General scalar products in the arbitrary six-vertex model

G A P Ribeiro
Departamento de Física, Universidade Federal de São Carlos, 13565-905 São Carlos-SP, Brazil
E-mail: pavan@df.ufscar.br

Received 16 September 2011
Accepted 21 October 2011
Published 18 November 2011

Online at stacks.iop.org/JSTAT/2011/P11015
doi:10.1088/1742-5468/2011/11/P11015

Abstract. In this work we use the algebraic Bethe ansatz to derive the general scalar product in the asymmetric six-vertex model for generic Boltzmann weights. We performed this calculation using only the unitarity property, the Yang–Baxter algebra and the Yang–Baxter equation. We have derived a recurrence relation for the scalar product. The solution of this relation was written in terms of the domain wall partition functions. In turn, these partition functions were also obtained for generic Boltzmann weights, which provided us with an explicit expression for the general scalar product.

Keywords: algebraic structures of integrable models, quantum integrability (Bethe ansatz)
1. Introduction

It is well known that the quantum inverse scattering method is a powerful tool to solve exactly quantum integrable models as well as their classical counterparts [1, 2]. Based on this approach one can construct the Bethe states by the action of pseudo-particle creation operators on a pseudo-vacuum state. These operators are the off-diagonal matrix elements of the monodromy matrix \( T_A(\lambda) \). In turn the monodromy matrix elements are the generators of the Yang–Baxter algebra given by the following quadratic relation:

\[
R_{12}(\lambda, \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda, \mu).
\]

The monodromy matrix elements act on the space of states of the quantum physical system and the \( R \)-matrix elements play the role of the structure constants of the above Yang–Baxter algebra (1). The latter algebra is associative thanks to the Yang–Baxter equation [3]:

\[
R_{12}(\lambda, \mu)R_{13}(\lambda, \gamma)R_{23}(\mu, \gamma) = R_{23}(\mu, \gamma)R_{13}(\lambda, \gamma)R_{12}(\lambda, \mu).
\]

The trace of the monodromy matrix over the auxiliary space \( T(\lambda) = \text{Tr}_A [T_A(\lambda)] \) is the row-to-row transfer matrix. This matrix constitutes a family of commuting operators \([T(\lambda), T(\mu)] = 0\), provided that the \( R \)-matrix is invertible. This condition is granted by the unitarity relation

\[
R_{21}(\lambda, \mu)R_{12}(\mu, \lambda) = I,
\]

where \( I \) is the identity matrix.

Taking the action of the transfer matrix over the Bethe states, one is able to obtain the eigenvalues of the transfer matrix and its related conserved quantities. The eigenvalue

doi:10.1088/1742-5468/2011/11/P11015
After obtaining the eigenvalues, one of the most challenging problems is to compute the scalar products and correlation functions. This was accomplished, by means of the algebraic Bethe ansatz, for models descending from the $R$-matrix of the symmetric six-vertex model \[5\]–\[7\].

Recently, it was realized that the algebraic formulation of the Bethe states of the transfer matrix can be done using only the Yang–Baxter algebra (1), the Yang–Baxter equation (2) and the unitarity property (3) satisfied by the $R$-matrix \[8\]. This way the on-shell properties (eigenvalues and the Bethe ansatz equations) as well as the off-shell Bethe vectors are obtained in terms of arbitrary Boltzmann weights.

This idea has been shown to be valid also in the computation of the monodromy matrix elements of the arbitrary six-vertex model in the $F$-basis and the scalar product for arbitrary Boltzmann weights \[9\].

In this paper we shall obtain the scalar product as well as its associated domain wall partition functions by means of the algebraic Bethe ansatz for the completely asymmetric six-vertex model for arbitrary Boltzmann weights. Besides the scalar product for the asymmetric six-vertex model, which is a non-trivial extension of the previous known case, we would also like to show that all this can be accomplished using only the Yang–Baxter algebra, the Yang–Baxter equation and the unitarity relation for the most general six-vertex model.

The strategy of working with general Boltzmann weights for the most asymmetric model brings new insights into the algebraic construction of Bethe states as well as their computation of scalar products. This is because the lack of symmetry restricts the Yang–Baxter equations and relations we are allowed to use. Therefore, this works as a guide towards the solution.

The outline of the paper is as follows. In section 2, we define the asymmetric six-vertex model and list the main relations needed in this work. In section 3, we derive a recurrence relation for the scalar product and sketch its solution. In section 4, we derive the partition function with domain wall boundary conditions which will help in writing a closed formula for the scalar product. Our conclusions are given in section 6.

2. The asymmetric six-vertex model

We define the $R$-matrix of the asymmetric six-vertex model as \[3,10\]

$$R(\lambda, \mu) = \begin{pmatrix}
    a_+ (\lambda, \mu) & 0 & 0 & 0 \\
    0 & b_+ (\lambda, \mu) & c_+ (\lambda, \mu) & 0 \\
    0 & c_- (\lambda, \mu) & b_- (\lambda, \mu) & 0 \\
    0 & 0 & 0 & a_- (\lambda, \mu)
\end{pmatrix}.$$ \[4\]

This $R$-matrix should satisfy the unitarity property (3), which results in the following set of relations for its matrix elements:

$$c_-(\lambda, \mu)b_-(\mu, \lambda) + b_-(\lambda, \mu)c_+(\mu, \lambda) = 0,$$ \[5\]

$$b_+(\lambda, \mu)c_-(\mu, \lambda) + c_+(\lambda, \mu)b_+(\mu, \lambda) = 0.$$ \[6\]
General scalar products in the arbitrary six-vertex model

\[ a_+(\lambda, \mu)a_+(\mu, \lambda) = 1, \]  
\[ a_-(\lambda, \mu)a_-(\mu, \lambda) = 1, \]  
\[ c_-(\lambda, \mu)c_-(\mu, \lambda) + b_-(\lambda, \mu)b_+(\mu, \lambda) = 1, \]  
\[ c_+(\lambda, \mu)c_+(\mu, \lambda) + b_+(\lambda, \mu)b_-(\mu, \lambda) = 1. \]  

Moreover the above \( R \)-matrix satisfies the Yang–Baxter equation (2). This provides us with an additional set of functional relations among the Boltzmann weights given by

\[ b_-'c''_+ + c_-c'_+ = a_+c'_+a''_+, \]  
\[ c_-'a''_+ + b_-c'_+ = c_-b'_+a''_+, \]  
\[ b_-'c_+ + c_-c'_+ = a_+b'_+c''_+, \]  
\[ c_-'b''_+ + b_-c'_+ = b'_+b''_+, \]  
\[ b_-'c_+ + c_-c''_+ = -a_-c_-a''_+, \]  
\[ b_-a'_+ + c_-c''_+ = -b_-c''_+, \]  
\[ b_-'c'_+ + c_-'c''_+ = a_+b'_+c''_+, \]  
\[ b_-c'_+ + c_-'c''_+ = a_+c_-'a''_+, \]  
\[ c_-'b''_+ + b_-c'_+ = b'_+b''_+, \]  
\[ b_-a'_+ + c_-c''_+ = -b_-c''_+, \]  
\[ c_-'c''_+ + c_-'c''_+ = b'_+b''_+, \]  
\[ b_-'c'_+ + c_-'c''_+ = a_+c_-'a''_+, \]  

where \( a_\pm = a_\pm(\lambda, \mu), a_\pm(\lambda, \mu) = a_\pm(\lambda, \gamma) \) and \( a_\pm(\mu, \gamma) \) and likewise for the other functions.

