Ten Points on a Cubic

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Abstract. The 16-year old Blaise Pascal found an incidence relation that holds when six points lie on a conic. A century later, Braikenridge and Maclaurin extended Pascal’s result to a straightedge construction that characterizes when six points lie on a conic. Nearly 400 years later, we develop a straightedge construction to check whether ten points lie on a cubic curve.

1. INTRODUCTION. A conic is a curve consisting of the points in the plane which satisfy a degree two equation. As we show later in the paper, there is a conic passing through any five points. This is not true for every set of six points. In 1639, when he was 16 years old, Blaise Pascal found a beautiful straightedge construction to test whether six points lie on a conic.

Figure 1. Three constructed points are collinear precisely when six points lie on a conic.

Pascal’s result naturally leads to a similar question for points on a cubic curve. For any nine points there is a cubic curve through them. This is not true for every set of ten points.

Pascal’s construction determines whether six points lie on a conic using only a straightedge. Our main result is in the same spirit.

Theorem 1. Given ten points in the plane, there exists a straightedge construction that produces three points such that the original ten points lie on a cubic curve if and only if the three points are collinear.

The next section gives details about Pascal’s theorem and the following section describes additional historical and geometric background material. The construction to check whether 10 points lie on a cubic is given in Section 4. We justify the construction in Section 5. Along the way we’ll meet several mathematicians from history and learn about the power of a point. Our explanation uses several geometric constructions involving conics whose proofs are deferred until Section 6. We end the paper with some pointers to the literature and several fun problems. Among these is an exercise that describes a computer-aided proof of our construction.
2. PASCAL’S CONSTRUCTION. Given six points in the plane, Pascal connected the points to form a (possibly nonconvex) hexagon whose three pairs of opposite sides extend to meet in three points. Pascal discovered that these three auxiliary points are collinear when the six original points lie on a conic.

**Theorem 2 (Pascal).** If six distinct points $A, B, C, a, b,$ and $c$ lie on a conic, then the lines $Ab$, $Bc$, and $Ca$ meet the lines $aB$, $bC$, and $cA$ in three new points and these new points are collinear.

The line through the three new points is called the Pascal line and is depicted as a dotted line in Figure 1. Each relabeling of the six points gives rise to a Pascal line. Though there are $6! = 720$ ways to reorder the points, these give rise to only 60 different Pascal lines. This arrangement of 60 lines, known as Pascal’s Hexagrammum Mysticum, exhibits some amazing combinatorics. It has been studied by many important mathematicians, including Arthur Cayley, the Reverend T.P. Kirkman, Julius Plücker, Jakob Steiner, and George Salmon. See Conway and Ryba’s papers [8, 9] for details, including a reference to the hand-drawn diagram of all 60 lines due to Anne and Elizabeth Linton [17], twin sisters who completed doctoral studies together at the University of Pennsylvania in 1921.

The Scottish mathematicians William Braikenridge and Colin Maclaurin established the converse to Pascal’s theorem almost a hundred years after Pascal’s discovery: if three lines meet another set of three lines in nine points with three of the points lying on yet another line, then the remaining six points lie on a conic. Thus Pascal’s construction precisely characterizes when six points lie on a conic.

As stated, Pascal’s construction is degenerate for special configurations of the original six points. For example, the construction may involve finding the intersection of two lines and, in some special cases, these lines may turn out to be parallel. However, the set of special configurations is a set of measure zero. Moreover, there are ways to interpret the construction to handle many of these instances. Some of these difficulties are dealt with using projective geometry. For example, the intersection of parallel lines is given a meaning in the projective plane, as we describe at the end of the next section.

Similarly, our construction that checks whether ten points lie on a cubic is degenerate for certain special configurations which form a set of measure zero. As with Pascal’s theorem, some of the degenerate configurations can be addressed using projective geometry while others need to be handled using ad-hoc techniques. In spite of this, we can extend our construction to give an algorithm that will determine whether any set of 10 points lies on a cubic curve. The extra work required is quite involved so we omit the details.

Our construction starts with ten given points in the Euclidean plane. If the coordinates of all ten points lie in a field $k$, then all the objects we build in our construction are also defined over $k$. In particular, if all ten points have real coordinates, then the constructed objects are real as well. Though the proofs described here use plane geometry, the result is valid for points with complex coordinates. See Problem 9 for details.

3. HISTORICAL AND GEOMETRIC BACKGROUND. Books I through VI of Euclid’s *Elements* treat ruler and compass constructions, which remain a mainstay of high school geometry in America today. Two classical problems that cannot be solved by ruler and compass are to double the cube—to construct a line segment of length $\sqrt[3]{2}$ given a segment of length 1—and to trisect a general angle. Today these impossibility results are usually proved using Galois theory, but the first proof dates to 1837, just five years after Evariste Galois was killed in a duel. That proof [26], given by the 23-year-old mathematician Laurent Wantzel, was ignored and forgotten for over 80 years.
There are many variants of the constructibility problems. In some we use a rusty compass that only has one setting. In another we replace the compass by the ring left from a coffee cup: we have no compass but are given a single circle. Our favorite is to toss the compass away completely and just consider straightedge-only constructions! Using a straightedge we are allowed to draw lines joining two known points and construct points by intersecting lines. This requires us to produce incidence relations—three collinear points or three concurrent lines—to characterize geometric properties. It may be surprising that even with this reduced material there are many beautiful results.

While a straightedge construction cannot use angles or distances directly, many incidence conditions are equivalent to angle or distance constraints. The best known of these are Menelaus’s and Ceva’s theorems.

The first theorem, due to Menelaus of Alexandria (70–140 A.D.) says that three points on the three (extended) edges of a triangle are collinear precisely when the product of three oriented length ratios is $-1$, as illustrated in Figure 2 (left).

Ceva’s theorem was first proved by Yusuf Al-Mu’taman ibn Hūd, an eleventh-century king of Saragossa in present-day Spain, and later proved and popularized by Giovanni Ceva in 1678. Ceva’s theorem dualizes Menelaus’s theorem, interchanging lines and points. Consider three lines, one through each vertex of a triangle. Ceva’s theorem asserts that the three lines are concurrent precisely when the product of the three oriented length ratios is 1, as depicted in Figure 2 (right).

