SKEW RIBBON PLANE PARTITIONS: CALCULUS AND ASYMPTOTICS

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Abstract. Plane partitions have been widely studied in Mathematics since MacMahon. See, for example, the works by Andrews, Macdonald, Stanley, Sagan and Krattenthaler. The Schur process approach, introduced by Okounkov and Reshetikhin, and further developed by Borodin, Corwin, Corteel, Saveliev and Vuletic, has been proved to be a powerful tool in the study of various kinds of plane partitions. The exact enumerations of ordinary plane partitions, shifted plane partitions and cylindric partitions could be derived from two summation formulas for Schur processes, namely, the open summation formula and the cylindric summation formula.

In this paper, we establish a new summation formula for Schur processes, called the complete summation formula. As an application, we obtain the generating function and the asymptotic formula for the number of ribbon plane partitions, which can be viewed as plane partitions “shifted at the two sides”. We prove that the order of the asymptotic formula depends only on the width of the ribbon, not on the profile (the skew zone) itself. By using the same methods, the generating function and the asymptotic formula for the number of symmetric cylindric partitions are also derived.

1. Introduction

An ordinary plane partition (resp. a defective plane partition) is a filling $\omega = (\omega_{i,j})$ of the quarter plane $\Lambda = \{(i,j) \mid i,j \geq 1\}$ (resp. of a connected area of the quarter plane $\Lambda$) with nonnegative integers such that rows and columns decrease weakly, and the size $|\omega| = \sum \omega_{i,j}$ is finite. The enumeration of various defective plane partitions has been widely studied (see [2, 11, 12, 23, 24, 25, 26, 34, 35, 36, 37, 38]). In particular, the generating functions for the following five types of defective plane partitions (see Fig. 1) have been obtained since MacMahon:

(A) the ordinary plane partitions (MacMahon [21, 22], Stanley [34]);
(B) the skew plane partitions (Sagan [33]);
(C) the skew shifted plane partitions (Sagan [33]);
(D) the symmetric plane partitions (Andrews [2], Macdonald [20]);
(E) the cylindric partitions (Gessel and Krattenthaler [11], Borodin [6]).

In the literature there are three approaches to deriving the generating functions for various defective plane partitions: (1) Determinant evaluation and nonintersecting lattice paths [1, 2, 3, 4, 5, 11, 17]; (2) Hook lengths and combinatorial proofs [22, 33, 55]; (3) Schur functions and Schur processes [30, 31, 54].
The Schur process was first introduced by Okounkov and Reshetikhin in 2001. Later, they used the Schur process to derive the cyclic symmetry of the topological vertex by considering a certain type of plane partitions. This result was further developed by Iqbal et al., for providing a short proof of the Nekrasov-Okounkov formula. Borodin used the Schur process to derive the generating function for cylindric partitions, introduced by Gessel and Krattenthaler. The Macdonald process, which is a \((q,t)\)-generalization of the Schur process, was first introduced by Vuletic, and further developed by Corteel, Savelief, Vuletic and Langer in the study of weighted cylindric partitions and plane overpartitions. Finally, a survey of Macdonald processes was published by Borodin and Corwin.

The Schur process approach is shown to be a powerful tool in the study of various kinds of defective plane partitions. In fact, the above generating functions for defective plane partitions (A-E) are specializations of two general summation formulas for Schur processes, namely, the open summation formula and the cylindric summation formula. Formulas have been developed by Okounkov, Reshetikhin, Borodin, Corteel, Savelief, Vuletic and Langer. For convenience, they are also reproduced in Theorem.

In the present paper we establish a new summation formula for Schur processes, called the complete summation formula. As an application, we obtain the generating functions for the skew ribbon plane partitions and the symmetric cylindric partitions (see Fig. 1 (F)/(G)/(H)).
Let us reproduce some classical formulas in this introduction (see, e.g., \cite{20,32,33,39}). The generating functions for ordinary plane partitions, shifted plane partitions and symmetric plane partitions are the following respectively:

\[
\sum_{\omega \in \text{PP}} z^{|\omega|} = \prod_{k=1}^{\infty} \prod_{i=1}^{\infty} \frac{1}{1 - z^i - z^j} = \prod_{k=1}^{\infty} (1 - z^k)^{-k};
\]

\[
\sum_{\omega \in \text{ShiftPP}} z^{|\omega|} = \prod_{k=1}^{\infty} \frac{1}{1 - z^k} \prod_{1 \leq i < j \leq \infty} \frac{1}{1 - z^i + z^j};
\]

\[
\sum_{\omega \in \text{SPP}} z^{|\omega|} = \prod_{k=1}^{\infty} \frac{1}{1 - z^k} \prod_{1 \leq i < j \leq \infty} \frac{1}{1 - z^2(i+j-1)}.
\]

As a byproduct of our complete summation formula (2.2) we can further derive the following generating function for ribbon plane partitions of width \(m\) (see Fig. 1 (F) and Section 3 for the definition).

**Theorem 1.1.** Let \(RPP_m\) be the set of all ribbon plane partitions \(\omega\) of width \(m\) (i.e., skew ribbon plane partitions with profile \(\delta = (-1)^{m-1}\)). Then

\[
\sum_{\omega \in \text{RPP}_m} z^{|\omega|} = \prod_{k \geq 1} \frac{1}{1 - z^k} \prod_{k \geq 0} \prod_{1 \leq i < j \leq m-1} \frac{1}{1 - z^{2k+i+j}}.
\]

Inspired by the works of Dewar, Murty and Kotěšovec \cite{10,16}, we establish some useful theorems for asymptotic formulas in \cite{14} (see Section 4). Furthermore, the following asymptotic formula for the number of ribbon plane partitions can also be obtained.

