EXISTENCE OF A MARTINGALE WEAK SOLUTION TO THE EQUATIONS OF NON-STATIONARY MOTION OF NON-NEWTONIAN FLUIDS WITH A STOCHASTIC PERTURBATION

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Abstract. In this paper, we consider the stochastic incompressible non-Newtonian fluids driven by a cylindrical Wiener process \( W \) with shear rate dependent on viscosity in a bounded Lipschitz domain \( D \in \mathbb{R}^n \) during the time interval \((0, T)\). For \( q > \frac{2n+2}{n+2} \) in the growth conditions (1.2), we prove the existence of a martingale weak solution with \( \nabla \cdot u = 0 \) by using a pressure decomposition which is adapted to the stochastic setting, the stochastic compactness method and the \( L^\infty \)-truncation.

1. Introduction

Let \( D \in \mathbb{R}^n \) \((n \geq 2)\) be a bounded Lipschitz domain. For the time interval \((0, T)\), we set \( Q := (0, T) \times D \). In this paper, we consider the following equations:

\[
\begin{aligned}
\frac{du}{dt} + \nabla \cdot (u \otimes u - S + pI)dt &= f dt + \Phi(u)dW, \\
\nabla \cdot u &= 0, \\
u|_{\partial D} &= 0, \\
u|_{t=0} &= u_0,
\end{aligned}
\] (1.1)

where \( S = \{S_{ij}\} \) is the deviatoric stress tensor, \( p \) the pressure, \( u \) the velocity, \( f \) the external force and \( W \) a cylindrical Wiener process with values in a Hilbert space. \( \Phi \) satisfy the linear growth assumption (see Sect.2 for details).

The stress \( S \) may depend on both \((x, t)\) and the “rate of strain tensor” \( \mathbb{D} = \{\mathbb{D}_{ij}\} \), which is defined by \( \mathbb{D}_{ij} = \mathbb{D}_{ij}(u) := \frac{1}{2} (\partial_j u_i + \partial_i u_j), \ i, j = 1, \ldots, n \). We refer to [2], [4] and [27] about the continuum mechanical background. As far as we know, the fluids with shear dependent viscosity are often used in engineering practice. So it’s meaningful to study this kind of fluid. In this paper, \( S \) is assumed to be a function of the shear rate and the constitutive relations reads as

\[
S = \nu(\mathbb{D}_{II})\mathbb{D},
\]

where \( \mathbb{D}_{II} = \frac{1}{2} \mathbb{D} : \mathbb{D} \) is the second invariant of \( \mathbb{D} \). Here are some examples of precise constructions of \( S \): for \( q \in (1, +\infty) \), constant \( \nu_0 \),

\[
S = \nu_0(\mathbb{D}_{II})^{\frac{q-2}{2}} \mathbb{D},
\]

\[
S = \nu_0(1 + \mathbb{D}_{II})^{\frac{q-2}{2}} \mathbb{D}.
\]

For details, see [2] [6] [22]. If \( q \in (1, 2) \), we say the non-Newtonian fluids is pseudoplastic or shear thinning (for example, ketchup); if \( q = 2 \), it’s Newtonian fluids; if \( q \in (2, +\infty) \), we say the non-Newtonian fluids is dilatant or shear thickening (for example, batter). The
following two constitutive laws which are also of interest in engineering practice are given by

\[ S = \nu_0 (D_1)^{2/n} + \nu_\infty D, \]
\[ S = \nu_0 (1 + D_1)^{2/n} + \nu_\infty D, \]

where \( \nu_0 \) and \( \nu_\infty \) are positive constants and \( q \in [1, +\infty) \). An extensive list for specific \( q \)-values for different fluids can be found in [6].

For \( q \in [1, +\infty) \), we assume the deviatoric stress tensor \( S \) satisfy the following conditions in this paper: \( S : Q \times M_{\text{sym}}^n \to M_{\text{sym}}^n \) is a Carathéodory function. \( \forall \xi \in M_{\text{sym}}^n \) (vector space of all symmetric \( n \times n \) matrices \( \xi = \{\xi_{ij}\} \)). We equip \( M_{\text{sym}}^n \) with scalar product \( \xi : \eta \) and norm \( \|\xi\| := (\xi : \xi)^{\frac{1}{2}} \), for almost all \( (x, t) \in Q \),

\[ |S(x, t, \xi)| \leq C_0 \|\xi\|^{q-1} + \eta_1, \]

where \( C_0 > 0, \eta_1 \geq 0, \eta_1 \in L^{q'}(Q), 1/q + 1/q' = 1; \forall \xi \in M_{\text{sym}}^n \), for almost all \( (x, t) \in Q \),

\[ S(x, t, \xi) : \xi \geq C_0 \|\xi\|^q - \eta_2, \]

where \( C_0 > 0, \eta_2 \geq 0, \eta_2 \in L^1(Q); \forall \xi, \eta \in M_{\text{sym}}^n (\xi \neq \eta) \), for almost all \( (x, t) \in Q \),

\[ (S(x, t, \xi) - S(x, t, \eta) : (\xi - \eta) > 0. \]

The flow of a homogenous incompressible fluid without stochastic part is described by the following equations:

\[ \begin{cases} 
\partial_t u + \nabla \cdot (u \otimes u - S + pD) = -\nabla \cdot f, \\
\nabla \cdot u = 0, \\
u|_{\partial D} = 0, \\
u|_{t=0} = u_0.
\end{cases} \tag{1.5} \]

In the late sixties, Lions and Ladyshenskaya in [28, 29, 30, 31] started the mathematical discussion of power-law model. In [28], Ladyzhenskaya achieved the existence and uniqueness of weak solutions and in [31] Lions achieved these results for \( q \geq \frac{2n+2}{n+2} \). They showed the existence of a weak solution in the space \( L^q((0, T); W^{1,q}_{0,\text{div}}(D)) \cap L^\infty((0, T); L^2(D)) \). In this particular case that \( u \otimes u : D(u) \in L^1(Q) \) follows from parabolic interpolation, the proof of existence is based on monotone operators and compactness arguments. In [32], Málek, Nečas and Ružička proved the existence for \( q \in [2, \frac{4n}{n+2}] \) under the assumption that \( D \in \mathbb{R}^3 \) is a bounded domain with \( C^3 \)-boundary and that \( S(D) \) has the form \( S(D) = \partial_3 \Phi(D_{11}) \). In [13], Wolf improved this result to the case \( p > \frac{2n+2}{n+2} \) by using \( L^\infty \)-truncation.

In the fluid motion, apart from the force \( f \), there might be further quantities with a influence on the motion. This influence usually is small and can be shown by adding a stochastic part to the equation. The stochastic part to the equation can be understood as a turbulence. This type of equation is often used in fluid mechanics since they model the phenomenon of perturbation. So it’s very interesting to study the stochastic fluids. In SPDES, we consider two concepts: strong (pathwise) solutions and weak (martingale) solutions. Strong solutions means that the underlying probability space and the Wiener process are given in advance. While martingale solutions means that the combination of these stochastic elements and the fluid variables is the solution of the problem and the original equations are satisfied in the sense of distributions. Clearly, the existence of strong solutions implies the existence of martingale solutions. There are many research results on the stochastic Newton flow dating back to the 1970’s with the initial work of Bensoussan and Temam [3]. For example, the existence of strong solutions and martingale solutions to the stochastic incompressible Navier-Stokes equations is established by Da Prato-Zabczyk [13], Breckner [8], Menaldi-Sritharan [34], Glatt-Holtz-Ziane [22].
Taniguchi [41], Kapuściński-Peszat [39, 42], Kim [26], Kapuściński-Gatarek [29], Mikulevičius-Rozovskii [35, 36], Brzeźniak-Motyl [11] and the references therein; for the stochastic incompressible MHD equations, the existence of solutions is considered in [40] and the references therein. For the stochastic incompressible non-Newtonian flow, there are only a few results. Recently, Breit [17] proved the existence of a martingale weak solution of the stochastic Navier-Stokes equations of the model $S(\mathbb{D}(u)) = (1 + \mathbb{D}u)\nu^{-2}\mathbb{D}u$.

In this paper, we will prove the existence of martingale solutions of the stochastic equations [14] with $S = \nu(\mathbb{D}_t)\mathbb{D}$, which is the general form of $S(\mathbb{D}(u)) = (1 + \mathbb{D}u)\nu^{-2}\mathbb{D}u$.

Comparing with the work in [43], we face the essential challenge of establishing sufficient compactness in order to be able to pass to the limit in the class of solutions. In general it is not possible to get any compactness in $\omega$ as no topological structure on the sample space $\Omega$. That is, even if a space $\mathcal{X}$ is compactly embedded in another space $\mathcal{Y}$, it is not usually the case that $L^2(\Omega, \mathcal{X})$ is compactly embedded in $L^2(\Omega, \mathcal{Y})$. As such, Aubin-Lions Lemma or Arzelà-Ascoli Theorem, which classically make possible the passage to the limit in the nonlinear terms, cannot be directly applied in the stochastic setting. To overcome this difficulty, it is classical to rather concentrate on compactness of the set of laws of the approximations (the Prokhorov Theorem, which is used to obtain compactness in the collection of probability measures associated to the approximate solutions) and apply the Skorokhod embedding Theorem, which provides almost sure convergences of a sequence of random variables that have the same laws as the original ones, but relative to a new underlying stochastic basis. However, the Skorokhod embedding Theorem is restricted to metric spaces but the structure of the stochastic non-Newtonian equations naturally leads to weakly converging sequences. For this, we apply the Jakubowski-Skorokhod Theorem which is valid on a large class of topological spaces (including separable Banach spaces with weak topology). Compared with the work in [17], the biggest difference is that we use the cut-off function to prove the approximated equations for $S = \nu(\mathbb{D}_t)\mathbb{D}$, which is the general form of $S$ in [17] also hold on the new probability space, rather than using a general and elementary method that was recently introduced in [38].

The rest of the paper is organized as follows. In Sect 2, we formulate some stochastic background and give our main Theorem. In Sect 3, we reconstructed the pressure which disappears in the weak formulation. In Sect 4, we use Galerkin method adding a large power of $u$ to study auxiliary problem. In Sect 5, we prove the main theorem.

2. Hypotheses and Main Theorem

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a stochastic basis, where $\mathcal{F}_t$ is a nondecreasing family of sub-$\sigma$-fields of $\mathcal{F}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leq s \leq t \leq T$. Assume that filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathcal{F}$.

The process $W$ is a cylindrical Wiener process, i.e., $W(t) = \sum_{k \geq 1} \beta_k(t) e_k$, with $(\beta_k)_{k \geq 1}$ being mutually independent real-valued standard Wiener processes relative to $\mathcal{F}_t$ and \{e_k\}_{k \geq 1} a complete orthonormal system in a separable Hilbert space $U$. Since $W$ don’t actually converge on $U$, we define $U_0 \supset U$ by $U_0 = \{v = \sum_{k \geq 1} \alpha_k e_k; \sum_{k \geq 1} \alpha_k^2 / k^2 < \infty\}$. The norm of $U_0$ is given by $\|v\|_{U_0}^2 = \sum_{k \geq 1} \alpha_k^2 / k^2, v = \sum_{k \geq 1} \alpha_k e_k$. Then the embedding $U \hookrightarrow U_0$ is Hilbert-Schmidt and the trajectories of $W$ are $\mathbb{P}$-a.s. continuous with values in $U_0$. Note that

$$\int_0^t \psi(r)dW(r)$$

where $\psi \in L^2(\Omega; L^2(U, L^2(D)))$ is progressively measurable, defines a $\mathbb{P}$-almost surely continuous $L^2(\Omega)$ valued $\mathcal{F}_t$-martingale. Furthermore, we can multiply the Itô’s integral
Theorem 2.1. Assume that \( S \) is called a martingale weak solution to (1.1), and \( \mu \) is the initial datum. Let \( t \) respectively. A system \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) is well-defined.

In this paper, the mapping \( \Phi(z) : U \to L^2(D) \) is defined by \( \Phi(z)e_k = g_k(z(\cdot)), \forall z \in L^2(D) \). We assume that \( g_k \in C(\mathbb{R} \times D) \) and satisfy the following condition:

\[
\sum_{k \geq 1} |g_k(\xi)| \leq c(1 + |\xi|), \forall \xi \in \mathbb{R}^n, \tag{2.1}
\]

\[
\sum_{k \geq 1} |\nabla g_k(\xi)|^2 \leq c, \forall \xi \in \mathbb{R}^n, \tag{2.2}
\]

and additionally implies

\[
\sup_{k \geq 1} k^2 |g_k(\xi)|^2 \leq c(1 + |\xi|^2), \forall \xi \in \mathbb{R}^n. \tag{2.3}
\]

Now, we are ready to give a precise definition of the martingale weak solutions.

Definition 2.1. Let \( \mu_0, \mu_f \) be Borel probability measures on \( L^2_{\text{div}}(D) \) and \( L^2(Q) \) respectively. A system

\[
((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), u, u_0, f, W)
\]

is called a martingale weak solution to (1.1), and \( S \) satisfy (1.2), (1.3), and (1.4) with the initial datum \( \mu_0 \) and \( \mu_f \) if the following conditions are satisfy:

1. \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) is a stochastic basis with a complete right-continuous filtration,
2. \( W \) is an \( \mathcal{F}_t \)-cylindrical Wiener process,
3. \( u \in L^2(\Omega; L^\infty(0, T; L^2(D))) \cap L^2(\Omega; L^q(0, T; W^1_{\text{div}}(D))) \) is progressively measurable,
4. \( u_0 \in L^2(\Omega; L^2(D)) \) with \( \mu_0 = \mathbb{P} \circ u_0^{-1} \),
5. \( f \in L^2(\Omega; L^2(Q)) \) is adapted to \( \mathcal{F}_t \) and \( \mu_f = \mathbb{P} \circ f^{-1} \),
6. \( \forall \varphi \in C^\infty_{\text{div}}(D) \) and \( \forall t \in [0, T] \), it holds that \( \mathbb{P} \)-a.s.

\[
\int_D (u(t) - u_0) \cdot \varphi dx = \int_0^t \int_D u \otimes u : \nabla(\varphi) - S(x, r, \nabla(u)) : \nabla(\varphi) dx dr
+ \int_0^t \int_D f \cdot \varphi dx dr + \int_0^t \int_D \Phi(u) \cdot \varphi dx dr + \int_0^t \int_D \Phi(u) \cdot \varphi dx dr.
\]

Next, we state our main result.

