KPZ-TYPE FLUCTUATION BOUNDS FOR INTERACTING DIFFUSIONS IN EQUILIBRIUM

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Abstract. We study the fluctuations in equilibrium of a class of Brownian motions interacting through a potential. For a certain choice of exponential potential, the distribution of the system coincides with differences of free energies of the stationary semi-discrete or O’Connell-Yor polymer.

We show that for Gaussian potentials, the fluctuations are of order \( N^{1/4} \) when the time and system size coincide, whereas for a class of more general convex potentials \( V \) the fluctuations are of order at most \( N^{1/3} \).

In the O’Connell-Yor case, we recover the known upper bounds for the fluctuation exponents using a dynamical approach, without reference to the polymer partition function interpretation.

1. Introduction

The O’Connell-Yor polymer \([20]\), also known as the semi-discrete polymer, is a central model of the Kardar-Parisi-Zhang (KPZ) universality class. The model is an ensemble of up-right paths in a random environment formed by independent standard Brownian motions \( B_1, \ldots, B_N \), with partition function

\[
Z_{N,t}(\beta) = \int_{0 \leq s_1 \leq \ldots \leq s_{N-1} \leq t} e^{\beta \sum_{j=1}^N B_j(s_j) - B_j(s_{j-1})} \, ds_1 \cdots ds_{N-1},
\]

where we use the conventions \( s_0 = 0 \) and \( s_N = t \). This model, along with a stationary version also introduced in [20] (see Section 2.1), has been an object of intense study over the past decade. Using the stationary version, O’Connell and Moriarty [22] computed the limiting free energy density

\[
\frac{1}{N} \log Z_{N,t}(\beta).
\]

Seppäläinen and Valkó [27] showed that the fluctuations of the free energy are of order \( N^{1/2} \) for both the stationary and non-stationary models when \( N \) and \( t \) are tuned in a certain characteristic direction (otherwise the fluctuations are Gaussian and of order \( (\max\{N,t\})^{1/2} \)). O’Connell [21] introduced a multidimensional diffusion process to Dyson Brownian motion such that the law of \( \log Z_{N,t} \) is equal to that of the “leading particle” of the process, and used this to give a contour integral expression. An alternate contour representation was used by Borodin, Corwin and Ferrari [6] to show that the free energy (the logarithm of (1.1)) asymptotically has Tracy-Widom fluctuations, confirming the expectation that the model belongs to the KPZ universality class. Virág [34] shows
convergence of a suitably centered and rescaled version of $\log Z_{N,t}(1)$ to the KPZ fixed point of Matetski, Remenik and Quastel [18].

It has been noted by several authors (see for example [11,21,24,29]) that the sequence $v_j(t) = \log Z_{j,t}$ for $j = 1, \ldots, N$ satisfies a system of stochastic differential equations of the form:

$$dv_j = -V'(v_j - v_{j-1})dt + dB_j,$$

$$V(x) = e^{-\beta x}.$$  (1.2)

In this setting, the implication of the Burke property discovered by O’Connell and Yor for their polymer model [20], is that the equations $(v_j(t))_{1 \leq j \leq N}$ have an invariant measure of product form. The “zero temperature” case, corresponding to the limit $\beta \to \infty$ of the system (1.2), has been studied by Sasamoto-Spohn as well as Ferrari, Spohn and Weiss [9,10,11,24]. In this formal limit, the system consists of Brownian motions reflected off each other. Ferrari, Spohn and Weiss’s results imply that for various classes of initial data, the distribution of the system has explicit expressions in terms of contour integrals that can be analyzed to find limiting distributions given by the Airy process. More recently, Nica, Remenik and Quastel [19] showed that the scaling limit of the time-dependent system is described by the KPZ fixed point.

Systems such as (1.2), as well as the one we study below are the totally asymmetric analog of the classical system of interacting Brownian motions models studied in [28], for example (see [13] as well for the case $\beta = \infty$):

$$dv_j = (V'(v_{j+1} - v_j) - V'(v_j - v_{j-1}))dt + dB_j.$$  (1.3)

For general convex $V$, this is sometimes known as the Ginzburg-Landau system [36]. Chang and Yau [7] famously derived the fluctuations for such processes out of equilibrium. In this symmetric case, the fluctuations are of order $N^{1/4}$.

Diehl, Gubinelli and Perkowski [8] study the weakly asymmetric case, where (1.3) is replaced by

$$dv_j = (pV'(v_{j+1} - v_j) - qV'(v_j - v_{j-1}))dt + dB_j$$

with $p - q = 1/N$ and show that after suitably rescaling, the field $u_j := v_j - v_{j-1}$, converges to a solution of the stochastic Burgers equation. Their result holds for a class of convex potentials with $V'$ Lipschitz. The recent work [14] specifically addresses the O’Connell-Yor case and proves convergence to the stochastic Burgers equation in the intermediate disorder regime.

In this paper, we study the magnitude of the equilibrium fluctuations of the system (1.2) for a class of convex potentials $V$ which includes the O’Connell-Yor $V(x) = e^{-x}$, but in general are not expected to be explicitly solvable. In particular, we recover the upper bounds for the known exponents for the O’Connell-Yor model [27,24] entirely through a dynamical approach without appealing to the polymer representation (1.1).

We consider the long-term behavior with fixed shape parameter $\theta$. We find that for the special case of quadratic potentials, the system can be solved exactly, and the fluctuations are Gaussian of order $N^{1/4}$ as in the symmetric case (1.3). For a class of
potentials satisfying an exponential growth condition at \(-\infty\), we obtain an upper bound of order \(N^{\frac{1}{3}}\) for the standard deviation.

2. Definition of model and statement of main results

Consider the following system of interaction diffusions

\[
\begin{align*}
\frac{du_1}{dt} &= -V'(u_1) dt + dB_0^{-\theta} + dB_1 \\
\frac{du_j}{dt} &= (V'(u_{j-1}) - V'(u_j)) dt + dB_j - dB_{j-1}, \quad 2 \leq j \leq N
\end{align*}
\] (2.1)

Here \(B_1, \ldots, B_N\) are independent standard Brownian motions on \(\mathbb{R}_+\), while

\[B_0^{-\theta}(t) = B_0(t) - \theta t\]

is an independent Brownian motion with drift \(-\theta < 0\). We are interested in the case where the initial data

\[u(0) = (u_1(0), \ldots, u_N(0))\]

is distributed according to the unique invariant measure for (2.1), which will eventually be seen to be iid with density proportional to \(e^{-(V(u)+\theta u)}\).

**Definition 1.** We say \(V\) is of O’Connell-Yor type if \(V\) is a smooth convex function which satisfies the following estimates for some constants \(0 < c, C < \infty\)

\[
\begin{align*}
|V^{(k)}| &\leq CV, \quad k = 1, 2, 3 \\
V(x) &\geq c|x|^2, \quad x \leq -C \\
0 &\leq -V'(x) \leq C(x + 1), \quad x \geq 0,
\end{align*}
\] (2.2)

As a consequence of the above assumptions we have for any \(\theta > 0\) that,

\[V(x) + \theta x \geq c'|x| - C'\] (2.3)

for some positive \(c', C' > 0\).

The following exhibits a large class of potentials satisfying the above assumptions.

**Proposition 1.** Let \(\mu\) be a finite positive measure supported inside \((a, A)\) for \(0 < a < A < \infty\). Then the potential

\[V_\mu(x) = \int e^{-\lambda x} \mu(d\lambda)\]

satisfies the conditions (2.2).

**Proof.** Differentiating under the integral sign, we find

\[0 \leq -V'_\mu(x) \leq A \int e^{-\lambda x} \mu(d\lambda),\]

and

\[0 \leq V''_\mu(x) = \int \lambda^2 e^{-\lambda x} \mu(d\lambda) \leq A^2 V_\mu(x).\]

Similarly,

\[|V'''_\mu(x)| \leq \int \lambda^3 e^{-\lambda x} \mu(d\lambda) \leq A^3 V_\mu(x).\]
Moreover, if $x > 0$, then
\[
V_\mu(-x) = \int e^{\lambda x} \mu(dx) \geq e^{ax} \int \mu(dx) \gg x^2.
\]

We remark that potentials of the form $V(x) = V_\mu(x) + \varepsilon \rho(x)$ for $\rho \in C^\infty_0(\mathbb{R})$ and $\varepsilon$ sufficiently small (depending on $\mu$ and $\rho$) also satisfy the above assumptions, as well as the additional conditions of Theorem 1 below.

2.1. Link with the O’Connell-Yor polymer. Let us now explain the connection between the model introduced in [20] and the system (2.1) in equilibrium. The stationary semi-discrete polymer is a polymer model in a random environment, defined by a variant of the partition function (1.1). Consider a collection $B_0, B_1, \ldots, B_N$ of $N + 1$ two-sided Brownian motions, and a parameter $\theta > 0$. We then define
\[
Z_{N,t}^\theta = \int_{-\infty < s_0 \leq s_1 \leq \ldots \leq s_{N-1} \leq t} e^{\theta s_0 - B_0(s_0) + \sum_{j=1}^N B_j(s_j) - B_j(s_{j-1})} ds_0 \cdots ds_{N-1},
\]
where $s_N = t$ as in (1.1), but $s_0$ is now a variable of integration. A simple computation using Itô’s formula shows that the quantities
\[
\begin{align*}
    u_{OY}^1(t) &:= \log Z_{1,t}^\theta + B_0(t) - \theta t, \\
    u_{OY}^j(t) &:= \log Z_{j,t}^\theta - \log Z_{j-1,t}^\theta, \quad 2 \leq j \leq N
\end{align*}
\]
satisfy the stochastic differential equations (2.1) with
\[
V(x) = e^{-x}.
\]
In particular, we note the following relation involving the free energy:
\[
\log Z_{N,t}^\theta = \sum_{j=1}^N u_{OY}^j(t) - B_0(t) + \theta t. \tag{2.4}
\]
The main result of [20], the Burke property, implies that $u_{OY}^j$ has a product form invariant measure. That is, if $u(0) = (u_1(0), \ldots, u_N(0))$ is an iid vector with the distribution $(\log \frac{1}{X_j})_{1 \leq j \leq N}$, where $X_j$ is a Gamma($\theta$) random variable, then $u(t)$ has the same distribution at later times. In particular,
\[
\log Z_{N,t=0}^\theta = \sum_{j=1}^N u_{OY}^j(0),
\]
has the distribution of an iid sum.

\footnote{These random variables have density $1_{\{x > 0\}} x^{\theta-1}e^{-x} \frac{dx}{\Gamma(\theta)}$.}
2.2. The height function. The main object of study in this paper is the analogue of the free energy in (2.4). Namely:

$$W^\theta_{N,t} := \sum_{j=1}^{N} u_j(t) - B_0(t) + \theta t.$$  (2.5)

Here the $u_j$ are solved by the system (2.1) with any fixed potential $V$ that satisfies the assumptions in (2.2).

When there is no risk of confusion, we simply denote $W^\theta_{N,t}$ by $W_{N,t}$.

2.3. Statement of results. We begin by introducing some notation we need to state our results. For $\theta > 0$, define

$$Z(\theta) = \int_{\mathbb{R}} e^{-\theta x - V(x)} \, dx.$$  

Then, for $k \geq 0$, we set

$$\psi^V_k(\theta) := d_{k+1}^{\theta+1} \log Z(\theta).$$  (2.6)

Note that $\psi^V_1(\theta) > 0$, being the variance of a random variable with the following distribution. For $\theta > 0$, we let $\nu_\theta$ be the distribution on $\mathbb{R}$ with density

$$f_\theta(x) = \frac{1}{Z(\theta)} e^{-\theta x - V(x)}.$$  

Note that if $V$ is of O’Connell-Yor type, then $Z(\theta) < \infty$ for any $\theta > 0$ due to (2.3). If $V$ is only assumed to be convex, we require that the growth at $-\infty$ be sufficient for $Z(\theta)$ to be finite. For example,

$$V(-x) \geq c|x|^\alpha$$

for some $\alpha > 1$ suffices. As explained in Section 5, the system (2.1) has the (unique) product invariant probability distribution

$$\omega_\theta(x) \, dx = \prod_{j=1}^{N} \nu_\theta(dx_j).$$  (2.7)

Our main result concerns the fluctuations of the height function (2.5) in equilibrium, that is, when the initial data $u(0) = \bar{u}_0$ is distributed according to $\omega_\theta$, independently of the Brownian motions $B_0, B_1, \ldots, B_N$. We denote by $\text{Var}^\theta$ the variance in equilibrium, i.e. with the initial data $u(0)$ distributed according to $\omega_\theta$ for $\theta > 0$.

From (2.5) and the invariance of the measure, we have

$$\text{Var}(W^\theta_{N,t}) \leq 2\text{Var}\left(\sum_{j=1}^{N} u_j(0)\right) + 2t = O(N + t).$$  (2.8)

if the initial data is distributed according to $\omega_\theta$. In general, the order of this bound cannot be improved: as a consequence of Proposition 9, we have the lower bound

$$\text{Var}(W^\theta_{N,t}) \geq |N\psi^V_1(\theta) - t|,$$

so (2.8) is of the correct order if $N \gg t$ or $N \ll t$. However, for special values of $N$ and $t$, there is cancellation between the sum and the Brownian motion term in (2.5). For
example, we will see below (see Proposition 2) that in the special case $V(x) = \frac{x^2}{2}$, and $t = N$, the variance can be computed exactly and the fluctuations are of order $N^{\frac{1}{4}}$.

The main result of Seppäläinen and Valkó [27] for the O’Connell-Yor polymer implies that if $t$ and $N$ are suitably chosen properly (see (2.9)), the fluctuations are of order $N^{\frac{1}{3}}$, a growth rate characteristic of the Kardar-Parisi-Zhang universality class. Our main result is a non-perturbative argument which extends the variance upper bound in [27] to a large class of potentials. Seppäläinen and Valkó’s proof relies on the polymer interpretation explained in Section 2.1, which is not available for potentials other than $e^{-βx}$, $β > 0$.

**Theorem 1.** Suppose $V$ is convex, with $V''' \leq 0$, and that or $V$ is of O’Connell-Yor type. Suppose that $N$, $t$, $θ$ are chosen so that

$$|t - Nψ_V^1(θ)| \leq A N^{\frac{α}{2}},$$  \hspace{1cm} (2.9)

for some $A > 0$. In addition assume there is a constant $c_0 > 0$ such that

$$(e^{cx}V''(x))' \leq 0.$$  \hspace{1cm} (2.10)

Then, there exists $C > 0$ such that

$$\text{Var}^θ(W_{N,N}^θ) \leq CN^{\frac{α}{2}}.$$  \hspace{1cm} (2.11)

We remark here that the assumption that $V$ be of O’Connell-Yor type is used to show that the system (2.1) is well-posed and that the solutions are differentiable with respect to various parameters. Our argument can likely be extended to any $V$ satisfying the other conditions in the theorem if well-posedness is known. Note that condition (2.10) precludes $V'$ from being uniformly Lipschitz. This condition is automatically satisfied for the class of examples in Proposition 1 with $c_0 = c$. Indeed, in that case we have

$$(e^{cx}V_μ(x)''') = \int_c^A (c - x)x^2 e^{-λx} μ(dλ) \leq 0.$$ 

If, instead of the characteristic direction condition (2.9), we assume $|t - Nψ_V^1(θ)| \geq N^α$ with $α > 2/3$, then the fluctuations are of order at least $N^{α/2}$. Indeed, we have the following result:

**Corollary 1.** Suppose there is $γ_0 > 0$ and $α > 2/3$ such that

$$N^{-α}|t - Nψ_V^1(θ)| \to γ_0 > 0,$$

then

$$\frac{W_{N,N}^θ + Nψ_V^1(θ)}{N^{\frac{α}{2}}}$$

converges to a standard Gaussian random variable with variance $γ_0/ψ_V^1(θ)$.

