The Barton Expansion
and the
Path Integral Approach
in Thermal Field Theory

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It has been shown how on-shell forward scattering amplitudes (the “Barton expansion”) and quantum mechanical path integral (QMPI) can both be used to compute temperature dependent effects in thermal field theory. We demonstrate the equivalence of these two approaches and then apply the QMPI to compute the high temperature expansion for the four-point function in QED, obtaining results consistent with those previously obtained from the Barton expansion.

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I. INTRODUCTION

Temperature dependent effects can be computed in thermal field theory in a variety of ways. In the real-time formalism, directly resorting to the Feynman diagram approach results in awkward sums over so-called Matsubara frequencies [1]. It is possible to perform
an analytic continuation in the time variable in this formalism to imaginary time, and to then convert the sum over Matsubara frequencies into a contour integral. The temperature dependent effects in one-loop processes are thereby related to an on-shell forward scattering amplitude weighted by a statistical factor. This is the so-called “Barton expansion” \[2,3\]. An alternate approach involves regulating the one-loop generating functional using the zeta function \[4,5\] and then computing matrix elements of the form \( < x | e^{-Ht} | y > \) that arise in this procedure by a quantum mechanical path integral (QMPI). The QMPI is over a space compactified in the temporal direction; in place of a sum over Matsubara frequencies, a sum over winding numbers is encountered \[6\].

These two approaches to computing radiative effects in thermal field theory (the Barton expansion and the QMPI) are apparently quite disparate and there is no obvious connection between the two techniques. In this paper, we demonstrate that in fact they are equivalent. A calculation of the one-loop amplitude with four external photons has been computed in thermal QED using the Barton expansion \[7\]; in this paper we reconsider this computation using the QMPI and demonstrate that the results of these approaches are compatible.

II. THE BARTON EXPANSION AND THE QMPI

Let us initially consider a model in which the term in the action bilinear in the quantum field \( \Phi \) is of the form.

\[
L^{(2)} = -\Phi \left[ \frac{1}{2} (p - A)^2 + V \right] \Phi, \tag{2.1}
\]

where \( p = -i \partial \) and \( A_\mu \) and \( V \) are functions of the external fields. The one-loop generating functional is hence given by

\[
\Gamma^{(1)} [A_\mu, V] = \log \text{sdet}^{-1/2} \left[ \frac{1}{2} (p - A)^2 + V \right]. \tag{2.2}
\]

In Euclidean space, a representation of \( \Gamma^{(1)} \) is provided by Schwinger’s proper-time integral \[8\].
\[ \Gamma^{(1)}[A_\mu, V] = \frac{1}{2} \int_0^\infty dt \text{Str} e^{-Ht} \]  

\( H \equiv \frac{1}{2} (p - A)^2 + V \), or, upon regulating \( \Gamma^{(1)} \) using the \( \zeta \)-function \[ \prod \frac{1}{2} \Gamma(s) \int_0^\infty dt t^{s-1} \text{Str} e^{-Ht} \]  

\[ \equiv \frac{1}{2} \zeta'(0). \]  

An expansion due to Schwinger \[ \prod \] can be used in (2.3) or (2.4) to extract contributions to \( \Gamma^{(1)} \) to any order in \( A_\mu \) and \( V \), thereby allowing us to compare one-loop Green’s functions. The Str operation, if performed in momentum space, results in a loop-momentum integral while integrals over the parameters \( \alpha_i \) are direct analogues of Feynman parameters integrals. In fact, from (2.3), the expression for the \( N \)-point function is identical to that obtained from the Feynman diagram approach once the denominators from individual propagators are combined. However, we note that the regulated expression for \( N \)-point function obtained by using the Schwinger expansion of (2.3) in conjunction with the regulated one-loop generating functional of (2.4) cannot be obtained by evaluating Feynman diagrams using an action that has been regulated in some way; the functional determinant in (2.2) is what has been regulated and hence the designation “operator regularization” is used in \[ \prod \].

In thermal field theory, fields are periodic in Euclidean time if they are Bosonic and antiperiodic if they are Fermionic; hence time-like momentum variables in these two cases are of the form \( n \,(2\pi/\beta) \) and \( (n + 1/2) \,(2\pi/\beta) \) respectively \[ \prod \]. A sum over \( n \) (the “Matsubara sum”) replaces the usual integral over the time-like component of the loop-momentum. It is possible to convert this sum into a contour integral \[ \prod \] and then to extract the temperature dependence \( (T \equiv 1/\beta) \) by considering a forward on-shell elastic scattering amplitude with an appropriate statistical weighting factor. To illustrate this, let us consider the particular
case where in (2.1), \( A_\mu = 0 \) and \( V = \frac{1}{2} F(x) \), a background scalar field in \( D \) dimensions.

