Some new sharp bounds for the spectral radius of a nonnegative matrix and its application

Jun He, Yan-Min Liu, Jun-Kang Tian and Xiang-Hu Liu

Abstract

In this paper, we give some new sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix. Using these bounds, we obtain some new and improved bounds for the signless Laplacian spectral radius of a graph or a digraph.

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. Let $N = \{1, \ldots, n\}$, for $i \in N$. We assume that $d_i$ is the degree of vertex $v_i$. Let $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the degree diagonal matrix of the graph $G$ and $A(G) = (a_{ij})$ be the adjacency matrix of the graph $G$. Then the matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of the graph $G$. The largest modulus of eigenvalues of $Q(G)$ is denoted by $\rho(G)$, which is also called the signless Laplacian spectral radius of $G$.

Let $\overrightarrow{G} = (V, E)$ be a digraph with vertex set $V(\overrightarrow{G}) = \{v_1, \ldots, v_n\}$ and arc set $E(\overrightarrow{G})$. Let $d'_i$ be the out-degree of vertex $v_i$, $D(\overrightarrow{G}) = \text{diag}(d'_1, d'_2, \ldots, d'_n)$ be the out-degree diagonal matrix of the digraph $\overrightarrow{G}$, and $A(\overrightarrow{G}) = (a_{ij})$ be the adjacency matrix of the digraph $\overrightarrow{G}$. Then the matrix $Q(\overrightarrow{G}) = D(\overrightarrow{G}) + A(\overrightarrow{G})$ is called the signless Laplacian matrix of the digraph $\overrightarrow{G}$. The largest modulus of eigenvalues of $Q(\overrightarrow{G})$ is denoted by $\rho(\overrightarrow{G})$, which is also called the signless Laplacian spectral radius of $\overrightarrow{G}$.

In recent decades, there are many bounds on the signless Laplacian spectral radius of a graph (digraph) [1–3]. Let $m_i = \frac{\sum_{j \sim i} d_j}{d_i}$ be the average degree of the neighbours of $v_i$ in $G$ and $m'_i = \frac{\sum_{j \sim i} d'_j}{d'_i}$ be the average out-degree of the out-neighbours of $v_i$ in $\overrightarrow{G}$. In this paper, we assume that the graph (digraph) is simple and connected (strong connected).

In 2013, Maden, Das, and Cevik [4] obtained the following bounds for the signless Laplacian spectral radius of a graph:

$$\rho(G) \leq \max_{i \neq j} \left\{ \frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2} \right\}.$$ (1)
In 2016, Xi and Wang [5] obtained the following bounds for the signless Laplacian spectral radius of a digraph:

\[
\rho(\overrightarrow{G}) \leq \max_{i \sim j} \left\{ \frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_j^+}}{2} \right\}.
\] (2)

In this paper, we improve the bounds for the signless Laplacian spectral radius of a graph (digraph) that are given in (1) and (2).

2 Main result

In this section, some upper and lower bounds for the spectral radius of a nonnegative irreducible matrix are given. We need the following lemma.

Lemma 2.1 ([6]) Let A be a nonnegative matrix with the spectral radius \( \rho(A) \) and the row sum \( r_1, r_2, \ldots, r_n \). Then \( \min_{1 \leq i \leq n} r_i \leq \rho(A) \leq \max_{1 \leq i \leq n} r_i \). Moreover, if the matrix A is irreducible, then the equalities hold if and only if \( r_1 = r_2 = \cdots = r_n \).

Theorem 2.1 Let \( A = (a_{ij}) \) be an irreducible and nonnegative matrix with \( a_{ii} = 0 \) for all \( i \in N \) and the row sum \( r_1, r_2, \ldots, r_n \). Let \( B = A + M \), where \( M = \text{diag}(t_1, t_2, \ldots, t_n) \) with \( t_i \geq 0 \) for any \( i \in N \), \( s_i = \sum_{j=1}^{n} a_{ij}t_j \), \( s_i = s_i - a_{ii}r_i \). Let \( \rho(B) \) be the spectral radius of B and let

\[
f(i, j) = \frac{t_i + t_j + s_i}{r_i} + \frac{\sqrt{(t_i - t_j + s_j)^2 + 4s_js_i}}{r_i},
\]

for any \( i, j \in N \). Then

\[
\min_{1 \leq i \leq n} \max_{j \neq i} \left\{ f(i, j), a_{ij} \neq 0 \right\} \leq \rho(B) \leq \max_{1 \leq i \leq n} \min_{j \neq i} \left\{ f(i, j), a_{ij} \neq 0 \right\}.
\] (3)

Moreover, either of the equalities in (3) holds if and only if \( t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j} \) for any distinct \( i, j \in N \).

