The Cylinder braiding of the quantum Weyl group of $sl_2$

Reinhard Häring-Oldenburg
Mathematisches Institut, Bunsenstr. 3-5, 37073 Göttingen, Germany
email: haering@cfgauss.uni-math.gwdg.de

August 13, 1996

Abstract

It is shown that the quantum Weyl group of $sl_2$ contains an element that is a cylinder twist, i.e. it gives rise to representations of the braid group of Coxeter type B.

1 Introduction

Every Coxeter graph defines a braid group that is an infinite covering of its Coxeter group. T. tom Dieck initiated in [1] the systematic study of these braid groups and their quotient algebras for all root systems.

The Coxeter group $W_{A_n}$ of type $A_n$ is the permutation group and the braid group $Z_{A_n}$ is Artin’s braid group. For type $B_n$ the Weyl group $W_{B_n}$ is a semi direct product of the permutation group $W_{A_n}$ with $\mathbb{Z}_2^n$.

Definition 1 The braid group $Z_{B_n}$ of Coxeter type B is generated by $\tau_0, \tau_1, \ldots, \tau_{n-1}$ with relations

\begin{align*}
\tau_i \tau_j &= \tau_j \tau_i \quad \text{if } |i - j| > 1 \\
\tau_i \tau_j \tau_i &= \tau_i \tau_j \tau_i \quad \text{if } i, j \geq 1, |i - j| = 1 \\
\tau_0 \tau_1 \tau_0 &= \tau_1 \tau_0 \tau_1 \tau_0 \\
\tau_0 \tau_i &= \tau_i \tau_0 \quad i \geq 2
\end{align*}

$\tau_0$ is called the cylinder twist.

Generators $\tau_i, i \geq 1$ satisfy the relations of Artin’s braid group.

$Z_{B_n}$ may be graphically interpreted (cf. figure 1) as symmetric braids or cylinder braids: The symmetric picture shows it as the group of braids with $2n$ strands (numbered $-n, \ldots, -1, 1, \ldots, n$) which are fixed under a 180 degree rotation about the middle axis. In the cylinder picture one adds a single fixed line (indexed 0) on the left and obtains $Z_{B_n}$ as the group of braids with $n$ strands that may surround this fixed line. The generators $\tau_i, i \geq 0$ are mapped to the diagrams $X_i^{(G)}$ given in figure 1.

The braid group $Z_{B_n}$ has applications in the theory of knots in the solid torus and in low dimensional physical systems with boundaries. These applications motivate the search for tensor representations of $Z_{B_n}$ on $n$ fold tensor product spaces $V^\otimes n$. It is natural
to ask for extensions of the tensor representations of $\mathbb{Z}_A^*$ given by quantum group $R$ matrices. I.e., we are looking for an endomorphism $F$ of $V$ such that the quantum braid matrix $B := P(\pi \otimes \pi)R$ ($P$ is the flip operator on $V \otimes V$ and $R$ is the R-matrix of the quantum group. $\pi$ is a representation on $V$.) fulfills $F_1B_{12}F_1B_{12} = B_{12}F_1B_{12}F_1$ on $V^{\otimes n}$.

Subscripts indicate the spaces in which the matrices act. $F$ is called the cylinder twist matrix.

T. tom Dieck has found such extensions for the defining representations of the quantum groups of series $A, B, C$ in [3]. These solutions can be extended by cabling to higher representations as shown in [3]. Naturally the question arises if these matrices come from an element in the quantum group. In [3] we have (from the point of view of universal operators) shown that this is the case for the quantum group of $sl_2$. The present paper is a more detailed description of this result with calculations taking place in the quantum Weyl group. We show that there is an element $t$ in the quantum Weyl group of $sl_2$ that gives rise to a cylinder twist matrix $F = \pi(t)$ in every representation $\pi$. This allows to calculate the representing matrices in all dimensions. Taking the quantum Weyl group as starting algebra highlights the Hopf algebraic content.

