Influence of the electron–phonon interfacial conductance on the thermal transport at metal/dielectric interfaces

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Abstract
Thermal boundary conductance at a metal-dielectric interface is a quantity of prime importance for heat management at the nanoscale. While the boundary conductance is usually ascribed to the coupling between metal phonons and dielectric phonons, in this work we examine the influence of a direct coupling between the metal electrons and the dielectric phonons. The effect of electron–phonon processes is generally believed to be resistive and tends to decrease the overall thermal boundary conductance as compared to the phonon–phonon conductance $\sigma_p$. Here, we find that the effect of a direct electron-phonon interfacial coupling $\sigma_e$ is to enhance the effective thermal conductance between the metal and the dielectric. Resistive effects turn out to be important only for thin films of metals that have a low electron–phonon coupling strength. Two approaches are explored to reach these conclusions. First, we present an analytical solution of the two-temperature model to compute the effective conductance which accounts for all the relevant energy channels, as a function of $\sigma_e$, $\sigma_p$ and the electron–phonon coupling factor $G$. Second, we use numerical resolution to examine the influence of $\sigma_e$ on two realistic cases: a gold film on silicon or silica substrates. We point out the implications for the interpretation of time-resolved thermoreflectance experiments.

Keywords: nanoscale heat transfer, electron–phonon conductance, analytical theory

1. Introduction
Interfacial thermal transport is of major importance in heat application management. Indeed, ohmic contacts are ubiquitous in microelectronics and experience increasing load levels. As a result, a fundamental understanding of the cooling kinetics of heated metals or semiconductors at ohmic contacts is a prerequisite for designing electronic components that can support large heat flux densities [1]. Thermal relaxation of microelectronic components is primarily governed by the boundary conductance, since, as the density of an interface increases, thermal losses in the bulk become negligible compared to interfacial resistance. While the boundary resistance a priori involves energy transfer between metal electrons or metal phonons and the phonons in the substrate, most of the models developed so far consider only the latter channel of energy. These include the acoustic mismatch model (AMM), where phonon transmission at an interface is assumed to be controlled by the difference in the acoustic impedances of the two media [2] and the diffuse mismatch model (DMM) which posits that interfacial roughness destroys any correlation between incident and reflected phonons [3].

The boundary resistance of metal-diamond interfaces at room temperature was first experimentally determined by Stoner and Maris in 1993 [4], using time-resolved thermoreflectance (TTR). This work was followed by the characterization of various interfaces, extending the early measurements to higher temperatures [5–8]. More recently, TTR has been applied to investigate heat transfer across metal-graphite [9] and metal-carbon nanotube interfaces as well [10]. In parallel, theoretical progress has been achieved through molecular dynamics [11, 12] which ignores electronic degrees
The direct route studied in this work and the bulk metal, characterized by theoretical predictions exist. Diffusive transport represented by dots plays a role in both cases.

of freedom, or treats them in the spirit of the two-temperature model. A recurrent conclusion of this body of work is that the Kapitza conductances measured between two solids depart significantly from the classical AMM and DMM models, which often yield comparable values. For instance, the thermal conductance of one of the most studied systems, the Pb-diamond interface, is found to be typically sevenfold higher than the DMM prediction.

Following the early room temperature measurements of Stoner and Maris, several authors have proposed to relate this excess of conductance to electron–phonon scattering taking place across the interface. Indeed, there are at least two mechanisms according to which electrons may heat up the substrate, as represented in figure 1: The first one has been extensively studied and involves an indirect coupling between the electrons and the substrate through phonon–phonon processes. This mechanism is only operative when the electrons are not in equilibrium with the lattice. In this situation, the effect of the electron–phonon process is to introduce an additional resistance $1/\sqrt{Gk_p}$ where $G$ is the electron–phonon coupling factor and $k_p$ the phononic thermal conductivity of the metal. Hence, it is found that the indirect electron–phonon processes deteriorate the interfacial energy transfer, as compared to the case where electronic degrees of freedom are neglected.

There is another energy channel that has recently been unveiled from the experimental side and which has been previously predicted theoretically. This mechanism is a direct coupling between electrons and phonons taking place at the interface, with a heat flux $\sigma_e(T_e - T_s)$ that depends on the temperature difference and on the electron–phonon interfacial conductance $\sigma_e$. Throughout the 2000s decade and until recently, such electron–phonon transfer mechanism has been disregarded essentially because experiments where the metal is replaced by a Bi semi-metal conclude that the values of the interfacial conductance seem to be independent of the nature of the electronic transport properties of the film. However, there are actually two hints that the direct electron–phonon channel could play a significant role. First, the above mentioned models for the conductance $\sigma_e$ predict values that are comparable or even largely exceed the typical values for the phonon–phonon conductance $\sigma_p$. Second, recent transient thermoreflectance experiments on an Au film concluded on the existence of an electron-substrate energy transfer, with an estimate of $\sigma_e$ in the range of 100–1000 MW m$^{-2}$ K$^{-1}$, compatible with the theoretical predictions. The consequences of this direct channel of energy have not been discussed in the literature.

In this article, we undertake a theoretical description of the competition between the three different channels represented in figure 1. We show that electron–phonon processes taking place in the bulk of the metal induce a small, commonly negligible resistance, while the effect of the direct coupling is to enhance interfacial heat transfer. Significantly, we find that the effect of a direct coupling persists over long time scales on the nanosecond range, even though the electrons and the phonons in the metal have long been allowed to equilibrate. Indeed, if the largest values of $\sigma_e$ are experimentally reported in situations where electrons are out of equilibrium with the metal lattice, a non vanishing value $\sigma_e > 100$ MW m$^{-2}$ K$^{-1}$ is still measured in case of negligible electron–phonon nonequilibrium in the metal. This indicates that this channel of interfacial transfer should be operative in the TTR measurements of metal/dielectric interfaces and may explain the large discrepancies observed between the DMM predictions and the TTR measurements.

The article is organized as follows: In section 2, we first motivate the study by calculating the interfacial electron–phonon conductance $\sigma_e$ for different interfaces using Sergeev’s model. Using analytical calculations on the two-temperature model, we examine in section 3 the influence of a finite $\sigma_e$ on the temperature decay of a heated metal. In section 4, we illustrate the effect of $\sigma_e$ for two specific systems: gold/silicon and gold/silica interfaces. We show that for electron–phonon conductance larger than the phonon–phonon conductance, it is actually the dominant mechanism governing the temperature decay. Our conclusions are given in section 5.

2. Theoretical estimates for electron–phonon and phonon–phonon conductances

In this section, we motivate our study by estimating, for various metal/non-metal systems, the electron–phonon and phonon–phonon interfacial conductances. In both cases, we rely on theoretical predictions.

