Continuum Electrodynamics of a Piecewise-Homogeneous Linear Medium

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The energy–momentum tensor and the tensor continuity equation serve as the conservation laws of energy, linear momentum, and angular momentum for a continuous flow. Previously, we derived equations of motion for macroscopic electromagnetic fields in a homogeneous linear dielectric medium that is draped with a gradient-index antireflection coating (J. Math Phys. 55, 042901 (2014)). These results are consistent with the electromagnetic tensor continuity equation in the limit that reflections and the accompanying surface forces are negligible thereby satisfying the condition of an unimpeded flow in a thermodynamically closed system. Here, we take the next step and derive equations of motion for the macroscopic fields in the limiting case of a piecewise-homogeneous simple linear dielectric medium. The presence of radiation surface forces on the interface between two different homogeneous linear materials means that the energy–momentum formalism must be modified to treat separate homogeneous media in which the fields are connected by boundary conditions at the interfaces. We demonstrate the explicit separation of the total momentum into a field component and a material motion component, we derive the radiation pressure that transfers momentum from the field to the material, we derive the electromagnetic continuity equations for a piecewise homogeneous dielectric, and we provide a lucid reinterpretation of the Jones and Richards experiment.

I. INTRODUCTION

The energy–momentum tensor is an innate and compelling aspect of energy and momentum conservation in a continuous flow [1]. Recently [2–5], we used global conservation principles to construct the total energy–momentum tensor for a thermodynamically closed system consisting of a quasimonochromatic optical pulse and a homogeneous simple linear medium that is draped with a gradient-index antireflection coating. Regarding the total energy–momentum tensor and the tensor continuity equation as fundamental, we derived equations of motion for the macroscopic fields. The formulation of continuum electrodynamics that was derived in our previous work [2–7] was limited to homogeneous materials with a gradient-index antireflection coating. In this article, we develop the theory of continuum electrodynamics for the more usual situation of a piecewise-homogeneous linear dielectric medium. One of the major differences with the Maxwell theory is that the Fresnel relations can no longer be derived from the application of Stoke’s theorem to the Faraday and Maxwell–Ampère Laws. Instead, we derive the Fresnel relations from the electromagnetic wave equation and conservation of energy. We obtain the field and material components of the total momentum, derive the radiation pressure, and derive the electromagnetic continuity equations for a piecewise homogeneous medium. We provide an interpretation of the Jones and Richards [8] measurement of the optical force on a mirror immersed in a dielectric fluid.

II. EQUATIONS OF MOTION

We take as a given that propagation of the electromagnetic field is characterized by the wave equation

\[ \nabla \times (\nabla \times A) + \frac{n^2}{c^2} \frac{\partial^2 A}{\partial t^2} = 0 \]  (2.1)

in a limit in which absorption can be neglected. Dispersion is treated parametrically for the arbitrarily long quasimonochromatic fields that are considered here. The electromagnetic wave equation, Eq. (2.1), is a mixed second-order differential equation that can be written in terms of first-order differential equations. To that end, we define the macroscopic magnetic field

\[ B = \nabla \times A \]  (2.2)

and a second macroscopic field

\[ \Pi = n \frac{\partial A}{\partial t} . \]  (2.3)

A Maxwell–Ampère-like law

\[ \nabla \times B + \frac{n}{c} \frac{\partial \Pi}{\partial t} = 0 \]  (2.4)

results from the substitution of the definitions of the macroscopic fields, Eqs. (2.2) and (2.3), into the wave equation, Eq. (2.1). Two more equations of motion are derived from the definitions of the fields. Thompson’s Law

\[ \nabla \cdot B = 0 \]  (2.5)

