JOINT OPTimal DECISIONs ON PRICING AND WARRANTY POLICY OF DUOPOLY SUPPLY CHAIN WITH ONE COMMON RETAILER

Soumita KUNDU
Department of Mathematics, Sister Nibedita Government General Degree College for Girls, Kolkata-700027, India
kundu.soumita21@gmail.com

Tripti CHAKRABARTI
Department of Applied Mathematics, University of Calcutta, Kolkata-700009, India
tripti.chakrabarti@gmail.com

Received: June 2018 / Accepted: March 2020

Abstract: Early researches related to the interaction between manufactures for complementary products, mainly considered price as only a dimension of competition. With the increasing competition in capturing the market share, manufactures cannot compete by only lowering price. In this paper, we assume that besides the price, the manufactures choose warranty as the competitive strategy of two different but substitutable products in a duopoly supply chain with one common retailer. Furthermore, two cases are considered (i) only one manufacturer adopts warranty policy as a competitive strategy against the other, (ii) both manufacturers offer warranty on their product, to study under which situation offering a warranty becomes more profitable for a manufacturer while the other competitive manufacturer has already adopted warranty policy. The profit functions of the manufacturers and the retailer are then maximized under manufacturers’ cooperative and non-cooperative strategies. We then compare the scenarios under different decision strategies numerically, which gives some insights on changes of key parameters to help the decision makers to capture the market.

Keywords: Warranty, Pricing, Supply Chain Management, Quality, Stackleberg Game.

MSC: 91B24.
1. INTRODUCTION

With rapid trends in business globalization and current competitive environment, the marketing strategies of the business have to be renovated to face the challenges of the global competitive marketplace. Today, various brands of a single kind of product (e.g., smart-phone manufactured by Samsung, Vivo, Apple etc) are often sold by the same retailer. Thus, the business models have experienced significant changes to improve customer service reputation in highly competitive market.

The competition among the companies was mainly concerned with prices, but in this modern age of social networking, the trust and support of the customers play a vital role in the business world. Thus, a good reputation of a business in terms of quality of the product and consumer service becomes crucial to its survival. Customer can forecast the durability of the product based on its length of warranty (Boulding and Kirmani [4]). To avoid the risk whether the product will serve as expected or not, the majority of the customers favor to buy a product from a manufacturing company who offers a warranty period guaranteeing replacement, refunding or repairing of the product during this period. As a result, the manufacturer can explore the market strategy that offer warranty on their product, e.g., Hyundai, Acura, Audi, Mercedes-Benz in the automobile market, Hewlett-Packard, Panasonic, Samsung, Cannon in the electronics market.

Therefore, it becomes important for manufacturers to decide on how to set the optimal wholesale price for their product because the demand of the product not only depends on its own price but also on the price of its complementary product. It is also observed that the manufacturer adopts some marketing strategies such as warranty for competition. As a result, it turns out to be more challenging for the manufacturers to decide how to set warranty period and wholesale price to increase their profit individually and for retailer, it becomes a crucial task to set their retail prices to satisfy the customer demand. We have addressed this issue by considering the demand of each product decreasing with its own price and the competitor’s warranty period and increasing with its own warranty period and the competitor’s price, which corresponds to reality in many practical situations. To examine the situation under which offering a warranty becomes more economical for a manufacturer while the other competitive manufacturer has already adopted warranty policy, we consider two scenarios (i) one manufacturer offers warranty on his product and the other does not (case 1) (ii) both manufacturers offer warranty as the competitive strategy (case 2).

2. LITERATURE REVIEW

Dealing with warranty policy for products has gained much interest from researchers. Regarding the agreement of warranty policies, manufacturers adopt different types of warranties such as (i) free replacement warranty policy, (ii)
money back warranty (full refund) policy, (iii) outsourcing maintenance service policy, (iv) Pro-rata warranty (replacement at a cost or refunding a fraction of its purchasing price) policy. Boom [3] discussed a situation where a monopolist supplier reimburses the risk-averse consumers by three types of warranty rules (a) no warranty, (b) money back guarantee, (c) renewing free replacement. Rinsaka and Sandoh [13] considered the case in which the manufacturer replaces the product or system with a new one for its first failure but minimal repairs are conducted with the succeeding failures during the warranty period. Asgharizadeh and Murthy [2] developed a game theoretic model where repairs are carried out by an external agent under a service contract when the equipments fail. Alqahtani and Gupta [1] studied a renewable two-dimensional Pro-Rata warrantee policy for end-of-life products.

In the present market, it is observed that multiple brands of a single type of product (e.g., sunglass made by Ray Ban, Gucci, Oakley etc) is often sold by the same retailer. In this situation, price, discount, warranty, or other service contracts are significant sale factors in capturing market share. There are numerous studies involving pricing problem (e.g., Choi [5]; Raju et al. [12]; Zhao et al. [19]; Tsay and Agrawal [17]). Choi [5] developed three types of pricing games of different power structures between two manufactures and a retailer in a two-echelon supply chain to examine how channel profits split among the channel members. Choi [6] extended this monopoly common retailer channel model by introducing price competition between duopoly common retailers where each manufacturer sells the same product to both retailers. Luo et al. [10] investigated the price competition between two manufacturers and a retailer in which the retailer sells differentiated brands, a good brand and an average brand, supplied by two manufacturers.

To solve the problem of gaining the market share, many researchers have focused on both price and warranty/service/capacity/location as the dimensions of competition (e.g., Wei et al. [18]; Tsay and Agrawal [16]; Hall and Porteus [7]; Iyer [8]; Tsao and Su [15]). In this study, we consider a pricing and warranty period decision problem in a supply chain consisting of two competing manufacturer and a common retailer. Lu et al. [9] examined a pricing and warranty decisions problem in a two-echelon dual supply chain model. Taleizadeh et al. [14] analyzed two markets with different level of willingness to pay for product with a common manufacturer at both markets who offers warranty as a competing factor when a third party distributor acts as a gray market. However, most of the studies which consider warranty as the effective strategy to boost the sales tend to ignore warranty cost as the function of product quality, and consider warranty cost as the function of length of warranty period and failure rate. But warranty cost mainly emerges due to poor quality level of a product. In the recent years, industries are continuously trying to reduce warranty costs by increasing product quality.
3. ASSUMPTIONS and NOTATIONS

To develop the model, we make the following assumptions and notations

3.1. Assumptions

- The model structure is developed for two different but substitutable products consisting of two manufacturers and a common retailer.
- Warranty cost of manufacturer depends on quality level of product and warranty period.
- Manufacturer bears a quality improvement cost to lessen the warranty cost.
- The manufacturer is more powerful in making decision than retailer.

3.2. Notations

\[\alpha_{pi}\] The market potential of the product produced by manufacturer \(i\) (\(\alpha_{pi} > 0\))

\[w_{ij}\] The wholesale price per unit by the manufacturer \(i\) in case \(j\)

\[p_{ij}\] The retail price per unit product produced by manufacturer \(i\) in case \(j\)

\[T_{Li,j}\] The warranty period offered by the manufacturer \(i\) in case \(j\)

\[\beta_c\] The price sensitivity factor (\(\beta_c > 0\))

\[\beta_t\] The warranty period sensitivity factor (\(\beta_t > 0\))

\[\eta_c\] The degree of price competition between the manufacturers (\(\eta_c > 0\))

\[\eta_t\] The degree of warranty period competition between the manufacturers (\(\eta_t > 0\))

\[q_i\] The quality level of the product produced by the manufacturer \(i\) (\(q_i \in [0,1]\))

\[c_i\] The production cost per unit of the manufacturer \(i\)

4. MODEL FORMULATION

In this paper, we develop a two-echelon supply chain model, where a common retailer sells two complementary products produced by two manufacturers indexed by \(i \in \{1, 2\}\). Thus, it leads to a competition between manufacturers. Besides the price, to attract the customers, the manufacturers provide a free repair warranty policy as a competitive strategy against each other. Manufacturer \(i\) faces the warranty cost \(C_{Li} = \lambda_i T_{Li} q_i^{-\delta_i}\), which is convex and decreasing with respect to the quality level \(q_i\) for any \(\delta_i > 0\) (i.e., \(\frac{\partial C_{Li}}{\partial q_i} < 0\), \(\frac{\partial^2 C_{Li}}{\partial q_i^2} > 0\)) (Noll [11]). We also see that this cost function \(C_{Li}\) is increasing and convex with respect to warranty period \(T_{Li}\) for any \(\gamma_i > 1\) (i.e., \(\frac{\partial^2 C_{Li}}{\partial T_{Li}^2} > 0\)). To reduce the warranty cost, manufacturer \(i\) expends cost \(C_{m,i}(q_i) = c_{m,i} \frac{q_i}{1-q_i}\) in improving his product quality level, which is increasing and convex with respect to \(q_i\), (i.e., \(\frac{\partial C_{m,i}}{\partial q_i} > 0\), \(\frac{\partial^2 C_{m,i}}{\partial q_i^2} > 0\)) \(\text{lim}_{q_i \to 0} C_{m,i} = 0\) and \(\text{lim}_{q_i \to 1} C_{m,i} = \infty\) in the range \(q_i \in [0,1]\).
4.1. Only one manufacturer offers warranty (Case 1)

In this situation, only the manufacturer 1 offers warranty on his product. We consider that the demand function for a product is decreasing with respect to its own retail price and increasing with respect to the complementary product’s retail price. On the other hand, increasing warranty period offered by manufacturer 1, increases manufacturer 1’s demand and decreases manufacturer 2’s demand. Thus, we design the demand functions of manufacturers respectively as follows

\[
D_{11}(p_{11}, p_{21}, T_{L11}) = \alpha_{p_1} - (\beta_c + \eta_c)p_{11} + \eta_c p_{21} + (\beta_t + \eta_t)T_{L11},
\]

(1)

and

\[
D_{21}(p_{11}, p_{21}, T_{L11}) = \alpha_{p_2} - (\beta_c + \eta_c)p_{21} + \eta_c p_{11} - \eta_t T_{L11}.
\]

(2)

The profit functions of two manufacturers and the retailer can be written respectively as follows

\[
TP_{m1} = (w_{11} - c_1 - c_m_1 \frac{q_1}{1 - q_1} - \lambda_1 q_1^{-\delta_1}T_{L11}^{\omega_1}) D_{11},
\]

(3)

\[
TP_{m2} = (w_{21} - c_2) D_{21}
\]

(4)

\[
TP_r = (p_{11} - w_{11}) \{\alpha_{p_1} - (\beta_c + \eta_c)p_{11} + \eta_c p_{21} + (\beta_t + \eta_t)T_{L11}\}
\]

\[
+ (p_{21} - w_{21}) \{\alpha_{p_2} - (\beta_c + \eta_c)p_{21} + \eta_c p_{11} - \eta_t T_{L11}\}.
\]

(5)

4.1.1. Decentralized decision

In decentralized decision making, considering the reality, we assume that the manufacturers are more powerful in decision making than the retailer, i.e., the manufacturers act as leaders and the common retailer is their follower. Based on the reaction of the retailer on retail prices, the manufacturers make decisions on their wholesale prices and warranty periods. To determine the retailer best response on retail price, we first optimize retailer profit function for the given manufacturers’ decision variables. That is

\[
\max TP_r (p_{11}, p_{21}|w_{11}, w_{21}, T_{L11}).
\]

(6)

The optimal values of \(p_{11}\) and \(p_{21}\) are obtained by solving \(\frac{\partial TP_r}{\partial p_{11}} = 0\) and \(\frac{\partial TP_r}{\partial p_{21}} = 0\) as follows

\[
p_{11}^* = \frac{w_{11}}{2} + \frac{(\beta_c + \eta_c)(\beta_t + \eta_t) - \eta_c \eta_t T_{L11} + (\beta_t + \eta_t)\alpha_{p_1} + \eta_c \alpha_{p_2}}{2\beta_c(\beta_c + 2\eta_c)}
\]

(7)

and

\[
p_{21}^* = \frac{w_{21}}{2} + \frac{(\beta_t + \eta_t)\eta_c - (\beta_c + \eta_c)\eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L11} + \frac{\eta_c \alpha_{p_1} + (\beta_c + \eta_c)\alpha_{p_2}}{2\beta_c(\beta_c + 2\eta_c)}
\]

(8)
Note that
\[
\frac{\partial^2 TP_{r_1}}{\partial p_1^2} = -2(\beta_c + \eta_c) < 0, \quad \frac{\partial^2 TP_{r_1}}{\partial p_2^2} = -2(\beta_c + \eta_c) < 0
\]
and
\[
\frac{\partial^2 TP_{r_1}}{\partial p_{11}^2} - \frac{\partial^2 TP_{r_1}}{\partial p_{21}^2} - \frac{\partial^2 TP_{r_1}}{\partial p_{12}^2} = 4(\beta_c + \eta_c)^2 - 4\eta_c^2 > 0.
\]
That is \(TP_{r_1}\) is a concave function of \(p_{11}\) and \(p_{21}\). Now the manufacturers make decisions, taking into account the retailer’s best response on retail prices, with the objective of maximizing their own profit. We develop two decision models by considering the manufacturers’ cooperative and noncooperative decision strategies.

