THE $\ell_1$-ANALYSIS WITH REDUNDANT DICTIONARY IN PHASE RETRIEVAL

BING GAO

ABSTRACT. This article presents new results concerning the recovery of a signal from magnitude only measurements where the signal is not sparse in an orthonormal basis but in a redundant dictionary. To solve this phaseless problem, we analyze the $\ell_1$-analysis model. Firstly we investigate the noiseless case with presenting a null space property of the measurement matrix under which the $\ell_1$-analysis model provide an exact recovery. Secondly we introduce a new property (S-DRIP) of the measurement matrix. By solving the $\ell_1$-analysis model, we prove that this property can guarantee a stable recovery of real signals that are nearly sparse in highly overcomplete dictionaries.

Keywords Compressed sensing, Phase retrieval, Sparse recovery, $\ell_1$-analysis

Mathematics Subject Classification 94A12

1. Introduction

1.1. Phase Retrieval. Phase retrieval is the process of recovering signals from phaseless measurements. It is of fundamental importance in numerous areas of applied physics and engineering [9], [11]. In general form, phase retrieval problem is to estimate the original signal $x_0 \in \mathbb{H}^n$ ($\mathbb{H} = \mathbb{C}$ or $\mathbb{R}$) from

$$|Ax| = |Ax_0| + e,$$

where $A = [a_1, \ldots, a_m]^\top \in \mathbb{H}^{m \times n}$ is the measurement matrix and $e = [e_1, \ldots, e_m] \in \mathbb{H}^m$ is an error term. While only the magnitude of $Ax_0$ is available, it is important to note that the setup naturally leads to ambiguous solutions. For example, if $\hat{x} \in \mathbb{H}^n$ is a solution to (1.1), then any multiplication of $\hat{x}$ and a scalar $c \in \mathbb{H}$ ($|c| = 1$) is also a solution to (1.1). Hence, these global ambiguities are considered acceptable for this problem. In this paper, we recover the signal $x_0$ actually means that we reconstruct $x_0$ up to a unimodular constant.

It is known that, when $\mathbb{H} = \mathbb{R}$, at least $2n - 1$ measurements are needed to recover a signal $x \in \mathbb{R}^n$ [3]. For the complex case, the minimum number of measurements are proved to be at least $4n - 4$ when $n$ is in the form of $n = 2^k + 1, k \in \mathbb{Z}_+$. However, for a general dimension $n$, the same question is still open. About the minimum number of observations, more details can be found in [4], [16]. To reduce the measurement numbers, priori information must be given. The most common priori information is sparsity, which means that only few elements in the target signal $x_0$ is nonzero. Here we say a signal is $k$-sparse if there are at most $k$ non-zero elements in the signal. In view of sparse signals, phase retrieval is also known as compressive phase retrieval, which have many applications in data acquisition [12], [14]. The compressive phase retrieval problem is in fact the magnitude-only compressive sensing problem. For this problem, Wang and Xu explored the minimum number of measurements
and extended the null space property in compressed sensing to phase retrieval [16]. In [15], Voroniski and Xu gave the definition of strong restricted isometry property (Definition 2.2), and then many conclusions in compressed sensing can be extended to compressive phase retrieval, such as instance optimality [10]. The above conclusions hold just for signals which are sparse in the standard coordinate basis. However, there are many examples in which a signal of interest is not sparse in an orthonormal basis but sparse in some transform basis. In recent years, many researchers laid special stress on analysing these dictionary-sparse signals in compressed sensing [7], [1], [13]. However, the phase retrieval literature is lacking on this subject. We will focus on this problem in this paper.

1.2. The $\ell_1$-analysis with redundant dictionary. At normal state, sparsity is expressed not in terms of an orthonormal basis but in terms of an overcomplete dictionary. That is to say, the signal $x_0 \in \mathbb{H}^n$ can be expressed as $x_0 = Dz$, where $D \in \mathbb{H}^{n \times N}$ is a frame and $z \in \mathbb{H}^N$ is a sparse vector. In this paper, we use $D^*$ to represent the adjoint conjugate of $D$ when $\mathbb{H} = \mathbb{C}$, while when $\mathbb{H} = \mathbb{R}$, we use $D^\intercal$ to represent the transpose of $D$.

In compressed sensing, to reconstruct the signal $x_0$, the most commonly used model is the $\ell_1$-analysis model

\[
\min \|D^*x\|_1 \quad \text{subject to} \quad \|Ax - Ax_0\|_2^2 \leq \epsilon^2,
\]

where $\epsilon$ is the upper bound of the noise. In [7], Candès, Eldar, Needell and Randall proved that when $D$ is a tight frame and $D^*x_0$ is almost $k$-sparse, the $\ell_1$-analysis (1.2) can guarantee a stable recovery provided that the measurement matrix is Gaussian random matrix with $m = O(k \log(n/k))$.

For the phase retrieval problem, we also analyze the $\ell_1$-analysis model

\[
\min \|D^*x\|_1 \quad \text{subject to} \quad \||Ax| - |Ax_0||_2^2 \leq \epsilon^2,
\]

where $\epsilon$ is the upper bound of the noise level.

In this paper, we aim to explore the conditions under which the $\ell_1$-analysis model (1.3) can generate an accurate or a stable solution to (1.1). Note that when $D = I$, this problem is reduced to the traditional phase retrieval and the $\ell_1$-analysis model is reduced to

\[
\min \|x\|_1 \quad \text{subject to} \quad \||Ax| - |Ax_0||_2^2 \leq \epsilon^2.
\]

For this case, when $\mathbb{H} = \mathbb{R}$, Gao, Wang and Xu provided a detailed analysis of (1.4) in [10] and had the conclusion that a $k$-sparse signal can be stably recovered by $O(k \log(n/k))$ Gaussian random measurements. Then a natural question that comes to mind is whether this conclusion still holds for a general frame $D$.

1.3. Organization. The rest of the paper is organized as follows. In section 2, we give notations and recall some previous conclusions. In section 3, for noiseless case ($\epsilon = 0$), we analyze the null space of the measurement matrix and give sufficient and necessary conditions for (1.3) to achieve an exact solution, which will be discussed in real and complex case separately. In general, it’s hard to check whether a matrix satisfies the null space property or not. So in section 4, we introduce a new property (S-DRIP) (Definition 4.1) on the measurement matrix, which is a natural generalization of the DRIP (see [7] for more details). Using this property, we prove that when the measurement matrix is real Gaussian random matrix with $m \geq O(k \log(n/k))$ the $\ell_1$-analysis (1.3) can guarantee a stable recovery
of real signals which are $k$-sparse under a redundant dictionary. Last, some proofs are given in the Appendix.