The key observation behind our approach is the following: The $F$ matrices of tom Dieck are all triangular with respect to the counter diagonal. Hence taking out the quantum Weyl element as a factor one is left with a upper (or lower) triangular matrix. Such matrices may be representations matrices of an element that is contained in one of the Borel sub algebras. Thus a simpler ansatz may be used.

\[
\begin{array}{cccccc}
\cdots & -3 & -2 & -1 & 1 & 2 & 3 & \cdots & \tau_0 & \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
1 \\
0 \\
1 \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\\end{array} & \cdots \\
\cdots & & & & & & & & & \\
\end{array}
\]

Figure 1: The graphical interpretation of the generators as symmetric tangles (on the left) and as cylinder tangles (on the right).

Preliminaries: Our notation for quantum groups is close to [3]. The quantum group $\mathcal{U}_q(sl_2)$ has relations $[H, X] = 2X, [H, Y] = -2Y, [X, Y] = (K^2 - K^{-2})/(q^{1/2} - q^{-1/2})$, where $K := q^{H/4}, q := e^h$. It follows that we have $KX = q^{1/2}XK, KY = q^{-1/2}YK$ and $Y^mH^n = (H + 2m)^nY^m$. It is convenient to use also $E := KX, F := K^{-1}Y$. One has $E^n = q^{-(n(n-1))/4}K^nX^n, F^n = q^{-(n(n-1))/4}K^{-n}Y^n$. The coproduct is defined on generators by $\Delta(X) := X \otimes K + K^{-1} \otimes X, \Delta(Y) := Y \otimes K + K^{-1} \otimes Y, \Delta(H) = H \otimes 1 + 1 \otimes H$ and the antipode by $S(H) = -H, S(X) = -q^{1/2}X, S(Y) = -q^{-1/2}Y$.

The associated quantum Weyl group $\mathcal{W}_q(sl_2)$ is the Hopf algebra extension given by an additional generator $w$ with relations $wX = -q^{1/2}Xw, wY = -q^{-1/2}Yw, wH = -Hw$. It follows that $wE = -q^{1/2}Ew, wF = -q^{-1/2}Ew$. The Weyl element obeys $\epsilon(w) = 1, w^2 = 1$, where $v$ is the ribbon element and $\epsilon$ is 1 in odd dimensional and $-1$ in even dimensional irreducible representations.
The universal $R$ matrix for both $\mathcal{U}_q(sl_2)$ and $\overline{\mathcal{U}}_q(sl_2)$ is given by

$$R = q^{H \otimes H/4} \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]!} q^{n(n-1)/4} E^n \otimes F^n$$

(5)

Here we have used the usual quantum factorial defined from the quantum number $[n] := (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$. We have to calculate the antipode of $w$ because there are different results stated in the literature [3,4]. We write $R = \sum_k \alpha_k \otimes \beta_k$ and introduce $c_n := \frac{(1-q^{-1})^n}{[n]!} q^{n(n-1)/4}$, $u := \sum_k S(\beta_k)\alpha_k$.

$$1 = \epsilon(w^{-1}) = m(S \otimes \text{id})\Delta(w^{-1}) = m(S \otimes \text{id})(w^{-1} \otimes w^{-1})R$$

$$= m(S \otimes \text{id})(R_{2,1}(w^{-1} \otimes w^{-1})) = \sum_k m(S \otimes \text{id})(\beta_k w^{-1} \otimes \alpha_k w^{-1})$$

$$= \sum_k S(w)^{-1} S(\beta_k)\alpha_k w^{-1} = S(w^{-1})uw^{-1}$$

$$S(w) = uw^{-1}$$

(6)

Essential for our calculations is the formula for the coproduct of the Weyl element $\Delta(w) = R^{-1}(w \otimes w)$ and a simple implication $(w \otimes w)R = R_{2,1}(w \otimes w)$.

2 Construction of the Cylinder twist

In this section we construct a solution of the cylinder braid equation

$$R_{2,1}t_2Rt_1 = t_1R_{2,1}t_2R$$

(7)

As motivated in the introduction we use the ansatz $t = wz$.

