IDENTITIES OF THE JONES MONOID $J_5$

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Abstract. Jones monoids $J_n$, for $1 < n$, is a family of monoids relevant in knot theory. The purpose of this paper is to characterize the identities satisfied by the Jones monoid $J_5$.

1. Introduction

Let $1 < n$. The Kauffman monoid $K_n$ is the monoid generated by $h_1, \ldots, h_{n-1}, c$, subject to the following relations:

$$h_i h_j = h_j h_i, \quad \forall 1 \leq i, j \leq n-1, \text{ if } |i - j| \geq 2,$$

$$h_i h_j h_i = h_i, \quad \forall 1 \leq i, j \leq n-1, \text{ if } |i - j| = 1,$$

$$h_i^2 = ch_i = h_i c, \quad \forall 1 \leq i \leq n-1.\tag{1}$$

L. H. Kauffman in [8] invented these monoids, $K_n$, as geometric objects. (The name was suggested in [3].) Kauffman monoids play an important role in several parts of mathematics such as knot theory, low-dimensional topology, topological quantum field theory, quantum groups, etc. Jones monoid $J_n$ is the monoid generated by $h_1, \ldots, h_{n-1}$, subject to the first and second relations in (1) and the relation $h_i^2 = h_i$, for all $1 \leq i \leq n-1$. Jones monoid is a class of diagram monoids like Kauffman monoids. (The name was suggested in [10] to honor the contribution of V. F. R. Jones to the theory).

Chen et al. in [5], provide an algorithm for checking identities in $K_3$. Kitov and Volkov in [9], extend this algorithm to the Kauffman monoid $K_4$ and also find a polynomial time algorithm for checking identities in the Jones monoid $J_4$. They prove that the Kauffman monoids $K_3$ and $K_4$ satisfy exactly the same identities. By delivering an identity, they show that $K_4$ and $K_5$ do not satisfy the same identities. In the present paper, we follow this line of research and characterize the identities of the monoid $J_5$. This characterization helps us to explain that for an identity when $J_4$ satisfies it and $J_5$ does not satisfy it.

The paper is organized as follows. We begin by recalling background on monoids, identities and Jones monoids, so as to make the paper accessible to as broad an audience as possible. Also, in a separated section we investigate the Jones monoid $J_5$ and give some lemmas which we need in the

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proof of the characterization. We then present in the following section our characterization of identities in $J_5$.

2. Preliminaries

2.1. Monoids. For standard notation and terminology relating to semigroups and monoids, we refer the reader to [1] Chapter 5, [3] Chapters 1-3 and [11] Appendix A]. Let $M$ a finite monoid. Let $a, b \in M$. We say that $a \mathcal{R} b$ if $aM = bM$, $a \mathcal{L} b$ if $Ma = Mb$ and $a \mathcal{H} b$ if $a \mathcal{R} b$ and $a \mathcal{L} b$. Also, we say that $a \mathcal{J} b$, if $MaM = MbM$. The relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{J}$ are Green relations and all of them are equivalence relations first introduced by Green [7]. An important property of finite monoids is the stability property that $J_m \cap Mm = L_m$ and $J_m \cap mM = R_m$, for every $m \in M$. A finite monoid is aperiodic if and only if its $\mathcal{H}$-relation is trivial.

An element $e$ of $M$ is called idempotent if $e^2 = e$. The set of all idempotents of $M$ is denoted by $E(M)$. An idempotent $e$ of $M$ is the identity of the monoid $eMe$. The group of units $G_e$ of $eMe$ is called the maximal subgroup of $M$ at $e$.

An element $m$ of $M$ is called (von Neumann) regular if there exists an element $n \in M$ such that $mmn = m$. Note that an element $m$ is regular if and only if $m \mathcal{L} e$, for some $e \in E(M)$, if and only if $m \mathcal{R} f$, for some $f \in E(M)$. A $\mathcal{J}$-class $J$ is regular if all its elements are regular, if and only if $J$ has an idempotent, if and only if $J^2 \cap J \neq \emptyset$.

2.2. Identities. Let $X$ be a countably infinite set. We call $X$ an alphabet and each element $x \in X$ an letter. Let $X^+$ be the set of all finite, non-empty words $x_1 \cdots x_n$ with $x_1, \ldots, x_n \in X$. The set $X^+$ forms a semigroup under concatenation which is called the free semigroup over $X$. The monoid $X^* = (X^+)^1$ is called the free monoid over $X$.

Let $t = x_1 \cdots x_n$ be a word of $X^+$ with $x_1, \ldots, x_n \in X$. The set $\{x_1, \ldots, x_n\}$ is called the content of $t$ and is denoted $c(t)$ while the number $n$ is referred to as the length of $t$ and is denoted $|t|$. If $x \in c(t)$, we say that a letter $x$ occurs in a word $t$. We say that a word $s \in X^+$ occurs in $t$ if $t = t_1st_2$ for some $t_1, t_2 \in X^*$. Let $u = u_1 \cdots u_m$ be a word in $X^*$ with $u_1, \ldots, u_m \in X$. We say that $u$ is a subword of the word $t$, if $t$ can be written $t = u'_0u_1u'_1 \cdots u_mu'_m$ for some words $u'_0, u'_1, \ldots, u'_m \in X^*$. For a subset $Y$ of the set $X$, let $t_Y$ be the longest subword of $t$ with $c(t_Y) \subseteq Y$.

An identity is an expression $t_1 = t_2$ with $t_1, t_2 \in X^*$. Let $M$ be a monoid. We say that the identity $t_1 = t_2$ holds in $M$ or $M$ satisfies the identity $t_1 = t_2$ if $\phi(t_1) = \phi(t_2)$ for every homomorphism $\phi: X^* \rightarrow M$ and we denote it by $\frac{t_1}{t_2} = \frac{t}{M}$.
2.3. Jones monoids. Let $1 < n$. Jones monoid $J_n$ is the monoid generated by $h_1, \ldots, h_{n-1}$, subject to the following relations:

$$
\begin{align*}
\text{if } |i - j| \geq 2, & \quad h_i h_j = h_j h_i, \forall i, j \in \{1, \ldots, n - 1\}, \\
\text{if } |i - j| = 1, & \quad h_i h_j h_i = h_i, \forall i, j \in \{1, \ldots, n - 1\}, \\
& \quad h_i^2 = h_i, \forall i \in \{1, \ldots, n - 1\}.
\end{align*}
$$

(2)

Note that $J_n$ is not a submonoid of $K_n$.

The Kauffman monoids $K_n$ and Jones monoids $J_n$ may be presented by geometric definitions with a series of diagram monoids (see [2]). In the current paper, we only deal with the Jones monoid $J_n$. Hence, we only mention a version of these geometric definitions which led to defining the Jones monoids.

Let $[n] := \{1, \ldots, n\}$ and $[n]' := \{1', \ldots, n'\}$ be two disjoint copies of the set of the first $n$ positive integers. Let $B_n$ be the set of all partitions $\pi$ of the $2n$-element set $[n] \cup [n]'$ into 2-element blocks. Such a pair can be represented by a wire diagram as shown in Figure 1. We draw a rectangular chip with $2n$ pins and represent the elements of $[n]$ by pins on the left hand side of the chip (left pins) while the elements of $[n]'$ are represented by pins on the right hand side of the chip (right pins). Usually we omit the numbers $1, 2, \ldots, n$. Now, for $\pi \in B_n$, we represent each block of the partition $\pi$ is represented by a line referred to as a wire. Thus, each wire connects two points; it is called an $l$-wire if it connects two left points, an $r$-wire if it connects two right points, and a $t$-wire if it connects a left point with a right point. Thus, each wire connects two pins. The wire diagram in Figure 1 corresponds to the pair \{\{1, 2\}, \{3, 5\}, \{4, 1'\}, \{2', 5'\}, \{3', 4'\}\}.

