Quantum mechanics in finite dimensional Hilbert space

A. C. de la Torre, D. Goyeneche
Departamento de Física, Universidad Nacional de Mar del Plata
Funes 3350, 7600 Mar del Plata, Argentina
dltorre@mdp.edu.ar
(March 31, 2022)

Abstract

The quantum mechanical formalism for position and momentum of a particle in a one dimensional cyclic lattice is constructively developed. Some mathematical features characteristic of the finite dimensional Hilbert space are compared with the infinite dimensional case. The construction of an unbiased basis for state determination is discussed.

I. INTRODUCTION

In the quantum mechanical description of physical systems, it is often assumed a continuous set of states requiring an infinite dimensional Hilbert space. There are however many physical systems whose states belong to a discrete and finite set, with a quantum mechanical description formalized in a finite dimensional Hilbert space. The best known example of this is the quantum treatment of angular momentum, indeed, a paradigm for quantum mechanics, presented in every text book. A less known example of discrete quantum mechanics, presented in this work, involves the description of position and momentum observables in a finite dimensional Hilbert space.

In this case, the position observable does not take values in a continuous set but instead it can take values on a lattice. One important practical motivation for studying such systems is that any computer simulation of position and momentum will necessarily involve a finite number of sites. On the other side, a highly speculative motivation is that the possible existence of a fundamental length scale, that is, a measure of length below which the concepts of distance and localization become meaningless, can make a discrete quantum mechanics more appropriated than a continuous one.
In order to make this work useful from the didactic point of view, the formalism of quantum mechanics in a finite dimensional Hilbert space will be presented in a constructive way where all steps are logically connected. This work may therefore be a useful complement to any textbook where quantum mechanics in infinite dimensional Hilbert space is developed. Finally, another didactic feature of this work is that finite dimensional quantum mechanics requires many interesting mathematical tools such as some finite sums and the Discrete Fourier Transform that are not usually presented at the undergraduate level. Furthermore, the important differences between finite and infinite dimensional Hilbert spaces are emphasized and the limit when the dimension becomes infinite is considered.

II. NOTATION AND DEFINITIONS

In this work we will consider a particle in a one dimensional periodic lattice with \( N \) sites and lattice constant \( a \). The quantum mechanical treatment of this system requires an \( N \) dimensional Hilbert space \( \mathcal{H} \). Given any two elements of this space \( \Phi \) and \( \Psi \) we will denote their internal product by \( \langle \Phi, \Psi \rangle \). We will use operators of the form \( A = \Psi \langle \Phi, \cdot \rangle \), where the dot indicates a space holder to be occupied by the Hilbert space element upon which the operator acts. The corresponding hermitian conjugate is \( A^\dagger = \Phi \langle \Psi, \cdot \rangle \).

Although we don’t need to choose any particular representation for the abstract Hilbert space \( \mathcal{H} \), it may be convenient, for didactic reasons, to specialize the formalism in a three or four dimensional Hilbert space whose elements \( \Psi \) are column vectors of complex numbers. In this case \( \langle \Psi, \cdot \rangle \) represents a complex conjugate row vector and operators are square matrices. This special representation is recommended for clarity but it is important to emphasize to the students that the formalism of quantum mechanics can be construed in the abstract Hilbert space and a particular representation is never required. The mathematical beauty of quantum mechanics is, indeed, most apparent in the abstract formulation.

Any basis \( \{ \varphi_k \} \) in the Hilbert space will have \( N \) elements labeled by an index \( k \) running through the values \( -j, -j + 1, -j + 2, \ldots, j - 1, j \), with \( j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \) corresponding to the dimensions \( N = 2j + 1 = 2, 3, 4, 5 \ldots \). This choice of labels has some advantage and some disadvantage. The main virtue of this symmetric labeling is that it corresponds to the physical concepts of position and momentum that can take positive and negative values. The main shortcoming of it, is that there are many summations and results that are usually given in books with integer indices running from 0 to \( N - 1 \). In order to cure this deficiency, we give in Appendix A some of these sums with the symmetric index. Furthermore, with this notation we must keep in mind that for even \( N \), \( j \) takes half-odd-integer values and this
may be relevant in modular mathematics.