Lemma 1 Equation (3) holds with $t = wz$ if

$$\Delta(z) = z_2w_2R_{2,1}w_2^{-1}z_1$$

(8)

Proof: We express $R_{2,1}$ using $w$.

$$\Rightarrow w_1w_2Rw_1^{-1}w_2^{-1}w_2z_2Rw_1z_1 = w_1z_1w_1w_2Rw_1^{-1}w_2^{-1}w_2z_2R$$

$$\Rightarrow Rw_1^{-1}z_2Rw_1z_1 = z_1w_1Rw_1^{-1}z_2R$$

$$\Rightarrow Rw_1^{-1}z_2w_1w_2R_{2,1}w_1^{-1}w_2^{-1}w_2z_1 = z_1w_1Rw_1^{-1}z_2R$$

$$\Rightarrow Rz_2w_2R_{2,1}w_2^{-1}z_1 = z_1w_1Rw_1^{-1}z_2R$$

$$\Rightarrow R\Delta(z) = \Delta'(z)R$$

The last line holds because $R$ intertwines between the coproduct and the opposite coproduct.

If one had made the ansatz $t = w^{-1}z$ the condition would be $\Delta(z) = z_2w_2^{-1}R_{2,1}w_2z_1$ and $t = zw$ would lead to $\Delta(z) = z_1w_1R_{2,1}w_1^{-1}z_2$.

Note that $t' := w^{-2}t = w^{-1}z$ is another solution of (3) because $w^2$ is central.

Remark 1 If $z$ is a solution of (3) then so is $\tilde{z} := K^{\alpha}zK^{\alpha}$ where $\alpha$ is an arbitrary number. The computation is straightforward from the fact that $R$ and $K^{\alpha} \otimes K^{\alpha}$ commute.
Remark 2 If \( z \) is a solution of (8) then so is \( S(u)^{-1}zu \). To prove this we first note that \( uw = S^2(w)u = S(uw^{-1})u = S(w^{-1})S(u)u = uw^{-1}S(u)u = Cwu^{-1} = wS(u) \) where \( C := uS(u) = S(u)u \) is Drinfeld’s Casimir operator. A consequence is \( Cw^{-1}w = wu \). We first investigate the behaviour of \( zuu \):

\[
(1 \otimes zuu)R_{2,1}(uzu \otimes w^{-1}) = (uu^{-1} \otimes zuCwu^{-1})R_{2,1}(uzu \otimes uu^{-1}w^{-1}) \\
= (u \otimes u)(1 \otimes Czu)R_{2,1}(zu \otimes C^{-1}w^{-1}u) = (u \otimes u)(1 \otimes zu)R_{2,1}(z \otimes w^{-1})(u \otimes u) \\
= (u \otimes u)\Delta(z)(u \otimes u) = R_{2,1}R\Delta(u)\Delta(z)\Delta(u)R_{2,1}R = \Delta(uzu)(R_{2,1}R)^2
\]

Here we have used that \( u \otimes u \) and \( R \) commute.

\[
(1 \otimes C^{-1}uzu)R_{2,1}(C^{-1}uzu \otimes w^{-1}) = (C^{-1} \otimes C^{-1})(1 \otimes zuw)R_{2,1}(uzu \otimes w^{-1}) \\
= (C^{-1} \otimes C^{-1})(R_{2,1}R)^2\Delta(uzu) = \Delta(C^{-1})\Delta(uzu) = \Delta(C^{-1}uzu)
\]

Now, \( C^{-1}uzu = S(u)^{-1}zu \) and the claim is shown.

Remark 3 The element \( t \) gives not only rise to representations of \( ZB_n \) but also of the braid group of the affine series \( C_n^{(1)} \). Tensor representations of this braid group need another element \( \overline{F} \) such that \( (1 \otimes \overline{F})B(1 \otimes \overline{F})B = B(1 \otimes \overline{F})B(1 \otimes \overline{F}) \) in \( V \otimes V \). If \( \overline{F} = \pi(\overline{F}) \) this is equivalent to \( R_{2,1}R_{2,1} = \overline{R}_{2,1}R_{2,1} \) which is (by permuting the tensor factors) equivalent to \( R_{2,1}R_{2,1} = \overline{R}_{2,1}R_{2,1} \). Now, assume that \( t \) is any solution of (8) and multiply (8) from the left with \( (w_1w_2)^{-1} \) and from the right with \( w_1w_2 \). This shows that \( \overline{t} = w^{-1}tw \) provides a solution of the above equation.

