THE REPETITION PROPERTY FOR SEQUENCES ON TORI
GENERATED BY POLYNOMIALS OR SKEW-SHIFTS

MICHAEL BOSHERNITZAN AND DAVID DAMANIK

Abstract. The repetition property of a sequence in a metric space, a notion introduced by us in an earlier paper, is of importance in the spectral analysis of ergodic Schrödinger operators. It may be used to exclude eigenvalues for such operators. In this paper we study the question of when a sequence on a torus that is generated by a polynomial or a skew-shift has the repetition property. This provides classes of ergodic Schrödinger operators with potentials generated by skew-shifts on tori that have, contrary to earlier belief, no eigenvalues.

1. Introduction

This short paper contains results that are complementary to those obtained by us in a recent paper [4]. There, the following notion was introduced. (Here and in what follows, we write \(\mathbb{Z}^+ = \{1, 2, 3, \ldots\}\).)

Definition 1. A sequence \(\{\omega_n\}_{n \geq 0}\) in a metric space \(\Omega\) has the repetition property if for every \(\varepsilon > 0\) and \(r \in \mathbb{Z}^+\), there exists \(q \in \mathbb{Z}^+\) such that \(\text{dist}(\omega_n, \omega_{n+q}) < \varepsilon\) for \(n = 0, 1, 2, \ldots, rq\).

An alternative way to state this definition that involves fewer parameters and is sometimes more convenient to use is the following: A sequence \(\{\omega_n\}_{n \geq 0}\) in a metric space \(\Omega\) has the repetition property if and only if for every \(r \in \mathbb{Z}^+\), there exists \(q \in \mathbb{Z}^+\) such that \(\text{dist}(\omega_n, \omega_{n+q}) < 1/r\) for \(n = 0, 1, 2, \ldots, rq\).

Sequences of particular interest are given by forward orbits corresponding to some map \(T : \Omega \to \Omega\), that is, \(O_+^T(\omega) = \{T^n \omega\}_{n \geq 0}\). For a given map \(T\), the set \(\text{PRP}(T)\) consists of the points \(\omega \in \Omega\) for which \(O_+^T(\omega)\) has the repetition property. Following [4], we say that \(T\) has the topological repetition property (TRP) if \(\text{PRP}(T) \neq \emptyset\), the metric repetition property (MRP) relative to some \(T\)-invariant measure \(\mu\) if \(\mu(\text{PRP}(T)) > 0\), and the global repetition property (GRP) if \(\text{PRP}(T) = \Omega\). Obviously, we always have (GRP) \(\Rightarrow\) (MRP) (relative to any non-zero \(T\)-invariant measure) \(\Rightarrow\) (TRP). Also, if \(T\) is continuous and \((\Omega, T)\) is minimal, then (TRP) implies that \(\text{PRP}(T)\) is residual; compare Corollary [1] below.

If \((\Omega, T)\) is a symbolic flow, the repetition property for a forward orbit \(O_+^T(\omega)\) is related to the initial critical exponent; see [2] for a precise definition and further references. Namely, \(\omega \in \text{PRP}(T)\) if and only if the initial critical exponent of \(\omega\) is infinite, that is, \(\omega\) has arbitrarily high powers as prefixes. For the prominent class of Sturmian symbolic flows, (TRP) and (MRP) relative to the unique shift-invariant probability measure are equivalent and they hold if and only if the so-called slope

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of the Sturmian flow has unbounded partial quotients. It is also of interest to note
that (GRP) fails whenever the symbolic flow \((\Omega, T)\) is aperiodic; compare [3]. The
repetition property in the symbolic setting and related issues will be studied further
in [5].

The repetition property for forward orbits plays an important role in the study
of the eigenvalue problem for Schrödinger operators in \(L^2(\mathbb{Z})\),

\[(H_\omega \psi)(n) = \psi(n + 1) + \psi(n - 1) + V_\omega(n)\psi(n),\]

with dynamically defined potentials, that is, \(V_\omega(n) = f(T^n \omega)\) with an invertible
map \(T : \Omega \to \Omega\) and a continuous function \(f : \Omega \to \mathbb{R}\). Namely, in the particular
situation where \(\Omega\) is compact and \(T\) is a minimal homeomorphism, the size of the
set \(PRP(T)\) was related in [4], for a generic continuous function \(f\), to the size of the
set of \(\omega\)'s for which \(H_\omega\) has no eigenvalues: If \(T\) has (TRP) (resp., (MRP) relative
to some \(T\)-ergodic measure \(\mu\)), then there is a residual set \(\mathcal{F} \subset C(\Omega)\) such that for
each \(f \in \mathcal{F}\), \(H_\omega\) has no eigenvalues for \(\omega\)'s from a residual (resp., full \(\mu\)-measure)
subset of \(\Omega\).