From (2.3) then,

\[
\Gamma^{(1)}[F] = \frac{1}{2} \int_0^\infty dt \text{tr} e^{-\frac{1}{2}(p^2 + F)t},
\]

which to second order in \( F \) gives, using (2.3)

\[
\simeq \frac{1}{2} \int_0^\infty dt \text{tr} (-t)^2 \int_0^1 du e^{-(1-u)p^2t} F e^{-up^2t} F.
\]

Computing the functional trace in (2.7) in momentum space as in [8,5] we find after integrating over \( t \) that

\[
\simeq \frac{1}{2} \int d^Dp \int d^Dq \left( \frac{2\pi}{D} \right) \int_0^1 du \frac{\tilde{F}(p) \tilde{F}(-p)}{u(1-u)(p+q)^2 + uq^2}.
\]

In (2.8) we see that upon integrating over \( u \) the Feynman integral for the contributions of the two-point functions to the one-loop generating functional is recovered,

\[
\simeq \frac{1}{2} \int d^Dp \tilde{F}(p) \tilde{F}(-p) \int \frac{d^Dq}{(2\pi)^D q^2} \frac{1}{(q+p)^2}.
\]

In thermal field theory this becomes

\[
\simeq \frac{1}{2} \int d^{D-1}p \sum_m \tilde{F}(\vec{p}, \frac{2\pi m}{\beta}) \tilde{F}(-\vec{p}, -\frac{2\pi m}{\beta}) \left( \frac{1}{\beta} \sum_n \int \frac{d^{D-1}q}{(2\pi)^{D-1}}} \frac{1}{q^2 + \left( \frac{2\pi m}{\beta} \right)^2 + \left( \frac{2\pi m}{\beta} (n + m) \right)^2} \right).
\]

The sum over \( n \) in (2.10) can be converted into a contour integral using the formula [3]

\[
\frac{2\pi i}{\beta} \sum_{n=-\infty}^{n=\infty} f \left( \frac{2\pi in}{\beta} \right) = \frac{1}{2} \int_{-\infty}^{\infty} dz \left[ f(iz) + f(-iz) \right] + \int_{-\infty+\delta}^{\infty+\delta} dz \left[ f(iz) + f(-iz) \right] e^{i\beta z} - 1,
\]

provided \( f(iz) \) has no singularities along the imaginary \( z \) axis. All the temperature dependence in (2.11) resides in the second term; hence we find that the temperature dependence in (2.10) comes from
\[ \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{1}{2\pi i} \int_{-\infty+\delta}^{\infty+\delta} N(z) \left[ \frac{1}{|q^2-z^2|} - \frac{1}{(p+q)^2-(p_0+z)^2} \right] \]  

\[ (N(z) \equiv (e^{\beta z} - 1)^{-1}) \]  

Performing a partial fraction expansion in (2.12) we obtain
\[ \int d^{D-1}q \frac{1}{2\pi i} \int_{-\infty+\delta}^{\infty+\delta} dz \ N(z) \left[ \frac{1}{2|q|} \left( \frac{1}{z-|q|} - \frac{1}{z+|q|} \right) \right] \]

\[ \left[ \frac{1}{2|\vec{p}+q|} \left( \frac{1}{z+p_0-|\vec{p}+q|} - \frac{1}{z+p_0+|\vec{p}+q|} \right) \right] \]

Since \( p_0 = 2\pi im/\beta \), we see that \( N(z) = N(z+p_0) \). This permits us to close the contour integral in \( z \) on the right half plane, and then to use the residue theorem to obtain
\[ \int d^{D-1}q \frac{N(|\vec{q}|)}{2|\vec{q}|} \left[ \frac{1}{(q+p)^2} + \frac{1}{(q-p)^2} \right] \bigg|_{q_0=|\vec{q}|} \]  

(2.14)

This is the so-called “Barton amplitude” \[2,3\]; viz. an expression the on-shell forward scattering amplitude with two external fields, weighted by the statistical factor \( N(|\vec{q}|) \). This can be generalized to handle more external legs, vertices with momentum dependence, and internal particles with Fermi statistics \[3\]. In particular, the non-linear four-point interaction of photons has been analysed using this approach \[4\].

An alternate procedure for examining thermal Green’s functions is to use the QMPI. In this approach, we use the fact that a path integral can be used to represent the matrix element in (2.3) and (2.4),
\[
\langle x | \exp \left[ \frac{1}{2} (p-A)^2 + V \right] t | y \rangle = 
\]

\[ \mathcal{P} \int D\vec{q}(\tau) \exp \left\{ \int_0^t d\tau \left[ -\frac{1}{2}q^2(\tau) + iA(\vec{q}(\tau)) \cdot \vec{q}(\tau) - V(\vec{q}(\tau)) \right] \right\} , \]  

(2.15)

(\( \mathcal{P} \) is the path ordering)

where “per” means invariance under the replacement \( q_0(\tau) \to q_0(\tau) + \beta \).