The monodromy matrix \( T_\lambda(\lambda, \{\nu_\lambda\}) = R_{\lambda L}(\lambda, \nu_L) \cdots R_{\lambda 1}(\lambda, \nu_1) \) satisfies the fundamental relation (1) thanks to the Yang–Baxter equation (2). For the six-vertex model, the monodromy matrix can be written as a \( 2 \times 2 \) matrix with operator-valued entries:

\[ T(\lambda, \{\nu_\lambda\}) = \begin{pmatrix} A(\lambda, \{\nu_\lambda\}) & B(\lambda, \{\nu_\lambda\}) \\ C(\lambda, \{\nu_\lambda\}) & D(\lambda, \{\nu_\lambda\}) \end{pmatrix}. \]  

These operators have to obey 16 commutation rules which follow from (1). We only list the commutation rules used in this work:

\[ B(\lambda)B(\mu) = \frac{a_-(\lambda, \mu)}{a_+(\lambda, \mu)}B(\mu)B(\lambda), \]  
\[ A(\lambda)B(\mu) = \frac{a_+(\mu, \lambda)}{b_-(\mu, \lambda)}B(\mu)A(\lambda) - \frac{c_+(\mu, \lambda)}{b_-(\mu, \lambda)}B(\lambda)A(\mu), \]  
\[ D(\lambda)B(\mu) = \frac{a_-(\lambda, \mu)}{b_-(\lambda, \mu)}B(\mu)D(\lambda) - \frac{c_-(\lambda, \mu)}{b_-(\lambda, \mu)}B(\lambda)D(\mu), \]
General scalar products in the arbitrary six-vertex model

\[
C(\lambda)C(\mu) = \frac{a_-(\mu, \lambda)}{a_+(\mu, \lambda)} C(\mu)C(\lambda),
\]
\[
C(\lambda)A(\mu) = \frac{a_+(\lambda, \mu)}{b_-(\lambda, \mu)} A(\mu)C(\lambda) - \frac{c_-(\lambda, \mu)}{b_-(\lambda, \mu)} A(\lambda)C(\mu),
\]
\[
C(\lambda)D(\mu) = \frac{a_-(\mu, \lambda)}{b_-(\mu, \lambda)} D(\mu)C(\lambda) - \frac{c_+(\mu, \lambda)}{b_-(\mu, \lambda)} D(\lambda)C(\mu),
\]
\[
C(\lambda)B(\mu) = \frac{b_+(\lambda, \mu)}{b_-(\lambda, \mu)} B(\mu)C(\lambda) + \frac{c_-(\lambda, \mu)}{b_-(\mu, \lambda)} (A(\mu)D(\lambda) - A(\lambda)D(\mu)).
\]

Additionally, the structure of the \( R \)-matrix (4) implies that both ferromagnetic states \(|\uparrow\rangle = \bigotimes_{j=1}^{L} (\downarrow,\downarrow)_j \) and \(|\downarrow\rangle = \bigotimes_{j=1}^{L} (\uparrow,\downarrow)_j \) are intrinsic eigenstates of the transfer matrix \( T(\lambda, \{\nu_k\}) = A(\lambda, \{\nu_k\}) + D(\lambda, \{\nu_k\}) \). Therefore, they play the role of pseudo-vacuum states.

One can see immediately that the action of the monodromy matrix over the pseudo-vacuum state, e.g. \(|\uparrow\rangle\), results in a triangular matrix:

\[
T(\lambda, \{\nu_k\})|\uparrow\rangle = \begin{pmatrix} a(\lambda, \{\nu_k\})|\uparrow\rangle & \# \\ 0 & d(\lambda, \{\nu_k\})|\uparrow\rangle \end{pmatrix},
\]

where \( a(\lambda, \{\nu_k\}) \) and \( d(\lambda, \{\nu_k\}) \) correspond to some fixed representation and the symbol \# stands for a non-null state. Therefore, one sees that the \( B(\lambda, \{\nu_k\}) \) and \( C(\lambda, \{\nu_k\}) \) operators are the creation and annihilation operators over the state \(|\uparrow\rangle\) [4], which means

\[
A(\lambda, \{\nu_k\})|\uparrow\rangle = a(\lambda, \{\nu_k\})|\uparrow\rangle \quad B(\lambda, \{\nu_k\})|\uparrow\rangle = \#, \quad \quad (32)
\]
\[
C(\lambda, \{\nu_k\})|\uparrow\rangle = 0 \quad D(\lambda, \{\nu_k\})|\uparrow\rangle = d(\lambda, \{\nu_k\})|\uparrow\rangle, \quad \quad (33)
\]
or alternatively

\[
\langle \uparrow | A(\lambda, \{\nu_k\}) = \langle \uparrow | d(\lambda, \{\nu_k\}) \quad \langle \uparrow | B(\lambda, \{\nu_k\}) = 0, \quad \quad (34)
\]
\[
\langle \uparrow | C(\lambda, \{\nu_k\}) = \# \quad \langle \uparrow | D(\lambda, \{\nu_k\}) = \langle \uparrow | d(\lambda, \{\nu_k\}). \quad \quad (35)
\]

The action of the creation operators over the pseudo-vacuum is the algebraic version of the famous Bethe ansatz and is given by

\[
|\Psi_M\rangle = B(\lambda_M, \{\nu_k\}) \cdots B(\lambda_1, \{\nu_k\})|\uparrow\rangle. \quad \quad (36)
\]

In order to obtain the eigenvalues of the transfer matrix one has to commute the operators \( A(\lambda, \{\nu_k\}) \) and \( D(\lambda, \{\nu_k\}) \) with \( B(\lambda, \{\nu_k\}) \). In doing so we need to use the relations (20), (24) and (25) for the operator \( A(\lambda, \{\nu_k\}) \) and the relations (12), (24) and (26) for the operator \( D(\lambda, \{\nu_k\}) \) [8], which results in (see the appendix)

\[
A(\lambda, \{\nu_k\}) \prod_{i=1}^{M} B(\lambda_i, \{\nu_k\}) = \prod_{i=1}^{M} \frac{a_+(\lambda_i, \lambda)}{b_-(\lambda_i, \lambda)} B(\lambda_i, \{\nu_k\}) A(\lambda, \{\nu_k\})
\]
\[
- \sum_{j=1}^{M} \frac{c_+(\lambda_j, \lambda)}{b_-(\lambda_j, \lambda)} \prod_{i=1}^{M} \frac{a_+^{(\theta)}(\lambda_i, \lambda_j)}{b_-(\lambda_i, \lambda_j)} B(\lambda, \{\nu_k\}) \prod_{i=1}^{M} B(\lambda_i, \{\nu_k\}) A(\lambda_j, \{\nu_k\}), \quad \quad (37)
\]

