Duality Twisted Reductions of Double Field Theory of Type II Strings

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Abstract

We study duality twisted reductions of the Double Field Theory (DFT) of the RR sector of massless Type II theory, with twists belonging to the duality group $Spin^+(10,10)$. We determine the action and the gauge algebra of the resulting theory and determine the conditions for consistency. In doing this, we work with the DFT action constructed by Hohm, Kwak and Zwiebach, which we rewrite in terms of the Mukai pairing: a natural bilinear form on the space of spinors, which is manifestly $Spin(n,n)$ invariant. If the duality twist is introduced via the $Spin^+(10,10)$ element $S$ in the RR sector, then the NS-NS sector should also be deformed via the duality twist $U = \rho(S)$, where $\rho$ is the double covering homomorphism between $Pin(n,n)$ and $O(n,n)$. We show that the set of conditions required for the consistency of the reduction of the NS-NS sector are also crucial for the consistency of the reduction of the RR sector, owing to the fact that the Lie algebras of $Spin(n,n)$ and $SO(n,n)$ are isomorphic. In addition, requirement of gauge invariance imposes an extra constraint on the fluxes that determine the deformations.
1 Introduction

Double Field Theory (DFT) is a field theory defined on a doubled space, where the usual coordinates conjugate to momentum modes are supplemented with dual coordinates that are conjugate to winding modes [1–4]. DFT was originally constructed on a doubled torus, with the aim of constructing a manifestly T-duality invariant theory describing the massless excitations of closed string theory [1, 2]. Later, this action was shown to be background independent [3], allowing for more general doubled spaces than the doubled torus. Obviously, the dual coordinates might not have the interpretation of being conjugate to winding modes on such general spaces. Construction of DFT builds on earlier work, see [5–13]. For reviews of DFT, see [14–17].

On a general doubled space of dimension 2n, the DFT action has a manifest $O(n, n)$ symmetry, under which the standard coordinates combined with the dual ones transform linearly as a vector. The doubled coordinates must satisfy a set of constraints, called the weak and the strong constraint and the theory is consistent only in those frames in which these constraints are satisfied. It is an important challenge to relax these constraints, especially the strong one, as in any such frame the DFT becomes a rewriting of standard supergravity, related to it by an $O(n, n)$ transformation. Even in this case, DFT has the virtue of exhibiting already in ten
dimensions (part of) the hidden symmetries of supergravity, that would only appear upon dimensional reduction in its standard formulation. This virtue should not to be underestimated, as it provides the possibility of implementing duality twisted reductions of ten dimensional supergravity with duality twists belonging to a larger symmetry group, that would normally be available only in lower dimensions.

Duality twisted reductions (or generalized Scherk-Schwarz reductions) are a generalization of Kaluza-Klein reductions, which introduces into the reduced theory mass terms for various fields, a non-Abelian gauge symmetry and generates a scalar potential for the scalar fields \[\text{(18)}\]. This is possible if the parent theory has a global symmetry \(G\), and the reduction anzats for the fields in the theory is determined according to how they transform under \(G\). It is natural to study duality twisted reductions of DFT, as it comes equipped with the large duality group \(O(n, n)\), and indeed this line of work has been pursued by many groups so far \[\text{(19,23)}\]. In \[\text{(19,20)}\] it was shown that the duality twisted reductions of DFT gives in 4 dimensions the electric bosonic sector of gauged \(\mathcal{N} = 4\) supergravity \[\text{(24)}\]. A curious fact which was noted in these works was that the weak and the strong constraint was never needed to be imposed on the doubled internal space. This (partial) relaxation of the strong constraint made the twisted reductions of DFT even more attractive. Later, in \[\text{(21)}\], this was made more explicit, as they showed that the set of conditions to be satisfied for the consistency of the twisted reduction are in one-to-one correspondence with the constraints of gauged supergravity, constituting a weaker set of constraints compared to the strong constraint of DFT. Following this, in \[\text{(25)}\], it was shown that the weakening of the strong constraint in the twisted reductions of DFT implies that even non-geometric gaugings of half-maximal supergravity (meaning that they cannot be T-dualized to gauged supergravities arising from conventional compactifications of ten-dimensional supergravity) has an uplift to DFT. Such non-geometric gaugings also arise from compactifications of string theory with non-geometric flux (see, for example \[\text{(26,28)}\]) and the relation of such compactifications with twisted compactifications of DFT was explored in various papers, including \[\text{(20,32)}\]. We should also note that, the results of \[\text{(21)}\] was also obtained by \[\text{(33)}\], by considering the duality twisted reductions of the DFT action they constructed in terms of a torsionful, flat generalized connection, called the Weitzenböck connection\[^1\].

In all of the works cited above, only the reduction of the DFT of the NS-NS sector of massless string theory was studied\[^2\]. The fundamental fields in this sector are the generalized metric (comprising of the Riemannian metric and the B-field) and the generalized dilaton. In a frame in which there is no dependence on the dual coordinates, this sector becomes the NS-NS sector of string theory. We will hereafter refer to this frame as the ”supergravity frame”. On the other hand, the DFT of the RR sector of Type II string theory has also been constructed by Hohm, Kwak and Zwiebach \[\text{(36,37)}\] (an alternative formulation of the RR sector, called the semi-covariant formulation is given in the papers \[\text{(38,39)}\]). Likewise, in the supergravity frame,

\[^1\]This formulation of the DFT action has the added advantage that it already includes an extra term, which has to be added by hand in the original formulation. This extra term is needed in order to match the 4 dimensional half-maximal gauged supergravity with the theory that results from the duality twisted reduction of the DFT action.

\[^2\]An exception is the work of \[\text{(22)}\], where they also include the reduction of the RR sector. However, their methods are different from ours, as they perform the twisted reduction in the semi-covariant formalism of DFT \[\text{(34,35)}\].
this action reduces to the action of the democratic formulation of the RR sector of Type II supergravity. The fundamental fields of this sector are two SO$(10, 10)$-spinor fields, $S$ and $\chi$. The latter is a spinor field which encodes the massless p-form fields of Type II theory. It has to have a fixed chirality, depending on whether the theory is to describe the DFT of the massless Type IIA theory or Type IIB theory. The field $S$ is the spinor representative of the generalized metric, that is, under the double covering homomorphism between $Pin(n, n)$ and $O(n, n)$, it projects to the generalized metric of the NS-NS sector. The action of this sector has manifest $Spin(10, 10)$ invariance (not $Pin(n, n)$) in order to preserve the fixed chirality of $\chi$. The action has to be supplemented by a self-duality condition, which further reduces the duality group to $Spin^+(10, 10)$.

The aim of this paper is to study the duality twisted reductions of the DFT of the RR sector of massless Type II theory, with twists belonging to the duality group $Spin^+(10, 10)$. We study how the action and the gauge transformation rules reduce and determine the conditions for the consistency of the reduction and the closure of the gauge algebra. We also construct the Dirac operator associated with the $Spin^+(10, 10)$ covariant derivative that arises in the RR sector. In finding the reduced theory, we find it useful to rewrite the action of $[36, 37]$ in terms of the Mukai pairing, which is a natural bilinear form on the space of spinors [41–43]. The advantage of this reformulation is that the Mukai pairing is manifestly $Spin(n, n)$ invariant. If the duality twist is introduced via the $Spin^+(10, 10)$ element $S$ in the RR sector, the consistency requires that the NS-NS sector should also be deformed, via a duality twisted anzats introduced by $U = \rho(S)$. Here, $\rho$ is the double covering homomorphism between $Pin(n, n)$ and $O(n, n)$. The fact that Lie algebras of $Spin(n, n)$ and $SO(n, n)$ are isomorphic plays a crucial role in all the calculations. We show that the set of conditions required for the consistency of the reduction of the NS-NS sector are also crucial for the consistency of the reduction of the RR sector. In addition, the deformed RR sector is gauge invariant only when the Dirac operator is nilpotent, which in turn imposes an extra constraint on the fluxes that determine the deformations. The fact that such a constraint should arise in the presence of RR fields has already been noted in [19] and was verified in [22].

The plan of the paper is as follows. Section 2 is a preliminary section on spin representations and the spin group. Most of the material needed in the calculations for the reduction is reviewed in this section. In the first part of section 3 we present a brief review of both sectors of DFT, with a special emphasis on the RR sector. As the DFT of the RR sector reduces to the democratic formulation of Type II theory in the supergravity frame, we start this section by a brief review of the democratic formulation of Type II supergravity. The rewriting of the action of $[36, 37]$ in terms of the Mukai pairing is also explained in this section. Section 4 is the main section, where we study the reduction of the action and the gauge algebra and discuss the conditions for consistency and closure of the gauge algebra. We finish with a discussion of our results in section 5.
2 Preliminaries on Spin Representations and The Spin Group

The purpose of this preliminary section is to review the material, which we will need in the later sections of the paper. We closely follow [44].

Let \( V \) be an even dimensional (m=2n) real vector space with a symmetric non-degenerate bilinear form (a metric) \( Q \) on it. Then the orthogonal group \( O(V, Q) \) is the space of automorphisms of \( V \) preserving \( Q \):

\[
O(V, Q) = \{ A \in Aut(V): Q(Av, Aw) = Q(v, w), \ \forall v, w \in V \} \tag{2.1}
\]

If we restrict this set to the automorphisms of determinant 1, then we get the subgroup \( SO(V, Q) \).

The corresponding orthogonal Lie algebras \( so(Q) = o(Q) \) are then the endomorphisms \( A : V \to V \) such that

\[
Q(Av, w) + Q(v, Aw) = 0 \tag{2.2}
\]

for all \( v, w \) in \( V \). The standard methodology in constructing the spin representations of the orthogonal Lie algebra is to embed it in the Clifford algebra on \( V \) associated to the bilinear form \( Q \) and use the well-known isomorphisms between the Clifford algebras and the matrix algebras.

Given the vector space \( V \) and the metric \( Q \), one can define the Clifford algebra \( C = Cl(V, Q) \) as the universal algebra which satisfies the property

\[
\{v, w\} \equiv v.w + w.v = 2Q(v, w) \tag{2.3}
\]

Here \( \cdot \) is the product on the Clifford algebra. \( Cl(V, Q) \) is an associative algebra with unit 1 and as such it determines a Lie algebra, with bracket \( [a, b] = a.b - b.a \). Clifford algebras enjoy nice isomorphisms with various matrix algebras (the form of which depends on \( V \) and \( Q \)) under which the Clifford product becomes the matrix multiplication. If \( e_1, \ldots, e_m \) form a basis of \( V \), then the unit element 1 and the products \( e_I = e_{i_1} \cdots e_{i_k} \), for \( I = \{i_1 < i_2 < \cdots < i_k\} \) form a basis for the \( 2^m \) dimensional algebra \( Cl(V, Q) \). The images of these basis elements (of \( V \)) under the isomorphisms with the matrix algebras are usually called \( \Gamma \)-matrices in the physics literature. The Clifford algebra is a \( Z_2 \) graded algebra and it splits as \( C = C^{\text{even}} \oplus C^{\text{odd}} \), where \( C^{\text{even}} \) is spanned by products of an even number of elements in \( V \) and \( C^{\text{odd}} \) is spanned by an odd number of elements of \( V \). The space \( C^{\text{even}} \) is also a subalgebra and it has half the dimension of \( C \), that is, it is an algebra of dimension \( 2^{m-1} \).

The orthogonal Lie algebra \( so(Q) \) embeds in the even part of the Clifford algebra as a Lie subalgebra via the map (for a proof, see [44])

\[
\psi \circ \varphi^{-1} : so(Q) \to C^{\text{even}}, \text{ where } \psi : \wedge^2 V \to Cl(V, Q),
\]

\[
\psi(a \wedge b) = \frac{1}{2}(a.b - b.a) = a.b - Q(a, b) \tag{2.4}
\]

and \( ^3 \)

\[
\varphi : \wedge^2 V \longrightarrow so(Q) \subset End(V) \tag{2.5}
\]

\[
a \wedge b \longmapsto \varphi_a \wedge b \tag{2.6}
\]

\(^3\)Here we identify the dual space \( V^* \) with \( V \) via the bilinear form \( Q \) and hence \( \wedge^2 V \subset End(V) = V \otimes V^* \cong V \otimes V \).
where \( \varphi_{a \land b} \) is given by
\[
\varphi_{a \land b}(v) = Q(b, v)a - Q(a, v)b, \quad a, b \in V.
\] (2.7)

Our main interest lies in bilinear forms, which are non-degenerate and are of signature \((n, n)\). Then a maximally isotropic subspace is of dimension \(n\). (Recall that a maximally isotropic subspace of \(V\) is a subspace of maximum possible dimension, on which \(Q\) restricts to the zero-form.) Let \(W\) be such a subspace and let \(W'\) be the orthogonal complement of \(W\) with respect to the bilinear form \(Q\), so that \(V = W \oplus W'\). The exterior algebra
\[
\wedge^* W = \wedge^0 W \oplus \cdots \oplus \wedge^n W
\] (2.8)
carries a representation of the Clifford algebra \(C\) and hence the orthogonal Lie algebra \(so(Q)\), which is a Lie subalgebra of \(C\). In other words, there exists an isomorphism of algebras between \(C\) and \(End(\wedge^* W)\). The ismorphism operates as follows: for \(w \in W\) and \(w' \in W'\) one has
\[
w + w' \mapsto l(w) + l'(w') \in End(\wedge^* W)
\] (2.9)
where
\[
l(w)\alpha = w \land \alpha \quad \text{and} \quad l'(w')\alpha = i(w')\alpha.
\] (2.10)
Here \(\alpha \in \wedge^* W\) and
\[
(w')^2(w) = 2Q(w, w').
\] (2.11)
It is straightforward to see that this defines a representation of the algebra \(Cl(V, Q)\) by verifying that
\[
(l(w))^2 = (l'(w'))^2 = 0
\] (2.12)
\[
\{l(w), l'(w')\} = 2Q(w, w')I.
\] (2.13)

This representation of the Clifford algebra carried by \(\wedge^* W\) is called the spin representation. This is an irreducible representation as a representation of the Clifford algebra, however it is reducible as a representation of the orthogonal Lie algebra \(so(Q) \cong so(n, n)\), which lies in \(C\). The invariant subspaces of the spin representation under the action of \(so(n, n)\) are denoted by \(S^+\) and \(S^-\) and corresponds to the decomposition of the exterior algebra into the sum of even and odd exterior powers. Hence we have
\[
S^+ = \wedge^{\text{even}} W, \quad S^- = \wedge^{\text{odd}} W
\] (2.14)
and
\[
S = S^+ \oplus S^-
\] (2.15)
where \(S = \wedge^* W\) is the spin representation. The elements of \(S^+\) and \(S^-\) are called chiral spinors.

