Quantum phase transition of a two-dimensional quadrupolar system

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Ensembles with long-range interactions between particles are promising for revealing strong quantum collective effects and many-body phenomena. Here we study the ground-state phase diagram of a two-dimensional Bose system with quadrupolar interactions using a diffusion Monte Carlo technique. We predict a quantum phase transition from a gas to a solid phase. The Lindemann ratio and the condensate fraction at the transition point are \( \gamma = 0.269(4) \) and \( n_0/n = 0.031(4) \), correspondingly. We observe the strong rotonization of the collective excitation branch in the vicinity of the phase transition point. Our results can be probed using state-of-the-art experimental systems of various nature, such as quasi-two-dimensional systems of quadrupolar excitons in transition metal dichalcogenide (TMD) tri-layers, quadrupolar molecules, and excitons or Rydberg atoms with quadrupole moments induced by strong magnetic fields.

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Manipulation of the short-range interactions in ultracold quantum gases has proven to be an efficient and productive way to generate novel many-body phases [1–3]. Even more evolved scenarios are realized in gases with long-range interactions such as dipolar ones [4–6]. Dipolar particles interact with each other via anisotropic and long-range forces, which drastically changes the structure of many-body phases in these systems both in the free space and lattices [4–7]. Remarkable progress in experiments with ultracold gases of large-spin atoms [8–11] and polar molecules [12–14] have opened fascinating prospects for the experimental observation of novel quantum phases, which are induced by the character of the dipolar interaction. Examples include, in particular, rotonization [15–19], crystallization [20–22], and supersolidity for both dilute [23–32] and dense [20, 21, 33, 34, 36–38] dipolar systems. However, the interactions between atomic dipoles are typically weak. This fact has stimulated the exploration of novel platforms with both strong interparticle interactions and sufficient tunability. Examples include long-lived excitons in solid-state systems [39–46]. Remarkable advances in experiments with monolayers of semiconducting transition metal dichalcogenides (TMDs) [47, 48] make them interesting for revealing unconventional quantum phenomena [49–53] in the regimes that are beyond what can be achieved with ultracold gases. TMD systems host long-lived excitons since the overlap between wavefunctions of electrons and holes locating in separate layers is suppressed, and the separation results in the appearance of the exciton dipole moment as it was predicted [39, 40]. Dipolar excitons in solid-state systems might manifest rotonization [20, 38, 54, 55] and supersolidity [38].

Quadrupolar interactions present a peculiar example of non-local interactions between particles [56–58], which can be fine-tuned using external fields. This makes quadrupolar systems a promising platform for performing the quantum simulation and revealing novel many-body phases and unconventional quantum states [56–58]. Experimental realizations of quadrupolar ensembles include quadrupolar molecules, whose interaction is induced and tuned by external fields, and quadrupolar excitons in solid-state systems. Quadrupolar species of particles acquiring electric quadrupole moments, such as Cs_2 [59] or Sr_2 [60, 61], are available in experiments. Moreover they are stable against collapse and ultracold chemical reactions at high densities, which are shortcomings for experiments with dipolar molecules [12–14]. Recent studies of TMD systems [62] have shown the rich many-body physics, which is induced by the nature of quadrupolar interactions. Besides, quadrupoles play an essential role in astrophysical objects in ultrastrong magnetic fields, e.g., on the surface of neutron stars [63]. However, a detailed microscopic study and ab initio simulations of the quadrupole many-body system is still lacking.

Here we predict a quantum phase transition from a gas to a crystal in a single-component two-dimensional (2D) Bose system with centrally symmetric quadrupolar interactions at zero temperature. We employ a diffusion Monte Carlo (DMC) technique for calculating the parameters of the phase transition and to study effects of strong correlations in the gas phase. Our results are in the quantitative agreement with predictions based on the quantum hydrodynamic model. We observe a roton-maxon character of the collective excitation branch. The predicted results can be probed in state-of-the-art experiments with ultracold atoms (e.g., Rydberg atoms),
molecular ensembles and TMD systems.

The Hamiltonian of a homogeneous system of \( N \) bosons with the quadrupolar interaction is as follows:

\[
\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \frac{Q^2}{\epsilon} \sum_{j<k}^{N} \frac{1}{|\mathbf{r}_j - \mathbf{r}_k|^5},
\]

(1)

where \( m \) is the particle mass, \( \mathbf{r}_i \) is the 2D position of \( i \)-th particle, \( Q \) is the quadrupolar moment and \( \epsilon \) is the dielectric constant.

