ALGEBRAIC COMPACTNESS OF $\prod M_{\alpha}/\bigoplus M_{\alpha}$

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Abstract: In this note, we are working within the category $R\text{Mod}$ of (unitary, left) $R$-modules, where $R$ is a countable ring. It is well known (see e.g. Kiełpiński & Simson [5], Theorem 2.2) that the latter condition implies that the (left) pure global dimension of $R$ is at most 1. Given an infinite index set $A$, and a family $M_{\alpha} \in R\text{Mod}$, $\alpha \in A$ we are concerned with the conditions as to when the $R$-module

$$\prod_{\alpha \in A} M_{\alpha}/\bigoplus_{\alpha \in A} M_{\alpha}$$

is or is not algebraically compact. There are a number of special results regarding this question and this note is meant to be an addition to and a generalization of the set of these results. Whether the module in the title is algebraically compact or not depends on the numbers of algebraically compact and non-compact modules among the components $M_{\alpha}$.

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Given an (infinite) cardinal $\kappa$, an $R$-module $M$ is $\kappa$-compact, if, every system of $\leq \kappa$ linear equations over $M$ (with unknowns $x_j$ and almost all $r_{ij} = 0$):

$$\sum_{j \in J} r_{ij} x_j = m_i \in M, \quad i \in I, \quad r_{ij} \in R, \quad |I|, |J| \leq \kappa \quad (1)$$

has a solution in $M$ whenever all its finite subsystems have solutions (in $M$). A module is (algebraically) compact if it is $\kappa$-compact, for every cardinal $\kappa$. It is well-known that if $M \in R\text{Mod}$ is $\kappa$-compact, for some $\kappa \geq |R|$, then $M$ is algebraically compact. Algebraic compactness of $M$ is equivalent to pure injectivity and this in turn is equivalent to $\text{Pext}_R^1(X, M) = 0$, for every $X \in R\text{Mod}$.

Recall that $\prod / \coprod$ is a special case of a more general construction of the reduced product $\prod M_\alpha / \mathcal{F}$, where $\mathcal{F}$ is the cofinite filter on $A$. Given a subset $B \subseteq A$, then $\mathcal{F} \cap B$ and $\mathcal{F} \cap (A \setminus B)$ are cofinite filters on $B$ and on $A \setminus B$ respectively, if $\mathcal{F}$ is the cofinite filter on $A$. One can now easily prove the following isomorphism (alternatively use Theorem 1.10 in [2]):

$$\prod_{\alpha \in A} M_\alpha / \bigoplus_{\alpha \in A} M_\alpha \cong \prod_{\alpha \in B} M_\alpha / \bigoplus_{\alpha \in B} M_\alpha \times \prod_{\alpha \in A \setminus B} M_\alpha / \bigoplus_{\alpha \in A \setminus B} M_\alpha. \quad (2)$$

The proof of the following result is straightforward, since it uses a powerful classical result of Mycielski.

**Proposition 1.** For every countable index set $B$,

$$\prod / \prod = \prod_{\alpha \in B} M_\alpha / \bigoplus_{\alpha \in B} M_\alpha$$

is an algebraically compact $R$-module.

**Proof.** Since $B$ is countable, there is a countable family of cofinite subsets of $B$ with empty intersection. By a classical result of Mycielski [6, Theorem 1], $\prod / \prod$ is $\aleph_0$-compact. This is equivalent to its algebraic compactness, since the rings we consider here are countable. □

Note that this result need not hold true, if $R$ is uncountable. For instance, if $K$ is a field and $R = K[[X, Y]]$ is the two-variable power series algebra, then $R^\mathbb{N} / R^{(\mathbb{N})}$ is not algebraically compact (see [4], Theorem 8.42).
Lemma 2. Assume that pure global dimension of $R$ is $\leq 1$. If $E : 0 \rightarrow A \rightarrow^* B \rightarrow C \rightarrow 0$ is a pure exact sequence and $B$ is pure injective, then $C$ is likewise pure injective (algebraically compact).

Proof. Given an arbitrary $X \in R\text{Mod}$, the segment of the $\text{Pext}^1_R(X, E)$ exact sequence we are interested in is as follows: $\ldots \rightarrow \text{Pext}^1_R(X, B) \rightarrow \text{Pext}^1_R(X, C) \rightarrow \text{Pext}^2_R(X, A) \rightarrow \ldots$ Since $\text{puregld } R \leq 1$ we have $\text{Pext}^2_R(X, A) = 0$. Since $B$ is pure injective, we have $\text{Pext}^1_R(X, B) = 0$. These facts now force $\text{Pext}^1_R(X, C) = 0$, i.e. $C$ is pure injective. \[\square\]

Proposition 3. Let $\text{puregld } R \leq 1$ and let $A$ be an arbitrary (infinite) index set; if every $M_\alpha$, $\alpha \in A$ is algebraically compact, then $\prod_{\alpha \in A} M_\alpha/\bigoplus_{\alpha \in A} M_\alpha$ is algebraically compact.

Proof. It is well known that $\prod = \bigoplus_{\alpha \in A} M_\alpha$ is a pure submodule of $\prod = \prod_{\alpha \in A} M_\alpha$ and that $\prod$ is algebraically compact iff all the components $M_\alpha$ are algebraically compact. Appeal to Lemma 2 completes the proof. \[\square\]

Theorem 4. Given any index set $A$, let $B \subseteq A$ be (at most) a countable set and $\forall \alpha \in B$, $M_\alpha$ is not algebraically compact, while $\forall \alpha \in A \setminus B$, $M_\alpha$ is algebraically compact. Then

$$\prod / \prod = \prod_{\alpha \in A} M_\alpha/\bigoplus_{\alpha \in A} M_\alpha$$

is algebraically compact.

