THE U(1)-KEPLER PROBLEMS

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Abstract. Let \( n \geq 2 \) be a positive integer. To each irreducible representation \( \sigma \) of \( U(1) \), a \( U(1) \)-Kepler problem in dimension \( (2n - 1) \) is constructed and analyzed. This system is super integrable and when \( n = 2 \) it is equivalent to a MICZ-Kepler problem. The dynamical symmetry group of this system is \( \hat{U}(n, n) \), and the Hilbert space of bound states \( H(\sigma) \) is the unitary highest weight representation of \( \hat{U}(n, n) \) with highest weight

\[
\left( \frac{-1}{2}, \ldots, \frac{-1}{2}, \frac{1}{2} + \bar{\sigma}, \frac{1}{2}, \ldots, \frac{1}{2} \right)
\]

when \( \bar{\sigma} \geq 0 \) or

\[
\left( \frac{-1}{2}, \ldots, \frac{-1}{2}, \frac{-1}{2} + \bar{\sigma}, \frac{1}{2}, \ldots, \frac{1}{2} \right)
\]

when \( \bar{\sigma} \leq 0 \). (Here \( \bar{\sigma} \) is the infinitesimal character of \( \sigma \).) Furthermore, it is shown that the correspondence between \( \sigma^* \) (the dual of \( \sigma \)) and \( H(\sigma) \) is the theta-correspondence for dual pair \( (U(1), U(n, n)) \) in \( Sp(4n, \mathbb{R}) \).

1. Introduction

The Kepler problem is a well-known physics problem in dimension three about two bodies which attract each other by a force proportional to the inverse square of their distance. What is less known about the Kepler problem is the fact that it is super integrable\(^1\) at both the classical and the quantum level, and belongs to a big family of super integrable models. One interesting such family is the family of MICZ-Kepler problems\(^1\)\(^2\).

In our recent research on the Kepler problem\(^3\)\(^4\)\(^5\)\(^6\), a dominant theme is the construction of new super integrable models of Kepler type on the one hand and the exhibition of the close relationship of these super integrable models with certain unitary highest weight modules for a real non-compact Lie group of Hermitian type on the other hand. Depending on the interests of the readers, one may view this investigation either as a journey to discover new super integrable models or as an effort to better understand the geometry of certain unitary highest weight modules. Here we continue this theme.

In Ref.\(^6\), a new family of super integrable models of the Kepler type — the \( O(1) \)-Kepler problems — has been constructed and analyzed, and their intimate

\(^1\)A physics model is called super integrable if the number of independent symmetry generators is bigger than the number of degree of freedom. For the Kepler problem, the degree of freedom is 3 and the number of independent symmetry generators is 5.
relationship with the theta-correspondence was uncovered. The purpose here is to construct and analyze their complex analogues — the U(1)-Kepler problems.

Recall that, in the construction of the O(1)-Kepler problems in dimension $n$, the canonical bundle

$$O(1) \to S^{n-1} \to \mathbb{R}P^{n-1}$$

plays a pivotal role. Here the corresponding bundle is

$$U(1) \to S^{2n-1} \to \mathbb{C}P^{n-1}.$$

Later we shall demonstrate that, a model with $n=2$ constructed in this paper is equivalent to a MICZ-Kepler problem. That is why the models constructed here are called the U(1)-Kepler problems.

1.1. Statement of the Main Results. Before stating our main result, let us fix some notations:

- $n$ — an integer which is at least 2;
- $\sigma$ — an irreducible representation of $U(1)$, it also denotes the underlying representation space of $\sigma$;
- $\sigma^*$ — the dual of $\sigma$, it also denotes the underlying representation space of $\sigma^*$;
- $\bar{\sigma}$ — the infinitesimal character of $\sigma$, so it is an integer;
- $\hat{U}(n)$ — the nontrivial double cover of $U(n)$;
- $\hat{U}(n)\times\hat{U}(n)$ — the double cover of $U(n) \times U(n)$ that corresponds to the homomorphism $\pi_1(U(n) \times U(n)) = \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2$ sending $(a,b)$ to $a+b$ mod 2;
- $\check{U}(n,n)$ — the double cover of $U(n,n)$ such that $(\check{U}(n,n), U(n)\times U(n))$ is the double cover of $(U(n,n), U(n) \times U(n))$;
- $\kappa$ — a half integer;
- $l$ — a nonnegative integer;
- $R^{\kappa}_{\tilde{n}}$ — the highest weight module of $\check{U}(n)$ with highest weight

$$(\kappa + l, \kappa, \cdots, \kappa_{n-1});$$

- $R^{\kappa}_{\tilde{n}}$ — the highest weight module of $\check{U}(n)$ with highest weight

$$(\kappa, \cdots, -\kappa_{n-1}, -(\kappa + l)).$$

We are now ready to state the main results on the U(1)-Kepler problems.

**Theorem 1.** Let $n \geq 2$ be an integer, $\sigma$ an irreducible representation of $U(1)$, $\sigma^*$ the dual of $\sigma$, and $\bar{\sigma}$ the infinitesimal character of $\sigma$. For integer $l \geq 0$, the highest weight modules of $\check{U}(n)$ with highest weight $(1/2 + l, 1/2, \cdots, 1/2)_{n-1}$

$$(1/2, \cdots, 1/2, -(1/2 + l))$$

respectively) is denoted by $R^{1/2}_{\tilde{n}}$ ($R^{\kappa}_{\tilde{n}}$ respectively).

For the $(2n-1)$-dimensional U(1)-Kepler problem with magnetic charge $\sigma$, the following statements are true:
1) The bound state energy spectrum is

\[ E_I = -\frac{1/2}{(I + \frac{n+|\bar{\sigma}|}{2})^2} \]

where \( I = 0, 1, 2, \ldots \). 

