ON WEAKLY MAXIMAL REPRESENTATIONS
OF SURFACE GROUPS

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Abstract. We introduce and study a new class of representations of surface groups into Lie groups of Hermitian type, called weakly maximal representations. They are defined in terms of invariants in bounded cohomology and extend considerably the scope of maximal representations studied in [17, 14, 12, 13, 58, 44, 5, 4, 31]. We prove that weakly maximal representations are discrete and injective and we describe the structure of the Zariski closure of their image. We show moreover that they admit an elementary topological characterization in terms of bi-invariant orders; in particular if the target group is Hermitian of tube type, the order can be described in terms of the causal structure on the Shilov boundary. Furthermore we study the relation between the subset of weakly maximal representations and other geometrically significant subsets of the representations variety.

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1. Introduction

Given a compact oriented surface \( \Sigma \) of negative Euler characteristic, possibly with boundary, a general theme is to study the space of representations \( \text{Hom}(\pi_1(\Sigma), G) \) of the fundamental group \( \pi_1(\Sigma) \) of \( \Sigma \) into a semisimple Lie group \( G \), and in particular to distinguish subsets of geometric significance. Classical examples include holonomy representations of geometric structures, such as the set of Fuchsian representations in \( \text{Hom}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{R})) \) or the set of convex real projective structures in \( \text{Hom}(\pi_1(\Sigma), \text{PSL}(3, \mathbb{R})) \). In recent years these studies have been extended to include more general semisimple Lie groups \( G \). Prominent examples of geometrically significant subsets of representation varieties with these more general targets include Hitchin components \([48, 23, 51, 38, 39]\), positive representations \([29, 30]\), maximal representations \([34, 35, 75, 45, 17, 14, 13, 58, 44, 36, 5, 4, 6, 31]\) and Anosov representations \([41, 51, 40]\). Even though these subsets exhibit several common properties, which has lead to summarizing their study under the terminology "higher Teichmüller theory," they are defined and investigated by very different methods.
The present article is concerned with an extension of higher Teichmüller theory in the Hermitian context that goes beyond maximal representations. More precisely, we call a Lie group $G$ of Hermitian type if it is semisimple, connected, with finite center and no compact factors and if the associated symmetric space $\mathcal{X}$ admits a non-vanishing $G$-invariant real two-form $\omega_\mathcal{X}$. Any such two-form is the imaginary part of a $G$-invariant Kähler metric on $\mathcal{X}$, and we will always normalize $\omega_\mathcal{X}$ in such a way that the minimal holomorphic sectional curvature of this metric is $-1$. The differential form $\omega_\mathcal{X}$ gives rise in a familiar way (see for example [17, §2.1.1]) to a continuous cohomology class $\kappa_G \in H^2_c(G, \mathbb{R})$, called the Kähler class, corresponding to a bounded continuous cohomology class $\kappa^b_G \in H^2_{cb}(G, \mathbb{R})$ called the bounded Kähler class. We can also use $\omega_\mathcal{X}$ to associate to every representation $\rho : \pi_1(\Sigma) \to G$ an invariant $T(\rho)$ called the Toledo invariant of $\rho$ in the following way: if $\Sigma$ is a surface with universal covering $\tilde{\Sigma} \to \Sigma$, let $f : \tilde{\Sigma} \to \mathcal{X}$ be an arbitrary $\rho$-equivariant smooth map. Let $c$ be the bounded singular cochain whose value on a singular simplex $\sigma \subset \Sigma$ with lift $\tilde{\sigma} \subset \tilde{\Sigma}$ is

$$c(\sigma) := \int_{\tilde{\sigma}} f^*(\omega_\mathcal{X});$$

since every component of $\partial \Sigma$ is a circle, we can find a cocycle $c'$ (boundedly) cohomologous to $c$ vanishing on $\partial \Sigma$. Then $T(\rho)$ is the evaluation of $c'$ on the relative fundamental class $[\Sigma, \partial \Sigma]$ (see §2 or [17] for a more “functorial” definition). In this way we obtain a continuous invariant

$$T : \text{Hom}(\pi_1(\Sigma), G) \to \mathbb{R}$$

on the representation variety, that satisfies the Milnor–Wood type inequality

$$|T(\rho)| \leq 2\|\kappa^b_G\| \cdot |\chi(\Sigma)|,$$

(1.1)

where $\|\cdot\|$ denotes the canonical norm in the second continuous bounded cohomology (see [17] §2.1.1). Recall that a representation is maximal if its Toledo invariant satisfies $T(\rho) = 2\|\kappa^b_G\| \cdot |\chi(\Sigma)|$. The starting point for our generalization is the observation (see §2) that the inequality (1.1) can be refined into the chain of inequalities

$$|T(\rho)| \leq 2\|\rho^*(\kappa^b_G)\| \cdot |\chi(\Sigma)| \leq 2\|\kappa^b_G\| \cdot |\chi(\Sigma)|.$$

In particular, a representation is maximal if and only if it satisfies both

$$T(\rho) = 2\|\rho^*(\kappa^b_G)\| \cdot |\chi(\Sigma)| \quad \text{and} \quad \|\rho^*(\kappa^b_G)\| = \|\kappa^b_G\|.$$
Representations satisfying \( \| \rho^*(\kappa_G^h) \| = \| \kappa_G^h \| \) are called \textit{tight}; these have been investigated in much greater generality in \cite{16, 12, 43}. Here we are interested in representations satisfying the other equality. Namely:

**Definition 1.1** (\cite[Chapter 8, Definition 2.1]{57}). A representation \( \rho : \pi_1(\Sigma) \to G \) is \textit{weakly maximal} if it satisfies

\[
T(\rho) = 2\| \rho^*(\kappa_G^h) \| \cdot |\chi(\Sigma)|.
\]

**Remark 1.2.** By definition a weakly maximal representation has non-negative Toledo number. Analogously one can define weakly minimal representations. Since the composition of a representation with an orientation reversing outer automorphism changes the sign of the Toledo number the theory is completely analogous.

The scope of this paper is to develop a theory of weakly maximal representations, showing in particular that some key properties of maximal representations are preserved. More specifically:

- We show the discreteness of the range and the injectivity of weakly maximal representations with non-zero Toledo invariant; in addition we examine the structure of the Zariski closure of the range of such representations (see §1.1).
- For a Hermitian Lie group \( G \) of tube type, we give a characterization of weakly maximal representations with non-zero Toledo invariant in terms of a natural order on \( \pi_1(\Sigma) \) and on the universal covering \( \tilde{G} \), arising from the causal structure on the Shilov boundary (see §1.2).
- We illustrate topological properties of the space of weakly maximal representations and its relation with other subsets of the representation variety (see §1.3).
- We provide non-trivial examples of weakly maximal representations (see §12).

1.1. **Structure Theorems for Weakly Maximal Representations.**

Some of the properties of maximal representations established by three of the authors in \cite{17} follow from the fact that maximal representations are in particular weakly maximal. More precisely we have (see Corollary 7.3):

**Theorem 1.3.** Let \( \rho : \pi_1(\Sigma) \to G \) be a weakly maximal representation and \( T(\rho) \neq 0 \). Then \( \rho \) is faithful with discrete image.

An important step in the proof of Theorem 1.3 is the realization that a representation \( \rho \) is weakly maximal if and only if there exists \( \lambda \geq 0 \)
such that

\[(1.3) \quad \rho^*(\kappa^b_G) = \lambda \kappa^b_{\Sigma},\]

where \(\kappa^b_G\) if the bounded fundamental class of \(\Sigma\) (see (3.1)). This characterization was in effect first established in [57, Corollary 3.4] for compact surfaces and in [17, Cor. 4.15] in the general case; a different proof using Bavard’s duality was later obtained in [19] (for a more thorough description of the relation with Calegari’s work, see [13]).

The constant \(\lambda\) appearing in (1.3) is related to the Toledo invariant by the relation

\[T(\rho) = \lambda |\chi(\Sigma)|.\]

An essential ingredient in the proof of Theorem 1.3 as well as of Theorem 1.6 below, is the fact that \(\lambda\), and hence \(T(\rho)\), are rational. We prove in § 4 that this poses severe restrictions on the kernel and range of \(\rho\).

If \(\partial \Sigma = \emptyset\), asserting the rationality of \(\lambda\) presents no difficulty: in fact, with this normalization, the Toledo invariant of any representation \(\rho\) has the property that \(T(\rho) \in q_G^{-1}\mathbb{Z}\), where \(q_G\) is a natural number depending only on \(G\) (namely, \(q_G\) is the smallest integer such that \(q_G\kappa^b_G\) is an integral class, see Remark 6.3). On the other hand, if \(\partial \Sigma \neq \emptyset\), we know that the image of \(T\) is the whole closed interval \([-\|\kappa^b_G\| \cdot |\chi(\Sigma)|, \|\kappa^b_G\| \cdot |\chi(\Sigma)|]\). However, for weakly maximal representation we can prove the following (see Theorem 7.2):

**Theorem 1.4.** There is a natural number \(\ell_G\) depending only on \(G\), such that for every weakly maximal representation \(\rho : \pi_1(\Sigma) \to G\) we have \(T(\rho) \in \frac{|\chi(\Sigma)|}{\ell_G} \mathbb{Z}\). In particular, \(T\) takes only finitely many values on the set of weakly maximal representations.

**Remark 1.5.** The integer \(\ell_G\) depends in an explicit way on \(q_G\) and on the degree of non-integrality of the restriction \(\kappa^b_G|_H\) to various connected semisimple subgroups \(H\) of \(G\), as well as on the cardinality of their center (see the proof of Corollary 6.4).

Of course not all features of maximal representations hold for the weakly maximal ones. For example, the Zariski closure of the image of a weakly maximal representation need not be reductive (see § 12.6). We can however still give a structure theorem for weakly maximal representations as follows. Given a closed subgroup \(L < G\), we show that there exists a unique maximal normal subgroup of \(L\), that we call the *Kähler radical* \(\text{Rad}_{\kappa^b_G}(L)\) of \(L\), on which \(\kappa^b_G|_L\) vanishes. Since \(\text{Rad}_{\kappa^b_G}(L)\) contains for instance the solvable radical of \(L\), the quotient
$L/\text{Rad}_{\mathcal{G}}(L)$ is semisimple. We hence establish the following as part of Theorem 7.2:

**Theorem 1.6.** Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal representation into a Lie group $G := G(\mathbb{R})^\circ$ of Hermitian type, where $G$ is a connected algebraic group defined over $\mathbb{R}$. Let $L := L(\mathbb{R})$, where $L$ is the Zariski closure of $\rho(\pi_1(\Sigma))$ in $G$, and let $H = L/\text{Rad}_{\mathcal{G}}(L)$ be the quotient of $L$ by its Kähler radical. Assume that $T(\rho) \neq 0$. Then:

1. $H$ is adjoint of Hermitian type and all of its simple factors are of tube type.
2. The composition $\pi_1(\Sigma) \to L \to H$ is faithful with discrete image.

Recall that a Hermitian Lie group is of tube type if the associated symmetric space is of tube type, that is biholomorphic to $\mathbb{R}^n + iC$, for some $n \in \mathbb{N}$, where $C \subset \mathbb{R}^n$ is an open convex cone.

**Remark 1.7.** In the above Theorems 1.3 and 1.6 it is essential that the Toledo invariant is non-zero. However the class of weakly maximal representations with $T(\rho) = 0$ is also of interest. This is precisely the set where $\rho^*(\kappa_{\mathcal{G}}^b) = 0$. In the case when $G = \text{PU}(1,n)$ such representations have been studied and characterized in [11] as representations that preserve a totally real subspace of complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^n$.

1.2. **Geometric Description of Weakly Maximal Representations.** Theorem 1.6 shows that the study of weakly maximal representations with arbitrary Hermitian targets can be reduced to the case in which the target is of tube type. At any rate, the geometric content of the definition of weak maximality is clearly mysterious, as it relies on the knowledge of the norm of a class in bounded cohomology which is difficult to compute. We show that if $G$ is of tube type, one can use the existence of a $G$-invariant causal structure on the corresponding Shilov boundary, to give a geometric characterization of weakly maximal representations. This brings in techniques from [3], where a connection between bounded cohomology and invariant orders on Lie groups was established (see also [25, 20] for a related circle of ideas).

Recall that the Shilov boundary $\hat{\mathcal{S}}$ is the unique closed $G$-orbit in the boundary $\partial \mathcal{D}$ of the bounded domain realization $\mathcal{D}$ of the symmetric space associated to $G$. It can be identified with $G/Q$ for a specific maximal parabolic $Q < G$. A causal structure $\mathcal{C}$ on $\hat{\mathcal{S}}$ is given by a family of closed proper convex cones with non-empty interior $\mathcal{C}_x \subset T_x \hat{\mathcal{S}}$, and the causal structure is invariant if $g_\ast \mathcal{C}_x = \mathcal{C}_{g_\ast x}$ for all $x \in \hat{\mathcal{S}}$ and all $g \in G$. The existence of such an invariant causal structure on $\hat{\mathcal{S}}$
(unique up to taking inverses) is equivalent to the property that \( G \) is of tube type, \([50]\). For more information on the classification of invariant causal structures on general homogeneous spaces and on the study of their associated orders see \([56, 53, 46, 47]\).

In order to simplify the discussion we assume that the tube type group \( G \) is adjoint simple. Moreover we denote by \( \hat{G} = \tilde{G}/\pi_1(G)^{tor} \) the unique central \( \mathbb{Z} \)-extension of \( G \), which acts effectively on the universal covering \( \tilde{R} \) of \( \tilde{S} \). The causal structure \( \mathcal{C} \) on \( \tilde{S} \) lifts to a \( \hat{G} \)-invariant causal structure on \( \tilde{R} \), that in turn defines a \( \hat{G} \)-invariant partial order \( \leq \) on \( \tilde{R} \) as follows (Lemma 9.5):

- if \( x, y \in \tilde{R} \), we say that \( x \leq y \) if there exists a causal curve (Definition 9.2) in \( \tilde{R} \) from \( x \) to \( y \), and
- \( x < y \) if \( x \leq y \) and, in addition, \( x \neq y \).

This allows to define a bi-invariant partial order on \( \hat{G} \) by setting

\[
g \preceq h \quad \text{if and only if} \quad gx \leq hx \quad \text{for all} \quad x \in \tilde{R}.
\]

The set of positive elements of \( \hat{G} \) is the pointed conjugacy invariant submonoid

\[
\hat{G}^+ := \{ g \in \hat{G} : g \succeq e \}
\]

and completely determines the causal order on \( \hat{G} \). The dominant set \( \hat{G}^{++} \) (in the sense of \([27, 3]\)) of the bi-invariant order on \( \hat{G} \) is of special importance to establish the link both between orders and weakly maximal representations (Theorem 1.12) and between orders and bounded cohomology: it is defined as

\[
\hat{G}^{++} := \{ g \in \hat{G}^+ : \text{for all} \ h \in G \text{ there exists} \ n \in \mathbb{N} \text{ with} \ g^n \succeq h \},
\]

and is a conjugacy-invariant ideal in \( \hat{G}^+ \). We show that the set of dominant elements consists essentially of the set of strictly positive elements in \( \hat{G} \) in the following sense (cf. Theorem 9.4):

**Theorem 1.8.** If \( \hat{G} \) is of tube type then

\[
\hat{G}^{++} = \{ g \in \hat{G} : gx > x \text{ for all} \ x \in \tilde{R} \}.
\]

1.2.1. **Orders, Weakly Maximal Representations and Causal Representations.** We now provide an interpretation of weakly maximal representations in terms of dominant sets. In the case of a surface \( \Sigma \) with non-empty boundary, it entails to defining a canonical order on the subgroup \( \Lambda < \pi_1(\Sigma) \) of homologically trivial loops and, roughly speaking, characterizing weakly maximal representations as those that map “large” dominant elements in \( \Lambda \) to dominant elements in \( \hat{G} \).
To define an order on $\pi_1(\Sigma)$, observe that, in the case of $\widehat{\text{PU}}(1,1)$, $\hat{S}$ is the unit circle in $\mathbb{C}$ and $\hat{R}$ is identified with $\mathbb{R}$. Then $\widehat{\text{PU}}(1,1) = \tilde{\text{PU}}(1,1)$ is a subgroup of the group of monotone increasing homeomorphisms of $\mathbb{R}$ commuting with integer translations and the order defined in (1.4) is given by

$$\tilde{\text{PU}}(1,1)^+ = \{ g \in \tilde{\text{PU}}(1,1) : g(x) \geq x, \text{ for all } x \in \mathbb{R} \}.$$ 

The pullback to $\pi_1(\Sigma)$ of this order gives

$$\pi_1(\Sigma)^+ := \{ \gamma \in \pi_1(\Sigma) : \tilde{\rho}(\gamma) \in \tilde{\text{PU}}(1,1)^+ \},$$ 

where $\tilde{\rho}_h : \pi_1(\Sigma) \to \hat{G}$ is a lift of a holonomy representation associated to a complete hyperbolic structure with finite area on the interior of $\Sigma$. This induces a canonical order on the group $\Lambda := [\pi_1(\Sigma), \pi_1(\Sigma)]$ of homologically trivial loops, that is an order independent of the chosen lift as well as of the hyperbolization. If $\partial \Sigma = \emptyset$, we refer the reader to §10. In any case, this order on $\Lambda$ is invariant under the action of the corresponding mapping class group.

