THE CANONICAL IDEAL AND THE DEFORMATION THEORY OF CURVES WITH AUTOMORPHISMS

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ABSTRACT. The deformation theory of curves is studied by using the canonical ideal. The deformation problem of curves with automorphisms is reduced to a deformation problem of linear representations.

1. Introduction

The deformation theory of curves with automorphisms is an important generalization of the classical deformation theory of curves. This theory is related to the lifting problem of curves with automorphisms since one can consider liftings from characteristic $p > 0$ to characteristic zero in terms of a sequence of local Artin-rings.

J. Bertin and A. Mézard in [4], following Schlessinger’s [32] approach introduced a deformation functor $D_{gl}$ and studied it in terms of Grothendieck’s equivariant cohomology theory [12]. In Schlessinger’s approach to deformation theory, we want to know the tangent space to the deformation functor $D_{gl}(k[\epsilon])$ and the possible obstructions to lift a deformation over an Artin local ring $\Gamma$ to a small extension $\Gamma' \to \Gamma$. The reader who is not familiar with deformation theory is refereed to section 2.1 for terminology and references to the literature. The tangent space of the global deformation functor $D_{gl}(k[\epsilon])$ can be identified as Grothendieck’s equivariant cohomology group $H^1(G, X, \mathcal{T}_X)$, which is known to be equal to the invariant space $H^1(X, \mathcal{T}_X)^G$. Moreover, a local local-global theorem is known, which can be expressed in terms of the short exact sequence:

\[
0 \longrightarrow H^1(X/G, \pi_*^G(\mathcal{T}_X)) \longrightarrow H^1(G, X, \mathcal{T}_X) \longrightarrow H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \longrightarrow 0
\]

The lifting obstruction can be seen as an element in

\[
H^2(G, X, \mathcal{T}_X) \cong \bigoplus_{i=1}^r H^2(G_{x_i}, \mathcal{F}_{X,x_i}).
\]

In the above equations $x_1, \ldots, x_r \in X$ are the ramified points, $G_{x_i}$ are the corresponding isotropy groups and $\mathcal{F}_{X,x_i}$ are the completed local tangent spaces, that is $\mathcal{F}_{X,x_i} = k[[t_i]]\frac{d}{dt_i}$, where $t_i$ is a local uniformizer at $x_i$. The space $k[[t_i]]\frac{d}{dt_i}$ is
where $I$ there is the following short exact sequence:

$$0 \to I_X \to \text{Sym}H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^\otimes n) \to 0,$$

where $I_X$ is generated by elements of degree 2 and 3. Also if $X$ is not a non-singular quintic of genus 6 or $X$ is not a trigonal curve, then $I_X$ is generated by elements of degree 2.

For a proof of this theorem we refer to [11, 31]. The ideal $I_X$ is called the canonical ideal and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}^{g-1}$.

For curves that satisfy the assumptions of Petri’s theorem and their canonical ideal is generated by quadrics, we prove in section 3 the following relative version of Petri’s theorem.

**Proposition 2.** Let $f_1, \ldots, f_r \in S := \text{Sym}H^0(X, \Omega_X) = k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal $I_X$ of a curve $X$ defined over an algebraically closed field $k$. Any deformation $\mathcal{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in \text{Sym}H^0(\mathcal{X}_A, \Omega_{\mathcal{X}_A/A}) = A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $m_A$ of $A$.

This approach allows us to replace several of Grothendieck’s equivariant cohomology constructions in terms of linear algebra. Let us mention that in general, it is not so easy to perform explicit computations with equivariant Grothendieck cohomology groups and usually, spectral sequences or a complicated equivariant Chech cohomology is used, see [3, 23 sec.3].

Let $i : X \to \mathbb{P}^{g-1}$ be the canonical embedding. In proposition 27 we prove that elements $[f] \in H^1(X, \mathcal{T}_X)^G = D_{gl}k[e]$ correspond to cohomology classes in $H^1(G, M_g(k)/\mathbb{I}_g)$, where $M_g(k)/\mathbb{I}_g$ is the space of $g \times g$ matrices with coefficients in $k$, modulo the vector subspace of scalar multiples of the identity matrix.

Furthermore, in our setting the obstruction to liftings is reduced to an obstruction to the lifting of the linear canonical representation

$$\rho: G \to \text{GL}(H^0(X, \Omega_X))$$

and a compatibility criterion involving the defining quadratic equations of our canonically embedded curve, namely in section 4 we will prove the following:

**Theorem 3.** Consider an epimorphism $\Gamma' \to \Gamma \to 0$ of local Artin rings. A deformation $x \in D_{gl}(\Gamma)$ can be lifted to a deformation $x' \in D_{gl}(\Gamma')$ if and only if the representation $\rho_\Gamma : G \to \text{GL}_g(\Gamma)$ lifts to a representation $\rho_{\Gamma'} : G \to \text{GL}_g(\Gamma')$
and moreover there is a lifting $X_{\Gamma'}$ of the embedded deformation of $X_{\Gamma}$ which is invariant under the lifted action of $\rho_{\Gamma'}$.

**Remark 4.** The liftability of the representation $\rho$ is a strong condition. In proposition 31 we give an example of a representation $\rho : G \rightarrow \text{GL}_2(k)$, for a field $k$ of positive characteristic $p$, which can not be lifted to a representation $\tilde{\rho} : G \rightarrow \text{GL}_2(R)$ for $R = W(k)[[\zeta_p]]$, meaning that a lifting in some small extension $R/m_R^{i+1} \rightarrow R/m_R^i$ is obstructed. Here $R$ denotes the Witt ring of $k$ with a primitive $p^h$ root of unity added, which has characteristic zero. In our counterexample $G = C_q \rtimes C_m$, $q = p^h$, $(m,p) = 1$.

**Remark 5.** One can always pass from the local lifting problem of $\rho : G \rightarrow \text{Aut}\Gamma[[t]]$ to a global lifting problem, by considering the Harbater-Katz-Gabber (HKG for short) compactification $X$ of the local action. Then one can consider the the criterion involving the linear representation $\rho : G \rightarrow \text{Gl}(H^0(X, \Omega_X))$. Notice that in [26] the canonical ideal for HKG-curves is explicitly described.

**Remark 6.** In [4] we will use the tools developed in this article to show that certain automorphisms of the Hermitian curve do not lift even in possible characteristic. This is expected since the Hermitian curve is the unique curve with an extreme size of its automorphism group, see [35].

**Remark 7.** The invariance of the canonical ideal $I_{X_{\Gamma}}$ under the action of $G$ can be checked using Gauss elimination and echelon normal forms, see [24] see 2.2.

**Remark 8.** The canonical ideal $I_{X_{\Gamma}}$ is determined by $r$ quadratic polynomials which form a $\Gamma[G]$-invariant $\Gamma$-submodule $V_{\Gamma}$ in the free $\Gamma$-module of symmetric $g \times g$ matrices with entries in $\Gamma$. When we pass from a deformation $x \in D_{g,\text{gl}}(\Gamma)$ to a deformation $x' \in D_{\text{gl}}(\Gamma')$ we ask that the canonical ideal $I_{X_{\Gamma'}}$ is invariant under the lifted action, given by the representation $\rho_{\Gamma'} : G \rightarrow \text{GL}_g(\Gamma')$. In definition 4.1 we introduce an action $T(g)$ on the vector space of symmetric $g \times g$ matrices, and the invariance of the canonical ideal is equivalent to the invariance under the $T$-action of the $\Gamma'$-submodule $V_{\Gamma'}$ generated by the quadratic polynomials generating $I_{X_{\Gamma'}}$. Therefore, we can write one more representation

$$\rho^{(1)} : G \rightarrow \text{GL}(\text{Tor}_T^2(k, I_X)).$$

Set $r = \binom{g-2}{2}$. Liftings of the representations $\rho, \rho^{(1)}$ defined in eq. 2, 3 in $\text{GL}_g(\Gamma)$ resp. $\text{GL}_r(\Gamma)$ will be denoted by $\rho_{\Gamma}$ resp. $\rho_{\Gamma}^{(1)}$.

Notice that if the representation $\rho_{\Gamma}$ lifts to a representation $\rho_{\Gamma'}$ and moreover there is a lifting $X_{\Gamma'}$ of the relative curve $X_{\Gamma}$ so that $X_{\Gamma'}$ has an ideal $I_{X_{\Gamma'}}$ which is $\rho_{\Gamma'}$ invariant, then the representation $\rho_{\Gamma}^{(1)}$ also lifts to a representation $\rho_{\Gamma}^{(1)}$, see also [24] prop. 5

The deformation theory of linear representations $\rho, \rho^{(1)}$ gives rise to cocycles $D_{\sigma}, D_{\sigma}^{(1)}$ in $H^1(G, M_g(k)), H^1(G, M^{(g-2)}(k))$, while the deformation theory of curves with automorphisms introduces a cocycle $B_{\sigma}[f]$ corresponding to $[f] \in H^1(X, \mathcal{F}_X)^G$. We will introduce a compatibility condition in section 4.2 among these cocycles, using the isomorphism

$$\psi : M_g(k)/\langle \mathbb{I} \rangle \xrightarrow{\cong} H^0(X, i^* \mathcal{F}_{p_{g-1}}) \hookrightarrow \text{Hom}_{\mathcal{S}}(I_X, S/I_X) = H^0(X, \mathcal{N}_{X/p_{g-1}})$$

$B \mapsto \psi_B$
Proposition 9. The following compatibility condition is satisfied
\[
\psi_{D_\sigma} - \psi_{B_\sigma} [f] = D^{(1)}_{\sigma^{-1}}.
\]

The structure of the article is as follows. In section 2.2 we will present together the deformation theory of linear representations \( \rho : G \to \text{GL}(V) \) and the deformation theory of representations of the form \( \rho : G \to \text{Aut}k[[t]] \). The deformation theory of linear representations is a better-understood object of study, see [28], which played an important role in topology [20] and also in the proof of Fermat’s last theorem, see [29]. The deformation theory of representations in \( \text{Aut}k[[t]] \) comes out from the study of local fields and it is related to the deformation problem of curves with automorphisms after the local global theory of Bertin Mézard. There is also an increased interest related to the study of Nottingham groups and \( \text{Aut}k[[t]] \), see [3], [9], [25]. It seems that the similarities between these two deformation problems are known to the expert, see for example [30, prop. 3.13]. For the convenience of the reader and in order to fix the notation, we also give a detailed explanation and comparison of these two deformation problems.

In section 3 we revise the theory of relative canonical ideals and the work of the first author together with H. Charalambous and K. Karagiannis [6] aiming at the deformation problem of curves with automorphisms. More precisely a relative version of Petri’s theorem is proved, which implies that the relative canonical ideal is generated by quadratic polynomials.

In section 4 we study both the obstruction and the tangent space problem of the deformation theory of curves with automorphisms using the relative canonical ideal point of view. In this section theorem 3 is proved.

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2. Deformation theory of curves with automorphisms

2.1. Global deformation functor. Let \( \Lambda \) be a complete local Noetherian ring with residue field \( k \), where \( k \) is an algebraically closed field of characteristic \( p \geq 0 \). Let \( \mathcal{C} \) be the category of local Artin \( \Lambda \)-algebras with residue field \( k \) and homomorphisms the local \( \Lambda \)-algebra homomorphisms \( \phi : \Gamma' \to \Gamma \) between them, that is \( \phi^{-1}(m_\Gamma) = m_{\Gamma'} \). The deformation functor of curves with automorphisms is a functor \( D_{\text{gl}} \) from the category \( \mathcal{C} \) to the category of sets
\[
D_{\text{gl}} : \mathcal{C} \to \text{Sets}, \Gamma \mapsto \left\{ \begin{array}{l}
\text{Equivalence classes} \\
\text{of deformations of} \\
\text{couples } (X, G) \text{ over } \Gamma
\end{array} \right\}
\]
defined as follows. For a subgroup \( G \) of the group \( \text{Aut}(X) \), a deformation of the couple \( (X, G) \) over the local Artin ring \( \Gamma \) is a proper, smooth family of curves \( X_{\Gamma} \to \text{Spec}(\Gamma) \) parametrized by the base scheme \( \text{Spec}(\Gamma) \), together with a group homomorphism \( G \to \text{Aut}_{\Gamma}(X_{\Gamma}) \), such that there is a \( G \)-equivariant isomorphism \( \phi \) from the fibre
over the closed point of $\Gamma$ to the original curve $X$:

$$\phi : X_\Gamma \otimes_{\text{Spec}(\Gamma)} \text{Spec}(k) \to X.$$ 

Two deformations $X_1^\Gamma, X_2^\Gamma$ are considered to be equivalent if there is a $G$-equivariant isomorphism $\psi$ that reduces to the identity in the special fibre and making the following diagram commutative:

$$\begin{array}{ccc}
X_1^\Gamma & \xrightarrow{\psi} & X_2^\Gamma \\
\downarrow & & \downarrow \\
\text{Spec} \Gamma & & \text{Spec} \Gamma
\end{array}$$

Given a small extension of Artin local rings

$$(5) \quad 0 \to E \cdot k \to \Gamma' \to \Gamma \to 0$$

and an element $x \in D_{gl}(\Gamma)$ we have that the set of lifts $x' \in D_{gl}(\Gamma')$ extending $x$ is a principal homogeneous space under the action of $D_{gl}(k[\epsilon])$ and such an extension $x'$ exists if certain obstruction vanishes. It is well known, see section 2.2, that similar behavior have the deformation functors of representations.

