Convergence of Numerical Approximations to Non-linear Continuity Equations with Rough Force Fields

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Abstract

We prove quantitative regularity estimates for the solutions to non-linear continuity equations and their discretized numerical approximations on Cartesian grids when advected by a rough force field. This allows us to not only recover the known optimal regularity for linear transport equations but also to obtain the convergence of a wide range of numerical schemes. Our proof is based on novel commutator estimates, quantifying and extending to the non-linear case the classical commutator approach of the theory of renormalized solutions.

1. Introduction

1.1. The Model

One of the main goals of this article is to study the convergence of some simple numerical schemes for the solution of the non-linear equation

$$\frac{\partial}{\partial t} u(t, x) + \text{div} (a(t, x) f(u(t, x))) = 0, \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}^d, \quad (1.1)$$

in the case where the velocity field $a$ only belongs to $L^p_{\text{loc}}(\mathbb{R}_+, W^{1,p}(\mathbb{R}^d))$ and is hence not smooth.

The density $u$ can model a large variety of agents or objects, from molecules to micro-organisms and individuals (in pedestrian models for instance). Eq. (1.1) combines a classical advection through the velocity $a$ with non-linear effects through the flux $f \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$; it is thus a hybrid between a linear advection equation and a scalar conservation law.

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A good example for \( f \) is \( f(u) = u (u_c - u)_+ \), where \( u_c \) is a critical density. Eq. (1.1) then ensures that \( u \leq u_c \) at all times. This is an important feature when relatively large agents are considered in comparison to the length scale over which one calculates the density. In such a case, the maximal density of agents (where they all touch each other) may be of the same order of magnitude as the average density under consideration. This is usually the case for crowd motion models. We refer to [40,41] for examples of such congestion effects.

We will not consider any particular coupling between \( a \) and \( u \) in this article. Since we do not study uniqueness, we only need to assume that some Sobolev regularity is obtained on \( a \). This makes our estimates compatible with a wide range of models. We give two examples, the first of which is coupling through the Poisson equation

\[
a = \nabla c, \quad -\Delta c = g(u),
\]

This is commonly used for so-called chemotaxis models, which perform to the dynamics of micro-organisms directed by chemical signals. In this case one considers only one chemical whose density is given by \( c(t, x) \); the micro-organisms try to follow the gradient of the chemical. Such a model has been studied in [21].

A variant of this is the Hamilton-Jacobi coupling

\[
a = \nabla c, \quad -\Delta c + |\nabla c|^2 = g(u),
\]

which has been implemented for pedestrian models as in [24].

Eq. (1.1) includes, as a special case for \( f = Id \), the classical continuity equation

\[
\partial_t u (t, x) + \text{div} (a(t, x) u(t, x)) = 0.
\]

The results presented here hence also apply to the case of (1.4). The non-linearity in (1.1) restricts many of the techniques that are available for (1.4), which is one of the recurring difficulties encountered in this article.

For simplicity, we call (1.1) the non-linear continuity equation and (1.4) the linear continuity equation; this emphasizes their main structural difference but of course in most applications both (1.1) and (1.4) are part of a larger non-linear system which couples \( a \) and \( u \).

1.2. An Example of Application: Compactness and Explicit Regularity Estimates for Eq. (1.1)

The key difficulty in many complex, nonlinear models is the possible instability in the density \( u \); the main challenge is to control how oscillations in \( u \) can develop in (1.1), especially for a rough velocity field such as \( a \in L^2_t H^2_x \), given by typical viscosity bounds. This puts the propagation of regularity on (1.1) at the center of our proposed work for convection models.

Unfortunately, it is, in general, not possible to propagate any kind of Sobolev regularity on \( u \), when often leads to an implicit or convoluted argument.

As a first illustration of the method introduced here, we present new explicit regularity estimates for solutions to (1.1). Define the semi-norms for \( 0 < \theta < 1 \):

\[
\|u\|_{p, \theta}^p = \sup_{\|h\| \leq 1/2} \log \|h\|^{-\theta} \int_{\mathbb{R}^d} \frac{\|u(x) - u(y)\|_{L^p}^p}{(\|x - y\| + h)^{d\theta}} |u(t, x) - u(t, y)| dxdy.
\]
Obviously the semi-norms are decreasing in $\theta$:

$$\|u\|_{p,\theta} \leq \|u\|_{p,\theta'}, \text{ if } \theta' \leq \theta,$$

and they are only semi-norms as $\|u\|_{p,\theta}$ vanishes if $u$ is a constant. We may define the corresponding spaces

$$W^p_{\log,\theta} = \{u \in L^p \mid \|u\|_{p,\theta} < \infty\}.$$

Those semi-norms measure intermediary regularity ($\log$ of a derivative) between $L^p$ spaces and Sobolev spaces $W^{s,p}$ as per the following proposition:

**Proposition 1.** For any $s > 0$, $0 < \theta < 1$, and any $1 \leq p \leq \infty$, one has the embeddings $W^{s,p} \subset W^p_{\log,\theta} \subset L^p$ which are compact on any smooth bounded domain of $\mathbb{R}^d$. For $\theta = 0$, $\|u\|_{p,0} \leq C \|u\|_{L^p}$. In addition, for $p = 2$,

$$\|u\|_{L^2,\theta}^2 + \|u\|_{L^2}^2 \sim \sup_h \int_{\mathbb{R}^d} \frac{\log \left( \frac{1}{|\xi|} + h \right) + 1}{|\log h|^\theta} |\mathcal{F} u(\xi)|^2 d\xi$$

$$\leq \int_{\mathbb{R}^d} (\log(1 + |\xi|))^{1-\theta} |\mathcal{F} u(\xi)|^2 d\xi,$$

where $\mathcal{F} u$ denotes the Fourier transform of $u$.

Let us further observe that for $\theta = 1$, $\|u\|_{p,1}$ is dominated by the $L^p$ norm

$$\|u\|_{p,1} \leq C \|u\|_{L^p}.$$

Prop. 1 implies that any sequence $u_n$ with $\sup \|u_n\|_{p,\theta} < \infty$ for some $\theta < 1$ is compact locally in $L^p$, but one has a converse property in the following precise sense:

**Proposition 2.** Assume that $u_n$ is compact in $L^p$, then

$$\lim \sup_{h \to 0} |\log h|^{-1} \sup_n \int_{\mathbb{R}^d} \frac{\mathbb{E}_{|x-y| \leq 1}}{(|x-y| + h)^d} |u_n(t,x) - u_n(t,y)|^p dx dy = 0.$$

The proof of Prop. 2 is straightforward and given in [9], for example. In view of this we could have removed the $\sup h$ in our definition of the semi-norms, leading, for example, to

$$\|u\|_{p,h}^p = |\log h|^{-1} \sup_n \int_{\mathbb{R}^d} \frac{\mathbb{E}_{|x-y| \leq 1}}{(|x-y| + h)^d} |u_n(t,x) - u_n(t,y)|^p dx dy,$$

and just studied, in general, how this semi-norm behaves as $h \to 0$. The estimates are actually completely similar and would then lead to very general compactness results, but to keep the analysis more transparent, we chose to look at the more specific behaviors in $\|u\|_{p,\theta}$ for $\theta < 1$.

There has recently been an increase in interest in spaces such as $W^p_{\log,\theta}$, which differ from classical Sobolev or $L^p$ spaces by a log scale; see for instance [13].

The semi-norms are at the critical scale where regularity is propagated for Eq. (1.1). We have, where $p^*$ denotes the dual exponent of $p$, $1/p^* + 1/p = 1$, the following:
Theorem 3. Assume that \( a \) belongs to the Besov space \( L^1([0, T), B^1_{p,q}({\mathbb R}^d)) \) for some \( p, q \geq 1 \) with \( \text{div} \, a \in L^\infty([0, T) \times {\mathbb R}^d) \). Any entropy solution \( u \in L^\infty([0, T], L^p \cap L^p_{r,1}) \), in the usual sense that \( \forall \phi \in C^2 \) convex, \( \exists q \in C^1 \) s.t.,

\[
\partial_t (\phi(u(t, x))) + \text{div}_x (a(t, x) q(u(t, x))) + (\phi'(u) f(u) - q(u)) \text{div}_x a \leq 0,
\]

satisfies the regularity estimate for any \( t \leq T \) and any \( \theta \geq \max(1/p^*, 1 - 1/q) \)

\[
\|u\|_{1,\theta} \leq e^C \|\text{div} \, a\|_{L^\infty} \|f'\|_{L^\infty} \int_0^t \|\text{div} \, a(s, \cdot)\|_{p,p(\theta-1/p^*)} ds
+ C \|\nabla a\|_{L^1([0, T], B^0_{p,q}({\mathbb R}^d))} \|f'\|_{L^\infty} \|u\|_{L^p_{r,1}} ^{p^*} + \|u^0\|_{1,\theta}.
\]

This implies the simple estimate, for \( a \in L^1([0, T], W^{1,p}({\mathbb R}^d)) \) with \( 1 < p \leq 2 \) and \( u \in L^\infty([0, T], L^1 \cap L^r({\mathbb R}^d)) \) for \( r > p^* \),

\[
\|u\|_{1,1/2} \leq e^C \|\text{div} \, a\|_{L^\infty} \|f'\|_{L^\infty} \int_0^t \|\text{div} \, a(s, \cdot)\|_{p,1-p/2} ds
+ C \|\nabla a\|_{L^1([0, T], L^p({\mathbb R}^d))} \|f'\|_{L^\infty} \|u\|_{L^p_{r,1}} \|u\|_{L^p_{r,1}} + \|u^0\|_{1,1/2}.
\]

Remark 4. This regularity result only applies to the entropy solution to the equation. More than simply obtaining bounds on the solution itself, the proof uses a doubling of variables argument (in the spirit of Kruzkov) and requires manipulation of non-linear quantities of \( u \).

Remark 5. Notice that the regularity provided by Th. 3 involves the semi-norm \( \|\text{div} \, a(s, \cdot)\|_{p,p(\theta-1/p^*)} \) of \( \text{div} \, a \). This is the equivalent of the classical fact that the compactness of the solution \( u \) requires the compactness of \( \text{div} \, a \), but is expressed here in a fully quantitative manner. See the comments after Prop. 2 about how such semi-norms can be directly related to any compactness in \( \text{div} \, a \).

Remark 6. Notice that the first part of our result actually works perfectly even if \( p = 1 \). As a matter of fact, there is nothing special in the main part of the analysis about \( p = 1 \) vs \( p > 1 \). Roughly speaking, as long as \( \nabla a \in B^0_{p,q} \) with \( q < \infty \), then we propagate some regularity (depending on the value of \( q \)). The issue with \( p = 1 \) appears when we are trying to “translate” this main estimate into something that depends on the Sobolev regularity \( W^{1,p} \) of \( a \). Then for \( 1 < p \leq 2 \), \( W^{1,p} \subset B^1_{p,2} \), but in general, one has at best \( W^{1,1} \subset B^1_{1,\infty} = BV \). We further observe here that if \( a \in BV \) only for the linear transport equation, no explicit regularity is available so far.

For technical reasons it is often more convenient to work with a smooth kernel in the definition of the semi-norms. Define

\[
K_h(x - y) = \frac{\phi(x - y)}{(|x - y| + h)^d}
\]  (1.5)
for some smooth function \( \phi \) with compact support in \( B(0, 2) \) and such that \( \phi = 1 \) inside \( B(0, 1) \). We can then take the variant definition

\[
\|u\|_{p, \theta}^p = \sup_{h \leq 1/2} |\log h|^{-\theta} \int_{\mathbb{R}^{2d}} K_h (x - y) |u(x) - u(y)|^p \, dx \, dy.
\]

A first rougher version of Th. 3 had been derived in [9]. The main estimate in the proof however was \( L^2 \) based, leading to non-optimal estimates where \( a \in W^{1,p}_x \) with \( p \neq 2 \). It was moreover essentially non compatible with a discretized setting such as the numerical schemes that we are mostly concerned with here. We have completely revisited the approach by identifying precisely the cancellations at the heart of Th. 3. This lets us obtain the optimal regularity in a much more general setting and to identify the critical Besov spaces for \( a \).

Quantitative regularity estimates were first obtained for linear advection or continuity equations in [20]. The method there is based on bounds along the characteristics and very different from the one followed here. This characteristics method was later used in [10, 12, 16, 19, 34] under various extended assumptions (singular integrals or force field with less than a derivative but with the right structure).

A more similar looking estimate has been obtained in [15]. This last estimate relies on a duality method which is only compatible with linear continuity equations but can then be more carefully tailored to the problem.

Still dealing with the linear case, it was observed in [38] that the critical multilinear estimates recently obtained in [44] actually allow a better propagation of regularity by improving the commutator estimates that we are proving here; in that case, one can essentially use \( \theta = 0 \) in Th. 3. The proof in [44] is however, very intricate.

All those explicit estimates propagate some form of a log of a derivative, just like Th. 3. In general, this is the best that one can hope for in the presence of a Sobolev force field, as was proved in [3, 33].

Another approach to quantitative estimates for the transport equation has been derived in [45] and relies on propagating weak distances (Wasserstein distances typically). In some sense this is a dual theory (weak distances vs. stronger regularity).

We further explain the connections between the present quantitative estimates and the classical theory of renormalized solutions for linear continuity equations when we state our commutator estimate in Section 3.