Theorem 2.1. Assume that \( q > \frac{2n+2}{n+2} \), \( S \) satisfies (1.2), (1.3), and (1.4). \( \Phi \) satisfies (2.1) and (2.2). And further suppose that

\[
\int_{L^2_{\text{div}}(D)} \|v\|_{L^2(D)}^\beta d\mu(v) < \infty, \quad \int_{L^2(Q)} \|g\|_{L^2(Q)}^\beta d\mu_f(g) < \infty \tag{2.4}
\]

with \( \beta := \max\{\frac{2n+2}{n+2}, \frac{2n+2}{n+2} \} \). Then there exists a martingale weak solution to (1.1) in the sense of Definition 2.1.

3. Pressure Decomposition

In the present section we are going to introduce a pressure method generalizes [43] to the stochastic case. Here the pressure \( p \) will be decomposed into four part \( p_1, p_2, p_h \) and \( p_5 \). We show a-priori estimates for the components \( p_1, p_2, p_h \) and \( p_5 \).
Theorem 3.1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a stochastic basis, $v \in L^2(\Omega; L^\infty(0, T; L^2(D)))$ adapted to $\mathcal{F}_t$. Assume $H_1 + H_2 \in L^\alpha(\Omega; L^\alpha(Q))$ adapted to $\mathcal{F}_t$ for some $\alpha > 1$, $H_1 \in L^\alpha(\Omega \times Q, \mathbb{P} \otimes L^{\alpha(t,q)+1})$ and $H_2, \nabla H_2 \in L^\alpha(\Omega \times Q, \mathbb{P} \otimes L^{\alpha+1})$. Moreover, let $v_0 \in L^2(\Omega; L^2(\mathbb{D}))$ and $\Phi \in L^2(\Omega; L^\infty(0, T; L^2(U, L^2(D))))$ progressively measurable such that
\[
\int_D (v(t) - v_0) \cdot \varphi dx + \int_0^t \int_D (H_1 + H_2) : \nabla \varphi dx dr = \int_0^t \int_D \Phi \cdot \varphi dx dW(r)
\] (3.1)
holds for all $\varphi \in C^{0,\alpha}_q(D)$. Then there are functions $p_1$, $p_2$, $p_h$ and $p_\Phi$ adapt to $\mathcal{F}_t$ such that

1. $\Delta p_h = 0$ and the following estimates are satisfied for $\theta := \min\{2, \alpha\}$:
\[
E \left( \int_Q |p_1|^{\alpha_1} dx dt \right)^\beta \leq c E \left( \int_Q |H_1|^{\alpha_1} dx dt \right)^\beta,
\]
\[
E \left( \int_Q |p_2|^{\alpha_2} dx dt \right)^\beta \leq c E \left( \int_Q |H_2|^{\alpha_2} dx dt \right)^\beta,
\]
\[
E \left( \int_0^T \int_D |\nabla p_2|^{\alpha_2} dx dt \right)^\beta \leq c E \left( \int_Q |H_2|^{\alpha_2} + |\nabla H_2|^{\alpha_2} dx dt \right)^\beta,
\]
\[
E \left( \int_Q |p_1 + p_2|^{\alpha} dx dt \right)^\beta \leq c E \left( \int_Q |H_1 + H_2|^{\alpha} dx dt \right)^\beta,
\]
\[
E \left( \sup_{t\in(0,T)} \int_D |p_h|^\theta dx \right)^\beta \leq c E \left( 1 + \sup_{t\in(0,T)} \int_D |v|^2 dx + \sup_{t\in(0,T)} \int_D |H_1 + H_2|^\alpha dx dt \right)^\beta,
\]
\[
E \left( \sup_{t\in(0,T)} \int_D |p_h|^\theta dx \right)^\beta \leq c E \left( \sup_{t\in(0,T)} \int_D |v|^2 dx + \sup_{t\in(0,T)} \int_D |H_1 + H_2|^\alpha dx dt \right)^\beta,
\]
for all $1 \leq \beta < \infty$ and $D' \subset \subset D$.

2. for all $\varphi \in C^{0,\alpha}_q(D)$, it holds that
\[
\int_D (v(t) - v_0 - \nabla p_h(t)) \cdot \varphi dx + \int_0^t \int_D (H_1 + H_2) : \nabla \varphi dx dr
\]
\[
= \int_0^t \int_D (p_1 + p_2) \text{div} \varphi dx dr + \int_0^t \int_D p_\Phi(t) \text{div} \varphi dx + \int_0^t \int_D \Phi \cdot \varphi dx dW(r).
\]
Moreover, we have $p(t) = p_h(t) + p_\Phi(t) + \int_0^t (p_1 + p_2) dr \in L^\theta(\Omega; L^\infty(0, T; L^\theta(D)))$ and $p_1(0) = p_2(0) = p_h(0) = p_\Phi(0) = 0$ $\mathbb{P}$-a.s.

Proof. Let $v$ be a weak solution to (3.1) for all $\varphi \in W^{1,\theta}_{0,\text{div}}(D), 1/\theta + 1/\theta' = 1$. Then by De Rham’s theorem (see [21]), there exists a unique function $p(t) \in L^\theta_0(D)$ with $p(0) = 0$, such that
\[
\int_D (v(t) - v_0) \cdot \varphi dx = \int_D p(t) \text{div} \varphi dx + \int_0^t \int_D (H_1 + H_2) : \nabla \varphi dx dr = \int_0^t \int_D \Phi \cdot \varphi dx dW(r),
\]
for all $\varphi \in W^{1,\theta}_0(D)$.

By using the Bogovskiĭ-operator $\text{Bog}_D$ (see [7]) and let $\mathcal{B} = \text{Bog}_D(\varphi - (\varphi)_D)$ where $(\varphi)_D = \frac{1}{|D|} \int_D \varphi \, dx$, then we can get

$$
\int_D p(t) \varphi \, dx = \int_D (v(t) - v_0) \cdot \mathcal{B}(\varphi) \, dx + \int_0^t \int_D (H_1 + H_2) : \nabla \mathcal{B}(\varphi) \, dx \, dr - \int_0^t \int_D \Phi \cdot \mathcal{B}(\varphi) \, dx \, dW(r).
$$

Hence, we have

$$
p(t) = \mathcal{B}^*(v(t) - v_0) + \int_0^t (\nabla \mathcal{B})^*(H_1 + H_2) \, dr - \int_0^t \mathcal{B}^* \Phi \, dW(r),
$$

where $\mathcal{B}^*$ denotes the adjoint of $\mathcal{B}$ with respect to the $L^2(D)$ inner product.

Since $\theta := \min\{2, \alpha\}$, using the continuity of $\mathcal{B}^*$ on $L^2(D)$, $(\nabla \mathcal{B})^*$ on $L^\alpha(D)$ and the Burkholder-Davis-Gundy inequality, one has

$$
E \left( \sup_{(0,T)} \int_D |\mathcal{B}^*(v(t) - v_0)|^2 \, dx \right) \lesssim E \left( \sup_{(0,T)} \int_D |\nabla \mathcal{B}^*(H_1 + H_2)|^\alpha \, dx \right) + \int_0^T \mathcal{B}^* \Phi \, dW(r).
$$

Then $p \in L^\theta(\Omega; L^\infty(0, T; L^\theta(D)))$.

Let $\Delta_D^{-2}$ be the solution operator to the bi-Laplace equation with respect to zero boundary values for function and gradient. Let $p_0 = \Delta \Delta_D^{-2} \Delta p$ and $p_h = p - p_0$. Using the continuity of the operator $\Delta \Delta_D^{-2} \Delta$ from $L^\theta(D)$ to $L^\theta(D)$ (see [37]), we have

$$
E \left( \sup_{t \in (0,T)} \int_D |p_0|^\theta \, dx \right) \lesssim E \left( \sup_{t \in (0,T)} \int_D |v|^2 \, dx \right) + \int_D |v_0|^2 \, dx + \int_Q |H_1 + H_2|^\alpha \, dx \, dt + \int_0^T \|\Phi\|_{L^2(\Omega; L^\alpha(D))}^2 \, dt.
$$

$$
E \left( \sup_{t \in (0,T)} \int_D |p_h|^\theta \, dx \right) \lesssim E \left( \sup_{t \in (0,T)} \int_D |v|^2 \, dx \right) + \int_D |v_0|^2 \, dx + \int_Q |H_1 + H_2|^\alpha \, dx \, dt + \int_0^T \|\Phi\|_{L^2(\Omega; L^\alpha(D))}^2 \, dt.
$$

Note that $p_0(t) \in \Delta W^{2,\theta}_0(D)$ is uniquely determined as the solution to the following equation:

$$
\int_D p_0(t) \Delta \varphi \, dx = \int_0^t \int_D (H_1 + H_2) : \nabla^2 \varphi \, dx \, dr - \int_0^t \int_D \Phi \cdot \nabla \varphi \, dx \, dW(r),
$$

for all $\varphi \in C_0^\infty(D)$.

From [37], we know that $p_1 \in \Delta W^{2,\alpha_1}_0(D)$ and $p_2 \in \Delta W^{2,\alpha_2}_0(D)$ are the unique solutions (defined $\mathbb{P} \otimes \mathcal{L}^1$-a.e.) such that

$$
\int_D p_1(t) \Delta \varphi \, dx = \int_D H_1 : \nabla^2 \varphi \, dx,
$$

$$
\int_D p_2(t) \Delta \varphi \, dx = \int_D H_2 : \nabla^2 \varphi \, dx,
$$

(3.6)
for all $\varphi \in C_0^\infty(D)$. Then we have
\begin{equation*}
\int_D (p_1(t) + p_2(t)) \Delta \varphi dx = \int_D (H_1 + H_2) : \nabla^2 \varphi dx,
\end{equation*}
for all $\varphi \in C_0^\infty(D)$ and $p_1 + p_2 \in \Delta W^{2,\beta}_0(D)$. From Lemma 2.3 in \[43\], it follows that
\begin{equation*}
\int_D |p_1|^{a_1} dx \leq c \int_D |H_1|^{a_1} dx,
\end{equation*}
\begin{equation*}
\int_D |p_2|^{a_2} dx \leq c \int_D |H_2|^{a_2} dx,
\end{equation*}
\begin{equation*}
\int_D |p_1 + p_2|^{a} dx \leq c \int_D |H_1 + H_2|^{a} dx \quad \mathbb{P} \otimes \mathcal{L}^1 - a.e..
\end{equation*}
These imply
\begin{equation*}
E \left( \int_Q |p_1|^{a_1} dx dt \right)^\beta \leq cE \left( \int_Q |H_1|^{a_1} dx dt \right)^\beta,
\end{equation*}
\begin{equation*}
E \left( \int_Q |p_2|^{a_2} dx dt \right)^\beta \leq cE \left( \int_Q |H_2|^{a_2} dx dt \right)^\beta,
\end{equation*}
\begin{equation*}
E \left( \int_0^T \int_{D'} |\nabla p_2|^{a_2} dx dt \right)^\beta \leq cE \left( \int_Q |H_2|^{a_2} + |\nabla H_2|^{a_2} dx dt \right)^\beta,
\end{equation*}
\begin{equation*}
E \left( \int_Q |p_1 + p_2|^{a} dx dt \right) \leq cE \left( \int_Q |H_1 + H_2|^{a} dx dt \right).\end{equation*}

Let $p_\Phi := p_\Phi(t) - \int_0^t (p_1 + p_2) dr \in \Delta W^{2,\beta}_0(D)$. From (3.5), (3.6) and (3.7), it follows that $p_\Phi$ is the unique solution to
\begin{equation*}
\int_D p_\Phi(t) \Delta \varphi dx = \int_0^t \int_D \Phi \cdot \nabla \varphi dx dW(r),
\end{equation*}
for all $\varphi \in C_0^\infty(D)$. Since $p_\Phi(t) \in \Delta W^{2,\beta}_0(D)$, by Weyl’s Lemma, for all $\varphi \in C_0^\infty(D)$, we have
\begin{equation*}
\int_D p_\Phi(t) \varphi dx = \int_0^t \int_D \Phi \cdot \nabla (\Delta^{-1} \Delta \varphi) dx dW(r).
\end{equation*}
Then $p_\Phi = \int_0^t \mathcal{D}^* \Phi dW(r), \quad \mathbb{P} \otimes \mathcal{L}^{n+1} - a.e.,$ where $\mathcal{D} = \nabla \Delta^{-1} \Delta : L^2(D) \to W^{1,2}_0(D), \quad \mathcal{D}^* : L^2(D) \to L^2(D)$. Using the Burkholder-Davis-Gundy inequality, we obtain
\begin{equation*}
E \left( \sup_{t \in (0,T)} \int_D |p_\Phi|^2 dx \right) \leq cE \left( \sup_{t \in (0,T)} \|\mathcal{D}^* \Phi\|_{L^1(U, L^2(D))}^2 dt \right)
\leq cE \left( \sup_{t \in (0,T)} \|\Phi\|_{L^2(U, L^2(D))}^2 dt \right). \tag{3.8}
\end{equation*}

Finally, we can infer that $p_\Phi(t) := p_\Phi(t) + \int_0^t (p_1 + p_2) dr$ solves (3.5) and there holds $\tilde{p}_\Phi(t) \in \Delta W^{2,\beta}_0(D)$ which implies $p_0(t) := p_\Phi(t) + \int_0^t (p_1 + p_2) dr$. Then, we get the equation claimed in (2) of Theorem 3.1.

