Global dynamics above the ground state energy for the combined power-type nonlinear Schrödinger equations with energy-critical growth at low frequencies

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Abstract

We consider the nonlinear Schrödinger equations with combined-type local interactions with energy-critical growth, and we study the solutions slightly above the ground state threshold at low frequencies, so that we obtain a so-called nine-set theory developed by Nakanishi-Schlag [20, 21].

1 Introduction

We consider the following nonlinear Schrödinger equation:

\[ i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{p-1} \psi + |\psi|^{\frac{4}{d-2}} \psi = 0, \quad \text{(NLS)} \]

where \( \psi = \psi(x,t) \) is a complex-valued function on \( \mathbb{R}^d \times \mathbb{R} \) \( (d \geq 3) \), \( \Delta \) is the Laplace operator on \( \mathbb{R}^d \) and \( p \) satisfies that

\[ 2_* := 2 + \frac{4}{d} < p + 1 < 2_*' := 2 + \frac{4}{d-2}. \quad \text{(1.1)} \]

We denote the mass and the Hamiltonian of \( \text{(NLS)} \) by \( M \) and \( H \), respectively:

\[ M(u) := \frac{1}{2} \| u \|_{L^2}^2, \quad \text{(1.2)} \]
\[ H(u) := \frac{1}{2} \| \nabla u \|_{L^2}^2 - \frac{1}{p+1} \| u \|_{L^{p+1}}^{p+1} - \frac{1}{2^*} \| u \|_{L^{2^*}}^{2^*}. \quad \text{(1.3)} \]

The Cauchy problem for \( \text{(NLS)} \) is locally well-posed in \( H^1 \) (see, e.g., Proposition 3.1 in [24]), and the mass and the Hamiltonian are conserved quantities for the flow defined by \( \text{(NLS)} \). Furthermore, for any solution \( \psi \) of finite variance, we have the “virial identity”:

\[ \frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |\psi(x,t)|^2 \, dx = 8K(\psi(t)), \quad \text{(1.4)} \]

where

\[ K(u) := \| \nabla u \|_{L^2}^2 - \frac{d(p-1)}{2(p+1)} \| u \|_{L^{p+1}}^{p+1} - \| u \|_{L^{2*}}^{2*}. \quad \text{(1.5)} \]
It is easy to see that for any non-trivial functions \( u \),
\[
\mathcal{H}(u) > \frac{1}{2} \mathcal{K}(u). \tag{1.6}
\]

A standing wave to (NLS) of frequency \( \omega \) is a solution to (NLS) of the form \( e^{i\omega t}u \), where \( u \) is a solution to the elliptic equation
\[
\omega u - \Delta u - |u|^{p-1}u - |u|^4d - 2u = 0, \quad u \in H^1(\mathbb{R}^d) \setminus \{0\}. \tag{\omega-SP}
\]
Moreover, a ground state of (\omega-SP) is a solution to (\omega-SP) of the minimal action, where by the action, we mean a functional \( S_\omega \) defined by
\[
S_\omega := \omega M + \mathcal{H}. \tag{1.7}
\]
In [2, 3], the same authors showed the existence of a ground state via the variational problem
\[
m_\omega := \inf \{ S_\omega(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{K}(u) = 0 \}. \tag{1.8}
\]
More precisely, we proved that for any \( d \geq 4 \), any \( p \in (2^*, 2^*) \) and any \( \omega > 0 \), the variational problem for \( m_\omega \) has a minimizer; any minimizer for \( m_\omega \) becomes a ground state of (\omega-SP) and vice versa; and
\[
m_\omega = \inf \{ I_\omega(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{K}(u) \leq 0 \} \tag{1.9}
\]
where
\[
I_\omega(u) := S_\omega(u) - \frac{2}{d(p-1)} \mathcal{K}(u), \tag{1.10}
\]
\[
J_\omega(u) := S_\omega(u) - \frac{1}{2} \mathcal{K}(u). \tag{1.11}
\]
It is worthwhile noting that \( I_\omega \) does not include the \( L^{p+1} \)-norm and that \( J_\omega \) does not the \( H^1 \)-norm.

We will see that for any \( \omega > 0 \), there exists a positive, radial ground state of frequency \( \omega \) (see Proposition 2.1 below). Furthermore, if \( \omega \) is sufficiently small, then it is unique (see Proposition 2.4 below). Our aim in this paper is to study the behavior of solutions to (NLS) starting from certain initial data above the “ground state threshold” (see 1.13 below), in the spirit of Nakanishi and Schlag [20, 21]. We only consider the positive, radial ground states at low frequencies. We will deal with high-frequency cases in a forthcoming paper.

Throughout this paper, we use the symbol \( \Phi_\omega \) to denote the positive, radial ground state of (\omega-SP). Moreover, \( \mathcal{O}(\Phi_\omega) \) denotes the orbit
\[
\mathcal{O}(\Phi_\omega) := \{ e^{i\theta} \Phi_\omega : \theta \in \mathbb{R} \}. \tag{1.12}
\]
Now, we state our main result.

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Theorem 1.1. Assume $d \geq 4$ and $(1.1)$. Then, there exists $\omega_* > 0$ such that for any $\omega \in (0, \omega_*)$, there exists a positive function $\varepsilon_\omega : [0, \infty) \to (0, \infty)$ with the following property: Set
\[
\tilde{PW}_\omega := \{ u \in H^1(\mathbb{R}^d) : S_\omega(u) < m_\omega + \varepsilon_\omega(M(u)) \}. \tag{1.13}
\]
Then, any radial solution $\psi$ starting from $\tilde{PW}_\omega$ exhibits one of the following scenarios:

(i) Scattering both forward and backward in time;
(ii) Finite time blowup both forward and backward in time;
(iii) Scattering forward in time, and finite time blowup backward in time;
(iv) Finite time blowup forward in time, and scattering backward in time;
(v) Trapped by $O(\Phi_\alpha)$ forward in time, and scattering backward in time;
(vi) Scattering forward in time, and trapped by $O(\Phi_\alpha)$ backward in time;
(vii) Trapped by $O(\Phi_\alpha)$ forward in time, and finite time blowup backward in time;
(viii) Finite time blowup forward in time, and trapped by $O(\Phi_\alpha)$ backward in time;
(ix) Trapped by $O(\Phi_\alpha)$ both forward, and backward in time.

Here, “scattering forward in time” means that the maximal lifespan of a solution $\psi$ is infinity and there exists $\phi \in H^1(\mathbb{R}^d)$ such that
\[
\lim_{t \to \infty} \| \psi(t) - e^{it\Delta} \phi \|_{H^1} = 0; \tag{1.14}
\]
“blowup forward in time” means that the maximal lifespan of a solution is finite; and “trapped by $O(\Phi_\alpha)$ forward in time” means that the maximal lifespan of a solution is infinity and the solution stays in some neighborhood of $O(\Phi_\alpha)$ in $H^1(\mathbb{R}^d)$ after some time. The terms corresponding to “backward in time” are used in a similar manner.

We give a proof of Theorem 1.1 in Section 3 below. In particular, we show how to construct the function $\varepsilon_\omega$ (see (3.9)). Moreover, we can verify that all of the nine scenarios actually happen in a way similar to [21].

The equation (NLS) reminds us of the following ones
\[
i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{p-1} \psi = 0, \tag{NLS'}
\]
\[
i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{d/2} \psi = 0. \tag{NLS''}
\]
These equations are invariant under the scalings below, respectively:
\[
\psi(x, t) \mapsto \lambda^{-\frac{2}{p-1}} \psi(\frac{x}{\lambda}, \frac{t}{\lambda^2}), \tag{1.15}
\]
\[
\psi(x, t) \mapsto \lambda^{-\frac{d}{2}} \psi(\frac{x}{\lambda}, \frac{t}{\lambda^2}). \tag{1.16}
\]
On the other hand, there is no scaling which leaves (NLS) invariant.
The stationary problems corresponding to (NLS†) and (NLS‡) are: respectively,
\[
\begin{align*}
\omega v - \Delta v - |v|^{p-1} v &= 0, \\
\Delta w + |w|^{\frac{4}{d-2}} w &= 0.
\end{align*}
\]  
(\omega-SP†)  (SP‡)

Any solution \(v\) to (\omega-SP†) gives rise to a standing wave \(e^{i\omega t}v\) to (NLS†). On the other hand, any solution to (SP‡) also solves the equation (NLS‡). Here, we define the “scaling-operator” \(T_\omega\) to be that for any function \(f\) on \(\mathbb{R}^d\),
\[
T_\omega f(x) := \omega^{\frac{1}{p-1}} f \left( \frac{x}{\sqrt{\omega}} \right).
\]  
(1.17)

Then, putting \(u := T_\omega v\), we see that \(u\) solves
\[
\begin{align*}
u - \Delta u - |u|^{p-1} u &= 0. \\
\end{align*}
\]  
(SP†)

It is well known that the equation (SP†) has a unique positive solution up to translations which is radially symmetric with respect to some point. We denote the positive solution symmetric about the origin by \(U\). On the other hand, the equation (SP‡) possesses a solution of the form
\[
W(x) := \left( \frac{\sqrt{d(d-2)}}{1 + |x|^2} \right)^{\frac{d-2}{2}}.
\]  
(1.18)

Here, a positive, \(C^2\)-solution of (SP‡) is unique up to dilations and translations (see Corollary 8.2 in [7]). Following Brezis-Nirenberg [6], we introduce the variational problem
\[
\sigma := \inf \left\{ \|\nabla u\|^2_{L^2} : u \in \dot{H}^1(\mathbb{R}^d), \|u\|_{L^{2^*}} = 1 \right\}.
\]  
(1.19)

Then, we know that a minimizer is given by \(u = W/\|W\|_{L^{2^*}}\) and
\[
\sigma^\frac{d}{2} = \|\nabla W\|^2_{L^2} = \|W\|^2_{L^{2^*}}.
\]  
(1.20)

In our context of defining (1.9), the problem (1.19) is equivalent to
\[
m^\dagger := \inf \left\{ I^\dagger(u) : u \in \dot{H}^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{K}^\dagger(u) \leq 0 \right\},
\]  
(1.21)

where
\[
I^\dagger(u) := \mathcal{H}^\dagger(u) - \frac{1}{2^*} \mathcal{K}^\dagger(u) = \frac{1}{2} \|\nabla u\|^2_{L^2}
\]  
(1.22)

with
\[
\begin{align*}
\mathcal{H}^\dagger(u) &= \frac{1}{2} \|\nabla u\|^2_{L^2} - \frac{1}{2^*} \|u\|_{L^{2^*}}^2, \\
\mathcal{K}^\dagger(u) &= \|\nabla u\|^2_{L^2} - \|u\|_{L^{2^*}}^2.
\end{align*}
\]  
(1.23)

Then, we can verify that
\[
m^\dagger = I^\dagger(W) = \mathcal{H}^\dagger(W) = \frac{1}{d} \sigma^\frac{d}{2}.
\]  
(1.25)
Furthermore, in [2], we proved that for any \( d \geq 4 \), any \( p \in (2, 2^*) \) and any \( \omega > 0 \),
\[
m_\omega < \frac{1}{d} \sigma^2.
\tag{1.26}
\]

Now, we are going back to the equation \((\omega-\text{SP})\). If \( u \) is a solution to \((\omega-\text{SP})\), then \( v := T_\omega u \) satisfies that
\[
v - \Delta v - |v|^{p-1} v - \omega 2^{*-(p+1)} |v|^{\frac{d}{p-2}} v = 0.
\tag{\omega-\text{SP}}
\]

We should mention that if \( \omega \) is small enough, it is expected that \((\omega-\text{SP})\) has properties similar to \((\text{SP}^\dag)\). Indeed, this is the heart of our analysis.

We briefly review results for \((\text{NLS}^\dag)\) and \((\text{NLS}^\ddag)\) which is used in this paper. In [15], Kenig and Merle studied the equation \((\text{NLS}^\ddag)\) in the dimensions \( d = 3, 4, 5 \), and proved the scattering of radial solutions starting from the set
\[
P_{\text{W}}^\ddag := \left\{ u \in H^1(\mathbb{R}^d) : \mathcal{H}^\ddag(u) < \mathcal{K}^\ddag(W), \|\nabla u\|_{L^2}^2 < \|\nabla W\|_{L^2}^2 \right\}.
\tag{1.27}
\]
Their result is extended to the higher dimensional cases by Killip and Visan [17]. We summarize these results as:

**Theorem 1.2** ([15, 17]). Assume that \( d \geq 3 \). Then, the set \( P_{\text{W}}^\ddag \) is invariant under the flow defined by \((\text{NLS}^\ddag)\). Furthermore, any non-trivial, radial solution \( \psi \) to \((\text{NLS})\) starting from \( P_{\text{W}}^\ddag \) exists globally in time, and satisfies that
\[
\|\nabla \psi\|_{\text{St}(\mathbb{R})} < \infty,
\tag{1.28}
\]
\[
\mathcal{H}^\ddag(\psi) \geq \inf_{t \in \mathbb{R}} \mathcal{K}^\ddag(\psi(t)) > 0,
\tag{1.29}
\]
where \( \text{St}(\mathbb{R}) := L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^2(\mathbb{R}, L^{2^*}(\mathbb{R}^d)) \).

The result for \((\text{NLS}^\dag)\) corresponding to Theorem 1.2 is derived by Duyckaerts, Holmer and Roudenko [9, 12] for \( d = p = 3 \). Furthermore, the first and the fourth authors extended their result to the general dimensions \( d \geq 1 \) and the powers \( p \) satisfying \((1.1)\) in [1]; the result says that we have either the scattering or the blowup, if the solutions start from the set
\[
P_{\text{W}}^\dag := \left\{ u \in H^1(\mathbb{R}^d) : \mathcal{H}^\dag(u) < \left( \frac{\mathcal{M}(U)}{\mathcal{M}(u)} \right)^{\frac{1}{p-\sigma_p}} \mathcal{H}^\dag(U) \right\},
\tag{1.30}
\]
where
\[
\mathcal{H}^\dag(u) := \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}}^{p+1},
\tag{1.31}
\]
and \( \sigma_p \) is the “scaling-exponent” given by
\[
\sigma_p := \frac{d}{2} - \frac{2}{p - 1}.
\tag{1.32}
\]
The condition (1.1) on $p$ implies that $0 < s_p < 1$. Furthermore, we can classify the behavior of the solutions by the sign of the functional $K^\dagger$ defined by

$$K^\dagger(u) := \|\nabla u\|_{L^2}^2 - \frac{d(p-1)}{2(p+1)} \|u\|_{L^{p+1}}^{p+1}.$$  \hfill (1.33)

Note here that for any $\omega > 0$, the replacement of $U$ with $T_\omega U$ in (1.30) leaves $P^\dagger$ unchanged. Moreover, introducing the set $PW^\dagger_\omega$ defined by

$$PW^\dagger_\omega := \{ u \in H^1(\mathbb{R}^d) : S^\dagger_\omega(u) < S^\dagger_\omega(U) \},$$  \hfill (1.34)

we can express $P^\dagger$ as the union of $PW^\dagger_\omega$ over all $\omega > 0$:

$$P^\dagger = \bigcup_{\omega > 0} PW^\dagger_\omega.$$  \hfill (1.35)

Here, $S^\dagger_\omega$ denotes the action for ($\omega$-SP$^\dagger$), i.e.,

$$S^\dagger_\omega(u) := \omega M(u) + \mathcal{H}^\dagger(u).$$  \hfill (1.36)

In particular, we simply write $S^\dagger_1$ by $S^\dagger$.  

In [21], Nakanishi and Schlag considered the equation (NLS$^\dagger$) in the case $d = p = 3$. They developed a method to analyze the behavior of (radial) solutions starting from the set

$$PW^{\dagger,\varepsilon} := \{ u \in H^1(\mathbb{R}^3) : \mathcal{H}^\dagger(u) < \left\{ \frac{M(U)}{M(u)} \right\}(\mathcal{H}^\dagger(U) + \varepsilon) \}$$  \hfill (1.37)

for some $\varepsilon > 0$. Clearly, the set $PW^{\dagger,\varepsilon}$ is an enlargement of $P^\dagger$. In particular, they proved that there exists $\varepsilon > 0$ such that all solutions starting from $PW^{\dagger,\varepsilon}$ exhibit one of the same nine scenarios as Theorem 1.1 above. This result motivated our study.

We note that the way to define $P^\dagger$ (see (1.30)) is based on the scaling-invariant nature of $U$ (cf. [1]). Due to lack of such a scaling property, it is not appropriate to define the corresponding “potential well” for our equation (NLS$^\dagger$) by simply replacing $\mathcal{H}^\dagger$ and $U$ with $\mathcal{H}$ and $\Phi_\omega$, respectively in $P^\dagger$. Instead, from the viewpoint of (1.35), we consider the frequency-wise “potential well” $PW_\omega$ defined by

$$PW_\omega := \{ u \in H^1(\mathbb{R}^d) : S_\omega(u) < m_\omega \},$$  \hfill (1.38)

This set is closely related to a variational nature of $\Phi_\omega$. Indeed, the definition of $m_\omega$ (see (1.8)) implies that if $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ and $K(u) = 0$, then $S_\omega(u) \geq m_\omega$, so that we can split $PW_\omega$ into three parts according to the sign of the functional $K$:

$$PW_\omega = PW_{\omega,+} \cup \{0\} \cup PW_{\omega,-},$$  \hfill (1.39)

where

$$PW_{\omega,+} := \{ u \in H^1(\mathbb{R}^d) : S_\omega(u) < m_\omega, K(u) > 0 \},$$  \hfill (1.40)

$$PW_{\omega,-} := \{ u \in H^1(\mathbb{R}^d) : S_\omega(u) < m_\omega, K(u) < 0 \}.$$  \hfill (1.41)

The following theorem follows from the result in [2, 3] together with [15].
Theorem 1.3. Assume $d \geq 4$ and $(\omega)$, and let $\omega > 0$. Then, $PW_{\omega,+}$ and $PW_{\omega,-}$ are invariant under the flow defined by $(\text{NLS})$. Furthermore, we have that:

(i) Any radial solution $\psi$ to $(\text{NLS})$ starting from $PW_{\omega,+}$ satisfies that

$$\|\langle \nabla \rangle \psi \|_{S_t(\mathbb{R})} < \infty,$$  \hspace{1cm} (1.42)

$$\mathcal{H}(\psi) \geq \frac{1}{2} \inf_{t \in \mathbb{R}} \mathcal{K}(\psi(t)) > 0.$$  \hspace{1cm} (1.43)

In particular, $\psi$ scatters both forward and backward in time.

(ii) Any radial solution $\psi$ to $(\text{NLS})$ starting from $PW_{\omega,-}$ blows up in a finite time both forward and backward in time.

We see that the main theorem in this paper (Theorem 1.1) is an extension of Theorem 1.3.

This paper is organized as follows. In Section 2 we give properties of ground state of $(\omega\text{-SP})$ which are used in this paper. In Section 3 we prove Theorem 1.1. The main ingredient of the proof is Theorem 3.1. We assign preliminaries for the proof of Theorem 3.1 to Section 4, Section 5, Section 6, and Section 7. In Section 4 we think about the decomposition of solution around the ground state in the same viewpoint of Nakanishi and Schlag [21]. In Section 5 we prove the “ejection lemma” for the equation $(\text{NLS})$. In Section 6 we introduce the distance function used in [21], and derive some variational lemmas. In Section 7 we prove “one-pass theorem”. In Section 8 we finally prove Theorem 3.1. This paper also has appended sections: In Section A we state properties of linearized operator around a ground state of $(\text{NLS})$. In Section B we record a small-data and a perturbation theories for $(\text{NLS})$.

Notation Besides the notation introduced above, we use the followings:

(i) As mentioned above, we use $\Phi_\omega$ and $U$ to denote the positive, radial grand states of $(\omega\text{-SP})$ and $(\text{SP}^\dagger)$, respectively. Moreover, we use $\Phi_\omega'$ to denote the derivative of $\Phi_\omega$ with respect to $\omega$ (see Proposition 2.4 below).

(ii) The daggered symbols are related to $(\text{SP}^\dagger)$:

$$L_+^\dagger := 1 - \Delta - pU^{p-1},$$  \hspace{1cm} (1.44)

$$L_-^\dagger := 1 - \Delta - U^{p-1}.$$  \hspace{1cm} (1.45)
(iii) Functionals with tilde are related to the rescaled equation $\omega$-SP:

\[
\tilde{K}_\omega(u) := \|\nabla u\|_{L^2}^2 - \frac{d(p-1)}{2(p+1)} \|u\|_{L^{p+1}}^{p+1} - \omega \frac{2^{\frac{2}{r}-p+1}}{p-1} \|u\|_{L^{2^*}}^{2^*},
\]

(1.46)

\[
\tilde{\mathcal{I}}_\omega(u) := \frac{1}{2} \|u\|_{L^2}^2 + \frac{s_p}{d} \|\nabla u\|_{L^2}^2 + \omega \frac{2^{\frac{2}{r}-p+1}}{p-1} \frac{1-s_p}{d} \|u\|_{L^{2^*}}^{2^*},
\]

(1.47)

\[
\tilde{L}_{\omega,+} := 1 - \Delta - p\langle T_\omega \Phi_\omega \rangle^{p-1} - \omega \frac{2^{\frac{2}{r}-p+1}}{p-1} (2^* - 1) \langle T_\omega \Phi_\omega \rangle^{\frac{4}{2^*-2}},
\]

(1.48)

\[
\tilde{L}_{\omega,-} := 1 - \Delta - (T_\omega \Phi_\omega)^{p-1} - \omega \frac{2^{\frac{2}{r}-p+1}}{p-1} (T_\omega \Phi_\omega)^{\frac{4}{2^*-2}}.
\]

(1.49)

(iv) We use $\langle \cdot, \cdot \rangle_{L^2}$ to denote the inner product in $L^2(\mathbb{R}^d)$:

\[
(u, v)_{L^2} := \int_{\mathbb{R}^d} u(x)\overline{v(x)} \, dx.
\]

We also use $L^2_{real}(\mathbb{R}^d)$ to denote the real Hilbert space of complex-valued functions in $L^2(\mathbb{R}^d)$ which is equipped with

\[
(u, v)_{L^2_{real}} := \Re \int_{\mathbb{R}^d} u(x)\overline{v(x)} \, dx.
\]

Furthermore, $H^1_{\text{real}}(\mathbb{R}^d)$ denotes the real Hilbert space of functions in $H^1(\mathbb{R}^d)$ equipped with the inner product

\[
(u, v)_{H^1_{\text{real}}} := (u, v)_{L^2_{\text{real}}} + \langle \nabla u, \nabla v \rangle_{L^2_{\text{real}}}.
\]

(iv) We use $\langle v, u \rangle_{H^{-1}, H^1}$ to denote the duality pair of $u \in H^1_{\text{real}}(\mathbb{R}^d)$ and $v \in H^{-1}_{\text{real}}(\mathbb{R}^d)$:

\[
\langle v, u \rangle_{H^{-1}, H^1} := ((1 - \Delta)^{-\frac{1}{2}} v, (1 - \Delta)^{\frac{3}{2}} u)_{L^2_{\text{real}}}.
\]

(v) Let $I$ be an interval, and let $q \in [2, 2^*]$. We use the following Strichartz-type spaces:

\[
St(I) := L^\infty_t L^2_x(I) \cap L^{\frac{d+2}{2}}_t L^{2^*}_x(I),
\]

(1.50)

\[
V_q(I) := L^{\frac{(d+2)(q-2)}{2}}_t L^{\frac{2d(d+2)(q-2)-8}{(d+2)(q-2)-8}}_x(I), \quad V(I) := V_{2^*}(I),
\]

(1.51)

\[
W_q(I) := L_{t,x}^{\frac{(d+2)(q-2)}{2}}, \quad W(I) := W_{2^*}(I).
\]

(1.52)

Note here that Sobolev’s embedding shows $|\nabla|^{-s_q-1} V_q(I) \hookrightarrow W_q(I)$. Moreover, by Strichartz’ estimate, we mean the following estimate: for any appropriate space-time function $u$, any $t_0 \in I$ and any pair $(q, r) \in [2, 2^*] \times [2, \infty]$ with $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{q})$,

\[
\|u\|_{St(I)} \lesssim \|u(t_0)\|_{L^2} + \|\frac{\partial u}{\partial t}\|_{L^q_t L^{q'}(I)} + \|\Delta u\|_{L^{q''}_t L^{q'''}(I)},
\]

(1.53)

where $q'$ and $r'$ denote the Hölder conjugates of $q$ and $r$ respectively.
For a space-time function $u$, we define

$$F[u] := |u|^{p-1}u + |u|^q u, \quad F^d[u] := |u|^{\frac{q}{d}} u,$$

$$e[u] := i \frac{\partial u}{\partial t} + \Delta u + F[u], \quad e^d[u] := i \frac{\partial u}{\partial t} + \Delta u + F^d[u].$$

### 2 Properties of Ground State

In this section, we state properties of ground state of $(\omega\text{-SP})$ in a series of propositions. Let us begin with the following:

**Proposition 2.1.** Assume $d \geq 4$ and (1.1). Let $\omega > 0$, and let $Q_\omega$ be a ground state of $(\omega\text{-SP})$. Then, there exist $\theta \in \mathbb{R}$, $y \in \mathbb{R}^d$ and a positive, radial ground state $\Phi_\omega$ of $(\omega\text{-SP})$ such that $\Phi_\omega \in C^2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, $x \cdot \nabla \Phi_\omega < 0$ and $Q_\omega = e^{i\theta} \Phi_\omega(\cdot - y)$.

**Proof of Proposition 2.1.** Let $Q_\omega$ be a ground state of $(\omega\text{-SP})$. Then, by the definition of the functional $I_\omega$, we see that $|Q_\omega|$ is also a ground state of $(\omega\text{-SP})$. We also see that $|Q_\omega| \in W^{2,q}_{\text{loc}}(\mathbb{R}^d)$ for any $2 \leq q < \infty$ (see Theorem 2.3 in [5]). Furthermore, the Schauder theory shows that $Q_\omega \in C^2(\mathbb{R}^d)$ and $\lim_{|x| \to \infty} |Q_\omega(x)| = 0$. Since the equation $(\omega\text{-SP})$ contains a “mass term”, the solution decays exponentially. On the other hand, it follows from Theorem 9.10 in [18] that for any compact set $K$ in $\mathbb{R}^d$, there exists a constant $C$ depending only on $K$ and $\omega$ such that for any $x \in K$,

$$|Q_\omega(x)| \geq C \int_K |Q_\omega(y)| \, dy. \quad (2.1)$$

Hence, $|Q_\omega|$ is a positive ground state of $(\omega\text{-SP})$. We see from the result of Gidas-Ni-Nirenberg [10] that there exist a positive radial function $\Phi_\omega$ with $\frac{x}{|x|} \cdot \nabla \Phi_\omega(x) < 0$ for
any \( x \in \mathbb{R}^d \setminus \{0\} \), and \( y \in \mathbb{R}^d \) such that \( |Q_\omega| = \Phi_\omega(\cdot - y) \). We define
\[
\text{sgn} Q_\omega := \frac{Q_\omega}{|Q_\omega|},
\tag{2.2}
\]
so that \( Q_\omega = |Q_\omega| \text{sgn} Q_\omega \). We shall show that \( \text{sgn} Q_\omega \) is constant, which together with \( |\text{sgn} Q_\omega| \equiv 1 \) completes the proof. Since
\[
\Re \left[ |\text{sgn} Q_\omega \nabla \text{sgn} Q_\omega(x) | \right] = \Re \left[ \frac{Q_\omega \nabla Q_\omega}{|Q_\omega|^2} - \frac{|Q_\omega| \nabla |Q_\omega|}{|Q_\omega|^2} \right] = 0,
\tag{2.3}
\]
we have
\[
\| \nabla Q_\omega \|_L^2 = \| \nabla ( |Q_\omega| \text{sgn} Q_\omega ) \|_L^2
\]
\[
= \int_{\mathbb{R}^d} |\nabla Q_\omega(x)|^2 \, dx + \int_{\mathbb{R}^d} |Q_\omega(x)|^2 |\nabla \text{sgn} Q_\omega(x)|^2 \, dx
\]
\[
+ \int_{\mathbb{R}^d} |\nabla Q_\omega(x)|^2 \Re \left[ \text{sgn} Q_\omega \nabla \text{sgn} Q_\omega(x) \right] \, dx
\]
\[
= \int_{\mathbb{R}^d} |\nabla Q_\omega(x)|^2 \, dx + \int_{\mathbb{R}^d} |Q_\omega(x)|^2 |\nabla \text{sgn} Q_\omega(x)|^2 \, dx.
\tag{2.4}
\]
Hence, if \( \nabla \text{sgn} Q_\omega \neq 0 \), then we would have
\[
m_\omega = S_\omega(Q_\omega) > S_\omega(|Q_\omega|) = S_\omega(\Phi_\omega) = m_\omega.
\tag{2.5}
\]
However, this is a contradiction. Thus, \( \text{sgn} Q_\omega \) must be constant.

We see from Proposition 2.1 that a ground state of \((\omega\text{-SP})\) is essentially positive and radial. Next, we give a decay property of a positive ground state.

**Proposition 2.2.** Assume \( d \geq 4 \) and (1.1). Let \( \Phi_\omega \in C^2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d) \) be a positive, radial ground state of \((\omega\text{-SP})\). Then, there exist positive constants \( C(\omega) > 0 \) and \( \delta(\omega) > 0 \) depending on \( \omega \) such that
\[
|\Phi_\omega(x)| + |\nabla \Phi_\omega(x)| + |\Delta \Phi_\omega(x)| \leq C(\omega) e^{-\delta(\omega)|x|}
\tag{2.6}
\]
for any \( x \in \mathbb{R}^d \). In particular, \( \Phi_\omega \in H^2(\mathbb{R}^d) \).

**Proof of Proposition 2.2.** We can prove the proposition in a way similar to [4].

**Proposition 2.3.** Assume \( d \geq 4 \) and (1.1). Let \( \Phi_\omega \) and \( U \) be positive, radial ground states of \((\omega\text{-SP})\) and \((\text{SP}^1)\), respectively. Then, we have
\[
\lim_{\omega \downarrow 0} \| T_\omega \Phi_\omega - U \|_{H^1} = 0.
\tag{2.7}
\]

**Proof of Proposition 2.3.** We can prove the proposition in a way similar to [14].
Remark 2.1. We see from Proposition 2.3 that for any \( q \in [2, 2^*], \)
\[
\lim_{\omega \to 0} \omega^{\frac{2^* - 2}{2^* - 4}} \| \Phi_{\omega} \|_{L^q}^q = \| U \|_{L^q}^q.
\] (2.8)

Proposition 2.4. Assume \( d \geq 4 \) and [11]. Let \( \Phi_{\omega} \) be a positive, radial ground state of \((\omega, SP).\) Then, there exists \( \omega_1 > 0 \) with the following properties:

(i) For any \( \omega \in (0, \omega_1), \) a positive, radial solution of \((\omega, SP)\) is unique.

(ii) The mapping \( \omega \in (0, \omega_1) \mapsto \Phi_{\omega} \in H^1(\mathbb{R}^d) \) is continuously differentiable.

(iii) For any \( \omega \in (0, \omega_1), \)
\[
\frac{d}{d\omega} M(\Phi_{\omega}) = (\Phi_{\omega}, \Phi'_\omega)_{L^2_{\text{rad}}} < 0.
\] (2.9)

In order to prove the claim (i) in Proposition 2.4, we need the following result due to Weinstein (see Proposition 2.8 in [26]):

Lemma 2.5. Assume \( d \geq 1 \) and \( p \) satisfies \( 0 < p - 1 < \frac{4}{d - 2}. \) Then, we have
\[
\text{Ker } L^\dagger_+ = \text{span}\{ \partial_1 U, \ldots, \partial_d U \},
\] (2.10)
where \( L^\dagger_+ \) is the operator defined by (1.43), and \( \partial_1 U, \ldots, \partial_d U \) denote the partial derivatives of \( U. \)

Now, we give a proof of the claim (i):

Proof of (i) in Proposition 2.4. Suppose for contradiction that for any \( k \in \mathbb{N}, \) there exists \( \omega_k \in (0, \frac{1}{k}) \) such that the equation \((\omega, SP)\) has two different positive radial solutions \( U_{\omega_k} \) and \( V_{\omega_k}. \) Put \( \tilde{U}_k(x) := T_{\omega_k} U_{\omega_k}, \tilde{V}_k(x) := T_{\omega_k} V_{\omega_k} \) and
\[
\tilde{u}_k := \frac{\tilde{U}_k - \tilde{V}_k}{\| \tilde{U}_k - \tilde{V}_k \|_{H^1}}.
\] (2.11)

Then, \( \tilde{u}_k \) satisfies the equation
\[
\tilde{u}_k - \Delta \tilde{u}_k - \frac{\tilde{U}_k^p - \tilde{V}_k^p}{\| \tilde{U}_k - \tilde{V}_k \|_{H^1}} - \frac{2^* - (p + 1)}{p - 1} \frac{\tilde{G}_k^{2^* - 1} - \tilde{V}_k^{2^* - 1}}{\| \tilde{U}_k - \tilde{V}_k \|_{H^1}} = 0.
\] (2.12)

Moreover, it follows from \( \tilde{u}_k \in H^1_{\text{rad}}(\mathbb{R}^d) \) and \( \| \tilde{u}_k \|_{H^1} = 1 \) for any \( k \in \mathbb{N} \) that there exists a radial function \( \tilde{u} \in H^1(\mathbb{R}^d) \) such that, passing to a subsequence, we have
\[
\lim_{k \to \infty} \tilde{u}_k = \tilde{u} \quad \text{weakly in } H^1(\mathbb{R}^d),
\] (2.13)
and for any \( 2 < q < 2^*, \)
\[
\lim_{k \to \infty} \tilde{u}_k = \tilde{u} \quad \text{strongly in } L^q(\mathbb{R}^d).
\] (2.14)

We shall show that \( \tilde{u} \equiv 0. \) First note that it follows from Proposition 2.3 and (2.14) that for any \( \phi \in C_c^\infty(\mathbb{R}^d), \)
\[
\lim_{k \to \infty} \langle \frac{\tilde{U}_k^p - \tilde{V}_k^p}{\| \tilde{U}_k - \tilde{V}_k \|_{H^1}}, \phi \rangle_{H^{-1}, H^1} = p \int_{\mathbb{R}^d} U^{p-1} \tilde{u} \phi \, dx.
\] (2.15)
Similarly, for a given function \( \phi \in C_\infty^\infty(\mathbb{R}^d) \), we ha
\[
\lim_{k \to \infty} \omega_k^{2^*-(p+1)/p} \left\langle \frac{\tilde{U}_k^{2^*-1} - \tilde{V}_k^{2^*-1}}{\|U_k - V_k\|_{H^1}}, \phi \right\rangle_{H^{-1},H^1} = 0. \tag{2.16}
\]
Combining (2.12) with (2.13), (2.15) and (2.16), we find that for any \( \phi \in C_\infty^\infty(\mathbb{R}^d) \),
\[
\left\langle L_+^\dagger \tilde{u}, \phi \right\rangle_{H^{-1},H^1} = \left\langle \tilde{u} - \Delta \tilde{u} - pU^{p-1}\tilde{u}, \phi \right\rangle_{H^{-1},H^1} = 0. \tag{2.17}
\]
Since \( \tilde{\tilde{u}} \) is a radial \( H^1 \)-solution of \( L_+^\dagger \tilde{u} = 0 \), we find that \( \tilde{\tilde{u}} \in H^2_{rad}(\mathbb{R}^d) \). Hence, it follows from Lemma 2.5 (there is no radial function in the kernel of \( L_+^\dagger + \tilde{\tilde{u}} \)) that \( \tilde{\tilde{u}} \equiv 0 \).

Now, multiplying the equation (2.12) by \( \tilde{\tilde{u}}_k \) and then integrating it over \( \mathbb{R}^d \), we obtain
\[
\|\tilde{\tilde{u}}_k\|_{H^1}^2 = \int_{\mathbb{R}^d} \frac{\tilde{U}_k^p - \tilde{V}_k^p}{\|U_k - V_k\|_{H^1}} \tilde{\tilde{u}}_k \ dx + \omega_k^{2^*-(p+1)/p} \int_{\mathbb{R}^d} \frac{\tilde{U}_k^{2^*-1} - \tilde{V}_k^{2^*-1}}{\|U_k - V_k\|_{H^1}} \tilde{\tilde{u}}_k \ dx. \tag{2.18}
\]
Then, it follows from (2.14) and \( \tilde{\tilde{u}} \equiv 0 \) that the right-hand side of (2.18) tends to 0 as \( k \to \infty \), whereas the left-hand side is identically 1 by \( \|\tilde{\tilde{u}}_k\|_{H^1} = 1 \). This is a contradiction.

Thus, we have completed the proof of the claim (i).

Next, we mention how to prove the claim (ii) in Proposition 2.4.

**Proof of (ii) in Proposition 2.4**
The claim (ii) follows from the result by Shatah and Strauss [22] (see also [16]).

Finally, we move on to the proof of the claim (iii) in Proposition 2.4. We notice that the claim (iii) is an analogy to that
\[
\frac{d}{d\omega} M(T_{\omega^{-1}} U) = \frac{1}{2} \frac{d}{d\omega} (\omega^{-sp}\|U\|_{L^2}^2) = -\frac{sp}{2} \omega^{-sp-1}\|U\|_{L^2}^2 < 0. \tag{2.19}
\]
Hence, it is convenient to introduce a function \( U' \) defined by
\[
U'(x) := \frac{d}{d\omega} T_{\omega^{-1}} U(x) \bigg|_{\omega=1} = \frac{1}{p-1} U(x) + \frac{1}{2} x \cdot \nabla U(x). \tag{2.20}
\]
Then, we can verify that \( U' \) obeys
\[
L_+^\dagger U' = -U. \tag{2.21}
\]
Differentiation of the both sides of the equation (\( \omega^{-SP} \)) with respect to \( \omega \) yields
\[
\omega \Phi'_\omega - \Delta \Phi'_\omega - p\Phi'_\omega^{p-1} \Phi'_\omega - (2^* - 1)\Phi'_\omega \Phi'_\omega^{2^*-1} \Phi'_\omega = -\Phi'_\omega. \tag{2.22}
\]
Furthermore, we see from (2.22) that
\[
\tilde{L}_{\omega,+}(\omega T_{\omega} \Phi'_\omega) = -T_{\omega} \Phi'_\omega, \tag{2.23}
\]
where \( \tilde{L}_{\omega,+} \) is the operator defined by (1.48). We state a property of the operator \( \tilde{L}_{\omega,+} \).
Lemma 2.6. There exist $\omega_0 > 0$ and $C > 0$ such that for any $\omega \in (0, \omega_0)$ and any $f \in H^2_{rad}(\mathbb{R}^d)$,
\[
\|\tilde{L}_{\omega,k} + f\|_{L^2} \geq C\|f\|_{H^1}.
\] (2.24)

Proof of Lemma 2.6. Suppose for contradiction that for any $k \in \mathbb{N}$, there exist $\omega_k \in (0, \frac{1}{k})$ and $f_k \in H^2_{rad}(\mathbb{R}^d)$ with $\|f_k\|_{H^1} = 1$ such that
\[
\|\tilde{L}_{\omega_k} + f_k\|_{L^2} < \frac{1}{k}.
\] (2.25)

Then, we can take $f_\infty \in H^1_{rad}(\mathbb{R}^d)$ such that, passing to some subsequence,
\[
\lim_{k \to \infty} f_k = f_\infty \quad \text{weakly in } H^1(\mathbb{R}^d),
\] (2.26)
and for any $2 < q < 2^*$,
\[
\lim_{k \to \infty} f_k = f_\infty \quad \text{strongly in } L^q(\mathbb{R}^d).
\] (2.27)

Furthermore, for any $\phi \in C^\infty_c(\mathbb{R}^d)$, we have
\[
\left|\langle L_{\omega,k}^+, f_\infty, \phi \rangle_{H^{-1},H^1} \right| \\
\leq \left| \langle \tilde{L}_{\omega_k} + f_k, \phi \rangle_{H^{-1},H^1} \right| + \left| \langle \tilde{L}_{\omega_k} + (f_\infty - f_k), \phi \rangle_{H^{-1},H^1} \right| \\
+ p\left| \langle \{ (T_{\omega_k} \Phi_{\omega_k})^{q-1} - U^{q-1} \} f_\infty, \phi \rangle_{H^{-1},H^1} \right| \\
+ \omega_k^{\frac{2^*-(p+1)}{p-1}} (2^* - 1) \left| \langle (T_{\omega_k} \Phi_{\omega_k})^{\frac{p-1}{q}} f_\infty, \phi \rangle_{H^{-1},H^1} \right|. 
\] (2.28)

This together with the hypothesis (2.25), (2.26), (2.27), Proposition 2.3 and $\lim_{k \to \infty} \omega_k = 0$ shows that $L^1_{\omega,k} f_\infty = 0$ in the distribution sense. Since $(\operatorname{Ker} L^1_{\omega,k})_{H^1_{rad}} = \{0\}$ (see Lemma 2.5), we conclude that $f_\infty = 0$. Furthermore, this fact $f_\infty = 0$ together with $\|f_k\|_{H^1} = 1$, (2.24) and $\lim_{k \to \infty} \omega_k = 0$ gives us that
\[
\lim_{k \to \infty} (\tilde{L}_{\omega_k} + f_k, f_k)_{L^2} = 1.
\] (2.29)

However, it follows from the hypothesis (2.25) that
\[
\lim_{k \to \infty} |(\tilde{L}_{\omega_k} + f_k, f_k)_{L^2}| \leq \lim_{k \to \infty} \|\tilde{L}_{\omega_k} + f_k\|_{L^2} = 0.
\] (2.30)

This is a contradiction. Thus, we have proved that (2.24) holds. \qed

Lemma 2.7.
\[
\lim_{\omega \to 0} \|\omega T_{\omega} \Phi'_\omega - U'\|_{H^1} = 0.
\] (2.31)

Proof of Lemma 2.7. Let $\omega_0 > 0$ and $C > 0$ be constants given in Lemma 2.6. Furthermore, let $\{\omega_n\}$ be a sequence in $(0, \omega_0)$ with $\lim_{n \to \infty} \omega_n = 0$. Then, we find from
Proposition 2.4 and Proposition 2.5 that there exists a number $N$ such that for any $n \geq N$,

\[ C\|\omega T_\omega \Phi_{\omega_n}'\|_{H^1} \leq \|\tilde{L}_{\omega_n} + (\omega T_\omega \Phi_{\omega_n}')\|_{L^2} = \|T_{\omega_n} \Phi_{\omega_n}\|_{H^1} \leq 2\|U\|_{H^1}. \tag{2.32} \]

Thus, $\{\omega T_\omega \Phi_{\omega_n}'\}$ is bounded in $H^1(\mathbb{R}^d)$ and therefore we can take $g_\infty \in H^1(\mathbb{R}^d)$ such that, passing to some subsequence,

\[ \lim_{n \to \infty} \omega T_\omega \Phi_{\omega_n}' = g_\infty \quad \text{weakly in } H^1(\mathbb{R}^d). \tag{2.33} \]

This together with Proposition 2.3 also shows that for any $\phi \in C_0^\infty(\mathbb{R}^d)$,

\[ \lim_{n \to \infty} \langle \tilde{L}_{\omega_n} + (\omega T_\omega \Phi_{\omega_n}'), \phi \rangle_{H^{-1},H^1} = \langle L_+^1 g_\infty, \phi \rangle_{H^{-1},H^1}. \tag{2.34} \]

On the other hand, we see from (2.23) and Proposition 2.3 that

\[ \lim_{n \to \infty} \tilde{L}_{\omega_n} + (\omega T_\omega \Phi_{\omega_n}) = -U \quad \text{strongly in } H^1(\mathbb{R}^d). \tag{2.35} \]

Putting (2.21) and (2.33) together, we find that $L_+^1 g_\infty = -U$. Furthermore, since $(\ker L_+^1)|_{H^1_{rad}} = \{0\}$ (see Lemma 2.5), this identity together with (2.21) shows that $g_\infty = U'$. It remains to show that

\[ \lim_{n \to \infty} \|\omega T_\omega \Phi_{\omega_n}'\|_{H^1} = \|U'\|_{H^1}. \tag{2.36} \]

We see from (2.23), Proposition 2.2 Proposition 2.3 (2.33) with $g_\infty = U'$ and (2.21) that

\[ \lim_{n \to \infty} \|\omega T_\omega \Phi_{\omega_n}'\|_{H^1}^2 \leq - \lim_{n \to \infty} \langle T_\omega \Phi_{\omega_n}, \omega T_\omega \Phi_{\omega_n}' \rangle_{L^2_{real}} 
+ \lim_{n \to \infty} p(\langle T_\omega \Phi_\omega \rangle)^{p-1} \omega T_\omega \Phi_{\omega_n}', \omega T_\omega \Phi_{\omega_n}' \rangle_{L^2_{real}} 
+ \lim_{n \to \infty} \omega^{2^*-p+1} (2^*-1) \langle T_\omega \Phi_\omega \rangle^{2^*/2-1} \omega T_\omega \Phi_{\omega_n}', \omega T_\omega \Phi_{\omega_n}' \rangle_{L^2_{real}} \]

\[ = -\langle U, U' \rangle_{L^2_{real}} + p(U^{p-1}U', U')_{L^2_{real}} \]

\[ = (L_+^1 U', U')_{L^2_{real}} + p(U^{p-1}U', U')_{L^2_{real}} = \|U'\|_{H^1}^2. \tag{2.37} \]

Thus, we have completed the proof. \hfill \square

Now, we are in a position to prove the claim (iii) in Proposition 2.4.