Proof Let \( R = \text{diag}(r_1, r_2, \ldots, r_n) \). Since the matrix A is nonnegative irreducible, the matrix \( R^{-1}BR \) is also nonnegative and irreducible. By the famous Perron-Frobenius theorem [6], there is a positive eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \) corresponding to the spectral radius of \( R^{-1}BR \).

Upper bounds: Let \( x_p > 0 \) be an arbitrary component of \( x \), \( x_q = \max\{x_k, 1 \leq k \leq n\} \). Obviously, \( p \neq q, a_{pq} \neq 0 \). By \( R^{-1}BRx = \rho(B)x \), we have

\[
\rho(B)x_p = t_px_p + \sum_{k=1, k \neq p}^{n} \frac{a_{pk}r_kx_k}{r_p} \leq t_px_p + \frac{x_q}{r_p} \sum_{k=1}^{n} a_{pk}r_k \leq t_px_p + \frac{x_qs_p}{r_p}.
\] (4)

Similarly, we have

\[
\rho(B)x_q = t_qx_q + \sum_{k=1, k \neq q}^{n} \frac{a_{qk}r_kx_k}{r_q} \leq \left( t_q + \frac{s_q - a_{qp}r_q}{r_q} \right)x_q + \frac{a_{qp}r_q}{r_q}x_p.
\] (5)
By (4), (5), and \( \rho(B) - t_p > 0, \rho(B) - t_q > 0 \), we have

\[
\left( \rho(B) - t_p \right) \left( \rho(B) - t_q - \frac{s_q - a_{qp}r_p}{r_q} \right) \leq \frac{s_p a_{qp}}{r_q}.
\]

Therefore,

\[
\rho(B) \leq \frac{t_p + t_q + \frac{s_p a_{qp}}{r_q} + \sqrt{(t_p - t_q - \frac{s_q}{r_q})^2 + \frac{4 s_p a_{qp}}{r_q}}}{2}.
\]

(6)

This must be true for every \( p \neq q \). Then

\[
\rho(B) \leq \min_{j \neq q} \frac{t_j + t_q + \frac{s_j}{r_j} + \sqrt{(t_j - t_q - \frac{s_j}{r_j})^2 + \frac{4 s_j a_{ij}}{r_j}}}{2}.
\]

(7)

This must be true for any \( q \in N \). Then

\[
\rho(B) \leq \max_{1 \leq i \leq n} \min_{j \neq i} \left\{ \frac{t_i + t_j + \frac{s_j}{r_j} + \sqrt{(t_i - t_j - \frac{s_j}{r_j})^2 + \frac{4 s_j a_{ij}}{r_j}}}{2}, a_{ij} \neq 0 \right\}.
\]

(8)

Lower bounds: Let \( x_p > 0 \) be an arbitrary component of \( x \), \( x_q = \min \{ x_k \mid 1 \leq k \leq n \} \). Obviously, \( p \neq q \), \( a_{pq} \neq 0 \). By \( R^{-1}BRx = \rho(B)x \), we have

\[
\rho(B)x_p = t_p x_p + \sum_{k=1, k \neq p}^n a_{pk} r_k x_k \geq t_p x_p + \frac{x_q}{r_p} \sum_{k=1}^n a_{pk} r_k \geq t_p x_p + \frac{x_q s_p}{r_p}.
\]

(9)

Similarly, we have

\[
\rho(B)x_q = t_q x_q + \sum_{k=1, k \neq q}^n a_{qk} r_k x_k \geq \left( t_q + \frac{s_q - a_{qp} r_p}{r_q} \right) x_q + \frac{a_{qp} r_p}{r_q} x_p.
\]

