Rotation Domains and Stable Baker Omitted Value

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Abstract
A Baker omitted value, in short bov of a transcendental meromorphic function $f$ is an omitted value such that there is a disk $D$ centered at the bov for which each component of the boundary of $f^{-1}(D)$ is bounded. Assuming all the iterates $f^n$ to be analytic in a neighborhood of its bov, this article proves a number of results on the rotation domains of the function. The number of $p$-periodic Herman rings is shown to be finite for each $p \geq 1$. It is also proved that every Julia component intersects the boundaries of at most finitely many Herman rings. Further, if the bov is the only limit point of the critical values then it is shown that $f$ has infinitely many repelling fixed points. If a repelling periodic point of period $p$ is on the boundary of a $p$-periodic rotation domain then the periodic point is shown to be on the boundary of infinitely many Fatou components. As a corollary we have shown that if $D$ is a $p$-periodic rotation domain of $f$ and $f^p$ is univalent in a neighbourhood of the boundary $\partial D$ of $D$ then $f$ has no repelling $p$-periodic point on $\partial D$. Under additional assumptions on the critical points, a sufficient condition is found for a Julia component to be a singleton. As a consequence, it is proved that if the boundary of a wandering domain $W$ accumulates at some point of the plane under the iteration of $f$ then each limit of $f^n$ on $W$ is either a parabolic periodic point or in the $\omega$-limit set of recurrent critical points. Using the same ideas, the boundaries of rotation domains are shown to be in the $\omega$-limit set of recurrent critical points.

Keywords Baker omitted value · Recurrent critical point · Rotation domain · Wandering domain

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1 Introduction

A function \( f : \mathbb{C} \to \hat{\mathbb{C}} \) with exactly one essential singularity, chosen to be at \( \infty \) is called general transcendental meromorphic if it has either at least two poles or one pole which is not an omitted value (i.e., there is at least one pre-image of the pole). The Fatou set of \( f \) (also called the stable set) denoted by \( \mathcal{F}(f) \) is defined as the set of all points at a neighborhood of which the sequence of functions \( \{f^n\}_{n \geq 0} \) is defined and normal in the sense of Montel. The complement of the Fatou set is called the Julia set. It is denoted by \( \mathcal{J}(f) \). By the definition of the Julia set, \( \infty \in \mathcal{J}(f) \). It is well known that \( \{f^n\}_{n \geq 0} \) is normal in a neighborhood of a point whenever \( f^n \) is defined and analytic for all \( n \) in the neighborhood [3].

A point \( b \in \mathbb{C} \) is called an omitted value of \( f \) if \( f(z) \neq b \) for any \( z \in \mathbb{C} \). A special type of omitted value, introduced by Chakra et al. in [5] is the concern of this article.

Definition 1.1 A Baker omitted value \( b \in \mathbb{C} \) of a meromorphic function \( f \) is an omitted value of \( f \) such that there is a disk \( D \) with center at \( b \) for which each component of the boundary of \( f^{-1}(D) \) is bounded.

The set of singular values is the closure of the union of the critical values and the asymptotic values. A critical value is the image of a critical point. On the other hand, a point \( a \in \hat{\mathbb{C}} \) is called an asymptotic value of \( f \) if there exists a curve \( \eta : [0, \infty) \to \mathbb{C} \) with \( \lim_{t \to \infty} \eta(t) = \infty \) such that \( \lim_{t \to \infty} f(\eta(t)) = a \). An omitted value is an asymptotic value.

The dynamics (the Fatou and the Julia set) of meromorphic functions, for which the set of singular values is finite or bounded has been investigated extensively. Some references can be found in [3]. In these studies the nature of the whole set of singular values (finite or bounded) is of primary importance. Instead of the whole set of singular values, one may look at few singular values with a particular property. How a specific type of singular values influences the dynamics of a function can be a different way to look at the subject. Omitted values are a special type of asymptotic values in the sense that every singularity lying over it is direct. More precisely, if \( \tilde{N} \) is a component of \( f^{-1}(N) \) for a sufficiently small neighborhood \( N \) of an asymptotic value \( a \) then \( f(z) \neq a \) for any \( z \in \tilde{N} \). A detailed discussion on the classification of singularities can be found in [12]. These are known to control a number of aspects of the dynamics of a function, as reported in [10,11,16]. A Baker omitted value is always a limit point of critical values (See Lemma 2.4 in the next section). This gives that the functions with a bov have infinitely many singular values. It is not known whether such a function can have an unbounded set of singular values or not. The well studied functions \( z \mapsto \lambda e^z \) and \( z \mapsto \lambda \tan z \) have omitted values. In each of these cases, the pre-image of a sufficiently small neighborhood of an omitted value is connected and simply connected. This fact has been crucial in the investigation in many different ways. But this is not the case for a Baker omitted value. In fact, the pre-image of a sufficiently small neighborhood of a bov is an infinitely connected domain (See Lemma 2.1 [5]). Thus, the investigation of dynamics of functions with bov is a new direction in transcendental dynamics. This has been initiated in [5].
Definition 1.2 The bov of a meromorphic function is called stable if the sequence of its iterates is defined in a neighborhood of the bov.

As remarked earlier, the stable bov is in the Fatou set of the function.

Let $\mathcal{M}_S$ denote the class of all general transcendental meromorphic functions with stable bov.

A Fatou component is a maximal connected subset of the Fatou set. It is very important to note that for every function in $\mathcal{M}_S$, all but one Fatou component are bounded (See Lemma 2.4 (2) [9] or Lemma 2.5(1) in the next section). This fact has been used in [9] to determine the connectivity of all Fatou components and is fundamental to many proofs of this article.

A Fatou component $V$ is called $p$-periodic if $p$ is the smallest non-negative integer such that $V_p \subseteq V$ where $V_k$ denotes the Fatou component containing $f^k(V)$ for $k \geq 0$ where $V_0$ is taken as $V$. If $V$ is not periodic but $V_m$ is periodic for some $m > 0$ then $V$ is called pre-periodic. A periodic Fatou component can be an attracting domain, a parabolic domain, a Baker domain, a Herman ring or a Siegel disk. The last two types of Fatou components (Herman rings and Siegel disks) are known as rotation domains.

More details on Fatou components can be found in [3].

Rotation domains are special in the sense that $f^p$ is conformally conjugate to an irrational rotation of an annulus (Herman ring) or the unit disk (Siegel disk) on a $p$-periodic rotation domain. In fact, a $p$-periodic Fatou component $V$ is a Herman ring (or a Siegel disk) if there exists a conformal map $\phi : V \to \{z : 1 < |z| < r\}$ (or $\phi : V \to \{z : |z| < 1\}$ respectively) such that $\phi(f^p(\phi^{-1}))(z) = e^{2\pi i \theta}z$ for some irrational $\theta$. A $p$-periodic rotation domain is an uncountable union of disjoint Jordan curves each of which is invariant under $f^p$. These curves are indeed the pre-images of the concentric circles centered at the origin under $\phi$.

Meromorphic functions with finitely many singular values cannot have infinitely many Herman rings. This is shown by Zheng [19] who also proved the existence of a function with infinitely many Herman rings. Though a function in $\mathcal{M}_S$ has infinitely many singular values, we have proved that there cannot be infinitely many Herman rings of a particular period. In view of the known fact that the period of a Herman ring of a meromorphic function with an omitted value is larger than two [16], we consider Herman rings of period at least three.

Theorem 1.1 Let $f \in \mathcal{M}_S$ and $p \geq 3$. Then the number of $p$-periodic Herman rings is finite.