In the case where $V(x)$ is quadratic, the system (2.1) is linear and admits an explicit solution as a Gaussian process. In this case, we have the following.

**Proposition 2** (The Gaussian case). Let $V(x) = \frac{x^2}{2}$. Then, for each $θ > 0$, we have

$$N^{-\frac{1}{4}} \text{Var}^θ(W_{N,N}) \to \sqrt{\frac{2}{π}}.$$
Moreover, the normalized quantity

\[ \sqrt{\pi W_{N,N}^\theta} \sqrt{2 \cdot N^{1/4}} \]

converges in distribution to a standard Gaussian random variable.

3. Outline of the paper

The strategy of proof is inspired by coupling arguments appearing in works of Balázs-Cator-Seppäläinen for the corner growth model with exponential weights [1], Balázs-Seppäläinen for the ASEP [2], Balázs-Komjáthy-Seppäläinen on zero-range and deposition processes [3][4], Balázs-Seppäläinen-Quastel on the KPZ equation [5] and Seppäläinen on various models of last passage percolation and polymers in random environments [25][26]. All of these models are expected to belong to the KPZ universality class, and in some cases this expectation has been confirmed by rigorous results such as the existence of asymptotic random matrix fluctuations (Tracy-Widom) fluctuations.

The difference between the model we consider here and those mentioned thus far – other than the discrete nature of the state space in most of these works – is that, without the polymer or particle system interpretation, we do not have access to quantities which play the role of the occupation length or second class particles when we work with perturbations of the initial data. For our argument, we require those perturbations to have certain monotonicity and convexity properties. This is achieved by generating the initial data from running the equation for a long initial time and using ergodicity. We gave an alternative proof of the result of Seppäläinen and Valkó in [16].

In Section 4, we discuss the well-posedness of the equations (2.1). Since the “drift” terms \( V'(u_j) \) appearing in the equation can have super-linear growth, the existence of global solutions does not follow directly from the most basic existence theorems for SDEs in the case where \( V \) is of O'Connell-Yor type. However, our assumptions on \( V(x) \) imply that it is strongly confining for negative \( x \) and sublinear for \( x > 0 \). Two further facts which simplify the analysis are the constant diffusion coefficients and the triangular nature of the system, that is, for each \( k \), the first \( k \) equations form a closed system.

In Section 5, we discuss the invariant measure for the system, proving in particular that it is unique. Consequently, the mean ergodic theorem can be applied to show that the time averages of observables converge to their expectation under the invariant measure.

In Section 6, we introduce the couplings of the initial data for different parameters that we use in our argument. The initial data we use is obtained by running the equation for some time. This produces data that is monotone and convex in the parameter \( \theta \).

In Section 7, we derive two representations for the variance in terms of derivatives of the height function with respect to the initial data and the parameter in the equations, respectively. For example, we have

\[ \text{Var}(W_{N,t}^\theta) = N\psi^V(\theta) - t + 2\mathbb{E}^\theta[\partial_\eta W_{N,t}(\eta, \theta)|_{\eta=\theta}] \]

Here \( W_{N,t}(\eta, \theta) \) denotes the height function at time \( t \), with initial data sampled from \( \omega_\eta \).

(In this informal outline, we ignore the issue of how to couple the different initial data as \( \eta \) varies.)
In Section 8 we analyze the Gaussian case in which $V(x)$ is quadratic. In this case $\partial_\theta W_{N,t}$ can be computed explicitly, and the fluctuations of $W_{N,t}^\theta$ are of order $N^{1/4}$.

In Section 9, we prove our main results. Given our representation for the variance above, we must show that the term (note that $N\psi_1^V(\theta) - t = O(N^{2/3})$ under the characteristic directions assumption on $N$ and $t$)

$$E^\theta[\partial_\theta W_{N,t}(\eta, \theta)|_{\eta=\theta}]$$

is of order $N^{1/4}$. The key point is that the full derivative

$$\frac{d}{d\theta} W_{N,t}(\theta, \theta)$$

is easier to compute than $\partial_\theta W_{N,t}(\theta, \theta)$. Roughly, we will show $\theta \to W_{n,t}(\theta, \theta)$ is convex, and so the derivative can be bounded in terms of difference quotients:

$$\frac{d}{d\theta} W_{N,t}(\theta, \theta) \leq \frac{W_{N,t}(\theta, \theta) - W_{N,t}(\lambda, \lambda)}{\lambda - \theta}.$$

We show that the difference between the full derivative and the partial derivative is $O(N^{1/2})$. We use the above inequality together with Cauchy-Schwarz and so the quantities appearing on the right-hand side can be related back to the variance of $W_{N,t}$. As a consequence, we derive the inequality

$$V \leq CN^{1/3}(V^{1/2} + N^{1/4})$$

for $V = \text{Var}^\theta(W_{N,t}^\theta)$.

4. Well-posedness

The existence of local-in-time solutions to the system (2.1) follows from standard results on stochastic differential equations. We will need to differentiate global solutions of the equation, so we provide some details of the construction here.

We remark that in this section and subsequent discussions on matters related to well-posedness we will not track the dependence in $N$ of constants, as the dependence is not important for this purpose.

The system (2.1) has the form of an $N$-dimensional stochastic differential equation:

$$dx = b(x)dt + \sigma dB,$$  \hspace{1cm} (4.1)

with smooth coefficients

$$b(x) = (-\theta - V'(x_1), V'(x_1) - V'(x_2), \ldots, V'(x_{N-1}) - V'(x_N))^T \in \mathbb{R}^N,$$  \hspace{1cm} (4.2)

and $\sigma \in M_{N \times (N+1)}(\mathbb{R})$ is given by

$$\sigma = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ \vdots & \vdots \\ -1 & 1 \\ -1 & 1 \end{pmatrix},$$  \hspace{1cm} (4.3)

and the Brownian motion $B$ is $B = (B_0, B_1, \ldots, B_N)^T$. 

We start by considering, for a fixed continuous \( f : \mathbb{R}_+ \rightarrow \mathbb{R}^N \), the equations
\[
x(t) = x(0) + \int_0^t b(x(s))ds + f(t).
\]
(4.4)

This is a system of integral equations with right-hand side \( F(x,t) = \int_0^t b(x(s))ds + f(t) \), where \( F \) locally Lipschitz in \( x \) (the Lipschitz constant does not depend on \( f \)). By classical results (see for example [33, Theorem 1, Chapter 21]) there exists, for each initial data \( u(0) = u_0 \) and each continuous \( f \), a time \( \tau = \tau_\infty(f) > 0 \) and a unique solution \( u(t) = u(t,u_0,f) \in \mathbb{R}^N \) of (4.4) on \((0,\tau)\). Moreover, if \( \tau_\infty(f) < \infty \), then \( \lim_{t \to \tau^-} |u(t)| = \infty \).

We now let
\[
f(t) = \sigma(B_0, B_1, \ldots, B_N)^T \in \mathbb{R}^N,
\]
(4.5)
and show

**Proposition 3.** Let \( u_0 \in \mathbb{R}^N \) and \( f \) be as in (4.5), where \( \sigma \) is defined in (4.3).

1. The \( u(t,u_0,f = \sigma B) \) is a solution of the stochastic system (2.1).
2. For every \( u_0 \in \mathbb{R}^N \) and almost every realization of the Brownian motions, we have
\[
\tau_\infty(\sigma B) = \infty.
\]
(4.6)
That is, the solution of the stochastic differential equation are global-in-time.
3. The solution satisfies the Markov property.

**Proof.** The first point follows immediately from (4.4). In the case where \( V'(x) \) is Lipschitz, the second and third point are classical, see for example [32, Theorems 5.1.1 and 5.1.5]. The second will be proved in Section 4.2. The third point follows by (2), as addressed in Section 4.4.

\[\square\]

### 4.1. Generator

We begin by remarking that system (2.1) is an Itô diffusion with generator
\[
L = \frac{1}{2} (\partial_1)^2 + \frac{1}{2} \sum_{j=2}^N (\partial_j - \partial_{j-1})^2 + \frac{1}{2} (\partial_N)^2
\]
\[
+ (-V'(u_1) - \theta) \partial_1 + \sum_{j=2}^N [V'(u_{j-1}) - V'(u_j)] \partial_j.
\]
(4.7)

This generator \( L \) has the form
\[
L = \frac{1}{2} \text{tr}(a \nabla^2) + b \cdot \nabla,
\]
where \( a \in M_{N \times N}(\mathbb{R}) \):
\[
a \equiv \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & 0 & \cdots & 0 & 2 \\
-1 & -1 & \cdots & -1 & 2
\end{pmatrix}
\]
and $b$ is as in \[(1.2).\]

**Proposition 4.** The symmetric matrix $a$ is positive definite.

**Proof.** From \[(4.7),\] all eigenvalues of the matrix $a$ are nonnegative:

$$\frac{1}{2} \xi^t a \xi = e^{-x \cdot \xi} \left\{ \frac{1}{2} (\partial_i)^2 + \frac{1}{2} \sum_{j=2}^N (\partial_j - \partial_{j-1})^2 + \frac{1}{2} (\partial_N)^2 \right\} e^{x \cdot \xi} \geq 0.$$  

Moreover, denoting by $D_n$ the determinant of the matrix $a$ in dimension $n$, we have for $n \geq 2$

$$D_n = 2D_{n-1} - D_{n-2}, \quad D_1 = 2, \quad D_2 = 3.$$  

from which one easily verifies $D_n = n + 1 > 0$. \hfill $\square$

Since the principal part of $L$ has constant coefficients, its analysis reduces to that of a perturbation of the Laplacian, for which various results are more readily available in the literature [33], [31, Chapter 7.3]. Indeed we have the following:

**Proposition 5.** Denote by $a^{1/2}$ the positive square root of the symmetric matrix. We let

$$A = a^{-\frac{1}{2}}.$$  

Define the operator $\tilde{L}$ by

$$\tilde{L} := \frac{1}{2} \Delta + \tilde{b}(x) \cdot \nabla, \quad (4.8)$$

where

$$\tilde{b}(x) = Ab(A^{-1} x). \quad (4.9)$$

Let $u$ be twice continuously differentiable on $\mathbb{R}^N$ and define

$$v(x) := u(Ax),$$

then

$$(Lv)(x) = (\tilde{L}u)(Ax).$$

**Proof.** By direct computation, we have

$$\partial_i v(x) = \partial_i u(Ax) = \sum_k A_{ki}(\partial_k u)(Ax) = (A \nabla u)_i(Ax), \quad (4.10)$$

$$\partial_{ij}^2 v(x) = \partial_{ij}^2 u(Ax) = \sum_{k,l} A_{ki}A_{lj}(\partial_{kl}^2 u)(Ax). \quad (4.11)$$

Taking the the inner product of \[(4.10)\] with $b(x)$, we find

$$\sum_i b_i(x) \partial_i v(x) = \sum_{i,k} b_i(x) A_{ik}(\partial_k u)(Ax)$$

$$= \sum_k \left( \sum_i A_{ki}b_i(A^{-1} Ax) \right)(\partial_k u)(Ax)$$

$$= [\tilde{b} \cdot (\nabla u)](Ax).$$
Multiplying (4.11) by $a_{ij}$ and summing over $1 \leq i, j \leq N$ we obtain
\[
\sum_{i,j} a_{ij} \partial_{ij}^2 u(x) = \sum_{k,l} \sum_{i,j} (a^{-\frac{1}{2}})_{kl} a_{ij} (a^{-\frac{1}{2}})_{jl} (\partial_{kl}^2 u)(Ax)
\]
\[
= \sum_{k,l} \delta_{kl} (\partial_{kl}^2 u)(Ax) = (\Delta u)(Ax).
\]
The result follows immediately by adding the last two displayed equations. □

The operator (4.8) is the generator of the diffusion process $(Au(t))_i, i = 1, \ldots, N$. By ellipticity, we have for $x \in \mathbb{R}^N$:
\[
c||x||_2 \leq ||Ax||_2 \leq C||x||_2,
\]
so $x$ and $Ax$ have comparable sizes at infinity.

4.2. Lyapunov functions. In this section, we establish some bounds for the local solutions in the previous section which imply in particular that the solutions to the SDE system introduced above are global. For notational convenience we note that we can add a positive constant to the potential $V(x)$ without changing the assumptions or affecting the proof of well-posedness. We replace $V$ by $V + 2C'$ so that $V(x) + \theta x \geq c'|x| + C'$ (see (2.3)). In particular, the function $\psi$ introduced below is positive.

**Proof of Proposition 3.2.** Define the non-negative (see discussion above) function $\psi$ by,
\[
\psi(x) = \sum_{j=1}^{N} (V(x_j) + \theta x_j).
\]

Applying the vector field part of $L$, that is $b(x) \cdot \nabla$, to $\psi$, we obtain
\[
\begin{align*}
(-V'(x_1) - \theta) \partial_1 \psi(x) + \sum_{j=2}^{N} (V'(x_{j-1}) - V'(x_j)) \partial_j \psi(x) \\
= -(V'(x_1) + \theta)^2 + \sum_{j=2}^{N} (V'(x_{j-1}) - V'(x_j))(V'(x_j) + \theta) \\
= -\theta^2 - \theta V'(x_N) - (V'(x_1) + \theta)V'(x_1) + \sum_{j=2}^{N} (V'(x_{j-1}) - V'(x_j))V'(x_j) \\
= -\frac{1}{2}(V'(x_1) + \theta)^2 - \frac{1}{2} \sum_{j=2}^{N} (V'(x_j) - V'(x_{j-1}))^2 - \frac{1}{2}(V'(x_N) + \theta)^2.
\end{align*}
\]
In particular, this is nonpositive. For the diffusion part $L - b(x) \cdot \nabla$, using (2.2), we have
\[
\frac{1}{2} (\partial_1^2 + \sum_{j=2}^{N} (\partial_j - \partial_{j-1})^2 + \partial_N^2) \psi(x) = \sum_{j=1}^{N} V''(x_j) \leq C_1 \psi(x).
\]
Putting everything together, we find that
\[ L\psi(x) \leq C_1\psi(x). \]  
(4.13)

Applying Itô’s formula with
\[ \tau_M = \inf\{t : |u(t)| > M\}, \]
we have
\[ \mathbb{E}[\psi(u(t \wedge \tau_M))] - \psi(u(0)) = \mathbb{E}\left[ \int_0^{t \wedge \tau_M} L\psi(u(s)) \, ds \right] \leq C_1 \int_0^t \mathbb{E}[\psi(u(s)), s < \tau_M] \, ds \]

Using Gronwall’s inequality, we have
\[ \mathbb{E}[\psi(u(t \wedge \tau_M))] \leq \psi(u(0))e^{C_1t}. \]  
(4.14)

Note that
\[ \psi(x) \geq cx_+, \]
\[ \psi(x) \geq c(x_-)^p, \quad p \geq 2. \]  
(4.15)

so by (4.14) we have for each \( u \in \mathbb{R}^N \)
\[ P(\tau_\infty < \infty | u_0 = u) \leq \frac{1}{c_M} \mathbb{E}[\psi(u(t)) | u_0 = u], \]

where \( c_M = \inf\{\psi(x) : |x| \geq M\} \). Taking \( M \to \infty \) we find \( P(\tau_\infty = \infty | u_0 = u) = 1. \)

4.3. Estimates for higher moments. In justifying various limits in the computations in the rest of the paper, we will need bounds of the moments of \( |u(t)| \) and \( V(u(t)) \), which we derive in this section. Here and below, \( |u(t)| = \sqrt{\sum_i u_i(t)^2} \) denotes the Euclidean norm. We will continue to use the Lyaponuv function \( \psi \) defined above in (4.12).

