SAMUEL MULTIPLICITIES AND BROWDER SPECTRUM OF OPERATOR MATRICES

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Abstract. In this paper, we first point out that the necessity of Theorem 4 in [8] does not hold under the given condition and present a revised version with a little modification. Then we show that the definitions of some classes of semi-Fredholm operators, which use the language of algebra and first introduced by X. Fang in [8], are equivalent to that of some well-known operator classes. For example, the concept of shift-like semi-Fredholm operator on Hilbert space coincide with that of upper semi-Browder operator. For applications of Samuel multiplicities we characterize the sets of

$$\bigcap_{C \in \mathcal{B}(K, H)} \sigma_{ab}(M_C), \bigcap_{C \in \mathcal{B}(K, H)} \sigma_{sb}(M_C) \quad \text{and} \quad \bigcap_{C \in \mathcal{B}(K, H)} \sigma_b(M_C),$$

respectively, where $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ denotes a 2-by-2 upper triangular operator matrix acting on the Hilbert space $H \oplus K$.

1. Introduction

Throughout this paper, let $H$ and $K$ be separable infinite dimensional complex Hilbert spaces and $B(H, K)$ the set of all bounded linear operators from $H$ into $K$, when $H = K$, we write $B(H, H)$ as $B(H)$. For $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$, we have $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(H \oplus K)$. For $T \in B(H, K)$, let $R(T)$ and $N(T)$ denote the range and kernel of $T$, respectively, and denote $\alpha(T) = \dim N(T)$, $\beta(T) = \dim K/R(T)$. If $T \in B(H)$, the ascent $\text{asc}(T)$ of $T$ is defined to be the smallest nonnegative integer $k$ which satisfies that $N(T^k) = N(T^{k+1})$. If such $k$ does not exist, then the ascent of $T$ is defined as infinity. Similarly, the descent $\text{des}(T)$ of $T$ is defined as the smallest nonnegative integer $k$ for which $R(T^k) = R(T^{k+1})$ holds. If such $k$ does not exist, then $\text{des}(T)$ is defined as infinity, too. If the ascent and the descent of $T$ are finite, then they are equal (see [3]). For $T \in B(H)$, if $R(T)$ is closed and $\alpha(T) < \infty$, then $T$ is said to be a upper semi-Fredholm operator, if $\beta(T) < \infty$, which implies that $R(T)$ is closed, then $T$ is said to be a lower semi-Fredholm operator. If $T \in B(H)$ is either upper or lower semi-Fredholm operator, then $T$ is said to

2000 Mathematical Subject Classification. Primary 47A10; Secondary 47A53.

Key words and phrases. Samuel multiplicities, Operator matrices, Upper semi-Browder operator, Upper semi-Browder spectrum, Browder operator, Browder spectrum.

This work is supported by the NSF of China (Grant Nos. 10771034 and 10771191).
be a semi-Fredholm operator. If both \( \alpha(T) < \infty \) and \( \beta(T) < \infty \), then \( T \) is said to be a Fredholm operator. For a semi-Fredholm operator \( T \), its index \( \text{ind}(T) \) is defined by \( \text{ind}(T) = \alpha(T) - \beta(T) \).

In this paper, the sets of invertible operators, left invertible operators and right invertible operators on \( H \) are denoted by \( G(H), G_l(H) \) and \( G_r(H) \), respectively, the sets of all Fredholm operators, upper semi-Fredholm operators and lower semi-Fredholm operators on \( H \) are denoted by \( \Phi(H), \Phi_+(H) \) and \( \Phi_-(H) \), respectively, the sets of all Browder operators, upper semi-Browder operators and lower semi-Browder operators on \( H \) are defined, respectively, by

\[
\Phi(H) := \{ T \in \Phi(H) : \text{asc}(T) = \text{des}(T) < \infty \},
\]

\[
\Phi_{ab}(H) := \{ T \in \Phi_+(H) : \text{asc}(T) < \infty \},
\]

\[
\Phi_{sb}(H) := \{ T \in \Phi_-(H) : \text{des}(T) < \infty \}.
\]

Moreover, for \( T \in B(H) \), we introduce its corresponding spectra as following [19];

- the spectrum: \( \sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin G(H) \} \),
- the left spectrum: \( \sigma_l(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin G_l(H) \} \),
- the right spectrum: \( \sigma_r(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin G_r(H) \} \),
- the essential spectrum: \( \sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H) \} \),
- the upper semi-Fredholm spectrum: \( \sigma_{SF+}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(H) \} \),
- the lower semi-Fredholm spectrum: \( \sigma_{SF-}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(H) \} \),
- the Browder spectrum: \( \sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi_b(H) \} \),
- the upper semi-Browder spectrum: \( \sigma_{ab}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{ab}(H) \} \),
- the lower semi-Browder spectrum: \( \sigma_{sb}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{sb}(H) \} \).