2) There is a natural unitary action of \( \tilde{\text{U}}(n,n) \) on the Hilbert space \( \mathcal{H}(\sigma) \) of bound states, which extends the manifest unitary action of \( \text{U}(n) \) — the diagonal of \( \text{U}(n) \times \text{U}(n) \). In fact, \( \mathcal{H}(\sigma) \) is the unitary highest weight module of \( \tilde{\text{U}}(n,n) \) with highest weight

\[ \left( -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2} + \bar{\sigma}, \frac{1}{2}, \ldots, \frac{1}{2} \right) \]

when \( \bar{\sigma} \geq 0 \) or

\[ \left( -\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{1}{2} + \bar{\sigma}, \frac{1}{2}, \ldots, \frac{1}{2} \right) \]

when \( \bar{\sigma} \leq 0 \).

3) When restricted to the maximal compact subgroup \( \text{U}(n) \times \text{U}(n) \), the above action yields the following orthogonal decomposition of \( \mathcal{H}(\sigma) \):

\[ \mathcal{H}(\sigma) = \bigoplus_{I=0}^{\infty} \mathcal{H}_I(\sigma) \]

where, as an irreducible \( \text{U}(n) \times \text{U}(n) \)-module,

\[ \mathcal{H}_I(\sigma) \cong \begin{cases} \mathcal{R}_{1/2}^{1/2} \otimes \mathcal{R}_{1/2}^{1/2} & \text{when } \bar{\sigma} \geq 0 \\ \mathcal{R}_{1/2}^{1/2} \otimes \mathcal{R}_{1/2}^{1/2} & \text{when } \bar{\sigma} \leq 0. \end{cases} \]

4) \( \mathcal{H}_I(\sigma) \) in part 3) is the energy eigenspace with eigenvalue \( E_I \) in part 1).

5) The correspondence between \( \sigma^* \) and \( \mathcal{H}(\sigma) \) is the theta-correspondence\(^2\) for dual pair \( (\text{U}(1), \text{U}(n,n)) \subset \text{Sp}(4n, \mathbb{R}) \).

For readers who are familiar with the Enright-Howe-Wallach classification diagram \(^7\) for the unitary highest weight modules, we would like to point out that, the unitary highest weight module identified in part 2) of this theorem occurs at the rightmost nontrivial reduction point of the classification diagram. Note that, part 3) of the above theorem is a multiplicity free K-type formula.

In section 2, we introduce the models and show that when \( n = 2 \) they are equivalent to the MICZ-Kepler problems. In section 3, we give a detailed analysis of the models and finish the proof of Theorem 1.

2. The models

Let \( n \geq 2 \) be an integer, \( \mathbb{C}^*_n = \mathbb{C}^n \setminus \{0\} \), and \( \sigma \) an irreducible unitary representation of \( \text{U}(1) \). Denote by \( \bar{\sigma} \) be the infinitesimal character of \( \sigma \).

Consider the principal bundle

\[ \text{U}(1) \to \mathbb{C}^*_n \to \mathbb{C}P^n \]

\(^2\)See Ref. \(^8\) for details on reductive dual pairs and theta-correspondence.
where \( \mathbb{C}P^n \cong \mathbb{R}_+ \times \mathbb{C}P^{n-1} \) is the quotient space of \( \mathbb{C}^n \) under the equivalence relation \( Z \sim \alpha \cdot Z, \alpha \in \text{U}(1) \). We shall assume the Euclidean metric on \( \mathbb{C}^n \) and the resulting quotient Riemannian metric on \( \mathbb{C}P^n \).

We use \((\rho, \Phi)\) to denote the polar coordinates on \( \mathbb{C}P^n \) and \( \gamma_\sigma \) to denote the vector bundle associated to representation \( \sigma \). Then \( \rho([Z]) = |Z| \), \( \Phi \) is a coordinate on \( \mathbb{C}P^{n-1} \), and \( \gamma_\sigma \) is a hermitian line bundle over \( \mathbb{C}P^n \) with a natural hermitian connection \( \mathcal{A} \). Here is a technical lemma which will be used later.

**Lemma 2.1.** 1) Let \( Z \in \mathbb{C}^n \) and \( \rho = |Z| = \sqrt{\overline{Z} \cdot Z} \). Then we have the following identity for the Euclidean metric on \( \mathbb{C}^n \):

\[
|dZ|^2 = d\rho^2 + \rho^2 \left( ds_{FS}^2 + \left( \frac{\text{Im}(\overline{Z} \cdot dZ)}{|Z|^2} \right)^2 \right)
\]

where “Im” in \( \text{Im}(\overline{Z} \cdot dZ) \) stands for “the imaginary part of”, and \( ds_{FS}^2 \) is the Fubini-Study metric on \( \mathbb{C}P^{n-1} \), i.e.,

\[
ds_{FS}^2 = \frac{|dZ|^2}{|Z|^2} - \frac{|Z \cdot d\overline{Z}|^2}{|Z|^4}.
\]

Consequently, the quotient metric on \( \mathbb{C}P^n \) is

\[
ds_{\mathbb{C}P^n}^2 = d\rho^2 + \rho^2 ds_{FS}^2.
\]

2) If we take the invariant Riemannian metric on \( \text{U}(n) \) as the restriction of the Euclidean metric on \( \mathbb{C}^{n^2} \), then the resulting quotient metric on \( \mathbb{C}P^{n-1} = \frac{\text{U}(n)}{\text{U}(n-1) \times \text{U}(1)} \) is twice of the Fubini-Study metric. Consequently,

\[
\Delta_\mathcal{A}|_{\mathbb{C}P^{n-1}} = 2 (c_2[\text{U}(n)] - c_2[\text{U}(1)])|_\sigma
\]

where \( \Delta_\mathcal{A}|_{\mathbb{C}P^{n-1}} \) is the (non-negative) Laplace operator acting on sections of \( \gamma_\sigma|_{\mathbb{C}P^{n-1}} \), \( c_2[\text{U}(n)] \) is the Casimir operator of \( \text{U}(n) \), and \( c_2[\text{U}(1)]|_\sigma \) is the value of the Casimir operator of \( \text{U}(1) \) at \( \sigma \).

