A GENERALIZATION OF HALL’S THEOREM FOR
k-UNIFORM k-PARTITE HYPERGRAPHS

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ABSTRACT. In this paper we prove a generalized version of Hall’s
theorem for hypergraphs. More precisely, let $\mathcal{H}$ be a $k$-uniform
$k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$.
such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique
perfect matching. In this case, we give a necessary and sufficient
condition for having a matching of size $t = |V_1|$ in $\mathcal{H}$. Some relevant
results and counterexamples are given as well.

1. Introduction

We refer to [7] and [6] for elementary backgrounds in graph and hypergraph
theory respectively.

Let $G$ be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A matching in $G$, is a set $M$ of pairwise disjoint edges of $G$. A
matching $M$ is said to be a perfect matching, if every $x \in V(G)$ lies in
one of elements of $M$. A matching $M$ in $G$, is maximum whenever for
every matching $M'$, $|M'| \leq |M|$.

For every set of vertices $A$, $N(A)$ which is called the neighborhood of
$A$ is the set of vertices which are adjacent with at least one element of $A$.
The following theorem is known as Hall’s theorem in bipartite graphs.

**Theorem 1.1.** ([7] Theorem 5.2) Let $G$ be a bipartite graph with bipar-
tition $(X,Y)$. Then $G$ contains a matching that saturates every vertex
in $X$, if and only if

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq X.$$
A vertex cover in $G$, is a subset $C$ of $V(G)$ such that for every edge $e$ of $G$, $e$ intersects $C$. A vertex cover $C$ is called a minimum vertex cover, if for every vertex cover $C'$, $|C| \leq |C'|$. The following theorem is known as König’s theorem in graph theory.

**Theorem 1.2.** ([7] Theorem 5.3) In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Let $V$ be a finite nonempty set. A hypergraph $H$ on $V$ is a collection of nonempty subsets of $V$ such that $\bigcup_{e \in H} e = V$. Each subset is said to be a hyperedge and each element of $V$ is called a vertex. We denote the set of vertices and hyperedges of $H$ by $V(H)$ and $E(H)$, respectively. Two vertices $x, y$ of a hypergraph are said to be adjacent whenever they lie in a hyperedge.

A matching in the hypergraph $H$ is a set $M$ of pairwise disjoint hyperedges of $H$. A perfect matching is a matching such that every $x \in V(H)$ lies in one of its elements. A matching $M$ in $H$ is called a maximum matching whenever for every matching $M'$, $|M'| \leq |M|$.

In a hypergraph $H$, a subset $C$ of $V(H)$ is called a vertex cover if every hyperedge of $H$ intersects $C$. A vertex cover $C$ is said to be minimum if for every vertex cover $C'$, $|C| \leq |C'|$. We denote the number of hyperedges in a maximum matching of the hypergraph $H$ by $\alpha'(H)$ and the number of vertices in a minimum vertex cover of $H$ by $\beta(H)$.

A hypergraph $H$ is said to be simple or a clutter if non of its two distinct hyperedges contains another. A hypergraph is called $t$-uniform (or $t$-graph), if all its hyperedges have the same size $t$. A hypergraph $H$ is said to be $r$-partite ($r \geq 2$), whenever $V(H)$ can be partitioned to $r$ subsets such that for every two vertices $x, y$ in one part, $x$ and $y$ are not adjacent. If $r = 2, 3$, the hypergraph is said to be bipartite and tripartite respectively.

Several researches have been done about matching and existence of perfect matching in hypergraphs (see for instance [1], [9], [12]). Also some attempts have been produced in generalization of Hall’s theorem and König’s theorem to hypergraphs (see [2], [3], [4], [5], [10], [11]).

**Definition 1.3.** Let $H$ be a $k$-uniform hypergraph with $k \geq 2$. A subset $e \subseteq V(H)$ of size $k - 1$ is called a submaximal edge if there is a hyperedge containing $e$. For a submaximal edge $e$, define the neighborhood of $e$ as the set $N(e) := \{v \in V(H) | e \cup \{v\} \in E(H)\}$. 
For a set \( A \) consisting of submaximal edges of \( \mathcal{H} \), \( \{ v \in V(\mathcal{H}) | \exists e \in A, v \in N(e) \} \) is denoted by \( N(A) \).

