A COMPUTATIONAL REDUCTION FOR MANY BASE CASES IN PROFINITE TELESCOPIC ALGEBRAIC $K$-THEORY

DANIEL G. DAVIS

ABSTRACT. For primes $p \geq 5$, $K(Ku_p)$—the algebraic $K$-theory spectrum of $(Ku_p)_0^0$, Morava $K$-theory $K(1)$, and Smith-Toda complex $V(1)$, Ausoni and Rognes conjectured (alongside related conjectures) that $L_{K(1)}S^0_{\text{unit}}(Ku_p)_0^0$ induces a map $K(L_{K(1)}S^0) \wedge v_2^{-1}V(1) \to K(Ku_p)_{hZ_p} \wedge v_2^{-1}V(1)$ that is an equivalence. Since the definition of this map is not well understood, we consider $K(L_{K(1)}S^0) \wedge v_2^{-1}V(1) \to (K(Ku_p) \wedge v_2^{-1}V(1))_{hZ_p}$, which is induced by $i$ and also should be an equivalence. We show that for any closed $G < Z_p^\times$, $\pi_*((K(Ku_p) \wedge v_2^{-1}V(1))_{hG})$ is a direct sum of two pieces given by (co)invariants and a coinduced module, for $K(Ku_p)_*(V(1))[v_2^{-1}]$. When $G = Z_p^\times$, the direct sum is, conjecturally, $K(L_{K(1)}S^0)*(V(1))[v_2^{-1}]$ and, by using $K(L_p_*(V(1))[v_2^{-1}], where $L_p = ((Ku_p)_0^0)_{hZ_p/(p-1)^2}$, the summands simplify. The Ausoni-Rognes conjecture suggests that in

\[(\cdot)_{hZ_p} \wedge v_2^{-1}V(1) \simeq (K(Ku_p) \wedge v_2^{-1}V(1))_{hZ_p},\]

$K(Ku_p)$ fills in the blank; we show that for any $G$, the blank can be filled by $(K(Ku_p))_{hG}$, a discrete $Z_p^\times$-spectrum built out of $K(Ku_p)$.

1. Introduction

1.1. Motivation for our work. Let $p$ be any prime, with $Z_p$ the $p$-adic integers, let $K(1)$ denote the first Morava $K$-theory spectrum, and let $L_{K(1)}(S^0)$ be the Bousfield localization of the sphere spectrum. Also, let $Ku_p$ be $p$-complete complex $K$-theory, so that

$$\pi_*(Ku_p) = Z_p[u^{\pm 1}],$$

where $\pi_0(Ku_p) = Z_p$ and $|u| = 2$, and let $Z_p^\times$ denote the group of units in $Z_p$. By $[16]$, $Z_p^\times$ — as the group of $p$-adic Adams operations — acts on the commutative $S^0$-algebra $Ku_p$ by maps of commutative $S^0$-algebras. Given a commutative $S^0$-algebra $A$, the algebraic $K$-theory spectrum of $A$, $K(A)$, is a commutative $S^0$-algebra, so that $K(Ku_p)$ is a commutative $S^0$-algebra, and by the functoriality of $K(-)$, $Z_p^\times$ acts on $K(Ku_p)$ by maps of commutative $S^0$-algebras.

For the rest of this paper, we let $p \geq 5$. Let $V(1)$ be the type 2 Smith-Toda complex $S^0/(p,v_1)$. Then there is a $v_2$-self-map $v: \Sigma^dV(1) \to V(1)$, where $d$ is some positive integer (see $[20]$ Theorem 9), and hence, $v$ induces a sequence

$$V(1) \to \Sigma^{-d}V(1) \to \Sigma^{-2d}V(1) \to \cdots$$

of maps of spectra, and we set

$$v_2^{-1}V(1) = \colim_{j \geq 0} \Sigma^{-jd}V(1).$$
the mapping telescope associated to \( v \). In [4] paragraph containing (0.1)], [5] Conjecture 4.2], and [3] page 46; Remark 10.8], Christian Ausoni and John Rognes conjectured that the \( K(1) \)-local unit map

\[
i: L_{K(1)}(S^0) \to KU_p
\]

induces a weak equivalence

\[
(1.1) \quad K(L_{K(1)}(S^0)) \wedge v_2^{-1}V(1) \to K(KU_p)^{h\mathbb{Z}_p^\times} \wedge v_2^{-1}V(1),
\]

where

\[
K(KU_p)^{h\mathbb{Z}_p^\times} = (K(KU_p))^{h\mathbb{Z}_p^\times}
\]

is a continuous homotopy fixed point spectrum that is formed with respect to a continuous action of the profinite group \( \mathbb{Z}_p^\times \) on \( K(KU_p) \).

**Remark 1.2.** The above conjecture is a collection of \( n = 1 \) instances of a more general conjecture made by Ausoni and Rognes for every positive integer and every prime (for more information, see the references mentioned above).

One difficulty with making progress on this conjecture is that there is no published construction of \( K(KU_p)^{h\mathbb{Z}_p^\times} \) and, according to [13] Remark 1.5], the only models for it, currently, are a “candidate definition” that uses condensed spectra (in the sense of Clausen-Scholze) in the setting of \( \infty \)-categories (the author learned of this construction from Jacob Lurie) and, possibly, a pyknotic version of this construction (in the framework of [7]). Thus, due to the lack of a robust model for the map in (1.1), the conjecture is difficult to approach computationally.

If \( G \) is any profinite group and \( X \) is a discrete \( G \)-spectrum (as in [8]; the crux of this concept is that for every \( k, \ell \geq 0 \), the set of \( \ell \)-simplices of the pointed simplicial set \( X_k \) is a discrete \( G \)-set), then there is a continuous homotopy fixed point spectrum \( X^{hG} \) [8] Section 3.1)] (and we use this notation for the rest of this paper). Thus, to address the above difficulty, the author showed in [13] Section 1.2] that \( K(KU_p) \wedge v_2^{-1}V(1) \), with \( v_2^{-1}V(1) \) equipped with the trivial \( \mathbb{Z}_p^\times \)-action, can be realized as a discrete \( \mathbb{Z}_p^\times \)-spectrum – written as \( C_p^\text{dis} \) in [13], and hence, one can form

\[
(K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times} := (C_p^\text{dis})^{h\mathbb{Z}_p^\times}
\]

and, by [13] Theorem 1.8], the map \( i \) induces a canonical map

\[
i': K(L_{K(1)}(S^0)) \wedge v_2^{-1}V(1) \to (K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times}.
\]

**Remark 1.3.** According to [13] Remark 1.5], the relationship between the target of \( i' \) and \( K \wedge v_2^{-1}V(1) \), where \( K \) denotes the aforementioned candidate model for \( K(KU_p)^{h\mathbb{Z}_p^\times} \), is unclear.

Now we make some observations to understand the relationship between the map \( i' \) and the conjectural equivalence in (1.1). If \( X \) is a discrete \( \mathbb{Z}_p^\times \)-spectrum and \( Y \) is a finite spectrum with trivial \( \mathbb{Z}_p^\times \)-action, then \( X \wedge Y \) is a discrete \( \mathbb{Z}_p^\times \)-spectrum and, by [14] Remark 7.16],

\[
(1.4) \quad (X \wedge Y)^{h\mathbb{Z}_p^\times} \simeq X^{h\mathbb{Z}_p^\times} \wedge Y.
\]

More generally, if \( \{X_i \}_{i \in I} \) is a diagram of discrete \( \mathbb{Z}_p^\times \)-spectra indexed by a cofiltered category \( I \), then the equivalence

\[
(hom_i X_i) \wedge Y \simeq \lim_i (X_i \wedge Y)
\]
implies that it is natural to make the definition
\[ ((\text{holim } X_i) \land Y)^{h\mathbb{Z}_p^\times} := (\text{holim } X_i \land Y)^{h\mathbb{Z}_p^\times} = \text{holim}(X_i \land Y)^{h\mathbb{Z}_p^\times}, \]
where the last step applies [6, Section 4.4], and thus, we have
\[ (1.5) \quad ((\text{holim } X_i) \land Y)^{h\mathbb{Z}_p^\times} \simeq (\text{holim } X_i)^{h\mathbb{Z}_p^\times} \land Y, \]
because
\[ \text{holim}(X_i \land Y)^{h\mathbb{Z}_p^\times} \simeq \text{holim}(X_i)^{h\mathbb{Z}_p^\times} \land Y = (\text{holim } X_i)^{h\mathbb{Z}_p^\times} \land Y. \]

Also, by [13], for each \( j \geq 0 \), \( K(KU_p) \land \Sigma^{-jd}V(1) \) can be realized as a discrete \( \mathbb{Z}_p^\times \)-spectrum, and hence, there is \( (K(KU_p) \land \Sigma^{-jd}V(1))^{h\mathbb{Z}_p^\times} \). Then since each \( \Sigma^{-jd}V(1) \) is a finite spectrum with trivial \( \mathbb{Z}_p^\times \)-action, the pattern in [14] and [15] suggests that there should be an equivalence
\[ (1.6) \quad K(KU_p)^{h\mathbb{Z}_p^\times} \land \Sigma^{-jd}V(1) \simeq (K(KU_p) \land \Sigma^{-jd}V(1))^{h\mathbb{Z}_p^\times}. \]

Here and elsewhere, we place a "?" over a relation to indicate that it is not known to be true, but it is desired and expected to some degree.

