Quantitative Fundamental Theorem of Algebra

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Abstract

Using subresultants, we modify a recent real-algebraic proof due to Eisermann of the Fundamental Theorem of Algebra ([FTA]) to obtain the following quantitative information: in order to prove the [FTA] for polynomials of degree \(d\), the Intermediate Value Theorem ([IVT]) is requested to hold only for real polynomials of degree at most \(d^2\). We also explain that the classical proof due to Laplace requires [IVT] for real polynomials of exponential degree. These quantitative results highlight the difference in nature of these two proofs.

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1 Introduction

Let \((\mathbb{R}, \leq)\) be an ordered field. The fact that \(\mathbb{R}\) admits an order compatible with the field structure implies that \(\operatorname{char}(\mathbb{R}) = 0\) and therefore \(\mathbb{R}\) has an infinite number of elements. It also implies that \(-1\) is not a square in \(\mathbb{R}\) and, consequently, \(\mathbb{R}[T]/(T^2 + 1) = \mathbb{R}[i] = \mathbb{C}\) is an algebraic field extension of \(\mathbb{R}\) of degree 2.

We consider the following properties on \((\mathbb{R}, \leq)\).

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• [IVT] (*Intermediate Value Theorem*): for every polynomial \( F \in \mathbb{R}[X] \) and every \( a, b \) in \( \mathbb{R} \) with \( a < b \) and \( F(a)F(b) < 0 \), there exists \( c \in \mathbb{R} \) with \( a < c < b \) such that \( F(c) = 0 \).

• [NnS] (*Non-negative elements are Squares*): for every \( a \in \mathbb{R} \) with \( a \geq 0 \), there exists \( c \in \mathbb{R} \) such that \( a = c^2 \).

• [OD] (*An odd degree polynomial has a root*): for every polynomial \( F \in \mathbb{R}[X] \) of odd degree, there exists \( c \in \mathbb{R} \) such that \( F(c) = 0 \).

• [FTA] (*Fundamental Theorem of Algebra*): for every polynomial \( F \in \mathbb{C}[Z] \setminus \mathbb{C} \), there exists \( z \in \mathbb{C} \) such that \( F(z) = 0 \) (i.e., \( \mathbb{C} \) is an algebraically closed field).

If \( \leq \) denotes the usual order over the real numbers, \((\mathbb{R}, \leq)\) and \((\mathbb{R}_{\text{alg}}, \leq)\) are typical examples of ordered fields satisfying all the properties above (where \( \mathbb{R}_{\text{alg}} \) is the set of real algebraic numbers), meanwhile \((\mathbb{Q}, \leq)\) is a typical example of an ordered field satisfying none of the properties above.

The next theorem is a classical result in real algebraic geometry (see for instance [2, Chapter 1]).

**Theorem 1** Let \((\mathbb{R}, \leq)\) be an ordered field. The following conditions are equivalent:

a) \((\mathbb{R}, \leq)\) satisfies [IVT].

b) \((\mathbb{R}, \leq)\) satisfies [NnS] and [OD].

c) \((\mathbb{R}, \leq)\) satisfies [FTA].

If \((\mathbb{R}, \leq)\) satisfies these conditions, then it is easy to see that the field order \( \leq \) on \( \mathbb{R} \) is unique and \( \mathbb{R} \) is said to be a **real closed field**.

We sketch briefly a proof of Theorem 1 which is essentially Laplace’s proof [5].

**Sketch of the proof of Theorem 1** Proving that [IVT] implies [NnS] is very simple: for \( a = 0 \) we take \( c = 0 \); and for \( a > 0 \) we consider the polynomial \( F = X^2 - a \in \mathbb{R}[X] \) and notice that \( F(0) < 0 \) and \( F(a + 1) > 0 \), then [IVT] ensures the existence of a \( c \in \mathbb{R} \) such that \( F(c) = 0 \), or equivalently, \( a = c^2 \). In fact, adding the condition \( c \geq 0 \), it is easy to prove the uniqueness of such \( c \).

In a similar way, in order to prove that [IVT] implies [OD] we only need to note that an odd degree polynomial necessarily changes its sign when evaluated at \( a \) and \( -a \) with \( a \in \mathbb{R} \) big enough.

The proof that [NnS] and [OD] imply [FTA] is much more sophisticated. To prove that \( F \in \mathbb{C}[Z] \setminus \mathbb{C} \) has a root in \( \mathbb{C} \), it is enough to prove that the polynomial \( \overline{\overline{F}} \in \mathbb{R}[Z] \setminus \mathbb{R} \) has a root \( w \) in \( \mathbb{C} \) (where \( \overline{\overline{F}} \) means the polynomial obtained from \( F \) by usual conjugation in \( \mathbb{C} \) of the coefficients of \( F \)); in this case either \( w \) or \( \overline{w} \) is a root of \( F \). Now, in order to show that an arbitrary polynomial \( G \in \mathbb{R}[Z] \setminus \mathbb{R} \) of degree \( d \) has a root in \( \mathbb{C} \), the proof proceeds by induction on the highest value of \( k \) such that \( 2^k \) divides \( d \). In the base case, which is \( k = 0 \) (and therefore odd \( d \)), the existence of a root of \( G \) in \( \mathbb{R} \subset \mathbb{C} \) is ensured by [OD]. For \( k \geq 1 \) (and therefore even \( d \)), the existence of a root of \( G \) in \( \mathbb{C} \) is
ensured by a clever argument involving [NnS] and the fact that every polynomial in $R[Z]$ of degree $\binom{d}{2}$ has a root in $C$. Note that the highest power of 2 dividing $\frac{1}{2}d(d-1) = 2^{k-1}$ and then the inductive hypothesis holds.

Finally, assuming [FTA], it is possible to prove that the irreducible elements in the unique factorization domain $R[X]$ have degree 1 or 2 and that the irreducible monic elements in $R[X]$ of degree 2 are positive when evaluated at any $r \in R$. From these facts, [IVT] holds easily. □

The main concern in the present work is the following question: assuming that [IVT] holds for $(R, \leq)$, if we take a fixed value of $d \in Z_{\geq 1}$ and we only want to prove that every polynomial in $C[Z] \setminus C$ of degree less than or equal to $d$ has a root in $C$, which is the highest degree of a polynomial in $R[X]$ for which we need the Intermediate Value Theorem to hold?

With the aim of stating our problem precisely, we consider for each $d \in Z_{\geq 1}$, the following properties on $(R, \leq)$.

- [IVT]$_d$: for every polynomial $F \in R[X]$ with $\deg F \leq d$ and every $a, b$ in $R$ with $a < b$ and $F(a)F(b) < 0$, there exists $c \in R$ with $a < c < b$ such that $F(c) = 0$.
- [FTA]$_d$: for every polynomial $F \in C[Z] \setminus C$ with $\deg F \leq d$, there exists $z \in C$ such that $F(z) = 0$.

We can now restate our main concern as follows:

Given $d \in Z_{\geq 1}$, which is the lowest value of $\alpha(d) \in Z_{\geq 1}$ for which $[IVT]_{\alpha(d)}$ implies $[FTA]_d$?

In order to evaluate from this new quantitative point of view the proof of Theorem 1 we sketched, we define the following functions:

**Notation 2** Let $\beta, \gamma : Z_{\geq 1} \to Z_{\geq 1}$ defined as follows:

\[
\beta(d) = \begin{cases} 
  d & \text{if } d \text{ is odd}, \\
  \beta(\binom{d}{2}) & \text{if } d \text{ is even},
\end{cases}
\]

\[
\gamma(d) = \max_{1 \leq e \leq d} \{\beta(2e)\}.
\]

Note that $\gamma(1) = \beta(2) = 1$ and for $d \geq 2$ we have that $\gamma(d) \geq \beta(4) = 15$. Note also that, meanwhile $\gamma$ is a non-decreasing function, the behavior of $\beta$ is rather chaotic.

First, we have that $[FTA]_1$ holds even under no assumptions on $(R, \leq)$. Then, for a fixed $d \geq 2$ and a polynomial $F \in C[Z]$ with $e = \deg F \leq d$, in order to be able to apply the proof of Theorem 1 we need to ensure [NnS] and the fact that the Intermediate Value Theorem holds for polynomials in $R[X]$ of degree $\beta(2e)$. Since $[IVT]_2$ implies [NnS], we have that

$[IVT]_{\gamma(d)}$ implies $[FTA]_d$.

The final conclusion is that $\alpha(d) \leq \gamma(d)$.
Now we want to exhibit explicit bounds for $\gamma$. It is possible to prove that for $d \in \mathbb{Z}, d \geq 4$, if $d = 2^k s$ with $k \in \mathbb{Z}_{\geq 0}$ and odd $s \in \mathbb{Z}_{\geq 1}$ then

$$\frac{8}{3} \left(\frac{3}{4}2^{k-1}s\right)^{2k} \leq \beta(d) \leq 2 \left(2^{k-1}s\right)^{2k} \leq 2 \left(\frac{d}{2}\right)^d.$$ 

Then, for $d \in \mathbb{Z}, d \geq 4$, we have

$$\gamma(d) \leq 2d^{2d}.$$ 

Also, by taking $k' = \lfloor \log_2 d \rfloor$, since $2^{k'} \leq d < 2^{k'+1}$,

$$\left(\frac{3}{8}\right)^{d-1} d^d = \frac{8}{3} \left(\frac{3}{4}d\right)^d < \frac{8}{3} \left(\frac{3}{4}2^k\right)^{2k'+1} \leq \beta(2^{k'+1}) \leq \gamma(d).$$

In this way, we know that $\gamma$ is bounded from below and above by exponential functions. This leads to an exponential upper bound for $\alpha$, which cannot be avoided as long as we keep attached to the proof we sketched of Theorem 1.

