UNCOUNTABLY MANY PERMUTATION STABLE GROUPS

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Abstract. In a 1937 paper B.H. Neumann constructed an uncountable family of 2-generated groups. We prove that all of his groups are permutation stable by analyzing the structure of their invariant random subgroups.

1. Introduction

Let $S(n)$ denote the symmetric group of degree $n \in \mathbb{N}$ with the bi-invariant Hamming metric $d_n$ given by

$$d_n(\sigma, \tau) = 1 - \frac{1}{n} |\text{Fix}(\sigma^{-1}\tau)|.$$

Let $G$ be a finitely generated group. An almost-homomorphism of $G$ is a sequence of set theoretic maps $f_n : G \to S(n)$ satisfying

$$d_n(f_n(g)f_n(h), f_n(gh)) \xrightarrow{n \to \infty} 0, \forall g, h \in G.$$

The almost-homomorphism $f_n$ is close to a homomorphism if there is a sequence of group homomorphisms $\rho_n : G \to S(n)$ satisfying

$$d_n(\rho_n(g), f_n(g)) \xrightarrow{n \to \infty} 0, \forall g \in G.$$

Definition 1.1. The group $G$ is permutation stable (or $P$-stable for short) if every almost-homomorphism of $G$ is close to a homomorphism.

Until quite recently only few groups were known to be permutation stable: free groups (trivially), finite groups [GR09] and abelian groups [AP15]. The situation was changed with [BLT19]. That work established a connection between stability and invariant random subgroups of a given amenable group $G$.

Recall that an invariant random subgroup $\mu$ of $G$ is a conjugation invariant probability measure on the space of all subgroups of $G$. The invariant random subgroup $\mu$ is co-sofic if $\mu$ is a limit of invariant random subgroups supported on finite index subgroups.

Theorem 1.2 (BLT19). A finitely generated amenable group $G$ is permutation stable if and only if every invariant random subgroup of $G$ is co-sofic.

This new viewpoint enabled the authors of [BLT19] to deduce that some groups whose invariant random subgroups are easy to analyze are permutation stable, namely polycyclic-by-finite as well as the Baumslag–Solitar groups $B(1, n)$ for all $n \in \mathbb{Z}$. That work motivated a deeper analysis of the invariant random subgroups of more complicated groups: the lamplighter group, as well as any wreath product of finitely generated abelian groups, is shown in [LL19] to be permutation stable, while [Zhe19] showed that the Grigorchuk group is such. The goal of this paper is to go further and prove...
Theorem 1.3. There exist uncountably many 2-generated permutation stable groups.

B.H. Neumann [Neu37] gave an explicit construction of an uncountable family of pairwise non-isomorphic 2-generator groups. We actually prove that all of those groups are permutation stable.

The B.H. Neumann family of groups is constructed as follows. Let \( \mathcal{P} = (n_i)_{i \in \mathbb{N}} \) be any monotone increasing sequence of odd integers with \( n_1 \geq 5 \). For all \( i \in \mathbb{N} \) write \( n_i = 2r_i + 1 \) and denote \( \llbracket r_i \rrbracket = \{ n \in \mathbb{Z} : |n| \leq r_i \} \) so that \( |\llbracket r_i \rrbracket| = n_i \). Let

\[
A(\mathcal{P}) = \prod_{i \in \mathbb{N}} \text{Alt}(\llbracket r_i \rrbracket)
\]

where \( \text{Alt}(\llbracket r_i \rrbracket) \) is the group of all even permutations of the set \( \llbracket r_i \rrbracket \) for each \( i \in \mathbb{N} \). The group \( A(\mathcal{P}) \) is profinite.

The B.H. Neumann group \( G(\mathcal{P}) \) associated to the sequence \( \mathcal{P} \) is the countable subgroup of \( A(\mathcal{P}) \) generated by the two elements \( \tau = (\tau_i)_{i \in \mathbb{N}} \) and \( \sigma = (\sigma_i)_{i \in \mathbb{N}} \), where each permutation \( \tau_i \) is the 3-cycle

\[
\tau_i = (-1,0,1) \in \text{Alt}(\llbracket r_i \rrbracket)
\]

and each permutation \( \sigma_i \) is the full \( n_i \)-cycle

\[
\sigma_i = (-r_i,-r_i+1,\ldots,-1,0,1,\ldots,r_i-1,r_i) \in \text{Alt}(\llbracket r_i \rrbracket).
\]

B.H. Neumann showed — see also [LW93] — that the groups \( G(\mathcal{P}) \) and \( G(\mathcal{P}') \) are isomorphic if and only if the two sequences \( \mathcal{P} \) and \( \mathcal{P}' \) as above coincide. So his family of groups is indeed uncountable. It has already been observed in [LW93] — see also [LW93] — that all these groups are amenable. So their stability, which is a purely group theoretic property, can be studied via their invariant random subgroups by Theorem 1.2. We will prove

**Theorem 1.4.** All invariant random subgroups of each B.H. Neumann group \( G(\mathcal{P}) \) are co-sofic.

In this work, as in [LL19], the proof of our Theorem 1.4 uses methods of ergodic theory, and relies on the Lindenstrauss pointwise ergodic theorem for amenable groups [Lin01].

**Outline of the paper.** The paper is organized as follows.

In §2 we recall in a slightly modified way the construction of the B.H. Neumann groups \( G = G(\mathcal{P}) \). We analyze their structure and provide two short exact sequences to be used later. For instance, we will show that the derived subgroup \( G' \) of \( G \) is mapped onto \( \text{Alt}_{\text{fin}}(\mathbb{Z}) \), the group of all even permutations of the set \( \mathbb{Z} \) with finite support.

In §3 we will recall some basic definitions and facts about invariant random subgroups, as well as the notion of Weiss approximations relying on the pointwise ergodic theorem.

In §4 we recall Vershik’s theorem which gives a complete classification of the invariant random subgroups of the group \( \text{Sym}_{\text{fin}}(\mathbb{Z}) \). This classification will be used in the proof of the main theorem.

The proof will be given in §6. One main ingredient of the proof is the explicit construction done in §5 of a Weiss approximation to any subgroup of \( G \) contained in the derived subgroup \( G' \). The other main ingredient has to do with showing that every invariant random subgroup of \( G \) not entirely contained in the derived subgroup \( G' \) is already supported on finite index subgroups.
An interesting aspect of the proof is that $G$ admits a certain normal subgroup $U$ so that $G/U$ is not residually finite. This means that the atomic invariant random subgroup of $G$ supported on the this normal subgroup $U$ cannot be approximated “from above” by invariant random subgroups corresponding to finite index subgroups containing $U$. Instead, it can only be approximated “from the side”, so to speak, using invariant random subgroups which do not come from $G/U$, see §5.

