General structure of neutral $\rho$ meson self energy and its spectral properties in hot and dense magnetized medium

Snigdha Ghosh\textsuperscript{a,}\textsuperscript{b} Arghya Mukherjee\textsuperscript{b,}\textsuperscript{c,}\textsuperscript{d} Pradip Roy\textsuperscript{b,}\textsuperscript{d} and Sourav Sarkar\textsuperscript{c,}\textsuperscript{d}

\textsuperscript{a}Indian Institute of Technology Gandhinagar, PalaJ, Gandhinagar 382355, Gujarat, India
\textsuperscript{b}Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata - 700064, India
\textsuperscript{c}Variable Energy Cyclotron Centre, 1/AF Bidhannagar, Kolkata 700 004, India and
\textsuperscript{d}Homi Bhabha National Institute, Training School Complex, Anushaktinagar, Mumbai - 400085, India

The one loop self energy of the neutral $\rho$ meson is obtained for the effective $\rho\pi\pi$ interaction at finite temperature and density in presence of a constant background magnetic field of arbitrary strength. In our approach, the $eB$-dependent vacuum part of the self energy is extracted by means of dimensional regularization where the ultraviolet divergences corresponding to the pure vacuum self energy manifest as the pole singularities of gamma as well as Hurwitz zeta functions. This improved regularization procedure consistently reproduces the expected results in the vanishing magnetic field limit and can be used quite generally in other self energy calculations dealing with arbitrary magnetic field strength. In presence of the external magnetic field, the general Lorentz structure for the in-medium vector boson self energy is derived which can also be implemented in case of the gauge bosons such as photons and gluons. It has been shown that with vanishing perpendicular momentum of the external particle, essentially two form factors are sufficient to describe the self energy completely. Consequently, two distinct modes are observed in the study of the effective mass, dispersion relations and the spectral function of $\rho^0$ where one of the modes possesses two fold degeneracy. For large baryonic chemical potential, it is observed that the critical magnetic field required to block the $\rho^0 \to \pi^+\pi^-\pi^0$ decay channel increases significantly with temperature. However, in case of smaller values reaching down to vanishing chemical potential, the critical field follows the opposite trend. The requirement of smaller critical field makes the stability of $\rho^0$ against the $\pi^+\pi^-$ decay more likely to be observed in heavy-ion collision experiments in LHC.

I. INTRODUCTION

In non-central heavy-ion collisions at LHC, the relative motion of the ions themselves can generate strong decaying magnetic pulse of the order $eB \sim 15m_e^2$ ($B \sim 5 \times 10^{15}$Tesla) \cite{najork}. While some of the studies support rapid decrease in the magnitude $\sim 15m_e^2$ \cite{Imamura:2002ud, Ghosh:2017vot}, an adiabatic decay is expected \cite{Roy:2013owa} due to the high conductivity of the produced medium. In spite of the ambiguities, the intensity of the produced magnetic field being much larger than the typical QCD scale, the possibility of magnetic modifications of different properties of the produced extreme state of matter cannot be refuted completely. In general, high intensity magnetic fields can play a significant role in many astrophysical and cosmological phenomena \cite{Liu:2010xk, Alford:2010zn, Blandford:1982ni, Rees:1984, Raffelt:1987yu}. Moreover, the magnetic influence on the properties of magnetars adds to the motivation of studying high density matter in presence of extreme magnetic fields \cite{Liu:2010xk, Alford:2010zn, Blandford:1982ni, Rees:1984, Raffelt:1987yu}.

The study of $\rho$ meson properties like the effective mass and dispersion relations are important in the context of magnetic field induced vacuum superconductivity \cite{Alford:2010zn, Liu:2010xk}. Using NJL model in presence of magnetic background, Liu et.al. have shown that the charged rho condensation in vacuum occurs at critical magnetic field $eB_c \sim 0.2$GeV$^2$ \cite{Alford:2010zn, Liu:2010xk}. Generalization of the study to finite temperature and density shows that the condensation survives even in presence of finite temperature and density \cite{Liu:2010xk}. At vanishing chemical potential, the corresponding critical magnetic field is observed to lie in the range 0.2 -0.6 GeV$^2$ for temperatures in between 0.2-0.5 GeV. However, the neutral $\rho$ meson in vacuum, having no trivial Landau shifts in the energy eigenvalue, shows a slow decrease in the effective mass in weak magnetic field region. Thus, if neutral rho condensation is possible, extremely large magnetic field values will be required to observe the condensation. It should be mentioned here that it has been shown using NJL model that the effective mass of $\rho^0$ meson in fact increases at higher values of magnetic fields showing no possibility of condensation \cite{Alford:2010zn}. In this scenario, $\rho^0 \to \pi^+\pi^-\pi^0$ decay may serve as an important probe to observe the influence of the magnetic field. As argued in Ref. \cite{Alford:2010zn}, even if point like $\rho^0$ meson is considered without any influence by magnetic field, there exists a critical value of the external magnetic field for which the $\rho^0$ to $\pi^+\pi^-$ decay stops due to the trivial enhancement of the charged pion mass. Later the magnetic modification arising from the loop corrections are taken into account at weak \cite{Alford:2010zn, Liu:2010xk} as well as at strong field limits \cite{Brambilla:2009pg} at zero temperature. An immediate generalization

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* snigdha.physics@gmail.com, snigdha.ghosh@iitgn.ac.in
arghya.mukherjee@saha.ac.in
pradipk.roy@saha.ac.in
sourav@vecc.gov.in
of the previous works will be to incorporate the medium effects of the $\rho^0$ meson which may reflect in the modification of the decay rate and the required critical magnetic field. It should be noted here that apart from being important in the study of dense hadronic matter at extreme conditions usually expected to be present within compact stars, the incorporation of the medium effects is also essential for the proper estimation of pion production in non-central heavy ion collisions.

In this work we focus on the temperature and density modifications of neutral $\rho$ meson properties in presence of a static homogeneous magnetic background. The one loop self energy of $\rho$ meson is calculated for the effective $\rho\pi\pi$ and $\rho NN$ interaction with magnetically modified pion and nucleon propagators corresponding to general field strength. After decomposing the self energy in terms of the form factors, the decay width for $\rho^0 \rightarrow \pi^+ \pi^-$ channel is obtained. It should be mentioned here that the spectral properties of rho meson in presence of finite temperature and magnetic field have been studied in our earlier work \[31\]. However, unlike the previous case, dimensional regularization technique is used here to extract the ultraviolet divergence as pole singularities of gamma and Hurwitz zeta functions\[32\]. Also, instead of considering only the spin averaged thermal self energy contribution, the general Lorentz structure has been addressed in detail. Apart from the technical differences, the density dependence arising from the charged nucleon loop serves as the most important extension of the previous study. Its importance can be understood as follows. It is well known that the general expression of decay width is related to the imaginary part of the self energy. Now, as far as the $\rho^0 \rightarrow \pi^+ \pi^-$ decay is concerned, the invariant mass regime of interest does not allow the nucleon loop to directly contribute to the imaginary part as the unitary cut threshold of $NN$ loop begins at much higher value. However, it should be noted that in the rest frame of the decaying particle, the decay width depends on its effective mass. The contribution from the nucleon loop incorporates significant modification in the effective mass of $\rho^0$ which in turn influences the decay. As we shall see, the critical field required to stabilize the neutral $\rho$ against the $\pi^+ \pi^-$ decay has a non-trivial dependence on the baryonic chemical potential.

The article is organized as follows. In Sec. II the vacuum self energy of $\rho$ is discussed followed by evaluation of the in-medium $\rho$ self-energy at zero magnetic field in Sec. III. Next in Sec. IV the in-medium self energy at non-zero external magnetic field is presented. Sec. V is devoted to the discussion of the general Lorentz structure of the in-medium self energy function in presence of a constant background magnetic field. After addressing the Lorentz structure of the interacting $\rho$ propagator in Sec. VI, the analytic structure of the self energy is discussed in Sec. VII. Sec. VIII contains the numerical results. Finally we summarize and conclude in Sec. IX. Some of the relevant calculational details are provided in the Appendix.