\[\text{doi:10.1088/1742-5468/2011/11/P11015}\]
where \( \prod_{j=1}^{M} b_{-}(\lambda_{j}) \prod_{i \neq j} B(\lambda_{i}, \{ \nu_{k} \}) D(\lambda, \{ \nu_{k} \}) \)

\[ \prod_{i=1}^{M} \frac{a_{-}(\lambda_{i}, \lambda_{j})}{b_{-}(\lambda_{i}, \lambda_{j})} \prod_{i \neq j} B(\lambda_{i}, \{ \nu_{k} \}) B(\lambda_{j}, \{ \nu_{k} \}) D(\lambda, \{ \nu_{k} \}), \tag{38} \]

where \( a_{+}^{(\theta)}(\lambda_{i}, \lambda_{j}) = a_{+}(\lambda_{i}, \lambda_{j}) \theta_{>}^{(\lambda_{i}, \lambda_{j})} \) and

\[ \theta_{>}^{(\lambda_{i}, \lambda_{j})} = \begin{cases} \frac{a_{-}(\lambda_{i}, \lambda_{j})}{a_{+}(\lambda_{i}, \lambda_{j})}, & i > j, \\ 1, & i \leq j. \end{cases} \tag{39} \]

Therefore, we obtain

\[ T(\lambda)|\Psi_{M}\rangle = \Lambda_{M}(\lambda, \{ \nu_{k} \})|\Psi_{M}\rangle + \sum_{j=1}^{M} \Gamma_{j}(\lambda) B(\lambda, \{ \nu_{k} \}) \prod_{i \neq j} B(\lambda_{i}, \{ \nu_{k} \}) |\uparrow\rangle, \tag{40} \]

where the eigenvalues \( \Lambda_{M}(\lambda, \{ \nu_{k} \}) \) are given by

\[ \Lambda_{M}(\lambda, \{ \nu_{k} \}) = a(\lambda, \{ \nu_{k} \}) \prod_{i=1}^{M} \frac{a_{+}(\lambda_{i}, \lambda)}{b_{-}(\lambda_{i}, \lambda)} + d(\lambda, \{ \nu_{k} \}) \prod_{i=1}^{M} \frac{a_{-}(\lambda_{i}, \lambda)}{b_{-}(\lambda_{i}, \lambda)}, \tag{41} \]

and

\[ \Gamma_{j}(\lambda) = a(\lambda_{j}, \{ \nu_{k} \}) \frac{c_{+}(\lambda_{j}, \lambda)}{b_{-}(\lambda_{j}, \lambda)} \prod_{i=1}^{M} \frac{a_{+}^{(\theta)}(\lambda_{j}, \lambda_{i})}{b_{-}(\lambda_{i}, \lambda_{j})} + d(\lambda_{j}, \{ \nu_{k} \}) \frac{c_{-}(\lambda, \lambda_{j})}{b_{-}(\lambda, \lambda_{j})} \prod_{i=1}^{M} \frac{a_{+}^{(\theta)}(\lambda_{j}, \lambda_{i})}{b_{-}(\lambda_{i}, \lambda_{j})}. \tag{42} \]

Finally, we have to impose the coefficient \( \Gamma_{j}(\lambda) \) to vanish for arbitrary \( \lambda \) in order to cancel the unwanted terms. Using the unitarity relation (5), one sees that the parameters \( \lambda_{j} \) have to satisfy the Bethe ansatz equations

\[ \frac{a(\lambda_{j}, \{ \nu_{k} \})}{d(\lambda_{j}, \{ \nu_{k} \})} = \prod_{i \neq j} \frac{a_{+}^{(\theta)}(\lambda_{j}, \lambda_{i})}{b_{-}(\lambda_{i}, \lambda_{j})} \tag{43} \]

Similarly the dual Bethe state can be written as follows:

\[ \langle \Psi_{M} | = \langle \uparrow | C(\lambda_{1}, \{ \nu_{k} \}) \cdots C(\lambda_{M}, \{ \nu_{k} \}). \tag{44} \]

However, in order to obtain the transfer matrix eigenvalues one should look for commutation rules between the operators \( A(\lambda, \{ \nu_{k} \}) \) and \( D(\lambda, \{ \nu_{k} \}) \) with \( C(\lambda, \{ \nu_{k} \}) \). Here, one has to use the relations (16), (27) and (28) for the operator \( A(\lambda, \{ \nu_{k} \}) \) and the relations (14), (27) and (29) for the operator \( D(\lambda, \{ \nu_{k} \}) \), such that

\[ \prod_{i=1}^{M} C(\lambda_{i}, \{ \nu_{k} \}) A(\lambda, \{ \nu_{k} \}) = \prod_{i=1}^{M} \frac{a_{+}(\lambda_{i}, \lambda)}{b_{-}(\lambda_{i}, \lambda)} A(\lambda_{i}, \{ \nu_{k} \}) \prod_{i=1}^{M} C(\lambda_{i}, \{ \nu_{k} \}) \]

\[ - \sum_{j=1}^{M} \frac{c_{-}(\lambda_{j}, \lambda)}{b_{-}(\lambda_{j}, \lambda)} \prod_{i=1}^{M} \frac{a_{+}^{(\theta)}(\lambda_{j}, \lambda_{i})}{b_{-}(\lambda_{i}, \lambda_{j})} A(\lambda_{j}, \{ \nu_{k} \}) \prod_{i=1}^{M} C(\lambda_{i}, \{ \nu_{k} \}) C(\lambda, \{ \nu_{k} \}), \tag{45} \]
\[
\prod_{i=1}^{M} C(\lambda_i, \{\nu_k\}) D(\lambda, \{\nu_k\}) = \prod_{i=1}^{M} \frac{a_-(\lambda, \lambda_i)}{b_-(\lambda, \lambda_i)} D(\lambda, \{\nu_k\}) \prod_{i=1}^{M} C(\lambda_i, \{\nu_k\}) \\
- \sum_{j=1}^{M} \frac{c_+(\lambda, \lambda_j)}{b_-(\lambda, \lambda_j)} \prod_{i=1}^{M} \frac{a_+(\theta, \lambda_i)}{b_-(\lambda_j, \lambda_i)} D(\lambda_j, \{\nu_k\}) \prod_{i=1, i \neq j}^{M} C(\lambda_i, \{\nu_k\}) C(\lambda, \{\nu_k\}).
\] (46)

3. The scalar product

The general scalar product is defined as

\[
S_M(\{\lambda\}_{i=1}^{M}, \{\mu\}_{i=1}^{M}) = \langle \uparrow| C(\mu_1) \cdots C(\mu_M) B(\lambda_M) \cdots B(\lambda_1)| \uparrow \rangle
\] (47)

where we have dropped the dependence on \(\nu_k\). In the particular case that both sets of parameters \(\lambda_j\) and \(\mu_j\) satisfy the Bethe ansatz equations (43), the product (47) becomes the norm of Bethe eigenstates [5].

