Temporal Semi-discretizations of a Backward Semilinear Stochastic Evolution Equation

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Abstract
This paper studies the convergence of three temporal semi-discretizations for a backward semilinear stochastic evolution equation. For general terminal value and general coefficient with Lipschitz continuity, the convergence of the first two temporal semi-discretizations is established, and an explicit convergence rate is derived for the third temporal semi-discretization. The third temporal semi-discretization is applied to a general stochastic linear quadratic control problem, and the convergence of a temporally semi-discrete approximation to the optimal control is established.

Keywords Backward semilinear stochastic evolution equation · Brownian motion · Discretization · Stochastic linear quadratic control

Mathematics Subject Classification 49M25 · 65C30 · 60H35 · 65K10

1 Introduction

In the literature, Bismut [3] first introduced the finite dimensional linear backward stochastic differential equations (BSDEs, for short) to study the stochastic optimal control problems. Later, Pardoux and Peng [33] studied the general finite dimensional BSDEs with Lipschitz nonlinearity, and Hu and Peng [21] established the well-posedness for the backward semilinear stochastic evolution equations with Lipschitz nonlinearity. Since then a considerable number of papers have been published for the applications of the BSDEs to stochastic optimal control, partial differential equations and mathematical finance; see [26, 32, 34, 36, 42] and the references cited therein. We particularly refer the reader to [10–18] and the references therein for the
applications of the backward stochastic partial differential equations to the stochastic optimal control problems.

By now, the numerical solutions of the finite-dimensional BSDEs have been extensively studied. We particularly introduce several works as follows. For backward–forward SDEs, Ma et al. [31] proposed a four-step scheme, Zhang [43] and Bouchard and Touzi [4] analyzed two Euler type schemes, and Chassagneux [6] studied a class of linear multistep methods. The above four works all require that the coefficients are deterministic. For a class of nonlinear BSDEs with particular terminal value and sufficiently smooth deterministic coefficients, Zhao et al. [44] proposed a stable multistep scheme. For the nonlinear BSDEs with general terminal value and general coefficients, Hu et al. [20] analyzed three schemes with some restrictions on the regularity of the underlying solution, and these restrictions might be difficult to verify. We also refer the reader to the references cited in the above papers for more related works. Additionally, because of the close connections between the stochastic evolution equations and the backward stochastic evolution equations, we refer the reader to [1, 5, 7, 8, 22–25, 27, 41] and the references therein, for the numerical analysis of the stochastic evolution equations.

Compared with the numerical analysis of the finite-dimensional BSDEs, the numerical analysis of the backward stochastic semilinear evolution equations is very limited. Wang [40] analyzed a discretization for a backward semilinear stochastic parabolic equation; since this discretization uses the eigenvectors of the Laplace operator, its application appears to be limited. Recently, Li and Xie [28] analyzed a spatial semi-discretization for a backward semilinear stochastic parabolic equation with general filtration, using the standard piecewise linear finite element method. To our best knowledge, no numerical analysis of temporal semi-discretizations is available for a backward semilinear stochastic evolution equation in an infinite-dimensional Hilbert space.

The immaturity of the numerical analysis of the backward semilinear stochastic evolution equations motivates us to study the temporal semi-discretizations for the equation

$$\left\{ \begin{array}{l} \mathrm{d}p(t) = -(A p(t) + f(t, p(t), z(t))) \mathrm{d}t + z(t) \mathrm{d}W(t), \quad 0 \leq t \leq T, \\ p(T) = p_T, \end{array} \right. \quad (1)$$

where $0 < T < \infty$, $W(\cdot)$ is a one-dimensional real Brownian motion, and $p_T$ and $f$ are given. One key difficulty in the numerical analysis of the backward semilinear stochastic evolution Eq. (1) is that the process $z$ is generally of low temporal regularity. In this paper, we analyze three Euler type temporal semi-discretizations for Eq. (1). For the first two semi-discretizations, the process $z$ is discretized by the piecewise constant processes, and we prove that the two semi-discretizations are convergent. More precisely, we obtain the error bound

$$c(\tau^{1/2} + \|z - P_\tau z\|_{L^2(0,T;H)}),$$

where $P_\tau z$ is the optimal approximation of $z$ in the space of piecewise constant processes. Hence, if the process $z$ indeed possesses higher temporal regularity, then an explicit convergence rate will readily be derived. For the third semi-discretization,
the process $z$ is not discretized, and an explicit convergence rate is derived. Finally, we apply the third semi-discretization to a general stochastic linear quadratic control problem, and establish the convergence of a temporally semi-discrete approximation, with reasonable regularity assumption on the data. To sum up, our main contributions lie in the following aspects.

- This work, to our best knowledge, provides the first numerical analysis of temporal semi-discretizations for an infinite-dimensional semilinear BSDE.
- Our analysis, compared with most of the numerical analysis of the finite-dimensional BSDEs, neither requires the terminal value to be generated by a forward stochastic evolution equation, nor requires the coefficient to be deterministic. In addition, it requires only some reasonable regularity assumptions on the data, and imposes no regularity restriction on the solution.
- In the literature, the numerical analysis of the stochastic optimal control problems governed by the SPDEs is very limited; see [9, 28, 37, 38, 45]. Our analysis for the temporal semi-discretization of the general stochastic linear quadratic control problem, as far as we know, appears to be the first numerical analysis of such kinds of problems where the noise is multiplicative and the diffusion term contains the control variable.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries. Section 3 gives three temporal semi-discretizations and their error estimates. Section 4 applies the third temporal semi-discretization to a stochastic linear quadratic control problem. Finally, Sect. 5 concludes this paper.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space, on which a one-dimensional Brownian motion $W(\cdot)$ is defined. Let $\mathbb{F} := \{\mathcal{F}_t \mid t \geq 0\}$ be the filtration generated by $W(\cdot)$ and augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. We use $\mathbb{E}$ to denote the expectation and use $\mathbb{E}_t$ to denote the conditional expectation with respect to $\mathcal{F}_t$ for each $t \geq 0$. For any separable Hilbert space $X$ with norm $\| \cdot \|_X$, we write the Hilbert space $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; X)$ as $L^2(\Omega; X)$, and use $\| \cdot \|_X$ to denote its norm. Moreover, define

$$L^2_{\mathbb{F}}(0, T; X) := \left\{ \varphi : [0, T] \times \Omega \to X \mid \varphi \text{ is } \mathbb{F}\text{-progressively measurable and } \int_0^T \| \varphi(t) \|^2_X \, dt < \infty \right\},$$

and let $L^2_{\mathbb{F}}(\Omega; C([0, T]; X))$ be the space of all $\mathbb{F}$-progressively measurable processes $\varphi$ with continuous paths in $X$ such that

$$\| \varphi \|_{C([0,T]; X)} := \left( \mathbb{E} \sup_{t \in [0,T]} \| \varphi(t) \|^2_X \right)^{1/2} < \infty.$$
The space $L^2_F(\Omega; C([0, T]; X))$ is a Banach space with respect to the above norm $\|\cdot\|_{C([0, T]; X)}$.

Let $H$ be a real separable Hilbert space with inner product $(\cdot, \cdot)_H$. Assume that

$$A : \text{Domain}(A) \subset H \rightarrow H$$

is a linear operator satisfying the following properties:

- $A$ is self-adjoint, i.e.,
  $$(Av, w)_H = (v, Aw)_H \text{ for all } v, w \in \text{Domain}(A);$$
- $A$ is surjective, and there exists a positive constant $\delta$ such that
  $$(-Av, v)_H \geq \delta \|v\|^2_H \text{ for all } v \in \text{Domain}(A);$$
- Domain$(A)$ is dense in $H$, and Domain$(A)$, equipped with the norm $\|A\cdot\|_H$, is compactly embedded into $H$.

It is evident that $A$ will generate an analytic contractive semigroup $\{e^{tA} \mid t \geq 0\}$ on $H$. For each $0 \leq \gamma \leq 1$, define

$$H^\gamma := \{(-A)^{-\gamma}v \mid v \in H\}$$

and endow this space with the norm

$$\|v\|_{H^\gamma} := \|(-A)^{-\gamma}v\|_H \forall v \in H^\gamma.$$ 

In the sequel, we will use $[\cdot, \cdot]$ to denote the usual inner product of the Hilbert space $L^2(\Omega; H)$.

For any two Banach spaces $B_1$ and $B_2$, $\mathcal{L}(B_1, B_2)$ is the space of all bounded linear operators from $B_1$ to $B_2$, and $\mathcal{L}(B_1, B_1)$ is abbreviated to $\mathcal{L}(B_1)$. We denote by $I$ the identity mapping.

Finally, for the data $f$ and $p_T$ in Eq. (1), we make the following assumptions.

**Hypothesis 2.1** We assume that (i)–(iii) hold:

(i) The functional $f : [0, T] \times \Omega \times H \times H \rightarrow H$ satisfies that

$$f(\cdot, v, w) \in L^2_F(0, T; H) \text{ for all } v, w \in H.$$ 

(ii) There exists a positive constant $C_L$ such that, $\mathbb{P}$ almost surely for almost every $t \in [0, T]$,

$$\|f(t, p_1, z_1) - f(t, p_2, z_2)\|_H \leq C_L(\|p_1 - p_2\|_H + \|z_1 - z_2\|_H) \quad (2)$$

for all $p_1, p_2, z_1, z_2 \in H$.

(iii) $p_T \in L^2(\Omega; H^{1/2})$. 