II. $\rho^0$ SELF ENERGY IN THE VACUUM

The effective Lagrangian for $\rho\pi\pi$ and $\rho NN$ interaction is \[33\]

$$\mathcal{L}_{int} = -g_{\rho\pi\pi} \partial_{\mu} \bar{\rho} \cdot (\partial_{\nu} \pi \times \partial^\nu \pi) - g_{\rho NN} \bar{\Psi} \left[ \gamma^\mu - \frac{\kappa_\rho}{2m_N} \sigma^\mu^\nu \partial_\nu \right] \vec{\tau} \cdot \vec{\rho} \Psi$$

where, $\Psi = \begin{bmatrix} p_n \\ \bar{n} \end{bmatrix}$ is the nucleon isospin doublet, $\sigma^\mu^\nu = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and the components of $\vec{\tau}$ correspond to the Pauli isospin matrices. It is understood that, the derivative within the square bracket in the above equation acts only on the $\rho$ field. The value of the coupling constants are given by $g_{\rho\pi\pi} = 20.72$ GeV$^{-2}$, $g_{\rho NN} = 3.25$ and $\kappa_\rho = 6.1$ with $m_N = 939$ MeV as the mass of the nucleons. The metric tensor in this work is taken as $g_{\mu\nu} = diag(1, -1, -1, -1)$. Using Eq. (1), the one-loop vacuum self energy of $\rho^0$ is obtained as

$$\Pi^\mu^\nu_{\text{pure-vac}} = (\Pi^\mu^\nu_{\pi})_{\text{pure-vac}} + (\Pi^\mu^\nu_{N})_{\text{pure-vac}}$$

FIG. 1. Feynman diagram for the one-loop self energy of neutral $\rho$ meson.
where, \((\Pi^\mu_\pi)_{\text{pure-vac}}\) and \((\Pi^\mu_N)_{\text{pure-vac}}\) are respectively the contributions from the \(\pi\pi\)-loop and \(NN\)-loop which are given by

\[
(\Pi^\mu_\pi)_{\text{pure-vac}}(q) = i \int \frac{d^4k}{(2\pi)^4} N^\mu_\pi(q,k) \Delta_F(k,m_\pi) \Delta_F(p = q + k, m_\pi)
\]

(3)

\[
(\Pi^\mu_N)_{\text{pure-vac}}(q) = -i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \Gamma^\mu(q)S_p(p = q + k, m_N)\Gamma^\mu(-q)S_p(k, m_N) \right] + \Gamma^\mu(q)S_n(p = q + k, m_N)\Gamma^\mu(-q)S_n(k, m_N)
\]

(4)

where,

\[
\Delta_F(k, m_\pi) = \frac{-1}{k^2 - m_\pi^2 + i\epsilon}
\]

(5)

is the vacuum Feynman propagator for the charged pion. \(S_p\) and \(S_n\) are respectively the vacuum Feynman propagators for proton and neutron and are given by

\[
S_p(k, m_N) = S_n(k, m_N) = -(k + m_N)\Delta_F(k, m_N).
\]

(6)

The second rank tensor \(N^\mu_\pi(q,k)\) and the vector \(\Gamma^\mu(q)\) in Eqs. (3) and (4) contain the factors coming from the interaction vertices:

\[
N^\mu_\pi(q,k) = g^2_{\pi\pi}(q^4k^4 + (q \cdot k)^2q^\mu q^\nu - q^2(q \cdot k)(q^\mu k^\nu + q^\nu k^\mu))
\]

(7)

\[
\Gamma^\mu(q) = g_{\rho NN} \left[ \gamma^\mu - i \frac{\kappa_\rho}{2m_N} \sigma^{\mu\nu} q_\nu \right].
\]

(8)

The evaluations of the momentum integrals in Eqs. (3) and (4) are briefly sketched in Appendix B and the final results can be read off from Eqs. (B10) and (B11).

\[
(\Pi^\mu_\pi)_{\text{pure-vac}}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left( \frac{-g^2_{\rho\pi\pi} q^2}{32\pi^2} \right) \int_0^1 dx \Delta_F \left[ \frac{1}{\varepsilon} - \gamma_E + 1 - \ln \left( \frac{\Delta_F}{4\pi\Lambda_\pi} \right) \right] \bigg|_{\varepsilon \to 0}
\]

(9)

\[
(\Pi^\mu_N)_{\text{pure-vac}}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left( \frac{g^2_{\rho NN}}{2\pi^2} \right) \int_0^1 dx \left[ 2x(1 - x) + \kappa_\rho + \frac{\kappa^2_\rho}{2} - \frac{\kappa^2_\rho}{4m^2_N} \Delta_F \right] \times \\
\left\{ \frac{1}{\varepsilon} - \gamma_E - \ln \left( \frac{\Delta_N}{4\pi\Lambda_N} \right) \right\} - \frac{\kappa^2_\rho}{4m^2_N} \Delta_F \bigg|_{\varepsilon \to 0}
\]

(10)

where \(\Delta_F\) and \(\Delta_N\) are defined in Eqs. (B3) and (B7). As can be seen from the above equations, the vacuum self energy is divergent and scale dependent which renormalizes the bare \(\rho^0\) mass to its physical mass after adding proper vacuum counter terms in the Lagrangian. The particular Lorentz structure in the above equations renders the self energy transverse to the \(\rho^0\) momentum i.e. \(q_\mu \Pi^\mu_\text{vac} = 0\).

## III. \(\rho^0\) SELF ENERGY IN THE MEDIUM

In order to calculate the \(\rho^0\) self energy at finite temperature and density, we employ the real time formalism of finite temperature field theory where all the two point correlation functions such as the propagator and the self energy become \(2 \times 2\) matrices in the thermal space. However, they can be put in a diagonal form where the diagonal elements can be obtained from any one component (say the 11-component) of the said \(2 \times 2\) matrix. The 11-components of real time thermal pion and nucleon propagators are

\[
D^{11}(k) = \Delta_F(k, m_\pi) + \eta(k \cdot u) [\Delta_F(k, m_\pi) - \Delta_F(k, m_\pi)]
\]

(11)

\[
S_{11}^{p,n}(k) = S_{p,n}(k, m_N) - \eta(k \cdot u) [S_{p,n}(k, m_N) - \gamma^0 S_{p,n}^\dagger(k, m_N) \gamma^0]
\]

(12)

where \(\eta(x) = \Theta(x)f(x) + \Theta(-x)f(-x)\) and \(\eta^0(x) = \Theta(x)f^+(x) + \Theta(-x)f^-(x)\) in which \(f(x)\) and \(f^\pm(x)\) are respectively the Bose-Einstein and Fermi-Dirac distribution functions corresponding to pions and nucleons:

\[
f(x) = \left[ e^{x/T} - 1 \right]^{-1}, \quad f^\pm(x) = \left[ e^{(x \mp \mu_B)/T} - 1 \right]^{-1}.
\]

(13)
Here, $\Theta(x)$ is the unit step function, $u^\mu$ is the medium four-velocity; $T$ and $\mu_B$ are respectively the temperature and baryon chemical potential of the medium. In the local rest frame (LRF) of the medium, $u^\mu_{\text{LRF}} \equiv (1, \vec{0})$.

For the evaluation of the 11-component of the thermal self energy matrix, the vacuum pion and nucleon propagators in Eqs. (3) and (4) are replaced by the respective 11-components of the thermal propagators given in Eqs. (11) and (12) as \[33\]

\[
(\Pi^{\mu\nu})_{11}(q) = i \int \frac{d^4k}{(2\pi)^4} N_{\pi}^{\mu\nu}(k) D^{11}(k, m_\pi) D^{11}(p = q + k, m_\pi)
\]

\[
(\Pi^{\mu\nu}_{N})_{11}(q) = -i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \Gamma^{\mu} q S^{11}_{N}(k, m_N) \Gamma^{\nu} q S^{11}_{N}(p = q + k, m_N) + \Gamma^{\nu} q S^{11}_{N}(k, m_N) \Gamma^{\mu} q S^{11}_{N}(p = q + k, m_N) \right].
\]

