Higher dimensional Lax pairs
of lower dimensional chaos and turbulence systems

Sen-yue Lou*

CCAST (World Laboratory), PO Box 8730, Beijing 100080, P. R. China
Physics Department of Shanghai Jiao Tong University, Shanghai 200030, P. R. China
Physics Department of Ningbo University, Ningbo 315211, P. R. China

Abstract

In this letter, a definition of the higher dimensional Lax pair for a lower dimensional system which may be a chaotic system is given. A special concrete (2+1)-dimensional Lax pair for a general (1+1)-dimensional three order autonomous partial differential equation is studied. The result shows that any (1+1)-dimensional three order semi-linear autonomous system (no matter it is integrable or not) possesses infinitely many (2+1)-dimensional Lax pairs. Especially, every solution of the KdV equation and the Harry-Dym equation with their space variable being replaced by the field variable can be used to obtain a (2+1)-dimensional Lax pair of any three order (1+1)-dimensional semi-linear equation.

In the nonlinear mathematical physics, if an (n+1)-dimensional nonlinear system can be considered as a consistent condition of an (n+1)-dimensional linear system, then many types of interesting properties of the nonlinear system can be obtained by analyzing the linear system. The linear system is called as the Lax pair of the original nonlinear system [1].

In this letter, we use the concept of the Lax pair under a more general meaning. If the compatible condition

\[ [L_1, L_2] \psi \equiv (L_1 L_2 - L_2 L_1) \psi = 0, \]  

(1)

of a pair of linear equations

\[ L_1(x_1, x_2, ..., x_n, t, u(x_1, x_2, ..., x_m, t)) \psi(x_1, x_2, ..., x_n, t) \equiv L_1 \psi = 0, \]  

(2)

\[ L_2(x_1, x_2, ..., x_n, t, u(x_1, x_2, ..., x_m, t)) \psi(x_1, x_2, ..., x_n, t) \equiv L_2 \psi = 0, \quad m < n, \]  

(3)

*Email: sylou@mail.sjtu.edu.cn
is only a \( \psi \)-independent nonlinear equation of \( u(x_1, x_2, ..., x_m, t) \equiv u \)

\[
F(u) = 0,
\]

(4)
then we call the system (2) and (3) the \((n+1)\)-dimensional Lax pair of the \((m+1)\)-dimensional nonlinear equation (4) (or equivalently (1)).

To find some nontrivial examples, we restrict ourselves to discussion the case for

\[
L_1 \equiv \frac{\partial^2}{\partial x \partial y} - F_y(t, y, u, u_x), \quad u = u(x, t)
\]

(5)

\[
L_2 \equiv \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} - G(t, y, u, u_x, u_{xx}) \frac{\partial}{\partial x},
\]

(6)
where \( F \equiv F(t, y, u, u_x), \ G \equiv G(t, y, u, u_x, u_{xx}) \) should be selected appropriately such that Eq. (1) is only a \((1+1)\)-dimensional partial differential equation of \( u \). For the notation simplicity later, the undetermined function in the equation (5) is written as a partial derivative form.

In the next step we try to find some autonomous evolution models

\[
u_t = H_0(u, u_x, u_{xx}, u_{xxx}, ...,)
\]

(7)
which are given by the compatibility conditions of (2) and (3) with (5) and (6)

\[
\psi_{xyt} = \psi_{txy}
\]

(8)
or equivalently (1) with (5) and (6).

Using the relations (5), (6) and (7), the compatibility condition (8) has the following form

\[
W_1(t, y, u, u_x, ...) \psi + W_2(t, y, u, u_x, ...) \psi_x + (3F_{u_x} u_{xx} - G + 3F_u u_x)y \psi_{xx} = 0,
\]

(9)
where \( W_1(t, y, u, u_x, ...) \equiv W_1 \) and \( W_2(t, y, u, u_x, ...) \equiv W_2 \) are two complicated functions of the indicated variables and \( F_{u_x} \) and \( F_u \) denotes the partial derivatives of \( F \) with respect to \( u_x \) and \( u \).