**Definition 1.9.** Let $G$ be an adjoint simple Hermitian Lie group of tube type. We say that a representation $\rho : \pi_1(\Sigma) \to G$ is causal if given any lift $\tilde{\rho} : \pi_1(\Sigma) \to \hat{G}$ and any $\gamma \in \Lambda^+$, then $\tilde{\rho}(\gamma) \in \hat{G}^+$. Observe that any two lifts of $\rho$ coincide on $\Lambda$.

By definition hyperbolizations are causal representations. In fact:

**Theorem 1.10.** Let $\Sigma$ be a surface of negative Euler characteristic. A representation $\rho : \pi_1(\Sigma) \to \text{PU}(1,1)$ with non-zero Toledo invariant is causal if and only if it is a hyperbolization.

The analogous result for representations into a general adjoint simple Hermitian Lie group of tube type does not necessarily hold. More precisely, it is always true that a causal representation is weakly maximal, but a weakly maximal representation maps only “sufficiently large” dominant elements in $\Lambda$ into dominant element. This is made precise by defining, for every $q \in \{0,1,2,\ldots\}$ the following set

$$\Lambda_q^+ := \{ \gamma \in \Lambda : \tilde{\rho}_h(\gamma)(x) > x + q \text{ for all } x \in \mathbb{R} \}.$$ 

**Definition 1.11.** Let $q \in \mathbb{N}$. We say that a representation $\rho : \pi_1(\Sigma) \to G$ is $q$-causal if given any lift $\tilde{\rho} : \pi_1(\Sigma) \to \hat{G}$ and any $\gamma \in \Lambda_{q}^+$, then $\tilde{\rho}(\gamma) \in \hat{G}^+$. 


The causal representations are hence exactly the 0-causal ones. Notice that, since $\tilde{\text{PU}}(1,1)_{q+1}^+ \subseteq \text{PU}(1,1)_q^+$, any $q$-causal representation is automatically $(q+1)$-causal.

The generalization of Theorem 1.10 then reads:

**Theorem 1.12.** Let $G$ be an adjoint simple Hermitian Lie group of tube type and let $\Sigma$ be a surface of negative Euler characteristic. There exists $q = q(\Sigma, G) \in \mathbb{N}$ such that a representation $\rho : \pi_1(\Sigma) \to G$ with non-zero Toledo invariant is weakly maximal if and only if it is $q$-causal.

**Remark 1.13.** The set $\Lambda^{++}$ corresponding to the set of positive fixed point free elements in $\tilde{\text{PU}}(1,1)$ can be directly described in terms of the winding number introduced and studied by Chillingworth, [21, 22], and subsequently by Huber, [49]. Namely, given a nowhere vanishing vector field $X$ in the interior of $\Sigma$, let $W_X(\gamma)$ denote the winding number of the closed geodesic in the free homotopy class of $\gamma \in \pi_1(\Sigma)$ with respect to $X$. Then

$$\Lambda^{++}_q = \{ \gamma \in \Lambda : W_X(\gamma) > q \}.$$

1.2.2. **Orders and Bounded Cohomology.** If $p : \hat{G} \to G$ denotes the covering homomorphism one knows that $p^*(\kappa^b_G)$ is trivial as an ordinary continuous class on $\hat{G}$ and there is a unique well defined continuous homogeneous quasimorphism $f_{\hat{G}} : \hat{G} \to \mathbb{R}$ with

$$[df_{\hat{G}}] = p^*(\kappa^b_G)$$

(see [17] or §3 for details). We revisit in the special case of the Shilov boundary the theory developed for general causal covers in [1] and provide a new self-contained proof of the following result (see [1, Corollary 5.14]) establishing the relation between the order on $\hat{G}$ and $f_{\hat{G}}$.

**Theorem 1.14.** Let $G$ be a connected adjoint simple Hermitian Lie group of tube type and let $\hat{G}$ be the unique connected $\mathbb{Z}$-covering of $G$ endowed with the causal order in (1.4). Then for every dominant element $g \in \hat{G}^{++}$ and every $h \in \hat{G}$, we have

$$\lim_{n \to \infty} \min \frac{\{ p \in \mathbb{Z} : g^n \succcurlyeq h^n \}}{n} = \frac{f_{\hat{G}}(h)}{f_{\hat{G}}(g)}.$$

1.3. **Comparison with Other Classes of Representations.** The representation variety $\text{Hom}(\pi_1(\Sigma), G)$ is identified with

$$\{(h_1, \ldots, h_{2g}) : \prod_{i=1}^{g} [h_{2i-1}, h_{2i}] = e \} \subset G^{2g}$$
if $\partial \Sigma = \emptyset$ and with $G^{2g+n-1}$ if $\partial \Sigma$ has $n \geq 1$ boundary components. If $G$ is real algebraic, then $\Hom(\pi_1(\Sigma), G)$ is real algebraic as well, and, if $G$ is a Lie group of Hermitian type, then $\Hom(\pi_1(\Sigma), G)$ has finitely many connected components.

The set of weakly maximal representations $\Hom_{wm}(\pi_1(\Sigma), G)$ decomposes as a disjoint union

$$\text{Hom}_{wm}(\pi_1(\Sigma), G) = \text{Hom}^*(\pi_1(\Sigma), G) \sqcup \text{Hom}_0(\pi_1(\Sigma), G)$$

of the set $\text{Hom}^*_{wm}(\pi_1(\Sigma), G)$ of weakly maximal representations with non-zero Toledo invariant and the set of representations with vanishing Toledo invariant

$$\text{Hom}_0(\pi_1(\Sigma), G) := \{ \rho : \pi_1(\Sigma) \to G : \rho^*(\kappa_G^b) = 0 \}.$$ 

The latter contains $\Hom(\pi_1(\Sigma), L)$ for every closed subgroup $L < G$ for which $\kappa_G^b|_L = 0$. In particular, representations in $\text{Hom}_0(\pi_1(\Sigma), G)$ are not necessarily injective, and their images are not necessarily discrete (see Remark 1.7), while, according to Theorem 1.3, $\text{Hom}^*_{wm}(\pi_1(\Sigma), G)$ is contained in the set $\text{Hom}_{d,i}(\pi_1(\Sigma), G)$ of injective homomorphisms with discrete image. Concerning the topological properties of the sets introduced so far, we have (cf. Corollary 11.4):

**Theorem 1.15.** The following

$$\text{Hom}_{d,i}(\pi_1(\Sigma), G) \leftarrow \text{Hom}_{max}(\pi_1(\Sigma), G) \rightarrow \text{Hom}^*(\pi_1(\Sigma), G) \rightarrow \text{Hom}_{wm}(\pi_1(\Sigma), G)$$

is a diagram of $\text{Aut}(\pi_1(\Sigma))$-invariant closed subsets of $\Hom(\pi_1(\Sigma), G)$.

If $G$ is real algebraic, $\text{Hom}_{max}(\pi_1(\Sigma), G)$ is a real semi-algebraic subset of $\text{Hom}(\pi_1(\Sigma), G)$, [17, Corollary 14], but we do not have such precise information on the other sets appearing in Theorem 1.15.

A prominent role in higher Teichmüller theory is played by Anosov representations, a notion introduced by F. Labourie in his study of Hitchin representations [51], then studied for Hermitian Lie groups in [12, 14] and in greater generality in [41]. The property of a representation to be Anosov is defined with respect to a parabolic subgroup $P < G$; for example, if $\partial \Sigma = \emptyset$, maximal representations $\rho : \pi_1(\Sigma) \to G$ are Anosov with respect to the parabolic group $Q$ which is the stabilizer in $G$ of a point in the Shilov boundary $\tilde{S}$ of the bounded symmetric
domain realization of $X$. We will call such an Anosov representation Shilov-Anosov.

In the case in which $\partial \Sigma = \emptyset$, if we denote by $\text{Hom}_{\tilde{S}-\text{An}}(\pi_1(\Sigma), G)$ the set of such representations and by $\text{Hom}_{\tilde{S}-\text{An}}^*(\pi_1(\Sigma), G)$ the subset of Shilov-Anosov representations with positive Toledo invariant, we can hence write the inclusions

$$\text{Hom}_{\text{max}}(\pi_1(\Sigma), G) \subset \text{Hom}_{\tilde{S}-\text{An}}(\pi_1(\Sigma), G) \subset \text{Hom}_{\tilde{S}-\text{An}}^*(\pi_1(\Sigma), G),$$

where $\text{Hom}_{\text{max}}(\pi_1(\Sigma), G)$ is a union of components and $\text{Hom}_{\tilde{S}-\text{An}}^*(\pi_1(\Sigma), G)$ is an open subset of $\text{Hom}(\pi_1(\Sigma), G)$, [12, 14, 11].

On the other hand we prove in Corollary 11.5 the following:

**Theorem 1.16.** If $G$ is of tube type and $\partial \Sigma = \emptyset$, then

$$\text{Hom}_{\tilde{S}-\text{An}}(\pi_1(\Sigma), G) \subset \text{Hom}_{\text{wm}}(\pi_1(\Sigma), G).$$

If $G = \text{Sp}(2n, \mathbb{R})$ with $n \geq 2$, we will see that $\text{Hom}_{\tilde{S}-\text{An}}(\pi_1(\Sigma), G)$ is open but not closed and since the set of weakly maximal representations is closed (see Corollary 11.3), it provides a natural framework in which to study limits of Shilov-Anosov representations. We refer the reader to §12.3 for examples of weakly maximal representations into $\text{Sp}(2n, \mathbb{R})$ for $n \geq 6$, that are not Shilov-Anosov, but are the limit of Shilov-Anosov representations.

For a closed surface $\Sigma$ and a Hermitian group $G$ of tube type we can summarize our discussion above and results from [51, 16] in the following diagram, where we denote by $\text{Hom}_{\text{red}}(\pi_1(\Sigma), G)$ the set of representations with reductive Zariski closure and by $\text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), G)$ the Hitchin components in case $G$ is locally isomorphic to $\text{Sp}(2n, \mathbb{R})$ (and the empty set otherwise). Then we have the following inclusions:

$$\text{Hom}_{\tilde{S}-\text{An}} \subset \text{Hom}_{\text{wm}} \subset \text{Hom}_{\text{Hitchin}} \subset \text{Hom}_{\text{max}}^* \subset \text{Hom}_{\tilde{S}-\text{An}}^* \subset \text{Hom}_{\text{wm}}^* \subset \text{Hom}_{\text{d,i}} \subset \text{Hom}_{\text{tight}} \subset \text{Hom}_{\text{red}}.$$

2. **The Toledo Invariant**

We recall here the definition of the Toledo invariant in a general context and indicate how the Milnor–Wood inequality follows from known isometric isomorphisms in bounded cohomology.
Let $\Sigma$ be a compact oriented surface with (possibly empty) boundary $\partial \Sigma$. Let $G$ be a locally compact group, $\kappa \in H^2_{cb}(G, \mathbb{R})$ a fixed continuous bounded class and $\rho : \pi_1(\Sigma) \to G$ a homomorphism. Recall that there is an isometric isomorphism

$$g_\Sigma : H^2_0(\pi_1(\Sigma), \mathbb{R}) \to H^2_0(\Sigma, \mathbb{R})$$

whose existence in general is due to Gromov [37]; in our case, the universal covering $\tilde{\Sigma}$ is contractible and the existence and isometric property of $g_\Sigma$ are easily established. Next, the inclusion of pairs $(\Sigma, \emptyset) \hookrightarrow (\Sigma, \partial \Sigma)$ gives rise to a map

$$j_{\partial \Sigma} : H^2_{\partial}(\Sigma, \partial \Sigma, \mathbb{R}) \to H^2_0(\Sigma, \mathbb{R})$$

where the left hand side refers to bounded relative cohomology. Since every connected component of $\partial \Sigma$ is a circle and hence has amenable fundamental group, the map $j_{\partial \Sigma}$ is an isometric isomorphism, [7, Theorem 1]. We hence define

$$T_\kappa(\rho) := \langle (j_{\partial \Sigma})^{-1} g_\Sigma(\rho^*(\kappa)), [\Sigma, \partial \Sigma] \rangle,$$

where $[\Sigma, \partial \Sigma]$ is the relative fundamental class. Since $g_\Sigma$ and $j_{\partial \Sigma}$ are isometries, we deduce that

$$|T_\kappa(\rho)| \leq \|\rho^*(\kappa)\| |[\Sigma, \partial \Sigma]|_1,$$

where the second factor refers to the norm in relative $\ell^1$-homology. Since $|[\Sigma, \partial \Sigma]|_1 = 2|\chi(\Sigma)|$ and the pullback is norm decreasing, then

$$|T_\kappa(\rho)| \leq \|\rho^*(\kappa)\| |\chi(\Sigma)| \leq \|\kappa\| |\chi(\Sigma)|.$$

In view of (2.1), we give the following definition:

**Definition 2.1.** A representation $\rho : \pi_1(\Sigma) \to G$ is $\kappa$-weakly maximal if $T_\kappa(\rho) = \|\rho^*(\kappa)\| |\chi(\Sigma)|$.

**Remark 2.2.** If $G$ is of Hermitian type and $\kappa = \kappa^b_G$ is the bounded Kähler class, then $\|\kappa^b_G\| = \text{rank}(G)/2$ (see [17, § 2.1]) and one obtains for the corresponding Toledo invariant $T(\rho)$ the familiar Milnor–Wood inequality

$$|T(\rho)| \leq \text{rank}(G) |\chi(\Sigma)|.$$

This will be used in the examples in §12, but otherwise never in the paper the explicit computation of the norm is used.
3. A Characterization of Weakly Maximal Representations

In this section we develop the general framework for the study of weakly maximal representations and establish some of their basic properties. In particular we show that weak maximality of a representation \( \rho : \pi_1(\Sigma) \to G \) is reflected in a relation between its lift to appropriate central extensions of \( \pi_1(\Sigma) \) and \( G \) and canonically defined quasimorphisms on those central extensions (cf. Proposition 3.2).

3.1. The Central Extension of \( \pi_1(\Sigma) \). Let \( h \) be any complete hyperbolic structure on the interior \( \Sigma^o \) of \( \Sigma \) compatible with the fixed orientation and let \( \rho_h : \pi_1(\Sigma) \to \mathrm{PU}(1,1) \) be the corresponding holonomy representation. Then

\[
\kappa^b : = \rho^*_h(\kappa^b_{\mathrm{PU}(1,1)}) \in H^2_b(\pi_1(\Sigma), \mathbb{R})
\]

is independent of the choice of \( h \) and is called the (real) bounded fundamental class of \( \Sigma \), [17 § 8.2]. If \( \hat{\Gamma} \) denotes the central extension

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \hat{\Gamma} \xrightarrow{p_{\Sigma}} \pi_1(\Sigma) \longrightarrow 0
\]

corresponding to the positive generator of \( H^2(\pi_1(\Sigma), \mathbb{Z}) \) if \( \partial \Sigma = \emptyset \) and \( \hat{\Gamma} = \pi_1(\Sigma) \) otherwise, then \( \rho_h \) lifts to a homomorphism \( \tilde{\rho}_h : \hat{\Gamma} \to \mathrm{PU}(1,1) \), where we think of the universal covering \( \mathrm{PU}(1,1) \) as contained in the group \( \mathrm{Homeo}_Z^+ (\mathbb{R}) \) of increasing homeomorphisms of the real line \( \mathbb{R} \) commuting with integer translations. Denote by \( \tau : \mathrm{PU}(1,1) \to \mathbb{R} \) the Poincaré translation quasimorphism defined by \( \tau(f) := \lim_{n \to \infty} \frac{f^n(x) - x}{n} \) for \( x \in \mathbb{R} \). Then the composition \( \tau \circ \tilde{\rho}_h \) is a homogeneous quasimorphism which is \( \mathbb{Z} \)-valued

\[
\tau \circ \tilde{\rho}_h : \hat{\Gamma} \to \mathbb{Z},
\]

since every element in the image of \( \rho_h \) has a fixed point. Moreover, the homogeneous quasimorphism \( \tau \) corresponds in real bounded cohomology to the pullback, via the projection \( p : \mathrm{PU}(1,1) \to \mathrm{PU}(1,1) \), of the real bounded cohomology class \( \kappa^b_{\mathrm{PU}(1,1)} \)

\[
[d\tau] = p^*(\kappa^b_{\mathrm{PU}(1,1)}),
\]

and hence, again in real bounded cohomology,

\[
[d(\tau \circ \tilde{\rho}_h)] = p^*_\Sigma(\kappa^b_{\Sigma}).
\]

While \( \tau \circ \tilde{\rho}_h \) depends on the hyperbolization \( \rho_h \) and on its lift \( \tilde{\rho}_h \), its restriction \( (\tau \circ \tilde{\rho}_h)|\Lambda \) to \( \Lambda := [\hat{\Gamma}, \hat{\Gamma}] \) is independent of the above choices, (as the equivalence in the Proposition 3.2 below shows).
3.2. The Central Extension for a Locally Compact Group $G$.