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Under the above hypothesis, Eq. (1) admits a unique mild solution \((p, z)\), and

\[
(p, z) \in \left( L^2_F(\Omega; C([0, T]; H^{1/2})) \cap L^2_F(0, T; H^1) \right) \times L^2_F(0, T; H^{1/2}).
\]  

(3)

Remark 2.1 By [21, Theorem 3.1], [17, Theorem 4.4], and the basic theory of BSDEs (see, e.g., [34, Chapter 5]), the proof of (3) is straightforward; see also [30, Theorem 4.10].

3 Three Temporal Semi-discretizations

Let \( J \) be a positive integer and define \( t_j := j \tau \) for each \( 0 \leq j \leq J \), where \( \tau := T/J \).

Define

\[
\mathcal{X}_\tau := \{ V : [0, T] \times \Omega \to H \mid V(t_j) \in L^2(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; H) \text{ and } V \text{ is constant on } [t_j, t_{j+1}) \text{ for each } 0 \leq j < J \}.
\]

For any \( V \in \mathcal{X}_\tau \), we denote \( V(t_j) \), \( 0 \leq j \leq J \), by \( V_j \) for convenience. For each \( 0 \leq j < J \), define \( \delta W_j := W(t_{j+1}) - W(t_j) \), and define \( \mathcal{T}_t^j : L^2(\Omega; H) \to L^2(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; H) \) by

\[
\mathcal{T}_t^j v := \frac{1}{\tau} \mathbb{E}_{t_j} (v \delta W_j) \quad \forall v \in L^2(\Omega; H).
\]  

(4)

We also let \( \mathcal{P}_\tau \) be the \( L^2(\Omega; L^2(0, T; H)) \)-orthogonal projection onto \( \mathcal{X}_\tau \); more concretely, for any \( v \in L^2(\Omega; L^2(0, T; H)) \),

\[
(\mathcal{P}_\tau v)(t) := \frac{1}{\tau} \mathbb{E}_{t_j} \int_{t_j}^{t_{j+1}} v(s) \, ds
\]

for all \( t_j \leq t < t_{j+1} \) with \( 0 \leq j < J \). In the rest of this paper, \( c \) denotes a generic positive constant, independent of \( \tau \), and its value may differ in different places.

Now we present three temporal semi-discretizations of Eq. (1). The first semi-discretization seeks \((P, Z) \in \mathcal{X}_\tau \times \mathcal{X}_\tau \) by

\[
\begin{align*}
P_j &= p_T, \quad \text{(5a)} \\
Z_j &= \mathcal{T}_t^j \left( P_{j+1} + \int_{t_j}^{t_{j+1}} f(t, P_{j+1}, Z_j) \, dt \right), \quad 0 \leq j < J, \quad \text{(5b)} \\
P_j - \mathbb{E}_{t_j} P_{j+1} &= \tau AP_j + \mathbb{E}_{t_j} \int_{t_j}^{t_{j+1}} f(t, P_{j+1}, Z_j) \, dt, \quad 0 \leq j < J. \quad \text{(5c)}
\end{align*}
\]
The second semi-discretization seeks \((P, Z) \in X_\tau \times X_\tau\) by
\[
\begin{align*}
P_J &= p_T, \\
Z_j &= I_j P_{j+1}, \quad 0 \leq j < J, \\
P_j - \mathbb{E}_j P_{j+1} &= \tau A P_j + \mathbb{E}_j \int_{t_j}^{t_{j+1}} f(t, P_{j+1}, Z_j) \, dt, \quad 0 \leq j < J. 
\end{align*}
\]

The third semi-discretization seeks \((P, Z) \in X_\tau \times L^2_{\mathbb{F}}(0, T; H)\) by
\[
\begin{align*}
P_J &= p_T, \\
P_j - P_{j+1} &= \tau A P_j + \int_{t_j}^{t_{j+1}} f(t, P_{j+1}, Z(t)) \, dt - \int_{t_j}^{t_{j+1}} Z(t) \, dW(t), \quad 0 \leq j < J. 
\end{align*}
\]

The main results of this section are the following three theorems.

**Theorem 3.1** Assume that Hypothesis 2.1 holds and \(\tau < 1/C_L\). Let \((p, z)\) and \((P, Z)\) be the solutions of (1) and (5), respectively. Then
\[
\begin{align*}
\max_{0 \leq j < J} \left\| p(t_j) - P_j \right\|_H + \| z - Z \|_{L^2(0, T; H)} &
\leq c\left(\frac{\tau^{1/2}}{2} + \| (I - P_\tau) z \|_{L^2(0, T; H)} \right).
\end{align*}
\]

**Theorem 3.2** Assume that Hypothesis 2.1 holds. Let \((p, z)\) and \((P, Z)\) be the solutions of (1) and (6), respectively. Then the error estimate (8) still holds.

**Theorem 3.3** Let \((p, z)\) and \((P, Z)\) be the solutions of (1) and (7), respectively. Then, under the conditions of Theorem 3.1, we have
\[
\begin{align*}
\max_{0 \leq j < J} \left\| p(t_j) - P_j \right\|_H + \| z - Z \|_{L^2(0, T; H)} &
\leq c\tau^{1/2}.
\end{align*}
\]

We only provide a complete proof of Theorem 3.1, since the proofs of Theorems 3.2 and 3.3 are similar (see Remark 3.4). To this end, we proceed as follows.

### 3.1 Preliminary Results

We present some standard estimates as follows. For any \(0 < t \leq T\) and \(0 \leq \beta \leq \gamma \leq 1\), we have (see, e.g., [35, Theorem 6.13, Chapter 2])
\[
\begin{align*}
\| e^{tA} \|_{L(H^\beta, H^\gamma)} &\leq c t^{\beta - \gamma}, \\
\| I - e^{tA} \|_{L(H^\gamma, H^\beta)} &\leq c t^{\gamma - \beta}.
\end{align*}
\]

By [39, Theorem 7.3] we have, for any \(0 \leq \beta \leq 1\),
\[
\| e^{m\tau A} - (I - \tau A)^{-m} \|_{L(H^\beta, H)} \leq c \tau^\beta m^{\beta - 1} \quad \forall m > 0.
\]
For any $v \in H^{1/2}$ and $g \in L^2(0, T; H)$, we have the following estimates:

$$
J - 1 \sum_{j=0}^{J-1} \left\| w_j - e^{(T-t_j)A} v - \int_{t_j}^T e^{(s-t_j)A} g(t) \, dt \right\|^2_H \leq c \tau \left( \| v \|_{H^{1/2}} + \| g \|_{L^2(0,T;H)}^2 \right) \tag{13}
$$

and, for any $0 \leq j < J$,

$$
\max_{j \leq k < J} \| w_k \|_{H^{1/2}} + \left( \sum_{k=j}^{J-1} \tau \| w_k \|_{H^1}^2 \right)^{1/2} \leq c \left( \| v \|_{H^{1/2}} + \| g \|_{L^2(t_j,T;H)} \right), \tag{14}
$$

where $\{w_j\}_{j=0}^{J-1}$ is defined by

$$
w_j := (I - \tau A)^{-(J-j)} v + \sum_{k=j}^{J-1} (I - \tau A)^{-(k-j+1)} \int_{t_k}^{t_{k+1}} g(t) \, dt \quad \forall 0 \leq j < J.
$$

In addition, for any $v \in L^2(\Omega; H)$ and $w \in L^2(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; H)$ with $0 \leq j < J$, the following properties are easily verified by (4):

$$
\mathcal{I}_t^j w = 0 \quad \mathbb{P}\text{-a.s.}, \tag{15}

(I - \delta W_j \mathcal{I}_t^j)(\delta W_j w) = 0 \quad \mathbb{P}\text{-a.s.}, \tag{16}

[v - \delta W_j \mathcal{I}_t^j v, \delta W_j w] = 0, \tag{17}

\| v - \delta W_j \mathcal{I}_t^j v \|_H^2 + \| \delta W_j \mathcal{I}_t^j v \|_H^2 = \| v \|_H^2. \tag{18}
$$

**Remark 3.1** The estimates (13) and (14) are standard; see, e.g., [39, Chapter 12].

### 3.2 Three Temporal Semi-discretizations of a Backward Linear Stochastic Evolution Equation

This subsection studies the convergence of three temporal semi-discretizations for the following backward linear stochastic evolution equation: a

$$
\begin{cases}
    dp(t) = -(Ap + g)(t) \, dt + z(t) \, dW(t), & 0 \leq t \leq T, \tag{19a} \\
    p(T) = p_T, & \tag{19b}
\end{cases}
$$

where $g \in L^2_{\mathbb{F}}(0, T; H)$ and $p_T \in L^2(\Omega; H)$. The main results are the following three lemmas.

