Results for the heat transfer of a fin with exponential-law temperature-dependent thermal conductivity and power-law temperature-dependent heat transfer coefficients

1 Preliminaries and problem formulation

It is noticeable that thermal studying of both solid and porous fins with regard to the differences in profiles and thermo-physical properties have been mainly focused by researchers [1,2]. In many engineering applications such as conveying flow of electricity through a conductor, nuclear rods and many other heating accessories for thermal producers, fins should be considered where conductive rate at temperature makes the model nonlinear and exponentially challenging to reach exact solution [3–11]. Kern and Kraus [12] represented its extensive surfaces and industrial applications. Also, it is difficult to obtain the accurate closed form solutions of these kinds of nonlinear problems especially when, heat transfer and thermal conductivity factors are variable and large temperature differences exist. Numerous numerical techniques and analytical methods have been carried out to solve these problems. Aziz and Hug [13] and Benzie [14] are pioneers in solving such problems, but when the factors vary linearly, techniques of perturbation ideas were applied. The differential equation and boundary condition of a fin with linear temperature-dependent heat transfer coefficient are in the following form [15]:

\[ (1 + \beta \theta) \frac{d^2 \theta}{dx^2} - M^2 \theta^{n+1} + \beta \left( \frac{d \theta}{dx} \right)^2 = 0, \]

BCs: \( \theta(0) = 0, \theta(1) = 1 \).

This problem has been solved in the case of fixed heat transfer coefficient and thermal conductivity is changing linearly with respect to the temperature \( n = 0 \) in Eq. (1) by using semi-analytical methods such as polynomial method, homotopy perturbation method (HPM), homotopy analysis method (HAM), differential transform method (DTM) and adomian decomposition method (ADM) [16–21]. Khani et al. [15] investigated the solutions of HPM, ADM...
and HAM when \( M \) rises to a large number. Lesnic and Heggies [22], Chang [23] and Chowdhury et al. [24] studied this equation for a fixed thermal conductivity \((\beta = 0)\) in Eq. (1) using DM, ADM, HPM and HAM. Moreover, a lot of research works related to the problem Eq. (1) have been reported by focusing on its different aspects with different techniques, see refs [23–31] and references therein.

In this article, we study this nonlinear fin, but by considering exponential-law temperature-dependent for the factor of thermal conductivity. The problem on desk is presented as:

\[
\frac{d}{dx} \left( k(T) \frac{dT}{dx} \right) - \frac{h(T)P}{A} (T - T_f) = 0, \quad 0 \leq x \leq L, \tag{2}
\]

\[
\frac{dT}{dx} (0) = 0, \quad T(L) = T_b, \tag{3}
\]

where \( L \) is the length of fin, \( P \) is its perimeter and \( T_b \) is the base temperature. Among the different types of boundary conditions, Dirichlet condition, Neumann condition and the Robin condition, we assumed Neumann condition which means it lacks heat transfer at the tip of the fin. Also, Dirichlet condition prescribes temperature \( T_b \) to the base of the fin.

It is important to emphasize that we consider the exponential-law temperature-dependent thermal conductivity in this work and also, as in other nonlinear models for heat transfer of the fin, the power-law temperature-dependent heat transfer factor is assumed, in other words, we have

\[
K(T) = K_0 \left[ \frac{\exp\left(\beta \frac{T - T_a}{T_b - T_a}\right) - 1}{\exp(\beta) - 1} \right], \quad \beta > 0, \tag{4}
\]

\[
h(T) = h_b \left( \frac{T - T_a}{T_b - T_a} \right)^n, \tag{5}
\]

where \( K_0 \) and \( h_b \) are their coefficients at the base temperature, respectively. The exponent \( n \) in Eq. (4) explains the mode of heat transfer. They are usually \( \frac{1}{4}, \frac{1}{2}, \frac{1}{2} \), \( 2, 3 \) \( 5, 25 \). Dimensionless parameters are as follows:

\[
M^2 = \frac{h_b P L^2}{K_0 A}, \quad x = \frac{X}{L}, \quad \theta = \frac{T - T_a}{T_b - T_a} \tag{6}
\]