2.1. Electron–phonon conductance $\sigma_e$

There are three analytical models predicting the electron–phonon interfacial conductance at a metal/dielectric interface. Huberman and Overhauser first propose that metal electrons couple with joint vibrational modes at the interface leading to a significant enhancement of the energy transfer,
as compared with direct phonon–phonon coupling. Later, Sergeev developed a field theory to predict the effect of the boundaries on the inelastic scattering of conduction electrons [15]. Finally, Mahan derived a theory where metal electrons may transfer heat to the substrate through the image charges created by the vibrating ions of the non-metal [17]. This model should mainly apply to ionic crystals or polar semi-conductors, rather than to apolar non-metals. In both Huberman and Mahan’s models the conductance is predicted to be constant at high temperatures, while in Sergeev’s model, it is found to increase linearly with temperature\textsuperscript{1}, in agreement with experimental observations [23]. Because it is the sole model to capture this temperature dependence, we will consider only Sergeev’s model in the following.

As far as we know, there has been no attempt to quantify the values of the electron–phonon interferential conductance predicted by Sergeev’s model for real cases at ambient temperatures. Here, we determine the values of $\sigma_e$ for different metal/dielectric interfaces. We do so by using input data from density functional theory (DFT) calculations predicting the electron–phonon coupling of several noble metals. To proceed, we start with the expression

$$\sigma_e = \frac{3\pi\hbar}{35\xi(3)k_B} \frac{\gamma u_L}{\tau_{e-p}} \left[ 1 + 2 \left( \frac{u_L}{u_T} \right)^3 \right],$$

where $\hbar$ is the reduced Planck constant, $k_B$ is the Boltzmann constant, $\xi$ is the Riemann zeta function, $\gamma$ is the Sommerfeld constant of the metal, $u_L$ and $u_T$ are respectively the metal longitudinal and transverse sound velocities and $\tau_{e-p}$ is the electron–phonon energy relaxation time of the bulk metal. One immediately obtains $\sigma_e \sim T$ because at a high temperature $1/\tau_{e-p} \sim T_c$ [27] while other factors have weak or no temperature dependence. Here, we estimate the electron–phonon energy relaxation time using its relation with the electron–phonon coupling factor [27,28]:

$$G = \frac{\pi^2 m_e c_s^2 n_e}{6 \tau_{e-p} T_c},$$

where $m_e$ is the effective mass of the electron, $c_s$ is the speed of sound and $n_e$ is the number density of free electrons. Besides, one has:

$$\gamma_T \equiv \frac{G}{C_e} = \frac{3\hbar\lambda(\omega^2)}{\pi k_B T_c},$$

where $\lambda$ is the electron–phonon coupling constant, so that the electron–phonon conductance equation (1) may be cast as

$$\sigma_e = \frac{54\gamma_T^2 \hbar^2 \lambda(\omega^2)}{35\pi^2 k_B^2 m_e c_s n_e} T_c \left[ 1 + 2 \left( \frac{u_L}{u_T} \right)^3 \right].$$

To calculate $\sigma_e$, we have used the values of $\lambda(\omega^2)$ from the DFT calculations of Lin et al [27] for noble metals. The effective mass $m_e^*$ is taken as equal to the electron mass $m_e$, except for Platinum, for which we have used $m_e^* = 13m_e$ [29]. For Bismuth, we have calculated the value of $\lambda(\omega^2)$\textsuperscript{1}

\footnotetext{1 Such temperature dependence stems from the inelastic nature of electron scattering at the interface.}

from the thermodynamic analysis of Giret et al [30,31], who reported an electron–phonon coupling factor $G = 1.45 \times 10^{16}$ W m$^{-2}$ K$^{-1}$. Table 1 gathers the values of the electron–phonon interfacial conductance for common metals and the semi-metal Bi.

| Metal | $n$ (mol m$^{-3}$) | $u_L$ (m s$^{-1}$) | $u_T$ (m s$^{-1}$) | $n_e$ | $\gamma$ | $\lambda(\omega^2)$ | $\sigma_e$ (100 MW m$^{-2}$ K$^{-1}$) |
|-------|------------------|-----------------|-----------------|------|-------|-----------------|---------------------|
| Al    | 100 000          | 6240            | 3040            | 18.1 | 1.3   | 185.9           | 2.64                |
| Au    | 97 970           | 3390            | 1290            | 5.9  | 0.67  | 23 $\pm$ 4      | 3.08                |
| Bi    | 46 794           | 1543            | 1107            | 14.1 | 0.08  | 229.3           | 0.02                |
| Cr    | 138 269          | 6980            | 4100            | 16.6 | 1.57  | 128             | 6.2                 |
| Pb    | 55 990           | 2350            | 970             | 13.2 | 2.93  | 45 $\pm$ 5      | 17.8                |
| Pt    | 110 872          | 4174            | 1750            | 3.2  | 6.54  | 142.5           | 186                 |

Table 1. Physical parameters for the metals: molar density $n$ (mol m$^{-3}$), longitudinal and transverse sound velocities $u_L$ and $u_T$ (m s$^{-1}$), free electron number density $n_e$ (10$^{27}$ m$^{-3}$). Sommerfeld’s constant $\gamma$ (mJ mol$^{-1}$ K$^{-2}$), factor $\lambda(\omega^2)$ (meV$^2$) and electron–phonon conductance $\sigma_e$ (100 MW m$^{-2}$ K$^{-1}$) at room temperature.

2.2. Phonon–phonon conductance $\sigma_p$

Here we use the diffuse mismatch model [3], as we will briefly discuss now. Quite generally, the phonon–phonon thermal boundary conductance is given by

$$\sigma_p = \frac{1}{2} \sum_p \int_0^{\omega_{1,p}} v_{1,p} \hbar \omega D_{1,p}(\omega) \frac{\partial f_{eq}(\omega, T)}{\partial T} \frac{\cos \theta}{\cos \theta d(\cos \theta) d\omega},$$

where the sum runs over the polarizations $p$, $\omega_{1,p}$ is the cutoff frequency, $v_{1,p}$ is the mode group velocity, $D_{1,p}(\omega)$ is the density of states, $f_{eq}$ is the equilibrium Bose–Einstein distribution and $\sigma_p(\omega, \cos \theta)$ is the phonon transmission coefficient. If interfacial scattering is supposed to be diffuse, the transmission coefficient no longer depends on the angle of incidence and from detailed balance it follows that

$$\alpha = \sum_p \frac{1}{v_{1,p}^2} = \frac{1}{\sum_p v_{1,p}^2} + \frac{1}{\sum_p v_{2,p}^2},$$

where the density of states is taken to be given by Debye’s model. Note that the transmission coefficient is assumed to be polarization independent [32]. To compute $\sigma_p$, we have used the acoustic properties of the different metals and dielectrics provided in tables 1 and 2.

