N=2-Supersymmetric Dynamics of a Spin-1/2 Particle in an Extended External Field

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Abstract

One considers the quantum dynamics of a charged spin-$\frac{1}{2}$ particle in an extended external electromagnetic field that arises from the reduction of a 5-dimensional Abelian gauge theory. The non-relativistic regime of the reduced 4D-dynamics is worked out and one identifies the system as a sector of an N=2-supersymmetric quantum-mechanical dynamics. The full supersymmetric model is studied and one checks the algebra of the fermionic charges; the conclusion is that no central charge drops out. The possible rôle of the extra external fields, a scalar and a magnetic-like field, is discussed.
I. INTRODUCTION

Supersymmetric quantum field theory at low energies (non-relativistic limit) raises special interest for a great deal of reasons, [6], [9], [13], [17]. If Supersymmetry is a symmetry of Nature, what we see today must be its low-energy remnant through the breaking realised by means of some mechanism [12], [14], [15]. In this limit, the underlying field theory should approach a Galilean invariant supersymmetric field theory and, by the Bargmann superselection rule [8], such a field theory should be equivalent to a supersymmetric Schrödinger equation in each particle number sector of the theory.

In this paper, we shall consider a particle of mass $m$ and spin-1/2 in an external field, [4], [10], [2], [3]. We build up a superspace action for this model, read off the supersymmetry transformations of the component coordinates and then obtain the supersymmetric charge operators [1]. We analyse their algebra and pursue the investigation of the possible existence of a central charge in the system.

We set our discussion by first considering the case of (1+2) dimensions. The Dirac action in such a space-time may be written as

$$ L = \overline{\Psi} (i\gamma^\mu D_\mu - m) \Psi, $$

where $\eta = (1, -1, -1)$ and $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, $\gamma^2 = \sigma_1$, $\gamma_3 = i\gamma_0 \gamma_1$, $D_\mu \equiv \partial_\mu + ieA_\mu$. From the equation of motion,

$$ (i\gamma^\mu D_\mu - m) \Psi = 0, $$

$$ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, $$

and with the approximation

$$ E + m - e\phi \cong 2m, $$

$$ \psi_2 \cong -\frac{(D_3 + iD_1)}{2m} \psi_1. $$
the Pauli Hamiltonian in 1+2 \cite{5} takes over the form:

\[ H = \frac{1}{2m} \nabla^2 - \frac{ie}{2m} \nabla \cdot \vec{A} - \frac{ie}{m} \frac{\nabla}{\cdot} \vec{A} - \frac{e^2}{2m} |\vec{A}|^2 - e\phi. \]

The minimal coupling does not generate here any coupling term of the type spin-magnetic field as it is the case for the Pauli Hamiltonian in D=1+3. We can also show that this coupling term cannot appear from a non-minimal coupling of the type:

\[ D_\mu = \partial_\mu + ieA_\mu + ig\tilde{F}_\mu, \]

\[ \tilde{F}_i = \frac{1}{2} \epsilon_{ijk} F^{jk}. \]

In (1+3)D, we see that the bosonic degree of freedom, \( x^i(t) \) (position), matches with the fermionic degree of freedom, \( S^i(t) \) (spin). This makes possible the construction of superfields and then of a manifestly supersymmetric Lagrangian with linearly realised supersymmetry \cite{1}. However, in (1+2)D, the same does not occur; angular momentum is not a vector any longer, the number of degrees of freedom of \( x^i \) and the spin do not match anymore and then we cannot in this way build up superfields nor a supersymmetric Lagrangian in superspace. We can try to work in (1+3)D, starting from Dirac, taking the non-relativistic limit and making a dimensional reduction to (1+2)D. But, we see that it is not possible to drag into (1+2)D the analogue of the coupling of spin-magnetic field that exists in (1+3)D \cite{5}. We will try to study this model in (1+3)D with N=1 and N=2 supersymmetry; we shall however start from (1+4)D, where we may propose a matching between coordinates and 5D spin and then carry out the dimmensional reduction to 4D, getting to an extended eletromagnetic-like field.