### 2.2. Lifting of representations.

Let $G : \mathcal{C} \to \text{Groups}$ be a group functor, see [8, ch. 2]. In this article, we will be mainly interested in two group functors. The first one, $\text{GL}_{g}$, will be represented by the group scheme $G_g = \Lambda[x_{11}, \ldots, x_{gg}, \det(x_{ij})^{-1}]$, that is $\text{GL}_g(\Gamma) = \text{Hom}_\Lambda(G_g, \Gamma)$. The second one is the group functor from the category of rings to the category of groups $\mathcal{N} : \Gamma \mapsto \text{Aut} \Gamma[[t]]$.

We also assume that each group $G(\Gamma)$ is embedded in the group of units of some ring $\mathcal{R}(\Gamma)$ depending functorially on $\Gamma$. This condition is asked since our argument requires us to be able to add certain group elements. We also assume that the additive group of the ring $\mathcal{R}(\Gamma)$ has the structure of direct product $\Gamma^I$, while $\mathcal{R}(\Gamma) = \mathcal{R}(\Lambda) \otimes_\Lambda \Gamma$. Notice, that $I$ might be an infinite set, but since all rings involved are Noetherian $\Gamma^I$ is flat, see [27, 4F].

A representation of the finite group $G$ in $G(\Gamma)$ is a group homomorphism

$$\rho : G \to G(\Gamma),$$

where $\Gamma$ is a commutative ring.

**Remark 10.** Consider two sets $X, Y$ acted on by the group $G$. Then every function $f : X \to Y$ is acted on by $G$, by defining $\sigma f : X \to Y$, sending $x \mapsto \sigma f \sigma^{-1}(x)$. This construction will be used throughout this article.

More precisely we will use the following actions

**Definition 11.**

1. Let $M_g(\Gamma)$ denote the set of $g \times g$ matrices with entries in ring $\Gamma$. An element $A \in M_g(\Gamma)$ will be acted on by $g \in G$ in terms of the action

$$T(g)A = \rho(g^{-1})^t A \rho(g^{-1}).$$

This is the natural action coming from the action of $G$ on $H^0(X, \Omega_{X/k})$ and on the quadratic forms $\omega^t A \omega$. We raise the group element in $-1$ in order to have a left action, that is $T(gh)A = T(g)T(h)A$. Notice also that $T(g)$ restricts to an action on the space $\mathcal{S}_g(\Gamma)$ of symmetric $g \times g$ matrices with entries in $\Gamma$. 

The adjoint action on elements \( A \in M_g(\Gamma) \), comes from the action to the tangent space of the general linear group.

\[
\text{Ad}(g)A = \rho(g)A\rho(g^{-1}).
\]

Actions on elements which can be seen as functions between \( G \)-spaces as in remark 10. This action will be denoted as \( f \mapsto \sigma f \).

**Examples**

1. Consider the groups \( \text{GL}_g(\Gamma) \) consisted of all invertible \( g \times g \) matrices with coefficients in \( \Gamma \). The group functor \( \Gamma \mapsto \text{GL}_g(\Gamma) = \text{Hom}(R, \Gamma) \), is representable by the affine \( \Lambda \)-algebra \( R = \mathbb{k}[x_{11}, \ldots, x_{gg}, \det((x_{ij}))^{-1}] \), see [36, 2.5]. In this case the ring \( R(\Gamma) \) is equal to \( \text{End}(\Gamma^g) \), while \( I = \{ i, j \in \mathbb{N} : 1 \leq i, j, \leq g \} \).

We can consider the subfunctor \( \text{GL}_{g,I_g} \) consisted of all elements \( f \in \text{GL}_g(\Gamma) \), which reduce to the identity modulo the maximal ideal \( \mathfrak{m}_\Gamma \). The tangent space \( T_{I_g} \text{GL}_g \) of \( \text{GL}_g \) at the identity element \( I_g \), that is the space \( \text{Hom}(\text{Spec}k[\epsilon], \text{Spec}R) \) or equivalently the set \( \text{GL}_{g,I_g}(k[\epsilon]) \) consisted of \( f \in \text{Hom}(R, k[\epsilon]) \), so that \( f \equiv I_g \mod \langle \epsilon \rangle \). This set is a vector space according to the functorial construct ion given in [29, p. 272] and can be identified to the space of \( \text{End}(k^g) = M_g(k) \), by identifying

\[
\text{Hom}(R, k[\epsilon]) \ni f \mapsto I_g + \epsilon M, M \in M_g(k).
\]

The later space is usually considered as the tangent space of the algebraic group \( \text{GL}_g(k) \) at the identity element or equivalently as the Lie algebra corresponding to \( \text{GL}_g(k) \).

The representation \( \rho : G \to \text{GL}_g(\Gamma) \) equips the space \( T_{I_g} \text{GL}_g = M_g(k) \) with the adjoint action, which is the action described in remark 10 when the endomorphism \( M \) is seen as an operator \( V \to V \), where \( V \) is a \( G \)-module in terms of the representation \( \rho \):

\[
G \times M_g(k) \to M_g(k)
\]

\[
(g, M) \mapsto \text{Ad}(g)(M) = gMg^{-1}.
\]

In order to make clear the relation with the local case below, where the main object of study is the automorphism group of a completely local ring we might consider the completion \( \hat{R}_i \) of the localization of \( R = \mathbb{k}[x_{11}, \ldots, x_{gg}, \det((x_{ij}))^{-1}] \) at the identity element. We can now form the group \( \text{Aut}\hat{R}_i \) of automorphisms of the ring \( \hat{R}_i \) which reduce to the identity modulo \( \mathfrak{m}_{\hat{R}_i} \). The later automorphism group is huge but it certainly contains the group \( G \) acting on \( \hat{R}_i \) in terms of the adjoint representation. We have that elements \( \sigma \in \text{Aut}\hat{R}_i \otimes k[\epsilon] \) are of the form

\[
\sigma(x_{ij}) = x_{ij} + \epsilon \beta(x_{ij}), \text{ where } \beta(x_{ij}) \in \hat{R}_i.
\]

Moreover, the relation

\[
\sigma(f \cdot g) = fg + \epsilon \beta(fg) = (f + \epsilon \beta(f))(g + \epsilon \beta(f))
\]

implies that the map \( \beta \) is a derivation and

\[
\beta(fg) = f\beta(g) + \beta(f)g.
\]
Therefore, \( \beta \) is a linear combination of \( \frac{\partial}{\partial x_{ij}} \), with coefficients in \( \hat{R} \), that is

\[
\beta = \sum_{0 \leq i, j \leq g} a_{i,j} \frac{\partial}{\partial x_{ij}}
\]

**Remark 12.** In the literature of Lie groups and algebras, the matrix notation \( M_{g}(k) \) for the tangent space is frequently used for the Lie algebra-tangent space at identity, instead of the later vector field-differential operator approach, while in the next example the differential operator notation for the tangent space is usually used.

2. Consider now the group functor \( \Gamma \mapsto N(\Gamma) = \text{Aut}\Gamma[[t]] \). An element \( \sigma \in \text{Aut}\Gamma[[t]] \) is fully described by its action on \( t \), which can be expressed as an element in \( \Gamma[[t]] \). When \( \Gamma \) is an Artin local algebra then an automorphism is given by

\[
\sigma(t) = \sum_{\nu=0}^{\infty} a_{i} t^{\nu}, \text{ where } a_{i} \in \Gamma, a_{0} \in m_{\Gamma} \text{ and } a_{1} \text{ is a unit in } \Gamma.
\]

If \( a_{1} \) is not a unit in \( \Gamma \) or \( a_{0} \not\in m_{\Gamma} \) then \( \sigma \) is an endomorphism of \( \Gamma[[t]] \). In this way \( \text{Aut}\Gamma[[t]] \) can be seen as the group of invertible elements in \( \Gamma[[t]] = \text{End}\Gamma[[t]] = \mathcal{A}(\Gamma) \). In this case, the set \( I \) is equal to the set of natural numbers, where \( \Gamma^I \) can be identified as the set of coefficients of each powerseries.

\[
\text{Aut}(k[e][[t]]) = \left\{ t \mapsto \sigma(t) = \sum_{\nu=1}^{\infty} a_{i} t^{\nu} : a_{i} = \alpha_{i} + \epsilon \beta_{i}, \alpha_{i}, \beta_{i} \in k, \alpha_{1} \neq 0 \right\}
\]

Exactly as we did in the general linear group case let us consider the subfunctor \( \Gamma \mapsto \mathcal{N}_{2}(\Gamma) \), where \( \mathcal{N}_{2}(\Gamma) \) consists of all elements in \( \text{Aut}\Gamma[[t]] \) which reduce to the identity mod \( m_{\Gamma} \).

Such an element \( \sigma \in \mathcal{N}_{2}(k[e]) \) transforms \( f \in k[[t]] \) to a formal powerseries of the form

\[
\sigma(f) = f + \epsilon F_{\sigma}(f),
\]

where \( F_{\sigma}(f) \) is fully determined by the value of \( \sigma(t) \). The multiplication condition \( \sigma(f_{1}f_{2}) = \sigma(f_{1})\sigma(f_{2}) \) implies that

\[
F_{\sigma}(f_{1}f_{2}) = f_{1}F_{\sigma}(f_{2}) + F_{\sigma}(f_{1})f_{2},
\]

that is \( F_{\sigma} \) is a \( k[[t]] \)-derivation, hence an element in \( k[[t]] \frac{d}{dt} \).

The local tangent space of \( \Gamma[[t]] \) is defined to be the space of differential operators \( f(t)\frac{d}{dt} \), see [4], [7], [22]. The \( G \) action on the element \( \frac{d}{dt} \) is given by the adjoint action, which is given as a composition of operators, and is again compatible with the action given in remark [10].

\[
\begin{align*}
\Gamma[[t]] & \xrightarrow{\rho(\sigma^{-1})} \Gamma[[t]] \xrightarrow{\frac{d}{dt}} \Gamma[[t]] \xrightarrow{\rho(\sigma)} \Gamma[[t]] \\
t & \xrightarrow{\rho(\sigma^{-1})(t)} \xrightarrow{\frac{d\rho(\sigma^{-1})(t)}{dt}} \xrightarrow{\rho(\sigma)} \left( \frac{d\rho(\sigma^{-1})(t)}{dt} \right)
\end{align*}
\]
So the \( G \)-action on the local tangent space \( k[[t]] \frac{dd}{dt} \) is given by
\[
f(t) \frac{dd}{dt} \mapsto - \text{Ad}(\sigma)\left( f(t) \frac{dd}{dt} \right) = \rho(\sigma)(f(t)) \cdot \rho(\sigma) \left( \frac{d\rho^{-1}(\sigma)(t)}{dt} \right) \frac{dd}{dt},
\]
see also [22, lemma 1.10], for a special case.

| \( \mathcal{G}(\Gamma) \) | \( \mathcal{H}(\Gamma) \) | tangent space | action |
|--------------------------|------------------|--------------|--------|
| GL\(_g\)(\( \Gamma \))   | End\(_g\)(\( \Gamma \)) | \( k[x] \frac{dx}{x} \mapsto \mathcal{H}(\( \Gamma \)) \) | \( M \mapsto \text{Ad}(\sigma)(M) \) |
| Aut\(_\Gamma([t])\)      | End(Aut\(_\Gamma([t])\)) | \( f(t) \frac{dx}{x} \mapsto \text{Ad}(\sigma)(f(t)) \cdot \text{Ad}(\sigma)(\frac{dx}{x}) \) |

Table 1. Comparing the two group functors

Motivated by the above two examples we can define

**Definition 13.** Let \( \mathcal{G}_I \) be the subfunctor of \( \mathcal{G} \), defined by
\[
\mathcal{G}_I(\Gamma) = \{ f \in \mathcal{G}(\Gamma) : f = I \text{ mod } m_\Gamma \}.
\]
The tangent space to the functor \( \mathcal{G} \) at the identity element is defined as \( \mathcal{G}_I(k[[t]]) \), see [29]. Notice, that \( \mathcal{G}_I(k[[t]]) \cong \mathcal{H}(k) \), is \( k \)-vector space, acted on in terms of the adjoint representation, given by
\[
G \times \mathcal{G}_I(\Gamma) \rightarrow \mathcal{G}_I(\Gamma)
\]
\[
(\sigma, f) \mapsto \rho(\sigma) \cdot f \cdot \rho(\sigma)^{-1}.
\]
If \( \mathcal{H}(\Gamma) \) can be interpreted as an endomorphism ring, then the above action can be interpreted in terms of the action on functions as described in remark 10.

We will define the tangent space in our setting as \( \mathcal{T} = \mathcal{H}(k) \), which is equipped with the adjoint action.

