Continuous-Time and Event-Triggered Online Optimization for Linear Multi-Agent Systems

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Abstract—This paper studies the decentralized online convex optimization problem for heterogeneous linear multi-agent systems. Agents have access to their time-varying local cost functions related to their own outputs, and there are also time-varying coupling inequality constraints among them. The target of each agent is to minimize the global cost function by selecting appropriate local actions only through communication between neighbors. We design a distributed controller based on the saddle-point method which achieves constant regret bound and sublinear fit bound. Moreover, to diminish the communication overhead, another distributed controller is developed with an event-triggered communication scheme and it is shown that the above bounds are still achieved in the case of discrete communications with no Zeno behavior. A numerical example is given to verify the proposed algorithms.

I. INTRODUCTION

Convex optimization has been broadly studied as a pretty effective method in research fields involving optimization and decision-making, such as automatic control systems [1], communication networks [2], and machine learning [3]. Early convex optimization works were based on fixed cost functions and static constraints. However, in practice, optimization costs and constraints of many problems may be time-varying and unknown in advance [4]. This motivates online convex optimization (OCO) which requires the decision maker to take actions at each instant based on previous information. A generally used performance indicator of online convex optimization is regret, that is, the difference between the cumulative cost of the selected action and that of the best ideal action made when knowing the global information beforehand. Another performance indicator is fit, which measures the degree of violation of static/time-varying inequality constraints. For more details, a recent survey can be referenced [5].

The OCO framework was introduced by [6], where the online gradient descent algorithm was analyzed based on projection. For time-varying convex cost functions and static constraints, the algorithm was proved to achieve $O(\sqrt{T})$ static regret bound. With the increase of data scale and problem complexity in recent years, distributed online convex optimization has also been widely studied in recent years [7]. In the continuous-time setting, the saddle-point algorithm proposed in [8] under constant constraints is shown to achieve sublinear regret and fit bounds. The authors of [9] generalized this result to the problem of time-varying constraints. In the discrete-time setting, the authors of [10] proposed a distributed mirror descent algorithm to solve OCO with feasible set constraints. The authors of [11], [12] used distributed primal-dual algorithms to solve OCO with static independent and coupled inequality constraints. Considering time-varying coupling constraints, the paper [13] proposed a novel distributed online primal-dual dynamic mirror descent algorithm to realize sublinear dynamic regret and constraint violation. A gradient-free distributed bandit online algorithm was proposed in [14], which is applicable to scenarios where it is strenuous to get the gradient information of cost functions. In the presence of aggregative variables in local cost functions, the authors of [15] developed an online distributed gradient tracking algorithm with true or stochastic/noisy gradients and showed the sublinear dynamic regret bound.

In actual physical systems, the implementation of optimization strategies must take into account the complicated dynamics of each agent. Along this line, only a few works have investigated online convex optimization with physical systems in recent years. For continuous-time multi-agent systems with high-order integrators, the authors of [16] used Proportion-Integration control idea to solve the distributed OCO problem. The authors of [17] considered the online convex optimization problem of linear systems, but did not consider any constraints. The online convex optimization problem of linear time-invariant (LTI) system was studied in [18] based on the behavioral system theory, where a proposed data-driven algorithm that does not rely on the model achieves sublinear convergence. However, the above two papers for linear systems only provide centralized algorithms. The distributed setup for online optimization algorithm with linear systems is yet to be studied. Furthermore, the restricted communication bandwidth and energy in practical physical systems limit the frequency of communication between agents [19]. Therefore, discrete communication mechanisms such as periodic [20] and event-triggered communication [21], [22] are used in static distributed optimization algorithms. However, it is rarely considered in distributed online optimization research.

The main contributions of this paper are as follows.

- Compared with the centralized OCO algorithms for linear systems with no constraints [17], [18], this paper studies the distributed online optimization of heterogeneous multi-agent systems with time-varying coupled
inequality constraints for the first time. Agents only rely on the information of themselves and their neighbors to make decisions and achieve constant regret bound and $O(\sqrt{T})$ fit bound. In comparison, most existing algorithms [13], [24] about distributed online optimization with coupled inequality constraints only achieve inferior sublinear regret bounds.

