On the energy of a null cone

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Abstract
We derive a formula for the Bondi mass aspect in terms of the asymptotic data of the Bondi–Sachs metric in the affine gauge. We prove the positivity of the total energy of a regular null cone in agreement with a recent result of Chruściel and Paetz.

Keywords: the Bondi mass aspect, the Trautman–Bondi mass, null cone
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1. Introduction

Schoen and Yau [1, 2] proved that the ADM energy in general relativity is positive under reasonable assumptions about the energy–momentum tensor and metric tensor. The proof was simplified by Witten [3] and his work was amended by Nester [4]. Using Witten’s method, a similar result was obtained for the Trautman–Bondi energy $E$ of a null surface [5–8]. Recently, Chruściel and Paetz [9] presented a proof of the positivity of the Trautman–Bondi energy of a regular null cone in an asymptotically flat spacetime. This property was suggested earlier by the present author and Korbicz in a framework of the Hamiltonian formalism on a cone [10]. However, paper [10] contains a condition that is too restrictive if the cone is to be regular at its vertex. Here an amended version of our approach in [10] is presented. In many respects it coincides with that in [9], but some proofs and formulas are simpler.

We consider the metric in the Bondi coordinates adapted to a foliation of spacetime by null cones, but instead of the luminosity distance we use the affine parameter along null generators of these cones. In section 2 we derive a formula for the Bondi mass aspect in terms of the asymptotic data of the metric. In section 3 we express the total energy of a regular cone as an integral of quantities that are either non-negative or become such if the energy dominant condition is satisfied.
2. The Bondi mass aspect in the affine gauge

In the Bondi–Sachs approach to gravitational radiation [11, 12], spacetime is asymptotically flat in future null directions and it can be foliated by null surfaces \( u = \text{const} \) having the structure of the Minkowskian light cone, at least in the asymptotic region. In adapted coordinates \( x^\mu = u, r, x^A \), where \( \mu = 0, 1, 2, 3 \) and \( A = 2, 3 \), the metric takes the form

\[
g = du \left( g_{00} du + 2g_{01} dr + 2\omega_A dx^A \right) + g_{AB} dx^A dx^B. \tag{1}
\]

Coordinates \( u \) and \( r \) are interpreted, respectively, as a retarded time and a distance from a center and \( x^A \) are coordinates on the two-dimensional sphere \( S_2 \). We assume that \( g \) has the signature \(+−−−\), so metric \( g_{AB} \) is negative definite. Note that \( g \) corresponds to \( \tilde{g} \) in [10].

Within this approach, the following expansions are assumed for large values of coordinate \( r \)

\[
g_{00} = 1 - 2Mr^{-1} + O(r^{-2}), \tag{2}
\]
\[
g_{01} = 1 + O(r^{-2}), \tag{3}
\]
\[
\omega_A = \psi_A + \kappa_A r^{-1} + O(r^{-2}), \tag{4}
\]
\[
g_{AB} = -s_{AB}b^2 + n_{AB}r + m_{AB} + O(r^{-1}). \tag{5}
\]

Here \( s_{AB} \) is the standard metric of \( S_2 \) and all coefficients are functions of \( u \) and \( x^A \). A lack of a term proportional to \( r^{-1} \) in (3) follows from minimal assumptions about the Ricci tensor, which ensures that the total energy–momentum vector is well defined [13].

Coordinate \( r \) is not yet fully defined. In the original Bondi coordinates it is the luminosity distance \( r' \) which satisfies

\[
r' = \left( \frac{\sigma}{\sigma_0} \right)^2, \tag{6}
\]

where \( \sigma^2 = \det g_{AB} \) and \( \sigma_0^2 = \det s_{AB} \). Let quantities related to the Bondi coordinates be denoted by a prime. Function \( M' \) is called the Bondi mass aspect and the total energy of the cone is given by

\[
E(u) = \frac{1}{4\pi} \int_{S_2} M' \sigma_s d^2x. \tag{7}
\]

In these coordinates \( s'^{AB}n'_{AB} = 0 \) and \( n'_{AB}u \) is equivalent to the Bondi news function (this is why we use letter \( n \) in the symbol \( n_{AB} \)).

