Existence of random attractors for strongly damped wave equations with multiplicative noise unbounded domain

Abdelmajid Ali Dafallah∗1, Qiaozhen Ma2, Ahmed Eshag Mohamed2

1Department of Mathematics, AL Salam University, El Muglad, Sudan
2Department of Mathematics, Northwest Normal University, Lanzhou 730070, P.R. China

Abstract
In this paper, we establish the existence of a random attractor for a random dynamical system generated by the non-autonomous wave equation with strong damping and multiplicative noise when the nonlinear term satisfies a critical growth condition.

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1. Introduction
In this work, we consider the asymptotic behavior of non-autonomous stochastic strong damped wave equations with multiplicative noise defined on

$$u_{tt} - \alpha \Delta u_t - \Delta u + g(u)u_t + f(u) = q(x,t) + cu \circ dW,$$  \hspace{1cm} (1.1)
with initial data

$$u(\tau,x) = u_\tau(x) \quad u_t(\tau,x) = u_{1,\tau}(x) \quad x \in \mathbb{R}^n, \ t \geq \tau, \ \tau \in \mathbb{R},$$  \hspace{1cm} (1.2)

where $-\Delta$ is the Laplacian operator with respect to the variable $x \in \mathbb{R}^n$, with $n \leq 3$, $u = u(t,x)$ is a real-valued function on $\mathbb{R}^n \times [\tau, \infty)$, $\alpha$ and $c$ are positive constants.

The given function $q(x,t) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, $W(t)$ are independent two sided real-valued Wiener processes on probability space. Here we show the following conditions.

(a) The function $g \in C^1(\mathbb{R})$ is not identically equal to zero and satisfies the following condition:

$$-\beta < g(s) \leq \beta_1 \leq \beta_2 < +\infty,$$  \hspace{1cm} (1.3)

where $\beta$, $\beta_1$ and $\beta_2$ are positive constants.

*Corresponding Author.
Email addresses: majid_dafallah@yahoo.com (A.D. Abdelmajid), mazh@nwnu.edu.cn (Q. Ma), ahmedesag@gmail.com (E.M. Ahmed)
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(b) We assume that the nonlinear function \( f \in C^2(\mathbb{R}) \) satisfies the following conditions:

\[
|f'(s)| \leq C_1(1 + |s|^p), \quad \text{with} \quad \begin{cases} 
0 \leq p < \infty, & \text{when } n = 1, 2 \\
0 \leq p < 2, & \text{when } n = 3,
\end{cases}
\]

(1.4)

and

\[
\lim_{|s| \to \infty} \sup_{s} \frac{f(s)}{s} \leq 0, \quad \forall \ s \in \mathbb{R},
\]

(1.5)

\[
\lim_{|s| \to \infty} \inf_{s} \frac{sf(s) - c_3 F(s)}{s^2} \geq 0,
\]

(1.6)

where \( C_1, C_2, C_3 \) are positive constants.

(c) \( q(x, t) \) is external force term satisfying the following conditions:

\[
\int_{-\infty}^{\tau} e^{\sigma r} ||q(x, r)||^2 dr < \infty, \quad \forall \tau \in \mathbb{R},
\]

(1.7)

and

\[
\lim_{k \to \infty} \int_{-\infty}^{\tau} \int_{|x| \geq k} e^{\sigma r} |q(x, r)|^2 dx dr = 0, \quad \forall \tau \in \mathbb{R}.
\]

(1.8)

The asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics and the theory has been greatly developed over the last decade or so. In the deterministic case the global attractor, a compact invariant and attracting set, occupy a central position see, for example, Temam [26]. In this paper, we study random attractors of equation (1.1) when the forcing term is time dependent. In this case, we want to introduce two parametric spaces to describe the dynamics of the equations: one is responsible for deterministic forcing and the other is responsible for stochastic perturbations. Existence and upper semi-continuity of the global attractor, pullback attractor (or kernel sections) for deterministic autonomous and non-autonomous dynamical systems were studied widely related with this problem (see, e.g., [6, 7, 16, 18, 20–22, 24, 32, 33, 35]).

In order to study the corresponding random dynamical system (RDS), some authors have introduced a different notion of an attractor from the view of stochastic partial differential equations, for example, see G.Da Prato, J. Zabczyk [11], Morimoto [23], L. Arnold [2] and H. Crauel, F. Fladoli Crauel [10], Duan, Lu, and Schmalfuß [12] and T. Caraballo, J. Langa [4], in which the authors studied the existence and the upper semi-continuity of attractors for deterministic and random dynamical systems, respectively. They obtained a general criteria for the existence and upper semi-continuous of attractors. For non-autonomous stochastic evolution equations with the time-dependent external term and multiplicative noise, Wang [29] established a useful theory about the existence and upper semi-continuity of random attractors by introducing two parametric spaces and giving some applications to non-autonomous stochastic reaction-diffusion equations and wave equations, see also [1, 3–5, 13, 19, 28, 30, 31, 34] for more details.

Note that the stochastic equation (1.1) is defined in unbounded domains. Since Sobolev embeddings are not compact on unbounded domains, we have an extra difficulty to prove our main results.

This article is organized as follows. In Section 2 we recall some basic concepts related to RDS and a random attractor for RDS. In Section 3, we first provide some basic settings about (1.1) and (1.2) and show that it generates an RDS in proper function space and existence and uniqueness of solutions. We devote the Section 4 to uniform energy estimates on the solutions of (1.1) and (1.2) defined on \( \mathbb{R}^n \) when \( t \to \infty \) with the purpose of proving
the existence of a bounded random absorbing set and the asymptotic compactness of the RDS associated with the equation. In Section 5, we prove the existence of a random attractor.

2. Preliminaries

In this section, we recall some basic concepts related to RDS and a random attractor for RDS [9,10,15,27], which are important for getting our main results. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X,d)$ be a Polish space with the Borel $\sigma$-algebra $B(X)$. Then, the distance between $x \in X$ and $B \subseteq X$ is denoted by $d(x,B)$. If $B \subseteq X$ and $C \subseteq X$, the Hausdorff semi-distance from $B$ to $C$ is denoted by $d(B,C) = \sup_{x \in B} d(x,C)$.

**Definition 2.1.** $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, $\theta_0$ is the identity on $\Omega$, $\theta_{s+t} = \theta_t \circ \theta_s, \forall (s,t) \in \mathbb{R}$ and $\theta_0 P = P, \forall t \in \mathbb{R}$.

**Definition 2.2.** A mapping $\Phi(t,\tau,\omega,x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$ is called continuous cocycle on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$, if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t,s \in \mathbb{R}^+$, the following conditions are satisfied:

i) $\Phi(t,\tau,\omega,x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable mapping,

ii) $\Phi(0,\tau,\omega,x)$ is identity on $X$,

iii) $\Phi(t+s,\tau,\omega,x) = \Phi(t,\tau+s,\theta_s\omega,x) \circ \Phi(s,\tau,\omega,x)$,

iv) $\Phi(t,\tau,\omega,x) : X \rightarrow X$ is continuous.

**Definition 2.3.** Let $2^X$ be the collection of all subsets of $X$. A set valued mapping $(\tau,\omega) \mapsto D(t,\omega) : \mathbb{R} \times \Omega \rightarrow 2^X$ is called measurable with respect to $\mathcal{F} \in \Omega$, if $D(t,\omega)$ is a (usually closed) nonempty subset of $X$ and the mapping $\omega \in \Omega \mapsto d(X,B(\tau,\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$. Let $B = \{B(t,\omega) \in D(t,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be called a random set.

**Definition 2.4.** A random bounded set $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ of $X$ is called tempered with respect to $\{\theta(t)\}_{t \in \Omega}$, if for p-a.e $\omega \in \Omega$, $\lim_{t \to \infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0$, $\forall \beta > 0$, where

$$d(B) = \sup_{x \in B} \|x\|_X.$$

**Definition 2.5.** Let $\mathcal{D}$ be a collection of random subset of $X$ and $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then $K$ is called an absorbing set of $\Phi \in \mathcal{D}$, if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathcal{D}$, there exists, $T = T(\tau,\omega,B) > 0$ such that

$$\Phi(t,\tau,\omega,B(\tau,\omega)) \subseteq K(\tau,\omega), \forall t \geq T.$$

**Definition 2.6.** Let $\mathcal{D}$ be a collection of random subset of $X$, the $\Phi$ is said to be $\mathcal{D}$-pullback asymptotically compact in $X$ if for p-a.e $\omega \in \Omega$, $\{\Phi(t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^\infty$ has a convergent subsequence in $X$ when $t_n \rightarrow \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$.