According to Eq. (1), the problem and its boundary conditions in dimensionless form could be rewritten as follows:

\[
\frac{d}{dx} \left[ (\exp(\beta \theta) - 1) \frac{d\theta}{dx} \right] - M^2 \theta^{n+1} = 0, \quad 0 \leq x \leq 1 \tag{7}
\]

or equivalently

\[
\beta \exp(\beta \theta) \left( \frac{d\theta}{dx} \right)^2 + \frac{d^2\theta}{dx^2} (\exp(\beta \theta) - 1) - M^2 \theta^{n+1} = 0, \tag{8}
\]

\[
0 \leq x \leq 1
\]

\[
\frac{d\theta}{dx} (0) = 0, \quad \theta(1) = 1. \tag{9}
\]

Generally, there are many numerical and semi-analytical methods to deal with the boundary value problems arisen from the heat transfer of a fin and other kinds of problems such as MHD flow of Newtonian and non-Newtonian fluid. In ref. [32], homotopy analysis method has been applied to analyze concentration flux dependent on radiative MHD Casson flow with Arrhenius activation energy. Radiative bioconvection nanofluid squeezing flow has been discussed by a semi-numerical study with the DTM-Padé approach [33]. Abbas et al. [34] considered artificial neural networks for parametric analysis and minimization of entropy generation in bioinspired magnetized non-Newtonian nanofluid pumping. Also, readers are referred to some related works refs [35,36]. On the other hands, there are some valuable studies which present exact closed-form solutions for some of these models in some especial cases [37–40]. The other main aim we seek in this work is to provide exact closed-form solution for problems (8)–(9).

## 2 Accurate closed form solution

We have the following relation by changing variable \( u = \frac{d\theta}{dx} \):

\[
\frac{d^2\theta}{dx^2} = \frac{du}{dx} \cdot \frac{du}{d\theta} = u \cdot \frac{du}{d\theta}. \tag{10}
\]

Therefore, Eq. (8) is changed to the following equation:

\[
(\exp(\beta \theta) - 1) u u_x + (\beta \exp(\beta \theta) u^2 - M^2 \theta^{n+1}) d\theta = 0. \tag{11}
\]

This equation can be modified to a differentiable one by multiplying each side by \( \exp(\beta \theta) - 1 \), i.e.,

\[
(\exp(\beta \theta) - 1)^2 u u_x + (\exp(\beta \theta) - 1)(\beta \exp(\beta \theta) u^2 - M^2 \theta^{n+1}) d\theta = 0. \tag{12}
\]

Now, we look for a function such that the derivatives with respect to \( u \) and \( \theta \) be

\[
(\exp(\beta \theta) - 1)^2 u \quad \text{and} \quad (\exp(\beta \theta) - 1)(\beta \exp(\beta \theta) u^2 - M^2 \theta^{n+1}),
\]
respectively. Then the solution is easily obtained as:

\[ M\theta^{n+2}(-\beta\theta)^{n-2}\Gamma(n + 2, -\beta\theta) + \frac{M\theta^{n+2}}{n + 2} - \frac{d}{dx}e^{\theta}\theta = C, \quad (13) \]

where \( C \) is the integral constant, after replacing \( u \) by \( \frac{d\theta}{dx} \). Eq. (13) is converted into the following equation:

\[ M\theta^{n+2}(-\beta\theta)^{n-2}\Gamma(n + 2, -\beta\theta) + \frac{M\theta^{n+2}}{n + 2} - \left( \frac{d}{dx} \right)^2e^{\theta}\theta = C, \quad (14) \]

where parameter \( C \) is reachable by the first boundary condition as follows:

\[ C = M\theta_0^{n+2}(-\beta\theta_0)^{n-2}\Gamma(n + 2, -\beta\theta_0) + \frac{M\theta_0^{n+2}}{n + 2}. \]