To compute the scalar product (47), we must calculate the action of the operator \(B(\lambda)\) over the state (44) (or, alternatively, the action of \(C(\mu)\) over the state (36)). For this purpose we have to use repeatedly the relation (30), which results in

\[
\langle \uparrow| C(\mu_1) \cdots C(\mu_M) B(\lambda_M) = \prod_{i=1}^{M} \frac{b_+(\mu_i, \lambda_M)}{b_-(\mu_i, \lambda_M)} \langle \uparrow| B(\lambda_M) C(\mu_1) \cdots C(\mu_M) \\
+ \sum_{j=1}^{M} \frac{c_-(\mu_j, \lambda_M)}{b_-(\mu_j, \lambda_M)} \langle \uparrow| \prod_{i=1}^{j-1} C(\mu_i) \\
\times (A(\lambda_M)D(\mu_j) - A(\mu_j)D(\lambda_M)) \prod_{i=j+1}^{M} C(\mu_i),
\] (48)

where the first term vanishes thanks to (34) and we set \(\lambda_i = \mu_{2M+1-i}, i = 1, \ldots, M\) for convenience.

After that, one has to compute the action of \(A(\lambda)D(\mu_j) - A(\mu_j)D(\lambda)\) over the \(C(\mu_i)\), where one should use the algebraic relations (45) and (46) along the same lines as [2, 5], such that

\[
\langle \uparrow| C(\mu_1) \cdots C(\mu_M) B(\mu_{M+1}) = \sum_{j,k=1}^{M+1, k \neq k} \langle \uparrow| A(\mu_k) D(\mu_j) \\
\times \frac{c_-(\mu_j, \mu_{M+1})}{c_+(\mu_{M+1}, \mu_k)} c_+(\mu_{M+1}, \mu_k) \frac{a_+^{(\theta)}(\mu_j, \mu_k)}{a_+^{(\theta)}(\mu_{M+1}, \mu_k)} a_+^{(\theta)}(\mu_{M+1}, \mu_k),
\] (49)
where we have also used the functional relation (16) and the unitarity relations (7)–(9) for simplification.

So we multiply the previous expression by additional $B(\mu_i)$ operators,

$$S_M(\{\mu\}_{i=M+1}^{2M}\{\mu\}_{i=1}^{M}) = \langle \uparrow | C(\mu_1) \cdots C(\mu_M)B(\mu_{M+1}) \prod_{i=M+2}^{2M} B(\mu_i) | \uparrow \rangle.$$  

This results in a recurrence relation for the scalar product given by

$$S_M(\{\mu\}_{i=M+1}^{2M}\{\mu\}_{i=1}^{M}) = \sum_{j,k=1, j\neq k}^{M+1} \frac{a(\mu_k)d(\mu_j)c_-(\mu_j; \mu_{M+1})c_+(\mu_{M+1}; \mu_k)a_+(\mu_j; \mu_k)}{a_+(\mu_j; \mu_{M+1})a_+(\mu_{M+1}; \mu_k)b_-(\mu_j; \mu_k)}$$

$$\times \prod_{i=1}^{M+1} a_+(\mu_i; \mu_k) a_+(\mu_j; \mu_i) b_-^{(\mu_i, \mu_k)} b_-^{(\mu_j, \mu_i)} S_{M-1}(\{\mu\}_{i=M+2}^{2M}\{\mu\}_{i=1}^{M}).$$  

(50)

One can iterate the above recursion relation (50) and represent the resulting expression as follows:

$$S_M(\{\lambda\}|\{\mu\}) = \sum_{\{\mu\} = \{\lambda^+\} \cup \{\mu^-\}} \prod_{i=1}^{n_+} a(\lambda_i^+) d(\mu_i^+)^n \prod_{i=1}^{n_-} a(\mu_i^-) d(\lambda_i^-) K_M(\{\lambda^+\}|\{\mu^+\}) K_M(\{\lambda^-\}|\{\mu^-\}),$$  

(51)

where we sum with respect to all partitions of the set $\{\lambda\} \cup \{\mu\}$ into two disjoint sets $\{\lambda^+\} \cup \{\mu^-\}$ and $\{\lambda^-\} \cup \{\mu^+\}$ of $M$ of elements. If the number of elements in the sets $\{\lambda^+\}$ and $\{\mu^+\}$ is $n_+ = |\lambda^+| = |\mu^+|$, then we have $n_- = |\lambda^-| = |\mu^-| = M - n_+$ elements in the sets $\{\lambda^-\}$ and $\{\mu^-\}$.

Notice that any coefficient $K_M$ is determined only by terms arising from algebraic relations among monodromy matrix elements. Therefore, $K_M$ is independent of the representation, i.e. independent of the choice of the functions $a(\lambda, \{\nu_k\})$ and $d(\lambda, \{\nu_k\})$ [2].

In view of that one can fix the coefficients $K_M$ using any special representation. Let us consider the case where $L = M$ and

$$a_M(\lambda) = \prod_{i=1}^{M} a_+^{(\lambda, \nu_i)}, \quad d_M(\lambda) = \prod_{i=1}^{M} b_-(\lambda, \nu_i),$$  

(52)

where we have fixed the inhomogeneities as follows:

$$\nu_i = \begin{cases} 
\lambda_i^+, & i = 1, \ldots, n_+ \\
\mu_i^{-n_+}, & i = n_+ + 1, \ldots, M.
\end{cases}$$  

(53)

Using the fact that the unitarity relation (5) implies that $b_-(\lambda, \lambda) = 0$, one sees that $d_M(\lambda) = 0$ for any $\lambda \in \{\lambda^+\} \cup \{\mu^-\}$ and $d_M(\lambda) \neq 0$ for any $\lambda \in \{\lambda^-\} \cup \{\mu^+\}$. Consequently,
General scalar products in the arbitrary six-vertex model

there is only one non-vanishing term in the sum (51), which allows us to write

$$\prod_{i=1}^{n_+} a_M(\lambda_i^+) d_M(\mu_i^+) \prod_{i=1}^{n_-} a_M(\lambda_i^-) d_M(\mu_i^-) K_M(\{\lambda^+\}, \{\lambda^-\}; \{\mu^+\}, \{\mu^-\})$$

$$= \langle \uparrow_M | C(\mu_1) \cdots C(\mu_M) B(\lambda_M) \cdots B(\lambda_1) | \uparrow_M \rangle,$$

where $| \uparrow_M \rangle$ is a state with $M$ spins up.