**Theorem 1.2.** Let \(\text{RPP}_m(n)\) be the number of ribbon plane partitions \(\omega\) of width \(m\) and size \(n\). Then,

\[
\text{RPP}_m(n) \sim C(m) \times \frac{1}{n} \exp \left( \pi \sqrt{\frac{m^2 + m + 2}{6m}} \right),
\]

where \(C(m)\) is a constant with respect to \(n\) given by the following expression:

\[
C(m) = \left( \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} \sin \left( \frac{i+j}{2m} \pi \right) \right)^{-1} \sqrt{\frac{m^2 + m + 2}{2(m^2 - 3m + 14)}} \frac{1}{\sqrt{3m}}.
\]

The proofs of Theorems 1.1 and 1.2 are given in Section 5. For example, for \(m = 3\) in Theorems 1.1 and 1.2, the generating function and asymptotic formula for ribbon plane partitions of width \(m = 3\) (see Fig. 2, case RPPa) are

\[
\sum_{\omega \in \text{RPP}_3} z^{|\omega|} = \prod_{k \geq 1} \frac{1}{(1 - z^k)(1 - z^{6k-3})};
\]

\[
\text{RPP}_3(n) \sim \frac{\sqrt{7}}{24} \exp \left( \pi \sqrt{\frac{m}{3}} \right) n.
\]

In fact, our Theorems 1.1 and 1.2 can be extended to skew ribbon plane partitions (see Sections 3 and 4). The asymptotic formulas for two other skew ribbon plane partitions, together with some ordinary plane partitions (PP), cylindric partitions (CP) and symmetric cylindric partitions (SCP) are also reproduced next. The proofs of those asymptotic formulas can be found in Sections 5 and 6 for RPP and SCP respectively, and in \cite{14} for PP and CP.
The skew ribbon plane partitions have some nice properties. (1) The order of the asymptotic formula depends only on the width of the ribbon, not on the profile (the skew zone) itself. The similar property holds for ordinary plane partitions. We may think that this is natural by intuition. However, the cylindric partitions (CP and SCP) show that this is not always the case. (2) The asymptotic formula for ribbon plane partitions gives already good approximative values for the numbers of RPP, even for small integer \( n \). While the asymptotic formula for PP needs a large integer \( n \) to produce an acceptable value, as shown in the following table.

| \( n \) | \#PPa | \#PPb | \#PPc |
|---|---|---|---|
| 5 | 21 | 319 | 3032 |
| Asymptotic | \( \sim 319 \) | \( \sim 2449 \) | \( \sim 17062 \) |
| 10 | 319 | 319 | 319 |
| Asymptotic | \( \sim 2449 \) | \( \sim 17062 \) | \( \sim 103112 \) |
| 15 | 319 | 319 | 319 |
| Asymptotic | \( \sim 2449 \) | \( \sim 17062 \) | \( \sim 103112 \) |
| 20 | 319 | 319 | 319 |
| Asymptotic | \( \sim 2449 \) | \( \sim 17062 \) | \( \sim 103112 \) |

It is amazing how the orders of the asymptotic formulas for CP and SCP differ. For the CP, the exponents of \( n \) in the denominator are always 1, but the exponents
of \( e \) differ. While for the SCP, the exponents of \( n \) differ, but the exponents of \( n \) are constant. Let us summarize these observations in the following table.

| n^\( \text{const} \) | \( e^\text{const} \sqrt{n} \) | Fast Convergence |
|---------------------|------------------|-----------------|
| PP                  | Yes              | Yes             |
| CP                  | Yes              | No              |
| SCP                 | No               | Yes             |
| RPP                 | Yes              | Yes             |

The rest of the paper is arranged in the following way. In Section 2 we establish the complete summation formula for Schur processes. The basic notation and the trace generating functions for skew ribbon plane partitions can be found in Section 3. After recalling some useful theorems for asymptotic formulas in Section 4, we compute the generating functions and the asymptotic formulas for the numbers of skew ribbon plane partitions and symmetric cylindric partitions, in Sections 5 and 6 respectively.

2. Summation formulas for skew Schur functions

For the definitions and basic properties of skew Schur functions we refer to the books [20, 39, 40]. Let

\[
\Psi(X, Y) = \prod_{i,j}(1 - x_iy_j)^{-1},
\]

\[
\Phi(X) = \prod_i(1 - x_i)^{-1} \prod_{i < j}(1 - x_ix_j)^{-1},
\]

where \( X = \{x_1, x_2, \ldots \} \) and \( Y = \{y_1, y_2, \ldots \} \) are two alphabets. Each \( \pm 1 \)-sequence \( \delta = (\delta_i)_{1 \leq i \leq h} \) of length \( h \geq 1 \) is called a profile. Let \( |\delta|_1 \) (resp. \( |\delta|_-1 \)) be the number of letters 1 (resp. -1) in \( \delta \). Therefore, \( h = |\delta|_1 + |\delta|_-1 \). The following theorem contains three fundamental summation formulas for skew Schur functions, namely, the open summation formula (2.1), the cylindric summation formula (2.3) and the complete summation formula (2.2). The open and cylindric formulas have already been derived by Okounkov, Reshetikhin, Borodin, Corteel, Saveliev, Vušetić and Langer [6, 7, 8, 13, 18, 19, 28, 29, 42, 43]. For convenience, they are also reproduced next. Our main contribution is the complete summation formula (2.2).

**Theorem 2.1.** Let \( h \) be a positive integer, \( \delta = (\delta_i)_{1 \leq i \leq h} \) be a profile of length \( h \), and \( Z^1, \ldots, Z^h \) be a sequence of alphabets. Write \( Z_\delta^i := \sum_{\delta_i = -1} Z^i, Z_\delta^i := \sum_{\delta_i = 1} Z^i \) and \( Z := Z_\delta^h + Z_\delta^1 = \sum_{1 \leq i \leq h} Z^i \). For a sequence of partitions \( \lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^h \), let \( s^i_\delta \) denote the skew Schur function \( s_{\lambda^i/\lambda^{i-1}} \) if \( \delta_i = 1 \) and \( s_{\lambda^{i-1}/\lambda^i} \) if \( \delta_i = -1 \). We have

\[
(2.1) \sum_{\lambda^1, \ldots, \lambda^{h-1}} \prod_{i=1}^h s^i_\delta(Z^i) = \prod_{1 \leq i < j \leq h, \delta_i > \delta_j} \Psi(Z^i, Z^j) \times \sum_{\gamma} s_{\lambda^0/\gamma}(Z^h) s_{\lambda^h/\gamma}(Z^1);
\]

\[
(2.2) \sum_{\lambda^0, \ldots, \lambda^h} z^{\lambda^h} \prod_{i=1}^h s^i_\delta(Z^i) = \prod_{1 \leq i < j \leq h, \delta_i > \delta_j} \Psi(Z^i, Z^j) \times \Phi(Z^h) \prod_{k \geq 1} \frac{\Phi(z^k Z)}{1 - z^k};
\]
Lemma 2.3. 