\section*{II. N=1-D=5 REDUCED TO N=1-D=4}

We consider the following action in D=4+1,

\[ L = \bar{\Psi}(i\Gamma^\mu D_\mu - m)\Psi; \]
where we define:

\[ \hat{\mu} \in \{0, 1, 2, 3, 4\}, \quad \eta = (+, -, -, -), \quad (2) \]

\[ x^{\hat{\mu}} = (t, x, y, z, w), \quad F_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} A_{\hat{\nu}} - \partial_{\hat{\nu}} A_{\hat{\mu}}, \]

\[ D_{\hat{\mu}} = \partial_{\hat{\mu}} + i e A_{\hat{\mu}}, \quad A_{\hat{\mu}} = (A_i, \Lambda), \quad \{\Gamma^\hat{\mu}, \Gamma^\hat{\nu}\} = 2\eta^{\hat{\mu}\hat{\nu}}, \]

\[ \partial^i \Leftrightarrow \vec{\nabla}, \quad \partial_0 \Leftrightarrow \frac{\partial}{\partial t}, \quad E^i \Leftrightarrow \vec{E}, \quad B^i \Leftrightarrow \vec{B}, \]

\[ F_{0i} = \mathcal{E}, \quad F_{4i} = \mathcal{B}_i, \quad F_{ij} = \epsilon_{ijk} B_k, \quad F_{0i} = E_i, \]

where \( i, j, k \in \{1, 2, 3\} \). Notice that the scalar, \( \mathcal{E} \), and the vector, \( \vec{B} \), accompany the vectors \( \vec{E} \) and \( \vec{B} \), later on to be identified with the electric and magnetic fields, respectively. Our explicit representation for the gamma-matrices is given in the Appendix. We set:

\[ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \overline{\Psi} = \Psi^\dagger \Gamma^0 \]

We now take the equation of motion for \( \Psi \) from the Euler-Lagrange equations (1). We suppose a stationary solution,

\[ \Psi(x, t) = \exp(i\epsilon t)\chi(x); \]

here, we are considering that the external field does not depend on \( t \), and \( A_0 = 0 \). Then, proposing the reduction prescription:

\[ \partial_4(A_{\mu}) = 0, \]

we obtain

\[ \vec{E} \Leftrightarrow F_{0i} \equiv \partial_0 A_i - \partial_i A_0 = 0; \quad (3) \]
\[ E \equiv F_{04} \equiv \partial_0 A_4 - \partial_4 A_0 = 0; \]

\[ F_{ij} = \partial_i A_j - \partial_j A_i \equiv -\epsilon_{ijk} B_k; \]

\[ F_{i4} = \partial_i A_4 - \partial_4 A_i \equiv \partial_i \Lambda \equiv -B_i. \]

Then, considering the non-relativistic limit of the reduced theory, the equations of motions read:

\[ (\epsilon + m) I_2 \chi_1 + (-i\sigma^i(\partial_i + ieA_i) + iI_2 eA_4) \chi_2 = 0. \]  \hspace{1cm} (4)

\[ (i\sigma^i(\partial_i + ieA_i) + iI_2 eA_4) \chi_1 + (-\epsilon + m) I_2 \chi_2 = 0. \]

Therefore, we see that \( \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \) looses two degrees of freedom, described by the "weak" spinor, \( \chi_2 \). So the Pauli-like Hamiltonian we get to reads as below:

\[ H = -\frac{1}{2m} \left[ (\vec{\nabla} - ie \vec{A})^2 + e \vec{\sigma} (\vec{\nabla} \times \vec{A}) ight] - e \vec{\sigma} \vec{\nabla} (A_4) - e^2 (A_4)^2, \]  \hspace{1cm} (5)

where we used (2).

If we define, \( P^i = -i \vec{\nabla}^i (\hbar = 1) \), then the Hamiltonian (5) will become

\[ H = \frac{1}{2m} \left[ (p_i - ieA_i)^2 - eB_iS_i + eB_iS_i - \frac{e^2}{2} \Lambda^2 \right]; \]

the corresponding Lagrangian is given by:

\[ L = \frac{1}{2}(\dot{x}_i)^2 + \frac{i}{2} \bar{\psi}_i \dot{\psi}_i + eA_i \dot{x}_i + \frac{ie}{2} B_i \epsilon_{ijk} \dot{\psi}_j \psi_k - \frac{ie}{2} B_i \epsilon_{ijk} \bar{\psi}_j \dot{\psi}_k - \frac{e^2 \Lambda^2}{2}, \]

where the dot is a derivative with respect to \( t \). We can also write
\[
L = \frac{1}{2}(\dot{x}_i)^2 + \frac{i}{2}\psi_i \dot{\psi}_i + eA_i \dot{x}_i, \tag{7}
\]

\[
-eB_i S_i + eB_i S_i - \frac{\epsilon^2 \Lambda^2}{2},
\]

where we define the spin by the product below:

\[
S_i = -\frac{i}{2} \epsilon_{ijk} \psi_j \psi_k.
\]

III. N=1-SUPERSYMMETRIC ACTION

To render more systematic our discussion, we think it is advisable to set up a superfield approach. We can define the N=1-supersymmetric model in analogy with the model presented above, eq.(7). We start defining the superfields by

\[
\Phi_i(t, \theta) = x_i(t) + i\theta \psi_i(t) \quad \Sigma(t, \theta) = \xi(t) + \theta R(t), \tag{8}
\]

\[
\Lambda(x) = A_4(x) \quad \Lambda(\Phi) = \Lambda(x) + i(\partial_j \Lambda(x)) \theta \psi_j.
\]