Now notice that there is the isomorphism
\[ (1.7) \quad K(KU_p)^{h\mathbb{Z}_p^\times} \land v_2^{-1}V(1) \cong \text{colim}_{j \geq 0}(K(KU_p)^{h\mathbb{Z}_p^\times} \land \Sigma^{-jd}V(1)) \]
and, by [15] Theorem 1.7, there is an equivalence
\[ (1.8) \quad (K(KU_p) \land v_2^{-1}V(1))^{h\mathbb{Z}_p^\times} \simeq \text{colim}_{j \geq 0}(K(KU_p) \land \Sigma^{-jd}V(1))^{h\mathbb{Z}_p^\times}. \]

Thus, (1.6) - (1.8) imply that there should be an equivalence
\[ (1.9) \quad K(KU_p)^{h\mathbb{Z}_p^\times} \land v_2^{-1}V(1) \simeq (K(KU_p) \land v_2^{-1}V(1))^{h\mathbb{Z}_p^\times}, \]
and this observation suggests that if (1.9) holds and \( i' \) is a weak equivalence, then one should be able to prove that the map in (1.3) is a weak equivalence, and thereby verify the conjecture of Ausoni and Rognes. This potentially fruitful strategy for proving this conjecture involves computing
\[ \pi_*((K(KU_p) \land v_2^{-1}V(1))^{h\mathbb{Z}_p^\times}), \]
and thus, in this paper, we make progress on this computation by showing that it is a direct sum of two pieces given by invariants and coinvariants involving the \( \mathbb{Z}_p^\times \)-action on \( \pi_*((K(KU_p) \land v_2^{-1}V(1)). \) Additionally, with
\[ L_p := (KU_p)^{h\mathbb{Z}/((p-1)\mathbb{Z})} \]
(as in [4]), the \( p \)-complete Adams summand and a commutative \( S^0 \)-algebra, where \( \mathbb{Z}/((p-1)\mathbb{Z}) \) is the usual subgroup of \( \mathbb{Z}_p^\times \), we show that the direct sum can be expressed as invariants and coinvariants of the \( \mathbb{Z}_p \)-action on \( \pi_*((K(KU_p) \land v_2^{-1}V(1)). \)

Given a profinite group \( G \) and a discrete \( G \)-spectrum \( X \), if \( H \) is any closed subgroup of \( G \), then \( H \) is a profinite group, \( X \) is a discrete \( H \)-spectrum (by restriction of the \( G \)-action), and hence, there is the continuous homotopy fixed point spectrum \( X^{hH} \). Our work for the above computation is in line with this multiplicity of possibilities: our result is not just for the \( \mathbb{Z}_p^\times \)-homotopy fixed points, but is
for the homotopy fixed points of any closed subgroup (though the aforementioned presentation involving \(L_p\) is only for the case \(G = H = \mathbb{Z}_p^\times\)).

1.2. The main results. In (1.9) above, we said that there should be an equivalence

\[ (-)^{h\mathbb{Z}_p^\times}v_2^{-1}V(1) \cong (K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times}, \]

where the blank “\(\_\)” can be filled in with \(K(KU_p)\). One of our intermediate steps in obtaining the results mentioned above is to give a way to fill in this blank with a discrete \(\mathbb{Z}_p^\times\)-spectrum that is related to \(K(KU_p)\).

**Theorem 1.10.** Let \(p \geq 5\) and let \(G\) be a closed subgroup of \(\mathbb{Z}_p^\times\). There is a discrete \(\mathbb{Z}_p^\times\)-spectrum \((K(KU_p))^{\text{dis}}_G\) with the property that for each \(j \geq 0\), there is an equivalence

\[ ((K(KU_p))^{\text{dis}}_G)^{hG} \wedge \Sigma^{-j\delta}V(1) \cong (K(KU_p) \wedge \Sigma^{-j\delta}V(1))^{hG}, \]

and

\[ ((K(KU_p))^{\text{dis}}_G)^{hG} \wedge v_2^{-1}V(1) \cong (K(KU_p) \wedge v_2^{-1}V(1))^{hG}. \]

The spectrum \((K(KU_p))^{\text{dis}}_G\) is defined in Definition 3.2 with \(O\) specified at the beginning of Section 4, the first equivalence in Theorem 1.10 is Theorem 4.2, and the last equivalence follows immediately from the first one and the general version of (1.8) that is stated in (2.2).

After the following prefatory remarks, we state our result that for any closed subgroup \(G\) in \(\mathbb{Z}_p^\times\), \(\pi_\ast((K(KU_p) \wedge v_2^{-1}V(1))^{hG})\) can be reduced to a direct sum.

Recall that \(\mathbb{Z}_p\) is the pro-\(p\) completion of \(\mathbb{Z}\) and \(\mathbb{Z}\) can be regarded as a subset of \(\mathbb{Z}_p\) in a way that makes the inclusion \(\mathbb{Z} \hookrightarrow \mathbb{Z}_p\) a ring homomorphism. We define

\[ C_{p-1} := \mathbb{Z}/((p-1)\mathbb{Z}) \]

to be the cyclic group of order \(p - 1\) and recall that

\[ \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times C_{p-1} \]

(since \(p \geq 5\)). If \(M\) is a \(\mathbb{Z}[\mathbb{Z}_p]\)-module (so that \(\mathbb{Z}_p\) acts on \(M\)), then \(M\) is naturally a \(\mathbb{Z}[\mathbb{Z}]\)-module, and \(M_g\) denotes the coinvariants. If \(K\) is a closed subgroup of a profinite group \(H\) and \(A\) is a discrete \(K\)-module, we let \(\text{Coind}_K^H(A)\) denote the coinduced discrete \(H\)-module of continuous \(K\)-equivariant functions \(H \to A\).

Let \(P(v_2) = \mathbb{F}_p[v_2]\) denote the polynomial algebra over \(\mathbb{F}_p\) generated by the periodic element \(v_2 \in \pi_{2p-2}(V(1))\). Also, \(P(v_2^{-1}) = \mathbb{F}_p[v_2, v_2^{-1}]\) is the algebra of Laurent polynomials on \(v_2\). To help manage the typography in the upcoming text, for a closed subgroup \(G\) of \(\mathbb{Z}_p^\times\), we let

\[ \mathbb{K}(p, G) := ((K(KU_p))^{\text{dis}}_G)^{hG} \wedge v_2^{-1}V(1) \]

and

\[ \mathcal{E}(p, G) := H^1_\ast(G, \pi_\ast(K(KU_p) \wedge v_2^{-1}V(1))), \]

a graded continuous cohomology group with coefficients in the stated discrete \(G\)-module.

**Theorem 1.11.** Let \(p \geq 5\) and let \(G\) be any closed subgroup of \(\mathbb{Z}_p^\times\). There is an isomorphism

\[ \pi_\ast((K(KU_p) \wedge v_2^{-1}V(1))^{hG}) \cong \pi_\ast(\mathbb{K}(p, G)), \]
where the right-hand side is the middle term in a short exact sequence

\[ 0 \to \mathcal{E}(p, G)_{t+1} \to \pi_*(\mathbb{Z}V(p, G)) \to (\pi_*(K(U_p) \wedge v_2^{-1}V(1)))^G \to 0 \]

of \(P(v_2^{\pm 1})\)-modules. In particular, in each degree \(t\), where \(t \in \mathbb{Z}\), this sequence is a split exact sequence of \(\mathbb{F}_p\)-modules and there is an isomorphism

\[
\pi_t((K(U_p) \wedge v_2^{-1}V(1))^hG) \\
\cong ((\text{Coind}_{G}^{\mathbb{Z}}(\pi_{t+1}(K(U_p) \wedge v_2^{-1}V(1))))^{G_{p-1}}) \oplus (\pi_t(K(U_p) \wedge v_2^{-1}V(1)))^G
\]

of abelian groups, where in the direct sum, the left summand is isomorphic to \(\mathcal{E}(p, G)_{t+1}\).

The proof of this result is broken up into six steps:

- in Section 2 we use various homotopy fixed point spectral sequences to present \(\pi_*(\mathbb{Z}V(p, G))\) as the middle term in a colimit of short exact sequences;
- Section 3 makes some recollections of several constructions that are needed to go further;
- for each \(j \geq 0\), \((K(U_p) \wedge \Sigma^{-jd}V(1))^hG\) is the continuous homotopy fixed points of, not literally, \(K(U_p) \wedge \Sigma^{-jd}V(1)\), but a discrete \(\mathbb{Z}_p^\times\)-spectrum equivalent to this \(\mathbb{Z}_p^\times\)-spectrum, and in Section 4 we study the role of \(V(1)\) in the construction of this discrete \(\mathbb{Z}_p^\times\)-spectrum and its associated homotopy fixed point spectral sequence (and thereby prove Theorem 1.10 of which the first isomorphism in Theorem 1.11 is an immediate consequence);
- Section 5 shows that each of the just-mentioned spectral sequences is isomorphic to a spectral sequence in the category of \(P(v_2)\)-modules;
- in Section 6 we obtain the desired short exact sequence of \(P(v_2^{\pm 1})\)-modules and the chief desideratum is shown to be a direct sum with its second summand as specified in Theorem 1.11 and
- we obtain the isomorphism between \(\mathcal{E}(p, G)_{t+1}\) and the expression involving \(\mathbb{Z}\)-coinvariants of \(C_{p-1}\)-invariants (in every integral degree \(t\) in Section 7)

**Remark 1.12.** In Lemma 7.21 we show that in Theorem 1.11 for each integer \(t\),

\[
\mathcal{E}(p, G)_{t+1} \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[\mathbb{Z}_p]]]}((\text{Coind}_{G}^{\mathbb{Z}}(\pi_{t+1}(K(U_p) \wedge v_2^{-1}V(1))))^{G_{p-1}}),
\]

where \(\mathbb{Z}_p\) is regarded as a \(\mathbb{Z}_p[[\mathbb{Z}_p]]\)-module by giving \(\mathbb{Z}_p\) the trivial \(\mathbb{Z}_p\)-group action. We give this result in case this form of \(\mathcal{E}(p, G)_{t+1}\) is easier to compute than the \(\mathbb{Z}\)-coinvariants of Theorem 1.11. We point out that “\(\otimes_{\mathbb{Z}_p[[\mathbb{Z}_p]]]}\)” above denotes the usual tensor product (for the category of abstract \(\mathbb{Z}_p[[\mathbb{Z}_p]]\)-modules) and not a completed tensor product (formed in some category of topological \(\mathbb{Z}_p[[\mathbb{Z}_p]]\)-modules).