We follow the convention $x^y = y^{-1}xy$ for conjugation.

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2. The B.H. Neumann Groups

We recall in detail the definition of the B.H. Neumann groups and present some of their properties to be used later.

Basic notations. For each $r \in \mathbb{N}$ write

$$[r] = \{m \in \mathbb{Z} : |m| \leq r\}$$

so that $[r]$ is a set of integers of size $2r + 1$. Let $U_r$ denote the group of all even permutations of the set $[r]$. So $U_r$ is naturally isomorphic to the alternating group $A_{2r+1}$. Let $U_\infty$ denote the group $\text{Alt}_{\text{fin}}(\mathbb{Z})$ of all even permutations of the set $\mathbb{Z}$ with finite support. Note that $U_\infty = \varprojlim U_r$ in a natural way. Therefore $U_\infty$ is a simple group.

Let $\mathcal{P} = (n_i)_{i \in \mathbb{N}}$ be a monotone sequence of odd integers satisfying $n_1 \geq 5$. For each $i \in \mathbb{N}$ denote $n_i = 2r_i + 1$ and $A_i(\mathcal{P}) = U_{r_i}$. Consider the profinite group

$$A(\mathcal{P}) = \prod_{i \in \mathbb{N}} A_i(\mathcal{P})$$

Let $G = G(\mathcal{P})$ be the countable subgroup of $A(\mathcal{P})$ generated by the two elements $\tau = (\tau_i)_{i \in \mathbb{N}}$ and $\sigma = (\sigma_i)_{i \in \mathbb{N}}$, where for each $i \in \mathbb{N}$ we have

$$\tau_i = (-1, 0, 1) \in A_i(\mathcal{P})$$

and

$$\sigma_i = (-r_i, -r_i + 1, \ldots, -1, 0, 1, \ldots, r_i - 1, r_i) \in A_i(\mathcal{P}).$$

The group $G(\mathcal{P})$ is the B.H. Neumann group associated to the sequence $\mathcal{P}$.

Sets of generators. For each $r \in \mathbb{N}$ the following collection of 3-cycles

$$C_r = \{(-r, -r + 1, -r + 2), \ldots, (-1, 0, 1), \ldots, (r - 2, r - 1, r)\}$$

of size $2r - 1$ is well known to generate the finite alternating group $U_r$. For each $i \in \mathbb{N}$ consider the following generating set

$$C_i(\mathcal{P}) = C_{r_i} = \{\tau_i^{\sigma_i^j} : j \in [r_i - 1]\}$$

for the group $A_i(\mathcal{P})$. In particular $A_i(\mathcal{P})$ is generated by the two elements $\tau_i$ and $\sigma_i$ so that $G(\mathcal{P})$ surjects onto each coordinate of $A(\mathcal{P})$. The following standard fact
of elementary group theory implies that \( G(P) \) is a dense subgroup of the profinite group \( A(P) \).

**Lemma 2.1.** Let \( \Gamma_1, \ldots, \Gamma_1 \) be a family of pairwise non-isomorphic simple groups. If \( H \) is a subgroup of the direct product \( \Gamma = \prod_{i=1}^{r} \Gamma_i \) surjecting onto each coordinate then \( H = \Gamma \).

**Local finiteness.** For each \( i \in \mathbb{N} \), let \( L_i(P) \) denote the subgroup of \( G(P) \) given by
\[
L_i(P) = \langle \tau^{p^j} : j \in [r_i - 1] \rangle.
\]
The projection of \( L_i(P) \) to any coordinate \( A_j(P) \) with \( j \leq i \) is surjective for it contains the generating set \( C_j(P) \), while if \( j > i \) the image of the projection to \( A_j(P) \) is isomorphic to \( A_i(P) \) in a natural way. In any case, the support of every such projection is contained inside \( [r_i] \).

In fact, for each \( i \in \mathbb{N} \) the projection of the subgroup \( L_i(P) \) onto \( \prod_{j=1}^{r} A_j(P) \) is an isomorphism. It is surjective according to Lemma [2.1]. To see that this projection is injective, note that every element \( g = (g_k)_{k \in \mathbb{N}} \) belonging to \( L_i(P) \) satisfies \( g_j = g_i \) for all \( j \geq i \). It follows that each subgroup \( L_i(P) \) is finite.

The subgroup \( N = N(P) \) of \( G(P) \) given by
\[
N(P) = \bigcup_{i \in \mathbb{N}} L_i(P).
\]
is locally finite. It is also the normal closure of the element \( \tau \) in the group \( G(P) \). Therefore \( G/N \) is a cyclic group generated by the image of \( \sigma \). This is an infinite cyclic group since \( \sigma \) is of infinite order and hence no non-trivial power of it can belong to \( N \). Moreover \( G \) is the semi-direct product of \( N \) and of the cyclic group generated by \( \sigma \).

It is easy to detect the elements of \( G(P) \) belonging to the subgroup \( N(P) \). An element \( g = (g_k)_{k \in \mathbb{N}} \in G(P) \) belongs to \( N(P) \) if and only if there is some \( i \in \mathbb{N} \) such that the support of \( g_k \) is contained in \( [i] \) for all \( k \in \mathbb{N} \). For every element \( g \in N \), let \( i = \min \{k \in \mathbb{N} : g_k \neq 1 \} \). The element \( g \) acts diagonally in all coordinates \( A_i(P) \) with \( j \geq i(g) \), namely \( g_j = g_i(g) \) for all such \( j \), provided that we identify \( A_i(g)(P) \) with a subgroup of \( A_j(P) \) in the natural way.

**The tail map.** The tail (or the diagonal part) of the element \( g = (g_k)_{k \in \mathbb{N}} \in N \) is
\[
t(g) = g_{i(g)} \in A_i(g)(P) = U_{r_{i(g)}} \subset U_{\infty}.
\]
We consider the tail \( t(g) \) as an even permutation of the set \( \mathbb{Z} \) with finite support by naturally regarding the set \( [r_i] \) as a subset of \( \mathbb{Z} \). It is easy to see that the map \( t : N \to U_{\infty} \) is a homomorphism. Let us summarize.