The supercharge operators and the covariant derivatives are given by:

\[
Q = \partial_\theta + i\theta \partial_t \quad D = \partial_\theta - i\theta \partial_t \quad H = i\partial_t. \tag{9}
\]

Then, the N=1-supersymmetric Lagrangian that generates, Lagrangian (3) can be written in superfields as:

\[
\mathcal{L} = \frac{i}{2} \dot{\Phi}_i D\Phi_i + ie(D\Phi_i)A_i(\Phi) + \frac{1}{2} \Sigma D \Sigma +
\]

\[
-e\Sigma \Lambda(\Phi) + \frac{ie}{2} \epsilon_{ijk} \partial_j \Lambda(\Phi) \Phi_j D\Phi_k,
\]

where we have set \( m = 1 \).

The commutators and anticommutators for the superfield components are

\[
[\psi_i, \psi_j]_+ = \delta_{ij}, \quad [\xi, \xi]_+ = -2i, \quad [x_i, p_j] = i\delta_{ij}, \tag{11}
\]

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\[ [\psi_i, \xi]_+ = 0; \]

the other commutators vanish.

The component field \( R \) does not have dynamics; then, we use the equation of motion to remove it from the Lagrangian:

\[ R = e\Lambda. \quad (12) \]

The supersymmetric action is then

\[ S = \int dt d\theta \mathcal{L} \]

We have written the supersymmetric Lagrangian in components, (11), under the form,

\[ \mathcal{L} = K + \theta L \]

where we define \( K \) and \( L \) as follows:

\[
K = -\frac{1}{2} \dot{x}_i \dot{\psi}_i - e\psi_i A^i + \frac{\xi R}{2} - e\xi \Lambda - e \frac{1}{2} \epsilon_{ijk} B_i \dot{x}_j \psi_k \\
L = \frac{1}{2} \dot{x}_i \dot{x}_i - \frac{1}{2} \dot{\psi}_i \dot{\psi}_i + e\dot{x}_i A_i - \frac{ie}{2} F_{ij} \psi_i \psi_j + \\
+ \frac{i}{2} e\xi \dot{\xi} + \frac{R^2}{2} - eR \Lambda + ie\xi \psi_j B_j + \\
+ \frac{e}{2} \epsilon_{ijk} B_i \dot{x}_j \dot{\psi}_k - \frac{ie}{2} \epsilon_{ijk} B_j \psi_j \psi_k - \frac{ie}{2} \epsilon_{ijk} \psi_k \partial_i B_r x_j \psi_k \\
\]

Also from \( L \) in (13), we can read the Hamiltonian:

\[
H = \frac{1}{2} (\dot{p}^i - e A^i)^2 + \frac{ie}{2} \epsilon_{ijk} B_k \psi_i \psi_j + \\
- \frac{ie}{2} \epsilon_{ijk} B_i \dot{\psi}_j \psi_k - \frac{ie}{2} \epsilon_{rji} (\partial_r B_k) \dot{x}_j \psi_k \psi_i + \frac{ie}{2} \epsilon_{ijk} \psi_i \psi_k + \\
- \frac{e}{2} \epsilon_{jki} B_j x_k (p_i - e A^i) + \frac{e^2}{8} \epsilon_{jki} \epsilon_{mln} B_j B_k x_m x_n + \frac{e^2 \Lambda^2}{2}; \\
\]

in this equation, we have eliminated the component-field without dynamical character.
A. The supersymmetric charge

Acting with the supercharge operator (11) on the superfields (8), we can obtain the
supersymmetry transformations of components fields; they are:
\[ \delta \psi^i = \epsilon \dot{x}^i, \quad \delta \xi = -i \epsilon R, \quad \delta R = -\epsilon \dot{\xi}, \quad \delta x^i = -i \epsilon \psi^i. \] (15)

From the supersymmetric transformations and the Lagrangian (13), we can analytically
calculate the supercharge, through the Noether’s theorem. The charge operator comes out
to be
\[ Q = \psi_i \dot{x}_i + \frac{1}{2} (1 - i) \xi R - \epsilon \xi \Lambda. \] (16)

The supercharge algebra reads
\[ [Q, Q]_+ = 2H \quad \text{or} \quad Q^2 = H, \]
where \( H \) is the Hamiltonian of eq. (14).