2. Notations and previous results

Throughout this paper, we use $D \in \mathbb{H}^{n \times N}$ as a frame with full column rank. Let
\[
\Sigma^N_k := \{ x \in \mathbb{H}^N : \| x \|_0 \leq k \}
\]
and
\[
D\Sigma^N_k := \{ x \in \mathbb{H}^N : \exists z \in \Sigma^N_k, x = Dz \}.
\]
Suppose the target signal $x_0$ is in the set $D\Sigma^N_k$, which means that $x_0$ can be represented as $x_0 = Dz_0$, where $z_0 \in \Sigma^N_k$.

The best $k$-term approximation error is defined as
\[
\sigma_k(x)_1 := \min_{z \in \Sigma_k} \| x - z \|_1.
\]
We use $[m]$ to represent the set $\{1, 2, \ldots, m\}$. Suppose $T \subseteq [m]$ is a subset of $[m]$. We use $T^c$ to represent the complement set of $T$ and $|T|$ to denote the cardinal number of $T$. Let $A_T := [a_j, j \in T]^\top$ denote the sub-matrix of $A$ where only rows with indices in $T$ are kept.

Definition 2.1 (DRIP). [7] Fix a dictionary $D \in \mathbb{R}^{n \times N}$ and a matrix $A \in \mathbb{R}^{m \times n}$. The matrix $A$ satisfies the DRIP with parameters $\delta$ and $k$ if
\[
(1 - \delta) \| Dz \|_2^2 \leq \| ADz \|_2^2 \leq (1 + \delta) \| Dz \|_2^2
\]
holds for all $k$-sparse vectors $z \in \mathbb{R}^N$.

The paper [7] shows that the Gaussian matrices and other random compressed sensing matrices satisfy the DRIP of order $k$ provided the number of measurements $m$ on the order of $O(k \log(n/k))$.

Definition 2.2 (SRIP). [15] We say the matrix $A = [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times n}$ has the Strong Restricted Isometry Property of order $k$ and constants $\theta_-, \theta_+ \in (0, 2)$ if
\[
\theta_- \| x \|_2^2 \leq \min_{|I| \leq m, |I| \geq m/2} \| A_I x \|_2^2 \leq \max_{|I| \leq m, |I| \geq m/2} \| A_I x \|_2^2 \leq \theta_+ \| x \|_2^2
\]
holds for all $k$-sparse signals $x \in \mathbb{R}^n$.

This property was first introduced in [15]. Voroninski and Xu also proved that the Gaussian random matrices satisfy SRIP with high probability. More details can be found in the following theorem.

Theorem 2.1. [15] Suppose that $t > 1$ and $A \in \mathbb{R}^{m \times n}$ is a random Gaussian matrix with $m = O(tk \log(n/k))$. Then there exist $\theta_-, \theta_+$, with $0 < \theta_- < \theta_+ < 2$, such that $A$ satisfies SRIP of order $tk$ and constants $\theta_-, \theta_+$, with probability $\exp(-cm^2)$, where $c > 0$ is an absolute constant and $\theta_-, \theta_+$ are independent with $t$. 
3. The Null Space Property

In this section, for \( x_0 \in D_{\Sigma_k}^N \), we consider the noiseless situation
\[
\text{(3.5)} \quad \min \| D^* x \|_1 \quad \text{subject to} \quad |Ax| = |Ax_0|.
\]
Similarly as the traditional compressed sensing problem, we analyze the null space of the measurement matrix \( A \) to explore conditions under which (3.5) can obtain \( c x_0 \) (\(|c| = 1\)).

3.1. The Real Case. We first restrict the signals and measurements to the field of real numbers. The next theorem provides a sufficient and necessary condition for the exact recovery of (3.5).

**Theorem 3.1.** For given matrix \( A \in \mathbb{R}^{m \times n} \) and dictionary \( D \in \mathbb{R}^{n \times N} \), we claim that the following properties are equivalent.

(A) For any \( x_0 \in D_{\Sigma_k}^N \),
\[
\arg\min_{x \in \mathbb{R}^n} \{\| D^* x \|_1 : |Ax| = |Ax_0|\} = \{\pm x_0\}.
\]

(B) For any \( T \subseteq [m] \), it holds
\[
\| D^*(u + v) \|_1 < \| D^*(u - v) \|_1
\]
for all
\[
u \in \mathcal{N}(A_T) \setminus \{0\}, \quad v \in \mathcal{N}(A_{T^c}) \setminus \{0\}
\]
satisfying
\[
u + v \in D_{\Sigma_k}^N.
\]

**Proof.** (B)\( \Rightarrow \) (A). Assume (A) is false, namely, there exists a solution \( \hat{x} \neq \pm x_0 \) to (3.5). As \( \hat{x} \) is a solution, we have
\[
|A\hat{x}| = |Ax_0|
\]
and
\[
\| D^* \hat{x} \|_1 \leq \| D^* x_0 \|_1.
\]
Denote \( a_j, j = 1, \ldots, m \) as the rows of \( A \). Then (3.6) implies that there exists a subset \( T \subseteq [m] \) satisfying
\[
j \in T, \quad \langle a_j, x_0 + \hat{x} \rangle = 0, \\
j \in T^c, \quad \langle a_j, x_0 - \hat{x} \rangle = 0,
\]
i.e.,
\[
A_T(x_0 + \hat{x}) = 0, \quad A_{T^c}(x_0 - \hat{x}) = 0.
\]
Define
\[
u := x_0 + \hat{x}, \quad v := x_0 - \hat{x}.
\]
As \( \hat{x} \neq \pm x_0 \), we have \( u \in \mathcal{N}(A_T) \setminus \{0\}, \quad v \in \mathcal{N}(A_{T^c}) \setminus \{0\} \) and \( u + v = 2x_0 \in D_{\Sigma_k}^N \). Then from (B), we know
\[
\| D^* x_0 \|_1 < \| D^* \hat{x} \|_1,
\]
which contradicts with (3.7).

(A)\( \Rightarrow \) (B). Assume (B) is false, which means that there exists a subset \( T \subseteq [m], \)
\[
u \in \mathcal{N}(A_T) \setminus \{0\}, \quad v \in \mathcal{N}(A_{T^c}) \setminus \{0\},
\]
such that
\[ u + v \in D\Sigma_k^N \]
and
\[
\|D^*(u + v)\|_1 \geq \|D^*(u - v)\|_1.
\]
(3.9)
Let \( x_0 := u + v \in D\Sigma_k^N \) be the signal we want to recover. Set \( \tilde{x} := u - v \) and we have \( \tilde{x} \neq \pm x_0 \). Then from (3.9) we have

\[
\|D^*\tilde{x}\|_1 \leq \|D^*x_0\|_1.
\]
(3.10)
Let \( a_j, j = 1, \ldots, m \) denote the rows of \( A \). Then from the definition of \( x_0 \) and \( \tilde{x} \), we have
\[
2\langle a_j, u \rangle = \langle a_j, x_0 + \tilde{x} \rangle,
\]
\[
2\langle a_j, v \rangle = \langle a_j, x_0 - \tilde{x} \rangle.
\]
By (3.8), the subset \( T \) satisfies
\[
j \in T, \quad \langle a_j, x_0 \rangle = -\langle a_j, \tilde{x} \rangle
\]
and
\[
j \in T^c, \quad \langle a_j, x_0 \rangle = \langle a_j, \tilde{x} \rangle,
\]
which implies
\[
|Ax_0| = |A\tilde{x}|.
\]
Putting (3.10) and (3.11) together, we know \( \tilde{x} \) is a solution to model (3.5). However, \( \tilde{x} \neq \pm x_0 \) contradicts with (A). \( \square \)

