The N-steps Invasion Percolation Model

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Abstract

A new kind of invasion percolation is introduced in order to take into account the inertia of the invader fluid. The inertia strength is controlled by the number \( N \) of pores (or steps) invaded after the perimeter rupture. The new model belongs to a different class of universality with the fractal dimensions of the percolating clusters depending on \( N \). A blocking phenomenon takes place in two dimensions. It imposes an upper bound value on \( N \). For pore sizes larger than the critical threshold, the acceptance profile exhibits a permanent tail.

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1 Introduction

When a nonviscous liquid is slowly injected into a porous medium already filled with a viscous fluid the predominant forces on the interface are of capillary nature. These forces are such as to make the injected fluid spontaneously displace the viscous one. The interface between the fluids advances pore by pore, being its dynamics determined by the capillary rule: the smaller pore is invaded first.

Invasion percolation \cite{1} is a theoretical model used to describe fluid-fluid displacement. As it was pointed out invasion percolation is a kind of self-organizing criticality \cite{2,3}. The system exhibits scale invariant behaviour in space and time achieving the critical state without a fine tuning mechanism to a particular parameter. There is now a strong evidence that this critical state corresponds to the critical standard percolation \cite{1,4}.

Under some modifications, the invasion percolation model has been successfully applied to describe the fingering phenomena in soils \cite{5} and fluid flowing with a privileged direction \cite{6}. In its original formulation, the invasion percolation assumes that all the pores situated in the perimeter of the cluster exchange informations in such a way that, at growth stage $t$, no matter how far they are, only an unique pore is invaded - that one with the smallest size. After this pore has been invaded the fluid flow stops, a new perimeter is determined and the process continues. Recently, a multiple invasion percolation model \cite{7,8} was proposed permitting that not just one but many pores belonging to the actual perimeter can be simultaneously invaded. But here again, the fluid does not exhibit inertia. By this we mean the tendency of the fluid to proceed further, invading the sites surrounding that pore of the perimeter where the invasion process is taking place. We implement this idea by proposing a model in which, at the growth stage $t$, the fluid occupies not only the smaller perimeter site $j$ but also invades (one by one and always following the capillary rule) $N-1$ additional pores on $j$'s neighborhood. At $t+1$, a new perimeter is determined and the process continues. We can say that the fluid walks (occupies) $N$ steps (pores) before to loose its inertia. We call this process the N-steps invasion percolation. The ordinary invasion percolation corresponds to the case $N=1$.

The N-steps invasion percolation model exhibits very different behaviours in
two and three dimensions. In two dimensions and for large $N$, the walks can be easily blocked since hindrances to the growth are set everywhere by parts of the own cluster. The walks are most of the time incomplete. In such walks, the number of steps actually executed is always equal or smaller than the external (and fixed) parameter $N$. The mean number of steps $\langle n \rangle$ is bounded and its maximum value depends on the lattice geometry. For an infinite square lattice we find that $\langle n \rangle$ cannot be greater than 35. The clusters have fractal dimensions very close to that of the ordinary invasion percolation but very different mean coordination numbers. The acceptance profiles show a strong dependence of the critical threshold with $\langle n \rangle$.

In three dimensions, the possibility of blocking is so small that $\langle n \rangle$ always coincides with $N$. The difference between the two and three dimensional behaviours is reminiscent of what happens in many growth models like, e.g., in the invasion percolation model with trapping \cite{1}. The calculated fractal dimensions obtained from a careful finite size scaling analysis, indicate that the N-steps model and the ordinary invasion percolation belong to different classes of universality.

2 The Model

Before treating the N-steps invasion percolation model, let us briefly recall how the ordinary invasion percolation is simulated. Assign a random number $r$, uniformly distributed in the range $[0,1]$ to each lattice site and choose the central site as the seed of the growth. The perimeter sites of the cluster are identified as the growth sites. At each growth step, occupy an unique perimeter site - that one with the smallest associated random number. The growth process is interrupted after the cluster reaches the lattice boundary. It is important to note that in the ordinary invasion percolation the occupation of one site corresponds to one growth step.