Carnot’s theorem, like Pascal’s theorem, characterizes when six points lie on a conic, but the result involves products of oriented distance ratios like Menelaus’s and Ceva’s theorems. Carnot drew three lines through pairs of the six points, producing the triangle $ABC$. Labeling the points $a_1, a_2, b_1, b_2, c_1,$ and $c_2$ as in Figure 3, Carnot observed that the six points lie on a conic precisely when a product of six distance ratios equals 1. Lazare Carnot is best known as the “Organizer of Victory” for the French Revolutionary Army at the end of the 18th century. He retired to write about mathematics and military tactics after being exiled for his revolutionary activities.

An important incidence theorem involving cubic curves is the Eight Implies Nine Theorem: Given any eight points, there is a special ninth point so that every cubic through the eight given points passes through the ninth point. Using a straightedge to construct the ninth point from the eight given points was a problem considered by several people in the 19th century, including Michel Chasles [7], Arthur Cayley [6], A. S. Hart [15], and Thomas Weddle [27]. Recently, Qingchun Ren, Jürgen Richter-Gebert, and Bernd Sturmfels [21] took a modern approach to that problem in this MONTHLY. The Eight Implies Nine Theorem is a consequence of the Cayley-Bacharach theorem. The Cayley-Bacharach theorem plays a vital role in our work, and it is stated in Problem 7 in the last section of our paper. The curious reader can see more details in
the excellent survey paper by Eisenbud, Green, and Harris [11], who trace the history of the theorem and prove several versions of it.

The key step in our construction is to realize the 10 points as a subset of 16 points that are the intersection of two degree-4 curves, each the union of two conics. The Cayley-Bacharach theorem implies that the 10 points lie on a cubic precisely when the 6 remaining points lie on a conic, which we test using Carnot’s theorem. This requires us to show that the product of six distance ratios equals 1.

A serious complication we face is that when using a straightedge we can only find two of the 6 extra points together with two lines that contain the remaining four points. Furthermore, we have no way to measure distances in order to directly evaluate the product of distance ratios. Remarkably, we are able to overcome these difficulties using elementary methods from Euclidean geometry, such as the power of a point and similar triangles. These methods reduce the computation to evaluating a product of three distance ratios, which we interpret as a collinearity condition using Menelaus’s theorem.

The combination of both projective and Euclidean tools is a feature of our approach to constructive problems. Though our main application is to cubic curves in the Euclidean plane, we find it convenient to work in the projective plane. Jacques-Salomon Hadamard [14] wrote “It has been written\(^1\) that the shortest and best way between two truths of the real domain often passes through the imaginary one.” Similarly, the best way between two truths in Euclidean geometry may pass through the projective domain.

The projective plane \(\mathbb{P}^2\) extends the Euclidean plane by adding points at infinity. The points in \(\mathbb{P}^2\) have the form \([a : b : c]\), where \(a\), \(b\), and \(c\) are scalars, not all equal to zero, and we impose the equivalence relation

\[
[\lambda a : \lambda b : \lambda c] = [a : b : c]
\]

for all nonzero scalars \(\lambda\). The points \((x, y)\) in the usual Euclidean plane correspond to points \([x : y : 1]\) in \(\mathbb{P}^2\) and the remaining points, each of the form \([x : y : 0]\), make up the line at infinity, whose equation is \(z = 0\). There is a natural way to extend any curve in the usual plane to \(\mathbb{P}^2\) by adding one or more points at infinity. Two different lines in \(\mathbb{P}^2\) always meet in a point: the point lies at infinity if the lines are parallel. Circles are those special conics that meet the line at infinity at the two distinguished points \(I = [1 : i : 0]\) and \(J = [1 : -i : 0]\).

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\(^1\)Apparently, Hadamard was paraphrasing Paul Painlevé [19], the French mathematician and statesman who served twice as Minister of War and twice as Prime Minister of France.
In 1872 Felix Klein announced his Erlangen program for geometry, classifying geometries by the type of transformations that act on the underlying space and properties invariant under those maps. Projective space $\mathbb{P}^2$ admits projective transformations, multiplication of points by an invertible $3 \times 3$ matrix $M \in GL_3(\mathbb{C})$. Such maps are well-defined since $M(\lambda v) = \lambda (Mv)$ and they send any collection of collinear points to new collinear points since multiplication by an invertible matrix preserves linear dependence.

Multiplying by a matrix can be interpreted as changing the basis of the underlying space, so projective geometry is mainly concerned with properties that are independent of the choice of coordinates on our space, such as incidence relations. Projective transformations do not preserve distances or angles, so there is a common perception that distances and angles are not proper objects of study in projective geometry. In contrast, our point of view in this paper carefully blends projective and Euclidean geometry, using distances to prove results in Euclidean geometry and interpreting those results in terms of projective objects.

4. MAIN CONSTRUCTION. We state and discuss our construction in this section, but in the interests of brevity, we postpone the proof until the next section.

Étienne Bézout [3] showed that when two plane curves defined by the vanishing of polynomials of degrees $d$ and $e$ meet in finitely many points, then there must be $de$ points, counted appropriately. Technically, points of tangency count with multiplicity and we allow points at infinity, but both kinds of points only occur for a set of curves of measure zero and we ignore them in our discussion. The $16$ points of intersection of two degree-four curves form a geometric object called a complete intersection, which is heavily studied in commutative algebra. The Cayley-Bacharach theorem implies that ten of the points lie on a cubic precisely when the remaining six points lie on a conic. See Problem 7 for details. This observation prompted our initial work on the construction.

To start, we build two degree-four curves through the ten given points using conic curves. Recall that any five points lie on a conic: substituting the coordinates of a point into the general degree-2 equation,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

gives a linear condition on the six coefficients of the conic and the homogeneous system defined by five such equations has at least a one-dimensional solution space by the Rank-Nullity theorem. Indeed, almost all sets of five points impose five independent conditions and determine a unique irreducible conic—a circle, ellipse, or hyperbola. All of our figures show ellipses, but that is just because they make nicer pictures.