![Figure 1. Wire diagram for an element of $B_5$](image)

The multiply of two wire diagrams in $B_n$, we shortcut the right pins of the first chip with the corresponding left pins of the second chip. Thus, we obtain a new chip whose left pins are the left pins of the first chip, right pins are the right pins of the second chip whose wires are sequences of consecutive wires of the factors, see Figure 2 (for more detail refer to [2]).

It is easy to see that the above defined multiplication in $B_n$ is associative and that the chip corresponds to the pair \{\{1, 1'\}, \{2, 2'\}, \ldots, \{n, n'\}\} is the identity element with respect to the multiplication. The monoid $B_n$ is known as the Brauer monoid [4]. The Jones monoid $J_n$ is the submonoid of $B_n$ consisting of all elements of $B_n$ that have a representation as a chip whose
wires do not cross. The element $h_i$ in $J_n$, for $1 \leq i \leq n-1$, is the chip

\[
\{\{i, i + 1\}, \{i', (i + 1)\}', \{j, j\}' \mid \text{for all } j \neq i, i + 1\},
\]

see Figure 3. These chips satisfy the relations (2). Note that the cardinality of $J_n$ is equal to \(\frac{1}{n+1}\binom{2n}{n}\).

3. The Jones monoid $J_5$

The Jones monoid $J_5$ is aperiodic and regular and has three $J$-classes named $A_1 = \{1\}, A_2$ and $A_3$. The elements of the $J$-class $A_2$ have one $l$-wire and one $r$-wire, and the elements of the $J$-class $A_3$ have two $l$-wire and two $r$-wire. The elements $h_1, h_2, h_3$ and $h_4$ are in $A_2$, and the elements $h_1h_3$, $h_1h_4$ and $h_2h_4$ are in $A_3$. Since $h_ih_jh_i = h_i$ for all $1 \leq i, j \leq 4$ with $|i - j| = 1$ and $h_i^2 = h_i$ for all $1 \leq i \leq 4$, for every $a \in A_2 \setminus \{h_1, h_2, h_3, h_4\}$, there exist integers $1 \leq i, j \leq 4$ such that

\[
a = \begin{cases} 
h_ih_{i+1}\cdots h_{j-1}h_j & \text{if } i < j; \\
h_ih_{i-1}\cdots h_{j+1}h_j & \text{if } i > j. 
\end{cases}
\]

The $J$-class $A_2$ has four $R$-classes and four $L$-classes (see Figure 4). In Figure 4, the rows corresponding to the $R$-classes and the columns to the $L$-classes contained in $A_2$.

If $a \in A_3$, then there exist elements $a_1, a_2 \in J_5$ and $b \in \{h_1h_3, h_1h_4, h_2h_4\}$ such that $a = a_1ba_2$. The $J$-class $A_3$ has five $R$-classes and five $L$-classes (see Figure 5). In Figure 5, the rows corresponding to the $R$-classes and the columns to the $L$-classes contained in $A_3$. 
Lemma 3.1. Let $2 < n_1, n_2$ and let $a = a_1 \cdots a_{n_1}$ and $b = b_1 \cdots b_{n_2}$, for some elements $a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2} \in \{h_1, h_2, h_3, h_4\}$, such that $a_1 \cdots a_{n_1-1} \in A_2$, $b_1 \cdots b_{n_2-1} \in A_2$ and $a_1 = b_1$. If one of the following conditions hold:

\begin{align*}
\text{(∗1)} & \quad \{a_{n_1-1}, a_{n_1}\} = \{h_1, h_3\} \quad \text{and} \quad \{b_{n_2-1}, b_{n_2}\} = \{h_2, h_4\}; \\
\text{(∗2)} & \quad a_{n_1-1} = b_{n_2-1} = h_4 \quad \text{and} \quad \{a_{n_1}, b_{n_2}\} = \{h_1, h_2\}; \\
\text{(∗3)} & \quad a_{n_1-1} = b_{n_2-1} = h_1 \quad \text{and} \quad \{a_{n_1}, b_{n_2}\} = \{h_3, h_4\}; \\
\text{(∗4)} & \quad a_{n_1-1} = b_{n_2} = h_4, \quad a_{n_1} = h_1 \quad \text{and} \quad b_{n_2-1} = h_2; \\
\text{(∗5)} & \quad a_{n_1-1} = b_{n_2} = h_1, \quad a_{n_1} = h_4 \quad \text{and} \quad b_{n_2-1} = h_3,
\end{align*}

then the elements $a$ and $b$ are not in the same $R$-class.

Also, if one of the following conditions hold:

\begin{align*}
\text{(#1)} & \quad \{a_{n_1-1}, b_{n_2-1}\} = \{h_1, h_2\} \quad \text{and} \quad a_{n_1} = b_{n_2} = h_4; \\
\text{(#2)} & \quad \{a_{n_1-1}, b_{n_2-1}\} = \{h_3, h_4\} \quad \text{and} \quad a_{n_1} = b_{n_2} = h_1; \\
\text{(#3)} & \quad a_{n_1-1} = b_{n_2} = h_4, \quad a_{n_1} = h_2 \quad \text{and} \quad b_{n_2-1} = h_1; \\
\text{(#4)} & \quad a_{n_1-1} = b_{n_2} = h_1, \quad a_{n_1} = h_3 \quad \text{and} \quad b_{n_2-1} = h_4,
\end{align*}

then the elements $a$ and $b$ are in the same $R$-class.
Proof. (∗1): We have the following cases subject to the element $a_1$:

(1) ($a_1 = h_1$): since $h_1h_2h_3h_1 = h_1h_2h_1h_3 = h_1h_3$, we have $a = h_1h_3$. Also, as $h_1h_2h_3h_4h_2 = h_1h_2h_3h_2h_4 = h_1h_2h_4$, we have $b = h_1h_2h_4$. Now, as $h_1h_3$ and $h_1h_2h_4$ are not in the same $R$-class, the elements $a$ and $b$ are not in the same $R$-class.

(2) ($a_1 = h_3$): we have $a = h_2h_3h_1$. Also, as $h_2h_3h_4h_2 = h_2h_3h_2h_4 = h_2h_4$, we have $b = h_2h_4$. Now, as $h_2h_3h_1$ and $h_2h_4$ are not in the same $R$-class, the result follows.

Figure 5. The elements of the $J$-class $A_3$
(3) \((a_4 = h_3)\): since \(h_3 h_2 h_1 h_3 = h_3 h_2 h_3 h_1 = h_1 h_3\), we have \(a = h_1 h_3\). Also, we have \(b = h_3 h_3 h_4\). Now, as \(h_1 h_3\) and \(h_3 h_2 h_4\) are not in the same \(\mathcal{R}\)-class, the result follows.