We define \( \varphi_2(x) \) by
\[ \varphi_2(x) = \sum_{j=1}^N (x_j)_+^2, \]
where we take the squares of the positive parts. Applying the generator to \( \varphi(x) \), we obtain, for \( x > 0 \)
\[ L\varphi(x) = (-V'(x_1) - \theta)(x_1)_+ + \sum_{j=2}^N \left[ V'(x_{j-1}) - V'(x_j) \right] (x_j)_+ + N \]
\[ \leq -V'(x_1)(x_1)_+ - \sum_{j=2}^N V'(x_j)(x_j)_+ + C_4 \]
\[ \leq C_3\varphi_2(x) + C_4. \]

In the second step we have used \( V'(x) \leq 0 \) and \( -V'(x)x_+ \leq C(x_+^2 + 1) \), which follows from our assumptions on \( V \). We thus have
\[ L\varphi_2(x) \leq C_3\varphi_2(x) + C_4, \]
and so applying Itô’s formula as above, we find
\[ \mathbb{E}[\varphi_2(u(t))] \leq e^{C \delta t}(\varphi_2(u(0)) + C_4) \] (4.17)

Combining this with (4.16) and using
\[ |x|^2 \leq x_+^2 + x_-^2, \]
we now see that the second moment is finite: for any \( t \geq 0 \) and \( u_0 \in \mathbb{R}^N \), we have
\[ \mathbb{E}[|u(t)|^2 | u(0) = u_0] \leq C(\psi(u_0) + \varphi_2(u_0))e^{Ct}. \] (4.18)

Letting
\[ \varphi_p(x) = \sum_{j=1}^N (x_j)^p \]
for \( p \geq 3 \), and repeating the argument in the previous proof, we find
\[ L \varphi_p(x) \leq C(p) (\varphi_p(x) + \varphi_{p-2}(x)), \]
leading inductively to the estimate
\[ \mathbb{E}[\varphi_p(u(t)) | u(0) = u_0] \leq C(\varphi_p(u_0) + 1)e^{Ct}. \]

Using
\[ |x|^p \leq x_+^p + x_-^p \leq C\psi(x) + \varphi_p(x), \]
we then get
\[ \mathbb{E}[|u(t)|^p | u(0) = u_0] \leq C(p)e^{Ct}(\psi(u_0) + \varphi_p(u_0)). \] (4.19)

From the equation, we get an estimate for the maximum over compact time intervals \( t \in [0, T] \).

**Proposition 6.** For each \( T > 0 \) and \( p \geq 2 \), we have
\[ \mathbb{E}[\sup_{t \leq T} |u(t)|^p | u(0) = u_0] \leq C(N, p) (\psi(u_0) + \varphi_p(u_0))e^{Ct}, \] (4.20)
where \( |u(t)| = \sqrt{\sum_i u_i(t)^2} \) denotes the Euclidean norm. Moreover, for any \( 1 \leq j \leq N \) and \( p \geq 1 \), we have
\[ \mathbb{E}[V(u_j(t))^p | u(0) = u_0] \leq C(N, p, \theta) (|V(u_0)|^q + |u_0|^q + 1)e^{Ct}, \] (4.21)
where \( q = 2^N p \) and \( C(N, p, \theta) \) is bounded for \( \theta \) is bounded intervals. The term \( |V(u_0)| \) denotes the Euclidean norm of the vector obtained by applying \( V \) component-wise to \( u_0 \).

**Proof.** Recall Doob’s maximal inequality for a Brownian motion \( B \):
\[ \mathbb{E}[\sup_{t \leq T} |B_t|^p] \leq C(p) T^{\frac{p}{2}}. \] (4.22)

Using this, (4.19) and the first equation (2.1), we have for any \( t \geq 0 \)
\[ \left| \int_0^t (-V'(u_1) - \theta) ds \right| \leq |u_1(t)| + |u_1(0)| + |B_1(t)| + |B_0(t)|. \]
Combining the last inequality with (4.22), we have
\[
\sup_{0 \leq t \leq T} \left| \int_0^t -V'(u_1) \, ds \right| \leq \left| \int_0^T -V'(u_1) \, ds \right| + \theta T \\
\leq |u_1(T)| + |u_1(0)| + |B_1(T)| + |B_0(T)| + \theta T.
\]

Combining the last inequality with (4.22), we have
\[
\mathbb{E}[\sup_{t \leq T} |u_1(t)|^p \mid u(0) = u_0] \leq C(p) \left( |u_1(0)|^p + \mathbb{E} \left| \int_0^T -V'(u_1(s)) \, ds \right|^p \right) \\
+ C(p) \mathbb{E}[\sup_{t \leq T} (|B_1(t)|^p + |B_0(t)|^p)] \\
\leq C(p) \left( |u_1(0)|^p + \mathbb{E}[|u_1(T)|^p] + T^p + T^{\frac{p}{2}} \right) \\
\leq C(p) (\psi(u_0) + \varphi_p(u_0)) e^{CT}.
\]

Similarly, for \( j \geq 2 \) using once again that \(-V' \geq 0\):
\[
\sup_{t \leq T} \left| \int_0^t -V'(u_j) \, ds \right| = \left| \int_0^T -V'(u_j) \, ds \right| \\
\leq \left| \int_0^T -V'(u_{j-1}) \, ds \right| + |u_j(T)| + |u_j(0)| \\
+ |B_j(T)| + |B_{j-1}(T)|.
\]

For the inequality in the second step, we have used the equation (2.1) for \( u_1 \). Using the equation (2.1) again, we find
\[
\mathbb{E}[\sup_{t \leq T} |u_j(t)|^p] \leq C(p) \left( |u_j(0)|^p + \mathbb{E}[|u_j(T)|^p] + \mathbb{E} \left| \int_0^T -V'(u_{j-1}) \, ds \right|^p + T^{\frac{p}{2}} + T \right) \\
\leq C(p) \left( (\psi(u_0) + \varphi_p(u_0)) e^{CT} + \mathbb{E} \left| \int_0^T -V'(u_{j-1}) \, ds \right|^p + T^{\frac{p}{2}} + T \right).
\]

Proceeding by induction on \( j \), we find:
\[
\mathbb{E}[\sup_{t \leq T} |u_j(t)|^p] \leq C(N, p) (\psi(u_0) + \varphi_p(u_0)) e^{CT},
\]
the stated inequality of (4.20).

We now turn to (4.21). For \( p = 1 \) and \( j = 1 \), (4.21) follows from (4.19) and (4.11). For \( p \geq 2 \), we have
\[
L(V(x_1) + \theta x_1)^p = -p(V'(x_1) + \theta)^2(V(x_1) + \theta x_1)^{p-1} \\
+ p(p-1)(V'(x_1) + \theta)^2(V(x_1) + \theta x_1)^{p-2} \\
+ pV''(x_1)(V(x_1) + \theta x_1)^{p-1}.
\]
So we have by Itô’s formula:

\[
\mathbb{E}[(V(u_1(t)) + \theta u_1(t))^p] \leq (V(u_1(0)) + \theta u_1(0))^p \\
+ p(p - 1) \int_0^t \mathbb{E}[(V'(u_1(s)) + \theta)^2(V(u_1(s)) + \theta u_1(s))^{p-2}] ds \\
+ p \int_0^t \mathbb{E}[V''(u_1(s))(V(u_1(s)) + \theta u_1(s))^{p-1}] ds \\
\leq C(p, \theta) \int_0^t (\mathbb{E}[(V(u_1(s)) + \theta u_1(s))^p] + \mathbb{E}[|u_1(s)|^p]) ds
\]

Above we used \(V''(x) - V'(x) \leq C(V(x) + \theta x)\) (recall we added a constant to \(V\) so the right side of the inequality is positive). By Gronwall’s inequality and (4.20) we obtain the estimate (4.21) for every \(p\) and \(j = 1\).

For \(j \geq 2\) and \(p \geq 1\), we have

\[
L(V(x_j)) = p(V'(x_{j-1}) - V'(x_j))V'(x_j)V(x_j)^{p-1} \\
+ p(V''(x_j)V(x_j)^{p-1} + p(p - 1)(V'(x_j))^2V(x_j)^{p-2}.
\]

Using the estimate

\[
V'(x_{j-1})V'(x_j) \leq \frac{1}{2}(V''(x_{j-1}))^2 + \frac{1}{2}(V'(x_j))^2,
\]

we find:

\[
LV(x_j)^p \leq \frac{p}{2}((V'(x_{j-1}))^2 - (V'(x_j))^2)V(x_j)^{p-1} + pV''(x_j)V(x_j)^{p-1} \\
+ p(p - 1)(V'(x_j))^2V(x_j)^{p-2}.
\]

Every term on the right side is bounded above by \(C|V(x_j)|^p\) except for,

\[
V'(x_{j-1})^2V(x_j)^{p-1} \leq (V'(x_{j-1}))^{2p} + (V(x_j))^p \leq C(V(x_{j-1}))^{2p} + (V(x_j))^p.
\]

Hence,

\[
\mathbb{E}[V(u_j(t))^p] \leq V(u_j(0))^p + C(p, j) \int_0^t \mathbb{E}[V(u_j(s))^p] ds \\
+ C(p, j) \int_0^t \mathbb{E}[|u_{j-1}(s)|^{2p}] ds.
\]

The estimate (4.21) now follows by induction and Gronwall’s inequality.

4.4. Semigroup. Since the time of existence for the equations is infinite with probability one by (4.6), by [31, Theorem 7.3.10] \(Au(t)\) is a Markov process, that is, the family \(\{P^t\}_{t \geq 0}\) of operators defined for a bounded, measurable function \(f\), by

\[
P^t f(x) = \mathbb{E}[f(Au(t)) \mid Au = x]
\]

is a semigroup with \(P^0 1 = 1\) for all \(t \geq 0\). Moreover, we have

\[
P^t f(x) - f(x) = \int_0^t (P^s \tilde{L}) f(x) ds
\]
for \( f \in C^2_b(\mathbb{R}^N) \), twice differentiable functions with bounded, continuous derivatives. Denote by \( p(t, x, dy) \) the representing the functional \( f \mapsto P^t f(x) \)

\[
P^t f(x) = \int f(y) p(t, x, dy).
\]

Then, by classical parabolic regularity results [15, Theorem 8.11.1, 8.12.1], \( p(t, x, dy) \) has a smooth density with respect to the Lebesgue measure:

\[
p(t, x, dy) = p(t, x, y) dy.
\]

Here \( p(t, x, y) \) is the fundamental solution of the Cauchy problem

\[
\begin{align*}
\frac{\partial}{\partial t} v(t, x) &= (\tilde{L} v)(t, x), \\
v(0, x) &= f(x).
\end{align*}
\]

(4.25)

Since \( \{P^t\}_{t \geq 0} \) is a semigroup, we have

\[
p(t + s, x, y) = \int p(t, x, z) p(s, z, y) d z.
\]

(4.26)

From (4.24) with \( f = g(A^{-1} \cdot) \) and \( x = Au_0 \), we have

\[
\mathbb{E}[g(u(t)) | u(0) = u_0] = \int g(y) q(t, u_0, y) dy =: Q^t g(u_0),
\]

where

\[
q(t, u_0, y) = (\det A)^{-N} p(t, A u_0, Ay).
\]

Thus \( u(t) \) is a Markov process with semigroup \( Q^t \) and invariant measure \( \omega_\theta \):

\[
\int Q^t g(x) \omega_\theta(dx) = \int g(x) \omega_\theta(dx),
\]

(4.27)

for all \( g \) bounded and measurable.

5. Invariant measure and ergodicity

In this section, we discuss the uniqueness and ergodicity properties of the measure \( \omega_\theta \). We obtain the uniqueness of the invariant measure from the following result [33, Ch. 31, p. 254], which is formulated for perturbations of the Laplacian such as \( \tilde{L} \).

**Theorem 2.** Let \( D = \frac{1}{2} \Delta + B \cdot \nabla \) with no explosion. Assume \( B(x) \) is \( C^\infty(\mathbb{R}^N; \mathbb{R}^N) \). Define the formal adjoint \( D^* \) of \( D \) by

\[
D^* = \frac{1}{2} \Delta - \nabla \cdot B.
\]

Suppose there exists a smooth function \( \varphi \) such that \( D^* \varphi = 0, \varphi \geq 0 \). Then \( \mu(A) = \int_A \varphi(y) dy \) defines a unique invariant distribution for the process.

In our case, we have the following:

**Lemma 1.**

\[
L^* \omega = 0,
\]

where \( L^* \) denotes the adjoint of the generator \( L \) in (4.7). Consequently, if we define

\[
\tilde{\omega}(x) = \det(A)^{-1} \omega(A^{-1} x) = D^{1/2} \omega(a^{1/2} x),
\]

for \( f \in C^2_b(\mathbb{R}^N) \), twice differentiable functions with bounded, continuous derivatives. Denote by \( p(t, x, dy) \) the representing the functional \( f \mapsto P^t f(x) \)

\[
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\]

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\[
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\]

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\[
\begin{align*}
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v(0, x) &= f(x).
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\]

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\[
\mathbb{E}[g(u(t)) | u(0) = u_0] = \int g(y) q(t, u_0, y) dy =: Q^t g(u_0),
\]

where

\[
q(t, u_0, y) = (\det A)^{-N} p(t, A u_0, Ay).
\]

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\[
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\]

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\[
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\[
\tilde{\omega}(x) = \det(A)^{-1} \omega(A^{-1} x) = D^{1/2} \omega(a^{1/2} x),
\]
we have
\[ \tilde{L}^* \tilde{\omega} = 0. \]  
(5.1)

Proof. By direct computation, we have
\[
L^* = \frac{1}{2} \text{tr}(a \cdot \nabla^2) - \nabla \cdot b
\]
\[
= \frac{1}{2} (\partial_1)^2 + \frac{1}{2} \sum_{j=2}^{N} (\partial_j - \partial_{j-1})^2 + \frac{1}{2} (\partial_N)^2
\]
\[
+ (V''(u_1) + \theta) \partial_1 - \sum_{j=2}^{N} [V'(u_{j-1}) - V'(u_j)] \partial_j + \sum_{j=1}^{N} V''(u_j).
\]

Next, recalling the definition of \( \omega_\theta \), we have
\[
(\partial_j - \partial_{j-1})^2 \omega_\theta = (-V''(u_j) - (\theta + V'(u_j))^2) \omega_\theta, \quad i \in \{1, N\},
\]
\[
(\partial_j - \partial_{j-1})^2 \omega_\theta = (-V''(u_j) - V''(u_{j-1})) \omega_\theta + (V'(u_{j-1}) - V'(u_j))^2 \omega_\theta, \quad j \geq 2,
\]
and
\[
(V'(u_{j-1}) - V'(u_j)) \partial_j \omega_\theta = -(V'(u_j) + \theta)(V'(u_{j-1}) - V'(u_j)) \omega_\theta.
\]

Next,
\[
\frac{1}{2} \sum_{j=2}^{N} (V'(u_j) - V'(u_{j-1}))^2 = -\sum_{j=2}^{N} V'(u_j) (V'(u_{j-1}) - V'(u_j))
\]
\[
+ \frac{(V'(u_1))^2}{2} - \frac{(V'(u_N))^2}{2},
\]  
(5.2)
\[
\frac{1}{2} \sum_{j=2}^{N} (-V''(u_j) - V''(u_{j-1})) = -\frac{V''(u_1)}{2} + \frac{V''(u_N)}{2} - \sum_{j=2}^{N} V''(u_j),
\]  
(5.3)
\[
\sum_{j=2}^{N} (V'(u_j) + \theta)(V'(u_{j-1}) - V'(u_j)) = \sum_{j=2}^{N} V'(u_j) (V'(u_{j-1}) - V'(u_j))
\]
\[
- \theta (V'(u_N) - V'(u_1)).
\]  
(5.4)

Summing (5.2) and (5.3), we have
\[
\left(\frac{\partial_1^2}{2} + \frac{1}{2} \sum_{j=2}^{N} (\partial_j - \partial_{j-1})^2 + \frac{\partial_N^2}{2} + \sum_{j=1}^{N} V''(u_j)\right) \omega_\theta
\]
\[
= -\sum_{j=2}^{N} V'(u_j) (V'(u_{j-1}) - V'(u_j)) \omega_\theta
\]
\[
+ ((\theta + V'(u_1))^2 - \theta V'(u_1) + \theta V'(u_N)) \omega_\theta.
\]  
(5.5)