The analytic thermal self energy function of $\rho^0$ denoted by a bar $\text{Re} \Pi^{\mu\nu}_{\pi,N}(q^0, \vec{q}) = \text{Re} \Pi^{\mu\nu}_{\pi}(q^0, \vec{q}) + \text{Re} \Pi^{\mu\nu}_{N}(q^0, \vec{q})$ is related to the above quantities by the relations \[33\]

\[
\text{Re} \Pi^{\mu\nu}_{\pi,N}(q^0, \vec{q}) = \text{Re} \Pi^{\mu\nu}_{\pi}(q^0, \vec{q}) (16)
\]

\[
\text{Im} \Pi^{\mu\nu}_{\pi,N}(q^0, \vec{q}) = \text{Im} \Pi^{\mu\nu}_{\pi}(q^0, \vec{q}) (17)
\]

where, $\text{sign}(x) = \Theta(x) - \Theta(-x)$. Rewriting Eqs. (11) and (12) as \[33\]

\[
D^{11}(k) = \Delta_F(k, m_\pi) + 2\pi i \delta(k \cdot u) \delta(k^2 - m_\pi^2) \]

\[
S^{11}_{\pi,N}(k) = (k + m_N) \left[ \Delta_F(k, m_N) - 2\pi i \delta(k \cdot u) \delta(k^2 - m_\pi^2) \right]
\]

and substituting into Eqs. (14) and (15) and performing the $dk^4$ integration (using the Dirac delta functions) followed by using Eqs. (16) and (17) we get the real parts as

\[
\text{Re} \Pi^{\mu\nu}_{\pi}(q^0, \vec{q}) = \text{Re} (\Pi^{\mu\nu})_{\text{pure-vac}}(q) + \int \frac{d^4k}{(2\pi)^4} \mathcal{P} \left\{ \frac{f(\omega_k)}{2\omega_k} \left\{ \frac{N^{\mu\nu}_{\pi}(k^0 = -\omega_k)(q^0 - \omega_k)^2 - (\omega_k)^2}{(q^0 - \omega_k)^2 - (\omega_k)^2} + \frac{N^{\mu\nu}_{\pi}(k^0 = \omega_k)(q^0 + \omega_k)^2 - (\omega_k)^2}{(q^0 + \omega_k)^2 - (\omega_k)^2} \right\} \right\}
\]

\[
\text{Re} \Pi^{\mu\nu}_{N}(q^0, \vec{q}) = \text{Re} (\Pi^{\mu\nu})_{\text{pure-vac}}(q) - \int \frac{d^4k}{(2\pi)^4} \mathcal{P} \left\{ \frac{1}{2\Omega_p} \frac{f^-(\Omega_k)N^{\mu\nu}_N(k^0 = -\Omega_k)(q^0 + \Omega_k)^2 - (\Omega_k)^2}{(q^0 + \Omega_k)^2 - (\Omega_k)^2} + \frac{f^+(\Omega_k)N^{\mu\nu}_N(k^0 = \Omega_k)(q^0 - \Omega_k)^2 - (\Omega_k)^2}{(q^0 - \Omega_k)^2 - (\Omega_k)^2} \right\}
\]

and the imaginary parts as

\[
\text{Im} \Pi^{\mu\nu}_{\pi}(q^0, \vec{q}) = -\text{sign}(q^0) \tanh \left( \frac{\beta q^0}{2} \right) \pi \int \frac{d^4k}{(2\pi)^4} \frac{1}{4\omega_k \omega_p} \left\{ \right.
\]

\[
\left. \left\{ \frac{N^{\mu\nu}_{\pi}(k^0 = -\omega_k)(q^0 - \omega_k - \omega_p) + N^{\mu\nu}_{\pi}(k^0 = \omega_k)(q^0 + \omega_k + \omega_p)}{\omega_k} + \left\{ f(\omega_k) + f(\omega_p) + 2f(\omega_k)f(\omega_p) \right\} \right\}
\]

\[
\text{Im} \Pi^{\mu\nu}_{N}(q^0, \vec{q}) = -\text{sign}(q^0) \tanh \left( \frac{\beta q^0}{2} \right) \pi \int \frac{d^4k}{(2\pi)^4} \frac{1}{4\Omega_k \Omega_p} \times
\]

\[
\left\{ \frac{1}{2} \frac{f^-(\Omega_k) - f^+(\Omega_k) + 2f^-(\Omega_k)f^+(\Omega_k)}{\Omega_k} N^{\mu\nu}_N(k^0 = -\Omega_k) \delta(q^0 - \Omega_k - \Omega_p) + f^-(\Omega_k) - f^+(\Omega_k) + 2f^+(\Omega_k)f^-(\Omega_k) N^{\mu\nu}_N(k^0 = \Omega_k) \delta(q^0 + \Omega_k + \Omega_p) + f^-(\Omega_k) - f^+(\Omega_k) + 2f^+(\Omega_k)f^-(\Omega_k) N^{\mu\nu}_N(k^0 = -\Omega_k) \delta(q^0 - \Omega_k - \Omega_p) + f^-(\Omega_k) - f^+(\Omega_k) + 2f^+(\Omega_k)f^-(\Omega_k) N^{\mu\nu}_N(k^0 = \Omega_k) \delta(q^0 + \Omega_k + \Omega_p) \right\}
\]

where, $\mathcal{P}$ denotes the Cauchy Principal value integration, $\omega_k = \sqrt{m_\pi^2 + k^2}$, $\Omega_k = \sqrt{m_N^2 + k^2}$ and $N_{\pi}(q, k)$ is defined in Eq. (33).
IV. $\rho^0$ SELF ENERGY IN THE MAGNETIZED MEDIUM

In presence of the external magnetic field $\vec{B} = B\hat{z}$, the propagations of the charged pion and proton are modified. One of the possible ways to incorporate the effect of external magnetic field is the Schwinger proper time formalism in which the 11-components of charged pion and proton propagators respectively become \[36, 37\]

$$D_{\mu}^{1l}(k) = \Delta_B(k, m_{\pi}) + \eta(k \cdot u) [\Delta_B(k, m_\pi) - \Delta_B^*(k, m_\pi)]$$ \hspace{1cm} (24)

$$S_{\mu}^{1l}(k) = S_B(k, m_N) - \eta(k \cdot u) [S_B(k, m_N) - \gamma^0 S_B^*(k, m_N) \gamma^0]$$ \hspace{1cm} (25)

where, $\Delta_B(k, m_{\pi})$ and $S_B(k, m_N)$ denote the momentum space vacuum (zero temperature) Schwinger proper time propagators for charged pion and proton respectively \[36\].

$$\Delta_B(k) = i \int_0^\infty ds \exp \left[ is \left\{ k_\parallel^2 + \frac{\tan(eBs)}{eBs} k_\perp^2 - m^2_N \right\} \right]$$ \hspace{1cm} (26)

$$S_B(k) = i \int_0^\infty ds \exp \left[ is \left\{ k_\parallel^2 + \frac{\tan(eBs)}{eBs} k_\perp^2 - m^2_N \right\} \right] \left( k_\parallel + m_N \right) \left\{ 1 - \gamma^1 \gamma^2 \tan(eBs) \right\} + \frac{k_\perp}{\sqrt{2}} \sec(eBs) \right) .$$ \hspace{1cm} (27)

In the above equations, $e = |e|$ is the charge of the proton; the four-vector $k$ is decomposed into $k = (k_\parallel + k_\perp)$ where $k_\parallel^\mu = g^{\mu\nu} k_\nu$ and $k_\perp^\mu = g^{\mu\nu} k_\nu$, corresponding to the decomposition of the metric tensor $g^{\mu\nu} = (g^{\mu\nu} + g_\perp^{\mu\nu})$, with $g_{\parallel}^{\mu\nu} = \text{diag}(1, 0, 0, -1)$ and $g_{\perp}^{\mu\nu} = \text{diag}(0, -1, -1, 0)$. The above decomposition can be done in a Lorentz covariant way by introducing another four-vector $b^\nu = \frac{1}{B} G^{\mu\nu} u_\nu$ \hspace{1cm} (28)

where $G^{\mu\nu} = \frac{1}{\gamma} e^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual of the electromagnetic field tensor $F^{\mu\nu}$. In the local rest frame of the medium, $b^\nu_{\text{LRF}} \equiv (0, 0, 0, 1)$, which is the direction of the external magnetic field. Using $b^\nu$, we can write

$$g_\parallel^{\mu\nu} = (u^\mu u^\nu - b^\mu b^\nu) \quad \text{and} \quad g_\perp^{\mu\nu} = (g^{\mu\nu} - u^\mu u^\nu + b^\mu b^\nu) .$$ \hspace{1cm} (29)