is obtained from the divergence of Eq. (2.2) with the vector identity that the divergence of the curl of any vector
is zero. The curl of Eq. (2.3)
\[ \nabla \times \Pi - \frac{n}{c} \frac{\partial B}{\partial t} = \frac{n}{n} \times \Pi \]  
(2.6)
is our variant of the Faraday Law. Taking the divergence of Eq. (2.4) and integrating with respect to time, we obtain the Gauss-like law
\[ \nabla \cdot \Pi = -\frac{n}{n} \times \Pi . \]  
(2.7)
A constant of integration has been suppressed in the absence of charges. Note that each field equation is algebraically equivalent to its counterpart in the macroscopic Maxwell equations
\[ \nabla \times \mathbf{B} - \frac{n^2}{c} \frac{\partial \mathbf{E}}{\partial t} = 0 \]  
(2.8)
\[ \nabla \cdot \mathbf{B} = 0 \]  
(2.9)
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]  
(2.10)
\[ \nabla \cdot (n^2 \mathbf{E}) = 0 \]  
(2.11)
if we define \( \Pi = -n \mathbf{E} \), which we have every right to do under the auspices of Maxwellian continuum electrodynamics. However, the different sets of motional equations for macroscopic fields have different tensorial and relativistic properties. This means that Maxwellian continuum electrodynamics, which is fundamentally a vector theory, admits improper tensor transformations of coordinates for the coupled equations \[ \text{Refs. [2–7]}. \]  
Here, we consider a quasimonochromatic optical pulse that passes from one simple linear medium into a second such medium through a planar interface at normal incidence. In this limit of piecewise-homogeneous media, we need only retain the homogeneous parts of Eqs. (2.4)–(2.7) to obtain
\[ \nabla \times \mathbf{B} + \frac{n}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]  
(2.12)
\[ \nabla \cdot \mathbf{B} = 0 \]  
(2.13)
\[ \nabla \times \Pi - \frac{n}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]  
(2.14)
\[ \nabla \cdot \Pi = 0 , \]  
(2.15)
where the fields are connected by boundary conditions at the interface between different homogeneous linear materials. While the derivation of the equations of motion for macroscopic fields in a homogeneous medium from Eqs. (2.4)–(2.7) is obvious, the usage of Eqs. (2.12)–(2.15) for piecewise-homogeneous matter remains to be investigated.

### III. THE FRESNEL RELATIONS

First, we demonstrate how our results apply to the most obvious issue relating to piecewise-homogeneous matter, the Fresnel relations. Because the equations of motion for macroscopic fields in a piecewise-homogeneous linear medium have changed, there is a question that arises as to how the Fresnel relations survive. Applying Stokes’s theorem to the macroscopic Maxwell curl equations, Eqs. (2.8) and (2.10), it was long ago found that the tangential components of the electric field \( \mathbf{E} \) and the magnetic auxiliary field \( \mathbf{H} = \mathbf{B} \) are continuous at a planar interface between linear homogeneous dielectrics. Similarly, the normal components of the displacement field \( \mathbf{D} = n^2 \mathbf{E} \) and the magnetic field \( \mathbf{B} \) were shown to be continuous by the divergence theorem applied to the Maxwell divergence equations, Eqs. (2.9) and (2.11). The simultaneous continuity of the transverse \( \mathbf{E} \) and \( \mathbf{H} \) fields and the normal parts of the \( \mathbf{D} \) and \( \mathbf{B} \) fields leads to the Fresnel relations. Because Eqs. (2.6) and (2.7) are inhomogeneous, the transverse and normal components of the macroscopic field \( \mathbf{H} \) are not continuous at the material interface, even though the appearance of Eqs. (2.14) and (2.15) might seem to suggest otherwise.

We treat the propagation of a quasimonochromatic optical pulse through a piecewise-homogeneous simple linear medium. The material is initially stationary in the laboratory frame of reference and is rigidly attached to a support. We take as a given that propagation of the electromagnetic field is characterized by the wave equation
\[ \nabla \times (\nabla \times \mathbf{A}) + \frac{n^2(r)}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 . \]  
(3.1)
The pulse is sufficiently monochromatic that dispersion can be neglected in accordance with the characterization of a simple linear medium. Further, the simple linear material with its support is assumed to be arbitrarily massive so that n(r) can be treated as a time-independent function of space in the laboratory frame of reference. In order to not overly complicate matters, we assume normal incidence and adopt the plane-wave limit for the field. The vector potential, in the plane-wave limit,