Several examples were considered in [4]. For most of them, \(\Omega\) is either the circle
\(\mathbb{T}\) or a \(k\)-dimensional torus \(\mathbb{T}^k\). Here, we denote \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\). The metric on \(\mathbb{T}^k\)
is given by \(\text{dist}(x, y) = \sum_{j=1}^{k} |x_j - y_j|\), where \((\tau) = \min\{|\hat{\tau} - p| : p \in \mathbb{Z}\}\)
with any representative \(\hat{\tau} \in \mathbb{R}\) of \(\tau \in \mathbb{T}\). For a shift on a torus (i.e., \(T\omega = \omega + \alpha\)), it
is easily seen that (GRP) holds. Moreover, we showed that almost every interval
exchange transformation has (MRP) relative to Lebesgue measure. On the other
hand, almost every interval exchange transformation is weakly mixing\(^1\). This is
a deep result with significant contributions by Katok, Stepin, Veech, Nogueira,
Rudolph and others; the final result is due to Avila and Forni [1]. This provides a
class of examples that have (MRP) and are weakly mixing.

As explained above, one obtains as a consequence results on the absence of
eigenvalues for Schrödinger operators with potentials generated by shifts on a torus
or by an interval exchange transformation that hold for almost every point in the
space in question. To a certain extent this confirms what was expected for such
operators based on earlier results.

For Schrödinger operators with potentials generated by a skew-shift on a torus,
however, it was expected that there should be plenty of eigenvalues; see, for ex-
ample, Bourgain’s recent book [6] and the discussion in [4]. Alas, we were able to
give a complete study of (TRP), (MRP), and (GRP) for skew-shifts on the two-
dimensional torus, \((\omega_1, \omega_2) \mapsto (\omega_1 + \alpha, \omega_2 + \omega_1)\), and proved in particular that all
three properties hold for Lebesgue almost every \(\alpha\). Hence we obtained the cor-
responding absence of eigenvalue result for the associated Schrödinger operators
which was surprising given earlier expectations for these operators. Possible exten-
sions to higher-dimensional tori were only indicated. One of the main goals of the
present paper is to work out these extensions in detail.

The class of skew-shifts we are interested in is given by maps \(T : \mathbb{T}^k \to \mathbb{T}^k\),
\(k \geq 2\), that have an irrational rotation in the first component and, for \(j \geq 2\), the \(j\)-th component is given by a rational linear combination of \(\omega_1, \ldots, \omega_j\). For example,
for \(\alpha\) irrational, the following map fits this profile:

\[(1) \quad T : \mathbb{T}^k \to \mathbb{T}^k, \quad (\omega_1, \omega_2, \ldots, \omega_k) \mapsto (\omega_1 + \alpha, \omega_2 + \omega_1, \ldots, \omega_k + \omega_{k-1}).\]

\(^1\)The permutation of intervals is assumed to be irreducible and not a rotation.
We refer the reader to the paper [8] by Furstenberg, which studies skew-shifts of this (and more general) form from an ergodic theory point of view. It was shown there that normalized Lebesgue measure on $\mathbb{T}^k$ is the unique $T$-invariant probability measure. Thus, in the context of the map (1), (MRP) will always be understood to be relative to this measure.

While our arguments will also apply to more general skew-shifts on the torus, we will consider for definiteness the specific map (1). It turns out that the size of $PRP(\mathbb{T})$ depends sensitively on both the dimension $k$ of the torus and Diophantine properties of the irrational number $\alpha$. The following result gives almost complete information as to which of the the properties (TRP), (MRP) (relative to Lebesgue measure), and (GRP) hold.

**Theorem 1.** Suppose that $\alpha \in \mathbb{T}$ is irrational and the map $T : \mathbb{T}^k \to \mathbb{T}^k$ is given by $T(\omega_1, \omega_2, \ldots, \omega_k) = (\omega_1 + \alpha, \omega_2 + \omega_1, \ldots, \omega_k + \omega_{k-1})$.

(a) For every $k \geq 2$, the following are equivalent:
   - $T$ has (TRP),
   - $\liminf_{q \to \infty} q^{k-1} \langle q\alpha \rangle = 0$.

(b) For $k = 2$, the following are equivalent:
   - $T$ has (TRP),
   - $T$ has (MRP),
   - $T$ has (GRP),
   - $\liminf_{q \to \infty} q \langle q\alpha \rangle = 0$.

