Abstract. A function in a class $\mathcal{F}(X)$ is said to be subdifferentially determined in $\mathcal{F}(X)$ if it is equal up to an additive constant to any function in $\mathcal{F}(X)$ with the same subdifferential. A function is said to be subdifferentially representable if it can be recovered from a subdifferential. We identify large classes of lower semicontinuous functions that possess these properties.

Keywords: Dini derivative, $ACG_*$ function, Henstock-Kurzweil integral, radial subderivative, subdifferential determination, subdifferential representation.

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1 Introduction

In this paper we are interested in two fundamental links between functions and subdifferentials: the subdifferential determination and the subdifferential representation of functions. A function in a class $\mathcal{F}(X)$ is said to be subdifferentially determined in $\mathcal{F}(X)$ if it is equal up to an additive constant to any function in $\mathcal{F}(X)$ with the same subdifferential. A function is said to be subdifferentially representable if it can be expressed in terms of a subdifferential, or put another way, if it can be recovered from a subdifferential.

The subdifferential determination of extended-real-valued lower semicontinuous convex functions defined on Hilbert spaces was brought to light by Moreau [20]. His result was later extended to general Banach spaces by Rockafellar [25]. In the non-convex case, the first works are due to Rockafellar [26] for the class of (Clarke) regular locally Lipschitz functions, and to Poliquin [22] for the class of primal lower nice functions with possibly extended-real values. Since then, this property has been considered for various classes of functions; let us mention the works of Correa-Jofré [9], Qi [23, 24], Birge-Qi [3], Thibault-Zagrodny [30, 31, 32], Borwein-Moors [7], Thibault-Zlateva [33], Bernard-Thibault-Zagrodny [2], Zajíček [34] and our recent work [17]. The subdifferential representation of extended-real-valued lower semicontinuous convex functions defined on a Banach space was established by Rockafellar [25]; different proofs of this result are proposed by Taylor [28], Thibault [29] and Ivanov-Zlateva [13], and a refined version by Benoist-Daniilidis [1]. Few results exist for non-convex functions; let us mention Qi [23] and Birge-Qi [3].

In the present article we study the subdifferential determination and the subdifferential representation properties with respect to an abstract subdifferential. Our abstract subdifferential recovers the Clarke, the Michel-Penot and the Ioffe subdifferentials in any Banach space, and the elementary subdifferentials (proximal, Fréchet, Hadamard, ...), as well as their viscosity and limiting versions, in appropriate Banach spaces. The subdifferential determination property is considered for different classes of functions lying between the class $LL(X)$ of locally Lipschitz functions and the class $Lsc(X)$ of extended-real-valued lower semicontinuous functions. The subclasses of subdifferentially determined functions are defined according to the continuity properties of their lower Dini derivative, independently of the subdifferential. In this introduction we give a brief overview of the above-mentioned works and their connections with the present work.

The class $LL(X)$ of locally Lipschitz functions is the natural framework for many applications in nonsmooth analysis and optimization. However, further assumptions are in general necessary to get sharpened
results. The most widely used assumptions are \textit{regularity} and \textit{semismoothness}: the regularity introduced by Clarke [8] makes it possible to obtain equality in the calculus rules for the (Clarke) subdifferential, the semismoothness proposed by Mifflin [18, 19] allows implementable algorithms for nonsmooth constrained optimization. The common feature of these assumptions is the postulation of a relationship between the lower Dini derivative (hereinafter called the \textit{radial subderivative}) of $f$ at $\bar{x}$,

$$f^{r}(\bar{x}; u) := \liminf_{t \downarrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t},$$

and the \textit{Clarke subdifferential} of $f$ at $\bar{x}$,

$$\partial C f(\bar{x}) := \{ x^* \in X^* : \langle x^*, u \rangle \leq f^o(\bar{x}; u), \forall u \in X \},$$

where

$$f^o(\bar{x}; u) := \limsup_{x \to \bar{x}} \frac{f(x + tu) - f(x)}{t}.$$ 

More precisely, a locally Lipschitz function $f : X \to \mathbb{R}$ is called \textit{regular at a point} $\bar{x}$ provided for every $u \in X$,

$$f^{r}(\bar{x}; u) = \max\{ \langle x^*, \bar{x} \rangle : x^* \in \partial C f(\bar{x}) \} (= f^o(\bar{x}; u)) \quad (1)$$

and \textit{semismooth at $\bar{x}$} provided for every $u \in X$,

$$x_n \rightharpoonup u \bar{x} \text{ and } x^*_n \in \partial C f(x_n) \Rightarrow \langle x^*_n, u \rangle \to f^{r}(\bar{x}; u), \quad (2)$$

where $x_n \rightharpoonup u \bar{x}$ indicates that $x_n = \bar{x} + t_n u_n$ with $t_n \searrow 0$ and $u_n \to u$. These two properties are independent: there are regular functions that are not semismooth, and semismooth functions that are not regular. The functions that are both regular and semismooth are characterized by the submonotonicity of their (Clarke) subdifferential; lower-$C_1$ functions are examples of such functions (see Spingarn [27]). It is well known that the properties of regularity and semismoothness can be reformulated in terms of the continuity of the radial subderivative; indeed, (1) and (2) are respectively equivalent to

$$f^{r}(\bar{x}; u) = \limsup_{x \to \bar{x}} f^{r}(x; u), \quad (1b)$$

$$f^{r}(\bar{x}; u) = \lim_{x \to u \bar{x}} f^{r}(x; u) \quad (2b)$$

(see for example [26] and [9]). There is one significant difference between the formulas (1b)-(2b) and the definitions (1)-(2), namely, the former can be extended beyond the Clarke subdifferential. In this paper we will consider a property that generalizes both: a locally Lipschitz function $f$ is said to be \textit{upper semismooth at a point} $\bar{x}$, \textit{in the direction} $u$ [17], if

$$f^{r}(\bar{x}; u) = \limsup_{x \to u \bar{x}} f^{r}(x; u). \quad (3)$$

Two stronger notions are naturally associated with the Clarke subdifferential of locally Lipschitz functions. We say that a locally Lipschitz function $f$ is \textit{strictly differentiable at $\bar{x}$, in the direction} $u$ [6], if

$$\lim_{t \searrow 0} \frac{f(x + tu) - f(x)}{t} \text{ exists;}$$
equivalently (see the proof of [8, Proposition 2.2.4]),

\[ f^\alpha(\bar{x}; u) = f^\alpha(\bar{x}; u) = -f^\alpha(\bar{x}; -u). \]

We say that \( f \) is strictly differentiable at \( \bar{x} \), if \( f \) is strictly differentiable at \( \bar{x} \) in every direction \( u \); equivalently, \( \partial_C f(\bar{x}) \) is a singleton ([8, Proposition 2.2.4]).

Let us now go back to the subdifferential determination property in the framework of locally Lipschitz functions. This property has been demonstrated for the Clarke subdifferential by Rockafellar [26] on \( \mathbb{R}^n \) for everywhere regular functions (Corollary 3 of Theorem 2); by Correa-Jofré [9] on a Banach space, for densely Gateaux differentiable and everywhere regular functions (Proposition 4.1) and for densely Gateaux differentiable and everywhere semismooth functions (Proposition 5.4) (note that the hypothesis of dense Gateaux differentiability is automatically satisfied on \( \mathbb{R}^n \) by Rademacher’s theorem, on separable Banach spaces by Christensen’s theorem, on smooth spaces by Preiss’ theorem); by Qi [23] on \( \mathbb{R}^n \) for the so-called primal functions, i.e. the functions that are strictly differentiable almost everywhere and \( D \)-representable (in fact, in \( \mathbb{R}^n \) the functions that are strictly differentiable almost everywhere are always \( D \)-representable, see [4, Corollary 3.10] or [5, Corollary 4.2]); examples of primal functions are the functions almost everywhere regular [26, Corollary 2 of Theorem 2] and the functions everywhere semismooth [24, Theorem 1]. All these results have been generalized by Borwein-Moors ([7, Theorem 3.8]), still for the Clarke subdifferential, on arbitrary Banach spaces, for the so-called essentially smooth functions, i.e. the functions \( f \) such that for each \( u, f \) is strictly differentiable in the direction \( u \) almost everywhere (in the sense of Haar). The proof of [7, Theorem 3.8] lies in the fact that the Clarke subdifferential mapping of such functions is a minimal weak*-cuscó. In Subsection 5.4, we study the subdifferential determination property in the class \( \text{CLACG}(G) \) of all real-valued Continuous functions defined on a nonempty open convex subset \( G \) of a Banach space \( X \), whose restrictions to Line segments \( [a, b] \subset G \) are ACG (see Section 2 for the definition). This class includes the locally Lipschitz functions defined on \( G \). In this class, we consider the subclass \( \text{CLACG}^{2ad}(G) \) of densely almost upper semismooth functions, i.e. the functions \( f \) satisfying the following property: for every \( \bar{x}, u \in X \) with \( [\bar{x}, \bar{x} + u] \subset G \) there exist sequences \( \bar{x}_n \rightarrow \bar{x}, u_n \rightarrow u \) such that for almost all \( x_n \in [\bar{x}_n, \bar{x}_n + u_n] \), \( f^\alpha(x_n; u_n) \) is finite and \( f \) is upper semismooth at \( x_n \) in the direction \( u_n \). We show that this subclass contains the Borwein-Moors essentially smooth functions (Proposition 5.3) and is subdifferentially determined for the abstract subdifferential in the class \( \text{CLACG}(U) \) (Theorem 5.2). This theorem appears as a continuous variant of our main theorem which concerns lower semicontinuous functions.

