EMBEDDINGS OF HOMOGENEOUS SPACES IN PRIME CHARACTERISTICS

NIELS LAURITZEN

Let $X$ be a projective algebraic variety over an algebraically closed field $k$ admitting a homogeneous action of a semisimple linear algebraic group $G$. Then $X$ can be canonically identified with the homogeneous space $G/G_x$, where $x$ is a closed point in $X$ and $G_x$ the stabilizer group scheme of $x$. A group scheme over a field of characteristic 0 is reduced so in this case, $X$ is isomorphic to a generalized flag variety $G/P$, where $P$ is a parabolic subgroup. In [2][7][6][5] the geometry of $X$ in prime characteristic has been studied and it has been shown that a lot of strange phenomena occur when $G_x$ is non-reduced. The simplest example of a projective homogeneous $G$-space (for $G = \text{SL}_3(k)$, char $k = p > 0$) not isomorphic to a generalized flag variety is the divisor $x_0 y_0^p + x_1 y_1^p + x_2 y_2^p = 0$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Since projective homogeneous spaces with non-reduced stabilizers are quite algebraic by construction, we give in §2 of this paper a simple geometric approach for their construction, involving only scheme theoretic images under partial Frobenius morphisms. We choose to do this focusing on the “unseparated incidence variety”. In this case the geometric approach completely determines the cohomology of effective line bundles. The reader unfamiliar with the general concept of projective homogeneous spaces in prime characteristic might find this section useful.

The main topic of this paper is the study of embeddings of homogeneous projective spaces. Let $X$ be a projective homogeneous $G$-space. In §3.2 we show that $X$ can be realized as the $G$-orbit of the $B$-stable line in $\mathbb{P}(L(\lambda))$, where $L(\lambda)$ denotes the simple $G$-representation of a certain highest weight $\lambda$. This approach leads in §3.3 to examples of some strange embeddings of homogeneous spaces in characteristic 2 - one lying on the boundary of Hartshorne’s conjecture on complete intersections (a so-called Hartshorne variety in the terms of [8]).

Recall that an ample line bundle $L$ on an algebraic variety $X$ is called normally generated [10] if the multiplication map

$$H^0(X, L)^{\otimes n} \rightarrow H^0(X, L^{\otimes n})$$

is surjective for all $n \geq 1$. For generalized flag varieties the very ampleness of ample line bundles follows from normal generation of ample line bundles [1][9] (In [10] one can find a nice and short proof of the fact that an ample normally generated line bundle $L$ is very ample). In the general setting of projective homogeneous spaces, normal generation of ample line bundles is an open question.

In §4 we compute the line bundles on projective homogeneous spaces and in §4.2 we use a simple “diagonal” construction to prove that ample line bundles on projective homogeneous $G$-spaces are very ample. In view of [5] this proves the existence of a counterexample to Kodaira

1991 Mathematics Subject Classification. Primary: 14M17; Secondary: 20G05.

Key words and phrases. Generalized flag variety, Frobenius subcover, non-reduced stabilizer group scheme, simple $G$-representations, Hartshorne variety, line bundles, Kodaira vanishing.
type vanishing in prime characteristic with a very ample line bundle answering Raynaud’s question ([11], Remarques et questions 3).

The paper is organized as follows

\textbf{Contents}

1. Preliminaries \hspace{1cm} 2
   1.1. \(G\)-spaces
   1.2. The Frobenius subcover
   1.3. The Frobenius kernel
   1.4. The diagonal action

2. The unseparated incidence variety \hspace{1cm} 3
   2.1. The incidence variety
   2.2. The unseparated incidence variety
   2.3. Cohomology of effective line bundles

3. Structure of projective homogeneous spaces \hspace{1cm} 5
   3.1. Parabolic subgroup schemes
   3.2. The action of \(G\) on \(\mathbb{P}(L(\lambda))\)
   3.3. Exceptional parabolic subgroup schemes

4. Line bundles \hspace{1cm} 7
   4.1. Characters
   4.2. Ample line bundles

References \hspace{1cm} 10

It is a pleasure to acknowledge inspiring discussions with and e-mails from H. H. Andersen, W. J. Haboush, J. Kollár, V. B. Mehta and V. Srinivas. I am indebted to Kollár and Mehta for pointing out the usefulness of the simple geometric viewpoint in §2 and to Haboush for telling me about the beautiful little book [8] containing Zak’s classification of Severi varieties.