We will adopt a very useful notation for the principal $N^{th}$ root of the identity defined by

$$\omega = e^{\frac{2\pi i}{N}}.$$  \hspace{1cm} (1)

Integer powers of this quantity build a cyclic group with the important property

$$1 = \omega^{Nn} = \omega^{(2j+1)n}, \forall n = 0, \pm1, \pm2, \cdots.$$  \hspace{1cm} (2)

### III. POSITION AND MOMENTUM

The position of the particle in the lattice can take any value (eigenvalue) $ax$ where $a$ has units of length and the discrete number $x$ can take any value in the set $\{ -j, -j+1, \cdots, j-1, j \}$. The state of the particle in each position is represented by a Hilbert space element $\varphi_x$ and the set $\{ \varphi_x \}$ is a basis in $\mathcal{H}$. In the spectral decomposition, we can write the position operator $X$ as

$$X = \sum_{x=-j}^{j} ax \varphi_x \langle \varphi_x, \cdot \rangle,$$  \hspace{1cm} (3)

that clearly satisfies $X \varphi_x = ax \varphi_x$. We can now construct a translation operator $T$ with the property

$$T \varphi_x = \begin{cases} 
\varphi_{x+1}, & x \neq j \\
(-1)^{N-1} \varphi_{-j} = \omega^{Nj} \varphi_{-j}, & x = j
\end{cases}.$$  \hspace{1cm} (4)

We will later explain the reason for defining this operator periodic for odd dimension but antiperiodic for even dimension. This operator is given by

$$T = \sum_{x=-j}^{j-1} \varphi_{x+1} \langle \varphi_x, \cdot \rangle + (-1)^{N-1} \varphi_{-j} \langle \varphi_j, \cdot \rangle,$$  \hspace{1cm} (5)

with its hermitian conjugate

$$T^\dagger = \sum_{x=-j}^{j-1} \varphi_x \langle \varphi_{x+1}, \cdot \rangle + (-1)^{N-1} \varphi_j \langle \varphi_{-j}, \cdot \rangle.$$  \hspace{1cm} (6)

It is straightforward to check that this operator is unitary, $TT^\dagger = T^\dagger T = 1$, and therefore its eigenvalues are complex numbers of unit modulus and their eigenvectors build a basis (see Appendix B). Let then $\{ \phi_p \}$ and $\{ \lambda_p \}$ for $p = -j, -j+1, \cdots, j-1, j$, be the eigenvectors and eigenvalues of $T$. In order to determine them, we expand $\varphi_p$ in terms of $\{ \varphi_x \}$ and consider
\[ T \sum_{x=-j}^{j} \langle \varphi_x, \phi_p \rangle \varphi_x = \lambda_p \sum_{x=-j}^{j} \langle \varphi_x, \phi_p \rangle \varphi_x. \]  

From Eq. (4) we get

\begin{align*}
\langle \varphi_{x-1}, \phi_p \rangle &= \lambda_p \langle \varphi_x, \phi_p \rangle \quad \text{for } x \neq j, \\
\langle \varphi_j, \phi_p \rangle \omega^{Nj} &= \lambda_p \langle \varphi_{-j}, \phi_p \rangle.
\end{align*}  

(8)

Up to an arbitrary phase, that can be absorbed in the definition of \( \phi_p \), and considering the normalization of \( \phi_p \), the solution of the above equations is

\[ \langle \varphi_x, \phi_p \rangle = \frac{1}{\sqrt{N}} \omega^{px}, \quad \text{and} \quad \lambda_p = \omega^{-p}. \]  

(9)

The two bases \( \{ \varphi_x \} \) and \( \{ \phi_p \} \) are then related by

\[ \varphi_x = \frac{1}{\sqrt{N}} \sum_{p=-j}^{j} \omega^{-px} \phi_p, \]  

(10)

and

\[ \phi_p = \frac{1}{\sqrt{N}} \sum_{x=-j}^{j} \omega^{px} \varphi_x. \]  

(11)

Except for the symmetric index and a different factor, this is essentially the Discrete Fourier Transform. Notice that if we had not defined the translation operator antisymmetric when \( N \) is even, then we would not have obtained such a simple relation in Eq. (9) above and we would have obtained different expressions for \( N \) even or odd. In other words, we choose to define the translation operator in a way to obtain a simple relation between the bases. The complication in the definition of the translation operator is also related to our choice of using symmetric indices. If we had chosen indices running from 0 to \( N - 1 \), then a translation operator periodic for all \( N \) would have lead to the two bases related by the Discrete Fourier Transform (also expressed in terms of asymmetric indices).