- Compared with current continuous-time online optimization algorithms [16], [8], [9], this paper introduces an event-triggered mechanism to reduce the communication overhead. In the case of discrete communication, the constant regret bound and $O(\sqrt{T})$ fit bound are still achieved.

The rest of the paper is organized as follows. Preliminaries are provided in Section II. In Section III, the heterogeneous multi-agent system under investigation is described mathematically, the online convex optimization problem is defined and some useful lemmas are given. Following that, the control laws with continuous and event-triggered communication are proposed, respectively, and the constant regret bound and sublinear fit bound are established in Section IV. Then, a simulation example is given to verify the validity of the algorithms in Section V. Finally, the conclusion is discussed in Section VI.

II. PRELIMINARIES

A. Notations

Let $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\mathbb{R}^{m \times n}$ be the sets of real numbers, real vectors of dimension $n$, non-negative real vectors of dimension $n$, and real matrices of dimension $m \times n$, respectively. The $n \times n$ identity matrix is denoted by $I_n$. The $n \times 1$ all-one and all-zero column vectors are denoted by $1_n$ and $0_n$, respectively. For a matrix $A \in \mathbb{R}^{m \times n}$, $A^\top$ is its transpose and $\text{diag}(A_1, \ldots, A_n)$ denotes a block diagonal matrix with diagonal blocks of $A_1, \ldots, A_n$. For a vector $x$, $\|x\|$ is its 1-norm, $\|x\|_2$ is its 2-norm, and $\text{col}(x_1, \ldots, x_n)$ is a column vector by stacking vectors $x_1, \ldots, x_n$. $A \otimes B$ represents the Kronecker product of matrices $A$ and $B$. Let $P_X(x)$ be the Euclidean projection of a vector $x \in \mathbb{R}^n$ onto the set $X \subseteq \mathbb{R}^n$, i.e., $P_X(x) = \text{argmin}_{v \in X} \|x - v\|^2$. For simplicity, let $[\cdot]_+$ denote $P_{\mathbb{R}^+_2}(\cdot)$. Define the set-valued sign function $\text{sgn}(\cdot)$ as follows:

\[
\text{sgn}(x) := \partial \|x\|_1 = \begin{cases} -1, & \text{if } x < 0, \\ [-1, 1], & \text{if } x = 0, \\ 1, & \text{if } x > 0. 
\end{cases}
\]

B. Graph Theory

For a system with $N$ agents, its communication network is modeled by an undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is a node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set. If information exchange can occur between any node to any other node in $\mathcal{V}$, then $G$ is called connected.

III. PROBLEM FORMULATION

Consider a multi-agent system consisting of $N$ heterogeneous agents indexed by $i = 1, \ldots, N$, and the $i$th agent has following linear dynamics:

\[
\begin{align*}
\dot{x}_i &= A_i x_i + B_i u_i, \\
y_i &= C_i x_i,
\end{align*}
\]

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^{p_i}$ are the state, input and output variables, respectively. $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$, and $C_i \in \mathbb{R}^{p_i \times n_i}$ are the state, input and output matrices, respectively.

Each agent $i$ has an output set $\mathcal{Y}_i \subseteq \mathbb{R}^{p_i}$ such that the output variable $y_i \in \mathcal{Y}_i$. $f_i(t, \cdot) : \mathbb{R}^{p_i} \rightarrow \mathbb{R}$ and $g_i(t, \cdot) : \mathbb{R}^{p_i} \rightarrow \mathbb{R}^q$ are the private cost and constraint functions for agent $i$. Denote $p := \sum_{i=1}^N p_i$, $\mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N \subseteq \mathbb{R}^p$, $y := \text{col}(y_1, \ldots, y_N) \in \mathbb{R}^p$, and $f(t, y) := \sum_{i=1}^N f_i(t, y_i)$. The goal of this paper is to design a distributed controller $u_i(t)$ for each agent by using only local knowledge in order that all agents jointly minimize the sum of the cost functions over a period of time $[0, T]$ with coupled inequality constraints:

\[
\begin{align*}
\min_{y \in \mathcal{Y}} & \quad \int_0^T f(t, y) \, dt, \\
\text{s.t.} & \quad \sum_{i=1}^N g_i(t, y_i) \leq 0.
\end{align*}
\]

Let $y^* = (y_1^*, \ldots, y_N^*) \in \mathcal{Y}$ denote the optimal solution for problem (2) when the time-varying cost and constraint functions are known beforehand.