**Lemma 3.1** Assume that $(p, z)$ is the solution of (19) with $p_T \in L^2(\Omega; H^{1/2})$ and $g \in L^2_{\mathbb{F}}(0, T; H)$. 

\[ \Box \]
Define \((P, Z) \in \mathcal{X}_\tau \times \mathcal{X}_\tau\) by

\[
P_j = p_T, \quad Z_j = I_t^j (P_{j+1} + \int_{t_j}^{t_{j+1}} g(t) \, dt), \quad 0 \leq j < J, \tag{20a}
\]

\[
P_j - \mathbb{E}_{t_j} P_{j+1} = \tau A P_j + \mathbb{E}_{t_j} \int_{t_j}^{t_{j+1}} g(t) \, dt, \quad 0 \leq j < J. \tag{20c}
\]

Then the following estimates hold: for any \(0 \leq j < J\),

\[
\|P(t_j) - P_j\|_H \leq c t^{1/2}\left((J-j)^{-1/2} \|p_T\|_{H^{1/2}} + \|g\|_{L^2(0,T;H)}\right), \tag{21}
\]

\[
\left(\sum_{j=0}^{J-1} \|P-j+1\|_{L^2(t_j,j+1;H)}^2\right)^{1/2} \leq c t^{1/2}\left(\|p_T\|_{H^{1/2}} + \|g\|_{L^2(0,T;H)}\right), \quad \tag{22}
\]

\[
\|z - Z\|_{L^2(0,T;H)} \leq c t^{1/2}\left(\|p_T\|_{H^{1/2}} + \|g\|_{L^2(0,T;H)}\right) + \|P - \mathcal{P}_T\|_{L^2(0,T;H)}. \tag{23}
\]

**Lemma 3.2** Define \((P, Z) \in \mathcal{X}_\tau \times \mathcal{X}_\tau\) by

\[
P_j = p_T, 
Z_j = I_t^j P_{j+1}, \quad 0 \leq j < J, 
P_j - \mathbb{E}_{t_j} P_{j+1} = \tau A P_j + \mathbb{E}_{t_j} \int_{t_j}^{t_{j+1}} g(t) \, dt, \quad 0 \leq j < J. 
\]

Then the three estimates in Lemma 3.1 still hold, under the conditions of Lemma 3.1.

**Lemma 3.3** Define \((P, Z) \in \mathcal{X}_\tau \times L^2_p(0,T;H)\) by

\[
P_j = p_T, 
P_j - P_{j+1} = \tau A P_j + \int_{t_j}^{t_{j+1}} g(t) \, dt - \int_{t_j}^{t_{j+1}} Z(t) \, dW(t), \quad 0 \leq j < J.
\]

Then, under the conditions of Lemma 3.1, the error estimates (21) and (22) in Lemma 3.1 still hold, and

\[
\|z - Z\|_{L^2(0,T;H)} \leq c t^{1/2}\left(\|p_T\|_{H^{1/2}} + \|g\|_{L^2(0,T;H)}\right). \tag{24}
\]

Since the proofs of Lemmas 3.2 and 3.3 are similar to (and simpler than) that of Lemma 3.1, we only prove the latter. To this end, we first present some standard properties of the solution \((p, z)\) to Eq. (19) as follows:

- for any \(0 \leq t \leq T\), we have

\[
p(t) = \mathbb{E}_t \left(e^{(T-t)A} p_T + \int_t^T e^{(r-t)A} g(r) \, dr\right) \quad \mathbb{P}\text{-a.s.}; \quad \tag{25}
\]
for any \( 0 \leq s \leq t \leq T \), we have
\[
p(s) - p(t) = \int_s^t (Ap + g)(r) \, dr - \int_s^t z(r) \, dW(r) \quad \mathbb{P}\text{-a.s.,}
\]
(26)
\[
p(s) - e^{(r-s)A} p(t) = \int_s^t e^{(r-s)A} g(r) \, dr - \int_s^t e^{(r-s)A} z(r) \, dW(r) \quad \mathbb{P}\text{-a.s.;}
\]
(27)

- for \( p_T \in L^2(\Omega; H^{1/2}) \) and \( g \in L^2_B(0; T; H) \), we have
\[
\| p \|_{L^2(0,T; H^1)} + \| z \|_{L^2(0,T; H^{1/2})} \leq c \left( \| p_T \|_{H^{1/2}} + \| g \|_{L^2(0,T; H)} \right).
\]
(28)

**Remark 3.2** The above properties are standard and easily verified by the Galerkin method and the basic properties of the finite-dimensional BSDEs (see, e.g., [34, Chapter 5]).

Then we present two technical lemmas, which can be proved by straightforward calculations.

**Lemma 3.4** For any \( 0 \leq j < J \),
\[
\sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)} \right\|_{L(H)}^2 \, dt \leq c \tau.
\]
(29)

**Proof** We have
\[
\sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)} \right\|_{L(H)}^2 \, dt
\]
\[
= \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| e^{(t-t_j)A} - e^{(t_{k+1}-t_j)A} + e^{(t_{k+1}-t_j)A} - (I - \tau A)^{-(k-j+1)} \right\|_{L(H)}^2 \, dt
\]
\[
\leq 2 \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| e^{(t-t_j)A} - e^{(t_{k+1}-t_j)A} \right\|_{L(H)}^2 \, dt +
\]
\[
2 \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| e^{(t_{k+1}-t_j)A} - (I - \tau A)^{-(k-j+1)} \right\|_{L(H)}^2 \, dt
\]
\[
= \mathbb{I}_1 + \mathbb{I}_2.
\]

For \( \mathbb{I}_1 \) we have
\[
\mathbb{I}_1 = 2 \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| (I - e^{(t_{k+1}-t)A}) e^{(t-t_j)A} \right\|_{L(H)}^2 \, dt
\]
\[
\leq 2 \int_{t_j}^{t_{j+1}} \left\| I - e^{(t_{j+1}-t)A} \right\|_{L(H)}^2 \left\| e^{(t-t_j)A} \right\|_{L(H)}^2 \, dt
\]
+ 2 \sum_{k=j+1}^{J-1} \int_{t_k}^{t_{k+1}} \| I - e^{(t_{k+1} - t)A} \|_{\mathcal{L}(H^1, H)}^2 \| e^{(t-t_j)A} \|_{\mathcal{L}(H, H^1)}^2 dt \\
\leq c\tau,

by the following two estimates:

\int_{t_j}^{t_{j+1}} \| I - e^{(t_{j+1} - t)A} \|_{\mathcal{L}(H)}^2 \| e^{(t-t_j)A} \|_{\mathcal{L}(H)}^2 dt \\
\leq c \int_{t_j}^{t_{j+1}} dt \quad \text{(by (10) and (11))}
\leq c\tau

and

\sum_{k=j+1}^{J-1} \int_{t_k}^{t_{k+1}} \| I - e^{(t_{k+1} - t)A} \|_{\mathcal{L}(H^1, H)}^2 \| e^{(t-t_j)A} \|_{\mathcal{L}(H, H^1)}^2 dt \\
\leq c \sum_{k=j+1}^{J-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 (t - t_j)^{-2} dt \quad \text{(by (10) and (11))}
\leq c\tau^2 \sum_{k=j+1}^{J-1} \int_{t_k}^{t_{k+1}} (t - t_j)^{-2} dt
\leq c\tau.

For \( I_2 \), by (12) we obtain

\| I_2 \| \leq c \sum_{k=j}^{J-1} \tau (k - j + 1)^{-2} \leq c\tau.

Combining the above estimates of \( I_1 \) and \( I_2 \) yields (29) and thus completes the proof.

\( \square \)

**Lemma 3.5** Let \((p, z)\) be the solution to Eq. (19) with \( g \in L^2_\Omega(0, T; H) \) and \( p_T \in L^2(\Omega; H^{1/2}) \). Then

\[
\sum_{k=0}^{J-1} \left( \int_{t_k}^{t_{k+1}} \| p(t) \|_{H^1} dt \right)^2 \leq c\tau \left( \| p_T \|_{H^{1/2}}^2 + \| g \|_{L^2(0, T; H)}^2 \right). \tag{30}
\]

**Proof** Let

\[\eta(t) := e^{(T-t)A} p_T + \int_t^T e^{(s-t)A} g(s) \, ds \quad \forall 0 \leq t \leq T.\]

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It is standard that
\[
\|\eta\|_{L^2(0,T;H^1)} \leq c \left( \|p_T\|_{H^{1/2}} + \|g\|_{L^2(0,T;H)} \right). \tag{31}
\]

By (25) we have, for any \(0 \leq t < T\),
\[
\|p(t)\|_{H^1} = \|\mathbb{E}_t \eta(t)\|_{H^1} \leq \|\eta(t)\|_{H^1},
\]
so that
\[
\sum_{k=0}^{J-1} \left( \int_{t_k}^{t_{k+1}} \|p(t)\|_{H^1} \, dt \right)^2 \leq \tau \int_{t_k}^{t_{k+1}} \|\eta(t)\|_{H^1}^2 \, dt = \tau \|\eta\|_{L^2(0,T;H^1)}^2,
\]
which, together with (31), proves the desired estimate (30). \(\square\)

Finally, we are in a position to prove Lemma 3.1 as follows. \textbf{Proof of Lemma 3.1.} Firstly, let us prove (21). Let \(0 \leq j < J\) be arbitrary but fixed. From (20), it is easily verified that
\[
P_j = \mathbb{E}_{t_j} \left( (I - \tau A)^{-J-j} p_T + \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} (I - \tau A)^{-k-j+1} g(t) \, dt \right) \quad \mathbb{P}\text{-a.s.} \tag{32}
\]

Hence, by (25) we obtain
\[
p(t_j) - P_j = \Pi_1 + \Pi_2 \quad \mathbb{P}\text{-a.s.,}
\]
where
\[
\Pi_1 := \mathbb{E}_{t_j} \left( e^{(T-t_j)A} - (I - \tau A)^{-J-j} \right) p_T,
\]
\[
\Pi_2 := \mathbb{E}_{t_j} \left( \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left( e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)} \right) g(t) \, dt \right).
\]

For \(\Pi_1\) we have
\[
\|\Pi_1\|_H = \|\mathbb{E}_{t_j} \left( e^{(T-t_j)A} - (I - \tau A)^{-J-j} \right) p_T\|_H \\
\leq \|e^{(T-t_j)A} - (I - \tau A)^{-J-j}\|_{L(H^{1/2},H)} \|p_T\|_{H^{1/2}}.
\]
\[ \leq c \tau^{1/2} (J - j)^{-1/2} \| p_T \|_{H^{1/2}} \quad \text{(by (12))}. \]

For \( \mathbb{I}_2 \) we have

\[
\| \mathbb{I}_2 \|_H = \left\| \mathbb{E}_j \left( \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} (e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)}) g(t) \, dt \right) \right\|_H
\leq \left\| \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} (e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)}) g(t) \, dt \right\|_H
\leq \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \|e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)}\|_H \|g(t)\|_H \, dt
\leq \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \|e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)}\|_{\mathcal{L}(H)} \|g(t)\|_H \, dt
\leq \left( \sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \|e^{(t-t_j)A} - (I - \tau A)^{-(k-j+1)}\|_{\mathcal{L}(H)}^2 \, dt \right)^{1/2} \|g\|_{L^2(t_j, T; H)}
\leq c \tau^{1/2} \|g\|_{L^2(t_j, T; H)} \quad \text{(by Lemma 3.4)}.
\]

Combining the above estimates of \( \mathbb{I}_1 \) and \( \mathbb{I}_2 \) then yields (21).