3.2. The Complex Case. We now consider the same problem in complex case which means that the signals and measurements are all in the complex number field. We say that \( S = \{S_1, \ldots, S_p\} \) is any partition of \([m]\) if

\[
S_j \in [m], \quad \bigcup_{j=1}^p S_j = [m] \quad \text{and} \quad S_l \cap S_j = \emptyset, \quad \forall \ l \neq j.
\]
Set \( S := \{c \in \mathbb{C}, |c| = 1\} \). The next theorem is a generalization of Theorem 3.1.  

**Theorem 3.2.** For given matrix \( A \in \mathbb{C}^{m \times n} \) and dictionary \( D \in \mathbb{C}^{n \times N} \), we claim that the following properties are equivalent.

(A) For any given \( x_0 \in D\Sigma_k^N \),
\[
\arg\min_{x \in \mathbb{C}^n} \{\|D^*x\|_1 : |Ax| = |Ax_0|\} = \{cx_0, c \in S\}.
\]
(B) Suppose \( S_1, \ldots, S_p \) is any partition of \([m]\). For any given \( \eta_j \in \mathcal{N}(A_{S_j}) \setminus \{0\} \), if
\[
\frac{c_1 - c_l}{c_j - c_l} = \frac{\eta_j - \eta_l}{c_j - c_l} \in D\Sigma_k^N \setminus \{0\}, \quad j, l \in \{2 : p\}, \quad j \neq l
\]
(3.12)
holds for some pairwise distinct \( c_1, \ldots, c_p \in S \), we have
\[
\|D^*(\eta_j - \eta_l)\|_1 < \|D^*(c_j\eta_j - c_l\eta_l)\|_1.
\]
Proof. \((B) \Rightarrow (A)\). Suppose the statement \((A)\) is false. That is to say, there exists a solution \(\hat{x} \notin \{cx_0, c \in S\}\) to \((3.5)\) which satisfies
\[(3.13)\]
\[\|D^*\hat{x}\|_1 \leq \|D^*x_0\|_1\]
and
\[(3.14)\]
\[|Ax_0| = |A\hat{x}|.
\]
Denote \(a_j^*, j = 1, \ldots, m\) as the rows of \(A\). From \((3.11)\) we have
\[\langle a_j, c j x_0 \rangle = \langle a_j, \hat{x} \rangle,
\]
with \(c_j \in S, j = 1, \ldots, m\). We can define an equivalence relation on \([m]\), namely \(j \sim l\) when \(c_j = c_l\). This equivalence relation leads to a partition \(S = \{S_1, \ldots, S_p\}\) of \([m]\). For any \(S_j\), we have
\[A S_j (c j x_0) = A S_j \hat{x}.
\]
Set \(\eta_j := c_j x_0 - \hat{x}\). Then we have \(\eta_j \in \mathcal{N}(A S_j) \setminus \{0\}\)
and
\[
\frac{\eta_j - \eta_l}{c_1 - c_l} = \frac{\eta_j - \eta_l}{c_1 - c_j} = x_0 \in D \Sigma_k^N \quad \forall j, l \in [2 : p], j \neq l.
\]
By the condition \((B)\), we can get
\[\|D^*(\eta_j - \eta_l)\|_1 < \|D^*(c_j \eta_j - c_j \eta_l)\|_1,
\]
i.e.,
\[\|D^*(c_j - c_l)x_0\|_1 < \|D^*(c_j - c_l)\hat{x}\|_1.
\]
That is equivalence to
\[\|D^*x_0\|_1 < \|D^*\hat{x}\|_1,
\]
which contradicts with \((3.13)\).

\((A) \Rightarrow (B)\). Assume \((B)\) is false, namely, there exists a partition \(S = \{S_1, \ldots, S_p\}\) of \([m]\), \(\eta_j \in \mathcal{N}(A S_j) \setminus \{0\}, j \in [1 : p]\) and some pairwise distinct \(c_1, \ldots, c_p \in S\) satisfying \((3.12)\) but
\[\|D^*(\eta_{j_0} - \eta_{l_0})\|_1 \geq \|D^*(c_{l_0} \eta_{j_0} - c_{j_0} \eta_{l_0})\|_1
\]
holds for some distinct \(j_0, l_0 \in [1 : p]\). Set
\[\tilde{x} := c_{l_0} \eta_{j_0} - c_{j_0} \eta_{l_0}, \quad c_{l_0} \neq c_{j_0},
\]
\[x_0 := \eta_{j_0} - \eta_{l_0} \in D \Sigma_k^N.
\]
Then we have
\[\tilde{x} \notin \{c x_0, c \in S\}
\]
and
\[(3.15)\]
\[\|D^*\tilde{x}\|_1 \leq \|D^*x_0\|_1.
\]
Let \(a_j^*, j = 1, \ldots, m\) denote the rows of \(A\). From \(\eta_j \in \mathcal{N}(A S_j) \setminus \{0\}\), we have
\[\langle a_k, \eta_{j_0} \rangle = 0 \quad \text{and} \quad \langle a_k, \eta_{l_0} \rangle = 0, \quad \forall k \in S_{l_0} \cup S_{j_0}.
\]
The definition of \(x_0\) and \(\tilde{x}\) implies
\[(3.16)\]
\[|\langle a_k, x_0 \rangle| = |\langle a_k, \tilde{x} \rangle|, \quad \forall k \in S_{l_0} \cup S_{j_0}.
\]
For \(k \notin S_{l_0} \cup S_{j_0}\), we might as well suppose \(k \in S_t\) \((t \neq l_0, j_0)\), i.e., \(\langle a_k, \eta_k \rangle = 0\). From
\[
\frac{\eta_j - \eta_l}{c_1 - c_l} = \frac{\eta_j - \eta_l}{c_1 - c_j},
\]
we can get
\[ \eta_j - \eta_l = \frac{\eta_m - \eta_n}{c_m - c_n}, \]
for any \( j, l, m, n \) are distinct integers.