As is discussed in \[10,13\] (2.15) can be rewritten as
\[ \sum_{n=-\infty}^{n=\infty} e^{i\delta x} \mathcal{P} \int D\vec{q}(\tau) \exp \left\{ \int_0^t d\tau \left[ -\frac{1}{2}q^2(\tau) + iA(\vec{q}(\tau)) \cdot \vec{q}(\tau) - V(\vec{q}(\tau)) \right] \right\} , \]

(2.16)
with \( \delta_n = 0 \) for Bosons and \( \delta_n = n\pi \) for Fermions.

We will restrict ourselves to examining (2.16) to zeroth order in \( V \) and second order in \( A_\mu \) (higher orders can be treated in a similar fashion). If

\[
(2\pi)^{D-1} \sqrt{\beta} A_\mu(q(\tau_i)) = \epsilon_{i\mu} \exp - (ik_i \cdot q(\tau_i)),
\]

then we have

\[
\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \sum_{n=-\infty}^{\infty} e^{i\delta_n} \int Dq(\tau) \exp \left\{ \int_0^t d\tau \left[ -\frac{1}{2} q^2(\tau) - \gamma(\tau) \cdot q(\tau) \right] \right\},
\]

(2.18)

where \( \gamma(\tau) = i \sum_{i=1}^2 \left[ \delta(\tau - \tau_i) k_i + \delta(\tau - \tau_i) \epsilon_i \right] \) provided we keep only terms linear in \( \epsilon_i \).

Evaluating this path integral and extracting a high temperature expansion for the two-point function from it is outlined in [11]. However, it is not apparent how the results one obtains from this approach are at all related to what is extracted from the Schwinger expansion of (2.5) in conjunction with (2.3), and hence the Barton expansion. Indeed, the Schwinger expansion of the left side of (2.15) to second order in \( A \) is

\[
I_{xy} = \langle x | \int_0^1 du e^{-\frac{1}{2}(1-u)p^2t} \left( -\frac{tA^2}{2} \right) e^{-\frac{1}{2}up^2t} + \int_0^1 du u e^{-\frac{1}{2}(1-u)p^2t} \left( \frac{1}{2} \right) (p \cdot A + A \cdot p) t e^{-\frac{1}{2}up(1-v)p^2t} \times \left( \frac{1}{2} \right) (p \cdot A + A \cdot p) t e^{-\frac{1}{2}wp^2t} \right] y >;
\]

(2.19)

we can also expand (2.16) to second order in \( A \) to obtain

\[
J_{xy} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \sum_{n=-\infty}^{\infty} e^{i\delta_n} \int Dq(\tau) \exp \int_0^t d\tau \left[ -\frac{1}{2} \dot{q}^2(\tau) \right] \left[ i \dot{q}(\tau_1) \cdot A(q(\tau_1)) \right] \left[ i \dot{q}(\tau_2) \cdot A(q(\tau_2)) \right].
\]

(2.20)

In order to demonstrate the equivalence of \( I_{xy} \) and \( J_{xy} \), we split up the path integral in (2.20) so that
is valid in the limit \( \epsilon_i \to 0 \).

The path integrals in (2.21) can be evaluated using the result

\[
J_{xy} = \int_0^1 d\tau_1 \int_0^{\tau_2} d\tau_2 \sum_{n=-\infty}^{\infty} e^{i\delta n} \int d\Delta b_1 \int dD b_2 \int Dq(\tau) \int Dq(\tau) \int Dq(\tau) \int Dq(\tau)
\]

\[
\times e^{-\frac{1}{2} \left[ \int_0^{\tau_2} d\tau + \int f_{\tau_1}^1 d\tau + \int f_{\tau_1}^1 d\tau \right] q^2(\tau)} \left[ i \left( \frac{q(\tau_1 + \epsilon_1) - q(\tau_1 - \epsilon_1)}{2\epsilon_1} \right) \cdot A(b_1) \right] \cdot \left( \frac{q(\tau_2 + \epsilon_2) - q(\tau_2 - \epsilon_2)}{2\epsilon_2} \right) \cdot A(b_2)
\]

In (2.21) we have replaced the time derivative \( \dot{q}(\tau_1) \) by \( (q(\tau_1 + \epsilon_1) - q(\tau_1 - \epsilon_1)) / 2\epsilon_1 \), which is valid in the limit \( \epsilon_i \to 0 \).

The path integrals in (2.21) can be evaluated using the result

\[
\int Dq(\tau) \exp \int_{\tau_2}^{\tau_1} d\tau \left( -\frac{1}{2} \dot{q}^2(\tau) - \gamma(\tau) \cdot q(\tau) \right)
\]

\[
= \frac{1}{[2\pi (\tau_1 - \tau_2)]^{2\pi}} \exp \left\{ -\frac{(b_1 - b_2)^2}{2(\tau_1 - \tau_2)} \int_{\tau_2}^{\tau_1} d\tau \left[ b_1 (\tau - \tau_2) + b_2 (\tau_1 - \tau) \right] \cdot \gamma(\tau) \right\}
\]

where

\[
G(\tau, \tau'; \tau_2, \tau_1) \equiv \frac{1}{2} \left[ \tau - \tau' - \frac{\tau_1 + \tau_2}{2(\tau_1 - \tau_2)} (\tau + \tau') + \frac{\tau \tau'}{\tau_1 - \tau_2} + \frac{\tau_1 \tau_2}{\tau_1 - \tau_2} \right].
\]