Secondly, let us prove (22). For any \( 0 \leq j < J \), by (26) we have

\[ p(t) - p(t_{j+1}) = \int_t^{t_{j+1}} (Ap + g)(s) \, ds - \int_t^{t_{j+1}} z(s) \, dW(s), \; t_j \leq t < t_{j+1}, \]

and so

\[
\left\| p - p(t_{j+1}) \right\|_{L^2(t_j, t_{j+1}; H)}^2
\leq 2 \int_{t_j}^{t_{j+1}} \left\| \int_t^{t_{j+1}} (Ap + g)(s) \, ds \right\|_H^2 \, dt + 2 \int_{t_j}^{t_{j+1}} \left\| \int_t^{t_{j+1}} z(s) \, dW(s) \right\|_H^2 \, dt
\leq 2 \tau^2 \|Ap + g\|_{L^2(t_j, t_{j+1}; H)}^2 + 2 \tau \|z\|_{L^2(t_j, t_{j+1}; H)}^2.
\]

It follows that

\[
\sum_{j=0}^{J-1} \left\| p - p(t_{j+1}) \right\|_{L^2(t_j, t_{j+1}; H)}^2 \leq 2 \tau^2 \|Ap + g\|_{L^2(0, T; H)}^2 + 2 \tau \|z\|_{L^2(0, T; H)}^2,
\]

and hence by (28) we obtain

\[
\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} \left\| p(t) - p(t_{j+1}) \right\|_H^2 \, dt \leq c \tau \left( \|p_T\|_{H^{1/2}}^2 + \|g\|_{L^2(0, T; H)}^2 \right). \tag{33}
\]
Using the above estimate and (21) yields

\[
\sum_{j=0}^{J-1} \left\| p - P_{j+1} \right\|_{L^2(t_j, t_{j+1}; H)}^2
= \sum_{j=0}^{J-1} \left\| p - p(t_{j+1}) + p(t_{j+1}) - P_{j+1} \right\|_{L^2(t_j, t_{j+1}; H)}^2
\leq 2 \sum_{j=0}^{J-1} \left\| p(t_{j+1}) \right\|_{L^2(t_j, t_{j+1}; H)}^2 + 2 \sum_{j=0}^{J-1} \left\| p(t_{j+1}) - P_{j+1} \right\|_{L^2(t_j, t_{j+1}; H)}^2
\leq c \tau \left( \| p_T \|_{H^{1/2}}^2 + \| g \|_{L^2(0, T; H)}^2 \right),
\]

which implies the desired estimate (22).

Thirdly, let us prove (23). Fix 0 ≤ j < J. By (26) we have

\[
p(t_{j+1}) + \int_{t_j}^{t_{j+1}} g(t) \, dt = p(t_j) - \int_{t_j}^{t_{j+1}} \mathcal{A}p(t) \, dt + \int_{t_j}^{t_{j+1}} z(t) \, dW(t) \quad \mathbb{P}\text{-a.s.,}
\]

so that, \( \mathbb{P}\text{-a.s.,} \)

\[
(I - \mathbb{E}_{t_j} - \delta W_j \mathcal{I}_t^j) \left( p(t_{j+1}) + \int_{t_j}^{t_{j+1}} g(t) \, dt \right)
= (I - \mathbb{E}_{t_j} - \delta W_j \mathcal{I}_t^j) \left( p(t_j) - \int_{t_j}^{t_{j+1}} \mathcal{A}p(t) \, dt + \int_{t_j}^{t_{j+1}} z(t) \, dW(t) \right)
= - \delta W_j \mathcal{I}_t^j p(t_j) - (I - \mathbb{E}_{t_j} - \delta W_j \mathcal{I}_t^j) \int_{t_j}^{t_{j+1}} \mathcal{A}p(t) \, dt + (I - \delta W_j \mathcal{I}_t^j) \int_{t_j}^{t_{j+1}} z(t) \, dW(t)
= - (I - \mathbb{E}_{t_j} - \delta W_j \mathcal{I}_t^j) \int_{t_j}^{t_{j+1}} \mathcal{A}p(t) \, dt + (I - \delta W_j \mathcal{I}_t^j) \int_{t_j}^{t_{j+1}} z(t) \, dW(t) \quad \text{(by (15))}.
\]

It follows that

\[
(I - \mathbb{E}_{t_j} - \delta W_j \mathcal{I}_t^j) \left( p(t_{j+1}) + \int_{t_j}^{t_{j+1}} g(t) \, dt \right) + (I - \mathbb{E}_{t_j}) \int_{t_j}^{t_{j+1}} \mathcal{A}p(t) \, dt
= \delta W_j \mathcal{I}_t^j \int_{t_j}^{t_{j+1}} \mathcal{A}p(t) \, dt + (I - \delta W_j \mathcal{I}_t^j) \int_{t_j}^{t_{j+1}} z(t) \, dW(t) \quad \mathbb{P}\text{-a.s.,}
\]

which further implies

\[
(I - \mathbb{E}_{t_j}) \left( p(t_{j+1}) + \int_{t_j}^{t_{j+1}} (\mathcal{A}p + g)(t) \, dt \right)
= \delta W_j \mathcal{I}_t^j \left( p(t_{j+1}) + \int_{t_j}^{t_{j+1}} (\mathcal{A}p + g)(t) \, dt \right) + (I - \delta W_j \mathcal{I}_t^j) \int_{t_j}^{t_{j+1}} z(t) \, dW(t) \quad \mathbb{P}\text{-a.s.} \quad \text{(35)}
\]
By (34) we also have
\[
\int_{t_j}^{t_{j+1}} z(t) \, dW(t) = (I - \mathbb{E}_{t_j}) \left( p(t_{j+1}) - p(t_j) + \int_{t_j}^{t_{j+1}} (Ap + g)(t) \, dt \right) = (I - \mathbb{E}_{t_j}) \left( p(t_{j+1}) + \int_{t_j}^{t_{j+1}} (Ap + g)(t) \, dt \right) \quad \mathbb{P}\text{-a.s.},
\]
which, together with (20b) and (16), implies $\mathbb{P}\text{-a.s.}$
\[
\int_{t_j}^{t_{j+1}} (z - Z)(t) \, dW(t) = (I - \mathbb{E}_{t_j}) \left( p(t_{j+1}) + \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right) - \delta W_j \mathcal{I}_t \left( P_{j+1} + \int_{t_j}^{t_{j+1}} g(t) \, dt \right)
\]
\[
= \delta W_j \mathcal{I}_t \left( p(t_{j+1}) - P_{j+1} + \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right) + (I - \delta W_j \mathcal{I}_t) \int_{t_j}^{t_{j+1}} z(t) \, dW(t) \quad \text{(by (37))}
\]
\[
= \delta W_j \mathcal{I}_t \left( p(t_{j+1}) - P_{j+1} + \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right) + (I - \delta W_j \mathcal{I}_t) \int_{t_j}^{t_{j+1}} (z - \mathcal{P}_T z)(t) \, dW(t).
\]

Hence,
\[
\left\| \int_{t_j}^{t_{j+1}} (z - Z)(t) \, dW(t) \right\|_H^2 = \left\| \delta W_j \mathcal{I}_t \left( p(t_{j+1}) - P_{j+1} + \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right) \right\|_H^2 + \left\| (I - \delta W_j \mathcal{I}_t) \int_{t_j}^{t_{j+1}} (z - \mathcal{P}_T z)(t) \, dW(t) \right\|_H^2
\]
\[
\leq \left\| p(t_{j+1}) - P_{j+1} + \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right\|_H^2 + \left\| \int_{t_j}^{t_{j+1}} (z - \mathcal{P}_T z)(t) \, dW(t) \right\|_H^2 \quad \text{(by (18))}
\]
\[
\leq 2 \left\| p(t_{j+1}) - P_{j+1} \right\|_H^2 + 2 \left\| \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right\|_H^2 + \left\| (I - \mathcal{P}_T) z \right\|_{L^2(t_j, t_{j+1}; H)}^2.
\]