Let us now discuss the subdifferential determination property for the class of extended-real-valued lower semicontinuous functions. In this general context, the conditions (1b) and (3) are no longer suitable. In fact, we must take into account the value \( f^\alpha(x; \bar{x} - x) \) which now does not necessarily tend towards 0 when \( x \) tends towards \( \bar{x} \): we must integrate this factor in the formulas. Similar phenomena occur every time we are dealing with unbounded values; see, for example, the definition (10) below of the closure \( \bar{\partial} f \) of the subdifferential \( \partial f \) of a lower semicontinuous function \( f \), the discussion in [14, Subsection 2.1] on the closure of the convex subdifferential, and the references therein. Thus, the conditions (1b) and (3) are respectively replaced by

\[
\begin{align*}
  f^\alpha(\bar{x}; u) &= \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}} f^\alpha(x; u + \alpha(\bar{x} - x)), \\
  f^\alpha(\bar{x}; u) &= \inf_{\alpha \geq 0} \limsup_{x \to u \bar{x}} f^\alpha(x; u + \alpha(\bar{x} - x)).
\end{align*}
\]

Clearly, the conditions (4) and (5) reduce to (1b) and (3), respectively, when the function \( f \) is Lipschitz around \( \bar{x} \).
In [17] we have shown that the following lower semicontinuous functions $f$ satisfy (4) at every point of every segment $[\bar{x}, \bar{x} + u] \subset \text{dom } f$: the convex functions, the directionally approximately convex functions and the \( \partial \)-subdifferentially and directionally stable functions of Thibault-Zagrodny [31]. More precisely, all these functions belong to the subclass $\text{Lsc}^{\gamma}(X)$ of the functions $f \in \text{Lsc}(X)$ with convex domain $\text{dom } f$ such that for every $[\bar{x}, \bar{x} + u] \subset \text{dom } f$ and for all points $\bar{x}_t \in [\bar{x}, \bar{x} + u]$, $f^r(\bar{x}_t; u) < +\infty$ and (4) is satisfied. Functions in $\text{Lsc}^{\gamma}(X)$ have a powerful continuity property along line segments, namely, the restriction of every $f \in \text{Lsc}^{\gamma}(X)$ to any line segment $[\bar{x}, \bar{x} + u] \subset \text{dom } f$ is continuous at the endpoints and locally Lipschitz at every point in $[\bar{x}, \bar{x} + u]$ (Proposition 4.1). Finally, we show that the class $\text{Lsc}^{\gamma}(X)$ is subdifferentially determined in $\text{Lsc}(X)$ (Theorem 4.2 (1)), thus extending the special case $\gamma = 0$ in [31, Theorem 4.1]. A slight variant of Theorem 4.2 (1), with a more complicated formulation, had been previously established in [17, Theorem 11].

Next we consider the classes $\text{LC}(X)$ and $\text{LACG}(X)$ of extended-real-valued lower semicontinuous functions whose restrictions to line segments are, respectively, continuous and ACG. In these (slightly) smaller classes, we expect to identify subclasses of subdifferentially determined functions larger than the subclass $\text{Lsc}^{\gamma}(X)$. This is indeed the case! We sketch the plan for the class $\text{LACG}(X)$. We denote by $\text{LACG}^{\gamma}(X)$ the subclass of those functions $f \in \text{LACG}(X)$ with convex domain $\text{dom } f$ such that for every $[\bar{x}, \bar{x} + u] \subset \text{dom } f$ and for almost all points $\bar{x}_t \in [\bar{x}, \bar{x} + u]$, $f^r(\bar{x}_t; u)$ is finite and (5) is satisfied. We show that $\text{LACG}^{\gamma}(X)$ is indeed larger than $\text{Lsc}^{\gamma}(X)$, contains the lower semicontinuous $\partial$-essentially directionally smooth functions of Thibault-Zagrodny [32] (Proposition 5.1) and is subdifferentially determined in $\text{LACG}(X)$ (Theorem 4.2 (3)). This extends [32, Theorem 3.4].

Finally, we identify a large class of lower semicontinuous functions which can be recovered from their (abstract) subdifferential via an integration formula, namely, the functions $f \in \text{LACG}_*(X)$ with convex domain $\text{dom } f$ such that for every $[\bar{x}, \bar{x} + u] \subset \text{dom } f$ and for almost all points $\bar{x}_t \in [\bar{x}, \bar{x} + u]$, (5) is satisfied (Theorem 4.3). This subclass of subdifferentially representable functions contains, among other things, the extended-real-valued lower semicontinuous convex functions considered by Rockafellar [25] and the primal functions on $\mathbb{R}^n$ considered by Qi [23, Theorem 9]. A detailed description of this subclass is given in Subsection 5.3.

It should be noticed that some interesting results mentioned above are not recovered by our approach: the subdifferential determination of primal lower nice functions and the like, studied in Poliquin [22], Thibault-Zagrodny [30] and Bernard-Thibault-Zagrodny [2]; the local subdifferential determination of regular directionally Lipschitz functions established by Thibault-Zlateva [33]; the subdifferential determination and the subdifferential representation of locally Lipschitz functions, in finite dimensional spaces, for the Michel-Penot subdifferential, given by Birge-Qi [3]; the subdifferential determination of the locally Lipschitz functions that are essentially smooth on a generic line parallel to a generic direction, in Asplund spaces, for the Clarke subdifferential, proven by Zajiček [34, Proposition 7.5].

This paper is a continuation of our works [16, 17]. In [16], we have established a formula linking the radial subderivative to other subderivatives and subdifferentials. In [17], we have proved a simple version of the subdifferential determination property without resorting to measure and integration theories. Here we propose a more precise statement of the subdifferential determination problem and we provide new contributions based this time on measure and integration theories. Moreover, we establish the subdifferential representation property for a large class of extended-real-valued lower semicontinuous functions. As in our paper [17], the technique for demonstrating the main theorems in the present paper relies on our formula linking subderivative and subdifferential [16] that reduces the original subdifferential problem on a Banach space to a problem involving the radial subderivative on the real line. The theory of ACG functions and
Henstock-Kurzweil integrals is then used to address this reduced problem. The relevant definitions and facts from this theory are gathered in Section 2. Subderivatives and subdifferentials are described in Section 3. The main results are stated and proved in Section 4. The last section contains examples and variants.

2 Links between functions and subderivatives on the real line

In this section, we have compiled the relevant facts concerning the ACG functions and the Henstock-Kurzweil integral in connection with our subject. Most of these facts have been taken from Gordon’s textbook [10], which offers a thorough analysis of this integral.

Throughout the section, \([a, b]\) denotes a compact interval of \(\mathbb{R}\), \(\varphi : \mathbb{R} \rightarrow [\pm \infty, +\infty]\) an extended-real-valued function and \(\text{dom } \varphi := \{x \in \mathbb{R} : \varphi(x) < +\infty\}\) its effective domain. The lower and upper right-hand Dini derivatives of \(\varphi\) at \(t_0 \in \text{dom } \varphi\) are respectively given by

\[
D_+ \varphi(t_0) := \liminf_{t \searrow 0} \frac{\varphi(t_0 + t) - \varphi(t_0)}{t}, \quad D^+ \varphi(t_0) := \limsup_{t \searrow 0} \frac{\varphi(t_0 + t) - \varphi(t_0)}{t}.
\]

For points \(t_0 \not\in \text{dom } \varphi\), the Dini derivatives are defined to be \(-\infty\). We say that \(\varphi\) is right-differentiable at \(t_0\) if

\[
\lim_{t \searrow 0} \frac{\varphi(t_0 + t) - \varphi(t_0)}{t}
\]

exists in \(\mathbb{R}\), or equivalently, if \(D_+ \varphi(t_0)\) is finite and \(D^+ \varphi(t_0) = D^+ \varphi(t_0)\).

Let \(S \subset [a, b]\). The function \(\varphi\) is absolutely continuous (AC) on \(S\) if for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(\sum_{i=1}^{n} |\varphi(d_i) - \varphi(c_i)| < \varepsilon\) whenever \(\{[c_i, d_i] : 1 \leq i \leq n\}\) is a finite collection of non-overlapping intervals that have endpoints in \(S\) and satisfy \(\sum_{i=1}^{n} (d_i - c_i) < \delta\). The function \(\varphi\) is generalized absolutely continuous (ACG) on \(S\) if \(\varphi|_S\) is continuous on \(S\) and \(S\) can be written as a countable union of sets on each of which \(\varphi\) is AC.

A property is said to hold almost everywhere in \(S\), or for almost all \(t \in S\), if it holds in \(S\) except for a set of (Lebesgue) measure zero. As in [10], we say that a property holds nearly everywhere in \(S\), or for nearly all \(t \in S\), if it holds in \(S\) except for a countable set.

**Theorem 2.1** (Subderivative test for monotonicity). Let \(\varphi : \mathbb{R} \rightarrow [\pm \infty, +\infty]\), \(a \in \text{dom } \varphi\) and \(b \in \mathbb{R}\) with \(b > a\). Then \(\varphi\) is nonincreasing on \([a, b]\) in each of the following situations:

1. \(\varphi\) is lower semicontinuous on \([a, b]\) and \(D^+ \varphi(t) \leq 0\) everywhere in \([a, b]\),
2. \(\varphi\) is continuous on \([a, b]\) and \(D^+ \varphi(t) \leq 0\) nearly everywhere in \([a, b]\),
3. \(\varphi\) is ACG on \([a, b]\) and \(D^+ \varphi(t) \leq 0\) almost everywhere in \([a, b]\).