Maximally connected subsets of the Julia set are referred as Julia components. Each component of the boundary of a Herman ring is always contained in a Julia component. But it is non-trivial to decide whether there can be infinitely many Herman rings sharing their boundaries with a common Julia component. The following result is an answer to this for all the functions with stable bov.

Theorem 1.2 Let $f \in \mathcal{M}_S$ and $J$ be a Julia component of $f$. Then the number of Herman rings whose boundary components are contained in $J$ is finite.

If the bov is the only limit point of the critical values for a function in $\mathcal{M}_S$, then the number of critical values lying in the Julia set is finite. Every Julia component $J$
containing a boundary component of a \( p \)-periodic Herman ring is invariant under \( f^p \). This is because every boundary component of such a Herman ring is invariant under \( f^p \) and \( f^k(J) \) is connected for all \( k \), the latter following from Lemma 2.5(5). This means that if the forward orbit of a critical value intersects the boundary of some Herman ring, then it intersects only finitely many Julia components. Therefore the number of Julia components intersecting the forward orbit of any such critical value is finite (as such critical values are finite in number). Since every Julia component intersects the boundaries of at most finitely many Herman rings by the previous theorem, we have the following corollary.

**Corollary 1.1** Let \( f \in \mathcal{M}_S \). If the bov is the only limit point of the critical values, then the number of Herman rings whose boundary intersects the forward orbit of a critical value is finite.

A fixed point \( z_0 \) of \( f \) is called weakly repelling if \( |f'(z_0)| > 1 \) or \( f'(z_0) = 1 \). These are related to the connectedness of the Julia set. The existence of weakly repelling fixed points for transcendental meromorphic functions is well-known in the presence of multiply connected Fatou components. This is proved for wandering domains in [4] whereas [7] deals with the case of immediate attracting and parabolic domains. These results ensure at least one weakly repelling fixed point. We prove the existence of infinitely many such fixed points for functions with a stable bov.

**Proposition 1.1** If \( f \in \mathcal{M}_S \) then it has infinitely many weakly repelling fixed points. Further, if the bov is the only limit point of the critical values then \( f \) has infinitely many repelling fixed points.

The existence of periodic points on the boundary of invariant rotation domains of rational functions is investigated by Imada [15]. He has proved that the boundary of an invariant rotation domain does not contain any periodic point except cremer points (i.e., irrationally indifferent periodic point not corresponding to a Siegel disk). Any such result for transcendental meromorphic functions is apparently not known. As a corollary to the following theorem we have shown that repelling fixed points cannot be on the boundary of invariant rotation domains whenever a function has stable bov and is injective in a neighborhood of the rotation domain. The theorem to follow says that if the boundary of a \( p \)-periodic rotation domain contains a repelling \( p \)-periodic point then the topology of the boundary of \( D \) is very complicated.

**Theorem 1.3** Let \( f \in \mathcal{M}_S \) and \( D \) be a \( p \)-periodic rotation domain of \( f \). If the boundary \( \partial D \) of \( D \) contains a repelling \( p \)-periodic point \( z_0 \) then, for each \( k \geq 1 \) there is a component \( D_{-k} \) of \( f^{-pk}(D) \) different from \( D \) such that \( z_0 \in \partial D_{-k} \).

The proof of the following corollary depends on the ideas developed in the proof of Theorem 1.3. It is given in Sect. 3.

**Corollary 1.2** Under the assumption of the above Theorem, if \( f^p \) is univalent in a neighborhood of the boundary of one of its \( p \)-periodic rotation domains \( D \) then there is no repelling \( p \)-periodic point on the boundary of \( D \).

Though the functions in \( \mathcal{M}_S \) have infinitely many critical values, only finitely many of them can be in the Julia set whenever the bov is the only limit point of the critical
values. The next two results assume that the bov is the only limit point of the critical values. This assumption, already made in Theorem 1.1 is necessary to make sense of recurrence of singular values in a natural way.

The \( \omega \)-limit set \( \omega(c) \) of a critical point \( c \) is the set of all accumulation points of its forward orbit, i.e., \( \omega(c) = \{ w : f^{n_k}(c) \to w \text{ as } k \to \infty \text{ for some subsequence } n_k \} \). This set is always closed. A critical point is said to be recurrent if it is in its own \( \omega \)-limit set. The possible presence of either infinitely many critical points or an asymptotic value, or both makes the study of recurrent singular values difficult for transcendental functions. In fact, the definition of a recurrent singular value itself requires extra considerations. However, the situation is amenable for functions in \( M_S \) whenever the bov is the only limit point of critical values. The importance of recurrent critical points is well-known in rational dynamics (See for example [13]). The next two results demonstrate the influence of recurrent critical points on Julia components, wandering domains and rotation domains. Wandering Julia components of a rational function are studied by Guizhen et al. in [14]. They have proved that each such Julia component, except countably many is either non-separating or its complement has two components. Following is a sufficient condition for Julia component of some transcendental functions to be a singleton.

**Theorem 1.4** For \( f \in M_S \) let,

1. the bov be the only limit point of critical values,
2. the number of critical points corresponding to each critical value lying in the Julia set is finite, and
3. every Fatou component containing a singular value is pre-periodic.

If \( J \) is a Julia component whose forward orbit accumulates at a point in \( \mathbb{C} \) which is neither a parabolic periodic point nor in the \( \omega \)-limit set of any recurrent critical point, then \( J \) is a singleton.

Each limit of the sequence of iterates of a transcendental meromorphic function on its wandering domain is known to be in the derived set of the postsingular set(i.e., the union of forward orbits of all the singular values) [18]. Under the assumption of the above theorem, \( U \), the Fatou component containing the bov is not wandering giving that its grand orbit cannot contain any wandering domain. Further, every other possible wandering domain of \( f \in M_S \) is simply connected by Theorem 3.1(1) [9]. This gives that the boundary of each wandering domain is connected. With a condition on the boundary of the wandering domain, we have shown that each limit of \( f^n \) on its wandering domain is a parabolic periodic point or in the \( \omega \)-limit set of recurrent critical points.

**Corollary 1.3** Let \( J \) be the boundary of a wandering domain of a function in \( M_S \) satisfying all the assumptions of Theorem 1.4. If its forward orbit \( \{ f^n(J) \}_{n>0} \) accumulates at a point \( w \in \mathbb{C} \) then \( w \) is either a parabolic periodic point or is in the \( \omega \)-limit set of recurrent critical points.

**Proof** Each wandering domain of \( f \) is simply connected and so \( J \) is connected and not a singleton. By Theorem 1.4, \( w \) is either a parabolic periodic point or is in the \( \omega \)-limit set of recurrent critical points.
The forward orbit of recurrent critical points are known to be dense in the boundary of rotation domains of rational maps (See for example Theorem 1.3 [13]). The main idea of the proof was to show a kind of backward contraction of pull backs of disks disjoint from the $\omega$-limit sets of recurrent critical points and parabolic periodic points. For transcendental meromorphic functions, it is known that the boundary of each rotation domain is contained in the closure of the forward orbits of all the singular values. Since each map with stable bov has a single asymptotic value and that is in its Fatou set, the boundary is contained in the closure of the forward orbit of critical points. These critical points turn out to be recurrent and we have the following for functions with stable bov with some additional assumptions.

**Theorem 1.5** For $f \in \mathcal{M}_S$, let
1. the bov be the only limit point of critical values,
2. the number of critical points corresponding to each critical value lying in the Julia set is finite, and
3. every Fatou component containing a singular value is pre-periodic.

Then the boundary of each rotation domain is contained in the $\omega$—limit set of the recurrent critical points.

The next section discusses some known and new results required for the proofs later. Section 3 presents all the proofs. A class of examples are provided in the last section.