Partition the ten given points $K_1, K_2, \ldots, K_{10}$ into two sets of five points, $S_1 = \{K_1, K_2, K_3, K_4, K_5\}$ and $S_2 = \{K_6, K_7, K_8, K_9, K_{10}\}$. There are conics $C_1$ and $C_2$ passing through the points in $S_1$ and $S_2$, respectively. The union $C = C_1 \cup C_2$ is a degree-four curve passing through the ten given points.

Now swap two points of $S_1$ for two points of $S_2$ to make a second partition of the ten points into two sets of five, say $T_1 = \{K_5, K_4, K_6, K_7\}$ and $T_2 = \{K_1, K_2, K_8, K_9, K_{10}\}$, so that $S_1$ and $T_1$ share three points in common. Again there is a conic $D_i$ through the points in each $T_i$ and their union $D = D_1 \cup D_2$ passes through all ten given points, as illustrated in Figure 4.

The two degree-four curves $C$ and $D$ meet in $16$ points, including our $10$ given points. The remaining six points will be referred to as the residual points. In Construction 3 of Section 6 we show how to use a straightedge to construct the fourth point of intersection of two conics if we know their other three intersection points. Since the
Figure 4. Two conics $C_1$ and $C_2$ meet two other conics $D_1$ and $D_2$ in 16 points, the ten original points and six more residual points.

Conics $C_i$ and $D_i$ share 3 known points in common—the three points in $S_i \cap T_i$—we can construct their fourth point of intersection $P_i$. The points $P_1$ and $P_2$ are two of the six residual points.

Two of the remaining residual points lie on the intersection of conics $C_1 \cap D_2$ and the other two lie on $C_2 \cap D_1$. Each conic in these pairs is determined by five of our given points, which include two of the four points of the pair’s intersection. Unfortunately it is not possible to construct the remaining two common points using only a straightedge. However, in Construction 2 of Section 6 we show how to use a straightedge to construct the line passing through the remaining two points of intersection, as depicted in the middle portion of Figure 9. We call this line the coradical axis of the two conics with respect to the two known points, though sometimes we omit mention of the two points when they are the only two known points of intersection of the two conics. When the two conics are circles and meet in two real points the line through these two points is called the radical axis of the circles. The line passing through their remaining two points of intersection $I = [1 : i : 0]$ and $J = [1 : -i : 0]$ is the coradical axis, which explains our choice of terminology. We draw the line $L_Q = Q_1 Q_2$, which is the coradical axis of $C_1$ and $D_2$. Similarly we draw the coradical axis of $C_2$ and $D_1$, which is the line $L_R = R_1 R_2$.

The three lines $L_P = P_1 P_2$, $L_Q$, and $L_R$ form a triangle with vertices $P = L_Q \cap L_R$, $Q = L_P \cap L_R$, and $R = L_P \cap L_Q$, as in Figure 5.

We construct two points $U \in L_Q$ and $V \in L_R$ that can be used to test whether the ten points lie on a cubic. As illustrated in Figure 5, the two conics $C_1$ and $D_1$ intersect in four points, one of which is the residual point $P_1$. The other three points of intersection are the known points in $S_1 \cap T_1 = \{ K_3, K_4, K_5 \}$. We choose one of these three intersection points not on the line $P P_1$ and rename it $G$. We relabel the other two intersection points $A$ and $B$. In Construction 1 of Section 6 we show how to use a straightedge to construct the second point of intersection of a line with a conic if we already know their other point of intersection. Using this technique we construct the second point of intersection $W$ of the line $PG$ with the conic $C_1$. Similarly, construct the second point of intersection $Z$ of the line $PG$ with the conic $D_1$. The line $L_P =$
Figure 5. An important triangle.

$P_1P_2$ meets $C_1$ in $P_1$ and an additional point $X$, and meets $D_1$ in $P_2$ and an additional point $Y$. Now define $U$ to be the point where the line $WX$ meets $L_Q$ and $V$ to be the point where the line $YZ$ meets $L_R$.

**Theorem 3.** The 10 general points lie on a cubic precisely when the three points $U$, $V$, and $P_2$ are collinear.

Theorem 3 shows that we can check whether 10 general points lie on a cubic curve using just a straightedge. We prove Theorem 3 in the next section.

Our construction described here works for ten general points. The construction described here may fail when the 10 original points are in special positions. In particular, if the construction calls for us to intersect two lines and the two lines are equal, then the point of intersection is not well-defined. Similarly if we try to form the line through two points but the points are equal, then the line is not well-defined. The degenerate configurations for which such problems occur possess additional geometric structure. We may exploit this extra structure using our straightedge to settle the question of whether the ten points lie on a cubic by ad-hoc means. As a simple example, if the seven points of $|S_1 \cup T_1|$ lie on a single conic, then $C_1 = D_1$ and so the fourth point of intersection $P_1$ of $C_1 \cap D_1$ is not well defined. Besides just making a less fateful choice of $S_1$ and $T_1$, there is a simpler way to proceed. By Bézout’s theorem, the cubic would have to be a union of the conic $C_1$ and a line, so the ten points lie on a cubic precisely when the points off the conic $C_1$ are collinear, which we can check using our straightedge. The other special cases are similar. In summary, in all cases we may determine whether ten points lie on a cubic using only a straightedge.

**Example 4.** We start with ten points: $K_1 = (1, 0)$, $K_2 = (2, 2)$, $K_3 = (0, 0)$, $K_4 = (0, -1)$, $K_5 = (2, -3)$, $K_6 = (-1, -1)$, $K_7 = (6, -15)$, $K_8 = (\frac{1}{4}, -\frac{1}{8})$, $K_9 = (6, 14)$, and $K_{10} = (\frac{1}{4}, -\frac{5}{8})$.