Part of this work was done while the author was visiting Max-Planck-Institut für Mathematik in Bonn. I thank the Institute for its hospitality and an inspiring environment.

1. Preliminaries

Let \(k\) be an algebraically closed field of prime characteristic \(p\). In the following, an algebraic variety \(X\) is assumed to be an algebraic variety over \(k\) and a morphism to be a morphism of \(k\)-varieties. We let \(X(A) = \text{Mor}_k(\text{Spec } A, X)\) denote the set of \(A\)-points of \(X\), where \(A\) is a \(k\)-algebra. Let \(G\) be a simply connected and semisimple algebraic group. We will assume that \(p > 3\) if \(G\) has a component of type \(G_2\) and \(p > 2\) if \(G\) has a component of type \(B_n, C_n\) or \(F_4\).
1.1. **G-spaces.** A G-space is an algebraic variety \(X\) endowed with a morphism \(G \times X \to X\) inducing an action of \(G(A)\) on \(X(A)\) for all \(k\)-algebras \(A\). A G-space \(X\) is called homogeneous if the action \(G(k) \times X(k) \to X(k)\) on \(k\)-points is transitive. A \(k\)-point \(x \in X(k)\) gives a natural morphism \(G \to X\). The fiber product \(G_x = G \times X \text{Spec}(k)\) is easily seen to be a closed subgroup scheme of \(G\). It is called the stabilizer group scheme of \(x\).

1.2. **The Frobenius subcover.** An algebraic variety \(X\) gives rise to a new algebraic variety \(X^{(n)}\) with the same underlying point space as \(X\), but where the \(k\)-multiplication is twisted via the ringhomomorphism: \(a \mapsto a^n\). The \(n\)-th order Frobenius homomorphism induces a natural morphism \(F^n_X : X \to X^{(n)}\). As \(X\) is reduced, \(X^{(n)}\) can be identified with \(k\)-subalgebra of \(p^n\)-th powers of regular functions on \(X\). We call \(X^{(n)}\) the \(n\)-th Frobenius subcover of \(X\). Recall that \(X\) is said to be defined over \(\mathbb{F}_p\) if there exists an \(\mathbb{F}_p\)-variety \(X'\), such that \(X \cong X' \times \text{Spec} \mathbb{F}_p \text{Spec} k\). If \(X\) is defined over \(\mathbb{F}_p\), then \(X\) is isomorphic to \(X^{(n)}\) (the isomorphism is given locally by \(f \otimes a \mapsto f \otimes a^n\), where \(a \in k\)).

1.3. **The Frobenius kernel.** Now \(G^{(n)}\) is an algebraic group and \(F^n_G : G \to G^{(n)}\) is a homomorphism of algebraic groups. The kernel of \(F^n_G\) is called the \(n\)-th Frobenius kernel of \(G\) and denoted \(G_n\). Let \(X\) be a homogeneous \(G\)-space and \(x\) a closed point of \(X\). It is easy to see that \(X^{(n)}\) is a \(G\)-homogeneous space through the homomorphism \(F^n_G\). If \(H = G_x\) then \(G_{F^n_G(x)} = G_n H\).

1.4. **The diagonal action.** Now let \(X\) and \(Y\) be homogeneous \(G\)-spaces with distinguished closed points \(x\) and \(y\) and let \(H = G_x\) and \(K = G_y\). The product \(X \times Y\) becomes a \(G\)-space through the diagonal action and \(G_{(x,y)} = H \cap K\).

2. **The unseparated incidence variety**

In this section we give a quite explicit geometric description (in 2.2) of certain projective homogeneous spaces for \(\text{SL}_n\) occurring as divisors in \(\mathbb{P}^n \times \mathbb{P}^n\).

2.1. **The incidence variety.** Let \(n > 1\) and \(G = \text{SL}_{n+1}(k)\). The natural action of \(G\) on \(V = k^{n+1}\) makes \(\mathbb{P}(V)\) and \(\mathbb{P}(V^*)\) into homogeneous spaces for \(G\). We fix points \(x_1 \in \mathbb{P}(V)\) and \(x_2 \in \mathbb{P}(V^*)\), such that \(G_{x_1} = P_1\) and \(G_{x_2} = P_2\) are appropriate parabolic subgroups containing the subgroup of upper triangular matrices \(B\) in \(G\). The orbit \(Y\) of \((x_1, x_2)\) in \(\mathbb{P}(V) \times \mathbb{P}(V^*)\) is a projective homogeneous space for \(G\) isomorphic to \(G/P\), where \(P = P_1 \cap P_2\). Notice that the points of \(Y\) are just pairs of incident lines and hyperplanes and that \(Y \cong Z(s)\), where \(s\) is the section \(x_0y_0 + \cdots + x_ny_n\) of \(\mathcal{O}(1) \times \mathcal{O}(1)\).