B. Equations of motion for the extended Electromagnetism in \( D=4+1 \)

We start by considering the equations of motion
\[ \epsilon^{\hat{\mu} \hat{\nu} \hat{\gamma} \hat{\delta}} \partial_{\hat{\nu}} F_{\hat{\gamma} \hat{\delta}} = 0 \quad \text{and} \quad \partial_{\hat{\gamma}} F^{\hat{\gamma} \hat{\delta}} = \rho_{\hat{\delta}}; \] where \( F_{\hat{\gamma} \hat{\delta}} \) is the field strength defined in (2) and \( \epsilon^{\hat{\mu} \hat{\nu} \hat{\gamma} \hat{\delta}} \)
is the Levi-Civita tensor in 5 dimensions. Then, we have:
\[ \bar{\nabla} \cdot \bar{B} = J^5, \]
\[ \bar{\nabla} \times \bar{B} = 0, \]
\[ \bar{\nabla} \cdot \bar{B} = 0, \]
\[ \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \]
\[ \bar{\nabla} \cdot \bar{E} = J^0, \]
\[ \bar{\nabla} \cdot \mathcal{E} = -\frac{\partial \bar{B}}{\partial t}, \]
\[ \bar{\nabla} \times \bar{B} = \frac{\partial \bar{E}}{\partial t} + J^i. \] (17)
The Lorentz force, calculated using the fact that the Hamiltonian in D=4+1 is

\[ H = \frac{1}{2m} \left( \vec{p} + e \vec{A} \right)^2 + eA^0, \]  

reads as below:

\[ m \frac{\partial^2 x^5}{\partial t^2} = e \vec{B} \vec{v} + e \mathcal{E}, \]  

\[ m \frac{\partial^2 \vec{x}}{\partial t^2} = e \vec{v} \times \vec{B} + e \vec{B} \dot{x}_5 + e \vec{E}. \]

IV. A TOY MODEL WITH N=2-SU.SY.

The superspace coordinates are \((t, \theta_1, \theta_2)\) with \(\theta_1\) and \(\theta_2\) real, or \((t, \theta, \bar{\theta})\) with \(\theta = \theta_1 + i\theta_2\) and \(\bar{\theta}\) is its complex conjugate. \(\theta, \bar{\theta}\) are complex Grassmannian variables.

The translations

\[ \delta t = ie^* \theta + i e \theta, \quad \delta \theta^* = ie^* \quad \text{and} \quad \delta \theta = -i e, \]

define a susy. transformation in superspace, where \(e\) is a complex Grassmannian parameter. The differential representation for the supercharge operators is given below:

\[ Q_\theta = \partial_\theta + i \bar{\theta} \partial_t, \quad Q_{\bar{\theta}} = \partial_{\bar{\theta}} + i \theta \partial_t, \]  

\[ [Q_\theta, Q_{\bar{\theta}}]_+ = 0, \quad [Q_\theta, Q_\theta]_+ = 0, \]  

\[ [Q_\theta, Q_{\bar{\theta}}]_+ = 2i \frac{\partial}{\partial t} = 2H, \]

where \(H\) will be identified with the Hamiltonian.
If we are able to define an invariant action, $S$, for a translation in superspace $(t, \theta, \bar{\theta})$ [1], [13], where the transformations parameter is a Grassmannian parameter, we say that this action has an $N=2$-supersymmetry. In this way, we can define $N=2$-superfields [16] as we discuss in the sequel.

The complex chiral superfield ($\bar{D}\Phi_i = 0$) admits the following $\theta$-expansion:

$$\Phi_i = x_i(t) + \theta \psi_i(t) + i\bar{\theta} \dot{x}_i(t). \quad (23)$$

$\psi_i$ is a Grassmannian variable and $x_i$ is a commutative complex variable.

We have also a real superfield ($\Sigma = \Sigma^*$),

$$\Sigma = \xi(t) + i\theta \chi(t) + i\bar{\theta} \epsilon(t) + \bar{\theta} \epsilon R(t), \quad (24)$$

where $\xi$, $R$ are real component fields; $\chi$ and $\chi^*$ are complex components fields; $\chi$ is a Grassmannian variable and $R$ is a commutative variable. From the susy. transformations of these superfields, the component coordinates [1] can be shown to transform as follows:

$$\delta x_i = -ie\psi_i \quad \delta \psi_i = -2\epsilon \dot{x}_i \quad \delta \xi = \epsilon^* \chi + \epsilon \chi^* \quad (25)$$

$$\delta \dot{x}_i = -ie\dot{\psi}_i \quad \delta \chi^* = \epsilon^* (R + i\dot{\xi}) \quad \delta \chi = \epsilon (-R + i\dot{\chi})$$

$$\delta R = -ie^* \dot{\chi} + ie \dot{\chi}^*.$$ 

With these superfields, we define the $N=2$-action for a model similar to the previous one by means of the expression,