The \( \psi \)-independent condition of (8) requires that three terms of (9) should all be zero. Vanishing the last term of (9) yields

\[
G = 3F_{u_x} u_{xx} + 3F_u u_x + G_0(t, u, u_x, u_{xx}),
\]

(10)
where \( G_0(t, u, u_x, u_{xx}) \) is a \( y \)-independent arbitrary function of the indicated variables. Substituting (10) into (9), we find that \( W_2 \) becomes zero identically while \( W_1 = 0 \) is simplified to

\[
W_{11}(t, y, u, u_x, ..., u_{k,x}) + F_{yu_x} H_{0u_x u_{k+1,x}} u_{k+1,x} = 0, \quad k > 3,
\]

(11)
where \( u_{k,x} \) denotes the \( k \)th order derivative of \( u \) with respect to \( x \), \( k \) is the highest derivative order of \( u \) included in \( H_0(u, u_x, u_{xx}, u_{xxx}, ...,) \equiv H_0 \) and \( W_{11} \) is \( u_{K,x} (K \geq k + 1) \) independent
complicated function. Because (11) should be true for the general solution \( u \) of (7), two terms of (11) should all be zero. For \( F_{yu_x} \neq 0 \), vanishing second term of (11), we have the result that \( H \) is \( u_{k,x} \) independent for \( k \geq 4 \).

For \( k = 3 \), (8) with (10) has the form

\[
W_{11}(t, y, u, u_x, u_{xx}, u_{xxx}) + F_{yu_x}(H_{0u_{xxx}} + 1)u_{xxxx} = 0. \tag{12}
\]

Vanishing the second term of (12) yields the result that the only possible form of \( H_0 \) can be written as

\[
H_0 = -u_{xxx} + H(u, u_x, u_{xx}) \tag{13}
\]

for \( F_{yu_x} \neq 0 \). Substituting (13) back to (12) yields

\[
W_{12}(t, y, u, u_x, u_{xx}) + (F_{yu_x}H_{u_{xxx}} - 3F_yF_{u_x} + 3F_{yu_{xx}}u_x + 3F_{yu_xu_x}u_{xx} - F_yG_{0u_{xx}})u_{xxx} = 0. \tag{14}
\]

Because \( W_{12}(t, y, u, u_x, u_{xx}) \) is \( u_{xxx} \) independent, the second term of (14) should be also zero. Vanishing the coefficients of \( u_{xxx} \) of (14) and partial integrating once with respect to \( u_{xx} \) we obtain

\[
\frac{3}{2}F_{yu_xu_x}u_{xx}^2 + 3(F_{yu_{xx}}u_x - F_yF_{u_x})u_{xx} - F_yG_0 + F_{yu_x}H + F_1(t, y, u, u_x) = 0, \tag{15}
\]

where \( F_1(t, y, u, u_x) \equiv F_1 \) is an integrating function. Because \( F \) and \( F_1 \) in (15) are \( y \)-dependent while all other functions in (15) are all \( y \)-independent, we can conclude that the only possible solution of (15) has the form

\[
G_0 = \frac{3}{2}\Gamma_2(t, u, u_x)u_{xx}^2 + \Gamma_3(t, u, u_x)u_{xx} + \Gamma_1(t, u, u_x)H + \Gamma_4(t, u, u_x), \tag{16}
\]

for general \( H \) with

\[
F_{yu_x} = F_y\Gamma_1(t, u, u_x), \tag{17}
\]

\[
F_{yu_xu_x} = F_y\Gamma_2(t, u, u_x), \tag{18}
\]

\[
3(F_{yu_{xx}}u_x - F_yF_{u_x}) = F_y\Gamma_3(t, u, u_x), \tag{19}
\]

\[
F_1(t, y, u, u_x) = F_y\Gamma_4(t, u, u_x), \tag{20}
\]

and \( \Gamma_1(t, u, u_x) \), \( \Gamma_2(t, u, u_x) \), \( \Gamma_3(t, u, u_x) \) and \( \Gamma_4(t, u, u_x) \) being four unknown functions which will be determined later.