**Proof.**

(1) First, we observe that \([a, b] \subset \text{dom } \varphi\). Indeed, let \(c \in [a, b]\) and let \(\mu \in \mathbb{R}\) such that \(\mu \leq \varphi(c) - \varphi(a)\). Applying the mean value inequality [15, Lemma 4.1], we get a point \(x_0 \in [a, c]\) such that \(\mu \leq (c-a)D_+ \varphi(x_0) \leq 0\). Consequently, \(\varphi(c) - \varphi(a) \leq 0\), hence \(c \in \text{dom } \varphi\). Now, let \(c, d \in [a, b]\), \(c < d\). Then, \(c, d \in \text{dom } \varphi\). Applying again the mean value inequality, we get a point \(x_0 \in [c, d]\) such that \(\varphi(d) - \varphi(c) \leq (d-c)D_+ \varphi(x_0) \leq 0\). Hence, \(\varphi(d) \leq \varphi(c)\).

(2) This is a special case of [17, Proposition 3].

(3) See, e.g., [10, Theorem 6.25]. \(\square\)
Let $\omega(\varphi, [c, d]) := \sup\{\varphi(y) - \varphi(x) : c \leq x \leq y \leq d\}$ denote the oscillation of the function $\varphi$ on the interval $[c, d]$. The function $\varphi : [a, b] \to \mathbb{R}$ is absolutely continuous in the restricted sense ($AC_r$) on $S \subset [a, b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n \omega(\varphi, [c_i, d_i]) < \varepsilon$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in $S$ and $\sum_{i=1}^n (d_i - c_i) < \delta$. The function $\varphi$ is generalized absolutely continuous in the restricted sense ($ACG_r$) on $S \subset [a, b]$ if $\varphi|_S$ is continuous on $S$ and $S$ can be written as a countable union of sets on each of which $\varphi$ is $AC_r$.

If $\varphi$ is AC on the interval $[a, b]$, then $\varphi$ is $AC_r$ on $[a, b]$; if $\varphi$ is continuous on $[a, b]$ and AC on every interval $[c, d] \subset [a, b]$, then $\varphi$ is $ACG_r$ on $[a, b]$ since $\varphi$ is $AC_r$ on each of the sets $\{a\}, \{b\}$ and $[a+1/n, b-1/n], n \in \mathbb{N}$. Clearly, $ACG_r$ implies AC. The converse is not true: an $ACG$ function is not necessarily differentiable almost everywhere [10, Example 6.20] while an $ACG_r$ function is:

**Fact 2.2** ([10, Corollary 6.19, Exercise 7.9, Theorem 6.22]). Let $\varphi : [a, b] \to \mathbb{R}$ be continuous.

1. If $\varphi : [a, b] \to \mathbb{R}$ is $ACG_r$ on $[a, b]$, then $\varphi$ is $AC$ on $[a, b]$ and differentiable almost everywhere on $[a, b]$. The converse is not true.

2. If $\varphi$ is right-differentiable nearly everywhere on $[a, b]$, then $\varphi$ is $ACG_r$ on $[a, b]$.

A function $\varphi : [a, b] \to \mathbb{R}$ is *Henstock-Kurzweil integrable* (HK-integrable) on $[a, b]$ with integral $\Phi_{a,b} \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists a positive function $\delta : [a, b] \to ]0, +\infty[$ such that

$$\left|\sum_{i=1}^n \varphi(t_i)(d_i - c_i) - \Phi_{a,b}\right| < \varepsilon$$

whenever $\Pi := \{(t_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is a finite collection of non-overlapping tagged intervals (i.e. $t_i \in [c_i, d_i]$) that satisfy $(d_i - c_i) < \delta(t_i)$ for all $i$. If it exists, the integral $\Phi_{a,b}$ is uniquely defined and we write

$$\Phi_{a,b} = \int_a^b \varphi(t)dt := \lim_{HK,\Pi} \sum_{i=1}^n \varphi(t_i)(d_i - c_i).$$

Moreover, if $\varphi$ is HK-integrable on $[a, b]$, then $\varphi$ is HK-integrable on every subinterval of $[a, b]$ (see, e.g., [11, Theorem 4] or [10, Theorem 9.8]), hence for every $x \in [a, b]$, the so-called indefinite HK-integral

$$\Phi_{a,x} = \int_a^x \varphi(t)dt$$

exists. It turns out that a function $\Phi$ is an indefinite HK-integral on $[a, b]$ if and only if $\Phi$ is $ACG_r$ on $[a, b]$. More precisely:

**Fact 2.3** ([10, Theorem 9.17 and Exercise 11.7]). Let $\varphi, \Phi : [a, b] \to \mathbb{R}$ be two functions. The following statements are equivalent:

(a) $\varphi$ is HK-integrable on $[a, b]$ and $\Phi(x) - \Phi(a) = \int_a^x \varphi(t)dt$ for all $x \in [a, b]$;

(b) $\Phi$ is $ACG_r$ on $[a, b]$ and $\Phi'(t) = \varphi(t)$ almost everywhere in $[a, b]$.

In view of Fact 2.3, we can consider integrating functions $\varphi$ that are only defined and finite almost everywhere in $[a, b]$: such a function $\varphi : [a, b] \to \mathbb{R}$ is declared HK-integrable on $[a, b]$ if there exists a finite-valued HK-integrable function $\psi : [a, b] \to \mathbb{R}$ such that $\varphi(t) = \psi(t)$ almost everywhere in $[a, b]$. Then $\int_a^x \varphi(t)dt := \int_a^x \psi(t)dt$ and the integral thus defined is independent of the chosen function $\psi$. 
Theorem 2.4 (Subderivative representation of functions). A function \( \varphi : [a, b] \rightarrow \mathbb{R} \) can be represented through its subderivative via the HK-integration formula

\[
\varphi(x) - \varphi(a) = \int_a^x D_+ \varphi(t) \, dt, \quad \forall x \in [a, b],
\]

if and only if \( \varphi \) is ACG\(_*\) on \([a, b]\).

Proof. By Fact 2.3, (a) \( \Rightarrow \) (b), if \( \varphi \) can be represented as an indefinite HK-integral on \([a, b]\), then \( \varphi \) is ACG\(_*\) on \([a, b]\). Conversely, if \( \varphi \) is ACG\(_*\) on \([a, b]\), then by Fact 2.2 (1), its derivative \( \varphi'(t) \) exists almost everywhere in \([a, b]\), hence \( D_+ \varphi(t) = \varphi'(t) \) almost everywhere in \([a, b]\). It therefore follows from Fact 2.3, (b) \( \Rightarrow \) (a), that the function \( t \mapsto D_+ \varphi(t) \) is HK-integrable on \([a, b]\) and \( \varphi(x) - \varphi(a) = \int_a^x D_+ \varphi(t) \, dt \).

\[ \square \]

3 Links between subderivatives and subdifferentials

From now on, \( X \) is a real Banach space, \( B_X \) its unit ball, \( X^* \) the topological dual, and \( \langle ., . \rangle \) the duality pairing. For \( x, y \in X \), we let \( [x, y] := \{x + t(y - x) : t \in [0, 1]\} \); the sets \([x, y]\) and \([x, y]\) are defined accordingly. Set-valued operators \( T : X \rightrightarrows X^* \) are identified with their graph \( T \subseteq X \times X^* \) and we write \( \text{dom} T := \{x \in X : T(x) \neq \emptyset\} \). All extended-real-valued functions \( f : X \rightarrow ]-\infty, +\infty] \) are assumed to be lower semicontinuous (lsc) and proper, which means that the set dom \( f := \{x \in X : f(x) < \infty\} \) is non-empty. A net \((x_\nu)\_\nu \subseteq X\) is said to converge to \( \bar{x} \) in the direction \( v \in X \), written \( x_\nu \rightharpoonup v \bar{x} \), if there are two nets \( t_\nu \searrow 0 \) (that is, \( t_\nu \to 0 \) with \( t_\nu > 0 \)) and \( v_\nu \to v \) such that \( x_\nu = \bar{x} + t_\nu v_\nu \) for all \( \nu \). Observe that for \( v = 0 \), \( x_\nu \rightharpoonup v \bar{x} \) simply means \( x_\nu \to \bar{x} \).

