Testing for Basins of Wada

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Nonlinear systems often give rise to fractal boundaries in phase space, hindering predictability. When a single boundary separates three or more different basins of attraction, we say that the set of basins has the Wada property and initial conditions near that boundary are even more unpredictable. Many physical systems of interest with this topological property appear in the literature. However, so far the only approach to study Wada basins has been restricted to two-dimensional phase spaces. Here we report a simple algorithm whose purpose is to look for the Wada property in a given dynamical system. Another benefit of this procedure is the possibility to classify and study intermediate situations known as partially Wada boundaries.

Sometimes a physical property can be labeled with different discrete values depending on parameters. Suppose a space has \( N_3 \geq 3 \) disjoint regions \( S_j \), where each \( S_j \) represents a different value or state. For example the sets might be basins of attraction in the phase space of a dynamical system. Numerical and experimental investigations of the property in question can be intricate when the boundaries between the sets are fractal. It may be difficult to predict the state transition of the system as the initial condition is disturbed. This unpredictability increases when the boundaries of \( S_j \) are not only fractal but also possess the Wada property; that is, each point on the boundary of any of these regions is in fact on the boundary of all of them. In other words, the sets share the same boundary.

This situation emerges for a variety of systems of high interest in physics such as the basins of the forced damped pendulum1, escapes in tokamaks2, dyes in open hydrodynamical flows3, the Duffing equation with a periodic forcing4, the Hénon-Heiles system5, and the use of Newton’s method to find complex roots6.

The problem first arose as an investigation of regions in the plane: Can three or more open, disjoint, connected regions \( S_j \) in the plane have the same boundary? This question was answered in the affirmative by L.E.J. Brouwer in7. In8 K. Yoneyama gave an example that he attributes to Mr. Wada, his Ph.D. supervisor, Takeo Wada. Hocking and Young9 used the term lakes of Wada, a pun on water, so it was natural to extend the wordplay to dynamical systems by introducing the term basins of Wada10. The papers11,12 applied this concept to dynamical systems and devised a method to identify Wada basins. It is remarkable how improbable the original topological examples appeared, seeming unrelated to anything real, and yet surprisingly appear to be common in dynamical systems including analytic systems like the forced damped pendulum.

The Nusse-Yorke (NY) method12 observes that when the unstable manifold of a boundary saddle point \( q \) crosses three or more different basins, then the point \( q \) is a Wada point, as is every point in the stable manifold of \( q \) and in its closure. It can be difficult to show that the closure of that stable manifold is the entire boundary. The procedure for carrying that out is based on a concept called basin cells, which requires detailed knowledge of stable and unstable manifolds. The NY method is an adequate approach for basins in two dimensional phase spaces. However, the method is not suitable in many cases.

The Wada property appears in diverse situations such as the parameter space of the Hénon map13 or the one-dimensional phase space of a competition model in ecology14. We can also find experimental and theoretical examples where the Wada property seems very likely to be present but the absence of a proper method of characterization prevents its study15,16. We present a special case where the basins of the dynamical system share their boundaries, but the different basins are not connected. These disconnected Wada basins can be illustrated as follows. Figure 1(a): divide a disk in six sectors and color three alternate sectors with different colors; Fig. 1(b): divide each empty sector into three sectors and color
Figure 1. Disconnected Wada set. Three different stages to build a Wada set using three disconnected regions. The basins (colors) share the same boundary but each colored set is disconnected.

The central one with the color that is not at the left nor at the right; Fig. 1(c): repeat the second step indefinitely until filling the whole disk. The boundary of the different colors displays the Wada property by construction, but the basins are not connected; indeed the set of Wada points on the boundaries possesses a Cantor set structure. There is a rich literature documenting systems with disconnected Wada sets: the experiment of light scattered by reflecting balls\textsuperscript{17}, the Newton method to find complex roots\textsuperscript{6}, chaotic scattering in more than two dimensions\textsuperscript{18}, etc.

A Grid Approach

We propose a simple method, straightforward to implement, to test the Wada property in all kind of systems. Furthermore, our method/terminology allows us to classify intermediate situations where only some basins and some of the boundary present the Wada property, which receives the name of partially Wada basins\textsuperscript{19,20}.

We first need some assumptions. We will discuss the notation for the case of a two dimensional space though it is easily adapted to other cases.