**Proof.** 1) Let \( Z = (Z_1, \cdots, Z_n) \in \mathbb{C}^n \). The Euclidean metric on \( \mathbb{C}^n \) is just \( |dZ|^2 = dZ \cdot d\overline{Z} \). Since \( \rho^2 = |Z|^2 = Z \cdot \overline{Z} \), \( 2d\rho d\rho = Z \cdot d\overline{Z} + \overline{Z} \cdot dZ \), and

\[
d\rho^2 = \frac{2 |Z \cdot d\overline{Z}|^2 + (Z \cdot d\overline{Z})^2 + (\overline{Z} \cdot dZ)^2}{4|Z|^2}
\]

A computation shows that

\[
|dZ|^2 = d\rho^2 + \rho^2 \left( ds_{FS}^2 + \left( \frac{\text{Im}(\overline{Z} \cdot dZ)}{|Z|^2} \right)^2 \right).
\]

The right \( \text{U}(1) \) action on \( \mathbb{C}^n \) generates a vector field on \( \mathbb{C}^n \) whose value at point \( Z_0 \in \mathbb{C}^n \) is \((Z_0, iZ_0) \in T_{Z_0} \mathbb{C}^n \). Let \( V_{Z_0} \) be the orthogonal complement of span\( \mathbb{R} \{iZ_0\} \) in \( \mathbb{C}^n = \mathbb{R}^{2n} \), then

\[
V_{Z_0} = \{ W \in \mathbb{C}^n \mid \text{Im}(\overline{Z}_0 \cdot W) = 0 \}.
\]

To finish the proof of part 1), we just need to check that quadratic form

\[
\left( \text{Im}(\overline{Z} \cdot dZ) \right)^2|_{Z_0}
\]

is the quotient metric on \( S^{2n-1}/\text{U}(1) \). On \( \mathbb{C}P^1 = S^2 \), it is \( \frac{1}{4} \) times the standard round metric.

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\( \text{Im} \) is the quotient metric on \( S^{2n-1}/\text{U}(1) \). On \( \mathbb{C}P^1 = S^2 \), it is \( \frac{1}{4} \) times the standard round metric.
vanishes when restricted to \( \{ Z_0 \} \times V_{Z_0} \). Indeed, if \( v := (Z_0, W) \in \{ Z_0 \} \times V_{Z_0} \), then
\[
(Im(\bar{Z} \cdot dZ))^2 \bigg|_{Z_0} \langle v, v \rangle = (Im(\bar{Z}_0 \cdot W))^2 = 0.
\]

2) The Euclidean metric on \( \mathbb{C}^{n^2} \) is \( \text{tr}(dM^\dagger \cdot dM) \), so the Riemannian metric on \( \mathbb{U}(n) \) is \( ds_{\mathbb{U}(n)}^2 = \text{tr}(dg^\dagger \cdot dg) \). Since the action of \( \mathbb{U}(n) \) on \( \mathbb{C}P^{n-1} = \frac{U(n)}{U(n-1) \times \mathbb{U}(1)} \) is transitive and both the Fubini-Study metric and the quotient metric here are \( \mathbb{U}(n) \) invariant, we just need to verify the statement at point \( p := [1, 0, \ldots, 0] \in \mathbb{C}P^{n-1} \).

In the following, if \( X \) is a Riemannian manifold, we shall use \( \langle \cdot , \cdot \rangle \big|_X \) to denote the Riemannian inner product on a tangent space of \( X \).

Let \( I \in \mathbb{U}(1) \) be the identity matrix. Let \( u, v \in T_I \mathbb{U}(n) \) be two tangent vectors which are orthogonal to \( I \) (2.4) then implies that
\[
\langle u, v \rangle|_{\mathbb{U}(n)} = 2 \text{Re}(a^\dagger b). \quad \text{Let}
\]
\[
\hat{p} = (1, 0, \ldots, 0)^T \in S^{2n-1},
\]
and \( \hat{u}, \hat{v} \) be the image of \( u, v \) respectively under the linearization at \( I \) of the quotient map
\[
\mathbb{U}(n) \to S^{2n-1}, \quad A \mapsto A\hat{p}.
\]
Then
\[
\hat{u} = \left( \hat{p}, \begin{pmatrix} 0 \\ a \end{pmatrix} \right), \quad \hat{v} = \left( \hat{p}, \begin{pmatrix} 0 \\ b \end{pmatrix} \right),
\]
so \( \langle \hat{u}, \hat{v} \rangle|_{S^{2n-1}} = \text{Re}(a^\dagger b) \). Therefore,
\[
(2.4) \quad \langle u, v \rangle|_{\mathbb{U}(n)} = 2 \langle \hat{u}, \hat{v} \rangle|_{S^{2n-1}}.
\]

Let \( \hat{u}, \hat{v} \) be the image of \( u, v \) respectively under the linearization at \( I \) of the quotient map
\[
\mathbb{U}(n) \to \mathbb{C}P^{n-1}, \quad A \mapsto [A\hat{p}].
\]
One can see that \( u, v \) are respectively the horizontal lift of \( \hat{u}, \hat{v} \), then, by definition,
\[
\langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle|_{\mathbb{U}(n)}.
\]
Here \( \langle \cdot , \cdot \rangle \big|_{\mathbb{U}(n)} \) denote the quotient metric on \( \mathbb{C}P^{n-1} = \frac{U(n)}{U(n-1) \times \mathbb{U}(1)} \).