Throughout this article, the Fatou component containing the bov is denoted by $U$. The disk $\{z : |z - a| < \delta\}$ is denoted by $D_\delta(a)$ for $\delta > 0$ and $a \in \mathbb{C}$. All the functions considered in this article are in $\mathcal{M}_S$ unless stated otherwise.

## 2 Preliminary Results

A non-empty connected and closed subset of $\hat{\mathbb{C}}$ is called a continuum. A continuum $K$ is called full if $\hat{\mathbb{C}} \setminus K$ is connected. If $K$ does not contain the point at $\infty$ and is not full then its complement has at least one bounded component. We say $K$ surrounds a point $z$ if a bounded component of $\hat{\mathbb{C}} \setminus K$ contains $z$. Clearly a full continuum does not surround any point. A Julia component is called full if it is a full continuum. Otherwise, it is called non-full. Non-full continua are also called separating. The following lemma is a generalization of Lemma 1, [11] and is to be used frequently. The proof is essentially the same. For this, we require a lemma from [10].

**Lemma 2.1** If $f$ is a meromorphic function with an omitted value and $D$ is a bounded domain then the closure of $f(D)$ cannot contain any omitted value.

**Lemma 2.2** Let $f$ be a transcendental meromorphic function and $K$ be a separating continuum not intersecting the backward orbit of $\infty$. If $K$ surrounds a point of $J(f) \setminus K$ then there is an $n \geq 0$ such that $f^n(K)$ surrounds a pole of $f$. Further, if $f$ has an omitted value then $f^{n+1}(K)$ surrounds the set of all omitted values of $f$.

**Proof** Consider a point $z \in J(f) \setminus K$ surrounded by $K$. Then there is a component $V$ of $\hat{\mathbb{C}} \setminus K$ containing $z$. Note that there is an $n$ such that $f^n$ is analytic on $V$ and
\( f^n(V) \) contains a pole, say \( w \). By the Maximum Modulus Principle, \( f^n(K) \) surrounds \( w \). The set \( f^{n+1}(V) \) contains a neighborhood of \( \infty \) and by Lemma 2.1, the closure of \( f^{n+1}(V) \) does not contain any omitted value. This gives that \( f^{n+1}(K) \) surrounds the set of all omitted values of \( f \).

\[ \square \]

It is well-known that for a meromorphic function, each pre-image component of every neighbourhood of an omitted value is unbounded. Further, the map is not one-one on any such pre-image component. This leads to the following lemma.

**Lemma 2.3** Let \( f \) be a meromorphic function with an omitted value. If \( V \) is a \( p \)-periodic component of \( f \) and \( f^p : V \to V \) is one-one then \( V \) does not contain any omitted value of \( f \). In particular, rotation domains do not contain any omitted value.

We put Lemma 2.5 and Lemma 2.3(1) of [9] together as a lemma that exhibits the influence of the bov on all other singular values of the function.

**Lemma 2.4** Let \( f \) be a meromorphic function with bov. Then,

1. The bov is a limit point of its critical values.
2. The function \( f \) has only one asymptotic value and that is the bov.

Some useful observations on the dynamics of functions with stable bov are made in Lemma 2.4 and Theorem 1.3 of [9]. We collect them here.

**Lemma 2.5** Let \( f \in \mathcal{M}_S \) and \( U \) be the Fatou component containing the bov. Then,

1. The pre-image of \( U \) is the only unbounded Fatou component of \( f \). Further, it is infinitely connected.
2. If \( U' \) is a Fatou component such that \( U'_k = U \) for some \( k \geq 1 \) then \( U' \) is infinitely connected.
3. If \( U \) is invariant then it is completely invariant.
4. There are infinitely many poles of \( f \).
5. All the components of \( J(f) \cap \mathbb{C} \) are bounded. In other words, every Julia component intersecting the backward orbit of \( \infty \) is a singleton. Consequently, for every non-singleton Julia component \( J \), \( f^k(J) \) is connected for all \( k \geq 1 \).
6. If \( U \) is unbounded then it is completely invariant. Consequently, \( f \) has no Herman ring.

We say a set \( A \) surrounds another set \( S \) if a bounded complementary component of \( A \) contains \( S \). The following two definitions appearing in [8] are very important for analyzing the arrangement of Herman rings. For a Herman ring \( H \), \( H_n \) denotes the Herman ring containing \( f^n(H) \).

**Definition 2.1** [Innermost ring with respect to a set] Given a Herman ring \( H \), we say \( H_k \) is innermost with respect to a set \( A \) if \( H_k \) surrounds \( A \) but does not surround any \( H_i \) for \( i \neq k \).

Taking \( K \) as an \( f^p \)-invariant Jordan curve in a \( p \)-periodic Herman ring \( H \) in Lemma 2.2, it is observed that \( H_n \) surrounds a pole for some \( n \). We choose \( n \) to be the smallest such number. Further, if \( H \) is taken to be the innermost Herman ring with respect to the bov then the first \( n \) forward iterates of \( H \) turn out to be crucial.
Definition 2.2 [Basic chain] Given a Herman ring $H$, the ordered set of rings $\{H_1, H_2, H_3, \ldots, H_k\}$ is called the basic chain, where $H_1$ is the innermost ring with respect to the bov and $k$ is the smallest natural number such that $H_k$ surrounds a pole. The number $k$ is called the length of the basic chain.

The basic chain of a cycle is also referred as the basic chain of a Herman ring contained in the cycle.

We collect some known facts about basic chains. Recall that $U$ denotes the Fatou component containing the bov. A finite sequence of rings $\{H_j, H_{j+1}, H_{j+2}, \ldots, H_{j+m}\}$ is called a chain whenever $H_j$ surrounds $U$ but not any pole and $m$ is the smallest natural number such that $H_{j+m}$ surrounds a pole.

Lemma 2.6 Let $f$ be a meromorphic function having a bov.

1. Every cycle of Herman rings has a unique basic chain.
2. The length of every chain is less than or equal to that of the basic chain.
3. For every $p$-cycle of Herman rings, the length of the basic chain $l_C$ satisfies $2 \leq l_C \leq p - 1$.

Proof 1. This is evident from the definition of the basic chain.
2. This is Lemma 2.3 of [8].
3. Since the innermost ring $H_1$ with respect to the bov never surrounds a pole by Remark 2.10 of [6], the length of the basic chain is at least two. If the length of the basic chain is equal to the period of the Herman ring then it follows from the definition of the basic chain that there is only one $H_1$-relevant pole, i.e., the total number of distinct poles surrounded by any of the Herman rings of the cycle is 1. However the number of $H$-relevant poles of every Herman ring of a function with an omitted value is at least two by Lemma 2.11, [6]. Hence the length of the basic chain corresponding to a $p$-cycle of Herman rings is at most $p - 1$. ☐

We continue to reveal the connection of $U$, the Fatou component containing the bov to the possible Herman rings of the function. Recall that the basic chain of a cycle of Herman rings $C$ is the ordered set $\{H_1, H_2, H_3, \ldots, H_{l_C}\}$ of rings belonging to $C$ where $H_1$ is the innermost ring with respect to the bov and $l_C$ is the smallest natural number such that $H_{l_C}$ surrounds a pole. Here $l_C$ is the length of the basic chain. Since $U$ is not itself a Herman ring (as a Herman ring cannot contain any omitted value), it is surrounded by $H_1$. Therefore $U_i$ is surrounded by $H_1$ for all $1 \leq i \leq l_C$. Let $S_C$ denote the set $\{U_1, U_2, U_3, \ldots, U_{l_C}\}$ where $U_1 = U$.

Lemma 2.7 Let $f \in M_S$ and $U$ be the Fatou component containing the bov.