It is convenient to rewrite Hamiltonian (1) in a dimensionless form by expressing all the distances in units of \( r_0 = \sqrt{mQ^2/\hbar^2\epsilon} \) and energies in units of \( E_0 = \hbar^2/mr_0^2 \). The characteristic quadrupole length \( r_0 \) is directly proportional to the quadrupole-quadrupole s-wave scattering length, \( a_s/r_0 = (e^{2\pi\hbar}/3)^{2/3} = 0.706383 \) with \( \gamma_E = 0.577\ldots \) the Euler constant. We calculate the zero-temperature phase diagram of the system in terms of the dimensionless density \( nr_0^2 \), where \( n \) is the 2D density of the system.

In order to find the system properties we resort to the DMC technique [65] based on solving the Schrödinger equation in imaginary time and allowing one to obtain the exact ground-state energy. The convergence is significantly improved by using an importance sampling for which we chose the trial wave function in the Nosanow-Jastrow product form [20]. Using the standard prescription, each particle in the solid phase is localized close to its lattice site by a one-body Gaussian term of variable width. An infinite width is used in the gas phase which results in a wave function having translational invariance. We chose the two-body Jastrow term as

\[
f_2(x) = \begin{cases} C_1K_0(2x^{-3/2}/3), & x \leq x_c, \\ C_2 \exp(-C_3/x - C_5/(\bar{L} - x)), & x_c \leq x \leq \bar{L}/2, \\ 1, & \bar{L}/2 \leq x, \end{cases}
\]

(2)

where \( x = r/r_0 \), \( \bar{L} = L/r_0 \) and \( x_c \) is the variational parameter (matching point between the two-body scattering solution at short distances and the phononic long-range decay [64]). Coefficients \( C_1, C_2, C_3 \) are fixed by the condition of the continuity of the function and its first derivative.

The thermodynamic limit is then reached by increasing the number of particles while keeping the density \( n = N/(L_x \times L_y) \) fixed and performing extrapolation to \( N \to \infty \) [67]. We simulate systems containing \( N = 100, 144, 256, 484 \) and 1156 particles a simulation box of size \( L_x \times L_y \) with periodic boundary conditions. We use a square box with equal sides \( L_x = L_y \) for simulation of the gas phase and a rectangular box commensurate with an elementary cell of a triangular lattice for the solid phase.

We find a quantum phase transition from a gas to a solid phase at zero temperature. In order to demonstrate its presence, we calculate the lowest energy in a state with translational symmetry (i.e. gas) and a state in which the translational symmetry is broken (i.e. solid). The phase transition density is obtained as the crossing between the energies of the two states. The two equations of state are shown in Fig. 1, where for convenience the energy \( E \) is scaled with the system area \( S \) and density as \( (E/S)/n^{7/2} \).

For small values of the dimensionless density \( nr_0^2 \), the energetically favorable state is a gas whereas the solid phase remains metastable. As the density is increased, the system experiences a first-order quantum phase transition to a triangular lattice phase. We estimate the transition density to be \( nr_0^2 = 2.10(7) \) (see Fig. 1) with the width of the coexistence of the phases \( \Delta n = 0.026(4) \). The Lindemann ratio quantifies the fluctuations of particles in a crystal and is defined as follows

\[
\gamma = \sqrt{\sum_{i=1}^{N} (|\mathbf{r}_i - \mathbf{r}_i^{\text{lat}}|^2/b^2)},
\]

(3)

where \( b = (4\sqrt{3}/3)^{1/4} \) is the triangular lattice period. We find the Lindemann ratio to be \( \gamma = 0.269(4) \) at the transition point. In the limit of high density, the potential energy dominates and the energy gradually ap-
The order parameter differentiating two phases is the height of the peak in the static structure factor

$$S(k) = \int \langle \hat{\rho}(r)\hat{\rho}(s) \rangle e^{i k (r-s)} d\mathbf{r} d\mathbf{s} / N,$$  \hspace{1cm} (4)