The exponential value of $\gamma(d)$ plays a significant role in the bounds obtained in a recent joint work of the authors with Henri Lombardi, giving a new constructive proof for Hilbert 17-th problem and Positivstellensatz and providing elementary recursive degree bounds [6]. Exploring other algebraic proofs of [FTA] from a quantitative point of view might be a first step in the improvement of the results of [6]. This hope is part of our motivation in this paper.

Recently, Michael Eisermann found a new proof of the Fundamental Theorem of Algebra which is also based on [IVT], and valid in any real closed field, but in opposition to Laplace’s proof which is purely algebraic, has a large real-algebraic geometry flavor. This new proof of Eisermann can be seen as a real-algebraic adaptation of one of the classical proofs of the Fundamental Theorem of Algebra using winding numbers and homotopy (see [4, Chapter 8]). One of the main ingredients of Eisermann’s proof is the Cauchy index of two polynomials which, roughly speaking, is the number of jumps from $-\infty$ to $+\infty$ minus the number of jumps from $+\infty$ to $-\infty$ that the function associated to their quotient has in a given interval. From the fact that the base ordered field $(\mathbb{R}, \leq)$ satisfies [IVT], it follows an inversion formula which implies that Cauchy indices can be computed by counting sign variations on Sturm chains. Another of the main ingredients of Eisermann’s proof is the fact that the winding number of a complex function on a rectangle, which counts the number of zeros of the function in the given rectangle, can be computed in a completely real-algebraic way by means of Cauchy indices. One of the most intricate steps in Eisermann’s proof is to prove that if a polynomial does not vanish in a rectangle, then the winding number is zero. This is achieved by a clever cancellation of terms using a suitable division of the rectangle under consideration. Finally, by means of algebraic homotopy-like tools, the proof follows by computing the winding number in a well-known special case. Then the conclusion follows.

In this paper, we introduce the notion of $(\sigma, \tau)$-chain, which is a generalization of the notion of Sturm chain and prove that they can be used to compute Cauchy indices. Since subresultant polynomial sequences fit in the definition of $(\sigma, \tau)$-chain, we are able to use the Structure Theorem of Subresultants, to adapt the proof of the Fundamental Theorem of Algebra by Eisermann in a quantitative way. By
doing so, we obtain a better control of the degrees of some intermediate polynomials relevant to the proof; this will be further explained in Section 3. We prove the following theorem which is our main result.

**Theorem 3** For \( d \in \mathbb{Z}_{\geq 1} \)

\[
\text{[IVT]}_{d^2} \text{ implies } \text{[FTA]}_d.
\]

In other words, Theorem 3 is equivalent to saying that \( \alpha(d) \leq d^2 \). Since \( d \mapsto d^2 \) is a polynomial function (actually \( d^2 \leq \gamma(d) \) for all \( d \in \mathbb{Z}_{\geq 1} \)), our result highlights the difference in nature between Laplace’s proof and our modification of Eisermann’s proof.

## 2 Preliminaries

### 2.1 Cauchy index

As said in the introduction, the Cauchy index of two polynomials \( Q \) and \( P \) on an interval is, roughly speaking, the number of jumps from \(-\infty\) to \(+\infty\) minus the number of jumps from \(+\infty\) to \(-\infty\) that the function associated to their quotient \( \frac{Q}{P} \) has in this interval. We recall now the precise definition of Cauchy index following [3, Section 3].

**Definition 4** Let \( x \in \mathbb{R} \) and \( P, Q \in \mathbb{R}[X] \).

- If \( P, Q \in \mathbb{R}[X] \setminus \{0\} \), the polynomials \( P \) and \( Q \) can be written uniquely as

\[
P = (X - x)^\mu(x) \tilde{P},
\]
\[
Q = (X - x)^\nu(x) \tilde{Q},
\]

with \( \mu(x), \nu(x) \in \mathbb{Z}_{\geq 0} \) and \( \tilde{P}(x) \neq 0, \tilde{Q}(x) \neq 0 \).

For \( \varepsilon \in \{+, -\} \), define

\[
\text{Ind}^\varepsilon_x(Q, P) = \begin{cases} 
\frac{1}{2} (\varepsilon^\mu(x) - \varepsilon^\nu(x)) \text{sign}(\tilde{Q}(x)\tilde{P}(x)) & \text{if } \mu(x) > \nu(x), \\
0 & \text{otherwise.}
\end{cases}
\]

- If \( P = 0 \) or \( Q = 0 \), define

\[
\text{Ind}^\varepsilon_x(Q, P) = 0.
\]

- The Cauchy index of \( Q, P \) at \( x \) is

\[
\text{Ind}_x(Q, P) = \text{Ind}^+_x(Q, P) - \text{Ind}^-_x(Q, P).
\]
We illustrate this notion considering the graph of the function \( \frac{Q}{P} \) around \( x \) in each different case.

![Graphs showing different cases of \( \text{Ind}_x(Q, P) \).]

\[
\begin{align*}
\text{Ind}_x(Q, P) &= 0 & \text{Ind}_x(Q, P) &= 1 & \text{Ind}_x(Q, P) &= -1 & \text{Ind}_x(Q, P) &= 0
\end{align*}
\]

**Definition 5** Let \( a, b \in \mathbb{R} \) and \( P, Q \in \mathbb{R}[X] \).

- If \( a < b \) and \( P, Q \neq 0 \), the Cauchy index of \( Q, P \) on the interval \( [a, b] \) is
  \[
  \text{Ind}^b_a(Q, P) = \text{Ind}^+_a(Q, P) + \sum_{x \in (a, b)} \text{Ind}_x(Q, P) - \text{Ind}^-_b(Q, P),
  \]
  where the sum is well-defined since only roots \( x \) of \( P \) in \( (a, b) \) contribute.
- If \( a > b \) and \( P, Q \neq 0 \),
  \[
  \text{Ind}^b_a(Q, P) = -\text{Ind}^a_b(Q, P).
  \]
- In every other case,
  \[
  \text{Ind}^b_a(Q, P) = 0.
  \]

In the following picture we consider again the graph of the function \( \frac{Q}{P} \), this time in \([a, b]\).

![Graphs showing different cases of \( \text{Ind}_a(Q, P) \) and \( \text{Ind}_b(Q, P) \).]

\[
\begin{align*}
\text{Ind}^b_a(Q, P) &= 1 + 0 + 1 = 2 & \text{Ind}^b_a(Q, P) &= -1 - 1 - \frac{1}{2} = -\frac{5}{2}
\end{align*}
\]

Note that with this precise definition of the Cauchy index, the Cauchy index of two polynomials on an interval belongs to \( \frac{1}{2} \mathbb{Z} \) and is not necessarily an integer number.
Remark 6 If both $P$ and $Q$ are multiplied by $S \in \mathbb{R}[X] \setminus \{0\}$, it is clear that $\text{Ind}^b_a(Q, P) = \text{Ind}^b_a(QS, PS)$, so when $P \neq 0$ the Cauchy index is associated to the rational function $\frac{Q}{P}$ rather than to the two polynomials $Q, P$. However, when $P = 0$, it is convenient for us to define also the Cauchy index, even if the rational function $\frac{Q}{P}$ does not make sense. This is the reason why we use the notation $\text{Ind}^b_a(Q, P)$ in all cases.

Remark 7 Even though it is not reflected in the notation, the field $\mathbb{R}$ plays a fundamental role in the definition of the Cauchy index. For instance, consider $P = X^2 - 2, Q = 1 \in \mathbb{Q}[X] \subset \mathbb{R}[X]$. If we take $\mathbb{R} = \mathbb{Q}$ we have $\text{Ind}_{2}^{1}(Q, P) = 0$, meanwhile if we take $\mathbb{R} = \mathbb{R}$ we have $\text{Ind}_{2}^{1}(Q, P) = 1$.

Remark 8 Cauchy index is invariant by linear rescaling: given any linear function $\ell : [a, b] \to \mathbb{R}$, and $P, Q \in \mathbb{R}[X]$,

$$\text{Ind}^b_{\ell(a)}(Q, P) = \text{Ind}^b_a(Q \circ \ell, P \circ \ell).$$

Remark 9 Cauchy index is additive on intervals: given any $a, c_1, \ldots, c_k, b \in \mathbb{R}$ and $P, Q \in \mathbb{R}[X]$,

$$\text{Ind}^b_a(Q, P) = \text{Ind}^b_{c_1}(Q, P) + \sum_{1 \leq i \leq k-1} \text{Ind}^b_{c_i+c_{i+1}}(Q, P) + \text{Ind}^b_{c_k}(Q, P).$$

2.2 Winding number

Assuming that $F$ does not vanish on $\partial \Gamma$, the classical geometric definition of winding number is the number of counterclockwise turns around 0 of the image of $F$ on $\partial \Gamma$. We now recall the algebraic definition of winding number following [3]. From now, we consider the usual identification $\mathbb{C} \sim \mathbb{R}^2$. We also specify some useful notation.

Notation 10 For $F \in \mathbb{C}[X,Y]$, we denote $F_{\text{re}}$ and $F_{\text{im}}$ the real and imaginary part of $F$, i.e. the unique polynomials in $\mathbb{R}[X,Y]$ such that the identity

$$F(X,Y) = F_{\text{re}}(X,Y) + iF_{\text{im}}(X,Y)$$

in $\mathbb{C}[X,Y]$ holds.