The framework, terminology and notation are the same as in our works [16, 17]. Let be given a lsc function \( f : X \rightarrow ]-\infty, +\infty] \), a point \( \bar{x} \in \text{dom} f \) and a direction \( u \in X \). We recall that the (lower right Dini) radial subderivative, its upper version and its upper strict version (the Clarke subderivative) are respectively defined by

\[
\begin{align*}
&f^r(\bar{x}; u) := \liminf_{t \searrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t}, \quad f^r_+(\bar{x}; u) := \limsup_{t \searrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t}, \\
&f^o(\bar{x}; u) := \limsup_{t \searrow 0} \frac{f(x + tu) - f(x)}{t},
\end{align*}
\]

whereas the (lower right Dini-Hadamard) directional subderivative and its upper strict version (the Clarke-Rockafellar subderivative) are respectively given by:

\[
\begin{align*}
&f^d(\bar{x}; u) := \liminf_{t \searrow 0} \frac{f(\bar{x} + tu') - f(\bar{x})}{t}, \\
&f^\uparrow(\bar{x}; u) := \sup_{\delta > 0} \limsup_{t \searrow 0} \inf_{u' \in B_\delta(u)} \frac{f(x + tu') - f(x)}{t}.
\end{align*}
\]

For points \( \bar{x} \notin \text{dom} f \), all the subderivatives are defined to be \(-\infty\).
Besides these classical subderivatives, we shall also consider upper semicontinuous regularizations of the radial subderivative, in the directional sense and in the full sense:

\[ f^\hat{}(\bar{x}; u) := \inf_{\alpha \geq 0} \limsup_{x \to u} f'(x; u + \alpha(\bar{x} - x)), \]  
\[ f^{\hat{\hat{\alpha}}}(\bar{x}; u) := \inf_{\alpha \geq 0} \limsup_{x \to u} f'(x; u + \alpha(\bar{x} - x)). \]  

(7a) (7b)

It turns out that the regularized subderivatives \( f^\hat{} \) and \( f^{\hat{\hat{\alpha}}} \) can be expressed in terms of any bivariate function \( f' \) lying between the subderivatives \( f^d \) and \( f^\uparrow \) (point (3) below):

**Fact 3.1** ([16, Proposition 4, Proposition 7 and Theorem 3]). Let \( f : X \to ]-\infty, +\infty] \) be lsc on a Banach space \( X, \bar{x} \in \text{dom } f \) and \( u \in X \).

1. If \( f \) is convex, then \( f'_{+}(\bar{x}; u) = f^{\hat{\hat{\alpha}}}(\bar{x}; u) \).
2. If \( f(\bar{x}) = \liminf_{t \searrow 0} f(\bar{x} + tu) \), then \( f^\uparrow_{+}(\bar{x}; u) \leq f^\hat{}(\bar{x}; u) \).
3. For any \( f': X \times X \to \mathbb{R} \) such that \( f^d \leq f' \leq f^\uparrow \),

\[ f^\hat{}(\bar{x}; u) = \inf_{\alpha \geq 0} \limsup_{x \to u} f'(x; u + \alpha(\bar{x} - x)), \]
\[ f^{\hat{\hat{\alpha}}}(\bar{x}; u) = \inf_{\alpha \geq 0} \limsup_{x \to u} f'(x; u + \alpha(\bar{x} - x)). \]

(8a) (8b)

The relationships between the regularized and the classical subderivatives are visualized on the following diagram where \( \rightarrow \) means \( '\leq' \), \( \Rightarrow \) means \( '\leq \) provided \( f(\bar{x}) = \liminf_{t \searrow 0} f(\bar{x} + tu) \)' , and \( \overset{*}{\rightarrow} \) means \( '\leq \) provided \( f \) is continuous relative to its domain:\n
\[ f^*(\bar{x}; u) \rightarrow f^*_{+}(\bar{x}; u) \overset{*}{\Rightarrow} f^\hat{}(\bar{x}; u) \rightarrow f^{\hat{\hat{\alpha}}}(\bar{x}; u) \overset{**}{\Rightarrow} f^*(\bar{x}; u) \]

\[ f^d(\bar{x}; u) \overset{\uparrow}{\Rightarrow} f^\uparrow(\bar{x}; u) \]

(9)

If \( f \) is locally Lipschitz at \( \bar{x} \) relative to its domain, one has \( f^*(\bar{x}; u) = f^d(\bar{x}; u) \), \( f^\hat{}(\bar{x}; u) = \limsup_{x \to u} f^*(x; u) \) and \( f^{\hat{\hat{\alpha}}}(\bar{x}; u) = \limsup_{x \to u} f^*(x; u) = f^\circ(\bar{x}; u) = f^\uparrow(\bar{x}; u) \), so the above diagram becomes a line:

\[ f^*(\bar{x}; u) = f^d(\bar{x}; u) \rightarrow f^*_{+}(\bar{x}; u) \rightarrow f^\hat{}(\bar{x}; u) \rightarrow f^{\hat{\hat{\alpha}}}(\bar{x}; u) = f^\circ(\bar{x}; u) = f^\uparrow(\bar{x}; u). \]

(9Lip)

If in addition \( f \) is regular at \( \bar{x} \) in the sense of Clarke, i.e. \( f^*(\bar{x}; u) = f^\circ(\bar{x}; u) \), then all the subderivatives coincide. But in general the inequality \( f^\hat{}(\bar{x}; u) \leq f^{\hat{\hat{\alpha}}}(\bar{x}; u) \) is strict: for \( f : x \in \mathbb{R} \mapsto -|x|, \bar{x} = 0 \) and \( u \neq 0 \),

\[ f^\hat{}(0; u) = \lim_{x \to u} f^*(x; u) = -|u| < f^{\hat{\hat{\alpha}}}(0; u) = \limsup_{x \to 0} f^*(x; u) = |u|. \]

Next, given a lsc function \( f : X \to ]-\infty, +\infty] \) and a point \( \bar{x} \in \text{dom } f \), we recall that the Moreau-Rockafellar subdifferential (the subdifferential of convex analysis) and the Clarke subdifferential are respectively defined by

\[ \partial_{MRF}(\bar{x}) := \{ x^* \in X^* : \langle x^*, y - \bar{x} \rangle + f(\bar{x}) \leq f(y), \forall y \in X \}, \]
\[ \partial_{C}(f)(\bar{x}) := \{ x^* \in X^* : \langle x^*, u \rangle \leq f^\uparrow(\bar{x}; u), \forall u \in X \}. \]

All the classical subdifferentials (proximal, Fréchet, Hadamard, Ioffe, Michel-Penot, ...) lie between these two objects, and for a lsc convex \( f \), all these subdifferentials coincide.
In the sequel, we call subdifferential any operator \( \partial \) that associates a set-valued mapping \( \partial f : X \rightrightarrows X^* \) to each function \( f \) on \( X \) so that
\[
\partial_{MR} f \subset \partial f \subset \partial_C f,
\]
and the following Separation Principle is satisfied:

\[\text{(SP)} \quad \text{For any lsc } f, \varphi \text{ with } \varphi \text{ convex Lipschitz near } \bar{x} \in \text{dom } f, \text{ if } f + \varphi \text{ admits a local minimum at } \bar{x}, \text{ then } 0 \in \partial f(\bar{x}) + \partial \varphi(\bar{x}), \text{ where} \]

\[
\hat{\partial} f(\bar{x}) := \{ \bar{x}^* \in X^* : \text{there is a net } ((x_\nu, x_\nu^*))_\nu \subset \partial f \text{ with } \]
\[
(x_\nu, f(x_\nu)) \to (\bar{x}, f(\bar{x})), \quad x_\nu^* \rightharpoonup \bar{x}^*, \quad \limsup \nu \langle x_\nu^*, x_\nu - \bar{x} \rangle \leq 0 \}. \quad (10)
\]

The Clarke subdifferential, the Michel-Penot (moderate) subdifferential and the Ioffe subdifferential satisfy (SP) in any Banach space. The elementary subdifferentials (proximal, Fréchet, Hadamard, . . . ), as well as their viscosity and limiting versions, satisfy (SP) in appropriate Banach spaces. See, e.g. [12, 15, 21] and the references therein.

Subdifferentials satisfying the Separation Principle (SP) are densely defined:

**Fact 3.2** ([15, Theorem 5.1]). Let \( X \) be a Banach space, \( f : X \to \mathbb{R} \) be lsc and \( \bar{x} \in \text{dom } f \). Then, there exists a sequence \((x_n, x_n^*)\) such that \( x_n \to \bar{x}, \ f(x_n) \to f(\bar{x}) \) and \( \limsup \nu \langle x_n^*, x_n - \bar{x} \rangle \leq 0 \).

We call subderivative associated to a subdifferential \( \partial f \) at a point \( \bar{x} \in \text{dom } f \) in the direction \( u \in X \) the support function of the set \( \partial f(\bar{x}) \) in the direction \( u \), which we denote by
\[
f^0(\bar{x}; u) := \sup \{ \langle \bar{x}^*, u \rangle : \bar{x}^* \in \partial f(\bar{x}) \}.
\]

A key feature is that the regularized subderivatives \( f^2 \) and \( f^{2\alpha} \) can also be expressed in terms of \( f^0 \) for any subdifferential \( \partial \):

**Fact 3.3** ([16, Theorem 3]). Let \( f : X \to \mathbb{R} \) be lsc on a Banach space \( X \), \( \bar{x} \in \text{dom } f \) and \( u \in X \). For any subdifferential \( \partial \),
\[
\begin{align*}
f^2(\bar{x}; u) &= \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}} f^0(x; u + \alpha(\bar{x} - x)), \quad (11a) \\
f^{2\alpha}(\bar{x}; u) &= \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}} f^0(x; u + \alpha(\bar{x} - x)). \quad (11b)
\end{align*}
\]

As a straightforward consequence, we obtain a variant of [17, Proposition 7]:

**Theorem 3.4** (Subdifferential representation of the radial subderivative). Let \( f : X \to \mathbb{R} \) be lsc on a Banach space \( X \) and let \( \partial \) be a subdifferential. Then, for any \( \bar{x} \in \text{dom } f \) and \( u \in X \),
\[
\begin{align*}
f^r(\bar{x}; u) &= f^2(\bar{x}; u) \iff f^r(\bar{x}; u) = \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}} f^0(x; u + \alpha(\bar{x} - x)), \quad (12a) \\
f^r(\bar{x}; u) &= f^{2\alpha}(\bar{x}; u) \iff f^r(\bar{x}; u) = \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}} f^0(x; u + \alpha(\bar{x} - x)). \quad (12b)
\end{align*}
\]
A lsc function \( f : X \to [-\infty, +\infty] \) is declared upper semismooth (respectively, strictly upper semismooth) at a point \( \bar{x} \in \text{dom} \ f \) in the direction \( u \in X \) provided \( f^\circ(\bar{x}; u) = f^\circ(\bar{x}; u) \) (respectively, \( f^\circ(\bar{x}; u) = f^\circ(\bar{x}; u) \) — the definitions used in \cite{17} are slightly less demanding, with \( \preceq \) instead of \( = \). Examples of upper semismooth functions are the locally Lipschitz Mifflin semismooth functions; examples of strictly upper semismooth functions are the locally Lipschitz Clarke regular functions, the proper lsc (approximately) convex functions, the lower-C^1 functions, the Thibault-Zagrodny directionally stable functions. See \cite{17} and the last section of the present paper for further discussion. By \textbf{Theorem 3.4}, a function \( f \) is upper semismooth (respectively, strictly upper semismooth) at a point \( \bar{x} \in \text{dom} \ f \) in the direction \( u \) if and only if its radial subderivative \( f^\circ(\bar{x}; u) \) at \( \bar{x} \) in the direction \( u \) can be recovered from a subdifferential through the formula \((12a)\) (respectively, the formula \((12b)\)).