(c) If $k \geq 3$, then $T$ does not have (GRP).

(d) If $k \geq 3$, then $T$ does not have (TRP) for Lebesgue almost every $\alpha$.

(e) If $k = 3$, then $T$ has (MRP) for $\alpha$’s from a residual subset of $\mathbb{T}$.

(f) If $k \geq 4$, then $T$ does not have (MRP).

In particular, parts (a) and (e) of Theorem 1 yield new classes of Schrödinger operators generated by skew-shifts that (surprisingly) have empty point spectrum.

Since the components of $T^n \omega$ are given by the projection to $T$ of polynomials in $n$ with coefficients that depend on $\alpha$ and $\omega_1, \ldots, \omega_k$, we will first study the repetition property for sequences on the circle $\mathbb{T}$ that are generated by polynomials. This is done in Section 3. Skew-shift orbits are then investigated in Section 4 where we prove Theorem 1.

2. General Preliminary Results

This section contains several results that hold in the general setting and which will be useful later when we specialize to the circle or the $k$-dimensional torus.

**Lemma 1.** Assume that $X$ is a topological space, $\Omega$ is a metric space, and $f_n : X \to \Omega$, $n \in \mathbb{Z}_+$ are continuous. Then the set

$$RPX = \{ x \in X : \text{the sequence} \{ f_n(x) \}_{n \geq 0} \text{has the repetition property} \}$$

is a $G_\delta$ set in $X$.

**Proof.** Denote for $r, q \in \mathbb{Z}_+$ and $\varepsilon > 0$,

$$H(r, q, \varepsilon) = \{ x \in X : \max_{0 \leq n \leq rq} \text{dist}(f_n(x), f_{n+q}(x)) < \varepsilon \}.$$
Since each set $H(r, q, \varepsilon)$ is open in $X$ and
\[
RPX = \bigcap_{r \geq 1} \bigcup_{k \geq 1} H(r, q, \frac{1}{k}),
\]
it follows that $RPX$ is a $G_\delta$ set in $X$. \hfill \square

**Corollary 1.** If $\Omega$ is a metric space and $T : \Omega \rightarrow \Omega$ is continuous, then $PRP(T)$ is a $G_\delta$ subset of $\Omega$.

**Proof.** Let $X = \Omega$, $f_n(\omega) = T^n \omega$ for $\omega \in \Omega$ and $n \geq 0$, and apply the previous lemma. \hfill \square

Next we recall the topological lemma from Boshernitzan’s appendix to Cheung’s paper [7]. It will play a crucial role in the the proof of Theorem 4 below, which in turn is an essential ingredient to the proof of Theorem 3(e).

**Lemma 2.** Assume that $L$ is a $G_\delta$ subset of a $\sigma$-compact metric space $K$, $P$ is a Polish space, $H$ is an $F_\sigma$ subset of $W = P \times K$. Denote, for $p \in P$,
\[K(p) = \{ k \in K : (p, k) \in H \}\]
and
\[P_0 = \{ p \in P : K(p) \subset L \}.
\]
If $P_0$ is dense in $P$, then $P_0$ is a residual subset of $P$.

**Proof.** See [7, Lemma A.1]. \hfill \square

The following notion will prove to be useful in our study of the repetition property for sequences generated by polynomials and skew-shifts.

**Definition 2.** A family of sequences $\{\omega_n^{(\gamma)}\}_{n \geq 0}$ in metric spaces $\Omega^{(\gamma)}$, $\gamma \in \Gamma$, has the joint repetition property if each of them has the repetition property and for each finite subfamily, $q = q(\varepsilon, r)$ can be chosen uniformly for all sequences in the finite subfamily.

We now state three lemmas about the joint repetition property. The proofs are so simple that we omit them.

**Lemma 3.** Suppose $\Omega, \Omega'$ are metric spaces, $\{\omega_n\}_{n \geq 0}$ is a sequence in $\Omega$, and $\{\omega'_n\}_{n \geq 0}$ is a sequence in $\Omega'$. Then, the sequences $\{\omega_n\}_{n \geq 0}$ and $\{\omega'_n\}_{n \geq 0}$ have the joint repetition property if and only if the sequence $\{(\omega_n, \omega'_n)\}_{n \geq 0}$ in $\Omega \times \Omega'$ has the repetition property.