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4Note that the usual interior product defined on the subspaces \(\wedge^k(V)\) can be extended to the whole exterior algebra by linearity.

5Note that \(Q\) allows one to identify \(V\) with the dual space \(V^*\) and under the decomposition \(V = W \oplus W'\) the subspace \(W\) is identified with \(W^{\vee}\) and \(W'\) is identified with \(W^{\vee}\), hence \((w')^2\) is in \(W^{\vee}\) and contraction with \((w')^2\) is well-defined.
Inside the Clifford algebra lies an important group, the group Pin(Q), which in fact turns out to be the double covering group of O(Q). In order to define it, one needs the following anti-involution $x \mapsto x^*$ on the Clifford algebra determined by

$$ (v_1 \cdots v_k)^* = (-1)^k v_k \cdots v_1 $$

(2.16)

for any $v_1, \ldots, v_k$ in $V$. This is the composite of the main automorphism $\tau : C \rightarrow C$ and the main involution $\alpha : C \rightarrow C$ determined by

$$ \tau(v_1 \cdots v_k) = v_k \cdots v_1 $$

(2.17)

$$ \alpha(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k $$

(2.18)

for $v_1, \ldots, v_k$ in $V$. Note that $(x + y)^* = x^* + y^*$ and $(x \cdot y)^* = y^* \cdot x^*$, which follows from $\tau(x \cdot y) = \tau(y) \cdot \tau(x)$ and $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.

Now the group Pin(Q) is defined as a certain subgroup of the multiplicative group of $C(Q)$:

$$ \text{Pin}(Q) = \{ x \in C(Q) : x \cdot x^* = \pm 1 \text{ and } x \cdot V \cdot x^{-1} \subset V \}. $$

(2.19)

Each element in $\text{Pin}(Q)$ determines an endomorphism $\rho(x)$ of $V$ by

$$ \rho : \text{Pin}(Q) \rightarrow O(Q) $$

$$ \rho(x) : v \mapsto x \cdot v \cdot x^{-1}. $$

(2.20)

(2.21)

One can show that $\rho$ is a surjective homomorphism, which preserves the metric $Q$ and its kernel is $\{+1, -1\}$ (for a proof, see [44]). If we further demand that $x$ lies in the even part of the Clifford algebra, then the group becomes the spinor group Spin(Q):

$$ \text{Spin}(Q) = \{ x \in C(Q)^{\text{even}} : x \cdot x^* = \pm 1 \text{ and } x \cdot V \cdot x^* \subset V \}, $$

(2.22)

It is easy to see that

$$ \text{Spin}(Q) = \text{Pin}(Q) \cap C(Q)^{\text{even}} = \rho^{-1}(SO(Q)). $$

Restricting further to the elements in Spin(Q), which satisfies $x \cdot x^* = +1$, we obtain the subgroup Spin$^+(Q)$.

The Lie algebra of the group Spin(Q) is a subalgebra of the Clifford algebra with the usual bracket. It can be shown that this subalgebra is nothing but the Lie algebra so(Q). In other words, the derived homomorphism

$$ \rho' : \text{spin}(Q) \rightarrow \text{so}(Q) $$

(2.23)

is in fact an isomorphism of the Lie algebras and the right hand side of

$$ \rho'(x)(v) = [x, v], $$

(2.24)
evaluated in the Clifford algebra (regarding $so(Q)$ and $V$ as subspaces of the Clifford algebra) coincides with the standard action of $so(Q)$ on $V$.

**Spinorial Action of $so(n,n)$ and $Spin(n,n)$ on exterior forms:** Let us choose a basis $e_M = \{e^1, \cdots, e^n, e_1, \cdots, e_n\} = \{e^i, e_i\}$ of $V$ such that

$$Q(e^i, e_j) = \delta^i_j, \quad Q(e_i, e_j) = Q(e^i, e^j) = 0, \quad \forall i, j. \quad (2.25)$$

With respect to this basis $Q$ is represented by the matrix $\eta$

$$\eta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}. \quad (2.26)$$

and the definition (2.3) becomes

$$e_M.e_N + e_N.e_M = 2\eta_{MN}. \quad (2.27)$$

Obviously the elements $\{e^1, \cdots, e^n\}$ span an isotropic subspace $W$ and the elements $\{e_1, \cdots, e_n\}$ span the orthogonal complement $W'$. The metric $Q$ allows us to identify $W$ with $W'$, that is, we can raise and lower indices with $\eta$: $e^M = \eta^{MN}e_N$. Looking back at the maps (2.4), (2.7), one can calculate that

$$e_M \wedge e_N \in \wedge^2 V \mapsto \rightarrow T_{MN} \in so(n,n) \quad (2.28)$$

where the generators $T_{MN}$ of $so(n,n)$ are endomorphisms of $V$ represented by the antisymmetric matrices

$$(T_{MN})^L_K = -\eta_{KM}\delta^L_N + \eta_{KN}\delta^L_M = -2\eta_K[\delta_N]^L. \quad (2.29)$$

Under the isomorphism $\psi \circ \varphi^{-1}$, $T_{MN}$ is mapped to

$$T_{MN} \leftarrow \frac{1}{4}(e_M.e_N - e_N.e_M) \equiv \frac{1}{2}e_{MN}. \quad (2.30)$$

Note that the standard action of $so(Q)$ on $V$ and its action on $V$ within the Clifford algebra (when we regard both $so(Q)$ and $V$ as subspaces of $C$) agree, as it should. That is, we have

$$T_{MN}(e_K) = e_L(T_{MN})^L_K = \frac{1}{2}[e_{MN}, e_K] \quad (2.31)$$

where the bracket on the right hand side above is evaluated in the Clifford algebra. Let us note that we obtain the more familiar elements $T^{MN}$ by raising the indices of $T_{MN}$ by $\eta$:

$$(T^{PQ})_L^K = (T_{MN})^K_L\eta^{MP}\eta^{NQ} = \eta^K_L\eta^{PQ} - \eta^K_Q\eta^P_L = 2\eta^K[\delta_P^Q]. \quad (2.32)$$

It can be shown that $T^{MN}$ satisfy the following commutation relations:

$$[T^{MN}, T^{KL}] = \eta^{MK}T^{LN} - \eta^{NK}T^{LM} - \eta^{ML}T^{KN} + \eta^{NL}T^{KM}. \quad (2.33)$$

Now that we know the Clifford algebra elements corresponding to the generators of the orthogonal Lie algebra, we can immediately calculate the spinorial action of each generator on forms in the exterior algebra $\wedge^\bullet W$. For this purpose, it is useful to divide the Lie algebra elements $T_{MN}$ into 3 groups: $T^{mn}, T_{mn}, T^m_n$. This corresponds to the decomposition $\wedge^2 V = \wedge^2 V_0 \oplus \wedge^2 V_1 \oplus \wedge^2 V_2$.\[7]
\( \wedge^2(W \oplus W') \cong \wedge^2(W) \oplus \wedge^2(W') \oplus \text{End}(W) \). The spinorial action of these elements on differential forms can now be easily read off from (2.11):

\[
T^{mn} : \alpha \mapsto \frac{1}{2} e^m \wedge e^n \wedge \alpha, \quad (2.34)
\]

\[
T_{mn} : \alpha \mapsto \frac{1}{2} i_{e_m} i_{e_n} \alpha, \quad (2.35)
\]

\[
T^m_n : \alpha \mapsto \frac{1}{4} (e^m \wedge i_{e_n} \alpha - i_{e_m} (e^m \wedge \alpha)) = \frac{1}{2} (\delta^m_n + e^m \wedge i_{e_n} \alpha). \quad (2.36)
\]

Here, it is important to note that \( i_{e_m} e^n = 2 \delta^m_n \), due to the factor 2 in (2.11). It is more common to work with the basis elements \( \psi_M = \frac{1}{\sqrt{2}} e_M \) which satisfies \( \{ \psi_m, \psi^n \} = \delta^m_n \), so that one has \( i_{\psi_m} \psi^n = \delta^m_n \). Then we have:

\[
T^{mn} : \alpha \mapsto \psi^m \wedge \psi^n \wedge \alpha, \quad (2.37)
\]

\[
T_{mn} : \alpha \mapsto i_{\psi_m} i_{\psi_n} \alpha, \quad (2.38)
\]

\[
T^m_n : \alpha \mapsto -\frac{1}{2} \delta^m_n + \psi^m \wedge i_{\psi_n} \alpha. \quad (2.39)
\]

In this case one should also write the spinor \( \alpha \in \wedge^* W \) in terms of the basis elements: \( \psi^I = \psi^{i_1} \cdots \psi^{i_k} \).

By exponentiating the Lie algebra elements \( T_{MN} \) in the fundamental representation, one obtains the identity component \( SO^+(n,n) \) of \( SO(n,n) \). A general group element in the identity component is of the form \( \exp \left[ \frac{1}{2} \Omega^{MN} T_{MN} \right] \). A simple computation shows that any such element can be written in terms of the matrices given below, where \( h_B, h_\beta, h_A \) corresponds to the exponentiation of \( T^{mn}, T_{mn}, T^m_n \), respectively:

\[
h_B = \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \quad B^T = -B, \quad (2.41)
\]

\[
h_\beta = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad \beta^T = -\beta \quad (2.42)
\]

\[
h_A = \begin{pmatrix} e^A & 0 \\ 0 & e^{-(A)^T} \end{pmatrix}, \quad (2.43)
\]

Here we have named \( B_{kl} = \Omega_{[kl]}, \beta^{kl} = \Omega^{[kl]}, A^I_k = \frac{1}{2} (\Omega^I_k - \Omega^I_k) \). On the other hand, exponentiation of the generators in the spin representation gives the corresponding elements \( S_B, S_\beta, S_A \) in the identity component \( Spin^+(n,n) \) of the spinor group \( Spin(n,n) \), which act on the differential forms.
forms as follows:

\[ S_B: \alpha \mapsto e^{-B} \wedge \alpha = (1 - B + \frac{1}{2} B \wedge B - \ldots) \wedge \alpha, \]  
\[ S_\beta: \alpha \mapsto e^\beta \alpha = (1 + i_\beta + \frac{1}{2} i_\beta^2 + \cdots) \alpha, \]  
\[ S_A: \alpha \mapsto \frac{1}{\sqrt{\det}} (e^A)^* \alpha. \]  

These transformation rules follow immediately from \([2.34],[2.35]\). Here \( B = \frac{1}{2} B_{kl} e^k \wedge e^l = \frac{1}{2} B_{kld} e^k \wedge \psi^d \), \( \beta = \frac{1}{4} \beta_{kl} e^k \wedge e^l = \frac{1}{2} \beta^i \psi_i \wedge \psi^l \), and \( r = e^A \). Also, \( i_\beta (\alpha) = \frac{1}{4} \beta^i \psi_i \alpha \) and \( r^* (\alpha) = r^j \psi_j \wedge i_\psi \alpha \), which is the usual action of \( GL^+ V \) on forms, where \( GL^+ V \) is the space of (orientation preserving) linear transformations on \( V \) of strictly positive determinant. Note that all these elements satisfy \( SS^* = 1 \), that is, they lie in the component \( Spin^+(n,n) \).

