A vectorial binary Darboux transformation for the first member of the negative part of the AKNS hierarchy

Folkert Müller-Hoissen

Institut für Theoretische Physik, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany

E-mail: folkert.mueller-hoissen@phys.uni-goettingen.de

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Abstract
Using bidifferential calculus, we derive a vectorial binary Darboux transformation for the first member of the ‘negative’ part of the AKNS hierarchy. A reduction leads to the first ‘negative flow’ of the NLS hierarchy, which in turn is a reduction of a rather simple nonlinear complex PDE in two dimensions, with a leading mixed third derivative. This PDE may be regarded as describing geometric dynamics of a complex scalar field in one dimension, since it is invariant under coordinate transformations in one of the two independent variables. We exploit the correspondingly reduced vectorial binary Darboux transformation to generate multi-soliton solutions of the PDE, also with additional rational dependence on the independent variables, and on a plane wave background. This includes rogue waves.

Keywords: AKNS hierarchy, binary Darboux transformation, rogue wave, soliton, bidifferential calculus

1. Introduction

The main subject of this work is the third-order nonlinear PDE

\[
\left( \frac{f_u}{f} \right)_t + 2 (f^*)_x = 0,
\]

1 (Some figures may appear in colour only in the online journal)
where $f$ is a complex function of two independent real variables $x$ and $t$, and $f^*$ is the complex conjugate of $f$. A subscript denotes a partial derivative with respect to one of the independent variables. An evident property of (1.1) is the following.

**Proposition 1.1.** If $f(x, t)$ solves (1.1), then also $f(\sigma(x), t)$, with an arbitrary differentiable function $\sigma(x)$. □

This expresses the fact that (1.1) is invariant under coordinate transformations $x \mapsto \sigma(x)$ in one dimension, and $f$ can be regarded as a scalar. A generalization of (1.1) to higher dimensions is the system

\[
\frac{\partial}{\partial t} \left( \int^{-1} \frac{\partial}{\partial t} \frac{\partial}{\partial x^\mu} f \right) + 2 \frac{\partial}{\partial x^\mu} (f^* f) = 0 \quad \mu = 1, \ldots, m.
\]

(1.2)

It behaves as the components of a covector (tensor of type $(0,1)$) under general coordinate transformations in $m$ dimensions, if $f$ is a scalar, also depending on a parameter $t$. This system thus defines dynamics of a scalar field on an $m$-dimensional differentiable manifold. Obviously, the following holds.

**Proposition 1.2.** If $f(x, t)$ solves (1.1), then $f(\sigma(x^1, \ldots, x^m), t)$, with an arbitrary differentiable function $\sigma$ of real independent variables $x^\mu$, $\mu = 1, \ldots, m$, solves (1.2). □

(1.2) can also be written as

\[
\frac{\partial}{\partial t} \left( \int^{-1} \frac{\partial}{\partial t} \frac{\partial}{\partial x^\mu} f \right) + 2 d(f^* f) = 0,
\]

where $d$ is the exterior derivative on the $m$-dimensional differentiable manifold.

(1.1) arises, via a reduction, from the first ‘negative flow’ of the AKNS hierarchy, which is related to the complex sine-Gordon equation [5]. In [6] a relation with the sharp line self-induced transparency (SIT) equations has been observed, also see [7, 8]. Furthermore, it is connected with a two-component Camassa–Holm equation [9, 10]. Soliton solutions have been found in [11], using Hirota’s bilinear method. Other methods have been applied in particular in [7] (see section 5.1 therein), [8], and [12] (see (4.28) therein).

For real $f$, (1.1) is among the simplest completely integrable PDEs and it has the peculiar property mentioned above. In this work, we derive a binary Darboux transformation (see, e.g. [13]) for (more precisely, a reduction of) (1.1) with complex $f$, by using a general result of bidifferential calculus, which is recalled in section 4.1 in a self-contained way (see, e.g. [14] for an introduction to bidifferential calculus and references), and demonstrate its use for finding soliton solutions. We proceed beyond the class of ‘simple solitons’, which are rational expressions built from trigonometric and hyperbolic functions of linear combinations of the independent variables. There are also regular solutions which additionally depend *rationally* on the variables $x$ and $t$, as in the case of the related nonlinear Schrödinger (NLS) equation (see, in particular, [15–17]).

A binary Darboux transformation for (1.1) has already been obtained in [18–20]\(^2\). The advantages of a *vectorial* binary Darboux transformation, involving a ‘spectral matrix’ instead of a spectral parameter in the underlying linear system, are summarized in remark 3.4 below.

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\(^2\) These authors write (1.1) in the form $u_t + 2u \partial_x \partial_x^{-1}(|u|^2) = u$ (hence $x$ and $t$ are exchanged relative to our notation). But the term on the right hand side should actually be multiplied by an arbitrary function of $t$, since this is the freedom in the definition of an inverse of $\partial_x$. It seems that the authors of [20] wanted to fix the freedom by demanding $u \rightarrow 0$ as $|x| \rightarrow \infty$. But how can this then be reconciled with exact solutions in their section 6, having a non-zero constant background? Throughout our work, there will be no need for introducing the inverse of a differential operator.

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Vectorial binary Darboux transformations for the prominent NLS equation, for example, appeared in [16, 22, 23] (also see [21, 22] for corresponding treatments of Davey–Stewartson equations, which possess a reduction to the NLS equation). Such an efficient solution-generating method has been obtained by now for many completely integrable equations. It is automatically available if the corresponding equation possesses a bidifferential calculus formulation, see section 4, and many examples can be found in particular in [22] and references cited there. For a different framework, see [24].

In section 2 we recall some useful Lie point symmetries of (1.1) and traveling wave solutions in terms of elementary Jacobi elliptic functions (cf [25] and references cited there for corresponding solutions of the ‘positive’ part of the NLS hierarchy).

Section 3 presents our version of an n-fold binary Darboux transformation for a reduction of (1.1), which we then exploit to find multi-soliton solutions. This includes solitons superposed on a plane wave background. Section 4 presents a derivation of the binary Darboux transformation, more generally for the first ‘negative flow’ of the AKNS hierarchy. Finally, section 5 contains some concluding remarks.

### 2. Some symmetries of the PDE and traveling wave solutions

(1.1) admits the following symmetry transformations:

- \( x \mapsto \sigma(x) \), see proposition 1.1.
- \( t \mapsto \pm t + \alpha, \alpha \in \mathbb{R} \).
- \( t \mapsto \pm |\beta| t, f \mapsto \beta f, \beta \in \mathbb{C}, \beta \neq 0 \).
- \( f \mapsto e^{i \phi_0} f, \phi_0 \in \mathbb{R} \).
- Complex conjugation of \( f \).

In the following, solutions will typically be presented modulo these symmetries.

Let us assume that \( f \) is real and, in some coordinate \( x \), has the form

\[
f(x, t) = f(x \pm ct),
\]

with a real constant \( c > 0 \). Then (1.1) reduces to the ODE\(^3\)

\[
f''/f + 2c^2 f^2 = k,
\]

with a real constant \( k \). Exclusively in this section, a prime indicates a derivative with respect to the argument of the function \( f \). Solutions of this equation are provided by the Jacobi elliptic functions \( cn \) and \( dn \) (see [27], for example). Indeed,

\[
f_{cn} = \sqrt{c} \sqrt{m} \ cn \left( \frac{1}{\sqrt{c}} (x \pm ct) \mid m \right)
\]

solves the ODE with \( k = (2m - 1)/c \). We note that

\[
f_{cn} = \sqrt{c} \ sech \left( \frac{1}{\sqrt{c}} (x \pm ct) \right) \quad \text{if} \quad m = 1.
\]

We will recover this solitary wave as a single soliton solution in section 3. Furthermore,

\[
f_{dn} = \sqrt{c} \ dn \left( \frac{1}{\sqrt{c}} (x \pm ct) \mid m \right)
\]

\(^3\) This is equation (7.7) in [26].
satisfies the ODE with \( k = (2 - m)c \). We note that
\[
    f_{dn} = \sqrt{c} \ \text{sech} \left( \frac{1}{\sqrt{c}} (x \pm ct) \right) \quad \text{if} \quad m = 1.
\]

### 3. A binary Darboux transformation and soliton solutions

(1.1) will be treated in the following form,
\[
    a_i = (f^* f)_x, \quad f_{xt} + 2af = 0. \tag{3.1}
\]

This system is invariant under a coordinate transformation \( x \mapsto x' \) if the function \( a \) transforms as \( a \mapsto a' = (\partial x/\partial x')a \).