**2.3. Deforming representations.** We can now define the deformation functor \( F_\rho \) for any local Artin algebra \( \Gamma \) with maximal ideal \( m_\Gamma \) in \( \mathcal{C} \) to the category of sets:

\[
F_\rho : \Gamma \in \text{Ob}(\mathcal{C}) \mapsto \left\{ \begin{array}{l}
\text{liftings of } \rho : G \rightarrow \mathcal{H}(k) \\
\text{to } \rho_\Gamma : G \rightarrow \mathcal{H}(\Gamma) \text{ modulo } \ker(\mathcal{H}(\Gamma) \rightarrow \mathcal{H}(k)) \\
\text{conjugation by an element of } \ker(\mathcal{H}(\Gamma) \rightarrow \mathcal{H}(k))
\end{array} \right\}
\]

Let
\[
0 \rightarrow \langle E \rangle = E \cdot \Gamma' = E \cdot k \xrightarrow{\phi} \Gamma' \xrightarrow{i} \Gamma \twoheadrightarrow 0
\]
be a small extension in \( \mathcal{C} \), that is the kernel of the natural onto map \( \phi \) is a principal ideal, generated by \( E \) and \( E \cdot m_{\Gamma'} = 0 \). In the above diagram \( i : \Gamma \rightarrow \Gamma' \) is a section, which is not necessarily a homomorphism. Since the kernel of \( \phi \) is a principal ideal \( E \cdot \Gamma' \) annihilated by \( m_{\Gamma'} \), it is naturally a \( k = \Gamma'/m_{\Gamma'} \)-vector space, which is one dimensional.

**Lemma 14.** For a small extension as given in eq. (7) consider two liftings \( \rho_1^\Gamma, \rho_2^\Gamma \) of the representation \( \rho_\Gamma \). The map
\[
d : G \twoheadrightarrow \mathcal{T} := \mathcal{H}(k) \\
\sigma \mapsto d(\sigma) = \frac{\rho_1^\Gamma(\sigma)\rho_2^\Gamma(\sigma)^{-1} - I_{\Gamma'}}{E}
\]
is a cocycle.

Proof. We begin by observing that 
\[ \phi \left( \rho^1_{\Gamma'}(\sigma) \rho^2_{\Gamma'}(\sigma)^{-1} - \mathbb{I}_{\Gamma'} \right) = 0, \]

thus 
\[ \rho^1_{\Gamma'}(\sigma) \rho^2_{\Gamma'}(\sigma)^{-1} = \mathbb{I}_{\Gamma'} + E \cdot d(\sigma), \]

where \( d(\sigma) \in \mathcal{T} \).

Also, we compute that 
\[ \mathbb{I}_{\Gamma'} + E \cdot d(\sigma \tau) = \rho^1_{\Gamma'}(\sigma \tau) \rho^2_{\Gamma'}(\sigma \tau)^{-1} \]
\[ = \rho^1_{\Gamma'}(\sigma) \rho^1_{\Gamma'}(\tau) \rho^2_{\Gamma'}(\tau)^{-1} \rho^2_{\Gamma'}(\sigma)^{-1} \]
\[ = \rho^1_{\Gamma'}(\tau) (\mathbb{I}_{\Gamma'} + Ed(\sigma)) \rho^2_{\Gamma'}(\tau)^{-1} \]
\[ = \rho^1_{\Gamma'}(\tau) \rho^2_{\Gamma'}(\tau)^{-1} + E \cdot \rho^1_{\Gamma'}(\tau) d(\sigma) \rho^2_{\Gamma'}(\tau)^{-1} \]
\[ = \mathbb{I}_{\Gamma'} + E \cdot d(\tau) + E \cdot \rho_k(\tau) d(\sigma) \rho_k(\tau)^{-1}, \]

since \( E \) annihilates \( m_{\Gamma'} \), so the values of both \( \rho^1_{\Gamma'}(\tau) \) and \( \rho^2_{\Gamma'}(\tau) \) when multiplied by \( E \) are reduced modulo the maximal ideal \( m_{\Gamma'} \). We, therefore, conclude that 
\[ d(\sigma \tau) = d(\tau) + \rho_k(\tau) d(\sigma) \rho_k(\tau)^{-1} = d(\tau) + \text{Ad}(\tau) d(\sigma). \]

Similarly if \( \rho^1_{\Gamma}, \rho^2_{\Gamma} \) are equivalent extensions of \( \rho_{\Gamma} \), that is 
\[ \rho^1_{\Gamma}(\sigma) = (\mathbb{I}_{\Gamma'} + EQ) \rho^2_{\Gamma}(\sigma)(\mathbb{I}_{\Gamma'} + EQ)^{-1}, \]

then 
\[ d(\sigma) = Q - \text{Ad}(\sigma)Q, \]

that is \( d(\sigma) \) is a coboundary. This proves that the set of liftings \( \rho_{\Gamma'} \) of a representation \( \rho_{\Gamma} \) is a principal homogeneous space, provided it is non-empty.

The obstruction to the lifting can be computed by considering a naive lift \( \rho_{\Gamma'} \) of \( \rho_{\Gamma} \) (that is we don’t assume that \( \rho_{\Gamma'} \) is a representation) and by considering the element 
\[ \phi(\sigma, \tau) = \rho_{\Gamma'}(\sigma) \circ \rho_{\Gamma'}(\tau) \circ \rho_{\Gamma'}(\sigma \tau)^{-1}, \]

for \( \sigma, \tau \in G \) which defines a cohomology class as an element in \( H^2(G, \mathcal{T}) \). Two naive liftings \( \rho^1_{\Gamma}, \rho^2_{\Gamma} \) give rise to cohomologous elements \( \phi^1, \phi^2 \) if their difference \( \rho^1_{\Gamma} - \rho^2_{\Gamma} \) reduce to zero in \( \Gamma' \). If this class is zero, then the representation \( \rho_{\Gamma'} \) can be lifted to \( \Gamma' \).

Examples Notice that in the theory of deformations of representations of the general linear group, this is a classical result, see [29, prop. 1], [28, p.30] while for deformations of representations in \( \text{Aut}\Gamma[[t]] \), this is in [7, H].

The functors in these cases are given by 
\[ F : \text{Ob}(\mathcal{C}) \ni \Gamma \mapsto \begin{cases} 
\text{liftings of } \rho : G \to \text{GL}_n(k) \\
\text{to } \rho_{\Gamma} : G \to \text{GL}_n(\Gamma) \text{ modulo} \\
\text{conjugation by an element} \\
of \ker(\text{GL}_n(\Gamma) \to \text{GL}_n(k)) 
\end{cases} \]

\[ D_P : \text{Ob}(\mathcal{C}) \ni \Gamma \mapsto \begin{cases} 
\text{lifts } G \to \text{Aut}(\Gamma[[t]]) \text{ of } \rho \text{ modulo} \\
\text{conjugation with an element} \\
of \ker(\text{Aut}\Gamma[[t]] \to \text{Aut}k[[t]]) 
\end{cases} \]
Let $V$ be the $n$-dimensional vector space $k$, and let $\text{End}_A(V)$ be the Lie algebra corresponding to the algebraic group $GL(V)$. The space $\text{End}_A(V)$ is equipped with the adjoint action of $G$ given by:

$$\text{End}_A(V) \to \text{End}_A(V)$$

$$e \mapsto (g \cdot e)(v) = \rho(g)(e(\rho(g)^{-1})(v))$$

The tangent space of this deformation functor equals to

$$F(k[\epsilon]) = H^1(G, \text{End}_A(V)),$$

where the later cohomology group is the group cohomology group and $\text{End}_A(V)$ is considered as a $G$-module with the adjoint action.

More precisely, if

$$0 \to \langle E \rangle \to \Gamma' \xrightarrow{\phi} \Gamma \to 0$$

is a small extension of local Artin algebras then we consider the diagram of small extensions

$$\begin{array}{ccc}
G & \xrightarrow{\rho_T} & \text{GL}_n(\Gamma) \\
\downarrow{\rho_T} & & \downarrow{\phi} \\
\text{GL}_n(\Gamma') & \xrightarrow{\rho_T'} & \text{GL}_n(\Gamma)
\end{array}$$

where $\rho_T'$, $\rho_T^2$ are two liftings of $\rho_T$ in $\Gamma'$.

We have the element

$$d(\sigma) := \frac{1}{F}(\rho_T^1(\sigma)\rho_T^2(\sigma)^{-1} - \mathbb{I}_n) \in H^1(G, \text{End}_n(k)).$$

To a naive lift $\rho_T$ of $\rho_T$ we can attach the 2-cocycle $\alpha(\sigma, \tau) = \rho_{T'}(\sigma)\rho_{T'}(\tau)\rho_{T'}(\sigma\tau)^{-1}$ defining a cohomology class in $H^2(G, \text{End}_n(k))$.

The following proposition shows us that a lifting is not always possible.

**Proposition 15.** Let $k$ be an algebraically closed field of positive characteristic $p > 0$, end let $R = W(k)[\zeta_q]$ be the Witt ring of $k$ with a primitive $q = p^h$ root adjoined. Consider the group $G = C_q \rtimes C_m$, where $C_m$ and $C_q$ are cyclic groups of orders $m$ and $q$ respectively and $(m, p) = 1$. Assume that $\sigma$ and $\tau$ are generators for $C_m$ and $C_q$ respectively and moreover

$$\sigma \tau \sigma^{-1} = \tau^a$$

for some integer $a$ (which should satisfy $a^n \equiv 1 \mod q$.) There is a linear representation $\rho : G \to \text{GL}_2(k)$, which can not be lifted to a representation $\rho_R : G \to \text{GL}_2(R)$.

**Proof.** Consider the field $F_p \subset k$ and let $\lambda$ be a generator of the cyclic group $F_p^\times$. The matrices

$$\sigma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfy

$$\sigma^{p-1} = 1, \tau^n = 1, \sigma \tau \sigma^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \sigma^a$$

and generate a subgroup of $\text{GL}_2(k)$, isomorphic to $C_q \rtimes C_m$ for $m = p - 1$, giving a natural representation $\rho : G \to \text{GL}_2(F_p) \subset \text{GL}_2(k)$. 

Suppose that there is a faithful representation \( \tilde{\rho} : G \to \text{GL}_n(R) \) which gives a faithful representation of \( \tilde{\rho} : G \to \text{GL}_n(\text{Quot}(R)) \). Since \( \tilde{\rho}(\tau) \) is of finite order after a Quot(\( R \)) linear change of basis we might assume that \( \tilde{\rho}(\tau) \) is diagonal with \( q \)-roots of unity in the diagonal (we have considered \( R = W(k)[\zeta] \) so that the necessary diagonal elements exist in Quot(\( R \)). We have

\[
\tilde{\rho}(\tau) = \text{diag}(\lambda_1, \ldots, \lambda_n).
\]

At least one of the diagonal elements say \( \lambda = \lambda_{i_0} \) in the above expression is a primitive \( q \)-th root of unity. Let \( E \) be an eigenvector, that is

\[
\tilde{\rho}(\tau)E = \lambda E.
\]

The equality \( \tau \sigma = \sigma \tau^a \) implies that \( \sigma E \) is an eigenvector of the eigenvalue \( \lambda^a \). This means that \( n \) should be greater than the order of \( a \mod q \) since we have at least as many different (and linearly independent) eigenvectors as the different values \( \lambda, \lambda^a, \lambda^{a^2}, \ldots \).

Since, for large prime \( (p > 3) \) we have \( 2 = n < p - 1 \) the representation \( \rho \) can not be lifted to \( R \).

**Local Actions** By the local-global theorems of J.Bertin and A. Mézard \cite{BertinMezard} and the formal patching theorems of D. Harbater, K. Stevenson \cite{HarbaterStevenson1}, \cite{HarbaterStevenson2}, the study of the functor \( \text{D}^{gl} \) can be reduced to the study of the deformation functors \( \text{D}_P \) attached to each wild ramification point \( P \) of the cover \( X \to X/G \), as defined in eq. \( (9) \). The theory of automorphisms of formal powerseries rings is not as well understood as is the theory of automorphisms of finite dimensional vector spaces, i.e. the theory of general linear groups.

As in the theory of liftings for the general linear group, we consider small extensions

\[
1 \to \langle E \rangle \to \Gamma' \xrightarrow{\phi} \Gamma \to 1
\]

An automorphism \( \rho^\Gamma(\sigma) \in \text{Aut}\Gamma[[t]] \) is completely described by a powerseries

\[
\rho^\Gamma(\sigma)(t) = f_\sigma = \sum_{\nu=1}^{\infty} a^\Gamma_\nu(\sigma)t^\nu,
\]

where \( a^\Gamma_\nu(\sigma) \in \Gamma \). Given a naive lift

\[
\rho'^\Gamma(\sigma)(t) = \sum_{\nu=1}^{\infty} a'^\Gamma_\nu(\sigma)t^\nu,
\]

where \( a'^\Gamma_\nu(\sigma) \in \Gamma' \) we can again form a two cocycle

\[
\alpha(\sigma, \tau) = \rho'^\Gamma(\sigma) \circ \rho'^\Gamma(\tau) \circ \rho'^\Gamma(\sigma \tau)^{-1}(t),
\]

defining a cohomology class in \( H^2(G, \mathcal{T}_k[[t]]) \). The naive lift \( \rho'^\Gamma(\sigma) \) is an element of \( \text{Aut}\Gamma'[[t]] \) if and only if \( \alpha \) is cohomologous to zero.

Suppose now that \( \rho'^\Gamma_1, \rho'^\Gamma_2 \) are two lifts in \( \text{Aut}\Gamma'[[t]] \). We can now define

\[
d(\sigma) := \frac{1}{t} \left( \rho'^\Gamma_1(\sigma)\rho'^\Gamma_2(\sigma)^{-1} - 1\right) \in H^1(G, \mathcal{T}_k[[t]])
\]
3. Relative Petri’s theorem.