**Proposition 2.2.** The group \( G = G(P) \) satisfies
\begin{enumerate}
\item \( G \) is a residually finite group generated by the two elements \( \sigma \) and \( \tau \),
\item the normal closure \( N = N(P) \) of the element \( \tau \) in \( G \) is locally finite and \( G = N \rtimes \langle \sigma \rangle \),
\item \( G \) is locally finite-by-cyclic and hence amenable, and
\item there is a surjective tail homomorphism \( t : N \to U_{\infty} \).
\end{enumerate}
Proof. Everything was already noticed before, except for the fact that $t$ is surjective. To see this, consider some element $h \in U_\infty$. The support of $h$ is contained in $[r_i]$ for some $i \in \mathbb{N}$ sufficiently large. Then $h = t(g)$ for any element $g \in L_i(P)$ whose projection on the $A_i(P)$ coordinate coincides with $h$. □

We would like to extend the tail map $t$ to the entire group $G(P)$. To this end it is desirable to replace the range of $t$ by a larger group. Let $\tilde{\sigma}$ be the permutation of the set $\mathbb{Z}$ given by $\tilde{\sigma}(x) = x + 1$ for all integers $x \in \mathbb{Z}$. Let $V$ denote the following subgroup of $\text{Sym}(\mathbb{Z})$

$$V = U_\infty \rtimes \langle \tilde{\sigma} \rangle.$$  

**Proposition 2.3.** Consider the group $G = G(P)$.

1. The tail map $t$ naturally extends to a surjective homomorphism $\tilde{t} : G \to V$.
2. Let $U(P)$ denote the kernel $\ker \tilde{t} = \ker t$, i.e the subgroup of $N$ consisting of all elements with trivial tail. Then $U(P) = \bigoplus_{i \in \mathbb{N}} A_i(P)$.

Proof. Recall that $G = N \rtimes \langle g \rangle$. It clear from the definitions that the map $\tilde{t} : G \to V$ given by $\tilde{t}(\sigma^k) = \tilde{\sigma}^k$ and $\tilde{t}(g) = t(g)$ for every element $g \in N$ is a surjective homomorphism. Statement (1) follows.

To see Statement (2) we consider the group $L_i(P)$ as defined above. We know that the projection of $L_i(P)$ on the coordinates $\prod_{j=1}^{i} A_j(P)$ is an isomorphism. The tail $t(g)$ of every element $g = (g_k)_{k \in \mathbb{N}} \in L_i(P)$ is equal to $g_i$. It follows that the kernel of the projection of $L_i(P)$ to the coordinate $A_i(P)$ is equal to $\oplus_{j=1}^{i-1} A_j(P)$. Call this last group $R_i$ so that $R_i = L_i(P) \cap \ker t$. Now we have $N = \bigcup_{i=1}^{\infty} L_i(P)$ and so $\ker \tilde{t} = U(P) = \bigcup_i R_i$. It follows that $U(P) = \bigoplus_{i \in \mathbb{N}} A_i(P)$. □

The structure of the group $U(P)$ clearly depends on the sequence $P$, while the group $V = U_\infty \rtimes \langle \tilde{\sigma} \rangle$ is independent of $P$. As $U_\infty$ and $U(P)$ are both perfect groups, the derived subgroup of $G(P)$ is $[G(P), G(P)] = N(P)$. The subgroup $U(P)$ is generated by all the finite normal subgroups of $N(P)$ and is therefore characteristic. It follows that any isomorphism from $G(P)$ to $G(P')$ would necessary take $U(P)$ to $U(P')$. The groups $U(P)$ to $U(P')$ are isomorphic if and only if $P = P'$. This reproves B.H. Neumann’s result saying that the groups $G(P)$ with different $P$’s are non-isomorphic, so that there are uncountably many such 2-generated groups.

The following commutative diagram depicts three short exact sequences involving the B.H. Neumann group $G(P)$. The diagonal sequence is the abelianization map.
Remark 2.4. Consider the situation where $\mathcal{P}'$ is an infinite subsequence of $\mathcal{P}$. The kernel of the natural map $G(\mathcal{P}) \to G(\mathcal{P}')$, obtained by deleting the coordinates of $\mathcal{P}$ which are not in $\mathcal{P}'$, is exactly
\[ \ker(G(\mathcal{P}) \to G(\mathcal{P}')) = \bigoplus_{i \in \mathbb{N}} A_i(\mathcal{P}) \leq U(\mathcal{P}). \]

The group $G(\mathcal{P}')$ can be formally defined in exactly the same way also when the sequence $\mathcal{P}'$ is finite. In that case the group $G(\mathcal{P}')$ is finite, and unlike what happens when $\mathcal{P}'$ is infinite, the kernel of the natural map $G(\mathcal{P}) \to G(\mathcal{P}')$ has finite index in $G(\mathcal{P})$. For a discussion of the finite index subgroups of $G(\mathcal{P})$ see [LPS96].

On finite index subgroups. We give a pure group-theoretic property of the B.H. Neumann group $G = G(\mathcal{P})$ that will be of crucial importance when we classify all invariant random subgroups of $G$ in $\mathfrak{G}$.

Proposition 2.5. If $H$ is a subgroup of $G$ with $[\mathcal{H} : \bar{t}(H)] < \infty$ then $[G : H] < \infty$.

Proof. Let $H$ be a subgroup $G$ and assume that $\bar{t}(H)$ has finite index in $\mathcal{H}$. The subgroup $U_\infty$ of $\mathcal{H}$ is simple and in particular has no finite index subgroups. Therefore $\bar{t}(H) = U_\infty \rtimes (\bar{\sigma}_k^-)$ for some $k \in \mathbb{N}$. The proof will proceed in five steps.

Step 1. There exists some $i_H \in \mathbb{N}$ such that for all $i > i_H$ the projection of the subgroup $H$ to the coordinate $A_i(\mathcal{P})$ is surjective.

Indeed, the image $\bar{t}(H)$ of $H$ contains the subgroup $U_\infty$. This means that the subgroup $H$ contains for each $j \in \mathbb{Z}$ an element $h_j$ belonging to $N$ whose tail satisfies $\bar{t}(h_j) = t(h_j) = t(\sigma_{j+1}^+)$. Moreover, as $\bar{\sigma}_k^- \in \bar{t}(H)$ for some $k \in \mathbb{N}$, the subgroup $H$ contains an element $h$ satisfying $\bar{t}(h) = \bar{\sigma}_k^-$. As the tails of $h_j$ and $\tau_{j+1}^+$ are the same, for each $j \in [k-1]$ there exists an element $u_j \in U(\mathcal{P})$ with $\tau_{j+1}^+ = u_jh_j$. Similarly there exists an element $u \in U(\mathcal{P})$ with $h = u\bar{\sigma}_k^-$. Choose $i_H \in \mathbb{N}$ to be sufficiently large such that the $u_j$’s as well as $u$ are all contained in $\bigoplus_{i=1}^{i_H} A_i(\mathcal{P})$.