For a semi-Fredholm operator \( T \in B(H) \), its shift Samuel multiplicity \( s_{mul}(T) \) and backward shift Samuel multiplicity \( b.s_{mul}(T) \) are defined ([5-8]), respectively, by

\[
s_{mul}(T) = \lim_{k \to \infty} \frac{\beta(T^k)}{k},
\]

\[
b.s_{mul}(T) = \lim_{k \to \infty} \frac{\alpha(T^k)}{k}.
\]

Moreover, it has been proved that \( s_{mul}(T), b.s_{mul}(T) \in \{ 0, 1, 2, \ldots, \infty \} \) and \( \text{ind}(T) = b.s_{mul}(T) - s_{mul}(T) \). These two invariants refine the Fredholm index and can be regarded as the stabilized dimension of the kernel and cokernel [8].

**Definition 1.1** ([8]). A semi-Fredholm operator \( T \in B(H) \) is called a pure shift semi-Fredholm operator if \( T \) has the form \( T = U^n P \), where \( n \in \mathbb{N} \) or \( n = \infty \), \( U \) is the unilateral
shift, and $P$ is a positive invertible operator. Analogously, $T$ is called a pure backward shift semi-Fredholm operator if its adjoint $T^*$ is a pure shift semi-Fredholm operator. Here $U^\infty$ denotes the direct sum of countably (infinite) many copies of $U$.

**Definition 1.2** ([8]) A semi-Fredholm operator $T \in B(H)$ is called a shift-like semi-Fredholm operator if $\text{b.s.}\_\text{mul}(T) = 0$; $T$ is called a shift semi-Fredholm operator if $\text{N}(T) = 0$. Analogous concepts for backward shifts can also be defined. $T$ is called a stationary semi-Fredholm operator if $\text{b.s.}\_\text{mul}(T) = 0$ and $\text{s.}\_\text{mul}(T) = 0$.

It follows from Definition 1.1 that $T$ is a shift semi-Fredholm operator iff $T$ is a left invertible operator, and that $T$ is a backward shift semi-Fredholm operator iff $T$ is a right invertible operator.

In ([8], Theorem 4 and Corollary 18), Fang gave the following $4 \times 4$ upper-triangular representation theorem: An operator $T \in B(H)$ is semi-Fredholm iff $T$ can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$,

$$T = \begin{pmatrix}
T_1 & * & * & *
0 & T_2 & * & *
0 & 0 & T_3 & *
0 & 0 & 0 & T_4
\end{pmatrix},$$

where $\dim H_4 < \infty$, $T_1$ is a pure backward shift semi-Fredholm operator, $T_2$ is invertible, $T_3$ is a pure shift semi-Fredholm operator, $T_4$ is a finite nilpotent operator. Moreover, $\text{ind}(T_1) = \text{b.s.}\_\text{mul}(T)$ and $\text{ind}(T_3) = -\text{s.}\_\text{mul}(T)$.

The following example shows that the representation theorem is not accurate.

**Example 1.3.** Let $H$ be the direct sum of countably many copies of $\ell^2 := \ell^2(\mathbb{N})$, that is, the elements of $H$ are the sequences $\{x_j\}_{j=1}^\infty$ with $x_j \in \ell^2$ and $\sum_{j=1}^\infty \|x_j\|^2 < \infty$. Let $V$ be the unilateral shift on $\ell^2$, i.e.,

$$V : \ell^2 \to \ell^2, \quad \{z_1, z_2, \ldots\} \mapsto \{0, z_1, z_2, \ldots\},$$

and the operators $T_1$ and $T_3$ be defined by

$$T_1 : H \to H, \quad \{x_1, x_2, \ldots\} \mapsto \{V^*x_1, V^*x_2, \ldots\}$$

and

$$T_3 : H \to H, \quad \{x_1, x_2, \ldots\} \mapsto \{Vx_1, Vx_2, \ldots\}.$$
Now, we consider the operator
\[ T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} : H \oplus H \to H \oplus H. \]

Note that \( T_1 \) is a pure backward shift semi-Fredholm operator, \( T_3 \) is a pure shift semi-Fredholm operator, so \( T \) satisfies the conditions of Fang’s \( 4 \times 4 \) triangular representation theorem, but, since \( \alpha(T_1) = \alpha(T) = \beta(T) = \dim(H/R(T_3)) = \infty \), so \( T \) is not a semi-Fredholm operator.