To evaluate the performance of designed control laws over time, we define two performance indicators: network regret and network fit. According to the previous definition, $y^*$ is the optimal output when the agents know all the information of network in the period of $[0, T]$. But in reality, agents can only make decisions based on their own and neighbors’ current and previous information. Regret is described as the gap between the cumulative action $\int_0^T f(t, y(t)) \, dt$ incurred by $y(t)$ and the optimal output $y^*$, i.e.,

\[
\mathcal{R}^T := \int_0^T \left( f(t, y(t)) - f(t, y^*) \right) \, dt.
\]

In order to judge the fitness of output trajectories $y(t)$ to the constraints (or in other words, the degree of violation of the constraints), we define fit as the projection of the cumulative constraints onto the nonnegative orthant:

\[
\mathcal{F}^T := \left\| \int_0^T \sum_{i=1}^N g_i(t, y_i) \, dt \right\|_+.
\]

This definition implicitly allows strictly feasible decisions to compensate for violations of constraints at certain times. This is reasonable when the variables can be stored or preserved, such as the average power constraints [23]. By $g_i(t, \cdot) : \mathbb{R}^{p_i} \rightarrow \mathbb{R}$, one can define $g_{i,j}(t, \cdot) : \mathbb{R}^{p_i} \rightarrow \mathbb{R}$ as the $j$th component of $g_i(t, \cdot)$, i.e.,

$g_i(t, \cdot) = \text{col}(g_{i,1}(t, \cdot), \ldots, g_{i,q}(t, \cdot))$. Further, define $F_j^T :=$
\[ j^T 0 \sum_{i=1}^N g_{i,j}(t, y_i) \, dt, j = 1, \ldots, q \] as the \( j \)th component of the constraint integral. It can be easily deduced that
\[
\mathcal{F}^T = \left( \sum_{j=1}^q \left[ F^T_j \right] \right)^{\frac{1}{2}}.
\]

**Assumption 1:** The communication network \( \mathcal{G} \) is undirected and connected.

**Assumption 2:** Each set \( \mathcal{V}_i \) is convex and compact. For \( t \in [0, T] \), functions \( f_i(t, y_i) \) and \( g_i(t, y_i) \) are convex, integrable and bounded on \( \mathcal{V}_i \), i.e., \( |f_i(t, y_i)| \leq K_f \) and \( g_i(t, y_i) \leq K_g \) with constants \( K_f > 0 \) and \( K_g > 0 \).

**Assumption 3:** The set of feasible outputs \( \mathcal{Y}_i(t) := \{ y : y \in \mathcal{Y}, \sum_{i=1}^N g_i(t, y_i) \leq 0, t \in [0, T] \} \) is non-empty.

**Definition 1** ([25]): Let \( \mathcal{S} \subseteq \mathbb{R}^n \) be a closed convex set. Then, for any \( x \in \mathcal{S} \) and \( y \in \mathbb{R}^n \), the projection of \( y \) over set \( \mathcal{S} \) at the point \( x \) can be defined as
\[
\Pi_{\mathcal{S}}[x, y] = \lim_{\xi \to \infty} \frac{P_2(x + \xi y) - x}{\xi}.
\]

**Lemma 1** ([23]): Let \( \mathcal{S} \subseteq \mathbb{R}^n \) be a convex set. If \( x, y \in \mathcal{S} \), one has
\[
(x - y)^T \Pi_{\mathcal{S}}[x, y] \leq (x - y)^T y, \forall y \in \mathbb{R}^n. \tag{5}
\]

**Assumption 4:** \( (A_i, B_i) \) is controllable, and
\[
\text{rank}(C_iB_i) = p_i, i = 1, \ldots, N.
\]

**Lemma 2** ([26]): Under Assumption 4, the matrix equations
\[
C_iB_iK_{\alpha_i} = C_iA_i, \tag{6a}
\]
\[
C_iB_iK_{\beta_i} = I_p, \tag{6b}
\]
have solutions \( K_{\alpha_i}, K_{\beta_i} \).