**Corollary 3.1.** Let the assumptions of Theorem 3.1 be satisfied. There exists $\Phi_p \in L^2(\Omega; L^\infty(0,T; L_2(U, L^2_{loc}(D))))$ progressively measurable such that
\begin{equation*}
\int_D p_\Phi(t) \text{div} \varphi dx = \int_0^t \int_D \Phi_p \cdot \varphi dx dW(r), \quad \forall \varphi \in C_0^\infty(D).
\end{equation*}
Let \( D' \subset D \), then \( \Phi \) satisfies \( \| \Phi \|_{L^2(D')} \leq c(D') \| \Phi \|_{L^2(D)} \), \( \forall k \), that is, it holds that
\[
P \otimes L^1\text{-a.e.}
\]
\[
\| \Phi_p \|_{L^2(U;L^2(D'))} \leq c(D') \| \Phi \|_{L^2(U;L^2(D))}.
\]
If we assume that \( \Phi \) satisfies (2.1) and (2.2), then there holds
\[
\| \Phi_p(v_1) - \Phi_p(v_2) \|_{L^2(U;L^2(D'))} \leq c(D') \| v_1 - v_2 \|_{L^2(D)}, \quad \forall v_1, v_2 \in L^2(D).
\]

**Proof.** From the proof of Theorem 3.1, it follows that
\[
\int_D p_\Phi(t) \text{div} \varphi \, dx = \int_0^t \int_D \Phi \cdot \nabla (\Delta^{-2} \text{div} \varphi) \, dx \, dW(r)
\]
\[
= \sum_k \int_0^t \int_D \Phi_{e_k} \cdot \nabla (\Delta^{-2} \text{div} \varphi) \, dx \, d\beta_k
\]
\[
= \sum_k \int_0^t \int_D \nabla \Delta^{-2} \text{div} \Phi_{e_k} \cdot \varphi \, dx \, d\beta_k
\]
\[
= \int_0^t \int_D \nabla \Delta^{-2} \text{div} \Phi \cdot \varphi \, dx \, dW(r),
\]
for all \( \varphi \in C^\infty(D) \). Let \( \Phi_p = \nabla \Delta^{-2} \text{div} \Phi \). Then we can get the first claim. By using the local regularity theory for the bi-Laplace equation in \( \mathbb{P} \), we can prove the rest results. \( \Box \)

4. **The approximated System**

Let us consider the following approximate system:
\[
\begin{cases}
 du + \nabla \cdot (u \otimes u - S + pI) \, dt + \varepsilon |u|^{q-2} u \, dt &= f \, dt + \Phi(u) \, dW, \\
u|_{t=0} = u_0,
\end{cases}
\]
for \( \varepsilon > 0 \), depending on the law \( \mu_0 \) on \( L^2(D) \) and \( \mu_f \) on \( L^2(Q) \).

Assume that \( f \) is adapted to \( \mathcal{F}_t \) (otherwise enlarge it) and \( f \in L^2(Q; L^2(D)) \) with \( \mu_f = \mathbb{P} \circ f^{-1} \) and \( u_0 \in L^2(Q; L^2(D)) \) with \( \mu_0 = \mathbb{P} \circ u_0^{-1} \). For the purpose of control the nonlinear term \( u \otimes u : \nabla u \), we add the term \( \varepsilon |u|^{\tilde{q}-2} u \) and choose \( \tilde{q} \geq \max\{2q', 3\} \) such that the solution is an admissible test function. Notice that \( \frac{2}{q} + \frac{1}{p} \leq 1 \) and \( \frac{1}{q-1} + \frac{1}{2} \leq 1 \). Let
\[
V_{q,\tilde{q}} = L^2(Q; L^\infty(0, T; L^2(D))) \cap L^q(\Omega \times Q; \mathbb{P} \otimes L^{n+1}) \cap L^q(\Omega; L^2(0, T; W^2_{0,\text{div}}(D))).
\]

From the appendix of \( \mathbb{P} \), we know that there exist a sequence \( \{\lambda_k\} \subset \mathbb{R} \) and a sequence of functions \( \{w_k\} \subset W^{\ell,2}_{0,\text{div}}(D), \ell \in \mathbb{N} \) such that
(a) \( w_k \) is an eigenvector to the eigenvalue \( \lambda_k \) of the Stokes-operator in the sense that
\[
\langle w_k, \varphi \rangle_{W^{\ell,2}_{0,\text{div}}} = \lambda_k \int_D w_k \cdot \varphi \, dx, \quad \forall \varphi \in W^{\ell,2}_{0,\text{div}}(D),
\]
(b) \( \int_D w_k w_m \, dx = \delta_{km}, \forall k, m \in \mathbb{N}, \)
(c) \( 1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \) and \( \lambda_k \to \infty, \)
(d) \( (\frac{w_k}{\sqrt{\lambda_k}}, \frac{w_m}{\sqrt{\lambda_m}})_{W^{\ell,2}_{0,\text{div}}} = \delta_{km}, \forall k, m \in \mathbb{N}, \)
(e) \( \{w_k\} \) is a basis of \( W^{\ell,2}_{0,\text{div}}(D). \)

Now, we use Galerkin approximation to separate space and time. Then approximate equations (1.1) becomes an ordinary stochastic differential equation. By using the classical existence theorems for SDEs from \( \mathbb{P} \), \( \mathbb{R} \), and \( \mathbb{I} \), we can prove the existence of approximated solution. To this end, choosing \( \ell > 1 + \frac{2}{q} \), such that \( W^{\ell,2}_{0,\text{div}}(D) \to W^{1,\infty}(D) \). We are finding an approximated solution:
\[
u^N = \sum_{k=1}^N c_k^N w_k = C^N \cdot w^N,
\]
where $C^N = (c^N_i) : \Omega \times (0,T) \to \mathbb{R}^N$ and $w^N = (w_1, w_2, \cdots , w_N)$.

Let $P^N : L^2_{\text{div}}(D) \to X := \text{span}\{w_1, w_2, \cdots , w_N\}$ be the orthogonal projection, i.e.,

$$P^N v = \sum_{k=1}^N \langle v, w_k \rangle_{L^2} \cdot w_k.$$ 

Therefore, we would like to solve the system

$$\int_D du^N \cdot w_k dx + \int_D S(x,t,\mathcal{D}(u^N)) : \mathcal{D}(w_k) dx dt + \varepsilon \int_D |u^N|^\alpha - 2 u^N \cdot w_k dx dt$$

$$= \int_D u^N \otimes u^N : \nabla w_k dx dt + \int_D f \cdot w_k dx dt + \int_D \Phi(u^N) \cdot w_k dx dW(t),$$

$$u^N(0) = P u_0,$$

$P$-a.s. for $k = 1, 2, \cdots , N$ and for a.e. $t$.

Assume that $W^N(s) = \sum_{k=1}^N \beta_k \epsilon_k(s) = \beta^N(s) \cdot e^N$. Then it turned out to solving the following ordinary stochastic differential equation:

$$\begin{cases}
dC^N = A(t,C^N)dt + B(C^N)d\beta^N_t, \\
C^N(0) = C_0,
\end{cases}$$

where

$$A(t,C^N) = \left( - \int_D S(x,t,C^N,\mathcal{D}(u^N)) : \mathcal{D}(w_k) dx + \int_D (C^N \cdot w^N) \otimes (C^N \cdot w^N) : \nabla w_k dx \right)_{k=1}^N$$

$$- \left( \varepsilon \int_D |C^N \cdot w^N|^\alpha - 2 (C^N \cdot w^N) \cdot w_k dx \right)_{k=1}^N + \left( \int_D f \cdot w_k dx \right)_{k=1}^N,$$

$$B(C^N) = \left( \int_D \Phi(C^N \cdot w^N) e_l \cdot w_k dx \right)_{k,l=1}^N,$$

$$C_0 = \left( \langle v_0, w_k \rangle_{L^2(D)} \right)_{k=1}^N.$$ 

In order to make use of the classical existence theorems for SDEs, we need to prove that $A$ and $B$ satisfy globally Lipschitz continuous condition and growth condition in the following. Note that

$$A(t,C^N) - A(t,\hat{C}^N) \cdot (C^N - \hat{C}^N)$$

$$= - \int_D (S(x,t,\mathcal{D}(u^N)) - S(x,t,\mathcal{D}(\hat{u}^N)) : \mathcal{D}(u^N) - \mathcal{D}(\hat{u}^N)) dx$$

$$+ \int_D (u^N \otimes u^N - \hat{u}^N \otimes \hat{u}^N) : (\mathcal{D}(u^N) - \mathcal{D}(\hat{u}^N)) dx$$

$$- \varepsilon \int_D (||u^N||^\alpha - 2 u^N - ||\hat{u}^N||^\alpha - 2 \hat{u}^N)(u^N - \hat{u}^N) dx$$

$$\leq \int_D (u^N \otimes u^N - \hat{u}^N \otimes \hat{u}^N) : (\mathcal{D}(u^N) - \mathcal{D}(\hat{u}^N)) dx.$$}

Here we have used the monotonicity assumption \([1.4]\). If $|C^N| \leq R$ and $|\hat{C}^N| \leq R$, then

$$(A(t,C^N) - A(t,\hat{C}^N)) \cdot (C^N - \hat{C}^N) \leq c(R,N)|C^N - \hat{C}^N|^2.$$ 

This implies weak monotonicity in the sense of (3.1.3) in \([39]\) by using Lipschitz continuity $B$ for $C^N$, cf \([21]\) and \([22]\). By virtue of $\int_D u^N \otimes u^N : \mathcal{D}(u^N) dx = 0$, \([1.3]\) and Hölder’s inequality, we have

$$A(t,C^N) \cdot C^N = - \int_D S(x,t,\mathcal{D}(u^N)) : \mathcal{D}(u^N) dx - \varepsilon \int_D |u^N|^\alpha dx + \int_D f(t) \cdot u^N dx$$
Itô’s formula

Lemma 4.1. **Proof.**

Then there holds uniformly in $c$

By using Hölder’s inequality and (2.1), one has

From (1.3), (4.5) and Korn’s inequality, it follows that

Next, we will get a priori estimate.

Lemma 4.1. *Under the assumption of (1.2), (1.3) and (1.4) with $q \in (1, \infty)$, (2.1), (2.2), $\tilde{q} \geq \{2q', 3\}$ and

\[
\int_{L^2(D)} |v|^2 d\mu_0(v) < \infty, \quad \int_{L^2(Q)} |g|^2 d\mu_f(g) < \infty, \tag{4.4}
\]

then there holds uniformly in $N$:

\[
E \left( \sup_{t \in (0,T)} \int_D |u^N(t)|^2 dx + \int_Q |\nabla u^N|^q dx dt + \varepsilon \int_Q |u^N|^{\tilde{q}} dx dt \right) \leq c \left(1 + \int_{L^2(D)} |v|^2 d\mu_0(v) + \int_{L^2(Q)} |g|^2 d\mu_f(g) \right),
\]

where $c$ is independent of $\varepsilon$.

*Proof.* Since $du^N = \sum_{k=1}^N dc^N_k \cdot w_k$, $\int_D u^N \otimes u^N : Du^N dx = 0$, $\int_D w_k w_m dx = \delta_{km}$, $\forall k, m \in \mathbb{N}$, and

\[
dc^N_k = -\int_D S(x, t, Du^N) : Du^N dx - \varepsilon \int_D |u^N|^{\tilde{q}-2} u^N \cdot w_k dx + \int_D u^N \otimes u^N : \nabla w_k dx + \int_D f \cdot w_k dx dt + \int_D \Phi(u^N) \cdot w_k dx \omega N(t),
\]

Itô’s formula $f(X) = \frac{1}{2} |X|^2$ yields

\[
\frac{1}{2} \|u^N(t)\|_{L^2(D)}^2 = \frac{1}{2} \|C^N(0)\|_{L^2(D)}^2 + \sum_{k=1}^N \int_0^t \int_D C^N_k d(C^N_k)(r) dx dr + \frac{1}{2} \sum_{k=1}^N \int_0^t d\langle C^N_k \rangle(r) dx dr
\]

\[
= \frac{1}{2} \|u_0\|_{L^2(D)}^2 + \int_0^t \int_D S(x, r, Du^N) : Du^N dx dr - \varepsilon \int_0^t \int_D |u^N|^{\tilde{q}} dx dr
\]

\[
+ \int_0^t \int_D f \cdot u^N dx dr + \int_0^t \int_D u^N \cdot \Phi(u^N) dx \omega N(r) \tag{4.5}
\]

From (1.3), (1.5) and Korn’s inequality, it follows that

\[
E \left( \int_D |u^N(t)|^2 dx + \int_0^t \int_D |\nabla u^N|^q dx dt + \varepsilon \int_0^t \int_D |u^N|^{\tilde{q}} dx dt \right)
\]

\[
\leq c \left[1 + I_1 + I_2 + I_3 + E \left(\|v_0\|_{L^2(D)}^2\right)\right],
\]
where
\[
I_1 = E \left( \int_0^t \int_D f \cdot u^N \, dx \, dr \right), \\
I_2 = E \left( \int_0^t \int_D u^N \cdot \Phi(u^N) \, dW^N(r) \right), \\
I_3 = E \left( \sum_{i=1}^N \int_0^t \int_D \Phi(u^N) e_i^2 \, dx \, dr \right).
\]

By using Young’s inequality, we have
\[
I_1 \leq \delta E \left( \int_0^t \int_D |u^N(t)|^2 \, dx \, dr \right) + c(\delta) E \left( \int_0^t \int_D |f|^2 \, dx \, dr \right) \text{ for } \forall \delta > 0.
\]

It is clear that \( I_2 = 0 \). Thanks to (2.1), we deduce that
\[
I_3 \leq E \left( \sum_{i=1}^N \int_0^t \int_D |g_i(u^N)|^2 \, dx \, dr \right)
\leq E \left( 1 + \int_0^t \int_D |u^N|^2 \, dx \, dr \right).
\]

Then, by interchanging the time-integral and the expectation value and using Gronwall’s inequality, we obtain
\[
E \left( \sup_{t \in (0,T)} \int_D |u^N(t)|^2 \, dx \right) + E \left( \int_Q |\nabla u^N|^q \, dx \, dt \right)
\leq cE \left( 1 + \int_D |u_0|^2 \, dx + \int_Q |f|^2 \, dx \, dt \right).
\tag{4.6}
\]

Similarly, we have
\[
E \left( \sup_{t \in (0,T)} \int_D |u^N(t)|^2 \, dx \right) \leq cE \left( 1 + \int_D |u_0|^2 \, dx + \int_Q |f|^2 \, dx \, dt + \int_0^T \int_D |u^N|^2 \, dx \, dt \right)
+ E \left( \sup_{t \in (0,T)} \left| \int_0^t \int_D u^N \cdot \Phi(u^N) \, dW^N(t) \right| \right).
\tag{4.7}
\]

Using Burkholder-Davis-Gundy inequality, Hölder’s inequality, Young’s inequality and (2.1), one has
\[
E \left( \sup_{t \in (0,T)} \left| \int_0^t \int_D u^N \cdot \Phi(u^N) \, dW^N(r) \right| \right)
\leq cE \left( \sup_{t \in (0,T)} \left| \sum_i \int_0^t \int_D u^N \cdot \Phi(u^N) e_i \, dx \, \beta_i(r) \right| \right)
\leq E \left( \sup_{t \in (0,T)} \left| \sum_i \int_0^t \int_D u^N \cdot g_i(u^N) \, dx \, \beta_i(r) \right| \right)
\leq cE \left[ \int_0^T \sum_i \left( \int_D u^N \cdot g_i(u^N) \, dx \right)^2 \, dt \right]^{\frac{1}{2}}
\leq cE \left[ \int_0^T \left( \sum_i \int_D |u^N|^2 \, dx \cdot \int_D |g_i(u^N)|^2 \, dx \right) \, dt \right]^{\frac{1}{2}}
\]

\[
\leq \delta E \left( \sup_{t \in (0,T)} \int_D |u^N|^2 \, dx \right) + c(\delta) E \left( 1 + \int_0^T \int_D |u^N|^2 \, dx \, dt \right).
\]