Definition 11 Let $x_0, x_1, y_0, y_1 \in \mathbb{R}$ with $x_0 < x_1$ and $y_0 < y_1$ and let $\Gamma \subset \mathbb{R}^2$ be the rectangle $\Gamma = [x_0, x_1] \times [y_0, y_1]$. For $F \in \mathbb{C}[X,Y]$ the winding number of $F$ on $\partial \Gamma$ is defined as

$$w(F \mid \partial \Gamma) = \frac{1}{2} \left( \text{Ind}^{x_1}_{x_0}(F_{\text{re}}(X, y_0), F_{\text{im}}(X, y_0)) + \text{Ind}^{y_1}_{y_0}(F_{\text{re}}(x_1, Y), F_{\text{im}}(x_1, Y)) \right)$$

$$+ \text{Ind}^{x_1}_{y_1}(F_{\text{re}}(X, y_1), F_{\text{im}}(X, y_1)) + \text{Ind}^{y_1}_{x_0}(F_{\text{re}}(x_0, Y), F_{\text{im}}(x_0, Y))) .$$
Notice that it follows from the definition of winding number that we are going through $\partial \Gamma$ following the counterclock sense. The idea behind this algebraic definition is to count one half of a turn each time this curve crosses the $X$-axis from quadrant IV to I or from quadrant II to III, and minus one half of a turn each time it crosses the $X$-axis from quadrant I to IV or from quadrant III to II. Since these crossings coincide with jumps of the rational function $\frac{F_{\text{re}}}{F_{\text{im}}}$ from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ respectively, the Cauchy index is an appropriate algebraic tool to count the number of turns counterclockwise, which is (when $F$ does not vanish on $\partial \Gamma$) the classical definition of the winding number.

Along the paper we will follow the convention of using $X, Y$ and $T$ for real variables, i.e. variables that will only be eventually evaluated at elements of $\mathbb{R}$, and $Z$ for a complex variable, i.e. a variable that will be eventually evaluated at arbitrary elements of $\mathbb{C}$.

To $F \in \mathbb{C}[Z]$ we associate $\bar{F}(X, Y) = F(X + iY) \in \mathbb{C}[X, Y]$. Abusing slightly notation, we denote $F_{\text{re}}, F_{\text{im}}, w(F \mid \partial \Gamma)$ for $\bar{F}_{\text{re}}, \bar{F}_{\text{im}}, w(\bar{F} \mid \partial \Gamma)$.

**Example 12** (See [3, Proposition 4.4]) Let $\Gamma = [x_0, x_1] \times [y_0, y_1] \subset \mathbb{R}^2$. For $z \in \mathbb{C}$, we have

$$w(Z - z \mid \partial \Gamma) = \begin{cases} 
1 & \text{if } z \text{ is in the interior of } \Gamma, \\
1/2 & \text{if } z \text{ is in one of the edges of } \Gamma, \\
1/4 & \text{if } z \text{ is in one of the vertices of } \Gamma, \\
0 & \text{if } z \text{ is in the exterior of } \Gamma.
\end{cases}$$

**Lemma 13** Let $\Gamma = [x_0, x_1] \times [y_0, y_1] \subset \mathbb{R}^2$ and consider a grid partition of $\Gamma$ into a finite number of rectangles $\Gamma_1, \ldots, \Gamma_s$. For $F \in \mathbb{C}[X, Y]$, we have

$$w(F \mid \partial \Gamma) = \sum_{1 \leq i \leq s} w(F \mid \partial \Gamma_i).$$

**Proof:** After replacing the winding number of $F$ on $\partial \Gamma_1, \ldots, \partial \Gamma_s$, along each edge in the interior of $\Gamma$ by its definition we have to add and subtract the Cauchy index of the same couple of polynomials,
which adds up to zero. Meanwhile, using the additivity of Cauchy index on intervals (Remark 9), adding on the remaining edges we obtain the winding number of $F$ on $\partial \Gamma$.

Finally, the following lemma will play an important role at the end of Section 3 when applying homotopy tools.

**Lemma 14** Let $x_0, x_1, y_0, y_1, t_0, t_1 \in \mathbb{R}$ with $x_0 < x_1, y_0 < y_1$ and $t_0 < t_1$. Let $\Gamma_T = [x_0, x_1] \times [y_0, y_1], \Gamma_Y = [x_0, x_1] \times [t_0, t_1], \Gamma_X = [y_0, y_1] \times [t_0, t_1] \subset \mathbb{R}^2$. For $F \in \mathbb{C}[X,Y,T]$, we have

$$- \ w(F(X,Y,t_0) \mid \partial \Gamma_T) + \ w(F(X,y_0,T) \mid \partial \Gamma_Y) - \ w(F(x_0,Y,T) \mid \partial \Gamma_X) + \ w(F(X,Y,t_1) \mid \partial \Gamma_T) - \ w(F(X,y_1,T) \mid \partial \Gamma_Y) + \ w(F(x_1,Y,T) \mid \partial \Gamma_X) = 0.$$ 

Therefore, if

$$w(F(X,y_0,T) \mid \partial \Gamma_Y) = w(F(x_0,Y,T) \mid \partial \Gamma_X) = w(F(X,y_1,T) \mid \partial \Gamma_Y) = w(F(x_1,Y,T) \mid \partial \Gamma_X) = 0$$

then

$$w(F(X,Y,t_0) \mid \partial \Gamma_T) = w(F(X,Y,t_1) \mid \partial \Gamma_T).$$

**Proof:** Consider the rectangular parallelepiped $[x_0, x_1] \times [y_0, y_1] \times [t_0, t_1] \subset \mathbb{R}^3$. After replacing each winding number by its definition, along each edge of this parallelepiped we have to add and subtract the Cauchy index of the same couple of polynomials; therefore obtaining 0 as the final result.

\[ \square \]
2.3 The intermediate value property for polynomials of bounded degree

Our main goal in this paper is to prove that $[\text{IVT}]_{d^2}$ implies $[\text{FTA}]_d$. So, from now, we take a fixed value of $d \in \mathbb{Z}_{\geq 1}$ and we suppose that $(\mathbb{R}, \leq)$ is an ordered field satisfying $[\text{IVT}]_{d^2}$ but not necessarily $[\text{IVT}]$. Since $[\text{FTA}]_1$ holds even under no assumptions on $(\mathbb{R}, \leq)$, we suppose $d \geq 2$.

Note that the current assumption on $(\mathbb{R}, \leq)$ is rather subtle, since for instance, it is only for $P \in \mathbb{R}[X]$ with $\deg P \leq d^2$ that we can claim that if $P$ has no roots on an interval $I \subset \mathbb{R}$, then $P$ has constant sign (different from 0) on $I$.

The purpose of this section is to reexamine some results from [3] concerning the Cauchy index and the winding number and prove that they still hold in the present setting, despite the fact that our hypotheses are weaker than in [3]. More explicitly, in [3] the assumption is that $\mathbb{R}$ is a real closed field, and therefore it satisfies $[\text{IVT}]$, meanwhile, we only suppose $[\text{IVT}]_{d^2}$. Nevertheless, in the results reviewed in this section, following the proof in [3] or a slight variation of it, it turns out that the Intermediate Value Theorem is applied to polynomials of degree less than or equal to $d^2$. This is enough to ensure that these results are still valid.

We introduce the following useful notation.

**Notation 15** Let $x \in \mathbb{R}$ and $P, Q \in \mathbb{R}[X]$, we denote the sign variation of $(P, Q)$ at $x$ by

$$\text{Var}_x(P, Q) = \frac{1}{2} \left| \text{sign}(P(x)) - \text{sign}(Q(x)) \right|.$$  

For $a, b \in \mathbb{R}$, we denote by $\text{Var}^b_a(P, Q)$ the sign variation of $(P, Q)$ at $a$ minus the sign variation of $(P, Q)$ at $b$; namely,

$$\text{Var}^b_a(P, Q) = \text{Var}_a(P, Q) - \text{Var}_b(P, Q).$$

We first recall the following property known as the inversion formula.

**Proposition 16** Let $a, b \in \mathbb{R}$ and $P, Q \in \mathbb{R}[X]$ with $\deg P, \deg Q \leq d^2$ and such that $P$ and $Q$ have no common root in $[a, b]$. Then

$$\text{Ind}^b_a(Q, P) + \text{Ind}^b_a(P, Q) = \text{Var}^b_a(P, Q).$$

*Proof:* See [3, Theorem 3.9]. \(\square\)

Next proposition shows the additivity of the winding number with respect to the product of complex polynomials.

**Proposition 17** Let $\Gamma = [x_0, x_1] \times [y_0, y_1] \subset \mathbb{R}^2$ and $F, G \in \mathbb{C}[X, Y]$ with $\deg FG \leq d^2$ and such that $F$ and $G$ do not vanish at $\partial \Gamma$. Then

$$w(FG \mid \partial \Gamma) = w(F \mid \partial \Gamma) + w(G \mid \partial \Gamma).$$
The proof of Proposition 17 uses the next lemma as an auxiliary result. It can be checked that for $P, Q, R, S \in \mathbb{R}[X]$, the condition $\deg(PR - QS), \deg(PS + QR) \leq d^2$ implies $\deg P, \deg Q, \deg R, \deg S \leq d^2$.