Now, we can prove the following improved \( 4 \times 4 \) upper-triangular representation theorem:

**Theorem 1.4.** An operator \( T \in B(H) \) is semi-Fredholm iff \( T \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \oplus H_3 \oplus H_4 \),
\[ T = \begin{pmatrix} T_1 & * & * & * \\ 0 & T_2 & * & * \\ 0 & 0 & T_3 & * \\ 0 & 0 & 0 & T_4 \end{pmatrix}, \]

where \( \dim H_4 < \infty \), \( T_1 \) is a pure backward shift semi-Fredholm operator, \( T_2 \) is invertible, \( T_3 \) is a pure shift semi-Fredholm operator and \( \min\{\text{ind}(T_1), -\text{ind}(T_3)\} < \infty \), \( T_4 \) is a finite nilpotent operator. Moreover,

1. \( \text{ind}(T_1) = b.s.\text{mul}(T) \), \( \text{ind}(T_3) = -s.\text{mul}(T) \);
2. \( \text{ind}(T) = +\infty \) iff \( \text{ind}(T_1) = +\infty \);
3. \( \text{ind}(T) = -\infty \) iff \( \text{ind}(T_3) = -\infty \);
4. \( \text{ind}(T) \) is finite iff both of \( \text{ind}(T_1) \) and \( \text{ind}(T_3) \) are finite.

Theorem 1.4 can be described as \( 3 \times 3 \) triangular representation form which may be more convenient for the study of operator theory, that is,

**Theorem 1.5.** An operator \( T \in B(H) \) is semi-Fredholm if and only if \( T \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \oplus H_3 \)
\[ T = \begin{pmatrix} T_1 & T_{12} & T_{13} \\ 0 & T_2 & T_{23} \\ 0 & 0 & T_3 \end{pmatrix} : H_1 \oplus H_2 \oplus H_3 \to H_1 \oplus H_2 \oplus H_3, \]
where \( \dim H_3 < \infty \), \( T_1 \) is a right invertible operator, \( T_3 \) is a finite, nilpotent operator, \( T_2 \) is a left invertible operator, and \( \min\{\text{ind}(T_1), -\text{ind}(T_2)\} < \infty \). Moreover, \( \text{ind}(T_1) = \alpha(T_1) = b.s._{-\text{mul}}(T) \), \( \text{ind}(T_2) = -\beta(T_2) = -s_{\text{mul}}(T) \) and \( \text{ind}(T) = \alpha(T_1) - \beta(T_2) \).

The next lemma is useful for the proofs of our results below, especially in Section 2.

**Lemma 1.6** [19]. Let \( A \in B(H), B \in B(K) \) and \( C \in B(K, H) \).

1. If \( A \in \Phi_b(H) \), then \( B \in \Phi_{ab}(K) \) iff \( M_C \in \Phi_{ab}(H \oplus K) \) for some \( C \in B(K, H) \).

2. If \( M_C \in \Phi_{ab}(H \oplus K) \) for some \( C \in B(K, H) \), then \( A \in \Phi_{ab}(H) \).

3. If \( A \in \Phi_{ab}(H) \) and \( B \in \Phi_{ab}(K) \), then \( M_C \in \Phi_{ab}(H \oplus K) \) for any \( C \in B(K, H) \).

4. If \( B \in \Phi_b(K) \), then \( A \in \Phi_{ab}(H) \) iff \( M_C \in \Phi_{ab}(H \oplus K) \) for some \( C \in B(K, H) \); \( A \in \Phi_{ab}(H) \) iff \( M_C \in \Phi_{ab}(H \oplus K) \) for some \( C \in B(K, H) \).

