ON THE ULTRAMETRICITY PROPERTY IN RANDOM FIELD
ISING MODELS

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ABSTRACT. In this paper we show that the ultrametricity property remains valid
in the Random Field Ising Model for any independent disorder whenever the field
strength is a small perturbation.

1. Introduction

In Statistical Mechanics, the Random Field Ising Model (RFIM) [10] is consid-
ered one of the simplest non-trivial models that belongs to a class of disordered
systems in which the disorder is coupled to the order parameter of the system. This
model is under intensive investigation and has been studied from several aspects.
For example, it is expected that many properties, as the Parisi ultrametricity (see
[13, 14]) and the Ghirlanda-Guerra identities (see [1, 9]), in disordered spin models
should not depend on the particular distribution of the coupling constants. These
properties are known to hold in several mean-field spin glass models, such as the
Sherrington-Kirkpatrick model [15] and generic mixed $p$-spin models. The ultra-
metricity property was predicted by Parisi in [14] as an attempt to describe the
expected behavior of the model and it still remains an unsolved math problem. On
the other hand, in [9] it was proven rigorously that the Ghirlanda-Guerra identities
hold (in the thermodynamic limit) in some approximate sense; for some specific
choice of perturbed parameters [17]. Results involving the ultrametricity property
in spin glass models can be found in [3, 11, 12, 13, 18].

The main goal of this paper is to remove the hypothesis of Gaussian disorder and
and to show that the Ghirlanda-Guerra identities are universal under mild conditions.
That is, we prove that these identities hold in the RFIM with any independent
disorder in the case that the field strength is a small perturbation. This result
combined with the main theorem of Panchenko (2011) [13] establishes ultrametricity
under these assumptions. We believe that this work is the first to present the
validity of this property in the RFIM. Furthermore, we also believe that the chaos
phenomena (see, e.g., [4, 6, 7]) in a non-Gaussian environment can also be validated
for this model by slightly modifying the argument we use here.

The rest of the paper proceeds as follows: in Section 2 we present the model and
state the main result of paper. In Section 3 the proof of the main result is given

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in details. Finally, we close this paper with the proof of the basic tool (Proposition 3.2: a generalization of the Gaussian integration by parts) in Appendix.

2. Statement of the result

Given \( n \geq 1 \), let \( V_n = \mathbb{Z}^d \cap [1, n]^d \), \( d \geq 1 \), be a finite subset of vertices of \( d \)-dimensional hypercubic lattice with cardinality denoted by \( |V_n| \). The (random) Gibbs measure of the ferromagnetic RFIM on the set of spin configurations \( \{-1, 1\}^{V_n} \) is given by

\[
G_n(\{\sigma\}) = \frac{1}{Z_n} \exp \left( \beta \sum_{\langle xy \rangle} \sigma_x \sigma_y + h \sum_x g_x \sigma_x \right),
\]

where \( \langle xy \rangle \) denotes the set of ordered pairs in \( V_n \) of nearest neighbors, \( \beta > 0 \) and \( h > 0 \), called inverse temperature and field strength respectively, the partition function \( Z_n \) enters the definition of \( G_n \) as a normalizing factor and the \( g_x \)'s are independent and not necessarily identically distributed random variables (that collectively are called the disorder) with zero-mean and unit-variance such that the field strength is a small perturbation with the following decay rate,

\[
\lim_{n \to \infty} h = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} \mathbb{E}(|g_x|^3 : |g_x| \geq \varepsilon h^{-1}) = 0, \quad \forall \varepsilon > 0.
\]

For a function \( f : (\{-1, 1\}^{V_n})^m \to \mathbb{R}, m \geq 1 \), we define

\[
\langle f \rangle = \langle f(\sigma^1, \ldots, \sigma^m) \rangle := \int f(\sigma^1, \ldots, \sigma^m) \, dG_n(\sigma^1) \cdots dG_n(\sigma^m).
\]

The randomness of the \( g_x \)'s will be represented by the measure \( \gamma \) on \( \mathbb{R}^{|V_n|} \). Following the notation of Talagrand [16], we write

\[
\nu(f) := \mathbb{E}\langle f \rangle = \int \langle f \rangle_{g=u} \, d\gamma(u),
\]

averaging on the realizations of the disorder, where \( \langle \cdot \rangle_{g=u} \) is the Gibbs expectation defined by setting \( g_x \) in \( \langle \cdot \rangle \) to be \( u_x \), for each \( x \in V_n \).