### 2. The Numerical Scheme and Main Results

We now turn to the main results of this article concerning the convergence of numerical schemes for Eq. (1.1). Numerical schemes for advection equations with rough force fields have mostly been studied in the context of compressible fluid dynamics where the density satisfies the continuity equation with an only \( H^1 \) velocity field. We refer in particular to schemes for the compressible Stokes system with for instance [26, 27, 29], or the Navier-Stokes system with [28, 30].
Compressible Fluid dynamics models typically involve the linear continuity equation (1.4) on the density. One of the major difficulties in proving the convergence of such numerical schemes is to obtain the compactness of the density. The convergence of schemes for the compressible Navier-Stokes system is in large part still an open question. We hope that the new quantitative estimates that we introduce can prove useful.

In addition to the linear continuity equation (1.4), Eq. (1.1) also contains the classical one-dimensional scalar conservation law

$$\partial_t u + \partial_x f(u) = 0. \quad (2.1)$$

The well posedness theory for (2.1) is now well understood since the work of Kruzkov [36]. The analysis of numerical schemes for conservation laws of which (2.1) is a very simple case is also classical and well-developed; we refer for example to [39,47].

Eq. (2.1) exhibits shocks in finite time so that it only propagates up to BV regularity or in general $W^{s,1}$ with $s < 1$. One of the challenges of our study was to find regularity estimates which are compatible both with linear advection equations with rough force fields and with shocks from conservation laws.

In general our non-linear continuity equation could be seen as a conservation law with time and space dependent fluxes. Although there are some results for such systems with discontinuous fluxes, see e.g. [7,35,42], they do not seem to be applicable in a case such as ours where only Sobolev bounds are known in the absence of any other strong structure on the flux.

Qualitative results of convergence for discretized linear advection equations have already been obtained in [14,48]. In the linear case, the first quantitative estimates were recently obtained in [43], based on the estimates developed at the continuous level in [45].

Before describing more in detail the schemes that we consider, we want to emphasize here that we focus on schemes on a Cartesian grid. It is in general an important and essentially still open question to determine, for a given non-Cartesian grid, how much of the regularity can be propagated on the solution even if the velocity field is smooth. We mention the following example, which seems to be widely known (we thank T. Gallouët and R. Herbin for pointing it out):

**Example 7.** Consider the two dimension velocity field $a = (1, 0)$ and the non-cartesian grid for the discretization parameter $\delta x$ whose cell numbered $(i, j)$ is given by $[i\delta x, (i + 1)\delta x) \times [j\delta x, (j + 1)\delta x)$ if $j$ is even and $[i\delta x/2, (i + 1)\delta x/2) \times [j\delta x, (j + 1)\delta x)$ if $j$ is odd. This simply means that we alternate a row with horizontal discretization $\delta x$, with another row with discretization $\delta x/2$.

Consider the upwind discretization $u^n$ over such a grid to the linear transport equation $\partial_t u + a \cdot \nabla u = 0$ with constant coefficients.

It is straightforward to check that the discrete BV norm of $u^n$ is not, in general, propagated by the discretized system. In fact there is a critical threshold of regularity in this case: a discrete $W^{s,1}$ norm is propagated iff $s < 1/2$. 

We actually hope that the techniques introduced here can later prove useful in such non-cartesian grids, but at the present time, we completely leave such cases out of our analysis.

### 2.1. Description of the Schemes Under Consideration

The discrete solution is given by a set of values \( u^n_i \) representing an approximation of the continuous solution at time \( t_n = n \delta t \) over the various points of the grid at \( i = (i_1, \ldots, i_d) \) with \( i_1 \ldots i_d = 1 \ldots N \). We assume a grid length equal to \( \delta x \) and denote by \( x_i \) the points at the center of each mesh.

We will make abundant use of the discrete \( L^p \) norms which we normalize by the grid size

\[
\| u^n \|_{L^p} = \delta x^d \sum_{i \in \mathbb{Z}^d} |u^n_i|^p.
\]

For convenience, we denote \( i + [\tau]_k \) for \( k = 1 \ldots d \), the index where coordinate \( i_k \) of \( i \) is shifted by \( \tau \). Thus, for example, \( x_{i + [1]_k} \) is simply the center of the next mesh in direction \( k \).

This lets us easily define discrete Sobolev norms per

\[
\| a^n \|_{d, W^{1,p}} = \delta x^d \sum_{i \in \mathbb{Z}^d} \sum_{k=1}^d |a^n_i - a^n_{i+[1]_k}|^p.
\]

We consider fairly general explicit schemes of the form

\[
u^n_{i+1} = \sum_{m \in \mathbb{Z}^d} b_{i,m}(a^n_m, u^n_m), \tag{2.2}
\]

where the \( b_{i,m} \) are functions normalized so that \( b(a, 0) = 0 \). The non-linear dependence on the velocity field \( a \) can, for instance, follow from upwinding. The velocity field itself is discretized so that for any \( n \) and \( i \), \( a^n_m \) is a vector of \( \mathbb{R}^d \):

\[
a^n_m = (a^n_{m,1}, \ldots, a^n_{m,d}).
\]

We do not explicitly distinguish the boundary conditions in the scheme, and we a priori allow ourselves to work on an unbounded grid, hence the fact that \( m \) is summed over all \( \mathbb{Z}^d \) in (2.2). Of course in most practical settings the grid is truncated, meaning that \( a^n_i \) and \( u^n_i \) have compact support in \( i \).

We also point out that the function \( b_{i,m} \) could be chosen differently from one time step to another, i.e. \( b_{i,m} = b^n_{i,m} \), without any difference in the proofs. In order to avoid additional indices in the notation as much as possible, we will just write \( b_{i,m} \).

The key assumption on the \( b_{i,m} \) is that \( b_{i,m}(a, u) \) is increasing in \( u \). This ensures that the scheme is monotone and entropic. For simplicity, we also assume that \( b_{i,i}(a, u) - u/2 \) is increasing in \( u \), which can always be ensured by choosing the appropriate CFL condition.
Most explicit schemes require a CFL condition to be monotone and this in turns typically demands a uniform bound on the velocity field

$$\sup_{i, n} |a_i^n| < \infty.$$  

We do not directly use such a bound but again it is likely to be indirectly imposed through the previous monotonicity condition.

We are asking that the scheme be conservative, which means that $b_i, m$ can be expressed as a difference of two fluxes $F_k$ in each direction $k = 1 \ldots d$:

$$b_i, m(a^n_m, u^n_m) = u^n_m \delta_i = m + \frac{\delta t}{\delta x} \sum_{k=1}^d \left( F^k_{i+[1]k - m}(a^n_m, u^n_m) - F^k_{i - m}(a^n_m, u^n_m) \right).$$  

(2.3)

The fluxes $F_j$ are obviously defined up to a constant function and we normalize them so that, for any $a \in \mathbb{R}^d$, $u \in \mathbb{R}_+$

$$\sum_j F_j^k(a, u) = a^k f(u),$$  

(2.4)

where $a^k$ is the $k$ coordinate of the vector $a$.

The conservative form of the scheme implies, for instance, the conservation of mass

$$\sum_{i, m \in \mathbb{Z}^d} b_i, m(a^n_m, u^n_m) = \sum_{i \in \mathbb{Z}^d} u^n_i.$$  

(2.5)

In general we do not ask that the divergence of $a$ be exactly discretized by the scheme. Instead we impose the following condition: there exists a Lipschitz function $\tilde{f}$ and uniformly bounded $D^n_i$ s.t. for any constant $U$

$$\sum_{j \in \mathbb{Z}^d} b_i, j(a^n_j, U) = U + \delta t D^n_i \tilde{f}(U).$$  

(2.6)

The uniform bound on the $D^n_i$ obviously correspond to the bound $\|\text{div} a\|_{L^\infty}$. We also impose that

$$\|\tilde{f}\|_{W^{1, \infty}} \leq C \|f\|_{W^{1, \infty}}.$$  

(2.7)

This general expression of the discretization allows for many (but not all) of the classical schemes such as Lax-Friedrichs, upwind schemes, etc.. In particular multi-point schemes are included in the formulation. We only impose that too much weight not be given to far away points. This translates into a simple moment condition on the flux : there exists a constant $C$ and $0 < \gamma \leq 1$ s.t.

$$\sum_m |i - m|^\gamma |F^k_{i - m}(a^n_m, u^n_m)| \leq C \|f\|_{L^\infty} \|a^n\|_{L^p} \|u^n\|_{L^{p^*}}.$$  

(2.8)

We now introduce the discretized version of our semi-norms, namely

$$\|u^n\|_{\alpha, p, \theta}^B = \sup_{h \geq \delta x^\alpha} \log h^{-\theta} \delta x^{2d} \sum_{i, j} K^n_{i - j} |u^n_i - u^n_j|.$$  

(2.9)
where the discrete kernel is directly obtained from $K_h(x)$ given by (1.5) through

$$K^h_i = K_h(i \delta x).$$

The main difference with respect to the continuous semi-norms is that we do not take the supremum over all possible values of $h$. This is because they do not make sense below a certain size depending on the discretization length $\delta x$, as will appear at some point in the proof. This is why we limit the supremum to those of $h \geq \delta x^\alpha$. Those semi-norms still provide compactness whenever $\alpha > 0$ and $\theta < 1$.

2.2. Main Result

We are able to obtain the exact equivalent of Theorem 3.

**Theorem 8.** Assume that $u^n_i$ is a solution to the recursive scheme given by (2.2)-(2.3) with functions $b_{i,j}(a, u)$ increasing in $u$ and s.t. $b_{i,j}(a, u) - u/2$ increasing in $u$. Assume moreover that the scheme satisfies (2.4) and (2.5) together with the discretization of the divergence provided that (2.6) with the bound (2.7) hold; we refer to the discussion in [39] for example for a full justification of this. This means that if $u^0 \in l^q$ then $u^n \in l^q$ with comparable norm. Hence, in such a case, provided that $a$ has the required regularity, the right hand-side is bounded.

Note that other than the norm $\|u^n\|_{l^q}$, the right hand-side in the estimate just above also involves Sobolev norms of $a$ and the corresponding semi-norms of $D^m$ which is a discretization of $\text{div} \ a$. It is in particular interesting to compare this estimate to the one in Th. 3. One can actually find the same type of terms with continuous norms replaced by discrete ones, the time integrals replaced by discrete sums.
From the previous remark, this result easily implies the compactness of the solutions to the scheme, with, for example

**Corollary 11. (Convergence of the scheme)** Consider a sequence of solutions $u_{i}^{n,m}$ for schemes satisfying the assumptions of Theorem 8 with a grid size $\delta x_m \to 0$ and a time step $\delta t_m \to 0$. Assume that the following quantities are uniformly in $m$, namely for some $q > p^{*}$ and some $\theta > 0$ and $\alpha > 0$:

\[
\sup_{m} \delta t_{m} \sum_{n \leq T/\delta t_{m}} \| a_{n,m} \|_{d,W^{1,p}} < \infty, \quad \sup_{m} \sup_{n} \| u_{n,m} \|_{q} < \infty,
\]

\[
\sup_{m} \sup_{n,i} | D_{i}^{n,m} | < \infty, \quad \sup_{m} \sup_{n} \| D_{n,m} \|_{\alpha,p,\theta} < \infty, \quad \sup_{m} \sup_{n} \| u_{0,m} \|_{\alpha,1,\theta} < \infty.
\]

Consider the sequence of functions $\tilde{u}^{m}$ on $[0, T] \times \mathbb{R}^d$ piecewise constant and equal to $u_{i}^{n,m}$ on the time interval $[n \delta t_{m}, (n+1) \delta t_{m})$ times the cube centered at $x_i = i \delta x_m$ and of size $\delta x_m$. Then the sequence $\tilde{u}^{m}$ is compact in $L^1_{\text{loc}}$.

**Remark 12.** With a minor variation, the assumption that

\[
\sup_{m} \sup_{n} \| D_{n,m} \|_{\alpha,p,\theta} < \infty, \quad \sup_{m} \sup_{n} \| u_{0,m} \|_{\alpha,1,\theta} < \infty
\]

can be replaced by assuming appropriate compactness on the initial data $u_{0,m}$ and the discrete divergence $D_{i}^{n,m}$.

There is a natural question following Corollary 11: given that we have an explicit regularity on the solution, could not we also obtain explicit rates of convergence? This is connected to the issue of whether we can have uniqueness (and in fact quantitative stability estimates) on either the limiting equation or its discretization. In the purely linear case where $f(\xi) = \xi$, this is actually equivalent to our regularity estimates and it would not be difficult to obtain a quantitative version of Corollary 11.

In the non-linear case ($f$ not linear), such a quantitative stability would require more work but should still be feasible. It would imply using our regularity estimates to regularize the equation then using a Kruzhkov-like contraction on the regularized equation.

The main motivation for our work, however, comes from systems where the velocity field is not given but is instead coupled back to $u$. We mentioned two such examples in the introduction, where $a$ solves a Poisson (1.2) or Hamilton-Jacobi equation (1.3) involving $u$. In those cases, and with a proper discretization of the corresponding coupled equation, it remains straightforward to prove uniform regularity on $a$. Therefore Corollary 11 still applied (mostly directly) in those complex non-linear settings. However we do not see how to obtain uniqueness (at least in any simple, general manner that would not be strongly dependent on the system).
3. The Main Commutator Estimate

3.1. The Estimate

The key point in the proof of the results of this article is a commutator estimate, quantifying the basic one introduced in [25]. This estimate is important in itself and is likely to be of further use.