Summing (5.4) and (5.5), only the term
\[
(\theta + V'(u_1))^2 \omega_\theta = -(V'(u_1) + \theta) \partial_1 \omega_\theta
\]
remains, and this finishes the proof of \( L^* \omega = 0 \).
For (5.1), a proof identical to that of Proposition 5 with \( u := \omega(A^{-1} \cdot) \) shows that
\[
(\tilde{L}^* \tilde{\omega})(Ax) = (L^* \omega)(x) = 0.
\]
\[\square\]

Applying Theorem 2 to \( D = \tilde{L} \) it follows that the unique invariant measure for the diffusion with generator \( \tilde{L} \) has density equal to
\[
| \det(A) |^{-N} \omega(A^{-1} \cdot).
\]
Thus the unique invariant measure for the diffusion with generator \( L \) has density \( \omega \). We will also have use for the following ergodic theorem for diffusions. This follows from [35, XIII.1, Theorems 1 & 2]:

**Theorem 3.** Let \( P^t \) be a Markov semigroup with a unique invariant distribution \( \mu \). Then for any \( f \in L^2(\mathbb{R}^N, \mu) \), the limit
\[
f^* = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau P^T f \, dT
\]
eexists in \( L^2(\mathbb{R}^N, \mu) \) and equals the constant
\[
f^* = \int f(u) \, \mu(du).
\]

**Proof.** The existence of the limit is proved in [35]. Moreover, we have \( P^t f^* = f^* \, \omega_\theta \)-almost everywhere. The fact that \( f^* \) is constant is a consequence of ergodicity: since \( \mu \) is the unique invariant probability measure, the Markov process is ergodic [12, Corollary 5.5], and it follows that the only functions such that \( P^t g = g \) are constants. Since \( P^t f^* = f^* \) is invariant by construction, it must be constant. \[\square\]

Let \( f \in L^2(\mathbb{R}^d, \omega) \). Applying Theorem 3 to \( f \) and the semigroup \( Q^t \), we conclude that
\[
\int \frac{1}{\tau} \int_0^\tau \mathbb{E}[f(u(T)) \mid u(0) = u] \, dT \to \int f(x) \omega_\theta(\, dx) \int \omega_\theta(du) \to 0
\]
as \( \tau \to \infty \).

**5.1. Some expectations in equilibrium.** We denote the expectation with respect to (2.7) by \( \mathbb{E}^\theta \). More precisely, by invariance, if \( u(0) \sim \omega_\theta \), then \((u_j(t, \theta, \theta))_{1 \leq j \leq N}\) has distribution \( \omega_\theta \) for any \( t \geq 0 \), so for any \( f : \mathbb{R}^N \to \mathbb{R} \), we have
\[
\int \mathbb{E}[f(u_1(t), \ldots, u_N(t)) \mid u(0) = u] \omega_\theta(\, du) = \int f(u_1, \ldots, u_N) \omega_\theta(\, du).
\]
We denote the expectation on the right by \( \mathbb{E}^\theta[f] \). Thus, using the notation (2.6), we have
\[
\mathbb{E}^\theta[u_j] = -\psi_0^V(\theta),
\]
\[
\text{Var}^\theta(u_j) = \psi_1^V(\theta).
\]
In particular, we have
\[
\mathbb{E}^\theta[W^\theta_{N,t}] = -N\psi_0^V(\theta) + \theta t. \tag{5.6}
\]
Lemma 2. Suppose that \(|t - N\psi^V_1(\theta)| \leq CN^{\frac{2}{3}}\) and \(|\lambda - \theta| \leq AN^{-\frac{1}{4}}\). Then, we have

\[ \mathbb{E}^\theta[W^\theta_{N,t}] - \mathbb{E}^\lambda[W^\lambda_{N,t}] = O(N^{\frac{1}{4}}). \]

Proof. Applying a second order Taylor expansion to (5.6), we find

\[ \psi_0^V(\lambda) = \psi_0^V(\theta) + \psi_1^V(\theta)(\lambda - \theta) + O(\lambda - \theta)^2, \]

we have

\[
\begin{align*}
\mathbb{E}^\theta[W^\theta_{N,t}] - \mathbb{E}^\lambda[W^\lambda_{N,t}] &= N(\psi_0^V(\theta) - \psi_0^V(\lambda)) + t(\theta - \lambda) \\
&= -N\psi_1^V(\theta)(\lambda - \theta) + t(\lambda - \theta) + O(N^{\frac{1}{4}}) \\
&= O(N^{\frac{1}{4}}).
\end{align*}
\]

\[ \Box \]

6. Initial data and coupling

For our later purposes, it will be useful to consider the system (2.1) run from some initial data \(\vec{x}(0)\) with the parameter \(\eta > 0\) for an initial time \(T\), and then with a different parameter \(\theta\) on \([T, T + t]\).

In other words, we replace \(B_0^{-\theta}(s)\) by \(B_0^{-\eta}(s)\) for \(s \in [0, T]\) and \(B_0^{-\theta}(s)\) for \(s \in [T, T + t]\). That is, we consider the solutions \(u_j(t, \eta, \theta)\) of the system:

\[
\begin{align*}
\begin{cases}
\text{du}_1(s, \eta, \theta) &= -V'(u_1(s, \eta, \theta))ds + dB_0^{-\eta} + dB_1, & 0 \leq s \leq T, \\
\text{du}_1(s, \eta, \theta) &= -V'(u_1(s, \eta, \theta))ds + dB_0^{-\theta} + dB_1, & T \leq s \leq T + t, \\
\text{du}_j(s, \eta, \theta) &= (V'(u_{j-1}(s, \eta, \theta)) - V'(u_j(s, \eta, \theta)))ds + dB_j - dB_{j-1}, & j \geq 2. 
\end{cases}
\end{align*}
\] (6.1)

For a function \(f(t), t \geq 0\), we define

\[ f^T(t) = f(t + T). \]

Consider the system \((u_j(t, \eta, \theta))_{1 \leq j \leq N}, \ t \geq 0, \) with initial data \(\vec{x}(0)\) and let

\[ u_j^T(t, \eta, \theta) = u_j(t + T, \eta, \theta), \quad t \geq 0. \]
We frequently omit the dependence on $\eta$ and $\theta$ from the notation, and denote $u^T_j(t) = u^T_j(t, \eta, \theta)$. With this notation, note that (6.1) is equivalent to
\[
\begin{align*}
\frac{d}{dt} u^T_1(t) &= -V'(u^T_1) \, dt + dB^\theta_0 + dB^T_1, \quad T \leq s \leq T + t, \\
\frac{d}{dt} u^T_j(t) &= (V'(u^T_{j-1}) - V'(u^T_j)) \, dt + dB^T_j - dB^T_{j-1}, \quad j \geq 2
\end{align*}
\] (6.2)
with the initial data
\[ u^T(0) = (u_1(T, \eta, \theta), \ldots, u_N(T, \eta, \theta)). \]
We note here that $u^T(0)$ does not in fact depend on $\theta$.

Now define
\[ R^T_{N,t}(\eta, \theta) := \sum_{j=1}^N u^T_j(t, \eta, \theta). \]
Correspondingly, we define
\[ W^T_{N,t}(\eta, \theta) := R^T_{N,t}(\eta, \theta) - B^T_0(t) + \theta t. \] (6.3)
Now note that if $\eta = \theta$ the height function
\[ W^T_{N,t}(\theta, \theta) = \sum_{j=1}^N u^T_j(t) - (B_0(T + t) - B_0(T)) + t\theta, \quad t \geq 0 \] (6.4)
has initial data given by
\[ W^T_{N,t=0}(\theta, \theta) = \sum_{j=1}^N u^T_j(0). \]

Thus, if $u_0 \sim \omega_\theta$, the process $W^T_{N,t}(\theta, \theta)$ same distribution as $W^\theta_{N,t}$, the height function (2.5) with initial data given by $\omega_\theta$:
\[ \mathbb{E}[W^T_{N,t}(\theta, \theta) \mid u_0 \sim \omega_\theta] = \mathbb{E}[W^\theta_{N,t}]. \] (6.5)
We will eventually take the limit $T \to \infty$ and use ergodicity.

6.1. **Differentiating with respect to parameters.** We now consider the partial derivatives of $W^T_t(\eta, \theta)$ with respect to $\eta$, $\theta$. These can be expressed as directional (or Malliavin) derivatives obtained by perturbing the Brownian motion $B_0$ in the directions $\int_0^1 1_{[0,T]}$ and $\int_0^1 1_{[T,T+t]}$. Indeed, assuming $W^T_{N,t}(\eta, \theta)$ is differentiable, we have for $h \in \mathbb{R}$,
\[ W^T_{N,t}(\eta + h, \theta) = W^T_{N,t}(\eta, \theta) + B_0 - h \int_0^T 1_{[0,T]}, \]
where to define the quantity on the right, we replace the Brownian motion in (6.1) by $B_0 - \int_0^1 h 1_{[0,T]}$. With the same notation, we also have
\[ W^T_{N,t}(\eta, \theta + h) = W^T_{N,t}(\eta, \theta) - B_0 - h \int_0^T 1_{[T,T+t]}. \]
Differentiating in \( h \), we obtain the formulas
\[
\partial_\theta W_{N,t}^T(\eta, \theta) = -DW_{N,t}^T(\int_0^1 \mathbb{1}_{[T,T+t]}),
\]
\[
\partial_\eta W_{N,t}^T(\eta, \theta) = -DW_{N,t}^T(\int_0^1 \mathbb{1}_{[0,T]}),
\]
where \( DF(v) \) denotes a directional derivative in the direction \( v \). Taking expectations, the distributions of the perturbed \( W_{N,t}^T \) are also expressible using the Cameron-Martin-Girsanov formula. For example:
\[
\mathbb{E}[W_{N,t}^T(\eta, \theta, B_0) \leftarrow B_0 - h \int_0^1 \mathbb{1}_{[T,T+t]}] = \mathbb{E}[W_{N,t}^T(\eta, \theta, B_0)e^{-h(B_0(T+t)-B_0(T)-\frac{h^2}{2}t)}].
\]
Assuming sufficient integrability to justify differentiation under the integral, we obtain the following integration by parts formula:
\[
\mathbb{E}[(B_0(T+t) - B_0(T))W_{N,t}^T(\eta, \theta)] = -\mathbb{E}[\partial_\theta W_{N,t}^T(\eta, \theta)].
\] (6.6)

The interested reader will find more information on the terminology “Malliavin derivative” in [17, Section 1.2], but note that we will not be using any refined properties of these objects, other than the ones that can be obtained directly from the equations.

6.2. **Existence of derivatives.** In this section we show that the partial derivatives of \( W_{N,t}^T(\eta, \theta) \) with respect to \( \eta \) and \( \theta \) exist and that moreover they have signs. The standard argument proceeding by considering the linear system satisfied by the formal derivatives is much simplified by the triangular nature of the system, and the convexity of \( V'' \), which implies monotonicity.

**Proposition 7.** The first partial derivatives of \( W_{N,t}^T(\eta, \theta) \),
\[
\partial_\theta W_{N,t}^T = \frac{\partial}{\partial \theta} W_{N,t}^T, \quad \partial_\eta W_{N,t}^T = \frac{\partial}{\partial \eta} W_{N,t}^T,
\]
exist and are continuous.

**Lemma 3.** There exists a version of \( u_j^T \) such that the partial derivatives
\[
h_j(t) := h_j(t, \eta, \theta) = \frac{\partial}{\partial \theta} u_j(t, \eta, \theta), \quad j = 1, \ldots, N
\] (6.7)
exist. Here, \((h_j^T(s))_{1 \leq j \leq n}\) solves the linear system of ODEs
\[
\begin{cases}
\frac{dh_1^T(s)}{ds} = (-V''(u_1^T(s))h_1^T(s) - 1) ds \\
\frac{dh_j^T(s)}{ds} = -V''(u_j^T(s))h_j^T(s) ds + V''(u_{j-1}^T(s))h_{j-1}^T(s) ds, \quad j = 2, \ldots, n \\
h_j^T(0) = 0, \quad j = 1, \ldots, N
\end{cases}
\] (6.8)
for \( s \in [0, t] \).

**Proof.** The equations (6.8) follow by formally differentiating (2.1) with respect to \( \theta \) once one shows that \( u_\theta(t) \) is differentiable. Let \(|h| < 1\) and consider first the quantity
\[
\Delta_{1,\theta,h}(t) = \frac{1}{h}(u_1^T(t, \eta, \theta + h) - u_1^T(t, \eta, \theta)),
\] (6.9)
where the initial data $u_T^T(0, \cdot)$ is held fixed. From the first equation in (6.8), we obtain
\[
\begin{aligned}
d\Delta_{1,\theta,h} & = (-F_1 \Delta_{1,\theta,h} - 1)dt, \\
\Delta_{1,\theta,h}(0) & = 0,
\end{aligned}
\] (6.10)
where
\[
F_1(t) = \int_0^1 V''(\tau u_T^T(t, \eta, \theta + h) + (1 - \tau)u_T^T(t, \eta, \theta)) d\tau.
\]

The linear equation (6.9) can be solved as
\[
\Delta_{1,\theta,h}(t) = -\int_0^t e^{-\int_s^t F_1(\tau) d\tau} ds \leq 0.
\] (6.11)

In particular, since $V'' \geq 0$, we have
\[
\sup_{s \leq t} |\Delta_{1,\theta,h}(s)| = \sup_{s \leq t} (-\Delta_{1,\theta,h}(s)) \leq t
\] (6.12)
for any $h > 0$. Moreover, from the equation (6.10), we have
\[
\sup_{t' \leq t} \left(-\int_0^{t'} F_1 \Delta_{1,h} ds\right) \leq t + \Delta_{1,\theta,h}(t)
\] (6.13)
\[
\leq t.
\]

In particular, we first see $\theta \mapsto u_T^T(t, \eta, \theta)$ is Lipschitz, and then, taking $h \to 0$ in (6.11), we see that $\partial_{\theta}u_T^T$ exists and is bounded by $t$ at time $t$. Moreover,
\[
\partial_{\theta}u_T^T(t, \eta, \theta) = -\int_0^t e^{-\int_0^\tau F_1(v) dv} \left(\int_0^\tau V''(u_T^T(v, \eta, \theta)) dv\right) ds,
\] (6.14)
so $\theta \mapsto \partial_{\theta}u_T^T(\theta, \cdot)$ is continuous, with the bound
\[
\sup_{s \leq t} |\partial_{\theta}u_T^T(s, \eta, \theta)| \leq t.
\] (6.15)

We define
\[
\Delta_{j-1,\theta,h}(t) := \frac{1}{h}(u_{j-1}^T(t, \eta, \theta + h) - u_{j-1}^T(t, \eta, \theta)).
\]
Then, we have
\[
\begin{aligned}
d\Delta_{j,\theta,h} & = (-F_j \Delta_{j,\theta,h} + F_{j-1} \Delta_{j-1,\theta,h})dt, \\
\Delta_{j,\theta,h}(0) & = 0,
\end{aligned}
\] (6.16)
where
\[
F_j(t) = \int_0^1 V''(\tau u_j(t, \theta + h) + (1 - \tau)u_j(t, \theta)) d\tau \geq 0.
\]

The linear equation (6.16) can be solved explicitly, giving:
\[
\Delta_{j,h}(t) = \int_0^t e^{-\int_0^\tau F_j(v) dv} \left(\int_0^\tau F_{j-1}(\nu) dv\right) ds.
\] (6.17)