For every index $i > i_H$ the projection of each element $h_j$ to the coordinate $A_i(\mathcal{P})$ is equal to $\tau_{i+1}^+$, while the projection of the element $h$ is equal to $\bar{\sigma}_k^-$. We claim that this projection contains the full generating set $C_i(\mathcal{P})$ for the group $A_i(\mathcal{P})$ of all alternating symmetries of $[r_i]$. Indeed, this projection contains $\tau_{j+1}^+$ for every $j \in \mathbb{Z}$, for if $l = ak + b$ for some $a, b \in \mathbb{Z}$ with $0 \leq b < k$ then
\[ \tau_i^{\sigma_j^+} = \left( \tau_i^{\sigma_j^+} \right)^{(\sigma_j^-)^a} \in C_i(\mathcal{P}). \]

We conclude that for all $i > i_H$ the projection of the subgroup $H$ to the coordinate $A_i(\mathcal{P})$ is surjective.

Step 2. The kernel $H_0$ of the projection from $H$ to $\prod_{i=1}^{i_H} A_i(\mathcal{P})$ has finite index in $H$. For all $i > i_H$ the projection of $H_0$ to the coordinate $A_i(\mathcal{P})$ is still surjective.

For $i > i_H$, let $M_i$ be the kernel of the projection from $H$ to the coordinate $A_i(\mathcal{P})$. Then $M_i$ is a normal subgroup of $H$ with $H/M_i \cong A_i(\mathcal{P})$, while $H_0$ is a normal subgroup of $H$ such that all of the Jordan–Holder components of $H/H_0$ are simple groups of order less than $|A_i(\mathcal{P})|$. It follows that $H_0M_i = H$ so that $H_0$ is mapped onto the coordinate $A_i(\mathcal{P})$ as required.

We can therefore replace $H$ by $H_0$ and assume from now on that our subgroup $H$ is mapped trivially on the first $i_H$ coordinates and onto all the others.
Step 3. The subgroup $K = H \cap U(P)$ is normal in $U(P)$.
To this end, let $x \in K$ and $y \in U(P)$. Note that the first $i_H$-many coordinates of $x$ are trivial by our assumption on $H$. Since $x$ is also in $U(P)$, there is some $i_x \in \mathbb{N}$ so that for every $i > i_x$ the coordinate of $x$ in $A_i(P)$ is trivial. We claim that there exists an element $h \in H$ whose coordinates in $A_i(P)$ for all $i_H < i \leq i_x$ are equal to those of the element $y$. This follows from Step 2. Indeed, the subgroup $H$ is mapped onto each of the coordinates $A_{i_H+1}(P), \ldots, A_{i_x}(P)$. Since these are different simple groups $H$ is also mapped onto their product, by Lemma 2.1. Hence such an element $h$ does exist.

Now, for this element $h$, we have that $x^y = x^h \in K^h$. Since $U(P) \triangleleft G(P)$ clearly $K < H$. Therefore $x^y \in K$. We conclude that $K$ is normal in $U(P)$.

Step 4. $K = \oplus_{i \in \mathbb{N}} U_i(P')$ for some subsequence $P'$ of $P$.
This follows from $H \cap U(P) < U(P)$ and $U = \oplus_{i \in \mathbb{N}} A_i(P)$.

Step 5. The subsequence $P'$ is co-finite in $P$.
Once we prove this, it follows that $H \cap U(P)$ is of finite index in $U(P)$ and altogether $H$ is of finite index in $G$.

Assume towards contradiction that the complementary sequence $P'' = P \setminus P'$ is infinite. Consider the map $G(P) \to G(P'')$ obtained by deleting all the coordinates $A_i(P)$ which are contained in the subgroup $H$, see Remark 2.3. The image of $H$ in the group $G(P'')$ via this map is isomorphic to

$$Q = H/(\oplus_{i \in \mathbb{N}} U_i(P')) = H/K = H/(H \cap U(P)).$$

This means that $Q$ is at the same time isomorphic to $i(H)$, which is a finite index subgroup of $V$. This is impossible as $G(P'')$ is residually finite while $V$ is not. Hence $P''$ is finite. This completes the proof.

Folner sets. We end Section 2 by constructing an explicit Folner sequence in the B.H. Neumann group $G(P)$.

**Proposition 2.6.** Denote $L_i = L_i(P)$. The subsets

$$F_i = F_i(P) = \{ \sigma^j l : j \in [r_i - 1] \}, l \in L_i$$

form a Folner sequence in the group $G(P)$.

**Proof.** To see that the subsets $F_i$ form a Folner sequence it will suffice to verify that both quantities $|\sigma F_i \setminus F_i|$ and $|\tau F_i \setminus F_i|$ are $o(|F_i|)$. In the case of the generator $\sigma$ we have that

$$\frac{|\sigma F_i \setminus F_i|}{|F_i|} = \frac{2|L_i|}{2(2r_i - 1)|L_i|} = \frac{2}{2r_i - 1}$$

as required. In the case of the generator $\tau$, note that

$$\tau F_i = \tau \{ \sigma^j : j \in [r_i - 1] \} L_i = \{ \sigma^j \tau^\sigma : j \in [r_i - 1] \} L_i = F_i$$

as $\tau^\sigma \in L_i$ for all $j \in [r_i - 1]$. In particular $|\tau F_i \setminus F_i| = 0$ for all $i \in \mathbb{N}$. This concludes the proof.

3. Invariant random subgroups and Weiss approximation

We first recall the notion of invariant random subgroups. We then discuss Weiss approximations introduced in [1]. Those approximations can be used in conjunction with the pointwise ergodic theorem for amenable groups to show that certain invariant random subgroups are co-sofic.
Invariant random subgroups. We recall the basic theory of invariant random subgroups, mainly in order to set the notations. For proofs and more details — see [ACV14] as well as [BLT19, LL19].

Let $G$ be a countable discrete group. Consider the power set $\text{Pow}(G) = \{0, 1\}^G$ equipped with the Tychonoff product topology. The space $\text{Pow}(G)$ is compact. The group $G$ acts on its power set $\text{Pow}(G)$ by homeomorphisms via conjugation. We denote this action by $c_g$, so that given an element $g \in G$ and a subset $A \subset G$ we have

$$c_g A = A g^{-1} = g A g^{-1}.$$  

Consider the following subset of $\text{Pow}(G)$

$$\text{Sub}(G) = \{H \leq G : \text{H is a subgroup of } G\}.$$  

The space $\text{Sub}(G)$ is called the Chabauty space of the group $G$. It is a closed subset of $\text{Pow}(G)$ and hence is compact. It is clear that $\text{Sub}(G)$ is preserved by the conjugation action $c_g$ of $G$. Let $\text{Sub}_f(G)$ denote the subset of $\text{Sub}(G)$ consisting of all finite index subgroups.

Let $\mathcal{M}(G)$ be the space of all Borel probability measures on the set $\text{Sub}(G)$. This space is weak-* compact according to the Banach–Alaoglu theorem. The conjugation action of $G$ on its Chabauty space $\text{Sub}(G)$ gives rise to a corresponding push-forward action of $G$ on the space $\mathcal{M}(G)$. We continue using the notation $c_g$ for this push-forward action.