**Proof of (iii) in Proposition 2.4.** We see from Proposition 2.3 (2.7) and (2.20) that

\[ \lim_{\omega \to 0} \omega^{p+1} \frac{d}{d\omega} \mathcal{M}(\Phi_\omega) = \lim_{\omega \to 0} \omega^{p+1} \langle \Phi_\omega, \Phi_\omega' \rangle_{L^2_{real}} = \lim_{\omega \to 0} \langle T_\omega \Phi_\omega, \omega T_\omega \Phi_\omega' \rangle_{L^2_{real}} = (U, U')_{L^2_{real}} = -\frac{sp}{2}\|U\|_{L^2}^2 < 0, \tag{2.38} \]

which gives us the desired result. \hfill \square
3 Proof of Theorem 1.1

In order to prove Theorem 1.1 for a given \( \varepsilon \geq 0 \), we consider the set
\[
A^\varepsilon_\omega := \{ u \in H^1(\mathbb{R}^d) : S_\omega(u) < m_\omega + \varepsilon, M(u) = M(\Phi_\omega) \}. 
\]
(3.1)

Then, a key fact to prove Theorem 1.1 is the following:

**Theorem 3.1.** Assume \( d \geq 4 \) and (1.1). Then, there exists \( \omega_* > 0 \) with the following property: for any \( \omega \in (0, \omega_*) \), there exists a positive constant \( \varepsilon(\omega) \) such that all radial solutions starting from \( A^\varepsilon_\omega \) exhibit one of the same nine scenarios in Theorem 1.1.

We give the proof of this theorem in Section 8.

In order to prove Theorem 1.1, we also use the following fact:

**Lemma 3.1.** Assume \( d \geq 4 \) and (1.1). Then, we have the followings:

(i) \( m_\omega \) is strictly increasing with respect to \( \omega > 0 \). Furthermore, \( m_\omega \) is differentiable at any \( \omega \in (0, \omega_1) \), and
\[
\frac{dm_\omega}{d\omega} = M(\Phi_\omega). 
\]
(3.2)

(ii) \( \frac{m_\omega}{\omega} \) is differentiable and strictly decreasing on \( (0, \omega_1) \).

(iii) Let \( 0 < \alpha < \beta < \omega_1 \). Then, we have that
\[
M(\Phi_\beta) < \frac{m_\beta - m_\alpha}{\beta - \alpha} < M(\Phi_\alpha). 
\]
(3.3)

**Proof of Lemma 3.1.** Let \( 0 < \alpha < \beta \). Then, since \( K(\Phi_\beta) = 0 \), we see that
\[
m_\alpha \leq S_\alpha(\Phi_\beta) = \alpha M(\Phi_\beta) + \mathcal{H}(\Phi_\beta) < \beta M(\Phi_\beta) + \mathcal{H}(\Phi_\beta) = m_\beta. 
\]
(3.4)

Furthermore, since \( S'(\Phi_\omega) = 0 \) and \( \Phi_\omega \) is differentiable with respect to \( \omega \) (see (ii) of Proposition 2.4), we see that for any \( \omega \in (0, \omega_1) \),
\[
\frac{dm_\omega}{d\omega} = \frac{d}{d\omega} S_\omega(\Phi_\omega) = M(\Phi_\omega) + S'_\omega(\Phi_\omega) \frac{d\Phi_\omega}{d\omega} = M(\Phi_\omega). 
\]
(3.5)

Thus, we have proved the first claim.

Next, we shall prove the second claim. The differentiability of \( \frac{m_\omega}{\omega} \) follows from the first claim. Furthermore, we conclude from (ii) of Proposition 2.4 that for any \( \omega \in (0, \omega_1) \),
\[
\frac{d}{d\omega} \left( \frac{m_\omega}{\omega} \right) = \frac{d}{d\omega} \left( \frac{S_\omega(\Phi_\omega)}{\omega} \right) = \frac{S'_\omega(\Phi_\omega) \frac{d\Phi_\omega}{d\omega} - S_\omega(\Phi_\omega)}{\omega^2} = -\frac{S_\omega(\Phi_\omega)}{\omega^2} < 0. 
\]
(3.6)

Thus, \( \frac{m_\omega}{\omega} \) is strictly decreasing.

Finally, we shall prove the last claim. It follows from the mean value theorem and (3.2) that for any \( 0 < \alpha < \beta < \omega_1 \), there exists \( \theta \in (0, 1) \) such that
\[
\frac{m_\beta - m_\alpha}{\beta - \alpha} = M(\Phi_{\alpha + \theta(\beta - \alpha)}). 
\]
(3.7)
On the other hand we see from (iii) of Proposition 2.4 that for any \( \theta \in (0, 1) \),
\[
\mathcal{M} (\Phi_\beta) < \mathcal{M}(\Phi_{\alpha + \theta (\beta - \alpha)}) < \mathcal{M}(\Phi_\alpha).
\] (3.8)

Putting (3.7) and (3.8) together, we obtain the desired result (3.3).

We see from (2.8) and (iii) of Proposition 2.4 that there is a strictly decreasing function \( \alpha : (\mathcal{M}(\Phi_{\omega_1}), \infty) \to (0, \omega_1) \) such that \( \mathcal{M}(\Phi_\alpha) \) is decreasing. Let \( \omega_* \) be the constant given by Theorem 3.1. Then, for each \( \omega \in (0, \omega_*) \), we define the positive function \( \varepsilon_\omega : [0, \infty) \to (0, \infty) \) to be that for any \( M \geq 0 \),
\[
\varepsilon_\omega(M) := \begin{cases} 
  m_{\omega_*} - m_\omega - (\omega_* - \omega)\mathcal{M}(\Phi_{\omega_*}) & \text{if } M \leq \mathcal{M}(\Phi_{\omega_*}), \\
  \varepsilon(\alpha(M)) + m_{\alpha(M)} - m_\omega - (\alpha(M) - \omega)M & \text{if } \mathcal{M}(\Phi_{\omega_*}) < M < \mathcal{M}(\Phi_\omega), \\
  \varepsilon(\omega) & \text{if } M = \mathcal{M}(\Phi_\omega), \\
  \varepsilon(\alpha(M)) + (\omega - \alpha(M))M - (m_\omega - m_{\alpha(M)}) & \text{if } M > \mathcal{M}(\Phi_\omega),
\end{cases}
\] (3.9)
where \( \varepsilon(\alpha(M)) \) and \( \varepsilon(\omega) \) are the positive constants given by Theorem 3.1. It follows from Lemma 3.1 that this function \( \varepsilon_\omega \) is positive.

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let \( \omega_* \) be the frequency found in Theorem 3.1 and let \( \omega \in (0, \omega_*) \). Furthermore, let \( \varepsilon_\omega \) be the positive function defined by (3.9), and let \( \tilde{P}W_\omega \) be the set defined by (1.13).

We consider a solution \( \psi \) to (NLS) starting from \( \tilde{P}W_\omega \). If \( \mathcal{M}(\psi) \leq \mathcal{M}(\Phi_{\omega_*}) \), then we can verify that \( \psi \in PW_{\omega_*} \) (see Figure 2 above). Hence, it follows from Theorem 1.3 that \( \psi \) exhibits the scenario (i) or (ii) in Theorem 1.1. On the other hand, if \( \mathcal{M}(\Phi_{\omega_*}) < \)
\( \mathcal{M}(\psi) < \mathcal{M}(\Phi_\omega) \), then we can verify that \( \psi \in A^{\varepsilon(\alpha)}_\alpha \), where \( \alpha := \alpha(\mathcal{M}(\psi)) \). Similarly, if \( \mathcal{M}(\psi) \geq \mathcal{M}(\Phi_\omega) \), then \( \psi \in A^{\varepsilon(\alpha)}_\alpha \). Hence, it follows from Theorem 3.1 that \( \psi \) exhibits one of the nine scenarios in Theorem 1.1. \( \square \)

4 Decomposition around ground state

Let \( \omega_1 \) be the frequency given by Proposition 2.4, \( \omega \in (0, \omega_1) \), and let \( \psi \) be a solution to \( \text{NLS} \) with the maximal existence-interval \( I_{\text{max}} \). Assume that

\[
\mathcal{M}(\psi) = \mathcal{M}(\Phi_\omega).
\]

Then, in view of the orbital stability theory, we consider a decomposition of the form

\[
\psi(x, t) = e^{i\theta(t)}(\Phi_\omega(x) + \eta(x, t)),
\]

where \( \theta(t) \) is some function of \( t \in I_{\text{max}} \) to be chosen later, and \( \eta(x, t) \) is some function of \( (x, t) \in \mathbb{R}^d \times I_{\text{max}} \). Note here that we do not have the orthogonality \( \langle \Phi_\omega, \eta(t) \rangle_{L^2_{\text{real}}} = 0 \).

Since

\[
\mathcal{M}(\psi) = \mathcal{M}(\Phi_\omega) + \mathcal{M}(\eta(t)) + \langle \Phi_\omega, \eta(t) \rangle_{L^2_{\text{real}}},
\]

the condition (4.1) implies that for any \( t \in I_{\text{max}} \),

\[
\langle \Phi_\omega, \eta(t) \rangle_{L^2_{\text{real}}} = -\mathcal{M}(\eta(t)).
\]

We consider the linearized operator \( \mathcal{L}_\omega \) around \( \Phi_\omega \) which is defined by

\[
\mathcal{L}_\omega u := \omega u - \Delta u - \frac{p + 1}{2} \Phi^p_{\omega^{-1}} u - \frac{p - 1}{2} \Phi^p_{\omega^{-1}} u - \frac{2^* - 2}{2} \Phi^\frac{4}{2^* - 2} u - \frac{2^* - 2}{2} \Phi^\frac{4}{2^* - 2} u.
\]

\[
= (\omega - \Delta - p \Phi^p_{\omega^{-1}} - (2^* - 1) \Phi^\frac{4}{2^* - 2}) \Re[u] + i(\omega - \Delta - \Phi^p_{\omega^{-1}} - \Phi^\frac{4}{2^* - 2}) \Im[u].
\]

The operator \( \mathcal{L}_\omega \) is self-adjoint in \( L^2_{\text{real}}(\mathbb{R}^d) \), and for any \( u, v \in H^1(\mathbb{R}^d) \),

\[
[\mathcal{S}''_\omega(\Phi_\omega) u] v = \langle \mathcal{L}_\omega u, v \rangle_{H^{-1}, H^1}.
\]

It is convenient to introduce the operators \( L_{\omega,+} \) and \( L_{\omega,-} \):

\[
L_{\omega,+} := \omega - \Delta - p \Phi^p_{\omega^{-1}} - (2^* - 1) \Phi^\frac{4}{2^* - 2},
\]

\[
L_{\omega,-} := \omega - \Delta - \Phi^p_{\omega^{-1}} - \Phi^\frac{4}{2^* - 2}.
\]

Then, we have

\[
\mathcal{L}_\omega u = L_{\omega,+} \Re[u] + i L_{\omega,-} \Im[u].
\]

Moreover, since \( \Phi_\omega \) is a solution to \( \omega\text{-SP} \), we can verify that

\[
\mathcal{L}_\omega \Phi_\omega = L_{\omega,+} \Phi_\omega = -(p - 1) \Phi^p_{\omega} - (2^* - 2) \Phi^{2^*-1}_{\omega},
\]

\[
\mathcal{L}_\omega (i \Phi_\omega') = L_{\omega,-} \Phi_\omega = 0,
\]

\[
\mathcal{L}_\omega \Phi_\omega' = L_{\omega,+} \Phi_\omega' = -\Phi_\omega.
\]
Inserting the decomposition $\psi(t) = e^{i\theta(t)}(\Phi_\omega + \eta(t))$ into the equation (NLS), we obtain the equation for $\eta$:

$$\frac{\partial \eta}{\partial t} + i\mathcal{L}_\omega \eta(t) \equiv -i\mathcal{L}_\omega \eta(t) - i\{\frac{d\theta}{dt}(t) - \omega\}(\Phi_\omega + \eta(t)) + iN_\omega(\eta(t)),$$

where $N_\omega(\eta)$ denotes the higher order term of $\eta$, i.e.,

$$
N_\omega(\eta) := |\Phi_\omega + \eta|^{p-1}(\Phi_\omega + \eta) - |\Phi_\omega|^{p-1}\Phi_\omega - \frac{p+1}{2}\Phi_\omega^{p-1}\eta - \frac{p-1}{2}\Phi_\omega^{-1}\eta
$$

$$
+ |\Phi_\omega + \eta|^{\frac{4}{1+p}}(\Phi_\omega + \eta) - |\Phi_\omega|^{\frac{4}{1+p}}\Phi_\omega - \frac{2^s}{2}\Phi_\omega^{-1}\eta - \frac{2^{-s}}{2}\Phi_\omega^{s-2}\eta
$$

$$= O(|\eta|^{\min\{2,p\}}).$$

Thus, the operator $i\mathcal{L}_\omega$ relates to the behaviour of the remainder $\eta(t)$. Unfortunately, $i\mathcal{L}_\omega$ is not symmetric in $L^2_{\text{real}}(\mathbb{R}^d)$, and therefore we do not have the orthogonality of eigen-functions in this space. Thus, we need to work in the symplectic space $(L^2(\mathbb{R}^d), \Omega)$ instead of $L^2_{\text{real}}(\mathbb{R}^2)$, where $\Omega$ is the symplectic form defined by

$$\Omega(f, g) := \Im \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, dx = \langle f, ig \rangle_{L^2_{\text{real}}}. \quad (4.15)$$

**Proposition 4.1.** Assume $d \geq 4$ and (1.1). Let $\omega_1$ be the frequency given by Proposition 2.4 Then, there exists $\omega_2 \in (0, \omega_1)$ such that for any $\omega \in (0, \omega_2)$, $-i\mathcal{L}_\omega$ has a positive eigen-value $\mu$, as an operator in $L^2_{\text{real}}(\mathbb{R}^d)$. Furthermore, the eigen-value $\mu$ satisfies that

$$-\mu^2 = \inf \left\{ \frac{(L_{\omega_-}^1 u, u)_{H^{-1}, H^1}}{(u, u)_{L^2_{\text{real}}}} : u \in H^1(\mathbb{R}^d), (u, \Phi_\omega)_{L^2_{\text{real}}} = 0 \right\}. \quad (4.16)$$

**Proof of Proposition 4.1** We prove the proposition following the exposition in Section 2 of [S]. What we need to prove is that there exist a function $u \in H^1(\mathbb{R}^d)$ and $\nu < 0$ such that

$$L_{\omega,-}L_{\omega,+}u = \nu u. \quad (4.17)$$

Indeed, putting $f_1 = -\Re[u]$ and $f_2 = -\sqrt{-\nu}(L_{\omega,-})^{-1}\Re[u]$, we see from (4.14) and (4.17) that

$$-i\mathcal{L}_\omega(f_1 + if_2) = -iL_{\omega,+}f_1 + L_{\omega,-}f_2
$$

$$= i(L_{\omega,-})^{-1}\nu\Re[u] - L_{\omega,-}\sqrt{-\nu}(L_{\omega,-})^{-1}\Re[u] = \sqrt{-\nu}(f_1 + if_2). \quad (4.18)$$

Furthermore, we can verify that the problem (4.17) is equivalent to that there exist $u \in H^1(\mathbb{R}^d), \nu < 0$ and $\alpha \in \mathbb{R}$ such that

$$
\begin{cases}
L_{\omega,+}u = \nu(L_{\omega,-})^{-1}u + \alpha \Phi_\omega,
\quad (u, \Phi_\omega)_{L^2_{\text{real}}} = 0.
\end{cases}
$$

$$\begin{cases}
L_{\omega,+}u = \nu(L_{\omega,-})^{-1}u + \alpha \Phi_\omega,
\quad (u, \Phi_\omega)_{L^2_{\text{real}}} = 0.
\end{cases}$$
Note here that \( \text{Ker } L_{\omega,-} = \text{span} \{ \Phi_\omega \} \) (see Lemma 4.11) and \( -\Phi_\omega = L_{\omega,+} \Phi_\omega \) (see (4.12)).

The problem (4.19) leads us to the following minimizing problem

\[
\nu_\omega := \inf \left\{ \frac{\langle L_{\omega,+} u, u \rangle_{H^{-1}, H^1}^\prime}{\langle (L_{\omega,-})^{-1} u, u \rangle_{L^2_{\text{real}}}^\prime} : u \in H^1(\mathbb{R}^d), \ (u, \Phi_\omega)_{L^2_{\text{real}}} = 0 \right\}.
\]  

(4.20)

We can verify that any minimizer of (4.20) satisfies the equation (4.19) with \( \nu = \nu_\omega \) for some \( \alpha \in \mathbb{R} \). Thus, it suffices to show that \( -\infty < \nu_\omega < 0 \) and the existence of a minimizer.

First, we shall show that \( -\infty < \nu_\omega \). We take any \( u \in H^1(\mathbb{R}^d) \) with \( (u, \Phi_\omega)_{L^2_{\text{real}}} = 0 \). Then, we see from Proposition 2.2 that

\[
\frac{\langle L_{\omega,+} u, u \rangle_{H^{-1}, H^1}}{\langle (L_{\omega,-})^{-1} u, u \rangle_{L^2_{\text{real}}}^\prime} = \frac{\langle L_{\omega,-} u, u \rangle_{H^{-1}, H^1}}{\parallel (L_{\omega,-})^{-1} u \parallel_{L^2}^2} - \frac{\langle (p - 1) \Phi_{\omega}^{-1} + (2^* - 2) \Phi_{\omega}^{-2} \rangle u, u \rangle_{H^{-1}, H^1}}{\parallel (L_{\omega,-})^{-1} u \parallel_{L^2}^2} - \frac{\parallel (L_{\omega,-})^{-1} u \parallel_{L^2}^2}{\parallel (L_{\omega,-})^{-1} u \parallel_{L^2}^2}
\]

(4.21)

where \( C(\omega) \) is some positive constant depending on \( \omega \). Moreover, it is easy to see that

\[
\parallel u \parallel_{L^2}^2 = \langle (L_{\omega,-})^{-1} u, u \rangle_{L^2_{\text{real}}} \leq \langle (L_{\omega,-})^{-1} u, u \rangle_{L^2_{\text{real}}} \parallel (L_{\omega,-})^{-1} u \parallel_{L^2}_{L^2}^2 \leq \frac{1}{2C(\omega)} \parallel (L_{\omega,-})^{-1} u \parallel_{L^2}^2 + \frac{C(\omega)}{2} \parallel (L_{\omega,-})^{-1} u \parallel_{L^2}^2.
\]

(4.22)

Putting the estimates (4.21) and (4.22) together, we find that \( \nu_\omega > -C(\omega)^2 > -\infty \).

Next, we shall show that \( \nu_\omega \) is negative for any sufficiently small \( \omega > 0 \). To this end, we introduce projections \( \Pi_\omega \) and \( \Pi \):

\[
\Pi_\omega u := u - \frac{(u, \Phi_\omega)_{L^2_{\text{real}}}}{\parallel \Phi_\omega \parallel_{L^2}^2} \Phi_\omega, \quad \Pi u := u - \frac{(u, U)_{L^2_{\text{real}}}}{\parallel U \parallel_{L^2}^2} U.
\]

(4.23)

Then, we see from substitution of variables, Proposition 2.3 and Lemma 2.7 that

\[
\omega^{-2} \nu_\omega = \omega^{-2} \inf \left\{ \frac{\langle L_{\omega,+} \Pi_\omega \Phi_\omega, \Pi_\omega \Phi_\omega \rangle_{H^{-1}, H^1}}{\langle (L_{\omega,-})^{-1} \Pi_\omega \Phi_\omega, \Pi_\omega \Phi_\omega \rangle_{L^2_{\text{real}}}^\prime} : u \in H^1(\mathbb{R}^d), \ (u, \Phi_\omega)_{L^2_{\text{real}}} = 0 \right\}
\]

\[
\leq \omega^{-2} \frac{\langle L_{\omega,+} \Pi_\omega \Phi_\omega, \Pi_\omega \Phi_\omega \rangle_{H^{-1}, H^1}}{\langle (L_{\omega,-})^{-1} \Pi_\omega \Phi_\omega, \Pi_\omega \Phi_\omega \rangle_{L^2_{\text{real}}}^\prime} = \frac{\langle \tilde{L}_{\omega,+} (\omega T_{\omega} \Pi_\omega \Phi_\omega), \omega T_{\omega} \Pi_\omega \Phi_\omega \rangle_{H^{-1}, H^1}}{\langle (\tilde{L}_{\omega,-})^{-1} (\omega T_{\omega} \Pi_\omega \Phi_\omega), \omega T_{\omega} \Pi_\omega \Phi_\omega \rangle_{L^2_{\text{real}}}^\prime}
\]

(4.24)

\[
\rightarrow \frac{\langle L_{\omega,+}^\dagger \Pi U', \Pi U' \rangle_{H^{-1}, H^1}}{\langle (L_{\omega,-})^{-1} \Pi U', \Pi U' \rangle_{L^2_{\text{real}}}^\prime} \quad \text{as } \omega \downarrow 0,
\]

where \( \tilde{L}_{\omega,+}, \tilde{L}_{\omega,-}, L_{\omega}^\dagger \) and \( L_{\omega}^\dagger \) are operators defined by \( (15.18), (15.19), (15.21) \) and \( (15.5) \), respectively. Thus, it suffices to show that

\[
\frac{\langle L_{\omega,+}^\dagger \Pi U', \Pi U' \rangle_{H^{-1}, H^1}}{\langle (L_{\omega,-})^{-1} \Pi U', \Pi U' \rangle_{L^2_{\text{real}}}^\prime} < 0.
\]

(4.25)
Note here that we see from (2.20) and integration by parts that

$$\Pi U' = U' + \frac{sp}{2} U.$$  \hfill (4.26)

Moreover, it follows from (2.20) and (2.21) that

$$L_+^\dagger \Pi U' = L_+^\dagger U' - \frac{(U', U)_{L^2_{\text{real}}} L_+^\dagger U}{\|U\|^2_{L^2}} = -U - \frac{(p-1)sp}{2} U^p.$$  \hfill (4.27)

Using (4.26), (4.27), (2.20) and integration by parts, we obtain that

$$\langle L_+^\dagger \Pi U', \Pi U' \rangle_{H^{-1}, H^1} = \langle -U - \frac{(p-1)sp}{2} U^p, U' \rangle_{H^{-1}, H^1} - \frac{sp}{2} \|U\|_{L^2}^2 - \frac{(p-1)s^2}{4} \|U\|_{L^{p+1}}^{p+1}.$$  \hfill (4.28)

Since Lemma A.1 implies that $\Pi U'$ is bounded in $H^1(\mathbb{R}^d)$ and $\Pi U' \in L^2_{\text{real}}$, we find from (4.28) that (4.25) holds.

Finally, we shall show the existence of a minimizer $u$ for $\nu_\omega$. We can take a sequence $\{u_n\}$ in $H^1(\mathbb{R}^d)$ such that

$$\langle u_n, \Phi_\omega \rangle_{L^2_{\text{real}}} = 0,$$  \hfill (4.29)

$$\langle (L_{\omega,-})^{-1} u_n, u_n \rangle_{L^2_{\text{real}}} = 1,$$  \hfill (4.30)

$$\lim_{n \to \infty} \langle L_{\omega,+} u_n, u_n \rangle_{H^{-1}, H^1} = \nu_\omega.$$  \hfill (4.31)

Note here that Lemma A.1 shows that the square root of $L_{\omega,-}$ is well-defined. We see from (4.31), Proposition 2.2, the Cauchy-Schwartz inequality, (A.1) and (4.30) that for any sufficiently large number $n$,

$$\|u_n\|^2_{H^1} \leq (1 + \omega^{-1}) \langle L_{\omega,+} u_n, u_n \rangle_{H^{-1}, H^1} + \int_{\mathbb{R}^d} \{ p(\Phi_\omega(x))^{p-1} + (2^* - 1)(\Phi_\omega(x))^{\frac{4}{d-2}} \} |u_n(x)|^2 \, dx$$

$$\leq (1 + \omega^{-1}) \nu_\omega + C_1(\omega)(\|L_{\omega,-}\|^\frac{1}{2} u_n, (L_{\omega,-})^{-\frac{1}{2}} u_n)_{L^2_{\text{real}}},$$

$$\leq (1 + \omega^{-1}) \nu_\omega + C_1(\omega)(\|L_{\omega,-}\|^\frac{1}{2} u_n \|L^2_{\text{real}}\|\|L_{\omega,+}\|^{-\frac{1}{2}} u_n \|L^2\|$$

$$= (1 + \omega^{-1}) \nu_\omega + C_1(\omega)(\|L_{\omega,-} u_n, u_n\|_{L^2_{\text{real}}}^\frac{1}{2} \|L_{\omega,-}\|^{-1} u_n, u_n\|_{L^2_{\text{real}}} \|L^2\|$$

$$\leq (1 + \omega^{-1}) \nu_\omega + C_2(\omega) \|u_n\|^2_{H^1},$$

where $C_1(\omega)$ and $C_2(\omega)$ are some constants depending only on $d$, $p$ and $\omega$. This implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$ and therefore we can extract a subsequence of $\{u_n\}$ (still
denoted by the same symbol) and a function \( u_\infty \in H^1(\mathbb{R}^d) \) such that
\[
\lim_{n \to \infty} u_n = u_\infty \quad \text{weakly in } H^1(\mathbb{R}^d),
\] (4.33)
\[
(u_\infty, \Phi_\omega)_{L^2_{\text{real}}} = 0,
\] (4.34)
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \Phi_\omega(x)^q |u_n(x)|^2 \, dx = \int_{\mathbb{R}^d} \Phi_\omega(x)^q |u_\infty(x)|^2 \, dx,
\] (4.35)
where \( q \) indicates \( p - 1 \) or \( \frac{4}{d-2} \). Here, (4.34) follows from (4.29) and (4.33), and (4.35) follows from (4.33) and Proposition 2.2. Furthermore, we can verify that
\[
(L_{\omega,+} u_\infty, u_\infty)_{H^{-1}, H^1} = \liminf_{n \to \infty} (L_{\omega,+} u_n, u_n)_{H^{-1}, H^1} = \nu_\omega < 0.
\] (4.36)
In particular, the limit \( u_\infty \) is non-trivial. We also see from Lemma A.1, (4.11), (4.29) and (4.30) that for any number \( n \),
\[
\|(L_{\omega,-})^{-1} u_n\|_{H^1}^2 \lesssim (u_n, (L_{\omega,-})^{-1} u_n)_{L^2_{\text{real}}} \leq 1,
\] (4.37)
where the implicit constant depends on \( \omega \). Hence, there exists a subsequence of \( \{ u_n \} \) (still denoted by the same symbol) and a function \( v_\infty \in H^1(\mathbb{R}^d) \) such that
\[
\lim_{n \to \infty} (L_{\omega,-})^{-1} u_n = v_\infty \quad \text{weakly in } H^1(\mathbb{R}^d).
\] (4.38)
Furthermore, we see from (4.11) that for any test function \( \phi \in C_\infty^0(\mathbb{R}^d) \),
\[
\langle (L_{\omega,-})^{-1} u_\infty, \phi \rangle_{H^{-1}, H^1} = \lim_{n \to \infty} \langle (L_{\omega,-})^{-1} u_n, L_{\omega,-} \phi \rangle_{L^2_{\text{real}}} = (u_\infty, \phi)_{L^2_{\text{real}}},
\] (4.39)
This together with (4.34) shows that
\[
v_\infty = (L_{\omega,-})^{-1} u_\infty.
\] (4.40)
We also see from (4.40) and Lemma A.1 that
\[
((L_{\omega,-})^{-1} u_\infty, u_\infty)_{L^2_{\text{real}}} = (v_\infty, L_{\omega,-} v_\infty)_{L^2_{\text{real}}} > 0.
\] (4.41)
Furthermore, it follows from (4.38), (4.40), (4.29) and (4.30) that
\[
((L_{\omega,-})^{-1} u_\infty, u_\infty)_{L^2_{\text{real}}} \leq \liminf_{n \to \infty} ((L_{\omega,-})^{-1} u_n, u_n)_{L^2_{\text{real}}} = 1,
\] (4.42)
so that we can take \( \theta \geq 1 \) such that \( ((L_{\omega,-})^{-1} \theta u_\infty, \theta u_\infty)_{L^2_{\text{real}}} = 1 \). Suppose here that \( \theta > 1 \). Then, we see from (4.34) and (4.36) that
\[
v_\omega \leq (L_{\omega,+} \theta u_\infty, \theta u_\infty)_{H^{-1}, H^1} < (L_{\omega,+} u_\infty, u_\infty)_{H^{-1}, H^1} \leq v_\omega.
\] (4.43)
This is a contradiction. Thus, we have shown that
\[
((L_{\omega,-})^{-1} u_\infty, u_\infty)_{L^2_{\text{real}}} = 1.
\] (4.44)
Then, the same argument as (4.43) shows that
\[ \nu_\omega = \langle L_\omega, u_\infty, u_\infty \rangle_{H^{-1}, H^1}. \]  
(4.45)

This together with (4.34) and (4.44) shows that \( u_\infty \) is a minimizer for the problem (4.20).

In what follows, we always assume that \( \omega \in (0, \omega_2) \) (\( \omega_2 \) is the frequency given in Proposition 4.1), so that we can take a positive eigen-value \( \mu \) of \(-iL_\omega \) in \( L^2_{\text{real}}(\mathbb{R}^d) \). Let \( U_+ \) be an eigen-function corresponding to \( \mu \), and put
\[ U_- := U_+. \]  
(4.46)

Then, we have
\[ -iL_\omega U_- = iL_\omega U_+ = -\mu U_+ = -\mu U_- . \]  
(4.47)

Hence, \( U_- \) is an eigen-function of \(-iL_\omega \) corresponding to \(-\mu \). We assume that \( U_+ \) and \( U_- \) are normalized in the following sense:
\[ \Omega(U_+, U_-) = 1, \quad \Omega(U_-, U_+) = -1. \]  
(4.48)

It is obvious that
\[ \Omega(U_+, U_+) = \Omega(U_-, U_-) = 0. \]  
(4.49)

Furthermore, it follows from (4.11), (4.12) and \( L_\omega U_+ = \pm i\mu U_+ \) that
\[ \Omega(i\Phi_\omega, U_+) = \Omega(\Phi_\omega', U_-) = 0. \]  
(4.50)

Now, we shall determine the function \( \theta(t) \). We apply the symplectic decomposition corresponding to the discrete modes of \(-iL_\omega \) to the remainder \( \eta(t) \):
\[ \eta(t) = \lambda_+(t)U_+ + \lambda_-(t)U_- + a(t)i\Phi_\omega + b(t)\Phi_\omega' + \gamma(t), \]  
(4.51)

where
\[ \Omega(\gamma(t), U_+) = \Omega(\gamma(t), U_-) = \Omega(\gamma(t), i\Phi_\omega) = \Omega(\gamma(t), \Phi_\omega') = 0. \]  
(4.52)

We see from (4.48), (4.49) and (4.50) that the coefficients are as follows:
\[ \lambda_+(t) = \Omega(\eta(t), U_-), \quad \lambda_-(t) = -\Omega(\eta(t), U_+), \]  
(4.53)

\[ a(t) = \frac{\Omega(\eta(t), \Phi_\omega')}{\langle \Phi_\omega, \Phi_\omega' \rangle_{L^2_{\text{real}}}}, \quad b(t) = -\frac{\Omega(\eta(t), i\Phi_\omega)}{\langle \Phi_\omega, \Phi_\omega' \rangle_{L^2_{\text{real}}}}. \]  
(4.54)

Here, the denominator in (4.54) is non-zero (see (iii) of Proposition 2.4). We require that \( \Phi_\omega \) does not appear in the decomposition (4.51), i.e., \( a(t) \equiv 0 \). To this end, we choose \( \theta(t) \) so that
\[ \Omega(e^{-i\theta(t)}\psi(t), \Phi_\omega') \equiv 0. \]  
(4.55)
Then, it follows from \( \Omega(\Phi_\omega, \Phi'_\omega) = 0 \) that \( \Omega(\eta(t), \Phi'_\omega) \equiv 0 \), and therefore \( a(t) \equiv 0 \). Furthermore, the choice of \( \theta(t) \) has room of an integer multiple of \( \pi \). Hence, in addition to (4.55), we can choose \( \theta(t) \) so that
\[
(e^{-i\theta(t)} \psi(t), \Phi'_\omega)_{L^2_{\text{real}}} < 0. \tag{4.56}
\]
This choice of \( \theta(t) \) plays an important role in the argument below (see (6.10)).

Note that it follows from Hölder’s inequality that
\[
\lambda^2_\pm(t) = \Omega(\eta(t), \mathcal{U}_\pm) \leq \|\mathcal{U}_\pm\|_{L^2}^2 \|\eta(t)\|_{L^2}^2. \tag{4.57}
\]

We shall derive the equation for \( \theta \). We see from (4.55) that
\[
0 = \frac{d}{dt} \Omega(\Phi_\omega + \eta(t), \Phi'_\omega) = (\frac{d\eta}{dt}(t), i\Phi'_\omega)_{L^2_{\text{real}}}. \tag{4.58}
\]
Putting (4.12) and (4.58) together, we obtain
\[
0 = (-L_\omega \eta(t) - \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\Phi_\omega + \eta(t)) + N_\omega(\eta(t)), \Phi'_\omega)_{L^2_{\text{real}}} = \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\Phi_\omega + \eta(t), \Phi'_\omega)_{L^2_{\text{real}}}. \tag{4.59}
\]
Furthermore, using (4.55), we obtain the equation for \( \lambda \):
\[
\left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\Phi_\omega + \eta(t), \Phi'_\omega)_{L^2_{\text{real}}} = -M(\eta(t)) + (N_\omega(\eta(t)), \Phi'_\omega)_{L^2_{\text{real}}}. \tag{4.60}
\]
Next, we shall derive equations for \( \lambda_+ \) and \( \lambda_- \). It follows from (4.55), (4.13), (4.50) and \( L_\omega \mathcal{U}_\pm = \pm i\mu \mathcal{U}_\pm \) that
\[
\frac{d\lambda_\pm}{dt}(t) = \pm \Omega \left( \frac{d\eta}{dt}(t), \mathcal{U}_\pm \right) \nonumber = \pm \Omega (-iL_\omega \eta(t) - i\left\{ \frac{d\theta}{dt}(t) - \omega \right\}\eta(t) + iN_\omega(\eta(t)), \mathcal{U}_\pm) 
\]
\[
= \pm (\eta(t), i\mu \mathcal{U}_\pm)_{L^2_{\text{real}}} + \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\eta(t) - N_\omega(\eta(t)), \mathcal{U}_\pm)_{L^2_{\text{real}}} 
\]
\[
= \pm \mu \lambda_\pm(t) + \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\eta(t) - N_\omega(\eta(t)), \mathcal{U}_\pm)_{L^2_{\text{real}}}. \tag{4.61}
\]
Thus, we have obtained the equation for \( \lambda_\pm \):
\[
\frac{d\lambda_\pm}{dt}(t) = \mu \lambda_\pm(t) - \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\eta(t) - N_\omega(\eta(t)), \mathcal{U}_\pm)_{L^2_{\text{real}}}, \tag{4.62}
\]
\[
\frac{d\lambda_-}{dt}(t) = -\lambda_- + \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\eta(t) - N_\omega(\eta(t)), \mathcal{U}_+)_{L^2_{\text{real}}}. \tag{4.63}
\]
Note here that it follows from \( iL_\omega \mathcal{U}_\pm = \pm \mu \mathcal{U}_\pm \), (4.62), (4.63) and (4.48) that
\[
\langle L_\omega (\lambda_+(t)\mathcal{U}_+ + \lambda_-(t)\mathcal{U}_-), \frac{d\lambda_+}{dt}(t)\mathcal{U}_+ + \frac{d\lambda_-}{dt}(t)\mathcal{U}_- \rangle_{H^{-1}, H^1} 
\]
\[
= -\mu \lambda_-(t) \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\eta(t) - N_\omega(\eta(t)), \mathcal{U}_-)_{L^2_{\text{real}}} 
\]
\[
+ \mu \lambda_+(t) \left\{ \frac{d\theta}{dt}(t) - \omega \right\}(\eta(t) - N_\omega(\eta(t)), \mathcal{U}_+)_{L^2_{\text{real}}}. \tag{4.64}
\]
Now, we put
\[ \Gamma(t) := b(t)\Phi'_\omega + \gamma(t). \]  
(4.65)

Since \( a(t) \equiv 0 \), the decomposition (4.51) is rewritten by
\[ \eta(t) = \lambda_+(t)\mathcal{U}_+ + \lambda_-(t)\mathcal{U}_- + \Gamma(t). \]  
(4.66)

It is easy to check that
\[ \Omega(\Gamma(t), \mathcal{U}_+) = \Omega(\Gamma(t), \mathcal{U}_-) = \Omega(\Gamma(t), \Phi'_\omega) = 0. \]  
(4.67)

We also see from (4.4) and (4.50) that
\[ \langle \Phi_\omega, \Gamma(t) \rangle_{L^2_{\text{real}}} = -M(\eta(t)). \]  
(4.68)

Moreover, it follows from (4.1), Taylor’s expansion around \( \Phi_\omega \), \( S'_\omega(\Phi_\omega) = 0 \), (4.6), \( L_\omega \mathcal{U}_\pm = \pm i\mu \mathcal{U}_\pm \) and (4.48) that
\[ H(\psi) - H(\Phi_\omega) + \mu \cdot |\mathcal{U}_\pm| \sim \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1}, H^1} + O(\|\eta(t)\|_{H^1}^{\min\{3, p+1\}}). \]  
(4.69)

We set
\[ \|\eta(t)\|_E^2 := \frac{\mu}{2}(\lambda^2_\pm(t) + \lambda^2_\mp(t)) + \frac{1}{2} \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1}, H^1}. \]  
(4.70)

Then, we have
\[ H(\psi) - H(\Phi_\omega) + \mu \cdot |\mathcal{U}_\pm| \sim \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1}, H^1} + O(\|\eta(t)\|_{H^1}^{\min\{3, p+1\}}). \]  
(4.71)

**Lemma 4.2.** We have
\[ \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1}, H^1} \sim \|\Gamma(t)\|_{H^1}^2. \]  
(4.72)

**Proof of Lemma 4.2.** Lemma A.5 together with (4.67) gives the desired result. \( \square \)

**Lemma 4.3.**
\[ \|\Gamma(t)\|_{H^1} \lesssim \|\eta(t)\|_{H^1} \sim \|\eta(t)\|_E, \]  
(4.73)

where the implicit constant depends on \( \mu \) and \( \|\mathcal{U}_\pm\|_{H^1} \).  