It can be checked that the above elements \( h_B, h_\beta, h_A \) and the corresponding \( S_B, S_\beta, S_A \) satisfy \( \rho(S) = h \), by verifying that \([2.21]\) is satisfied. In other words, one can verify that

\[ e_N h_M^N = S.e_M.S^{-1}. \]  

Multiplying both sides with \( \eta^{KM} \) and using the identity \( \eta^{NP}(h^{-1})^K_P = \eta^{KM} h_M^N \) we also have

\[ (h^{-1})^N_M e^N = S.e^N.S^{-1}. \]  

Note that the right hand-side remains the same if we change \( S \to -S \), which reflects the fact that the kernel of the homomorphism \( \rho \) is \( \{1,-1\} \), that is, \( \rho(S) = \rho(-S) = h \). Obviously, these relations also hold for the Gamma matrices \( \Gamma_M \), which are the matrix images of the Clifford algebra generators \( e_M \) under the isomorphisms with the matrix algebras. Under such an isomorphism the Clifford multiplication becomes matrix multiplication and we have \[^8\]

\[ \Gamma_N h_M^N = ST^M S^{-1}, \quad (h^{-1})^M_N \Gamma^N = ST^M S^{-1}. \]  

Before we move on to the discussion of some important elements of \( Pin(n,n) \), which do not lie in \( Spin^+(n,n) \), we would like to make a remark. Note that the description of spinors as forms in an exterior algebra that we have discussed above is very useful and one can take this idea one step further by demanding that \( W \) is the cotangent space at a point \( p \) of an \( n \)-dimensional smooth manifold \( M \), \( W = T^*_p M \). Then the orthogonal complement is naturally identified with the tangent space \( W' = T_p M \). Then \( V = T^*_p M \oplus T_p M \) is a section of the bundle \( T^* \oplus T \). All the linear algebra discussed above can be transported to the whole bundle, as it is known that (for example, see \([43]\)) the \( SO(n,n) \) bundle \( T^* \oplus T \) on an orientable manifold always carries a \( Spin(n,n) \) structure. Then the spinor fields becomes sections of the exterior bundle \( \wedge^* T^* M \), which are smooth differential forms on \( M \), which are also called polyforms in the physics literature due to the fact that they are not necessarily homogenous forms. This is the setting in generalized complex geometry \([43],[45]\), where the identification of \( Spin(n,n) \) spinor fields with smooth differential forms plays a crucial role.

\[ ^7 \text{Note that the determinant term arises from exponentiation of the trace term which appears in } [2.36]. \]

\[ ^8 \text{Here, we abuse the notation by calling the matrix image of the Clifford algebra element } S \text{ also } S. \]
Let us now move on to the discussion of some other important elements in \(Pin(n,n)\), that will be needed in the rest of this paper. So far, our aim has been to understand the spinorial action of the orthogonal Lie algebra (which is isomorphic to the Spinor Lie algebra) on the exterior algebra \(\Lambda^*W\). At the group level, this has given us only the identity components of the orthogonal group and the Spinor group. In implementing the duality twisted reduction, we will only need such elements (connected to the identity element), as the real symmetry group of the RR sector of the DFT action is \(Spin^+(n,n)\) (see section 3.2). However, in constructing this part of the DFT action, one needs more. For example, the spinor representative of the generalized metric \(H\) is in \(Spin^-(n,n)\), as the generalized metric itself must be in \(SO^-(n,n)\) due to the Lorentzian signature of the Riemannian metric encoded in \(H\) \([36]\). In order to understand such elements, one needs the elements of \(O(n,n)\) which interchanges \(e^i \leftrightarrow e_i\) and keeps all other basis elements of \(V\) fixed, possibly up to a sign. Let us define the following \(O(n,n)\) elements:

\[
\Lambda_i^\pm = (\psi^i \mp \psi_i), \tag{2.51}
\]

where we have used the normalized \(\psi^M = \frac{1}{\sqrt{2}}e^M\) so that \(\Lambda_i^\pm = \pm 1\), rather than \(\pm 2\). Note that \((\Lambda_i^\pm)^2 = \mp 1\), so we have \((\Lambda_i^\pm)^{-1} = -\Lambda_i^\mp\) and \((\Lambda_i^-)^{-1} = \Lambda_i^+\). One can easily verify the following by using the Clifford algebra relations

\[
\Lambda_i^\pm .e^M, (\Lambda_i^\pm)^{-1} = \begin{cases} 
  e_i & \text{if } e^M = e^i \\
  e^i & \text{if } e^M = e_i \\
  \pm e^M & \text{otherwise}
\end{cases} \tag{2.52}
\]

Therefore, \(\rho(\Lambda_i^\pm) = h_i^\pm\), as we have claimed.

From the elements \(\Lambda_i^\pm\) one can construct a very important element in the Pinor group, which projects to the following matrix \(J\) in \(O(d,d)\):

\[
J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.53}
\]

Obviously, \(J\) swaps \(e^i \leftrightarrow e_i\) for all \(i\). On the other hand, \(h_i^\pm = \rho(\Lambda_i^\pm)\) interchanges \(e^i\) with \(e_i\), while keeping all other basis elements fixed, possibly up to a sign \(e^M \leftrightarrow \pm e^M, \: M \neq i\). Therefore, to construct the Pinor group element that projects to \(J\), we need the product of all such elements, with some extra care to determine the overall sign. With a bit of work, one can show that the Pinor group element \(C\), which satisfies \(\rho(C) = J\) is

\[
C = C^+ \equiv \Lambda_1^+ \ldots \Lambda_d^+, \tag{2.54}
\]

in even dimensions and

\[
C = C^- \equiv \Lambda_1^- \ldots \Lambda_d^- \tag{2.55}
\]
in odd dimensions.

Note that \((\Lambda_i^+)^2 = -1\) and \((\Lambda_i^-)^2 = 1\) for all \(i\). This implies that

\[
C^+(C^+)^* = 1 \quad (2.56)
\]

and

\[
C^-(C^-)^* = -1. \quad (2.57)
\]

On the other hand, with a bit of care with the ordering of the elements one can calculate that

\[
C^2 = (-1)^{\sum_1^d (d-k)} I = (-1)^{(d-1)/2} I
\]

which gives

\[
C^{-1} = (-1)^{(d-1)/2} C, \quad (2.59)
\]

both for \(C^+\) and \(C^-\). It is straightforward to check that \(C\) indeed satisfies (both in odd and even dimensions)

\[
C \Gamma^M C^{-1} = J^M_N \Gamma^N 
\]

(note that \(J^{-1} = J\)), so indeed

\[
\rho(C) = J, \quad (2.60)
\]

as we have claimed. Since \(C\) and \(C^{-1}\) just differ by a sign, we also have \(\rho(C^{-1}) = J\) as a result of which we have

\[
C^{-1} \Gamma^M C = J^M_N \Gamma^N . \quad (2.62)
\]

It is appropriate to call this element of the Pinor group the charge conjugation matrix, as it satisfies the same Gamma matrix relations as the standard charge conjugation matrix in quantum field theory. By the help of it, it is possible to define the action of a dagger operator in the Clifford algebra as

\[
S^\dagger \equiv C \tau(S) C^{-1}. \quad (2.63)
\]

Obviously, one has \((S_1 \cdot S_2)^\dagger = S_2^\dagger \cdot S_1^\dagger\) (which follows immediately from \(\tau(S_1 \cdot S_2) = \tau(S_2) \cdot \tau(S_1)\)) and it can be checked that \(C^\dagger = C^{-1}\) (as \(\tau(C) = 1\) both in even and odd dimensions). It is also straightforward to verify that \(S \in \text{Pin}(n,n)\) implies \(S^\dagger \in \text{Pin}(n,n)\). Also note that \(\tau(S) = S^* = \pm S^{-1}\), when \(S \in Spin^\pm(n,n)\) so we have

\[
S^\dagger = C S^* C^{-1} = \pm C S^{-1} C^{-1}, \quad S \in Spin^\pm(n,n). \quad (2.64)
\]

The following facts can be proved without much effort (for details, see [36])

\[
\rho(\tau(S)) = \rho(S)^{-1} \quad \text{and} \quad \rho(S^\dagger) = \rho(S)^T. \quad (2.65)
\]

**A bilinear form on the space of spinors: Mukai pairing:** The last thing we would like to discuss is the natural inner product on the Clifford module \(\wedge^\bullet W\). Later in section (3.3), we will utilize this inner product in order to rewrite the DFT action of the RR sector of Type II theory. Recall the map \(\tau : v_1 \cdot \cdots \cdot v_k \mapsto v_k \cdot \cdots \cdot v_1\) we defined above. It represents a transpose map in the Clifford algebra which, from the point of view of the spin module, arises from the following bilinear form on \(\langle \,,\, \rangle : S \otimes S \to \wedge^\bullet W:\)

\[
\langle \chi_1, \chi_2 \rangle = (\tau(\chi_1) \wedge \chi_2)_{\text{top}} = \sum_j (-1)^j (\chi_1^{2j} \wedge \chi_2^{n-2j} + \chi_1^{2j+1} \wedge \chi_2^{n-2j-1}), \quad \chi_1, \chi_2 \in \wedge^\bullet W. \quad (2.66)
\]
where \( (\cdot)_\text{top} \) means that the top degree component of the form should be taken and the superscript \( k \) denotes the \( k \)-form component of the form. This bilinear form is known as the Mukai pairing and it behaves well under the action of the Spin group \([43]\):

\[
\langle S\chi_1, S\chi_2 \rangle = \pm \langle \chi_1, \chi_2 \rangle, \quad S \in \text{Spin}^\pm(n, n).
\]

This bilinear form is non-degenerate and it is symmetric in dimensions \( n \equiv 0, 1 \pmod{4} \) and is skew-symmetric otherwise:

\[
\langle \chi_1, \chi_2 \rangle = (-1)^{n(n-1)/2} \langle \chi_2, \chi_1 \rangle.
\]

In particular, it is skew-symmetric for \( n = 10 \), which is the relevant dimension in constructing the DFT action for Type II strings. Also importantly, the bilinear form is zero on \( S^+ \times S^- \) and \( S^- \times S^+ \) for even \( n \) and it is zero on \( S^+ \times S^+ \) and \( S^- \times S^- \) for odd \( n \). More details can be found in \([43]\).

Now assume that there exists an inner product on the vector space \( W \). This also induces a non-degenerate bilinear form on \( \wedge^* W \) taking values in \( \wedge^n W \):

\[
(\chi_1, \chi_2) = \chi_1 \wedge \star \chi_2 = \sum_j \chi_1^j \wedge \star \chi_2^j
\]

where \( \star \) is the Hodge duality operator with respect to the inner product on \( W \). It is possible to show that this bilinear form is related to the Mukai pairing in the following way:

\[
(\chi_1, \chi_2) = (\chi_1, C^{-1} \chi_2) = (\tau(\chi_1) \wedge C^{-1} \chi_2)_{\text{top}},
\]

where the charge conjugation matrix presented in \([2.54, 2.55]\) should be written in terms of an orthonormal basis with respect to the inner product on \( W \).

### 3 Democratic Formulation of Type II Theories and the Double Field Theory Extension

#### 3.1 Democratic Formulation

The aim of this subsection is to give a brief review of the democratic formulation of the bosonic sector of (massless) Type IIA and Type IIB supergravity theories \([46, 47]\) (also see \([48]\)). The (bosonic) matter content of these two theories are as follows:

\[
\begin{align*}
\text{IIA} & : \quad \{ g, B_2, \phi, C_1, C_3 \} \quad (3.1) \\
\text{IIB} & : \quad \{ g, B_2, \phi, C_0, C_3, C_5 \} \quad (3.2)
\end{align*}
\]

The NS-NS sector, which only involves the metric \( g \), the Kalb-Ramond field \( B_2 \) (which is a 2-form field) and the dilaton \( \phi \) is common to both Type IIA and Type IIB (as well as to other 3 perturbative superstring theories) and is given as

\[
S_{\text{NS-NS}} = \int e^{-2\phi} \left[ R + \frac{1}{2} (d\phi \wedge \star d\phi) - \frac{1}{2} (H^{(3)} \wedge \star H^{(3)}) \right],
\]

(3.3)
where $H_3 = dB_2$. In order to write down the Lagrangian for the RR sector in the democratic formulation, one first defines the following modified RR potentials:

\[
\begin{align*}
D_0 & \equiv C_0, & D_1 & \equiv C_1, \\
D_2 & \equiv C_2 + B_2 \wedge D_0, & D_3 & \equiv C_3 + B_2 \wedge C_1, \\
D_4 & \equiv C_4 + \frac{1}{2} B_2 \wedge C_2 + \frac{1}{2} B_2 \wedge B_2 \wedge C_0.
\end{align*}
\]

Now introduce

\[
D \equiv \sum_{p=0}^{8} D_p, \quad F \equiv e^{-B_2} \sum_{p=0}^{8} dD_p = \sum_{p=0}^{8} F_{p+1}.
\]

The indices run from 0 to 8, as we have also included the electromagnetic duals of the gauge potentials $D_p$. The electromagnetic duals $D_{8-p}$ of $D_p$ are the potential fields obtained by solving the field equations for the latter. This ensures that $F$ defined as above satisfies

\[
F_{10-p} = (-1)^{[\frac{p-1}{2}]} * F_p
\]

where $[\frac{p-1}{2}]$ is the first integer greater than or equal to $\frac{p-1}{2}$. Note that $D$ is a section of the exterior bundle $\wedge^\bullet T^*M$, where $M$ is the manifold on which the RR fields live. We can also decompose

\[
D = D^+ + D^-
\]

where $D^+$ involves k-forms of even degree ($k=0,2,4,6,8$), whereas $D^-$ involves forms of odd degree. Then $D^+$ and $D^-$ are sections of the bundles $\wedge^{\text{even}} T^*M$ and $\wedge^{\text{odd}} T^*M$, respectively. Obviously, there is a corresponding decomposition of the differential form $F = F^+ + F^-$. Now consider the following simple actions:

\[
\begin{align*}
S_{\text{IIA RR}} &= \frac{1}{4} \int F^+ \wedge * F^+ \\
S_{\text{IIB RR}} &= \frac{1}{4} \int F^- \wedge * F^- \\
\end{align*}
\]

It can be shown that the actions given above are equivalent to the standard action of Type IIA and Type IIB supergravity theories, which also involve some complicated Chern-Simons type terms, in the following sense [36, 46, 47]: If one applies the duality relations (3.6) to the field equations derived from the actions (3.8), (3.9), then one obtains exactly the same field equations that one would have derived from the standard actions. The field equations for lower degree form fields match directly in the two formulations. On the other hand, the field equations (in the democratic formulation) for the higher degree fields which are absent in the standard formulation becomes, after applying (3.6), the Bianchi identities for the lower degree fields in the standard formulation.

