Convergence analysis of the shrinking approximants for fixed point problem and generalized split common null point problem

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Abstract
In this paper, we compute a common solution of the fixed point problem (FPP) and the generalized split common null point problem (GSCNPP) via the inertial hybrid shrinking approximants in Hilbert spaces. We show that the approximants can be easily adapted to various extensively analyzed theoretical problems in this framework. Finally, we furnish a numerical experiment to analyze the viability of the approximants in comparison with the results presented in (Reich and Tuyen in Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 114:180, 2020).

MSC: 47H05; 47H10; 47J25; 49M30; 54H25

Keywords: Shrinking approximants; Strong convergence; Fixed point problem; Demimetric operator; Generalized null point problem

1 Introduction
The triplet \((\Xi, \langle \cdot, \cdot \rangle, \| \cdot \|)\) represents a real Hilbert space, the inner product, and the induced norm, respectively. For an operator \(U : K \to K\), \(\text{Fix}(U)\) denotes the set of all fixed points of the operator \(U\), where \(K\) is a nonempty closed convex subset of \(\Xi\). Recall that the operator \(U\) is called \(\eta\)-demimetric [46], where \(\eta \in (-\infty, 1)\), if \(\text{Fix}(U) \neq \emptyset\) and

\[
(p - q, (I_d - U)p) \geq \frac{1}{2}(1 - \eta)\| (I_d - U)p \|^2 \quad \text{for all } p \in K \text{ and } q \in \text{Fix}(U),
\]

where \(I_d\) denotes the identity operator.

The \(\eta\)-demimetric operator is equivalently defined by

\[
\|Up - q\|^2 \leq \|p - q\|^2 + \eta \|p - Up\|^2 \quad \text{for all } p \in K \text{ and } q \in \text{Fix}(U).
\]

The class of \(\eta\)-demimetric operators plays a prominent role in metric fixed point theory and has been analyzed in various instances of fixed point problems [47, 48, 50]. We remark that various nonlinear operators have been analyzed in connections with variational inequality problems, fixed point problems, equilibrium problems, convex feasibility
problems, signal processing, and image reconstruction [3–6, 11, 12, 14–19, 23, 24, 26, 28–30, 32, 34, 35, 37, 42, 43, 53–57]. In 2007, Aoyama et al. [2] suggested a Halpern [33] type approximants for an infinite family of nonexpansive operators satisfying the AKTT-Condition \( \sum_{k=1}^{\infty} \sup_{p \in X} \| U_{k+1} p - U_k p \| < \infty \) for any bounded subset \( X \) of \( \Xi \). The following construction of operator \( S_k \) for a countably infinite family of \( \eta \)-demimetric operators does not require the AKTT-Condition and hence improves the performance of the approximants:

\[
Q_{k,1} = I d,

Q_{k,k} = \lambda_k U_k' Q_{k,k+1} + (1 - \lambda_k) I d,

Q_{k,k-1} = \lambda_{k-1} U_{k-1}' Q_{k,k} + (1 - \lambda_{k-1}) I d,

\vdots

Q_{k,m} = \lambda_m U_m' Q_{k,m+1} + (1 - \lambda_m) I d,

\vdots

Q_{k,2} = \lambda_2 U_2' Q_{k,3} + (1 - \lambda_2) I d,

S_k = Q_{k,1} = \lambda_1 U_1' Q_{k,2} + (1 - \lambda_1) I d,
\]

where \( 0 \leq \lambda_m \leq 1 \) and \( U_m' = \rho p + (1 - \rho)((1 - \gamma) I d + \gamma U_m)p \) for all \( p \in K \) with \( U_m \) being \( \eta \)-demimetric operator, \( \rho \in (0,1) \), and \( 0 < \gamma < 1 - \eta \). It is well known in the context of operator \( S_k \) that each \( U_m' \) is nonexpansive and the limit \( \lim_{k \to \infty} Q_{k,m} \) exists. Moreover,

\[
Sp = \lim_{k \to \infty} S_k p = \lim_{k \to \infty} Q_{k,1} p \quad \text{for all } p \in K.
\]

This implies that \( \text{Fix}(S) = \bigcap_{k=1}^{\infty} \text{Fix}(S_k) \) [36, 49].

The following concept of a split convex feasibility problem (SCFP) is presented in [20]:

Let \( H \) and \( W \) be nonempty closed convex subsets of real Hilbert spaces \( \Xi_1 \) and \( \Xi_2 \), respectively. In SCFP, we compute

\[
p \in H \quad \text{such that } Vp \in W,
\]

where \( V : \Xi_1 \to \Xi_2 \) is a bounded linear operator. The SCFP is a particular case of the following split common null point problem (SCNPP) of maximal monotone operators:

Let \( A_1 \subseteq \Xi_1 \times \Xi_1 \) and \( A_2 \subseteq \Xi_2 \times \Xi_2 \) be two monotone operators such that \( \Gamma = A_1^{-1}(0) \cap V^{-1}(A_2^{-1}(0)) \neq \emptyset \). In SCNPP, we compute \( p \in \Gamma \). Some interesting results on the SCNPP
via iterative approximants can be found in [13, 21, 22, 44]. It is worth mentioning that the concept of SCNPP has been extended to the concept of a generalized split common null point problem (GSCNPP) in Hilbert spaces [39, 40]. In GSCNPP, we compute

\[ p \in \Gamma := A_1^{-1}(0) \cap V_1^{-1}(A_2^{-1}(0)) \cap \cdots \cap V_1^{-1}(V_2^{-1} \cdots (V_{N-1}^{-1}(A_N^{-1}(0)))) , \]  

(1.2)

where \( A_j : \mathcal{X}_j \to 2^{\mathcal{X}_j}, j \in \{1, 2, \ldots, N\} \), is a finite family of maximal monotone operators, and \( V_j : \mathcal{X}_j \to \mathcal{X}_{j+1}, j \in \{1, 2, \ldots, N-1\} \), is a finite family of bounded linear operators such that \( V_j \neq 0 \).

From the perspective of optimization, problem (1.2) has been analyzed via different iterative approximants. A variant of the classical CQ-algorithm, essentially due to Byrne [13], is employed in [39], whereas shrinking projection approximants are analyzed in [40] to obtain the strong convergence results in Hilbert spaces. It is therefore natural to ask whether we can devise strongly convergent approximants to compute a solution of GSCNPP and fixed point point problem of an infinite family of operators without employing the AKTT-Condition.

To answer the above question, we consider the following GSCNPP and FPP:

\[ p \in \Omega = \Gamma \cap \text{Fix}(S) \neq \emptyset . \]

(1.3)

For the computation of a solution of problem (1.3), we employ hybrid shrinking approximants embedded with the inertial extrapolation technique, essentially due to Polyak [38] (see also [1, 7–10]), in Hilbert spaces.

The rest of the paper is organized as follows: In Sect. 2, we present mathematical preliminaries. We establish strong convergence results of the approximants and their variant, namely the Halpern-type approximants in Sect. 3. In Sect. 4, we elaborate the adaptability of the approximants for various extensively analyzed theoretical problems in this framework. Section 6 provides a numerical experiment to analyze the viability of the approximants in comparison with the existing results.

2 Preliminaries

We start this section with mathematical preliminary notions. We always assume that \( K \) is a nonempty closed convex subset of a real Hilbert space \( \mathcal{X}_1 \).