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Proof of Lemma 4.3. We see from (4.75) that
\[
\|\Gamma(t)\|_{H^1}^2 = \|\eta(t) - \lambda_+(t)U_+ - \lambda_-(t)U_-\|_{H^1}^2,
\]
\[
\lesssim \|\eta(t)\|_{H^1}^2 + \lambda_+^2(t)\|U_+\|_{H^1}^2 + \lambda_-^2(t)\|U_-\|_{H^1}^2
\]
\[
\lesssim (1 + 2\|U_+\|_{H^1}^4)\|\eta(t)\|_{H^1}^2, \tag{4.74}
\]
which proves the inequality in (4.73).

Moreover, it follows from (4.77) that there exists a constant 0 < \(\delta\) such that if \(\|\eta(t)\|_E \leq \delta\), then
\[
\|\eta(t)\|_E^2 \lesssim \mu\|U_+\|_{L^2}^2\|\eta(t)\|_{H^1}^2 + \|\Gamma(t)\|_{H^1}^2.
\]
\[
\lesssim \mu\|U_+\|_{L^2}^2\|\eta(t)\|_{H^1}^2 + (1 + 2\|U_+\|_{H^1}^4)\|\eta(t)\|_{H^1}^2. \tag{4.76}
\]

Putting (4.75) and (4.76) together, we have completed the proof. \(\Box\)

Now, we shall introduce a distance \(d_\omega\) from the ground state \(\Phi_\omega\) (see (21)). We see from (4.71) and Lemma 4.3 that there exists a constant 0 < \(\delta_E(\omega)\) such that if \(\|\eta(t)\|_E \leq 4\delta_E(\omega)\), then
\[
\left| \mathcal{H}(\psi(t)) - \mathcal{H}(\Phi_\omega) + \frac{\mu}{2}(\lambda_+(t) + \lambda_-(t))^2 - \|\eta(t)\|_E^2 \right| \leq \frac{\|\eta(t)\|_E^2}{10}. \tag{4.77}
\]

Fix such a constant \(\delta_E(\omega)\), and fix a smooth cut-off function \(\chi\) on \([0, \infty)\) such that
\[
\chi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases} \tag{4.78}
\]

Then, we define \(d_\omega(\psi(t))\) by
\[
d_\omega(\psi(t))^2 := \|\eta(t)\|_E^2 + \chi \left( \frac{\|\eta(t)\|_E}{2\delta_E(\omega)} \right) C_\omega(\psi(t)), \tag{4.79}
\]
where
\[
C_\omega(\psi(t)) = \mathcal{H}(\psi(t)) - \mathcal{H}(\Phi_\omega) + \frac{\mu}{2}(\lambda_+(t) + \lambda_-(t))^2 - \|\eta(t)\|_E^2. \tag{4.80}
\]

We rephrase (4.77) as follows: if \(\|\eta(t)\|_E \leq 4\delta(\omega)\), then
\[
|C_\omega(\psi(t))| \leq \frac{\|\eta(t)\|_E^2}{10}. \tag{4.81}
\]

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We can define the distance \( d_\omega \) for any function \( u \in H^1(\mathbb{R}^d) \) satisfying \( M(u) = M(\Phi_\omega) \). Note that the distance is continuous.

It is convenient to introduce new parameters \( \lambda_1(t) \) and \( \lambda_2(t) \) defined by

\[
\lambda_1(t) := \frac{\lambda_+(t) + \lambda_-(t)}{2}, \quad \lambda_2(t) := \frac{\lambda_+(t) - \lambda_-(t)}{2}.
\]

We see from (4.62) and (4.63) that

\[
\frac{d\lambda_1}{dt}(t) = \mu \lambda_2(t) + \frac{1}{2} \left\{ \left( \frac{d\theta}{dt}(t) - \omega \right) \eta(t) - N_\omega(\eta(t)), \mathcal{U}_+ - \mathcal{U}_- \right\}_{L_2^{\text{real}}},
\]

which follows from (4.82). We also introduce the real-valued functions \( f_1 \) and \( f_2 \):

\[
f_1 := \frac{\mathcal{U}_+ + \mathcal{U}_-}{2} = \Re[\mathcal{U}_+], \quad f_2 := \frac{\mathcal{U}_+ - \mathcal{U}_-}{2i} = \Im[\mathcal{U}_+].
\]

When we emphasise the dependence on \( \omega \), we use the notation \( \mu_\omega, f_1,\omega \), and \( f_2,\omega \) instead of \( \mu, f_1 \) and \( f_2 \).

Then, we have

\[
\eta(t) = 2\lambda_1(t)f_1 + 2i\lambda_2(t)f_2 + \Gamma(t).
\]

We also see from (4.87) that

\[
\mathcal{L}_\omega \mathcal{U}_+ = i\mu \mathcal{U}_+.
\]

Furthermore, it follows from (4.87) that

\[
\|f_1\|_{L_\infty} + \|f_2\|_{L_\infty} < \infty.
\]

**Lemma 4.4.** We have that \( f_2 \not\in \text{span}\{\Phi_\omega\} = \text{Ker} \mathcal{L}_{\omega,-} \).

**Proof of Lemma 4.4.** Suppose for a contradiction that \( f_2 = k\Phi_\omega \) for some \( k \neq 0 \). Then, it follows from (4.11) that

\[
0 = k\mathcal{L}_\omega(i\Phi_\omega) = \mathcal{L}_\omega(if_2) = \frac{1}{2} \mathcal{L}_\omega(\mathcal{U}_+ - \mathcal{U}_-) = \frac{1}{2} i\mu \mathcal{U}_+ + \frac{1}{2} i\mu \mathcal{U}_- = i\mu f_1.
\]

Furthermore, we see from (4.87) and (4.89) that

\[
-\mu f_2 = \mathcal{L}_{\omega,+}f_1 = 0.
\]

Thus, \( f_1 = f_2 \equiv 0 \). This is a contradiction. \( \square \)

**Lemma 4.5.** We have the following orthogonality:

\[
(\Phi_\omega, f_1)_{L_2} = (\Phi'_\omega, f_2)_{L_2} = 0.
\]

Furthermore, we have

\[
(f_1, f_2)_{L_2} > 0
\]

and

\[
(\Phi_\omega, f_2)_{L_2} \neq 0.
\]
Proof of Lemma 4.4. Since $L_{\omega,-}\Phi_\omega = 0$ and $L_{\omega,-}$ is self-adjoint in $L^2(\mathbb{R}^d)$, we find from (4.87) that
\[
(\Phi_\omega, f_1)_{L^2} = \mu^{-1} \langle L_{\omega,-} f_2, \Phi_\omega \rangle_{H^{-1},H^1} = \mu^{-1} \langle L_{\omega,-} \Phi_\omega, f_2 \rangle_{L^2} = 0. \tag{4.94}
\]
On the other hand, it follows from (4.87), (4.12) and the self-adjointness of $L_{\omega,+}$ in $L^2(\mathbb{R}^d)$ that
\[
(\Phi'_\omega, f_2)_{L^2} = -\mu^{-1} \langle L_{\omega,+} f_1, \Phi'_\omega \rangle_{H^{-1},H^1} = -\mu^{-1} \langle L_{\omega,+} \Phi'_\omega, f_1 \rangle_{L^2} = \mu^{-1} \langle \Phi_\omega, f_1 \rangle_{L^2}, \tag{4.95}
\]
which together with (4.94) proves (4.91).

Next, we shall prove (4.92). It follows from (4.87) that
\[
(f_1, f_2)_{L^2} = \mu^{-1} \langle L_{\omega,-} f_2, f_2 \rangle_{H^{-1},H^1}. \tag{4.96}
\]
Since $f_2 \notin \text{Ker } L_{\omega,-}$ (see Lemma 4.4), Lemma A.4 together with (4.96) shows the desired result.

Finally, we prove (4.93). Suppose for contradiction that $(\Phi_\omega, f_2)_{L^2} = 0$. Then, we see from (4.10), the self-adjointness of $L_{\omega,+}$ and (4.87) that
\[
((p-1)\Phi'_\omega + (2^*-2)\Phi^{2^*-1}_\omega, f_1)_{L^2} = -\langle L_{\omega,+} \Phi_\omega, f_1 \rangle_{L^2} = \mu (\Phi_\omega, f_2)_{L^2} = 0. \tag{4.97}
\]
Hence, it follows from Lemma A.3 that $\langle L_{\omega,+} f_1, f_1 \rangle_{H^{-1},H^1} \geq 0$. However, it must follow from (4.87) and (4.92) that
\[
\langle L_{\omega,+} f_1, f_1 \rangle_{H^{-1},H^1} = -\mu \langle f_2, f_1 \rangle_{L^2} < 0. \tag{4.98}
\]
This is a contradiction. Thus, we find that (4.93) holds.

The relation (4.93) in Lemma 4.5 allows us to choose $f_2$ so that
\[
(\Phi_\omega, f_2)_{L^2} < 0. \tag{4.99}
\]

Lemma 4.6. For any constant $C > 0$, there exists $\omega(C) > 0$ such that for any $\omega \in (0,\omega(C))$,
\[
\mu |(\Phi_\omega, f_2)_{L^2}| \geq C |(\Phi^{2^*-1}_\omega, f_1)_{L^2}|. \tag{4.100}
\]

Proof of Lemma 4.6. Suppose for contradiction that there exists a constant $C_0 > 0$ with the following property: for any number $n$, there exists $\omega_n \in (0,\frac{1}{n})$ such that
\[
\mu_{\omega_n} |(\Phi_{\omega_n}, f_{2,\omega_n})_{L^2}| < C_0 |(\Phi^{2^*-1}_{\omega_n}, f_{1,\omega_n})_{L^2}|, \tag{4.101}
\]
where $f_{1,\omega_n}$ and $f_{2,\omega_n}$ are functions given by (4.85) for $\omega_n$. We consider the functions $f_{1,n}$ and $f_{2,n}$ defined by
\[
f_{1,n} := \omega_n^{-s_p} T_{\omega_n} f_{1,\omega_n}, \quad f_{2,n} := \omega_n^{-s_p} T_{\omega_n} f_{2,\omega_n}. \tag{4.102}
\]
It follows from (4.104) that
\[ \bar{L}_{\omega_n,+} f_{1,n} = -\mu_{\omega_n} \omega_n^{-1} f_{2,n}, \quad \bar{L}_{\omega_n,-} f_{2,n} = \mu_{\omega_n} \omega_n^{-1} f_{1,n}, \] (4.103)
where \( \bar{L}_{\omega,+} \) and \( \bar{L}_{\omega,-} \) are the operators defined by (1.38) and (1.49), respectively. We put
\[ g_n := \frac{f_{1,n}}{\|f_{1,n}\|_{H^1}}, \quad h_n := \frac{f_{2,n}}{\|f_{1,n}\|_{H^1}}. \] (4.104)
Then, we can take a real-valued function \( g \) in \( H^1(\mathbb{R}^d) \) such that
\[ \lim_{n \to \infty} g_n = g \quad \text{weakly in} \quad H^1(\mathbb{R}^d). \] (4.105)
We shall show that \( g \) is non-trivial. Suppose for contradiction that \( g \) was trivial. Then, it follows from Lemma 1.35 (4.103), Proposition 2.3 and (4.105) that
\[ 0 \geq \lim_{n \to \infty} \frac{-\omega_n^{-p} \mu_{\omega_n}}{\|f_{1,n}\|_{H^1}^2} (f_{1,\omega_n}, f_{2,\omega_n})_{L^2} = \lim_{n \to \infty} \frac{-\omega_n^{-p} \mu_{\omega_n}}{\|f_{1,n}\|_{H^1}^2} (f_{1,n}, f_{2,n})_{L^2} \]
\[ = \lim_{n \to \infty} \frac{1}{\|f_{1,n}\|_{H^1}^2} \langle \bar{L}_{\omega_n,+} f_{1,n}, f_{1,n} \rangle_{H^{-1},H^1} \]
\[ = 1 - \lim_{n \to \infty} \int_{\mathbb{R}^d} \left\{ p(T_{\omega_n} \Phi_{\omega_n})^{p-1} |g_n|^2 + \omega_n \frac{2^* - (p+1)}{p-1} (2^* - 1)(T_{\omega_n} \Phi_{\omega_n}) \frac{2}{p-1} |g_n|^2 \right\} dx \]
\[ = 1. \]
This is a contradiction, and therefore \( g \) is non-trivial.

Next, we shall show that there exists a constant \( C_0 > 0 \) such that for any number \( n \) and any real-valued function \( f \in H^1(\mathbb{R}^d) \) with \( (f, T_{\omega_n} \Phi_{\omega_n})_{L^2} = 0 \),
\[ \langle \bar{L}_{\omega_n,-} f, f \rangle_{H^{-1},H^1} \geq C_0 \|f\|_{H^1}^2. \] (4.107)
Suppose for contradiction that we could take a subsequence of \( \{\omega_n\} \) (still denoted by the same symbol) and a sequence \( \{f_n\} \) in \( H^1(\mathbb{R}^d) \) such that
\[ \lim_{n \to \infty} \langle \bar{L}_{\omega_n,-} f_n, f_n \rangle_{H^{-1},H^1} = 0, \quad \|f_n\|_{H^1} = 1, \quad (f_n, T_{\omega_n} \Phi_{\omega_n})_{L^2} = 0. \] (4.108)
Furthermore, we can take \( f_0 \in H^1(\mathbb{R}^d) \) such that \( \lim_{n \to \infty} f_n = f_0 \) weakly in \( H^1(\mathbb{R}^d) \). Then, we see from the weak lower semicontinuity and Proposition 2.3 that
\[ 0 = \lim_{n \to \infty} \langle \bar{L}_{\omega_n,-} f_n, f_n \rangle_{H^{-1},H^1} \]
\[ = \lim_{n \to \infty} \left\{ \|f_n\|_{H^1}^2 - \int (T_{\omega_n} \Phi_{\omega_n})^{p-1} |f_n|^2 dx - \omega_n \frac{2^* - (p+1)}{p-1} \int (T_{\omega_n} \Phi_{\omega_n}) \frac{2}{p-1} |f_n|^2 dx \right\} \]
\[ \geq \langle L_{f_0}^+, f_0 \rangle_{H^{-1},H^1}. \] (4.109)
Moreover, it follows from Proposition 4.3 and the hypothesis (4.108) that

\[
(f_0, U)_{L^2} = \lim_{n \to \infty} (f_n, (T_{\omega_n} \Phi_{\omega_n}))_{L^2} = 0. \tag{4.110}
\]

Hence, we conclude from (4.109) and the positivity of \( L \) so that

\[
0 \geq \langle L_{\omega_n}^+ f_0, f_0 \rangle \gtrsim \|f_0\|_{H^1}^2, \tag{4.111}
\]

so that \( f_0 \) is trivial. However, the same argument as (4.106) yields that \( f_0 \) is non-trivial. Thus, we arrive at a contradiction and therefore (4.107) holds.

We shall show that \( \{h_n\} \) is bounded in \( H^1(\mathbb{R}^d) \). We see from (4.107) and (4.103) that

\[
C_0 \|f_{2,n}\|_{H^1}^2 \leq \langle \tilde{L}_{\omega_n} - f_{2,n}, f_{2,n} \rangle_{H^{-1},H^1} = \mu_{\omega_n} \omega_n^{-1} (f_{1,n}, f_{2,n})_{L^2} \leq C \|f_{1,n}\|_{H^1}^2. \tag{4.112}
\]

Dividing the both sides above by \( \|f_{1,n}\|_{H^1} \), we find that \( \{h_n\} \) is bounded in \( H^1(\mathbb{R}^d) \).

We shall show that the sequence \( \{\mu_{\omega_n} \omega_n^{-1}\} \) is bounded. It follows from (4.108) that

\[
\left| \mu_{\omega_n} \omega_n^{-1} \right| = \frac{|\langle \tilde{L}_{\omega_n} - f_{2,n}, f_{1,n} \rangle_{H^{-1},H^1}|}{\|f_{1,n}\|_{L^2}^2}. \tag{4.113}
\]

Furthermore, it follows from (4.105) and the boundedness of \( \{g_n\} \) and \( \{h_n\} \) in \( H^1(\mathbb{R}^d) \) that

\[
\frac{\|g_n\|_{H^1}^2}{\sup_{n \in \mathbb{N}} \|g_n\|_{L^2}^2} \leq \frac{\sup_{n \in \mathbb{N}} \|h_n\|_{H^1}^2}{\|f_{1,n}\|_{L^2}^2} \lesssim 1. \tag{4.114}
\]

Thus, we find that \( \{\mu_{\omega_n} \omega_n^{-1}\} \) is bounded.

Since \( \{h_n\} \) is bounded in \( H^1(\mathbb{R}^d) \), there exists \( \tilde{h} \in H^1(\mathbb{R}^d) \) such that

\[
\lim_{n \to \infty} h_n = \tilde{h} \quad \text{weakly in } H^1(\mathbb{R}^d). \tag{4.115}
\]

Moreover, we can take a subsequence of \( \{\mu_{\omega_n} \omega_n^{-1}\} \) (still denoted by the same symbol) and \( \nu_s \in [0, \infty) \) such that \( \lim_{n \to \infty} \mu_{\omega_n} \omega_n^{-1} = \nu_s \). We shall show that \( \nu_s \neq 0 \) and \( \tilde{h} \) is non-trivial. Recall here that \( \mu_{\omega_n} \) is a positive eigen-value of \( -iL_{\omega_n} \). We find from the proof of Proposition 4.1 (see (4.18) and (4.20)) that

\[
- \mu_{\omega_n}^2 = \inf \left\{ \frac{\langle L_{\omega_n}^+, u, u \rangle_{H^{-1},H^1}}{\|(L_{\omega_n}^-)^{-1} u, u\|_{L^2_{\text{real}}}} : u \in H^1(\mathbb{R}^d), (u, \Phi_{\omega_n})_{L^2_{\text{real}}} = 0 \right\}. \tag{4.116}
\]

Furthermore, it follows from the estimates (4.24) and (4.25) in the proof of Proposition 4.1 shows that

\[
- \nu_s^2 = \lim_{n \to \infty} \omega_n^{-2} (\mu_{\omega_n}^2) = \frac{\langle L_{\omega_n}^+, \Pi U', \Pi U' \rangle_{H^{-1},H^1}}{\|(L_{\omega_n}^-)^{-1} \Pi U', \Pi U'\|_{L^2_{\text{real}}}} < 0, \tag{4.117}
\]

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where II is the projection given by (4.24). Thus, we find that \( \nu_s \neq 0 \). Next, suppose for contradiction that \( h \) was trivial. Then, we see from (4.103) and (4.105) that for any \( \varphi \in C_{0}^{\infty}(\mathbb{R}^d) \),

\[
0 = \lim_{n \to \infty} \langle \tilde{L}_{\omega_n, -h_n}, \varphi \rangle_{H^{-1}, H^1} = \lim_{n \to \infty} \langle \mu_{\omega_n} g_n, \varphi \rangle_{L^2_{\text{real}}} = \langle \nu_s g, \varphi \rangle_{L^2_{\text{real}}}. \tag{4.118}
\]

However, this contradicts that \( \nu_s \neq 0 \) and \( g \) is non-trivial. Thus, we find that \( h \) is non-trivial.

Note that \( g \) and \( h \) satisfies that

\[
L^\dagger_{\nu} g = -\nu_s h, \quad L^\dagger_{\nu} h = \nu_s g. \tag{4.119}
\]

Then, applying the same proof of (4.93) (use Lemma 2.2 in [8] instead of Lemma A.3), we find that

\[
\langle L^1_{\nu} U, h \rangle_{H^{-1}, H^1} = (p - 1)(U^p, h)_{L^2} \neq 0. \tag{4.120}
\]

We see from (4.87) and the scaling that

\[
- \omega_n^{-1} \mu_{\omega_n} (\Phi_{\omega_n}, f_{2, \omega_n})_{L^2} = \langle \tilde{L}_{\omega_n, + T_{\omega_n} \Phi_{\omega_n}, f_{1, n}} \rangle_{H^{-1}, H^1}. \tag{4.121}
\]

Moreover, it follows from Proposition 2.3 that

\[
\lim_{n \to \infty} \tilde{L}_{\omega_n, + T_{\omega_n} \Phi_{\omega_n}} = L^1_{\nu} U \quad \text{strongly in } H^{-1}(\mathbb{R}^d). \tag{4.122}
\]

This together with (4.115) yields

\[
\lim_{n \to \infty} \langle \tilde{L}_{\omega_n, + T_{\omega_n} \Phi_{\omega_n}, h_n} \rangle_{H^{-1}, H^1} = \langle L^1_{\nu} U, h \rangle_{H^{-1}, H^1}. \tag{4.123}
\]

We find from (4.121), (4.123) and (4.120) that

\[
\lim_{n \to \infty} \|f_{1, n}\|_{H^{-1}_{\omega_n} \omega_n^{-1} \mu_{\omega_n}}(\Phi_{\omega_n}, f_{2, \omega_n})_{L^2} = \lim_{n \to \infty} \|\langle \tilde{L}_{\omega_n, + T_{\omega_n} \Phi_{\omega_n}, h_n} \rangle_{H^{-1}, H^1} - \langle L^1_{\nu} U, h \rangle_{H^{-1}, H^1} \|
\geq 1,
\tag{4.124}
\]

On the other hand, it follows from Proposition 2.3 and (4.105) that

\[
\lim_{n \to \infty} \|f_{1, n}\|_{H^{-1}_{\omega_n} \omega_n^{-1} \mu_{\omega_n}}(\Phi_{\omega_n}^{2^* - 1}, f_{1, \omega_n})_{L^2} = \lim_{n \to \infty} \|((T_{\omega_n} \Phi_{\omega_n})^{2^* - 1}, g_n)_{L^2} \|
\leq 1. \tag{4.125}
\]

Since \( \frac{2^* - 2}{p - 1} > 1 \), we conclude from (4.124) and (4.125) that for any sufficiently large number \( n \),

\[
\mu_{\omega_n} (\Phi_{\omega_n}, f_{2, \omega_n})_{L^2} \geq \omega_n^{-\frac{2^* - 2}{p - 1} + 1} \|((T_{\omega_n} \Phi_{\omega_n})^{2^* - 1}, g_n)_{L^2} \| \geq C_0 (\Phi_{\omega_n}^{2^* - 1}, f_{1, \omega_n})_{L^2}, \tag{4.126}
\]

where \( C_0 \) is the constant given in the hypothesis (4.101). However, this contradicts (4.101). Thus, the desired result (4.101) holds. \( \Box \)
**Lemma 4.7.** Assume \( d \geq 4 \) and \((1.1)\). Let \( \omega \in (0, \omega_2) \) (\( \omega_2 \) is the frequency given by Proposition \( \text{Proposition 4.1} \)), and let \( \psi \) be a solution to \((\text{NLS})\) with the maximal existence-interval \( I_{\max} \). Assume that

\[
\mathcal{M}(\psi) = \mathcal{M}(\Phi_\omega),
\]

and for any \( t \in I_{\max} \),

\[
d_\omega(\psi(t)) \leq \delta E(\omega).
\]

Then, the followings hold for any \( t \in I_{\max} \):

\[
\frac{1}{2} \| \eta(t) \|_E^2 \leq d_\omega(\psi(t))^2 \leq \frac{3}{2} \| \eta(t) \|_E^2;
\]

\[
d_\omega(\psi(t))^2 = \mathcal{H}(\psi) - \mathcal{H}(\Phi_\omega) + 2\mu \lambda_1^2(t),
\]

\[
\frac{d}{dt} d_\omega(\psi(t))^2 = 4\mu^2 \lambda_1(t) \lambda_2(t) + 4\mu \lambda_1(t) \Omega \left\{ \left\{ \frac{d\theta}{dt} - \omega \right\} \eta(t) - N_\omega(\eta(t)), f_2 \right\}.
\]

Furthermore, if we have that for any \( t \in I_{\max} \),

\[
\mathcal{S}_\omega(\psi) < m_\omega + \frac{1}{2} d_\omega(\psi(t))^2
\]

then for any \( t \in I_{\max} \),

\[
d_\omega(\psi(t)) \sim |\lambda_1(t)|.
\]

**Proof of Lemma 4.7.** First, we shall show that for any \( t \in I_{\max} \),

\[
\| \eta(t) \|_E \leq 4\delta E(\omega).
\]

Suppose for contradiction that \( \| \eta(t_0) \|_E > 4\delta E(\omega) \) for some \( t_0 \in I_{\max} \). Then, it follows from the definition \( (4.79) \) that \( d_\omega(\psi(t_0))^2 = \| \eta(t_0) \|_E^2 \geq 4\delta E(\omega) \). However, this contradicts \( (4.128) \). Thus, we have proved \( (4.134) \). Then, \( (4.129) \) follows from \( (4.81) \). Indeed,

\[
\frac{1}{2} \| \eta(t) \|_E^2 \leq \| \eta(t) \|_E^2 - |C_\omega(\psi(t))| \\
\leq d_\omega(\psi(t))^2 \leq \| \eta(t) \|_E^2 + |C_\omega(\psi(t))| \leq \frac{3}{2} \| \eta(t) \|_E^2.
\]

Next, we shall prove \( (4.130) \). We see from \( (4.128) \) and \( (4.129) \) that

\[
\| \eta(t) \|_E^2 \leq 2d_\omega(\psi(t)) \leq 2\delta E(\omega).
\]

Hence, we have

\[
d_\omega(\psi(t))^2 = \| \eta(t) \|_E^2 + C_\omega(\psi(t)) = \mathcal{H}(\psi) - \mathcal{H}(\Phi_\omega) + \frac{\mu}{2} (\lambda_+^2(t) + \lambda_-^2(t)),
\]

so that \( (4.130) \) holds. Furthermore, it follows from \( (1.30) \) and \( (4.83) \) that the equality \( (4.131) \) holds.
Finally, we shall prove (4.133). We see from (4.134) and (4.131) that
\[
\|\eta(t)\|_E^2 = \mathcal{H}(\psi) - \mathcal{H}(\Phi_\omega) + \frac{\mu}{2}(\lambda_+(t) + \lambda_-(t))^2 - C_\omega(\psi(t))
\]
\[
\leq \mathcal{H}(\psi) - \mathcal{H}(\Phi_\omega) + \frac{\mu}{2}(\lambda_+(t) + \lambda_-(t))^2 + \frac{\|\eta(t)\|_E^2}{10},
\]
so that
\[
\frac{9}{10}\|\eta(t)\|_E^2 \leq \mathcal{H}(\psi) - \mathcal{H}(\Phi_\omega) + 2\mu\lambda_1(t)^2.
\] (4.139)
Moreover, it follows from (4.127) and (4.132) that
\[
\mathcal{H}(\psi) - \mathcal{H}(\Phi_\omega) < \frac{1}{2}d_\omega(\psi(t))^2.
\] (4.140)
Putting (4.129), (4.139) and (4.140) together, we obtain that
\[
d_\omega(\psi(t))^2 \leq \frac{3}{2}\|\eta(t)\|_E^2 < \frac{5}{6}d_\omega(\psi(t))^2 + \frac{10}{3}\mu\lambda_1(t)^2.
\] (4.141)
Hence, we have
\[
d_\omega(\psi(t))^2 < 200\mu\lambda_1(t)^2.
\] (4.142)
On the other hand, we see from (4.57), Lemma 4.3 and (4.129) that
\[
\mu\lambda_1(t)^2 < 2\mu(\lambda_+^2(t) + \lambda_-^2(t)) \leq 4\mu\|\mathcal{U}_+\|^2_2\|\eta(t)\|_E^2 \leq d_\omega(\psi(t))^2;
\] (4.143)
where the implicit constant depends on \(\mu\) and \(\|\mathcal{U}_+\|^2_2\). Combining (4.142) and (4.143), we obtain (4.133).

We see from (4.11) and (4.10) that
\[
2\omega\Phi_\omega - 2\Delta\Phi_\omega - \frac{d(p - 1)}{2}\Phi_\omega^p - 2^*\Phi_\omega^{2^*-1}
\]
\[
= (2 - s_p)L_{\omega,-}\Phi_\omega + s_pL_{\omega,+}\Phi_\omega - (1 - s_p)(2^* - 2)\Phi_\omega^{2^*-1}
\]
\[= s_pL_{\omega,+}\Phi_\omega - (1 - s_p)(2^* - 2)\Phi_\omega^{2^*-1} = -s_p(p - 1)\Phi_\omega^p - (2^* - 2)\Phi_\omega^{2^*-1}.
\] (4.144)
Furthermore, this together with (4.86), (4.87) and (4.68) shows that
\[
\mathcal{K}'(\Phi_\omega)\eta(t)
\]
\[
= 2(\omega\Phi_\omega - \Delta\Phi_\omega - \frac{d(p - 1)}{4}\Phi_\omega^p - \frac{2^*}{2}\Phi_\omega^{2^*-1}, \eta(t))_{H^{-1},H^1} - 2\omega(\Phi_\omega, \eta(t))_{L^2_{\text{real}}}
\]
\[
= 2s_p\lambda_1(t)(L_{\omega,+}\Phi_\omega, f_1)_{L^2} - 2(1 - s_p)(2^* - 2)\lambda_1(t)(\Phi_\omega^{2^*-1}, f_1)_{L^2}
\]
\[
- s_p((p - 1)\Phi_\omega^p + (2^* - 2)\Phi_\omega^{2^*-1}, \Gamma(t))_{L^2_{\text{real}}}
\]
\[
- (1 - s_p)(2^* - 2)(\Phi_\omega^{2^*-1}, \Gamma(t))_{L^2_{\text{real}}} + \omega M(\eta(t))
\] (4.145)
\[
= -2\mu s_p\lambda_1(t)(\Phi_\omega, f_2)_{L^2} - 2(1 - s_p)(2^* - 2)\lambda_1(t)(\Phi_\omega^{2^*-1}, f_1)_{L^2}
\]
\[
- s_p(p - 1)(\Phi_\omega^p, \Gamma(t))_{L^2_{\text{real}}} - (2^* - 2)(\Phi_\omega^{2^*-1}, \Gamma(t))_{L^2_{\text{real}}} + \omega M(\eta(t)).
\]

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5 Ejection lemma

Let $\omega_2$ be the frequency given by Proposition 4.1. It follows from the argument in the previous section that for any $\omega \in (0, \omega_2)$ and any solution $\psi$ satisfying $M(\psi) = M(\Phi_\omega)$, we can decompose $\psi$ into the form (4.2) with (4.51), (4.55) and (4.56), and define the distance function $d_\omega(\psi(t))$ by (4.79).

Lemma 5.1 (Ejection lemma). Assume $d \geq 4$ and $I_\omega$. Let $\omega \in (0, \omega_2)$, and let $\delta_E(\omega) \ll 1$ be a positive constant satisfying (4.77). Then, there exist constants $\delta_X \in (0, \delta_E(\omega))$, $1 \leq A_s \leq 1 \leq B_s \leq 1$, $1 \leq C_s \sim 1$ and $T_s \sim 1$ with the following properties: for any $t_0 \in \mathbb{R}$ and any solution $\psi$ of (NLS) defined around $t_0$ satisfying

\begin{align*}
M(\psi) &= M(\Phi_\omega), \\
0 < R_0 := d_\omega(\psi(t_0)) < \delta_X, \\
S_\omega(\psi) &< m_\omega + \frac{R_0^2}{2},
\end{align*}

we can extend $\psi$ as long as $d_\omega(\psi(t)) \leq \delta_X$. Furthermore, assume that there exists $T > t_0$ such that

\begin{equation}
R_0 \leq \min_{t \in [t_0, T]} d_\omega(\psi(t)),
\end{equation}

and define

\begin{equation}
T_X := \inf \{ t \in [t_0, T] : d_\omega(\psi(t)) = \delta_X \},
\end{equation}

where we interpret $T_X = T$ if $d_\omega(\psi) < \delta_X$ on $[t_0, T]$. Then, for any $t \in [t_0, T_X]$,\n
\begin{align*}
A_s e^{\mu(t-t_0)} R_0 &\leq d_\omega(\psi(t)) \leq B_s e^{\mu(t-t_0)} R_0, \\
\|\eta(t)\|_{H^1} &\sim s \lambda_1(t) \sim s \lambda_2(t) \sim e^{\mu(t-t_0)} R_0, \\
|\lambda_-(t)| + \|\Gamma(t)\|_{H^1} &\lesssim R_0 + \left( e^{\mu(t-t_0)} R_0 \right)^{\min(2,p+1)} \frac{1}{2}, \\
s K(\psi(t)) &\gtrsim \left( e^{\mu(t-t_0)} - C_s \right) R_0,
\end{align*}

where $s$ is some constant with $s = 1$ or $s = -1$. Moreover, $d_\omega(\psi(t))$ is increasing on the region $\{ t \in [t_0, T_X] : t_0 + T_s R_0^{\min(1,p-1)} \leq t \}$; and

\begin{equation}
|d_\omega(\psi(t)) - R_0| \lesssim R_0^{\min(2,p)}
\end{equation}

on the region $\{ t \in [t_0, T_X] : t_0 \leq t \leq t_0 + T_s R_0^{\min(1,p-2)} \}$.

Proof of Lemma 5.1. Let $\delta_X < \delta_E(\omega)$ be a small constant to be chosen later, and let $\psi$ be a solution satisfying (5.1), (5.2) and (5.3). Since the equation (NLS) is invariant under the time translation, it suffices to consider the case where $t_0 = 0$. 

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Suppose that $d_\omega(\psi(t_1)) < \delta_X$ for some $t_1 > 0$. Then, we see from Lemma \ref{lem:4.3} and Lemma \ref{lem:4.7} that
\[
\|\eta(t_1)\|_{H^1} \lesssim d_\omega(\psi(t_1)) < \delta_X. \tag{5.11}
\]
Hence, we have
\[
\|\psi(t_1)\|_{H^1} \lesssim \|\Phi_\omega\|_{H^1} + 1. \tag{5.12}
\]
Moreover, it follows from Strichartz’ estimate and \ref{eq:5.11} that for any $t_2 > t_1$,
\[
\|\langle \nabla \rangle e^{i(t-t_1)\Delta} \psi(t_1)\|_{\dot{H}^s(t_1,t_2)} \lesssim \|\langle \nabla \rangle e^{i(t-t_1)\Delta} \Phi_\omega\|_{\dot{H}^s(t_1,t_2)} + \|\eta(t_1)\|_{H^1}
\lesssim \|\langle \nabla \rangle e^{i(t-t_1)\Delta} \Phi_\omega\|_{\dot{H}^s(t_1,t_2)} + \delta_X. \tag{5.13}
\]
Here, for the quantity $\|\Phi_\omega\|_{H^1} + 1$, the well-posedness theory determines a size $\delta(\omega) > 0$ with the following property (see, e.g., \cite{[3]}): for any $t_2 > t_1$ with
\[
\|\langle \nabla \rangle e^{i(t-t_1)\Delta} \psi(t_1)\|_{\dot{H}^s(t_1,t_2)} \leq \delta(\omega),\tag{5.14}
\]
the solution $\psi$ exists on $[t_1, t_2]$. We choose $\delta_X \ll \delta(\omega)$. Then, \ref{eq:5.13} together with
\[
\lim_{\tau \downarrow t_1} \|\langle \nabla \rangle e^{i(t-t_1)\Delta} \Phi_\omega\|_{\dot{H}^s(t_1,\tau)} = 0 \tag{5.15}
\]
shows that $\psi$ extends beyond $t_1$. Thus, we find that $\psi$ extends as long as $d_\omega(\psi(t)) < \delta_X$.

Now, in addition to \ref{eq:5.11}, \ref{eq:5.12} and \ref{eq:5.13}, we assume \ref{eq:5.4}. Then, it follows from the definition of $T_X$ that for any $t \in [0, T_X]$,
\[
d_\omega(\psi(t)) \leq \delta_X < \delta_E(\omega). \tag{5.16}
\]
Hence, we see from Lemma \ref{lem:4.3} and Lemma \ref{lem:4.7} that for any $t \in [0, T_X]$,
\[
\|\Gamma(t)\|_{H^1} \lesssim \|\eta(t)\|_{H^1} \sim \|\eta(t)\|_E \sim d_\omega(\psi(t)) \sim |\lambda_1(t)|, \tag{5.17}
\]
\[
\frac{d}{dt}d_\omega(\psi(t))^2 = 4\mu^2 \lambda_1(t) \lambda_2(t) + 4\mu \lambda_1(t) \Omega \left( \left\{ \frac{d}{dt}(t - \omega) \right\} \eta(t) - N_\omega(\eta(t), f_2) \right). \tag{5.18}
\]
In particular, we deduce from \ref{eq:4.37}, \ref{eq:5.16} and \ref{eq:5.17} that for any $t \in [0, T_X]$,
\[
|\lambda_-(t)| + \|\Gamma(t)\|_{H^1} \lesssim \|\eta(t)\|_{H^1} \sim |\lambda_1(t)| \lesssim \delta_X \ll 1. \tag{5.19}
\]
Furthermore, we find from \ref{eq:5.4} and \ref{eq:5.17} that for any $t \in [0, T_X]$,
\[
0 < R_0 \lesssim |\lambda_1(t)|, \tag{5.20}
\]
which together with the continuity of $\lambda_1(t)$ shows that for any $t \in [0, T_X]$,
\[
s := \text{sgn}[\lambda_1(t)] \equiv \text{sgn}[\lambda_1(0)]. \tag{5.21}
\]
We further choose $\delta_X$ so small that
\[
\delta_X \ll \frac{|(\Phi_\omega, \Phi'_\omega)_{L^2}|}{2 \|\Phi_\omega\|_{L^2}}. \tag{5.22}
\]
Then, we find from (5.11) that for any $t \in [0, T_X]$, 
\[
\|\eta(t)\|_{L^2} \leq \|\eta(t)\|_{H^1} \leq \min \left\{ 1, \frac{\left\| (\Phi_\omega, \Phi'_\omega) \right\|_{L^2}}{2 \|\Phi'_\omega\|_{L^2}} \right\},
\]
(5.23)
Furthermore, (5.23) implies that for any $t \in [0, T_X]$, 
\[
\left| (\Phi_\omega + \eta(t), \Phi'_\omega) \right| \geq \left| (\Phi_\omega, \Phi'_\omega) \right| - \|\eta(t)\|_{L^2} \|\Phi'_\omega\|_{L^2} \geq \frac{1}{2} \left| (\Phi_\omega, \Phi'_\omega) \right|.
\]
(5.24)
It follows from $N_\omega(\eta(t)) = O(\|\eta(t)\|^{\min\{2,p\}})$ (see (1.14)) that for any function $u \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, 
\[
\left| (N_\omega(\eta(t)), u) \right|_{L^2_{real}} \leq \|N_\omega(\eta(t))\|_{L^{\min\{2,p\}} \cap L^{\infty}} \|u\|_{L^{\max\{1,p-1\}}} \lesssim \|\eta(t)\|_{L^{\min\{2,p\}}} \|u\|_{L^2 \cap L^\infty}.
\]
(5.25)
We deduce from the equation (4.60) for $\theta(t)$, (5.24), (5.21) and (5.23) that for any $t \in [0, T_X]$, 
\[
\left| \frac{d\theta}{dt}(t) - \omega \right| \lesssim \frac{\|\eta(t)\|_{L^2}^2 + \|\eta(t)\|_{L^{\min\{2,p\}} \cap L^\infty} \|\Phi'_\omega\|_{L^2 \cap L^\infty}}{\left| (\Phi_\omega, \Phi'_\omega) \right|} \lesssim \|\eta(t)\|_{H^1}^{\min\{2,p\}},
\]
(5.26)
where the implicit constant depends on $\omega$. Furthermore, it follows from (5.26), (5.25) and (5.23) that for any $t \in [0, T_X]$, 
\[
\left| \Omega \left( \begin{array}{c} \frac{d\theta}{dt}(t) - \omega \\ \eta(t) - N_\omega(\eta(t)) \end{array} \right) \right| \lesssim \|\eta(t)\|_{H^1}^{\min\{2,p\}+1} \|\eta(t)\|_{H^1}^{\min\{2,p\}} \lesssim \|\eta(t)\|_{H^1}^{\min\{2,p\}},
\]
(5.27)
where the implicit constant depends on $f_2$ and $\omega$. Hence, we find form (5.27) and (5.17) that for any $t \in [0, T_X]$, 
\[
\left| \Omega \left( \begin{array}{c} \frac{d\theta}{dt}(t) - \omega \\ \eta(t) - N_\omega(\eta(t)) \end{array} \right) \right| \lesssim \lambda_1(t)^{\min\{2,p\}},
\]
(5.28)
where the implicit constant depends on $f_2$ and $\omega$. Similarly, we can verify that for any $t \in [0, T_X]$, 
\[
\left| \Omega \left( \begin{array}{c} \frac{d\theta}{dt}(t) - \omega \\ \eta(t) - N_\omega(\eta(t)), \mathcal{U}_\pm \end{array} \right) \right| \lesssim \lambda_1(t)^{\min\{2,p\}},
\]
(5.29)
where the implicit constant depends on $\mathcal{U}_\pm$ and $\omega$.