where we have used the property (17) in the first equality. Since $0 \leq j < J$ is arbitrary, summing over $j$ from 0 to $J - 1$ leads to
\[
\sum_{j=0}^{J-1} \left\| \int_{t_j}^{t_{j+1}} (z - Z)(t) \, dW(t) \right\|_H^2
\]
\[
\leq 2 \sum_{j=0}^{J-1} \left\| p(t_{j+1}) - P_{j+1} \right\|_H^2 + 2 \sum_{j=0}^{J-1} \left\| \int_{t_j}^{t_{j+1}} (Ap)(t) \, dt \right\|_H^2 + \left\| (I - \mathcal{P}_T) z \right\|_{L^2(0, T; H)}^2
\]
\[
\leq 2 \sum_{j=0}^{J-1} \left\| p(t_{j+1}) - P_{j+1} \right\|_H^2 + 2 \sum_{j=0}^{J-1} \left( \int_{t_j}^{t_{j+1}} \left\| Ap(t) \right\|_H \, dt \right)^2 + \left\| (I - \mathcal{P}_T) z \right\|_{L^2(0, T; H)}^2
\]
\[
= 2 \sum_{j=0}^{J-1} \left\| p(t_{j+1}) - P_{j+1} \right\|_H^2 + 2 \sum_{j=0}^{J-1} \left( \int_{t_j}^{t_{j+1}} \left\| p(t) \right\|_H \, dt \right)^2 + \left\| (I - \mathcal{P}_T) z \right\|_{L^2(0, T; H)}^2.
\]
which, together with the equality
\[
\sum_{j=0}^{J-1} \left| \int_{t_j}^{t_{j+1}} (z - Z)(t) \, dW(t) \right|^2_H = \sum_{j=0}^{J-1} \left\| z - Z \right\|^2_{L^2(t_j, t_{j+1}; H)} = \left\| z - Z \right\|^2_{L^2(0,T; H)},
\]
implies
\[
\| z - Z \|^2_{L^2(0,T; H)} \leq 2 \sum_{j=0}^{J-1} \left\| p(t_{j+1}) - P_{j+1} \right\|^2_H + 2 \sum_{j=0}^{J-1} \left( \int_{t_j}^{t_{j+1}} \left\| p(t) \right\|_{L^2} \, dt \right)^2 + \left\| (I - \mathcal{P}_\tau) z \right\|^2_{L^2(0,T; H)},
\]
(36)

By (32) and the fact
\[
p(t) = \mathbb{E}_t \left( e^{(T-t)A} p_T + \int_t^T e^{(s-t)A} g(s) \, ds \right), \quad 0 \leq t \leq T,
\]
we can use (13) to conclude that
\[
\sum_{j=0}^{J-1} \left\| p(t_{j+1}) - P_{j+1} \right\|^2_H \leq c\tau \left( \left\| p_T \right\|^2_{L^2(0,T; H)} + \left\| g \right\|^2_{L^2(0,T; H)} \right).
\]
(37)

Finally, combining (36), (37) and (30) proves (23) and thus concludes the proof of Lemma 3.1.

### 3.3 Proof of Theorem 3.1

For any \( 0 \leq j < J \), since (18) implies
\[
\tau^{1/2} \left\| I^j v \right\|_H = \left\| \delta W_j I^j v \right\|_H \leq \left\| v \right\|_H \quad \forall v \in L^2(\Omega; H),
\]
we obtain
\[
\left\| I^j \right\|_{L(L^2(\Omega; H))} \leq \tau^{-1/2} \quad \forall 0 \leq j < J.
\]

By the above estimate, (2) and the condition \( \tau < 1/C_L^2 \), a straightforward contraction argument proves that the temporal semi-discretization (5) admits a unique solution \((P, Z)\). In the sequel, we will assume that \( \tau \) is sufficiently small; otherwise, the error estimate (8) is evident. We split the rest of the proof into the following four steps.

**Step 1** We present some preliminary notations and estimates. Let
\[
\mathcal{M} := \tau + \left\| (I - \mathcal{P}_\tau) z \right\|^2_{L^2(0,T; H)}. \tag{38}
\]
Define \((\widetilde{P}, \widetilde{Z}) \in \mathcal{X}_r \times \mathcal{X}_r\) by

\[
\begin{align*}
\widetilde{P}_j &= p_T, \quad (39a) \\
\widetilde{Z}_j &= \mathcal{I}_r^j \left( \widetilde{P}_{j+1} + \int_{t_j}^{t_{j+1}} f(t, p(t), z(t)) \, dt \right), \quad 0 \leq j < J, \quad (39b) \\
\widetilde{P}_j - \mathbb{E}_{t_j} \widetilde{P}_{j+1} &= \tau A \widetilde{P}_j + \mathbb{E}_{t_j} \int_{t_j}^{t_{j+1}} f(t, p(t), z(t)) \, dt, \quad 0 \leq j < J. \quad (39c)
\end{align*}
\]

In view of \(p_T \in H^{1/2}\) and the fact

\[f(\cdot, p(\cdot), z(\cdot)) \in L^2_{T^2}(0, T; H),\]

by Lemma 3.1 we obtain

\[
\max_{0 \leq k < J} \left\| p(t_k) - \widetilde{P}_k \right\|_H + \left( \sum_{k=0}^{J-1} \left\| p - \widetilde{P}_{k+1} \right\|_{L^2(t_k, t_{k+1}; H)}^2 \right)^{1/2} + \left\| z - \widetilde{Z} \right\|_{L^2(0, T; H)} \leq c \mathcal{M}^{1/2}. \quad (40)
\]

Letting \(E^P := P - \widetilde{P}\) and \(E^Z := Z - \widetilde{Z}\), from (5) and (39) we conclude that

\[
\begin{align*}
E^P_j &= 0, \quad (41a) \\
E^Z_j &= \mathcal{I}_r^j \left( E^P_{j+1} + \int_{t_j}^{t_{j+1}} G(t, E^P_{j+1}, E^Z_j) \, dt \right), \quad 0 \leq j < J, \quad (41b) \\
E^P_j - \mathbb{E}_{t_j} E^P_{j+1} &= \tau A E^P_j + \mathbb{E}_{t_j} \int_{t_j}^{t_{j+1}} G(t, E^P_{j+1}, E^Z_j) \, dt, \quad 0 \leq j < J, \quad (41c)
\end{align*}
\]

where

\[
G(t, E^P_{j+1}, E^Z_j) := f(t, E^P_{j+1} + \widetilde{P}_{j+1}, E^Z_j + \widetilde{Z}_j) - f(t, p(t), z(t)) \quad (42)
\]

for all \(t_j \leq t < t_{j+1}\) with \(0 \leq j < J\). We have, for any \(0 \leq k < J\),

\[
\begin{align*}
\int_{t_k}^{t_{k+1}} \left\| G(t, E^P_{k+1}, E^Z_k) \right\|_H^2 \, dt \\
&= \int_{t_k}^{t_{k+1}} \left\| f(t, E^P_{k+1} + \widetilde{P}_{k+1}, E^Z_k + \widetilde{Z}_k) - f(t, p(t), z(t)) \right\|_H^2 \, dt \quad \text{(by (44))} \\
&\leq c \int_{t_k}^{t_{k+1}} \left\| E^P_{k+1} + \widetilde{P}_{k+1} - p(t) \right\|_H^2 + \left\| E^Z_k + \widetilde{Z}_k - z(t) \right\|_H^2 \, dt \quad \text{(by (2))} \\
&\leq c \left( \tau \left\| E^P_{k+1} \right\|_H^2 + \left\| E^Z_k \right\|_{L^2(t_k, t_{k+1}; H)}^2 + \left\| p - \widetilde{P}_{k+1} \right\|_{L^2(t_k, t_{k+1}; H)}^2 + \left\| z - \widetilde{Z} \right\|_{L^2(t_k, t_{k+1}; H)}^2 \right) \quad (43).
\end{align*}
\]

Hence, for each \(0 \leq j < J\),

\[
\sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \left\| G(t, E^P_{k+1}, E^Z_k) \right\|_H^2 \, dt
\]

\[\square\] Springer
\[
\begin{align*}
&\leq c\left(\sum_{k=j}^{J-1} \|E_{k+1}^P\|_H^2 + \|E^Z\|_{L^2(t_j, T; H)}^2 + \sum_{k=j}^{J-1} \|p - \bar{P}_{k+1}\|_{L^2(t_k, t_{k+1}; H)}^2 + \|z - Z\|_{L^2(0, T; H)}^2\right),
\end{align*}
\]

which, together with the fact \(E_j^P = 0\) and (40), leads to

\[
\begin{align*}
&\sum_{k=j}^{J-1} \int_{t_k}^{t_{k+1}} \|G(t, E_{k+1}^P, E_j^Z)\|_H^2 \, dt \\
&\leq c\left(\|E^P\|_{L^2(t_j, T; H)}^2 + \|E^Z\|_{L^2(t_j, T; H)}^2 + \mathcal{M}\right).
\end{align*}
\] (43)

**Step 2** Let us prove that, for any \(0 \leq j < J\),

\[
\|E^Z\|_{L^2(t_j, T; H)} \leq c\left(\mathcal{M}^{1/2} + \|E_j^P\|_{H^{1/2}} + \|E_j^P\|_{L^2(t_j, T; H)} + \sqrt{T - t_j} \|E^Z\|_{L^2(t_j, T; H)}\right).
\] (44)

For each \(0 \leq j < J\), define

\[
\eta_j := E_{j+1}^P - E_j^P + \tau A E_j^P + \int_{t_j}^{t_{j+1}} G(t, E_{j+1}^P, E_j^Z) \, dt.
\] (45)

Using (17), (41b) and the fact

\[
[\tau A E_j^P - E_j^P, E_j^Z \delta W_j] = 0,
\]

we obtain

\[
[\eta_j - E_j^Z \delta W_j, E_j^Z \delta W_j] = 0 \quad \text{for all} \quad 0 \leq j < J.
\]

For any \(0 \leq k \neq j < J\), since (43c) implies \(\mathbb{E}_t \eta_j = 0\ \mathbb{P}\)-a.s., it is easily verified that

\[
[\eta_j - E_j^Z \delta W_j, E_k^Z \delta W_k] = 0.
\]