**Proposition 13.** Let $1 < p < \infty$, $\exists C < \infty$ depending only on $p$ and the dimension $s.t. \forall a \in W^{1, p}([\mathbb{R}^d])$ with $1 \leq p \leq 2$ and $\forall g \in L^{2p^*}$ with $1/p^* = 1 - 1/p$,

$$\int_{\mathbb{R}^{2d}} \nabla K_h(x - y) (a(x) - a(y)) |g(x) - g(y)|^2 \, dx \, dy \leq C \|\nabla a\|_{B_p^q} \|\log h\|^{1-1/q} \|g\|_{L^2 p^*}^2$$

$$+ C \|\text{div} \, a\|_{L^\infty} \int_{\mathbb{R}^{2d}} K_h(x - y) |g(x) - g(y)|^2 \, dx \, dy.$$ 

In particular, using $q = 2$,

$$\int_{\mathbb{R}^{2d}} \nabla K_h(x - y) (a(x) - a(y)) |g(x) - g(y)|^2 \, dx \, dy \leq C \|\nabla a\|_{L^p} \|\log h\|^{1/2} \|g\|_{L^2 p^*}^2$$

$$+ C \|\text{div} \, a\|_{L^\infty} \int_{\mathbb{R}^{2d}} K_h(x - y) |g(x) - g(y)|^2 \, dx \, dy.$$ 

3.2. The Connection with the Classical Theory of Renormalized Solutions

In essence, Prop. 13 is a quantified version of the classical commutator estimate at the heart of the theory of renormalized solutions, which, for this reason, we describe in further details. The theory was introduced by DiPerna and Lions in [25] to handle the well posedness of weak solutions to the linear continuity equation (1.4), which, we, recall is

$$\partial_t u + \text{div} \, (a(t, x) \, u(t, x)) = 0.$$ 

If $a \in L^1_t L^p_x$, a weak solution $u \in L^{\infty}_t L^q_x$ to (1.4) is said to be renormalized iff, for any $\beta \in C^1(\mathbb{R})$ with $|\beta(\xi)| \leq C |\xi|$, $\beta(u)$ is a solution to

$$\partial_t \beta(u) + \text{div} \, (a(t, x) \, \beta(u)) + \text{div} \, a(\beta'(u) \, u - \beta(u)) = 0.$$ 

An equation for a given $a$ is renormalized iff all weak solutions are renormalized. This is now an important property which directly implies uniqueness: if $u$ is a weak solution with $u(t = 0) = 0$, then $|u|$ is also a weak solution and hence

$$\int_{\mathbb{R}^d} |u(t, x)| \, dx = \int_{\mathbb{R}^d} |u(t = 0, x)| \, dx = 0.$$
This also indirectly implies the compactness of any sequence of solutions \( u_n \). Essentially, one combines the uniqueness with the renormalization property at the limit to prove that

\[
w - \lim \beta(u_n) = \beta(w - \lim u_n),
\]

where \( w - \lim \) denotes the weak limit in the appropriate \( L^q \) space.

DiPerna and Lions proved in [25] that if \( a \in L^1_t W^{1,1}_x \), then any bounded solution \( u \) is renormalized. The proof relies on the following simple but powerful idea: given a weak solution \( u \), consider a smooth convolution kernel \( K_\varepsilon \) and convolve the equation with \( K_\varepsilon \) to get

\[
\partial_t K_\varepsilon \ast u + \text{div} (a K_\varepsilon \ast u) = R_\varepsilon.
\]

Of course \( K_\varepsilon \ast u \) cannot be also a solution and there is a remainder term which can be rewritten as

\[
R_\varepsilon = \int_{\mathbb{R}^d} \text{div}_x \left( (a(x) - a(y)) K_\varepsilon(x - y) u(y) \right) dy \\
= \int_{\mathbb{R}^d} (a(x) - a(y)) \nabla K_\varepsilon(x - y) u(y) dy + \int_{\mathbb{R}^d} \text{div}_x a(x) K_\varepsilon(x - y) u(y) dy \\
= C_\varepsilon + D_\varepsilon.
\]

For a fixed \( a \in L^1_t W^{1,1}_x \) and \( u \in L^\infty \), one can then prove that \( R_\varepsilon \) converges to 0 in \( L^1 \).

It is then straightforward to write an equation on \( \beta(u_\varepsilon) \) and pass to the limit as \( \varepsilon \to 0 \).

This idea was then extended to include \( a \in L^1_t BV_x \), first in [11] in the kinetic context and then in the seminal [4] in the general case; we also refer to [37]. Those require the use of specific kernels, based on a quadratic form in \( \mathbb{R}^d \) that is adapted to the singular part of \( \nabla a \).

In general, without any additional structure, \( BV \) seems to be the critical space here as proved in [23]. If some additional structure is available, then one may be able to work with less. Typical examples are found in dimension 2, in [2, 17, 18, 31], or with some phase space structure in [16, 34].

As one can readily see, the quantity that we bound explicitly in Prop. 13 is very close to the commutator estimate (3.1). In fact this proposition could be used to directly give an explicit bound in \( \varepsilon \) in (3.1) for the particular \( K_\varepsilon \) that we use. It is in this sense that we talk of a quantified commutator estimate.

The fact that renormalized solutions are connected to some regularity of the solutions had been noticed, for instance, in [6], but this regularity had not been quantified until [20], as we mentioned earlier. As we explained, [20] involves a quantitative regularity estimate at the level of the characteristics. The approach that we follow here is very different and it is a nice feature to be able to quantify this commutator exactly.

For more on renormalized solutions, we refer to the surveys in [5, 22].
3.3. Useful Technical Lemmas

The proof of Prop. 13 requires the use of two classical lemmas. The first relates to the difference between \( a(x) - a(y) \) and \( \nabla a \).

**Lemma 14.** There exists a bounded function \( \psi \) s.t. \( \psi \) is \( W^{1,1} \) on \( B(0,1) \times S^{d-1} \) and for any \( a \in (BV)_{loc}^d \)

\[
a_i(x) - a_i(y) = |x - y| \int_{B(0,1)} \psi \left( z, \frac{x - y}{|x - y|} \right) \cdot \nabla a_i(x + |x - y| z) \frac{dz}{|z|^{d-1}} + |x - y| \int_{B(0,1)} \psi \left( z, \frac{x - y}{|x - y|} \right) \cdot \nabla a_i(y + |x - y| z) \frac{dz}{|z|^{d-1}}.
\]

Moreover, for some given constant \( \alpha \),

\[
\int_{B(0,1)} \psi \left( z, \frac{x - y}{|x - y|} \right) \frac{dz}{|z|^{d-1}} = \alpha \frac{|x - y|}{|x - y|}.
\]

The proof of Lemma 14 is straightforward; it consists in integrating \( \nabla v \) over a curve between \( x \) and \( y \) and then averaging the resulting estimate over all such curves. We refer to [16] for a detailed proof.

We need another technical result to control slightly “delocalized” convolutions of \( \nabla a \).

**Lemma 15.** For any \( 1 < p < \infty \), any \( L \in W^{s,1} \) for some \( s > 0 \) with compact support and \( \int L = 0 \), there exists \( C > 0 \) s.t. for any \( u \in L^p (\mathbb{R}^d) \),

\[
\int_{h_0}^1 \|L_r * u\|_{L^p} \frac{dr}{r} \leq C |\log h_0|^{1-1/q} \|u\|_{B^0_{p,q}}, \tag{3.2}
\]

where \( L_r(x) = r^{-d} L(z/r) \) and the constant \( C \) depends only on the \( W^{s,1} \) norm and the size of the support of \( L \). As a consequence, for \( p \leq 2 \),

\[
\int_{h_0}^1 \|L_r * u\|_{L^p} \frac{dr}{r} \leq C |\log h_0|^{1/2} \|u\|_{L^p}. \tag{3.3}
\]

The proof is again classical and is given in the appendix for the sake of completeness.

3.4. The Proof of Proposition 13

Observe that by the definition of \( K_h \),

\[
\nabla K_h = \frac{x - y}{(|x - y| + h)^{d+1}} \chi(x - y)
\]

for some smooth function \( \chi(x - y) = \chi(|x - y|) \) with support in \( |x - y| \leq 2 \) and with \( \chi = 1 \) if \( |x - y| \leq 1 \).
Using Lemma 14, one obtains
\[ \int_{\mathbb{R}^d} \nabla K_h (x - y) (a (x) - a (y)) |g (x) - g (y)|^2 \, dx \, dy \]
\[ = 2 \int_{\mathbb{R}^d} \frac{(x - y) \chi(x - y)}{|x - y| + h} \frac{d}{dz} \int_{B(0,1)} \psi \left( z, \frac{x - y}{|x - y|} \right) \nabla a (x + |x - y| z) \frac{dz}{|z|^{d-1}} \]
\[ |g (x) - g (y)|^2 \, dx \, dy, \]
where by the symmetry of the expression in \( x \) and \( y \), both terms in Lemma 14 lead to the same expression.

The main idea of the proof is to try to replace some of the key expressions in the above formula by their average. Thanks to Lemma 15, the difference with the average naturally produces the desired scale.

We first introduce the average of \( \psi \) as given by Lemma 14 and decompose accordingly:
\[ \int_{\mathbb{R}^d} \frac{(x - y) \chi(x - y)}{|x - y| + h} \frac{d}{dz} \int_{B(0,1)} \psi \left( z, \frac{x - y}{|x - y|} \right) \nabla a (x + |x - y| z) \frac{dz}{|z|^{d-1}} \]
\[ |g (x) - g (y)|^2 \, dx \, dy = I + \tilde{\alpha} J, \]
where
\[ I = \int_{\mathbb{R}^d} \frac{(x - y) \chi(x - y)}{|x - y| + h} \frac{d}{dz} \int_{B(0,1)} \left( \psi \left( z, \frac{x - y}{|x - y|} \right) - \tilde{\alpha} \frac{x - y}{|x - y|} \right) \nabla a (x + |x - y| z) \frac{dz}{|z|^{d-1}} \]
\[ |g (x) - g (y)|^2 \, dx \, dy, \]
and
\[ J = \int_{\mathbb{R}^d} \frac{(x - y)_i \chi(x - y) (x - y)_j}{|x - y| + h} \frac{d}{dz} \int_{B(0,1)} \partial_i a_j (x + |x - y| z) \frac{dz}{|z|^{d-1}} \]
\[ |g (x) - g (y)|^2 \, dx \, dy, \]
where we used Einstein convention of summation.

The constant \( \tilde{\alpha} \) is chosen so that
\[ \int_{B(0,1)} \left( \psi \left( z, \frac{x - y}{|x - y|} \right) - \tilde{\alpha} \frac{x - y}{|x - y|} \right) \frac{dz}{|z|^{d-1}} = 0, \]
which is always possible thanks to Lemma 14. Both terms \( I \) and \( J \) rely on appropriate cancellations that allow to use Lemma 15 but on different terms. As such we have to handle them separately.

**Control of \( I \).** Denote, for simplicity,
\[ \tilde{\psi} (z, \omega) := \psi (z, \omega) - \tilde{\alpha} \omega, \quad L (z, \omega) := \tilde{\psi} (z, \omega) \frac{1_{|z| \leq 1}}{|z|^{d-1}}. \]
One can see easily that for any fixed $\omega$, $L$ is compactly supported and belongs to $W^{s,1}$ for any $s < 1$; it hence satisfies the assumptions of Lemma 15, uniformly in $\omega$.

Now observe that
\[
\int_{B(0,1)} \frac{\tilde{\psi}(z, w)}{|z|^{d-1}} \cdot \nabla a(x + rz) \, dz = L_r(\cdot, \omega) \ast \nabla a,
\]
where $L_r(z, \omega) = r^{-d} L(z/r, \omega)$. On the other hand, by the spherical changes of variables,
\[
I = \int_0^1 \int_{S^{d-1}} \int_{\mathbb{R}^d} \frac{r^d}{(r + h)^{d+1}} \int_{B(0,1)} \frac{\tilde{\psi}(z, \omega)}{|z|^{d-1}} \nabla a(x + rz) \, dz
\]
\[
|g(x) - g(x - r\omega)|^2 \, dx \, d\omega \, dr
\]
\[
\leq \int_{S^{d-1}} \int_0^1 \int_{\mathbb{R}^d} \frac{r^d}{(r + h)^{d+1}} |L_r(\cdot, \omega) \ast \nabla a| \, |g(x) - g(x - r\omega)|^2.
\]

By a Hölder estimate,
\[
I \leq \int_0^1 \int_{S^{d-1}} \frac{r^d}{(r + h)^{d+1}} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \|g(\cdot) - g(\cdot - r\omega)\|_{L^2^{p^*}} \, d\omega \, dr.
\]

Of course,
\[
\|g(\cdot) - g(\cdot - r\omega)\|_{L^2^{p^*}} \leq 2 \|g\|_{L^{2p^*}},
\]
so that
\[
I \leq 2 \|g\|_{L^{2p^*}} \int_{S^{d-1}} \int_0^1 \frac{r^d}{(r + h)^{d+1}} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \, dr
\]
\[
= 2 \|g\|_{L^{2p^*}} \int_{S^{d-1}} \left( \int_0^h + \int_h^1 \right) \frac{r^d}{(r + h)^{d+1}} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \, d\omega
\]
\[
\leq \frac{C}{h^{d+1}} \|g\|_{L^{2p^*}} \int_{S^{d-1}} \int_0^h \frac{r^d}{r} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \, dr \, d\omega
\]
\[
+ 4 \|g\|_{L^{2p^*}} \int_{S^{d-1}} \int_h^1 \frac{1}{r} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \, dr \, d\omega.
\]

Of course, as $B_{0,p}^0 \subset L^p$ if $p \leq 2$ and $B_{0,p,2}^0 \subset L^p$ if $p \geq 2$, one has that $\|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \leq C \log^\theta |r| \|\nabla a\|_{B_{0,p,q}^0}$ with $\theta = (\max(1/2, 1/p) - 1/q) +< 1 - 1/q$. Therefore the first term in the right-hand side can be simplified as follows:
\[
\frac{C}{h^{d+1}} \int_0^h \frac{r^d}{r} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \, dr \leq \frac{C}{h^{d+1}} \|\nabla a\|_{B_{0,p,q}^0} \int_0^h \frac{r^d}{r} \log^\theta |r| \, dr
\]
\[
\leq C \|\log h\|^\theta \|\nabla a\|_{B_{0,p,q}^0},
\]
leading to
\[
I \leq C \|\log h\|^\theta \|g\|_{L^{2p^*}} \|\nabla a\|_{B_{0,p,q}^0} + 4 \|g\|_{L^{2p^*}} \int_{S^{d-1}} \int_h^1 \frac{1}{r} \|L_r(\cdot, \omega) \ast \nabla a\|_{L^p} \, dr.
\]
We now recall that $L$ satisfies the assumption of Lemma 15 uniformly in $\omega$. We hence deduce thus for some constant $C$,

\[
I \leq C |\log h|^\theta \|g\|_{L^p} \|\nabla a\|_{B^0_{p,q}} + C \|\nabla a\|_{B^0_{p,q}} |\log h|^{1-1/q} \|g\|_{L^2}.
\]

since $\theta < 1 - 1/q$.