By systematically using (2.22), (2.21) becomes

\[
J_{xy} = -\int_0^1 d\tau_1 \int_0^{\tau_2} d\tau_2 \int d\Delta b_1 db_2 \sum_n e^{i\delta n} \left[ \frac{1}{2\pi (t - \tau_1)} \frac{1}{2\pi (\tau_1 - \tau_2)} \frac{1}{2\pi (\tau_1 - \tau_2)} \right] D^{\mu_1} A(b_1) D^{\mu_2} (b_2)
\]

\[
\times \exp \left[ -\frac{1}{2} \left( \frac{(x^*-b_1)^2}{t-\tau_1} + \frac{(b_1 - b_2)^2}{\tau_1 - \tau_2} + \frac{(b_2 - y)^2}{\tau_2} \right) \right]
\]

\[
\times \left\{ \frac{\delta^{\mu_1\mu_2}}{4\epsilon_1 \epsilon_2} G(\tau_2 + \epsilon_2, \tau_1 - \epsilon_1; \tau_2, \tau_1) + \frac{1}{4} \left[ \frac{x^*-b_1}{t-\tau_1} + \frac{b_1 - b_2}{\tau_1 - \tau_2} \right] \frac{b_1 - b_2}{\tau_1 - \tau_2} \frac{b_2 - y}{\tau_2} \right\}
\]

(2.24)
(we have defined $x^* \equiv (x^0 + n\beta, \vec{x})$). Using the Fourier transform

$$e^{-x^2/2t} \equiv \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot x - \frac{i}{4} p^2 t},$$

we can rewrite (2.24) in the form

$$= - \int_0^t \! dt_1 \int_0^{\tau_1} \! dt_2 \int \! db_1 \, db_2 A_{\mu_1} (b_1) \, A_{\mu_2} (b_2) \int \! \frac{dp_1}{(2\pi)^D} \frac{dp_2}{(2\pi)^D} \frac{dp_3}{(2\pi)^D} e^{-\frac{i}{2} \left( p_1^2 (t - \tau_1) + p_2^2 (\tau_1 - \tau_2) + p_3^2 \tau_2 \right)}$$

$$\times \sum_n e^{i\delta_n} \left\{ \frac{\delta_{\mu_1 \mu_2}}{4\epsilon_1 \epsilon_2} G (\tau_2 + \epsilon_2, \tau_1 - \epsilon_1; \tau_2, \tau_1) + \frac{1}{4} \left[ \frac{1}{t - \tau_1} \left( i \frac{\partial}{\partial p_1} \right) + \frac{1}{\tau_1 - \tau_2} \left( i \frac{\partial}{\partial p_2} \right) \right]_{\mu_1} \right\}$$

$$\times \left[ \frac{1}{\tau_1 - \tau_2} \left( i \frac{\partial}{\partial p_2} \right) + \frac{1}{\tau_2} \left( i \frac{\partial}{\partial p_3} \right) \right]_{\mu_2} e^{- \left[ p_1 \cdot (x^* - b_1) + p_2 \cdot (b_1 - b_2) + p_3 \cdot (b_2 - y) \right]}$$

(2.26)

Integrating by parts with respect to the derivatives occurring in (2.26) leaves us with

$$= - \frac{1}{4} \int_0^t \! dt_1 \int_0^{\tau_1} \! dt_2 \int \! \frac{dp_1}{(2\pi)^D} \frac{dp_2}{(2\pi)^D} \frac{dp_3}{(2\pi)^D} \sum_n e^{i\delta_n} e^{-ip_1 \cdot x^* + i\epsilon_3 \cdot y} \int \! \frac{db_1}{(2\pi)^D} e^{i(p_1 - p_2) \cdot b_1} A_{\mu_1} (b_1)$$

$$\times \left\{ \frac{\delta_{\mu_1 \mu_2}}{\epsilon_1 \epsilon_2} \left[ G (\tau_2 + \epsilon_2, \tau_1 - \epsilon_1; \tau_2, \tau_1) + \frac{1}{\tau_1 - \tau_2} \right] \right\}$$

$$- (p_1 + p_2)_{\mu_1} (p_1 + p_3)_{\mu_2} \right\} e^{-\frac{i}{2} \left[ p_1^2 (1 - \sigma_1) + p_2^2 (\sigma_1 - \sigma_2) + p_3^2 \sigma_2 \right]}$$

(2.27)

From the definition of $G$ in (2.23), we see that

$$G (\tau_2 + \epsilon_2, \tau_1 - \epsilon_1; \tau_2, \tau_1) = \frac{1}{2} \left[ \tau_1 - \tau_2 - \epsilon_1 - \epsilon_2 \right] - \frac{1}{2} \left[ (\tau_1 - \tau_2 - \epsilon_1 - \epsilon_2) - \frac{\epsilon_1 \epsilon_2}{\tau_1 - \tau_2} \right]$$