$$S = \int dt d\theta d\bar{\theta} \mathcal{L}, \quad (26)$$

$$\mathcal{L} = \frac{i}{8} \frac{d\Phi_i}{dt} \Phi_i^* + \frac{i}{8} \frac{d\Phi_i^*}{dt} \Phi_i + \frac{ie}{4} A_i(\Phi) \Phi_i - \frac{ie}{4} \Phi_i (A_i(\Phi))^*$$

$$- \frac{1}{4} (D_\theta \Sigma)(D_\theta \Sigma)^* + \frac{e\sqrt{2}}{4} \Sigma \Lambda(\Phi_j) + \frac{e\sqrt{2}}{4} (\Lambda(\Phi))^* \Sigma^*$$

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\begin{equation}
+ \frac{ie}{4} \epsilon_{ijk} \Lambda_i(\Phi) \Phi_j \Phi_k^* - \frac{ie}{4} \epsilon_{ijk} \Phi_k^* \Phi^*_j \Lambda_i(\Phi)^*,
\end{equation}

where

\[ A_i(\Phi) = A_i(x) + \theta \psi_j \partial_j A_i(x) + i \theta^* \theta \dot{x}_j \partial_j A_i(x), \]

\[ \Lambda(\Phi) = \Lambda(x) + \theta \psi_j \partial_j \Lambda(x) + i \theta^* \theta \dot{x}_j \partial_j \Lambda(x), \]

\[ B_i \equiv A_i \equiv \frac{\partial \Lambda(x)}{\partial x_i}, \quad \Lambda_{ik} \equiv \frac{\partial^2 \Lambda(x)}{\partial k \partial x_i}, \quad A_{ji} \equiv \frac{\partial A_j(x)}{\partial x_i}. \]

To work out the supersymmetric Lagrangian in terms of component s, we start off from:

\[ L = K + \theta V_{\bar{\theta}} + \bar{\theta} V_\theta + \bar{\theta} \bar{L}. \]

Integrating eq. (26) on \( \bar{\theta} \) and \( \theta \), we get

\[ S = \int dt L, \]

where we can write

\[ \bar{L} = L + \frac{dY}{dt}, \]

the term \( Y \) is a surface term, \( \bar{L} \) is a total derivative of surface term added to \( L \), and

\[ V_\theta = -\frac{i}{4} \dot{x}_i \psi_i^* - \frac{i}{4} e A_i \psi_i^* + \frac{i}{4} e x_i (\partial_j A_i)^* \psi_j^* + \frac{i}{4} (R \chi + i \dot{\xi} \chi) + \frac{i}{4} e \sqrt{2} \Lambda \chi - \frac{1}{4} e \sqrt{2} (\partial_i \Lambda)^* \xi \psi_i^* + \frac{i}{4} e \sqrt{2} \Lambda^* \chi - \frac{i}{4} e \epsilon_{ijk} \partial_i \Lambda x_j \psi_k^* + \frac{i}{4} e \epsilon_{ijk} x_j \chi (\partial_i \partial_p \Lambda)^* \psi_p^* + \frac{i}{4} e \epsilon_{ijk} x_k (\partial_i \Lambda)^* \psi_j^* \]

\[ V_{\bar{\theta}} = \frac{i}{4} x_i^* \dot{\psi}_i - \frac{i}{4} e A_i^* \psi_i + \frac{i}{4} e x_i \partial_j A_i \psi_j + \frac{i}{4} (R \chi^* - i \dot{\xi} \chi^*) + \frac{i}{4} e \sqrt{2} \Lambda^* \chi^* + \frac{i}{4} e \sqrt{2} \Lambda \chi^* - \frac{i}{4} e \epsilon_{ijk} (\partial_i \Lambda)^* x_j^* \psi_k^* + \]
\[ \frac{1}{4} \sqrt{2} \partial_i \Lambda \xi \psi_i + \frac{i}{4} \epsilon_{ijk} x^*_k \partial_i \partial_p \Lambda \psi_p + \frac{i}{4} \epsilon_{ijk} x^*_k \partial_i \Lambda \psi_j. \]