\section{Links betweens functions and subdifferentials}

This section is devoted to the study of the subdifferential determination and the subdifferential representation of a function. We write \( \text{Lsc}(X) \) for the class of all lsc functions on \( X \) and \( \text{LC}(X) \) (\( \text{LACG}_*(X) \), \( \text{LACG}(X) \), respectively) for the subclass of \( \text{Lsc}(X) \) consisting of lsc functions \( f \) whose restrictions to Line segments \([a, b] \subset \text{dom} \ f \) are Continuous (\( \text{ACG}_* \), \( \text{ACG} \), respectively).

For the subdifferential determination issue, we consider three subclasses of \( \text{Lsc}(X) \) depending on the degree of regularity of the radial subderivative of the functions, namely, the classes of strictly, nearly and almost upper semismooth functions. They are respectively defined by:

\[
\text{Lsc}^{\circ\circ}(X) = \{ g \in \text{Lsc}(X) : \text{dom} \ g \text{ is convex and } \forall [\bar{x}, \bar{x} + u] \subset \text{dom} \ g, \bar{x}_t = \bar{x} + tu, \\
g^\circ(\bar{x}_t; u) < +\infty \text{ and } g^{\circ\circ}(\bar{x}_t; u) = g^\circ(\bar{x}_t; u) \text{ for all } t \in [0, 1[\}.
\]

\[
\text{LC}^{\circ\circ}(X) = \{ g \in \text{LC}(X) : \text{dom} \ g \text{ is convex and } \forall [\bar{x}, \bar{x} + u] \subset \text{dom} \ g, \bar{x}_t = \bar{x} + tu, \\
g^\circ(\bar{x}_t; u) \text{ is finite and } g^{\circ\circ}(\bar{x}_t; u) = g^\circ(\bar{x}_t; u) \text{ for nearly all } t \in [0, 1[\}.
\]

\[
\text{LACG}^{\circ\circ}(X) = \{ g \in \text{LACG}(X) : \text{dom} \ g \text{ is convex and } \forall [\bar{x}, \bar{x} + u] \subset \text{dom} \ g, \bar{x}_t = \bar{x} + tu, \\
g^\circ(\bar{x}_t; u) \text{ is finite and } g^{\circ\circ}(\bar{x}_t; u) = g^\circ(\bar{x}_t; u) \text{ for almost all } t \in [0, 1[\}.
\]

One has \( \text{Lsc}^{\circ\circ}(X) \subset \text{LC}^{\circ\circ}(X) \subset \text{LACG}^{\circ\circ}(X) \). The first inclusion follows from \textbf{Proposition 4.1} below. To prove the second inclusion, let \( g \in \text{LC}^{\circ\circ}(X) \) and \([\bar{x}, \bar{x} + u] \subset \text{dom} \ g\). The function \( \varphi : t \in [0, 1] \mapsto g(\bar{x} + tu) \) is continuous, so by \textbf{Fact 3.1} (2) and the definition of \( \text{LC}^{\circ\circ}(X) \),

\[
D^+ \varphi(t) = g^\circ(\bar{x}_t; u) \leq g^{\circ\circ}(\bar{x}_t; u) = D^+ \varphi(t) \text{ for nearly all } t \in [0, 1[.
\]

Therefore, \( D^+ \varphi(t) \) is finite and \( D^+ \varphi(t) = D^+ \varphi(t) \) nearly everywhere on \([0, 1] \), which means that \( \varphi \) is right-differentiable nearly everywhere on \([0, 1] \). We conclude that \( \varphi \) is \( \text{ACG}_* \) on \([0, 1] \) by \textbf{Fact 2.2} (2). Hence \( g \in \text{LACG}^{\circ\circ}(X) \). A discussion of these classes of functions, with examples, comments and variants, is given in the last section.

We say that a function \( g : X \to ]-\infty, +\infty[ \) is radially Lipschitz continuous at a point \( x \in \text{dom} \ g \) in the direction \( u \in X \) if the restriction of \( g \) to any open line segment \([\bar{x}, \bar{x} + u]\) containing \( x \) is locally Lipschitz at \( x \), namely, there exist \( t_0 > 0 \) and \( \lambda > 0 \) such that

\[
y, z \in [x - t_0u, x + t_0u[ \implies g(z) - g(y) \leq \lambda \|z - y\|.
\]
Proposition 4.1 (Radial Lipschitz continuity of functions in $\text{Lsc}^{\infty}(X)$). The restriction of every $g \in \text{Lsc}^{\infty}(X)$ to any line segment $[\bar{x}, \bar{x} + u] \subset \text{dom} g$ is continuous at the endpoints and locally Lipschitz at every $x \in [\bar{x}, \bar{x} + u]$; in particular, $g^r(x; u)$ is finite for all $x \in [\bar{x}, \bar{x} + u]$.

Proof. Let $x \in [\bar{x}, \bar{x} + u]$. We show that $g$ is locally Lipschitz at $x$ relative to $[\bar{x}, \bar{x} + u]$. Note that $[\bar{x}, \bar{x} + u] \equiv [\bar{y}, \bar{y} - u]$ for $\bar{y} = \bar{x} + u$. By definition of the space $\text{Lsc}^{\infty}(X)$ we have $g^{\infty}(x; v) = g^r(x; v) < +\infty$ for $v = \pm u$.

Hence, there exists $\alpha \in \mathbb{R}$ such that, for $v = \pm u$, $g^{\infty}(x; v) \leq \alpha$. Let $x_t = x + tv$. Since $v + \alpha(x - x_t) = (1 - \alpha t)v$, it follows that for any $\alpha \geq 0$,

$$
\limsup_{t \to 0} g^r(x_t; v) = \limsup_{t \to 0} (1 - \alpha t)g^r(x_t; v) = \limsup_{t \to 0} g^r(x_t; v + \alpha(x - x_t)) \leq \limsup_{x' \to x} g^r(x'; v + \alpha(x - x')).
$$

Hence, for $v = \pm u$,

$$
\limsup_{t \to 0} g^r(x_t; v) \leq g^{\infty}(x; v) \leq \alpha.
$$

We derive that there exists $t_0 > 0$ such that for $v = \pm u$,

$$
g^r(x'; v) \leq \alpha \quad \text{for all} \quad x' \in [x - t_0 u, x + t_0 u]. \tag{13}
$$

Let $y, z$ be arbitrary points in $[x - t_0 u, x + t_0 u]$. There exist $t_1 \in [0, t_0]$ and $v \in \{u, -u\}$ such that $z = y + t_1 v$. Consider the lsc function $t \mapsto \varphi(t) = g(y + tv) - \alpha t$. Using (13), we see that

$$
D_+ \varphi(t) = g^r(y + tv; v) - \alpha \leq 0 \quad \text{for all} \quad t \in [0, t_1].
$$

The Monotonicity Theorem 2.1 (1) then shows that $\varphi(t_1) \leq \varphi(0)$, in other words,

$$
g(z) - g(y) \leq \alpha \lambda_1 = (\lambda/\|u\|)\|z - y\|.
$$

Thus the local Lipschitz property holds as long as $y$ and $z$ belong to $[x - t_0 u, x + t_0 u]$. From this it follows that $-\lambda \leq g^r(x; u) \leq \lambda$, hence $g^r(x; u)$ is finite.

We now prove that $g$ is continuous at $\bar{x}$ relative to $[\bar{x}, \bar{x} + u]$. The argument is similar. Since $g^{\infty}(\bar{x}; u) = g^r(\bar{x}; u) < +\infty$, there exists $\lambda \in \mathbb{R}$ such that $g^{\infty}(\bar{x}; u) \leq \lambda$. As above, we infer that

$$
\limsup_{t \to 0} g^r(\bar{x} + tu; u) \leq g^{\infty}(\bar{x}; u) \leq \lambda.
$$

So there exists $t_0 > 0$ such that

$$
g^r(\bar{x} + tu; u) \leq \lambda \quad \text{for all} \quad t \in [0, t_0].
$$

Then, we derive from the Monotonicity Theorem 2.1 (1) that, for every $t \in [0, t_0]$,

$$
g(\bar{x} + tu) - g(\bar{x}) \leq \lambda t,
$$

proving that $g$ is continuous at $\bar{x}$ relative to $[\bar{x}, \bar{x} + u]$. Since $[\bar{x}, \bar{x} + u] = [\bar{y}, \bar{y} - u]$ for $\bar{y} = \bar{x} + u$, the continuity of $g$ at $\bar{x} + u$ relative to $[\bar{x}, \bar{x} + u]$ follows as well. \qed

Typical examples of functions in $\text{Lsc}^{\infty}(X)$ are the proper lsc convex functions (see the last section). As an illustration of Proposition 4.1, consider the proper lsc convex function $g$ defined on $X = \mathbb{R}$ by $g(x) = -\sqrt{x}$ for $x \in [0, 1]$ and $g(x) = +\infty$ otherwise. Then $g$ is continuous on $[0, 1]$, locally Lipschitz at every point in $[0, 1]$ but not locally Lipschitz at $\bar{x} = 0$. 