(2.28)

So that in the limit $\epsilon_1, \epsilon_2 \to 0$, (2.27) reduces to

$$= \frac{t^2}{4} \int_0^1 \! d\sigma_1 \int_0^{\sigma_1} \! d\sigma_2 \int \! \frac{dp_1}{(2\pi)^D} \frac{dp_2}{(2\pi)^D} \frac{dp_3}{(2\pi)^D} \sum_n e^{i\delta_n} e^{-ip_1 \cdot x^* + i\epsilon_3 \cdot y} A_{\mu_1} (p_1 - p_2) \tilde{A}_{\mu_2} (p_2 - p_3)$$

$$\times \left\{ -\delta_{\mu_1 \mu_2} \frac{\delta (\sigma_1 - \sigma_2)}{t} + (p_1 + p_2)_{\mu_1} (p_2 + p_3)_{\mu_2} \right\} e^{-\frac{i}{2} \left[ p_1^2 (1 - \sigma_1) + p_2^2 (\sigma_1 - \sigma_2) + p_3^2 \sigma_2 \right]}$$

(provided $\tau_i = \sigma_i t$ and $\tilde{A}_\mu$ is the Fourier transform of $A_\mu$. The Poisson resummation formula

8
\[
\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i m} f(\mu)
\] 

(2.30)
can now be used to show that if \( \delta_n = n \pi \epsilon \) (\( \epsilon = 0 \) for Bosons and \( \epsilon = 1 \) for Fermions) then

\[
\sum_{m=-\infty}^{\infty} e^{i\delta_n} e^{-ip_1^\mu (n\beta)} = \sum_{m=-\infty}^{\infty} \frac{2\pi}{\beta} \delta \left( \frac{p_0^1 - \frac{2\pi(n+\epsilon/2)}{\beta}}{\beta} \right).
\] 

(2.31)
Substitution of (2.31) into (2.23) leads to

\[
J_{xy} = \frac{t^2}{4} \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \frac{dp_1 dp_2 dp_3}{(2\pi)^D} \sum \frac{2\pi}{\beta} \delta \left( \frac{p_0^1 - \frac{2\pi(n+\epsilon/2)}{\beta}}{\beta} \right) e^{-ip_1^x+ip_3^y} \times \tilde{A}_{\mu_1} (p_1 - p_2) \tilde{A}_{\mu_2} (p_2 - p_3) \left\{ -\delta_{\mu_1\mu_2} \frac{\delta (\sigma_1 - \sigma_2)}{t} + (p_1 + p_2)_{\mu_1} (p_2 + p_3)_{\mu_2} \right\} \times e^{-\frac{t}{\beta} [p_1^2 (1 - \sigma_1) + p_2^2 (\sigma_1 - \sigma_2) + p_3^2 \sigma_2]}
\] 

(2.32)
It is immediately apparent that (2.32) is identical to (2.13) upon insertion of a complete set of momentum states into the later, as in [8,5], provided these momentum states have Matsubara frequencies associated with their time-like component. One can easily generalize this result to show that one can obtain the same result for the matrix element \( M_{xy} = \langle x | e^{t/2(p - A)^2 / 2 + V} | y \rangle \) whether one uses the Schwinger expansion or the quantum mechanical path integral for \( T > 0 \); consequently the Barton expansion and the expansion of the quantum mechanical path integral are equivalent procedures, despite their apparent differences.

Let us now consider how the four-point function can be computed in quantum electrodynamics [12] and compare the result with that obtained in [7] where the Barton expansion was employed.

### III. The Four Point Function

The one-loop generating function for Green's functions with external photon lines in quantum electrodynamics is
\[ i\Gamma^{(1)} [A_{\mu}] = \ln \det^{1/2} \left[ \frac{1}{2} (\dot{\phi} - A)^2 + m^2 \right] \] (3.1)

If \( \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \) and \( \sigma_{\mu\nu} = i \frac{1}{2} [\gamma_\mu, \gamma_\nu] \), then

\[ (\dot{\phi} - A)^2 = (p - A) + \frac{1}{2} \sigma_{\mu\nu} F^{\mu\nu} \quad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu) \] (3.2)

so that in accordance with (2.13) we must compute

\[
\text{tr} < x | \exp \left[ -\frac{1}{2} (p - A)^2 + \frac{1}{4} \sigma_{\mu\nu} F^{\mu\nu} \right] t | y >
\]

\[
= \text{tr} \sum_{n=-\infty}^{\infty} e^{i \pi n} P \int_{q(0) = y} Dq(\tau) \exp \left\{ \int_0^t d\tau \left[ -\frac{1}{2} \hat{q}^2 (\tau) + iA (q(\tau)) \cdot \dot{q}(\tau) - \frac{1}{4} \sigma_{\alpha\beta} F^{\alpha\beta} (q(\tau)) \right] \right\}.
\] (3.3)