Denote

$$\text{IRS}(G) = \{\mu \in \mathcal{M}(G) : c_g \mu = \mu \text{ for all } g \in G\}.$$  

Note that $\text{IRS}(G)$ is a weak-* closed and hence a compact subset of $\mathcal{M}(G)$. An element $\mu \in \text{IRS}(G)$ is called an invariant random subgroup. Let $\text{IRS}_f(G)$ denote the subspace consisting of all $\mu \in \text{IRS}(G)$ satisfying $\mu(\text{Sub}_f(G)) = 1$.

Definition 3.1. The invariant random subgroup $\mu \in \text{IRS}(G)$ is co-sofic if

$$\mu \in \text{IRS}_f(G) \text{ weak-*}.$$  

The space $\text{IRS}(G)$ is a Choquet simplex [Phe01, §12]. This means that every invariant random subgroup $\mu \in \text{IRS}(G)$ can be decomposed as a convex combination of ergodic invariant random subgroups in a unique way.

Chabauty spaces of subgroups and quotients. Let $H$ be any subgroup of $G$. There is a continuous restriction map given by

$$\cdot|_H : \text{Sub}(G) \to \text{Sub}(H), \quad L \mapsto L|_H = L \cap H \quad \forall L \leq G.$$  

The restriction map is $H$-equivariant for the conjugation action of the subgroup $H$. Pushing-forward via the restriction determines a map

$$\cdot|_H : \text{IRS}(G) \to \text{IRS}(H), \quad \mu \mapsto \mu|_H \quad \forall \mu \in \text{IRS}(G).$$  

Let $Q$ be a quotient of $G$ admitting a surjective homomorphism $\pi : G \to Q$. There is a corresponding map $\pi : \text{Sub}(G) \to \text{Sub}(Q)$ of subgroups taking every subgroup $H \leq G$ to its image $\pi(H)$ in $Q$. The map $\pi : \text{Sub}(G) \to \text{Sub}(Q)$ is $G$-equivariant and Borel measurable. We obtain a push-forward map

$$\pi_* : \text{IRS}(G) \to \text{IRS}(Q), \quad \mu \mapsto \pi_* \mu \in \text{IRS}(Q) \quad \forall \mu \in \text{IRS}(G).$$
Weiss approximable subgroups. Let $G$ be a discrete group and $H$ be a fixed subgroup of $G$. A transversal for $H$ in $G$ is a subset $T_0$ consisting of one element from each coset $tH$ of $H$ in $G$. A finite-to-one transversal $T$ for $H$ in $G$ is a disjoint union of finitely many transversals for $H$ in $G$. That is to say, there is some $k \in \mathbb{N}$ so that $T$ consists of exactly $k$ elements from each coset of $H$.

**Definition 3.2.** The subgroup $H$ is Weiss approximable in $G$ if there are finite index subgroups $K_i$ of $G$ with finite-to-one transversals $F_i$ to $N_G(K_i)$ such that

$$d_{M(G)} \left( \frac{1}{|F_i|} \sum_{f \in F_i} \delta_{fK_i f^{-1}}, \frac{1}{|F_i|} \sum_{f \in F_i} \delta_{fHf^{-1}} \right) \xrightarrow{i \to \infty} 0$$

where $d_{M(G)}$ is any compatible metric on the space $M(G)$. We will say that the sequence $(K_i, F_i)$ is a Weiss approximation for $H$ in $G$.

We now present an explicit condition for a given sequence $(K_i, F_i)$ to constitute a Weiss approximation for the subgroup $H$.

**Proposition 3.3.** Let $H$ be a subgroup of $G$. If a given sequence $K_i$ of finite index subgroups of $G$ with finite-to-one transversals $F_i$ to $N_G(K_i)$ satisfies

$$p_i(g) = \frac{1}{|F_i|} \sum_{f \in F_i} \delta_{gfK_i f^{-1}} \xrightarrow{i \to \infty} 0$$

for all elements $g \in G$ then $(K_i, F_i)$ is a Weiss approximation for the subgroup $H$.

*Proof.* This is precisely [LL19, Proposition 3.7].

The following result will be our main tool in establishing that a given invariant random subgroup is co-sofic.

**Theorem 3.4.** Let $G$ be an amenable group and $\mu \in \text{IRS}(G)$. Let $F_i$ be a fixed Folner sequence in the group $G$. If $\mu$-almost every subgroup $H$ admits a Weiss approximation $(K_i, F_i)$ with some sequence of finite index subgroups $K_i$ of $G$ then $\mu$ is co-sofic.

See [LL19, Theorem 3.10] for a detailed proof of our Theorem 3.4. The proof crucially relies on the pointwise ergodic theorem for amenable groups due to Lindenstrauss [Lin01]. For the reader’s convenience we reproduce the main ideas below.

*Proof outline for Theorem 3.4.* Making use of the ergodic decomposition, we may assume without loss of generality that the invariant random subgroup $\mu$ is ergodic.

Consider the probability measure preserving action of the amenable group $G$ on the Borel space $(\text{Sub}(G), \mu)$. The pointwise ergodic theorem [Lin01] implies that for $\mu$-almost every subgroup $H \in \text{Sub}(G)$ the sequence of probability measures

$$\nu_i = \frac{1}{|F_i|} \sum_{f \in F_i} \delta_{gHf}$$

converges to the invariant random subgroup $\mu$ as $i \to \infty$.

By our assumption $\mu$-almost every subgroup $H \in \text{Sub}(G)$ is Weiss approximable with respect to some sequence $(K_i, F_i)$. Therefore there exists a subgroup $H$ which is Weiss approximable and at the same time the conclusion of the pointwise ergodic
theorem applies to $H$. It follows from Definition 3.2 that the sequence of invariant random subgroups

$$\mu_i = \frac{1}{|F_i|} \sum_{f \in F_i} \delta_{c_f \in IRS_{F_i}(G)}$$

satisfies $d_G(\nu_i, \mu_i) \rightarrow 0$. Hence also $\mu_i$ converges to the invariant random subgroup $\mu$ as $i \rightarrow \infty$. We conclude that $\mu$ is co-sofic. □

While the ergodic theorem of [Lin01] requires the Folner sequence $F_i$ to be so-called tempered, this can always be achieved by passing to a subsequence, so that this issue can be safely ignored for our purposes.

4. INVARIANT RANDOM SUBGROUPS OF $V$

Consider the group $\text{Sym}_{\text{fin}}(\mathbb{Z})$ of all finitely supported permutations of the set $\mathbb{Z}$ as well as its index two subgroup $U_\infty = \text{Alt}_{\text{fin}}(\mathbb{Z})$ consisting of even permutations.