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Let \( \mathcal{F}(X) \) be a class of lsc functions on \( X \). We say that a subclass \( \mathcal{G}(X) \subset \mathcal{F}(X) \) is subdifferentially determined in \( \mathcal{F}(X) \) if for every \( g \in \mathcal{G}(X) \), \( f \in \mathcal{F}(X) \) and \( \Omega \subset X \) open convex with \( \Omega \cap \text{dom } f \neq \emptyset \), one has
\[
\partial f(x) \subset \partial g(x) \quad \text{for all } x \in \Omega \implies f = g + \text{Const} \text{ on } \Omega \cap \text{dom } f.
\]

Each version of the Monotonicity Theorem 2.1 naturally leads to a corresponding version for the subdifferential determination property:

**Theorem 4.2** (Subdifferential determination of functions). Let \( X \) be a Banach space.

1. The class \( \text{Lsc}^{2\alpha}(X) \) is subdifferentially determined in \( \text{Lsc}(X) \).
2. The class \( \text{LC}^{2\alpha}(X) \) is subdifferentially determined in \( \text{LC}(X) \).
3. The class \( \text{LACG}^{2\alpha}(X) \) is subdifferentially determined in \( \text{LACG}(X) \).

**Proof.** The structure of the proof is the same for each case and is similar to that of [17, Theorem 10]. We give the details for the case (1) and a sketch for the other (simpler) cases.

**Case (1).** Let \( g \in \text{Lsc}^{2\alpha}(X) \) and \( f \in \text{Lsc}(X) \), and let \( \Omega \subset X \) be an open convex subset with \( \Omega \cap \text{dom } f \neq \emptyset \). Assume
\[
\partial f(x) \subset \partial g(x) \quad \text{for all } x \in \Omega.
\] (14)

Without loss of generality, we may consider that \( \Omega \cap \text{dom } f \) contains two distinct points. By Fact 3.2, the set \( \Omega \cap \text{dom } \partial f \) is dense in \( \Omega \cap \text{dom } f \), it therefore also contains two distinct points. Observe that by (14) \( \Omega \cap \text{dom } \partial f = \Omega \cap \text{dom } \partial f \cap \text{dom } g \subset \Omega \cap \text{dom } f \cap \text{dom } g \).

**First step.** Let \( \bar{x} \in \Omega \cap \text{dom } \partial f \) and \( \bar{y} \in \Omega \cap \text{dom } f \cap \text{dom } g \) with \( \bar{x} \neq \bar{y} \). Note that \( [\bar{x}, \bar{y}] \subset \Omega \cap \text{dom } g \). Put \( u := \bar{y} - \bar{x} \) and let \( t \in [0, 1] \). Then \( \bar{x}_t := \bar{x} + tu \in [\bar{x}, \bar{x} + u] = [\bar{x}, \bar{y}] \subset \Omega \cap \text{dom } g \). If \( \bar{x}_t \not\in \text{dom } f \), \( f^\prime(\bar{x}_t; u) = -\infty \), hence \( f^\prime(\bar{x}_t; u) \leq g^\prime(\bar{x}_t; u) \). Otherwise, \( \bar{x}_t \in \Omega \cap \text{dom } f \cap \text{dom } g \), so by Fact 3.3
\[
\begin{align*}
f^{2\alpha}(\bar{x}_t; u) &= \inf_{\alpha \geq 0} \lim_{x \to \bar{x}_t} f^\alpha(x; u + \alpha(\bar{x}_t - x)) \\
g^{2\alpha}(\bar{x}_t; u) &= \inf_{\alpha \geq 0} \lim_{x \to \bar{x}_t} g^\alpha(x; u + \alpha(\bar{x}_t - x)),
\end{align*}
\]
which entails from (14) that \( f^{2\alpha}(\bar{x}_t; u) \leq g^{2\alpha}(\bar{x}_t; u) \). But \( f^\prime(\bar{x}_t; u) \leq f^{2\alpha}(\bar{x}_t; u) \) by definition of \( f^{2\alpha}(\bar{x}_t; u) \) and \( g^{2\alpha}(\bar{x}_t; u) = g^\prime(\bar{x}_t; u) \) by assumption on \( g \). Hence \( f^\prime(\bar{x}_t; u) \leq g^\prime(\bar{x}_t; u) \) also in this case. Thus, we have just shown that
\[
f^\prime(\bar{x}_t; u) \leq g^\prime(\bar{x}_t; u) \quad \text{for all } t \in [0, 1].
\] (15)

**Second step.** By Proposition 4.1, \( g^\prime(\bar{x}_t; u) \) is finite for every \( t \in [0, 1] \). On the other hand, \( g^\prime(\bar{x}; u) < +\infty \) and since \( \bar{x} \in \Omega \cap \text{dom } \partial g \), we infer that \( g^\prime(\bar{x}; u) = g^{2\alpha}(\bar{x}; u) \geq g^\alpha(\bar{x}; u) > -\infty \), so \( g^\prime(\bar{x}; u) \) is finite as well. Rewriting (15) with the functions \( \varphi : t \in [0, 1] \mapsto f(\bar{x}_t) \) and \( \gamma : t \in [0, 1] \mapsto g(\bar{x}_t) \), we get
\[
D_+\varphi(t) \leq D_+\gamma(t) \quad \text{for all } t \in [0, 1].
\] (16)

Since \( D_+\gamma(t) \) is finite everywhere on \( [0, 1] \), one has \( D_+\gamma(t) \leq D_+\varphi(t) - D_+\gamma(t) \) everywhere on \( [0, 1] \), hence (16) entails
\[
D_+(\varphi - \gamma)(t) \leq 0 \quad \text{for all } t \in [0, 1].
\]
Note that \( \varphi \) is lsc and \( \gamma \) is continuous by Proposition 4.1, so the function \( \varphi - \gamma \) is lsc. Applying the Monotonicity Theorem 2.1 (1), we obtain that \( (\varphi - \gamma)(1) \leq (\varphi - \gamma)(0) \). Finally we have proved that
\[
f(\bar{y}) - f(\bar{x}) \leq g(\bar{y}) - g(\bar{x}) \quad \text{for all } \bar{x} \in \Omega \cap \text{dom } \partial f \text{ and } \bar{y} \in \Omega \cap \text{dom } f \cap \text{dom } g.
\] (17)
This inequality can be extended to all \( \bar{x} \in \Omega \cap \text{dom} \, f \) and \( \bar{y} \in \Omega \cap \text{dom} \, f \cap \text{dom} \, g \). Indeed, by Fact 3.2, there is a sequence \((\bar{x}_n)_n \subset \Omega \cap \text{dom} \, \partial f\) such that \( \bar{x}_n \to \bar{x} \) and \( f(\bar{x}_n) \to f(\bar{x}) \). Since \( g \) is lower semicontinuous at \( \bar{x} \), passing to the limit in the inequality

\[
f(\bar{y}) - f(\bar{x}_n) \leq g(\bar{y}) - g(\bar{x}_n),
\]

we see that (17) holds for all \( \bar{x} \in \Omega \cap \text{dom} \, f \). From this we derive that every point \( \bar{x} \) in \( \Omega \cap \text{dom} \, f \) belongs to \( \Omega \cap \text{dom} \, g \), so \( \Omega \cap \text{dom} \, f \cap \text{dom} \, g = \Omega \cap \text{dom} \, f \). We conclude that

\[
f(\bar{y}) - f(\bar{x}) \leq g(\bar{y}) - g(\bar{x}) \text{ for all } \bar{x}, \bar{y} \in \Omega \cap \text{dom} \, f,
\]

hence in fact

\[
f(\bar{y}) - f(\bar{x}) = g(\bar{y}) - g(\bar{x}) \text{ for all } \bar{x}, \bar{y} \in \Omega \cap \text{dom} \, f,
\]

which means that \( f = g + \text{const} \) on \( \Omega \cap \text{dom} \, f \).

**Case (2) (Case (3), respectively).** Let \( g \in \text{LC}^{2n}(X) \) and \( f \in \text{LC}(X) \) \((g \in \text{LACG}^{2n}(X) \) and \( f \in \text{LACG}(X) \), respectively\), and let \( \Omega \subset X \) be an open convex subset with \( \Omega \cap \text{dom} \, f \neq \emptyset \). Assume that the inclusion (14) holds.