For \( \delta \) sufficiently small, this together with (1.6) yield Lemma 4.1. \qed

**Lemma 4.2.** Assume that (1.2)-(1.4) with \( q \in (1, \infty) \), (2.1), (2.2), \( \tilde{q} \geq \{2q', 3\} \) and (4.4) hold. Then

(1) There exists a martingale weak solution \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), \mathcal{V}, \mathcal{V}_0, \mathcal{F}, \mathbb{W})\) to (4.1) in the sense that:

(a) \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) is a stochastic basis with a complete right-continuous filtration;
(b) \(\mathbb{W}\) is an \(\mathcal{F}_t\)-cylindrical Wiener process;
(c) \(\mathcal{V}\) is independent of \(\mathcal{V}_0\); where
(d) \(\mathcal{V}_0 \in L^2(\Omega; L^2(D))\) with \(\mu_0 = \mathbb{P} \circ \mathcal{V}_0^{-1}\);
(e) \(\mathcal{F}\) is adapted to \(\mathcal{F}_t\) and \(\mu_f = \mathbb{P} \circ \mathcal{F}^{-1}\);
(f) \(\forall \phi \in C_0^{\infty}(D)\) and \(\forall t \in [0,T]\), it holds that \(\mathbb{P}\)-a.s.

\[
\int_0^t \int_D (\mathbb{V}(t) - \mathbb{V}_0) \cdot \phi \, dx + \varepsilon \int_0^t \int_D |\nabla \mathbb{V}|^2 \phi \, dx \, dt - \int_0^t \int_D \mathbb{V} \otimes \mathbb{V} : \mathbb{D}(\phi) + S(x, r \mathbb{D}(\mathcal{V})) : \mathbb{D}(\phi) \, dx \, dr
\]

\[
= \int_0^t \int_D \mathbb{F} \cdot \phi \, dx \, dr + \int_0^t \int_D \mathbb{F}(\mathcal{V}) \cdot \phi \, dx \, dr,
\]

(2) There holds

\[
\mathbb{E} \left( \sup_{t \in (0,T)} \int_D |\mathbb{V}(t)|^2 \, dx + \int_Q |\nabla \mathbb{V}|^2 \, dx \, dt + \varepsilon \int_Q |\mathbb{V}|^2 \, dx \, dt \right)
\]

\[
\leq c \left( 1 + \int_{L^2(D)} \|v\|_{L^q(D)} \, d\mu_0(v) + \int_{L^2(Q)} \|g\|_{L^q(Q)} \, d\mu_f(g) \right),
\]

where \( c \) is independent of \( \varepsilon \).

**Proof.** Let \( S(u) = \varepsilon|u|^{\tilde{q} - 2} u \), from Lemma 4.1, we know that there exist functions \( u \in V_{\tilde{q}, \tilde{q}} \) and functions \( \tilde{S} \) and \( \tilde{S} \), such that

\[
u^N \to u \quad \text{in} \quad L^q(\Omega; L^q(0,T; W_{0,\text{div}}^{1,q}(D))),
\]

\[
u^N \to u \quad \text{in} \quad L^{\tilde{q}}(\Omega; L^{\tilde{q}}(Q)),
\]

\[
S(\nu^N) \to \tilde{S} \quad \text{in} \quad L^{\tilde{q}}(\Omega; L^{\tilde{q}}(Q)),
\]

\[
S(x, t, \mathbb{D}(\nu^N)) \to \tilde{S} \quad \text{in} \quad L^{\tilde{q}}(\Omega; L^{\tilde{q}}(Q)),
\]

\[
u^N \otimes u^N \to \tilde{U} \quad \text{in} \quad L^{\tilde{q}/2}(\Omega; L^{\tilde{q}/2}(Q)),
\]

\[
\Phi(\nu^N) \to \tilde{\Phi} \quad \text{in} \quad L^{2}(\Omega; L^{2}(0,T; L_2(U, L^{q}(D)))
\]

In order to prove

\[
\tilde{U} = u \otimes u, \quad \tilde{\Phi} = \Phi(u),
\]

we will use some compactness arguments similar to the ideas from [23 Sec.4]. Let \( \mathcal{P}_t^N \) denotes the projection from \( W_{0,\text{div}}^{1,2}(D) \) into \( \mathcal{X}_N \). By using (1.4), we have

\[
\int_D u^N \cdot \phi \, dx = \int_D u^N(t) \cdot \mathcal{P}_t^N \phi \, dx
\]
For the stochastic term, using (2.1), (2.2) and Lemma 4.1, for any $\vartheta > 0$, where

$$W_5 \in L^{q_0}((0,T;\mathbb{P}) \otimes \mathcal{L}^{n+1}), \quad q_0 := \min \left\{ q', \frac{q'}{2}, \frac{q'}{4} \right\} > 1,$$

uniformly in $N$. Let

$$\mathcal{H}(t,\varphi) = \int_0^t \left( H_1^N + H_2^N \right) : \nabla \mathcal{P}_t^N \varphi dxdr, \quad \varphi \in C^{\infty}_{0,\text{div}}(\Omega).$$

By the fact $W^{\tilde{\ell},0}(D) \hookrightarrow W^{\tilde{\ell},0}(D)$ for $\tilde{\ell} \geq \ell + n(1 + \frac{2}{q_0})$ and (4.16), we have

$$E \left( \| \mathcal{H} \|_{W^{1,q_0}(0,T;W^{\tilde{\ell},0}(D))} \right) \leq c.$$

For the stochastic term, using (2.1), (2.2) and Lemma 4.1, for any $\vartheta > 2$, one has

$$E \left( \left\| \int_s^t \Phi(u^N) dW^N(r) \right\|_{L^2(D)}^{\vartheta} \right) \leq c(t-s)^{\frac{\vartheta}{2}}.$$

Thanks to the Kolmogorov continuity criterion [14], we can infer that for any $\lambda \in [0, 1/2)$,

$$E \left( \left\| \int_0^t \Phi(u^N) dW^N(r) \right\|_{C^\lambda([0,T];L^2(D))} \right) \leq c,$$

Then

$$E \left( \left\| u^N \right\|_{C^\lambda([0,T];W^{-\tilde{\ell},q_0}(D))} \right) \leq c,$$

and

$$E \left( \left\| u^N \right\|_{W^{\lambda,q_0}(0,T;W^{-\tilde{\ell},q_0}(D))} \right) \leq c,$$

for some $\lambda > 0$. Note that an interpolation with $L^{q_0}(0,T;W^{1,q_0}_{\text{div}}(D))$ yields for some $\kappa > 0$

$$E \left( \left\| u^N \right\|_{L^{q_0}(0,T;L^{\kappa}_{\text{div}}(D))} \right) \leq c.$$  

Now, we prepare the setup for our compactness method. Define the path space of $(u^N, W, u_0, f)$ by

$$\mathcal{V} := L^\gamma(0,T;L^\gamma(D)) \times C([0,T],U_0) \times L^2(\text{div})(D) \times L^2(Q).$$

Let us denote by $\mu_{u^N}$ the law of $u^N$ on $\mathcal{V}$. By $\mu_W$, we denote the law of $W$ on $C([0,T],U_0)$. The joint law of $u^N$, $W$, $u_0$ and $f$ on $\mathcal{V}$ is denoted by $\mu^N$.

**Proposition 4.1.** The set $\{\mu^N| N \in \mathbb{N}\}$ is tight on $\mathcal{V}$.

**Proof.** In order to prove the tightness of $\mu^N$, we need the following three steps.

Step 1: Tightness of $\mu_{u^N}$. On account of $L^\tilde{q} \hookrightarrow L^{\tilde{q}_0}$, if $-\frac{\tilde{q}}{\gamma} < -\frac{\tilde{q}_0}{\gamma}$, we can use Theorem 5.2 [11] to obtain

$$W^{\kappa,q_0}(0,T;L^{q_0}_{\text{div}}(D)) \cap V_{\tilde{q},\tilde{q}_0} \hookrightarrow L^\gamma(0,T;L^\gamma_{\text{div}}(D))$$
compactly for all \( q_0 < \gamma < \tilde{q} \). We consider the ball \( B_R \) in the space \( W^{\kappa, q_0}(0, T; L_{div}^{q_0}(D)) \cap V_{q,\tilde{q}} \) and let \( B_R^c \) be the complement of the ball. Using Lemma 4.1 and (4.19), we have

\[
\mu_u(N(B_R^c)) = \mathbb{P}(\|u\|_{W^{\kappa,q_0}(0, T; L_{div}^{q_0}(D))} + \|u\|_{V_{q,\tilde{q}}} \geq R) \leq \frac{1}{R} E \left( \sup_{[0,T]} \|W(r)\|_{U_0} \right) \leq \frac{c}{R}.
\]

Then, there exists \( R(\eta) \) such that

\[
\mu_u(N(B_R(\eta))) \geq 1 - \frac{\eta}{4},
\]

for a fixed \( \eta > 0 \). These yield the tightness of \( \mu_u\).

Step 2: Tightness of \( \mu_W \). We consider the ball \( B_R \) in the space \( C([0, T]; U_0) \) and let \( B_R^c \) be the complement of the ball. Then

\[
\mu_W(B_R^c) = \mathbb{P}(\|W\|_{C([0, T]; U_0)} \geq R) \leq \frac{1}{R} E \left( \sup_{[0,T]} \|W(r)\|_{U_0} \right) \leq \frac{c}{R}.
\]

Then, there exists \( R(\eta) \) such that

\[
\mu_W(B_R(\eta)) \geq 1 - \frac{\eta}{4},
\]

for a fixed \( \eta > 0 \). These imply the tightness of \( \mu_W \).

Step 3: Tightness of \( \mu_0, \mu_f \). We consider the ball \( B_R \) in the space \( L_{div}^2(D) \) and let \( B_R^c \) be the complement of the ball. Therefore

\[
\mu_0(B_R^c) = \mathbb{P}(\|u_0\|_{L_{div}^2(D)} \geq R) \leq \frac{1}{R} E \left( \|u_0\|_{L_{div}^2(D)} \right) \leq \frac{c}{R}.
\]

Then, there exists \( R(\eta) \) such that

\[
\mu_0(B_R(\eta)) \geq 1 - \frac{\eta}{4},
\]

for a fixed \( \eta > 0 \). These yield the tightness of \( \mu_0 \).

We consider the ball \( B_R \) in the space \( L^2(Q) \) and let \( B_R^c \) be the complement of the ball. Then we have

\[
\mu_f(B_R^c) = \mathbb{P}(\|u_f\|_{L^2(Q)} \geq R) \leq \frac{1}{R} E \left( \|u_f\|_{L^2(Q)} \right) \leq \frac{c}{R}.
\]

Then, there exists \( R(\eta) \) such that

\[
\mu_f(B_R(\eta)) \geq 1 - \frac{\eta}{4},
\]

for a fixed \( \eta > 0 \). These imply the tightness of \( \mu_f \).

So we can find a compact subset \( \mathcal{V}_\eta \subset \mathcal{V} \) such that \( \mu^N(\mathcal{V}_\eta) \geq 1 - \eta \). Thus, \( \{\mu^N|N \in \mathbb{N}\} \) is tight in the same space.

\( \square \)

Thanks to Prokhorov’s Theorem in [24], we can infer that \( \mu^N \) is also relatively weakly compact. Then \( \mu_n \to \mu \) weakly. By the Skorohod representation theorem in [24], we know that the following result.

**Proposition 4.2.** There exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with \( \mathcal{V} \)-valued Borel measurable random variables \((\bar{\mu}^N, \bar{\mu}_0^N, \bar{\mu}_0^N, \bar{\mu}^N) \) and \((\bar{\mu}, \bar{\mu}_0, \bar{\mu}, \bar{\mu}) \) such that the following hold:

\( \blacklozenge \) The laws of \((\bar{\mu}^N, \bar{\mu}_0^N, \bar{\mu}_0^N, \bar{\mu}^N) \) and \((\bar{\mu}, \bar{\mu}_0, \bar{\mu}, \bar{\mu}) \) under \( \mathbb{P} \) coincide with \( \mu^N \) and \( \mu \).

\( \blacklozenge \)

\( \bar{\mu}^N \to \bar{\mu} \) in \( L^1(0, T; L^1(D)) \) \( \mathbb{P} \)-a.s.,

\( \bar{\mu}_0^N \to \bar{\mu}_0 \) in \( C([0, T], U_0) \) \( \mathbb{P} \)-a.s.,
\[ \pi_0^N \rightarrow \pi_0 \text{ in } L^2(D; \mathbb{P}) \text{ a.s.,} \]

\[ \mathcal{F}^N \rightarrow \mathcal{F} \text{ in } L^2(0, T; L^2(D)) \text{ a.s.} \]

**Theorem**: The convergence in \[\text{L}^2\] and \[\text{L}^1\] still hold for the corresponding functions defined on \((\Omega, \mathcal{F}, \mathbb{P})\). Moreover, we have

\[ \int_{[0,T]} \left( \sup_{t \in [0,t]} ||W^N(t)||^\alpha_{U_0} \right) d\mathbb{P} = \int_{\Omega} \left( \sup_{t \in [0,T]} W(t)^\alpha \right) d\mathbb{P} \text{ for all } \alpha < \infty. \]

By Vitali’s convergence Theorem, for all \( \gamma < \beta \), we have

\[ W^N \rightarrow W \text{ in } L^2(\Omega; C([0,T], U_0)), \]

\[ \pi^N \rightarrow \pi \text{ in } L^1(\Omega \times Q; \mathbb{P} \times \mathcal{L}^{n+1}), \]

\[ \pi_0^N \rightarrow \pi_0 \text{ in } L^2(\Omega \times D; \mathbb{P} \times \mathcal{L}^{n+1}), \]

\[ \mathcal{F}^N \rightarrow \mathcal{F} \text{ in } L^2(\Omega \times Q; \mathbb{P} \times \mathcal{L}^{n+1}), \]

after choosing a subsequence.