Now we focus on the case \(G = \mathbb{Z}_p^\times\): our result in this case – Corollary 1.13 below – consists of three isomorphisms, and the first one is an immediate consequence of Theorem 1.11. As alluded to earlier, the last two isomorphisms involve \(K(L_p)\), and so we note that \(\pi_*(L_p) = \mathbb{Z}_p[v_2^{\pm 1}]\) and \(L_p \simeq E(1)_p\), the \(p\)-completed first Johnson-Wilson spectrum. Also, we make explicit the following, which was implicitly referred to earlier: after taking \(C_{p-1}\)-homotopy fixed points to form \(L_p\), there is a residual action by \(\mathbb{Z}_p\) on \(L_p\) through morphisms of commutative \(S^0\)-algebras, and hence, \(K(L_p)\) carries a \(\mathbb{Z}_p\)-action. The telescope \(v_2^{-1}V(1)\) is given the trivial \(\mathbb{Z}_p\)-action, and \(K(L_p) \wedge v_2^{-1}V(1)\) is equipped with the diagonal \(\mathbb{Z}_p\)-action.
Corollary 1.13. Let $p \geq 5$. There are isomorphisms
\[ \pi_*((K(U_p) \wedge v_2^{-1}V(1))^h \mathbb{Z}_p) \]
\[ \cong \left( (\pi_{*+1}(K(U_p) \wedge v_2^{-1}V(1)))^C_{p-1} \right) \mathbb{Z} \oplus (\pi_*(K(U_p) \wedge v_2^{-1}V(1)))^Z \mathbb{Z}_p \]
\[ \cong (\pi_{*+1}(L_p) \wedge v_2^{-1}V(1))) \mathbb{Z} \oplus (K(L)_*(V(1))[v_2^{-1}])^Z \mathbb{Z}_p \]
\[ \cong (\mathbb{Z}_p \otimes \mathbb{Z}_p)[v_2^{-1}][\pi_{*+1}(K(L_p) \wedge v_2^{-1}V(1))]) \oplus (K(L)_*(V(1))[v_2^{-1}])^Z \mathbb{Z}_p. \]

Remark 1.14. As discussed in more detail in Section [13], the Ausoni-Rognes conjecture suggests that for $p \geq 5$, the direct sum in Corollary 1.13 is expressed in three different, but isomorphic, ways – is a conjectural description of
\[ \pi_*(K(LK(1)(S^0)) \wedge v_2^{-1}V(1)) \cong K(LK(1)(S^0)_*(V(1))[v_2^{-1}]), \]
and it seems that it would be helpful to have a more explicit form of this direct sum. We note that [11] page 4: Theorem 1.5 describes a strategy for computing $\pi_*(K(LK(1)(S^0)) \wedge V(1))$ and gives a result that begins making progress on this strategy.

The second isomorphism of Corollary 1.13 comes from the first one and the fact that there is a $\mathbb{Z}_p$-equivariant isomorphism
\[ (\pi_*(K(U_p) \wedge v_2^{-1}V(1)))^C_{p-1} \cong K(L)_*(V(1))[v_2^{-1}], \]
which is deduced in Section [8] from the fact that $K(L_p)$ and $K(U_p)^h C_{p-1}$ are equivalent after $p$-completion (for this equivalence, see [25], the sentence above Remark 4.4; a proof is in [3] pages 11-12). The third isomorphism in the corollary is an application of Remark 1.12.

1.3. Considerations for the future, terminology, and notation. In our discussion of [12], we saw that proving that $i'$ is a weak equivalence would be a substantial step towards verifying the Ausoni-Rognes conjecture (more precisely, the instances described earlier of this general conjecture), and by Corollary 1.13 this step can be done by showing that $i'$ induces an isomorphism
\[ K(LK(1)(S^0)_*(V(1))[v_2^{-1}] \]
\[ \cong (\pi_{*+1}(K(U_p) \wedge v_2^{-1}V(1)))^C_{p-1} \mathbb{Z} \oplus (\pi_*(K(U_p) \wedge v_2^{-1}V(1)))^Z \mathbb{Z}_p. \]

In [2], Theorem 8.3, under the assumption of two hypotheses, there is a description of the graded abelian group $\pi_*(K(U_p) \wedge V(1))$ as a certain type of module (see [ibid.] for the details), and progress in verifying this description was made by [10] page 2; Theorem 4.5.

Also by Corollary 1.13, we see that another and perhaps easier way to take the aforementioned step is to prove that $i'$ induces an isomorphism
\[ K(LK(1)(S^0)_*(V(1))[v_2^{-1}] \cong (\pi_{*+1}(K(L_p) \wedge v_2^{-1}V(1)))^Z \mathbb{Z}_p. \]

We make a comment related to computing more explicitly the right-hand side of this conjectural isomorphism. By [9] (as conjectured in [4] page 5), there is a localization cofiber sequence
\[ K(\mathbb{Z}_p) \rightarrow K(\ell_p) \rightarrow K(L_p) \rightarrow \Sigma K(\mathbb{Z}_p), \]
where \( t_p \) is the \( p \)-complete connective Adams summand, with \( \pi_*(t_p) = \mathbb{Z}_p[v_1] \). Thus, there is the cofiber sequence

\[
K(\mathbb{Z}_p) \wedge V(1) \to K(t_p) \wedge V(1) \to K(L_p) \wedge V(1) \to \Sigma K(\mathbb{Z}_p) \wedge V(1),
\]

and as stated in [24, page 1267], from explicit computations of \( K(\mathbb{Z}_p)_*(V(1)) \) (known by [11]; see also [2, page 664]) and \( K(t_p)_*(V(1)) \) [4, theorem 9.1], the long exact sequence for this cofiber sequence yields calculations of \( K(L_p)_*(V(1)) \), and some information about this is in [24, Example 5.3].

The author did not push the computation of the last-mentioned “right-hand side” further and one reason is a lack of knowledge about the \( \mathbb{Z}_p \)-action on \( K(L_p)_*(V(1)) \). In this vein, we note that [4, Remark 1.4] mentions a gap in understanding of how a certain Adams operation on \( K(\ell_p) \) acts on a particular class in \( K_{2p-1}(\ell_p) \) (we refer the reader to [ibid.] for the details).

In this paper, we always work in the category \( Sp_\Sigma \) of symmetric spectra of simplicial sets, so that “spectrum” always means symmetric spectrum (except for a few places in this introduction, where the context makes the meaning clear). We let

\[
(-)_f : Sp_\Sigma \to Sp_\Sigma, \quad Z \mapsto Z_f
\]
denote a fibrant replacement functor, so that given the spectrum \( Z \), there is a natural map \( Z \to Z_f \) that is a trivial cofibration, with \( Z_f \) fibrant. If \( K \) is any group and \( X \) is a \( K \)-spectrum, then \( X_f \) is also a \( K \)-spectrum and the trivial cofibration \( X \to X_f \) is \( K \)-equivariant.

Given a spectrum \( Z \) and an integer \( t \), \( \pi_t(Z) \) denotes \([S^t, Z]\), the set of morphisms \( S^t \to Z \) in the homotopy category of symmetric spectra, where here, \( S^t \) denotes a fixed cofibrant and fibrant model for the \( t \)-th suspension of the sphere spectrum. Outside of this introduction, “holim” denotes the homotopy limit for \( Sp_\Sigma \), as defined in [19, Definition 18.1.8]. If \( Z^* \) is a cosimplicial spectrum that is objectwise fibrant, then by “the homotopy spectral sequence for holim\(\Delta Z^*\),” we mean the conditionally convergent spectral sequence

\[
E_2^{s,t} = H^s(\pi_t(Z^*)) \Rightarrow \pi_{t-s}(\operatorname{holim}_\Delta Z^*),
\]

where \( \pi_t(Z^*) \) is the usual cochain complex associated to the cosimplicial abelian group \( \pi_t(Z^*) \).

2. Step I: A Reduction to a Colimit of Short Exact Sequences

Let \( G \) be any closed subgroup of \( \mathbb{Z}_p^\times \). If \( M \) is a discrete \( G \)-module, then we let \( H_*(G, M) \) denote the continuous cohomology groups of \( G \) with coefficients in \( M \). By [13, Theorem 1.7], there is a strongly convergent homotopy spectral sequence \( \{E_2^{s,t}\}_{t \geq 1} = \{E_2^{s,t}\} \) that has the form

\[
E_2^{s,t} = H_*(G, \pi_t(KKU_p) \wedge V(1)[v_2^{-1}]) \Rightarrow \pi_{t-s}(\Sigma (K(KU_p) \wedge v_2^{-1}V(1))^{hG}),
\]

with \( E_2^{s,t} = 0 \), for all \( s \geq 2 \), \( t \in \mathbb{Z} \). Since the \( E_2 \)-page has only two nontrivial columns, there is a short exact sequence

\[
0 \to E_2^{1,t+1} \to \pi_t((KKU_p) \wedge v_2^{-1}V(1))^{hG}) \to E_2^{0,t} \to 0,
\]

for each \( t \in \mathbb{Z} \).

By [13, Theorem 1.7], there is an equivalence of spectra

\[
(K(KU_p) \wedge v_2^{-1}V(1))^{hG} \simeq \operatorname{colim}_{j \geq 0} (KKU_p) \wedge \Sigma^{-jd}V(1))^{hG}.
\]
This result, coupled with the fact that $H^*_r(G, -)$ commutes with colimits of discrete $G$-modules indexed by directed posets, implies that for every $t \in \mathbb{Z}$, the three nontrivial terms in (2.1) satisfy the following:

$$E_{2}^{1,t+1} \cong \text{colim}_{j \geq 0} H^1_j(G, \pi_{t+1}(K(KU_p) \wedge \Sigma^{-jd}V(1))),$$

$$\pi_t((K(KU_p) \wedge v_2^{-1}V(1))^{hG}) \cong \text{colim}_{j \geq 0} \pi_t((K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG}),$$

$$E_{2}^{0,t} \cong \text{colim}_{j \geq 0} (\pi_t(K(KU_p) \wedge \Sigma^{-jd}V(1)))^G.$$

Also, for each $j \geq 0$, by [13] Remark 1.20, Theorem 7.6, (8.3), there is a strongly convergent homotopy spectral sequence $\{E_r^{*,*}\}$ having the form

$$jE_{2}^{s,t} = H^s_r(G, \pi_t(K(KU_p) \wedge \Sigma^{-jd}V(1))) \Longrightarrow \pi_{t-s}((K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG}),$$

with $jE_{2}^{s,t} = 0$, for all $s \geq 2$, $t \in \mathbb{Z}$, so that there is a short exact sequence

$$(2.3) \quad 0 \rightarrow jE_{2}^{1,t+1} \rightarrow \pi_t((K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG}) \rightarrow jE_{2}^{0,t} \rightarrow 0,$$

where $t \in \mathbb{Z}$.