Recall that a functor $F : \mathcal{C} \to \text{Sets}$ can be extended to a functor $\hat{F} : \hat{\mathcal{C}} \to \text{Sets}$ by letting for every $R \in \text{Ob}(\hat{\mathcal{C}})$, $\hat{F}(R) = \lim F(R/m_R^{n+1})$. An element $\hat{u} \in \hat{F}(R)$ is called a formal element, and by definition it can be represented as a system of elements $\{u_n \in F(R/m_R^{n+1})\}_{n \geq 0}$, such that for each $n \geq 1$, the map $F(R/m_R^{n+1}) \to F(R/m_R^n)$ induced by $R/m_R^{n+1} \to R/m_R^n$ sends $u_n \mapsto u_{n-1}$. For $R \in \text{Ob}(\hat{\mathcal{C}})$ and a formal element $\hat{u} \in \hat{F}(R)$, the couple $(R, \hat{u})$ is called a formal couple. It is known that there is a 1-1 correspondence between $\hat{F}(R)$ and the set of morphisms of functors $h_R := \text{Hom}_{\hat{\mathcal{C}}}(R, -) \to F$, see [34] lemma 2.2.2. The formal element $\hat{u} \in \hat{F}(R)$ will be called versal if the corresponding morphism $h_R \to F$ is smooth. For the definition of a smooth map between functors, see [34] def. 2.2.4. The ring $R$ will be called versal deformation ring.

Schlessinger [32, 3.7] proved that the deformation functor $D$ for curves without automorphisms, admits a ring $R$ as versal deformation ring. Schlessinger calls the versal deformation ring the hull of the deformation functor. Indeed, since there are no obstructions to liftings in small extensions for curves, see [32, rem. 2.10], the hull $R$ of $D_{\Lambda}$ is a powerseries ring over $\Lambda$, which can be taken as an algebraic extension of $W(k)$. Moreover $R = \Lambda[\{x_1, \ldots, x_{3g-3}\}]$, as we can see by applying [3 cor. 3.3.5], when $G$ is the trivial subgroup of the automorphism group. In this case the quotient map $f : X \to \Sigma = X/\{\text{Id}\} = X$ is the identity. Indeed, for the equivariant deformation functor, in the case of the trivial group, there are no ramified points and the short exact sequence in eq. (1) reduces to an isomorphism of the first two spaces. We have $\dim_k H^1(X/G, \pi^G_1(\mathcal{F}_X)) = \dim_k H^1(X, \mathcal{F}_X) = 3g-3$.

The deformation $\mathcal{X} \to \text{Spec} R$ can be extended to a deformation $\mathcal{X} \to \text{Spec} A$ by Grothendieck’s effectivity theorem, see [34] th. 2.5.13, [13].

The versal element $\hat{u}$ corresponds to a deformation $\mathcal{X} \to \text{Spec} R$, with generic fibre $\mathcal{X}_0$ and special fibre $\mathcal{X}_\eta$. The couple $(R, \hat{u})$ is called the versal [34] def. 2.2.6] element of the deformation functor $D$ of curves (without automorphisms). Moreover, the element $u$ defines a map $h_{R/A} \to D$, which by definition of the hull is smooth, so every deformation $X_A \to \text{Spec} A$ defines a homomorphism $R \to A$, which allows us to see $A$ as an $R$-algebra.

Indeed, for the Artin algebra $A \to A/m_A = k$ we consider the diagram

$$h_{R/A} = \text{Hom}_R(R, A) \to h_{R/A}(k) \times_{D(k)} D(A)$$

This section aims to prove the following

**Proposition 16.** Let $f_1, \ldots, f_r \in k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal of a curve $X$ defined over an algebraic closed field $k$. Any deformation $\mathcal{X}_A$ is given by quadratic polynomials $\hat{f}_1, \ldots, \hat{f}_r \in A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $m_A$ of $A$.

For $n \geq 1$, we write $\Omega_{\mathcal{X}/R}^n$ for the sheaf of holomorphic polydifferentials on $\mathcal{X}$. By [17] lemma II.8.9] the $R$-modules $H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^n)$ are free of rank $d_{n,g}$ for all $n \geq 1$, with $d_{n,g}$ given by eq. (10)

$$d_{n,g} = \begin{cases} g, & \text{if } n = 1 \\ (2n-1)(g-1), & \text{if } n > 1. \end{cases}$$
Indeed, by a standard argument using Nakayama’s lemma, see [17, lemma II.8.9], [21] we have that the $R$-module $H^0(X, \Omega^{\otimes n}_{X/R})$ is free. Notice that to use Nakayama’s lemma we need the deformation over $R$ to have both a special and generic fibre and this was the reason we needed to consider a deformation over the spectrum of $R$ instead of the formal spectrum.

**Lemma 17.** For every Artin algebra $A$ the $A$-module $H^0(X_A, \Omega^{\otimes n}_{X_A/A})$ is free.

**Proof.** This follows since $H^0(X, \Omega_X/R)$ is a free $R$-module and [17, prop. II.8.10], which asserts that $\Omega_X/A \cong g^\ast(\Omega_X/R)$, where $g'$ is shown in the next commutative diagram:

$$
X_A = \mathcal{X} \times_{\text{Spec}R} \text{Spec}A \xrightarrow{g'} \mathcal{X}.
$$

We have by definition of the pullback

$$
g'^\ast(\Omega_X/R)(X_A) = (g')^{-1}\Omega_X/R(X_A) \otimes (g')^{-1}O_X(X_A) O_X(X_A)
$$

and by definition of the fiber product $O_X = O_X \otimes R A$. Observe also that since $A$ is a local Artin algebra the schemes $X_A$ and $\mathcal{X}$ share the same underlying topological space so

$$g'^{-1}(\Omega_X/R(X_A)) = \Omega_X/R(\mathcal{X}).$$

So $H^0(X_A, \Omega_{X_A/A})$ is a free $A$-module of the same rank as $H^0(X, \Omega_{X/R})$. The proof for $H^0(X_A, \Omega^{\otimes n}_{X_A/A})$ follows in the same way. □

We select generators $W_1, \ldots, W_g$ for the symmetric algebra

$$\text{Sym}(H^0(X, \Omega_{X/R})) = R[W_1, \ldots, W_g].$$

Similarly, we write

$$\text{Sym}(H^0(\mathcal{X}_\eta, \Omega_{\mathcal{X}_\eta/L})) = L[\omega_1, \ldots, \omega_g]$$
and $\text{Sym}(H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k})) = k[w_1, \ldots, w_g]$, where

$$\omega_i = W_i \otimes R L \quad w_i = W_i \otimes R k \text{ for all } 1 \leq i \leq g.$$  

We have the following diagram relating special and generic fibres.

$$
\begin{array}{ccc}
\text{Spec}(k) \times_{\text{Spec}(R)} \mathcal{X} = \mathcal{X}_0 & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & \text{Spec}(R)
\end{array}
$$

Our article is based on the following relative version of Petri’s theorem.
Theorem 18. Diagram (12) induces a deformation-theoretic diagram of canonical embeddings

\[ 0 \rightarrow I_{X_0} \subset S_L := L[\omega_1, \ldots, \omega_g] \rightarrow H^0(\mathcal{X}_0, \Omega^0, \mathcal{X}/L) \rightarrow 0 \]

\[ 0 \rightarrow \tilde{I}_{\mathcal{X}} \subset S_R := R[W_1, \ldots, W_g] \rightarrow H^0(\mathcal{X}_0, \Omega^0, \mathcal{X}/R) \rightarrow 0 \]

\[ 0 \rightarrow I_{X_0} \subset S_k := k[w_1, \ldots, w_g] \rightarrow H^0(\mathcal{X}_0, \Omega^0, \mathcal{X}/k) \rightarrow 0 \]

where \( I_{X_0} = \ker \phi_0, I_{\mathcal{X}} = \ker \phi, I_{X_0} = \ker \phi_0, \) each row is exact and each square is commutative. Moreover, the ideal \( I_{\mathcal{X}} \) can be generated by elements of degree 2 as an ideal of \( S_R \).

The commutativity of the above diagram was proved in [6] by H. Charalambous, K. Karagiannis and the first author. For proving that \( I_{\mathcal{X}} \) is generated by elements of degree 2 as in the special and generic fibers we argue as follows: Since \( L \) is a field it follows by Petri’s Theorem, that there are elements \( f_1, \ldots, f_r \in S_L \) of degree 2 such that

\[ I_{X_0} = (\tilde{f}_1, \ldots, \tilde{f}_r). \]

Now we choose an element \( c \in R \) such that \( f_i := cf_i \in S_R \) for all \( i \) and notice that \( \deg(f_i) = \deg(\tilde{f}_i) = 2 \).

1. Assume first that the element \( c \in R \) is invertible in \( R \). Consider the ideal \( I = (f_1, \ldots, f_r) \) of \( S_R \). We will prove that \( I = I_{\mathcal{X}} \). Consider the multiplicative system \( R^* \). We will prove first \( I \subset I_{\mathcal{X}} = \ker \phi \). Indeed, using the commuting upper square every element \( a = \sum_{i=1}^r a_i f_i \in I \) maps to \( \sum_{i=1}^r a_i f_i \otimes_R 1 \) which in turn maps to 0 by \( \phi \). The same element maps to \( \phi(a) \) and \( \phi(a) \otimes_R 1 \) should be zero. Since all modules \( H^0(\mathcal{X}, \Omega^0) \) are free \( \phi(a) = 0 \) and \( a \in I_{\mathcal{X}} \).

Since the family \( \mathcal{X} \rightarrow \text{Spec} R \) is flat we have that \( I_{\mathcal{X}} \otimes_R L = I_{\mathcal{X}_0} \), that is we apply the \( \otimes_R L \) functor on the middle short exact sequence of eq. (13). The ideal \( I = I_{X_0} \cap S_R = (I_{\mathcal{X}} \otimes_R L) \cap S_R \). By [2] prop. 3.11ii] this gives that

\[ I = \bigcup_{s \in R^*} (I_{\mathcal{X}} : s) \subset I_{\mathcal{X}}, \]

so \( I_{\mathcal{X}} = I \). In the above formula \( (I_{\mathcal{X}} : s) = \{ x \in S_R : xs \in I_{\mathcal{X}} \} \).

2. From now on we don’t assume that the element \( c \) is an invertible element of \( R \).

Let \( \tilde{g} \) be an element of degree 2 in \( I_{X_0} \), we will prove that we can select an element \( g \in I_{\mathcal{X}} \) such that \( g \otimes 1_k = \tilde{g}, \) so that \( g \) has degree 2.

Let us choose a lift \( \tilde{g} \in S_R \) of degree 2 by lifting each coefficient of \( g \) from \( k \) to \( R \). This element is not necessarily in \( I_{\mathcal{X}} \). We have \( \phi(g) \otimes 1_k = \phi_0(g \otimes 1_k) = \phi_0(\tilde{g}) = 0 \). Let \( e_1, \ldots, e_{3g-3} \) be generators of the free \( R \)-module \( H^0(\mathcal{X}, \Omega^0) \) and choose \( e_1, \ldots, e_{3g-3} \in S_R \) such that \( \phi(e_i) = \tilde{e}_i \). Let us write \( \phi(\tilde{g}) = \sum_{i=1}^{3g-3} \lambda_i \tilde{e}_i \), with \( \lambda_i \in R \). Since \( \phi_0(\tilde{g}) = 0 \) we have that all \( \lambda_i \in m_R \) for all \( 1 \leq i \leq 3g - 3 \). This
means that the element \( g = \tilde{g} - \sum_{i=1}^{3g-3} \lambda_i \bar{e}_i \in S_R \) reduces to \( \tilde{g} \) modulo \( m_R \) and also 
\[
\phi(g) = \phi(\tilde{g}) - \sum_{i=1}^{3g-3} \lambda_i \bar{e}_i = 0, \quad \text{so} \quad g \in I_{\mathcal{X}}.
\]

Let \( g_1, \ldots, g_n \in I_{\mathcal{X}_0} \) be elements of degree 2 such that 
\[
I_{\mathcal{X}_0} = (\bar{g}_1, \ldots, \bar{g}_n)
\]
and, using the previous construction, we take \( g_i \) lifts in \( I_{\mathcal{X}} \triangleleft S_R \), i.e. such that \( g_i \otimes 1_k = \tilde{g}_i \) and also assume that the elements \( g_i \) have also degree 2.

We will now prove that the elements \( g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L \in S_L \) generate the ideal \( I_{\mathcal{X}_n} \). By the commutativity of the diagram in eq. (13) we have \( (g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L) \subset I_{\mathcal{X}_n} = \ker \phi_n \). Observe that any linear relation 
\[
\sum_{\nu=1}^s (a_\nu g_\nu \otimes_{S_R} 1_L) = 0, \quad \text{with} \quad a_\nu \in L
\]
gives rise to a relation for some \( c \in R \)
\[
\sum_{\nu=1}^s c \cdot a_\nu g_\nu = 0, \quad c \cdot a_\nu \in S_R,
\]
which implies that \( c \cdot a_\nu \in m_R \).

We will prove that the elements \( g_i \otimes_{S_R} 1_L \) are linear independent.

**Lemma 19.** Let \( \bar{v}_1, \ldots, \bar{v}_n \in k^m \) be linear independent elements and \( v_1, \ldots, v_n \) be lifts in \( R^m \). Then
\[
\sum_{\nu=1}^n a_\nu v_\nu = 0 \quad a_\nu \in R,
\]
implies that \( a_1 = \cdots = a_n = 0 \).

