On the Ristic-Balakrishnan distribution: bivariate extension and characterizations

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Abstract

Over the last few decades, a significant development has been made towards the augmentation of some well-known lifetime distributions by various strategies. These newly developed models have enjoyed a considerable amount of success in modeling various real life phenomena. Motivated by this, Ristic & Balakrishnan (2012) developed a special class of univariate distributions (see Ristic- Balakrishnan (2012)). Henceforth we call this family of distribution as RB-G family of distributions. The RB-G family has the same parameters of the G distribution plus an additional positive shape parameter \( a \). Several RB-G distribution can be obtained from a specified \( G \) distribution. For \( a = 1 \), the baseline \( G \) distribution is a basic exemplar of the RB-G family with a continuous crossover towards cases with various shapes. In this article we focus our attention on the characterization of this family and discuss some structural properties of the bivariate RB-G family of distributions which are not discussed in detail in Ristic and Balakrishnan (2012).

1 Introduction

The statistics literature is filled with hundreds of continuous univariate distributions. In the last two decades, considerable amount of work has been done on introducing various univariate and bivariate non-normal models and then discussing their properties, fit and applications; for elaborate details, one may refer to the books by Kotz, Balakrishnan and Johnson (2000) and Balakrishnan and Lai (2009). There has been an increased interest in defining new generated classes of univariate continuous distributions by introducing additional shape parameters to the baseline model motivated by the need to fit various observed phenomena, specially in those situations, where the baseline probability distribution fails to fit them adequately. One such model that has been studied extensively in the literature is the Ristic- Balakrishnan \( G \) family of distributions ( henceforth RB in short); see Ristic- Balakrishnan (2012) for pertinent details.

The RB- \( G(a) \) (where \( a > 0 \) is the parameter) pdf (probability density function), cdf (cumulative distribution function) and the hazard function are given, respectively, by

\[
f(x) = \frac{1}{\Gamma(a)} (-\log G(x))^{a-1} g(x), \quad x \in \mathbb{R},
\]

\[
F(x) = 1 - \frac{\int_0^{-\log G(x)} t^{a-1} \exp(-t) dt}{\Gamma(a)} = 1 - \frac{\gamma(a, -\log G(x))}{\Gamma(a)}, \quad x \in \mathbb{R},
\]

and

\[
h(x) = \frac{(-\log G(x))^{a-1} g(x)}{\gamma(a, -\log G(x))},
\]
where $G(x)$ is any baseline cdf and $\gamma(a,x)$ is the upper incomplete gamma function.

An extensive survey on the univariate RB-$G$ model is given in Cordeiro et al. (2015) while the bivariate and subsequently multivariate generalization of such gamma-generated models are discussed in (Ristic and Balakrishnan (2016)). In this work we focus mainly on the characterizations of the univariate RB-$G$ family via hazard function, moments of truncated order statistics and many other strategies. Further, we consider conditional specification approach to construct a class of bivariate RB-$G$ type distributions in which both of the conditional distributions (i.e., $Y$ given $X = x$ and $X$ given $Y = y$) belong to the univariate RB-$G$ family with appropriate parameters. The major objective of this article is two fold: one is to characterize the univariate RB-$G$ family and the other is to provide a bivariate extension of such a family, considering the situation in which both the conditionals belong to the univariate RB-$G$ distribution with appropriate parameters.

The article is outlined as follows: In section 2 we discuss the construction and characterization of bivariate RB-$G$ family of distributions and discuss some stochastic properties of the assumed model. Section 3 represents one type of characterization via Lorenz ordering. In section 4 we discuss the closure property of the RB-$G$ family of distributions via sample extremum. In section 5 we consider characterizations based on two truncated moments. Section 6 represents characterization based on truncated moment of the first order statistic. Characterization of RB-$G$ distribution in terms of hazard function is presented in section 7. Some discussion on the estimation of the model parameters via the method of maximum likelihood are discussed in section 8. Finally, some concluding remarks are made in section 9.