In the construction of the N-steps invasion percolation cluster we define one growth step, taking place at the growth stage $t$ (which shall not be confused with time, see below for details), as being composed by the following procedures. First, as in the ordinary invasion percolation, the smallest perimeter site is
occupied. Then, starting from this site, additional $N - 1$ sites are sequentially invaded. In each invasion, the capillary rule is obeyed, i.e., among the few sites surrounding the actual growth site, only the smallest empty site is invaded. In terms of a fluid flowing language, these supplementaries $N - 1$ invasions would correspond to the inertia of the fluid - that is, after the perimeter rupture has occurred, the fluid cannot stop instantaneously. It walks $N - 1$ steps further. It may happen that this occupation of exactly $N$ sites is not allowed by restrictions imposed by the growth process itself and the lattice geometry. Many times, an incomplete walk can be found in a ’cul de sac’. In such cases, of course, a smaller number of sites is in fact invaded. Any way, the next thing to do is now to determine the new cluster’s perimeter. This completes what we called one growth step. The growth stage is then updated to $t + 1$ and the process is repeated until the cluster touches one of the lattice frontiers.

3 Numerical Simulations

As mentioned above, sometimes at a growth stage $t$, the presence of already occupied sites prevents the walk from completing all $N$ steps, and the walk is blocked. This means that the number of steps actually executed is a wild function of the growth stage $t$ and, more important, it is always smaller or equal to $N$. On average, this would bring the mean number of steps too far from $N$. We are then faced with the problem of having an external (and fixed) parameter $N$ disconnected from the really performed mean number of steps. In order to bring the values of these two quantities as closed as possible, we devised the following compensation mechanism in our simulations. Any time a walk has performed say $\bar{N}$ steps ($\bar{N} < N$) then the debt $N - \bar{N}$ is recorded and, in the next growth stage, a total of $N + (N - \bar{N})$ number of steps will be permitted. So, sometimes the number of steps actually executed can be larger than $N$, sometimes smaller and the mean number of steps will be fluctuating around $N$. This scheme resembles those used in systems with self organized criticality. In our case, it is the mean number of steps which is spontaneously tuned to the value $N$. As we shall see later, this tuning is always possible in three dimensions but not in two.
Let \( n(t) \) be the number of steps really executed and \( n_e(t) \) the expected number of steps to be executed at growth stage \( t \) (\( t \) is an integer such that \( t \geq 1 \)). The debt is defined as \( D(t) = n_e(t) - n(t) \) with \( n_e(t = 1) = N \). This debt should be paid in the next growth stage \( t + 1 \) by executing a longer walk, i.e., one with an expected number of steps \( n_e(t + 1) = N + D(t) \) steps. The quantities \( n(t) \) and \( n_e(t) \) are plotted in Fig. 1 for one typical realization. From the fluctuating character of \( n(t) \) we conclude that the relevant parameter to be measured is the mean number of steps \( \langle n \rangle \). If, from a total of \( S \) realizations, the number of growth steps in the \( i \)th realization is \( T_i \) then

\[
\langle n \rangle = \frac{1}{S} \sum_{i=1}^{S} \frac{\sum_{t=1}^{T_i} n_i(t)}{T_i}
\]  

(1)

where \( n_i(t) \) stays for the number of steps really executed at the growth stage \( t \) in the \( i \)th realization.

We performed numerical simulations of the N-steps invasion percolation model on two and three dimensional lattices. For the square and the honeycomb lattices, sizes of 201, 401, 801 and 1601 were used and the mean number of steps \( \langle n \rangle \) (averaged over 400 – 2000 experiments) determined for several values of \( N \). Even for small \( N \), we observe the presence of blocking. In order to measure how frequent are these blockings for fixed \( N \), we calculate the quantity

\[
f_b = \frac{1}{S} \sum_{i=1}^{S} \frac{\text{total number of blockings occurred in the } i \text{th realization}}{T_i}
\]  

(2)

As can be seen in Fig. 2 (a), this fraction of blocking increases with \( N \) and reaches 100 % around \( N = 35 \) for the square lattice. Our finite lattice size analysis, indicates that, as the thermodynamic limit of an infinite lattice is approached, the curve extrapolates to a cusp. Fig. 2 (b) shows the dependence of the mean number of steps \( \langle n \rangle \) with \( N \). They coincide until the upper bound \( N_{\text{max}} = 35 \). This means that, no matter how much \( N \) is bigger than 35, the ultimate N-steps invasion percolation model is that one with \( N = 35 \). We conclude that the model is well defined only for \( N \in [1, 35] \) where \( \langle n \rangle \) and \( N \) coincide.
This blocking phenomenon is also found in the honeycomb lattice, but it is more frequent than that observed for the square lattice. As a result, the breakdown is slightly smaller, i.e., $N_{\text{max}} = 30$.