**Proof.** We have \( n \leq m \). We write the elements \( v_1, \ldots, v_n \) (resp. \( \bar{v}_1, \ldots, \bar{v}_n \)) as columns and in this way we obtain an \( m \times n \) matrix \( J \) (resp. \( \bar{J} \)). Since the elements are linear independent in \( k^m \) there is an \( n \times n \) minor matrix with an invertible determinant. Without loss of generality, we assume that there is an \( n \times n \) invertible matrix \( Q \) with coefficients in \( k \) such that \( Q \cdot J^t = \begin{pmatrix} I_n & A \end{pmatrix} \), where \( A \) is an \((m-n) \times n\) matrix. We now get lifts \( Q, J \) and \( A \) of \( Q, J \) and \( A \) respectively, with coefficients in \( R \), i.e.
\[
Q \cdot J^t = \begin{pmatrix} I_n & A \end{pmatrix} \mod m_R.
\]
The columns of \( J \) are lifts of the elements \( \bar{v}_1, \ldots, \bar{v}_n \). It follows that \( Q \cdot J^t = \begin{pmatrix} I_n & A \end{pmatrix} + \begin{pmatrix} C & D \end{pmatrix} \), where \( C, D \) are matrices with entries in \( m_R \). The determinant of \( I_n + C \) is \( 1 + m \), for some element \( m \in m_R \), and this is an invertible element in the local ring \( R \). Similarly, the matrix \( Q \) is invertible. Therefore,
\[
J^t = \begin{pmatrix} Q^{-1}(I_n + C) & Q^{-1}(A + D) \end{pmatrix}
\]
has the first \( n \times n \) block matrix invertible and the desired result follows.

**Remark 20.** It is clear that over a ring where \( 2 \) is invertible, there is an 1-1 correspondence between symmetric \( g \times g \) matrices and quadratic polynomials. Indeed, a quadratic polynomial can be written as
\[
f(w_1, \ldots, w_g) = \sum_{1 \leq i, j \leq g} a_{ij} w_i w_j = w^t A w,
\]
where $A = (a_{ij})$. Even if the matrix $A$ is not symmetric, the matrix $(A + A^t)/2$ is and generates the same quadratic polynomial

$$w^tAw = w^t\left(\frac{A + A^t}{2}\right)w.$$  

Notice that the map

$$A \mapsto \frac{A + A^t}{2}$$

is onto the space of symmetric matrices and has as kernel the space of antisymmetric matrices.

A minimal set of quadratic generators is given by a set of polynomials $f_1, \ldots, f_r$, with $f_i = w^t A w$, where the symmetric polynomials are linearly independent.

By the general theory of Betti tables we know that in the cases the canonical ideal is generated by quadratic polynomials, the dimension of this set of matrices equals $\binom{g-2}{2}$, see [10, prop. 9.5]. Therefore we begin on the special fibre with the $s = \binom{g-2}{2}$ generators $g_1, \ldots, g_s$ elements. As we have proved in theorem [18] we can lift them to elements $g_1, \ldots, g_s \in I_{X_0}$ so that for $J := \{g_1, \ldots, g_s\}$ we have

(i) $J \otimes_R L = I_{X_0}$,

(ii) $J \otimes_R k = I_{X_0}$.

In this way we obtain the linear independent elements $g_1 \otimes_S R, 1_L, \ldots, g_s \otimes_S R, 1_L$ in $I_{X_0}$. We have seen that the $s = \binom{g-2}{2}$ linear independent quadratic elements generate also $I_{X_0}$.

By following Lemma 5 (ii) of [13] we have the next lemma.

**Lemma 21.** Let $G$ be a set of polynomials in $S_R$ such that $\langle G \rangle \otimes_R L = I_{X_0}$ and $\langle G \rangle \otimes_R k = I_{X_0}$. Then $I_{X_0} = \langle G \rangle$.

Essential for the proof of lemma [21] was that the ring $R$ has a generic fibre. The deformation theory is concerned with deformations over local Artin algebras which do not have generic fibres. But by tensoring with $A$ in the middle sequence of eq. [13] we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I_{X_A} & \longrightarrow & S_A := A[W_1, \ldots, W_g] & \phi & \bigoplus_{n=0}^{\infty} H^0(X_A, \Omega_{X_A/A}^n) & \longrightarrow & 0 \\
\oplus_A A/m_A & \downarrow & \phi_{n=0} & & \oplus_A A/m_A & \downarrow & \phi_{n=0} & & \oplus_A A/m_A \\
0 & \longrightarrow & I_{X_0} & \longrightarrow & S_k := k[w_1, \ldots, w_g] & \phi_0 & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k}^n) & \longrightarrow & 0
\end{array}
\]

Indeed, since $H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k}^n)$ is free the left top arrow in the above diagram is injective. Moreover the relative canonical ideal $I_{X_A}$ is still generated by quadratic polynomials in $S_A$.

### 3.1. Embedded deformations.

Let $Z$ be a scheme over $k$ and let $X$ be a closed subscheme of $Z$. An embedded deformation $X' \rightarrow \text{Spec} k[\epsilon]$ of $X$ over $\text{Spec} k[\epsilon]$ is a
closed subscheme $X' \subset Z' = Z \times \text{Spec} [\epsilon]$ fitting in the diagram:

$$\begin{array}{ccc}
Z & \longrightarrow & Z \times \text{Spec} [\epsilon] \\
\downarrow & & \downarrow \\
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
\text{Spec} & \longrightarrow & \text{Spec} [\epsilon]
\end{array}$$

Let $\mathcal{I}$ be the ideal sheaf describing $X$ as a closed subscheme of $Z$ and

$$\mathcal{N}_{X/Z} = \mathcal{H}om_Z(\mathcal{I}, \mathcal{O}_X) = \mathcal{H}om_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X),$$

be the normal sheaf. In particular for an affine open set $U$ of $X$ we set $B' = \mathcal{O}_{Z'}(U) = B \oplus \epsilon B$, where $B = \mathcal{O}_Z(U)$ and we observe that describing the sheaf of ideals $\mathcal{I}'(U) \subset B'$ is equivalent to giving an element

$$\phi_U \in \text{Hom}_{\mathcal{O}_Z(U)}(\mathcal{I}(U), \mathcal{O}_Z(U)/\mathcal{I}(U)),$$

see [18, prop. 2.3].

In this article, we will take $Z = \mathbb{P}^{g-1}$ and consider the canonical embedding $f : X \rightarrow \mathbb{P}^{g-1}$. We will denote by $N_f$ the sheaf $\mathcal{N}_{X/\mathbb{P}^{g-1}}$. Let $\mathcal{I}_X$ be the sheaf of ideals of the curve $X$ seen as a subscheme of $\mathbb{P}^{g-1}$. Since the curve $X$ satisfies the conditions of Petri’s theorem it is fully described by certain quadratic polynomials $f_1 = A_1, \ldots, f_r = A_r$, which correspond to a set $g \times g$ matrices $A_1, \ldots, A_r$, see [24]. The elements $f_1, \ldots, f_r$ generate the ideal $I_X$ corresponding to the projective cone $C(X)$ of $X$, $C(X) \subset \mathbb{A}^g$.

We have

$$H^0(X, N_f) = \text{Hom}_S(I_X, \mathcal{O}_X).$$

Assume that $X$ is deformed to a curve $X_\Gamma \rightarrow \text{Spec} \Gamma$, where $\Gamma$ is a local Artin algebra, $X_\Gamma \subset \mathbb{P}^{g-1}_\Gamma = \mathbb{P}^{g-1} \times \text{Spec} \Gamma$. Our initial curve $X$ is described in terms of the homogeneous canonical ideal $I_X$, generated by the elements $\{w^t A_1 w, \ldots, w^t A_r w\}$. For a local Artin algebra $\Gamma$ let $\mathcal{S}_g(\Gamma)$ denote the space of symmetric $g \times g$ matrices with coefficients in $\Gamma$. The deformations $X_\Gamma$ are expressed in terms of the ideals $I_{X_\Gamma}$, which by the relative Petri’s theorem are also generated by elements $w^t A_1 w, \ldots, w^t A_r w$, where $A_1^\Gamma$ is in $\mathcal{S}_g(\Gamma)$. This essentially fits with Schlessinger’s observation in [23], where the deformations of the projective variety are related to the deformations of the affine cone, notice that in our case all relative projective curves are smooth and the assumptions of [23, th. 2] are satisfied. We can thus replace the sheaf theoretical description of eq. (14) and work with the affine cone instead.

**Remark 22.** A set of quadratic generators $\{w^t A_1 w, \ldots, w^t A_r w\}$ is a minimal set of generators if and only if the elements $A_1, \ldots, A_r$ are linear independent in the free $\Gamma$-module $\mathcal{S}_g(\Gamma)$ of rank $(g+1)g/2$.

**3.1.1. Embedded deformations and small extensions.** Let

$$0 \rightarrow \langle E \rangle \rightarrow \Gamma' \xrightarrow{\pi} \Gamma \rightarrow 0$$

be a small extension and a curve $\mathbb{P}^{g-1}_{\Gamma'} \supset X_{\Gamma'} \rightarrow \text{Spec} \Gamma'$ be a deformation of $X_\Gamma$ and $X$. The curve $X_{\Gamma'}$ is described in terms of quadratic polynomials $w^t A_{r'} w$, where
$A_i^{r'} \in \mathcal{I}_g(\Gamma')$, which reduce to $A_i^{r}$ modulo $\langle E \rangle$. This means that

$$A_i^{r'} \equiv A_i^{r} \mod \ker(\pi) \text{ for all } 1 \leq i \leq r$$

and if we select a naive lift $i(A_i^{r})$ of $A_i^{r}$, then we can write

$$A_i^{r'} = i(A_i^{r}) + E \cdot B_i, \text{ where } B_i \in \mathcal{I}_g(k).$$

The set of liftings of elements $A_i^{r'}$ of elements $A_i^{r}$, for $1 \leq i \leq r$ is a principal homogeneous space, under the action of $H^0(X, N_f)$, since two such liftings $\{A_i^{(1)}(\Gamma'), 1 \leq i \leq r\}, \{A_i^{(2)}(\Gamma'), 1 \leq i \leq r\}$ differ by a set of matrices in $\{B_i(\Gamma') = A_i^{(1)}(\Gamma') - A_i^{(2)}(\Gamma'), 1 \leq i \leq r\}$ with entries in $\langle E \rangle \cong k$, see also [18, thm. 6.2].

Define a map $\phi : \langle A_1, \ldots, A_r \rangle \rightarrow \mathcal{I}_g(k)$ by $\phi(A_i) = B_i(\Gamma')$ and we also define the a corresponding map on polynomials $\hat{\phi}(A_i) = w^r \phi(A_i)w$. we obtain a map $\hat{\phi} \in \text{Hom}_S(I_X, O_X) = H^0(X, N_f)$, see also [18, th. 6.2], where $S = S_k$. Obstructions to such liftings are known to reside in $H^1(X, \mathcal{N}_{X/\mathbb{P}^{g-1}} \otimes_k \ker \pi)$, which we will prove it is zero, see remark 23.

3.1.2. Embedded deformations and tangent spaces. Let us consider the $k[\epsilon]/k$ case. Since $i : X \hookrightarrow \mathbb{P}^{g-1}$ is non-singular we have the following exact sequence

$$0 \rightarrow \mathcal{F}_X \rightarrow i^* \mathcal{F}_{\mathbb{P}^{g-1}} \rightarrow \mathcal{N}_{X/\mathbb{P}^{g-1}} \rightarrow 0$$

which gives rise to

$$
\begin{array}{cccccc}
0 & \rightarrow & H^0(X, \mathcal{F}_X) & \rightarrow & H^0(X, i^* \mathcal{F}_{\mathbb{P}^{g-1}}) & \rightarrow & H^0(X, \mathcal{N}_{X/\mathbb{P}^{g-1}}) \\
& & & \downarrow \delta & & \\
& & H^1(X, \mathcal{F}_X) & \rightarrow & H^1(X, i^* \mathcal{F}_{\mathbb{P}^{g-1}}) & \rightarrow & H^1(X, \mathcal{N}_{X/\mathbb{P}^{g-1}}) \rightarrow 0
\end{array}
$$

Remark 23. In the above diagram, the last entry in the bottom row is zero since it corresponds to a second cohomology group on a curve. By Riemann-Roch theorem we have that $H^0(X, \mathcal{F}_X) = 0$ for $g \geq 2$. Also, the relative Petri theorem implies that the map $\delta$ is onto. We will give an alternative proof that $\delta$ is onto by proving that $H^1(X, i^* \mathcal{F}_{\mathbb{P}^{g-1}}) = 0$. This proves that $H^1(X, \mathcal{N}_{X/\mathbb{P}^{g-1}}) = 0$ as well, so there is no obstruction in lifting the embedded deformations.

Each of the above spaces has a deformation theoretic interpretation, see [16, p.96]:

- The space $H^0(X, i^* \mathcal{F}_{\mathbb{P}^{g-1}})$ is the space of deformations of the map $i : X \hookrightarrow \mathbb{P}^{g-1}$, that is both $X, \mathbb{P}^{g-1}$ are trivially deformed, see [34, p. 158, prop. 3.4.2(ii)]
- The space $H^0(X, \mathcal{N}_{X/\mathbb{P}^{g-1}})$ is the space of embedded deformations, where $\mathbb{P}^{g-1}$ is trivially deformed see [18, p. 13, Th. 2.4).
- The space $H^1(X, \mathcal{F}_X)$ is the space of all deformations of $X$.