If \( f(x, t) \) is allowed to be complex, then (1.1) is most likely not completely integrable [28]. But the reduction of (3.1), obtained by restricting the function \( a \) to be real, possesses a Lax pair (cf. remark 3.4 below) and is thus completely integrable in this sense. Although the first of equation (3.1) requires \( a_i \) to be real, this does not exclude an imaginary part of \( a \). As pointed out in [28], there is thus another reduction of (3.1), where \( a = a_1 + ia_2 \), with real functions \( a_j \), \( j = 1, 2, a_2 \neq 0, a_2 = 0 \), and this does not pass the Painlevé test of integrability.

In the following we will only consider the integrable reduction of (3.1), i.e. we will assume that \( a \) is real. We next formulate the main result of this work. A derivation and proof is postponed to section 4.

**Theorem 3.1.** Let \( a_0, f_0 \) be a solution of (3.1) with real \( a_0 \). Let \( n \)-component column vectors \( \eta_i \), \( i = 1, 2 \), be solutions of the linear system
\[
    \Gamma \eta_{1x} = a_0 \eta_1 + f_0^* \eta_2, \quad \Gamma \eta_{2x} = -a_0 \eta_2 + f_0 \eta_1, \tag{3.2}
\]
\[
    \eta_{1t} = -\frac{1}{2} \Gamma \eta_1 + f_0^* \eta_2, \quad \eta_{2t} = \frac{1}{2} \Gamma \eta_2 - f_0 \eta_1, \tag{3.3}
\]
where \( \Gamma \) is an invertible constant \( n \times n \) matrix satisfying the spectrum condition \( \text{spec}(\Gamma) \cap \text{spec}(-\Gamma^+) = \emptyset \). Furthermore, let \( \Omega \) be an invertible solution of the Lyapunov equation
\[
    \Gamma \Omega + \Omega \Gamma^+ = \eta_1 \eta_1^+ + \eta_2 \eta_2^+, \tag{3.4}
\]
where \( ^+ \) denotes Hermitian conjugation (transposition and complex conjugation). Then
\[
    a = a_0 - (\eta_1^+ \Omega^{-1} \eta_1)_x, \quad f = f_0 - \eta_1^+ \Omega^{-1} \eta_2 \tag{3.5}
\]
is also a solution of (3.1). As a consequence, \( f \) solves (1.1). \( \square \)

An application of this theorem essentially reduces to solving the linear system for a given solution \( a_0, f_0 \) of (3.1) and a constant \( n \times n \) matrix \( \Gamma \), since it is well-known that, under the stated spectrum condition, (3.4) has a unique solution \( \Omega \) and there are concrete expressions for it. In order to obtain a more explicit expression for the generated solution \( f \) of (1.1), however, one needs to evaluate the inverse of the matrix \( \Omega \), which is getting more and more difficult with increasing \( n \), of course.

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4 Via the above coordinate invariance, it can then be (at least locally) achieved that \( a_2 \) is a real constant, different from zero.

5 If \( n > 2 \), \( \Omega \) can be decomposed into block submatrices and the inverse can be computed using Schur complements, which are (special) quasideterminants. The latter are used in the alternative iterative approach to a binary Darboux transformation in [18–20].
Without restriction of generality, \( \Gamma \) can be restricted to Jordan normal form. We also note that \( \Omega \) in the preceding theorem is Hermitian (also see (4.18)) and consequently \( \det(\Omega) \) is real. If \( f_0 \) is a regular solution of (1.1) on \( \mathbb{R}^2 \), a solution \( f \) generated via the above theorem can only be singular if \( \Omega \) is not invertible somewhere on \( \mathbb{R}^2 \), i.e. if \( \det(\Omega) \) has a zero.

**Proposition 3.2.** If \( a_0, f_0 \) with \( f_0 \neq 0 \) and real \( a_0 \), solve (3.1), then the linear system (3.2) and (3.3) is equivalent to

\[
\eta_{1x} - \frac{f_{0x}}{f_0} \eta_{1t} - \left( \frac{1}{4} \Gamma^2 + \frac{f_{0t}}{2f_0} \Gamma - |f_0|^2 \right) \eta_1 = 0, \quad (3.6)
\]
\[
\Gamma \eta_{1x} - \frac{f_{0x}}{f_0} \eta_{1t} - \left( a_0 + \frac{f_{0t}}{2f_0} \Gamma \right) \eta_1 = 0, \quad (3.7)
\]
\[
\eta_2 = \frac{1}{f_0} \left( \eta_{1r} + \frac{1}{2} \Gamma \eta_1 \right). \quad (3.8)
\]

**Proof.** (3.8) is the first equation in (3.3), solved for \( \eta_2 \). The second of (3.3) is then equivalent to (3.6). By use of (3.8), the first of (3.2) takes the form (3.7). As a consequence of these equations, the second of (3.2) is satisfied iff \( a_0, f_0 \) solve (3.1).

**Remark 3.3.** If \( f_{0x} = 0 \), the second of equation (3.1) requires \( a_0 = 0 \), (3.2) then restricts \( \eta_1 \) and \( \eta_2 \) to not depend on \( x \). Since any function independent of \( x \) solves (1.1), such an \( f_0 \) is not a useful ‘seed’ for the binary Darboux transformation in theorem 3.1.

**Remark 3.4.** Writing (3.2) and (3.3) in the form

\[
\begin{pmatrix}
\eta_1 \\
\eta_2 
\end{pmatrix}_x = \begin{pmatrix}
a_0 \Gamma^{-1} & f_{0x} \Gamma^{-1} \\
f_{0x} \Gamma^{-1} & -a_0 \Gamma^{-1}
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 
\end{pmatrix}, \quad \begin{pmatrix}
\eta_1 \\
\eta_2 
\end{pmatrix}_t = \begin{pmatrix}
-\frac{1}{2} \Gamma & f_0 I_n \\
-f_0 I_n & \frac{1}{2} \Gamma
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 
\end{pmatrix},
\]

where \( I_n \) is the \( n \times n \) identity matrix, constitutes a ‘Lax pair’ for (3.1) with real \( a \), since its integrability condition is equivalent to \( a_0, f_0 \) satisfying (3.1) and \( a_0 = a_0^* = a_0 \). It should be noticed that the usual spectral parameter is promoted to a matrix \( \Gamma \), which is typical for a vectorial generalization of a (binary) Darboux transformation. It has the effect that there is no need to consider iterations of Darboux transformations (in contrast to the older approach taken, e.g. in [18–20]). One obtains the result of an \( n \)-fold elementary binary Darboux transformation right away in a single step and efficient matrix methods can be used to elaborate concrete solutions. A particular advantage lies in the fact that important classes of completely integrable equations are directly obtained by choosing the ‘spectral matrix’ to be a (non-diagonal) Jordan matrix (see [16] for an early example). This concerns, in particular, the rogue wave solutions of the nonlinear Schrödinger equation (see [23] and references cited there). In some approaches, they can only be obtained indirectly, starting from a solution obtained by a multiple application of an elementary binary Darboux transformation, each time with a different value of the spectral parameter (which then corresponds, in the vectorial setting, to choosing a diagonal spectral matrix with distinct eigenvalues), and then taking suitable coincidence limits of the spectral parameter values (and possibly other parameters).