If \( \sigma^1, \sigma^2, \ldots \) are independent and identically distributed configurations under Gibbs measure (11), known as replicas, the overlap between two replicas \( \sigma^l, \sigma^s \) is defined as

\[
R_{l,s} := \frac{1}{|V_n|} \sum_{x \in V_n} \sigma^l_x \sigma^s_x, \quad \forall l, s \geq 1.
\]

Note that \( |R_{l,s}| \leq 1, R_{l,l} = 1 \) and that the infinite random array \( R = (R_{l,s})_{l,s \geq 1} \) is symmetric, non-negative definite and weakly exchangeable. Following [8], an infinite random array \( R \) with such properties is known as Gram-de Finetti matrix. The array
$R$ is said to satisfy the Ghirlanda-Guerra identities (see [2, 9]) if for any $m \geq 2$ and any bounded measurable function $f = f((R_{l,s})_{1 \leq l,s \leq m})$,

$$
\nu(fR_{1,m+1}) - \frac{1}{m} \nu(f) \nu(R_{1,2}) - \frac{1}{m} \sum_{s=2}^{m} \nu(fR_{1,s}) \rightarrow 0.
$$

(5)

For any $\beta > 0$, let

$$
H_n := \frac{1}{|V_n|} \sum_{x \in V_n} g_x \sigma_x.
$$

For technical reasons we will assume that

$$
\mathbb{E}|\langle H_n \rangle - \nu(H_n)| \rightarrow 0.
$$

(6)

The main result of the paper is the following.

**Theorem 1** (Ultrametricity in the RFIM). Under assumptions (2) and (6), the array $R$ defined in (4) is ultrametric,

$$
\mathbb{P}(R_{2,3} \geq \min\{R_{1,2}, R_{1,3}\}) = 1.
$$

The rest of this paper is devoted to the proof of Theorem 1.

3. Proof

Since $R = (R_{l,s})_{l,s \geq 1}$ is a Gram-de Finetti matrix, in this section we show that the Gibbs measure of the RFIM, with mild assumptions: (2) and (6), satisfies the Ghirlanda-Guerra identities (5), implying automatically the ultrametricity property (7) in the RFIM (see, for example, Panchenko (2011) [13]). The major step of the proof of Theorem 1 shows that, as in [3, 5], a generalization of the Gaussian integration by parts suffices.

Our first main tool will be the following proposition. Its proof appears in Auffinger and Chen (2016) [3], Lemma 2.2.

**Proposition 3.1** (Univariate generalized Gaussian integration by parts). Let $y$ be a random variable such that its first $k \geq 2$ moments match those of a Gaussian random variable. Suppose that $f \in C^{k+1}(\mathbb{R})$. For any $K \geq 1$ and $k \geq 2$;

$$
|\mathbb{E}yf(y) - \mathbb{E}f'(y)| \leq \frac{2(\|f^{(k-1)}\|_{\infty} +\|f^{(k)}\|_{\infty})}{(k-1)!} \mathbb{E}(|y|^k : |y| \geq K) + \frac{(k+1)K}{k!} \|f^{(k)}\|_{\infty} \mathbb{E}|y|^k.
$$

Our second main tool will be the following proposition. Its proof is presented in Appendix. This result is new and can be seen as a generalization of Proposition 3.1 for the bivariate case.

**Proposition 3.2** (Bivariate generalized Gaussian integration by parts). Let $x, y$ be two independent random variables such that their first $k \geq 2$ moments match those
of a Gaussian random variable. Suppose that \( f \in C^{k+2}(\mathbb{R}^2) \). For any \( K_1, K_2 \geq 1 \) and \( k \geq 2 \);