**Definition 2.7.** Let $\mathcal{D}$ be a collection of random subset of $X$ and $A = \{A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then $A$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for $\Phi$, if the following conditions are satisfied:

For all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

i) $A(\tau,\omega)$ is compact, and $\omega \mapsto d(x,A(\omega))$ is measurable for every $x \in X$,

ii) $A(\tau,\omega)$ is invariant, that is

$$\Phi(t,\tau,\omega,A(\tau,\omega)) = A(\tau + t,\theta_{t}\omega), \forall t \geq \tau.$$
iii) $A(\tau, \omega)$ attracts every set in $\mathcal{D}$, that is for every $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D},$

$$\lim_{t \to \infty} d_X(\Phi(t, \tau, \theta_t \omega, B(\tau, \theta_t \omega)), A(\tau, \omega)) = 0,$$

where $d_X$ is the Hausdorff semi-distance given by

$$d_X(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} ||y - z||_X, \forall (Y, Z) \in X.$$

**Lemma 2.8.** Let $\mathcal{D}$ be a neighborhood-closed collection of $(\tau, \omega)$-parameterized families of nonempty subsets of $X$ and $\Phi$ be a continuous cocycle on $X$ over $\mathbb{R}$ and $(\Omega, F, P, (\theta_t)_{t \in \mathbb{R}})$. Then $\Phi$ has a pullback $\mathcal{D}$-attractor $A$ in $\mathcal{D}$ if and only if $\Phi$ is pullback $\mathcal{D}$-asymptotically compact in $X$ and $\Phi$ has a closed, $F$-measurable pullback $\mathcal{D}$-absorbing set $K \in \mathcal{D}$, the unique pullback $\mathcal{D}$-attractor $A = A(\tau, \omega)$ is given by

$$A(\tau, \omega) = \bigcap_{r \geq 0} \bigcup_{i \geq r} \Phi(\tau - t, \theta_t \omega, K(\tau - t, \theta_t \omega)) \tau \in \mathbb{R}, \omega \in \Omega.$$

3. Existence and uniqueness of solutions

In this section, we study the existence and uniqueness of solutions for the system (1.1) and (1.2) on an unbounded set of $\mathbb{R}^n$, $(n = 3)$.

Let $A = -\Delta, D(A) = H^1(\mathbb{R}^n) \cap H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. It is self-adjoint, positive and linear, and the eigenvalue $\{\lambda_i\}_{i \in N}$ of $A$ satisfies, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_i \leq \cdots, \lambda_i \to +\infty (i \to +\infty)$.

Define the set of Hilbert space $V_{2r} = D(A^r), r \in \mathbb{R}$, with standard inner products and norms, respectively, $(\cdot, \cdot)_{D(A^r)} = (A^r \cdot, A^r \cdot), \| \cdot \|_{D(A^r)} = \| A^r \cdot \|, ((u, v)) = \int_{\mathbb{R}^n} \nabla u \nabla vdx, \| \nabla u \| = ((u, u))^{\frac{1}{2}}, \forall u, v \in H^1(\mathbb{R}^n)$.

Especially, $(\cdot, \cdot)$ and $\| \cdot \|$ denote the $L^2(\mathbb{R}^n)$ inner product and norm respectively as,

$$(u, v) = \int_{\mathbb{R}^n} uvdx, \| u \| = ((u, u))^{\frac{1}{2}}, \forall u, v \in L^2(\mathbb{R}^n).$$

Thus we have $D(A^r) \to D(A^s)$, for $r > s$, and the continuous embedding $D(A^r) \to L^{\frac{2n}{2n-r}}(\mathbb{R}^n), \forall r \in [0, \frac{n}{2})$.

Let $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, endowed with norms on $E$ and $E_\nu = H_{2\nu+1} \times H_{2\nu}$, $\| \varphi \|_{E}^2 = \| \varphi \|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2$.

For a convenient study of dynamical behavior of the problem (1.1) and (1.2), we need to convert the stochastic system into deterministic with a random parameter, then show that it generates an RDS.

Consider Ornstein-Uhlenbeck process driven by the Brownian motion, which satisfies the Itô differential equation

$$dz + \mu zdt = dW, \mu > 0,$$

and hence the solution is given by

$$\theta_t \omega(s) = \omega(t+s) - \omega(t),$$

$$z(\theta_t \omega) = z(t, \omega) = -\mu \int_0^t e^{sz}(\theta_t \omega)sds, s \in \mathbb{R}, \omega \in \Omega. \quad (3.1)$$

From [33], it is known that the random variable $|z(\omega)|$ is tempered and there is an invariant set $\Omega \subseteq \Omega$ of full $P$ measure, such that $z(\theta_t \omega) = z(t, \omega)$ is continuous in $t$ for every
\( \omega \in \bar{\Omega} \). For convenience, we shall write \( \bar{\Omega} \) as \( \Omega \).

For the Ornstein-Uhlenbeck process \( z(\theta_t \omega) \) in (3.1) (see [8,14]), we have

\[
\begin{align*}
\lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{|t|} &= 0, \\
\lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^{0} z(\theta_s \omega) ds &= E[z(\theta_s \omega)] = 0, \\
\lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^{0} |z(\theta_s \omega)|^2 ds &= E[|z(\theta_s \omega)|^2] = \frac{1}{2\delta}. \\
\end{align*}
\]

By (3.2), there exists \( T_1(\omega) > 0 \), such that, for all \( t \geq T_1(\omega) \),

\[
\int_{-t}^{0} z(\theta_s \omega) ds < \frac{2}{\sqrt{\pi \delta}} t, \quad \int_{-t}^{0} |z(\theta_s \omega)|^2 ds < \frac{1}{2\delta} t.
\]

To define a cocycle associated with the problem (1.1) and (1.2), let \( v = u_t + \varepsilon u - cu z(\theta_t \omega) \), then we get

\[
\begin{align*}
\frac{du}{dt} &= v - \varepsilon u + cu z(\theta_t \omega), \\
\frac{dv}{dt} &= (\varepsilon - \alpha A)v - (\varepsilon^2 - \varepsilon \alpha A + A)u + g(u)(v - \varepsilon u + cu z(\theta_t \omega)) + c((3\varepsilon - \alpha A)u - \varepsilon u - cu z(\theta_t \omega)) z(\theta_t \omega) - f(u) + q(x,t), \\
&= u(x,\tau) = u_r(x), \quad v(x,\tau) = v_r(x) = u_1(x) + \varepsilon u_r(x) - cu_r(x) z(\theta_t \omega). \\
\end{align*}
\]

We define

\[
\psi_1 = u, \quad \psi_2 = \frac{du}{dt} + \varepsilon u - cu z(\theta_t \omega),
\]

where \( \varepsilon \) is a given positive constant, then the system (3.4) can be rewritten as the following equivalent system with random coefficients, but without multiplicative noise on \( E \)

\[
\begin{align*}
\frac{d\psi_1}{dt} &= \psi_2 - \varepsilon \psi_1 + c\psi_1 z(\theta_t \omega), \\
\frac{d\psi_2}{dt} &= (\varepsilon - \alpha A)\psi_2 - (\varepsilon^2 - \varepsilon \alpha A + A)\psi_1 - g(\psi_1)(\psi_2 - \varepsilon \psi_1 + c\psi_1 z(\theta_t \omega)) + c((3\varepsilon - \alpha A)\psi_1 - \varepsilon \psi_1 - c\psi_1 z(\theta_t \omega)) z(\theta_t \omega) - f(\psi_1) + q(x,t), \\
\psi_1(x,\tau) &= u_r(x), \quad \psi_2(x,\tau) = u_1(x) + \varepsilon u_r(x) - cu_r(x) z(\theta_t \omega), \\
\end{align*}
\]

then the random differential equation (3.6) can be written as

\[
\begin{align*}
\psi' + L\psi &= Q(\psi_1, t, \omega) \\
\psi_{\tau} &= (\psi_1(x,\tau), \psi_2(x,\tau)) = (u_r(x), u_1(x) + \varepsilon u_r(x) - cu_r(x) z(\theta_t \omega))^T, \\
\psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad L = \begin{pmatrix} \varepsilon I \\ -\varepsilon I + \alpha A \end{pmatrix}, \\
Q(\psi, t, \omega) &= \begin{pmatrix} -g(\psi_1) \psi_1 + c((3\varepsilon - \alpha A)\psi_1 - \varepsilon \psi_1 - c\psi_1 z(\theta_t \omega)) z(\theta_t \omega) - f(\psi_1) + q(x,t) \\ c\psi_1 z(\theta_t \omega) \end{pmatrix}.
\end{align*}
\]

Consider the equation (3.7), we know that \( -L \) is the infinitesimal generator of semigroup \( e^{-Lt} \) on \( E \) for \( t > 0 \), by the assumptions (1.3)-(1.8). It is easy to check that \( Q(\psi, t, \omega) : E \to E \) is locally Lipschitz continuous with respect to \( \psi \), then by the classical semigroup
Assume that the conditions (1.3)-(1.8) hold, for each \( R \) and \( \psi \) any \( \| \cdot \| \). It is clear that the norm \( \| \cdot \| \) is equivalent to the usual norm \( \| \cdot \| \). Thus, like as (3.6), it yields

\[
\psi(t, \tau, \omega, \psi_\tau) = e^{-L(t-\tau)}\psi_\omega + \int_\tau^t e^{-L(t-r)}Q(\psi, r, \omega)dr.
\]

(3.8)

In this case, \( \psi(t, \tau, \omega, \psi_\tau) \) is called mild solution of (3.7), \( \psi(t, \tau, \omega, \psi_\tau) \) is jointly continuous into \( t \) and measurable in \( \omega \), moreover \( (u, u_t) \in C([\tau, +\infty, E]) \), \( \forall \tau > 0 \).