Figure 2 gathers the electron–phonon and phonon–phonon conductances for the different systems considered. It is apparent that in most cases $\sigma_e$ is much larger than $\sigma_p$. This implies that for most of the metal/dielectric interfaces, $\sigma_e$ may not be neglected and should be accounted for in the analysis of thermoreflectance data. Qualitatively, one expects that compared to the bare phonon–phonon conductance, the apparent thermal conductance of the systems discussed should be enhanced, as will be investigated quantitatively in the next section.
sections. Moreover, note that even Bismuth which is a semi-metal and not considered a good metal is predicted to have a significant $\sigma_e$. This suggests that the relevant energy channel discussed here should be present not only in metals, but also in semi-metals despite the fact that the number of free electrons is relatively small. Hence, even materials which display poor electronic transport properties such as semi-metals, could be affected by electron–phonon interfacial coupling. This is because for these materials the electron–phonon relaxation time takes values slightly larger than but comparable with those of good metals.

It should be kept in mind however, that our estimates of the transport coefficient $\sigma_e$ come from a model which may have some limitations. In particular, Sergeev’s prediction yields a conductance which depends only on the bulk properties of the metal and not on the nature of the substrate. Also, it should be noted that for gold systems, Sergeev’s model overestimates by a factor of two the electron–phonon conductance reported experimentally [23]. The solid lines are the values obtained from equations (17) and (18). The blue dashed line shows Sergeev’s predictions for gold, as given by equation (4).

choose any specific value or model, but rather explore a wide range of values for $\sigma_e$.

3. Analytical predictions

In this section we derive analytical expressions for the characteristic cooling time of a heated metal/dielectric interface, taking into account not only the phonon–phonon conductance, but also the electron–phonon conductance. The analytical solutions may allow us to understand quantitatively the effect of $\sigma_e$ on the effective interfacial conductance which accounts for the different channels and analyze the limiting behaviours.

3.1. Model and parameters

The situation relevant to the TTR experiments is schematically depicted in figure 4: a metal film with thickness $h$ on a semi-infinite dielectric substrate. To keep calculations tractable,

2 The dielectric substrate is not infinite, but has a finite length $H$ in the $x$-direction. However, the associated time scale $H^2/\alpha_s$ is so large that it is generally irrelevant in experiments.
we assume that the problem is 1D, with variation occurring only perpendicular to the interface (x-direction). In principle, the cooling kinetics should be described by the solution of the Boltzmann transport equation, but the long time regime of the order of 1000 ps that we discuss here is generally well captured by the two-temperature model where the electrons and phonons are each characterized by a temperature, subject to diffusion and relaxation [21, 28, 33]. The evolution equations are

\[ c_e \partial_t T_e = k_e \partial_x^2 T_e - G(T_e - T_p), \]

\[ c_p \partial_t T_p = k_p \partial_x^2 T_p + G(T_e - T_p), \]

\[ c_s \partial_t T_i = k_i \partial_x^2 T_i. \]

Here, we use subscripts \( i = e, p \) and \( s \) for the metal electrons, the metal phonons and the substrate phonons respectively, \( c_i \) and \( k_i \) are the heat capacities and thermal conductivities, \( G \) is the coupling factor. Unless otherwise mentioned, we will consider that all coefficients are constant and in particular independent of temperature. All quantities are given in SI units, unless indicated otherwise. As regards the boundary conditions, we assume a zero energy flux at the air-metal interface, while the metal-dielectric interface is characterized by two boundary conductances,

for \( x = 0 \), \[-k_e \partial_x T_e = -k_p \partial_x T_p = 0, \]

for \( x = h \), \[-k_i \partial_x T_i = \alpha_e(T_e - T_i) + \alpha_p(T_p - T_i), \]

\[ = -k_e \partial_x T_e - k_p \partial_x T_p. \]

Here \( \alpha_p \) and \( \alpha_e \) are the phonon–phonon and electron–phonon conductances respectively and the second equation expresses the continuity of the energy flux across the interface. Finally, the initial conditions are

\[ T_e(x) = T^i(x), \quad T_p = T_i = 0. \]  \hspace{1cm} (7)

\( T_e, T_p \) and \( T_i \) are thus the differences with respect to the temperature \( T_0 \) of the system prior to any laser heating\(^3\). The initial temperature profile \( T^i(x) \) is exponential with decay length \( \delta \), the penetration depth of the laser in the metal film:

\[ T^i(x) = T^i(0) e^{-x/\delta}, \quad T^i(0) = \frac{h}{\delta(1 - e^{-h/\delta})}. \]

The normalization constant \( T^i(0) \) is such that the average temperature increase in the film is unity\(^4\). Below, we focus on the electron temperature at the surface of the film and its long-time behavior\(^5\).

Table 3 lists the various quantities needed in the problem, from which one can construct the relaxation, diffusion and conductance times. The relaxation time \( \tau_i = c_i/G \) is related to the electron–phonon coupling in the bulk of the metal film, \( \tau_{di} = h^2/\alpha_i \) is the typical time for diffusion over the film and \( \tau_{ci} = c_i/\alpha_i \) characterizes the cooling of the carriers induced by the interfacial conductances. Taking ratios between these characteristic times leads to dimensionless parameters: the \( R \) number compares the electron and phonon relaxation times, the Biot numbers \( B_{e,p} \) compare the diffusion and conductance times and the \( M_{e,p} \) numbers compare the relaxation and diffusion times. Below we focus on the influence of the two interface conductances, with \( \alpha_p \) taken in the broad range \( 10^5 \)–\( 10^{11} \) and \( \alpha_p \) in the range \( 10^2 \)–\( 10^6 \) W m\(^{-2}\) K\(^{-1}\), which should encompass most cases of experimental relevance. All other parameters are kept fixed to a value representative of real systems, those default values are also given in table 3.

3.2. Influence of the electron–phonon conductance \( \sigma_e \)

We now investigate how, within the two-temperature model, a finite electron–phonon conductance \( \sigma_e \) affects the temperature decay. We first present a simplified model, whose analytical solution is straightforward.

3.2.1. Fast diffusion approximation. To obtain compact formulas, we consider here the two-temperature model in the fast diffusion (FD) approximation: the temperature is

\(^3\) Because all coefficients are assumed to be constant, \( T_0 \) is irrelevant.

\(^4\) The normalization constant is irrelevant when the long-time decay is exponential but will play a role when the decay is power-law.