We now turn to the construction of the central extension (depending on a “rational” class) and of the associated quasimorphism for a general locally compact group. The relation with §3.1 is outlined in §3.3.

Definition 3.1. We say that a bounded cohomology class $\kappa$ is rational if there is an integer $n \geq 1$ such that $n\kappa$ is representable by a bounded $\mathbb{Z}$-valued Borel cocycle.

Let $\kappa$ be rational and let $c : G^2 \to \mathbb{Z}$ be a normalized bounded Borel cocycle representing $nk\kappa$. Endow $G \times \mathbb{Z}$ with the group structure defined by the Borel map

$$(g_1, n_1)(g_2, n_2) := (g_1g_2, n_1 + n_2 + c(g_1, g_2))$$

and let $G_{nk}$ denote the Borel group $G \times \mathbb{Z}$ endowed with the unique compatible locally compact group topology [52]. This gives the topological central extension

$$0 \to \mathbb{Z} \to G_{nk} \xrightarrow{p_{nk}} G \to e.$$  

Then $f_{nk}'(g, m) = \frac{1}{n}m$ is a Borel quasimorphism such that $df_{nk}'$ represents $p_{nk}'(\kappa)$. Its homogenization $f_{nk} : G_{nk} \to \mathbb{R}$ is a continuous homogeneous quasimorphism [17, Lemma 7.4] such that

$$[df_{nk}] = p_{nk}^*(\kappa),$$

and

$$f_{nk}(i(m)) = \frac{1}{n}m.$$  

The homogeneous quasimorphisms $f_{nk}$ are well defined in the following sense. Let $\zeta = nk\kappa$ be the smallest integer multiple of $\kappa$ representable by a $\mathbb{Z}$-valued bounded Borel cocycle and let $\ell \in \mathbb{Z}$, $\ell \neq 0$. Then the map $G \times \mathbb{Z} \to G \times \mathbb{Z}$ defined by $(g, m) \mapsto (g, \ell m)$ induces a continuous homomorphism that identifies $G_{\zeta}$ with a closed subgroup of finite index in $G_{nk\kappa}$. Via this identification, the quasimorphism $f_{\zeta}$ is the restriction to $G_{\zeta}$ of $f_{nk\kappa}$.

The following result describes the property of being weakly maximal in terms of quasimorphisms and will be essential in the sequel.

Proposition 3.2. Let $G$ be a locally compact group and $\kappa \in H^2_b(G, \mathbb{R})$ a rational class with $nk\kappa$ integral. Let $\rho : \pi_1(\Sigma) \to G$ be a homomorphism and $\tilde{\rho} : \tilde{\Gamma} \to G_{nk\kappa}$ some lift, where $\tilde{\Gamma}$ and $G_{nk\kappa}$ are defined respectively in (3.2) and (3.6). The following are equivalent:

1. $\rho$ is $\kappa$-weakly maximal;
2. there exists $\lambda \geq 0$ such that $\rho^*(\kappa) = \lambda\kappa_{\Sigma}$.  

(3) there exists \( \lambda \geq 0 \) and \( \psi \in \text{Hom}(\hat{\Gamma}, \mathbb{R}) \) such that
\[ f_{nk} \circ \tilde{\rho} = \lambda (\tau \circ \tilde{\rho}_h) + \psi; \]
(4) there exists \( \lambda \geq 0 \) such that
\[ (f_{nk} \circ \tilde{\rho})|_{\Lambda} = \lambda (\tau \circ \tilde{\rho}_h)|_{\Lambda}. \]

Remark 3.3. The constant \( \lambda \geq 0 \) that appears in Proposition 3.2 is nothing but \( \lambda = \frac{T_\kappa(\rho)}{\chi(\Sigma)} \), as it can be easily seen from the definition of the Toledo invariant in §2.

Proof of Proposition 3.2. The equivalence of (1) and (2) is [13, Corollary 4.15].

Observe that, modulo modifying the lift \( \tilde{\rho} \) by a homomorphism with values in \( \mathbb{Z} \hookrightarrow G_{nk} \), the diagram
\[ \begin{array}{ccc}
\hat{\Gamma} & \overset{\tilde{\rho}}{\longrightarrow} & G_{nk} \\
p\Sigma \downarrow & & \downarrow p_{nk} \\
\pi_1(\Sigma) & \overset{\rho}{\longrightarrow} & G
\end{array} \]
commutes. This, together with the relation (3.5) and the fact that \( p_\Sigma^* \) induces an isomorphism in real bounded cohomology, proves the equivalence of (2) and (3).

Finally, by taking coboundaries on both sides of the equality in (4), one obtains that the bounded classes \([d(f_{nk} \circ \tilde{\rho})]\) and \(\lambda [d(\tau \circ \tilde{\rho}_h)]\) coincide on \( \Lambda \). Since \( \hat{\Gamma}/\Lambda \) is amenable, we deduce the equality in \( \hat{\Gamma} \), which in turns implies (3). \( \square \)

3.3. The Hermitian Case. Assume that \( G \) is of Hermitian type and almost simple. We apply the above discussion to the bounded Kähler class \( \kappa_G^b \), which is rational, [9, §5]. If \( \kappa = n\kappa_G^b \) is any integer multiple represented by an integral cocycle, the topological central extension
\[ 0 \rightarrow \mathbb{Z} \overset{i}{\rightarrow} G_\kappa \overset{p_\kappa}{\rightarrow} G \rightarrow e \]
is not trivial. Since \( \pi_1(G) \) is isomorphic to \( \mathbb{Z} \) modulo torsion, there is a unique connected central \( \mathbb{Z} \)-extension \( \hat{G} \) and, as a result, the connected component of the identity \((G_\kappa)^o\) is isomorphic to \( \hat{G} \). We denote by \( f_{\hat{G}} : \hat{G} \rightarrow \mathbb{R} \) the continuous homogeneous quasimorphism corresponding to \( f_\kappa \) under this isomorphism and hence obtain that
\[ (3.10) \quad [df_{\hat{G}}] = p^*(\kappa_G^b), \]
where \( p = p|_{(G_\kappa)^o} : \hat{G} \rightarrow G \) is the projection. For example, if \( G = \text{PU}(1,1) \), the bounded Kähler class \( \kappa_{\text{PU}(1,1)}^b \) is already integral as it is
the image of the bounded Euler class $e^b$ under the change of coefficients $H^2_{cb}(\text{PU}(1,1),\mathbb{Z}) \to H^2_{cb}(\text{PU}(1,1),\mathbb{R})$. Then $\tilde{G}$ is the universal covering $\text{PU}(1,1)$ and $f_{\tilde{G}}$ is the Poincaré translation quasimorphism.

4. A General Framework for Injectivity and Discreteness

In this section we give two results on weakly maximal representations: one concerns injectivity with general locally compact targets, and the other discreteness of the image in the case of Hermitian targets. In both cases we use the characterization of weakly maximal representations in §3. However, while the characterization holds also if the Toledo invariant vanishes, both results in this section are obtained under the assumption that $T_\kappa(\rho) \neq 0$ and discreteness requires in addition that $T_\kappa(\rho)$ is rational.

**Proposition 4.1.** Let $\rho : \pi_1(\Sigma) \to G$ be $\kappa$-weakly maximal and assume that $T_\kappa(\rho) \neq 0$. Then $\rho$ is injective.

**Proof.** Applying Proposition 3.2, we have that

$$f_\kappa \circ \tilde{\rho} = \lambda (\tau \circ \tilde{\rho}_h) + \psi$$

with $\lambda = \frac{T_\kappa(\rho)}{|\chi(\Sigma)|} \neq 0$. Since $p^{-1}_\Sigma(\ker \rho) = \tilde{\rho}^{-1}(i(\mathbb{Z}))$, we deduce from $\lambda \neq 0$ that $\tau \circ \tilde{\rho}_h$ is a homomorphism on $p^{-1}_\Sigma(\ker \rho)$. A formal argument using the central extension (3.2) restricted to $\ker \rho < \pi_1(\Sigma)$, the commutativity of the diagram

$$\begin{CD}
\tilde{\Gamma} @>\tilde{\rho}_h>> \text{PU}(1,1) \\
p_\Sigma @>>\rho_h>> \pi_1(\Sigma) @>\rho_h>> \text{PU}(1,1)
\end{CD}$$

and (3.4) implies that

$$(\rho_h|_{\ker \rho})^*(\kappa^b_{\text{PU}(1,1)}) = 0.$$ 

Since this implies that $\rho_h(\ker \rho)$ is elementary (see next lemma), we conclude that $\ker \rho$ is trivial.

We provide a proof of the following easy lemma for ease of reference.

**Lemma 4.2.** Let $\pi : \Delta \to \text{PU}(1,1)$ be a homomorphism such that $\pi^*(\kappa^b_{\text{PU}(1,1)}) = 0$. Then $\pi(\Delta)$ is elementary, that is $\pi(\Delta)$ has a finite orbit in $\mathbb{D}$. 

\[ \Box \]
Proof. Since \( \pi^*(\kappa^b_{\text{PU}(1,1)}) = 0 \), we conclude from the exactness of the sequence

\[
\text{Hom}(\Delta, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H^0_b(\Delta, \mathbb{Z}) \xrightarrow{} H^0_b(\Delta, \mathbb{R})
\]

that \( \pi^*(e^b) = \delta(\chi) \) for some homomorphism \( \chi : \Delta \to \mathbb{R}/\mathbb{Z} \). In other words, if we consider \( \chi \) as a homomorphism into the group of rotations in \( \text{PU}(1,1) \), we have that \( \chi^*(e^b) = \pi^*(e^b) \). In particular \( (\pi|_{\Delta, \Delta})^*(e^b) = 0 \). Hence, by Ghys’ theorem [32, Theorem 6.6], \( [\Delta, \Delta] \) has a fixed point in \( \partial \mathbb{D} \). There are then two cases, depending on whether or not \( [\Delta, \Delta] \) has infinitely many fixed points. In the first case \( [\pi(\Delta), \pi(\Delta)] \) is trivial, that is \( \pi(\Delta) \) is abelian. In the second case \( \pi(\Delta) \) has a finite orbit in \( \partial \mathbb{D} \). In either cases \( \pi(\Delta) \) is elementary. \( \square \)

The following proposition gives a criterion to insure the discreteness of the image of a \( \kappa \)-weakly maximal representation. In Theorem 7.2 we are going to show that such criterion is always satisfied if \( \kappa \) is the bounded Kähler class.

**Proposition 4.3.** Let \( G \) be a Lie group of Hermitian type and let \( \kappa \in H^2_b(G, \mathbb{R}) \) be a rational class. Assume that \( \rho : \pi_1(\Sigma) \to G \) is \( \kappa \)-weakly maximal and that \( T_\kappa(\rho) \in \mathbb{Q}^\times \). Then \( \rho \) has discrete image.

**Remark 4.4.** In the case in which \( \partial \Sigma = \emptyset \), we have automatically that \( T_\kappa(\rho) \in \mathbb{Q} \), so that in this case the condition just reads \( T_\kappa(\rho) \neq 0 \).

**Proof.** Let \( \Gamma := \pi_1(\Sigma) \). It will be enough to show that \( \rho([\Gamma, \Gamma]) \) is discrete. In fact \( \rho([\Gamma, \Gamma]) \) is a normal subgroup of \( \rho(\Gamma) \) and, if discrete, also of \( \overline{\rho(\Gamma)} \); thus \( \overline{\rho(\Gamma, \Gamma)} \) centralizes the connected component of the identity \( \overline{\rho(\Gamma)} \) of \( \overline{\rho(\Gamma)} \). If \( \overline{\rho(\Gamma)} \) were not trivial, we could find \( \rho(\gamma) \in \rho(\Gamma) \cap \overline{\rho(\Gamma)} \) with \( \rho(\gamma) \neq e \): indeed \( \overline{\rho(\Gamma)} \) is an open subgroup of \( \overline{\rho(\Gamma)} \) and \( \rho(\Gamma) \) is dense in \( \overline{\rho(\Gamma)} \). But since, by Proposition 4.1, \( \rho \) is injective, this would imply that \( \gamma \) centralizes \( [\Gamma, \Gamma] \). This is a contradiction, which shows that \( \rho(\Gamma) \) is discrete.

To show the discreteness of \( \rho([\Gamma, \Gamma]) \), we retain the notation of § 3.3 and we define \( L := \overline{\rho([\Gamma, \Gamma])} \) and \( L' := p^{-1}(L) \). Observe that (3.6) and (3.9) imply that \( p^{-1}(\rho([\Gamma, \Gamma])) = \overline{\rho([\Gamma, \Gamma])} \ker p \). Since \( p \) is a covering, then \( p^{-1}(\rho([\Gamma, \Gamma])) = p^{-1}(\rho([\Gamma, \Gamma])) \), so that

\[
L' = \overline{\rho([\Gamma, \Gamma])} \ker p \tag{4.1}
\]

We apply now the implication (1)⇒(3) in Proposition 3.2 to obtain that

\[
f_{\text{nk}} \circ \rho = \lambda(\tau \circ \tilde{\rho}_n) + \psi,
\]
where \( \lambda = \frac{T_\ast \rho}{\chi(S)} \) and \( \psi \in \text{Hom}(\hat{\Gamma}, \mathbb{R}) \). Since \( \tau \circ \tilde{\rho} \) is \( \mathbb{Z} \)-valued, we deduce that

\[
fn_\kappa(\tilde{\rho}(\hat{\Gamma}, \hat{\Gamma})) = \lambda \tau(\tilde{\rho}_h(\hat{\Gamma})) \subset \lambda \tau(\tilde{\rho}_h(\hat{\Gamma})) \subset \lambda \mathbb{Z},
\]

which, together with (3.8) implies that

\[
f_{\kappa}(\tilde{\rho}(\hat{\Gamma}, \hat{\Gamma})) \ker p \subset \frac{1}{n} \mathbb{Z} + \lambda \mathbb{Z}.
\]

Since \( \lambda \in \mathbb{Q} \), then \( \frac{1}{n} \mathbb{Z} + \lambda \mathbb{Z} \) is discrete in \( \mathbb{R} \). Taking into account (4.1), this implies that \( L' \subset \mathbb{R} \) is a discrete subset. Hence, if we denote by \((L')^o\) the identity component of \( L' \), then

\[
f_{\kappa}|_{(L')^o} = 0.
\]

This implies easily that \( \kappa|_{L^o} \) vanishes as a bounded real class: in fact, because of (4.2) and (3.7),

\[
0 = df_{\kappa}|_{(L')^o} = p^*(\kappa)|_{(L')^o} = (p|_{(L')^o})^*(\kappa|_{L^o}),
\]

where we used that, since \( p : G_{nk} \to G \) is an open map, \( p((L')^o) = L^o \).

Consider now the subgroup \( \Delta := \rho^{-1}(\rho([\Gamma, \Gamma]) \cap L^o) \) of \( \Gamma \). Then \( (\rho|\Delta)^*(\kappa|_{L^o}) = 0 \) and hence, by hypothesis and definition of \( \kappa \)-weak maximality, \( \lambda \kappa|_{\Delta} = 0 \). Since by hypothesis \( \lambda \neq 0 \), then it must be that \( \kappa|_{\Delta} = 0 \). Thus \( \rho_h|\Delta : \Delta \to PU(1,1) \) satisfies the hypotheses of Lemma 4.2 and hence is elementary. Since \( \Delta \lhd \pi_1(\Sigma) \), this implies \( \Delta \) is trivial and hence \( \rho([\Gamma, \Gamma]) \cap L^o \) is trivial as well.

But since \( L \) is a Lie group (as a Lie subgroup of \( G \)), \( L^o \) is open in \( L \) and \( \rho([\Gamma, \Gamma]) \cap L^o \) is dense in \( L^o \). Hence \( L^o \) is trivial, and thus \( L \), and consequently \( \rho([\Gamma, \Gamma]) \), is discrete. \( \square \)

5. ON THE RADICAL DEFINED BY A BOUNDED CLASS

In this section, given a locally compact group \( L \) and a bounded rational class \( \kappa \in H^2_{ob}(L, \mathbb{R}) \), we show the existence of a largest normal closed subgroup \( \text{Rad}_k(L) \) on which the restriction of the class vanishes. We show moreover that the class \( \kappa \) comes from a bounded real class on the quotient \( L/\text{Rad}_k(L) \), the radical of which is trivial. If \( L \) is a connected Lie group, the quotient \( L/\text{Rad}_k(L) \) is adjoint semisimple without compact factors.

**Proposition 5.1.** Let \( L \) be a locally compact second countable group and let \( \kappa \in H^2_{ob}(L, \mathbb{R}) \) be a rational class. There is a unique largest normal subgroup \( N \lhd L \) with \( \kappa|_N = 0 \) which, in addition, is closed.

This relies on the following:
Lemma 5.2. Let $f : L \to \mathbb{R}$ be a continuous homogeneous quasimorphism.

(1) There is a unique largest normal subgroup $N_1 \lhd L$ with $f|_{N_1} = 0$.

(2) There is a unique largest normal subgroup $N_2 \lhd L$ on which $f$ is a homomorphism.

