Bi-Hamiltonian structures for integrable systems on regular time scales

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Abstract

A construction of the bi-Hamiltonian structures for integrable systems on regular time scales is presented. The trace functional on an algebra of $\delta$-pseudo-differential operators, valid on an arbitrary regular time scale, is introduced. The linear Poisson tensors and the related Hamiltonians are derived. The quadratic Poisson tensors is given by the use of the recursion operators of the Lax hierarchies. The theory is illustrated by $\Delta$-differential counterparts of Ablowitz-Kaup-Newell-Segur and Kaup-Broer hierarchies.

1 Introduction

The concept of integrable systems on regular time scales can build bridges between field systems and lattice systems. This concept provides us not only a unified approach to study on discrete intervals with uniform step size (i.e. lattice $\hbar\mathbb{Z}$) and continuous intervals but also an extended approach to study on discrete intervals with non-uniform step size (for instance $q$-discrete numbers $\mathbb{K}_q$) or combination of continuous and discrete intervals.

The approach of time scales allows the unification of such classes of nonlinear evolution equations like field soliton systems \cite{1, 2, 3, 4, 5}, lattice soliton systems \cite{6, 7, 8, 9}, $q$-discrete soliton systems \cite{10, 11, 12, 13} and others. The above approach was initiated in \cite{14} where the Gelfand-Dickey construction was extended. The theory was further developed in \cite{15}, where systematic construction of $(1 + 1)$-dimensional integrable systems on regular time scales was presented, and a very effective tool, classical $R$-matrix formalism, was utilized. The $R$-matrix

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formalism provides a construction of infinite hierarchies of mutually commuting vector fields. In [15], we examined the general classes of admissible Lax operators, presented examples of integrable systems on time scales that can be written in an explicit form. We also explained the source of constraints first observed in [14].

The greatest advantage of the classical $R$-matrix formalism is that it allows the construction of the bi-Hamiltonian structures and conserved quantities. The goal of this work is to present bi-Hamiltonian structures for $\Delta$-differential integrable systems on regular time scales. Thus the main result of this article is the formulation of an appropriate trace form on the algebra of $\delta$-pseudo-differential operators, that is valid on an arbitrary regular time scale and in particular the real case recovers the trace form of pseudo-differential operators [2]. If the appropriate constraints are taken into consideration then the trace form recovers also the one of shift operators [7].

In Section 2, we give a brief review of the concept of time scales, including $\Delta$-derivative and $\Delta$-integrals. In Section 3, we fix the class of the $\Delta$-differential evolution equations under consideration. Besides we define appropriate functionals and their variational derivatives. In Sections 4 and 5, we describe the algebra of $\delta$-pseudo-differential operators, the Lax hierarchies and the constraints that appear naturally between the dynamical fields of admissible finite-field Lax operators. In Section 6, in order to find the bi-Hamiltonian structures, we introduce a trace functional on the algebra of $\delta$-pseudo-differential operators in terms of which we construct the linear Poisson tensors and the related Hamiltonians. The quadratic Poisson tensors are reconstructed in the frame of the recursion operators [15] of the Lax hierarchies. Finally, in Section 7, the theory is illustrated by bi-Hamiltonian formulation of finite-field integrable hierarchies on regular time scales which are $\Delta$-differential counterparts of Ablowitz-Kaup-Newell-Segur (AKNS) and Kaup-Broer hierarchies.

2 Calculus on time scales

A time scale $T$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ [16, 17, 18, 19]. For the definition of the derivative on time scales, we use forward and backward jump operators $\sigma, \rho : T \to T$ defined by

$$\sigma(x) = \inf \{ y \in T : y > x \} \quad \rho(x) = \sup \{ y \in T : y < x \}.\$$

We set in addition $\sigma(\max T) = \max T$ if there exists a finite $\max T$, and $\rho(\min T) = \min T$ if there exists a finite $\min T$. The jump operators $\sigma$ and $\rho$ allow the classification of points on a time scale in the following way: $x$ is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(x) = x$, $\sigma(x) > x$, $\rho(x) = x$, $\rho(x) < x$, $\sigma(x) = \rho(x) = x$ and $\rho(x) < x < \sigma(x)$, respectively. Moreover, we define the graininess function $\mu : T \to T$ as follows

$$\mu(x) = \sigma(x) - x.$$ 

Besides, $T^\kappa$ denotes a set consisting of $T$ except for a possible left-scattered maximal point. Set $x_* = \min T$ if there exists a finite $\min T$, and set $x_* = -\infty$ otherwise. Also set $x^* = \max T$ if there exists a finite $\max T$, and set $x^* = \infty$ otherwise.

Let $f : T \to \mathbb{R}$ be a function on a time scale $T$. Delta derivative of $f$ at $x \in T^\kappa$, denoted by $\Delta f(x)$, is defined as

$$\Delta f(x) = \lim_{s \to x} \frac{f(\sigma(x)) - f(s)}{\sigma(x) - s}, \quad s \in T,$$
provided that the limit exists. A function on a time scale is said to be \(\Delta\)-smooth if it is infinitely \(\Delta\)-differentiable at all points from \(\mathbb{T}^\kappa\).

If functions \(f, g : \mathbb{T} \to \mathbb{R}\) are \(\Delta\)-differentiable, then their product is also \(\Delta\)-differentiable and the following Lebniz-like rule holds

\[
\Delta(fg)(x) = g(x)\Delta f(x) + f(\sigma(x))\Delta g(x) = f(x)\Delta g(x) + g(\sigma(x))\Delta f(x)
\]

\(x \in \mathbb{T}^\kappa.\) \(\tag{1}\)

Besides, if \(f\) is \(\Delta\)-differentiable function, then

\[
f(\sigma(x)) = f(x) + \mu(x)\Delta f(x).\] \(\tag{2}\)

If \(x \in \mathbb{T}\) is right-dense, then \(\mu(x) = 0\) and the relation (2) is trivial.

The shift operator \(E\) is defined by the formula

\[
Ef(x) = f(\sigma(x)) \quad x \in \mathbb{T}.
\]

Moreover the relation (2) implies that

\[E = 1 + \mu\Delta.\] \(\tag{3}\)

In particular, if point \(x \in \mathbb{T}\) lies within some continuous interval, being part of a time scale, or if the time scale \(\mathbb{T} = \mathbb{R}\), then \(\Delta\)-derivative is ordinary derivative with respect to \(x\), i.e. \(\Delta = \partial_x\). If \(x \in \mathbb{T}\) is such that \(\mu(x) \neq 0\), then \(\Delta = \frac{1}{\mu}(E - 1)\). This is the case when \(x\) is an isolated point, for instance \(\mathbb{T} = \mathbb{Z}\) or \(\mathbb{K}_q\).