The above facts allow us to conclude that spectral sequence $\{E_r^{*,*}\}$ is the colimit over $\{j \geq 0\}$ of the spectral sequences $\{E_r^{*,*}\}$, and hence, the short exact sequence in (2.1) has the form of the short exact sequences in (2.3). More explicitly, there is a commutative diagram

$$\begin{array}{cccc}
0 & \rightarrow & E_{2}^{1,s+1} & \rightarrow \pi_s((K(KU_p) \wedge v_2^{-1}V(1))^{hG}) & \rightarrow & E_{2}^{0,s} & \rightarrow & 0 \\
& \cong & & \cong & & \cong & & \\
0 & \rightarrow & \text{colim}_{j \geq 0} jE_{2}^{1,s+1} & \rightarrow \text{colim}_{j \geq 0} \pi_s((K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG}) & \rightarrow & \text{colim}_{j \geq 0} jE_{2}^{0,s} & \rightarrow & 0 \\
\end{array}$$

in which the rows are exact and the columns are isomorphisms.

3. Step II: A Recollection of Various Constructions with Spectra

To go further, we need to better understand spectral sequence $\{E_r^{*,*}\}$, for each $j \geq 0$, and to do this, we need to recall several constructions. In this section, $H$ is an arbitrary profinite group.

Given a spectrum $Z$, let $\text{Sets}(H, Z)$ be the $H$-spectrum whose $k$th pointed simplicial set $\text{Sets}(H, Z)_k$ has $l$-simplices $\text{Sets}(H, Z)_{k,l}$ equal to the $H$-set $\text{Sets}(H, Z_{k,l})$ of all functions $H \rightarrow Z_{k,l}$, for each $k, l \geq 0$, where the $H$-action on $\text{Sets}(H, Z_{k,l})$ is defined by

$$\begin{align*}
(h \cdot f)(h') &= f(h'h), \\
h, h' &\in H.
\end{align*}$$

As explained in [13] Section 2, given any $H$-spectrum $X$, there is a cosimplicial $H$-spectrum $\text{Sets}(H^{*+1}, X)$, where for each $n \geq 0$, the spectrum of $n$-cosimplices of $\text{Sets}(H^{*+1}, X)$ is obtained by applying $\text{Sets}(H, -)$ iteratively $n+1$ times to $X$.

**Definition 3.2.** Let $X$ be an $H$-spectrum and let $\mathcal{O} = \{N_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of open normal subgroups of $H$ ordered by inclusion, over a directed poset $\Lambda$. Following [13] Definition 4.4],

$$X^\text{dis}_\mathcal{O} := \text{colim}_{\lambda \in \Lambda} \text{holim}_\Delta \text{Sets}(H^{*+1}, X_f)^{N_\lambda},$$
where the colimit is formed in spectra (this definition is slightly more general than that of \(X_N^\text{dis}\) in \cite{ibid.}: \(N\) satisfies several hypotheses that we do not require from \(O\)). Each spectrum \(\operatorname{holim}_\Delta \operatorname{Sets}(H^{\bullet+1}, X_f)^{N}\) is an \(H/N_\lambda\)-spectrum, and hence, a discrete \(H\)-spectrum, via the canonical projection \(H \to H/N_\lambda\), so that \(X_N^\text{dis}\) is a discrete \(H\)-spectrum. Also, \(X_N^\text{dis}\) is a fibrant spectrum (this follows from \cite{ibid.} steps taken between (4.12) and (4.13)) and the fact that a homotopy limit of fibrant spectra is again fibrant.

Let \(O\) be as in Definition \ref{definition} By \cite{ibid.} Lemma 4.7, proof of Theorem 4.9, for any \(H\)-spectrum \(X\), there is a zigzag

\[
\Gamma^\bullet_{H} X \xrightarrow{\sim} \lim_{\Delta} \operatorname{Sets}(H^{\bullet+1}, X_f) \overset{\phi_X}{\leftarrow} X_O^\text{dis}
\]

of \(H\)-equivariant maps, where \(i_X\) is a weak equivalence of spectra and \(\phi_X\) is induced by the inclusions \(\operatorname{Sets}(H^{\bullet+1}, X_f)^{N}\) \(\to\) \(\operatorname{Sets}(H^{\bullet+1}, X_f)\).

Now suppose that \(X\) is a discrete \(H\)-spectrum. As in \cite{8} Sections 2.4, 3.2, there is a cosimplicial spectrum \(\Gamma^\bullet_{H} X\), where for each \(n \geq 0\), the spectrum of \(n\)-cosimplices of \(\Gamma^\bullet_{H} X\) satisfies the isomorphism

\[
(\Gamma^\bullet_{H} X)^n \cong \lim_{U \triangleleft H^n} \prod_{H^n/U} X,
\]

where \(H^n\) is the \(n\)-fold cartesian product of copies of \(H\) (\(H^0\) is the trivial group \(\{e\}\)) and the colimit is over all the open normal subgroups of \(H^n\). By \cite{8} Theorem 3.2.1 and \cite{14} page 330, Remark 7.5, if \(H = G\), a closed subgroup of \(Z_p^\times\), then

\[
X^{hG} \cong \lim_{\Delta} \Gamma^\bullet_{G} X_{\text{fib}},
\]

where \(X_{\text{fib}}\) is any discrete \(G\)-spectrum that is fibrant as a spectrum and is equipped with a \(G\)-equivariant map \(X \xrightarrow{\sim} X_{\text{fib}}\) that is a weak equivalence of spectra.

4. Step iii: the role of \(V(1)\) in the spectral sequences \(\{iE^r_{*,*}\}\)

Now we focus on understanding the part played by \(V(1)\) in spectral sequence \(\{iE^r_{*,*}\}\), where \(j \geq 0\) and \(G\) is any closed subgroup of \(Z_p^\times\). Let

\[O = \{p^mZ_p\}_{m \geq 0},\]

where each \(p^mZ_p\) is the open normal subgroup of \(Z_p^\times\) that corresponds to \((p^mZ_p) \times \{e\} \triangleleft Z_p \times C_{p-1}\).

In the introduction, we noted that \(K(KU_p) \wedge v^{-1}_2 V(1)\) is realized by the discrete \(Z_p^\times\)-spectrum \(C_p^\text{dis}\), which we can now define:

\[C_p^\text{dis} := \lim_{j \geq 0} \left( (K(KU_p) \wedge \Sigma^{-jd} V(1)) f \right)^\text{dis}_{O}.\]

By \cite{13} Remark 1.20, (8.1)], spectral sequence \(\{E^r_{*,*}\}\) is the homotopy spectral sequence for \(\operatorname{holim}_\Delta \Gamma^\bullet_{G} C_p^\text{dis}\). In Section \ref{section} we noted that there is the isomorphism

\[
\{E^r_{*,*}\} \cong \lim_{j \geq 0} \{iE^r_{*,*}\}
\]

of spectral sequences; for each \(j\), \(\{E^r_{*,*}\}\) is the homotopy spectral sequence for

\[
\operatorname{holim}_\Delta \Gamma^\bullet_{G} \left( (K(KU_p) \wedge \Sigma^{-jd} V(1)) f \right)^\text{dis}_{O}.\]
Fix any \( j \geq 0 \). To increase readability and when the additional intuition carried by the original notation is not needed, we will sometimes use the abbreviation

\[
\mathbb{K}_j := K(KU_p) \wedge \Sigma^{-jd}V(1).
\]

Since the fibrant replacement morphism \( \mathbb{K}_j \rightarrow (\mathbb{K}_j)_f \) is a weak equivalence of spectra that is \( \mathbb{Z}_p^\infty \)-equivariant, the induced map \((\mathbb{K}_j)_f^{\text{dis}} \rightarrow ((\mathbb{K}_j)_f)_f^{\text{dis}}\) is a weak equivalence that is \( \mathbb{Z}_p^\infty \)-equivariant, by [13, Remark 1.20, paragraph after (8.4)]. If \( X \) is a discrete \( G \)-spectrum, then for each \( n \geq 0 \), the spectrum of \( n \)-cosimplices of \( \Gamma^*_G X \) is obtained by applying iteratively \( n \) times to \( X \) a functor that preserves weak equivalences of spectra, by [8, Lemma 2.4.1]. Thus, the induced morphism

\[
\Gamma^*_G((\mathbb{K}_j)_f^{\text{dis}}) \xrightarrow{\sim} \Gamma^*_G(((\mathbb{K}_j)_f)_f^{\text{dis}})
\]

is an objectwise weak equivalence of cosimplicial spectra, so spectral sequence \( \lbrace J_E^r \rbrace \) is isomorphic to the homotopy spectral sequence for

\[
\operatorname{holim}_{\Delta} \Gamma^*_G((\mathbb{K}_j)_f^{\text{dis}}).
\]

Hence, we shift our focus to this latter spectral sequence.