The product of $B$ operators overturns all $M$ spins resulting in the state $| \psi_M \rangle$. Therefore, we have

$$\langle \uparrow_M | C(\mu_1) \cdots C(\mu_M) B(\lambda_M) \cdots B(\lambda_1) | \uparrow_M \rangle$$

$$= Z_M^{(C)}(\{\mu\}; \{\lambda^+\} \cup \{\mu_-\}) Z_M^{(B)}(\{\lambda\}; \{\lambda^+\} \cup \{\mu_-\}),$$

where the functions $Z_M^{(B,C)}$ are defined by

$$Z_M^{(B)}(\{\lambda\}; \{\lambda^+\} \cup \{\mu_-\}) = \langle \downarrow_M | B(\lambda_M) \cdots B(\lambda_1) | \downarrow_M \rangle,$$

$$Z_M^{(C)}(\{\mu\}; \{\lambda^+\} \cup \{\mu_-\}) = \langle \uparrow_M | C(\mu_1) \cdots C(\mu_M) | \downarrow_M \rangle.$$

Finally, the coefficient $K_M$ can be written in terms of the above functions as follows:

$$K_M \left( \begin{array}{l} \{\lambda^+\} \\ \{\mu^+\} \\ \{\lambda^-\} \\ \{\mu^-\} \end{array} \right) = \frac{Z_M^{(C)}(\{\mu\}; \{\lambda^+\} \cup \{\mu_-\}) Z_M^{(B)}(\{\lambda\}; \{\lambda^+\} \cup \{\mu_-\})}{\prod_{i=1}^{n_+} a_M(\lambda_i^+) d_M(\mu_i^+) \prod_{i=1}^{n_-} a_M(\lambda_i^-) d_M(\lambda_i^-)}.$$

The functions (56) and (57) are usually called domain wall partition functions [5] and play an important role in the calculation of scalar products and correlation function. We shall compute these partition functions in section 4 in order to write a closed formula for the scalar product (51).

4. The domain wall partition function

In this section we derive a recurrence relation for the partition function for the asymmetric six-vertex model with domain wall boundary conditions for arbitrary Boltzmann weights. This way we will proceed along the same lines of [11] and define a couple of auxiliary one-point boundary correlation functions as follows:

$$G_{M,N}^{(B)} = \frac{1}{Z_M^{(B)}} \langle \downarrow_M | B(\lambda_M) \cdots B(\lambda_{N+1}) p_1^+ B(\lambda_N) B(\lambda_{N-1}) \cdots B(\lambda_1) | \uparrow_M \rangle,$$

$$H_{M,N}^{(B)} = \frac{1}{Z_M^{(B)}} \langle \downarrow_M | B(\lambda_M) \cdots B(\lambda_{N+1}) p_1^- B(\lambda_N) p_1^+ B(\lambda_{N-1}) \cdots B(\lambda_1) | \uparrow_M \rangle,$$

where $p_1^\pm = \frac{1}{2}(1 \pm \sigma_1^\pm)$ are the local spin up and down projectors and $G_{M,M}^{(B)} = 1$. The above boundary correlations are related by

$$G_{M,N}^{(B)} = \sum_{j=1}^N H_{M,j}^{(B)},$$

$$G_{M,N}^{(B)} = H_{M,N}^{(B)} + G_{M,N-1}^{(B)}.$$

doi:10.1088/1742-5468/2011/11/P11015
One can use the above relations in order to write the domain wall partition function
\[ Z_M^{(B)}(\{\lambda\}; \{\nu\}) = \sum_{j=1}^{M} \langle \psi_M | B(\lambda_M) \cdots B(\lambda_{j+1}) p_1^+ B(\lambda_j) p_1^- B(\lambda_{j-1}) \cdots B(\lambda_1) | \uparrow_M \rangle. \] (63)

To derive the recurrence relation for the partition function we use the two-site model decomposition. More specifically, one has to decompose the monodromy matrix into two parts and introduce two monodromy matrices:
\[ T_A(\lambda) = T_A^{(2)}(\lambda) T_A^{(1)}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \] (64)

such that
\[ T_A^{(1)} = R_{A1}(\lambda, \nu_1) = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{pmatrix}, \] (65)
\[ T_A^{(2)} = R_{AM}(\lambda, \nu_M) \cdots R_{A2}(\lambda, \nu_2) = \begin{pmatrix} A_2(\lambda) & B_2(\lambda) \\ C_2(\lambda) & D_2(\lambda) \end{pmatrix}, \] (66)

which act on the states \( |\uparrow_i\rangle, i = 1, 2 \) given by
\[ |\uparrow_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\uparrow_2\rangle = \bigotimes_{j=2}^{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \] (67)

In particular, the required monodromy matrix elements are readily obtained:
\[ B(\lambda) = A_2(\lambda) B_1(\lambda) + B_2(\lambda) D_1(\lambda), \] (68)
\[ B_1(\lambda) = c_+(\lambda, \nu_1) \sigma_+, \] (69)
\[ D_1(\lambda) = b_-(\lambda, \nu_1) p_1^+ + a_-(\lambda, \nu_1) p_1^- \] (70)

and the action of the operator \( A_2(\lambda) \) on the state \( |\uparrow_2\rangle \) is given by
\[ A_2(\lambda) |\uparrow_2\rangle = \prod_{i=2}^{M} a_+(\lambda, \nu_i) |\uparrow_2\rangle. \] (71)

Using (68)–(70), we can rewrite (63) in terms of the operators \( A_2(\lambda) \) and \( B_2(\lambda) \) as follows:
\[ Z_M^{(B)}(\{\lambda\}; \{\nu\}) = \sum_{j=1}^{M} c_+(\lambda_j, \nu_1) \prod_{i=1}^{j-1} b_-(\lambda_i, \nu_1) \prod_{i=j+1}^{M} a_-(\lambda_i, \nu_1) \times \langle \psi_2 | B_2(\lambda_M) \cdots B_2(\lambda_{j+1}) A_2(\lambda_j) B_2(\lambda_{j-1}) \cdots B_2(\lambda_1) | \uparrow_2 \rangle. \] (72)
At this point one should use the algebraic relations (24) and (37) \[11\] in order to obtain the recurrence relation for the domain wall partition function:

\[
Z_M^{(B)}(\{\lambda\}_{i=1}^M; \{\nu\}_{i=1}^M) = \sum_{j=1}^M c_+(\lambda_j, \nu_1) \prod_{i=1 \atop i \neq j}^M b_-(\lambda_i, \nu_1) \prod_{i=1}^M a_+(\lambda_j, \nu_i) \\
\times \prod_{i=1 \atop i \neq 1}^M \frac{a_+^{(\theta)}(\lambda_i, \lambda_j)}{b_-(\lambda_i, \lambda_j)} Z_{M-1}^{(B)}(\{\lambda\}_{i=1}^M; \{\nu\}_{i=1}^M).
\]

(73)

Iterating the above relation \(M - 1\) times, one gets the solution as a sum over all permutations \(P\) of \(\{\lambda\}_{i=1}^M\):

\[
Z_M^{(B)}(\{\lambda\}_{i=1}^M; \{\nu\}_{i=1}^M) = \sum_P \prod_{i=1}^M c_+(\lambda_P, \nu_i) \prod_{i=1 \atop i < j}^M a_+^{(\theta)}(\lambda_{P_i}, \lambda_{P_j}) b_-(\lambda_{P_i}, \nu_i) b_-(\lambda_{P_j}, \nu_i).
\]

(74)