\(^5\) In experiments, what is probed is actually a combination of electron and phonon temperatures.
assumed to be uniform within the film, implying that both
electrons and phonons diffusion are neglected. With the vector
notation $T = [T_e, T_p]^T$, where $T$ denotes the transpose, the
equations are
\[
\partial_t T = -M \tau, \quad M = \begin{bmatrix}
\frac{1}{\tau_e} + \frac{1}{\tau_p} & -\frac{1}{\tau_e} \\
-\frac{1}{\tau_p} & \frac{1}{\tau_p} + \frac{1}{\tau_e}
\end{bmatrix}.
\]
The decay is found to be exponential, with a characteristic
decay time $\tau$ given by
\[
\tau = \frac{2}{Tr - \sqrt{Tr^2 - 4D}},
\]
\[
Tr = Tr(M) = \frac{1}{\tau_e} + \frac{1}{\tau_p} + \frac{1}{\tau ce} + \frac{1}{\tau cp},
\]
\[
D = \text{Det}(M) = \frac{1}{\tau_e \tau_{cp}} + \frac{1}{\tau_p \tau_{ce}} + \frac{1}{\tau_{ce} \tau_{cp}}.
\]
To get a qualitative understanding of the $\tau$ dependence on $\sigma_e$, we
first consider two limiting values for $\sigma_e \rightarrow 0$ and $\sigma_e \rightarrow \infty$, which are denoted as $\tau_{max}$ and $\tau_{min}$. Physically, $\tau_{max}$
represents the metal’s cooling time in the absence of any
electron–phonon conductance, while $\tau_{min}$ is the cooling time
in the limit of high electron–phonon conductance. On physical
grounds, one can expect that upon increasing $\sigma_e$, $\tau$ decreases
monotonically from $\tau_{max}$ to $\tau_{min}$. To simplify even further, we
use the fact that $\tau_e$ is often small compared to all others time
involved in the problem, i.e. $\tau_e \ll \tau_p \tau_{ce}$, $\tau_{cp}$. Then
\[
\tau_{max} = \tau_{cp} \left[ 1 + \frac{\tau_p}{\tau_p} + O(\tau_e^2) \right],
\]
\[
\tau_{min} = \frac{\tau_p \tau_{cp}}{\tau_p + \tau_{cp}} + O(\tau_e).
\]
Compared to equation (8), both expressions are numerically
accurate and no further term in the expansion is necessary. As
regards the ratio $r$, the above approximations yields
\[
r = \frac{\tau_{min}}{\tau_{max}} = \frac{1}{(1 + \tau_e/\tau_p)(1 + \tau_{cp}/\tau_p)},
\]
\[
\simeq \frac{1}{1 + \tau_{cp}/\tau_p} \quad \text{for } \tau_e \ll \tau_p.
\]
This shows that at low $\sigma_p$ values, $\tau$ depend largely on $\sigma_e$. For
instance\textsuperscript{6}, taking $\sigma_p = 10^3$ yields $r \simeq 1/100$, i.e. one obtains
a 100-fold reduction in $\tau$ when $\sigma_e$ increases from 0 to $\infty$.

For conductances $\sigma_p$ smaller than $10^3$, $\tau$ is actually rather
well approximated by the simple expression
\[
\frac{\tau - \tau_{min}}{\tau_{max} - \tau_{min}} = \frac{1}{1 + \sigma_e/\sigma_p}.
\]
Deviations amount to only a few percents for $\sigma_p < 10^3$ and
occur only at very large $\sigma_p$, where $\sigma_p$ should be replaced
with a function $g(\sigma_p)$ which increases slightly less than $\sigma_p$
\textsuperscript{6} We remind that conductances and other physical quantities are expressed in
SI units, unless otherwise indicated.

(see below). The resulting cooling time is shown in figure 5,
which demonstrates a strong effect of the electron–phonon
interfacial conductance $\sigma_e$, especially for low values of $\sigma_p$.
A useful approximation for the following may be obtained, if
we consider that $\tau_{min} \simeq \frac{\tau_{cp}}{\tau_p} \tau_p \tau_{cp}$ and $\tau_{max} \simeq \tau_{cp}$, which is a
reasonable approximation given that $\tau_{cp} \gg \tau_p$ with the default
parameters. This yields
\[
\tau \simeq \tau_p + \tau_{cp},
\]
\[
\tau_{cp} = \frac{1}{1 + \sigma_e/\sigma_p}.
\]

3.2.2. Full two-temperature model. The FD model neglects
diffusion of the energy carriers in the metal, which
may induce additional resistance to the heat flow. Here, we go
one step further as compared with the FD model and account
explicitly for the electron and phonon diffusion. As detailed in
appendix A, the long-time behavior of the temperature decay
depends on whether the substrate temperature is constant or
not. In the latter case, the decay is power-law, without any
characteristic time scale. In the former case, the decay may
be characterized by a single decay time $\tau$. This is the reason
why we use this cold substrate approximation and determine
analytically how the characteristic decay time $\tau$ depends
on $\sigma_p$ and $\sigma_e$. We find that if $\sigma_e \gg \sigma_p$, the temperature decay
is much faster than that expected for zero $\sigma_e$.

Using $h$ and $\tau_e$ as the length and time units, the relevant
equations, boundary and initial conditions read as:
\[
\partial_t T_e = \frac{1}{M_e} \partial_{xx} T_e - (T_e - T_p),
\]
\[
R \partial_t T_p = \frac{1}{M_p} \partial_{xx} T_p + (T_e - T_p),
\]
\[
\partial_x T_e = 0, \quad \partial_x T_p = 0, \quad \text{for } x = 0,
\]
\[
-\partial_x T_e = B_e T_e, \quad -\partial_x T_p = B_p T_p, \quad \text{for } x = 1,
\]
\[
T_e = T^t, \quad T_p = 0, \quad \text{for } t = 0.
\]
There are thus six dimensionless parameters: $B_e$, $B_p$, $M_e$, $M_p$, $R$ and $h$.

The general solution of the previous set of equations is not exponential. However, in order to characterize
the long time decay, we consider Laplace transforms with
respect to time and take its expansion close to zero frequency.
As further explained in the appendices A and B, this gives an
estimate of the longest relaxation time characterizing the long
time cooling kinetics. Even if this relies on an approximation,
we believe that it can reflect faithfully the influence of $\sigma_e$.

For the two-temperature model considered here, the
characteristic decay time $\tau$ can be obtained analytically using
a symbolic computation software, but the expression is much
too large to be reported. As shown in figure 5 for the default
parameters and a wide range of $\sigma_p$ and $\sigma_e$, large $\sigma_e$ can
result in a significant decrease of the decay time. This trend
may be captured analytically as detailed in the appendix B.
Finally, as visible in figure 5, the FD model provides a
very reasonable approximation to the full two-temperature
model, showing that even when ‘neglecting’ diffusion, one
can recover the qualitative behavior. This suggests that
the effective conductance characterizing the different energy
channels present at a metal/non metal interface, does not depend
primarily on the diffusion of the energy carriers in the metal.
3.3. Effective thermal boundary conductance

From the solution of the two models analyzed above, we can predict the effect of the different electron–phonon processes on the effective thermal boundary conductance $\sigma_{\text{eff}}$. This latter quantity is the apparent interfacial conductance extracted from a TTR signal using a one-temperature model. Because the one-temperature model ignores the electronic degrees of freedom, the effective conductance depends on both the electron–phonon coupling constant and the conductance $\sigma_e$. Here, we discuss this dependence. The effective conductance is computed using

$$\sigma_{\text{eff}} = \frac{q}{(T_p - T_0)},$$

(13)

where $q = -h \frac{\partial T}{\partial x}$ is the interfacial heat flux density. Note that here we are assuming the phonon temperature to be uniform, in the spirit of the FD model as we concluded that phonon diffusion within the metal film does not change significantly the metal cooling kinetics.