For each \( n \geq 0 \), the spectrum of \( n \)-cosimplices of \( \Gamma^*_G((\mathbb{K}_j)_f^{\text{dis}}) \) satisfies

\[
\left(\Gamma^*_G((\mathbb{K}_j)_f^{\text{dis}})\right)^n \cong \operatorname{colim}_{U \subset G^n} \prod_{m \geq 0} \operatorname{holim}_{\Delta} \operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (\mathbb{K}_j)_f^{\text{dis}} Z_p^m.
\]

Now choose any \( m \geq 0 \). Again at the level of \( n \)-cosimplices, we have

\[
\left(\operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (\mathbb{K}_j)_f^{\text{dis}} Z_p^m\right)^n \cong \prod_{\Delta} \left( K(KU_p) \wedge \Sigma^{-jd}V(1) \right)_f^{\text{dis}} Z_p^m /
\]

\[
\left(\operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (K(KU_p))_f^{\text{dis}} Z_p^m \wedge \Sigma^{-jd}V(1)\right).
\]

where the isomorphism is as in [13, proof of Lemma 2.1] and the second step applies the fact that smashing with a finite spectrum commutes with any product.

If \( Z^*: \Delta \rightarrow Sp^\Sigma \) is a cosimplicial spectrum and \( Z' \) is any spectrum, then there is the functor

\[
(-) \wedge Z': Sp^\Sigma \rightarrow Sp^\Sigma, \quad Y \mapsto Y \wedge Z',
\]

and we let \( Z^* \wedge Z' \) denote the cosimplicial spectrum \((-) \wedge Z' \circ Z^* \). Then we have

\[
\operatorname{holim}_{\Delta} \operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (\mathbb{K}_j)_f^{\text{dis}} Z_p^m
\]

\[
\cong \operatorname{holim}_{\Delta} \operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (K(KU_p))_f^{\text{dis}} Z_p^m \wedge \Sigma^{-jd}V(1)_f
\]

\[
\cong \operatorname{holim}_{\Delta} \operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (K(KU_p))_f^{\text{dis}} Z_p^m \wedge \Sigma^{-jd}V(1),
\]

where the last step is because \( \Sigma^{-jd}V(1) \) is a finite spectrum.

Our last conclusion implies that for each \( n \geq 0 \), we have

\[
\left(\Gamma^*_G((\mathbb{K}_j)_f^{\text{dis}})\right)^n
\]

\[
\cong \operatorname{colim}_{U \subset G^n} \prod_{m \geq 0} \operatorname{holim}_{\Delta} \left(\operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (K(KU_p))_f^{\text{dis}} Z_p^m \wedge \Sigma^{-jd}V(1)_f\right)
\]

\[
\cong \left(\operatorname{holim}_{\Delta} \operatorname{Sets}(\mathbb{Z}_p^\infty)^{m+1}, (K(KU_p))_f^{\text{dis}} Z_p^m \wedge \Sigma^{-jd}V(1)\right)
\]

\[
\cong \left(\Gamma^*_G(K(KU_p))_f^{\text{dis}}\right)^n \wedge \Sigma^{-jd}V(1),
\]
where the second step uses that the smash product commutes with colimits and finite products (which are weakly equivalent to finite coproducts). This shows that there is a zigzag of objectwise weak equivalences between the following two cosimplicial spectra:

\[(4.1) \Gamma_G((K(KU_p) \wedge \Sigma^{-jd}V(1))^{\text{dis}}) \simeq ((\Gamma_G(K(KU_p))^{\text{dis}}) \wedge \Sigma^{-jd}V(1))_f.\]

If \(Z^\bullet\) is a cosimplicial spectrum that is objectwise fibrant, we let \(hss(Z^\bullet)\) denote the associated homotopy spectral sequence. We have shown that there are isomorphisms

\[
\{E_r^{*,*}\} \cong hss\left((\Gamma_G((K(KU_p) \wedge \Sigma^{-jd}V(1))^{\text{dis}})\right)
\cong hss\left(((\Gamma_G(K(KU_p))^{\text{dis}}) \wedge \Sigma^{-jd}V(1))_f\right)
\]

of spectral sequences; the first isomorphism was obtained earlier in this section and the second one is by (4.1), which also yields the following result.

**Theorem 4.2.** Let \(p \geq 5\). If \(G\) is a closed subgroup of \(\mathbb{Z}_p^\times\) and \(j \geq 0\), then

\[(K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG} \simeq ((K(KU_p))^{\text{dis}})^{hG} \wedge \Sigma^{-jd}V(1).\]

**Proof.** We have

\[
(K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG} := ((K(KU_p) \wedge \Sigma^{-jd}V(1))^{\text{dis}})^{hG} \\
\simeq \text{holim}_\Delta \Gamma_G((K(KU_p) \wedge \Sigma^{-jd}V(1))^{\text{dis}}) \\
\simeq \text{holim}_\Delta ((\Gamma_G(K(KU_p))^{\text{dis}}) \wedge \Sigma^{-jd}V(1))_f \\
\simeq \text{holim}_\Delta \Gamma_G(K(KU_p))^{\text{dis}} \wedge \Sigma^{-jd}V(1),
\]

where each step is justified by [13, end of Section 1.2], (3.3), (4.1), and the fact that \(\Sigma^{-jd}V(1)\) is a finite spectrum, respectively, and the last expression above is equivalent to the right-hand side in the desired result (again, by (3.3)). \(\square\)

5. **Step IV: Each Spectral Sequence is One of \(P(v_2)\)-Modules**

In this section, \(j \geq 0\) and, as usual, \(G\) is any closed subgroup of \(\mathbb{Z}_p^\times\).

Since \(p \geq 5\), \(V(1)\) is a homotopy commutative and homotopy associative ring spectrum. Then by Theorem 4.2

\[
\pi_*((K(KU_p) \wedge \Sigma^{-jd}V(1))^{hG}) \cong \pi_*(((K(KU_p))^{\text{dis}})^{hG} \wedge \Sigma^{-jd}V(1))
\]

is a right \(\pi_*\text{(V(1))-module, and hence, it is a P(v_2)-module. This observation suggests that spectral sequence}

\[
\mathfrak{S}G_j := hss\left(((\Gamma_G(K(KU_p))^{\text{dis}}) \wedge \Sigma^{-jd}V(1))_f\right)
\]

is one of \(P(v_2)\)-modules, and now we show that this is the case.

If \(Z^\bullet\) is a cosimplicial spectrum, let \(\prod^\ast Z^\bullet\) be its cosimplicial replacement. Also, let

\[
C^\bullet := \Gamma_G(K(KU_p))^{\text{dis}},
\]

so that

\[
\text{holim}_\Delta \left((\Gamma_G(K(KU_p))^{\text{dis}}) \wedge \Sigma^{-jd}V(1)\right)_f = \text{Tot}(\prod^\ast (C^\bullet \wedge \Sigma^{-jd}V(1))_f).
\]

For each \(l \geq 0\), let

\[
(5.1) F_l \rightarrow \text{Tot}_l\left((\prod^\ast (C^\bullet \wedge \Sigma^{-jd}V(1))_f\right) \rightarrow \text{Tot}_{l-1}\left((\prod^\ast (C^\bullet \wedge \Sigma^{-jd}V(1))_f\right)
\]
be a homotopy fiber sequence (when \( l = 0 \), the last term above is \( * \), the trivial spectrum) and to conserve space, let

\[ \mathbb{T}_l(\cdot) := \text{Tot}_l(\prod^*(\cdot)) \quad \text{and} \quad C^*_l := (C^* \land \Sigma^{-jd}V(1))_f. \]

Then \( \mathcal{H} \mathcal{S}_j \) is the spectral sequence obtained from the exact couple formed from the long exact sequences

\[ \cdots \to \pi_t(F_l) \to \pi_t(\mathbb{T}_l(C^*_l)) \to \pi_t(\mathbb{T}_{l-1}(C^*_l)) \to \pi_{t-1}(F_l) \to \cdots \]

associated to the above homotopy fiber sequences.

As done earlier, we now exploit the fact that smashing with a finite spectrum commutes with products and homotopy limits. Notice that for each \( l \), where the first and last steps are by [15, Proposition 3.10], thus, in the stable objects of cardinality less than \( l \) where the middle two products are indexed over all length \( n \), we have

\[ (\prod^*(C^* \land \Sigma^{-jd}V(1))_f)^n = \prod_{([n] \to \cdot \to [n])} (C^{jn} \land \Sigma^{-jd}V(1))_f \]

\[ \simeq \left( \prod_{([n] \to \cdot \to [n])} C^{jn} \right) \land \Sigma^{-jd}V(1) \]

\[ = (\prod^* C^*)^n \land \Sigma^{-jd}V(1), \]

where the middle two products are indexed over all length \( n \) compositions in the category \( \Delta \), so that

\[ \prod^*(C^* \land \Sigma^{-jd}V(1))_f \simeq \left( \prod^* C^* \right) \land \Sigma^{-jd}V(1)_f, \]

which depicts a zigzag of objectwise weak equivalences between cosimplicial spectra. Then for each \( l \geq 0 \), with \( \Delta^{(l)} \) equal to the full subcategory of \( \Delta \) consisting of objects of cardinality less than \( l + 2 \), and – given a cosimplicial spectrum \( Z^* \) – using \( \text{holim}_{\Delta^{(l)}} Z^* \) to denote \( \text{holim}_{\Delta^{(l)}} (\Delta^{(l)} \hookrightarrow \Delta \xrightarrow{\beta} Sp^\Sigma) \), we have

\[ \text{Tot}_l(\prod^*(C^* \land \Sigma^{-jd}V(1))_f) \]

\[ \simeq \text{holim}_{\Delta^{(l)}} \prod^*(C^* \land \Sigma^{-jd}V(1))_f \simeq \text{holim}_{\Delta^{(l)}} \left( \prod^* C^* \right) \land \Sigma^{-jd}V(1)_f \]

\[ \simeq \left( \text{holim}_{\Delta^{(l)}} \prod^* C^* \right) \land \Sigma^{-jd}V(1) \simeq \text{Tot}_l\left( \prod^* C^* \right) \land \Sigma^{-jd}V(1), \]

where the first and last steps are by [15, Proposition 3.10]. Thus, in the stable homotopy category, for \( l \geq 0 \), we can regard the homotopy fiber sequence in (5.1) as having the form

\[ (5.2) \quad F_l \to \text{Tot}_l\left( \prod^* C^* \right) \land \Sigma^{-jd}V(1) \to \text{Tot}_{l-1}\left( \prod^* C^* \right) \land \Sigma^{-jd}V(1). \]