1. There is a bounded region $\Omega$ containing $N_A \geq 3$ disjoint regions $S_j$ where $j = 1, \ldots, N_A$.
2. There is a rectangular grid $G$ covering $\Omega$. We typically use a $1000 \times 1000$ grid. Hence $\Omega$ is covered by a set $P = \{box_1, \ldots, box_K\}$ of grid boxes (whose interiors do not intersect each other). Here $K$ would be $10^4$ for that usual grid.
3. For each point $x$ in $\Omega$, it is possible to determine to which set $S_j$ it belongs to. In other words, there is a function $C$ with $C(x) = j$ if $x \in S_j$ and $C(x) = 0$ if $x$ is in none of the sets $S_j$. If the sets are basins, the trajectory for each $x \in \Omega$ leads to an attractor labelled by $C(x)$. Notice that we do not impose different labels for each attractor. It is possible to merge several attractors into the same category. For any rectangular box denoted as box we define $C(box) = C(x)$ where $x$ is the point at the center of box. If it does not go to an attractor in our collection of numbered attractors, then $C(box) = 0$, such events are reported at the end of the run. For convenience we will refer to this numerical value $C$ as the color of the grid box. Of course other points in the same box might lead to different attractors.
4. We define $b(box)$ to be the collection of grid boxes consisting of box and all the grid boxes that have at least one point in common with box, so in dimension two, $b(box)$ is a $3 \times 3$ collection of boxes with box being the central box.
5. For each box, we determine the number of different (non-zero) colors in $b(box)$ and write $M(box)$ for that number.
6. In each box, with $M(box) = 1, N_A$, that is, which is not in the interior nor in the Wada boundary, we accomplish the following procedure. We select the two closest boxes in $b(box)$ with different colors and trace a line segment between them. We compute the color of the middle point of the segment. In case that the color newly computed completes all colors inside $b(box)$, then $M(box) = N_A$ and the algorithm stops. Otherwise, we compute two new points: one in the middle of the leftmost and central point, and another in the middle of the rightmost and central point. In the second step, four points interspersed with the previous five points would be calculated. In the third step, we would compute eight points interspersed with the previous nine. The procedure keeps on until $M(box) = N_A$ or the number of calculated points in that segment reaches some maximum value previously set up. A major computational advantage of this method is that the refinement is made in a one dimensional subspace (the segment linking the two points), no matter the dimension of $\Omega$.
7. Next we define $G_m$ to be the set of all the original grid boxes box, for which $M(box) = m$. 


For $m = 1$, all the boxes inside the ball $b(box_i)$ have the same color as they all lead to the same attractor. In fact $G_1$ represents points that are in the interior of a basin. A grid box $box_j$ is in the set $G_2$ (boundary of two) if there are two different colors inside the ball $b(box_j)$, a grid box is in the set $G_3$ (boundary of three) if there are three different colors inside the ball and so forth. To account for the evolution of these sets as the algorithm progresses, we call $G_n$ the set $G_n$ at step $q$.

Then, we will say that the system is Wada if $\lim_{q \to \infty} \sum_{m=2}^{N-1} \#G_m^q = 0$. This simply means that the grid boxes are either in the interior $G_1$ or in the Wada boundary $G_N$ after a sufficient number of steps $q$.

The basic idea underlying the whole process is that if three basins are Wada, then it is always possible to find a third color between the other two colors (similar reasoning can be done for Wada basins with more than three colors). Notice also that if a boundary separates two basins we will only see those two basins at all resolutions.

To illustrate the iterative process we represent in Fig. 2 our example of a partial Wada set and compute the boundary set for three grid boxes $box_1$, $box_2$, and $box_3$ on a regular rectangular grid forming a partition $P$ of the phase space. The first iteration for $box_1$ shows that it belongs to the interior region $G_1$ because its surrounding 8 boxes have the same color. On the other hand, $box_2$ and $box_3$ are in the boundary of two attractors, i.e., they are adjacent to boxes whose color is different. In the next step the algorithm classifies $box_2$ still in $G_2$ (boundary of two), while $box_3$ is now classified in $G_3$ (boundary of three). Ideally the process would keep on forever redefining the sets $G_1$, $G_2$, and $G_3$ at each step, though in practice we can impose some stopping condition. This plot constitutes an example of partially Wada basins.
\[ W_m = \lim_{q \to \infty} \frac{\#G_m^q}{\sum_{j=2}^{N_A} \#G_j^q} \tag{1} \]

where \( m \in [2, N_A] \). This parameter \( W_m \in [0, 1] \) takes a value zero if the system has no grid boxes that are in the boundary separating \( m \) basins and it takes a value one if all the boxes in the boundary separate \( m \) basins. Thus, if \( W_{N_A}=1 \) the system is said to be Wada. Partial Wada occurs when \( 0 < W_m < 1 \) with \( m \geq 3 \). As we will see, \( W_m \) is also useful to test the global numerical convergence.

