LECTURES ON PAINLEVE PROPERTY FOR SEMI-SIMPLE
FROBENIUS MANIFOLDS

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Abstract. These notes are based on a sequence of five lectures given to graduate students. The main goal is to prove the so-called Painleve property for semi-simple Frobenius manifolds.

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1. Introduction

The Painleve property for semi-simple Frobenius manifolds can be stated as follows. Let $U$ be a contractible open subset of the configuration space $Z_N = \{u \in \mathbb{C}^N : u_i \neq u_j \text{ for } i \neq j\}$.

As we will see later on, in order to define a semi-simple Frobenius structure on $U$ we have to choose a holomorphic 1-form $\sum_{i=1}^{N} \eta_i(u) du_i$ on $U$ such that

(i) $\eta_i(u) \neq 0$ for all $i$ and for all $u \in U$.

(ii) The 1-form is closed, i.e., $\eta_{ab} := \partial \eta_a / \partial u_b$ is symmetric in $a$ and $b$.

(iii) The functions $\eta_i(u)$ $(1 \leq i \leq N)$ satisfy the following system of PDEs

$$(\partial u_1 + \cdots + \partial u_N) \eta_i = 0 \quad (1 \leq i \leq N)$$

$$(D + u_1 \partial u_1 + \cdots + u_N \partial u_N) \eta_i = 0 \quad (1 \leq i \leq N)$$

$$\frac{\partial \eta_{ij}}{\partial u_k} - \frac{1}{2} \left( \frac{\eta_{ij} \eta_{kj}}{\eta_j} + \frac{\eta_{ij} \eta_{ik}}{\eta_k} + \frac{\eta_{ki} \eta_{ji}}{\eta_i} \right) = 0 \quad (k \neq i \neq j \neq k),$$

where $D$ is some constant and $\partial u_a := \partial / \partial u_a$.

The geometric interpretation of the above conditions is the following. Let us define a diagonal bi-linear pairing on the tangent bundle $TU$ via

$$(\partial u_i, \partial u_j) = \delta_{ij} \eta_j(u).$$

Condition (i) is equivalent to requiring that the metric is non-degenerate. Condition (ii) is equivalent to requiring that the translation vector field $e = \partial u_1 + \cdots + \partial u_N$ is flat with respect to the Levi–Civita connection $\nabla^{L.C.}$ of the pairing. Finally, the system of PDEs in (iii) is equivalent to requiring that the metric is translation invariant, conformal invariant, and flat. The main goal of these lectures is to prove the following theorem.

**Theorem 1.1.** Suppose that $U$ is lifted to the universal cover $T$ of $Z_N$. Then every 1-form $\sum_{i=1}^{N} \eta_i(u) du_i$ satisfying the above conditions (i)–(iii) extends to a global meromorphic 1-form on $T$.

Recall that a system of differential equations defined in some domain $T$ is said to have the Painleve property on $T$ if the so-called movable singularities (singularities depending on the initial conditions) are at most poles. That is why the statement of Theorem 1.1 is usually referred to as the Painleve property for semi-simple Frobenius manifolds.
Theorem 1.1 is a corollary of the theory of isomonodromic deformations. In fact, there are two possible ways to obtain a proof. They correspond to the fact that a semi-simple Frobenius manifold can be obtained as a solution to two different Riemann–Hilbert (RH) problems (see [3]). The first RH problem consists of finding a connection on $\mathbb{P}^1$ with one Fuchsian singularity at, say $\infty$, and one irregular singular point at, say 0. The second one is a RH problem for a Fuchsian connection. The two connections are related via a formal Laplace transform, so in principle one could switch between the two languages. Using the first RH problem, the proof of Theorem 1.1 follows from the results of T. Miwa in [12], while the second RH problem reduces the proof to the results of Malgrange in [9]. Therefore, if one wants to understand the statement of Theorem 1.1 then the main task is to understand either the results of Miwa or the results of Malgrange.

In these lectures we are going to pursue the second approach based on Malgrange results. The most difficult part in [9] is a certain theorem about existence of a meromorphic trivialization of a family of vector bundles on $\mathbb{P}^1$ (see Theorem 4.1 in these notes). On the other hand, A. Bolibruch found an elementary proof, so combining the work of Bolibruch and Malgrange we can obtain an argument that requires only basic knowledge of complex geometry (e.g. chapter 0 in [7]) and ordinary differential equations (e.g. [1]).

Summarizing, our lectures include Bolibruch’s proof of the Birkhoff–Grothendieck theorem with parameters, Malgrange’s proof of the Painleve property for the Schlesinger equations, and finally we check that Theorem 1.1 is a corollary of the Painleve property for the Schlesinger equations. The last part can be found also in [2] [10]. We have also included a very interesting theorem of Manin (see [10]) classifying the initial conditions for the Schlesinger equations such that the corresponding solution of the Schlesinger equations comes from a semi-simple Frobenius manifold.

We have to admit that many of the arguments in these lectures although elementary are a bit cumbersome. They could be made more elegant if one is willing to use some more advanced (but standard) techniques in complex geometry. Fortunately, this goal is achieved by C. Sabbah. For more details we refer to the excellent book [13].

2. Levelt’s theory for Fuchsian connections

The main goal of this lecture is to prove the existence of weak Levelt solutions for Fuchsian systems. We follow closely [2].

2.1. Fuchsian systems. Let $D = \{ \lambda : |\lambda| < r_0 \}$ be the open disk of radius $r_0$ and $B_0(\lambda) \in \mathfrak{gl}(\mathbb{C}^p)$ be a $p \times p$-matrix whose entries depend holomorphically on $\lambda \in D$. We will be interested in the system of ODEs defined by

$$\frac{\partial y}{\partial \lambda}(\lambda) = B(\lambda)y(\lambda), \quad B(\lambda) := B_0(\lambda)/\lambda.$$ 

Systems of this type are said to be Fuchsian in a neighborhood of 0.
Let us fix a small sector $S$ in $D$ containing the open interval $(0, r_0)$, e.g.,

$$S = \{ \lambda \in D - \{0\} : -\epsilon < \text{Arg}(\lambda) < \epsilon \}$$

where $0 < \epsilon < 2\pi$ is fixed arbitrary. Furthermore, let us fix a reference point $\lambda_0 \in (0, r_0) \subset S$ and denote by $X$ the space of holomorphic functions $y : S \to \mathbb{C}^p$ that solve the above system. The general theory of ODEs implies that $X$ is a finite dimensional vector space of dimension $p$. More precisely

$$X \cong \mathbb{C}^p, \quad y \mapsto y(\lambda_0).$$

Since the coefficients of the linear system are holomorphic in $D - \{0\}$, every solution $y \in X$ can be extended analytically along any path in $D - \{0\}$. In particular, we have a linear map

$$\sigma : X \to X$$

corresponding to analytic continuation along a loop based at $\lambda_0$ that goes once around $\lambda = 0$ in counter-clockwise direction.

### 2.2. Fuchsian singularities are regular

The following result is well known in the theory of ODEs. Nevertheless, we give our own proof as well. For a different argument, which is shorter but yields a slightly weaker result, see [2], Theorem 4.1 and Lemma 4.1.

**Proposition 2.1.** Every solution $y \in X$ has the form

$$y(\lambda) = \sum_{\rho} \sum_{k=0}^{p-1} y_{\rho,k}(\lambda)\lambda^{\rho}(\log \lambda)^k,$$

where the first sum is over all eigenvalues $\rho$ of $B_0(0)$ and $y_{\rho,k}(\lambda)$ are $\mathbb{C}^p$-valued functions analytic for all $\lambda \in D$.

**Proof.** We will prove that the system has a fundamental matrix whose columns have the above form. Using a constant gauge transformation $y(\lambda) \mapsto Cy(\lambda)$ we can reduce the general case to the case when $B_0(0)$ is in Jordan normal form. Moreover, we may assume that the Jordan blocks are ordered in such a way that $B_0(0) = R + N^{(0)}$, where $R$ and $N^{(0)}$ have the following properties. Both

$$R = \text{diag}(R_1, \ldots, R_s) \quad \text{and} \quad N^{(0)} = \text{diag}(N_1^{(0)}, \ldots, N_s^{(0)})$$

are block-diagonal. The block $R_i = \rho_i I_i$, where $I_i$ is an identity matrix of size the multiplicity of $\rho_i$ as an eigenvalue of $B_0(0)$ and

$$\text{Re}(\rho_1) > \cdots > \text{Re}(\rho_s).$$

The block $N_i^{(0)}$ ($1 \leq i \leq s$) is an upper-triangular nilpotent matrix whose size is the same as the the size of $I_i$. Note that the commutator $[R, N^{(0)}] = 0$. 

Claim 2.2. There exists a formal solution

\[ Y(\lambda) = U(\lambda)\lambda^R \lambda^N, \]

where

\[ U(\lambda) = 1 + U_1 \lambda + U_2 \lambda^2 + \cdots, \quad U_k \in \mathfrak{gl}(\mathbb{C}^p) \]

and \( N \) is upper-triangular nilpotent matrix of the form

\[ N = N^{(0)} + N^{(1)} + \cdots, \quad [R, N^{(k)}] = k N^{(k)}. \]

Proof. Put \( B_0(\lambda) = B_{0,0} + B_{0,1}\lambda + \cdots \), substitute \( Y(\lambda) \) in the differential equation, and compare the coefficients in front of the powers of \( \lambda^k \). For \( k = 0 \) we get \( R + N^{(0)} = B_{0,0} \), which is true by definition. For \( k > 0 \) we get

\[ kU_k + [U_k, R + N^{(0)}] + N^{(k)} = B_{0,k} + \sum_{i=1}^{k-1} \left( B_{0,k-i} U_i - U_i N^{(k-i)} \right). \]

The linear operator

\[ \text{ad}_R : \mathfrak{gl}(\mathbb{C}^p) \to \mathfrak{gl}(\mathbb{C}^p), \quad x \mapsto [R, x] \]

is diagonalizable, i.e., we have a decomposition

\[ \mathfrak{gl}(\mathbb{C}^p) = \bigoplus_{a \in \text{spec}(R)} \mathfrak{gl}_a(\mathbb{C}^p), \]

where \( \text{spec}(R) \) denotes the set of eigenvalues of \( \text{ad}_R \) and for \( a \in \text{spec}(R) \)

\[ \mathfrak{gl}_a(\mathbb{C}^p) = \{ x : [R, x] = ax \} \]

is the corresponding eigen-subspace. Let us denote by

\[ \pi_a : \mathfrak{gl}(\mathbb{C}^p) \to \mathfrak{gl}_a(\mathbb{C}^p) \]

the projection map defined via the above decomposition. Let us assume that we have determined \( U_1, \ldots, U_{k-1} \) and \( N^{(1)}, \ldots, N^{(k-1)} \). Then \( U_k = \sum_{a \in \text{spec}(R)} \pi_a(U_k) \) and \( N^{(k)} \in \mathfrak{gl}_k(\mathbb{C}^p) \) are defined by projecting via \( \pi_a \) the above recursion relation and solving for \( \pi_a(U_k) \) and \( \pi_a(N^{(k)}) \). There are two cases. First, if \( a = k \), then we set \( \pi_k(U_k) = 0 \). Note that since \( N^{(0)} \) commutes with \( R \), we have \( \pi_k([U_k, N^{(0)}]) = [\pi_k(U_k), N^{(0)}] = 0 \) and \( \pi_k(N^{(k)}) = N^{(k)} \). Therefore, we can uniquely solve for \( N^{(k)} \). The second case is if \( a \neq k \), then \( \pi_a(N^{(k)}) = 0 \) and

\[ \pi_a(kU_k + [U_k, R + N^{(0)}]) = (k - a - \text{ad}_{N^{(0)}}) \pi_a(U_k). \]

Since \( N^{(0)} \) is nilpotent, the linear operator \( \text{ad}_{N^{(0)}} \) is also nilpotent. Therefore the linear operator \( k - a - \text{ad}_{N^{(0)}} \) is invertible, so we can uniquely solve for \( \pi_a(U_k) \). \( \square \)

It remains to prove that the formal series \( U(\lambda) \) is convergent. Note that \( U(\lambda) \) satisfies the following differential equation

\[ (\lambda \partial_\lambda - \text{ad}_{R + N^{(0)}}) U = (U \alpha(\lambda) + \beta(\lambda) U), \]

(1)
where
\[ \alpha(\lambda) = - \sum_{i=1}^{\infty} N(i) \lambda^i \quad \beta(\lambda) = \sum_{i=1}^{\infty} B_{0,i} \lambda^i. \]

Let us fix an integer \( k > 0 \), such that the set \( \text{spec}(R) \) does not contain any integers \( \ell > k \). Note that \( N(\ell) = 0 \) for all \( \ell > k \), so \( \alpha(\lambda) \) is polynomial in \( \lambda \). Let us write the formal series in the form
\[ U(\lambda) = U_{\leq k}(\lambda) + \lambda^k V(\lambda), \quad U_{\leq k}(\lambda) = 1 + \sum_{i=1}^{k} U_i \lambda^i, \]
where \( V(\lambda) = \sum_{j=1}^{\infty} U_j + \lambda^j \). Then \( V(\lambda) \) satisfies the following differential equation
\[ (\lambda \partial_{\lambda} + k - \text{ad}_{R+N(0)}) V = V \alpha(\lambda) + \beta(\lambda)V + \gamma(\lambda), \]
where
\[ \gamma(\lambda) = \lambda^{-k} \left( U_{\leq k}(\lambda) \alpha(\lambda) + \beta(\lambda)U_{\leq k}(\lambda) - (\lambda \partial_{\lambda} - \text{ad}_{R+N(0)})U_{\leq k}(\lambda) \right) \]

By definition \( U_{\leq k}(\lambda) \) satisfies the differential equation (1) up to terms of order \( O(\lambda^{k+1}) \). Therefore, \( \gamma(\lambda) \) is analytic at \( \lambda = 0 \) and \( \gamma(0) = 0 \). It is enough to prove that the differential equation (2) has a solution \( V_{\text{hol}}(\lambda) \) analytic at \( \lambda = 0 \). Indeed, the linear operator \( k - \text{ad}_{R+N(0)} \) is invertible, so after substituting the Taylor series of \( V_{\text{hol}}(\lambda) \) in the differential equation we get that the Taylor series must coincide with the formal series \( V(\lambda) \).

In order to construct a holomorphic solution, we use the standard idea to identify \( V_{\text{hol}}(\lambda) \) with the fixed point of a certain integral operator. Let us fix a closed disk \( D_r = \{ \lambda : |\lambda| \leq r \} \) with radius \( r < r_0 \). Let us define a sequence of holomorphic \( \mathfrak{gl}(\mathbb{C}^p) \)-valued functions
\[ V_n : D_r \to \mathfrak{gl}(\mathbb{C}^p), \quad n = 0, 1, 2, \ldots \]
as follows. Put \( V_0(\lambda) = 0 \) and let \( V_{n+1}(\lambda) \) be such that
\( (\lambda \partial_{\lambda} + k - \text{ad}_{R+N(0)}) V_{n+1} = V_n \alpha(\lambda) + \beta(\lambda)V_n + \gamma(\lambda). \)

Note that
\[ V_{n+1}(\lambda) = \int_0^1 e^{k-\text{ad}_{R+N(0)}(t\lambda)} \left( V_n(t\lambda) \alpha(t\lambda) + \beta(t\lambda)V_n(t\lambda) + \gamma(t\lambda) \right) \frac{dt}{t}. \]
The convergence of the integral follows from the fact that if we choose \( k \) sufficiently large the real part of the eigenvalues of \( k - \text{ad}_{R+N(0)} \) will be positive. Therefore \( V_{n+1}(\lambda) \) is an analytic function for all \( \lambda \in D_r \).

In order to prove that the sequence \( V_n \) is convergent we introduce the following norm. Let \( |\cdot| : \mathfrak{gl}(\mathbb{C}^p) \to \mathbb{R}_{\geq 0} \) be the standard matrix norm
\[ |A| = \sup_{|v| \neq 0} \frac{|Av|}{|v|}, \]
where $|v| = \sqrt{|v_1|^2 + \cdots + |v_p|^2}$ is the standard Euclidean norm of $v \in \mathbb{C}^p$. If $A : D_r \to \mathfrak{gl}(\mathbb{C}^p)$ is holomorphic, then we define

$$|A|_r = \sum_{i=0}^{\infty} |A_i|r^i,$$

where $A(\lambda) = \sum_{i=0}^{\infty} A_i \lambda^i$ is the Taylor series expansion. Let $B_r$ be the space of those holomorphic maps $A$ for which $|A|_r < \infty$. It is known (see [6]) that $B_r$ is a Banach algebra. Using the Cauchy inequality it is easy to prove that if $A(\lambda)$ is holomorphic for all $\lambda \in D$ then $A \in B_r$.

**Claim 2.3.** Suppose $k > |\text{ad}_{R+N(0)}|$. Then the map

$$F : B_r \to B_r, \quad F(A)(\lambda) := \int_0^1 t^{k-\text{ad}_{R+N(0)}}A(t\lambda)\frac{dt}{t}$$

is a bounded linear operator of norm less or equal to 1, i.e., $\|F(A)\|_r \leq |A|_r$.

**Proof.** Put $A(\lambda) = \sum_{i=0}^{\infty} A_i \lambda^i$. Then the coefficient in front of $\lambda^i$ in $F(A)$ is

$$F(A)_i = \int_0^1 t^{k+i-1-\text{ad}_{R+N(0)}}A_idt.$$

Using that

$$|t^{k+i-1-\text{ad}_{R+N(0)}}| = t^{k+i-1}|t^{-\text{ad}_{R+N(0)}}| \leq t^{k+i-1}t^{-|\text{ad}_{R+N(0)}|}, \quad 0 \leq t \leq 1$$

we get

$$|F(A)_i| \leq \frac{|A_i|}{k+i-|\text{ad}_{R+N(0)}|} \leq |A_i|.$$  \quad \square

Note that $V_{n+1} = F(V_n \alpha + \beta V_n + \gamma)$. Therefore

$$\|V_{n+1} - V_n\|_r \leq (\|\alpha\|_r + \|\beta\|_r) \|V_n - V_{n-1}\|_r.$$  

Since $\alpha(0) = \beta(0) = 0$ we can always choose $r$ so small that $\|\alpha\|_r + \|\beta\|_r < 1$. Then the above inequality shows that $\{V_n\}$ is a Cauchy sequence in $B_r$, so the limit $V_{hol} = \lim_{n \to \infty} V_n$ exists and it gives a solution to the differential equation [2].

Finally, note that the series $U(\lambda)$ must be analytic for all $\lambda \in D$, because the fundamental matrix $Y(\lambda) = U(\lambda)\lambda^{R_N}$ extends analytically along any path inside $D - \{0\}$.

**Corollary 2.4.** If $B_0(0)$ is nilpotent, then the matrix of the monodromy of the Fuchsian system with respect to a basis of $X$ given by the columns of the fundamental matrix $Y(\lambda)$ satisfying the initial condition $Y(\lambda_0) = 1$ is $e^{2\pi \sqrt{-1}B_0(0)}$.  \quad \square
2.3. Levelt evaluations. Let us denote by \( \mathcal{O}[S] \) the space of holomorphic maps \( y : S \to \mathbb{C}^p \), such that
\[
\lim_{\lambda \to 0} \frac{y(\lambda)}{|\lambda|^m} = 0
\]
for some integer \( m \). Such functions are also sometimes said to be of moderate growth at \( \lambda = 0 \). The key to Levelt’s theory is the map
\[
\varphi : \mathcal{O}[S] \to \mathbb{Z} \cup \{\infty\}
\]
defined by
\[
\varphi(y) := \max \left\{ m \in \mathbb{Z} \mid \lim_{\lambda \to 0} \frac{y(\lambda)}{|\lambda|^\ell} = 0 \text{ for all } \ell < m \right\}
\]
for all \( y \in \mathcal{O}[S] \setminus \{0\} \) and \( \varphi(0) = \infty \). Note that according to Proposition 2.1 the space of solutions \( X \subset \mathcal{O}[S] \).
Therefore, if we knew that the lemma holds for constant polynomials, then we would get \( a_{i,m} = 0 \) for all \( i \) – contradiction with the definition of \( m \).

Let us assume that \( a_i \in \mathbb{C} \) are constants. Using induction on \( m \) it is easy to prove that if \( \lambda_m < \cdots < \lambda_1 < \epsilon \) is any sequence of real numbers, then

\[
\lim_{x \to +\infty} \left( \sum_{i=1}^{n} \frac{a_i e^{\sqrt{-1} \theta_i x}}{\sqrt{-1} \theta_i + \lambda_1} \cdots (\sqrt{-1} \theta_i + \lambda_m) \right) e^{\lambda x} = 0, \quad \forall \lambda < \lambda_m.
\]

Indeed, the starting point of the induction is \( m = 0 \) and the statement is true by definition. Suppose the statement is true for \( m \) and that \( \lambda_{m+1} < \lambda_m \). Let us pick \( \lambda' \) in the open interval \((\lambda_{m+1}, \lambda_m)\). Using the inductive assumption we get

\[
\left| \sum_{i=1}^{n} \frac{a_i e^{\sqrt{-1} \theta_i + \lambda_{m+1}} y}{\sqrt{-1} \theta_i + \lambda_1} \cdots (\sqrt{-1} \theta_i + \lambda_m) \right| \leq C' e^{(\lambda_{m+1} - \lambda') y}, \quad \forall y \geq 0
\]

for some constant \( C' \) depending on the choice of \( \lambda' \). Integrating the function inside the absolute value on the LHS for \( y \) from 0 to \( x \) and using the above inequality to estimate the absolute value of the integral, we get the following inequality

\[
\left| \sum_{i=1}^{n} \frac{a_i (e^{\sqrt{-1} \theta_i + \lambda_{m+1}} x - 1)}{(\sqrt{-1} \theta_i + \lambda_1) \cdots (\sqrt{-1} \theta_i + \lambda_m)(\sqrt{-1} \theta_i + \lambda_{m+1})} \right| \leq C' e^{(\lambda_{m+1} - \lambda') x - 1} = \frac{e^{(\lambda_{m+1} - \lambda') x - 1}}{\lambda_{m+1} - \lambda'}.
\]

If \( \lambda < \lambda_{m+1} \) is any given number, then we multiply the above inequality by \( e^{(\lambda_{m+1} - \lambda') x} \), and let \( x \to +\infty \).

To complete the proof of the lemma we proceed as follows. Let us choose a sequence of \( n \) numbers \( 0 < \lambda_n < \cdots < \lambda_1 < \epsilon \) and define the matrix \( C \) with entries

\[
C_{im} := \frac{1}{(\sqrt{-1} \theta_i + \lambda_1) \cdots (\sqrt{-1} \theta_i + \lambda_m)}, \quad 1 \leq i, m \leq n.
\]

Note that for \( \lambda_1 = \cdots = \lambda_n \) the determinant of \( C \) turns into a Vandermonde determinant, which is not 0 according to the assumption \( \theta_i \neq \theta_j \) for \( i \neq j \). Therefore choosing \( \lambda_1 \) sufficiently close to \( \lambda_n \) we may guarantee that \( C \) is invertible. On the other hand if we define

\[
g_m(x) = \sum_{i=1}^{n} a_i e^{\sqrt{-1} \theta_i x} C_{im}, \quad 1 \leq m \leq n,
\]

then according to the above fact \( \lim_{x \to +\infty} g_m(x) = 0 \). However, since \( C \) is invertible, we can solve the above equations and express each \( a_i e^{\sqrt{-1} \theta_i x} \) as a linear combination of \( g_m(x) \) with constant coefficients. Therefore \( \lim_{x \to +\infty} a_i e^{\sqrt{-1} \theta_i x} = 0 \). This however is possible only if \( a_i = 0 \).