2 Characterization via conditional specification approach

Let us suppose that the random variable $X$ for given $Y = y$ for each fixed $Y = y$ is distributed as RB-$G$ with parameter $\delta(y)$ and the parent distribution $G$ and the random variable $Y$ for given $X = x$ for each fixed $X = x$ is distributed as RB with parameter $\psi(x)$ and the parent distribution $G$. Also, let $f(x)$ and $f(y)$ be the marginal distributions of the random variables $X$ and $Y$, respectively. Then the joint distribution $f(x,y)$ of the random variables $X$ and $Y$ can be written as

$$f(x,y) = f(x) \left[ \frac{1}{\Gamma(\psi(x))} (-\log G(y))^{\psi(x)-1} g(y) \right] = f(y) \left[ \frac{1}{\Gamma(\delta(y))} (-\log G(x))^{\delta(y)-1} g(x) \right].$$

Our conditional density of $X$ given $Y = y$ can be rewritten in the following form

$$f(x|y) = [\Gamma(\delta(y))]^{-1} \left[ -\log G(x) \right]^{-1} \exp \left[ \delta(y) \log (-\log G(x)) + \log g(x) \right].$$

This can be expressed in the form of $\xi_1 = 2$ parameter family of densities (Ref. Definition 4.1, Arnold et al. (1999)) of the form

$$f(x|y) = r_1(x) \beta_1 \left( \theta_1(y) \right) \exp \left( \sum_{i=1}^{2} \theta_i(y) q_{i1}(x) \right).$$

In our case, we have the following:

- $r_1(x) = (\log \left[ -\log G(x) \right])^{-1}$,
- $\beta_1 \left( \theta_1(y) \right) = \left[ \Gamma(\delta(y)) \right]^{-1}$,
- $\theta_1(y) = \delta(y), \quad \theta_2(y) = 1$,
- $q_{11}(x) = \log (-\log G(x)), \quad q_{12}(x) = 1$. 


Similarly, the other conditional density can be rewritten in the same form but with different parametric configuration and replacing \(x\) by \(y\) in appropriate places. If the above holds true, then according to Theorem 4.1, of Arnold et al. (1999) the bivariate density \(f(x, y)\) will be of the form

\[
f(x, y) = r_1(x)r_2(y)\exp \left[ (q_1(x))^T M q_2(y) \right],
\]

(5)

where \(T\) stands for transpose. Also,

\[
q_1^T(x) = (q_{10}(x), q_{11}(x), q_{12}(x)),
\]

and

\[
q_2^T(y) = (q_{20}(y), q_{21}(y), q_{22}(y)),
\]

where \(q_{10}(x) = q_{20}(y) = 1\) and \(M\) is a \(3 \times 3\) matrix of constants subject to \(\int_0^\infty \int_0^\infty f(x, y)\,dx\,dy = 1\). (Ref. Equation 4.5 of Arnold et al. (1999), page 76). In our case we can write \(M\) as

\[
M = \begin{bmatrix}
m_{00} & m_{01} & m_{02} \\
m_{10} & m_{11} & m_{12} \\
m_{20} & m_{21} & m_{22}
\end{bmatrix}.
\]

Also, in our case

\[
q_1(t) = q_2(t) = \begin{bmatrix} 1 \\ \log (-\log G(t)) \\ \log g(t) \end{bmatrix}.
\]

Then from (2), we can write the joint density \(f(x, y)\) as follows:

\[
f(x, y) = [\log (-\log G(x))]^{-1} [\log (-\log G(y))]^{-1} \exp \left( m_{00} - m_{01} \log \log G(y) + m_{02} \log g(y) \right)
\]

\[
- m_{10} \log \log G(x) + m_{11} \log \log G(x) \log \log G(y) - m_{12} \log \log G(x) \log g(y)
\]

\[
- m_{20} \log g(x) - m_{21} \log \log G(y) \log g(x) + m_{22} \log g(x) \log g(y).
\]

(6)

Observe that for model (6), independence will be achieved if the following holds true: \(m_{11} = m_{12} = m_{21} = m_{22} = 0\) and \(m_{11} > 0, m_{20} > 0, m_{01} > 0, m_{02} > 0\). Note that sometimes the joint density might lead us to some nonstandard models (in the sense that they might have a valid joint density but might not produce valid marginal densities and vice versa). Hence, we do need appropriate constraints on the choice of \(m_{ij}, i, j = 1, 2\).

**Some observations:**

- For various choices of \(\delta(y)\) and \(\psi(x)\) functions, one can obtain various bivariate probability distribution models.

- Note that in the above model, \(m_{11} = m_{12} = m_{21} = m_{22} = 0\) implies independence.

- From the elements of the matrix \(M\), one can establish the following relationships among the elements \(m_{ij}, i, j = 1, 2\) which will indicate whether we will have positive or negative dependence. The simple way to look at it is by the expression of the determinant of the matrix \(M\), which is

\[
|M| = m_{00} (m_{11}m_{22} - m_{12}m_{21}) - m_{01} (m_{10}m_{22} - m_{12}m_{20}) + m_{02} (m_{10}m_{21} - m_{11}m_{20}).
\]

From this, we can say the following (with the assumption that all \(m_{ij} > 0\)):

- One will have positive dependence iff \(\frac{m_{02}}{m_{12}} < \frac{m_{01}}{m_{10}} < \frac{m_{00}}{m_{11}}\).
– One will have negative dependence iff \( \frac{m_{12}}{m_{12}} > \frac{m_{20}}{m_{10}} > \frac{m_{22}}{m_{12}} \).