We shall show that 
\[
\lambda_1(0) \sim \lambda_\pm(0) \sim s R_0,
\]
(5.30)
where the implicit constants depend on $\omega$. We see from (5.2) and (5.4) that 
\[
\frac{d}{dt} d_\omega(\psi(t))^2 \bigg|_{t=t_0=0} = 2R_0 \frac{d}{dt} d_\omega(\psi(t)) \bigg|_{t=0} \geq 0,
\]
(5.31)
which together with (5.18) yields

$$0 \leq \mu^2 \lambda_1(0) \lambda_2(0) + \mu \lambda_1(0) \left| \Omega \left( \left\{ \frac{d\theta}{dt}(0) - \omega \right\} \eta(0) - N_\omega(\eta(0)), f_2 \right) \right|. \quad (5.32)$$

Combining (5.32) with (5.28), we obtain that

$$0 \leq \mu^2 \text{sgn}[\lambda_1(0)] |\lambda_1(0)| |\lambda_2(0)| + C \mu |\lambda_1(0)|^{\min\{3,p+1\}}$$

for some constant $C > 0$ depending on $f_2$ and $\omega$. Furthermore, it follows from (5.33) that

$$-C |\lambda_1(0)|^{\min\{2,p\}} \leq \mu \lambda_2(0). \quad (5.34)$$

Suppose here that $\text{sgn}[\lambda_1(0)] = 1$. Then, we see from (5.34) and (5.19) that

$$0 < \lambda_1(0) \lesssim \lambda_1(0) - \frac{C}{\mu} |\lambda_1(0)|^{\min\{2,p\}} \lesssim \lambda_1(0) + \lambda_2(0) \leq 2 \lambda_+(0), \quad (5.35)$$

so that $\text{sgn}[\lambda_+(0)] = 1$. Suppose next that $\text{sgn}[\lambda_1(0)] = -1$. Then, (5.34) becomes $\mu \lambda_2(0) \leq C |\lambda_1(0)|^{\min\{2,p\}}$. This together with (5.19) shows that

$$2 \lambda_+(0) = \lambda_1(0) + \lambda_2(0) \leq \lambda_1(0) + \frac{C}{\mu} |\lambda_1(0)|^{\min\{2,p\}} \sim \lambda_1(0). \quad (5.36)$$

Thus, we conclude that

$$\text{sgn}[\lambda_1(0)] = \text{sgn}[\lambda_+(0)], \quad (5.37)$$

which together with (5.17) implies (5.30).

Next, we shall prove (5.7). Since we have (5.17), it suffices to show that for any $t \in [0,T_X]$

$$\lambda_1(t) \sim \lambda_+(t) \sim s R_0 e^{\mu t}. \quad (5.38)$$

Let $\alpha > 1$ be a constant satisfying $|\lambda_1(0)| \leq \frac{1}{2} \alpha R_0$, and consider

$$T_\alpha := \sup \{ T \in [0,T_X] : |\lambda_1(t)| \leq (\alpha R_0 e^{\mu t}), \forall t \in [0,T] \}. \quad (5.39)$$

Then, we have $T_\alpha > 0$. We would like to show that $T_\alpha = T_X$. Suppose for contradiction that $T_\alpha < T_X$. Then, we could take $\tau \in [0,T_X]$ such that

$$\alpha R_0 e^{\mu \tau} = |\lambda_1(\tau)| \lesssim \delta_X. \quad (5.40)$$

Using the equation (4.62) for $\lambda_+$, we have that

$$\frac{d}{dt}(e^{-\mu t} \lambda_+(t)) = e^{-\mu t} \left( \frac{d\lambda_+}{dt}(t) - \mu \lambda_+(t) \right)$$

$$= -e^{-\mu t} \left( \left\{ \frac{d\theta}{dt}(t) - \omega \right\} \eta(t) - N_\omega(\eta(t)), f_2 \right)_{L^2_{real}}. \quad (5.41)$$

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Furthermore, integrating the equation (5.41), and then using (5.29) and (5.18), we find that for any \( t \in [0, T_X] \),

\[
|\lambda_+(t) - e^{\mu t} \lambda_+(0)| = \left| e^{\mu t} \int_0^t e^{-\mu s} \left( \frac{d\vartheta}{dt}(s) - \omega \right) \eta(s) - N_\omega(\eta(s)), \ U_\omega \right|_{L_2} \, ds \nonumber
\]

\[
\lesssim \int_0^t e^{\mu(t-s)} |\lambda_1(s)|^{\min \{2, p \}} \, ds,
\]

where the implicit constant depends on \( U_\omega \) and \( \omega \). Similarly, we have

\[
|\lambda_-(t) - e^{-\mu t} \lambda_-(0)| \lesssim \int_0^t e^{-\mu(t-s)} |\lambda_1(s)|^{\min \{2, p \}} \, ds.
\] (5.43)

Then, we find from (5.42), (5.43), (5.30) and (5.19) that for any \( t \in [0, T_X] \),

\[
|\lambda_1(t)| \leq |\lambda_+(t)| + |\lambda_-(t)|
\]

\[
\lesssim e^{\mu t} |\lambda_+(0)| + e^{-\mu t} |\lambda_-(0)| + \int_0^t e^{\mu(t-s)} |\lambda_1(s)|^{\min \{2, p \}} \, ds
\]

\[
\lesssim e^{\mu t} R_0 + R_0 + e^{\mu t} \int_0^t e^{-\mu s} |\lambda_1(s)|^{\min \{2, p \}} \, ds.
\] (5.44)

This together with (5.40) shows that

\[
|\lambda_1(t)| \lesssim e^{\mu t} R_0 + \frac{1}{\mu \min \{1, p - 1 \}} (\alpha R_0 e^{\mu t})^{\min \{2, p \}}
\]

\[
\leq e^{\mu t} R_0 + \frac{1}{\mu \min \{1, p - 1 \}} \delta_X \min \{1, p - 1 \} \alpha R_0 e^{\mu t}.
\] (5.45)

Hence, if \( \frac{1}{\mu \min \{1, p - 1 \}} \delta_X \min \{1, p - 1 \} \ll 1 \), then (5.45) implies that \( |\lambda_1(t)| \leq \frac{1}{2} \alpha R_0 e^{\mu t} \). However, this contradicts (5.40). Thus, we have proved that \( T_\alpha = T_X \), and therefore for any \( t \in [0, T_X] \),

\[
|\lambda_1(t)| \lesssim R_0 e^{\mu t}.
\] (5.46)

We also see from (5.42), (5.43) and (5.46) that for any \( t \in [0, T_X] \),

\[
|\lambda_+(t) - e^{\mu t} \lambda_+(0)| \lesssim (R_0 e^{\mu t})^{\min \{2, p \}},
\] (5.47)

\[
|\lambda_-(t) - e^{-\mu t} \lambda_-(0)| \lesssim (R_0 e^{\mu t})^{\min \{2, p \}}.
\] (5.48)

Furthermore, we find from (5.30), (5.47) and (5.48) that (5.38) holds. Combining (5.17) and (5.38), we also obtain (5.40).

We shall prove (5.3). It follows from (5.38) that for any \( t \in [0, T_X] \),

\[
|\lambda_+(t)| + |\lambda_-(t)| \lesssim |\lambda_+(t)| + |\lambda_1(t)| \sim R_0 e^{\mu t}.
\] (5.49)

Furthermore, (5.48) together with (5.49) shows that

\[
|\lambda_-(t)| \lesssim e^{-\mu t} |\lambda_-(0)| + (R_0 e^{\mu t})^{\min \{2, p \}} \lesssim R_0 + (R_0 e^{\mu t})^{\min \{2, p \}}.
\] (5.50)
In order to complete the proof of (5.8), we employ the “nonlinear energy projected onto \( U \pm \) plane”:

\[
E_{\{U_+, U_-\}}(t) := S_\omega \{ \Phi_\omega + \lambda_+(t)U_+ + \lambda_-(t)U_- \} - S_\omega (\Phi_\omega). \tag{5.51}
\]

The second order Taylor’s expansion around \( \Phi_\omega \) together with \( L_\omega U_\pm = \pm \imath \mu U_\pm \) and (4.48) shows that

\[
E_{\{U_+, U_-\}}(t) = \frac{1}{2} \langle L_\omega (\lambda_+(t)U_+ + \lambda_-(t)U_-), \lambda_+(t)U_+ + \lambda_-(t)U_- \rangle_{H^{-1}, H^1}
+ O(\| \lambda_+(t)U_+ + \lambda_-(t)U_- \|_{H^1}^{3, p+1}) \tag{5.52}
\]

\[
= \frac{1}{2} \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1}, H^1}
+ O(\| \eta(t) \|_{H^1}^{3, p+1}) + O(\| \lambda_+(t)U_+ + \lambda_-(t)U_- \|_{H^1}^{3, p+1}) \nonumber
\]

\[
\sim \| \lambda_+(t) \|_{H^1}^2 + O(\| \eta(t) \|_{H^1}^{3, p+1}) + O(\| \lambda_+(t)U_+ + \lambda_-(t)U_- \|_{H^1}^{3, p+1}). \tag{5.53}
\]

We find from (4.69), (5.1), (5.52) and Lemma 4.2 that

\[
S_\omega (\psi) - S_\omega (\Phi_\omega) - E_{\{U_+, U_-\}}(t)
= \frac{1}{2} \langle \mathcal{L}_\omega \Gamma(t), \Gamma(t) \rangle_{H^{-1}, H^1}
+ O(\| \eta(t) \|_{H^1}^{3, p+1}) + O(\| \lambda_+(t)U_+ + \lambda_-(t)U_- \|_{H^1}^{3, p+1}) \tag{5.53}
\]

\[
\sim \| \Gamma(t) \|_{H^1}^2 + O(\| \eta(t) \|_{H^1}^{3, p+1}) + O(\| \lambda_+(t)U_+ + \lambda_-(t)U_- \|_{H^1}^{3, p+1}).\nonumber
\]
Moreover, it follows from \( S'_\omega(\Phi_\omega) = 0 \), (4.61), (4.64) and (5.29) that
\[
\left| \frac{d}{dt} E\left\{ |u_+|, |u_-| \right\}(t) \right|
\]
\[
= \left| S'_\omega(\Phi_\omega + \lambda(t)u_+ + \lambda(t)u_-) \right| \left\{ \frac{d\lambda_+}{dt}(t)u_+ + \frac{d\lambda_-}{dt}(t)u_- \right\}
\]
\[
\lesssim \left| S''_\omega(\Phi_\omega)(\lambda(t)u_+ + \lambda(t)u_-) \right| \left\{ \frac{d\lambda_+}{dt}(t)u_+ + \frac{d\lambda_-}{dt}(t)u_- \right\}
\]
\[\leq \left\| \lambda(t)u_+ + \lambda(t)u_- \right\|_{L^1} \left\| \frac{d\lambda_+}{dt}(t)u_+ + \frac{d\lambda_-}{dt}(t)u_- \right\|_{H^1}
\]
\[\leq \mu (|\lambda(t)| + |\lambda(t)|) + \left( |\lambda(t)| + |\lambda(t)| \right)^{2} \left( \left\| \frac{d\lambda_+}{dt}(t) \right\| + \left\| \frac{d\lambda_-}{dt}(t) \right\| \right)
\]

Here, the equations (4.62) and (4.63) together with (5.26) and (5.25) show that for any \( t \in [0, T_*] \),
\[
\left| \frac{d\lambda_+}{dt}(t) \right| \leq \mu |\lambda_+(t)| + |\lambda_+(t)| -\omega \left\| \eta(t) \right\|_{L^2} \left\| \lambda(t) u_+ \right\|_{L^2} + \left\| \left( N_\omega(\eta(t)), u_+ \right) \right\|_{L^1}^{\text{real}}
\]
\[
\lesssim \left| \lambda_+(t) \right| + \left| \lambda_- \right| + \left\| \eta(t) \right\|_{L^2}^{\text{min}(3,p+1)} + \left\| \eta(t) \right\|_{L^2}^{\text{min}(2,p)}
\]
\[\leq \left| \lambda_+(t) \right| + \left| \lambda_- \right| + \left\| \lambda_+(t) u_+ + \lambda_- (t) u_- + \Gamma(t) \right\|_{L^2}^{\text{min}(2,p)}
\]
\[\leq \left| \lambda_+(t) \right| + \left| \lambda_- \right| + \left\| \Gamma(t) \right\|_{H^1}^{\text{min}(2,p)},
\]
where the implicit constant depends on \( \omega, \mu \) and \( u_+ \). Putting (4.64) and (5.55) together, we find that for any \( t \in [0, T_*] \),
\[
\left| \frac{d}{dt} E\left\{ |u_+|, |u_-| \right\}(t) \right|
\]
\[
\lesssim \left\| \Gamma(t) \right\|_{H^1}^{\text{min}(2,p)} \left( \left| \lambda_+(t) \right| + \left| \lambda_- \right| \right) + \left( \left| \lambda_+(t) \right| + \left| \lambda_- \right| \right)^{2} \left( \left\| \frac{d\lambda_+}{dt}(t) \right\| + \left\| \frac{d\lambda_-}{dt}(t) \right\| \right)
\]
\[\lesssim \left( \left| \lambda_+(t) \right| + \left| \lambda_- \right| \right) \left( \left| \lambda_+(t) \right| + \left| \lambda_- \right| \right)^{2},
\]
where the implicit constant depends on \( \omega, \mu \) and \( u_+ \). Combining (5.53) with (5.56), we
obtain that for any \( t \in [0, T_X] \),

\[
\| \Gamma(t) \|^2_{H^1} \lesssim S_\omega(\psi) - S_\omega(\Phi_\omega) - E_{\{u_+, u_-\}}(0) - (E_{\{u_+, u_-\}}(t) - E_{\{u_+, u_-\}}(0)) + O(\| \eta(t) \|_{H^1}^{\min\{3,p+1\}}) + O(\| \lambda_+(t) U_+ + \lambda_-(t) U_- \|_{H^1}^{\min\{3,p\}}) \]

\[
\lesssim \| \Gamma(0) \|^2_{H^1} + \int_0^t \left| \frac{d}{dt} E_{\{u_+, u_-\}}(t') \right| dt' + \sup_{0 \leq t' \leq t} \| \Gamma(t') \|_{H^1}^{\min\{3,p+1\}} \| (|\lambda_+(t')| + |\lambda_-(t')|) \|_{H^1}^{\min\{3,p+1\}} \]

\[
\lesssim \| \Gamma(0) \|^2_{H^1} + \int_0^t \| \Gamma(t') \|_{H^1}^{\min\{2,p\}} (|\lambda_+(t')| + |\lambda_-(t')|) dt' + \int_0^t (|\lambda_+(t')| + |\lambda_-(t')|) dt' \]

\[
+ \sup_{0 \leq t' \leq t} (|\lambda_+(t')| + |\lambda_-(t')|) \]

Suppose here that \( p < 2 \). Note that \( p + 1 < \frac{2}{2-p} \). Then, it follows from (5.57), Young’s inequality, (5.17), (5.49) and (5.38) that for any \( t \in [0, T_X] \),

\[
\sup_{0 \leq t' \leq t} \| \Gamma(t') \|^2_{H^1} \lesssim \| \Gamma(0) \|^2_{H^1} + \int_0^t (|\lambda_+(t')| + |\lambda_-(t')|)^{\frac{2}{2-p}} dt' \]

\[
+ \int_0^t (|\lambda_+(t')| + |\lambda_-(t')|)^{p+1} dt' + \sup_{0 \leq t' \leq t} (|\lambda_+(t')| + |\lambda_-(t')|) \]

\[
\lesssim |\lambda_1(0)|^2 + \int_0^t |\lambda_1(t')|^{p+1} dt' + \sup_{0 \leq t' \leq t} |\lambda_1(t')|^{p+1} \]

\[
\lesssim R_0^2 + \int_0^t (R_0 e^{\mu t'})^{p+1} dt' + \sup_{0 \leq t' \leq t} (R_0 e^{\mu t'})^{p+1} \]

\[
\lesssim R_0^2 + (R_0 e^{\mu t'})^{p+1}. \]

This together with (5.34-5.36) gives us the desired result (5.58) for \( p < 2 \). Similarly, we can prove (5.8) for \( p \geq 2 \).

We shall prove that there exists \( T_x \sim 1 \) such that \( d_\omega(\psi(t)) \) is strictly increasing on
that for any \( t \in [0, T_X] \),
\[
|\lambda_1(t)| \frac{d}{dt} d_\omega(\psi(t)) \sim d_\omega(\psi(t)) \frac{d}{dt} d_\omega(\psi(t)) = \frac{1}{2} \frac{d}{dt} d_\omega(\psi(t))^2 \geq 2\mu^2|\lambda_1(t)|s_\lambda_2(t) - C_1\mu|\lambda_1(t)|^{\min(3,p+1)},
\]
so that
\[
\frac{d}{dt} d_\omega(\psi(t)) \gtrsim \mu^2 s_\lambda_2(t) - C_1\mu|\lambda_1(t)|^{\min(2,p)},
\]
where \( C_1 > 0 \) is constant depending on \( \omega \) and \( f_2 \). Here, multiplying the equation by \( s \), integrating the resulting equation, and using \( 5.29 \) and \( 5.38 \), we obtain that
\[
s_\lambda_2(t) = s_\lambda_2(0) + \mu \int_0^t s_\lambda_1(t') dt' - \frac{1}{2} \int_0^t \left\{ \frac{d\theta}{dt}(t') - \omega \right\} \eta(t') - N_\omega(\eta(t')), \ U_+ + \U_- \right)_{\text{real}} dt' \geq s_\lambda_2(0) + C_2R_0(e^{\mu t} - 1) - C_3 \int_0^t (R_0e^{\mu t'})^{\min(2,p)} dt',
\]
where \( C_2 \) and \( C_3 \) are some positive constants depending on \( \omega \) and \( \U_+ \). Here, we see from \( 5.34 \) and \( 5.30 \) that
\[
\mu s_\lambda_2(0) \gtrsim -|\lambda_1(0)|^{\min(2,p)} \sim -R_0^{\min(2,p)},
\]
where the implicit constants depend on \( \omega \) and \( f_2 \). Moreover, it follows from \( 5.38 \) and \( 5.19 \) that for any \( t \in [0, T_X] \),
\[
|\lambda_1(t)| \sim R_0e^{\mu t} \lesssim \delta_X,
\]
so that, taking \( \delta_X \) sufficiently small depending on \( \omega \), \( \mu \), and \( \U_+ \), we have
\[
C_3 \int_0^t (R_0e^{\mu t'})^{\min(2,p)} dt' \leq \frac{C_2}{2} \mu \int_0^t R_0e^{\mu t'} dt' = \frac{C_2}{2} R_0(e^{\mu t} - 1).
\]
Putting the estimates \( 5.61 \), \( 5.62 \) and \( 5.64 \) together, we find that for any \( t \in [0, T_X] \),
\[
\mu s_\lambda_2(t) \geq \frac{C_2}{2} R_0(e^{\mu t} - 1) - C_4R_0^{\min(2,p)} \geq \frac{C_2}{2} R_0\mu t - C_4R_0^{\min(2,p)},
\]
for some constants \( C_4 > 0 \) depending on \( \omega \), \( \mu \), and \( f_2 \). Hence, if we choose \( T_* \geq \frac{4C_4}{\mu C_2} \), then for any \( t \in [T_*R_0^{\min(1,p-1)}, T_X] \),
\[
\mu s_\lambda_2(t) \geq T_*C_2\mu R_0^{\min(2,p)}.
\]
Furthermore, it follows from \( 5.48 \) and \( 5.38 \) that
\[
|\lambda_2(t)| \geq |\lambda_1(t)| - 2|\lambda_-(t)| \geq |\lambda_1(t)| - 2e^{-\mu t} \lambda_-(0) - C_5(e^{\mu t}R_0)^{\min(2,p)} \geq C_1R_0e^{\mu t} - 2R_0e^{-\mu t} - C_5(R_0e^{\mu t})^{\min(2,p)}.
\]
for some constant $c_1 \in (0,1)$ independent of $R_0$. Hence, if $t \geq \frac{1}{2
u} \log \frac{4}{c_1}$, then we have
\[
|\lambda_2(t)| \geq \frac{c_1}{10} R_0 e^{\mu t} \sim |\lambda_1(t)|.
\]
(5.68)

Thus, we see from (5.60) and (5.63) that for any $t \geq \frac{1}{2
u} \log \frac{4}{c_1}$,
\[
\frac{d}{dt} d_\omega(\psi(t)) > 0.
\]
(5.69)

On the other hand, if $t \leq \frac{1}{2
u} \log \frac{4}{c_1}$, then we have
\[
|\lambda_1(t)| \sim R_0 e^{\mu t} \lesssim R_0.
\]
(5.70)

Hence, choosing $T_*$ suitably, and using (5.60) and (5.66), we conclude that for any $t \in [T_* R_0^{\min(1,p^{-1})}, T_X]$ with $t \leq \frac{1}{2
u} \log \frac{4}{c_1}$,
\[
\frac{d}{dt} d_\omega(\psi(t)) \geq \frac{\mu^3 C_2 T_*}{2} R_0^{\min(2,p)} - C_1 |\lambda_1(t)|^{\min(2,p)} > 0,
\]
(5.71)

and therefore we have obtained the desired result.

We shall prove (5.10). It follows from the fundamental theorem of calculus that
\[
|d_\omega(\psi(t)) - R_0| \lesssim \int_0^t |\frac{d}{ds} d_\omega(\psi(s))| \, ds.
\]
(5.72)

Here, it follows from (5.17), (5.18), (5.28) and (5.38) that
\[
|\lambda_1(t)| \frac{d}{dt} d_\omega(\psi(t)) \sim |2d_\omega(\psi(t)) \frac{d}{dt} d_\omega(\psi(t))| = \frac{1}{2} d_\omega(\psi(t))^2
\]
\[
\lesssim |\lambda_1(t)||\lambda_2(t)| + |\lambda_1(t)|^{\min(3,p+1)}
\]
\[
= |\lambda_1(t)||\lambda_+ - \lambda_0| + |\lambda_0|^{\min(3,p+1)} \lesssim |\lambda_1(t)|^2,
\]
(5.73)

where the implicit constants depend on $\omega, \mu$ and $f_2$. Furthermore, this together with (5.38) gives us that for any $t \in [0, T_* R_0^{\min(1,p^{-1})}]
\[
\left| \frac{d}{dt} d_\omega(\psi(t)) \right| \lesssim R_0 e^{\mu t}.
\]
(5.74)

Putting (5.72) and (5.74) together, we find that for any $t \in [0, T_* R_0^{\min(1,p^{-1})}]$,
\[
|d_\omega(\psi(t)) - R_0| \lesssim \int_0^t R_0 e^{\mu s} ds \lesssim T_* R_0^{\min(2,p)} e^{\mu T_* R_0^{\min(1,p^{-1})}} \lesssim R_0^{\min(2,p)},
\]
(5.75)

where the implicit constants depend on $\omega, \mu$ and $f_2$. Thus, we have proved (5.10).

Finally, we shall prove (5.9). Taylor’s expansion of $K$ around $\Phi_\omega$ together with $\mathcal{K}(\Phi_\omega) = 0$ and (4.145) shows that
\[
\mathcal{K}(\psi(t)) = \mathcal{K}(\Phi_\omega + \eta(t)) = \mathcal{K}(\Phi_\omega)\eta(t) + O(\|\eta(t)\|_{\dot{H}^1}^2)
\]
\[
= -2\mu s_p \lambda_1(\Phi_\omega, f_2) L_2 - 2(1 - s_p)(2^* - 2)\lambda_1(\Phi_\omega^{2^*-1}, f_1) L_2
\]
\[
- s_p (p - 1)(\Phi_\omega^{p}, \Gamma(t))_{L^2_{real}} - (2^* - 2)(\Phi_\omega^{2^*-1}, \Gamma(t))_{L^2_{real}}
\]
\[
+ \omega \mathcal{M}(\eta(t)) + O(\|\eta(t)\|_{\dot{H}^1}^2).
\]
(5.76)
Furthermore, multiplying the both sides above by $s = \text{sgn}[\lambda_1]$, and using $(\Phi_\omega, f_2)_{L^2} < 0$ (see (4.99), Lemma 4.6 and (5.17)), we find that for any sufficiently small $\omega > 0$ and any $t \in [0, T_X]$,

$$sK(\psi(t)) \gtrsim \mu(\Phi_\omega, f_2)_{L^2} |\lambda_1(t)|$$

$$- \|\Phi_\omega\|_{L^{p+1}}^p \|\Gamma(t)\|_{L^{p+1}} - \|\Phi_\omega\|_{L^2}^{2p-1} \|\Gamma(t)\|_{L^2} - |\lambda_1(t)|^2,$$

(5.77)

where the implicit constant is independent of $\omega$. Remember that $|\lambda_1(t)| \lesssim \delta_X \ll 1$ for $t \in [0, T_X]$ (see (5.17)). Then, we conclude from (5.77), (5.38) and the proved result (5.8) that there exists a constant $C_\omega > 0$ depending on $\omega$ such that

$$sK(\psi(t)) \gtrsim R_0 e^{\mu t} - C_\omega R_0,$$

(5.78)

where the implicit constant depends on $\omega$. Thus, we have completed the proof.  

\[\square\]

6 \textbf{Modified distance function}

We continue to consider the decomposition (4.2) with (4.1), (4.51) and (4.55). We introduce two distances between the solution $\psi$ and the orbit of $\Phi_\omega$ at a time $t$:

$$\text{dist}_{H^1}(\psi(t), \Omega(\Phi_\omega)) := \inf_{\theta \in \mathbb{R}} \|\psi(t) - e^{i \theta} \Phi_\omega\|_{H^1},$$

(6.1)

$$\text{dist}_{L^2}(\psi(t), \Omega(\Phi_\omega)) := \inf_{\theta \in \mathbb{R}} \|\psi(t) - e^{i \theta} \Phi_\omega\|_{L^2}.$$

(6.2)

**Lemma 6.1.** If $\text{dist}_{H^1}(\psi(t), \Omega(\Phi_\omega)) \ll 1$, then

$$\|\eta(t)\|_E \lesssim \text{dist}_{H^1}(\psi(t), \Omega(\Phi_\omega)) \leq \|\eta(t)\|_E.$$

(6.3)

Similarly, if $\text{dist}_{L^2}(\psi(t), \Omega(\Phi_\omega)) \ll 1$, then

$$\|\eta(t)\|_{L^2} \sim \text{dist}_{L^2}(\psi(t), \Omega(\Phi_\omega)).$$

(6.4)

**Proof of Lemma 6.1.** Note first that

$$\text{dist}_{H^1}(\psi(t), \Omega(\Phi_\omega)) \leq \|\psi(t) - e^{i \theta_0(t)} \Phi_\omega\|_{H^1} = \|\eta(t)\|_{H^1} \lesssim \|\eta(t)\|_E.$$

(6.5)

Moreover, we can take a continuous function $\theta_0(t)$ of $t$ such that

$$\|\psi(t) - e^{i \theta_0(t)} \Phi_\omega\|_{H^1} = \text{dist}_{H^1}(\psi(t), \Omega(\Phi_\omega)).$$

(6.6)

Put $\eta_0(t) := \psi(t) - e^{i \theta_0(t)} \Phi_\omega$, so that $\|\eta_0(t)\|_{H^1} = \text{dist}_{H^1}(\psi(t), \Omega(\Phi_\omega))$. Then, it follows from $\Omega(\Phi_\omega, \Phi'_\omega) = (\Phi_\omega, i \Phi'_\omega)_{L^2_{\text{real}}} = 0$ and (4.55) that

$$\Omega(e^{-i \theta_0(t)}\eta_0(t), \Phi_\omega') = \Omega(e^{-i \theta_0(t)}\psi(t), \Phi'_\omega) - \Omega(e^{i \theta_0(t) - \theta(t)} \Phi_\omega, \Phi'_\omega)$$

$$= - \langle ie^{i \theta_0(t) - \theta(t)} \Phi_\omega, \Phi'_\omega \rangle_{L^2_{\text{real}}}$$

$$= \sin (\theta_0(t) - \theta(t))(\Phi_\omega, \Phi'_\omega)_{L^2_{\text{real}}}.$$

(6.7)
Hence, we have
\[
| \sin(\theta_0(t) - \theta(t))|||\Phi_\omega, \Phi'_\omega||_{L^2_{\text{real}}}| \leq ||\eta_0(t)||_{L^2}||\Phi'_\omega||_{L^2}
\]
\[
\leq \text{dist}_{H^1}(\psi(t), O(\Phi_\omega))||\Phi'_\omega||_{L^2}.
\]
(6.8)
We see from (6.8) and dist\(_{H^1}(\psi(t), O(\Phi_\omega)) \ll 1\) that
\[
\inf_{k \in \mathbb{Z}} |\theta_0(t) - \theta(t) + k\pi| \lesssim \text{dist}_{H^1}(\psi(t), O(\Phi_\omega)),
\]
(6.9)
where the implicit constant depends on \(\Phi_\omega\). Here, we can eliminate the case where \(k\) is odd in the infimum above. Indeed, it follows from the choice of \(\theta(t)\) (see (4.50)) that
\[
0 > (e^{-i\theta(t)}\psi(t), \Phi'_\omega)_{L^2_{\text{real}}}
\]
\[
= \cos(\theta_0(t) - \theta(t))(\Phi_\omega, \Phi'_\omega)_{L^2_{\text{real}}} + (e^{-i\theta(t)}\eta_0(t), \Phi'_\omega)_{L^2_{\text{real}}},
\]
(6.10)
so that
\[
- \cos(\theta_0(t) - \theta(t))(\Phi_\omega, \Phi'_\omega)_{L^2_{\text{real}}} > (e^{-i\theta(t)}\eta_0(t), \Phi'_\omega)_{L^2_{\text{real}}}.
\]
Thus, supposing for contradiction that \(\theta_0(t) - \theta(t)\) lay near an odd multiple of \(\pi\), we have
\[
\frac{1}{2}(\Phi_\omega, \Phi'_\omega)_{L^2_{\text{real}}} \geq (e^{-i\theta(t)}\eta_0(t), \Phi'_\omega)_{L^2_{\text{real}}}.
\]
(6.12)
However, this together with (iii) of Proposition 2.4 shows
\[
1 \leq \frac{1}{2}|(\Phi_\omega, \Phi'_\omega)_{L^2_{\text{real}}}| < ||\eta_0(t)||_{L^2}||\Phi'_\omega||_{L^2} \leq \text{dist}_{H^1}(\psi(t), O(\Phi_\omega))||\Phi'_\omega||_{L^2},
\]
(6.13)
which contradicts that dist\(_{H^1}(\psi(t), O(\Phi_\omega)) \ll 1\). Since dist\(_{H^1}(\psi(t), O(\Phi_\omega)) \ll 1\) implies that \(\theta_0(t) - \theta(t)\) lies near a multiple of \(2\pi\), we see from Lemma 4.3 and (6.9) that
\[
||\eta(t)||_{L^2} \lesssim ||\eta(t)||_{H^1} = ||e^{-i\theta(t)}\psi(t) - \Phi_\omega||_{H^1}
\]
\[
\leq |e^{i(\theta_0(t) - \theta(t))} - 1||\Phi_\omega||_{H^1} + ||\eta_0(t)||_{H^1}
\]
\[
\lesssim |\cos(\theta_0(t) - \theta(t)) - 1| + |\sin(\theta_0(t) - \theta(t))| + ||\eta_0(t)||_{H^1}
\]
\[
\lesssim \inf_{k \in \mathbb{Z}} |\theta_0(t) - \theta(t) - 2k\pi| + ||\eta_0(t)||_{H^1}
\]
\[
\lesssim \text{dist}_{H^1}(\psi(t), O(\Phi_\omega)).
\]
(6.14)
Putting (6.5) and (6.14) together, we obtain the desired result (6.3). The same argument as the above shows (6.4).

We find from (6.3) in Lemma 6.1 that there exists a function \(\tilde{d}_\omega(\psi(t))\) of \(t\) with the following properties:
\[
\tilde{d}_\omega(\psi(t)) \sim \text{dist}_{H^1}(\psi(t), O(\Phi_\omega));
\]
(6.15)
and if $\tilde{d}_\omega(\psi(t)) \ll 1$, then
\begin{equation}
\tilde{d}_\omega(\psi(t)) = d_\omega(\psi(t)),
\end{equation}
where $d_\omega(\psi(t))$ is the function given by (1.79). We are also able to define the “modified distance” $\tilde{d}_\omega$ for any function $u \in H^1(\mathbb{R}^d)$ satisfying $\mathcal{M}(u) = \mathcal{M}(\Phi_\omega)$. Note that $\tilde{d}_\omega$ is continuous.

**Lemma 6.2.** For any $\delta > 0$, there exist $\varepsilon_0(\delta) > 0$ and $\kappa_1(\delta) > 0$ with the following properties: $\varepsilon_0(\delta)$ and $\kappa_1(\delta)$ are non-decreasing with respect to $\delta$; and for any radial function $u \in H^1(\mathbb{R}^d)$ satisfying
\begin{equation}
\mathcal{M}(u) = \mathcal{M}(\Phi_\omega),
\end{equation}
\begin{equation}
\mathcal{S}_\omega(u) < m_\omega + \varepsilon_0(\delta),
\end{equation}
\begin{equation}
\tilde{d}_\omega(u) \geq \delta,
\end{equation}
we have
\begin{equation}
|\mathcal{K}(u)| \geq \kappa_1(\delta)
\end{equation}

**Remark 6.1.** The conclusion (6.20) is stronger than the corresponding one in Lemma 3.4 of [21]. This modification is due to the existence of the energy-critical term.

**Proof of Lemma 6.2.** We suppose for contradiction that there existed $\delta_0 > 0$ with the following property: for any $n \in \mathbb{N}$, there exists a radial function $u_n \in H^1(\mathbb{R}^d)$ satisfying
\begin{equation}
\mathcal{M}(u_n) = \mathcal{M}(\Phi_\omega),
\end{equation}
\begin{equation}
\mathcal{S}_\omega(u_n) < m_\omega + \frac{1}{n},
\end{equation}
\begin{equation}
\tilde{d}_\omega(u_n) \geq \delta_0,
\end{equation}
\begin{equation}
|\mathcal{K}(u_n)| \leq \frac{1}{n}.
\end{equation}
Moreover, we see from (5.24) and (6.22) that
\begin{equation}
\frac{s_p}{d} \|\nabla u_n\|_L^2 \leq \mathcal{I}_\omega(u_n) = \mathcal{S}_\omega(u_n) - \frac{2}{d(p-1)} \mathcal{K}(u_n) \leq m_\omega + 2,
\end{equation}
so that
\begin{equation}
\|u_n\|_{H^1}^2 \leq \mathcal{M}(\Phi_\omega) + \frac{d}{s_p} m_\omega + \frac{2d}{s_p}.
\end{equation}
Hence, we can take $u_\infty \in H^1(\mathbb{R}^d)$ and subsequence of $\{u_n\}$ (still denoted by the same symbol $\{u_n\}$) such that
\begin{equation}
\lim_{n \to \infty} u_n = u_\infty \quad \text{weakly in } H^1(\mathbb{R}^d)
\end{equation}
and for any \(2 < q < 2^*,\)
\[
\lim_{n \to \infty} u_n = u_\infty \quad \text{strongly in } L^q(\mathbb{R}^d).
\] (6.28)

We shall show that \(u_\infty\) is non-trivial. Suppose for contradiction that \(u_\infty\) was trivial. Then, it follows from (6.28) that
\[
\lim_{n \to \infty} \|u_n\|_{L^{q+1}} = 0.
\] (6.29)

Moreover, (6.24) together with (6.29) shows that, passing to some subsequence, we have
\[
\lim_{n \to \infty} \|\nabla u_n\|_{L^2}^2 = \lim_{n \to \infty} \|u_n\|_{L^{2^*}}^{2^*}.
\] (6.30)

Recall here the variational value \(\sigma\) (see (1.19)). We see from the definition of \(\sigma\) (or Sobolev’s embedding) and (6.30) that
\[
\lim_{n \to \infty} \|\nabla u_n\|_{L^2}^2 \geq \sigma \lim_{n \to \infty} \|u_n\|_{L^{2^*}}^{2^*} = \sigma \lim_{n \to \infty} \|\nabla u_n\|_{L^2}^2,\] (6.31)

which together with (6.30) yields
\[
\lim_{n \to \infty} \|u_n\|_{L^{2^*}}^{2^*} \geq \sigma \frac{d}{d^*}.
\] (6.32)

Combining (6.32), (6.22) and (6.24), we find that
\[
\frac{1}{d} \sigma \frac{d}{d^*} \leq \lim_{n \to \infty} \frac{1}{d} \|u_n\|_{L^{2^*}}^{2^*} \leq \lim_{n \to \infty} \mathcal{J}_\omega(u_n) \leq \lim_{n \to \infty} \left\{ \mathcal{S}_\omega(u_n) + \frac{1}{2} |K(u_n)| \right\} \leq m_\omega.
\] (6.33)

However, this contradicts (1.26) and therefore \(u_\infty\) is non-trivial.

Next, we shall show that \(u_\infty = e^{i\theta_0} \Phi_\omega\) for some \(\theta_0\). We see from (6.22) and (6.24) that
\[
\mathcal{I}_\omega(u_\infty) \leq \lim_{n \to \infty} \mathcal{I}_\omega(u_n) \leq \lim_{n \to \infty} \left\{ \mathcal{S}_\omega(u_n) + \frac{2}{d(p-1)} |K(u_n)| \right\} \leq m_\omega.
\] (6.34)

Here, suppose for a contradiction that \(K(u_\infty) > 0\). Then, the Brezis-Lieb lemma together with (6.24) shows
\[
\lim_{n \to \infty} K(u_n - u_\infty) = - \lim_{n \to \infty} \left\{ K(u_n) - K(u_n - u_\infty) \right\} = -K(u_\infty) < 0,
\] (6.35)

so that for any sufficiently large \(n \in \mathbb{N},\)
\[
K(u_n - u_\infty) < 0.
\] (6.36)

Hence, we can take \(\lambda_n < 1\) such that \(K(\lambda_n^{\frac{d}{d^*}}(u_n - u_\infty)(\lambda_n^*) = 0\) (cf. Lemma 2.1 in [3]).