Set
\[ y_0 := \frac{\eta_{j0} - \eta_t}{c_{j0} - c_t} = \frac{\eta_0 - \eta_t}{c_0 - c_t}. \]

Then we have
\[ \eta_{j0} = (c_{j0} - c_t)y_0 + \eta_t, \]
\[ \eta_0 = (c_0 - c_t)y_0 + \eta_t. \]

So \( \tilde{x} \) and \( x_0 \) can be rewritten as
\[ \tilde{x} = c_{j0}\eta_{j0} - c_{j0}\eta_0 = c_t(c_{j0} - c_0)y_0 + (c_t - c_j)\eta_t, \]
\[ x_0 = \eta_{j0} - \eta_0 = (c_{j0} - c_0)y_0. \]

Then \( \langle a_k, \eta_t \rangle = 0 \) implies
\[ |\langle a_k, \tilde{x} \rangle| = |\langle a_k, x_0 \rangle|, \quad \text{for } k \in S_t. \]

Using a similar argument, we can prove that the claim is also true for other subset \( S_j \). So we have
\[ (3.17) \quad |\langle a_k, \tilde{x} \rangle| = |\langle a_k, x_0 \rangle|, \quad \forall k. \]
Combining (3.15) and (3.17), we know \( \tilde{x} \) is also a solution to (3.16). However, \( \tilde{x} \notin \{cx_0, c \in S \} \)
contradicts with (A).

\[ \square \]

**Remark 3.1.** If we chose \( D = I \), the null space property in Theorem 3.1 and Theorem 3.2
is consistent with the null space property which was introduced in paper [16].

By the Theorem 3.1 and Theorem 3.2, we know that it is possible to find a good measurement matrix to obtain an exact solution by solving the model (3.5). But in general, condition (B) is difficult to be checked. So in section 4 we provide another property of the measurement matrix which can be satisfied by Gaussian random matrix.

### 4. S-DRIP and Stable Recovery

In compressed sensing, for any tight frame \( D \), [7] has the conclusion that a signal \( x_0 \in D\Sigma_k^N \)
can be approximately reconstructed using \( \ell_1 \)-analysis (1.2) provided the measurement matrix satisfies DRIP and \( D^*x_0 \) decays rapidly. While in phase retrieval, when \( H = \mathbb{R} \), Gao, Wang and Xu proved that if the measurement matrix satisfies SRIP, then the \( \ell_1 \)-analysis (1.4) can provide a stable solution to traditional phase retrieval problem [10]. Next we combine this two results to explore the conditions under which the \( \ell_1 \)-analysis model (1.3) can guarantee a stable recovery.

We first impose a natural property on the measurement matrix, which is a combination of DRIP and SRIP.

**Definition 4.1 (S-DRIP).** Let \( D \in \mathbb{R}^{n \times N} \) be a frame. We say the measurement matrix \( A \) obeys the S-DRIP of order \( k \) with constants \( \theta_-, \theta_+ \in (0 : 2) \) if
\[ \theta_- \|Dv\|_2^2 \leq \min_{I \subseteq [m], |I| \geq m/2} \|A_I Dv\|_2^2 \leq \max_{I \subseteq [m], |I| \geq m/2} \|A_I Dv\|_2^2 \leq \theta_+ \|Dv\|_2^2 \]
holds for all \( k \)-sparse signals \( v \in \mathbb{R}^N \).
Thus a matrix $A \in \mathbb{R}^{m \times n}$ satisfying S-DRIP means that any $m' \times n$ submatrix of $A$, with $m' \geq m/2$ satisfies DRIP with appropriate parameters.

In fact any matrix $A \in \mathbb{R}^{m \times n}$ obeying
\begin{equation}
\mathbb{P}[c_- \| Dv \|_2^2 \leq \min_{I \subseteq [m], |I| \geq m/2} \| A_I Dv \|_2^2 \leq \max_{I \subseteq [m], |I| \geq m/2} \| A_I Dv \|_2^2 \leq c_+ \| Dv \|_2^2] \geq 1 - 2e^{-\gamma m}
\end{equation}
(0 < $c_- < c_+$ and $\gamma$ is a positive number constant) for fixed $Dv \in \mathbb{R}^n$ will satisfy the S-DRIP with high probability. This can be seen by a standard covering argument (see the proof of Theorem 2.1 in [15]). In [15], Voroninski and Xu proved that Gaussian random matrix satisfies (4.18) in Lemma 4.4. So we have the following conclusion.

**Corollary 4.1.** Gaussian random matrix $A \in \mathbb{R}^{m \times n}$ with $m = \mathcal{O}(tk \log(n/k))$ satisfies the S-DRIP of order $tk$ with constants $\theta_-, \theta_+ \in (0: 2)$.

For $x_0 \in D\Sigma_k^N$, we return to consider the solving model
\begin{equation}
\min \| D^* x \|_1 \quad \text{subject to} \quad \| Ax \| - \| Ax_0 \|_2 \leq \epsilon^2,
\end{equation}
where $\epsilon$ is the error bound. Here all signals and matrices are all restricted to the real number field. The next theorem tells under what conditions the solution to (4.19) is stable.

**Theorem 4.1.** Assume that $D \in \mathbb{R}^{n \times N}$ is a tight frame and $x_0 \in D\Sigma_k^N$. The matrix $A \in \mathbb{R}^{m \times n}$ satisfies the S-DRIP of order $tk$ ($t$ is a positive integer) and level $\theta_-, \theta_+ \in (0: 2)$, with
\[ t \geq \max\{1 \, 2\theta_--\theta_+^2, \frac{1}{2\theta_++\theta_+^2}\}. \]
Then the solution $\hat{x}$ to (4.19) satisfies
\[ \min\{\| \hat{x} - x_0 \|_2, \| \hat{x} + x_0 \|_2\} \leq c_1 \epsilon + c_2 \frac{2\sigma_k(D^* x_0)_1}{\sqrt{k}}, \]
where $c_1 = \frac{\sqrt{2(1+\delta)}}{1-\sqrt{t(t-1)\delta}}$, $c_2 = \frac{\sqrt{2\delta + \sqrt{t(\sqrt{(t-1)/t-\delta})}}}{t(\sqrt{(t-1)/t-\delta})} + 1$. Here $\delta$ is a constant satisfying $\delta \leq \max\{1 - \theta_-, \theta_+ - 1\} \leq \sqrt{\frac{t-1}{t}}$.

We first give a more general lemma, which is the key to prove Theorem 4.1.

**Lemma 4.1.** Let $D \in \mathbb{R}^{n \times N}$ be an arbitrary tight frame, $x_0 \in D\Sigma_k^N$ and $\rho > 0$. Suppose that $A \in \mathbb{R}^{m \times n}$ is a measurement matrix satisfying the DRIP with $\delta = \frac{\delta^2}{tk} \leq \sqrt{\frac{t-1}{t}}$ for some $t > 1$. Then for any $D^* \hat{x} \in \{D^* x \in \mathbb{R}^N : \| D^* x \|_1 \leq \| D^* x_0 \|_1 + \rho, \| Ax - Ax_0 \|_2 \leq \epsilon\}$, we have
\[ \| \hat{x} - x_0 \|_2 \leq c_1 \epsilon + c_2 \frac{2\sigma_k(D^* x_0)_1}{\sqrt{k}} + c_2 \cdot \frac{\rho}{\sqrt{k}}, \]
where $c_1 = \frac{\sqrt{2(1+\delta)}}{1-\sqrt{t(t-1)\delta}}$, $c_2 = \frac{\sqrt{2\delta + \sqrt{t(\sqrt{(t-1)/t-\delta})}}}{t(\sqrt{(t-1)/t-\delta})} + 1$.