**Control of $J$.** The general idea is similar to $I$; trying to identify convolution with a kernel of vanishing average to gain regularity.

For this reason, we decompose again $J = J_1 + J_2$,

where we subtracted the right average in $J_1$ as

\[
J_1 = \int_{\mathbb{R}^{2d}} \frac{|x - y| \chi}{(|x - y| + h)^{d+1}} \left( \frac{(x - y)}{|x - y|} \otimes \frac{(x - y)}{|x - y|} - \tilde{C} I_d \right) : \nabla a (x + |x - y| z) \frac{dz}{|z|^{d-1}} |g(x) - g(y)|^2 \, dx \, dy,
\]

where $A : B$ denotes the total contraction of two matrices $\sum_{i,j} A_{ij} B_{ij}$ and where $\tilde{C}$ is again chosen s.t.

\[
\int_{S^{d-1}} (\omega_i^2 - \tilde{C}) \, d\omega = 0.
\]

This leaves, as $J_2$,

\[
J_2 = \tilde{C} \int_{\mathbb{R}^{2d}} \int_{B(0,1)} \frac{|x - y| \chi}{(|x - y| + h)^{d+1}} I_d : \nabla a (x + |x - y| z) \frac{dz}{|z|^{d-1}} |g(x) - g(y)|^2 \, dx \, dy.
\]

This term can be immediately bounded as

\[
I_d : \nabla a = \text{div} \, a,
\]

so that

\[
J_2 \leq C \|\text{div} \, a\|_{L^\infty} \int_{\mathbb{R}^{2d}} \frac{|g(x) - g(y)|^2}{(|x - y| + h)^d} \, dx \, dy
\]

(3.4)

We now turn to $J_1$, where we need to use a slight variant of spherical coordinates by writing $x - y = -r \, w$ for $r \in \mathbb{R}$ and $1/4 \leq w \leq 1$ instead of $w \in S^{d-1}$ with
\(|w| = 1\) as usual. Indeed, for a fixed \(x \in \mathbb{R}^d\) we use first spherical coordinates \(w = s \omega\) to calculate

\[
\int_0^1 \int_{1/2 \leq |w| \leq R} \Phi(x + r w) r^{d-1} dw dr
= \int_0^1 \int_0^R \int_{S^{d-1}} \Phi(x + s r \omega) (r s)^{d-1} dr d\omega ds
= \int_0^1 \int_{1/2}^R \int_{S^{d-1}} \Phi(x + \tilde{r} \omega) \tilde{r}^{d-1} d\tilde{r} d\omega ds
= \int_{|x-y| \leq R} \Phi(y) \int_{\max(1,|x-y|)/R}^1 \frac{ds}{s} dy,
\]

with the change of variables \(r \rightarrow \tilde{r} = rs\) for a fixed \(s\) and a final use of spherical coordinates \(y = x + \tilde{r} \omega\), therefore defining the smooth function \(W_R(r) = \int_{\max(1/2,r/R)}^1 s^{-1} ds\) and for any \(\Phi\) by taking \(\Phi(y) = \Phi(y)/W_R(|x - y|)\)

\[
\int_{|x-y| \leq R} \tilde{\Phi}(y) dy = \int_0^R \int_{1/2 \leq |w| \leq R} \tilde{\Phi}(x + r w) \frac{r^{d-1}}{W_R(r |w|)} dw dr. \tag{3.5}
\]

Denote, accordingly,

\[
\tilde{L}(w) = \left( \frac{w \otimes w}{|w|^2} - \tilde{C} I_d \right) \mathbb{1}_{1/2 \leq w \leq 1}.
\]

This allows as to rewrite

\[
J_1 = \int_0^1 \int_{|w| \leq 1} \int_{B(0,1)} \frac{r^d \chi(r)}{W_1(r |w|)} (r + h)^{d+1} \tilde{L}(w) : \nabla a(x + r z) \frac{dz}{|z|^{d-1}}
|g(x) - g(x + r w)|^2 \, dx \, dw \, dr.
\]

Observe that \(W_1(u) = \tilde{w}\) is constant for \(u < 1/2\) that is, since \(|w| \leq 1\), \(W_1(r |w|) = \tilde{w}\) if \(r < 1/2\). Obviously the integral \(J_1\) is bounded for \(r > 1/2\) so that

\[
J_1 \leq C + \frac{1}{\tilde{w}} \int_0^1 \int_{|w| \leq 1} \int_{B(0,1)} \frac{r^d \chi(r)}{(r + h)^{d+1}} \tilde{L}(w) : \nabla a(x + r z) \frac{dz}{|z|^{d-1}}
|g(x) - g(x + r w)|^2 \, dx \, dw \, dr.
\]

Now denote, for simplicity,

\[
A_r(x) = \int_{B(0,1)} \nabla a(x + r z) \frac{dz}{|z|^{d-1}},
\]

and let us expand \(|g(x) - g(y)|^2 = g^2(x) + g^2(y) - 2g(x)g(y)\) so that

\[
J_1 \leq C + \frac{J_x - 2 J_{xy} + J_y}{\tilde{w}},
\]
with
\[ J_x = \int_0^1 \int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{r \chi(r)}{(r + h)^{d+1}} L(w) : A_r(x) g^2(x) \, dx \, dw \, dr, \]
\[ J_{xy} = \int_0^1 \int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{r \chi(r)}{(r + h)^{d+1}} L(w) : A_r(x) g(x) g(x + rw) \, dx \, dw \, dr, \]
\[ J_y = \int_0^1 \int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{r \chi(r)}{(r + h)^{d+1}} L(w) : A_r(x) g^2(x + rw) \, dx \, dw \, dr. \]

First, note that since \( \int \tilde{L}(w) \, dw = 0 \), one simply has that
\[ J_x = 0. \]

The term \( J_y \) can be controlled through Lemma 15 as one can identify a convolution
\[ J_y = \int_{\mathbb{R}^d} \int_0^1 r^d \chi(r) \frac{A_r(x)}{(r + h)^{d+1}} \tilde{L}(w) \star |g|^2 \, dx \, dr, \]
\[ = \int_{\mathbb{R}^d} \int_0^1 r^d \chi(r) \frac{A_r(x)}{(r + h)^{d+1}} L_r \star A_r(x) |g|^2 \, dx \, dr, \]
since \( \tilde{L} \) is even. Now, by Hölder inequality, one has that
\[ J_y \leq \int_0^1 \frac{1}{r + h} \| \tilde{L}_r \star A_r \|_{L^p} \| g^2 \|_{L^{p^*}} \, dr. \]

Of course, denoting \( B(z) = \|z\| \leq 1 |z|^{1-d} \), one has, by the definition of \( A_r \),
\[ \| \tilde{L}_r \star A_r \|_{L^p} = \| \tilde{L}_r \star (B_r \star \nabla a) \|_{L^p} = \| B_r \star (\tilde{L}_r \star \nabla a) \|_{L^p} \]
\[ \leq \| B_r \|_{L^1} \leq \| \tilde{L}_r \star \nabla a \|_{L^p}. \]

Finally, by Lemma 3 applied to the exponent \( p \), we deduce that
\[ J_y \leq C \left\| g^2(t, \cdot) \right\|_{L^{p^*}} \| \nabla a \|_{B^0_{p,q}} |\log h|^{1-1/q}. \]

Again we identify the convolution in \( J_{xy} \) as
\[ J_{xy} = \int_{\mathbb{R}^d} \int_0^1 r^d \chi(r) \frac{L_r \star g(x)}{(r + h)^{d+1}} \, dx \, dr, \]
and still by Lemma 15, we get
\[ |J_{xy}| \leq C \left\| g(t, \cdot) \right\|_{L^{2p^*}}^2 |\log h|^{1-1/q} \| \nabla a \|_{B^0_{p,q}}. \]

Collecting all estimates and recalling that, for \( p \leq 2, L^p \subset B^0_{p,2} \), concludes the proof of the proposition.
4. Proof of Proposition 1 and Theorem 3

4.1. Proof of Proposition 1

The embedding \( W_{\text{log}, \theta}^p \subset L^p \) is straightforward from the definition. For the embedding \( W_s^p \subset W_{\text{log}, \theta}^p \), note that if \( u \in W_s^p \),

\[
\|u(\cdot + z) - u(\cdot)\|_{L^p} \leq C |z|^s \|\nabla u\|_{L^p},
\]

which can easily be obtained by interpolation from the \( s = 1 \) case

\[
\|u(\cdot + z) - u(\cdot)\|_{L^p} = \left\| \int_0^1 z \nabla u(\cdot + sz) ds \right\|_{L^p} \leq \int_0^1 |z| \|\nabla u(\cdot + sz)\|_{L^p} ds = |z| \|\nabla u\|_{L^p}.
\]

On the other hand,

\[
\|u\|_{p, \theta}^p = \sup_h |\log h|^{-\theta} \int |z| K_h(z) \|u(\cdot + z) - u(\cdot)\|_{L^p} dz \leq \|\nabla u\|_{L^p} \sup_h |\log h|^{-\theta} \int |z|^s K_h(z) dz,
\]

while finally,

\[
\int |z|^s K_h(z) dz \leq \int_{|z| \leq 2} \frac{|z|^s}{|z|^d} dz \leq C,
\]

concluding the bound.

Note that with similar calculations,

\[
\int K_h(z) dz \sim \int_{|z| \leq 2} \frac{1}{(h + |z|)^d} dz \sim |\log h|,
\]

which implies that

\[
\|u\|_{p,1}^p \leq 2 \|u\|_{L^p}^p \sup_h |\log h|^{-1} \int K_h(z) dz \leq C \|u\|_{L^p}^p,
\]

so that the semi-norms do not carry any special information when \( \theta = 1 \).

Define now

\[
\tilde{K}_h = \frac{K_h}{\|K_h\|_{L^1}} \sim \frac{K_h}{\log h}.
\]

The kernel \( \tilde{K}_h \) is normalized and, moreover, for any \( \delta > 0 \), as \( h \to 0 \),

\[
\int_{|z| \geq \delta} \tilde{K}_h(z) dz \longrightarrow 0,
\]
such that $\tilde{K}_h$ is a classical convolution kernel. Observe that
\[
\|u - \tilde{K}_h \ast u\|_{L^p} \leq \|K_h\|_{L^1} \int_{\mathbb{R}^d} K_h(z) \|u(\cdot + z) - u(\cdot)\|_{L^p} \, dz
\]
\[
\leq \left(\|K_h\|_{L^1} \int_{\mathbb{R}^d} K_h(z) \|u(\cdot + z) - u(\cdot)\|_{L^p} \, dz\right)^{1/p}
\]
\[
\leq C |\log h|^{\theta - 1} \|u\|_{p, \theta}.
\]
This proves, by the Rellich criterion, that for any sequence $u_n$ s.t. $\|u_n\|_{p, \theta} + \|u_n\|_{L^p}$ is uniformly bounded, $u_n$ is locally compact in $L^p$.

Finally let us calculate for $p = 2$, using the Fourier transform, that
\[
\int_{z \in \mathbb{R}^d} K_h(z) \|u(\cdot + z) - u(\cdot)\|^2_{L^2} \, dz = \int_{\mathbb{R}^d} K_h(z) \int_{\mathbb{R}^d} \left| e^{iz \cdot \xi} - 1 \right|^2 |\mathcal{F} u(\xi)|^2 \, d\xi \, dz.
\]
This leads to calculate
\[
\int_{\mathbb{R}^d} K_h(z) \left| e^{iz \cdot \xi} - 1 \right|^2 \, dz \sim \int_{|z \cdot \xi| \geq 1} K_h(z) + \int_{|z \cdot \xi| \leq 1} K_h(z) |z \cdot \xi|^2
\]
\[
\sim \left| \log \left( \frac{1}{|\xi| + h} \right) \right| + 1,
\]
concluding the proof.

4.2. Proof of Theorem 3

The proof of Th. 3 mostly follows the steps of [9], the main improvement being the more precise Proposition 13.