Lemma 18 Let $a, b \in \mathbb{R}$ and $P, Q, R, S \in \mathbb{R}[X]$ with $\deg(PR - QS), \deg(PS + QR) \leq d^2$ and such that $P$ and $Q$ have no common root in $[a, b]$ and $R$ and $S$ have no common root in $[a, b]$. Then

$$\text{Ind}_a^b (PR - QS, PS + QR) = \text{Ind}_a^b (P, Q) + \text{Ind}_a^b (R, S) + \frac{1}{2} \text{sign}((PS + QR)QS)(a) - \frac{1}{2} \text{sign}((PS + QR)QS)(b).$$

Remark 19 In [3, Theorem 4.5] there is a statement with a slightly different formula and no assumption on polynomials $P, Q, R, S \in \mathbb{R}[X]$. We observed that this formula does not hold for the case $a = 0, b = 1, P = 1, Q = X, R = X - 1, S = X$. Notice that in this example, $P, Q, R, S$ actually meet our extra assumptions, but since $PS + QR = QS = X^2$, if we deal with the rational function $PS + QR/QS$ as in [3, Theorem 4.5], there is a simplification which is the cause of the problem. For this reason, we work with polynomials and not rational functions; but then the extra assumptions of not having common roots are necessary, since common factors would not modify the Cauchy indices but could modify the signs involved in the formula in Lemma 18. To illustrate this situation, $a = 0, b = 1, P = X - 1, Q = X(X - 1), R = X - 1, S = X$ would be a counterexample if we made no assumptions.

Proof of Lemma 18. If $Q = 0$, $S = 0$ or $PS + QR = 0$, the result is immediate. If $P = 0$ or $R = 0$, the result follows from Proposition 16 taking into account that for $x, y \in \{-1, 0, 1\}$ with $(x, y) \neq (0, 0)$, the identity $\text{sign}(xy) = 1 - |x - y|$ holds. In every other case, from the invariance by linear rescaling and the additivity on intervals of Cauchy index (Remarks 8 and 9) we can suppose that $a$ is the only possible root of $P, Q, R, S$ or $PS + QR$ on $[a, b]$ and finish the proof following the ideas in [3, Theorem 4.5].

Proof of Proposition 17. The proof can be done in a similar way to [3, Corollary 4.6 and Corollary 4.7]. After replacing each winding number by its definition, we apply Lemma 18 one time on each side of $\partial \Gamma$. For instance, on the bottom side, we take $a = x_0, b = x_1, P = F_{re}(X, y_0), Q = F_{im}(X, y_0), R = G_{re}(X, y_0)$ and $S = G_{im}(X, y_0)$. The identity in the lemma is obtained after checking that on each vertex of $\partial \Gamma$, signs cancel after being added on one side and subtracted on the other side.

From Example 12 and Proposition 17 the following result is easily deduced.

Example 20 For $e \in \mathbb{Z}_{\geq 1}$ with $e \leq d^2$,

$$w(Z^e | \partial \Gamma) = e,$$

if $\Gamma \subset \mathbb{R}^2$ is a rectangle containing 0 in its interior.
Finally, we recall the property saying that the winding number vanishes in a small rectangle around a non-zero of a polynomial.

**Proposition 21** Let \((x, y) \in \mathbb{R}^2\) and \(F \in \mathbb{C}[X, Y]\) with \(\deg F \leq d^2\) and such that \(F(x, y) \neq 0\). Then there exists \(\delta \in \mathbb{R}, \delta > 0\) such that for every \(\Gamma \subset [x-\delta, x+\delta] \times [y-\delta, y+\delta] \subset \mathbb{R}^2\), \(F\) does not vanish at \(\Gamma\) and \(w(F \mid \partial \Gamma) = 0\).

**Proof:** First, since [IVT],\(d^2\) holds, it is easy to see that for every \(a \in \mathbb{R}\) with \(a \geq 0\) and every \(n \in \mathbb{Z}_{\geq 1}, n \leq d^2\), there is a unique \(c \in \mathbb{R}\) such that \(c \geq 0\) and \(c^n = a\), which we note as \(c = a^{1/n}\). Then the proof can be done as in [3, Lemma 5.2]. □

### 2.4 Subresultant polynomials

Let \(D\) be an integral domain. The subresultant polynomial sequence of two polynomials \(P, Q \in D[X]\) is a sequence of polynomials in \(D[X]\) which contains the classical Sylvester resultant of \(P, Q\); more specifically, the last subresultant polynomial, which actually belongs to \(D\), coincides up to sign with the Sylvester resultant. Even though the subresultant polynomials of \(P\) and \(Q\) are defined in a completely different way, they are closely related to the polynomials appearing in the remainder sequence of \(P\) and \(Q\), as reflected in the Structure Theorem of Subresultants (Theorem 24). It can be proved that the behavior of their coefficients is better controlled than the behavior of the coefficients of the polynomials in the remainder sequence, and for this reason, they constitute a widely used tool in gcd computation, real root counting and many other problems in computational algebra. In Section 3, we will use subresultants in the particular case of \(D = \mathbb{R}[Y]\) and the good behavior of their coefficients implies a good control of the degree in \(Y\) (Proposition 25), which will be a key point to obtain our main result.

We include now some definitions and properties concerning subresultants. We refer the reader to [1] for proofs and details.

**Definition 22** Let \(P, Q \in D[X] \setminus \{0\}\) with \(\deg P = p \geq 1\) and \(\deg Q = q < p\).

- For \(0 \leq j \leq q\), the Sylvester-Habicht matrix \(\text{SyHa}_j(P, Q) \in D^{(p+q-2j) \times (p+q-j)}\) is the matrix whose rows are the polynomials

\[
X^{q-j-1}P, \ldots, P, Q, \ldots, X^{p-j-1}Q,
\]

expressed in the monomial basis \(X^{p+q-j-1}, \ldots, X, 1\).

- For \(0 \leq j \leq q\), the \(j\)-th subresultant polynomial of \(P\) and \(Q\), \(s\text{Res}_j(P, Q) \in D[X]\) is the polynomial determinant of \(\text{SyHa}_j(P, Q)\), i.e.

\[
s\text{Res}_j(P, Q) = \sum_{0 \leq i \leq j} \det(\text{SyHa}_{j,i}(P, Q)) \cdot X^i \in D[X]
\]
where $\text{SyHa}_{j,i}(P,Q) \in \mathbb{D}^{(p+q-2j)\times(p+q-2j)}$ is the matrix obtained by taking the $p+q-2j-1$ first columns and the $(p+q-j-i)$-th column of $\text{SyHa}_j(P,Q)$. By convention, we extend this definition with

\[ \text{sRes}_p(P,Q) = P \in \mathbb{D}[X], \]
\[ \text{sRes}_{p-1}(P,Q) = Q \in \mathbb{D}[X], \]
\[ \text{sRes}_j(P,Q) = 0 \in \mathbb{D}[X] \quad \text{for } q < j < p - 1. \]

- For $0 \leq j \leq q$, the $j$-th signed subresultant coefficient of $P$ and $Q$, $\text{sR}_j(P,Q) \in \mathbb{D}$ is the coefficient of $X^j$ in $\text{sRes}_j(P,Q)$. By convention, we extend this definition with

\[ \text{sR}_p(P,Q) = 1 \in \mathbb{D} \quad (\text{even if } P \text{ is not monic}), \]
\[ \text{sR}_j(P,Q) = 0 \in \mathbb{D} \quad \text{for } q < j \leq p - 1. \]

- For $0 \leq j \leq p$, $\text{sRes}_j(P,Q)$ is said to be defective if $\deg \text{sRes}_j(P,Q) < j$ or, equivalently, if $\text{sR}_j(P,Q) = 0$.

We will also use the following notation.

**Notation 23** Let $P, Q \in \mathbb{D}[X] \setminus \{0\}$ with $\deg P = p \geq 1$ and $\deg Q = q < p$. Let $(d_0, \ldots, d_s)$ be the sequence of degrees of the non-defective subresultant polynomials of $P$ and $Q$ in decreasing order (note that $d_0 = p$ and $d_1 = q$). For $1 \leq i \leq s$,

\[ T_{d_{i-1}-1}(P,Q) = \text{lcoeff}(\text{sRes}_{d_{i-1}-1}(P,Q)) \in \mathbb{D} \setminus \{0\}. \]

We extend this notation with $T_p(P,Q) = 1 \in \mathbb{D} \setminus \{0\}$.

The following theorem is one of the most important results in the theory of subresultants.

**Theorem 24 (Structure Theorem of Subresultants)** Let $P, Q \in \mathbb{D}[X] \setminus \{0\}$ with $\deg P = p \geq 1$ and $\deg Q = q < p$. Let $(d_0, \ldots, d_s)$ be the sequence of degrees of the non-defective subresultant polynomials of $P$ and $Q$ in decreasing order and let $d_{-1} = p + 1$. Then

- for $1 \leq i \leq s$,

\[ \text{sRes}_{d_{i-1}-2}(P,Q) = \cdots = \text{sRes}_{d_{i+1}}(P,Q) = 0 \in \mathbb{D}[X] \]

and $\text{sRes}_{d_{i-1}-1}(P,Q)$ and $\text{sRes}_{d_i}(P,Q)$ are proportional. More precisely,

\[ \text{sR}_{d_i}(P,Q) \cdot \text{sRes}_{d_{i-1}-1}(P,Q) = T_{d_{i-1}-1}(P,Q) \cdot \text{sRes}_{d_i}(P,Q) \in \mathbb{D}[X] \]

with

\[ \text{sR}_{d_i}(P,Q) = (-1)^{\frac{1}{2}(d_{i-1}-d_i)(d_{i-1}-d_i-1)} T_{d_{i-1}-1}(P,Q)^{d_{i-1}-d_i} \frac{\text{sR}_{d_{i-1}}(P,Q)^{d_{i-1}-d_i-1}}{\text{sR}_{d_{i-1}}(P,Q)^{d_{i-1}-d_i-1}} \in \mathbb{D}. \]

This implies $\deg \text{sRes}_{d_{i-1}-1}(P,Q) = d_i$. 
• for \(1 \leq i \leq s\),

\[
T_{d_i - 2}(P, Q) \cdot sR_{d_i - 1}(P, Q) \cdot sResP_{d_i - 1}(P, Q) = -\text{Rem} \left( T_{d_i - 1}(P, Q) \cdot sR_d(P, Q) \cdot sResP_{d_i - 1}(P, Q) \right) \in D[X]
\]

and

\[
\text{Quot} \left( T_{d_i - 1}(P, Q) \cdot sR_d(P, Q) \cdot sResP_{d_i - 1}(P, Q) \right) \in D[X]
\]

(where Rem and Quot means the remainder and quotient in the euclidean division in \(\text{ff}(D)[X]\) of the first polynomial by the second polynomial).

