The same remains obviously true for any microscopic system: the appearance of weak correlations between the quantum state of the system and the fluctuations of the Bohmian positions creates a relaxation process with a rate \(\gamma_L^2 \tau_c\), which remains negligible over a time equal to the age of the Universe. We therefore recover the standard equation of evolution of the density operator.

For a macroscopic system, the situation may be completely different: we have seen in §2.1 that the localization term itself grows quadratically with the number of particles involved, so that the second order
rate of localization $\gamma_2^2 \tau_c$ is now multiplied by the fourth power of the number of particles (assuming that $\tau_c$ is independent of $N$). Since the Avogadro number is very large, one can easily obtain situations where the rate becomes very fast, and where the motional narrowing condition is actually no longer valid. This corresponds to situations where the von Neumann projection postulate may be applied and where the measurement apparatuses can be treated classically.

### 3.3 Partial traces

Assume that the complete system $S$ is made of two subsystems $S_A$ and $S_B$, which are localized in two disconnected regions of space $\mathcal{V}_A$ and $\mathcal{V}_B$, and contain $N_A$ and $N_B$ particles respectively, and have no mutual interaction. The two density functions $N_A(\mathbf{r})$ and $N_B(\mathbf{r})$ then have non-overlapping supports, so that both the Hamiltonian and the localization operator are then the sum of two terms:

$$H(t) = H_A(t) + H_B(t)$$

$$\mathcal{T}(t) = \mathcal{T}_A(t) + \mathcal{T}_B(t)$$

In the space of states of a single particle, we choose a basis $\{|u_i\rangle\}$ such that each of these states is localized, either in $\mathcal{V}_A$, or $\mathcal{V}_B$ (its wave function is zero in the other volume). A basis in the space of states of $S$ can be obtained with states where the occupation number $n_i$ of each $|u_i\rangle$ is specified, that is with the ensemble of kets:

$$|u_1 : n_1; u_2 : n_2; \ldots; u_P : n_P \rangle = |n_A, n_B\rangle$$

where $n_A$ is a condensed notation for all the occupation numbers of the states localized in $\mathcal{V}_A$, and similarly $n_B$ a condensed notation for the occupation numbers of the states localized in $\mathcal{V}_B$. The matrix elements of the density operator $\rho$ describing $S$ are:

$$\langle n_A, n_B | \rho(t) | n_A', n_B' \rangle$$

Any operator $A$ acting in $\mathcal{V}_A$ but not $\mathcal{V}_B$ changes the value of $n_A$ but not that of $n_B$. The average $\langle A \rangle$ of $A$ can therefore be obtained from the spatial trace $\rho_A$ of $\rho$ over region $\mathcal{V}_B$ defined as:

$$\langle n_A | \rho_A(t) | n_A' \rangle = \sum_{n_B} \langle n_A, n_B | \rho(t) | n_A', n_B \rangle$$

where the sum over $n_B$ is taken over all possible values of the occupation number of the states localized in $\mathcal{V}_B$. The time evolution of $\rho_A(t)$ is obtained by taking the spatial trace of $[24]$. The terms in $H_A(t)$ and $\mathcal{T}_A(t)$ give the same effect as in (29), with indices $A$ added to the operators. Moreover, as usual the term in $H_B(t)$ vanishes (the partial trace of the commutator is zero). As for the term in $\mathcal{T}_B(t)$, it leads to:

$$\frac{d}{dt} \bigg|_{t_B} \langle n_A | \rho_A(t) | n_A' \rangle = \gamma_L \sum_{n_B, n_A', n_B''} \left\{ \langle n_A, n_B | \mathcal{T}_B(t) | n_A'', n_B'' \rangle \langle n_A'', n_B'' | \rho(t) | n_A', n_B \rangle + \langle n_A, n_B | \rho(t) | n_A'', n_B'' \rangle \langle n_A'', n_B'' | \mathcal{T}_B(t) | n_A', n_B \rangle \right\}$$