\[
\left| \mathbb{E}xyf(x, y) - \mathbb{E} \frac{\partial^2 f(x, y)}{\partial x \partial y} \right| \\
\leq \frac{2}{(k - 1)!} \left( \left\| \frac{\partial^{k-1} f}{\partial y^{k-1}} \right\|_\infty + \left\| \frac{\partial^k f}{\partial y^k} \right\|_\infty \right) \mathbb{E}(|x| : |x| \geq K_1) \mathbb{E}(|y|^k : |y| \geq K_2) \\
+ \frac{2}{(k - 1)!} \left( \left\| \frac{\partial^k f}{\partial x^k \partial y} \right\|_\infty + \left\| \frac{\partial^{k+1} f}{\partial x^{k+1} \partial y} \right\|_\infty \right) \mathbb{E}(|x|^k : |x| \geq K_2) \\
+ \frac{2(k + 1)K_1}{k!} \left( K_2 \left\| \frac{\partial^k f}{\partial y^k} \right\|_\infty \mathbb{E}(|y|^k) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y} \right\|_\infty \mathbb{E}(|x|^k) \right) \\
+ \frac{(k + 1)K_1}{k!} \left( \left\| \frac{\partial^k f}{\partial y^k} \right\|_\infty \mathbb{E}(|y|^{k+1} : |y| \geq K_2) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y} \right\|_\infty \mathbb{E}(|x|^k) \right) \\
+ \frac{(k + 1)}{k!} \left( K_2 \left\| \frac{\partial^k f}{\partial y^k} \right\|_\infty \mathbb{E}(|x| : |x| \geq K_1) \mathbb{E}(|y|^k) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y} \right\|_\infty \mathbb{E}(|x|^{k+1} : |x| \geq K_1) \right).
\]

In order to prove that the Ghirlanda-Guerra identities hold in the RFIM, we enunciate and prove the following preliminary result.

**Lemma 3.3.** Under hypothesis of Theorem 1, for any \( \beta > 0 \),

\[
\nu(|H_n - \nu(H_n)|) \to 0.
\]

**Proof.** Let \( \langle \sigma_x; \sigma_y \rangle := \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle \) be the truncated two-point correlation. A straightforward computation shows that

\[
\left| \frac{\partial \langle \sigma_x; \sigma_y \rangle}{\partial g_y} \right| \leq 2h, \quad \left| \frac{\partial^2 \langle \sigma_x; \sigma_y \rangle}{\partial g_x g_y} \right| \leq 6h^2 \quad \text{and} \quad \left| \frac{\partial^3 \langle \sigma_x; \sigma_y \rangle}{\partial g_x^2 g_y} \right| \leq 24h^3.
\]

Let \( \langle \cdot \rangle_{g_x = u, g_y = v} \) be the Gibbs expectation defined by setting \( g_x \) and \( g_y \) in \( \langle \cdot \rangle \) to be \( u \) and \( v \) respectively, and \( F_{x,y}(u, v) := \langle \sigma_x; \sigma_y \rangle_{g_x = u, g_y = v} \). A generalized Gaussian integration by parts (see Proposition 32), with \( f_{x,y}(u, v) = \mathbb{E}F_{x,y}(u, v), k = 2 \) and \( K_1 = K_2 = \varepsilon h^{-1} \), for any \( \varepsilon > 0 \), gives

\[
\left| \mathbb{E}g_x g_y f_{x,y} - \mathbb{E} \frac{\partial^2 f_{x,y}}{\partial g_x \partial g_y} \right| \leq 4h(1 + 3h) \mathbb{E}(|g_x| : |g_x| \geq \varepsilon h^{-1}) \mathbb{E}(|g_y|^2 : |g_y| \geq \varepsilon h^{-1}) \\
+ 12h^2(1 + 4h) \mathbb{E}(|g_x|^2 : |g_x| \geq \varepsilon h^{-1}) + 18\varepsilon(\varepsilon + 4h^2) \\
+ 9\varepsilon h \mathbb{E}(|g_y|^3 : |g_y| \geq \varepsilon h^{-1}) + 4h \\
+ 9h(\varepsilon + 4h^2) \mathbb{E}(|g_x|^3 : |g_x| \geq \varepsilon h^{-1}).
\]

Dividing this inequality by \( |V_n|^2 \) and summing over all \( x, y \in V_n \), the triangle inequality and the inequalities \( 0 \leq \frac{1}{|V_n|^2} \sum_{x,y} \mathbb{E} \frac{\partial^2 f_{x,y}}{\partial g_x \partial g_y} \leq 6h^2, \langle H_n^2 \rangle - \langle H_n \rangle^2 \geq 0, \)
give
\[
\mathbb{E}(\langle H_n^2 \rangle - \langle H_n \rangle^2) \leq 4h(1 + 3h) \left( \frac{1}{|V_n|} \sum_x \mathbb{E}(|g_x|^2 : |g_x| \geq \varepsilon h^{-1}) \right)^2 \\
+ 12h^2(1 + 4h) \frac{1}{|V_n|} \sum_x \mathbb{E}(|g_x|^2 : |g_x| \geq \varepsilon h^{-1}) \\
+ 18\varepsilon(\varepsilon + 4h^2) + 6h^2 \\
+ 9h(\varepsilon + 4h^2) \frac{1}{|V_n|} \sum_x \mathbb{E}(|g_x|^3 : |g_x| \geq \varepsilon h^{-1})
\]