Both $N_1$ and $N_2$ are closed.

Proof. Clearly (2) implies (1) with $N_1 = \ker(f|_{N_2})$.

Let now $M_1, M_2$ be normal subgroups of $L$ such that $f|_{M_i} : M_i \to \mathbb{R}$ is a homomorphism. For $m_1 \in M_1$ and $m_2 \in M_2$, let $\chi(m_1 m_2) := f(m_1) + f(m_2)$. We claim that $\chi$ is well defined. Indeed, if $m_1 m_2 = m'_1 m'_2$ with $m_i, m'_i \in M_i$, then $(m'_1)^{-1} m_1 = m'_2 m_2^{-1} \in M_1 \cap M_2$. Thus $f((m'_1)^{-1} m_1) = f(m'_2 m_2^{-1})$, which implies, taking into account that $f$ is a homomorphism on $M_1$ and $M_2$, that

$$-f(m'_1) + f(m_1) = f(m'_2) - f(m_2).$$

This shows that $\chi$ is well defined. Next we claim that $\chi$ is a homomorphism. If $m_1, m'_1 \in M_i$, and since $M_1$ is normal in $L$, we have

$$\chi((m_1 m_2)(m'_1 m'_2)) = \chi(m_1 (m_2 m'_1 m_2^{-1}) m_2 m'_2) = f(m_1 (m_2 m'_1 m_2^{-1})) + f(m_2 m'_2) = f(m_1) + f(m_2 m'_1 m_2^{-1}) + f(m_2) + f(m'_2).$$

Since $f$ is a homogeneous quasimorphism, we have $f(m_2 m'_1 m_2^{-1}) = f(m'_1)$, which implies that $\chi : M_1 M_2 \to \mathbb{R}$ is a homomorphism. Since $f$ is a quasimorphism, we have in particular that for all $m_i \in M_i$,

$$|f(m_1 m_2) - \chi(m_1 m_2)| = |f(m_1 m_2) - f(m_1) - f(m_2)| \leq C,$$

for some constant $C$. Thus the homogeneous quasimorphism $f|_{M_1 M_2}$ is at finite distance from the homomorphism $\chi$ and hence $f|_{M_1 M_2} = \chi$.

This shows the existence of a unique largest normal subgroup $N_2 \lhd L$ on which $f$ is a homomorphism. By continuity of $f$, the subgroup $N_2$ is closed. \hfill \Box

Proof of Proposition 5.1. Let

$$\begin{align*}
0 & \to \mathbb{Z} \xrightarrow{i} L_{nk} \xrightarrow{p} L \xrightarrow{e} e
\end{align*}$$

be the topological central extension in (3.6) and $f_{nk} : L_{nk} \to \mathbb{R}$ the continuous homogeneous quasimorphism such that $[df_{nk}] = p^*(\kappa) \in H^2_{cb}(L_{nk}, \mathbb{R})$. It is an easy verification that, given a subgroup $N \lhd L$, the property $\kappa|_N = 0$ is equivalent to the property that $f_{nk}$ is a
homomorphism. Together with Lemma 5.2 this concludes the proof of the proposition. □

**Definition 5.3.** Let \( \kappa \in H^2_{cb}(L, \mathbb{R}) \) be a rational class. Denote by \( \text{Rad}_\kappa(L) \) the normal closed subgroup of \( L \) given by Proposition 5.1 and call it the \( \kappa \)-radical of \( L \).

**Corollary 5.4.** Let \( L \) be a locally compact second countable group and let \( \kappa \in H^2_{cb}(L, \mathbb{R}) \) be a rational class.

1. \( \text{Rad}_\kappa(L) \supset \text{Rad}_a(L) \), where \( \text{Rad}_a(L) \) denotes the amenable radical of \( L \).
2. Let \( \pi : L \to L/\text{Rad}_\kappa(L) \) denote the canonical projection. Then there is a unique class \( u \in H^2_{cb}(L/\text{Rad}_\kappa(L), \mathbb{R}) \) with \( \pi^*(u) = \kappa \). The restriction of \( u \) to any non-trivial closed normal subgroup of \( L/\text{Rad}_\kappa(L) \) does not vanish.

**Proof.** (1) follows from the fact that the bounded cohomology of an amenable group vanishes.

(2) The first assertion follows from the exactness of the short sequence

\[
0 \to H^2_{cb}(L/N, \mathbb{R}) \to H^2_{cb}(L, \mathbb{R}) \to H^2_{cb}(N, \mathbb{R}),
\]

where \( N \triangleleft L \) is any closed normal subgroup \([18, \text{Theorem 4.1.1}]\). The second assertion follows from the first and the maximality of \( \text{Rad}_\kappa(L) \).

We denote by \( \hat{H}^2_{cb}(L, \mathbb{Z}) \) the cohomology of the complex of integer valued bounded Borel cochains on \( L \). If \( \varkappa \in \hat{H}^2_{cb}(L, \mathbb{Z}) \), we denote by \( \varkappa^\mathbb{R} \) the image of \( \varkappa \) in \( H^2_{cb}(L, \mathbb{R}) \).

The above discussion applied to a general connected Lie group \( L \) has the following nice consequences.

**Corollary 5.5.** Let \( L \) be a connected Lie group and \( \varkappa \in \hat{H}^2_{cb}(L, \mathbb{Z}) \). Then \( H := L/\text{Rad}_a(L) \) is connected adjoint of Hermitian type and a direct product \( H = H_1 \times \cdots \times H_n \) of simple non-compact factors.

**Proof.** Since the quotient \( L/\text{Rad}_a(L) \) of \( L \) by its amenable radical is adjoint semisimple without compact factors, so is \( H \) (Corollary 5.1(1)). Let \( H = H_1 \times \cdots \times H_n \) be the direct product decomposition into simple factors. Let \( u \in H^2_{cb}(H, \mathbb{R}) \) be such that \( \pi^*(u) = \varkappa^\mathbb{R} \), where \( \pi : L \to H \) is the projection in (5.1). According to Corollary 5.2, \( u|_{H_j} \neq 0 \) and hence \( H^2_{cb}(H_j, \mathbb{R}) \neq 0 \): thus \( H_j \) is of Hermitian type for every \( 1 \leq j \leq n \) and hence so is \( H \). □
The following definition will be needed in the next section.

**Definition 5.6.** Let $L$ be a connected Lie group which admits a closed Levi factor and $\zeta \in \hat{H}^2_{cb}(L, \mathbb{Z})$. A $\zeta$-Levi factor of $L$ is a connected semisimple subgroup $S$ with finite center such that $\pi(S) = L/\text{Rad}_\zeta(L)$, where $\pi$ is as in (5.1), and $\ker(\pi|_S)$ is the center $Z(S)$ of $S$.

If $L_0$ is a Levi factor of $L$, a $\zeta$-Levi factor of $L$ is nothing but the product of the almost simple factors in $L$ whose image via $\pi$ is non-trivial.

### 6. Rationality Questions

The class $u$ in Corollary 5.4(2) is a priori only a real class. We show in this section that if $L$ is a closed subgroup of a group $G$ of Hermitian type and $\kappa$ is the restriction of a bounded rational class on $G$, then $u$ has nice integrality properties.

If in particular $\kappa = \kappa^b|_L$ is the restriction of the bounded Kähler class, then the class $u$ is the linear combination of the bounded Kähler classes of the individual simple factors of $L/\text{Rad}_{\kappa^b}(L)$ with rational coefficients whose denominators are bounded by an integer depending only on $G$.

If $G$ is a Lie group of Hermitian type, $L < G$ a closed subgroup and $\zeta \in \hat{H}^2_{cb}(G, \mathbb{Z})$, we denote by $\zeta^R_L$ the restriction of $\zeta^R$ to $L$.

**Proposition 6.1.** Let $G$ be a Lie group of Hermitian type and let $L < G$ be a closed connected subgroup that admits a closed Levi factor $S$. Let $\zeta \in \hat{H}^2_{cb}(G, \mathbb{Z})$ and let $\pi : L \to H := L/\text{Rad}_\zeta(L)$ denote the canonical projection. Let $p_j : H \to H_j$ be the projection onto the simple factors of $H$ (see Corollary 5.5) and let $\zeta_{H_j}$ be a generator of $\hat{H}^2_{cb}(H_j, \mathbb{Z})$, for $j = 1, \ldots, n$. If $u \in H^2_{cb}(H, \mathbb{R})$ is such that $\pi^*(u) = \zeta^R_L$ (see Corollary 5.4(2)), then

\begin{equation}
(6.1) \quad u = \sum_{j=1}^n \lambda_j p_j^*(\zeta_{H_j}^R)
\end{equation}

with

\begin{equation}
(6.2) \quad \lambda_j \in \frac{1}{|Z(S)|} \mathbb{Z},
\end{equation}

where $Z(S)$ denotes the center of the $\zeta$-Levi factor of $L$.

**Proof.** According to [17, Proposition 7.7 (3)], the set $\{p_j^*(\zeta_{H_j}) : 1 \leq j \leq n\}$ is a basis of $\hat{H}^2_{cb}(H, \mathbb{Z})$ corresponding to the basis $\{p_j^*(\zeta_{H_j}^R) :
1 \leq j \leq n \} \text{ of } H^2_{\text{cb}}(H, \mathbb{R}). \text{ It follows that}

\[ u = \sum_{j=1}^{n} \lambda_j p_j^*(\mathcal{R}_H), \]

with \( \lambda_j \in \mathbb{R} \).

In order to show that the \( \lambda_j \) are rational with an universal bound on the denominator, we consider the following diagram

\[
\begin{array}{c}
\hat{H}^2_{\text{cb}}(H, \mathbb{Z}) \xrightarrow{(\pi|_S)_{\mathbb{Z}}} \hat{H}^2_{\text{cb}}(S, \mathbb{Z}) \xrightarrow{\text{Res}} \hat{H}^2_{\text{cb}}(L, \mathbb{Z}),
\end{array}
\]

where \( \text{Res} \) is the restriction map in cohomology and the vertical arrows are the change of coefficients.

If \( (\pi|_S)^* \) were surjective, again [17, Proposition 7.7 (3)], the commutativity of (6.3) and the fact that \( (\pi|_S)^* \) is an isomorphism would readily imply the integrality of the \( \lambda_j \). This is however not necessarily true and the following lemma identifies explicitly the nature of the map \( (\pi|_S)^* \).

**Lemma 6.2.** Let \( \omega : S \to H \) be a surjective homomorphism between connected semisimple Lie groups with finite center. Then the map

\[
\omega_{\mathbb{Z}}^* : \hat{H}^2_{\text{cb}}(H, \mathbb{Z}) \longrightarrow \hat{H}^2_{\text{cb}}(S, \mathbb{Z})
\]

is injective and

\[
\text{Image}(\omega_{\mathbb{Z}}^*) \supset |\ker \omega| \hat{H}^2_{\text{cb}}(S, \mathbb{Z}),
\]

where \( |\ker \omega| \) denotes the cardinality of \( \ker \omega \) and \( \hat{H}^2_{\text{cb}}(S, \mathbb{Z}) \) is considered as a \( \mathbb{Z} \)-module.

We postpone the proof of the lemma and use its conclusion with \( \omega = \pi|_S \). If \( Z(S) = \ker(\pi|_S) \) denotes the center of \( S \), the same argument as above, applied to \( |Z(S)| \mathcal{R}_S \in (\pi|_S)^* \hat{H}^2_{\text{cb}}(H, \mathbb{Z}) \) shows that \( |Z(S)| u \) is, in fact, in \( \hat{H}^2_{\text{cb}}(H, \mathbb{Z}) \) and hence its coordinates \( |Z(S)| \lambda_j \) are integers, \( j = 1, \ldots, n \). \( \square \)

**Proof of Lemma 6.2** If \( M \) is any connected semisimple Lie group with finite center and maximal compact \( K_M \), Wigner's theorem asserts that the restriction map

\[
\hat{H}^2_c(M, \mathbb{Z}) \longrightarrow \hat{H}^2_c(K_M, \mathbb{Z})
\]
is an isomorphism \[59\]. This, together with the fact that the comparison map
\[ \hat{H}^2_{cb}(M, \mathbb{Z}) \to \hat{H}^2_c(M, \mathbb{Z}) \]
is an isomorphism \[17\), Proposition 7.7], implies the isomorphism
\[ j : \hat{H}^2_{cb}(M, \mathbb{Z}) \to \hat{H}^2_c(K_M, \mathbb{Z}) . \]
Using now the long exact sequence in cohomology associated to the short exact sequence
\[ 0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0 , \]
and the fact that the terms of positive degree and real coefficients vanish, we obtain that the connecting homomorphism
\[ \text{Hom}_c(K_M, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_{KH}} \hat{H}^2_c(K_M, \mathbb{Z}) \]
is an isomorphism.
Given \( \omega : S \to H \) as in the statement of the lemma, let \( K_H < H \) be a maximal compact subgroup. Then \( K_S := \omega^{-1}(K_H) \) is a maximal compact in \( S \). One verifies then that the diagram
\[ \begin{array}{ccc}
\hat{H}^2_{cb}(H, \mathbb{Z}) & \xrightarrow{\omega^*} & \hat{H}^2_{cb}(S, \mathbb{Z}) \\
\delta_{KH}^{-1} \circ j & \downarrow & \delta_{KS}^{-1} \circ j \\
\text{Hom}_c(K_H, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\omega^*} & \text{Hom}_c(K_S, \mathbb{R}/\mathbb{Z}) ,
\end{array} \]
where the vertical arrows are isomorphisms and the bottom one is the precomposition with \( \omega|_{K_S} \), is commutative. Then the assertion of the lemma follows from the fact that \( \ker \omega \subset K_S \) and that \( \omega|_{K_S} : K_S \to K_H \) is surjective. \( \square \)

Recall that the bounded Kähler class \( \kappa^b_G \in \hat{H}^2_{cb}(G, \mathbb{R}) \) of a Lie group \( G \) of Hermitian type is rational \([9, \S 5]\).

**Definition 6.3.** Let \( G \) be a Lie group of Hermitian type and \( \kappa^b_G \in \hat{H}^2_{cb}(G, \mathbb{R}) \) its bounded Kähler class. The *Kähler radical* \( \text{Rad}_{\kappa^b_G}(L) \) of a closed subgroup \( L < G \) is the \( \kappa^b_G|_L \)-radical of \( L \).

If \( L \) admits a closed Levi factor, a *Kähler-Levi factor* of \( L \) is a \( \kappa^b_G|_L \)-Levi factor.

**Corollary 6.4.** Let \( G \) be a Lie group of Hermitian type and let \( L < G \) be a closed connected subgroup that admits a closed Levi factor. Then the group \( H := L/\text{Rad}_{\kappa^b_G}(L) \) is connected adjoint of Hermitian type,
and admits a direct product decomposition $H = H_1 \times \cdots \times H_n$ of simple non-compact factors.

Moreover if $\pi : L \to H$ and $p_j : H \to H_j$ are the canonical projections, then there exists an integer $\ell_G \geq 1$ depending only on $G$ such that

$$\kappa_{G, L}^b = \pi^* \left( \sum_{j=1}^n \nu_j p_j^* (\kappa_{H_j}^b) \right),$$

where $\nu_j \in \frac{1}{\ell_G} \mathbb{Z}$.

Remark 6.5. Let $H$ be a Lie group of Hermitian type. We denote by $q_H$ the smallest integer such that there exists $\alpha \in H^2_{cb}(H, \mathbb{Z})$ with $q_H \kappa_H^b = \alpha^\mathbb{R}$. If in addition $H$ is simple and $\alpha_H$ is a generator of $H^2_{cb}(H, \mathbb{Z})$, then there exists an integer $m_H$ such that $\alpha = m_H \alpha_H$. It follows that

$$\alpha^\mathbb{R} = \frac{q_H}{m_H} \kappa_H^b.$$

Proof of Corollary 6.4. We apply Proposition 6.1 to $\alpha \in H^2_{cb}(G, \mathbb{Z})$ such that $\alpha^\mathbb{R} = q_G \kappa_G^b$. Then (6.1) and (6.2), together with (6.4), show that

$$\kappa_{G, L}^b = \pi^* \left( \sum_{j=1}^n \nu_j p_j^* (\kappa_{H_j}^b) \right),$$

with

$$\nu_j \in \frac{q_{H_j}}{q_G m_{H_j} |Z(S)|} \mathbb{Z},$$

where $|Z(S)|$ is the cardinality of the center of a Kähler-Levi factor $S$.

Since there are only finitely many possible conjugacy classes of connected semisimple subgroups of $G$, we obtain the result. $\Box$

7. Structure of Weakly Maximal Representations

In § 4 the discreteness of a $\kappa$-weakly maximal representation was proven under the assumption that the Toledo invariant $T_\kappa(\rho)$ is rational. In this section we prove that if $\kappa$ is the bounded Kähler class, this is always the case and that the representation into the quotient of its Zariski closure by the Kähler radical is also discrete and injective. The definition and properties of the Kähler radical will be essential to show that the Toledo invariant of the representation into the quotient is also non-zero.