2.2. **The unseparated incidence variety.** Let \(X\) be the \(G\)-orbit of of \((x_1, F^r(x_2))\) in \(\mathbb{P}(V) \times \mathbb{P}(V^*)^{(r)}\). By 1.3 and 1.4,

\[ X \cong G/\bar{P} \]
where for any $k$-algebra $A$

$$\tilde{P}(A) = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \\ 0 & a & \ldots & a & * \end{pmatrix} \in \text{SL}_n(A) | a \in A, a^p^n = 0 \right\}$$

There is a natural equivariant morphism

$$\varphi : \mathbb{P}(V) \times \mathbb{P}(V^*) \to \mathbb{P}(V) \times \mathbb{P}(V^*)^{(r)}$$

and $X$ is the scheme theoretic image $\varphi(Y)$. The induced morphism $\varphi : Y \to X$ is the natural morphism

$$G/P \to G/\tilde{P}$$

given by the inclusion $P \subseteq \tilde{P}$. Since the sheaf of ideals of the scheme theoretic image is the kernel of the comorphism of $Y \to \mathbb{P}(V) \times \mathbb{P}(V^*)^{(r)}$ ([3], Exercise II.3.11 (d)), $X$ is the zero scheme of the section $\bar{s} = x_0^{p^r} y_0 + \cdots + x_n^{p^r} y_n$ of $\mathcal{O}(p^r) \times \tilde{O}(1)$. Using the isomorphism from 1.2, we get that $X = Z(\bar{s})$ is isomorphic to its scheme theoretic image $Z(\bar{s}) \subseteq \mathbb{P}(V) \times \mathbb{P}(V^*)$, where

$$\bar{s} = x_0^{p^r} y_0 + \cdots + x_n^{p^r} y_n$$

is a section of $\mathcal{O}(p^r) \times \tilde{O}(1)$.

### 2.3. Cohomology of effective line bundles.

Let $a, b \in \mathbb{Z}$. The restriction to $Y$ of the line bundle $\mathcal{O}(a) \times \mathcal{O}(b)$ on $\mathbb{P}^n \times \mathbb{P}^n$ will be denoted $L(a, b)$. The restriction to $X$ of the line bundle $\mathcal{O}(a) \times \tilde{O}(b)$ on $\mathbb{P}^n \times (\mathbb{P}^n)^{(r)}$ will be denoted $L(a, \tilde{b})$. Notice that the isomorphism in 1.2 maps $L(a, \tilde{b})$ to $L(a, b)$. By 2.2 there is an exact sequence

$$0 \to \mathcal{O}(-p^r) \times \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_X \to 0$$

For the line bundle $L = L(a, \tilde{b})$ on $X$, we therefore get the exact sequence

$$0 \to \mathcal{O}(a - p^r) \times \mathcal{O}(b - 1) \to \mathcal{O}(a) \times \mathcal{O}(b) \to L \to 0$$

Now assume that $a, b \geq 0$ ($L(a, \tilde{b})$ is effective). Then tracing through the long exact sequence and using the Künneth formula, we get $H^i(X, L) = 0$, if $1 \leq i < n - 1$ along with the following exact sequences:

$$0 \to H^0(\mathbb{P}^n, \mathcal{O}(a - p^r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(b - 1)) \to H^0(\mathbb{P}^n, \mathcal{O}(a)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(b)) \to H^0(X, L) \to 0$$

and

$$0 \to H^{n-1}(X, L) \to H^n(\mathbb{P}^n, \mathcal{O}(a - p^r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(b - 1)) \to 0$$

By Serre duality one has

$$H^{n-1}(X, L) \cong H^0(\mathbb{P}^n, \mathcal{O}(p^r - a - n - 1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(b - 1))$$

so that the higher cohomology of $L$ vanishes if $a > p^r - n - 1$.