### 3.2 Double Field Theory Extension

In the previous section, we have seen that the (modified) RR fields form sections of the bundles $\wedge^{\text{odd}} T^*M$ and $\wedge^{\text{even}} T^*M$ for Type IIA and Type IIB, respectively. We have also seen in section 2
that fibers of these bundles, when $T_p^*M$ is regarded as an isotropic subspace of the doubled vector space $T_pM \oplus T_p^*M$ at a given point $p \in M$, are in fact modules for the Clifford algebra $Cl(n, n)$ (when $M$ is $n$ dimensional) and carry the irreducible spin representation for the isomorphic Lie algebras $so(n, n)$, $spin(n, n)$ and the corresponding Lie groups. This structure on the fibers can be transported to the whole bundle $T \oplus T^*$ on any orientable manifold $M$. This immediately tells us that the modified RR fields transform in the spin representation of the group $SO(n, n)$ or $Spin(n, n)$. In fact, the main motivation of constructing the democratic formulation in the first place was to show the invariance of the RR sector under the orthogonal group \[46\].

In order to achieve this, one reduces Type IIA or Type IIB on a $(10 - d)$-dimensional torus. The invariance of the scalar and vector fields in $d$ dimensions under $O(d, d)$ had already been a well-known fact. The vectors transform in the fundamental representation of $SO(d, d)$, whereas scalar fields form the coset $SO(d, d)/SO(d) \times SO(d)$ and transform non-linearly. In \[46\], it was shown that the RR sector couples to the vector and scalar fields through the $Spin(d, d)$ matrix, which projects, under the homomorphism $\rho : Spin(d, d) \to SO(d, d)$ onto the $SO(d, d)$ element that parameterizes the $SO(d, d)/SO(d) \times SO(d)$ scalar coset. They also show that the reduced action can be put in a form in which the $Spin(d, d)$ invariance is manifest. As a result, it was established that the $d$ dimensional theory obtained by dimensional reduction on an $(10 - d)$-dimensional torus was invariant under $SO(d, d)$, not only in the NS-NS sector, but also in the RR sector.

Double Field Theory (DFT) of Type II strings is an extension of massless Type II string theories, in which the duality symmetry $SO(d, d)$ is already manifest in $d = 10$ dimensions without the requirement of dimensional reduction.\[10\] The main purpose of this section is to give a brief overview of DFT and in particular, to review how the sector of DFT describing the RR fields is an extension of the democratic formulation of Type II theories, in the sense that it reduces exactly to it in a particular frame. In what follows, we will keep the dimension $d$ general, rather than fixing it to $d = 10$, unless it is inevitable.

The main idea in DFT is to allow the (massless) fields in string theory to depend on "dual coordinates", in addition to the usual coordinates of the space-time manifold on which the string propagates. For backgrounds admitting non-trivial cycles, e.g. for toroidal backgrounds, the dual coordinates are interpreted as being conjugate to the winding degrees of freedom, in the same way space coordinates and momenta are conjugate variables in classical field theory. This idea in DFT is inspired by closed string field theory, where all string fields naturally depend on both the usual coordinates and the dual coordinates. DFT aims to realize this in the sector of massless fields in order to construct a manifestly T-duality invariant action describing this sector. In string theory, momentum and winding modes combine to transform as a vector under the T-duality group $O(d, d)$. Therefore, in DFT one demands the same behavior from

9Note that we are restricting ourselves to $SO(d, d)$ here. In fact the whole Type II theory is invariant under the bigger group $O(d, d)$, which also involves the T-duality transformations between the Type IIA and Type IIB theories, given by the standard Buscher transformation rules in the NS-NS sector. In the RR sector, this corresponds to changing the chirality of the spinor state, which is fixed at the outset in the democratic formulation. Also note that we prefer to keep the dimension $d$ general, rather than fixing it to $d = 10$

10In fact, the DFT of the NS-NS sector of the massless Type II theories is invariant under the larger group $O(d, d)$. However, the RR sector is only invariant under $Spin^+(d, d)$.\[14\]
the space-time and dual coordinates, that is, they form an $O(d, d)$ vector transforming as:

$$X'^M = h^M_N X^N, \quad X^M = \left( \tilde{x}_i \atop x^i \right)$$  \hspace{1cm} (3.10)

Here $\tilde{x}_i$ are the dual coordinates and $h^M_N$ is a general $O(d, d)$ matrix. In what follows we will always decompose the indices $M$ labelling the $O(d, d)$ representation as $M = (i, \bar{i})$, where $i$ and $\bar{i}$ label representations of the $GL(d)$ subgroup of $O(d, d)$. We will raise and lower indices by the $O(d, d)$ invariant metric $\eta$, so that $X^M = \eta_{MN} X^N$. Although the theory is formally doubled by the introduction of the dual coordinates, the existence of an $O(d, d)$ invariant constraint, called the strong constraint, makes sure that there is always a choice of a frame in which all the fields and gauge parameters depend only on half of the coordinates. The constraint is $O(d, d)$ invariant and is given below:

$$\partial^M \partial_M A = \eta^{MN} \partial_M \partial_N A = 0, \quad \partial^M A \partial_M B = 0, \quad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$  \hspace{1cm} (3.11)

where $A$ and $B$ represent any fields or parameters of the theory. To be more precise, the first of the above constraints is called the weak constraint and follows from the level matching constraint in closed string theory. The second constraint is stronger and is called the strong constraint. Regarding the partial derivatives as a coordinate basis for the tangent space, the strong constraint implies that all vector fields are sections of a restricted tangent bundle in the sense that at each point the tangent space is restricted to a maximally isotropic subspace with respect to the metric $\eta$.

Let us now present the DFT action, in its generalized metric formulation, which was first constructed by Hohm, Hull and Zwiebach for the NS-NS sector \[4\], and then by Hohm, Kwak and Zwiebach for the RR sector \[36\]. These actions can also be presented in terms of a generalized vielbein, as was first done in \[7\].

$$S = \int dxd\tilde{x} (\mathcal{L}_{\text{NS-NS}} + \mathcal{L}_{\text{RR}}),$$  \hspace{1cm} (3.12)

where

$$\mathcal{L}_{\text{NS-NS}} = e^{-2d} \mathcal{R}(\mathcal{H}, d)$$  \hspace{1cm} (3.13)

and

$$\mathcal{L}_{\text{RR}} = \frac{1}{4} (\partial \chi)^\dagger \mathcal{S} \partial \chi.$$  \hspace{1cm} (3.14)

This action has to be implemented by the following self-duality constraint

$$\partial \chi = -\mathcal{K} \partial \chi, \quad \mathcal{K} \equiv C^{-1} \mathcal{S}.$$  \hspace{1cm} (3.15)

We will call the first term in the above action the DFT action of the NS-NS sector of string theory, whereas the second term will be referred to as the DFT action of the RR sector. The reason for this terminology is that in the frame $\partial \tilde{\phi} = 0$ (which we call the "supergravity frame"), which solves the strong constraint trivially, the first term reduces to the standard NS-NS action for the massless fields of string theory and the second term reduces to the RR sector of the democratic formulation of Type II supergravity theories, discussed in section (3.1). It is in this sense that this action is an extension of the democratic formulation of Type II theory.
The term $\mathcal{R}(\mathcal{H},d)$ in (3.13) is the generalized Ricci scalar and its explicit form can be found in [4]. It is defined in terms of the generalized metric $\mathcal{H}$ and the generalized dilaton $d$. These are $O(d,d)$ covariant tensors (in fact the dilaton is invariant) depending on both the space-time and dual coordinates. Their precise form is as below:

$$
\mathcal{H}_{MN} = \left( \begin{array}{cc} H^{ij} & H^i_j \\ H_i^j & H_{ij} \end{array} \right) = \left( \begin{array}{cc} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{array} \right), \quad e^{-2d} = \sqrt{g}e^{-2\phi}. \tag{3.16}
$$

where $g = |\det g|$. $\mathcal{H}$ is a symmetric $O(d,d)$ matrix and as such it satisfies $\mathcal{H}_{MP}\eta^{PQ}\mathcal{H}_{QR} = \eta^{MR}$. The Ramond-Ramond sector couples to the NS-NS sector via $S$, where $S$ is the spinor field which projects to the generalized metric $\mathcal{H}$ under the homomorphism $\rho$ of section 2, that is, $\rho(S) = \mathcal{H}$. In Lorentzian signature, the generalized metric $\mathcal{H}$ is in the coset $SO^{-(d,d)}[11]$ and there are subtleties in lifting this to an element $\mathcal{H}$ of $Spin^{-(d,d)}$ (for a detailed discussion, see [30]). So, in [30] the following viewpoint was adopted: it is the spin field $\mathcal{H}$ which projects to the generalized metric $\mathcal{H}$ under the homomorphism $\rho$ of section 2, that is, $\rho(S) = \mathcal{H}$. The field $S$ satisfies $S^\dagger = S$, which immediately implies that $\mathcal{H}$ is symmetric, as it has to be.

The other dynamical field in the DFT of the RR sector is the spinor field $\chi$, which encodes all the (modified) p-form fields in the RR sector. The field $\chi$, being a spinor field, transforms in the spinor representation of $Spin(d,d)$. Its chirality has to be fixed at the outset, so that it is either an element of $S^+$ or $S^-$ (see section 2). If we demand that the doubled manifold $M^{doub}$ is spin and the physical manifold $M$ sits in it in such a way that at each point $p \in M$, the cotangent space $T^*pM$ is an isotropic subspace of the whole cotangent space $T^*M^{doub}$ with respect to the metric $\eta$, then $\chi$ forms a section of the exterior bundle $\wedge^{even}T^*M$ or $\wedge^{odd}T^*M$, depending on its fixed chirality. Therefore, when restricted to the physical manifold, that is, in the frame $\hat{\partial}^i = 0$, it encodes all the RR fields of either the Type IIA or the Type IIB theory, depending on how its chirality has been fixed. More generally, all the independent fields, including $\chi$, might depend both on the physical coordinates and the dual ones. The operator $\hat{\phi}$ in the action (3.14), which differentiates $\chi$ is the generalized Dirac operator defined as

$$
\hat{\phi} \equiv \frac{1}{\sqrt{2}} \Gamma^M \partial_M = \frac{1}{\sqrt{2}} (\Gamma^i \partial_i + \Gamma_i \hat{\partial}^i). \tag{3.17}
$$

The self-duality constraint (3.15) makes sure that the p-form fields encoded by the spinor field $\chi$ obey the self-duality relations in the previous section. It should be noted that (3.15) is consistent only if $K^2 = 1$. On the other hand,

$$
K^2 = C^{-1}SC^{-1}S = CSCS = -C^2 = -(1)^{(d-1)/2}, \tag{3.18}
$$

[11] When the space-time metric $g$ is positive definite, so is the generalized metric $\mathcal{H}$ and hence its components form a matrix that lies in $SO^+(d,d)$. In this case the corresponding spin group element $S$ is in $Spin^+(d,d)$. However, when the (semi-)Riemannian metric $g$, has Lorentzian signature, then $\mathcal{H}$ is in $SO^-(d,d)$ and correspondingly $S$ lives in $Spin^-(d,d)$. Here, $SO^-(d,d)$ is the component of $SO(d,d)$ connected to the identity. It is also a subgroup, whereas its complement, $SO^+(d,d)$ is a coset of $SO^+(d,d)$. 
[12] From this section on, we will always work with the Gamma matrices $\Gamma_M$, which are the matrix images of the Clifford algebra generators $e_M$. 

16
where we have used (2.64), (2.59) and the facts that $S \in \text{Spin}^-(n, n)$ and $S^\dagger = S$. As a result, consistency of the self-duality equation imposes that $d(d-1)/2$ should be odd, that is $d \equiv 2, 3 \pmod{4}$. These are exactly the dimensions for which the Mukai pairing is anti-symmetric. This fact will play a crucial role in section (3.3).