• \(sResP_{d_{s-1}}(P, Q) \in D[X]\) and \(sResP_d(P, Q) \in D[X]\) are the greatest common divisor of \(P\) and \(Q\) in \(\text{ff}(D)[X]\) multiplied by elements in \(D\). In addition, if \(d_s > 0\) then

\[
sResP_{d_{s-1}}(P, Q) = \cdots = sResP_0(P, Q) = 0 \in D[X].
\]

---

**Proof:** See [1, Chapter 8]. \(\square\)

As said before, in Section 3 we will use subresultants in the particular case of \(D = \mathbb{R}[Y]\) and we will need some degree bounds which we develop here.
Proposition 25 Let \( P, Q \in \mathbb{R}[X,Y] \setminus \{0\} \) with \( \deg_X P = p \geq 1 \), \( \deg_X Q = q < p \) and total degree \( \deg P, \deg Q \leq d \) (with \( d \geq p \)). We consider subresultants with respect to variable \( X \) (this is to say, following Definition \( \text{[22]} \) we take \( D = \mathbb{R}[Y] \)). For \( 0 \leq j \leq q \) and \( 0 \leq i \leq j \), the degree in \( Y \) of the coefficient of \( X^i \) in \( \text{sRes}_j(P, Q) \in \mathbb{R}[Y][X] \) is bounded by \( d^2 \).

Proof: Let \( P = \sum_{0 \leq i \leq p} a_i(Y)X^i \) and \( Q = \sum_{0 \leq i \leq q} b_i(Y)X^i \); then \( \deg_Y a_i(Y), \deg_Y b_i(Y) \leq d - i \). By definition,

\[
\text{sRes}_j(P, Q) = \sum_{0 \leq i \leq j} \det(\text{SyHa}_{j,i}(P, Q)) \cdot X^i \in \mathbb{R}[Y][X],
\]

where \( \text{SyHa}_{j,i}(P, Q) \in \mathbb{R}[Y]^{(p+q-2j) \times (p+q-2j)} \) is the matrix obtained by taking the \( p + q - 2j - 1 \) first columns and the \( (p + q - j - i) \)-th column of \( \text{SyHa}_{j,i}(P, Q) \in \mathbb{R}[Y]^{(p+q-2j) \times (p+q-2j)} \). By defining \( a_i(Y) = 0 \) if \( i \geq p + 1 \) or \( i \leq -1 \) and \( b_i(Y) = 0 \) if \( i \geq q + 1 \) or \( i \leq -1 \) we have that for \( 1 \leq k \leq p + q - 2j \) and \( 1 \leq \ell \leq p + q - j \),

\[
(\text{SyHa}_j(P, Q))_{k\ell} = \begin{cases} a_{p+k-\ell}(Y) & \text{if } k \leq q - j, \\ b_{p+2q-2j+1-k-\ell}(Y) & \text{if } k \geq q - j + 1. \end{cases}
\]

The proof can be completed by bounding the degree of any possible nonzero product of entries of \( \text{SyHa}_{j,i}(P, Q) \) with one element per row and column. We obtain that the degree in \( Y \) of the coefficient of \( X^i \) in \( \text{sRes}_j(P, Q) \in \mathbb{R}[Y][X] \) is bounded by

\[
d(p + q - 2j) - pq + j^2 + j - i \leq d(p + q) - pq \leq d^2.
\]

\[\square\]

3 Counting complex roots

In this section we introduce \((\sigma, \tau)\)-chains, develop suitable generalizations of results from \([3]\) and prove Theorem \([3]\). As said before, till the end of the paper, we take a fixed value of \( d \in \mathbb{Z}_{\geq 2} \) and we suppose that \((\mathbb{R}, \leq)\) is an ordered field satisfying \([\text{IVT}]_{\mathbb{Q}}\) but not necessarily \([\text{IVT}]\).

3.1 \((\sigma, \tau)\)-chains and Cauchy index

A Sturm chain \((\mathbb{R})\) is a finite sequence of univariate polynomials such that for every polynomial with exception of the first one and the last one, and for each real root \( a \) of this polynomial, the previous and next polynomial evaluated at \( a \) have opposite sign. A very nice property of Sturm chains is that if the last polynomial in the chain has no real roots, then the Cauchy index of the quotient of the second polynomial by the first polynomial in the chain can be easily computed with a simple sign changing counting rule (see \([3], \text{Theorem 3.11}\)].

We introduce now the notion of \((\sigma, \tau)\)-chain, which is a generalization of the notion of Sturm chain. Then, in Proposition \([30]\) and Corollary \([31]\) we develop a modified sign changing counting rule, so that we can still use \((\sigma, \tau)\)-chains to compute Cauchy indices.
The benefit of this generalization is that, later on, the subresultant polynomial sequence will fit in this definition for some pair \((\sigma, \tau)\), which is an essential ingredient for the proof of Theorem 3 since we can use good degree bounds on subresultants (see Proposition 25).

**Definition 26** Let \(I\) be an interval of \(\mathbb{R}\), \(n \in \mathbb{Z}_{\geq 1}\) and \(\sigma, \tau \in \{-1, 1\}^{n-1}\) with \(\sigma = (\sigma_1, \ldots, \sigma_{n-1})\) and \(\tau = (\tau_1, \ldots, \tau_{n-1})\).

A sequence of polynomials \((S_0, \ldots, S_n) \in \mathbb{R}[X]\) is a \((\sigma, \tau)\)-chain with respect to \(I\) if for \(1 \leq i \leq n - 1\) there exists polynomials \(A_i, B_i, C_i \in \mathbb{R}[X]\) such that

1. \(A_i S_{i+1} + B_i S_i + C_i S_{i-1} = 0\),
2. for every \(x \in I\), \(\text{sign}(A_i(x)) = \sigma_i\),
3. for every \(x \in I\), \(\text{sign}(C_i(x)) = \tau_i\).

A sequence of polynomials \((S_0, \ldots, S_n) \in \mathbb{R}[X]\) is a Sturm \((\sigma, \tau)\)-chain with respect to \(I\) if it is a \((\sigma, \tau)\)-chain with respect to \(I\) and \(S_{n-1}\) and \(S_n\) have no common root on \(I\).

A sequence of polynomials \((S_0, \ldots, S_n) \in \mathbb{R}[X]\) is a good Sturm \((\sigma, \tau)\)-chain with respect to \(I\) if it is a \((\sigma, \tau)\)-chain with respect to \(I\) and \(S_n\) have no root on \(I\).

Note that for \(n = 1\), taking \([-1, 1]^0 = \{\bullet\}\), any sequence \((S_0, S_1)\) in \(\mathbb{R}[X]\) is a \((\bullet, \bullet)\)-chain with respect to \(I\).

Note also that if a sequence of polynomials \((S_0, \ldots, S_n)\) in \(\mathbb{R}[X]\) is a \((\sigma, \tau)\)-chain with respect to \(I\), then for every \(0 \leq m \leq n - 1\), \((S_m, \ldots, S_n)\) is a \((\sigma', \tau')\)-chain with respect to \(I\), with \(\sigma' = (\sigma_{m+1}, \ldots, \sigma_{n-1})\) and \(\tau' = (\tau_{m+1}, \ldots, \tau_{n-1})\). The analogous statements hold also for Sturm \((\sigma, \tau)\)-chains and good Sturm \((\sigma, \tau)\)-chains.