**Manufacturers’ noncooperative decision (MNC) strategy**

In this situation, two manufacturers maximize their profits non-cooperatively and make their decisions on wholesale prices and warranty periods independently subject to the constraints imposed by equations in (7) and (8). Hence, the manufacturers’ decision problem is formulated as follows

\[
\begin{align*}
\max_{(w_{11}, T_{L_{11}})} & \quad TP_{m_{11}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*(w_{11}, w_{21}, T_{L_{11}}), p_{21}^*(w_{11}, w_{21}, T_{L_{11}})) \\
\max_{w_{21}} & \quad TP_{m_{21}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*(w_{11}, w_{21}, T_{L_{11}}), p_{21}^*(w_{11}, w_{21}, T_{L_{11}})) \\
\text{subject to} & \quad (7) \text{ and } (8).
\end{align*}
\]

(9)

The partial derivatives of \(TP_{m_{11}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*, p_{21}^*)\) with respect to \(w_{11}, T_{L_{11}}\) and \(TP_{m_{21}}(w_{11}, w_{21}, T_{L_{11}}, p_{11}^*, p_{21}^*)\) with respect to \(w_{21}\) are respectively as follows

\[
\frac{\partial TP_{m_{11}}}{\partial w_{11}} = -(\beta_c + \eta_c)w_{11} + \frac{1}{2}\eta_c w_{21} + \frac{1}{2}(\beta_c + \eta_c)T_{L_{11}} + \frac{1}{2}\beta_c + \eta_c)\lambda_1 T_{L_{11}} q_1^{-\delta_1} + \frac{1}{2}\left\{\alpha_{p_1} + (\beta_c + \eta_c)\left(c_1 + c_{m_1}\frac{q_1}{1 - q_1}\right)\right\}
\]

(10)

\[
\frac{\partial TP_{m_{11}}}{\partial T_{L_{11}}} = \frac{1}{2}(\beta_c + \eta_c)\left(w_{11} - c_1 - c_{m_1}\frac{q_1}{1 - q_1} - \lambda_1 T_{L_{11}} q_1^{-\delta_1}\right)
\]

(11)

\[
\frac{\partial TP_{m_{21}}}{\partial w_{21}} = \frac{1}{2}\eta_c w_{11} - (\beta_c + \eta_c) w_{21} + \frac{1}{2}\eta T_{L_{11}} + \frac{1}{2}\left\{\alpha_{p_2} + (\beta_c + \eta_c)c_2\right\}
\]

(12)
Proposition 1. The profit function $TP_{m_{11}}$ under decentralized MNC strategy is a concave function in $w_{11}$ and $T_{L_{11}}$ if $\left(\gamma_{1} - 1\right)\{\alpha_{p_{1}} - (\beta_{c} + \eta_{c})w_{11}^{mnc} + \eta_{c}w_{11}^{mnc}\} + (\gamma_{1} + 1)(\beta_{i} + \eta_{c})T_{L_{11}}^{mnc} > 0$ and $\left(\gamma_{1} - 1\right)\{\alpha_{p_{1}} - (\beta_{c} + \eta_{c})w_{11}^{mnc} + \eta_{c}w_{11}^{mnc} + (\beta_{i} + \eta_{c})T_{L_{11}}^{mnc}\} > 0$.

Proof. The profit function $TP_{m_{11}}$ under decentralized MNC strategy would be concave in $w_{11}$ and $T_{L_{11}}$ if at the stationary point $(w_{11}^{mnc}, T_{L_{11}}^{mnc})$, the Hessian matrix of $TP_{m_{11}}$ is negative definite. Here, at $(w_{11}^{mnc}, T_{L_{11}}^{mnc})$
\[
\frac{\partial^2 TP_{m11}}{\partial w_{11}^2} = -(\beta_c + \eta_c) < 0,
\]
\[
\frac{\partial^2 TP_{m11}}{\partial T_{L11}^2} = -\frac{(\gamma_1 - 1)(\beta_t + \eta_t)\{\alpha_{p1} - (\beta_c + \eta_c)w_{11}^{mnc\ast} + \eta_c w_{21}^{mnc\ast}\}}{2(\beta_c + \eta_c)T_{L11}^{mnc\ast}} - \frac{(\gamma_1 + 1)(\beta_c + \eta_c)^2}{2(\beta_c + \eta_c)} < 0.
\]

if \((\gamma_1 - 1)\{\alpha_{p1} - (\beta_c + \eta_c)w_{11}^{mnc\ast} + \eta_c w_{21}^{mnc\ast}\} + (\gamma_1 + 1)(\beta_t + \eta_t)T_{L11}^{mnc\ast} > 0\) holds.

\[
\frac{\partial^2 TP_{m11}}{\partial w_{21}^2} \frac{\partial^2 TP_{m11}}{\partial T_{L11}^2} - \frac{\partial^2 TP_{m11}}{\partial w_{11} T_{L11}} \frac{\partial^2 TP_{m11}}{\partial T_{L11} w_{11}} = -(\beta_t + \eta_t)^2 + (\beta_c + \eta_c)
\]
\[
\times \left[\frac{(\gamma_1 - 1)(\beta_t + \eta_t)\{\alpha_{p1} - (\beta_c + \eta_c)w_{11}^{mnc\ast} + \eta_c w_{21}^{mnc\ast}\}}{2(\beta_c + \eta_c)T_{L11}^{mnc\ast}} + \frac{(\gamma_1 + 1)(\beta_c + \eta_c)^2}{2(\beta_c + \eta_c)}\right] > 0
\]

if \((\gamma_1 - 1)\{\alpha_{p1} - (\beta_c + \eta_c)w_{11}^{mnc\ast} + \eta_c w_{21}^{mnc\ast}\} + (\beta_t + \eta_t)T_{L11}^{mnc\ast} > 0\) holds. This completes the proof.

**Proposition 2.** The profit function \(TP_{m21}\) under decentralized MNC strategy is a concave function in \(w_{21}\).

**Proof.** Here at \(w_{21} = w_{21}^{mnc\ast}\),
\[
\frac{\partial^2 TP_{m21}}{\partial w_{21}^2} = -(\beta_c + \eta_c) < 0.
\]

Hence, the profit function \(TP_{m21}\) under decentralized MNC strategy is a concave function in \(w_{21}\). This completes the proof. \(\square\)

**Manufacturers’ cooperative (MC) decision strategy**

In this situation, two manufacturers cooperate and make decisions jointly to find their maximum total profit after seeing the retailer’s reaction on retail prices. After optimization, their joint profit would be divided between the two manufacturers. Hence, the manufacturers’ decision problem is formulated as follows.

\[
\max_{(w_{11}, w_{21}, T_{L11})} \left[TP_{m11} + TP_{m21}\right](w_{11}, w_{21}, T_{L11}, p_{11}^{s1}(w_{11}, w_{21}, T_{L11}), p_{21}^{s1}(w_{11}, w_{21}, T_{L11}))
\]

subject to (7) and (8).

(18)

The partial derivatives of \(TP_{m11} + TP_{m22}\) with respect to \(w_{11}, T_{L11}\) and \(w_{21}\) are respectively as follows:
The profit function \( S \) is a concave function of \((\beta_c + \eta_t)w_{11} + \eta_t w_{21}\) where \( w_{11} = \frac{1}{2}\eta_t (w_{21} - c_2) \) and \( \alpha_{p_1} - \frac{(\beta_c + \eta_t) w_{11}^{\text{mcs}} + \eta_t w_{21}^{\text{mcs}}}{(TP_{m_{11}}^{\text{mcs}})^{\gamma - 1}} (= 0 \) and \( \frac{(TP_{m_{11}}^{\text{mcs}})}{\partial T_{L_{11}}^{\text{mcs}}} = 0, \) we obtain the optimal values of \( w_{11}, w_{21}, T_{L_{11}} \). Analytically it is difficult to solve this equation. We solve the equation numerically by using Matlab2013 software. Let the solution be \( w_{11} = w_{11}^{\text{mcs}}, w_{21} = w_{21}^{\text{mcs}}, \) and \( T_{L_{11}} = T_{L_{11}}^{\text{mcs}} \).

**Proposition 3.** The profit function \((TP_{m_{11}} + TP_{m_{21}}))(w_{11}, w_{21}, T_{L_{11}})\) is a concave function of \((\beta_c + \eta_t)w_{11} + \eta_t w_{21}\) where \( w_{11} = \frac{1}{2}\eta_t (w_{21} - c_2) \) and \( \alpha_{p_1} - \frac{(\beta_c + \eta_t) w_{11}^{\text{mcs}} + \eta_t w_{21}^{\text{mcs}}}{(TP_{m_{11}}^{\text{mcs}})^{\gamma - 1}} (= 0 \) and \( \frac{(TP_{m_{11}}^{\text{mcs}})}{\partial T_{L_{11}}^{\text{mcs}}} = 0, \) we obtain the optimal values of \( w_{11}, w_{21}, T_{L_{11}} \). Analytically it is difficult to solve this equation. We solve the equation numerically by using Matlab2013 software. Let the solution be \( w_{11} = w_{11}^{\text{mcs}}, w_{21} = w_{21}^{\text{mcs}}, \) and \( T_{L_{11}} = T_{L_{11}}^{\text{mcs}} \).
\[
\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial T_{L11}^2} \bigg|_{atS_1} = -\frac{1}{2} \lambda_1 \gamma_1 (\gamma_1 - 1) q_i^{-\delta_1} \left\{ \alpha_{p1} - (\beta_c + \eta_c) w_{11}^{mc} + \eta_c w_{21}^{mc} \right\}
\times (T_{L11}^{mc})^{\gamma_1 - 2} - \frac{1}{2} \lambda_1 \gamma_1 (\gamma_1 + 1)(\beta_i + \eta_i) q_i^{-\delta} (T_{L11}^{mc})^{\gamma_1 - 1} = u_1 \text{(say)},
\]
\[
\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial w_{11}^2} \bigg|_{atS_1} = \frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial T_{L11} \partial w_{11}} \bigg|_{atS_1} = \eta_c,
\]
\[
\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial w_{11} \partial T_{L11}} \bigg|_{atS_1} = \frac{1}{2} \beta_i + \eta_i + \frac{1}{2} (\beta_c + \eta_c) \lambda_1 \gamma_1 q_i^{-\delta_1} (T_{L11}^{mc})^{\gamma_1 - 1} = u_2 \text{(say)},
\]
\[
\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial w_{21} \partial T_{L11}} \bigg|_{atS_1} = -\frac{1}{2} \eta_i - \frac{1}{2} \eta_c \lambda_1 \gamma_1 q_i^{-\delta_1} (T_{L11}^{mc})^{\gamma_1 - 1} = u_3 \text{(say)}.
\]