An important ingredient in DFT is the generalized Lie derivative $\mathcal{L}$, which determines the gauge transformations of the DFT and the C-bracket, which determines how the gauge algebra closes [2]. Let us define $\xi^M = (\hat{\xi}_i, \xi^i)$ as the $O(d, d)$ vector which generates the following gauge transformations.

$$
\delta \xi H_{MN} = \mathcal{L}_\xi H_{MN} \equiv \xi^P \partial_P H_{MN} + (\partial_M \xi^P - \partial^P \xi_M) H_{PN} + (\partial_N \xi^P - \partial^P \xi_N) H_{MP},
$$

$$
\delta d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M
$$
in the NS-NS sector and

$$
\delta \xi \chi = \mathcal{L}_\xi \chi = \xi^M \partial_M \chi + \frac{1}{\sqrt{2}} \partial \xi^M \Gamma_M \chi
$$

$$
\delta \xi \chi = \xi^M \partial_M \chi + \frac{1}{2} \partial_N \xi_M \Gamma^N \Gamma_M \chi.
$$
in the RR sector, where $\Gamma^{PQ} = \frac{1}{2} [\Gamma^P, \Gamma^Q]$, as in (2.30).

It was shown in [2, 4] (for the NS-NS sector) and in [36] (in the RR sector) that the DFT action is invariant under these gauge transformations. The gauge transformations in the RR sector were determined by demanding that they leave the action invariant as well as demanding compatibility with the gauge transformation rules in the NS-NS sector.

In the frame $\tilde{\partial}^i = 0$, the gauge parameter $\xi^M = (\xi_i, \xi^i)$ combines the diffeomorphism parameter $\xi^i(x)$ and the Kalb-Ramond gauge parameter $\xi_i(x)$. The double field theory version of the abelian gauge symmetry of p-form gauge fields is

$$
\delta \lambda \chi = \partial \lambda = \frac{1}{\sqrt{2}} \Gamma^M \partial_M \lambda,
$$

where $\lambda$ is a space-time dependent spinor.

These gauge transformations form a gauge algebra with respect to the C-bracket, which is the $O(d, d)$ covariantization of the Courant bracket in generalized geometry [43, 45]. The C-bracket of two $O(d, d)$ vectors is given as

$$
[\xi_1, \xi_2]_C^M = 2\xi_1^N \partial_N \xi_2^M - \xi_1^P \partial_P \xi_2^M
$$

(3.23)

The gauge transformations above satisfy

$$
[\delta \xi_1, \delta \xi_2] = -\delta [\xi_1, \xi_2]_C
$$

$$
[\delta \lambda, \delta \xi] = \delta \xi_\lambda
$$

(3.24)
We would like to emphasize that the strong constraint is crucial in proving the closure of the gauge algebra.

The DFT action presented in (3.14) is invariant under the following transformations:

\[ S(X) \rightarrow S'(X') = (S^{-1})^t S(X) S^{-1}, \quad \chi(X) \rightarrow \chi(X') = S\chi(X) \quad (3.25) \]

Here \( S \in Spin^+(d,d) \) and \( X' = hX \), where \( h = \rho(S) \in SO(d,d) \). The dilaton is invariant. The duality group is broken to \( Spin(d,d) \) as the full \( Pin(d,d) \) does not preserve the fixed chirality of the spinor field \( \chi \). Also, a general \( Spin(d,d) \) transformation does not preserve the self-duality constraint (3.15) and the duality group is further reduced to the subgroup \( Spin^+(d,d) \). The transformation of \( S \) implies the following transformation rule for the generalized metric \( H = \rho(S) \):

\[ H(X) \rightarrow H'(X') = (h^{-1})^T H(X) h^{-1} . \quad (3.26) \]

These transformation rules will dictate our duality twisted reduction ansatz in section 4.

### 3.3 The DFT Action of the RR Sector Rewritten With the Mukai Pairing

In this section, we rewrite the DFT action of the RR sector in terms of the Mukai pairing reviewed in Section 2. Writing the action in this form will simplify the calculations, when we study the duality twisted reduction of the action. Besides, the fact that the DFT action (3.14) is an extension of the democratic formulation of supergravity theory becomes explicit in this reformulation.

Recall that the DFT action (3.14), which was constructed in [36] reduces to (3.8) or (3.9) in the supergravity frame \( \tilde{\partial}^i = 0 \), depending on the chirality of \( \chi \). Here, we will start with the supergravity actions (3.8) or (3.9) and show that they extend to the action (3.14), rewritten with the Mukai pairing.

The actions (3.8) or (3.9) are quite simple; in fact they just involve the inner product of \( F^\pm \in S^\pm \) with itself, where the inner product is the natural inner product (2.69). In section 2, we stated how this inner product is related with the Mukai pairing, see (2.70). Therefore these Lagrangians can also be written as

\[ L_{IIA} = \frac{1}{4} \langle F^+, C^{-1}F^+ \rangle, \quad (3.27) \]

and

\[ L_{IIB} = \frac{1}{4} \langle F^-, C^{-1}F^- \rangle, \quad (3.28) \]

where \( \langle , \rangle \) is the Mukai pairing in (2.66). As a matter of fact, we could just as well write

\[ L = \frac{1}{4} \langle F, C^{-1}F \rangle \quad (3.29) \]

with \( F = F^+ + F^- \), as the Mukai pairing is already zero on \( S^+ \times S^+ \) and \( S^- \times S^+ \) for even \( d \) and is zero on \( S^+ \times S^- \) and \( S^- \times S^- \) for odd \( d \). Hence, there is no need to fix the chirality in

\[ ^{13} \text{The transformation of } \chi \text{ in (3.25) implies that } \partial \chi \rightarrow S\partial \chi \text{ and we have } C^{-1}(S^{-1})^t = SC^{-1} \text{ only for } S \in Spin^+(d,d). \]

\[ ^{14} \text{Note that } C^{-1}F^\pm \in S^\pm \text{ in even dimensions and } C^{-1}F^\pm \in S^\mp \text{ in odd dimensions.} \]
this case; the Mukai pairing already picks up the desired combinations. Recall that the charge conjugation matrix has to be written in terms of an orthonormal basis with respect to the metric on \( M \). Alternatively, we can write \( C \) as in (2.54, 2.55) and compensate that by pulling back the differential form \( F \) with the spin representative \( S_g^{-1} \) of the inverse metric \( g^{-1} \). This gives us

\[
L = \frac{1}{4} \langle F, C^{-1}S_g^{-1}F \rangle. \tag{3.30}
\]

Now, it follows from (3.5) that \( F = S_b \phi_x \), where \( S_b \) is as in (2.44) and \( \phi \) is the spinor field encoding the modified gauge potentials \( D_p \), see (3.4), (3.5). Writing (3.30) in terms of \( \phi \) we have

\[
L = \frac{1}{4} \langle S_b \phi_x, C^{-1}S_g^{-1}S_b \phi \rangle. \tag{3.31}
\]

Now we use the invariance property (2.67), which gives

\[
L = \frac{1}{4} \langle \phi, S_b^{-1}C^{-1}S_g^{-1}S_b \phi \rangle. \tag{3.32}
\]

Note that the + sign has to be picked in (2.67) as \( S \) in [36], so our action becomes

\[
L = \frac{1}{4} \langle \phi, C^{-1}S \phi \rangle. \tag{3.34}
\]

When \( \chi = \chi(x) \) and \( S = S(x) \), this action is just a rewriting of the supergravity actions (3.8) and (3.9) in the democratic formulation. On the other hand, when \( \chi = \chi(x, \bar{x}) \) and \( S = S(x, \bar{x}) \), the action (3.34) is equivalent to (3.14) of [36, 37]. Note that, in the first case we have \( \phi(x) = \psi^i \partial_i \chi(x) + \psi_i \bar{\partial} \chi(x, \bar{x}) \), whereas in the DFT extension we have \( \phi(x, \bar{x}) = \psi^i \partial_i \chi(x, \bar{x}) \) and \( \bar{\partial} \chi \).

Let us discuss the transformation properties of this action under (3.25). First of all, note that under \( \chi \to S \chi \) we have \( \phi \to S \phi \). Indeed,

\[
\phi \to \psi^M (h^{-1})_N^M \partial_N (S \chi) = SS^{-1} \psi^M S (h^{-1})_N^M \partial_N \chi
= Sh^M \psi^P (h^{-1})_N^M \partial_N \chi = S \psi^P \partial_P \chi = S \phi. \tag{3.35}
\]

Here \( h = \rho(S) \) and we have used (2.49) (recall that \( \psi^M = 1/\sqrt{2} \Gamma^M \)). As a result, under (3.25), the Lagrangian (3.34) transforms as

\[
L \to \langle S \phi, C^{-1}(S^{-1})^\dagger S \phi \rangle = \langle S \phi, \pm S C^{-1} S \phi \rangle, \quad S \in Spin^\pm(10,10), \tag{3.36}
\]

where we have used (2.64). Now the invariance property (2.67) of the Mukai pairing immediately implies that the Lagrangian is invariant under the whole \( Spin(10,10) \). As we noted above, the democratic action (without introducing the dual coordinates) is already in the form (3.34). However, this action is not invariant under \( Spin(10,10) \) unless we introduce the dual coordinates. Note that, for Riemannian \( g \), this operator is just \( S_e = S_e S_e^\dagger \), where \( g = ec \) and \( S_e \) as is in (2.46) with \( A = e \). For Lorentzian metric, it is a bit more involved, for details see [36]. For our purposes, it is sufficient to know that \( S_g \in Spin^-(10,10) \) and it satisfies \( S_g^{-1} = S_g^\dagger \) and \( S_g = S_g^\dagger \).
coordinates. Indeed, as can be seen from our discussion above, \( \chi \rightarrow S\chi \) implies \( \partial\chi \rightarrow S\partial\chi \) only when the dual coordinates are introduced.

Recall that the self-duality relation (3.15) involved the spin element \( K \in Pin(d,d) \), which we defined as \( K = C^{-1}S \). Consistency imposed \( K^2 = 1 \), which implied that \( d \) has to satisfy \( d \equiv 2, 3 \pmod{4} \), since \( K^2 = -(1)^{d(d-1)/2} \), see (3.18). It is possible to rewrite (3.34) as

\[
L = \frac{1}{4} \langle \partial\chi, K\partial\chi \rangle. \tag{3.37}
\]

Note that for even \( d \), \( K \in Spin^- (d,d) \). Using the invariance property (2.67) we then have (for even \( d \))

\[
L = -\frac{1}{4} \langle K\partial\chi, K^2\partial\chi \rangle. \tag{3.38}
\]

Now we use (3.18) to write

\[
L = (-1)^{d(d-1)/2} \frac{1}{4} \langle K\partial\chi, \partial\chi \rangle \tag{3.39}
\]

It is an important consistency check that the right hand side above can be written as

\[
\frac{1}{4} \langle \partial\chi, K\partial\chi \rangle, \tag{3.40}
\]

which follows immediately from (2.68).

When we impose the constraint (3.15) in the action (3.37), we get

\[
L = -\frac{1}{4} \langle \partial\chi, \partial\chi \rangle, \tag{3.41}
\]

which becomes identically zero for \( d \equiv 2, 3 \pmod{4} \) due to the antisymmetry property of the Mukai pairing in these dimensions. These are exactly the dimensions in which it is consistent to impose the constraint (3.15). This is the usual case with constrained actions and as usual, one must impose the constraint only to the equations of motion, not the action itself.

4 Duality Twisted Reductions of DFT: Gauged Double Field Theory

In the previous section we reviewed the action of DFT describing both the NS-NS and R-R sectors of massless string theory. The DFT action of the NS-NS sector has global \( Pin(d,d) \) symmetry. When one includes the RR sector, this symmetry group is reduced to \( Spin(d,d) \) due to the chirality condition and is further reduced to \( Spin^+(d,d) \) due to the existence of the self-duality constraint (3.15). This global symmetry group makes it possible to implement a duality twisted anzats in the dimensional reduction of the DFT action. More precisely, the transformation rule (3.25) for the fundamental fields in the theory make it possible to introduce the following duality twisted dimensional reduction anzats:

\[
S(X,Y) = (S^{-1})^T(Y)S(X)S^{-1}(Y) \tag{4.1}
\]

\[
\chi(X,Y) = S(Y)\chi(X) \tag{4.2}
\]
Here, $X$ denote collectively the coordinates of the reduced theory, whereas $Y$ denote the internal coordinates, which are to be integrated out. The twist matrix $S(Y)$ belongs to the duality group $Spin^+(d,d)$ and encodes the whole dependence on the internal coordinates.

The above anzats for the spinor fields implies the following anzats in the NS-NS sector:

$$H_{MN}(X,Y) = U^A_M(Y)H_{AB}(X)U^B_N(Y). \tag{4.3}$$

The duality twisted dimensional reduction of the DFT action of the NS-NS sector with the anzats (4.3) has already been studied by several groups [19–21], and the resulting theory was dubbed Gauged Double Field Theory (GDFT) [21]. For the details of the reduction of the action and the gauge transformations of the dimensionally reduced theory, we refer the reader to these papers. Here, we also study the duality twisted reduction of the DFT action describing the RR sector.