In the first term inside the summation, $n_A'' = n_A$, while in the second term $n_A'' = n_A'$, so that the right hand side of this equation is equal to:

$$\gamma_L \sum_{n_B, n_B''} \left\{ \langle n_B | \mathcal{T}_B(t) | n_B'' \rangle \langle n_A, n_B'' | \rho(t) | n_A', n_B \rangle + \langle n_A, n_B | \rho(t) | n_A', n_B'' \rangle \langle n_B'' | \mathcal{T}_B(t) | n_B \rangle \right\}$$
where the two terms become identical as soon as the two dummy variables \( n_B \) and \( n_B' \) are interchanged. We therefore obtain:

\[
\frac{d}{dt} \left| L_B \langle n_A | \rho_A(t) | n_A' \rangle \right| = 2\gamma_L \sum_{n_B, n_B'} \langle n_B | T_B(t) | n_B' \rangle \langle n_A, n_B' | \rho(t) | n_A', n_B \rangle
\]

(37)

(i) If the matrix elements of the density operator of \( S \) factorize:

\[
\langle n_A, n_B | \rho(t) | n_A', n_B' \rangle = \langle n_A | \rho_A(t) | n_A' \rangle \times \langle n_B | \rho_B(t) | n_B' \rangle
\]

we get:

\[
\frac{d}{dt} \left| L_B \langle n_A | \rho_A(t) | n_A' \rangle \right| = 2\gamma_L \langle n_A | \rho_A(t) | n_A' \rangle \cdot \text{Tr} \left\{ T_B(t) \rho_B(t) \right\}
\]

(38)

But we have seen in (30) that the trace in the right-hand side vanishes. If two subsystems occupy different regions of space, and if they are uncorrelated, each partial density operator evolves independently (as is the case in the absence of the localization term).

(ii) If the density operator of \( S \) does not factorize, the preceding simplification does not occur. Let us first study the evolution of the density matrices in a single realization of an experiment. If the two systems \( S_A \) and \( S_B \) are entangled, the dynamical collapse acting on \( S_B \) may affect the state of \( S_A \), in the same way as the standard von Neumann collapse postulate can change at the same time the state of two remote entangled systems. Mathematically, the origin of this mutual effect of the two subsystems is the anticommutator that contains \( L_B(t) \) in (29), while the Hamiltonian appears in a commutator. Therefore, in the partial trace, while the two terms in \( H_B \) cancel each other, the two terms in \( L_B(t) \) add to provide the double of each contribution.

The quantum nonlocality then manifests itself in two ways. The first also occurs in standard dBB theory, where the motion of the Bohmian positions of the whole system are guided in the configuration space by the wave function in this space. The second is due to the nonlocal effect of the collapse term in the equation of evolution. This effect is necessary to recover the results provided by the usual von Neumann reduction postulate in standard quantum mechanics.

(iii) Nevertheless, if many realizations of the experiment are performed, one has to consider the average of the localization operator \( T_B \) over these realizations. We have discussed in §2.2 the conditions under which the quantum equilibrium is obtained. If this is the case, \( N_\Phi(r, t) \) and \( N_\rho(r, t) \) are constantly equal, and the average of \( T_B \) vanishes. Therefore, there can be no influence of an experiment performed in region \( B \) on the density operator in region \( A \), which automatically ensures the non-signaling property necessary to obtain a model that is compatible with relativity. We recover the relation obtained by Valentini between quantum equilibrium and the no-signaling condition, which is thus also valid within our non-standard model.