Every continuous function \(f : \mathbb{T} \to \mathbb{R}\) possesses \(\Delta\)-antiderivative \(F : \mathbb{T} \to \mathbb{R}\) such that \(\Delta F(x) = f(x)\) holds for all \(x \in \mathbb{T}^\kappa\). Thus we define \(\Delta\)-integral from \(a\) to \(b\) of \(f\) by

\[
\int_a^b f(x) \, \Delta x = F(b) - F(a) \quad a, b \in \mathbb{T}.
\]

Notice that, for every continuous function \(f\) we have

\[
\int_x^{\sigma(x)} f(x) \, \Delta x = \mu(x)f(x).
\]

Hence, it is clear that the \(\Delta\)-integral is determined by local properties of a time scale.

In particular, when the points \(a\) and \(b\) lie within continuous interval, being part of a time scale, then (1) is an ordinary Riemann integral. If all the points between \(a\) and \(b\) are isolated, then \(b = \sigma^n(a)\) for some \(n \in \mathbb{Z}_+\) and \(\Delta\)-integral is a sum (this follows immediately from (5)), i.e.

\[
\int_a^b f(x) \, \Delta x = \sum_{i=1}^{n-1} \mu(\sigma^i(a))f(\sigma^i(a)).
\]

For more complicated time scales which are combinations of continuous intervals, isolated points, etc., the integrals can be constructed by appropriate gluing of Riemann integrals and sums.

The integration by parts formula follows from the Leibniz-like rule (1) as

\[
\int_a^b f(x)\Delta g(x) \, \Delta x = f(x)g(x)|_a^b - \int_a^b g(\sigma(x))\Delta f(x) \, \Delta x,
\]

\(\tag{6}\)
where \( f \) and \( g \) are continuous functions. The generalization of (4) to the improper integral is clear. Thus, we define \( \Delta \)-integral over an whole time scale \( T \) by

\[
\int_T f(x) \Delta x := \int_{x_*}^{x^*} f(x) \Delta x = \lim_{x \to x_*} F(x) - \lim_{x \to x_*} F(x)
\]

provided that this integral converges, i.e. the limits exist.

For our purposes, we demand the time scales where the forward jump operator \( \sigma : T \to T \) is invertible. A time scale \( T \) is called regular if \( \sigma(\rho(x)) = x \) and \( \rho(\sigma(x)) = x \) for all \( x \in T \). The first condition implies that \( \sigma \) is 'onto' and the second condition implies that \( \sigma \) is 'one-to-one'. Thus on a regular time scale \( \sigma^{-1}(x) = x \). Actually, a time scale is regular if and only if each point of \( T \setminus \{x_*, x^*\} \) is either two-sided dense or two-sided scattered and the point \( x_* = \min T \) is right dense and the point \( x^* = \max T \) is left-dense [14]. For instance \( T = [-1, 0] \cup \{1/k : k \in \mathbb{N}\} \cup \{k/(k+1) : k \in \mathbb{N}\} \cup [1, 2] \) is a regular time scale.

Let us consider some particular examples of regular time scales:

**The real case,** \( T = \mathbb{R} \). We have \( \sigma(x) = x \) and \( \mu(x) = 0 \) for all \( x \in \mathbb{R} \). In this case \( \Delta \)-derivative and \( \Delta \)-integral are such that

\[
\Delta f(x) = \partial_x f(x) \quad \text{and} \quad \int_{\mathbb{R}} f(x) \Delta x = \int_{-\infty}^{+\infty} f(x) \, dx.
\]

**The lattice case,** \( T = h\mathbb{Z} \). Let \( h \) be a positive parameter. In this case \( \sigma(x) = x + h \) and \( \mu(x) = h \), where \( x \in h\mathbb{Z} \). \( \Delta \)-derivative and \( \Delta \)-integral have the form

\[
\Delta f(x) = \frac{1}{h} (f(x+h) - f(x)) \quad \text{and} \quad \int_{h\mathbb{Z}} f(x) \Delta x = h \sum_{n \in \mathbb{Z}} f(nh).
\]

**The \( q \)-discrete numbers,** \( T = \mathbb{K}_q := q^\mathbb{Z} \cup \{0\} \) \((q > 1)\). For \( x \in \mathbb{K}_q \), one finds that \( \sigma(x) = qx \) and \( \mu(x) = (q - 1)x \). Then

\[
\Delta f(x) = \frac{f(qx) - f(x)}{(q - 1)x}
\]

where \( x \neq 0 \), and

\[
\int_{\mathbb{K}_q} f(x) \Delta x = \sum_{n \in \mathbb{Z}} q^n(q - 1)f(q^n).
\]

3 Delta-differential systems

Consider \( N \)-tuple \( u := (u_1, \ldots, u_N)^T \) of dynamical fields \( u_k : T \to \mathbb{R} \) being \( \Delta \)-smooth functions on a regular time scale \( T \). Let

\[
\mathcal{C} = \{ \Lambda u_k : k = 1, \ldots, N; \Lambda \in S \},
\]

where

\[
S = \{ \Delta^{i_1} \Delta^{j_1} \cdots \Delta^{i_n} \Delta^{j_n} : n \in \mathbb{N}_0, i_1, j_1, \ldots, i_n, j_n \in \mathbb{N} \}.
\]
and $\Delta^\dagger$ is defined by (10). Therefore $S$ is the set of all possible strings of $\Delta$ and $\Delta^\dagger$ operators. Note that $\Delta$ and $\Delta^\dagger$ do not commute.

What we mean by a $\Delta$-differential system, is a system of evolution equations

$$u_t = K[u],$$

(7)

where $t \in \mathbb{R}$ is an evolution parameter (time), $u_t := \frac{\partial u}{\partial t}$ and $K := (K_1, K_2, \ldots)^T$ with $K_i$ being finite order polynomials of elements from $C$, with coefficients that might be time independent ($\Delta$-smooth) functions.

Additionally, we assume that all fields $u$ with their $\Delta$-derivatives are rapidly decaying functions as $x$ goes to $x_s$ or $x^*$. Then, the functionals have the following form

$$F(u) = \int_T f[u] \Delta x,$$

(8)

where $f[u]$ are polynomial functions of $C$. Clearly two densities give equivalent functionals (8) if they differ modulo exact $\Delta$-derivatives. Having defined the class of evolution systems (7) and functionals (8), we further proceed in a standard way, that is we define the duality map, Poisson tensors, etc.