Therefore we have

\[ T \phi_p = \omega^{-p} \phi_p, \]  

(12)

or equivalent,

\[ T = \sum_{p=-j}^{j} \omega^{-p} \phi_p \langle \phi_p, \cdot \rangle. \]  

(13)

We can now construct an hermitian operator \( P \) as a superposition of projectors in the basis \( \{ \phi_p \} \).
\[ P = \sum_{p=-j}^{j} gp\phi_{p}\langle\phi_{p},\cdot\rangle, \quad (14) \]

where \( g \) is a real constant to be determined later. Clearly, this operator is hermitian and satisfies the eigenvalue equation \( P\phi_{p} = gp\phi_{p} \). From this equation and from Eq.\((12)\), and doing the power expansion of the exponential we prove that

\[ T = \exp \left( -i \frac{2\pi P}{N g} \right). \quad (15) \]

We can now assign a physical interpretation to the operator \( P \). This last Eq.\((13)\), together with Eq.\((4)\) means that \( P \) is the generator of translations in the position observable. We identify this observable \( P \), as is done in classical mechanics, with the momentum. If the position observable takes value in a lattice with lattice constant \( a \), then, the momentum observable must also assume values in a lattice with lattice constant \( g \). In the next section we will see that these values must be related by \( ga = 2\pi/N \), that is, the momentum lattice is the reciprocal lattice of the position.

In an identical manner as was done before, we can now construct a unitary operator \( B \) that “boosts” the momentum states

\[ B\phi_{p} = \begin{cases} 
\phi_{p+1}, & p \neq j \\
(-1)^{N-1}\phi_{-j} = \omega^{Nj}\phi_{-j}, & p = j
\end{cases}, \quad (16) \]

and show that

\[ B\varphi_{x} = \omega^{x}\varphi_{x}, \quad (17) \]

and

\[ B = \sum_{x=-j}^{j} \omega^{x}\varphi_{x}\langle\varphi_{x},\cdot\rangle, \quad (18) \]

and also that

\[ B = \exp \left( i \frac{2\pi X}{N a} \right). \quad (19) \]

**IV. COMMUTATION RELATION AND THE LIMIT \( N \rightarrow \infty \)**

Every quantum mechanics textbook emphasizes that the position and the momentum operators satisfy the commutation relation \([X, P] = i\) (we use units such that \( \hbar = 1 \)). However, in most cases it is not mentioned that this commutation relation is false in a
finite dimensional Hilbert space. It becomes clear that this must be so because one can prove that the commutation relation \([X, P] = i\) implies that the operators \(X\) and \(P\) are unbound. However, in a finite dimensional Hilbert space all operators are bounded; therefore such a commutation relation is impossible in a finite dimensional Hilbert space. An explicit calculation of the commutator in the position representation, that is, in terms of the basis \(\{\varphi_x\}\) results in

\[
[X, P] = ag \sum_{k=-j}^{j} \sum_{s=-j}^{j} \sum_{r=-j}^{j} k(s - r) \frac{1}{N} \exp \left( i \frac{2\pi}{N} k(s - r) \right) \varphi_s \langle \varphi_r, \cdot \rangle .
\] (20)

The sum over \(k\) can be performed but it is advantageous to leave it unperformed. We can now see that in the continuous limit, where \(N \to \infty, a \to 0\) and \(g \to 0\) but \(agN \to \text{constant}\), the above commutator approaches the value \(i\), provided that lattice constants satisfy \(agN = 2\pi\). In this limit, the sums over discrete indices \(k, s, r\), become integrals over continuous variables \(\kappa, \sigma, \rho\), according to the scheme

\[
\sqrt{\frac{2\pi}{N}} k \to \kappa , \quad \sqrt{\frac{2\pi}{N}} s \to \sigma , \quad \sqrt{\frac{2\pi}{N}} r \to \rho ,
\]

\[
\sqrt{\frac{2\pi}{N}} \varphi_s \to \varphi(\sigma) , \quad \sqrt{\frac{2\pi}{N}} \varphi_r \to \varphi(\rho) , \quad \sum_{-j}^{j} \to \int_{-\infty}^{\infty} .
\]