Because in the FD model discussed previously, the long time decay of the metal lattice temperature is exponential, with a time constant $\tau$ the effective conductance equation (13) is given by: $\sigma_{\text{eff}} = c \rho h / \tau$. Using equation (12) for the characteristic time, we predict the following expression of the conductance:

$$1/\sigma_{\text{eff}} = 1/Gh + 1/(\sigma_e + \sigma_p).$$

(14)

This equation has a simple interpretation: the effect of the electron–phonon processes is equivalent to add in series the bulk resistance $1/Gh$ with the total interfacial resistance $1/(\sigma_e + \sigma_p)$. This latter resistance comprises two parallel interfacial resistances quantifying electron–phonon and phonon–phonon processes at the interface. At this point, it is important to realize that the expression of the bulk resistance $1/Gh$ differs from the expression $1/\sqrt{Gh \kappa_p}$ derived by Majumdar and Reddy [16]. The difference stems from the different boundary conditions considered for the electrons: Majumdar and Reddy [16] assumed an insulating boundary condition for the electrons, as they were not allowed to directly exchange energy with the substrate. Here, because we consider the possibility of a direct exchange at the interface, the boundary condition is not adiabatic. This difference of boundary condition changes the expression of the internal resistance governed by electron–phonon processes in the metal thin film.

Finally, we discuss the traditional separation between equilibrium and non-equilibrium contributions to the effective conductance [16, 21]. The expression of the conductance equation (14) has been derived under non-equilibrium conditions. If we had assumed equilibrium, we would have obtained the intuitive expression of the interfacial conductance $\sigma_e + \sigma_p$ which expresses the fact that the two energy channels -electrons and phonons- act in parallel to transmit heat to the substrate. Hence, we find that the non-equilibrium contribution consistent with an isothermal boundary condition is

$$\sigma_{\text{neq}} = Gh,$$

(15)

and is found to be larger for thick metal films $h > 100\,\text{nm}$ and metals having a large electron–phonon coupling $G > 10^{13}\,\text{W m}^{-2}\text{K}^{-1}$. For these systems, the non-equilibrium conductance is $\sigma_{\text{neq}} > 10^{10}\,\text{W m}^{-2}\text{K}^{-1}$ and the corresponding resistance is practically negligible, as compared with the phonon–phonon and electron–phonon interfacial resistances. On the other hand, for thin Au films with $h < 10\,\text{nm}$ the non-equilibrium conductance is $\sigma_{\text{neq}} < 250\,\text{MW m}^{-2}\text{K}^{-1}$ and may become comparable with the interfacial conductances $\sigma_p$ and $\sigma_e$. We conclude then that in metal/non metal superlattices with a small period, non-equilibrium effects can not be neglected and should contribute to enhance the thermal resistance.

We now discuss the effect of a finite $\sigma_e$ on the value of the effective conductance equation (14). All the expressions of the characteristic time derived so far clearly indicate that the electron–phonon conductance $\sigma_e$ has a strong effect on the long time decay of the temperature of the metal and may significantly accelerate the cooling kinetics. To further illustrate this point, we have considered the relative enhancement of the metal cooling induced by the electron–phonon conductance. This enhancement is simply defined by the ratio $\chi = \sigma_{\text{eff}}(\sigma_e) / \sigma_{\text{eff}}(\sigma_e = 0)$ and as such may be considered as a simple estimate of the increase of the apparent thermal boundary conductance induced by $\sigma_e$. Figure 6 shows that the relative enhancement may reach one or two orders of magnitude, depending on the value of the phonon–phonon conductance $\sigma_p$. The enhancement is most significant when the electron–phonon conductance $\sigma_e$ becomes greater than $\sigma_p$, a situation that might be common, as discussed in section 2. Again, we conclude that the largest enhancements are observed when the metal/substrate interface transmits poorly the phonons, which corresponds to the case of acoustically mismatched solids.
predicted by the fast diffusion approximation equation (14) with default parameters.

As regards the heat capacity of the substrate, we used experimental values obtained from standard expression [34].

We have discussed above the effect of the interfaces characterizing the metal and the dielectric. Here, we consider two realistic cases, which allows to take into account the temperature dependence of the transport coefficients. We focus on the gold/silicon and gold/silica systems, for which values for the different transport and thermodynamic quantities are known [23].

|       |       |       |       |       |       |
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4. Two realistic cases: gold/silicon and gold/silica interfaces

4.1. Methods and parameters

We have discussed above the effect of \( \sigma_e \) using a set of typical values for the different transport and thermodynamic quantities characterizing the metal and the dielectric. Here, we consider two realistic cases, which allows to take into account the temperature dependence of the transport coefficients. We focus on the gold/silicon and gold/silica systems, for which values for the electron-phonon interfacial conductance are known [23].

The electron-phonon coupling factor was set to the value \( G = 2.5 \times 10^{10} \text{ MW m}^{-2} \text{ K}^{-1} \) predicted by DFT calculations when the electronic temperature is below 5000 K [27]. The values of the other thermophysical parameters at 300 K are given in table 4. The heat capacity of the gold electrons is calculated with \( C_e = \gamma T_e \) where \( \gamma = 65.64 \text{ J m}^{-3} \text{ K}^{-2} \) is Sommerfeld’s constant of gold, while the vibrational heat capacity is obtained from standard expression [34].

The electron-phonon conductance for silica and silicon was extracted from [23], using the following expressions

\[
\sigma_e^{\text{Au/Si}} = 25.18 + 0.363 T, \tag{17}
\]

\[
\sigma_e^{\text{Au/SiO}_2} = 96.12 + 0.189 T, \tag{18}
\]

where temperature is given in K and conductances in MW m\(^{-2}\) K\(^{-1}\). Equations (17) and (18) provide a reasonable description of the experimental data, as shown in figure 3.

The equations of the two-temperature model are solved using a finite-difference scheme, with perfectly matched layers at the boundaries so as to simulate a quasi-infinite substrate, whose thickness is fixed at 50 μm. Gradients are estimated using finite differences with three successive lattice points, while the condition \( h(T(x^+,t) - T(x^-,t)) = k \delta T \) is enforced at the interface using two points to evaluate the local gradient. The typical lattice step is 5 nm; the time step is chosen so that the maximal temperature increment between two successive time steps is \( \Delta T = \max_{x,t} |T(x,t_{n+1}) - T(x,t_n)| = 0.01 \text{ K} \) if \( t < 1 \text{ ps} \); \( \Delta T = 0.5 \text{ K} \) otherwise unless the temperature difference becomes \( |T - T_0| < 2 \text{ K} \), when a finer criterion \( \Delta T = 0.05 \text{ K} \) is retained. Here and in all the following \( T_0 \) is the initial temperature of the system, prior to any heating. The initial value of the electronic temperature is \( T_e = T_0 + 2000 \text{ K} \), the metal penetration depth is \( \delta = 20 \text{ nm} \) and the film thickness is fixed to \( h = 100 \text{ nm} \).