The Hessian matrix \( H_1 \) of \( TP_{m11} + TP_{m21} \) at the stationary point \( S_1 (w_{11}^{mc}, w_{21}^{mc}, T_{L11}^{mc}) \)

\[
H_1 = \begin{pmatrix}
\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial w_{11}^2}
& \frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial T_{L11} \partial w_{11}}
& \frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial T_{L11}^2}

\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial w_{11} \partial T_{L11}}
& \frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial T_{L11}^2}

\frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial w_{21} \partial T_{L11}}
& \frac{\partial^2 (TP_{m11} + TP_{m21})}{\partial T_{L11}^2}

\end{pmatrix}
\bigg|_{atS_1}
\]

The profit function \( TP_{m11} + TP_{m21} \) will be concave function if the principal minors of \( H_1 \) are alternatively negative and positive, i.e., if the \( i^{th} \) order principal minor \( D_i \) of \( H_1 \) takes the sign \((-1)^i\). Here,

\[
D_1 = - (\beta_c + \eta_c) < 0,
\]
\[
D_2 = \begin{vmatrix}
- (\beta_c + \eta_c) & \eta_c \\
\eta_c & - (\beta_c + \eta_c)
\end{vmatrix}
= (\beta_c + \eta_c)^2 - \eta_c^2 > 0
\]

and

\[
D_3 = |H_1| = (\beta_c + \eta_c)^2 u_1 + 2 \eta_c u_2 u_3 + (\beta_c + \eta_c) u_3^2 + (\beta_c + \eta_c) u_2^2 - \eta_c^2 u_1 < 0
\]

if \((\beta_c + \eta_c)^2 u_1 + 2 \eta_c u_2 u_3 + (\beta_c + \eta_c) u_3^2 + (\beta_c + \eta_c) u_2^2 - \eta_c^2 u_1 < 0 \) holds. This completes the proof. \( \square \)

4.1.2. Centralized decisions

In this decision case, both the manufacturers and their common retailer cooperate to maximize the total profit of the supply chain. The total profit function
under this scenario is

$$TP_{c_1} = TP_{m_{11}} + TP_{m_{21}} + TP_{r_1}$$

$$= (p_{11} - c_1 - c_{m_1} \frac{q_1}{1 - q_1} - \lambda_1 T_{L_{11}} \eta_1 q_1^{-\delta_1})D_{11} + (p_{21} - c_2)D_{12}. \quad (22)$$

Hence, the channel members’ decision problem is formulated as follows

$$\max_{(p_{11}, p_{21}, T_{L_{11}})} TP_{c_1}(p_{11}, p_{21}, T_{L_{11}}). \quad (23)$$

The partial derivatives of $TP_{c_1}(p_{11}, p_{21}, T_{L_{11}})$ with respect to $p_{11}, T_{L_{11}},$ and $p_{21}$ are respectively as follows:

$$\frac{\partial TP_{c_1}}{\partial p_{11}} = -2(\beta_c + \eta_c)p_{11} + 2\eta_c p_{21} + \{(\beta_t + \eta_t)T_{L_{11}} + (\beta_c + \eta_c)T_{L_{11}}\}$$

$$= \left\{ \alpha_{p_1} + (\beta_c + \eta_c)c_1 - \beta_c c_2 + c_{m_1}(\beta_c + \eta_c)\frac{q_1}{1-q_1} \right\}, \quad (24)$$

$$\frac{\partial TP_{c_1}}{\partial p_{21}} = 2\eta_t p_{21} - 2(\beta_c + \eta_c)p_{21} - \{(\beta_t + \eta_t)T_{L_{11}} + \beta_c T_{L_{11}}\}$$

$$= \left\{ \alpha_{p_2} + (\beta_c + \eta_c)c_2 - \beta_c c_1 - c_{m_1}(\beta_c + \eta_c)\frac{q_1}{1-q_1} \right\}. \quad (25)$$

and

$$\frac{\partial TP_{c_1}}{\partial T_{L_{11}}} = -\eta_t(p_{21} - c_2) + (\beta_t + \eta_t)\{p_{11} - c_1 - c_{m_1}(\beta_c + \eta_c)\frac{q_1}{1-q_1} - \lambda_1 q_1^{-\delta_1}T_{L_{11}}\}$$

$$= \lambda_1 q_1^{-\delta_1}T_{L_{11}}^{-1} \left\{ \alpha_{p_1} - (\beta_c + \eta_c)p_{11} + \eta_t p_{21} + (\beta_t + \eta_t)T_{L_{11}} \right\}. \quad (26)$$

Solving equations $\frac{\partial TP_{c_1}}{\partial p_{11}} = 0, \frac{\partial TP_{c_1}}{\partial p_{21}} = 0,$ and $\frac{\partial TP_{c_1}}{\partial T_{L_{11}}} = 0,$ we obtain the optimal values of $p_{11}, p_{21}, T_{L_{11}}$. Analytically it is difficult to solve these equations. We solve the equation numerically by using Matlab2013 software. Let the solution be $p_{11} = p_{11}^*, p_{21} = p_{21}^*, T_{L_{11}} = T_{L_{11}}^*$.

**Proposition 4.** The profit function $TP_{c_1}(p_{11}, p_{21}, T_{L_{11}})$ is a concave function if $4(\beta_c + \eta_c)^2u_4 + 4\eta_u u_5 u_6 + 2(\beta_c + \eta_c)u_2^2 + 2(\beta_c + \eta_c)u_2^2 - 4u_4^2 < 0$ where $u_4 = -2(\beta_t + \eta_t)T_{L_{11}}^{-1} \left\{ \alpha_{p_1} - (\beta_c + \eta_c)p_{11} + \eta_t p_{21} + (\beta_t + \eta_t)T_{L_{11}} \right\}. \quad (27)$

$u_2 = \lambda_1 T_{L_{11}}^{-1}$ and $u_6 = -\eta_t - \lambda_1 T_{L_{11}}^{-1}$.

**Proof.** The second order partial derivatives of $TP_{c_1}$ at stationary point $S_2 = (p_{11}^*, p_{21}^*, T_{L_{11}}^*)$ are

$$\frac{\partial^2 TP_{c_1}}{\partial p_{11}^2} \mid_{at S_2} = -2(\beta_c + \eta_c), \quad \frac{\partial^2 TP_{c_1}}{\partial p_{21}^2} \mid_{at S_2} = -2(\beta_c + \eta_c),$$

$$\frac{\partial^2 TP_{c_1}}{\partial p_{11} \partial p_{21}} \mid_{at S_2} = -2(\beta_c + \eta_c),$$

$$\frac{\partial^2 TP_{c_1}}{\partial p_{11} \partial T_{L_{11}}} \mid_{at S_2} = -2(\beta_c + \eta_c),$$

$$\frac{\partial^2 TP_{c_1}}{\partial p_{21} \partial T_{L_{11}}} \mid_{at S_2} = -2(\beta_c + \eta_c).$$
The Hessian matrix $H_2$ of $TP_{c_1}$ at the stationary point $S_2 (p^*_{i1}, p^*_{i2}, T^*_{L_{i1}})$

$$
H_2 = \begin{pmatrix}
\frac{\partial^2 TP_{c_1}}{\partial p_{i1}^2} & \frac{\partial^2 TP_{c_1}}{\partial p_{i1} \partial p_{i2}} & \frac{\partial^2 TP_{c_1}}{\partial p_{i2}^2} \\
\frac{\partial^2 TP_{c_1}}{\partial p_{i1} \partial p_{i2}} & \frac{\partial^2 TP_{c_1}}{\partial p_{i2}^2} & \frac{\partial^2 TP_{c_1}}{\partial p_{i1} \partial T_{L_{i1}}} \\
\frac{\partial^2 TP_{c_1}}{\partial p_{i1} \partial T_{L_{i1}}} & \frac{\partial^2 TP_{c_1}}{\partial p_{i2} \partial T_{L_{i1}}} & \frac{\partial^2 TP_{c_1}}{\partial p_{i2} \partial T_{L_{i1}}}
\end{pmatrix}
$$

The profit function $TP_{c_1}$ will be concave function if the principal minors of $H_2$ are alternatively negative and positive, i.e., if the $i^{th}$ order principal minor $D_i$ of $H_2$ takes the sign $(-1)^i$. Here,

$$
D_{11} = -2(\beta_c + \eta_c) < 0
$$

$$
D_{21} = \begin{vmatrix} -2(\beta_c + \eta_c) & 2\eta_c \\ 2\eta_c & -2(\beta_c + \eta_c) \end{vmatrix} = 4(\beta_c + \eta_c)^2 - 4\eta_c^2 > 0
$$

$$
D_3 = |H_2| = 4(\beta_c + \eta_c)^2 u_4 + 4\eta_c u_5 u_6 + 2(\beta_c + \eta_c) u_5^2 + 2(\beta_c + \eta_c) u_6^2 - 4u_4 \eta_c^2 < 0
$$

if $4(\beta_c + \eta_c)^2 u_4 + 4\eta_c u_5 u_6 + 2(\beta_c + \eta_c) u_5^2 + 2(\beta_c + \eta_c) u_6^2 - 4u_4 \eta_c^2 < 0$ holds. This completes the proof. \(\square\)

4.2. Both manufacturers offer warranty (Case 2)

In this case, we assume that demand function of each manufacturer $i$ is symmetric between two complementary products and is expressed as

$$
D_i(p_{i2}, p_{k2}, T_{L_{i2}}, T_{L_{k2}}) = \alpha_{p_i} - (\beta_c + \eta_c) p_{i2} + \eta_c p_{k2} + (\beta_t + \eta_t) T_{L_{i2}} - \eta_t T_{L_{k2}}, \quad (27)
$$

where $i \in \{1, 2\}$ and $k = 3 - i$. The profit functions of two manufacturers and the retailer can be written respectively as follows

$$
TP_{m_{i2}} = \left( w_{i2} - c_i - c_m \frac{q_i}{1 - q_i} - \lambda^*_i q_i \delta, T_{L_{i2}} \right) D_{i2}
$$

(28)
and
\[ TP_{r2} = \sum_{i=1}^{2} (p_{i2} - w_{i2})D_{i2}. \] (29)

4.2.1. Decentralized decisions

In this decentralized decision making, the manufacturers and the retailer operate independently and the manufacturers make decisions first as Stackelberg leader and then the retailer reacts as their follower. So, we first determine the optimal values of \( p_{12} \) and \( p_{22} \) for given \( w_{12}, w_{22}, T_{L_{12}} \) and \( T_{L_{22}} \) to maximize the retailer’s profit function, that is
\[
\max_{p_{12}, p_{22}} TP_{r2}(p_{12}, p_{22} | w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}}). \quad (30)
\]
The optimal values of \( p_{12} \) and \( p_{22} \) are obtained by solving \( \frac{\partial TP_{r2}}{\partial p_{12}} = 0 \) and \( \frac{\partial TP_{r2}}{\partial p_{22}} = 0 \) as follows
\[
p^*_{12} = \frac{w_{12}}{2} + \frac{(\beta_c + \eta_c)(\beta_t + \eta_t) - \eta_c \eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L_{12}} + \frac{(\beta_t + \eta_t)\eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L_{22}} + \frac{(\beta_c + \eta_c)\alpha_{p_1} + \eta_c \alpha_{p_2}}{2\beta_c(\beta_c + 2\eta_c)} \quad (31)
\]
and
\[
p^*_{22} = \frac{w_{22}}{2} + \frac{(\beta_t + \eta_t)\eta_t - (\beta_c + \eta_c) \eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L_{12}} + \frac{(\beta_c + \eta_c)(\beta_t + \eta_t) - \eta_t \eta_t}{2\beta_c(\beta_c + 2\eta_c)} T_{L_{22}} + \frac{\eta_c \alpha_{p_1} + \eta_c \alpha_{p_2}}{2\beta_c(\beta_c + 2\eta_c)}. \quad (32)
\]
Note that \( \frac{\partial^2 TP_{r2}}{\partial p_{12}^2} \frac{p_{12}}{\partial p_{22}^2} = -2(\beta_c + \eta_c) < 0, \frac{\partial^2 TP_{r2}}{\partial p_{22}^2} = -2(\beta_c + \eta_c) < 0 \) and \( \frac{\partial^2 TP_{r2}}{\partial p_{12}^2} \frac{\partial^2 TP_{r2}}{\partial p_{22}^2} \frac{\partial^2 TP_{r2}}{\partial p_{12}^2} \frac{\partial^2 TP_{r2}}{\partial p_{22}^2} = 4(\beta_c + \eta_c)^2 - 4\eta_c^2 > 0 \). That is \( TP_{r2} \) is a concave function of \( p_{12} \) and \( p_{22} \). Now, observing the retailer’s best response on retail prices, the manufacturers decide to offer wholesale prices and warranty periods with the purpose of maximizing their own profit. We establish two decision models by considering the manufacturers’ cooperative and noncooperative decision strategies.