**Lemma 4.** Suppose $\Omega, \Omega'$ are metric spaces, $g : \Omega \rightarrow \Omega'$ is uniformly continuous, $\{\omega_n\}_{n \geq 0}$ is a sequence in $\Omega$ with the repetition property, and the sequence $\{\omega'_n\}_{n \geq 0}$ in $\Omega'$ is given by $\omega'_n = g(\omega_n)$ for $n \geq 0$. Then, the sequences $\{\omega_n\}_{n \geq 0}$ and $\{\omega'_n\}_{n \geq 0}$ have the joint repetition property.

**Lemma 5.** Suppose $\Omega, \Omega'$ are metric spaces and $\{\omega_n\}_{n \geq 0}$ is a sequence in $\Omega$ with the repetition property. Then, the set of sequences $\{\omega'_n\}_{n \geq 0}$ in $\Omega'$ so that $\{\omega_n\}_{n \geq 0}$ and $\{\omega'_n\}_{n \geq 0}$ have the joint repetition property is closed with respect to uniform convergence.

3. **Sequences on the Circle**

In this section we study sequences on the circle, that is, we let $\Omega = \mathbb{T}$. We will be especially interested in sequences generated by polynomials.
3.1. **Preservation Properties.** In this subsection we exhibit a number of operations performed on a given sequence that preserve the repetition property.

We begin with operations that obviously preserve the repetition property.

**Lemma 6.** Suppose \( \{ \omega_n \}_{n \geq 0} \) is a sequence in \( T \) that has the repetition property. Then, the following sequences have the repetition property as well:

- (a) \( \{ \omega_{n+l} \}_{n \geq 0} \) for every \( l \in \mathbb{Z}^+ \),
- (b) \( \{ \omega_{nl} \}_{n \geq 0} \) for every \( l \in \mathbb{Z}^+ \),
- (c) \( \{ l\omega_n \}_{n \geq 0} \) for every \( l \in \mathbb{Z} \).

In fact, the sequences listed above have the joint repetition property.

**Proof.** This is readily verified. \( \square \)

Note that parts (a) and (b) of this lemma are not specific to the circle and hold in a general metric space.

The next lemma addresses the following question: Which sequences can be added to any given sequence without destroying the repetition property? This is related to having the joint repetition property with any given sequence that has the repetition property. Here we only treat those sequences we need in the sequel. In the appendix we investigate this issue in more depth.

**Lemma 7.** Suppose \( \{ \omega_n \}_{n \geq 0} \) is a sequence in \( T \) that has the repetition property. Then, for every \( \alpha \in T \), \( \{ \omega_n \}_{n \geq 0} \) and \( \{ \alpha n \}_{n \geq 0} \) have the joint repetition property. Consequently, for every \( \alpha, \beta \in T \), \( \{ \omega_n \}_{n \geq 0} \) and \( \{ \omega_n + \alpha + \beta \}_{n \geq 0} \) have the joint repetition property.

**Proof.** Since \( \{ \omega_n \}_{n \geq 0} \) has the repetition property, there exists, for every small \( \varepsilon > 0 \) and \( r \in \mathbb{Z}^+ \), an integer \( \tilde{q} \geq 1 \) such that

\[
\langle \omega_{n+\tilde{q}} - \omega_n \rangle < \varepsilon^2 \quad \text{for } n = 0, 1, 2, \ldots, \left\lfloor \frac{r+2}{\varepsilon} \right\rfloor \tilde{q}.
\]

Select an integer \( 1 \leq d \leq \frac{1}{\varepsilon} \) such that \( \langle d\tilde{q}\alpha \rangle < \varepsilon \). Set \( q = d\tilde{q} \). Then we have

\[
\langle \omega_{n+q} - \omega_n \rangle < d\varepsilon^2 \leq \varepsilon \quad \text{for } n = 0, 1, 2, \ldots, rq
\]

and

\[
\langle (n+q)\alpha - \alpha n \rangle = \langle q\alpha \rangle = \langle d\tilde{q}\alpha \rangle < \varepsilon \quad \text{for every } n \in \mathbb{Z}.
\]

It follows that \( \{ \omega_n \}_{n \geq 0} \) and \( \{ \alpha n \}_{n \geq 0} \) have the joint repetition property. This implies that \( \{ \omega_n \}_{n \geq 0} \) and \( \{ \omega_n + \alpha n \}_{n \geq 0} \) have the joint repetition property.

Clearly, the addition of \( \beta \) does not affect the distances in question so that the asserted joint repetition property for \( \{ \omega_n \}_{n \geq 0} \) and \( \{ \omega_n + \alpha n + \beta \}_{n \geq 0} \) follows immediately. \( \square \)

Note again that there is an immediate extension of the first part of the lemma to sequences in a general metric space that have the repetition property. Any such sequence has the joint repetition property with the sequence \( \{ \alpha n \}_{n \geq 0} \) in \( T \) for any \( \alpha \in T \). The result in this generality will be used in the proof of Theorem 5 below.