Here and below we drop the $\theta$ dependence from $\Delta_{j,h}$ for simplicity of notation. From the last equation we deduce the implication
\[
\Delta_{j-1,h} \leq 0 \Rightarrow \Delta_{j,h} \leq 0.
\] (6.18)
Thus \( F_{j-1} \Delta_{j-1} \leq 0 \) for all \( j \geq 2 \), and consequently:

\[
\sup_{s \leq t} |\Delta_{j,h}(s)| \leq \sup_{t' \leq t} \left( - \int_0^{t'} F_{j-1} \Delta_{j-1,h} \, ds \right) = - \int_0^{t'} F_{j-1} \Delta_{j-1,h} \, ds.
\]

On the other hand, from (6.16), we have for \( j \geq 2 \):

\[
- \int_0^{t'} F_j \Delta_{j,h} \, ds = \Delta_{j,h}(t) - \int_0^{t'} F_{j-1} \Delta_{j-1,h} \, ds \leq - \int_0^{t'} F_{j-1} \Delta_{j-1,h} \, ds,
\]

and so by induction from (6.13)

\[
\sup_{s \leq t} |\Delta_{j,h}(s)| \leq t, \quad j = 1, \ldots, N. \tag{6.19}
\]

The family \( \theta \mapsto u_j(\cdot, \eta, \theta) \) is continuous in the space \( C([0, t], \mathbb{R}) \) with the uniform norm. In particular, almost surely \( u_j^T(s, \eta, \theta + h) \to u_j^T(s, \eta, \theta) \) uniformly for \( s \) in compact sets as \( h \to 0 \). Assuming the almost sure existence and continuity in \( \theta \) of \( \partial_\theta u_j^T \), we can now take the limit on the right-hand side in (6.17) to find that \( \partial_\theta u_j^T \) exists and is given by

\[
\partial_\theta u_j^T(t, \eta, \theta) = \int_0^t e^{-\int_s^t V''(u_j^T(\tau, \eta, \theta)) \, d\tau} V''(u_{j-1}^T(s, \eta, \theta)) \partial_\theta u_{j-1}^T(s, \eta, \theta) \, ds, \tag{6.20}
\]

which shows that \( \theta \mapsto \partial_\theta u_j^T(\cdot, \theta) \) is continuous, with the bound

\[
\sup_{s \leq t} |\partial_\theta u_j^T(s, \eta, \theta)| \leq t. \tag{6.21}
\]

Taking the limit in the equations (6.14), (6.20), and taking the \( h \to 0 \) limit, we obtain (6.8). \( \square \)

**Lemma 4.** There is a version of \( (u_j^T)_{1 \leq j \leq N} \) such that the partial derivatives

\[
k_j(t) := k_j(t, \eta, \theta) = \frac{\partial}{\partial \eta} u_j(t, \eta, \theta), \quad j = 1, \ldots, N \tag{6.22}
\]

exist. Moreover, \( (k_j(s))_{1 \leq j \leq n} \) solves the system of ODEs

\[
\begin{cases}
dk_1(s) = (-V''(u_1(s)) k_1(s) - 1_{[0,T]}(s)) \, ds \\
dk_j(s) = -V''(u_1(s)) k_j(s) \, ds + V''(u_{j-1}(s)) k_{j-1}(s) \, ds, \quad j = 2, \ldots, N \\
k_j(0) = 0, \quad j = 1, \ldots, N,
\end{cases} \tag{6.23}
\]

for \( s \in [0, T + t] \).

**Proof.** The proof proceeds as for Lemma 3, considering the quantities

\[
\tilde{\Delta}_{j,h}(t) = \frac{1}{h} (u_j(t, \eta + h, \theta) - u_j(t, \eta, \theta)),
\]
which satisfy the linear equations

\[
\begin{cases}
    d\tilde{\Delta}_{1,h} = (-\tilde{F}_1 \tilde{\Delta}_{1,h} - 1_{[0,T]}(s)) dt, \\
    d\tilde{\Delta}_{j,h} = (-\tilde{F}_j \tilde{\Delta}_{j,h} + \tilde{F}_{j-1} \tilde{\Delta}_{j-1}) dt, \\
    \tilde{\Delta}_{j,h}(0) = 0,
\end{cases}
\]  

(6.24)

where

\[
\tilde{F}_j(s) = \int_0^1 V''(\tau u_j(s, \eta + h, \theta) + (1 - \tau) u_j(s, \eta, \theta)) d\tau \geq 0.
\]

The triangular system (6.24) has the explicit solution

\[
\tilde{\Delta}_{1,h}(s) = \int_0^{s \wedge T} e^{-\int_0^s \tilde{F}_1(\tau) d\tau} d\tau \leq 0,
\]

\[
\tilde{\Delta}_{j,h}(s) = \int_0^s e^{-\int_0^s \tilde{F}_j(\tau) d\tau} \tilde{F}_{j-1} \tilde{\Delta}_{j-1}(v) dv \leq 0.
\]

Here the final inequality on the second line follows by induction starting from the final inequality on the first line. Using these solutions, we show by induction as in the proof of Lemma 3 that \( \eta \mapsto u_j(t, \eta, \theta) \) is continuous, and then that \( \partial_{\eta} u_j(s, \eta, \theta) \) exists and is continuous. Moreover \( k_j := \partial_{\eta} u_j(s, \eta, \theta) \) have the expressions:

\[
k_1(s) = -\int_0^{s \wedge T} e^{-\int_0^s V''(u_j(\tau)) d\tau} dv,
\]

(6.25)

\[
k_j(s) = \int_0^s e^{-\int_0^s V''(u_j(\tau)) d\tau} V''(u_{j-1}(v)) k_{j-1}(v) dv,
\]

which shows that \( \partial_{\eta} u_j(s, \eta, \theta) \) is continuous in \( \theta \) and \( \eta \). Moreover, we have the bounds

\[
\sup_{s \leq t} |\partial_{\eta} u_j(s, \eta, \theta)| \leq t.
\]

(6.26)

\[\Box\]

6.3. Monotonicity.

**Lemma 5.** For \( t > 0 \), we have

\[
\partial_{\theta} W_{N,T}^T(\eta, \theta) = \sum_{j=1}^N h_j^T(t) + t,
\]

(6.27)

\[
\partial_{\eta} W_{N,T}^T(\eta, \theta) = \sum_{j=1}^N k_j^T(t),
\]

where \( h_j^T \) is defined in (6.17) and \( k_j^T \) is defined in (6.23). If \( V'' > 0 \), then

\[
h_j^T(t) \leq 0, \quad t > 0,
\]

(6.28)

and

\[
k_j(s) \leq 0, \quad s > 0.
\]

(6.29)
We also have the inequalities:
\[
\partial_\theta W_{N,t}(\eta, \theta) \geq 0. \quad (6.30)
\]
\[
\partial_\eta W_{N,t}(\eta, \theta) \leq 0. \quad (6.31)
\]

**Proof.** The expressions for \(\partial_\eta W_{N,t}\) and \(\partial_\theta W_{N,t}\) follow directly by differentiating the defining expression (6.4). The inequality (6.28) follows from (6.11) and (6.18). The inequalities (6.29) for the \(k_j(s)\) follow similarly. Summing the equations in (6.8), we obtain
\[
d\partial_\theta W_{N,t} = -V''(u_T^N(t, \theta, \theta))h_N^T(t)dt,
\]
so we have
\[
W_{N,t}(\eta, \theta) \geq W_{N,0}(\eta, \theta) = 0.
\]
For (6.30), simply use (6.29). \(\square\)

### 6.4. Second derivatives

The following result asserts that the second derivatives of \(W_{N,t}\) have a sign. In particular, \(W_{N,t}\) is convex in both variables \(\eta\) and \(\theta\).

**Lemma 6.** Let \(V\) be a potential of O’Connell-Yor type or with globally Lipschitz derivative. Assume that,
\[
V'' > 0, \quad \text{and} \quad V^{(3)} < 0. \quad (6.33)
\]

Then
\[
\partial_\theta^2 W_{N,t}(\eta, \theta) \geq 0, \quad (6.34)
\]
\[
\partial_\eta^2 W_{N,t}(\eta, \theta) \geq 0, \quad (6.35)
\]
\[
\partial_\theta \partial_\eta W_{N,t}(\eta, \theta) \geq 0. \quad (6.36)
\]

In particular, for each \(j = 1, \ldots, N\) and \(s \in [0, t]\), the functions
\[
\eta \mapsto W_{j,s}^{T}(\eta, \theta), \quad \theta \mapsto W_{j,s}^{T}(\eta, \theta), \quad \lambda \mapsto W_{j,s}^{T}(\lambda, \lambda)
\]
are convex.

**Proof.** Define
\[
\Delta'_{1, \theta, h}(t) = \frac{1}{h}(h_1^T(t, \theta + h) - h_1^T(t, \theta)).
\]
Then we have
\[
d\Delta'_{1, \theta, h} = (-V''(u_1^T(\theta, t)))\Delta'_{1, \theta, h} - G_1 \cdot \Delta_{1, \theta, h}dt,
\]
where
\[
G_1 = \partial_\theta u_1^T(\theta + h, t) \cdot \int_0^1 V'''(su_1^T(\theta, t) + (1 - s)u_1^T(\theta + h, t))ds \geq 0.
\]

This linear equation is easily solved, leading to
\[
\Delta'_{1, \theta, h}(t') = -\int_0^{t'} e^{-\int_0^s V''(u_1^T(\theta, \tau))d\tau}G_1(s)\Delta_{1, \theta, h}(s)ds \geq 0. \quad (6.38)
\]
Together with (6.15), we have, almost surely for some \( \theta' \) in \([\theta, \theta + h]\) or \([\theta + h, \theta]\):
\[
\sup_{t' \leq t} |su_1^T(t', \theta) + (1 - s)u_1^T(t', \theta + h)| \leq 2 \sup_{t' \leq t} |u_1^T(t', \theta')| + h \sup_{t' \leq t} |\partial_\theta u_1^T(t', \theta')| \leq 2 \sup_{t' \leq t} |u_1^T(t', \theta)| + t < \infty.
\]

Since \( V''(u_1^T(t, \eta, \theta)) \) is continuous, this implies the existence of \( C(\omega, t) \) independent of \( h \) such that, almost surely \( C \) is bounded for \( t \) on compact intervals and:
\[
\sup_{t' \leq t} |V''(su_1^T(\theta, t) + (1 - s)u_1^T(\theta + h, t))| \leq C(\omega, t).
\]

Thus from (6.12), (6.15), we first obtain that almost surely
\[
\frac{d}{dt} \frac{\partial_\theta u_1(t)}{\partial_\theta u_1(\theta, t)} = -V''(u_1^T(\theta, t))m_1 - V'''(u_1^T(t))(\partial_\theta u_1^T(\theta, t))^2,
\]

with solution
\[
m_1(t, \eta, \theta) = -\int_0^t e^{-\int_0^s V''(u_1^T(\tau, \eta, \theta)) d\tau} V'''(u_1^T(s))(\partial_\theta u_1^T)^2 \, ds.
\]

For later purposes, we will prove the following uniform estimate in \(|h| \leq 1\) estimate on the difference quotients:
\[
\mathbb{E}[\sup_{t' \leq t} |\Delta_{j, \theta, h}(t')|^p] \leq C(p, N, \theta)e^{C_{\psi}(\psi(u_0)^q + |u_0|^q)}, \quad p \geq 1. \tag{6.39}
\]

The constants and the exponent \( q = q(N, p) \) are bounded for \( p, N, \theta \) in bounded regions. We first prove this for \( j = 1 \). Taking absolute values in (6.37) and using \( G_1 \geq 0 \), \( \Delta_{1, \theta, h} \leq 0 \):
\[
\sup_{t' \leq t} \Delta_{1, \theta, h}(t') \leq -\int_0^t G_1 \Delta_{1, \theta, h} \, ds.
\]

Then, using also (6.12), we find
\[
\mathbb{E}[\sup_{t' \leq t} |\Delta_{1, \theta, h}(t')|^p] \leq \mathbb{E}\left[ \int_0^t G_1 \Delta_{1, \theta, h} \, ds \right]^p \leq \int_0^t s^{2p} \mathbb{E}[|G_1|^p] \, ds \leq C \int_0^t s^{2p} \int_0^1 \mathbb{E}[|V''(\tau u_1^T(\theta, s) + (1 - \tau)u_1^T(\theta + h, s))|^p] \, d\tau \, ds \tag{6.40}
\]

\[
\leq C \int_0^t s^{2p} \int_0^1 \mathbb{E}[|V(u_1^T(\theta, s) + (1 - \tau)u_1^T(\theta + h, s))|^p] \, d\tau \, ds 
\leq C \int_0^t s^{2p}(\mathbb{E}[|V(u_1^T(\theta, s))|^p] + \mathbb{E}[|V(u_1^T(\theta + h, s))|^p]) \, ds 
\leq e^{C_\psi}(\varphi_q(u(0)) + \psi(u(0))^q).
\]
Here in the second step we have used the bound $|V''| \leq CV$ from (2.2) and in the third step we have used convexity. The final inequality is from (4.21). Using (6.37), we have

$$\left| \int_0^t V''(u_1^T(s)) \Delta_{1,h,\theta}^t \, ds \right| \leq |\Delta_{1,h,\theta}(t)| + \left| \int_0^t G_1 \Delta_{1,h,\theta} \, ds \right|$$

and so also

$$\mathbb{E} \left| \int_0^t V''(u_1^T(s)) \Delta_{1,h,\theta}^t \, ds \right|^p \leq C_{p,t} (\varphi_q(u(0)) + \psi(u(0))^\eta).$$

Similarly, defining

$$\Delta_{j,h,\theta}^t = \frac{1}{h} (h_j^T(\theta + h) - h_j^T(\theta)),$$

we see that $\Delta_{j,h,\theta}^t$ satisfies the linear equation

$$\begin{cases}
    d\Delta_{j,h,\theta}^t = -V''(u_j^T(\theta)) \Delta_{j,h,\theta}^t + V''(u_{j-1}^T(\theta)) \Delta_{j-1,h,\theta}^t - G_j \Delta_{j,h,\theta}^t + G_{j-1} \Delta_{j-1,h,\theta}^t, \\
    \Delta_{j,h,\theta}^t(0) = 0.
\end{cases} \quad (6.41)$$

Here we have used the notation

$$G_j = \partial_\theta u_j^T(\theta + h, t) \cdot \int_0^1 V''(su_j^T(\theta, t) + (1-s)u_j^T(\theta + h, t)) \, ds.$$ 

This equation is also easily solved explicitly:

$$\Delta_{j,h,\theta}^t(t) = -\int_0^t e^{-\int_0^s V''(u_j(\tau)) \, d\tau} V''(u_{j-1}^T(s)) \Delta_{j-1,h,\theta}^t \, ds + \int_0^t e^{-\int_0^s V''(u_j(\tau)) \, d\tau} (G_j \Delta_{j,h,\theta}^t + G_{j-1} \Delta_{j-1,h,\theta}^t) \, ds. \quad (6.42)$$

Using this we now complete a short induction argument that $\partial_\theta u_j(\eta, \theta, t)$ is Lipschitz in $\theta$ and the derivative exists. Assume the existence of $C_{j-1}(\omega, t)$ independent of $h$ such that $C_{j-1} < \infty$ almost surely and

$$\sup_{t' \leq t} |\Delta_{j-1,h}^t(t')| \leq C_{j-1}(\omega, t) < \infty.$$ 

By (6.21), we have

$$\sup_{t' \leq t} |su_j^T(\theta, t') + (1-s)u_h^T(\theta + h, t')| \leq 2 \sup_{t' \leq t} |u_j^T(\theta, t')| + h \sup_{t' \leq t} |\partial_\theta u_j^T(\theta', t')|$$

$$\leq 2 \sup_{t' \leq t} |u_j^T(\theta, t')| + tj < \infty.$$ 

Thus by (6.16), we have

$$\sup_{|h| \leq 1} (|V''(u_{j-1}^T(t))| + |G_{j-1} \Delta_{j-1,h,\theta}| + |G_j \Delta_{j,h,\theta}|) < \infty,$$

and then

$$\sup_{t' \leq t} |\Delta_{j,h,\theta}^t(t')| \leq C_j(\omega, t),$$

So almost surely, $\theta \mapsto \partial_j u_j(t, \eta, \theta)$ is locally Lipschitz.
Taking the limit in (6.42), we find that \( m_j^T(\theta, t) := \partial_\theta^2 u_j \) exists and is continuous in \( \theta \), and moreover solves the equation

\[
dm_j = -V''(u_j^T(\theta))m_j^T + V''(u_{j-1}^T(\theta))m_{j-1}^T - V''(u_j^T(\theta))\partial_\theta u_j^T(\theta) + V''(u_{j-1}(\theta))\partial_\theta u_{j-1}(\theta).
\]

This completes the induction argument for existence of the second derivatives.