On the other hand, one can see that \( \hat{u} \) and \( \hat{v} \) are respectively the horizontal lift of \( u \) and \( v \) in fiber bundle \( S^{2n-1} \to \mathbb{C}P^{n-1} \), then, by definition,
\[
\langle u, v \rangle_{FS} = \langle \hat{u}, \hat{v} \rangle|_{S^{2n-1}}.
\]
Eq. (2.4) then implies that \( \langle u, v \rangle = 2 \langle \hat{u}, \hat{v} \rangle_{FS} \). In view of the fact that the Laplace operator gets multiplied by \( 1/a \) if the Riemannian metric gets multiplied by a number \( a \), Eq. (2.3) follows from Lemma [A.1] in appendix [A].

\( \square \)

We are now ready to introduce the notion of \( \mathbb{U}(1) \)-Kepler problems.
Definition 2.2. Let $n \geq 2$ be an integer and $\sigma$ an irreducible representation of $U(1)$. The $U(1)$-Kepler problem in dimension $(2n - 1)$ with magnetic charge $\sigma$ is the quantum mechanical system for which the wave functions are smooth sections of $\gamma_\sigma$, and the hamiltonian is

$$
H = -\frac{1}{8\rho} \Delta_A \frac{1}{\rho} + \frac{\bar{\sigma}^2 + (n - \frac{5}{4})}{8\rho^2} - \frac{1}{\rho^2}
$$

where $\Delta_A$ is the (non-positive) Laplace operator on $\mathbb{C} \mathbb{P}^n$ twisted by $\gamma_\sigma$, $\bar{\sigma}$ is the infinitesimal character of $\sigma$, and $\rho(|Z|) = |Z|$.

Note that $\mathbb{R}^2_\rho$ and $\mathbb{C} \mathbb{P}^2$ are diffeomorphic. We use $(r, \Theta)$ to denote the polar coordinates on $\mathbb{R}^2_\rho$ and $(\rho, \Phi)$ to denote the polar coordinates on $\mathbb{C} \mathbb{P}^2$. Let $\pi: \mathbb{C} \mathbb{P}^2 \to \mathbb{R}^2_\rho$ be the diffeomorphism such that $\pi(\rho, \Phi) = (\rho^2, \Phi)$, then

$$
\pi^*(d\rho^2 + r^2 d\Theta^2) = 4\rho^2 (d\rho^2 + \rho^2 ds^2_{\mathbb{C} \mathbb{P}^2}) \quad \text{and} \quad \pi^*(\text{vol}_{\mathbb{R}^2_\rho}) = (2\rho)^3 \text{vol}_{\mathbb{C} \mathbb{P}^2}.
$$

Let $\gamma(\bar{\sigma})$ be the pullback of $\gamma_\sigma$ by $\pi^{-1}$. It is clear that $\gamma(\bar{\sigma})$ is a hermitian line bundle over $\mathbb{R}^2_\rho$ with a natural hermitian connection $A$. Recall from Refs. [12, 13] that the MICZ Kepler problem with magnetic charge $\mu \in \frac{1}{2} \mathbb{Z}$ is the quantum mechanical system for which the wave functions are smooth sections of $\gamma(\bar{\sigma})$ ($\mu = \bar{\sigma}/2$), and the hamiltonian is

$$
\hat{h}_\mu = -\frac{1}{2} \Delta_A + \frac{\mu^2}{2r^2} - \frac{1}{r}
$$

where $\Delta_A$ is the (non-positive) Laplace operator on $\mathbb{R}^3_\rho$ twisted by $\gamma(\bar{\sigma})$ and $r(x) = |x|$. We are now ready to state the following

Proposition 2.3. The $U(1)$-Kepler problem in dimension three with magnetic charge $\sigma$ is equivalent to the MICZ-Kepler problem with magnetic charge $\bar{\sigma}/2$.

Proof. Let $\Psi_i$ ($i = 1$ or 2) be a wave-section for the MICZ-Kepler problem with magnetic charge $\bar{\sigma}/2$, and

$$
\psi_i(\rho, \Phi) := (2\rho)^{\frac{\bar{\sigma}}{2}} \pi^*(\Psi_i)(\rho, \Phi) = (2\rho)^{\frac{\bar{\sigma}}{2}} \Psi_i(\rho^2, \Phi).
$$

Then it is not hard to see that

$$
\int_{\mathbb{C} \mathbb{P}^2} \overline{\psi_1} \psi_2 \text{vol}_{\mathbb{C} \mathbb{P}^2} = \int_{\mathbb{C} \mathbb{P}^2} \pi^*(\overline{\Psi_1}) \pi^*(\Psi_2) \pi^*(\text{vol}_{\mathbb{R}^2_\rho}) = \int_{\mathbb{R}^3_\rho} \overline{\Psi_1} \Psi_2 \text{vol}_{\mathbb{R}^3_\rho}
$$

and

$$
\int_{\mathbb{C} \mathbb{P}^2} \overline{\psi_1} H \psi_2 \text{vol}_{\mathbb{C} \mathbb{P}^2} = \int_{\mathbb{C} \mathbb{P}^2} \pi^*(\overline{\Psi_1}) \frac{1}{\rho^2} H \rho^2 \pi^*(\Psi_2) \pi^*(\text{vol}_{\mathbb{R}^2_\rho}) = \int_{\mathbb{R}^3_\rho} \overline{\Psi_1} \hat{h}_{\bar{\sigma}/2} \Psi_2 \text{vol}_{\mathbb{R}^3_\rho}.
$$

Here we have used the fact that

$$
\frac{1}{\rho^2} H \rho^2 = -\frac{1}{8\rho^{5/2}} \left( \frac{1}{\rho^2} \partial_\rho^2 \rho^2 \partial_\rho - \frac{1}{\rho^2} \Delta_A |_{\mathbb{C} \mathbb{P}^1} \right) \rho^{1/2} + \frac{\bar{\sigma}^2 + 3/4}{8\rho^4} - \frac{1}{\rho^2}
$$

$$
= -\frac{1}{2} \left( \frac{1}{\rho^2} \partial_\rho^2 \rho^2 \partial_\rho - \frac{1}{r^2} \Delta_A |_{\mathbb{C} \mathbb{P}^1} \right) + \frac{\bar{\sigma}^2}{8r^2} - \frac{1}{r}
$$

$$
= -\frac{1}{2} \Delta_A + \frac{(\bar{\sigma}/2)^2}{2r^2} - \frac{1}{r} = \hat{h}_{\bar{\sigma}/2}.
$$