The final preservation property we wish to address here is the multiplication of a given sequence by a rational number. The following lemma exhibits a class of sequences for which the repetition property is preserved under such an operation.
Lemma 8. Suppose \( \{\omega_n\}_{n \geq 0} \) is a sequence in \( \mathbb{R} \) whose projection to \( T \) has the repetition property and which satisfies a linear recurrence relation of the following form: there exist integers \( k \geq 1 \) and \( a_1, \ldots, a_k \) with \( |a_k| = 1 \) such that

\[
\omega_n + a_1 \omega_{n-1} + a_2 \omega_{n-2} + \cdots + a_k \omega_{n-k} = 0.
\]

Then, for every \( r \in \mathbb{Q} \), the projection of the sequence \( \{r\omega_n\}_{n \geq 0} \) to \( T \) has the repetition property, jointly with the original sequence.

Proof. By Lemma 6.(c), it suffices to consider the case \( r = m^{-1} \), where \( m \) is an integer \( \geq 2 \). By the recurrence relation, any \( k \) consecutive values determine the entire sequence. This is of course also true for the sequence \( \{m^{-1}\omega_n\}_{n \geq 0} \) in \( \mathbb{R} \).

By assumption, \( \{\omega_n \mod 1\}_{n \geq 0} \) has the repetition property in \( T = \mathbb{R}/\mathbb{Z} \). Consequently, \( \{m^{-1}\omega_n \mod m^{-1} \}_{n \geq 0} \) has the repetition property in \( \mathbb{R}/(m^{-1}\mathbb{Z}) \) with, in fact, the same \( q(\varepsilon, r) \) as the original sequence. To see that \( \{m^{-1}\omega_n \mod 1\}_{n \geq 0} \) has the repetition property, notice that when passing from \( \mathbb{R}/(m^{-1}\mathbb{Z}) \) to \( \mathbb{R}/\mathbb{Z} \), we can in principle draw from \( m \) choices for each \( n \). After fixing \( k \) values, the rest of the sequence is effectively determined.

Now consider almost-repetitions of \( m^{-1}\omega_n \mod m^{-1} \) in \( \mathbb{R}/(m^{-1}\mathbb{Z}) \) with \( r \) sufficiently large (larger than \( m^k \)) and apply the pigeonhole principle to see that \( m^{-1}\omega_n \mod 1 \) in \( \mathbb{R}/\mathbb{Z} \) must have almost-repetitions as well. More precisely, find a repeated cell of size \( m^{-1} \) first and then use the recursion relation to see that the resulting pattern must repeat in both directions.

In this way, we can map every triple \( (\varepsilon, r, q(\varepsilon, r)) \) describing an almost-repetition for the original sequence \( \{\omega_n \mod 1\}_{n \geq 0} \) to a new triple for the derived sequence \( \{m^{-1}\omega_n \mod 1\}_{n \geq 0} \). With these derived almost-repetitions, one may check that \( \{m^{-1}\omega_n \mod 1\}_{n \geq 0} \) has the repetition property in \( T \), jointly with the projection of \( \{\omega_n\}_{n \geq 0} \) to \( T \).

\[\square\]

3.2. Characterization of Polynomials With the Repetition Property. Consider the sequence \( \omega_n = p(n) \mod 1 \) in \( T \), where \( p(n) = \sum_{k=0}^{d} a_k n^k \) is a polynomial. If this sequence has the repetition property, we say that \( p \) has the repetition property. Let us denote the terms of \( p \) by \( p_k \), that is, \( p_k(n) = a_k n^k \).

Theorem 2. A polynomial \( p \) has the repetition property if and only if its terms \( p_k \), \( 0 \leq k \leq d \) have the joint repetition property.

Proof. That \( p \) has the repetition property if \( p_k \), \( 0 \leq k \leq d \) have the joint repetition property is easy to verify.