By the adjunction formula ([3], Proposition II.8.20) we get $\omega_X \cong \mathcal{O}(p^r - n - 1) \times \mathcal{O}(-n)$. A line bundle $L = L(a, \tilde{b})$ on $X$ is ample if and only if $a, b > 0$. Kodaira type vanishing
(vanishing higher cohomology for \( L \otimes \omega_X \), where \( L \) is ample) for \( X \) amounts to the fact that \( a + p^t - n - 1 > p^t - n - 1 \), when \( a > 0 \).

The unseparated incidence variety admits a lifting to a flat \( \mathbb{Z} \)-scheme. There are projective homogeneous spaces for \( SL_4 \), which do not admit a lifting to a flat \( \mathbb{Z} \)-scheme ([2], \S 6).

3. STRUCTURE OF PROJECTIVE HOMOGENEOUS SPACES

A projective homogeneous \( G \)-space \( X \) is determined through its stabilizer group scheme \( G_x \) at some closed point \( x \in X \). Notice that since \( X \) is projective, Borel’s fixed point theorem implies that \( G_x \) contains a Borel subgroup \( B \). We introduce some more notation. Let \( T \) be a maximal torus of \( G \) contained in the Borel subgroup \( B \). Denote by \( R = R(G, T) \) the roots of \( G \) w.r.t. \( T \). Let the roots \( R(B, T) \) of \( B \) be the positive roots \( R^+ \) in \( R \) and \( S \subseteq R^+ \) the simple roots of \( R \). Let \( X(T) \) be the characters of \( T \) and \( Y(T) \) the one parameter subgroups. The usual pairing \( X(T) \times Y(T) \to \mathbb{Z} \) is denoted \( \langle \cdot, \cdot \rangle \). The coroot in \( Y(T) \) corresponding to \( \alpha \in R \) is denoted \( \alpha' \). The root subgroup corresponding to \( \alpha \in R \) is denoted by \( U_\alpha \). Let \( \{ X_\alpha \}_{\alpha \in R} \) and \( \{ H_\alpha \}_{\alpha \in S} \) be the Chevalley basis for \( \text{Lie}(G) \). The monomials

\[
\prod_{\alpha \in R^+} X_{-\alpha}^{n_\alpha} \prod_{\alpha \in S} \left( H_\alpha^{m_\alpha} \right) \prod_{\alpha \in R^+} X_\alpha^{n_\alpha}
\]

where \( n'_\alpha, m_\alpha, n_\alpha \in \mathbb{N} \), form a basis for the \( k \)-algebra of distributions \( \text{Dist}(G) \) ([4], II.1.12).

Recall that a subgroup scheme \( H \subseteq G \) is uniquely determined by its subalgebra \( \text{Dist}(H) \subseteq \text{Dist}(G) \). A \( k \)-basis for \( \text{Dist}((U_\alpha)_n) \) is given by \( 1, X_\alpha, X_\alpha^{(2)}, \ldots, X_\alpha^{(p^n-1)} \).

3.1. Parabolic subgroup schemes. Let \( \tilde{P} \) be a subgroup scheme containing \( B \). Since \( P = \tilde{P}_{\text{red}} \) is a parabolic subgroup (the nil-radical is a Hopf ideal), it follows that \( \tilde{P} \) is a connected group scheme. In particular we get for \( \alpha \in R^- \) that \( \text{Dist}(\tilde{P}) \cap \text{Dist}(U_\alpha) = \text{Dist}((U_\alpha)_{n_\alpha}) \) for a suitable \( n_\alpha \), where \( 0 \leq n_\alpha \leq \infty \) with the convention \( (U_\alpha)_\infty = U_\alpha \). The subalgebra \( \text{Dist}(\tilde{P}) \) is determined completely by \( (n_\alpha)_{\alpha \in R^-} \). With the assumptions given in \S 1 on \( p = \text{char} \ k \) it follows by ([2], Proposition 1.6) that \( \text{Dist}(\tilde{P}) \) is uniquely determined by \( (n_\alpha)_{\alpha \in -S} \).