We still have to determine the partition function \(Z_M^{(C)}\). The calculation goes in a similar way as above. Therefore, we just present the final result for the recurrence relation:

\[
Z_M^{(C)}(\{\mu\}_{i=1}^M; \{\nu\}_{i=1}^M) = \sum_{j=1}^M c_- (\mu_j, \nu_M) \prod_{i=1 \atop i \neq j}^M b_- (\mu_i, \nu_M) \prod_{i=1}^M a_+ (\mu_j, \nu_i) \\
\times \prod_{i=1 \atop i \neq 1}^M \frac{a_+^{(\theta)}(\mu_i, \mu_j)}{b_-(\mu_i, \mu_j)} Z_{M-1}^{(C)}(\{\mu\}_{i=1}^M; \{\nu\}_{i=1}^M),
\]

(75)

whose solution obtained by iteration is given as a sum over all permutations \(\bar{P}\) of \(\{\mu\}_{i=1}^M\):

\[
Z_M^{(C)}(\{\mu\}_{i=1}^M; \{\nu\}_{i=1}^M) = \sum_P \prod_{i=1}^M c_- (\mu_P, \nu_i) \prod_{i=1 \atop i < j}^M a_+^{(\theta)}(\mu_{P_i}, \mu_{P_j}) b_-(\mu_{P_i}, \nu_i) b_-(\mu_{P_j}, \nu_i).
\]

(76)

5. Explicit expression

Using the special choice for the inhomogeneities (53) and the recurrence relations (73) and (75), we can write a simplified expression for the coefficient \(K_M\) as follows:

\[
K_M \left( \{\lambda^+\}; \{\mu^+\} \right) = \prod_{j=1 \atop j \neq 1}^{n_+} \prod_{k=1}^{n_-} \frac{a_+ (\mu_j^+, \mu_k^-)}{b_-(\mu_j^+, \mu_k^-)} \prod_{j,k=1 \atop j < k}^{n_+} \frac{a_- (\lambda_j^+, \lambda_k^+)}{a_+ (\lambda_j^+, \lambda_k^-)} \\
\times \frac{Z_M^{(C)}(\{\mu^+\}; \{\lambda^+\}) Z_M^{(B)}(\{\lambda^-\}; \{\mu^-\})}{\prod_{j,k=1 \atop j \neq 1}^{n_+} b_-(\mu_j^+, \lambda_k^+) \prod_{j,k=1 \atop j \neq 1}^{n_-} b_-(\lambda_j^-, \mu_k^-)}.
\]

(77)
Finally, if we substitute the expression (74), (76) and (77) in (51), we obtain an explicit expression for the scalar product for arbitrary Boltzmann weights:

\[
S_M(\{\lambda\}|\{\mu\}) = \sum_{\{\lambda\}=\{\lambda^+\} \cup \{\lambda^-\}} \frac{\prod_{i=1}^{n_+} a(\lambda_i^+)d(\mu_i^+) \prod_{k=1}^{n_2} b(\mu_k^-)}{\prod_{j,k=1}^{n_2} b(\mu_j^+, \lambda_k^-) \prod_{j,k=1}^{n_2} b(\lambda_j^-, \mu_k^-)} \times \prod_{j=1}^{n_2} \prod_{k=1}^{n_2} \frac{a_+(\mu_j^+, \mu_k^-) a_-(\lambda_j^-, \lambda_k^+)}{b_-(\mu_j^+, \mu_k^-) b_-(\lambda_j^-, \lambda_k^+)} \times \prod_{i=1}^{n_+} \frac{a_{+}(\mu_i^+, \mu_i^-)}{b_{-}(\mu_i^+, \mu_i^-)} \times \prod_{i=1}^{n_+} \frac{a_{+}(\lambda_i^-, \lambda_i^+)}{b_{-}(\lambda_i^-, \lambda_i^+)}.
\] (78)

6. Conclusion

In this paper we obtained an explicit expression for the scalar product and the domain wall partition function for the arbitrary asymmetric six-vertex model. We have presented the algebraic Bethe ansatz for the asymmetric six-vertex model under the perspective of working with arbitrary Boltzmann weights. Using the main ingredients of the algebraic Bethe ansatz, we have been able to derive a recurrence relation for the scalar product. We managed to write the solution of this recurrence relation in terms of the domain wall partition functions. Then, we obtained a closed formula for the required partition functions. Finally, we have obtained an explicit expression for the general scalar product in terms of the arbitrary Boltzmann weights. We have done all that using only the Yang–Baxter algebra, the unitarity relation and the Yang–Baxter relations. It remains as an open question whether or not the formula (78) (or similar formula [9]) can be recast in the determinant form (or any other equally useful form) using only the Yang–Baxter relations and the unitarity. However, it was recently shown that a similar formula can be written in the determinant form providing that the explicit Boltzmann weights are assigned and that certain polynomial identities among the Boltzmann weights are fulfilled [9]. We expect that the strategy of working with general Boltzmann weights would be helpful in dealing with other integrable models. In particular, we hope this should be fruitful for integrable models solvable by a non-nested Bethe ansatz.

Acknowledgments

The author thanks F Göhmann for introducing him to this topic and M J Martins for discussions. This work has been supported by FAPESP and CNPq.
Appendix: Two-and three-particle state

In this appendix we work out in detail the commutation of the operator $A(\lambda)$ with two and three $B$ operators in order to exemplify the use of Yang–Baxter relations (11)–(22). In fact, to pass the $A$ operator over any number of $B$’s one needs only to use the Yang–Baxter relation (20).

### A.1. $M = 2$

Let us start with the two $B$ operator cases and use the commutation rule (25) twice, which results in

$$A(\lambda)B(\lambda_2)B(\lambda_1) = \frac{a_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda)} B(\lambda_2)B(\lambda_1) A(\lambda)$$

$$- \frac{c_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda_2)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda_2)} B(\lambda)B(\lambda_1) A(\lambda_2)$$

$$- \left[ \frac{a_+ (\lambda_2, \lambda) c_+ (\lambda_1, \lambda)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda)} B(\lambda_2)B(\lambda) \right] \text{equation (24)}$$

$$- \frac{c_+ (\lambda_2, \lambda_1) c_+ (\lambda_1, \lambda_2)}{b_- (\lambda_2, \lambda_1) b_- (\lambda_1, \lambda_2)} B(\lambda)B(\lambda_2) A(\lambda_1). \quad (A.1)$$