Proof. Let $F(X)$ be the left-hand side of (2.9). Then, by (2.8) we have

\begin{align*}
F(X) &= \sum_{\mu, \tau} z^{\mu \tau} s_{\mu \tau}(zX) = \sum_{\tau} z^{\tau} \sum_{\mu} s_{\mu \tau}(zX)
&= \Phi(zX) \sum_{\tau} z^{\tau} \sum_{\rho} s_{\tau \rho}(zX) = \Phi(zX) F(zX).
\end{align*}
Hence, we obtain
\[ \sum_{\mu, \tau} z^{|\mu|} s_{\mu/\tau}(X) = \prod_{k \geq 1} \Phi(z^k X) \times F(0). \]

Since
\[ F(0) = \sum_{\mu, \tau} z^{|\mu|} s_{\mu/\tau}(0) = \sum_{\mu} z^{|\mu|} = \prod_{k \geq 1} \frac{1}{1 - z^k}, \]
then (2.9) is proved. \( \square \)

**Lemma 2.4.** We have
\[ (2.10) \sum_{\lambda, \mu, \gamma} z^{|\lambda|} s_{\mu/\gamma}(X)s_{\lambda/\gamma}(Y) = \Phi(Y) \prod_{k \geq 1} \frac{\Phi(z^k (X + Y))}{1 - z^k}. \]

**Proof.** By Lemma 2.3 we have
\[ \sum_{\lambda, \mu} z^{|\mu|} \sum_{\gamma} s_{\mu/\gamma}(X)s_{\lambda/\gamma}(Y) \]
\[ = \sum_{\mu} z^{|\mu|} \sum_{\gamma} s_{\mu/\gamma}(X) \sum_{\lambda} s_{\lambda/\gamma}(Y) \]
\[ = \Phi(Y) \sum_{\mu} z^{|\mu|} \sum_{\gamma} s_{\mu/\gamma}(X) \sum_{\tau} s_{\gamma/\tau}(Y) \]
\[ = \Phi(Y) \sum_{\mu} z^{|\mu|} \sum_{\tau} s_{\mu/\tau}(X + Y) \]
\[ = \Phi(Y) \prod_{k \geq 1} \frac{\Phi(z^k (X + Y))}{1 - z^k}. \] \( \square \)

**Remark.** Formula (2.10) is similar to the following formula stated in Macdonald’s book [20] (see p. 94, ex. 28(a)), which has two free partitions \( \lambda \) and \( \gamma \):
\[ (2.11) \sum_{\lambda, \gamma} z^{|\lambda|} s_{\lambda/\gamma}(X)s_{\lambda/\gamma}(Y) = \prod_{k \geq 1} \frac{\Phi(z^k X, Y)}{1 - z^k}. \]

Now we are ready to give the proof of Theorem 2.2.

**Proof of the Theorem 2.2.** Let \( F(X^0, X^1, \ldots, X^{h-1}, Y^0, Y^1, \ldots, Y^{h-1}) \) be the left-hand side of (2.3). By (2.7) we have
\[ F(X^0, X^1, \ldots, X^{h-1}, Y^0, Y^1, \ldots, Y^{h-1}) \]
\[ = \sum_{\lambda^1, \ldots, \lambda^{h-2}, \mu^0, \ldots, \mu^{h-1}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i)s_{\lambda^{i+1}/\mu^i}(Y^i) \]
\[ \times s_{\lambda^{h-2}/\mu^{h-2}}(X^{h-2})s_{\lambda^{h-1}/\mu^{h-1}}(Y^{h-1}) \sum_{\lambda^{h-1}} s_{\lambda^{h-1}/\mu^{h-2}}(Y^{h-2})s_{\lambda^{h-1}/\mu^{h-1}}(X^{h-1}) \]
\[ = \Psi(Y^{h-2}, X^{h-1}) \sum_{\lambda^1, \ldots, \lambda^{h-2}, \mu^0, \ldots, \mu^{h-1}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i)s_{\lambda^{i+1}/\mu^i}(Y^i) \]
\[ \times s_{\lambda^{h-2}/\mu^{h-2}}(X^{h-2})s_{\lambda^{h-1}/\mu^{h-1}}(Y^{h-1}) \sum_{\lambda^{h-1}} s_{\mu^{h-2}/\lambda^{h-1}}(X^{h-1})s_{\mu^{h-1}/\lambda^{h-1}}(Y^{h-2}) \]
The connected area $\Delta$ is defined to be $\Delta := \Lambda \backslash (\bigcup_{i=1}^{r-1} \{(c, d) \in \Lambda : c = \sum_{j=1}^{r-i} a_{r-j} \leq c \leq \sum_{j=1}^{r-i} a_{r-j} + 1 \leq d \leq \sum_{j=1}^{i} b_{i}\})$

$\Delta_1 = \bigcup_{i=1}^{r-1} \{(c, d) \in \Lambda : c = \sum_{j=1}^{r-i} a_{r-j} \leq c \leq \sum_{j=1}^{r-i} a_{r-j} + 1 \leq d \leq \sum_{j=1}^{i} b_{i}\}$,

$\Delta_2 = \{ (c, d) \in \Lambda : c - d > \sum_{i=0}^{r} a_{i} \}$,

$\Delta_3 = \{ (c, d) \in \Lambda : d - c > \sum_{i=1}^{r} b_{i} \}$.

The connected area $\Delta$ is defined to be $\Delta := \Lambda \backslash (\Delta_1 \cup \Delta_2 \cup \Delta_3)$. For example, with the profile $\delta = (1, -1, -1, 1, -1, 1, -1, 1)$, the four areas $\Delta_1, \Delta_2, \Delta_3, \Delta$ are illustrated in Fig. 3.