The element \( w_2R_{2,1}w_2^{-1} \) that occurs in (8) is explicitly:

\[
w_2R_{2,1}w_2^{-1} = q^{-H \otimes H/4} \sum_{n=0}^{\infty} \frac{(1 - q^{-1})^n}{[n]!} q^{n(n-1)/4}(-q^{1/2})^n F^n \otimes F^n
\]

This shows that we may assume that \( z \) is an element of the Borel sub-algebra generated by \( H, Y \). In order to reproduce the first factor in (8) from a coproduct it seems to be adequate to make the further factorisation \( z = q^{-H^2/8}z \). Note that the factor \( q^{-H^2/8} \) already occurred in (8) in the connection with Lusztig’s automorphisms.

\[
\Delta(z) = \left(q^{-H^2/8} \otimes q^{-H^2/8}\right) q^{-H \otimes H/4} \Delta(z)
\]

The right-hand side of (8) becomes

\[
(1 \otimes q^{-H^2/8})(1 \otimes z)q^{-H \otimes H/4} \sum_{n=0}^{\infty} \frac{(1 - q^{-1})^n}{[n]!} q^{n(n-1)/4}(-q^{1/2})^n \\
\left(F^n \otimes F^n\right)(1 \otimes q^{-H^2/8})(1 \otimes z) \\
= \left(q^{-H^2/8} \otimes q^{-H^2/8}\right)(1 \otimes z)q^{-H \otimes H/4} \\
\sum_{n=0}^{\infty} \frac{(1 - q^{-1})^n}{[n]!} q^{n(n-1)/4}(-q^{1/2})^n q^{-n^2/2}\left((q^{-Hn/2}F^n) \otimes F^n\right)(1 \otimes 1) 
\]

Here we have used \( F^n q^{-H^2/8} = q^{-(H+2n)^2/8} F^n \).
Cancelling the leftmost factors in (14) = (13) and introducing the shortcut $B_n := (1-q^{-1})^n q^{n(n-1)/4} (-q^{1/2})^n q^{-n^2/2}$ we arrive at the following equation which has to be solved

$$\Delta(\hat{\xi}) = q^H \otimes H/4 (1 \otimes \hat{\xi}) q^{-H \otimes H/4} \sum_{n=0}^{\infty} B_n ((q^{-Hn/2} F_n) \otimes F_n)(\hat{\xi} \otimes 1) \quad (12)$$

It seems to be difficult to go on with a general ansatz $\sum_{i,j} c_{i,j} K^i F^j$ for $\hat{\xi}$. Experiments show that the following ansatz works:

$$\hat{\xi} = \sum_{m=0}^{\infty} \beta_m q^{\alpha H m} Y^m \quad (13)$$

Here $\alpha, \beta_m$ are coefficients which are yet to be determined. It is important not to use $F^m$ in (13) because this would cause the coproduct to produce unbalanced $K$ factors.

$$\Delta(\hat{\xi}) = \sum_{m} \beta_m (q^{\alpha H m} \otimes q^{\alpha H m})(Y \otimes K + K^{-1} \otimes Y)^m$$

$$= \sum_{m} \sum_{i=0}^{m} \frac{[m]!}{[i]! [m-i]!} q^{i(m-i)/2} (q^{\alpha H m} \otimes q^{\alpha H m})(Y \otimes K)^i (K^{-1} \otimes Y)^{m-i}$$

$$= \sum_{m} \sum_{i=0}^{m} \beta_m \frac{[m]!}{[i]! [m-i]!} (q^{\alpha H m} \otimes q^{\alpha H m}) q^{(i-m)H/4 Y^i} \otimes q^{iH/4 Y^{m-i}} \quad (14)$$