The continuous limit is then given by

\[
[X, P] \to i \frac{agN}{2\pi} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\rho \frac{-1}{2\pi} \int_{-\infty}^{\infty} d\kappa i\kappa(\sigma - \rho) \exp (i\kappa(\sigma - \rho)) \varphi(\sigma) \langle \varphi(\rho), \cdot \rangle .
\] (21)

The sum over \(k\), that was left unperformed, assumes in the continuous limit a simple form. Indeed, the integral over \(\kappa\) is a well known representation of Dirac’s delta function. Therefore

\[
[X, P] \to i \frac{agN}{2\pi} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\rho \delta(\sigma - \rho) \varphi(\sigma) \langle \varphi(\rho), \cdot \rangle
\]

\[
= i \frac{agN}{2\pi} \int_{-\infty}^{\infty} d\sigma \varphi(\sigma) \langle \varphi(\rho), \cdot \rangle = i \frac{agN}{2\pi} 1 ,
\] (22)

where we have used the completeness relation. Therefore the usual commutation relation for the continuous case is recovered, provided that

\[
agN = 2\pi .
\] (23)

V. STATE AND TIME EVOLUTION

At any instant of time, the state of the particle will be determined by a Hilbert space element \(\Psi\). We can represent this state in the position or momentum representation, that is, expanded in the bases \(\{\varphi_x\}\) or \(\{\phi_p\}\).
\[ \Psi = \sum_{x=-j}^{j} c_x \varphi_x = \sum_{p=-j}^{j} d_p \phi_p , \quad (24) \]

where the complex coefficients \(c_x\) and \(d_p\) are related by the Discrete Fourier Transformation

\[ d_p = \frac{1}{\sqrt{N}} \sum_{x=-j}^{j} \omega^{-px} c_x , \quad c_x = \frac{1}{\sqrt{N}} \sum_{p=-j}^{j} \omega^{px} d_p , \quad (25) \]

and their absolute values squared \(|c_x|^2\) and \(|d_p|^2\) represent the probability distributions for position and for momentum. Let \(\Psi(t_0)\) be the state of the system at some instant \(t_0\), that we can choose to be \(t_0 = 0\). In Schrödinger’s picture, this state will evolve according to the time evolution unitary operator given in terms of the hamiltonian \(H\) as

\[ U_t = \exp(-iHt) . \quad (26) \]

This description of the time evolution is equivalent to Schrödinger’s equation if time is represented by a continuous variable. However in some cases it may be convenient to assume that also time takes discrete values giving preference to the formulation with the time evolution operator above. If the state is given in the position or in the momentum representation, the coefficients of Eq.(24) will become functions of time \(c_x(t)\) and \(d_p(t)\). Let us consider for instance the case of a free particle with hamiltonian \(H = P^2/2m\). In the momentum representation the coefficients are simply given by

\[ d_p(t) = d_p(0) \exp \left( -i \frac{g^2 p^2}{2m} t \right) = d_p(0) \omega^{-p^2 \frac{t}{\tau}} , \quad (27) \]

where we have introduced a time scale \(\tau\) defined by

\[ \tau = \frac{2ma}{g} , \quad (28) \]

whereas in the position representation we have

\[ c_r(t) = \sum_{x=-j}^{j} c_x(0) \frac{1}{N} \sum_{p=-j}^{j} \omega^{(p(r-x)-p^2 \frac{t}{\tau})} . \quad (29) \]

The second summation in Eq.(29) is a Discrete Fourier Transform that becomes, in the continuous limit, a Fourier Integral Transform of a gaussian function with a very well known result. In our discrete case, this summation can not be evaluated in general. This is an example of the difficulties encountered in the discrete case. It took Gauß four years working “with all efforts”\(^1\) in order to evaluate a similar summation (the so called “Gauß sum”) for some special values of the parameters involved. In any case we can see that the state is periodic, \(\Psi(t + T) = \Psi(t)\) with period \(T = N\tau\) if \(N\) es odd and \(T = 4N\tau\) if \(N\) es even.
VI. STATE DETERMINATION AND UNBIASED BASES

At an early stage in the development of quantum mechanics, Pauli raised the question whether the knowledge of the probability density functions for position and momentum where sufficient in order to determine the state of a particle. Since position and momentum are all the (classically) independent variables of the system, it was, erroneously, guessed that this Pauli problem could have an affirmative answer. Indeed, many examples of Pauli partners, that is, different states with identical probability distributions, where found. A review of these issues, with references to the original papers can be found in refs. Considering the similar problem in classical statistic, we should not be surprised to find out that the Pauli question can not have a positive answer. The marginal probability distribution functions of two random variables uniquely determine the combined distribution function only in the case when they are uncorrelated, that is, when they are independent random variables. Position and momentum are, however, always correlated; that is indeed the essence of Heisenberg’s uncertainty principle, and therefore we should not expect that their distributions uniquely determine the quantum state.