**Definition 1.4.** Let \( \mathcal{H} \) be a hypergraph, and \( \emptyset \neq V' \subseteq V(\mathcal{H}) \). The subhypergraph generated on \( V' \) is
\[
< V' > := \{ e \cap V' | e \in E(\mathcal{H}), e \cap V' \neq \emptyset \}.
\]

If \( \mathcal{H} \) is a \( k \)-uniform \( k \)-partite hypergraph with parts \( V_1, V_2, \ldots, V_k \), it is clear that the subhypergraph generated on the union of every \( k - 1 \) distinct parts is a \( (k - 1) \)-uniform \( (k - 1) \)-partite hypergraph.

Let \( \mathfrak{A} = (A_1, \ldots, A_n) \) be a family of subsets of a set \( E \). A subset \( \{x_1, \ldots, x_n\} \neq E \) of \( E \) is said to be a transversal (or SDR) for \( \mathfrak{A} \), if for every \( i (1 \leq i \leq n) \), \( x_i \in A_i \). A partial transversal (partial SDR) of length \( l \) \( (1 \leq l \leq n - 1) \) for \( \mathfrak{A} \), is a transversal for a subfamily of \( \mathfrak{A} \) with \( l \) sets.[8]

The following theorem is known as Hall’s theorem in combinatorics.

**Theorem 1.5.** ([8] Theorem 4.1) The family \( \mathfrak{A} = (A_1, \ldots, A_n) \) of subsets of a set \( E \) has a transversal if and only if
\[
| \bigcup_{i \in I'} A_i | \geq |I'|, \quad \forall I' \subseteq \{1, \ldots, n\}.
\]

**Corollary 1.6.** ([8] Corollary 4.3) The family \( \mathfrak{A} = (A_1, \ldots, A_n) \) of subsets of a set \( E \) has a partial transversal of length \( l(> 0) \) if and only if
\[
| \bigcup_{i \in I'} A_i | \geq |I'| - n + l, \quad \forall I' \subseteq \{1, \ldots, n\}.
\]

2. The main results

Now we are ready to present our first theorem.

**Theorem 2.1.** Let \( \mathcal{H} \) is a \( k \)-uniform \( k \)-partite hypergraph with some ordering on parts, as \( V_1, V_2, \ldots, V_k \) such that the subhypergraph generated on \( \bigcup_{i=1}^{k-1} V_i \) has a unique perfect matching \( M \). Then \( \mathcal{H} \) has a matching of size \( t = |V_1| \), if and only if for every subset \( A \) of \( M \), \( |N(A)| \geq |A| \).

**Proof.** Let \( t = |V_1| \) and let the elements of \( M \) are \( e_1, \ldots, e_t \). Let \( \mathcal{H} \) has a matching of size \( t \) with elements \( e_1, \ldots, e_t \). By uniqueness of \( M \), let \( M = \{e_1 - V_k, \ldots, e_t - V_k\} \). Therefore
\[
(N(e_1), \ldots, N(e_t)) = (N(e_1 - V_k), \ldots, N(e_t - V_k)).
\]
Then the family \((N(e_1), \ldots, N(e_t))\) has an SDR. Then by Theorem 1.3
\[
|\bigcup_{i \in I} N(e_i)| \geq |I|, \quad \forall I \subseteq \{1, \ldots, t\}
\]
and therefore for every subset \(A\) of \(M\), \(|N(A)| \geq |A|\).

Conversely let for every subset \(A\) of \(M\), we have \(|N(A)| \geq |A|\). Now \((N(e_1), \ldots, N(e_t))\) is a family such that
\[
|\bigcup_{i \in I} N(e_i)| \geq |I|, \quad \forall I \subseteq \{1, \ldots, t\}.
\]
Therefore by Theorem 1.3, the mentioned family has an SDR. That is, there are distinct elements \(x_1, \ldots, x_t\) of \(V^k\) such that \(x_j \in N(e_j)\). Now for every \(1 \leq j \leq t\), \(e_j \cup \{x_j\}\) is a hyperedge of \(H\) and these hyperedges are pairwise disjoint. Then they form a matching of size \(t\) for \(H\). \(\square\)

**Corollary 2.2.** Let \(H\) be a \(k\)-uniform \(k\)-partite hypergraph with some ordering on parts as \(V_1, V_2, \ldots, V_k\) where \(|V_1| = |V_2| = \cdots = |V_k|\), such that the subhypergraph generated on \(\bigcup_{i=1}^{k-1} V_i\) has a unique perfect matching \(M\). Then \(H\) has a perfect matching if and only if for every subset \(A\) of \(M\), \(|N(A)| \geq |A|\).