In the reduction of the NS-NS sector, it is also possible to introduce the following anzats for the generalized dilaton [21]

$$d(X,Y) = d(X) + \rho(Y). \tag{4.4}$$

This then leads to an overall conformal rescaling in the NS-NS sector

$$\mathcal{L}_{NS-NS} \rightarrow e^{-2\rho(Y)}\mathcal{L}_{NS-NS}. \tag{4.5}$$

This overall factor contributes to the volume factor, when one integrates out the $Y$ coordinates in order to define the GDFT action of the NS-NS sector [21]:

$$S_{GDFT} = v \int d^N X e^{-2d(\mathcal{R} + \mathcal{R}_f)} \tag{4.6}$$

where $\mathcal{R}_f$ is determined by the fluxes $f_{ABC}$ and $\eta_A$, as we will discuss in the next subsection and $v$ is defined as

$$v = \int d^dYe^{-2\rho(Y)}. \tag{4.7}$$

In the presence of the RR fields, the GDFT action will be of the form

$$S_{GDFT} = v \int d^N X [(\mathcal{L}_{NS-NS} + \mathcal{L}_{RR}) + (\mathcal{L}_{NS-NS} + \mathcal{L}_{RR})_{def}] \tag{4.8}$$

In order to induce the overall $\rho$-dependent factor in the RR sector, it is necessary to modify the anzats (4.2) as follows

$$\chi(X,Y) = e^{-\rho(Y)}S(Y)\chi(X). \tag{4.9}$$

In the next two subsections, we will study the GDFT action arising from the introduction of the anzats (4.1, 4.3, 4.4) and (4.9). Before we move on, let us clarify a point. Comparing (4.3) with (3.26), we see that we have $U = h^{-1}$, where $h = \rho(S) \in SO^+(d,d)$. In other words we have $U = (\rho(S))^{-1} = \rho(S^{-1})$. The reason that we have made this naming (rather than naming $\rho(S) = U$) is to make sure that our notation is consistent with that of the papers mentioned above, especially that of [21]. Again, following [21], we make a distinction between the indices of the parent theory and the indices of the resulting theory, which we label by $M$ and $A$, respectively.
4.1 Review of the Reduction of the NS-NS Sector

In the duality twisted reduction of the NS-NS sector, there are two main conditions to be imposed on the twist matrix $U$: Firstly, one demands that the Lorentzian coordinates $X$ remain untwisted, which is ensured if the following condition is satisfied by all the $X$ dependent fields of the resulting GDFT:

$$
(U^{-1})^M_A \partial_M g(X) = \partial_A g(X). \tag{4.10}
$$

The second condition is

$$
\partial^P (U^{-1})^M_A \partial_P g(X) = 0. \tag{4.11}
$$

This is trivially satisfied if one works with twist matrices such that a given coordinate and its dual are either both external or both internal. If the ansatz involves a non-zero $\rho(Y)$ in (4.4), a condition similar to (4.11) has to be imposed also on $\rho$:

$$
\partial^P \rho \partial_P g(X) = 0. \tag{4.12}
$$

As was shown in [19–21], all the information about the twist matrix $U$ is encoded in the entities $f_{ABC}$ and $\eta_A$ that we will define below. These entities, which we will refer to as "fluxes", as is usual in the literature, determine both the deformation of the action and that of the gauge algebra. The situation is entirely the same in the RR sector as we will discuss shortly. The fluxes are defined as

$$
f_{ABC} = 3\Omega_{[ABC]}, \quad \eta_A = \partial_M (U^{-1})^M_A - 2(U^{-1})^M_A \partial_M \rho \tag{4.13}
$$

where $\rho$ is as in (4.4) and

$$
\Omega_{ABC} = -(U^{-1})^M_A \partial_M (U^{-1})^N_B U^D_N \eta_{CD}. \tag{4.14}
$$

Note that $\Omega_{ABC}$ are antisymmetric in the last two indices: $\Omega_{ABC} = -\Omega_{ACB}$. We also make the following definition

$$
f_A = -\partial_M (U^{-1})^M_A = \Omega^C_A. \tag{4.15}
$$

It can be shown that the conditions (4.10) and (4.11) imply that the following has to be satisfied:

$$
f^A_{\quad BC} \partial_A g(X) = 0, \quad f^A \partial_A g(X) = 0. \tag{4.16}
$$

Note that the second condition in (4.16) and (4.12) imply together that $\eta_A$ should also satisfy

$$
\eta^A \partial_A g(X) = 0. \tag{4.17}
$$

These constraints are crucial for the closure of the gauge algebra. In addition, one also needs that all the fluxes $f_{ABC}$ and $\eta_A$ must be constant. This ensures that the $Y$ dependence in the GDFT is completely integrated out. Also, the weak and the strong constraint has to be imposed on the external space so that

$$
\partial_A \partial^A V(X) = 0, \quad \partial_A V(X) \partial^A W(X) = 0 \tag{4.18}
$$
for any fields or gauge parameters $V, W$ that has dependence on the coordinates of the external space only. Finally, the following Jacobi identity and the orthogonality condition should be satisfied for the closure of the gauge algebra:

$$f_{E|AB} f_{C|D}^E = 0,$$

and\textsuperscript{16}

$$\eta^A f_{ABC} = 0. \quad (4.20)$$

To summarize, for the consistency of the reduction of the DFT of the NS-NS sector one needs the conditions (4.10-4.12) and (4.16-4.20). In addition, the fluxes $f_{ABC}$ and $\eta_A$ must be constant. These are the only conditions that have to be satisfied in order to obtain a consistent GDFT.

Surprisingly, it is not necessary to impose the strong constraint in the internal space, that is, one does not need to impose

$$\partial^P U^A_M \partial_P U^B_N. \quad (4.21)$$

Therefore, the duality twisted anzats (4.1)-(4.4) allows for a relaxation of the strong constraint on the total space.

### 4.2 Reduction of The RR sector

Our aim here is to study the reduction of (3.34) and the constraint (3.15). Recall the main relation (2.49), which we rewrite here for $U = \rho(S^{-1})$:

$$S^{-1} \Gamma^M S = (U^{-1})^M_A \Gamma^A \quad (4.22)$$

Now, we plug in the anzats

$$\chi(X, Y) = e^{-\rho(Y)} S(Y) \chi(X) \quad (4.23)$$

in $\partial \chi(X, Y)$ in (3.34) to get

$$\sqrt{2} \partial \chi(X, Y) = \Gamma^M \partial_M \chi(X, Y) = \Gamma^M \partial_M (e^{-\rho(Y)} S(Y) \chi(X))$$

$$= e^{-\rho(Y)} \left\{-\Gamma^M S \partial_M \rho(Y) + \Gamma^M S \partial_M + \Gamma^M S(S^{-1} \partial_M S)\right\} \chi(X)$$

$$= e^{-\rho(Y)} S(Y) \Gamma^A (- (U^{-1})^M_A \partial_M \rho(Y) + \partial_A + (U^{-1})^M_A S^{-1} \partial_M S) \chi(X) \quad (4.24)$$

where, in passing from the second line to the third, we have used (4.10) and (4.22).

Recall that the Lie algebras of $Spin(d, d)$ and $O(d, d)$ are isomorphic. This gives us the important property:

$$\Gamma^A (U^{-1})^M_A S^{-1} \partial_M S = \frac{1}{4} \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C, \quad (4.25)$$

which we prove now.

As $U$ and $S$ are in the connected component of the orthogonal group and the Spinor group, they can be written as

$$[\exp \left( \frac{1}{2} \Lambda_{PQ} (Y) T^{PQ} \right)], \quad (4.26)$$

\textsuperscript{16}This condition does not appear in [21], as they constrain $\eta_A = 0$.\textsuperscript{23}
where the generators $T_{MN}$ are in the fundamental representation for the $SO^{\pm}(D,D)$ matrix $U$, whereas it is in the spinor representation for $S$, see section [2]. Therefore, we have

$$(U^{-1})^M_A = \left[ \exp \left( \frac{i}{2} \Lambda_{PQ}(Y) T^{PQ} \right) \right]^M_A , \quad \text{and} \quad S = \left[ \exp \left( \frac{i}{2} \Lambda_{PQ}(Y) \frac{1}{2} \Gamma^{PQ} \right) \right] (4.27)$$

Now we prove (4.25) starting from the right hand side:

$$\frac{1}{4} \Omega_{ABC} \Gamma^B \Gamma^C = - \frac{1}{4} (U^{-1})^M_A (U^D_N \partial_M (U^{-1})^N_B) \eta_{CD} \Gamma^B \Gamma^C$$

$$= - \frac{1}{4} (U^{-1})^M_A (U \partial_M U^{-1})^D_B \eta_{CD} \Gamma^B \Gamma^C$$

$$= - \frac{1}{4} (U^{-1})^M_A \frac{1}{2} \partial_M \Lambda_{PQ}(T^{PQ})^D_B \eta_{CD} \Gamma^B \Gamma^C$$

$$= - \frac{1}{4} (U^{-1})^M_A \frac{1}{2} \partial_M \Lambda_{PQ}(\eta^{DP} \delta^Q_B - \eta^{DP} \delta^P_B) \eta_{CD} \Gamma^B \Gamma^C$$

$$= - \frac{1}{4} (U^{-1})^M_A \frac{1}{2} (\partial_M \Lambda_{CB} - \partial_M \Lambda_{BC}) \Gamma^B \Gamma^C$$

$$= \frac{1}{2} (U^{-1})^M_A \partial_M \Lambda_{BC} \frac{1}{4} (\Gamma^B \Gamma^C - \Gamma^C \Gamma^B)$$

$$= (U^{-1})^M_A S^{-1} \partial_M S$$ (4.28)

which immediately implies (4.25). As a result, we have:

$$\phi \chi(X,Y) = \frac{1}{\sqrt{2}} \Gamma^M \partial_M \chi(X,Y) = e^{-\rho(Y)} S(Y) \nabla \chi(X),$$

where we have defined

$$\nabla \chi(X) \equiv (\phi + \frac{1}{4 \sqrt{2}} \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C) \chi(X) - \frac{1}{\sqrt{2}} \Gamma_A (U^{-1})^M_A \partial_M \rho(Y) \chi(X).$$

Here, one might be puzzled that it is $\Omega_{ABC}$ rather than the $f_{ABC}$ and $\eta_A$ which appear in the reduced Lagrangian. After all, it is $f_{ABC}$ and $\eta_A$ and not $\Omega_{ABC}$, which are constrained to be constant by the consistency requirement of the reduction of the NS-NS sector. However, the following can be shown by using the commutation relations in the Clifford algebra:

$$\frac{1}{4} \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C \chi(X) = \frac{1}{12} f_{ABC} \Gamma^A \Gamma^B \Gamma^C \chi(X) - \frac{1}{2} f_B \Gamma^B \chi(X)$$

Indeed

$$\frac{1}{4} \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C = \frac{1}{12} \left( \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C + \Omega_{BCA} \Gamma^B \Gamma^C \Gamma^A + \Omega_{CBA} \Gamma^C \Gamma^B \Gamma^A \right)$$

$$= \frac{1}{12} \left( \Omega_{ABC} + \Omega_{BCA} + \Omega_{CAB} \right) \Gamma^A \Gamma^B \Gamma^C$$

$$+ \frac{1}{12} \left( \Omega_{BCA} (2 \eta^{AC} \Gamma^B - 2 \eta^{AB} \Gamma^C) + \Omega_{CAB} (2 \eta^{AC} \Gamma^B - 2 \eta^{CB} \Gamma^A) \right)$$

$$= \frac{1}{12} f_{ABC} \Gamma^A \Gamma^B \Gamma^C - \frac{1}{2} f_B \Gamma^B$$ (4.32)

where we have used the definitions (4.13) and (4.15) and the Clifford algebra identity

$$\Gamma^A \Gamma^B \Gamma^C = \Gamma^B \Gamma^C \Gamma^A - 2 \eta^{AC} \Gamma^B + 2 \eta^{B \Lambda} \Gamma^C.$$ (4.33)
Plugging this back in (4.30), we get
\[
\nabla \chi(X) \equiv (\phi + \frac{1}{2\sqrt{2}} f_{ABC} \Gamma^A \Gamma^B \Gamma^C + \frac{1}{2\sqrt{2}} \eta_B \Gamma^B) \chi(X) \\
= (\phi + \frac{1}{6} f_{ABC} \psi^A \psi^B \psi^C + \frac{1}{2} \eta_B \psi^B) \chi(X).
\] (4.34)

The Dirac operator \(\nabla\) is the same as the Dirac operator introduced in [15, 49], where they study backgrounds with non-geometric fluxes within the context of flux formulation of DFT and \(\beta\)-supergravity, respectively, without performing any duality twisted reduction (see also the associated papers [19, 50] and [22]). It was shown in [49] that the Bianchi identities for the NS-NS fluxes are satisfied, only when this Dirac operator is nilpotent. We will discuss this condition of nilpotency further at the end of this subsection, when it reappears as a condition to be satisfied for the gauge invariance of the GDFT of the RR sector.