4 Effect of the localization term on the densities and currents

We now study the effect of the localization term on the density of particles and on their current. As before, for the sake of simplicity, we assume that the particles are spinless.
4.1 Evolution of the one-body density

We begin with the study of a pure state. The wave function is symmetric with respect to the exchange of particles. The one body density is then:

\[ D_\Phi (r) = N \int d^3r_2...d^3r_N \left| \overline{\Phi} (r_1 = r, r_2, ..., r_N; t) \right|^2 \]  \hspace{1cm} (40)

According to (19), the contribution of the localization term to its time evolution is given by:

\[ \frac{d}{dt} \left| \overline{\Phi} \right|_{loc} = 2\gamma_L N \int d^3r_2...d^3r_N \left[ \sum_{n=1}^N \Delta (r_n, t) - N < \Delta > \right] \left| \overline{\Phi} (r_1 = r, r_2, ..., r_N; t) \right|^2 \]  \hspace{1cm} (41)

In (41), the term \( n = 1 \) merely introduces a term proportional to \( \Delta (r,t) D_\Phi (r) \), which depends only on the single particle density; all the other terms contain the position correlation function \( D^{ll}_\Phi (r,r') \) of two particles at points \( r \) and \( r' \):

\[ D^{ll}_\Phi (r,r') = N (N-1) \int d^3r_3...d^3r_N \left| \overline{\Phi} (r_1 = r, r_2 = r', ..., r_N; t) \right|^2 \]  \hspace{1cm} (42)

Equation (41) then provides:

\[ \frac{d}{dt} \left| \overline{\Phi} \right|_{loc} = 2\gamma_L \left[ \Delta (r,t) D_\Phi (r) + \int d^3r' D^{ll}_\Phi (r,r') \Delta (r',t) - N < \Delta > D_\Phi (r) \right] \]  \hspace{1cm} (43)

We do not get a closed equation for the evolution of the single particle density; the right hand side of this equation contains the two-particle density, as in the usual BBGKY hierarchy, and despite of the fact that the localization term is a single-particle operator. This is because the localization term is non-Hermitian, which introduces anticommutators instead of commutators.

Since:

\[ \int d^3r \ D^{ll}_\Phi (r,r') = (N - 1) \ D_\Phi (r') \]  \hspace{1cm} (44)

we can check the particle conservation rule:

\[ \int d^3r \left| \frac{d}{dt} \right|_{loc} \overline{\Phi} (r) = 2\gamma_L \left[ \int d^3r \Delta (r,t) \ D_\Phi (r) + (N-1) \int d^3r' \ D_\Phi (r') \Delta (r',t) - N^2 < \Delta > \right] \]

\[ = 2\gamma_L \left[ N \int d^3r \Delta (r,t) \ D_\Phi (r) - N^2 < \Delta > \right] = 0 \]  \hspace{1cm} (45)

If the system is not described by a pure state, but by a density operator \( \rho \), the time evolution of the density \( D_\rho (r,t) \) is obtained by the same calculation, with the simple substitution:

\[ \left| \overline{\Phi} (r_1 = r, r_2, ..., r_N; t) \right|^2 \Rightarrow \langle r_1 = r, r_2, ..., r_N | \rho (t) | r_1 = r, r_2, ..., r_N \rangle \]  \hspace{1cm} (46)

in all the equations, including (42), which becomes the definition of the two body density \( D^{ll}_\rho (r,r') \). The time evolution of \( D_\rho (r,t) \) is therefore given by:

\[ \frac{d}{dt} \left| \Phi \right|_{loc} = 2\gamma_L \left[ \Delta (r,t) \ D_\rho (r) + \int d^3r' \ D^{ll}_\rho (r,r') \Delta (r',t) - N < \Delta > D_\rho (r) \right] \]  \hspace{1cm} (47)
4.2 Mean field, role of the spatial correlations

When the distance between \( r \) and \( r' \) becomes very large, the correlation function \( D_{II}^{(r,r')} \) factorizes. We can therefore introduce a function \( F(r,r') \) by setting:

\[
D_{II}^{(r,r')} = D_\Phi (r) D_\Phi (r') [1 - F(r,r')] \quad (48)
\]

with:

\[
F(r,r') \to 0 \quad (49)
\]

In (48) we have chosen to write a minus sign before \( F(r,r') \) because, if two neighbor systems exchange particles, their local densities are anticorrelated; \( F \) is then positive. Relation (44) provides:

\[
\int d^3 r' D_\Phi (r') F(r,r') = 1 \quad (50)
\]

The function \( F(r,r') \) may change sign in general, but positive values dominate this integral.