From Theorem 3.1, the solution \( \psi(t, \tau, \omega, \psi_\tau) \) of (3.7) can define a continuous RDS over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}}) \).

\[
\Phi_\tau(t, \omega) : \mathbb{R} \times \Omega \times E \mapsto E, t \geq \tau,
\]

\[
\psi(\tau, \omega) = (u_\tau(\omega), v_\tau(\omega))^\top \mapsto (u(t, \omega), v(t, \omega))^\top = \psi(t, \omega),
\]

(3.9)

it is easy to see that

\[
\Phi(t, \omega) : \psi(\tau, \omega) + (0, cuz(\theta_\tau \omega))^\top \mapsto \psi(t, \omega) + (0, cuz(\theta_t \omega))^\top.
\]

(3.10)

We will also use the transformation

\[
\varphi_1 = u, \quad \varphi_2 = u_t + \varepsilon u.
\]

(3.11)

Thus, like as (3.6), it yields

\[
\begin{cases}
\varphi' + H\varphi = F_1(\varphi, t, \omega), \\
\varphi_\tau = (u_\tau, v_\tau)^\top = (u_\tau(x), u_1(\tau(x) + \varepsilon u_\tau(x)))^\top,
\end{cases}
\]

(3.12)

where

\[
H\varphi = \begin{pmatrix}
\varepsilon(\varepsilon - \alpha A)u + Au - (\varepsilon - \alpha A)u
\end{pmatrix},
\]

and

\[
F_1(\varphi, t, \omega) = \begin{pmatrix}
g(u)(v - \varepsilon u) - f(u) + q(x, t) + cuz(\theta_t \omega)
\end{pmatrix}.
\]

We introduce the isomorphism \( T_\varphi \varphi = (\varphi_1, \varphi_2 - \varepsilon \varphi_1)^\top \), \( \varphi = (\varphi_1, \varphi_2)^\top \in E \) which has inverse isomorphism \( T_{-\varepsilon} \psi = (\varphi_1, \varphi_2 + \varepsilon \varphi_1)^\top \), it follows that \( (\theta, \varphi) \) with mapping

\[
\Psi = T_\varphi \Phi(t, \omega) T_{-\varepsilon} = \Psi(t, \omega)
\]

(3.13)

is an RDS corresponding, such that the two RDS are equivalent.

4. Uniform estimates of solutions

In this section, we will show the existence of a random absorbing set for the RDS \( \{\varphi(t, \tau, \omega, \varphi_\tau), t \geq 0\} \) in the space \( E \) and uniform estimates for the solutions of (3.4) defined on \( \mathbb{R}^n \). For this purpose, we introduce a new weight inner product and norm in the Hilbert space \( E \) that is, \( \langle \varphi, \tilde{\varphi} \rangle_E = \mu \langle A_1^{\frac{1}{2}}u_1, A_2^{\frac{1}{2}}u_2 \rangle + \langle v_1, v_2 \rangle \) and \( \| \varphi \|^2 = \langle \varphi, \varphi \rangle_E \) for any \( \varphi = (u_1, v_1)^\top, \tilde{\varphi} = (u_2, v_2)^\top \in E \), where \( \mu \) is chosen as

\[
\mu = \frac{4 + (\alpha \lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1}{4 + 2(\alpha \lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1} \in \left( \frac{1}{2}, 1 \right).
\]

(4.1)

It is clear that, the norm \( \| \cdot \|^2 \) is equivalent to the usual norm \( \| \cdot \|^2_{H^1 \times L^2} \) of \( E \). Let \( \varphi = (u, v)^\top = (u, u_t + \varepsilon u - cuz(\theta_\omega))^\top \), where \( \varepsilon \) is chosen as

\[
\varepsilon = \frac{\alpha \lambda_1 + \beta_1}{4 + 2(\alpha \lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1},
\]

(4.2)
and then the system (3.6) can be rewritten as

$$
\begin{cases}
\varphi' + H\varphi = F_1(t, \varphi, \omega), \\
\varphi_t = (u_t, v_t)^\top = (u_t(x), u_{1_t}(x) + \varepsilon u_t(x) - cu_t z(\theta_t \omega))^\top,
\end{cases}
$$

(4.3)

where

$$
H\varphi = \begin{pmatrix}
\varepsilon (v - u) \\
\varepsilon (\varepsilon - \alpha A)u + Au - (\varepsilon - \alpha A)v
\end{pmatrix},
$$

and

$$
F_1(t, \varphi, \omega) = \begin{pmatrix}
-cuz(\theta_t \omega) \\
-g(u)(v - \varepsilon u - cuz(\theta_t \omega)) + c((3\varepsilon - \alpha A)u - (1 + \varepsilon)u - cuz(\theta_t \omega)) z(\theta_t \omega) - f(u) + q(x, t)
\end{pmatrix}.
$$

Lemma 4.1. Suppose $-\alpha \lambda_1 < \beta_1 \leq \beta_2 < +\infty$ and

$$
\beta_2 \geq \beta_1 + \min\left[\frac{1}{\alpha}, \frac{\alpha \lambda_1 + \beta_1}{2}\right],
$$

(4.4)

for any $\varphi = (u, v) \in E$, it follows that

$$
(H\varphi, \varphi)_E \geq \sigma \|
\varphi\|^2_E + \frac{\alpha_1}{2} \|
\nabla \varphi\|^2_E + \frac{\beta_1}{2} \|
\varphi\|^2_E,
$$

(4.5)

where

$$
\sigma = \frac{\alpha \lambda_1 + \beta_1}{\gamma_1 + \sqrt{\gamma_1 \gamma_2}}
$$

(4.6)

Lemma 4.2. Assume that (1.3)-(1.8) hold, then there exists closed tempered random absorbing set $\tau(\omega)$ and a bounded ball $B_0(\tau, \omega) \subset E$, centered at 0 with a radius $\rho(\omega) > 0$, $B_\rho(0, \rho(\omega)) \in \mathcal{D}(E)$, such that for any bounded non-random set $B \in \mathcal{D}(E)$, there exists $T = T(\tau, \omega, B) \geq 0$, for all $\tau \in \mathbb{R}, \omega \in \Omega, B \in \mathcal{D}$, such that the mild solution $\varphi \in B_0$ of system (4.3) with initial value $\varphi_0 = (u_t, v_t) \in B$ satisfies

$$
\|\varphi(t, \tau, \omega, \varphi(\theta_t \omega))\|^2_E \leq \rho^2(\omega),
$$

(4.7)

$$
\Phi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)) \subseteq B_0(\tau, \omega), \quad t \geq T.
$$

Proof. For any $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$, suppose that $\varphi(s) = (u(s), v(s)) \in E$, $t \geq s \geq \tau$ be a mild solution of (4.3), taking the inner product of (4.3) with $\varphi(s) = (u, v) = (u, u_t + \varepsilon u - cuz(\theta_t \omega))^\top$. According to Lemma 4.1, we find that

$$
\frac{1}{2} \frac{d}{dt} \|
\varphi\|^2_E + \sigma \|
\varphi\|^2_E + \frac{\alpha \lambda_1 + \beta_1}{2} \|
\varphi\|^2_E \leq (F_1(t, \varphi, \omega), \varphi).
$$

(4.8)

We estimate the right hand side of (4.8)

$$
(F_1(\varphi, t, \omega), \varphi) = c((uz(\theta_t \omega), u)) - \varepsilon (g(u)(v - \varepsilon u - cuz(\theta_t \omega)), v) - \varepsilon g(u)(u, v) - cuz(\theta_t \omega) + c((3\varepsilon u - cuz(\theta_t \omega)) z(\theta_t \omega), v) - (f(u) - q(x, t), v),
$$

(4.9)

we deal with the term in (4.9) one by one as follows. For the first term of the right hand side of (4.9)

$$
c((uz(\theta_t \omega), u)) \leq c|z(\theta_t \omega)||\nabla u|^2.
$$

(4.10)

By (1.3), we have

$$
-g(u)(v - \varepsilon u - cuz(\theta_t \omega), v) = -g(u)(v, v) + \varepsilon g(u)(u, v) - cuz(\theta_t \omega) + c((3\varepsilon u - cuz(\theta_t \omega)) z(\theta_t \omega) - (f(u) - q(x, t), v),
$$