The right-hand side of (12) becomes

$$\sum_{s,t} \beta_s \beta_t \sum_{n} B_n q^H \otimes H/4 (1 \otimes q^{\alpha H s Y^s}) q^{-H \otimes H/4}$$

$$\left( q^{-Hn/2} (K^{-1} Y)^n \otimes (K^{-1} Y))^n \right) (q^{\alpha H t Y^t} \otimes 1)$$

$$= \sum_{s,t} \beta_s \beta_t \sum_{n} B_n q^{-n(n-1)/2} (q^{-H s/2} \otimes q^{\alpha H s Y^s}) (q^{-H n/2} K^{-n} Y^n q^{\alpha H t Y^t} \otimes K^{-n} Y^n)$$

$$= \sum_{s,t} \beta_s \beta_t \sum_{n} B_n q^{-n(n-1)/2} q^{-sn/2+2n at}$$

$$\left( q^{-H s/2} q^{-H n/2} K^{-n} q^{\alpha H t Y^n+t} \otimes q^{\alpha H s K^{-n} Y^n+n} \right)$$

$$= \sum_{s,t} \beta_s \beta_t \sum_{n} B_n q^{-n(n-1)/2} q^{-sn/2+2n at}$$

$$\left( q^{H(-(s-n-n+2n at)/2 Y^{n+t}} \otimes q^{H(2as-n+2)/2 Y^{n+n}} \right) \quad (15)$$

Since $H Y^j$ is a basis of the Borel sub-algebra we can make a term by term comparison of the coefficients of $Y^a \otimes Y^b$. We start by investigating the first few terms:

| $a$ | $b$ | $i$ | $m$ | Coeff. (14) | $n$ | $t$ | $s$ | Coeff. (13) | Conclusion |
|-----|-----|-----|-----|-------------|-----|-----|-----|-------------|------------|
| 0   | 0   | 0   | 1   | $\beta_0$  | 0   | 0   | 0   | $\beta_0 B_0$ | $\beta_0 = 1$ |
| 0   | 1   | 0   | 1   | $\beta_1 q^{H(\alpha-1/4)} \otimes q^{\alpha H}$ | 0   | 0   | 1   | $\beta_1 q^{-H/2} \otimes q^{\alpha H}$ | $\alpha = -1/4$ |
| 1   | 0   | 1   | 1   | $\beta_1 (q^{-H/4} \otimes 1)$ | 0   | 1   | 0   | $\beta_1 q^{-H/4} \otimes 1$ | $-$ |
Now that we have determined $\alpha$ and $\beta_0$ we can consider the general case. The coefficient of $Y^a \otimes Y^b$ in (14) is

$$\beta_{a+b} \frac{[a+b]!}{[a]! [b]!} (q^{-H(a+b)/4} \otimes q^{-H(a+b)/4}) (q^{-bH/4} \otimes q^{aH/4})$$

$$= \beta_{a+b} \frac{[a+b]!}{[a]! [b]!} (q^{-H(a+4b)/2} \otimes q^{-Hb/4})$$

In (15) we set $t = a - n \geq 0, s = b - n \geq 0$ and obtain the coefficient:

$$\sum_{n} B_n \beta_{a-n} \beta_{b-n} q^{-n(n-1)/2-(b-n)n/2-n(a-n)/2}$$

$$q^{H(n-b-n-n/2-(a-n)/2)/2} \otimes q^{H(-b-n-2-n/2)/2}$$

$$= \sum_{n} B_n \beta_{a-n} \beta_{b-n} q^{n^2/2-n(a+b-1)/2} q^{H(-b-a/2)/2} \otimes q^{-Hb/4}$$

The terms involving $H$ are equal. We are left with

$$\beta_{a+b} \frac{[a+b]!}{[a]! [b]!} = \sum_{n} B_n \beta_{a-n} \beta_{b-n} q^{n^2/2-n(a+b-1)/2}$$

Since we have $s = b - n, t = a - n \geq 0$ the sum over $n$ runs only from 0 to min$(a, b)$. We set $b = 1$ and obtain

$$\beta_{a+1} [a+1] = \beta_a \beta_1 + B_1 \beta_{a-1} q^{1/2-a/2}$$

$$= \beta_a \beta_1 + \beta_{a-1} (q^{-1} - 1) q^{(1-a)/2}$$

It remains to show that (17) holds for $b > 1$. To this end we first simplify the recursion formula by defining $\beta'_a := \beta_a [a]!$:

$$\beta'_{a+1} = \beta'_1 \beta'_a + \beta'_{a-1} (q^{-a} - 1)$$

(18)