at the reciprocal lattice period $k_L = 2\pi \sqrt{n(4/3)^{1/4}}$ of the triangular crystal, where $\hat{\rho}(\mathbf{r})$ is the density operator and $\langle \cdots \rangle$ denotes ground state averaging. The characteristic feature of a crystalline phase is that the value of $S(k_L)$ is linearly proportional to the number of particles and the peak becomes macroscopic in the thermodynamic limit. This should be contrasted to the behavior in the gas phase in which the static structure factor always remains finite, see Fig. 2 for characteristic examples. In that case, $S(k)$ is a monotonous function of momentum at low densities and it becomes non-monotonous (i.e. a peak is formed) in the regime of strong quantum correlations. The height of the peak increases linearly as the density is incremented and the phase transition from the gas to the crystal happens when the critical value, $\max_k S(k) = 1.6(1)$, is reached. There is a discontinuity in the order parameter, $S(k_L)$, across the phase transition point which is typical behavior for the first-order phase transition. At the same time, the low-momentum behavior, $S(k) = \hbar k / (2mc)$, is more similar in the two phases which reflects a relatively minor change of the speed of sound $c$ across the transition (compare the dashed lines in Fig 2b).

The appearance of the short-range ordering in the gas phase in the vicinity of the critical density can be seen from the snapshots shown in the inset of Fig. 2b. The snapshot of the gas phase indicates the formation of a local triangular lattice with vacancies and dislocations, whereas a defect-free triangular lattice is observed in the ground state of the solid phase.

The coherence properties are quantified by the condensate fraction which is reported in Fig. 3. We have verified that in a finite-size system, the long-range behavior of the one-body density matrix $g_1(r)$ is well reproduced by the hydrodynamic (HD) theory of Ref. [68]. We use the HD theory for the extrapolation of the OBDM in order to obtain the condensate fraction according to $n_0/n = \lim_{r \to \infty} g_1(r)$. We observe a strong condensate depletion as the density $nr_0^2$ is increased, so the value $n_0/n$ becomes fairly small close to the gas-solid transition. Even if the value of a few percent might seem very small, conceptually it is important that $n_0/n$ experiences a sudden jump to zero value in the solid phase. In other words, the condensate fraction is another order parameter and together with the height of the structure factor both order parameters are discontinuous across the first-order phase transition. Both order parameters have exactly the same critical point and as a consequence a supersolid (simultaneous presence of both order parameters) is absent in the present system.

The rotonization of the excitation spectrum is yet another non-trivial effect that might be present in strongly-
of the spectrum in the quadrupolar system by analyzing rotonization [20, 21, 38]. Here we make evident the rotonization of the collective excitation branch may potentially lead to the spontaneous establishment of crystalline order and formation of supersolid. One of the possible mechanisms for supersolidity of dipolar systems is its formation near the gas-solid phase transition [19]. Indeed, rotonization of the collective excitation in dipolar systems is its formation near the gas-solid phase transition [20].

As a possible realization of our model, we analyze typical experimental schemes shown in Fig. 5. We assume a semiconductor structure consisting of three layers separated by barriers. In configuration (a), a quadrupole can be formed out of two holes in the middle layers and two electrons each one in the outer layer. Pauli exclusion principle does not allow the holes to be close and their density profile forms a ring. Assuming a thin ring of radius $R$ and a total charge $-2q$ in the central layer and two point-like $+q$ charges in the lateral layers, the quadrupole moment is equal to

$$Q = 3qD^2 \sqrt{1 + 2\alpha(\alpha - 1)/3},$$ \hspace{1cm} (6)

where $D$ is the distance between the centers of the central and lateral layers, $\alpha = R^2/D^2$ and the hole charge is $q = e > 0$. Configuration (b) is obtained for the specular...
case with the inverted charges, $q = -e$. Configuration (c) assumes only a single charge $2q = -e$ in the central layer ($\alpha = 0$ in Eq. (6)) and half-charges $q = +e/2$ in the outer layers. The physical realizations are based on ZnSe quantum wells [77] with interlayer separation $D = 6$ nm and MoS$_2$/MoTe$_2$ monolayers [78, 79] with $D = 1.667$ nm, according to the schemes shown in Fig. 5. The critical densities of the gas-solid phase transition are realistic and correspond to (a) $n_c = 1.8 \cdot 10^{14}$ cm$^{-2}$, (b) $n_c = 2.6 \cdot 10^{12}$ cm$^{-2}$ and (c) $n_c = 1.3 \cdot 10^{13}$ cm$^{-2}$.