**Example 27** Here are important examples illustrating the definitions. Item b) plays a key role in the proof of Lemma 34:

a) Let \(S_0, S_1 \in \mathbb{R}[X] \setminus \{0\}\) with \(\deg S_0 = p \geq 1\), \(\deg S_1 = q < p\) and \(S_0, S_1\) coprime. Let \((d_0, d_1, \ldots, d_s)\) be the sequence of degrees of the non-defective subresultant polynomials of \(S_0, S_1\) in decreasing order (note that \(d_0 = p\), \(d_1 = q\), and \(d_{-1} = p + 1\)). Finally, for \(2 \leq i \leq s\), we define

\[S_i = \text{sResP}_{d_{i-1} - 1}(S_0, S_1) \in \mathbb{R}[X]\]

(note that the above identity also holds for \(i = 0, 1\)). Since \(S_0\) and \(S_1\) are coprime in \(\mathbb{R}[X]\), by the Structure Theorem of Subresultants (Theorem 27) we have that \(S_s \in \mathbb{R}\) and \(d_s = 0\). Also, defining for \(1 \leq i \leq s - 1\)

\[A_i = T_{d_{i-2} - 1}(S_0, S_1) \cdot \text{sR}_{d_{i-1}}(S_0, S_1) \in \mathbb{R} \setminus \{0\}\]

\[B_i = -\text{Quot}(T_{d_{i-1} - 1}(S_0, S_1) \cdot \text{sR}_{d_i}(S_0, S_1) \cdot S_{i-1}, S_i) \in \mathbb{R}[X],\]

\[C_i = T_{d_i - 1}(S_0, S_1) \cdot \text{sR}_{d_i}(S_0, S_1) \in \mathbb{R} \setminus \{0\},\]

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we have
\[ A_i S_{i+1} + B_i S_i + C_i S_{i-1} = 0. \]

We define \( \sigma_i = \text{sign}(A_i) \) and \( \tau_i = \text{sign}(C_i) \) for \( 1 \leq i \leq s-1 \) and we have that \((S_0, S_1, \ldots, S_s)\) is a good Sturm \((\sigma, \tau)\)-chain.

b) Let \( S_0, S_1 \in \mathbb{R}[X, Y] \setminus \{0\} \) with \( \deg_X S_0 = p \geq 1, \) \( \deg_X S_1 = q < p \) and \( S_0, S_1 \) coprime in the unique factorization domain \( \mathbb{R}[X, Y] \). Let \( (d_0, d_1, \ldots, d_s) \) be the sequence of degrees of the non-defective subresultant polynomials in decreasing order (note that \( d_0 = p \) and \( d_1 = q \)), and \( d_{-1} = p + 1 \); where all the subresultants are defined considering \( X \) as the main variable (this is to say, \( D = \mathbb{R}[Y] \) in Definition 22). Finally, for \( 2 \leq i \leq s \), we define
\[ S_i = s\text{ResP}_{d_{i-1}}(S_0, S_1) \in \mathbb{R}[Y][X] \]
(note that the above identity also holds for \( i = 0, 1 \)). Since \( S_0 \) and \( S_1 \) are coprime in \( \mathbb{R}[X, Y] \), by the Structure Theorem of Subresultants (Theorem 24) we have that \( S_s \in \mathbb{R}[Y] \). Also, defining for \( 1 \leq i \leq s-1 \)
\[
A_i = T_{d_{i-2}}(S_0, S_1) \cdot s\text{Res}_{d_{i-1}}(S_0, S_1) \in \mathbb{R}[Y] \setminus \{0\} \\
B_i = -\text{Quot}(T_{d_{i-1}-1}(S_0, S_1) \cdot s\text{Res}_{d_i}(S_0, S_1) \cdot S_{i-1}, S_i) \in \mathbb{R}[Y][X], \\
C_i = T_{d_{i-1}-1}(S_0, S_1) \cdot s\text{Res}_{d_i}(S_0, S_1) \in \mathbb{R}[Y] \setminus \{0\},
\]

we have
\[ A_i S_{i+1} + B_i S_i + C_i S_{i-1} = 0. \]

Take an interval \([b, b']\) such that \( S_s, A_i \) and \( C_i \) have constant sign different from 0 on \([b, b']\). We define then \( \sigma = (\sigma_1, \ldots, \sigma_{s-1}), \tau = (\tau_1, \ldots, \tau_{s-1}) \in \{-1, 1\}^{s-1} \) by choosing any \( c \in [b, b'] \) and taking \( \sigma_i = \text{sign}(A_i(c)) \) and \( \tau_i = \text{sign}(C_i(c)) \) for \( 1 \leq i \leq s-1 \). In this way, since also \( S_s \in \mathbb{R}[Y] \) does not vanish on \([b, b']\), we have that

\[- \text{ for any } x \in \mathbb{R}, (S_0(x, Y), \ldots, S_s(x, Y)) \in \mathbb{R}[Y] \text{ is a good Sturm } (\sigma, \tau)\text{-chain with respect to } [b, b'], \]
\[- \text{ for any } y \in [b, b'], (S_0(X, y), \ldots, S_s(X, y)) \in \mathbb{R}[X] \text{ is a good Sturm } (\sigma, \tau)\text{-chain with respect to } \mathbb{R}. \]

**Lemma 28** Let \( I \) be an interval of \( \mathbb{R} \), \( n \in \mathbb{Z}_{\geq 1} \) and \( \sigma, \tau \in \{-1, 1\}^{n-1} \). If a sequence of polynomials \((S_0, \ldots, S_n)\) in \( \mathbb{R}[X] \) is a Sturm \((\sigma, \tau)\)-chain with respect to \( I \), then for every \( 1 \leq m \leq n \), \( S_{m-1} \) and \( S_m \) have no common root on \( I \).

**Proof:** The proof can be easily done by reverse induction on \( m = n, \ldots, 1 \), taking into account that conditions 1 and 2 from Definition 26 imply that for \( m < n \), any common root of \( S_{m-1} \) and \( S_m \) would also be a root of \( S_{m+1} \). \( \square \)

We introduce some more useful notation.
Notation 29 Let \( a, b \in \mathbb{R} \), \( n \in \mathbb{Z}_{\geq 1} \), \((S_0, \ldots, S_n)\) in \( \mathbb{R}[X] \) and \( \sigma, \tau \in \{ -1, 1 \}^{n-1} \). We define
\[
\epsilon(\sigma, \tau)_i = \prod_{1 \leq j \leq i-1} \sigma_j \tau_j
\]
for \( 1 \leq i \leq n \) and
\[
\text{Var}(\sigma, \tau)_a^b(S_0, \ldots, S_n) = \sum_{1 \leq i \leq n} \epsilon(\sigma, \tau)_i \text{Var}_a^b(S_{i-1}, S_i).
\]

Note that it is always the case that \( \epsilon(\sigma, \tau)_1 = 1 \).

The following result is a generalization of [3, Theorem 3.11].

Proposition 30 Let \( a, b \in \mathbb{R} \) with \( a < b \), \( I = [a, b] \), \( n \in \mathbb{Z}_{\geq 1} \) and \( \sigma, \tau \in \{ -1, 1 \}^{n-1} \). If \((S_0, \ldots, S_n)\) is a Sturm \((\sigma, \tau)\)-chain with respect to \( I \) and \( \deg S_0, \ldots, \deg S_n \leq d^2 \), then
\[
\text{Ind}_a^b(S_1, S_0) + \epsilon(\sigma, \tau)_n \text{Ind}_a^b(S_{n-1}, S_n) = \text{Var}(\sigma, \tau)_a^b(S_0, \ldots, S_n).
\]

Proof: By Lemma 28 we know that for every \( 1 \leq m \leq n \), \( S_{m-1} \) and \( S_m \) have no common root on \( I \). We proceed then by induction on \( n \). For \( n = 1 \), the result holds by Proposition 16.

Now we take \( n \geq 2 \). Let \( x \) be a root of \( S_1 \) on \( I \) (and therefore not a root of \( S_0 \) nor \( S_2 \)). The identity
\[
A_1 S_2 + B_1 S_1 + C_1 S_0 = 0
\]
implies that \( C_1(x)S_0(x) = -A_1(x)S_2(x) \neq 0 \) and then \( \text{sign}(S_0(x)) = -\sigma_1 \tau_1 \text{sign}(S_2(x)) \). From this we deduce
\[
\text{Ind}_a^b(S_0, S_1) = -\sigma_1 \tau_1 \text{Ind}_a^b(S_2, S_1).
\]
We consider \( \sigma' = (\sigma_2, \ldots, \sigma_{n-1}) \), \( \tau' = (\tau_2, \ldots, \tau_{n-1}) \) and we apply the inductive hypothesis to the Sturm \((\sigma', \tau')\)-chain \((S_1, \ldots, S_n)\). For \( 2 \leq i \leq n \) we have that \( \epsilon(\sigma, \tau)_i = \sigma_1 \tau_1 \epsilon(\sigma', \tau')_{i-1} \).

Finally, using Proposition 16
\[
\text{Ind}_a^b(S_1, S_0) + \epsilon(\sigma, \tau)_n \text{Ind}_a^b(S_{n-1}, S_n) = \text{Ind}_a^b(S_1, S_0) + \text{Ind}_a^b(S_0, S_1) + \sigma_1 \tau_1 \text{Ind}_a^b(S_2, S_1) + \sigma_1 \tau_1 \epsilon(\sigma', \tau')_{n-1} \text{Ind}_a^b(S_{n-1}, S_n) = \text{Var}(\sigma, \tau)_a^b(S_0, \ldots, S_n)
\]
as we wanted to prove. \( \square \)

Corollary 31 Let \( a, b \in \mathbb{R} \) with \( a < b \), \( I = [a, b] \), \( n \in \mathbb{Z}_{\geq 1} \) and \( \sigma, \tau \in \{ -1, 1 \}^{n-1} \). If \((S_0, \ldots, S_n)\) is a good Sturm \((\sigma, \tau)\)-chain with respect to \( I \) and \( \deg S_0, \ldots, \deg S_n \leq d^2 \), then
\[
\text{Ind}_a^b(S_1, S_0) = \text{Var}(\sigma, \tau)_a^b(S_0, \ldots, S_n).
\]

Proof: Since \((S_0, \ldots, S_n)\) is a good Sturm \((\sigma, \tau)\)-chain with respect to \( I \), \( S_n \) has no roots on \( I \) and
\[
\text{Ind}_a^b(S_{n-1}, S_n) = 0,
\]
therefore the claim holds by Proposition 30. \( \square \)
3.2 Quantitative Main Lemma

Our next goal is to prove a quantitative adaptation of the Main Lemma (see [3, Lemma 5.3]): using [IVT] and subresultants we want to prove that if $F \in \mathbb{C}[X,Y]$ with $\deg F \leq d$ does not vanish on $\Gamma$, then $w(F \mid \partial \Gamma) = 0$.