\[ \mathbf{A}(z, t) = \frac{1}{2} \left( \mathbf{\hat{A}}(z, t) e^{-i(\omega t - k_d z)} + \mathbf{\hat{A}}^*(z, t) e^{i(\omega t - k_d z)} \right) \]

can be written in terms of an envelope function \( \mathbf{\hat{A}}(z, t) \) and a carrier wave with center frequency \( \omega_d \). Here, \( k_d \) is the amplitude of the wave vector \( k_d = (n\omega_d/c)\mathbf{\hat{e}}_z \) that is associated with the center frequency of the field, and \( \mathbf{\hat{e}}_z \) is a unit vector in the direction of propagation along the \( z \) axis. The vector potential amplitude \( \mathbf{\hat{A}} \) is not slowly varying if there is a backward propagating field component. Figure 1 shows a one-dimensional representation of the amplitude of the incident field \( \mathbf{\hat{A}}_0(z) = \left( \mathbf{\hat{A}}(z, t_0) \cdot \mathbf{\hat{A}}^*(z, t_0) \right)^{1/2} \) about to enter a simple linear medium with \( n_2 = 1.40 \) at normal incidence from the vacuum \( n_1 = 1 \). Figure 2 presents a time-domain numerical solution of the wave equation at a later time \( t_1 \) depicted by \( \mathbf{\hat{A}}_1(z) = \left( \mathbf{\hat{A}}(z, t_1) \cdot \mathbf{\hat{A}}^*(z, t_1) \right)^{1/2} \). The reflected field and the refracted field have separated and the refracted field is entirely inside the medium. The reflected pulse has not propagated as far as it would have propagated in the vacuum due to the reduced speed of light \( c/n \) in the material. In addition, the spatial extent of the reflected pulse in the medium is \( w_t = n_1w_i/n_2 \) in terms of the width \( w_i \) of the incident pulse. As shown in Fig. 2, the amplitudes of the reflected and refracted fields are different from the amplitude of the incident field.

We would like a formula for determining how much of the incident pulse goes into the reflected field and how much is refracted. Although the Fresnel relations are the necessary formulas, their use is problematic at this point because their provenance from the macroscopic Maxwell equations, Eqs. (2.8)–(2.11), is suspect. Because the macroscopic field \( \mathbf{\Pi} \) is not continuous at a step index boundary, we present a derivation of the Fresnel relations that is based on the wave equation and conservation of energy. Applying Stokes’s theorem to the wave equation, Eq. (3.1), we have

\[ \oint_C (\nabla \times \mathbf{\hat{A}}) \cdot dl = \oint_S (\nabla \times (\nabla \times \mathbf{\hat{A}})) \cdot \mathbf{\hat{n}} \, da . \]  \hspace{1cm} (3.2)

Consider a thin right rectangular box or “Gaussian pillbox” that straddles the interface between the two mediums with the large surfaces parallel to the interface. Then \( S \) is the surface of the pillbox, \( da \) is an element of area on the surface, and \( \mathbf{\hat{n}} \) is an outwardly directed unit vector normal to \( da \). There is no contribution to the surface integral from the large surfaces for our normally incident field because \( \nabla \times (\nabla \times \mathbf{\hat{A}}) \) is orthogonal to \( \mathbf{\hat{n}} \). The contributions from the smaller surfaces can be neglected as the box becomes arbitrarily thin. Then

\[ \oint_C (\nabla \times \mathbf{\hat{A}}) \cdot dl = 0 . \]  \hspace{1cm} (3.3)

We choose the closed contour \( C \) in the form of a rectangular Stokesian loop with sides that bisect the two large surfaces and two of the small surfaces on opposite sides of the pillbox. Here, \( dl \) is a directed line element that lies on the contour, \( C \). Then \( C \), like \( S \), straddles the material interface. For normal incidence in the plane-wave limit, the field \( \nabla \times \mathbf{\hat{A}} \) can be oriented along the long sides of the contour \( C \). Performing the contour integration in Eq. (3.3), the contribution from the short sides of the contour are neglected as the loop is made vanishingly thin and we obtain