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6 Nevertheless, the classical iterative (forward, backward and binary) Darboux transformations can also be formulated in bidifferential calculus, see [14].
3.1. Some results about the Lyapunov equation

If \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n) \), with \( \gamma_i \neq -\gamma_j^* \), \( i, j = 1, \ldots, n \), the solution of (3.4) is the Cauchy-like matrix

\[
\Omega = \left( \frac{\eta_i \eta_j^* + \eta_j \eta_i^*}{\gamma_i + \gamma_j^*} \right).
\]

In the following subsections we will also consider the case where \( \Gamma \) is a Jordan matrix. Therefore we recall some results from [23]. For fixed \( n > 1 \), let \( \eta_i = (\eta_{i1}, \eta_{i2}, \ldots, \eta_{in})^T \), \( i = 1, 2 \). For \( 1 \leq k \leq n \), let \( \Omega_{(k)} \) be the solution of (3.4), where \( \Gamma \) is the \( k \times k \) lower triangular Jordan matrix

\[
\Gamma_{(k)} = \begin{pmatrix}
\gamma & 0 & \ldots & 0 \\
1 & \gamma & \ddots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \gamma
\end{pmatrix},
\]

and \( \eta_i \) is replaced by \( \eta_{(k)} := (\eta_{i1}, \eta_{i2}, \ldots, \eta_{ik})^T \), \( i = 1, 2 \).

**Proposition 3.5.** For \( 1 \leq k \leq n - 1 \), we have

\[
\Omega_{(k+1)} = \begin{pmatrix} \Omega_{(k)} & B_{k+1} \\ B_{k+1}^T & \omega_{k+1} \end{pmatrix},
\]

where

\[
B_{k+1} := \frac{1}{\kappa} K_{(k)}^{-1} \left( \eta_{(k)} \eta_{1,k+1}^* + \eta_{2(k)} \eta_{2,k+1}^* - \Omega_{(k)} (0, \ldots, 0, 1)^T \right),
\]

\[
\omega_{k+1} := \frac{1}{\kappa} \left( |\eta_{1,k+1}|^2 + |\eta_{2,k+1}|^2 - 2 \text{Re}[(0, \ldots, 0, 1)B_{k+1}] \right),
\]

and \( K_{(k)} \) is the Jordan matrix \( \Gamma_{(k)} \) with \( \gamma \) replaced by \( \kappa \).

This proposition allows to recursively compute the solution of the Lyapunov equation (3.4) with a Jordan matrix \( \Gamma_{(r)} \). The inverse of \( \Omega_{(r)} \) can also be recursively computed via

\[
\Omega_{(k+1)}^{-1} = \begin{pmatrix}
\Omega_{(k)}^{-1} - S_{\Omega_{(r)}}^{-1}B_{k+1}B_{k+1}^T\Omega_{(k)}^{-1} & -S_{\Omega_{(r)}}^{-1}B_{k+1}\Omega_{(k)}^{-1} \\
-S_{\Omega_{(r)}}^{-1}B_{k+1}^T\Omega_{(k)}^{-1} & S_{\Omega_{(r)}}^{-1}
\end{pmatrix},
\]

with the scalar Schur complement

\[
S_{\Omega_{(r)}} = \omega_{k+1} - B_{k+1}^T\Omega_{(r)}^{-1}B_{k+1}.
\]

**Example 3.6.** For \( n = 2 \), we obtain

\[
\Omega_{(2)} = \frac{1}{\kappa} \sum_{i=1}^{2} \begin{pmatrix}
|\eta_{i1}|^2 & \eta_{i1}(\eta_{2i} - \kappa^{-1}\eta_{i1})^* \\
\eta_{i1}(\eta_{2i} - \kappa^{-1}\eta_{i1}) & |\eta_{i1}|^2 + \kappa^{-2}|\eta_{i1}|^2
\end{pmatrix},
\]

7 A superscript T indicates the transpose of a matrix.
Its determinant is
\[
\det(\Omega(2)) = \kappa^{-4}(|\eta_{11}|^2 + |\eta_{21}|^2)^2 + \kappa^{-2} |\det(\eta_1, \eta_2)|^2,
\]
where \(\det(\eta_1, \eta_2) = \eta_{11}\eta_{22} - \eta_{12}\eta_{21}\).

**Proposition 3.7.** Let \(\Gamma\) be a lower triangular \(n \times n\) Jordan matrix with eigenvalue \(\gamma\) and \(\text{Re}(\gamma) \neq 0\). If the first component of one of the vectors \(\eta_i, i = 1, 2\), is different from zero, then the solution of (3.4) is invertible. \(\square\)

Unfortunately, to our knowledge, a convenient formula for the determinant of the solution of the Lyapunov equation with an \(n \times n\) Jordan matrix \(\Gamma\) is only available in the case where its right hand side is a rank one matrix.

### 3.2. Zero seed solutions

If \(f_0 = 0\), we choose \(a_0 = -1/2\). The linear system for \(\eta\) then has the solutions
\[
\eta_1 = \exp\left(-\frac{1}{2}(\Gamma^{-1}x + \Gamma t)\right)v, \quad \eta_2 = \exp\left(\frac{1}{2}(\Gamma^{-1}x + \Gamma t)\right)w,
\]
where \(v, w\) are constant \(n\)-component column vectors. The ansatz
\[
\Omega = e^{-\frac{1}{2}(\Gamma^{-1}x + \Gamma t)}Xe^{-\frac{1}{2}(\Gamma^\dagger^{-1}x + \Gamma^\dagger t)} + e^{\frac{1}{2}(\Gamma^{-1}x + \Gamma t)}Ye^{\frac{1}{2}(\Gamma^\dagger^{-1}x + \Gamma^\dagger t)},
\]
with constant \(n \times n\) matrices \(X, Y\), solves (3.4) if
\[
\Gamma X + X\Gamma^\dagger = vv^\dagger, \quad \Gamma Y + Y\Gamma^\dagger = ww^\dagger.
\]
According to theorem 3.1,
\[
f = v^\dagger e^{-\frac{1}{2}(\Gamma^\dagger^{-1}x + \Gamma^\dagger t)}\Omega^{-1} e^{\frac{1}{2}(\Gamma^{-1}x + \Gamma t)}w
= \frac{1}{\det(\Omega)} v^\dagger e^{-\frac{1}{2}(\Gamma^\dagger^{-1}x + \Gamma^\dagger t)} \text{adj}(\Omega) e^{\frac{1}{2}(\Gamma^{-1}x + \Gamma t)}w,
\]
where adj takes the adjugate of a matrix, represents (an infinite set of) exact solutions of (1.1). Here we dropped a global minus sign, since \(f \leftrightarrow -f\) is a symmetry of (1.1). We also note that \(\Gamma \leftrightarrow -\Gamma\) together with \((x, t) \leftrightarrow -(x, t)\) amounts to \(f \leftrightarrow -f\). Furthermore, \(\Gamma \leftrightarrow -\Gamma\) together with exchange of \(v\) and \(w\) means \(f \leftrightarrow -f^*\).

#### 3.2.1. Simple multi-soliton solutions

These are obtained by choosing \(\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)\), where \(\gamma_i^* \neq -\gamma_j, i, j = 1, \ldots, n\). The solutions of the Lyapunov equation (3.13) are then the Cauchy-like matrices
\[
X = \begin{pmatrix} v_j w_j^* \\ \gamma_i + \gamma_j^* \end{pmatrix}, \quad Y = \begin{pmatrix} w_j v_j^* \\ \gamma_i + \gamma_j^* \end{pmatrix}.
\]
Substituting these expressions in (3.12) and (3.14) provides us with an exact solution of (1.1) for any \(n \in \mathbb{N}\).