If we insert (48) into (43), the first term in the right hand side cancels the term in \( N < \Delta > D_\Phi (r) \).

Using (50), we get:

\[
\frac{d}{dt} \bigg|_{loc} D_\Phi (r) = 2\gamma_L \left[ \Delta (r,t) - \int d^3 r' F(r,r') D_\Phi (r') \Delta (r',t) \right] D_\Phi (r) \\
= 2\gamma_L D_\Phi (r) \int d^3 r' F(r,r') D_\Phi (r') [\Delta (r,t) - \Delta (r',t)] \quad (51)
\]

The “local” character of this equation of evolution depends on the properties of \( F(r,r') \), in particular whether it tends to zero sufficiently rapidly when the difference of positions increases.

(i) In mean-field theory, one merely assumes that \( F(r,r') \) vanishes. The first line of relation (51) then becomes:

\[
\frac{d}{dt} \bigg|_{loc} D_\Phi (r) = 2\gamma_L \Delta (r,t) D_\Phi (r) \quad (52)
\]

Mean field theory merely predicts that \( D_\Phi (r) \) increases at points where \( \Delta (r,t) \) is positive, decreases at points where this function is negative.

(ii) Beyond mean-field theory, if \( F(r,r') \) has a small range \( l \) (range of correlations in the system), the localization term depends only on the values of \( \Delta (r',t) \) in a small domain around \( r \). In the limit of a very small range where:

\[
F(r,r') \sim \delta (r - r') \quad (53)
\]

The right-hand side of (51) vanishes. More generally, if \( \Delta (r',t) \) is constant in the domain where \( F(r,r') \) is not zero, relation (51) shows that the evolution of the density introduced by the localization process vanishes. If \( \Delta (r',t) \) varies in space over a distance \( l \), the integral can approximated by:

\[
- \int_{r' \leq l} d^3 r' F(r,r') D_\Phi (r') (r' - r) \cdot \nabla \Delta (r,t) \quad (54)
\]

If the product \( F(r,r') D_\Phi (r') \) varies linearly as a function of \( r' \) within its range \( l \):

\[
F(r,r') D_\Phi (r') = F(r,0) D_\Phi (0) + (r' - r) \cdot \nabla (FD) \quad (55)
\]
we obtain:

\[
\frac{d}{dt} D_{\Phi}(\mathbf{r}) = -\gamma L \frac{4\pi l^5}{15} (FD) \cdot \nabla \Delta (\mathbf{r},t)
\]  

(56)

This expression varies very rapidly with the range of correlation \( l \).

(iii) But \( F(\mathbf{r}, \mathbf{r'}) \) can also have a large range. For instance, just after a measurement has been performed, the correlation function between two different positions of the pointer vanishes, while the product of one-body densities does not. In this case, the transfer of density due to the localization term is not local. This is a necessary feature to eliminate MDQS efficiently, and to obtain a projection after measurement.

We conclude from this discussion that the localization term has little effect in most (ordinary) situations. But, if a MDQS appears for some reason (Schrödinger cat, etc.), it is promptly reduced to one of its components by the localization term.

4.3 Evolution of the local momentum

We have only studied the direct effects of the localization term on the local density, but of course this term also has indirect effects: by localizing the wave functions in space, it changes their Fourier transform, and therefore the average value of the velocities. At later times, this change will also modify the average positions of the particles.