In order to be able to work with subresultants, we need to consider separately for $F \in \mathbb{C}[X,Y]$ the degrees with respect to $X$ and $Y$ of $F_{\text{re}}$ and $F_{\text{im}}$, and we wish each of these two degrees of $F_{\text{re}}$ to drop with respect to the respective degree of $F_{\text{im}}$. Since $(iF)^{\text{re}} = -F_{\text{im}}$ and $(iF)^{\text{im}} = F_{\text{re}}$, up to multiplication by $i$, it will be enough for our purposes if these degrees are different. We will also need some degree control on some auxiliary subresultant polynomials which will play a key role in our proof. For these reasons, we introduce the following definition.

**Definition 32** Let $F \in \mathbb{C}[X,Y]$. We say that $F$ is well-controlled if the following conditions are satisfied:

1. $F_{\text{re}}, F_{\text{im}} \neq 0$,
2. $\deg_X F_{\text{im}} \neq \deg_X F_{\text{re}}$ and $\deg_Y F_{\text{im}} \neq \deg_Y F_{\text{re}}$.

For a well-controlled $F$, we denote by $F^X$ the unique polynomial in $\{F, iF\} \subset \mathbb{C}[X,Y]$ such that $\deg_X F_{\text{im}}^X > \deg_X F_{\text{re}}^X$. Similarly, we denote by $F^Y$ the unique polynomial in $\{F, iF\} \subset \mathbb{C}[X,Y]$ such that $\deg_Y F_{\text{im}}^Y > \deg_Y F_{\text{re}}^Y$.

**Example 33** Let $F \in \mathbb{C}[Z] \setminus \mathbb{C}$ with $F$ monic. Then $F$ is well-controlled.

Now we are ready to prove our quantitative version of the Main Lemma [3, Lemma 5.3]. We stress the fact that it is in the proof of the Main Lemma [34] that subresultant polynomials play a key role to keep control of the degree of intermediate auxiliary polynomials and we use the hypothesis [IVT]$_{d^2}$ for dealing with a polynomial of degree $\leq d$.

**Lemma 34** (Quantitative Main Lemma) Let $\Gamma = [x_0, x_1] \times [y_0, y_1] \subset \mathbb{R}^2$ and $F \in \mathbb{C}[X,Y]$ with $\deg F \leq d$ and such that $F$ does not vanish on $\Gamma$. If $F$ is well-controlled, then $w(F \mid \partial \Gamma) = 0$.

**Proof:** We will produce in several steps a suitable grid partition of $\Gamma$ into a finite number of rectangles $\Gamma_1, \ldots, \Gamma_s$ and we will prove that $w(F \mid \partial \Gamma_i) = 0$ for $1 \leq i \leq s$. Then the result will follow from Lemma [13]. Let $G$ be a greatest common divisor of $F_{\text{re}}$ and $F_{\text{im}}$ in the unique factorization domain $\mathbb{R}[X,Y]$ and let $F^X, F^Y \in \mathbb{C}[X,Y]$ be as in Definition [32].

**First step.** We decompose $F_{\text{im}}^X = GS_0$ and $F_{\text{re}}^X = GS_1$ and note that we have $\deg S_0, \deg S_1 \leq d$ and $d \geq p := \deg_X S_0 > q := \deg_X S_1$. 

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We consider $S_0$ and $S_1$ as elements of $\mathbb{R}[Y][X]$ and we take the subresultant polynomial sequence with respect to the variable $X$ (as in Section 2.3 taking $D = \mathbb{R}[Y]$)

$$s\text{ResP}_p(S_0, S_1) = S_0, s\text{ResP}_{p-1}(S_0, S_1) = S_1, \ldots, s\text{ResP}_0(S_0, S_1) \in \mathbb{R}[Y][X].$$

For $0 \leq j \leq p$ and $0 \leq i \leq j$, the degree in $Y$ of the coefficient of $X^i$ in $s\text{ResP}_j(S_0, S_1) \in \mathbb{R}[Y][X]$ is bounded by $d^2$. This is so by Proposition 25 for $0 \leq j \leq q$ and by definition of subresultant polynomials for $q < j \leq p$.

We take $(d_0, d_1, \ldots, d_s)$ as the sequence of degrees of the non-defective subresultant polynomials in decreasing order (note that $d_0 = p$ and $d_1 = q$), and $d_{-1} = p + 1$, and we define $\mathcal{Y}_1$ as the union of the sets of roots in $[y_0, y_1]$ of the polynomials $S_s, T_{d_{-1}-1}(S_0, S_1), \ldots, T_{d_{s-2}-1}(S_0, S_1), s\text{ResP}_{d_0}(S_0, S_1), \ldots, s\text{ResP}_{d_{s-1}}(S_0, S_1) \in \mathbb{R}[Y] \setminus \{0\}$ (note that $T_{d_{-1}-1}(S_0, S_1) = s\text{ResP}_{d_0}(S_0, S_1) = 1$, therefore these two polynomials actually add no roots to the set $\mathcal{Y}_1$; note also that in the particular case $s = 1$, $\mathcal{Y}_1$ is just the set of roots in $[y_0, y_1]$ of the polynomial $S_1$).

For uniformity reasons in exposition we define $\mathcal{Y}_2 = \{y_0, y_1\}$ and we also define

$$\mathcal{Y}_3 = \{y \in [y_0, y_1] \mid F^{X}_{\text{re}}(X, y) = 0 \in \mathbb{R}[X] \text{ or } F^{X}_{\text{im}}(X, y) = 0 \in \mathbb{R}[X]\}.$$

Finally, we define

$$\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3.$$

We think of $\mathcal{Y}$ as the set of the $Y$-coordinates of bad behaving points in $\Gamma$. Suppose $\mathcal{Y} = \{b_1, \ldots, b_\ell\}$ with $y_0 = b_1 < \cdots < b_\ell = y_1$.

Second step. We proceed as in the first step, but replacing polynomial $F^X$ by $F^Y$ and the role of variables $X$ and $Y$, to produce a set $\mathcal{X} \subset [x_0, x_1]$, which we think of as the set of the the $X$-coordinates of bad behaving points in $\Gamma$. Suppose $\mathcal{X} = \{a_1, \ldots, a_k\}$ with $x_0 = a_1 < \cdots < a_k = x_1$.

Third step. We take $Z = \mathcal{X} \times \mathcal{Y} \subset \Gamma$. For each $z = (a, b) \in Z$, since $F(a, b) \neq 0$, by Proposition 21 there exist $\delta_z > 0$ such that the winding number of $F$ vanishes on any rectangle contained in $[a - \delta_z, a + \delta_z] \times [b - \delta_z, b + \delta_z]$. So we take $\delta > 0$, with $\delta \leq \delta_z$ for every $z \in Z$ and such that

$$x_0 = a_1 < a_1 + \delta < a_2 - \delta < a_2 + \delta < a_3 - \delta < \cdots < a_{k-1} + \delta < a_k - \delta < a_k = x_1$$

and

$$y_0 = b_1 < b_1 + \delta < b_2 - \delta < b_2 + \delta < b_3 - \delta < \cdots < b_{\ell-1} + \delta < b_{\ell} - \delta < b_{\ell} = y_1.$$

We divide intervals $[x_0, x_1]$ and $[y_0, y_1]$ using all these numbers above, and finally we use these divisions of these intervals to obtain a grid partition of $\Gamma = [x_0, x_1] \times [y_0, y_1]$. 

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Now that the grid partition is defined, we have to prove that the winding number of $F$ vanishes on each rectangle in the grid. Take $\Gamma' = [a, a'] \times [b, b'] \subset \Gamma$ as one of this rectangles. Then either there is a single point of $Z$ in $\Gamma'$ or there is no point of $Z$ in $\Gamma'$. In the first case, $w(F|\partial \Gamma') = 0$ by Proposition 21 and the election of $\delta$. In the second case, then either $[a, a'] \cap X = \emptyset$ or $[b, b'] \cap Y = \emptyset$.

Let us suppose first that $[b, b'] \cap Y = \emptyset$ holds and prove that $w(F^X|\partial \Gamma') = 0$; then $w(F|\partial \Gamma') = 0$ as well either because $F^X = F$ or because $F^X = iF$ and by Proposition 17.

Since $F$ is well-controlled of degree at most $d$ and the polynomials $T_{d-1}(S_0, S_1), \ldots, T_{d-2}(S_0, S_1), sR_{d_0}(S_0, S_1), \ldots, sR_{d_{s-1}}(S_0, S_1) \in R[Y]$ are coefficients of subresultant polynomials of $S_0$ and $S_1$ with respect to variable $X$, their degree in $Y$ is bounded by $d^2$. Since none of these polynomials vanishes on $[b, b']$, using the notation and results from Example 27(b), for $1 \leq i \leq s - 1$ we have that $A_i$ and $C_i$ have constant sign different from 0 on $[b, b']$ and

- for any $x \in [a, a']$, $(S_0(x, Y), \ldots, S_s(x, Y)) \in R[Y]$ is a good Sturm $(\sigma, \tau)$-chain with respect to $[b, b']$ with all its elements with degree bounded by $d^2$,
- for any $y \in [b, b']$, $(S_0(X, y), \ldots, S_s(X, y)) \in R[X]$ is a good Sturm $(\sigma, \tau)$-chain with respect to $[a, a']$ with all its elements with degree bounded by $d \leq d^2$.