5. If \( M_C \in \Phi_b(H \oplus K) \) for some \( C \in B(K, H) \), then \( A \in \Phi_{ab}(H) \) and \( B \in \Phi_{ab}(K) \).

6. If two of \( A, B \) and \( M_C \) are Browder, then so is the third.

**Proposition 1.7.** Let \( T \in B(H) \). Then \( T \) is upper semi-Browder iff \( T \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \),

\[
T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix},
\]

where \( \dim(H_1) < \infty \), \( T_1 \) is nilpotent, \( T_2 \) is left invertible, and \( \beta(T_2) = s_{\text{mul}}(T) = -\text{ind}(T) \).

**Proof.** Necessity. Suppose that \( T \) is upper semi-Browder. Then we can assume \( p = \text{asc}(T) < \infty \). Let \( H_1 = N(T^p) \). Note that \( T \) is upper semi-Fredholm, so \( \dim H_1 < \infty \). Let \( H = H_1 \oplus H_1^\perp \), we have

\[
T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} : H_1 \oplus H_1^\perp \rightarrow H_1 \oplus H_1^\perp.
\]

That \( T_1 \) is nilpotent is clear. Moreover, since the fact that \( \dim H_1 < \infty \) implies \( T_1 \in \Phi_b(H_1) \), it follows from Lemma 1.6 (1) that \( T_2 \in \Phi_{ab}(H_1^\perp) \). A direct calculation shows that \( T_2 \) is injective, thus, \( T_2 \) is left invertible. From Theorem 1.5, it is clear that \( \beta(T_2) = s_{\text{mul}}(T) = \text{ind}(T_2) \).

Sufficiency follows from Lemma 1.6 immediately.

**Proposition 1.8.** Let \( T \in B(H) \). Then \( T \) is lower semi-Browder iff \( T \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \),

\[
T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix},
\]

where \( \dim(H_2) < \infty \), \( T_1 \) is right invertible, \( T_2 \) is nilpotent, and \( \alpha(T_1) = b.s._{-\text{mul}}(T) = \text{ind}(T) \).
Proof. Necessity. If \( T \) is lower semi-Browder, then we can assume \( p = \text{des}(T) < \infty \). Denote \( H_1 = R(T^p) \) and \( H_2 = H_1^\perp \). Note that \( T^p \) is lower semi-Browder, so \( \dim H_2 < \infty \). Let \( H = H_1 \oplus H_2 \), we have

\[
T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2.
\]

That \( T_1 \) is surjective and \( T_2^p = 0 \) is evident. Note that \( \dim H_2 < \infty \) implies \( T_2 \in \Phi_b(H_2) \), it follows from Lemma 1.6 that \( T_1 \in \Phi_{sb}(H_1) \), and so \( T_1 \) is right invertible. From Theorem 1.5, we have \( \alpha(T_1) = \text{ind}(T_1) = \text{b.s.\_mul}(T) \).

Sufficiency follows from Lemma 1.6.

Combining Theorem 1.5, Propositions 1.7 and 1.8, we have the following theorem immediately.

**Theorem 1.9.** Let \( T \in B(H) \). Then

1. \( T \) is a shift-like semi-Fredholm operator iff \( T \) is an upper semi-Browder operator.
2. \( T \) is a backward shift-like semi-Fredholm operator iff \( T \) is a lower semi-Browder operator.
3. \( T \) is a stationary semi-Fredholm operator iff \( T \) is a Browder operator.

### 2. Applications of Samuel multiplicities

In ([8-12]), Fang studied Samuel multiplicities and presented some applications. In this section, by using Samuel multiplicities, we characterize the sets

\[
\bigcap_{C \in B(K,H)} \sigma_{sb}(M_C), \bigcap_{C \in B(K,H)} \sigma_{sb}(M_C)
\]

and \( \bigcap_{C \in B(K,H)} \sigma_b(M_C) \) completely, where \( M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) is a \( 2 \times 2 \) upper triangular operator matrix defined on \( H \oplus K \). For the study advances of \( 2 \times 2 \) upper triangular operator matrix, see ([1-4], [13-19]).