□
3. The Dynamical Symmetry Analysis

Let \( \psi \) be an eigensection of \( H \) in Eq. (2.5) with eigenvalue \( E \), so \( \psi \) is square integrable with respect to volume form \( \mathrm{vol}_{\mathbb{C}P^n} \), and

\[
(3.1) \quad \left( -\frac{1}{8\rho} \Delta_4 + \frac{1}{\rho} + \frac{\sigma^2}{8\rho^4} + \frac{(n-\frac{5}{2})}{\rho^2} - \frac{1}{\rho^2} \right) \psi = E\psi.
\]

We shall solve this eigenvalue problem by separating the angles from the radius.

In view of the branching rule for \((U(n), U(n-1) \times U(1))\), the Frobenius reciprocity theorem implies that, as modules of \( U(n) \),

\[
(3.2) \quad L^2(\gamma_{\sigma}|_{\mathbb{C}P^{n-1}}) = \bigoplus_{p,q \in \mathbb{N}} \mathcal{R}_{p,q}^{\sigma}
\]

where \( \mathcal{R}_{p,q}^{\sigma} \) is the irreducible and unitary representation of \( U(n) \) with the highest weight \((p, 0, \cdots, 0, -q)\) subject to condition \( p - q = \bar{\sigma} \). For nonnegative integers \( p, q \) with \( p - q = \bar{\sigma} \), we introduce the notation

\[
l = \begin{cases} 
q & \text{if } \sigma \geq 0 \\
p & \text{if } \sigma \leq 0,
\end{cases}
\]

and rewrite \( \mathcal{R}_{p,q}^{\sigma} \) as \( \mathcal{R}_{l}^{\sigma} \), then Eq. (3.2) becomes

\[
(3.3) \quad L^2(\gamma_{\sigma}|_{\mathbb{C}P^{n-1}}) = \bigoplus_{l=0}^{\infty} \mathcal{R}_{l}^{\sigma}.
\]

Let \( \{ Y_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{I}(l) \} \) be a minimal spanning set for \( \mathcal{R}_{l}^{\sigma} \). Write \( \psi(x) = \hat{R}_{kl}(\rho)Y_{\mathbf{m}}(\Phi) \). After separating out the angular variables with the help of Eq. (2.3) in Lemma 2.1, Eq. (3.1) becomes

\[
\left( -\frac{1}{8\rho^{2n-1}} \partial_\rho \rho^{2n-2} \partial_\rho + \frac{(l + |\sigma|)^2}{2\rho^4} + \frac{(n-1)(l + |\sigma|) + \frac{1}{4}(n-\frac{5}{2})}{\rho^2} - \frac{1}{\rho^2} \right) \hat{R}_{kl} = E\hat{R}_{kl}.
\]

where \( \hat{R}_{kl} \in L^2(\mathbb{R}^+, \rho^{2n-2} d\rho) \). Let \( R_{k(l+|\sigma|)}(t) = \hat{R}_{kl}(\sqrt{t})/t^{3/4} \), then we have \( R_{k(l+|\sigma|)} \in L^2(\mathbb{R}^+, t^n dt) \) and

\[
\left( -\frac{1}{2t^n} \partial_t t^n \partial_t + \frac{(l + |\sigma|)^2}{2t^2} + \frac{(n-1)(l + |\sigma|) - 1}{t} \right) R_{k(l+|\sigma|)} = ER_{k(l+|\sigma|)}.
\]

By quoting results from appendix A in Ref. [6], we have

\[
E_{k(l+|\sigma|)} = -\frac{1/2}{(k + l + \frac{|\sigma| + n}{2} - 1)^2}
\]

where \( k = 1, 2, 3, \cdots \). Let \( I = k - 1 + l \), then the bound energy spectrum is

\[
E_I = -\frac{1/2}{(I + \frac{n+|\sigma|}{2})^2}
\]

where \( I = 0, 1, 2, \cdots \). Part 1) of Theorem 1 is proved. Moreover, since \( \hat{R}_{kl}(\rho) = \rho^{2} R_{k(l+|\sigma|)}(\rho^2) \), we have