(4.11)
For the fourth term on the right-hand side of (4.9), the last term of (4.11), using the Cauchy-Schwartz inequality and Young’s inequality, we obtain
\[
c(3\varepsilon + \beta_2)u - v + cuz(\theta t\omega), v) z(\theta t\omega)
\leq c(3\varepsilon + \beta_2)z(\theta t\omega) + c^2|z(\theta t\omega)|^2(u, v) - c|z(\theta t\omega)||u||v|^2
\leq \frac{1}{2}c(3\varepsilon + \beta_2)z(\theta t\omega) + c^2|z(\theta t\omega)|^2(||u||^2 + ||v||^2) - c|z(\theta t\omega)||u||v|^2
\leq \frac{1}{2\sqrt{\lambda_1}}c(3\varepsilon + \beta_2)|z(\theta t\omega)| + c^2|z(\theta t\omega)|^2 \|\varphi\|_E^2 - c|z(\theta t\omega)||u||v|^2,
\]
\[
- ac(Au, v)z(\theta t\omega) \leq ac(\nabla uz(\theta t\omega), \nabla v) \leq \frac{ca\sqrt{\lambda_1}}{2}|z(\theta t\omega)||\varphi|_E^2,
\]
\[
(q(x, t), v) \leq \frac{1}{2(\alpha_1 + \beta_1 - 2\beta_2 - \varepsilon)} \|q(x, t)||v||^2 + \frac{(\alpha_1 + \beta_1 - 2\beta_2 - \varepsilon)}{2\sqrt{\lambda_1}}||v||^2.
\]
Let \(\tilde{F}(u) = \int_{\mathbb{R}^n} F(u)dx\), then using (1.5) and (1.6), and the Hölder inequality, we get that
\[
- (f(u), v) = - \left( f(u), \frac{d}{dt} + \varepsilon u - c u z(\theta t\omega) \right)
\leq - \frac{d}{dt} \tilde{F}(u) - \varepsilon (f(u), u) + (f(u), c u z(\theta t\omega)).
\]
Due to (1.6) and Poincaré inequality, there exists positive constants \(\theta_1, \theta_2\), such that
\[
(f(u), u) - C_3 \tilde{F}(u) + \theta_1 \|\nabla u\|^2 + \theta_2 \geq 0.
\]
According to [17], follows (1.5), for each given \(C_{\theta_3} > 0\),
\[
(f(u), u) + \theta_3 \|\nabla u\|^2 + C_{\theta_3} \geq 0.
\]
Then, by (4.14)-(4.17) we obtain
\[
- (f(u), v) \leq - \frac{d}{dt} \tilde{F}(u) - \varepsilon C_3 \tilde{F}(u) + (\varepsilon \theta_1 - \theta_3 c|z(\theta t\omega)|) \|\nabla u\|^2
\leq - \varepsilon \theta_2 - C_{\theta_3} c|z(\theta t\omega)|.
\]
Thus, we show that
\[
(F_1(t, \varphi, \omega), \varphi) \leq (\alpha_1 c|z(\theta t\omega)| + \frac{c^2|z(\theta t\omega)|^2}{\sqrt{\lambda_1}} - \sigma_1) \|\varphi\|_E^2 - \frac{d}{dt} \tilde{F}(u) - \varepsilon C_3 \tilde{F}(u)
\leq - \varepsilon \theta_2 - C_{\theta_3} c|z(\theta t\omega)| + \frac{(\alpha_1 + \beta_1)}{2}(\|q(x, t)||v||^2),
\]
where \(\alpha_1\) depends on \(1, \frac{1}{2\sqrt{\lambda_1}}, \frac{1}{2\lambda_1}((3\varepsilon + \beta_2), \frac{ca\sqrt{\lambda_1}}{2}, \frac{\beta_1}{2\sqrt{\lambda_1}}\), and \(\sigma_1 = \min\{\varepsilon \theta_1, \frac{\varepsilon}{2}\}\). Hence we conclude that
\[
\frac{d}{dt} \left( ||\varphi||_E^2 + 2\tilde{F}(u) \right) \leq -2 \left( \sigma - \alpha_1 c|z(\theta t\omega)| - \frac{c^2|z(\theta t\omega)|^2}{\sqrt{\lambda_1}} + \sigma_1 \right) ||\varphi||_E^2
\leq -2 \varepsilon \theta_2 - 2C_{\theta_3} \tilde{F}(u) + r_0(t, \theta t\omega).
\]
Put
\[
r_0(t, \theta t\omega) = \kappa (1 + c|z(\theta t\omega)| + \|q(x, t)||^2),
\]
where \(\kappa\) is a positive constant depends only on \(\varepsilon, \theta_2, C_{\theta_3}, \frac{\alpha_1 + \beta_1}{2\sqrt{\lambda_1}}\), such that
\[
\frac{d}{dt} \left( ||\varphi||_E^2 + 2\tilde{F}(u) \right) \leq -2 \left( \sigma - \alpha_1 c|z(\theta t\omega)| - \frac{c^2|z(\theta t\omega)|^2}{\sqrt{\lambda_1}} + \sigma_1 \right) ||\varphi||_E^2
\leq -\sigma ||\varphi||_E^2 + r_0(t, \theta t\omega).
\]
By using Gronwall’s inequality in (4.21) on \(\tau, t\), and then replacing \(\omega\) to \(\theta_{-\omega}\) yield
\[
||\varphi||_E^2 \leq \left( ||\varphi||_E^2 + 2\tilde{F}(u) \right) e^{-\sigma \int_0^t \tilde{F}(u(t)) dt}
\leq \left( ||\varphi||_E^2 + 2\tilde{F}(u) \right) e^{-2C_{\theta_3} \int_0^t \tilde{F}(u(t)) dt}
\]
\[
+ \int_t^e r_0(t, \theta_{-\omega}) e^{-2C_{\theta_3} \int_0^t \tilde{F}(u(t)) dt} (\tau - t) d\tau,
\]
where \(\tilde{\sigma} = \min\{\sigma, \frac{\varepsilon}{2}, \sigma_1\}\), and \(y(t, \omega) = ||\varphi||_E^2 + 2\tilde{F}(u) \geq ||\varphi||_E^2 \geq 0\).
Using (1.4) and Young inequality, there exists a constant \(c_4 \geq 0\), such that
\[
\tilde{F}(u) = \int_{\mathbb{R}^n} F(u)dx \leq c_4 \left( 1 + \|u_r\|^2 + \|u_r\|_{p+2}^2 \right).
\]
Due to $\varphi_\tau = (u_\tau, v_\tau)^T \in B(\tau, \theta_{-t}\omega)$ and $B \in \mathcal{D}$, which is tempered with respect to the norm of $E$, since $\varphi_\tau(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there exists $T = T(\tau, \omega, B) > 0$, such that, for all $t \geq T$.

$$\begin{align*}
\left\{ e^{-\sigma t} \left( \|\varphi_\tau\|^2_E + \tilde{F}(u_\tau) \right) \right\} & \leq \left( \left. \lim_{t \to -\infty} e^{-\sigma t} \right| \|\varphi_\tau\|^2_E + \tilde{F}(u_\tau) \right| = 0.
\end{align*}$$

(4.24)

Thus, by Lemma 3.1 and for any set $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, $\varphi_\tau = (u_\tau(x), u_1(x) + \varepsilon u_\tau(x) - cu_\tau \rho(\theta_{-t}\omega)) \in \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D(\mathcal{E})$. We deduce that

$$\begin{align*}
\lim_{t \to \infty} \sup(t \to \infty) \|\varphi(\theta_{-t}\omega)\|^2_E + 2\varepsilon C_\alpha(\|u_\tau\|^2 + \|u_\tau\|_{H^1}^2) e^{-2c_m(\theta_{-t}\omega)}|\tau-\xi|^2 \|\theta_{-t}\omega\|^2 (t-\tau) & = 0,
\end{align*}$$

$$\begin{align*}
\lim_{t \to \infty} \int_0^t \|\varphi(\theta_{-t}\omega)\|^2_E e^{-2c_m(\theta_{-t}\omega)}|\tau-\xi|^2 \|\theta_{-t}\omega\|^2 (s-t) \, ds & < \infty.
\end{align*}$$

(4.25)

When $q(x, t)$ only satisfying (1.7) and (1.8) which is a tempered random variable, then by (4.23)-(4.25), for any non-random bounded $B \subseteq E$ with radius $r(\omega)$, there exists a random variable $T = T(\tau, B, \omega) > 0$, such that for any $\varphi_\tau \in B_0(\omega)$, $\{\varphi \in E : \|\varphi(\theta_{-t}\omega)\|_E \leq \rho^2(\omega)\}$ is closed measurable absorbing ball in $D(E)$, $\forall \, t \geq T$.

The proof is completed. 

Next we construct the uniform estimates on the tail parts of the solutions for large space variables when time is sufficiently large in order to prove the pullback asymptotic compactness of the cocycle associated with equation (3.4) on the unbounded domain $\mathbb{R}^n$.