Let $\lambda$ and $\mu$ be two integer partitions. We write $\lambda \succ \mu$ or $\mu \prec \lambda$ if $\lambda/\mu$ is a horizontal strip (see [18, 19, 20, 29, 30, 39]).

3. Definitions for skew ribbon plane partitions

In this section we give the definition and the trace generating function of skew ribbon plane partitions. Each profile $\delta$ is associated with a connected area $\Delta := \Delta(\delta)$ of the quarter plane $\Lambda$ in a unique manner. For a given profile $\delta = 1^{a_0} (-1)^{b_1} 1^{a_1} (-1)^{b_2} \ldots 1^{a_{r-1}} (-1)^{b_r}$ with $a_0, b_r \geq 0, a_i, b_i \geq 1$ for $1 \leq i \leq r - 1$, let

$\Delta_1 = \bigcup_{i=1}^{r-1} \{(c, d) \in \Lambda : \sum_{j=1}^{r-i} a_{r-j} \leq c \leq \sum_{j=1}^{r-i} a_{r-j} + 1 \leq d \leq \sum_{j=1}^{i} b_{i}\}$,

$\Delta_2 = \{(c, d) \in \Lambda : c - d > \sum_{i=0}^{r} a_{i}\}$,

$\Delta_3 = \{(c, d) \in \Lambda : d - c > \sum_{i=1}^{r} b_{i}\}$.

The connected area $\Delta$ is defined to be $\Delta := \Lambda \backslash (\Delta_1 \cup \Delta_2 \cup \Delta_3)$. For example, with the profile $\delta = (1, -1, -1, 1, -1, 1, -1, 1)$, the four areas $\Delta_1, \Delta_2, \Delta_3, \Delta$ are illustrated in Fig. 3.

Let $\lambda$ and $\mu$ be two integer partitions. We write $\lambda \succ \mu$ or $\mu \prec \lambda$ if $\lambda/\mu$ is a horizontal strip (see [18, 19, 20, 29, 30, 39]).
Definition 3.1. Let $\delta = (\delta_i)_{1 \leq i \leq h}$ be a profile. A skew ribbon plane partition (RPP) with profile $\delta$ is a filling $\omega = (\omega_{i,j})$ of $\Delta(\delta)$ with nonnegative integers such that the size $|\omega| = \sum_{(i,j)} \omega_{i,j}$ is finite, and the rows and columns are weakly decreasing, i.e.,

$$\omega_{i,j} \geq \omega_{i,j+1}, \quad \omega_{i,j} \geq \omega_{i+1,j}$$

whenever these numbers are well-defined.

The set of all RPP with profile $\delta$ is denoted by $\text{RPP}_\delta$. Recall that the Schur process for plane partitions was first introduced by Okounkov and Reshetikhin [29] (see also [18, 19, 30]); the main idea was to read the plane partitions along the diagonals. When reading the RPP $\omega$ with profile $\delta$ along the diagonals from left to right, we obtain a sequence of integer partitions $(\lambda_0, \lambda_1, \ldots, \lambda_h)$ such that $\lambda_i - 1 \prec \lambda_i$ (resp. $\lambda_i - 1 \succ \lambda_i$) if $\delta_i = 1$ (resp. $\delta_i = -1$), and $|\omega| = \sum_{i=0}^{h} |\lambda_i|$. For simplicity, we identify the skew ribbon plane partition $\omega$ and the sequence of integer partitions by writing $\omega = (\lambda_0, \lambda_1, \ldots, \lambda_h)$.

The width of the skew ribbon plane partition $\omega$ is defined to be $h + 1$.

For example, with the RPP $\omega$ given in Fig. 3, we obtain a sequence of partitions: $(4,1) \prec (5,4) \succ (5,2) \succ (3) \prec (4,1) \succ (2) \prec (2,2) \succ (2,1) \prec (5,2,1)$. Hence, $\omega = ((4,1),(5,4),(5,2),(3),(4,1),(2),(2,2),(2,1),(5,2,1))$ is an RPP of width 9 with profile $\delta = (1,-1,-1,1,-1,1,1)$.

![Fig. 3. A skew ribbon plane partition](image)

For a sequence of parameters $u_i$ ($i \geq 0$), write $U_j = u_0 u_1 \cdots u_{j-1}$ ($j \geq 0$). Let $\text{RPP}_\delta(\lambda^0, \lambda^h)$ denote the set of the skew ribbon plane partitions $\omega = (\lambda^0, \lambda^1, \ldots, \lambda^h)$ starting from $\lambda^0$ and ending at $\lambda^h$ with profile $\delta$. Let $Z^i = \{U_i^{-\delta_i}\}$ in Theorem 2.1 we obtain the following trace generating functions for skew ribbon plane partitions.

**Theorem 3.1.** Let $\delta = (\delta_i)_{1 \leq i \leq h}$ be a profile. We have

$$\sum_{\omega \in \text{RPP}_\delta(\lambda^0, \lambda^h)} \prod_{i=0}^{h} u_i^{\lambda_i} = U_{h+1}^{\lambda^h} \prod_{1 \leq i < j \leq h} \frac{1}{1 - U_i^{-\delta_i} U_j}$$

(3.1)
\[ \sum_{\omega \in \text{RPP}} \prod_{i=0}^{h} u_i^{(\lambda_i)} = \prod_{\delta_i > \delta_j} \frac{1}{1 - U_i^{-1} U_j} \times \Phi(\{ U_i : \delta_i = -1 \}) \prod_{k \geq 1} \frac{\Phi(\{ U_i^{-1} U_k^k : 1 \leq i \leq h \})}{1 - U_k^{-1}} \]

(3.2)

\[ \sum_{\omega \in \text{RPP}(\lambda^0 = \lambda^h)} \prod_{i=0}^{h} u_i^{(\lambda_i)} = \prod_{\delta_i > \delta_j} \frac{1}{1 - U_i^{-1} U_j} \times \prod_{k \geq 1} \frac{\Psi(\{ U_i U_k^{k+1} : \delta_i = -1 \}, \{ U_j^{-1} : \delta_j = 1 \})}{1 - U_k^{k+1}} \]

(3.3)

The above theorem implies many classical results on various defective plane partitions, including the trace generating function of ordinary plane partitions (PP) (Stanley [35]), and the generating functions of symmetric plane partitions (SPP) (Andrews [2], Macdonald [20]), skew plane partitions (Sagan [33]) and skew shifted plane partitions (Sagan [33]).