Manufacturers’ noncooperative decision (MNC) strategy

In this situation, two manufacturers maximize their profits independently and make their decisions on wholesale prices and warranty periods individually, based on the reaction of the retailer. Hence, the manufacturers’ decision problem is formulated, as follows.
\[
\begin{align*}
\max_{(w_{12}, T_{L_{12}})} & \quad TP_{m_{12}}(w_{12}, T_{L_{12}}, w_{22}, T_{L_{22}}, p^*_{12}(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}}), p^*_{22}(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}})) \\
\max_{(w_{22}, T_{L_{22}})} & \quad TP_{m_{22}}(w_{12}, T_{L_{12}}, w_{22}, T_{L_{22}}, p^*_{12}(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}}), p^*_{22}(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}})) \\
& \text{subject to (31) and (32).}
\end{align*}
\]
The partial derivatives of $TP_{m_{i2}}$ with respect to $w_{i2}$ and $T_{Li2}$ are respectively as follows:

\[
\frac{\partial TP_{m_{i2}}}{\partial w_{i2}} = - (\beta_c + \eta_c) w_{i2} + \frac{1}{2} \eta_c w_{k2} + \frac{1}{2} \left\{ (\beta_t + \eta_t) T_{Li2} + (\beta_c + \eta_c) \lambda_i q_i^{- \delta_i} T_{Li2}^{\gamma_i} \right\} \\
- \frac{1}{2} \eta T_{k2} + \frac{1}{2} \left\{ \alpha_{pi} + (\beta_c + \eta_c) \left( c_i + c_m, \frac{q_i}{1 - q_i} \right) \right\}
\]

(34)

and

\[
\frac{\partial TP_{m_{i2}}}{\partial T_{Li2}} = \frac{1}{2} (\beta_t + \eta_t) \left( w_{i2} - c_i - c_m, \frac{q_i}{1 - q_i} - \lambda_i q_i^{- \delta_i} T_{Li2}^{\gamma_i} \right) \\
- \frac{1}{2} \lambda_i \gamma_i q_i^{- \delta_i} T_{Li2}^{\gamma_i - 1} \left\{ \alpha_{pi} - (\beta_c + \eta_c) w_{i2} + \eta_c w_{k2} + (\beta_t + \eta_t) T_{Li2} - \eta T_{k2} \right\}
\]

(35)

where $i \in \{1, 2\}$, and $k = 3 - i$. Equating the above partial derivatives to zero, we have

\[
w_{i2}^{mnc*} = \frac{2(\beta_c + \eta_c) \{ \alpha_{pi} + (\beta_c + \eta_c) c_i \} + \eta_c \{ \alpha_k + (\beta_c + \eta_c) c_k \}}{4(\beta_c + \eta_c)^2 - \eta_c^2} \\
+ \frac{2(\beta_c + \eta_c)^2 c_m, q_i}{(1 - q_i) \{ 4(\beta_c + \eta_c)^2 - \eta_c^2 \}} + \frac{2(\beta_c + \eta_c) \eta_c c_m, q_i}{(1 - q_i) \{ 4(\beta_c + \eta_c)^2 - \eta_c^2 \}} \\
+ \left\{ \frac{2(\beta_c + \eta_c)(\beta_t + \eta_t) \left( 1 + \frac{1}{\gamma_k} \right) - \eta T_{Li2}}{4(\beta_c + \eta_c)^2 - \eta_c^2} \right\} \text{ and }
\]

\[
T_{Li2}^{mnc*} = \left\{ \frac{q_k^{\delta_k}(\beta_t + \eta_t)}{\lambda_i \gamma_i (\beta_c + \eta_c)} \right\} \frac{1}{\gamma_k - 1}.
\]

(36)

The corresponding retail prices under MNC strategy respectively are as follows:

\[
p_{i2}^{mnc*} = \frac{w_{i2}^{mnc*}}{2} + \frac{(\beta_c + \eta_c)(\beta_t + \eta_t) - \eta_c \eta T_{Li2}^{mnc*}}{2 \beta_c (\beta_c + 2 \eta_c)} + \frac{(\beta_t + \eta_t) \eta_c - (\beta_c + \eta_c) \eta T_{Li2}^{mnc*}}{2 \beta_c (\beta_c + 2 \eta_c)} \\
+ \frac{(\beta_c + \eta_c) \alpha_{pi} + \eta_c \alpha_{pi}}{2 \beta_c (\beta_c + 2 \eta_c)}
\]

(38)

where $w_{i2}^{mnc*}$ and $T_{Li2}^{mnc*}$ are given in Equations (36) and (37).
Proposition 5. The profit function $TP_{m_{12}}$ under decentralized MNC strategy is a concave function in $w_{12}$ and $T_{L_{12}}$ if $(\gamma_{i} - 1)\{p_{1} - (\beta_{c} + \eta_{c})w_{12}^{mnc} + \eta_{c}w_{k2}^{mnc} - \eta_{T_{L_{k2}}}T_{L_{k2}}^{mnc}\} + (\gamma_{i} + 1)(\beta_{t} + \eta_{t})T_{L_{k2}}^{mnc} > 0$ and $(\gamma_{i} - 1)\{p_{1} - (\beta_{c} + \eta_{c})w_{12}^{mnc} + \eta_{c}w_{k2}^{mnc} + (\beta_{t} + \eta_{t})T_{L_{k2}}^{mnc} - \eta_{T_{L_{k2}}}T_{L_{k2}}^{mnc}\} > 0$, where $i \in \{1, 2\}$ and $k = 3 - i$.

Proof. The profit function $TP_{m_{11}}$ under decentralized MNC strategy would be concave in $w_{12}$ and $T_{L_{12}}$ if at the stationary point $(w_{12}^{mnc}, T_{L_{12}}^{mnc})$, the Hessian matrix of $TP_{m_{12}}$ is negative definite. Here, at $(w_{12}^{mnc}, T_{L_{12}}^{mnc})$

\[
\frac{\partial^{2}TP_{m_{12}}}{\partial w_{12}^{2}} = -(\beta_{c} + \eta_{c}) < 0,
\]

\[
\frac{\partial^{2}TP_{m_{12}}}{\partial T_{L_{12}}^{2}} = -(\beta_{t} + \eta_{t})T_{L_{k2}}^{mnc} < 0.
\]

if $(\gamma_{i} - 1)\{p_{1} - (\beta_{c} + \eta_{c})w_{12}^{mnc} + \eta_{c}w_{k2}^{mnc} - \eta_{T_{L_{k2}}}T_{L_{k2}}^{mnc}\} + (\gamma_{i} + 1)(\beta_{t} + \eta_{t})T_{L_{k2}}^{mnc} > 0$ holds.

\[
\frac{\partial^{2}TP_{m_{11}}}{\partial w_{12}^{2}} \frac{\partial^{2}TP_{m_{11}}}{\partial T_{L_{12}}^{2}} = \frac{\partial^{2}TP_{m_{11}}}{\partial w_{12}^{2}} T_{L_{12}}^{mnc} \frac{\partial^{2}TP_{m_{11}}}{\partial T_{L_{12}}^{2}} = (\beta_{c} + \eta_{c})\left[\frac{\gamma_{i} - 1}{2(\beta_{c} + \eta_{c})T_{L_{k2}}^{mnc}} \left(\beta_{t} + \eta_{t}\right) + (\beta_{t} + \eta_{t})^{2}\right] - (\beta_{t} + \eta_{t})^{2} > 0
\]

if $(\gamma_{i} - 1)\{p_{1} - (\beta_{c} + \eta_{c})w_{12}^{mnc} + \eta_{c}w_{k2}^{mnc} + (\beta_{t} + \eta_{t})T_{L_{k2}}^{mnc} - \eta_{T_{L_{k2}}}T_{L_{k2}}^{mnc}\} > 0$ holds, where $i \in \{1, 2\}$ and $k = 3 - i$ This completes the proof. □

Manufacturers’ cooperative (MC) decision strategy

In this strategy, two manufacturers operate jointly and agree to make decisions jointly in order to maximize their total profit, subject to the constraints imposed by equations in (31), and (32). Hence, the manufacturers’ decision problem is formulated as follows

\[
\max_{(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}})} (TP_{m_{12}} + TP_{m_{22}})\begin{pmatrix} w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}}, p_{1}, p_{2} \end{pmatrix}
\]

subject to (31) and (32).