First, note that if \( T \in B(H) \), then \( T \) is bounded below iff \( T \) is left invertible, thus, Theorem 1 of [14] can be rewritten as follows:

**Lemma 2.1** [14]. For any given \( A \in B(H) \) and \( B \in B(K) \), \( M_C \) is left invertible for some \( C \in B(K,H) \) iff \( A \) is left invertible and

\[
\alpha(B) \leq \beta(A) \quad \text{if} \, R(B) \text{ is closed,} \quad \beta(A) = \infty \quad \text{if} \, R(B) \text{ is not closed.}
\]

**Lemma 2.2** [4]. For any given \( A \in B(H) \) and \( B \in B(K) \),

\[
\bigcap_{C \in B(K,H)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{ \lambda \in \mathbb{C} : \alpha(B - \lambda) \neq \beta(A - \lambda) \}.
\]
One of the main results in this section is:

**Theorem 2.3.** For any given \( A \in B(H) \) and \( B \in B(K) \), \( M_C \in \Phi_{ab}(H \oplus K) \) for some \( C \in B(K, H) \) iff \( A \in \Phi_{ab}(H) \) and

\[
\begin{cases}
  s\text{-}mul(A) = \infty & \text{if } B \notin \Phi_+(K), \\
  b.s.\text{-}mul(B) \leq s\text{-}mul(A) & \text{if } B \in \Phi_+(K).
\end{cases}
\]

**Proof.** We first claim that if \( B \notin \Phi_+(K) \), then

\[
M_C \in \Phi_{ab}(H \oplus K) \quad \text{for some } C \in B(K, H) \iff A \in \Phi_{ab}(H) \text{ and } s\text{-}mul(A) = \infty.
\]

To do this, suppose \( M_C \in \Phi_{ab}(H \oplus K) \). Then from Lemma 1.6 we have \( A \in \Phi_{ab}(H) \). If \( s\text{-}mul(A) < \infty \), then \( A \in \Phi(H) \), since \( \text{ind}(A) = \alpha(A) - \beta(A) = b.s.\text{-}mul(A) - s\text{-}mul(A) \). Hence it is easy to show that \( B \in \Phi_+(K) \), which is in a contradiction. Thus, \( s\text{-}mul(A) = \infty \).

Conversely, suppose that \( A \in \Phi_{ab}(H) \) and \( s\text{-}mul(A) = \infty \), which implies \( \beta(A) = \infty \). It follows from Proposition 1.7 that \( A \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \)

\[
A = \begin{pmatrix}
A_1 & A_{12} \\
0 & A_2
\end{pmatrix},
\]

where \( \text{dim}(H_1) < \infty \), \( A_1 \) is nilpotent, and \( A_2 \) is a left invertible operator. Noting that \( \beta(A) = \infty \), we have \( \beta(A_2) = \infty \). Hence it follows from Lemma 2.1 that there exists some \( C_0 \in B(K, H_2) \) such that \( \begin{pmatrix} A_2 & C_0 \\ 0 & B \end{pmatrix} \) is left invertible. Now consider operator

\[
M_C = \begin{pmatrix}
A & C \\
0 & B
\end{pmatrix} = \begin{pmatrix}
A_1 & A_{12} & 0 \\
0 & A_2 & C_0 \\
0 & 0 & B
\end{pmatrix},
\]

where \( C = \begin{pmatrix} 0 \\ C_0 \end{pmatrix} \in B(K, H) \). By Lemma 1.6, it is easy to check that \( M_C \in \Phi_{ab}(H \oplus K) \).

Next, We claim that if \( B \in \Phi_+(K) \), then

\[
(3) \quad M_C \in \Phi_{ab}(H \oplus K) \quad \text{for some } C \in B(K, H) \iff A \in \Phi_{ab}(H) \text{ and } b.s.\text{-}mul(B) \leq s\text{-}mul(A).
\]

To this end, suppose \( M_C \in \Phi_{ab}(H \oplus K) \), which implies \( A \in \Phi_{ab}(H) \). By Proposition 1.8, we have that \( A \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \)

\[
A = \begin{pmatrix}
A_1 & A_{12} \\
0 & A_2
\end{pmatrix},
\]

where \( \text{dim}(H_1) < \infty \), \( A_1 \) is nilpotent, and \( A_2 \) is a left invertible operator.
\[
A = \begin{pmatrix}
A_1 & A_{12} \\
0 & A_2
\end{pmatrix},
\]