Recall the nearest point projector \( \Pi_K^{\mathcal{X}_1} \) of \( \mathcal{X}_1 \) onto \( K \subset \mathcal{X}_1 \) is defined so that for every \( p \in \mathcal{X}_1 \), we have a unique \( \Pi_K^{\mathcal{X}_1} p \) in \( K \) such that

\[ \| p - \Pi_K^{\mathcal{X}_1} p \| \leq \| p - q \| \quad \text{for all } q \in K . \]

Note here that the nearest point projector has the following properties:

(i) \[ \| \Pi_K^{\mathcal{X}_1} p - \Pi_K^{\mathcal{X}_1} q \|^2 \leq (p - q, \Pi_K^{\mathcal{X}_1} p - \Pi_K^{\mathcal{X}_1} q) \] for all \( p, q \in K \) (firmly nonexpansive);

(ii) \[ (p - \Pi_K^{\mathcal{X}_1} p, \Pi_K^{\mathcal{X}_1} p - q) \geq 0 \] for all \( p \in \mathcal{X}_1 \) and \( q \in K \) (characterization property).
Let the sets $D(A_1) = \{ p \in \Xi_1 \mid A_1 p \neq \emptyset \}$, $\mathcal{R}(A_1) = \{ u \in \Xi_1 \mid (\exists p \in \Xi_1) u \in A_1(p) \}$, $\mathcal{G}(A_1) = \{(p,u) \in \Xi_1 \times \Xi_1 \mid u \in A_1(p) \}$, and $\mathcal{Z}(A_1) = \{ p \in \Xi_1 \mid 0 \in A_1(p) \}$ denote the domain, range, graph, and zeros of a set-valued operator $A_1 \subseteq \Xi_1 \times \Xi_1$, respectively. If a set-valued operator $A_1$ satisfies $(p - q, t - w) \geq 0$ for all $(p,t),(q,w) \in \text{gra}(A_1)$, then $A_1$ is called monotone. Recall also that a monotone operator $A_1$ is called maximal monotone if its graph is not strictly contained in the graph of any other monotone operator on $\Xi_1$. The well-defined single-valued operator $J_{\theta}^{A_1} := ((\text{Id} + \theta A_1))^{-1} : \mathcal{R}(\text{Id} + \theta A_1) \to D(A_1)$ is known as the resolvent of $A_1$, where $\theta > 0$. The resolvent operator $J_{\theta}^{A_1}$ is closely related to $A_1$ as $q \in A_1^{-1}(0)$ if and only if $q = J_{\theta}^{A_1}(q)$.

**Lemma 2.1** ([46]) Let $K \subset \Xi_1$, and let $U : K \to \Xi_1$ be an $\eta$-demimetric operator with $\eta \in (-\infty, 1)$. Then $\text{Fix}(U)$ is closed and convex.

**Lemma 2.2** ([50]) Let $K \subset \Xi_1$, and let $U : K \to \Xi_1$ be an $\eta$-demimetric operator with $\eta \in (-\infty, 1)$ and $\text{Fix}(U) \neq \emptyset$. Let $\gamma$ be a real number such that $0 < \gamma < 1 - \eta$ and set $M = (1 - \gamma)\text{Id} + \gamma U$. Then $M$ is a quasinonexpansive operator of $K$ into $\Xi_1$.

**Lemma 2.3** ([45]) Let $\Xi_1$ be a real Hilbert space, and let $(d_k)$ be a sequence in $\Xi_1$. Then:

(i) If $d_k \to d$ and $\|d_k\| \to \|d\|$ as $k \to \infty$, then $d_k \to d$ as $k \to \infty$ (the Kadec–Klee property);

(ii) If $d_k \to d$ as $k \to \infty$, then $\|d\| \leq \liminf_{k \to \infty} \|d_k\|$.

**Lemma 2.4** ([27]) Let $A_1 \subseteq \Xi_1 \times \Xi_1$ be a maximal monotone operator. Then for $\theta \geq \tilde{\theta} > 0$, we have

$$\|p - J_{\theta}^{A_1} p\| \leq 2\|p - J_{\tilde{\theta}}^{A_1} p\| \quad \text{for all } p \in \Xi_1.$$ 

**Lemma 2.5** ([31]) Let $K \subset \Xi_1$. Then the operator $\text{Id} - U$ satisfies the demiclosedness principle with respect to the origin, that is, $(\text{Id} - U)(d) = 0$, provided that there exists a sequence $(d_k)$ in $K$ that converges weakly to some $d$ and $((\text{Id} - U)d_k)$ converges strongly to $0$.

**Lemma 2.6** ([52]) Let $K \subset \Xi_1$, and let $(U_{m}^\ast)$ be a sequence of nonexpansive operators such that $\bigcap_{k=1}^{\infty} \text{Fix}(U_{m}) \neq \emptyset$ and $0 \leq \beta_m \leq b < 1$. Then for a bounded subset $D$ of $K$, we have

$$\limsup_{k \to \infty, p \in D} \|S_p - S_k p\| = 0.$$

### 3 Convergence analysis of the approximants

For the computation of a solution of (1.3), we propose the following approximants:
Algorithm 1 Hybrid Shrinking Approximants (Algo.1)

Initialization: Suppose $H_0 = W_0 = \Xi_1$. Assume nonincreasing sequences $\rho_k, \tau_k \subset (0, 1)$, $\mu_k \in [0, 1)$, $\gamma_k \in (0, \infty)$, and let $V_0$ be the identity operator on $\Xi_1$ and $\bar{V}_{j-1} = \{V_{j-1}, V_{j-2}, \ldots, V_0\}$ for all $j \in \{1, 2, \ldots, N\}$. Choose the inertial parameter

$$\mu_k = \begin{cases} \min\{\frac{\tau_k}{\|d_k - d_{k-1}\|}, \mu\} & \text{if } d_k \neq d_{k-1}; \\ \mu & \text{otherwise,} \end{cases}$$

where $\mu \in [0, 1)$, and $(\tau_k)$ is a sequence of positives satisfying $\sum_{k=1}^{\infty} \tau_k < \infty$.

Step 0. Choose arbitrarily $d_0, d_1 \in \Xi_1$ and set $k \geq 1$;

Iterative Steps: Given $d_k \in \Xi_k$, calculate $a_k, b_k,$ and $c_k$ as follows:

Step 1. Compute

$$\begin{align*}
  a_k &= (1 + \mu_k)d_k - \mu_k d_{k-1}; \\
  b_k &= \rho_k a_k + (1 - \rho_k) S_k a_k; \\
  c_k &= \tau_k \bar{V}_{j-1} b_k + (1 - \tau_k) (\bar{J}_{1/k} (\bar{V}_{j-1} b_k)).
\end{align*}$$

The approximants abort if $a_k = b_k = c_k = d_k$, and then $d_k$ is the required approximation. Otherwise,

Step 2. Compute

$$\begin{align*}
  H_{k+1} &= \{ p \in \Xi_1 : \| b_k - p \|^2 \leq \| d_k - p \|^2 + \mu_k^2 \| d_k - d_{k-1} \|^2 + 2 \mu_k \langle d_k - p, d_k - d_{k-1} \rangle \}, \\
  W_{k+1} &= \{ p \in \Xi_1 : \| c_k - \bar{V}_{j-1} p \| \leq \| \bar{V}_{j-1} b_k - \bar{V}_{j-1} p \| \}, \\
  d_{k+1} &= \cap_{k=1}^{\infty} H_{k+1} \cap W_{k+1} \quad \forall k \geq 1.
\end{align*}$$

Fix $k = k + 1$ and reiterate Step 1.

We assume the following control conditions on the approximants:

(C1) $\sum_{k=1}^{\infty} \mu_k \| d_{k-1} - d_k \| < \infty$;
(C2) $0 < a \leq \rho_k \leq b < 1$;
(C3) $\lim \inf_{k \to \infty} \tau_k > 0$;
(C4) $\min\{\inf_k [\theta_{k, k}]\} \geq m > 0$.