(4) \((a_4 = h_4)\): since \(h_4 h_3 h_2 h_1 h_3 = h_4 h_3 h_2 h_3 h_1 = h_4 h_3 h_1 = h_4 h_3 h_1\), we have \(a = h_4 h_3 h_1\). Also, as \(h_4 h_3 h_2 h_4 = h_4 h_3 h_4 h_2 = h_4 h_2\), we have \(b = h_4 h_2\). Now, as \(h_4 h_3 h_1\) and \(h_4 h_2\) are not in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

\((\ast 2)\): If \(a_1 = h_4\), then we have \(a_1 \cdots a_{n_1-1} = b_1 \cdots b_{n_2-1} = h_4\). As, \(h_1 h_4\) and \(h_2 h_4\) are not in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class. If \(a_1 \neq h_4\), then there exist integers \(1 \leq i_1 < n_1 - 1\) and \(1 \leq i_2 < n_2 - 1\) such that \(a_{i_1} = h_3, b_{i_2} = h_3\) and \(a_{i_1+1} \cdots a_{n_1-1} = b_{i_2+1} \cdots b_{n_2-1} = h_4\). Then, we have \(\{a_{i_1} \cdots a_{n_1}, b_{i_2} \cdots b_{n_2}\} = \{h_3 h_4 h_1, h_3 h_4 h_2\} = \{h_3 h_1 h_4, h_3 h_4 h_2\}\). Now, by the previous part, the result follows.

\((\ast 3)\): Similarly the previous part, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

\((\ast 4)\): we have the following cases subject to the element \(a_1\):

1. \((a_1 = h_1)\): we have \(a = h_1 h_2 h_3 h_4 h_1 = h_1 h_2 h_3 h_1 h_4 = h_1 h_2 h_1 h_3 h_4 = h_1 h_3 h_4\) and \(b = h_1 h_2 h_4\). Now, as \(h_1 h_3 h_4\) and \(h_1 h_2 h_4\) are not in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

2. \((a_1 = h_2)\): we have \(a = h_2 h_3 h_4 h_1\) and \(b = h_2 h_4\). Now, as \(h_2 h_3 h_4 h_1\) and \(h_2 h_4\) are not in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

3. \((a_1 = h_3)\): we have \(a = h_3 h_4 h_1\) and \(b = h_3 h_2 h_4\). Now, as \(h_3 h_4 h_1\) and \(h_3 h_2 h_4\) are not in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

4. \((a_1 = h_4)\): we have \(a = h_4 h_1\) and \(b = h_4 h_3 h_2 h_4 = h_4 h_3 h_4 h_2 = h_4 h_2\). Now, as \(h_4 h_1\) and \(h_4 h_2\) are not in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

\((\ast 5)\): Similarly Part \((\ast 4)\), the elements \(a\) and \(b\) are not in the same \(\mathcal{R}\)-class.

\((\#1)\): By symmetry, we may assume that \(a_{n_1-1} = h_1\) and \(b_{n_2-1} = h_2\). We have the following cases subject to the element \(a_1\):

1. \((a_1 = h_1)\): we have \(a = h_1 h_4\) and \(b = h_1 h_2 h_4\). Now, as \(h_1 h_4\) and \(h_1 h_2 h_4\) are in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are in the same \(\mathcal{R}\)-class.

2. \((a_1 = h_2)\): we have \(a = h_2 h_1 h_4\) and \(b = h_3 h_4\). Now, as \(h_2 h_1 h_4\) and \(h_2 h_4\) are in the same \(\mathcal{R}\)-class, the result follows.

3. \((a_1 = h_3)\): we have \(a = h_3 h_2 h_1 h_4\) and \(b = h_3 h_2 h_4\). Now, as \(h_3 h_2 h_1 h_4\) and \(h_3 h_2 h_4\) are in the same \(\mathcal{R}\)-class, the result follows.

4. \((a_1 = h_4)\): since \(h_4 h_3 h_2 h_1 h_4 = h_4 h_3 h_2 h_3 h_1 = h_4 h_3 h_4 h_2 h_1 = h_4 h_2 h_1\), we have \(a = h_4 h_2 h_1\). Also, as \(h_4 h_3 h_2 h_4 = h_4 h_3 h_4 h_2 = h_4 h_2\), we have \(b = h_4 h_2\). Now, as \(h_4 h_2 h_1\) and \(h_4 h_2\) are in the same \(\mathcal{R}\)-class, the elements \(a\) and \(b\) are in the same \(\mathcal{R}\)-class.

Similarly Part \((\#1)\), other Parts \((\#2), (\#3)\) and \((\#4)\) hold.
Similarly Lemma 3.1 we have the following lemma.

**Lemma 3.2.** Let $2 < n_1, n_2$ and let $a = a_1 \cdots a_{n_1}$ and $b = b_1 \cdots b_{n_2}$, for some elements $a_1, \ldots , a_{n_1}, b_1, \ldots , b_{n_2} \in \{ h_1, h_2, h_3, h_4 \}$, such that $a_2 \cdots a_{n_1} \in A_2, b_2 \cdots b_{n_2} \in A_2$ and $a_{n_1} = b_{n_2}$. If one of the following conditions hold:

- $(*)$ $\{ a_1, a_2 \} = \{ h_1, h_3 \}$ and $\{ b_1, b_2 \} = \{ h_2, h_4 \}$;
- $(*)$ $\{ a_1, b_1 \} = \{ h_1, h_2 \}$ and $a_2 = b_2 = h_4$;
- $(*)$ $\{ a_1, b_1 \} = \{ h_3, h_4 \}$ and $a_2 = b_2 = h_1$;
- $(*)$ $a_1 = b_2 = h_4$, $a_2 = h_2$ and $b_1 = h_1$;
- $(*)$ $a_1 = b_2 = h_1$, $a_2 = h_3$ and $b_1 = h_4$,

then the elements $a$ and $b$ are not in the same $\mathcal{L}$-class.

Also, if one of the following conditions hold:

- $(#1)$ $a_1 = b_1 = h_4$ and $\{ a_2, b_2 \} = \{ h_1, h_2 \}$;
- $(#2)$ $a_1 = b_1 = h_1$ and $\{ a_2, b_2 \} = \{ h_3, h_4 \}$;
- $(#3)$ $a_1 = b_2 = h_4$, $a_2 = h_1$ and $b_1 = h_2$;
- $(#4)$ $a_1 = b_2 = h_1$, $a_2 = h_4$ and $b_1 = h_3$,

then the elements $a$ and $b$ are in the same $\mathcal{L}$-class.

4. Characterization of the identities of the monoid $\mathcal{J}_5$

Let $w$ and $v$ be words of $X^*$.

**Lemma 4.1.** If $w \overset{\mathcal{T}_5}{=} v$, then we have $c(w) = c(v)$.

*Proof.* Suppose the contrary, that there exists an letter $x \in c(w) \setminus c(v)$. Substituting for all letters in $c(v)$ the value $h_1$, we obtain $v = h_1$. Substituting for all letters in $c(w) \setminus \{ x \}$ the value $h_1$ and the letter $x$ the value $h_3$ we obtain $w = h_3$ or $w \in A_3$. Hence, $\mathcal{J}_5$ does not satisfy the identity $w = v$. Also, if there exists an letter $x \in c(v) \setminus c(w)$, we have a similar contradiction. Thus, we have $c(w) = c(v)$. \hfill $\Box$

By Lemma 4.1 if $\mathcal{J}_5$ satisfies the identity $w = v$, then $w = 1$ if and only if $v = 1$. Then, we assume that $w, v \neq 1$ and $c(w) = c(v)$.