Let $C_1$ be the conic through $K_1$, $K_2$, $K_3$, $K_4$, and $K_5$; $C_2$ the conic through $K_6$, $K_7$, $K_8$, $K_9$, and $K_{10}$; $D_1$ the conic through $K_3$, $K_4$, $K_5$, $K_6$, and $K_7$; and $D_2$ the conic through $K_1$, $K_2$, $K_8$, $K_9$, and $K_{10}$. Conics $C_1$ and $D_1$ share three known points and meet in a fourth point $P_1 = (\frac{1}{11}, -\frac{5}{11})$ and conics $C_2$ and $D_2$ also share three known points and meet in a fourth point $P_2 = (\frac{7}{17}, \frac{10}{17})$. Conics $C_1$ and $D_2$ share two known points.
and their coradical axis is $18x - y = 4$. Similarly, the coradical axis of conics $C_2$ and $D_1$ is $13x + y = 2$. The two coradical axes meet in $P = (\frac{6}{17}, -\frac{16}{17})$, the first coradical axis $L_Q$ meets $P_1P_2$ in $R = (\frac{13}{39}, -\frac{2}{39})$, and the second coradical axis $L_R$ meets $P_1P_2$ in $Q = (\frac{15}{31}, -\frac{1}{31})$. Conic $C_1$ meets $L_P$ in $P_1$ and $X = (-\frac{3}{17}, -\frac{18}{17})$ and conic $D_1$ meets $L_P$ in $P_1$ and $Y = (-\frac{3}{31}, -\frac{33}{31})$. The conics $C_1$ and $D_1$ meet at $G = K_3$, $A = K_4$, $B = K_5$, and $P_1$. The line $PG$ meets $C_1$ at $W = (-\frac{3}{31}, \frac{8}{31})$ and meets $D_1$ at $Z = (-12, 32)$. The line $WX$ meets the line $L_Q$ at $U = (\frac{3}{31}, \frac{34}{31})$ and the two lines $YZ$ and $L_R$ meet at $V = (\frac{15}{46}, -\frac{103}{46})$. Now we can check that $P_2, U$, and $V$ all lie on the line $33x - y = 13$, which tells us that the 10 points we started with lie on a cubic curve. In fact, they all lie on the curve $x^3 - x - y^2 - y = 0$.

The reader will notice that the denominators of the coordinates become larger as we construct more lines and points. This behavior is typical of these constructions. In fact, this example was meticulously constructed, involving very careful choices combined with a computer search among over a billion sets of points, to find an example involving rational coefficients expressible using small integers. More typical examples generate coefficient explosion, requiring numerators and denominators several tens or hundreds of digits long for the constructed points.

5. JUSTIFYING THE CONSTRUCTION. In this section we revisit our main construction and prove Theorem 3. We break the proof into five steps: (1) reduce to the case where two of the conics are circles, which forces some of our lines to be parallel; (2) characterize when the six residual points lie on a conic in terms of a product of oriented distance ratios using Carnot's theorem; (3) cancel some terms in the product using the power of a point; (4) restate the product using similar triangles; (5) use Menelaus's theorem to interpret this product as a collinearity result.

Step 1: Reduce to the case where two of the conics are circles. Projective transformations are invertible linear maps so they preserve collinearity of points and concurrency of lines. As a consequence, all incidence results remain valid upon application of a projective transformation. Additionally, whether there exists a conic through six points is not affected by applying a projective transformation. This follows from Pascal’s theorem, which characterizes when six points lie on a conic in terms of an incidence of lines. Using the Cayley-Bacharach theorem, we reduced the question of whether ten points lie on a cubic curve to the question of whether six residual points lie on a conic. Thus the property of ten points lying on a cubic is also unchanged by projective transformations.

The two conics $C_1$ and $D_1$ meet in three of the ten given points, $A$, $B$, and $G$. There exists an invertible projective transformation sending $A$ to $I = [1 : i : 0]$ and $B$ to $J = [1 : -i : 0]$, carrying the line $AB$ to the line at infinity and the two conics $C_1$ and $D_1$ to circles. Since incidence results are preserved under projective transformations, it is enough to consider the case where $C_1$ and $D_1$ are circles.

Transforming the conics to circles has a geometric side benefit: the lines $WX$ and $YZ$ become parallel. This is a consequence of the following lemma.

**Intersecting Circles Lemma.** Let two circles $C$ and $D$ meet at $S$ and $T$, as in Figure 6. If a line through $S$ meets $C$ and $D$ at $X$ and $Y$, respectively, and a line through $T$ meets $C$ and $D$ at $W$ and $Z$, respectively, then $WX$ and $YZ$ are parallel lines.

**Proof.** Draw the segment $ST$, forming two cyclic quadrilaterals, $STWX$ and $SYZT$. 
Because the opposite angles of cyclic quadrilaterals are complementary,
\[ \angle SXW + \angle WTS = 180^\circ \quad \text{and} \quad \angle STZ + \angle ZYS = 180^\circ. \]
Additionally, we have
\[ \angle WTS + \angle STZ = 180^\circ. \]
Adding the first two equations and subtracting the second shows that \( \angle SXW + \angle ZYS = 180^\circ \), so the two lines \( WX \) and \( YZ \) make the same angle with the line \( XY \) and hence they are parallel.

**Step 2: Characterize when the six residual points lie on a conic in terms of a product of distance ratios using Carnot’s theorem.** The six residual points \( P_1, P_2, Q_1, Q_2, R_1, \) and \( R_2 \) lie on a triangle with vertices \( P, Q, \) and \( R \). Carnot’s theorem characterizes when the six residual points lie on a conic in terms of a product of oriented distance ratios. We rearrange terms in the equation displayed in Figure 3 to get the following statement of Carnot’s theorem.

**Theorem 5 (Carnot).** The six residual points \( P_1, P_2, Q_1, Q_2, R_1, \) and \( R_2 \), lie on a conic if and only if
\[
\frac{|QP_1||Q P_2||RQ_1||RQ_2||PR_1||PR_2|}{|RP_1||RP_2||PQ_1||PQ_2||Q R_1||QR_2|} = 1. \tag{1}
\]

We use the power of a point, a special result that holds for circles, to evaluate this product, even though we do not know the exact location of four of the residual points.