The local current of particles at point \( \mathbf{r} \) is:

\[
J_{\Phi}(\mathbf{r},t) = \frac{\hbar}{2im} \sum_{p=1}^{N} \int d^3r_1 \ldots \int d^3r_{p-1} \int d^3r_{p+1} \ldots \int d^3r_N
\Phi^* (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p = \mathbf{r}, \ldots \mathbf{r}_N; t) \nabla_r \Phi (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p = \mathbf{r}, \ldots \mathbf{r}_N; t) + \text{c.c.}
\]  

(57)

where c.c. means complex conjugate. The time derivative of this current induced by the localization process is obtained by using equation (19). The derivative of \( \Phi (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p = \mathbf{r}, \ldots \mathbf{r}_N; t) \) introduces the expression:

\[
\begin{align*}
\frac{\hbar\gamma L}{2im} & \sum_{p=1}^{N} \nabla_r \left[ \Delta (\mathbf{r},t) + \sum_{n \neq p} \Delta (\mathbf{r}_n,t) - N\Delta \right] \Phi (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p = \mathbf{r}, \ldots \mathbf{r}_N; t) \\
& + \frac{\hbar\gamma L}{2im} \left[ \nabla_r \Delta (\mathbf{r},t) + \sum_{n \neq p} \Delta (\mathbf{r}_n,t) - N\Delta \right] \Phi (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p = \mathbf{r}, \ldots \mathbf{r}_N; t) \\
& + \frac{\hbar\gamma L}{2im} \nabla_r \Delta (\mathbf{r},t) \left[ \Phi (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p = \mathbf{r}, \ldots \mathbf{r}_N; t) \right]
\end{align*}
\]  

(58)

The first term in the right hand side of the second line reconstructs the particle current \( J(\mathbf{r},t) \), multiplied by \( \Delta (\mathbf{r},t) \); the rest of the second line introduces a new integral. The term in the third line introduces the single particle density \( \Phi \), which is real; since the whole term is multiplied by \( i \) in (57), this term disappears when the real part is taken. If we combine these terms with those resulting from the derivative
of \(\Phi^*\), we obtain:

\[
\frac{d}{dt} J_\Phi (r) = 2 \gamma_L [\Delta(r, t) - \Delta] J_\Phi (r)
\]

\[
+ \frac{\hbar \gamma_L}{2im} \sum_{p=1}^N \int d^3r_1 \cdots \int d^3r_{p-1} \int d^3r_{p+1} \cdots \int d^3r_N \sum_{n \neq p} [\Delta(r_n, t) - \Delta] \nabla_r \Phi + \text{c.c.} \quad (59)
\]

The first line of this equation adds an exponential increase or decrease of the particle current with a fluctuating rate, which can be positive or negative. The second line couples the current \(J_\Phi (r)\) to another integral. Since \(\gamma_L\) is very small, the influence of the localization process on the current of particles remains very small in most situations, except when the difference \([\Delta(r_n, t) - \Delta]\) can take on very large values, as is the case during a measurement process (§2.1).

In standard GRW or CSL theory, the random localization process is a pure Markov process. The corresponding changes of the momentum of an object are then given by a random walk with no memory, and no preferred direction [35]. In the model of the present article, the localization process has a non-zero memory arising from the statistical properties of \(\Delta(r, t)\). These properties depend on the complicated nonlinear relative motion of the Bohmian positions and the wave function. Of course, for macroscopic objects, the corresponding time constants are very large, due to the very small value of the localization time constant \(\gamma_L\). Nevertheless, on very long time scales such as those often considered in astrophysics, it may be that observable effects are predicted. Such predictions are nevertheless difficult to make, since it is not easy to evaluate the spatial and temporal scales of the fluctuations of the localization source \(\Delta(r, t)\).

5 Possible interpretations

In terms of possible interpretations of quantum mechanics, the model is relatively robust: it can remain compatible with very different points of view.

One can consider that the Bohmian variables are just a mathematical tool to introduce the stochastic reduction of the state vector, which then represents physical reality, in the line of GRW and CSL theories. Indeed, we have not assumed that Bohmian variables directly provide the results of position measurements, but that the results are determined by the quantum density in space provided by the state vector: they depend on the values of \(D_\Phi (r)\), not on those of the Bohmian positions. One may also consider that \(D_\Phi (r)\) gives a direct description of physical reality or ordinary 3D space, coming back to a fluid representation of matter, as envisaged initially by Schrödinger when he introduced his equation. This view is just the opposite of the usual dBB theory, where it is assumed that the observations reveal the values of the Bohmian positions, which therefore directly represent the physical reality. Here, the position in the configuration space is just a mathematical variable that drives the wave function and the associated quantum density in space; it plays a role that is analogous to the “subquantum medium” acting on the state vector in Fényes-Nelson theories [36,37]. One can even combine this new dynamics with the Everett interpretation; if one includes the memory registers of the observers into \(\Psi\), one obtains a sort of “Everett interpretation with projection”, predicting the existence of a “single world”.