The dimension of the space $H^1(X, \mathcal{F}_X)$ can be computed using Riemann-Roch theorem on the dual space $H^0(X, \Omega_X^{2})$ and equals $3g - 3$. In next section we will give a linear algebra interpretation for the spaces $H^0(X, \mathcal{N}_{X/\mathbb{P}^{g-1}}), H^0(X, i^* \mathcal{F}_{\mathbb{P}^{g-1}})$ allowing us to compute its dimensions.
3.2. Some matrix computations. We begin with the Euler exact sequence (see. [17] II.8.13, [37] p. 581 and [19] MO)

\[ 0 \to \mathcal{O}_{p_{g-1}} \to \mathcal{O}_{p_g} \oplus \mathcal{O} \to \mathcal{O}_{p_{g-1}} \to 0. \]

We restrict this sequence to the curve \( X \):

\[ 0 \to \mathcal{O}_X \to i^* \mathcal{O}_{p_{g-1}} \oplus \omega_X \to i^* \mathcal{O}_{p_{g-1}} \to 0. \]

We now take the long exact sequence in cohomology \((16)\)

\[ 0 \to k = H^0(X, \mathcal{O}_X) \to H^0(X, i^* \mathcal{O}_{p_{g-1}} \oplus \omega_X) \to H^0(X, i^* \mathcal{O}_{p_{g-1}}) \to 0. \]

The spaces involved above have the following dimensions:

- \( i^* \mathcal{O}_{p_{g-1}}(1) = \Omega_X \) (canonical bundle)
- \( \dim H^0(X, i^* \mathcal{O}_{p_{g-1}}(1) \oplus \omega_X) = g \cdot \dim H^0(X, \Omega_X) = g^2 \)
- \( \dim H^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X) = g \)
- \( \dim H^1(X, i^* \mathcal{O}_{p_{g-1}}(1) \oplus \omega_X) = g \cdot \dim H^0(X, \mathcal{O}_X) = g \)

We will return to the exact sequence given in eq. \((16)\) and the above dimension computations in the next section.

3.2.1. Study of \( H^0(X, N_f) \). By relative Petri theorem the elements \( \phi(A_i) \) are quadratic polynomials not in \( I_X \), that is elements in a vector space of dimension \((g + 1)g/2 - \binom{g-2}{2} = 3g - 3\), where \((g + 1)g/2\) is the dimension of the symmetric \(g \times g\) matrices and \( \binom{g-2}{2} \) is the dimension of the space generated by the generators of the canonical ideal, see [10] prop. 9.5.

The set of matrices \( \{A_1, \ldots, A_r\} \) can be assumed to be linear independent but this does not mean that an arbitrary selection of quadratic elements \( \omega^t B_i \omega \in \mathcal{O}_X \) will lead to a homomorphism of rings. Indeed, the linear independent elements \( A_i \) might satisfy some syzygies, see the following example where the linear independent elements

\[ x^2 = (x \ y)^t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

\[ xy = (x \ y)^t \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

satisfy the syzygy

\[ y \cdot x^2 - x \cdot xy = 0. \]

Therefore, a map of modules \( \phi \), should be compatible with the syzygy and satisfy the same syzygy. This is known as the fundamental Grothendieck flatness criterion, see [33] 1.1 and also [1] lem. 5.1, p. 28].

Proposition 24. The map

\[ \psi : M_g(k) \to \text{Hom}_S(I_X, S/I_X) = H^0(X, \mathcal{O}_X) \]

\[ B \mapsto \psi_B : \omega^t A_i \omega \mapsto \omega^t (A_i B + B^t A_i) \omega \mod I_X \]

identifies the vector space \( M_g(k)/(\mathbb{Z}_p) \) to \( H^0(X, i^* \mathcal{O}_{p_{g-1}}) \subset H^0(X, \mathcal{O}_X) \). The map \( \psi \) is equivariant, where \( M_g(k) \) is equipped with the adjoint action

\[ B \mapsto \rho(g) B \rho(g^{-1}) = \text{Ad}(g) B, \]

that is

\[ \psi_B = \psi_{\text{Ad}(g)} B. \]
Proof. Recall that the space $H^0(X, i^*\mathcal{F}_{\mathbb{P}^{g-1}})$ can be identified to the space of deformations of the map $f$, where $X, \mathbb{P}^{g-1}$ are both trivially deformed. By [33] a map $\phi \in \text{Hom}_S(I_X, S/I_X) = \text{Hom}_S(I_X, \mathcal{O}_X)$ gives rise to a trivial deformation if there is a map

$$w_j \mapsto w_j + \epsilon \delta_j(w),$$

where $\delta_j(w) = \sum_{\nu=1}^g b_{j,\nu} w_\nu$. The map can be defined in terms of the matrix $B = (b_{j,\nu})$,

$$w \mapsto w + \epsilon Bw$$

so that for all $\tilde{A}_i, 1 \leq i \leq r$

$$\nabla \tilde{A}_i \cdot Bw = \phi(\tilde{A}_i) = \phi(w^t A_i w) \mod I_X.$$  

But for $\tilde{A}_i = w^t A_i w$ we compute $\nabla \tilde{A}_i = w^t A_i$, therefore eq. (17) is transformed to

$$w^t A_i Bw = w^t B_i w \mod I_X,$$

for a symmetric $g \times g$ matrix $B_i$ in $\mathcal{F}_g(k[\epsilon])$. Therefore if 2 is invertible according to remark [20] we replace the matrix $A_i B$ appearing in eq. (18) by the symmetric matrix $A_i B + B^t A_i$. Since we are interested in the projective algebraic set defined by homogeneous polynomials the 1/2 factor of remark [21] can be omitted.

For every $B \in M_g(k)$ we define the map $\psi_B \in \text{Hom}_S(I_X, S/I_X) = \text{Hom}_S(I_X, \mathcal{O}_X)$ given by

$$\tilde{A}_i = \omega^t A_i \omega \mapsto \omega^t (A_i B + B^t A_i) \omega \mod I_X,$$

and we have just proved that the functions $\psi_B$ are all elements in $H^0(X, i^*\mathcal{F}_{\mathbb{P}^{g-1}})$. The kernel of the map $\psi : B \mapsto \psi_B$ consists of all matrices $B$ satisfying:

$$A_i B = -B^t A_i \mod I_X \text{ for all } 1 \leq i \leq \binom{g-2}{2}.$$  

This kernel seems to depend on the selection of the elements $A_i$. This is not the case. We will prove that the kernel consists of all multiples of the identity matrix. Indeed,

$$\dim H^0(X, i^*\mathcal{F}_X) = g^2 - \ker \psi.$$  

We now rewrite the spaces in eq. (16) by their dimensions we get

$$\begin{array}{cccccc}
\circ & (0) & \rightarrow & (1) & \rightarrow & (g^2) \\
& f_1 & & f_2 & & f_3 \\
& \circ & (g) & \rightarrow & (?,) & \rightarrow & (0)
\end{array}$$

So

- $\dim \ker f_2 = \dim \text{Im} f_1 = 1$
- $\dim \ker f_3 = \dim \text{Im} f_2 = g^2 - 1$
- $\dim \text{Im} f_3 = (g^2 - \dim \ker \psi) - (g^2 - 1) = 1 - \dim \ker \psi$

It is immediate that $\dim \ker \psi = 0$ or 1. But obviously $\mathbb{I}_g \in \ker \psi$, and hence

$$\dim \ker \psi = 1.$$  

Finally $\dim \text{Im} f_3 = 0$, i.e. $f_3$ is the zero map and we get the small exact sequence,

$$0 \rightarrow k = H^0(X, \mathcal{O}_X) \rightarrow H^0(X, i^*\mathcal{F}_{\mathbb{P}^{g-1}}) \rightarrow H^0(X, i^*\mathcal{F}_{\mathbb{P}^{g-1}} \oplus g) \rightarrow H^0(X, i^*\mathcal{F}_{\mathbb{P}^{g-1}}) \rightarrow 0.$$
It follows that
\[ \dim H^0(X, i^* \mathcal{F}_{g-1}) = g^2 - 1. \]
We have proved that \( \psi : M_g(k)/(I_g) \to H^0(X, i^* \mathcal{F}_{g-1}) \) is an isomorphism of vector spaces. We will now prove it is equivariant.

Using remark \([10]\) we have that the action of the group \( G \) on the function
\[ \psi_B : A_i \mapsto A_iB + B^iA_i, \]
seen as an element in \( H^0(X, i^* \mathcal{F}_{g-1}) \) is given:
\[
A_i \mapsto T(\sigma^{-1}A_i) \psi_B \rightarrow T(\sigma) (\rho(\sigma)^i A_i \rho(\sigma)B + B^i \rho(\sigma)^i A_i \rho(\sigma))
= (A_i \rho(\sigma) B \rho(\sigma^{-1}) + (\rho(\sigma) B \rho(\sigma^{-1}))^i A_i)
\]
\[ \square \]

**Corollary 25.** The space \( H^0(X, i^* \mathcal{F}_{g-1})^G \) is generated by the elements \( B \neq \{\lambda I_g : \lambda \in k\} \) such that

\[ \rho(\sigma)B \rho(\sigma^{-1})B^{-1} = [\rho(\sigma), B] \in \langle A_1, \ldots, A_r \rangle \text{ for all } \sigma \in \text{Aut}(X). \]

**Remark 26.** This construction allows us to compute the space \( H^1(X, i^* \mathcal{F}_{g-1}) \). Indeed, we know that \( f_1 \) is isomorphism and hence \( f_2 \) is the zero map, on the other hand \( f_3 \) is surjective, it follows that \( H^1(X, i^* \mathcal{F}_{g-1}) = 0 \). This provides us with another proof of the exactness of the sequence

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X, i^* \mathcal{F}_{g-1}) & \longrightarrow & H^0(X, \mathcal{N}_{X/\mathcal{F}_{g-1}}) & \xrightarrow{\delta} & H^1(X, \mathcal{F}_X) & \longrightarrow & 0
\end{array}
\]

### 3.3. Invariant spaces.

Let
\[ 0 \to A \to B \to C \to 0 \]
be a short exact sequence of \( G \)-modules. We have the following sequence of \( G \)-invariant spaces
\[
0 \to A^G \to B^G \to C^G \xrightarrow{\delta_G} H^1(G, A) \to \cdots
\]
where the map \( \delta_G \) is computed as follows: an element \( c \) is given as a class \( b \) \ modulo \( A \) and it is invariant if and only if \( gb = a \) \ in \( A \). The map \( G \ni g \mapsto a \) \ is the cocycle defining \( \delta_G(c) \in H^1(G, A) \).

Using this construction on the short exact sequence of eq. \([20]\) we arrive at

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X, i^* \mathcal{F}_{g-1})^G & \longrightarrow & H^0(X, \mathcal{N}_{X/\mathcal{F}_{g-1}})^G & \xrightarrow{\delta} & H^1(X, \mathcal{F}_X)^G
\end{array}
\]

\[ \delta_G \]

\[
\begin{array}{ccccccc}
\delta_G & \longrightarrow & H^1(G, H^0(X, i^* \mathcal{F}_{g-1})) & \longrightarrow & \cdots
\end{array}
\]

We will use eq. \([20]\) in order to represent elements in \( H^1(X, \mathcal{F}_X) \) as elements \( [f] \in H^0(X, \mathcal{N}_{X/\mathcal{F}_{g-1}})/H^0(X, i^* \mathcal{F}_{g-1}) = H^0(X, \mathcal{N}_{X/\mathcal{F}_{g-1}})/\text{Im}\psi \).