That \( \theta_0 = \theta(0) \) is the dimensionless temperature of the fin at the tip, by interchanging \( C \) into, we have:

\[ -\left( \frac{d}{dx} \right)^2e^{\theta}\theta + \frac{1}{2} \left( \frac{d}{dx} \right)^2e^{\theta}\theta + \frac{1}{2} \left( \frac{d}{dx} \right)^2 = M\theta_0^{n+2}(-\beta\theta_0)^{n-2}\Gamma(n + 2, -\beta\theta_0) + \frac{M\theta_0^{n+2}}{n + 2}. \]

or equally,

\[ \frac{1}{2}(\exp(\beta\theta) - 1)\left( \frac{d}{dx} \right)^2 = \frac{M^2}{(-\beta)^{n+2}} \left( \Gamma(n + 2, -\beta\theta_0) - \Gamma(n + 2, -\beta\theta) \right) + \frac{M^2}{n + 2}(\theta_0^{n+2} - \theta^{n+2}), \quad (16) \]

where the function \( \Gamma(a, z) \) is the incomplete gamma function which is defined by the integral

\[ \Gamma(a, z) = \int_z^\infty \exp(-t)dt. \]

Eq. (16) can be represented as

\[ dx = \frac{(\exp(\beta\theta) - 1)d\theta}{\sqrt{2}M\sqrt{\frac{\Gamma(n + 2, -\beta\theta_0) - \Gamma(n + 2, -\beta\theta)}{(-\beta)^{n+2}} + \frac{\theta_0^{n+2}}{n + 2} - \frac{\theta^{n+2}}{n + 2}}}. \quad (17) \]

After integration from both sides of Eq. (17) and imposition of the notation, we have

\[ \sqrt{2}Mx = \int_0^\theta \frac{\exp(\beta z) - 1}{\sqrt{\frac{\Gamma(n + 2, -\beta\theta_0) - \Gamma(n + 2, -\beta\theta)}{(-\beta)^{n+2}} + \frac{\theta_0^{n+2}}{n + 2} - \frac{\theta^{n+2}}{n + 2}}}dz. \quad (18) \]

In order to deal with Eq. (19) easily, let us define the right hand side as a new non-algebraic function of definite integral:

\[ F(\theta; \theta_0, n, \beta) = \int_{\theta_0}^\theta \frac{(\exp(\beta z) - 1)dz}{\sqrt{\frac{\Gamma(n + 2, -\beta\theta_0) - \Gamma(n + 2, -\beta\theta)}{(-\beta)^{n+2}} + \frac{\theta_0^{n+2}}{n + 2} - \frac{\theta^{n+2}}{n + 2}}}. \quad (19) \]

The function \( F(\theta; \theta_0, n, \beta) \) can be treated as identical to the other familiar functions by current powerful computer software such as Maple and Mathematica. Then, we can obviously rewrite the solution as the following form:

\[ \sqrt{2}Mx = F(\theta; \theta_0, n, \beta) \]

Or the other hand, \( \theta_0 \) is an unknown parameter in Eq. (20). But, it can be disclosed indeed by:

\[ \sqrt{2}M = F(1; \theta_0, n, \beta). \quad (21) \]

Now, the exact closed form solution is represented by Eq. (20), when \( \theta_0 \) is determined through Eq. (21) for any given \( M, n \) and \( \beta \). There would be no difficulty to work with non-algebraic function \( F(\theta; \theta_0, n, \beta) \) as the same as other known function for everyone familiar with the software programs. It is important to announce that the multiplicity of solution to the root \( \theta_0 \) in solving the nonlinear Eq. (21) proves the existence of the multiplicity of the solutions to the problem Eq. (8).