**Remark 1:** Assumption 2 is reasonable since the output variables in practice, such as voltage, often have a certain range. The cost and constraint functions are not requested to be differentiable, which can be dealt with by using subgradients. The controllability in Assumption 4 is quite standard in dealing with the problem for linear systems.

**IV. MAIN RESULTS**

**A. Continuous Communication**

For agent \( i \), to solve the online convex optimization problem (2), we can formulate the time-varying Lagrangian
\[
\mathcal{H}_i(t, y_i, \mu_i) = f_i(t, y_i) + \mu_i^T g_i(t, y_i) - K_\mu h_i, \tag{7}
\]
where \( \mu_i \in \mathbb{R}^q_+ \) is the local Lagrange multiplier for agent \( i \), \( K_\mu > 0 \) is the preset parameter and \( h_i := \frac{1}{2} \sum_{j=1}^N a_{ij} \| \mu_i - \mu_j \| \) is a metric of \( \mu_i \)’s disagreement [27]. Notice that \( f_i(t, \cdot) \), \( g_i(t, \cdot) \) are convex and \( \mu_i \geq 0 \), hence (7) is convex with respect to \( y_i \). Let us denote by \( \mathcal{H}_i^{\mu_i}(t, y_i, \mu_i) \) a subgradient of \( \mathcal{H}_i \) with respect to \( y_i \), i.e.,
\[
\mathcal{H}_i^{\mu_i}(t, y_i, \mu_i) \in \partial f_i(t, y_i) + \mu_i^T \partial g_i(t, y_i), \tag{8}
\]
for simplicity, define
\[
\mathcal{H}(t, y, \mu) := f(t, y) + \mu^T g(t, y) - K_\mu h(\mu), \tag{10}
\]
where \( y = \text{col}(y_1, \ldots, y_N), f(t, y) = \sum_{i=1}^N f_i(t, y_i), \mu = \text{col}(\mu_1, \ldots, \mu_N), g(t, y) = \text{col}(g_1(t, y_1), \ldots, g_N(t, y_N)) \), and \( h(\mu) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \| \mu_i - \mu_j \|_1 \). It can be easily verified that \( \mathcal{H}(t, y, \mu) = \sum_{i=1}^N \mathcal{H}_i(t, y_i, \mu_i) \).

A controller following modified Arrow-Hurwicz algorithm for the \( i \)th agent is proposed as
\[
u_i = -K_{\alpha_i}x_i + K_{\beta_i} (\Pi_{\mathcal{Y}_i}[y_i, -\varepsilon \mathcal{H}_i^{\mu_i}(t, y_i, \mu_i)]), \tag{11a}
\]
\[
\mu_i = \Pi_{\mathbb{R}^q_+}[\mu_i, \varepsilon \mathcal{H}_i^{\mu_i}(t, y_i, \mu_i)], \tag{11b}
\]
where \( \varepsilon > 0 \) is the step size, \( K_{\alpha_i}, K_{\beta_i} \) are feedback matrices that are the solutions of (6), and the initial value \( \mu_i(0) = 0 \).

Substituting the controller (11) into the system (1), the system dynamics of the \( i \)th agent is
\[
\dot{x}_i = (A_i - B_iK_{\alpha_i})x_i + B_iK_{\beta_i} (\Pi_{\mathcal{Y}_i}[y_i, -\varepsilon \mathcal{H}_i^{\mu_i}(t, y_i, \mu_i)]), \tag{12a}
\]
\[
\dot{\mu}_i = \Pi_{\mathbb{R}^q_+}[\mu_i, \varepsilon \mathcal{H}_i^{\mu_i}(t, y_i, \mu_i)], \tag{12b}
\]
\[
y_i = C_ix_i. \tag{12c}
\]

For the subsequent analysis, consider the following energy function with any \( \tilde{y} \in \mathcal{Y} \) and \( \tilde{\mu} \in \mathbb{R}^N_+ \):
\[
V(\tilde{y}, \tilde{\mu}) = \frac{1}{2} \| y - \tilde{y} \|^2 + \frac{1}{2} \| \mu - \tilde{\mu} \|^2. \tag{13}
\]

The following lemma establishes the relationship between the above energy function and time-varying Lagrangian (7) along the dynamics (12).