The Lagrangian in terms of the physical coordinates reads as below:

\[ L = \frac{1}{2} \dot{x}_i \dot{x}_i^* - \frac{i}{8} \dot{\psi}_i \psi_i^* + \frac{i}{8} \dot{\psi}_i \psi_i^* + \frac{1}{2} e A_i \dot{x}_i^* - \frac{i}{4} e \partial_j A_i \psi_j \psi_i^* \]

\[ \frac{1}{2} e \dot{x}_i A_i^* + \frac{1}{4} i e (\partial_i A_j)^* \psi_j \psi_i^* + \frac{1}{4} i \dot{\chi} \chi^* - \frac{1}{4} i \dot{\chi} \chi^* \]  \hspace{1cm} (31)

\[ - \frac{1}{4} R^2 - \frac{1}{4} \xi^2 + \frac{1}{4} e \sqrt{2} (i \xi \dot{x}_i B_i + i B_i \psi_i + R \Lambda) \]

\[ + \frac{1}{4} e \sqrt{2} (\Lambda^* R + i (B_i)^* \psi_i^* \chi^* - i (B_i)^* \dot{x}_i^* \xi) \]

\[ + \frac{1}{4} e \epsilon_{kji} (2 B_k x_j \dot{x}_i^* - i B_k \psi_j \psi_i^* - i \partial_j B_k x_j \psi_r \psi_i^*) \]

\[ + \frac{1}{4} e \epsilon_{kji} \left( 2 \dot{x}_i (B_k)^* x_j^* + i (B_k)^* \psi_i \psi_j^* + i (\partial_j B_k)^* x_j^* \psi_i \psi_r^* \right) ; \]

also, the surface terms collected in \( Y \) are as follows:

\[ Y = -\frac{1}{8} x_i \dot{x}_i^* - \frac{1}{8} \dot{x}_i x_i^* - e \frac{1}{4} A_i \dot{x}_i^* - e \frac{1}{4} x_i A_i^* \]  \hspace{1cm} (32)

\[ - \frac{e}{4} \Lambda_i x_k x_k^* \epsilon_{ijk} - e \frac{1}{4} x_k \dot{x}_k^* \Lambda^* \epsilon_{ijk} \]

A. Supersymmetric charges from the Lagrangian

The Hamiltonian reads:

\[ H = \dot{x}_i \Pi_{x_i} + \Pi_{x_i} \dot{x}_i^* + \dot{\psi}_i \Pi_{\psi_i} - \Pi_{\psi_i} \dot{\psi}_i^* + \]

\[ + \dot{\chi} \Pi_{\chi} - \Pi_{\chi} \dot{\chi} + \dot{\xi} \Pi_{\xi} - L; \]

so,
\[
H = \left( \Pi_{x_i} - \frac{eA_i}{2} \right) \left( \Pi_{x_i} - \frac{eA_i^*}{2} \right) + \\
\frac{1}{8} i e \left( \partial_j A_i - (\partial_i A_j)^* \right) \psi_j \psi_i^* + \\
\frac{i}{4} \epsilon_{kji} B_k \psi_j \psi_i^* + \frac{i}{4} \epsilon_{kji} (\partial_r B_k)^* x_j^* \psi_r^* \psi_i + \frac{i}{4} \epsilon_{kji} B_i \chi \psi_i \\
\epsilon_{kji} \left( \Pi_{x_i} - \frac{eA_i}{2} \right) B_k^* x_j^* + \frac{\epsilon^2}{4} \epsilon_{kji} \epsilon_{npq} B_k B_i^* x_j x_p^* \\
\frac{i}{2} \sqrt{2} \epsilon_i B_i^* \left( \Pi_{x_i} - \frac{eA_i^*}{2} \right) + \frac{\epsilon^2}{8} B_i^* B_i - \frac{\Pi_i^2}{2} - \frac{R^2}{8} + C.C.,
\]

where we take from the Lagrangian (31) canonical moments \(\Pi_j\).

Using Noether’s theorem for the supersymmetry transformations, we read the charges and we neglect the surface terms, which shall be considered in the next section. The idea is that here we are not interested in topological solutions.

\[
Q_\theta = \left( \frac{5}{4} i \Pi_{x_i} - \frac{3}{8} i e A_i - \frac{5}{16} e \sqrt{2} \xi \Lambda_i^* + \frac{1}{8} i e x_j A_{ij} \right) + \\
\frac{1}{8} i e \epsilon_{kij} x_j \Lambda_k^* + \frac{1}{8} i e \epsilon_{kij} x_j x_r \Lambda_{kj}^* + \frac{1}{8} i e A_{ij}^* x_j \\
- \frac{1}{4} i e \epsilon_{jki} x_k \Lambda_j - \frac{1}{8} i e \epsilon_{jrk} \Lambda_{ji}^* x_r x_k + \frac{1}{4} i e \epsilon_{jrk} \Lambda_{ij}^* x_r x_k \psi_i^* \\
+ (2\Pi_\xi + \frac{i}{2} e \sqrt{2} \Lambda^* + \frac{i}{2} e \sqrt{2} \Lambda - \frac{i}{8} e \sqrt{2} x_i \Lambda_i) \chi \\
Q_\theta = \left( -\frac{5}{4} i \Pi_{x_i} + \frac{3}{8} i e A_i^* - \frac{5}{16} e \sqrt{2} \xi \Lambda_i^* - \frac{1}{8} i e x_j^* A_{ij}^* \right) + \\
\frac{1}{8} i e \epsilon_{kij} x_j^* \Lambda_k - \frac{1}{8} i e \epsilon_{kij} x_j^* x_r \Lambda_{kj} - \frac{1}{8} i e A_{ij}^* x_j^* + \\
\frac{1}{4} i e \epsilon_{jki} x_k \Lambda_j + \frac{1}{8} i e \epsilon_{jrk} \Lambda_{ji}^* x_r x_k - \frac{1}{4} i e \epsilon_{jrk} \Lambda_{ij} x_r x_k \psi_i
\]
\[(2\Pi_\xi - \frac{i}{2}e\sqrt{2}\Lambda - \frac{i}{2}e\sqrt{2}\Lambda^* + \frac{i}{8}e\sqrt{2}x_i^\star\Lambda_i^*)\chi^* \]