One can construct \( \tilde{P} \neq G \) as follows: The maximal parabolic subgroup \( P(\gamma) \) corresponding to a simple root \( \gamma \in S \) is the parabolic subgroup with roots generated by \( S \setminus \{ \gamma \} \). The parabolic subgroup \( P \) is the intersection \( P = P(\alpha_1) \cap \cdots \cap P(\alpha_m) \) for certain simple roots \( S' = \{ \alpha_1, \ldots, \alpha_m \} \subseteq S \). It is easy to see that \( \tilde{P} \subseteq G_n(P(\alpha_i)) \), for \( n \) sufficiently big. Let \( n_i \) be the minimal \( n \) with this property. Then

\[
\tilde{P} = G_{n_1} P(\alpha_1) \cap \cdots \cap G_{n_m} P(\alpha_m)
\]

In the notation above \( \text{Dist}(\tilde{P}) \) is determined uniquely by \( n_{\alpha_1} = n_1, \ldots, n_{\alpha_m} = n_m \) and \( n_\alpha = \infty \) if \( \alpha \notin S' \).

3.2. The action of \( G \) on \( \mathbb{P}(L(\lambda)) \). Recall that the simple \( G \)-representations are parametrized by dominant weights \( X(T)_+ \). Let \( L(\lambda) \) denote the simple \( G \)-representation associated with \( \lambda \in X(T)_+ \). Then \( L(\lambda) \) is generated by a \( B \)-stable line of (highest) weight \( \lambda \).
Proposition 1. Let \( S = \{ \alpha_1, \ldots, \alpha_l \} \) and \( \nu_p \) denote the \( p \)-adic valuation, such that \( \nu_p(0) = \infty \). Let \( L(\lambda) \) be the simple representation of highest weight \( \lambda \in X(T)_+ \) and \( n_i = \nu_p(\langle \lambda, \alpha_i^\vee \rangle) \). Then the stabilizer of the \( B \)-stable line \( x \in \mathbb{P}(L(\lambda)) \) for the natural action of \( G \) is

\[
G_x = G_{n_1} P(\alpha_1) \cap \cdots \cap G_{n_l} P(\alpha_l)
\]

Proof. Let \( v \) be a generator for \( x \in \mathbb{P}(L(\lambda)) \). We compute the algebra of distributions \( \text{Dist}(G_x) \). For the induced action of \( \text{Dist}(G) \) on \( L(\lambda) \) we have

\[
\text{Dist}(G_x) = \{ X \in \text{Dist}(G) | Xv = 0 \}
\]

By 3.1 it suffices to show for a simple root \( \alpha_i \in S \), that

\[
X_{\alpha_i}^{(2)} v = X_{\alpha_i}^{(\nu_i)} v = \cdots = X_{\alpha_i}^{(p_{n_i} - 1)} v = 0
\]

and

\[
X_{\alpha_i}^{(p_{n_i})} v \neq 0
\]

where the last condition is void in the case \( n_i = \infty \). Since \( L(\lambda) \) is a highest weight module generated by \( v \) and \( \alpha_i \) is a simple root, it suffices to prove that

\[
X_{\alpha_i}^{(n)} X_{-\alpha_i}^{(n)} v = 0
\]

to conclude that \( X_{\alpha_i}^{(n)} v = 0 \). In \( \text{Dist}(G) \) we have the following commutation formula for \( \alpha \in R^+ \):

\[
X_{\alpha}^{(m)} X_{-\alpha}^{(n)} = \sum_{j=0}^{\min(m,n)} X_{\alpha}^{(n-j)} \left( H_{\alpha} - m - n + 2j \right) X_{\alpha}^{(m-j)}
\]

From this formula it follows that

\[
X_{\alpha_i}^{(n)} X_{-\alpha_i}^{(n)} v = \left( \frac{\langle \lambda, \alpha_i^\vee \rangle}{n} \right) v
\]

When \( n_i = \infty \) it follows that \( X_{\alpha_i}^{(n)} v = 0 \) for \( n > 0 \). Assume now that \( n_i < \infty \). Since \( \binom{m}{n} \equiv 0 \) (mod \( p \)) if \( 0 < n < p^r \) and \( \not\equiv 0 \) (mod \( p \)) if \( n = p^r \), when \( \nu_p(m) = r \), the result follows. \( \square \)