**Proposition 2.7.** If \( y \in X \), then \( \varphi(\sigma y) = \varphi(y) \).

**Proof.** Recalling Proposition 2.4 we write the solution as

\[
y(\lambda) = \sum_{i=1}^{n} y_i(\lambda) \lambda^p, \quad y_i(\lambda) = \sum_{k=0}^{p-1} y_{i,k}(\lambda) (\log \lambda)^k
\]
where \( y_{i,k}(\lambda) \) are analytic at \( \lambda = 0 \). We may further assume that \( \text{Re}(\rho_1) \leq \cdots \leq \text{Re}(\rho_n) \). Let us write the solution as
\[
y(\lambda) = \lambda^{\rho_1} \left( f(\lambda) + \sum_j y_j(\lambda) \lambda^{\rho_j - \rho_1} \right),
\]
where the sum is over all \( j \), s.t., that \( \text{Re}(\rho_j) > \text{Re}(\rho_1) \) and
\[
f(\lambda) = \sum_i y_i(\lambda) \lambda^{\rho_i - \rho_1},
\]
where the sum is over all \( i \), s.t., \( \text{Re}(\rho_i) = \text{Re}(\rho_1) \). Let us assume that \( y_{1,k}(0) \neq 0 \) for at least one \( k \). Otherwise, we can replace \( \rho_1 \) with an exponent with a larger real part. Note that \( \varphi(y) = \lceil \text{Re}(\rho_1) \rceil \), where \( \lceil x \rceil \) is the largest integer that does not exceed \( x \). Indeed, by definition we have that if \( \ell < \lceil \text{Re}(\rho_1) \rceil \), then \( \lim y(\lambda)/|\lambda|^{\ell} = 0 \), so \( \varphi(y) \geq \lceil \text{Re}(\rho_1) \rceil \). If the inequality is strict then we can find \( \epsilon > 0 \), such that
\[
\text{Re}(\rho_1) - \epsilon < \text{Re}(\rho_1) < \text{Re}(\rho_1) + 2\epsilon < \varphi(y)
\]
and \( \epsilon < \text{Re}(\rho_j) - \text{Re}(\rho_1) \) for all \( j \) for which \( \text{Re}(\rho_j) \neq \text{Re}(\rho_1) \). We have
\[
\frac{y(\lambda)}{|\lambda|^{|\text{Re}(\rho_1)|+\epsilon+\ell}} = \frac{\lambda^{\rho_1}}{|\lambda|^{|\text{Re}(\rho_1)|+\epsilon}} \left( \frac{f(\lambda)}{|\lambda|^\ell} + \sum_j y_j(\lambda) \frac{\lambda^{\rho_j - \rho_1}}{|\lambda|^\ell} \right).
\]
If \( \ell < \epsilon \), then the LHS has limit 0 as \( \lambda \to 0 \), while the limit of the first factor on the RHS is \( \infty \) and the limit of the sum over \( j \) is 0. Therefore, we must have
\[
\lim_{\lambda \to 0} \frac{f(\lambda)}{|\lambda|^\ell} = 0, \quad \forall \ell < \epsilon.
\]
If we put \( \lambda = e^{-x} \), \( x \in \mathbb{R}_{>0} \) and let \( x \to +\infty \) we get that
\[
\sum_i \sum_{k=0}^{p-1} y_{i,k}(0) x^k e^{\sqrt{-1} \theta_i x} \sqrt{-1} \theta_i = \rho_i - \rho_1
\]
satisfies the condition of Lemma \([2.6]\) so it must be 0, which contradicts the choice of \( \rho_1 \).

Note also that we have
\[
\sigma y(\lambda) = \sum_{i=1}^n \sum_{k=0}^{p-1} y_{i,k}(\lambda) e^{2\pi \sqrt{-1} \rho_i \lambda} (\log \lambda + 2\pi \sqrt{-1})^k.
\]
Therefore, choosing \( k \) to be the largest integer such that \( y_{1,k}(0) \neq 0 \) we get
\[
\varphi(y) = \varphi(y_{1,k}(\lambda) \lambda^{\rho_1} (\log \lambda)^k) = \varphi(y_{1,k}(\lambda) e^{2\pi \sqrt{-1} \rho_1 \lambda} (\log \lambda + 2\pi \sqrt{-1})^k) \leq \varphi(\sigma y),
\]
where in the last equality we used Lemma \([2.5]\) Part a). Similarly \( \varphi(y) \leq \varphi(\sigma^{-1} y) \) for all \( y \in X \). Finally we get
\[
\varphi(y) \leq \varphi(\sigma y) \leq \varphi(\sigma^{-1}(\sigma y)) = \varphi(y). \quad \square
\]
2.4. Weak Levelt solutions. The eigenvalues of $\sigma$ can be written uniquely as

$$e^{2\pi \sqrt{-1}\rho_i}, \quad 0 \leq \text{Re}(\rho_i) < 1, \quad 1 \leq i \leq s.$$ 

Let

$$X = X_1 \oplus \cdots \oplus X_s, \quad X_i := \{ y \in X : (\sigma - e^{2\pi \sqrt{-1}\rho_i})^n y = 0 \text{ for all } n \gg 0 \}$$

be the decomposition of $X$ into generalized eigensubspaces.

Using Lemma 2.5 we get that $\varphi(X)$ is a finite set. Let us define the set

$$\{ \infty, k_{i1}^1, \ldots, k_{im_i}^i \} := \varphi(X_i), \quad 1 \leq i \leq s,$$

where in addition we assume that $k_{i1}^1 > \cdots > k_{im_i}^i$. Put

$$X_i^\ell = \{ y \in X \mid \varphi(y) \geq k_{i\ell}^i \}, \quad 1 \leq i \leq s, \quad 1 \leq \ell \leq m_i.$$

According to Lemma 2.5 the sets $X_i^\ell$ are vector subspaces of $X_i$, so we have a strictly increasing filtration (in particular we see that $\varphi$ could take only finitely many values on $X_i$)

$$X_1^1 \subset X_2^2 \subset \cdots \subset X_i^{m_i} = X_i$$

Using Proposition 2.7 we get that the above filtration is $\sigma$-invariant.

A weak Levelt solution $Y(\lambda)$ is by definition a fundamental matrix whose columns are split into $s$ groups

$$Y(\lambda) = [Y_1(\lambda) \cdots Y_s(\lambda)],$$

where the columns in $Y_i(\lambda)$ represent a basis of $X_i$ with the following property. We can split $Y_i(\lambda)$ into $m_i$ groups

$$Y_i(\lambda) = [Y_{i,1}(\lambda) \cdots Y_{i,m_i}(\lambda)]$$

such that

(i) The columns in $Y_{i,\ell}(\lambda)$ represent a basis of the quotient subspace $X_i^\ell/X_i^{\ell-1}$.

(ii) The matrix of the linear operator in $X_i^\ell/X_i^{\ell-1}$ induced by $\sigma$ in the basis represented by the columns of $Y_{i,\ell}(\lambda)$ is upper-triangular.

Let $G$ be the matrix of $\sigma$ with respect to the basis of $X$ given by the columns of a weak Levelt solution $Y(\lambda)$. Note that the matrix $G$ is block-diagonal

$$G = \text{diag}(G_1, \ldots, G_s)$$

where each block is a square matrix of size $\dim_{\mathbb{C}}(X_i)$. Each block $G_i$ has a natural block-matrix form corresponding to the filtration $X_1^1 \subset \cdots \subset X_i^{m_i}$

$$G_i = \begin{bmatrix}
G_{i1}^{11} & G_{i1}^{12} & \cdots & G_{i1}^{1m_i} \\
0 & G_{i2}^{22} & \cdots & G_{i2}^{2m_i} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & G_{im_i}^{m_i m_i}
\end{bmatrix},$$

where the size of the block $G_{iab}^{ab}$ is $\dim_{\mathbb{C}}(X_a^a/X_{a-1}^a) \times \dim_{\mathbb{C}}(X_b^b/X_{b-1}^b)$. The definition of a weak Levelt solution implies that $G_{iab}^{ab} = 0$ for $a > b$ (: the filtration
is \( \sigma \)-invariant) and that the block \( G_{i\ell} \) has the form of an upper-triangular matrix with all diagonal entries being equal to \( e^{2\pi \sqrt{-1} \rho_i} \) (i.e. the matrix of the linear map in \( X_i^\ell/X_i^{\ell-1} \) induced by \( \sigma \) is upper-triangular).

2.5. Levelt’s theorem. Let \( Y(\lambda) \) be a weak Levelt solution. Let us write the monodromy matrix \( G = e^{2\pi \sqrt{-1} T E} \), where \( E \) has the same block-matrix structure as \( G \). Namely,

\[
E = \text{diag}(E_1, \ldots, E_s)
\]
is block-diagonal and each block \( E_i \) has the form

\[
E_i = \begin{bmatrix}
E_{i1}^{11} & E_{i1}^{12} & \cdots & E_{i1}^{1m_i} \\
0 & E_{i2}^{22} & \cdots & E_{i2}^{2m_i} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E_{im_i}^{m_i}
\end{bmatrix},
\]

where \( E_{i\ell} = \rho_i I_{i\ell} + N_{i\ell} \) is upper-triangular matrix whose diagonal entries are all equal to \( \rho_i \). We have denoted by \( I_{i\ell} \) the identity matrix of size \( \text{dim}_C(X_i^\ell/X_i^{\ell-1}) \) while by \( N_{i\ell} \) we have denoted the strictly upper-triangular part of \( E_{i\ell} \).

Let us define also the matrix \( K \) with the same block-diagonal structure as \( G \) and \( E \), i.e.,

\[
K = \text{diag}(K_1, \ldots, K_s)
\]
where the block \( K_i \) is given by the diagonal matrix

\[
K_i = \text{diag}(k_{i1}^{11} I_{1i}, \ldots, k_{im_i}^{m_i} I_{mi_i}).
\]
The main result of this lecture can be stated as follows.

**Theorem 2.8** (Levelt). Suppose that \( Y(\lambda) \) is a weak Levelt solution and that \( K \) and \( E \) are the matrices defined as above. Then

\[
Y(\lambda) = U(\lambda) \lambda^K \lambda^E,
\]
where \( U(\lambda) \) is holomorphic and invertible at \( \lambda = 0 \).

**Proof.** Our argument follows [2]. Note that the analytic continuation of \( Y(\lambda) \) and \( \lambda^E \) around \( \lambda = 0 \) are respectively \( Y(\lambda) G \) and \( \lambda^E G \). Therefore, the holomorphic branch of

\[
U(\lambda) := Y(\lambda) \lambda^{-E} \lambda^{-K}
\]
defined in the sector \( S \subset D - \{0\} \) extends analytically to the entire punctured disc \( D - \{0\} \). Using Proposition 2.1 we get that \( U(\lambda) \) has at most a finite order pole at \( \lambda = 0 \).

Let us prove that \( U(\lambda) \) is holomorphic at \( \lambda = 0 \). Let us denote by

\[
r = \max_{1 \leq j \leq s} \text{Re}(\rho_j)
\]

Since \( r < 1 \) we can find a real number \( \epsilon > 0 \), such that
We claim that \( \lim_{\lambda \to 0} U(\lambda) \lambda^{r + 2\epsilon} = 0 \). This clearly implies that \( U(\lambda) \) does not have a pole at \( \lambda = 0 \). To prove that the limit is 0 we write

\[
U(\lambda) \lambda^{r + 2\epsilon} = Y(\lambda) \lambda^{-K + \epsilon} \exp \left( (r - \lambda^K E \lambda^{-K}) \log \lambda \right) \lambda^\epsilon.
\]

Note that the first two factors on the RHS give a matrix obtained from \( Y(\lambda) \) by multiplying each column in \( Y \) by \( \lambda^{-K + \epsilon} \). Since the Levelt evaluation of every column in \( Y_{i,\ell} \) is at least \( k_i^\ell \) we get that the limit of \( Y(\lambda) \lambda^{-K + \epsilon} \) is 0. Since \( K \) and \( E \) have the same block-diagonal structure we get that 3rd and the 4th factor give a matrix which is also block-diagonal and the \( i \)-th block is

\[
\lambda^\epsilon + r - \rho_i \lambda^{K_i N_i \lambda^{-K_i}} \log \lambda,
\]

where \( N_i \) is strictly upper triangular. Since \( K_i \) is diagonal with decreasing diagonal entries the matrix \( \lambda^{K_i N_i \lambda^{-K_i}} \) is holomorphic at \( \lambda = 0 \). Therefore the limit of (3) is 0.

It remains only to prove that \( U(0) \) is invertible. Substituting \( Y(\lambda) = U(\lambda) \lambda^K \lambda E \) in the differential equation we get

\[
\lambda U'(\lambda) + U(\lambda) L(\lambda) = B_0(\lambda) U(\lambda),
\]

where \( L(\lambda) = K + \lambda^K E \lambda^{-K} \). As we discussed above the matrix \( \lambda^K E \lambda^{-K} \) is holomorphic at \( \lambda = 0 \). In fact \( L(0) \) is block-diagonal and the \( i \)-th block is

\[
\begin{pmatrix}
(k_i^1 + \rho_i) I_i^1 + N_i^{11} & 0 & \cdots & 0 \\
0 & (k_i^2 + \rho_i) I_i^2 + N_i^{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & (k_i^{m_i} + \rho_i) I_i^{m_i} + N_i^{m_im_i}
\end{pmatrix}.
\]

Since \( U(0)L(0) = B_0(0)U(0) \), we get that \( L(0) \) is a linear operator in \( \text{Ker}(U(0)) \). If we assume that \( U(0) \) is not invertible, then \( L(0) \) has a non-zero eigenvector \( c \in \text{Ker}(U(0)) \). Let us denote by \( y_c(\lambda) = Y(\lambda)c \).

Let us split the vector-column \( c \) in the following way

\[
c = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}, \quad c_i = \begin{bmatrix} c_{i,1} \\ \vdots \\ c_{i,m_i} \end{bmatrix},
\]

where the length of the subcolumn \( c_{i,\ell} \) is the same as the dimension of \( X_i^\ell/X_i^{\ell-1} \). Since \( L(0) \) is block-diagonal and upper triangular, we get that there exists a unique pair \((i, \ell)\) for which \( c_{i,\ell} \neq 0 \) and that the eigenvalue of \( c \) is \( \rho_i + k_i^\ell \). Note that

\[
y_c(\lambda) = Y_{i,\ell} c_{i,\ell}
\]

is a linear combination of elements in \( X_i^\ell \) that project to a basis in \( X_i^\ell/X_i^{\ell-1} \). Therefore \( \varphi(y_c) = k_i^\ell \).
On the other hand, let us denote by $R$ the diagonal part of $E$ and write $E = R + N$. Therefore
\[ y_c(\lambda) = U(\lambda) \lambda^K \lambda^N (- K_{c\lambda}) \lambda^\rho + k^\ell, \]
where we used that $K + R$ is the diagonal part of $L(0)$. Furthermore, using that $\lambda^K N \lambda^{\bar{K}} = L(\lambda) - K - R$ is a holomorphic nilpotent matrix we get
\[ U(\lambda) \lambda^K \lambda^N (- K_{c\lambda}) \lambda^\rho + k^\ell, \]
where we used that $K + R$ is the diagonal part of $L(0)$. Furthermore, using that $\lambda^K N \lambda^{\bar{K}} = L(\lambda) - K - R$ is a holomorphic nilpotent matrix we get
\[ U(\lambda) \lambda^K \lambda^N (- K_{c\lambda}) \lambda^\rho + k^\ell \]
where $m$ is an integer such that $N^m = 0$. However $(L(0) - K - R)c = U(0)c = 0$, so we get that $\varphi(y_c) \geq 1 + k^\ell - \text{contradiction}.
\]

3. Vector bundles on $\mathbb{P}^1$

Let us assume that $E \to \mathbb{P}^1 \times \tilde{\Pi}$ is a holomorphic vector bundle of rank $p$, where
\[ \tilde{\Pi} = \{ u = (u_1, \ldots, u_N) \in \mathbb{C}^N \mid |u_i - u_i^\circ| < \tilde{\delta}_i, \ 1 \leq i \leq N \}, \]
is the polydisc with center $u^\circ := (u_1^\circ, \ldots, u_N^\circ)$ and polyradius $\tilde{\delta} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_N)$. The main goal of this lecture is to prove the existence of Birkhoff factorization for the transition matrix of $E$. We follow Bolibruch [2].

3.1. Transition function. We will be interested in transition functions of $E$ of the following type. Let us fix a point $b \in \mathbb{C} \subset \mathbb{P}^1$, real numbers $0 < r < R$, and a polydisc
\[ \Pi = \{ u \in \mathbb{C}^N \mid |u_i - u_i^\circ| < \delta_i, \ 1 \leq i \leq N \}, \]
where $0 < \delta_i < \tilde{\delta}_i$ for all $i$. The discs
\[ D_b = \{ \lambda \in \mathbb{C} \mid |\lambda - b| < R \}, \quad D_\infty = \{ \lambda \in \mathbb{P}^1 \mid |\lambda - b| > r \} \]
give an open cover of $\mathbb{P}^1$. The open subsets $D_\nu \times \Pi, \nu = b, \infty$, are Stein and contractible, so according to the Grauert–Oka principle $E|_{D_\nu \times \Pi}$ is trivial. Let us define raw vectors
\[ e_\nu = (e_{\nu, 1}, \ldots, e_{\nu, p}), \quad e_{\nu, i} \in \Gamma(D_\nu \times \Pi, E), \]
such that $\{e_{\nu, i}\}_{i=1}^p$ is a trivializing frame for $E|_{D_\nu \times \Pi}$. On the intersection the two frames are related by a holomorphic invertible matrix
\[ e_\infty(\lambda, u) = e_b(\lambda, u) M(\lambda, u), \quad (\lambda, u) \in D_\infty \times \Pi, \]
where $D_\infty = D_b \cap D_\infty$ and
\[ M : D_\infty \times \Pi \to \text{GL}(\mathbb{C}^p) \]
is a holomorphic map. Choosing different trivialization frames \( \tilde{e}_b = e_b U \) and \( \tilde{e}_\infty = e_\infty W \), where

\[
U : D_b \to \text{GL}(\mathbb{C}^p) \quad \text{and} \quad W : D_\infty \to \text{GL}(\mathbb{C}^p)
\]

are holomorphic maps, yields a new transition matrix \( \tilde{e}_\infty = \tilde{e}_b \tilde{M} \), where

\[
\tilde{M}(\lambda, u) = U(\lambda, u)^{-1} M(\lambda, u) W(\lambda, u), \quad (\lambda, u) \in D_{b\infty} \times \Pi.
\]

Our main goal can be stated as follows. We would like to prove that after decreasing \( \Pi \) if necessary and removing an analytic hypersurface from \( \Pi \) we can always arrange that

\[
\tilde{M} = \text{diag}((\lambda - b)^{k_1}, \ldots, (\lambda - b)^{k_p}),
\]

where \( k_1 \geq \cdots \geq k_p \) is a decreasing sequence of integers.

3.2. GAGA reduction.

**Definition 3.1.** We say that a map \( M : D_{b\infty} \times \Pi \to \mathfrak{gl}(\mathbb{C}^p) \) is \( \Pi \)-rational if the entries of \( M(\lambda, u) \) are quotients of polynomials in \( O(\Pi)[\lambda] \), where \( O(\Pi) \) is the ring of holomorphic functions on \( \Pi \).

We would like to reduce the general analytic problem to an algebraic one. More precisely we would like to prove the following proposition

**Proposition 3.2.** Decreasing the size of \( \Pi \) if necessary, we can find a transition matrix

\[
M : D_{b\infty} \times \Pi \to \text{GL}(\mathbb{C}^p)
\]

such that

(i) \( M \) is \( \Pi \)-rational.

(ii) The zeroes of \( \det(M(\lambda, u)) \) and the poles of \( M(\lambda, u) \) for \( (\lambda, u) \in \mathbb{P}^1 \times \Pi \) are independent of \( u \).

Let us introduce the following notation. If \( \Pi \) is an open polydisc, then we denote by \( \overline{\Pi} \) the corresponding closed polydisc. If \( X \subset \mathbb{P}^1 \times \overline{\Pi} \) is an open subset, then we define

\[
H(X) := \{ \phi : \overline{X} \to \mathfrak{gl}(\mathbb{C}^p) \mid \phi \text{ is continuous in } \overline{X} \text{ and holomorphic in } X \},
\]

and

\[
H^0(X) := \{ \phi \in H(X) \mid \phi(x) \text{ is invertible for all } x \in X \}.
\]

Recall that \( H(X) \) is a Banach algebra with norm

\[
\|A\| = \sup_{(\lambda, u) \in \overline{X}} |A(\lambda, u)|,
\]

where \( | \cdot : \mathfrak{gl}(\mathbb{C}^p) \to \mathbb{R}_{\geq 0} \) is the matrix norm

\[
|A| := \sup_{v \in \mathbb{C}^p \setminus \{0\}} |Av|/|v|,
\]

where for \( w \in \mathbb{C}^p \) we denote by \( |w| = (|w_1|^2 + \cdots + |w_p|^2)^{1/2} \) the Euclidean norm of \( w \)
**Lemma 3.3.** There exists an $\epsilon > 0$, depending on $r$, and $R$ such that if $B \in H(D_{b\infty} \times \Pi)$ has norm $\|B\| < \epsilon$, then $1 + B \in H^0(D_{b\infty} \times \Pi)$ and we have a factorization

$$1 + B = UW, \quad U \in H^0(D_b \times \Pi), \quad W \in H^0(D_{\infty} \times \Pi).$$

**Proof.** The Laurent series expansion gives a decomposition

$$H(D_{b\infty} \times \Pi) = H(D_b \times \Pi) \bigoplus H(D_{\infty} \times \Pi)(\lambda - b)^{-1}, \quad B = B_+ + B_-.$$  

Let $pr_\pm$ be the corresponding projection maps $B \mapsto B_\pm$. We have

$$pr_+(B)(\lambda, u) = \frac{1}{2\pi \sqrt{-1}} \int_{|\zeta - b|=R} \frac{B(\zeta, u)d\zeta}{\zeta - \lambda}$$

and

$$pr_-(B)(\lambda, u) = -\frac{1}{2\pi \sqrt{-1}} \int_{|\zeta - b|=r} \frac{B(\zeta, u)d\zeta}{\zeta - \lambda}.$$  

It is easy to check that $\|pr_\pm(B)\| \leq C \|B\|$ for some constant $C$ that depends on $r$ and $R$. Using these estimates and choosing $\epsilon$ sufficiently small ($\epsilon < 1/C$ works) we can prove that the series

$$w = \sum_{n=1}^{\infty} (-pr_0 \circ B)^n 1 = -B_- + (BB_-)_- - (BB_-)_- + \cdots$$

is convergent. Note that $(Bw)_- + w + B_- = 0$ therefore, $(1 + B)(1 + w) = 1 + u$, with $u = B_+ + (Bw)_+$.

Decreasing $\epsilon$ if necessary ($\epsilon < 1/(4C)$ works) we can arrange that $1 + u$ and $1 + w$ are invertible, so the lemma follows with $U = 1 + u$ and $W = (1 + w)^{-1}$. \(\square\)

**Proof of Proposition 3.2.** Let us fix positive numbers $0 < r' < r'' < r < R < R'' < R'$ and polydiscs $\Pi \subset \Pi'' \subset \Pi' \subset \Pi$ with center at $u^0$.

Here $\Pi'$ is chosen arbitrary, while the sizes of $\Pi''$ and $\Pi$ will be specified later on.

Let us pick an arbitrary transition matrix

$$M' : D'_{b\infty} \times \Pi' \to GL(\mathbb{C}^p), \quad D'_{b\infty} := \{r' < |\lambda - b| < R'\}.$$  

Note that $M' \in H^0(D'_{b\infty} \times \Pi''\prime)$ where $D''_{b\infty} = \{r'' < |\lambda - b| < R''\}$. The Laurent series expansion of $M'(\lambda, u_0)^{-1}$ at $\lambda = b$ is uniformly convergent for $r'' \leq |\lambda - b| \leq R''$, while $M'(\lambda, u)^{-1}$ is uniformly continuous for $(\lambda, u) \in \overline{D''_{b\infty} \times \Pi''\prime}$. Therefore by truncating the Laurent series expansion of $M'(\lambda, u_0)^{-1}$ appropriately and choosing $\Pi''$ sufficiently small, we can find a Laurent polynomial $P \in \mathfrak{g}((\mathbb{C}^p)[[\lambda - b]^{\pm 1}]]$, s.t., $|M'P - 1|_{r''', R''', \Pi''} < \epsilon$, where the norm is in the space $H(D''_{b\infty} \times \Pi'\prime)$. Recalling Lemma 3.3 we find $U_1 \in H^0(D'_b \times \Pi''\prime)$ and $W_1 \in H^0(D''_{\infty} \times \Pi'\prime)$, s.t., $M'P = U_1W_1$, i.e.,

$$M' = U_1W_1P^{-1}.$$
Similarly, we can find $Q \in \mathfrak{gl}(\mathbb{C}^p)\{[\lambda - b]^{\pm 1}\}$, s.t., $QW_1P^{-1} = U_2W_2$ with $U_2 \in H^0(D_b \times \Pi)$ and $W_2 \in H^0(D_\infty \times \Pi)$ for some sufficiently small polydisc $\Pi \subset \Pi'$.

Therefore we get

$$M' = U_1Q^{-1}U_2W_2.$$ 

We claim that the matrix $M := Q^{-1}U_2$ has the required properties. Condition (ii) is easy to verify. Let us prove that $U_2$ is $\Pi$-rational. We have

$$U_2 = QW_1P^{-1}W_2^{-1}.$$ 

Let $g(\lambda) \in \mathbb{C}[\lambda]$ be a common denominator for the entries of $Q$ and $P^{-1}$. The matrix $g^2U_2 = (gQ)W_1(gP^{-1})W_2^{-1}$ is holomorphic for all $\lambda \in \mathbb{C}$, because $U_2$ is holomorphic in $D_b \times \Pi$ while $W_1$ and $W_2$ are holomorphic in $D_\infty \times \Pi$. Moreover, since $W_1$ and $W_2$ are holomorphic at $\lambda = \infty$, the matrix $g^2U_2$ has at most a finite order pole at $\lambda = \infty$, so it must be polynomial in $\lambda$. \hfill $\Box$

### 3.3. Existence and uniqueness of Birkhoff factorization.

Let $\Pi$ be a polydisc with center $u^0$ and $\Theta_0 \subset \Pi$ be an analytic hypersurface with finitely many irreducible components. Since $\Pi$ is Stein and contractible, there exists a holomorphic function $f_0 \in \mathcal{O}(\Pi)$ such that $\Theta_0$ is the zero locus of $f_0$. Suppose that we have a transition matrix

$$M : D_{b\infty} \times (\Pi - \Theta_0) \to \text{GL}(\mathbb{C}^p),$$

such that

(i) $M$ is $\Pi$-rational.