4. Useful theorems for asymptotic formulas

Inspired by the works of Dewar, Murty and Kotešovec [10, 16], we have established some useful theorems for asymptotic formulas in [14]. The purpose of this section is to restate these results. Define

\[ \psi_n(v, r, b; p) := v \sqrt{\frac{p(1-p)}{2\pi}} \frac{r^{b+1-p/2}}{n^{b+1-p/2}} \exp(n^{p}r^{1-p}) \]

for \( n \in \mathbb{N}, v, b \in \mathbb{R}, r > 0, 0 < p < 1 \).

**Theorem 4.1** ([14]). Let \( t_1 \) and \( t_2 \) be given positive integers with \( \gcd(t_1, t_2) = 1 \). Suppose that

\[ F_1(q) = \sum_{n=0}^{\infty} a_{t_1 n} q^{t_1 n} \quad \text{and} \quad F_2(q) = \sum_{n=0}^{\infty} c_{t_2 n} q^{t_2 n} \]

are two power series such that their coefficients satisfy the asymptotic formulas

\[ a_{t_1 n} \sim t_1 \psi_{t_1 n}(v_1, r_1, b_1; p), \]
\[ c_{t_2 n} \sim t_2 \psi_{t_2 n}(v_2, r_2, b_2; p), \]

with \( 0 < p < 1, r_1 > 0, r_2 > 0, v_1, b_1, v_2, b_2 \in \mathbb{R} \). Then, the coefficients \( d_n \) in the product

\[ F_1(q)F_2(q) = \sum_{n=0}^{\infty} d_n q^n \]

satisfy the following asymptotic formula

\[ d_n \sim \psi_n(v_1 v_2, r_1 + r_2, b_1 + b_2; p). \]

(4.2)
Theorem 4.2. Let \( m \) be a positive integer. Suppose that \( x_i \) and \( y_i \) \((1 \leq i \leq m)\) are positive integers such that \( \gcd(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = 1 \). Then, the coefficients \( d_n \) in the following infinite product

\[
\prod_{i=1}^{m} \prod_{k \geq 0} \frac{1}{1 - q^{x_i+k}} = \sum_{n=0}^{\infty} d_n q^n
\]

have the following asymptotic formula

\[
d_n \sim v \frac{1}{2\sqrt{2\pi}} r^{b+1/4} \exp(\sqrt{nr}),
\]

where

\[
v = \prod_{i=1}^{m} \frac{\Gamma(y_i/x_i)}{\sqrt{2\pi i^2 x_i}}, \quad r = \sum_{i=1}^{m} \frac{2\pi^2}{3x_i}, \quad b = \sum_{i=1}^{m} \left( \frac{y_i}{2x_i} - \frac{1}{4} \right).
\]

5. Formulas for skew ribbon plane partitions

Let \( u_i = z \) for \( i \geq 0 \). We then derive the generating function for \( \text{RPP}_\delta \):

\[
\sum_{\omega \in \text{RPP}_\delta} z^{[\omega]} = \prod_{1 \leq j < i \leq m \atop \delta_i < \delta_j} \frac{1}{1 - z^{i-j}} \times \Phi(\{z^i : \delta_i = -1\})
\]

\[
\times \prod_{k \geq 1} \Phi(\{z^{(h+1)k+i} : \delta_i = -1\} + \{z^{(h+1)k-j} : \delta_j = 1\})
\]

where \( \Phi \) is the generating function for \( \text{RPP}_\delta \).

The right-hand side of the above identity can be further simplified. For each profile \( \delta = (\delta_i)_{1 \leq i \leq m-1} \), let

- \( W_1(\delta) = \{i \mid \delta_i = -1\} \cup \{m-i \mid \delta_i = 1\} \)
- \( W_2(\delta) = \{i+j \mid 1 \leq i < j \leq m-1, \delta_i = \delta_j = -1\}
- \cup \{2m-i-j \mid 1 \leq i < j \leq m-1, \delta_i = \delta_j = 1\}
- \cup \{2m+i-j \mid 1 \leq i < j \leq m-1, \delta_i < \delta_j\}
- \cup \{j-i \mid 1 \leq i < j \leq m-1, \delta_i > \delta_j\}

Theorem 5.1. The generating function for the skew ribbon plane partitions with profile \( \delta = (\delta_i)_{1 \leq i \leq m-1} \) is

\[
\sum_{\omega \in \text{RPP}_\delta} z^{[\omega]} = \prod_{k \geq 0} \left( \prod_{t \in W_1(\delta)} \frac{1}{1 - z^{mk+t}} \right) \left( \prod_{t \in W_2(\delta)} \frac{1}{1 - z^{2mk+t}} \right).
\]

By using the above generating function, our next theorem gives asymptotic formulas for the number of skew ribbon plane partitions. For simplicity, write

\[
\epsilon(\delta) = \sum_{\delta_i = -1} i - \sum_{\delta_i = 1} j.
\]

Theorem 5.2. Let \( m \geq 2 \) be a positive integer and \( \delta = (\delta_j)_{1 \leq j \leq m-1} \) be a profile of length \( m-1 \). We denote by \( \text{RPP}_\delta(n) \) the number of skew ribbon plane partitions \( \omega \) with profile \( \delta \) and size \( n \). Then,

\[
\text{RPP}_\delta(n) \sim C_1(\delta)C_2(m) \times \frac{1}{n} \exp\left(\pi \sqrt{\frac{(m^2 + m + 2)m}{6m}}\right),
\]
where $C_1(\delta)$ and $C_2(m)$ are two constants with respect to $n$:

$$C_1(\delta) = 2^{-\frac{|\delta|_1}{2}} \times \prod_{t \in W_1(\delta)} \frac{\Gamma\left(\frac{t}{m}\right)}{m} \prod_{t \in W_2(\delta)} \frac{\Gamma\left(\frac{t}{2m}\right)}{2},$$

$$C_2(m) = \left(2^{m^2-3m+14\pi^2m^2-m} \right)^{\frac{1}{4}} \times \sqrt{\frac{m^2 + m + 2}{3}}.$$