We can further manipulate the third term of the above expression using the commutation rule (24) and the unitarity relation (5) as indicated. This provides us with the following expression:

$$A(\lambda)B(\lambda_2)B(\lambda_1) = \frac{a_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda)} B(\lambda_2)B(\lambda_1) A(\lambda)$$

$$- \frac{c_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda_2)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda_2)} B(\lambda)B(\lambda_1) A(\lambda_2)$$

$$- \left[ \frac{b_- (\lambda_2, \lambda_1) a_- (\lambda_2, \lambda) c_+ (\lambda_1, \lambda) + c_- (\lambda_2, \lambda_1) c_+ (\lambda_2, \lambda) b_- (\lambda_1, \lambda)}{b_- (\lambda_2, \lambda_1) b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda)} \right]$$

$$\times B(\lambda)B(\lambda_2) A(\lambda_1). \quad (A.2)$$

Finally, one can see that the resulting expression for the third term coincides with the left-hand side of the Yang–Baxter relation (20). After substituting (20) in (A.2), we obtain

$$A(\lambda)B(\lambda_2)B(\lambda_1) = \frac{a_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda)} B(\lambda_2)B(\lambda_1) A(\lambda)$$

$$- \frac{c_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda_2)}{b_- (\lambda_2, \lambda) b_- (\lambda_1, \lambda_2)} B(\lambda)B(\lambda_1) A(\lambda_2)$$

$$- \frac{c_+ (\lambda_1, \lambda) a_- (\lambda_2, \lambda_1)}{b_- (\lambda_1, \lambda) b_- (\lambda_2, \lambda_1)} B(\lambda)B(\lambda_2) A(\lambda_1). \quad (A.3)$$

DOI: 10.1088/1742-5468/2011/11/P11015
A.2. \( M = 3 \)

Now we turn to the case where we have three \( B \) operators. We use the previous result (A.3) and the commutation rule (24), such that

\[
A(\lambda)B(\lambda_3)B(\lambda_2)B(\lambda_1) = \frac{a_+(\lambda_3, \lambda) a_+(\lambda_2, \lambda) a_+(\lambda_1, \lambda)}{b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda) b_-(\lambda_1, \lambda)} B(\lambda_3)B(\lambda_2)B(\lambda_1)A(\lambda) \\
- \frac{c_+(\lambda_3, \lambda) a_+(\lambda_2, \lambda_3) a_+(\lambda_1, \lambda_3)}{b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda_3) b_-(\lambda_1, \lambda_3)} B(\lambda)B(\lambda_3)B(\lambda_1)A(\lambda_3) \\
- \frac{c_+(\lambda_2, \lambda) a_-(\lambda_3, \lambda_2) a_+(\lambda_1, \lambda_2)}{b_-(\lambda_2, \lambda) b_-(\lambda_3, \lambda_2) b_-(\lambda_1, \lambda_2)} B(\lambda)B(\lambda_3)B(\lambda_1)A(\lambda_2) \\
+ \left[ -\frac{c_+(\lambda_1, \lambda) a_+(\lambda_3, \lambda) a_+(\lambda_2, \lambda)}{b_-(\lambda_1, \lambda) b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda)} \right] B(\lambda_3)B(\lambda_2)B(\lambda) \quad \text{equation (24)} \\
+ \frac{c_+(\lambda_3, \lambda) c_+(\lambda_1, \lambda_3) a_+(\lambda_2, \lambda_3)}{b_-(\lambda_3, \lambda) b_-(\lambda_1, \lambda_3) b_-(\lambda_2, \lambda_3)} B(\lambda) B(\lambda_2)B(\lambda_3) \quad \text{equation (24)} \\
+ \frac{c_+(\lambda_2, \lambda) c_+(\lambda_1, \lambda_2) a_-(\lambda_3, \lambda_2)}{b_-(\lambda_2, \lambda) b_-(\lambda_1, \lambda_2) b_-(\lambda_3, \lambda_2)} B(\lambda)B(\lambda_3)B(\lambda_2) \right] A(\lambda_1). \quad \text{(A.4)}
\]

Again one should use the commutation rule (24) and unitarity relation (5) as indicated. So the last term of (A.4) can be rewritten as follows:

\[
- \frac{c_+(\lambda_1, \lambda) a_-(\lambda_3, \lambda) a_-(\lambda_2, \lambda)}{b_-(\lambda_1, \lambda) b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda)} \frac{c_+(\lambda_3, \lambda) c_+(\lambda_1, \lambda_3) a_-(\lambda_2, \lambda_3)}{b_-(\lambda_3, \lambda) b_-(\lambda_1, \lambda_3) b_-(\lambda_2, \lambda_3)} \\
+ \frac{c_+(\lambda_2, \lambda) c_-(\lambda_2, \lambda_1) a_-(\lambda_3, \lambda_2)}{b_-(\lambda_2, \lambda) b_-(\lambda_2, \lambda_1) b_-(\lambda_3, \lambda_2)} B(\lambda) B(\lambda_3)B(\lambda_2)A(\lambda_1). \quad \text{(A.5)}
\]

At this point, we have to simplify the term inside the square bracket \((I_1)\). This can be done by joining the first and the last term in (A.5) and using equation (20) a couple of times, such that

\[
I_1 = \frac{[b_-(\lambda_2, \lambda_1) a_-(\lambda_2, \lambda) c_+(\lambda_1, \lambda)] a_-(\lambda_3, \lambda) b_-(\lambda_3, \lambda_2)}{b_-(\lambda_1, \lambda) b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda_3) b_-(\lambda_3, \lambda_2)} \quad \text{equation (20)} \\
+ \frac{[a_-(\lambda_3, \lambda_2) b_-(\lambda_3, \lambda) c_+(\lambda_2, \lambda)] c_-(\lambda_2, \lambda_1) b_-(\lambda_1, \lambda)}{b_-(\lambda_1, \lambda) b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda) b_-(\lambda_2, \lambda_1) b_-(\lambda_3, \lambda_2)} \\
- \frac{c_+(\lambda_3, \lambda) c_+(\lambda_1, \lambda_3) a_-(\lambda_2, \lambda_3)}{b_-(\lambda_3, \lambda) b_-(\lambda_1, \lambda_3) b_-(\lambda_2, \lambda_3)}, \quad \text{(A.6)}
\]

\[\text{doi:10.1088/1742-5468/2011/11/P11015} \]
and

\[
I_1 = \frac{a_-(\lambda_3, \lambda)}{b_-(\lambda_3, \lambda)} \left[ \frac{b_-(\lambda_2, \lambda_1)a_-(\lambda_2, \lambda)c_+ (\lambda_1, \lambda)}{b_-(\lambda_2, \lambda_1) b_-(\lambda_2, \lambda) b_-(\lambda_1, \lambda)} + \frac{c_-(\lambda_2, \lambda_1)c_+ (\lambda_2, \lambda)b_-(\lambda_1, \lambda)}{b_-(\lambda_2, \lambda_1) b_-(\lambda_2, \lambda) b_-(\lambda_1, \lambda)} \right]
\]