**n = 1.** In this case, (3.14) becomes
\[
f = 2 \text{Re}(\gamma) v w \left(|v|^2 e^{-(\gamma^{-1}x + \gamma t)} + |w|^2 e^{\gamma^{-1}x + \gamma^* t}\right)^{-1}.
\]
If \(\gamma, v, w\) are real, this can be rewritten, up to a global sign, as
\[
f = \gamma \text{sech}(\gamma^{-1}x + \gamma t + \alpha), \quad (3.15)
\]
where \( \alpha = \ln(|w/v|) \). The solitary wave has the form of the bright soliton of the NLS equation and the solitary wave of the modified \( \text{KdV} \) (m\( \text{KdV} \)) equation.

**n = 2.** (3.14) yields

\[
\begin{align*}
    f &= \frac{1}{2\Re(\gamma_1)(\gamma_1 + \gamma_2^*) \det(\Omega)} e^{-\Re(\gamma_1)(t+x)/|\gamma_1|^2} + \Im(\gamma_2)(t-x/|\gamma_2|^2) \\
    &\quad \times \left( (\gamma_2^* - \gamma_1^*) |v_1|^2 v_2^* w_2 + (\gamma_1 + \gamma_2^*) v_2^* |w_1|^2 w_2 e^{2\Re(\gamma_1)(t+x)/|\gamma_1|^2} \right) \\
    &\quad - 2\Re(\gamma_1) v_1^* w_1 |w_2|^2 e^{(\gamma_1 + \gamma_2^*) t + (\gamma_1 - \gamma_2^*) x} \\
    &\quad + \frac{1}{2\Re(\gamma_2)(\gamma_1 + \gamma_2) \det(\Omega)} e^{-\Re(\gamma_2)(t+x)/|\gamma_2|^2} + \Im(\gamma_1)(t-x/|\gamma_1|^2) \\
    &\quad \times \left( (\gamma_1^* - \gamma_2^*) v_2^* |v_2|^2 w_1 + (\gamma_1^* + \gamma_2) v_1^* w_1 |w_2|^2 e^{2\Re(\gamma_2)(t+x)/|\gamma_2|^2} \right) \\
    &\quad - 2\Re(\gamma_2) v_2^* w_2 |w_1|^2 e^{(\gamma_1 + \gamma_2) t + (\gamma_1 - \gamma_2^*) x},
\end{align*}
\]

where

\[
\det(\Omega) = \frac{e^{-\Re(\gamma_1 + \gamma_2) t - \Re(\gamma_1^{-1} + \gamma_2^{-1}) x}}{4\Re(\gamma_1) \Re(\gamma_2)} |\gamma_1 + \gamma_2|^2 \left| \begin{vmatrix} |\gamma_1 + \gamma_2|^2 & v_2^* w_2 e^{(\gamma_1 + \gamma_2) t - \gamma_1^{-1} x} - v_1 w_2 e^{\gamma_1 t + \gamma_2^{-1} x} \\ v_1 w_2^* e^{(\gamma_1 + \gamma_2) t + \gamma_1^{-1} x} & |v_1|^2 |w_2|^2 \end{vmatrix} \right|,
\]

which were able to express in an explicitly non-negative form.

**Proposition 3.8.** The two-soliton solution is regular if \( \gamma_1, \gamma_2 \neq 0, \gamma_1 \neq \gamma_2, \gamma_1 \neq -\gamma_2^* \), \( \{v_1, w_1\} \neq \{0\} \) and \( \{v_2, w_2\} \neq \{0\} \).

**Proof.** Assuming \( v_2 w_1 w_2 \neq 0 \), for a zero of \( \det(\Omega) \) we would need

\[
e^{\phi_1 - \phi_2} = \frac{v_1 w_2}{v_2 w_1}, \quad e^{\phi_1 + \phi_2} = -\frac{v_1 v_2^*}{w_1 w_2^*},
\]

where \( \phi_k = \gamma_k t + \gamma_k^{-1} x, k = 1, 2 \). Hence

\[
e^{2\Re(\phi_2)} = -\frac{|v_2|^2}{|w_2|^2},
\]

which contradicts the positivity of the real exponential function. If any of \( v_2, w_1, w_2 \) is zero, a zero of \( \det(\Omega) \) is only possible if either \( v_1 \) and \( w_1 \), or \( v_2 \) and \( w_2 \) are zero, which we excluded.

**Example 3.9.** Choosing \( n = 2, \gamma_1 = 1, \gamma_2 = 2 \) and \( v_1 = v_2 = w_1 = w_2 = 1 \), (3.14) yields the special real two-soliton solution

\[
f = 6 \cosh \left( \frac{2t + \frac{1}{2} x}{2} \right) - 2 \cosh \left( t + x \right) \cosh \left( 3t + \frac{1}{2} x \right) + 9 \cosh \left( \frac{t - \frac{1}{2} x}{2} \right) - 8.
\]

A plot is shown in figure 1.

**3.2.2. Solitons associated with non-diagonal Jordan matrices.** In contrast to simple solitons, solutions determined by (3.14), with \( \Gamma \) chosen as a non-diagonal Jordan matrix, depend also rationally on \( x \) and \( t \) (cf [16] for the NLS case).
For \( n = 2 \) and \( \Gamma = \Gamma_{(2)} \), we have

\[
\eta_1 = e^{-\frac{1}{2}(x+\gamma t)} \left( \frac{v_1}{\gamma t} (x - \gamma^2 t) + v_2 \right),
\]

\[
\eta_2 = e^{\frac{1}{2}(x+\gamma t)} \left( -\frac{w_1}{\gamma t} (x - \gamma^2 t) + w_2 \right).
\]

Using example 3.6, we obtain

\[
f = \frac{1}{4 \text{Re}(\gamma)} \left[ \text{Re}(\gamma) v_1^* w_1^*(x/\gamma - t) - \text{Re}(\gamma) (v_1 w_2 - v_2 w_1)^* + v_1^* w_1^* \right] e^{x/\gamma + \gamma t}
\]

\[
- v_1^2 \left[ \text{Re}(\gamma) v_1 w_1 (x/\gamma - t) - \text{Re}(\gamma) (v_1 w_2 - v_2 w_1) - v_1 w_1 \right] e^{-x/\gamma + \gamma t},
\]

where

\[
\text{det}(\Omega) = \frac{1}{16 \text{Re}(\gamma)^4} \left( \left( |v_1|^2 e^{-x/\gamma - \gamma t} + |w_1|^2 e^{x/\gamma + \gamma t} \right)^2 + 4 \text{Re}(\gamma)^2 |v_1 w_2 - v_2 w_1 - v_1 w_1 (x/\gamma - t)|^2 \right).
\]

**Example 3.10.** Choosing \( \gamma = v_1 = v_2 = w_1 = w_2 = 1 \), yields

\[
f = 4 \frac{\cosh(x+t) + (x-t) \sinh(x+t)}{1 + 2(x-t)^2 + \cosh(2(x+t))}.
\]

A plot is shown in figure 2.

It is straightforward to work out solutions associated with Jordan matrices with \( n > 2 \) and, more generally, solutions where \( \Gamma \) is composed of different Jordan blocks, representing a non-linear superposition of corresponding elementary solitons.
3.3. Solutions with a plane wave background

Choosing as the ‘seed’ \( f_0 \) the plane wave solution
\[
  f_0 = Ce^{i(\alpha x - \beta t)},
\]
(3.16)
with a complex constant \( C \neq 0 \) and real constants \( \alpha \) and \( \beta \), \( (3.1) \) determines \( a_0 = -\alpha \beta/2 \).