According to the compatibility condition \( F_{uu_x} = F_{u_{x}u} \) and the fact that \( F \) is the only \( y \)-dependent function, we can obtain the unique possible solution of (17)-(19):

\[
\Gamma_1(t, u, u_x) = u_{x}^{-1}, \tag{21}
\]
\[ \Gamma_2(t, u, u_x) = 0, \]  
\[ \Gamma_3(t, u, u_x) = -3G_{1ux}(t, u, u_x), \]  
\[ F = G_1(t, u, u_x) + u_x F_0(t, y, u), \]  
\[ F_{0u}(t, y, u) = \frac{1}{2} F_0(t, y, u)^2 + \Gamma_5(t, u), \]  

where \( G_1(t, u, u_x) \) is an arbitrary function of \( \{ t, u, u_x \} \) and \( \Gamma_5(t, u) \) is an arbitrary function of \( \{ t, u \} \). Using the relations (16), (21)-(25), Eq. (12) is simplified to

\[
\frac{F_{0ty}(t, y, u)}{F_{0y}(t, y, u)} - [2u_x^3 \Gamma_5(t, u) + \Gamma_4(t, u, u_x) + 3G_{1u}(t, u, u_x) u_x^2] F_0(t, y, u) = 0,
\]

\[
-\frac{u_{xx} [3u_x (2 \Gamma_5(t, u) + G_{1uxx}(t, u, u_x)) + \Gamma_{4xx}(t, u, u_x) + 6G_{1u}(t, u, u_x) + u_x^{-1} \Gamma_4(t, u, u_x)]}{\Gamma_4(t, u, u_x) - 2u_x^2 \Gamma_{5u} - 3u_x^2 G_{1ux}(t, u, u_x) = 0.}
\]

Because \( F_0(t, y, u) \) is \( y \)-dependent, and other functions in (26) are \( y \) and \( u_{xx} \) independent, (26) is true only for

\[
F_{0ty}(t, y, u) = F_{0y}(t, y, u) (\Gamma_6(t, u) F_0(t, y, u) + \Gamma_7(t, u)),
\]

\[
3u_x (2 \Gamma_5(t, u) + G_{1uxx}(t, u, u_x)) + \Gamma_{4xx}(t, u, u_x) + 6G_{1u}(t, u, u_x) + u_x^{-1} \Gamma_4(t, u, u_x) = 0, \tag{28}
\]

\[
\Gamma_7(t, u) - u_x \Gamma_{4u}(t, u, u_x) - 2u_x^2 \Gamma_{5u} - 3u_x^2 G_{1ux}(t, u, u_x) = 0, \tag{29}
\]

\[
2u_x^2 \Gamma_5(t, u) + \Gamma_4(t, u, u_x) u_x + 3G_{1u}(t, u, u_x) u_x^2 - \Gamma_6(t, u) = 0, \tag{30}
\]

where \( \Gamma_6(t, u) \) and \( \Gamma_7(t, u) \) are arbitrary functions of \( \{ t, u \} \). The general solution of (28)-(30) reads

\[
\Gamma_4(t, u, u_x) = -3u_x G_{1u}(t, u, u_x) - 2u_x^2 \Gamma_5(t, u) + u_x^{-1} \Gamma_6(t, u), \tag{31}
\]

\[
\Gamma_7(t, u) = \Gamma_{6u}(t, u). \tag{32}
\]