It is important to note that, the coordinate space Schwinger propagator contains a gauge dependent translationally non-invariant phase factor. However, for the one-loop graphs containing equally charged particle in the loop, the phase factor gets canceled and the momentum space propagator is sufficient for the calculation of the self energy. The proper time integral in Eqs. (26) and (27) can be performed in order to express the propagators as a sum over discrete Landau levels as

$$\Delta_B(k) = - \sum_{l=0}^{\infty} \frac{2(-1)^l e^{-\alpha_k} L_l(2\alpha_k)}{k_\parallel^2 - m_{\pi}^2 - (2l + 1) eB + i\epsilon}$$ \hspace{1cm} (30)

$$S_B(k) = - \sum_{l=0}^{\infty} \left[ \frac{(-1)^l e^{-\alpha_k} D_l(k)}{k_\parallel^2 - M_l^2 - 2leB + i\epsilon} \right]$$ \hspace{1cm} (31)

where,

$$D_l(k) = (k_\parallel + m_N) \left[ (1 + i\gamma^1 \gamma^2) L_l(2\alpha_k) - (1 - i\gamma^1 \gamma^2) L_{l-1}(2\alpha_k) \right] - 4k_\perp L_{l-1}(2\alpha_k)$$ \hspace{1cm} (32)

with $\alpha_k = -k_\perp^2 / eB$. Here, $L_l^a(z)$ denotes the generalized Laguerre polynomial with $L_{l-1}^0(z) = 0$ and $L_l(z) = L_l(0)$. We now rewrite Eqs. (24) and (25) using Eqs. (30) and (31) as

$$D_{\mu}^{1l}(k) = \sum_{l=0}^{\infty} 2(-1)^l e^{-\alpha_k} L_l(2\alpha_k) \left[ \frac{-1}{k_\parallel^2 - m_{\pi}^2 + i\epsilon} + 2\pi i \eta(k \cdot u) \delta \left( k_\parallel^2 - m_{\pi}^2 \right) \right]$$ \hspace{1cm} (33)

$$S_{\mu}^{1l}(k) = \sum_{l=0}^{\infty} (-1)^l e^{-\alpha_k} D_l(k) \left[ \frac{-1}{k_\parallel^2 - M_l^2 + i\epsilon} - 2\pi i \eta(k \cdot u) \delta \left( k_\parallel^2 - M_l^2 \right) \right]$$ \hspace{1cm} (34)

where we have defined the Landau level dependent “dimensionally reduced effective masses” (as a consequence of dimensional reduction) of pion and proton as

$$m_l = \sqrt{m_{\pi}^2 + (2l + 1)eB} \quad \text{and} \quad M_l = \sqrt{m_N^2 + 2leB} .$$ \hspace{1cm} (35)
We now replace the 11-component of the charged pion and proton propagators in Eqs. (13) and (15) as $D^{11} \rightarrow D_{D}^{11}$, $S_{p}^{11} \rightarrow S_{N}^{11}$ i.e by the respective magnetized ones given in Eqs. (33) and (34) and then perform the $dk^0$ integrations (using the Dirac delta functions). Following Eqs. (13) and (17) we get the thermal self energy functions under external magnetic field which we will denote by a double bar to distinguish them from the thermal self energy functions in the absence of magnetic field. Their explicit expressions are given by

$$\text{Re} \Pi^{\mu \nu}_\pi (q^0, \vec{q}) = \frac{\infty}{\sum \sum} \int \frac{d^3k}{(2\pi)^3} \frac{\text{Re} \Pi^{\mu \nu}_\pi (q^0, \vec{q})}{2\omega} \left\{ \frac{\Lambda^{\mu \nu}_{\pi, \pi, n, l}(k^0 = -\omega_k) + \Lambda^{\mu \nu}_{\pi, \pi, n, l}(k^0 = \omega_k)}{(q^0 - \omega_k)^2 - (\omega_p)^2} \right\} + \text{Re} (\Pi^{\mu \nu}_\pi)_\text{vac} (q, eB)$$

$$\text{Re} \Pi^{\mu \nu}_N (q^0, \vec{q}) = \frac{1}{2} \text{Re} \Pi^{\mu \nu}_N (q^0, \vec{q}) - \sum \sum \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega^4} \left\{ \left\{ f^{-1} (q^0) \Lambda^{\mu \nu}_{\pi, p, n, l}(k^0 = -\omega_k) \right\} + \left\{ f^{-1} (q^0) \Lambda^{\mu \nu}_{\pi, p, n, l}(k^0 = \omega_k) \right\} \right\} + \text{Re} (\Pi^{\mu \nu}_p)_\text{vac} (q, eB)$$

$$\text{Im} \Pi^{\mu \nu}_\pi (q^0, \vec{q}) = -\text{sign} (q^0) \tanh \left( \frac{\beta q^0}{2} \right) \pi \sum \sum \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega^4} \left\{ \left\{ f^{-1} (q^0) \Lambda^{\mu \nu}_{\pi, p, n, l}(k^0 = -\omega_k) \right\} + \left\{ f^{-1} (q^0) \Lambda^{\mu \nu}_{\pi, p, n, l}(k^0 = \omega_k) \right\} \right\}$$

$$\text{Re} (\Pi^{\mu \nu}_p)_\text{vac} (q, eB)$$

$$\text{Im} \Pi^{\mu \nu}_N (q^0, \vec{q}) = \frac{1}{2} \text{Im} \Pi^{\mu \nu}_N (q^0, \vec{q}) - \text{sign} (q^0) \tanh \left( \frac{\beta q^0}{2} \right) \pi \sum \sum \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega^4} \left\{ \left\{ f^{-1} (q^0) \Lambda^{\mu \nu}_{\pi, p, n, l}(k^0 = -\omega_k) \right\} + \left\{ f^{-1} (q^0) \Lambda^{\mu \nu}_{\pi, p, n, l}(k^0 = \omega_k) \right\} \right\}$$

where,

$$\Lambda^{\mu \nu}_{\pi, n, l}(q^0, \vec{q}) = 4(1)^{n+l} e^{-\alpha_n - \alpha_p} L_i(2\alpha_n) L_n(2\alpha_p) N^{\mu \nu}_\pi (q, k)$$

$$\Lambda^{\mu \nu}_{p, n, l}(q^0, \vec{q}) = g^2 \frac{2}{NN} (1)^{n+l} e^{-\alpha_n - \alpha_p} \text{Tr} [\Gamma^\mu (q) D_n(q + k) \Gamma^\nu (-q) D_t(k)]$$

$$\omega_k = \sqrt{k^2 + m^2} = \sqrt{k^2 + m^2 + (2l + 1)eB}$$

$$\Omega_l = \sqrt{k^2 + M^2} = \sqrt{k^2 + m^2 + 2leB}$$

The first terms on the RHS of Eqs. (37) and (39) are the contributions from the neutron-neutron loop which are not affected by the external magnetic field. The last terms on the RHS of Eqs. (39) and (41) are the contributions from the magnetic field which depend on the external magnetic field but independent of temperature. Their explicit forms are given by

$$\text{Re} (\Pi^{\mu \nu}_\pi)_\text{vac} (q, eB) = \text{Re} \sum \sum \int \frac{d^4k}{(2\pi)^4} \Lambda^{\mu \nu}_{\pi, n, l} \Delta F_k (q^0, k) \Delta F_{q^0 + k^0, m^0} (44)$$

$$\text{Re} (\Pi^{\mu \nu}_p)_\text{vac} (q, eB) = \text{Re} \sum \sum \int \frac{d^4k}{(2\pi)^4} \Lambda^{\mu \nu}_{p, n, l} \Delta F_k (q^0, k) \Delta F_{q^0 + k^0, m^0} (45)$$