Explicitly stated in our case, the Pauli question is: can we find the set of \( N \) complex numbers \( \{c_x\} \) that determine the state in Eq.(24) with the knowledge of the sets \( \{|c_x|^2\} \) and \( \{|d_p|^2\} \) related by Eq.(25)? Let us notice that the state has an arbitrary phase and is normalized; therefore we only need to find \( 2N - 2 \) real numbers in order to determine the state. The known numbers \( \{|c_x|^2\} \) and \( \{|d_p|^2\} \) are not independent because the numbers of each set are probabilities and they should add to 1. We have therefore \( 2N - 2 \) equations at our disposal in order to find \( 2N - 2 \) unknown. However, the equations are not linear and they are not sufficient for an unambiguous determination of the state. There is another very important feature in these equations. We will see that not every set of data \( \{|c_x|^2\} \) and \( \{|d_p|^2\} \) are compatible. The equations have solution only if the position and momentum data satisfy a number of relations. These constraint on the data is just Hiesenberg’s uncertainty principle and is a consequence of the relations in Eq.(25). These concepts are clarified by an example with \( N = 2 \).

Let \( \{\varphi_-, \varphi_+\} \) and \( \{\phi_-, \phi_+\} \) be the position and momentum bases in two dimensional Hilbert space. Their internal product is given by Eq.(9). An arbitrary state, normalized and with a phase fixed is determined by \( 2N - 2 = 2 \) numbers \( 0 \leq \varrho \leq 1 \) and \( 0 \leq \alpha \leq 2\pi \):

\[
\psi = \varrho e^{i\alpha} \varphi_- + \sqrt{1 - \varrho^2} \varphi_+ .
\]  

The independent data on position is \( |\langle \varphi_-, \psi \rangle|^2 = \varrho^2 \), that directly determines \( \varrho \) and the
independent data on momentum is $|\langle \phi_-, \psi \rangle|^2 = \varpi^2$. With this last data we must determine $\alpha$. Using that $\langle \phi_-, \varphi_\pm \rangle = \exp(\pm i\pi/4)/\sqrt{2}$, we get after some algebra that

$$\sin \alpha = \frac{\varpi^2 - 1/2}{\varrho \sqrt{1 - \varrho^2}}. \quad (31)$$

This equation can only have solution if

$$\left| \frac{\varpi^2 - 1/2}{\varrho \sqrt{1 - \varrho^2}} \right| \leq 1, \quad (32)$$

that, after squaring and arranging, becomes

$$(\varpi^2 - 1/2)^2 + (\varrho^2 - 1/2)^2 \leq (1/2)^2. \quad (33)$$

This relation is indeed the uncertainty principle: if $\varrho^2 = 0$ or 1, that is, exact localization in $\varphi_+$ or $\varphi_-$, then $\varpi^2 = 1/2$, that is, maximal spread in momentum and, vise versa, exact momentum ($\varpi^2 = 0$ or 1) implies maximal spread in position ($\varrho^2 = 1/2$). Now, even if the data is consistent with the uncertainty principle, there is an ambiguity in the solution of Eq.(31) because if $\alpha$ is a solution then $\pi - \alpha$ is also a solution. This ambiguity *can not* be solved with the given data and requires more experimental information. We will next consider what observables can we measure in order to determine the state without ambiguity.