**Proof.** By the definitions of $W_1(\delta)$ and $W_2(\delta)$ we have

- $\#W_1(\delta) = m$;
- $\#W_2(\delta) = \left(\begin{array}{c} m - 1 \\ 2 \end{array}\right)$;
- \[ \sum_{t \in W_1(\delta)} t = m(|\delta|_1 + 1) + \epsilon(\delta); \]
- \[ \sum_{t \in W_2(\delta)} t = (m - 2)\epsilon(\delta) + 2m\left(\frac{|\delta|_1}{2}\right) + 2m \sum_{1 \leq i < j \leq m - 1} 1. \]

Hence,

$$\sum_{t \in W_1(\delta)} \frac{1}{m} + \sum_{t \in W_2(\delta)} \frac{1}{2m} = \frac{m}{m} + \frac{1}{2} \left(\begin{array}{c} m - 1 \\ 2 \end{array}\right) = \frac{m^2 + m + 2}{4m}. $$

Furthermore,

$$\sum_{t \in W_1(\delta)} \frac{t}{m} + \sum_{t \in W_2(\delta)} \frac{t}{2m} = 1 + |\delta|_1 + \left(\begin{array}{c} |\delta|_1 \\ 2 \end{array}\right) + \sum_{1 \leq i < j \leq m - 1} 1 + \frac{\epsilon(\delta)}{2}. $$

If we exchange any two adjacent letters in $\delta$, the sum of the last two terms doesn’t change, therefore it is equal to $\frac{1}{2}\left(\begin{array}{c} m \\ 2 \end{array}\right) - \left(\begin{array}{c} |\delta|_1 + 1 \\ 2 \end{array}\right)$. Then we obtain

$$\sum_{t \in W_1(\delta)} \frac{t}{m} + \sum_{t \in W_2(\delta)} \frac{t}{2m} = \frac{m^2 - m + 4}{4} $$

and

$$\sum_{t \in W_1(\delta)} \left(\frac{t}{2m} - \frac{1}{4}\right) + \sum_{t \in W_2(\delta)} \left(\frac{t}{4m} - \frac{1}{4}\right) = \frac{1}{4}. $$

By Theorems 4.2 and 5.1 the number of RPP with profile $\delta$ and size $n$ is asymptotic to

$$v \cdot \frac{1}{2\sqrt{2\pi}} \frac{r^{b+1/4}}{n^{b+3/4}} \exp(\sqrt{nr}),$$

where

$$v = \prod_{t \in W_1(\delta)} \left(\frac{\Gamma(t/m)}{\sqrt{m\pi} m^{t/m}}\right) \prod_{t \in W_2(\delta)} \left(\frac{\Gamma(t/2m)}{\sqrt{2m\pi} (2m)^{t/2m}}\right)$$

$$= 2^{-\frac{|\delta|_1}{2} - \frac{1}{4} \left(\frac{m-1}{2}\right)} m^{1/2} \pi (-m^2 + m - 2)/4.$$
Since \( \delta_{m} \) rems 5.1 and 5.2 for skew ribbon plane partitions when taking the profile
This achieves the proof.

Proof of Theorems 1.1 and 1.2. Take \( (C_{1}, C_{2}) \) and that respectively:

\[
\begin{align*}
&\Gamma(z) = \frac{\pi}{\sin(z\pi)}, \\
&\prod_{j=1}^{m-1} \sin(\frac{i\pi}{m}) = \frac{m}{\prod_{j=1}^{m-1} \sin(\frac{i\pi}{m})},
\end{align*}
\]

Their profiles, generating functions and asymptotic formulas are given in Fig. 2. Therefore Theorem 5.1 implies Theorem 1.1. Since \( \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(z\pi)} \) and

\[
\prod_{j=1}^{m-1} \sin(\frac{i\pi}{m}) = \frac{m}{\prod_{j=1}^{m-1} \sin(\frac{i\pi}{m})},
\]

we have

\[
\prod_{t \in W_{1}(\delta)} \Gamma\left(\frac{t}{m}\right) = \sqrt{\frac{\pi^{m-1}}{\prod_{j=1}^{m-1} \sin(\frac{j\pi}{m})}} = \sqrt{\frac{(2\pi)^{m-1}}{m}},
\]

and

\[
\prod_{t \in W_{2}(\delta)} \Gamma\left(\frac{t}{2m}\right) = \pi^{(m-1)(m-2)/4} \left( \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} \sin\left(\frac{i+j}{2m}\pi\right) \right)^{-1}.
\]

Since \( \delta = (-1)^{m-1} \), by Theorem 5.2 we verify that

\[
C_{1}(\delta) = 2 \cdot \frac{z^{(x)}}{m} \cdot \prod_{t \in W_{1}(\delta)} \Gamma\left(\frac{t}{m}\right) \prod_{t \in W_{2}(\delta)} \Gamma\left(\frac{t}{2m}\right)
\]

\[
= \frac{\pi^{(m^{2}-m)/4}}{\sqrt{m}} \left( \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} \sin\left(\frac{i+j}{2m}\pi\right) \right)^{-1},
\]

and that \( C_{1}(\delta)C_{2}(m) = \) is equal to \( C(m) \) given in Theorem 1.2.