We first reformulate (17) in terms of $\beta'_a$:

$$\beta'_{a+b} = \sum_{n=0}^{\min(a,b)} B_n \frac{[a]! [b]!}{[a-n]! [b-n]!} \beta'_{a-n} \beta'_{b-n} q^{n^2/2-n(a+b-1)/2}$$

$$= \sum_{n=0}^{\min(a,b)} \left[ \begin{array}{c} a \\ n \end{array} \right] \left[ \begin{array}{c} b \\ n \end{array} \right] [n]! \beta'_{a-n} \beta'_{b-n} q^{n^2/2-n(a+b-1)/2} q^{n(n-1)/4} (q^{-1} - 1)^n q^{n^2/2-n^2/2}$$

$$= \sum_{n=0}^{\min(a,b)} B_n^{a,b} \beta'_{a-n} \beta'_{b-n}$$

(19)

$$B_n^{a,b} := \left[ \begin{array}{c} a \\ n \end{array} \right] \left[ \begin{array}{c} b \\ n \end{array} \right] [n]! q^{-n(a+b)/2} q^{3n/4+n^2/4} (q^{-1} - 1)^n$$

(20)

We are done if we can show that the right hand side of (19) is actually only a function of $a + b$. This will follow from the fact that the substitutions $a \rightarrow a + 1$ and $b \rightarrow b + 1$
The calculation of $\sum_{n=0}^{\infty} B_n^{a+1,b}$ using the formula for the q-binomial coefficients:

$$\begin{align*}
\binom{a+1}{n} &= q^{-n/2} \binom{a}{n} + q^{(a+1-n)/2} \binom{a}{n-1} \\
B_n^{a+1,b} &= q^{-n} B_n^{a,b} + q^{(a+1-n)/2} \binom{a}{n-1} [n]_q q^{-n(a+1+b)/2} q^{3n/4+n^2/4}(q^{-1}-1)^n \\
&= q^{-n} B_n^{a,b} + q^{(a+1-n)/2} \binom{a}{n-1} [n]_q [b-n+1] \\
&= q^{-n} B_n^{a,b} + q^{(1-n)/2} \binom{a}{n-1} (q-1)[b-n+1] \\
&= q^{-n} B_n^{a,b} + q^{(1-n)/2} \binom{a}{n-1} (q-1)[b-n+1] B_n^{a,b} \\
&= q^{-n} (B_n^{a,b} + (q^{n-b} - q) B_n^{a,b}) \\
B_n^{a,b+1} &= q^{-n} (B_n^{a,b} + (q^{n-a} - q) B_n^{a,b})
\end{align*}$$

We now consider the right-hand side of (19). It is convenient to set $B_n^{a,b} = \beta_n = 0$ for negative $n$. Doing this we don’t have to care about summation ranges and can freely shift the summation variable as we do in the third step and in the first summand in the fifth step of the following calculation:

$$\begin{align*}
\sum_n B_n^{a+1,b} \beta_{a+1-n} \beta_{b-n} &= \\
&= \sum_n q^{-n} (B_n^{a,b} + (q^{n-b} - q) B_n^{a,b}) \beta_{a-n+1} \beta_{b-n} \\
&= \sum_n q^{-n-1} (B_n^{a,b} + (q^{n+1-b} - q) B_n^{a,b}) \beta_{a-n} \beta_{b-n-1} \\
&= \sum_n q^{-n-1} B_n^{a,b} \beta_{a-n} \beta_{b-n-1} + \sum_n q^{-n} B_n^{a,b} \beta_{a-n} (q^{n-b} - 1) \beta_{b-n-1} \\
&= \sum_n q^{-n} B_n^{a,b} \beta_{a-n+1} \beta_{b-n} + \sum_n q^{-n} B_n^{a,b} \beta_{a-n} (\beta_{b-n+1} - \beta_{b-n}) \\
&= \sum_n q^{-n} B_n^{a,b} \beta_{a-n+1} \beta_{b-n} + \sum_n q^{-n} B_n^{a,b} \beta_{a-n} \beta_{b-n+1} + \sum_n q^{-n} B_n^{a,b} \beta_{a-n} \beta_{b-n} \\
&= \sum_n q^{-n} B_n^{a,b} \beta_{a-n+1} \beta_{b-n} + \sum_n q^{-n} B_n^{a,b} \beta_{a-n} \beta_{b-n+1} + \sum_n q^{-n} B_n^{a,b} \beta_{a-n+1} \beta_{b-n}
\end{align*}$$

The calculation of $\sum_n B_n^{a,b+1} \beta_{a-n} \beta_{b+1-n}$ would give the same terms: Exchanging $a$ and $b$ interchanges the first two summands and leaves the third invariant. The proof is complete.