The proof is essentially a straightforward calculation. We use the equation (1.1) to propagate $\|u\|_{1, \theta}$ and we try to put the corresponding right hand-side under the form of the commutator estimate provided by Prop. 13. We still have to be careful to justify our otherwise formal calculations, which is where the notion of entropy solution is critical.

First of all, by Kruzkov’s doubling of variables, see [36], any entropy solution $u$ to (1.1) satisfies in the sense of distributions that
\[
\partial_t |u(t, x) - u(t, y)| + \text{div}_x (a(t, x) F(u(t, x), u(t, y))
\]
\[
+ \text{div}_y (a(t, y) F(u(t, x), u(t, y)) + G(u(t, x), u(t, y)) \text{div}_x a(t, x)
\]
\[
+ G(u(t, y), u(t, x)) \text{div}_y a(t, y) \leq 0,
\]
where
\[
F(\xi, \zeta) = (f(\xi) - f(\zeta)) \text{sign}(\xi - \zeta) = F(\zeta, \xi),
\]
\[
G(\xi, \zeta) = f(\xi) \text{sign}(\xi - \zeta) - F(\xi, \zeta) = \bar{G}(\xi, \zeta) \frac{1}{2} F(\xi, \zeta), \quad \text{with}
\]
\[
\bar{G}(\xi, \zeta) = \frac{f(\xi) + f(\zeta)}{2} \text{sign}(\xi - \zeta) = -\bar{G}(\zeta, \xi).
\]
Note that, up to adding a constant in $f$, we may assume that $f(0) = 0$, thus normalizing $\tilde{G}$ s.t. $\tilde{G}(0, 0) = 0$.

For any fixed $h$, $K_h(x - y)$ is a smooth, compactly supported function which we may hence use as a test function for (4.1), giving

$$\frac{d}{dt} \int_{\mathbb{R}^2d} K_h(x - y) |u(t, x) - u(t, y)| \, dx \, dy$$

$$\leq \int_{\mathbb{R}^2d} \nabla K_h(x - y) (a(x) - a(y)) F(u(x), u(y)) \, dx \, dy$$

$$+ \int_{\mathbb{R}^2d} K_h(x - y) \frac{\text{div} a(x) + \text{div} a(y)}{2} F(u(x), u(y)) \, dx \, dy$$

$$+ \int_{\mathbb{R}^2d} K_h(x - y) (\text{div} a(x) - \text{div} a(y)) \tilde{G}(u(x), u(y)) \, dx \, dy.$$ 

Using the bound on the divergence, the second term in the r.h.s. can simply be bounded by

$$\| f' \|_{L^\infty} \| \text{div} a \|_{L^\infty} \int_{\mathbb{R}^2d} K_h(x - y) |u(t, x) - u(t, y)| \, dx \, dy,$$

since $F(\xi, \zeta) \leq \| f' \|_{L^\infty} |\xi - \zeta|$, while by Hölder’s estimate, the third term in the r.h.s. is bounded by

$$\left( \int_{\mathbb{R}^2d} K_h(x - y) (\text{div} a(x) - \text{div} a(y))^p \right)^{1/p}$$

$$\times \left( \int_{\mathbb{R}^2d} K_h(x - y) |\tilde{G}(u(x), u(y))|^p^* \right)^{1/p^*}$$

$$\leq \| f' \|_{L^\infty} \| \log h \|^\theta \| u \|_{L^{p^*}} \| \text{div} a \|_{p, p (\theta - 1/p^*)},$$

simply by using that $\tilde{G}(\xi, \zeta) \leq \| f' \|_{L^\infty} \frac{\xi + \zeta}{2}$, since $f(0) = 0$.

The combination of those two bounds yields that

$$\frac{d}{dt} \| u \|_{1, \theta} \leq \| f' \|_{L^\infty} \| \text{div} a \|_{L^\infty} \| u \|_{1, \theta} + \| f' \|_{L^\infty} \| u \|_{L^{p^*}} \| \text{div} a \|_{p, p (\theta - 1/p^*)} + C,$$

where $C$ is the commutator

$$C = \sup_h |\log h|^{-\theta} \int_{\mathbb{R}^2d} \nabla K_h(x - y) (a(x) - a(y)) F(u(x), u(y)) \, dx \, dy.$$ 

Therefore all the difficulty lies in obtaining an explicit quantitative estimate on this commutator. This is where Prop. 13 comes in, leading to improved, more precise results with respect to [9].

We still need an additional step to put $C$ in precisely the right form for Prop. 13. As this is going to be used as well for the numerical scheme, and we put the corresponding estimate in a lemma as follows:
Lemma 16. Assume that \( f \in W^{1,\infty}(\mathbb{R}_+) \), and that for some \( 1 < p, q < \infty \), we have \( u \in L^{p^*} \) with \( 1/p^* = 1 - 1/p \), and that \( a \in B^{1}_{p,q} \). Then, provided \( \theta \geq 1 - 1/q \),
\[
C = \sup \log h^{-\theta} \int_{\mathbb{R}^2} \nabla K_h(x-y) (a(x) - a(y)) F(u(x), u(y)) \, dx \, dy
\]
\[
\leq C \| \text{div} \, a \|_{L^\infty} \| f' \|_{L^\infty} \| u \|_{1, \theta} + C \| \nabla a \|_{B^{0}_{p,q}} \| f' \|_{L^\infty} \| u \|_{L^{p^*,1}},
\]
where we recall that \( F(\xi, \zeta) = (f(\xi) - f(\zeta)) \text{sign} (\xi - \zeta) = F(\zeta, \xi) \).

**Proof.** Just as in [9], we use the repartition function of \( u \):
\[
\kappa(t, x, \xi) = \mathbb{1}_{0 \leq \xi \leq u(t, x)},
\]
which, from the definition of \( F \) in (4.2), implies the simple representation
\[
F(u(x), u(y)) = \int_{0}^{\infty} f'(\xi) |\kappa(x, \xi) - \kappa(y, \xi)|^2 \, d\xi.
\]
This lets us simply write
\[
C = \sup \log h^{-\theta} \int_{0}^{\infty} f'(\xi)
\int_{\mathbb{R}^2} \nabla K_h(x-y) (a(x) - a(y)) |\kappa(t, x, \xi) - \kappa(t, y, \xi)|^2 \, dx \, dy \, d\xi.
\]
We may now directly use Prop. 13 to find that, provided \( \theta \geq 1 - 1/q \),
\[
C \leq C \| \text{div} \, a \|_{L^\infty} \| f' \|_{L^\infty} \| u \|_{1, \theta} + C \| \nabla a \|_{B^{0}_{p,q}} \int_{0}^{\infty} f'(\xi) \| \kappa(t, \xi) \|^2_{L^{2p^*}} \, d\xi.
\]
It is now straightforward to estimate
\[
\int_{0}^{\infty} f'(\xi) \| \kappa(t, \xi) \|^2_{L^{2p^*}} \, d\xi \leq \| f' \|_{L^{\infty}} \int_{0}^{\infty} |\{ u(t, \cdot) \geq \xi \}|^{1/p^*} \, d\xi
\]
\[
= \| f' \|_{L^{\infty}} \| u \|_{L^{p^*,1}},
\]
where \( L^{p^*,1} \) denotes the corresponding Lorentz space. \( \square \)

From the previous Lemma we finally obtain that if \( \theta \geq \max(1/p^*, 1 - 1/q) \),
\[
\frac{d}{dt} \| u \|_{1, \theta} \leq C \| f' \|_{L^{\infty}} \| \text{div} \, a \|_{L^{\infty}} \| u \|_{1, \theta} + \| f' \|_{L^{\infty}} \| u \|_{L^{p^*}} \| \text{div} \, a \|_{p, p(\theta-1/p^*)}
\]
\[
+ C \| \nabla a \|_{B^{0}_{p,q}} \| f' \|_{L^{\infty}} \| u \|_{L^{p^*,1}}.
\]
A Gronwall estimate concludes the first statement of Th. 3. The embeddings \( L^p \subset B^{0}_{p,2} \) for \( p \leq 2 \) and \( L^r \cap L^1 \subset L^{p^*,1} \) for \( r > p^* \) conclude the second statement.
5. The Numerical Scheme: Proof of Theorem 8

We briefly recall here the notations of the numerical scheme for the convenience of the readers. By (2.2), our numerical scheme reads

$$u_{i}^{n+1} = \sum_{m \in \mathbb{Z}^{d}} b_{i,m}(a_{m}^{n}, u_{m}^{n}),$$

with the functions $b$ expressed as differences of fluxes by (2.3):

$$b_{i,m}(a_{m}^{n}, u_{m}^{n}) = u_{m}^{n} \delta_{i=m} + \frac{\delta t}{\delta x} \sum_{k=1}^{d} \left( F_{i+1,k}^{k}(a_{m}^{n}, u_{m}^{n}) - F_{i-k}^{k}(a_{m}^{n}, u_{m}^{n}) \right).$$

The fluxes are normalized by (2.4) or

$$\sum_{j} F_{j}^{k}(a, u) = a^{k} f(u),$$

while we define the discrete divergence by (2.6), repeated here:

$$\sum_{j \in \mathbb{Z}^{d}} b_{i,j}(a_{j}^{n}, U) = U + \delta t D_{i}^{n} f(U),$$

where we imposed by (2.7) that $\| f \|_{W^{1,\infty}} \leq C \| f \|_{W^{1,\infty}}$.

The goal of this section is to obtain bounds on the discrete semi-norms defined by (2.9) or

$$\| u^{n} \|_{p,\alpha,p,\theta} = \sup_{h \geq \delta x} | \log h |^{-\theta} \delta x^{2d} \sum_{i,j} K_{i-j}^{h} |u_{i}^{n} - u_{j}^{n}|,$$

where $K_{i}^{h} = K_{h}(i \delta x)$ with $K_{h}$ given by (1.5), i.e.

$$K_{h}(x) = \frac{\phi(x)}{(|x| + h)^{d}}.$$

The control is $\| u^{n} \|_{p,\alpha,p,\theta}$ in terms of the discrete Sobolev norm of $a$ gives

$$\| a^{n} \|_{d,W^{1,p},\delta x} = \delta x^{d} \sum_{i \in \mathbb{Z}^{d}} \sum_{k=1}^{d} |a_{i}^{n} - a_{i+1}^{n}|^{p},$$

and the moments of the flux given by (2.8) or

$$\sum_{m} |i - m|^{\gamma} |F_{i-m}^{k}(a_{m}^{n}, a_{m}^{n})| \leq C \| f \|_{L^{\infty}} \| a^{n} \|_{lp} \| u^{n} \|_{lp^{*}}.$$
5.1. Discrete Equivalent of Kruzkov Inequality Eq. (4.1)

The aim is here is to prove an equivalent of Kruzkov argument to propagate strong norms on the scheme. We start with

Lemma 17. Denoting $s_{i,j}^{n+1} = \text{sign}(u_{i}^{n+1} - u_{j}^{n+1})$, we have, for any discrete solution $u$ to the scheme (2.2) with the fluxes given by (2.3) and the discrete divergence defined by (2.6),

$$
\sum_{i,j \in \mathbb{Z}^d} |u_{i}^{n+1} - u_{j}^{n+1}| K_{i-j}^{h} \leq \sum_{i,j} K_{i-j}^{h} |u_{i}^{n} - u_{j}^{n}| + A_{h} + B_{h} + D_{h}, \tag{5.1}
$$

where we denote

$$
A_{h} = \frac{\delta t}{\delta x} \sum_{i,j,m,k=1}^{d} s_{m,j}^{n} K_{i-j}^{h} \left( F_{i+[1]k-m}(a_{m}^{n}, u_{m}^{n}) - F_{i-[1]k-m}(a_{m}^{n}, u_{j}^{n}) \right)
$$

$$
B_{h} = \frac{\delta t}{\delta x} \sum_{i,j,m,k=1}^{d} s_{m,j}^{n} K_{i-j}^{h} \left( F_{j+[1]k-m}(a_{m}^{n}, u_{i}^{n}) - F_{j-[1]k-m}(a_{m}^{n}, u_{m}^{n}) \right)
$$

and

$$
D_{h} = \delta t \sum_{i,j} s_{i,j}^{n+1} K_{i-j}^{h} \left( D_{i}^{n} \tilde{f}(u_{i}^{n}) - D_{j}^{n} \tilde{f}(u_{j}^{n}) \right).
$$

Note that since we use $C$ as a notation for a generic constant, we prefer to avoid using the notation $C_{h}$ above.