**Lemma 3:** If Assumptions 1-4 hold and \( \tilde{\mu} := 1_N \otimes \gamma, \forall \gamma \in \mathbb{R}_+^q \), then for any \( T \geq 0 \), the trajectories of heterogeneous linear multi-agent system (1) with control protocol (11) satisfy
\[
\int_0^T \left( \mathcal{H}(t, y, \tilde{\mu}) - \mathcal{H}(t, \tilde{y}, \tilde{\mu}) \right) dt \leq \frac{V(\tilde{y}, \tilde{\mu})(y(0), 0)}{\varepsilon}. \tag{14}
\]

**Proof:** Calculating the time derivative of the energy function (13) together with (12) yields
\[
\dot{V}(\tilde{y}, \tilde{\mu}) = (\mu - \tilde{\mu})^T \dot{\mu} + (y - \tilde{y})^T \dot{y} = \sum_{i=1}^{N} (\mu_i - \tilde{\mu}_i)^T \Pi_{R_q} \left[ \mu_i, \varepsilon H_{\mu}^i(t, y_i, \mu_i) \right] + \sum_{i=1}^{N} (y_i - \tilde{y}_i)^T \Pi_{y_i} \left[ y_i, -\varepsilon H_{y}^i(t, y_i, \mu_i) \right] \\
\leq \sum_{i=1}^{N} (\mu_i - \tilde{\mu}_i)^T \left( \varepsilon H_{\mu}^i(t, y_i, \mu_i) \right) + \sum_{i=1}^{N} (y_i - \tilde{y}_i)^T \left( -\varepsilon H_{y}^i(t, y_i, \mu_i) \right) \\
\leq \varepsilon \sum_{i=1}^{N} \left( H_{i}(t, y_i, \mu_i) - H_{i}(t, y_i, \tilde{\mu}_i) \right) + \varepsilon \sum_{i=1}^{N} \left( H_{i}(t, \tilde{y}_i, \mu_i) - H_{i}(t, y_i, \mu_i) \right) \\
= \varepsilon \left( H(t, \tilde{y}, \mu) - H(t, y, \tilde{\mu}) \right),
\]

where the second equation holds in view of (6) and (12), the first inequality holds because of (5), and the last inequality holds since the Lagrangian (7) is convex with respect to \( y_i \) and concave with respect to \( \mu_i \).

Integrating (15) from 0 to \( T \) on both sides leads to that

\[
\int_{0}^{T} \left( H(t, y, \mu) - H(t, \tilde{y}, \tilde{\mu}) \right) dt \\
\leq -\frac{1}{\varepsilon} \int_{0}^{T} \dot{V}(\tilde{y}, \tilde{\mu})(y(t), \mu(t)) dt \\
= -\frac{1}{\varepsilon} \left( V(\tilde{y}, \tilde{\mu})(y(0), \mu(0)) \right).
\]

Because the energy function (13) is always nonnegative and \( \mu(0) = 0 \), the conclusion (14) can be obtained.

We now state the main result about the regret and fit bounds of continuous communication controller (11), whose proof and the following proofs are omitted here which will be presented in the full paper.

**Theorem 1:** Suppose that Assumptions 1-4 hold. Then for any \( T \geq 0 \) and \( \varepsilon > 0 \) in control protocol (11), by choosing \( K_\mu \geq NK_y \), the following regret and fit bounds hold:

\[
R^T \leq \frac{\|y(0) - y^*\|^2}{2\varepsilon},
\]

\[
F^T \leq \sqrt{\frac{N\|y(0) - y^*\|}{\varepsilon}} + 2N\sqrt{\frac{K_I}{\varepsilon} \sqrt{T}}.
\]

**Remark 2:** Theorem 1 means that \( R^T = O(1) \) and \( F^T = O(\sqrt{T}) \) under continuous communication, which match the bounds of the centralized algorithm [23]. In comparison, explicit bounds on both the regret and fit with a sublinear growth are obtained in [13], [24] for single-integrator multi-agent systems, i.e., \( R^T = O(T^{\max\{1, 1/\kappa\}}) \), \( F^T = O(T^{\max\{1, 1/\kappa\}}) \) in [13] and \( R^T = O(T^{\max\{1/2, 1/\kappa\}}) \), \( F^T = O(T^{\max\{1/2, 1/\kappa\}}) \) in [24] for \( \kappa \in (0, 1) \). Theorem 1 achieves stricter regret bound than [13], [24] under more complex system dynamics.