It is important to note that, the above quantities respectively contain the divergent pure vacuum contributions $\left( \Pi^{\mu \nu}_\pi \right)_\text{vac} (q)$ and $\frac{1}{2} \left( \Pi^{\mu \nu}_N \right)_\text{vac} (q)$ in a nontrivial way (as the above equations seem to appear non-perturbative
in eB). In contrast, for the case of weak magnetic field expansion of the Schwinger propagator, the pure vacuum contribution to the self energy trivially decouples from the magnetic field dependent terms. Since we are working with the full propagator including all the Landau levels, we have to properly regularize the above expressions in order to extract the pure vacuum contributions from these quantities. We use dimensional regularization in which the ultraviolet divergence appear as the pole of Gamma and Hurwitz zeta function the details of which are provided in the Appendices. Here, we take the transverse momentum of $\rho^0$ to be zero i.e. $q_\perp = 0$ which makes substantial simplifications of the analytic calculations. The final result can be read off from Eqs. (48) and (49)

\[
\begin{align*}
(P_{\mu\nu})_{\text{vac}}^\pi (q_\parallel, eB) &= (P_{\mu\nu})_{\text{pure-vac}}^\pi (q_\parallel) + (P_{\mu\nu})_{eB\text{-vac}}^\pi (q_\parallel, eB) \\
(P_{\mu\nu})_{\text{vac}}^p (q_\parallel, eB) &= \frac{1}{2} \left( (P_{\mu\nu})_{\text{pure-vac}}^\pi (q_\parallel) + (P_{\mu\nu})_{eB\text{-vac}}^\pi (q_\parallel, eB) \right)
\end{align*}
\]

where, the scale dependent divergent pure-vacuum parts are completely decoupled as the first term on the RHS of the above equation; the scale independent and finite “eB-dependent vacuum contribution” to the real part of the self energy functions are

\[
\begin{align*}
(P_{\mu\nu})_{eB\text{-vac}}^\pi (q_\parallel, eB) &= \frac{-g_{\mu\nu}^2 g_\parallel^2}{32 \pi^2} \int_0^1 dx \left\{ \ln \left( \frac{\Delta_{\pi}(q_\perp = 0)}{2 eB} \right) - 1 \right\} \Delta_{\pi}(q_\perp = 0)(q_\parallel^2 g_{\mu\nu}^\pi - q_\parallel^\mu q_\parallel^\nu) \\
&- (q_\parallel^2 g_{\mu\nu}^\pi - q_\parallel^\mu q_\parallel^\nu) 2 eB \left\{ \ln \Gamma \left( z_\pi + \frac{1}{2} \right) - \ln \sqrt{2\pi} \right\} \\
&+ q_\parallel^2 g_\parallel^\mu \left\{ \Delta_{\pi}(q_\perp = 0) + \frac{eB}{2} - \frac{1}{2} \Delta_{\pi}(q_\perp = 0) \right\} \left\{ \psi \left( z_\pi + \frac{1}{2} \right) + \psi \left( z_\pi + x + \frac{1}{2} \right) \right\} \right\}
\end{align*}
\]

\[
\begin{align*}
(P_{\mu\nu})_{eB\text{-vac}}^p (q_\parallel, eB) &= \frac{g_{\mu\nu}^2 \Lambda_{\parallel}}{4 \pi^2} \int_0^1 dx \left\{ \ln \left( \frac{\Delta_N(q_\perp = 0)}{2 eB} \right) \right\} 2eB \left\{ z_N - \frac{m_N^2}{2 eB} \right\} \left\{ \psi(z_N + x) + z_N \right\} \\
&+ \ln \Gamma(z_N + x) - \ln \sqrt{2\pi} \right\} - \kappa_\tau \left\{ (q_\parallel^2 g_{\mu\nu}^\tau - q_\parallel^\mu q_\parallel^\nu) \left\{ \psi(z_N + x) + \frac{1}{2 z_N} \right\} + q_\parallel^2 g_{\mu\nu}^\tau \psi(z + x) \right\} \\
&+ \frac{\kappa_\tau^2}{4 m_N^2} 2 eB \left\{ \frac{1}{2} \ln \Gamma(z_N) - \ln \Gamma(z_N + x) \right\} \\
&- \frac{2 q_\parallel^2 g_{\mu\nu}^\tau}{eB} \left\{ \frac{m_N^2}{eB} - z_N \right\} \left\{ \psi(z + x) + \frac{1}{2} \right\} + \frac{\kappa_\tau^2}{4 m_N^2} \left( q_\parallel^2 g_{\mu\nu}^\tau - q_\parallel^\mu q_\parallel^\nu \right) \Delta_N(q_\perp = 0) \right\}.
\end{align*}
\]

Eqs. (48) and (49) imply that the vacuum counter terms are sufficient to renormalize the theory and thus the external magnetic field does not create additional divergences. For $q_\perp = 0$, the $d^2 k_\perp$ integrals in Eqs. (49)-(50) can be analytically performed (see Appendix B) and we finally get,

\[
\begin{align*}
\text{Re} \Pi_{\mu\nu}^\pi(q_0, q_\perp) &= \text{Re} (\Pi_{\mu\nu}^\pi)_{\text{pure-vac}}^\pi(q_\parallel) + \sum_{n=0}^{\infty} \sum_{l=-(n-1)}^{(n-1)} \int_0^{\infty} \frac{dk_\perp}{2\pi} \left\{ \frac{f(\omega_p^l)}{2 \omega_p^l} \left\{ \frac{N_{\mu\nu,\parallel}^{\tau}(k_0 = -\omega_p^l)}{(q^0 + \omega_p^l)^2 - (\omega_p^l)^2} + \frac{N_{\mu\nu,\parallel}^{\tau}(k_0 = \omega_p^l)}{(q^0 - \omega_p^l)^2 - (\omega_p^l)^2} \right\} \right\} \\
&+ \text{Re} (\Pi_{\mu\nu}^\pi)_{eB\text{-vac}}^\pi(q_\parallel, eB)
\end{align*}
\]

\[
\begin{align*}
\text{Re} \Pi_{\mu\nu}^\pi(q_0, q_\perp) &= \text{Re} (\Pi_{\mu\nu}^\pi)_{\text{pure-vac}}^\pi(q_\parallel) + \sum_{n=0}^{\infty} \sum_{l=-(n-1)}^{(n-1)} \int_0^{\infty} \frac{dk_\perp}{2\pi} \left\{ \frac{1}{2 \omega_p^l} \left\{ \frac{f(\Omega_p^l)N_{\mu\nu,\parallel}^{\tau}(k_0 = -\Omega_p^l)}{(q^0 - \Omega_p^l)^2 - (\Omega_p^l)^2} + \frac{f(\Omega_p^l)N_{\mu\nu,\parallel}^{\tau}(k_0 = \Omega_p^l)}{(q^0 + \Omega_p^l)^2 - (\Omega_p^l)^2} \right\} \right\} \\
&+ \text{Re} (\Pi_{\mu\nu}^\pi)_{eB\text{-vac}}^\pi(q_\parallel, eB)
\end{align*}
\]
\[ \text{Im} \tilde{\Pi}_\pi^\mu (q^0, q_z) = -\text{sign} (q^0) \tanh \left( \frac{\beta q^0}{2} \right) \pi \sum_{n=0}^{\infty} \sum_{l=(n-1)}^{(n+1)} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{1}{4\omega_k^2 \omega_{k+h}^2} \left\{ 1 + f(\omega_k^0) + f(\omega_{k+h}^0) + 2f(\omega_k^0)f(\omega_{k+h}^0) \right\} \]

\[ \left\{ \tilde{N}_{\pi,n}^\mu (k^0 = -\omega_k^0) \delta(q^0 - \omega_k^0) + \tilde{N}_{\pi,n}^\mu (k^0 = \omega_k^0) \delta(q^0 + \omega_k^0) \right\} + \left\{ f(\omega_k^0) + f(\omega_{k+h}^0) + 2f(\omega_k^0)f(\omega_{k+h}^0) \right\} \]

\[ \text{Im} \tilde{\Pi}_N^\mu (q^0, q_z) = \frac{1}{2} \text{Im} \tilde{\Pi}_N^\mu (q^0, q_z) - \text{sign} (q^0) \tanh \left( \frac{\beta q^0}{2} \right) \pi \sum_{n=0}^{\infty} \sum_{l=(n-1)}^{(n+1)} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{1}{4\Omega_k^0 \Omega_{k+l}^0} \times \]

\[ \left[ \left\{ 1 - f^-(\Omega_k^0) - f^+(\Omega_{k+h}^0) + 2f^+(\Omega_k^0)f^+(\Omega_{k+h}^0) \right\} \tilde{N}_{\pi,n}^\mu (k^0 = -\Omega_k^0) \delta(q^0 - \Omega_k^0 - \Omega_{k+h}^0) \right] \]