Now, we are going to show that the approximated equations also hold on the new probability space. To this end, we define

\[ \xi^N(t) = \int_D (u^N(t) - u_0) \cdot \varphi dx - \int_0^T \int_D u^N \otimes u^N : \nabla \mathcal{P}_N \varphi dx dt + \int_0^T \int_D S(u^N) \cdot \mathcal{P}_N \varphi dx dt \]

\[ + \int_0^T \int_D S(x, r, \mathcal{D}(u^N)) : \mathbb{D} \left( \mathcal{P}_N \varphi - f \cdot \mathcal{P}_N \varphi \right) dx dt - \int_0^T \int_D \Phi(u^N) \cdot \mathcal{P}_N \varphi dx dt \]

\[ Z^N = \int_0^T ||\xi^N(t)||^2_{W_{\text{div}}^N(D)} dt. \]

Of course

\[ Z^N = 0, \mathbb{P} \text{ a.s.} \]

Let

\[ \xi^N(t) = \int_D (\pi^N(t) - \pi_0^N) \cdot \varphi dx - \int_0^T \int_D \pi^N \otimes \pi^N : \nabla \mathcal{P}_N \varphi dx dt + \int_0^T \int_D S(\pi^N) \cdot \mathcal{P}_N \varphi dx dt \]

\[ + \int_0^T \int_D S(x, r, \mathcal{D}(\pi^N)) : \mathbb{D} \left( \mathcal{P}_N \varphi - \mathcal{F}^N \cdot \mathcal{P}_N \varphi \right) dx dt - \int_0^T \int_D \Phi(\pi^N) \cdot \mathcal{P}_N \varphi dx dt \]

\[ Y^N = \int_0^T ||\xi^N(t)||^2_{W_{\text{div}}^N(D)} dt. \]

We want to verify that

\[ \mathcal{E}Y^N = 0. \]

To this end, we have the following Proposition:

**Proposition 4.3**: \( Y^N = 0, \mathbb{P} \) a.s., that is, \((\pi^N, \pi_0^N, \mathcal{F}^N, W^N)\) satisfies the equation \((4.1)\).

**Proof**: The difficulty comes from \( Z_n \) is not expressed as a deterministic function of \((u^N, W^N)\) because of the presence of the stochastic integral. By Theorem 2.4 and Corollary 2.5 in \[10\], we can infer that

\[ \mathcal{L}(\pi^N, \pi_0^N, \mathcal{F}^N, W^N, \xi^N) = \mathcal{L}(u^N, u_0^N, f^N, W^N, \xi^N). \]

\[ (4.24) \]
Here \( \mathcal{L}(f) \) is the probability distribution of \( f \). Note that \( Y^N \) is continuous as a function of \( \xi^N \). In view of (4.24) and the continuity of \( Y^N \), one deduces that the distribution of \( Y^N \) is equal to the distribution of \( Z^N \) on \( \mathbb{R}_+ \), that is,
\[
\mathbb{P}\phi(Y^N) = E\phi(Z^N),
\]
(4.25)
for any \( \phi \in C_b(\mathbb{R}_+) \), where \( C_b(X) \) is the space of continuous bounded functions defined on \( X \). Now, let \( \varepsilon > 0 \) be an arbitrary number and \( \phi_\varepsilon \in C_b(\mathbb{R}_+) \) defined by
\[
\phi_\varepsilon = \begin{cases} 
\frac{\varepsilon}{2}, & 0 \leq y < \varepsilon; \\
1, & y \geq \varepsilon.
\end{cases}
\]

One can check that
\[
\mathbb{P}(Y^N \geq \varepsilon) = \int_\Omega 1_{[\varepsilon, \infty)} Y^N d\mathbb{P} \leq \int_\Omega 1_{[0, \varepsilon]} Y^N d\mathbb{P} + \int_\Omega 1_{[\varepsilon, \infty)} Y^N d\mathbb{P},
\]
Hence by the definition of \( \mathbb{E}\phi_\varepsilon(Y^N) \), we can infer that
\[
\mathbb{P}(Y^N \geq \varepsilon) \leq \mathbb{E}\phi_\varepsilon(Y^N),
\]
which together with (4.25) imply that
\[
\mathbb{P}(Y^N \geq \varepsilon) \leq E\phi_\varepsilon(Z^N),
\]
By the fact that \((u^N, u_0^N, f^N, W^N)\) satisfies the Galerkin equation, from the above inequality, it holds that
\[
\mathbb{P}(Y^N \geq \varepsilon) \leq E\phi_\varepsilon(Z^N) = 0,
\]
(4.26)
for any \( \varepsilon > 0 \). Since \( \varepsilon > 0 \) is arbitrary, from (4.25), we can infer that
\[
Y^N = 0, \quad \mathbb{P} - a.s.
\]
(4.27)
It follows from (4.26) that \((\pi^N, \pi_0^N, f^N, W^N)\) satisfies the equation (4.1). \( \square \)

Since \( W^N \) has the same law as \( W \), there exists a collection of mutually independent real-valued \( \mathcal{F}_t \)-Wiener process \( \{\beta^N_k\}_k \) such that \( W^N = \sum_k \beta^N_k e_k \), i.e., there exists a collection of mutually independent real-valued \( \mathcal{F}_t \)-Wiener process \( \{\beta_k\}_k \geq 1 \) such that \( W = \sum_k \beta_k e_k \). We denote \( \mathcal{W}^N := \sum_{k=1}^N e_k \beta^N_k \). Proposition (4.3) means the equations
\[
\int_D d\pi^N \cdot w_k dx + \int_D S(x, t, \mathbb{D}(\pi^N)) : \mathbb{D}(w_k) dxdt + \varepsilon \int_D |\pi^N|^{q-2} \pi^N \cdot w_k dxdt \\
= \int_D \pi^N \otimes \pi^N : \nabla(w_k) dxdt + \int_D f(w_k) dxdt + \int_D S(\pi^N) \cdot w_k dx \mathcal{W}^N(t),
\]
(4.28)
\[
\pi^N(0) = \mathcal{P}^N \pi_0,
\]
\((k = 1, 2, \cdots, N)\) holds on the new probability space \((\Omega, \mathcal{F}, \mathbb{P})\). At the same time, we have
\[
\pi^N \to \pi \quad \text{in} \quad L^q(\Omega; L^q(0, T; W_{0, \text{div}}^1(D))), \]
(4.29)
\[
\pi^N \to \pi \quad \text{in} \quad L^q(\Omega; L^q(Q)),
\]
(4.30)
\[
S(\pi^N) \to S(\pi) \quad \text{in} \quad L^{q'}(\Omega; L^{q'}(Q)),
\]
(4.31)
\[
S(x, t, \mathbb{D}(\pi^N)) \to S(\pi) \quad \text{in} \quad L^{q'}(\Omega; L^{q'}(Q)),
\]
(4.32)
\[
S(x, t, \mathbb{D}(\pi^N)) \to S(\pi) \quad \text{in} \quad L^{q'}(\Omega; L^{q'}(0, T; W_{0, \text{div}}^{-1,q'}(D))),
\]
(4.33)
\[
\pi^N \otimes \pi^N \to \pi \otimes \pi \quad \text{in} \quad L^{q/2}(\Omega; L^{q/2}(Q)),
\]
(4.34)
\[
\Phi(\pi^N) \to \Phi(\pi) \quad \text{in} \quad L^2(\Omega; L^2(0, T; L_2(U, L^2(D)))),
\]
(4.35)
By using (4.20) - (4.29), one has
\[
\int_D (\overline{u}(t) - \overline{u}_0) \cdot \varphi dx + \int_0^t \int_D \tilde{S} \cdot \nabla \varphi dxdr + \int_0^t \int_D S(\overline{u}) \cdot \varphi dxdr = \int_0^t \int_D \overline{\sigma} \otimes \overline{u} : \nabla \varphi dxdr \\
+ \int_0^t \int_D \overline{f} \cdot \varphi dxdr + \int_0^t \int_D \Phi(\overline{u}^N) \cdot \varphi dxd\overline{W}(r),
\]
(4.36)
for all \( \varphi \in C_{0, \text{div}}^\infty(D) \). It’s worth noting that the limits in the stochastic term is gained by
\[
\overline{W}^N \to \overline{W} \quad \text{in} \quad C([0, T], U_0),
\]
\[
\Phi(\overline{u}^N) \to \Phi(\overline{u}) \quad \text{in} \quad L^2(0, T; L^2(D))
\]
in probability. By using Lemma 2.1 in [16], we have
\[
\int_0^t \Phi(\overline{u}^N) d\overline{W}^N(s) \to \int_0^t \Phi(\overline{u}) d\overline{W}(s) \quad \text{in} \quad L^2(0, T; L^2(D)),
\]
in probability. Finally, we prove
\[
\overline{S} = S(x, t, \mathbb{D}(\overline{u})).
\]
(4.37)
It follows from equation (4.36), \( \int_D \overline{u} \otimes \overline{u} : \mathbb{D}(\overline{u}) dx = 0 \) and Itô’s formula that
\[
\frac{1}{2} \| \overline{u}(t) \|^2_{L^2(D)} = \frac{1}{2} \| \overline{u}_0 \|^2_{L^2(D)} - \int_0^t \int_D \tilde{S} \cdot \mathbb{D}(\overline{u}) dxdr - \int_0^t \int_D S(\overline{u}) \cdot \overline{u} dxdr \\
+ \int_0^t \int_D \overline{f} \cdot \overline{u} dxdr + \int_0^t \int_D \Phi(\overline{u}) \cdot \overline{u} dxd\overline{W}(r) \\
+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_D |\Phi(\overline{u}) e_i|^2 dxdr.
\]
Similarly,
\[
\frac{1}{2} \| \overline{u}^N(t) \|^2_{L^2(D)} = \frac{1}{2} \| \overline{u}^N_0 \|^2_{L^2(D)} - \int_0^t \int_D S(x, r, \mathbb{D}(\overline{u}^N)) : \mathbb{D}(\overline{u}^N) dxdr \\
- \int_0^t \int_D S(\overline{u}^N) \cdot \overline{u}^N dxdr + \int_0^t \int_D \tilde{f} \cdot \overline{u}^N dxdr \\
+ \int_0^t \int_D \overline{u}^N \cdot \Phi(\overline{u}^N) dxd\overline{W}^{N,N}(r) + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_D |\Phi(\overline{u}^N) e_i|^2 dxdr.
\]
Subtracting these two equality and applying expectation, we get
\[
\mathbb{E} \left( \int_0^T \int_D \left( S(x, r, \mathbb{D}(\overline{u}^N)) - S(x, r, \mathbb{D}(\overline{u})) \right) : \mathbb{D}(\overline{u}^N - \overline{u}) dxdr \right) \\
+ \mathbb{E} \left( \int_0^T \int_D \left( S(\overline{u}^N) - S(\overline{u}) \right) : \mathbb{D}(\overline{u}^N - \overline{u}) dxdr \right) \\
= \frac{1}{2} \mathbb{E} \left( \int_D (|\overline{u}(T)|^2 - |\overline{u}^N(T)|^2) dx + \int_D (|\overline{u}^N_0|^2 - |\overline{u}_0|^2) dx \right) \\
+ \mathbb{E} \left( \int_0^T \int_D \left( \overline{S} - S(x, r, \mathbb{D}(\overline{u}^N)) \right) : \mathbb{D}(\overline{u}) dxdr - \int_0^T \int_D S(x, r, \mathbb{D}(\overline{u})) : \mathbb{D}(\overline{u}^N - \overline{u}) dxdr \right)
\]
\[ + \mathbb{E} \left( \int_0^T \int_D (S(\pi) - S(\pi^N)) : \pi dx dr - \int_0^T \int_D S(\pi) \cdot (\pi^N - \pi) dx dr \right) \]
\[ + \mathbb{E} \left( \int_0^T \int_D (\pi^N - \pi) dx dr \right) + \mathbb{E} \left( \frac{1}{2} \sum_{i=1}^N \int_0^T \int_D |\Phi(\pi^N) e_i|^2 dx dr \right) \]
\[ - \mathbb{E} \left( \frac{1}{2} \sum_{i=1}^N \int_0^T \int_D |\Phi(\pi) e_i|^2 dx dr \right). \]

By using (1.4), (4.29) and \( \lim \inf_{N \to \infty} \mathbb{E} \left[ \int_D \left( |\pi^N(T)|^2 - |\pi(T)|^2 \right) dx \right] \geq 0 \) which follows from the lower semi-continuity and weak convergence of \( \pi^N(T) \), we can infer that
\[ \lim_{N \to \infty} \mathbb{E} \left[ \int_0^T \int_D \left( S(x, r, D(\pi^N)) - S(x, r, D(\pi)) \right) : D(\pi^N - \pi) dx dr \right] \]
\[ \leq \frac{1}{2} \lim_{N \to \infty} \mathbb{E} \left[ \sum_{i=1}^N \int_0^T \int_D \left( |\Phi(\pi^N) e_i|^2 - |\Phi(\pi) e_i|^2 \right) dx dr \right]. \]

By (4.20), (4.21), (2.1) and (2.2), we have
\[ \mathbb{E} \left( \sum_{i=1}^N \int_0^T \int_D |\Phi(\pi^N) e_i|^2 dx dr \right) \to \mathbb{E} \left( \sum_{i=1}^N \int_0^T \int_D |\Phi(\pi) e_i|^2 dx dr \right), \]

after letting \( N \to \infty \). Then
\[ \lim_{N \to \infty} \mathbb{E} \left[ \int_0^T \int_D \left( S(x, r, D(\pi^N)) - S(x, r, D(\pi)) \right) : D(\pi^N - \pi) dx dr \right] = 0. \]

Thanks to (1.4) and the monotonicity of \( S \), we have
\[ D(\pi^N) \to D(\pi) \quad \mathbb{P} \otimes \mathcal{L}^{n+1} - \text{a.e.}. \]

This implies (4.37) and we complete the proof of Lemma 4.2. \( \square \)

**Corollary 4.1.** Let the assumptions of Lemma 4.2 be satisfied and
\[ \int_{L^2_{\text{div}}(D)} \|v\|_{L^2(D)}^\beta d\mu_0(v) < \infty, \int_{L^2(Q)} \|g\|_{L^2(Q)}^\beta d\mu_f(g) < \infty \]
for some \( \beta \geq 2 \). Then there exists a martingale weak solution to (4.1) such that
\[ E \left( \sup_{t \in (0, T)} \int_D |\pi(t)|^2 dx + \int_Q |\nabla \pi|^q dx dt + \varepsilon \int_Q |\pi|^\beta dx dt \right)^{\beta/2} \]
\[ \leq c E \left( 1 + \int_{L^2_{\text{div}}(D)} \|v\|_{L^2(D)}^2 d\mu_0(v) + \int_{L^2(Q)} \|g\|_{L^2(Q)}^2 d\mu_f(g) \right)^{\beta/2}, \]

where \( c \) is independent of \( \varepsilon \).
Proof. It follows from (4.35) that
\[
\frac{1}{2} E \left( \sup_{t \in (0,T)} \int_D |u_N^N(t)|^2 dx \right)^{\beta/2} + E \left( \int_Q |\nabla u_N|^q dx dt + \varepsilon \int_Q |u_N|^q dx dt \right)^{\beta/2} 
\leq E \left( 1 + \int_D |u_0|^2 dx + \int_0^T \int_D |f||u_N| dxdr \right)^{\beta/2} 
+ E \left( \sup_{t \in (0,T)} \left| \int_0^T u_N \cdot \Phi(u_N) dx dr \right| \right)^{\beta/2} 
+ E \left( \sum_{i=1}^N \int_0^T |\Phi(u_N e_i)|^2 dr \right)^{\beta/2}.
\]