An interesting feature of the proof of the rationality of the Toledo invariant is that it depends upon showing first that the single factors of the quotient by the Kähler radical are of tube type.
Let $G$ be a Lie group of Hermitian type and $\kappa^b_G \in H^2_{cb}(G, \mathbb{R})$ its bounded Kähler class. For ease of notation, we denote the Toledo invariant $T_{\kappa^b_G}$ by

$$T : \text{Hom}(\pi_1(\Sigma), G) \to \mathbb{R}.$$ 

**Definition 7.1.** A homomorphism $\rho : \pi_1(\Sigma) \to G$ is weakly maximal if

$$T(\rho) = \|\rho^*(\kappa^b_G)\| |\chi(\Sigma)|,$$

that is if $\rho$ is $\kappa^b_G$-weakly maximal in the sense of Definition 2.1.

**Theorem 7.2.** Let $G = G(\mathbb{R})^c$ be of Hermitian type, where $G$ is a connected semisimple algebraic group defined over $\mathbb{R}$, and let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal homomorphism. Let $L$ be the connected component of the real points of the Zariski closure of the image of $\rho$ and let $\Gamma := \rho^{-1}(L)$. Assume that $T(\rho) \neq 0$. Then:

1. the group $L/\text{Rad}_{\kappa^b_G}(L)$ is Hermitian of tube type;
2. there is an integer $\ell_G \geq 1$ depending only on $G$ (see Corollary 6.4), such that
   $$T(\rho) \in \frac{|\chi(\Sigma)|}{\ell_G} \mathbb{Z};$$
3. the composition
   $$\Gamma \xrightarrow{\rho|_\Gamma} L \xrightarrow{\pi} L/\text{Rad}_{\kappa^b_G}(L)$$
   is injective with discrete image.

Using that in a surface group there are no finite normal subgroups, one obtains immediately:

**Corollary 7.3.** Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal homomorphism with $T(\rho) \neq 0$. Then $\rho$ is injective with discrete image.

An important role in the proof is played by the generalized Maslov cocycle. Recall that if $H$ is a connected Lie group of Hermitian type and $\tilde{S}$ the Shilov boundary of the associated bounded symmetric domain, the generalized Maslov cocycle $\beta_{\tilde{S}} : \tilde{S}^3 \to \mathbb{R}$ is a bounded alternating $H$-invariant cocycle constructed by J.-L. Clerc in [24]. We will use it in two ways: on the one hand it represents the bounded Kähler class in a particular resolution useful to implement pullbacks in bounded cohomology (see (7.2)); on the other, through the relation (7.4) with the Hermitian triple product

$$\langle \langle \cdot, \cdot, \cdot \rangle \rangle : \tilde{S}^{(3)} \to \mathbb{R}^\times \backslash \mathbb{C}^\times,$$

it gives a criterion to detect when $H$ is of tube type, [9, 15].
We refer the reader to [8, 9] and [15] § 4.2 for details.

**Proof of Theorem 7.2.** We use heavily in this proof techniques developed in [18, 8, 9, 15], to which we refer the reader for details.

According to Corollary 6.4, the group $H := L/\text{Rad}_{\kappa_G}(L) = H_1 \times \cdots \times H_n$ is of Hermitian type and $\kappa_G|_L = \pi^*(u)$, where

$$u = \sum_{j=1}^{n} \nu_j p_j^*(\kappa_H^b),$$

and $p_j : H \to H_j$ and $\pi : L \to H$ are the canonical projections. We set aside for the moment that the $\nu_j$ are rational, and will pick it up towards the end of the proof.

Let $\xi := \pi \circ \rho|_{\Gamma} : \Gamma \to L \to H$ be the composition of $\pi$ with $\rho|_{\Gamma}$. Observe that $\Gamma$ is of finite index in $\pi_1(\Sigma)$. It follows that, since $\rho$ is weakly maximal, that is $\rho^*(\kappa_G^b) = \lambda \kappa_{\Sigma}^b$ for $\lambda = \frac{T(\rho)}{|\chi(\Sigma)|}$, then

$$\xi^*(u) = \lambda \kappa_{\Sigma'}^b,$$

where $\Sigma' \to \Sigma$ is the finite covering corresponding to $\Gamma < \pi_1(\Sigma)$.

As usual, to realize the pullback $\xi^*$ in bounded cohomology, we use boundary maps. According to [8], this is possible since if $(\mathcal{B}_{\text{alt}}^\infty(\check{S}\bullet))$ denotes the complex of bounded alternating Borel cocycles on $\check{S}$, then the class $[\beta_{\check{S}}]$ defined by the generalized Maslov cocycle corresponds to the bounded Kähler class $\kappa_H^b$ under the canonical map

$$H^\bullet(\mathcal{B}_{\text{alt}}^\infty(\check{S}\bullet)^H) \longrightarrow H_{\text{cb}}^\bullet(H, \mathbb{R}).$$

Likewise, if $\check{S} = \check{S}_1 \times \cdots \times \check{S}_n$ is the decomposition into a product, where $\check{S}_j$ is the Shilov boundary of $H_j$ and $\check{p}_j : \check{S} \to \check{S}_j$ is the projection, a standard cohomological argument shows that the diagram

$$
\begin{array}{ccc}
\mathcal{H}_{\text{cb}}^\bullet(H_j, \mathbb{R}) & \xrightarrow{p_j^*} & \mathcal{H}_{\text{cb}}^\bullet(H, \mathbb{R}) \\
\uparrow & & \uparrow \\
\mathcal{H}^\bullet(\mathcal{B}_{\text{alt}}^\infty(\check{S}_j)^{H_j}) & \xrightarrow{\check{p}_j^*} & \mathcal{H}_{\text{cb}}^\bullet(\mathcal{B}_{\text{alt}}^\infty(\check{S}\bullet)^H)
\end{array}
$$

commutes. It follows that $u$ is represented, again via (7.2), by the bounded Borel cocycle on $\check{S}^3$ defined by

$$(x, y, z) \longmapsto \sum_{j=1}^{n} \nu_j \beta_{\check{S}_j}(x_j, y_j, z_j).$$

To recall the existence of the boundary map, endow the interior of $\Sigma'$ with a complete hyperbolic metric of finite area so that $\Gamma$ is identified...
with a lattice in PU(1, 1). Since ξ(Γ) is Zariski dense in H, there is a Γ-equivariant measurable map \( \varphi : \partial \mathbb{D} \to \hat{S} \). Because of the product structure of \( \hat{S} \), the map \( \varphi \) has the form

\[
\varphi_1 \times \cdots \times \varphi_n : \partial \mathbb{D} \to \hat{S}_1 \times \cdots \times \hat{S}_n,
\]

where \( \varphi_j : \partial \mathbb{D} \to \hat{S}_j \) is measurable and equivariant with respect to \( p_j \circ \xi \).

Consequently, according to [8] the pullback \( \xi^*(u) \) is represented by the following bounded measurable Γ-invariant cocycle

\[
(x, y, z) \mapsto \sum_{j=1}^n \nu_j \beta_{\hat{S}_j}(\varphi_j(x), \varphi_j(y), \varphi_j(z)),
\]

for almost all \((x, y, z) \in (\partial \mathbb{D})^3\).

Using now again that the bounded fundamental class of \( \Sigma \) is represented by \( \beta_{\partial \mathbb{D}} \), it follows from (7.1) that

\[
(7.3) \quad \lambda \beta_{\partial \mathbb{D}}(x, y, z) = \sum_{j=1}^n \nu_j \beta_{\hat{S}_j}(\varphi_j(x), \varphi_j(y), \varphi_j(z))
\]

for almost all \((x, y, z) \in (\partial \mathbb{D})^3\).

Recall now that the Hermitian triple product is an \( H \)-invariant real algebraic map on \((\hat{S})^3\) whose relation with the Maslov cocycle is given by

\[
(7.4) \quad \langle \langle x, y, z \rangle \rangle_{\hat{S}} \equiv e^{i\pi d_{\hat{S}}(x, y, z)} \mod \mathbb{R}^\times,
\]

where \( d_{\hat{S}} \) is an integer given in terms of the root system of \( H \), [15]. We will use this relation for the Hermitian triple product on each of the \( H_j \). To uniformize the exponents, since \( \nu_j \in \mathbb{Q}^\times \), we can pick an integer \( m \geq 1 \) such that \( mv_j = n_j d_j \) for some \( n_j \in \mathbb{Z}, 1 \leq j \leq n \). Multiplying (7.3) by \( m \), exponentiating and using that \( \beta_{\partial \mathbb{D}} \) takes only \( \pm 1/2 \) as values, we obtain

\[
(7.5) \quad e^{i\pi m \lambda(\pm 1/2)} = \prod_{j=1}^n \langle \langle \varphi_j(x), \varphi_j(y), \varphi_j(z) \rangle \rangle_{\hat{S}_j}^{n_j},
\]

for almost every \((x, y, z) \in (\partial \mathbb{D})^3\). If \( x_j, y_j, z_j \in \hat{S} \), we define now

\[
R((x_j), (y_j), (z_j)) := \prod_{j=1}^n \langle \langle x_j, y_j, z_j \rangle \rangle_{\hat{S}_j}^{n_j}.
\]

Because of (7.5) the function \( R \) takes at most two values on \( \varphi(\partial \mathbb{D})^3 \); since the latter set is Zariski dense in \( \hat{S}^3 \) and \( R \) is rational, it takes on at most two values on \( \hat{S}^3 \). The factors \((x_j, y_j, z_j)_{\hat{S}_j}^{n_j}\) are pairwise
independent rational functions, therefore each of these factors itself can take only finitely many values. As previously recalled, it follows from [15] that the corresponding groups are of tube type, thus showing assertion (1).

Furthermore, since $\beta_{S_j}$ takes only values in $\frac{1}{2}Z$ [26, Theorem 4.3], it follows from (7.3) and Corollary 6.4 that $\lambda \in \frac{1}{\ell_0}Z$. This shows assertion (2).

The third assertion follows from Proposition 4.1 (2) and Proposition 4.3. □

8. Orders and Quasimorphisms

In this section we present the basic facts and definition concerning some bi-invariant orders on groups, namely orders “sandwiched” by a homogeneous quasimorphism. The main point is a theorem that reconstructs the quasimorphism (up to a multiplicative constant) from the knowledge of the order. This material is taken from [3, 2], where it is presented in a more elaborate form.

Let $G$ be a group.

**Definition 8.1.** A bi-invariant (partial) order on $G$ is a relation $\preceq$ on $G \times G$ such that:

1. $g \preceq h$ and $h \preceq g$ implies that $g = h$;
2. $g \preceq h$ and $h \preceq k$ implies that $g \preceq k$, and
3. if $g \preceq h$ then $agb \preceq ahb$ for all $a, b \in G$.

Let

$$G^+ = \{g \in G : g \succeq e\}$$

denote the set of positive elements. Then $G^+$ is a pointed conjugacy invariant submonoid (where pointed means that $G^+ \cap (G^+)^{-1} = (e)$). Conversely, every pointed conjugacy invariant submonoid of $G$ is the set of positive elements of a unique bi-invariant order on $G$.

**Example 8.2.** We can define a bi-invariant order on the group $G := \text{Homeo}^+(\mathbb{R})$ of orientation preserving homeomorphisms on the real line, by saying that $f \succeq g$ if $f(t) \geq g(t)$ for all $t \in \mathbb{R}$. Note that if $G$ consisted of all homeomorphisms of $\mathbb{R}$, the order would have been only right invariant.

**Example 8.3.** Let $f : G \to \mathbb{R}$ be a homogeneous quasimorphism with defect

$$\|df\|_\infty := \sup_{x, y \in G} |f(gh) - f(g) - f(h)|.$$
For every $C \in \mathbb{R}$, let us define
\[
N_C(f) := \{ g \in G : f(g) \geq C \}.
\]
Then for every $C \geq \|df\|_{\infty}$, the set $N_C(f) \cup \{e\}$ is a conjugacy invariant submonoid that is pointed if $C > 0$, and hence gives rise to a bi-invariant order on $G$.

We will be concerned with orders that are "sandwiched" by quasimorphisms in the sense of [3]. Before giving the definition we consider the following:

**Lemma 8.4.** Assume that $G^+$ is the set of positive elements of a bi-invariant order and that $f : G \to \mathbb{R}$ is a homogeneous quasimorphism such that
\[
N_C(f) \subset G^+
\]
for some $C \in \mathbb{R}$. Then we have
\[
G^+ \subset N_0(f).
\]

**Proof.** We need to show that if $g \in G^+$, then $f(g) \geq 0$. We may assume that $f(g) \neq 0$, otherwise we are done. Since $f$ is homogeneous, for $p \in \mathbb{N}$ large enough we have that $|f(g^p)| = |pf(g)| \geq |C|$. If $f(g)$ were to be negative, then $f(g^{-p}) = -pf(g) \geq |C| \geq 0$, and hence by hypothesis $g^{-p} \in G^+$. Since $G^+$ is a semigroup, it would contain both $g^p$ and $g^{-p}$. But the fact that $G^+$ is pointed implies that $g^p = e$ and hence $f(g) = 0$, that is a contradiction. \(\square\)

**Definition 8.5.** We say that a bi-invariant order satisfying the assumption (and hence the conclusion) of Lemma 8.4 is $C$-sandwiched by the quasimorphism $f$. The constant $C$ is called a sandwiching constant.

Note that if $C$ is a sandwiching constant, then any $C' > C$ is also a sandwiching constant. In particular we can always take $C > 0$.

Given an order on $G$, we recall from [27] the definition of the set of dominant elements
\[
G^{++} := \{ g \in G^+ : \text{for every } h \in G \text{ there exists } p \geq 1 \text{ with } g^p \succeq h \}.
\]
We may also refer to $G^{++}$ with a slight abuse of terminology as to the dominant set. We notice the following properties:

**Lemma 8.6.** With the above notation:
(1) $G^{++} \cap (G^{++})^{-1} = \emptyset$;
(2) $G^{++} \cdot G^+ = G^+ \cdot G^{++} = G^{++}$;
(3) $G^{++}$ is conjugacy invariant.
Proof. Properties (1) and (3) are clear. It follows from bi-invariance, that for every \( h \in G^+ \) and \( p \geq 1 \),
\[
(gh)^p = gh(gh)^{p-1} \geq g(h)^{p-1},
\]
and thus, by recurrence, \( (gh)^p \geq g^p \). This implies that \( G^+ \cdot G^+ = G^+ \). \( \square \)

The set of dominant elements will play an essential role in showing that a quasimorphism sandwiching an order is essentially determined by the order. The following characterization is taken from [2, Lemma 2.10]:

**Lemma 8.7.** Assume that \( G^+ \) is sandwiched by a homogeneous quasimorphism \( f \). Then
\[
G^{++} = \{ g \in G^+ : f(g) > 0 \}.
\]

**Proof.** The quasimorphism property implies that for all \( g, h \in G \) and for all \( p \geq 1 \)
\[
 pf(g) \geq f(g^p h^{-1}) + f(h) - \|df\|_\infty.
\]
Choose \( h \in G \) such that \( f(h) \geq \|df\|_\infty \). If \( g \in G^{++} \), there exists \( p \geq 1 \) such that \( g^p \geq h \). Hence \( g^p h^{-1} \geq e \) and, by Lemma 8.4, \( f(g^p h^{-1}) \geq 0 \). Thus \( f(g) > 0 \) and \( G^{++} \subseteq \{ g \in G^+ : f(g) > 0 \} \).

Conversely, to see that \( \{ g \in G^+ : f(g) > 0 \} \subseteq G^{++} \), we need to show that if \( g \in G^+ \) with \( f(g) > 0 \), then, given \( h \in G \), there exists \( p \geq 1 \) such that \( g^p \geq h \) or, equivalently, \( g^p h^{-1} \geq e \). Because \( G^+ \) is sandwiched by \( f \), and hence \( N_C(f) \subset G^+ \) for some \( C \geq 0 \), it will be enough to show that \( f(g^p h^{-1}) \geq C \). But the quasimorphism property implies that
\[
 f(g^p h^{-1}) \geq pf(g) - f(h) - \|df\|_\infty,
\]
and, since \( f(g) > 0 \), there exists \( p \geq 1 \) such that
\[
 pf(g) - f(h) - \|df\|_\infty \geq C.
\]
\( \square \)

**Remark 8.8.** If \( G^+ \) is sandwiched by a homogeneous quasimorphism \( f \) with \( C > 0 \), it follows from Lemma 8.7 that \( N_C(f) \subset G^{++} \).