We can choose a smooth function $\rho$ defined on $\mathbb{R}^+$, such that $0 \leq \rho(s) \leq 1$, for all $s \in \mathbb{R}^+$ and

$$\rho(s) = \begin{cases} 0, & \text{for } 0 \leq s \leq 1, \\ 1, & \text{for } s \geq 2. \end{cases}$$

(4.26)

Then there exist constant $\mu_1$ and $\mu_2$, such that $|\rho^\prime(s)| \leq \phi_1, |\rho^\prime\prime(s)| \leq \phi_2$, for any $s \in \mathbb{R}^+$. Given $k \geq 1$, denote by $H_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\mathbb{R}^n \setminus H_k$ the complement of $H_k$. To prove the asymptotic compactness of the RDS, we prove the following lemma.

Lemma 4.3. Let conditions (1.3)-(1.8) hold. $B=B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \in \mathcal{D}$ and $\varphi_\tau(\omega) = B(\tau, \omega)$. Then for every $\eta > 0$ and $P$-a.e $\omega \in \Omega$, there exists $T = T(\tau, B, \omega, \eta) > 0$ and $K = K(\tau, \omega, \eta) \geq 1$, such that $\varphi(t, \omega, \varphi_\tau(\omega))$ is a solution of (3.4) satisfying, for all $t \geq T, k \geq K$,

$$\|\varphi(\tau, t - \theta_{-t}\omega, \varphi_\tau(\theta_{-t}\omega))\|^2_E (\mathbb{R}^n \setminus H_k) \leq \eta.$$

(4.27)

Proof. Taking the inner product of the second equation of (3.4) with $\rho(x^2)v$ in $L^2(\mathbb{R}^n)$, we find that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} |v|^2 \, dx = \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \left( (\varepsilon - g(u)) - \varepsilon Av - c\varepsilon \varepsilon u - c\varepsilon \varepsilon u + (\varepsilon \alpha - 1)Au \right) v dx$$

(4.28)

$$+ \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \left( |c(3\varepsilon - g(u))u - v - cu_\tau \rho(\theta_{-t}\omega)| z(\theta_{-t}\omega) - c\alpha Au z(\theta_{-t}\omega) - f(u) + q(x, t) \right) v dx.$$

In order to estimate the left hand side, we are substituting $v$ in the first term of (3.4), so it follows that

$$\int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \varepsilon (\varepsilon - g(u)) v dx$$

(4.29)

$$= \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \left( \varepsilon - g(u) \right) u \left[ \frac{du}{dt} + \varepsilon u - cu_\tau \rho(\theta_{-t}\omega) \right] \, dx$$

$$\leq \varepsilon (\varepsilon - \beta_1) \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \left[ \frac{1}{2} \frac{d}{dt} |u|^2 + (\varepsilon - c |z(\theta_{-t}\omega)|) |u|^2 \right] \, dx.$$
\[
(\varepsilon \alpha - 1) \int_{\mathbb{R}^n} (-\Delta u) \rho \left[ \frac{|x|^2}{k^2} \right] v dx \\
= (\varepsilon \alpha - 1) \int_{\mathbb{R}^n} (\nabla u) \nabla \left[ \rho \left[ \frac{|x|^2}{k^2} \right] \left[ \frac{d u}{d t} + \varepsilon u - c u z(\theta_\omega) \right] \right] dx \\
= (\varepsilon \alpha - 1) \int_{\mathbb{R}^n} \nabla u \left[ \frac{2x}{k^2} \right] v dx \\
+ (\varepsilon \alpha - 1) \int_{\mathbb{R}^n} (\nabla u) \left[ \rho \left[ \frac{|x|^2}{k^2} \right] \nabla \left[ \frac{1}{2} \frac{d u}{d t} + \varepsilon u - c u z(\theta_\omega) \right] \right] dx \\
\leq (\varepsilon \alpha - 1) \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{k^2} \right] \left[ \frac{d u}{d t} + \varepsilon |\nabla u|^2 - c |\nabla u|^2 z(\theta_\omega) \right] dx \\
+ (\varepsilon \alpha - 1) \sqrt{2} \phi_1 (||\nabla u||^2 + ||v||^2) ,
\]
and
\[
-\alpha \int_{\mathbb{R}^n} (-\Delta v) \rho \left[ \frac{|x|^2}{k^2} \right] v dx \\
= -\alpha \int_{\mathbb{R}^n} (\nabla v) \nabla \left[ \rho \left[ \frac{|x|^2}{k^2} \right] v \right] dx \\
\leq \alpha f_k \int_{\mathbb{R}^n} \nabla v \left[ \frac{2x}{k^2} \right] \rho \left[ \frac{|x|^2}{k^2} \right] v dx - \alpha f_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{k^2} \right] |\nabla v|^2 dx \\
\leq \alpha f_k \int_{r < |z| < \sqrt{2} k} \frac{2x}{k^2} \phi_1 \nabla v ||v|| dx - \alpha f_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{k^2} \right] |\nabla v|^2 dx \\
\leq \alpha f_k \phi_1 (||\nabla v||^2 + ||v||^2) - \alpha f_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{k^2} \right] |\nabla v|^2 dx .
\]

For the nonlinear term, according to (1.6) and (1.7), (4.20), and applying Young inequality, after detailed computations, we obtain
\[
- \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] f(u) v dx \\
\geq - \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] F(u) dx - \varepsilon k \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] F(u) dx \\
+ (\varepsilon \theta_1 - \theta \varepsilon c |z(\theta_\omega)|) \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |\nabla u|^2 dx \\
+ (\varepsilon \theta_2 - \theta \varepsilon c |z(\theta_\omega)|) \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |v|^2 dx .
\]

By the Cauchy-Schwartz inequality, the Young inequality and \( ||\nabla v||^2 \geq \lambda_1 ||v||^2 \), we deduce that
\[
\int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] g(x,t) v dx \leq \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |g(x,t)|^2 4\alpha \lambda_1 dx + \alpha \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |\nabla v|^2 dx , \tag{4.33}
\]
and
\[
\alpha c z(\theta_\omega) \int_{\mathbb{R}^n} (\nabla u) \nabla \left[ \rho \left[ \frac{|x|^2}{r^2} \right] v \right] dx \leq \alpha c |z(\theta_\omega)| \int_{\mathbb{R}^n} (\nabla u) \nabla \left[ \rho \left[ \frac{|x|^2}{r^2} \right] v \right] dx \tag{4.34}
\]
and
\[
\alpha c z(\theta_\omega) \int_{\mathbb{R}^n} (-\nabla u) \rho \left[ \frac{|x|^2}{r^2} \right] v dx \leq \alpha c |z(\theta_\omega)| \int_{\mathbb{R}^n} (\nabla u) \nabla \left[ \rho \left[ \frac{|x|^2}{r^2} \right] v \right] dx \\
+ (\varepsilon \theta_1 - \theta \varepsilon c |z(\theta_\omega)|) \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |\nabla u|^2 dx \\
\leq \alpha c z(\theta_\omega) \sqrt{2} \phi_1 (||\nabla u||^2 + ||v||^2) + \frac{\alpha c |z(\theta_\omega)|}{2} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |v|^2 dx \\
+ \frac{\alpha \lambda_1 |z(\theta_\omega)|}{2} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] |\nabla u|^2 dx . \tag{4.35}
\]
Collecting (4.29)-(4.35), from (4.28), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v|^2 + \varepsilon(\varepsilon - \beta_1)|u|^2 + (1 - \alpha \varepsilon)|\nabla u|^2 + 2 \tilde{F}(u) \right) dx \\
\leq - (\varepsilon - c|z(\theta_1 \omega)|) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v|^2 + \varepsilon(\varepsilon - \beta_1)|u|^2 + (1 - \alpha \varepsilon - \tilde{c}_\theta \varepsilon)|\nabla u|^2 + \varepsilon k \tilde{F}(u) \right) dx \\
+ \frac{\sqrt{}}{K} \phi_1 \left[ \alpha \left( |\nabla v|^2 + |v|^2 \right) + (1 - \alpha \varepsilon)(|\nabla u|^2 + |v|^2) + \alpha c|z(\theta_1 \omega)|(|\nabla u|^2 + |v|^2) \right] \\
+ \frac{1}{2} \left( c(3\varepsilon + \beta_2)|z(\theta_1 \omega)| + c^2|z(\theta_1 \omega)|^2 \right) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |u|^2 + |v|^2 \right) dx \\
+ \frac{1}{4(\alpha \lambda_1 - \beta_2)} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left| g(x, t) \right|^2 dx + \left( \tilde{c}_\theta \varepsilon - c_{\theta_1 \varepsilon}|z(\theta_1 \omega)| \right) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) dx. 
\tag{4.36}
\]