Integrating (27) once with respect to \( y \) leads to

\[
F_{0t}(t, y, u) = \frac{1}{2} \Gamma_6(t, u) F_0(t, y, u)^2 + \Gamma_{6u}(t, u) F_0(t, y, u) + \Gamma_8(t, u), \tag{33}
\]

where \( \Gamma_8(t, u) \) is a further integrating function. Because one function, \( F_0(t, y, u) \), should satisfy two equations (25) and (33), the compatibility condition \( F_{tu} = F_{ut} \) yields the constraints

\[
\Gamma_8(t, u) = \Gamma_6(t, u) \Gamma_5(t, u) + \Gamma_{6uu}(t, u), \tag{34}
\]

and

\[
\Gamma_{5u}(t, u) = \Gamma_{6uu}(t, u) + \Gamma_6(t, u) \Gamma_{5u}(t, u) + 2 \Gamma_5(t, u) \Gamma_{6u}(t, u). \tag{35}
\]

Finally, we write down the obtained results here. A \((1+1)\)-dimensional three order equation

\[
u_t = -u_{xxx} + H(u, u_x, u_{xx}) \tag{36}
\]
with an arbitrary nonlinear interaction term \(H(u, u_x, u_{xx})\) possesses a (2+1)-dimensional Lax pair

\[
\psi_{xy} - u_x F_0(t, y, u) \psi = 0, \tag{37}
\]

\[
\psi_t + \psi_{xx} - [3F_0(t, y, u)u_{xx} + 3u_x^2 F_0u(t, y, u) + H_1(t, u, u_x, u_{xx})] \psi_x = 0, \tag{38}
\]

where

\[
H_1(t, u, u_x, u_{xx}) = u_x^{-1}(H(u, u_x, u_{xx}) + \Gamma_6(t, u)) - 2u_x^2 \Gamma_5(t, u). \tag{39}
\]

In the Lax pair equations (37) and (38), the function \(F_0(t, y, u) \equiv F_0\) is determined by a pair of the consistent Riccati equations

\[
F_{0t}(t, y, u) = \frac{1}{2} \Gamma_6(t, u) F_0(t, y, u)^2 + \Gamma_6u(t, u) F_0(t, y, u) + \Gamma_6(t, u) \Gamma_5(t, u) + \Gamma_6u, \tag{40}
\]

\[
F_{0u}(t, y, u) = \frac{1}{2} F_0(t, y, u)^2 + \Gamma_5(t, u) \tag{41}
\]

while the consistent condition of (40) and (41) gives the constraint equation (35) for the functions \(\Gamma_6(t, u)\) and \(\Gamma_5(t, u)\).

It is known that for the usual Lax pair of a given nonlinear model, there may be infinitely many Lax pairs\(^2\). The similar situation appears for the extended higher dimensional Lax pairs. For instance, for the given evolution system (36), the arbitrariness in the selections of the functions \(\Gamma_5(t, u)\) and \(\Gamma_6(t, u)\) and the solutions of \(F_0(t, y, u)\) means that the evolution equation (36) possesses infinitely many different kinds of Lax pairs. To give out some more concrete results, we may fix the functions \(\Gamma_6(t, u)\) and \(\Gamma_5(t, u)\). The first interesting selection is

\[
\Gamma_6(t, u) = \Gamma_5(t, u). \tag{42}
\]

Under the selection (42), (36) becomes the well known Korteweg de-Vries (KdV) equation

\[
\Gamma_5t(t, u) = \Gamma_5uuu(t, u) + 3\Gamma_5(t, u) \Gamma_5u(t, u). \tag{43}
\]

with the independent variables \(\{t, u\}\) and the linearized system of (40) and (41) by using the Cole-Hopf transformation is just the Lax pair of the KdV equation (after neglecting the variable \(y\) in (40) and (41)). Now substituting every known special solution of the KdV equation and the related solution of the pseudopotential \(F_0\) of (40) and (41) into (37) and (38), we get a concrete Lax pair of the system (36). The simplest trivial solution of the KdV equation is

\[
\Gamma_5(t, u) = \Gamma_6(t, u) = 0, \tag{44}
\]

and the related solution of (40) and (41) reads

\[
F_0(t, y, u) = -\frac{2}{u + q(y)}, \tag{45}
\]
where \( q(y) \) is an arbitrary function of \( y \).