Consequently,

\[
\left[\sum_{k=j}^{J-1} \eta_k - \sum_{k=j}^{J-1} E_k^Z \delta W_k, \sum_{k=j}^{J-1} E_k^Z \delta W_k\right] = 0 \quad \forall 0 \leq j < J.
\]

It follows that, for any \(0 \leq j < J\),

\[
\left\|\sum_{k=j}^{J-1} E_k^Z \delta W_k\right\|_H^2 = \left[\sum_{k=j}^{J-1} \eta_k, \sum_{k=j}^{J-1} E_k^Z \delta W_k\right].
\]
\[
\begin{align*}
&= \left[ E_j^P - E_j^P + \sum_{k=j}^{j-1} \tau A E_k^P + \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} G(t, E_{k+1}^P, E_k^Z) \, dt, \sum_{k=j}^{j-1} E_k^Z \delta W_k \right] \quad \text{(by (47))} \\
&= \left[ E_j^P + \sum_{k=j}^{j-1} \tau A E_k^P + \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} G(t, E_{k+1}^P, E_k^Z) \, dt, \sum_{k=j}^{j-1} E_k^Z \delta W_k \right] \\
&\leq \left( \| E_j^P \|_H + \tau \sum_{k=j}^{j-1} \| E_k^P \|_{H^1} + \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} \| G(t, E_{k+1}^P, E_k^Z) \|_H \, dt \right) \left( \sum_{k=j}^{j-1} \| E_k^Z \delta W_k \|_H \right),
\end{align*}
\]
which, together with the identity
\[
\left\| \sum_{k=j}^{j-1} E_k^Z \delta W_k \right\|_H^2 = \sum_{k=j}^{j-1} \| E_k^Z \delta W_k \|_H^2 = \sum_{k=j}^{j-1} \tau \| E_k^Z \|_H^2 = \| E_j^Z \|_{L^2(t_j, T; H)}^2,
\]
implies
\[
\| E_j^Z \|_{L^2(t_j, T; H)} \leq \| E_j^P \|_H + \tau \sum_{k=j}^{j-1} \| E_k^P \|_{H^1} + \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} \| G(t, E_{k+1}^P, E_k^Z) \|_H \, dt \\
&\leq \| E_j^P \|_H + \sqrt{T - t_j} \| E_j^P \|_{L^2(t_j, T; H)} + \sqrt{T - t_j} \left( \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} \| G(t, E_{k+1}^P, E_k^Z) \|_H^2 \, dt \right)^{1/2}.
\]
(46)

For any \( 0 \leq j < J \), it is easily verified by (41) that
\[
E_j^P = \mathbb{E}_t \left( (I - \tau A)^{-j} E_j^P + \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} (I - \tau A)^{-(k-j+1)} G(t, E_{k+1}^P, E_k^Z) \, dt \right),
\]
and so using (14) gives
\[
\max_{j \leq k < J} \| E_j^P \|_{H^{1/2}}^2 + \| E_j^P \|_{L^2(t_j, T; H^1)}^2 \leq c \left( \| E_j^P \|_{H^{1/2}}^2 + \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} \| G(t, E_{k+1}^P, E_k^Z) \|_H^2 \, dt \right),
\]
(47)

Combining (46) and (47) yields, for any \( 0 \leq j < J \),
\[
\| E_j^Z \|_{L^2(t_j, T; H)} \leq c \| E_j^P \|_{H^{1/2}} + c \sqrt{T - t_j} \left( \sum_{k=j}^{j-1} \int_{t_k}^{t_{k+1}} \| G(t, E_{k+1}^P, E_k^Z) \|_H^2 \, dt \right)^{1/2},
\]
so that from (43) we conclude the desired estimate (44).

**Step 3** Let \( c^* \) be a particular constant \( c \) in the inequality (44), and set
\[
j^* := \min \left\{ 0 \leq j < J \mid c^* \sqrt{T - t_j} \leq 1/2 \right\}.
\]
From (44) it follows that
\[
\| E^Z \|_{L^2(t_j, T; H)}^2 \leq c( \| E_p^P \|_{H^{1/2}}^2 + \| E_p^P \|_{L^2(t_j, T; H)}^2 + M) \quad \forall j^* \leq j < J, \tag{48}
\]
and so by (43) and (47) we infer that
\[
\| E_p^P \|_{H^{1/2}}^2 \leq c( \| E_p^P \|_{H^{1/2}}^2 + \| E_p^P \|_{L^2(t_j, T; H)}^2 + M) \quad \forall j^* \leq j < J.
\]
Since \( H^{1/2} \) is continuously embedded into \( H \), we then obtain
\[
\| E_p^P \|_{H^{1/2}}^2 \leq c( \| E_p^P \|_{H^{1/2}}^2 + \| E_p^P \|_{L^2(t_j, T; H^{1/2})}^2 + M) \quad \forall j^* \leq j < J,
\]
and therefore using the discrete Gronwall’s inequality yields
\[
\max_{j^* \leq j \leq J} \| E_p^P \|_{H^{1/2}}^2 \leq c( \| E_p^P \|_{H^{1/2}}^2 + M),
\]
which, together with (48), leads to
\[
\max_{j^* \leq j < J} \| E_p^P \|_{H^{1/2}} + \| E^Z \|_{L^2(t_j^*, T; H)} \leq c \| E_p^P \|_{H^{1/2}} + c M^{1/2}.
\]
Hence, by the estimate \( \| E_p^P \|_{H^{1/2}} \leq c M^{1/2} \) (in fact \( E_p^P = 0 \)), we obtain
\[
\max_{j^* \leq j < J} \| E_p^P \|_{H^{1/2}} + \| E^Z \|_{L^2(t_j^*, T; H)} \leq c M^{1/2}. \tag{49}
\]

Step 4 Note that \( j/(J - j^*) \) is independent of \( \tau \). Repeating the argument in Steps 2 and 3 several times (not greater than \( j/(J - j^*) \)) proves
\[
\max_{0 \leq j < J} \| E_p^P \|_{H^{1/2}} + \| E^Z \|_{L^2(0, T; H)} \leq c M^{1/2}, \tag{50}
\]
which, together with (40) and the fact that \( H^{1/2} \) is continuously embedded into \( H \), yields the desired estimate (8). This completes the proof of Theorem 3.1.

Remark 3.3 Assume that \( (P, Z) \) is the solution to (5) and that \( f \) satisfies (i) and (ii) in Hypothesis 2.1. Using the techniques in the proof of Theorem 3.1, we can easily obtain the following stability estimate:
\[
\max_{0 \leq j \leq J} \| P_j \|_{H^{1/2}} + \| Z \|_{L^2(0, T; H)} \leq c \left( \| p_T \|_{H^{1/2}} + \| f(\cdot, 0, 0) \|_{L^2(0, T; H)} \right),
\]
provided that \( p_T \in H^{1/2} \). Moreover, we can use the estimate (50) and the stability estimate of \( \tilde{P} \) to further derive the stability estimate of \( P \) for \( p_T \in H \).
Remark 3.4 Following the proof of Theorem 3.1, we can easily prove Theorems 3.2 and 3.3 by Lemmas 3.2 and 3.3, respectively.

4 Application to a Stochastic Linear Quadratic Control Problem

4.1 Continuous Problem

We are concerned with the following stochastic linear quadratic control problem:

\[
\min_{u \in L^2_F(0, T; H)} \frac{1}{2} \| y - y_d \|^2_{L^2(0, T; H)} + \frac{\nu}{2} \| u \|^2_{L^2(0, T; H)},
\]

subject to the state equation

\[
\begin{cases}
  d y(t) = (A y + \alpha_0 y + \alpha_1 u)(t) \, dt + (\alpha_2 y + \alpha_3 u)(t) \, dW(t), & 0 \leq t \leq T, \\
  y(0) = 0,
\end{cases}
\]

where \(0 < \nu < \infty\), \(y_d \in L^2_F(0, T; H)\) and \(\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in L^2_F(0, T; \mathbb{R}) \cap L^\infty(\Omega \times (0, T))\).

It is standard that problem (51) admits a unique solution \(\bar{u}\). Let \(\bar{y}\) be the state with respect to the control \(\bar{u}\), and let \((\bar{p}, \bar{z})\) be the solution of the backward stochastic evolution equation

\[
\begin{cases}
  d \bar{p}(t) = -(A \bar{p} + \alpha_0 \bar{p} + \bar{y} - y_d + \alpha_2 \bar{z})(t) \, dt + \bar{z}(t) \, dW(t), & 0 \leq t \leq T, \\
  \bar{p}(T) = 0.
\end{cases}
\]

Applying the celebrated Itô’s formula to \([y(\cdot), \bar{p}(\cdot)]\) yields

\[
\int_0^T [(\bar{y} - y_d)(t), y(t)] \, dt = \int_0^T [(\alpha_1 \bar{p} + \alpha_3 \bar{z})(t), u(t)] \, dt
\]

for all \(u \in L^2_F(0, T; H)\), where \(y\) is the state with respect to the control \(u\). Using the above equality, we readily conclude the first-order optimality condition of problem (51):

\[
\bar{u} = -\nu^{-1}(\alpha_1 \bar{p} + \alpha_3 \bar{z}).
\]

Noting that \((\bar{p}, \bar{z})\) is the solution to (53), we have

\[
(\bar{p}, \bar{z}) \in \left( L^2_F(\Omega; C([0, T]; H^{1/2})) \cap L^2_F(0, T; H^1) \right) \times L^2_F(0, T; H^{1/2}).
\]
and so by (54) we get

\[ \tilde{u} \in L^2_F(0, T; H^{1/2}). \]

Since \( \tilde{y} \) is the state with respect to the control \( \tilde{u} \), we then obtain

\[ \tilde{y} \in L^2_F(\Omega; C([0, T]; H^{1/2})) \cap L^2_F(0, T; H^1). \] (56)

**Remark 4.1** The first-order optimality condition (54) follows from [2, 3]. For the theoretical analysis of the stochastic linear quadratic control problems in infinite dimensions, we refer the reader to [30] and the references therein.