Let \( \bar{x} \in \Omega \cap \text{dom} \, \partial f = \Omega \cap \text{dom} \, \partial f \cap \text{dom} \, \partial g \) and \( \bar{y} \in \Omega \cap \text{dom} \, f \cap \text{dom} \, g \) with \( \bar{x} \neq \bar{y} \). Note that \([\bar{x}, \bar{y}] \subset \Omega \cap \text{dom} \, g\). As in Case (1), we let \( u := \bar{y} - \bar{x} \) and for \( t \in [0,1] \), \( \bar{x}_t := \bar{x} + tu \in \Omega \cap \text{dom} \, g \). If \( \bar{x}_t \notin \text{dom} \, f \), \( f^+_t(\bar{x}_t; u) = -\infty \), hence \( f^+_t(\bar{x}_t; u) \leq g^+(\bar{x}_t; u) \). Otherwise, \( \bar{x}_t \in \Omega \cap \text{dom} \, f \cap \text{dom} \, g \), so proceeding as in Case (1), we derive from Fact 3.3 and (14) that \( f^+(\bar{x}_t; u) \leq g^+(\bar{x}_t; u) \). But by Fact 3.1 (2), \( f^+_t(\bar{x}_t; u) \leq f^+(\bar{x}_t; u) \), and by assumption on \( g \), \( g^+(\bar{x}_t; u) = g^+(\bar{x}_t; u) \in \mathbb{R} \) nearly (almost, respectively) everywhere on \([0,1]\). Hence finally

\[
g^+(\bar{x}_t; u) \text{ is finite and } f^+_t(\bar{x}_t; u) \leq g^+(\bar{x}_t; u) \text{ for nearly (almost, resp.) all } t \in [0,1]. \tag{18}
\]

Rewriting (18) with the functions \( \varphi : t \in [0,1] \mapsto f(\bar{x}_t) \) and \( \gamma : t \in [0,1] \mapsto g(\bar{x}_t) \), we get

\[
D_+ \gamma(t) \text{ is finite and } D^+ \varphi(t) \leq D_+ \gamma(t) \text{ for nearly (almost, resp.) all } t \in [0,1], \tag{19}
\]

hence,

\[
D^+(\varphi - \gamma)(t) \leq 0 \text{ for nearly (almost, resp.) all } t \in [0,1]. \tag{20}
\]

Applying the appropriate version of the Monotonicity Theorem 2.1 we conclude that \( \varphi(1) - \gamma(1) \leq \varphi(0) - \gamma(0) \). Thus we have proved that

\[
f(\bar{y}) - f(\bar{x}) \leq g(\bar{y}) - g(\bar{x}) \text{ for all } \bar{x} \in \Omega \cap \text{dom} \, \partial f \text{ and } \bar{y} \in \Omega \cap \text{dom} \, f \cap \text{dom} \, g.
\]

The rest of the proof goes in the same way as in Case (1). \( \blacksquare \)

We now proceed with the subdifferential representation issue. To this end, we introduce the class of almost upper semismooth functions in the restricted sense:

\[
\text{LACG}^{2n}_+(X) = \{ g \in \text{LACG}_+(X) : \text{dom} \, g \text{ is convex and } \forall [\bar{x}, \bar{x} + u] \subset \text{dom} \, g, \; \bar{x}_t = \bar{x} + tu, \; g^+(\bar{x}_t; u) = g^+(\bar{x}_t; u) \text{ for almost all } t \in [0,1] \}.
\]

One has \( \text{LC}^{2n}(X) \subset \text{LACG}^{2n}_+(X) \subset \text{LACG}^{2n}(X) \). Indeed, we have already observed that each function in \( \text{LC}^{2n}(X) \) belongs to \( \text{LACG}_+(X) \) so the first inclusion holds. The second inclusion follows from the fact that for any \( g \in \text{LACG}^{2n}_+(X) \) and \( [\bar{x}, \bar{x} + u] \subset \text{dom} \, g \), the function \( \varphi : t \in [0,1] \mapsto g(\bar{x} + tu) \) is differentiable almost everywhere on \([0,1]\) by Fact 2.2 (1), so \( g^+(\bar{x}_t; u) \) is finite for almost all \( t \in [0,1] \).
Theorem 4.3 (Subdifferential representation of functions). Let $X$ be a Banach space. Any $g \in \text{LACG}^{a}(X)$ can be represented through its subdifferential via the integration formula
\[
g(\bar{x} + u) - g(\bar{x}) = \int_{0}^{1} g^{r}(\bar{x}_{t}; u)dt, \quad \forall [\bar{x}, \bar{x} + u] \subset \text{dom} \, g, \ \bar{x}_{t} = \bar{x} + tu,
\]
where $g^{r}(\bar{x}_{t}; u) = \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}_{t}} g^{\partial}(x; u + \alpha(\bar{x}_{t} - x))$ for every $t \in [0, 1]$. 

Proof. The function $\varphi : t \mapsto g(\bar{x}_{t})$ is $\text{ACG}_{a}$ on $[0, 1]$ and $D_{+}\varphi(t) = g^{r}(\bar{x}_{t}; u)$. It therefore follows from the Subderivative Representation Theorem 2.4 that
\[
g(\bar{x} + u) - g(\bar{x}) = \int_{0}^{1} g^{r}(\bar{x}_{t}; u)dt.
\]
But $g^{r}(\bar{x}_{t}; u) = g^{n}(\bar{x}_{t}; u)$ almost everywhere on $[0, 1]$ and by Fact 3.3,
\[
g^{n}(\bar{x}_{t}; u) = \inf_{\alpha \geq 0} \limsup_{x \to \bar{x}_{t}} g^{\partial}(x; u + \alpha(\bar{x}_{t} - x)).
\]

5 Examples, comments and variants

5.1 The space $\text{Lsc}^{\infty}(X)$

Let $g : X \to \mathbb{R}$ be locally Lipschitz on a open convex subset $U \subset X$. Then, for every $x \in U$ and $u \in X$, $g^{\partial}(x; u)$ is finite and $g^{\partial}(x; u) = g^{n}(x; u)$ (see the diagram (9Lip)). So the equality $g^{r}(x; u) = g^{\infty}(x; u)$ for every $u \in X$ is equivalent to the (Clarke) regularity of $g$ at $x$, i.e. $g^{r}(x; u) = g^{\partial}(x; u)$ for every $u \in X$. In other words, the locally Lipschitz functions in $\text{Lsc}^{\infty}(X)$ are precisely the (Clarke) regular functions.

Besides the locally Lipschitz regular functions, the space $\text{Lsc}^{\infty}(X)$ contains the proper lsc convex functions, the proper lsc approximately convex functions (hence also the lower-C$^{1}$ functions) and (more generally) the directionally stable functions in the sense of Thibault-Zagrodny [30]. See [17] for proofs and discussion.

We don’t know whether the space $\text{Lsc}^{\infty}(X)$ contains the lsc radially Lipschitz continuous functions which are regular in the sense of Rockafellar, i.e. $g^{d}(x; u) = g^{g}(x; u)$ for every $u \in X$. We recall that for a convex lsc $g$ one has, for every $x, x + u \in \text{dom} \, g$,
\[
g^{d}(x; u) = g^{g}(x; u) \leq g^{r}(x; u) = g^{\infty}(x; u) < +\infty
\]
where the inequality $\leq$ may be strict.

5.2 The space $\text{LC}^{\infty}(X)$

The space $\text{LC}^{\infty}(X)$ contains the Mifflin semismooth functions like $x \in \mathbb{R} \mapsto -|x|$ (see [17]). It also contains non-locally Lipschitz functions like $x \in \mathbb{R} \mapsto -\sqrt{|x|}$ or $x \in \mathbb{R} \mapsto \sqrt{|x|}$, and even non-absolutely continuous functions like $f : \mathbb{R} \to \mathbb{R}$ given by
\[
f(x) := \begin{cases} 
x \sin(1/x) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}.
\]
These functions are not in $\text{Lsc}^{\infty}(X)$.
5.3 The space $\text{LACG}^a_*(X)$

The following classes of locally Lipschitz functions are considered in Thibault-Zagrodny [32]:
- the segmentwise essentially smooth functions [32, p. 2305], that is, locally Lipschitz functions $g$ defined on a nonempty open convex subset $\Omega \subset X$, such that for every $\bar{x}, u \in X$ with $[\bar{x}, \bar{x} + u] \subset \Omega$,
\[
g^0(\bar{x} + tu; u) = -g^0(\bar{x} + tu; -u) \quad \text{for almost all} \ t \in [0, 1],
\]  
(21)
- the segmentwise essentially subregular functions [32, Definition 4.6], that is, locally Lipschitz functions $g$ such that, instead of (21) one has
\[
g^r(\bar{x} + tu; u) = g^r(\bar{x} + tu; u) \quad \text{for almost all} \ t \in [0, 1].
\]  
(22)

In fact, (21) and (22) are equivalent (see the proof of [6, Lemma 2.1]), so the two classes are identical. They contain the class of arcwise essentially smooth functions previously studied by Borwein-Moors [6]. A remarkable feature of these classes is that they are stable by composition, addition and multiplication. For more details, see [6, 32, 34] and the references therein.