From previous example it is clear that we need further information besides the distribution of position and of momentum in order to determine the state of the particle. This will involve an observable depending on both, position and momentum because any observable depending on only one of them will not bring new independent information. Some candidates may be $X + P$ or the correlation $XP + PX$ or any function $F(X, P)$ symmetric or antisymmetric under the exchange $X \leftrightarrow P$. Perhaps the best choice of an observable that provides information on the system not available in the knowledge of position and momentum distributions is an observable whose associated basis is *unbiased* to the position and to the momentum bases. Two bases in a Hilbert space are unbiased when they are as different as possible in the sense that any element of one basis has the same “projection” on *all* elements of the other basis. More precisely, the modulus of their internal product is a constant for all pairs. Unbiased bases are candidates for the quantum mechanical description of classically independent variables like position and momentum; indeed we have from Eq.(3) $|\langle \varphi_x, \phi_p \rangle| = 1/\sqrt{N} \forall x, p$. This leads us to the search of a basis $\{\eta_s\}$ unbiased to $\{\varphi_x\}$ and to $\{\phi_p\}$.

The importance of unbiased bases associated to non commuting observables was recognized by Schwinger long ago but the existence and explicit construction of maximal sets of
mutually unbiased bases for any dimension is still an open problem. When the dimension $N$ is a prime number, a set of $N + 1$ mutually unbiased bases was presented and this was extended to the case when $N$ is a power of a prime number. In order to find unbiased bases we will follow the method given by Bandyiopadhyay et al. We have seen that the eigenvectors $\{\phi_p\}$ of the operator $T$ that produces a translation or shift on the basis $\{\varphi_x\}$ build an unbiased basis to $\{\varphi_x\}$. This result is generalized in reference where it is shown that, if $N$ is prime, the eigenvectors of the unitary operators $T, B, TB, TB^2, \ldots TB^{N-1}$ build a set of $N + 1$ mutually unbiased bases where $T$ and $B$ are the translation operators for position and momentum defined in Eqs. and . In our case we want to find a set of only three unbiased bases and therefore we just consider the first three operators that provide unbiased bases for any $N$ (prime or not). The first two operators provide the bases $\{\phi_p\}$ and $\{\varphi_x\}$, that are related by the Discrete Fourier Transform and are clearly unbiased. One can easily prove, with Eqs. and that the operator $TB$ is a shift operator for both bases $\{\phi_p\}$ and $\{\varphi_x\}$ and therefore its eigenvectors $\{\eta_s\}$ build a basis unbiased to both of them. We have indeed

$$TB\varphi_x = \begin{cases} \omega^x\varphi_{x+1}, & x \neq j \\ \omega^{-2j^2}\varphi_{-j}, & x = j \end{cases} \quad (34)$$

and

$$TB\phi_p = \begin{cases} \omega^{-(p+1)}\phi_{p+1}, & p \neq j \\ \omega^{-2j^2}\phi_{-j}, & p = j \end{cases} \quad (35)$$

The eigenvectors of $TB$ in the position representation are found by expanding $\eta_s$ in the basis $\{\varphi_x\}$ and using Eqs. and the relation

$$TB\eta_s = \omega^s\eta_s \quad (36)$$

In this calculation one must use with care the modular mathematics defined in Eq. This results in

$$\eta_s = \frac{1}{\sqrt{N}} \sum_{x=-j}^{j} \omega^{\frac{1}{2}x^2-(s+\frac{1}{2})x}\varphi_x \quad (37)$$

With a similar calculation we obtain the eigenvectors of $TB$ in momentum representation

$$\eta_s = \frac{1}{\sqrt{N}} \sum_{p=-j}^{j} \omega^{-\frac{1}{2}p^2-(s+\frac{1}{2})p}\phi_p \quad (38)$$

Clearly we see that $\{\eta_s\}$ is unbiased to $\{\varphi_x\}$ and to $\{\phi_p\}$. The analytical calculation of Discrete Fourier Transforms is much more difficult than the Fourier Integral Transform and
therefore it is of mathematical interest that, from the last two equations we can obtain the
Discrete Fourier Transform for a family of sequences. If we equate Eqs.(37) and (38) and
we expand $\varphi_x$ in terms of $\{\phi_p\}$ we obtain

$$
\frac{1}{\sqrt{N}} \sum_{x=-j}^{j} \omega^{b x^2 - b x} \omega^{-px} = \omega^{-\frac{b}{2} p^2 - b p}, \begin{cases} 
  b = 0, \pm 1, \pm 2, \cdots, \text{N even}, \\
  b = \pm 1/2, \pm 3/2, \cdots, \text{N odd}.
\end{cases}
$$

(39)

We rewrite this result in terms of the asymmetric indices more usual in the mathematical
literature as,

$$
\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (-1)^n \omega^{\frac{b}{2} n^2 - b mn} = (-1)^m \omega^{-\frac{b}{2} m^2 - bm}, \forall b \text{ integer}.
$$

(40)

The unbiased basis found is of course not unique. It was generated by the operator $TB$
but there are many other operators whose eigenvectors build unbiased basis to $\{\varphi_x\}$ and to
$\{\phi_p\}$. Indeed, any combination $T^n B^m$ or $B^m T^n$ where $n$ and $m$ are not divisors of $N$
could do the job.