Theorem 3.1 Any approximants defined via Algorithm 1, under control conditions (C1)–(C4), converge strongly to an element in $\Omega$.

Proof We divide the proof into different steps for understanding.

Step 1. We show that the approximants $(d_k)$ defined in Algorithm 1 are stable. Claim: $H_k$ and $W_k$ are closed and convex subsets of $\Xi_1$ for all $k \geq 0$.

Consider, for each $k \geq 0$, the following representation of the subsets $H_k$ and $W_k$:

$$H_k = \{ p \in \Xi_1 : \langle b_k - \bar{V}_{j-1} p, b_k - \bar{V}_{j-1} p \rangle \leq \frac{1}{2} (\| b_k \|^2 - \| d_k \|^2 + \mu_k^2 \| d_k - d_{k-1} \|^2 + 2 \mu_k \langle d_k - p, d_k - d_{k-1} \rangle) \}. $$
\[ W_k = \left\{ p \in \Xi_1 : (\tilde{V}_{j-1}b_k - c_k, \tilde{V}_{j-1}p) \leq \frac{1}{2}(\| \tilde{V}_{j-1}b_k \|^2 - \| c_k \|^2) \right\} , \]

\[ = \left\{ p \in \Xi_1 : (\tilde{V}_{j-1}^* (\tilde{V}_{j-1}b_k - c_k), p) \leq \frac{1}{2}(\| \tilde{V}_{j-1}b_k \|^2 - \| c_k \|^2) \right\} . \]

The claim follows from the above representations of closed and convex subsets \( H_k \) and \( W_k \) of \( \Xi_1 \) for all \( k \geq 1 \). Further, the sets \( \Gamma \) and \( \text{Fix}(S) \) (from Lemma 2.1) are closed and convex. Hence we have that \( \Omega \) is nonempty, closed, and convex. Let \( q \in \Omega, \Omega \subset H_0 = \Xi_1 \). Now it follows from Algorithm 1 that

\[
\| a_k - q \|^2 = \| (1 + \mu_k)d_k - \mu_k(d_{k-1} - q) \|^2 \\
\leq \| d_k - q \|^2 + \mu_k^2\| d_k - d_{k-1} \|^2 + 2\mu_k\langle d_k - q, d_k - d_{k-1} \rangle. \quad (3.1)
\]

From (3.1) and Lemma 2.2 we obtain

\[
\| b_k - q \|^2 = \| \rho_ka_k + (1 - \rho_a)S_tak - q \|^2 \\
\leq \rho_a \| a_k - q \|^2 + (1 - \rho_a)\| S_tak - q \|^2 + \rho_a(1 - \rho_a)\| (ld - S_t)a_k \|^2 \\
\leq \rho_a \| a_k - q \|^2 + (1 - \rho_a)\| a_k - q \|^2 + \rho_a(1 - \rho_a)\| (ld - S_t)a_k \|^2 \\
\leq \| a_k - q \|^2 + \rho_a(1 - \rho_a)\| (ld - S_t)a_k \|^2 \\
\leq \| d_k - q \|^2 + \mu_k^2\| d_k - d_{k-1} \|^2 + 2\mu_k\langle d_k - q, d_k - d_{k-1} \rangle. \quad (3.2)
\]

This shows that \( \Omega \) is contained in \( H_k \) for all \( k \geq 1 \). Now assume that \( \Omega \subset W_k \) for some \( k \geq 1 \). Using the nonexpansiveness of \( J_{\theta,j}^A \), (3.1), and (3.2), we get

\[
\| c_k - \tilde{V}_{j-1}q \|^2 = \| \tau_k(\tilde{V}_{j-1}b_k - \tilde{V}_{j-1}q) + (1 - \tau_k)(J_{\theta,j}^A(\tilde{V}_{j-1}b_k) - \tilde{V}_{j-1}q) \|^2 \\
\leq \tau_k\| \tilde{V}_{j-1}b_k - \tilde{V}_{j-1}q \|^2 + (1 - \tau_k)\| J_{\theta,j}^A(\tilde{V}_{j-1}b_k) - J_{\theta,j}^A(\tilde{V}_{j-1}q) \|^2 \\
\leq \tau_k\| \tilde{V}_{j-1}b_k - \tilde{V}_{j-1}q \|^2 + (1 - \tau_k)\| \tilde{V}_{j-1}b_k - \tilde{V}_{j-1}q \|^2 \\
\leq \| \tilde{V}_{j-1}b_k - \tilde{V}_{j-1}q \|^2. \quad (3.3)
\]

It follows from estimate (3.3) that \( \Omega \subset W_{k+1} \), and hence \( \Omega \subset H_{k+1} \cap W_{k+1} \). Consequently, by Step 1 the approximants \( (d_k) \) defined in Algorithm 1 are stable.

**Step 2.** We next show that \( \lim_{k \to \infty} \| d_k - d_1 \| \) exists.

Observe that

\[
\| d_{k+1} - d_1 \| \leq \| q - d_1 \| \quad \text{for all } q \in \Omega \subset H_{k+1},
\]

since \( d_{k+1} = \Pi_{H_{k+1} \cap W_{k+1}}d_1 \). In particular,

\[
\| d_{k+1} - d_1 \| \leq \| \Pi_{\Xi_1} d_1 - d_1 \|. 
\]

These estimates establish the boundedness of the approximants \( (\| d_k - d_1 \|) \). Since \( d_k = \Pi_{H_k \cap W_k}d_1 \) and \( d_{k+1} = \Pi_{H_{k+1} \cap W_{k+1}}d_1 \in H_{k+1} \), we have

\[
\| d_k - d_1 \| \leq \| d_{k+1} - d_1 \|. 
\]
This yields that the approximants \( \|d_k - d_1\| \) are nondecreasing, and hence
\[
\lim_{k \to \infty} \|d_k - d_1\| \text{ exists.} \tag{3.4}
\]

**Step 3.** We now show that \( \tilde{q} \in \Omega \).

We first compute
\[
\|d_{k+1} - d_k\|^2 = \|d_{k+1} - d_1\|^2 + \|d_k - d_1\|^2 - 2\langle d_k - d_1, d_{k+1} - d_1 \rangle
\]
\[
= \|d_{k+1} - d_1\|^2 + \|d_k - d_1\|^2 - 2\langle d_k - d_1, d_{k+1} - d_k + d_k - d_1 \rangle
\]
\[
= \|d_{k+1} - d_1\|^2 - \|d_k - d_1\|^2 - 2\langle d_k - d_1, d_{k+1} - d_k \rangle
\]
\[
\leq \|d_{k+1} - d_1\|^2 - \|d_k - d_1\|^2.
\]

By (3.3) the above computation yields
\[
\lim_{k \to \infty} \|d_{k+1} - d_k\| = 0. \tag{3.5}
\]

In view of the control condition (C1), we get
\[
\lim_{k \to \infty} \|a_k - d_k\| = \lim_{k \to \infty} \mu_k \|d_k - d_{k-1}\| = 0. \tag{3.6}
\]

As a consequence of estimates (3.5) and (3.6), we also obtain that
\[
\lim_{k \to \infty} \|a_k - d_{k+1}\| = 0. \tag{3.7}
\]

Since \( d_{k+1} \in H_{k+1} \), we have
\[
\|b_k - d_{k+1}\| \leq \|d_k - d_{k+1}\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - d_{k+1}, d_k - d_{k-1} \rangle.
\]

This estimate, in the light of estimate (3.5) and the control condition (C1), yields that
\[
\lim_{k \to \infty} \|b_k - d_{k+1}\| = 0. \tag{3.8}
\]

Similarly, we infer from estimates (3.5) and (3.8) that
\[
\lim_{k \to \infty} \|b_k - d_k\| = 0, \tag{3.9}
\]

and from the estimates (3.6) and (3.9) that
\[
\lim_{k \to \infty} \|b_k - a_k\| = 0. \tag{3.10}
\]