4.2. Results

We present now the results obtained by numerically solving the two-temperature model, using the physical parameters detailed above. As in section 3, we focus on the electronic temperature of the metal at the air/metal surface, since this is a simple measure of the metal reflectivity, which is probed in the time-resolved thermoreflectance experiments.

Figure 7(a) displays the temporal evolution of the surface electron temperature for the gold/silicon system. For each temperature, curves with the electron-phonon conductance switched off (\( \sigma_e = 0 \)) are also shown. The difference between the two cases is striking. Clearly, discarding the electron-phonon channel results in an electron cooling which is much slower. After 1000 ps, the electronic temperature is almost twice larger in the case of zero \( \sigma_e \). Figure 7(a) thus illustrates the key point of this work: in the time interval relevant to thermoreflectance measurements 100–1000 ps, the...
electron–phonon interfacial coupling has a strong effect on the temperature decay. Therefore it should be taken into account, along with the phonon–phonon conductance. It should be acknowledged at this point, that for the nanosecond time regime we are considering here, electrons and phonons in the metal are practically at equilibrium as illustrated in figure 8. A slight difference between electron and phonon temperature is however visible when $\sigma_e = 0$, as a result of the adiabatic boundary condition obeyed by the electrons in this case.

To quantify the effect of $\sigma_e$ on the temperature decay, we introduce a time $\tau_{1/2}$ defined as the time after which the electronic temperature has cooled down to half its initial value. $\tau_{1/2}$ is a simple estimate characterizing the cooling kinetics, that may be used in experiments. Figure 9 displays $\tau_{1/2}$ as a function of the initial temperature $T_0$. The effect of the interfacial electron–phonon coupling is clearly seen: in presence of this channel, the electron cooling is typically a factor two to three faster than in its absence. In terms of apparent thermal boundary conductance, this corresponds to a change by an amount ranging from 200 to 300%. Again, we see that interfacial electron–phonon coupling should contribute significantly to the cooling process and to the apparent value of the thermal boundary conductance. Note also that figure 9 shows cooling times that are practically independent of the initial temperature $T_0$, in both cases where $\sigma_e$ has been set to zero or not. This may certainly be explained by the use of a constant phonon–phonon conductance in the model. The use of a temperature dependent electron–phonon conductance $\sigma_e$ does not change the dependence on $T_0$.

We have also characterized the effect of $\sigma_e$ on the cooling kinetics of the gold/silica system. As illustrated in figure 7(b), the temperature decay is weakly dependent on the interfacial
electron–phonon conductance. This is confirmed by figure 9 where $\tau_{1/2}$ is seen to be practically insensitive to the presence of the additional interfacial energy channel. This relative difference between the cases of gold/silicon and gold/silica may be explained by two factors. The first is the difference in relative magnitude of the two conductances: while $\sigma_e$ is twice larger than $\sigma_p$ for gold/silicon, they are nearly equal in the case of gold/silica. The other factor is the difference of thermal conductivities of the substrates, with silicon having a conductivity two orders of magnitude higher than silica. As a result, any energy flux coming from the metal’s electrons is conveyed away from the interface. In contrast, heat in the silica remains confined in the vicinity of the interface, slowing down any energy transfer coming from the metal’s electrons and resulting in the relative insensitivity to $\sigma_e$ in the cooling kinetics of the gold/silica system.

As a final illustration of the relevance of the electron–phonon boundary conductance, we consider for the case of a gold/silicon interface a conductance $\sigma_e$ varying in a wide range. The time $\tau_{1/2}$ is shown in figure 10, where the abscissa axis has been normalized by the ‘experimental’ value given by equation (17). The decrease of the cooling time $\tau_{1/2}$ is very sharp for low $\sigma_e$, suggesting that even a moderate electron/substrate coupling may lead to an enhanced thermal conductance. For larger values of $\sigma_e$, the decrease is somewhat less pronounced, but still leading to a threefold faster cooling rate for the highest value studied. Again, this roughly corresponds to a factor three in the apparent thermal boundary resistance. Note that the corresponding value of $\sigma_e \approx 1300 \text{ MW m}^{-2} \text{ K}^{-1}$ may be realistic to describe metals other than gold, as we discussed in section 2.

5. Conclusion

In summary, we used a combination of analytical work and numerical simulations of the two-temperature model to probe the influence of a direct interfacial electron–phonon energy transfer in the thermal transport between a metal, or a semi-metal and a dielectric substrate. We have shown that Sergeev’s model predicts for most metal/non metal systems considered here an electron–phonon conductance $\sigma_e$ which is larger or comparable with the phonon–phonon conductance $\sigma_p$. Perhaps surprisingly, even semi-metals such as Bismuth, with poor electronic transport properties compared to those of noble metals, may involve a significant $\sigma_e$, but which remains smaller than $\sigma_p$. This suggests that electrons of materials which are poor electronic conductors may participate in interfacial heat transport.

To assess the effect of a finite electron–phonon interfacial conductance, we investigated the cooling of a metal thin film instantaneously heated by a strong laser pulse. This situation is of direct relevance to time-resolved thermoreflectance technique, which is used to measure the thermal boundary resistance between the metal and the substrate. The effective interfacial conductance consists in a non-equilibrium contribution $Gh$ and an equilibrium conductance $\sigma_p + \sigma_e$. The non-equilibrium conductance is found to be important for metals having a relatively low value of $G$ and for thin films a situation relevant to metal-dielectric superlattices or core-shell nanoparticles. The effect of $\sigma_e$ is found to enhance the apparent thermal boundary conductance as compared with $\sigma_p$, up to two orders of magnitude for acoustic mismatched interfaces, characterized by low values of $\sigma_p$ on the order of $10 \text{ MW m}^{-2} \text{ K}^{-1}$. This may explain the large discrepancies reported between the apparent conductances measured for solids having a large Debye temperature ratio and the classical DMM model [7]. Numerical simulations of the two-temperature model on two realistic cases confirm our analytical findings: while $\sigma_e$ has only a mild effect on the cooling of gold/silica system, it significantly speeds up the cooling of the gold/silicon system. Those distinct behaviors originate in the difference in the phonon–phonon conductance $\sigma_p$ and the thermal conductivities of the substrate.

On the theoretical side, one final remark is in order. We emphasized the strong limitations of Sergeev’s model to predict the electron–phonon conductance $\sigma_e$. In particular, Sergeev’s model disregards the nature of the substrate, in disagreement with the experimental results [23] and seems to overestimate the interfacial electron–phonon conductance, in the few cases when comparison with experiments is possible. Hence, there is a clear need to develop theoretical models for the electron–phonon conductance that would account for the nature of the non-metal substrate. Given the small number of $\sigma_e$ values that have been obtained experimentally so far, measurements for a variety of metal/substrate systems would be invaluable in this endeavour [37].