**Step 3: Cancel some terms in the product using the power of a point.** Jakob Steiner\(^2\) introduced the power of a point in a long article in 1826, when he was 30 years old [23]. Remarkably, Steiner “was born to a poor peasant family that could hardly afford to send him to school; he could not even write before the age of fourteen” (quoted material from Fried [12]; see Burckhardt [5] for more on the life of Jakob Steiner). Given a point \( X \) and a circle \( C \), draw a line through \( X \) that intersects the circle

\(^2\)There seems to be some controversy about the spelling of Steiner’s first name. The authoritative version of his collected works [23] gives the author’s name spelled with a k and the subject’s name spelled with a c. Perhaps this confusion is common among people whose work is important enough to be translated into many languages.
at $A$ and $B$. The power $\text{POP}(X, C)$ of the point $X$ with respect to the circle $C$ is defined to be the product of oriented distances $|XA| \cdot |XB|$. The Intersecting Secants Theorem says that the power of a point does not depend on the line drawn through $X$; if another line through $X$ meets the circle at $C$ and $D$ then,

$$|XA| \cdot |XB| = |XC| \cdot |XD|.$$

The Intersecting Secants Theorem follows easily from considerations about angles and similar triangles. Indeed, referencing Figure 7, the two inscribed angles $\angle ADC$ and $\angle ABC$ are equal since both are half the central angle $\angle AOC$ supported at the center $O$ of the circle. Then the triangles $XAD$ and $XCB$ are similar so

$$\frac{|XA|}{|XD|} = \frac{|XC|}{|XB|},$$

and hence $|XA| \cdot |XB| = |XC| \cdot |XD|$.

Now we modify Carnot’s condition (1) using the power of a point several times. The careful reader may want to consult Figure 5. We have

$$\frac{|RQ_1|}{|RQ_2|} = \frac{|RP_1|}{|RP_2|} = \frac{|PG|}{|PZ|} = \frac{|PR_1|}{|PR_2|} = \frac{|PQ_1|}{|PQ_2|} = \frac{|POP(R, C_1)|}{|POP(P, D_1)|} = \frac{|POP(Q, D_1)|}{|POP(P, C_1)|} = \frac{|POP(Q, C_1)|}{|POP(P, D_1)|} = \frac{|QZ_1|}{|QZ_2|} = \frac{|PZ|}{|PW|}.$$

Replacing terms in (1), canceling three terms, and reordering the remaining products gives the equivalent condition

$$\frac{|QP_2|}{|QY|} \frac{|RX|}{|RP_2|} \frac{|PZ|}{|PW|} = 1.$$

**Step 4: Restate the product using similar triangles.** We imagine a third line parallel to $WX$ and $YZ$ through $P$, intersecting $P_2R$ at the point $M$, as in Figure 8.

Now we claim that

$$\frac{|PZ|}{|PW|} = \frac{|MY|}{|MX|}.$$

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Figure 7. The power of a point is well-defined: $|XA| \cdot |XB| = |XC| \cdot |XD|$. 

\[\text{Figure 7. The power of a point is well-defined: } |XA| \cdot |XB| = |XC| \cdot |XD|.\]
Figure 8. Parallel lines giving rise to similar triangles.

To see this, draw a line through \( P \) parallel to the line containing \( M, X \) and \( Y \). This line hits \( XW \) in \( X' \) and \( YZ \) in \( Y' \), as shown in Figure 8. Now

\[
\frac{|PZ|}{|PW|} = \frac{|PY'|}{|PX'|} = \frac{|MY|}{|MX'|}
\]

where the first equality comes from the similarity of the triangles \( PZY' \) and \( PWX' \). Substituting (3) in (2), rearranging terms, and flipping some oriented distances, gives

\[
\frac{|QP_2| \cdot |RX| \cdot |MY|}{|RP_2| \cdot |XM| \cdot |YQ|} = 1.
\] (4)

Now define \( U = XW \cap PR \) and \( V = YZ \cap PQ \). As depicted in Figure 8, triangle \( RXU \) is similar to \( RMP \) and triangle \( QYV \) is similar to \( QMP \) so

\[
\frac{|RX|}{|XM|} = \frac{|RU|}{|UP|} \quad \text{and} \quad \frac{|MY|}{|YQ|} = \frac{|PV|}{|VQ|}.
\]

Substituting in the previous equation and changing the orientation of the denominator of the first term, we get

\[
\frac{|QP_2| \cdot |RU| \cdot |PV|}{|P_2R| \cdot |UP| \cdot |VQ|} = -1.
\] (5)

**Step 5: Use Menelaus’s theorem to interpret this product as a collinearity result.**

Equation (5) is precisely the product of distance ratios in Menelaus’s theorem, so the six residual points lie on a conic precisely when the three points \( P_2, U, \) and \( V \) are collinear. By the Cayley-Bacharach theorem, this collinearity occurs if and only if the ten original points lie on a cubic curve. This completes the proof of Theorem 3. Note that \( U \) and \( V \) were constructed using only a straightedge and without knowledge of the precise locations of \( Q_1, Q_2, R_1, \) and \( R_2 \). \( \Box \)

**Remark 6.** A careful analysis of our proof shows that we also proved the following result. Suppose that \( PQR \) is a triangle with sides lying on the lines \( \mathcal{L}_P, \mathcal{L}_Q, \) and \( \mathcal{L}_R \). Let \( G \) be an arbitrary point and let \( W \) and \( Z \) be points on the line \( PG \). Fix points \( P_1, X \) and \( Y \) on \( \mathcal{L}_P \). Let \( U = WX \cap \mathcal{L}_Q \) and \( V = YZ \cap \mathcal{L}_R \). Suppose that \( C_1 \) is any
conic through $P_1, G, W,$ and $X$ and $\mathcal{D}_1$ is any conic through $P_1, G, Y,$ and $Z$. Then the unique conic $\mathcal{E}$ passing through the two points where $\mathcal{L}_Q$ meets $\mathcal{C}_1$, the two points where $\mathcal{L}_R$ meets $\mathcal{D}_1$, and $P_1$ always passes through the point $P_2 = UV \cap \mathcal{L}_P$. In Algebraic Geometry, we would say that the collection of conics $\mathcal{E}$ (varying as we vary $\mathcal{C}_1$ and $\mathcal{D}_1$) is a family of conics with a base point $P_2$.

6. GEOMETRIC CONSTRUCTIONS. To perform our constructions we need to solve three problems from constructive synthetic geometry involving conics and lines. Figure 9 illustrates the three problems we consider: intersecting a conic with a line through a point on the conic (left image), constructing the coradical axis of two conics sharing two known common points (center image), and finding the fourth point of intersection of two conics with three given points in common (right image). We give three constructions, each solving one of these problems.