The term contributing to the path integral in (3.3) that is of fourth order in the background field \( A_\mu \) is

\[
\Lambda = \text{tr} \sum_{n=-\infty}^{\infty} e^{i \pi n} \int_{q(0) = y} Dq(\tau) \exp \left\{ \int_0^t d\tau \left[ \frac{1}{2} \dot{q}^2 (\tau) + \frac{1}{2} A_1 \dot{q}_1 \cdot \dot{A}_1 + \frac{1}{2} A_2 \dot{q}_2 \cdot \dot{A}_2 + \frac{1}{2} A_3 \dot{q}_3 \cdot \dot{A}_3 + \frac{1}{2} A_4 \dot{q}_4 \cdot \dot{A}_4 + \text{perm.} \right] \right\} \]

\[
+ \frac{1}{4} \left( F_{1\alpha\beta} F_{2\gamma\delta} F_{3\gamma\alpha} - F_{1\alpha\beta} F_{3\gamma\alpha} F_{2\gamma\delta} - F_{2\alpha\beta} F_{1\gamma\delta} F_{3\gamma\alpha} - F_{2\alpha\beta} F_{3\gamma\alpha} F_{1\gamma\delta} \right) \]

\[
+ \frac{1}{16} \left( F_{1\alpha\beta} F_{2\alpha\beta} F_{3\gamma\delta} F_{4\gamma\delta} - F_{1\alpha\beta} F_{2\alpha\beta} F_{4\gamma\delta} F_{3\gamma\delta} - F_{1\alpha\beta} F_{3\gamma\delta} F_{2\gamma\delta} F_{4\gamma\delta} - F_{1\alpha\beta} F_{4\gamma\delta} F_{2\gamma\delta} F_{3\gamma\delta} \right) \].
\] (3.4)

(We have let \( q_i \equiv q(\tau_i) \) and \( A_{\mu i} \equiv A_{\mu} (q(\tau_i)) \), and “perm.” refers to permutations of 1, 2, 3, 4. The trace identities

\[
\text{tr} \sigma_{\alpha\beta} = 0
\]

\[
\text{tr} \sigma_{\alpha\beta} \sigma_{\gamma\delta} F^{\alpha\beta} G_{\gamma\delta} = -8 \text{tr} (FG)
\]

\[
\text{tr} \sigma_{\alpha\beta} \sigma_{\gamma\delta} \sigma_{\lambda\rho} F^{\alpha\beta} G_{\gamma\delta} H_{\lambda\rho} = -32i \text{tr} (FGH)
\]

\[
\text{tr} \sigma_{\alpha\beta} \sigma_{\gamma\delta} \sigma_{\lambda\rho} \sigma_{\mu\nu} F^{\alpha\beta} G_{\gamma\delta} H_{\lambda\rho} I_{\mu\nu} = 64 \text{tr} (FGHI - FHI - GI - IFGH)
\]

\[
+16 \left[ \text{tr} (FG) (HI) + (FH) (GI) + (FI) (GH) \right]
\]

have also been used.) In order to evaluate the path integral in (3.4), we use the plane wave expansion of (2.17). The standard result of (2.22) and the formula

\[\text{tr} \sigma_{\alpha\beta} \sigma_{\gamma\delta} F^{\alpha\beta} G_{\gamma\delta} = -8 \text{tr} (FG)\]
\[
\sum_{n=-\infty}^{n=\infty} e^{i\pi n^2 t^2/2} = \frac{\sqrt{2\pi t}}{\beta} \sum_{n=-\infty}^{n=\infty} e^{-2\pi^2 t (n+1/2)^2/\beta^2},
\] (3.5)

together lead to the contribution of the \( \zeta \)-function to the four-point process to be