### 4.2 Temporally Semi-discrete Problem

The temporally semi-discrete problem reads as follows:

\[
\min_{U \in \mathcal{X}_\tau} \frac{1}{2} \|Y - y_d\|_{L^2(0,T;H)}^2 + \frac{v}{2} \|U\|_{L^2(0,T;H)}^2,
\] (57)

subject to the discrete state equation

\[
\begin{cases}
Y_{j+1} - Y_j = \tau AY_{j+1} + \int_{t_j}^{t_{j+1}} (\alpha_0 Y + \alpha_1 U)(t) \, dt + \\
\int_{t_j}^{t_{j+1}} (\alpha_2 Y + \alpha_3 U)(t) \, dW(t), & 0 \leq j < J,
\end{cases}
\] (58)

where \( Y \in \mathcal{X}_\tau \). The main result of this section is the following error estimate.

**Theorem 4.1** Assume that \( y_d \in L^2_F(0, T; H) \). Let \( \tilde{u} \) and \( \tilde{U} \) be the solutions to problems (51) and (57), respectively. Then

\[
\|\tilde{u} - \tilde{U}\|_{L^2(0,T;H)} \leq c(\tau^{1/2} + \|(I - P_\tau)\tilde{u}\|_{L^2(0,T;H)}).
\] (59)

**Remark 4.2** Recently, Li and Xie [28] have analyzed a spatial semi-discretization for a stochastic linear quadratic control problem with general filtration. For a special case of problem (57), Li and Zhou [29] obtained the temporal accuracy \( O(\tau^{1/2}) \) for rough data. For other related works, we refer the reader to [9, 37, 38, 45].

The main task of the rest of this subsection is to prove the above theorem. To this end, we proceed as follows. For any \( v \in L^2_F(0, T; H) \), we use \( S_\tau v \) to denote the solution to discretization (58) with \( U \) being replaced by \( v \). A routine argument (see, e.g., [27, Theorem 3.14]) gives

\[
\max_{0 \leq j \leq J} \|(S_\tau v)_j\|_H \leq c\|v\|_{L^2(0,T;H)}.
\] (60)
For any \( P, Z \in \mathcal{X}_\tau \) and \( g, v \in L^2_\mathcal{P}(0, T; H) \), define
\[
\mathcal{S}(P, Z, g, v) := \sum_{j=0}^{J-1} \left( \int_{t_j}^{t_{j+1}} \left[ (\alpha_1 P_{j+1} + \alpha_3 Z)(t), v(t) \right] \, dt - \int_{t_j}^{t_{j+1}} (\alpha_0 P_{j+1} + g + \alpha_2 Z)(t) \, dt, \int_{t_j}^{t_{j+1}} (\alpha_2 S_\tau v + \alpha_3 v)(t) \, dW(t) \right). 
\]

In the sequel we will always assume
\[
\tau < \frac{1}{\|\alpha_2\|_L^\infty(\Omega \times (0,T))},
\]
to ensure that the later discretizations (62) and (67) each admit a unique solution (see the proof of Theorem 3.1). One form of the first-order optimality condition of problem (57) is as follows.

**Lemma 4.1** Assume that \( \bar{U} \) is the solution to problem (57). Let \((\bar{P}, \bar{Z}) \in \mathcal{X}_\tau \times L^2_\mathcal{P}(0, T; H)\) be the solution to the discretization
\[
\begin{aligned}
\bar{P}_j &= 0, \\
\bar{P}_j - \bar{P}_{j+1} &= \tau A \bar{P}_j + \int_{t_j}^{t_{j+1}} (\alpha_0 \bar{P}_{j+1} + S_\tau \bar{U} - y_d + \alpha_2 \bar{Z})(t) \, dt \\
&\quad - \int_{t_j}^{t_{j+1}} \bar{Z}(t) \, dW(t), \quad 0 \leq j < J.
\end{aligned}
\]

Then
\[
\nu \int_0^T [\bar{U}(t), U(t)] \, dt + \mathcal{S}(\bar{P}, \bar{Z}, S_\tau \bar{U} - y_d, U) = 0 \quad \forall U \in \mathcal{X}_\tau.
\]

**Proof** Following the proof of [29, Lemma 4.19], we can easily obtain
\[
\int_0^T \left[ (S_\tau \bar{U} - y_d)(t), (S_\tau v)(t) \right] \, dt = \mathcal{S}(\bar{P}, \bar{Z}, S_\tau \bar{U} - y_d, v)
\]
for all \( v \in L^2_\mathcal{P}(0, T; H) \). By this equality, a straightforward calculation yields (63). \(\square\)

**Remark 4.3** Note that (62) is not a natural adjoint equation of the discrete state Eq. (58), and hence the first-order optimality condition (63) is unusual. We can also use the temporal semi-discretizations (5) and (6) to form the first-order optimality condition of problem (63); however, we observe that the temporal semi-discretization (7) appears to be more suitable for the numerical analysis of problem (57).

**Lemma 4.2** Let \( \bar{u} \) be the solution to (51), and let \( \bar{y} \) be the state with respect to \( \bar{u} \). Then
\[
\|\|\bar{y} - S_\tau \bar{u}\|\|_{L^2(0,T;H)} \leq c \tau^{1/2}.
\]
**Proof** Fix $0 \leq j < J$. By definition we have
\[ d\bar{y}(t) = (A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) \, dt + (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) \, dW(t), \quad 0 \leq t \leq T, \]
so that
\[ \bar{y}(t) - \bar{y}(t_j) = \int_{t_j}^{t} (A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) \, dt + \int_{t_j}^{t} (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) \, dW(t), \quad t_j \leq t \leq T. \]
It follows that for any $t_j \leq t \leq t_{j+1}$,
\[
\left\| \bar{y}(t) - \bar{y}(t_j) \right\|_H^2 \\
\leq 2\left\| \int_{t_j}^{t} (A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) \, dt \right\|_H^2 + 2\left\| \int_{t_j}^{t} (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) \, dW(t) \right\|_H^2 \\
= 2\left\| \int_{t_j}^{t} (A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) \, dt \right\|_H^2 + 2\int_{t_j}^{t} \left\| (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) \right\|_H^4 \, dt \\
\leq 2(t - t_j) \int_{t_j}^{t} \left\| (A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) \right\|_H^2 \, dt + 2\int_{t_j}^{t} \left\| (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) \right\|_H^2 \, dt,
\]
which implies
\[
\left\| \bar{y} - \bar{y}(t_j) \right\|_{L^2(t_j, t_{j+1}; H)}^2 \\
\leq \tau^2 \left\| A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u} \right\|_{L^2(t_j, t_{j+1}; H)}^2 + 2\tau \left\| \alpha_2 \bar{y} + \alpha_3 \bar{u} \right\|_{L^2(t_j, t_{j+1}; H)}^2.
\]
Hence,
\[
\sum_{j=0}^{J-1} \left\| \bar{y} - \bar{y}(t_j) \right\|_{L^2(t_j, t_{j+1}; H)}^2 \\
\leq \tau^2 \left\| A\bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u} \right\|_{L^2(0, T; H)}^2 + 2\tau \left\| \alpha_2 \bar{y} + \alpha_3 \bar{u} \right\|_{L^2(0, T; H)}^2.
\]
By (56) and the fact $\bar{u} \in L^2_p(0, T; H)$, we then obtain
\[
\sum_{j=0}^{J-1} \left\| \bar{y} - \bar{y}(t_j) \right\|_{L^2(t_j, t_{j+1}; H)}^2 \leq c\tau,
\]
so that the desired estimate (65) follows from
\[
\max_{0 \leq j < J} \left\| \bar{y}(t_j) - (S_T \bar{u})_j \right\|_H \leq c\tau^{1/2} \quad \text{(see [27, Theorem 3.14])}.
\]
This completes the proof. \qed
Finally, we are in a position to prove Theorem 4.1 as follows. 

**Proof of Theorem 4.1.** Let $\bar{y}$ be the state with respect to the control $\bar{u}$, and let $(\bar{p}, \bar{z})$ be the solution to Eq. (53). Similar to (33), we have

\[
\left( \sum_{j=0}^{J-1} \left\| \bar{p} - \bar{p}(t_{j+1}) \right\|_{L^2(t_j, t_{j+1}; H)}^2 \right)^{1/2} \leq c\tau^{1/2}. \tag{66}
\]

We divide the rest of the proof into the following four steps.