(ii) The zeroes of $\det(M(\lambda, u))$ and the poles of $M(\lambda, u)$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ are independent of $u$.

According to the previous section such a transition matrix exists provided we choose $\Pi$ sufficiently small and $\Theta_0 = \emptyset$.

Note that condition (i) implies that the points $(\lambda, u) \in D_b \times \Pi$ for which $M(\lambda, u)$ is not holomorphic form an analytic hypersurface $Z_\infty(M)$. More precisely $Z_\infty(M)$ is the union of all irreducible hypersurfaces $V \subset D_b \times \Pi$ such that there exists an entry $m = g/f$ ($g, f \in \mathcal{O}(D_b \times \Pi)$) of $M$ for which $\text{ord}_V(f) > \text{ord}_V(g)$, where $\text{ord}_V(h)$ denotes the order of vanishing of the holomorphic function $h \in \mathcal{O}(D_b \times \Pi)$ along $V$.

**Lemma 3.4.** Every irreducible component of $Z_\infty(M)$ has either the form $\{b'\} \times \Pi$ for some $b' \in D_b - D_{b\infty}$ or $D_b \times \Theta'_0$, where $\Theta'_0$ is an irreducible component of $\Theta_0$.

**Proof.** Let $V$ be an irreducible component of $Z_\infty(M)$. Condition (ii) implies that

$$V \cap D_b \times (\Pi - \Theta_0) = \bigcup_{i=1}^{s}\{b_i\} \times (\Pi - \Theta_0),$$

where $s$ is a finite number and $b_i \in D_b$.

Since $\Theta_0 = \emptyset$, all irreducible components of $Z_\infty(M)$ have a finite intersection with $D_b \times \Pi$. Therefore, $V \cap D_b \times \Pi$ consists of finitely many irreducible components $V_i$, each of which is either of the form $\{b'_i\} \times \Pi$ or $D_b \times \Theta'_0$, where $\Theta'_0$ is an irreducible component of $\Theta_0$. This proves the lemma.

**Corollary 3.5.** If $M$ is $\Pi$-rational, then $M(\lambda, u)$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$. The zeroes of $\det(M(\lambda, u))$ and the poles of $M(\lambda, u)$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ are independent of $u$. Thus, $M$ is $\Pi$-rational and $M(\lambda, u)$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$. The zeroes of $\det(M(\lambda, u))$ and the poles of $M(\lambda, u)$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ are independent of $u$. Thus, $M$ is $\Pi$-rational.

**Proof.** By Lemma 3.4, every irreducible component of $Z_\infty(M)$ has either the form $\{b'\} \times \Pi$ or $D_b \times \Theta'_0$, where $\Theta'_0$ is an irreducible component of $\Theta_0$. Therefore, $M(\lambda, u)$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$. The zeroes of $\det(M(\lambda, u))$ and the poles of $M(\lambda, u)$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ are independent of $u$. Thus, $M$ is $\Pi$-rational.

**Theorem 3.6.** Let $M : D_{b\infty} \times (\Pi - \Theta_0) \to \text{GL}(\mathbb{C}^p)$ be a transition matrix. Then $M$ is $\Pi$-rational if and only if $\det(M(\lambda, u))$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$.

**Proof.** By Lemma 3.4, $M$ is $\Pi$-rational if and only if $\det(M(\lambda, u))$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$. This proves the theorem.
for some $b_i \in D_b$. Since $M(\lambda, u)$ is holomorphic and invertible for $(\lambda, u) \in D_{b_0} \times (\Pi - \Theta_0)$ we have $b_i \in D_b - D_{b_0}$ and
\[ V \subset \left( \bigcup_{i=1}^{s} \{b_i\} \times \Pi \right) \bigcup D_b \times \Theta_0. \]

The RHS of the above inclusion relation is an analytic hypersurface, so $V$ must be one of its irreducible components. \hfill \Box

**Proposition 3.5.** a) There exists an analytic hypersurface $\Theta \subset \Pi$ that contains $\Theta_0$ and has finitely many irreducible components, such that
\[ M(\lambda, u) = U(\lambda, u)(\lambda - b)^K W(\lambda, u), \]
where

(i) $U$ and $W$ are $\Pi$-rational.

(ii) $U(\lambda, u)$ (resp. $W(\lambda, u)$) is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta)$ (resp. $D_{\infty} \times (\Pi - \Theta)$).

(iii) $K = \text{diag}(k_1, \ldots, k_p)$, where $k_1 \geq \cdots \geq k_p$ are integers.

b) If
\[ M(\lambda, u) = U_i(\lambda, u)(\lambda - b)^{K(i)} W_i(\lambda, u), \quad i = 1, 2, \]
are two factorizations satisfying the conditions in part a), then $K^{(1)} = K^{(2)}$.

**Proof.** a) We split the proof into two cases.

Case 1: If $\det(M(\lambda, u)) \neq 0$ for all $(\lambda, u) \in (D_b - \{b\}) \times (\Pi - \Theta_0)$. We may assume that $M(\lambda, u) = L(\lambda, u)(\lambda - b)^K$, where $L(\lambda, u)$ is holomorphic for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ and $K = \text{Diag}(k_1, \ldots, k_p)$, where $k_1 \geq k_2 \geq \cdots \geq k_p$ are integers. This can be always achieved by first multiplying $M$ from the right by matrices of the type $(\lambda - b_0)^{K_0}(\lambda - b)^{-K_0}$, so that we clear all the poles of $M(\lambda, u)$ from $D_b \times (\Pi - \Theta_0)$, and finally multiply by a constant permutation matrix to arrange that the entries of $K$ are decreasing. Moreover, according to Lemma 3.4 there exists an integer $n$, such that $L(\lambda, u)f_0(u)^n$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$.

The Taylor’s series expansion of $L$ has the form
\[ L(\lambda, u) = L_0(u) + L_1(u)(\lambda - b) + L_2(u)(\lambda - b)^2 + \cdots. \]
Let us denote by $m_i(u)$, $1 \leq i \leq p$, the columns of the matrix $L_0(u)$. We may assume that $m_1 \neq 0$, otherwise we can factor out $(\lambda - b)$ from the first column of $L(\lambda, u)$ and increase $k_1$ by one. We can also assume that $\det(L_0(u)) = 0$, otherwise the matrix $L(\lambda, u)$ is invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta)$, where $\Theta \subset \Pi$ is the union of $\Theta_0$ and the zero locus of $\det(L_0(u))$, and this is already a Birkhoff factorization, so there is nothing to prove.

Let us denote by $i$ the maximal index, s.t., some $i \times i$ minor of $L_0(u)$ contained in the first $i$-columns is not identically 0 for $u \in \Pi - \Theta_0$. If there are several such minors, then we choose one of them, write it in the form $g(u)/f_0(u)^n$ for some $g \in \mathcal{O}(\Pi)$ and denote by $\Theta \subset \Pi$ the analytic hypersurface defined by the zero locus
of $g(u) f_0(u)$. There are functions $s_1(u), \ldots, s_i(u)$, holomorphic for $u \in \Pi - \Theta$ and meromorphic along $\Theta$, s.t.,

$$m_{i+1}(u) = s_1(u)m_1(u) + \cdots + s_i(u)m_i(u).$$

Let

$$W(\lambda, u) = 1 - \sum_{a=1}^{i} s_a(u)(\lambda - b)^{-k_a+k_{i+1}} E_{a,i+1},$$

where $E_{a,i+1}$ is the matrix with only one non-zero entry, which is equal to 1 and it is in row $a$ and column $i+1$. Note that $k_a \geq k_{i+1}$, so $W(\lambda, u)$ is holomorphic and invertible for $(\lambda, u) \in D_\infty \times (\Pi - \Theta)$ and meromorphic along $D_\infty \times \Theta$. It is easy to check that $M(\lambda, u)W(\lambda, u) = \tilde{L}(\lambda, u)(\lambda - b)^{\tilde{K}}$, where $\tilde{K} = \text{Diag}(\tilde{k}_1, \ldots, \tilde{k}_p)$ satisfies $\tilde{k}_j = k_j$ for $j \neq i$ and $\tilde{k}_{i+1} > k_{i+1}$. Multiplying if necessary $W$ by a constant permutation matrix from the right we can arrange that $\tilde{k}_1 \geq \cdots \geq \tilde{k}_p$. Note that

$$\det(L(\lambda, u)) = \det(\tilde{L}(\lambda, u))(\lambda - b)^{\sum_{i=1}^{p}(\tilde{k}_i - k_i)},$$

so the order of vanishing of $\det(L(\lambda, u))$ at $\lambda = b$ decreases strictly. Repeating the above procedure finitely many times we will eventually get a matrix $L(\lambda, u)$, s.t., $\det(L(\lambda, u)) \neq 0$ at $\lambda = b$, which as explained above would give a Birkhoff factorization.

**Case 2:** general case. Just like in Case 1, multiplying $M(\lambda, u)$ from the right by an appropriate holomorphic invertible matrix defined for all $(\lambda, u) \in D_\infty \times \Pi$ and by $(\lambda - b)^m \text{Id}$ with $m \gg 0$, we may reduce the proof to the case when $M(\lambda, u)$ is holomorphic for all $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ and meromorphic along $D_b \times \Theta_0$. We argue by induction on the number of zeroes of $\det(M(\lambda, u))$ in $D_b \times (\Pi - \Theta_0)$. If there are no zeroes, then $M(\lambda, u)$ is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ and there is nothing to prove.

Let $b_1 \in D_b$ be a zero. Let us choose a small disc $D_1 = \{|\lambda - b_1| < R_1\}$ with center $b_1$, s.t., $D_1 \subset D_b$ and $D_1$ does not contain other zeroes of $\det(M(\lambda, u))$. Let us recall Case 1 for $M$ and the covering of $\mathbb{P}^1$ given by the discs $D_1$ and $D_1^\infty := \{|\lambda - b_1| > r_1\}$, where $0 < r_1 < R_1$. We get a Birkhoff factorization

$$M(\lambda, u) = M_1(\lambda, u)(\lambda - b_1)^{K_1}W_1(\lambda, u),$$

where

(i) $M_1$ and $W_1$ are $\Pi$-rational.

(ii) $M_1(\lambda, u)$ (resp. $W_1(\lambda, u)$) is holomorphic and invertible for all $(\lambda, u) \in \tilde{D}_1 \times (\Pi - \Theta_1)$ (resp. $\tilde{D}_1^\infty \times (\Pi - \Theta_1)$) for some analytic hypersurface $\Theta_1 \subset \Pi$ with $\Theta_0 \subset \Theta_1$.

(iii) $K_1$ is a diagonal matrix with decreasing integer entries.

Note that

(4)

$$M_1(\lambda, u) = M(\lambda, u)W_1(\lambda, u)^{-1}(\lambda - b_1)^{-K_1},$$
is holomorphic for \((\lambda, u) \in D_b \times (\Pi - \Theta_1)\) and invertible for \((\lambda, u) \in D_b \times (\Pi - \Theta_1)\). The zeroes of \(\text{det}(M_1(\lambda, u))\) for \((\lambda, u) \in D_b \times (\Pi - \Theta_1)\) are first of all in 
\((D_b - D_1) \times (\Pi - \Theta_1)\) and then by expecting the RHS of (4), we get that the they are contained in the set of zeroes of \(\text{det}(M(\lambda, u))\). Note that if \(\lambda = b_1\) is the only zero of \(\text{det}(M(\lambda, u))\) for \((\lambda, u) \in D_b \times (\Pi - \Theta_1)\), then we are done, because \(M_1(\lambda, u)\) will be holomorphic and invertible for \((\lambda, u) \in D_b \times (\Pi - \Theta_1)\). Otherwise, let \(b_2 \in D_b\) be a 2nd zero of \(\text{det}(M(\lambda, u))\) and let \(m > 0\) be an integer such that the diagonal entries of \(K_1\) are greater than \(-m\). We get that the number of zeroes of \(\text{det}\left(M_1(\lambda, u)(\lambda - b_2)^{K_1+m}\right)\) in \(D_b \times (\Pi - \Theta_1)\) is at least 1 less than the number of zeroes of \(\text{det}(M(\lambda, u))\). Using the inductive assumption we get a Birkhoff factorization

\[
M_1(\lambda, u)(\lambda - b_2)^{K_1+m} = U(\lambda, u)(\lambda - b)^K W'(\lambda, u),
\]

where

(i) \(U\) and \(W'\) are \(\Pi\)-rational.
(ii) \(U(\lambda, u)\) (resp. \(W'(\lambda, u)\)) is holomorphic and invertible for all \((\lambda, u) \in D_b \times (\Pi - \Theta_1)\) (resp. \(D_\infty \times (\Pi - \Theta)\)) for some analytic hypersurface \(\Theta \subset \Pi\) with \(\Theta_1 \subset \Theta\).
(iii) \(K\) is a diagonal matrix with decreasing integer entries.

Therefore,

\[
M(\lambda, u) = U(\lambda, u)(\lambda - b)^K W'(\lambda, u) \left(\frac{\lambda - b_1}{\lambda - b_2}\right)^{K_1} \left(\frac{\lambda - b}{\lambda - b_2}\right)^m W_1(\lambda, u)
\]

which provides a Birkhoff factorization for all \(u \in \Pi - \Theta\).

b) Put \(K^{(i)} = \text{Diag}(k_1^{(i)}, \ldots, k_p^{(i)})\). We argue by induction on \(i\) that \(k_i^{(1)} = k_i^{(2)}\) for all \(i\). Assume that \(k_a^{(1)} = k_a^{(2)}\) for \(a = 1, 2, \ldots, i-1\) and \(k_i^{(1)} > k_i^{(2)}\). Comparing the two Birkhoff factorization, we get

\[
(U_2^{-1}U_1)_{a\ell} = (\lambda - b)^{k_a^{(2)} - k_i^{(2)}} (W_2W_1^{-1})_{a\ell},
\]

where \(A_{a\ell}\) denotes the \((a, \ell)\)-entry of the matrix \(A\). The LHS is analytic for \(\lambda \in D_b\). If \(k_a^{(2)} < k_i^{(1)}\), then the RHS is analytic in \(D_\infty\) and vanishes for \(\lambda = \infty\), so by Liouville’s theorem both sides must vanish. We get that \((U_2^{-1}U_1)_{a\ell} = 0\) for \(1 \leq \ell \leq i\) and \(a \geq i\), because according to our assumptions

\[
k_a^{(2)} \leq k_i^{(2)} < k_i^{(1)} \leq k_\ell^{(1)}.
\]

The first \(i\)-columns of \(U_2^{-1}U_1\) have non-zero entries only in the first \((i-1)\) places, therefore they must be linearly dependent. This however contradicts the fact that \(U_2^{-1}U_1\) is invertible for \((\lambda, u) \in D_b \times V\). Similarly, the assumption \(k_i^{(1)} < k_i^{(2)}\) would contradict the invertibility of \(W_2W_1^{-1}\), so \(k_i^{(1)} = k_i^{(2)}\). \(\square\)
4. Painleve property for the Schlesinger equations

In this lecture we prove two theorems of Malgrange, which will be used later on to prove the Painleve property for Frobenius manifolds. Our arguments follow closely Bolibruch [2] for Theorem 4.1 and Malgrange [9] for Theorem 4.4.

4.1. Vector bundles on $\mathbb{P}^1$. Using the results of Lecture 2 we will prove the following theorem of Malgrange (see [9]).

**Proposition 4.1** (Malgrange). Suppose that $T$ is a connected smooth analytic variety and $E$ is a vector bundle on $\mathbb{P}^1 \times T$, s.t., $E_{\mathbb{P}^1 \times \{t_0\}}$ and $E_{\{t_0\} \times T}$ are trivial for some $(b_0, t_0) \in \mathbb{P}^1 \times T$. Then

a) The subset

$$\Theta = \{ t \in T : E_{\mathbb{P}^1 \times \{t\}} \text{ is not trivial} \}$$

is either empty or it is an analytic hypersurface of $T$.

b) $E_{\mathbb{P}^1 \times (T - \Theta)}$ is trivial and meromorphic along $\mathbb{P}^1 \times \Theta$.

Let us clarify the meaning of being meromorphic in Proposition 4.1. It means that we can find a trivializing frame $\{ e_i \}_{i=1}^p$ for $E_{\mathbb{P}^1 \times (T - \Theta)}$, s.t., if $\{ e_i^U \}_{i=1}^p$ is a local frame for $E$ in a neighborhood $U$ of some point $p \in \mathbb{P}^1 \times \Theta$, then the transition function between the two frames is a $p \times p$ matrix whose entries are meromorphic functions on $U$ with poles along $U \cap (\mathbb{P}^1 \times \Theta)$.

**Proof.** a) We argue by induction on the dimension of $T$. If $T$ is 0-dimensional, then there is nothing to prove. Let us define the set

$$N = \{ t \in T : E_{\mathbb{P}^1 \times \{t\}} \text{ is trivial} \}.$$

**Claim 4.2.** If $t' \in N$, then there exists an open neighborhood $V$ of $t'$ in $T$ such that $E_{\mathbb{P}^1 \times V}$ is trivial.

**Proof.** Let $V$ be an open polydisc neighborhood of $t'$. We can find trivializations of $E_{|D_b \times V}$, s.t., the transition function $M(\lambda, t') = 1$. Indeed, using that $E_{\mathbb{P}^1 \times \{t'\}}$ is trivial we get that $M(\lambda, t') = U'(\lambda)W'(\lambda)$, where $U'(\lambda)$ (resp. $W'(\lambda)$) is holomorphic and invertible for $\lambda \in D_b$ (resp. $D_{\infty}$). Changing the trivialization frames of $E_{|D_b \times V}$ and $E_{|D_{\infty} \times V}$ via $U'$ and $W'$ we can transform the the transition matrix into $U'(\lambda)^{-1}M(\lambda, t)W'(\lambda)^{-1}$, which turns into 1 at $t = t'$.

Let us assume now that the transition matrix is such that $M(\lambda, t') = 1$. Decreasing $V$ if necessary, we can make $M(\lambda, t)$ sufficiently close to $M(\lambda, t')$. Recalling Lemma 3.3 we get a Birkhoff factorization $M(\lambda, t) = U(\lambda, t)W(\lambda, t)$, which implies that $E_{\mathbb{P}^1 \times V}$ is trivial.

The above claim shows that $N$ is an open subset.

**Claim 4.3.** The vector bundle $E_{\mathbb{P}^1 \times N}$ is trivial.

**Proof.** Let $\Sigma$ be the set of open subsets $V \subset T$ such that $E_{\mathbb{P}^1 \times V}$ is trivial. By definition $t_0 \in N$, so according to Claim 4.2 the set $\Sigma$ is non-empty. Using the inclusion of open subsets we can define a partial ordering on $\Sigma$. Clearly every
increasing chain \( V_1 \subset V_2 \subset \cdots \) in \( \Sigma \) is bounded by \( \cup_i V_i \in \Sigma \). Therefore, recalling the Zorn’s lemma the set \( \Sigma \) has a maximal element, say \( V \). If \( V \neq N \), then let \( t' \in N \) be a boundary point of \( V \). According to Claim 1.2 we can find an open neighborhood \( V' \in \Sigma \) that contains \( t' \). Let \( e' \) and \( e \) be raw vectors whose entries give trivializations of respectively \( E|_{\mathbb{P}^1 \times V'} \) and \( E|_{\mathbb{P}^1 \times V} \). Then \( e' = eU \), where \( U : \mathbb{P}^1 \times (V' \cap V) \to \text{GL}(\mathbb{C}^p) \) is a transition matrix. Since the entries of \( U(\lambda, u) \) are holomorphic for all \( \lambda \in \mathbb{P}^1 \), by Liouville’s theorem they must be constants independent of \( \lambda \), i.e., \( U(\lambda, u) = U(b_0, u) \). On the other hand by definition \( E|_{\{b_0\} \times T} \) is a trivial bundle, so we can factorize \( U(b_0, u) = A(u)A'(u)^{-1} \). Therefore \( eA(u) = e'A'(u) \) for \( u \in V \cap V' \), so we get that \( E|_{\mathbb{P}^1 \times (V \cup V')} \) is trivial. Since \( V \) is maximal we get \( V' \subseteq V \), which however contradicts the fact that \( t' \in V' \) is a boundary point of \( V \).

If \( N = T \), then we are done. Let us assume that \( N \neq T \). We have to show that \( T - N \) is an analytic subvariety of codimension 1. Let \( u^0 \in T \) be a boundary point of \( N \) and \( \Pi \) be a polydisc with center \( u^0 \). Let \( \Pi(\lambda, u) \) be the transition matrix for some trivializations \( E|_{D_{b,\Pi}} \), \( \nu = b, \infty \). According to Proposition 3.5 we may assume that \( \Pi \) is \( \Pi \)-rational. Decreasing \( \Pi \) if necessary, we get a Birkhoff factorization (see Proposition 3.5, part a)) \( \Pi(\lambda, u) = \Pi(\lambda, u)(\lambda - b)^K W(\lambda, u) \), where \( U(\lambda, u) \) (resp. \( W(\lambda, u) \)) is holomorphic and invertible for \( (\lambda, u) \in D_0 \times (\Pi - \Theta) \) (resp. \( D_\infty \times (\Pi - \Theta) \)). On the other hand, if \( V \subset (\Pi - \Theta) \cap N \subset N \) is an open subset, then \( E|_{\mathbb{P}^1 \times V} \) is trivial. Therefore, the transition function \( \Pi(\lambda, u) = \Pi'(\lambda, u)W'(\lambda, u) \). Comparing the two Birkhoff factorizations of \( \Pi \) and recalling Proposition 3.5, part b) we get that \( K = 0 \), which implies that \( E|_{\mathbb{P}^1 \times (\Pi - \Theta)} \) is trivial, i.e., \( \Pi - N \subset \Theta \).

If \( \Pi - N \neq \emptyset \), then \( N \cap \Theta \neq \emptyset \). Using the inductive assumption we get that \( E|_{\mathbb{P}^1 \times \Theta} \) is trivial on the complement of some analytic hypersurface \( \Theta_0 \subset \Theta \) (note that \( \Theta_0 \) contains the singular locus of \( \Theta \)) and therefore \( \Pi - \Theta_0 \subset N \). Using the Hartog’s extension theorem and that \( \Theta_0 \) is of complex codimension 2, it is easy to prove that \( E|_{\mathbb{P}^1 \times \Pi} \) is also trivial. Indeed, since the vector bundle \( E \) is trivial on \( D_\nu \times \Pi, \nu = b, \infty \), we can choose frames \( \{e_{b,i}\}_{i=1}^p \) and \( \{e_{\infty,i}\}_{i=1}^p \). Let \( M : D_{b,\infty} \times \Pi \to \text{GL}(\mathbb{C}^p) \) be the transition matrix, i.e., \( e_\infty = e_b M \). Since \( E \) is trivial on \( \mathbb{P}^1 \times (\Pi - \Theta_0) \), we can choose a frame of \( E \) on \( \mathbb{P}^1 \times (\Pi - \Theta_0) \). Therefore, we have a Birkhoff factorization

\[
\Pi(\lambda, u) = U(\lambda, u)W(\lambda, u),
\]

where \( U(\lambda, u) \) (resp. \( W(\lambda, u) \)) is holomorphic and invertible in \( D_0 \times (\Pi - \Theta_0) \) (resp. \( D_\infty \times (\Pi - \Theta_0) \)). On the other hand, since \( \Theta_0 \) has complex codimension 2 in \( \Pi \), the Hartog’s extension theorem implies that \( U(\lambda, u) \) (resp. \( W(\lambda, u) \)) extends analytically for all \( (\lambda, u) \in D_0 \times \Pi \) (resp. \( D_\infty \times \Pi \)). Moreover, the zero locus of \( \det(U(\lambda, u)) \) in \( D_0 \times \Pi \) is contained in the codimension 2 analytic subset \( D_0 \times \Theta_0 \). However, the zero locus of an analytic function is either empty or codimension 1. Therefore, \( U(\lambda, u) \) is invertible for \( (\lambda, u) \in D_0 \times \Pi \). Similar argument implies that \( W(\lambda, u) \) is invertible for \( (\lambda, u) \in D_\infty \times \Pi \). Hence the trivialization of \( E \) over \( \mathbb{P}^1 \times (\Pi - \Theta_0) \) extends analytically across \( \mathbb{P}^1 \times \Theta_0 \), i.e., \( \Pi \subset N \). This however
contradicts the fact that the center of $\Pi$ is a boundary point of $N$. Therefore, $\Pi - N = \Theta$ is an analytic hypersurface.

b) Let $e = (e_1, \ldots, e_p)$, $e_i \in \Gamma(\mathbb{P}^1 \times (T - \Theta), E)$ be a trivializing frame. Every other trivializing frame has the from $eC$, where $C : T - \Theta \to \text{GL}(\mathbb{C}^p)$. On the other hand, since $E_{\{b_0\} \times T}$ is trivial we get that we can always choose $C$ in such a way that $eC$ extends to a trivializing frame of $E_{\{b_0\} \times T}$. Therefore there exists a frame $e$ such that $e_{\{b_0\} \times (T - \Theta)}$ extends to a trivializing frame of $E_{\{b_0\} \times T}$. We claim that such a frame $e$ is meromorphic.