The partial derivatives of $TP_{m_{12}} + TP_{m_{22}}$ with respect to $w_{12}, w_{22}, T_{L_{12}}$ and $T_{L_{22}}$ are respectively as follows:
\[
\frac{\partial (TP_{m12} + TP_{m22})}{\partial w_{12}} = - (\beta_c + \eta_c)w_{12} + \eta_c w_{22} + \frac{1}{2} \left((\beta_t + \eta_t)T_{L12}\right) + (\beta_c + \eta_c)\lambda_1 q_1^{-\delta_1 T_{L12}} + \frac{1}{2} \left\{\alpha_{p1} + (\beta_c + \eta_c)c_1 + c_{m1}(\beta_c + \eta_c)\frac{q_1}{1 - q_1}\right\} - \frac{1}{2} \eta_c c_2 + c_{m2} \frac{q_2}{1 - q_2},
\]
\[
\frac{\partial (TP_{m12} + TP_{m22})}{\partial w_{22}} = \eta_c w_{12} - (\beta_c + \eta_c)w_{22} - \frac{1}{2} \left\{\eta T_{L12} + \eta_c \lambda_1 q_1^{-\delta_1 T_{L12}}\right\} + \frac{1}{2} \left\{(\beta_t + \eta_t)T_{L22} + (\beta_c + \eta_c)\lambda_2 q_2^{-\delta_2 T_{L22}}\right\} - \frac{1}{2} \eta_c \left\{c_1 + c_{m1} \frac{q_1}{1 - q_1}\right\} + \frac{1}{2} \left\{\alpha_{p2} + (\beta_c + \eta_c)c_2 + c_{m2}(\beta_c + \eta_c)\frac{q_2}{1 - q_2}\right\},
\]
\[
\frac{\partial (TP_{m12} + TP_{m22})}{\partial T_{L12}} = \frac{1}{2} \left(\beta_t + \eta_t\right) \left\{w_{12} - c_1 - c_{m1} \frac{q_1}{1 - q_1} - \lambda_1 q_1^{-\delta_1 T_{L12}}\right\} - \frac{1}{2} \lambda_1 \gamma_1 q_1^{-\delta_1 T_{L12}} - \frac{1}{2} \eta_t \left\{w_{22} - c_2 - c_{m2} \frac{q_2}{1 - q_2} - \lambda_2 q_2^{-\delta_2 T_{L22}}\right\}
\]
and
\[
\frac{\partial (TP_{m12} + TP_{m22})}{\partial T_{L22}} = \frac{1}{2} \left(\beta_t + \eta_t\right) \left\{w_{22} - c_2 - c_{m2} \frac{q_2}{1 - q_2} - \lambda_2 q_2^{-\delta_2 T_{L22}}\right\} - \frac{1}{2} \lambda_2 \gamma_2 q_2^{-\delta_2 T_{L22}} - \frac{1}{2} \eta_t \left\{w_{12} - c_1 - c_{m1} \frac{q_1}{1 - q_1} - \lambda_1 q_1^{-\delta_1 T_{L12}}\right\}
\]

Solving equations \(\frac{\partial (TP_{m12} + TP_{m22})}{\partial w_{12}} = 0, \frac{\partial (TP_{m12} + TP_{m22})}{\partial w_{22}} = 0, \frac{\partial (TP_{m12} + TP_{m22})}{\partial T_{L12}} = 0 \) and \(\frac{\partial (TP_{m12} + TP_{m22})}{\partial T_{L22}} = 0\), we obtain the optimal values of \(w_{12}, w_{22}, T_{L12}\) and \(T_{L22}\). Analytically it is difficult to solve these equation. We solve the equation
numerically by using Matlab2013 software. Let the solution be $w_{12} = w_{12}^{mc*}$, $w_{22} = w_{22}^{mc*}$, $T_{L_{12}} = T_{L_{12}}^{mc*}$ and $T_{L_{22}} = T_{L_{22}}^{mc*}$.

**Proposition 6.** The profit function $(TP_{m12} + TP_{m22})(w_{12}, w_{22}, T_{L_{12}}, T_{L_{22}})$ is a concave function if $(\beta_c + \eta_c)^2 u_{11} + 2 \eta_c u_T u_{10} + (\beta_c + \eta_c) u_{10}^2 - \eta_c^2 u_{11}^2 + (\beta_c + \eta_c) u_{10}^2 < 0$ and $\{(\beta_c + \eta_c)^2 - \eta_c^2\} (u_{11} u_{13} - u_{12}^2) + (\beta_c + \eta_c) u_{13} (u_{10}^2 + u_{12}^2) + (\beta_c + \eta_c) u_{11} (u_{10}^2 + u_{12}^2) + (u_{10}^2 + u_{12}^2 - 2(\beta_c + \eta_c) u_{10} u_{12} - 2 \eta_c u_T u_{10} u_{12} - 2 \eta_c u_T u_{10} u_{11} + 2 \eta_c u_T u_{10} u_{11} + 2 \eta_c u_T u_{10} u_{13} - 2(\beta_c + \eta_c) u_T u_{13} - 2 u_T u_{10} u_{13} > 0$, where

$w_T = \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \beta_1 \gamma_1 q_1^{-1} (T_{L_{12}}^{mc*})^{-1}, u_{10} = \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \beta_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1}$,

$u_{11} = - \frac{\lambda_1 \gamma_1 (\gamma_1 - 1) q_1^{-1}}{2} \{\alpha_{p_1} - (\beta_c + \eta_c) w_{12}^{mc*} + \eta_T T_{L_{12}}^{mc*} - \frac{1}{2} \lambda_1 \gamma_1 (\gamma_1 + 1)(\beta_c + \eta_c) q_1^{-1} (T_{L_{12}}^{mc*})^{-1}, u_{12} = \frac{1}{2}(\beta_c + \eta_c) \beta_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1} + \frac{1}{2} \lambda_1 \gamma_1 (\gamma_1 + 1)(\beta_c + \eta_c) q_1^{-1} (T_{L_{22}}^{mc*})^{-1} - \frac{1}{2} \lambda_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1} \}$

and $u_{13} = - \frac{\lambda_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-2}}{2} \{\alpha_{p_2} - (\beta_c + \eta_c) w_{22}^{mc*} + \eta_T T_{L_{22}}^{mc*} - \frac{1}{2} \lambda_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1} \}$

$= \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \lambda_1 \gamma_1 q_1^{-1} (T_{L_{12}}^{mc*})^{-1} = w_T (say),$

$= \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \beta_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1} = u_T (say),$

$= \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \lambda_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1} = u_T (say)$. 

Proof. The second order partial derivatives of $(TP_{m12} + TP_{m22})$ at stationary point $S_3 = (w_{12}^{mc*}, w_{22}^{mc*}, T_{L_{12}}^{mc*}, T_{L_{22}}^{mc*})$ are

$$\begin{align*}
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{12} \partial w_{12}} &= - (\beta_c + \eta_c), \\
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{12} \partial w_{22}} &= \eta_c, \\
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{12} \partial T_{L_{12}}} &= \frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial T_{L_{12}} \partial w_{12}} = 1, \\
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{22} \partial T_{L_{22}}} &= \frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial T_{L_{22}} \partial w_{22}} = \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \lambda_1 \gamma_1 q_1^{-1} (T_{L_{12}}^{mc*})^{-1} = u_T (say), \\
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{12} \partial T_{L_{22}}} &= \frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial T_{L_{22}} \partial w_{12}} = \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \beta_2 q_2^{-1} (T_{L_{22}}^{mc*})^{-1} = u_T (say), \\
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{22} \partial T_{L_{12}}} &= \frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial T_{L_{12}} \partial w_{22}} = \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \lambda_1 \gamma_1 q_1^{-1} (T_{L_{12}}^{mc*})^{-1} = u_T (say), \\
\frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial w_{22} \partial T_{L_{22}}} &= \frac{\partial^2 (TP_{m12} + TP_{m22})}{\partial T_{L_{22}} \partial w_{22}} = \frac{1}{2}(\beta_c + \eta_c) + \frac{1}{2}(\beta_c + \eta_c) \lambda_1 \gamma_1 q_1^{-1} (T_{L_{12}}^{mc*})^{-1} = u_T (say). 
\end{align*}$$
\[
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}}^2} \bigg|_{at\, S_3} = -\frac{\lambda_1 \gamma_1 (\gamma_1 - 1) q_1^{-\delta_1} (T_{L_{12}}^{mcs})^{\gamma_1 - 2}}{2} \\
\quad \times \{\alpha_{p_1} - (\beta_c + \eta_c) w_{12}^{mcs} + \eta_c w_{12}^{mcs} - \eta_c T_{L_{12}}^{mcs}\} \\
- \frac{1}{2} \lambda_1 \gamma_1 (\gamma_1 + 1) (\beta_t + \eta_t) q_1^{-\delta_1} (T_{L_{12}}^{mcs})^{\gamma_1 - 1} = u_{11} \text{(say)},
\]

\[
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} \partial T_{L_{22}}} \bigg|_{at\, S_3} = \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}} \partial T_{L_{12}}} \bigg|_{at\, S_3} = \frac{1}{2} \eta \lambda_1 \gamma_1 q_1^{-\delta_1} (T_{L_{12}}^{mcs})^{\gamma_1 - 1} + \frac{1}{2} \eta \lambda_2 \gamma_2 q_2^{-\delta_2} (T_{L_{22}}^{mcs})^{\gamma_2 - 1} = u_{12} \text{(say)},
\]

\[
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}}^2} \bigg|_{at\, S_3} = -\frac{\lambda_2 \gamma_2 (\gamma_2 - 1) q_2^{-\delta_2} (T_{L_{22}}^{mcs})^{\gamma_2 - 2}}{2} \\
\quad \times \{\alpha_{p_2} - (\beta_c + \eta_c) w_{22}^{mcs} + \eta_c w_{22}^{mcs} - \eta_c T_{L_{22}}^{mcs}\} \\
- \frac{1}{2} \lambda_2 \gamma_2 (\gamma_2 + 1) (\beta_t + \eta_t) q_2^{-\delta_2} (T_{L_{22}}^{mcs})^{\gamma_2 - 1} = u_{13} \text{(say)}.
\]

The Hessian matrix \(H_3\) of \((TP_{m_{12}} + TP_{m_{22}})\) at the stationary point \(S_3 (w_{12}^{mcs}, w_{22}^{mcs}, T_{L_{12}}^{mcs}, T_{L_{22}}^{mcs})\):

\[
H_3 = \\
\begin{pmatrix}
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12}^2} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12} \partial w_{22}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12} \partial T_{L_{12}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12} \partial T_{L_{22}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12} \partial T_{L_{12}}^2} \\
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{22} \partial w_{12}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{22}^2} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{22} \partial T_{L_{12}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{22} \partial T_{L_{22}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{22} \partial T_{L_{12}}^2} \\
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} w_{12}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} w_{22}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} T_{L_{12}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} T_{L_{22}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} T_{L_{12}}^2} \\
\frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}} w_{12}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}} w_{22}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}} T_{L_{12}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}} T_{L_{22}}} & \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{22}} T_{L_{12}}^2}
\end{pmatrix} \bigg|_{at\, S_3}
\]

The profit function \(TP_{m_{12}} + TP_{m_{22}}\) will be concave function if the principal minors of \(H_3\) are alternatively negative and positive, i.e., if the \(i^{th}\) order principal minor \(D_i\) of \(H_3\) takes the sign \((-1)^i\). Here,

\[
D_1 = \begin{vmatrix} \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{12}^2} - (\beta_c + \eta_c) \end{vmatrix} < 0,
D_2 = \begin{vmatrix} \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial w_{22}^2} - (\beta_c + \eta_c) \end{vmatrix} \begin{vmatrix} \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}}^2} - (\beta_c + \eta_c) \end{vmatrix} > 0
\]

and

\[
D_3 = \begin{vmatrix} \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} w_{12}} - (\beta_c + \eta_c) \end{vmatrix} \begin{vmatrix} \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} w_{22}} - (\beta_c + \eta_c) \end{vmatrix} \begin{vmatrix} \frac{\partial^2 (TP_{m_{12}} + TP_{m_{22}})}{\partial T_{L_{12}} T_{L_{12}}} - (\beta_c + \eta_c) \end{vmatrix} < 0
\]
if $(\beta_c + \eta_c)^2 u_{11} + 2\eta_c u_7 u_9 + (\beta_c + \eta_c) u_9^2 - \eta_c^2 u_{11} + (\beta_c + \eta_c) u_9^2 < 0$ holds.

$$|H_3| = \{(\beta_c + \eta_c)^2 (u_{11} u_{13} - u_{12}^2) + (\beta_c + \eta_c) u_{13} (u_9^2 + u_9^2)
+ (\beta_c + \eta_c) u_{11} (u_9^2 + u_9^2) + (u_9^2 u_9^2 + u_9^2 u_9^2)
- 2(\beta_c + \eta_c) u_9 u_{10} u_{12} - 2\eta_c u_8 u_9 u_{12}
- 2\eta_c u_7 u_9 u_{12} + 2\eta_c u_8 u_{10} u_{11} + 2\eta_c u_7 u_9 u_{13} - 2(\beta_c + \eta_c) u_7 u_8 u_{12} - 2u_7 u_8 u_{10} > 0,$$

if $\{(\beta_c + \eta_c)^2 - \eta_c^2\}(u_{11} u_{13} - u_{12}^2) + (\beta_c + \eta_c) u_{13} (u_9^2 + u_9^2) + (\beta_c + \eta_c) u_{11} (u_9^2 + u_9^2) + (u_9^2 u_9^2 + u_9^2 u_9^2)
- 2(\beta_c + \eta_c) u_9 u_{10} u_{12} - 2\eta_c u_8 u_9 u_{12} - 2\eta_c u_7 u_9 u_{12} + 2\eta_c u_8 u_{10} u_{11} + 2\eta_c u_7 u_9 u_{13} - 2(\beta_c + \eta_c) u_7 u_8 u_{12} - 2u_7 u_8 u_{10} > 0$ holds. This completes the proof. □

**Proposition 7.** Under decentralized MC strategy, the profit of each channel member is equal, that is $TP_{m12} = TP_{m22} = TP_{r2}$ and independent of $\eta_c$ and $\eta_t$ if two manufacturers are identical (that is, $\alpha_{p1} = \alpha_{p2}, c_1 = c_2, c_{m1} = c_{m2}, q_1 = q_2, \lambda_1 = \lambda_2, \gamma_1 = \gamma_2,$ and $\delta_1 = \delta_2$).