Combining this with the inequality
\[
\nu(|H_n - \langle H_n \rangle|) \leq \sqrt{\mathbb{E}(\langle H_n^2 \rangle - \langle H_n \rangle^2)} ,
\]
and after using the assumption \((2)\), and the arbitrariness of \(\varepsilon\), one finds that
\[
\nu(|H_n - \langle H_n \rangle|) \to 0 .
\]
Finally, the proof follows by using the assumption \((6)\). □

The Lemma \textbf{3.3} plays an important role in the proof of the next result.

\textbf{Lemma 3.4} (Ghirlanda-Guerra identities). Given \(m \geq 2\), let \(f = f((R_{l,s})_{1 \leq l,s \leq m}) : \mathbb{R}^{m(m-1)/2} \to [-1,1] \) be a bounded measurable function of the overlaps \((4)\) that not change with \(n\). Then, under assumption \((2)\), the identities \((5)\) are satisfied for all \(\beta > 0\).

\textbf{Proof.} Let \(\langle \cdot \rangle_{g_x=u}\) be the Gibbs expectation defined by setting \(g_x\) in \(\langle \cdot \rangle\) to be \(u\) and \(F_x(u) := \langle \sigma_x^1 f \rangle_{g_x=u}\). Using \((3)\), a straightforward computation shows that
\[
\frac{\partial^j F_x(u)}{\partial u^j} = h^j \left( \langle \sigma_x^1 \cdot \left( \sum_{s=1}^m \sigma_s^5 - m\sigma_x^{m+1} \right)^j f \rangle \right)_{g_x=u} , \quad j = 1, 2, \ldots .
\]

Since \(\left| \frac{\partial^j F_x(u)}{\partial u^j} \right| \leq (2mh)^j \|f\|_{\infty} \), a generalized Gaussian integration by parts (see Proposition \textbf{3.1}) with \(f_x(u) := \mathbb{E}F_x(u)\), \(k = 2\) and \(K = \varepsilon h^{-1}\), for any \(\varepsilon > 0\), gives
\[
\left| \mathbb{E}g_x f_x - \mathbb{E}\frac{df_x}{dg_x} \right| \leq 4mh(1 + 2mh)\|f\|_{\infty} \mathbb{E}(|g_x|^2 : |g_x| \geq \varepsilon h^{-1}) + 6\varepsilon m^2 h\|f\|_{\infty} .
\]
Dividing this inequality by $|V_n|$ and summing over all $x \in V_n$, the triangle inequality gives
\[
\left| \nu \left( H_n(\sigma^1)f \right) - h \nu \left( \sigma^1_x \cdot \left( \sum_{s=1}^{m} \sigma^s_x - m\sigma^{m+1}_x \right) f \right) \right| \leq 4mh(1 + 2mh)\|f\|_\infty \frac{1}{|V_n|} \sum_x \mathbb{E}(|g_x|^2 : |g_x| \geq \varepsilon h^{-1}) + 6\varepsilon m^2 h\|f\|_\infty.
\]

Therefore, from both the assumption (2) and arbitrariness of $\varepsilon$, it follows that
\[
\limsup_{n \to \infty} \sup_f \left| \nu \left( H_n(\sigma^1)f \right) - h \nu \left( \left( \sum_{s=1}^{m} R_{1,s} - mR_{1,m+1} \right) f \right) \right| = 0.
\]

Since $\nu(|H_n - \nu(H_n)|) \not\to 0$ (by Lemma 3.3), it is well-known (see e.g. [16], Section 2.12) that (8) is sufficient to guarantee the validity of the Ghirlanda-Guerra identities (5). The proof of lemma is complete. \qed

**Remark 3.5 (Self-averaging of the overlap).** Under assumption (2), for any $\beta > 0$, such that $\lim_{n \to \infty} h\sqrt{|V_n|} = \infty$, it follows that
\[
\mathbb{E} \left( (R_{1,2}^2 - \langle R_{1,2} \rangle^2) \right) = \nu(R_{1,2} - \langle R_{1,2} \rangle)^2 \not\to 0 \quad \text{and} \quad \nu(m(\sigma) - \langle m(\sigma) \rangle)^2 \not\to 0,
\]
where $m(\sigma) := \frac{1}{|V_n|} \sum_x \sigma_x$ define the magnetization. For more details, see Auffinger and Chen (2016) [3], Example 3.