To prove this, let us pick a point $u^0 \in \Theta$, a polydisc $\Pi$ with center $u^0$, and trivializations $e_b^\Pi$ and $e_\infty^\Pi$ of respectively $E|_{D_b \times \Pi}$ and $E|_{D_\infty \times \Pi}$ such that the transition function $M : D_{b_0} \to \text{GL}(\mathbb{C}^p)$ is $\Pi$-rational and we have a Birkhoff factorization

$$M(\lambda, u) = U(\lambda, u)(\lambda - b)^K W(\lambda, u),$$

such that $U$ (resp. $W$) is $\Pi$-rational, holomorphic, and invertible for $(\lambda, u) \in D_b \times (\Pi - \Theta)$ (resp. $D_\infty \times (\Pi - \Theta)$) where $\Theta' \subset \Pi$ is an analytic hypersurface. Such choices are possible according to Propositions 3.2 and 3.5 provided we choose $\Pi$ sufficiently small. As we have proved in a), $K = 0$ and $\Theta' = \Pi - N = \Theta$. Let $\tilde{e}_b^\Pi = e_b^\Pi U$ and $\tilde{e}_\infty^\Pi = e_\infty^\Pi W^{-1}$ be trivializing frames of respectively $E|_{D_b \times (\Pi - \Theta)}$ and $E|_{D_\infty \times (\Pi - \Theta)}$. Note that these frames agree on the intersection $D_{b_0} \cap (\Pi - \Theta)$, so we get a trivializing frame $\tilde{e}^\Pi$ of $E|_{\mathbb{P}^1 \times (\Pi - \Theta)}$. Therefore, there exists a transition matrix $C : \Pi - \Theta \to \text{GL}(\mathbb{C}^p)$, such that $e^\Pi = eC$. Let $f \in \mathcal{O}(\Pi)$ be the function whose zero locus defines the hypersurface $\Theta \cap \Pi$. Since both $U(\lambda, u)$ and $W(\lambda, u)^{-1}$ are meromorphic along respectively $D_b \times \Theta$ and $D_\infty \times \Theta$, there exists an integer $n > 0$ such that $f(u)^n U(\lambda, u)$ (resp. $f(u)^n W(\lambda, u)^{-1}$) extends analytically for all $(\lambda, u) \in D_b \times \Pi$ (resp. $D_\infty \times \Pi$). Therefore $e^\Pi f^n = eC f^n$ is a raw vector whose entries are holomorphic section of $E$ on $\mathbb{P}^1 \times \Pi$. Restricting to $\{b_0\} \times \Pi$ and recalling that $e_{\{b_0\} \times \Pi}$ is a trivializing frame, we get that $H(u) := C(u)f(u)^n$ is holomorphic for all $u \in \Pi$, i.e., $C$ is meromorphic along $\Theta$. Therefore

$$e_b^\Pi = eC(u)U(\lambda, u)^{-1}, \quad e_\infty^\Pi = eC(u)W(\lambda, u)$$

and we get that the transition matrices $C(u)U(\lambda, u)^{-1}$ and $C(u)W(\lambda, u)$ are meromorphic along respectively $D_b \times \Theta$ and $D_\infty \times \Theta$.

\[\square\]

4.2. The Schlesinger equations. Let $\nabla^\circ$ be a Fuchsian connection on the trivial vector bundle $\mathbb{P}^1 \times \mathbb{C}^p$. Written in coordinates

$$\nabla^\circ = d - A^\circ(\lambda)d\lambda,$$

where

$$A^\circ(\lambda) = \frac{A^0_1}{\lambda - u_1^\circ} + \cdots + \frac{A^0_N}{\lambda - u_N^\circ},$$

where $A^0_i$ are $p \times p$ matrices and $u_i^\circ$ are the finite poles of $\nabla^\circ$. Let us also assume that $\sum_{i=1}^N A^0_i \neq 0$, so that the connection has a Fuchsian singularity at $\lambda = \infty$. 
The Schlesinger equations are the following system of differential equations

\[
\frac{\partial A_i}{\partial u_j} = \frac{[A_j, A_i]}{u_j - u_i}, \quad 1 \leq i \neq j \leq N,
\]

\[
\sum_{j=1}^{N} \frac{\partial A_i}{\partial u_j} = 0, \quad 1 \leq i \leq N,
\]

\[
A_i(u^o) = A_i^o, \quad 1 \leq i \leq N,
\]

where \( u^o = (u_1^o, \ldots, u_N^o) \). Here

\[
A_i(u_1, \ldots, u_N) \in \mathfrak{gl}(\mathbb{C}^p), \quad 1 \leq i \leq N,
\]

is a set of matrix-valued functions that should be thought as deformations of the coefficients of the Fuchsian connection \( \nabla^o \). It is easy to check that the Schlesinger equations are compatible (integrable). Therefore the solution exists for all \( u = (u_1, \ldots, u_N) \) sufficiently close to \( u^o \).

The main goal of this lecture is to prove that the local solutions extend to global meromorphic functions. More precisely, let

\[
Z_N = \{ u \in (\mathbb{P}^1)^{N+1} : u_i \neq u_j \text{ for } i \neq j \text{ and } u_{N+1} = \infty \}
\]

be the configuration space of \( N \) points in \( \mathbb{C} \). Every point \( u \in Z_N \) corresponds to a punctured sphere \( \mathbb{P}^1 - \{u_1, \ldots, u_{N+1}\} = \mathbb{C} - \{u_1, \ldots, u_N\} \).

Let us denote by \( T \) the universal cover of \( Z_N \). The point \( u^o \in Z_N \) will be fixed as a base point and we identify \( T \) as the set of pairs \( (u, [c]) \) such that \( u \in Z_N \) and \( [c] \) is the homotopy class of a path \( c \) in \( Z_N \) from \( u^o \) to \( u \). A small neighborhood of \( u^o \) in \( Z_N \) has a natural lift to a small neighborhood of \( t^o := (u^o, [1]) \in T \), where \([1] \) is the trivial path from \( u^o \) to \( u^o \). In particular, every solution of the Schlesinger equations (with the specified initial condition) is defined and analytic in a neighborhood of \( t^o \in T \). The main goal of this lecture is to prove the following theorem due to Malgrange [9].

**Theorem 4.4 (Malgrange).** If \( \{A_i(u)\}_{i=1}^N \) is a solution to the Schlesinger equations, then each \( A_i \) extends to a meromorphic function on \( T \).

**4.3. Malgrange’s vector bundle \( E \).** The Fuchsian connection \( \nabla^o \) determines a monodromy representation

\[
\mu : \pi_1(\mathbb{C} - \{u_1^o, \ldots, u_N^o\}, b^o) \to \text{GL}(\mathbb{C}^p),
\]

where \( b^o \) is a reference point. The representation is defined as follows. Let \( Y^o(\lambda) \) be the fundamental solution of \( \nabla^o \) defined in a neighborhood of \( b^o \) such that \( Y^o(b^o) = 1 \). If \( \gamma \) is a closed path in \( \mathbb{C} - \{u_1^o, \ldots, u_N^o\} \) then the analytic continuation of \( Y^o(\lambda) \) along \( \gamma \) has the form \( Y^o(\lambda)\mu(\gamma) \).
Let us denote by $D_i \subset \mathbb{P}^1 \times T$, $1 \leq i \leq N + 1$, the hypersurface consisting of points $(\lambda, u, [c])$ such that $\lambda = u_i$ ($u_{N+1} := \infty$). Let $\mathcal{C} := \mathbb{P}^1 \times T - \bigcup_{i=1}^{N+1} D_i$. The projection map

$$\pi : \mathcal{C} \to T, \quad (\lambda, u, [c]) \mapsto (u, [c])$$

is a smooth fibration with fiber diffeomorphic to $\pi^{-1}(t^0) = \mathbb{C} - \{u_1^0, \ldots, u_N^0\}$. Since $\pi_k(T) = \pi_k(Z_N) = 0$ for $k > 1$ and $\pi_1(T) = \{1\}$, we get that $T$ is a contractible space. Using the long exact sequence of homotopy groups we get that the natural inclusion

$$(\mathbb{C} - \{u_1^0, \ldots, u_N^0\}, b^0) \to (\mathcal{C}, (b^0, t^0))$$

induces an isomorphism between the fundamental groups. Therefore the monodromy representation $\mu^0$ of $\nabla^0$ induces a representation

$$\mu : \pi_1(\mathcal{C}, (b^0, t^0)) \to \text{GL}(\mathbb{C}^p).$$

There exists a unique vector bundle $E \to \mathcal{C}$ of rank $p$ equipped with a flat connection $\nabla$ such that the monodromy representation of $\nabla$ is equivalent to the given representation (5). We will refer to $E \to \mathcal{C}$ as the Malgrange’s vector bundle. The equivalence between the monodromy representation of $\nabla$ and (5) means that there exists a raw vector $f^0 = (f_1^0, \ldots, f_p^0)$ whose entries form a basis of the fiber $E_{b^0, t^0}$ such that the parallel transport with respect to $\nabla$ along a closed loop $\gamma$ based at $(b^0, t^0)$ transforms $f^0$ into $f^0 \mu(\gamma)$.

For the reader’s convenience let us recall the construction of $E$. We choose a covering of $\mathcal{C}$ by open balls $\{B_i\}_{i \in \mathcal{C}}$ that have contractible connected intersections. This can be achieved by choosing a Riemannian metric on $\mathcal{C}$ and letting $B_i$ be the ball with center $i \in C$ of radius $r_i$, where $r_i$ is the injectivity radius of $\mathcal{C}$ at the point $i$. It is known that if $x', x'' \in B_i$, then there exists a unique geodesic in $\mathcal{C}$ from $x'$ to $x''$ whose length is the distance between $x'$ and $x''$. Moreover, such a geodesic is entirely in $B_i$. If $B_i \cap B_j \neq \emptyset$, then we choose a smooth path $\gamma_{ij}$ in $B_i \cup B_j$ between the centers of $B_i$ and $B_j$. Let us also fix $B_0$ to be the ball with center the base point $(b^0, t^0)$. Let us also fix a path $\gamma_i$ from $B_0$ to $B_i$ consisting of paths $\gamma_{ab}$. Then we define $E|_{B_i} := B_i \times \mathbb{C}^p$ and let $e^i = (e_1^i, \ldots, e_p^i)$ be the trivializing frame corresponding to the standard basis of $\mathbb{C}^p$. On the overlaps $B_i \cap B_j \neq \emptyset$ the bundles are glued via

$$e^j = e^i g_{ij}, \quad g_{ij} = \mu(\gamma_i^{-1} \circ \gamma_{ji} \circ \gamma_j),$$

where $\mu$ is the given monodromy representation (5). Since $g_{ij}$ are constants, the standard flat connections given by the de Rham differential on $B_i \times \mathbb{C}^p$ glue together, so the bundle $E$ is naturally equipped with a flat connection.

4.4. Extension of $E$. Recall that $Y^0(\lambda)$ is the fundamental solution of $\nabla^0$ defined in a neighborhood of a fixed reference point $\lambda = b^0$. $Y^0(\lambda)$ is uniquely determined by requiring that it satisfies the initial condition $Y^0(b^0) = 1$. For
every singular point \( u_i^0 \) \((1 \leq i \leq N)\) of \( \nabla^o \) let us fix a sector with vertex at \( u_i^0 \) of the following form

\[
\{ \lambda \in \mathbb{C} : 0 < |\lambda - u_i^0| < R_i^0, -\varepsilon < \text{Arg}(\lambda - u_i^0) < \varepsilon \},
\]

where \( R_i^0 \) is sufficiently small so that the disc with center \( u_i^0 \) and radius \( R_i^0 \) does not contain other singular points \( u_j^0 \) and \( 0 < \varepsilon < 2\pi \). Let us fix a path \( \gamma_i^0 \) \((1 \leq i \leq N + 1)\) in \( \mathbb{C} - \{u_1^0, \ldots, u_N^0\} \) from \( b^0 \) to a point \( u_i^0 + \lambda_i^0 \) in the above sector, e.g., \( \lambda_i^0 := R_i^0/2 \). Let us extend analytically \( Y^o(\lambda) \) along \( \gamma_i^0 \). We get an analytic solution of \( \nabla^o \) defined in the above sector. Finally, let us choose an invertible matrix \( S_i^0 \in \text{GL}(\mathbb{C}^p) \), such that \( Y^o(\lambda)S_i^0 \) is a weak Levelt solution for the Fuchsian singularity of \( \nabla^o \) at \( \lambda = u_i^0 \). We have

\[
Y^o(\lambda)S_i^0 = U_i^0(\lambda)(\lambda - u_i^0)^{K_i}(\lambda - u_i^0)^{E_i},
\]

where the matrix

\[
E_i = \text{diag}(E_i^1, \ldots, E_i^p)
\]
is block diagonal with each block corresponding to an eigenvalue of \( E_i \), the block \( E_i^j = \rho_i^j I + N_i^j \), where \( N_i^j \) is an upper-triangular nilpotent matrix and the eigenvalue \( \rho_i^j \) satisfies

\[
0 \leq \text{Re}(\rho_i^j) < 1,
\]

\( K_i = \text{diag}(K_i^1, \ldots, K_i^p) \) has the same block diagonal structure as \( E_i \) with each block \( K_i^j \) being a diagonal matrix with decreasing integer entries, and \( U_i^0(\lambda) \) is holomorphically invertible in a neighborhood of \( \lambda = u_i^0 \).

It is convenient to extend our notation for the singular points of \( \nabla^o \) in order to include also the singularity at \( \lambda = u_{N+1}^0 = \infty \). The above statements remain the same except that we have to replace everywhere \( \lambda - u_i^0 \) with \( \lambda^{-1} \). In particular, the fundamental solution takes the forms

\[
Y^o(\lambda)S_{N+1}^0 = U_{N+1}^o(\lambda)^{-K_{N+1}}\lambda^{-E_{N+1}}.
\]

The vector bundle \( E \) can be extended across the divisors \( D_i \) \((1 \leq i \leq N + 1)\) as follows. Let us take a tubular neighborhood

\[
T_i = \{(\lambda, u, [c]) : |\lambda - u_i| < R_i(u)\} \subset \mathbb{P}^1 \times T,
\]

where \( R_i : Z_N \to \mathbb{R}_{>0} \) is a smooth function satisfying

\[
R_i(u) < |u_j - u_i|, \quad \text{for all} \quad 1 \leq i \neq j \leq N,
\]

and

\[
R_{N+1}(u) > |u_j|, \quad \text{for all} \quad 1 \leq j \leq N.
\]

Using parallel transport with respect to the flat connection \( \nabla \) we construct a multivalued flat frame \( f = (f_1, \ldots, f_p) \) of \( E \) whose value at a point \((\lambda, t) \in C\)

\[
f(\lambda, t) = (f_1(\lambda, t), \ldots, f_p(\lambda, t)), \quad f_i(\lambda, t) \in E_{\lambda, t}
\]
depends on the choice of a reference path in $\mathcal{C}$ from $(b^o, t^o)$ to $(\lambda, t)$: the component $f_1(\lambda, t)$ is obtained from $f_i^0 \in E_{b^o, t^o}$ via a parallel transport along the reference path. Let us trivialize $E_{|T_i - D_i}$ via the frame

$$f(\lambda, t)S_i^0(\lambda - u_i)^{-E_i(\lambda - u_i)^{-K_i}}, \quad (\lambda, t) \in T_i - D_i,$$

where $t = (u_i, [c]) \in T$ and the path specifying the value of $f(\lambda, t)$ is chosen as follows. We identify $\mathbb{C} - \{u_1^\circ, \ldots, u_N^\circ\}$ with the fiber $\mathcal{C}_{t^o} := \pi^{-1}(t^o)$. Note that the path $\gamma_i^o \subset \mathcal{C}_{t^o}$ connects the reference point $(b^o, t^o)$ with the point $(u_i^0 + R_i^o/2, t^o) \in T_i$ (provided we define $R_i^o := R_i(u^o)$). The path that we would like to select consists of two pieces the path $\gamma_i^o$ and any path in $T_i - D_i$ connecting the end point of $\gamma_i^o$ and $(\lambda, t)$. The analytic continuation of $f(\lambda, t^o)$ and $Y^o(\lambda)$ along a closed loop around $\lambda = u_i^o$ are respectively $f(\lambda, t^o)M_i$ and $Y^o(\lambda)M_i$, where $M_i$ is such that $M_iS_i^o = S_i^o e^{2\pi\sqrt{-1}E_i}$. The monodromy of $f(\lambda, t)S_i^0$ around $D_i$ cancels out the monodromy of $(\lambda - u_i)^{-E_i(\lambda - u_i)^{-K_i}}$ around $D_i$. Hence the frame \([8]\) provides a holomorphic trivialization of $E_{|T_i - D_i}$. We extend $E$ across $D_i$ in the obvious way: on the overlap of $T_i$ and $T_i - D_i$ we identify the standard frame of $T_i \times \mathbb{C}^p$ with the frame \([8]\) of $E_{|U_i - D_i}$.

4.5. **Proof of Theorem 4.4.** We are going to construct a multivalued analytic function $Y(\lambda, t)$ with values in $\text{GL}(\mathbb{C}^p)$ defined for all $(\lambda, t) \in \mathcal{C}$ such that

1. $Y(\lambda, t^o) = Y^o(\lambda)$.
2. The 1-form $\omega := dY(\lambda, t)Y^{-1}(\lambda, t)$ is a meromorphic 1-form on $\mathbb{P}^1 \times T$ of the form

$$\sum_{i=1}^{N} \frac{A_i(t)}{\lambda - u_i} (d\lambda - du_i),$$

where $A_i$ is a $\text{gl}(\mathbb{C}^p)$-valued meromorphic function on $T$ and $u_i : T \to \mathbb{C}$ is the $i$th component of the projection map $T \to \mathbb{C}^N$.

If we manage to do this then Theorem 4.4 follows immediately. Indeed, the 1st condition implies that $A_i(t^o) = A_i^o$. While the fact that $A_i(t)$ satisfy the Schlesinger equations follows from the fact that $\omega$ is a 1-form satisfying

$$d\omega + \omega \wedge \omega = d(dY Y^{-1}) + dYY^{-1} \wedge dYY^{-1} = 0.$$

The matrix-valued function $Y(\lambda, t)$ is constructed by comparing two trivializing frames of $E$. The first one is the multivalued flat frame

$$f(\lambda, t) = (f_1(\lambda, t), \ldots, f_p(\lambda, t)), \quad f_i(\lambda, t) \in E_{\lambda, t},$$

defined by the parallel transport with respect to $\nabla$ with initial value $f(b^o, t^o) := f^o$. Recall that $f^o$ is the frame of $E_{b^o, t^o}$ that we fixed so that the monodromy representation of $\nabla$ coincides with the monodromy representation \([5]\).

The 2nd frame will be constructed by using Theorem 4.1 which guarantees the existence of a meromorphic trivialization of $E$. Let us check that the conditions of Theorem 4.1 are satisfied. By definition, $D_{N+1} = \{\infty\} \times T$ and $E_{|D_{N+1}}$ is trivial.

**Claim 4.5.** *The restriction $E_{|\mathbb{P}^1 \times t^o}$ is trivial.*
Proof. We will prove that \( f(\lambda, t^o)Y^o(\lambda)^{-1} \) is a trivializing frame. By definition the monodromy of the frame \( f(\lambda, t^o) \) and the monodromy of the matrix \( Y^o(\lambda)^{-1} \) cancel each other. Therefore the above frame provides a trivialization of \( E|_{\mathbb{P}^1 \times T^o} \) on \( \mathbb{C} - \{u_1^o, \ldots, u_N^o\} \). Let us check that the trivialization extends analytically in a neighborhood of \( \lambda = u_i^o \) for all \( 1 \leq i \leq N + 1 \). Let us assume that \( 1 \leq i \leq N \). The case \( i = N + 1 \) is the same but one has to use slightly different notation. By definition the trivializing frame of \( E|_{\mathbb{P}^1 \times T^o} \) in a neighborhood of \( \lambda = u_i^o \) is given by

\[
 f(\lambda, t^o)S_i^o(\lambda - u_i^o)^{-E_i}(\lambda - u_i^o)^{-K_i}.
\]

However, recalling the definition of \( S_i^o \) we get that the above frame coincides with

\[
 f(\lambda, t^o)Y^o(\lambda)^{-1}U_i^o(\lambda).
\]

According to Levelt’s theorem \( U_i^o(\lambda) \) is holomorphically invertible at \( \lambda = u_i^o \). Therefore the frame \( f(\lambda, t^o)Y^o(\lambda)^{-1} \) extends holomorphically and it remains a frame at the point \( \lambda = u_i^o \). \( \square \)

According to Theorem 4.1, there exists an analytic hypersurface \( \Theta \subset T \), such that \( E|_{\mathbb{P}^1 \times (T - \Theta)} \) is a trivial vector bundle. Let

\[
 \bar{e} = (\bar{e}_1, \ldots, \bar{e}_p), \quad \bar{e}_i \in \Gamma(\mathbb{P}^1 \times (T - \Theta), E)
\]

be a trivializing frame. We may further assume that \( \bar{e}(\lambda, t^o) = f(\lambda, t^o)Y^o(\lambda)^{-1} \).

The frame that we need in order to define \( Y(\lambda, u) \) is slightly different. The necessary modification is constructed as follows. In the tubular neighborhood \( T_{N+1} \) we have

\[
 f(\lambda, t)S_{N+1}^o\lambda^{E_{N+1}}\lambda^{K_{N+1}} = \bar{e}(\lambda, t)\bar{U}(\lambda, t), \quad \forall (\lambda, t) \in T_{N+1} - T_{N+1} \cap (\mathbb{P}^1 \times \Theta),
\]

where \( \bar{U}(\lambda, t) \) is holomorphic and invertible for all \( (\lambda, t) \in T_{N+1} - T_{N+1} \cap (\mathbb{P}^1 \times \Theta) \) and meromorphic along \( T_{N+1} \cap (\mathbb{P}^1 \times \Theta) \). The Taylor series expansion at \( \lambda = \infty \) yields

\[
 \bar{U}(\lambda, t) = \bar{U}_0(t) + \bar{U}_1(t)\lambda^{-1} + \bar{U}_2(t)\lambda^{-2} + \cdots,
\]

where \( \bar{U}_0(t) \) is holomorphic and invertible for all \( t \in T - \Theta \) and meromorphic along \( \Theta \). The frame that we need is

\[
 e(\lambda, t) = \bar{e}(\lambda, t)\bar{U}_0(t)^{-1}\bar{U}_0(t^o).
\]

Note that the above frame is holomorphic for all \( t \in T - \Theta \) and meromorphic along \( \Theta \).

Let us define \( Y(\lambda, t) \in \text{GL}(\mathbb{C}^p) \) as the transition matrix

\[
 f(\lambda, t) = e(\lambda, t)Y(\lambda, t), \quad (\lambda, t) \in \mathcal{C} - (\mathcal{C} \cap (\mathbb{P}^1 \times \Theta)).
\]

Note that at \( t = t^o \) we have \( Y(\lambda, t^o) = Y^o(\lambda) \). Therefore, we need to check that the 1-form \( \omega = dY Y^{-1} \) has the required properties.

To begin with, note that \( \omega \) is single valued and analytic on \( \mathcal{C} \). Indeed, the monodromy of \( Y(\lambda, t) \) is the same as the monodromy of \( f(\lambda, t) \), i.e., under the analytic continuation along a closed loop \( \gamma \) the value of \( Y(\lambda, t) \) changes into \( Y(\lambda, t)\mu(\gamma) \).
However, $\mu(\gamma)$ is independent of $\lambda$ and $t$, so the value of $\omega$ remains the same. Since being analytic is a local property and locally $Y(\lambda, t)$ is analytic the same is true for $\omega$.