Unfortunately, the luminosity distance is not convenient for proving \( E \geq 0 \). It is better to assume that

\[
g_{01} = 1. \tag{8}
\]

In this case \( r \) is the affine parameter along null geodesics tangent to \( u^\mu \partial_\mu \). A transformation between the luminosity distance \( r' \) and the affine distance \( r \) is defined up to a function \( a'(u, x^A) \)

\[
r = r' + \int_{r'}^{\infty} \left( 1 - g_{01} \right) dr'' + a'(u, x^A). \tag{9}
\]
If \( a' \neq 0 \) then instead of (2) one obtains
\[
g'_{00} = a - 2Mr^{-1} + O(r^{-2}),
\]
where \( a \) is a function of \( u \) and \( x^A \). A relation between \( M \) and the Bondi mass aspect \( M' \) is described by the following proposition in which \( \dot{a}_a \) is denoted by a dot.

**Theorem 2.1.** Metric
\[
g = du \left( g_{00} du + 2dr + 2\omega_A dx^A \right) + g_{AB} dx^A dx^B
\]
is equivalent to the Bondi–Sachs metric with the luminosity distance if it satisfies conditions (4), (5) and
\[
g_{00} = 1 + \frac{1}{2} \dot{h}^A_A - \frac{2}{r} M + O(r^{-2}).
\]

The Bondi mass aspect is given by
\[
M' = M + 4 \dot{h}^A_A - \frac{1}{4} \dot{h}_{AB} \dot{h}^{AB} + \frac{1}{16} h^A_A \dot{h}^B_B.
\]

If the Einstein equation \( R_{11} = T_{11} \) is satisfied then
\[
m^A_A = \frac{1}{4} h_{AB} h^{AB} - \lim_{r \to \infty} r^4 T_{11}.
\]

**Proof.** Given the metric (11) function \( g'_{00} \) and the Bondi mass aspect \( M' \) can be determined from the equality
\[
g'_{00} du + 2 g'_{01} dr' + 2\omega'_{A} dx^A = g_{00} du + 2dr + 2\omega_A dx^A,
\]
where primes correspond to the luminosity gauge. It follows from (15) that
\[
g'_{01} = \frac{1}{r'}, \quad g'_{00} = g_{00} - 2r' \frac{\dot{r}'}{r'}.
\]

In virtue of (6) equation (16) yields
\[
g'_{00} = g_{00} - 2\frac{\sigma_u}{\sigma_r}.
\]

From (5) and (6) one obtains
\[
\sigma \approx \sigma_s r^2 \left( 1 + \frac{1}{2r} h^A_A \right), \quad r' \approx r + \frac{1}{4} h^A_A
\]
for large values of \( r \). One consequence of (17), (18) and condition (2) for \( g'_{00} \) is equation (12). Another one is that the Bondi mass aspect is given by
\[
M' = \frac{1}{2} \lim_{r \to \infty} r' \left( 1 - g_{00} + 2\frac{\sigma_u}{\sigma_r} \right)
\]
or, equivalently, by

\[ M' = \frac{1}{4\sigma_r} \lim_{r \to \infty} \left( \sigma_r \left( 1 - g_{00} \right) + 2\sigma_{uu} \right) = \frac{1}{4\sigma_r} \lim_{r \to \infty} f + \frac{1}{8} n^A A. \]  

(20)

where

\[ f = 2\sigma_r r + g_{00} \sigma_r + 2\sigma_{uu}. \]  

(21)

Expression (20) will be useful in a proof of \( E \geq 0 \) in the next section. In order to obtain (13) we calculate one more term in expansions (18)

\[ \sigma \approx \sigma_r \left( r^2 + \frac{1}{2} m^A A + \frac{1}{2} \tilde{m}^A A + \frac{1}{16} \left( n^A A \right)^2 \right), \quad r' \approx r + \frac{1}{4} n^A A + \frac{1}{4} r \tilde{m}^A A. \]  

(22)

where

\[ \tilde{m}^A A = m^A A - \frac{1}{2} n_{AB} n^{AB} + \frac{1}{16} \left( n^A A \right)^2. \]  

(23)

It follows from (20)–(23) that

\[ M' = M + \frac{1}{4} \tilde{m}^A A. \]  

(24)

and hence (13) is obtained. One can further transform this expression if the Einstein equation \( R_{11} = T_{11} \) (see (28)) is satisfied. Expanding both sides of this equation into powers of \( r^{-1} \) yields (14) as the first non-trivial equality.

\[ \square \]

Remark 1. In order to eliminate \( m^A A \) from (13) one can use (14) or the equality

\[ \tilde{m}^A A = \frac{1}{2} n_{AB} n^{AB} + \lim_{r \to \infty} r^3 \left( 2T_{01} - T \right) \]  

(25)

which follows from the Einstein equation \( R_{01} = T_{01} - T/2 \).