**Appendix**

**Proof of Proposition 3.2.** Let $g(x, y) := xf(x, y)$. Applying Taylor’s Theorem to $g$ for $(k-1)$-th and k-th orders,
\[
yg(x, y) = g(x, 0)y - \frac{\partial^{k-1}g(x, 0)}{\partial y^{k-1}} \frac{y^k}{(k-1)!} + \sum_{n=1}^{k-1} \frac{\partial^n g(x, 0)}{\partial y^n} \frac{y^{n+1}}{n!}
\]
\[
+ \frac{\partial^{k-1}g(x, a(y))}{\partial y^{k-1}} \frac{y^k}{(k-1)!},
\]
\[
= g(x, 0)y + \sum_{n=0}^{k-1} \frac{\partial^n g(x, 0)}{\partial y^n} \frac{y^{n+1}}{n!} + \frac{\partial^k g(x, b(y))}{\partial y^k} \frac{y^{k+1}}{k!}
\]
and using Taylor’s Theorem for $\frac{\partial g(x, y)}{\partial y}$ for the $(k-1)$-th order,
\[
\frac{\partial g(x, y)}{\partial y} = \sum_{n=1}^{k-1} \frac{\partial^n g(x, 0)}{\partial y^n} \frac{y^{n+1}}{(n-1)!} + \frac{\partial^k g(x, c(y))}{\partial y^k} \frac{y^{k+1}}{k!},
\]
where \(a(y), b(y), c(y)\) are some functions depending only on \(y\). From (9) and (11),

\[
\begin{align*}
y g(x, y) - \frac{\partial g(x, y)}{\partial y} &= g(x, 0) y - \frac{\partial^{k-1} g(x, 0)}{\partial y^{k-1}} \frac{y^k}{(k - 1)!} + \sum_{n=1}^{k-1} \frac{\partial^n g(x, 0)}{\partial y^n} h_n(y) \\
&\quad + \frac{\partial^{k-1} g(x, a(y))}{\partial y^{k-1}} \frac{y^k}{(k - 1)!} - \frac{\partial^k g(x, c(y))}{\partial y^k} \frac{y^{k-1}}{(k - 1)!},
\end{align*}
\]

where \(h_n(y) := \left(\frac{y^{n+1}}{n!} - \frac{y^{n-1}}{(n-1)!}\right)\). From (10) and (11),

\[
\begin{align*}
y g(x, y) - \frac{\partial g(x, y)}{\partial y} &= g(x, 0) y + \sum_{n=1}^{k-1} \frac{\partial^n g(x, 0)}{\partial y^n} h_n(y) \\
&\quad + \frac{\partial^k g(x, b(y))}{\partial y^k} \frac{y^{k+1}}{k!} - \frac{\partial^k g(x, c(y))}{\partial y^k} \frac{y^{k-1}}{(k - 1)!}.
\end{align*}
\]

On the other hand, again, using Taylor’s Theorem to \(\frac{\partial f(x, y)}{\partial y}\) for \((k - 1)\)-th and \(k\)-th orders,

\[
\begin{align*}
x \frac{\partial f(x, y)}{\partial y} &= \frac{\partial f(0, y)}{\partial y} x - \frac{\partial^k f(0, y)}{\partial x^k \partial y} \frac{x^k}{(k - 1)!} + \sum_{n=1}^{k-1} \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} \frac{x^{n+1}}{n!} \\
&\quad + \frac{\partial^k f(\tilde{a}(x), y)}{\partial x^k \partial y} \frac{x^k}{(k - 1)!},
\end{align*}
\]

\[
\begin{align*}
= \frac{\partial f(0, y)}{\partial y} x + \sum_{n=1}^{k-1} \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} \frac{x^{n+1}}{n!} + \frac{\partial^k f(\tilde{b}(x), y)}{\partial x^k \partial y} \frac{x^{k+1}}{k!},
\end{align*}
\]

and using Taylor’s Theorem to \(\frac{\partial^2 f(x, y)}{\partial x \partial y}\) for the \((k - 1)\)-th order,