3.3. Exceptional parabolic subgroup schemes. The action of \( G \) on \( \mathbb{P}(L(\lambda)) \) gives a lot of examples of exceptional parabolic subgroup schemes in characteristic 2 - parabolic subgroup schemes which are not the intersection of thickenings of the maximal parabolic subgroups as in §3.1. In this section \( k \) is assumed to be of characteristic 2. Recall that the simple module \( L(\lambda) \) is a quotient of the Weyl module \( V(\lambda) \). The Weyl module \( V(\lambda) \) is the base extension to \( k \) of the minimal admissible \( \mathbb{Z} \)-form in the simple representation of highest weight \( \lambda \) for the complex semisimple Lie algebra corresponding to \( G \). Let \( K(G) \) denote the Grothendieck group of \( G \). In the examples below, the decompositions in \( K(G) \) of Weyl modules were computed using Jantzen’s sum formula ([4], II.8). Example 2 was discovered using a computer program, developed by A. Buch, for computations in modular representation theory.
Example 1. Let $G$ be of type $B_2$ with positive roots $\alpha$, $\beta$, $\alpha + \beta$ and $2\alpha + \beta$, where $\beta$ is the long simple root. Let $\omega$ be the fundamental weight dual to $\beta$. In $K(G)$ we have

$$V(\omega) = L(\omega) + L(0)$$

Let $v$ be a highest weight vector of $L(\omega)$. To determine $G_x$, where $x = kv \in L(\omega)$, we notice that $X_{-\alpha} v = 0$, $X_{-\alpha - \beta} v = 0$ (this is because 0 is not a weight of $L(\omega)$), $X_{-2\alpha - \beta} v \neq 0$. This means in the notation of 3.1 that Dist($G_x$) is given by $n_{-\alpha} = 0$, $n_{-\beta} = \infty$, $n_{-\alpha - \beta} = 1$, $n_{-2\alpha - \beta} = 0$.

Example 2. Let $G$ be of type $C_4$ with simple roots and fundamental dominant weights numbered as below

$$\begin{array}{cccc}
\omega_1 & \omega_2 & \omega_3 & \omega_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4
\end{array}$$

In $K(G)$ we have

$$V(\omega_4) = L(\omega_4) + L(\omega_2) + L(0)$$

and furthermore $\dim L(\omega_4) = 16$, while $\dim V(\omega_4) = 42$. Let $v$ be a highest weight vector in $L(\omega_4)$ and $x = kv \in \mathbb{P}(L(\omega_4))$. The stabilizer $G_x$ is given by Dist($G_x$), which is determined by the table

$$\begin{array}{cccc}
\alpha \in R^+ & n_{-\alpha} & \alpha \in R^+ & n_{-\alpha} \\
1000 & \infty & 1100 & \infty \\
1110 & \infty & 0100 & \infty \\
0110 & \infty & 0010 & \infty \\
0001 & 0 & 0011 & 1 \\
0111 & 1 & 1111 & 1 \\
0021 & 0 & 0121 & 1 \\
1121 & 1 & 0221 & 0 \\
1221 & 1 & 2221 & 0
\end{array}$$

The orbit $X = G/G_x$ of $x = [v]$ has dimension 10 and we get an example of a variety lying on the boundary of Hartshorne’s conjecture [8] ($10 = \frac{2}{3}15$)

$$X \hookrightarrow \mathbb{P}(L(\omega_4)) = \mathbb{P}^{15}$$

I do not know whether $X \subseteq \mathbb{P}^{15}$ is a complete intersection. One may check in accordance with Zak’s result [8] on linear normality that the restriction map

$$H^0(\mathbb{P}^{15}, \mathcal{O}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is surjective.

4. **Line bundles**

In this section we classify the line bundles on projective homogeneous $G$-spaces following [6]. When $G$ is simply connected, all line bundles are homogeneous induced by a character on $G_x$. 
4.1. Characters. Let \( X \) be a projective homogeneous \( G \)-space. Suppose that \( G_x \) is the stabilizer group scheme at a closed point \( x \in X(k) \). Let \( B \) be the Borel subgroup contained in \( G_x \). The character lattice \( \chi(B) = \chi(T) \) is
\[
\mathbb{Z} \omega_{\alpha_1} + \cdots + \mathbb{Z} \omega_{\alpha_l}
\]
where \( \omega_\alpha \) is the fundamental dominant weight associated with the simple root \( \alpha \in S \). The restriction homomorphism \( \chi(H) \to \chi(T) \) is injective for any subgroup scheme \( H \supseteq T \). Recall that for a maximal parabolic subgroup \( P(\alpha) \), we have \( \chi(P(\alpha)) = \mathbb{Z} \omega_\alpha \).