In view of Young’s inequality, we obtain
\[
E \left( \int_0^T \int_D |f||u_N| dxdr \right)^{\beta/2} \leq c(\delta) E \left( \int_Q |f|^2 dx dt \right)^{\beta/2} + \delta E \left( \int_0^T \left( \int_D |u_N|^2 dx \right)^{\beta/2} dr \right)
\leq c(\delta) E \left( \int_Q |f|^2 dx dt \right)^{\beta/2} + \delta E \left( \sup_{t \in (0,T)} \int_D |u_N|^2 dx \right)^{\beta/2}.
\]

By the Burkholder-Davis-Gundy inequality, Hölder’s inequality and Young’s inequality, one deduces that
\[
E \left( \sup_{t \in (0,T)} \left| \int_0^t \int_D u_N \cdot \Phi(u_N) dx dr \right| \right)^{\beta/2} 
= E \left( \sup_{t \in (0,T)} \left| \sum_i \int_0^t \int_D u_N \cdot g_i(u_N) dx dr \right| \right)^{\beta/2} 
\leq c E \left( \int_0^T \sum_i \left( \int_D u_N \cdot g_i(u_N) dx \right)^2 dt \right)^{\beta/4} 
\leq c E \left( \int_0^T \left( \sum_i \int_D |u_N|^2 dx \cdot \int_D |g_i(u_N)|^2 dx \right) dt \right)^{\beta/4} 
\leq c(\delta) E \left( 1 + \int_0^T \int_D |u_N|^2 dx dr \right)^{\beta/2} + \delta E \left( \sup_{t \in (0,T)} \int_D |u_N|^2 dx \right)^{\beta/2}.
\]

So we have
\[
E \left( \sup_{t \in (0,T)} \left( \int_D |u_N|^2 dx \right)^{\beta/2} \right) + E \left( \int_Q |\nabla u_N|^q dx dt + \varepsilon \int_Q |u_N|^q dx dt \right)^{\beta/2} 
\leq c E \left( 1 + \int_D |u_0|^2 dx + \int_0^T |f|^2 dx dt \right) + c E \left( \int_0^T \left( \int_D |u_N|^2 dx \right)^{\beta/2} dt \right).
\]

We apply Gronwall’s inequality to get
\[
E \left( \sup_{t \in (0,T)} \int_D |u_N|^2 dx \right)^{\beta/2} + E \left( \int_Q |\nabla u_N|^q dx dt + \varepsilon \int_Q |u_N|^q dx dt \right)^{\beta/2} 
\leq c E \left( 1 + \int_D |u_0|^2 dx + \int_0^T |f|^2 dx dt \right)^{\beta/2},
\]

which gives the claimed inequality. \hfill \Box
5. Non-stationary Flows

In this section, we approximate the original equation by some equations satisfying the assumptions in Section 4. By using Lemma 4.1, we get a solution to this approximated system, meanwhile we get a priori estimates and a weak convergent subsequence. Finally, we use the \( L^\infty \)-truncation to pass to the limit in the nonlinear stress deviator.

5.1. A priori estimates and weak convergence. Let’s consider the equation:

\[
\begin{aligned}
du + \nabla \cdot (u \otimes u - S + pI)dt + \frac{1}{m}|u|^\beta - 2udt &= fdt + \Phi(u)dW, \\
u|_{t=0} &= u_0.
\end{aligned}
\]  

From Lemma 4.1 and Lemma 4.2 for \( \varepsilon = \frac{1}{m} \), it follows that there exists a martingale weak solution \( ((\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P}), u^m, u_0^m, f^m, W) \) to (5.1) with \( u^m \in V_{q, \beta}, \mu_0 = \mathbb{P} \circ (u_0^m)^{-1} \) and \( \mu_f = \mathbb{P} \circ (f^m)^{-1} \). For simplicity, we omit the overline. Then, there holds

\[
\int_D (u^m(t) - u^m) \cdot \varphi dx + \frac{1}{m} \int_0^t \int_D |u^m|^{\beta/2} u^m dxr + \int_0^t \int_D S(x, r, D(u^m)) : D(\varphi) dxr = \int_0^t \int_D u^m \otimes u^m : D(\varphi) dxr + \int_0^t \int_D f^m \cdot \varphi dxr + \int_0^t \int_D \Phi(u^m) \cdot \varphi dxdW(r),
\]

for all \( \varphi \in C_0^\infty(D) \).

From [24] (beginning of the proof of Thm 2.7 on p.9) we know that the probability space and the Brownian motion \( W \) can be chosen independently of \( m \). By using Lemma 4.1, we obtain the uniform estimates for \( u^m \):

\[
u^m \in L^2(\Omega; L^\infty(0, T; L^2(D))) \cap L^q(0, T; W_0^{1, q}(D)).
\]

It follows from Corollary 4.1 and [24] that

\[
E \left( \sup_{t \in (0, T)} \left( \int_D |u^m|^2 dx \right)^{\beta/2} \right) + E \left( \int_Q |\nabla u^m|^q dxdt + \int_Q \frac{1}{m} |u^m|^{\beta/2} dxdt \right)^{\beta/2} \leq c(\beta).
\]

With a parabolic interpolation and the choice of \( \beta \), we have

\[
E \left( \sup_{t \in (0, T)} \int_Q |u^m|^{r_0} dxdt \right) \leq c,
\]

for all \( r_0 := q\frac{\beta + 2}{n} \), uniformly in \( m \). By using (5.2), (5.3) and the assumption \( q > \frac{2n+2}{m+2} \), we obtain

\[
E \left( \int_Q |u^m \otimes u^m|^q dxdt + \int_Q |\nabla (u^m \otimes u^m)|^q dxdt \right) \leq c,
\]

for some \( q_0 > 1 \). After passing to subsequence, one has

\[
u^m \to u \quad \text{in} \quad L^{\frac{\beta}{\beta q}}(\Omega; L^q(0, T; W_0^{1, q}(D))),
\]

\[
u^m \to u \quad \text{in} \quad L^{\beta}(\Omega; L^\gamma(0, T; L^2(D))), \quad \forall \gamma < \infty,
\]

\[
\frac{1}{m} |u^m|^\beta - 2u^m \to 0 \quad \text{in} \quad L^{\frac{\beta}{\beta q}}(\Omega; L^q(Q)),
\]

\[
S(x, t, D(u^m)) \to \tilde{S} \quad \text{in} \quad L^{\gamma}(\Omega; L^q(Q)),
\]

\[
S(x, t, D(u^m)) \to \tilde{S} \quad \text{in} \quad L^{\gamma}(\Omega; L^q(0, T; W^{-1, q}(D))),
\]

\[
u^m \otimes u^m \to U \quad \text{in} \quad L^{\gamma}(\Omega; L^{q_0}(0, T; W^{1, q_0}(D))),
\]

\[
\Phi(u^m) \to \tilde{\Phi} \quad \text{in} \quad L^\beta(\Omega; L^\gamma(0, T; L^2(U, L^2(D)))), \quad \forall \gamma < \infty.
\]
Moreover, we have
\[ u \in L^2(\Omega; L^\infty(0, T; L^2(D))), \]
\[ \tilde{\Phi} \in L^2(\Omega; L^\infty(0, T; L^2(U, L^2(D)))). \]

Let
\[ H_1^m := S(x, t, \nabla(u^m)), \]
\[ H_2^m := \nabla \Delta^{-1} f^m + \nabla \Delta^{-1} \left( \frac{1}{m} |u^m|^{\tilde{q}-2} u^m \right) + u^m \otimes u^m, \]
\[ \Phi^m := \Phi(u^m). \]

From Theorem 3.1 and Corollary 3.1, we know that there exist the functions \( p_h^m, p_1^m, p_2^m \) which are adapted to \( \mathcal{F}_t \) and \( \Phi_p^m \) which is progressively measurable such that
\[
\int_D (u^m(t) - u_0^m - \nabla p_h^m(t)) \cdot \varphi dx + \int_0^t \int_D (H_1^m - p_1^m I) : \nabla \varphi dx dr
= \int_0^t \text{div}(H_2^m - p_2 I) \cdot \varphi dx dr + \int_0^t \Phi^m \cdot \varphi dx dW(r) + \int_0^t \Phi_p^m \cdot \varphi dx dW(r). \tag{5.12}
\]

Using the continuity of \( \nabla \Delta^{-1} \) from \( L^{\tilde{q}}(D) \) to \( W^{1, \tilde{q}}(D) \), we have
\[
H_1^m \in L^{2 \tilde{q}'}(\Omega; L^{\tilde{q}'}(Q)), \tag{5.13}
H_2^m \in L^{\tilde{q}}(\Omega; L^{\tilde{q}}(0, T; W^{1, \tilde{q}}(D))), \tag{5.14}
\Phi^m \in L^{2}(\Omega; L^{\infty}(0, T; L_2(U, L^2(D)))), \tag{5.15}
\]
uniformly in \( m \). Thanks to the estimates of Theorem 3.1 and Corollary 3.1, we obtain the following uniform bounds for the pressure functions:
\[
p_h^m \in L^{2}(\Omega; L^{\gamma}(0, T; L^2(D))), \tag{5.16}
p_1^m \in L^{2 \tilde{q}'}(\Omega; L^{\tilde{q}'}(Q)), \tag{5.17}
p_2^m \in L^{\tilde{q}}(\Omega; L^{\tilde{q}}(0, T; W^{1, \tilde{q}}(D))), \tag{5.18}
\Phi_p^m \in L^{2}(\Omega; L^{\infty}(0, T; L_2(U, L^2(D)))), \tag{5.19}
\]
uniformly in \( m \).

For the pressure function \( p_h^m \), since \( \Delta p_h^m = 0 \), by using regularity theory for harmonic functions and theorem 3.1, one has
\[
p_h^m \in L^{2}(\Omega; L^{\gamma}(0, T; W^{k, \infty}_{loc}(D))), \tag{5.20}
\]
for all \( k \in \mathbb{N} \). Therefore, for arbitrary \( \gamma < \infty \), we obtain the following convergence:
\[
p_h^m \rightharpoonup p_h \quad \text{in} \quad L^{2}(\Omega; L^{\gamma}(0, T; W^{k, \gamma}_{loc}(D))), \tag{5.21}
p_1^m \rightharpoonup p_1 \quad \text{in} \quad L^{2 \tilde{q}'}(\Omega; L^{\tilde{q}'}(Q)), \tag{5.22}
p_2^m \rightharpoonup p_2 \quad \text{in} \quad L^{\tilde{q}}(\Omega; L^{\tilde{q}}(0, T; W^{1, \tilde{q}}(D))), \tag{5.23}
\Phi_p^m \rightharpoonup \Phi_p \quad \text{in} \quad L^{2}(\Omega; L^{\gamma}(0, T; L_2(U, L^2(D)))), \tag{5.24}
\]
after passing to subsequences.
5.2. Approximate to $u \otimes u$ and $\Phi(u)$. In this subsection, we show that the limit functions in (5.5) satisfy $U = u \otimes u$ and $\hat{\Phi} = \Phi(u)$ by using the tightness of $u^m$. It follows from (5.1)-(5.3) that

$$E \left( \left\| \int_0^t \Phi(u^m)dW(r) \right\|_{W^{1,q_0}(0,T;W^{-1,q_0}_0(D))} \right) \leq c.$$

We can deal with the stochastic term similar to (4.17). By using (5.2) with $r_0 > 2$ and (2.1), we have

$$E \left( \left\| \int_0^t \Phi(u^m)dW(r) \right\|_{C^\Lambda([0,T];L^2(D,\nu \otimes \nu))} \right) \leq c(1 + \int_{\Omega \times Q} |u^m|^\Lambda \, dx dt \, d\mathbb{P}) \leq c,$$

for $\Lambda \in [0,1/2)$. Combining the both inequality above, we obtain

$$E \left( \left\| u^m \right\|_{C^\Lambda([0,T];W^{-1,q_0}_0(D))} \right) \leq c, \quad (5.25)$$

and also for some $\lambda > 0$

$$E \left( \left\| u^m \right\|_{W^{\lambda,q_0}(0,T;W^{-1,q_0}_0(D))} \right) \leq c, \quad (5.26)$$

On account of (5.2), an interpolation with $L^{q_0}(0,T;W^{1,q_0}_0(D))$ shows

$$E \left( \left\| u^m \right\|_{W^{\lambda,q_0}(0,T;L^q_0(D))} \right) \leq c, \quad (5.27)$$

for some $\kappa > 0$.

Next, we prepare the setup for our compactness method. We define the path space of $(u^m, p^m_h, p^m_1, p^m_2, \Phi^m_p, W, u_0, f)$ by

$$\mathcal{V} = L^\gamma(0,T;L^1_0(D)) \times L^\gamma(0,T;L^1_0(D)) \times (L^\gamma(Q),w) \times (L^{q_0}(0,T;W^{-1,q_0}_0(D)),w) \times (L^\gamma(0,T;L^2(D)),w) \times C([0,T],U_0) \times L^2(D) \times L^2(Q),$$

where $w$ refers to the weak topology. Let us denote by $\nu_{u^m}$, $\nu_{p^m_h}$, $\nu_{p^m_1}$, $\nu_{p^m_2}$, $\nu_{\Phi^m}$, respectively, the law of $u^m$, $p^m_h$, $p^m_1$, $p^m_2$ and $\Phi^m_p$. By $\nu_W$, we denote the law of $W$ on $C([0,T],U_0)$. The joint law of $u^m$, $p^m_h$, $p^m_1$, $p^m_2$, $\Phi^m_p$, $W$, $u_0$ and $f$ on $\mathcal{V}$ is denoted by $\nu_{u^m}$.