**Proposition 2** A solution of $R_{2,1}t_2Rt_1 = t_1 R_{2,1} t_2 R$ is given by

$$t = wq^{-H^2/8} \sum_{m=0}^{\infty} \beta_m q^{-Hm/4} Y^m$$

where $\beta_0 = 1$, $\beta_1$ is arbitrary and

$$\beta_{a+1} = (\beta_a \beta_1 + \beta_{a-1} (q^{-1}-1) q^{(1-a)/2})/[a+1]$$
From this expressions $t$ can be calculated in all irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. We give the matrices in the 2, 3 and 4 dimensional representations. We use the standard basis in the order of decreasing weights.

$$
\begin{pmatrix}
-\beta q^{-1/2} & -q^{-3/4} \\
q^{-1/4} & 0
\end{pmatrix}
$$

(23)

$$
\begin{pmatrix}
(1 - q + q^2 q^2) q^2 & q^{-7/4} (q + 1) \beta q^{-1} \\
-q^{-5/4} (q + 1)^{1/2} & -q^{-1} 0
\end{pmatrix}
$$

(24)

$$
\begin{pmatrix}
-q^{-7/2} \beta (1 + q - 2q^2 + q^2 q^2) q^{-15/4} q^{-1} \\
q^{-13/4} (1 - q + q^2 q^2) q^{-5/2} (1 + q) \beta q^{-7/4} \\
-q^{-5/2} \beta q^{-7/4} \\
q^{-9/4} 0 0 0
\end{pmatrix}
$$

(25)

$$
\gamma := (1 + q + q^2) q^{1/2}
$$

(26)

It can easily be checked that these matrices indeed fulfil (7).

It should be noted that the proposition leads to a second infinite series of tensor representations of the braid group $\mathbb{Z}_B^n$ because there is a second series of irreducible representations of the quantum Weyl group $\mathcal{U}_q(\mathfrak{sl}_2)$. These representations are not irreducible as representations of $\mathcal{U}_q(\mathfrak{sl}_2)$.

We calculate the inverse of $\hat{z}$.

**Lemma 3**

$$
\hat{z}^{-1} = \sum_{m=0}^{\infty} \alpha_m q^{-Hm/4} Y^m
$$

(27)

$$
\alpha_0 = 1
$$

(28)

$$
\alpha_a = - \sum_{m=0}^{a-1} \alpha_{a-1-m} \beta_m q^{-m(a-1-m)/2}
$$

(29)

Proof:

$$
1 = \hat{z} \hat{z}^{-1} = \sum_{m,n} \alpha_n \beta_m q^{-Hm/4} Y^m q^{-Hn/4} Y^n
$$

$$
= \sum_{m,n} \alpha_n \beta_m q^{-Hm/4} q^{-(H+2m)n/4} Y^{m+n}
$$

$$
= \sum_{m,n} \alpha_n \beta_m q^{-nm/2} q^{-H(m+n)/4} Y^{m+n}
$$

Looking at the term with $Y^0$ one obtains $\alpha_0 = 1$. We now consider the terms with $q^{-Hn/4} Y^a, a \geq 1$. We substitute $n = a - m$. Since $n \geq 0$ we obtain a restriction for the $m$ summation $0 \leq m \leq a$ The coefficient that should vanish is

$$
\sum_{m=0}^{a} \alpha_{a-m} \beta_m q^{-m(a-m)/2}
$$

Isolation of $\alpha_a$ yields the formula given in the lemma.
3 Properties of the cylinder twist

In this section we try to analyse the algebraic properties of \( t \) and try to fit them in a broader framework.