We used the first construction to find the points $W, X, Y,$ and $Z$ in Section 4. We used Construction 2 to draw the two lines $L_R$ and $L_Q$, each through two residual points whose precise locations were unknown. The third construction allowed us to locate the other two residual points $P_1$ and $P_2$.

Construction 1. Given five points $P_1, \ldots, P_5$ lying on a unique conic and another point $Q$, we construct the second point of intersection of the conic with the line $P_1Q$. To do this we just use the given line $P_1Q$ as one of the edges of the hexagon in Pascal’s theorem, as illustrated in Figure 10. As we draw the hexagon, we construct the points $Q_1 = P_1P_2 \cap P_4P_3$ and $Q_2 = P_1Q \cap P_3P_4$. Then we construct $Q_3$ as $Q_1Q_2 \cap P_2P_3$ and the point $R = P_5Q_3 \cap P_1Q$ lies at the intersection of the line and the conic by Pascal’s theorem.

Construction 2. Given two points $S$ and $T$ on the intersection of two conics $\mathcal{C}$ and $\mathcal{D}$, three additional points $X, W,$ and $W'$ on $\mathcal{C}$ and three more points on $\mathcal{D}$, we construct the coradical axis of the two conics with respect to $S$ and $T$. Recall that the coradical axis is the line that passes through the two unknown points of intersection. Call these points $I'$ and $J'$.

If we use a projective transformation to send $I'$ and $J'$ to $I = [1:i:0]$ and $J = [1:-i:0]$, respectively, then the two conics $\mathcal{C}$ and $\mathcal{D}$ are transformed to circles and the coradical axis is sent to the line at infinity. The Intersecting Circles Lemma produces pairs of parallel lines, which must intersect on the line at infinity, as illustrated in Figure 6. Intersecting two parallel lines yields a point on the line at infinity. Doing this twice we have two such points; connect them to produce the coradical axis, the line at infinity, $IJ$.

Applying projective transformations preserves incidence constructions so applying the incidence constructions of the previous paragraph to our original conics produces the coradical axis $I'J'$. In particular, we use Construction 1 to find $Y = XS \cap \mathcal{D}$ and $Z = WT \cap \mathcal{D}$. By the Intersecting Circles Lemma, $M = XW \cap YZ$ lies on the coradical axis. Replacing $W$ with $W'$, the same construction produces a second point $M'$ on
the coradical axis, allowing us to draw the line $MM'$ through the two remaining points of intersection of $C$ and $D$.

**Construction 3.** Given three points $S$, $T$, and $U$ on two conics $C$ and $D$, two additional points on $C$, and two additional points on $D$, we construct the fourth point of intersection of the two conics. To do this, use Construction 2 to draw the coradical axis of $C$ and $D$ with respect to $S$ and $T$. This line passes through $U$, the third known point of intersection of the conics. Using Construction 1, we can find the intersection of this line with one of the conics, producing the fourth point of intersection of $C$ and $D$.

7. **EXTENSIONS AND EXERCISES.** We close with some fun exercises (including several open research problems), pointers to further reading, and comments about how this work connects to related topics. Two excellent books in geometry are Coxeter and Greitzer [10] and Richter-Gebert [22]. The geometry chapter in Zeitz [28] contains many challenging and enjoyable problems. The Cayley-Bacharach theorem played a key role in our construction; see Problem 7 for a statement of the theorem. The reader may enjoy two related papers in this MONTHLY. Bashelor et al. [2] treats the enumerative study of conics. Traves [25] gives another application of the Cayley-Bacharach theorem to cubics.

**Problem 1.** Use the power of a point to prove Menelaus’s theorem and Carnot’s theorem. There are solutions at the wonderful Cut the Knot website.³

**Problem 2.** Explain why the $6! = 720$ labelings of 6 points on a conic only give rise to 60 different Pascal lines.

**Problem 3.** Given five points on a conic, use Pascal’s theorem to show that you can construct the tangent line to the conic at each of the five points.

**Problem 4.** Prove the Intersecting Circles Lemma using the power of a point. Give yet another proof using only Pascal’s theorem. Thus the Intersecting Circles Lemma admits an algebraic proof which avoids relying on angles and similar triangles.

³https://www.cut-the-knot.org/pythagoras/PPower.shtml
Problem 5. Given a curve defined by a polynomial equation \( g(x, y) = 0 \) in the Euclidean plane, there is a standard way to extend the curve to all of \( \mathbb{P}^2 \). If the polynomial \( g \) has degree \( d \), we homogenize \( g \) by multiplying each term in the polynomial by a power of \( z \) to ensure that it too has total degree \( d \). The homogenization \( G(x, y, z) \) vanishes at all points in the plane where \( g \) vanished since \( G(x, y, 1) = g(x, y) \). Show that the value of \( G \) at a point \([x : y : z]\) is not well-defined in general but the set of points in \( \mathbb{P}^2 \) where \( G(x, y, z) = 0 \) is well-defined. This set is the extension of our original curve to \( \mathbb{P}^2 \). Show that circles can be characterized as the conics that meet the line at infinity \( z = 0 \) in the two special points \( I = [1 : i : 0] \) and \( J = [1 : -i : 0] \). Finally, show that the tangent lines to a circle at \( I \) and \( J \) both pass through the center of the circle.

Remark 7. Let \( V_n \) denote the vector space of homogeneous polynomials of degree \( n \) in \( \mathbb{C}[x, y, z] \). For a set \( S \) of points in \( \mathbb{P}^2 \), let \( \mathbb{I}_n(S) \) denote the subspace of \( V_n \) consisting of polynomials that vanish on \( S \). Requiring a polynomial to vanish at \( m \) points can impose up to \( m \) linearly independent conditions on \( V_n \); however, the points may impose fewer linearly independent conditions if the points lie in special position or if \( m \) is larger than the dimension of \( V_n \). If \( S \) consists of \( m \) points, we say that \( S \) imposes \( \dim V_n - \dim \mathbb{I}_n(S) \) linearly independent conditions on \( V_n \). We also say that the failure of \( S \) to impose conditions on \( V_n \) is \( m - (\dim V_n - \dim \mathbb{I}_n(S)) \).