**Proof.** Under symmetrical condition of two complementary products, at stationary point $S_3$ we have

$$\frac{\partial (TP_{m12} + TP_{m22})}{\partial w_{22}} \bigg|_{\text{at} S_3} = \frac{\partial (TP_{m12} + TP_{m22})}{\partial w_{12}} \bigg|_{\text{at} S_3} = 2(\beta_c + 2\eta_c)(w_{m2} - w_{m2})$$

$$-\lambda(\beta_c + 2\eta_c) q \gamma^{-\delta}(T_{m12}^{*} - T_{m22}^{*}) - (\beta_c + 2\eta_c)(T_{m12}^{*} - T_{m22}^{*}) = 0$$

(44)

and

$$\frac{\partial (TP_{m12} + TP_{m22})}{\partial T_{12}} \bigg|_{\text{at} S_3} = \frac{\partial (TP_{m12} + TP_{m22})}{\partial T_{22}} \bigg|_{\text{at} S_3} = (\beta_c + 2\eta_c)(w_{m2} - w_{m2})$$

$$-\lambda(\beta_c + 2\eta_c) q \gamma^{-\delta}(T_{m12}^{*} - T_{m22}^{*}) - \lambda q^{-\delta}(\eta_c w_{m2}^{*} - (\beta_c + \eta_c) w_{m2}^{*})T_{m12}^{*} - 1$$

$+$ $(\beta_c + \eta_c) w_{m2}^{*} - \eta_c w_{m2}^{*} T_{m12}^{*} - 1 + (\beta_c + \eta_c)(T_{m12}^{*} - T_{m22}^{*})$

$-$ $\eta_c T_{m12}^{*} - 2 - T_{m22}^{*} + \alpha(T_{m12}^{*} - T_{m22}^{*}) = 0$

(45)

where $\alpha_{p1} = \alpha_{p2} = \alpha_{p}(\text{say}), c_1 = c_2 = c(\text{say}), c_{m1} = c_{m2} = c_m(\text{say}), q_1 = q_2 = q(\text{say}), \lambda_1 = \lambda_2 = \lambda(\text{say}), \gamma_1 = \gamma_2 = \gamma(\text{say})$ and $\delta_1 = \delta_2 = \delta(\text{say})$. Now, from (44) we can write

$$((\beta_c + 2\eta_c)(w_{m2} - w_{m2}) = (T_{m12}^{*} - T_{m22}^{*}) g(T_{m12}^{*} - T_{m22}^{*})(\text{say})$$

(46)

$$= 2\eta_c, w_{m2}^{*} - (\beta_c + \eta_c) w_{m2}^{*}$$

$$- (T_{m12}^{*} - T_{m22}^{*}) g(T_{m12}^{*} - T_{m22}^{*})$$

(47)
Thus, from Equations (48) and (46) we can conclude that
\[ T_{w_{1z}} \sim T_{w_{2z}} \]
and 2\( g(T_{w_{1z}}, T_{w_{2z}}) = (\beta_i + 2\eta_i) + \lambda q^{-\delta}(\beta_c + 2\eta_c)g_1(T_{w_{1z}}, T_{w_{2z}}) \). Hence from (45), we get
\[
(T_{mc} - T_{mc}^*) \left[ \frac{\beta_i + 2\eta_i}{\beta_c + 2\eta_c} g(T_{w_{1z}}, T_{w_{2z}}) - \lambda q^{-\delta}(\beta_i + 2\eta_i)g_1(T_{w_{1z}}, T_{w_{2z}}) 

- \lambda q^{-\delta}(\beta_i + \eta_i)T_{mc} - \lambda q^{-\delta}(\beta_i + \eta_i)T_{mc}^* \bigg) \right] = 0.
\]
Thus, from Equations (48), and (46) we can conclude that \( T_{mc}^* = T_{mc}^* \) and \( w_{1z} = w_{2z} \) is a solution of the Equations (40)-(43). Now if \( T_{mc}^* = T_{mc}^* = T_{mc}^*(\text{say}) \) and \( w_{1z} = w_{2z} = w_{2z}^*(\text{say}) \) is the optimal solution of the manufacturers, then from Equations (31) and (32), we get
\[
p_{mc} = p_{2z}^* = \frac{1}{2} \left( \frac{\alpha_p}{\beta_c} + \frac{\beta_c}{\beta_c} \right) = p_{2z}^*(\text{say})
\]
and equating the partial derivative \( \frac{\partial(T_{mc} + T_{mc}^*)}{\partial w_{1z}} \) (given in expression (40) ) to zero we get
\[
-\frac{\beta_c}{2} \left( w_{2z} - c - c_m \frac{q}{1-q} - \lambda q^{-\delta}T_{mc}^* \right) + \frac{1}{2}(\alpha_p - \beta_c w_{2z}^* + \beta_i T_{mc}^*) = 0. \tag{49}
\]
Hence the manufacturer’s optimal profit becomes
\[
T_{mc}^* = \left( p_{2z}^* - w_{2z}^* \right)(\alpha_p - \beta_c w_{2z}^* + \beta_i T_{mc}^*)
\]
and the retailer optimal profit becomes
\[
T_{m_{2z}} = \left( p_{2z}^* - w_{2z}^* \right)(\alpha_p - \beta_c w_{2z}^* + \beta_i T_{mc}^*)
\]
\[
= \frac{1}{2} \left( \frac{\alpha_p}{\beta_c} + \frac{\beta_c}{\beta_c} \right)(\alpha_p - \beta_c w_{2z}^* + \beta_i T_{mc}^*)
\]
\[
= \frac{1}{2} \left( w_{2z}^* - c - c_m \frac{q}{1-q} - \lambda q^{-\delta}T_{mc}^* \right)(\alpha_p - \beta_c w_{2z}^* + \beta_i T_{mc}^*)
\]
\[\text{from Equation (49))}\]
This completes the proof. \( \square \)
4.2.2. Centralized decisions

In this case, both the manufacturers and their common retailer cooperate and together they make the decision that maximizes the overall supply chain profit. The total profit function under this scenario is

\[ TP_{c_2} = TP_{m_{12}} + TP_{m_{22}} + TP_r \]

\[ = \sum_{i=1}^{2} \left( p_{i2} - c_i - c_{m_i} \frac{q_i}{1 - q_i} - \lambda_i q_i^{-\delta_i} T_{L_{12}} \right) D_{i2}. \]  

(50)

Hence, the channel members’ decision problem is formulated as follows

\[ \max_{(p_{12}, p_{22}, T_{L_{12}}, T_{L_{22}})} TP_{c_2}(p_{12}, p_{22}, T_{L_{12}}, T_{L_{22}}). \]

(51)

The partial derivatives of \( TP_{m_{12}} + TP_{m_{22}} \) with respect to \( p_{12}, p_{22}, T_{L_{12}} \) and \( T_{L_{22}} \) are respectively as follows:

\[ \frac{\partial TP_{c_2}}{\partial p_{12}} = -2(\beta_c + \eta_c)p_{12} + \{ (\beta_t + \eta_t)T_{L_{12}} + (\beta_c + \eta_c)\lambda_1 q_1^{-\delta_1} T_{L_{12}} \} \]

\[ - (\eta_c T_{L_{22}} + \eta_c \lambda_2 q_2^{-\delta_2} T_{L_{22}}^\gamma) + \left\{ \alpha_{p_1} + (\beta_c + \eta_c) c_1 + c_{m_1} (\beta_c + \eta_c) \frac{q_1}{1 - q_1} \right\} \]

\[ - \eta \left\{ c_2 + c_{m_2} \frac{q_2}{1 - q_2} \right\}. \]

(52)

\[ \frac{\partial TP_{c_2}}{\partial p_{22}} = 2\eta_c p_{12} - 2(\beta_c + \eta_c)p_{22} - \{ \eta_c T_{L_{12}} + \eta_c \lambda_1 q_1^{-\delta_1} T_{L_{12}}^\gamma \} \]

\[ + \{ (\beta_t + \eta_t)T_{L_{22}} + (\beta_c + \eta_c)\lambda_2 q_2^{-\delta_2} T_{L_{22}}^\gamma \} - \eta \left\{ c_1 + c_{m_1} \frac{q_1}{1 - q_1} \right\} \]

\[ + \left\{ \alpha_{p_2} + (\beta_c + \eta_c) c_2 + c_{m_2} (\beta_c + \eta_c) \frac{q_2}{1 - q_2} \right\}. \]

(53)

\[ \frac{\partial TP_{c_2}}{\partial T_{L_{12}}} = (\beta_t + \eta_t) \left\{ p_{12} - c_1 - c_{m_1} \frac{q_1}{1 - q_1} - \lambda_1 q_1^{-\delta_1} T_{L_{12}}^\gamma \right\} \]

\[ - \lambda_1 q_1^{-\delta_1} T_{L_{12}}^\gamma \left\{ \alpha_{p_1} - (\beta_c + \eta_c) p_{12} + \eta_c p_{22} + (\beta_t + \eta_t) T_{L_{12}} - \eta c T_{L_{22}} \right\} \]

\[ - \eta \left\{ p_{22} - c_2 - c_{m_2} \frac{q_2}{1 - q_2} - \lambda_2 q_2^{-\delta_2} T_{L_{22}}^\gamma \right\} \]

(54)

and

\[ \frac{\partial TP_{c_2}}{\partial T_{L_{12}}} = (\beta_t + \eta_t) \left\{ p_{22} - c_2 - c_{m_2} \frac{q_2}{1 - q_2} - \lambda_2 q_2^{-\delta_2} T_{L_{22}}^\gamma \right\} \]

\[ - \lambda_2 q_2^{-\delta_2} T_{L_{22}}^\gamma \left\{ \alpha_{p_2} - (\beta_c + \eta_c) p_{22} + \eta_c p_{12} + (\beta_t + \eta_t) T_{L_{22}} - \eta c T_{L_{12}} \right\} \]

\[ - \eta \left\{ p_{12} - c_1 - c_{m_1} \frac{q_1}{1 - q_1} - \lambda_1 q_1^{-\delta_1} T_{L_{12}}^\gamma \right\}. \]  

(55)
Proof. where

Proposition 8. The profit function \( TP_c(p_{12}, p_{22}, T_{L_{12}}, T_{L_{22}}) \) is a concave function if \( 4(\beta_c + \eta_c)^2u_{18} + 4\eta_c u_{14}u_{16} + 2(\beta_c + \eta_c)u_{16}^2 - 4\eta_c u_{18} + 2(\beta_c + \eta_c)u_{14}^2 < 0 \) and

\[
\left\{ (\beta_c + \eta_c)^2 \eta_c \right\} (u_{18}u_{20} - u_{19}^2) + 2(\beta_c + \eta_c)u_{20}(u_{16}^2 + u_{14}^2) + 2(\beta_c + \eta_c)u_{18}(u_{17}^2 + u_{15}^2) + (u_{15}^2u_{16} + u_{14}^2u_{17}) - 4(\beta_c + \eta_c)u_{16}u_{17}u_{19} - 4\eta_c u_{15}u_{16}u_{19} - 4\eta_c u_{14}u_{17}u_{19} + 4\eta_c u_{15}u_{17}u_{20} - 4(\beta_c + \eta_c)u_{14}u_{15}u_{19} - 2u_{14}u_{15}u_{16}u_{17} > 0,
\]

where \( u_{14} = (\beta_c + \eta_c) + (\beta_c + \eta_c)\lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} \), \( u_{15} = -\eta_c - \eta_c\lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-1} \), \( u_{16} = -\eta_c - \eta_c\lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} \), \( u_{17} = (\beta_c + \eta_c) + (\beta_c + \eta_c)\lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-1} \), \( u_{18} = -\lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} \), \( u_{19} = \eta_c - \eta_c\lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} - \eta_c\lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-1} \), \( u_{20} = -\lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-2} \), \( \{ \alpha_{p_2} - (\beta_c + \eta_c)q_2^{\delta_2} + \eta_c p_{22}^* - \eta_T T_{12}^* \} - \lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} \), \( \{ \alpha_{p_2} - (\beta_c + \eta_c)q_2^{\delta_2} - \eta_c p_{22}^* - \eta_T T_{12}^* \} - \lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-1} \).