In conclusion, we have obtained the ground-state phase diagram of two-dimensional bosons interacting via quadrupolar potential at zero temperature. Energetic, structural and coherent properties have been studied in the vicinity of the gas-solid quantum phase transition. We have demonstrated that the excitation spectrum experiences a strong rotonization in the gas phase close to the critical density. We have found an agreement with quantum hydrodynamical calculations for the calculation of the condensate fraction. Our predictions can be probed in experiments with TMD systems and ultracold gases, where the technique for the observation of roton phenomena recently has been developed. Promising candidates for the creation of such phases are quadrupolar excitons in TMD layer structures [55, 62], where the quantum phase transition for the two-component systems has been observed [62], and Rydberg atomic ensembles.

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[1] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. **80**, 885 (2008).

[2] I. Bloch, J. Dalibard, and S. Nascimbéne, Quantum simulations with ultracold quantum gases, Nat. Phys. **8**, 267 (2012).

[3] C. Gross and I. Bloch, Quantum simulations with ultracold atoms in optical lattices, Science **357**, 995 (2017).

[4] M.A. Baranov, Theoretical progress in many-body physics with ultracold dipolar gases, Phys. Rep. **464**, 71 (2008).

[5] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, The physics of dipolar bosonic quantum gases, Rep. Prog. Phys. **72**, 126401 (2009).

[6] M.A. Baranov, M. Delmonte, G. Pupillo, and P. Zoller, Condensed matter theory of dipolar quantum gases, Chem. Rev. **112**, 5012 (2012).

[7] C. Treffzger, C. Menotti, B. Capogrosso-Sansone, and M. Lewenstein, Ultracold dipolar gases in optical lattices, J. Phys. B: At. Mol. Opt. Phys. **44**, 193001 (2011).

[8] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, Bose-Einstein condensation of chromium, Phys. Rev. Lett. **94**, 160401 (2005).

[9] M. Lu, N.Q. Burdict, S.H. Youn, and B.L. Lev, Strongly dipolar Bose-Einstein condensate of dysprosium, Phys. Rev. Lett. **107**, 190401 (2011).

[10] M. Lu, N.Q. Burdick, and B.L. Lev, Quantum degenerate dipolar Fermi gas, Phys. Rev. Lett. **108**, 215301 (2012).

[11] K. Aikawa, A. Frisch, M. Mark, S. Baier, A. Rietzler, R. Grimm, and F. Ferlaino, Bose-Einstein condensation of erbium, Phys. Rev. Lett. **108**, 210401 (2012).

[12] For a review, see L.D. Carr, D. DeMille, R.V. Krems, and J. Ye, Cold and ultracold molecules: science, technology and applications, New J. Phys. **11**, 055049 (2009).

[13] O. Dulieu and C. Gabbannini, Cold and ultracold molecules: science, technology and applications, Rep. Prog. Phys. **72**, 086401 (2009).

[14] S.A. Moses, J.P. Covey, M.T. Miecnikowski, D.S. Jin, and J. Ye, New frontiers for quantum gases of polar molecules, Nat. Phys **13**, 13 (2017).

[15] L. Santos, G.V. Shlyapnikov, and M. Lewenstein, Roton-maxon spectrum and stability of trapped dipolar Bose-Einstein condensates, Phys. Rev. Lett. **90**, 250403 (2003).

[16] R.M. Wilson, S. Ronen, J.L. Bohn, and H. Pu, Manifestations of the roton mode in dipolar Bose-Einstein condensates, Phys. Rev. Lett. **100**, 245302 (2008).

[17] R.M. Wilson, C. Ticknor, J.L. Bohn, and E. Timmermans, Roton immiscibility in a two-component dipolar Bose gas, Phys. Rev. A **86**, 033606 (2012).

[18] A. Bondjemâa and G.V. Shlyapnikov, Two-dimensional dipolar Bose gas with the roton-maxon excitation spectrum, Phys. Rev. A **87**, 025601 (2013).

[19] A.K. Fedorov, I.L. Kurbakov, Y.E. Shchadilova, and Yu.E. Lozovik, Two-dimensional Bose gas of tilted dipoles: Roton instability and condensate depletion, Phys. Rev. A **90**, 043616 (2014).

[20] G.E. Astrakharchik, J. Boronat, I.L. Kurbakov, and Yu.E. Lozovik, Quantum phase transition in a two-dimensional system of dipoles, Phys. Rev. Lett. **98**, 060405 (2007).