We now begin a second induction argument to obtain the estimates (6.39) in \( L^p \) for the \( \Delta_{j,h} \). From (6.42) we see that

\[
\sup_{t' \leq t} |\Delta'_{j,h,\theta}(t')| \leq \sup_{t' \leq t} |\Delta'_{j-1,h,\theta}| \int_0^t V''(u_{j-1}(s)) \, ds
\]

\[
+ \int_0^t G_j \Delta_{j,h,\theta} \, ds + \int_0^t G_{j-1} \Delta_{j-1,h,\theta} \, ds
\]

(6.43)

Similar to (6.40) we have for any \( p \) that

\[
\mathbb{E}\left[ \int_0^t G_j \Delta_{j,h,\theta} \, ds \right]^p \leq C e^{C p t} (|u_0|^q + (\psi(u_0))^q),
\]

for some \( q \), and similarly for the \( j - 1 \) term on the right of the inequality (6.43). For the first term on the right of (6.43) we have by Schwarz, the estimates (4.21), the assumption \( V'' \leq CV \), and the induction assumption,

\[
\mathbb{E} \left[ \sup_{t' \leq t} |\Delta'_{j-1,h,\theta}| \int_0^t V''(u_{j-1}(s)) \, ds \right]^p
\]

\[
\leq C t^{\frac{p}{2}} \mathbb{E} \left[ \sup_{t' \leq t} |\Delta'_{j-1,h,\theta}|^{2p} \right]^{1/2} \int_0^t \mathbb{E} |V(u_{j-1}(s))|^{2p} \, ds
\]

\[
\leq C e^{C p t} (|u_0|^q + (\psi(u_0))^q)
\]

(6.44)

for some \( q \). This completes the induction argument for the estimates (6.39).

Repeating the same calculations as above and using Lemma 4 show that the derivatives \( \partial_\eta^3 u_j \) are continuous in \( \eta \), and solve the equations:

\[
d\partial_\eta^2 u_1 = (-V''(u_1)\partial_\eta^2 u_1 - V'''(u_1)(\partial_\eta u_1)^2)dt,
\]

\[
d\partial_\eta^2 u_j = (-V''(u_j)\partial_\eta^2 u_j - V'''(u_j)(\partial_\eta u_j)^2)dt
\]

\[
+ (V''(u_{j-1})\partial_\eta^2 u_{j-1} + V'''(u_{j-1})(\partial_\eta u_{j-1})^2)dt, \quad j = 2, \ldots, N,
\]

Similarly, the mixed derivatives \( \partial_\eta^3 u_j \) exist for \( j = 1, \ldots, N \) and are continuous in \( \eta \) and \( \theta \), and satisfy the equations:

\[
d\partial_\eta^2 u_1 = (-V''(u_1)\partial_\eta^2 u_1 - V'''(u_1)\partial_\eta u_1 \partial_\theta u_1)dt,
\]

\[
d\partial_\eta^2 u_j = (-V''(u_j)\partial_\eta^2 u_j - V'''(u_j)(\partial_\eta u_j)\partial_\theta u_j)dt
\]

\[
+ (V''(u_{j-1})\partial_\eta^2 u_{j-1} + V'''(u_{j-1})(\partial_\eta u_{j-1})\partial_\theta u_{j-1})dt, \quad j = 2, \ldots, N.
\]
We also record uniform integrability estimates for the difference quotients
\[
\Delta_{j,h}^t(t) := \frac{1}{h}(\partial_{\eta} u(t, \eta + h) - \partial_{\eta} u(t, \eta)),
\]
\[
\Delta_{j,\eta,h}^t(t) := \frac{1}{h}(\partial_{\eta} u(t, \eta + h) - \partial_{\eta} u(t, \eta)).
\]

There are constants such that for \( j = 1, \ldots, N \) and \( p \geq 1 \)
\[
\mathbb{E}[\sup_{t' \leq t} |\Delta_{j,h}^t(t')|^p] + \mathbb{E}[\sup_{t' \leq t} |\Delta_{j,\eta,h}^t(t')|^p] \leq C(p, N, \theta)e^{C_{\rho,t}(|u_0|^p + (\psi(u_0))^p)}.
\]

The proof is identical to the corresponding estimates (6.39) for the difference quotients of \( \partial_{\theta} u_j \) in \( \theta \).

It follows that \( \partial_{\theta} W_{t,N}^T \) is differentiable in \( \theta \) for \( t \geq 0 \). Differentiating the equation (6.32), we obtain the equation
\[
d\partial_{\theta}^2 W_{s,N}^T = -V'(u_N^T(s))(h_N^T(s))^2 \, ds - V''(u_N^T(s))\partial_{\theta} h_N^T(s) \, ds
\geq -V''(u_N^T(s))\partial_{\theta} h_N^T(s) \, ds
= -V''(u_N^T(s))\partial_{\theta}^2 W_{N,s}^T \, ds + V''(u_N^T(s))\partial_{\theta}^2 W_{N-1,s}^T \, ds
\]
for \( N \geq 2 \), while
\[
d\partial_{\theta}^2 W_{1,s}^T = -V'(u_1^T(s))(h_1^T(s))^2 \, ds - V''(u_1^T(s))\partial_{\theta} h_1^T(s) \, ds
\geq -V''(u_1^T(s))\partial_{\theta}^2 W_{1,s}^T \, ds.
\]

Since \( \partial_{\theta}^2 W_{j,0}^T = 0 \), we obtain
\[
\partial_{\theta}^2 W_{1,t}^T \geq 0
\]
and thus
\[
\partial_{\theta}^2 W_{j,t}^T \geq 0
\]
for \( j = 2, \ldots, N \). The same proof applies to \( \partial_{\theta}^2 W_{N,t}^T \) and shows that this quantity is also non-negative.

For \( \partial_{\theta,\eta} W_{t}^T(\eta, \theta) \), since the product \( k_N h_N \) is non-negative by (6.28), (6.29), we have
\[
d\partial_{\theta,\eta}^2 W_{s,N}^T = -V'(u_N^T(s))h_N^T(s)h_N^T(s) \, ds - V''(u_N^T(s))\partial_{\theta} h_N^T(s) \, ds
\geq -V''(u_N^T(s))\partial_{\theta} h_N^T(s) \, ds
= -V''(u_N^T(s))\partial_{\theta,\eta}^2 W_{N,s}^T \, ds + V''(u_N^T(s))\partial_{\theta,\eta}^2 W_{N-1,s}^T \, ds,
\]
from which the stated bound follows when combined with the initial condition \( \partial_{\theta,\eta}^2 W_{j,0}^T = 0 \).

6.5. Estimate for the product of derivatives. In the proof of our main result, we compare \( \partial_{\theta} W_{N,t}^T(\eta, \theta) \) to the quantity
\[
\frac{d}{d\theta} W_{N,t}^T(\theta, \theta) = (\partial_{\theta} W_{N,t}^T(\eta, \theta) + \partial_{\eta} W_{N,t}^T(\eta, \theta))|_{\theta=\eta}.
\]
Taking the square of the last equation, we obtain

\[
\left( \frac{d}{d\theta} W_{N,t}^T(\theta, \theta) \right)^2 = \left( \partial_{\theta} W_{N,t}^T(\theta, \eta) \right)_{\theta=\eta}^2 + \left( \partial_{\eta} W_{N,t}^T(\theta, \eta) \right)_{\theta=\eta}^2 + 2 \left[ \partial_{\theta} W_{N,t}^T(\eta, \theta) \cdot \partial_{\eta} W_{N,t}^T(\theta, \eta) \right]_{\theta=\eta}.
\]

(6.46)

The cross-term is negative, but can be controlled in case the potential \( V \) behaves like \( e^{-cx} \) for \( c > 0 \) as captured by the condition (6.48) in Lemma 7. Taking the second derivative of \( W_{N,t}^T(\theta, \theta) \), we obtain

\[
\frac{d^2}{d\theta^2} W_{N,t}^T(\theta, \theta) = \frac{\partial^2}{\partial\theta^2} W_{N,t}^T(\eta, \theta) |_{\eta=\theta} + \frac{\partial^2}{\partial\eta^2} W_{N,t}^T(\eta, \theta) |_{\eta=\theta} + 2 \cdot \frac{\partial^2}{\partial\theta \partial\eta} W_{N,t}^T(\eta, \theta) |_{\eta=\theta}.
\]

The mixed derivative term turns out to be equal to final term in (6.46) in the O’Connell-Yor case \( V(x) = e^{-x} \). In the general case, we have the following:

**Lemma 7.** Let \( V \) be a potential of O’Connell-Yor type with \( V'' \geq 0 \), \( V''' \leq 0 \). Suppose moreover that there is a constant \( c_0 \) such that \( x \mapsto e^{c_0 x} V''(x) \) is non-increasing. Then, for any \( N \geq 1 \), we have

\[
c_0 \left[ \partial_{\theta} W_{N,t}^T(\eta, \theta) \cdot \partial_{\eta} W_{N,t}^T(\eta, \theta) \right]_{\theta=\eta} + \frac{\partial^2}{\partial\theta \partial\eta} W_{N,t}^T(\eta, \theta) |_{\theta=\eta} \geq 0.
\]

(6.47)

Note that the assumption that \( x \mapsto e^{c_0 x} V''(x) \) is non-increasing is equivalent to the following inequality between the second and third derivatives of \( V \)

\[
c_0 V''(x) + V'''(x) \leq 0.
\]

(6.48)

**Proof.** Define the quantity,

\[
A_n(t) = c_0 \left[ \partial_{\theta} W_{n,t}^T(\eta, \theta) \cdot \partial_{\eta} W_{n,t}^T(\eta, \theta) \right]_{\theta=\eta} + \frac{\partial^2}{\partial\theta \partial\eta} W_{n,t}^T(\eta, \theta) |_{\theta=\eta}
\]

We will prove by induction that \( A_n(t) \geq 0 \) for every \( n \) by calculating the differential equation that the \( A_n \)'s satisfy. In what follows, for brevity, we suppress the argument \( (\eta, \theta) \) of \( W_{n,t}^T(\eta, \theta) \) with the understanding that after the derivatives are calculated, they are evaluated on the diagonal \( \eta = \theta \).

We first calculate the derivative of the left-hand side of (6.47). Since

\[
\frac{d}{d\theta} W_{n,t}^T = -V''(u_{n,t}(t)) h_{n,t}(t) \quad \text{and} \quad \frac{d}{d\eta} W_{n,t}^T = -V''(u_{n,t}(t)) k_{n,t}(t),
\]

(6.49)

using the product rule, we get

\[
d \left( \frac{\partial}{\partial\theta} W_{n,t}^T \cdot \frac{\partial}{\partial\eta} W_{n,t}^T \right) = -V''(u_{n,t}(t)) \left[ k_{n,t}(t) \frac{\partial}{\partial\theta} W_{n,t}^T + h_{n,t}(t) \frac{\partial}{\partial\eta} W_{n,t}^T \right].
\]

(6.50)

Applying the chain and product rule to (6.49) gives
As in the case case

Next, for every

Using the initial condition

We now start the induction. When $n = 1$, (6.51) reads

$$
\frac{d}{dt} A_1(t) = -V''(u_1^T(t)) A_1(t) - [c_0 V''(u_1^T(t)) + V^{(3)}(u_1^T(t))] h_1^T(t) k_1^T(t)
$$

Since $x \mapsto e^{c_0 V''(x)}$ is non-increasing by assumption, and $h_1$, $k_1$ are both negative by Lemmas 3 and 4 the last term of (6.51) is positive, giving

$$
\frac{d}{dt} A_1(t) \geq -V''(u_1^T(t)) A_1(t).
$$

Since $A_1(0) = 0$, we see that

$$
A_1 \geq 0.
$$

Next, for $n > 1$, notice that (6.51) can be rewritten as

$$
\frac{d}{dt} A_n(t) = -V''(u_n^T(t)) [A_n(t) - A_{n-1}(t)] - (c_0 V''(u_n^T(t)) + V^{(3)}(u_n^T(t))) h_n^T(t) k_n^T(t)
$$

$$
\quad = -V''(u_n^T(t)) A_n(t) + V''(u_n^T(t)) A_{n-1}(t) - [c_0 V''(u_n^T(t)) + V^{(3)}(u_n^T(t))] h_n^T(t) k_n^T(t).
$$

As in the case case $n = 1$, since $k_n^T$ and $h_n^T$ are negative by (6.28), (6.29), $V'' \geq 0$ and $x \mapsto e^{c_0 V''(x)}$ is non-increasing, we see that the contribution of the second term of the last line of (6.52) is positive. By the induction assumption as well as the assumption that $V'' \geq 0$, we see that in fact the entire last line of (6.52) is positive so that

$$
\frac{d}{dt} A_n(t) \geq -V''(u_n^T(t)) A_n(t).
$$

Using the initial condition $A_n(0) = 0$, we conclude that

$$
A_n(t) \geq 0,
$$

for every $n$. \qed
The above lemma depends critically on the property $x \mapsto e^{c_0 x} V(x)$ is non-decreasing. If this property is changed to $x \mapsto e^{c_0 x} V''(x)$ is non-increasing, then the reverse inequality in the conclusion, $A_N \leq 0$, holds. Note that this implies that the O’Connell-Yor potential gives $A_N \equiv 0$ with the choice $c_0 = 1$.