- If \( m_{12} > 0 \) and \( m_{21} > 0 \) and \( m_{22} \leq 0 \), then still (6) is a legitimate joint density. However, if any of \( m_{12} \) and/or \( m_{21} \) is 0, then the model is improper in the sense that it is no longer integrable, although nonnegative.

- If both \( \delta(y) \) and \( \psi(x) \) are linear functions of \( y \) and \( x \) respectively, for example, \( \delta(y) = a_{01} + a_{02}y \) and \( \psi(x) = a_{03} + a_{04}x \) with the condition that \( a_{02} > 0, a_{04} > 0 \), and \( a_{01} = a_{02} \neq 0 \), then we get the joint density of an exponential family of distributions. However, if those restrictions are replaced by other possibilities, we might get the joint density for a truncated exponential.

- If \( \delta(y) = \left( \frac{y}{\sigma_1} \right)^2 + 1 \) and \( \psi(x) = \left( \frac{x}{\sigma_2} \right)^2 + 1 \), for some non-negative constants \( \sigma_1, \sigma_2 \), then (6) will produce one of those models which Bhattacharya (1943) identified as nonstandard models with normal conditional distributions.

- If \( \delta(y) = b_0y^2 + 1 \) and \( \psi(x) = c_0x^2 + 1 \), where \( b_0 > 0 \) and \( c_0 > 0 \) are some constants, then \( f(x, y) \) will produce a bivariate distribution with normal conditionals.

Since both of the conditionals are in Gamma family and can be written (we already have used that representation) in the form of Equation (4.32) of Arnold et al. (1999), then the joint density is of the form (Equation 4.33, page 83, Arnold et al. (1999)) we will have the following:

1. The conditional distribution of \( X \) given \( Y = y \) is

\[
X|Y = y \sim \text{Gamma} \left( m_{20} + m_{22} \log g(y) + m_{21} \log (-\log G(y)) \right), m_{10} + m_{11} \log (-\log G(y)) + m_{12} \log g(y)) \).
\]

Similarly, the conditional distribution of \( Y \) given \( X = x \) is

\[
Y|X = x \sim \text{Gamma} \left( m_{02} + m_{22} \log g(x) + m_{12} \log (-\log G(x)) \right), m_{01} + m_{11} \log (-\log G(x)) + m_{21} \log g(x)) \).
\]

Consequently, using known results for gamma distribution we may verify that the general \( k \)-th order moment \( (k \geq 2) \) will be

\[
E(Y^k | X = x) \equiv \frac{(m_{20} + m_{22} \log g(y) + m_{21} \log (-\log G(y)) + k - 1) \cdots (m_{02} + m_{22} \log g(x) + m_{12} \log (-\log G(x)))}{(m_{10} + m_{11} \log (-\log G(y)) + m_{12} \log g(y))}.
\]

An analogous expression for the \( k \)-th order conditional moment for the other conditional distribution can be easily obtained.

2. The marginal density of \( X \) will be

\[
f_X(x) = \frac{\Gamma (m_{02} + m_{22} \log g(x) + m_{12} \log (-\log G(x)))}{[m_{01} + m_{11} \log (-\log G(x)) + m_{21} \log g(x)]^{m_{02} + m_{22} \log g(x) + m_{12} \log (-\log G(x))}} \times \exp \left( m_{00} + m_{10} \log (-\log G(x)) + m_{20} \log g(x) \right), x > 0.
\]

Similarly, one can find an analogous expression for the density of \( Y \).
2.1 Some distributional properties

- **Shape of the distribution:** A critical point of a function with two variables is a point where the partial derivatives of first order are equal to zero. There are two reasons as to why it is important to find the critical points of a bivariate probability distribution: (a) To determine the shape of the distribution in order to find its flexibility in fitting a data which is exhibiting a similar shape pattern, and (b) To identify the number of and the locations of modes of the density. In order to identify the location of the mode of the density (6), we consider the first derivatives of \( \log f(x, y) \) with respect to \( x \) and \( y \) and then equate to zero. This results in the following two equations:

\[
\frac{\partial f(x, y)}{\partial x} = \left(- \log G(x)\right)^{-1} \frac{g(x)}{G(x)} \left[- \left(\log (- \log G(x))\right)^{-2} - m_{10} + m_{11} \log (\log G(y)) - m_{12} \log g(y)\right] + \frac{g'(x)}{g(x)} \left[-m_{20} - m_{21} \log (\log G(y)) + m_{22} \log g(y)\right] = 0.
\]