Problem 6. (a) Which sets of three points fail to impose 1 condition on \( V_1 \)?
(b) Which sets of six points fail to impose 1 condition on \( V_2 \)?
(c) Which sets of six points fail to impose 2 conditions on \( V_2 \)?

Problem 7. The Cayley-Bacharach theorem in the projective plane deals with the intersection of two curves in the projective plane. Eisenbud, Green, and Harris [11] give many statements of the Cayley-Bacharach theorem in their very beautiful paper. One of their formulations of the Cayley-Bacharach theorem, Theorem CB5, is as follows. Suppose \( X \) and \( Y \) are plane curves of degrees \( d \) and \( e \), respectively, which meet in a set \( \Gamma \) of \( de \) points. Partition \( \Gamma = \Gamma' \cup \Gamma'' \) as the disjoint union of two sets of points \( \Gamma' \) and \( \Gamma'' \). Put \( s = d + e - 3 \) and let \( k \leq s \) be a nonnegative integer. Then the dimension of the degree-\( k \) polynomials vanishing on \( \Gamma' \) (modulo the polynomials vanishing on all of \( \Gamma \)) equals the failure of \( \Gamma'' \) on the space of degree-\( (s - k) \) polynomials \( V_{s-k} \):

\[
\dim \mathbb{I}_s(\Gamma') - \dim \mathbb{I}_k(\Gamma) = \text{failure of } \Gamma'' \text{ on } V_{s-k}.
\]

(a) Use the Cayley-Bacharach theorem in the case \( d = e = 3 \) and \( k = 2 \) to prove Pascal’s theorem.
(b) Use the Cayley-Bacharach theorem in the case \( d = e = 4 \) and \( k = 3 \) to prove that if two degree-4 curves meet in 16 points then there is a cubic through 10 of the points if and only if there is a conic through the remaining 6 points. This result is the key that unlocks our construction.

Research Problem 1. It is natural to wonder whether there is a similar straightedge construction that checks whether 15 points lie on a degree-4 curve. Take two degree-5 curves through the 15 given points. These two curves meet in 25 points — the 15 given points and 10 residual points. Then the Cayley-Bacharach theorem with \( d = e = 5 \) and \( k = 4 \) says that the 15 given points lie on a quartic precisely when the 10 residual points lie on a cubic. We would like to test the later condition using the construction of this article. However, it is not clear how to choose the two quintics in such a way that we can get sufficient information about the location of the ten residual points. It
is an open research problem to overcome this obstruction and obtain a straightedge construction that checks whether 15 points lie on a quartic.

More generally, we want to determine if a set \( \Gamma' \) of \( \binom{m+1}{2} \) given points lies on a curve of degree \( k = m - 1 \). Two degree-\( m \) curves, both through the points in \( \Gamma' \), meet in a set \( \Gamma = \Gamma' \cup \Gamma'' \) of \( m^2 \) points. Then the Cayley-Bacharach theorem implies that there is a degree \( m - 1 \) curve through the given \( \binom{m+1}{2} \) points if and only if there is a degree \( m - 2 \) curve through the set \( \Gamma'' \) of \( \binom{m}{2} \) residual points.

Here are two open problems in synthetic geometry.

**Research Problem 2.** Given 9 points sitting on a unique cubic curve, construct the third point of intersection of the cubic with a line through two of the points using a straightedge.

**Research Problem 3.** Given 9 points sitting on a unique cubic curve and a subset consisting of five of these points sitting on a unique conic, construct the sixth point of intersection of the cubic with the conic using a straightedge.

Algebraic methods can be used to give a second proof of Theorem 3. Below we describe the bracket method, which can be used to prove results in incidence geometry.

**Remark 8.** Given a collection of distinct points \( A, B, C, \ldots \) in \( \mathbb{P}^2 \) with fixed representatives for their vectors of projective coordinates, the bracket \( [ABC] \) denotes the determinant of the \( 3 \times 3 \) matrix whose columns are the projective coordinates of points \( A, B, \) and \( C \). Quotients of brackets, for example \( [ABC]/[CDE] \), are invariant under projective transformation since if we multiply each column in a matrix by \( M \), then the determinant is multiplied by \( \det(M) \) and this factor cancels from both terms in the quotient. If all the points do not lie on a single line, then the collection of values of all possible quotients of the brackets completely determine the positions of the points, up to projective transformation. Perhaps this is why these numbers are called determinants. Since brackets determine the relative locations of points, they have been used to study incidence relations.

**Problem 8.**
(a) Show that if \( P \) is the intersection of the lines \( AB \) and \( CD \), then \( P \) is a nonzero scalar multiple of both \( [CDB]A - [CDA]B \) and \( [ABD]C - [ABC]D \).
(b) If \( E \) is a fifth point, show that
\[
[BCD][ADE] - [ACD][BDE] - [ABD][CDE] = 0.
\]
This is an example of a Grassmann-Plücker relation among \( 3 \times 3 \) determinants.
(c) If \( A, B, \) and \( D \) are not collinear, show that \( C, D, \) and \( E \) are collinear precisely when
\[
[BCD][ADE] = [ACD][BDE].
\] (6)
(d) The locus \( \{ X \in \mathbb{P}^2 : [ABX] = 0 \} \) is the line through \( A \) and \( B \). Similarly, \( [ABX][CDX] = 0 \) represents two lines whose union contains \( A, B, C, \) and \( D \). Another reducible conic with the same property is \( [ACX][BDX] = 0 \). In fact, when \( A, B, C, \) and \( D \) are not collinear, any conic through the four points has an equation of the form \( \gamma [ABX][CDX] + \delta [ACX][BDX] = 0 \) for constants \( \gamma \) and \( \delta \). Show that the conic through \( A, B, C, D, \) and a new point \( E \) passes through a sixth point \( F \) when
\[
[ACE][BDE][ABF][CDF] = [ABE][CDE][ACF][BDF].
\] (7)
Remark 9. Each of the conic or collinear constraints in our construction imposes an equality of bracket monomials of the form (6) or (7). Choosing an appropriate subset of these equations, multiplying all their left-hand sides together and all their right-hand sides together, and canceling terms appearing in both sides, leaves just the equality