[21] H.P. Büchler, E. Demler, M. Lukin, A. Micheli, N. Prokof’ev, G. Pupillo, and P. Zoller, Strongly correlated 2D quantum phases with cold polar molecules: Controlling the shape of the interaction potential, Phys. Rev. Lett. **98**, 060404 (2007).

[22] J. Boning, A. Filinov, and M. Bonitz, Crystallization of an exciton superfluid, Phys. Rev. B **84**, 075130 (2011).

[23] Z.-K. Lu, Y. Li, D.S. Petrov, and G.V. Shlyapnikov, Stable dilute supersolid of two-dimensional dipolar bosons, Phys. Rev. Lett. **115**, 075303 (2015).

[24] L. Chomaz, R.M.W. van Bijnen, D. Petter, G. Faraoni, S. Baier, J.H. Becher, M.J. Mark, F. Waechtler, L. Santos, and F. Ferlaino, Observation of roton mode population in a dipolar quantum gas, Nat. Phys. **14**, 442 (2018).

[25] D. Petter, G. Natale, R.M.W. van Bijnen, A. Patscheider, M.J. Mark, L. Chomaz, and F. Ferlaino, Probing the
roton excitation spectrum of a stable dipolar Bose gas, Phys. Rev. Lett. 122, 183401 (2019).

[26] M. Wenzel, F. Böttcher, T. Langen, I. Ferrier-Barbut, and T. Pfau, Striped states in a many-body system of tilted dipoles, Phys. Rev. A 96, 053630 (2017).

[27] F. Böttcher, J.-N. Schmidt, M. Wenzel, J. Hertkorn, M. Guo, T. Langen, and T. Pfau, Transient supersolid properties in an array of dipolar quantum droplets, Phys. Rev. X 9, 011051 (2019).

[28] L. Chomaz, D. Petter, P. Ilzhöfer, G. Natale, A. Trautmann, C. Politì, G. Durastante, R.M.W. van Bijnen, A. Patscheider, M. Söhmnsen, M.J. Mark, and F. Ferlaino, Long-lived and transient supersonic behaviors in dipolar quantum gases, Phys. Rev. X 9, 021012 (2019).

[29] L. Tanzl, E. Lucioni, F. Fanà, J. Catani, A. Fioretti, C. Gabbanini, R. N. Bisset, L. Santos, and G. Modugno, Observation of a dipolar quantum gas with metastable supersonid properties, Phys. Rev. Lett. 122, 130405 (2019).

[30] F. Cinti and M. Boninsegni, Classical and quantum filaments in the ground state of trapped dipolar Bose gases, Phys. Rev. A 96, 013627 (2017).

[31] D. Baillie and P.B. Blakie, Droplet crystal ground states of a dipolar Bose gas, Phys. Rev. Lett. 121, 195301 (2018).

[32] S. M. Roccuzzo and F. Ancilotto, Supersolid behavior of a dipolar Bose-Einstein condensate confined in a tube, Phys. Rev. A 99, 041601(R) (2019).

[33] F. Cinti, P. Jain, M. Boninsegni, A. Micheli, P. Zoller, and G. Pupillo, Supersolid droplet crystal in a dipole-blockaded gas, Phys. Rev. Lett. 105, 135301 (2010).

[34] N. Henkel, R. Nath, and T. Pohl, Three-dimensional roton excitations and supersonid formation in Rydberg-excited Bose-Einstein condensates, Phys. Rev. Lett. 104, 195302 (2010).

[35] M. Boninsegni, A. B. Kuklov, L. Pollet, N. V. Prokof’ev, S. Saccani, S. Moroni, and M. Boninsegni, Phase diagram of soft-core bosons in two dimensions, Phys. Rev. B 83, 092506 (2011).

[36] J.L. Kurbakov, Yu.E. Lozovik, G.E. Astrakharchik, and J. Boronat, Quasiequilibrium supersolid phase of a two-dimensional dipolar crystal, Phys. Rev. B 82, 014508 (2010).

[37] Yu.E. Lozovik and V.I. Yudson, Feasibility of superfluidity of paired spatially separated electrons and holes; a new superconductivity mechanism, Sov. Phys. JETP Lett. 22, 274 (1975).

[38] S.I. Shevchenko, Phase diagram of systems with pairing of spatially separated electrons and holes, Phys. Rev. Lett. 72, 3242 (1994).