6.6. **Application of the ergodic theorem.** Defining the functions $f, g : \mathbb{R}^N \to \mathbb{R}$ by

\[ f(u) = \mathbb{E}[(W_{N,t}^\theta)^2 | u(0) = u], \]
\[ g(u) = \mathbb{E}[W_{N,t}^\theta | u(0) = u]. \]

Recall the definition of the semigroup $Q$ in Section 4.4 and note that,

\[ Q^T f(u) = \mathbb{E}[f(u(T)) | u(0) = u] = \mathbb{E}^\theta[(W_{N,t}^\theta)^2 | u(0) = u], \]
\[ Q^T g(u) = \mathbb{E}[g(u(T)) | u(0) = u] = \mathbb{E}^\theta[W_{N,t}^\theta | u(0) = u]. \]

Moreover, by invariance, we have

\[ \int f(u) \omega_\theta(du) = \mathbb{E}^\theta[(W_{N,t}^\theta)^2], \]
\[ \int g(u) \omega_\theta(du) = \mathbb{E}^\theta[W_{N,t}^\theta]. \]

By (4.18), we have (for some $N$-dependent constants; we only need the following bounds for integrability considerations):

\[ |g(u)| \leq C e^{C_1 t} (|u| + \psi(u)) \in L^2(\mathbb{R}^d, \omega_\theta), \]
\[ |f(u)| \leq C e^{C_0 t} (|u|^2 + \psi(u)) \in L^2(\mathbb{R}^d, \omega_\theta), \]  \hspace{1cm} (6.53)

for $\theta > 0$. The bounds (4.18) are for $V$ of O’Connell-Yor type. The same bounds hold without $\psi$ on the right side for $V'$ Lipschitz. By the mean ergodic theorem, Theorem 3, we have

\[ \frac{1}{\tau} \int_0^\tau Q^T g(u) dT \to \mathbb{E}_{\omega_\theta}[g(u)] = \mathbb{E}^\theta[W_{N,t}^\theta], \]
\[ \frac{1}{\tau} \int_0^\tau Q^T f(u) dT \to \mathbb{E}_{\omega_\theta}[f(u)] = \mathbb{E}^\theta[(W_{N,t}^\theta)^2], \]

the limit being in the $L^2(\mathbb{R}^d, \omega_\theta)$ norm:

\[ \mathbb{E}^\theta\left[\frac{1}{\tau} \int_0^\tau Q^T f(u) dT - \mathbb{E}^\theta[(W_{n,t}^\theta)^2]\right]^2 \to 0 \]

as $\tau \to \infty$. Passing to a subsequence, we may assume that

\[ \frac{1}{\tau_n} \int_0^{\tau_n} Q^T g(u) dT \to \mathbb{E}^\theta[W_{N,t}^\theta], \]
\[ \frac{1}{\tau_n} \int_0^{\tau_n} Q^T f(u) dT \to \mathbb{E}^\theta[(W_{N,t}^\theta)^2] \]  \hspace{1cm} (6.54)

for $\omega_\theta$-almost every $u$. Since $\omega_\theta$ is absolutely continuous with respect to Lebesgue measure, the statement is true for Lebesgue almost-every $u$. Repeating the preceding argument and passing to subsequences each time, we obtain the following:
Proposition 8. Let \( \lambda, \theta > 0 \). Then, for any sequence \( \tau_n \to \infty \), there is a subsequence \( \tau_{n_k} \) such that the following quantities converge for almost every initial vector \( u \in \mathbb{R}^d \):

\[
\frac{1}{\tau_{n_k}} \int_0^{\tau_{n_k}} \mathbb{E}[(W^T_{N,t}(\theta))^k \mid u(0) = u] dT \to \mathbb{E}^\theta[(W^\theta_{N,t})^k],
\]

(6.56)

\[
\frac{1}{\tau_{n_k}} \int_0^{\tau_{n_k}} \mathbb{E}[(W^T_{N,t}(\lambda, \lambda))^k \mid u(0) = u] dT \to \mathbb{E}^\lambda[(W^\lambda_{N,t})^k],
\]

(6.57)

for \( k = 1, 2 \).

Moreover, there is a \( c' > 0 \) independent of \( N \) such that the convergence \( (6.56) \) holds in \( L^2(\omega_\nu) \) for \( |\theta - \nu| \leq c' \), and the convergence in \( (6.57) \) holds in \( L^2(\omega_\nu) \) if \( |\lambda - \nu| \leq c' \).

Proof. Given the discussion preceding the Proposition, only the assertion regarding convergence in \( L^2 \) requires comment. We already know that the convergence \( (6.56) \) holds in \( L^2(\omega_\theta) \). The Radon-Nikodym derivative of \( \omega_\nu \) with respect to \( \omega_\theta \) is bounded by

\[
C_{N\nu} |\nu - \theta| \sum_{\tau} |u_{\tau}| \in L^4(\omega_\theta)
\]

(6.58)

if \( c' \) is small enough. The claim now follows by Hölder’s inequality.

Remark. Note that for any \( h \) we have

\[
\int \left| \frac{1}{\tau} \int_0^\tau \mathbb{E}[h(u(T)) \mid u_0 = u] dT \right|^p d\omega_\theta(u) \leq \frac{1}{\tau} \int_0^\tau \int |\mathbb{E}[h(u(T)) \mid u_0 = u]|^p d\omega_\theta(u)
\]

\[
\leq \frac{1}{\tau} \int_0^\tau \mathbb{E}^\theta[|h(u(T)|^p)] dT
\]

\[
= \mathbb{E}^\theta[|h(u)|^p].
\]

(6.59)

By \( (6.53) \) we see that \( f \) and \( g \) are in \( L^p(\omega_\theta) \) for any \( p < \infty \). Therefore, the \( L^p(\omega_\theta) \) norms of the Cesàro means are uniformly bounded and so the convergence holds in \( L^p(\omega_\theta) \) for any \( p < \infty \). By the above argument, the convergence in the above proposition holds in any \( L^p(\omega_\nu) \) for \( |\nu - \theta| \leq c' \) (note that the \( c' \) doesn’t depend on \( p \)).

We also have the following consequence of the previous result:

Lemma 8. There is a subsequence \( \tau_n, n \to \infty \) such that the quantity

\[
\frac{1}{\tau_n} \int_0^{\tau_n} \frac{d}{d\theta} \mathbb{E}[W^T_{N,t}(\theta, \theta) \mid u(0) = u] dT
\]

converges to \(-N\psi_1(\theta) + t\) for almost every \( u \in \mathbb{R}^N \).

Moreover, the Cesàro averages are uniformly bounded in \( L^2(\omega_\nu) \) for \( |\nu - \theta| \) sufficiently small, and thus the convergence also holds in \( L^1(\omega_\nu) \) for the subsequence \( \tau_{n_k} \) in the previous statement.

Proof. Let \( F(\theta) = F(T, \theta) := \mathbb{E}[W^T_{N,t}(\theta, \theta)] \). By convexity, we have for \( \theta^+, \theta^- \) satisfying \( \theta^- < \theta < \theta^+ \),

\[
F(\theta^-) - F(\theta) \leq \frac{d}{d\theta} F(\theta) \leq \frac{F(\theta^+) - F(\theta)}{\theta^+ - \theta}.
\]

(6.60)
Taking subsequential limits, we find
\[ -n(\psi_0(\theta^-) - \psi_0(\theta)) + t(\theta^- - \theta) \leq \liminf_{n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} \frac{d}{d\theta} F(T, \theta) \, d\theta \leq \limsup_{n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} \frac{d}{d\theta} F(T, \theta) \, d\theta \leq \frac{-n(\psi_0(\theta^+) - \psi_0(\theta)) + t(\theta^+ - \theta)}{\theta^+ - \theta}. \]

Choosing further subsequences for parameters \( \theta_n^+ \uparrow \theta \) and \( \theta_n^- \downarrow \theta \), we find the subsequence \( \tau_n \) for which the almost everywhere convergence holds by a diagonal argument.

For the statement about convergence in \( L^1 \), we first note that, by (6.60), it suffices to estimate \( E[W_{T,N,t}(\theta', \theta) \mid u(0) = u] \) for \( \theta' \) close to \( \theta \), uniformly in \( T \). By the conditional Jensen’s inequality, this is bounded uniformly in \( L^2(\omega_{\theta'}) \). Estimating \( d\omega_{\psi_{\theta'}} \) as in (6.58), we obtain a uniform bound in \( L^2(\omega_{\theta'}) \) by Hölder’s inequality. Convergence in \( L^1 \) statement follows since this uniform bound implies uniform integrability. \( \square \)

7. Two formulas for the variance

**Proposition 9** (Representations for the variance). For each \( \theta > 0 \) and \( t \geq 0 \), we have
\[ \text{Var}(W_{T,N,t}(\theta, \theta)) = \text{Var}(R_{T,N,t}(\theta, \theta)) - t + 2E[\partial_\theta W_{T,N,t}(\eta, \theta) \mid \eta = \theta]. \quad (7.1) \]

and
\[ \text{Var}(W_{T,N,t}(\theta, \theta)) = \text{Var}(R_{T,N,t}(\theta, \theta)) - t + 2\frac{d}{d\theta} E[W_{T,N,t}(\theta, \theta)] - 2E[\partial_\eta W_{T,N,t}(\eta, \theta) \mid \eta = \theta]. \quad (7.2) \]

Both formulas hold for any distribution of the initial date \( u_0 = u(0) \).

The equations (7.1) and (7.2) should be compared with the following expressions found by Seppäläinen and Valkó for the variance in the terms of the expectation of the first jump \( \sigma_0 \) in the O’Connell-Yor ensemble, taken with respect to a Gibbs measure corresponding to the polymer partition function (see [27, Theorem 3.6]):
\[ \text{Var}(\log Z_{N,t}^\theta) = N\psi_1(\theta) - t + 2E[\sigma_0^+] \]
\[ = t - N\psi_1(\theta) + 2E[\sigma_0^-]. \]

**Proof.** Since the parameters \( \eta \) and \( \theta \) will be fixed throughout, we will abbreviate \( W_{T,N,t}(\theta, \theta) \), \( R_{T,N,t}(\theta) \) by \( W_{T,N,t} \) and \( R_{T,N,t} \) respectively. Taking the variance of both sides of (6.3) and rearranging, we have
\[ \text{Var}(R_{T,N,t}) = \text{Var}(W_{T,N,t}) + t + 2\text{Cov}(W_{T,N,t}, B_0^T(t)), \]
or equivalently:
\[ \text{Var}(W_{T,N,t}) = \text{Var}(R_{T,N,t}) - t - 2\text{Cov}(W_{T,N,t}, B_0^T(t)). \]

Consider the term
\[ \text{Cov}(W_{T,N,t}, B_0^T(t)) = E[W_t^T(B_0(T + t) - B_0(T))]. \]
The formula (7.1) follows at once by applying (6.6). Noting that
\[ \frac{d}{d\theta} W_{N,t}^{T}(\theta, \theta) = \partial_{\theta} W_{N,t}^{T}(\eta, \theta)|_{\eta=\theta} + \partial_{\theta} W_{N,t}^{T}(\eta, \theta)|_{\eta=\theta}, \]
we obtain (7.2). The interchange of derivatives and expectation is readily justified using (6.19).

7.1. Variance comparison. In this section we show that perturbations in the parameter (in equilibrium) change the variance by \(O(N)\) times the magnitude of the perturbation.

**Lemma 9.** Let \(\lambda > \theta\). Then,
\[ \text{Var}^{\theta}(W_{N,t}) - \text{Var}^{\lambda}(W_{N,t}) \leq N(\psi_{1}^{V}(\theta) - \psi_{1}^{V}(\lambda)). \]
If \(\lambda > \theta\) and \(|\lambda - \theta| \leq c'\) where \(c'\) is the constant from Lemma 8 then,
\[ \text{Var}^{\theta}(W_{N,t}) - \text{Var}^{\lambda}(W_{N,t}) \geq -N(\psi_{1}^{V}(\theta) - \psi_{1}^{V}(\lambda)). \]
Hence, if \(|\lambda - \theta| \leq c'\) we have \(|\text{Var}^{\theta}(W_{N,t}) - \text{Var}^{\lambda}(W_{N,t})| \leq N|\psi_{1}^{V}(\theta) - \psi_{1}^{V}(\lambda)|\]

**Proof.** WLOG, assume \(\lambda > \theta\). For any \(\tau > 0\), we have by invariance
\[
\text{Var}^{\theta}(W_{N,t}) = \text{Var}^{\theta}(R_{N,t}(\theta)) - t + 2\mathbb{E}^{\theta}[\partial_{\theta} W_{N,t}^{T}(\theta, \theta)]
= \text{Var}^{\theta}(R_{N,t}(\theta)) - \text{Var}^{\lambda}(R_{N,t}(\lambda))
+ \text{Var}^{\lambda}(R_{N,t}(\lambda)) - t + 2\int_{0}^{\tau} \mathbb{E}^{\theta}[\partial_{\theta} W_{N,t}^{T}(\theta, \theta)] dT
= N(\psi_{1}^{V}(\theta) - \psi_{1}^{V}(\lambda))
+ \text{Var}^{\lambda}(R_{N,t}(\lambda)) - t + 2\int_{0}^{\tau} \mathbb{E}^{\theta}[\partial_{\theta} W_{N,t}^{T}(\theta, \theta)] dT.
\]
Next we use the monotonicity of \(\theta \mapsto \partial_{\theta} W(\theta, \theta)\) implied by Lemma 6 to find that the integral term is bounded above by
\[ \frac{2}{\tau} \int_{0}^{\tau} \mathbb{E}^{\theta}[\partial_{\theta} W_{N,t}^{T}(\lambda, \lambda)] dT \]
By the estimates (6.21), we have \(\mathbb{E}[\partial_{\lambda} W_{N,t}(\lambda, \lambda) | u(0) = u] \in L^{2}(\omega_{\theta})\), so applying Theorem 3 we can choose a sequence \(\tau_{n}\) such that, for almost every \(u \in \mathbb{R}^{N}:
\text{Var}^{\lambda}(R_{N,t}(\lambda)) - t + 2\int_{0}^{\tau_{n}} \mathbb{E}^{\theta}[\partial_{\lambda} W_{N,t}^{T}(\lambda, \lambda)] dT \to \text{Var}^{\lambda}(R_{N,t}(\lambda)) - t + 2\mathbb{E}^{\lambda}[\partial_{\lambda} W_{N,t}^{\lambda}]
= \text{Var}^{\lambda}(W_{N,t}^{\lambda}).
\]
The upper bound in the statement of the Lemma follows from this (with no restriction on \(\lambda > \theta\)).
For the lower bound, we begin again with
\[ \text{Var}^{\theta}(W_{N,t}) = N\psi_{1}^{V}(\theta) - t + 2\int_{0}^{\tau} \mathbb{E}^{\theta}[\partial_{\theta} W_{N,t}^{T}(\eta, \theta)|_{\eta=\theta}] dT. \]
We now rewrite the integrand as,
\[ \mathbb{E}^{\theta} \left[ \partial_{\theta} W_{N,t}^T(\eta, \theta) \right] = \mathbb{E}^{\theta} \left[ \frac{d}{d\theta} W_{N,t}^T(\theta, \theta) \right] + \mathbb{E}^{\theta} \left[ \partial_{\eta} W_{N,t}^T(\eta, \lambda) \right] \]

In the inequality we again used the monotonicity of the partial derivatives. Now, as long as \(|\lambda - \theta| \leq c'\), we have by Lemma 8 and the argument above that,
\[ \lim_{\tau \to \infty} 2 \tau \int_0^\tau \mathbb{E}^{\theta} \left[ \frac{d}{d\theta} W_{N,t}^T(\theta, \theta) \right] - \mathbb{E}^{\theta} \left[ \frac{d}{d\lambda} W_{N,t}^T(\lambda, \lambda) \right] + \mathbb{E}^{\theta} \left[ \partial_{\lambda} W_{N,t}^T(\eta, \lambda) \right] dT = -2N \left( \psi_1(\theta) - \psi_1(\lambda) \right) + \frac{\pi}{2} \psi_2(\lambda). \]

Hence,
\[ \text{Var}^\theta(W_{N,t}) - \text{Var}^\lambda(W_{N,t}) \geq -N \left( \psi_1(\theta) - \psi_1(\lambda) \right). \]

This yields the claim.

\[ \square \]

8. The Gaussian case

8.1. Variance of order \( N^{1/2} \). The following theorem gives exact formulas for \( h_j = \partial_{\theta} u_j \) and \( \text{Var}[W_{N,t}] \) in the case when \( V'' = 1 \). It also implies that when \( t = N \), \( W_{N,t} \) has fluctuations of order \( N^{1/4} \).