**B. Event-triggered Communication**

The above continuous-time control law, which requires each agent to know the real-time Lagrange multipliers of neighbors, may cause excessive communication overhead. In this section, an event-triggered protocol is proposed to avoid continuous communication.

For agent \( i \), suppose that \( t_i^l \) is its \( l \)-th communication instant and \( \{t_i^1, \ldots, t_i^{l+1}, \ldots\} \) is its communication instant sequence. Define \( \tilde{\mu}_i(t) := \mu_i(t_l) \), \( \forall t \in [t_l^l, t_l^{l+1}) \) as the available information of its neighbors and \( e_i := \tilde{\mu}_i(t) - \mu_i(t) \) as the measurement error. It can be known that \( e_i = 0 \) at any instant \( t^l_i \).

In event-triggered communication, agents cannot know the real-time information of their neighbors, which may destroy the stability of the control systems and the optimality of the optimization results. Therefore, the design of controllers and triggering mechanisms is crucial. An event-triggered control law is proposed as

\[
u_i = -K_{\alpha_i}x_i + K_{\beta_i} \left( \Pi_{y_i}[y_i, -\varepsilon H_{y}^i(t, y_i, \mu_i)] \right),
\]

\[
\dot{\mu}_i = \Pi_{R_q}[\mu_i, \varepsilon g_i(t, y_i) - 2\varepsilon K_\mu \sum_{j=1}^{N} a_{ij} \text{sgn}(\tilde{\mu}_i - \tilde{\mu}_j)],
\]

where \( \varepsilon > 0 \) is the step size, \( K_{\alpha_i}, K_{\beta_i} \) are feedback matrices that are solutions of (6), \( a_{ij} \) is the weight corresponding to the edge \((j, i)\), and the initial value \( \mu_i(0) = 0 \). Note that \( 0 \) is chosen for the sign function in (19b) when its argument is zero.

Substituting controller (19) into (1), the system dynamics of the \( i \)-th agent becomes

\[
\dot{x}_i = (A_i - B_i K_{\alpha_i})x_i + B_i K_{\beta_i} \left( \Pi_{y_i}[y_i, -\varepsilon H_{y}^i(t, y_i, \mu_i)] \right),
\]

\[
\dot{\mu}_i = \Pi_{R_q}[\mu_i, \varepsilon g_i(t, y_i) - 2\varepsilon K_\mu \sum_{j=1}^{N} a_{ij} \text{sgn}(\tilde{\mu}_i - \tilde{\mu}_j)],
\]

\[
y_i = C_i x_i.
\]

The communication instant is chosen as

\[
t_i^{l+1} := \inf_{t > t_i^l} \left\{ t \| \varepsilon_i(t) \| \geq \frac{1}{6N\sqrt{q}} \sum_{j=1}^{N} a_{ij}\| \tilde{\mu}_i - \tilde{\mu}_j \| + \frac{\sigma}{3NK_y \sqrt{q}} \right\},
\]

where \( \sigma \) and \( \iota \) are prespecified positive real numbers.

The following lemma is a modification of Lemma 3 under event-triggered communication.

**Lemma 4:** If Assumptions 1-4 hold and \( \tilde{\mu} = 1_N \otimes \gamma, \forall \gamma \in \mathbb{R}^q \), then for any \( T \geq 0 \), the trajectories of heterogeneous linear multi-agent system (1) with control protocol (20) satisfy

\[
\int_{0}^{T} \left( H(t, y, \tilde{\mu}) - H(t, \tilde{y}, \mu) \right) dt \leq \frac{V(\tilde{y}, \tilde{\mu})(y(0), 0)}{\varepsilon} + \frac{\sigma}{\iota}.
\]
We now state the main result about the regret and fit bounds of event-triggered communication controller (20).