**Proposition 5.1.** The set $\{ u^m | m \in \mathbb{N} \}$ is tight on $\mathcal{V}$.

**Proof.** In order to prove the tightness of $u^m$, we need the following five steps.

**Step 1:** Tightness of $\nu_{u^m}$. Since $q > \frac{2n+2}{n+2}$, by using Remark 1.2 in (13) and Theorem 5.2 in (1), we have

$$W^{\infty,q_0}(0,T;L^q_0(D)) \cap L^\infty(0,T;L^2(D)) \cap L^\infty(0,T;W^{1,q}_0(D)) \hookrightarrow L^\gamma(0,T;L^1_0(D))$$

compactly for all $\gamma < q \frac{n+2}{n+2}$. Choosing a ball $B_R$ in the space $W^{\infty,q_0}(0,T;L^q_0(D)) \cap L^\infty(0,T;L^2(D)) \cap L^\infty(0,T;W^{1,q}_0(D))$ and using (13) and (22.7), we obtain

$$\nu_{u^m}(B_R^c) = \mathbb{P}(\left\| u^m \right\|_{W^{\infty,q_0}(0,T;L^q_0(D))} + \left\| u^m \right\|_{L^\infty(0,T;W^{1,q}_0(D))} + \left\| u^m \right\|_{L^\infty(0,T;L^2(D))}) \geq R) \leq \frac{1}{R} E \left( \left\| u^m \right\|_{W^{\infty,q_0}(0,T;L^q_0(D))} + \left\| u^m \right\|_{L^\infty(0,T;W^{1,q}_0(D))} + \left\| u^m \right\|_{L^\infty(0,T;L^2(D))} \right) \leq \frac{c}{R},$$

where $B_R^c$ is the complement of $B_R$. Then we can find $R(\eta)$ such that

$$\nu_{u^m}(B_R(\eta)) \geq 1 - \frac{\eta}{8},$$

for a fixed $\eta > 0$. These imply the tightness of $\nu_{u^m}$.

**Step 2:** Tightness of $\nu_{p^m_h}$. It follows from local regularity theory for harmonic function and Lebesgue dominate convergence Theorem (cf. (4.3)) that

$$L^\infty(0,T;L^2(D)) \cap \{ \Delta v(t) = 0 \text{ for a.e. } t \} \hookrightarrow L^\gamma(0,T;L^1_0(D)).$$
Proposition 5.2. Choosing balls \( B_R \) such that the following hold:

\[
\nu_p(B_R^c) = \mathbb{P}(\|p_m^n\|_{L^\infty(0,T;L^2(D))} \geq R) \leq \frac{1}{R} E \left[ \|p_m^n\|_{L^\infty(0,T;L^2(D))} \right] \leq \frac{c}{R}
\]

where \( B_R^c \) is the complement of \( B_R \). Hence, we can find \( R(\eta) \) such that

\[
\nu_p^n(B_{R(\eta)}) \geq 1 - \frac{\eta}{8},
\]

for a fixed \( \eta > 0 \). This yield that the law of \( p_m^n \) is also tight.

Step 3: Tightness of \( \nu_{p_2}^* \) and \( \nu_{p_2}^* \). Since the reflexivity of the corresponding spaces, choosing balls \( B_{R_1} \) in the space \( L^2(Q) \), \( B_{R_2} \) in the space \( L^0([0,T];W^{1,\delta_0}(D)) \), \( B_R \) in the space \( L^\infty([0,T];L_2(U,L^2(D))) \), respectively, and by using \( 5.17 \)-\( 5.19 \), we have

\[
\nu_{p_1}(B_{R_1}^c) = \mathbb{P}(\|p_1^n\|_{L^\infty(0,T;L^2(D))} \geq R_1) \leq \frac{1}{R_1} E \left[ \|p_1^n\|_{L^\infty(0,T;L^2(D))} \right] \leq \frac{c}{R_1},
\]

\[
\nu_{p_2}(B_{R_2}^c) = \mathbb{P}(\|p_2^n\|_{L^0([0,T];W^{1,\delta_0}(D))} \geq R_2) \leq \frac{1}{R_2} E \left[ \|p_2^n\|_{L^0([0,T];W^{1,\delta_0}(D))} \right] \leq \frac{c}{R_2},
\]

\[
\nu_{p_2}(B_R^c) = \mathbb{P}(\|p_2^n\|_{L^\infty([0,T];L_2(U,L^2(D)))} \geq R) \leq \frac{1}{R} E \left[ \|p_2^n\|_{L^\infty([0,T];L_2(U,L^2(D)))} \right] \leq \frac{c}{R}.
\]

Then we can find compact sets for \( p_1^n, p_2^n \) and \( \Phi_p^n \) with measures greater than \( 1 - \frac{\eta}{8} \) (or equal).

Step 4: Tightness of \( \nu_{W} \). The law \( \nu_{W} \) is tight as it coincides with the law of \( W \) which is a Radon measure on the Polish space \( C([0,T],U_0) \). Then there exists a compact subset \( C_\eta \subset C([0,T],U_0) \) such that \( \nu_{W}(C_\eta) \geq 1 - \frac{\eta}{8} \).

Step 5: Tightness of \( \mu_0 \) and \( \mu_f \). By the same argument, we can find compact subsets of \( L^2(\Omega,D) \) and \( L^2(Q) \) such that \( \mu_0 \) and \( \mu_f \) are smaller than \( 1 - \frac{\eta}{8} \).

So, we can find a compact subset \( \mathcal{V}_\eta \subset \mathcal{V} \) such that \( \nu^m(\mathcal{V}_\eta) \geq 1 - \eta \). Hence, \( \{\nu^m, m \in \mathbb{N}\} \) is tight in the same space. \( \square \)

By using the Jakubowski-Skorohod Theorem in [23], we obtain the following result.

**Proposition 5.2.** There exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( \mathcal{V} \)-valued Borel measurable random variables \((\pi_0^n, \pi_1^n, \pi_2^n, \Phi_p^n, \mathcal{W}_m^n, \nu_0^n, \mathcal{J}_m^n)\) and \((\pi, \mathcal{P}_h, \mathcal{P}_1, \mathcal{P}_2, \Phi_p, \mathcal{W}, \nu_0, \mathcal{J})\) such that the following hold:

1. The laws of \((\pi_0^n, \pi_1^n, \pi_2^n, \Phi_p^n, \mathcal{W}_m^n, \nu_0^n, \mathcal{J}_m^n)\) and \((\pi, \mathcal{P}_h, \mathcal{P}_1, \mathcal{P}_2, \Phi_p, \mathcal{W}, \nu_0, \mathcal{J})\) under \( \mathbb{P} \) coincide with \( \nu_m \) and \( \nu := \lim_{m \to \infty} \nu_m \).

2. The strong convergence:

\[
\pi_0^n \to \pi_0 \quad \text{in} \quad L^2(D) \quad \mathbb{P} \text{-a.s.},
\]

\[
\pi_1^n \to \pi_1 \quad \text{in} \quad L^1(0,T;L^\gamma(D)) \quad \mathbb{P} \text{-a.s.},
\]

\[
\pi_2^n \to \pi_2 \quad \text{in} \quad L^1(0,T;L^\gamma_{loc}(D)) \quad \mathbb{P} \text{-a.s.},
\]

\[
\mathcal{W}_m^n \to \mathcal{W} \quad \text{in} \quad C([0,T],U_0) \quad \mathbb{P} \text{-a.s.},
\]

\[
\mathcal{J}_m^n \to \mathcal{J} \quad \text{in} \quad L^2(0,T;L^2(D)) \quad \mathbb{P} \text{-a.s.}
\]

3. The weak convergence:

\[
\pi_1^n \to \pi_1 \quad \text{in} \quad L^1(Q) \quad \mathbb{P} \text{-a.s.},
\]

\[
\pi_2^n \to \pi_2 \quad \text{in} \quad L^0([0,T];W^{1,\delta_0}(D)) \quad \mathbb{P} \text{-a.s.},
\]

\[
\Phi_p^n \to \Phi_p \quad \text{in} \quad L^r(0,T;L_2(U,L^2(D))) \quad \mathbb{P} \text{-a.s.}
\]
\[ (4) \int_{\Omega} \left( \sup_{t \in [0, T]} \| W(t) \|_{U_0}^2 \right) d\mathbb{P} = \int_{\Omega} \left( \sup_{t \in [0, T]} \| W(t) \|_{U_0}^2 \right) d\mathbb{P}, \]

for all \( \alpha < \infty \).

By virtue of the equality of laws, we obtain the weak convergence:

\[ p_1^n \to p_1 \quad \text{in} \quad L^q(\Omega; L^q(Q)), \]
\[ p_2^n \to p_2 \quad \text{in} \quad L^q(\Omega; L^q(0, T; W^{1,q}(D))), \]
\[ \bar{p}_p^n \to \bar{p}_p \quad \text{in} \quad L^q(\Omega; L^q(0, T; L_2(U, L^2(D)))). \]

By Vitali’s convergence Theorem, we get the strong convergence:

\[ W^n \to W \quad \text{in} \quad L^2(\Omega; C([0, T], U_0)), \] (5.28)
\[ p^n \to p \quad \text{in} \quad L^q(\Omega \times Q; \mathbb{P} \otimes \mathcal{L}^{n+1}), \] (5.29)
\[ p_0^n \to p_0 \quad \text{in} \quad L^2(\Omega \times D; \mathbb{P} \otimes \mathcal{L}^{n+1}), \] (5.30)
\[ \nabla \bar{p}_h^n \to \nabla \bar{p}_h \quad \text{in} \quad L^q(\Omega \times (0, T) \times D'; \mathbb{P} \otimes \mathcal{L}^{n+1}), \] (5.31)
\[ f^n \to f \quad \text{in} \quad L^2(\Omega \times Q; \mathbb{P} \otimes \mathcal{L}^{n+1}), \] (5.32)

for all \( \gamma < q \frac{m-2}{m} \) and all \( D' \subset \subset D \), after choosing a subsequence. For the harmonic pressure \( h \) and all \( n \), applying local regularity theory for harmonic maps above, one has for all \( t \in [0, T] \),

\[ \int_{D} (\tilde{p}_n(t) - p_0^n) - \nabla \bar{p}_h^n(t)) \cdot \varphi dx + \int_{0}^{t} \left( (\overline{H}_1)^n - \overline{p}_1^n I \right) \cdot \nabla \varphi dx dr \]
\[ = \int_{0}^{t} \int_{D} \text{div}(\overline{H}_2^n - \overline{p}_2 I) \cdot \varphi dx dr + \int_{0}^{t} \int_{D} \Phi(\overline{H}_2^n) \cdot \varphi dx dr + \int_{0}^{t} \int_{D} \overline{p}_p^n \cdot \varphi dx dr, \]

\( \mathbb{P} \otimes \mathcal{L}^1 \)-a.e. for all \( \varphi \in C^\infty_0(D) \), where

\[ \overline{H}_1^n := S(x, t, D(\tilde{p}_n)), \]
\[ \overline{H}_2^n := \tilde{p}_n \otimes \tilde{p}_n + \nabla \Delta^{-1} \left( \frac{1}{m} \tilde{p}_n \otimes \tilde{p}_n \right) + \nabla \Delta^{-1} f^n. \]

**Remark 5.1.** Here we use the test-functions \( \varphi \in C^\infty_0(D) \), instead of \( \varphi \in C^\infty_{0, \text{div}}(D) \).

Using Lemma 2.1 in [16] and the convergence [5.28]-[5.35], we obtain the limit equation:

\[ \int_{D} (\tilde{p}(t) - p_0 - \nabla \bar{p}_h(t)) \cdot \varphi dx + \int_{0}^{t} \int_{D} \left( H_1 - \overline{p}_1 I \right) \cdot \nabla \varphi dx dr \]
\[ = \int_{0}^{t} \int_{D} \text{div}(H_2 - \overline{p}_2 I) \cdot \varphi dx dr + \int_{0}^{t} \int_{D} \Phi(H) \cdot \varphi dx dr + \int_{0}^{t} \int_{D} \overline{p}_p \cdot \varphi dx dr, \] (5.36)
for all $\varphi \in C_0^\infty(D)$, where

$$\overline{H}_1 := \overline{S}, \overline{H}_2 := \overline{u} \otimes \overline{u} + \nabla \Delta^{-1} f.$$  

It remains to show $\overline{S} = S(x, t, \mathcal{D}(\overline{u}))$. Let

$$\overline{G}_1^n := S(x, t, \mathcal{D}(\overline{\varphi}^n)) - \overline{S},$$

$$\overline{G}_2^n := \overline{u}^n \otimes \overline{u}^n - \overline{u} \otimes \overline{u} + \nabla \Delta^{-1} \left( \frac{1}{m} |\overline{\varphi}^n|^q \overline{\varphi}^n \right) + \nabla \Delta^{-1} (f^n - f),$$

$$\overline{\Psi}^n := (\Phi(\overline{\varphi}^n), -\Phi(\overline{u})), \quad \overline{\Psi}_\varphi^n := (\Phi_\varphi(\overline{\varphi}^n), -\Phi_\varphi(\overline{u})),$$

$$\overline{v}_h^n := \overline{v}_1^n - \overline{v}_h, \quad \overline{v}_1^n := \overline{v}_1^n - \overline{v}_1, \quad \overline{v}_2^n := \overline{v}_2^n - \overline{v}_2.$$  

Then the following convergence hold:

$$\overline{\varphi}^n - \overline{u} \to 0 \quad \text{in} \quad L^2(0, T; W^{1,q}_0(D)), \quad (5.37)$$

$$\overline{\varphi}^n - \overline{u} \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.38)$$

$$\overline{G}_1^n \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.39)$$

$$\overline{G}_2^n \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.40)$$

$$\overline{\Psi}^n - \overline{\Psi} \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.41)$$

where $\overline{\Psi} = (\Phi(\overline{u}), -\Phi(\overline{u}))$. For the pressure functions, we have

$$\overline{v}_h^n \to 0 \quad \text{in} \quad L^2(0, T; W^{1,\gamma}_0(D)), \quad (5.42)$$

$$\overline{v}_1^n \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.43)$$

$$\overline{v}_2^n \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.44)$$

$$\overline{\Psi}_\varphi^n - \overline{\Psi}_\varphi \to 0 \quad \text{in} \quad L^2(0, T; L^2(D)), \quad (5.45)$$

Moreover, we obtain

$$\overline{v}_h^n \in C_0(0, T; L^2(D)), \quad (5.46)$$

$$\overline{\Psi}^n \in C_0(0, T; L^2(D)), \quad (5.47)$$

$$\overline{\Psi}_\varphi^n \in C_0(0, T; L^2(D)), \quad (5.48)$$

uniformly in $m$.