Let $Y$ be a non empty subset of $c(w)$. By Lemma 4.1 we have $c(w_Y) = c(v_Y)$. There exist letters $y_1, \ldots , y_r, z_1, \ldots , z_s$ in $Y$ such that $w_Y = y_1 \cdots y_r$ and $v_Y = z_1 \cdots z_s$.

**Lemma 4.2.** If $w_Y \overset{\mathcal{R}}{=} v_Y$, then we have $y_1 = z_1$ and $y_r = z_s$.

*Proof.* Let $a_1, a_2 \in A_3$ such that $a_1$ and $a_2$ are not in the same $\mathcal{R}$-class. Substituting the letter $y_1$ the value $a_1$, the letter $z_1$ the value $a_2$, we obtain that $w$ and $v$ are not in the same $\mathcal{R}$-class. Then, $\mathcal{J}_5$ does not satisfy the identity $w_Y = v_Y$, a contradiction.

Similarly, we have $y_r = z_s$. \hfill $\Box$

Throughout the remainder of this section before the main theorem, we assume that the following conditions hold for the subset $Y$ as follows:

1. $y_1 = z_1$;
Lemma 4.3. If $w_Y \not\preceq v_Y$, then each word of length 2 occurs in $w_Y$ if and only if it occurs in $v_Y$.

Proof. We prove the result by contradiction.

Let $xy$ be a word of length 2 occurs in $w_Y$ and does not occur in $v_Y$. Since $xy$ occurs in $w_Y$, there exist letters $y_i, y_{i+1} \in Y$, for some integer $1 \leq i \leq r - 1$ such that $y_i = x$ and $y_{i+1} = y$. We have two cases as follows:

$(x \neq y)$: If $|c(w_Y)| = 2$, then there exists an integer $1 \leq j < s$ such that $z_1 = \cdots = z_j = y$ and $z_{j+1} = \cdots = z_s = x$. Substituting the letter $y_i$ the value $h_3$ and the letter $y_{i+1}$ the value $h_1h_2$, we obtain that $w_Y \in A_3$, because $h_3h_1h_2 \in A_3$. Now, as $v_Y = h_1h_2h_3 \in A_2$, there is a contradiction. Hence, we suppose that $|c(w_Y)| > 2$. Again, by substituting the letter $y_i$ the value $h_3$, the letter $y_{i+1}$ the value $h_1h_2$ and for all letters in $c(w_Y) \setminus \{y_i, y_{i+1}\}$ the value $h_2$, we again obtain that $w_Y \in A_3$. Also, since $h_2h_3h_2 = h_2$, $h_3h_2h_3 = h_3$, $zy = h_2h_1h_2 = h_2$ for all $z \in c(w_Y) \setminus \{y_i, y_{i+1}\}$ and $xy$ does not occur in $v_Y$, one of the following conditions hold:

$(z_1 = x)$: $v_Y = h_3h_2$ or $v_Y = h_3$;
$(z_1 = y)$: $v_Y = h_1h_2$ or $v_Y = h_1h_2h_3$;
$(z_1 \not\in \{x, y\})$: $v_Y = h_2h_3$ or $v_Y = h_2$.

Thus, we have $v_Y \in A_2$, a contradiction.

$(x = y)$: First suppose that $|c(w_Y)| = 1$. Substitute the letter $y_i$ the value $h_1h_2h_3$. We have $w_Y \in A_3$ and $v_Y \in A_2$, because $x^2$ does not occur in $v_Y$. A contradiction. Hence, we suppose that $|c(w_Y)| > 1$. Substituting the letter $y_i$ the value $h_3h_2h_1$ and for all letters in $c(w_Y) \setminus \{y_i\}$ the value $h_2$, we obtain that $w_Y \in A_3$, because $(h_3h_2h_1)^2 \in A_3$. Also, since $h_3h_2h_1h_2 = h_3h_2$ and $x^2$ does not occur in $v_Y$, one of the following conditions hold:

$(z_1 \neq y_i)$: $v_Y = h_2h_1$ or $v_Y = h_2$;
$(z_1 = y_i)$: $v_Y = h_3h_2h_1$ or $v_Y = h_3h_2$.

Thus, we have $v_Y \in A_2$, a contradiction.

Lemma 4.4. Let $x, y, z, t \in Y$ with $xy \neq zt$. Suppose that there exist words $w_1, w_2, v_1$ and $v_2$ in $Y^*$ such that $w_Y = w_1xyw_2$ and $v_Y = v_1ztv_2$. If one of the following conditions hold:

1. if $xy$ and $zt$ do not occur in $w_1x$ and $v_1z$, and $x = z$;
2. if $y$ does not occur in $w_1x$ and $v_1z$, and $y = t$;
3. if $xy$ and $zt$ do not occur in $yw_2$ and $tv_2$, and $y = t$;
4. if $x$ does not occur in $yw_2$ and $tv_2$, and $x = z$,

then $J_5$ does not satisfy the identity $w_Y = v_Y$.

Proof. (1) Since $x = z$ and $xy \neq zt$, by symmetry we may assume that one of the following conditions holds:
(I) $xy = xx$, $zt = xt$ and $x \neq t$;  
(II) $zt = xt$ and $|[x,y,t]| = 3$.

For every case, we substitute as follows:

(I) substitute the letter $x$ the value $h_2h_3h_4$ and the letter $t$ the value $h_1h_2$. If there is another letter in $c(w_Y) \setminus \{x,t\}$, substitute it the value $h_3h_2$. By Lemma 3.1 (2), we obtain that $w_Y$ and $v_Y$ are in different $\mathcal{R}$-classes.

(II) substitute the letter $x$ the value $h_3h_4$, the letter $y$ the value $h_1h_2$ and the letter $t$ the value $h_2$. If there is a letter in $c(w_Y) \setminus \{x,y,t\}$, substitute it the value $h_3h_2$. Again, by Lemma 3.1 (2), we obtain that $w_Y$ and $v_Y$ are in different $\mathcal{R}$-classes.

(2) Substitute the letter $x$ the value $h_3h_4$, the letter $z$ the value $h_4h_3$, the letter $y$ the value $h_2h_3h_1$ and the other letters the value $h_3$. Since $y_1 = z_1$, one of the following conditions holds:

(I) $w_1xy = h_3h_4h_2h_3h_1$ and $v_1zy = h_3h_4h_3h_2h_3h_1 = h_3h_1$;

(II) $w_1xy = h_1h_3h_4h_2h_3h_1 = h_1h_2h_3h_1$ and $v_1zy = h_4h_3h_2h_3h_1 = h_4h_3h_1$.

In both cases, by Lemma 3.1 (1), we obtain that $w_Y$ and $v_Y$ are in different $\mathcal{R}$-classes.

(3) As $y = t$ and $xy \neq zt$, by symmetry we may assume that one of the following conditions holds:

(I) $xy = xx$, $zt = xx$ and $x \neq z$;

(II) $zt = yz$ and $|[x,y,z]| = 3$.