The integration by parts formula (6) leads us to the relation

$$\int_T \Delta(f)g \Delta x = -\int_T f \Delta E^{-1}(g) \Delta x =: \int_T f \Delta^\dagger(g) \Delta x.$$  

(9)

Thus the adjoint of $\Delta$-derivative is given by

$$\Delta^\dagger = -\Delta E^{-1}.$$  

(10)

Note that

$$E^{-1} = 1 + \mu \Delta^\dagger.$$  

Besides, by the use of (9), one finds that

$$(E\mu)^\dagger = \mu(1 + \mu \Delta)^\dagger = \mu - \mu \Delta E^{-1}\mu = \mu - (E - 1)E^{-1}\mu = E^{-1}\mu.$$  

(11)

Consequently, the variational derivative of a functional in the form (8) is defined by

$$\frac{\delta F}{\delta u_k} = \sum_{\Lambda \in S} \Lambda^\dagger \frac{\partial f[u]}{\partial (\Lambda u_k)} \quad k = 1, \ldots, N.$$  

(12)

Notice that $\frac{\delta}{\delta u} \Delta = 0$, therefore the definition of variational derivative (12) is consistent with the definition of functionals (8).

4 $\delta$-pseudo-differential operators

We introduce $\delta$ operator acting on $\Delta$-smooth functions $u : \mathbb{T} \to \mathbb{R}$ by

$$\delta u := \Delta u + E\mu \delta,$$

(13)
which is consistent with the Leibniz-like rule (1). By the use of (13), we have

$$\delta^{-1} u = E^{-1}u\delta^{-1} + \delta^{-1}\Delta u\delta^{-1}$$

$$= E^{-1}u\delta^{-1} + E^{-1}\Delta u\delta^{-2} + E^{-1}\Delta^2 u\delta^{-3} + \ldots .$$

The generalized Leibniz rule for the $\delta$-pseudo-differential operators takes the form

$$\delta^n f = \sum_{k=0}^{\infty} S^n_k f \delta^{n-k} \quad n \in \mathbb{Z}, \quad (14)$$

where

$$S^n_k = \Delta^k E^{n-k} + \ldots + E^{n-k} \Delta^k \quad \text{for} \quad n \geq k \geq 0,$$

is a sum of all possible strings of length $n$, containing exactly $k$ times $\Delta$ and $n-k$ times $E$;

$$S^n_k = E^{-1} \left( \Delta^k E^{n+1} + \ldots + E^{n+1} \Delta^k \right) \quad \text{for} \quad n < 0 \quad \text{and} \quad k \geq 0$$

consists of the factor $E^{-1}$ times the sum of all possible strings of length $k-n-1$, containing exactly $k$ times $\Delta^k$ and $-n-1$ times $E^{-1}$; in all remaining cases $S^n_k = 0$. Besides, we have the recurrence relations

$$S^{n+1}_k = S^n_k E + S^n_{k-1} \Delta \quad \text{for} \quad n \geq 0 \quad (15)$$

and

$$S^{n-1}_k = \sum_{i=0}^{k} S^n_{k-i} E^{-1} \Delta^i \quad \text{for} \quad n < 0. \quad (16)$$

**Lemma 4.1** For all $n \in \mathbb{Z}$, the relation

$$\sum_{k \geq 0} (-\mu)^k S^n_k = (E - \mu \Delta)^n = 1 \quad (17)$$

holds.

The proof is postponed to Appendix.

When $x \in \mathbb{T}$ is a dense point, i.e. $\mu(x) = 0$, then the rule (14) is of the form

$$\delta^n f = \sum_{k=0}^{\infty} \binom{n}{k} \Delta^k f \delta^{n-k} \quad n \in \mathbb{Z}, \quad (18)$$

where $\binom{n}{k}$ is a binomial coefficient such that $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$, and particularly when $x$ is inside of some interval then $\Delta = \partial_x$. Thus, in this case we recover the generalized Leibniz formula for pseudo-differential operators.

For $x \in \mathbb{T}$ such that $\mu(x) \neq 0$ it is more convenient to deal with the operator $\xi := \mu \delta$ instead of $\delta$. By the use of (13), the generating rule yields

$$\xi u = (E - 1)u + Eu \xi,$$

and hence it follows that

$$\xi^n f = \sum_{k=0}^{\infty} \binom{n}{k} (E - 1)^k E^{n-k} f \xi^{n-k} \quad n \in \mathbb{Z}, \quad (19)$$

The important fact is that the operator $A = \sum_i a_i \delta^i$ has a unique $\xi$-representation $A = \sum_i a_i \xi^i$, and nonnegative (negative) order terms with respect to $\delta$ transform into nonnegative (negative) order terms with respect to $\xi$. 
5 Lax hierarchies

We define the associative algebra of $\delta$-pseudo-differential operators

$$
\mathfrak{g} = \mathfrak{g}_{\geq k} \oplus \mathfrak{g}_{< k} = \left\{ \sum_{i \geq k} u_i(x)\delta^i \right\} \oplus \left\{ \sum_{i < k} u_i(x)\delta^i \right\},
$$

equipped with a Lie bracket given by the commutator $[A, B] = AB - BA$, where $A, B \in \mathfrak{g}$. The classical $R$-matrices following from the decomposition of $\mathfrak{g}$ into Lie subalgebras are

$$
R = \frac{1}{2}(P_{\geq k} - P_{< k}) = P_{\geq k} - \frac{1}{2} = \frac{1}{2} - P_{< k},
$$

(20)

where $k = 0$ or $1$ and projections are such that

$$
A_{\geq k} = \sum_{i \geq k} a_i\delta^i \quad \text{for} \quad A = \sum_i a_i\delta^i.
$$

(21)

According to the classical $R$-matrices (20) we have two Lax hierarchies of commuting evolution equations [15]

$$
L_{tn} = \left[ (L^N)^{\frac{k}{\mu}} , L \right] \quad k = 0, 1 \quad n \in \mathbb{N},
$$

(22)

generated, in general, by fractional powers of some Lax operator $L$. The general admissible finite-field Lax operators have the form [15]

$$
L = u_N\delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1\delta + u_0 + \delta^{-1}u_{-1} + \sum_s \psi_s\delta^{-1}\varphi_s,
$$

(23)

where for $k = 0$ the field $u_N$ is a nonzero time-independent field and $u_{-1} = 0$. Analysing (22) one finds that

$$
(-\mu)^{N+k-1}L_{tn} \bigg|_{\delta = \frac{1}{\mu}} = 0,
$$

for details see [15]. Hence, there arises natural constraint between dynamical fields from (23) given by

$$
\sum_{i=-k}^{N+k-1} (-\mu)^{N+k-1-i}u_i + (-\mu)^{N+k}\sum_s \psi_s\varphi_s = a,
$$

(24)

where $a$ is time-independent function (for $k = 1$ nonzero when $\mu = 0$). Notice that, the constraint (24) is compatible with the dynamics of Lax hierarchies (22). Using (24) one can eliminate one dynamical-field. The convenient choice is to eliminate the field $u_{N-1}$ for $k = 0$ and the field $u_N$ for $k = 1$.