For example, consider the three skew ribbon plane partitions (RPPa)-(RPPc) given in Fig. 2. Their profiles, generating functions and asymptotic formulas are respectively:

(a) Fig. 2, case RPPa. \( \delta = (1, 1) \), \( W_{1}(\delta) = \{3, 2, 1\} \), \( W_{2}(\delta) = \{3\} \),

\[
\sum_{\omega \in \text{RPPa}} z^{\vert\omega\vert} = \prod_{k \geq 0} \frac{1}{(1-z^{k+1})(1-z^{6k+3})},
\]

\[
\text{RPPa}(n) \sim \frac{\sqrt{\pi}}{24} \exp\left(\frac{\pi^{2}}{24n}\right).
\]

(b) Fig. 2, case RPPb. \( \delta = (1, -1) \), \( W_{1}(\delta) = \{3, 2, 2\} \), \( W_{2}(\delta) = \{1\} \),

\[
\sum_{\omega \in \text{RPPb}} z^{\vert\omega\vert} = \prod_{k \geq 0} \frac{1}{(1-z^{3k+3})(1-z^{3k+2})(1-z^{6k+1})},
\]
\[
\text{RPP}_b(n) \sim \sqrt{2\alpha} \times \frac{\sqrt{7} \exp(\pi \frac{\sqrt{2n}}{n})}{24},
\]

where

\[
\alpha = 2^{-\frac{1}{12}} \sqrt{3\pi^{-\frac{2}{3}} \Gamma(\frac{2}{3})^2 \Gamma(\frac{1}{6})} = 0.8908 \ldots
\]

(c) Fig. 2, case RPPc.

\[
\delta = (-1,1), \quad W_1(\delta) = \{3,1,1\}, \quad W_2(\delta) = \{5\},
\]

\[
\sum_{\omega \in \text{RPP}_c} z^{||\omega||} = \prod_{k \geq 0} \frac{1}{(1 - z^{2k+3})(1 - z^{2k+1})^2(1 - z^{2k+5})},
\]

\[
\text{RPP}_c(n) \sim \sqrt{2\alpha^{-1}} \times \frac{\sqrt{7} \exp(\pi \frac{\sqrt{2n}}{n})}{24}.
\]

6. *Formulas for symmetric cylindric partitions*

Cylindric partitions were introduced by Gessel and Krattenthaler in 1997 [11]. They obtained the generating function for cylindric partitions of some given shape that satisfy certain row bounds as some summation of determinants related to \(q\)-binomial coefficients. Later, Borodin gave an equivalent definition [6] and obtained the generating function for cylindric partitions. A cylindric partition (CP) with profile \(\delta\) is an RPP

\[
\omega = (\lambda^0, \lambda^1, \ldots, \lambda^{h-1}, \lambda^h)
\]

with profile \(\delta\) such that \(\lambda^0 = \lambda^h\). The size of such partition is defined by \(|\omega| = \sum_{i=0}^{h-1} |\lambda^i|\) (notice that \(\lambda^h\) is not counted here). The following generating function for cylindric partitions, first proved by Borodin [6], later by Tingley [41] and Langer [19], can be obtained by letting \(u_i = z (0 \leq i \leq h - 1)\) and \(u_h = 1\) in (3.3).

**Theorem 6.1** (Borodin [6]). Let \(\delta = (\delta_i)_{1 \leq i \leq h}\) be a profile. Then the generating function for the cylindric partitions with profile \(\delta\) is

\[
\sum_{\omega \in \text{CP}_\delta} z^{||\omega||} = \prod_{k \geq 0} \prod_{t \in W_3(\delta)} \frac{1}{1 - z^{h+k+t}},
\]

where

\[
W_3(\delta) = \{h\} \cup \{j - i : i < j, \; \delta_i > \delta_j\} \cup \{h + i - j : i < j, \; \delta_i < \delta_j\}.
\]

A symmetric cylindric partition (SCP) with profile \(\delta = (\delta_1, \delta_2, \ldots, \delta_h)\) is an RPP

\[
\omega = (\lambda^h, \lambda^{h-1}, \ldots, \lambda^1, \lambda^0, \lambda^1, \ldots, \lambda^{h-1}, \lambda^h)
\]

with profile \((-\delta_h, -\delta_{h-1}, \ldots, -\delta_2, -\delta_1, \delta_1, \delta_2, \ldots, \delta_{h-1}, \delta_h)\). Notice that a symmetric cylindric partition is always a cylindric partition, and when \(\lambda^h = \emptyset\), the SCP \(\omega\) becomes an SyPP. The size of the symmetric cylindric partition \(\omega\) is defined by

\[
|\omega| = |\lambda^0| + 2 \sum_{i=1}^{h} |\lambda^i|.
\]

**Theorem 6.2.** The generating function for symmetric cylindric partitions with profile \(\delta\) is

\[
(6.1) \sum_{\omega \in \text{SCP}_\delta} z^{||\omega||} = \prod_{i<j} \frac{1}{1 - z^{2(j-i)}} \Phi \{\{z^{2i-1} : \delta_i = -1\}\}
\]
Then we obtain the following result.

\[ \Phi(z^{2(h+1)k}) \left( \frac{\{ z^{-2i+1} : \delta_i = 1 \} + \{ z^{-2i-1} : \delta_i = -1 \}}{1 - z^{2(h+1)k}} \right). \]

Proof. By (3.1), we have

\[ \sum_{\omega \in SCP_4} z^{\omega} = \sum_{\lambda^0, \lambda^h} z^{-|\lambda^0|} \sum_{\omega' \in RPP_4(\lambda^0, \lambda^h)} z^{2|\omega'|} \]

\[ = \prod_{\delta_i < j} \frac{1}{1 - z^{2(j-1)}} \]

\[ \times \sum_{\lambda, \mu} z^{(2h+1)|\mu| - |\lambda|} \sum_{\gamma \subseteq \lambda, \mu} s_{\lambda/\gamma}(\{ z^{2i} : \delta_i = 1 \}) s_{\mu/\gamma}(\{ z^{-2i} : \delta_i = 1 \}) \]

\[ \times \sum_{\lambda, \mu} z^{(2h+1)|\mu|} \sum_{\gamma \subseteq \lambda, \mu} s_{\lambda/\gamma}(\{ z^{2i-1} : \delta_i = 1 \}) s_{\mu/\gamma}(\{ z^{-2i+1} : \delta_i = 1 \}). \]

By Lemma 2.4, this is equal to the right hand side of (6.1). \( \square \)

The right-hand side of the above identity can be further simplified. For each profile \( \delta = (\delta_i)_{1 \leq i \leq m-1} \), let

\[ W_4(\delta) = \{ 2m - 1 \} \cup \{ 2i - 1 \ | \ \delta_i = -1 \} \cup \{ 2m - 2i \ | \ \delta_i = 1 \}; \]

\[ W_5(\delta) = \{ 2i + 2j - 2 \ | \ 1 \leq i < j \leq m - 1, \ \delta_i = \delta_j = 1 \} \]

\[ \cup \{ 4m - 2i - 2j \ | \ 1 \leq i < j \leq m - 1, \ \delta_i = \delta_j = 1 \} \]

\[ \cup \{ 2(2m - 1) + 2i - 2j \ | \ 1 \leq i < j \leq m - 1, \ \delta_i < \delta_j \} \]

\[ \cup \{ 2j - 2i \ | \ 1 \leq i < j \leq m - 1, \ \delta_i > \delta_j \}. \]

Then we obtain the following result.