Like as above, for every case, we substitute as follows:

(I) substitute the letter $x$ the value $h_4h_3h_2$ and the letter $z$ the value $h_2h_3$. If there is another letter in $c(w_Y) \setminus \{x,z\}$, substitute it the value $h_2h_3$. By Lemma 3.2 (2), we obtain that $w_Y$ and $v_Y$ are in different $\mathcal{L}$-classes.

(II) substitute the letter $x$ the value $h_2h_1$, the letter $y$ the value $h_4h_3$ and the letter $z$ the value $h_2$. If there is a letter in $c(w_Y) \setminus \{x,y,z\}$, substitute it the value $h_2h_3$. Again, by Lemma 3.2 (2), we obtain that $w_Y$ and $v_Y$ are in different $\mathcal{L}$-classes.

(4) Substitute the letter $y$ the value $h_3h_4$, the letter $t$ the value $h_4h_3$, the letter $x$ the value $h_1h_3h_2$ and the other letters the value $h_3$. Since $y_r = z_s$, one of the following conditions holds:

(I) $xyw_3 = h_1h_3h_2h_3h_4h_3 = h_1h_3$ and $xtv_2 = h_1h_3h_2h_4h_3$;

(II) $xyw_2 = h_1h_3h_2h_4h_3 = h_1h_3h_4$ and $xtv_2 = h_1h_3h_2h_4h_3 = h_1h_3h_2h_4$.

In both cases, by Lemma 3.2 (1), we obtain that $w_Y$ and $v_Y$ are in different $\mathcal{L}$-classes.

\begin{lemma}
Let $x,y,z,t \in Y$ such that $x \neq z$ and $y \neq t$. Suppose that there exist words $w_1, w_2, v_1$ and $v_2$ in $Y^*$ such that $w_Y = w_1xyw_2$ and $v_Y = v_1ztv_2$. Let $C_1 = \{w_1x, v_1z\}$, $C_2 = \{yw_2, tv_2\}$ and let $C \in \{C_1, C_2\}$. If $xy$ and $zt$ do not occur in the elements of $C$, and one of the following states holds:

(\textbf{xt}): $xt$ does not occur in the elements of $C$;

(\textbf{zy}): $zy$ does not occur in the elements of $C$;

\end{lemma}
(\(xu, zu'\)): there exists a subset \(\{x, y, z, t\} \subseteq Y' \subseteq Y\) such that \(Y' = Y_1 \cup Y_2\), for some subsets \(Y_1\) and \(Y_2\), with the following conditions:

1. \(Y_1 \cap Y_2 = \emptyset\);
2. if \(u \in Y_1\), then \(xu\) does not occur in \(o_{Y'}\), for \(o \in C\);
3. if \(u' \in Y_2\), then \(zu'\) does not occur in \(o_{Y'}\), for \(o \in C\);
4. \(xy\) and \(zt\) does not occur in \(o_{Y'}\), for \(o \in C\),

(\(uy, u't\)): there exists a subset \(\{x, y, z, t\} \subseteq Y' \subseteq Y\) such that \(Y' = Y_1 \cup Y_2\), for some subsets \(Y_1\) and \(Y_2\), with the following conditions:

1. \(Y_1 \cap Y_2 = \emptyset\);
2. if \(u \in Y_1\), then \(uy\) does not occur in \(o_{Y'}\), for \(o \in C\);
3. if \(u' \in Y_2\), then \(u't\) does not occur in \(o_{Y'}\), for \(o \in C\);
4. \(xy\) and \(zt\) does not occur in \(o_{Y'}\), for \(o \in C\),

then \(J_5\) does not satisfy the identity \(w_y = v_y\).

**Proof.** Since \(x \neq z\) and \(y \neq t\), by symmetry, we may assume that one of the following states holds:

(\(xx, zz\)): \(x = y, z = t\) and \(x \neq z\);

(\(xx, zt\)): \(x = y\) and \(\{x, z, t\}\) = 3;

(\(xy, zx\)): \(x = y\) and \(\{x, y, z\}\) = 3;

(\(xy, yx\)): \(x = t\) and \(\{x, y, z\}\) = 3;

(\(xy, zt\)): \(\{x, y, z, t\}\) = 4.

For every state (\(xx, zz\), (\(xx, zt\), (\(xy, zx\), (\(xy, yx\)), (\(xy, zt\) and every states of the lemma, we define a homomorphism \(\phi: c(\omega_{Y'})^* \rightarrow J_5\) as follows:

(\(xx, zz\), (\(xt\)): \(\phi(x) = h_1h_2h_1, \phi(z) = h_2h_3h_2\) and for every \(u \in c(\omega_{Y'}) \setminus \{x, z\}\), \(\phi(u) = h_2h_3\).

(\(xx, zz\), (\(zy\)): \(\phi(x) = h_1h_2h_3, \phi(z) = h_2h_3h_4\) and for every \(u \in c(\omega_{Y'}) \setminus \{x, z\}\), \(\phi(u) = h_3h_2\).

(\(xx, zz\), (\(xu, zu'\)): \(\phi(x) = h_3h_2h_1, \phi(z) = h_2h_3h_4\), for every \(u \in Y_1 \setminus \{x\}\), \(\phi(u) = h_3\), for every \(u' \in Y_2 \setminus \{z\}\), \(\phi(u') = h_2\) and for every \(u'' \in Y \setminus Y'\), \(\phi(u'') = 1\).

(\(xx, zz\), (\(uy, u't\)): \(\phi(x) = h_1h_2h_3, \phi(z) = h_4h_2h_2\), for every \(u \in Y_1 \setminus \{x\}\), \(\phi(u) = h_3\), for every \(u' \in Y_2 \setminus \{z\}\), \(\phi(u') = h_2\) and for every \(u'' \in Y \setminus Y'\), \(\phi(u'') = 1\).

(\(xx, zt\), (\(xt\)): \(\phi(x) = h_3h_2h_1, \phi(z) = h_2, \phi(t) = h_4h_3\) and for every \(u \in c(\omega_{Y'}) \setminus \{x, z, t\}\), \(\phi(u) = h_2h_3\).

(\(xx, zt\), (\(zy\)): \(\phi(x) = h_1h_2h_3, \phi(z) = h_3h_4, \phi(t) = h_2\) and for every \(u \in c(\omega_{Y'}) \setminus \{x, z, t\}\), \(\phi(u) = h_3h_2\).

(\(xx, zt\), (\(xu, zu'\)): \(\phi(x) = h_3h_2h_1, \phi(t) = h_2h_3\), for every \(u \in Y_1 \setminus \{x, z\}\), \(\phi(u) = h_3, \phi(u') = h_2\) and for every \(u'' \in Y \setminus Y'\), \(\phi(u'') = 1\). If \(z \in Y_1\) then \(\phi(z) = h_3h_4\), otherwise, \(\phi(z) = h_2h_3h_4\).

(\(xx, zt\), (\(uy, u't\)): \(\phi(x) = h_1h_2h_3, \phi(z) = h_3h_2\), for every \(u \in Y_1 \setminus \{x, t\}\), \(\phi(u) = h_3\), for every \(u' \in Y_2 \setminus \{z, t\}\), \(\phi(u') = h_2\) and for every \(u'' \in Y \setminus Y'\), \(\phi(u'') = 1\). If \(t \in Y_1\) then \(\phi(t) = h_4h_3\), otherwise, \(\phi(t) = h_4h_3h_2\).