But one can also prefer the usual approach of the dBB theory, and consider that the Bohmian positions represent the beables [38,39] of the physical system. The advantage of the model is then to get rid of all the empty waves of the dBB theory, and of the relative difficulty to attribute them a status [40]. If one sees the wave function as similar to a Lagrangian or Hamiltonian in classical mechanics [21], it seems
preferable to keep only the effective part of this wave function, eliminating all empty waves that have accumulated in the past.

One can also take an intermediate point of view, and consider for instance that what represents physical reality is the spatial density $N_B(r, t)$. In this view, the problem of the “long tails” occurring in the usual spontaneous theories becomes irrelevant, since no particular physical meaning is attributed to the exponentially vanishing empty waves.

Whatever status is eventually attributed to the state vector $\Psi$, it remains clear that it is less disconnected from physical reality than with a dynamics having no collapse mechanism. Nevertheless, the various mathematical components of this dynamics can be interpreted in different ways, leading to various ontologies.

6 Discussion and conclusion

We have seen that the attraction of the Bohmian densities can be used to obtain a reasonable model of spontaneous collapse of the state vector. In addition to the standard “pilot wave” of the dBB theory, we have introduced a “pilot density” for the wave. The corresponding dynamic is deterministic: the stochasticity of the initial position in configuration space is sufficient to reproduce the standard prediction of the Born rule for probabilities. The localization term tends to constantly adapt the wave function in order to obtain a better match between the quantum density and the density of positions in space. In QSMS situations, all the empty waves disappear, so that the wave function and the Bohmian positions progress together in time, while in usual dBB theory they are disconnected. For instance, it has been claimed that the dBB theory is really “a many-world theory with a superfluous configuration appended to one of the worlds”, and that “pilot-wave theories are parallel-universes theories in a state of chronic denial” [41]; for discussions of these claims, see for instance [42] and [43]. Clearly, if one accepts the present model where the dynamics of the state vector is coupled to the Bohmian positions, this discussion is settled.

The mechanism of the projection in the dynamics is actually based on the cohesion of macroscopic objects. Because standard theory predicts that, even if such objects reach in QSMS, their constituent particles remain strongly correlated spatially, the mechanism of our model projects them into a single localization. In fact, only macroscopic objects that do not break spontaneously into several parts acquire in this way a unique spatial localization: the “moon is there even if nobody looks” [44], and the reason why the center of the moon occupies a well defined point on its orbit before any measurement is the internal cohesion of the moon. By contrast, objects than can be split into several components without breaking any energy barrier can go through QSMS without collapsing; for instance, Bose-Einstein gaseous condensates that are split into two remote parts can give rise later to quantum interference effects, as discussed in more detail in [22].

The localization term also provides a sharp transition between the quantum and classical regime. For a molecule or cluster containing $N$ particles in a volume smaller than $a_L$, any superposition of several quantum states localized at a distance larger that $a_L$ is projected into one single component of this superposition with a rate $\Gamma$ that varies quadratically as a function of $N$:

$$\Gamma \simeq \gamma_L N^2$$

With the choice (21) of constants, and if the number of particles is about $10^{12}$, this relation predicts a localization time of 1 s. Since the number density $n$ of solid (or liquid) physical objects is of the order of
$10^{30}$ atoms/cubic meter, the same relation can be expressed in terms of the size of the object:

$$\Gamma \simeq \gamma_L n^3 l^6$$

(61)

For an object of size $l < a_L$ containing $n$ particles per unit volume, this rate varies proportionally to the sixth power of the size $l$; the localization time is 1 second if $l \simeq 1 \mu m$. For an object of size $l > a_L$ containing $N$ particles, the rate becomes:

$$\Gamma \simeq \gamma_L n a_L^3 n l^3$$

(62)

One can therefore consider that $l = a_L$ and $N = 10^{12}$ provide the border between standard quantum and classical behavior of physical objects.