### 3 Fin efficiency and effectiveness

According to ref. [31], fin efficiency is the ratio of the real heat transfer rate to the ideal heat transfer rate if the entire fin were at the base temperature,

\[ \eta = \frac{Q_{\text{actual}}}{Q_{\text{ideal}}} = \int_0^1 \frac{ph(T)(T - T_0)dX}{pLh(T_0 - T_0)} = \int_0^1 \theta^{n+1}(x)dx, \quad (22) \]

hence

\[ \eta = \int_0^1 \{F^{-1}(\sqrt{2}Mx; \theta_0, n, \beta)\}^{n+1}dx, \quad (23) \]

where \( F^{-1} \) is the inverse function of \( F \). Also, the rate of heat transferred by a fin to the rate of heat transferred without the fin is fin effectiveness that is given, in dimensionless form, by

\[ \epsilon = \omega \int_0^1 \theta^{n+1}(x)dx = \omega \int_0^1 \{F^{-1}(\sqrt{2}Mx; \theta_0, n, \beta)\}^{n+1}dx, \quad (24) \]

where \( \omega \) is the fin length to the fin thickness ratio.
4 Main results

In the previous sections, exact closed form solution of the nonlinear fin problem formulated by Eqs. (8) and (9) has been developed and represented by the form of Eq. (20) and augmented to Eq. (21). The implicit solution Eq. (20) can be easily obtained by computer’s software mentioned before, we have used Mathematica in this article.

As it can be shown, Figure 1 illustrates the effect of fin parameter $M$ on the temperature. With regard to Eq. (5), it can be proven that when fin parameter increases, the mean tip end temperature and the mean temperature decline. At the end of the fin where $x = 0$, when the ratio $h_b/k_b$ increases, the temperature along the fin has lower figure. The inverse condition would happen if this ratio decreases.

Temperature distribution along the fin has been shown in Figure 2 for different values of $\beta$ for $n = 1/3$ and $M = 1$. In Figure 3, temperature distribution has been drawn for different values of $n$ for $\beta = -1/2$ and $M = 2$. Furthermore, Figure 4 stands for temperature distribution when both $n$ and $\beta$ change simultaneously.

![Figure 1: Diagram of $\theta(x)$ versus $x$ for temperature distributions with $n = \beta = \frac{1}{3}$.](image1)

![Figure 2: Diagram of $\theta(x)$ versus $x$ for temperature distributions with $M = 1$ and $n = \frac{1}{3}$.](image2)

![Figure 3: Diagram of $\theta(x)$ versus $x$ for temperature distributions with $M = 1$ and $\beta = \frac{1}{2}$.](image3)

![Figure 4: Diagram of $\theta(x)$ versus $x$ for temperature distributions with $M = \frac{1}{2}$.](image4)

![Figure 5: Diagram of fin efficiency versus $M$ for different $n$ with $\beta = \frac{1}{2}$.](image5)
with $M = \frac{1}{2}$. The effect of different values of thermo-geometric parameter on fin efficiency is shown in Figure 5. Regarding correlation between the mean temperature and fin efficiency, both of them have identical treatment.

5 Conclusion

The present study solves the nonlinear fin problem with exponentially temperature-dependent thermal conductivity exactly and presents exact analytical solution of the problem in implicit form. To this aim, we have reduced the order of differential equation and then converted into a total differential equation by multiplying a proper integration operator, after that we resolved it by imposing boundary conditions. The problem has been assumed that transfer coefficient is power-law temperature dependent. Depending on different values of the parameters of the model $n, M$ and $\beta$, we have extracted at least one solution for the considered problem. The existence of multiple solutions can be a new research line for future works. Furthermore, the exact analytical expression for fin efficiency has been obtained and illustrated graphically.

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References

[1] Gorla RS, Darvishi MT, Khani F. Effect of variable thermal conductivity on natural convection and radiation in porous. Thermal Energy Power Eng. 2013;2:79–85.

[2] Moradi A, Hayat T, Alsaedi A. Convection-radiation thermal analysis of triangular porous fins with temperature-dependent thermal conductivity by DTM. Energy Convers Manag. 2014 Jan 1;77:70–7.

[3] Mao A, Luo J, Li Y, Wang R, Li G, Guo Y. Engineering design of thermal quality clothing on a simulation-based and lifestyle-oriented CAD system. Eng Comput. 2011 Oct 1;27(4):405.