We check that

\[ [Q_\theta, Q_\theta]_+ = 2H_1 \quad (35) \]

\[ [Q_\theta, Q_\theta]_+ = \frac{5}{32} i e x_k A_{k[i,j]} - \left( \frac{5}{8} \right)^2 e\sqrt{2}\xi\Lambda_{[i,j]} \]

\[ - \frac{5}{16} i e \Lambda_{[i,k]} x^\star_n \epsilon_{kjn} - \frac{5}{16} i e \Lambda_{[k,j]} x^\star_n \epsilon_{kin} \]

\[ - \frac{5}{32} i e x_k x_r \epsilon_{nkr} (\Lambda_{nji} - 2\Lambda_{jmi} - \Lambda_{nij} + 2\Lambda_{mij})\psi_i\psi_j \]

\[ [Q_\theta, Q_\theta]_+ = \left( -\frac{5}{32} i e x_k A^\star_{k[i,j]} - (\frac{5}{8})^2 e\sqrt{2}\xi A^\star_{[i,j]} \right) \]

\[ + \frac{5}{16} i e \Lambda^\star_{[i,k]} x_n \epsilon_{kin} + \frac{5}{16} i e \Lambda^\star_{[k,j]} x_n \epsilon_{kjn} + \]

\[ \frac{5}{32} i e x_k x^\star_r \epsilon_{nkr} (\Lambda^\star_{nij} - 2\Lambda^\star_{mj} - \Lambda^\star_{nji} + 2\Lambda^\star_{jmi})\psi^\star_i\psi^\star_j , \]

where the subscript \([i,j]\) stands for anti-symmetrisation over the indices i,j. If we impose that the algebra of charges are the ones in (22), we need that

\[ A_{k[i,j]} \equiv 0 \quad \Lambda_{[i,j]} \equiv 0 \quad \Lambda^*_{k[i,j]} \equiv 0 \quad (36) \]

\[ A^\star_{k[i,j]} \equiv 0 \quad \Lambda^\star_{[i,j]} \equiv 0 \quad \Lambda^*^\star_{k[i,j]} \equiv 0 , \]

which means that the funtions have continuous second derivatives. We then see that central charges in the N=2 - su.sy. algebra might appear only if we have potential configurations with non-continuous second derivatives, like what happens in the presence of Dirac-string monopoles.
B. Supersymmetric charges from the Lagrangian with the surface terms

If we now consider the surface terms coming from \( Y \), we get to the Lagrangian

\[
\tilde{L} = L + \frac{dY}{dt}.
\]

We now consider the contributions arising from the surface terms, since they might lead to topological configurations that could induce the appearance of central charges in the \( N=2 \)-susy. algebra.

\[
Q_{\bar{\theta}} = \left( \frac{4}{3} i \Pi_{x^i} - \frac{1}{3} i e A_i - \frac{1}{3} \sqrt{2} \xi A^*_i + \frac{1}{3} i e x_j A_{ji} \right)
\]

\[
\frac{1}{3} i e \epsilon_{ijk} x_k \Lambda_j^* + \frac{1}{3} i e \epsilon_{rjk} x_k x_j^* \Lambda_{ri}^* - \frac{1}{3} i e \epsilon_{jki} x_k \Lambda_j
\]

\[\psi_i^* + (2 \Pi \xi + \frac{e}{2} \sqrt{2} \Lambda^* + \frac{e}{2} \sqrt{2} \Lambda) \chi \]

\[
Q_{\theta} = \left( -\frac{4}{3} i \Pi_{x^i} + \frac{1}{3} i e A^*_i - \frac{1}{3} \sqrt{2} \xi A_i - \frac{1}{3} i e x_j A^*_{ji} \right)
\]