We will now try to find the physical meaning of the basis $\{\eta_s\}$. That is, we would
like to find an hermitian operator $S(X, P)$ with $\{\eta_s\}$ as eigenvectors corresponding to the
eigenvalues $h_s$ with $h = ag$. That is,

$$
TB = \exp (iS(X, P))
$$

(41)

In terms of the operators $X$ and $P$, and using the relation $agN = 2\pi$, we have

$$
\exp (iS(X, P)) = \exp (-iaP) \exp (igX).
$$

(42)

Notice that here we can not use the Baker-Campbell-Hausdorff relation $e^P e^X = e^{P+X-\frac{1}{2}}$
that is valid in the $N \rightarrow \infty$ case, where the commutator $[X, P]$ is a constant. If this
were possible, then the operator $S$ would be simply equal to $gX - aP$. This is one of the
subtle differences of finite and infinite dimensional Hilbert spaces. It is possible to find
the eigenvectors of the operator $X - P$ but the basis so obtained is not unbiased to either $\{\varphi_x\}$
nor $\{\phi_p\}$ however it becomes unbiased in the infinite dimensional limit\textsuperscript{[1]}. The relation of
$S$ to $X$ and $P$ is not simple but we can prove that $S$ is antisymmetric under the exchange
$P \leftrightarrow X$ and $a \leftrightarrow g$. Indeed, from the hermitian conjugate of Eq.(12) we have

$$
\exp (-iS(gX, aP)) = \exp (-igX) \exp (iaP) = \exp (iS(aP, gX)).
$$

(43)
VII. CONCLUSION

In this work we have presented the quantum mechanical formalism for position and momentum of a particle in a one dimensional cyclic lattice in a way that may be useful for a didactic complement of the infinite dimensional case presented in quantum mechanics text books. In doing this, several mathematical subtleties related to the difference between infinite and finite dimensional Hilbert spaces, and of modular mathematics, arise. We have discussed the physical and mathematical relevance of unbiased bases and, as consequence from the construction of such a basis, the Discrete Fourier Transform for a family of sequences is given.

It is a strongly recommended exercise to reproduce all this work in terms of the asymmetric indices running from 0 to \( N - 1 \). One can see thereby the need to define the translation operator always cyclic in order to get the position and momentum bases related by the Discrete Fourier Transform. The calculations of the eigenvectors of the operator \( T \) are useful exercises for the modular mathematics.

This work received partial support from “Consejo Nacional de Investigaciones Científicas y Técnicas” (CONICET), Argentina.

VIII. APPENDIX A

All sums appearing in this appendix can be derived from the fundamental expression

\[
\sum_{k=0}^{N-1} z^k = \frac{1 - z^N}{1 - z}, \tag{44}
\]

for any complex number \( z \), that in terms of the symmetric index becomes

\[
\sum_{k=-j}^{j} z^k = \frac{z^{N/2} - z^{-N/2}}{z^{1/2} - z^{-1/2}} \quad \text{for} \quad \begin{cases} j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \\ N = 2j + 1 = 2, 3, 4, 5, \ldots \\ z \text{ complex} \end{cases}, \tag{45}
\]

If \( z \) takes the special values \( z = \exp(i \frac{2\pi r}{N}) = \omega^r \) with \( r \) an arbitrary number, then

\[
\sum_{k=-j}^{j} \omega^{kr} = \frac{\sin(\pi r)}{\sin(\frac{\pi}{N} r)}. \tag{46}
\]

In our case, the number \( r \) will often assume integer or half-odd-integer values. For these cases we have,
\[
\sum_{k=-j}^{j} \omega^{kr} = \begin{cases} 
(-1)^{n(N-1)}N & \text{for } r = nN, \ n = 0, \pm 1, \pm 2, \cdots \\
0 & \text{for } r = \pm 1, \pm 2, \pm 3, \cdots \neq nN \\
\frac{2\omega^r}{1-\omega^2} & \text{for } r = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \cdots 
\end{cases} \tag{47}
\]