In view of the control condition (C2), consider the variant of estimate (3.2)
\[
a(1 - b) \| (Id - S_k) a_k \|^2 \leq \|d_k - q\|^2 - \|b_k - q\|^2
\]
\[
\leq (\|d_k - q\| + \|b_k - q\|) \|d_k - b_k\|.
\]
Letting $k \to \infty$ and using (3.9) and (C2), we have

$$
\lim_{k \to \infty} \|a_k - S_k a_k\| = 0. \tag{3.11}
$$

Observe that

$$
\|b_k - S_k b_k\| \leq \|b_k - a_k\| + \|a_k - S_k a_k\| + \|S_k a_k - S_k b_k\|
\leq 2\|b_k - a_k\| + \|a_k - S_k a_k\|.
$$

The above computation, in view of estimates (3.10) and (3.11), yields

$$
\lim_{k \to \infty} \|b_k - S_k b_k\| = 0. \tag{3.12}
$$

Note that $d_{k+1} = \Pi_{H_k \cap W_k}^\Xi (d_k) \in W_k$. Therefore we have

$$
\|c_k - \bar{V}_{j-1} d_{k+1}\| \leq \|\bar{V}_{j-1} b_k - \bar{V}_{j-1} d_{k+1}\| \leq \|\bar{V}_{j-1}\| \|b_k - d_{k+1}\|.
$$

Employing estimate (3.8), the above computation yields

$$
\lim_{k \to \infty} \|c_k - \bar{V}_{j-1} d_{k+1}\| = 0. \tag{3.13}
$$

Reasoning as above, we infer from estimates (3.8) and (3.13) that

$$
\lim_{k \to \infty} \|c_k - \bar{V}_{j-1} b_k\| = 0 \tag{3.14}
$$

and from estimates (3.9) and (3.14) that

$$
\lim_{k \to \infty} \|c_k - \bar{V}_{j-1} d_k\| = 0. \tag{3.15}
$$

Since $(d_k)$ is bounded, there exists a subsequence $(d_{k_i})$ of $(d_k)$ such that $d_{k_i} \to \tilde{q} \in \Xi_1$ as $t \to \infty$. Therefore $b_{k_i} \to \tilde{q}$ and $c_{k_i} \to \bar{V}_{j-1} \tilde{q}$ as $t \to \infty$. From the definition of $\bar{V}_{j-1}$ we have $\bar{V}_{j-1} d_{k_i} \to \bar{V}_{j-1} \tilde{q}$ as $t \to \infty$ for all $j \in \{1, 2, \ldots, N\}$.

Using (3.14), we estimate that

$$
\lim_{t \to \infty} \|\bar{V}_{j-1} c_{k_i} - J^A_{\tilde{q}, \tilde{g}} (\bar{V}_{j-1} c_{k_i})\| = 0
$$

for all $j \in \{1, 2, \ldots, N\}$. Then from Lemma 2.4 and (C4) we obtain the inequality

$$
\|\bar{V}_{j-1} c_{k_i} - J^A_{\tilde{q}} (\bar{V}_{j-1} c_{k_i})\| \leq 2 \|\bar{V}_{j-1} c_{k_i} - J^A_{\tilde{q}, \tilde{g}} (\bar{V}_{j-1} c_{k_i})\|.
$$

This estimate implies that

$$
\lim_{t \to \infty} \|\bar{V}_{j-1} c_{k_i} - J^A_{\tilde{q}, \tilde{g}} (\bar{V}_{j-1} c_{k_i})\| = 0 \tag{3.16}
$$
for all \( j \in \{1, 2, \ldots, N\} \). By Lemma 2.5 we have \( \tilde{V}_{j-1} \tilde{q} \in \text{Fix}(T_J^{ \alpha_j}) \) for all \( j \in \{1, 2, \ldots, N\} \), that is, \( \tilde{q} \in \Gamma \). It remains to show that \( \tilde{q} \in \text{Fix}(S) \). Observe that

\[
\| b_k - Sb_k \| \leq \| b_k - Sb_k \| + \| Sb_k - Sb_k \| \\
\leq \| b_k - Sb_k \| + \sup_{p \in D} \| Sp - Sp \|. 
\]

Using (3.12) and Lemma 2.6, this estimate implies that \( \lim_{k \to \infty} \| b_k - Sb_k \| = 0 \). This, together with the fact that \( b_k \to \tilde{q} \), implies by Lemma 2.5 that \( \tilde{q} \in \text{Fix}(S) = \bigcap_{k=1}^{\infty} \text{Fix}(S_k) \). Hence \( \tilde{q} \in \Omega \).

**Step 4.** The final part is showing that \( d_k \to q = \Pi_\Omega^{\tilde{q}} d_1 \).

Since \( q = \Pi_\Omega^{\tilde{q}} d_1 \) and \( \tilde{q} \in \Omega \), Lemma 2.3 implies that

\[
\| q - d_1 \| \leq \| \tilde{q} - d_1 \| \leq \liminf_{t \to \infty} \| d_k - d_1 \| \\
\leq \limsup_{t \to \infty} \| d_k - d_1 \| \leq \| q - d_1 \|. 
\]

Using the uniqueness of \( q \) yields the equality \( \tilde{q} = q \). From Step 2 it follows that \( \| d_k - d_1 \| \leq \| q - d_1 \| \), and from Lemma 2.3 we obtain \( \lim_{k \to \infty} d_k = \tilde{q} = q = \Pi_\Omega^{\tilde{q}} d_1 \). \( \square \)

We first apply Theorem 3.1 to the following problem:

\[
\Omega := \bigcap_{j=1}^{s} A_j^{-1}(0) \cap V^{-1} \left( \bigcap_{j=s+1}^{N} A_j^{-1}(0) \right) \cap \text{Fix}(S),
\]

where \( A_j \subseteq \Xi_1 \times \Xi_1, j \in \{1, 2, \ldots, s\} \), and \( A_\ell \subseteq \Xi_1 \times \Xi_1, \ell = \{s+1, s+2, \ldots, N\} \), are finite families of maximal monotone operators, and \( S_k : \Xi_1 \to \Xi_1 \) is an infinite family of \( \eta \)-demimetric operators.

**Corollary 3.2** Assume that \( \Omega \neq \emptyset \). Then the approximants initialized by arbitrary \( d_0, d_1 \in \Xi_1 \) and \( H_0 = W_0 = \Xi_1 \) with the nonincreasing sequences \( \rho_k, \tau_k \subset (0, 1), \mu_k \in [0, 1) \), and \( \gamma_k \in (0, \infty) \) for \( k \geq 1 \) defined as

\[
\begin{align*}
a_k &= (1 + \mu_k)d_k - \mu_k d_{k-1}, \\
b_k &= \rho_k a_k + (1 - \rho_k)S_k a_k, \\
c_k &= \tau_k \tilde{V}_{j-1} b_k + (1 - \tau_k)(\sigma_j \tilde{V}_{j-1} b_k), \\
H_{k+1} &= \{ p \in \Xi_1 : \text{cond} \}, \\
W_{k+1} &= \{ p \in \Xi_1 : \| c_k - \tilde{V}_{j-1} p \| \leq \| \tilde{V}_{j-1} b_k - \tilde{V}_{j-1} p \| \}, \\
d_{k+1} &= \Pi_{H_{k+1} \cap W_{k+1}}^{\tilde{q}} d_1 \quad \forall k \geq 1,
\end{align*}
\]

under the control conditions (C1)–(C4), converge strongly to an element in \( \Omega \).