Theorem 4. If \( V(x) = x^2/2 \), then for all \( n \in \mathbb{N} \) and \( t \geq 0 \),
\[ h_N(t) = -\sqrt{\frac{\pi}{2}} e^{-t} \quad \text{and} \quad \text{Var}(W_{n,t}) = (n-t) \left[ 1 + \psi_1(t) \right] + 2 \frac{\rho^n}{(n-1)!} e^{-t}. \]

Moreover,
\[ \lim_{n \to \infty} \frac{\text{Var}(W_{N,t})}{\sqrt{N}} = \sqrt{\frac{2}{\pi}}. \]

Proof. By (6.8), the functions \( h_j \) are determined by \( h_j(0) = 0 \) and
\[ dh_1 = (-1 - h_1) dt \quad \text{and} \quad dh_j = (h_{j-1} - h_j) dt \quad \text{for} \ j \geq 2. \]

We first show that the functions \( h_n \) in (8.1) satisfy these differential equations. When \( n = 1 \), (8.1) gives \( h_1(t) = e^{-t} - 1 \) which agrees with (8.4). When \( n = j \geq 2 \), integration by parts of (8.1) gives
\[ h_j(t) = \frac{j^{j-1}}{j-1!} e^{-t} + h_{j-1}(t). \]
which implies (8.5). We now check the variance formula (8.2). Since
\[ \partial_\theta W_{N,t} = \sum_{j=1}^{N} h_j(t) + t, \]
we have
\[ \text{d}\partial_\theta W_{N,t} = -h_N \text{d}t. \]
Thus, using (8.1), we have
\[ \partial_\theta W_{N,t} = \int_{0}^{t} h_N(s) \text{d}s \]
\[ = \int_{0}^{t} \int_{0}^{s} \frac{r^{N-1}}{(N-1)!} e^{-r} \text{d}r \text{d}s \]
\[ = \int_{0}^{t} \int_{r}^{t} \frac{r^{N-1}}{(N-1)!} e^{-r} \text{d}s \text{d}r \]
\[ = \int_{0}^{t} (t-r) \frac{r^{N-1}}{(N-1)!} e^{-r} \text{d}r \]
\[ = -th_N(t) + Nh_{N+1}(t) \]
\[ = (N-t)h_N(t) + n [h_{N+1}(t) - h_N(t)] \]
\[ = (N-t)h_N(t) + \frac{t^N}{(N-1)!} e^{-t}. \]
The last equality follows from (8.6). By the variance formula (7.1),
\[ \text{Var}[W_{N,t}] = N - t + 2\mathbb{E}[\partial_\theta W_{N,t}] \]
\[ = (N-t) [1 + 2h_N(t)] + 2 \frac{t^N}{(N-1)!} e^{-t}. \]
Finally, to prove (8.3), set \( t = N \) in (8.2) and use Stirling’s approximation to get
\[ \frac{\text{Var}[W_{N,t=N}]}{\sqrt{N}} = \frac{2N^N e^{-N}}{(N-1)! \sqrt{N}} = 2 \left( \frac{N}{e} \right)^N \frac{\sqrt{N}}{N!} \to \sqrt{\frac{2}{\pi}}. \]
\[ q.e.d. \]

8.2. Convergence to a normal distribution. Here we prove Proposition 2. When \( V(x) = x^2/2 \), the equations (2.1) read
\[ du_1 = (-\theta - u_1) dt + dB_0 + dB_1 \]
\[ du_j = (u_{j-1} - u_j) dt + dB_j - dB_{j-1} \]
These equations have explicit solutions
\[ u_1(t) = \int_0^t e^{-\int_s^t ds} (-\theta ds + dB_0 + dB_1) + e^{-t}u_1(0) \]
\[ = \theta(e^{-t} - 1) + \int_0^t e^{-(t-s)} dB_0 + \int_0^t e^{-(t-s)} dB_1 + e^{-t}u_1(0), \]
\[ u_j(t) = \int_0^t e^{-\int_s^t dr} (u_{j-1}(s)ds + dB_j - dB_{j-1}) + e^{-t}u_j(0) \]
\[ = \int_0^t e^{-(t-s)} u_{j-1}(s)ds + \int_0^t e^{-(t-s)} dB_j - \int_0^t e^{-(t-s)} dB_{j-1} + e^{-t}u_j(0) \text{ for } j \geq 2. \]

Induction on the explicit solutions now implies that when the initial \( u_j(0) \) are Gaussian, then \( \{u_j(t) : t \geq 0\} \) is a Gaussian process for all \( j \in \mathbb{N} \). The inductive argument goes as follows: in the last line, the 2nd and 3rd integrals clearly lead to Gaussian processes. An integration by parts applied to the first integral along with the inductive hypothesis that \( u_{j-1} \) is a Gaussian process finally shows first integral is also a Gaussian process. Thus
\[ W_{N,t} = \sum_{j=1}^N u_j(t) + \theta t - B_0(t) \]
is a Gaussian process for every \( N \in \mathbb{N} \) as well. In particular, the distribution of \( W_{N,N} \) is completely determined by its expectation and variance. Now Theorem 4 implies that \( W_{N,N}/N^{1/4} \) is a centered Gaussian whose variance approaches \( \sqrt{2/\pi} \). Thus we have the convergence in distribution:
\[ \frac{W_{N,N}}{N^{1/4}} \xrightarrow{d} \mathcal{N} \left( 0, \sqrt{\frac{2}{\pi}} \right). \]

9. Proof of main results

**Proof of Theorem 1.** Let
\[ \lambda = \theta + c'N^{-\varepsilon/2}, \quad \mu = \theta - c'N^{-\varepsilon/2}. \]
where \( c' > 0 \) is the constant from Lemma 8. We choose the initial data \( u_0 \) to be distributed according to \( \omega_\theta \). By the variance representation (7.1), we have using invariance of \( \omega_\theta \):
\[ \text{Var}^\theta(W^\theta_{N,t}) = \text{Var}^\theta(W^T_{N,t}(\theta, \theta)) \]
\[ = \text{Var}^\theta(R^T_{N,t}(\theta)) - t + 2\mathbb{E}^\theta[\partial_\theta W^T_{N,t}] \]
\[ \leq \text{Var}^\theta(R^T_{N,t}(\theta)) - t + 2\mathbb{E}^\theta[(\partial_\theta W^T_{N,t})^2]^{\frac{1}{2}} \]
Averaging over intervals \([0, \tau_n]\) and using (8), we obtain
\[ \text{Var}^\theta(W^\theta_{N,t}) \leq N\psi_1(\theta) - t + \limsup_{\tau_n \to \infty} \frac{2}{\tau_n} \int_0^{\tau_n} \mathbb{E}^\theta[(\partial_\theta W^T_{N,t})^2]^{\frac{1}{2}} dT. \]
We now examine the term

\[
\frac{1}{\tau_n} \int_0^{\tau_n} \mathbb{E}^\theta[(\partial_\theta W^T_{N,t}(\theta, \theta))^2]^{1/2} \, dT \leq \left( \frac{1}{\tau_n} \int_0^{\tau_n} \mathbb{E}^\theta[(\partial_\theta W^T_{N,t}(\theta, \theta))^2] \, dT \right)^{1/2}.
\] (9.4)

Using (6.46), we have

\[
\mathbb{E}^\theta[(\partial_\theta W^T_{N,t}(\theta, \theta))^2] \leq \mathbb{E}^\theta\left[\left( \frac{d}{d\theta} W^T_{N,t}(\theta, \theta) \right)^2 \right] - 2\mathbb{E}^\theta[(\partial_\theta W^T_{N,t}(\eta, \theta) \cdot \partial_\eta W^T_{N,t}(\theta, \eta)|_{\eta=\theta}].
\]

The \(\partial_\theta \times \partial_\eta\) cross term on the right is difficult to compute directly, but we will show then it is of order \(N\) by comparing it to the second derivative of \(W^T_{N,t}(\theta, \theta)\). By (6.47), we have

\[
\mathbb{E}^\theta[(\partial_\theta W^T_{N,t}(\theta, \theta))^2] \leq \mathbb{E}^\theta\left[\left( \frac{d}{d\theta} W^T_{N,t}(\theta, \theta) \right)^2 \right] + \frac{2}{\epsilon_0} \mathbb{E}^\theta[\partial_{\theta\eta} W^T_{N,t}(\eta, \theta)|_{\eta=\theta}].
\] (9.5)

Next, we have

\[
\mathbb{E}^\theta\left[\frac{d^2}{d\theta^2} W^T_{N,t}(\theta, \theta)\right] = \mathbb{E}^\theta[\partial_{\theta\theta}^2 W^T_{N,t}(\theta, \theta)] + \mathbb{E}^\theta[\partial_{\theta\eta}^2 W^T_{N,t}(\theta, \theta)] + 2\mathbb{E}^\theta[\partial_{\eta\eta} W^T_{N,t}(\theta, \theta)].
\]

The first two terms of the last equation are non-negative, so we have

\[
2\mathbb{E}^\theta[\partial_{\theta\eta}^2 W^T_{N,t}(\theta, \eta)|_{\theta=\eta}] \leq \mathbb{E}^\theta\left[\frac{d^2}{d\theta^2} W^T_{N,t}(\theta, \theta)\right].
\] (9.6)

Now compute:

\[
\frac{d^2}{d\theta^2} \mathbb{E}^\theta[W^T_{N,t}(\theta, \theta)] = \mathbb{E}^\theta\left[\frac{d^2}{d\theta^2} W^T_{N,t}(\theta, \theta)\right] - \sum_{i=1}^N \text{Cov}_{\omega_\theta} \frac{d}{d\theta} \mathbb{E}[W^T_{N,t}(\theta, \theta) | u(0) = u_i, u_i) \right.
\]

\[
+ \int \mathbb{E}[W^T_{N,t}(\theta, \theta) | u(0) = u] \frac{d^2}{d\theta^2} \omega_\theta(u) \, du.
\] (9.9)

The interchange of derivatives implicit in (9.7) follows from the bounds in (6.19) and the uniform integrability in (6.39) as well as (6.45).

We examine the term (9.9). We have

\[
\frac{d^2}{d\theta^2} \omega_\theta(u) = \left( \sum_{i=1}^N u_i \right)^2 \omega_\theta(u) := w(u) \omega_\theta(u).
\]

By (2.2), we have \(w \in L^1(\mathbb{R}^N, \omega_\theta)\).

Recall now from Section 6.6 that the averages over \([0, \tau_n]\) of the function \(Q_T f(u) = \mathbb{E}[W^T_{N,t}(\theta, \theta) | u(0) = u\] converge to \(\mathbb{E}^\theta[W^T_{N,t}]\) in \(L^2(\omega_\theta)\). Hence, the averages,

\[
\frac{1}{\tau_n} \int_0^{\tau_n} \mathbb{E}[W^T_{N,t}(\theta, \theta) | u(0) = u] \frac{d^2}{d\theta^2} \omega_\theta(u) \, du \, dT
\]

converge to

\[
\mathbb{E}^\theta[W^T_{N,t}] \int \frac{d^2}{d\theta^2} \omega_\theta(u) \, du = 0.
\]
As for (9.8), by convexity, by Lemma 8 that the Cesàro averages of the function
\[ \frac{d}{d\theta} \mathbb{E}[W_{N,t}^T(\theta, \theta) \mid u(0) = u] \]
are bounded in $L^{3/2}(\omega_\theta)$. Since $u_i \in L^p(\omega_\theta)$ for any $p > 0$ we see by uniform integrability and by Lemma 8 that the Cesàro means of the terms in (9.8) converge, up to passing to appropriate subsequences, to
\[ - \sum_{i=1}^N \text{Cov}_{\omega_\theta} \left( \frac{d}{d\theta} \mathbb{E}^\theta [W_{N,t}(\theta, \theta)], u_i \right) = 0. \]

It follows that
\[ \limsup_{\tau_n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} 2 \mathbb{E}^\theta [\partial_{\theta\eta} W_{N,t}^T(\theta, \eta) \mid \theta = \eta] \, dT \leq \limsup_{\tau_n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} \mathbb{E}^\theta \left[ \frac{d^2}{d\theta^2} W_{N,t}^T(\theta, \theta) \right] dT \]
\[ = \limsup_{n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} \frac{d^2}{d\theta^2} \mathbb{E}^\theta [W_{N,t}^T(\theta, \theta)] dT. \]

Since for the initial data we have $u(0) \sim \omega_\theta$, the quantity on the right is
\[ \frac{d^2}{d\theta^2} \mathbb{E}^\theta [W_{N,t}^T(\theta, \theta)] = -N \psi_2(\theta) = O(N), \]
and so
\[ \limsup_{\tau_n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} 2 \mathbb{E}^\theta [\partial_{\theta\eta} W_{N,t}^T(\theta, \eta) \mid \theta = \eta] \, dT \leq \limsup_{\tau_n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} \mathbb{E}^\theta \left[ \frac{d^2}{d\theta^2} W_{N,t}^T(\theta, \theta) \right] dT \leq O(N). \]

By convexity, we have,
\[ \mathbb{E}^\theta \left[ \left( \frac{d}{d\theta} W_{N,t}^T(\theta, \theta) \right)^2 \right] \leq C \left\{ \frac{1}{(\lambda - \theta)^2} \mathbb{E}^\theta [ (W_{N,t}^T(\lambda, \lambda) - W_{N,t}^T(\theta, \theta))^2 ] + \frac{1}{(\mu - \theta)^2} \mathbb{E}^\theta [ (W_{N,t}^T(\mu, \mu) - W_{N,t}^T(\theta, \theta))^2 ] \right\}. \]

We now show just how to estimate the first term on the right side of the previous inequality, the second is identical. We have,
\[ \frac{1}{(\lambda - \theta)^2} \mathbb{E}^\theta [ (W_{N,t}^T(\lambda, \lambda) - W_{N,t}^T(\theta, \theta))^2 ] \]
\[ \leq \frac{C}{(\lambda - \theta)^2} \left( \text{Var}^\theta (W_{N,t}^T(\theta, \theta)) + \mathbb{E}^\theta [ W_{N,t}^T(\lambda, \lambda) - \mathbb{E}^\lambda [ W_{N,t}^T(\lambda, \lambda) ] ]^2 \right) \]
\[ + \frac{C}{(\lambda - \theta)^2} \left( \mathbb{E}^\lambda [ W_{N,t}^T(\lambda, \lambda) ] - \mathbb{E}^\theta [ W_{N,t}^T(\lambda, \lambda) ] \right)^2. \]
Note that we added and subtracted the term $\mathbb{E}^\lambda[W_{N,t}^T(\lambda, \lambda)] = \mathbb{E}^\lambda[W_{N,t}]$ which is independent of $T$. From Proposition 8 and Lemma 2 we have,

$$\limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{(\lambda - \theta)^2} \mathbb{E}^\theta\left[\left(W_{N,t}^T(\lambda, \lambda) - W_{N,t}^T(\theta, \theta)\right)^2\right]dT \leq \frac{C}{(\lambda - \theta)^2} \left(\text{Var}^\theta(W_{N,t}^\theta) + \text{Var}^\lambda(W_{N,t}^\lambda) + CN^{4/3}\right)$$

where we used Lemma 9 in the second inequality. By the same argument,

$$\limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{(\mu - \theta)^2} \mathbb{E}^\theta\left[\left(W_{N,t}^T(\mu, \mu) - W_{N,t}^T(\theta, \theta)\right)^2\right]dT \leq \frac{C'}{(\lambda - \theta)^2} \left(\text{Var}^\theta(W_{N,t}^\theta) + N^{2/3}\right) + CN^{4/3}$$

Combining the last two inequalities with (9.5) and (9.11), we obtain the following inequality for $V := \text{Var}^\theta(W_{N,t}^\theta)$:

$$V \leq \frac{CN^{1/3}}{2} \left(V^{1/2} + N^{2/3}\right).$$

Since $V \geq 0$, the result (2.11) follows from this inequality.

9.1. Characteristic direction.

Proof of Corollary 1. The proof is the same as for the corresponding result for the inverse Gamma polymer [25, Corollary 2.2]. Recall that we assume

$$N^{-\alpha}|t - N\psi_1(\theta)| \to \gamma_0 > 0,$$

where $\gamma_0 > 0$. Assuming for example that $t < N\psi_1(\theta)$, we have

$$W_{N,t}(\theta, \theta) = \sum_{j=1}^m u_j(t) - B_0(t) + \theta t + \sum_{j=m+1}^N u_j(t).$$

Letting $m = \lfloor t/\psi_1(\theta) \rfloor$ and dividing by $N^{\alpha/2}$, the upper bound (2.11) for the variance shows that the first term converges to 0 in probability after re-centering, while the second term converges to a Gaussian by the classical central limit theorem.

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