We put the proof of this Lemma in the Appendix.
Remark 4.1. When \( D = I \), which corresponds to the case of standard compressed sensing, this result is consistent with Lemma 2.1 in [10].

Remark 4.2. Here the DRIP constant is better than the constant given in [2]. In [2], Baker established a generated DRIP constant for compressed sensing. He proved that signals with \( k \)-sparse \( D \)-representation can be reconstructed if the measurement matrix satisfies DRIP with constant \( \delta_{2k} < \frac{2}{3} \). We extended his approach to get a better bound \( \delta_{tk} \leq \sqrt{\frac{t-1}{t}} \). As [7] shows, in the special case \( D = I \), for any \( t \geq 4/3 \), the condition \( \delta_{tk} \leq \sqrt{\frac{t-1}{t}} \) is sharp for stable recovery in the noisy case. So it is not difficult to show that for any tight frame \( D \), the condition \( \delta_{tk} \leq \sqrt{\frac{t-1}{t}} \) is also sharp when \( t \geq 4/3 \).

Proof of the Theorem 4.1. As \( \hat{x} \) is the solution to (4.19), we have

\[
\|D^* \hat{x}\|_1 \leq \|D^* x_0\|_1
\]

and

\[
\|A \hat{x} - |Ax_0|\|_2^2 \leq \epsilon^2.
\]

Denote \( a_j^T, j \in \{1, \ldots, m\} \) as the rows of \( A \) and divide \( \{1, \ldots, m\} \) into two groups:

\[ T = \{ j \mid \text{sign}(\langle a_j, \hat{x} \rangle) = \text{sign}(\langle a_j, x_0 \rangle) \}, \]

\[ T^c = \{ j \mid \text{sign}(\langle a_j, \hat{x} \rangle) = -\text{sign}(\langle a_j, x_0 \rangle) \}. \]

Then either \( |T| \geq m/2 \) or \( |T^c| \geq m/2 \). Without loss of generality, we suppose \( |T| \geq m/2 \). Then (4.21) implies that

\[
\|A_T \hat{x} - A_T x_0\|_2^2 \leq \|A_T \hat{x} - A_T x_0\|_2^2 + \|A_{T^c} \hat{x} + A_{T^c} x_0\|_2^2 \leq \epsilon^2.
\]

Combining (4.20) and (4.22), we have

\[
D^* \hat{x} \in \{ D^* x \in \mathbb{R}^N : \|D^* x\|_1 \leq \|D^* x_0\|_1, \|A_T x - A_T x_0\|_2 \leq \epsilon \}.
\]

Recall that \( A \) satisfies S-DRIP of order \( tk \) with constants \( \theta_- \), \( \theta_+ \in (0 : 2) \). Here

\[
t \geq \max\{\frac{1}{2\theta_- - \theta_+^2}, \frac{1}{2\theta_+ - \theta_-^2}\} \geq 1.
\]

So \( A_T \) satisfies DRIP of order \( tk \) with

\[
\delta_{tk}^{A_T} \leq \max\{1 - \theta_-, \theta_+ - 1\} \leq \sqrt{\frac{t-1}{t}}.
\]

Combining (4.23), (4.24) and Lemma 4.1 we obtain

\[
\|\hat{x} - x_0\|_2 \leq c_1 \epsilon + c_2 \frac{2\sigma_k(D^* x_0)_1}{\sqrt{k}},
\]

where \( c_1 \) and \( c_2 \) are defined as before in the Theorem 4.1.

If \( |T^c| \geq \frac{m}{2} \), we can get the corresponding result

\[
\|\hat{x} + x_0\|_2 \leq c_1 \epsilon + c_2 \frac{2\sigma_k(D^* x_0)_1}{\sqrt{k}}.
\]

Then we have proved the theorem. \( \square \)
In problem (1.1), suppose \( x_0 \in D \Sigma N_k \) and \( D^* x_0 \in \mathbb{R}^N \) decays rapidly. From Theorem 4.1 and Corollary 4.1, we conclude that the \( \ell_1 \)-analysis (4.19) can provide a stable solution to problem (1.1) if we use as many as \( O(k \log(n/k)) \) Gaussian random measurements.

5. Acknowledgments

My deepest gratitude goes to Professor Zhiqiang Xu, my academic supervisor, for his guidance and many useful discussions.

6. Appendix

The following two lemmas are useful in the proof of Lemma 4.1.

**Lemma 6.1.** (Sparse Representation of a Polytope [6, 17]): Suppose \( \alpha > 0 \) is a constant and \( s > 0 \) is an integer. Set

\[
T(\alpha, s) := \{ v \in \mathbb{R}^n : \|v\|_\infty \leq \alpha, \|v\|_1 \leq s\alpha \}.
\]

For any \( v \in \mathbb{R}^n \), set

\[
U(\alpha, s, v) := \{ u \in \mathbb{R}^n : \text{supp}(u) \subseteq \text{supp}(v), \|u\|_0 \leq s, \|u\|_1 = \|v\|_1, \|u\|_\infty \leq \alpha \}.
\]

Then \( v \in T(\alpha, s) \) if and only if \( v \) is in the convex hull of \( U(\alpha, s, v) \). In particular, any \( v \in T(\alpha, s) \) can be expressed as

\[
v = \sum_{i=1}^{M} \lambda_i u_i \quad \text{and} \quad 0 \leq \lambda_i \leq 1, \sum_{i=1}^{M} \lambda_i = 1,
\]

\[u_i \in U(\alpha, s, v)\].

**Lemma 6.2.** (Lemma 5.3 in [5]): Suppose \( m \geq r \), \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \) and \( \sum_{i=1}^{r} a_i \geq \sum_{i=r+1}^{m} a_i \). Then for all \( \alpha \geq 1 \), we have

\[
\sum_{j=r+1}^{m} a_j^\alpha \leq \sum_{i=1}^{r} a_i^\alpha.
\]

Now we are ready to prove Lemma 4.1.

**Proof of the Lemma 4.1** We assume that the tight frame \( D \in \mathbb{R}^{n \times N} \) is normalized, i.e., \( DD^* = I \) and \( \|y\|_2 = \|D^* y\|_2 \) for all \( y \in \mathbb{R}^n \). For a subset \( T \subseteq \{1, 2, \ldots, N\} \), we denote \( D_T \) as the matrix \( D \) restricted to the columns indexed by \( T \) (replacing other columns by zero vectors).