Proof. The second order partial derivatives of \( TP_c \) at stationary point \( S_4 = (p_{12}^*, p_{22}^*, T_{L_{12}}^*, T_{L_{22}}^*) \) are

\[
\left. \frac{\partial^2 TP_c}{\partial p_{12}^2} \right|_{at S_4} = -2(\beta_c + \eta_c), \quad \left. \frac{\partial^2 TP_c}{\partial p_{22}^2} \right|_{at S_4} = -2(\beta_c + \eta_c),
\]

\[
\left. \frac{\partial^2 TP_c}{\partial p_{12} \partial p_{22}} \right|_{at S_4} = \left. \frac{\partial^2 TP_c}{\partial p_{22} \partial p_{12}} \right|_{at S_4} = 2\eta_c,
\]

\[
\left. \frac{\partial^2 TP_c}{\partial T_{L_{12}}^2} \right|_{at S_4} = \left. \frac{\partial^2 TP_c}{\partial T_{L_{22}}^2} \right|_{at S_4} = (\beta_c + \eta_c) + (\beta_c + \eta_c)\lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} = u_{14}(say),
\]

\[
\left. \frac{\partial^2 TP_c}{\partial T_{L_{12}} \partial p_{12}} \right|_{at S_4} = \left. \frac{\partial^2 TP_c}{\partial T_{L_{22}} \partial p_{22}} \right|_{at S_4} = -\eta_c - \eta_c\lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-1} = u_{15}(say),
\]

\[
\left. \frac{\partial^2 TP_c}{\partial T_{L_{22}} \partial p_{12}} \right|_{at S_4} = \left. \frac{\partial^2 TP_c}{\partial T_{L_{12}} \partial p_{22}} \right|_{at S_4} = -\eta_c - \eta_c\lambda_1 q_1^{\delta_1}(T_{L_{12}}^*)^{\gamma_1-1} = u_{16}(say),
\]

\[
\left. \frac{\partial^2 TP_c}{\partial p_{12} \partial T_{L_{22}}^2} \right|_{at S_4} = \left. \frac{\partial^2 TP_c}{\partial p_{22} \partial T_{L_{12}}^2} \right|_{at S_4} = (\beta_c + \eta_c) + (\beta_c + \eta_c)\lambda_2 q_2^{\delta_2}(T_{L_{22}}^*)^{\gamma_2-1} = u_{17}(say).
\]
\[
\frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}^2} \bigg|_{atS_4} = -\lambda_1 \gamma_1 (\gamma_1 - 1) q_1^{-\delta_1} (T_{L_{12}}^*)^{\gamma_1 - 2} \{\alpha_{p_1} - (\beta_c + \eta_c)p_{12}^* + \eta_c p_{22}^* \\
- \eta_c T_{L_{12}}^*\} - \lambda_1 \gamma_1 (\gamma_1 + 1)(\beta_i + \eta_i)q_i^{-\delta_1} (T_{L_{12}}^*)^{\gamma_1 - 1} = u_{18} \text{(say)},
\]

\[
\frac{\partial^2 TP_{c2}}{\partial T_{L_{12}} \partial T_{L_{22}}} \bigg|_{atS_4} = \frac{\partial^2 TP_{c2}}{\partial T_{L_{22}} \partial T_{L_{12}}} \bigg|_{atS_4} = \eta_c \lambda_1 q_1^{-\delta_1} (T_{L_{12}}^*)^{\gamma_1 - 1} + \eta_c \lambda_2 q_2^{-\delta_2} (T_{L_{22}}^*)^{\gamma_2 - 1} = u_{19} \text{(say)},
\]

\[
\frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}^2} \bigg|_{atS_4} = -\lambda_2 \gamma_2 (\gamma_2 - 1) q_2^{-\delta_2} (T_{L_{22}}^*)^{\gamma_2 - 2} \{\alpha_{p_2} - (\beta_c + \eta_c)p_{22}^* + \eta_c p_{12}^* \\
- \eta_c T_{L_{22}}^*\} - \lambda_2 \gamma_2 (\gamma_2 + 1)(\beta_i + \eta_i)q_2^{-\delta_2} (T_{L_{22}}^*)^{\gamma_2 - 1} = u_{20} \text{(say)}.
\]

The Hessian matrix $H_4$ of $TP_{c2}$ at the stationary point $(p_{12}^*, p_{22}^*, T_{L_{12}}^*)$

\[
H_4 = \begin{pmatrix}
\frac{\partial^2 TP_{c2}}{\partial T_{L_{12}}^2} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{12}} \partial T_{L_{22}}} & \cdots & \frac{\partial^2 TP_{c2}}{\partial T_{L_{L2}}^2} \\
\frac{\partial^2 TP_{c2}}{\partial T_{L_{12}} \partial T_{L_{22}}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{22}}^2} & \cdots & \frac{\partial^2 TP_{c2}}{\partial T_{L_{L2}}^2} \\
\frac{\partial^2 TP_{c2}}{\partial T_{L_{L2}} \partial T_{L_{12}}} & \frac{\partial^2 TP_{c2}}{\partial T_{L_{L2}}^2} & \cdots & \frac{\partial^2 TP_{c2}}{\partial T_{L_{L2}}^2} \\
\end{pmatrix} \bigg|_{atS_4}
\]

The profit function $TP_{c2}$ will be concave function if the principal minors of $H_4$ are alternatively negative and positive, i.e., if the $i^{th}$ order principal minor $D_i$ of $H_4$ takes the sign $(-1)^i$. Here,

\[
D_1 = -2(\beta_c + \eta_c) < 0,
\]

\[
D_2 = \begin{vmatrix}
-2(\beta_c + \eta_c) & 2\eta_c \\
2\eta_c & -2(\beta_c + \eta_c)
\end{vmatrix} = 4(\beta_c + \eta_c)^2 - 4\eta_c^2 > 0
\]

and

\[
D_3 = \begin{vmatrix}
-2(\beta_c + \eta_c) & 2\eta_c & u_{14} \\
2\eta_c & -2(\beta_c + \eta_c) & u_{16} \\
u_{14} & u_{16} & u_{18}
\end{vmatrix} = 4(\beta_c + \eta_c)^2 u_{18} + 4\eta_c u_{14} u_{16} + 2(\beta_c + \eta_c)u_{14}^2 - 4\eta_c^2 u_{18} + 2(\beta_c + \eta_c)u_{14}^2 < 0
\]

if $4(\beta_c + \eta_c)^2 u_{18} + 4\eta_c u_{14} u_{16} + 2(\beta_c + \eta_c)u_{14}^2 - 4\eta_c^2 u_{18} + 2(\beta_c + \eta_c)u_{14}^2 < 0$ holds.

\[
|H_4| = 4 \{(\beta_c + \eta_c)^2 - \eta_c^2\} (u_{18} u_{20} - u_{19}^2) + 2(\beta_c + \eta_c)u_{20}(u_{16}^2 + u_{14}^2) + 2(\beta_c + \eta_c)u_{14}(u_{16}^2 + u_{18}^2) + 4(\beta_c + \eta_c) u_{14} u_{16} u_{17} u_{19}
\]

\[
- 4(\beta_c + \eta_c) u_{15} u_{16} u_{19} - 4(\beta_c + \eta_c) u_{14} u_{17} u_{19} + 4(\beta_c + \eta_c) u_{15} u_{17} u_{18} + 4(\beta_c + \eta_c) u_{14} u_{16} u_{20}
\]

\[
= 4(\beta_c + \eta_c) u_{14} u_{15} u_{16} u_{19} - 2u_{14} u_{15} u_{16} u_{17} > 0,
\]

if $4 \{(\beta_c + \eta_c)^2 - \eta_c^2\} (u_{18} u_{20} - u_{19}^2) + 2(\beta_c + \eta_c)u_{20}(u_{16}^2 + u_{14}^2) + 2(\beta_c + \eta_c)u_{14}(u_{16}^2 + u_{18}^2) + 4(\beta_c + \eta_c) u_{14} u_{16} u_{17} u_{19} + 4(\beta_c + \eta_c) u_{15} u_{16} u_{19} - 4(\beta_c + \eta_c) u_{14} u_{17} u_{19} + 4(\beta_c + \eta_c) u_{15} u_{17} u_{18} + 4(\beta_c + \eta_c) u_{14} u_{16} u_{20}$
$\eta_c u_15 u_17 u_18 + 4 \eta_c u_14 u_16 u_20 - 4 (\beta_c + \eta_c) u_14 u_15 u_19 - 2 u_14 u_15 u_16 u_17 > 0$ holds.