\[ (\nabla \times \mathbf{\hat{A}})_1 \cdot \Delta l_1 + (\nabla \times \mathbf{\hat{A}})_2 \cdot \Delta l_2 = 0 \]  \hspace{1cm} (3.4)

from the long sides, 1 and 2, of the contour.
For linearly polarized radiation, we can write the vector potential of the incident, reflected, refracted, and transmitted waves as

\[
A_i = \hat{e}_x \tilde{A}_i e^{-i(\omega t - k_1 z)} \tag{3.5}
\]

\[
A_r = \hat{e}_x \tilde{A}_r e^{-i(\omega t + k_1 z)} \tag{3.6}
\]

\[
A_t = \hat{e}_x \tilde{A}_t e^{-i(\omega t - k_2 z)} \tag{3.7}
\]

\[
A_T = \hat{e}_x \tilde{A}_T e^{-i(\omega t - k_1 z)}, \tag{3.8}
\]

where \(k_1 = n_1 \omega / c\), \(k_2 = n_2 \omega / c\), and \(\hat{e}_x\) is a unit polarization vector. It is understood that we use the real part of complex fields and neglect double frequency terms in field products. For convenience, the scalar amplitudes are taken to be real. Using the fact that the line elements \(A_1\) and \(\Delta l_2\) in Eq. (3.4) are equal and opposite, we obtain a relation

\[
n_1(\tilde{A}_i - \tilde{A}_r) = n_2 \tilde{A}_t \tag{3.9}
\]

between the amplitudes of the incident, reflected, and refracted fields. In order to derive boundary conditions, we need another such relation.

For a stationary simple linear material, the electromagnetic energy

\[
U = \int_\sigma \frac{1}{2} \left( \frac{n^2}{c^2} \left( \frac{\partial A}{\partial t} \right)^2 + (\nabla \times A)^2 \right) dv \tag{3.10}
\]

is conserved. Here, the volume of integration, which includes all fields present, has been extended to all-space \(\sigma\). The total energy is invariant in time by virtue of being conserved. The total energy at time \(t_0\), the incident energy \(U(t_0) = U_i\), is equal to the total energy at a later time \(t_1\), \(U(t_1) = U_r + U_t\), which is the sum of the reflected energy \(U_r\) and the refracted energy \(U_t\) when the refracted field is entirely within the medium.

In terms of the incident, reflected, and transmitted energy, the energy balance \(U(t_0) = U(t_1)\) is

\[
U_i = U_r + U_t. \tag{3.11}
\]

Substituting Eqs. (3.5)–(3.7) into the formula for the energy, Eq. (3.10), and expressing the energy balance, Eq. (3.11), in terms of the amplitudes of the incident, reflected, and transmitted vector potential results in

\[
\int_\sigma n_1^2 \tilde{A}_i^2 dv = \int_\sigma n_1^2 \tilde{A}_r^2 dv + \int_\sigma n_2^2 \tilde{A}_t^2 dv. \tag{3.12}
\]

In order to facilitate the integration of Eq. (3.12), we choose the incident pulse to be rectangular with a nominal width of \(\omega\). The pulse has a finite rise time and a finite fall time to reduce ringing, but the short transition region can be neglected compared to the arbitrarily large width of the pulse. The refracted pulse has a width of \(n_1 w_1 / n_2\) due to the change in the velocity of light between the two media. Then, evaluating the integrals of Eq. (3.12) results in

\[
n_1^2 \tilde{A}_i^2 = n_2^2 \tilde{A}_r^2 + n_2 n_1 \tilde{A}_t^2. \tag{3.13}
\]

Grouping terms of like refractive index, the previous equation

\[
n_1 \left( \tilde{A}_i^2 - \tilde{A}_r^2 \right) = n_2 \tilde{A}_t^2 \tag{3.14}
\]

becomes more suggestive as

\[
n_1 \left( \tilde{A}_i - \tilde{A}_r \right) \left( \tilde{A}_i + \tilde{A}_r \right) = n_2 \tilde{A}_t^2 \tag{3.15}
\]

by factoring the binomial. The second-order equation can be written as two first-order equations. Substituting Eq. (3.9) into Eq. (3.15), we have the unique decomposition

\[
n_1 \left( \tilde{A}_i - \tilde{A}_r \right) \left( \tilde{A}_i + \tilde{A}_r \right) = n_2 \tilde{A}_t^2 \tag{3.16}
\]

\[
(\tilde{A}_i + \tilde{A}_r) = \tilde{A}_i \tag{3.17}
\]

of Eq. (3.13). We eliminate \(\tilde{A}_t\) from Eq. (3.16) using Eq. (3.17) to obtain

\[
\frac{\tilde{A}_r}{\tilde{A}_i} = \frac{n_1 - n_2}{n_1 + n_2}. \tag{3.18}
\]