It is clear that in the case of $\mathbb{T} = \mathbb{R}$ the Lax hierarchies (22) yield field soliton systems and in particular for Lax operators in the form (23) one recovers the results of [3, 4]. In the case of $\mathbb{T} = \mathbb{Z}$ one obtains lattice soliton systems that are equivalent to the ones considered in [9]. Notice that in [9] the $R$-matrix for $k = 1$ has a slightly different form, this follows from the fact that the construction in [9] is by means of shift operators. Similarly for $\mathbb{T} = \mathbb{K}_q$ the Lax hierarchies lead to $q$-discrete soliton systems, etc.

\[1\] Notice that in the formulae of Theorem 3.5 from [15] there are misprints in degrees of powers of $\mu$. 

6 Hamiltonian structures

Let $A = \sum_i a_i \delta^i$ be a $\delta$-pseudo-differential operator. We define the trace form by

$$\text{Tr} A := - \int_T \frac{1}{\mu} A_{<0}|_{\delta = -\frac{1}{\mu}} \Delta x \equiv \int_T \sum_{i<0} (-\mu)^{-i-1} a_i \Delta x, \quad (25)$$

where $A_{<0}$ is the projection onto negative terms. To show that the substitution $\delta = -\frac{1}{\mu}$ in (25) is well-posed, we state the following proposition.

**Proposition 6.1** Let the $\delta$-differential operators $A$ and $B$ be such that $(AB)_{<0} = AB$, then

$$\int_T \frac{1}{\mu} AB|_{\delta = -\frac{1}{\mu}} \Delta x = \int_T \frac{1}{\mu} A|_{\delta = -\frac{1}{\mu}} B|_{\delta = -\frac{1}{\mu}} \Delta x. \quad (26)$$

**Proof.** It is enough to consider $A = a \delta^m$ and $B = b \delta^n$ such that $m + n < 0$, thus

$$\text{Tr}(AB) = - \int_T \frac{1}{\mu} a \delta^m b \delta^n|_{\delta = -\frac{1}{\mu}} \Delta x = - \int_T \frac{1}{\mu} \sum_{k \geq 0} S_k^m b \delta^{m+n-k}|_{\delta = -\frac{1}{\mu}} \Delta x$$

$$= \int_T a \sum_{k \geq 0} (-\mu)^{k-m-n-1} S_k^m b \Delta x = \int_T ab(-\mu)^{-m-n-1} \Delta x,$$

where the equality (17) is used. Consequently (26) follows. \qed

Roughly speaking, the above proposition implies that the multiplication operation in the algebra $\mathfrak{g}$ of $\delta$-pseudo-differential operators commutes with the substitution $\delta = -\frac{1}{\mu}$ given in the trace form (25).

Note that, in particular for the points $x \in \mathbb{T}$ such that $\mu(x) = 0$, the trace form (25) turns out to be

$$\text{Tr} A = \int_T a_{-1} \Delta x,$$

where $A = \sum_i a_i \delta^i$. Thus when $T = \mathbb{R}$, we recover the trace formula for the algebra of pseudo-differential operators [2]. For the case $\mu(x) \neq 0$, the trace form (25) within the algebra of $\xi$-operators is given by

$$\text{Tr} A := - \int_T \frac{1}{\mu} A_{<0}|_{\xi = -1} \Delta x \equiv - \int_T \frac{1}{\mu} \sum_{i<0} (-1)^i a_i' \Delta x,$$

where $A = \sum_i a_i' \xi^i$.

**Theorem 6.2** The inner product on $\mathfrak{g}$ defined by the bilinear map

$$(\cdot, \cdot)_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K} \quad (A, B)_\mathfrak{g} := \text{Tr}(AB), \quad (27)$$

in terms of the trace (25), is nondegenerate, symmetric and ad-invariant, i.e.

$$(A, [B, C])_\mathfrak{g} + ([B, A], C)_\mathfrak{g} = 0.$$
Proof. The nondegeneracy of \((27)\) follows immediately from the definition of the trace.

In order to show that \((25)\) is symmetric, it is enough to consider the monomials \(A = a\delta^m\) and \(B = b\delta^n\). Then, if \(m, n \geq 0\), we have \(\text{Tr}(AB) = \text{Tr}(BA) = 0\). If \(m, n < 0\) the symmetricity immediately follows from \((26)\). Thus, it remains to prove the case when one of the operators \(A\) and \(B\) is of positive order and the other one is of negative order.

Without loss of generality, let \(m > 0\) and \(n < 0\). We consider the cases \(\mu(x) = 0\) and \(\mu(x) \neq 0\), separately.

For \(\mu(x) = 0\), we have

\[
\text{Tr}(AB) = \text{Tr}(a\delta^m b\delta^n) = \text{Tr}\left(\sum_{k=0}^{m} (m_k) a\Delta^k b\delta^{m-n-k}\right) = \int_{T} \left(\frac{m}{m+n+1}\right) a\Delta^{m+n+1} b \Delta x.
\]

The converse formula for \((18)\) has the form

\[
u\delta^n = \sum_{k=0}^{\infty} \delta^{n-k} \left(\begin{array}{c} n \\ k \end{array}\right) \Delta^k u.
\]  

Hence

\[
\text{Tr}(BA) = \text{Tr}(b\delta^n a\delta^m) = \text{Tr}\left(\sum_{k=0}^{m} (m_k) b\delta^{m-n-k} \Delta^k a\right) = \int_{T} \left(\frac{m}{m+n+1}\right) b\Delta^{m+n+1} a \Delta x
\]

\[= \int_{T} \left(\frac{m}{m+n+1}\right) a\Delta^{m+n+1} b \Delta x = \text{Tr}(AB),\]

where we make use of \((9)\).