Moreover, using Lemma 4.1 and \(|\nabla u|^2 \geq \lambda_1 |u|^2|, when \(\varepsilon\) is small enough, need

\[
\begin{align*}
\varrho &= \min \left\{ \varepsilon, \frac{\varepsilon k}{2} \right\}, \\
1 - \alpha \varepsilon - \varepsilon \theta_1 &\geq 1 - \alpha \varepsilon, \\
\alpha_1 &= \min \left\{ \frac{\sqrt{}}{k} \phi_1 (1 - \alpha \varepsilon, \alpha) \right\}, \\
\Upsilon(t, \omega) &= \frac{1}{4(\alpha \lambda_1 - \beta_2)} |q(x, t)|^2 + \left( \tilde{c}_\theta \varepsilon - C_{\theta_1 \varepsilon}|z(\theta_1 \omega)| \right).
\end{align*}
\tag{4.37}
\]

By all the above inequalities, we can write that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v|^2 + \varepsilon(\varepsilon - \beta_1)|u|^2 + (1 - \alpha \varepsilon)|\nabla u|^2 + 2 \tilde{F}(u) \right) dx \\
\leq - (\varrho - c|z(\theta_1 \omega)|) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v|^2 + \varepsilon(\varepsilon - \beta_1)|u|^2 + (1 - \alpha \varepsilon)|\nabla u|^2 + \tilde{F}(u) \right) dx \\
+ \frac{1}{2} \left( c(3\varepsilon + \beta_2)|z(\theta_1 \omega)| + c^2|z(\theta_1 \omega)|^2 \right) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |u|^2 + |v|^2 \right) dx \\
+ \alpha_1 \left( |\nabla v|^2 + |\nabla u|^2 + |v|^2 \right) + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \Upsilon(t, \omega) dx. 
\tag{4.38}
\]

Setting

\[
X = |v|^2 + \varepsilon(\varepsilon - \beta_1)|u|^2 + (1 - \alpha \varepsilon)|\nabla u|^2. 
\tag{4.39}
\]

So, it follows that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left[ X(t, \tau, \omega, X_\tau(\omega)) + \tilde{F}(u) \right] dx \\
\leq -2 \left( \varrho - \frac{1}{2} \left( c(3\varepsilon + \beta_2)|z(\theta_1 \omega)| + c^2|z(\theta_1 \omega)|^2 \right) \right) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left[ X(t, \tau, \omega, X_\tau(\omega)) + \tilde{F}(u) \right] dx \\
+ \alpha_1 \left( |\nabla v|^2 + |\nabla u|^2 + |v|^2 \right) + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \Upsilon(t, \omega) dx. 
\tag{4.40}
\]

Integrating (4.40) over \([\tau, t]\), we find that, for all \(t \geq \tau\)

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left[ X(t, \tau, \omega, X_\tau(\omega)) \right] dx \\
\leq \varepsilon e^{2\tau(t-\tau)} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left[ X_\tau(\omega) + \tilde{F}(u_\tau) \right] dx \\
+ \alpha_1 \int_{\tau}^{t} e^{2\tau(r-\tau)} \left( |\nabla v(r, \tau, \omega, v_\tau)|^2 + |\nabla u(r, \tau, \omega, u_\tau)|^2 + |v(r, \tau, \omega, v_\tau)|^2 \right) dr \\
+ \int_{\tau}^{t} e^{2\tau(r-\tau)} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \Upsilon(r, \theta_1 \omega) dx dr, 
\tag{4.41}
\]
Then we complete the proof.

\[
\int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] \left[ |X(t, \tau, \theta_{-\tau}, X_r(\theta_{-\tau}))|^2_E + 2\tilde{F}(u(t, \tau, \theta_{-\tau}, u_r)) \right] dx \\
\leq e^{2\Gamma(t-\tau)} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] \left[ |X_r(\theta_{-\tau})|^2_E + \tilde{F}(u_r) \right] dx \\
+ \int_{\tau-t}^{0} e^{2\Gamma(r-t)} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] \Upsilon(r, \theta_r, \omega) dx dr \\
+ \alpha_1 \int_{\tau-t}^{0} e^{2\Gamma(r-t)} \left( ||\nabla v(r, \tau, \theta_r, \omega, v_r)||^2 + ||v(r, \tau, \theta_r, \omega, v_r)||^2 \right) dr.
\]

(4.42)

1. such that for all \( \bar{\omega} \geq \bar{\tau} \) by (4.24). Given positive integer \( q \), there are \( \bar{k}_1 = \bar{k}_1(\tau, \omega, B) \), for all \( t \geq \bar{T}_1 \),

\[
\lim_{t \to \infty} e^{-\Gamma \tau} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] \left[ |X_r(\theta_{-\tau})|^2_E + \tilde{F}(u_r) \right] dx \leq 2\zeta.
\]

(4.43)

Together with the Lemma 4.1 and (1.8), there are \( \bar{T}_2 = \bar{T}_2(\tau, B, \omega) > 0 \) and \( \bar{k}_1 = \bar{k}_1(\tau, \omega) \geq 1 \) such that for all \( t \geq \bar{T}_2 \) and \( k \geq \bar{k}_1 \),

\[
\int_{\tau-t}^{0} e^{2\Gamma(r-t)} \int_{\mathbb{R}^n} \rho \left[ \frac{|x|^2}{r^2} \right] \Upsilon(r, \theta_r, \omega) dx dr \leq \zeta.
\]

(4.44)

From Lemma 4.2, there are \( \bar{T}_3 = \bar{T}_3(\tau, B, \omega) > 0 \) and \( \bar{k}_2 = \bar{k}_2(\tau, \omega) \geq 1 \), such that for all \( t \geq \bar{T}_3 \) and \( k \geq \bar{k}_2 \),

\[
\alpha_1 \int_{\tau-t}^{0} e^{2\Gamma(r-t)} \left( ||\nabla v(r, \tau, \theta_r, \omega, v_r)||^2 + ||v(r, \tau, \theta_r, \omega, v_r)||^2 \right) dr \leq \zeta.
\]

(4.45)

Setting

\[
\begin{align*}
\bar{T} &= \max\{\bar{T}_1, \bar{T}_2, \bar{T}_3\}, \\
\bar{k} &= \{\bar{k}_1, \bar{k}_2\},
\end{align*}
\]

(4.46)

by (4.43)-(4.45), we arrive at

\[
||X(t, \tau, \theta_{-\tau}, X_r(\theta_{-\tau}))||^2_{E(\mathbb{R}^n \setminus H_\omega)} \leq 4\zeta.
\]

(4.47)

Then we complete the proof. \( \square \)

Now we derive uniform estimates on the high-mode parts of the solution in bounded domains \( H_{2k} = \{ x \in \mathbb{R}^n : |x| < 2k \} \), these estimates will also be used to establish pullback asymptotic compactness. Denote \( q(s) = I - p(s) \), where \( p(s) \) is the cut-off function defined by (4.24). Given positive integer \( k \), we define two new variables \( \tilde{u} \) and \( \tilde{v} \) by

\[
\begin{align*}
\tilde{u}(t, \tau, \omega, u_r) &= q \left[ \frac{|x|^2}{k^2} \right] u(t, \tau, \omega, u_r), \\
\tilde{v}(t, \tau, \omega, v_r) &= q \left[ \frac{|x|^2}{k^2} \right] v(t, \tau, \omega, v_r).
\end{align*}
\]

(4.48)
Suppose that the condition (1.3)-(1.8) hold. Let $B = (4.54)$

\begin{equation}
\tilde{\phi} \text{ results with}
\end{equation}

Suppose that the problem has a family of eigenfunctions $(3.4)$ satisfies, for all $t$

\begin{equation}
\tau_1 \leq \tau_2 \leq \cdots \leq \tau_i \leq \cdots, \tau_i \to +\infty \ (i \to +\infty),
\end{equation}

such that $(e_i)_{i \in \mathbb{N}}$, is an orthonormal basis of $L^2(2k)$ for given $n$, let

\begin{equation}
X_n = \{e_1, \cdots, e_n\} \text{ and } P_n : L^2(2k) \to X_n
\end{equation}

be the projection operator.