It is clear that the model differs from the GRW and CSL theories in several respects. Beyond the fact that the dynamics is not stochastic, already mentioned in the introduction, another difference is that the localization process is no longer a single particle process, where each of them is localized independently, but results from a collective effect between the particles; this creates correlations between them, as remarked at the end of §1.3. This collective character introduces a sharper transition between the quantum and classical regimes, due to the quadratic term in $N$ in (60). This in turn is a consequence of the fact that the number of particles enters twice in the model, once in the Bohmian number $N_B(\mathbf{r}, t)$, and once in the integral of the quantum operator $\Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r})$. In QSMDS situations, our localization term is therefore “stronger” than those of GRW and CSL. Nevertheless, in usual situations, it is “softer”: for instance, in a bulk piece of matter located in a single region of space, the Bohmian density coincides almost perfectly with $D_\Phi(\mathbf{r})$, and in our model the collapse term has practically no effect. By contrast, it is constantly active in GRW and CSL theories. In other words, in experiments such that of Ref [46], our model would be compatible with the observation of zero heating. We also note that the term that we added to the standard Schrödinger equation is not only non-linear, but also non-local: when the two components of the wave function of a macroscopic object begin to separate in space, one component is transferred in space to the other at some finite distance. This feature is necessary to fit with the quantum predictions in Bell type experiments.

For macroscopic systems, the appearance of position uniqueness is not necessarily the only effect predicted by the model. We have seen in §4.3 that another effect of the localization term may be to change the local current of the particles, which will have an indirect effect on the density. This is not surprising since a space localization of the wave function implies an increase of the width of its Fourier transform, which corresponds to higher velocities. This violates the usual momentum conservation rule; similarly, the GRW and CSL theories predict a spontaneous heating effect that seems to violate the energy conservation rule [45]. Whether or not our model predict slow changes of the momentum of macroscopic objects, for instance on a cosmological time scale, remains to be studied.

We have also seen in §2.2 that, within this model, the Born rule is no longer a postulate; it emerges from the dynamics, and it is actually not an exact rule. It nevertheless remains a fantastically good approximation, with an accuracy better than $10^{-16}$ if the preparation and the measurement of quantum system are separated by more than 1s.

Needless to say, the class of models we have discussed is, in a sense, very naive. It is neither relativistic nor expressed in terms of a plausible field theory. Its purpose is just to indicate a range of possibilities, which might be exploited in a second stage to construct a more credible theory. This range is relatively broad, for several reasons. The first obvious reason is that there exist a large domain of possible values for the two constants $a_L$ and $\gamma_L$ that introduce no contradiction with the known experimental facts; this robustness of the model is both a strength and a weakness, since having too much flexibility amounts
to reducing the predictive power of a theory. A second reason is that the equations are not particularly plausible: they are just the simplest version of a dynamics where position variables are added to the state vector, and where this vector is attracted towards these additional positions. The localization function $A_{L}(r - r')$ that we have introduced in (3) is arbitrary, and the choice of a Gaussian has no particular justification. Even the form of the operator $L(t)$ could be changed: it might be possible to re-introduce at this stage the effect of gravitation by choosing another localization operator, more in the line of the ideas of Refs. [12]-[14]; such a possibility remains to be explored. Finally, we have not specified the nature of the particles on which the localization process should apply: they could be for instance, nucleons only, or nucleons and electrons, or quarks, etc. In any case, at this stage, even the order of magnitude of the constants is not determined. As mentioned in the introduction, the main merit of this class of models is only to show that they various quantum descriptions of physical reality remain compatible with known experimental data.

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