Moreover, this implies that
\[
m_\omega \leq \mathcal{I}_\omega(\lambda_n^{\frac{d}{d^*}}(u_n - u_\infty)(\lambda_n^*)) \leq \mathcal{I}_\omega(u_n - u_\infty)
= \mathcal{I}_\omega(u_n) - \mathcal{I}_\omega(u_\infty) + o_n(1)
= \mathcal{S}_\omega(u_n) - \frac{2}{d(p-1)} K(u_n) - \mathcal{I}_\omega(u_\infty) + o_n(1),
\] (6.37)
which together with (6.22) and (6.24) shows
\[ m_{\omega} \leq m_{\omega} - I_{\omega}(u_{\infty}). \] (6.38)
Since \( I_{\omega}(u_{\infty}) > 0 \), this is a contradiction and therefore \( K(u_{\infty}) \leq 0 \). Furthermore, since any minimizer of \( I_{\omega} \) is also a ground state (see Proposition 1.2 in [2]), we conclude from the inequality (6.34), \( K(u_{\infty}) \leq 0 \) \((u_{\infty} \neq 0)\) and Proposition 2.1 with the radial assumption that \( u_{\infty} = e^{i\theta_0} \Phi_{\omega} \) for some \( \theta_0 \in \mathbb{R} \). In particular, the condition (6.23) prevents \( u_{\infty} \) from existing. Indeed, it follows from the weak convergence (6.27), (6.34) and \( I_{\omega}(u_{\infty}) = m_{\omega} \) that
\[ \lim_{n \to \infty} \| u_n \|_{H^1} = \| u_{\infty} \|_{H^1}, \] (6.39)
so that
\[ \lim_{n \to \infty} u_n = u_{\infty} \quad \text{strongly in } H^1(\mathbb{R}^d). \] (6.40)
Using (6.23) and (6.40), we have
\[ \delta_0 \leq \inf_{\theta} \| u_n - e^{i\theta} \Phi_{\omega} \|_{H^1} \leq \| u_n - e^{i\theta_0} \Phi_{\omega} \|_{H^1} = \| u_n - u_{\infty} \|_{H^1} = o_n(1). \] (6.41)
However, this is contradiction. Thus, there is no sequence \( \{ u_n \} \) satisfying (6.21)–(6.24) and therefore the lemma holds. \( \square \)

**Lemma 6.3.** For any \( \delta > 0 \), there exists \( \kappa_2(\delta) > 0 \) such that for any \( u \in H^1(\mathbb{R}^d) \) with \( J_{\omega}(u) \leq m_{\omega} - \delta \), we have
\[ K(u) \geq \kappa_2(\delta). \] (6.42)

**Proof of Lemma 6.3.** Suppose for contradiction that there exists \( \delta_0 > 0 \) with the following property: for any \( n \in \mathbb{N} \), we can take \( u_n \) in \( H^1(\mathbb{R}^d) \) such that \( J_{\omega}(u_n) \leq m_{\omega} - \delta_0 \) and \( K(u_n) \leq \frac{1}{4} \). If we had \( \liminf_{n \to \infty} S(u_n) \leq m_{\omega} - \frac{1}{2} \delta_0 \), then we could extract a subsequence of \( \{ u_n \} \) (still denoted by the same symbol \( \{ u_n \} \)) such that \( K(u_n) \geq 1 \) (see [3]). However, this contradicts \( \lim_{n \to \infty} K(u_n) = 0 \). Thus, we can take a subsequence \( \{ u_n \} \) (still denoted by the same symbol \( \{ u_n \} \)) such that \( m_{\omega} - \frac{3}{4} \delta_0 \leq S(u_n) \) for any \( n \in \mathbb{N} \). In this case, we have that for any \( n \in \mathbb{N} \),
\[ \frac{1}{2} K(u_n) = S_{\omega}(u_n) - J_{\omega}(u_n) \geq m_{\omega} - \frac{3}{4} \delta_0 - (m_{\omega} - \delta_0) = \frac{1}{4} \delta_0, \] (6.43)
which also contradicts \( \lim_{n \to \infty} K(u_n) = 0 \). Thus, we have completed the proof. \( \square \)

### 7 One-pass theorem

Let \( \delta_X, A_*, B_* \) and \( C_* \) be constants given by the ejection lemma (Lemma 5.1), and put
\[ \delta_S := \frac{A_* \delta_X}{2 B_* C_*}. \] (7.1)
Note that $\delta_S < \delta_X < \delta_E(\omega)$, where $\delta(E)$ is the constant used in the definition of $d_\omega(\psi(t))$ (see (4.79)). Moreover, for a given $\delta > 0$, let $\varepsilon_0(\delta)$ be the constant determined by Lemma 6.2. We may assume that $\delta_X$ is so small that

$$s_{\mu t}(\Phi_\omega, f_2)_{L^2} \gg (||\Phi_\omega||_{L^{p+1}}^p + ||\Phi_\omega||_{L^{2^*}}^{2^*-1}) \delta_X^{\min(1,p-1)} + \delta_X.$$  \hspace{2cm} (7.2)

**Theorem 7.1** (One-pass theorem). There exists $\omega_* > 0$ with the following property: for any $\omega \in (0, \omega_*)$, there exist positive constants $\varepsilon_*, \delta_*$ and $R_*$ such that

$$\sqrt{\varepsilon_*} \ll R_* \ll \delta_* \ll \delta_S, \quad \varepsilon_* \leq \varepsilon_0(\delta_*),$$  \hspace{2cm} (7.3)

and for any radial solution $\psi$ to (NLS), with the maximal life-span $T_{\text{max}} > 0$, satisfying the conditions

$$\mathcal{M}(\psi) = \mathcal{M}(\Phi_\omega),$$  \hspace{2cm} (7.4)

$$S_\omega(\psi) < m_\omega + \varepsilon,$$  \hspace{2cm} (7.5)

$$\tilde{d}_\omega(\psi(0)) < R$$  \hspace{2cm} (7.6)

for some $\varepsilon \in (0, \varepsilon_*)$ and some $R \in (\sqrt{2\varepsilon}, R_*)$, one has either

(i) $\tilde{d}_\omega(\psi(t)) < R + R^{\min(3,p+1)}$ for any $t \in [0, T_{\text{max}}]$; or

(ii) there exists $t_* > 0$ such that $\tilde{d}_\omega(\psi(t)) \geq R + R^{\min(3,p+1)}$ for any $t \in [t_*, T_{\text{max}}]$.

**Proof of Theorem 7.1**. Let $\omega > 0$ be a constant to be chosen later. We prove the theorem by contradiction, so suppose that for any $\varepsilon_*, \delta_*, R_* > 0$ satisfying that $\sqrt{\varepsilon_*} \ll R_* \ll \delta_* \ll \delta_S$ and $\varepsilon_* \leq \varepsilon_0(\delta_*)$, there exists a radial solution $\psi$, with the maximal life-span $T_{\text{max}} > 0$, such that it satisfies the conditions (7.1), (7.3) and (7.6), while both (i) and (ii) fail.

The failure of (i) together with (7.4) shows that there exists $t_2 > t_1 > 0$ such that

$$\tilde{d}_\omega(\psi(t_1)) = R < \delta_*,$$  \hspace{2cm} (7.7)

$$\tilde{d}_\omega(\psi(t_2)) = R + R^{\min(3,p+1)} < \delta_*,$$  \hspace{2cm} (7.8)

and for any $t \in (t_1, t_2)$,

$$R < \tilde{d}_\omega(\psi(t)) < R + R^{\min(3,p+1)} \ll \delta_S < \delta_X.$$  \hspace{2cm} (7.9)

Moreover, the failure of (ii) shows that we can take $t_3 > t_2$ such that

$$R < \tilde{d}_\omega(\psi(t_3)) < R + R^{\min(3,p+1)},$$  \hspace{2cm} (7.10)

and for any $t \in (t_2, t_3)$

$$\tilde{d}_\omega(\psi(t)) > \tilde{d}_\omega(\psi(t_3)).$$  \hspace{2cm} (7.11)

We see from (7.9) and (7.11) that for any $t \in [t_1, t_3]$,

$$\tilde{d}_\omega(\psi(t)) \geq R = \tilde{d}_\omega(\psi(t_1)).$$  \hspace{2cm} (7.12)

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This together with (7.17) and $\sqrt{2\varepsilon} < R$ shows that for any $t \in [t_1, t_3]$,
\[
S_\omega(\psi) < m_\omega + \frac{1}{2} \tilde{d}_\omega(\psi(t))^2.
\]  
(7.13)

We consider a time $t \in [t_1, t_3]$ for which $\tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$. We see from (6.10), Lemma 4.7 and (7.12) that
\[
R \leq d_\omega(\psi(t)) \sim |\lambda_1(t)|.
\]  
(7.14)

Thus, $\text{sign} [\lambda_1(t)]$ is continuous (constant) as long as $\tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$. On the other hand, it follows from Lemma 6.2 that $\text{sign} [\mathcal{K}(\psi(t))]$ is continuous (constant) as long as $\tilde{d}_\omega(\psi(t)) \geq \delta_*$. We shall show that for any $t \in [t_1, t_3]$ satisfying $\delta_* \leq \tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$,
\[
\text{sign} [\lambda_1(t)] = \text{sign} [\mathcal{K}(\psi(t))].
\]  
(7.15)

Note here that the ejection lemma (Lemma 5.1) together with (7.8) shows that $\tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$. We see from the ejection lemma (Lemma 5.1) that $\tilde{d}_\omega(\psi(t))$ is constant as long as $\tilde{d}_\omega(\psi(t)) \geq \delta_*$. We shall show that for any $t \in [t_1, t_3]$ satisfying $\delta_* \leq \tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$, $\tilde{d}_\omega(\psi(t))$ must reach $\delta_X$ before $t_3$. Put
\[
T_X := \inf \{t \in [t_1, t_3]: \tilde{d}_\omega(\psi(t)) = \delta_X\}.
\]  
(7.16)

Then, $T_X < t_3$, and there exists $t_S \in (t_1, T_X)$ such that $\tilde{d}_\omega(\psi(t_S)) = \delta_S$. Since both $\text{sign} [\lambda_1(t)]$ and $\text{sign} [\mathcal{K}(\psi(t))]$ are constant when $\delta_* \leq \tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$, it suffices to show that $\text{sign} [\lambda_1(\psi(T_X))] = \text{sign} [\mathcal{K}(\psi(T_X))]$. We see from the ejection lemma (Lemma 5.1) that $\text{sign} [\lambda_1(\cdot)]$ is constant on $[t_1, T_X]$, and for any $t \in [t_1, T_X],$
\[
A_* e^{\mu(t-t_1)} R \leq d_\omega(\psi(t)) \leq B_* e^{\mu(t-t_1)} R,
\]  
(7.17)

\[
\text{sign} [\lambda_1(t)] \mathcal{K}(\psi(t)) \geq (e^{\mu(t-t_1)} - C_*) R.
\]  
(7.18)

In particular, (7.17) together with (6.16) shows that
\[
\delta_X \leq B_* e^{\mu(T_X-t_1)} R \leq \frac{B_*}{A_*} e^{\mu(T_X-t_S)} \tilde{d}_\omega(\psi(t_S)) = \frac{B_*}{A_*} e^{\mu(T_X-t_S)} \delta_S.
\]  
(7.19)

Taking the logarithm in (7.19) and using the relation $\delta_S = \frac{A_* \delta_X}{2B_* C_*}$, we find that
\[
T_X - t_S \geq \frac{1}{\mu} \log \left( \frac{A_* \delta_X}{B_* \delta_S} \right) > \frac{1}{\mu} \log C_*.
\]  
(7.20)

Since $t_1 < t_S$, we find from (7.18) and (7.20) that $\text{sign} [\lambda_1(T_X)] = \text{sign} [\mathcal{K}(\psi(T_X))]$. Hence, (7.15) holds, and therefore the following function $\mathcal{G}: [t_1, t_3] \to \{+1, -1\}$ is well-defined:
\[
\mathcal{G}(t) := \begin{cases} 
\text{sign} [\lambda_1(t)] & \text{if } \tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega), \\
\text{sign} [\mathcal{K}(\psi(t))] & \text{if } \tilde{d}_\omega(\psi(t)) \geq \delta_*.
\end{cases}
\]  
(7.21)

Furthermore, since $\text{sign} [\lambda_1(t)]$ is constant as long as $\tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega)$, and since $\text{sign} [\mathcal{K}(\psi(t))]$ is constant as long as $\tilde{d}_\omega(\psi(t)) \geq \delta_*$, the function $\mathcal{G}(t)$ is constant on $[t_1, t_3]$.
\[ s := \mathcal{S}(t) \in \{1, -1\}. \] When \( s = 1 \), we have the uniform boundedness of the family \( \{\psi(t)\}_{t \in [t_1, t_3]} \) in \( H^1(\mathbb{R}^d) \):

\[
\sup_{t \in [t_1, t_3]} \|\psi(t)\|_{H^1} \lesssim 1, \tag{7.22}
\]

where the implicit constant may depend on \( \omega \). Indeed, if \( \tilde{d}_\omega(\psi(\tau)) \geq \delta_S \) for some \( \tau \in [t_1, t_3] \), then (7.21) shows \( \inf_{t \in [t_1, t_3]} K(\psi(t)) \geq 0 \). Hence, we see from (1.10) that for any \( t \in [t_1, t_3] \),

\[
\frac{\omega}{2} \|\psi(t)\|_{L^2}^2 + \frac{\kappa_\omega}{d} \|\nabla \psi(t)\|_{L^2}^2 \leq \mathcal{I}_\omega(\psi(t)) \leq \mathcal{S}_\omega(\psi) \leq m_\omega + 1. \tag{7.23}
\]

On the other hand, if \( \tilde{d}_\omega(\psi(t)) \leq \delta_S \) for any \( t \in [t_1, t_3] \), then we see from Lemma [4.3] and Lemma 1.4 that for any \( t \in [t_1, t_3] \),

\[
\|\psi(t)\|_{H^1}^2 \lesssim \|\Phi_\omega\|_{H^1}^2 + \|\eta(t)\|_{H^1}^2 \lesssim \|\Phi_\omega\|_{H^1}^2 + \tilde{d}_\omega(\psi(t))^2 \lesssim \|\Phi_\omega\|_{H^1}^2 + \delta_S. \tag{7.24}
\]

Thus, we have verified that (7.22) holds.

Now, we find from the ejection lemma that there exists \( t'_1 \in (t_1, t_2) \) such that

\[
\tilde{d}_\omega(\psi(t'_1)) - R \gg R^{\min(2, p)}, \tag{7.25}
\]

and \( \tilde{d}_\omega(\psi(t)) \) increases for \( t > t'_1 \) until it reaches \( \delta_X \); and further there exists \( t'_2 \in (t_2, t_3) \) such that

\[
\tilde{d}_\omega(\psi(t'_2)) = \delta_X, \tag{7.26}
\]

and for any \( t \in (t_1, t'_2) \),

\[
\delta_X \geq \tilde{d}_\omega(\psi(t)) \sim \|\eta(t)\|_{H^1} \sim |\lambda_1(t)| \sim e^{\mu(t-t_1)} R, \tag{7.27}
\]

\[
\|\Gamma(t)\|_{H^1} \lesssim \tilde{d}_\omega(\psi(t_1)) + |\lambda_1(t)|^{\min(3, p+1)} \tag{7.28}
\]

\[
s\mathcal{K}(\psi(t)) \gtrsim (e^{\mu(t-t_1)} - C_s) R. \tag{7.29}
\]

Arguing from \( t_3 \) backward in time, we are also able to obtain a time interval \( (t'_2, t_3) \subset (t'_2, t_3) \) such that

\[
\tilde{d}_\omega(\psi(t'_2)) = \delta_X, \tag{7.30}
\]

and for any \( t \in (t'_2, t_3) \),

\[
\delta_X \geq \tilde{d}_\omega(\psi(t)) \sim \|\eta(t)\|_{H^1} \sim |\lambda_1(t)| \sim e^{\mu(t-t_3)} \tilde{d}_\omega(\psi(t_3)), \tag{7.31}
\]

\[
\|\Gamma(t)\|_{H^1} \lesssim \tilde{d}_\omega(\psi(t_3)) + |\lambda_1(t)|^{\min(3, p+1)} \tag{7.32}
\]

\[
s\mathcal{K}(\psi(t)) \gtrsim (e^{\mu(t-t_3)} - C_s) \tilde{d}_\omega(\psi(t_3)) \tag{7.33}
\]

and \( \tilde{d}_\omega(\psi(t)) \) decreases at least in the region \( \tilde{d}_\omega(\psi(t)) \geq 2\tilde{d}_\omega(\psi(t_3)) \).

Suppose here that there exists a time \( \tau \in (t'_2, t'_2) \) such that \( \tilde{d}_\omega(\psi(\tau)) \) is a local minimum and \( \tilde{d}_\omega(\psi(\tau)) < \delta_X \). Then, we can apply the ejection lemma (Lemma 5.1) from \( \tau \) both forward and backward in time to obtain an open interval \( I_\tau \subset (t'_2, t'_2) \) such that

\[
\tilde{d}_\omega(\psi(\inf I_\tau)) = \tilde{d}_\omega(\psi(\sup I_\tau)) = \delta_X, \tag{7.34}
\]
and for any $t \in I_\tau$,
\[
\delta_X \geq \tilde{d}_\omega(\psi(t)) \sim \|\eta(t)\|_{H^1} \sim |\lambda_1(t)| \sim e^{\mu|t-\tau|}\tilde{d}_\omega(\psi(\tau)),
\]
\[
\|\Gamma(t)\|_{H^1} \lesssim \tilde{d}_\omega(\psi(\tau)) + |\lambda_1(t)|^{\min(2,p+1)},
\]
\[
sK(\psi(t)) \gtrsim (e^{\mu|t-\tau|} - C_*)\tilde{d}_\omega(\psi(\tau)),
\]
and $\tilde{d}_\omega(\psi(t))$ is monotone in the region $\tilde{d}_\omega(\psi(t)) \geq 2\tilde{d}_\omega(\psi(\tau))$. Note that for any distinct local minimum points $\tau_1$ and $\tau_2$ of $\tilde{d}_\omega(\psi(t))$ in $(t_1', t_2'')$, the monotonicity away from them implies that the intervals $I_{\tau_1}$ and $I_{\tau_2}$ are either disjoint or identical.

Figure 3: The case where there exists a local minimum point $\tau \in (t_1', t_2'')$ such that $\tilde{d}_\omega(\psi(\tau)) < \delta_*$

It follows from the above observation that we can take disjoint open subintervals $I_1, \ldots, I_n$ of $[t_1, t_3]$ ($n \geq 2$) with the following property: for any $1 \leq j \leq n$, there exists $\tau_j \in I_j$ such that $\tilde{d}_\omega(\psi(\tau_j)) < \delta_*$;

\[
\tilde{d}_\omega(\psi(\inf I_j)) = \tilde{d}_\omega(\psi(\sup I_j)) = \delta_X
\]

1Note that $I_1 = (t_1', t_2')$. Moreover, if $\tilde{d}_\omega(\psi(t)) \geq \delta_*$ on $[t_2', t_3'']$, then $n = 2$ and $I_2 = (t_2', t_3)$.
except for \( j = 1, n \); and for any \( t \in I_j \)
\[
\delta_X \geq \tilde{d}_\omega(\psi(t)) \sim ||\eta(t)||_{H^1} \sim |\lambda_1(t)| \sim e^{\mu|t-\tau_j|} \tilde{d}_\omega(\psi(\tau_j)), \tag{7.39}
\]
\[
||\Gamma(t)||_{H^1} \lesssim \tilde{d}_\omega(\psi(\tau_j)) + |\lambda_1(t)|^{\min(2,p+1)}, \tag{7.40}
\]
\[
s\mathcal{K}(\psi(t)) \gtrsim (e^{\mu|t-\tau_j|} - C_\ast) \tilde{d}_\omega(\psi(\tau_j)). \tag{7.41}
\]

Since the number of intervals \( n \) is finite, we can take the implicit constants in (7.39) and (7.41) uniformly in \( j \). Put 
\[ I'_j := [t_1, t_3] \setminus \bigcup_{j=1}^n I_j. \]
Then, for any \( t \in I'_j \), we have
\[
s\mathcal{K}(\psi(t)) > \delta_\ast, \tag{7.42}
\]
and (7.38) that for any \( 1 \leq j \leq n \),
\[
\mu \tilde{d}_\omega(\psi(\tau_j)) |I_j| = \tilde{d}_\omega(\psi(\tau_j)) \left\{ \log e^{\mu|\sup I_j - \tau_j|} + \log e^{\mu|\inf I_j - \tau_j|} \right\}
\lesssim \delta_X \frac{\delta_X}{\tilde{d}_\omega(\psi(\tau_j))} \ll \delta_X \sim \delta_X \frac{\delta_X}{C_\ast}. \tag{7.43}
\]

When \( s = +1 \), we can derive a contradiction following the argument in Section 4.2 of [21]. Indeed, almost same computation shows that
\[
\frac{d}{dt} \exists \int_{\mathbb{R}^d} \frac{M|x|}{M + |x|} \frac{x}{|x|} \cdot \nabla \psi(x,t) \bar{\psi}(x,t) \, dx \geq 2\mathcal{K} \left( \frac{M}{M + |x|} \psi(t) \right) - CM^{1 - \frac{(d-2)(p-1)}{2}}, \tag{7.44}
\]
where \( C \) is some positive constant independent of \( M \). Here, we remark that our modification in Lemma 6.2 (see Remark 6.1) helps us to treat a general power \( p \in (2_\ast, 2^\ast) \). Furthermore, the main difference from [21] is that we need Lemma 6.2. Let us explain this point. We see from Taylor’s expansion of \( \mathcal{K} \) around \( \Phi_\omega \), (4.145), (4.134), \( s = 1, \)
where (7.22) and (4.99) that \( \omega \) large that for any sufficiently small \( \omega > \) Using Lemma 4.6, Hölder’s inequality, (7.39), (7.40) and Proposition 2.2, we find that

\[
\begin{align*}
&\kappa(\Phi_\omega + \eta(t)) - \frac{|x|}{M + |x|} (\Phi_\omega + \eta(t)) \\
&\kappa'_{(\Phi_\omega)} \left\{ \eta(t) - \frac{|x|}{M + |x|} (\Phi_\omega + \eta(t)) \right\} \\
&+ O(\|\eta(t) - \frac{|x|}{M + |x|} (\Phi_\omega + \eta(t))\|_{H^1}^2) \\
&= 2\mu s_p |(\Phi_\omega, f_2)_{L^2} |\lambda_1(t)| - 10(\|\Phi_\omega\|_{L^{p+1}}^p + \|\Phi_\omega\|_{L^{2^*}}^{2^*-1}) \|\Gamma(t)\|_{H^1} \\
&- \frac{10}{M} \|x\|^{\frac{1}{p}} \Phi_\omega \|_{L^{p+1}}^p \|\eta(t)\|_{L^{p+1}} - \frac{10}{M} \|x\|^{\frac{1}{2^*-1}} \Phi_\omega \|_{L^{2^*}}^{2^*-1} \|\eta(t)\|_{L^{2^*}} \\
&- \frac{2\omega}{M} \|x\| \|\Phi_\omega\|_{L^2} \|\eta(t)\|_{L^2} - C(\|\eta(t)\|_{H^1}^2 + \frac{1}{M^2} \|x\| \|\Phi_\omega\|_{H^1}^2) \\
&\geq s_p \mu |(\Phi_\omega, f_2)_{L^2} |\lambda_1(t)| \\
&- C_1(\omega) \tilde{a}_\omega(\psi(t_j)) - 10(\|\Phi_\omega\|_{L^{p+1}}^p + \|\Phi_\omega\|_{L^{2^*}}^{2^*-1}) \delta_X \frac{\min(1, p-1)}{2} \|\lambda_1(t)\| \\
&- \frac{1}{M} C_2(\omega) |\lambda_1(t)| - C\delta_X |\lambda_1(t)| - \frac{1}{M^2} C_3(\omega),
\end{align*}
\]

We need Lemma 4.6 to estimate the first two terms on the right-hand side of (7.45). Using Lemma 4.6, Hölder’s inequality, (7.39), (7.40) and Proposition 2.2, we find that for any sufficiently small \( \omega > 0 \),

\[
\begin{align*}
\kappa(\Phi_\omega + \eta(t)) - \frac{|x|}{M + |x|} (\Phi_\omega + \eta(t)) \\
&\geq s_p \mu |(\Phi_\omega, f_2)_{L^2} |\lambda_1(t)| \\
&- 10(\|\Phi_\omega\|_{L^{p+1}}^p + \|\Phi_\omega\|_{L^{2^*}}^{2^*-1}) \|\Gamma(t)\|_{H^1} \\
&- \frac{10}{M} \|x\|^{\frac{1}{p}} \Phi_\omega \|_{L^{p+1}}^p \|\eta(t)\|_{L^{p+1}} - \frac{10}{M} \|x\|^{\frac{1}{2^*-1}} \Phi_\omega \|_{L^{2^*}}^{2^*-1} \|\eta(t)\|_{L^{2^*}} \\
&- \frac{2\omega}{M} \|x\| \|\Phi_\omega\|_{L^2} \|\eta(t)\|_{L^2} - C(\|\eta(t)\|_{H^1}^2 + \frac{1}{M^2} \|x\| \|\Phi_\omega\|_{H^1}^2) \\
&\geq s_p \mu |(\Phi_\omega, f_2)_{L^2} |\lambda_1(t)| \\
&- C_1(\omega) \tilde{a}_\omega(\psi(t_j)) - 10(\|\Phi_\omega\|_{L^{p+1}}^p + \|\Phi_\omega\|_{L^{2^*}}^{2^*-1}) \delta_X \frac{\min(1, p-1)}{2} \|\lambda_1(t)\| \\
&- \frac{1}{M} C_2(\omega) |\lambda_1(t)| - C\delta_X |\lambda_1(t)| - \frac{1}{M^2} C_3(\omega),
\end{align*}
\]

where \( C \) is some constant independent of \( \omega \), and \( C_1(\omega) \), \( C_2(\omega) \) and \( C_3(\omega) \) are some positive constants depending on \( \omega \). Since \( \tilde{a}_\omega(\psi(t)) \geq R \) on \([t_1, t_3] \), we can choose \( M \) so large that

\[
M^{-1} C_1(\omega) \ll s_p \mu |(\Phi_\omega, f_2)_{L^2}|, \\
M^{-2} C_3(\omega) \leq C_1(\omega) \tilde{a}_\omega(\psi(t_j)).
\]
Then, (7.40) together with (7.2) and (7.39) shows that for any sufficiently small $\omega > 0$ and any $t \in I_j$,

$$
\mathcal{K}\left( \frac{M}{M + |x|} \psi(t) \right) \geq e^{\mu(t - \tau^*)} \tilde{d}_\omega(\psi(\tau_j)) - 2C_{**}(\omega) \tilde{d}_\omega(\psi(\tau_j)),
$$

(7.49)

where $C_{**}(\omega)$ is some constant depending on $\omega$, and the implicit constant may depend on $\omega$. Thus, we have obtained the following estimate for the right-hand side of (7.44) in the “hyperbolic region” $\bigcup_{j=1}^n I_j$: for any sufficiently large $M$ and any $t \in I_j (1 \leq j \leq n)$,

$$
2\mathcal{K}\left( \frac{M}{M + |x|} \psi(t) \right) - CM^1 \cdot \frac{(d-2)(p-1)}{2} \geq e^{\mu(t - \tau^*)} \tilde{d}_\omega(\psi(\tau_j)) - 3C_{**}(\omega) \tilde{d}_\omega(\psi(\tau_j)).
$$

(7.50)

Thus, Lemma 4.6 plays an important role to derive the estimate in the hyperbolic region.

We also refer to estimates in the “variational region” $I'$. It follows from (7.42) that for any $t \in I'$,

$$
\mathcal{K}(\psi(t)) \geq \kappa_1(\delta_*).
$$

(7.51)

Choosing $\varepsilon_* > 0$ so small that $\varepsilon_* < \frac{1}{2}\kappa_1(\delta_*)$, and using (7.50) and (7.51), we obtain that for any $t \in I'$,

$$
\mathcal{J}_\omega\left( \frac{M}{M + |x|} \psi(t) \right) \leq \mathcal{J}_\omega(\psi(t)) = S_\omega(\psi) - \frac{1}{2} \mathcal{K}(\psi(t)) < M_\omega - \varepsilon_*.
$$

(7.52)

which together with Lemma 6.3 gives us that for any $t \in I'$

$$
\mathcal{K}\left( \frac{M}{M + |x|} \psi(t) \right) \geq \kappa_2(\varepsilon_*).
$$

(7.53)

Furthermore, if $M$ is sufficiently large depending on $\kappa_2(\varepsilon_*)$, then we estimate the right-hand side of (7.44) as follows:

$$
2\mathcal{K}\left( \frac{M}{M + |x|} \psi(t) \right) - CM^1 \cdot \frac{(d-2)(p-1)}{2} \geq \kappa_2(\varepsilon_*).
$$

(7.54)

Thus, we also obtained an estimate in the variational region. Using the estimates (7.44), (7.50) and (7.54), we see that for any sufficiently large $M > 0$,

\[
\left[ \int_{\mathbb{R}^d} Ma\left( \frac{|x|}{M} \right) \frac{x}{|x|} \cdot \Im \left[ \psi(x, t) \nabla \psi(x, t) \right] \right]_{t_1}^{t_3}
= \int_{t_1}^{t_3} \frac{d}{dt} \int_{\mathbb{R}^d} Ma\left( \frac{|x|}{M} \right) \frac{x}{|x|} \cdot \Im \left[ \psi(x, t) \nabla \psi(x, t) \right] \, dx \, dt
\geq \sum_{j=1}^n \int_{I_j} \left\{ e^{\mu(t - \tau_j)} \tilde{d}_\omega(\psi(\tau_j)) - C_{**}(\omega) \tilde{d}_\omega(\psi(\tau_j)) \right\} \, dt + \int_{I'} \kappa_2(\varepsilon_*^2) \, dt
\geq \sum_{j=1}^n \frac{1}{2} \left\{ e^{\mu\sup_{I_j} I_j - \tau_j} + e^{\mu\inf_{I_j} I_j - \tau_j} - 2 \right\} \tilde{d}_\omega(\psi(\tau_j)) - C_{**}(\omega) \sum_{j=1}^n |I_j| \tilde{d}_\omega(\psi(\tau_j)).
\]
Here, it follows from (7.38) and (7.39) that for any \(1 \leq j \leq n\),
\[
\delta_X = \tilde{d}_{\omega}(\psi(\sup I_j)) \sim e^{\mu|\sup I_j - \tau_j|} \tilde{d}_{\omega}(\psi(\tau_j)),
\]
(7.56)
\[
\delta_X = \tilde{d}_{\omega}(\psi(\inf I_j)) \sim e^{\mu|\inf I_j - \tau_j|} \tilde{d}_{\omega}(\psi(\tau_j)).
\]
(7.57)
Moreover, we see from (7.43) that if \(\delta_X\) is sufficiently small,
\[
|I_j| \tilde{d}_{\omega}(\psi(\tau_j)) \leq \frac{1}{2\mu C_{**}(\omega)} \delta_X.
\]
(7.58)
Thus, combining (7.55) with (7.56), (7.57), \(\tilde{d}_{\omega}(\psi(\tau_j)) \leq \delta_* \ll \delta_X\) and (7.58), we obtain
\[
\left[ \int_{\mathbb{R}^d} Ma\left( \frac{|x|}{M} \right) \frac{x}{|x|} : \Im \left[ \psi(x,t) \psi(x,t) \right] \right]_{t_1}^{t_3} \geq \sum_{j=1}^{n} \frac{2}{\mu} \{ \delta_X - \delta_* \} - \frac{1}{\mu} \sum_{j=1}^{n} \delta_X \geq n\delta_X \geq \delta_X.
\]
(7.59)
Thus, the same computation as (4.18) in [21] shows that
\[
\left| \int_{\mathbb{R}^d} Ma\left( \frac{|x|}{M} \right) \frac{x}{|x|} : \Im \left[ \psi(x,t) \psi(x,t) \right] \right|_{t_1}^{t_3} \ll \delta_X.
\]
(7.60)
Thus, we have arrived at a contradiction.

We can also deal with the case \(s = -1\) in a way similar to Section 4.1 of [21], and complete the proof.

\[\square\]

8 Proof of Theorem 3.1

Our goal in this section is to prove Theorem 3.1. By the time reversality, it suffices to show that there are only three possibilities (scattering, blowup, trapping) forward in time.

Let \(\psi\) be a radial solution to (NLS). We use \(I_{\max}(\psi)\) to denote the maximal existence-interval of \(\psi\), and set \(I_{\max}(\psi) := \sup I_{\max}(\psi)\) and \(I_{\min}(\psi) := \inf I_{\max}(\psi)\). Furthermore, since the set \(A^\omega_{\omega}\) is invariant under the flow defined by (NLS) for any \(\varepsilon > 0\) and any \(\omega \in (0, \omega_1)\) (see (3.1)), we use the convention \(\psi \in A^\omega_{\omega}\) if \(\psi(t) \in A^\omega_{\omega}\) for any \(t \in I_{\max}(\psi)\).

For given \(\omega > 0\), \(R > 0\) and \(\varepsilon > 0\), we define \(S^\omega_{\omega,R}\) by
\[
S^\omega_{\omega,R} := \left\{ \psi : \psi \text{ is a radial solution to (NLS) such that } 0 \in I_{\max}(\psi), \psi \in A^\omega_{\omega}, \text{ and } \tilde{d}_{\omega}(\psi(t)) \geq R \text{ for any } t \in [0, T_{\max}(\psi)]. \right\}
\]
(8.1)
Let \(\omega_* > 0\) be a constant given by the one-pass theorem (Theorem 7.1), and let \(\omega \in (0, \omega_*)\). Then, we can take a constant \(0 < \delta_{E}(\omega) \ll 1\) satisfying (4.77). Furthermore, the
Lemma 5.1 determines the constants \( \delta_X \in (0, \delta_E(\omega)) \) and \( \delta_S \) (see (7.2)). Then, the one-pass theorem shows that there exists constants \( 0 < \varepsilon_* \ll R_* \ll \delta_* \ll \delta_S \) such that: \( \varepsilon_* \leq \varepsilon_0(\delta_*) \) (see Lemma 6.2 for the definition of \( \varepsilon_0(\delta_*) \)); and for any \( \varepsilon \in (0, \varepsilon_*) \), any \( R \in (2\varepsilon, R_*) \) and any solution \( \psi \) to (NLS) with \( \psi \in A^\varepsilon_\omega \), the following alternative holds:

**Case 1.** There exists \( t_0 \in I_{\mathrm{max}}(\psi) \) such that \( \tilde{d}_\omega(\psi(t_0)) < R \) and \( \tilde{d}_\omega(\psi(t)) < R + R^{\min(3,p+1)/2} \) for all \( t \in [t_0, T_{\mathrm{max}}(\psi)] \); or

**Case 2.** There exists \( t_1 \in I_{\mathrm{max}}(\psi) \) such that for all \( t_1 \leq t < T_{\mathrm{max}}(\psi) \),

\[
\tilde{d}_\omega(\psi(t)) \geq R. \tag{8.2}
\]

In Case 1, it follows from \( R + R^{\min(3,p+1)/2} < \delta_X \ll 1 \) (6.16) and the ejection lemma (Lemma 5.1) that \( T_{\mathrm{max}}(\psi) = \infty \). Hence, \( \psi \) is trapped by \( O(\Phi_\omega) \) forward in time.

Next, we consider Case 2. In this case, we have that \( \psi(\cdot, t) \in S^\varepsilon_{\omega,R} \) and

\[
\varepsilon \leq \min \left\{ \frac{\tilde{d}_\omega(\psi(t))}{2}, \varepsilon_0(\delta_*) \right\} \tag{8.3}
\]

for any \( t \in [t_1, T_{\mathrm{max}}(\psi)) \). In Section 8.1 and Section 8.2 we will prove that the solution \( \psi \) scatters or blows up in a finite time. To this end, we make several preparations. The first one is to show the following:

**Lemma 8.1.** Let \( \varepsilon \in (0, \varepsilon_*) \), \( R \in (2\varepsilon, R_*) \), and let \( \psi \in S^\varepsilon_{\omega,R} \). Then, for any \( T \in (0, T_{\mathrm{max}}(\psi)) \), there exists \( \tau \in [T, T_{\mathrm{max}}(\psi)) \) such that

\[
\tilde{d}_\omega(\psi(\tau)) \geq \delta_X. \tag{8.4}
\]

**Proof of Lemma 8.1.** Suppose for contradiction that there exists \( T_0 \in (0, T_{\mathrm{max}}(\psi)) \) such that for any \( t \in [T_0, T_{\mathrm{max}}(\psi)) \),

\[
d_\omega(\psi(t)) = \tilde{d}_\omega(\psi(t)) < \delta_X, \tag{8.5}
\]

where the equality follows from (6.16) and \( \delta_X \ll 1 \). Then, we see from the ejection lemma (Lemma 5.1) that \( T_{\mathrm{max}}(\psi) = \infty \). Let \( L \gg \frac{1}{\varepsilon_*} \), and consider a time \( t_L \in [T_0, T_0 + L \log \frac{\delta_X}{\varepsilon_*}] \) such that \( R_L := d_\omega(\psi(t_L)) \) is the minimum of \( d_\omega(\psi(t)) \) over the interval \( [T_0, T_0 + L \log \frac{\delta_X}{\varepsilon_*}] \). Note that \( R_L \geq R > \varepsilon_* \). If \( T_0 \leq t_L \leq T_0 + \frac{L}{2} \log \frac{\delta_X}{\varepsilon_*} \), then we see from the ejection lemma (Lemma 5.1) that

\[
d_\omega(\psi(t_L + \frac{L}{2} \log \frac{\delta_X}{\varepsilon_*})) \sim e^{\mu(\frac{L}{2} \log \frac{\delta_X}{\varepsilon_*})} R_L > \left( \frac{\delta_X}{\varepsilon_*} \right)^{\frac{\mu L}{\varepsilon_*}} \gg \delta_X. \tag{8.6}
\]

However, this contradicts (8.5). Similarly, when \( t_L \geq T_0 + \frac{L}{2} \log \frac{\delta_X}{\varepsilon_*} \), applying the ejection lemma backward in time, we reaches a contradiction. Thus, we have found that (8.4) holds.

Next, we show that the solution \( \psi \) stays away from \( \Phi_\omega \) after some time.
Lemma 8.2. Let \( \varepsilon \in (0, \varepsilon_*) \), \( R \in (2\varepsilon, R_*) \), and let \( \psi \in S^\varepsilon_{\omega, R} \). Then, there exists a time \( T_0 \in [0, T_{\max}(\psi)) \) such that for any \( t \in [T_0, T_{\max}(\psi)) \),

\[
\tilde{d}_\omega(\psi(t)) \geq R_*. 
\]  
(8.7)

**Proof of Lemma 8.2.** It suffices to consider the case where there exists a time \( \tau_0 \in [0, T_{\max}(\psi)) \) such that

\[
R \leq \tilde{R}_* := \tilde{d}_\omega(\psi(\tau_0)) < R_*.
\]  
(8.8)

Here, it follows from Lemma 8.1 that we may assume that

\[
\sqrt{1 + 4R_* - \frac{1}{2}} \leq \tilde{R}_* < R_*.
\]  
(8.9)

Then, we see from the one-pass theorem (Theorem 7.1) that either

\[
\sup_{t \geq \tau_0} \tilde{d}_\omega(\psi(t)) \leq \tilde{R}_* + \frac{\min(3, p+1)}{2},
\]  
(8.10)

or there exists \( T_0 \geq \tau_0 \) such that

\[
\inf_{t \geq T_0} \tilde{d}_\omega(\psi(t)) \geq \tilde{R}_* + \frac{\min(3, p+1)}{2} \geq \tilde{R}_* + \frac{\tilde{R}_*^2}{2} \geq R_*.
\]  
(8.11)

We find from Lemma 8.1 that the former case (8.10) never happens. Thus, the latter case (8.11) only happens, and the proof is completed.

Lastly, we introduce a sign function which will determine the scattering or blowup.

**Lemma 8.3.** Let \( \varepsilon \in (0, \varepsilon_*) \) and \( R \in (2\varepsilon, R_*) \). Then, there exists a function \( \mathcal{S} : S^\varepsilon_{\omega, R} \to \{+1, -1\} \) such that

\[
\mathcal{S}(\psi) = \begin{cases} 
\text{sign}[\lambda_1(t)] & \text{if } \tilde{d}_\omega(\psi(t)) \leq \delta_E(\omega), \\
\text{sign}[K(\psi(t))] & \text{if } \tilde{d}_\omega(\psi(t)) \geq \delta_*.
\end{cases}
\]  
(8.12)

In addition, there exists a positive constant \( C(\omega) \) which is independent of \( \varepsilon \) and \( R \) and satisfies that

\[
\sup \{ \|\psi\|_{L_t^\infty H_x^1([0, T_{\max}(\psi))]} : \psi \in S^\varepsilon_{\omega, R}, \mathcal{S}(\psi) = +1 \} \leq C(\omega).
\]  
(8.13)

**Proof of Lemma 8.3.** We introduced a sign function having a similar property in the proof of Proposition 7.1 (see (7.21)). Applying the same argument and Lemma 8.1, we can prove the existence of a desired function.

We divide \( S^\varepsilon_{\omega, R} \) into two parts according to the sign of \( \mathcal{S} \):

\[
S^\varepsilon_{\omega, R, \pm} := \{ \psi \in S^\varepsilon_{\omega, R} : \mathcal{S}(\psi) = \pm 1 \text{ on } [0, T_{\max}(\psi)) \}.
\]  
(8.14)
8.1 Analysis on $S^\varepsilon_{\omega,R,-}$

In this section, we shall prove that any solution $\psi \in S^\varepsilon_{\omega,R,-}$ blows up forward in time.

**Proposition 8.4.** Assume that $0 < \varepsilon < \min\{\varepsilon_0(R_*), \varepsilon_*\}$, where $\varepsilon_0(R_*)$ is a constant given by Lemma 6.2. Let $R \in (2\varepsilon, R_*)$, and let $\psi \in S^\varepsilon_{\omega,R,-}$. Then, the maximal lifespan $T_{\max}(\psi)$ is finite: the finite time blowup forward in time.

**Proof of Proposition 8.4.** We see from Lemma 8.2 and Lemma 6.2 that there exist a time $T_0 \in [0, T_{\max}(\psi))$ and $\kappa_1(R_*) > 0$ such that for any $t \in [T_0, T_{\max}(\psi))$,

$$|K(\psi(t))| \geq \kappa_1(R_*).$$  \hspace{1cm} (8.15)

We also see from Lemma 8.1 that there exists a time $\tau \in [T_0, T_{\max}(\psi))$ such that

$$\tilde{d}_\omega(\psi(\tau)) \geq \delta \geq \delta_*.$$  \hspace{1cm} (8.16)

Hence, Lemma 8.3 together with $S(\psi) = -1$ shows that

$$K(\psi(\tau)) < 0.$$  \hspace{1cm} (8.17)

Putting (8.15) and (8.17) together, we find that for any $t \in [T_0, T_{\max}(\psi))$,

$$K(\psi(t)) \leq -\kappa_1(R_*).$$  \hspace{1cm} (8.18)

Then, the same proof as Theorem 1.3 in [2] is available. Thus, we find that $T_{\max}(\psi) < \infty$. $\Box$

8.2 Analysis on $S^\varepsilon_{\omega,R,+}$

In this section, we shall prove that any solution $\psi \in S^\varepsilon_{\omega,R,+}$ scatters forward in time. Lemma 8.2 together with time-translation allows us to restrict ourselves to the solutions in $S^\varepsilon_{\omega,R,+}$. Then, Lemma 6.2 determines constants $\varepsilon_0(R_*) > 0$ and $\kappa_1(R_*) > 0$ such that if $\varepsilon \in (0, \varepsilon_0(R_*))$, then any solution $\psi \in S^\varepsilon_{\omega,R,+}$ obeys

$$\inf_{[0,T_{\max}(\psi))} |K(\psi(t))| \geq \kappa_1(R_*).$$  \hspace{1cm} (8.19)

Furthermore, this together with Lemma 8.1 and Lemma 8.3 shows that for any $\varepsilon \in (0, \varepsilon_0(R_*))$ and any $\psi \in S^\varepsilon_{\omega,R,+}$,

$$\inf_{[0,T_{\max}(\psi))} K(\psi(t)) \geq \kappa_1(R_*).$$  \hspace{1cm} (8.20)

The purpose of this section is to prove the following:

**Proposition 8.5.** For any $\omega \in (0, \omega_*)$, there exists $\varepsilon \in (0, \varepsilon_*)$ such that for any $\psi \in S^\varepsilon_{\omega,R,+}$,

$$\|\psi\|_{W^{p+1}([0,T_{\max}(\psi)) \cap W([0,T_{\max}(\psi)))} < \infty.$$  \hspace{1cm} (8.21)

In particular, the solution $\psi$ scatters forward in time.
Proof of Proposition 8.5. We divide the proof into several parts: In sections 8.2.1 we suppose for contradiction that the proposition was false, and extract some sequence of non-scattering solutions in $S^{ε,ω,R∗,+}$. In Section 8.2.2, we apply the profile decomposition to the sequence, and obtain the “linear profiles”. In Section 8.2.3 we introduce the “non-linear profiles”, and investigate their fundamental properties. Furthermore, in Section 8.2.4 we show the existence of a nonlinear profile whose Strichartz norm diverges. Finally, in Section 8.2.5 we derive a contradiction by showing the existence of the “critical element”, in the spirit of Kenig-Merle [15].