$$[VP_2 P][UY P_2] = [VYP_2][UP_2 P].$$

Using (6), we find that $P_2$, $U$ and $V$ are collinear (as long as $[PP_2 Y] \neq 0$ and all the brackets that we canceled are not zero, which requires a large collection of triples of points to be noncollinear). This is precisely the conclusion of Theorem 3! This computational proof was found using MATLAB [18] to set up an integer programming problem, which we then passed to the optimization solver Gurobi [13]. Our optimization problem involved 552 variables and 22,022 constraints, which Gurobi solved in just under 33 seconds on a five-year-old laptop. We used the conclusion of Theorem 3 to set up the optimization problem, so this second proof only verifies the result in Theorem 3; this method is not immediately applicable in the search for geometric results.

This computational proof of Theorem 3 works for points with complex coordinates and gives another demonstration that our construction is valid for such points. Unfortunately, this computational approach does not give any intuition for why the result is true. This tendency of computer-assisted proofs to lead to results that cannot be easily understood by a human is one of the central problems with deep neural networks and other advanced tools in artificial intelligence today.

Remark 10. Here is a matrix algebra approach to checking whether six points lie on a conic. Evaluating the homogenized conic equation $ax^2 + bxy + cxz + dy^2 + eyz + fz^2 = 0$ at the point $P_i = [x_i : y_i : z_i]$ $(i \in \{1, \ldots, 6\})$, produces a linear condition on the coefficients of the conic, $a, \ldots, f$. Together these six linear equations form a matrix equation $Av = 0$, where $A$ is a $6 \times 6$ matrix and $v = [a, b, c, d, e, f]^T$. There is a conic through the six points if and only if this matrix $A$ has a nonzero nullspace. This occurs precisely when $\text{det}(A) = 0$. The Fundamental Theorem of Invariant Theory [24, Theorem 3.2.1] implies that this determinant condition can be written as a polynomial in the brackets $[P_i P_j P_k]$, each a determinant of a $3 \times 3$ matrix. The determinant of $A$ is a degree-4 polynomial in the brackets with 720 terms. But the Grassmann-Plücker relations among the brackets allow us to reduce the number of terms. One can check that the polynomial evaluates to a scalar multiple of the expression (7).

Research Problem 4. The same approach can be used to check whether ten points lie on a cubic. In this case the matrix $A$ is a $10 \times 10$ matrix and in 1870 Reiss [20] wrote down an expression for $\text{det}(A)$ as a degree-10 polynomial in the brackets. The expansion of the determinant has $10! = 3,628,800$ terms but, in a masterwork, Reiss used the Grassmann-Plücker relations to rewrite this polynomial as an expression with just 20 terms. It took a modern computer over 8 hours to check Reiss’s calculation using Gröbner bases. Suzanne Apel [1] developed an algorithm to express this polynomial times a product of brackets as an expression in the Grassmann-Cayley algebra, which can be interpreted as a massive incidence structure. Unfortunately, it is hard to get any intuition about why that structure implies the presence of a cubic through the ten points. The reader might want to search for a short bracket polynomial expression for the determinant of the $15 \times 15$ matrix that determines whether 15 points lie on a degree-4 curve. This is an open research problem!
Remark 11. It is fun to consider incidence structures with special properties. For instance, imagine a finite set of points for which every line through a pair of the points also contains a third point. One such collection consists of the nine points of inflection of a cubic curve. In fact, there are 12 lines passing through pairs (necessarily, triples) of the nine points, forming the beautiful incidence pattern depicted in Figure 11. However, at least one of the points here has to have complex coordinates: the Sylvester-Gallai theorem says that the only such point configuration in $\mathbb{R}^2$ has all the points lying on a common line!

8. CONCLUSION. Pascal’s original observation about conics was stated solely in terms of an incidence of lines. The natural generalization of Pascal’s theorem to cubic curves asks for a straightedge construction to check whether 10 points lie on a cubic. This problem has lain unsolved for almost 400 years. The restriction to only use a straightedge seems to preclude many advanced techniques. Indeed, when we started this project, it seemed possible that the constraints were so confining that no straightedge construction even existed. Working under these strong constraints forced us to explore seemingly forbidden avenues. This is a common experience in mathematics: difficult problems push us to follow unexpected paths to a solution.

Each generation builds on the results of their predecessors, solving old problems and generating new ones. Our solution relies on the work of many mathematicians—including Menelaus, Steiner, Cayley, and Bacharach—to interweave measurement with results from projective geometry. We expect the combined power of these tools would be a surprise even to their originators. The key ideas came from a diverse group of people from vastly different backgrounds spanning thousands of years. Menelaus, Steiner, and Carnot spoke different languages, ate different foods, listened to different music, and worshipped differently, but they shared a common cultural connection—the love of mathematics. Mathematics has the power to bring people from many different cultures and eras together.

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Yet Another Integral Representation of Catalan’s Constant

In [1, p. 456], Seán Stewart proves that

\[ \int_{0}^{\pi/4} \log \left( \frac{\cos x + \sin x}{\cos x - \sin x} \right) \, dx = C, \]

where \( C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965594... \) is the Catalan constant. Here we derive this nice formula in a different short way. Our new proof is based on the following representation (see e.g., [1, p. 450]):

\[ J := - \int_{0}^{\pi/4} \log \tan s \, ds = C. \]

By using the change of variable \( s = -t + \pi/4 \), and the formula \( \sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \) and \( \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \), respectively,

\[ J = - \int_{0}^{\pi/4} \log \left( \frac{\sin(-t + \pi/4)}{\cos(-t + \pi/4)} \right) \, dt \]
\[ = - \int_{0}^{\pi/4} \log \left( \frac{-\sqrt{2} \sin t + \sqrt{2} \cos t}{\sqrt{2} \cos t - \sqrt{2} \sin(-t)} \right) \, dt \]
\[ = \int_{0}^{\pi/4} \log \left( \frac{\cos t + \sin t}{\cos t - \sin t} \right) \, dt. \]

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