Then we again have to use equation (20) twice:

\[
I_1 = \frac{c_+ (\lambda_1, \lambda) a_-(\lambda_3, \lambda) a_-(\lambda_2, \lambda_1)}{b_-(\lambda_1, \lambda) b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda_1)}
\]

\[
- \frac{c_+ (\lambda_3, \lambda) \left[ c_-(\lambda_2, \lambda_1)c_+ (\lambda_2, \lambda_3)b_-(\lambda_1, \lambda_3) + b_-(\lambda_2, \lambda_1) a_-(\lambda_2, \lambda_3)c_+ (\lambda_1, \lambda_3) \right]}{b_-(\lambda_2, \lambda_1) b_-(\lambda_2, \lambda_3) b_-(\lambda_1, \lambda_3)}
\]

\[
= \frac{a_-(\lambda_2, \lambda_1)}{b_-(\lambda_2, \lambda_1)} \left[ \frac{c_+ (\lambda_1, \lambda) a_-(\lambda_3, \lambda)}{b_-(\lambda_1, \lambda)} - \frac{c_+ (\lambda_1, \lambda_3) c_+ (\lambda_3, \lambda)}{b_-(\lambda_1, \lambda_3) b_-(\lambda_3, \lambda)} \right]
\]

and

\[
I_1 = \frac{a_-(\lambda_2, \lambda_1)}{b_-(\lambda_2, \lambda_1)} \left[ \frac{b_-(\lambda_3, \lambda_1) a_-(\lambda_3, \lambda)c_+ (\lambda_1, \lambda)}{b_-(\lambda_3, \lambda_1) b_-(\lambda_3, \lambda) b_-(\lambda_1, \lambda)} \right]
\]

\[
= \frac{a_-(\lambda_2, \lambda_1) a_-(\lambda_3, \lambda_1) c_+ (\lambda_1, \lambda)}{b_-(\lambda_2, \lambda_1) b_-(\lambda_3, \lambda_1) b_-(\lambda_1, \lambda)}
\]

Finally, we can substitute the final result for the term \(I_1\) (A.11) in equation (A.4):

\[
A(\lambda) B(\lambda_3) B(\lambda_2) B(\lambda_1) = \frac{a_+ (\lambda_3, \lambda) a_+ (\lambda_2, \lambda) a_+ (\lambda_1, \lambda)}{b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda) b_-(\lambda_1, \lambda)} B(\lambda_3) B(\lambda_2) B(\lambda_1) A(\lambda)
\]

\[
- \frac{c_+ (\lambda_3, \lambda) a_+ (\lambda_2, \lambda_3) a_+ (\lambda_1, \lambda_3)}{b_-(\lambda_3, \lambda) b_-(\lambda_2, \lambda_3) b_-(\lambda_1, \lambda_3)} B(\lambda) B(\lambda_2) B(\lambda_1) A(\lambda_3)
\]

\[
- \frac{c_+ (\lambda_2, \lambda) a_-(\lambda_3, \lambda_2) a_+ (\lambda_1, \lambda_2)}{b_-(\lambda_2, \lambda) b_-(\lambda_3, \lambda_2) b_-(\lambda_1, \lambda_2)} B(\lambda) B(\lambda_3) B(\lambda_1) A(\lambda_2)
\]

\[
- \frac{c_+ (\lambda_1, \lambda) a_-(\lambda_3, \lambda_1) a_-(\lambda_2, \lambda_1)}{b_-(\lambda_1, \lambda) b_-(\lambda_3, \lambda_1) b_-(\lambda_2, \lambda_1)} B(\lambda) B(\lambda_3) B(\lambda_2) A(\lambda_1)
\]

This expression coincides with the formula (37) for the case \(M = 3\). Note that we have only used the commutation rules (24) and (25), the unitarity relation (5) and the Yang–Baxter relation (20). It is remarkable that the same applies to any \(M\) values, where we again would need only the mentioned relations.

Similarly, the expression (38) can be obtained using the commutation rules (24) and (26), the unitarity relation (5) and the Yang–Baxter relation (12).
Alternatively, one could have obtained the formula (37) in a shorter way. Note that the first term in (37) is obtained straightforwardly using the first term of the algebraic relation (25) M times. On the other hand, the second term of (37) can be very complicated, as we have seen above. The coefficient of the term not containing the first \( B \) operator, \( B(\lambda_M) \), is very simple though. This coefficient is obtained using the second term of (25) in order to exchange the \( \Lambda(\lambda) \) and \( B(\lambda_M) \) and in all other steps one has to use only the first term of (25) [4], which results in the following expression:

\[
A(\lambda, \{\nu_k\}) \prod_{i=1}^{M} B(\lambda_i, \{\nu_k\}) = \prod_{i=1}^{M} \frac{a_+ (\lambda_i, \lambda)}{b_-(\lambda_i, \lambda)} \prod_{i=1}^{M} B(\lambda_i, \{\nu_k\}) A(\lambda, \{\nu_k\}) \\
- \frac{c_+ (\lambda_M, \lambda)}{b_- (\lambda_M, \lambda)} \prod_{i=1}^{M} \frac{a_+ (\lambda_i, \lambda)}{b_- (\lambda_i, \lambda)} B(\lambda, \{\nu_k\}) \prod_{i=1}^{M} B(\lambda_i, \{\nu_k\}) A(\lambda_M, \{\nu_k\}) \\
+ \text{sum of other terms not containing } B(\lambda_j), \quad j = 1, \ldots, M - 1. \tag{A.13}
\]

However, one could have done similar analysis for any \( B(\lambda_j) \). In doing so, one has to use the relation (24) in order to move \( B(\lambda_j) \) to the leftmost position:

\[
B(\lambda_M) \cdots B(\lambda_{j+1}) B(\lambda_j) \cdots B(\lambda_1) = B(\lambda_j) \prod_{\substack{i=1 \atop i \neq j}}^{M} B(\lambda_i) \theta_>(\lambda_i, \lambda_j) \tag{A.14}
\]

and then repeat the above-described procedure. This would result in the missing terms in (A.13) and finally we should obtain the expression (37).

References

[1] Takhtadzhan L A and Faddeev L D, 1979 Russ. Math. Surv. 34 11
[2] Korepin V E, Bogoliubov N M and Izergin A G, 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press)
[3] Baxter R J, 1982 Exactly Solved Models in Statistical Mechanics (New York: Academic)
[4] Takhtajan L A, 1985 Introduction to Algebraic Bethe Ansatz (Lecture Notes in Physics vol 242) (Berlin: Springer) p 175
[5] Faddeev L D, How algebraic Bethe ansatz works for integrable models, 1998 Quantum Symmetries (Les Houches, 1995) (Les Houches Summerschool Proc. vol. 64) ed A Connes et al (Amsterdam: North-Holland) pp 149–219
[6] Korepin V E, 1982 Commun. Math. Phys. 86 391
[7] Izergin A G and Korepin V E, 1984 Commun. Math. Phys. 94 67
[8] Izergin A G and Korepin V E, 1985 Commun. Math. Phys. 99 271
[9] Kitanine N, Maillet J M and Terras V, 1999 Nucl. Phys. B 554 647
[10] Kitanine N, Maillet J M and Terras V, 2000 Nucl. Phys. B 567 554
[11] Kitanine N, Maillet J M, Slavnov N A and Terras V, 2004 Int. J. Mod. Phys. A 19 248
[12] Melo C S and Martins M J, 2009 Nucl. Phys. B 806 567
[13] Martins M J and Zuparic M, 2011 Nucl. Phys. B 851 565
[14] Baxter R J, 2002 J. Stat. Phys. 108 1
[15] Baxter R J, 1971 Studies Appl. Math. 50 51
[16] Bogoliubov N M, Pronko A G and Zvonarev M B, 2002 J. Phys. A: Math. Gen. 35 5525