1. If $H$ is a $p$-periodic Herman ring and the length of its basic chain is $l$ then for each $n$ there is an $i \in \{1, 2, 3, \ldots, l\}$ such that $H_n$ surrounds $U_i$ where $U_1 = U$.
2. Let $C'$ and $C''$ be two $p$-cycles of Herman rings. If $l_{C'} > l_{C''}$ then $S_{C'} \supset S_{C''}$ or $S_{C'} = S_{C''}$ or $S_{C'} \subset S_{C''}$ respectively.

Proof 1. Let $\{H = H_1, H_2, \ldots, H_p\}$ be a $p$-cycle of Herman rings and $H_1$ be the innermost ring with respect to $U$. Since the length of the basic chain is $l$, $H_i$
surrounds $U_i$ for $i = 1, 2, \ldots, l$. Further, $H_l$ surrounds a pole by the definition of the basic chain. By Lemma 2.2, $H_{l+1}$ surrounds the bov. But the bov is in $U = U_1$ which is not a Herman ring by Lemma 2.3. Hence $H_{l+1}$ surrounds $U_1$.

If $H_{l+1}$ surrounds a pole then there is an $l' \geq 1$ such that $H_{l+1+l'}$ surrounds $U_1$ but not any pole. If $k$ is the smallest natural number for which $H_{l+1+l'+k}$ surrounds a pole then it follows from the Maximum Modulus Principle that $H_{l+1+l'+j}$ surrounds $U_j$ for all $j \leq k$. Note that $\{H_{l+1+l'+j} : 0 \leq j \leq k\}$ is a chain. By Lemma 2.6(2), the basic chain is the longest chain. In other words, $k + 1 \leq l$. Now $H_{l+1+l'+k+1}$ surrounds $U_1$ by Lemma 2.2. This argument can be continued with $H_{l+1+l'+k+1}$ instead of $H_{l+1}$ for finitely many times to complete the proof.

2. Each cycle of Herman rings contains a ring which is the innermost with respect to the bov. Let $H_1$ and $G_1$ be such innermost rings of the $p$-cycles $C'$ and $C''$ respectively. Both $H_1$ and $G_1$ surround $U_1$. Using Lemma 2.7(1), we have $\mathcal{S}_{C'} = \{U_1, U_2, U_3, \ldots, U_{l_C'}\}$ and $\mathcal{S}_{C''} = \{U_1, U_2, U_3, \ldots, U_{l_C''}\}$. The proof now follows.

### 3 Proofs of the Results

Here is the proof of the first result of this article.

**Proof of Theorem 1.1** If $f$ has no Herman ring then there is nothing to prove. Else $U$, the Fatou component containing the bov is bounded by Lemma 2.5 (6)). Further $f^{-1}(U)$ is unbounded, infinitely connected and all its complementary components are bounded by Lemma 2.5(1). We write $\mathbb{C} \setminus f^{-1}(U) = \bigcup_{i=1}^{\infty} B_i$ and choose $B_1$ such that it contains $U$.

Suppose on the contrary that $\{C_n\}_{n>0}$ is the set of all the $p$-cycles of Herman rings. Let $l_n$ be the length of the basic chain of $C_n$. Since $l_n \leq p - 1$ for all $n$ by Lemma 2.6(3), $\max\{l_n : n > 0\} = l$ is a finite number. If $C$ is a $p$-cycle of Herman rings with $l_C = l$ then $\mathcal{S}_{C_n} \subseteq \mathcal{S}_C$ for all $n$ by Lemma 2.7(2). Recall that $\mathcal{S}_C$ denotes the set $\{U_1, U_2, U_3, \ldots, U_{l_C}\}$ where $U_1 = U$. Here more than one cycle with maximum length of basic chain are not ruled out and for each such cycle $C$, $\mathcal{S}_C$ is the same set. Let $K = \bigcup\{B_i : B_i \text{ contains at least one element of } \mathcal{S}_C\}$. Every $p$-periodic Herman ring belongs to $C_n$ for some $n$ and therefore surrounds an element of $\mathcal{S}_{C_n}$ by Lemma 2.7(1). Since every Herman ring is different from $f^{-1}(U_1)$ and $\mathcal{S}_{C_n} \subseteq \mathcal{S}_C$, every $p$-periodic Herman ring is in $K$. In other words, for each $p$ there is a compact set $K$ such that all the $p$-periodic Herman rings are contained in $K$.

Each $C_n$ contains a Herman ring which is innermost with respect to the bov. Let such a ring be denoted by $H^p_l$. After passing to a subsequence, if required, we can find a sequence of innermost rings $\{H^p_l\}_{n>0}$ such that either $H^p_{l+1}$ surrounds $H^p_l$ for all $n$, or $H^p_{l+1}$ is surrounded by $H^p_l$ for all $n$. Let $A_n$ be a topological annulus bounded by two $f^p$-invariant Jordan curves, one contained in $H^p_l$ and the other in $H^p_{l+1}$. Without loss of generality we assume that $A_i \cap A_j = \emptyset$ for $i \neq j$. Observe that $A_n \subset K$ for all $n$.

If $f^p : A_n \to \mathbb{C}$ is analytic then $f^{kp}(A_n) = A_n$ for all $k$ and $\{f^{kp}\}_{k>0}$ becomes normal in $A_n$. But this is not possible as $A_n$ intersects the Julia set of $f$. Hence $f^p$
has a singularity in $A_n$. Each such singularity $z$ must satisfy $f^k(z) = \infty$ for some $1 \leq k \leq p$. Since no innermost ring (with respect to the bov) surrounds a pole, each singularity of $f^p$ in $A_n$ must be an element of $\{z : f^k(z) = \infty, 1 < k \leq p\}$. This means that there is an integer $k'$, $1 < k' \leq p$ such that $f^{k'}$ has a pole $w_n$ in $A_n$ for infinitely many values of $n$. Since $w_n \in B_1$ for all $n$, $\{w_n\}_{n>0}$ has a limit point say $w$. Without loss of generality assume $w_n \to w$ as $n \to \infty$. Thus $w$ is an essential singularity of $f^{k'}$. In other words, there exists an $l < k'$ such that $f^l(w) = \infty$. Hence $f^l(w_n) \to \infty$ as $n \to \infty$. The line segment joining $w_n$ and $w_{n+1}$ contains at least one point $z_n$ of $H^n_1$. Since each $A_n$ surrounds the bov and $w_n \to w$, $z_n \to w$ as $n \to \infty$. This implies that $f^l(z_n) \to \infty$ as $n \to \infty$. In other words, there is an unbounded sequence of $p$-periodic Herman rings, namely $\{f^l(H^n_1)\}_{n>0}$. But this is not possible as all such Herman rings are contained in the bounded set $K$. Thus the number of $p$-cycles of Herman rings is finite. $\square$

**Remark 3.1** In the proof, it is important to note that $K$ contains all the Herman rings whose length of the basic chain is $l$ irrespective of their periods. This gives rise to a slightly more generalized version of Theorem 1.1; for a given $l$, the number of Herman rings (of a function with stable bov), the length of whose basic chains is $l$, is finite.

We need a definition to prove Theorem 1.2. The outer (or inner) boundary of a Herman ring $H$ is the boundary of the unbounded (or bounded respectively) component of $\hat{C} \setminus H$. By saying a Herman ring surrounds a pole we mean that its inner boundary surrounds the pole.

**Proof of Theorem 1.2** First we make two useful observations on the Julia components meeting the boundary of a Herman ring. Let $J$ be a Julia component intersecting the boundaries (inner or outer) of more than one Herman ring. Note that a Julia component cannot intersect both the boundaries of a Herman ring.