\[
\begin{align*}
\frac{\partial^2 f(x, y)}{\partial x \partial y} &= \sum_{n=1}^{k-1} \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} \frac{x^{n-1}}{(n - 1)!} + \frac{\partial^{k+1} f(\tilde{c}(x), y)}{\partial x^k \partial y} \frac{x^{k-1}}{(k - 1)!},
\end{align*}
\]

where \(\tilde{a}(x), \tilde{b}(x), \tilde{c}(x)\) are some functions depending only on \(x\). From (14) and (16),

\[
\begin{align*}
x \frac{\partial f(x, y)}{\partial y} - \frac{\partial^2 f(x, y)}{\partial x \partial y} &= \frac{\partial f(0, y)}{\partial y} x - \frac{\partial^k f(0, y)}{\partial x^k \partial y} \frac{x^k}{(k - 1)!} + \sum_{n=1}^{k-1} \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} h_n(x) \\
&\quad + \frac{\partial^k f(\tilde{a}(x), y)}{\partial x^k \partial y} \frac{x^k}{(k - 1)!} - \frac{\partial^{k+1} f(\tilde{c}(x), y)}{\partial x^k \partial y} \frac{x^{k-1}}{(k - 1)!}.
\end{align*}
\]
From (15) and (16) we have

\[ \frac{x}{\partial f(x, y)} \frac{\partial f(x, y)}{\partial y} - \partial^2 f(x, y) = \frac{\partial f(0, y)}{\partial y} + \sum_{n=1}^{k-1} \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} h_n(x) \]
\[ + \frac{\partial^{k+1} f(\tilde{b}(x), y)}{\partial x^k \partial y} \frac{x^{k+1}}{k!} - \frac{\partial^{k+1} f(\tilde{c}(x), y)}{\partial x^k \partial y} \frac{x^{k-1}}{(k-1)!} \]

Summing (12) and (17) one get that

\[ xyf(x, y) - \frac{\partial^2 f(x, y)}{\partial x \partial y} = f(x, 0)xy + \frac{\partial f(0, y)}{\partial y} x - \frac{\partial^{-1} f(x, 0)}{\partial y^{-1}} \frac{xy^k}{(k-1)!} \]
\[ - \frac{\partial^k f(0, y)}{\partial x^{-k} \partial y} \frac{x^k}{(k-1)!} \]
\[ + \sum_{n=1}^{k-1} \left( \frac{\partial^n f(x, 0)}{\partial y^n} xh_n(y) + \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} h_n(x) \right) \]
\[ + \frac{\partial^{k-1} f(x, a(y))}{\partial y^{k-1}} \frac{xy^k}{(k-1)!} - \frac{\partial^{k-1} f(x, c(y))}{\partial y^{k-1}} \frac{xy^{k-1}}{(k-1)!} \]
\[ + \frac{\partial^{k} f(\tilde{a}(x), y)}{\partial x^{-k} \partial y} \frac{x^k}{(k-1)!} - \frac{\partial^{k+1} f(\tilde{c}(x), y)}{\partial x^k \partial y} \frac{x^{k-1}}{(k-1)!} \]

Summing (13) and (18),

\[ xyf(x, y) - \frac{\partial^2 f(x, y)}{\partial x \partial y} = f(x, 0)xy + \frac{\partial f(0, y)}{\partial y} x \]
\[ + \sum_{n=1}^{k-1} \left( \frac{\partial^n f(x, 0)}{\partial y^n} xh_n(y) + \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} h_n(x) \right) \]
\[ + \frac{\partial^{k} f(x, b(y))}{\partial y^{k}} \frac{xy^{k+1}}{k!} - \frac{\partial^{k} f(x, c(y))}{\partial y^{k}} \frac{xy^{k-1}}{(k-1)!} \]
\[ + \frac{\partial^{k+1} f(\tilde{b}(x), y)}{\partial x^k \partial y} \frac{x^{k+1}}{k!} - \frac{\partial^{k+1} f(\tilde{c}(x), y)}{\partial x^k \partial y} \frac{x^{k-1}}{(k-1)!} \]