**Proof.** Using the definition of the scheme, we have to calculate

$$
\sum_{i,j \in \mathbb{Z}^d} |u_{i}^{n+1} - u_{j}^{n+1}| K_{i-j}^{h} = \sum_{i,j} s_{i,j}^{n+1} K_{i-j}^{h} \sum_{m \in \mathbb{Z}^d} (b_{i,m}(a_{m}^{n}, u_{m}^{n}) - b_{j,m}(a_{m}^{n}, u_{m}^{n})) .
$$

We introduce the terms that we need to obtain the right cancellations and write

$$
\sum_{i,j \in \mathbb{Z}^d} |u_{i}^{n+1} - u_{j}^{n+1}| K_{i-j}^{h}
$$

$$
= \sum_{i,j} s_{i,j}^{n+1} K_{i-j}^{h} \sum_{m \in \mathbb{Z}^d} \left( b_{i,m}(a_{m}^{n}, u_{m}^{n}) - b_{i,m}(a_{m}^{n}, u_{j}^{n}) + \frac{1}{2} (u_{j}^{n} - u_{i}^{n}) \delta_{m-i} \right)
$$

$$
+ \sum_{i,j} s_{i,j}^{n+1} K_{i-j}^{h} \left( \sum_{m} (b_{i,m}(a_{m}^{n}, u_{j}^{n}) - b_{j,m}(a_{m}^{n}, u_{m}^{n})) - u_{j}^{n} + u_{i}^{n} \right) .
$$

(5.2)
where we have used that
\[
\sum_{m} \frac{1}{2} (u^n_j - u^n_i) \delta_{m-i} + \sum_{m} \frac{1}{2} (u^n_j - u^n_i) \delta_{m+i} = u^n_j - u^n_i.
\]

We start with the last term in (5.2) and use the discretized expression of the divergence of \( a \) given by Eq. (2.6) to find that
\[
\sum_{m} b_{i,m}(a^n_m, u^n_j) = u^n_j + \delta t D^n_i f(u^n_j), \quad \sum_{m} b_{j,m}(a^n_m, u^n_i) = u^n_i + \delta t D^n_j f(u^n_i).
\]

This implies that
\[
\sum_{i,j} s_{i,j}^{n+1} K_{i-j}^h \left( \sum_{m} (b_{i,m}(a^n_m, u^n_j) - b_{j,m}(a^n_m, u^n_i)) - u^n_j + u^n_i \right)
= \delta t \sum_{i,j} s_{i,j}^{n+1} K_{i-j}^h \left( D^n_i f(u^n_j) - D^n_j f(u^n_i) \right) = D_h.
\]

For the first terms in the right-hand side of (5.2), we recall that \( b_{i,m}(a, u) \) is increasing in \( u \) s.t. \( b_{i,m}(a^n_m, u^n_j) - b_{i,m}(a^n_m, u^n_i) \) has the sign of \( u^n_m - u^n_j \) and, in particular, for \( i \neq m \),
\[
s_{i,j}^{n+1} (b_{i,m}(a^n_m, u^n_m) - b_{i,m}(a^n_m, u^n_j)) \leq s_{m,j}^{n} (b_{i,m}(a^n_m, u^n_m) - b_{i,m}(a^n_m, u^n_j)).
\]

Similarly, \( b_{i,m}(a, u) - u/2 \) is increasing in \( u \), which lets us handle, in the case \( m = i \),
\[
s_{i,j}^{n+1} \left( b_{i,i}(a^n_i, u^n_i) - b_{i,i}(a^n_i, u^n_j) + \frac{1}{2} (u^n_j - u^n_i) \right)
\leq s_{i,j}^{n} \left( b_{i,i}(a^n_i, u^n_i) - b_{i,i}(a^n_i, u^n_j) + \frac{1}{2} (u^n_j - u^n_i) \right).
\]

We hence have, in either case,
\[
\sum_{i,j,m} s_{i,j}^{n+1} K_{i-j}^h \left( b_{i,m}(a^n_m, u^n_m) - b_{i,m}(a^n_m, u^n_j) + \frac{1}{2} (u^n_j - u^n_i) \delta_{m-i} \right)
\leq \sum_{i,j,m} s_{m,j}^{n} K_{i-j}^h \left( b_{i,m}(a^n_m, u^n_m) - b_{i,m}(a^n_m, u^n_j) + \frac{1}{2} (u^n_j - u^n_i) \delta_{m-i} \right).
\]

We can finally use the definition of the flux in (2.3), giving that
\[
b_{i,m}(a^n_m, u^n_m) - u^n_i \delta_{m-i} = \frac{\delta t}{\delta x} \sum_{k=1}^{d} (F^n_{i+[1]k-m}(a^n_m, u^n_m) - F^n_{i-m}(a^n_m, u^n_m)).
\]
Hence
\[
\sum_{i,j,m} s_{i,j}^{n+1} K_{i-j}^h \left( b_{i,m}(a^n_m, u^n_m) - b_{i,m}(a^n_m, u^n_m) + \frac{1}{2}(u^n_j - u^n_i) \delta_{m-i} \right)
\leq \frac{\delta t}{\delta x} \sum_{i,j,m} \sum_{k=1}^d s_{m,j}^n K_{i-j}^h \left( F_{i+1}^k(a^n_m, u^n_m) - F_{i-m}^k(a^n_m, u^n_m) \right)

- F_{i+1}^k(a^n_m, u^n_m) + F_{i-m}^k(a^n_m, u^n_m) + \frac{1}{2} \sum_{i,j} s_{m,j}^n K_{i-j}^h (u^n_j - u^n_i)
\leq A_h + \frac{1}{2} \sum_{i,j} K_{i-j}^h |u^n_j - u^n_i|.
\]

We can perform the same calculations on the next term and find, just exchanging the role of \(i\) and \(j\), that
\[
\sum_{i,j,m} s_{i,j}^{n+1} K_{i-j}^h \left( b_{j,m}(a^n_m, u^n_m) - b_{i,m}(a^n_m, u^n_m) + \frac{1}{2}(u^n_j - u^n_i) \delta_{m-i} \right)
\leq \frac{\delta t}{\delta x} \sum_{i,j,m} \sum_{k=1}^d s_{m,j}^n K_{i-j}^h \left( F_{j+1}^k(a^n_m, u^n_m) - F_{j-m}^k(a^n_m, u^n_m) \right)

- F_{j+1}^k(a^n_m, u^n_m) + F_{j-m}^k(a^n_m, u^n_m) + \frac{1}{2} \sum_{i,j} s_{m,j}^n K_{i-j}^h (u^n_j - u^n_i)
\leq B_h + \frac{1}{2} \sum_{i,j} K_{i-j}^h |u^n_j - u^n_i|.
\]

Combining those inequalities with (5.3) and (5.2), we indeed conclude that
\[
\sum_{i,j} |u_i^{n+1} - u_j^{n+1}| K_{i-j}^h \leq A_h + B_h + D_h + \sum_{i,j} K_{i-j}^h |u^n_j - u^n_i|.
\]

□

Our next steps are to bound \(A_h\), \(B_h\) and \(D_h\). While this follows the main ideas in the proof of Theorem 3, we still present the full arguments, for convenience.

5.2. Control of \(D_h\) and \(A_h + B_h\)

We obtain the exact equivalent of the continuous case, namely,

**Lemma 18.** Under the assumptions of Lemma 17, and assuming that (2.7) holds (\(\tilde{f}\) Lipschitz), then, for any \(h \geq \delta x^\alpha\) and any \(0 < \alpha \leq 1\),
\[
\delta x^{2d} | \log h |^{-\theta} D_h \leq C \delta t \| f \|_{W^{1,\infty}} \left( \| u \|_{L^p}^p \| D^n \|_{\alpha, p, p} \| \Phi_{\theta-1/p} \| + \| D^n \|_{L^\infty} \| u^n \|_{\alpha, 1, \theta} \right),
\]

(5.4)
where we recall that
\[ D_h = \delta t \sum_{i,j} s_{i,j}^{n+1} K_{i-j}^h \left( D_i^n \tilde{f}(u_i^n) - D_j^n \tilde{f}(u_j^n) \right). \]

**Remark 19.** The requirement \( h \geq \delta x^\alpha \) comes directly from the definition of the discrete semi-norm \( \| \cdot \|_{\alpha,p,\theta} \), where we take the supremum only over values of \( h \) larger than \( \delta x^\alpha \). Therefore to control \( \delta x^{2d} \log h |^{\theta} D_h \) in terms of those semi-norms, we need to impose the same condition.

**Proof.** This is the simplest term to handle with
\[ D_h \leq \delta t \sum_{i,j} K_{i-j}^h |D_i^n - D_j^n| \tilde{f}(u_i^n) + \delta t \sum_{i,j} K_{i-j}^h |D_i^n| |\tilde{f}(u_i^n) - \tilde{f}(u_j^n)|. \]

We now use the Lipschitz bound on \( \tilde{f} \) to find
\[ D_h \leq \delta t \| \tilde{f} \|_{W^{1,\infty}} \left( \sum_{i,j} K_{i-j}^h |D_i^n - D_j^n| |u_i^n| + \| D^n \|_{L^\infty} \sum_{i,j} K_{i-j}^h |u_i^n - u_j^n| \right). \]

We recall the definition of the discrete \( l^p \) norms:
\[ \| v \|_{l^p} = \delta x^d \sum_i |v_i|^p. \]

Hence, by a discrete Hölder estimates, one obtains that
\[ \sum_{i,j} K_{i-j}^h |D_i^n - D_j^n| |u_i^n| = \delta x^{-2d} \sum_{i,j} K_{i-j}^h |D_i^n - D_j^n| |u_i^n| \]
\[ \leq \delta x^{-2d} \left( \delta x^{2d} \sum_{i,j} K_{i-j}^h |u_i^n|^p \right)^{1/p^*} \left( \delta x^{2d} \sum_{i,j} K_{i-j}^h |D_i^n - D_j^n|^p \right)^{1/p^*}. \]

Observe that, from the definition of \( K_h \) and straightforward comparison to an integral,
\[ \delta x^d \sum_j K_{i-j}^h \leq \int \frac{dx}{|x| \leq C} \frac{\delta x^d}{(|x| + h)^d} \leq C |\log h| + \frac{\delta x^d}{h^d} \leq C |\log h|. \]

since \( h \geq \delta x^\alpha \) with \( \alpha \leq 1 \).

Thus
\[ D_h \leq \delta t \| \tilde{f} \|_{L^\infty} \delta x^{-2d} \left( \| u \|_{l^p^*} |\log h| \right)^{1/p^*} \left( \delta x^{2d} \sum_{i,j} K_{i-j}^h |D_i^n - D_j^n|^p \right)^{1/p^*} \]
\[ + \| D^n \|_{L^\infty} \delta x^{-2d} \sum_{i,j} K_{i-j}^h |u_i^n - u_j^n|. \]
We now recall the definition of our discretized semi-norms from (2.9):
\[
\|v\|_{\alpha, p, \theta}^p = \sup_{h \geq \delta x^\alpha} |\log h|^{-\theta} \delta x^{2d} \sum_{i,j} K^h_{i-j} |v_i - v_j|.
\]
Of course one may first recognize that
\[
\delta x^{2d} |\log h|^{-\theta} \sum_{i,j} K^h_{i-j} |u^n_i - u^n_j| \leq \|u^n\|_{\alpha, 1, \theta},
\]
provided that \(h \geq \delta x^\alpha\), as is imposed by the Lemma for this reason.

On the other hand,
\[
|\log h|^{-\theta + 1/p^*} \left( \delta x^{2d} \sum_{i,j} K^h_{i-j} |D^n_i - D^n_j|^p \right)^{1/p} = \left( \delta x^{2d} |\log h|^{-\theta + p/p^*} \sum_{i,j} K^h_{i-j} |D^n_i - D^n_j|^p \right)^{1/p} \leq \|D^n\|_{\alpha, p, \theta - p/p^*},
\]
again since \(h \geq \delta x^\alpha\). This implies that
\[
\delta x^{2d} |\log h|^{-\theta} D_h \leq \delta t \|\tilde{f}\|_{W^{1,\infty}} \left( \|u^n\|_{l^p} \|D^n\|_{\alpha, p, \theta - p/p^*} + \|u^n\|_{\alpha, 1, \theta} \right).
\]
To conclude we only need to use (2.7) to control the Lipschitz norm of \(\tilde{f}\) by the Lipschitz norm of \(f\). □

We now turn to the terms \(A_h, B_h\), which we treat together to use cancellations in the sum \(A_h + B_h\).

**Lemma 20.** Under the assumptions of Lemma 17, and assuming the moments condition (2.8) on the flux, we have, for any \(\gamma \leq 1\), and provided that \(h \geq \delta x^\alpha\) with \(\alpha \leq 1\),
\[
A_h + B_h \leq C \delta t \frac{\delta x^{\gamma - 2d}}{h^{1+\gamma}} \|f\|_{W^{1,\infty}} \|a^n\|_{l^p} \|u^n\|_{l^{p^*}} + \frac{\delta t}{\delta x} \sum_{i,j} \sum_{1}^{d} \sum_{k=1}^{d} s^n_{i,j} (F^h_{i-[1]k-j} - F^h_{i-j}) (a^n_{i,k} - a^n_{j,k}) (f(u^n_i) - f(u^n_j)),
\]
where we recall that
\[
A_h = \frac{\delta t}{\delta x} \sum_{i,j,m} \sum_{k=1}^{d} s^n_{i,j} K^h_{i-j} \left( F^k_{i+[1]k-m}(a^n_{i,j}) - F^k_{i-[1]k-m}(a^n_{i,j}) \right) + F^k_{i-m}(a^n_{i,j}) + F^k_{i-m}(a^n_{i,j}),
\]
\[
B_h = \frac{\delta t}{\delta x} \sum_{i,j,m} \sum_{k=1}^{d} s^n_{i,j} K^h_{i-j} \left( F^k_{j+[1]k-m}(a^n_{i,j}) - F^k_{j-[1]k-m}(a^n_{i,j}) \right) + F^k_{j-m}(a^n_{i,j}) + F^k_{j-m}(a^n_{i,j}).
\]
Remark 21. The estimate in this lemma is the reason that we need to impose \( h \geq \delta x^\alpha \) in the definition of the discrete semi-norms. The issue is specifically the term \( \delta x^\gamma / h^{1+\gamma} \), which appears in the first term of the right-hand side. This means that we have to impose \( h \geq \delta x^\alpha \), and of course with \( \alpha \) such that \( \delta x^\gamma / h^{1+\gamma} << 1 \).