1. If two boundary components of two different Herman rings intersect $J$ then both of these cannot be the inner boundaries of the respective Herman rings. In other words, all the boundary components of Herman rings contained in $J$ are outer, with a possible exception.
2. Two Herman rings whose outer boundaries are contained in a Julia component cannot surround a common pole. As mentioned earlier, this means that the inner boundaries of these Herman rings cannot surround the same pole.

In order to prove this theorem by the method of contradiction, suppose that $J$ contains the boundary components of infinitely many Herman rings. By the observation (1) above, all except possibly one such boundary components are outer. Let $\{J_n\}_{n>0}$ be the sequence of all such outer boundaries of distinct Herman rings, say $H^n$. Since $J$ is bounded, the number of all poles surrounded by some sub-continuum of $J$ is finite. By the observation (2) above, the Herman ring $H^n$ does not surround any pole for infinitely many values of $n$. Without loss of generality we assume that $H^n$ does not surround any pole for any $n$.

For each $n$, there is a $k_n$ such that $f^i(H^n)$ does not surround any pole for $0 \leq i < k_n$ but $f^{k_n}(H^n)$ surrounds a pole, by Lemma 2.2. The outer boundary of $f^{k_n}(H^n)$ is the $f^{k_n}$-image of the outer boundary of $H^n$ by the Maximum Modulus Principle.
Since each point of the backward orbit of \( \infty \) is a singleton Julia component, and in particular does not intersect the boundary of any Herman ring, \( f^k(J) \) is bounded for each \( k \). If \( p = \min\{p(n) : H^n \text{ is } p(n) \text{-periodic} \} \) then \( p \geq 3 \) and \( f^p(J) \subseteq J \). Consequently there is a Julia component \( J^* \in \{ J, f(J), f^2(J), \ldots, f^{p-1}(J) \} \) containing the outer boundary of \( f^{kn}(H^n) \) for infinitely many values of \( n \). Since \( J^* \) is bounded, the number of all poles surrounded by any of its sub-continuum is finite. Hence one pole is surrounded by at least two Herman rings (their inner boundaries). But this is not possible by the observation (2), leading to a contradiction. 

We now present the proofs of Proposition 1.1 and Theorem 1.3.

**Proof of Proposition 1.1** Let \( U \) be the Fatou component containing the bov and \( U_{-1} = f^{-1}(U) \) be the pre-image of \( U \). It follows from Lemma 2.5(1) that \( U_{-1} \) is infinitely connected and all its complementary components are bounded. Let \( \{B_i\}_{i=1}^\infty \) be the set of all such components. Note that functions with a bov has infinitely many poles, otherwise \( \infty \) will be an asymptotic value, which is not true as bov is the only asymptotic value (Lemma 2.4(2)). Thus \( B_j \) contains at least one pole for infinitely many values of \( j \). Let \( \{\gamma_j\}_{j \geq 0} \) be an infinite sequence of Jordan curves in \( U_{-1} \) such that \( \gamma_j \) surrounds \( B_j \) but not any other complementary component of \( U_{-1} \). Note that no \( \gamma_j \) contains any pole and at most one \( \gamma_j \) surrounds the bov. Then by Corollary 2.9, [1], \( f \) has infinitely many weakly repelling fixed points. Since bov is the only limit point of the critical values, there are finitely many critical values which qualify to be in some parabolic domain. This gives that \( f \) has at most finitely many parabolic periodic points, and in particular finitely many parabolic fixed points. Hence \( f \) has infinitely many repelling fixed points.

The main idea of the next proof is that the branch of \( f^{-p} \) fixing a repelling \( p \)-periodic point of \( f \) is different from the branch of \( f^{-p} \) fixing the rotation domain whose boundary contains the periodic point.

**Proof of Theorem 1.3** Since \( z_0 \) is a repelling \( p \)-periodic point, \( |(f^p)'(z_0)| > 1 \). Let \( g \) be the inverse branch of \( f^p \) defined on a neighborhood \( N \) of \( z_0 \) such that \( g(z_0) = z_0 \). Since \( |(f^{-p})'(z_0)| < 1 \), \( z_0 \) is an attracting fixed point of \( g \). Let \( N' \subseteq N \) be a neighbourhood of \( z_0 \) such that \( g(N') \subseteq N' \). The existence of such \( N' \) is evident from the fact that analytic functions are locally conformally conjugate to linear maps at their attracting fixed points. As \( z_0 \in \partial D \), there exists \( z \in D \cap N' \). Define \( z_n = g^n(z) \) such that \( z_n \in N' \) for \( n \geq 1 \). Clearly \( z_n \rightarrow z_0 \) as \( n \rightarrow \infty \). Let \( \gamma \) be an \( f^p \)-invariant Jordan curve containing \( z \) and contained in \( D \). If \( z_n \in D \) for infinitely many values of \( n \) then each such \( z_n \) must be on \( \gamma \). This is because \( D \) is a \( p \)-periodic rotation domain and it follows from its definition that all the points in \( D \) whose \( f^{np} \)-image is \( z \) are on \( \gamma \). But \( \gamma \) is at a positive distance from \( z_0 \) contradicting \( z_n \rightarrow z_0 \) as \( n \rightarrow \infty \). Thus, there is a natural number \( n_0 \) such that \( z_{n_0} \in D \) but \( z_n \notin D \) for any \( n > n_0 \).

Let \( D_{-k} \) be the component of \( f^{-kp}(D) \) containing \( z_{n_0+k} \) for \( k \geq 1 \). Then \( D_{-k} \) intersects \( g^k(N' \cap D) \). It is important to note that \( D_{-k} \) is not in the periodic cycle of \( D \) for any \( k \). Now \( z_{n_0+1} \in D_{-1} \setminus D \). Since \( D \) and \( D_{-1} \) are Fatou components, they are either equal or disjoint. Thus \( D_{-1} \cap D = \emptyset \). If \( D_{-2} \cap D_{-1} \neq \emptyset \) or \( D_{-2} \cap D \neq \emptyset \) then applying \( f^p \) on these sets we get \( D_{-1} \cap D \neq \emptyset \), which is just shown to be impossible. Therefore \( D_{-2} \) is disjoint from \( D \) as well as from \( D_{-1} \). Inductively, it can be shown
that $D_{-k} \cap D_{-i} = \emptyset$ for all $0 \leq i \leq k - 1$. It is seen that $z_0 \in \partial D_{-k}$ for all $k$ completing the proof.

**Proof of Corollary 1.2** Let $g$ be the branch of $f^{-P}$ fixing $z_0$ and defined on $N$ as mentioned in the above proof. Choose $N$ such that it is in the region of univalence of $f^P$. Consider an $f^P$-invariant Jordan curve $\gamma$ in a neighborhood of $\partial D$ where $f^P$ is univalent such that $\gamma \cap N \neq \emptyset$. As observed in the proof, there is a point $z$ in $D \cap N$ such that $g(z) \in D_{-1} \cap N$. However there is also a point on $\gamma$ whose image under $f^P$ is $z$. This point is clearly on $\gamma$ which is in the region of univalence of $f^P$. This contradicts the univalence of $f^P$ in a neighborhood of the boundary of $D$. □

The proof of Theorem 1.4 is based on some ideas developed in [13]. We start by stating two lemmas proved in the same paper. We replace the unit disk by $D_R(a)$ and $D_r(0)$ by $D_{R}(a)$ in the original form of Lemma 2.1, [13]. This is not any loss of generality. A hyperbolic domain is an open connected subset of $\mathbb{C}$ whose complement contains at least three points. The unit disk is equipped with a hyperbolic distance and is the universal cover of every hyperbolic domain. Hence every hyperbolic domain inherits a distance from that of the unit disk via the universal cover. This distance is known as the hyperbolic distance of the hyperbolic domain. For details, one may refer to Chapter 3 [2]. If a non-constant function $h$ is analytic at a point $z_0$ then it is locally conjugate to a monomial $z \mapsto z^m$ for some $m > 0$ at $z_0$. This $m$ is known as the local degree of $h$ at $z_0$. If $m > 1$ then $z_0$ is a critical point of $h$ and its multiplicity is $m - 1$. For two domains $A$ and $B$, a map $h : A \rightarrow B$ is called proper of degree $k$ if the number of pre-images of each point of $B$ counting multiplicities is $k$.