\[
\hat{R}_{kl}(\rho) = c(k, l + \frac{|\sigma|}{2}) \rho^{2l+|\sigma|+3/2} L_{k-1}^{2l+|\sigma|+n-1} \left( \frac{2\rho^2}{I + \frac{n+|\sigma|}{2}} \right) \exp \left( -\frac{\rho^2}{I + \frac{n+|\sigma|}{2}} \right).
\]
For each integer $I \geq 0$, we let $\mathcal{H}_I(\sigma)$ be the linear span of 
\[ \{ \tilde{R}_{kl} Y_{lm} \mid m \in \mathcal{I}(l), k - 1 + l = I \}, \]
then
\begin{equation}
\mathcal{H}_I(\sigma) \cong \bigoplus_{l=0}^{I} \mathcal{H}_l(\sigma)
\end{equation}
is the eigenspace of $H$ with eigenvalue $E_I$, and the Hilbert space of bound states
admits the following orthogonal decomposition into the eigenspaces of $H$:
\[ \mathcal{H}(\sigma) = \bigoplus_{I=0}^{\infty} \mathcal{H}_I(\sigma). \]
Part 4) of Theorem 1 is then clear. We shall show that, $U(n)$
as irreducible representations of $U(n)$, we define its twist $\tilde{\psi}$ of $\psi$
where $\tilde{\psi}(Z) = c_I \frac{1}{|Z|^2} \psi(\sqrt{\frac{n_I}{2}} |Z|)$
where $c_I$ is the unique constant such that
\begin{equation}
\int_{C^n} |\tilde{\psi}|^2 = \int_{C^n} |\psi|^2.
\end{equation}
We use $\tilde{\mathcal{H}}_I(\sigma)$ to denote the span of all such $\tilde{\psi}$’s, $\tilde{\mathcal{H}}(\sigma)$ to denote the Hilbert
space direct sum of $\tilde{\mathcal{H}}_I(\sigma)$. We write the linear map sending $\psi$ to $\tilde{\psi}$ as
\begin{equation}
\tau : \mathcal{H}(\sigma) \to \tilde{\mathcal{H}}(\sigma).
\end{equation}
Then $\tau$ is a linear isometry.
Since $H \psi = E_I \psi_I$, after re-scaling: $|Z| \to \sqrt{\frac{n_I}{2}} |Z|$, we have
\begin{equation}
\left( -\frac{(2/n_I)^2}{8r} \Delta_A + \frac{2/n_I}{8r^4} \right) \psi(\sqrt{\frac{n_I}{2}} |Z|) = E_I \psi_I(\sqrt{\frac{n_I}{2}} |Z|),
\end{equation}
where $r = |Z|$. Multiplying by $(n_I r)^2$, we obtain
\begin{equation}
\frac{1}{r^{3/2}} \left( -\frac{r}{2} \left( \Delta_A - \frac{2/n_I}{r^2} \right) \right) \tilde{\psi}_I(Z) = n_I^2 E_I r^2 \tilde{\psi}_I(Z)
\end{equation}
where $r = |Z|$. Applying Lemma A.1, the previous equation becomes
\begin{equation}
-\frac{1}{2} r^2 \tilde{\psi}_I = \left( -\frac{1}{2} \left( \frac{1}{r^{2n_I - 2}} \partial_{r} r^{2n_I - 2} \partial_{r} r^{2} - \frac{\Delta |Z|^{2n_I - 2} + n - \frac{5}{2}}{r^2} \right) - 2n_I \right) \tilde{\psi}_I.
\end{equation}
\[ \psi_I = -\frac{1}{2} \left( \frac{1}{r^{2n-1}} \partial_r r^{2n-1} \partial_r - \frac{\Delta |\sigma|^{2n-1}}{r^2} \right) \tilde{\psi}_I, \]

where \( \Delta \) is the (negative definite) standard Laplace operator on \( \mathbb{C}^n \), and \( \tilde{\psi}_I \) is viewed as a \( \sigma \)-valued function on \( \mathcal{C}_n^\sigma \). Then

\[ \left( -\frac{1}{2} \Delta + \frac{1}{2} r^2 \right) \tilde{\psi}_I = (2J + |\tilde{\sigma}| + n) \tilde{\psi}_I. \]

Denote the collection of all smooth sections of \( \gamma_\sigma \) by \( C^\infty(\gamma_\sigma) \), and the collection of all smooth complex-valued functions on \( \mathbb{C}^n \) by \( C^\infty(\mathbb{C}_n^\sigma) \). Note that an element in \( \sigma^* \otimes C^\infty(\gamma_\sigma) \) can be viewed as a smooth map \( F: \mathcal{C}_n^\sigma \to \text{End}(\sigma) \) such that \( F(x \cdot g^{-1}) = \rho_{\sigma}(g) \circ F(x) \) for any \( g \in U(1) \). Denote by \( \text{Tr} \) the map sending \( x \in \mathbb{C}^n \) to the trace of \( F(x) \) and by \( \text{Tr} \) the linear map sending \( F \in \sigma^* \otimes C^\infty(\gamma_\sigma) \) to \( \text{Tr} F \in C^\infty(\mathbb{C}_n^\sigma) \).

The left \( U(n) \)-action on \( \mathbb{C}^n \) induces natural actions on both \( C^\infty(\gamma_\sigma) \) (hence on \( \sigma^* \otimes C^\infty(\gamma_\sigma) \)) and \( C^\infty(\mathcal{C}_n^\sigma) \) with respect to which \( \text{Tr} \) is equivariant: For any \( g \in U(n) \),

\[ \text{Tr} \left( L_g^* F \right)(x) = \text{Tr} \left( L_g^* F(x) \right) = \text{Tr} (F(g \cdot x)) = \text{Tr} F(g \cdot x) = L_g^* \text{Tr} F(x). \]

The right \( U(1) \)-action on \( \mathbb{C}^n \) induces a natural action on \( C^\infty(\mathcal{C}_n^\sigma) \). The action of \( U(1) \) on \( \sigma^* \) yields a natural \( U(1) \)-action on \( \sigma^* \otimes C^\infty(\gamma_\sigma) \) for which

\[ (g \cdot F)(x) = F(x) \circ \rho_{\sigma}(g^{-1}) \quad \text{for any} \ g \in U(1). \]

It is clear that \( \text{Tr} \) is also \( U(1) \)-equivariant: For any \( g \in U(1) \),

\[ \text{Tr} (g \cdot F)(x) = \text{Tr} ((g \cdot F)(x)) = \text{Tr} (F(x) \circ \rho_{\sigma}(g^{-1})) = \text{Tr} (\rho_{\sigma}(g^{-1}) \circ F(x)) = \text{Tr} (F(x \cdot g)) = \text{Tr} F(x \cdot g) = R_g^* (\text{Tr} F)(x), \]

where \( R_g \) stands for the right action of \( g \) on \( \mathcal{C}_n^\sigma \).