Assume now that \( p \) has the repetition property. By Lemma 6(b), the polynomials \( p^{(l)} \) given by \( p^{(l)}(n) = p(\ell n) \), \( 0 \leq l \leq d \) have the joint repetition property. Moreover, by taking a finite number of discrete derivatives of \( p \), we eventually obtain the zero polynomial. In this way, we find a linear recursion relation between \( p(n), \ldots, p(n - d - 1) \) with constant coefficients whose absolute values are given by the entries in the Pascal triangle. Consequently, \( p(n) \) obeys the assumptions of Lemma 6.

\[\text{It is determined only up to integers, which however are irrelevant for the repetition property in } T.\]
Consider the matrix $A = (a_{i,j})_{0 \leq i,j \leq d}$ given by $a_{i,j} = i^j$. Then
\[
\begin{pmatrix}
 p(n \cdot 0) \\
p(n \cdot 1) \\
p(n \cdot 2) \\
\vdots \\
p(n \cdot d)
\end{pmatrix} = A
\begin{pmatrix}
a_0 \\
a_1 n \\
a_2 n^2 \\
\vdots \\
a_d n^d
\end{pmatrix},
\]

Notice that $A$ a Vandermonde matrix and hence its invertibility follows from the well-known formula for the determinant of a Vandermonde matrix. Of course, $A^{-1}$ has only rational entries.

Since
\[
\begin{pmatrix}
a_0 \\
a_1 n \\
a_2 n^2 \\
\vdots \\
a_d n^d
\end{pmatrix} = A^{-1}
\begin{pmatrix}
p(n \cdot 0) \\
p(n \cdot 1) \\
p(n \cdot 2) \\
\vdots \\
p(n \cdot d)
\end{pmatrix},
\]

it therefore follows from Lemma 8 that $p_k$, $0 \leq k \leq d$ have the joint repetition property.  

Theorem 3 suggests investigating the repetition property for homogeneous polynomials. This problem is addressed in the following theorem.

**Theorem 3.** A homogeneous polynomial $p_k$ of the form $p_k(n) = a_k n^k$ has the repetition property if and only if $\liminf_{q \to \infty} q^{k-1} \langle a_k q \rangle = 0$.

**Proof.** Write $\omega_n = p_k(n) \mod 1$. We have
\[
\text{dist}(\omega_{n+q}, \omega_n) = \sum_{j=0}^{k-1} a_k \binom{k}{j} n^j q^{k-j},
\]

This shows immediately that the repetition property follows from the existence of $q_m \to \infty$ with $\lim_{m \to \infty} q_{m-1} \langle a_k q_m \rangle = 0$.

Conversely, assume that $p_k$ has the repetition property. By Lemma 5(a), $\tilde{p}_k$ given by
\[
\tilde{p}_k(n) = p_k(n+1) = a_k \sum_{j=0}^{k} \binom{k}{j} n^j
\]

has the repetition property as well. Thus, by Theorem 2 and Lemma 3 we find that $n \mapsto a_k n^j$, $0 \leq j \leq k$ have the joint repetition property. For $j = 0$ and $j = 1$, this statement is obvious. Let us explore what information can be gleaned from the case $j = 2$: for every $\varepsilon > 0$, we have $\langle 2a_k q n + a_k q^2 \rangle < \varepsilon$ for some $q_l \to \infty$ and $0 \leq n \leq q_l$. Evaluating this for $n = 0$, we find that $\langle a_k q^2 \rangle < \varepsilon$. Now vary $n$. Each time we increase $n$, we shift in the same direction by $\langle 2a_k q \rangle$. If $\varepsilon > 0$ is sufficiently small, it follows that we cannot go around the circle completely and hence we have $\langle 2a_k q n \rangle = n \langle 2a_k q \rangle$ for every $0 \leq n \leq q_l$. Taking $\varepsilon$ to zero, this implies $\liminf_{q \to \infty} q \langle a_k q \rangle = 0$. Using this, we can consider the case $j = 3$ and argue in a similar way to find that $\liminf_{q \to \infty} q^2 \langle a_k q \rangle = 0$. Carrying on inductively, we arrive at the desired statement, $\liminf_{q \to \infty} q^{k-1} \langle a_k q \rangle = 0$. 

□
For the remainder of this section, we will be concerned with a special class of polynomials; namely, \( p(n) = \alpha n^3 + \beta n^2 \). The study of the repetition property for these polynomials is somewhat more involved as they present a borderline case. We have the following theorem:

**Theorem 4.** There are a residual subset \( P_0 \) of \( \mathbb{T} \) and a subset \( G \) of \( \mathbb{T} \) of full Lebesgue measure such that for \( \alpha \in P_0 \) and \( \beta \in G \), the polynomial \( p(n) = \alpha n^3 + \beta n^2 \) has the repetition property.