For \(\mu(x) \neq 0\), we pass to the calculations in terms of \(\xi\)-pseudo-differential operators. Let \(A = a\xi^m\) and \(B = b\xi^n\) with \(m > 0\) and \(n < 0\). We have

\[
\text{Tr}(AB) = \text{Tr}(a\xi^m b\xi^n) = \text{Tr}\left(\sum_{k=0}^{m} (m_k) a(E-1)^k E^{m-n-k} \xi^{m+n-k}\right)
\]

\[= -\int_{T} \frac{1}{\mu} \sum_{k=m+n+1}^{m} (m_k) (-1)^{m+n-k} a(E-1)^k E^{m-k} b \Delta x.
\]

The converse formula for \((19)\) is given by

\[
f\xi^n = \sum_{k=0}^{\infty} \xi^{n-k} \left(\begin{array}{c} n \\ k \end{array}\right) (E^{-1} - 1)^k E^{k-n} f
\]

Let \(f(E)\) be a polynomial function of \(E\). Then by \((11)\), it follows that

\[
\left(\frac{1}{\mu} f(E)\right)^\dagger = \frac{1}{\mu} f(E^{-1}).
\]
Therefore

\[ \text{Tr}(BA) = \text{Tr}(b\xi^m a\xi^n) = \text{Tr} \left( \sum_{k=0}^{m} \binom{m}{k} b\xi^{m+k-n} (E^{-1} - 1)^k E^{k-m} a \right) \]

\[ = - \int_{T} \frac{1}{\mu} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-n-k} b (E^{-1} - 1)^k E^{k-m} a \Delta x \]

\[ = - \int_{T} \frac{1}{\mu} \sum_{k=m+n+1}^{m} \binom{m}{k} (-1)^{m+n-k} a (E-1)^k E^{m-k} \Delta x = \text{Tr}(AB). \]

The symmetricity of the trace functional on the algebra of \( \xi \)-pseudo-differential operators implies the symmetricity of the trace functional on the algebra of \( \delta \)-pseudo-differential operators for \( \mu(x) \neq 0 \).

Hence, the inner product (27) is symmetric. Finally, the adjoint invariance of (27) follows from the fact that the inner product (27) is symmetric and the multiplication operation defined on the algebra \( g \) of \( \delta \)-pseudo-differential operators is associative. \( \square \)

For the next proposition, we have to consider \((1 + \mu \delta)^{-1}\) and its expansion such that it is valid for all points of \( T \) including the case \( \mu = 0 \). Besides, we assume that the expansion of \((1 + \mu \delta)^{-1}\) is given by nonnegative order terms in \( \delta \)-pseudo-differential operators. We derive the following expansion

\[ (1 + \mu \delta)^{-1} := \sum_{k=0}^{\infty} (-\delta)^k (\mu^k + \Delta \mu^{k+1}) \equiv \sum_{k=0}^{\infty} (-\delta)^k \frac{E \mu^{k+1}}{\mu}, \quad (29) \]

which can be verified by multiplying both sides of the expression (29) with \((1 + \mu \delta)\) from right-hand side. Hence

\[ (1 + \mu \delta)^{-1}(1 + \mu \delta) = \sum_{k=0}^{\infty} (-\delta)^k \frac{E \mu^{k+1}}{\mu} + \sum_{k=0}^{\infty} (-\delta)^k E \mu^{k+1} \delta \]

\[ = \sum_{k=0}^{\infty} (-\delta)^k \frac{E \mu^{k+1}}{\mu} - \sum_{k=0}^{\infty} (-\delta)^{k+1} \mu^{k+1} - \sum_{k=0}^{\infty} (-\delta)^k \Delta \mu^{k+1} \]

\[ = \frac{E \mu}{\mu} - \Delta \mu + \sum_{k=1}^{\infty} (-\delta)^k \left( \frac{E \mu^{k+1}}{\mu} - \mu^k - \Delta \mu^{k+1} \right) = 1, \]

where (3) and the converse formula (28) are used.

**Proposition 6.3** The alternative formula of the trace form (25) is given by

\[ \text{Tr} A = \int_{T} \frac{E^{-1} \mu}{\mu} \text{res} \left( A(1 + \mu \delta)^{-1} \right) \Delta x, \quad (30) \]

where

\[ \text{res} A := a_{-1} \quad \text{for} \quad A = \sum_{i} a_i \delta^i. \]
Proof. We first calculate the residue. Thus

$$\text{res} \left( (A (1 + \mu \delta)^{-1}) \right) = \text{res} \left( \sum_{k=0}^{\infty} \sum_{i} (-1)^k a_i \delta^{-k} \left( \frac{(E \mu)^{k+1}}{\mu} \right) \right)$$

$$= \text{res} \left( \sum_{i<0} (-1)^{-i-1} a_i \delta^{-1} \left( \frac{(E \mu)^{-i}}{\mu} + \ldots \right) \right) = \text{res} \left( \sum_{i<0} (-1)^{-i-1} a_i \frac{\mu^{-i} \delta^{-1} + \ldots}{E^{-1} \mu} \right)$$

$$= - \sum_{i<0} \frac{(-\mu)^{-i}}{E^{-1} \mu} a_i.$$

Substituting the residue into the alternative trace form (30), we obtain

$$\text{Tr} A = \int_{\mathbb{T}} \frac{E^{-1} \mu}{\mu} \text{res} \left( (A (1 + \mu \delta)^{-1}) \right) \Delta x = \int_{\mathbb{T}} \sum_{i<0} (-\mu)^{-i-1} a_i \Delta x.$$

Hence the trace forms (25) and (30) are equivalent. □

Notice that, the definition of the trace form given by (30) is very similar to the trace formula introduced in [14]. However the trace form in [14] is valid only either on $\mathbb{T} = \mathbb{R}$ or on regular-discrete time scales, i.e. on such regular time scales where all points are isolated and $\mu \neq 0$. One of the main contribution of this article is to generalize the trace form given in [14]. The trace form (25) is valid for an arbitrary regular time scale. In particular when $\mathbb{T} = \mathbb{R}$, the trace form (25) implies the standard trace formula for the algebra of pseudo-differential operators. Thus, for the use of the alternative trace form (30), we have to choose that $(1 + \mu \delta)^{-1}$ expands into nonnegative order $\delta$-operators (29) and the expansion is also valid when $\mu = 0$.