**Theorem 6.3.** The generating function for symmetric cylindric partitions with profile \( \delta = (\delta_i)_{1 \leq i \leq m-1} \) is

\[ \sum_{\omega \in SCP_4} z^{\omega} = \prod_{k \geq 0} \left( \prod_{t \in W_4(\delta)} \frac{1}{1 - z^{2(m-1)k+t}} \right) \left( \prod_{t \in W_5(\delta)} \frac{1}{1 - z^{2(2m-1)k+t}} \right). \]

By the definitions of \( W_4 \) and \( W_5 \), it is easy to verify that \( |W_4(\delta)| = m \) and \( |W_5(\delta)| = \binom{m-1}{2} \). Hence,

\[ \sum_{t \in W_4(\delta)} \frac{1}{2m - 1} \sum_{t \in W_5(\delta)} \frac{1}{2(2m - 1)} = \frac{m^2 + m + 2}{4(2m - 1)}. \]

By Theorems 4.2 and 6.3, we obtain the following asymptotic formula for the number of SCP with size \( n \).

**Theorem 6.4.** Let \( m \geq 2 \) be a positive integer and \( \delta = (\delta_j)_{1 \leq j \leq m-1} \) be a profile of length \( m - 1 \). Let \( SCP_\delta(n) \) denote the number of symmetric cylindric partitions with profile \( \delta \) and size \( n \). Then

\[ SCP_\delta(n) \sim v \sqrt{\frac{1}{8\pi n^{6+3/4}}} \exp(\sqrt{n^3}), \]
where $r, b, v$ are given below:

$$
r = \frac{(m^2 + m + 2)\pi^2}{6(2m - 1)},
$$

$$
b = \sum_{t \in W_4(\delta)} \frac{t}{2(2m - 1)} - \frac{1}{4} + \sum_{t \in W_5(\delta)} \frac{t}{4(2m - 1)} - \frac{1}{4},
$$

$$
v = 2^{-\frac{m}{2}} \sum_{t \in W_4(\delta)} t - \frac{1}{4}(m^2 - 3m + 2)\pi - \frac{1}{4}(m^2 - m + 2)(2m - 1)^2b
\times \prod_{t \in W_4(\delta)} \Gamma\left(\frac{t}{2m - 1}\right) \prod_{t \in W_5(\delta)} \Gamma\left(\frac{t}{2(2m - 1)}\right).
$$

Remark. The coefficient $r$ depends only on the width of the symmetric cylindric partitions, not on the profile itself, while the coefficient $b$ depends on the profile. It is interesting to compare these phenomena with the asymptotic formula for cylindric partitions [14]. There, the coefficient $b$ depends only on the width, and $r$ depends on the profile.

For example, consider the three symmetric cylindric partitions (SCP$a$)-(SCP$c$) given in Fig. 2. Their profiles, generating functions and asymptotic formulas are respectively:

(a) Fig. 2, case SCP$a$. $\delta = (-1, -1)$. $W_4(\delta) = \{1, 3, 5\}$ and $W_5(\delta) = \{4\}$.

$$
\sum_{\omega \in \text{SCP}a} z^{\omega} = \prod_{k \geq 0} \frac{1}{(1 - z^{4k + 1})(1 - z^{4k + 3})(1 - z^{4k + 7})(1 - z^{10k + 4})},
$$

$$
\text{SCP}a(n) \sim \Gamma\left(\frac{4}{5}\right) \Gamma\left(\frac{2}{5}\right) \Gamma\left(\frac{3}{5}\right) 2^{-\frac{3}{10}} 5^{3/5} \pi - \frac{1}{5} \times \frac{1}{n^{17/20}} \exp\left(\pi \sqrt{\frac{7n}{15}}\right).
$$

(b) Fig. 2, case SCP$b$. $\delta = (1, -1)$. $W_4(\delta) = \{3, 4, 5\}$ and $W_5(\delta) = \{2\}$.

$$
\sum_{\omega \in \text{SCP}b} z^{\omega} = \prod_{k \geq 0} \frac{1}{(1 - z^{4k + 3})(1 - z^{4k + 7})(1 - z^{4k + 11})(1 - z^{10k + 2})},
$$

$$
\text{SCP}b(n) \sim \Gamma\left(\frac{1}{5}\right) \Gamma\left(\frac{3}{5}\right) \Gamma\left(\frac{4}{5}\right) \pi - \frac{1}{10} \times \frac{1}{n^{21/20}} \exp\left(\pi \sqrt{\frac{7n}{15}}\right).
$$

(c) Fig. 2, case SCP$c$. $\delta = (1, 1)$. $W_4(\delta) = \{2, 4, 5\}$ and $W_5(\delta) = \{6\}$.

$$
\sum_{\omega \in \text{SCP}c} z^{\omega} = \prod_{k \geq 0} \frac{1}{(1 - z^{4k + 2})(1 - z^{4k + 6})(1 - z^{4k + 10})(1 - z^{10k + 6})},
$$

$$
\text{SCP}c(n) \sim \Gamma\left(\frac{2}{5}\right) \Gamma\left(\frac{3}{5}\right) \Gamma\left(\frac{4}{5}\right) \pi - \frac{1}{6} \times \frac{1}{n^{23/20}} \exp\left(\pi \sqrt{\frac{7n}{15}}\right).
$$

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