(10)

By (9), (10), and \( \rho(B) - t_p > 0, \rho(B) - t_q > 0 \), we have

\[
\left( \rho(B) - t_p \right) \left( \rho(B) - t_q - \frac{s_q - a_{qp} r_p}{r_q} \right) \geq \frac{s_p a_{qp}}{r_q}.
\]

(11)

Therefore,

\[
\rho(B) \geq \frac{t_p + t_q + \frac{s_p a_{qp}}{r_q} + \sqrt{(t_p - t_q - \frac{s_q}{r_q})^2 + \frac{4 s_p a_{qp}}{r_q}}}{2}.
\]

(12)

This must be true for every \( p \neq q \). Then

\[
\rho(B) \geq \max_{j \neq q} \frac{t_j + t_q + \frac{s_j}{r_j} + \sqrt{(t_j - t_q - \frac{s_j}{r_j})^2 + \frac{4 s_j a_{ij}}{r_j}}}{2}.
\]

(13)

This must be true for all \( q \in N \). Then

\[
\rho(B) \geq \min_{1 \leq i \leq n} \max_{j \neq i} \left\{ \frac{t_i + t_j + \frac{s_j}{r_j} + \sqrt{(t_i - t_j - \frac{s_j}{r_j})^2 + \frac{4 s_j a_{ij}}{r_j}}}{2}, a_{ij} \neq 0 \right\}.
\]

(14)
From (4), (5), and $x_p > 0$ as an arbitrary component of $x$, we get $x_k = x_q = x_p$ for all $k$. Then we see easily that the right equality holds in (8) for $t_i + \frac{s_i}{t_i} = t_j + \frac{s_j}{t_j}$ for any distinct $i, j \in N$. The proof of the left equality in (3) is similar to the proof of the right equality, and we omit it here.

Thus, we complete the proof. □

3 Signless Laplacian spectral radius of a graph

In this section, we will apply Theorem 2.1 to obtain some new results on the signless Laplacian spectral radius $\rho(G)$ of a graph.

Theorem 3.1 Let $G = (V, E)$ be a simple connected graph on $n$ vertices. Then

$$\min_{1 \leq i \leq n} \max_{i \neq j} \left\{ \frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2} \right\} \leq \rho(G) \leq \max_{1 \leq i \leq n} \min_{i \neq j} \left\{ \frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2} \right\}. \quad (15)$$

Moreover, one of the equalities in (15) holds if and only if $G$ is a regular graph.

Proof We apply Theorem 2.1 to $Q(G)$. Let $t_i = 0$ for any $i \in N$. Then $f(i, j) = \frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2}$. Thus (15) holds.

And the equality holds in (15) for regular graphs if and only if $G$ is a regular graph. □

Remark 3.1 Obviously, we have

$$\max_{1 \leq i \leq n} \min_{i \neq j} \left\{ \frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2} \right\} \leq \max_{i \neq j} \left\{ \frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2} \right\}. \quad (16)$$

That is to say, our upper bound in Theorem 3.1 is always better than the upper bound (1) in [4].

Theorem 3.2 Let $G = (V, E)$ be a simple connected graph on $n$ vertices. Then

$$\rho(G) \geq \min_{1 \leq i \leq n} \max_{i \neq j} \left\{ \frac{d_i + d_j + m_j - d_i/d_j + \sqrt{(d_i - d_j - m_j + d_i/d_j) + 4d_i}}{2} \right\} \quad (16)$$

and

$$\rho(G) \leq \max_{1 \leq i \leq n} \min_{i \neq j} \left\{ \frac{d_i + d_j + m_j - d_i/d_j + \sqrt{(d_i - d_j - m_j + d_i/d_j) + 4d_i}}{2} \right\}. \quad (17)$$

Moreover, one of the equalities in (16), (17) holds if and only if $G$ is a regular graph or a bipartite semi-regular graph.