(\(xy, zx\), (\(xt\)): \(\phi(x) = h_3h_2h_1, \phi(y) = h_3, \phi(z) = h_2\) and for every \(u \in c(\omega_{Y'}) \setminus \{x, y, z\}\), \(\phi(u) = h_2h_3\).
((xy, zx), (zy)): $\phi(x) = h_2h_3$, $\phi(y) = h_1h_2$, $\phi(z) = h_3h_4$ and for every $u \in c(w'Y) \setminus \{x, y, z\}$, $\phi(u) = h_3h_2$.

((xy, zx), (xu, zu')): $\phi(x) = h_2h_1$, $\phi(y) = h_3$, for every $u \in Y_1 \setminus \{z\}$, $\phi(u) = h_3$, for every $u' \in Y_2 \setminus \{z, x\}$, $\phi(u') = h_2$ and for every $u'' \in Y \setminus Y'$, $\phi(u'') = 1$. If $z \in Y_1$ then $\phi(z) = h_3h_4$, otherwise, $\phi(z) = h_2h_3h_4$.

((xy, zx), (uy, u't)): $\phi(x) = h_4h_3$, $\phi(y) = h_1h_2$, for every $u \in Y_1 \setminus \{y, x\}$, $\phi(u) = h_3$, for every $u' \in Y_2 \setminus \{y, z\}$, $\phi(u') = h_2$ and for every $u'' \in Y \setminus Y'$, $\phi(u'') = 1$. If $y \in Y_1$ then $\phi(y) = h_1h_2h_3$, otherwise, $\phi(y) = h_1h_2$.

((xy, yx), (xt)): $\phi(x) = h_3h_2h_3h_1$, $\phi(y) = h_3h_2$ and for every $u \in c(w'Y) \setminus \{x, y\}$, $\phi(u) = h_3h_2$.

((xy, yx), (xu, zu')): $\phi(x) = h_2h_1$, $\phi(y) = h_1h_2$, for every $u \in Y_1 \setminus \{y\}$, $\phi(u) = h_3$, for every $u' \in Y_2 \setminus \{y, z\}$, $\phi(u') = h_2$ and for every $u'' \in Y \setminus Y'$, $\phi(u'') = 1$.

((xy, yx), (uy, u't)): $\phi(x) = h_4h_3$, $\phi(y) = h_1h_2$, for every $u \in Y_1 \setminus \{x\}$, $\phi(u) = h_3$, for every $u' \in Y_2 \setminus \{y, z\}$, $\phi(u') = h_2$ and for every $u'' \in Y \setminus Y'$, $\phi(u'') = 1$.

((xy, zt), (xt)): $\phi(x) = h_2h_1$, $\phi(y) = h_3$, $\phi(z) = h_2$, $\phi(t) = h_4h_3$ and for every $u \in c(w'Y) \setminus \{x, y, z, t\}$, $\phi(u) = h_2h_3$.

((xy, zt), (zy)): $\phi(x) = h_3$, $\phi(y) = h_1h_2$, $\phi(z) = h_3h_4$, $\phi(t) = h_2$ and for every $u \in c(w'Y) \setminus \{x, y, t\}$, $\phi(u) = h_3h_2$.

((xy, zt), (xu, zu')): $\phi(y) = h_3h_2$, $\phi(t) = h_2$, for every $u \in Y_1 \setminus \{y, z\}$, $\phi(u) = h_3h_2$, for every $u' \in Y_2 \setminus \{y, z\}$, $\phi(u') = h_2$ and for every $u'' \in Y \setminus Y'$, $\phi(u'') = 1$. If $x \in Y_1$ then $\phi(x) = h_3h_2h_1$, otherwise, $\phi(x) = h_2h_1$. Also, if $z \in Y_1$ then $\phi(z) = h_3h_4$, otherwise, $\phi(z) = h_2h_3h_4$.

((xy, zt), (uy, u't)): $\phi(x) = h_2h_3$, $\phi(y) = h_3h_2$, for every $u \in Y_1 \setminus \{x, y, t\}$, $\phi(u) = h_3h_2$, for every $u' \in Y_2 \setminus \{y, z, t\}$, $\phi(u') = h_2$ and for every $u'' \in Y \setminus Y'$, $\phi(u'') = 1$. If $y \in Y_1$ then $\phi(y) = h_1h_2h_3$, otherwise, $\phi(y) = h_1h_2$. Also, if $t \in Y_1$ then $\phi(t) = h_3h_2$, otherwise, $\phi(t) = h_2h_3h_2$.

By the substitutions for the state (xx, zz) subject to the conditions of the states (xt), (zy), (xu, zu') and (uy, u't), the elements $\phi(xx)$ and $\phi(zz)$ are in $A_3$, and the elements $\phi(w_1x)$ and $\phi(v_1z)$ are in $A_2$, for $C = C_1$. Also, for $C = C_2$, the elements $\phi(yw_2)$ and $\phi(tv_2)$ are in $A_2$. These substitutions satisfy the conditions of Lemma 3.19*(1) or Lemma 3.20*(1). Therefore, $\mathcal{J}_5$ does not satisfy the identity $w_Y = v_Y$, for the state (xx, zz) and $C \in \{C_1, C_2\}$. We have same result for other states (xx, zt), (xy, zx), (xy, yx) and (xy, zt) by others substitutions as above. Hence, we conclude that $\mathcal{J}_5$ does not satisfy the identity $w_Y = v_Y$. 

**Theorem 4.6.** Let $w$ and $v$ be words of $X^*$. The monoid $\mathcal{J}_5$ satisfies the identity $w = v$ if and only if $c(w) = c(v)$ and for every non empty subset $Y$ of $c(w)$, the following conditions hold:

1. the first letter of $w_Y$ and $v_Y$ are equal;
2. the last letter of $w_Y$ and $v_Y$ are equal;
(3) each word of length 2 occurs in \( w_Y \) if and only if it occurs in \( v_Y \);
(4) let \( x, y, z, t \in Y \) with \( xy \neq zt \). Suppose that there exist words \( w_1, w_2, v_1 \) and \( v_2 \) in \( Y^* \) such that \( w_Y = w_1xyw_2 \) and \( v_Y = v_1ztv_2 \). Let \( C_1 = \{ w_1x, v_1z \} \), \( C_2 = \{ yw_2, tv_2 \} \) and let \( C \in \{ C_1, C_2 \} \). The following conditions hold:

(a) if \( xy \) and \( zt \) do not occur in the elements of \( C_1 \), then \( x \neq z \);
(b) if \( y \) does not occur in the elements of \( C_1 \), then \( y \neq t \);
(c) if \( xy \) and \( zt \) do not occur in the elements of \( C_2 \), then \( y \neq t \);
(d) if \( x \) does not occur in the elements of \( C_2 \), then \( x \neq z \);
(e) if \( x \neq z \), \( y \neq t \) and, \( xy \) and \( zt \) do not occur in the elements of \( C \) then the following conditions hold:

(i) \( zt \) occurs in the elements of \( C \);
(ii) \( zy \) occurs in the elements of \( C \);
(iii) there does not exist a subset \( \{ x, y, z, t \} \subseteq Y' \subseteq Y \) such that \( Y' = Y_1 \cup Y_2 \), for some subsets \( Y_1 \) and \( Y_2 \), with the following conditions:

(A) \( Y_1 \cap Y_2 = \emptyset \);
(B) if \( u \in Y_1 \), then \( xu \) does not occur in \( oY' \), for \( o \in C \);
(C) if \( u' \in Y_2 \), then \( zu' \) does not occur in \( oY' \), for \( o \in C \);
(D) \( xy \) and \( zt \) does not occur in \( oY' \), for \( o \in C \);

(iv) there does not exist a subset \( \{ x, y, z, t \} \subseteq Y' \subseteq Y \) such that \( Y' = Y_1 \cup Y_2 \), for some subsets \( Y_1 \) and \( Y_2 \), with the following conditions:

(A) \( Y_1 \cap Y_2 = \emptyset \);
(B) if \( u \in Y_1 \), then \( uv \) does not occur in \( oY' \), for \( o \in C \);
(C) if \( u' \in Y_2 \), then \( u't \) does not occur in \( oY' \), for \( o \in C \);
(D) \( xy \) and \( zt \) does not occur in \( oY' \), for \( o \in C \).