\[
\Gamma(s) \zeta^{(4)}(s) = \int_0^\infty dt^{s-1} \left( \frac{1}{(2\pi)^{3/2}} \right) \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi t}}{\beta} e^{-2\pi^2 t (n+1/2)^2/\beta^2} \times \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \left[ \left( \frac{2\pi}{t} \right)^4 \delta \left( \sum k_i \right) \exp \left( \frac{i}{2} k_i \cdot k_j \frac{G_{ij}}{t} \right) \right] \times \left\{ 4 \left[ \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 \tilde{G}_{12} \tilde{G}_{34} + \text{perm.} \right. \right. \\
+ \left( \epsilon_1 \cdot \epsilon_2 \tilde{G}_{12} \right) \left( k_i \tilde{G}_{i3} \epsilon_3 \right) \left( k_j \tilde{G}_{j4} \epsilon_4 \right) + \text{perm.} \right. \\
+ \left( k_i \tilde{G}_{i4} \epsilon_1 \right) \left( k_j \tilde{G}_{j2} \epsilon_2 \right) \left( k_m \tilde{G}_{m4} \epsilon_4 \right) \\
+ \frac{1}{2} \left[ \text{tr} \left( f_1 f_2 f_3 \right) \left( k_i \tilde{G}_{i4} \epsilon_4 \right) + \text{perm.} \right] \\
+ \frac{1}{4} \left[ \text{tr} \left( f_1 f_2 f_3 f_4 - f_1 f_3 f_4 f_2 - f_1 f_4 f_2 - f_1 f_4 \right) \text{tr} \left( f_2 f_3 \right) \right] \\
\left. \right\} \right. \\
\left. \right. \\
\left. \right. \left. \right. \\
\left. \right. \\
\text{where } f_{i\alpha\beta} \equiv -i \left( k_{i\alpha} \epsilon_{\beta} - k_{i\beta} \epsilon_{\alpha} \right), \ G_{ij} = \frac{1}{2} \left( \sigma_i - \sigma_j \right) - \frac{1}{2} \left( \tau_i \pm \tau_j \right) + \tau_i \tau_j / t, \ G_{ij} = \frac{d^2}{d\tau_i d\tau_j} G_{ij} \right. \text{ and } \\
\tilde{G}_{ij} = \frac{d^2}{d\tau_i d\tau_j} G_{ij}. \right. \text{ The rescaling } \tau_i = \sigma_i t \text{ allows us to integrate over } t; \text{ we can then sum over } \\
n \text{ using } [3,4] \\
\sum_{n=-\infty}^{\infty} \left[ \nu^2 + \left( n + \frac{\theta}{2\pi} \right)^2 \right]^{2s} = \frac{\sqrt{\pi}}{\Gamma(s)} \left[ \frac{\Gamma\left( s - \frac{1}{2} \right)}{\nu^{2s-1}} + 4 \sum_{m=1}^{\infty} \cos(m\theta) \left( \frac{\nu m}{\nu} \right)^{s-\frac{1}{2}} \right] \\
\times K_{s-\frac{1}{2}}(2\pi \nu m),
\] (3.7)

so that we are dealing with expressions of the form

\[
\sum_{n=-\infty}^{\infty} \Gamma \left( s + \frac{3}{2} + \epsilon \right) \left[ \frac{\Lambda^2}{2} + \frac{2\pi^2}{\beta^2} \left( n + \frac{1}{2} \right) \right]^{-\left( s + \frac{3}{2} + \epsilon \right)} \\
= \left( \frac{2\pi^2}{\beta^2} \right)^{-(s+\frac{1}{2}+\epsilon)} \sqrt{\pi} \left\{ \Gamma \left( s + 1 + \epsilon \right) \left( \frac{\beta \Lambda}{2\pi} \right)^2 s + 2 + 2 \epsilon \right. + 4 \sum_{m=1}^{\infty} (-1)^m \left( \frac{m\pi}{\beta \Lambda / 2\pi} \right)^{s+1+\epsilon} \\
K_{s+1+\epsilon} \left( m\beta \Lambda \right) \right\},
\] (3.8)

(Here we have \( \epsilon = -1, \ 0, \ 1 \) and \( \Lambda^2 = -k_i g_{ij} k_j, \) with \( g_{ij} \equiv \frac{1}{4} G(\sigma_i t, \sigma_j t). \) Using the integral representation [3]

\[
K_a(z) = \int_0^\infty dt \cosh(at) e^{-z \cosh(t)},
\] (3.9)
allows us to evaluate the sum over \( m \) in (3.8). The Mellin formula \([14]\)

\[
\frac{1}{e^\lambda + 1} = \int_{C} \frac{dz}{2\pi i} \zeta_+ (z) \Gamma (z) \lambda^{-z}
\]

\[
\zeta_+ (z) \equiv \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = (1 - 2^{1-z}) \zeta (z) ; \text{C is the contour} \) Re \( z > 1 - 2\epsilon \)
\]

(3.10)
can then be employed; by using it we can integrate over the parameter \( t \) used in (3.9) since \([15]\)

\[
\int_{0}^{\infty} \frac{dt}{(\cosh t)^a} = \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{a}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)}.
\]

(3.11)

A final simplification involves using the formula \([15]\)

\[
\Gamma (2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma (z) \Gamma (z + \frac{1}{2});
\]

(3.12)
at last we get

\[
\frac{d}{ds} \bigg|_{s=0} \frac{\Gamma (s + 3/2 + \epsilon)}{\Gamma (s)} \sum_{n=-\infty}^{\infty} \left[ \frac{\Lambda^2}{2} + \frac{2\pi^2}{\beta^2} (n + 1/2)^2 \right]^{-(s+3/2+\epsilon)}
\]