From studies of the kinetic growth walk model \cite{9} we know that the blocking phenomenon is especially acute in two dimensions but it is irrelevant in higher dimensions.

For the three-dimensional case we use the simple cubic lattice with $L = 51, 75, 101, 151$ and $201$ (400 - 2000 experiments). Here, the blocking is so rare that $\langle n \rangle$ and $N$ coincide for the whole interval $1 \leq N \leq 100$.

### 4 Cluster Structure

The clusters of ordinary invasion percolation are fractal objects in the sense that their mean mass $\langle M \rangle$ scales with their gyration radius $R_g$ with a non-integer exponent \cite{13}. This exponent is the fractal dimension $D_F$. The values of $D_F$ are known from several studies: $D_F = 1.89$ \cite{1, 14} for two dimensional systems and $D_F = 2.52$ \cite{1, 14} for the three dimensional case.

The fractal dimension and the mean coordination number \cite{7} are two simple ways of characterizing the cluster’s structure. We determined these quantities for the N-steps invasion percolation model defined on the square and simple cubic lattices. The estimated fractal dimensions as a function of $\langle n \rangle$ are shown in the Table 1.

For the square lattice only a small dependence is detected. On the other hand, universality is definitely broken for the cubic lattice, with $D_F$ varying from 2.52 to 2.77. These results are shown in Fig. 3.

The mean coordination number $\langle z \rangle$ of the clusters is the number of first neighbours averaged over all sites of the clusters. For the ordinary site invasion percolation this quantity has values: $2.51(1)$ \cite{3} for the square lattice and $2.31(1)$ for the simple cubic lattice. Our estimated values are shown in the Table 1. The mean coordination number increases with $\langle n \rangle$ but quickly stabilizes. These considerable changes of $\langle z \rangle$ with $\langle n \rangle$ indicate that some visual differences between the clusters should exist. This can be seen in Fig. 4. We note the formation of globules, i.e., regions with higher densities, as a result of increasing $\langle n \rangle$. 

5 The acceptance profile

The acceptance profile $a(r)$ concept was introduced by Wilkinson and Willemse to study ordinary invasion percolation. They defined $a(r)$ as the ratio between the number of random numbers in the interval $[r, r + dr]$ which were accepted into the cluster to the number of random numbers in that range which became available. In the limit of an infinite lattice the acceptance profile tends to a step function with the discontinuity located at the ordinary percolation threshold $p_c$.

We determined the acceptance profile $a(r)$ for the $N$-steps invasion percolation as a function of $\langle n \rangle$ for both the square and simple cubic lattices. In the Fig.5(a) we show how the finite size of the lattice affects $a(r)$. As the lattice size is increased, the acceptance profile develops a plateau up to a threshold $r_c \sim 0.35$ (indicating that all the small random numbers which became available were accepted into the cluster). After this value, however, a tail appears and remains finite even in thermodynamic limit. This happens here due to the fact that the dynamic of the model forces invasion of some larger pores.

Fig.5(b) shows the acceptance profile for many values of $\langle n \rangle$ for the square lattice. The threshold $r_c$ of the plateau clearly depends on the mean number of steps, i.e., $r_c(\langle n \rangle)$. For $\langle n \rangle = 1$ the acceptance profile tends to a step function with $r_c(\langle n \rangle = 1) \simeq 0.59$ and the ordinary invasion percolation behaviour is recovered. Increasing $\langle n \rangle$ diminishes $r_c$ until the ultimate value $r_c(35) \sim 0.24$. A similar behaviour was observed for the acceptance profile of the simple cubic lattice, the only difference being the probable inexistence of the upper limited value of $N$ due the absence of blocking.