Set \( h := \hat{x} - x_0 \). Let \( T_0 \) denote the index set of the largest \( k \) coefficients of \( D^* x_0 \) in magnitude. Then

\[
\|D^* x_0\|_1 + \rho \geq \|D^* \hat{x}\|_1 = \|D^* x_0 + D^* h\|_1
\]

\[
= \|D^*_{T_0} x_0 + D^*_{T_0} h + D^*_{T_0^c} x_0 + D^*_{T_0} h\|_1
\]

\[
\geq \|D^*_{T_0} x_0\|_1 - \|D^*_{T_0} h\|_1 - \|D^*_{T_0^c} x_0\|_1 + \|D^*_{T_0} h\|_1,
\]
which implies
\[
\|D^*_0 h\|_1 \leq \|D^*_0 h\|_1 + 2\|D^*_0 x_0\|_1 + \rho \\
= \|D^*_0 h\|_1 + 2\sigma_k(D^* x_0)_1 + \rho.
\]

Suppose \(S_0\) is the index set of the \(k\) largest entries in absolute value of \(D^* h\). We get
\[
\|D^* h\|_1 \leq \|D^* h\|_1 \leq \|D^*_0 h\|_1 + 2\sigma_k(D^* x_0)_1 + \rho \\
\leq \|D^* h\|_1 + 2\sigma_k(D^* x_0)_1 + \rho.
\]

Set
\[
\alpha := \frac{\|D^* h\|_1 + 2\sigma_k(D^* x_0)_1 + \rho}{k}.
\]

We divide \(D^*_0 h\) into two parts \(D^*_0 h = h^{(1)} + h^{(2)}\), where
\[
h^{(1)} := D^*_0 h \cdot I_{\{i;|D^*_0 h(i)| > \alpha/(t-1)\}}, \quad h^{(2)} := D^*_0 h \cdot I_{\{i;|D^*_0 h(i)| \leq \alpha/(t-1)\}}.
\]

Then a simple observation is that \(\|h^{(1)}\|_1 \leq \|D^*_0 h\|_1 \leq \alpha k\). Set
\[
\ell := |\text{supp}(h^{(1)})| = \|h^{(1)}\|_0.
\]

Since all non-zero entries of \(h^{(1)}\) have magnitude larger than \(\alpha/(t-1)\), we have
\[
\alpha k \geq \|h^{(1)}\|_1 = \sum_{i \in \text{supp}(h^{(1)})} |h^{(1)}(i)| \geq \sum_{i \in \text{supp}(h^{(1)})} \frac{\alpha}{t-1} = \ell \cdot \frac{\alpha}{t-1},
\]
which implies \(\ell \leq (t-1)k\).

Note that
\[
\|h^{(2)}\|_1 = \|D^*_0 h\|_1 - \|h^{(1)}\|_1 \leq k\alpha - \ell \cdot \frac{\alpha}{t-1} = (k(t-1) - \ell) \cdot \frac{\alpha}{t-1},
\]
\[
\|h^{(2)}\|_\infty \leq \frac{\alpha}{t-1}.
\]

Then in Lemma [6.1] by setting \(s := k(t-1) - \ell\), we can express \(h^{(2)}\) as a weighted mean:
\[
h^{(2)} = \sum_{i=1}^{M} \lambda_i u_i,
\]
where \(0 \leq \lambda_i \leq 1\), \(\sum_{i=1}^{M} \lambda_i = 1\), \(\|u_i\|_0 \leq k(t-1) - \ell\), \(\|u_i\|_\infty \leq \alpha/(t-1)\) and \(\text{supp}(u_i) \subseteq \text{supp}(h^{(2)})\). Thus
\[
\|u_i\|_2 \leq \sqrt{\|u_i\|_0 \cdot \|u_i\|_\infty} = \sqrt{k(t-1) - \ell \cdot \|u_i\|_\infty} \\
\leq \sqrt{k(t-1) \cdot \|u_i\|_\infty} \\
\leq \alpha \sqrt{k/(t-1)}.
\]
Recall that $\alpha = \frac{\|D^*_S h\|_1 + 2\sigma_k(D^* x_0)_1 + \rho}{k}$. Then

$$
\|u_i\|_2 \leq \alpha \sqrt{k/(t-1)}
$$

\[
\leq \frac{\|D^*_S h\|_2}{\sqrt{t-1}} + \frac{2\sigma_k(D^* x_0)_1 + \rho}{\sqrt{k(t-1)}}
\]

$$
\leq \frac{\|D^*_S h + h^{(1)}\|_2}{\sqrt{t-1}} + \frac{2\sigma_k(D^* x_0)_1 + \rho}{\sqrt{k(t-1)}}
$$

(6.25)

$$
z = \sum^{M}_{i=1} \lambda_i \beta_i = D^*_S h + h^{(1)} + \mu \cdot u_j,
\quad j = 1, \ldots, M.
$$

Then for any fixed $i \in [M]$,

$$
M \sum^{M}_{j=1} \lambda_j \beta_j - d \beta_i = D^*_S h + h^{(1)} + \mu \cdot h^{(2)} - d \beta_i
$$

$$
= (1 - \mu - d)(D^*_S h + h^{(1)}) - d\mu u_i + \mu D^* h.
$$

For $\sum^{M}_{i=1} \lambda_i = 1$, we have the following identity

(6.27)

\[
(2d-1) \sum_{1 \leq i < j \leq M} \lambda_i \lambda_j \|AD(\beta_i - \beta_j)\|_2^2 = \sum^{M}_{i=1} \lambda_i \|AD(\sum^{M}_{j=1} \lambda_j \beta_j - d \beta_i)\|_2^2 - \sum^{M}_{i=1} \lambda_i (1-d)^2 \|AD\beta_i\|_2^2.
\]

In (6.26), we chose $d = 1/2$ and $\mu = \sqrt{t(t-1) - (t-1)} < 1/2$. Then (6.27) implies

$$
0 = \sum^{M}_{i=1} \lambda_i \|AD(\sum^{M}_{j=1} \lambda_j \beta_j - d \beta_i)\|_2^2 - \sum^{M}_{i=1} \frac{\lambda_i}{4} \|AD\beta_i\|_2^2
$$

$$
= \sum^{M}_{i=1} \lambda_i \|AD \left( \frac{1}{2} - \mu \right)(D^*_S h + h^{(1)}) - \frac{\mu}{2} u_i + \mu D^* h \right)\|_2^2 - \sum^{M}_{i=1} \frac{\lambda_i}{4} \|AD\beta_i\|_2^2
$$

$$
= \sum^{M}_{i=1} \lambda_i \|AD \left( \frac{1}{2} - \mu \right)(D^*_S h + h^{(1)}) - \frac{\mu}{2} u_i \right)\|_2^2
$$

$$
+ 2 \left\langle AD \left( \frac{1}{2} - \mu \right)(D^*_S h + h^{(1)}) - \frac{\mu}{2} h^{(2)} \right), \mu ADD^* h \right\rangle + \mu^2 \|ADD^* h\|_2^2 - \sum^{M}_{i=1} \frac{\lambda_i}{4} \|AD\beta_i\|_2^2
$$