This completes the proof. \( \square \)

5. NUMERICAL ANALYSIS

In this section, we compare the optimal solutions for different scenarios with the following numerical data: $\alpha_{p_1} = \alpha_{p_2} = 50; c_1 = c_2 = 10; c_{m_1} = c_{m_2} = 0.5; \beta_c = 7.5; \eta_c = 0.45; \beta_t = 6.2; \eta_t = 0.3; \delta_1 = \delta_2 = 0.5; \lambda_1 = \lambda_2 = 0.25; \gamma_1 = \gamma_2 = 2.$

In Table 1, we observe that in both cases 1, and 2, centralized decision policy is the better strategy for overall supply chain than the decentralized decision policies. Table 1 also indicates that, in case 1 when only the manufacturer 1 adopts warranty policy, the retail price of product 1 is the highest in MC model, followed by MNC model, and centralized model. For product 2, the retail price is the highest in MNC, followed by MC model and centralized model. In case 2, when both the manufacturers adopt warranty policy, the optimal decisions on pricing and warranty strategies of two manufactures are the same under the identical manufacturer assumption, and the retail price of each product is highest in MC model, followed by MNC model and centralized model.

| Case (j) | Models | $p_{1j}$ | $p_{2j}$ | $TP_{rj}$ | $T_{L_{1j}}$ | $T_{L_{2j}}$ | $TP_{m_{1j}}$ | $TP_{m_{2j}}$ | Total Profit |
|----------|--------|---------|---------|----------|-------------|-------------|--------------|--------------|--------------|
|          | MNC    | 23.33   | 22.55   | 264.98   | 19.06       | 1.156       | 259.94       | 18.43        | 260.09       | 785.01       |
| Centralized | MC    | 23.46   | 22.51   | 260.15   | 19.30       | 1.169       | 259.55       | 18.34        | 260.76       | 780.46       |
|          | MNC    | 23.34   | 23.34   | 275.25   | 19.06       | 1.156       | 259.67       | 19.00        | 1.156       | 259.67       | 794.59       |
| Centralized | MC    | 23.47   | 23.47   | 259.89   | 19.31       | 1.169       | 259.89       | 19.31        | 1.169       | 259.89       | 779.68       |
|          | MNC    | 19.31   | 19.31   | 1.169    | 1.169       | 1.169       | 1.169        | 1.169        | 1.169        | 1039.57      |
| Centralized | MC    | 19.31   | 19.31   | 1.169    | 1.169       | 1.169       | 1.169        | 1.169        | 1.169        | 1039.57      |

Table 1: Optimal results for different scenarios

We study the changes of optimal profits of the two manufacturers and their common retailer by changing the model parameters under different decision strategies (Tables 2-3) to help decision makers take proper marketing decision strategy and examine when manufacturer 2 generates more profit by offering a warranty period on his product. Based on the optimal solutions provided in Tables 1-3, it is also observed that the retailer makes more profit in MNC strategy than in MC strategy for case 2. In case 1, MNC decision strategy can yield more profit for manufacturer 1 while manufacturer 2 is better off in MC decision strategy. As compared with case 1, the retailer makes more profit in case 2 under MNC decision strategy. From Tables 2-3, we observe the following features and managerial insights:

Table 2 shows that while $\beta_c$ increases, the optimal profits of manufacturers and retailer decrease in MNC model and MC model for both cases 1 and 2. The
profit of manufacturer 2 in case 2 will be higher than his profit in case 1, as long as $\beta_c \leq 7.20$ in MNC and MC models. We also see that in Case 1, as $\beta_c$ increases above a certain level ($\geq 7.60$ for MNC model and $\geq 7.20$ for MC model) manufacturer 2’s profit is greater than the manufacturer 1’s profit, which indicates that as price sensitivity coefficient increases, it becomes unprofitable to adopt warranty policy.

With increase in $\beta_t$, the optimal profits of manufacturers and retailer increases in MNC model (for case-2) and MC model (for both cases 1 and 2) but in case-1 the profits of manufacturer 2 and retailer decrease in MNC model (see Table 2). The manufacturer 2 generates more profit by offering warranty period in MNC model if $\beta_t \geq 6.30$ and in MC model if $\beta_t \geq 6.40$.

When $\eta_c$ increases, the optimal profits of the retailer and manufacturer 2 increase but optimal profit of manufacturer 1 decreases in all model structures for case 1. But an opposite behavior in the optimal profits of channel members is recorded in all model structures for case 1 when $\eta_t$ increases. In case 2, with the increasing value of $\eta_c$, in MNC model, the optimal profit of retailer increases but the optimal profit of each manufacturer decreases and with increasing value of $\eta_t$ the optimal profit of each manufacturer and their common retailer increase. The optimal profit of each channel member remains unchanged when the sensitivity of MC model in case 2 is investigated for the changes in $\eta_c$ and $\eta_t$, which supports Proposition 4.2.1.

Table 3 shows that the optimal profit of manufacturer $i$ of all model structures for both cases is concave with respect to his product quality level $q_i$, i.e., initial increment of $q_i$ reduces his warranty cost and increases profit, but after a certain level increase in increment of $q_i$ increases his quality improvement cost and hence, decreases the profit. The optimal profit of manufacturer $k$ of all model structures for both cases increases with increasing value of product quality level $q_i$, and the optimal profit of the retailer is also concave with respect to $q_i$.

With the increase in $\lambda_i$, optimal profits of manufacturer $i$ and the retailer decrease but optimal profit of manufacturer $k$ increases in all model structures for both cases except retailer’s optimal profit of MNC model in case 1, which increases with increasing value of $\lambda_i$. 
26 S. Kundu, et al. / Joint Optimal Decisions on Pricing and Warranty Policy

Table 2: Optimal profits of channel members for changing the values of \( \beta_c, \beta_t, \eta_c \) and \( \eta_t \) under different scenarios.

| Parameter | MNC | MC |
|-----------|-----|----|
| \( \beta_c \) | 0.40 | 0.30 | 0.26 | 0.24 |
| \( \beta_t \) | 0.80 | 0.60 | 0.40 | 0.30 | 0.20 |
| \( \eta_c \) | 0.35 | 0.31 | 0.29 | 0.27 |
| \( \eta_t \) | 0.50 | 0.48 | 0.46 | 0.44 | 0.42 |

Table 3: Optimal profits of channel members for changing the values of \( q_i \) and \( \lambda_k \) under different scenarios, where \( i \in \{1, 2\} \) and \( k = 3 - i \).

| Parameter | MNC | MC |
|-----------|-----|----|
| \( q_1 \) | 0.20 | 0.30 | 0.40 | 0.50 |
| \( q_2 \) | 0.60 | 0.70 | 0.80 | 0.90 |
| \( \lambda_1 \) | 0.20 | 0.30 | 0.40 | 0.50 |
| \( \lambda_2 \) | 0.60 | 0.70 | 0.80 | 0.90 |

Table 4: Comparison of optimal results with Wei et al.’s [18] model when both manufacturers adopt warranty policy.

| Models | \( TP_{m12} \) | \( TP_{m22} \) | \( TP_{r2} \) | Total Profit |
|--------|---------------|---------------|---------------|-------------|
| Our MNC | 9354.50 | 9354.50 | 9305.10 | 28014.1 |
| Model MC | 9250.40 | 9250.40 | 7751.2 | 9305.10 |
| Wei’s MNC | 7553.80 | 7553.80 | 4835.70 | 19943.3 |
| Model MC | 8350.60 | 8350.60 | 10105.0 | 26806.2 |
Table 4, we have compared our model with the model in Wei et al. [18] using their numerical data as follows: \( \alpha_{p_1} = \alpha_{p_2} = 100; c_1 = c_2 = 30; \eta_c = .25; \beta_c + \eta_c = .30; \eta_t = 0.2; \beta_t = .3 \) and remaining parameters of our model remain unchanged. We observed that in our model when both manufacturers adopt warranty policy, profits of manufacturers and retailer are higher than that of Wei et al.’s [18] model. Because Wei et al. [18] expressed the product’s demand function as decreasing function of its selling price, as well as its complementary product’s selling price and increasing function of its warranty period and its complementary product’s warranty period. So, in order to maximize the market demand, the manufacturers decrease product’s price and increase the warranty period which amplify warranty cost and result in lower values of profits. Differing from their study, in this model we consider the demand of each product decreasing with its own selling price and the competitor’s warranty period and increasing with its own warranty period and the competitor’s product selling price, that corresponds with reality in many practical situations.

6. CONCLUSION

In this article, we studied the importance of price and warranty in the interactions between two manufacturers and their common retailer for two complementary products under decentralized and centralized decision strategies. We consider that the demand of products to depend not only on price but also on warranty period. The role of warranty as a competitive strategy has been explored by examining the model through two different scenarios (i) only one manufacturer offers warranty on his product, (ii) both manufacturers offer warranty on their product. We observed that as price sensitivity factor increases, the adoption of warranty policy would become more unprofitable for the manufacture but with the increase of warranty period sensitivity factor, the manufacturer inclines to adopt the warranty policy. Numerical analysis also reveals that in case 1 if a manufacturer adopts warranty policy, then he will be more profitable under MNC decision strategy. In both cases, the retailer always earns more profit under MNC decision strategy as compared to the MC strategy. Since under MC strategy the manufacturers make decision jointly instead of independently and retailer acts as their follower. We also find that manufacturer profit function is concave with respect to product quality level.

The proposed model could be extended in many aspects such as developing the model under stochastic demand pattern, introducing competitive strategies among multiple retailers and incorporating some contract mechanisms (e.g., price discount contract, revenue sharing contract, wholesale price sharing contract etc) to coordinate the supply chain.

Acknowledgement: The authors thank the anonymous reviewers for their positive comments and suggestions that improved the presentation of the paper.
REFERENCES

[1] AY Alqahtani and SM Gupta. Renewable two-dimensional pro-rata warranty analysis for reverse supply chain. In Proceedings of Annual Meeting of the Decision Sciences Institute (DSI), Austin, TX, 2016.

[2] E Ashgarizadeh and dan DNP Murthy. Service contracts: A stochastic model. Mathematical and Computer Modelling, 31(10-12):11–20, 2000.

[3] Anette Boom. Product risk sharing by warranties in a monopoly market with risk-averse consumers. Journal of economic behavior & organization, 33(2):241–257, 1998.

[4] William Boulding and Anna Kirmani. A consumer-side experimental examination of signaling theory: do consumers perceive warranties as signals of quality? Journal of consumer research, 20(1):111–123, 1993.

[5] S Chan Choi. Price competition in a channel structure with a common retailer. Marketing Science, 10(4):271–296, 1991.

[6] S Chan Choi. Price competition in a duopoly common retailer channel. Journal of retailing, 72(2):117–134, 1996.

[7] Joseph Hall and Evan Porteus. Customer service competition in capacitated systems. Manufacturing & Service Operations Management, 2(2):144–165, 2000.

[8] Ganesh Iyer. Coordinating channels under price and nonprice competition. Marketing Science, 17(4):338–355, 1998.

[9] Jye-Chyi Lu, Yu-Chung Tsao, and Chayakrit Charoensiriwath. Competition under manufacturer service and retail price. Economic Modelling, 28(3):1256–1264, 2011.

[10] Zheng Luo, Xu Chen, Jing Chen, and Xiaojun Wang. Optimal pricing policies for differentiated brands under different supply chain power structures. European Journal of Operational Research, 259(2):437–451, 2017.

[11] Juergen Noll. Comparing quality signals as tools of consumer protection: are warranties always better than advertisements to promote higher product quality? International Review of Law and Economics, 24(2):227–239, 2004.

[12] Jagmohan S Raju, Raj Sethuraman, and Sanjay K Dhar. The introduction and performance of store brands. Management science, 41(6):957–978, 1995.

[13] Koichiro Rinsaka and Hiroaki Sandoh. A stochastic model on an additional warranty service contract. Computers & Mathematics with Applications, 51(2):179–188, 2006.

[14] Ata Allah Taleizadeh, Shirin Hadadpour, Leopoldo Eduardo Cárdenas-Barrón, and Ali Akbar Shaikh. Warranty and price optimization in a competitive duopoly supply chain with parallel importation. International Journal of Production Economics, 185:76–88, 2017.

[15] Yu-Chung Tsao and Po-Yuan Su. A dual-channel supply chain model under price and warranty competition. International Journal of Innovative Computing, Information and Control, 8(3):2125–2135, 2012.

[16] Andy A Tsay and Narendra Agrawal. Channel dynamics under price and service competition. Manufacturing & Service Operations Management, 2(4):372–391, 2000.

[17] Andy A Tsay and Narendra Agrawal. Channel conflict and coordination in the e-commerce age. Production and operations management, 13(1):93–110, 2004.

[18] Jie Wei, Jing Zhao, and Yongjian Li. Price and warranty period decisions for complementary products with horizontal firms’ cooperation/noncooperation strategies. Journal of Cleaner Production, 105:86–102, 2015.

[19] Jing Zhao, Wansheng Tang, Ruqing Zhao, and Jie Wei. Pricing decisions for substitutable products with a common retailer in fuzzy environments. European Journal of Operational Research, 216(2):409–419, 2012.