Proof. We first perform a discrete integration by parts in \( A_h \) and \( B_h \). Essentially we recognize a discrete derivative of \( F \) in \( A_h \) and \( B_h \) and instead change it into a difference of the kernel \( K \) with

\[
A_h = \frac{\delta t}{\delta x} \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n K_{i-j}^h \left(F_{i+1}[1]_{k-m}(a^n_m, u^n_m) - F_{i+1}[1]_{k-m}(a^n_m, u^n_j)\right)
- F_{i-m}(a^n_m, u^n_m) + F_{i-m}(a^n_m, u^n_j))

= \frac{\delta t}{\delta x} \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n (K_{i-[1]_{k-j}}^h - K_{i-j}^h) \left(F_{i-m}(a^n_m, u^n_m) - F_{i-m}(a^n_m, u^n_m)\right),
\]

and by symmetry, the same calculations lead to

\[
B_h = \frac{\delta t}{\delta x} \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n (K_{i-[1]_{k-j}}^h - K_{i-j}^h) \left(F_{i-m}(a^n_m, u^n_m) - F_{i-m}(a^n_m, u^n_m)\right).
\]

Let us swap \( i \) and \( m \) in \( A_h \) and \( j \) and \( m \) in \( B_h \) to find

\[
A_h = \frac{\delta t}{\delta x} \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n (K_{i-[1]_{k-j}}^h - K_{i-j}^h) \left(F_{m-i}(a^n_i, u^n_i) - F_{m-i}(a^n_i, u^n_j)\right),
\]

\[
B_h = \frac{\delta t}{\delta x} \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n (K_{i+[1]_{k-m}}^h - K_{i-m}^h) \left(F_{m-j}(a^n_j, u^n_j) - F_{m-j}(a^n_j, u^n_m)\right).
\]

We recall here the moments condition (2.8) on the flux:

\[
\sum_{m} |i - m|^\gamma |F_{i-m}(a^n_m, u^n_m)| \leq C \| f \|_{W^{1,\infty}} \| a^n \|_{L^p} \| u^n \|_{L^p}.
\]

This ensures that \( F_{m-i} \) is small unless \( m \) is close to \( i \). This will allow us to replace \( K_{m-[1]_{k-j}}^h - K_{i-j}^h \) by \( K_{i-[1]_{k-j}}^h - K_{i-j}^h \) (and similarly for \( B_h \)). More precisely, we write

\[
\sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n \left(K_{m-[1]_{k-j}}^h - K_{m-j}^h\right) \left(F_{m-i}(a^n_i, u^n_i) - F_{m-i}(a^n_i, u^n_j)\right)
\]

\[
= \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n \left(K_{i-[1]_{k-j}}^h - K_{i-j}^h\right) \left(F_{m-i}(a^n_i, u^n_i) - F_{m-i}(a^n_i, u^n_j)\right)
\]

\[
+ \sum_{i,j,m} \frac{d}{k=1} s_{i,j}^n \left(K_{m-[1]_{k-j}}^h - K_{i-[1]_{k-j}}^h - K_{m-j}^h + K_{i-j}^h\right)
\]

\[
\left(F_{m-i}(a^n_i, u^n_i) - F_{m-i}(a^n_i, u^n_j)\right).
\]
We recall that

\[ K_{i-j}^h = \frac{\phi(\delta x (i - j))}{(\delta x |i - j| + h)^d}. \]

Using the straightforward bound twice,

\[ \frac{1}{|X|^k} - \frac{1}{|Y|^k} \leq C \left| \frac{|X - Y|}{|X|^{k+1}} + \frac{|X - Y|}{|Y|^{k+1}} \right|, \]

and we obtain that, for a given \( 0 < \gamma \leq 1, \)

\[ |K_{i-j}^h - K_{m-j}^h - K_{i-[1]_{k-j}}^h + K_{m-[1]_{k-j}}^h| \]
\[ \leq \frac{C \delta x^{1+\gamma} |i - m|^\gamma}{(\delta x |i - j| + h)^{d+1+\gamma}} + \frac{C \delta x^{1+\gamma} |i - m|^\gamma}{(\delta x |m - j| + h)^{d+1+\gamma}}. \]

We also recall that, because of compact support, we only have to calculate the expression for \( |j - i| + |j - m| \leq C \delta x^{-1}. \) Therefore, considering the second term in the right-hand side of (5.5), we have that

\[ \sum_{m,i,j \text{ s.t.} |j-i|+|j-m|\leq C \delta x^{-1}} |K_{i-j}^h - K_{m-j}^h - K_{i-[1]_{k-j}}^h | \]
\[ + K_{m-[1]_{k-j}}^h \left| F_{m-i}^k (a^n_i, u^n_i) \right| \]
\[ \leq \sum_{i,j, |j-i|\leq C \delta x^{-1}} \frac{C \delta x^{1+\gamma}}{(\delta x |i - j| + h)^{d+1+\gamma}} \sum_m |i - m|^\gamma \left| F_{m-i}^k (a^n_i, u^n_i) \right| \]
\[ + \sum_{m,j, |j-m|\leq C \delta x^{-1}} \frac{C \delta x^{1+\gamma}}{(\delta x |m - j| + h)^{d+1+\gamma}} \sum_i |i - m|^\gamma \left| F_{m-i}^k (a^n_i, u^n_i) \right|, \]

giving, by (2.8), that

\[ \sum_{m,i,j \text{ s.t.} |j-i|+|j-m|\leq C \delta x^{-1}} |K_{i-j}^h - K_{m-j}^h - K_{i-[1]_{k-j}}^h | \]
\[ + K_{m-[1]_{k-j}}^h \left| F_{m-i}^k (a^n_i, u^n_i) \right| \]
\[ \leq \sum_{i,j, |i-j|\leq C \delta x^{-1}} \frac{C \delta x^{1+\gamma}}{(\delta x |i - j| + h)^{d+1+\gamma}} \| f \|_{W^{1,\infty}} |a^n_i| |u^n_i| \]
\[ + \sum_{m,j, |j-m|\leq C \delta x^{-1}} \frac{C \delta x^{1+\gamma}}{(\delta x |m - j| + h)^{d+1+\gamma}} \| f \|_{W^{1,\infty}} \| a^n \|_{L^p} \| u^n \|_{L^p}. \]

We remark that since \(|x|^{-d-1-\gamma} \) is integrable at \( \infty, \)

\[ \sum_{i \neq 0} \frac{1}{(\delta x |i| + h)^{d+1+\gamma}} \leq \frac{C}{h^{1+\gamma} \delta x^d}. \]
and adding the term at $|i| = 0$,

$$
\sum_i \frac{1}{(\delta x |i| + h)^{d+1+y}} \leq \frac{C}{h^{1+y} \delta x^d} + \frac{1}{h^{1+y+d}} \leq \frac{C}{h^{1+y} \delta x^d},
$$

since $h \geq \delta x^\alpha$, with $\alpha \leq 1$. We simply deduce that

$$
\sum_{m, j, |j-m| \leq C \delta x^{-1}} \frac{1}{(\delta x |m - j| + h)^{d+1+y}} = \sum_{m \leq C \delta x^{-1}} \sum_{i} \frac{1}{(\delta x |i| + h)^{d+1+y}} \leq \frac{C}{h^{1+y} \delta x^{2d}},
$$

allowing us to conclude that

$$
\sum_{j, m, i, |i-j| + |m-j| \leq C \delta x^{-1}} |K_h^{i,j} - K_h^{m-j} - K_h^{m-[1]k-j} + K_h^{m-[1]k-j} | F_k^{m-j}(a^n_i, u^n_i) |
\leq C \delta x^{-2d} \delta x^{1+y} \frac{\|f\|_{W^{1,\infty}} \|a^n\|_{l^p} \|u^n\|_{l^{p^*}}}. 
$$

We leave as is the first part of the right-hand side in (5.5), and performing the same operation on $B_h$, we obtain that

$$
A_h + B_h 
\leq C \delta t \frac{\delta x^{\gamma-2d}}{h^{1+y}} \|f\|_{W^{1,\infty}} \|a^n\|_{l^p} \|u^n\|_{l^{p^*}} + \frac{\delta t}{\delta x} \sum_{i, j, m} \sum_{k=1}^d s_{i,j}^n (K_h^{i-[1]k-j} - K_h^{i-j}) \left( F_k^{m-i}(a^n_i, u^n_i) - F_k^{m-j}(a^n_i, u^n_j) \right) 
+ \frac{\delta t}{\delta x} \sum_{i, j, m} \sum_{k=1}^d s_{i,j}^n (K_h^{i+[1]k-j} - K_h^{i-j}) \left( F_k^{m-j}(a^n_j, u^n_i) - F_k^{m-j}(a^n_j, u^n_j) \right),
$$

losing a power of $\delta x$ in the first term in the right-hand side because of the ratio $\frac{\delta t}{\delta x}$.

We now observe some simplifications in the expression as the sum in $m$ is only carried over $F_k^{m-i}$ and $F_k^{m-j}$. The normalization of the flux (Eq. (2.4)) that we had chosen is equivalent to

$$
\sum_m F_k^{m-i}(a^n_i, u^n_i) = a^n_{i,k} f(u^n_i).
$$
where again we denote by \( a^n_{i,k} \) the \( k \) coordinate of the vector \( a^n_i \). This leads to

\[
A_h + B_h 
\leq C \delta t \frac{\delta x^{\gamma-2d}}{h^{1+\gamma}} \| f \|_{W^{1,\infty}} \| a^n \|_{L^p} \| u^n \|_{L^p^*}
\]

\[
+ \frac{\delta t}{\delta x} \sum_{i,j} \sum_{k=1}^d s^n_{i,j} (K^h_{i-[1]k-j} - K^h_{i-j}) \left( a^n_{i,k} f(u^n_i) - a^n_{j,k} f(u^n_j) \right)
\]

\[
+ \frac{\delta t}{\delta x} \sum_{i,j} \sum_{k=1}^d s^n_{i,j} (K^h_{i+[1]k-j} - K^h_{i-j}) \left( a^n_{j,k} f(u^n_j) - a^n_{j,k} f(u^n_j) \right)
\]

We now finally need to use the cancellation between \( A_h \) and \( B_h \) by grouping together \( a^n_{i,k} f(u^n_i) \) and \( a^n_{j,k} f(u^n_j) \) on the first part, and \( a^n_{i,k} f(u^n_i) \) and \( a^n_{j,k} f(u^n_j) \). However the indices in \( K^h_{i-[1]k-j} - K^h_{i-j} \) and \( K^h_{i-[1]k-j} - K^h_{i-j} \) do not exactly fit, so we have an extra term:

\[
A_h + B_h \leq C \delta t \frac{\delta x^{\gamma-2d}}{h^{1+\gamma}} \| f \|_{W^{1,\infty}} \| a^n \|_{L^p} \| u^n \|_{L^p^*}
\]

\[
+ \frac{\delta t}{\delta x} \sum_{i,j} \sum_{k=1}^d s^n_{i,j} (K^h_{i-[1]k-j} - K^h_{i-j}) \left( a^n_{i,k} - a^n_{j,k} \right) (f(u^n_i) - f(u^n_j))
\]

\[
+ \frac{\delta t}{\delta x} \sum_{i,j} \sum_{k=1}^d (K^h_{i+[1]k-j} - 2K^h_{i-j} + K^h_{i-[1]k-j}) s^n_{i,j} a^n_{j,k} (f(u^n_j) - f(u^n_j))
\]

To conclude the proof of the lemma, it is now enough to bound the last term in the right-hand side. Observe that, provided \( h \geq \delta x^\alpha \) with \( \alpha \leq 1 \), again by using the explicit formula for \( K^h \),

\[
\left| K^h_{i+[1]k-j} - 2K^h_{i-j} + K^h_{i-[1]k-j} \right| \leq \frac{C \delta x^2}{(\delta x |i-j| + h)d+2}
\]

We can again easily estimate such a sum (the calculations are similar to the ones we have just performed); it is possible to show that

\[
\sum_{i,j} \sum_{k=1}^d (K^h_{i+[1]k-j} - 2K^h_{i-j} + K^h_{i-[1]k-j}) s^n_{i,j} a^n_{j,k} (f(u^n_j) - f(u^n_j))
\]

\[
\leq C \| f \|_{W^{1,\infty}} \| a^n \|_{L^p} \| u^n \|_{L^p^*} \delta x^{-2d} \frac{\delta x^2}{h^2}
\]

Therefore

\[
A_h + B_h \leq C \delta t \delta x^{-2d} \left( \frac{\delta x^\gamma}{h^{1+\gamma}} + \frac{\delta x}{h^2} \right) \| f \|_{W^{1,\infty}} \| a^n \|_{L^p} \| u^n \|_{L^p^*}
\]

\[
+ \frac{\delta t}{\delta x} \sum_{i,j} \sum_{k=1}^d s^n_{i,j} (K^h_{i-[1]k-j} - K^h_{i-j}) \left( a^n_{i,k} - a^n_{j,k} \right) (f(u^n_i) - f(u^n_j)).
\]
Since \( \delta x \leq h \) and \( \gamma \leq 1 \), we have that \( \frac{\delta x^2}{h^2} \leq \frac{\delta x^{1+\gamma}}{h^{1+\gamma}} \), which concludes the proof of the lemma. \( \square \)

Our final step is to bound the remaining term:

\[
\frac{\delta t}{\delta x} \sum_{i,j} s_{i,j} (K_{i-[1]k-j}^h - K_{i-j}^h) (a_{i,k}^n - a_{j,k}^n) (f(u_i^n) - f(u_j^n)),
\]

from Lemma 20. This is given by Lemma 16, which we had also identified as the key lemma in the continuous case, yielding

**Lemma 22.** For any \( \theta \geq 1 - 1/p \) and any \( q > p^* \),

\[
\delta x^{-1} \sum_{i,j} s_{i,j} (K_{i-[1]k-j}^h - K_{i-j}^h) (a_{i,k}^n - a_{j,k}^n) (f(u_i^n) - f(u_j^n)) \\
\leq C \delta x^{-2d} \| \log h \|^\theta \| f \|_{W^{1,\infty}} (\| D^n \|_{l\infty} \| u^n \|_{d,1,\theta} + \| a^n \|_{d,W^{1,p}} \| u^n \|_{l\theta}).
\]

**Proof.** The idea is simply to construct continuous fields from the discrete ones so as to be able to directly apply Lemma 16.