(6.28)

$$
= \sum^{M}_{i=1} \lambda_i \|AD \left( \frac{1}{2} - \mu \right)(D^*_S h + h^{(1)}) - \frac{\mu}{2} u_i \right)\|_2^2
$$

$$
+ \mu(1 - \mu) \left\langle AD(D^*_S h + h^{(1)}), ADD^* h \right\rangle - \sum^{M}_{i=1} \frac{\lambda_i}{4} \|AD\beta_i\|_2^2.
$$
We next estimate the three terms in (6.28). First we give the following useful relation:

\[
\langle D(D_{S_0}^* h + h^{(1)}), Dh^{(2)} \rangle = \langle D(D_{S_0}^* h + h^{(1)}), D(D^* h - D_{S_0}^* h - h^{(1)}) \rangle
\]

\[
= \langle D(D_{S_0}^* h + h^{(1)}), h \rangle - \langle D(D_{S_0}^* h + h^{(1)}), D(D_{S_0}^* h + h^{(1)}) \rangle
\]

\[
= \langle D_{S_0}^* h + h^{(1)}, D^* h \rangle - \|D(D_{S_0}^* h + h^{(1)})\|_2^2
\]

(6.29)

\[
= \|D_{S_0}^* h + h^{(1)}\|_2 - \|D(D_{S_0}^* h + h^{(1)})\|_2^2.
\]

Noting that \(\|D_{S_0}^* h\|_0 \leq k, \|h^{(1)}\|_0 = \ell \leq (t-1)k\) and \(\|u_i\|_0 \leq s = k(t-1) - \ell\), we obtain

\[
\|D_{S_0}^* h + h^{(1)}\|_0 \leq \ell + k \leq t \cdot k, \quad \|\beta_i\|_0 \leq \|D_{S_0}^* h\|_0 + \|h^{(1)}\|_0 + \|u_i\|_0 \leq t \cdot k,
\]

and

\[
\|\left(\frac{1}{2} - \mu\right)(D_{S_0}^* h + h^{(1)}) - \frac{\mu}{2} u_i\|_0 \leq t \cdot k.
\]

Here we assume \(t \cdot k\) as an integer first. Since \(A\) satisfies the DRIP of order \(t \cdot k\) with constant \(\delta\), we can obtain

\[
\sum_{i=1}^{M} \lambda_i \|AD\left(\frac{1}{2} - \mu\right)(D_{S_0}^* h + h^{(1)}) - \frac{\mu}{2} u_i\|_2^2
\]

\[
\leq \sum_{i=1}^{M} \lambda_i (1 + \delta) \|D\left(\frac{1}{2} - \mu\right)(D_{S_0}^* h + h^{(1)}) - \frac{\mu}{2} u_i\|_2^2
\]

\[
= (1 + \delta) \left(\frac{1}{2} - \mu\right)^2 \|D(D_{S_0}^* h + h^{(1)})\|_2^2 + \frac{\mu^2}{4} \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 - \mu(\frac{1}{2} - \mu) \langle D(D_{S_0}^* h + h^{(1)}), Dh^{(2)} \rangle
\]

(6.29)

\[
= (1 + \delta) \left(\frac{1}{2} - \mu\right)^2 \|D(D_{S_0}^* h + h^{(1)})\|_2^2 + \frac{\mu^2}{4} \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 - \mu(\frac{1}{2} - \mu) \|D_{S_0}^* h + h^{(1)}\|_2^2
\]

\[
\langle AD(D_{S_0}^* h + h^{(1)}), ADD^* h \rangle = \langle AD(D_{S_0}^* h + h^{(1)}), Ah \rangle
\]

\[
\leq \sqrt{1 + \delta} \cdot \|D(D_{S_0}^* h + h^{(1)})\|_2 \cdot \epsilon
\]

and

\[
\sum_{i=1}^{M} \lambda_i \|AD\beta_i\|_2^2
\]

\[
= \sum_{i=1}^{M} \lambda_i \|AD(D_{S_0}^* h + h^{(1)} + \mu \cdot u_i)\|_2^2
\]

\[
= (1 - \delta) \sum_{i=1}^{M} \lambda_i \|D(D_{S_0}^* h + h^{(1)} + \mu \cdot u_i)\|_2^2
\]

(6.29)

\[
= (1 - \delta) \left(\|D(D_{S_0}^* h + h^{(1)})\|_2^2 + \mu^2 \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 + 2\mu \langle D(D_{S_0}^* h + h^{(1)}), Dh^{(2)} \rangle \right)
\]

\[
= (1 - \delta) \left(1 - 2\mu\right) \|D(D_{S_0}^* h + h^{(1)})\|_2^2 + \mu^2 \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 + 2\mu \|D_{S_0}^* h + h^{(1)}\|_2^2.
\]
Combining the above results with (6.25) and (6.28), we get

\[
0 \leq \frac{1}{2}(1+\delta)(\frac{1}{2} - \mu)\|D(D_{S_0}^\ast h + h^{(1)})\|_2^2 + \frac{1+\delta}{4}\mu^2 \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 - (1+\delta)\mu(\frac{1}{2} - \mu)\|D_{S_0}^\ast h + h^{(1)}\|_2^2 \\
+ \mu(1-\mu)\sqrt{1+\delta\|D(D_{S_0}^\ast h + h^{(1)})\|_2 \cdot \epsilon} \\
- \frac{1}{4}(1-\delta)(1-2\mu)\|D(D_{S_0}^\ast h + h^{(1)})\|_2^2 - \frac{1-\delta}{4}\mu^2 \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 - \frac{1-\delta}{2}\mu\|D_{S_0}^\ast h + h^{(1)}\|_2^2 \\
= \delta(\frac{1}{2} - \mu)\|D(D_{S_0}^\ast h + h^{(1)})\|_2^2 + (\mu^2(1+\delta) - \mu)\|D_{S_0}^\ast h + h^{(1)}\|_2^2 + \frac{\delta}{2}\mu^2 \sum_{i=1}^{M} \lambda_i \|Du_i\|_2^2 \\
+ \mu(1-\mu)\sqrt{1+\delta\|D(D_{S_0}^\ast h + h^{(1)})\|_2 \cdot \epsilon} \\
\leq \left((1+\delta)(\frac{1}{2} - \mu)^2 - \frac{1-\delta}{4}\right)z^2 + \frac{\delta}{2}\mu^2(z + R)^2 + \mu(1-\mu)\sqrt{1+\delta \cdot z \cdot \epsilon} \\
\leq \left((\mu^2 - \mu) + \delta \left(\frac{1}{2} - \mu + (1 + \frac{1}{2(t-1)\mu^2})\right)\right)z^2 + \left(\mu(1-\mu)\sqrt{1+\delta \cdot \epsilon} + \frac{\delta\mu R}{t-1}\right)z + \frac{\delta\mu^2 R^2}{2(t-1)} \\
= -t\left(2t - 1 - 2\sqrt{t(t-1)}\right)(\sqrt{\frac{t-1}{t}} - \delta)z^2 + \left(\mu^2 \sqrt{\frac{t}{t-1}} \sqrt{1+\delta \cdot \epsilon} + \frac{\delta\mu^2 R}{t-1}\right)z + \frac{\delta\mu^2 R^2}{2(t-1)} \\
= \frac{\mu^2}{t-1}\left(-t(\sqrt{\frac{t-1}{t}} - \delta)z^2 + (\sqrt{t(t-1)(1+\delta)\epsilon + \delta R}z + \frac{\delta R^2}{2}\right),
\]