Also,

\[
\frac{\partial f(x, y)}{\partial y} = \left(- \log G(y)\right)^{-1} \frac{g(y)}{G(y)} \left[- \left(\log (- \log G(y))\right)^{-2} - m_{01} + m_{11} \log (\log G(x)) - m_{21} \log g(x)\right] + \frac{g'(y)}{g(y)} \left[-m_{02} - m_{21} \log (\log G(x)) + m_{22} \log g(x)\right] = 0.
\]

It is clear from the above that for a baseline \( G \) distribution a numerical evaluation is required as analytical expressions are difficult to obtain.

- Next, we focus our attention to some dependence properties of the bivariate distribution in (6). There are various ways to describe and measure the dependence or association between two random variables. A distribution is said to be positive likelihood ratio dependent (PLRD) if its pdf \( f(x, y) \) satisfies \( \frac{f(x_1, y_1)f(x_2, y_2)}{f(x_1, y_2)f(x_2, y_1)} \geq 1 \), \( \forall x_1 > x_2 \) and \( y_1 > y_2 \). The quantity \( \frac{f(x_1, y_1)f(x_2, y_2)}{f(x_1, y_2)f(x_2, y_1)} \geq 1 \) measures "local" positive (or negative) likelihood ratio dependence at each point \( (x, y) \in \mathbb{R}^2 \), and its integral over the portion of \( \mathbb{R}^2 \), where \( x_1 > x_2 \) and \( y_1 > y_2 \) is a measure of "average" likelihood ratio dependence. For the bivariate density in (6), the above condition reduces to

\[
c_{22} \left[\left(a_2(x_1) - a_2(x_2)\right)\left(a_1(y_1) - a_1(y_2)\right)\right] \times \left[\frac{\log G(x_1)}{\log G(x_2)}\right]^{a_1(y_1)-a_1(y_2)} \times \left[\frac{\log G(y_1)}{\log G(y_2)}\right]^{a_2(x_1)-a_2(x_2)} \geq 1. \quad (7)
\]

- Next, since \( x_1 > x_2, y_1 > y_2 \) if both \( a_1(.) \) and \( a_2(.) \) are monotonically increasing functions then (7) holds (provided \( c_{22} \geq 0 \)). Hence the bivariate density in (6) will exhibit PLRD property.

This PLRD property of the density (6) implies the following:

- \( P(X \leq x | y = y) \) is non-increasing in \( y \) for all \( x \),
- \( P(Y \leq y | X = x) \) is non-increasing in \( x \) for all \( y \),
- \( P(Y > y | X > x) \) is non-decreasing in \( x \) for all \( y \),
- \( P(Y \leq y | X \leq x) \geq P(Y \leq y)P(X \leq x) \),
- \( P(Y > y | X > x) \geq P(Y > y)P(X > x) \).
3 Characterization via generalized Lorenz ordering

The expression for generalized Lorenz ordering (henceforth in short GL) is given by (for a random variable $X$) by

$$GL_X(p) = \int_0^p F_X^{-1}(t) dt,$$

for $p \in (0, 1)$. Also, we mention here a result by Ramos et al. (2000) which is as follows:

If $Z_i \sim \text{gamma}(\alpha_i, \beta_i)$ for $i = 1, 2$. Then, if $\alpha_1 \leq \alpha_2$ and $\alpha_1 \beta_1 \leq \alpha_2 \beta_2$, then $Z_2 \leq_{GL} Z_1$. In our case, according to Ristic-Balakrishnan model motivation, we consider the following:

Suppose $Z_1 \sim \text{gamma}(\delta_1, 1)$ then $X = F_X^{-1}(1 - \exp(-Z_1)) \sim RB(\delta_1)$. Similarly, $Z_2 \sim \text{gamma}(\delta_2, 1)$ then $X = F_X^{-1}(1 - \exp(-Z_2)) \sim RB(\delta_2)$.

Now, if we assume $\delta_1 \leq \delta_2$, then according to Ramos et al. (2000) result and noting the fact that Lorenz ordering is preserved under one-to-one transformation, assuming $X$ and $Y$ are one-to-one transformation of $Z_1$ and $Z_2$, the result will hold in this case also. In other words if $X \sim RB(\delta_1)$ and $Y \sim RB(\delta_2)$, with $\delta_1 \leq \delta_2$, then $Y \leq_{GL} X$. This is one type of characterization for the RB-G family.