**Results**

For all the results presented in this work, we use as initial partition a uniform grid of one million boxes, and the verification of the Wada property is subjected to that resolution. In order to illustrate the features of the described method, we will present an analysis of the results for the damped forced pendulum and the Newton method to find complex roots. They could be considered as paradigmatic examples of connected and disconnected Wada sets respectively. Nonetheless, we have also tested our method for the Duffing oscillator\(^4\), the Hénon-Heiles system\(^5\) and the magnetic pendulum\(^{21,22}\). In all of them, we have obtained values of \( W_j=1 \) which means they all share the Wada property.

The first system we will analyze is the forced damped pendulum defined by

\[ \ddot{x} + 0.2 \dot{x} + \sin x = 1.66 \cos t, \]

which constitutes a paradigmatic system with connected Wada basins\(^3\). After applying our method to the basin of attraction of Fig. 3(a), we find that all the boxes lie either in the boundary of the three basins or in the interior within the resolution of the method (see Fig. 3(b)). The histogram of Fig. 3(c) reflects the computational effort needed to test the Wada property in this system\(^{23}\). It shows that most of the points take less than five iterations to be labeled into the set boundary of three. Most importantly, the fractal structure of the boundary allows fast computation: as the number of steps \( q \) grows the number of remaining points \( N \) decreases exponentially (see Fig. 3(d)). At the end of the process, we find that \( W_j=1 \) which implies that the system is Wada.

A more challenging case arises when we increase the amplitude of the external forcing:

\[ \ddot{x} + 0.2 \dot{x} + \sin x = 1.73 \cos t. \]

For these parameters the damped forced pendulum has at least eight
attractors, and its basins are mixed complicatedly as shown in Fig. 4(a). When we apply our algorithm, we see that not all the boxes are classified as Wada (Fig. 4(b)). This is an example of partial Wada. The points in the boundary which are not Wada (points in \( G \) to \( G^7 \) in our example) increases the computational effort of the algorithm (Fig. 4(c)). The reason is due to nature of the process, the algorithm stops computing when \( M(box) = NA \) for each box or reach the maximum of allowed steps.

In order to establish a stopping rule we can use the parameter \( W_m \): the algorithm stops when \( \varepsilon (+) - (\) < \( \text{StepMax} \) for each box or reach the maximum of allowed steps.

Another option to deal with partial Wada is to merge some basins of attraction, considering two colors as only one single color for example. Making such a redefinition of the basins, we can say that the system is Wada if all the new basins share the same boundary (assuming we have at least three basins).

The second system under study is the Newton method to find complex roots, which provides examples of disconnected Wada sets. To find the roots of \( z^r = 1, z \in \mathbb{C} \) and \( r \in \mathbb{N} \), the Newton method iterates the map \( z_{n+1} = z_n - (z^r - 1)/(r z^{r-1}) \). This map has \( r \) different fractalized and disconnected Wada basins corresponding to the complex roots of unity. In Fig. 5(a) we depict the basin for \( r = 7 \) and in Fig. 5(b) we see that the algorithm yields to \( W_7 = 1 \), that is, all the boxes belong to the interior or to the boundary of the seven basins. Thus, our method can verify the Wada property for an arbitrary number of basins and for disconnected Wada sets too. An analysis of the computational effort for the Newton method varying \( r \) from 3 to 7 reflects that it grows with the number of basins (the maxima of the histograms in Fig. 5(c) shifts to the right as the number of basins increases).

Discussion

Fractal Wada boundaries seem common in nonlinear systems. However they can be overlooked or misinterpreted with a simple fractal boundary. We can also have an intermediate situation such as partial Wada. Our algorithm shows that in a computationally affordable time, the Wada basins can be detected for a given grid precision. Furthermore it is possible to apply the technique to disconnected Wada sets, high-dimensional problems, experimental settings and partially Wada systems.
A simple key idea drives the search: on the line segment between two points of different basins there is always a point belonging to another basin if the boundary is Wada. The indicator $W_m$ quantifies the Wada property and allows comparisons between basins. We can imagine for example a study of the measurement $W_m$ as a function of a parameter of the system under study. We can also imagine an optimization algorithm to implement a heuristic search for Wada basins. We believe that this algorithm can constitute a powerful tool in the study of dynamical systems in general and of the Wada property in particular.

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**Author Contributions**

A.D., A.W., M.A.F.S. and J.A.Y. devised the research. A.D. performed the numerical simulations. A.D., A.W., M.A.F.S. and J.A.Y. analyzed the results and wrote the paper.

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