[39] J.P. Eisenstein and A.H. MacDonald, Bose-Einstein condensation of excitons in bilayer electron systems, Nature (London) 432, 691 (2004).

[40] A.V. Gorbunov and V.B. Timofeev, Large-scale coherence of the Bose condensate of spatially indirect excitons, JETP Lett. 84, 329 (2006).

[41] D.W. Snoke, Coherence and optical emission from bilayer exciton condensates, Adv. Cond. Matt. Phys. 2011, 938609 (2010).

[42] A.A. High, J.R. Leonard, A.T. Hammack, M.M. Fogler, L.V. Butov, A.V. Kavokin, K.L. Campman, and A.C. Gossard, Spontaneous coherence in a cold exciton gas, Nature (London) 483, 584 (2012).

[43] Y. Shilo, K. Cohen, B. Laiikhtman, K. West, L. Pfeiffer, and R. Rapaport, Particle correlations and evidence for dark state condensation in a cold dipolar exciton fluid, Nat. Commun. 4, 2335 (2013).

[44] N.S. Voronova, I.L. Kurbakov, and Yu.E. Lozovik, Bose condensation of long-living direct excitons in an off-resonant cavity, Phys. Rev. Lett. 121, 235702 (2018).

[45] K.F. Mak and J. Shan, Photonics and optoelectronics of 2D semiconductor transition metal dichalcogenides, Nat. Photonics 10, 216 (2016).

[46] A.K. Geim and I.V. Grigorieva, Van der Waals heterostructures, Nature (London) 499, 419 (2013).

[47] M.M. Fogler, L.V. Butov, and K.S. Novoselov, High-temperature superfluidity with indirect excitons in van der Waals heterostructures, Nat. Commun. 5, 4555 (2014).

[48] P. Rivera, J.R. Schaibley, A.M. Jones, J.S. Ross, S. Wu, G. Avizian, P. Klement, K. Seyler, G. Clark, N.J. Ghimire, J. Yan, D.G. Mandrus, W. Yao, and X. Xu, Observation of long-lived interlayer excitons in monolayer MoSe2-WS2 heterostructures, Nat. Commun. 6, 6242 (2015).

[49] E.V. Calman, C.J. Dorow, M.M. Fogler, L.V. Butov, S. Hu, A. Mishchenko, and A.K. Geim, Control of excitons in multi-layer van der Waals heterostructures, Appl. Phys. Lett. 108, 101901 (2016).

[50] B. Miller, A. Steinhoff, B. Pano, F. Jahnke, A. Holleitner, and U. Wurstbauer, Long-lived direct and indirect interlayer excitons in van der Waals heterostructures, Nano Lett. 17, 5229 (2017).

[51] E.V. Calman, M.M. Fogler, L.V. Butov, S. Hu, A. Mishchenko, and A.K. Geim, Indirect excitons in van der Waals heterostructures at room temperature, Nat. Commun. 9, 1895 (2018).

[52] A.K. Fedorov, I.L. Kurbakov, and Yu.E. Lozovik, Roton-maxon spectrum and instability for weakly interacting dipolar excitons in a semiconductor layer, Phys. Rev. B 90, 165430 (2014).

[53] O. Cotlet, D.S. Wild, M.D. Lukin, and A. Imamoglu, Rotons in optical excitation spectra of monolayer semiconductors, Phys. Rev. B 101, 205409 (2020).

[54] S.G. Bhongale, L. Mathey, E. Zhao, S.F. Yelin, and M. Lemoshko, Quantum phases of quadrupolar Fermi gases in optical lattices, Phys. Rev. Lett. 110, 155301 (2013).

[55] M. Lahrz, M. Lemoshko, K. Sengstock, C. Becker, and L. Mathey, Detecting quadruple interactions in ultracold Fermi gases, Phys. Rev. A 89, 043616 (2014).

[56] M. Lahrz, M. Lemoshko, and L. Mathey, Exotic roton excitations in quadrupolar Bose-Einstein condensates New J. Phys. 17, 045005 (2014).

[57] J. Herbig, T. Kraemer, M. Mark, T. Weber, C. Chin, H.-C. Nägerl, and R. Grimm, Bose-Einstein condensation of Cesium, Science 12, 31 (2003).