where \(\dim(H_1) < \infty\), \(A_1\) is nilpotent, \(A_2\) is a left invertible operator, and \(\beta(A_2) = s_{mul}(A)\). Since the assumption that \(B \in \Phi_{+}(K)\), using Theorem 1.5, we know that \(B\) can be decomposed into the following form with respect to some orthogonal decomposition \(K = K_1 \oplus K_2 \oplus K_3\)

\[
B = \begin{pmatrix}
B_1 & * & * \\
0 & B_2 & * \\
0 & 0 & B_3
\end{pmatrix},
\]

where \(\dim K_3 < \infty\), \(B_1\) is a right invertible operator, \(B_2\) is a left invertible operator, \(B_3\) is a finite, nilpotent operator, and the parts marked by * can be any operators. Moreover, \(\text{ind}(B_1) = \alpha(B_1) = b.s_{mul}(B)\), \(\text{ind}(B_2) = -\beta(B_2) = -s_{mul}(B_1)\) and \(\text{ind}(B) = \alpha(B_1) - \beta(B_2)\). Therefore, \(M_C\) can be rewritten as the following form

\[
M_C = \begin{pmatrix}
A_1 & A_{12} & C_{11} & C_{12} & C_{13} \\
0 & A_2 & C_{21} & C_{32} & C_{23} \\
0 & 0 & B_1 & * & * \\
0 & 0 & 0 & B_2 & * \\
0 & 0 & 0 & 0 & B_3
\end{pmatrix} : H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3 \rightarrow H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3.
\]

Noting that \(\dim(H_1) < \infty\) and \(\dim(K_3) < \infty\), we have \(A_1 \in \Phi_b(H_1)\) and \(B_3 \in \Phi_b(K_3)\). Consequently, Lemma 1.6 leads to

\[
\begin{pmatrix}
A_2 & C_{21} & C_{32} \\
0 & B_1 & * \\
0 & 0 & B_2
\end{pmatrix} \in \Phi_{ab}(H_2 \oplus K_1 \oplus K_2),
\]

which implies

\[
\begin{pmatrix}
A_2 & C_{21} \\
0 & B_1
\end{pmatrix} \in \Phi_{ab}(H_2 \oplus K_1).
\]

Now we shall prove that

\[\beta(A_2) \geq \alpha(B_1).\]

If \(\beta(A_2) = \infty\), the above inequality obviously holds. On the other hand, if \(\beta(A_2) < \infty\), then \(A_2 \in \Phi(H_2)\), and hence \(B_1 \in \Phi_{+}(K_1)\). Thus,
0 ≥ \text{ind}\left( \begin{pmatrix} A_2 & C_{21} \\ 0 & B_1 \end{pmatrix} \right) = \text{ind}(A_2) + \text{ind}(B_1) = -\beta(A_2) + \alpha(B_1),

that is,

\alpha(B_1) ≤ \beta(A_2).

Therefore,

b.s._\text{mul}(B) ≤ s._\text{mul}(A).

Conversely, suppose \( A \in \Phi_{ab}(H) \), \( B \in \Phi_+(K) \) and \( \text{b.s._mul}(B) ≤ s._\text{mul}(A) \). Similar to the above arguments, we have

\[ A = \left( \begin{array}{ccc} A_1 & A_{12} \\ 0 & A_2 \end{array} \right) : H_1 \oplus H_2 \mapsto H_1 \oplus H_2 \]

and

\[ B = \left( \begin{array}{ccc} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{array} \right) : K_1 \oplus K_2 \oplus K_3 \mapsto K_1 \oplus K_2 \oplus K_3, \]

where \( \dim(H_1) < \infty \), \( A_1 \) is nilpotent, \( A_2 \) is a left invertible operator; \( \dim K_3 < \infty \), \( B_1 \) is a right invertible operator, \( B_2 \) is a left invertible operator, \( B_3 \) is a finite, nilpotent operator, and the parts marked by * can be any operators. Moreover, \( \beta(A_2) = s._\text{mul}(A) \) and \( \alpha(B_1) = \text{b.s._mul}(B) \). Since the assumption that \( \text{b.s._mul}(B) ≤ s._\text{mul}(A) \), we have \( \alpha(B_1) ≤ \beta(A_2) \). It follows from Lemma 2.1 that there exists a left invertible operator \( \widetilde{C} \in B(K_1, H_2) \) such that

\( \left( \begin{array}{ccc} A_2 & \widetilde{C} \\ 0 & B_1 \end{array} \right) \in B(H_2 \oplus K_1) \) is left invertible.