4 Characterization via closure property of sample extremum

**Theorem 4.1:** The Ristic-Balakrishnan G family of distributions is closed under minimization and maximization. In other words, for a random sample (i.i.d) of size $n$ drawn from (1), we can write the following:

$$X_{1:n} \sim \frac{\Gamma(n\delta + s_k)}{[\Gamma(\delta)]^n} RB(n\delta + s_k),$$

and

$$X_{n:n} \sim \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\Gamma(j\delta + s_k)}{[\Gamma(\delta)]^j} RB(j\delta + s_k),$$

where $X_{1:n} = \min_{1 \leq i \leq n} X_i$ and $X_{n:n} = \max_{1 \leq i \leq n} X_i$ are the smallest and largest order statistics respectively.

**Proof.** Let us consider

$$P(X_{1:n} > x) = [P(X_1 > x)]^n$$

$$= \left[ \frac{\Gamma(\delta, -\log G(x))}{\Gamma(\delta)} \right]^n$$

$$= \left[ \frac{1}{\Gamma(\delta)} \right]^n \sum_{k=0}^{\infty} (-1)^k \frac{(-\log G(x))^{\delta + k}}{k!(\delta + k)}$$

on using power series expansion of the incomplete gamma function

$$= \left[ \frac{1}{\Gamma(\delta)} \right]^n \sum_{k_1}^{\infty} \cdots \sum_{k_n=0}^{\infty} (-1)^{s_k} \frac{(-\log G(x))^{s_k + n\delta}}{p_k},$$

(8)

where $s_k = \sum_{i=1}^{n} k_i$ and $p_k = \prod_{i=1}^{n} k_i!$. From (7), it is easy to show that $X_{1:n} \sim \frac{\Gamma(n\delta + s_k)}{[\Gamma(\delta)]^n} RB(n\delta + s_k)$. Similarly the other part of the theorem can be established.
5 Characterization based on two truncated moments

In this section we present characterizations of the RB-G distribution in terms of a simple relationship between two truncated moments. The results derived here will employ an interesting theorem due to Glänzel (1987), which is given below. The advantage of the characterizations given here is that the cdf need not have a closed form and it is given as an integral whose integrand is in terms of the solution of a differential equation. This provides a bridge between probability and differential equation.

**Theorem 5.1.** Let $(\Omega, \mathcal{F}, P)$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty$, $b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function $F$ and let $q_1$ and $q_2$ be two real functions defined on $H$ such that

$$E[q_1(X) \mid X \geq x] = E[q_2(X) \mid X \geq x] \eta(x), \quad x \in H,$$

is defined with some real function $\eta$. Assume that $q_1, q_2 \in C^1(H), \eta \in C^2(H)$ and $F$ is twice continuously differentiable and strictly monotone function on the set $H$. Finally, assume that the equation $q_2 \eta = q_1$ has no real solution in the interior of $H$. Then $F$ is uniquely determined by the functions $q_1, q_2$ and $\eta$, particularly

$$F(x) = \int_x^a C \left| \frac{\eta'(u)}{\eta(u) q_2(u) - q_1(u)} \right| \exp(-s(u)) \, du,$$

where the function $s$ is a solution of the differential equation $s' = \frac{q_2'}{\eta q_2 - q_1}$ and $C$ is a constant, chosen to make $\int_H dF = 1$.

**Remarks 5.1.** (a) In Theorem 5.1, the interval $H$ need not be closed. (b) The goal is to have the function $\eta$ as simple as possible. (c) It is possible to state Theorem 5.1 based on two functions $q_1$ and $\eta$ by setting $q_2(x) \equiv 1$, however, the extra function gives more flexibility as far as applications are concerned.

**Proposition 5.1.** Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_2(x) = (G(x))^{-1}$ and $q_1(x) = q_2(x) (-\log(G(x)))$, for $x \in \mathbb{R}$. Then the pdf as given by (1) is true if and only if the function $\eta$ defined in Theorem 5.1 has the form

$$\eta(x) = \frac{a}{a+1} (-\log(G(x))), \quad x \in \mathbb{R}.$$

**Proof.** Let $X$ have pdf (1), then

$$(1 - F(x)) \ E[q_2(X) \mid X \geq x] = \frac{1}{a \Gamma(a)} (-\log(G(x)))^a,$$

and

$$(1 - F(x)) \ E[q_1(X) \mid X \geq x] = \frac{1}{(a+1) \Gamma(a)} (-\log(G(x)))^{a+1},$$

and finally
\[ \eta(x) q_2(x) - q_1 (x) = -\frac{1}{a+1} q_2 (x) (-\log (G (x))) < 0, \quad x \in \mathbb{R}. \]

Conversely, if \( \eta \) is given as above, then

\[ s'(x) = \frac{\eta'(x) q_2 (x)}{\eta(x) q_2 (x) - q_1 (x)} = \frac{a g(x)}{G(x) (-\log (G(x)))}, \quad x \in \mathbb{R}, \]

and hence

\[ s(x) = -a \log((-\log (G(x)))), \quad x \in \mathbb{R}. \]

Now, in view of Theorem 5.1, \( X \) has cdf (2) and pdf (1).