Subsequently, we eliminate \(\tilde{A}_r\) to get

\[
\frac{\tilde{A}_t}{\tilde{A}_i} = \frac{2n_1}{n_1 + n_2}. \tag{3.19}
\]

We see that the usual Fresnel relations can be derived without invoking Maxwell’s equations. Conservation of energy, by itself, is sufficient to derive Eq. (3.13). However, there are several ways that Eq. (3.13) can be decomposed into two first-order equations. The application of Stoke’s theorem to the wave equation guarantees uniqueness of the decomposition represented by Eqs. (3.16) and (3.17) and the Fresnel relations, Eqs. (3.18) and (3.19).

IV. MOMENTUM CONSERVATION IN PIECEWISE HOMOGENEOUS MEDIA

The correct form for the momentum of the electromagnetic field in a dielectric is the subject of the century-old Abraham–Minkowski controversy [10–16]. The currently accepted resolution of the controversy, due to Møller [17], Penfield and Haus [18], Pfeifer et al. [19], and others, is that the issue is undecided because neither the Abraham momentum nor the Minkowski momentum is the total momentum. If we adopt this viewpoint, then the
Abraham momentum and the Minkowski momentum are irrelevant. Here we use conservation of total energy and conservation of total momentum to show that each component of the total linear momentum, reflected, refracted or transmitted, and kinematic, has a definite expression in terms of the macroscopic fields.

The energy and momentum of an electromagnetic pulse in the vacuum

\[ U_v = \int_{\sigma} \frac{1}{2} (\Pi_v^2 + B_v^2) \, dv \quad (4.1) \]

\[ \vec{G}_v = \int_{\sigma} \vec{B}_v \times \frac{\Pi_v}{c} \, dv \quad (4.2) \]

are well-defined and settled. Here, \( v \) denotes a quantity that is based in the vacuum, \( n_1 = 1 \). The total energy and the total momentum of our system in the initial configuration at \( t_0 \) as shown in Fig. 1 are considered to be given quantities. Figure 3 shows the result of continuing the numerical solution of the wave equation, Eq. (3.1), until the pulse has propagated completely through the medium. The incident, reflected, and transmitted fields are in vacuum with well-defined energies so that we can write an energy balance equation

\[ U_i = U_r + U_T + U_{\text{kinematic}}. \]

Here, \( U_{\text{kinematic}} \) is the kinematic energy of a solid block of dielectric material. Writing the components of the energy balance equation in terms of the corresponding vector potential amplitudes, we have

\[ \int_{\sigma} \frac{\omega^2}{2c^2} \tilde{A}_v^2 \, dv = \int_{\sigma} \frac{\omega^2}{2c^2} \tilde{A}_i^2 \, dv + \int_{\sigma} \frac{\omega^2}{2c^2} \tilde{A}_T^2 \, dv + U_{\text{kinematic}}. \quad (4.3) \]

Conservation of linear momentum cause more problems than conservation of energy because linear momentum is a directed quantity that changes sign upon reflection. Surface reflection takes momentum from the field and transfers the momentum to the material through radiation pressure. Once the field has passed entirely through the surface, as in Fig. 2, there are no more surface forces and the block of material moves with constant velocity carrying a momentum \( \vec{G}_{\text{kinematic}} \). Note that \( \vec{G}_{\text{kinematic}} \) is not the kinetic momentum described by Barnett [19]. Here, in Fig. 3, all the fields have left the material and have well-defined momentums in the vacuum. Then, we can write