Since the stable model structure on \( Sp^\Sigma \) is proper [21, Theorem 5.5.2], by [19, Remark 19.1.6, Propositions 13.4.4 and 19.5.3], we can regard a homotopy fiber as a homotopy limit. For each \( l \geq 0 \), let

\[ \xi_l \rightsquigarrow \text{Tot}_l(\prod^* C^*) \xrightarrow{\alpha_l} \text{Tot}_{l-1}(\prod^* C^*) \xrightarrow{\beta_l} \Sigma F_l \]

be a homotopy fiber sequence (our names for the maps follow [27, (5.29)]): by an application of (\( - \land \Sigma^{-jd}V(1) \), we obtain the homotopy fiber sequence

\[ \xi_l \land \Sigma^{-jd}V(1) \xrightarrow{\alpha_l \land 1} \text{Tot}_l(\prod^* C^*) \land \Sigma^{-jd}V(1) \xrightarrow{\alpha_l \land 1} \text{Tot}_{l-1}(\prod^* C^*) \land \Sigma^{-jd}V(1). \]

By comparing this fiber sequence with (5.2), another application of commuting a homotopy limit with smashing with a finite spectrum yields

\[ F_l \simeq \xi_l \land \Sigma^{-jd}V(1), \quad l \geq 0. \]
It follows that \( H\mathfrak{S}_j \) is the spectral sequence obtained from the exact couple formed from the long exact sequences

\[
\cdots \to \pi_*(\tilde{F}_l \land \Sigma^{-jd}V(1)) \xrightarrow{(\eta_1^{*\lambda_1})_*} \pi_*(\tilde{T}_l(\mathbb{C}^\bullet) \land \Sigma^{-jd}V(1)) \to \cdots
\]

\[
(\eta_1^{*\lambda_1})_* \to \pi_*(\tilde{T}_{l-1}(\mathbb{C}^\bullet) \land \Sigma^{-jd}V(1)) \xrightarrow{(\beta_1^{*\lambda_1})_*} \pi_{*-1}(\tilde{F}_l \land \Sigma^{-jd}V(1)) \to \cdots
\]

(the top row ends with a morphism that is continued in the bottom row), where \( l \geq 0 \). As recalled earlier, \( V(1) \) is a homotopy commutative and homotopy associative ring spectrum, so that this long exact sequence is in the category of \( P(v_2) \)-modules. Thus, the associated exact couple and, consequently, spectral sequence \( H\mathfrak{S}_j \) live in the category of \( P(v_2) \)-modules.

6. Step v: the \( P(v_2) \)-module spectral sequences give a direct sum

As usual, \( G \) is any closed subgroup of \( \mathbb{Z}_p^\times \), and \( \text{Mod}_{P(v_2)} \) is the category of \( P(v_2) \)-modules. We recall from Section 2 that there is the isomorphism

\[
\pi_*((K(KU_p) \land v_2^{-1}V(1))^{hG}) \cong \colim_{j \geq 0} \pi_*((K(KU_p) \land \Sigma^{-jd}V(1))^{hG}),
\]

where the right-hand side is the middle term in the colimit

\[
\colim_{j \geq 0} (0 \to jE_2^{1,j+1} \to \pi_*((K(KU_p) \land \Sigma^{-jd}V(1))^{hG}) \to jE_2^{0,*} \to 0)
\]

of short exact sequences. For each \( j \geq 0 \), \( H\mathfrak{S}_j \) is a spectral sequence in \( \text{Mod}_{P(v_2)} \) and since it is isomorphic to spectral sequence \( \{E_2^{*,*}\} \), the associated short exact sequence (displayed above, inside the parentheses) is in \( \text{Mod}_{P(v_2)} \). It will be helpful to write out this short exact sequence explicitly: omitting the trivial terms on the ends and letting \( K \) denote \( K(KU_p) \), this sequence of \( P(v_2) \)-modules has the form

\[
H_c^1(G, \pi_{*+1}(K \land \Sigma^{-jd}V(1))) \to \pi_*((K_{d_1}^\text{hG} \land \Sigma^{-jd}V(1)) \to (\pi_* (K \land \Sigma^{-jd}V(1)))^G,
\]

where the middle term resulted from applying Theorem 4.2.

If \( Z \) is any spectrum, then the diagram \( \{\pi_*(Z \land \Sigma^{-jd}V(1))\}_{j \geq 0} \) is in \( \text{Mod}_{P(v_2)} \), so that the isomorphism

\[
\pi_* (Z \land v_2^{-1}V(1)) \cong \colim_{j \geq 0} \pi_* (Z \land \Sigma^{-jd}V(1))
\]

is in the category of \( P(v_2^{-1}) \)-modules (for example, see [6, Corollary 1.2]). The direct system of spectra \( \{\Sigma^{-jd}V(1)\}_{j \geq 0} \) induces a direct system

\[
\{((\Gamma^G_{KU_p}(K_{d_1}^\text{dis}) \land \Sigma^{-jd}V(1))_{f_{j \geq 0}}\}
\]

of cosimplicial spectra, and hence, a direct system \( \{H\mathfrak{S}_j\}_{j \geq 0} \) of homotopy spectral sequences. Thus, there is the direct system

\[
\{\pi_*(((\Gamma^G_{KU_p}(K_{d_1}^\text{dis}) \land \Sigma^{-jd}V(1))_{f_{j \geq 0}}\}
\]

of associated cochain complexes in \( \text{Mod}_{P(v_2)} \), the cohomology of which induces the direct system

\[
\{H_c^s(G, \pi_* (K(KU_p) \land \Sigma^{-jd}V(1)))\}_{j \geq 0}
\]

in \( \text{Mod}_{P(v_2)} \), for \( s = 0, 1 \). Therefore, the diagram

\[
\{0 \to jE_2^{1,j+1} \to \pi_*(((K(KU_p))_{d_1}^{\text{hG}} \land \Sigma^{-jd}V(1)) \to jE_2^{0,*} \to 0\}_{j \geq 0}
\]
of short exact sequences is in $\text{Mod}_{P(v_2)}$, so that the exact sequence
\[
0 \to \text{colim}_{j \geq 0} j^i E_2^{j+1} \to \text{colim}_{j \geq 0} j \pi_*(\{(K(KU_p))_{1}^{\text{dis}} G) \wedge \Sigma^{-jd} V(1)) \to \text{colim}_{j \geq 0} j^i E_2^{-j} \to 0
\]
is in the category of $P(v_2)$-modules, where the isomorphisms
\[
\text{colim}_{j \geq 0} j^i E_2^{j+1} \cong H^s\left[\text{colim}_{j \geq 0} j \pi_*(\{(K(KU_p))_{1}^{\text{dis}} G) \wedge \Sigma^{-jd} V(1))\right], \quad s = 0, 1,
\]
show that the two outer nontrivial terms in the exact sequence are indeed modules over $P(v_2)$. In particular, in every degree $t$, the sequence is one of $\mathbb{F}_p$-modules and is split exact, giving
\[
\pi_t((K(KU_p) \wedge v_2^{-1} V(1))^G)
\]
\[
\cong (\text{colim}_{j \geq 0} H^i_1 (G, \pi_{t+1}(K(KU_p) \wedge \Sigma^{-jd} V(1)))) \oplus \pi_t((K(KU_p) \wedge v_2^{-1} V(1))^G),
\]
an isomorphism of $\mathbb{F}_p$-modules.

7. Step vi: simplifying $H^1_c(G, \pi_*(K(KU_p) \wedge V(1))[v_2^{-1}])$

Now we work on reducing the first summand in the direct sum obtained at the end of the previous section to a more familiar object. Fix $j \geq 0$ and $t \in \mathbb{Z}$, and recall that
\[
\pi_t(\mathbb{K}_j) = \pi_t((K(KU_p) \wedge \Sigma^{-jd} V(1))
\]
is a finite abelian group (this fact is explained in [13] Section 1.2; the author did not play a role in the hard work behind the explanation, which was done by others, as noted by the references in [ibid.]) and, as a unitary $\mathbb{F}_p$-module, it is a $p$-torsion group (that is, $pm = 0$, for every element $m$). Notice that
\[
H^1_c(G, \pi_t(K(KU_p) \wedge \Sigma^{-jd} V(1))) \cong H^1_c(\mathbb{Z}_p^\times, \text{Coind}_{\mathbb{Z}_p^\times}^{\mathbb{K}_j}(\pi_t(\mathbb{K}_j)))
\]
\[
\cong \text{colim}_{N \in \mathbb{Z}_p^\times} H^1_c(\mathbb{Z}_p^\times, C_{(t,j)}^N),
\]
where the first isomorphism is by Shapiro’s Lemma, the second one is by [24] Proposition 6.10.4, (a) – with
\[
C_{(t,j)}^N := \text{Coind}_{GN, N}^{\mathbb{Z}_p^\times}(\pi_t(\mathbb{K}_j))^{N(G),}
\]
and each $C_{(t,j)}^N$ is a $\mathbb{Z}_p^\times/N$-module (by definition), which makes $C_{(t,j)}^N$ a discrete $\mathbb{Z}_p^\times$-module via the projection $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times/N$.

Let $N$ be fixed. As a set, $C_{(t,j)}^N$ is finite and, for every element $f$ in this abelian group, $pf = 0$. This last fact – together with $p - 1$ and $p$ being relatively prime – implies that the cohomology $H^*(C_{p-1}, C_{(t,j)}^N)$ for the $C_{p-1}$-module $C_{(t,j)}^N$ (by restriction of the $\mathbb{Z}_p^\times$-action) vanishes in positive degrees, so that in the Lyndon-Hochschild-Serre spectral sequence
\[
E_2^{p,q} = H^p_2(\mathbb{Z}_p, H^q(C_{p-1}, C_{(t,j)}^N)) \Longrightarrow H^{p+q}(\mathbb{Z}_p, C_{(t,j)}^N),
\]
we have
\[
E_2^{p,q} = \begin{cases} H^p_2(\mathbb{Z}_p, (C_{(t,j)}^N)^{C_{p-1}}), & q = 0; \\ 0, & q > 0. \end{cases}
\]
This gives
\[
H^1_c(\mathbb{Z}_p^\times, C_{(t,j)}^N) \cong H^1_c(\mathbb{Z}_p, (C_{(t,j)}^N)^{C_{p-1}}) \cong H^1(\mathbb{Z}, (C_{(t,j)}^N)^{C_{p-1}}) \cong ((C_{(t,j)}^N)^{C_{p-1}})_\mathbb{Z},
\]
14
where the third expression above is a non-continuous cohomology group and the second isomorphism is because \((C_{(t,j)}^{N})_{C_{p}^{-1}}\) is finite and \(p\)-torsion (for example, see [22, Example 4.6, Lemma 4.7]).