**Corollary 5.1.** Let \( X : \Omega \to \mathbb{R} \) be a continuous random variable and let \( q_2 (x) \) be as in Proposition 5.1. Then the pdf as given by (1) is valid if and only if there exist functions \( q_1 \) and \( \eta \) defined in Theorem 5.1 satisfying the differential equation

\[ \eta(x) q_2 (x) - q_1 (x) = -a g(x) G(x) (-\log (G(x))), \quad x \in \mathbb{R}. \]

**Remarks 5.2.** (a) The general solution of the differential equation in Corollary 1 is

\[ \eta(x) = (-\log (G(x)))^a \left[ -\int_{-\infty}^{\infty} \frac{a g(x)}{G(x)} (-\log (G(x)))^{a-1} (q_2 (x))^{-1} q_1 (x) dx + D \right], \quad x \in \mathbb{R}, \]

where \( D \) is a constant. One set of appropriate functions is given in Proposition 5.1 with \( D = 0 \).

(b) Clearly there are other triplets of functions \( (q_1, q_2, \eta) \) satisfying the conditions of Theorem 5.1. We presented one such triplet in Proposition 1.

### 6 Characterizations based on truncated moment of the 1st order statistic

Let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) be the corresponding order statistics from a random sample of size \( n \) of a continuous cdf \( F \). We briefly discuss here characterization results based on functions of the 1st order statistic. We like to mention here that the proof of Proposition 2 below is straightforward extension of that of Theorem 2.2 of Hamedani (2010). We give a short proof of it for the sake of completeness.

**Proposition 6.1.** Let \( X : \Omega \to \mathbb{R} \) be a continuous random variable with cdf \( F \). Let \( \psi(x) \) and \( q(x) \) be two differentiable functions on \( \mathbb{R} \) such that

\[ \lim_{x \to \infty} \psi(x) [1 - F(x)]^n = 0, \quad \int_{-\infty}^{\infty} \frac{q'(t)}{q(t) - \psi(t)} dt = \infty. \]
Then
\[ E [\psi (X_{1:n}) \mid X_{1:n} > t] = q(t) , \quad t \in \mathbb{R} \] (9)
implies
\[ F(x) = 1 - \exp \left\{ - \int_{-\infty}^{x} \frac{q'(t)}{n[q(t) - \psi(t)]} dt \right\} , \quad x \in \mathbb{R} . \] (10)

**Proof.** If (9) holds, then using integration by parts on the left hand side of (9) and the assumption \( \lim_{x \to \infty} \psi(x)[1 - F(x)]^n = 0 \), we have
\[
\int_{t}^{\infty} \psi'(x) (1 - F(x))^n dx = [q(t) - \psi(t)] (1 - F(t))^n .
\]
Differentiating both sides of the above equation with respect to \( t \), we arrive at
\[
\frac{f(t)}{1 - F(t)} = \frac{q'(t)}{n[q(t) - \psi(t)]} , \quad t \in \mathbb{R} . \] (11)

Now, integrating (11) from \(-\infty\) to \( x \), we have, in view of \( \int_{-\infty}^{\infty} \frac{q'(t)}{q(t) - \psi(t)} dt = \infty \), a cdf \( F \) given by (10).

**Remarks 5.3.** (a) Taking, for instance, \( \psi(x) = (\gamma(a, -\log(G(x))))^n \) and \( q(x) = \frac{1}{2}\psi(x) \) in Proposition 6.1, we arrive at (2). (b) the above Proposition holds with the random variable \( X \) in place of \( X_{1:n} \) with of course appropriate conditions.

### 7 Characterization based on hazard function

It is obvious that the hazard function, \( h_F \), of a twice differentiable distribution function, \( F \), satisfies the first order differential equation
\[
\frac{h'_F(x)}{h_F(x)} - h_F(x) = q(x) ,
\]
where \( q(x) \) is an appropriate integrable function. Although this differential equation has an obvious form since
\[
\frac{h'_F(x)}{h_F(x)} - h_F(x) = \frac{f'(x)}{f(x)} , \] (12)
for many univariate continuous distributions (12) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (7). Here, we present a characterization of the of RB-G model based on a nontrivial differential equation in terms of the hazard function.