\[ \left\{ f^-(\Omega_k^0) - f^+(\Omega_{k+h}^0) + 2f^+(\Omega_k^0)f^+(\Omega_{k+h}^0) \right\} \tilde{N}_{\pi,n}^\mu (k^0 = \Omega_k^0) \delta(q^0 + \Omega_k^0 + \Omega_{k+h}^0) \]

\[ \left\{ -f^-(\Omega_k^0) - f^+(\Omega_{k+h}^0) + 2f^+(\Omega_k^0)f^+(\Omega_{k+h}^0) \right\} \tilde{N}_{\pi,n}^\mu (k^0 = -\Omega_k^0) \delta(q^0 - \Omega_k^0 + \Omega_{k+h}^0) \]

\[ \left\{ -f^-(\Omega_k^0) - f^+(\Omega_{k+h}^0) + 2f^+(\Omega_k^0)f^+(\Omega_{k+h}^0) \right\} \tilde{N}_{\pi,n}^\mu (k^0 = \Omega_k^0) \delta(q^0 + \Omega_k^0 - \Omega_{k+h}^0) \right] \]

where, $\tilde{N}_{\pi,n}^\mu (q||k||)$ and $\tilde{N}_{\pi,n}^\mu (q||k||)$ can be read off from Eq. (55) and (58). The presence of Kronecker delta functions in the expressions of $\tilde{N}_{\pi,n}^\mu (q||k||)$ and $\tilde{N}_{\pi,n}^\mu (q||k||)$ has eliminated one of the double sums or in other words, the sum over index \( l \) now runs from \((n - 1)\) to \((n + 1)\).

V. LORENTZ STRUCTURE OF THE VECTOR BOSON SELF ENERGY IN MAGNETIZED MEDIUM

In this section, we will derive the tensorial decomposition of the massive vector boson self energy. We note that, the self energy $\Pi^{\mu\nu} (q)$ being a second rank tensor, has sixteen components which will mix among themselves with the change of frame. It is useful to use linearly independent basis tensors (constructed with the available vectors and tensors) to express $\Pi^{\mu\nu} (q)$ so that the form factors (corresponding to each basis) remain Lorentz invariant. This will also enable one to solve the Dyson-Schwinger equation in order to obtain the complete interacting vector boson propagator. In order to do so, we first note that the vector boson self energy satisfies the following constrain

\[ \Pi^{\mu\nu} (q) = \Pi^{\nu\mu} (q) \quad \text{and} \quad q_{\mu} \Pi^{\mu\nu} (q) = 0 . \]

Let us first consider the pure vacuum case i.e., zero temperature and zero external magnetic field. In this case, the only available vector is the momentum $q^\mu$ along with the metric tensor $g^{\mu\nu}$ so that $\Pi^{\mu\nu} (q)$ is a linear combination of $q^\mu q^\nu$ and $g^{\mu\nu}$ i.e $\Pi^{\mu\nu} (q) = (\alpha_1 g^{\mu\nu} + \alpha_2 q^\mu q^\nu)$). Imposing the constrains of Eq. (54), we get $\alpha_1 + \alpha_2 q^2 = 0$ which makes the only possible Lorentz structure of the self energy as

\[ \Pi^{\mu\nu} = \alpha_1 \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \]

where the Lorentz invariant form factor $\alpha_1 = \alpha_1 (q^2) = \frac{1}{4} \Pi^{\mu\nu}$. Note that, with $q^\mu$ and $g^{\mu\nu}$, the only possible Lorentz scalar that can be formed by contracting with $\Pi^{\mu\nu} (q)$ is the quantity $g_{\mu\nu} \Pi^{\mu\nu} = \Pi^{\mu\nu}$ implying the existence of only one form factor.

We now consider the case with finite temperature but zero magnetic field. In this case we have an additional four vector $u^\mu$ (medium four-velocity) along with $q^\mu$ and $g^{\mu\nu}$. This makes $\Pi^{\mu\nu}$ to be a linear combination of $g^{\mu\nu}, q^\mu q^\nu, u^\mu u^\nu, q^\mu u^\nu, q^\nu u^\mu$ and $q^\mu u^\mu$ i.e.

\[ \Pi^{\mu\nu} (q) = (\alpha_1 g^{\mu\nu} + \alpha_2 q^\mu q^\nu + \alpha_3 u^\mu u^\nu + \alpha_4 q^\mu u^\nu + \alpha_5 q^\nu u^\mu) \]

However, imposing the constrains in Eq. (54), we find the following relationship among the coefficients

\[ \alpha_5 = \alpha_4 \]

\[ \alpha_1 + \alpha_2 q^2 + \alpha_4 (q \cdot u) = 0 \]

\[ \alpha_3 (q \cdot u) + \alpha_4 q^2 = 0 \]

which makes only two of the coefficients independent. Choosing $\alpha_1$ and $\alpha_2$ as independent, we get,
\[ \Pi^{\mu\nu}(q) = \alpha_1 \left[ g^{\mu\nu} + \frac{q^2}{(q \cdot u)} u^\mu u^\nu - \frac{1}{(q \cdot u)} (q^\mu u^\nu + q^\nu u^\mu) \right] + \alpha_2 \left[ q^\mu q^\nu + \frac{q^4}{(q \cdot u)^2} u^\mu u^\nu - \frac{q^2}{(q \cdot u)} (q^\mu u^\nu + q^\nu u^\mu) \right] \]  

where the Lorentz invariant form factors \( \alpha_1 = 1(q^2, q \cdot u) \) and \( \alpha_2 = \alpha_2(q^2, q \cdot u) \) can be obtained by contracting both side of the above equations with \( g_{\mu\nu} \) and \( u_\mu u_\nu \) so that the form factors will become functions of the Lorentz scalars \( g_{\mu\nu}, \Pi^{\mu\nu} = \Pi^{\mu\nu}_g \) and \( u_\mu u_\nu \Pi^{\mu\nu} \). Note that, with \( q^\mu, u^\mu \) and \( g^{\mu\nu} \), only two possible Lorentz scalars that can be formed by contracting with \( \Pi^{\mu\nu}(q) \) are the quantities \( \Pi^{\mu\nu}_g \) and \( u_\mu u_\nu \Pi^{\mu\nu} \) implying the existence of only two form factors. Unlike the pure vacuum case given in Eq. (55), here the decomposition of \( \Pi^{\mu\nu} \) by contracting with \( \Pi^{\mu\nu}_g \) and \( u_\mu u_\nu \Pi^{\mu\nu} \) is not unique. As already mentioned, it is useful to construct linearly independent (and mutually orthogonal) basis tensors (note that the basis tensors within square brackets in Eq. (60) are not mutually orthogonal). One such choice of orthogonal tensor basis could be

\[ P_1^{\mu\nu} = \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \] and \[ P_2^{\mu\nu} = \left( \frac{\tilde{u}^\mu \tilde{u}^\nu}{u^2} \right) \]  

where

\[ \tilde{u}^\mu = u^\mu - \frac{(q \cdot u)}{q^2} q^\mu, \]  

which is constructed from \( u^\mu \) by subtracting out its projection along \( q^\mu \). It is easy to check that \( P_1^{\mu\nu} \) and \( P_2^{\mu\nu} \) satisfy all the properties of projection tensors i.e.

\[ g_{\alpha\beta} P_i^{\mu\alpha} \delta^{\beta\nu} = \delta_{ij} P^{\mu\nu}_i \] and \[ g_{\alpha\beta} g_{\mu\nu} P_i^{\mu\alpha} \delta^{\beta\nu} = \delta_{ij}. \]  

Therefore, \( \Pi^{\mu\nu} \) can be written as

\[ \Pi^{\mu\nu}(q) = \Pi_1 (q^2, q \cdot u) P_1^{\mu\nu} + \Pi_2 (q^2, q \cdot u) P_2^{\mu\nu} \]  

where the form factors are

\[ \Pi_1 (q^2, q \cdot u) = \left( \frac{\Pi_2}{u^2} - u_\mu u_\nu \Pi^{\mu\nu} \right) \] and \[ \Pi_2 (q^2, q \cdot u) = \left( \frac{1}{u^2} u_\mu u_\nu \Pi^{\mu\nu} \right) . \]  

Care should be taken when considering the special case like \( \vec{q} = 0 \) \[ \text{[25]} \]. To see this, let us consider \( q^i = |\vec{q}| n^i \) so that the spatial components of the projectors at \( \vec{q} = 0 \) become (in the LRF)

\[ P_1^{ij} = g^{ij} + n^i n^j \] and \[ P_2^{ij} = -n^i n^j \].  