Now we put the pieces together as \(j\) varies. Given a group \(K\), let \(\mathbb{Z}[K]\)-Mod be the category of \(K\)-modules, and let \(\text{Ab}\) denote the category of abelian groups.

Also, given mathematical expressions \(A\) and \(B\), notation of the form
\[
A \cong_{K/e/L} B \quad \text{or} \quad A \cong_{e/K/L} B
\]
means that (a) in \(\text{Ab}\), \(A \cong B\); (b) in expression \(A\), any colimits are in \(\mathbb{Z}[K]\)-Mod or \(\text{Ab}\) (signified by “\(e\)” in “\(e/K/L\)”), respectively, but these colimits can be formed in \(\text{Ab}\) or \(\mathbb{Z}[K]\)-Mod, respectively, since the forgetful functor \(\mathbb{Z}[K] \rightarrow \text{Ab}\) is a left adjoint; (c) part (b) explains the commuting of any colimits with the evident functor and this commuting underlies the isomorphism \(A \cong B\); and (d) \(L\) denotes a group, and in \(B\), any colimits are in \(\mathbb{Z}[L]\)-Mod, by which we mean \(\text{Ab}\), when \(L\) is “\(e\).” (To avoid any confusion, we note that if \(K = \mathbb{Z}\), then \(\mathbb{Z}[K]\)-Mod means \(\mathbb{Z}[\mathbb{Z}]\)-Mod.)

We have
\[
colim_{j \geq 0} H^j_c(G, \pi_t(K(KU_p) \otimes \Sigma^{-jd}V(1))) \cong \colim_{N <_a \mathbb{Z}_p^\times} \colim_{j \geq 0} \left( (C_{(t,j)}^{N})_{C_{p}^{-1}} \right)_{\mathbb{Z}}^{e/\mathbb{Z}_e} \cong \left( \colim_{N <_a \mathbb{Z}_p^\times} \colim_{j \geq 0} \left( (C_{(t,j)}^{N})_{C_{p}^{-1}} \right)_{\mathbb{Z}}^{e/\mathbb{Z}_e} \right)_{\mathbb{Z}}
\]

Again, let \(N <_a \mathbb{Z}_p^\times\) be fixed and, as is standard, given \(A \in \mathbb{Z}[GN/N]-\text{Mod}\), let
\[
\text{Ind}_{\mathbb{Z}[GN/N]}^{\mathbb{Z}[p^\times/N]}(A) = \mathbb{Z}[p^\times/N] \otimes_{\mathbb{Z}[GN/N]} A,
\]
and set
\[
P_j := (\pi_t(K_{j}))^{N \cap G}.
\]

Then there are isomorphisms
\[
C_{(t,j)}^{N} \cong \text{Hom}_{\mathbb{Z}[GN/N]-\text{Mod}}(\mathbb{Z}[\mathbb{Z}_p^\times/N], \mathbb{P}_j) \cong \text{Ind}_{\mathbb{Z}[GN/N]}^{\mathbb{Z}[p^\times/N]}(\mathbb{P}_j)
\]
of \(\mathbb{Z}[\mathbb{Z}_p^\times/N]-\text{modules}\), since \(\mathbb{Z}_p^\times/N\) is finite (for example, see [24, proof of Proposition 6.10.4]) and because \((\mathbb{Z}_p^\times/N)/(GN/N) \cong \mathbb{Z}_p^\times/GN\) is finite [24 Proposition 5.9], respectively. Hence, there are the following isomorphisms of \(\mathbb{Z}[\mathbb{Z}_p^\times/N]-\text{modules}\) (in the first use below of the “\(e/\mathbb{Z}_e\)” notation, part (c) of its meaning does not apply):
\[
\colim_{j \geq 0} C_{(t,j)}^{N} \cong \text{Ind}_{\mathbb{Z}[GN/N]}^{\mathbb{Z}[p^\times/N]}(\mathbb{P}_j) \cong \text{Ind}_{\mathbb{Z}[GN/N]}^{\mathbb{Z}[p^\times/N]}(\mathbb{P}_j)
\]
\[
\cong \text{Coind}_{\mathbb{Z}[GN/N]}^{\mathbb{Z}[p^\times/N]}(\mathbb{P}_j)
\]
These four isomorphisms are of \(\mathbb{Z}[\mathbb{Z}_p^\times]-\text{modules}\) (via the projection \(\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times/N\)) and, by [24 Proposition 6.10.4, (a)], we conclude that
\[
\colim_{j \geq 0} H^j_c(G, \pi_t(K(KU_p) \otimes \Sigma^{-jd}V(1))) \cong \left( \text{Coind}_{\mathbb{Z}[p^\times]}^{\mathbb{Z}[p^\times]}(\pi_t(K(KU_p) \otimes v_{1}^{-1}V(1))) \right)_{\mathbb{Z}}^{(C_{p}^{-1})}
\]
completing the proof of Theorem 1.11.

In case it is easier to compute \(\colim_{j \geq 0} H^j_c(G, \pi_t(K(KU_p) \otimes \Sigma^{-jd}V(1)))\) by not restricting the \(\mathbb{Z}_p\)-action to the \(\mathbb{Z}\)-action, as done on the right-hand side in the last isomorphism above, we take another look at each \(H^j_c(Z_{p}, (C_{(t,j)}^{N})_{C_{p}^{-1}})\) to obtain the
following result, which is the content of Remark \textbf{1.12}. Here (as in the remark), \( \mathbb{Z}_p \) is regarded as having the trivial \( \mathbb{Z}/p \) group action.

**Lemma 7.1.** When \( p \geq 5 \), \( G \) is any closed subgroup of \( \mathbb{Z}_p^\times \), and \( t \in \mathbb{Z} \), there is an isomorphism

\[
H^1_c(G, \pi_t(K(U_1) \wedge v_2^{-1}V(1))) \\
\cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[\mathbb{Z}_p]]} (\text{Coind}_{C_p}^{\mathbb{Z}_p^\times} (\pi_t(K(U_1) \wedge v_2^{-1}V(1))))^{C_p^{-1}}.
\]

*Proof.* Notice that every coefficient group \( (C_{(t,j)}^N)^{C_p^{-1}} \) is a finite discrete \( p \)-torsion \( \mathbb{Z}_p[[\mathbb{Z}_p]] \)-module. It is standard that there is a projective resolution

\[
0 \rightarrow \mathbb{Z}_p[[\mathbb{Z}_p]] \xrightarrow{1} \mathbb{Z}_p[[\mathbb{Z}_p]] \rightarrow \mathbb{Z}_p \rightarrow 0
\]

(for example, see [18] proof of Proposition 6) for any omitted details) that can be used to compute the cohomology group (for more information about this, see [26] Section 3.2). Thus, the cohomology group is the cohomology of the complex obtained by applying the functor \( \text{Hom}_{\mathbb{Z}_p[[\mathbb{Z}_p]]}^{\mathbb{Z}_p^\times} (-, (C_{(t,j)}^N)^{C_p^{-1}}) \) of continuous module homomorphisms to this resolution: we obtain that

\[
H^1_c(\mathbb{Z}_p, (C_{(t,j)}^N)^{C_p^{-1}}) \cong (C_{(t,j)}^N)^{C_p^{-1}} \text{Tor}^\mathbb{Z}_p (\pi^t, (C_{(t,j)}^N)^{C_p^{-1}}) \rightarrow (C_{(t,j)}^N)^{C_p^{-1}}
\]

\[
\cong H^0_c(\mathbb{Z}_p, (C_{(t,j)}^N)^{C_p^{-1}}) = \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[\mathbb{Z}_p]]} (C_{(t,j)}^N)^{C_p^{-1}}
\]

where the right-hand side of the second isomorphism is a continuous homology group (see [26] Section 3.3) and the last step is because \( (C_{(t,j)}^N)^{C_p^{-1}} \) is finite (and thus, a finitely generated object in the category of profinite \( \mathbb{Z}_p[[\mathbb{Z}_p]] \)-modules; see [24] Proposition 5.5.3). The isomorphism in the second step is not quite immediate, and it can be justified in a slick way: since \( \hat{Z} \) is an orientable discrete Poincaré duality group of dimension one and pro-\( p \) good (in the sense of [29] Section 3.1; by [22] Example 4.6, Lemma 4.7), the pro-\( p \) completion \( \hat{Z}_p \) is an orientable (profinite) Poincaré duality group at \( p \) of dimension one, by [29] Proposition 3.2 [here, for “orientable (profinite) Poincaré duality group at \( p \),” we use the definitions in [26] Section 4.4, page 394] and by [28] Remark 2.2, these are equivalent to those used in [29]), and the desired isomorphism follows.