Let us analyze the singularities of $\omega$ as a 1-form on $\mathbb{P}^1 \times T$. The possible singular locus is along the following divisors

$$D_i \ (1 \leq i \leq N + 1), \ \mathbb{P}^1 \times \Theta.$$ 

Let us fix $t \notin \Theta$ and look in a neighborhood of $\lambda = u_i$ for $1 \leq i \leq N$. We have

$$f(\lambda, t)S_i^\omega(\lambda - u_i)^{-E_i} = e(\lambda, t)U_i(\lambda, t),$$

where $U_i$ is holomorphic and invertible for all $(\lambda, t) \in T_i - T_i \cap (\mathbb{P}^1 \times T)$ and meromorphic along $T_i \cap (\mathbb{P}^1 \times T)$. In particular, the Taylor series expansion at $\lambda = u_i$ takes the form

$$U_i(\lambda, t) = U_{i,0}(t) + U_{i,1}(t)(\lambda - u_i) + \cdots,$$

where $U_{i,0}(t)$ is holomorphic and invertible for $t \in T - \Theta$ and meromorphic along $\Theta$. Recalling the definition of $Y(\lambda, t)$ we get

$$Y(\lambda, t)S_i^\omega = U_i(\lambda, t)(\lambda - u_i)^{-K_i} = U_0(t)\tilde{U}_0(t, \lambda),$$

where the branch of $\lambda^{-K}$ is independent of $\Theta$. According to Liouville’s theorem $\omega$ has a holomorphic form for $1 \leq i \leq N + 1$, so the value of $\omega$ remains the same. We have

$$Y(\lambda, t) = \tilde{U}_0(t^\omega)\tilde{U}_0(t, \lambda)^{-1}U(t, \lambda)^{-K_{N+1}}\lambda^{-E_{N+1}}.$$ 

Put

$$A_i(t, \lambda) := -\lambda - u_i(\partial_{u_i}Y(\lambda, t))Y(\lambda, t)^{-1}.$$ 

If $t \notin \Theta$ is fixed then $A_i$ is an analytic matrix-valued function on $\mathbb{C} - \{u_1, \ldots, u_N\}$. Near $\lambda = u_j$ with $1 \leq j \neq i \leq N$ we get

$$A_i(t, \lambda) = (u_i - u_j)(\partial_{u_i}U_{j,0}(t))U_{j,0}(t)^{-1} + O(\lambda - u_j),$$

which is analytic in a neighborhood of $\lambda = u_j$. Near $\lambda = u_i$ we get

$$A_i(t, \lambda) = -(\lambda - u_i)(\partial_{u_i}U_i(\lambda, t))U_i(\lambda, t)^{-1} +$$

$$U_i(\lambda, t)\left(K_i + (\lambda - u_i)^{K_i}E_i(\lambda - u_i)^{-K_i}\right)U_i(\lambda, t)^{-1}.$$ 

Using the special form of the matrices $E_i$ and $K_i$ we get that the above expression is analytic at $\lambda = u_i$. Finally at $\lambda = \infty$ we have

$$A_i(t, \lambda) = -(\lambda - u_i)\tilde{U}_0(t^\omega)\partial_{u_i}(\tilde{U}_0(t)^{-1}\tilde{U}_0(t, \lambda))\tilde{U}_0(t, \lambda)^{-1}\tilde{U}_0(t)\tilde{U}_0(t^\omega)^{-1},$$

and this again is analytic at $\lambda = \infty$. According to Liouville’s theorem $A_i(t, \lambda)$ is independent of $\lambda$. Setting $\lambda = u_i$ we get that

$$A_i(t) := A_i(t, \lambda) = U_{i,0}(t)C_iU_{i,0}(t)^{-1},$$

where $C_i$ is a constant upper triangular matrix. Moreover, we get that $A_i$ is meromorphic along $\Theta$. 

Similar argument shows that the matrix
\[ A(\lambda, t) := \frac{\partial_\lambda Y(\lambda, t)Y(\lambda, t)}{Y(\lambda, t)} \]
is holomorphic at \( \lambda = \infty \) and equal to 0 at \( \lambda = \infty \). While at \( \lambda = u_i \) we have
\[ A(\lambda, t) = \frac{A_i(t)}{\lambda - u_i} + \cdots, \]
where the dots stand for terms analytic at \( \lambda = u_i \). This implies that
\[ A(\lambda, t) - \sum_{i=1}^{N} \frac{A_i(t)}{\lambda - u_i} \]
is analytic for all \( \lambda \in \mathbb{P}^1 \) and vanishing at \( \lambda = \infty \). Recalling again Liouville’s theorem we get that
\[ A(\lambda, t) = \sum_{i=1}^{N} \frac{A_i(t)}{\lambda - u_i}. \]

Summarizing, we get that
\[ \omega = dY Y^{-1} = \sum_{i=1}^{N} \frac{A_i(t)}{\lambda - u_i} (d\lambda - du_i), \]
where \( A_i \) are meromorphic functions on \( T \). This completes the proof of Theorem 4.4. □

### 4.6. Levelt solution with parameters.

The proof of Theorem 4.4 has the following interesting corollary. Suppose that we have a Fuchsian connection \( \nabla^0 \) of the same form as in Section 4.2. Let \( Y^0(\lambda) \) be a fundamental solution defined in a neighborhood of a fixed reference point \( b^0 \in \mathbb{C} - \{u_1^0, \ldots, u_N^0\} \). Using the same notation as in Section 4.4, let us fix reference paths \( \gamma_i^0 \) connecting \( b^0 \) with a neighborhood of \( u_i^0 \) and invertible matrices \( S_i \in \text{GL}(\mathbb{C}^p) \) \( 1 \leq i \leq N + 1 \) such that \( Y^0(\lambda)S_i \) is a weak Levelt solution of the form (6).

The isomonodromic deformations of Schlesinger preserve the form of the Levelt solutions. Namely, let \( \{A_i(u)\}_{i=1}^{N} \) be the solution to the Schlesinger equations satisfying the initial condition \( A_i(u^0) = A_i^0 \) and defined for all \( u \) sufficiently close to \( u^0 \). Then the system
\[ \partial_\lambda Y(\lambda, u) = \left( \sum_{i=1}^{N} \frac{A_i(u)}{\lambda - u_i} \right) Y(\lambda, u) \]
\[ \partial_{u_i} Y(\lambda, u) = -\frac{A_i(u)}{\lambda - u_i} Y(\lambda, u), \quad 1 \leq i \leq N, \]
satisfying the initial condition \( Y(\lambda, u^0) = Y^0(\lambda) \) has a unique solution. Moreover, for \( \lambda \) close to \( u_i \) \( 1 \leq i \leq N \) we have
\[ Y(\lambda, u)S_i = U_i(\lambda, u)(\lambda - u_i)^{E_i}(\lambda - u_i)^{E_i}. \]
and for $\lambda$ close to $\infty$ we have
\[ Y(\lambda, u)S_{N+1} = U_{N+1}(\lambda, u)\lambda^{-K_{N+1}}\lambda^{-E_{N+1}}, \]
where the matrices $S_i, K_i,$ and $E_i$ ($1 \leq i \leq N + 1$) are independent of the deformation parameters $u$.

5. Tau-function of the Schlesinger equation

Recall that for a given Fuchsian connection $\nabla^0$ we have Malgrange’s vector bundle $E$ on $\mathbb{P}^1 \times T$. According to Theorem 4.1, $E$ is trivial in the complement of $\mathbb{P}^1 \times \Theta$, where $\Theta \subset T$ is the subset of all points $t$ such that $E|_{\mathbb{P}^1 \times \{t\}}$ is trivial.

The main goal of this lecture is to present a simple algorithm due to Bolibruch [2] that allows us to compute the equation defining $\Theta$ in terms of the solution of the corresponding Schlesinger equations.

5.1. Tau-function. The notion of tau-function of an isomonodromic deformation was introduced in the work of M. Jimbo, T. Miwa, and K. Ueno [8]. The key to the construction of the tau-function in our settings is the following 1-form
\begin{equation}
\omega = \frac{1}{2} \sum_{i=1}^{N} \sum_{j:j \neq i} \frac{\text{Tr}(A_i(u)A_j(u))}{u_i - u_j} (du_i - du_j),
\end{equation}
where $\{A_i(u)\}_{i=1}^{N}$ is a solution to the Schlesinger equations satisfying given initial condition $A_i(u^0) = A_i^0$. According to Theorem 4.4, $\omega$ is a meromorphic 1-form on $T$ with poles along the divisor $\Theta$. We are going to prove the following lemma.

Lemma 5.1. Suppose that $t^* \in \Theta$ and $\tau_i^* \in O_{T,t^*}$ ($1 \leq i \leq s$) are the holomorphic germs whose zero loci define the irreducible components of the germ of $\Theta$ at $t^*$. Then there are integers $r_i$ ($1 \leq i \leq s$) such that the 1-form $\omega - \sum_{i=1}^{s} r_i \frac{d\tau_i}{\tau_i}$ is holomorphic at $t^*$.

Following Bolibruch we prove this lemma by giving an algorithm that produces a set of meromorphic functions whose zero loci contain $\Theta$. This lemma implies the following theorem.

Theorem 5.2. There exists a meromorphic function $\tau$ on $T$ such that $\omega = d\log \tau$.

Proof. Let us define
\[ \tau(t) := \exp \left( \int_{t^*}^{t} \omega \right), \]
where the integral is along a path in $T - \Theta$. Note that this is a single valued holomorphic function on $T - \Theta$ because according to Lemma 5.1, the periods of $\omega$ along closed loops around $\Theta$ are integer multiples of $2\pi\sqrt{-1}$.

Let us prove that $\tau$ is meromorphic at $t^* \in \Theta$. Let us denote by $\tau_i^* \in O_{T,t^*}$ ($1 \leq i \leq s$) the functions that define the irreducible components of $\Theta$ at $t^*$. Let us take $U^* \subset T$ to be an open neighborhood of $t^*$ such that $\tau_i^*$ and $\omega - \sum_{i=1}^{s} r_i \frac{d\tau_i}{\tau_i}$ can
be represented respectively by holomorphic functions on $U^*$ and a holomorphic 1-form on $U^*$. Finally, let us pick a point $b \in U^* - \Theta$ and let $t \in U^* - \Theta$. Then we have
\[
\tau(t) = e^{\int_b^t \omega} = e^{\int_b^t \omega}.
\]

On the other hand
\[
\int_b^t \omega = \sum_{i=1}^s r_i (\log \tau_i^*(t) - \log \tau_i^*(b)) + \int_b^t \left( \omega - r_1 \frac{d\tau_1^*}{\tau_1^*} - \cdots - r_s \frac{d\tau_s^*}{\tau_s^*} \right).
\]

The integral on the RHS extends analytically for all $t \in U^*$, because the integrand is a holomorphic 1-form in $U^*$. Therefore up to a holomorphically invertible function in $U^*$ the function $\tau(t)$ equals $\tau_1^*(t) \cdots \tau_s^*(t)$. This proves that $\tau$ is meromorphic in $U^*$. To prove that $\tau$ is globally meromorphic we have to recall that $T$ is a contractible Stein manifold, so every function which is locally meromorphic must be globally meromorphic.

Function $\tau \in O_T(T - \Theta)$ having the properties in Theorem 5.2 is unique up to a non-zero constant factor. Indeed, if $\tau_1$ and $\tau_2$ are two such functions, then
\[
d(\tau_1(t)/\tau_2(t)) = (\tau_1(t)/\tau_2(t)) \left( \frac{d\tau_1(t)}{\tau_1(t)} - \frac{d\tau_2(t)}{\tau_2(t)} \right) = 0
\]
for all $t \in T - \Theta$. Every function $\tau(t)$ satisfying the properties in Theorem 5.2 is called tau-function of the isomonodromic deformation. In fact, it is a theorem due to Miwa [12] (in the case of generic monodromy data) and Malgrange [9] (in all cases) that the tau-function is analytic along $\Theta$.

**Remark 5.3.** Bolibruch claimed in his notes [2] that his algorithm implies the analyticity of the tau-function. However, there seems to be a missing justification. Namely Bolibruch’s algorithm produces a sequence of meromorphic functions, whose product according to the general theory should be holomorphic. However, using only the elementary approach pursued in these lectures, we could not justify the analyticity of the product.

**Remark 5.4.** There is a notion of tau-function more generally for isomonodromic deformations of connections with irregular singularities. Miwa proved the analyticity of the tau-function in full generality. However, in Miwa’s work there is a generality assumption about the monodromy data. Namely, the monodromy operators are diagonalizable.

5.2. **Bolibruch’s algorithm.** Let $t^* = (u^*, [c]) \in \Theta$ be a generic point, where $u^* = (u_1^*, \ldots, u_N^*)$. We will be interested only in a small neighborhood of $t^*$ in $T$, which is isomorphic via the covering map $T \to Z_N$ to a small neighborhood $U$ of $u^*$ in $Z_N$. Using this local bi-holomorphism we will sometimes write $u \in T$ for all $u \in U$. Let us recall also the notation from Section 4.4 involving the following data:

1. Fuchsian connection $\nabla^o$ on $\mathbb{P}^1 \times \{t^o\}$, where $t^o = (u^o, [1])$ is a fixed reference point.
(2) We fixed a reference point \( b^0 \) in \( \mathbb{P}^1 \), fundamental solution \( Y^0 \) of \( \nabla^0 \) such that \( Y^0(b^0) = 1 \), and a system of paths from \( b^0 \) to each singular point \( u^0_i \) that allows us to analytically continue \( Y^0(\lambda) \) in a neighborhood of each \( \lambda = u^0_i \).

(3) For each singular point we have chosen a constant matrix \( S_i \) such that
\[
Y^0(\lambda)S_i \text{ is a weak Levelt solution of the type } (6).
\]

This is the data necessary to define Malgrange’s bundle \( E \to \mathbb{P}^1 \times T \) and hence determines the analytic hypersurface \( \Theta \) as well.

The first step in Bolibruch’s algorithm is to construct an auxiliary Fuchsian system on \( \mathbb{P}^1 \times \{t^*\} \) that has an extra singular point. To avoid cumbersome notation let us assume that \( u^0_i \neq 0 \) (1 \( \leq i \leq N \)). Then for an extra singular point we choose 0. Let us denote by \( f = (f_1, \ldots, f_p) \) the multivalued flat frame of Malgrange’s bundle \( E \to \mathcal{C} \) (see Section 4.3). The frame \( f \) provides a trivialization of \( E|_{D_0 \times \{t^*\}} \) where \( D_0 \subset \mathbb{P}^1 \) is a small neighborhood of \( \lambda = 0 \). According to Proposition 5.5 there exists a trivializing frame \( \tilde{e} \) of \( E|_{(\mathbb{P}^1 \setminus \{0\}) \times \{t^*\}} \), and a matrix \( \tilde{U}^*(\lambda, t^*) \) holomorphic and invertible for all \( \lambda \in D_0 \), s.t.,
\[
\tilde{e}(\lambda, t^*) = f(\lambda, t^*) \tilde{U}^*(\lambda, t^*)^{-1} \lambda^{-K}, \quad \lambda \in D_0,
\]
where \( K = \text{diag}(k_1, \ldots, k_p) \) with \( k_1 \geq \cdots \geq k_p \) integers and we have fixed also a reference path in \( \mathcal{C} \subset \mathbb{P}^1 \times T \) from \( (\lambda^0, t^0) \), which specifies the value \( f(\lambda, t^*) \). Note that at least one \( k_i \neq 0 \), otherwise \( E|_{\mathbb{P}^1 \times \{t^*\}} \) would be trivial, which contradicts the definition of \( \Theta \). Permuting the entries of the frame \( f \) if necessary, we can arrange that the matrix \( \tilde{U}^*(0, t^*) \) has non-vanishing principal minors.

**Remark 5.5.** The matrix \( \tilde{U}^*(\lambda, t^*) \) depends analytically on \( t^* \) if we allow \( t^* \) to vary along a subset \( \Theta_K \subset \Theta \) along which the vector bundle \( E|_{\mathbb{P}^1 \times \{t^*\}} \cong \mathcal{O}(k_1 + \cdots + k_p) \). It is known that \( \Theta_K \) is a constructible subset of \( \Theta \): intersection of a closed and an open subsets.

**Lemma 5.6.** a) There exists a unique matrix \( \Gamma(\lambda, t^*) \) polynomial in \( \lambda^{-1} \) such that \( \Gamma(\lambda, t^*) \) is invertible for \( \lambda \neq 0 \) and
\[
\Gamma(\lambda, t^*) \lambda^K \tilde{U}^*(\lambda, t^*) = U^*(\lambda, t^*) \lambda^K,
\]
where \( U^*(\lambda, t^*) \) is holomorphic and invertible for all \( \lambda \in D_0 \).

b) The matrix
\[
\lambda^{-K}U^*(\lambda, t^*) \lambda^K
\]
is holomorphic and invertible for all \( \lambda \in D_0 \).

**Proof.** To avoid cumbersome notation let us redenote \( V(\lambda, t^*) := \tilde{U}^*(\lambda, t^*) \) and assume that \( K = \text{diag}(c_1 I_1, \ldots, c_s I_s) \), where \( c_1 > \cdots > c_s \) are the eigenvalues of \( K \) and \( I_j \) is the identity matrix of size \( m_j := \) the multiplicity of \( c_j \) in the sequence \( (k_1, \ldots, k_p) \). Given a matrix \( A \) of size \( p \times p \), then we denote by \( A^{lm} \), \( 1 \leq l, m \leq s \) the block in position \( (l, m) \), where the splitting of a \( A \) into blocks is according to the block-diagonal structure of \( K \). We argue by induction on the number of
blocks $s$. Moreover, we are going to prove that $\Gamma$ is block-lower triangular such that the block $\Gamma^{lm}$ is a polynomial in $\lambda^{-1}$ of degree $\leq c_m - c_l$ and the diagonal blocks $\Gamma^{mm}$ are identity matrices.

For $s = 1$, the statements are trivial. For $s > 1$ let us write $K = K' + K''$, where

$$K' = \text{diag}((c_1 - c_{s-1})I_1, \ldots, (c_{s-2} - c_{s-1})I_{s-2}, 0 \cdot I_{s-1}, 0 \cdot Is)$$

and

$$K'' = \text{diag}(c_{s-1}I_1, \ldots, c_{s-1}I_{s-2}, c_{s-1}I_{s-1}, c_sI_s).$$

We have

$$\lambda^KV(\lambda, t^*) = \lambda^{K'}V'(\lambda, t^*)\lambda^{K''},$$

where the matrix $V'(\lambda, t^*) = \lambda^{K''}V(\lambda, t^*)\lambda^{-K''}$ has the form

$$V'(\lambda, t^*) = \begin{bmatrix} A(\lambda, t^*) & B(\lambda, t^*)\lambda^m \\ C(\lambda, t^*)\lambda^{-m} & D(\lambda, t^*) \end{bmatrix},$$

where $m = c_{s-1} - c_s$, $A$ is a $(p - m_s) \times (p - m_s)$ matrix invertible at $\lambda = 0$, and $B(\lambda, t^*)$, $C(\lambda, t^*)$, and $D(\lambda, t^*)$ are holomorphic matrices, whose sizes are uniquely determined from the sizes of $A$ and $V'$. There exists a matrix polynomial

$$R(\lambda, t^*) = \sum_{j=0}^m R_j(t^*)\lambda^j,$$

where $R_j$ is a matrix of size $m_s \times (p - m_s)$, s.t.,

$$(C(\lambda, t^*) + R(\lambda, t^*)A(\lambda, t^*))\lambda^{-m}$$

is holomorphic and vanishing at $\lambda = 0$. The matrices $R_j(t^*)$ are uniquely determined by requiring that the coefficients in front of the non-positive powers of $\lambda$ vanish

$$C_j(t^*) + (R_0(t^*)A_j(t^*) + \cdots + R_j(t^*)A_0(t^*)) = 0, \quad 0 \leq j \leq m,$$

where $C_j(t^*)$ and $A_j(t^*)$ are the coefficients in front of $\lambda^j$ in the Taylor’s expansion at $\lambda = 0$ of respectively $C(\lambda, t^*)$ and $A(\lambda, t^*)$. The assumption about the principal minors of $V(0, t^*)$ implies that $A_0(t^*)$ is invertible (i.e., it is a principal minor of $V(0, t^*)$), so the equations for $R_j(t^*)$ ($0 \leq j \leq m$) can be solved uniquely. Note that the matrix

$$\Gamma_s(\lambda, t^*) := \lambda^{K'} \begin{bmatrix} I & 0 \\ R(\lambda, t^*)\lambda^{-m} & I \end{bmatrix} \lambda^{-K''}$$

is polynomial in $\lambda^{-1}$ and holomorphically invertible for $\lambda \neq 0$. Moreover, the non-zero off diagonal blocks have the form $\Gamma^{si}_s$ and they are polynomials in $\lambda^{-1}$ of degree at most $c_i - c_s$. We have

$$\Gamma_s(\lambda, t^*) \lambda^KV(\lambda, t^*) = \lambda^{K'}V''(\lambda, t^*)\lambda^{K''},$$
where
\[
V''(\lambda, t^*) = \begin{bmatrix}
A(\lambda, t^*) & B(\lambda, t^*)
\end{bmatrix}
\begin{bmatrix}
(C(\lambda, t^*) + R(\lambda, t^*)A(\lambda, t^*))\lambda^{-m} & \lambda^m \\
D(\lambda, t^*) + R(\lambda, t^*)B(\lambda, t^*)
\end{bmatrix}.
\]

Note that \(V''(\lambda, t^*)\) is holomorphic at \(\lambda = 0\). We claim that \(V''(0, t^*)\) has non-vanishing principal minors. In order to prove this we recall that an invertible matrix \(M\) has non-vanishing principal minors if and only if it admits a \(LDU\)-decomposition, i.e., \(M\) can be written as the product of lower-triangular, diagonal, and upper-triangular matrices. Put \(A_0 := A(0, t^*), B_0 := B(0, t^*), C_0 := C(0, t^*), D_0 := D(0, t^*)\) and note that \(R_0(t^*) = R(0, t^*) = -C_0A_0^{-1}\). We get
\[
V''(0, t^*) = \begin{bmatrix}
A_0 & 0 \\
D_0 - C_0A_0^{-1}B_0 & 0
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
-C_0A_0^{-1} & I
\end{bmatrix} V(0, t^*) \begin{bmatrix}
I & -A_0^{-1}B_0 \\
0 & I
\end{bmatrix},
\]

so the matrix \(V''(0, t^*)\) has a \(LDU\)-decomposition, because according to our assumptions \(V(0, t^*)\) has non-vanishing principal minors, which implies that \(V(0, t^*)\) has a \(LDU\)-decomposition.

Recalling the inductive assumption we find a matrix \(\Gamma'(\lambda, t^*)\), s.t., \(\Gamma'(\lambda, t^*)\) is polynomial in \(\lambda^{-1}\), invertible for \(\lambda \neq 0\), and
\[
\Gamma'(\lambda, t^*) \lambda K' V''(\lambda, t^*) = U^*(\lambda, t^*) \lambda K',
\]

where \(U^*(\lambda, t^*)\) is a matrix holomorphically invertible in a neighborhood of \(\lambda = 0\). We claim that the matrix \(\Gamma(\lambda, t^*) := \Gamma'(\lambda, t^*)\Gamma_s(\lambda, t^*)\) satisfies all the required properties. In fact, the only thing left to check is that the degree of the block \(\Gamma^m_l\) as a polynomial in \(\lambda^{-1}\) is at most \(c_m - c_l\). We have
\[
\Gamma^m_l = \sum_{k=m}^{l} (\Gamma')^k \Gamma_s^m.
\]

If \(l \leq s - 1\), then \(\Gamma_s^m \neq 0\) only for \(k = m\), so \(\Gamma^m_l = (\Gamma')^m\). Recalling the inductive assumption for \(\Gamma'\) we get that the degree in \(\lambda^{-1}\) does not exceed
\[
c_m - c_l' = (c_m - c_{l-1}) - (c_l - c_{l-1}) = c_m - c_l.
\]

The case \(l = m = s\) is trivial. If \(l = s\) and \(m = s - 1\). Then since the \((s-1, s-1)\)-block of \(\Gamma'\) with respect to the block-matrix structure of \(K'\) has the form
\[
\begin{bmatrix}
(\Gamma')^{s-1, s-1} \\
(\Gamma')^{s, s-1} \\
(\Gamma')^{s, s}
\end{bmatrix}
\]

and by inductive assumption this should be an identity matrix, we get that \((\Gamma')^{s, s-1} = 0\) and \((\Gamma')^{s, s}\) is an identity matrix. Therefore \(\Gamma^m_l = \Gamma_s^{s, s-1}\) so the degree of this matrix as polynomial in \(\lambda^{-1}\) as we proved above is \(\leq c_{s-1} - c_s = c_m - c_l\). Finally, if \(l = s\) and \(m \leq s - 2\), then \(\Gamma_s^{km} \neq 0\) either if \(k = m\) or if \(k = s\), i.e.,
\[
(10) \Gamma^{sm} = (\Gamma')^{sm} + \Gamma_s^{sm}.
\]
Note that the \((s - 1, m)\)-block of \(\Gamma'\) with respect to the block-matrix structure of \(K'\) has the form

\[
\begin{bmatrix}
(\Gamma')^{s-1,m} \\
(\Gamma')^{s,m}
\end{bmatrix}.
\]

The inductive assumption about the degree with respect to \(\lambda^{-1}\) implies that the degree of \((\Gamma')^{s,m}\) does not exceed \(c'_m - c'_{s-1} = c'_m - c_{s-1} < c_m - c_s\). Therefore, both terms on the RHS of (10) are polynomials in \(\lambda^{-1}\) of degree \(\leq c_m - c_s\), which completes the proof of the inductive assumption and hence of part a) as well.