Consider a set of cubes \( C_i \) of size \( \delta x \) s.t. \( C_i \) is centered at point \( x_i \).

We first define the field \( \tilde{u}^n(x) \) which is piecewise constant with \( \tilde{u}^n(x) = u_i^n \) within \( C_i \). We then construct a velocity field \( \tilde{a}^n(x) \) piecewise linear in each cube \( C_i \) and such that, for any \( i, j \),

\[
\frac{1}{\delta x^{2d}} \int_{C_i \times C_j} \nabla K_h(x - y) \cdot (\tilde{a}^n(x) - \tilde{a}^n(y)) = \sum_{k=1}^d K_{i-[1]k-j}^h - K_{i-j}^h \frac{\delta x}{\delta x} (a_{i,k}^n - a_{j,k}^n).
\]

As a consequence, the corresponding norms of \( a^n \) and \( u^n \) are dominated by the discrete norms

\[
\| \tilde{u}^n \|_{L^\gamma(\mathbb{R}^d)} \leq \| u^n \|_{l\gamma}, \quad \| \tilde{a}^n \|_{W^{1,p}(\mathbb{R}^d)} \leq \| a^n \|_{d,W^{1,p}}, \quad \| \text{div} \tilde{a}^n \|_{L^\infty} \leq \sup_i |D_i^n|.
\]

(5.6)

We recall that, by Eq. (4.2),

\[
F(\xi, \zeta) = (f(\xi) - f(\zeta)) \text{sign}(\xi - \zeta),
\]

so that one has the identity

\[
\delta x^{-1} \sum_{i,j} s_{i,j} (K_{i-[1]k-j}^h - K_{i-j}^h) (a_{i,k}^n - a_{j,k}^n) (f(u_i^n) - f(u_j^n)) \\
= \delta x^{-2d} \int_{\mathbb{R}^{2d}} \nabla K_h(x - y) \cdot (\tilde{a}^n(x) - \tilde{a}^n(y)) F(u^n(x), u^n(y)) \, dx \, dy,
\]

where \( \nabla K_h \) derives from \( \frac{K_{i-[1]k-j}^h - K_{i-j}^h}{\delta x} \).
We now apply Lemma 16 to find that, provided $\theta \geq 1 - 1/q$, 
\[
\delta x^{-1} \sum_{i,j,m,k=1}^{d} \delta_{i,j}^{n} (K_{i-1}^{h} - K_{i-j}^{h} + a_{i,k}^{n} - a_{j,k}^{n}) (f(u_{i}^{n}) - f(u_{j}^{n})) 
\leq C \delta x^{-2d} \log h^{\theta} \| f \|_{L^{\infty}} \left( \| \nabla \tilde{a} \|_{L^{\infty}} \| u \|_{1,\theta} + \| \nabla \tilde{a} \|_{B_{p,q}} \| \tilde{u} \|_{L^{p\cdot q,1}} \right).
\]

We choose $p = q$ so that by classical embeddings $\| \nabla \tilde{a} \|_{B_{p,q}} \leq \| \nabla \tilde{a} \|_{L^{p}}$. By the bounds from Eq. (5.6), we deduce the Lemma. \(\square\)

We are now ready to conclude our proof. By combining Lemma 22 with Lemma 20, we first obtain that
\[
A_{h} + B_{h} \leq C \delta t \frac{\delta x^{\gamma - 2d}}{h^{1+\gamma}} \| f \|_{W^{1,\infty}} \| a^{n} \|_{l_{p}} \| u^{n} \|_{l_{p^{*}}} 
+ C \delta x^{-2d} \log h^{\theta} \| f \|_{W^{1,\infty}} \left( \| u^{n} \|_{d,1,\theta} + \| a^{n} \|_{d,W^{1,p}} \| u^{n} \|_{l^{q}} \right).
\]

Given the bound on $D_{h}$ provided by Lemma 18, we may now directly calculate from Lemma 17 that
\[
\| u^{n+1} \|_{\alpha,1,\theta} = \sup_{h \geq \delta x^{\alpha}} | \log h |^{-\theta} \delta x^{2d} \sum_{i,j} \left( \sum_{i,j} K_{i-j}^{h} \| u_{i}^{n+1} - u_{j}^{n+1} \| + A_{h} + B_{h} + D_{h} \right)
\leq \sup_{h \geq \delta x^{\alpha}} | \log h |^{-\theta} \delta x^{2d} \left( \sum_{i,j} \left( \sum_{i,j} K_{i-j}^{h} \| u_{i}^{n} - u_{j}^{n} \| + A_{h} + B_{h} \right) \right)
\leq \| u^{n} \|_{\alpha,1,\theta} + C \delta t \sup_{h \geq \delta x^{\alpha}} \frac{\delta x^{\gamma}}{h^{1+\gamma}} \| f \|_{W^{1,\infty}} \| a^{n} \|_{l_{p}} \| u^{n} \|_{l_{p^{*}}} 
+ \delta t \| f \|_{W^{1,\infty}} \left( \| u^{n} \|_{l_{p^{*}}} \| D^{n} \|_{\alpha,p,p(1/p - 1)} + \| D^{n} \|_{l^{\infty}} \| u^{n} \|_{d,1,\theta} \right)
\leq \| u^{n} \|_{\alpha,1,\theta} + \| a^{n} \|_{d,W^{1,p}} \| u^{n} \|_{l^{q}} \right).
\]

This is where we finally see the absolute need for the requirement that $h \geq \delta x^{\alpha}$ in the discrete semi-norms. Otherwise we would of course just have $\sup_{h \geq 0} \frac{\delta x^{\gamma}}{h^{1+\gamma}} = \infty$, but here we simply have that $\sup_{h \geq \delta x^{\alpha}} \frac{\delta x^{\gamma}}{h^{1+\gamma}} = \delta x^{\gamma - \alpha(1+\gamma)}$ and
\[
\| u^{n+1} \|_{\alpha,1,\theta} \leq \| u^{n} \|_{\alpha,1,\theta} + C \delta t \delta x^{\gamma - \alpha(1+\gamma)} \| f \|_{W^{1,\infty}} \| a^{n} \|_{l_{p}} \| u^{n} \|_{l_{p^{*}}} 
+ \delta t \| f \|_{W^{1,\infty}} \left( \| u^{n} \|_{l_{p^{*}}} \| D^{n} \|_{\alpha,p,p(1/p - 1)} + \| D^{n} \|_{l^{\infty}} \| u^{n} \|_{d,1,\theta} \right)
\leq \| u^{n} \|_{\alpha,1,\theta} + \| a^{n} \|_{d,W^{1,p}} \| u^{n} \|_{l^{q}} \right).
\]

A discrete Gronwall estimate allows as to conclude the proof of Theorem 8.

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Appendix A: Proof of Lemma 15

The estimates presented here are classical and we refer, for instance, to [1,8] or [46].

Choose any family $\Psi_k \in \mathcal{S}(\mathbb{R}^d)$ s.t.:

- For $k \geq 1$, its Fourier transform $\hat{\Psi}_k$ is positive and compactly supported in the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$.
- It leads to a decomposition of the identity in the sense that there exists $\Psi_0$ with $\hat{\Psi}_0$ compactly supported in $\{|\xi| \leq 2\}$ s.t., for any $\xi$,

$$1 = \sum_{k \geq 0} \hat{\Psi}_k(\xi).$$

- The family is localized in $\mathbb{R}^d$ in the sense that, for all $s > 0$,

$$\sup_k \|\Psi_k\|_{L^1} < \infty, \quad \sup_k 2^{ks} \int_{\mathbb{R}^d} |z|^s |\Psi_k(z)| \, dz < \infty.$$  

Such a family can be used to define the usual Besov norms with

$$\|u\|_{B_{p,q}^s} = \left( \sum_{k=0}^{\infty} 2^{skq} \|\Psi_k * u\|_{L^p_{x}}^q \right)^{1/q} < \infty. \quad (5.7)$$

For this reason, it is useful to denote

$$U_k = \Psi_k * u.$$

Since $U_k$ is localized in frequency, one may easily relate all its Sobolev norms; for any $1 < p < \infty$, any $k \geq 1$, and any $\alpha$,

$$\|U_k\|_{W^{\alpha,p}} \leq C_p 2^{k\alpha} \|U_k\|_{L^p}. \quad (5.8)$$

We first give the bound that we use for $k \leq \lfloor \log_2 r \rfloor$. Since the kernel $L$ has 0 average,

$$L_r * U_k = \int_{\mathbb{R}^d} L_r (x - y) (U_k (y) - U_k (x)) \, dy.$$

Therefore

$$\|L_r * U_k\|_{L^p} \leq \int_{\mathbb{R}^d} L_r (z) \|U_k (\cdot) - U_k (\cdot + z)\|_{L^p} \, dz$$

$$\leq \int_{\mathbb{R}^d} L_r (z) |z|^s \|U_k\|_{W^{s,p}} \, dz.$$

Since $L$ has bounded moments, $\int |z|^s L_r(z) \, dz = r^s \int |z|^s L(z) \, dz$, yielding

$$\|L_r * U_k\|_{L^p} \leq C r^s 2^{ks} \|U_k\|_{L^p} \quad (5.9)$$

by inequality (5.8) for $\alpha = s$, and for a fixed constant $C$ depending only on $\int |z|^s L(z) \, dz$. 

For the case $k \geq |\log_2 r|$, we use that $L \in W^{s,1}$, and deduce that
\[ \|L_r \ast U_k\|_{L^p} \leq \|L_r\|_{W^{s,1}} \|U_k\|_{W^{-s,p}} \leq Cr^{-s}2^{-ks} \|U_k\|_{L^p}. \] (5.10)
by again using (5.8), but for $\alpha = -s$, where $C$ only depends on the $W^{s,1}$ norm of $L$.

Using now this decomposition and the two bounds, (5.9)-(5.10), we get
\[
\int_{h_0}^1 \|L_r \ast u\|_{L^p} \frac{dr}{r} = \sum_{k=0}^{\infty} \int_{h_0}^1 \|L_r \ast U_k\|_{L^p} \frac{dr}{r} \\
\leq C \sum_{k=0}^{\infty} \|U_k\|_{L^p} \left( \int_{h_0}^1 r^{-s}2^{ks} \frac{dr}{r} + \int_{\max(h_0,2^{-k})}^\infty r^{-s}2^{-ks} \frac{dr}{r} \right).
\]
This implies that
\[
\int_{h_0}^1 \|L_r \ast u\|_{L^p} \frac{dr}{r} \leq C \sum_{k=0}^{\infty} \|U_k\|_{L^p} + C \sum_{k=|\log_2 h_0|}^{\infty} \frac{2^{-ks}}{h_0^s} \|U_k\|_{L^p} (5.11)
\]
Now, simply bound
\[
\sum_{k=|\log_2 h_0|}^{\infty} \|U_k\|_{L^p} + \sum_{k>|\log_2 h_0|}^{\infty} \frac{2^{-ks}}{h_0^s} \|U_k\|_{L^p} \leq C \sum_{0}^{\infty} \|U_k\|_{L^p} = C \|u\|_{B^0_{p,1}},
\]
which gives (3.2) in the case $q = 1$.

Next, remark that
\[
\sum_{k>|\log_2 h_0|}^{\infty} \frac{2^{-ks}}{h_0^s} \|U_k\|_{L^p} \leq \sup_{k>|\log_2 h_0|} \|U_k\|_{L^p} \sum_{k>|\log_2 h_0|}^{\infty} \frac{2^{-ks}}{h_0^s} \leq \sup \|U_k\|_{L^p} \leq C \|u\|_{B^0_{p,\infty}}.
\]
On the other hand,
\[
\sum_{k=|\log_2 h_0|}^{\infty} \|U_k\|_{L^p} \leq \|\log_2 h_0\|^{1-1/q} \left( \sum_k \|U_k\|_{L^p}^q \right)^{1/q} \leq \|\log_2 h_0\|^{1-1/q} \|u\|_{B^0_{p,q}},
\]
implying (3.2) for general $q$.

We now recall the well-known embedding of $L^p$ into $B^0_{p,2}$ when $p \leq 2$, giving
\[
\sum_{k=0}^{\infty} \|U_k\|_{L^p} \leq C \sqrt{\|\log_2 h_0\|} \|u\|_{L^p}.
\]
Therefore, (5.11) yields
\[
\int_{h_0}^1 \|L_r \ast u\|_{L^p} \frac{dr}{r} \leq C \sqrt{\|\log_2 h_0\|} \|u\|_{L^p} + C \|u\|_{B^0_{p,\infty}},
\]
which proves (3.3).
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