Formulas given in this section agree with those in [10] if \( T^{\mu \nu} = 0 \) and \( n^A A = 0 \). The latter condition can be obtained by a shift of the affine distance. However, assumption \( n^A A = 0 \) is too restrictive (as pointed out by P Chrustciel) if a regular foliation of the complete cone \( u = \text{const} \) by surfaces \( r = \text{const} \) is considered in order to prove \( E \geq 0 \).

3. The total energy of a light cone

Let us assume that the metric satisfies conditions (1)–(5) and \( u = \text{const} \) is a complete future cone with a vertex at \( r = 0 \). For our purposes the following regularity conditions at the vertex are important

\[ \lim_{r \to 0} r^{-2} \sigma \neq 0, \quad \sigma_{uu} = \sigma_{rr} = 0, \quad \left| g_{00} - \omega_A \omega^A \right| < \infty \text{ at } r = 0. \]  

(26)

These conditions are preserved if we pass to the luminosity distance \( r' \) or the affine distance, provided that the latter vanishes at the vertex.

We intend to prove the following proposition which, to a large extent, overlaps with equation (31) in [9] and its consequence, saying that \( E \geq 0 \) \( (m_T B \geq 0 \) in the notation of [9]) if the dominant energy condition is satisfied. The proof of this proposition is based on the equations and method presented by the author and Korbicz in [10].
Proposition 3.1. Let metric (11) satisfy conditions (4), (5), (12), (26) and the Einstein equations $R_{11} = T_{11}$ and $g^{AB}R_{AB} = g^{AB}T_{AB} - T$. Then

$$16\pi E = -\int_0^\infty dr \int_{S_2} \left( \frac{1}{4} r^2 g_{AB,r} g^{AB} + \frac{1}{2} \sigma g_{AB} \omega^A_r \omega^B_r \right) d^2x$$

$$+ \int_0^\infty dr \int_{S_2} \left( r^2 T_{11} + \sigma \left( T - g^{AB}T_{AB} \right) \right) d^2x,$$  

(27)

where $g_{AB} = r^{-2}g_{AB}$. If the dominant energy condition is satisfied then $E \geq 0$.

Proof. The Einstein equations mentioned above have the following form

$$\frac{1}{4} g_{AB,r} g^{AB} - (\ln \sigma)_r = T_{11},$$  

(28)

$$R^{(2)} + \sigma^{-1} \left( 2 \sigma \omega^A_{,A} - g^{AB} \sigma_{,r} - 2 \sigma_{,u} \right)_r - \left( \omega^A_{,r} \right)_A - \frac{1}{2} g^{AB} \omega^A_r \omega^B_r = g^{AB}T_{AB} - T.$$  

(29)

Here, $\omega^A_{,r}$ is the covariant derivative with respect to $g_{AB}$, $R^{(2)}$ is the Ricci scalar of this metric, $T^{\mu\nu}$ is the energy–momentum tensor and $T = T^\rho_\rho$. Equations (28) and (29) reduce to equations (29) and (32) in [10] if $T^{\mu\nu} = 0$.

Let us integrate equation (29) over a part of the cone $u = \text{const}$ between $r_1$ and $r_2$ with the measure $\sigma dx^2 dx^3 dr$. Integrals of divergences of $\omega^A$ and $\omega^A_r$ vanish. The Gauss–Bonnet theorem for the negative definite metric $g_{AB}$ yields

$$\int_{S_2} R^{(2)} \sigma d^2x = -8\pi$$  

(30)

and it allows us to replace $R^{(2)} \sigma$ by $-2(\sigma_r)_r$. Hence

$$\int_{S_2} \left( \tilde{f}(r_2) - \tilde{f}(r_1) \right) d^2x = \int_{r_1}^{r_2} dr \int_{S_2} P \sigma d^2x,$$  

(31)

where

$$\tilde{f} = 2 \sigma_r + g^{11} \sigma_{,u} + 2 \sigma_{,u}$$  

(32)

and

$$P = T - g^{AB}T_{AB} - \frac{1}{2} g^{AB} \omega^A_r \omega^B_r$$  

(33)