Vershik [Ver12] had established a complete classification of all ergodic invariant random subgroups of the group $\text{Sym}_{\text{fin}}(\mathbb{Z})$. We recall this classification and apply it to study the invariant random subgroups of $V = U_\infty \rtimes \langle \bar{\sigma} \rangle$. Our description follows closely that of Thomas’ paper [Tho18].

**Random partitions.** Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider the product space $\mathbb{N}_0 \times \mathbb{Z}$. To every point $\omega \in \mathbb{N}_0 \times \mathbb{Z}$, we correspond the partition $\mathbb{Z} = B_0^\omega \sqcup B_1^\omega \sqcup B_2^\omega \sqcup \cdots$ where an integer $x \in \mathbb{Z}$ satisfies $x \in B_i^\omega$ if and only if $\omega(x) = i$. Some members $B_i^\omega$ of this partition may be empty. Alternatively, we may think of the point $\omega$ as describing a coloring of the set $\mathbb{Z}$ using the colors $\mathbb{N}_0$.

For $\omega \in \mathbb{N}_0^\mathbb{Z}$, let $\text{Sym}_{\text{fin}}(\omega)$ denote the subgroup of $\text{Sym}_{\text{fin}}(\mathbb{Z})$ preserving the partition (or, alternatively, the coloring) corresponding to $\omega$ and fixing $B_0^\omega$ pointwise. Namely, the color 0 $\in \mathbb{N}_0$ is special in that $\text{Sym}_{\text{fin}}(\omega)$ is required to fix every point $x \in \mathbb{Z}$ with $\omega(x) = 0$. In other words

$$\text{Sym}_{\text{fin}}(\omega) = \bigoplus_{i \in \mathbb{N}} \text{Sym}_{\text{fin}}(B_i^\omega).$$

In addition, there is a sign map

$$\text{sgn}_{\omega} : \text{Sym}_{\text{fin}}(\omega) \rightarrow \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$$

defined component-wise. Let

$$\varepsilon : \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

be the map taking an element $\oplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ to the sum of all of its components. A permutation $g \in \text{Sym}_{\text{fin}}(\omega)$ is even if and only if $\varepsilon \circ \text{sgn}_{\omega}(g) = 0$. The derived subgroup of $\text{Sym}_{\text{fin}}(\omega)$ is the subgroup $\text{Alt}_{\text{fin}}(\omega) = \ker(\varepsilon \circ \text{sgn}_{\omega})$.

We now describe a random model for such partitions. Let $\alpha$ be a probability vector of the form

$$\alpha = (\alpha_0; \alpha_1, \alpha_2, \ldots)$$

where $\sum_{i \in \mathbb{N}_0} \alpha_i = 1$, for all $i \in \mathbb{N}_0$ the $i$-th entry satisfies $0 \leq \alpha_i \leq 1$ and the sequence $(\alpha_i)_{i \in \mathbb{N}}$ is monotone decreasing in the sense that $\alpha_{i+1} \geq \alpha_i$ for all $i \in \mathbb{N}$. It is not required that $\alpha_0 \geq \alpha_1$. Consider the probability measure space $\Omega = (\mathbb{N}_0^\mathbb{Z}, \alpha^\mathbb{Z})$.
where $\alpha^\mathbb{Z}$ is the Bernoulli probability measure. This means that the color of each point is chosen independently at random with respect to the probability measure $\alpha$ at each coordinate. A random partition of the set $\mathbb{Z}$ corresponding to the probability vector $\alpha$ is obtained as described above with respect to a $\alpha^\mathbb{Z}$-random point $\omega \in \Omega$.

**Remark 4.1.** Let $i \in \mathbb{N}_0$ be such that $\alpha_i > 0$. Then for $\alpha^\mathbb{Z}$-almost every partition $\omega$ the member $B_i^\omega$ is infinite, see e.g. [Ver12, Tho18].

**Remark 4.2.** The Bernoulli measure $\alpha^\mathbb{Z}$ is $\text{Sym}(\mathbb{Z})$-invariant and the subgroup $\text{Alt}_{\text{fin}}(\mathbb{Z})$ is already acting ergodically. The proof of this fact is essentially the same as the well-known mixing argument for the ergodicity of the shift. See e.g. [Wal00, Theorem 1.30].

**Vershik’s classification theorem.** Roughly speaking, this theorem says that every invariant random subgroup of $\text{Sym}_{\text{fin}}(\mathbb{Z})$ arises from a random partition. More precisely (see also [Tho18]):

**Theorem 4.3** ([Ver12]). Let $\mu$ be an ergodic invariant random subgroup of $\text{Sym}_{\text{fin}}(\mathbb{Z})$. Then there is a probability vector $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ as above and a subgroup $S \leq \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ such that $\mu = f^S\alpha^\mathbb{Z}$, where

$$f^S : \Omega \to \text{Sub}(\text{Sym}_{\text{fin}}(\mathbb{Z})), \quad f^S : \omega \mapsto \text{sgn}^{-1}(S) \leq \text{Sym}_{\text{fin}}(\omega).$$

Vershik’s theorem deals with invariant random subgroups of $\text{Sym}_{\text{fin}}(\mathbb{Z})$ rather than of $\text{Alt}_{\text{fin}}(\mathbb{Z})$. Let us rapidly outline how to go from one group to the other.

**Corollary 4.4.** Let $\mu$ be an ergodic invariant random subgroup of $\text{Alt}_{\text{fin}}(\mathbb{Z})$. Then $\mu$ is obtained as in Theorem 4.3 with the subgroup $S \leq \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ satisfying $\varepsilon(S) = 0$.

In the following proof we rely on that fact that for any given group $G$ the space IRS$(G)$ is a Choquet simplex. Therefore every invariant random subgroup $\mu$ of $\text{Alt}_{\text{fin}}(\mathbb{Z})$ admits a unique ergodic decomposition, and likewise for $\text{Sym}_{\text{fin}}(\mathbb{Z})$.

**Proof of Corollary 4.4.** Consider the element $\bar{\rho} \in \text{Sym}_{\text{fin}}(\mathbb{Z})$ given by $\bar{\rho} = (1, 2)$, say. The alternating group $\text{Alt}_{\text{fin}}(\mathbb{Z})$ has index two in $\text{Sym}_{\text{fin}}(\mathbb{Z})$ and $\bar{\rho}$ represents the non-trivial coset.

It is easy to verify that the convex combination $\nu = \frac{1}{2}(\mu + \bar{\rho}_*\mu)$ is an invariant random subgroup of $\text{Sym}_{\text{fin}}(\mathbb{Z})$. If $\bar{\rho}_*\mu = \mu$ then Vershik’s Theorem 4.3 applied to the invariant random subgroup $\nu = \mu$ directly implies our Corollary.