8.2.1 Setup
For any $E > 0$, we define $ν(E)$ by
\[
ν(E) := \sup \left\{ \|ψ\|_{W^{p+1}(\{T_{\max}(ψ)\})} : ψ ∈ S^{ε,ω,R∗,+}, \mathcal{H}(ψ) ≤ E \right\}.
\] (8.22)
Furthermore, we put
\[
E∗ := \sup\{E > 0: ν(E) < ∞\}.
\] (8.23)
If $ψ ∈ S^{ε,ω,R∗,+}$ and $\mathcal{H}(ψ) < \mathcal{H}(Φ_ω)$, then we see from (8.20) that $ψ ∈ PW_{ω,+}$. Hence, it follows from Theorem 1.3 that
\[
E∗ ≥ \mathcal{H}(Φ_ω).
\] (8.24)
Thus, what we want to prove is that $E∗ > \mathcal{H}(Φ_ω)$. We prove this by contradiction, and therefore suppose that
\[
E∗ = \mathcal{H}(Φ_ω).
\] (8.25)
Then, we can take a sequence $\{ε_n\}$ of constants in $(0, ε_*)$ and a sequence $\{ψ_n\}$ of solutions such that
\[
\lim_{n→∞} ε_n = 0,
\] (8.26)
\[
ε_n < \min \left\{ ε_0(R_*), \frac{κ_1(R_*)}{10d(p−1) + 10} \right\},
\] (8.27)
\[
ψ_n ∈ S^{ε_0,R_{ω,R∗,+}}
\] (8.28)
\[
\|ψ_n\|_{W^{p+1}(\{T_{\max}(ψ_n)\})∩W((0,T_{\max}(ψ_n)))} = ∞.
\] (8.29)
In particular, we have that
\[
\inf_{n≥1} \inf_{t∈[0,T_{\max}(ψ_n)))} \tilde{d}_ω(ψ_n(t)) ≥ R_*,
\] (8.30)
\[
\lim_{n→∞} \mathcal{H}(ψ_n) = \mathcal{H}(Φ_ω) = E_*.
\] (8.31)
Furthermore, it follows from (8.20) and Lemma 8.3 that
\[ \inf_{n \geq 1} \inf_{t \in [0,T_{\max}(\psi_n))]} \mathcal{K}(\psi_n(t)) \geq \kappa_1(R_*), \]  
(8.32)

\[ \sup_{n \geq 1} \sup_{t \in [0,T_{\max}(\psi_n))]} \|\psi_n(t)\|_{H^1} \lesssim 1, \]  
(8.33)

where the implicit constant may depend on \( \omega \).

8.2.2 Linear profiles

We apply the profile decomposition (cf. Lemma 2.2. in [3]) to the sequence \( \{\psi_n\} \). Then, we can extract a subsequence (denoted by the same symbol \( \{\psi_n\} \)) with the following properties: there exists

- a family \( \{\{(x_1^n, t_1^n, \lambda_1^n), (x_2^n, t_2^n, \lambda_2^n), \ldots\}\} \) of sequences in \( \mathbb{R}^d \times \mathbb{R} \times (0, \infty) \) with
  \[ \lim_{n \to \infty} t_j^n = t_j^\infty \in \mathbb{R} \cup \{\pm \infty\}, \]  
  (8.34)

  \[ \lim_{n \to \infty} \lambda_j^n = \lambda_j^\infty \in (0,1,\infty), \]
  \[ \lambda_j^n = 1 \text{ if } \lambda_j^\infty = 1, \]
  \[ \lambda_j^n \leq 1 \text{ if } \lambda_j^\infty = 0, \]
  (8.35)

and for any \( j' \neq j \),

\[ \lim_{n \to \infty} \left\{ \frac{\lambda_j'^n}{\lambda_j^n} + \frac{x_j^n - x_j'^n}{\lambda_j^n} + \frac{|t_j^n - t_j'^n|}{(\lambda_j^n)^2} \right\} = \infty; \]

(8.36)

- a family \( \{\tilde{u}^1, \tilde{u}^2, \ldots\} \) of functions in \( H^1(\mathbb{R}^d) \) with
  \[ \tilde{u}^j \equiv 0 \text{ if } \lambda_j^\infty = \infty; \]  
  (8.37)

- a family \( \{w_1^n, w_2^n, \ldots\} \) of sequences in \( H^1(\mathbb{R}^d) \) satisfying that for any \( \dot{H}^1 \)-admissible pair \( (q,r) \),
  \[ \lim_{j \to \infty} \lim_{n \to \infty} \| |\nabla|^{-1} \langle \nabla \rangle e^{it\Delta} w_j^n \|_{L^q_t L^r_x(\mathbb{R})} = 0; \]
  (8.38)

such that for any \( k \in \mathbb{N} \),

\[ e^{it\Delta} \psi_n(0) = \sum_{j=1}^k \langle \nabla \rangle^{-1} |\nabla| G_1^n(e^{it\Delta} |\nabla|^{-1} \langle \nabla \rangle \tilde{u}^j) + e^{it\Delta} w_n^k \]

\[ = \sum_{j=1}^k g_j^n \sigma_j^n e^{i(t-t_j^n)\Delta} \tilde{u}^j + e^{it\Delta} w_n^k, \]

\[ \text{A pair } (q,r) \text{ is said to be } \dot{H}^1 \text{-admissible if } \frac{1}{q} = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{r} - \frac{1}{d} \right) \text{ and } (q,r) \in [2, \infty] \times [2, \infty]). \]
where

\[ G^j_n v(x, t) := \frac{1}{(\lambda^j_n)^{d/2}} \psi \left( \frac{x - x_n^j}{\lambda^j_n}, \frac{t - t_n^j}{(\lambda^j_n)^2} \right), \quad (8.40) \]

\[ g^j_n u(x) := \frac{1}{(\lambda^j_n)^{d/2}} u \left( \frac{x - x_n^j}{\lambda^j_n} \right), \quad (8.41) \]

\[ \sigma^j_n := \frac{\langle (\lambda^j_n)^{-1}\nabla \rangle^{-1} \nabla}{\lambda^j_n}. \quad (8.42) \]

Furthermore, we have that for any number \( k \) and \( s \in \{0, 1\} \),

\[ \lim_{n \to \infty} \left\{ \| \nabla^s \psi_n(0) \|_{L^2}^2 - \sum_{j=1}^{k} \left\| \nabla^s g^j_n \sigma^j_n e^{-i \frac{t_{nj}}{\lambda^j_n^2}} \Delta \tilde{u}^j_n \right\|_{L^2}^2 - \| \nabla^s w^k_n \|_{L^2}^2 \right\} = 0, \quad (8.43) \]

\[ \lim_{n \to \infty} \left\{ \mathcal{H}(\psi_n) - \sum_{j=1}^{k} \mathcal{H}(g^j_n \sigma^j_n e^{-i \frac{t_{nj}}{\lambda^j_n^2}} \Delta \tilde{u}^j_n) - \mathcal{H}(w^k_n) \right\} = 0, \quad (8.44) \]

\[ \lim_{n \to \infty} \left\{ \mathcal{S}_\omega(\psi_n) - \sum_{j=1}^{k} \mathcal{S}_\omega(g^j_n \sigma^j_n e^{-i \frac{t_{nj}}{\lambda^j_n^2}} \Delta \tilde{u}^j_n) - \mathcal{S}_\omega(w^k_n) \right\} = 0, \quad (8.45) \]

\[ \lim_{n \to \infty} \left\{ \mathcal{I}_\omega(\psi_n(0)) - \sum_{j=1}^{k} \mathcal{I}_\omega(g^j_n \sigma^j_n e^{-i \frac{t_{nj}}{\lambda^j_n^2}} \Delta \tilde{u}^j_n) - \mathcal{I}_\omega(w^k_n) \right\} = 0. \quad (8.46) \]

We see from Strichartz’ estimate, (8.33) and (8.43) that for any \( k \geq 1 \), there exists a number \( N(k) \) such that for any \( n \geq N(k) \),

\[ \| \langle \nabla \rangle e^{it \Delta} w^k_n \|_{S_t(\mathbb{R})} \lesssim \| w^k_n \|_{H^1} \lesssim 1. \quad (8.47) \]

Moreover, it follows from (8.39) and (8.47) that for any \( 2s < q \leq 2s^* \),

\[ \lim_{k \to \infty} \lim_{n \to \infty} \| e^{it \Delta} w_n^k \|_{W^q(\mathbb{R})} \leq \lim_{k \to \infty} \lim_{n \to \infty} \| e^{it \Delta} w_n^k \|_{W^{1-s}_q(\mathbb{R})} \| e^{it \Delta} w_n^k \|_{W^q(\mathbb{R})} \lesssim \lim_{k \to \infty} \lim_{n \to \infty} \| \nabla^{-1} \langle \nabla \rangle e^{it \Delta} w_n^k \|_{W^q(\mathbb{R})} = 0. \quad (8.48) \]

For each number \( j \), we define the operator \( \sigma^j_\infty \) by

\[ \sigma^j_\infty := \begin{cases} 1 & \text{if } \lambda^j_\infty = 1, \\ \| \nabla^{-1} \langle \nabla \rangle \| & \text{if } \lambda^j_\infty = 0. \end{cases} \quad (8.49) \]

In what follows, we observe fundamental properties of operators \( g^j_n \) and \( \sigma^j_n \).

**Lemma 8.6.** Let \( j \) be a number for which \( \lambda^j_\infty = 0 \). Then, we have that for any function \( f \in H^1(\mathbb{R}^d) \),

\[ \lim_{n \to \infty} \| \nabla \{ \sigma^j_n f - \sigma^j_\infty f \} \|_{L^2} = 0, \quad (8.50) \]

\[ \lim_{n \to \infty} \| g^j_n \sigma^j_n f \|_{L^2} = \lim_{n \to \infty} \| \lambda^j_n \sigma^j_n f \|_{L^2} = 0. \quad (8.51) \]
Proof of Lemma 8.6. The first claim follows from Perseval’s identity and Lebesgue’s convergence theorem. Indeed, since \( \lim_{n \to \infty} \lambda_n^j = \lambda_j^\infty = 0 \), we have

\[
\lim_{n \to \infty} \| \nabla \{ \sigma_n^j f - \sigma_j^\infty f \} \|_{L^2} = \lim_{n \to \infty} \left\| \nabla \left\{ \frac{(\lambda_{n}^j)^{-1}\nabla^{-1} - |\nabla|^{-1}}{\lambda_n^j} \right\} f \right\|_{L^2} = \lim_{n \to \infty} \left\{ \frac{|\xi|}{\sqrt{(\lambda_n^j)^2 + |\xi|^2}} - 1 \right\} \langle \xi \rangle \mathcal{F}[f] \right\|_{L^2} = 0. \tag{8.52}
\]

Similarly, we can verify the second claim. \( \square \)

Lemma 8.7. We have that for any numbers \( j \) and \( n \), any \( q \in [2, \infty) \) and any \( f \in L^q(\mathbb{R}^d) \),

\[
\| \lambda_n^j \sigma_n^j f \|_{L^q} + \| \nabla |\sigma_n^j f| \|_{L^q} \lesssim \| \langle \nabla \rangle f \|_{L^q}, \tag{8.53}
\]

where the implicit constant depends only on \( d \) and \( q \).

Proof of Lemma 8.7. We find from Mihlin’s multiplier theorem that

\[
\| \lambda_n^j \sigma_n^j f \|_{L^q} + \| \nabla |\sigma_n^j f| \|_{L^q} \lesssim \| \langle \nabla \rangle f \|_{L^q} + \| \sigma_n^j (\sigma_j^\infty)^{-1} |\nabla| \sigma_j^\infty f \|_{L^q}
\lesssim \| \langle \nabla \rangle f \|_{L^q} + \| \nabla |\sigma_j^\infty f| \|_{L^q} \lesssim \| \langle \nabla \rangle f \|_{L^q}. \tag{8.54}
\]

Thus, we have proved the lemma. \( \square \)

Lemma 8.8. Let \( j \) be a number for which \( \lambda_j^\infty = 0 \). Then, we have

\[
\lim_{n \to \infty} \| g_n^j \sigma_n^j \frac{-i \lambda_n^j}{(\lambda_n^j)^2} \Delta \tilde{u}^j \|_{L^{p+1}} = 0. \tag{8.55}
\]

Proof of Lemma 8.8. We see from the Gagliardo-Nirenberg inequality, Lemma 8.6 and Lemma 8.7 that

\[
\lim_{n \to \infty} \| g_n^j \sigma_n^j \frac{-i \lambda_n^j}{(\lambda_n^j)^2} \Delta \tilde{u}^j \|_{L^{p+1}} \lesssim \lim_{n \to \infty} \| \lambda_n^j \sigma_n^j \tilde{u}^j \|_{L^2}^{p+1 - \frac{d(p-1)}{2}} \| \nabla \sigma_n^j \tilde{u}^j \|_{L^2}^{\frac{d(p-1)}{2}} = 0. \tag{8.56}
\]

Thus, we have proved the lemma. \( \square \)

8.2.3 Nonlinear profiles

For each number \( j \geq 1 \), we define the “nonlinear profile” \( \tilde{\psi}^j \) (see Section 6 in [3] for the details) to be that if \( \lambda_j^\infty = \infty \), then \( \tilde{\psi}^j \equiv 0 \); and if \( \lambda_j^\infty \in \{0, 1\} \), then \( \tilde{\psi}^j \) is a solution to

\[
\tilde{\psi}^j(t) = e^{it \Delta}\tilde{u}^j + i \int_{-t \lambda_j^\infty \gamma}^t e^{i(t-t') \Delta} (\sigma_j^\infty)^{-1} \left\{ \left(\lambda_j^\infty\right)^{\frac{(d-2)\gamma^* - (p+1)}{2}} F[^4 \sigma_j^\infty \tilde{\psi}^j(t')] + F[^4 \sigma_j^\infty \tilde{\psi}^j(t')] \right\} dt'. \tag{8.57}
\]

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When \( -\frac{\ell_{n}}{(\lambda_{n})^{2}} \in \{ \pm \infty \} \), the equation (8.57) is interpreted as the final value problem at \( \pm \infty \). We simply write

\[ I_{j}^{\text{max}} := I_{\text{max}}(\tilde{\psi}^{j}), \quad T_{j}^{\text{max}} := T_{\text{max}}(\tilde{\psi}^{j}) \text{ and } T_{j}^{\text{min}} := T_{\text{min}}(\tilde{\psi}^{j}). \]

Moreover, for an interval \( I \) and each number \( j \geq 1 \), we define

\[ W^{j}(I) := \begin{cases} W_{p+1}(I) \cap W(I) & \text{if } \lambda_{j}^{\infty} \in \{1, \infty\}, \\ W(I) & \text{if } \lambda_{j}^{\infty} = 0. \end{cases} \] (8.58)

Our aim here is to show that there is at most one “bad” nonlinear profile; the other profiles are “good” in the sense (8.100) and (8.101) below.

We find from the construction of the nonlinear profiles that for any number \( j \),

- \( \tilde{\psi}^{j} \in C(I_{\text{max}}^{j}, H^{1}(\mathbb{R}^{d})) \);
- \( \sigma_{\infty}^{j} \tilde{\psi}^{j} \) satisfies
  \[
  \begin{cases}
  i\partial_{t}\sigma_{\infty}^{j} \psi + \Delta \sigma_{\infty}^{j} \psi + F[\sigma_{\infty}^{j} \psi] = 0 & \text{if } \lambda_{\infty}^{j} = 1, \\
  i\partial_{t}\sigma_{\infty}^{j} \psi + \Delta \sigma_{\infty}^{j} \psi + F\sigma_{\infty}^{j} \psi = 0 & \text{if } \lambda_{\infty}^{j} = 0;
  \end{cases}
  \]

(8.59)

- there exists a number \( N(j) \) such that for any \( n \geq N(j) \),

\[ -\frac{\ell_{n}^{j}}{(\lambda_{n})^{2}} \in I_{\text{max}}^{j} \] (8.60)

and

\[ \lim_{n \to \infty} \left\| \tilde{\psi}^{j} \left( -\frac{\ell_{n}^{j}}{(\lambda_{n}^{j})^{2}} \right) - e^{-\frac{\ell_{n}^{j}}{(\lambda_{n})^{2}} \Delta} \tilde{u}^{j} \right\|_{H^{1}} = 0; \] (8.61)

- if \( -\frac{\ell_{n}^{j}}{(\lambda_{n}^{j})^{2}} = \infty \), then \( T_{\text{max}}^{j} = \infty \) and for any \( T > T_{\text{min}}^{j} \),

\[ \| \sigma_{\infty}^{j} \tilde{\psi}^{j} \|_{W^{j}(\langle T, \infty \rangle)} < \infty; \] (8.62)

- if \( -\frac{\ell_{n}^{j}}{(\lambda_{n}^{j})^{2}} = -\infty \), then \( T_{\text{min}}^{j} = -\infty \) and for any \( T < T_{\text{max}}^{j} \),

\[ \| \sigma_{\infty}^{j} \tilde{\psi}^{j} \|_{W^{j}(\langle -\infty, T \rangle)} < \infty. \] (8.63)

It is convenient rewriting (8.61) in the form

\[
\begin{cases}
\lim_{n \to \infty} \left\| \tilde{\psi}^{j} \left( -\frac{\ell_{n}^{j}}{(\lambda_{n}^{j})^{2}} \right) - g_{n}^{j} \sigma_{\infty}^{j} e^{-\frac{\ell_{n}^{j}}{(\lambda_{n})^{2}} \Delta} \tilde{u}^{j} \right\|_{H^{1}} = 0 & \text{if } \lambda_{\infty}^{j} = 1, \\
\lim_{n \to \infty} \| \nabla g_{n}^{j} \sigma_{\infty}^{j} \tilde{\psi}^{j} \left( -\frac{\ell_{n}^{j}}{(\lambda_{n}^{j})^{2}} \right) - \sigma_{n}^{j} e^{-\frac{\ell_{n}^{j}}{(\lambda_{n})^{2}} \Delta} \tilde{u}^{j} \|_{L^{2}} = 0 & \text{if } \lambda_{\infty}^{j} = 0.
\end{cases}
\] (8.64)

Now, for each number \( j \), we define

\[ \psi_{n}^{j} := G_{n}^{j} \sigma_{n}^{j} \tilde{\psi}^{j}, \] (8.65)
and let \( I_{\text{max},n} \) be the maximal existence-interval of \( \psi_n^j \). Then, we see that

\[
\begin{aligned}
    i \frac{\partial \psi_n^j}{\partial t} + \Delta \psi_n^j &= \begin{cases} 
        -F(\psi_n^j) & \text{if } \lambda_n^j = 1, \\
        -(\sigma_n^j)^{-1} F^j[G_n^j \sigma_n^j \tilde{\psi}^j] & \text{if } \lambda_n^j = 0,
    \end{cases}
\end{aligned}
\tag{8.66}
\]

and

\[
I_{\text{max},n}^j := ((\lambda_n^j)^2 T_{\text{min}}^j + t_n^j, (\lambda_n^j)^2 T_{\text{max}}^j + t_n^j).
\tag{8.67}
\]

Furthermore, (8.60) shows that \( 0 \in I_{\text{max},n}^j \) for any \( n \geq N(j) \).

In what follows, we give properties of nonlinear profiles in a series of lemmas.

**Lemma 8.9.** Let \( j \) be a number for which \( \lambda_n^j = 0 \). Then, for any interval \( I \) satisfying

\[
\| \langle \nabla \rangle \tilde{\psi}^j \|_{S(I)} < \infty,
\tag{8.68}
\]

we have

\[
\lim_{n \to \infty} \| \langle \nabla \rangle \{ (\sigma_n^j)^{-1} F^j[G_n^j \sigma_n^j \tilde{\psi}^j] \} - \langle \nabla \rangle F^j[G_n^j \sigma_n^j \tilde{\psi}^j] \|_{L_{t,x}^{2(d+2)}(I_n^j)} = 0,
\tag{8.69}
\]

where \( I_n^j := ((\lambda_n^j)^2 \inf I + t_n^j, (\lambda_n^j)^2 \sup I + t_n^j) \).

**Proof of Lemma 8.9.** Since

\[
\lim_{n \to \infty} \lambda_n^j ((\lambda_n^j)^{-1} \nabla) F^j[\sigma_n^j \tilde{\psi}^j] = |\nabla| F^j[\sigma_n^j \tilde{\psi}^j] \quad \text{almost everywhere in } \mathbb{R}^d \times I,
\tag{8.70}
\]

Lebesgue’s convergence theorem implies that

\[
\begin{aligned}
    \lim_{n \to \infty} \| \langle \nabla \rangle (\sigma_n^j)^{-1} F^j[G_n^j \sigma_n^j \tilde{\psi}^j] - \langle \nabla \rangle F^j[G_n^j \sigma_n^j \tilde{\psi}^j] \|_{L_{t,x}^{2(d+2)}(I_n^j)} &= 0,
    \\
    &= \lim_{n \to \infty} \| \langle \nabla \rangle F^j[\sigma_n^j \tilde{\psi}^j] - \lambda_n^j ((\lambda_n^j)^{-1} \nabla) F^j[\sigma_n^j \tilde{\psi}^j] \|_{L_{t,x}^{2(d+2)}(I)} = 0.
\end{aligned}
\tag{8.71}
\]

On the other hand, we see from Lemma 8.7 \( \lambda_n^j = 0 \) and the hypothesis (8.68) that

\[
\lim_{n \to \infty} \| \langle \nabla \rangle F^j[G_n^j \sigma_n^j \tilde{\psi}^j] \|_{L_{t,x}^{2(d+2)}(I_n^j)} \lesssim \lim_{n \to \infty} \left( \lambda_n^j \right)^{(d-2)/2} \| \langle \nabla \rangle \tilde{\psi}^j \|_{S(I)}^p = 0.
\tag{8.72}
\]

Then, the desired result (8.69) follows from (8.71) and (8.72). \hfill \Box

**Lemma 8.10.** Let \( j \) and \( n \) be numbers, and let \( I \) be an interval on which \( \tilde{\psi}^j \) exists. Then, we have

\[
\| \langle \nabla \rangle \psi_n^j \|_{S(I_n^j)} \lesssim \| \langle \nabla \rangle \tilde{\psi}^j \|_{S(I)},
\tag{8.73}
\]

where \( I_n^j := ((\lambda_n^j)^2 \inf I + t_n^j, (\lambda_n^j)^2 \sup I + t_n^j) \).

**Proof of Lemma 8.10.** Since

\[
\begin{aligned}
    \| \langle \nabla \rangle \psi_n^j \|_{L_t^p L_x^q(I_n^j)} &\leq \| \lambda_n^j \sigma_n^j \tilde{\psi}^j \|_{L_t^p L_x^q(I)} + \| \nabla \sigma_n^j \tilde{\psi}^j \|_{L_t^p L_x^q(I)},
    \\
    \| \langle \nabla \rangle \psi_n^j \|_{L_t^p L_x^q(I_n^j)} &\leq \| \lambda_n^j \sigma_n^j \tilde{\psi}^j \|_{L_t^p L_x^q(I)} + \| \nabla \sigma_n^j \tilde{\psi}^j \|_{L_t^p L_x^q(I)}.
\end{aligned}
\tag{8.74}
\]

the claim follows from Lemma 8.7. \hfill \Box
Lemma 8.11 (cf. Lemma 6.8 in \[3\]). There exists a number $J_0$ such that $I_{\text{max}}^j = \mathbb{R}$ for any $j > J_0$; and for any $r \geq 2$,

$$\sum_{j > J_0} \| (\nabla ) \tilde{\psi}^j \|_{S(R)} \lesssim \sum_{j > J_0} \| \tilde{u}^j \|_{H^1} < \infty. \quad (8.76)$$

Proof of Lemma 8.11. It follows from (8.43) and the uniform boundedness \[8.33\] that

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} \| g_n^j \sigma_n^j e^{-\frac{i}{(\lambda_n)^2} \Delta} \tilde{u}^j \|_{L^2} = \lim_{n \to \infty} \| \nabla \sigma_n^j \tilde{u}^j \|_{L^2} = \| \tilde{u}^j \|_{H^1}. \quad (8.78)$$

Moreover, let $j$ be a number for which $\lambda_n^j = 1$. Then, we have that for any number $n$,

$$\| g_n^j \sigma_n^j e^{-\frac{i}{(\lambda_n)^2} \Delta} \tilde{u}^j \|_{H^1} = \| \tilde{u}^j \|_{H^1}. \quad (8.79)$$

Putting \[8.77\], \[8.78\] and \[8.79\] together, we obtain that

$$\sum_{j=1}^{\infty} \| \tilde{u}^j \|_{H^1}^2 \lesssim 1. \quad (8.80)$$

In particular, we have

$$\lim_{j \to \infty} \| \tilde{u}^j \|_{H^1} = 0. \quad (8.81)$$

The small-data theory (Lemma B.1 in the appendix) together with (8.81) implies that there exists a number $J_0$ such that for any $j \geq J_0$,

$$\| (\nabla ) \tilde{\psi}^j \|_{S(R)} \lesssim \| \tilde{u}^j \|_{H^1}. \quad (8.82)$$

Then, the claim (8.76) follows from (8.80) and (8.82).

Lemma 8.12. For any number $k$, there exists a number $N(k)$ such that for any $n \geq N(k)$,

$$\sum_{j=1}^{k} \mathcal{I}(g_n^j \sigma_n^j e^{-\frac{i}{(\lambda_n)^2} \Delta} \tilde{u}^j) + \mathcal{I}(w_n^k) \leq m_{\omega} - \frac{\kappa_1(R_\ast)}{10d(p-1)} \quad (8.83)$$

Proof of Lemma 8.12. It follows from (8.27), (8.32) and $\mathcal{S}_\omega(\psi_n) \leq m_{\omega} + \varepsilon_n$ that for any numbers $n$ and $k$,

$$m_{\omega} - \frac{\kappa_1(R_\ast)}{2d(p-1)} \geq m_{\omega} + \varepsilon_n - \frac{2}{d(p-1)} K(\psi_n(0)) \geq \mathcal{S}_\omega(\psi_n) - \frac{2}{d(p-1)} K(\psi_n(0)) = \mathcal{I}(\psi_n(0)). \quad (8.84)$$

This together with (8.46) gives us the desired result (8.83).
Lemma 8.13. For any number \( k \), there exists a number \( N(k) \) such that for any \( n \geq N(k) \) and any \( j \leq k \) for which the linear profile \( \tilde{u}^j \) is non-trivial,
\[
\mathcal{H}(g_n^j \sigma_n e^{-i \frac{t}{(\lambda_n)^2} \Delta } \tilde{u}^j) > \frac{1}{2} \mathcal{K}(g_n^j \sigma_n e^{-i \frac{t}{(\lambda_n)^2} \Delta } \tilde{u}^j) > 0, \tag{8.85}
\]
\[
\mathcal{H}(w_n^k) \geq \frac{1}{2} \mathcal{K}(w_n^k) \geq 0. \tag{8.86}
\]

Proof of Lemma 8.13. Since \( \mathcal{I}_\omega \) is non-negative, Lemma 8.12 together with (1.9) shows that there exists a number \( N(k) \) such that for any \( n \geq N(k) \) and any \( j \leq k \) for which \( \tilde{u}^j \) is non-trivial,
\[
\mathcal{K}(g_n^j \sigma_n e^{-i \frac{t}{(\lambda_n)^2} \Delta } \tilde{u}^j) > 0, \quad \mathcal{K}(w_n^k) \geq 0. \tag{8.87}
\]
This together with (1.6) gives us the desired result. \( \square \)

Lemma 8.14. There is at most one number \( j_0 \) such that
\[
\left\{ \begin{array}{ll}
\mathcal{S}_\omega(\sigma_{\infty} \tilde{\psi}^{j_0}) \geq m_\omega & \text{if } \lambda_{\infty} = 1, \\
\mathcal{H}^\dagger(\sigma_{\infty} \tilde{\psi}^{j_0}) \geq \mathcal{H}^\dagger(W) & \text{if } \lambda_{\infty} = 0.
\end{array} \right. \tag{8.88}
\]

Proof of Lemma 8.14. First, we consider a number \( k \) such that
\[
\mathcal{S}_\omega(g_n^k \sigma_n^k e^{-i \frac{t}{(\lambda_n)^2} \Delta } \tilde{u}^k) \leq \frac{2}{3} m_\omega. \tag{8.89}
\]
Then, it follows from (8.64) and (1.20) that for any sufficiently large number \( n \),
\[
\left\{ \begin{array}{ll}
\mathcal{S}_\omega(\sigma_{\infty} \tilde{\psi}^k) = \mathcal{S}_\omega(\sigma_{\infty} \tilde{\psi}^k( - \frac{t}{(\lambda_n)^2})) < m_\omega & \text{if } \lambda_{\infty} = 1, \\
\mathcal{H}^\dagger(\sigma_{\infty} \tilde{\psi}^k) = \mathcal{H}^\dagger(\sigma_{\infty} \tilde{\psi}^k( - \frac{t}{(\lambda_n)^2})) < \mathcal{H}^\dagger(W) & \text{if } \lambda_{\infty} = 0.
\end{array} \right. \tag{8.90}
\]
Thus, it suffices for the desired result to show that there is at most one number \( j_0 \) such that for any sufficiently large \( n \),
\[
\mathcal{S}_\omega(g_n^{j_0} \sigma_n^{j_0} e^{-i \frac{t}{(\lambda_n)^2} \Delta } \tilde{u}^{j_0}) \geq \frac{2}{3} m_\omega. \tag{8.91}
\]
We see from Lemma 8.13 that for any \( j \geq 1 \) and any sufficiently large \( n \),
\[
\mathcal{S}_\omega(g_n^j \sigma_n \tilde{u}^j) \geq 0. \tag{8.92}
\]
Hence, it follows from (8.45) and \( \mathcal{S}_\omega(\psi_n(0)) < m_\omega + \varepsilon_n \) that there are at most one linear profile satisfying (8.91). \( \square \)

Now, using Lemma 8.14 and reordering the indices, we may assume that for any \( j \geq 2 \),
\[
\left\{ \begin{array}{ll}
\mathcal{S}_\omega(\sigma_{\infty} \tilde{\psi}^j) < m_\omega & \text{if } \lambda_{\infty} = 1, \\
\mathcal{H}^\dagger(\sigma_{\infty} \tilde{\psi}^j) < \mathcal{H}^\dagger(W) & \text{if } \lambda_{\infty} = 0.
\end{array} \right. \tag{8.93}
\]
Lemma 8.15. Assume (8.93). Let \( j \geq 1 \) be a number for which the nonlinear profile \( \tilde{\psi}^j \) is non-trivial. Then, we can take a number \( N(\varepsilon, j) \) such that for any \( n \geq N(\varepsilon, j) \),

\[
\begin{cases}
K(\sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right)) > 0 & \text{if } \lambda^j_\infty = 1, \\
K^+(\sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right)) > 0 & \text{if } \lambda^j_\infty = 0.
\end{cases}
\]

(8.94)

In addition, if \( j \geq 2 \) and \( \lambda^j_\infty = 0 \), then for any \( n \geq N(\varepsilon, j) \),

\[
\left\| \nabla \sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right) \right\|_{L^2}^2 < \| \nabla W \|_{L^2}^2.
\]

(8.95)

Proof of Lemma 8.15. We see from (8.64) and Lemma 8.12 that there exists a number \( N(j) \) such that for any \( n \geq N(j) \),

\[
\begin{align*}
I_\omega(\sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right)) & \leq m_\omega - \kappa_1(R_\ast) \frac{100d(p-1)}{d} & \text{if } \lambda^j_\infty = 1, \\
I^+_\omega(\sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right)) & \leq m_\omega - \kappa_1(R_\ast) \frac{100d(p-1)}{d} & \text{if } \lambda^j_\infty = 0.
\end{align*}
\]

(8.96)

(8.97)

When \( \lambda^j_\infty = 1 \), (1.9) together with (8.96) implies the desired result. Assume that \( \lambda^j_\infty = 0 \). Then, (1.25) together with (1.26) and (8.97) shows that for any number \( n \geq N(j) \),

\[
K^+(\sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right)) > 0.
\]

(8.98)

Thus, we find that (8.94) holds. Furthermore, using (8.98), we have that

\[
\frac{1}{d} \left\| \nabla \sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right) \right\|_{L^2}^2 < \mathcal{H}^+(\sigma^j_\infty \tilde{\psi}^j \left( - \frac{t_n}{(\lambda^j_n)^2} \right)).
\]

(8.99)

Thus, (8.95) follows from (8.93) and (1.25). \( \square \)

We see from Lemma 8.15 and (8.93) that for any \( j \geq 2 \) for which \( \tilde{\psi}^j \) is non-trivial,

\[
\begin{cases}
\sigma^j_\infty \tilde{\psi}^j \in PW_{\omega,+} & \text{if } \lambda^j_\infty = 1, \\
\sigma^j_\infty \tilde{\psi}^j \in PW^+_+ & \text{if } \lambda^j_\infty = 0.
\end{cases}
\]

(8.100)

Furthermore, it follows from Theorem 1.3 and Lemma 8.11 that \( P_{\max}^j = \mathbb{R} \) for any \( j \geq 2 \), and

\[
\sup_{j \geq 2} \left\| \langle \nabla \rangle \tilde{\psi}^j \right\|_{L^2(\mathbb{R})} < \infty.
\]

(8.101)
8.2.4 Existence of bad profile

In the previous section, we have shown that there are at most one bad profile, and the candidate is $\tilde{\psi}^1$. Our main aim is to show that for any $T \in (T_{\min}^1, T_{\min}^1)$,

$$\| \sigma_1^1 \tilde{\psi}^1 \|_{W^1(T, T_{\max}^1)} = \infty. \quad (8.102)$$

Here, $W^1$ is the function space defined by (8.58). See Lemma 8.22 below for the full statement. Thus, $\tilde{\psi}^1$ is the bad profile. To this end, we shall observe properties of $\tilde{\psi}^1$ on an interval where

$$\| \sigma_1^1 \tilde{\psi}^1 \|_{W^1(I)} < \infty. \quad (8.103)$$

**Lemma 8.16.** Assume that $\lambda_1^1 \neq \infty$ Then, for any interval $I$ satisfying (8.103), we can take a constant $A(I) > 0$ such that

$$\| \langle \nabla \rangle \tilde{\psi}^1 \|_{L^1(I)} \leq A(I). \quad (8.104)$$

**Proof of Lemma 8.16.** When $I$ is a compact interval in $I_{\max}^1$, the claim follows from the well-posedness theory. Thus, we may assume that $\sup I = T_{\max}^1$ or $\inf I = T_{\min}^1$. We only consider the case where $\sup I = T_{\max}^1$ and $T_{\min}^1 < \inf I$. The same proof is applicable for the other cases. Since $\sup I = T_{\max}^1$, the blowup criterion (cf. Theorem 4.1 in [3] and Lemma 2.11 in [15]) together with (8.103) shows that $T_{\max}^1 = +\infty$ and

$$\| \sigma_1^1 \tilde{\psi}^1 \|_{W^1([\inf I, \infty))} < \infty. \quad (8.105)$$

We shall show that

$$\sup_{t \in [\inf I, \infty)} \| \tilde{\psi}^1(t) \|_{H^1} < \infty. \quad (8.106)$$

Suppose for contradiction that (8.106) was false. Then, we could take a sequence $\{t_n\}$ in $[\inf I, \infty)$ such that $\lim_{n \to \infty} t_n = \infty$ and

$$\lim_{n \to \infty} \| \tilde{\psi}^1(t_n) \|_{H^1} = \infty. \quad (8.107)$$

If $\lambda_1^1 = 1$, then $\tilde{\psi}^1$ is a solution to (NLS), and Strichartz’ estimate shows that for any $t_m < t_n$,

$$\| e^{-it_m \Delta} \tilde{\psi}^1(t_m) - e^{-it_n \Delta} \tilde{\psi}^1(t_n) \|_{H^1} = \sup_{t \in [t_m, t_n]} \left\| \int_{t_m}^{t_n} e^{-it \Delta} F(\tilde{\psi}^1(t)) \, dt \right\|_{H^1} \quad (8.108)$$

$$\lesssim \| \tilde{\psi}^1 \|_{W_{p+1}^1([t_m, \infty))} \| \langle \nabla \rangle \tilde{\psi}^1 \|_{W_{p+1}^1([t_m, \infty))} + \| \tilde{\psi}^1 \|_{W(\{t_m, \infty\})} \| \langle \nabla \rangle \tilde{\psi}^1 \|_{V([t_m, \infty))}.$$ 

Moreover, we see from (8.105) and $\lambda_1^1 = 1$ that

$$\lim_{T \to \infty} \| \tilde{\psi}^1 \|_{W_{p+1}^1([T, \infty)) \cap W([T, \infty))} = 0. \quad (8.109)$$

In particular, we have that for any $\delta > 0$, there exists $T(\delta) > 0$ such that

$$\| \tilde{\psi}^1 \|_{W_{p+1}^1([T(\delta), \infty)) \cap W([T(\delta), \infty))} \leq \delta. \quad (8.110)$$
An estimate similar to (8.108) together with (8.110) also yields that
\[
\| (\nabla) \tilde{\psi}^1 \|_{V_{p+1}([T(\delta),\infty))} < \|
\| \nabla \tilde{\psi}^1 \|_{V_{p+1}([T(\delta),\infty))} \leq \|
\| \nabla \tilde{\psi}^1 \|_{V_{p+1}([T(\delta),\infty))}
\]
(8.111)
Thus, we can take \( \delta_0 > 0 \) such that
\[
\| (\nabla) \tilde{\psi}^1 \|_{V_{p+1}([T(\delta),\infty))} \leq \| \tilde{\psi}^1 \|_{T(\delta)} < 1.
\]
(8.112)
Combining (8.108) with (8.109) and (8.112), we find that \( \{ e^{-it_n \Delta \tilde{\psi}^1(t_n)} \} \) is a Cauchy sequence in \( H^1(\mathbb{R}^d) \). Similarly, when \( \lambda_\infty^1 = 0 \), we can verify that \( \{ e^{-it_n \Delta \tilde{\psi}(t_n)} \} \) is a Cauchy sequence in \( H^1(\mathbb{R}^d) \). Thus, in both cases \( \lambda_\infty^1 = 1 \) and \( \lambda_\infty^1 = 0 \), we can take \( \phi_+ \in H^1(\mathbb{R}^d) \) such that
\[
\lim_{n \to \infty} \| \tilde{\psi}^1(t_n) \|_{H^1} = \| \phi_+ \|_{H^1} < \infty.
\]
(8.113)
However, this contradicts (8.107). Hence, we have proved (8.106).

Now, we are able to show (8.104). If \( \lambda_\infty^1 = 1 \), then \( e^{[\tilde{\psi}^1]} \equiv 0 \) (see (1.35) for the definition of \( e^{[\tilde{\psi}^1]} \)), and Lemma B.2 together with (8.105) and (8.106) gives us the desired estimate (8.104). If \( \lambda_\infty^1 = 0 \), then \( e^{[\xi]} \equiv 0 \). Moreover, it follows from (8.105) and (8.106) that
\[
\| \sigma_\infty^1 \tilde{\psi}^1 \|_{W([\inf I, \infty))} < \infty, \quad \sup_{t \in [\inf I, \infty)} \| \nabla \sigma_\infty^1 \tilde{\psi}^1 \|_{L^2} < \infty.
\]
(8.114)
Hence, we see from Lemma B.2 that
\[
\| (\nabla) \tilde{\psi}^1 \|_{St([\inf I, \infty))} = \| \nabla \sigma_\infty^1 \tilde{\psi}^1 \|_{St([\inf I, \infty))} < \infty,
\]
(8.115)
which completes the proof.

Since the estimate (8.38) is insufficient to control the remainder \( e^{it \Delta w_n^j} \) in the Strichartz space \( (\nabla)^{-1} St(\mathbb{R}) \), we need the following estimate:

**Lemma 8.17.** Assume (8.92), and let \( q \) denote \( p + 1 \) or \( 2^* \). Then, for any \( \delta > 0 \), any interval \( I \) satisfying (8.103), and any number \( j \geq 1 \), there exists a number \( K(\delta, I, j) \) with the following property: for any \( k \geq K(\delta, I, j) \), there exists a number \( N(\delta, I, j, k) \) such that for any \( n \geq N(\delta, I, j, k) \) and \( s \in \{0, 1\} \),
\[
\| \psi_n^j \|_{\| (\nabla)^s e^{it \Delta w_n^k} \|_{L_{t,x}^{2(d+2)(q-2)\frac{2(d+2)(q-2)}{(q-2)+4}}(I_n)}} \leq \delta,
\]
(8.116)
where \( I_n := ((\lambda_n^1)^2 \inf I + t_n^1, (\lambda_n^1)^2 \sup I + t_n^1) \).