Writing
\[
  \eta_1 = e^{-\frac{i}{2}(\alpha x - \beta t)} \tilde{\eta}_1, \quad \eta_2 = e^{\frac{i}{2}(\alpha x - \beta t)} \tilde{\eta}_2,
\]
(3.17)
the linear system, as rewritten in proposition 3.2, is converted into
\[
  \tilde{\eta}_{1tt} - \left( \frac{1}{4} \tilde{\Gamma}^2 - |C|^2 \right) \tilde{\eta}_1 = 0, \quad \text{i} \alpha \tilde{\eta}_{1t} + \tilde{\Gamma} \tilde{\eta}_{1x} = 0,
\]
(3.18)
\[
  \tilde{\eta}_2 = \frac{1}{C^*} \left( \tilde{\eta}_{1t} + \frac{1}{2} \tilde{\Gamma} \tilde{\eta}_1 \right),
\]
(3.19)
where
\[
  \tilde{\Gamma} := \Gamma + \text{i} \beta I_n.
\]
The system is solved by
\[
  \tilde{\eta}_1 = \cosh(\Theta) V, \quad \tilde{\eta}_2 = \frac{1}{2C^*} \left( \cosh(\Theta) \tilde{\Gamma} - 2R \sinh(\Theta) \right) V,
\]
(3.20)
with a constant \( n \)-component column vector \( V \) and
\[
  \Theta := \text{i} \alpha x \Gamma^{-1} R - t R + K = \text{i} \alpha x (\tilde{\Gamma} - \text{i} \beta I_n)^{-1} R - t R + K.
\]
(3.21)
Here \( R \) is a matrix root of
\[
  R^2 = \frac{1}{4} \tilde{\Gamma}^2 - |C|^2 I_n,
\]
(3.21)
which is assumed to be invertible, and $K$ is a constant $n \times n$ matrix that commutes with $\Gamma$ (and then also with $R$). The solution of (3.4) is given by

$$\Omega = \cosh(\Theta) X \cosh(\Theta^\dagger) + \frac{1}{4|C|^2} \cosh(\Theta) \tilde{\Gamma} X \tilde{\Gamma}^\dagger \cosh(\Theta^\dagger) + \frac{1}{|C|^2} \sinh(\Theta) RX R^\dagger \sinh(\Theta^\dagger),$$

where $X$ is the solution of the Lyapunov equation

$$\tilde{\Gamma} X + X \tilde{\Gamma}^\dagger = \Gamma X + X \Gamma^\dagger = V V^\dagger.$$

(3.22)

Essentially, for specified data, it remains to compute the inverse of $\Omega$ in order to find a more explicit form of this solution.

3.3.1. Simple solitons on a plane wave background. If $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)$, where $\gamma_i \neq -\gamma_j$, $i, j = 1, \ldots, n$, then $X$ is given by the Cauchy-like matrix

$$X = \left( \frac{v_i v_j^*}{\gamma_i + \gamma_j^*} \right) = \left( \frac{\tilde{\gamma}_i + \tilde{\gamma}_j^*}{\gamma_i + \gamma_j^*} \right).$$

Example 3.11. For $n = 1$, we find

$$\Omega = \frac{|V|^2}{8 \text{Re}(\tilde{\gamma})} \left( 4 |\cosh(\Theta)|^2 + \frac{1}{|C|^2} |\tilde{\gamma} \cosh(\Theta) - 2 r \sinh(\Theta)|^2 \right),$$

where now $\Theta = (i \alpha x / (\tilde{\gamma} - i \beta) - t) r + K$ with $r = \pm\sqrt{\frac{1}{4} \tilde{\gamma}^2 - |C|^2}$. (3.5) yields the solution

$$f = C e^{i(\alpha x - \beta t)} \left( 1 - \frac{\text{Re}(\tilde{\gamma}) \cosh(\Theta^\dagger) (\tilde{\gamma} \cosh(\Theta) - 2 r \sinh(\Theta))}{|C|^2 |\cosh(\Theta)|^2 + \frac{1}{4} |\tilde{\gamma} \cosh(\Theta) - 2 r \sinh(\Theta)|^2} \right).$$

(3.23)

Figure 3 shows a plot of the absolute value of $f$ for specified data. We note that $f = -f_0$ if $\tilde{\gamma} = 2|C|$ (and thus $r = 0$). Comparison of (3.23) with the focusing NLS solution (3.11) in [23] shows that, if $\tilde{\gamma}$ is real, we have a counterpart of a single Akhmediev breather if $|\tilde{\gamma}| < 2|C|$ and a Kuznetsov–Ma breather if $|\tilde{\gamma}| > 2|C|$. We refer to the references in [23] for the original literature on the NLS breathers.

For $n > 1$, we obtain (nonlinear) superpositions of the ‘elementary solutions’, given in the preceding example.

3.3.2. Solutions associated with non-diagonal Jordan matrices. Following [23], it is straightforward to elaborate the counterparts of the $n$th order Akhmediev and Kuznetsov–Ma breathers. They are obtained by choosing $\tilde{\Gamma}$ to be an $n \times n$ Jordan matrix, i.e. (3.9) with $k = \ldots$
Figure 3. Plots of the absolute value of $f$ for a single soliton on a plane wave background. Here we chose the data $\tilde{\gamma} = 3, C = \alpha = \beta = v = 1$ (real $r$, left plot), respectively $\tilde{\gamma} = 1, C = \alpha = \beta = v = 1$ (imaginary $r$, right plot), and $K = 0$.

$n$ and $\gamma$ replaced by $\tilde{\gamma}$. The corresponding solution of the Lyapunov equation can be taken from section 3.2.2. A root of (3.21) is given by the Toeplitz matrix

$$R = \frac{1}{2} \sqrt{\tilde{\gamma}^2 - 4|C|^2} \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ \tilde{\gamma} (\tilde{\gamma}^2 - 4|C|^2)^{-1} & 1 & \ldots & \ldots & \ldots \\ -2 (\tilde{\gamma}^2 - 4|C|^2)^{-2} & \ldots & \ldots & \ldots & \ldots \\ 2 \tilde{\gamma} (\tilde{\gamma}^2 - 4|C|^2)^{-3} & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 1 \end{pmatrix},$$

which commutes with the above Jordan matrix $\tilde{\Gamma}$, since this holds for any lower triangular $n \times n$ Toeplitz matrix. The entries in the first column of $R$ are the Taylor series coefficients of $\frac{1}{2} \sqrt{(\tilde{\gamma} + z)^2 - 4|C|^2}$ at $z = 0$.

The following result parallels that in section 3.1.2 of [23].

**Proposition 3.12.** Let $\tilde{\Gamma}$ be an $n \times n$ (lower triangular) Jordan matrix with $\text{Re}(\tilde{\gamma}) \neq 0$ and $\tilde{\gamma} \neq \pm 2|C|$. Then the solution of (1.1), obtained from the solution (3.20) of the linear system, is regular if the first component of the vector $V$ is different from zero. Furthermore, without restriction of generality one can set $V = (1, 0, \ldots, 0)^T$. \hfill \Box

### 3.3.3. A degenerate case.

The previously found solution of the linear system is only valid if the matrix $\frac{1}{2} \tilde{\Gamma}^2 - |C|^2 I_n$ is invertible. Now we drop this assumption.

**Example 3.13.** Let $n = 1$ and $\tilde{\gamma} = 2|C|$. Then we have

$$\tilde{\eta}_1 = c_0 + c_1 \left( \frac{\alpha x}{2|C|} + t \right), \quad \tilde{\eta}_2 = \frac{C}{|C|} \tilde{\eta}_1 + \frac{c_1}{C^2},$$

with complex constants $c_0, c_1$. The corresponding solution of (3.4) is

---

8 For the case of a $2 \times 2$ Jordan matrix $\Gamma$ and the NLS equation, also see [16].
Figure 4. Plot of the absolute value of the solution in example 3.13. Here we chose the data \( \alpha = \beta = C = c_0 = c_1 = 1. \)

\[
\Omega = \frac{1}{4|C|} (|\tilde{\eta}_1|^2 + |\tilde{\eta}_2|^2) = \frac{1}{4|C|} \left( 2 \left| \frac{c_1}{2|C|} + \frac{|c_1|^2}{2|C|^2} \right|^2 \right)
\]

\[
= \frac{1}{2|C|} \left( c_0 + c_1 \left( \frac{\alpha x}{2i|C|} + \beta t + \frac{1}{2|C|} \right) \right)^2 + \left| \frac{c_1}{4|C|^2} \right|^2.
\]

and we obtain

\[
f = Ce^{i(\alpha x - \beta t)} \left[ 1 - \frac{1}{|C|\Omega} \left( c_0 + c_1 \left( \frac{\alpha x}{2i|C|} + \beta t \right) \right)^2 
+ \frac{c_1}{|C|} \left( c_0^* + c_1^* \left( \frac{\alpha x}{-2i|C|} + \beta t \right) \right) \right].
\]

This quasi-rational solution is the counterpart of the Peregrine breather solution of the focusing NLS equation, which models a rogue wave. Also see figure 4.