4.2.1 Reduction of The Lagrangian

The reduced Lagrangian can be obtained easily. If we plug (4.29) and (4.1) in (3.34) we have
\[
e^{2\rho(Y)} L_{\text{red}} = \frac{1}{4} \langle S \nabla \chi(X), C^{-1}(S^{-1})^\dagger SS^{-1}S \nabla \chi(X) \rangle \\
= \frac{1}{4} \langle \nabla \chi(X), C^{-1}S \nabla \chi(X) \rangle \\
= \frac{1}{4} \langle \phi \chi(X), C^{-1}S \phi \chi(X) \rangle \\
+ \frac{1}{4} \langle \bar{\chi}, C^{-1}S \bar{\phi} \chi(X) \rangle + \frac{1}{4} \langle \bar{\phi} \chi(X), C^{-1}S \bar{\chi} \rangle \\
+ \frac{1}{4} \langle \bar{\chi}, C^{-1}S \bar{\chi} \rangle,
\] (4.35)

where \(\bar{\chi} = \frac{1}{12\sqrt{2}} f_{ABC} \Gamma^A \Gamma^B \Gamma^C + \frac{1}{2\sqrt{2}} \eta_A \Gamma^A \chi\). Note that, we have used (2.64) and (2.67) in passing from the first line to the second line.

The term (4.36) is the undeformed part of the Lagrangian. The two terms in (4.37) are equivalent as can be seen as follows:\(^{17}\)
\[
\langle \bar{\chi}, C^{-1}S \bar{\phi} \chi(X) \rangle = -\langle C^{-1}S \bar{\chi}, C^{-1}S \bar{\phi} \chi(X) \rangle = -\langle C^{-1}S \bar{\chi}, \bar{\phi} \chi(X) \rangle = \langle \bar{\phi} \chi(X), C^{-1}S \bar{\chi} \rangle.
\]

Here we have used the fact that \(K = C^{-1}S \in \text{Spin}^{-}(10,10)\), which explains the minus sign in applying (2.67) and that \(K^2 = 1\) and the Mukai pairing is skew-symmetric in 10 dimensions. Now the two terms in (4.37) add up to give:
\[
\frac{1}{2} \langle \bar{\chi}, C^{-1}S \bar{\phi} \chi \rangle = \frac{1}{2} \langle S_b \bar{\chi}, S_b C^{-1}S_b^\dagger S^{-1}S_b \bar{\phi} \chi \rangle = \frac{1}{2} \langle S_b \bar{\chi}, C^{-1}S_b S^{-1}S_b \bar{\phi} \chi \rangle,
\] (4.39)

where we have used (2.64) and the invariance property (2.67) along with the fact that \(S_b \in \text{Spin}^{+}(10,10)\). We have also plugged in the definition \(S = S_1 S_2^{-1} S_b\). Note that here \(b = b(X),\)

\(^{17}\)Note that \(\bar{\chi}\) and \(\chi\) have different chiralities. On the other hand, \(C^{-1}S \bar{\chi}\) and \(\bar{\chi}\) have the same chirality in 10 dimensions. Hence, \(\phi \chi\) and \(C^{-1}S \bar{\chi}\) have the same chirality.
\[ g = g(X) \text{ and } \chi = \chi(X), \text{ as all the } Y \text{ dependence in } S \text{ factorized out already in the first step. Also, } \text{requirement of constancy of } f_{ABC} \text{ and } \eta_A \text{ implies that } \bar{\chi} = \bar{\chi}(X). \]

One can similarly compute the term (4.38) and find that the reduced Lagrangian (4.35) has the form

\[ e^{2\rho(Y)} L_{\text{red}} = \frac{1}{4} \langle F(X), C^{-1} S^{-1} F(X) \rangle + \frac{1}{2} \langle F(X), C^{-1} S^{-1} \bar{\chi}_B \rangle + \frac{1}{4} \langle \bar{\chi}_B, C^{-1} S^{-1} \bar{\chi}_B \rangle. \]  

(4.40)

Here we have defined \( F(X) = S_b \partial_X(X) = e^{-B} \wedge \partial \chi(X) \) and \( \bar{\chi}_B = S_b \bar{\chi} = e^{-B} \wedge \bar{\chi} \). If the internal coordinates \( X \) include no dual coordinates, then this can be written as follows

\[ e^{2\rho(Y)} L_{\text{red}} = \frac{1}{4} F(X) \wedge * F(X) + \frac{1}{2} F(X) \wedge * \bar{\chi}_B + \frac{1}{4} \bar{\chi}_B \wedge * \bar{\chi}_B, \]  

(4.41)

where \( * \) is the Hodge duality operator with respect to the reduced metric \( g(X) \).

On the other hand, the constraint reduces to

\[ \nabla \chi(X) = -C^{-1} S \nabla \chi(X) \]  

(4.42)

as can be shown easily by recalling the definition \( S^\dagger = C \tau(S) C^{-1} \) and the fact that \( \tau(S) = S^* = S^{-1} \) for \( S \in Spin^+(n, n) \).

### 4.2.2 Reduction of The Gauge Algebra

In order to find the gauge transformation rules for the reduced theory we plug the anzatse

\[ \xi^M(X, Y) = (U^{-1})^M_A \hat{\xi}_A(X) \]  

(4.43)

\[ \chi(X, Y) = e^{-\rho(Y)} S(Y) \chi(X) \]  

(4.44)

in the gauge transformation rules (3.20), (3.22) of the parent theory. This gives us the following deformed gauge transformations for the spinor field \( \chi \):
\begin{align}
\delta \chi &= \xi^M \partial_M (e^{-\rho(Y)} S(Y) \chi) + \frac{1}{2} \partial_N \xi_M \Gamma^N \Gamma^M (e^{-\rho(Y)} S(Y) \chi) \\
&= (U^{-1})^M_A \xi^A \partial_M (e^{-\rho(Y)} S(Y) \chi) + \frac{1}{2} \partial_M (U^{-1}_N \xi^A \Gamma^M \Gamma^N (e^{-\rho(Y)} S(Y) \chi) \\
&= e^{-\rho(Y)} S(Y) \{ \xi^A \partial_A - \xi^A (U^{-1})^M_A \partial_M \rho(Y) + \xi^A (U^{-1})^M_A (S^{-1} \partial_M S) \} \\
&+ \frac{1}{2} U^{-1}_N \partial_M \xi^A (U^{-1})^M_B (U^{-1})^N_C \Gamma^B \Gamma^C + \frac{1}{2} (\partial_M U^{-1}_N) \xi^A (U^{-1})^M_B (U^{-1})^N_C \Gamma^B \Gamma^C \chi \\
&= e^{-\rho(Y)} S(Y) \{ \xi^A \partial_A + \frac{1}{2} \partial_B \xi^A \Gamma^B \Gamma^C - \xi^A (U^{-1})^M_A \partial_M \rho(Y) \} \\
&+ \left( \frac{1}{4} \Omega^A_{BC} - \frac{1}{2} \Omega^A_{B C} \Gamma^B \Gamma^C \right) \chi \\
&= e^{-\rho(Y)} S(Y) \left\{ \xi^A \partial_A + \frac{1}{2} \partial_B \xi^A \Gamma^B \Gamma^C + \frac{1}{2} \Omega^A_{B C} \Gamma^B \Gamma^C \right\} \\
&= e^{-\rho(Y)} S(Y) \left\{ \xi^A \partial_A + \frac{1}{2} \partial_B \xi^A \Gamma^B \Gamma^C + \frac{1}{2} \eta^A \xi^A \right\} \\
&= e^{-\rho(Y)} S(Y) \{ \delta \chi \} \quad (4.46)
\end{align}

where we have used (4.10), (4.13), (4.14), (4.22), (4.25) and Clifford algebra identities. In the last two lines we made the following definitions:

\begin{align}
\hat{\delta} \chi &= \delta \chi + \frac{1}{4} f^A_{BC} \xi^A \Gamma^B \Gamma^C \chi + \frac{1}{2} \eta^A \xi^A \\
\hat{\delta} \chi &= \xi^A \partial_A + \frac{1}{2} \partial_B \xi^A \Gamma^B \Gamma^C \chi \quad (4.47)
\end{align}

On the other hand, the deformation of the gauge transformation \( [3.22] \) is found by plugging in the anzats

\[ \lambda(X, Y) = e^{-\rho(Y)} S(Y) \tilde{\lambda}(X), \]

which then gives

\begin{align}
\delta \lambda &= \hat{\phi} (e^{-\rho(Y)} S(Y) \tilde{\lambda}) \\
&= \frac{e^{-\rho(Y)}}{\sqrt{2}} \left( - \Gamma^M \partial_M \rho(Y) S(Y) \tilde{\lambda} + \Gamma^M (\partial_M S) \tilde{\lambda} + \Gamma^M \partial_M \tilde{\lambda} \right) \\
&= \frac{e^{-\rho(Y)}}{\sqrt{2}} \Gamma^M \left( - \partial_M \rho(Y) + \partial_M + S^{-1} \partial_M S \right) \tilde{\lambda} \\
&= \frac{e^{-\rho(Y)}}{\sqrt{2}} S \left( (U^{-1})^M_A \Gamma^A \left( - \partial_M \rho(Y) + \partial_M + S^{-1} \partial_M S \right) \right) \tilde{\lambda} \\
&= \frac{e^{-\rho(Y)}}{\sqrt{2}} S \left( (U^{-1})^M_A \Gamma^A - \partial_M \rho(Y) \right) \tilde{\lambda} \\
&= \frac{e^{-\rho(Y)}}{\sqrt{2}} S (\Gamma^A \partial_A + \frac{1}{4} \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C - (U^{-1})^M_A \partial_M \rho(Y)) \tilde{\lambda} \\
&= \frac{e^{-\rho(Y)}}{\sqrt{2}} S (\Gamma^A \partial_A + \frac{1}{12} f_{ABC} \Gamma^A \Gamma^B \Gamma^C + \eta_A \Gamma^A) \tilde{\lambda} \\
&= e^{-\rho(Y)} S(Y) (\nabla \tilde{\lambda}) + e^{-\rho(Y)} S(Y) (\hat{\delta} \chi). \quad (4.51)
\end{align}
4.2.3 Consistency of the Reduced Theory

Now that we have the deformed action and the deformed gauge transformation rules, we can analyze the conditions under which the GDFT of the RR sector is consistent. Consistency is achieved if

1. $Y$ dependence drops both in the reduced action and the gauge algebra.

2. The reduced action is invariant under the deformed gauge transformation rules.

3. The gauge algebra closes.

One can show that the constraints that arise from the consistency of the reduction of the DFT of the NS-NS sector, that is, the constancy of the fluxes $f_{ABC}$ and $\eta_A$ and the conditions (4.10-4.12) and (4.16-4.20) are sufficient to satisfy the first and third items in the list above. When these conditions are satisfied, the deformed gauge transformations we found above close to form a gauge algebra as follows:

$$[\hat{\delta}_{\hat{\xi}_1}, \hat{\delta}_{\hat{\xi}_2}] \chi = -\hat{\delta}_{[\hat{\xi}_1, \hat{\xi}_2]} f^{A B C} \Gamma^B \Gamma^C \chi$$

where

$$[\hat{\xi}_1, \hat{\xi}_2] f^{A B C} \Gamma^B \Gamma^C \chi$$

and

$$\hat{\delta}_{\hat{\lambda}} \hat{\xi} = \hat{\xi} \Gamma^A \partial_A \hat{\lambda} + \frac{1}{2} \partial_B \hat{\xi}_B \Gamma^B \Gamma^C \hat{\lambda} + \frac{1}{4} f^{A B C} \hat{\xi}_A \Gamma^B \Gamma^C \hat{\lambda} + \frac{1}{2} \eta_A \hat{\xi}_A \hat{\lambda},$$

as can be verified by a tedious calculation. In addition to the conditions (4.10-4.12) and (4.16-4.20), one also needs the Clifford algebra identity (4.33) and the following two identities:

$$(S^{-1} \partial_A S)(S^{-1} \partial_B S) = -(\partial_A S^{-1})(\partial_B S)$$

$$\Gamma^A \Gamma^B \Gamma^C \Gamma^D = \Gamma^C \Gamma^D \Gamma^A \Gamma^B + 2\eta^{CB} \Gamma^A \Gamma^D - 2\eta^{DB} \Gamma^A \Gamma^C + 2\eta^{AC} \Gamma^D \Gamma^B - 2\eta^{DA} \Gamma^C \Gamma^B$$

The last identity follows directly from the Clifford algebra.