**Lemma 3.1** For every natural number $d$ and $r \in (0, 1)$, there exists $C(d, r) > 0$ such that for a given simply connected hyperbolic domain $V$ and a proper analytic map $g : V \rightarrow D_R(a)$ of degree at most $d$, each component of $g^{-1}(D_{R}(a))$ has diameter less than $C(d, r)$ with respect to the hyperbolic distance of $V$. Moreover, \( \lim_{r \rightarrow 0} C(d, r) = 0 \).

For a hyperbolic domain $A$ and $A' \subset A$, the diameter of $A'$ with respect to the hyperbolic distance of $A$ is denoted by $\text{diam}_A A'$. Let $|A|$ denote the diameter of $A$ with respect to the spherical distance of $\mathbb{C}$. The details of spherical distance can be found in Chapter 2, [2].

**Lemma 3.2** Let $W$ be a simply connected domain and $a \in W' \subset W \subset \Omega \subset \mathbb{C} \setminus \{0\}$ for two domains $W'$ and $\Omega$. If $\text{diam}_W W' \leq C$ then $|W'|_{s} \leq 2C'(e^{2C} - 1) \inf\{d(a, \partial \Omega), \frac{1}{|a|}\}$ where $C'$ is a universal constant.

The next lemma deals with the pre-image component of simply connected domains containing exactly one critical value under proper maps. Though this fact is known (for example Proposition 3.1, [17]), we choose to give a detailed proof.

**Lemma 3.3** Let $h : A \rightarrow B$ be a proper analytic map such that $B$ is simply connected and contains at most one critical value of $h$, then $A$ is simply connected.

**Proof** First note that, by the Riemann-Hurwitz formula (Theorem 5.4.1, [2]), $c(A) - 2 = d(c(B) - 2) + N$, where $d$ is the degree of $h$ and $N$ is the number of critical points
of $h$ in $A$ counting multiplicity. Here $c(A)$ and $c(B)$ denote the number of components of $\hat{C} \setminus A$ and $\hat{C} \setminus B$ respectively. Since $B$ is simply connected, $c(A) = 2 - d + N$.

If $B$ does not contain any critical value then $A$ does not contain any critical point and $h$ is one-one. In other words, $A$ is simply connected. If $B$ contains a critical value then it is the only critical value of $h$ in $B$, by the hypothesis of this lemma. Let $c$ be a critical point in $A$. If it is the only pre-image of the critical value in $A$ then the local degree of $h$ at $c$ is $d$ and hence $N = d - 1$. Thus $c(A) = 1$. If $A$ contains a pre-image of the critical value that is different from $c$ then the sum of the local degrees at all the pre-images of the critical value is $d$. This gives that $d - N \geq 1$. Consequently $c(A) \leq 1$ and $c(A) = 1$. \hfill $\Box$

Now we proceed to prove Theorem 1.4. For a proper map $g : A \to B$, let $\deg(g : A \to B)$ denote its degree.

**Proof of Theorem 1.4** Let $f$ satisfy all the hypotheses of the theorem. Suppose that $w \in \mathbb{C}$ is an accumulation point of $J$, i.e., there exists a sequence $n_k \in \mathbb{N}$ such that $\lim_{k \to \infty} f^{n_k}(J) = w$. Here, the limit is with respect to the Hausdorff distance.

Let $U$ be the Fatou component containing the bov. Let $K^0$ be the closure of a simply connected subset of $U$ containing all the critical values belonging to $U$ and the bov. By the first assumption, the number of Fatou components different from $U$ and containing some critical value is finite. Let $\{U^i, 1 \leq i \leq N\}$ be the set of all such Fatou components. Consider the closure $K^i$ of a simply connected domain in $U^i$ containing all the critical values in $U^i$. Set

$$B = \bigcup_{i=1}^{N} \bigcup_{k \geq 0} f^k(K^i).$$

(3.1)

Then $B$ is a forward invariant compact subset of the union of $\mathcal{F}(f)$ and the set of all parabolic periodic points of $f$. It is important to note here that $f \in \mathcal{M}_5$ has no Baker domain by Theorem 1.4 [9].

Note that $\hat{C} \setminus B$ is a backward invariant open set containing all the critical points of $\hat{f}$ belonging to the Julia set. The critical points belonging to the Fatou set may be in $\hat{C} \setminus B$, but this does not matter when we discuss pull backs of disks centered at a point of the Julia set which is not a parabolic periodic point. More precisely, the forward orbits of these critical points cannot accumulate at any point of the Julia set except possibly at parabolic periodic points. In view of the assumption, let $NC = \{c_1, c_2, \ldots, c_k\}$ be the set of all non-recurrent critical points of $f$ in the Julia set. Also, let $\deg(f, c_i)$ denote the local degree of $f$ at $c_i$ and $d = \prod_{i=1}^{k} \deg(f, c_i)$. Let $C_1 = N_0dC(d, \frac{2}{3})$ where $C(d, \frac{2}{3})$ is the constant as defined in the Lemma 3.1 and $N_0 - 1$ is the minimum number of open disks $D_{\frac{2}{3}}(z')$, $\frac{2}{3} < |z'| < 1$ whose union covers $\{z : \frac{2}{3} \leq |z| \leq 1\}$.

Note that $N_0 - 1$ is also the minimum number of open disks $D_{\frac{2}{3}}(z')$, $\frac{2}{3} < |z'| < r$ required to cover an annulus $\{z : \frac{2}{3} \leq |z - z_0| \leq r\}$.

Corresponding to each $c_i$, choose a repelling periodic point $w_i$ sufficiently close to $c_i$ such that the cycle of $w_i$ does not contain $w$, the accumulation point of $J$ and such that the set $\Omega = \hat{C} \setminus \{B \cup B_{i=1}^{k} \{f^n(w_i) : n \geq 1\}\}$ satisfies the following conditions.

1. $d_{\Omega}(c_i, f^n(c_i)) \geq C_1$ for all $n \geq 1$ and $1 \leq i \leq k$, 


2. \( d_\Omega(f(c_i), f(c_j)) \geq C_1 \) whenever \( f(c_i) \neq f(c_j) \), where \( d_\Omega \) is the hyperbolic distance of \( \Omega \).

Such a choice of \( \Omega \) is possible as the set of all repelling periodic points is dense in the Julia set. In fact, each \( w_i \) is chosen arbitrarily close to \( c_i \) so that the hyperbolic density at \( c_i \) with respect to \( \Omega \) (whose boundary contains \( w_i \)) is sufficiently high. This along with the non-recurrence of each \( c_i \) ensure the first condition. Similarly, by choosing \( w_i \) and \( w_j \) sufficiently near to \( c_i \) and \( c_j \) respectively, \( f(c_i) \) and \( f(c_j) \) can be made arbitrarily close to \( f(w_i) \) and \( f(w_j) \) respectively as a consequence of the continuity of \( f \). Since \( f(w_i), f(w_j) \in \partial \Omega \), the hyperbolic densities at \( f(c_i) \) and \( f(c_j) \) are arbitrarily high. As \( f(c_i) \neq f(c_j) \), the choice of the second condition is possible. Since \( \Omega \) does not contain any asymptotic value of \( f \) (by Lemma 2.4(2)), for every simply connected domain \( D \subset \Omega \) and every component \( D' \) of \( f^{-1}(D) \), \( f : D' \to D \) is a proper map. Considering a conformal conjugate of \( f \), if necessary we assume that \( 0, \infty \notin \Omega \). Note that \( \Omega \) is a hyperbolic domain containing \( J(f) \setminus \{z : z \text{ is a parabolic periodic point of } f\} \). In particular, \( w \in \Omega \) or is a parabolic periodic point.