**Proof of Lemma 8.17.** Let \( I \) be an interval where (8.103) holds. We use notation \( I^1 := I \) and \( I^j := \mathbb{R} \) for \( j \geq 2 \). Then, it follows from Lemma 8.10 and (8.101) that there exists a constant \( A(I) > 0 \) such that
\[
\sup_{j \geq 1} \| (\nabla) \tilde{\psi}^j \|_{St(I^j)} \leq A(I).
\]
(8.117)
For each number \( j \geq 1 \), we can take a sequence \( \{v_m^{j}\}_{m \geq 1} \) of smooth functions on \( \mathbb{R}^d \times \mathbb{R} \) with the following properties: for each \( m \geq 1 \), there exists \( T_m^I \) and \( R_m^I > 0 \) such that the support of \( v_m^{j} \) is contained in \( K_m^I := \{(x, t) : |x| \leq R_m^I, |t| \leq T_m^I \} \) and
\[
\lim_{m \to \infty} \| \langle \nabla \rangle (\tilde{\psi}^{j} - v_m^{j} ) \|_{W_{2s}(I) \cap V(I)} = 0. \tag{8.118}
\]
Here, since \( \lambda_{\infty}^j = \infty \) implies \( \tilde{\psi}^{j} \equiv 0 \), we may assume that \( \lambda_{\infty}^j \in \{0,1\} \) for all \( j \geq 1 \). Furthermore, for each \( j \geq 1 \), we can take a number \( N_1(j) \) such that for any number \( m \geq 1 \) and any \( n \geq N_1(j) \),
\[
\{ (x, t) : |x| \leq \lambda_n^j R_m^I, |t| \leq (\lambda_n^j)^2 T_m^I \} \subset K_m^I. \tag{8.119}
\]
We see from (8.117) and (8.118) that for each \( j \geq 1 \), there exists a number \( M_1(j) \) such that for any number \( m \geq 1 \),
\[
\| \langle \nabla \rangle v_m^{j} \|_{W_{2s}(I) \cap V(I)} \leq A(I) + 1. \tag{8.120}
\]
Moreover, it follows from (8.118) that for any \( \delta > 0 \) and any number \( j \geq 1 \), there exists a number \( M_2(\delta, I, j) \) such that for any \( m \geq M_2(\delta, I, j) \),
\[
\| \langle \nabla \rangle (\tilde{\psi}^{j} - v_m^{j} ) \|_{V_q(I)} \leq \min \left\{ \delta, \frac{\delta}{(A(I) + 1)^{q-3}} \right\}. \tag{8.121}
\]
Put \( m_0 := \max \{ M_1(j), M_2(\delta, I, j) \} \). Then, we see from Hölder’s inequality and Sobolev’s embedding that for any numbers \( j, k \) and \( n \), and \( s \in \{0,1\} \),
\[
\| \psi_n^{j} |\nabla|^s e^{it\Delta} u_n^k \|_{L_{t,x}^{2(d+2)(q-2)+4}(I_n)} \leq (\lambda_n^j)^{2-s-q-1}\|\sigma_n^{j}\tilde{\psi}^{j}(G_n^j)^{-1}|\nabla|^s e^{it\Delta} u_n^k \|_{L_{t,x}^{2(d+2)(q-2)+4}(I)} \tag{8.122}
\]
\[
\leq (\lambda_n^j)^{1-s-q-1}\|\tilde{\psi}^{j} - v_m^{j} \|_{W_q(I)} \|\nabla|^s e^{it\Delta} u_n^k \|_{W_{2s}(R)} + (\lambda_n^j)^{2-s-q-1}\|\sigma_n^{j} v_m^{j} (G_n^j)^{-1}|\nabla|^s e^{it\Delta} u_n^k \|_{L_{t,x}^{2(d+2)(q-2)+4}(I)}.
\]
Furthermore, we see from (8.47), Lemma 8.7 and (8.121) that if \( n \) is sufficiently large depending on \( k \), then the first term on the right-hand side of (8.122) is estimated as follows:
\[
(\lambda_n^j)^{1-s-q-1}\|\tilde{\psi}^{j} - v_m^{j} \|_{W_q(I)} \|\nabla|^s e^{it\Delta} u_n^k \|_{W_{2s}(R)} \approx \| \langle \nabla \rangle (\tilde{\psi}^{j} - v_m^{j} ) \|_{V_q(I)} \leq \delta. \tag{8.123}
\]
We consider the second term on the right-hand side of (8.122). When \( s = 0 \), we see from Hölder’s inequality, Lemma 8.7 (8.120), (8.47) and (8.38) that there exists a number \( K_1(\delta, I) \) with the following property: for any \( k \geq K_1(\delta, I) \), there exists a number
\( N_3(\delta, I, k) \) such that for any \( j \geq 1 \) and any \( n \geq N_3(\delta, I, k) \),

\[
(\lambda^j_n)^{2-s_q-1} \| \sigma^j_n v^j_{m_0} (G^j_n)^{-1} e^{it \Delta} u^k_n \|_{L_{t,x}^{\frac{2(d+2)(q-2)}{(d+2)(q-2)+4}}} (I)
\leq \lambda^j_n \| \sigma^j_n v^j_{m_0} \|_{W_2(\Omega)} \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})} \lesssim \| \nabla \sigma^j_n v^j_{m_0} \|_{W_2(\Omega)} \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})}
\lesssim (A(I) + 1) \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})}^{s_q-1} \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})} \leq (A(I) + 1) \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})}^{s_q-1} \leq \delta.
\] (8.124)

On the other hand, when \( s = 1 \), we see from \([8, 119]\), Hölder’s inequality, Sobolev’s embedding, Lemma \([8, 120]\, \([8, 47]\, \text{Lemma 2.5 of} \, [17] \, \text{and} \, (8.38) \) that there exists a number \( K_2(\delta, I, j) \) with the following property: for any \( k \geq K_2(\delta, I, j) \), there exists a number \( N_4(\delta, I, j, k) \) such that for any \( n \geq \max \{ N_1(j), N_4(\delta, I, j, k) \} \),

\[
(\lambda^j_n)^{2-s_q-1} \| \sigma^j_n v^j_{m_0} (G^j_n)^{-1} \nabla e^{it \Delta} u^k_n \|_{L_{t,x}^{\frac{2(d+2)(q-2)}{(d+2)(q-2)+4}}} (I)
\lesssim \| \nabla \sigma^j_n v^j_{m_0} \|_{V(I)} \| \nabla e^{it \Delta} u^k_n \|_{L^{\frac{2(d+2)(q-2)}{q}}(\Omega)}
\lesssim (A(I) + 1) \| \nabla e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})} \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})}^{s_q-1} \| e^{it \Delta} u^k_n \|_{W_q(\mathbb{R})} \leq \delta.
\] (8.125)

Thus, Putting the above estimates together, we obtain the desired result.

Now, for a number \( k \), we define an approximate solution \( \psi_n^{k-app} \) by

\[
\psi_n^{k-app}(t) := \sum_{j=1}^{k} \psi_n^j(t) + e^{it \Delta} u^k_n.
\] (8.126)

Assume \([8.93]\). Then, we have \( I_{max}^j = \mathbb{R} \) for all \( j \geq 2 \), which together with \([8.67]\) shows that \( I_{max,n}^j = \mathbb{R} \) for all \( j \geq 2 \). Hence, for any numbers \( k \) and \( n \), the maximal existence-interval of \( \psi_n^{k-app} \) is \( I_{max,n}^j := (\lambda_n^{j-1}, \lambda_n^{j+1}) \). Furthermore, we see from \([8.69]\) that \( 0 \in I_{max,n}^j \) for any sufficiently large number \( n \).

**Lemma 8.18.** Assume \([8.88]\), and let \( q \) denote \( p + 1 \) or \( 2^* \). Then, for any numbers \( j_0 \) and \( k \) with \( 2 \leq j_0 < k \), and any \( \delta > 0 \), there exists a number \( N(j_0, k, \delta) \) such that for any \( n \geq N(j_0, k, \delta) \),

\[
\| \sum_{j=j_0}^{k} \psi_n^j \|_{W_q(\mathbb{R})} \leq \sum_{j=j_0}^{k} \| \psi_n^j \|_{W_q(\mathbb{R})}^2 + \delta,
\] (8.127)

\[
\| \sum_{j=j_0}^{k} \psi_n^j \|_{L_t^\infty H_x^1(\mathbb{R})} \lesssim \sum_{j=j_0}^{k} \| \psi_n^j \|_{L_t^\infty H_x^1(\mathbb{R})}^2 + \delta.
\] (8.128)
Proof of Lemma 8.18  Let \( j_0 \) and \( k \) be numbers with \( 2 \leq j_0 < k \). Then, an elementary calculation shows that there exists a constant \( C(j_0, k) \) such that for any number \( n \),

\[
\| \sum_{j=j_0}^{k} \psi_n^{j} \|_{W_q^2(\mathbb{R})}^{(d+2)(q-2)} \leq \sum_{j=j_0}^{k} \| \psi_n^{j} \|_{W_q^2(\mathbb{R})}^{(d+2)(q-2)} + C(j_0, k) \sum_{j=2}^{k} \sum_{2 \leq j' \leq k} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi_n^{j'}|^{(d+2)(q-2)} \frac{1}{2} |\psi_n^{j'}| \cdot \| \psi_n^{j'} \|_{L^1(\mathbb{R})}^{-1}. \tag{8.129}
\]

Furthermore, the orthogonality \( \| \psi_n^{j} \|_{L^2(\mathbb{R})} \) shows that for any \( \delta > 0 \), we can take a number \( N(j_0, k, \delta) \) such that for any distinct numbers \( 2 \leq j, j' \leq k \) and any \( n \geq N(j_0, k, \delta) \),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi_n^{j}(x, t)|^{(d+2)(q-2)} \frac{1}{2} |\psi_n^{j}(x, t)| \cdot dxdt \leq \frac{\delta}{C(j_0, k)k^2}. \tag{8.130}
\]

Putting \( 8.129 \) and \( 8.130 \) together, we obtain the desired result \( 8.127 \). Similarly, we can prove \( 8.128 \).

Lemma 8.19. Assume \( 8.30 \). Then, for any interval \( I \) satisfying \( 8.103 \), we can take a constant \( B(I) \) with the following property: for any number \( k \), there exists a number \( N(k) \) such that for any \( n \geq N(k) \),

\[
\| \psi_n^{k} \|_{W_{p+1}(I_n) \cap W(I_n)} + \int \langle \nabla \rangle \psi_n^{k} \|_{L^2(I_n) \cap L^\infty_t L^2_x(I_n)} \leq B(I), \tag{8.131}
\]

where \( I_n := ((\lambda_n^1)^2 \inf I + t_n^1, (\lambda_n^1)^2 \sup I + t_n^1) \).

Proof of Lemma 8.19  We consider the first term on the right-hand side of \( 8.131 \). Let \( q \) denote \( p + 1 \) or \( 2^* \), and let \( J_0 \) be the number found in Lemma 8.11. Then, it follows from Lemma 8.10, Lemma 8.18 with \( \delta = 1 \), Lemma 8.16, 8.101, and Lemma 8.11 that for any number \( k \), there exists a number \( N(k) \) such that for any \( n \geq N(k) \),

\[
\| \psi_n^{k} \|_{W_{p+1}(I_n) \cap W(I_n)} \leq \| \psi_n^{1} \|_{W_{q}^2(I_n)} + \sum_{j=2}^{k} \| \psi_n^{j} \|_{W_q^2(\mathbb{R})} + 1 + \| e^{itI} W_n^{k} \|_{W_q^2(\mathbb{R})} \tag{8.132}
\]

\[
\leq \| \langle \nabla \rangle \tilde{\psi}_n^{1} \|_{W_q^2(I_n)} + \sum_{j=2}^{J_0} \| \langle \nabla \rangle \tilde{\psi}_n^{j} \|_{W_q^2(\mathbb{R})} + \sum_{j=J_0}^{\infty} \| \langle \nabla \rangle \tilde{\psi}_n^{j} \|_{W_q^2(\mathbb{R})} + 1 + \| e^{itI} W_n^{k} \|_{W_q^2(\mathbb{R})} \tag{8.132}
\]

\[
\leq A(I) \frac{(d+2)(q-2)}{2} + J_0 + 1.
\]

Thus, we have obtained the desired estimate. We can deal with the second term in a similar way. \( \square \)
Lemma 8.20. Assume \( \text{(8.93)} \), and let \( q \) denote \( p+1 \) or \( 2^* \). Then, for any \( \delta > 0 \) and any interval \( I \) satisfying \( \text{(8.103)} \), we can take a number \( K(\delta, I) \) with the following property: for any \( k \geq K(\delta, I) \), there exists a number \( N(\delta, I, k) \) such that for any \( n \geq N(\delta, I, k) \) and \( s \in \{0,1\}, \)
\[
\| \psi_{n}^{\text{k-app}} |q-2| \nabla \odot e^{it\Delta} u_{n}^{k} \|_{L_{t,x}^{2(2q-2)+1}(R)} \leq \delta, \tag{8.133}
\]
where \( I_{n} := ((\lambda_{n})^{2} \inf I + t_{n}^{1}, (\lambda_{n})^{2} \sup I + t_{n}^{1}) \).

Proof of Lemma 8.20. Let us begin with some preparation. We see from Lemma 8.10 and Lemma 8.11 that there exists a number \( K_{1}(\delta) \) such that
\[
\sum_{j = K_{1}(\delta)}^{\infty} \| \psi_{j,n}^{d+2(2q-2)} \|_{W_{q}(R)} \lesssim \sum_{j = K_{1}(\delta)}^{\infty} \| \langle \nabla \rangle \psi_{j}^{d} \|_{L_{t}^{2}(R)} \leq \delta. \tag{8.134}
\]
This together with Lemma 8.12 shows that for any \( k > K_{1}(\delta) \), there exists a number \( N_{1}(\delta, k) \) such that for any \( n \geq N_{1}(\delta, k) \),
\[
\| \sum_{j = K_{1}(\delta)+1}^{k} \psi_{j,n}^{d+2(2q-2)} \|_{W_{q}(R)} \lesssim \sum_{j = K_{1}(\delta)+1}^{k} \| \psi_{j,n}^{d+2(q-2)} \|_{W_{q}(R)} + \delta \lesssim \delta. \tag{8.135}
\]
Furthermore, we see from Hölder’s inequality, \( \text{(8.37)} \) and \( \text{(8.38)} \) that there exists a number \( K_{2}(\delta) \) with the following property: for any \( k \geq K_{2}(\delta) \), there exists a number \( N_{2}(\delta, k) \) such that for any \( n \geq N_{2}(\delta, k) \) and \( s \in \{0,1\}, \)
\[
\| e^{it\Delta} u_{n}^{k} \|_{L_{t,x}^{2(2q-2)+1}(R)} \lesssim \| e^{it\Delta} u_{n}^{k} \|_{W_{2,q}(R)} \lesssim \| \nabla \odot e^{it\Delta} u_{n}^{k} \|_{W_{2,q}(R)} \leq \delta, \tag{8.136}
\]
Now, we shall prove \( \text{(8.133)} \). First, we consider the case where \( q \leq 3 \). It follows from Hölder’s inequality, \( \text{(8.47)} \), \( \text{(8.135)} \) and \( \text{(8.136)} \) that for any \( k > \max\{K_{1}(\delta), K_{2}(\delta)\} \), any
\[ n \geq \max\{N_1(\delta, k), N_2(\delta, k)\} \quad \text{and} \quad s \in \{0, 1\}, \]

\[
\begin{align*}
&\|\psi_n^{k\text{-app}}\|_{L^{q-2}(\mathbb{R})}^2 \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \leq \|\psi_n^{k\text{-app}}\|_{L^{q-2}(\mathbb{R})}^2 \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \\
&\quad + \sum_{j=K_1(\delta)+1}^{K_1(\delta)} \|\psi_j^{k\text{-app}}\|_{W_q(\mathbb{R})} \|e^{it\Delta} u_n^k\|_{W_{q+4}(\mathbb{R})} \\
&\quad + \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \\
&\quad \lesssim C(\delta) K_1(\delta) \sup_{1 \leq j \leq K_1(\delta)} \|\psi_j^{k\text{-app}}\|_{L^{q+4}_t(\mathbb{R})} \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} + \delta,
\end{align*}
\]

where \(C(\delta)\) is some constant depending on \(d, q\) and \(\delta\). Moreover, Lemma 8.17 shows that we can take a number \(K_3(\delta, I)\) with the following property: for any \(k \geq K_3(\delta, I)\), there exists a number \(N_3(\delta, I, k)\) such that for any \(1 \leq j \leq K_1(\delta)\), any \(n \geq N_3(\delta, I, k)\) and \(s \in \{0, 1\},\)

\[
\|\psi_j^{k\text{-app}}\|_{L^{q+4}_t(\mathbb{R})} \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \leq \frac{\delta}{C(\delta) K_1(\delta)}. \tag{8.138}
\]

Putting (8.137) and (8.138) together, we obtain the desired result in the case \(q \leq 3\).

Next, we consider the case where \(q > 3\). In this case, we see from Hölder’s inequality and Lemma 8.19 that there exists a number \(B(I) > 0\) with the following property: for any number \(k\), there exists a number \(N_4(k)\) such that for any \(n \geq N_4(k)\) and \(s \in \{0, 1\},\)

\[
\begin{align*}
&\|\psi_n^{k\text{-app}}\|_{L^{q-2}(\mathbb{R})}^2 \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \leq \|\psi_n^{k\text{-app}}\|_{L^{q-2}(\mathbb{R})}^2 \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \\
&\quad \leq \|\psi_n^{k\text{-app}}\|_{L^{q-2}(\mathbb{R})}^2 \|e^{it\Delta} u_n^k\|_{L^{q+4}_t(\mathbb{R})} \frac{B(I)}{q-3}.
\end{align*}
\]

Then, we can obtain the desired estimate (8.139) in a way similar to the case \(q \leq 3\). \(\square\)

The following lemma enables us to control the error term of approximate solution:

**Lemma 8.21.** Assume (8.93). Then, for any \(\delta \in (0, 1)\) and any interval \(I\) satisfying (8.103), we can take numbers \(k_0\) (depending on \(\delta\) and \(I\)) and \(N(\delta, I)\) such that for any
n ≥ N(δ, I),
\[
\| \langle \nabla \rangle e[\psi_n^{k_0\text{-app}}]\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n) ≤ \delta,
\] (8.140)

where \( I_n := ((\lambda_n^1)^2 \inf I + t_n^1, (\lambda_n^1)^2 \sup I + t_n^1) \).

**Proof of Lemma 8.21** We see from Lemma 8.20 that there exists a number \( k_0 \) (depending on \( \delta \) and \( I \)) with the following property: there exists a number \( N_0(\delta, I) \) such that for any \( n ≥ N_0(\delta, I) \),
\[
\|\|\psi_n^{k_0\text{-app}}\| q - 2 e^{it\Delta} u_n^{k_0}\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n) + \|\|\nabla\| e^{it\Delta} u_n^{k_0}\| \| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n) ≤ \delta.
\] (8.141)

We also see from Lemma 8.19 that there exists a constant \( B(I) > 0 \) with the following property: there exists a number \( N_1(\delta, I) \) such that for any \( n ≥ N_1(\delta, I) \),
\[
\|\|\psi_n^{k_0\text{-app}}\| W_{p+1}(I_n) \cap W(I_n) + \|\|\psi_n^{k_0\text{-app}}\| W_{2\epsilon}(I_n) \cap L^d(I_n) ≤ B(I).
\] (8.142)

Furthermore, it follows from (8.48) that there exists a number \( N_2(\delta, I) \) such that for any \( n ≥ N_2(\delta, I) \),
\[
\| e^{it\Delta} u_n^{k_0}\| W_{p+1}(\mathbb{R}) \cap W(\mathbb{R}) ≤ \frac{\delta}{(1 + B(I))^{2\epsilon}}.
\] (8.143)

We rewrite the error term as follows:
\[
e[\psi_n^{k_0\text{-app}}] = \sum_{j=1}^{k_0} \left\{ i \frac{\partial \psi_n^j}{\partial t} + \Delta \psi_n^j \right\} + F\left[ \sum_{j=1}^{k_0} \psi_n^j \right]
+ F\left[ \psi_n^{k_0\text{-app}} \right] - F\left[ \psi_n^{k_0\text{-app}} - e^{it\Delta} u_n^{k_0} \right].
\] (8.144)

Then, we have
\[
\| \langle \nabla \rangle e[\psi_n^{k_0\text{-app}}]\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n)
≤ \| \langle \nabla \rangle \left( \sum_{j=1}^{k_0} \left\{ i \frac{\partial \psi_n^j}{\partial t} + \Delta \psi_n^j \right\} + F\left[ \sum_{j=1}^{k_0} \psi_n^j \right] \right)\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n)
+ \| \langle \nabla \rangle \left\{ F\left[ \psi_n^{k_0\text{-app}} \right] - F\left[ \psi_n^{k_0\text{-app}} - e^{it\Delta} u_n^{k_0} \right] \right\}\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n).
\] (8.145)

We consider the first term on the right-hand side of (8.145). It follows from (8.66) and Lemma 8.19 that there exists a number \( N_3(\delta, I) \) such that for any \( n ≥ N_3(\delta, I) \),
\[
\| \langle \nabla \rangle \left( \sum_{j=1}^{k_0} \left\{ i \frac{\partial \psi_n^j}{\partial t} + \Delta \psi_n^j \right\} + F\left[ \sum_{j=1}^{k_0} \psi_n^j \right] \right)\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n)
≤ \| \langle \nabla \rangle \left\{ \sum_{j=1}^{k_0} F\left[ \psi_n^j \right] - F\left[ \sum_{j=1}^{k_0} \psi_n^j \right] \right\}\| L^{\frac{2(d+2)}{d+4}}_{t,x} (I_n) + \delta.
\] (8.146)
Let $q$ denote $p + 1$ or $2^*$. Then, an elementary calculation shows that

$$
\| \sum_{j=1}^{k_0} |\psi_j^n|^{q-2} \sum_{j=1}^{k_0} \psi_j^n - \sum_{j=1}^{k_0} |\psi_j^n|^{q-2} \psi_j^n \|_{L_t^\infty L_x^2} \lesssim k_0 \sum_{j=1}^{k_0} \sum_{1 \leq k \leq k_0, k \neq j} \| |\psi_j^n|^{q-2} \psi_k^n \|_{L_t^\infty L_x^2} (I_n) \tag{8.147}
$$

and

$$
\| \nabla \left\{ \sum_{j=1}^{k_0} |\psi_j^n|^{q-2} \sum_{j=1}^{k_0} \psi_j^n - \sum_{j=1}^{k_0} |\psi_j^n|^{q-2} \psi_j^n \right\} \|_{L_t^\infty L_x^2} \lesssim k_0 \sum_{j=1}^{k_0} \sum_{1 \leq k \leq k_0, k \neq j} \| |\psi_j^n|^{q-2} \nabla \psi_k^n \|_{L_t^\infty L_x^2} (I_n) \tag{8.148}
$$

When $q > 3$, we must add the following term to the right-hand side of (8.148):

$$
\sum_{j=1}^{k_0} \sum_{1 \leq k \leq k_0, k \neq j} \| |\psi_j^n|^{q-3} \nabla \psi_k^n \|_{L_t^\infty L_x^2} (I_n) \tag{8.149}
$$

We see from the orthogonality condition (8.36) that there exists a number $N_4(\delta, I)$ such that for any distinct numbers $j, k \in \{1, \ldots, k_0\}$ and any $n \geq N_4(\delta, I)$,

$$
\| |\psi_j^n|^{q-2} \psi_k^n \|_{L_t^\infty L_x^2} + \| |\psi_j^n|^{q-2} \nabla \psi_k^n \|_{L_t^\infty L_x^2} \leq \frac{\delta}{k_0^2}. \tag{8.150}
$$

When $q > 3$, we also have that for any $i \in \{1, \ldots, k_0\}$, any distinct numbers $j, k \in \{1, \ldots, k_0\}$ and any $n \geq N_4(\delta, I)$,

$$
\| |\psi_i^n|^{q-3} \nabla \psi_j^n \psi_k^n \|_{L_t^\infty L_x^2} \leq \frac{\delta}{k_0^3}. \tag{8.151}
$$

Putting the estimates (8.147), (8.148), (8.149), (8.150) and (8.151) together, we find that for any $n \geq \max\{N_3(\delta, I), N_4(\delta, I)\}$,

$$
\| \nabla \left\{ \sum_{j=1}^{k_0} \left\{ i \frac{\partial \psi_j^n}{\partial t} + \Delta \psi_j^n \right\} + F \left[ \sum_{j=1}^{k_0} \psi_j^n \right] \right\} \|_{L_t^\infty L_x^2} \lesssim \delta. \tag{8.152}
$$

Next, we move on to the second term on the right-hand side of (8.145). We can verify
that for any number $n$,
\[
\| |\langle \nabla \rangle | q - 2 e^{it\Delta} \psi_n - e^{it\Delta} u_n k_0 | q - 2 \psi_n k_0 - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)} \lesssim \| |\langle \nabla \rangle | q - 2 e^{it\Delta} \psi_n - e^{it\Delta} u_n k_0 | q - 2 \psi_n k_0 - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)} + \| |\langle \nabla \rangle | q - 2 e^{it\Delta} \psi_n - e^{it\Delta} u_n k_0 | q - 2 \psi_n k_0 - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)} + \| |\langle \nabla \rangle | q - 2 e^{it\Delta} \psi_n - e^{it\Delta} u_n k_0 | q - 2 \psi_n k_0 - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)}.
\] (8.153)

where $q$ denotes $p + 1$ or $2^*$. When $q > 3$, we must add the following terms to the right-hand side of (8.153):
\[
\| |\langle \nabla \rangle | q - 3 e^{it\Delta} u_n k_0 \nabla \psi_n k_0 - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)} \] (8.154)
\[
\| e^{it\Delta} u_n k_0 | q - 3 \psi_n k_0 - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)} \] (8.155)

Furthermore, we can see from (8.141), Hölder’s inequality, (8.142), (8.143) and (8.47) that the right-hand side of (8.153) vanishes as $n$ tends to the infinity (cf. the proof of (3.10) in [17]). Thus, we conclude that
\[
\lim_{n \to \infty} \| |\langle \nabla \rangle | F[\psi_n k_0 - e^{it\Delta} u_n k_0 \rangle - e^{it\Delta} u_n k_0 \rangle \|_{L_{t,x} t^{(d+2)} (I_n)} = 0.
\] (8.156)

Putting (8.145), (8.152) and (8.156) together, we find that the desired estimate (8.140) holds.

**Lemma 8.22.** Assume (8.93). Then, for any $T \in I^1_{\text{max}} = (T^1_{\text{min}}, T^1_{\text{max}})$,
\[
\| \sigma_\infty^1 \tilde{\psi} \|_{W^1([T,T_{\text{max}}])} = \infty.
\] (8.157)

Furthermore, we have that $\lambda_\infty^1 \neq \infty$ and
\[
\left\{ \begin{array}{ll}
S_\omega(\sigma_\infty^1 \tilde{\psi}) \geq m_\omega & \text{if } \lambda_\infty^1 = 1, \\
H_\omega(\sigma_\infty^1 \tilde{\psi}) \geq \frac{1}{d} \sigma_\infty^1 > m_\omega & \text{if } \lambda_\infty^1 = 0.
\end{array} \right.
\] (8.158)

**Proof of Lemma 8.22.** Suppose for contradiction that (8.157) failed for some $T \in I^1_{\text{max}}$. Then, it follows from the well-posedness theory that $T^1_{\text{max}} = \infty$ and for any $\tau \in (T^1_{\text{min}}, \infty)$,
\[
\| \sigma_\infty^1 \tilde{\psi} \|_{W^1([T,\tau])} < \infty.
\] (8.159)

In order to derive a contradiction, we consider the approximate solution $\psi_n^{k_0}$ defined by (8.126). Note that for any numbers $k$ and $n$, the maximal existence interval of $\psi_n^{k_0}$ is identical to $T^1_{\text{max},n} := ((\lambda_n^1)^2 T^1_{\text{min}} + t_n^1, \infty)$.
First, we shall show that there exists a constant $B > 0$ with the following property: for any number $k$, there exists a number $N(k)$ such that for any $n \geq N(k)$,

$$\|\psi_n^{k-app}\|_{W^{p+1}((0,\infty)) \cap W((0,\infty))} \leq B. \quad (8.160)$$

If $-\frac{t_1}{(\lambda_n^1)^2} = -\infty$, then we see from (8.159) and (8.63) that Lemma 8.19 is available on the whole interval $\mathbb{R}$. Thus, we can take a constant $B_1 > 0$ with the following property: for any number $k$, there exists a number $N_1(k)$ such that for any $n \geq N_1(k)$,

$$\|\psi_n^{k-app}\|_{W^{p+1}(\mathbb{R}) \cap W((0,\infty))} \leq B_1. \quad (8.161)$$

On the other hand, if $-\frac{t_1}{(\lambda_n^1)^2} \neq -\infty$, then it follows from the construction of $\tilde{\psi}^1$ (see (8.57)) that $-\frac{t_1}{(\lambda_n^1)^2} \in (T_{min}^1, \infty) \cup \{\infty\}$. This implies that there exist $\tau_0 \in (T_{min}^1, \infty)$ and a number $N_0$ such that $\tau_0 < -\frac{t_1}{(\lambda_n^1)^2}$ for any $n \geq N_0$, so that $[0, \infty) \subset ((\lambda_n^1)^2 \tau_0 + t_1^1, \infty) \subset \bigcap_{\tau} T_{max,n}$. Furthermore, we see from Lemma 8.19 and (8.159) with $\tau = \tau_0$ that there exists a constant $B_2 > 0$ with the following property: for any number $k$, there exists a number $N_2(k)$ such that for any $n \geq \max\{N_0, N_2(k)\}$,

$$\|\psi_n^{k-app}\|_{W^{p+1}((0,\infty)) \cap W((0,\infty))} \leq B_2. \quad (8.162)$$

Thus, (8.161) and (8.162) give the desired result (8.160).

Now, we go on to the proof of (8.157). We first note that it follows from (8.39) and Lemma 8.7 that

$$\max_{n} \|\psi_n(0) - \psi_n^{k-app}(0)\|_{H^1} = \sum_{j=1}^{k} \left| \sum_{n} g_n^j \sigma_n^j e^{-i \frac{t_1}{(\lambda_n^1)^2} \Delta} \tilde{u}_j - \sum_{n} g_n^j \sigma_n^j \tilde{\psi}_j \left( - \frac{t_1}{(\lambda_n^1)^2} \right) \right|_{H^1} \lesssim \sum_{j=1}^{k} e^{-\frac{t_1}{(\lambda_n^1)^2} \Delta} \tilde{u}_j - \tilde{\psi}_j \left( - \frac{t_1}{(\lambda_n^1)^2} \right) \|_{H^1}. \quad (8.163)$$

This estimate together with (8.61) shows that for any number $k$, there exists a number $N_3(k)$ such that

$$\sup_{n \geq N_3(k)} \|\psi_n(0) - \psi_n^{k-app}(0)\|_{H^1} \leq 1. \quad (8.164)$$

We also see from (8.33) that for any number $n$,

$$\|\psi_n\|_{L^\infty_t H^1_x((0,\infty))} \lesssim 1. \quad (8.165)$$

Let $\delta > 0$ be a constant determined by the long-time perturbation theory (see Proposition 5.6 in [2]) together with (8.160), (8.164) and (8.165). Then, we see from Sobolev’s embedding, Strichartz estimate, (8.163) and (8.61) that for any number $k$, there exists a number $N_4(k)$ such that for any $n \geq N_4(k)$,

$$\|e^{it\Delta} \{\psi_n(0) - \psi_n^{k-app}(0)\}\|_{W^{p+1}((0,\infty)) \cap W((0,\infty))} \lesssim \|\nabla e^{it\Delta} \{\psi_n(0) - \psi_n^{k-app}(0)\}\|_{V^{p+1}((0,\infty)) \cap V((0,\infty))} \lesssim \delta. \quad (8.166)$$
Moreover, it follows from Lemma 8.21 that we can take numbers $k_0$ and $N_5$ such that for any $n \geq N_5$,

$$\|\langle \nabla \rangle e^{[\psi_n^{k_0\text{-app}}]}\|_{L^2_{t,x}([0,\infty))} \leq \delta. \quad (8.167)$$

Thus, applying the long-time perturbation theory (see Proposition 5.6 in [3]) to $\psi_n$ and $\psi_n^{k_0\text{-app}}$ with $n \geq \max\{N(k_0), N_3(k_0), N_4(k_0), N_5\}$, we find that

$$\|\langle \nabla \rangle \psi_n\|_{St([0,\infty))} < \infty. \quad (8.168)$$

However, this contradicts (8.29). Thus, we have shown that (8.157) holds.

8.2.5 Critical element and completion of proof

We shall show the existence of “critical element” (see Lemma 8.26 below). Furthermore, we will derive a contradiction by using it under the hypothesis (8.25), which completes the proof.

Let us begin by defining the functionals $S_j^\omega$ and $I_j^\omega$. For each $j \geq 1$, we define

$$S_j^\omega := \begin{cases} S_\omega & \text{if } \lambda_j^\infty = 1, \\ S_\omega & \text{if } \lambda_j^\infty = 0, \end{cases} \quad I_j^\omega := \begin{cases} I_\omega & \text{if } \lambda_j^\infty = 1, \\ I_\omega & \text{if } \lambda_j^\infty = 0. \end{cases} \quad (8.169)$$

Lemma 8.23. Assume (8.93). Then, for any $\delta > 0$ and any number $k \geq 2$, there exists a number $N(\delta,k)$ such that for any $n \geq N(\delta,k)$,

$$\sum_{j=2}^k I_j^\omega(\sigma_j^\infty \psi_j(t)) + I_\omega(w_n^k) \leq \delta. \quad (8.170)$$

Proof of Lemma 8.23. We see from (8.26), (8.28) and (8.45) that for any $\delta > 0$ and any number $k$, there exists a number $N_1(\delta,k)$ such that for any $n \geq N_1(\delta,k)$,

$$S_\omega(g_n^1 \sigma_n^1 e^{-i \frac{\lambda_n^1}{(\lambda_n^1)^2} \Delta} u^1) + \sum_{j=2}^k S_\omega(g_n^j \sigma_n^j e^{-i \frac{\lambda_n^j}{(\lambda_n^j)^2} \Delta} u^j) + S_\omega(w_n^k) \leq m_\omega + \delta. \quad (8.171)$$

If $\lambda_1^\infty = \lim_{n \to \infty} \lambda_n^1 = 0$, then Lebesgue’s convergence theorem shows that for each $t \in I_{\text{max}}^1$,

$$\lim_{n \to \infty} \|g_n^1 \sigma_n^1 \psi_n^1(t)\|_{L^2} = \lim_{n \to \infty} \left\| \frac{1 + |\xi|^2}{\sqrt{1 + (\lambda_n^1)^2}} \mathcal{F}[\psi_n^1(t)] \right\|_{L^2} = 0. \quad (8.172)$$

We see from Lemma 8.15 (8.86) in Lemma 8.13 (8.64), (8.171), (8.172) and (8.158) in Lemma 8.22 that for any $\delta > 0$ and any number $k$, there exists $N_2(\delta,k)$ such that for any
\( t \in \mathbb{R} \) and any \( n \geq N_2(\delta, k) \),
\[
\sum_{j=2}^{k} I_{\omega}(\sigma_{\infty}^l(t)) + I_{\omega}(w_n^k) \leq \sum_{j=2}^{k} S_{\omega}(\sigma_{\infty}^l(t)) + S_{\omega}(w_n^k)
\]
\[
\leq \sum_{j=2}^{k} S_{\omega}(g_n^l\sigma_n^l e^{(-t_n^l)(\lambda_n^l)^2} \Delta^l) + \delta + S_{\omega}(w_n^k)
\]
\[
\leq m_\omega - S_{\omega}(\sigma_{\infty}^l(t)) + 2 \delta \leq 2 \delta.
\] (8.173)

Thus, we have completed the proof.

Put \( \tau_1^\min := -\frac{t_n^l}{(\lambda_n^l)^2} \). Then, passing to a subsequence, we may assume that
\[
\tau_1^\infty := \lim_{n \to \infty} \tau_1^n \in \mathbb{R} \cup \{\pm \infty\}.
\] (8.174)

**Lemma 8.24.** Assume (8.93). Then, we have that \( \lambda_1^1 = 1 \). Furthermore, there exists a time \( T \in T_{\max}^1 \) such that
\[
\inf_{t \in [T, T_{\max}^1)} \bar{d}_{\omega}(\tilde{\psi}^1(t)) \geq \frac{R_*}{2}
\] (8.175)
and
\[
\inf_{t \in [T, T_{\max}^1)} \mathcal{K}(\tilde{\psi}^1(t)) \geq \frac{\kappa_1(R_*)}{2},
\] (8.176)

where \( \kappa_1(R_*) \) is the constant appearing in (8.32).

**Proof of Lemma 8.24.** Lemma 8.22 together with (8.62) shows that \( \tau_1^\infty \neq \infty \). If \( \tau_1^\infty = -\infty \), then it follows from (8.63) that \( T_{\min}^1 = -\infty \) and \( \|\sigma_1^\infty \tilde{\psi}^1\|_{W^1([-\infty, T])} < \infty \) for any \( T < T_{\max}^1 \). If \( \tau_1^\infty \in \mathbb{R} \), then we see from the construction of \( \tilde{\psi}^1 \) (see (8.57)) that \( \tau_1^\infty \in I_{\max}^1 \).

Put
\[
\tau_1^\min := \begin{cases} 
\frac{T_{\min}^1 + \tau_1^\infty}{2} & \text{if } T_{\min}^1 > -\infty, \\
\tau_1^\infty - 1 & \text{if } \tau_1^\infty \in \mathbb{R} \text{ and } T_{\min}^1 = -\infty, \\
-\infty & \text{if } \tau_1^\infty = -\infty.
\end{cases}
\] (8.177)

Then, \( T_{\min}^1 \leq \tau_1^\min \leq \tau_1^\infty \), and we can take a number \( N_1 \) such that for any \( n \geq N_1 \),
\[
\tau_1^\min < -\frac{t_n^l}{(\lambda_n^1)^2} < T_{\max}^1.
\] (8.178)

Note that \( 0 \in I_{\max, n}^1 \) for all \( n \geq N_1 \). Furthermore, for each \( T \in (\tau_1^\min, T_{\max}^1) \),
\[
\|\sigma_1^\infty \tilde{\psi}^1\|_{W^1([-\infty, T])} < \infty,
\] (8.179)
which together with Lemma 8.16 shows that for any \( T \in (\tau_1^\min, T_{\max}^1) \), there exists a constant \( A(T) > 0 \) such that
\[
\|\langle \nabla \rangle \tilde{\psi}^1\|_{\mathcal{S}_T(\tau_1^\min, T)} \leq A(T).
\] (8.180)
Sobolev’s embedding also gives us that
\[ \|\sigma_1 \tilde{\psi}\|_{W^1((\tau_{\min}^1, T))] \lesssim \|\langle \nabla \rangle \tilde{\psi}^1\|_{S(t;(\tau_{\min}^1, T))} \leq A(T). \] (8.181)

Now, we consider the approximate solution \( \psi_{k,\text{app}} \) defined by (8.120). Lemma 8.19 together with (8.181) shows that for any \( T \in (\tau_{\min}^1, T_{\max}^1) \), there exists \( B(T) > 0 \) with the following property: for any number \( k \), there exists a number \( N_2(k) \) such that for any \( n \geq N_2(k) \),
\[ \|\langle \nabla \rangle \psi_{k,\text{app}}\|_{W^2_n(I_n(T)) \cap L^\infty_T L^2(I_n(T))} \leq B(T), \] (8.182)
where
\[ I_n(T) := \left( (\lambda_n^1)^2 \tau_{\min}^1 + t_n^1, (\lambda_n^1)^2 T + t_n^1 \right]. \] (8.183)

Lemma 8.21 together with (8.181) shows that for any \( \delta \in (0, 1) \) and any \( T \in (\tau_{\min}^1, T_{\max}^1) \), there exists numbers \( k_0 \) (depending on \( \delta \) and \( T \)) and \( N_3(\delta, T) \) such that for any \( n \geq N_3(\delta, T) \),
\[ \|\langle \nabla \rangle e^{\psi_{k,\text{app}} \sigma_j(\tilde{\psi}^j)} \|_{L^2(t,\tilde{\psi}^j)} \leq \delta. \] (8.184)

We also see from (8.39), Lemma 8.7 and (8.61) that for any number \( k \) and any \( \gamma > 0 \), there exists a number \( N_4(k, \gamma) \) such that for any \( n \geq N_4(k, \gamma) \),
\[ \|\psi_n(0) - \psi_{k,\text{app}}(0)\|_{H^1} = \left\| \sum_{j=1}^k g_n^j \sigma_j^1 e^{-i \frac{\sqrt{2}}{(\lambda_n^1)^2} \tilde{\psi}^j} - \sum_{j=1}^k g_n^j \sigma_j^1 \tilde{\psi}^j \right\|_{H^1} \leq \gamma. \] (8.185)

Furthermore, it follows from Strichartz’ estimate and an estimate similar to (8.185) that for any number \( k \) and any \( \delta > 0 \), there exists a number \( N_5(k, \delta) \) such that for any \( n \geq N_5(k, \delta) \),
\[ \|\langle \nabla \rangle e^{it \Delta} \{ \psi_n(0) - \psi_{k,\text{app}}(0) \}\|_{H^1(t;\mathbb{R})} \lesssim \|\psi_n(0) - \psi_{k,\text{app}}(0)\|_{H^1} \leq \delta. \] (8.186)

The long-time perturbation theory (Lemma B.3) together with (8.33), (8.182), (8.184), (8.185) and (8.186) shows that for any \( T \in (\tau_{\min}^1, T_{\max}^1) \), there exists \( \gamma(T) > 0, \delta(T) > 0 \), \( C(T) > 0 \) and a function \( \tilde{c}: (0, \infty) \times (0, \infty) \to (0, \infty) \) with the following properties:
\[ \lim_{||\gamma, \delta|| \to 0} \tilde{c}(\gamma, \delta) = 0, \] (8.187)
and for any \( 0 < \delta < \min\{1, \gamma(T), \delta(T)\} \), we can take numbers \( k_0 \) (depending on \( \delta \) and \( T \)) and \( N_6(\delta, T) \) such that for any \( n \geq N_6(\delta, T) \),
\[ \|\langle \nabla \rangle \psi_n\|_{S(t;I_n(T))} \leq C(T), \] (8.188)
\[ \sup_{t \in I_n(T)} \|\psi_n(t) - \psi_{k,\text{app}}(t)\|_{H^1} \leq C(T)\tilde{c}(\delta), \] (8.189)
where we put $\bar{c}(\delta) := \bar{c}(\delta, \delta)$. Furthermore, it follows from \textbf{8.189}, Lemma \textbf{8.18}, Lemma \textbf{8.6} and Lemma \textbf{8.23} that for any $T \in (\tau_{\min}^1, T_{\max}^1)$, any $0 < \delta < \min\{1, \gamma(T), \delta(T)\}$ and the number $k_0$ determined by $T$ and $\delta$, we can take a number $N_\tau(\delta, T)$ such that for any $n \geq N_\tau(\delta, T)$,

$$
\| \psi_n((\lambda_n^1)^2T + t_n^1) - \psi_n^1((\lambda_n^1)^2T + t_n^1) \|_{H^1}^2 \\
\lesssim \| \psi_n((\lambda_n^1)^2T + t_n^1) - \psi_n^0 \|_{H^1}^2 + \| \psi_n^0 \|_{H^1}^2 + \| \psi_n^0(\cdot) \|_{H^1}^2 + \| \psi_n \|_{H^1}^2 + \| \psi_n \|_{H^1}^2 + \delta \lesssim C(T)^2 \bar{c}(\delta)^2 + \delta.
$$

(8.190)

Now, we shall show that $\lambda_\infty^1 = 1$. Suppose for contradiction that $\lambda_\infty^1 = 0$. Fix a time $T \in (\tau_{\min}^1, T_{\max}^1)$. Then, it follows from Lemma \textbf{8.6} that

$$
\lim_{n \to \infty} \| \psi_n(\lambda_n^1)^2T + t_n^1) \|_{L^2} = \lim_{n \to \infty} \| \psi_n^0 \|_{L^2} = 0.
$$

(8.191)

Furthermore, we see from \textbf{(8.8)}, $M(\psi_n) = M(\Phi_\omega)$, \textbf{(8.190)} and \textbf{(8.191)} that for any sufficiently small $\omega > 0$ and any $0 < \delta < \min\{1, \gamma(T), \delta(T)\}$,

$$
\omega^n ||U||_{L^2}^2 \lesssim ||\Phi_\omega||_{L^2}^2 = \lim_{n \to \infty} \| \psi_n((\lambda_n^1)^2T + t_n^1) \|_{L^2}^2 \lesssim C(T)^2 \bar{c}(\delta)^2 + \delta.
$$

(8.192)

Taking $\delta \to 0$, we deduce that $U \equiv 0$. However, this is a contradiction. Thus, we find that $\lambda_\infty^1 = 1$, so that $\lambda_\infty^1 \equiv 1$, $\tau_{\infty}^1 = -t_{\infty}^1$ and $\sigma_\infty^1 \equiv 1$.