\[
= \begin{cases} 
\frac{(2\pi^2)}{(\beta^2)} \sqrt{\pi} \left[ \frac{(2\pi)^2}{(\beta\Lambda)^4} + 4 \left( \frac{2\pi^2}{\beta\Lambda} \right)^2 \int_{C} \frac{dz}{2\pi i} \zeta_+ (z) 2^{z-2} (z+2) \Gamma^2 \left( \frac{z}{2} \right) (-\beta\Lambda)^{-z-2} \right] & (\epsilon = 1) \\
\frac{(2\pi^2)}{(\beta^2)} \sqrt{\pi} \left[ \frac{(2\pi)^2}{(\beta\Lambda)^4} + 4 \left( \frac{-2\pi^2}{\beta\Lambda} \right)^2 \int_{C} \frac{dz}{2\pi i} \zeta_+ (z) 2^{z-2} z\Gamma^2 \left( \frac{z}{2} \right) (-\beta\Lambda)^{-z-1} \right] & (\epsilon = 0) \\
\frac{(2\pi^2)}{(\beta^2)} \sqrt{\pi} \left[ \ln \left( \frac{2}{\Lambda^2} \right) - 4 \int_{C} \frac{dz}{2\pi i} \zeta_+ (z) 2^{z-2} \Gamma^2 \left( \frac{z}{2} \right) (-\beta\Lambda)^{-z} \right] & (\epsilon = -1)
\end{cases}
\]

(3.13)

With this result we see that (3.6) becomes
\[ \zeta^{(4)}(0) = \sqrt{\pi} \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3 \int_0^{\sigma_3} d\sigma_4 \frac{\delta (\sum k_i)}{(2\pi)^{7/2} \beta^3} \times \left\{ \left( \frac{2\pi^2}{\beta^2} \right)^{-1/2} \ln \left( \frac{2}{\Lambda^2} \right) - 4 \int_C \frac{dz}{2\pi i} \zeta_+(z) 2^{z-2} \Gamma(\frac{z}{2})(\beta \Lambda)^{-z} \right\} \]

\[ + \left( \frac{2\pi^2}{\beta^2} \right)^{3/2} \left[ \left( \frac{2\pi}{\beta \Lambda} \right)^4 + 4 \left( \frac{2\pi^2}{\beta \Lambda} \right)^2 \int_C \frac{dz}{2\pi i} \zeta_+(z) 2^{z-2} (z+2) \Gamma^2 \left( \frac{z}{2} \right) (-\beta \Lambda)^{-z-2} \right] \]

\[ + \left( \frac{2\pi^2}{\beta^2} \right)^{-\frac{5}{2}} \sqrt{\pi} \left[ \sum_{i} k_i \epsilon_1 \left( -\frac{1}{2} \epsilon (\sigma_i - \sigma_1) - \frac{1}{2} + \sigma_i \right) \cdots \sum_{j} k_j \epsilon_4 \left( -\frac{1}{2} \epsilon (\sigma_j - \sigma_4) - \frac{1}{2} + \sigma_j \right) + \text{perm.} \right] \]

\[ + \frac{1}{2} \text{tr} f_1 f_2 \sum_{i} k_i \epsilon_3 \left( -\frac{1}{2} \epsilon (\sigma_i - \sigma_3) - \frac{1}{2} + \sigma_i \right) \sum_{j} k_j \epsilon_4 \left( -\frac{1}{2} \epsilon (\sigma_j - \sigma_4) - \frac{1}{2} + \sigma_j \right) + \text{perm.} \]

\[ + \frac{1}{4} \text{tr} \left( f_1 f_2 f_3 \sum_{j} k_j \epsilon_4 \right) \left( -\frac{1}{2} \epsilon (\sigma_j - \sigma_4) - \frac{1}{2} + \sigma_j \right) + \text{perm.} \]

\[ + \frac{1}{16} \left( \text{tr} f_1 f_2 \text{tr} f_3 f_4 + \text{tr} f_1 f_3 \text{tr} f_2 f_4 + \text{tr} f_1 f_4 \text{tr} f_2 f_3 \right) \}

(3.14)

The integrals over \( \sigma_i \) eliminates the first term in (3.14). Closing the contour \( C \) with a semicircle at infinity on the left side of the complex plane allows us to evaluate the integrals over \( z \) in (3.14) by use of the residue theorem. The sum over the contributions of these residues gives rise to a power series in \( T^{-1} = \beta \), starting at \( T^2 \). This is consistent with (3) once we recall the dependence on \( \beta \) in the normalization of the wave function in (2.17). There is no dependence on \( \ln \beta \) since \( z \zeta_+ (z) \Gamma^2 (z/2) \) has only a single pole at \( z = 0, -2, -4, -6 \cdots \); this is consistent with (3) as well.

In [11] it is demonstrated that the vacuum polarization in thermal quantum electrodynamics is transverse; it is anticipated that one can similarly show that the four-point function in (3.14) is also gauge invariant.
IV. DISCUSSION

In this paper we have demonstrated how thermal effects in field theory can be computed perturbatively in the loop expansion by using either a so-called “Barton Expansion” or the quantum mechanical path integral. The two approaches are significantly different in their technical details; nevertheless we have demonstrated that the results obtained in [7] for the four point function at one loop order in thermal QED using the “Barton expansion” are consistent with what can be obtained using QMPI.

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