\[ \vec{G}_i = \vec{G}_r + \vec{G}_T + \vec{G}_{\text{kinematic}} \quad (4.4) \]

or

\[ \int_{\sigma} \alpha \tilde{A}_i^2 \hat{e}_z \, dv = - \int_{\sigma} \alpha \tilde{A}_r^2 \hat{e}_z \, dv + \int_{\sigma} \alpha \tilde{A}_T^2 \hat{e}_z \, dv + \vec{G}_{\text{kinematic}} \quad (4.5) \]

by conservation of linear momentum. Here, \( \alpha = \omega^2/(2c^3) \) is a useful combination of coefficients. Substituting Eq. (4.3) into Eq. (4.5) we find that the kinematic momentum of the material is

\[ \vec{G}_{\text{kinematic}} = \int_{\sigma} 2\alpha \tilde{A}_r^2 \hat{e}_z \, dv + U_{\text{kinematic}}/c. \quad (4.6) \]

Taking \( U_{\text{kinematic}}/c \) to be negligible, we find that the kinematic momentum is twice the momentum of the reflected field, but in the forward direction as determined by the direction of the incident field such that

\[ \vec{G}_{\text{kinematic}} = -2\vec{G}_r = - \int_{\sigma} 2\frac{\vec{B}_r \times \Pi_r}{c} \, dv. \quad (4.7) \]

The sign is a bit awkward, but necessary, because \( \vec{B}_r \times \Pi_r \) is in the backward (negative) direction while the kinematic momentum is in the forward direction.

Now we can return to the situation of Fig. 2 with the refracted pulse entirely within the medium. There is a linear momentum that is associated with the propagating field in the medium [20] although its composition in terms of what portion is field momentum and what portion is material momentum remains disputed [10]. Whatever the composition, the momentum that travels with the field through the material, \( \vec{G}_i \), must be equal to the well-defined momentum of the field

\[ \vec{G}_i(t_1) = \vec{G}_r(t_2) = \int_{\sigma} \frac{\vec{B}_T \times \Pi_T}{c} \, dv \quad (4.8) \]

that has exited the material through the antireflection coating. Applying conservation of energy, Eq. (3.10), we find that \( \Pi_t = \sqrt{n} \Pi_T, \vec{B}_t = \sqrt{n} \vec{B}_T, \) and

\[ \vec{G}_i(t_1) = \int_{\sigma} \frac{\vec{B}_t \times \Pi_t}{c} \, dv \quad (4.9) \]

is the total momentum that travels with the field inside the medium. When comparing Eqs. (4.8) and (4.9), recall that the refracted field is spatially narrower than the transmitted field. Substituting Eqs. (4.7)–(4.9) into Eq. (4.3), we find that the total momentum \( \vec{G}_{\text{tot}} \) is the
sum of well-defined quantities for the refracted momentum, the reflected momentum, and the kinematic momentum

$$G_{tot} = \int_{\sigma} \frac{B_t \times \Pi_t}{c} dv + \int_{\sigma} \frac{B_r \times \Pi_r}{c} dv - \int_{\sigma} \frac{2B_r \times \Pi_r}{c} dv.$$  \hspace{1cm} (4.10)

The identification of $G_{tot}$ with $G_3$ is proven by demonstrating conservation of the total momentum. Substituting the definitions of the fields, Eqs. (2.3) and (2.4), and the Fresnel relations, Eqs. (3.18) and (3.19), into Eq. (4.10) proves that the total momentum is conserved. The total momentum has a definite electromagnetic component

$$G_{em} = \int_{\sigma} \frac{B_t \times \Pi_t}{c} dv + \int_{\sigma} \frac{B_r \times \Pi_r}{c} dv$$  \hspace{1cm} (4.11)

that is associated with the propagating field. The material component

$$G_{mat} = - \int_{\sigma} \frac{2B_r \times \Pi_r}{c} dv$$  \hspace{1cm} (4.12)

is the momentum of the block of dielectric. In the next section, we will relate the kinematic movement of the block to the Fresnel surface force. Now we consider the general case of a dielectric block with index $n_2$ in an inviscid dielectric fluid of index $n_1$. The momentum of the field in the dielectric block is known to be

$$G_{em} = \int_{\sigma} \frac{B \times \Pi}{c} dv$$  \hspace{1cm} (4.13)

by Eq. (4.9) and by prior work [2][5]. Then repeating the above analysis, the total, electromagnetic, and material momentums are still given by the formulas, Eqs. (4.10)–(4.12), although the transmitted and reflected fields are different in accordance with the Fresnel relations, Eqs. (3.18) and (3.19).