Consider operator \( M_C = \left( \begin{array}{ccc} A & C \\ 0 & B \end{array} \right) : H \oplus K \mapsto H \oplus K \)

\[ = \left( \begin{array}{ccc} A_1 & A_{12} & 0 & 0 & 0 \\ 0 & A_2 & \widetilde{C} & 0 & 0 \\ 0 & 0 & B_1 & * & * \\ 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & B_3 \end{array} \right): H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3 \mapsto H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3, \]

where \( C = \left( \begin{array}{ccc} 0 & 0 & 0 \\ \widetilde{C} & 0 & 0 \end{array} \right) \in B(K_1 \oplus K_2 \oplus K_3, H_1 \oplus H_2) \). Using Lemma 1.6, it is easy to see that \( M_C \in \Phi_{ab}(H \oplus K) \).
By duality, we have

**Theorem 2.4.** For any given \( A \in B(H) \) and \( B \in B(K) \), \( M_C \in \Phi_{sb}(H \oplus K) \) for some \( C \in B(K, H) \) iff \( B \in \Phi_{sb}(K) \) and

\[
\begin{align*}
  b.s._{mul}(B) &= \infty & \text{if } A \not\in \Phi_{-}(H) \\
  b.s._{mul}(B) &\geq s._{mul}(A) & \text{if } A \in \Phi_{-}(H)
\end{align*}
\]

From Theorems 2.3 and 2.4, we obtain the following two corollaries, concerning perturbations of the upper semi-Browder spectrum and lower semi-Browder spectrum, respectively.

**Corollary 2.5.** For any given \( A \in B(H) \) and \( B \in B(K) \), we have

\[
\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \big\{ \lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B) \text{ and } s._{mul}(A - \lambda) < \infty \big\} \cup \\
\big\{ \lambda \in \Phi(A) \cap \Phi_{+}(B) : b.s._{mul}(B - \lambda) > s._{mul}(A - \lambda) \big\}.
\]

**Corollary 2.6.** For any given \( A \in B(H) \) and \( B \in B(K) \), we have

\[
\bigcap_{C \in B(K, H)} \sigma_{sb}(M_C) = \sigma_{sb}(B) \cup \big\{ \lambda \in \mathbb{C} : \lambda \in \sigma_{SF-}(A) \text{ and } b.s._{mul}(B - \lambda) < \infty \big\} \cup \\
\big\{ \lambda \in \Phi(B) \cap \Phi_{-}(A) : b.s._{mul}(B - \lambda) < s._{mul}(A - \lambda) \big\}.
\]

**Theorem 2.7.** For any given \( A \in B(H) \) and \( B \in B(K) \), the following statements are equivalent:

1. \( M_C \in \Phi_{b}(H \oplus K) \) for some \( C \in B(K, H) \);
2. \( A \in \Phi_{ab}(H) \), \( B \in \Phi_{sb}(K) \) and \( b.s._{mul}(B) = s._{mul}(A) \);
3. \( A \in \Phi_{ab}(H) \), \( B \in \Phi_{sb}(K) \) and \( \alpha(A) + \alpha(B) = \beta(A) + \beta(B) \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( M_C \in \Phi_{b}(H \oplus K) \). Then from Lemma 1.6, we have \( A \in \Phi_{ab}(H) \) and \( B \in \Phi_{sb}(K) \). Using Propositions 1.7 and 1.8, we have
\[ M_C = \begin{pmatrix}
A_1 & A_{12} & C_{11} & C_{12} \\
0 & A_2 & C_{21} & C_{32} \\
0 & 0 & B_1 & B_{12} \\
0 & 0 & 0 & B_2
\end{pmatrix} : H_1 \oplus H_2 \oplus K_1 \oplus K_2 \to H_1 \oplus H_2 \oplus K_1 \oplus K_2, \]

where \( \dim(H_1) < \infty \), \( A_1 \) is nilpotent, \( A_2 \) is a left invertible operator, \( \dim K_2 < \infty \), \( B_1 \) is a right invertible operator, \( B_2 \) is a finite, nilpotent operator. Moreover,

\[ \beta(A_2) = s._{\text{mul}}(A) \text{ and } \alpha(B_1) = b.s._{\text{mul}}(B). \]

In addition, it follows from Lemma 1.6 that

\[ \begin{pmatrix}
A_2 & C_{21} \\
0 & B_1
\end{pmatrix} \in \Phi_b(H_2 \oplus K_1). \]

Note the well-known fact that if \( M_C \in \Phi(H \oplus K) \), then \( A \in \Phi(H) \) if and only if \( B \in \Phi(K) \).