which is a quadratic inequality for \(z\). Recall that \(\delta < \sqrt{(t-1)/t}\). So by solving the above inequality we get

\[
z \leq \frac{(\sqrt{t(t-1)(1+\delta)\epsilon + \delta R}) + (\sqrt{2t(\sqrt{(t-1)/t} - \delta)\delta R^2})^1/2}{2t(\sqrt{(t-1)/t} - \delta)} \\
\leq \frac{\sqrt{t(t-1)(1+\delta)}}{t(\sqrt{(t-1)/t} - \delta)}\epsilon + \frac{2\delta + \sqrt{2t(\sqrt{(t-1)/t} - \delta)\delta R^2}}{2t(\sqrt{(t-1)/t} - \delta)}R.
\]

We know \(\|D_{S_0}^\ast h\|_1 \leq \|D_{S_0}^\ast h\|_1 + R\sqrt{k}\). In the Lemma 5.2 if we set \(m = N\), \(r = k\), \(\lambda = R\sqrt{k} \geq 0\) and \(\alpha = 2\), we can obtain

\[
\|D_{S_0}^\ast h\|_2 \leq \|D_{S_0}^\ast h\|_2 + R.
\]
\[ \|h\|_2 = \|D^*h\|_2 \]
\[ = \sqrt{\|D_{S_0}^*h\|_2^2 + \|D_{S_0}^*h\|_2^2} \]
\[ \leq \sqrt{\|D_{S_0}^*h\|_2^2 + (\|D_{S_0}^*h\|_2 + R)^2} \]
\[ \leq \sqrt{2\|D_{S_0}^*h\|_2^2 + R} \leq \sqrt{2z + R} \]
\[ \leq \frac{\sqrt{2(1 + \delta)}}{1 - \sqrt{t/(t - 1)\delta}} \epsilon + \left( \frac{\sqrt{2\delta} + \sqrt{t/\sqrt{(t - 1)/t - \delta})\epsilon}}{t(\sqrt{(t - 1)/t - \delta})} + 1 \right) R. \]

Substituting \( R \) into this inequality, we can get the conclusion. For the case where \( t \cdot k \) is not an integer, we set \( t^* := \lceil tk \rceil / k \), then \( t^* > t \) and \( \delta_{t^*k} = \delta_{tk} < \sqrt{\frac{t^* - 1}{t^*}} < \sqrt{\frac{t - 1}{t}} \). We can prove the result by working on \( \delta_{t^*k} \). \( \square \)

References

[1] Aldroubi, A., Chen, X., Powell, A.M.: Perturbations of measurement matrices and dictionaries in compressed sensing. Appl. Comput. Harmon. Anal. 33(2), 282-291 (2012)
[2] Baker, C.A.: A note on sparsification by frames. arXiv:1308.5249 (2013)
[3] Balan, R., Casazza, P., Edidin, D.: On signal reconstruction without phase. Appl. Comput. Harmon. Anal. 20(3), 345-356 (2006)
[4] Bodmann, B.G., Hammen, N.: Stable phase retrieval with low-redundancy frame. Adv. Comput. Math. 41(2), 317-331 (2015)
[5] Cai, T.T., Zhang, A.: Sharp RIP bound for sparse signal and low-rank matrix recovery. Appl. Comput. Harmon. Anal. 35(1), 74-93 (2013)
[6] Cai, T.T., Zhang, A.: Sparse representation of a polytope and recovery of sparse signals and low-rank matrices. IEEE Trans. Inf. Theory 60(1), 122-132 (2014)
[7] Candès, E.J., Eldar, Y.C., Needell, D., Randall, P.: Compressed sensing with coherent and redundant dictionaries. Appl. Comput. Harmon. Anal. 31(1), 59-73 (2011)
[8] Conca, A., Edidin, D., Hering, M., Vinzant, C.: An algebraic characterization of injectivity in phase retrieval. Appl. Comput. Harmon. Anal. 38(2), 346-356 (2015)
[9] Fienup, J.: Reconstruction of a complex-valued object from the modulus of its Fourier transform using a support constraint. J. Opt. Soc. Am. A 4(1), 118-123 (1987)
[10] Gao, B., Wang, Y., Xu, Z.Q.: Stable signal recovery from phaseless measurements. J. Fourier Anal. Appl. (2015). doi: 10.1007/s00041-015-9434-x
[11] Kohler, D., Mandel, L.: Source reconstruction from the modulus of the correlation function: a practical approach to the phase problem of optical coherence theory. J. Opt. Soc. Am. 63(2), 126-134 (1973)
[12] Moravec, M., Romberg, J., Baraniuk, R.: Compressive phase retrieval. Proc. of SPIE Vol. (2007). doi: 10.1117/12.736360
[13] Rauhut, H., Schnass, K., Vanderheydst, P.: Compressed sensing and redundant dictionaries. IEEE Trans. Infor. Theo. 54(5), (2008)
[14] Shechtman, Y., Eldar, Y.C., Szameit, A., Segev, M.: Sparsity based sub-wavelength imaging with partially incoherent light via quadratic compressed sensing. Opt. Express. 19(16), 14807-14822 (2011)
[15] Voroninski, V., Xu, Z.Q.: A strong restricted isometry property, with an application to phaseless compressed sensing. Appl. Comput. Harmon. Anal. (2015). doi:10.1016/j.acha.2015.06.004
[16] Wang, Y., Xu, Z.Q.: Phase retrieval for sparse signals. Appl. Comput. Harmon. Anal. 37(3), 531-544 (2014)
[17] Xu, G.W., Xu, Z.Q.: On the \( \ell_1 \)-norm invariant convex \( k \)-sparse decomposition of signals. J. Oper. Res. Soc. China 1(4), 537-541 (2013)
LSEC, Inst. Comp. Math., Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing, 100091, China

E-mail address: gaobing@lsec.cc.ac.cn