**Proposition 7.1.** Let \( X : \Omega \to \mathbb{R} \) be a continuous random variable. Then the pdf as given by (1) is true if and only if its hazard function \( h_F \) satisfies the differential equation

\[
 h_F'(x) - \frac{g'(x)}{g(x)} h_F(x) = g(x) \frac{d}{dx} \left\{ \frac{(- \log(G(x)))^{a-1}}{\gamma(a, - \log(G(x)))} \right\}, \quad x \in \mathbb{R}. \tag{13}
\]

**Proof:** If the pdf as given by (1) is true, then clearly (13) holds. Now, if (13) holds, then after dividing both sides of (13) by \( g(x) \), we arrive at

\[
 \frac{d}{dx} \left\{ (g(x))^{-1} h_F(x) \right\} = \frac{d}{dx} \left\{ \frac{(- \log(G(x)))^{a-1}}{\gamma(a, - \log(G(x)))} \right\},
\]

from which we have

\[
 h_F(x) = \frac{f(x)}{1 - F(x)} = \frac{g(x) (- \log(G(x)))^{a-1}}{\gamma(a, - \log(G(x)))}. \tag{14}
\]

Integrating both sides of (9) from \(-\infty\) to \( x \), we have

\[
 - \log((1 - F(x)) = - \log \left\{ \gamma(a, - \log(G(x))) \right\} / \Gamma(a),
\]

from which we obtain

\[
 1 - F(x) = \frac{\gamma(a, - \log(G(x)))}{\Gamma(a)}, \quad x \in \mathbb{R}.
\]

**Remarks 4.** For \( a = 2 \), equation (14) reduces to the following simple equation

\[
 h_F'(x) - \frac{g'(x)}{g(x)} h_F(x) = \frac{(g(x))^2}{(\gamma(2, - \log(G(x))))^2} \left\{ (\log(G(x)))^2 - \frac{\gamma(2, - \log(G(x)))}{G(x)} \right\},
\]

or

\[
 \frac{d}{dx} \left\{ (g(x))^{-1} h_F(x) \right\} = \frac{d}{dx} \left\{ \frac{(- \log(G(x)))}{\gamma(2, - \log(G(x)))} \right\}. \]
8 Estimation of model parameters

Here we consider method of maximum likelihood of estimation under the classical approach, which we describe below:

In this case, we consider (for more general set up) the bivariate RB-G distribution with the joint density as in (5). As pointed earlier, \( m_{01}, m_{02}, m_{10}, m_{11}, m_{12}, m_{21}, m_{22} \) are constrained to make the density integrable while \( m_{00} \) is evaluated, as a function of the other parameters, to make the integral equal to 1. For notational simplicity, we relabel of the model parameters by setting, \( m_{01} = \theta_1, m_{02} = \theta_2, m_{10} = \theta_3, m_{11} = \theta_4, m_{12} = \theta_5, m_{20} = \theta_6, m_{21} = \theta_7, m_{22} = \theta_8 \) and let

\[
\Psi(\theta) = \exp(-m_{00}) = \int_0^\infty \int_0^\infty r_1(x)r_2(y) \exp(\theta_1 q_{21}(x) + \theta_2 q_{22}(y) + \theta_3 q_{11}(x) + \theta_4 q_{11}(x)q_{21}(y) + \theta_5 q_{11}(x)q_{22}(y) + \theta_6 q_{12}(x)q_{21}(y) + \theta_7 q_{12}(x)q_{22}(y) + \theta_8 q_{12}(x)q_{22}(y)) \, dx \, dy.
\]

(15)

With this notation the log-likelihood of a sample of size \( n \) \(((X_1, Y_1), \cdots, (X_n, Y_n))\) from our density in (5) is

\[
\log L = \ell = -n \log \Psi(\theta) + \sum_{i=1}^n r_1(X_i) + \sum_{i=1}^n r_2(Y_i) + \theta_1 \sum_{i=1}^n q_{21}(X_i) + \theta_2 \sum_{i=1}^n q_{22}(Y_i) + \theta_3 \sum_{i=1}^n q_{11}(X_i) + \theta_4 \sum_{i=1}^n q_{11}(X_i)q_{21}(Y_i) + \theta_5 \sum_{i=1}^n q_{11}(X_i)q_{22}(Y_i) + \theta_6 \sum_{i=1}^n q_{12}(X_i)q_{21}(Y_i) + \theta_7 \sum_{i=1}^n q_{12}(X_i)q_{22}(Y_i) + \theta_8 \sum_{i=1}^n q_{12}(X_i)q_{22}(Y_i)
\]