Counterparts of higher order Peregrine breathers are obtained if \( \tilde{\Gamma} \) is a Jordan matrix with eigenvalue \( \tilde{\gamma} = 2|C|. \)

9 We follow the steps in section 3.2 of [23] and omit the analogous proofs.

Proposition 3.14. Let

\[
\mathcal{N} := \frac{1}{4} \tilde{\Gamma}^2 - |C|^2 I_n
\]

be nilpotent of degree \( N > 0. \) (3.18) is then solved by

\[
\tilde{\eta}_1 = \left( R_1(\mathcal{N},t) R_1(-\alpha^2 \Gamma^{-2} \mathcal{N},x) - i \alpha \Gamma^{-1} \mathcal{N} R_2(\mathcal{N},t) R_2(-\alpha^2 \Gamma^{-2} \mathcal{N},x) \right) \nu 
+ \left( R_1(\mathcal{N},t) R_2(-\alpha^2 \Gamma^{-2} \mathcal{N},x) + \frac{1}{\alpha} \Gamma R_2(\mathcal{N},t) R_1(-\alpha^2 \Gamma^{-2} \mathcal{N},x) \right) w,
\]

9 The second order Peregrine breather solution of the NLS equation appeared first in [16] to our knowledge, see section 4.1.2 therein.
with constant n-component vectors v, w, and

\[ R_1(N, t) = \sum_{k=0}^{N-1} \frac{\rho_k}{(2k)!} N^k, \]
\[ R_2(N, t) = \sum_{k=0}^{N-1} \frac{\rho_k+1}{(2k+1)!} N^k. \]

\( \tilde{\eta}_2 \) is given by (3.19).

Again, the corresponding solution of the Lyapunov equation is obtained via the results recalled in section 3.2.2.

**Proposition 3.15.** Let \( \tilde{\Gamma} \) be a (lower triangular) Jordan matrix with eigenvalue \( \tilde{\gamma} = 2 |C| \). Let \( \eta_1, \eta_2 \) be the solution of the linear system, given by proposition 3.14 and (3.17). The solution of (1.1), obtained via (3.5), is then regular if the first component of v is different from zero. Moreover, without restriction of generality, we can set \( v = (1, 0, \ldots, 0)^T \).

**4. Derivation of the binary Darboux transformation**

**4.1. Binary Darboux transformations in bidifferential calculus**

A graded associative algebra is an associative algebra \( \Omega = \bigoplus_{r \geqslant 0} \Omega^r \) over a field \( K \) of characteristic zero, where \( A := \Omega^0 \) is an associative algebra over \( K \) and \( \Omega^r \), \( r \geqslant 1 \), are \( A \)-bimodules such that \( \Omega^r \Omega^s \subseteq \Omega^{r+s} \). Elements of \( \Omega^r \) will be called \( r \)-forms. A bidifferential calculus is a unital graded associative algebra \( \Omega \), supplied with two \( K \)-linear graded derivations \( \bar{d}, d : \Omega \rightarrow \Omega \) of degree one (hence \( \bar{d} \Omega^r \subseteq \Omega^{r+1}, d \Omega^r \subseteq \Omega^{r+1} \)), and such that

\[ d^2 = \bar{d}^2 = \bar{d}d + dd = 0. \]  

We refer the reader to [14] for an introduction to this structure and an extensive list of references.

**Theorem 4.1.** Given a bidifferential calculus, let zero-forms \( \Delta \), \( \Gamma \) and one-forms \( \kappa \), \( \lambda \) satisfy

\[ \bar{d} \Delta + [\lambda, \Delta] = (d \Delta) \Delta, \]
\[ d \lambda + \lambda^2 = (d \lambda) \Delta, \]
\[ d \Gamma - [\kappa, \Gamma] = \Gamma d \Gamma, \]
\[ d \kappa - \kappa^2 = \Gamma d \kappa. \]  

Let zero-forms \( \theta \) and \( \eta \) be solutions of the linear equations

\[ \bar{d} \theta = A \theta + (d \theta) \Delta + \theta \lambda, \]
\[ \bar{d} \eta = -\eta A + \Gamma d \eta + \kappa \eta, \]  

where the one-form \( A \) satisfies

\[ dA = 0, \quad \bar{d}A = A^2. \]  

Furthermore, let \( \Omega \) be an invertible solution of the linear system

\[ \Gamma \Omega - \Omega \Delta = \eta \theta, \]
\[ d \Omega = (d \Omega) \Delta - (d \Gamma) \Omega + \kappa \Omega + \Omega \lambda + (d \eta) \theta. \]

Under suitable assumptions for \( \Delta \) and \( \Gamma \), these equations arise as integrability conditions of the linear system and ‘adjoint linear system’ given in (4.3), by use of (4.2). In any case, the integrability conditions are satisfied if (4.2) and (4.4) hold.

The equation obtained by acting with \( \bar{d} \) on (4.6) is satisfied as a consequence of the preceding equations.
Then
\[ A' := A - d(\theta \Omega^{-1} \eta) \] (4.7)
also solves (4.4).

Proof. Clearly, we have \( dA' = 0 \). Using (4.4), we obtain
\[ \dd A' - A'^2 = \dd d(\theta \Omega^{-1} \eta) + A d(\theta \Omega^{-1} \eta) + \theta \Omega^{-1} \eta A - d(\theta \Omega^{-1} \eta) \dd d(\theta \Omega^{-1} \eta). \]

With the help of the linear equations (4.3) and (4.6), we find
\[ \dd (\theta \Omega^{-1} \eta) = A \theta \Omega^{-1} \eta - \theta \Omega^{-1} \eta A + (d\theta) \Delta \Omega^{-1} \eta + \theta \Omega^{-1} \Gamma d\eta - \theta \Omega^{-1} (d\Omega) \Delta \Omega^{-1} \eta + \theta \Omega^{-1} (d\Gamma) \eta - \theta \Omega^{-1} (d\eta) \theta \Omega^{-1} \eta. \]
Eliminating \( \Gamma \) using (4.5), it becomes
\[ \dd (\theta \Omega^{-1} \eta) = A \theta \Omega^{-1} \eta - \theta \Omega^{-1} \eta A + d(\theta \Delta \Omega^{-1} \eta) + \theta \Omega^{-1} \eta d(\theta \Omega^{-1} \eta). \]
Inserting this in our first equation leads to \( \dd A' - A'^2 = 0 \).

The preceding theorem, and also the result stated next, remain true if the ingredients are matrices of forms with dimensions chosen in such a way that the required products are all defined. Furthermore, it will be sufficient to have the maps \( d \) and \( \dd \) defined on those matrices that appear in the theorem, but not necessarily on the whole of \( \Omega \).

Corollary 4.2. Let (4.2) hold and (4.3) with \( A = d\phi \), where the zero-form \( \phi \) is a solution of
\[ \dd d\phi = d\phi d\phi. \] (4.8)
If \( \Omega \) is an invertible solution of (4.5) and (4.6), then
\[ \phi' = \phi - \theta \Omega^{-1} \eta + K, \] (4.9)
where \( K \) is any \( d \)-constant (i.e. \( dK = 0 \)), solves the same equation.