Defining

\[ I_1 = I_2 := xyf(x, y) - \frac{\partial f(x, y)}{\partial x \partial y} - f(x, 0)xy - \frac{\partial f(0, y)}{\partial y} x \]
\[ - \sum_{n=1}^{k-1} \left( \frac{\partial^n f(x, 0)}{\partial y^n} xh_n(y) + \frac{\partial^{n+1} f(0, y)}{\partial x^n \partial y} h_n(x) \right) \]
Taking expectation of first inequality we have

\[ |I_1| \leq 2 \left\| \frac{\partial^{k-1}f}{\partial y^{k-1}} \right\|_{\infty (k-1)!} \frac{|x||y|^k}{k!} + 2 \left\| \frac{\partial^k f}{\partial x^{k-1} \partial y} \right\|_{\infty (k-1)!} \frac{|x|^k}{(k-1)!} + \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty (k-1)!} \frac{|x||y|^{k-1}}{(k-1)!} + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y^1} \right\|_{\infty (k-1)!} \frac{|x|^{k-1}}{(k-1)!} \]

and

\[ |I_2| \leq \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty} |x| \left( \frac{|y|^{k+1}}{k!} + \frac{|y|^{k-1}}{(k-1)!} \right) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y} \right\|_{\infty} \frac{|x|^{k+1}}{k!} + \frac{|x|^{k-1}}{(k-1)!} \].

Taking expectation of \(|I_1|\) on \(D := \{|x| \geq K_1, |y| \geq K_2\}\), with \(K_1, K_2 \geq 1\), for the first inequality we have

\[ \mathbb{E}(|I_1| : D) \leq \frac{2}{(k-1)!} \left( \left\| \frac{\partial^{k-1} f}{\partial y^{k-1}} \right\|_{\infty} + \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty} \right) \mathbb{E}(|x| : |x| \geq K_1) \mathbb{E}(|y|^k : |y| \geq K_2) \]

+ \[ \frac{2}{(k-1)!} \left( \left\| \frac{\partial^k f}{\partial x^k \partial y^1} \right\|_{\infty} + \left\| \frac{\partial^{k+1} f}{\partial x^{k-1} \partial y} \right\|_{\infty} \right) \mathbb{E}(|x|^k : |x| \geq K_2) \]

and taking expectation of \(|I_2|\) on the set \(|x| \leq K_1\) for the second inequality, we obtain

\[ \mathbb{E}(|I_2| : |x| \leq K_1) = \mathbb{E}(|I_2| : |x| \leq K_1, |y| \leq K_2) + \mathbb{E}(|I_2| : |x| \leq K_1, |y| \geq K_2) \]

\[ \leq \frac{(k+1)K_1}{k!} \left( K_2 \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty} \mathbb{E}(|y|^k) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y^1} \right\|_{\infty} \mathbb{E}(|x|^k) \right) \]

+ \[ \frac{(k+1)K_1}{k!} \left( \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty} \mathbb{E}(|y|^k : |y| \geq K_2) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y^1} \right\|_{\infty} \mathbb{E}(|x|^k) \right) \].

Analogously, taking expectation of \(|I_2|\) on the set \(|y| \leq K_2\) for the second inequality, it is proved that

\[ \mathbb{E}(|I_2| : |y| \leq K_2) \leq \frac{(k+1)K_1}{k!} \left( K_2 \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty} \mathbb{E}(|y|^k) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y^1} \right\|_{\infty} \mathbb{E}(|x|^k) \right) \]

+ \[ \frac{(k+1)K_1}{k!} \left( K_2 \left\| \frac{\partial^k f}{\partial y^k} \right\|_{\infty} \mathbb{E}(|x| : |x| \geq K_1) \mathbb{E}(|y|^k) + \left\| \frac{\partial^{k+1} f}{\partial x^k \partial y^1} \right\|_{\infty} \mathbb{E}(|x|^k : |x| \geq K_1) \right) \].

Since \(x, y\) are two random variables such that their first \(k \geq 2\) moments match those of a Gaussian random variable, it follows that \(\mathbb{E}h_n(x) = \mathbb{E}h_n(y) = 0, n = 1, \ldots, k-1\).
Then,
\[
\left| \mathbb{E}(xyf(x, y) - \mathbb{E} \frac{\partial^2 f(x, y)}{\partial x \partial y}) \right| = |\mathbb{E}(I_1)| = |\mathbb{E}(I_2)| = |\mathbb{E}(I_1 : D) + \mathbb{E}(I_2 : D^c)|
\]
\[
\leq \mathbb{E}(|I_1 : D|) + \mathbb{E}(|I_2| : |x| \leq K_1) + \mathbb{E}(|I_2| : |y| \leq K_2).
\]
Combining the above inequality with (21), (22) and (23), the proof follows. 

\[\square\]

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