We now consider the following Halpern-type variant of Algorithm 1:
Algorithm 2 Hybrid Shrinking Halpern Approximants (Algo.2)

Initialization: Choose arbitrarily $t, d_0, d_1 \in \mathcal{X}_1$, and $H_0 = W_0 = \mathcal{X}_1$, set $k \geq 1$ and non-increasing sequences $\rho_k, \tau_k \subset (0, 1)$, $\mu_k \in [0, 1)$, let $V_0$ be the identity operator on $\mathcal{X}_1$, and let $V_j = \{V_1, V_2, \ldots, V_N\}$ for all $j \in \{1, 2, \ldots, N\}$. Choose the inertial parameter

$$
\mu_k = \begin{cases} 
\min\{\frac{\tau_k}{\|d_k - d_{k-1}\|}, \mu\} & \text{if } d_k \neq d_{k-1}, \\
\mu & \text{otherwise},
\end{cases}
$$

where $\mu \in [0, 1)$, and $(\tau_k)$ is a sequence of positives satisfying $\sum_{k=1}^{\infty} \tau_k < \infty$.

Iterative Steps: Given $d_k \in \mathcal{X}_k$, calculate $b_k$ and $c_k$ as follows:

**Step 1.** Compute

$$
\begin{align*}
& a_k = (1 + \mu_k)d_k - \mu_k d_{k-1}, \\
& b_k = \rho_k t + (1 - \rho_k)S_1 a_k, \\
& c_k = \tau_k \bar{V}_j b_k + (1 - \tau_k)(\bar{J}_{i,j}(\bar{V}_j b_k)).
\end{align*}
$$

The approximants abort if $k > k_{\text{max}}$ for some chosen sufficiently large number $k_{\text{max}}$, and then $d_k$ is the required approximation. Otherwise,

**Step 2.** Compute

$$
\begin{align*}
& H_{k+1} = \{p \in \mathcal{X}_1 : \|b_k - p\|^2 \leq \rho_k \|t - p\|^2 + (1 - \rho_k)(\|d_k - p\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 \\
& \quad + 2\mu_k \langle d_k - p, d_k - d_{k-1}\rangle)\}, \\
& W_{k+1} = \{p \in \mathcal{X}_1 : \|c_k - \bar{V}_{j-1} p\| \leq \|\bar{V}_{j-1} b_k - \bar{V}_{j-1} p\|, \\
& d_{k+1} = \Pi_{H_{k+1} \cap W_{k+1}} t \quad \forall k \geq 1.
\end{align*}
$$

Fix $k = k + 1$ and reiterate Step 1.

**Theorem 3.3** Any approximants defined via Algorithm 2, under the control conditions (C1)–(C4), converge strongly to an element in $\Omega$.

**Proof** Observe that for each $k \geq 1$, the subsets $H_k$ have the following form:

$$
H_k = \{p \in \mathcal{X}_1 : \|b_k - p\|^2 \leq \rho_k \|t - p\|^2 + (1 - \rho_k)(\|d_k - p\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 \\
\quad + 2\mu_k \langle d_k - p, d_k - d_{k-1}\rangle)\}.
$$

Arguing similarly as in the proof of Theorem 3.1 (Steps 1–2), we deduce that $\Omega$, $H_k$, and $W_k$ are closed and convex. Moreover, $\Omega \subset H_{k+1} \cap W_{k+1}$ for all $k \geq 1$. Furthermore, the sequence $(d_k)$ is bounded, and

$$
\lim_{k \to \infty} \|d_{k+1} - d_k\| = 0. \quad (3.18)
$$
Since $d_{k+1} = \Pi_{H_k \cap W_k} (t) \in H_k$, we have
\[ \|b_k - d_{k+1}\|^2 \leq \rho_k \|t - d_{k+1}\|^2 + (1 - \rho_k)(\|d_k - d_{k+1}\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k (d_k - z, d_k - d_{k-1})). \]

Letting $k \to \infty$, using (3.18) along (C1)–(C2), and the boundedness of $(d_k)$, we obtain
\[ \lim_{k \to \infty} \|b_k - d_{k+1}\| = 0. \]

Similarly, we get
\[ \lim_{k \to \infty} \|b_k - d_k\| = 0. \]

Let $b_k = \rho_k t + (1 - \rho_k)S_k a_k$. An easy calculation along (C1)–C2) implies that
\[ \|S_k a_k - a_k\| \leq \frac{1}{(1 - \rho_k)} \|b_k - a_k\| + \frac{\rho_k}{(1 - \rho_k)} \|t - a_k\|. \]

This estimate implies that
\[ \lim_{k \to \infty} \|S_k a_k - a_k\| = 0. \]

The rest of the proof of Theorem 3.3 follows immediately from the proof of Theorem 3.1 and is therefore omitted. □

4 Applications

Our main result in the previous section has various interesting applications of great importance in the field. We present some of these applications.

4.1 Generalized split feasibility problems

In the context of generalized split feasibility problems [20], we recall that the indicator function $j_K$ is a proper lower semicontinuous convex function (PCLS), where $K \subset \Xi_1$. Therefore $\partial j_K$, the subdifferential of $j_K$, satisfies the maximal monotonicity such that $\partial j_K (p) = N^K_p$, where $N^K_p$ denotes the normal cone of $K$ at $u$. From this we can deduce that $\partial j_K$ coincides with $\Pi_{\Xi_1}^1$. Assume that
\[ \Theta := K_1 \cap V^{-1}_1 (K_2) \cap \cdots \cap V^{-1}_1 (V^{-1}_2 \cdots (V^{-1}_{N-1} (K_N))), \]

where $K_j \subset \Xi_j$, $j \in\{1, 2, \ldots, N\}$.

**Theorem 4.1** Assume that $\Omega = \Theta \cap \text{Fix}(S) \neq \emptyset$. Then the approximants initialized by arbitrary $d_0, d_1 \in \Xi_1$ and $H_0 = W_0 = \Xi_1$ with the nonincreasing sequences $\rho_k, \tau_k \subset (0, 1)$,
\( \mu_k \in [0, 1), \) and \( \gamma_k \in (0, \infty) \) for \( k \geq 1 \) defined as

\[
\begin{align*}
\mu_k &= (1 + \mu_k)d_k - \mu_k d_{k-1}, \\
b_k &= \rho_k a_k + (1 - \rho_k)S_k a_k, \\
c_k &= \tau_k \tilde{V}_{j-1} b_k + (1 - \tau_k) (\Pi_{k_j} \tilde{V}_{j-1} b_k), \\
H_{k+1} &= \{ z \in \Xi_1 : \\
&\quad \| b_k - z \|^2 \leq \| d_k - z \|^2 + \mu_k^2 \| d_k - d_{k-1} \|^2 + 2 \mu_k \langle d_k - z, d_k - d_{k-1} \rangle \}, \\
W_{k+1} &= \{ z \in \Xi_1 : \| c_k - \tilde{V}_{j-1} z \| \leq \| \tilde{V}_{j-1} b_k - \tilde{V}_{j-1} z \| \}, \\
d_{k+1} &= \Pi_{H_{k+1} \cap W_{k+1}} d_1 \quad \forall k \geq 1, \\
\end{align*}
\]

under the control conditions (C1)–(C4), converge strongly to an element in \( \Omega \).

### 4.2 Generalized split variational inequality problems

The well-known variational inequality problem deals with computation of a point \( p \in K \) such that

\[
\langle Ap, q - p \rangle \geq 0 \quad \forall q \in K,
\]

where \( A : K \to \Xi_1 \) is a nonlinear monotone operator defined with respect to \( K \subseteq \Xi_1 \).