**Lemma 1.** Let \( \alpha \in S \) be a simple root. Then
\[
\chi(G_n P(\alpha)) = \mathbb{Z} p^\langle \lambda, \alpha^\vee \rangle
\]

**Proof.** It follows from ([4], II.3.15, Remarks 2), that a character on the \( n \)-th Frobenius kernel \( G_n \) has to be trivial. Now the first isomorphism theorem for groups gives
\[
\chi(G_n P(\alpha)) = \chi(G_n P(\alpha)/G_n) = \chi(P(\alpha)/G_n \cap P(\alpha)) = \chi(P(\alpha)/P(\alpha)_n) = p^n \chi(P(\alpha))
\]

\( \square \)

Let \( x_\alpha : \mathbb{G}_a \to G \) be the root homomorphism associated with \( \alpha \in R \). There is a homomorphism ([4], p. 176)
\[
\varphi_\alpha : \text{SL}_2 \to G
\]
such that
\[
\varphi_\alpha \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) = x_\alpha(a) \quad \text{and} \quad \varphi_\alpha \left( \begin{array}{cc} 1 & 0 \\ a & 1 \end{array} \right) = x_{-\alpha}(a)
\]
and
\[
\alpha^\vee(t) = \varphi_\alpha \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right), \quad t \in \mathbb{G}_m
\]

We are now ready to prove

**Proposition 2.** Let \( G_x = G_{n_1} P(\alpha_1) \cap \cdots \cap G_{n_m} P(\alpha_m) \), where \( \alpha_1, \ldots, \alpha_m \in S \). Then
\[
\chi(G_x) = \mathbb{Z} p^{n_1} \omega_{\alpha_1} + \cdots + \mathbb{Z} p^{n_m} \omega_{\alpha_m}
\]

**Proof.** As \( G_x \subseteq G_{n_i} P(\alpha_i), \ i = 1, \ldots, m \), we get by lemma 1
\[
\chi(G_x) \supseteq \mathbb{Z} p^{n_1} \omega_{\alpha_1} + \cdots + \mathbb{Z} p^{n_m} \omega_{\alpha_m}
\]
Suppose on the other hand that \( \lambda = a_1 \omega_{\alpha_1} + \cdots + a_m \omega_{\alpha_m} \in \chi(G_x) \). Then \( \lambda \circ \varphi_\alpha \) is a character of \( \tilde{B} \subseteq \text{SL}_2(k) \), where for any \( k \)-algebra \( A \)
\[
\tilde{B}(A) = \{ \left( \begin{array}{cc} * & * \\ a & * \end{array} \right) \in \text{SL}_2(A) | a \in A, \ a^n = 0 \}
\]
and \( n = \langle \lambda, \alpha^\vee \rangle \). Therefore we get \( p^{n_i} | \langle \lambda, \alpha^\vee \rangle \), so that \( p^{n_i} | a_i \). \( \square \)
4.2. Ample line bundles. Let $\tilde{P}$ be a parabolic subgroup scheme and $\chi \in X(\tilde{P})$. The total space of the line bundle $L_{\tilde{P}}(\chi)$ induced by $\chi$ is $G \times \tilde{P} \mathbb{A}^1 = G \times \mathbb{A}^1 / \tilde{P}$, where $\tilde{P}$ acts on $G \times \mathbb{A}^1$ through $h.(g, a) = (g h, \chi(h^{-1})a)$. The natural morphism
\[
G \times \tilde{P} \mathbb{A}^1 \to G / \tilde{P}
\]
is equivariant, when $G$ acts on $G \times \tilde{P} \mathbb{A}^1$ through left multiplication. Since $G$ is simply connected ($\text{Pic} \ G = 0$) every line bundle is induced by a character.

Let $P = \tilde{P}_{\text{red}}$. A line bundle $L$ on $G/P$, where $P = P(\alpha_1) \cap \cdots \cap P(\alpha_r)$ for simple roots $\alpha_1, \ldots, \alpha_r \in S$, is induced by a character
\[
\chi \in X(P) = \mathbb{Z}\omega_{\alpha_1} + \cdots + \mathbb{Z}\omega_{\alpha_r} \subseteq X(B) = X(T)
\]
Then $L = L_{P}(\chi)$ is very ample on $G / P$ if and only if $\chi \in X(P)^{++} = \{ \lambda \in X(P) | \langle \lambda, \alpha_{\gamma} \rangle > 0, \ldots, \langle \lambda, \alpha_{\gamma} \rangle > 0 \}$. It is easy to see that $f^{*}L_{\tilde{P}}(\chi) = L_{P}(\chi)$, where $f : G / P \to G / \tilde{P}$ is the natural morphism.