Proof. By Proposition 1, the $R$-module $\prod_{\alpha \in B} M_\alpha/\bigoplus_{\alpha \in B} M_\alpha$ is algebraically compact. By Proposition 3, $\prod_{\alpha \in A \setminus B} M_\alpha/\bigoplus_{\alpha \in A \setminus B} M_\alpha$ is likewise algebraically compact. Now use isomorphism (2) to conclude that $\prod / \prod$ is algebraically compact. \[\square\]

Our main concern is the converse of Theorem 4: If $\prod / \prod$ is algebraically compact, can we conclude that at most countably many $M_\alpha$’s are not algebraically compact?

Every linear system (1) has a short-hand representation $\mu \cdot x = m$, where $\mu = (r_{ij})_{i \in I, j \in J}$ is the corresponding row-finite matrix (call it the system matrix) and $x = (x_j)_{j \in J}$, $m = (m_i)_{i \in I}$ are the corresponding column vectors. The rows of matrix $\mu$ (which are the left hand sides of equations (1)) may be viewed as elements of the free $R$-module $\bigoplus_{j \in J} Rx_j$. The cardinality of these $R$-modules is $|R|2^{|J|}$. Thus the cardinality of the
set of different matrices $\mu$ representing (left-hand-sides) of (1) is at most $(|R||J|)^{|I|} = |R||I|^{|J||I|}$. For purposes of algebraic compactness, it suffices to consider only $|I| = |J| = \max(|R|, \aleph_0)$, thus the latter cardinality is at most $\max(2^{|R|}, 2^\aleph_0)$; for countable rings this bound is $2^{\aleph_0}$. This is an important fact that we use in the proof of the next result.

**Proposition 5.** Let $|A| > \max(2^{|R|}, 2^\aleph_0)$ and $\forall \alpha \in A, M_\alpha$ is not algebraically compact. Then $\prod_i M_\alpha / \oplus M_\alpha$ is not algebraically compact.

**Proof.** For every $M_\alpha, \alpha \in A$, there is a system of equations of type (1)

$$S_\alpha : \sum_{j \in J} r_{ij}^\alpha x_j^\alpha = m_i^\alpha \in M_\alpha, \quad i \in I, \quad r_{ij} \in R, \quad |I| = |J| = \max(|R|, \aleph_0)$$

with the corresponding row finite system matrices $\mu_\alpha = (r_{ij}^\alpha)_{i \in I, j \in J}$ and the property that every finite subsystem is solvable, without the whole system being solvable. By the observation on the number of different system matrices $\mu_\alpha$, the number of different left hand sides of systems $S_\alpha$ is $\max(2^{|R|}, 2^\aleph_0)$. By the assumption on the cardinality of $A$, we conclude that there are $|A|$ many systems $S_\alpha$ with identical left hand sides. Without loss of generality we assume this is correct for all $\alpha \in A$, thus we consider systems (3) where the coefficients $r_{ij}^\alpha = r_{ij}$ do not vary by coordinates $\alpha \in A$. This coefficient uniformity enables a passage to the induced system in $\prod_i M_\alpha / \oplus M_\alpha$:

$$S : \sum_{j \in J} r_{ij}(x_j^\alpha)_{\alpha \in A} = (m_i^\alpha)_{\alpha \in A} \quad i \in I,$$

(bars denote the classes mod $\oplus_{\alpha \in A} M_\alpha$). Every finite subsystem of $S$ is equivalent to the set of coordinate finite subsystems of $S_\alpha$, for all but finitely many $\alpha \in A$. These have solutions, which will be the coordinates of the solutions of the original finite subsystem of $S$. But $S$ has no global solution, for if $x_j = (s_j^\alpha)_{\alpha \in A}, j \in J$ were global solutions of $S$, then $x_j^\alpha = s_j^\alpha, j \in J$ would provide global solutions of $S_\alpha$, for almost all $\alpha \in A$. This contradiction then completes the proof that $\prod_i M_\alpha / \oplus M_\alpha$ is not algebraically compact. \qed

As we have not succeeded in extending the latter result to all infinite $|A|$, we formulate the following
Conjecture. If $|A|$ is an uncountable index set of cardinality $\leq 2^{|R|}$ and all $M_\alpha \in R\text{Mod}$, $\alpha \in A$, are not algebraically compact, then $\prod/\bigoplus$ is not algebraically compact. If this is true then, for countable rings $R$, $\prod/\bigoplus$ is algebraically compact if and only if all but countably many $M_\alpha \in R\text{Mod}$, $\alpha \in A$ are algebraically compact.

Remarks. There are strong indications the conjecture is correct: Gerstner [3] proved that $\mathbb{Z}^A/\mathbb{Z}^{(A)}$ is algebraically compact, if $A$ is countable. A generalization follows for reduced powers of modules over countable rings: If $M \in R\text{Mod}$ is not algebraically compact, then use Lemma 1.2 in [1] to conclude that if $M^A/M^{(A)}$ is algebraically compact then $A$ must be countable. For Abelian groups, Rychkov [7] proved that $\prod/\bigoplus$ is algebraically compact if and only if $A$ is countable. In fact, if $S$ denotes a set of system matrices with the property that for every $M \in R\text{Mod}$ that is not algebraically compact, there is a $\mu \in S$ that is a system matrix for a system proving algebraic non-compactness of $M$, let $n$ denote minimal cardinality of all such systems. Close inspection of the proof of Proposition 1, ibid. seems to reveal that the RD-purity used there is not essential, namely that it may be replaced by purity (a condition always satisfied for Prüfer domains). In that case, if $|A| > \max(n, \aleph_0)$ and all $M_\alpha$, $\alpha \in A$ are non-compact implies that $\prod/\bigoplus$ is non-compact.

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