Observe that, one can define alternatively the following trace form

$$\text{Tr}' A := \int_{\mathbb{T}} \frac{1}{\mu} A_{\geq 0}|_{\delta^{-1} = \frac{1}{\mu}} \Delta x \equiv - \int_{\mathbb{T}} \sum_{i \geq 0} (-\mu)^{-i-1} a_i \Delta x,$$

valid on regular-discrete time scales, that is for $\mu \neq 0$. Choosing the expansion of $(1 + \mu \delta)^{-1}$ into negative order terms,

$$(1 + \mu \delta)^{-1} := - \sum_{k=1}^{\infty} (-\delta)^{-k} \frac{1}{\mu E \mu^{k-1}},$$

and $\mu \neq 0$, the formula (30) yields (31), i.e. one recovers from (30) the trace formula of shift operators, as in [14].

The shift operator can be introduced by the relation: $\mathcal{E} = 1 + \mu \delta$, then $\mathcal{E}^m u = \mathcal{E}^m u \mathcal{E}^m$ for $m \in \mathbb{Z}$. The expansion of the operator $A$ by means of shift operators $\mathcal{E}$, i.e. $A = \sum a_i \mathcal{E}^i$ (we assume that $\delta^{-1}$ expands into negative order terms of shift operator $\mathcal{E}$) allows us to obtain from (31) the standard trace form of the algebra of shift operators

$$\text{Tr}' A := \int_{\mathbb{T}} \frac{1}{\mu} a'_0 \Delta x.$$

Although the traces (25) and (31) are not equivalent in general, they are closely related to each other on regular-discrete time scales. To be more precise, on regular-discrete time
scales when applied to the constrained operators such that \( A|_{\delta = -\frac{1}{\mu}} = \text{const} \), the traces (25) and (31) are equal up to a constant, as

\[
A_{\geq 0}|_{\delta = -\frac{1}{\mu}} = -A_{<0}|_{\delta = -\frac{1}{\mu}} + \text{const}.
\]

Notice that, this is the case of the Lax operators (23) for which the constraints (24) are taken into consideration. Thus it is clear that in the case of lattice time scale, i.e. \( T = \mathbb{Z} \), the Hamiltonians defined below, as well as the following scheme, are equivalent to the one from [9]. By similar observations one also finds that for \( T = \mathbb{K}_q \) one recovers from (30) and (31) the trace form of \( q \)-discrete numbers, see the appendix of [12].

To sum up, the trace form (25) is valid on arbitrary regular time scales and in particular for \( T = \mathbb{R} \) give the standard form of pseudo-differential operators. Besides, if the appropriate constraints are taken into consideration, (25) also recovers the trace forms for \( T = \mathbb{Z} \) of ‘lattice’ shift operators and for \( T = \mathbb{K}_q \) of \( q \)-discrete numbers.

In order to define the Hamiltonian structures for (22) we first need to find adjoint of \( R \)-matrices (20), i.e \( R^\dagger \), such that 

\[
(A, RB)_{g} = (R^\dagger A, B)_{g}.
\]

Using the alternative definition of the trace form (30) one finds that

\[
R^\dagger = P^\dagger_{\geq k} - \frac{1}{2} \quad k = 0, 1,
\]

where

\[
P_{\geq k}^\dagger A = (A(1 + \mu \delta)^{-1})_{< -k} (1 + \mu \delta)
\]

and the projections are such that

\[
B_{< -k} = \sum_{i< -k} \delta^i b_i \quad \text{for} \quad B = \sum_i \delta^i b_i.
\]

Notice that the above projections are defined on the operators given in a different form than in (21).

The existence of the well-defined inner product (27) allows us to identify \( g \) with its dual \( g^* \). Let \( \mathcal{F}(g \cong g^*) \) be the space of smooth function on \( g \) consisting of functionals (8). The linear Poisson tensor has the form [20, 5]

\[
\pi_0 dH = [RdH, L] + R^\dagger [dH, L]
\]

\[
= [L, dH_{<k}] + ([dH, L] (1 + \mu \delta)^{-1})_{< -k} (1 + \mu \delta) \quad k = 0, 1,
\]

where \( H \in \mathcal{F}(g) \).

We do not present explicit form of the differentials \( dH \) with respect to Lax operators (23) as it would be cumbersome, but we explain how to construct them. We postulate that

\[
dH = \sum_{i=1}^{n} \delta^{i-N-k} \gamma_i,
\]

where \( n \) is the number of independent dynamical fields in (23) (the number of the rest of the dynamical fields after taking the constraint (24) into consideration). Thus, we look forward to express \( \gamma_i \)’s in terms of dynamical fields of (23) and their variational derivatives by the use of the assumption

\[
(dH, L_t)_{g} = \int_T \left( \sum_{i-k}^{N-k} \frac{\delta H}{\delta u_i} (u_i)_{t} + \sum_{s} \left( \frac{\delta H}{\delta \psi_s} (\psi_s)_{t} + \frac{\delta H}{\delta \phi_s} (\psi_s)_{t} \right) \right) \Delta x.
\]
Some useful formulae to calculate linear Poisson tensors $\pi_0$ for the examples are derived as

\[
P_{\geq 0}^\dagger(a\delta^{-1}b) = a\delta^{-1}b + \mu ab
\]
\[
P_{\geq 1}^\dagger(a\delta^{-1}b) = a\delta^{-1}b - \delta^{-1}ab
\]
\[
P_{\geq 1}^\dagger(\delta^{-1}a\delta^{-1}b) = \delta^{-1}a\delta^{-1}b + \delta^{-1}\mu ab.
\]

The case of the quadratic Poisson tensor $\pi_1$ is more delicate and for the construction there appears additional conditions on $R$ and $R^\dagger$ [9, 7, 20, 5]. The form of (32) with (33) does not allow us to proceed in a standard way anymore, thus we omit the construction of the quadratic Poisson tensor from the $R$-matrix scheme. In article [15], we have constructed recursion operators $\Phi$ for the Lax hierarchies (22), such that

\[
\Phi L_{tn} = L_{tn+N}.
\]

Thus, as we know $\pi_0$, the quadratic Poisson tensor $\pi_1$ can be reconstructed alternatively by

\[
\pi_1 = \Phi \pi_0.
\]

The recursion operator $\Phi$ is hereditary at least on the vector space spanned by the symmetries from the related Lax hierarchy. Therefore the Poisson tensors $\pi_0$ and $\pi_1$ are compatible [21, 5].