**Proof.** Set \( X = \mathbb{T}^2 \). Lemma \( \text{[1]} \) implies that

\[
H = \{ (\alpha, \beta) \in \mathbb{T}^2 : p(n) = \alpha n^3 + \beta n^2 \text{ does not have the repetition property} \}
\]

is an \( F_{\sigma} \) subset of \( \mathbb{T}^2 \).

Next we wish to apply Lemma \( \text{[2]} \). We set \( P = K = \mathbb{T} \), so that \( W = P \times K = \mathbb{T}^2 \), and let \( L \) be any \( G_\delta \) subset of \( \mathbb{T} \) of Lebesgue measure zero that contains the set of badly approximable numbers (those numbers that have bounded partial quotients).

Writing

\[
K(\alpha) = \{ \beta \in \mathbb{T} : (\alpha, \beta) \in H \} = \{ \beta \in \mathbb{T} : p(n) = \alpha n^3 + \beta n^2 \text{ does not have the repetition property} \}
\]

for \( \alpha \in \mathbb{T} \) and

\[
P_0 = \{ \alpha \in \mathbb{T} : K(\alpha) \subset L \},
\]

and observing that \( P_0 \) contains all rational numbers in \( \mathbb{T} \), it follows from Lemma \( \text{[2]} \) that \( P_0 \) is a residual subset of \( \mathbb{T} \). In other words, if we denote \( G = \mathbb{T} \setminus L \), then \( p(n) = \alpha n^3 + \beta n^2 \) has the repetition property for every \( \alpha \in P_0 \) and \( \beta \in G \). \( \square \)

4. **Proof of Theorem \( \text{[1]} \)**

Denote the \( j \)-th component of \( T^n \omega \) by \( t(n, j, \omega) \). Observe that \( n \mapsto T^n \omega \) has the repetition property if and only if \( n \mapsto t(n, j, \omega), 1 \leq j \leq k \) have the joint repetition property.

Clearly, we have

\[
t(n, 1, \omega) = n\alpha + \omega_1.
\]

Noting that \( t(n, 2, \omega) = t(n - 1, 1, \omega) + t(n - 1, 2, \omega) \), we find that \( t(n, 2, \omega) = t(n - 1, 1, \omega) + t(n - 2, 1, \omega) + \cdots + t(0, 1, \omega) + t(0, 2, \omega) \) and hence

\[
t(n, 2, \omega) = \frac{n(n-1)}{2}\alpha + n\omega_1 + \omega_2.
\]

Continuing in this fashion, we see that \( t(n, j, \omega) \) is a polynomial in \( n \) of degree \( j \) and its leading coefficient is a rational multiple of \( \alpha \), while the other coefficients are rational linear combinations of \( \alpha, \omega_1, \ldots, \omega_k \), but not pure multiples of \( \alpha \).

(a) Assume that \( T \) has (TRP). In particular, for some \( \omega \in \mathbb{T}^k \), the polynomial \( n \mapsto t(n, k, \omega) \) has the repetition property. Thus, by Theorem \( \text{[2]} \) and Lemma \( \text{[8]} \) the polynomial \( n \mapsto \alpha n^k \) has the repetition property, which is equivalent to \( \liminf_{q \to \infty} q^{k-1}(\rho q) = 0 \) by Theorem \( \text{[8]} \). Conversely, if the polynomial \( n \mapsto \alpha n^k \) has the repetition property, this is true jointly for all polynomials \( n \mapsto \alpha n^j, 0 \leq j \leq k \) (by the argument in the proof of Theorem \( \text{[8]} \)). It follows that \( 0 \in PRP(T) \) and hence \( T \) has (TRP).

(b) This was shown in \( \text{[4]} \) and is stated here for the sake of completeness.
(c) If \( k \geq 3 \), then \( n \mapsto t(n, 3, \omega) \) is a polynomial of degree 3 and the coefficient of \( n^2 \) is a rational linear combination of \( \alpha, \omega_1, \ldots, \omega_k \), but not a pure multiple of \( \alpha \). In particular, for some \( \omega \in T^k \), this coefficient is badly approximable and hence, by Theorems \( \mathbb{X} \) and \( \mathbb{X} \) this \( \omega \) does not belong to \( PRP(T) \). In other words, \( T \) does not have (GRP).

(d) This follows immediately from part (a).