By \( \text{Sol}(K, A) \) we denote the set of all solutions associated with the variational inequality problem. We consider the following problem:

\[
\Theta := \text{Sol}(K_1, A_1) \cap V_1^{-1}(\text{Sol}(K_2, A_2)) \cap \cdots \cap V_N^{-1}(\text{Sol}(K_N, A_N)) \neq \emptyset.
\]

**Theorem 4.2** Assume that \( \Omega = \Theta \cap \text{Fix}(S) \neq \emptyset \). Then the approximants initialized by arbitrary \( d_0, d_1 \in \Xi_1 \) and \( H_0 = W_0 = \Xi_1 \) with the nonincreasing sequences \( \rho_k, \tau_k \subset (0, 1), \mu_k \in [0, 1), \) and \( \gamma_k \in (0, \infty) \) for \( k \geq 1 \) defined as

\[
\begin{align*}
\mu_k &= (1 + \mu_k)d_k - \mu_k d_{k-1}, \\
b_k &= \rho_k a_k + (1 - \rho_k)S_k a_k; \\
c_k &= \tau_k \tilde{V}_{j-1} b_k + (1 - \tau_k) (\Pi_{k_j} \tilde{V}_{j-1} b_k), \\
H_{k+1} &= \{ z \in \Xi_1 : \\
&\quad \| b_k - z \|^2 \leq \| d_k - z \|^2 + \mu_k^2 \| d_k - d_{k-1} \|^2 + 2 \mu_k \langle d_k - z, d_k - d_{k-1} \rangle \}, \\
W_{k+1} &= \{ z \in \Xi_1 : \| c_k - \tilde{V}_{j-1} z \| \leq \| \tilde{V}_{j-1} b_k - \tilde{V}_{j-1} z \| \}, \\
d_{k+1} &= \Pi_{H_{k+1} \cap W_{k+1}} d_1 \quad \forall k \geq 1, \\
\end{align*}
\]

under the control conditions (C1)–(C4), converge strongly to an element in \( \Omega \).

**Proof** Let \( h_{A_j} \subset \Xi_j \times \Xi_j \) be defined by

\[
h_{A_j} p = \begin{cases} 
A_j p + N_{k_j}(p) & \text{if } p \in K_j, \\
\emptyset & \text{if } p \notin K_j,
\end{cases}
\]

where \( N_{k_j}(p) := \{ q \in \Xi_j : \langle t - p, q \rangle \leq 0 \text{ for all } t \in K_j \} , j = \{1, 2, \ldots, N\} \).
Note that \( h_{A_j} \) is maximal monotone [41] such that

\[
0 \in h_{A_j}(p) \iff p \in \text{Sol}(K_j, A_j) \iff p = \Pi_{K_j}(p - \theta_j A_j(p)).
\]

The rest of the proof now follows from Theorem 3.1. □

4.3 Generalized split minimization problems

Let the set of minimizers associated with the function \( \phi : \Xi_1 \to (-\infty, \infty] \) be denoted as

\[
\arg\min_{x \in \Xi_1} \phi(x) := \{ p \in \Xi_1 : \phi(p) \leq \phi(q) \forall q \in \Xi_1 \}.
\]

If \( \phi \) is a proper convex lower semicontinuous (PCLS) function, then \( \partial \phi \) is a maximal monotone operator. Moreover, \( q \in (\partial \phi)^{-1}0 \iff 0 \in \partial \phi(q) \) (see [25]). Now observe that

\[
\Theta := \arg\min_{x \in \Xi_1} \phi_1(x) \cap V_{-1}^{-1} \left( \arg\min_{x \in \Xi_2} \phi_2(x) \right) \cap \cdots \cap V_{-1}^{-1} \left( \arg\min_{x \in \Xi_N} \phi_N(x) \right) \neq \emptyset,
\]

where \( \phi_j : \Xi_j \to (-\infty, \infty] \) is as defined above.

**Theorem 4.3** Assume that \( \Omega = \Theta \cap \text{Fix}(S) \neq \emptyset \). Then the approximants initialized by arbitrary \( d_0, d_1 \in \Xi_1 \) and \( H_0 = W_0 = \Xi_1 \) with the nonincreasing sequences \( \rho_k, \tau_k \subset (0, 1), \mu_k \in [0, 1), \) and \( \gamma_k \in (0, \infty) \) for \( k \geq 1 \) defined as

\[
\begin{align*}
ak_k &= (1 + \mu_k)d_k - \mu_k d_{k-1}, \\
b_k &= \rho_k a_k + (1 - \rho_k)S_k a_k, \\
c_k &= \tau_k \tilde{V}_{j-1}b_k + (1 - \tau_k)\mathcal{J}_{j, k}^\partial \tilde{V}_{j-1}b_k, \\
H_{k+1} &= \{ p \in \Xi_1 : \| b_k - p \|^2 \leq \| d_k - p \|^2 + \mu_k^2 \| d_k - d_{k-1} \|^2 + 2\mu_k \langle d_k - p, d_k - d_{k-1} \rangle \}, \\
W_{k+1} &= \{ p \in \Xi_1 : \| c_k - \tilde{V}_{j-1}p \| \leq \| \tilde{V}_{j-1}b_k - \tilde{V}_{j-1}p \| \}, \\
d_{k+1} &= \Pi_{H_{k+1}} \cap W_{k+1} d_1 \quad \forall k \geq 1,
\end{align*}
\]

under the control conditions (C1)–(C4), converge strongly to an element in \( \Omega \).

4.4 Signal processing

This subsection deals with the case of signal recovery problem, which we aim to solve by applying Theorem 4.1. The following underdetermined formalism denotes the signal recovery problem:

\[
Vd = \kappa - \vartheta,
\]

where \( \kappa \in \mathbb{R}^M \) is the measured noise data with noise \( \vartheta \), \( d \in \mathbb{R}^N \) is the sparse original data for recovery, and \( V : \mathbb{R}^N \to \mathbb{R}^M (M < N) \) is the bounded linear observation matrix. Formalism (4.4) is equivalent to the well-known least absolute shrinkage and selection operator
Table 1 Comparison of Theorems 4.1 and 4.4 (Reich et al. [40])

|                  | No. of Iterations | CPU Time |
|------------------|-------------------|----------|
|                  | Test I    | Test II  | Test I    | Test II  |
| IGSNPP, Theorem 4.1 | 2841    | 6069    | 0.4969    | 6.4199   |
| GSNPP, Theorem 4.4 [40] | 3573    | 7903    | 0.7023    | 9.8853   |

Figure 1 Comparison of two algorithms for Numerical Test 1

LASSO problem [51] in the following convex constrained optimization formalism:

$$
\min_{d \in \mathbb{R}^N} \left\{ \frac{1}{2} \| Vd - \kappa \|^2 \right\} \text{ subject to } t \geq \|d\|_1.
$$

(4.5)

If we set $\Theta = K_1 \cap V^{-1}(K_2) \neq \emptyset$ with $K_1 = \{d \mid t \geq \|d\|_1\}$ and $K_2 = \{\kappa\}$, then the LASSO problem can be easily solved via Theorem 4.1. To conduct the numerical experiment, we generate (i) the matrix $V^{N \times M}$ from the standard normal distributions with zero mean and unit variance, (ii) $d$ having $m \ll N$ nonzero elements via a uniform distribution in $[-2, 2]$, and (iii) $\kappa$ from a Gaussian noise with signal-to-noise ratio $\text{SNR} = 40$. The approximants are initiated with randomly chosen $d_0$, $d_1$ and abort when the following mean square error
is satisfied:

$$E_k = \frac{1}{N} \| d_k - d^* \| < 10^{-4}.$$  

Here $d^*$ is called the estimated signal of $d$.

For Theorem 4.1, we choose $\mu_k = \frac{1}{(100k+1)^{1.04}}, \rho_k = \frac{1}{k^{1.02}}, t = m - 0.001$, and $\vartheta = 0$.