In [4] we were led by categorial considerations to the following axioms: A restricted Coxeter-B braided Hopf algebra is a ribbon Hopf algebra with an element \( v \in H \) such that

\[
R_{2,1}v_2 Rv_1 = v_1 R_{2,1}v_2 R \quad (30)
\]
\[
\epsilon(v) = 1 \quad (31)
\]
\[
\Delta(v) = R^{-1}v_2 Rv_1 \quad (32)
\]
\[
S(v) = v^2 v^{-1} \quad (33)
\]

Now, we consider the properties of \( t \).

Proposition 4

\[
R_{2,1}t_2 Rt_1 = t_1 R_{2,1}t_2 R \quad (34)
\]
\[
\Delta(t) = R^{-1}t_2 Rt_1 \quad (35)
\]
\[
\epsilon(t) = 1 \quad (36)
\]

Proof: The first equation has already been proven.

\[
\Delta(t) = \Delta(w) \Delta(z) = R^{-1}w_1 w_2 z_2 w_2 R_{2,1} w_2^{-1} z_1 = R^{-1}w_2 z_2 w_1 R_{2,1} w_2^{-1} w_1^{-1} w_1 z_1 = R^{-1}w_2 z_2 R w_1 z_1 = R^{-1}t_2 R t_1
\]

The third equation is trivial. \( \square \)

Since (33) follows from the remaining axioms we conclude that the quantum Weyl group of \( sl_2 \) is a restricted Coxeter-B braided Hopf algebra.

4 Outlook

A natural further challenge is to find a cylinder twist element \( t \) in the quantum Weyl groups associated to other Lie algebras. The Weyl element \( w \) has a natural generalisation as the longest element \( w_0 \) in the quantum Weyl group. However, generalising our construction of \( z \) would require to evaluate products of root vectors which is a highly non-trivial task. We would need a sort of quantum double construction of \( z \). Using the fact that \( w_0 \) maps a positive root vector to a multiple of a negative root vector and hence interchanges \( H \) and \( H^* \) in the quantum double construction one may write down a version of (8). To be more precise, consider the quantum double realised as in [6] on \( H^* \otimes H \). With dual bases \( \{ f^a \}, \{ e^a \} \) the \( R \) matrix is \( R = \sum_a f^a \otimes 1 \otimes 1 \otimes e^a \). We assume that there are scalars \( \lambda_a \) such that \( w(f^a \otimes 1) w^{-1} = \lambda_a (1 \otimes e^a) \). This implies \( w(1 \otimes e^a) w^{-1} = \lambda_a^{-1} (f^a \otimes 1) \) and \( (w \otimes w) R = R_{2,1} (w \otimes w) \). Furthermore, we assume \( \Delta(w) = R^{-1} (w \otimes w) \). Then the ansatz

\[
z = \sum_a \beta_a (1 \otimes e^a)
\]

turns equation (8) into

\[
\sum_a \beta_a (1 \otimes e^a \otimes 1 \otimes e^a_2) = \sum_{a,b,n} \lambda_n \beta_a \beta_b (1 \otimes e^n e^a \otimes 1 \otimes e^b e^n)
\]

Unfortunately, no natural solution suggests itself.

Furthermore, one should clarify possible connections with other occurrences of (7), especially Majid’s theory of braided Lie algebras [7].
References

[1] T. tom Dieck: Knotentheorien und Wurzelsysteme I, II, Math. Gottingensis 21 (1993) and Math Gottingensis 44 (1993)
[2] T. tom Dieck: On tensor representations of knot algebras, Math. Gottingensis 45 (1995)
[3] T. tom Dieck, R. Häring-Oldenburg: Quantum Groups and cylinder braiding
[4] R. Häring-Oldenburg: Tensor Categories of Coxeter Type B and QFT on the Half Plane, to appear in Journal of Math. Physics
[5] A. N. Kirillov, N. Reshetikhin: q-Weyl Group and a Multiplicative Formula for Universal R-Matrices, Comm. Math . Ph. 134
[6] S. Levendorski, Y. Soibelman: Algebras of Functions on Compact Quantum Groups, Schubert Cells and Quantum Tori, Comm. Math. Ph. 139
[7] S. Majid: Foundations of Quantum Group Theory, Cambridge 1995