The result in corollary 4.2 can be regarded as a reduction of that in theorem 4.1. Corollary 4.2 has been used in many previous applications of bidifferential calculus, see in particular [14, 22, 29]. Here we provided short proofs of the above general results. Below, we will use corollary 4.2 to deduce theorem 3.1.

4.2. An application

Let \( \mathcal{A} \) be a unital associative algebra over \( \mathbb{C} \), where the elements are allowed to depend on real variables \( x, t \). Let \( \text{Mat}(\mathcal{A}) \) be the algebra of all matrices over \( \mathcal{A} \), where the product of two matrices is defined to be zero whenever their dimensions do not fit. We choose
\[ \Omega = \text{Mat}(\mathcal{A}) \otimes \wedge \mathbb{C}^2, \] (4.10)
where \( \wedge \mathbb{C}^2 \) is the exterior algebra of the vector space \( \mathbb{C}^2 \). It is then sufficient to define \( d \) and \( d \) on \( \text{Mat}(\mathcal{A}) \), since they extend in an evident way to \( \Omega \), treating elements of \( \wedge \mathbb{C}^2 \) as \( d \)- and \( d \)-constants.

Let \( \xi_1, \xi_2 \) be a basis of \( \wedge^1 \mathbb{C}^2 \). For each \( m \in \mathbb{N} \), let \( J_m \) be a constant \( m \times m \) matrix over \( \mathcal{A} \). For an \( m \times n \) matrix \( F \) over \( \mathcal{A} \), let
\[ dF = F_i \xi_1 + \frac{1}{2} (J_m F - F J_n) \xi_2, \quad \dd F = \frac{1}{2} (J_m F - F J_n) \xi_1 + F_i \xi_2 \]
Then $d$ and $\tilde{d}$ satisfy the Leibniz rule on a product of matrices, and the conditions in (4.1) are satisfied. In the linear systems (4.3) we choose a $2 \times n$ matrix $\theta$ and an $n \times 2$ matrix $\eta$. Then $A$ has to be a $2 \times 2$ matrix of one-forms. Writing

$$A = A_1 \xi_1 + A_2 \xi_2, \quad \kappa = \kappa_1 \xi_1 + \kappa_2 \xi_2, \quad \lambda = \lambda_1 \xi_1 + \lambda_2 \xi_2,$$

with $2 \times 2$ matrices (over $\mathcal{A}$) $A_1$ and $A_2$. (4.3) reads

$$\frac{1}{2} (J_2 \theta - \theta J_n) = A_1 \theta + \theta \xi + \theta \lambda_1, \quad \theta_1 = A_2 \theta + \frac{1}{2} (J_2 \theta - \theta J_n) \Delta + \theta \lambda_2,$$

$$\frac{1}{2} (J_0 \eta - \eta J_2) = -\eta A_1 + \Delta \eta + \kappa_1 \eta, \quad \eta_2 = -\eta A_2 + \frac{1}{2} \Gamma (J_0 \eta - \eta J_2) + \kappa_2 \eta.$$

Choosing

$$\kappa_1 = \frac{1}{2} J_n, \quad \kappa_2 = \frac{1}{2} \Gamma J_n, \quad \lambda_1 = -\frac{1}{2} J_n, \quad \lambda_2 = \frac{1}{2} J_n \Delta,$$

the latter system simplifies to

$$\frac{1}{2} J_2 \theta = A_1 \theta + \theta \Delta, \quad \theta_1 = A_2 \theta + \frac{1}{2} J_2 \theta \Delta,$$

$$\frac{1}{2} \eta J_2 = \eta A_1 - \Gamma \eta, \quad \eta_2 = -\eta A_2 - \frac{1}{2} \Gamma \eta J_2,$$

which does not involve $J_n$ with $n \neq 2$ anymore. The conditions in (4.2) boil down to

$$\Delta_2 = \Delta_1 = 0, \quad \Gamma_2 = \Gamma_1 = 0,$$

so that $\Delta$ and $\Gamma$, which are $n \times n$ matrices over $\mathcal{A}$, have to be constant. (4.6) becomes

$$\Xi_2 \Delta = -\eta \theta, \quad \Xi_2 = \Gamma_1 = 0.$$

In addition, the $n \times n$ matrix $\Xi$ has to satisfy (4.5). Choosing $J_2 = \text{diag}(1, -1)$, where 1 stands for the identity element of $\mathcal{A}$, and writing

$$\phi = \begin{pmatrix} p & f \\ q & -\tilde{p} \end{pmatrix},$$

elaborating corollary 4.2, we have

$$A = d \phi = \begin{pmatrix} p_x & f_x \\ q_x & -\tilde{p}_x \end{pmatrix} \xi_1 + \begin{pmatrix} 0 & f \\ -q & 0 \end{pmatrix} \xi_2,$$

and (4.8) takes the form

$$\phi_{\alpha} = \frac{1}{2} \left[ J_2 \phi, \phi_{\alpha} - \frac{1}{2} J_2 \phi \right],$$

also see [8]. This results in the system

$$f_\alpha = f - p_x f - f \tilde{p}_x, \quad q_\alpha = q - q p_x - \tilde{p}_x q, \quad p_\alpha = (fq)_x, \quad \tilde{p}_\alpha = (qf)_x.$$

Introducing

$$a := p_x - \frac{1}{2} 1, \quad \tilde{a} := \tilde{p}_x - \frac{1}{2} 1,$$

it reads

$$f_\alpha = -af - f \tilde{a}, \quad q_\alpha = -aq - \tilde{a} q, \quad a_\alpha = (fq)_x, \quad \tilde{a}_\alpha = (qf)_x.$$

(4.11)

12 On a $1 \times 1$ matrix $f$, which is an element of $\mathcal{A}$, we have $df = f_x \xi_1$ and $\tilde{d}f = f_x \xi_2$. 


The constraint
\[ q = \pm f^\dagger, \quad a^\dagger = a, \quad \tilde{a}^\dagger = \tilde{a}, \]
reduces the last system to
\[ f_{a^\dagger} = -af - f\tilde{a}, \quad a_t = \pm (ff^\dagger)_x, \quad \tilde{a}_t = \pm (f^\dagger f)_x. \]

**Remark 4.3.** Instead, the reduction \( q = \pm f, \tilde{a} = a \) leads to
\[ f_{a^\dagger} = -af - fa, \quad a_t = \pm (f^2)_x. \]
Choosing the upper sign, this system may be regarded, as has been suggested in [8] (see equation (9) therein), as a matrix version of the SIT equations.

### 4.3. The commutative case

Let \( A \) now be the commutative algebra of functions on \( \mathbb{R}^2 \). Then we have
\[ \text{tr}(\theta\Omega^{-1}\eta) = \text{tr}(\eta\theta\Omega^{-1}) = \text{tr}((\Gamma\Omega - \Omega\Delta)\Omega^{-1}) = \text{tr}(\Gamma) - \text{tr}(\Delta), \]
where \( \Gamma \) and \( \Delta \) shall now be \( n \times n \) matrices over \( \mathbb{C} \), \( \theta \) and \( \eta \) of size \( k \times n \) and \( n \times k \), respectively. Hence
\[ \text{tr}(d(\theta\Omega^{-1}\eta)) = \text{tr}(\theta\Omega^{-1}\eta), \xi_1 = (\text{tr}(\Gamma) - \text{tr}(\Delta))\xi_1 = 0. \]
As a consequence,
\[ \text{tr}A = 0 \]
is a reduction that is consistent with the solution-generating method of theorem 4.1. The latter reduction means \( \tilde{a} = a \), so that (4.11) becomes
\[ a_t = (fq)_x, \quad f_{a^\dagger} = -2af, \quad q_{a^\dagger} = -2aq. \] (4.12)

This is the first ‘negative flow’ of the AKNS hierarchy (see, e.g. system (12) in [6] and also [7, 12], for example). Soliton solutions of it have, apparently, first been found in [11], using Hirota’s bilinear method.