The difference of approximates equation and limit equation read as

$$\int_D (\overline{\varphi}^n(t) - \overline{u}(t) + \overline{u}_0 - \overline{v}_0^n - \nabla \overline{v}_h^n(t)) \cdot \varphi dx + \int_0^t \int_D (\overline{G}_1^n - \overline{G}_2^n) \nabla \varphi dx dr$$

$$= \int_0^t \int_D \text{div}(\overline{G}_2^n - \overline{G}_2^n) \varphi dx dr + \int_0^t \int_D \overline{\Psi}^n \cdot \varphi dx dr (\overline{W}_h^n(r), \overline{W}(r))$$

$$+ \int_0^t \int_D \overline{\Psi}_\varphi^n \cdot \varphi dx dr (\overline{W}_h^n(r), \overline{W}(r))$$

for all $\varphi \in C_0^\infty(D)$. Define $\overline{v}^n = \overline{\varphi}^n - \nabla \overline{v}_h^n$ and denote $\overline{v}^{n,k} := \overline{v}^n - \overline{v}^k$, $m \geq k$. Similarly, we define $\overline{G}_1^n, \overline{G}_2^n, \overline{v}_1^n, \overline{v}_2^n, \overline{\Psi}_\varphi^n, \overline{W}_h^n$ and $\overline{W}_h^n$. Then, we have

$$\overline{v}^n \to 0 \quad \text{in} \quad L^2(0, T; W^{1,\alpha}_0(D)), \quad (5.50)$$

$$\overline{v}^n \to 0 \quad \text{in} \quad L^2(0, T; D') \otimes L^{n+1}, \quad (5.51)$$
By using Itô’s formula, we have

\[
\begin{align*}
\int_{D} (\overline{\psi}^{m,k} - \overline{\psi}_0^{n,k}) \cdot \varphi dx + \int_0^t \int_D (G_1^{m,k} - \overline{G}_1^{n,k}) : \nabla \varphi dxdr \\
= \int_0^t \int_D \text{div} (G_2^{m,k} - \overline{G}_2^{n,k}) : \varphi dxdr + \int_0^t \int_D \overline{\psi}^{m,k} \cdot \varphi dxdr (\overline{W}^m (r), \overline{W}^k (r)) \\
+ \int_0^t \int_D \Phi_{\varphi} \cdot \varphi dxdr (\overline{W}^m (r), \overline{W}^k (r)),
\end{align*}
\]

for all \(\varphi \in C_0^\infty (D)\).

### 5.3. \(L^\infty\)-truncation.

From density arguments, we are allowed to test the equations with \(\varphi \in W_0^{1,p} \cap L^\infty (D)\). Since the function \(\overline{\psi}(w, t, \cdot)\) does not belong to this class, the \(L^\infty\)-truncation is used to the deterministic problem in [43]. In this subsection, we apply the \(L^\infty\)-truncation to the stochastic setting.

Let

\[
h_L (s) := \int_0^s \Psi (\theta) d\theta, \quad H_L (\xi) := h_L (|\xi|), \quad \Psi_L := \sum_{l=1}^L \psi_{-l}, \quad \psi_5 := \psi (\delta s),
\]

for \(L \in \mathbb{N}_0\), where \(\psi \in C_0^\infty ([0, 2])\), \(\psi \equiv 0\) on \([0, 1]\), \(0 \leq \psi \leq 1\) and \(0 \leq -\psi' \leq 2\). Denote

\[
f_L (u) := \int_D \eta H_L (u) dx, \quad \text{for} \ \eta \in C_0^\infty (D).
\]

By using Itô’s formula, we have

\[
\begin{align*}
\int_D \eta H_L (\overline{\psi}^{m,k} (t)) dx \\
= f_L (\overline{\psi}^{m,k} (0)) + \int_0^t f_L (\overline{\psi}^{m,k}) d\overline{\psi}^{m,k} + \frac{1}{2} \int_0^t f_L'' (\overline{\psi}^{m,k}) d (\overline{\psi}^{m,k} (r)) \\
= \int_D \eta H_L (\overline{\psi}_0^{n} - \overline{\psi}_0^{k}) dx - \int_0^t \int_D \eta (G_1^{m,k} - \overline{G}_1^{n,k}) : \nabla (\Psi_L (|\overline{\psi}^{m,k}|) \overline{\psi}^{m,k}) dxdr \\
- \int_0^t \int_D (G_2^{m,k} - \overline{G}_2^{n,k}) : \nabla \eta \otimes (\Psi_L (|\overline{\psi}^{m,k}|) \overline{\psi}^{m,k}) dxdr \\
+ \int_0^t \int_D \eta \Psi_L (|\overline{\psi}^{m,k}|) \text{div} (G_2^{m,k} - \overline{G}_2^{n,k} : \overline{\psi}^{m,k} dxdr \\
+ \int_0^t \int_D \eta \Psi_L (|\overline{\psi}^{m,k}|) \overline{\psi}^{m,k} \cdot (\Phi (\overline{\psi}^{m,k}) d\overline{W}^m (r) - \Phi (\overline{\psi}^{k}) d\overline{W}^k (r)) dx \\
+ \int_0^t \int_D \eta \Psi_L (|\overline{\psi}^{m,k}|) \overline{\psi}^{m,k} \cdot (\Phi (\overline{\psi}^{m,k}) d\overline{W}^m (r) - \Phi (\overline{\psi}^{k}) d\overline{W}^k (r)) dx \\
+ \frac{1}{2} \int_0^t \int_D \eta \overline{\psi} \cdot \overline{\psi} (\overline{W}^m (r) - \overline{W}^k (r)) dx \\
+ \frac{1}{2} \int_0^t \int_D \eta \overline{\psi} \cdot \overline{\psi} (\overline{W}^m (r) - \overline{W}^k (r)) dx
\end{align*}
\]

\(\overline{\psi} [J_1] \to 0\) if \(m, k \to \infty\), gained by equation [5.24] and \(\overline{\psi}^m (0) - \overline{\psi}^k (0) = \overline{\psi}^m (0) - \overline{\psi}^k (0)\) (see Theorem [3.1] (2)). We are going to show that the expectation values of \(J_3\) and \(J_4\) vanish if \(m, k \to \infty\). By using the monotone operator theory, we obtain \(\mathbb{D} (\overline{\psi}^m) \to \mathbb{D} (\overline{\psi})\), a.e.. Clearly, \(\Psi_L (|\overline{\psi}^{m,k}|) \overline{\psi}^{m,k}\) are bounded in \(L^1\). By virtue of [5.29] and the construction of
Clearly, \( \Psi \) for all \( \gamma < \infty \) the properties of \( \Phi \). So we can estimate have

\[
J_7 \leq c \sum_{\ell=1}^{n} \int_{D}^{t} \left( \int_{0}^{1} (\Phi(\pi^m) - \Phi(\pi^k))d\pi^m \right)^{\ell}(r)dr
\]

By using (2.1), (2.2) and (5.29), we obtain

\[
\text{In view of (2.3), (5.28) and (5.29), we have }
\]

\[
\limsup_{m, k \to \infty} \left( J_7 \right) = 0.
\]

By using (2.1), (2.2) and (5.29), we obtain

\[
\mathbb{E}(J_{71}) \leq c \mathbb{E} \left( \int_{0}^{t} ||\Phi(\pi^m) - \Phi(\pi^k)||^2_{L_2(U, L^2(D))]dr} \right)
\]

\[
\leq c \mathbb{E} \left( \int_{0}^{t} \int_{D}^{t} |\pi^m - \pi^k|^2 dxdr \right) \to 0, \quad m, k \to \infty
\]

In view of (2.1), (2.2) and \( \pi^k \in L^2(\Omega \times Q; \mathbb{F} \otimes L^{n+1}) \) uniformly in \( k \), one deduces that

\[
\mathbb{E}(J_{72}) = \mathbb{E} \left( \int_{0}^{t} \sum_{i=1}^{\infty} \left( \int_{D}^{t} |g_i(\pi^k)|^2 \text{Var} (\bar{\beta}_i^m (1) - \bar{\beta}_i^k (1)) dx \right) dt \right)
\]

\[
\leq c \mathbb{E} \left( \int_{0}^{t} \left( \int_{D}^{t} \sup_i t^2 |g_i(\pi^k)|^2 dx \right) dt \right) \sum_{i} \frac{1}{t^2} \text{Var} (\bar{\beta}_i^m (1) - \bar{\beta}_i^k (1))
\]

\[
\leq c \mathbb{E} \left( \int_{0}^{t} \int_{D}^{t} (1 + |\pi^k|^2) dxdt \right) \cdot \mathbb{E} \left( ||\pi^m - \pi^k||^2_{L^2(0,T,U_0)} \right)
\]

\[
\to 0, \quad m, k \to \infty.
\]

From Corollary 3.1 and the usage of the cut-off function \( \eta \), we know that \( \Phi_\varrho \) inherits the properties of \( \Phi \). So we can estimate \( J_8 \) by the same method. Plugging all together, we have

\[
\limsup_{m} \mathbb{E} \left( \int_{Q} \eta(S(x, r, \mathbb{D}(\pi^m)) - \mathbb{S}) : \Psi_L(\mathbb{D}(\pi^m - \mathbb{\hat{\pi}}))d\mathbb{\pi} \pi^m dxdr \right)
\]

\[
\leq \limsup_{m} \mathbb{E} \left( \int_{Q} \eta(S(x, r, \mathbb{D}(\pi^m)) - \mathbb{S}) : \nabla \{ \Psi_L(\mathbb{D}(\pi^m - \mathbb{\hat{\pi}})) \} \otimes (\pi^m - \mathbb{\hat{\pi}}) dxdr \right)
\]

\[
+ \limsup_{m} \mathbb{E} \left( \int_{Q} \eta \pi^m_1 \text{div}(\Psi_L(\mathbb{D}(\pi^m - \mathbb{\hat{\pi}}))(\pi^m - \mathbb{\hat{\pi}})) dxdr \right).
\]
Since \( \text{div}(\nabla u - \nabla v) = 0 \), there holds

\[
\limsup_m E \left( \int_Q \eta \nabla^2 \text{div}(\Psi_L(\nabla u - \nabla v)(\nabla u - \nabla v)) \, dx \, dt \right)
= \limsup_m E \left( \int_Q \eta \nabla^2 \nabla \{ \Psi_L(\nabla u - \nabla v) \} \cdot (\nabla u - \nabla v) \, dx \, dt \right).
\]

Note that, for all \( \ell \in \mathbb{N}_0 \),

\[
|\nabla \{ \psi_{2^{-\ell}}(|\nabla u - \nabla v|) \} \cdot (\nabla u - \nabla v)| \leq |\psi_{2^{-\ell}}(|\nabla u - \nabla v|) \nabla (\nabla u - \nabla v)|
\leq -2^{-\ell} |\nabla u - \nabla v| |\psi^{\prime}(2^{-\ell} |\nabla u - \nabla v|)| |\nabla (\nabla u - \nabla v)|
\leq c |\nabla (\nabla u - \nabla v)|_{X_{A_\ell}},
\]

where \( A_\ell := \{ 2^\ell < |\nabla u - \nabla v| \leq 2^{\ell+1} \} \). This yields

\[
|\nabla \Psi_L(|\nabla u - \nabla v|)(\nabla u - \nabla v)| \leq \sum_{\ell=0}^L |\nabla \{ \psi_{2^{-\ell}}(|\nabla u - \nabla v|) \} \nabla (\nabla u - \nabla v)|
\leq c \sum_{\ell=0}^L |\nabla (\nabla u - \nabla v)|_{X_{A_\ell}} \leq c |\nabla (\nabla u - \nabla v)|.
\]

By using \( \text{(5.37)} \) and \( \text{(5.42)} \), we have

\[
\nabla \Psi_L(|\nabla u - \nabla v|)(\nabla u - \nabla v) \in L^q(I \times Q; \mathbb{T}^n \otimes \mathbb{L}^{n+1}),
\]

uniformly in \( L \) and \( m \). Then, we can conclude that

\[
\limsup_m E \left( \int_Q \eta(S(x,r,\mathbb{D}(\nabla u)) - \mathbb{S}) : \Psi_L(|\nabla u - \nabla v|) \mathbb{D}(\nabla u - \nabla v) \, dx \, dr \right) \leq K. \tag{5.56}
\]

In view of \( \text{(5.56)} \), using Cantor’s diagonalizing principle, there exists a subsequence with

\[
\sigma_{\ell,m_\ell} := E \left( \int_Q \eta(S(x,r,\mathbb{D}(\nabla u)) - \mathbb{S}) : \psi_{2^{-\ell}}(|\nabla u - \nabla v|) \mathbb{D}(\nabla u - \nabla v) \, dx \, dr \right) \rightarrow \sigma_\ell,
\]

for \( \ell \in \mathbb{N}_0 \), as \( \ell \rightarrow \infty \). From \( \text{(1.4)} \), we know that \( \sigma_\ell \geq 0 \) for all \( \ell \in \mathbb{N}_0 \) and \( \sigma_\ell \) is increasing in \( \ell \). Thanks to \( \text{(5.56)} \), we have

\[
0 \leq \sigma_0 \leq \sigma_1 + \sigma_2 + \cdots + \sigma_\ell \leq \frac{K \ell}{\ell},
\]

for all \( \ell \in \mathbb{N} \). Hence \( \sigma_0 = 0 \) and therefore

\[
E \left( \int_Q (S(x,r,\mathbb{D}(\nabla u)) - \mathbb{S}) : \psi_1(|\nabla u - \nabla v|) \mathbb{D}(\nabla u - \nabla v) \, dx \, dr \right) \rightarrow 0 \quad \text{as} \quad m \rightarrow 0.
\]

It follows from \( \text{(5.31)} \) that

\[
E \left( \int_Q (S(x,r,\mathbb{D}(\nabla u)) - \mathbb{S}) : \psi_1(|\nabla u - \nabla v|) \mathbb{D}(\nabla u - \nabla v) \, dx \, dr \right) \rightarrow 0 \quad \text{as} \quad m \rightarrow 0. \tag{5.57}
\]

Using \( \text{(5.39)} \) and the fact \( \psi_{2^{-N}}(|\nabla u - \nabla v|) \rightarrow 1 \) as \( m \rightarrow \infty \), one has

\[
\limsup_m E \left( \int_Q S(x,r,\mathbb{D}(\nabla u)) : \psi_1(|\nabla u - \nabla v|) \mathbb{D}(\nabla u) \, dx \, dr \right) = E \left( \int_Q \mathbb{S} : \mathbb{D}(\nabla u) \, dx \, dr \right). \tag{5.58}
\]

Lemma A.2 in \( \text{(43)} \) implies that \( \mathbb{S} = S(x,t,\mathbb{D}(\nabla u)) \). Then we complete the proof of Theorem 2.1.
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