We claim that $\bar{\rho}_*\mu \neq \mu$ is impossible. To see this, note that $\bar{\rho}_*\mu$ is again an ergodic invariant random subgroup of $\text{Alt}_{\text{fin}}(\mathbb{Z})$. So that the convex combination $\frac{1}{2}(\mu + \bar{\rho}_*\mu)$ is the $\text{Alt}_{\text{fin}}(\mathbb{Z})$-ergodic decomposition of $\nu$. On the other hand, Theorem 4.3 combined with Remark 4.2 implies that every ergodic $\text{Sym}_{\text{fin}}(\mathbb{Z})$-invariant random subgroup is $\text{Alt}_{\text{fin}}(\mathbb{Z})$-ergodic. Therefore the $\text{Sym}_{\text{fin}}(\mathbb{Z})$-ergodic decomposition of $\nu$ coincides with its ergodic decomposition regarded as an invariant random subgroup of $\text{Alt}_{\text{fin}}(\mathbb{Z})$. This eliminates the possibility that $\bar{\rho}_*\mu \neq \mu$, as required. □

Recall that $\bar{\sigma} \in \text{Sym}(\mathbb{Z})$ is the infinitely supported permutation acting via $\bar{\sigma}(x) = x + 1$ for all $x \in \mathbb{Z}$. Moreover recall the notation $U_\infty = \text{Alt}_{\text{fin}}(\mathbb{Z})$.

**Corollary 4.5.** Let $\mu$ be an ergodic invariant random subgroup of $U_\infty$. If $\mu$-almost every subgroup is normalized by $\bar{\sigma}^k u$ for some fixed $k \in \mathbb{N}$ and some element $u \in U_\infty$ depending on the subgroup, then either $\mu = \delta_{\{e\}}$ or $\mu = \delta_{U_\infty}$. 
Proof. Apply Corollary 4.4 to classify the invariant random subgroup $\mu$. Let $\alpha$ be the resulting probability vector and $l(\alpha) \in \mathbb{N} \cup \{\infty\}$ be the number of the non-zero entries in $\alpha$. We claim that $l(\alpha) = 1$. The conclusion follows immediately from this, for the only two possible probability vectors with $l(\alpha) = 1$ are either $\alpha_0 = (1;0,0,\ldots)$ or $\alpha_1 = (0;1,0,\ldots)$. The first one gives rise to the invariant random subgroup $\delta_{\{e\}}$ and the second one to $\delta_{U_{\infty}}$.

Assume towards contradiction that $l(\alpha) \geq 2$. This means that $\mu$-almost every subgroup preserves a random partition $\omega$ admitting at least two infinite members, see Remark 4.1. Such a subgroup is normalized by some $\bar{\sigma}$ invariant and permutes the blocks $B_1^\omega, B_2^\omega, \ldots$ among themselves. The probability that $\bar{\sigma}^k$ acts in such a structured way with respect to a random partition $\omega$ is zero. Since $\bar{\sigma}^k u$ differs from $\bar{\sigma}^k$ on at most a finite number of points, the probability with respect to $\mu$ of this happening is also zero. We arrive at a contradiction. $\square$

**Invariant random subgroups of $V$.** Recall that $V = U_{\infty} \rtimes \langle \bar{\sigma} \rangle$.

**Corollary 4.6.** Let $\mu$ be an ergodic invariant random subgroup of $V$. Then exactly one of the following two statements is true:

1. $\mu$-almost every subgroup satisfies $H \leq U_{\infty}$, or
2. $\mu$ is the atomic probability measure supported on the subgroup $U_{\infty} \rtimes \langle \bar{\sigma}^k \rangle$ for some $k \in \mathbb{N}$.

Proof. It is clear that the two possibilities are mutually exclusive. Assume that Condition (1) fails. As $\mu$ is ergodic, this means that $\mu$-almost every subgroup contains an element of the form $\bar{\sigma}^k u$ for a fixed $k \in \mathbb{N}$ and some $u \in U_{\infty}$ depending on that subgroup. The restriction of $\mu$ to $U_{\infty}$ is an invariant random subgroup of $U_{\infty}$. It follows from Corollary 4.4 that this restriction is equal to a convex combination $s\delta_{\{e\}} + (1-s)\delta_{U_{\infty}}$ for some $s \in [0,1]$. Write $\mu = s\mu' + (1-s)\delta_{U_{\infty}} \rtimes \langle \bar{\sigma}^k \rangle$ where $\mu'$-almost every subgroup has trivial intersection with $U_{\infty}$. This means that $\mu'$-almost every subgroup is cyclic and generated by an element of the form $\bar{\sigma}^k u$. Since $V$ admits only countably many such cyclic subgroups and each has an infinite conjugacy class, we conclude that $s = 0$. Condition (2) follows. $\square$

5. **Invariant random subgroups supported on the derived subgroup**

Let $\mathcal{P} = (n_1, n_2, \ldots)$ be a monotone increasing sequence of integers with $n_1 \geq 5$. Consider the corresponding B.H. Neumann group $G = G(\mathcal{P})$. In this section we construct a Weiss approximation to every subgroup of $G$ contained in its derived subgroup $N = N(\mathcal{P}) = [G(\mathcal{P}), G(\mathcal{P})]$.

Write $n_i = 2r_i + 1$. Recall the finite subgroups $L_i(\mathcal{P})$ defined in 3.2 for every $i \in \mathbb{N}$. In particular $N$ is the ascending union of the $L_i(\mathcal{P})$’s. Denote $L_i = L_i(\mathcal{P})$.

Consider the subgroups $G_i$ given for each $i \in \mathbb{N}$ by

$$G_i = \ker \left( G(\mathcal{P}) \to \prod_{j=1}^i A_j(\mathcal{P}) \right).$$

Since the projection of each $L_i$ to the product $\prod_{j=1}^i A_j(\mathcal{P})$ is an isomorphism it follows that $L_i$ is a complement to $G_i$ in the group $G$, namely for each $i \in \mathbb{N}$

$$L_i \cap G_i = \{e\} \quad \text{and} \quad L_i G_i = G.$$
Finally, consider the abelianization map $G \to \langle \bar{\sigma} \rangle \cong \mathbb{Z}$ and take
\[ D_i = \ker (G \to \mathbb{Z}/(2r_i - 1)\mathbb{Z}). \]
Clearly both $G_i$ and $D_i$ are finite index normal subgroups of $G$ for every $i \in \mathbb{N}$.