**Lemma 4.4.** Suppose that the condition (1.3)-(1.8) hold. Let $B = \{B(\omega)\}_{\omega \in \Omega} \in D$ and $\varphi(\tau, \omega) \in B(\tau, \omega)$. Then for every $\eta > 0$ and $P-a.e \omega \in \Omega$, there exists $T = T(B, \eta, \omega) > 0$

\begin{equation}
\|I - P_n\|_{E(2k)} \leq \eta.
\end{equation}

**Proof.** Let

\begin{equation}
\begin{cases}
\tilde{u}_{n,1} = P_n \tilde{u}, \\
\tilde{v}_{n,1} = P_n \tilde{v},
\end{cases}
\text{ and }
\begin{cases}
\tilde{u}_{n,2} = (I - P_n) \tilde{u}, \\
\tilde{v}_{n,2} = (I - P_n) \tilde{v}.
\end{cases}
\end{equation}

Multiplying the first equation of (4.50) with $(I - P_n)$, we get

\begin{equation}
\tilde{v}_{n,2} = \frac{d}{dt} \tilde{v}_{n,2} + \varepsilon \tilde{u}_{n,2} - c\tilde{u}_{n,2} \tilde{z}(\theta_1 \omega),
\end{equation}

then applying $(1 - P_n)$ to the second equation of (4.50) and taking the inner product for results with $\tilde{v}_{n,2}$ in $L^2(2k)$, we show that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\tilde{v}_{n,2}\|^2 = (\varepsilon - g(u)) \|\tilde{v}_{n,2}\|^2 - \alpha \|\nabla \tilde{v}_{n,2}\| - \alpha \left( \varepsilon \Delta \tilde{u}_{n,2} \right)^2 + 2\nabla v \nabla q \left( \frac{|x|^2}{k^2}, \tilde{v}_{n,2} \right)
\end{equation}

\begin{equation}
+ \varepsilon (\varepsilon - g(u)) \varepsilon \Delta (\tilde{u}_{n,2}, \tilde{v}_{n,2}) + (1 - \varepsilon \alpha) \left( \Delta \tilde{u}_{n,2} - u \Delta q \left( \frac{|x|^2}{k^2} \right) - 2\nabla u \nabla q \left( \frac{|x|^2}{k^2} \right), \tilde{v}_{n,2} \right)
\end{equation}

\begin{equation}
+ c (3\varepsilon - g(u))\varepsilon \Delta (\tilde{u}_{n,2}, \tilde{v}_{n,2}) \tilde{z}(\theta_1 \omega) + (I - P_n) q \left( \frac{|x|^2}{k^2} \right) (q(x, t) - f(u), \tilde{v}_{n,2})
\end{equation}

\begin{equation}
+ \alpha c \left( \Delta \tilde{u}_{n,2} - u \Delta q \left( \frac{|x|^2}{k^2} \right) - 2\nabla u \nabla q \left( \frac{|x|^2}{k^2} \right), \tilde{v}_{n,2} \right) \tilde{z}(\theta_1 \omega) - c \|\nabla \tilde{v}_{n,2}\| \tilde{z}(\theta_1 \omega). \end{equation}
Next, we need to estimate each term on the right-hand side of (4.54)
\[
\varepsilon(\varepsilon - g(u))(\bar{u}_{n,2}, \bar{v}_{n,2}) = \varepsilon(\varepsilon - g(u)) \left( \frac{d}{dt} \bar{u}_{n,2} + \varepsilon \bar{u}_{n,2} - c \bar{u}_{n,2} z(\theta_{t}\omega) \right)
\leq \varepsilon(\varepsilon - \beta_1) \left( \frac{d}{dt} \|\bar{u}_{n,2}\|^2 + (\varepsilon - c|z(\theta_{t}\omega)|) \|\bar{u}_{n,2}\|^2 \right),
\]
and
\[
(\varepsilon - 1)(-\Delta \bar{u}_{n,2}, \bar{v}_{n,2})
= (\varepsilon - 1)(-\Delta \bar{u}_{n,2}, \frac{d}{dt} \bar{u}_{n,2} + \varepsilon \bar{u}_{n,2} - c \bar{u}_{n,2} z(\theta_{t}\omega))
\leq (\varepsilon - 1)(\nabla \bar{u}_{n,2}, \nabla (\frac{d}{dt} \bar{u}_{n,2} + \varepsilon \bar{u}_{n,2} - c \bar{u}_{n,2} z(\theta_{t}\omega)))
\leq (\varepsilon - 1)(\nabla \bar{u}_{n,2}, \nabla (\frac{d}{dt} \bar{u}_{n,2} - (-\varepsilon + c|z(\theta_{t}\omega)|) \bar{u}_{n,2})
\leq (\varepsilon - 1)\frac{1}{2} \left( \frac{d}{dt} \|\nabla \bar{u}_{n,2}\|^2 - (\varepsilon - c|z(\theta_{t}\omega)|) \|\nabla \bar{u}_{n,2}\|^2 \right).
\]

For the nonlinear term, by (1.4)-(1.7) we have
\[
\begin{align*}
&\left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f(u, \bar{v}_{n,2}) \\
&\leq \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f(u, \frac{d}{dt} \bar{u}_{n,2} + \varepsilon \bar{u}_{n,2} - c \bar{u}_{n,2} z(\theta_{t}\omega))
\leq \frac{d}{dt} \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f(u, \bar{u}_{n,2}) - \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f_u(u, \bar{u}_{n,2})
+ (\varepsilon c - c|z(\theta_{t}\omega)) \right) \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f(u, \bar{u}_{n,2}).
\end{align*}
\]

By (1.4), using Hölder inequality and Gagliardo-Nirenberg inequality, the nonlinear term in (4.54) satisfies
\[
2\varepsilon \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f_u(u, \bar{u}_{n,2})
\leq c_1 2\varepsilon \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] (1 + \|u\|^{p'}) \|u\| \|\bar{u}_{n,2}\|
\leq M_1 ||u|| \|\bar{u}_{n,2}\| + M_2 ||u\|^{p} ||u\| ||\bar{u}_{n,2}\| ^{\frac{2}{p}} \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
\leq M_1 \lambda_{n+1}^{\frac{1}{2}} \|u\| \|\nabla \bar{u}_{n,2}\| + M_3 \|u\|^{p} \|u\| \|\bar{u}_{n,2}\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
\leq M_1 \lambda_{n+1}^{-\frac{1}{2}} \|u\| \|\nabla \bar{u}_{n,2}\| + M_3 \lambda_{n+1}^{-1} \|u\|^{p} \|u\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
\leq M_4 \|\nabla \bar{u}_{n,2}\| + M_5 \lambda_{n+1}^{-\frac{1}{2}} \|u\|^{p} \|u\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
\leq M_4 \|\nabla \bar{u}_{n,2}\| + M_6 \lambda_{n+1}^{-1} \|u\|^{p} \|u\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
+ \|u\|^{p} \|u\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}} (\|u\|^{p} \|u\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}).
\]

Similarly, we also have
\[
2\varepsilon \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] f(u, \bar{u}_{n,2})
\leq c_1 2\varepsilon \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] (1 + \|u\|^{p+1}) \|\bar{u}_{n,2}\|
\leq M_7 \|u\| \|\bar{u}_{n,2}\| + M_8 \|u\|^{p+1} \|\bar{u}_{n,2}\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
\leq M_7 \lambda_{n+1}^{-\frac{1}{2}} \|u\| \|\nabla \bar{u}_{n,2}\| + M_8 \lambda_{n+1}^{-1} \|u\|^{p+1} \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
\leq M_10 \|\nabla \bar{u}_{n,2}\| + M_11 \lambda_{n+1}^{-1} \|u\|^{p} \|u\| \|\nabla \bar{u}_{n,2}\| ^{\frac{2}{p}}
.
\]

By using the Cauchy-Schwartz inequality and the Young inequality, we have that
\[
\begin{align*}
&\left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] q(x, t, \bar{v}_{n,2}) \leq \frac{2\lambda_1}{2\lambda_1 + \delta + 1} \left( I - P_n \right) q \left[ \frac{|x|^2}{k^2} \right] q(x, t) ^{2}
+ \frac{2\lambda_1 + \delta + 1}{2} \|\bar{v}_{n,2}\| ^{2},
\end{align*}
\]
\[
\begin{align*}
c ((3\varepsilon + \beta_2) - cz(\theta_{t}\omega)) (\bar{u}_{n,2}, \bar{v}_{n,2}) z(\theta_{t}\omega)
\leq \frac{1}{2} ((3\varepsilon + \beta_2) c |z(\theta_{t}\omega)| + \frac{1}{2} c^2 |z(\theta_{t}\omega)|^2) (\|\bar{u}_{n,2}\|^2 + \|\bar{v}_{n,2}\|^2).
\end{align*}
\]
Next, we have
\[
\left(1 - u \right) \left[ \frac{\|x\|^2}{k^2} \right] - 2 \frac{\nabla u \nabla \eta}{k^2}, \bar{v}_{n,2} \leq \left(1 - u \right) \left[ \frac{\|x\|^2}{k^2} \right] + 2 \frac{\nabla u \nabla \eta}{k^2}, \bar{v}_{n,2} \right)
\]
(4.62)
\[
\leq f_{k<|x|<2\sqrt{k}} \left(1 - u \right) \left[ \frac{\|x\|^2}{k^2} \right] + 2 \frac{\nabla u \nabla \eta}{k^2}, \bar{v}_{n,2} \right)
\]
(4.63)
\[
\alpha(v\eta) \left[ \frac{\|x\|^2}{k^2} \right] - 2 \frac{\nabla v \nabla \eta}{k^2}, \bar{v}_{n,2} \right)
\]
(4.64)
By applying (4.55)-(4.63) and (4.54), after detailed computations we obtain
\[
\frac{d}{dt} \left( \frac{\|\bar{v}_{n,2}\|^2 + (1 - \varepsilon \alpha)\|\nabla \bar{u}_{n,2}\|^2}{2} + \left(2 - \frac{\|\bar{v}_{n,2}\|^2 + (1 - \varepsilon \alpha)\|\nabla \bar{u}_{n,2}\|^2}{2} \right) \right)
\]
(4.65)
where $\gamma = \min[\varepsilon_3, \varepsilon]$, note that $0 \leq p < 3, \lambda_n \to +\infty$ when $(n \to +\infty)$. Therefore, given $\eta > 0$, there exists $N = N(\eta) > 1$ and $K_1 = K_1(\eta) > 0$ such that for all $n \geq N, k \geq K_1$:

$$\frac{d}{dt} \left( \|\psi_{n,2}\|_E^2 + \left(2(I - P_n)q \left[ \frac{|x|^2}{k^2} \right] f(u), \bar{u}_{n,2} \right) \right) \leq 2\Gamma(t, \omega) \left( \|\psi_{n,2}\|_E^2 + \left(2(I - P_n)q \left[ \frac{|x|^2}{k^2} \right] f(u), \bar{u}_{n,2} \right) \right) + \mu \left( \|I - P_n\| \left[ \frac{|x|^2}{k^2} \right] q(x, t) \right) + 2\tau \|\nabla u\|_2^2 + 4\|\nabla v\|_2^2 + \frac{2}{k^2} \|v\|_2^2 + \frac{\bar{\rho}}{k^2} \|\bar{\rho}\|_2^2 \right)$$