**Proposition 27.** Let \([f] \in H^1(X, \mathcal{F}_X)^G\) be a class of a map \( f : I_X \to S/I_X \) modulo \( \text{Im}\psi \). For each element \( \sigma \in G \) there is a matrix \( B_{\sigma}[f] \), depending on \( f \), which defines a class in \( M_g(k)/(I_g) \) satisfying the cocycle condition in eq. \([22]\), such that
\[
\delta_G(f)(\sigma) : A_i \mapsto A_i (B_{\sigma}[f]) + (B^i_{\sigma}[f]) A_i \mod\langle A_1, \ldots, A_g \rangle.
\]
Proof. Let \([f] \in H^1(X, T_X)^G\), where \(f : \mathcal{X} \rightarrow S/I_X\) that is \(f \in H^0(X, \mathcal{X}_f)\). The \(\delta_G(f)\) is represented by an 1-cocycle given by \(\delta_G(f)(\sigma) = f - \sigma f\). Using the equivariant isomorphism of \(\psi : M_g(k) / \langle \mathfrak{l}_g \rangle \rightarrow H^0(X, i^* T_{\mathcal{X}_f})\) of proposition \([24]\) we arrive at the diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\psi^{-1}} & M_g(k) / \langle \mathfrak{l}_g \rangle \\
\sigma & \xrightarrow{\delta_G(f)(\sigma)} & B[f]_\sigma := \psi^{-1}(\delta_G(f)(\sigma))
\end{array}
\]

We will now compute

\[
\sigma f : A_i \xrightarrow{T^{(\sigma^{-1})}} T^{-1}A_i \xrightarrow{f} f(T^{-1}A_i) \xrightarrow{T^{(\sigma)}} T(\sigma)f(T^{-1}A_i).
\]

We set

\[
T^{-1}(A_i) = \rho(\sigma)^i A_i \rho(\sigma) = \sum_{\nu=1}^r \lambda_{i,\nu}(\sigma) A_i
\]

so

\[
(21) \quad \delta_G(f)(\sigma)(A_i) = \sum_{\nu=1}^r \lambda_{i,\nu}(\sigma) \cdot \rho(\sigma^{-1})^i f(A_\nu) \rho(\sigma^{-1}) - f(A_i)
\]

\[
= A_i B_\sigma[f] + B_\sigma[f]^i A_i \mod I_X
\]

for some matrix \(B_\sigma[f] \in M_g(k)\) such that for all \(\sigma, \tau \in G\) we have

\[
(22) \quad B_{\sigma \tau}[f] = B_\sigma[f] + \sigma B_\tau[f] \sigma^{-1} + \lambda(\sigma, \tau) \mathfrak{l}_g
\]

\[
= B_\sigma[f] + \text{Ad}(\sigma) B_\tau[f] + \lambda(\sigma, \tau) \mathfrak{l}_g.
\]

In the above equation we have used the fact that \(\sigma \mapsto B_\sigma[f]\) is a 1-cocycle in the quotient space \(M_g(k) / \mathfrak{l}_g\), therefore the cocycle condition holds up to an element of the form \(\lambda(\sigma, \tau) \mathfrak{l}_g\). \(\square\)

Remark 28. Let

\[
\lambda(\sigma, \tau) \mathfrak{l}_g = B_{\sigma \tau}[f] - B_\tau[f] - \text{Ad}(\sigma) B_\sigma[f].
\]

The map \(G \times G \rightarrow k\), \((\sigma, \tau) \mapsto \lambda(\sigma, \tau)\) is a normalized 2-cocycle (see \([39]\) p. 184)), that is

\[
0 = \lambda(\sigma, 1) = \lambda(1, \sigma)
\]

for all \(\sigma \in G\)

\[
0 = \text{Ad}(\sigma_1) \lambda(\sigma_2, \sigma_3) - \lambda(\sigma_1 \sigma_2, \sigma_3) + \lambda(\sigma_1, \sigma_2 \sigma_3) - \lambda(\sigma_1, \sigma_2)
\]

for all \(\sigma_1, \sigma_2, \sigma_3 \in G\)

\[
= \lambda(\sigma_2, \sigma_3) - \lambda(\sigma_1 \sigma_2, \sigma_3) + \lambda(\sigma_1, \sigma_2 \sigma_3) - \lambda(\sigma_1, \sigma_2)
\]

for all \(\sigma_1, \sigma_2, \sigma_3 \in G\)

For the last equality notice that the \(\text{Ad}\)-action is trivial on scalar multiples of the identity.

Proof. The first equation is clear. For the second one,

\[
\lambda(\sigma_1 \sigma_2, \sigma_3) \mathfrak{l}_g = B_{\sigma_1 \sigma_2 \sigma_3}[f] - B_{\sigma_1 \sigma_2}[f] - \text{Ad}(\sigma_1 \sigma_2) B_{\sigma_2}[f]
\]

and

\[
\lambda(\sigma_1, \sigma_2) \mathfrak{l}_g = B_{\sigma_1 \sigma_2}[f] - B_{\sigma_1}[f] - \text{Ad}(\sigma_1) B_{\sigma_2}[f].
\]
Hence
\[
\lambda(\sigma_1\sigma_2, \sigma_3)I_g + \lambda(\sigma_1, \sigma_2)I_g = B_{\sigma_1\sigma_2\sigma_3}[f] - \text{Ad}(\sigma_1\sigma_2)B_{\sigma_3}[f] - B_{\sigma_1}[f] - \text{Ad}(\sigma_1)B_{\sigma_2}[f]
\]
\[= B_{\sigma_1\sigma_2\sigma_3}[f] - B_{\sigma_1}[f] - \text{Ad}(\sigma_1)B_{\sigma_2}[f] + \text{Ad}(\sigma_1)B_{\sigma_2\sigma_3}[f] + \lambda(\sigma_1\sigma_2\sigma_3)I_g + \lambda(\sigma_1)B_{\sigma_2}[f] - \text{Ad}(\sigma_1)B_{\sigma_3}[f]
\]
\[= \lambda(\sigma_1\sigma_2\sigma_3)I_g + \lambda(\sigma_1)B_{\sigma_2}[f] - \text{Ad}(\sigma_1)B_{\sigma_3}[f]
\]
\[= \lambda(\sigma_1)\lambda(\sigma_2, \sigma_3)I_g + \lambda(\sigma_1, \sigma_2\sigma_3)I_g.
\]
\[\square\]

**Corollary 29.** If \( f(\omega^i A_i \omega) = \omega^i B_i \omega \), where \( B_i \in M_g(k) \) are the images of the elements defining the canonical ideal in the small extension \( \Gamma' \to \Gamma \), then the symmetric matrices defining the canonical ideal \( I_X(\Gamma') \) are given by \( A_i + E \cdot B_i \). Using proposition [27] we have

\begin{equation}
(f^* - f)(A_i) = \sum_{\nu=1}^r \lambda_{i,\nu}(\sigma) T(\sigma)(B_{\nu}) - B_i
\end{equation}

\[= (A_i B_{\sigma} [f] + B_{\sigma}^2 [f] A_i) \mod (A_1, \ldots, A_r)
\]

\[= \psi_{B_{\sigma}[f]} A_i.
\]

Therefore, using also eq. [21]

\begin{equation}
\sum_{\nu=1}^r \lambda_{i,\nu}(\sigma)(B_{\nu}) - T(\sigma^{-1})B_i = T(\sigma^{-1})\psi_{B_{\sigma}[f]} A_i.
\end{equation}

4. **On the deformation theory of curves with automorphisms**

Let \( 1 \to \langle E \rangle \to \Gamma' \to \Gamma \to 0 \) be a small extension of Artin local algebras and consider the diagram

\[
\begin{array}{ccc}
X_\Gamma & \longrightarrow & X_{\Gamma'} \\
\downarrow & & \downarrow \\
\text{Spec}(\Gamma) & \longrightarrow & \text{Spec}(\Gamma')
\end{array}
\]

Suppose that \( G \) acts on \( X_\Gamma \), that is every automorphism \( \sigma \in G \) satisfies \( \sigma(I_{X_\Gamma}) = I_{X_\Gamma} \). If the action of the group \( G \) is lifted to \( X_{\Gamma'} \), then we should have a lift of the representations \( \rho, \rho^{(1)} \) defined in eq. [2] to \( \Gamma' \) as well. The set of all such liftings is a principal homogeneous space parametrized by the spaces \( H^1(G, M_g(k)) \), \( H^1(G, M_r(k)) \), provided that the corresponding lifting obstructions in \( H^2(G, M_g(k)), H^2(G, M_r(k)) \) both vanish.

Assume that there is a lifting of the representation

\begin{equation}
\begin{array}{ccc}
G & \longrightarrow & GL_g(\Gamma') \\
\rho_{\Gamma'} & \downarrow & \text{mod}(E) \\
\rho_{\Gamma} & \downarrow & GL_g(\Gamma)
\end{array}
\end{equation}

This lift gives rise to a lifting of the corresponding automorphism group to the curve \( X_{\Gamma'} \) if

\[\rho_{\Gamma'}(\sigma)I_{X_{\Gamma'}} = I_{X_{\Gamma'}}, \quad \text{for all } \sigma \in G,
\]
that is if the relative canonical ideal is invariant under the action of the lifted
representation $\rho^{(1)}_{\Gamma'}$. In this case the free $\Gamma'$-modules $V_{\Gamma'}$, defined in remark [3] are
$G$-invariant and the $T$-action, as defined in definition [11] restricts to a lift of the
representation
\begin{equation}
\begin{array}{ccc}
\rho^{(1)}_{\Gamma'} & \rightarrow & GL_r(\Gamma') \\
\downarrow & & \downarrow \text{mod}(E) \\
G & \xrightarrow{\rho^{(1)}_G} & GL_r(\Gamma)
\end{array}
\end{equation}
In [24, sec. 2.2] we gave an efficient way to check this compatibility in terms of
linear algebra:
Consider an ordered basis $\Sigma$ of the free $\Gamma$-module $S_g(\Gamma)$ generated by the ma-
trices $\Sigma(\sigma_{ij}) = (\delta_{i,\nu,\mu})$, $1 \leq i \leq j \leq g$, ordered lexicographically, with elements
\[
\sigma(\sigma_{ij})_{\nu,\mu} = \begin{cases} 
\delta_{i,\nu}\delta_{j,\mu} + \delta_{i,\mu}\delta_{j,\nu}, & \text{if } i \neq j \\
\delta_{i,\nu}\delta_{i,\mu}, & \text{if } i = j.
\end{cases}
\]
For example, for $g = 2$ we have the elements
\[
\sigma(11) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma(22) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
For every symmetric matrix $A$, let $F(A)$ be the column vector consisted of the
coordinates of $A$ in the basis $\Sigma$. Consider the symmetric matrices $A_1^{\Gamma'}, \ldots, A_r^{\Gamma'}$, which exist since at the level of curves there is no obstruction of the embedded
deformation. For each $\sigma \in G$ the $(g + 1)g/2 \times 2r$ matrix
\begin{equation}
F_{\Gamma'}(\sigma) = \left[ F \left( A_r^{\Gamma'} \right), \ldots, F \left( A_1^{\Gamma'} \right), F \left( \rho_{\Gamma'}(\sigma)^t A_r^{\Gamma'} \rho_{\Gamma'}(\sigma) \right), \ldots, F \left( \rho_{\Gamma'}(\sigma)^t A_1^{\Gamma'} \rho_{\Gamma'}(\sigma) \right) \right].
\end{equation}
The automorphism $\sigma$ acting on the relative curve $X_\Gamma$ is lifted to an automorphism $\sigma$ of $X_\Gamma$, if and only if the matrix given in eq. (27) has rank $r$.

**Proposition 30.** The obstruction to lifting an automorphism of $X_\Gamma$ to $X_{\Gamma'}$, has a
global obstruction given by vanishing the class of
\[
A(\sigma, \tau) = \rho_{\Gamma'}(\sigma)\rho_{\Gamma'}(\tau)\rho_{\Gamma'}(\sigma\tau)^{-1}
\]
in $H^2(G, M_g(k))$ and a compatibility rank condition given by requiring that the
matrix $F_{\Gamma'}(\sigma)$ equals $r$ for all elements $\sigma \in G$.

4.1. **An example.** Let $k$ be an algebraically closed field of positive characteristic
$p > 0$. Consider the Hermitian curve, defined over $k$, given by the equation
\begin{equation}
H : y^p - y = \frac{1}{x^{p+1}},
\end{equation}
which has the group $\text{PGU}(3, p^2)$ as an automorphism group, [38 th. 7]. As an
Artin-Schreier extension of the projective line, this curve fits within the Bertin-
Mézard model of curves, and the deformation functor with respect to the subgroup
$\mathbb{Z}/p\mathbb{Z} \cong \text{Gal}(H/\mathbb{P}^1) = \{ y \mapsto y + 1 \}$ has versal deformation ring $W(k)[[\zeta]][[[x_1]]]$, where $\zeta$ is a primitive $p$ root of unity which resides in an algebraic extension of
$\text{Quot}(W(k))$ [3], [21]. Indeed, $m = p + 1 = 2p - (p - 1) = qp - l$, so in the notation
of [3] $q = 2$ and $t = p - 1$. 
The reduction of the universal curve in the Bertin-Mézard model modulo $m_W(k)[\zeta]$ is given by the Artin-Schrein equation:

\begin{equation}
X^p - X = \frac{x^{p-1}}{(x^2 + x_1x)^p}
\end{equation}

which has special fibre at the specialization $x_1 = 0$ the original Hermitian curve given in eq. (28).

The initial Hermitian curve admits the automorphism $\sigma: y \mapsto \zeta_1, x \mapsto \zeta x$, where $\zeta_{p+1}$ is a primitive $p+1$ root of unity. We will use the tools developed in this article in order to show that the automorphism $\sigma$ does not lift even in positive characteristic.

We set $a(x) = x^2 + x_1x$ and $\lambda = \zeta_1 - 1 \in W(k)[\zeta]$. In [21] the first author together with S. Karanikolopoulos proved that the free $R$-module $H_0(X, \Omega_{X/R})$ has basis $c = \{W_{N, \mu} = x^N a(x)^{p-1-\mu} X^{p-1-\mu} dx : \frac{\mu}{p} \leq N \leq \mu q - 2, 1 \leq \mu \leq p - 1 \}$.

From the form of the holomorphic differentials it is clear that the representation of $\langle \sigma \rangle$ on $H^0(H, \Omega_{H/k})$ is diagonal, since $a(x) = x^2 + x_1x$ reduces to $x^2$ for $x_1 = 0$. In our example, we have $q = \deg a(x) = 2$ so in the special fibre we have

$w_{N, \mu} = x^{N-2\mu} X^{p-1-\mu} dx$

$\sigma(w_{N, \mu}) = \zeta_{p+1}^{N-2\mu+1} w_{N, \mu}$

and

\begin{equation}
\sigma(w_{N, \mu}w_{N', \mu'}) = \zeta_{p+1}^{N+N'-2(\mu+\mu')} w_{N, \mu}w_{N', \mu'}.
\end{equation}

Thus, the action of $\sigma$ on holomorphic differentials on the special fibre is given by a diagonal matrix.