Thus, if \( \beta(A_2) = \infty \), then \( B_1 \not\in \Phi(K_1) \), and so \( \beta(A_2) = \alpha(B_1) = \infty \) since that \( B_1 \) is right invertible. Otherwise, if \( \beta(A_2) < \infty \), then both \( A_2 \) and \( B_1 \) are Fredholm. Consequently,

\[ 0 = \text{ind}(\begin{pmatrix}
A_2 & C_{21} \\
0 & B_1
\end{pmatrix}) = \text{ind}(A_2) + \text{ind}(B_1) = -\beta(A_2) + \alpha(B_1), \]

that is, \( \beta(A_2) = \alpha(B_1) \). Therefore, \( s._{\text{mul}}(A) = b.s._{\text{mul}}(B) \).

(2) \Rightarrow (1). Suppose that \( A \in \Phi_{ab}(H) \), \( B \in \Phi_{sb}(K) \) and that \( s._{\text{mul}}(A) = b.s._{\text{mul}}(B) \).

Then from Proposition 1.7 we have that \( A \) can be decomposed into the following form with respect to some orthogonal decomposition \( H = H_1 \oplus H_2 \)

\[ A = \begin{pmatrix}
A_1 & A_{12} \\
0 & A_2
\end{pmatrix}, \]

where and \( \dim(H_1) < \infty \), \( A_1 \) is nilpotent, and \( A_2 \) is a left invertible operator. By Proposition 1.8, \( B \in B(K) \) can be decomposed into the following form with respect to some orthogonal decomposition \( K = K_1 \oplus K_2 \)

\[ B = \begin{pmatrix}
B_1 & B_{12} \\
0 & B_2
\end{pmatrix}, \]

where \( \dim(K_2) < \infty \), \( B_1 \) is a right invertible operator, and \( B_2 \) is nilpotent. Moreover, \( s._{\text{mul}}(A) = \beta(A_2) \) and \( b.s._{\text{mul}}(B) = \alpha(B_1) \). Since the assumption that \( s._{\text{mul}}(A) = b.s._{\text{mul}}(B) \), \( \alpha(B_1) = \beta(A_2) \). Thus, we conclude from Theorem 1.5 that there exists some
operator $C_{12} \in B(K_1, H_2)$ such that
\[
\begin{pmatrix}
A_2 & C_{21} \\
0 & B_1
\end{pmatrix}
\] is invertible. Define $C \in B(K, H)$ as follows:
\[
C = \begin{pmatrix}
0 & 0 \\
C_{12} & 0
\end{pmatrix}.
\]
By Lemma 1.6, it is no hard to prove that $M_C \in \Phi_b(H \oplus K)$.

(2) $\iff$ (3). For this, it is sufficient to prove that if $A \in \Phi_{ab}(H)$ and $B \in \Phi_{sb}(K)$, then

\[
\alpha(A) + \alpha(B) = \beta(A) + \beta(B) \text{ if and only if } b.s.mul(B) = s.mul(A),
\]
which follows from Propositions 1.7 and 1.8 immediately. This completes the proof.

In [1], Cao has proved the equivalence of (1) and (3) of Theorem 2.7 by a different method, which seems to be more complicated.

The next corollary immediately follows from Theorem 2.7.

**Corollary 2.8.** For any given $A \in B(H)$ and $B \in B(K)$, we have
\[
\bigcap_{C \in G(K, H)} \sigma_b(M_C) = \sigma_{ab}(A) \cup \sigma_{sb}(B) \cup \\
\{ \lambda \in \Phi_{ab}(A) \cap \Phi_{sb}(B) : b.s.mul(B - \lambda) \neq s.mul(A - \lambda) \}
\]
\[
= \sigma_{ab}(A) \cup \sigma_{sb}(B) \cup \\
\{ \lambda \in \mathbb{C} : \alpha(A - \lambda) + \alpha(B - \lambda) \neq \beta(A - \lambda) + \beta(B - \lambda) \}.
\]

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