Appendix A. Characterizing the long-time temperature decay

This appendix focuses on the long-time behavior of the temperature decay. We find that when the substrate heating is
negligible, which we call the ‘cold substrate’ approximation, the long-time decay may be characterized by a single decay time $\tau$. On the other hand, if the substrate heating is significant the long-time decay is power-law, with no characteristic time. This distinction is essential when one wants to extract the thermal boundary conductance from the experimental curves. For simplicity, we use the simple one-temperature model to make this point, but it applies to the two-temperature model as well.

### A.1. Cold substrate

Here we assume that the temperature increase of the substrate is negligible, that is $T_s = 0$ at all time. Taking $h$ as the unit length and the phonon conductance time $\tau_{cp} = c_p h / \sigma_p$ as the unit time, the adimensionalized equations for the one-temperature model are

$$B_p \partial_x T_p = \partial_{xx}^2 T_p,$$

$$\partial_x T_p = 0 \quad \text{for} \quad x = 0,$n

$$B_p T_p + \partial_x T_p = 0 \quad \text{for} \quad x = 1,$n

$$T_p = T_i(x) \quad \text{for} \quad t = 0.$n

There are only two parameters: the Biot number $B_p = h \sigma_p / k_p$ and the $\delta/h$ ratio. We now introduce $\tilde{T}_p(x, s)$, the Laplace transform of $T_p$ with respect to time, which satisfies the equation

$$-\partial_{xx}^2 \tilde{T}_p + B_p \tilde{T}_p = B_p T_i(x),$$

whose solution is

$$\tilde{T}_p(x, s) = C_1 e^{\gamma x} + C_2 e^{-\gamma x} + \tilde{T}_{part}, \quad \gamma = \sqrt{B_p s},$$

$$\tilde{T}_{part} = \frac{B_p}{\gamma} \int_0^x \sinh \left[ \gamma (u - x) \right] T_i(u) du.$$

The constants $C_1$ and $C_2$ are obtained from the boundary conditions at $x = 0$ and $x = 1$. From now on, we focus on the temperature at the air/metal interface $\tilde{T}_p(s) = \tilde{T}_p(0, s)$ and consider the long-time behavior by taking a small-$s$ expansion. The result is $\tilde{T}_p(s) \simeq a_0 + a_1 s + O(s^2)$. Note that for a time dependent function $f(t)$, whose Laplace transform has the expansion $f(s) = a_0 + a_1 s + O(s^2)$, we can always characterize the long time decay of an effective time $\tau_{eff} = -a_1 / a_0$. If $f(t) \sim e^{-t/\tau}$ is a single exponential, one recovers $\tau_{eff} = \tau$. If on the other hand, $f(t)$ is a sum of exponential terms:

$$\tilde{T}_p(t) = \sum_n c_n e^{-t/\tau_n},$$

then the procedure yields

$$\tau_{eff} = \frac{\langle \tau^2 \rangle}{\langle \tau \rangle} \langle x \rangle \equiv \frac{\sum c_n x_n}{\sum c_n},$$

that is an average that takes into account the amplitude of each mode and gives more weight to the slow modes.

Following this procedure for the metal phonon temperature, we obtain the characteristic time $\tau$ for arbitrary penetration depth $\delta$

$$\tau = \left[ 3(B_p \delta - 1) (B_p (2 \delta^2 - 1) - 2) -2e^{\delta} (B_p (3B_p \delta^3 - 3(B_p + 1) \delta^2 + B_p + 3) + 3) \right]^{-1} \times \left[ 6 e^{\delta} (B_p (\delta - 1) - B_p \delta + 1) \right]^{-1}. \quad (21)$$

$$\tau = -\left[ -2 \cosh(2 \kappa (h^3 - 3h^2 \kappa + 3\delta^3)) \right.$$

$$+3\kappa (h - \delta) (h + \delta + 3h)$$

$$-3(\delta \kappa - 1) (h^2 \kappa + 2h - 2\delta^2)$$

$$\left. \times \left[ 6a_{cp}(e^{\delta/h} (h \kappa - \delta \kappa + 1) + \delta \kappa - 1) \right]^{-1}. \quad (22) \right.$$}

Equation (21) applies if length and time units are $h$ and $\tau_{cp}$ respectively. Equation (22) is given in terms of dimensional parameters and for simplicity of notations, we have introduced the inverse length $\kappa = \sigma_p / k_p$. This formula simplifies in the two limiting cases

$$\delta \rightarrow 0, \quad \frac{\tau}{\tau_{cp}} = 1 + \frac{B_p^2}{3(B_p + 1)}, \quad (23)$$

$$\delta \rightarrow \infty, \quad \frac{\tau}{\tau_{cp}} = 1 + \frac{B_p (5B_p + 8)}{12(B_p + 2)}.$$

The Biot number $B_p$ indicates the relative importance between the heat flux within the bulk of the film and the flux at the interface. If $B_p \ll 1$, the temperature is quasi-uniform within the film and the flux is limited by the interface resistance, then $\tau = \tau_{cp}$, whatever the initial profile. On the other hand, if $B_p \gg 1$, diffusion within the film is the limiting factor and $\tau \sim \kappa / \alpha_p = \tau_{cp}$ is the typical time of phonon diffusion over the film thickness $h$, with a prefactor that depends on the initial profile. Both limiting cases are visible in figure 11, which shows the characteristic time $\tau$ as a function of $\alpha_p$. Interestingly, with the default parameters, a typical value $\sigma_p = 10^8$ yields $B_p = 1$, meaning that none of the limits applies. And indeed, the difference is significant between $\tau_{cp}$ = 1000 ps and $\tau = 1167, 1256$ and 1361 ps for $\delta = 0, 20, \infty$ respectively.

### A.2. Heated substrate

Retracing the steps of the analysis above, one finds for the small-$s$ expansion of the surface temperature

$$\tilde{T}_p(s) = \frac{a_0 - a_1/s}{\sqrt{s}} + a_0 + a_1/s + a_2/s^2 + O(s^3), \quad (25)$$

When taking the inverse depth $\tau_{cp}$ into account, the terms with half-integer exponents yield power laws, which are dominant.
over other terms\(^8\). We thus expect for \(T_p(t)\) the approximation

\[
T_p(t) = \frac{a-1/2}{\sqrt{\pi}} t^{-1/2} - \frac{a1/2}{2\sqrt{\pi}} t^{-1/2} + \frac{3 a1/2}{4\sqrt{\pi}} t^{-1/2} + O(t^{-1})
\]

The key point is that the decay is power-law, since at the longest time, \(T_p(t) \approx a-1/2/\sqrt{\pi t}\). This is a general feature of diffusion on a half-line \(^{[38]}\). Furthermore, the coefficient \(a_{-1/2} = h c_p / \sqrt{c_r c_s}\) depends on the total energy injected and on the substrate properties, but not on the interfacial conductance \(\sigma_p\). In contrast with the cold substrate approximation treated above, it is not possible to define a single time scale that could characterize the temperature decay. Thus, those calculations with the one-temperature model show that two cases should be distinguished. If substrate heating is negligible, the decay may be characterized by an effective time \(\tau\) and presumably the decay time \(\tau\) can be easily extracted, from which, if all other parameters are known, \(\sigma_p\) can be deduced. If on the other hand, the cold substrate approximation does not apply, the temperature decay is power-law and at the longest time, completely independent of \(\sigma_p\). It might be possible, using fits of the experimental curve, to estimate the coefficients \(a_{1/2}, a_{-1/2}\) and subsequent and deduce \(\sigma_p\), but the process seems much more delicate. This conclusion applies \textit{a fortiori} to the two-temperature model.