To decide, using the tools developed in this article, whether the action lifts to the Artin local ring $k[\epsilon]$, we have to see first whether the diagonal representation can be lifted, that is whether we have the following commutative diagram:

\begin{equation}
\begin{array}{ccc}
\mathrm{GL}_g(k[\epsilon]) & \xrightarrow{\hat{\rho}} & \mathrm{GL}_g(k) \\
\downarrow \hat{\rho} & & \downarrow \rho \\
\langle \sigma \rangle & \xrightarrow{\rho} & \mathrm{GL}_g(k)
\end{array}
\end{equation}

Since $\rho(\sigma) = \text{diag}(\delta_1, \ldots, \delta_g)$ $= \Delta$ a possible lift will be given by $\hat{\rho}(\sigma) = \Delta + \epsilon B$, for some $g \times g$ matrix $B$ with entries in $k$. The later element should have order $p+1$, that is

$I_g = (\Delta + \epsilon B)^{p+1} = \Delta^{p+1} + \epsilon \Delta^p B,$

which in turn implies that $\Delta^p B = 0$ and since $\Delta$ is invertible $B = 0$. This means that the representation of the cyclic group generated by $\sigma$ is trivially deformed to a representation into $\mathrm{GL}_g(k[\epsilon])$.

The next step is to investigate whether the canonical ideal is kept invariant under the action of $\sigma$ for $x_1 \neq 0$. The canonical ideal for Bertin-Mézard curves was recently studied by H. Haralampous K. Karagiannis and the first author, [6].
Namely, using the notation of [6] we have
\[ a(x)^{p-i} = (x^2 + x_1 x)^{p-i} = \sum_{j=j_{\text{min}}}^{2(p-1)} c_{j,p-i} x^j \]
so by setting \( J = j + p - i \), \( p - i \leq J \leq 2(p - i) \) we have
\[ c_{j,p-i} = \begin{cases} \frac{1}{(p-i)} x_1^{2(p-i)-J} & \text{if } J \geq p - i \\ 0 & \text{if } J < p - i \end{cases} \]
This means that \( c_{2(p-i),p-i} = 1 \), \( c_{2(p-i)-1,p-i} = (p - i)x_1 \) and for all other values of \( J \), the quantity \( c_{j,p-i} \) is either zero or a monomial in \( x_1 \) of degree \( \geq 2 \).

It is proved in [6] that the canonical ideal is generated by two sets of generators \( G_1 \) and \( G_2 \) given by:
\[ G_1^c = \{ W_{N_1,\mu_1} W_{N_1',\mu_1'} - W_{N_2,\mu_2} W_{N_2',\mu_2'} \in S : W_{N_1,\mu_1} W_{N_1',\mu_1'}, W_{N_2,\mu_2} W_{N_2',\mu_2'} \in \mathbb{T}^2 \} \]
\[ G_2^c = \{ W_{N,\mu} W_{N',\mu'} - W_{N'',\mu''} W_{N''',\mu'''} \] 
\[ + \sum_{i=1}^{p-1} \sum_{j=j_{\text{min}}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{i,p-i} W_{N_j,\mu_j} W_{N_j',\mu_j'} \in S : \]
\[ N'' + N''' = N + N' + p - 1, \quad \mu'' + \mu''' = \mu + \mu' + p, \]
\[ N_j + N_j' = N + N' + j, \quad \mu_j + \mu_j' = \mu + \mu' + p - i \]
for \( 0 \leq i \leq p, j_{\text{min}}(i) \leq j \leq (p - i)q \).

The reduction modulo \( m_{W(k)c} \) of the set \( G_1^c \) is given by simply replacing each \( W_{n,\mu} \)
by \( w_{N,\mu} \) and does not depend on \( x_1 \). Therefore it does not give us any condition to deform \( \sigma \).

The reduction of the set \( G_2^c \) modulo \( m_{W(k)c} \) is given by
\[ G_2^c \otimes_R k = \{ w_{N,\mu} w_{N',\mu'} - w_{N'',\mu''} w_{N''',\mu'''} \] 
\[ + \sum_{j=j_{\text{min}}(1)}^{(p-1)q} c_{j,p-1} w_{N_j,\mu_j} w_{N_j',\mu_j'} \in S : \]
\[ N'' + N''' = N + N' + p - 1, \quad \mu'' + \mu''' = \mu + \mu' + p, \]
\[ N_j + N_j' = N + N' + j, \quad \mu_j + \mu_j' = \mu + \mu' + p - i \]
for \( j_{\text{min}}(1) \leq j \leq (p - 1)q \).

If we further consider this set modulo \( \langle x_1^2 \rangle \), that is if we consider the canonical curve as a family over first-order infinitesimals then, only the terms \( c_{2(p-1),p-1} = 1 \), \( c_{2(p-1)-1,p-1} = (p-1)x_1 \) survive.
Using eq. (30) and the definition of $G_p^2$, we have that for
\[ W = w_{N_1} w_{N_2} - w_{N_2} w_{N_3} - w_{N_3} W_{N_4} - w_{N_4} W_{N_5}. \]

Let
\[ \sigma(W) = w_{p+1}^{N+N'-2(\mu+\mu')} W. \]

Set
\[ W'' = w_{N_2(p-1)-1} w_{N_2(p-1)-1} m_{p-1}' \]

The automorphism lifts if and only if the element
\[ W' = W + x_1 W'' \]

we have
\[ \sigma(W') = \chi(\sigma)(W'). \]

But this is not possible since for
\[ \sigma(W'') = w_{p+1}^{N_2(p-1)-1 + N_2(p-1)-1 - 2(\mu_{p-1} + \mu_{p-1}')} + 2 = N + N' - 2(\mu + \mu') + 2 - 1. \]

4.2. A tangent space condition. All lifts of $X_G$ to $X_{\Gamma'}$ form a principal homogeneous space under the action of $H^0(X, \mathcal{X}_{/\mathbb{P}^{N'-1}})$. This paragraph aims to provide the compatibility relation given in eq. (31) by selecting the deformations of the curve and the representations.

Let $\{A_1^\Gamma, \ldots, A_r^\Gamma\}$ be a basis of the canonical Ideal $I_{X_G}$, where $X_G$ is a canonical curve. Assume also that the special fibre is acted on by the group $G$, and we assume that the action of the group $G$ is lifted to the relative curve $X_G$. Since $X_G$ is assumed to be acted on by $G$, we have the action
\[ T(\sigma^{-1})(A_i^\Gamma) = \rho_G(\sigma)^t A_i^\Gamma \rho_G(\sigma) = \sum_j \lambda_{i,j}^\Gamma(\sigma)A_j^\Gamma \]

where $\rho_G$ is a lift of the representation $\rho$ induced by the action of $G$ on $H^0(X_G, \Omega_{X/G})$, and $\lambda_{i,j}^\Gamma(\sigma)$ are the entries of the matrix of the lifted representation $\rho_G^{(1)}$. Notice that the matrix $\rho_G(\sigma) \in \text{GL}_g(\Gamma)$. We will denote by $A_1^\Gamma, \ldots, A_r^\Gamma \in \mathcal{F}_g(\Gamma')$ a set of liftings of the matrices $A_1^\Gamma, \ldots, A_r^\Gamma$. Since the couple $(X_G, G)$ is lifted to $(X_{\Gamma'}, G)$, there is an action
\[ T(\sigma^{-1})(A_i^\Gamma) = \rho_G(\sigma)^t A_i^\Gamma \rho_G(\sigma) = \sum_j \lambda_{i,j}^\Gamma(\sigma)A_j^\Gamma \]

where $\lambda_{i,j}^\Gamma(\sigma) \in \Gamma'$. All other liftings extending $X_G$ form a principal homogeneous space under the action of $H^0(X, \mathcal{X}_{/\mathbb{P}^{N'-1}})$ that is we can find matrices $B_1, \ldots, B_r \in \mathcal{F}_g(k)$, such that the set
\[ \{A_1^\Gamma + E \cdot B_1, \ldots, A_r^\Gamma + E \cdot B_r\} \]

forms a basis for another lift $I_{X_{\Gamma'}}$ of the canonical ideal of $I_{X_G}$. That is all lifts of the canonical curve $I_{X_G}$ differ by an element $f \in \text{Hom}_G(I_X, S/I_X) = H^0(X, \mathcal{X}_{/\mathbb{P}^{N'-1}})$ so that $f(A_i) = B_i$.

In the same manner, if $\rho_{\Gamma'}$ is a lift of the representation $\rho_G$ every other lift is given by
\[ \rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma), \]

where $\tau(\sigma)$ is a lift of the automorphism $\tau$.
where \( \tau(\sigma) \in M_g(k) \).

We have to find out when \( \rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma) \) is an automorphism of the relative curve \( X_{\Gamma'} \), i.e. when

\[
T(\rho_{\Gamma'}(\sigma^{-1}) + E \cdot \tau(\sigma^{-1})) (A_{i_j}^{T'} + E \cdot B_i) \in \text{span}_{\Gamma'} \{ A_i^{T'} + E \cdot B_1, \ldots, A_i^{T'} + E \cdot B_r \},
\]

that is

\[
(\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma))^t \left( A_{i_j}^{T'} + E \cdot B_i \right) (\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)) = \sum_{j=1}^{r} \tilde{\lambda}_{i_j}^{T'}(\sigma) \left( A_{i_j}^{T'} + E \cdot B_j \right),
\]

for some \( \tilde{\lambda}_{i_j}^{T'}(\sigma) \in \Gamma' \). Since

\[
T_{\Gamma'}(\sigma^{-1}) A_{i_j}^{T'} = \rho_{\Gamma'}(\sigma)^t A_{i_j}^{T'} \rho_{\Gamma'}(\sigma) \mod(E)
\]

we have that \( \tilde{\lambda}_{i_j}^{T'}(\sigma) = \lambda_{i_j}^{T'}(\sigma) \mod(E) \), therefore we can write

\[
\tilde{\lambda}_{i_j}^{T'}(\sigma) = \lambda_{i_j}^{T'}(\sigma) + E \cdot \mu_{ij}(\sigma),
\]

for some \( \mu_{ij}(\sigma) \in k \). We expand first the right-hand side of eq. \( \text{(33)} \) using eq. \( \text{(34)} \). We have

\[
\sum_{j=1}^{r} \tilde{\lambda}_{i_j}^{T'}(\sigma) \left( A_{i_j}^{T'} + E \cdot B_j \right) = \sum_{j=1}^{r} \left( \lambda_{i_j}^{T'}(\sigma) + E \cdot \mu_{ij}(\sigma) \right) \left( A_{i_j}^{T'} + E \cdot B_j \right)
\]

\[
= \sum_{j=1}^{r} \lambda_{i_j}^{T'}(\sigma) A_{i_j}^{T'} + E(\mu_{ij}(\sigma) A_j + \lambda_{ij}(\sigma) B_j).
\]

Here we have used the fact that \( E_{\Gamma'} = E_{\Gamma'} \) so \( E \cdot x = E \cdot (x \mod E_{\Gamma'}) \) for every \( x \in \Gamma' \).

We now expand the left-hand side of eq. \( \text{(33)} \).

\[
(\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma))^t \left( A_{i_j}^{T'} + E \cdot B_i \right) (\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)) = \rho_{\Gamma'}(\sigma)^t A_{i_j}^{T'} \rho_{\Gamma'}(\sigma)
\]

\[
+ E \cdot (\rho(\sigma))^t B_i \rho(\sigma) + \tau(\sigma) A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma).
\]

Set \( D_\sigma = \tau(\sigma) \rho(\sigma)^{-1} = d(\sigma) \) according to the notation of lemma \( \text{[14]} \) we can write

\[
\tau(\sigma)^t A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma)
\]

\[
= \rho(\sigma)^t (\rho(\sigma)^{-1})^t \tau(\sigma)^t A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma) \rho(\sigma)^{-1} \rho(\sigma)
\]

\[
= \rho(\sigma)^t (D_\sigma A_i) \rho(\sigma) + \rho(\sigma)^t (A_i D_\sigma) \rho(\sigma)
\]

\[
= T(\sigma^{-1}) \psi_{D_\sigma}(A_i).
\]

while eq. \( \text{(24)} \) implies that

\[
\rho(\sigma)^t B_i \rho(\sigma) - \sum_{j=1}^{r} \lambda_{ij}(\sigma^{-1}) B_j = -T(\sigma^{-1}) \psi_{B_i f}(A_i).
\]

For the above computations recall that for a \( g \times g \) matrix \( B \), the map \( \psi_B \) is defined by

\[
\psi_B(A_i) = A_i B + B^t A_i.
\]
Combining now eq. (37) and (38) we have that eq. (33) is equivalent to

\[ T(\sigma^{-1})(\psi_{D_\sigma}(A_i)) - T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i) = \sum_{j=1}^{r} \mu_{ij}(\sigma)A_j \]

\[ \left( \psi_{D_\sigma}(A_i) \right) - \psi_{B_\sigma[f]}(A_i) = \sum_{j=1}^{r} T(\sigma)\mu_{ij}(\sigma)A_j. \]

\[ = \sum_{j=1}^{r} \sum_{\nu=1}^{r} \mu_{ij}(\sigma)\lambda_{j\nu}(\sigma^{-1})A_\nu. \]

On the other hand the action \( T \) on \( A_1, \ldots, A_r \) is given in terms of the matrix \( (\lambda_{i,j}) \) while the right hand side of eq. (39) \((\mu_{i,j}(\sigma^{-1}))(\lambda_{i,j}(\sigma))\) corresponds to the derivation \( D^{(1)}(\sigma^{-1}) \) of the \( \rho_1 \)-representation. Equation (34) is now proved.

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