[4] Sobamowo MG, Kamiyo OM, Adeleye OA. Thermal performance analysis of a natural convection porous fin with temperature-dependent thermal conductivity and internal heat generation. Thermal Sci Eng Progress. 2017 Mar 1;1:39–52.

[5] Kim S, Huang CH. A series solution of the non-linear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient. J Phys D Appl Phys. 2007 Apr 19;40(9):2979.

[6] Khani F, Raji MA, Nejad HH. Analytical solutions and efficiency of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient. Commun Nonlinear Sci Numer Simul. 2009 Aug 1;14(8):3327–38.

[7] Ganji DD. The application of He’s homotopy perturbation method to nonlinear equations arising in heat transfer. Phys Lett A. 2006 Jul 10;355(4–5):337–41.

[8] Tari H, Ganji DD, Babazadeh H. The application of He’s variational iteration method to nonlinear equations arising in heat transfer. Phys Lett A. 2007 Mar 26;363(3):213–7.

[9] Heemserkj JP, Van Kuik FG, Knaap HF, Beenakker JJ. The thermal conductivity of gases in a magnetic field: The temperature dependence. Physica. 1974 Feb 1;71(3):484–514.

[10] Neek-Amal M, Moussavi R, Sepangi HR. Monte Carlo simulation of size effects on thermal conductivity in a two-dimensional Ising system. Phys A Statist Mech Appl. 2006 Nov 15;371(2):424–32.

[11] Mahmoud MA. Thermal radiation effects on MHD flow of a micropolar fluid over a stretching surface with variable thermal conductivity. Phys A Statist Mech Appl. 2007 Mar 1;375(2):401–10.

[12] Kern DQ, Kraus AD. Extended surface heat transfer. New York: McGrawHill; 1972.

[13] Aziz A, Hug SME. Perturbation solution for convecting fin with variable thermal conductivity. J Heat Transf Trans ASME. 1975;97:300–1.

[14] Aziz A, Hug E. Perturbation solution for convecting fin with variable thermal conductivity. J Heat Trans. 1995;97:300–10.

[15] Khani F, Raji MA, Nejad HH. Analytical solutions and efficiency of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient. Communications in Nonlinear Science and Numerical Simulation. 2009 Aug 1;14(8):3327–38.

[16] Ganji DD, Hosseini MJ, Shayegh J. Some nonlinear heat transfer equations solved by three approximate methods. Int Commun Heat Mass Transf. 2007 Oct 1;34(8):1003–16.

[17] Khani F, Raji MA, Hamedi-Nezhad S. A series solution of the fin problem with a temperature-dependent thermal conductivity. Commun Nonlinear Sci Numer Simulat. 2009 Jul 1;14(7):3007–17.

[18] Ganji DD, Afrouzi GA, Talarpshiti RA. Application of variational iteration method and homotopy-perturbation method for nonlinear heat diffusion and heat transfer equations. Phys Lett A. 2007 Sep 3;368(6):450–7.

[19] Domairry G, Fazeli M. Homotopy analysis method to determine the fin efficiency of convective straight fins with temperature-dependent thermal conductivity. Commun Nonlinear Sci Numer Simulat. 2009 Feb 1;14(2):489–99.
[20] Joneidi AA, Ganji DD, Babaeahi M. Differential transformation method to determine fin efficiency of convective straight fins with temperature dependent thermal conductivity. Int Commun Heat Mass Transf. 2009 Aug 1;36(7):757–62.

[21] Arslanurk C. A decomposition method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity. Int Commun Heat Mass Transf. 2005 May 1;32(6):831–41.

[22] Lesnic D, Heggs PJ. A decomposition method for power-law fin-type problems. Int Commun Heat Mass Transf. 2004 Jul 1;31(5):673–82.

[23] Chang MH. A decomposition solution for fins with temperature dependent surface heat flux. Int J Heat Mass Transf. 2005 Apr 1;48(9):1819–24.

[24] Chowdhury MS, Hashim I, Abdualziz O. Comparison of homotopy analysis method and homotopy-perturbation method for purely nonlinear fin-type problems. Commun Nonlinear Sci Numer Simulat. 2009 Feb 1;14(2):371–8.