Since we guaranteed form-invariance of \( A \) under the transformation given by corollary 4.2, writing
\[ \theta = \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right), \quad \eta = (\eta_1, \eta_2), \]
we have
\[
\begin{align*}
d\phi' &= A' = A - d(\theta\Omega^{-1}\eta) = \left( \begin{array}{cc} a' + \frac{1}{2} & f' \\ q_\omega & -a' - \frac{1}{2} \end{array} \right) \xi_1 + \left( \begin{array}{cc} 0 & f' \\ -q' & 0 \end{array} \right) \xi_2 \\
&= \left( \begin{array}{cc} a + \frac{1}{2} - (\theta_1\Omega^{-1}\eta_1)_x & (f - \theta_1\Omega^{-1}\eta_2)_x \\ (q - \theta_2\Omega^{-1}\eta_1)_x & -a - \frac{1}{2} - (\theta_2\Omega^{-1}\eta_2)_x \end{array} \right) \xi_1 + \left( \begin{array}{cc} 0 & f - \theta_1\Omega^{-1}\eta_2 \\ -q + \theta_2\Omega^{-1}\eta_1 & 0 \end{array} \right) \xi_2,
\end{align*}
\]
and hence
\[ a' = a - (\theta_1\Omega^{-1}\eta_1)_x = a + (\theta_2\Omega^{-1}\eta_2)_x, \quad f' = f - \theta_1\Omega^{-1}\eta_2, \quad q' = q - \theta_2\Omega^{-1}\eta_1, \]
satisfy the same equations as \( a, f, q \). Collecting the main results, we arrive at the following theorem, which expresses a binary Darboux transformation for the system (4.12).
Theorem 4.4. Let $a_0, f_0, q_0$ be a solution of (4.12). Let $\theta_i$ and $\eta_i, i = 1, 2$, be solutions of the linear system
\[
\begin{align*}
\theta_{1x} \Delta &= -a_0 \theta_1 - f_0 \theta_2, \\
\theta_{2x} \Delta &= a_0 \theta_2 - q_0 \theta_1, \\
\theta_{1t} &= -\frac{1}{2} \theta_1 \Delta + f_0 \theta_2, \\
\theta_{2t} &= -\frac{1}{2} \theta_2 \Delta - q_0 \theta_1, \\
\Gamma \eta_{1x} &= a_0 \eta_1 + q_0 \eta_2, \\
\Gamma \eta_{2x} &= -a_0 \eta_2 + f_0 \eta_1, \\
\eta_{1t} &= -\frac{1}{2} \Gamma \eta_1 + q_0 \eta_2, \\
\eta_{2t} &= -\frac{1}{2} \Gamma \eta_2 - f_0 \eta_1,
\end{align*}
\]
where $\Delta$ and $\Gamma$ are invertible constant $n \times n$ matrices. Let $\Omega$ be an invertible solution of the linear equations
\[
\begin{align*}
\Gamma \Omega - \Omega \Delta &= \eta_1 \theta_1 + \eta_2 \theta_2, \\
\Omega_{1x} \Delta &= -\eta_1 \theta_1 - \eta_2 \theta_2, \\
\Omega_{t} &= -\frac{1}{2} \eta_1 \theta_1 + \frac{1}{2} \eta_2 \theta_2.
\end{align*}
\]
Then
\[
a = a_0 - (\theta_1 \Omega^{-1} \eta_1), \quad f = f_0 - \theta_2 \Omega^{-1} \eta_2, \quad q = q_0 - \theta_2 \Omega^{-1} \eta_1,
\]
constitutes also a solution of (4.12). □

If we impose the condition
\[
q = \pm f^*,
\]
the system (4.12) reduces to (3.1), if we choose the plus sign. It remains to implement the above reduction in the solution-generating method.

4.4. The reduction $q = f^*$

Let us set
\[
q = f^*, \quad \theta = \eta^+, \quad \Delta = -\Gamma^+.
\]

Then theorem 4.4 implies theorem 3.1.

Proof of theorem 3.1. The linear system in theorem 4.4 reduces to (3.2) and (3.3), by using (4.17), and (4.13) becomes (3.4). If $\Gamma$ and $-\Gamma^+$ have no eigenvalue in common, i.e. $\text{spec}(\Gamma) \cap \text{spec}(-\Gamma^+) = \emptyset$, the Lyapunov equation (3.4) is known to have a unique solution $\Omega$. By taking its conjugate, we can then deduce that
\[
\Omega^+ = \Omega,
\]
which in turn implies that the equations
\[
\Omega \Gamma^+ = \eta_1 \eta^+_1 + \eta_2 \eta^+_2, \quad \Omega_{t} = -\frac{1}{2} \eta_1 \eta^+_1 + \frac{1}{2} \eta_2 \eta^+_2,
\]
constitute a solution of the linear system (4.12). □
resulting from (4.14), are satisfied as a consequence of the equations

\[
\Omega_x \Gamma^\dagger - \eta_x \eta^\dagger + \Gamma \Omega_x - \eta \eta_x^\dagger = 0,
\]

\[
\Omega_t \Gamma^\dagger + \frac{1}{2} \eta_1 \eta_1^\dagger - \frac{1}{2} \eta_2 \eta_2^\dagger \Gamma^\dagger = 0,
\]

obtained by differentiation of (3.4) with respect to \(x\), respectively \(t\), and using (3.3). Furthermore,

\[
(\eta_1^\dagger \Omega^{-1} \eta_1) = \eta_1^\dagger \Omega^{-1} \eta_1,
\]

so that

\[
(\eta_1^\dagger \Omega^{-1} \eta_2) = \eta_2^\dagger \Omega^{-1} \eta_1,
\]

and the last two equations in (4.15) indeed coincide if (4.17) holds. Furthermore, \(\eta_1^\dagger \Omega^{-1} \eta_1\) is real, so that \(a\) is real if \(a_0\) is real, which is required by the linear system.

\[
\square
\]

5. Conclusion

In this work we explored the nonlinear PDE (1.1), which is completely integrable (in the sense that a Lax pair exists) if the dependent variable is real. Expressing (1.1) as the equivalent system (3.1), in the complex case integrability apparently requires that the function \(a\) has to be real [28]. Accordingly, the vectorial binary Darboux transformation, which we generated from a universal binary Darboux transformation in bidifferential calculus, only works for (3.1) with the restriction to real \(a\).

We exploited the vectorial binary Darboux transformation to obtain multi-soliton solutions of (1.1), also on a plane wave background solution. This includes counterparts of the Akhmediev and Kuznetsov–Ma breathers and rogue wave solutions of the NLS equation. All these solutions can still be generalized by using proposition 1.1.

Little is known about the non-integrable sector of (1.1), which is (3.1) if the function \(a\) has a non-zero imaginary part. In this sector, the only exact solution we know is the fairly trivial one \(f = Ce^{i(\alpha x - \beta t)}\), where \(\alpha\) is any real function with non-vanishing derivative, \(\beta \neq 0\) is a real constant and \(C \neq 0\) a complex constant. That switching on an imaginary part of the function \(a\) in (3.1) apparently breaks complete integrability, is an interesting observation [28], which deserves further exploration.

More generally, we derived a vectorial binary Darboux transformation for the system (4.12), which is the first ‘negative flow’ of the AKNS hierarchy.

Changing the plus sign in (1.1) to a minus sign, leads us to a ‘defocusing’ version of it, using the familiar terminology for the corresponding versions of the NLS equation. In this case we have to deal with the reduction condition (4.16) with the choice of the minus sign and implement it in the binary Darboux transformation for the system (4.12). This has not been elaborated in this work.

Most likely, the simple multi-soliton solutions admit generalizations in terms of Jacobi elliptic functions. We have seen that there are even two such extensions of the one-soliton solution.

The plots in this work have been generated using Mathematica [31].
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**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

**ORCID iD**

Folkert Müller-Hoissen [https://orcid.org/0000-0001-6692-0772](https://orcid.org/0000-0001-6692-0772)

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