Let \( w \) be neither a parabolic periodic point nor in the \( \omega \)-limit set of any recurrent critical point. Then there exists \( D_r = \{z : |z - w| < r\} \) and \( D_{2r} = \{z : |z - w| < 2r\} \) such that \( diam_\Omega D_{2r} < C_1 \) and \( D_{2r} \) does not intersect the \( \omega \)-limit set of any recurrent critical point or any parabolic periodic point. Clearly, \( diam_\Omega(D_r) < diam D_{2r}(D_r) < C_1 \).

We claim that, for all \( n \) and for every connected component \( V_n' \) of \( f^{-n}(D_r) \),

\[
V_n' \text{ is simply connected, } deg(f^n : V_n' \to D_r) \leq d, \quad \text{ and } \quad diam_\Omega V_n' < C_1. \tag{3.2}
\]

To prove it, we proceed by induction on \( n \).

Let \( V_1' \) and \( V_1 \) be the components of \( f^{-1}(D_r) \) and \( f^{-1}(D_{2r}) \) respectively such that \( V_1' \subset V_1 \). By the choice of \( \Omega \), \( D_{2r} \) (and hence \( D_r \)) contains at most one critical value. It follows from Lemma 3.3 that \( V_1' \) and \( V_1 \) are simply connected. Now \( f : V_1' \to D_r \) is a proper map of degree at most \( d \) (It is in fact 1 if \( D_r \) does not contain any critical value). Consider the annulus \( A(w; \frac{2r}{3}, r) = \{z : \frac{2r}{3} \leq |z - w| \leq r\} \) and choose \( z_1, z_2, \ldots, z_{N_0-1} \) in it such that \( A(w; \frac{2r}{3}, r) \subset \bigcup_{i=1}^{N_0-1} D_{\frac{r}{3}}(z_i) \). Note that the disks \( D_{\frac{r}{3}}(z_i) \) and \( D_{\frac{r}{3}}(z_j) \) may intersect and that does not make any difference in this context. Since \( D_{\frac{r}{3}}(z_i) \subset D_{2r}, D_{\frac{r}{3}}(z_i) \) contains at most one critical value. It follows from Lemma 3.3 that each component \( \tilde{V} \) of \( f^{-1}(D_{\frac{r}{3}}(z_i)) \) is simply connected. By the choice of \( \Omega \), \( f : \tilde{V} \to D_{\frac{r}{3}}(z_i) \) is a proper map with degree at most \( d \). Since \( D_{\frac{r}{3}}(z_i) \subset D_{\frac{r}{3}}(z_i) \), Lemma 3.1 gives that the diameter of the component \( \tilde{V} \) of \( f^{-1}(D_{\frac{r}{3}}(z_i)) \), with respect to the hyperbolic distance of \( \tilde{V} \), is at most \( C(d, \frac{2}{3}) \). Since \( \tilde{V} \subset \Omega, diam_\Omega \tilde{V} < C(d, \frac{2}{3}) \). Again using Lemma 3.1 for \( D_{\frac{r}{3}}(w) \subset D_r \) and arguing similarly, we have that \( diam_\Omega \tilde{U} < C(d, \frac{2}{3}) \) for each component \( \tilde{U} \) of \( f^{-1}(D_{\frac{r}{3}}(w)) \). Note that \( D_r \subset D_{\frac{r}{3}}(w) \cup_{i=1}^{N_0-1} D_{\frac{r}{3}}(z_i) \). Since \( f : V_1' \to D_r \) is a proper map with
degree at most $d$, the pre-image of each of the above mentioned disks has at most $d$ components. Since the diameter of each such pre-image component with respect to $d_{\Omega}$ is less than $C(d, \frac{2}{3})$, $diam_{\Omega} V'_1 < dN_0 C(d, \frac{2}{3}) = C_1$. Thus the claim is proved for $n = 1$.

Assume that the claim is true for $n = m$. This implies that every connected component $V$ of $f^{-m}(D_r)$ is simply connected, $deg(f^m : V \to D_r) \leq d$ and $diam_{\Omega} V < C_1$. We shall be done by proving these for $n = m + 1$. Let $V'_{m+1}$ be a component of $f^{-m-1}(D_r)$. If $f(V'_{m+1}) = V'_{m}$ then $V'_{m}$ is a component of $f^{-m}(D_r)$, which, by the choice of $\Omega$ and the induction assumption, contains at most one critical value. It now follows from Lemma 3.3 that $V'_{m+1}$ is simply connected. Again by the choice of $\Omega$, each critical point $c \in NC$ appears at most once in $U_i = f^i(V'_{m+1})$ for $i = 0, 1, \ldots, m$. (If not then $c \in U_i \cap U_{i+k'}$ and $U_{i+k'}$ contains $c$ as well as $f^{k'}(c)$, which gives that $diam_{\Omega} U_{i+k'} \geq C_1$. But this is contrary to the induction assumption).

Thus,

$$deg(f^{m+1} : V'_{m+1} \to D_r) \leq d. \tag{3.4}$$

Now consider the same open cover $\{D_\frac{2}{3}(\varepsilon_i) : i = 1, 2, 3, \ldots, N_0 - 1\} \cup D_\frac{2}{3}$ of $D_r$. Using (3.4) and repeating the arguments as earlier, we get that

$$diam_{\Omega} V'_{m+1} < C_1. \tag{3.5}$$

This proves the claim for all $n$.

Suppose on the contrary that $J$ is not a singleton. Then its spherical diameter $|J|_s$ is positive. Let $z \in J$. Then $z \neq 0, \infty$ as $J \subset \Omega \subset \mathbb{C}$ by our earlier assumption. Choose a sufficiently small $C > 0$ such that $2C'(e^{2C} - 1) \frac{1}{|z|} < |J|_s$ where $C'$ is as mentioned in Lemma 3.2. Also, choose $0 < \rho < 1$ such that $C(d, \rho) < C$. This is possible as $lim_{r \to 0} C(d, r) = 0$ by Lemma 3.1. Note that $D_{\rho r} \subset D_r$. Since $f^{nk}(J) \to w$ as $k \to \infty$, there is an $n$ such that $f^n(J) \subset D_{\rho r}$. Let $W'$ and $W$ be the components of $f^{-n}(D_{\rho r})$ and $f^{-n}(D_r)$ respectively, each containing $J$. As already proved, $W'$ and $W$ are simply connected and $deg(f^n : W' \to D_{\rho r}) \leq d$. Therefore, $diam_{W} W' \leq C(d, \rho) < C$. By Lemma 3.2, $|W'|_s \leq 2C'(e^{2C} - 1) \frac{1}{|z|}$, which is less than $|J|_s$. But $J$ is properly contained in $W'$ as $f^n(J)$ is a compact subset of $D_{\rho r}$ and $f^n : W' \to D_{\rho r}$ is proper. However, this is not possible as $|J|_s > 0$. Therefore, $J$ is a singleton, and the proof completes.

Here is a useful remark.