**Theorem 1.** Let $X = G / \tilde{P}$ be a projective homogeneous space such that
\[
\tilde{P} \cong G_{n_1} P(\alpha_1) \cap \cdots \cap G_{n_r} P(\alpha_r)
\]
where $\alpha_1, \ldots, \alpha_r \in S$ are simple roots and $n_1, \ldots, n_r$ integers $\geq 0$. Let $\chi = a_1 p^n_{\omega_{\alpha_1}} + \cdots + a_r p^n_{\omega_{\alpha_r}} \in X(\tilde{P})$. Then $L_{\tilde{P}}(\chi)$ is very ample on $X$ if and only if $L_{P}(\chi) = f^{*}L_{\tilde{P}}(\chi)$ is very ample on $G / P$, where $P = P(\alpha_1) \cap \cdots \cap P(\alpha_r)$ and $f$ is the natural morphism $f : G / P \to G / \tilde{P}$.

**Proof.** Consider the natural diagram
\[
\begin{array}{ccc}
G / P & \longrightarrow & G / P(\alpha_1) \times \cdots \times G / P(\alpha_r) \\
\downarrow f & & \downarrow \\
G / \tilde{P} & \longrightarrow & (G / P(\alpha_1))^{(n_1)} \times \cdots \times (G / P(\alpha_r))^{(n_r)}
\end{array}
\]
By 1.3 and 1.4 it follows that $j$ is a closed immersion. Since $(G / P(\alpha_i))^{(n_i)} = G / G_{n_i} P(\alpha_i) \cong G / P(\alpha_i)$ it follows that ample line bundles on $(G / P(\alpha_i))^{(n_i)}$ are very ample. Since the natural morphism $G / P(\alpha_i) \to G / G_{n_i} P(\alpha_i)$ is a finite surjective morphism it follows ([3], Exercise III.5.7 (d)) that $L_{G_{n_i} P(\alpha_i)}(a_i p^{n_i}_{\omega_{\alpha_i}})$ is very ample if and only if $a_i > 0$. By the Segre embedding we have that
\[
L = L_{G_{n_1} P(\alpha_1)}(a_1 p^{n_1}_{\omega_{\alpha_1}}) \times \cdots \times L_{G_{n_r} P(\alpha_r)}(a_r p^{n_r}_{\omega_{\alpha_r}})
\]
is very ample. Now that $j$ is a closed immersion and $j^{*}L = L_{\tilde{P}}(\chi)$ the result follows. \qed
References

1. H. H. Andersen, *The Frobenius morphism on the cohomology of homogeneous vector bundles on G/B*, Ann. of Math. **112** (1980), 113–121.
2. W. J. Haboush and N. Lauritzen, *Varieties of unseparated flags*, Linear Algebraic Groups and Their Representations (Richard S. Elman, ed.), Contemp. Math., vol. 153, Amer. Math. Soc., 1993, pp. 35–57.
3. Robin Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.
4. J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, 1987.
5. N. Lauritzen, *The Euler characteristic of a homogeneous line bundle*, C. R. Acad. Sci. Paris **315** (1992), 715–718.
6. ___, *Line bundles on projective homogeneous spaces*, Ph.D. thesis, Univ. of Illinois, 1993, Aarhus Universitet, Matematisk Institut, preprint series 1993, no. 12.
7. ___, *Splitting properties of complete homogeneous spaces*, J. Alg. **162** (1993), 178–193.
8. R. Lazarsfeld and Van de Ven, *Topics in the geometry of projective space*, Birkhäuser, 1984.
9. V. Mehta and A. Ramanathan, *Schubert varieties in G/B × G/B*, Comp. Math. **67** (1988), 355–358.
10. D. Mumford, *Varieties defined by quadratic equations*, Questions on algebraic varieties (Rome) (E. Marchionna, ed.), Centro Internazionale Matematica Estivo, Edizioni Cremonese, 1970.
11. M. Raynaud, *Contre-exemple au “Vanishing Theorem” en caractéristique p > 0*, C. P. Ramanujam – A tribute, Tata Institute of Fundamental Research, Springer Verlag, 1978.