**Theorem 2:** Suppose that Assumptions 1-4 hold. Then for any $T \geq 0$ and $\varepsilon > 0$ in control protocol (19) with event triggering condition (21), by choosing $K_\mu \geq N K_\nu$, system (20) excludes the Zeno behavior and the following regret and fit bounds hold:

$$R^T \leq \frac{\|y(0) - y^*\|^2}{2\varepsilon} + \frac{\sigma}{t}, \quad (23)$$

$$F^T \leq \frac{\sqrt{N}}{\varepsilon} \|y(0) - y^*\| + \sqrt{2N\sigma} + 2N\sqrt{\frac{K_\mu}{\varepsilon}\sqrt{T}}. \quad (24)$$

**Remark 3:** Theorem 2 means that $R^T = O(1)$ and $F^T = O(\sqrt{T})$ still hold even under event-triggered communication. The bounds of regret and fit are determined by the communication frequency. Generally speaking, decreasing $\sigma$ and increasing $t$ will achieve smaller bounds on regret and fit, but simultaneously increase the number of communications, which results in a tradeoff between them.

**V. SIMULATION**

Consider a heterogeneous multi-agent system composed of 5 agents described by (1), where $x_i \in \begin{cases} R^2 & i = 1, 2, 3 \\ R^3 & i = 4, 5 \end{cases}$,

$$y_i = \begin{cases} (y_{i,a}, y_{i,b}) \in R^2 & i = 1, 2, 3 \\ (y_{i,a}, y_{i,b}, y_{i,c}) \in R^3 & i = 4, 5 \end{cases}, A_{1,2} = [1, 0; 0, 2], A_3 = [0; 2; -1, 1], A_{4,5} = [2, 1; 0, 0; 1, 1; 1, 0, 2], B_{1,2} = [0, 1; 1, 3], B_{3} = [2, 1; 1, 0], B_{4,5} = [1, 0, 0; 0, 1, 0; 0, 1, 0], C_{1,2} = [2; 0, 0; 1], C_3 = [2; 1; -1, 0]. C_{4,5} = [3, 0, 0; 0, 1; 0, 1, 2].$$

![Fig. 1: Communication network among five agents.](image)

The local objective functions are time-varying quadratic functions as follows:

$$f_1(t, y_1) = (y_{1,a} - \cos t - 1)^2 + (y_{1,b} - \cos 1.5t - 1.5)^2;$$

$$f_2(t, y_2) = 3(y_{2,a} - \cos t - 1)^2 + 3(y_{2,b} - \cos 1.7t - 1)^2;$$

$$f_3(t, y_3) = 2(y_{3,a} - \cos t - 1)^2 + (y_{3,b} - \cos 2t - 1)^2;$$

$$f_4(t, y_4) = 0.5(y_{4,a} - \cos t - 2)^2 + (y_{4,b} - \cos 1.2t - 1)^2 + (y_{4,c} - \cos 1.5t - 2)^2;$$

$$f_5(t, y_5) = 2(y_{5,a} - \cos t - 1)^2 + 3(y_{5,b} - \cos 1.5t - 1)^2 + (y_{5,c} - \cos 2t - 1.5)^2.$$
$K_{α3} = [-1, 1; 2, 0], \quad K_{α4,5} = [2, 1, 0; 0, 1, 1; 1, 0, 2],$
$K_{β1,2} = [-1.5, 1; 1, 0; 0, 1, 0, 0; -0.5, 0.5].$ The initial values $x_i(0)$ are randomly selected in $[-5, 5]$ and $μ(0) = 0$.

Fig. 2 illustrates that the continuous-time control law achieves constant regret bound and sublinear fit bound. The result is in accordance with that established in Theorem 1. Likewise, the similar result can be observed in Fig. 3 under event-triggered communication. Fig. 4 shows the communication moments of five agents with event-triggered control laws, from which one can observe that the communication among five agents is discrete and exhibits no Zeno behavior.

VI. CONCLUSION

In this paper, we studied distributed online convex optimization for heterogeneous linear multi-agent systems with time-varying cost functions and time-varying coupling inequality constraints. A distributed controller was proposed based on the saddle-point method, showing the constant regret bound and sublinear fit bound. In order to reduce the communication cost, an event-triggered communication scheme with no Zeno behavior was developed, which also achieves constant regret bound and sublinear fit bound. We may consider adaptive online algorithms that do not rely on any global information in the future.

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