Proof. If \( J_5 \) satisfies the identity \( w = v \), then by Lemmas 4.1, we have \( c(w) = c(v) \) and \( J_5 \) satisfies \( w_Y = v_Y \), for every subset \( Y \subseteq c(w) \). It is followed easily by substituting every letter in \( X \times Y \) value 1. Then, by 4.2, 4.3, 4.4 and 4.5, all Conditions (1), (2), (3) and (4) hold.

Now, suppose the contrary that \( J_5 \) does not satisfy the identity \( w = v \), \( c(w) = c(v) \) and the conditions of the theorem hold. Hence, there exists a homomorphism \( \phi: c(w)^* \to J_5 \) such that \( \phi(w) \neq \phi(v) \). Let \( Y = \{ y \in c(w) | \phi(y) \neq 1 \} \). It easily follows that \( \phi(w_Y) \neq \phi(v_Y) \). There exist letters \( y_1, \ldots, y_r, z_1, \ldots, z_s \) in \( Y \) such that \( w_Y = y_1 \ldots y_r \) and \( v_Y = z_1 \ldots z_s \). By Condition (3), we have \( \phi(w_Y), \phi(v_Y) \in A_2 \) or \( \phi(w_Y), \phi(v_Y) \in A_3 \). By Conditions (1) and (2), we have \( y_1 = z_1 \) and \( y_r = z_s \). Now, as \( J_5 \) is aperiodic, if \( \phi(w_Y), \phi(v_Y) \in A_2 \), then we have \( \phi(w_Y) = \phi(v_Y) \), a contradiction. Hence, we have \( \phi(w_Y), \phi(v_Y) \in A_3 \). Since \( \phi(w_Y) \neq \phi(v_Y) \) and \( J_5 \) is aperiodic, one or both of the following conditions holds:

(1) \( \phi(w_Y) \) and \( \phi(v_Y) \) are not in the same \( R \)-class;
(2) \( \phi(w_Y) \) and \( \phi(v_Y) \) are not in the same \( L \)-class.
First, we suppose that \( \phi(w_Y) \) and \( \phi(v_Y) \) are not in the same \( R \)-class. Since \( y_1 = z_1 \), we have \( \phi(y_1) = \phi(z_1) \). If \( \phi(y_1) \in A_3 \), then \( \phi(w_Y) \) and \( \phi(v_Y) \) are in the same \( R \)-class. Hence, we have \( \phi(y_1) \in A_2 \). As \( \phi(w_Y), \phi(v_Y) \in A_3 \), there exist integers \( 1 \leq i_1 < r \) and \( 1 \leq i_2 < s \) such that \( \phi(y_i \cdots y_{i_1}) \in A_2 \), \( \phi(z_i \cdots z_{i_2}) \in A_2 \) and \( \phi(z_1 \cdots z_{i_2+1}) \in A_3 \). If \( y_i y_{i_1} = z_i \cdots z_{i_2+1} \), then we have \( y_i = z_{i_2} \). It follows that \( \phi(y_i y_{i_1}) = \phi(z_1 \cdots z_{i_2}) \), because \( F \) is aperiodic, \( \phi(y_i y_{i_1}), \phi(z_1 \cdots z_{i_2}) \in A_2 \) and \( y_1 = z_1 \). Now, as \( y_{i_1+1} = z_{i_2+1}, \phi(w_Y) \) and \( \phi(v_Y) \) are in the same \( R \)-class, a contradiction. Then, we have \( y_i y_{i_1+1} \neq z_{i_2} z_{i_2+1} \).

If \( \phi(y_{i_1+1}) \in A_3 \), then \( y_{i_1+1} \) does not occur in \( y_1 \cdots y_{i_1} \) and \( z_1 \cdots z_{i_2} \) and by Condition (4).(b), we have \( y_{i_1+1} \neq z_{i_2+1} \). If \( z_{i_2+1} \) does not occur in \( y_1 \cdots y_{i_3} \) and \( z_1 \cdots z_{i_2} \), then the words \( w_{\{y_{i_1+1}, z_{i_2+1}\}} \) and \( v_{\{y_{i_1+1}, z_{i_2+1}\}} \) do not satisfy Condition (1). Hence, \( z_{i_2+1} \) occurs in one or both words \( y_1 \cdots y_{i_1} \) and \( z_1 \cdots z_{i_2} \) that causes \( \phi(z_{i_2+1}) \in A_2 \). Since \( \phi(z_1 \cdots z_{i_2+1}) \in A_3 \) and \( \phi(z_{i_2+1}) \in A_2 \), we have \( \phi(z_{i_2} z_{i_2+1}) \in A_3 \) and, thus, the word \( z_{i_2} z_{i_2+1} \) does not occur in \( y_1 \cdots y_{i_3} \) and \( z_1 \cdots z_{i_2} \). Now, as \( y_{i_1+1} \) does not occur in \( y_1 \cdots y_{i_1} \) and \( z_1 \cdots z_{i_2} \), by Condition (4).(a), we have \( y_i \neq z_{i_2} \). Now, as \( y_{i_1+1} \neq z_{i_2+1} \), Condition (4).(e).(i) or (4).(e).(ii) does not hold, a contradiction. Then, we have \( \phi(y_{i_1+1}) \in A_2 \). Similarly, we have \( \phi(z_{i_2+1}) \in A_2 \).

Now, as \( \phi(y_1 \cdots y_{i+1}), \phi(z_1 \cdots z_{i+1}) \in A_3 \) and \( \phi(y_1 \cdots y_{i_1}), \phi(z_1 \cdots z_{i_2}), \phi(z_{i_2} \cdots z_{i_2+1}) \in A_2 \), we have \( \phi(y_{i_1} y_{i_1+1}), \phi(z_{i_2} z_{i_2+1}) \in A_3 \). Hence, we have

\[
\phi(y_{i_1} y_{i_1+1}), \phi(z_{i_2} z_{i_2+1}) \in \{ \alpha_1 h_1 h_3 \alpha_2, \beta_1 h_1 h_4 \beta_2, \gamma_1 h_2 h_4 \gamma_2, \alpha_1' h_3 h_1 \alpha_2', \\
\beta_1' h_4 h_1 \beta_2', \gamma_1' h_4 h_2 \gamma_2' \} \text{ for some elements } \alpha_1, \alpha_2, \\
\beta_1, \beta_2, \gamma_1, \gamma_2, \alpha_1', \alpha_2', \beta_1', \beta_2', \gamma_1', \gamma_2' \in A_2 \cup A_1
\]

for which

\[
\begin{align*}
\alpha_1 h_1, & h_3 \alpha_2, \beta_1 h_1, h_4 \beta_2, \\
& \gamma_1 h_2, h_4 \gamma_2, \alpha_1' h_3, h_1 \alpha_2', \beta_1' h_4, h_1 \beta_2', \gamma_1' h_4, h_2 \gamma_2' \in A_2.
\end{align*}
\]