We now show how to reconstruct a quasimorphism from a sandwiched order. To this purpose, if \( g \in G^{++} \), \( h \in G \) and \( n \geq 1 \), we define
\[
 E_n(g, h) := \{ p \in \mathbb{Z} : g^p \geq h^n \}.
\]
By definition of dominants, the set \( E_n(g, h) \) is not empty. Moreover it is easy to see that it is bounded below: in fact, if \( g^{-k} \geq h^n \) for all
$k > 0$, then also $g^k \preceq h^{-n}$ for all $k > 0$, contradicting the fact that $g$ is a dominant. We may thus define

$$e_n(g, h) := \min E_n(g, h).$$

Since $g^{k+1} \succeq g^k$ for all $k \in \mathbb{Z}$, then

$$E_n(g, h) = [e_n(g, h), \infty) \cap \mathbb{Z}.$$ 

Observe that if $g^{p_1} \succeq h^n$ and $g^{p_2} \succeq h^m$, then $g^{p_1+p_2} \succeq h^{n+m}$. Thus

$$e_{n+m}(g, h) \leq e_n(g, h) + e_m(g, h)$$

and hence

$$e(g, h) := \lim_{n \to \infty} \frac{e_n(g, h)}{n}$$

exists. The idea that the function $e$ can be used to reconstruct quasimorphisms from an associated dominant set was developed in [3], where in particular the following reconstruction theorem was first established:

**Theorem 8.9 ([3]).** Let $G$ be a group with a bi-invariant order and let $G^+$ be the set of positive elements. If $G^+$ is $C$-sandwiched by a homogeneous quasimorphism $f$, then for every $n \in \mathbb{N}$,

$$-\|df\|_\infty - nf(h) + pf(g) \leq f(g^p h^{-n}) \leq pf(g) - nf(h) + \|df\|_\infty,$$ 

for every $g \in G^+$ and $h \in G$.

We immediately deduce from this the following:

**Corollary 8.10.** Let $G$ be an group with an bi-invariant order and let $G^+$ be the set of positive elements.

1. If $G^+$ is sandwiched by a homogeneous quasimorphism $f$, then

$$e(g, h) = \frac{f(h)}{f(g)}$$

for all $g \in G^+$ and $h \in G$.

2. If $G^+$ is sandwiched by homogeneous quasimorphisms $f_1, f_2$, there is $\lambda > 0$ such that $f_2 = \lambda f_1$.

**Proof of Theorem 8.9.** For all $p, n \in \mathbb{Z}$, the quasimorphism inequality reads

$$-\|df\|_\infty - nf(h) + pf(g) \leq f(g^p h^{-n}) \leq pf(g) - nf(h) + \|df\|_\infty.$$ 

Remark that if $g \in G^+$, then $f(g) > 0$.

The first inequality in (8.1) follows immediately from the fact that if $p \geq e_n(g, h)$, then

$$0 \leq f(g^p h^{-n}) \leq pf(g) - nf(h) + \|df\|_\infty.$$
To show the second inequality in (8.1), observe that for all \( p \in \mathbb{Z} \) such that the left hand side of (8.2) is \( \geq C \), that is for all \( p \in \mathbb{Z} \) such that
\[
(8.3) \quad p \geq \frac{n f(h) + C + \|df\|_\infty}{f(g)},
\]
we have that \( g^p h^{-n} \in N_C(f) \subset G^+ \). Thus \( p \geq e_n(g,h) \) and hence
\[
\frac{n f(h) + C + \|df\|_\infty}{f(g)} \geq e_n(g,h) - 1.
\]
\( \square \)

We want to apply Corollary 8.10 to the situation in which we have a homomorphism \( \rho : H \to G \) between two groups with bi-invariant orders \( H^+, G^+ \) each sandwiched by a homogeneous quasimorphism, respectively \( f_H \) and \( f_G \).

Under a certain “order” preserving hypothesis on \( \rho \), we would like to conclude that \( f_G \circ \rho \) and \( f_H \) are proportional. In general \( \rho^{-1}(G^+) \) does not give an order on \( H \) because of the presence of a kernel. We have however:

**Proposition 8.11.** Let \( G, H \) be groups with a bi-invariant order, \( \rho : H \to G \) be a homomorphism and let \( H^+, G^+ \) be sandwiched respectively by homogeneous quasimorphisms \( f_H \) and \( f_G \). Assume that either

1. \( \ker \rho = \{e\} \) and \( N_C(f_H) \subset \rho^{-1}(G^+) \) for some \( C \), or
2. \( N_C(f_H) \subset \rho^{-1}(G^{++}) \) for some \( C \).

Then \( f_H \) and \( f_G \circ \rho \) are proportional.

**Proof.** Because of Corollary 8.10, we need to show that both \( f_H \) and \( f_G \circ \rho \) sandwich a pointed conjugacy invariant submonoid in \( H \).

To this purpose observe that if \( M_G \subset G \) is any subset and for \( C' > 0 \)
\[
(8.4) \quad N_{C'}(f_G) \subset M_G,
\]
then
\[
N_{C'}(f_G \circ \rho) \subset \rho^{-1}(M_G).
\]
If \( \rho^{-1}(M_G) \) is a pointed conjugacy invariant submonoid, Lemma 8.4 implies that \( \rho^{-1}(M_G) \) is sandwiched by \( f_G \circ \rho \).

1. Since \( \ker \rho = \{e\} \), then \( \rho^{-1}(G^+) \) is a pointed conjugacy invariant submonoid. By hypothesis (8.4) holds with \( M_G = G^+ \).

2. Observe that \( \rho^{-1}(G^{++}) \cup \{e\} \) is a pointed conjugacy submonoid. It follows from Remark 8.8 that (8.4) holds with \( M_G = G^{++} \).
Since in both cases the hypotheses and Lemma 8.4 imply that the pointed conjugacy invariant submonoid is sandwiched by $f_H$, the assertion is proven. □

Example 8.12. Let $G = \text{Homeo}_+^+(\mathbb{R})$ be the group of increasing homeomorphisms of $\mathbb{R}$ which commute with integer translations with the order defined in Example 8.2. The following lemma shows that the Poincaré translation quasimorphism $\tau : G \to \mathbb{R}$ sandwiches this order (cf. [2, Prop. 2.16]).

Lemma 8.13. Let $g \in \text{Homeo}_+^+(\mathbb{R})$. Then the following assertions are equivalent:

1. $g(x) > x$ for all $x \in \mathbb{R}$;
2. $\tau(g) > 0$.

Assuming the lemma and taking into account Lemma 8.7, we deduce that

$$G^+ = \{g \in G : \tau(g) > 0\} = \{g \in G : g(x) > x \text{ for all } x \in \mathbb{R}\}$$

$$\subseteq G^+ \subseteq \{g \in G : \tau(g) \geq 0\}$$

In particular, the order is $C$-sandwiched by $\tau$ for any sandwiching constant $C > 0$.

Proof of Lemma 8.13. (2)⇒(1): If (1) fails, then either $g(x) - x < 0$ for all $x \in \mathbb{R}$ or $g(x) - x$ changes sign. In the first case $g^n(x) < x$ and thus $\tau(g) \leq 0$. In the second case, by the Intermediate Value Theorem there is $x_0 \in \mathbb{R}$ with $g(x_0) = x_0$ and thus $\tau(g) = 0$.

(1)⇒(2): If $g^n$ denotes the $n$-th iterate of $g$, then $\tau(g) = \lim_{n \to \infty} \frac{g^n(x) - x}{n}$ for every $x \in \mathbb{R}$. In view of (1), the sum

$$g^n(x) - x = \sum_{i=0}^{n-1} (g(g^i(x)) - g^i(x))$$

consists of positive terms. Thus if $\tau(g) = 0$, there is a subsequence $i_k \to +\infty$ with

$$\lim_{k \to \infty} (g(g^{i_k}(x)) - g^{i_k}(x)) = 0.$$ 

Thus, since $g$ commutes with integer translations, if $\{\cdot\}$ denotes the fractional part of a real number, then

$$\lim_{k \to \infty} (g(\{g^{i_k}(x)\}) - \{g^{i_k}(x)\}) = 0.$$ 

If now $y \in [0, 1]$ is an accumulation point of the sequence $\{g^{i_k}(x)\}_{k \geq 1}$, then $g(y) = y$, which contradicts (1). □
9. The Causal Ordering

In this section $G$ is a simple adjoint connected Lie group of Hermitian type and we assume that the associated bounded symmetric domain $\mathcal{D}$ is of tube type. Recall that the Shilov boundary $\tilde{S} \cong G/Q$ is a homogeneous space for $G$, where $Q$ is an appropriate maximal parabolic subgroup. It was established by Kaneyuki [50] that $\tilde{S}$ carries a $G$-invariant causal structure, unique up to inversion. The lift of this causal structure to the universal covering $\tilde{\mathcal{R}}$ of $\tilde{S}$ defines an order on $\tilde{\mathcal{R}}$ invariant under the action of an appropriate covering $\hat{G}$ of $G$. We use this order on $\tilde{\mathcal{R}}$ on the one hand to define an order on $\hat{G}$ and on the other to define explicitly an integral valued Borel quasimorphism on $\hat{G}$ whose homogenization is essentially the quasimorphism defined in §3. We then use this construction to show that the order on $\hat{G}$ is sandwiched by this quasimorphism and determines its set of dominant elements. A similar construction of quasimorphisms starting with a causal structure has been pointed out by Calegari in [20, 5.2.4] and in greater generality by Ben Simon and Hartnick in [1].

We need to recall some basic properties of Kaneyuki’s construction. Observe that any maximal compact subgroup $K$ of $G$ acts transitively on $\tilde{S}$. We denote by $M$ the stabilizer of the basepoint $o = eQ$, so that $\tilde{S} \cong K/M$. Consider now the holonomy representation of $\rho : M \to GL(V)$ on $V := T_{eQ}\tilde{S}$. According to [50], there exists an inner product $\langle \cdot, \cdot \rangle$ and an open cone $\Omega \subset V$ such that

$$\rho(M) = \{ g \in O(V, \langle \cdot, \cdot \rangle) \mid g\Omega = \Omega \}.$$  

The cone $\Omega$ is symmetric with respect to $\langle \cdot, \cdot \rangle$ in the sense of [28, p. 4]. In particular $\Omega$ is self-dual, i.e.

$$(9.1) \quad \Omega = \{ w \in V \mid \langle v, w \rangle > 0 \text{ for all } v \in \overline{\Omega} \setminus \{0\} \}.$$  

Furthermore, we deduce from [28, Prop. I.1.9] that $\rho(M)$ fixes a unit vector $e$ in $\Omega$. As a consequence, there exists a constant $k$ such that for all $w \in \overline{\Omega}$,

$$(9.2) \quad \langle e, w \rangle \geq k\|w\|.$$  

Since $\langle \cdot, \cdot \rangle$, $e$ and $\overline{\Omega}$ are $M$-invariant they give rise respectively to a $K$-invariant Riemannian metric $g$, a $K$-invariant vector field $v$, and a $K$-invariant causal structure $C$, all uniquely determined by their definition at the basepoint

$$(9.3) \quad g_o = \langle \cdot, \cdot \rangle, \quad v_o = e, \quad C_o = \overline{\Omega}.$$  

For future reference we record here the following observation that follows immediately from (9.2) and the fact that the objects in (9.3) are $K$-invariant.

**Lemma 9.1.** The cones $C_p$ are uniformly acute with respect to $g_p$, that is there exists $k > 0$ such that
\[ g_p(w, v_p) \geq k \|w\|, \]
for all $w \in C_p$ and for all $p \in \hat{S}$.

According to [50], the $K$-invariant causal structure defined above is $G$-invariant. In the sequel we will also need to consider various coverings of $(\hat{S}, \mathcal{C})$. To this end we recall [25] that $\pi_1(\hat{S}) \cong \mathbb{Z}$, from which it is easily deduced that also $\pi_1(G)$ has rank one. However, in general the map $\pi_1(G) \to \pi_1(\hat{S})$ is not surjective or, in other words, $Q$ may not be connected. To deal with this problem we introduce the finite covering $S' := G/Q^o$ of $\hat{S}$ and observe that the evaluation map
\[ G \to S' \]
\[ g \mapsto gQ^o \]
induces an isomorphism
\[(9.4) \quad \pi_1(G)/\pi_1(G)_{\text{tor}} \cong \pi_1(S'). \]
If $\hat{G}$ denotes the connected covering of $G$ associated to $\pi_1(G)_{\text{tor}}$, then $\hat{G}$ acts faithfully on the universal covering $\hat{R}$ of $S'$, covering the $G$-action on $S'$. Moreover it follows from (9.4) that the action of the center $Z(\hat{G})$ of $\hat{G}$ coincides with the $\pi_1(S')$-action. Now the causal structure $\mathcal{C}$ on $\hat{S}$ lifts to causal structures on $S'$ and $\hat{R}$, invariant under the corresponding automorphism groups. We will abuse notation and denote all these causal structures by $\mathcal{C}$. With this abuse of notation understood, Lemma 9.1 holds also for $S'$ and $\hat{R}$.

**Definition 9.2.** Let $M$ be any covering of $\hat{S}$. A curve $\gamma : [a, b] \to M$ is causal if it is piecewise $C^1$, with existing left and right tangents $\dot{\gamma}(t_\pm)$ and $\dot{\gamma}(t_-)$ at all points $t \in [a, b]$, and such that $\dot{\gamma}(t_\pm) \in C_{\gamma(t)}$ for all $t \in [a, b]$.

We define a relation on $\hat{R}$ as follows:

**Definition 9.3.** Let $x, y \in \hat{R}$. We say that $x \leq y$ if there is a causal curve from $x$ to $y$.

This relation is transitive and we will see (Lemma 9.5) that it gives a $\hat{G}$-invariant partial order on $\hat{R}$.
Let $x < y$ denote the relation $x \leq y$ and $x \neq y$ on $\tilde{R}$ and $f_{\hat{G}} : \hat{G} \to \mathbb{R}$ the homogeneous quasimorphism in (3.10) coming from the bounded Kähler class of $G$. The rest of this section will be devoted to the proof of the following:

**Theorem 9.4.** Let $G$ be a Hermitian Lie group of tube type. The relation $g \preceq h$ defined by $g(x) \leq h(x)$ for all $x \in \tilde{R}$, gives a bi-invariant order on $\hat{G}$ sandwiched by the quasimorphism $f_{\hat{G}}$. In addition, its set of dominant elements is

$$\hat{G}^+ = \{ g \in \hat{G} : g(x) > x \text{ for all } x \in \tilde{R} \}.$$ 

We now define a $K$-invariant 1-form $\alpha$ on $S'$ by

$$\alpha_p(x) := g_p(x, v_p),$$

for $x \in T_p S'$. (Here and in the sequel we abuse notation to denote the lifts of $v$ and $g$ from $\tilde{S}$ to $S'$ by the same letters.) Since $S'$ is symmetric and $\alpha$ is $K$-invariant we have, by Cartan’s lemma,

$$d\alpha = 0.$$ 

Furthermore, let $p : \tilde{R} \to S'$ be the universal covering map, $\tilde{\alpha} := p^*(\alpha)$, and fix a smooth function $\zeta : \tilde{R} \to \mathbb{R}$ such that $d\zeta = \tilde{\alpha}$. If $\gamma : [0, 1] \to \tilde{R}$ is a causal curve joining $x$ to $y$, then it follows from Lemma 9.1 that

$$\zeta(y) - \zeta(x) = \int_\gamma \tilde{\alpha} \geq k \text{ Length}(\gamma) \geq k d_{\tilde{R}}(x, y),$$

where $d_{\tilde{R}}$ refers to the Riemannian metric on $\tilde{R}$ corresponding to the one on $S'$. This gives immediately the following:

**Lemma 9.5.** The relation $\preceq$ is an order on $\tilde{R}$.

**Proof.** The relation is transitive since the concatenation of causal curves is causal. If now $x \leq y$ and $y \leq x$, it follows from (9.7) that $\zeta(y) - \zeta(x) \geq 0$ and $\zeta(x) - \zeta(y) \geq 0$, hence $d_{\tilde{R}}(x, y) = 0$. \(\square\)

We record for future purposes that, integrating over a geodesic path from $x$ to $y$ and using that $v$ is unit length, we have also the inequality

$$\zeta(y) - \zeta(x) \leq d_{\tilde{R}}(x, y).$$

We now exploit the fact that $\preceq$ is an order on $\tilde{R}$ but not on $S'$, where causal curves can be closed. However we first need to introduce some objects that stem from an alternative description of the vector field $v$. To this purpose observe that the action of $Z(K)$ on $S'$ gives a locally free $S^1$-action; the unit length vector field generating this action
is precisely $v_p$. [28 Proposition I.1.9]. Since $v_p \in C_p \subset C_p$, the integral curve
\begin{equation}
\gamma_p : [0, L] \to S'
\end{equation}
going through the point $p \in S'$ is a closed causal curve of length $L$ and hence $\gamma_p$ generates a subgroup of finite index in $\pi_1(S')$. Let $S''$ be the finite covering of $S'$ corresponding to this subgroup. For every $p \in S''$, the unique lift of $\gamma_p$ to a closed curve in $S''$ through $p$ is now a generator $Z$ of $\pi_1(S'')$ that is causal and has also length $L$.

**Lemma 9.6.** There exists a constant $D > 0$ depending only on the Riemannian metric $g(\cdot, \cdot)$ on $S''$ such that any two points in $S''$ can be joined by a causal path of length at most $D$.

**Proof.** First we show that there exists $\eta > 0$ such that every $q \in S''$ can be joined to the points in the open $\eta$-ball in $S''$ with center in $q$ by a causal path of length at most $L + 1$, where $L$ is defined in (9.9). Note that the causal curve will not necessarily be contained in the ball.