**Remark 3.2** Under the hypotheses of the above theorem, for every $\epsilon > 0$ and every non-singleton Julia component $J$, there is $n_0$ such that $J \subset \{f^{-n}(B)\}_\epsilon$ for all $n > n_0$. The $\epsilon$-neighborhood of a set $A$, denoted by $[A]_\epsilon$ is defined as $\cup_{a \in A} D_\epsilon(a)$. Since $J$ is disjoint from $f^{-n}(B)$ for every $n$ except parabolic periodic points, the above statement gives that $J \subset \{\partial f^{-n}(B)\}_\epsilon$. To prove this, suppose on the contrary that for a non-singleton Julia component $J$ and for an $\epsilon > 0$ there is a sequence $z_k \in J$ and an increasing sequence $n_k$ such that $z_k \notin \{\partial f^{-n_k}(B)\}_\epsilon \subset \{f^{-n_k}(B)\}_\epsilon$. Then consider a limit point $z^*$ of $\{z_k\}_{k>0}$. Since $f^{-n}(B) \supseteq f^{-n-1}(B)$ for all $n$ by the construction
of \( B, z_k \notin \{ f^{-j}(B)\}_\epsilon \) for any \( j = 1, 2, 3, \ldots n_k \). It is clear that \( z^*_k \notin \{ f^{-n_k}(B)\}_\epsilon \) for any \( k \). In other words \( z^* \) is not a limit point of \( \bigcup_{k>0} f^{-n_k}(B) \). Consequently, there is a disk \( D \) around \( z^* \) such that \( \{ f^{n_k}\}_{k>0} \) omits all the points of \( B \) on \( D \). However this is not possible as \( z^* \) is in the Julia set.

Let \( |A| \) denote the Euclidean diameter of the subset \( A \) of \( \hat{\mathbb{C}} \).

**Proof of Theorem 1.5** Let \( V \) be a \( p \)-periodic rotation domain of \( f \). If \( N \subset V \) is an open set then there exists an \( \epsilon > 0 \) such that \( |N-n| \geq \epsilon \) for all \( n \), where \( N-n \) is the component of \( f^{-n}(N) \) contained in a Fatou component belonging to the cycle containing \( V \). To see this, let \( \gamma_1 \) and \( \gamma_2 \) be two \( f^p \)-invariant Jordan curves contained in \( V \) and intersecting \( N \). Then \( \epsilon = \min_{1 \leq i \leq p} d_H(f^i(\gamma_1), f^i(\gamma_2)) > 0 \) where \( d_H \) denotes the Hausdorff distance. Since \( N \) intersects \( \gamma_1 \) and \( \gamma_2 \), \( N-n \) must intersect \( f^{-n}(\gamma_1) \) and \( f^{-n}(\gamma_2) \). Denoting the intersections by the same symbol, we have \( f^{-n}(\gamma_j) = f^{p-n}(\gamma_j) \) for \( j = 1, 2 \). Since \( d_H(f^i(\gamma_1), f^i(\gamma_2)) \geq \epsilon \) for \( 1 \leq i \leq p \),

\[
|N-n| \geq \epsilon \quad \text{for all } n.
\]

(3.6)

Note that the number of critical values not contained in the Fatou component containing the bov is finite. So \( f \) has at most finitely many parabolic periodic cycles.

Let \( z_0 \in \partial V \) such that \( z_0 \) is not contained in the \( \omega \)-limit set of the recurrent critical points. Since the number of parabolic cycles is finite, without loss of generality we assume that \( z_0 \) is not a parabolic periodic point. Let \( D_2r \) be a disk centered at \( z_0 \) contained in \( \mathbb{C} \setminus B \), where \( B \) is a forward invariant compact set containing all the singular values belonging to the Fatou set (See Equation 3.1 in the proof of Theorem 1.4 for the definition of \( B \)). Further, let \( D_2r \) be such that it does not intersect the \( \omega \)-limit set of any recurrent critical point or any parabolic periodic point. If \( D_{2n} \) is the component of \( f^{-np}(D_{2r}) \) intersecting the boundary of \( V \) and \( D_{-n} \) is the component of \( f^{-np}(D_r) \) contained in \( D_{-2n} \), then it follows from the proof of Theorem 1.4 that there is a \( C > 0 \) with \( \text{diam}_{D_{-2n}}(D_{-n}) < C \) for all \( n \). Let \( z_n \in D_{-n} \) such that \( f^{np}(z_n) = z_0 \). Note that \( z_n \in \partial V \) for all \( n \).

By Remark 3.2, for every \( \epsilon > 0 \) there is \( n_0 \) such that \( \partial V \subset \{ \partial f^{-np}(B) \}_\epsilon \) for all \( n > n_0 \). Since \( \partial B = \partial (\hat{\mathbb{C}} \setminus B) \subset \partial \Omega \) (\( \Omega \), as defined in the proof of Theorem 1.4, is \( \hat{\mathbb{C}} \setminus B \) except suitably chosen finitely many repelling periodic points), \( f^{-np}(\partial B) \subset f^{-np}(\partial \Omega) \) and consequently, \( \partial f^{-np}(B) \subset \partial f^{-np}(\Omega) \) for all \( n \). Therefore \( \partial V \subset \{ \partial f^{-np}(\Omega) \}_\epsilon \) for all \( n > n_0 \). In other words, \( d(z_n, \partial f^{-np}(\Omega)) \rightarrow 0 \) as \( n \rightarrow \infty \).

Now, it follows from Lemma 3.2 that \( |D_{-n}| \rightarrow 0 \) as \( n \rightarrow \infty \). However, considering an open set in \( D_{2r} \), it can be seen from Equation 3.6 that the Euclidean diameter of \( D_{-n} \) is positive for all \( n \) leading to a contradiction. Thus the boundary of each rotation domain is contained in the \( \omega \)-limit set of the recurrent critical points.

4 **Example**

Now we give a class of functions for which the hypotheses of the theorems discussed in this article are satisfied.
Let \( f_{\alpha, \beta, c, d}(z) = \frac{\alpha}{P(z) + e^z} + \beta \) where \( \alpha \neq 0, \beta \in \mathbb{C} \) and \( P(z) = cz^d \) for some \( c \in \mathbb{C} \setminus \{0\} \) and \( d \in \mathbb{N} \). It follows from Remark 2.3, [9] that 0 is the bov of \( \frac{1}{P(z) + e^z} \). The map \( z \mapsto \alpha z + \beta \) preserves the defining property of bov, i.e., each component of the boundary of the pre-image of all sufficiently small disk about the bov is bounded. Therefore \( \beta \) is the bov of \( f_{\alpha, \beta, c, d} \). As stated in the same remark, this is the only limit point of its critical values. For suitable choices of \( \alpha, \beta, c, d \), the function \( f_{\alpha, \beta, c, d} \) has stable bov satisfying the hypothesis of Theorem 1.1, Theorem 1.2, Proposition 1.1 and Theorem 1.3. This is discussed at length in Sect. 4, [9].

For \( c = d = 1 \) and \( \beta = 0 \), the function \( f_{\alpha, \beta, c, d}(z) \equiv f_{\alpha}(z) = \frac{\alpha}{z + e^z} \) is studied in [9]. The critical points are \( i \pi (2k + 1), k \in \mathbb{Z} \) and they correspond to distinct critical values. In other words, the number of critical points corresponding to each critical value is one. Thus, the first two hypotheses of Theorem 1.4 and Theorem 1.5 are satisfied. It is shown in Example 1, [9] that for \( 0 < \alpha < 0.05 \), the bov and all the critical values of \( f_{\alpha} \) are in an invariant attracting domain meeting the third hypothesis of the above theorems.

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