We recover the signals for the following two tests:

**Numerical Test 1** Choose $N = 512, M = 256$, and $m = 15$.

**Numerical Test 2** Choose $N = 1024, M = 512$, and $m = 30$.

From Table 1 and Figs. 1 and 2 we conclude that IGSNPP as in Theorem 4.1 reconstruct the original signal (A) faster than the algorithm for GSNPP as in Theorem 4.4 [40] in the compressed sensing. Moreover, the graph of error function values (B) and objective function values (C) generated by IGSNPP as in Theorem 4.1 converge faster as compared to the algorithm for GSNPP as in Theorem 4.4 [40].

### 5 Numerical experiment and results

In this section, we focus on numerical implementation of our proposed algorithm. Comparison with Reich et al. [40]) shows the effectiveness and efficiency of our proposed al-
algorithm. All codes were written in MATLAB R2020a and performed on a laptop Intel(R) Core(TM)i3-3217U @1.80 GHz, RAM 4.00 GB.

**Example 5.1** Let \( \mathbb{E}_1 = \mathbb{R}^2 \) and \( \mathbb{E}_2 = \mathbb{R}^4 \) with the inner product defined by \( (x, y) = xy \), for all \( x, y \in \mathbb{R}^2, \mathbb{R}^4 \) and the induced usual norm \( |\cdot| \).

Consider the following problem: find an element \( q \in \mathbb{R}^2 \) such that

\[
q \in \Omega = \Theta_1 \cap V^{-1}(\Theta_2) \cap \text{Fix}(S) \neq \emptyset,
\]

where

\[
\Theta_1 = \{ x \in \mathbb{R}^2 \mid \|x - a_1\|^2 \leq R_1^2 \}, \quad \Theta_2 = \{ x \in \mathbb{R}^4 \mid \|x - a_2\|^2 \leq R_2^2 \},
\]

and \( V : \mathbb{R}^2 \to \mathbb{R}^4 \) is a bounded linear matrix randomly generated in the closed interval \([-5, 5]\). Let the operators \( S_k : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( S_k(x) = (-k + 1)x_1, -(k + 1)x_2 \) for \( k = 1, 2 \). Then \( S_k \) is a \( \eta \)-demimetric operator with \( \eta_1 = \frac{1}{2} \) and \( \eta_2 = \frac{1}{2} \), respectively. It is easy to observe that \( \bigcap_{k=1}^{\infty} \text{Fix}(S_k) = \{0\} \) and \( \Theta := \{ \Theta_1 \cap V^{-1}(\Theta_2) \} = 0 \). Hence \( \Omega = \Theta \cap \text{Fix}(S) = 0 \).

Furthermore, the coordinate of the center \( a \) is randomly generated in the closed interval \([-1, 1]\), and the radii \( R_1 \) and \( R_2 \) are randomly generated in the closed intervals \([5, 9]\) and \([9, 17]\), respectively. The coordinates of the initial point \( d_0 \), \( d_1 \) are randomly generated in the closed interval \([-5, 5]\). Choose \( \mu = 0.9 \), \( m = 0.01 \), \( \rho_k = \frac{1}{100k+1} \), and \( \beta_1 = \frac{1}{100k+1} \). We provide a numerical test of the hybrid shrinking approximants defined in Theorem 4.1 (i.e., Theorem 4.1 with \( \mu_k \neq 0 \)) with the noninertial variant of Theorem 4.4 (i.e., Theorem 4.4, Reich et al. [40]). It is remarked that the function \( E_k \) is defined by

\[
E_k = \frac{1}{2} \left[ \|d_k - \Pi_{\Theta_1}^{\mathbb{R}^2}(d_k)\|^2 + \|Vd_k - \Pi_{\Theta_2}^{\mathbb{R}^4}(Vd_k)\|^2 \right] \quad \text{for} \ k \geq 1.
\]

Note that at the \( k \)-th step, \( E_k = 0 \), and then \( d_k \in \Theta \), which implies that \( d_k \) is a solution of this problem. The stopping criterion is defined as \( E_k < 10^{-5} \). The different choices of \( d_0 \), \( d_1 \) are given as follows:

- **Case I:** \( d_0 = [6, 8]^T \), \( d_1 = [3, 7]^T \).
- **Case II:** \( d_0 = [6.5, 7.2]^T \), \( d_1 = [-1.4, -9.7]^T \).
- **Case III:** \( d_0 = [3, -4.7]^T \), \( d_1 = [1.2, 4]^T \).

| \( \mu \) | No. of Iterations | CPU Time |
|---|---|---|
| 0.15 | Case I | Case II | Case III | Case I | Case II | Case III |
| Theorem 4.1 | 13 | 18 | 15 | 0.03426 | 0.04521 | 0.03218 |
| Theorem 4.1 | 12 | 17 | 14 | 0.04036 | 0.08146 | 0.05919 |
| Theorem 4.1 | 11 | 15 | 12 | 0.05731 | 0.09032 | 0.07984 |
| Theorem 4.1 | 10 | 14 | 11 | 0.09104 | 1.09821 | 0.09973 |
| Theorem 4.4 | 57 | 67 | 49 | 2.93910 | 3.12310 | 2.11351 |
Remark 5.2

(i) The example presented above serves for two purposes:
   - impact of different values of $\mu_k$ on our proposed algorithm
   - comparison with the noninertial ($\mu_k = 0$) type algorithm proposed by Reich et al. [40] given in Theorem 4.4.

(ii) The numerical results presented in Table 2 and Figs. 3–5 indicate that our proposed approximants are efficient, easy to implement, and do well for any values of $\mu_k \neq 0$ in both number of iterations and CPU time required.

(iii) We observe that the CPU time of Theorem 4.1 increases, but the number of iterations decreases as the parameter $\mu$ approaches 1.
(iv) We observe from the numerical implementation above and our proposed algorithm outperformed the noninertial version proposed by Reich et al. [40] given in Theorem 4.4 both in the number of iterations and CPU time required to reach the stopping criterion.

6 Conclusions

The problem for computing a common solution via unifying approximants, of a finite family of GSCNPP and the FPP for a countably infinite family of nonlinear operators has its own importance in the fields of monotone operator theory and fixed point theory. We proved that the approximants perform in an effective and efficient way when compared with the existing approximants, in particular, those studied in Hilbert spaces. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. Moreover, this framework has also been implemented to various instances of the split inverse problems. We would like to emphasize that the above mentioned problems occur naturally in many applications. Therefore iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

Acknowledgements

The authors wish to thank the anonymous referees for their comments and suggestions. Yasir Arfat was supported by the Petchra Pha Jom Klao PhD Research Scholarship from King Mongkut’s University of Technology Thonburi, Thailand (Grant No. 16/2562).

Funding

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research was funded by King Mongkut’s University of Technology North Bangkok, Contract No. KMUTNB-65-KNOW-28.

Availability of data and materials

Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.
Declarations

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
Conceptualization of the paper was carried out by YA, MA, and PK. Methodology by YA and MA. Formal analysis, investigation, and writing the original draft preparation by YA, MA and OS. Software and validation by OS, WK and KS. Writing, reviewing, and editing by YA, MA, and PK. Project administration by PK, WK, and KS. All authors read and approved the final manuscript.