[25] Moitsheki RJ, Hayat T, Malik MY. Some exact solutions of the fin problem with a power law temperature-dependent thermal conductivity. Nonlinear Analysis: Real World Applications. 2010 Oct 1;11(5):3287–94.

[26] Ndlouv PL, Moitsheki RJ. Analytical solutions for steady heat transfer in longitudinal fins with temperature-dependent properties. Math Problems Eng. 2013;2013:14. Article ID: 273052.

[27] Abbabsbandy S. The application of homotopy analysis method to nonlinear equations arising in heat transfer. Phys Lett A. 2006 Dec 18;360(1):109–13.

[28] Abbabsbandy S, Shivanian E. Exact analytical solution of a nonlinear equation arising in heat transfer. Phys Lett A. 2010 Jan 11;374(4):567–74.

[29] Shivanian E, Campo A. Exact, analytic temperature distributions of pin fins with constant thermal conductivity and power law type heat transfer coefficient. Heat Transf Asian Res. 2018 Jan;47(1):42–53.

[30] Anbarloei M, Shivanian E. Exact closed-form solution of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient. J Heat Transf. 2016;138:1–6. doi: 10.1115/1.4033809.

[31] Mosayebidorcheh S, Ganji DD, Farzinpoor M. Approximate solution of the nonlinear heat transfer equation of a fin with the power-law temperature-dependent thermal conductivity and heat transfer coefficient. Propulsion Power Res. 2014 Mar 1;3(1):41–7.

[32] Kohilavani Naganthran AZ, Basir MF, Shehzad N, Nazar R, Choudhary R, Balaji S. Concentration flux dependent on radiative MHD Casson flow with Arrhenius activation energy: homotopy analysis method (HAM) with an evolutionary algorithm. Int J Heat Technol. 2020 Dec;38(4):785–93.

[33] Zeeshan A, Arain MB, Bhatti MM, Alzahrani F, Bég OA. Radiative bioconvection nanofluid squeezing flow between rotating circular plates: Semi-numerical study with the DTM-Padé approach. Modern Phys Lett B. 2021 Dec 13;13:2150552.

[34] Abbas MA, Bég OA, Zeeshan A, Hobiny A, Bhatti MM. Parametric analysis and minimization of entropy generation in bioinspired magnetized non-Newtonian nanofluid pumping using artificial neural networks and particle swarm optimization. Thermal Sci Eng Progress. 2021 Aug 1;24:100930.

[35] Bhatti MM, Arain MB, Zeeshan A, Ellahi R, Doranehgard MH. Swimming of Gyrotactic Microorganism in MHD Williamson nanofluid flow between rotating circular plates embedded in porous medium: application of thermal energy storage. J Energy Storage. 2016;103511. doi: 10.1016/j.est.2021.103511.

[36] Bhatti MM, Zeeshan A, Bashir F, Sait SM, Ellahi R. Sinusoidal motion of small particles through a Darcy-Brinkman-Forchheimer microchannel filled with non-Newtonian fluid under electro-osmotic forces. J Taibah Univ Sci. 2021 Jan 1;15(1):514–29.

[37] Abbabsbandy S, Shivanian E. Exact analytical solution of the MHD Jeffery-Hamel flow problem. Meccanica. 2012 Aug;47(6):1379–89.

[38] Ellahi R, Shivanian E, Abbabsbandy S, Rahman SU, Hayat T. Analysis of steady flows in viscous fluid with heat/mass transfer and slip effects. Int J Heat Mass Transf. 2012 Nov 1;55(23–24):6384–90.

[39] Abbabsbandy S, Shivanian E, Hashim I. Exact analytical solution of forced convection in a porous-saturated duct. Commun Nonlinear Sci Numer Simulat. 2011 Oct 1;16(10):3981–9.

[40] Abbabsbandy S, Shivanian E. Exact closed form solutions to nonlinear model of heat transfer in a straight fin. Int J Thermal Sci. 2017 Jun 1;116:45–51.