Now, we are in a position to prove \textbf{(8.175)} and \textbf{(8.176)}. We first consider the case $\tau_{\infty}^1 = -\infty$. In this case, we have $t_{\infty}^1 = \infty$. Suppose for contradiction that \textbf{(8.175)} failed. Then, for any $T \in (\tau_{\min}^1, T_{\max}^1)$, we can take $T' \in [T, T_{\max}^1]$ such that

$$
\tilde{d}_\omega(\tilde{\psi}(T')) \leq \frac{2}{3} R_s.
$$

(8.193)

Since $\lim_{n \to \infty} t_n^1 = \infty$, we can take a number $N_8(T')$ such that $T' + t_n^1 \geq 0$ for any $n \geq N_8(T')$. Thus, we find from \textbf{(8.30)}, \textbf{(8.190)} with $\lambda_n^1 \equiv 1$ and \textbf{(8.193)} that

$$
R_s \leq \tilde{d}_\omega(\psi_n(T' + t_n^1)) \leq \tilde{d}_\omega(\psi_n^1(T' + t_n^1)) + \omega_\delta(1) = \tilde{d}_\omega(\psi_\omega(\tilde{\psi}(T'))) + o_\delta(1) \leq \frac{2}{3} R_s + o_\delta(1).
$$

(8.194)

(8.194)

However, this is impossible for sufficiently small $\delta$. Hence, the claim \textbf{(8.175)} is true. Suppose next that \textbf{(8.176)} failed. Then, for any $T \in (\tau_{\min}^1, T_{\max}^1)$, there exists $T' \in [T, T_{\max}^1)$ such that

$$
\mathcal{K}(\tilde{\psi}(T')) \leq \frac{2}{3} \kappa_1(R_s).
$$

(8.195)
Let \( N_\delta(T') \) be the number obtained above. Then, we find from (8.32), (8.190) with \( \lambda_n^1 \equiv 1 \) and (8.195) that

\[
\kappa_1(R_*) \leq K(\psi_n(T' + t_n^1)) \leq K(\psi_n^1(T' + t_n^1)) + o_\delta(1) \leq \frac{2}{3} \kappa_1(R_*) + o_\delta(1). \tag{8.196}
\]

However, for a sufficiently small \( \delta \), this is a contradiction. Hence, the claim (8.176) is true.

It remains to consider the case \( \tau_{1\infty}^1 \in \mathbb{R} \). In this case, we have that \( \tau_{1\min}^1 < \tau_{1\infty}^1 = -t_{1\infty}^1 < T_{1\max}^1 \) (see (8.178)). Let \( T' \in [ -t_{1\infty}^1, T_{1\max}^1 ] \) be a time for which (8.193) or (8.195) holds. Then, we can derive a contradiction as well as the case \( \tau_{1\infty}^1 = -\infty \).

\[ \text{Lemma 8.25. Assume (8.93). Then, we have } M(\tilde{\psi}_1) = M(\Phi_\omega). \]

\[ \text{Proof of Lemma 8.25. We see from } \lambda_1^\infty \equiv 1 \text{ (see Lemma 8.24), (8.43), (8.61), Lemma 8.23 and Lemma 8.6 that for any } \delta > 0, \text{ there exists a number } N(\delta) \text{ such that for any } n \geq N(\delta), \]

\[
\delta \geq | M(\psi_n(0)) - M(e^{-it_n^1 \Delta \tilde{\psi}_1^n}) | - M(g_{\omega}^2 e^{-it_n^2 \Delta \tilde{u}_2^n}) - M(w_{\omega}^2) \]

\[
\geq | M(\psi_n(0)) - M(e^{-it_n^1 \Delta \tilde{\psi}_1^n}) | - \delta \]

\[
\geq | M(\Phi_\omega) - M(\tilde{\psi}_1) | - 2 \delta. \tag{8.197}
\]

Since \( \delta \) is an arbitrary constant, (8.197) implies the desired result. \[ \square \]

\[ \text{Lemma 8.26. There exists a solution } \Psi \text{ to (NLS) such that} \]

\[
\Psi \in S_{\omega, R_*, +}^*, \tag{8.198}
\]

\[
\inf_{t \in [0, T_{max})} K(\Psi(t)) \geq \frac{\kappa_1(R_*)}{2}, \tag{8.199}
\]

\[
\mathcal{H}(\Psi) = E_*, \tag{8.200}
\]

\[
\| \Psi \|_{W_p((0,T_{max})) \cap W((0,T_{max}))} = \infty, \tag{8.201}
\]

\[
\{ \Psi(t) : t \in [0, T_{max}) \} \text{ is precompact in } H^1(\mathbb{R}^d), \tag{8.202}
\]

where \( T_{max} \) denotes the maximal lifespan of \( \Psi \).

\[ \text{Proof of Lemma 8.26. We see from Lemma 8.24 that } \lambda_1^\infty = 1 \text{ (hence } \sigma_1^\infty = 1 \text{ and } \tilde{\psi}^1 \text{ is a solution to (NLS)), and there exists } T_0 \in T_{max}^1 \text{ such that} \]

\[
\inf_{t \in [T_0, T_{max})} \tilde{d}_\omega(\tilde{\psi}_1(t)) \geq \frac{R_*}{2}, \quad \inf_{t \in [T_0, T_{max})} K(\tilde{\psi}_1(t)) \geq \frac{\kappa_1(R_*)}{2}. \tag{8.203}
\]

We also see from (8.26), (8.28), (8.31), (8.45) and (8.61) that

\[
S_{\omega}(\tilde{\psi}_1) \leq \omega M(\Phi_\omega) + E_\omega = m_\omega. \tag{8.204}
\]
This together with Lemma 8.25 and (8.20) shows that \( \tilde{\psi}^1(\cdot + T_0) \in S^{\epsilon_{\omega,R}}_{\omega,R} \). Furthermore, by Lemma 8.2, we may assume that \( \tilde{\psi}^1(\cdot + T_0) \in S^{\epsilon_{\omega,R}}_{\omega,R} \). Put \( \Psi(t) := \tilde{\psi}^1(t + T_0) \). Then, the maximal-lifespan of \( \Psi \) is \( T_{\max}(\Psi) := T_{\max} - T_0 \), and \( \Psi \in S^{\epsilon_{\omega,R}}_{\omega,R} \). Moreover, it follows from Lemma 8.1, Lemma 8.3 and (8.203) that \( S(\Psi) = +1 \), so that

\[
\Psi \in S^{\epsilon_{\omega,R}}_{\omega,R}, +1.
\] (8.205)

We also find from Lemma 8.22, (1.6) and (8.203) that

\[
\| \Psi \|_{W^{p+1}_{p+1}([0,T_{\max}(\Psi)]) \cap W([0,T_{\max}(\Psi)) = \infty, \\
\mathcal{H}(\Psi) \geq \frac{1}{2} \inf_{t \in [0,T_{\max}(\Psi)]} K(\Psi(t)) \geq \frac{\kappa_1(R_\omega)}{4}.
\] (8.206)

Since \( \mathcal{M}(\Psi) = \mathcal{M}(\tilde{\psi}^1) = \mathcal{M}(\Phi_\omega) \), the definition of \( E_* \) (see (8.23)) together with (8.204) and (8.206) shows the property (8.200). Now, suppose for contradiction that there existed a number \( j \geq 2 \) such that \( \tilde{\psi}^j \) is non-trivial. Without loss of generality, we may assume that \( \tilde{\psi}^2 \) is non-trivial. Then, it follows from (8.100) and Theorem 1.2 that

\[
\begin{cases}
\mathcal{H}(\tilde{\psi}^{2}) > 0 & \text{if } \lambda_\infty^2 = 1, \\
\mathcal{H}^1(\sigma_\infty^2 \tilde{\psi}^{2}) > 0 & \text{if } \lambda_\infty^2 = 0.
\end{cases}
\] (8.207)

We see from (8.64), Lemma 8.8 and (8.208) that

\[
0 = \lim_{n \to \infty} \mathcal{H}(\psi_n) = \lim_{n \to \infty} \mathcal{H}(g_n^2 \sigma_n^2 e^{-i (\Delta \tilde{\psi}^1)^2 / (\lambda_n^2)^2} \Delta \tilde{\psi}^1) - \mathcal{H}(w_n^2).
\] (8.209)

This together with (8.64) shows that

\[
\mathcal{H}(\Psi) = \mathcal{H}(\tilde{\psi}^1) = \lim_{n \to \infty} \mathcal{H}(e^{-i t_n^1 \Delta \tilde{\psi}^1}) \leq E_* - h_2.
\] (8.210)

However, this contradicts the proved property (8.200). Thus, we have found that \( \tilde{\psi}^j \equiv 0 \) for all \( j \geq 2 \). Then, we also have that \( \tilde{u}^j \equiv 0 \) for all \( j \geq 2 \).

We return to the decomposition (8.39): \( \psi_n(0) = e^{-i t_n^1 \Delta \tilde{u}^1} + w_n^1 \). It follows from (8.43), (8.64) and \( \mathcal{M}(\Psi) = \mathcal{M}(\Phi_\omega) = \mathcal{M}(\psi_n) \) that

\[
\lim_{n \to \infty} \| w_n^1 \|_{L^2} = 0.
\] (8.211)
Moreover, it follows from (8.21), (8.211) and (8.200) that
\[
\lim_{n \to \infty} \mathcal{H}(w^1_n) = 0,
\]
which together with (8.86) in Lemma 8.13 shows
\[
\lim_{n \to \infty} \|\nabla w^1_n\|_{L^2}^2 \leq \frac{d}{s_p} \lim_{n \to \infty} \left\{ \mathcal{H}(w^1_n) - \frac{2}{d(p-1)} K(w^1_n) \right\} \leq \frac{d}{s_p} \lim_{n \to \infty} \mathcal{H}(w^1_n) = 0.
\]
Thus, we find that
\[
\lim_{n \to \infty} \|w^1_n\|_{H^1} = 0.
\]

Finally, we shall prove the precompactness of \(\{\Psi(t)\}\) in \(H^1(\mathbb{R}^d)\). Take a sequence \(\{\tau_n\}\) in \([0, T_{\text{max}}(\Psi))\). By the continuity in time, it suffices to consider the case where \(\lim_{n \to \infty} \tau_n = T_{\text{max}}(\Psi)\). Applying the above argument to \(\{\Psi(t+\tau_n)\}\), we can take a subsequence of \(\{\tau_n\}\) (still denoted by the same symbol), a sequence \(\{t_n\}\) in \(\mathbb{R}\), a sequence \(\{w_n\}\) in \(H^1(\mathbb{R}^d)\) and a function \(\tilde{u} \in H^1(\mathbb{R}^d)\) such that
\[
\Psi(\tau_n) = e^{-it_n \Delta \tilde{u}} + w_n,
\]
\[
\lim_{n \to \infty} \|w_n\|_{H^1} = 0,
\]
\[
t_{\infty} := \lim_{n \to \infty} t_n \in \mathbb{R} \cup \{\infty\},
\]
and if \(t_\infty = \infty\), then \(T_{\text{min}}(\Psi) = -\infty\). We see from (8.216) and (8.217) that
\[
\lim_{n \to \infty} \|\Psi(\tau_n) - e^{-it_n \Delta \tilde{u}}\|_{H^1} = 0.
\]
Hence, in order to prove the precompactness, it suffices to show that \(t_\infty \in \mathbb{R}\). Suppose for contradiction that \(t_\infty = \infty\). Then, we find from Strichartz’ estimate and (8.219) that
\[
\lim_{n \to \infty} \|\nabla e^{i(t+\tau_n)\Delta} \Psi(\tau_n)\|_{V_{p+1}((-\infty,0) \cap V((-\infty,0))}
\]
\[
= \lim_{n \to \infty} \|\nabla e^{i(t+\tau_n)\Delta} \Psi(\tau_n)\|_{V_{p+1}((-\infty,-t_n) \cap V((-\infty,-t_n))}
\]
\[
\leq \lim_{n \to \infty} \|\nabla e^{i(t+\tau_n)\Delta} e^{-it_n \Delta \tilde{u}}\|_{V_{p+1}((-\infty,-t_n) \cap V((-\infty,-t_n))}
\]
\[
+ \lim_{n \to \infty} \|\nabla e^{i(t+\tau_n)\Delta} \{\Psi(\tau_n) - e^{-it_n \Delta \tilde{u}}\}\|_{V_{p+1}((-\infty,-t_n) \cap V((-\infty,-t_n))}
\]
\[
\lesssim \lim_{n \to \infty} \|\nabla e^{i(t+\tau_n)\Delta} \tilde{u}\|_{V_{p+1}((-\infty,-\tau_n) \cap V((-\infty,-\tau_n))} + \lim_{n \to \infty} \|\Psi(\tau_n) - e^{-it_n \Delta \tilde{u}}\|_{H^1} = 0.
\]
Thus, the small data theory (see Lemma B.1 below) shows that there exists a number \(N\) such that for any \(n \geq N\),
\[
\|\Psi\|_{W_{p+1}((-\infty,-\tau_n)) \cap W((-\infty,\tau_n))} = \|\Psi(\cdot + \tau_n)\|_{W_{p+1}((-\infty,0) \cap W((-\infty,0))} \lesssim 1,
\]
which together with \(\lim_{n \to \infty} \tau_n = T_{\text{max}}(\Psi)\) shows
\[
\|\Psi\|_{W_{p+1}((-\infty,T_{\text{max}}(\Psi)) \cap W((-\infty,T_{\text{max}}(\Psi))) \lesssim 1.
\]
However, this contradicts (8.201). Thus, we have \(t_\infty \in \mathbb{R}\) and therefore the precompactness holds. \(\Box\)
Now, we are finishing up with the proof of Proposition 8.5. Under the hypothesis (8.25), we have shown the existence of solution \( \Psi \) with the properties (8.198)–(8.202) in Lemma 8.26. However, the same argument as the proof of Proposition 5.3 in [15] shows that such a solution \( \Psi \) never exists, and therefore we have arrived at a contradiction. Thus, the hypothesis (8.25) is false, and we have proved Proposition 8.5.

A Linearized operator

We can verify that for any \( f \in H^1(\mathbb{R}^d) \),

\[
\left| \langle L_\omega, f, f \rangle_{H^{-1},H^1} \right| + \left| \langle L_\omega, - f, f \rangle_{H^{-1},H^1} \right| \lesssim (\omega + \| \Phi_\omega \|_{L^{p+1}} + \| \Phi_\omega \|_{L^{2^*}}^2) \| f \|^2_{H^1},
\]

(A.1)

where the implicit constant is independent of \( \omega \).

We see from (4.6) that for any \( u,v \in H^1(\mathbb{R}^d) \),

\[
\left\{ S''(\Phi_\omega)u \right\} v = \langle L_\omega, +\Re[u] + iL_\omega, -\Im[u], v \rangle_{H^{-1},H^1}.
\]

(A.2)

Lemma A.1. The operator \( L_{\omega,-} \) is non-negative and \( \text{Ker}\ L_{\omega,-} = \text{span}\{\Phi_\omega\} \); in particular, there exists \( \delta_\omega > 0 \) depending on \( \omega \) such that for any \( u \in H^1(\mathbb{R}^d) \) with \( (u, \Phi_\omega)_{L^2} \text{real} = 0 \),

\[
\langle L_{\omega,-}u, u \rangle_{H^{-1},H^1} \geq \delta_\omega \| u \|^2_{H^1}.
\]

(A.3)

Proof of Lemma A.1. We can prove in a way similar to [25].

Differentiating the equation \((\omega \text{SP})\) for \( \Phi_\omega \) with respect to \( x_j, 1 \leq j \leq d \), we have

\[
0 = \partial_j(\omega \Phi_\omega - \Delta \Phi_\omega - \Phi_\omega^p - \Phi_\omega^{2^*-1}) = L_{\omega,+} \partial_j \Phi_\omega.
\]

(A.4)

Lemma A.2. There exists \( \omega_3 > 0 \) such that for any \( \omega \in (0, \omega_3) \), we have

\[
\text{Ker}\ L_{\omega,+} = \text{span}\{\partial_1 \Phi_\omega, \ldots, \partial_d \Phi_\omega\}.
\]

(A.5)

Proof of Lemma A.2. We can prove this lemma by regarding the operator \( L_{\omega,+} \) as a perturbation of \( L_{\omega}^1 \) (see [11] and Lemma 2.5).

Now, we introduce the functional \( \mathcal{N}_\omega \) as

\[
\mathcal{N}_\omega(u) := \omega \| u \|^2_{L^2} + \| \nabla u \|^2_{L^2} - \| u \|_{L^{p+1}}^{p+1} - \| u \|_{L^{2^*}}^{2^*}.
\]

(A.6)

Then, we can verify that

\[
m_\omega = \inf \left\{ \mathcal{S}_\omega(u): u \in H^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{N}_\omega(u) = 0 \right\}.
\]

(A.7)

Lemma A.3. For any \( u \in H^1(\mathbb{R}^d) \) satisfying

\[
\left( (p-1)\Phi_\omega^p + (2^* - 2)\Phi_\omega^{2^*-1} \right)_{L^2} = 0,
\]

(A.8)

we have

\[
\langle L_{\omega,+}u, u \rangle_{H^{-1},H^1} \geq 0.
\]

(A.9)
Proof of Lemma A.3. The proof is similar to Lemma 2.3 in [20].

**Lemma A.4.** As an operator in \( L^2_{rad}(\mathbb{R}^d) \), \( L_{\omega,+} \) has only one negative eigenvalue which is non-degenerate and 0 is not an eigenvalue.

Proof of Lemma A.4. We can prove this lemma in a way similar to [25] and [11].

**Lemma A.5.** There exists \( \omega_0 > 0 \) such that for any \( 0 < \omega < \omega_0 \), we have the following:

Let \( \mu > 0 \) be a positive eigen-value of \( iL_{\omega} \) as an operator in \( L^2_{\text{real}}(\mathbb{R}^d) \), and let \( U_+ \) be a corresponding eigen-function. Then, we have:

(i) For any non-trivial, real-valued radial function \( g \in H^1(\mathbb{R}^d) \) with \( (g, \Im[U_+])_{L^2} = 0 \), we have

\[
\langle L_{\omega,+} g, g \rangle_{H^{-1},H^1} \sim \|g\|_{H^1}^2,
\]

where the implicit constant may depend on \( \omega \).

(ii) For any non-trivial, real-valued radial function \( g \in H^1(\mathbb{R}^d) \) with \( (g, \Phi_{\omega}' L_{\text{real}})_{L^2_{\text{real}}} = 0 \), we have

\[
\langle L_{\omega,-} g, g \rangle_{H^{-1},H^1} \sim \|g\|_{H^1}^2,
\]

where the implicit constant may depend on \( \omega \).

Proof of Lemma A.5. We prove the claim (i).

First, we shall show that for any \( \omega > 0 \) and any nontrivial, radial real-valued function \( g \in H^1(\mathbb{R}^d) \) satisfying \( (g, f_2)_{L^2_{\text{real}}} = 0 \),

\[
\langle L_{\omega,+} g, g \rangle_{H^{-1},H^1} > 0.
\]

Suppose for contradiction that there exists \( \omega > 0 \) and a nontrivial radial real-valued function \( g_- \in H^1(\mathbb{R}^d) \) such that

\[
(g_-, f_2)_{L^2_{\text{real}}} = 0,
\]

\[
\langle L_{\omega,+} g_-, g_- \rangle_{H^{-1},H^1} \leq 0.
\]

Since \( L_{\omega,+} \) is self-adjoint in \( L^2_{\text{real}}(\mathbb{R}^d) \), we see from (4.87) and (A.13) that

\[
\langle L_{\omega,+} g_-, f_1 \rangle_{H^{-1},H^1} = (g_-, L_{\omega,+} f_1)_{L^2_{\text{real}}} = -\mu (g_-, f_2)_{L^2_{\text{real}}} = 0.
\]

Moreover, it follows from (4.87) and (4.92) that

\[
\langle L_{\omega,+} f_1, f_1 \rangle_{H^{-1},H^1} = -(\mu f_2, f_1)_{L^2_{\text{real}}} = -\mu (f_1, f_2)_{L^2_{\text{real}}} < 0.
\]

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Note here that (A.15) and (A.16) show that $g_-$ and $f_1$ are linearly independent in $L^2_{real}(\mathbb{R}^d)$. We see from the hypothesis (A.14) and (A.15) that for any $a, b \in \mathbb{R}$,

$$\langle L_{\omega,+} \left( a f_1 + b \frac{g_-}{\|g_-\|_{L^2}} \right), \left( a f_1 + b \frac{g_-}{\|g_-\|_{L^2}} \right) \rangle_{H^{-1},H^1}$$

$$= a^2 \left( L_{\omega,+} f_1, f_1 \right)_{L^2_{rad}} + \frac{b^2}{\|g_-\|_{L^2}^2} \left( L_{\omega,+} g_-, g_- \right)_{H^{-1},H^1} + \frac{2ab}{\|g_-\|_{L^2}^2} \left( L_{\omega,+} g_-, f_1 \right)_{H^{-1},H^1}$$

$$\leq a^2 \left( L_{\omega,+} f_1, f_1 \right)_{L^2_{rad}} \leq 0.$$  

(A.17)

Put

$$e_1 := \frac{g_-}{\|g_-\|_{L^2}}, \quad e_2 := \frac{f_1 - (f_1, e_1)_{L^2_{rad}} e_1}{\|f_1 - (f_1, e_1)_{L^2_{rad}} e_1\|_{L^2}}.$$  

(A.18)

Then, it is clear that $\|e_1\|_{L^2} = \|e_2\|_{L^2} = 1$ and $(e_1, e_2)_{L^2_{rad}} = 0$. Moreover, we find from (A.17) that for any $\alpha, \beta \in \mathbb{R}$,

$$\langle L_{\omega,+} (\alpha e_1 + \beta e_2), \alpha e_1 + \beta e_2 \rangle_{H^{-1},H^1} \leq 0.$$  

(A.19)

Since Weyl’s essential spectrum theorem shows $\sigma_{ess}(L_{\omega,+}) = \sigma_{ess}(-\Delta + \omega^2) = [\omega^2, \infty)$, the min-max theorem (see Theorem 12.1 in [18]) together with (A.19) shows that the second eigen-value of $L_{\omega,+}$ is non-positive. However, this contradicts the fact that $L_{\omega,+}$ has only one non-positive eigen-value as an operator in $L^2_{rad}(\mathbb{R}^d)$ (see Lemma A.4). Thus, we have proved the claim (A.21).

Now, we are in a position to prove the claim (i). It follows from (A.1) and (2.8) that there exists $\omega_0 > 0$ such that for any $\omega \in (0, \omega_0)$ and any $g \in H^1(\mathbb{R}^d)$,

$$\langle L_{\omega,+} g, g \rangle_{H^{-1},H^1} \lesssim (\omega + \omega^{1 + \frac{d-2}{p+1}} \frac{4}{p} \|U\|_{L^{p+1}}^{p-1} + \omega^{(d-2)(p-1)} \|U\|_{L^{2^*}_{rad}}^{2^* - 2}) \|g\|_{H^1}^2.$$  

(A.20)

Thus, it suffices to show that

$$\inf_{\substack{g \in H^1_{rad}(\mathbb{R}^d, \mathbb{R}) \\|g\|_{H^1} = 1 \\langle g, f_2 \rangle_{L^2_{rad}} = 0}} \langle L_{\omega,+} g, g \rangle_{H^{-1},H^1} > 0,$$  

(A.21)

where $H^1_{rad}(\mathbb{R}^d, \mathbb{R})$ denotes the space of radial real-valued functions in $H^1(\mathbb{R}^d)$. Suppose for a contradiction that (A.21) failed. Then, we could take a sequence $\{g_n\}$ of radial real-valued functions in $H^1(\mathbb{R}^d)$ such that

$$\|g_n\|_{H^1} = 1,$$  

(A.22)

$$(g_n, f_2)_{L^2_{rad}} = 0,$$  

(A.23)

$$\lim_{n \to \infty} \langle L_{\omega,+} g_n, g_n \rangle_{H^{-1},H^1} = 0.$$  

(A.24)
Here, passing to some subsequence, we may assume that there exists a radial real-valued function \( g_{\infty} \in H^1(\mathbb{R}^d) \) such that
\[
\lim_{n \to \infty} g_n = g_{\infty} \quad \text{weakly in } H^1(\mathbb{R}^d).
\] (A.25)

If \( g_{\infty} = 0 \), then we see from (A.24) and (A.25) that for any \( \omega \in (0, 1) \),
\[
0 = \lim_{n \to \infty} \langle L_{\omega, +} g_n, g_n \rangle_{H^{-1}, H^1} = \lim_{n \to \infty} \langle L_{\omega, +} g_n, g_n \rangle_{L^2_{\text{real}}}
\]
\[
= \lim_{n \to \infty} \left\{ \omega \int_{\mathbb{R}^d} \|g_n\|^2 + \int_{\mathbb{R}^d} |\nabla g_n|^2 - p \int_{\mathbb{R}^d} \Phi_{\omega}^{-1} |g_n|^2 - (2^* - 1) \int_{\mathbb{R}^d} \Phi_{\omega}^\frac{2}{2^* - 2} |g_n|^2 \right\}
\]
\[
\geq \omega - \lim_{n \to \infty} \left\{ p \int_{\mathbb{R}^d} \Phi_{\omega}^{-1} |g_n|^2 + (2^* - 1) \int_{\mathbb{R}^d} \Phi_{\omega}^\frac{2}{2^* - 2} |g_n|^2 \right\}
\]
\[
= \omega,
\] (A.26)

which is a contradiction. Thus, we have found that \( g_{\infty} \) is nontrivial. Then, it follows from (A.24), the lower semicontinuity and (A.12) that
\[
0 = \lim_{n \to \infty} \langle L_{\omega, +} g_n, g_n \rangle_{H^{-1}, H^1} \geq \langle L_{\omega, +} g_{\infty}, g_{\infty} \rangle_{H^{-1}, H^1} > 0.
\] (A.27)

This absurd conclusion comes from the hypothesis that (A.21) failed. Thus, we have prove the claim (i).

Next, we prove the claim (ii). Let \( g \in H^1(\mathbb{R}^d) \) be a non-trivial, real-valued, radial function with \( (g, \Phi_{\omega}'_{L^2_{\text{real}}}) = 0 \). We write \( g \) in the form
\[
g = a \Phi_{\omega} + h,
\] (A.28)

where \( h \) satisfies \( (h, \Phi_{\omega})_{L^2_{\text{real}}} = 0 \). Note that the condition \( (g, \Phi_{\omega}')_{L^2_{\text{real}}} = 0 \) implies that
\[
a = -\frac{(h, \Phi_{\omega}')_{L^2_{\text{real}}}}{(\Phi_{\omega}, \Phi_{\omega}')_{L^2_{\text{real}}}}.
\] (A.29)

Hence, we find from \( (h, \Phi_{\omega})_{L^2_{\text{real}}} = 0 \) that
\[
\|g\|_{H^1}^2 = \langle g, g \rangle_{L^2_{\text{real}}} + \langle \nabla g, \nabla g \rangle_{L^2_{\text{real}}}
\]
\[
= a^2 \|\Phi_{\omega}\|_{H^1}^2 + \|h\|_{H^1}^2 - 2a \langle \nabla \Phi_{\omega}, \nabla h \rangle_{L^2_{\text{real}}}
\]
\[
\leq \frac{\|\Phi_{\omega}'\|_{L^2_{\text{real}}}^2 \|h\|_{L^2_{\text{real}}}^2}{(\Phi_{\omega}, \Phi_{\omega}')_{L^2_{\text{real}}}^2} \|\Phi_{\omega}\|_{H^1}^2 + \|h\|_{H^1}^2 + 2 \|\Phi_{\omega}'\|_{L^2_{\text{real}}} \|h\|_{L^2_{\text{real}}} \|\nabla \Phi_{\omega}\|_{L^2_{\text{real}}} \|\nabla h\|_{L^2_{\text{real}}}
\]
\[
\lesssim \|h\|_{H^1}^2.
\] (A.30)

Moreover, we see from \( L_{\omega, -} \Phi_{\omega} = 0 \) and Lemma [A.1] that
\[
\langle L_{\omega, -} g, g \rangle_{H^{-1}, H^1} = \langle L_{\omega, -} (a \Phi_{\omega} + h), a \Phi_{\omega} + h \rangle_{H^{-1}, H^1}
\]
\[
= \langle L_{\omega, -} h, h \rangle_{H^{-1}, H^1} \geq \|h\|_{H^1}^2.
\] (A.31)
Putting (A.30) and (A.31) together, we obtain

\[ \|g\|_{H^\frac{1}{2}}^2 \lesssim \langle L_\omega, -g, g \rangle_{H^{-1}, H^1}. \]  (A.32)

The opposite relation follows from (A.1). Hence, we have completed the proof. \(\square\)

## B Small-data and Perturbation theories

We record a small-data theory for \((\text{NLS})\) and \((\text{NLS}^\dagger)\):

**Lemma B.1.** Let \(\sigma_\infty\) denote the identity operator 1 or \(|\nabla|^{-1}\langle\nabla\rangle\). Then, for any \(A > 0\), there exists \(\delta_0 > 0\) with the following property: for any \(\psi_0 \in H^1(\mathbb{R}^d)\) satisfying \(\|\psi_0\|_{H^1} \leq A\) and \(\|\nabla e^{it\Delta} \sigma_\infty \psi_0\|_{V^1(\mathbb{R}) \cap W(\mathbb{R})} < \delta_0\), and for any \(t_\infty \in \mathbb{R} \cup \{\pm \infty\}\), there exists a function \(\psi \in C(\mathbb{R}, H^1(\mathbb{R}^d))\) such that:

- when \(\sigma_\infty = 1\), \(\sigma_\infty \psi\) is a global solution to \((\text{NLS})\) with \(\psi(t_\infty) = \psi_0\), and
- when \(\sigma_\infty = |\nabla|^{-1}\langle\nabla\rangle\), \(\sigma_\infty \psi\) is a global solution to \((\text{NLS}^\dagger)\) with \(\psi(t_\infty) = \psi_0\). Here, if \(t_\infty \in \{\pm \infty\}\), we regard \(\sigma_\infty \psi\) as a solution of the final value problem.

Furthermore, we have

\[ \|\nabla \sigma_\infty \psi\|_{V^1(\mathbb{R}) \cap W(\mathbb{R})} \lesssim \|\nabla e^{it\Delta} \sigma_\infty \psi_0\|_{V^1(\mathbb{R}) \cap W(\mathbb{R})}, \]  (B.1)

\[ \|\langle\nabla\rangle \psi\|_{S(I)} \lesssim \|\psi_0\|_{H^1}. \]  (B.2)

**Proof of Lemma B.1.** This lemma is just a combination of the small-data theories for \((\text{NLS})\) and \((\text{NLS}^\dagger)\). \(\square\)

We can control the norm of the full Strichartz space by a few particular norms:

**Lemma B.2.** Assume \(d \geq 3\), and let \(A_1, A_2, A_3 > 0\). Then, there exists a constant \(C(A_1, A_2, A_3) > 0\) with the following property:

(i) for any interval \(I\) and any space-time function \(u\) satisfying

\[ \|u\|_{L^\infty(I, H^1)} \leq A_1, \]  (B.3)

\[ \|u\|_{W^{p,1}(I) \cap W(I)} \leq A_2, \]  (B.4)

\[ \|\langle\nabla\rangle e^{\Delta} \psi\|_{L^{2(d+2)}_{t,x}(I)} \leq A_3, \]  (B.5)

we have

\[ \|\langle\nabla\rangle u\|_{S(I)} \leq C(A_1, A_2, A_3). \]  (B.6)

(ii) for any interval \(I\) and any space-time function \(u\) satisfying

\[ \|\nabla u\|_{L^\infty(I, L^2)} \leq A_1, \]  (B.7)

\[ \|u\|_{L^1(I)} \leq A_2, \]  (B.8)

\[ \|\nabla e^{\Delta} \psi\|_{L^{2(d+2)}_{t,x}(I)} \leq A_3, \]  (B.9)
we have
\[ \| \nabla u \|_{S(I)} \leq C(A_1, A_2, A_3). \] (B.10)

**Proof of Lemma B.2.** See the proof of Lemma 4.1 in [3]. □

Next, we shall introduce the long-time perturbation theory for the equation (NLS). In the higher dimensions \( d \geq 6 \), we need "exotic Strichartz spaces" to treat the non-Lipschitz nonlinearity. In the paper [3] by the same authors, we introduced the spaces \( ES \) and \( ES^* \) for which the following estimate holds: for any interval \( I \) and any \( t_* \in I \),
\[
\| u \|_{ES(I)} \lesssim \| e^{i(t-t_*)\Delta} u(t_*) \|_{ES(I)} + \| F[u] \|_{ES^*(I)} + \| \langle \nabla \rangle e[u] \|_{L^{\frac{2(d+2)}{d+4}}(I)}.
\] (B.11)

Now, we record the long-time perturbation theory.

**Lemma B.3.** Assume \( d \geq 3 \). Let \( I \) be an interval, \( \psi \in C(I, H^1(\mathbb{R}^d)) \) be a solution to (NLS), and let \( u \) be a function in \( C(I, H^1(\mathbb{R}^d)) \). Moreover, let \( A > 0 \), \( A' > 0 \), \( B > 0 \), and assume that
\[
\| u \|_{L^\infty_t H^1_x(I)} \leq A, \quad \| \psi \|_{L^\infty_t H^3_x(I)} \leq A', \quad \| \langle \nabla \rangle u \|_{V^{p+1}(I)} \leq B.
\] (B.12-14)

Then, there exists constants \( \gamma(A, A', B) > 0 \), \( \delta(A, A', B) > 0 \), \( C(A, A', B) > 0 \) and a continuous function \( \tilde{c} : (0, \infty) \times (0, \infty) \rightarrow (0, \infty) \) with the following properties:
\[
\lim_{|(\gamma, \delta)| \rightarrow 0} \tilde{c}(\gamma, \delta) = 0; \quad \text{and if}
\] (B.15)

\[
\| \psi(t_1) - u(t_1) \|_{H^1} \leq \gamma \] (B.16)
\[
\| \langle \nabla \rangle e[u] \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I)} \leq \delta, \] (B.17)
\[
\| \langle \nabla \rangle e^{i(t-t_1)\Delta} \{ \psi(t_1) - u(t_1) \} \|_{V^{p+1}(I)} \leq \delta \] (B.18)

for some \( 0 < \gamma < \gamma(A, A', B), \) \( 0 < \delta < \delta(A, A', B) \) and \( t_1 \in I \), then we have
\[
\| \langle \nabla \rangle \psi \|_{S(I)} \leq C(A, A', B), \] (B.19)
\[
\sup_{t \in I} \| \psi(t) - u(t) \|_{H^1} \leq C(A, A', B) \tilde{c}(\gamma, \delta). \] (B.20)

**Proof of Lemma B.3.** The proof is identical to [23], except that we use \( ES \) and \( ES^* \) as exotic Strichartz spaces (see [3]). □
### Table of notation

| Symbols | Description or equation number |
|---------|--------------------------------|
| \( M \) | (1.2) |
| \( H, H^\dagger, H^\ddagger \) | (1.3), (1.31), (1.24) |
| \( S, S^\dagger, S^\ddagger \) | (1.7), (1.36), (8.169) |
| \( \mathcal{K}, \mathcal{K}^\dagger, \mathcal{K}^\ddagger \) | (1.5), (1.46), (1.33), (1.23) |
| \( I, I^\dagger, I^\ddagger \) | (1.10), (1.47), (1.22), (8.169) |
| \( J \) | (1.11) |
| \( \Phi, U \) | positive, radial ground states of \( \omega \text{-SP} \) and \( \omega \text{-SP}^\dagger \) |
| \( W \) | (1.18) |
| \( m_\omega, \sigma, \nu_\omega \) | (1.8), (1.19), (4.20) |
| \( PW, PW^\dagger, PW_{\omega,+}, PW_{\omega,-} \) | (1.13), (1.38), (1.40), (1.41) |
| \( PW^\dagger, PW_{\omega,+}^\dagger \) | (1.30), (1.34), (1.37) |
| \( T \) | (1.27) |
| \( s_p \) | (1.32) |
| \( L, L_{\omega,+}, L_{\omega,-} \) | (4.5), (4.7), (4.8) |
| \( L^\dagger, L^- \) | (1.44), (1.45) |
| \( \tilde{L}, \tilde{L}_{\omega,+}, \tilde{L}_{\omega,-} \) | (1.48), (1.49) |
| \( \mu \) | (4.53) |
| \( \mathcal{U}_\pm \) | (4.15) |
| \( \lambda_1(t), \lambda_2(t) \) | (4.82) |
| \( \Gamma(t) \) | (4.79), (6.15) |
| \( d_\omega(\psi(t)), \tilde{d}_\omega(\psi(t)) \) | (1.77), (5.22), (7.1), (7.3) |
| \( \delta_E, \delta_X, \delta_S, \delta_\Sigma \) | (5.5) |
| \( T_X \) | (6.20) |
| \( \kappa(t) \) | (3.1), (8.1), (8.14) |
| \( A_{\omega, S}^{\Sigma}, S_{\omega,R}^{\Sigma}, S_{\omega,R,+}^{\Sigma} \) | (1.50), (1.51), (8.58) |
| \( St(I), V_q(I), V(I), W_q(I), W(I), W^j(I) \) | (8.40), (8.41) |
| \( G, g \) | (8.42), (8.49) |
| \( \sigma_n, \sigma_{n,\infty} \) | (8.34), (8.35), (8.36) |
| \( x_n, x_n^j, x_n^\dagger, \lambda'_n \) | (8.37), (8.38) |
| \( \tilde{w}_n^j, w_n^j \) | (8.57), (8.65) |
| \( \tilde{\psi}_n^j, \psi_n^j \) | (8.126) |

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