6 Conclusions

We proposed a new kind of invasion percolation model where the inertia of the invader fluid was taken into account. The additional number of steps (or pores) $N$ governs the impetus of the fluid. In two dimensions, the appearance of the blocking phenomenon, gives to $N$ an upper bound value $N_{max}$. This means that the proposed mechanism can only be implemented with $N$ varying from
1 to $N_{\text{max}}$. The $N_{\text{max}}$ values were estimated to be 35 and 30 for square and honeycomb lattices, respectively. There is a strong dependence between the mean coordination number $z$ and $N$. As $N$ is increased, we find that some globules are formed inside the clusters structure. For the simple cubic lattice, the fractal dimensions depend strongly on $N$ and put, definitely, the $N$-steps invasion percolation model in different classes of universality.

Let us now discuss in what sense we believe our model may contribute to a better understanding of fluid flow in porous media. Inertial forces in a porous medium are directly proportional to the square of the fluid velocity and inversely proportional to the pore’s diameter. The Reynolds number is given by the ratio between the inertial forces and viscous forces and, consequently, has a linear dependence with the fluid velocity. If this number is small (low velocity) then the Darcy’s law (which states that the pressure gradient is linearly proportional to the velocity) can be applied. However, if the fluid velocity is increased but still kept below the turbulent regime, then a departure of Darcy’s law can be experimentally measured. This departure is macroscopically well described by the Forchheimer equation \cite{13} which adds a quadratic velocity term to the Darcy’s law. It is believed that this quadratic term comes, essentially, from inertial forces contributions and it could be detected and measured by augmenting the fluid velocity. Recently \cite{16}, the Forchheimer equation has been investigated by numerical solutions of the continuous Navier-Stokes equation. In our model, the parameter $N$ can be related to the fluid velocity. This can be seen if we remember that the invaded volume by growth step (or the fluid flux) increases with $N$. To check this directly will require to find amenable definitions of average pressure gradients and velocities valid in the context of the invasion percolation models.

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Table 1

| $\langle n \rangle$ | $D_F$  | $\langle z \rangle$ | $\langle n \rangle$ | $D_F$  | $\langle z \rangle$ |
|-------------------|--------|---------------------|-------------------|--------|---------------------|
| 1                 | 1.89(1)| 2.51(1)             | 1                 | 2.52(2)| 2.31(1)             |
| 2                 | 1.89(1)| 2.65(1)             | 2                 | 2.52(2)| 2.42(1)             |
| 4                 | 1.90(1)| 2.84(1)             | 8                 | 2.58(2)| 2.51(1)             |
| 15                | 1.91(1)| 2.89(1)             | 15                | 2.64(3)| 2.52(1)             |
| 30                | 1.92(1)| 2.89(1)             | 30                | 2.69(3)| 2.52(1)             |
| 35                | 1.92(1)| 2.89(1)             | 45                | 2.75(3)| 2.53(1)             |
| 40                | 1.92(1)| 2.89(1)             | 60                | 2.77(3)| 2.53(1)             |
| 60                | 1.92(1)| 2.89(1)             | 100               | 2.77(3)| 2.53(1)             |
FIGURE CAPTION

Figure 1. Simulation of the number of executed steps \( n(t) \) and the number of expected steps \( n_e(t) \) as a function of the growth steps \( t \).

Figure 2. (a) The fraction of blocking \( f_b \) as a function of \( N \) for the square lattice, (b) The mean number of steps versus \( N \). The dotted line is the straight \( \langle n \rangle = N \).

Figure 3. Logarithm of the cluster average mass plotted versus the logarithm of the average gyration radius for the cubic and square lattices and for some values of \( \langle n \rangle \).

Figure 4. The upper figure shows a typical cluster of the ordinary invasion percolation model for a square lattice with \( L = 201 \). The lower is a cluster of the \( N \)-steps invasion percolation model with \( \langle n \rangle = 30 \) and \( L = 201 \).

Figure 5. The dependence of the acceptance profiles with the random number \( r \) as the lattice size \( L \) and the mean number of steps \( \langle n \rangle \) are changed (a) and (b), respectively).
Table 1. The fractal dimensions and mean coordination numbers of the clusters on the square and simple cubic lattices for several values of ⟨n⟩.
Figure 1

\[ n_e(t) = N + D(t-1) \]

\[ n(t) = N + D(t-1) - D(t) \]

Number of steps vs. Growth stage (t)
Figure 2
Figure 3
Figure 5