Hence, the Lax hierarchies (22) have bi-Hamiltonian structure

\[
L_{tn} = \pi_0 dH_n = \pi_1 dH_{n-N},
\]

where the related Hamiltonians are given by

\[
H_n(L) = \frac{N}{n + N} \text{Tr} \left( L^{\frac{n}{N}} \right).
\]

They are such that $dH_n = L^{\frac{n}{N}}$.

### 7 Examples

**$\Delta$-differential AKNS, $k = 0$.** The Lax operator (36) for $N = 1$, with the constraint (24) ($a = 0$) is of the form

\[
L = \delta + \mu \psi \varphi + \psi \delta^{-1} \varphi.
\]

The first and the second flows from the Lax hierarchy (22) are

\[
\psi_{t_1} = \mu \psi^2 \varphi + \Delta \psi,
\]
\[
\varphi_{t_1} = -\mu \varphi^2 \psi - \Delta \varphi.
\]

and

\[
\psi_{t_2} = \mu^2 \psi^3 \varphi^2 + 2 \psi^2 \varphi + \Delta^2 \psi + \Delta (\mu \psi^2 \varphi) + 2 \mu \psi \varphi \Delta \psi + \mu \psi^2 \Delta \varphi
\]
\[
\varphi_{t_2} = -\mu^2 \psi^3 \varphi^2 - 2 \psi^2 \varphi^2 - \Delta^2 \varphi - \Delta (\mu \psi^2 \varphi) - \mu \varphi^2 \Delta \psi - 2 \mu \psi \varphi \Delta \varphi.
\]

For Lax operator (36) the differential of an functional $H$, such that (35) is valid, is given by

\[
dH = \frac{1}{\varphi} \delta H \frac{1}{\psi} \Delta \left( \frac{1}{\varphi} \right) \Delta^{-1} A - \delta \frac{1}{\psi \varphi + \mu \psi \Delta \varphi} \Delta^{-1} A,
\]
where
\[ A = \psi \frac{\delta H}{\delta \psi} - \varphi \frac{\delta H}{\delta \varphi} \]
and \( \Delta^{-1} \) is a formal inverse of \( \Delta \). Then, one finds the linear Poisson tensor
\[ \pi_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
The recursion operator constructed in [15] has the form
\[ \Phi = \left( \Delta + 2 \mu \psi \varphi + 2 \psi \Delta^{-1} \varphi \right) \left( -\mu \varphi^2 - 2 \varphi \Delta^{-1} \varphi \right) + \left( -\Delta^{-1} + 2 \varphi \Delta^{-1} \varphi \right) \left( -\mu \varphi^2 - 2 \varphi \Delta^{-1} \varphi \right) \cdot \]
Hence, the quadratic Poisson tensor is
\[ \pi_1 = \Phi \pi_0 = \begin{pmatrix} -\mu \psi^2 - 2 \psi \Delta^{-1} \varphi & \Delta + 2 \mu \psi \varphi + 2 \psi \Delta^{-1} \varphi \\ -\Delta^{-1} + 2 \varphi \Delta^{-1} \varphi & -\mu \varphi^2 - 2 \varphi \Delta^{-1} \varphi \end{pmatrix}. \tag{39} \]
The skew-symmetricity of (39) follows from (3) and (10). The first three Hamiltonians are
\[
H_0 = \int_T \psi \varphi \Delta x \\
H_1 = \int_T \left( \frac{1}{2} \mu \psi^2 \varphi^2 + \varphi \Delta \psi \right) \Delta x \\
H_2 = \int_T \left( \frac{1}{3} \mu^2 \psi^3 \varphi^3 + \psi^2 \varphi^2 + \varphi \Delta^2 \psi + \mu \psi \varphi^2 \Delta \psi + \mu \psi^2 \varphi \Delta \varphi \right) \Delta x \\
\vdots
\]
Particularly, when \( T = \mathbb{R} \) the above bi-Hamiltonian hierarchy is exactly the bi-Hamiltonian field soliton AKNS hierarchy [4]. In this case the first nontrivial flow is the second one (38), i.e. the AKNS system. When \( T = \mathbb{Z} \) and \( T = \mathbb{K}_q \) we get the lattice and \( q \)-discrete counterparts of the AKNS hierarchy where the first nontrivial flow is (37). Besides, for \( T = \mathbb{Z} \) the system (37), together with its bi-Hamiltonian structure, is equivalent to the system considered in [9].

**\( \Delta \)-differential Kaup-Broer, \( k = 1 \).** Consider the following Lax operator with the constraint (24) \((a = 1)\)
\[ L = (1 + \mu v - \mu^2 w) \delta + v + \delta^{-1} w. \]
The first and the second flows are
\[
v_{t_1} = (1 + \mu v - \mu^2 w) \Delta v - \mu \Delta^{\dagger} (w + \mu vv - \mu^2 w^2), \\
w_{t_1} = -\Delta^{\dagger} (w + \mu vv - \mu^2 w^2)
\]
and
\[
v_{t_2} = u \Delta \left( v^2 + 2 \tilde{u} w + \tilde{u} \Delta v + \mu \Delta^{\dagger} (\tilde{u} w) \right) - \mu \Delta^{\dagger} \left( 2 \tilde{u} w + \mu \tilde{u} w \Delta v + u \Delta^{\dagger} (\tilde{u} w) \right), \\
w_{t_2} = -\Delta^{\dagger} \left( 2 \tilde{u} w + \mu \tilde{u} w \Delta v + \tilde{u} \Delta^{\dagger} (\tilde{u} w) \right),
\]
where
\[ \tilde{u} := 1 + \mu v - \mu^2 w. \]
The differentials are given in the form
\[ dH = \delta^{-1} \frac{\delta H}{\delta v} + \frac{\delta H}{\delta w} + \mu \frac{\delta H}{\delta v}. \]

Thus, the linear Poisson tensor \([31]\) is
\[ \pi_0 = \begin{pmatrix} \bar{u} \Delta \mu - \mu \Delta^\dagger \bar{u} & \bar{u} \Delta \\ -\Delta^\dagger \bar{u} & 0 \end{pmatrix}. \]

The recursion operator has the form \([15]\)
\[ \Phi = \begin{pmatrix} w + \bar{u} \Delta + R & \mu v - \mu w + (2 + \mu \Delta^\dagger) \bar{u} - R \mu \\
-\Delta^\dagger \bar{u} w \Delta^{-1} \bar{u}^{-1} & \Delta^\dagger \bar{u} + v - \mu w \Delta^\dagger \bar{u} w \Delta^{-1} \mu \bar{u}^{-1} \end{pmatrix}, \]
where
\[ R = \bar{u} \Delta v \Delta^{-1} \bar{u}^{-1} - \mu \Delta^\dagger \bar{u} w \Delta^{-1} \bar{u}^{-1}. \]