[58] S. Stellmer, B. Pasquiou, R. Grimm, and F. Schreck, Creation of ultracold Sr2 molecules in the electronic ground state, Phys. Rev. Lett. 109, 115302 (2012).

[59] G. Reinaudi, C. B. Osborn, M. McDonald, S. Kotochigova, and T. Zelevinsky, Optical production of stable ultracold 88Sr2 molecules, Phys. Rev. Lett. 109, 115302 (2012).
[62] Y. Slobodkin, Y. Mazuz-Harpaz, S. Refaely-Abramson, S. Gazit, H. Steinberg, R. Rapaport, Quantum phase transitions of tri-layer excitons in atomically thin heterostructures, arXiv.org:2004.06687.

[63] M. Ruderman, Matter in superstrong magnetic fields: the surface of a neutron star, Phys. Rev. Lett. 27, 1306 (1971).

[64] L. Reatto and G.V. Chester, Phonons and the properties of a Bose system, Phys. Rev. 155, 88 (1967).

[65] J. Boronat and J. Casulleras, Monte Carlo analysis of an interatomic potential for He, Phys. Rev. B 49, 8920 (1994).

[66] The macroscopic value of the Lindemann ratio is obtained in a linear extrapolation in $1/\sqrt{N}$ according to

$$\lim_{N \to \infty} \gamma(N) \approx \frac{[\sqrt{N_1} \gamma(N_1) - \sqrt{N_2} \gamma(N_2)]}{(\sqrt{N_1} - \sqrt{N_2})}$$

where we consider $N_1 = 34^2 = 1156$ and $N_2 = 22^2 = 484$ particles. A similar formula is used for taking the macroscopic limit of the energy per particle $E_0(N)/N$.

[67] We use quantum hydrodynamics for extrapolating the non-condensate fraction $n_0/n$ in the macroscopic limit, see details in Ref. [68].

[68] Yu.E. Lozovik, I.L. Kurbakov, G.E. Astrakharchik, and J. Boronat, Universal relation between the static structure factor and the condensate fraction, arXiv.org:2007.10050.

[69] L. Xing, Monte Carlo simulations of a two-dimensional hard-disk boson system, Phys. Rev. B 42, 8426 (1990).

[70] A.F. Andreev and I.M. Lifshitz, Quantum theory of defects in crystals, Sov. Phys. JETP 29, 1107 (1969).

[71] M. Boninsegni, A. B. Kuklov, L. Pollet, N. V. Prokof’ev, B. V. Svistunov, and M. Troyer, Fate of vacancy-induced supersolidity in $^4$He, Phys. Rev. Lett. 97, 080401 (2006).

[72] Y. Lutsyshyn, C. Cazorla, G. E. Astrakharchik, and J. Boronat, Properties of vacancy formation in hcp 4He crystals at zero temperature and fixed pressure, Phys. Rev. B 82, 180506(R) (2010).

[73] Grigory E. Astrakharchik, Konstantin V. Krutitsky, Maciej Lewenstein, Ferran Mazzanti, Jordi Boronat, Optical lattices as a tool to study defect-induced superfluidity, Phys. Rev. A 96, 033606 (2017).

[74] R. Rota, J. Boronat, Temperature dependence of the vacancy formation energy in solid 4He, Crystals 8, 344 (2018).

[75] W.R. Magro and D.M. Ceperley, Ground state of two-dimensional Yukawa bosons: Applications to vortex melting, Phys. Rev. B 48, 411 (1993).

[76] W.R. Magro and D.M. Ceperley, Ground-state properties of the two-dimensional Bose coulomb liquid, Phys. Rev. Lett. 73, 826 (1994).

[77] J. Puls, M. Rabe, A. Siarkos, and F. Henneberger, Excitonic properties of ZnSe/(Zn,Mg)Se quantum wells: A model study of the tensile-strain situation”, Phys. Rev. B 57, 14749 (1998).

[78] C. Robert, R. Picard, D. Lagarde, G. Wang, J. P. Echeverry, F. Cadiz, P. Renucci, A. Høgele, T. Amand, X. Marie, I. C. Gerber, and B. Urbaszek, Excitonic properties of semiconducting monolayer and bilayer MoTe$_2$, Phys. Rev. B 94, 155425 (2016).

[79] H.-P. Komsa and A. V. Krasheninnikov, Electronic structures and optical properties of realistic transition metal dichalcogenide heterostructures from first principles, Phys. Rev. B 88, 085318 (2013).