V. ELECTROMAGNETIC CONTINUITY EQUATIONS

The field imparts a surface force to the material due to the change of sign of the electromagnetic momentum upon reflection. By Newton’s third law, the material accelerates, increasing in momentum. By Newton’s second law, the material imposes an equal force on the electromagnetic field and momentum is extracted from the field. Clearly, the subsystems are open systems as momentum is removed from the field and transferred to the material by the surface force, but the total system is thermodynamically closed and the total linear momentum, as well as the total energy, is conserved.

Consider a quasimonochromatic pulse incident on an arbitrarily large homogeneous medium. The medium is draped with a gradient-index antireflection coating and the index changes sufficiently slowly that Helmholtz forces are negligible. Then a stationary medium remains stationary, the system is thermodynamically closed, and energy and momentum are conserved. The tensor continuity equation is

$$\partial_\beta T^{\alpha\beta} = 0,$$  \hspace{1cm} (5.1)

where

$$\partial_\beta = \left(\frac{n}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$  \hspace{1cm} (5.2)

is the material four-divergence operator [2][3][7][21].

$$T = \begin{bmatrix} \frac{(\Pi^2 + B^2)}{2} & (B \times \Pi)_1 & (B \times \Pi)_2 & (B \times \Pi)_3 \\ (B \times \Pi)_1 & W_{11} & W_{12} & W_{13} \\ (B \times \Pi)_2 & W_{21} & W_{22} & W_{23} \\ (B \times \Pi)_3 & W_{31} & W_{32} & W_{33} \end{bmatrix}$$  \hspace{1cm} (5.3)

is the total energy–momentum tensor [2][5], and

$$W_{ij} = -\Pi_i \Pi_j - B_i B_j + \frac{1}{2}(\Pi^2 + B^2)\delta_{ij}$$  \hspace{1cm} (5.4)

is the stress tensor [2][5].

The imposition of a step-index interface on the incident surface of the solid material is accompanied by a surface force $F$ due to Fresnel reflection. For a field of cross-sectional area $A$ with square temporal dependence,

$$\Delta G_{mat} = -2A \frac{B_t \times \Pi_t}{c} \frac{c \Delta t}{n_1}$$  \hspace{1cm} (5.5)

is found by integration of Eq. (4.7). Here, as in the preceding section, $n_1$ is the refractive index of the region from which the field originates, that is, the index of the dielectric fluid (or vacuum $n_1 = 1$) in which the dielectric block is immersed. Then

$$F = \frac{n_1}{c} \frac{\Delta G_{mat}}{\Delta t} = -2A \frac{B_t \times \Pi_t}{c}.$$  \hspace{1cm} (5.6)

The force must represent a source or sink of electromagnetic momentum and it must therefore have the same time dependence as the momentum continuity equation. For a field in the plane-wave limit that is normally incident on the block of material, the radiation pressure

$$\frac{F}{A} = \frac{1}{A} \frac{n_1 c \Delta G_{mat}}{\Delta t} = -2B_r \times \Pi_r$$  \hspace{1cm} (5.7)

acts on the incident surface at $z = 0$. Then the radiation pressure can be represented in terms of a force density as

$$f = (-2B_r \times \Pi_r)\delta(z).$$  \hspace{1cm} (5.8)

There is no source or sink of electromagnetic energy so we can write a four-force density

$$f_\alpha = (0, (-2B_r \times \Pi_r)\delta(z)).$$  \hspace{1cm} (5.9)