The second interesting selection is

\[
\Gamma_6(t, u) = \Gamma_5^{-1/2}(t, u).
\]  

(46)

In this case, (36) becomes the well known Harry-Dym (HD) equation

\[
\Gamma_5(t, u) = [\Gamma_5^{-1/2}(t, u)]_{uuu}.
\]  

(47)

with the independent variables \( \{t, u\} \) and the linearized system of (40) and (41) is the Lax pair of the HD equation. That means substituting every known special solution of the HD equation and the related solution of the psudopotential \( F_0 \) of (40) and (41) into (37) and (38) will yield a concrete Lax pair of the semi-linear system (36).

Generally, the evolution equation (36) is nonintegrable under the traditional meanings. In Ref. [3], the authors had claimed that there exist only six nonequivalent three order semi-linear integrable models under the usual meanings. In other words, the evolution equation (36) is chaotic or turbulent for most of the selections of \( H \). The results of this letter show us that no matter of the model (36) is integrable or not, it may have infinitely many \((2+1)\)-dimensional Lax pairs. A simple special nonintegrable example of (36) is the so-called KdV-Burgers (KdVB) equation

\[
u_t + uu_x - \nu u_{xx} + u_{xxx} = 0.
\]  

(48)

The KdVB is one of the possible candidate to describe the turbulence phenomena in fluid physics and plasma physics. Though one has not yet find any \((1+1)\)-dimensional Lax pair of the turbulence system KdVB equation, we can do find many types of \((2+1)\)-dimensional Lax pairs. Especially, every solution of the KdV equation and the HD equation can be used to form a \((2+1)\)-dimensional Lax pair of the KdVB equation.

In summary, a lower dimensional nonlinear system may have some (perhaps infinitely many) types of higher dimensional Lax pairs. In terms of a special example we show that any three order semi-linear equation (no mater it is integrable or not) possesses infinitely many Lax pairs. The next interesting important problem should be studied in the future work may be: What kinds of information about lower dimensional systems especially the lower dimensional turbulent and chaotic systems can be obtained from some types of special higher dimensional Lax pairs?

The work was supported by the Outstanding Youth Foundation and the National Natural Science Foundation of China (Grant. No. 19925522), the Research Fund for the Doctoral Program of Higher Education of China (Grant. No. 2000024832) and the Natural Science Foundation of Zhejiang Province, China. The author is in debt to thanks the helpful discussions with the
professors Q. P. Liu, G-x Huang and C-p Sun and the Drs. X-y Tang, S-l Zhang, C-l Chen and B. Wu.
References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons Nonlinear Evolution Equations and Inverse Scattering, London Mathematical Society Lecture Note Series 149 (1991 Cambridge University Press).

[2] S-Lou and X-b Hu, J. Math. Phys. 38(1997)6401.

[3] S. I. Svinolupov, V. V. Sokolov, and R. I. Yamilov, Sov. Math. Dokl., 28 (1983) 165; S. I. Svinolupov and V. V. Sokolov, Func. Anal. Appl. 16 (1982) 317.

[4] Y. Nakamura, H. Bailung and P. K. Shukla, Phys. Rev. Lett. 83 (1999) 1602; E. P. Raposo and D. Bazeia, Phys. Lett. A253 (1999) 151; G. Karch, Nonlinear Analysis: Theory Methods & Applications, 35 (1999)199;

[5] S-d Liu and S-k Liu, Sincia Sinica, A 9 (1991) 938 (in Chinese); N. Antar, and H. Demiray, Int. J. Engineering Sci. 37 (1999) 1859.