**Remark 2.3.** Theorem 2.1 implies Theorem 1.1 (Hall’s theorem) in case \(k = 2\).

**Remark 2.4.** In Theorem 2.1, if the hypothesis of uniqueness of perfect matching of subhypergraph generated on \(\bigcup_{i=1}^{k-1} V_i\) is removed, only one side of theorem will remains correct. That is, from this fact that for every subset \(A\) of \(M\), \(|N(A)| \geq |A|\), we conclude that \(H\) has a matching of size \(t = |V_1|\). The following example shows that the inverse case is not true in general.

**Example 2.5.** Assume the 3-uniform 3-partite hypergraph \(H\) with the following presentation.
Indeed, $H = \{ \{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{x_2, y_2, z_2\}, \{x_2, y_1, z_2\}\}$ where the parts of $H$ are

$V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $V_3 = \{z_1, z_2\}$.

In this case there is a perfect matching $M_1 = \{ \{x_2, y_1\}, \{x_1, y_2\}\}$ for subhypergraph generated on $V_1 \cup V_2$. Although the hypergraph $H$ has a matching $M' = \{ \{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$ of size 2, if $A = M_1$, we have $N(A) = \{z_2\}$. Therefore $|N(A)| \geq |A|$. Note that $M_1$ is not the unique perfect matching of subhypergraph generated on $V_1 \cup V_2$ because $M_2 = \{ \{x_1, y_1\}, \{x_2, y_2\}\}$ is also yet.

**Theorem 2.6.** Let $H$ be a $k$-uniform $k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a perfect matching $M$. If for every subset $A$ of $M$, we have $|N(A)| \geq |A| - p$ where $p$ is a fix integer and $1 \leq p \leq t - 1$, then $H$ has a matching of size $t - p$, where $t$ is the size of $V_1$.

*Proof.* Let the elements of $M$ be $\varepsilon_1, \ldots, \varepsilon_t$. $(N(\varepsilon_1), \ldots, N(\varepsilon_t))$ is a family such that the cardinality of the union of each $s$ terms is greater than or equal to $s - t + (t - p)$. Then by Corollary 1.4, the family $(N(\varepsilon_1), \ldots, N(\varepsilon_t))$ has a partial SDR of size $t - p$. That is, there are distinct elements $y_1, \ldots, y_{t-p}$ of $V_k$ such that $y_j \in N(\varepsilon_{i_j})$. Now for every $1 \leq j \leq t - p$, $\varepsilon_{i_j} \cup \{y_j\}$ is a hyperedge of $H$ and these hyperedges are pairwise disjoint. Then they form a matching of size $t - p$ for $H$. □

**Theorem 2.7.** Let $H$ be a $k$-uniform $k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$, and let $t = |V_1|$. Then $H$ has a matching of size $t$ if and only if $\alpha' = \beta = t$.

*Proof.* Let $H$ has a matching of size $t$. We show that $\alpha' = \beta = t$. Clearly $\beta \geq \alpha'$ because for covering each hyperedge of maximum matching, one
vertex is needed. But since there is a matching of size \( t \), then \( \alpha' \geq t \). Now \( V_1 \) is a minimal vertex cover of \( \mathcal{H} \) because each hyperedge has only one vertex in \( V_1 \) and each vertex of \( V_1 \) lies in a hyperedge. Therefore \( t \geq \beta \) which implies that \( \alpha' \geq \beta \). Then \( \alpha' = \beta \). The matching of size \( t \) is the maximum matching because it covers all vertices of \( V_1 \).

Conversely, if \( \alpha' = \beta = t \), it is clear that \( \mathcal{H} \) has a matching of size \( t \).

The following example shows that removing the condition \( t = |V_1| \) in Theorem 2.7 is not possible even if the subhypergraph generated on union of every \( k - 1 \) parts, has a perfect matching.

**Example 2.8.** Assume 3-uniform 3-partite hypergraph \( \mathcal{H} \) with the following presentation, where the parts of \( \mathcal{H} \) are

\[
V_1 = \{1, 2\}, \ V_2 = \{3, 4\}, \ V_3 = \{5, 6\}.
\]

Indeed \( \mathcal{H} = \{\{1, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}\} \).

In this hypergraph we have the matching \( \{\{1, 3, 5\}\} \) of size 1. But \( \alpha' \neq \beta \) because \( \alpha' = 1 \) and \( \beta = 2 \). Note that each one of subhypergraph generated on \( V_1 \cup V_2 \), \( V_2 \cup V_3 \) and \( V_1 \cup V_3 \) have a perfect matching.

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