Then the tensor continuity equation for the unimpeded flow of the electromagnetic field, Eq. (5.1), becomes

$$\partial_\beta T^{\alpha\beta} = f_\alpha$$  \hspace{1cm} (5.10)
for a piecewise homogeneous medium. Because the force density is a sink of the electromagnetic momentum density, there is an equal and opposite force that acts as a source for the kinematic momentum of the material. Using the results of Ref. [7] we can derive Newton’s second law

$$ F = M \frac{d\mathbf{v}}{d(t/n_1)} $$

for a material body of mass $M$ immersed in a dielectric fluid of index $n_1$. Then the momentum conservation law for the solid block of material is

$$ F = M \frac{n_1}{c} \frac{d\mathbf{v}}{dt} = \int_{\sigma} -2\mathbf{B}_r \times \Pi \delta(z) dv. \quad (5.11) $$

Note that the tensor continuity equation for a flow of non-interacting material particles that is based on a dust tensor that is used in Refs. [10] and [22], for example, does not apply here because we have posited a solid dielectric. The components of the tensor continuity equation, Eq. (5.10), are the energy continuity equation,

$$ \frac{n}{c} \frac{\partial}{\partial t} \left( \frac{1}{2} (\Pi^2 + \mathbf{B}^2) \right) + \nabla \cdot \left( \frac{\mathbf{B} \times \Pi}{c} \right) = 0, \quad (5.12) $$

and the momentum continuity equation,

$$ \frac{n}{c} \frac{\partial}{\partial t} (\mathbf{B} \times \Pi) + \nabla \cdot W = (-2\mathbf{B}_r \times \Pi_r) \delta(z). \quad (5.13) $$

The momentum continuity equation, Eq. (5.13), explicitly displays the relation between the change in momentum on the left-hand side and the effect of the force, acting as a momentum sink, on the right-hand side. The momentum continuity equation is not an exact composition of Eqs. (5.12) because boundary conditions impose additional constraints.

One of the enduring questions of the Abraham–Minkowski controversy is why the Minkowski momentum is so often measured experimentally while the Abraham form of momentum seems to be so favored in theoretical work. We now have the tools to answer that question. The Minkowski momentum is not measured directly, but inferred from a measured index dependence of the optical force on a mirror placed in a dielectric fluid [8, 10, 19]. Because the field is completely reflected at the mirror, the force on the mirror is

$$ \mathbf{F} = \frac{n}{c} \frac{d}{dt}(2c\mathbf{G}) = \frac{n}{c} \frac{d}{dt} \int_V 2\mathbf{B} \times \Pi \delta(z) dv. \quad (5.14) $$

The measured force on the mirror is directly proportional to the refractive index $n = n_1$ of the fluid [8, 10]. On the other hand, if we were to assume $\mathbf{F} = d\mathbf{G}/dt$, then we can write Eq. (5.14) as

$$ \mathbf{F} = \frac{1}{c} \frac{d}{dt} \int_V 2\mathbf{D} \times \Pi \delta(z) dv \quad (5.15) $$

using $\mathbf{D} = -n\Pi$. Then one might infer that the momentum of the field in the dielectric fluid is the Minkowski momentum. Instead, we see that the electromagnetic momentum that is obtained from an experiment that measures the optical force on a mirror depends on the theory that is used to interpret the results. However, based on the changes to continuum electrodynamics that are necessitated by conservation of energy and momentum by the propagation of light in a continuous medium, we find that Eq. (5.14) is the correct relation between the force on the mirror and the momentum of the field in a dielectric.

VI. CONCLUSION

The extraordinary persistence of theoretical and experimental inconsistencies surrounding the Abraham–Minkowski controversy [10, 16] regarding the energy–momentum tensor for light in a linear medium suggested the need to re-examine the role of conservation of energy and conservation of momentum in classical continuum electrodynamics. We found that it is necessary to give up the classical macroscopic Maxwell equations in order to preserve the tensor form of the energy and momentum conservation laws [2–6]. Then we must re-examine the body of work that has been built upon the historical forms of the macroscopic Maxwell equations. In this article, we derived equations of motion, boundary conditions, continuity equations and radiation forces for the limiting case of macroscopic fields $\mathbf{B}$ and $\Pi$ propagating through a piecewise-homogeneous linear dielectric medium.

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