For part b), it is enough to check that

\[
\lambda^{-K} \Gamma(\lambda, t^*) \lambda^K
\]

is holomorphically invertible at \(\lambda = 0\), but this follows easily if we recall the block-lower triangular structure of \(\Gamma\) combined with the degree estimates of the blocks \(\Gamma^{lm}\). \(\square\)

Let us define \(e(\lambda, t^*) = \tilde{e}(\lambda, t^*) \Gamma(\lambda, t^*)^{-1} C(t^*)^{-1}\), where \(C(t^*)\) is a constant invertible matrix. The choice of \(C(t^*)\) will be specified below. We have the following relation

\[
f(\lambda, t^*) = e(\lambda, t^*) C(t^*) U^*(\lambda, t^*) \lambda^K, \quad \lambda \in D_0.
\]

Let us denote by \(Y^*(\lambda)\) the multivalued function on \(\mathbb{P}^1 - \{0, u^*_1, \ldots, u^*_N, \infty\}\) (here \(u^* \in Z_N\) is the projection of \(t^*\)) with matrix values defined by

\[
f(\lambda, t^*) = e(\lambda, t^*) Y^*(\lambda).
\]

In particular, if \(\lambda\) is close to 0 we have

\[
Y^*(\lambda) = C(t^*) U^*(\lambda) \lambda^K.
\]

The local forms of \(Y^*(\lambda)\) near the remaining singularities are

\[
Y^*(\lambda) = U^*_i(\lambda) (\lambda - u^*_i)^{K_i} (\lambda - u^*_i) E_i S_i^{-1},
\]

if \(\lambda\) is close to \(u^*_i\) \((1 \leq i \leq N)\) and

\[
Y^*(\lambda) = U^*_{N+1}(\lambda, t^*) \lambda^{-K_{N+1}} \lambda^{-E_{N+1}} S_{N+1}^{-1},
\]

if \(\lambda\) is close to \(u^*_{N+1} := \infty\). Here the matrices \(U^*_j(\lambda)\) are holomorphically invertible near \(\lambda = u^*_j\) for \(1 \leq j \leq N + 1\). Changing the matrix \(C(t^*)\) if necessary we can arrange that the Taylor’s series of \(U^*_{N+1}(\lambda)\) has a constant term \(U^*_{N+1}(\infty) = 1\).

Using the above expansions, it is easy to verify that

\[
(11) \quad \partial_\lambda Y^*(\lambda) = \left(\frac{A^*_0}{\lambda} d\lambda + \sum_{i=1}^N \frac{A^*_i}{\lambda - u^*_i}\right) Y^*(\lambda).
\]
Shrinking the neighborhood $U$ of $t^*$ if necessary we may assume that the Schlesinger equations

$$d\hat{A}_i = \sum_{j \neq i, 0 \leq j \leq N} \frac{[\hat{A}_j, \hat{A}_i]}{u_j - u_i} (du_j - du_i), \quad 0 \leq i \leq N,$$

$$\hat{A}_i(0, u^*_1, \ldots, u^*_N) = A^*_i$$

have holomorphic solutions $\hat{A}_i(u_0, u_1, \ldots, u_N)$ defined for all $u_0$ close to 0 and for all $(u_1, \ldots, u_N) \in U$. Then we define

$$A_i(u_1, \ldots, u_N) := \hat{A}_i(0, u_1, \ldots, u_N).$$

In other words we have constructed an isomonodromic deformation that keeps the singular point 0 fixed. The system

$$\partial_\lambda Y(\lambda, u) = \left( \frac{A_0(u)}{\lambda} + \sum_{i=1}^N \frac{A_i(u)}{\lambda - u_i} \right) Y(\lambda, u),$$

$$\partial_{u_i} Y(\lambda, u) = -\frac{A_i(u)}{\lambda - u_i} Y(\lambda, u), \quad 1 \leq i \leq N,$$

has a unique solution $Y(\lambda, u)$ satisfying the initial condition $Y(\lambda, u^*) = Y^*(\lambda)$, where $u^* = (u^*_1, \ldots, u^*_N) \in Z_N$ is the projection of $t^*$. Finally, according to the remark of Section 4.6 the deformation $Y(\lambda, u)$ has the following local expansions

$$Y(\lambda, u) = U_{N+1}(\lambda, u)\lambda^{-K_{N+1}}\lambda^{-E_{N+1}}S_{N+1}^{-1}$$

for $\lambda$ near $\infty$,

$$Y(\lambda, u) = U_i(\lambda, u)(\lambda - u_i)^{K_i}(\lambda - u_i)^{E_i}S_i^{-1}, \quad 1 \leq i \leq N,$$

for $\lambda$ near $u_i$, and

$$Y(\lambda, u) = U(\lambda, u)\lambda^K$$

for $\lambda$ near 0.

Let us express the coefficients of the Fuchsian connection in terms of the coefficients of the Taylor's series expansion of $U(\lambda, u)$. Substituting

$$Y(\lambda, u) = (U_0(u) + U_1(u) \lambda + \cdots) \lambda^K$$

in the differential equations and comparing the coefficients in front of $\lambda$ we get the following relations

$$A_0(u) = U_0(u)KU_0(u)^{-1},$$

and

$$u_i\partial_{u_i}(U_0^{-1}U_1) = \frac{U_0^{-1}A_iU_0}{u_i}, \quad 1 \leq i \leq N,$$

and

$$U_0^{-1}U_1 + [U_0^{-1}U_1, K] = -\sum_{i=1}^N \frac{U_0^{-1}A_iU_0}{u_i}. $$
More generally, comparing the coefficients in front of $\lambda^{k-1}$ for $k > 0$ yields

$$u_i \partial_u (U_0^{-1} U_k) = (U_0^{-1} A_i U_0) \sum_{s=1}^k u_i^{-s} U_0^{-1} U_{k-s}, \quad 1 \leq i \leq N,$$

$$(k - \text{ad}_K)(U_0^{-1} U_k) = - \sum_{i=1}^N (U_0^{-1} A_i U_0) \sum_{s=1}^k u_i^{-s} U_0^{-1} U_{k-s}. \quad 1 \leq i \leq N.$$ 

5.3. **Gauge transformations.** Bolibruch has introduced gauge transformations of the following type. Let $g(u)$ be an entry of $U_0^{-1} U_1$, and denote its position by $(\alpha, \beta)$. Put

$$\Gamma_1(\lambda, u) := I - \frac{U_0(u) E_{\beta \alpha} U_0(u)^{-1}}{g(u) \lambda},$$

where $E_{\beta \alpha}$ is the matrix with 1 on position $(\beta, \alpha)$ and 0 elsewhere. The gauge transformation $Y_1(\lambda, u) = \Gamma_1(\lambda, u) Y(\lambda, u)$ yields an isomonodromic system of differential equations corresponding to a Schlesinger deformation

$$\partial_\lambda Y_1(\lambda, u) = \left( \frac{A^1_0(u)}{\lambda} + \sum_{i=1}^N \frac{A^1_i(u)}{\lambda - u_i} \right) Y_1(\lambda, u),$$

$$\partial_u Y_1(\lambda, u) = - \frac{A^1_i(u)}{\lambda - u_i} Y_1(\lambda, u), \quad 1 \leq i \leq N,$$

where the matrices $A^1_i$ are given by the following formulas. Put

$$N_{\alpha \beta} = U_0(u) E_{\beta \alpha} U_0(u)^{-1}$$

and note that this is a nilpotent matrix $N_{\alpha \beta}^2 = 0$. Then

$$A^1_i(u) = \left( I - \frac{N_{\alpha \beta}}{g(u) u_i} \right) A_i \left( I + \frac{N_{\alpha \beta}}{g(u) u_i} \right), \quad 1 \leq i \leq N,$$

and

$$A^0_1(u) = A_0 + \sum_{i=1}^N (A_i - A^1_i).$$

The local form of the expansion of $Y_1(\lambda, u)$ at the singular points $\lambda = u_i$, $1 \leq i \leq N + 1$ is the same as for $Y(\lambda, u)$. Let us define $U^1(\lambda, u)$ such that

$$Y_1(\lambda, u) =: U^1(\lambda, u) \lambda K^1,$$

$K^1 = \text{diag}(k^1_1, \ldots, k^1_p) = K - E_{\alpha \alpha} + E_{\beta \beta}$, i.e.,

$$k^1_i = \begin{cases} k_i - 1, & \text{if } i = \alpha, \\ k_i + 1, & \text{if } i = \beta, \\ k_i, & \text{otherwise}. \end{cases}$$

The properties of the matrix $U^1$ are summarized in the following lemma.
Lemma 5.7. The local expansion at $\lambda = 0$ of $U^1(\lambda, u)$ has the form

\[ U^1(\lambda, u) = U^1_0(u) + U^1_1(u)\lambda + U^1_2(u)\lambda^2 + \cdots, \]

where $U^1_0(u)$ is holomorphic and invertible for all $u \in U$ such that $g(u) \neq 0$.

Proof. By definition

\[ U^1(\lambda, u) = \Gamma^1(\lambda, u)U(\lambda, u)\lambda^{E_{\alpha\alpha} - E_{\beta\beta}}. \]

Substituting the expansions at $\lambda = 0$ we get a Laurent series that has a pole of order at most 2. It is easy to see that the coefficients in front of $\lambda^{-2}$ and $\lambda^{-1}$ are 0. While the remaining coefficients are as follows.

\[ U^1_0(u) = U_0 \left( \sum_{i: i \neq \alpha, \beta} E_{ii} + U_0^{-1}U_1 E_{\beta\beta} + \frac{1}{g}(E_{\beta\alpha} + \sum_{i: i \neq \alpha, \beta} (U_0^{-1}U_0 E_{\beta\alpha}) (U_0^{-1}U_1)_{\alpha i} + (U_0^{-1}U_2)_{\alpha \beta} E_{\beta\beta}) \right) \]

and

\[ U^1_k(u) = U_0 \left( U_0^{-1}U_{k-1} E_{\alpha\alpha} + \sum_{i: i \neq \alpha, \beta} U_0^{-1}U_0 E_{ii} + U_0^{-1}U_{k+1} E_{\beta\beta} + \frac{1}{g}(U_0^{-1}U_k)_{\alpha\alpha} E_{\beta\alpha} + \sum_{i: i \neq \alpha, \beta} (U_0^{-1}U_{k+1})_{\alpha i} E_{\beta\alpha} + (U_0^{-1}U_{k+2})_{\alpha \beta} E_{\beta\beta}) \right), \]

where we have denoted by $(U_0^{-1}U_0)_{ab}$ the $(a, b)$-entry of the matrix $U_0^{-1}U_0$.

We have to prove that the matrix $U^1_0$ is invertible. Note that

\[ U^1_0 = U_0 g^{E_{\alpha\alpha} - E_{\beta\beta}} \bar{U}_0^1 \]

where the matrix $\bar{U}_0^1$ has the form

\[
\begin{bmatrix}
1 & b_1 & \cdots & b_{\alpha-1} \\
\vdots & 1 & 0 & b_{\alpha-1} \\
\cdots & 1 & 1 & b_{\alpha-1} \\
1 & b_{\alpha+1} & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots \\
a_1 & \cdots & a_{\alpha-1} & -1 & a_{\alpha+1} & \cdots & a_{\beta-1} & x & a_{\beta+1} & \cdots & a_p \\
b_{\beta+1} & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
b_p & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix}
\]

The entries

\[ a_i := -(U_0^{-1}U_1)_{\alpha i}, \quad i \neq \alpha, \beta, \]
\[ b_i := (U_0^{-1} U_1)_{i\beta}, \quad i \neq \alpha, \beta, \]

and \( x = -(U_0^{-1} U_2)_{\alpha\beta} + g(U_0^{-1} U_1)_{\beta\beta}. \) The inverse of the matrix \( \tilde{U}_0^1 \) is straightforward to compute. The answer is the following

\[
(U_0^1)^{-1} = \begin{bmatrix}
1 & -b_1 & \cdots & \cdots & -b_{\alpha-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
a_1 & \cdots & a_{\alpha-1} & y & a_{\alpha+1} & \cdots & a_{\beta-1} & -1 & a_{\beta+1} & \cdots & a_p \\
-\beta_{\alpha+1} & 1 & \cdots & \cdots & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-\beta_{\beta+1} & \cdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
\end{bmatrix},
\]

where

\[ y = -x - \sum_{i: i \neq \alpha, \beta} a_i b_i. \]

The proof of the Lemma is complete.

**Lemma 5.8.** If the position \((\alpha, \beta)\) of \(g\) satisfies \(k_\alpha - k_\beta > 1\), then \(g(t^*) = 0\).

**Proof.** By construction \(U(\lambda, u^*) = C(t^*) U^*(\lambda)\). Therefore \(U_0(u^*)^{-1} U_1(u^*) = (U_0^*)^{-1} U_1^*, \) where \(U_i^*\) is the coefficient in front of \(\lambda^i\) of the Taylor’s series expansion of \(U^*(\lambda)\) at \(\lambda = 0\). Recalling Lemma 5.6, part b) we get that

\[
(U_0^*)_{ab} = 0 \quad \text{if} \quad k_a - k_b > \ell, 
\]

where \(A_{ab}\) denotes the \((a, b)\)-entry of \(A\). This implies that the non-zero entries of \((U_0^*)^{-1}\) are in positions \((a, b)\) such that \(k_a - k_b \leq 0\). We have

\[
g(t^*) = ((U_0^*)^{-1} U_1^*)_{\alpha\beta} = \sum_{m=1}^{p} ((U_0^*)^{-1})_{\alpha m} (U_1^*)_{m\beta}. 
\]

The only non-zero terms could be for \(m\) such that \(k_a - k_m \leq 0\) and \(k_m - k_\beta \leq 1\). However such \(m\) do not exist otherwise \(k_\alpha - k_\beta \leq 1\). \(\square\)

Let us determine how the 1-form

\[
\omega^* := \frac{1}{2} \sum_{i=0}^{N} \sum_{j: j \neq i} \frac{\text{tr}(A_i(u) A_j(u))}{u_i - u_j} (du_i - du_j)
\]

changes under the gauge transformation. Recall that we are working only with deformations that keep \(u_0\) fixed, i.e., \(u_0 = 0\).
Lemma 5.9. The 1-form
\[ \omega^1 := \frac{1}{2} \sum_{i=0}^{N} \frac{\text{tr}(A_1^i(u)A_1^j(u))}{u_i - u_j} (du_i - du_j) \]
satisfies
\[ \omega^1 = \omega^* + d \log g(u). \]

Proof. The form \( \omega^1 \) can be written also as
\[ \frac{1}{2} \sum_{i=1}^{N} du_i \text{Res}_{\lambda = u_i} \left( \sum_{j=0}^{N} \frac{A_1^j(u)}{\lambda - u_j} \right)^2 \]
The sum over \( j \) is precisely \( \partial_\lambda Y_1(\lambda, u) Y_1(\lambda, u)^{-1} \). Furthermore, recalling the gauge transformation and using some elementary properties of the trace operation we get
\[ \text{tr} \left( \partial_\lambda Y_1 Y_1^{-1} \right)^2 = \text{tr} \left( (\partial_\lambda \Gamma_1 \Gamma_1^{-1})^2 + 2\Gamma_1^{-1} \partial_\lambda \Gamma_1 \partial_\lambda YY^{-1} + (\partial_\lambda YY^{-1})^2 \right). \]
Substituting the formula for \( \Gamma_1 = 1 - (U_0 E_{\beta \alpha} U_0^{-1}) g^{-1} \lambda^{-1} \) we get
\[ \frac{1}{2} \text{tr} \left( \partial_\lambda Y_1 Y_1^{-1} \right)^2 - \frac{1}{2} \text{tr} \left( \partial_\lambda YY^{-1} \right)^2 = \frac{1}{g} \sum_{j=0}^{N} \text{tr} \left( U_0 E_{\beta \alpha} U_0^{-1} \frac{A_j(u)}{\lambda^2(\lambda - u_j)} \right). \]
The residue of the above function at \( \lambda = u_i \) is
\[ \text{tr} \left( U_0 E_{\beta \alpha} U_0^{-1} \frac{A_j}{u_i^2} \right) = \frac{(U_0^{-1} A_i U_0)_{\alpha \beta}}{u_i^2} = \partial_{u_i} g, \]
where in the last equality we used formula (17). \( \square \)

5.4. Proof of Lemma 5.11 Let us split the matrix \( K \) into a block diagonal form \( \text{diag}(c_1 I_1, \ldots, c_s I_s) \), where \( I_j \) is the identity matrix of size equal to the multiplicity of the number \( c_j \) in the sequence \( (k_1, \ldots, k_p) \). If \( A \) is a \( p \times p \) matrix, then we split it into blocks according to the block-diagonal structure of \( K \) and denote by \( A^{lm} \) the block in position \( (l, m) \).

Lemma 5.10. If \( c_l - c_m > 1 \), then at least one entry among the entries of the blocks \( (U_0^{-1} A_i U_0)^{lm} \), \( 1 \leq i \leq N \), is not identically 0.

Proof. Assume that this is not true, i.e., \( (U_0^{-1} A_i U_0)^{lm} = 0 \) for all \( i = 1, 2, \ldots, N \) and all \( l \) and \( m \), s.t., \( c_l - c_m > 1 \). Let us make a gauge transformation
\[ \tilde{Y}(\lambda, u) = \lambda^{-K} U_0(u)^{-1} Y(\lambda, u). \]
We get the following differential equation
\[ \partial_\lambda \tilde{Y}(\lambda, u) = \left( \sum_{i=1}^{N} \lambda^{-K} \frac{U_0^{-1} A_i U_0}{\lambda - u_i} \lambda^K \right) \tilde{Y}(\lambda, u). \]
Our assumption implies that the above system is Fuchsian at \( \lambda = 0 \) and that the coefficients \( \tilde{B}_0 \) in front of \( \lambda^{-1} \) is a nilpotent matrix. This would imply that the monodromy around \( \lambda = 0 \) is \( e^{2\pi \sqrt{-1} \tilde{B}_0} = 1 \) (see Corollary 2.3), i.e., \( \tilde{B}_0 = 0 \).

Therefore, the matrix \( \tilde{Y}(\lambda, u) \) is regular at \( \lambda = 0 \) for all \( u \) sufficiently close to \( u^* \).

Note that \( f Y(\lambda, u)^{-1} U_0(u) \) is a frame for the vector bundle \( E|_{\mathbb{P}^1 \times \{u\}} \) on \( \mathbb{P}^1 \setminus \{0\} \), while \( f \tilde{Y}(\lambda, u)^{-1} \) is a frame for \( E|_{\mathbb{P}^1 \times \{u\}} \) near \( \lambda = 0 \). The relation

\[
f Y(\lambda, u)^{-1} U_0(u) = (f \tilde{Y}(\lambda, u)^{-1}) \lambda K
\]

implies that \( E_{\mathbb{P}^1 \times \{u\}} \cong E_{\mathbb{P}^1 \times \{u^*\}} \) is a non-trivial vector bundle for all \( u \) in a neighborhood of \( u^* \), which contradicts the fact that \( \Theta \) is at most a hypersurface.

The above lemma implies that at least one entry in \((U_0^{-1}U_1)^{lm}\) with \( c_l - c_m > 1 \) is not identically 0. Let us choose such an entry \( g(u) \) and let \((\alpha, \beta)\) be its position. By definition \( k_\alpha = c_\ell \) and \( k_\beta = c_m \) so \( k_\alpha - k_\beta > 1 \). We apply the gauge transformation

\[
Y_1(\lambda, u) = \Gamma_1(\lambda, u) Y(\lambda, u), \quad \Gamma_1 = 1 - (U_0 E_{\beta\alpha} U_0^{-1}) g^{-1} \lambda^{-1}.
\]

The resulting matrix-valued function is a fundamental solution to a Fuchsian system that has the same type of expansion at \( \lambda = u_i \) for \( i = 1, \ldots, N+1 \) while at \( \lambda = 0 \) we have \( Y_1(\lambda, u) = U^1(\lambda, u) \lambda K^1 \) (see Section 5.3). Note that

\[
\text{Tr}((K^{1})^2) - \text{Tr}((K)^2) = 2 - 2k_\alpha + 2k_\beta \leq -2.
\]

Repeating this process we get a sequence of fundamental matrices

\[
Y_i(\lambda, u) = U^i(\lambda, u) \lambda K^i,
\]

satisfying \( \text{Tr}((K^i)^2) < \text{Tr}((K^{i-1})^2) \). Therefore, the sequence stops after finitely many steps when \( K^s = 0 \) for some \( s \). Let us denote by \( g_\ell(u) \) the non-zero entry of \((U_0^{-1})^{-1} U_1^{\ell-1}\) that we choose in order to construct \( Y_\ell \). The function \( g_1 := g \in O_{T,t^*} \) is holomorphic, but the remaining ones are meromorphic at \( t^* \), i.e., \( g_\ell \in \text{Frac}(O_{T,t^*}) \) (\( 2 \leq \ell \leq s \)), where for an integral domain \( R \) we denote by \( \text{Frac}(R) \) the quotient field of \( R \). Put \( g_\ell = b_\ell / h_\ell \), where \( b_\ell, h_\ell \in O_{T,t^*} \) are relatively prime (recall that \( O_{T,t^*} \) is a UFD). According to our construction \( h_\ell = 1 \) and for each \( \ell > 1 \) there exists an integer \( m_\ell \geq 0 \) such that \( h_\ell \) is a divisor of \( (b_1 \cdots b_{\ell-1})^{m_\ell} \).

Shrinking the neighborhood \( U \) of \( u^* \) if necessary we may assume that \( b_\ell \) are represented by holomorphic functions in \( U \). Put \( \tau^*(u) := b_1(u) \cdots b_\ell(u) \). Then since \( K^s = 0 \) we get that \( f Y_s(\lambda, u)^{-1} \) is a global trivializing frame for \( E|_{\mathbb{P}^1 \times \{u\}} \) for all \( u \) such that \( \tau^*(u) \neq 0 \). In particular, the analytic germ

\[
(\Theta, t^*) \subset \{ \tau^*(t) = 0 \}.
\]

On the other hand if \( \tilde{Y}(\lambda, u) \) is the fundamental solution for the Schlesinger deformation of \( \nabla^\circ \) satisfying the initial condition \( Y(\lambda, t^*) = Y^0(\lambda) \), then \( f \tilde{Y}(\lambda, u)^{-1} \) is also a global trivialization of \( E|_{\mathbb{P}^1 \times \{u\}} \) for all \( u \notin \Theta \). Therefore \( Y_s(\lambda, u) = C \tilde{Y}(\lambda, u) \) for some constant invertible matrix \( C \). Therefore the connection 1-form of the Fuchsian connection corresponding to \( Y_s \) is conjugated with the connection...
1-form corresponding to \( \tilde{Y} \) via \( C \). We get that \( \omega^s \) coincides with the 1-form \( \omega \). Recalling Lemma 5.9 we get
\[
\omega = \omega^s = \omega^s + \frac{d(g_1 \cdots g_s)}{g_1 \cdots g_s}.
\]
It remains only to factorize \( g_1(u) \cdots g_s(u) = \tau_1^*(u)^{r_1} \cdots \tau_s^*(u)^{r_s} h(u) \) where \( \tau_i^* = 0 \) are the local equations of the irreducible components of \( \Theta \) at \( t^* \) and \( h \) is relatively prime to \( \tau_1^* \cdots \tau_s^* \). Note that \( h(u^*) \neq 0 \), otherwise the form \( \omega \) will have a pole along the hypersurface \( \{ h = 0 \} \) which is not contained in \( \Theta \). Lemma 5.1 follows.

Since \( g_1 \) is holomorphic, if the algorithm stops on the first step we would get that the isomonodromic tau-function is analytic at \( t^* \). Therefore we get the following corollary.

**Corollary 5.11.** If \( E|_{\mathbb{P}^1 \times \{t^* \}} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{p-2} \) for a generic \( t^* \in \Theta \), then the isomonodromic tau-function is analytic on \( T \).

**Remark 5.12.** One can prove that if \( t^* \in \Theta \) is a smooth point and we choose \( g_1 \) appropriately, then \( g_2 \) is also holomorphic. The analysis of the poles of \( g_\ell \) for \( \ell > 2 \) however becomes more difficult.

**Remark 5.13.** According to Lemma 5.8 the subset \( \Theta_K \subset \Theta \) (see Remark 5.5) is contained in the zero locus of \( b_1 = g_1 \) (the function generated on the first step of the algorithm). In particular, if \( t^* \in \Theta \) is generic (so that \( \Theta_K \) is open in \( \Theta \)) then the first step of the algorithm already determines the germ of \( \Theta \) at \( t^* \).

### 6. Frobenius manifolds

The main goal of this lecture is to recall the notion of a semi-simple Frobenius manifold and to prove that semi-simple Frobenius manifolds can be classified by solutions of a certain system of PDEs. The general reference for more details is [12] (see also [10]).