(16)

Differentiating and subsequently the partial derivatives equal to zero yields the following likelihood equations

\[
\frac{\partial \Psi(\theta)}{\partial \theta_1} = 1 \frac{1}{n} \sum_{i=1}^n q_{21}(X_i) = 0
\]

(17)

\[
\frac{\partial \Psi(\theta)}{\partial \theta_2} = 1 \frac{1}{n} \sum_{i=1}^n q_{22}(Y_i) = 0
\]

(18)

\[
\frac{\partial \Psi(\theta)}{\partial \theta_3} = 1 \frac{1}{n} \sum_{i=1}^n q_{11}(X_i) = 0
\]

(19)

\[
\frac{\partial \Psi(\theta)}{\partial \theta_4} = 1 \frac{1}{n} \sum_{i=1}^n q_{11}(X_i)q_{21}(Y_i) = 0
\]

(20)

\[
\frac{\partial \Psi(\theta)}{\partial \theta_5} = 1 \frac{1}{n} \sum_{i=1}^n q_{11}(X_i)q_{22}(Y_i) = 0
\]

(21)

\[
\frac{\partial \Psi(\theta)}{\partial \theta_6} = 1 \frac{1}{n} \sum_{i=1}^n q_{12}(X_i)q_{21}(Y_i) = 0
\]

(22)

\[
\frac{\partial \Psi(\theta)}{\partial \theta_7} = 1 \frac{1}{n} \sum_{i=1}^n q_{12}(X_i)q_{22}(Y_i) = 0
\]

(23)
\[
\frac{\partial \Psi(\theta)}{\partial \theta} = \frac{1}{n} \sum_{i=1}^{n} q_{12}(X_i)q_{22}(Y_i)
\]  
(24)

If \(\Psi(\theta)\) is a simple analytic expression, the above set of equations can be easily solved (direct or iteratively). In case \(\Psi(\theta)\) is very nasty in nature (which by the way is true in most cases) still can be evaluated by numerical integration. A reasonable approach (for details, see Arnold et al. (1999)) would be in terms of choosing initial values of \(\theta_j, j = 1, 2, \cdots, 7\) (may be based on moment estimates) and then search for a value of \(\theta_8\) to make (24) valid. Next, taking this value of \(\theta_8\) with the previous values of \(\theta_k, k = 2, 3, \cdots, 7\) search for a value of \(\theta_1\) to make (17) valid and so on. This is without a doubt is heavily computer intensive but most likely more efficient than a direct search which might involve more numerical evaluations of \(\Psi(\bar{\theta})\) for various choices of \(\bar{\theta}\). After solving the likelihood equations, we may, with the help of numerical integration, write an approximation for the variance-covariance matrix of our estimate \(\hat{\theta}\). The Fisher information matrix corresponding to our model is the \(8 \times 8\) matrix \(I(\hat{\theta})\) with the \((i,j)\)-th element given by

\[
I_{i,j}(\hat{\theta}) = \frac{\Psi(\hat{\theta}) \left[ \frac{\partial^2 \Psi(\hat{\theta})}{\partial \theta_i \partial \theta_j} \right] - \left( \frac{\partial \Psi(\hat{\theta})}{\partial \theta_i} \right) \left( \frac{\partial \Psi(\hat{\theta})}{\partial \theta_j} \right)}{(\Psi(\hat{\theta}))^2}
\]

Then the estimated variance covariance matrix of \(\hat{\theta}\) is \(\sum(\hat{\theta}) = \frac{1}{n}, \) where \(\hat{\theta}\) is the solution to (17)-(24). Of course, the entries in \(I(\hat{\theta})\) must be computed numerically. It is to be noted that for specific choices of the baseline cdf \(G(.)\) there will be additional parameter choices, and that can also be evaluated with the above procedure.

9 Concluding remarks

In this paper, we discuss in brief, a bivariate extension of the univariate RB-G model and some associated structural properties via conditional specification. Most of the structural properties for the univariate RB-G model have been discussed in Bourguignon et al. (2016). Here we mainly focus on various useful characterizations of the univariate RB-G model along with the bivariate extension. We have also introduced some inferential strategies for estimating the model parameters under the maximum likelihood method. Inferential procedures for such bivariate models under the Bayesian paradigm (for specific members of \(G(.)\)) will be the subject matter of a different article as a future project.

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