The requirement of gauge invariance of the GDFT action of the RR sector imposes one more constraint on the fluxes. Recall that the DFT analogue of the p-form gauge transformation of the RR fields has been deformed as

$$\hat{\delta}_{\hat{\lambda}} \chi = \nabla \hat{\lambda}.$$
nilpotency has already appeared in [49]. Let us now work out the square of the Dirac operator. One can show that

\[ 2\nabla^2 \chi = \partial_A \partial^A \chi + \frac{1}{2} f_{ABC} \Gamma^A \Gamma^B \partial^C \chi - \eta_A \partial^A \chi - \frac{1}{4} f_{ABC} \eta^C \Gamma^A \Gamma^B \chi \\
+ \frac{1}{4} \eta_A \eta^A \chi + \frac{1}{16} f_{ABC} f_{DE} \Gamma^A \Gamma^B \Gamma^D \Gamma^E \Gamma^C \chi. \]  

(4.59)

When the conditions (4.16, 4.17, 4.18, 4.20) are satisfied, the first line of the above expression vanishes. On the other hand, applying the Jacobi identity (4.19), one can show that the last term of the second line can be rewritten as

\[ \frac{1}{16} f_{ABC} f_{DEF} \Gamma^A \Gamma^B \Gamma^D \Gamma^E \Gamma^C \chi = -\frac{1}{8} f_{ABC} f_{ABC} \chi - \frac{1}{8} f_{ABC} f_{ADEF} \Gamma^A \Gamma^B \Gamma^D \Gamma^E \Gamma^C \chi, \]  

(4.60)

where one uses in passing to the second line the fact that \( f_{ADEF} = 0 \). Then, we conclude that, up to the constraints that are required for the consistency of the GDFT of the NS-NS sector, we have

\[ \nabla^2 \chi = (2\eta_A \eta^A - f_{ABC} f^{ABC}) \chi = 0. \]  

(4.61)

Now let us consider the gauge invariance of the deformed action (4.35) under the deformed gauge transformations with parameter \( \hat{\xi} \), for which we need \( \hat{\delta}_\hat{\xi} K \) and \( \hat{\delta}_\hat{\xi} \nabla \chi \). In order to calculate the first, one first has to note that the anzats (4.1) implies the following anzats for \( K = C^{-1} S \):

\[ K(X,Y) = S(Y)K(X)S^{-1}(Y). \]  

(4.62)

Then using similar steps to above in the calculation of the deformed gauge transformations for \( \chi \), one finds\(^{18}\)

\[ \hat{\delta}_\hat{\xi} K = \hat{\xi}^A \partial_A K + \frac{1}{2} \{ \Gamma^{AB}, K \} (\partial_A \hat{\xi}^B + \frac{1}{2} f_{CAB} \hat{\xi}^C). \]  

(4.64)

On the other hand, one can compute

\[ \hat{\delta}_\hat{\xi} (\nabla \chi) = \hat{\xi}^A \partial_A (\nabla \chi) + \frac{1}{2} \partial_B \hat{\xi}^C \Gamma^B \Gamma^C \chi + \frac{1}{4} f_{CDE} \hat{\xi}^D \Gamma^C \Gamma^D + \frac{1}{2} \eta^B \hat{\xi}^B \} \nabla \chi. \]  

(4.65)

Plugging these in (4.35) one finds\(^{19}\) only using the constraints (4.10, 4.12) and (4.16, 4.20)

\[ \hat{\delta}_\hat{\xi} L_{\text{def}} = \hat{\xi}^A \partial_A L_{\text{def}} + \partial_A \hat{\xi}^A L_{\text{def}} + \eta^A \hat{\xi}^A L_{\text{def}}. \]  

(4.66)

Therefore, the deformed Lagrangian is gauge invariant only when the fluxes \( \eta^A \) vanish.\(^{20}\) Combined with (4.61), following from the requirement of nilpotency of the Dirac operator, we

\^\text{18}\text{One also needs the following identity in the proof, which had not been needed before:}

\[ S^{-1} \Gamma^{MN} S = \Gamma^{AB} (U^{-1})^M_A (U^{-1})^N_B. \]  

(4.63)

\^\text{19}\text{The details are similar to those in section 4.2.2 of [36].}

\^\text{20}\text{It has already been noted in [21] that these fluxes should vanish for the gauge invariance of the GDFT of the NS-NS sector. It was also pointed out that this can be circumvented by considering a modified reduction anzats, that involves a warp factor, as in [19]. It would be interesting to see whether the GDFT of the RR sector would remain gauge invariant also for non-vanishing \( \eta \) fluxes, by introducing such a warp factor.}
conclude that the requirement of gauge invariance of the GDFT of the RR sector brings in the extra condition

$$f_{ABC}f^{ABC} = 0. \quad (4.67)$$

The necessity of this extra constraint in the presence of RR fields had already been anticipated in [17, 15] and had been verified by the analysis of [22]. As was mentioned in section (1), the constraints of the GDFT (of the NS-NS sector) are in one-to-one correspondence with the constraints of half-maximal gauged supergravities. This extra condition we have found implies that the gauged theory in hand corresponds to a truncation of maximal supergravity [51]. We also note that the gauge invariance of the duality relations (4.42) can also be verified easily, and does not impose any extra constraints.

Before we finish, let us also comment on a possible modification of the anzats (4.9), which introduces gaugings associated with non-trivial RR fluxes. Note that the DFT action of the RR sector (3.14) is invariant under the global shift symmetry $\chi \rightarrow \chi + \alpha$, which would make it possible to introduce an anzats of the form $\chi(X,Y) = \chi(X) + \tilde{\alpha}(Y)$. However, the gauge transformation rules (3.20) has an explicit dependence on $\chi$, which then means that the $Y$ dependence arising from such an anzats would not drop from the reduced gauge transformation rules.\footnote{We also note that it is possible to modify the gauge transformation rules so as to be invariant under the global shift as is done in [40].} One can still consider introducing such an anzats by choosing the spinor field $\tilde{\alpha}(Y)$ appropriately. Indeed, one can take

$$\tilde{\alpha}(Y) = e^{-\rho(Y)}S(Y)\alpha, \quad (4.68)$$

where $\alpha$ is a constant spinor field. Then, the anzats (4.9) can be modified to

$$\chi(X,Y) = e^{-\rho(Y)}S(Y)(\chi(X) + \alpha). \quad (4.69)$$

When combined with the anzats (4.1), (4.4), (4.43) and (4.49), a reduction with the modified anzats (4.69) leads to a consistent theory, for which the $Y$ dependence drops from the action and the gauge transformations. Plugging (4.69) in $\Gamma^M \partial_M \chi(X,Y)$, one finds (we now take $\eta^A = 0$)

$$\phi\chi(X,Y) = \frac{1}{\sqrt{2}}\Gamma^M \partial_M (e^{-\rho(Y)}S(Y)\chi(X)) + \frac{1}{\sqrt{2}}\Gamma^M \partial_M (e^{-\rho(Y)}S(Y)\alpha)$$

$$= e^{-\rho(Y)}S(Y)\nabla\chi(X) + \frac{1}{\sqrt{2}}e^{-\rho(Y)}S(Y)(U^{-1})^A_B\Gamma_A\Gamma_B\Gamma^C\alpha$$

$$= e^{-\rho(Y)}S(Y)\{\nabla\chi(X) + \frac{1}{12\sqrt{2}}f_{ABC}\Gamma^A\Gamma^B\Gamma^C\alpha\}, \quad (4.70)$$

where we have used the identities (4.22), (4.25) and (4.31) and $\nabla$ is as before.\footnote{Note that we have not taken the terms associated with the derivative of $\rho$ into account, as they combine with $f_A$ in (4.31) to give $\eta_A$, which we take zero now.} The reduced theory in this case is

$$L_{\text{red}} = \frac{1}{4}\langle \nabla\chi + \frac{1}{12\sqrt{2}}f_{ABC}\Gamma^A\Gamma^B\Gamma^C\alpha, \mathcal{K}(\nabla\chi + \frac{1}{12\sqrt{2}}f_{ABC}\Gamma^A\Gamma^B\Gamma^C\alpha)\rangle. \quad (4.71)$$

The Lagrangian (4.71) has to be supplemented by the following duality relation

$$\nabla\chi + \frac{1}{12\sqrt{2}}f_{ABC}\Gamma^A\Gamma^B\Gamma^C\alpha = -\mathcal{K}(\nabla\chi + \frac{1}{12\sqrt{2}}f_{ABC}\Gamma^A\Gamma^B\Gamma^C\alpha). \quad (4.72)$$
The Lagrangian and the duality relation is invariant under the following gauge transformation
\[ \delta_{\hat{\xi}}\chi = \hat{\xi}^A \partial_A \chi + \frac{1}{2} \partial_B \hat{\xi}_C \Gamma^B \Gamma^C (\chi + \alpha) + \frac{1}{4} f^{A}_{BC} \hat{\xi}_A \Gamma^B \Gamma^C (\chi + \alpha). \]  

(4.73)

5 Conclusions and Outlook

In this paper, we studied the duality twisted reduction of the Double Field Theory of the RR sector of massless Type II theory. This sector of DFT has a global Spin\(^+(n,n)\) symmetry, which we have utilized to introduce the duality twisted anzats. We obtained the reduced action and the gauge transformation rules and showed that the gauge algebra closes. The fact that the Lie algebras of Spin\((n,n)\) and SO\((n,n)\) are isomorphic plays a crucial role in our analysis.

Our reduction anzats is determined by a Spin\(^+(n,n)\) element \(S\). Under the double covering homomorphism \(\rho\) between Spin\((n,n)\) and SO\((n,n)\), the twist element \(S\) projects to an element \(U \in SO^+(n,n)\). This then implies a duality twist in the accompanying reduction of the NS-NS sector of DFT, through the matrix \(U\). The duality twisted reduction of the NS-NS sector has already been studied by several groups [19–21]. As was shown in these works, the consistency of the reduced theory and its gauge algebra places restrictions on the fluxes determined by the twist matrix \(U\). It was shown in [21] that these constraints are in one-to-one correspondence with the constraints of half-maximal gauged supergravity. All these constraints are also crucial for the consistency of the GDFT of the RR sector. In addition, we have shown here that the requirement of gauge invariance in the RR sector imposes the extra constraint (4.67), which also appeared in [22]. It is known that any half-maximal gauged supergravity that satisfies this constraint can be uplifted to maximal gauged supergravity [51]. Therefore, the existence of this extra constraint can be seen as a sign that the reduction we have studied here should be related to duality twisted reductions of Exceptional Field Theory (EFT), which is a U-duality invariant extension of supergravity [52–54]. Indeed, the reduction of EFT on generalised parallelisable manifolds [55] (which corresponds to a reduction with a duality twisted anzats of the type we have considered here) gives rise to maximal gauged supergravity upon imposing a section constraint, which is the analogue of the strong constraint of DFT [56–58]. A flux formulation of (a particular type of) EFT is also available and geometric and non-geometric RR fluxes were studied also in this formulation [59]. For recent work on how to truncate such theories further to half-maximal gauged supergravities, see [60,61].

An interesting feature of our reduced action is the natural appearance of the nilpotent Dirac operator (4.34), associated with the spinorial covariant derivative acting in the RR sector. This Dirac operator has already appeared in various papers before (e.g. [15,22,49]). It was shown in [49] that the Bianchi identities for the NS-NS fluxes are satisfied, only when this Dirac operator is nilpotent and the same condition arises here from the analysis of the gauge invariance of the GDFT of the RR sector. Note that the flux dependent terms in the Dirac operator involves (products of) Gamma matrices. As we discussed in section [2], the spinorial action of these Clifford algebra elements on the spinor field \(\chi\) (which is equivalently a differential form) is by contraction, when they belong to the orthogonal complement of the vector subspace, whose exterior algebra carries the spinorial representation of the Clifford algebra. In other words, the Gamma matrices with a lower index act on the spinor fields by contraction. For certain choices
of twists, this gives the possibility of inducing 0-forms as deformation terms in the reduced action. We will explore this feature in [62], where we study massive deformations of Type IIA theory within DFT.

In analyzing the reduction of the DFT Lagrangian of the RR sector, we found it useful to rewrite it in terms of Mukai pairing, which is a $\text{Spin}(n,n)$ invariant bilinear form on the space of spinors. We believe that the Lagrangian, when written in the form (3.34) is worth further study. Note that (3.34) gives a non-vanishing $n$-form, which is a volume form when the underlying manifold $M$ is $n$ dimensional as in generalized geometry of Hitchin. As a Lagrangian for DFT, it gives us an $n$-form on the $2n$ dimensional doubled manifold. However, it is a very special $n$-form. Recall that the spinor field $\chi$ is a section of the restricted exterior bundle $\bigwedge^\bullet T^*\mathcal{M}_{\text{doub}}$, in the sense that at each point the cotangent space is restricted to a maximally isotropic subspace with respect to the metric $\eta$. This then implies that the $n$-form produced by the Lagrangian (3.34) belongs to a 1-dimensional subspace of the $(2n)!/(n)!(n)!$ dimensional space of all possible $n$-forms on a $2n$ dimensional manifold, as it can have components only along these $n$ restricted directions. (Note that the value of the form still depends on the $\tilde{x}$ coordinates of the manifold). Then, one can naturally identify this $n$-form with a scalar function (a 0-form), which then becomes the Lagrangian density to be integrated on the whole doubled manifold. It would be desirable to come up with a Lagrangian which produces a volume form for the whole doubled manifold. This obviously calls for a better understanding of the differential geometric features of doubled manifolds. We believe that this direction deserves further study and we hope to come back to these issues in future work.

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