This implies that the spatial components of self energy at vanishing three momentum

\[ \Pi^{ij}(q^0, \vec{q} = 0) = \Pi_1 g^{ij} + n^i n^j (\Pi_1 - \Pi_2) \]  

depend on the direction of \( \vec{q} \) even at \( |\vec{q}| = 0 \). This ambiguity is eliminated by setting additional constraint on the form factors as \( \Pi_1 (q^0, \vec{q} = 0) = \Pi_2 (q^0, \vec{q} = 0) \). Following the same strategy, we now construct suitable orthogonal tensor basis for the vector boson self energy at finite temperature under external magnetic field. In this case we have an additional four vector \( b^\mu \) (corresponding to the magnetic field direction) along with \( q^\mu, u^\mu \) and \( g^{\mu\nu} \). This makes the symmetric \( \Pi^{\mu\nu} \) to be a linear combination of seven tensors as

\[ \Pi^{\mu\nu}(q) = \alpha_1 g^{\mu\nu} + \alpha_2 q^\mu q^\nu + \alpha_3 u^\mu u^\nu + \alpha_4 b^\mu b^\nu + \alpha_5 (q^\mu u^\nu + q^\nu u^\mu) + \alpha_6 (q^\mu b^\nu + q^\nu b^\mu) + \alpha_7 (u^\mu b^\nu + u^\nu b^\mu) \]  

However, imposing the constrains in Eq. (61), we find the following relationship among the coefficients

\[ \alpha_1 + \alpha_2 q^2 + \alpha_5 (q \cdot u) + \alpha_6 (q \cdot b) = 0 \]  

\[ \alpha_3 + \alpha_5 q^2 + \alpha_7 (q \cdot b) = 0 \]  

\[ \alpha_4 (q \cdot b) + \alpha_6 q^2 + \alpha_7 (q \cdot u) = 0 \]  

which makes only (7-3=4) four of the coefficients independent. The Lorentz invariant form factors \( \alpha_i = \alpha_i (q^2, q \cdot u, q \cdot b) \) with \( i = 1,2, ..., 7 \) can be obtained by contracting both side of the above equations separately with \( g_{\mu\nu}, u_\mu u_\nu, b_\mu b_\nu \) and \( u_\mu b_\nu \) so that the form factors will become functions of the Lorentz scalars \( \Pi^{\mu\nu}_g, u_\mu u_\nu \Pi^{\mu\nu}, b_\mu b_\nu \Pi^{\mu\nu} \) and \( u_\mu b_\nu \Pi^{\mu\nu} \). Note that, with \( q^\mu, u^\mu, b^\mu \) and \( g^{\mu\nu} \), only four possible Lorentz scalars that can be formed by contracting with \( \Pi^{\mu\nu}(q) \)
are the quantities $\Pi^\mu_\nu$, $u_\mu u_\nu \Pi^{\mu\nu}$, $b_\mu b_\nu \Pi^{\mu\nu}$ and $u_\mu b_\nu \Pi^{\mu\nu}$ implying the existence of only four form factors. Like the finite temperature case, here the the decomposition of $\Pi^{\mu\nu}$ is also not unique. One convenient choice of tensor basis could be

\begin{align}
P_1^{\mu\nu} &= \left( g^{\mu\nu} - \frac{q^{\mu} q^{\nu}}{q^2} - \frac{\tilde{u}^{\mu} \tilde{u}^{\nu}}{\tilde{u}^2} - \frac{\tilde{b}^{\mu} \tilde{b}^{\nu}}{\tilde{b}^2} \right) \\
P_2^{\mu\nu} &= \left( \frac{\tilde{u}^{\mu} \tilde{u}^{\nu}}{\tilde{u}^2} \right) \\
P_3^{\mu\nu} &= \left( \frac{\tilde{b}^{\mu} \tilde{b}^{\nu}}{\tilde{b}^2} \right) \\
Q^{\alpha\beta} &= \frac{1}{\sqrt{\tilde{u}^2 \tilde{b}^2}} \left( \tilde{u}^{\alpha} \tilde{b}^{\beta} + \tilde{u}^{\beta} \tilde{b}^{\alpha} \right)
\end{align}

where $\tilde{u}^{\mu}$ is defined in Eq. (62) and $\tilde{b}^{\mu}$ is defined as

\begin{equation}
\tilde{b}^{\mu} = b^{\mu} - \frac{(q \cdot b)}{q^2} q^{\mu} - \frac{b \cdot \tilde{u}}{\tilde{u}^2} \tilde{u}^{\mu}.
\end{equation}

The basis tensors in Eqs. (72)-(75) satisfy the following relations:

\begin{align}
g_{\alpha\beta} g_{\mu\nu} P_1^{\mu\nu} P_2^{\alpha\beta} &= \delta_{ij} \\
g_{\alpha\beta} g_{\mu\nu} P_3^{\mu\nu} Q^{\alpha\beta} &= 0 \\
g_{\alpha\beta} g_{\mu\nu} Q^{\alpha\beta} Q^{\alpha\beta} &= 2 \\
g_{\alpha\beta} P_1^{\mu\alpha} P_2^{\nu\beta} &= \delta_{ij} P_1^{\mu\nu} \\
g_{\alpha\beta} Q^{\mu\alpha} Q^{\nu\beta} &= P_2^{\mu\nu} + P_3^{\mu\nu} \\
g_{\alpha\beta} P_1^{\mu\alpha} Q^{\nu\beta} &= g_{\alpha\beta} Q^{\mu\alpha} P_1^{\nu\beta} = 0 \\
g_{\alpha\beta} P_2^{\mu\alpha} Q^{\nu\beta} &= g_{\alpha\beta} Q^{\mu\alpha} P_2^{\nu\beta} = \frac{\tilde{u}^{\alpha} \tilde{b}^{\beta}}{\sqrt{\tilde{u}^2 \tilde{b}^2}} \\
g_{\alpha\beta} P_3^{\mu\alpha} Q^{\nu\beta} &= g_{\alpha\beta} Q^{\mu\alpha} P_3^{\nu\beta} = \frac{\tilde{u}^{\alpha} \tilde{b}^{\beta}}{\sqrt{\tilde{u}^2 \tilde{b}^2}}
\end{align}

Using the basis given in Eqs. (72)-(75), the self energy at finite temperature under external magnetic can be written as

\begin{equation}
\Pi^{\mu\nu}(q) = \Pi_\alpha P_1^{\mu\nu} + \Pi_\beta P_2^{\mu\nu} + \Pi_\gamma P_3^{\mu\nu} + \Pi_\delta Q^{\mu\nu}
\end{equation}

where the form factors are obtained as

\begin{align}
\Pi_\alpha &= \frac{1}{\tilde{u}^2} u_\mu u_\nu \Pi^{\mu\nu} \\
\Pi_\beta &= \frac{b_\mu b_\nu \Pi^{\mu\nu}}{\tilde{u}^2} + \frac{2 \tilde{b} \cdot \tilde{u}}{\tilde{u}^2} u_\mu u_\nu \Pi^{\mu\nu} - 2 \frac{b \cdot \tilde{u}}{\tilde{u}^2} u_\mu b_\nu \Pi^{\mu\nu} \\
\Pi_\gamma &= \frac{1}{\sqrt{\tilde{u}^2 \tilde{b}^2}} \left[ u_\mu b_\nu \Pi^{\mu\nu} - \frac{(b \cdot \tilde{u})}{\tilde{u}^2} u_\mu u_\nu \Pi^{\mu\nu} \right] \\
\Pi_\delta &= (\Pi^\mu_\mu - \Pi_\alpha - \Pi_\gamma)
\end{align}