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Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 March 2022 Accepted: 9 May 2022 Published online: 26 May 2022

References
1. Alvarez, F., Attouch, H.: An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Anal. 9, 3–11 (2001)
2. Aoyama, K., Kimura, Y., Takahashi, W., Toyoda, M.: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. 67, 2350–2360 (2007)
3. Arfat, Y., Kumam, P., Khan, M.A., Iyiola, O.S.: Multi-inertial parallel hybrid projection algorithm for generalized split null point problems. J. Appl. Math. Comput. (2021). https://doi.org/10.1007/s12190-021-01660-4
4. Arfat, Y., Kumam, P., Khan, M.A., Ngiam-sunthorn, P.S.: Parallel shrinking inertial extragradient approximants for pseudomonotone equilibrium, fixed point and generalized split null point problem. Ric. Mat. (2021). https://doi.org/10.1007/s11587-021-00647-4
5. Arfat, Y., Kumam, P., Khan, M.A., Ngiam-sunthorn, P.S.: Shrinking approximants for fixed point problem and generalized split null point problem in Hilbert spaces. Optim. Lett. (2021). https://doi.org/10.1007/s11590-021-01810-4
6. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiam-sunthorn, P.S.: An accelerated visco-Cesaro means Tseng type splitting method for fixed point and monotone inclusion problems. Carpath. J. Math. 38(2), 281–297 (2022)
7. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiam-sunthorn, P.S., Kaewkhao, A.: An inertially constructed forward-backward splitting algorithm in Hilbert spaces. Adv. Differ. Equ. 2021, 124 (2021)
8. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiam-sunthorn, P.S., Kaewkhao, A.: A parallel hybrid accelerated extragradient algorithm for pseudomonotone equilibrium, fixed point, and split null point problems. Adv. Differ. Equ. 2021, 364 (2021)
9. Arfat, Y., Kumam, P., Ngiam-sunthorn, P.S., Khan, M.A.A.: An inertial based forward-backward algorithm for monotone inclusion problems and split mixed equilibrium problems in Hilbert spaces. Adv. Differ. Equ. 2020, 453 (2020)
10. Arfat, Y., Kumam, P., Ngiam-sunthorn, P.S., Khan, M.A.A.: An accelerated projection based parallel hybrid algorithm for fixed point and split null point problems in Hilbert spaces. Math. Methods Appl. Sci. 1–19 (2021). https://doi.org/10.1002/mma.7405
11. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, Berlin (2017)
12. Bauschke, H.H., Matoušková, E., Reich, S.: Projection and proximal point methods: convergence results and counterexamples. Nonlinear Anal. 56, 715–738 (2004)
13. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. 20(1), 103–120 (2004)
14. Cai, G., Gilali, A., Iyiola, O.S., Shehu, Y.: A new double-projection method for solving variational inequalities in Banach spaces. J. Optim. Theory Appl. 178(1), 219–239 (2018)
15. Ceng, L.C., Coroian, I., Qin, X., Yao, J.C.: A general viscosity implicit iterative algorithm for split variational inclusions with hierarchical variational inequality constraints. Fixed Point Theory 20, 469–482 (2019)
16. Ceng, L.C., Li, X., Qin, X.: Parallel proximal point methods for systems of vector optimization problems on Hadamard manifolds without convexity. Optimization 69, 357–383 (2020)
17. Ceng, L.C., Petrusel, A., Qin, X., Yao, J.C.: A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. Fixed Point Theory 21, 93–108 (2020)
45. Takahashi, W.: Nonlinear Functional Analysis: Fixed Point Theory and Its Applications. Yokohama Publishers, 44.

44. Takahashi, S., Takahashi, W.: The split common null point problem and the shrinking projection method in Banach spaces. Fixed Point Theory Appl. 2015, 192 (2015).

43. Shehu, Y., Iyiola, O.S., Reich, S.: A modified inertial subgradient extragradient method for solving variational inequalities. Optim. Eng. 2019, 274 (2019).

42. Shehu, Y., Iyiola, O.S., Ogbuisi, F.U.: Iterative methods with inertial terms for nonexpansive mappings: application to variational inequalities. J. Fixed Point Theory Appl. 2018, 2071–2084 (2017).

41. Shehu, Y., Iyiola, O.S., Reich, S.: On the convergence of an inertial Tseng’s extragradient algorithm for solving variational inequalities and applications. J. Nonlinear Var. Anal. 2017, 301–318 (2017).

40. Reich, S., Tuyen, T.M.: Parallel iterative methods for solving the generalized split common null point problem in Hilbert spaces. J. Inequal. Appl. 2015, 505 (2015).

39. Reich, S., Tuyen, T.M.: Iterative methods for solving variational inequality problems in Hilbert spaces. J. Convex Anal. 2015, 339–357 (2015).

38. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. USSR Comput. Math. Math. Phys. 1964, 1–22 (1964).

37. Shehu, Y., Iyiola, O.S., Birgisdottir, A.: A modified inertial extragradient method for solving variational inequalities and applications. J. Nonlinear Var. Anal. 2020, 101–117 (2020).

36. Komiya, H., Takahashi, W.: Strong convergence theorem for an infinite family of demimetric mappings in a Hilbert space. J. Math. Anal. Appl. 2018, 67–71 (2018).

35. Jantakarn, K., Kaewcharoen, A.: A Bregman hybrid extragradient method for solving pseudomonotone equilibrium problems in Hilbert spaces. J. Math. Anal. Appl. 2019, 7–17 (2019).

34. Iyiola, O.S., Shehu, Y.: Alternated inertial method for nonexpansive mappings with applications. J. Nonlinear Convex Anal. 2018, 75–88 (2018).

33. Halpern, B.: Fixed point theory for nonexpansive mappings and related problems. Bull. Amer. Math. Soc. 1967, 1–25 (1967).

32. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990).

31. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990).

30. Dong, Q.L., Liu, L., Yao, Y.: Equilibrium problems in Hilbert spaces. J. Fixed Point Theory Appl. 2017, 139–151 (2017).

29. Dong, Q.L., Yao, Y.: A strongly convergent extragradient method for solving equilibrium problems in Hilbert spaces. J. Fixed Point Theory Appl. 2015, 119–130 (2015).

28. Cui, H.H., Su, M.: On sufficient conditions ensuring the norm convergence of an iterative sequence to zeros of accretive operators. Appl. Math. Comput. 2015, 67–71 (2015).

27. Cui, H.H., Zhang, H.X., Ceng, L.C.: Strong convergence theorem for an infinite family of demimetric mappings with variational inequality problems in a Hilbert space. J. Inequal. Appl. 2015, 201–216 (2015).

26. Cui, H.H., Ceng, L.C.: Iterative solution of the split common fixed point problem for strictly pseudo-contractive mappings. J. Fixed Point Theory Appl. 2016, 139–151 (2016).

25. Combettes, P.L., Pesquet, J.C.: Proximal splitting methods in signal processing. In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 185–212 (2011).

24. Combettes, P.L.: The convex feasibility problem in image recovery. Adv. Imaging Electron Phys. 1996, 155–453 (1996).

23. Chen, J.Z., Ceng, L.C., Qiu, Y.Q., Kong, Z.R.: Extra-gradient methods for solving split feasibility and fixed point problems. Fixed Point Theory Appl. 2018, 65 (2018).

22. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms 1994, 221–239 (1994).

21. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

20. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

19. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms 2012, 301–323 (2012).

18. Censor, Y., Reich, S.: The split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

17. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

16. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

15. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

14. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

13. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

12. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

11. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

10. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

9. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

8. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

7. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

6. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

5. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

4. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

3. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

2. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).

1. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications. Inverse Probl. 2005, 2071–2084 (2005).
56. Yao, Y., Liou, Y.C., Yao, J.C.: Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations. J. Nonlinear Sci. Appl. 10, 843–854 (2017)
57. Yao, Y., Yao, J.C., Liou, Y.C., Postolache, M.: Iterative algorithms for split common fixed points of demicontractive operators without prior knowledge of operator norms. Carpath. J. Math. 34, 459–466 (2018)