Hence
\[ \pi_1 = \Phi \pi_0 = \begin{pmatrix} \pi_{vv} & \bar{u} \Delta v + \bar{u} \Delta \bar{u} \Delta + \mu \bar{u} w \Delta - \mu \Delta^\dagger \bar{u} w \\
-\bar{u} \Delta^\dagger \bar{u} \Delta + \mu \bar{u} w \Delta - \mu \Delta^\dagger \bar{u} \Delta^\dagger \bar{u} + \mu \bar{u} w \Delta \mu - \mu \Delta^\dagger \mu \bar{u} w. \end{pmatrix}, \]
where
\[ \pi_{vv} = \bar{u} \Delta \mu v - \mu v \Delta^\dagger \bar{u} + \bar{u} \Delta \bar{u} - \bar{u} \Delta^\dagger \bar{u} + \bar{u} \Delta \bar{u} \Delta \mu - \mu \Delta^\dagger \bar{u} \Delta^\dagger \bar{u} + \mu \bar{u} w \Delta \mu - \mu \Delta^\dagger \mu \bar{u} w. \]

The Hamiltonians are
\[
H_0 = \int_{\mathbb{T}} w \Delta x \\
H_1 = \int_{\mathbb{T}} \left( vw - \frac{1}{2} \mu w^2 \right) \Delta x \\
H_2 = \int_{\mathbb{T}} \left( w^2 + v^2 w + (w + \mu vw - \mu^2 w^2) \Delta v - \frac{2}{3} \mu^2 w^3 \right) \Delta x \\
\vdots.
\]

When \( \mathbb{T} = \mathbb{R} \) the above construction recovers the field Kaup-Broer hierarchy with its bi-Hamiltonian structure \([3]\). As previously, the Kaup-Broer system is given only by the second flow \([11]\). In the lattice case, i.e. of \( \mathbb{T} = \mathbb{Z} \), the above bi-Hamiltonian hierarchy is equivalent to the relativistic Toda hierarchy considered in \([9]\) and, \([10]\) is equivalent to the relativistic Toda system.

### 8 Conclusions

We have presented a unified theory of the construction of the bi-Hamiltonian nonlinear evolution hierarchies such as field, lattice and \(q\)-discrete soliton hierarchies. Actually, we took advantage from the theory of time scales. Therefore, one can also consider the construction of soliton systems with spatial variable belonging to the spaces being partially continuous and discrete. This might be interesting from the point of view of applications. There are also
other approaches generalizing and unifying theory of soliton systems, presented in [22] and [23].

On the other hand, making use of the regular-discrete time scales only, the theory from the article can be considered as a discretization scheme of field soliton systems. In some special cases, introducing appropriately deformation parameter to some regular-discrete time scales, one can consider the quasi-classical limit of discrete soliton systems yielding dispersive field soliton equations. In particular for $T = h\mathbb{Z}$ the quasi-classical limit is given by $h \to 0$ and for $T = \mathbb{K}_q$ by $q \to 1$, see [15].

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**Appendix**

We verify the Lemma 4.1 by considering the positive and negative cases of $n$ separately by the use of induction. Note that $E - \mu \Delta = E^{-1} - \mu \Delta^\dagger = 1$.

Let $n \geq 0$. Assume that (17) holds for positive $n$. Then

$$(E - \mu \Delta)^{n+1} = (E - \mu \Delta)^n (E - \mu \Delta) = (E - \mu \Delta)^n E - \mu (E - \mu \Delta)^n \Delta$$

$$= \sum_{k=0}^{n} (-\mu)^k S_k^n E + \sum_{k=0}^{n} (-\mu)^{k+1} S_k^n \Delta = \sum_{k=0}^{n+1} (-\mu)^k S_k^n E + \sum_{k=0}^{n+1} (-\mu)^k S_{k-1}^n \Delta$$

$$= \sum_{k=0}^{n+1} (-\mu)^k (S_k^n E + S_{k-1}^n \Delta) = \sum_{k=0}^{n+1} (-\mu)^k S_{k+1}^n$$

where we used the fact that $S_{n+1}^n = S_{-1}^n = 0$ and the recurrence relation (15).

Let $n < 0$. First we show (15) for $n = -1$. Thus, using the recursive substitution we deduce

$$(E - \mu \Delta)^{-1} = (E^{-1} - \mu \Delta^\dagger) (E - \mu \Delta)^{-1} = E^{-1} - \mu (E - \mu \Delta)^{-1} \Delta^\dagger$$

$$= E^{-1} - \mu (E^{-1} - \mu (E - \mu \Delta)^{-1} \Delta^\dagger) \Delta^\dagger = E^{-1} - \mu E^{-1} \Delta^\dagger + \mu^2 (E - \mu \Delta)^{-1} \Delta^{\dagger 2}$$

$$= E^{-1} - \mu E^{-1} \Delta^\dagger + \mu^2 E^{-1} \Delta^{\dagger 2} - \mu^3 E^{-1} \Delta^{\dagger 3} + \ldots$$

$$= \sum_{k=0}^{\infty} (-\mu)^k E^{-1} \Delta^{\dagger k} = \sum_{k=0}^{\infty} (-\mu)^k S_k^{-1}.$$
Assume that \( (17) \) holds for negative \( n \). Then

\[
(E - \mu \Delta)^{n-1} = (E - \mu \Delta)^n (E - \mu \Delta)^{-1} = \sum_{k=0}^{\infty} (-\mu)^k S_k^n \sum_{i=0}^{\infty} (-\mu)^i E^{-1} \Delta^{i}
\]

\[
= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-\mu)^{k+i} S_k^n E^{-1} \Delta^{i}
= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-\mu)^k S_{k-i}^n E^{-1} \Delta^{i}
\]

\[
= \sum_{k=0}^{\infty} (-\mu)^k \sum_{i=0}^{k} S_{k-i}^n E^{-1} \Delta^{i}
= \sum_{k=0}^{\infty} (-\mu)^k S_k^{n-1},
\]

where we used \( (17) \) for \( n = -1 \) and the recurrence relation \( (16) \). Hence \( (17) \) holds for \( n - 1 \), which finishes the proof.

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