Taking into account that

$$F^X_{re}(X, b), F^X_{im}(X, b), F^X_{re}(X, b'), F^X_{im}(X, b') \neq 0 \in R[X],$$

$$F^X_{re}(a, Y), F^X_{im}(a, Y), F^X_{re}(b, Y), F^X_{im}(b, Y) \neq 0 \in R[Y],$$

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we conclude that
\[
2w(F^X | \partial \Gamma') = \text{Ind}_{a'}^{\partial} (F^{X}_{\text{re}}(X, b), F^{X}_{\text{im}}(X, b)) + \text{Ind}_{b'}^{\partial} (F^{X}_{\text{re}}(a', Y), F^{X}_{\text{im}}(a', Y)) \\
+ \text{Ind}_{a'}^{\partial} (F^{X}_{\text{re}}(X, b'), F^{X}_{\text{im}}(X, b')) + \text{Ind}_{b'}^{\partial} (F^{X}_{\text{re}}(a, Y), F^{X}_{\text{im}}(a, Y)) \\
= \text{Ind}_{a'}^{\partial} (S_1(X, b), S_0(X, b)) + \text{Ind}_{b'}^{\partial} (S_1(a', Y), S_0(a', Y)) \\
+ \text{Ind}_{a'}^{\partial} (S_1(X, b'), S_0(X, b')) + \text{Ind}_{b'}^{\partial} (S_1(a, Y), S_0(a, Y)) \\
= \text{Var}((\sigma, \tau)_{a'}^{\partial} (S_0(X, b), \ldots, S_0(X, b')) + \text{Var}((\sigma, \tau)_{b'}^{\partial} (S_0(a', Y), \ldots, S_0(a', Y)) \\
+ \text{Var}((\sigma, \tau)_{a'}^{\partial} (S_0(X, b'), \ldots, S_0(X, b')) + \text{Var}((\sigma, \tau)_{b'}^{\partial} (S_0(a, Y), \ldots, S_0(a, Y)) \\
= 0
\]
using Corollary [31].

In case that \([a, a'] \cap \mathcal{X} = \emptyset\) holds, we proceed in a similar way exchanging the role of \(X\) and \(Y\), to prove that \(w(F^Y | \partial \Gamma') = 0\), and then we have that \(w(F | \partial \Gamma') = 0\) again either because \(F^Y = F\) or because \(F^Y = iF\) and by Proposition [17].

3.3 The winding number counts the complex roots

From Example [12], Proposition [17] and the Quantitative Main Lemma [34] we deduce the following result.

**Theorem 35** Let \(\Gamma \subset \mathbb{R}^2\) be a rectangle and \(F \in \mathbb{C}[Z]\) with \(\deg F \leq d\) and such that \(F\) does not vanish at \(\partial \Gamma\). Then \(w(F | \partial \Gamma)\) counts the number of zeros of \(F\) in the interior of \(\Gamma\) with multiplicity.

**Proof:** Factorize \(F = a \cdot (Z - z_1) \cdot \ldots \cdot (Z - z_r) \cdot \tilde{F}\) with \(a \in \mathbb{C}, z_1, \ldots, z_r \in \Gamma \setminus \partial \Gamma\) and monic \(\tilde{F} \in \mathbb{C}[Z]\) with no roots in \(\Gamma\). If \(\tilde{F} = 1\) the result follows from Example [12] and Proposition [17]. Otherwise, since \(\tilde{F}\) is well-controlled (Example [33]) the result follows from Example [12] and Proposition [17] and the Main Lemma [34].

3.4 Quantitative Homotopy

The last ingredient for the proof of Theorem [3] is a quantitative homotopy tool similar [3] Theorem 5.4 and Corollary 5.5, Proposition 5.8 and Theorem 5.9]. Yet, since we need to deal with well-controlled polynomials, we have to divide the homotopy in two steps, one for the real part and one for the imaginary part.

**Theorem 36** Let \(F \in \mathbb{C}[Z]\), with \(F \neq 0\) and \(\deg F = e \leq d\). There exists \(r \in \mathbb{R}, r > 0\) such that if \(m \geq r\) and \(\Gamma = [-m, m] \times [-m, m]\), then \(w(F | \partial \Gamma) = e\).
Proof: If \( e = 0 \) there is nothing to prove, so we suppose \( e > 0 \). By Proposition\textsuperscript{17} we can also suppose that \( F \) is monic. Let
\[
F = Z^e + \sum_{j=0}^{e-1} (a_j + ib_j)Z^j
\]
with \( a_j, b_j \in \mathbb{R} \) for \( 0 \leq j \leq e - 1 \) and take \( G = F - Z^e \in \mathbb{C}[Z] \) collecting all the terms of degree less than \( e \) in \( F \).

We take the auxiliary polynomial
\[
K = (X + iY)^e + G_{re} \in \mathbb{C}[X, Y],
\]
note that in general \( K \) does not come from a polynomial in \( \mathbb{C}[Z] \) by means of the substitution \( Z = X + iY \). The idea of the proof is to obtain \( \Gamma \in \mathbb{R} \), \( r > 0 \) such that if \( m \geq r \) and \( \Gamma = [-m, m] \times [-m, m] \), then \( e = w(Z^e | \partial \Gamma) = w(K | \partial \Gamma) = w(F | \partial \Gamma) \) (see Example\textsuperscript{20}).

We suppose \( G_{re}, G_{im} \neq 0 \), and if this is not the case, the rest of the proof can be simplified. Actually, the only case where \( G_{re} = 0 \) or \( G_{im} = 0 \) is \( G \in \mathbb{R} \cup i\mathbb{R} \).

We define \( H_0, H_1 \in \mathbb{C}[X, Y, T] \) by
\[
H_0(X, Y, T) = (1 - T)(X + iY)^e + TK(X, Y) = (X + iY)^e + T G_{re}(X, Y)
\]
and
\[
H_1(X, Y, T) = (1 - T)K(X, Y) + TF(X + iY) = (X + iY)^e + G_{re}(X, Y) + i\ T G_{im}(X, Y).
\]
Take \( r = 1 + 2 \max \{|a_j + ib_j| | 0 \leq j \leq e - 1\} \). Proceeding similarly to \textsuperscript{3} Proposition 5.8, it can be proved that both \( H_0, H_1 \) do not vanish on \( \partial \Gamma \times [0, 1] \).

Now enlarge \( r \) if necessary so that \( X \pm m \) and \( Y \pm m \) are not factors of \( F_{re}(X, Y), G_{re}(X, Y) \) and \( G_{im}(X, Y) \). Then it can be verified that the polynomials \( H_0(X, -m, T), H_0(X, m, T), H_1(X, -m, T), H_1(X, m, T) \in \mathbb{C}[X, T] \) and \( H_0(-m, Y, T), H_0(m, Y, T), H_1(-m, Y, T), H_1(m, Y, T) \in \mathbb{C}[Y, T] \) are of degree at most \( d \) and well-controlled.

Finally, take \( \Gamma_X = \Gamma_Y = [-m, m] \times [0, 1] \) and \( \Gamma_T = \Gamma \). By the Main Lemma\textsuperscript{34}
\[
w(H_0(X, -m, T) | \partial \Gamma_Y) = w(H_0(X, m, T) | \partial \Gamma_Y) = w(H_1(X, -m, T) | \partial \Gamma_Y) = w(H_1(X, m, T) | \partial \Gamma_Y) = 0
\]
and
\[
w(H_0(-m, Y, T) | \partial \Gamma_X) = w(H_0(m, Y, T) | \partial \Gamma_X) = w(H_1(-m, Y, T) | \partial \Gamma_X) = w(H_1(m, Y, T) | \partial \Gamma_X) = 0.
\]
Therefore, by Lemma\textsuperscript{14} applied to \( H_0 \) and \( H_1 \) and the cube \([-m, m] \times [-m, m] \times [0, 1] \subset \mathbb{R}^3 \), we have
\[
e = w(Z^e | \partial \Gamma) = w(H_0(X, Y, 0) | \partial \Gamma_T) = w(H_0(X, Y, 1) | \partial \Gamma_T) = w(K | \partial \Gamma)
\]
and
\[
w(K | \partial \Gamma) = w(H_1(X, Y, 0) | \partial \Gamma_T) = w(H_1(X, Y, 1) | \partial \Gamma_T) = w(F | \partial \Gamma)
\]
as we wanted to prove. \( \square \)
3.5 Proof of Theorem \[\text{3}\]

We are now ready to deduce our main result.

Proof of Theorem \[\text{3}\] As mentioned before, since \([\text{FTA}]_1\) holds even under no assumptions on \((\mathbb{R}, \leq)\) we suppose \(d \geq 2\). Take \(F \in \mathbb{C}[Z] \setminus \mathbb{C}\) with \(\deg F = e \leq d\). By Theorem \(\text{36}\) there exists \(r \in \mathbb{R}, r > 0\) such that if \(m \geq r\) and \(\Gamma = [-m,m] \times [-m,m]\), then \(w(F|\partial \Gamma) = e \geq 1\). By Theorem \(\text{35}\), \(w(F|\partial \Gamma)\) counts the number of zeros of \(F\) in the interior \(\Gamma\) with multiplicity. This implies that there exists at least one \(z \in \Gamma \subset \mathbb{R}^2 \sim \mathbb{C}\) such that \(F(z) = 0\). \(\square\)

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