**Step 1** Let $(P, Z) \in \mathcal{X}_\tau \times L^2_{\mathbb{P}}(0, T; H)$ be the solution to the discretization

\[
\begin{cases}
P_J = 0, \\
P_j - P_{j+1} = \tau A P_j + \int_{t_j}^{t_{j+1}} \left( \alpha_0 P_j + \bar{y} - y_d + \alpha_2 Z(t) \right) \, dt - \int_{t_j}^{t_{j+1}} Z(t) \, dW(t), \quad 0 \leq j < J.
\end{cases} \tag{67}
\]

In view of (56) and the fact $y_d \in L^2_{\mathbb{P}}(0, T; H)$, we can use Theorem 3.3 to conclude that

\[
\max_{0 \leq j \leq J} \left\| \bar{p}(t_j) - P_j \right\|_H + \| \bar{z} - Z \|_{L^2(0, T; H)} \leq c\tau^{1/2}, \tag{68}
\]

which, together with (66), yields

\[
\left( \sum_{j=0}^{J-1} \left\| \bar{p} - P_{j+1} \right\|_{L^2(t_j, t_{j+1}; H)}^2 \right)^{1/2} \\
\leq \left( \sum_{j=0}^{J-1} \left\| \bar{p} - \bar{p}(t_{j+1}) \right\|_{L^2(t_j, t_{j+1}; H)}^2 \right)^{1/2} + \left( \sum_{j=0}^{J-1} \| P_{j+1} - \bar{p}(t_{j+1}) \|_{L^2(t_j, t_{j+1}; H)}^2 \right)^{1/2} \\
\leq c\tau^{1/2}. \tag{69}
\]

In addition, from (68) and (55) we conclude that

\[
\left( \tau \sum_{j=0}^{J-1} \| P_{j+1} \|_H^2 \right)^{1/2} + \| Z \|_{L^2(0, T; H)} \leq c. \tag{70}
\]

**Step 2** Let us prove

\[
v \left\| \bar{u} - \bar{U} \right\|_{L^2(0, T; H)}^2 \leq c\tau + c \| (I - \mathcal{P}_\tau) \bar{u} \|_{L^2(0, T; H)}^2 + I_1 + I_2 + I_3 + I_4. \tag{71}
\]
The basic idea is standard (see, e.g., [19, Theorem 3.4]). We first present three equalities. Inserting (74) and similarly we have

\[ I_1 := \sum_{j=0}^{j+1} \left[ \alpha_1(t)(P_{j+1} - \bar{P}(t)), (P_{j+1} - \bar{U}(t)) \right] dt, \]

\[ I_2 := \int_0^T \left[ (\alpha_3 Z - \alpha_3 \bar{Z})(t), (P_{j+1} - \bar{U}(t)) \right] dt, \]

\[ I_3 := -\sum_{j=0}^{j+1} \left[ \int_{t}^{T} (\alpha_0 P_{j+1} + \bar{y}_d + \alpha_2 Z)(t) dt, \int_{t}^{T} (\alpha_2 S(t(P_{j+1} - \bar{U}) + \alpha_3 (P_{j+1} - \bar{U}))(t)dW(t) \right], \]

\[ I_4 := -\int_0^T \left[ (\alpha_1 \tilde{\bar{P}} + \alpha_3 \bar{Z})(t), (\bar{u} - P_{j+1} - \bar{U}(t)) \right] dt. \]

The basic idea is standard (see, e.g., [19, Theorem 3.4]). We first present three equalities. Inserting \( v := P_{j+1} - \bar{U} \) into (64) gives

\[ \int_0^T \left[ (S, \bar{U} - y_d(t), (S, (P_{j+1} - \bar{U}))(t) \right] dt = J(\bar{P}, \bar{Z}, S, \bar{U} - y_d, P_{j+1} - \bar{U}), \]

and similarly we have

\[ \int_0^T \left[ (\bar{y} - y_d(t), (S, (P_{j+1} - \bar{U}))(t) \right] dt = J(P, \bar{Z}, S, \bar{U} - y_d, P_{j+1} - \bar{U}). \]

By definition, it is easily verified that

\[ J(P, \bar{Z}, S, \bar{U} - y_d, P_{j+1} - \bar{U}) - \int_0^T \left[ (\alpha_1 \tilde{\bar{P}} + \alpha_3 \bar{Z})(t), (\bar{u} - \bar{U}(t)) \right] dt = I_1 + I_2 + I_3 + I_4. \]

Next, by (54) we have

\[ v \int_0^T \left[ \bar{u}(t), (\bar{u} - \bar{U})(t) \right] dt = -\int_0^T \left[ (\alpha_1 \tilde{\bar{P}} + \alpha_3 \bar{Z})(t), (\bar{u} - \bar{U}(t)) \right] dt, \]

and inserting \( U := P_{j+1} - \bar{U} \) into (63) gives

\[ -v \int_0^T \left[ \bar{U}(t), (\bar{u} - \bar{U})(t) \right] dt = J(\bar{P}, \bar{Z}, S, \bar{U} - y_d, P_{j+1} - \bar{U}). \]

Summing up the above two equalities yields

\[ v \|\bar{u} - \bar{U}\|_{L^2(0,T;H)}^2 \]

\[ = -\int_0^T \left[ (\alpha_1 \tilde{\bar{P}} + \alpha_3 \bar{Z})(t), (\bar{u} - \bar{U})(t) \right] dt + J(\bar{P}, \bar{Z}, S, \bar{U} - y_d, P_{j+1} - \bar{U}). \]
Hence, the desired estimate (71) follows from

\[ \int_0^T [(S_T \bar{U} - \bar{y})(t), (S_T (\mathcal{P}_T \bar{u} - \bar{U}))(t)] \, dt \]

\[ \leq - \frac{1}{2} \| \bar{y} - S_T \bar{U} \|^2_{L^2(0,T;H)} + \frac{1}{2} \| S_T (\mathcal{P}_T \bar{u} - \bar{y}) \|^2_{L^2(0,T;H)} \]

\[ \leq \frac{1}{2} \| \bar{y} - S_T \mathcal{P}_T \bar{u} \|^2_{L^2(0,T;H)} \]

\[ \leq c \tau + c \| (I - \mathcal{P}_T) \bar{u} \|^2_{L^2(0,T;H)} \]

(by (67) and (62)).

**Step 3** Let us estimate $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_3$ and $\mathbb{I}_4$. For $\mathbb{I}_1$, by (69) we have

\[ \mathbb{I}_1 \leq \left( \sum_{j=0}^{J-1} \| \bar{p} - P_{j+1} \|^2_{L^2(t_j,t_{j+1};H)} \right)^{1/2} \| \mathcal{P}_T \bar{u} - \bar{U} \|^2_{L^2(0,T;H)} \]

\[ \leq c \tau^{1/2} \| \mathcal{P}_T \bar{u} - \bar{U} \|^2_{L^2(0,T;H)} . \]

For $\mathbb{I}_2$ we have

\[ \mathbb{I}_2 \leq c \| \bar{z} - Z \|^2_{L^2(0,T;H)} \| \mathcal{P}_T \bar{u} - \bar{U} \|^2_{L^2(0,T;H)} \]

\[ \leq c \tau^{1/2} \| \mathcal{P}_T \bar{u} - \bar{U} \|^2_{L^2(0,T;H)} \] (by (70)).

For $\mathbb{I}_3$ we have

\[ \mathbb{I}_3 \leq \sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} \left[ (a_0 P_{j+1} + \bar{y} - y_d + \alpha_2 Z)(t) \right] dt \| \mathcal{P}_T \bar{u} - \bar{U} \|^2_{L^2(0,T;H)} \]

\[ = \sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} \left[ (a_0 P_{j+1} + \bar{y} - y_d + \alpha_2 Z)(t) \right] dt \| \alpha_2 S_T (\mathcal{P}_T \bar{u} - \bar{U}) + \alpha_3 (\mathcal{P}_T \bar{u} - \bar{U}) \|^2_{L^2(t_j,t_{j+1};H)} \]
which implies the desired estimate (59). This completes the proof of Theorem 4.1.

Assume that

Remark 4.4

We can then apply the Young’s inequality with \( \varepsilon \) to obtain

\[
\nu \left\| \tilde{u} - \tilde{U} \right\|^2_{L^2(0,T;H)} \leq c\tau + c \left\| (I - \mathcal{P}_\tau) \tilde{u} \right\|^2_{L^2(0,T;H)},
\]

which implies the desired estimate (59). This completes the proof of Theorem 4.1. \( \square \)

**Remark 4.4** Assume that \( \alpha_3 = 0 \). We can estimate \( \mathbb{I}_4 \) as follows:

\[
\mathbb{I}_4 = \nu^{-1} \int_0^T \left[ (\alpha_1 \tilde{p})(t), (\alpha_1 \tilde{p} - \mathcal{P}_\tau (\alpha_1 \tilde{p}))(t) \right] \, dt \quad \text{(by (56))}
\]

\[
= \nu^{-1} \left\| (I - \mathcal{P}_\tau) (\alpha_1 \tilde{p}) \right\|^2_{L^2(0,T;H)}.
\]
Under the condition that $\alpha_1$ is sufficiently smooth in time, say $\alpha_1 \equiv 1$, we then obtain

$$I_4 = v^{-1} \sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} \| \tilde{p}(t) - (P_\tau \tilde{p})(t) \|^2_H \, dt$$

$$\leq v^{-1} \sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} \| \tilde{p}(t) - \tilde{p}(t_j) \|^2_H \, dt \quad \text{(by the definition of $P_\tau$)}$$

$$\leq c\tau \quad \text{(similar to (35))}.$$  

Using this estimate and the estimates of $I_1, I_2$ and $I_3$ in Step 3, we can finally conclude that

$$\| \tilde{u} - \tilde{U} \|_{L^2(0,T;H)} \leq c\tau^{1/2}.$$  

5 Conclusions

In this paper, we have analyzed three Euler type temporal semi-discretizations for a backward semilinear stochastic evolution equation with Lipschitz nonlinearity. With reasonable regularity assumptions on the data, we have established the convergence for the first two semi-discretizations and derived an explicit convergence rate for the third semi-discretization. In the numerical analysis, no regularity restriction is imposed on the solution, the coefficient is not necessarily deterministic, and the terminal value is not necessarily generated by a forward stochastic evolution equation. We have applied the third temporal semi-discretization to a general stochastic linear quadratic control problem and established the convergence for a temporally semi-discrete approximation of the optimal control.

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Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.

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