Then the result follows by the manipulations that preceded this lemma. To understand the abstract \( \mathbb{Z}_p[[\mathbb{Z}_p]] \)-module structure of the \( C_{p^{-1}} \)-fixed points of the coinduced module in the statement of the lemma (and of the various pieces involved in the manipulations), it is helpful to note that if \( H \) is an arbitrary profinite group, then a \( p \)-torsion discrete \( H \)-module is canonically a discrete, and hence abstract, \( \mathbb{Z}_p[[H]] \)-module.

\[\square\]

8. A further reduction in the case when \( G = \mathbb{Z}_p^\times \)

Let \( V(0) \) be the mod \( p \) Moore spectrum \( M(p) \), and more generally, for each integer \( i \geq 1 \), let \( M(p^i) \) be the mod \( p^i \) Moore spectrum. By restriction of the \( \mathbb{Z}_p^\times \)-action, \( C_{p^{-1}} \) acts on \( K(U_1) \), so that there is the homotopy fixed point spectrum

\[
K(U_1)^{hC_{p^{-1}}} = (K(U_1))^{hC_{p^{-1}}},
\]

16
and by [25] page 1267 (see [3] pages 11-12 for a proof), the canonical map
\[ \text{holim}_{i \geq 1}(K(L_p) \wedge M(p^i)) \xrightarrow{\cong} \text{holim}_{i \geq 1}(K(U_p)^{hC_{p-1}} \wedge M(p^i)) \]
is a weak equivalence. It follows that the morphism
\[ L_{V(0)}(K(L_p)) \xrightarrow{\cong} L_{V(0)}(K(U_p)^{hC_{p-1}}) \]
(between Bousfield localizations with respect to \( V(0) \)) is a weak equivalence, so that the natural map \( K(L_p) \to K(U_p)^{hC_{p-1}} \) is a \( V(0) \)-equivalence. The familiar cofiber sequence
\[ \Sigma^{2p-2}V(0) \xrightarrow{\iota_0} V(0) \xrightarrow{\iota_1} V(1) \]
induces the commutative diagram
\[
\begin{array}{ccc}
K(L_p) \wedge \Sigma^{2p-2}V(0) & \xrightarrow{\iota_0} & K(L_p) \wedge V(0) \\
\downarrow & & \downarrow \\
K(U_p)^{hC_{p-1}} \wedge \Sigma^{2p-2}V(0) & \to & K(U_p)^{hC_{p-1}} \wedge V(0) \\
\end{array}
\]
in which the rows are cofiber sequences. Since the leftmost and middle vertical maps are weak equivalences, the rightmost vertical map is a weak equivalence. Thus, for each \( j \geq 0 \), the map
\[ K(L_p) \wedge \Sigma^{-jd}V(1) \xrightarrow{\cong} K(U_p)^{hC_{p-1}} \wedge \Sigma^{-jd}V(1) \]
is a weak equivalence. We apply this conclusion in the following way.

There are the homotopy fixed point spectral sequences
\[ 
\bar{E}_{2}^{s,t} = H^s(C_{p-1}, \pi_t(K(U_p) \wedge v_2^{-1}V(1))) \Rightarrow \pi_{t-s}(K(U_p) \wedge v_2^{-1}V(1))^{hC_{p-1}} 
\]
and
\[ 
\bar{E}_{2}^{s,t} = H^s(C_{p-1}, \pi_t(K(U_p) \wedge \Sigma^{-jd}V(1))) \Rightarrow \pi_{t-s}(K(U_p) \wedge \Sigma^{-jd}V(1))^{hC_{p-1}}, 
\]
for each \( j \geq 0 \). Since each \( \pi_t(K(U_p) \wedge \Sigma^{-jd}V(1)) \) is \( p \)-torsion, both
\[ 
\bar{E}_{2}^{s,t} \cong \text{colim}_{j \geq 0} \bar{E}_{2}^{s,t}_{j} 
\]
and each \( \bar{E}_{2}^{s,t}_j \) vanish for \( s > 0 \), \( t \in \mathbb{Z} \), and \( j' \geq 0 \). As a consequence,
\[ 
\pi_*((K(U_p) \wedge v_2^{-1}V(1))^{hC_{p-1}}) \cong (\pi_*((K(U_p) \wedge v_2^{-1}V(1)))^{hC_{p-1}}, 
\]
\[ 
(K(U_p) \wedge v_2^{-1}V(1))^{hC_{p-1}} \cong \text{colim}_{j \geq 0}(K(U_p) \wedge \Sigma^{-jd}V(1))^{hC_{p-1}}, \] and
\[ 
\pi_*((K(U_p) \wedge \Sigma^{-jd}V(1))^{hC_{p-1}}) \cong (\pi_*((K(U_p) \wedge \Sigma^{-jd}V(1)))^{hC_{p-1}}, j \geq 0 
\]
(the above equivalence of spectra (the middle line) is a special case of [22] from [13] Theorem 1.7, but here, [ibid.] is not needed and the conclusion follows from the vanishing properties stated above and [23] Proposition 3.3). Therefore (for the following deductions, we do not need the second isomorphism in Ab displayed above (which is indexed by \( \{ j \mid j \geq 0 \} \)); we state it here because of its intrinsic
interest), we have the isomorphisms
\[
(\pi_*(K(KU_p) \wedge V_2^{-1}V(1)))^{C_p^{-1}} \cong \colim_{j \geq 0} \pi_*((K(KU_p) \wedge \Sigma^{-jd}V(1))^{hC_p^{-1}})
\]
\[
\cong \colim_{j \geq 0} \pi_*(K(KU_p)^{hC_p^{-1}} \wedge \Sigma^{-jd}V(1))
\]
\[
\cong \pi_*(K(L_p) \wedge v_2^{-1}V(1)) \cong K(L_p)_*(V(1))[v_2^{-1}].
\]

Each of the spectra $K(L_p)$ and $K(KU_p)^{hC_p^{-1}}$ have a natural action by $\mathbb{Z}_p$ and the map $K(L_p) \to K(KU_p)^{hC_p^{-1}}$ is $\mathbb{Z}_p$-equivariant; thus, each of the above four isomorphisms is $\mathbb{Z}_p$-equivariant.

**References**

[1] Gabe Angelini-Knoll. On topological Hochschild homology of the $K(1)$-local sphere. 30 pp., arXiv:1612.00548v3; January 8, 2021; accepted for publication in *Journal of Topology*.

[2] Christian Ausoni. On the algebraic $K$-theory of the complex $K$-theory spectrum. *Invent. Math.*, 180(3):611–668, 2010.

[3] Christian Ausoni and John Rognes. Algebraic $K$-theory of the fraction field of topological $K$-theory. 54 pp., arXiv:0911.3781; November 25, 2009.

[4] Christian Ausoni and John Rognes. Algebraic $K$-theory of topological $K$-theory. *Acta Math.*, 188(1):1–39, 2002.

[5] Christian Ausoni and John Rognes. The chromatic red-shift in algebraic $K$-theory. In *Guido’s Book of Conjectures*, Monographie de L’Enseignement Mathématique, volume 40, pages 13–15, 2008.

[6] Christian Ausoni and John Rognes. Algebraic $K$-theory of the first Morava $K$-theory. *J. Eur. Math. Soc. (JEMS)*, 14(4):1041–1079, 2012.

[7] Clark Barwick and Peter Haine. Pyknotic objects, I. Basic notions. 39 pages, arXiv:1904.09969v2; April 30, 2019.

[8] Mark Behrens and Daniel G. Davis. The homotopy fixed point spectra of profinite Galois extensions. *Trans. Amer. Math. Soc.*, 362(9):4983–5042, 2010.

[9] Andrew J. Blumberg and Michael A. Mandell. The localization sequence for the algebraic $K$-theory of topological $K$-theory. *Acta Math.*, 200(2):155–179, 2008.

[10] Andrew J. Blumberg and Michael A. Mandell. Localization for $THH(ku)$ and the topological Hochschild and cyclic homology of Waldhausen categories. *Mem. Amer. Math. Soc.*, 265(1286):v+100, 2020.

[11] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. *Astérisque*, (226):7–8, 57–143, 1994. $K$-theory (Strasbourg, 1992).

[12] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[13] Daniel G. Davis. A construction of some objects in many base cases of an Ausoni-Rognes conjecture. 32 pages, arXiv:2005.04199v4; December 12, 2020.

[14] Daniel G. Davis. Homotopy fixed points for $L_K(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra*, 206(3):322–354, 2006.

[15] Björn Ian Dundas and John Rognes. Cubical and cosimplicial descent. *J. Lond. Math. Soc. (2)*, 98(2):439–460, 2018.

[16] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*., pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[17] Paul G. Goerss and Michael J. Hopkins. André-Quillen (co)-homology for simplicial algebras over simplicial operads. In *Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999)*, pages 41–85. Amer. Math. Soc., Providence, RI, 2000.

[18] Hans-Werner Henn. A mini-course on Morava stabilizer groups and their cohomology. In *Algebraic topology*, volume 2194 of *Lecture Notes in Math.*., pages 149–178. Springer, Cham, Switzerland, 2017.

[19] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
[20] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. Ann. of Math. (2), 148(1):1–49, 1998.
[21] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. J. Amer. Math. Soc., 13(1):149–208, 2000.
[22] Peter Linnell and Thomas Schick. Finite group extensions and the Atiyah conjecture. J. Amer. Math. Soc., 20(4):1003–1051, 2007.
[23] Stephen A. Mitchell. Hypercohomology spectra and Thomason’s descent theorem. In Algebraic K-theory (Toronto, ON, 1996), pages 221–277. Amer. Math. Soc., Providence, RI, 1997.
[24] Luis Ribes and Pavel Zalesskii. Profinite groups. Springer-Verlag, Berlin, 2000.
[25] John Rognes. Algebraic K-theory of strict ring spectra. In Proceedings of the International Congress of Mathematicians, Seoul 2014, Volume II, pages 1259–1283. Kyung Moon Sa, Seoul, Korea, 2014.
[26] Peter Symonds and Thomas Weigel. Cohomology of p-adic analytic groups. In New horizons in pro-p groups, pages 349–410. Birkhäuser Boston, Boston, MA, 2000.
[27] R. W. Thomason. Algebraic K-theory and étale cohomology. Ann. Sci. École Norm. Sup. (4), 18(3):437–552, 1985.
[28] Thomas Weigel. Maximal l-Frattini quotients of l-Poincaré duality groups of dimension 2. Arch. Math. (Basel), 85(1):55–69, 2005.
[29] Thomas Weigel. On profinite groups with finite abelianizations. Selecta Math. (N.S.), 13(1):175–181, 2007.