(definition (32) coincides with (50) in [10]). Function $\tilde{f}$ vanishes at $r = 0$ in virtue of conditions (26). Since $g^{11} = \omega^A_{,u} g_{AA} - g_{00}$, $\tilde{f}$ can be approximated by function (13) for large values of $r$. In virtue of (20) equation (31) in the limit $r_1 \to 0$ and $r_2 \to \infty$ yields the following expression for the total energy

$$16\pi E = \frac{1}{2} \int_{S_2} \left( n^A \sigma_1 d^2x + \int_0^\infty dr \int_{S_2} P \sigma d^2x. \right)$$  

(34)

Equation (34) agrees with (48) in [10] if $n^A_{,u} = T_{au} = 0$.

In order to find sufficient conditions which ensure that the first term on the rhs of (34) is non-negative, let us write equation (28) in the following way
\[ \frac{1}{4} r^2 \hat{g}_{AB,r} \hat{g}^{AB} - \left( r^2 \ln(\hat{\sigma})_{,r} \right)_{,r} = r^2 T_{11}, \]  

(35)

where \( \hat{g}_{AB} = r^2 \hat{g}_{AB} \) and \( \sigma = r^2 \hat{\sigma} \). From (18) and (26) one obtains

\[ \lim_{r \to \infty} r^2 \left( \ln(\hat{\sigma})_{,r} \right) = -\frac{1}{2} n^A A, \quad \lim_{r \to 0} r^2 \left( \ln(\hat{\sigma})_{,r} \right) = 0. \]  

(36)

Hence, integrating (35) over \( r \) between 0 and \( \infty \) yields

\[ n^A A = 2 \int_0^\infty \left( T_{11} - \frac{1}{4} \hat{g}_{AB,r} \hat{g}^{AB} \right) \, dr. \]  

(37)

Substituting (33) and (37) into (34) yields (27). Since

\[ \hat{g}_{AB,r} \hat{g}^{AB} = -\hat{g}_{AB,r} \hat{g}^{AC} \hat{g}^{BD} \, \delta_{CD,r} \leq 0 \]  

(38)

condition \( T_{11} \geq 0 \) is sufficient to obtain \( n^A A \geq 0 \).

Let us consider the integral of \( P \) on the rhs of (34). Since \( g_{AB} \omega^A \omega^B \leq 0 \), in order to obtain \( P \geq 0 \) it is sufficient to ensure the positivity of the expression

\[ T - g^{AB} T_{AB} = 2T_{11} + g^{AB} T_{11} - 2\omega^A T_{1A} = T_{11} + g^{AB} \right), \]  

(39)

where

\[ v = 2\partial_t + \left( \omega^B \omega_B - g_{00} \right) \partial_r - 2\omega^A \partial_A, \quad v^A = -4\omega^A \omega^A \geq 0. \]  

(40)

Let \( \omega_A \omega^A \neq 0 \). Then \( v \) is timelike and future directed since \( \partial_t \) is null future directed and \( g(v, \partial_t) > 0 \). If \( T_{11} \) satisfies the dominant energy condition then vector \( \nu = T_{11} \nu^\beta \partial_\beta \) is non-spacelike and \( g(\nu, \nu) \geq 0 \). Hence, \( \nu \) is also future directed and \( T_{11} \nu^\beta \partial_\beta = g(\nu, \partial_t) \geq 0 \). If \( \omega_A \omega^A = 0 \) then the same result follows from the continuity. Thus, \( T - g^{AB} T_{AB} \geq 0 \) and \( P \geq 0 \).

If the energy dominant condition is satisfied, then also \( T_{11} \geq 0 \) since \( T_{11} = T_{10} k^0 k^r \), where \( k = \partial_r \). Hence, \( n^A A \geq 0 \) and the total energy \( E \) is non-negative.

\[ \square \]

4. Concluding remarks

The results of Korbicz and Tafel [10] on the energy of a null surface in the Bondi–Sachs formalism have been completed. We presented an expression for the Bondi mass aspect in terms of the asymptotical data corresponding to the metric in the affine gauge (proposition 2.1). Using only two of the Einstein equations, we confirmed a result of Chruściel and Paetz [9] on the positivity of the Trautman–Bondi energy of a regular null cone (proposition 3.1). Our proof is shorter and expression (27) for the total energy is simpler than that in [9].

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