#### 6.1. Definition

There are several ways to introduce the notion of a Frobenius manifold. We have chosen a set of axioms convenient for our purposes. Our definition is equivalent to (Definition 1.2 in [3]). Let \( M \) be a complex manifold and \( \mathcal{T}_M \) denotes the sheaf of holomorphic vector fields on \( M \). Let us assume that \( M \) is equipped with the following structures

(a) Each tangent space \( T_t M, t \in M \), is equipped with the structure of a *Frobenius algebra* depending holomorphically on \( t \). In other words, we have a commutative associative multiplication \( \bullet_t \) and symmetric non-degenerate bi-linear pairing \( (\ , \ )_t \) satisfying the Frobenius property
\[
(v_1 \bullet_t w, v_2) = (v_1, w \bullet_t v_2), \quad v_1, v_2, w \in T_t M.
\]

The pointwise multiplication \( \bullet_t \) defines a multiplication \( \bullet \) in \( \mathcal{T}_M \), i.e., a \( \mathcal{O}_M \)-bilinear map
\[
\mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M, \quad v_1 \otimes v_2 \mapsto v_1 \bullet v_2.
\]
The pairing $(\cdot, \cdot)_t$ determine a $\mathcal{O}_M$-bilinear pairing

$$(\cdot, \cdot)_t : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M.$$  

(b) There exists a global vector field $e \in \mathcal{T}_M$, called *unit vector field*, such that

$$\nabla^\text{L.C.}_v e = 0, \quad e \cdot v = v, \quad \forall v \in \mathcal{T}_M,$$

where $\nabla^\text{L.C.}$ is the Levi–Civita connection on $\mathcal{T}_M$ corresponding to the bi-linear pairing $(\cdot, \cdot)$.

(c) There exists a global vector field $E \in \mathcal{T}_M$, called *Euler vector field*, such that

$$E(v_1, v_2) - ([E, v_1], v_2) - (v_1, [E, v_2]) = (2 - D)(v_1, v_2),$$

for all $v_1, v_2 \in \mathcal{T}_M$ and for some constant $D \in \mathbb{C}$.

The above data allows us to define the so-called *structure connection* $\nabla$ on the vector bundle $\text{pr}_M^* \mathcal{T} \mathcal{M} \to M \times \mathbb{C}^*$, where $\text{pr}_M : M \times \mathbb{C}^* \to M, \quad (t, z) \mapsto t$ is the projection map. Namely,

$$\nabla_v := \nabla^\text{L.C.}_v - z^{-1}v \cdot, \quad v \in \mathcal{T}_M$$

$$\nabla_{\partial/\partial z} := \frac{\partial}{\partial z} - z^{-1}\theta + z^{-2}E \cdot,$$

$v \cdot$ and $E \cdot$ are $\mathcal{O}_M$-linear maps $\mathcal{T}_M \to \mathcal{T}_M$ corresponding to the Frobenius multiplication by respectively $v$ and $E$. The $\mathcal{O}_M$-linear map $\theta : \mathcal{T}_M \to \mathcal{T}_M$ is defined by

$$\theta(v) := \nabla^\text{L.C.}_v E - (1 - D/2)v.$$

The operator $\theta$ is sometimes called *Hodge grading operator*. Let us point out that the term $(1 - D/2)v$ in the definition of $\theta(v)$ is inserted so that $\theta$ becomes skew-symmetric with respect to the Frobenius pairing

$$(\theta(v_1), v_2) + (v_1, \theta(v_2)) = 0, \quad v_1, v_2 \in \mathcal{T}_M.$$  

**Definition 6.1.** The data $((\cdot, \cdot), \cdot, e, E)$ satisfying the conditions (a), (b), and (c) from above is said to be a *Frobenius structure* on $M$ of conformal dimension $D$ if the structure connection $\nabla$ is flat.

**6.2. Properties.** The following proposition is a direct consequence of the definition.

**Proposition 6.2.** Suppose that $(M, (\cdot, \cdot), \cdot, e, E)$ is a Frobenius structure. Then

a) The Levi–Civita connection $\nabla^\text{L.C.}$ is flat.

b) Let $t = (t_1, \ldots, t_N)$ be $\nabla^\text{L.C.}$-flat coordinates defined on a contractible open subset $U \subset M$. There exists a holomorphic function $F \in \mathcal{O}_M(U)$, such that

$$(\partial/\partial t_a \cdot \partial/\partial t_b, \partial/\partial t_c) = \frac{\partial^3 F}{\partial t_a \partial t_b \partial t_c}$$
and
\[ EF = (3 - D)F + H, \]
where \( H \) is a polynomial in \( t_1, \ldots, t_N \) of degree at most 2.

c) The Hodge grading operator is covariantly constant: \( \nabla^{L.C.}_v \theta = 0 \). In particular, in flat coordinates \( t = (t_1, \ldots, t_N) \) the matrix \( (\theta_{ab})^{N}_{a,b=1} \) of \( \theta \) defined by
\[ \theta(\partial/\partial t_b) = \sum_{a=1}^{N} \theta_{ab} \partial/\partial t_b \]
is constant.

d) The following identity holds
\[ [E, v \bullet w] - [E, v] \bullet w - v \bullet [E, w] = v \bullet w, \quad v, w \in \mathcal{T}_M. \]

Proof. Parts a) and b) are straightforward. We will prove c) and d) simultaneously. To begin with note that both c) and d) are \( \mathcal{O}_M \)-linear in \( v \) and \( w \).
Therefore, we may assume \( v \) and \( w \) are flat with respect to \( \nabla^{L.C.} \).

The flatness of \( \nabla \) implies that
\[ \nabla_{z\partial_z + E} \nabla_v w - \nabla_v \nabla_{z\partial_z + E} w - \nabla_{[E, v]} w = 0. \]

By definition
\[ \nabla_v = \nabla^{L.C.}_v - z^{-1} v \bullet \]
and
\[ \nabla_{z\partial_z + E} = z\partial_z + \nabla^{L.C.}_E - \theta. \]

Substituting these operators in the 0-curvature equation and using that \( v \) and \( w \) are flat we get a polynomial expression in \( z \) of degree 1 for which the coefficient in front of \( z^0 \) is
\[ \nabla^{L.C.}_v \theta(w) \]
and the coefficient in front of \( z^{-1} \) is
(18) \[ v \bullet w + [E, v] \bullet w - v \bullet \theta(w) + \theta(v \bullet w) - \nabla^{L.C.}_E(v \bullet w). \]

Therefore both expressions must vanish. The vanishing of \( \nabla^{L.C.}_v \theta(w) \) for all flat vector fields \( v \) and \( w \) is equivalent to the statement in c). For the 2nd expression, using the definition of \( \theta \) we get
\[ -v \bullet \theta(w) + \theta(v \bullet w) = v \bullet [E, w] - [E, v \bullet w] + \nabla^{L.C.}_E(v \bullet w). \]

Substituting this identity in (18) we get the identity of part d). \( \square \)

Note that locally the Frobenius structure is completely determined by the Euler vector field \( E \) and the holomorphic function \( F \). It is possible to state the definition in terms of \( F \) as well (see [3]). This leads to the so-called WDVV equations for \( F \). In many applications the Frobenius structures arise as solutions of the WDVV equations. However, in our lectures this point of view would not play an important role.
6.3. Example: quantum cohomology. Let $X$ be a smooth projective variety. Recall that a stable map $(\Sigma, z_1, \ldots, z_n; f)$ is a holomorphic map $f : \Sigma \to X$, where $\Sigma$ is a nodal Riemann surface, $z_i$ are marked points (pairwise distinct and nonsingular), such that the automorphism group of $(\Sigma, z_1, \ldots, z_n; f)$ is finite. The homology class $d = f_\ast[\Sigma] \in H_2(X, \mathbb{Z})$ is called the degree of the stable map. Let us denote by $M_{g,n}(X, d)$ the moduli space of stable maps $(\Sigma, z_1, \ldots, z_n; f)$ such that the arithmetic genus of $\Sigma$ is $g$, the number of marked points is $n$, and the degree of $f$ is $d$. This is a proper Deligne–Mumford stack equipped with a virtual fundamental cycle $[X_{g,n,d}]$ of dimension (over $\mathbb{C}$)

$$3g - 3 + n + (1 - g)D + \int_{[X]} c_1(TX),$$

where $D = \dim\mathbb{C}(X)$. The Gromov–Witten invariants of $X$ are defined by the following correlators

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,d} = \int_{[X_{g,n,d}]} \text{ev}^\ast(\alpha_1, \ldots, \alpha_n), \quad \alpha_i \in H^\ast(X; \mathbb{C}),$$

where

$$\text{ev} : M_{g,n}(X, d) \to X^n, \quad (\Sigma, z_1, \ldots, z_n; f) \mapsto (f(z_1), \ldots, f(z_n))$$

is the evaluation map.

In order to define quantum cohomology we need also to recall the definition of the Novikov ring. The degrees of stable maps form a cone in $H_2(X, \mathbb{Z})$ usually denoted by $\text{Eff}(X)$. The Novikov ring is by definition the formal group algebra of $\text{Eff}(X)$, i.e.,

$$\mathbb{C}[Q] := \left\{ \sum_{d \in \text{Eff}(X)} c_d Q^d \mid c_d \in \mathbb{C} \right\}.$$

Let us fix a set of ample line bundles $L_1, \ldots, L_r$ on $X$, such that $p_i := c_1(L_i)$ form a $\mathbb{C}$-basis of $H^{1,1}(X; \mathbb{C})$. Then the map

$$Q^d \mapsto Q_1^{(p_1,d)} \cdots Q_r^{(p_r,d)}$$

gives an embedding

$$\mathbb{C}[Q] \to \mathbb{C}[Q_1, \ldots, Q_r].$$

Let $H^{\text{ev}}(X; \mathbb{C}) := \bigoplus_{d=0}^{\infty} H^{2d}(X; \mathbb{C})$ and let us fix a homogeneous basis $\{\phi_i\}_{i=1}^N$ of $H^{\text{ev}}(X; \mathbb{C})$ such that $\phi_1 = 1$ and $\phi_{i+1} = p_i$ for $1 \leq i \leq r$. Put $t = \sum_i t_i \phi_i$. Then the genus-$0$ potential of $X$ is defined by

$$F^{(0)}(Q, t) = \sum_{n=0}^{\infty} \sum_{d \in \text{Eff}(X)} \frac{Q^d}{n!} (t, \ldots, t)_{0,n,d}.$$
The GW invariants satisfy the so-called divisor equation which implies that $\partial_{t_{i+1}} F^{(0)} = Q_i \partial Q_i F^{(0)}$ for all $1 \leq i \leq r$. Therefore, the genus-0 potential has the form

$$F^{(0)}(Q,t) = F(t_1, Q_1 e^{t_2}, \ldots, Q_r e^{t_{r+1}}, t_{r+2}, \ldots, t_N).$$

Let us fix $Q_1, \ldots, Q_r$ as complex parameters (e.g. set $Q_i = 1$ for all $i$). In many important examples, the formal series defining quantum cohomology is convergent on a domain

$$M = \{ t \in H^*(X; \mathbb{C}) \mid \text{Re}(t_{i+1}) < -R, \ 1 \leq i \leq r, \ |t_j| < \epsilon \ 1 \leq j \leq r \},$$

where $R > 0$ and $\epsilon > 0$ are real numbers. Let us introduce also the vector fields

$$e = \partial/\partial t_1, \quad E = \sum_{i=1}^N ((1 - d_i) t_i + r_i) \partial_{t_i},$$

where $d_i = \deg(\phi_i)/2$ and $r_i$ are the coordinates of $c_1(TX)$, i.e., $c_1(TX) = \sum_{i=1}^N r_i \phi_i$. If the domain of convergence $M$ exists, then the Poincare pairing, the vector fields $e$ and $E$, and the multiplication defined in terms of $F^{(0)}$ via the formulas of Proposition 6.2, part b), determine a Frobenius structure on $M$ of conformal dimension $D = \dim_{\mathbb{C}}(X)$.

6.4. Semi-simple Frobenius manifolds.

**Definition 6.3.** A Frobenius manifold $(M, (\cdot, \cdot), \cdot, e, E)$ is said to be semi-simple if there are local coordinates $u = (u_1, \ldots, u_N)$ defined in a neighborhood of some point on $M$ such that

$$\partial/\partial u_i \cdot \partial/\partial u_i = \delta_{ij} \partial/\partial u_j, \quad 1 \leq i, j \leq N.$$

The coordinates $u_i$ are called canonical coordinates.

As we will see now, canonical coordinates are unique up to permutation and constant shifts. To avoid cumbersome notation we put $\partial_{u_i} := \partial/\partial u_i$.

**Proposition 6.4.** Let $u = (u_1, \ldots, u_N)$ be canonical coordinates defined on some open subset $U \subset M$. Then

a) The Frobenius pairing takes the form

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij} \eta_j(u), \quad 1 \leq i, j \leq N,$$

where $\eta_j \in \mathcal{O}_M(U)$ and $\eta_j(u) \neq 0$ for all $u \in U$.

b) The unit vector field takes the form $e = \sum_{i=1}^N \partial_{u_i}$.

c) The 1-form $\sum_{i=1}^N \eta_i(u) du_i$ is closed.

d) There are constants $c_i$ ($1 \leq i \leq N$) such that

$$E = \sum_{i=1}^N (u_i + c_i) \partial_{u_i}.$$
Proof. a) If \( i \neq j \) then we have
\[
(\partial_{u_i}, \partial_{u_j}) = (\partial_{u_i}, \partial_{u_j}) = (e, \partial_{u_i} \bullet \partial_{u_j}) = 0.
\]
The fact that \( \eta_i(u) := (\partial_{u_i}, \partial_{u_i}) \neq 0 \) follows from the non-degeneracy of the Frobenius pairing.

b) Let \( e = \sum_{i=1}^N e_i(u) \partial_{u_i} \). Then
\[
\partial_{u_j} = \partial_{u_j} \bullet e = e_j(u) \partial_{u_j}.
\]
Therefore \( e_j(u) = 1 \) for all \( j \).

c) We have to check that \( \partial_{u_j} \eta_i = \partial_{u_i} \eta_j \). On the other hand
\[
\partial_{u_j} \eta_i = \partial_{u_j} (\partial_{u_i}, e) = (\nabla_{\partial_{u_j}}^{L.C.} \partial_{u_i}, e),
\]
where we used the Leibnitz rule and the fact the \( e \) is a flat vector field. It remains only to recall that the Levi–Civita connection is torsion free, so
\[
\nabla_{\partial_{u_j}}^{L.C.} \partial_{u_i} = \nabla_{\partial_{u_i}}^{L.C.} \partial_{u_j}.
\]
d) Put \( E = \sum_{i=1}^N E_i(u) \partial_{u_i} \). Let us recall Proposition 6.2 part d) with \( v = \partial_{u_i} \) and \( w = \partial_{u_j} \). For \( i \neq j \) we get
\[
(\partial_{u_i} E_j) \partial_{u_i} + (\partial_{u_j} E_i) \partial_{u_j} = 0.
\]
Hence \( \partial_{u_i} E_j = 0 \) for \( i \neq j \). If \( i = j \) then we get \( \partial_{u_i} E_i = 1 \). Therefore \( E_i(u) = u_i + c_i \) for some constant \( c_i \).

Part d) of the above proposition shows that in every canonical coordinate system up to some constant shifts the canonical coordinates coincide with the eigenvalues of the operator \( E \bullet \). Therefore, up to constant shifts and permutations the canonical coordinates are uniquely determined. From now on we will work only with canonical coordinates such that
\[
E = \sum_{i=1}^N u_i \partial_{u_i}.
\]

The question that we would like to answer now is the following. Let us assume that \( U \) is an open subset of the universal cover \( T \) of \( Z_N \) and \( \sum_{i=1}^N \eta_i(u) du_i \) is a closed 1-form on \( U \). The tangent bundle of \( T \) and hence of \( U \) as well is trivial, because \( T \) is a contractible Stein manifold, so according to the Grauert–Oka principle every holomorphic vector bundle on \( T \) is trivial. Alternatively, we can prove that \( \mathcal{T}_T \) is a free \( \mathcal{O}_T \)-module by using that the vector fields \( \partial_{u_i} \) of the configuration space \( Z_N \) lift naturally to vector fields on \( T \) and provide a global trivialization of \( \mathcal{T}_T \). Using the 1-form we define a pairing
\[
(\partial_{u_i}, \partial_{u_j}) = \delta_{ij} \eta_j(u).
\]
Let us also define multiplication
\[
\partial_{u_i} \bullet \partial_{u_j} = \delta_{ij} \partial_{u_j}
\]
and vector fields
\[ e = \sum_{i=1}^{N} \partial_{u_i}, \quad E = \sum_{i=1}^{N} u_i \partial_{u_i}. \]

The problem then is to classify all 1-forms \( \sum_{i=1}^{N} \eta_i(u) du_i \) such that the above data determines a Frobenius structure on \( U \). The answer is given by the following theorem.

**Theorem 6.5.** The closed 1-form \( \sum_{i=1}^{N} \eta_i(u) du_i \) determines a Frobenius structure on \( U \) of conformal dimension \( D \) if and only if the following conditions are satisfied

1. \( \eta_i(u) \neq 0 \) for all \( i \) and for all \( u \in U \).
2. \( e \eta_i(u) = 0 \) for all \( i \).
3. \( E \eta_i(u) = -D \eta_i(u) \).
4. For all \( k \neq i \neq j \neq k \) we have
   \[ \frac{\partial \eta_{ij}}{\partial u_k} = \frac{1}{2} \left( \frac{\eta_{ij} \eta_{kj}}{\eta_j} + \frac{\eta_{jk} \eta_{ik}}{\eta_k} + \frac{\eta_{ki} \eta_{lj}}{\eta_i} \right), \]

   where \( \eta_{ab}(u) := \partial_{u_a} \eta_b(u) \).

**Proof.** **Step 1.** Determine when does the 1-form \( \sum_{i=1}^{N} \eta_i(u) du_i \) defines data satisfying conditions (a), (b), and (c) in the definition of a Frobenius manifold.

In part (a), we would like the multiplication and the pairing to give a holomorphic family of Frobenius algebras. This is clearly satisfied for any choice of the 1-form. The requirement that the pairing is non-degenerate yields that \( \eta_i(u) \neq 0 \) for all \( i \) and for all \( u \in U \).

For condition (b), we would like to know when is \( e \) a flat vector field. Let \( \Gamma^k_{ij} \) be the Christoffel's symbols of the pairing \( g_{ij}(u) = \delta_{ij} \eta_j \). A straightforward computation yields
\[ \Gamma^j_{ij} = \frac{\eta_{ij}}{2\eta_j}, \quad 1 \leq i, j \leq N, \]
\[ \Gamma^j_{ii} = -\frac{\eta_{ij}}{2\eta_j}, \quad 1 \leq i \neq j \leq N, \]
and
\[ \Gamma^k_{ij} = 0, \quad k \neq i \neq j \neq k. \]

Using the above formulas we compute directly that
\[ \nabla^\text{LC} \partial_{u_i} e = \frac{e \eta_i}{2\eta_i} \partial_{u_i}. \]

Therefore \( e \) is a flat vector field if and only if \( e \eta_i = 0 \) for all \( i \).

Finally, for condition (c) to hold we must have \( E \eta_i = -D \eta_i \) for all \( i \). Therefore, the 1-form will define a data satisfying conditions (a), (b), and (c) if and only if the functions \( \eta_i(u) \) satisfy conditions (1), (2), and (3) in Theorem 6.5.

**Step 2.** When is the Levi–Civita connection flat?
The flatness of $\nabla^{L.C.}$ is equivalent to: the expression
$$2(\nabla^{L.C.}_{\partial u_i} \nabla^{L.C.}_{\partial u_k} \partial u_i)$$
is symmetric in $i$ and $j$. Using the Leibnitz rule we transform this expression into

$$\partial u_i \left( 2\Gamma^\ell_{jk} \eta_\ell - \sum_{a=1}^N 2\Gamma^a_{jk} \Gamma^a_i \eta_a \right).$$

Let us assume first that $i, j,$ and $k$ are pairwise distinct. Then we get

$$\delta_{ij} \left( \frac{\partial \eta_{jk}}{\partial u_i} \eta_{ji} - \eta_{ij} \eta_{ji} \frac{2}{\eta_j} \right) + \delta_{ik} \left( \frac{\partial \eta_{jk}}{\partial u_i} \eta_{ki} + \eta_{ij} \eta_{ki} \frac{2}{\eta_k} \right).$$

The last term is symmetric in $i$ and $j$, so a non-trivial condition will be obtained either if $\ell = i$ or $\ell = j$. Due to the symmetry between $i$ and $j$ we may assume that $\ell = j$. Then we get

$$\delta_{ij} \left( \frac{\partial \eta_{jk}}{\partial u_i} - \eta_{ij} \eta_{ji} \frac{2}{\eta_j} \right) + \delta_{ik} \left( \eta_{ij} \eta_{ki} \frac{2}{\eta_i} + \eta_{ij} \eta_{ji} \frac{2}{\eta_j} \right).$$

This is exactly the PDE given in condition (4).

There are 3 more cases to analyze. Indeed, since we may assume that $i \neq j$ we get that $k = i$ or $k = j$. Again exchanging the role of LHS and RHS provides a symmetry between $i$ and $j$, which allows us to assume that $k = i$. Therefore the remaining cases are: $(k, \ell) = (i, i), (i, j)$, or $k = i$ and $\ell \neq i, j$. The first case yields $\partial u_i \eta_{ij} = \partial u_j \eta_{ij}$, which is always satisfied because the 1-form $\sum_i \eta_i du_i$ is closed. The 2nd case $(k, \ell) = (i, j)$ yields

$$\partial u_i \eta_{ij} + \partial u_j \eta_{ij} = \frac{\eta_{ij} \eta_{ij}}{2\eta_i} + \frac{\eta_{ij} \eta_{ij}}{2\eta_j} + \frac{\eta_{ij} \eta_{ij}}{2\eta_i} - \sum_{a \neq i, j} \frac{\eta_{ia} \eta_{ja}}{2\eta_a}.$$ 

It is easy to see that this identity is a consequence of (2) and (4). In the last case if $k = i$ and $\ell \neq i, j$ we get

$$\partial u_i \eta_{i\ell} = \frac{\eta_{i\ell} \eta_{i\ell}}{2\eta_i} + \frac{\eta_{i\ell} \eta_{i\ell}}{2\eta_\ell} + \frac{\eta_{i\ell} \eta_{i\ell}}{2\eta_j}$$

which is equivalent to (4).

**Step 3.** It remains only to verify that under the conditions (1)–(4) the structure connection $\nabla$ is flat. The argument is similar to the argument in Step 2, so it will be left as an exercise. 

\[\square\]

7. **Painlevé property for semi-simple Frobenius manifolds**

7.1. **The second structure connection.** Let $U \subset Z_N$ be an open contractible neighborhood of some fixed point $u^0 \in Z_N$. Suppose that $U$ is equipped with a semi-simple Frobenius structure $(( , ), \bullet, e, E)$. Put $H = T_{u^0}U$ and let us trivialize the tangent bundle

$$TU \cong U \times H \cong U \times \mathbb{C}^N$$

$$2(\nabla_{\partial u_i} \nabla_{\partial u_k} \partial u_i)$$
using the Levi–Civita connection. In other words, we fix a basis $\{\phi_a\}_{a=1}^N$ of $H$ and let $\partial_{t_a} \in \mathcal{T}_U$ be the flat vector field on $U$ obtained by parallel transport with respect to the Levi–Civita connection. Then the isomorphisms (20) are given by the maps

$$(u, v) \in TU \mapsto (u, v_1\phi_1 + \cdots + v_N\phi_N) \in U \times H \mapsto (u, v_1, \ldots, v_N) \in U \times \mathbb{C}^N,$$

where $v \in T_uU$ and $v =: v_1\partial_{t_1} + \cdots + v_N\partial_{t_N}$. The isomorphism (20) identifies the structure connection of the Frobenius structure with the flat connection on the trivial bundle

$$(U \times \mathbb{C}^*) \times \mathbb{C}^N \to U \times \mathbb{C}^*$$

defined by

$$
\nabla_{\partial_{u_i}} = \partial_{u_i} - z^{-1}P_i(u), \quad 1 \leq i \leq N,
$$
$$
\nabla_{\partial_{\lambda}} = \partial_{\lambda} - z^{-1}\theta + z^{-2}\mathcal{E}(u),
$$

where $P_i : U \to \mathfrak{gl}(\mathbb{C}^N)$ is a holomorphic map whose $(a, b)$-entry $P_{iab}(u)$ is defined by the identity

$$
\partial_{u_i} \cdot \partial_{t_b} = \sum_{a=1}^N P_{iab}(u)\partial_{t_a},
$$

$\mathcal{E} = \sum_{i=1}^N u_iP_i(u)$, and $\theta$ is a constant matrix whose $(a, b)$-entry $\theta_{ab}$ is defined by

$$
\theta(\partial_{t_b}) = [\partial_{t_b}, E] - (1 - D/2)\partial_{t_b} =: \sum_{a=1}^N \theta_{ab}\partial_{t_a}.
$$

In order to justify the definition of the second structure connection we make the following heuristic argument. Suppose that the structure connection has a solution

$$J : U \times \mathbb{C}^* \to \mathbb{C}^N$$

given by a Laplace transform

$$J(u, z) = \frac{(-z)^{n-\frac{1}{2}}}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda/z}I^{(n)}(u, \lambda)d\lambda$$

along an appropriate contour $\Gamma \subset \mathbb{C}$ of some $\mathbb{C}^N$-valued function $I^{(n)}(u, \lambda)$ holomorphic for all $(u, \lambda) \in U \times \Gamma$. Here $n \in \mathbb{C}$ is an arbitrary number. Assuming that the Laplace transform works, we would get that $J(u, z)$ is a solution to the structure connection if and only if $I^{(n)}(u, \lambda)$ is a solution to the following connection

$$
\nabla^{(n)}_{\partial_{u_i}} = \partial_{u_i} + (\lambda - \mathcal{E})^{-1}P_i(u)(\theta - n - 1/2), \quad 1 \leq i \leq N,
$$
$$
\nabla^{(n)}_{\partial_{\lambda}} = \partial_{\lambda} - (\lambda - \mathcal{E})^{-1}(\theta - n - 1/2).
$$
This is a connection on 
\[(U \times \mathbb{C})' \times \mathbb{C}^N \to (U \times \mathbb{C})',\]
where
\[(U \times \mathbb{C})' = \{(u, \lambda) \in U \times \mathbb{C} \mid \det(\lambda - E) \neq 0\}.\]

**Proposition 7.1.** The connection \(\nabla^{(n)}\) is flat for all \(n \in \mathbb{C}\).

The proof is left as an exercise. The connection \(\nabla^{(n)}\) is called the second structure connection.