These functions are contained in a more sophisticated class of functions introduced by L. Thibault and D. Zagrodny in [32]: given a subdifferential $\partial$, a lsc function $g : X \to ]-\infty, +\infty]$ is called $\partial$-essentially directionally smooth (eds for short) on a nonempty open convex subset $\Omega \subset X$, provided that (simplified version) for every $u, v \in \Omega \cap \text{dom} \ g$ with $v \neq u$, the following properties hold:

(i) the function $t \mapsto g_{u,v}(t) := g(u + t(v - u))$ is finite and continuous on $[0, 1]$;
(ii) there are real numbers $0 = t_0 < \cdots < t_p = 1$ such that the function $t \mapsto g_{u,v}(t)$ is absolutely continuous on each closed interval included in $[0, 1] \setminus \{t_0, t_1, \ldots, t_p\}$;
(iii) for every $\mu > 0$ there exists a subset $T_\mu \subset [0, 1]$ of full Lebesgue measure (i.e. of Lebesgue measure 1) such that for every $t \in T_\mu$ and every sequence $((x_k, x^*_k))_k \subset \partial g$ with $x_k \to x(t) := u + t(v - u)$, there is some $w \in [x(t), v]$ for which
\[
\limsup_{k \to \infty} \langle x^*_k, w - x_k \rangle \leq \|w - x(t)\| \left(\|v - u\|^{-1}g^r_{u,v}(t; 1) + \mu\right).
\]

Proposition 5.1 (eds versus $\text{LACG}^a_*(X)$). Each eds function $g : X \to ]-\infty, +\infty]$ belongs to $\text{LACG}_*(X)$ and satisfies: for every $[\bar{x}, \bar{x} + u] \subset \text{dom} \ g$, $g^\sharp(\bar{x} + tu; u) = g^\ast(\bar{x} + tu; u)$ for almost all $t \in [0, 1]$. Hence, the class of eds functions is contained in $\text{LACG}^a_*(X)$.

Proof. Let $g : X \to ]-\infty, +\infty]$ be $\partial$-essentially directionally smooth on $\Omega$. Let $\bar{x} \in \Omega \cap \text{dom} \ g$ and $u \in X$ so that $\bar{x} + u \in \Omega \cap \text{dom} \ g$. We apply the above definition with the pair $(\bar{x}, \bar{x} + u)$ in lieu of $(u, v)$. For $t \in [0, 1]$, we set $\bar{x}_t := \bar{x} + tu$ and $g_{\bar{x}, u}(t) := g(\bar{x}_t)$. The conditions (i) and (ii) imply that $t \mapsto g_{\bar{x}, u}(t)$ is $\text{ACG}_*$ on $[0, 1]$ (see the observation before Fact 2.2), hence $g$ belongs to $\text{LACG}_*(X)$. Let $\mu > 0$. By (iii), there exists a subset $T_\mu \subset [0, 1]$ of full Lebesgue measure such that for every $t \in T_\mu$ and every sequence $((x_n, x^*_n))_n \subset \partial g$ with $x_n \to \bar{x}_t$ there is some $w \in [\bar{x}_t, \bar{x} + u]$ for which
\[
\limsup_{n \to \infty} \langle x^*_n, w - x_n \rangle \leq \|w - \bar{x}_t\| \left(\|u\|^{-1}g^r_{\bar{x}, u}(t; 1) + \mu\right).
\]  
(23)

Since $w \in [\bar{x}_t, \bar{x} + u]$ there exists $\tau > 0$ such that $w = \bar{x}_t + \tau u$. Then (23) can be rewritten as
\[
\limsup_{n \to \infty} \langle x^*_n, \tau u + \bar{x}_t - x_n \rangle \leq \tau \|u\| \left(\|u\|^{-1}g^r(\bar{x}_t; u) + \mu\right) = \tau (g^r(\bar{x}_t; u) + \mu \|u\|).
\]
Hence, for any $t \in T_\mu$ and every $((x_n, x_n^*))_n \subset \partial g$ with $x_n \to \bar{x}_t$ there is $\tau > 0$ such that
\[
\limsup_{n \to \infty} (x_n^*, u + \frac{x_t - x_n}{\tau}) \leq g^r(\bar{x}_t; u) + \mu\|u\|.
\]
Setting $\alpha := 1/\tau > 0$, we derive that for every $\mu > 0$ there exists a subset $T_\mu \subset [0,1]$ of full Lebesgue measure such that for every $t \in T_\mu$,
\[
\inf_{\alpha > 0} \limsup_{x \to \bar{x}_t} g^\partial(x; u + \alpha(\bar{x}_t - x)) \leq g^r(\bar{x}_t; u) + \mu\|u\|.
\]
Consider the subset $T \subset [0,1]$ of full Lebesgue measure defined by $T := \bigcap_{n \in \mathbb{N}} T_{1/n}$. Then, for every $t \in T$ it holds
\[
\inf_{\alpha > 0} \limsup_{x \to \bar{x}_t} g^\partial(x; u + \alpha(\bar{x}_t - x)) \leq g^r(\bar{x}_t; u). \tag{24}
\]
By Fact 3.3, the left-hand side of (24) is equal to $g^\natural(\bar{x}_t; u)$. Hence, $g^\natural(\bar{x}_t; u) = g^r(\bar{x}_t; u)$ for almost all $t \in [0,1]$. A fortiori, $g^\natural(\bar{x}_t; u) = g^r(\bar{x}_t; u)$ for almost all $t \in [0,1]$. The proof is complete. \qed

5.4 Continuous variant
When the functions are continuous, a more refined subdifferential determination property can be established with a simpler proof. Let $G \subset X$ be a nonempty open convex subset. We denote by $\text{CLACG}(G)$ the class of all real-valued Continuous functions on $G$ whose restrictions to Line segments $[a,b] \subset G$ are ACG, and we consider its subclass of densely almost upper semismooth functions defined by:
\[
\text{CLACG}^{\text{ad}}(G) := \{ g \in \text{CLACG}(G) : \forall x, x + u \in G \exists x_n \to x, u_n \to u : \forall n, g^\partial(x_n + tu_n; u_n) \in \mathbb{R} \text{ and } g^\natural(x_n + tu_n; u_n) = g^r(x_n + tu_n; u_n) \text{ for almost all } t \in [0,1] \}.
\]

**Theorem 5.2** (Subdifferential Determination – Continuous variant). Let $X$ be a Banach space. The class $\text{CLACG}^{\text{ad}}(G)$ is subdifferentially determined in $\text{CLACG}(G)$.

**Proof.** Let $g \in \text{CLACG}^{\text{ad}}(G)$ and $f \in \text{CLACG}(G)$, and let $\Omega \subset G$ be a nonempty open convex subset. Assume
\[
\partial f(x) \subset \partial g(x) \text{ for all } x \in \Omega.
\]
Let $\bar{x}, \bar{x} + u \in \Omega$. Let $x_n \to \bar{x}$ and $u_n \to u$ such that $g^\partial(x_n + tu_n; u_n) = g^r(x_n + tu_n; u_n) \in \mathbb{R}$ for almost all $t \in [0,1]$. Proceeding as in Theorem 4.2 (3), we derive that, for every $n \in \mathbb{N}$,
\[
f^\partial_r(x_n + tu_n; u_n) \leq g^r(x_n + tu_n; u_n) \in \mathbb{R} \text{ for almost all } t \in [0,1],
\]
which leads to
\[
f(x_n + u_n) - f(\bar{x}_n) \leq g(\bar{x}_n + u_n) - g(\bar{x}_n) \text{ for every } n \in \mathbb{N}.
\]
Since $f$ and $g$ are continuous, passing to the limit, we get $f(\bar{x} + u) - f(\bar{x}) \leq g(\bar{x} + u) - g(\bar{x})$. It then follows that $f - g$ is constant on $\Omega$. The proof is complete. \qed

Now, let $\mathcal{S}_e(G)$ denote the class of essentially smooth functions studied by Borwein-Moors [6, 7], that is, the locally Lipschitz functions $g$ on $G$ such that for each $u \in X$,
\[
B_u := \{x \in G : g^\partial(x; u) \neq -g^\partial(x; -u)\} \text{ is a Haar-null subset of } X. \tag{25}
\]
Proposition 5.3 $(S_e(G)$ versus $CLACG^{\text{ad}}(G))$. Let $G \subset X$ be a nonempty open convex subset of a Banach space $X$. Each $g \in S_e(G)$ satisfies: for every $u \in X$ there is a dense subset $D_u$ of $G$ such that for every $w \in D_u$ with $[w, w + u] \subset G$, $g^{\tilde{\tau}}(w + tu; tu) = g^r(w + tu; tu)$ for almost all $t \in [0, 1]$. Hence, $S_e(G) \subset CLACG^{\text{ad}}(G)$.

Proof. Let $g \in S_e(G)$, $0 \neq u \in X$, and let $W_u$ be a topological complement of $\text{span}\{u\}$. Applying Fact 5.4 given below to the Borel Haar null set $B_u$ defined in (25), we obtain that there is a dense set $S_u$ in $W_u$ such that for every $w \in S_u$,

$$g^\sigma(w + tu; u) = -g^\sigma(w + tu; -u)$$

for almost all $t \in \mathbb{R}$. (26)

Then the set $D_u := S_u + \text{span}\{u\}$ is a dense subset of $G$ such that for every $w \in D_u$ the relation (26) holds, or equivalently (see the discussion at the beginning of Subsection 5.3),

$$g^\sigma(w + tu; u) = g^r(w + tu; u)$$

for almost all $t \in \mathbb{R}$,

which implies (see the diagram (9))

$$g^{\tilde{\tau}}(w + tu; tu) = g^{\tilde{\tau}}(w + tu; u) = g^r(w + tu; u)$$

for almost all $t \in [0, 1]$.

In particular, $g \in CLACG^{\text{ad}}(G)$.

Fact 5.4 ([34, Lemma 2.4] and [7, Theorem 2.4]). Let $X$ be a Banach space, $0 \neq u \in X$, and let $W$ be a topological complement of $\text{span}\{u\}$. Let $B \subset X$ be a Borel Haar-null set. Then there exists a set $S \subset W$ dense in $W$ such that the set $\{t \in \mathbb{R} : w + tu \in B\}$ is Lebesgue-null for each $w \in S$.

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