Let $K''$ be an appropriate covering of $K$ that acts effectively and transitively on $S''$, and let $M''$ be the stabilizer of $q$. For a suitable generator $Y$ of the Lie algebra of $Z(K'')$, we have that

$$\gamma_q(t) = \text{Exp} tY = \exp tv_q,$$

where exp is the Riemannian exponential map. As a result, $\exp Lv_q = q$.

Let now $0 < \epsilon < 1$ be such that the ball $B_\epsilon(Lv_q)$ with radius $\epsilon$ in the tangent space at $q \in S''$ is contained in $C_q$ and let $\eta > 0$ such that $B_\eta(q) \subset \exp(B_\epsilon(Lv_q))$. Given now any $v \in C_q$, we observe that the curve $t \mapsto \exp tv$, for $t \geq 0$, is causal. Indeed, the causal structure is left invariant by the parallel transport, since it is realized by left multiplication of appropriate elements of $K''$. Thus for $v \in C_q$ of length 1 such that $\ell_v v \in B_\epsilon(Lv_q)$, the assignment $t \mapsto \exp tv$ from $[0, \ell_v]$ to $S''$ gives the causal curve joining $q$ to $\exp \ell_v v$.

Let then $d = \text{diam}(S'')$. Given two points $p, q \in S''$, choose a distance minimizing geodesic $c : [0, d(p, q)] \to S''$ parametrized by arc length. Then it follows from the above claim that for every $n \in \mathbb{N}$ with $n \frac{\eta}{2} \leq d(p, q)$, $c((n - 1)\eta/2)$ can be joined to $c(n\eta/2)$ by a causal path of length at most $L + 1$. Thus $p = c(0)$ can be joined to $q = c(d(p, q))$ by a causal path of length at most $D := \left(\left\lceil \frac{2d}{\eta} \right\rceil + 1\right)(L + 1)$.

With the aid of the invariant order on $\hat{R}$, we now proceed to the construction of the quasimorphism on $\hat{G}$. We start by observing that
since $Z \in \pi_1(S^n)$ can be represented by the closed causal curve $\gamma_p$ for every $p \in S^n$, then
\begin{equation}
Z(x) \geq x
\end{equation}
for every $x \in \hat{R}$.

Given $x, y \in \hat{R}$, let us define
\[ I(x, y) := \{ n \in \mathbb{Z} : Z^ny \geq x \} . \]

It follows immediately from (9.10) that if $m \in I(x, y)$, then $Z^{m+1}y \geq Zx \geq x$, which shows that $m+1 \in I(x, y)$. In order to study further properties of the set $I$, we will need the function $\zeta$ defined above, in particular the property that, for all $x \in \hat{R}$, and all $n \in \mathbb{Z}$,
\begin{equation}
\zeta(Z^n x) = nL + \zeta(x)
\end{equation}
since $L > 0$ is the integral of $\alpha$ over $\gamma_p$.

**Lemma 9.7.** With the above notation
\begin{enumerate}
\item $\iota(x, y) := \min I(x, y)$ is well defined and $I(x, y) = [\iota(x, y), \infty) \cap \mathbb{Z}$.
\item $\iota$ is invariant under the diagonal \(\hat{G}\)-action on $\hat{R} \times \hat{R}$.
\item $|L \iota(x, y) - (\zeta(x) - \zeta(y))| \leq D$ for all $x, y \in \hat{R}$, where $D$ is given by Lemma 9.6.
\end{enumerate}

**Proof.** (1) If $n \in I(x, y)$, then, by (9.7),
\[ \zeta(Z^ny) - \zeta(x) \geq 0 , \]
and thus, by using (9.11)
\[ nL - (\zeta(x) - \zeta(y)) \geq 0 . \]
Thus $n \geq \frac{1}{L}(\zeta(x) - \zeta(y))$, and since $n \in I(x, y)$ is arbitrary, then
\begin{equation}
L \iota(x, y) - (\zeta(x) - \zeta(y)) \geq 0 .
\end{equation}

(2) This follows immediately from the fact that the \(\hat{G}\)-action commutes with $Z$ and preserves the order.

(3) Let $c : [0, D] \to S^n$ be a causal curve of length at most $D$ joining $p(x)$ to $p(y)$, where $p : \hat{R} \to S^n$ is the canonical projection (see Lemma 9.6). Let $\tilde{c} : [0, D] \to \hat{R}$ be the unique continuous lift of $c$ with $c(0) = x$. Then $\tilde{c}(D) = Z^ny$ for some $n \in \mathbb{Z}$. Applying (9.8), we get
\[ \zeta(Z^ny) - \zeta(x) \leq d_R(Z^ny, x) \leq D , \]
which, taking into account (9.11), implies that
\[ nL + \zeta(y) - \zeta(x) \leq D , \]
that is
\[ L \iota(x, y) - (\zeta(x) - \zeta(y)) \leq D. \]
This inequality and (9.12) conclude the proof. \qed

The function \( \iota \) is an example of an abstract height function of a causal covering in the sense of [1], where a general theory of such functions is developed. It follows from this general theory that for any \( x \in \hat{R} \) the function \( R_x : \hat{G} \to \mathbb{Z} \) given by
\[ R_x(g) := \iota(gx, x) \]
is a quasimorphism and that all these quasimorphisms are mutually at bounded distance. In the present case it is actually easy to derive these properties directly:

**Lemma 9.8.** For all \( x, y \in \hat{R} \), all \( g, h \in \hat{G} \) and all \( n \in \mathbb{Z} \), we have:

1. \( R_x(Z^n) = n \);
2. \( 0 \leq R_x(g) + R_x(g^{-1}) \leq 2D/L \);
3. \( |R_x(gh) - R_x(g) - R_x(h)| \leq 3D/L \);
4. \( |R_x(g) - R_{g^{-1}}(g)| \leq 4D/L \).

**Proof.** (1) follows from the fact that \( Z(x) \geq x \) and \( Z^{-1}(x) \leq x \) for all \( x \in \hat{R} \).

(2) Using Lemma 9.7(3), we get
\[ L|\iota(x, g^{-1}x) + \iota(g^{-1}x, x)| \leq 2D, \]
which implies the right inequality since that \( \iota(x, g^{-1}x) = \iota(gx, x) \).

To see the left inequality observe that that if \( Z^ny \geq x \) and \( Z^mx \geq y \), then \( Z^{n+m}y \geq Z^m x \geq y \), then \( [9.10] \) implies that \( n + m \geq 0 \).

(3) Using repeatedly Lemma 9.7(2) and (3) we obtain:
\[
L|R_x(gh) - R_x(g) - R_x(h)| \\
= |L\iota(hx, g^{-1}x) - L\iota(x, g^{-1}x) - L\iota(hx, x)| \\
= |[L\iota(hx, g^{-1}x) - (\zeta(hx) - \zeta(g^{-1}x))] \\
+ [L\iota(x, g^{-1}x) - (\zeta(x) - \zeta(g^{-1}x))] \\
+ [L\iota(hx, x) - (\zeta(hx) - \zeta(x))]| \leq 3D.
\]
(4) Again from Lemma \[9.7(2)\] and (3)
\[
|L(R_x(g) - R_y(g)) - L(R_x(g) - \iota(gx, gy) + \iota(x, y) - R_y(g))| = \left|\left[\iota(gx, x) - (\zeta(gx) - \zeta(x))\right] - \left[\iota(gx, gy) - (\zeta(gx) - \zeta(gy))\right] + \left[\iota(x, y) - (\zeta(x) - \zeta(y))\right] - \left[\iota(gy, y) - (\zeta(gy) - \zeta(y))\right]\right| \leq 4D.
\]

We consider now the homogenization
\[
\psi(g) := \lim_{n \to \infty} \frac{R_x(g^n)}{n}
\]
of the quasimorphism $R_x$. Notice that, because of Lemma \[9.8(4)\], $\psi(g)$ does not depend on $x \in \hat{\mathbb{R}}$.

**Proposition 9.9.** The map $\psi : \hat{G} \to \mathbb{R}$ is a continuous quasimorphism satisfying the following properties:

1. $\psi(Z^n) = n$, for $n \in \mathbb{Z}$;
2. $\|\psi - R_x\|_{\infty} \leq 3D/L$;
3. $\psi$ sandwiches $G^+$, in fact $\{g \in G : \psi(g) \geq 5D/L\} \subset G^+$.

**Proof.** Since $\psi$ is the homogenization of $R_x$, then (1) and (2) follow respectively from Lemma \[9.8(1)\] and (3). The fact that $\psi$ is continuous follows from the fact that it is a homogeneous Borel quasimorphism \[17, Lemma 7.4\].

If now $\psi(g) \geq 5D/L$, then it follows from (2) that $R_x(g) \geq 2D/L$. Lemma \[9.8(2)\] then implies that $\iota(x, gx) = R_x(g^{-1}) \leq 0$ and hence $gx \geq x$ for all $x \in \hat{\mathbb{R}}$. The assertion follows now from Lemma \[8.4\].

The following proposition extends to a general Lie group of Hermitian type the statement in Lemma \[8.13\] for Homeo$^+(\mathbb{R})$ and the Poincaré translation quasimorphism $\tau$. The proof is also very similar to that of Lemma \[8.13\] where here the function $\zeta$ plays an important role.

**Proposition 9.10.** $\hat{G}^+ = \{g \in \hat{G} : g(x) > x \text{ for all } x \in \hat{\mathbb{R}}\}$.

**Proof.** Observe first of all that from the definition of $\psi$ and Lemma \[9.7(3)\], it follows that
\[
\psi(g) = \lim_{n \to \infty} \left(\frac{\zeta(g^n x) - \zeta(x)}{L n}\right).
\]
From Proposition 9.9 and Lemma 8.7 it follows that \( \hat{G}^{++} = \{ g \in \hat{G} : \psi(g) > 0 \} = \{ g \in G : g(x) \geq x \text{ for all } x \in \hat{R} \text{ and } \psi(g) > 0 \} \).

Hence we need to show the equivalence for an element \( g \in \hat{G} \) between

1. \( g(x) \geq x \text{ for all } x \in \hat{R} \text{ and } \psi(g) > 0 \), and
2. \( g(x) > x \text{ for all } x \in \hat{R} \).

To see that \( (1) \Rightarrow (2) \) it is immediate to verify that if \( (1) \) holds and \( (2) \) fails, then there exists a fixed point \( x_0 \in \hat{R} \) and hence, because of (9.13), \( \psi(g) = 0 \), a contradiction.

To see that \( (2) \Rightarrow (1) \) we show that if \( g(x) \geq x \text{ for all } x \in \hat{R} \) and \( \psi(g) = 0 \), then \( g \) has a fixed point in \( \hat{R} \).

Let now \( g \in \hat{G}^{+} \) and write

\[
\zeta(g^n x) - \zeta(x) = \sum_{i=0}^{n-1} [\zeta(g^{i+1}(x)) - \zeta(g^i(x))].
\]

Observe that since \( g \succeq e \), all summands are non-negative. If now \( \psi(g) = 0 \), then there exists a subsequence \( (i_n)_{n \geq 1} \) with

\[
\lim_{n \to \infty} \zeta(g^{i_n+1}(x)) - \zeta(g^{i_n}(x)) = 0.
\]

Let \( \mathcal{F} \subset \hat{R} \) be a relatively compact fundamental domain for the \( \langle Z \rangle \)-action on \( \hat{R} \). Then \( g^{i_n}(x) = Z^{t_n}(y_n) \) for some \( y_n \in \mathcal{F} \). By taking into account (9.11), we deduce that

\[
\lim_{n \to \infty} \zeta(g(y_n)) - \zeta(y_n) = 0.
\]

Since \( g \succeq e \), we deduce from (9.6) that \( \lim_{n \to \infty} d(g y_n, y_n) = 0 \) and hence any accumulation point of the sequence \( (y_n)_{n \geq 1} \) provides a fixed point for \( g \). \( \square \)

**Proof of Theorem 9.4.** This now follows from Propositions 9.9 and 9.10 and the fact that, up to multiplicative constant, there is only one continuous quasimorphism on \( \hat{G} \) since \( H_1^b(\hat{G}, \mathbb{R}) \simeq H_1^b(G, \mathbb{R}) \) is one dimensional. \( \square \)

10. **Causal Representations and Weakly Maximal Representations**

In this section we prove the characterization of weakly maximal representations in terms of orders in the case in which the target group is of tube type. More specifically, with the use of §§ 8 and 9, we complete the proofs of Theorems 1.10, 1.12 and 1.14 in one go.

We begin by giving the general definition of \( q \)-causal representation, including the case in which \( \partial \Sigma \neq \emptyset \). In the notation of § 8, given a
homomorphism \( \rho : \pi_1(\Sigma) \to G \) into a simple Hermitian Lie group of tube type, let \( \tilde{\rho} : \tilde{\Gamma} \to \widehat{G} \) denote the lift of \( \rho \) to the corresponding central extension and, for \( q \in \mathbb{N} \),

\[
\Lambda_q^{++} = \{ \gamma \in \Lambda : \tilde{\rho}_h(\gamma)(x) > x + q \text{ for all } x \in \mathbb{R} \}.
\]

**Definition 10.1.** We say that \( \rho : \pi_1(\Sigma) \to G \) is \( q \)-causal if \( \tilde{\rho}(\gamma) \in \widehat{G}^{++} \) for all \( \gamma \in \Lambda_q^{++} \).

It follows from Lemma 8.13 that if \( \rho_h : \pi_1(\Sigma) \to \text{PU}(1, 1) \) is a hyperbolization, then

\[
(10.1) \quad \Lambda_q^{++} = \{ \gamma \in \Lambda : \tau(\tilde{\rho}_h(\gamma)) > q \}.
\]

Because of Theorem 9.4, the orders on \( \Lambda \) and on \( \widehat{G} \) are sandwiched respectively by \( \tau \circ \tilde{\rho}_h \) and \( f_{\widehat{G}} \). If \( \rho \) is \( q \)-causal with \( q \geq 0 \), then (10.1) implies that \( \mathcal{N}_C(\tau \circ \tilde{\rho}_h) \subset \rho^{-1}(\widehat{G}^{++}) \) with \( C = q + 1/2 \). Then Proposition 8.11(2) implies that

\[
\lambda = T(\rho)/|\chi(\Sigma)|.
\]

Since \( \lambda \neq 0 \), the image of \( \rho \) is not elementary and hence there is a \( \Gamma \)-equivariant measurable map \( \varphi : \partial \mathbb{D} \to \partial \mathbb{D} \). It follows by a standard argument that if \( \varepsilon \) denotes the orientation cocycle on \( \partial \mathbb{D} \),

\[
\varepsilon(\varphi(x), \varphi(y), \varphi(z)) = \lambda \varepsilon(x, y, z),
\]

for almost every \( (x, y, z) \in (\partial \mathbb{D})^3 \). This implies that \( \lambda = \pm 1 \) and thus, since by assumption \( \lambda \geq 0 \), \( \lambda = 1 \). But then the Toledo invariant \( T(\rho) = |\chi(\Sigma)| \) is maximal and hence by [34, 10] \( \rho \) is a hyperbolization. This concludes the proof of Theorem 1.10.

Let now \( G \) be as in Theorem 1.12 and assume that \( \rho : \pi_1(\Sigma) \to G \) is weakly maximal. Then by Proposition 3.2 we have

\[
(f_{\widehat{G}} \circ \tilde{\rho})|_{\Lambda} = \lambda(\tau \circ \tilde{\rho}_h)|_{\Lambda}
\]

with \( \lambda = T(\rho)/|\chi(\Sigma)| \).

By Theorem 9.4 there exists a sandwiching constant \( C_G > 0 \) for the quasimorphism \( f_{\widehat{G}} \) with the causal ordering on \( \widehat{G} \), that is

\[
\{ g \in \widehat{G} : f_{\widehat{G}}(g) \geq C_G \} \subset \widehat{G}^{++}.
\]
We deduce that if \( \tau(\tilde{\rho}_h(\gamma)) \geq \frac{C_G |\chi(\Sigma)|}{T(\rho)} \), then \( \tilde{\rho}(\gamma) \in \hat{G}^{++} \).

According to Theorem 1.4, \( T(\rho) = m/\ell_G \) for some integer \( m \geq 1 \). Thus, setting \( q = [C_G |\chi(\Sigma)| \ell_G] \) and taking (10.1) into account, we obtain a non-trivial number depending only on \( \Sigma \) and on \( G \) such that \( \tilde{\rho}(\Lambda^+ q) \subset \hat{G}^{++} \).

Finally, Theorem 1.14 follows from Corollary 8.10(1) and Theorem 9.4.

11. WEAKLY MAXIMAL REPRESENTATIONS AND RELATIONS WITH OTHER REPRESENTATION VARIETIES

In this section we first show that the set of weakly maximal representations and the set of weakly maximal representations with non-zero Toledo number are closed. Then we examine the relationship of weakly maximal representations with Shilov-Anosov representations.