(e) We begin with \( t(n, 3, \omega) \), discuss the repetition property for this sequence in the circle, and then proceed to include the other components of \( T^n \omega \) in our consideration. As we saw in Lemma \( \mathbb{X} \), we can restrict our attention to the terms in \( t(n, 3, \omega) \) of degree 2 and 3. Computing these terms, we find that
\[
t(n, 3, \omega) = \frac{\alpha}{6} n^3 + \left( \frac{\alpha}{2} + \frac{\omega_1}{2} \right) n^2 + \text{lower order terms.}
\]
Applying Theorem \( \mathbb{X} \), we find that \( t(n, 3, \omega) \) has the repetition property for \( \alpha \)'s from a certain residual subset of \( T \) and \( \omega_1 \)'s from a certain full measure subset of \( T \). Recalling the specific form \( \mathbb{X} \) of \( t(n, 2, \omega) \), we can use Theorems \( \mathbb{X} \) and \( \mathbb{X} \) to see that \( t(n, 2, \omega) \) has the repetition property, jointly with \( t(n, 3, \omega) \), for \( \alpha \) and \( \omega_1 \) as above and every \( \omega_2 \). Since \( t(n, 1, \omega) \) has the form \( \mathbb{X} \), adding on the joint repetition property for \( t(n, 1, \omega) \) is then easy by the argument from the proof of Lemma \( \mathbb{X} \).

Putting everything together, we find that for \( \alpha \)'s from a certain residual subset of \( T \), \( \omega_1 \)'s from a certain full measure subset of \( T \), and every \( (\omega_2, \omega_3) \in T^2 \), \( t(n, j, \omega) \), \( 1 \leq j \leq 3 \) have the joint repetition property. Consequently, \( T \) has (MRP) for such \( \alpha \)'s.

(f) If \( k \geq 4 \), then \( n \mapsto t(n, 4, \omega) \) is a polynomial of degree 4 and the coefficient of \( n^3 \) is a rational linear combination of \( \alpha, \omega_1, \ldots, \omega_k \), but not a pure multiple of \( \alpha \).

Consider an \( \omega_i \) with non-zero coefficient in this linear combination. Fixing all the other entries of \( \omega \), we see that the coefficient satisfies the necessary condition for the repetition property in this component only for \( \omega_i \) from a set of zero Lebesgue measure. By Fubini, it follows that \( T \) does not have (MRP).

\( \square \)

### Appendix A. Universal Joint Repetition Property

The notion of joint repetition property was useful in our study above. Pushing a bit further, it is natural to introduce the following class of sequences:

**Definition 3.** A sequence \( \{\omega_n\}_{n \geq 0} \) in a metric space \( \Omega \) has the universal joint repetition property if for every sequence \( \{\hat{\omega}_n\}_{n \geq 0} \) in some metric space \( \check{\Omega} \) that has the repetition property, \( \{\omega_n\}_{n \geq 0} \) and \( \{\hat{\omega}_n\}_{n \geq 0} \) have the joint repetition property.

We consider it an interesting problem to find a characterization of the sequences that have the universal joint repetition property. Theorem \( \mathbb{X} \) below presents a first step in this direction.

Recall that \( a : \mathbb{Z} \to \mathbb{C} \) is called almost periodic if its translates form a relatively compact subset of \( \ell^\infty(\mathbb{Z}) \). It is a fundamental result (see, e.g., \( \mathbb{X} \)) that every almost periodic sequence can be approximated uniformly by finite linear combinations of exponentials \( e^{2\pi i a_n} \). The following theorem is an extension of Lemma \( \mathbb{X} \) in Section \( \mathbb{X} \).

**Theorem 5.** Every almost periodic sequence has the universal joint repetition property.

**Proof.** Let \( \{\omega_n\}_{n \geq 0} \) be a sequence in a metric space \( \Omega \) that has the repetition property. By Lemma \( \mathbb{X} \) (see the remark after the proof of this lemma), \( \{\omega_n\}_{n \geq 0} \)
and \( \{\alpha n\}_{n \geq 0} \) in \( \mathbb{T} \) have the joint repetition property for every \( \alpha \in \mathbb{T} \). Thus, by Lemmas 3 and 4, \( \{\omega_n\}_{n \geq 0} \) and \( \{ce^{2\pi i \alpha n}\}_{n \geq 0} \) in \( \mathbb{C} \) have the joint repetition property for every \( \alpha \in \mathbb{T} \) and \( c \in \mathbb{C} \).

Inductively, it follows that \( \{\omega_n\}_{n \geq 0} \) and any finite linear combination of exponentials have the joint repetition property. Since the latter lie densely in the almost periodic sequences with respect to uniform convergence, the theorem now follows from Lemma 5. \(\square\)

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Department of Mathematics, Rice University, Houston, TX 77005, USA
E-mail address: michael@rice.edu

Department of Mathematics, Rice University, Houston, TX 77005, USA
E-mail address: damanik@rice.edu