By Lemma 3.3 and by symmetry, we may assume that, there exist some elements \( \alpha, \beta, \gamma, \lambda \in A_2 \cup A_1 \) such that one of the following conditions holds:

(A1) \( \phi(y_{i_1}) = \alpha \alpha, \phi(y_{i_1+1}) = \beta \beta, \phi(z_{i_2}) = \gamma \gamma \) and \( \phi(z_{i_2+1}) = \delta \delta \), for some elements \( \{a, b\} = \{h_1, h_3\} \) and \( \{c, d\} = \{h_2, h_4\} \);

(A2) \( \phi(y_{i_1}) = \alpha h_1, \phi(y_{i_1+1}) = h_4 \beta, \phi(z_{i_2}) = \gamma h_1 \) and \( \phi(z_{i_2+1}) = h_3 \lambda \);

(A3) \( \phi(y_{i_1}) = \alpha h_1, \phi(y_{i_1+1}) = h_4 \beta, \phi(z_{i_2}) = \gamma h_3 \) and \( \phi(z_{i_2+1}) = h_1 \lambda \);

(A4) \( \phi(y_{i_1}) = \alpha h_1, \phi(y_{i_1+1}) = h_1 \beta, \phi(z_{i_2}) = \gamma h_2 \) and \( \phi(z_{i_2+1}) = h_4 \lambda \);

(A5) \( \phi(y_{i_1}) = \alpha h_4, \phi(y_{i_1+1}) = h_1 \beta, \phi(z_{i_2}) = \gamma h_4 \) and \( \phi(z_{i_2+1}) = h_2 \lambda \).

We obtain that \( y_1 y_{i_1+1} \neq z_{i_2} z_{i_2+1} \) and \( \phi(y_1 y_{i_1+1}), \phi(z_{i_2} z_{i_2+1}) \in A_3 \), the words \( y_1 y_{i_1+1} \) and \( z_{i_2} z_{i_2+1} \) do not occur in the words \( y_1 \cdots y_{i_1+1} \) and \( z_1 \cdots z_{i_2+1} \). By Condition (4).(a), we have \( y_i \neq z_{i_2} \). Also, by considering Conditions (A1), (A2), (A3), (A4) and (A5), we have \( y_{i_1+1} \neq z_{i_2+1} \).

First, suppose that Condition (A1) holds. If \( a = h_1 \), then by Condition (4).(e).(i), we have \( d = h_2 \). Then, we have \( b = h_3 \) and \( c = h_4 \). Also, if \( a = h_3 \), then we have \( b = h_1 \) and by Condition (4).(e).(ii), we have \( c = h_2 \). Then, we have \( d = h_4 \). Then, we have \( (a, c) = (h_1, h_4) \) or \( (b, d) = (h_1, h_4) \).
By Condition (4).(e).(iii), there exists a letter $u_1 \in Y$ such that $y_i u_1$ occurs in $y_1 \cdots y_i$ or $z_1 \cdots z_i$ and also $z_{i_2} u_1$ occurs in $y_1 \cdots y_{i_1}$ or $z_1 \cdots z_{i_2}$. Hence, we have $\phi(y_i u_1), \phi(z_{i_2} u_1) \in A_2$. If $a = h_1$ and $c = h_4$, then we have $y_i u_1 \in A_3$ or $z_{i_2} u_1 \in A_3$, a contradiction. Because, there does not exist an element $h \in A_2$ such that $h_1 h, h_4 h \in A_2$. Also, by Condition (4).(e).(iv), there exists a letter $u_2 \in Y$ such that $u_2 y_i u_1$ occurs in $y_1 \cdots y_i$ or $z_1 \cdots z_{i_2}$ and also $u_2 z_{i_2} u_1$ occurs in $y_1 \cdots y_{i_1}$ or $z_1 \cdots z_{i_2}$. Then, we have $\phi(u_2 y_i u_1), \phi(u_2 z_{i_2} u_1) \in A_2$. If $b = h_1$ and $d = h_4$, then we have $u_2 y_i u_1 + A_3$ or $u_2 z_{i_2} u_1 \in A_3$, a contradiction. Therefore, Condition (A1) does not hold. Also, if Conditions (A2) or (A5) holds, then Condition (4).(e).(i) fails. Similarly, if Condition (A3) or (A4) holds, then Condition (4).(e).(iv) fails. Therefore, $\phi(w_Y)$ and $\phi(v_Y)$ are in the same $\mathcal{R}$-class.

Similarly, by Lemma 3.2, $\phi(w_Y)$ and $\phi(v_Y)$ are in the same $\mathcal{L}$-class and, thus, $J_5$ satisfies the identity $w = v$. □

By Theorem 4.6, the following corollary easily follows.

**Corollary 4.7.** The monoid $J_5$ satisfies the identities $x^3 = x^2$ and does not satisfy the identity $x^2 = x$. Also, if $J_5$ satisfies the identity $w = v$, then we have $|w_x| \geq 2$ if and only if $|v_x| \geq 2$, for every $x \in X$.

Note that, in Theorem 4.6, if $xy$ and $zt$ do not occur in $w_1 x$ and $v_1 z$, and $y = t$, for a subset $Y \in c(w)$, $J_5$ may satisfy the identity $w = v$. Also, if $xy$ and $zt$ do not occur in $yw_2$ and $tv_2$, and $x = z$, $J_5$ may satisfy the identity $w = v$. For example, by Theorem 4.6, $J_5$ satisfies the following identities:

$$xu^2 xu^2 xu^2 xu^2 x^2 = xu^2 xu^2 xu^2 xu^2 x^2,$$

$$y^2 z^2 x^2 x^2 y^2 x^2 y^2 = y^2 z^2 x^2 x^2 y^2 x^2 y^2,$$

$$x^2 t^2 \cdot u^2 x^2 u^2 x^2 = x^2 t^2 \cdot u^2 x^2 u^2 x^2,$$

$$x^2 y^2 \cdot t^2 x^2 y^2 t^2 x^2 = x^2 y^2 \cdot t^2 x^2 y^2 t^2 x^2.$$

Also, note that, there exists an identity such that $xy \neq zt$, $x \neq z$, $y \neq t$ and, $xy$ and $zt$ do not occur in the elements of $C_1$ or $C_2$. For example, for $C_1$, we have the following identity that satisfies all conditions of Theorem 4.6

$$y x y x z x y z x z y z y^2 x^2 = y x y x z x y z x z y z y^2 x^2.$$

In [9], the authors characterize the identities of $J_4$. They show that for words $w$ and $v$ of $X^*$, the monoid $J_4$ satisfies the identity $w = v$ if $c(w) = c(v)$ and for every non empty subset $Y$ of $c(w)$, Conditions (1), (2) and (3) of Theorem 4.6 hold. Hence, Condition (4) of Theorem 4.6 has a key role to recognize the identities of $J_4$ which $J_5$ does not satisfy them.

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