Proof We apply Theorem 2.1 to $Q(G)$. Let $t_i = d_i$, $s_i = \sum_{j=1}^{n} a_{ij}r_j = d_im_i$ for any $1 \leq i \leq n$. Then $f(i, j) = \frac{d_i + d_j + m_j - d_i/d_j + \sqrt{(d_i - d_j - m_j + d_i/d_j) + 4d_i}}{2}$. Thus (16), (17) hold.
And the equality holds if and only if \( G \) is a regular graph or a bipartite semi-regular graph. \( \square \)

4 Signless Laplacian spectral radius of a digraph

In this section, we will apply Theorem 2.1 to obtain some new results on the signless Laplacian spectral radius \( \rho(G) \) of a digraph.

**Theorem 4.1** Let \( G = (V, E) \) be a strong connected digraph on \( n \) vertices. Then

\[
\min_{1 \leq i \leq n} \max_{j \neq i} \left\{ \frac{d^+_i + 2d^-_j - 1 + \sqrt{(d^+_i - 2d^-_j + 1)^2 + 4d^+_i}}{2} \right\} \\
\leq \rho(G) \leq \max_{i \neq j} \min_{1 \leq i \leq n} \left\{ \frac{d^+_i + 2d^-_j - 1 + \sqrt{(d^+_i - 2d^-_j + 1)^2 + 4d^+_i}}{2} \right\},
\]

(18)

Moreover, one of the equalities in (18) holds if and only if \( G \) is a regular digraph.

**Proof** We apply Theorem 2.1 to \( Q(G) \). Let \( t_i = 0 \) for any \( 1 \leq i \leq n \). Then \( f(i, j) = \frac{d^+_i + 2d^-_j - 1 + \sqrt{(d^+_i - 2d^-_j + 1)^2 + 4d^+_i}}{2} \). Then the inequality (18) holds.

And the equality holds in (18) if and only if \( G \) is a regular digraph. \( \square \)

**Remark 4.1** Obviously, we have

\[
\max_{i \neq j} \min_{1 \leq i \leq n} \left\{ \frac{d^+_i + 2d^-_j - 1 + \sqrt{(d^+_i - 2d^-_j + 1)^2 + 4d^+_i}}{2} \right\} \\
\leq \max_{i \neq j} \left\{ \frac{d^+_i + 2d^-_j - 1 + \sqrt{(d^+_i - 2d^-_j + 1)^2 + 4d^+_i}}{2} \right\}.
\]

That is to say, our upper bound in Theorem 4.1 is always better than the upper bound (2) in [5].

**Theorem 4.2** Let \( G = (V, E) \) be a strong connected digraph on \( n \) vertices. Then

\[
\rho(G) \geq \min_{1 \leq i \leq n} \max_{j \neq i} \left\{ \frac{d^+_i + d^-_j + m^-_j - d^+_i / d^+_j + \sqrt{(d^+_i - d^-_j - m^-_j + d^+_i / d^+_j)^2 + 4d^+_i}}{2} \right\}
\]

(19)

and

\[
\rho(G) \leq \max_{i \neq j} \min_{1 \leq i \leq n} \left\{ \frac{d^+_i + d^-_j + m^-_j - d^+_i / d^+_j + \sqrt{(d^+_i - d^-_j - m^-_j + d^+_i / d^+_j)^2 + 4d^+_i}}{2} \right\}.
\]

(20)

Moreover, one of the equalities in (19), (20) holds if and only if \( G \) is a regular digraph or a bipartite semi-regular digraph.
Proof. We apply Theorem 2.1 to $Q(\overrightarrow{G})$. Let $t_i = d_i^+; s_i = \sum_{j=1}^{n} a_{ij} r_j = d_i^+ m_i^+$ for any $1 \leq i \leq n$. Then $f(i,j) = \frac{d_i^+ r_i^+ - d_i^+ / d_j^+}{2} \sqrt{(d_i^+ - d_i^+ - m_i^+ + d_i^+ / d_j^+ + d_i^+ / d_j^+)^2}$. Thus (19), (20) hold.

One sees easily that the equality holds if and only if $\overrightarrow{G}$ is a regular digraph or a bipartite semi-regular digraph. □

5 Conclusion
In this paper, we give some new sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix. Using these bounds, we obtain some new and improved bounds for the signless Laplacian spectral radius of a graph or a digraph which are better than the bounds in [4, 5].

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Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
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