(4.66)

where $\mu, \lambda, c$ are positive constants, $\Gamma(t, \omega) = \gamma - \kappa \|z(\theta_{\omega})\| - c^2 \|z(\theta_{\omega})\|^2$. Applying Gronwall’s lemma over $[\tau - \tau, \tau]$ and replacing $\omega$ by $\theta_{\tau, \omega}$ on (4.66), for all $t \geq 0$, then we estimate every term on the right-hand side of the results.

For the first terms, by using condition (1.4), (4.59), (4.66) and estimate every term on the right-hand side of the results.

$$e^2 \int_{\tau - \tau}^{\tau} \Gamma(r, \omega) \left( \|\psi_{r,2}\|_E^2 + \left(2(I - P_n)q \left[ \frac{|x|^2}{k^2} \right] f(u), \bar{u}_{n,\tau} \right) \right) \leq 2\eta, \quad (4.67)$$

by conditions (1.7) and (1.8), $q(x, t) \in L^2(\mathbb{R}^n)$, there is $N_2 = N_2(\omega, \eta) \geq N_2 > 0$, such that for all $n \geq N_2$,

$$e \int_{-\infty}^{\tau} e^2 \int_{\tau - \tau}^{\tau} \Gamma(r, \omega) \left( \|\psi_{r,2}\|_E^2 + \left(2(I - P_n)q \left[ \frac{|x|^2}{k^2} \right] f(u), \bar{u}_{n,\tau} \right) \right) d\tau \leq \eta. \quad (4.68)$$

Due to Lemma 4.1, Lemma 4.2 and (4.66), there exist $T_2 = T_2(\omega, \eta) > 0$ and $K_2 = K_2(\omega, \eta) > 0$ such that for all $t \geq T_2$ and $k \geq K_2$:

$$\int_{\tau - \tau}^{\tau} e^2 \int_{\tau - \tau}^{\tau} \left( \|u(s, \tau, \theta_{\tau, \omega}, \bar{u}_{\tau})\| + \|u(s, \tau, \theta_{\tau, \omega}, \bar{u}_{\tau})\| \right) ds \leq \eta.$$  

(4.69)

For the fourth term on the right-hand side of results, by Lemmas 3.1, 4.1, 4.2, (19) Lemma 4.3 and (4.59), there exists $T_3 = T_3(\omega, \eta) > 0$, such that for all $t \geq T_3$:

$$c \eta \int_{\tau - \tau}^{\tau} e^2 \int_{\tau - \tau}^{\tau} \left( \|u(\tau, \tau, \theta_{\tau, \omega}, \bar{u}_{\tau})\| + \|v(\tau, \tau, \theta_{\tau, \omega}, \bar{u}_{\tau})\| \right) ds \leq \eta, \quad (4.70)$$

Let

$$T = \max\{T_1, T_2, T_3\} \quad K = \{K_1, K_2\}, \quad (4.71)$$

then by (4.67)-(4.70) and (4.66), there is $N_3 = N_3(\omega, \eta) \geq N_2 > 0$ such that for all $n \geq N_3, K \geq K_3$, we have for all $t > T$ and $n > N$:

$$\|\psi_{n,2}(\tau, \tau, \theta_{\tau, \omega}, \psi_{\tau})\|_E^2 + \left(2(I - P_n)q \left[ \frac{|x|^2}{k^2} \right] f(u), \bar{u}_{n,\tau} \right) \leq \eta(4 + \mathbb{R}(\tau, \omega)) \quad (4.72)$$

which implies formula (4.72) holds. Therefore, we conclude

$$\|\psi_{n,2}(t, \theta_{\tau, \omega}, \psi_{\tau}(\theta_{\tau, \omega}))\|_E^2 \leq \eta(4 + \mathbb{R}(\tau, \omega)).$$

Then we complete the proof. \qed
5. Random attractors

In this section, we prove the existence of a $\mathcal{D}$-random attractor for the random dynamical system $\Phi$ associated with the stochastic wave equation (3.7) on $\mathbb{R}^n$. We are now ready to apply the lemmas in Section 4 to prove the asymptotic compactness of solutions on $\mathbb{R}^n$. It follows from Lemma 4.1, that it is $\Phi$ has a closed random absorbing set in $\mathcal{D}$, which along with the $\mathcal{D}$-pullback asymptotic compactness will imply the existence of a unique $\mathcal{D}$-random attractor. The $\mathcal{D}$-pullback asymptotic compactness of $\Phi$ is given below and will be proved by using the uniform estimates on the tails of solutions.

**Lemma 5.1.** We assume that (1.3)-(1.8) hold. Then the random dynamical system $\phi$ of problem (3.4) is $\mathcal{D}$-pullback asymptotically compact in $E(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, $\omega \in \Omega$, the sequence $\{\varphi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\}$ has a convergent subsequence in $E(\mathbb{R}^n)$ provided $t_n \to \infty$ and $\varphi_{\tau, n} \in B(\tau - t_n, \theta_{-t_n} \omega)$.

**Proof.** Let $t_n \to \infty$, $B \in \mathcal{D}$ and $\varphi_{\tau, n} \in B(\tau - t_n, \theta_{-t_n} \omega)$. Then by Lemma 4.1, for P-a.e $\omega \in \Omega$, we have that $\{\varphi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\}$ is bounded in $E(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$, there exists $L_1 = L_1(\tau, \omega, B) > 0$, such that for all $\tilde{L} \geq L_1$,

$$\|\varphi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\|^2_{E(\mathbb{R}^n)} \leq \rho^2(\tau, \omega); \quad (5.1)$$

moreover, it follows from Lemma 4.3 that there exist $k_1 = k_1(\tau, \omega, \eta) > 0$ and $L_2 = L_2(\tau, B, \omega, \eta) > 0$, such that for all $\tilde{L} \geq L_2$,

$$\|\psi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\|^2_{E(\mathbb{R}^n \setminus H_{k_1})} \leq \eta. \quad (5.2)$$

Next, by using Lemma 4.4, there are $N = N(\tau, \omega, \eta) > 0$, $k_2 = k_2(\tau, \omega, \eta) > k_1$ and $L_3 = L_3(\tau, B, \omega, \eta) > 0$, such that for all $\tilde{L} \geq L_3$,

$$(I - P_n)\|\varphi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\|^2_{E(H_{2k_2})} \leq \eta. \quad (5.3)$$

By means of (4.49) and (5.1), we find that $\{P_n \varphi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\}$ is bounded infinite-dimensional space $P_n E(H_{2k_2})$, which associated with (5.3) implies $\{\psi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\}$ is precompact in $H_0^1(H_{2k_2}) \times L^2(H_{2k_2})$.

Note that $q \left[\frac{|x|^2}{k_2}\right] = 1$ for $x \in \mathbb{R}^n : |x| \leq k_2$, recalling (4.49), we find that $\{\varphi(\tau, \tau - t_n, \theta_{-\tau t_n}, \varphi_{\tau, n})\}$ is pre-compact in $E(H_{2k_2})$, which along with (5.2) show that the precompactness of this sequence in $E(\mathbb{R}^n)$, this completes the proof. The main result of this section can now be stated as follows.

**Theorem 5.2.** We assume that (1.3)-(1.8) hold. Then the continuous cocycle $\Phi$ associated with problem (3.4) has a unique $\mathcal{D}$-pullback attractor $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $\mathbb{R}^n$.

**Proof.** Notice that the continuous cocycle $\Phi$ has a closed random absorbing set $\{A(\omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$ by Lemma 4.2. On the other hand, by (3.9) and Lemma 5.1, the continuous cocycle $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $\mathbb{R}^n$. Hence the existence of a unique $\mathcal{D}$-random attractor for $\Phi$ follows from Lemma 2.8 immediately.

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