The first two cases correspond, with the asymmetric index, to the well known result \(\sum_{k=0}^{N-1} \omega^{kr} = N\delta_{r,nN}\) where we see that this choice leads to simpler mathematics. The third case above must be handled with care in numerical evaluations because the numerator is the fourth root of \(\exp(i\frac{2\pi}{N}u)\), with \(u\) an odd integer. Therefore it has four possible numerical results. The denominator has also two possible results. A formal derivative of Eq.(46) with respect to the parameter \(r\) leads to the result

\[
\sum_{k=-j}^{j} k\omega^{kr} = \frac{i}{2} \frac{\sin(\pi r) \cos(\frac{\pi}{N}r) - N \cos(\pi r) \sin(\frac{\pi}{N}r)}{\sin^2(\frac{\pi}{N}r)} \tag{48}
\]

Deriving again with respect to \(r\), we can obtain other summations involving higher powers of \(k\).

**IX. APPENDIX B**

In most textbooks it is proven that the non degenerate eigenvalues of an hermitian operator are real and their eigenvectors are orthogonal. We give here the corresponding proof for unitary operators.

Let \(T\) be an unitary operator and \(\lambda_k\) and \(\phi_k\) the non degenerate eigenvalues and normalized eigenvectors. Then, we will prove that, \(|\lambda_k|^2 = 1\) and \(\langle \phi_r, \phi_k \rangle = \delta_{rk}\).

From \(T\phi_k = \lambda_k \phi_k\) and \(T^\dagger T = 1\) it follows that

\[
|\lambda_k|^2 = \langle T\phi_k, T\phi_k \rangle = \langle \phi_k, T^\dagger T\phi_k \rangle = 1 . \tag{49}
\]

In order to prove the orthogonality consider that

\[
T\phi_k = \lambda_k \phi_k \rightarrow \langle \phi_r, T\phi_k \rangle = \lambda_k \langle \phi_r, \phi_k \rangle ,
\]

\[
T^\dagger \phi_r = \lambda_r^* \phi_r \rightarrow \langle T^\dagger \phi_r, \phi_k \rangle = \lambda_r \langle \phi_r, \phi_k \rangle .
\]

Subtracting both equations we get \(0 = (\lambda_k - \lambda_r) \langle \phi_r, \phi_k \rangle\). Since the eigenvalues are non degenerate, the product \(\langle \phi_r, \phi_k \rangle\) must vanish for \(k \neq r\).

Since \(T\) is bounded, the completeness of the eigenvectors can be proved in the usual way and therefore \(\{\phi_k\}\) is a basis.
REFERENCES

1 B. C. Berndt, R. J. Evans “The determinastion of Gauss sums” Bull. Amer. Math. Soc. 5, 107-129 (1981). Cited in: J. J. Benedetto *Harmonic Analysis and Applications* CRC Press, (1997).

2 W. Pauli. “Quantentheorie” *Handbuch der Physik* 24(1933).

3 Leslie E. Ballentine. “*Quantum Mechanics. A Modern Development*”. World Scientific (1998), Pg. 215.

4 S. Weigert. “Pauli problem for a spin of arbitrary length: A simple method to determine its wave function” Phys Rev. A 45,7688-7696 (1992).

5 S. Weigert. “How to determine a quantum state by measurements: The Pauli problem for a particle with arbitrary potential” Phys Rev. A 53,2078-2083 (1996).

6 J. Schwinger Proc. Natl. Acad. Sci. USA 46, 570 (1960).

7 I. D. Ivanović “Geometrical description of quantal state determination” J. Phys. A: Math. Gen. 14, 3241-3245 (1981).

8 W. K. Wooters “Quantum mechanics without probability amplitudes” Found. Phys. 16, 391-405, (1985).

9 W. K. Wooters, B. D. Fields “Optimal state-determination by mutually unbiased measurements” Ann. Phys. (NY) 191, 363-381 (1989).

10 S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, and F. Vatan, “A new proof for the existence of mutually unbiased bases” LANL eprint quant-ph/0103162.

11 A. C. de la Torre “Relativity of representations in quantum mechanics” Am. J. Phys. 70, 298-300, (2002).