On \( \sigma^* \otimes C^\infty(\gamma_\sigma) \), there is a natural inner product:

\[ \langle \alpha \otimes \psi, \beta \otimes \phi \rangle = \langle \alpha, \beta \rangle \int_{\mathbb{C}^2} \langle \psi, \phi \rangle. \]

On \( C^\infty(\mathcal{C}_n^\sigma) \), there is a natural inner product:

\[ \langle f, g \rangle = \int_{\mathbb{C}^n} \bar{f} g. \]

It is easy to see that \( \text{Tr} \) is an isometry: if we let \( v \) be an orthonormal basis of \( \sigma \), \( \hat{v} \) be the dual of \( v \), then an element in \( \sigma^* \otimes C^\infty(\gamma_\sigma) \) can be written as \( f v \hat{v} \). So

\[ \langle \text{Tr} (f v \hat{v}), \text{Tr} (g v \hat{v}) \rangle = \langle f, g \rangle = \int_{\mathbb{C}^n} \bar{f} g = \langle \hat{v}, \hat{v} \rangle \int_{\mathbb{C}^n} \langle f v, g v \rangle = \langle f v \hat{v}, g v \hat{v} \rangle. \]

Since \( \text{Tr} \) is an isometry, it must be injective. Actually, we have
Proposition 3.1. \( \text{Tr} \) induces a Hilbert space isomorphism

\[
\text{Tr}_*: \bigoplus_{\sigma \in U(1)} \sigma^* \otimes \tilde{H}(\sigma) \cong L^2(\mathbb{C}^n) = L^2(\mathbb{C}^n)
\]

which is equivariant with respect to both the \( U(n) \)-actions and the \( U(1) \)-actions.

Proof. Let \( \mathcal{H}_k \) be the \( k \)-th energy eigenspace of the \( 2n \)-dimensional isotropic harmonic oscillator with hamiltonian \(-\frac{1}{2}\Delta + \frac{1}{2}p^2\). In view of Eq. (3.9), we have the following map

\[
\iota : = \bigoplus_{2I + |\bar{\sigma}| = k} \sigma^* \otimes \tilde{H}(\sigma) \longrightarrow \mathcal{H}_k.
\]

Claim 3.2. \( \iota \) is an isomorphism.

Proof. Since \( \iota \) is injective, one just needs to verify the dimension equality:

\[
\sum_{2I + |\bar{\sigma}| = k} \dim \sigma^* \cdot \dim \tilde{H}(\sigma) = \dim \mathcal{H}_k.
\]

In view of Eq. (3.6),

\[
\dim \tilde{H}(\sigma) = \dim \mathcal{H}(\sigma) = \sum_{i=0}^I \dim \mathcal{H}_i(\sigma)
\]

\[
= \sum_{p+q = |\bar{\sigma}| \leq I} \dim \mathcal{R}_{p,q}.
\]

Using the dimension formula in representation theory to compute \( \dim \mathcal{R}_{p,q} \), one arrives at the following explicit form of the dimension equality:

\[
\sum_{2I + |p-q|=k} \sum_{p+q+n-1 \leq I} \binom{p+n-2}{n-2} \binom{q+n-2}{n-2} = \binom{2n+k-1}{2n-1}.
\]

To prove this identity, we use the method of generating functions. The details are as follows:

\[
\sum_{k=0}^{\infty} \sum_{p+q \leq k} \frac{p+q+n-1}{n-1} \binom{p+n-2}{n-2} \binom{q+n-2}{n-2} t^k
\]

\[
= \sum_{p+q \leq 2I} \frac{1}{1-t^2} \left( \binom{p+n-1}{n-1} \binom{q+n-2}{n-2} + \binom{p+n-2}{n-1} \binom{q+n-2}{n-2} \right)
\]

\[
= \frac{1}{1-t^2} \left( (1-t)^{-2n+1} + t(1-t)^{-2n+1} \right) = (1-t)^{-2n}
\]

\[
= \sum_{k \geq 0} \binom{2n+k-1}{2n-1} t^k.
\]

\( \square \)
As a consequence of the above claim, we have the following Hilbert space isomorphism:

$$\bigoplus_{\sigma \in U(1)} \sigma^* \otimes \mathcal{H}(\sigma) \cong \bigoplus_{\tau \in U(1), \tau \geq 0} \sigma^* \otimes \mathcal{H}_\tau(\sigma)$$

$$\cong \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

$$= L^2(C^n_+).$$

\[ \square \]

3.2. Proof of Theorem 1

Parts 1) and 4) have been proved. Let \((q^i, p_i)^\text{'}s\) be the canonical symplectic coordinates on \(T^*\mathbb{C}^n\), in terms of which, the canonical symplectic form can be written as

$$\omega = dq^i \wedge dp_i.$$ 

The complex structure \(I\) on \(\mathbb{C}^n\) is a diffeomorphism from \(\mathbb{C}^n\) to itself, so \(T^*I\) is a symplectomorphism from \((T^*\mathbb{C}^n, \omega)\) to itself. Introducing \(z^i = q^{2i-1} + \sqrt{-1}q^{2i}\) and \(w_i = p_{2i-1} + \sqrt{-1}p_{2i}\), one can check that \(T^*I(z^i, w_i) = \sqrt{-1}(z^i, w_i)\), so \(T^*I\) is a complex structure on \(T^*\mathbb{C}^n\) and \((z^1, \ldots, z^n, w_1, \ldots, w_n)\) are the standard complex coordinates on \(T^*\mathbb{C}^n\). Let \(z = (z^1, \ldots, z^n)^T\) and \(w = (w_1, \ldots, w_n)^T\), then

$$\omega = \text{Re} \left( dz^i \wedge dw_i \right)$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} dz \\ dw \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \wedge \begin{pmatrix} du \\ dv \end{pmatrix},$$

where \(u = \frac{z^i + iw_i}{\sqrt{2}}\) and \(v = \frac{z^i - iw_i}{\sqrt{2}}\). Therefore, \(\omega\) is always invariant under a complex linear transformation \(A\) on \(T^*\mathbb{C}^n = \mathbb{C}^n \oplus \mathbb{C}^n\) satisfying the following condition:

$$A^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The collection of all such \(A\)'s form the group \(U(n, n)\) — a subgroup of \(\text{Sp}(4n, \mathbb{R})\). Note that \(U(1)\) acts on \(\mathbb{C}^n\) from right, and the induced action on \(T^*\mathbb{C}^n\) can be identified with the diagonal subgroup of \(U(1) \times \cdots \times U(1)\), so it commutes with the action of \(U(n, n)\) on \(T^*\mathbb{C}^n\). It is now clear that Eq. (5.11) is really the decomposition in Eqs. (4.1) and (4.2) of Ref. [5] with the dual pair being \((U(1), U(n, n))\) in \(\text{Sp}(4n, \mathbb{R})\); consequently, \(\mathcal{H}(\sigma)\) is a unitary highest weight module of \(U(n, n)\). In view of the fact that \(\tau\) in Eq. (5.8) is an isometry, by pulling back the action of \(\tilde{U}(n, n)\) on \(\mathcal{H}(\sigma)\) via \(\tau\), we get the action of \(\tilde{U}(n, n)\) on \(\mathcal{H}(\sigma)\). Part 5) is then proved.

Since

$$\omega = \frac{1}{2i} (du^i, dv^T) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} du \\ dv \end{pmatrix},$$

we know that \(\omega\) is also invariant under the action of \(U(2n)\) on \(T^*\mathbb{C}^n = \mathbb{C}^n \oplus \mathbb{C}^n\). This \(U(2n)\), being the unitary group which leaves

$$|u|^2 + |v|^2 = |z|^2 + |w|^2$$

for all \(u, v, z, w\), is the correct group to use in this context. Therefore, the thesis has been proved.

\[ \square \]
invariant, is a maximal compact subgroup of Sp(4n, \mathbb{R}). It is not hard to see that
\begin{equation}
U(2n) \cap U(n,n) = U(n) \times U(n)
\end{equation}
with following identification:
\begin{equation}
U(2n) \ni \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \longleftrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in U(n,n).
\end{equation}
The natural left action of U(n) on \mathbb{C}^n induces a left action on \mathcal{T}^* \mathbb{C}^n. This induced action coincides with the restriction to the diagonal subgroup of the above left action of U(n) \times U(n) on \mathcal{T}^* \mathbb{C}^n. Therefore, the above unitary action of \tilde{U}(n,n) on the Hilbert space \mathcal{H}(\sigma) of bound states, extends the manifest unitary action of U(n) — the diagonal of U(n) \times U(n).

Since \mathcal{H}_k is invariant under the action of \tilde{U}(2n), in view of Eqs. (3.14) and (3.10), we conclude that the Hilbert space isomorphism
\begin{equation}
\iota : \bigoplus_{2I+|\overrightarrow{\sigma}|=k} \sigma^* \otimes \mathcal{H}_{\tilde{\sigma}}(\sigma) \rightarrow \mathcal{H}_k
\end{equation}
is an isomorphism of U(n) \times U(n)-modules, so \mathcal{H}_{\tilde{\sigma}}(\sigma) (hence \mathcal{H}_{{\tilde{\sigma}}}^k(\sigma)) is a U(n) \times U(n)-module. Since Eq. (3.10) is a decomposition of U(n)-modules, assuming that \overrightarrow{\sigma} \geq 0, by using the Littlewood-Richardson rule, there is a half integer \kappa such that
\begin{equation}
\mathcal{H}_k(\sigma) \cong \mathcal{R}_{I}^k \otimes \mathcal{R}_{I+|\overrightarrow{\sigma}|}^k
\end{equation}
as representations of U(n) \times U(n). Therefore, in view of Eq. (3.15), the isomorphism in Eq. (3.16) and some simple facts on harmonic isolators, we conclude that
\begin{equation}
2I + |\overrightarrow{\sigma}| + n = 2I + |\overrightarrow{\sigma}| + 2n\kappa,
\end{equation}
so we must have \kappa = 1/2. This proves part 3), and consequently part 2) when \overrightarrow{\sigma} \leq 0. By the similar arguments, the case when \overrightarrow{\sigma} \geq 0 can be proved, too.

**Appendix A. A formulae concerning the Laplace operators**

Let G be a compact connected reductive Lie group, X, B be manifolds, and \pi: X \to B be a principal G-bundle. For any x \in X, by linearizing \pi at x and map
\begin{equation}
G \rightarrow X, \quad g \rightarrow x \cdot g^{-1}
\end{equation}
at e_G, we arrive at short exact sequence
\begin{equation}
0 \rightarrow \mathfrak{g} \rightarrow T_x X \rightarrow T_{\pi(x)} B \rightarrow 0.
\end{equation}
We assume that X is equipped with a Riemannian metric g such that G becomes an isometric group of (X, g). This assumption, via the orthogonal projection, yields an equivariant splitting of (A.1) (i.e., a connection \mathcal{A} on the principal bundle) and consequently a quotient Riemannian metric on B.

Let \sigma be a unitary irreducible representation of G, V be its underlying representation space, and \xi be the corresponding hermitian vector bundle. We use \Delta to denote the (non-negative) Laplace operator on V-valued smooth functions on X and \Delta_A to denote the (non-negative) Laplace operator on smooth sections of \xi. Note that if f is a smooth section of \xi, then f is actually a smooth map from X to V such that f(x \cdot g^{-1}) = \sigma(g) \cdot f(x) for any (x, g) \in X \times G.

The following lemma can be proved by using the argument in appendix A.1 of Ref. [10]:
Lemma A.1. With the notations and background introduced above, we have

$$\Delta f = \Delta_A f + c_2([G])|_\sigma f$$

where $c_2([G])|_\sigma$ is the value of the Casimir operator of $G$ at representation $\sigma$.

For example, we can let $X = G'$ be a compact Lie group and $G$ a closed Lie subgroup of $G'$. Let $\xi$ be the vector bundle over $B = G'/G$ associated with irreducible rep. $\sigma$ of $G$. Since $\Delta = c_2([G])$, this lemma says that $\Delta_A = c_2([G']) - c_2([G])|_\sigma$ — an identity proved in appendix A.1 of Ref. [10].

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