**Proposition 5.1.** Fix an element $g \in N$. Then
\[ \frac{|\{ j \in [r_i] : g^{\sigma_j} \in L_i \}|}{|r_i|} \to 1. \]

**Proof.** Recall that $i = i(g) \in \mathbb{N}$ is the smallest index so that $g \in L_i(\mathcal{P})$. We will prove the stronger statement that
\[ \frac{|\{ j \in [r_i] : L_{i(g)}^{\sigma_j} \leq L_i \}|}{|r_i|} \to 1. \]
It follows from the definition of the groups $L_i$ that the condition $L_{i(g)}^{\sigma_j} \leq L_i$ will hold provided that $j + \|i(g)\| \subset [r_i]$. This last condition is satisfied whenever $j \in [r_i - i(g)]$. As the index $i(g)$ is being kept fixed, we have
\[ \frac{\|r_i - i(g)\|}{|r_i|} = \frac{2(r_i - i(g)) + 1}{2r_i + 1} \to 1 \]
as required. \hfill\□

The above Proposition 5.1 is to be compared with the notion of *adapted subsets* in the setting of metabelian groups \[\text{[LL19, Definition 7.2 and Lemma 10.3].}\]

We now construct the desired Weiss approximations with respect to the Folner sequence $F_i = F_i(\mathcal{P}) = \{ \sigma_j : j \in [r_i - 1] \}$, $L_i$ studied in Proposition 2.6.

**Proposition 5.2.** Let $H \leq N$ be any subgroup. Then $H$ has a Weiss approximation $(K_i, F_i)$ for some sequence $K_i \leq G$ of finite index subgroups.

**Proof.** Fix $i \in \mathbb{N}$ and consider the subgroup $K_i$ of $G$ given by
\[ K_i = (H \cap L_i) (G_i \cap D_i). \]
Since $G_i \cap D_i$ is a finite index normal subgroup of $G$ it follows that $K_i$ has finite index in $G$ as well. Moreover, since \{ $\sigma^k : k \in [r_i - 1]$ \} is a transversal to the normal subgroup $D_i$ and $L_i$ is a transversal to the normal subgroup $G_i$, the subset $F_i$ is a finite-to-one transversal to $G_i \cap D_i$ (actually, it is a transversal since $G_i D_i = G$, but this fact is not needed). We deduce that $F_i$ is a finite-to-one transversal to the subgroup $K_i$.

We claim that for all $i \in \mathbb{N}$, the condition $L_i \subset G \setminus (H \triangle K_i)$ holds. This condition is equivalent to
\[ L_i \cap H = L_i \cap K_i. \]
The inclusion $L_i \cap H \subset L_i \cap K_i$ is clear. For the converse, consider some $g \in L_i \cap K_i$ so that $g = hd$ for some pair of elements $h \in H \cap L_i$ and $d \in G_i \cap D_i$. As both $g$ and $h$ are contained in $L_i$ it follows that $d \in L_i$ as well. However $L_i \cap G_i = \{ e \}$ and $d \in G_i$ so that $d$ must be trivial. Hence $g = h \in L_i \cap H$.

We would like to apply Proposition 3.3 and deduce that $(K_i, F_i)$ is a Weiss approximation. In other words, we need to verify that every element $g \in G$ satisfies
\[ p_i(g) = \frac{|\{ f \in F_i : g^f \in K_i \triangle H \}|}{|F_i|} \to 0. \]
To this end, fix an element \( g \in G \). First consider the case where \( g \notin N \). Therefore \( g^f \notin H \) for all \( f \in G \). Similarly, provided \( i \in \mathbb{N} \) is sufficiently large \( g^f \notin D_i \), which implies \( g^f \notin K_i \) for all \( f \in G \). In particular \( \nu_i(g) = 0 \) for all \( i \in \mathbb{N} \) sufficiently large.

It remains to consider the more interesting case where \( g \in N \). Since by the above \( L_i \) belongs to \( G \setminus (H \triangle K_i) \), it will suffice to show that

\[
q_i(g) = \left| \frac{\{ f \in F_i : g^f \in L_i \}}{|F_i|} \right| \xrightarrow{i \to \infty} 1.
\]

Every element \( f \in F_i \) can be written as \( f = \sigma^j l \) where \( j \in [r_i - 1] \) and \( l \in L_i \). Observe that

\[
g^f \in L_i \iff g^{\sigma^j} l \in L_i \iff g^{\sigma^j} \in L_i.
\]

The element \( g \) being kept fixed, the probability that \( g^{\sigma^j} \in L_i \) as \( j \) ranges over the set \( [r_i - 1] \) tends to one as \( i \) goes to infinity, see Proposition 5.1. This completes the proof.

Returning to the remark made in the final paragraph of §1, note that a special case of Proposition 5.2 constructs a Weiss approximation \( (K_i, F_i) \) to the subgroup \( U = U(P) \). In this case, one can check that the proof gives

\[K_i = \ker(G(P) \to A_i(P)).\]

It is interesting to note that for all \( i \in \mathbb{N} \) sufficiently large, the finite index subgroups \( K_i \) do not contain the subgroup \( U \). Indeed, the only finite index subgroups containing \( U \) are the \( D_i \)'s, and they are clearly do not form a Weiss approximation to the subgroup \( U \).

6. Proof of the main theorem

Let \( \mu \in \text{IRS}(G(P)) \) be any invariant random subgroup. We now show that \( \mu \) is co-sofic, thereby completing the proof of Theorem 1.4. Our main result Theorem 1.3 follows, relying on the criterion [BLT19] reproduced here as Theorem 1.2.

We may assume without loss of generality that \( \mu \) is ergodic, see [LL19, Corollary 2.6] for clarification.

First consider the case where \( \mu \)-almost every subgroup \( H \) is contained in \( N \). Therefore \( \mu \)-almost every subgroup admits a Weiss approximation \( (K_i, F_i) \) with respect to our fixed Folner sequence \( F_i = F_i(P) \) of finite-to-one transversals as established in Proposition 5.2. The fact that \( \mu \) is co-sofic follows from Theorem 3.4 and relying on the Lindenstrauss pointwise ergodic theorem for amenable groups.

The second case to consider is where the projection of \( \mu \) to the abelianization of \( G \) is a non-trivial ergodic invariant random subgroup. Namely, the image of \( \mu \)-almost every subgroup \( H \) in the \( G/G' \) is equal to \( \langle \bar{\sigma} \rangle^k \) for some \( k \in \mathbb{N} \). We may therefore consider the resulting push-forward invariant random subgroup on \( V = U_\infty \times \langle \bar{\sigma} \rangle \). Vershik’s theorem and in particular its Corollary 4.1 implies that for \( \mu \)-almost every subgroup \( H \) its image in \( V \) has finite index. But according to the algebraic Proposition 2.5 this means that \( \mu \)-almost every subgroup has finite index in \( G \) to begin with. In other words \( \mu \in \text{IRS}_0(G) \), which is already a much stronger conclusion than \( \mu \) being co-sofic. The theorem is now proved.
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