**Appendix B. Two-temperature model in the cold substrate approximation**

Here, we apply to the full two-temperature model the analysis detailed in appendix A. To do so, let’s introduce the dimensionless parameters

\[
a^2 = M_e + M_p,
\]

\(8\) The terms \(a_0 + a_1\delta\) yields an exponential \(e^{-t/\tau_e}\), with \(\tau_e = -a_1/a_0\). For \(t > \tau_e\), it becomes quickly negligible with respect to the long-time tails.

\(9\) The coefficients \(a_{1/2}, a_{-1/2}\) and subsequent do depend on \(\sigma_p\).

The full expression for the characteristic time \(\tau\) is very unwieldy, in part because \(\tau\) involves powers of \(e^a\), an exponential term can not be easily approximated since \(a\) is neither small nor large: with the default parameters, \(a = 3.31\) is of order unity.

To get a qualitative understanding of the \(\tau\) dependence on \(\sigma_e\), we first consider the two limiting values for \(\sigma_e\), and \(\sigma_e \to \infty\), which are denoted as \(\tau_{\text{max}}\) and \(\tau_{\text{min}}\). Even after taking the limits, the expressions for \(\tau_{\text{max}}\) and \(\tau_{\text{min}}\) are still untractable. We therefore resort to two bold approximations. First, the limit \(\delta/t \to 0\) is taken, even if the value 0.2 is not so small. Second, a large-\(a\) expansion is made, even though \(e^a \simeq 27\) is not so large. The resulting expressions are quite simple

\[
\frac{\tau_{\text{max}}}{\tau_e} = 1 + \frac{c^2}{M_e} \left[ \frac{a^2}{B_p} + a - 1 + O \left( \frac{1}{a} \right) \right],
\]

\[
\frac{\tau_{\text{min}}}{\tau_e} = \frac{c^2}{3} + 4b + \frac{c^2}{3a} = \frac{3b + 2c^2}{3a^2} + O \left( \frac{1}{a^2} \right),
\]

and yields for the ratio \(r = \tau_{\text{min}}/\tau_{\text{max}}\)

\[
r = \frac{B_p M_e}{3a^2} \left[ 1 + \left( \frac{6b}{c^2} - 1 - B_p \right) \frac{B_p}{a} + O \left( \frac{1}{a^2} \right) \right].
\]

Our main conclusion follows from those approximate expressions. Large values of \(\sigma_e\) can induce, compared to the \(\sigma_e = 0\) case, a very pronounced decrease in \(\tau\). For instance, \(\sigma_p = 10^7\) yields \(r = 0.012\), that is a 100-fold reduction.

Table 5 compares exact numerical values for \(\tau_{\text{min}}, \tau_{\text{max}}\) and their approximation given by equations (30) and (31), which are surprisingly accurate, given the assumption on \(\delta\) and \(a\). Note that the approximate equation (31) predicts that \(\tau_{\text{min}}\) is independent of \(\sigma_p\). However, when taking the five-term expansion of \(\tau_{\text{min}}\), one recovers the slight decrease with \(\sigma_p\) which is observed for the exact \(\tau_{\text{min}}\). Finally, one can obtain a good approximation of \(\tau\) with the simple formula

\[
\frac{\tau - \tau_{\text{min}}}{\tau_{\text{max}} - \tau_{\text{min}}} = \frac{1}{1 + \frac{\sigma_p}{\sigma_{p,\text{max}}}}.
\]

**Table 5. Maximum and minimum decay times \(\tau_{\text{max}}\) and \(\tau_{\text{min}}\). App. and FD correspond to equations (30) and (9) for the former and equations (31) and (10) for the latter. The conductance \(\sigma_p\) is expressed in W m\(^{-2}\) K\(^{-1}\) and the times in ps.**

| \(\sigma_p\) | \(\tau_{\text{max}}\) App. | \(\tau_{\text{max}}\) FD | \(\tau_{\text{min}}\) App. | \(\tau_{\text{min}}\) FD |
|---|---|---|---|---|
| \(10^7\) | 10 288 | 10 314 | 10 100 | 90 |
| \(510^7\) | 2213 | 2234 | 20 280 | 88 |
| \(10^8\) | 1208 | 1224 | 10 010 | 86 |
| \(10^9\) | 322 | 315 | 101 78 | 78 |

\[
b^2 = \frac{(M_e^2 + R M_p^2)^2}{4(M_e + M_p)^3},
\]

\[
c^2 = \frac{M_e M_p (1 + R)}{M_e + M_p}.
\]
where \( g(\sigma_p) \) is a function plotted in figure 12. For low \( \sigma_p \ll 5 \times 10^7 \), \( g(\sigma_p) \approx \sigma_p \) while it becomes smaller at higher \( \sigma_p \).

Two side remarks are in order. First, the formulas given above for the coefficients \( a, b \) and \( c \) are valid for any dimensionless parameters but may be simplified in many cases. For instance, taking the default parameters of table 3, yields \( M_e = 1, M_p = 10, R = 1000 \), which suggests taking the limiting case \( R \gg 1 \) and \( M_e \ll M_p \) and leads to the simple expressions

\[
a^2 = M_p, \quad b^2 = \frac{M_p R^2}{4}, \quad c^2 = M_e R.
\]

Second, there is a simple limiting case for \( \tau_{\text{max}} \) as given by equation (30). In addition to \( R \gg 1 \), let’s assume that \( \tau_{\text{max}} \gg \tau_e \) and \( B_p \ll 1 \). With default parameters, the former approximation is well justified and the latter applies only for low \( \sigma_p \) values (\( B_p = 0.1 \) for \( \sigma_p = 10^7 \) for instance). Then the first term inside the brackets of equation (30) dominates over the others and one finds \( \tau_{\text{max}} = \tau_{\text{cp}} \), i.e. one recovers the characteristic time for the one-temperature model in the limit \( B_p \rightarrow 0 \), as given in appendix A.

Appendix C. Thermal conductivity for silicon and silica

We have used the following fits to describe the temperature dependence of the thermal conductivity:

\[
k_{\text{Si}} = 203.913 \, T^{-1.26}, \tag{34}
\]

\[
k_{\text{SiO}_2} = 0.624 \ln T - 2.19, \tag{35}
\]

where the temperature \( T \) is expressed in K and the conductivities in W m\(^{-1}\) K\(^{-1}\). The experimental values and fitting curves are displayed in figure 13.

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