\[-\frac{1}{3} i e \epsilon_{ijk} x_k \Lambda^*_j - \frac{1}{3} i e \epsilon_{rjk} x_k x_j^* \Lambda_{ri}^* + \frac{1}{3} i e \epsilon_{jki} x_k \Lambda_j
\]

\[\psi_i + (2 \Pi \xi - \frac{e}{2} \sqrt{2} \Lambda^* - \frac{e}{2} \sqrt{2} \Lambda) \chi^* \]

following that the charges algebra are,

\[
[Q_{\theta}, Q_{\bar{\theta}}]_+ = 2 \tilde{H}
\]

\[
[Q_{\theta}, Q_{\bar{\theta}}]_+ = \left( \frac{4}{9} e i x_k^* A_{k[j,i]} + \frac{4}{9} \sqrt{2} \xi A_{[i,j]} \right)
\]

\[+ \frac{4}{9} e i x_n x_k^* e_{rkn} \Lambda_{r[j,i]} \psi_i \psi_j \]

\[
[Q_{\theta}, Q_{\bar{\theta}}]_+ = \left( -\frac{4}{9} e i x_k^* A_{k[i,j]} + \frac{4}{9} \sqrt{2} \xi A^*_{[i,j]} \right)
\]
\[-\frac{4}{9} e^{ix_n^k x_k} \epsilon_{ijk} \Lambda^*_{[i,j]} \psi^*_j \psi^*_i,\]

where the Hamiltonian $\tilde{H}$ is the same obtained from lagrangian $\tilde{L}$ by Legendre transformation.

If we impose that the algebra of charges is the same as the one in (22), we need that

\begin{align*}
A_k^{[i,j]} &\equiv 0 \\
\Lambda^*_{[i,j]} &\equiv 0 \\
A^*_{r[j,i]} &\equiv 0
\end{align*}

(39)

\begin{align*}
A^{*}_{k[j,i]} &\equiv 0 \\
\Lambda^*_{[i,j]} &\equiv 0 \\
A^*_{r[j,i]} &\equiv 0;
\end{align*}

the algebra of commutation relations for the component coordinates are given as:

\begin{align*}
[x_i, \Pi_{x_k}] &\equiv i \delta_{ik} \\
[x^*_i, \Pi_{x^*_k}] &\equiv i \delta_{ik} \\
[\xi, \Pi_{\xi}] &\equiv i
\end{align*}

\begin{align*}
[\psi_i, \psi^*_k] &\equiv 8 \delta_{ik} \\
[\chi, \chi^*] &\equiv -4,
\end{align*}

(40)

\begin{align*}
[\psi_i, \Pi_{\psi_k}] &\equiv -i \\
[\psi^*_i, \Pi_{\psi^*_k}] &\equiv -i \\
[\chi, \Pi_{\chi}] &\equiv -i
\end{align*}

all other relations vanish. We can obtain the same algebra for the components of superfields, proceeding to the canonical quantization [11], where the momenta $\tilde{\Pi}_j$, follow now from the Lagrangian $\tilde{L}$.

**V. CONCLUSIONS**

We propose a supersymmetric quantum-mechanical system given by a non-relativistic particle in an external field. We formulate the problem in 5D, where the particle presents 4 fermionics degrees of freedom, and 4 bosonics degrees of freedom (N=1 - D=4 => N=2 - D=4). In dimensions smaller than 3+1, we do not have sufficient degrees of freedom in the fermionic sector. Then, we cannot write the model in terms of superfields.
Starting from 4+1, the consequence is that we drag another object from there, $B_i$. The interpretation of this object is not clear, but it follows from the extended Electromagnetism we have discussed. We obtain a simple and also extended (N=2) su.sy..

We observe that, although the charge in N=2-su.sy. changes with the surface terms in the Lagrangian, we do not have change the charges algebra of supersymmetry. In other words, surface terms do not give rise to a central charge in this model, except in the case the derivatives do not commute when applied on $A^\mu$ and $\Lambda$, or, in other words, when Dirac-like strings are present.

Here, we have set an extended eletromagnetic model to see how central charges may show up. Next, we wish to reset the turbulent system, decribed by a BRS-symmetry, as studied in [7] in terms of an N=2-D=2-supersymmetry. The instanton solution present in the system may be the source of a central charge in the N=2-algebra. Also, the realisation of an N=2-su.sy. for the turbulent system may ensure the topological stability of dual solutions that saturate the Bogomol’nyi bound [16], [17].

APPENDIX

We quote below the $\Gamma$-matrices representing the Clifford algebra of (1+4)D:

$$
\Gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}; \quad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad \Gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}; \quad \Gamma^4 = -i\Gamma^5 = \begin{pmatrix} 0 & i_2 \\ i_2 & 0 \end{pmatrix}.
$$

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