**7.2. Proof of Theorem 1.1.**

**Lemma 7.2.** Let \(\tilde{\Psi}\) be the matrix whose \((a, i)\)-entry is given by \(\tilde{\Psi}_{ai} = \partial_{u_i}/\partial u_i\).

Then
\[
\tilde{\Psi}^{-1}P_i\tilde{\Psi} = E_{ii}, \quad \tilde{\Psi}^{-1}E\tilde{\Psi} = \text{diag}(u_1, \ldots, u_N),
\]
where \(E_{ii}\) is the matrix whose entry in position \((i, i)\) is 1 and all other entries are 0.

**Proof.** We have
\[
\partial_{u_i} \cdot \partial_{t_b} = \partial_{u_i} \cdot \sum_{j=1}^{N} \frac{\partial u_j}{\partial t_b} \partial_{u_j} = \frac{\partial u_i}{\partial t_b} \partial_{u_i} = \sum_{a=1}^{N} \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i} \partial_{t_a}.
\]

Therefore
\[
P_{ab} = \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i}.
\]

Using this formula we find that the \((a, j)\)-entry of \(P_i\tilde{\Psi}\) is
\[
\sum_{b=1}^{N} \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i} \tilde{\Psi}_{bj} = \sum_{b=1}^{N} \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i} \frac{\partial u_j}{\partial t_b} = \delta_{ij} \frac{\partial t_a}{\partial u_i} = \delta_{ij} \tilde{\Psi}_{aj}.
\]

The latter is precisely the \((a, j)\)-entry of \(\tilde{\Psi}E_{ii}\). Therefore \(P_i\tilde{\Psi} = \tilde{\Psi}E_{ii}\). \(\square\)

**Lemma 7.3.** Let \(n \in \mathbb{C}\) be arbitrary. Then the matrix-valued functions
\[A_i^{(n)}(u) := P_i(u)(\theta - n - 1/2), \quad 1 \leq i \leq N,
\]
satisfy the Schlesinger equations.

**Proof.** We have to prove that the connection
\[
\tilde{\nabla}_{u_i} = \partial_{u_i} + \frac{A_i^{(n)}(u)}{\lambda - u_i}, \quad 1 \leq i \leq N
\]
\[
\tilde{\nabla}_{\lambda} = \partial_{\lambda} - \sum_{i=1}^{N} \frac{A_i^{(n)}(u)}{\lambda - u_i}
\]
is flat. However, using Lemma 7.2 we get

$$(\lambda - E)^{-1}P_i(\theta - n - 1/2) = \frac{A_i^{(n)}(u)}{\lambda - u_i}.$$  

Therefore $\tilde{\nabla}^{(n)} = \nabla^{(n)}$, so it remains only to recall Proposition 7.1. □

The proof of Theorem 1.1 can be given as follows. Let us choose $n \in \mathbb{C}$ such that the operator $\theta - n - 1/2$ is invertible. Then

$$\eta_i(u) = (\partial u_i, \partial u_i) = (P_i(u)e, e) = (A_i^{(n)}(u)(\theta - n - 1/2)^{-1}e, e).$$

According to Theorem 4.4 and Lemma 7.3 the functions $\eta_i(u)$ extend to meromorphic functions on $T$. □

### 7.3. Special initial conditions.

In this section we are going to prove a theorem of Manin [10] which answers the question of what kind of initial conditions for the Schlesinger equations determine a semi-simple Frobenius structures. Following Manin we introduce the following definition.

**Definition 7.4.** Let $H$ be a vector space equipped with a non-degenerate symmetric bi-linear pairing $(\ , \ )$ and a distinguished vector $e \in H$. Suppose also that we have a set of linear operators $\theta$, $\{P_i^o\}_{i=1}^N \in \mathfrak{gl}(H)$. The data $(H, (\ , \ ), e, \theta, \{P_i^o\}_{i=1}^N)$ is said to be a **special initial condition** if the following conditions are satisfied:

1. $\theta$ is skew-symmetric: $(\theta(a), b) + (a, \theta(b)) = 0$ for all $a, b \in H$.
2. $e$ is an eigenvector of $\theta$ with eigenvalue $D/2$.
3. The set $\{P_i^o\}_{i=1}^N$ is a complete set of orthogonal projectors of $H$, i.e.,
   a. $P_i^o P_j^o = \delta_{ij} P_j^o$ for all $1 \leq i, j \leq N$.
   b. $P_1^o + \cdots + P_N^o = 1$.
   c. $(P_i^o(a), b) = (a, P_i^o(b))$ for all $1 \leq i \leq N$ and for all $a, b \in H$.
   d. $P_i^o e \neq 0$ for all $1 \leq i \leq N$. □

Suppose that $(H, (\ , \ ), e, E)$ is a semi-simple Frobenius structure on some complex manifold $M$ and that $u^\circ \in M$ is a semi-simple point, i.e., a neighborhood of $u^\circ$ admits canonical coordinates. Then the data

$$H := T_{u^\circ} M, (\ , \ ), e, \theta := \nabla^{L,C} E - (1 - D/2), P_i^o = P_i(u^\circ), 1 \leq i \leq N;$$

is a special initial condition. In fact the only property that we did not check yet is that $e$ is an eigenvector of $\theta$. However

$$\theta(e) = [e, E] - (1 - D/2)e = e - (1 - D/2)e = (D/2)e,$$

where in the first equality we used that $e$ is flat and in the second equality we used that $e = \sum_i u_i \partial u_i$. and $E = \sum_i u_i \partial u_i$.

**Proposition 7.5.** Given a special initial condition $(H, (\ , \ ), e, \theta, \{P_i^o\}_{i=1}^N)$ and a point $u^\circ \in Z_N$, then there exists an open neighborhood $U \subset Z_N$ of $u^\circ$ and an isomorphism $T_{u^\circ} U \cong H$ such that the special initial condition is obtained from a semi-simple Frobenius structure on $U$. 

Proof. Let \( A_i^{(n)}(u), 1 \leq i \leq N \) be solutions to the Schlesinger equations such that
\[
A_i^{(n)}(u) = P_i^\circ(\theta - n - 1/2).
\]
If \( n + 1/2 \) is not an eigenvalue of \( \theta \), then we define
\[
P_i^{(n)}(u) = A_i^{(n)}(u)(\theta - n - 1/2)^{-1}.
\]

Lemma 7.6. The set \( \{P_i^{(n)}(u)\}_{i=1}^N \) is a complete set of orthogonal projections for all \( u \) sufficiently close to \( u^\circ \).

Proof. Let us fix a basis \( \{\phi_i\}_{i=1}^N \) of \( H \) and identify \( \mathfrak{gl}(H) \) with the space of \( p \times p \)-matrices. Let \( \mathcal{A} \) be the polynomial ring
\[
\mathcal{A} = \mathbb{C}[(u_i - u_j)^\pm 1 : 1 \leq i < j \leq N] \otimes \mathbb{C}[A_1, \ldots, A_N],
\]
where \( A_i = (A_{ij})_{a,b=1}^N \) are matrix variables. We define derivations \( \partial_{u_1}, \ldots, \partial_{u_N} \) of \( \mathcal{A} \) such that
\[
\partial_{u_i}A_j := \frac{[A_i, A_j]}{u_i - u_j}, \quad 1 \leq i \neq j \leq N,
\]
and if \( f \in \mathcal{A} \) depends only on \( u_1, \ldots, u_N \) then \( \partial_{u_i} \) is defined to be the usual derivative. It is easy to check that these differentiations pairwise commute, so \( \mathcal{A} \) becomes a \( \mathcal{D} \)-module for the ring \( \mathcal{D} \) of differential operators on \( Z_N \).

Let us define \( \mathcal{I} \subset \mathcal{A} \) to be the ideal generated by the relations corresponding to conditions (a)–(c) in Definition 7.4. More precisely, we replace \( P_i^\circ \) by \( A_i(\theta - n - 1/2)^{-1} \) and take the entries of the corresponding matrix identities as generators of \( \mathcal{I} \). Condition (a) yields generators given by the entries of
\[
R_{ij}(A_1, \ldots, A_N) = A_i(\theta - n - 1/2)^{-1}A_j - \delta_{ij}A_j, \quad 1 \leq i, j \leq N.
\]
Condition (b) gives the entries of
\[
R(A_1, \ldots, A_N) = A_1 + \cdots + A_N - \theta + n + \frac{1}{2}.
\]
Finally, condition (c) gives the entries of
\[
R_i(A_1, \ldots, A_N) = A_i(\theta - n - 1/2)^{-1} + (\theta + n + 1/2)^{-1}A_i^T, \quad 1 \leq i \leq N,
\]
where \( T \) is the transposition operation in \( \mathfrak{gl}(H) \) with respect to the pairing \( (\ , \) \).

We claim that in order to prove the lemma it is enough to check that \( \mathcal{I} \) is \( \mathcal{D} \)-invariant. Indeed, condition (a) in Definition 7.4 will be satisfied if
\[
R_{ij}(A_1^{(n)}(u), \ldots, A_N^{(n)}(u)) = 0.
\]
On the other hand, the Taylor series expansion of \( R_{ij}(A_1^{(n)}(u), \ldots, A_N^{(n)}(u)) \) at \( u = u^\circ \) has the form
\[
\sum_{m_1, \ldots, m_N=0}^\infty \frac{\partial_{u_1}^{m_1} \cdots \partial_{u_N}^{m_N} R_{ij}}{m_1! \cdots m_N!} (A_1^{(n)}(u^\circ), \ldots, A_N^{(n)}(u^\circ))(u_1 - u_1^\circ)^{m_1} \cdots (u_N - u_N^\circ)^{m_N},
\]
Lemma 7.7. If $n + \frac{1}{2}$ and $m + \frac{1}{2}$ are not eigenvalues of $\theta$, then $P_i^{(m)}(u) = P_i^{(n)}(u)$.
Proof. According to Lemma 7.6 the matrices $P_{i}(u)$ pairwise commute. Using that $A^{(n)}_{i}(u)$ satisfy the Schlesinger equations we get

$$dP_{i}^{(n)}(u) = \sum_{j:j \neq i} \frac{du_{j} - du_{i}}{u_{j} - u_{i}} \left(P_{j}^{(n)}(u)\theta P_{i}^{(n)}(u) - P_{i}^{(n)}(u)\theta P_{j}^{(n)}(u)\right).$$

Using these equations and the fact that $P_{i}^{(n)}(u)$ pairwise commute we get that the matrix-valued functions $\tilde{A}_{i}^{(n)}(u) := P_{i}^{(n)}(u)\left(\theta - n - \frac{1}{2}\right)$ $(1 \leq i \leq N)$ satisfy the Schlesinger equations. However the initial condition $\tilde{A}_{i}^{(n)}(u^{0}) = A_{i}^{(n)}(u^{0}).$ Therefore $\tilde{A}_{i}^{(n)}(u) = A_{i}^{(n)}(u).$ \qed

According to Lemma 7.7 the matrices $P_{i}(u) := P_{i}^{(n)}(u)$ are independent of $n,$ while Lemma 7.6 implies that they form a complete system of orthogonal projections.

**Lemma 7.8.** The 1-form

$$\sum_{i=1}^{N} \eta_{i}(u)du_{i}, \quad \eta_{i}(u) := (P_{i}(u)e, e), \quad 1 \leq i \leq N,$$

defines a Frobenius structure on every sufficiently small neighborhood $U$ of $u^{0}.$

**Proof.** Let us first check that the above 1-form is closed. We have

$$\eta_{ij}(u) := \partial_{u_{j}}\eta_{i} = \partial_{u_{j}}(P_{i}(u)e, e) = \frac{2}{D - 1 - 2n}(\partial_{u_{j}}A_{i}^{(n)}(u)e, e),$$

where we used that $P_{i}(u) = A_{i}^{(n)}(u)\left(\theta - n - 1/2\right)^{-1}$ and that $\theta(e) = (D/2)e.$ We have to prove that $\eta_{ij}(u) = \eta_{ji}(u).$ Let us assume that $i \neq j.$ Since $A_{i}^{(n)}(u)$ $(1 \leq i \leq N)$ satisfy the Schlesinger equations we get

$$\partial_{u_{j}}A_{i}^{(n)} = \frac{[A_{j}, A_{i}]}{u_{j} - u_{i}} = \partial_{u_{i}}A_{j}^{(n)},$$

which implies that $\eta_{ij} = \eta_{ji},$ so the 1-form is closed. To complete the proof we have to check that the 4 conditions of Theorem 6.5 are satisfied.

Note that the vectors $P_{i}^{0}e$ $(1 \leq i \leq N)$ form a basis of $H.$ Indeed, if $\sum_{i} \alpha_{i}P_{i}^{0}e = 0,$ then applying to both sides $P_{i}^{0}$ we get $\alpha_{i}P_{i}^{0}e = 0.$ By assumption $P_{i}^{0}e \neq 0,$ so $\alpha_{i} = 0.$ The matrix of the form $(\quad, \quad)$ is diagonal in the basis $P_{i}^{0}e$ with diagonal entries $\eta_{i}(u^{0}).$ Therefore $\eta_{i}(u^{0}) \neq 0$ for all $i$ otherwise the form will be degenerate. By continuity there exists a small neighborhood $U$ of $u^{0}$ such that $\eta_{i}(u) \neq 0$ for all $i$ and for all $u \in U.$

The second condition that we have to check is $e\eta_{i} = 0.$ This follows from the fact that

$$\sum_{j=1}^{N} \eta_{j}(u) = \left(\sum_{j=1}^{N} P_{j}(u)e, e\right) = (e, e).$$
is a constant independent of \( u \).

The third condition that we have to check is \( E\eta_i = -D\eta_i \). We have (see above)

\[
\eta_i(u) = \frac{2}{D - 1 - 2n} (A_i^{(n)}(u)e, e).
\]

Note that

\[
EA_i^{(n)}(u) = t_E dA_i^{(n)}(u) = t_E \sum_{j:i\neq i} \frac{du_j - du_i}{u_j - u_i} [A_j^{(n)}(u), A_i^{(n)}(u)] = [\theta, A_i^{(n)}(u)],
\]

where in the second equality we used the Schlesinger equations and in the third one we used that

\[
\sum_{j=1}^{N} A_j^{(n)}(u) = \sum_{j=1}^{N} P_j(u)(\theta - n - 1/2) = \theta - n - 1/2.
\]

Therefore

\[
E\eta_i = \frac{2}{D - 1 - 2n} ([\theta, A_i^{(n)}(u)]e, e).
\]

It remains only to use that \( \theta(e) = (D/2)e \) and that \( \theta \) is skew-symmetric with respect to the pairing.

Finally, the last condition that we have to check is

\[
\frac{\partial \eta_{ij}}{\partial u_k} = \frac{1}{2} \left( \frac{\eta_{ik}\eta_{jk}}{\eta_i} + \frac{\eta_{ij}\eta_{ki}}{\eta_j} + \frac{\eta_{kj}\eta_{ij}}{\eta_j} \right), \quad k \neq i \neq j \neq k.
\]

Let us explain how to express the LHS as a quadratic expression in the functions \( \eta_{ab} \). Recall that we have the following differential equation

\[
\partial_{u_j} P_i = \frac{1}{u_j - u_i} \left( P_j\theta P_e - P_i\theta P_j \right).
\]

Using the above differential equations and the fact that the operators \( P_a \) are self-adjoint and \( \theta \) is skew symmetric with respect to \( (\ , \ ) \) we get

\[
\eta_{ij} = (\partial_{u_j} P_i(u)e, e) = \frac{2}{u_i - u_j} (P_i(u)e, \theta P_j(u)e).
\]

The derivative \( \partial_{u_k} \eta_{ij} \) becomes

\[
\frac{2}{u_i - u_j} \left( \frac{(P_k\theta P_i e, \theta P_j e)}{u_k - u_i} - \frac{(P_k\theta P_j e, \theta P_i e)}{u_k - u_j} + \frac{(P_i\theta P_k e, \theta P_j e)}{u_i - u_k} - \frac{(P_j\theta P_k e, \theta P_i e)}{u_j - u_k} \right).
\]

Using the projection formula \( P_i x = (x, P_i e) \frac{P_i e}{\eta_i} \) we get

\[
\frac{(P_k\theta P_i e, \theta P_j e)}{u_k - u_i} = \frac{(P_i\theta P_k e, \theta P_j e) 1}{\eta_k} = \frac{\eta_{ik}\eta_{jk}}{4\eta_k} (u_k - u_j).
\]

Similar formulas hold for the remaining 3 terms above, so for the derivative \( \partial_{u_k} \eta_{ij} \) we get

\[
\frac{2}{u_i - u_j} \left( \frac{\eta_{ik}\eta_{jk}}{4\eta_k} (u_k - u_j) - \frac{\eta_{ik}\eta_{jk}}{4\eta_k} (u_k - u_i) + \frac{\eta_{ki}\eta_{ji}}{4\eta_i} (u_i - u_j) - \frac{\eta_{kj}\eta_{ij}}{4\eta_j} (u_j - u_i) \right).
\]
The above expression is precisely the RHS of (21).

□

The proof of the proposition can be completed as follows. Let us define the isomorphism

$$T_{u^0}U \cong H, \quad \partial_{u_i} \mapsto P^0_i e,$$

where slightly abusing the notation we have denoted by $\partial_{u_i}$ the tangent vector in $T_{u^0}U$ representing the value of the coordinate vector field $\partial_{u_i}$ at $u^0$. We claim that the special initial condition corresponding to the Frobenius structure defined by Lemma 7.8 coincides with the given special initial condition. The easiest way to see this is if we fix the basis of $H$ to be $\phi_i = P^0_i e$. Then for the given special initial condition we have: the matrix $P^0_j$ is $E_{jj}$ (the matrix with 1 on place $(j,j)$ and 0 elsewhere), the matrix of the pairing $(\ ,\ )$ is diagonal with diagonal entries $(P^0_i e, e) = \eta_i(u^0)$, the vector $e$ has coordinates $(1, \ldots, 1)$, and the matrix of $\theta$ becomes (see formula (22))

$$\theta_{ij} = (u_i^0 - u_j^0) \frac{\eta_{ij}(u^0)}{2\eta_i(u^0)}, \quad 1 \leq i, j \leq N.$$

Comparing with the special initial condition corresponding to the Frobenius structure we see that the only thing left to prove is that the Hodge grading operator $\tilde{\theta}|_{T_{u^0}U}$ coincides with $\theta$. Let us compute the matrix of $\tilde{\theta}$ in canonical coordinates. Note that $\tilde{\theta}_{ij} = 0$ for $i = j$ due to skew-symmetry. Let us assume that $i \neq j$. Then

$$\tilde{\theta}_{ij}(u)\eta_i(u) = (\partial_{u_i}, \nabla_{\partial_{u_j}} E) = \partial_{u_j}(\partial_{u_i}, E) - \sum_{k=1}^N \Gamma^k_{ij}(u)(\partial_{u_k}, E),$$

where $\Gamma^k_{ij}$ are the Christoffel’s symbols of the Frobenius pairing. Recalling the formulas for the Christoffel’s symbols (see Step 1 in the proof of Theorem 6.5) we get

$$\tilde{\theta}_{ij}(u)\eta_i(u) = (u_i - u_j) \frac{\eta_{ij}(u)}{2} \quad \Rightarrow \quad \tilde{\theta}_{ij}(u) = (u_i - u_j) \frac{\eta_{ij}(u)}{2\eta_i(u)}.$$

Restricting to $u = u^0$ we get that $\tilde{\theta}(u^0) = \theta$. □

7.4. The genus-1 potential. We would like to finish this lecture by deriving the relation between the genus-1 primary potential of the semi-simple Frobenius structure and the isomonodromic tau-function. Following Givental (see [5] and the references there in) we introduce the genus-1 potential as

$$F^{(1)}(u) = \frac{1}{2} \int \sum_{i=1}^N R^{ii}_1(u) du_i - \frac{1}{48} \log(\eta_1(u) \cdots \eta_N(u)),$$

where $R_1(u)$ is a matrix and $R^{ii}_1$ is the $(i,i)$-entry. The function is called genus-1 potential, because in the case of quantum cohomology of some manifold $X$, the
above formula coincides with the generating function of genus-1 Gromov–Witten invariants of \( X \).

In order to define the matrix \( R_1(u) \) we have to make a choice of square root and define \( \eta_i(u)^{1/2} \) for all \( i \). Let \( \Psi(u) \) be the matrix with entries

\[
\Psi_{ai}(u) = \bar{\Psi}_{ai}(u) \eta_i^{-1/2} = \frac{\partial t_a}{\partial u_i} \eta_i^{-1/2},
\]

where \( t = (t_1, \ldots, t_N) \) is a flat coordinate system. Dubrovin’s connection \( \nabla \) has a unique formal asymptotic solution near \( z = 0 \) of the form

\[
\Psi(u)(1 + R_1(u)z + R_2(u)z^2 + \cdots)e^{U/z}, \quad U = \text{diag}(u_1, \ldots, u_N).
\]

Substituting this formula in the differential equation \( \nabla_\partial J = 0 \) and recalling Lemma 7.2 we get

\[
z \partial_z R(u, z) + z^{-1}[U, R(u, z)] = V(u)R(u, z),
\]

where \( V(u) := \Psi(u)^{-1} \theta \Psi(u) \). Comparing the coefficients in front of \( z^k \) we get

\[
k R_k + [U, R_{k+1}] = VR_k, \quad k \geq 0.
\]

Since we work with a Frobenius structure defined on an open subset of \( T \), the diagonal entries of \( U \) are pairwise distinct, so the above recursion has a unique solution. In particular, for the \( (i, j) \)-entry of \( R_1 \) we get

\[
R_{ij}^1(u) = \frac{V_{ij}(u)}{u_i - u_j}, \quad \text{if } i \neq j
\]

and

\[
R_{ii}^1(u) = \sum_{j:j \neq i} V_{ij}(u)R_{ij}^1(u) = - \sum_{j:j \neq i} \frac{V_{ij}(u)V_{ji}(u)}{u_i - u_j}.
\]

By definition \( V \) is the matrix of the Hodge grading operator in the orthonormal basis \( e_i := \eta_i^{-1/2} \partial u_i \), i.e.,

\[
\theta(e_j) = \sum_{i=1}^N V_{ij}(u)e_i.
\]

On the other hand we have already computed the matrix of \( \theta \) in the canonical basis (see formula (24)). Therefore

\[
V_{ij}(u) = (u_i - u_j)\frac{\eta_{ij}(u)}{2\eta_i(u)^{1/2} \eta_j(u)^{1/2}}.
\]

Finally, for the 1-form \( \sum_{i=1}^N R_{ii}^1(u)du_i \) we get

\[
\sum_{i=1}^N \sum_{j:j \neq i} (u_i - u_j) \frac{\eta_{ij}(u)^2}{4\eta_i(u)\eta_j(u)} du_i = \frac{1}{8} \sum_{i=1}^N \sum_{j:j \neq i} (u_i - u_j) \frac{\eta_{ij}(u)^2}{\eta_i(u)\eta_j(u)} (du_i - du_j).
\]
Note that the above form is independent of the choice of a square root used in the definition of $R_1$. Let us compare this form with the 1-form $\omega$ defining the isomonodromic $\tau$-function of the second structure connection $\nabla^{(n)}$. We have
\[
\text{tr}(A_i^{(n)}(u)A_j^{(n)}(u)) = \text{tr}(P_i\theta P_j\theta) - (n + 1/2)\text{tr}(P_i\theta + \theta P_j).
\]
The first trace on the RHS is
\[
\eta_i^{-1}(P_i\theta P_j\theta P_i e, e) = -(u_i - u_j)^2 \frac{\eta_i^2}{4\eta_i \eta_j},
\]
where we used formula (23). The second trace is 0 because
\[
\text{tr}(P_i\theta) = \eta_i^{-1}(P_i\theta P_i e, e) = \eta_i^{-1}(\theta P_i e, P_i e) = 0,
\]
where the last equality uses the fact that $\theta$ is skew-symmetric with respect to the Frobenius pairing. We get
\[
\omega = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{\text{tr}(A_i^{(n)}(u)A_j^{(n)}(u))}{u_i - u_j} (du_i - du_j) = -\sum_{i=1}^{N} R_{ij}^2(u) du_i.
\]
Finally we get the following relation
\[
e^{-48F^{(1)}(u)} = \tau(u)^{24} \eta_1(u) \cdots \eta_N(u).
\]

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