Returning to our general framework in §3, let \( G \) be locally compact and second countable, \( \Gamma \) a discrete group and \( \kappa \in H^2_{\text{cb}}(G, \mathbb{R}) \) a fixed class. We define now a (Hausdorff) topology on \( H^2_b(\Gamma, \mathbb{R}) \) with respect to which the map

\[
\text{Hom}(\Gamma, G) \to H^2_b(\Gamma, \mathbb{R})
\]

\[
\rho \mapsto \rho^*(\kappa)
\]

will be continuous. To this purpose recall that \( H^2_b(\Gamma, \mathbb{R}) := \ker \delta^2 / \text{im} \delta^1 \), where

\[
0 \to \mathbb{R} \to \ell^\infty(\Gamma) \xrightarrow{\delta^1} \ell^\infty(\Gamma^2) \xrightarrow{\delta^2} \ell^\infty(\Gamma^3) \xrightarrow{\delta^3} \ldots
\]

denotes the inhomogeneous bar resolution. If we endow each \( \ell^\infty(\Gamma^n) \) with the weak-* topology as the dual of \( \ell^1(\Gamma^n) \), then:

**Lemma 11.1.** \( H^2_b(\Gamma, \mathbb{R}) \) is a Hausdorff topological vector space with the quotient weak-* topology.

**Proof.** It is clear that \( \ker \delta^2 \) is weak-* closed. Since \( H^2_b(\Gamma, \mathbb{R}) \) is a Banach space, \( \text{im} \delta^1 \) is closed and hence, by [51, Theorem 4.14], \( \text{im} \delta^1 \) is weak-* closed as well. \( \square \)

**Lemma 11.2.** The function in (11.1) is continuous with respect to the weak-* topology on \( H^2_b(\Gamma, \mathbb{R}) \).

**Proof.** Let \( c_\kappa : G^2 \to \mathbb{R} \) be a bounded continuous inhomogeneous cocycle representing \( \kappa \). Let \( (\rho_n)_{n \geq 1} \) be a sequence of elements in \( \text{Hom}(\Gamma, G) \) converging to \( \rho \). Setting

\[
c_n(x, y) := c_\kappa(\rho_n(x), \rho_n(y))
\]

\[
c(x, y) := c_\kappa(\rho(x), \rho(y))
\]
we have that $c_n \in \ell^\infty(\Gamma^2)$ (respectively $c \in \ell^\infty(\Gamma^2)$) represent $\rho_n^*(\kappa)$ (respectively $\rho^*(\kappa)$). In addition

(1) $c_n \to c$ pointwise on $\Gamma^2$, and

(2) $\|c_n\|_\infty$ and $\|c\|_\infty$ are bounded by $\|\kappa\|_\infty$.

Then if $f \in \ell^1(\Gamma^2)$, we have that

(1) $fc_n \to fc$ pointwise, and

(2) $|f(x,y)c_n(x,y)| \leq |f(x,y)| \|\kappa\|_\infty$,

then the Dominated Convergence Theorem implies that

$$\int_{\Gamma^2} fc_n \to \int_{\Gamma^2} fc$$

and shows therefore that $\rho_n^*(\kappa) \to \rho^*(\kappa)$ for the quotient topology in $H^2_b(\Gamma,\mathbb{R})$.

If now $\Gamma = \pi_1(\Sigma)$, then

**Corollary 11.3.** The set $\text{Hom}_{wm}(\pi_1(\Sigma), G)$ of weakly maximal representations is closed in $\text{Hom}(\pi_1(\Sigma), G)$.

**Proof.** It follows from Proposition 3.2 that $\text{Hom}_{wm}(\pi_1(\Sigma), G) = \{\rho \in \text{Hom}(\pi_1(\Sigma), G) : \rho^*(\kappa) = t\kappa_\Sigma^b \text{ for some } t \in [0, \infty)\}$.

Since $H^2_{ob}(\pi_1(\Sigma), \mathbb{R})$ with the weak-* topology is Hausdorff, the subset $\{t\kappa_\Sigma^b : t \in [0, \infty)\}$ is closed. The assertion follows then from the continuity of the map $\rho \mapsto \rho^*(\kappa)$.

Let now set

$\text{Hom}_{wm}^*(\pi_1(\Sigma), G) := \{\rho : \pi_1(\Sigma) \to G : \rho \text{ is weakly maximal and } T(\rho) \neq 0\}$.

**Corollary 11.4.** Let $G$ be a Lie group of Hermitian type. Then the set of weakly maximal representations with non-zero Toledo invariant is closed in $\text{Hom}(\pi_1(\Sigma), G)$.

**Proof.** By Theorem 7.2(2) we have

$$\text{Hom}_{wm}^*(\pi_1(\Sigma), G) = \left\{\rho \in \text{Hom}_{wm}(\pi_1(\Sigma), G) : T(\rho) \geq \frac{\|\chi(\Sigma)\|}{l_G} \right\},$$

which implies the claim since $\rho \mapsto T(\rho)$ is continuous, [17 Proposition 3.10], and $\text{Hom}_{wm}(\pi_1(\Sigma), G)$ is closed.

Finally, we turn to the relation with Anosov representations. Let $\partial \Sigma = \emptyset$ and realize $\pi_1(\Sigma)$ as cocompact lattice $\Gamma$ in $\text{PU}(1,1)$ via a hyperbolization. If $G$ is, say, simple of tube type with Shilov boundary $\tilde{S}$, then a Shilov-Anosov representation $\rho : \Gamma \to G$ implies the existence of a (unique) continuous equivariant map $\varphi : \partial \mathbb{D} \to \tilde{S}$ with the additional
property that for every \( x, y \in \partial \mathbb{D} \) with \( x \neq y \), \( \varphi(x) \) and \( \varphi(y) \) are transverse. Let \( (\partial \mathbb{D})^{3,+} \) be the connected set of distinct, positively oriented triples in \( (\partial \mathbb{D})^3 \) and \( \tilde{S}^{(3)} \) the set of triples of pairwise transverse points in \( \tilde{S}^3 \). Then \( \varphi \times \varphi \times \varphi: (\partial \mathbb{D})^{3,+} \to \tilde{S}^{(3)} \) must send \( (\partial \mathbb{D})^{3,+} \) into a connected component of \( \tilde{S}^{(3)} \).

Thus if \( \beta_{\tilde{S}}: \tilde{S}^3 \to \mathbb{R} \) denotes the generalized Maslov cocycle, \( \beta_{\tilde{S}} \circ \varphi^3 \) is a multiple of the orientation cocycle. This and Proposition 3.2 imply that the set \( \text{Hom}_{\text{wm}}(\pi_1(\Sigma), G) \) is contained in \( \text{Hom}_+^{\tilde{S}}(\pi_1(\Sigma), G) \).

**Corollary 11.5.**

\[ \text{Hom}_{\tilde{S}^{(3)}}^+(\pi_1(\Sigma), G) \subset \text{Hom}_+^{\tilde{S}}(\pi_1(\Sigma), G). \]

12. Examples

In this section we describe some examples of weakly maximal representations.

12.1. Maximal Representations. Any maximal representation is a weakly maximal representation. Concrete examples of maximal representations are described in [17, 12, 40].

12.2. Embeddings of \( \text{SL}(2, \mathbb{R}) \). Special examples of weakly maximal representations arise from embeddings of \( \text{SL}(2, \mathbb{R}) \). Consider a faithful representation \( \rho_0: \pi_1(\Sigma) \to \text{SL}(2, \mathbb{R}) \) with discrete image, and let \( \tau: \text{SL}(2, \mathbb{R}) \to G \) be an injective homomorphism into a Lie group of Hermitian type \( G \). Then the composition \( \tau \circ \rho_0: \pi_1(\Sigma) \to G \) is a weakly maximal representation.

More generally if \( H, G \) are Lie groups of Hermitian type, and \( \tau: H \to G \) is a homomorphism such that \( \tau^*(\kappa^b_G) \) is a multiple of \( \kappa^b_H \). Then the composition of a weakly maximal representation \( \rho: \pi_1(\Sigma) \to H \) with \( \tau \) is a weakly maximal representation into \( G \).

12.3. Shilov-Anosov Representations. We explained in §11 that any Shilov-Anosov representation into a Hermitian Lie group \( G \) of tube type with non-negative Toledo number is weakly maximal. Here we just give an example of such a representation into \( \text{Sp}(2n, \mathbb{R}) \). Let again \( \rho_i: \pi_1(\Sigma) \to \text{SL}(2, \mathbb{R}) \) \( i = 1, \ldots, n \) be faithful representations with discrete image. Consider the embedding \( \tau: \text{SL}(2, \mathbb{R})^n \to \text{Sp}(2n, \mathbb{R}) \) corresponding to a maximal polydisk in the bounded symmetric domain realization of \( \text{Sp}(2n, \mathbb{R}) \). Then \( \rho := \tau \circ (\rho_1, \ldots, \rho_n): \pi_1(\Sigma) \to \text{Sp}(2n, \mathbb{R}) \).
is a Shilov-Anosov representation, and hence a weakly maximal representation. Since the set of Shilov-Anosov representations is open, any small deformation of such a representation is also weakly maximal. Similarly to the construction in [12] one can explicitly construct bending deformations of the representation $\rho$ with Zariski dense image.

Note that $\rho$ if maximal if and only if the representations $\rho_i$ are orientation preserving for all $i = 1, \ldots, n$.

12.4. Cancelling Contributions. Let $G = G_1 \times \cdots \times G_n$ be a semisimple Lie group of Hermitian type. If $\rho : \pi_1(\Sigma) \to G$ is a maximal representation, then $\rho_i = p_i \circ \rho : \pi_1(\Sigma) \to G_i$, $i = 1, \ldots, n$ are maximal representations, where $p_i$ denotes the projection onto the $i$-th factor.

In order to illustrate that this does not hold true for weakly maximal representations consider an arbitrary representation $\rho_a : \pi_1(\Sigma) \to SL(2, \mathbb{R})$ and denote by $\rho_a : \pi_1(\Sigma) \to SL(2, \mathbb{R})$ the composition of $\rho_a$ with the outer automorphism of $SL(2, \mathbb{R})$ reversing the orientation. Let $\rho_0 : \pi_1(\Sigma) \to SL(2, \mathbb{R})$ be a faithful representation with discrete image, as before. Then the representation $\rho = (\rho_0, \rho_a, \rho_a) : \pi_1(\Sigma) \to SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is weakly maximal.

12.5. Limit of Shilov-Anosov Representations. Let $\partial \Sigma = \emptyset$. We have seen in Corollary 11.5 that any representation that is the limit of Shilov-Anosov representations is weakly maximal. By choosing appropriately $\rho_a$ in §12.4 it is easy to construct representations that are weakly maximal but not Shilov-Anosov. While we do not know whether the containment in (11.2) is strict, we give here an example to show that the containment

$$\text{Hom}^*_\text{S−An}(\pi_1(\Sigma), G) \subset \text{Hom}^*_\text{S−An}(\pi_1(\Sigma), G).$$

is strict, that is an example of a representation that is the limit of Shilov-Anosov representations but is not Shilov-Anosov.

To this purpose we consider the group $Sp(2n, \mathbb{R})$ and we denote for simplicity by $\mathcal{R}_d(n)$ the set of representations $\rho : \pi_1(\Sigma) \to Sp(2n, \mathbb{R})$ with Toledo invariant equal to $d$ and with $(\mathcal{R}_d(n))_{S−An} \subset \mathcal{R}_d(n)$ the subset of Shilov-Anosov representations.

If $d = (g-1)n$, that is if $\mathcal{R}_{(g-1)n}(n)$ consists of maximal representations, then by [12] there is the equality $(\mathcal{R}_{(g-1)n}(n))_{S−An} \equiv \mathcal{R}_{(g-1)n}(n)$. On the other hand, it is easy to construct a representation that has zero Toledo invariant but is not Shilov-Anosov (by taking for example $\rho_a \oplus \rho_a : \pi_1(\Sigma) \to SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \hookrightarrow Sp(4, \mathbb{R})$, where $\rho_a$ is any representation with dense image in $SL(2, \mathbb{R})$). Since for $n \geq 3$ the set $\mathcal{R}_0(n)$ is connected [31, Theorem 1.1 (1)] and $(\mathcal{R}_0(n))_{S−An}$ is open, this
shows that
\[(\mathcal{R}_0(n))_{\overline{\mathcal{S}}_{-\Lambda n}} \subsetneq (\mathcal{R}_0(n))_{\overline{\mathcal{S}}_{-\Lambda n}}.\]
We can use this fact as follows. Recall that the direct sum of symplectic vector spaces leads to a canonical homomorphism
\[\text{Sp}(2n_1, \mathbb{R}) \times \text{Sp}(2n_2, \mathbb{R}) \longrightarrow \text{Sp}(2(n_1 + n_2), \mathbb{R})\]
and consequently to a continuous map
\[\mathcal{R}_{d_1}(n_1) \times \mathcal{R}_{d_2}(n_2) \rightarrow \mathcal{R}_{d_1 + d_2}(n_1 + n_2)\]
\[(\rho_1, \rho_2) \mapsto \rho_1 \oplus \rho_2.\]
Observe moreover that \(\rho_1 \oplus \rho_2\) is Shilov-Anosov precisely if both \(\rho_1\) and \(\rho_2\) are. In particular if \(n_1 \geq 3\) we can consider a sequence \(\rho^{(k)}_1 \in (\mathcal{R}_0(n_1))_{\overline{\mathcal{S}}_{-\Lambda n}}\) whose limit \(\rho_1\) is not Shilov-Anosov. Then for any \(\rho_2 \in (\mathcal{R}_d(n_2))_{\overline{\mathcal{S}}_{-\Lambda n}},\) the sequence \(\rho^{(k)}_1 \oplus \rho_2\) converges to the representation \(\rho_1 \oplus \rho_2\) that is not Shilov-Anosov and has Toledo invariant equal to \(d_1\).

12.6. Weakly maximal representations with non-reductive Zariski closure. Let \(J_n\) be the \(2n \times 2n\) matrix with block entries \(J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\).

We consider the subgroup of \(\text{Sp}(2(n + 1), \mathbb{R})\) given by
\[Q = \left\{ \begin{pmatrix} A & b \\ c & d \end{pmatrix} : A \in \text{Sp}(2n, \mathbb{R}) \text{ and } c = t^b J_n A \right\},\]
which is the semidirect product of \(\text{Sp}(2n, \mathbb{R})\) and the \(n\)-dimensional Heisenberg group \(H_n\).

A map \(\rho : \Gamma \rightarrow Q\) with entries \(\pi(\gamma), b(\gamma), c(\gamma)\) and \(d(\gamma)\) is a homomorphism if and only if \(\pi : \Gamma \rightarrow \text{Sp}(2n, \mathbb{R})\) is a homomorphism, \(b : \Gamma \rightarrow \mathbb{R}^{2n}\) is a 1-cocycle (with the \(\Gamma\)-module structure on \(\mathbb{R}^{2n}\) is given by \(\pi\)) and
\[(12.1) \quad d(\gamma_1 \gamma_2) = t^b(\gamma_1) J_n \pi(\gamma_1) b(\gamma_2) + d(\gamma_1) + d(\gamma_2).\]

To reinterpret (12.1), we observe that there is a bilinear symmetric form on \(H^1(\Gamma, \pi)\) with values in \(H^2(\Gamma, \mathbb{R})\) obtained by composing the cup product \(H^1(\Gamma, \pi) \times H^1(\Gamma, \pi) \rightarrow H^2(\Gamma, \pi \otimes \pi)\) with the projection on trivial coefficients given by the invariant symplectic form; denoting by \(Q_\pi\) the corresponding quadratic form, we have that
\[(\gamma_1, \gamma_2) \mapsto t^b(\gamma_1) J_n \pi(\gamma_1) b(\gamma_2)\]
is a representative of \(Q_\pi([b])\) and the existence of a function \(d\) satisfying (12.1) amounts to \(Q_\pi([b]) = 0.\)
Let now $\Gamma = \pi_1(S)$, where $S$ is an oriented surface of genus $g$, and let $\rho_1 : \Gamma \to \text{Sp}(2n_1, \mathbb{R})$, where $\rho_1$ is in the Hitchin component of $\text{Sp}(2n_1, \mathbb{R})$, while $\rho_2$ is the precomposition of a Hitchin representation into $\text{Sp}(2n_2, \mathbb{R})$ with an orientation reversing automorphism of $\Gamma$. Identifying $H^2(\Gamma, \mathbb{R})$ with $\mathbb{R}$ by means of the chosen orientation, we have that $Q_{\rho_1}$ is positive definite while $Q_{\rho_2}$ is negative definite. Choose $[b_1] \in H^1(\Gamma, \rho_i)$ both non-zero such that

\begin{equation}
Q_{\rho_1}([b_1]) + Q_{\rho_2}([b_2]) = 0.
\end{equation}

Set $\pi := \rho_1 \oplus \rho_2$, $b := b_1 + b_2$ and let $d$ be a solution of (12.1), which exists by (12.2). Then $\rho : \pi_1(S) \to \text{Sp}(2(n+1), \mathbb{R})$ is a weakly maximal representation with $T(\rho) = n_1 - n_2$, and the real Zariski closure of its image is the semi direct product of $G_1 \times G_2 < \text{Sp}(2n, \mathbb{R})$ with $H_n$; here $G_i$ is the real Zariski closure of the image of $\rho_i$.

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