Analogous to the case of only finite temperature, care should be taken while considering the special case $q_\perp = 0$. To see this, let us consider $q_\perp = |\tilde{q}_\perp| n^i$ with $i = 1, 2$ so that the following components of self energy at vanishing $q_\perp$ become (in the LRF)

\begin{align}
\Pi_{ij}(q^0, q_\perp = 0, q_z) &= \Pi_{ij} - n_i n_j (\Pi_\alpha - \Pi_\gamma) \\
\Pi_{i3}(q^0, q_\perp = 0, q_z) &= \frac{q^0}{q_\parallel} n_i \Pi_\delta
\end{align}

which depend on the direction of $\tilde{q}_\perp$ even at $q_\perp = 0$. This ambiguity is eliminated by setting additional constraints on the form factors as

\begin{equation}
\Pi_\alpha(q^0, q_\perp = 0, q_z) = \Pi_\gamma(q^0, q_\perp = 0, q_z) \quad \text{and} \quad \Pi_\delta(q^0, q_\perp = 0, q_z) = 0.
\end{equation}
VI. THE INTERACTING $\rho$ MESON PROPAGATOR AND ITS LORENTZ STRUCTURE

Let us first consider the zero temperature and zero magnetic field case for which the complete interacting $\rho$ propagator $D^{\mu\nu}$ is obtained by solving the Dyson-Schwinger equation

$$D^{\mu\nu} = \Delta^{\mu\nu} - \Delta^{\mu\alpha}\Pi_{\alpha\beta}D^{\beta\nu} \tag{93}$$

where

$$\Delta^{\mu\nu} = \left( -g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{m_{\rho}^2} \right) \Delta_F(q, m_{\rho}) \tag{94}$$

is the free vacuum Feynman propagator and $\Pi^{\mu\nu}$ is the one-loop self energy of $\rho$ meson which has the Lorentz structure given in Eq. (55) as

$$\Pi^{\mu\nu} = \left( g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2} \right) \Pi \tag{95}$$

with the form factor $\Pi = \frac{1}{3} \Pi_{\mu}$. In order to solve Eq. (93), we rewrite it as

$$(D^{\mu\nu})^{-1} = (\Delta^{\mu\nu})^{-1} + \Pi^{\mu\nu} \tag{96}$$

where $(\Delta^{\mu\nu})^{-1} = (q^2 - m_{\rho}^2)g^{\mu\nu} - q^{\mu}q^{\nu}$ which satisfies $\Delta^{\mu\alpha}(\Delta_{\alpha\nu})^{-1} = g^{\mu\nu}$. Substituting $\Pi^{\mu\nu}$ from Eq. (95) in the above equation, we get the inverse of the complete propagator which can be inverted using the relation $D^{\mu\alpha}(D_{\alpha\nu})^{-1} = g^{\mu\nu}$ to obtain the complete propagator as

$$D^{\mu\nu}(q) = \left( -g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2} \right) \left( \frac{-1}{q^2 - m_{\rho}^2 + \Pi} \right) - \frac{q^{\mu}q^{\nu}}{q^2 m_{\rho}^2} \tag{97}$$

We now consider the case of finite temperature and zero magnetic field. As already mentioned in Sec. [III], in RTF of finite temperature field theory all the two point correlation functions become $2 \times 2$ matrices in thermal space. In this case the Dyson-Schwinger equation also becomes a matrix equation \[35\]

$$D^{\mu\nu} = \Delta^{\mu\nu} - \Delta^{\mu\alpha}\Pi_{\alpha\beta}D^{\beta\nu} \tag{98}$$

Each term of the above equation can be diagonalized in terms of the respective analytic functions (denoted by a bar) so that the above equation becomes an algebraic one

$$\overline{D}^{\mu\nu} = \overline{\Delta}^{\mu\nu} - \overline{\Delta}^{\mu\alpha}\bar{\Pi}_{\alpha\beta}\overline{D}^{\beta\nu} \tag{99}$$

where $\overline{\Delta}^{\mu\nu} = \Delta^{\mu\nu}$. The above equation can be rewritten as

$$\left( \overline{D}^{\mu\nu} \right)^{-1} = \left( \overline{\Delta}^{\mu\nu} \right)^{-1} + \bar{\Pi}^{\mu\nu} \tag{100}$$

In this case, the Lorentz structure of the thermal self energy function is given in Eq. [61] as

$$\bar{\Pi}^{\mu\nu}(q) = \Pi_1(q^2, q \cdot u)P_1^{\mu\nu} + \Pi_2(q^2, q \cdot u)P_2^{\mu\nu} \tag{101}$$

where the projection tensors and form factors are respectively defined in Eqs. [61] and [63]. Substituting the above equation in Eq. [100], we get the inverse of the complete propagator. In order to obtain the complete propagator, we write

$$\overline{D}^{\mu\nu} = A_1 P_1^{\mu\nu} + A_2 P_2^{\mu\nu} + \xi q^{\mu}q^{\nu} \tag{102}$$

and use the relation $\overline{D}^{\mu\alpha}(\overline{D}_{\alpha\nu})^{-1} = g^{\mu\nu}$ to extract $A_1$, $A_2$ and $\xi$. The final form of the complete interacting thermal propagator is obtained as

$$\overline{D}^{\mu\nu} = \frac{P_1^{\mu\nu}}{q^2 - m_{\rho}^2 + \Pi_1} + \frac{P_2^{\mu\nu}}{q^2 - m_{\rho}^2 + \Pi_2} - \frac{q^{\mu}q^{\nu}}{q^2 m_{\rho}^2} \tag{103}$$

Finally we consider the case with both finite temperature and external magnetic field. In this case we need to solve the Dyson-Schwinger equation

$$\left( \overline{D}^{\mu\nu} \right)^{-1} = \left( \overline{\Delta}^{\mu\nu} \right)^{-1} + \bar{\Pi}^{\mu\nu} \tag{104}$$
appears in the physical timelike region only if the loop particles have different masses and lie in the kinematic domain 
with their time reversed processes. If we restrict ourselves to the physical timelike kinematic regions defined in terms of decay or scattering. For example, Unitary cuts correspond to the decay of 

Using the above constraints, we get from Eqs. (86)-(89) Eq. (92), for the special case 

where the basis tensors and form factors are given in Eqs. (72)-(75) and (86)-(89). Substituting the above equation in Eq. (104), we get the inverse of the complete propagator. In order to obtain the complete propagator, we write 

and use the relation \( \overline{D}^{\mu\nu} (\overline{D}_{\alpha\nu})^{-1} = g^\mu_{\nu} \) to extract the coefficients as

\[
A_\alpha = \frac{1}{q^2 - m^2_{\rho} + \Pi_{\alpha}} \\
A_\beta = \frac{q^2 - m^2_{\rho} + \Pi_{\gamma}}{(q^2 - m^2_{\rho} + \Pi_{\gamma}) (q^2 - m^2_{\rho} + \Pi_{\beta}) - \Pi_{\delta}} \\
A_\gamma = \frac{q^2 - m^2_{\rho} + \Pi_{\delta}}{(q^2 - m^2_{\rho} + \Pi_{\beta}) (q^2 - m^2_{\rho} + \Pi_{\gamma}) - \Pi_{\delta}} \\
A_\delta = \frac{(q^2 - m^2_{\rho} + \Pi_{\beta}) (q^2 - m^2_{\rho} + \Pi_{\gamma}) - \Pi_{\delta}}{\Pi_{\delta}} \\
\xi = \frac{-1}{q^2 m^2_{\rho}}.
\]

**VII. ANALYTIC STRUCTURE OF THE SELF ENERGY**

In this work, we have considered the transverse momentum of the rho meson to be zero i.e. \( q_\perp = 0 \). As shown in Eq. (92), for the special case \( q_\perp = 0 \), the additional constraints to be imposed on the form factors are \( \Pi_\alpha(q_\perp = 0, q_\perp) = \Pi_\gamma(q_\perp = 0, q_\perp) = 0 \) and \( \Pi_\delta(q_\perp = 0, q_\perp) = 0 \).

Using the above constraints, we get from Eqs. (86)-(89)

\[
\Pi_\alpha = \Pi_\gamma = \frac{1}{2} \left( \Pi_\mu^{\mu} - \frac{1}{u^2} u_{\mu} u_{\nu} \Pi^{\mu\nu} \right) \\
\Pi_\beta = \frac{1}{u^2} u_{\mu} u_{\nu} \Pi^{\mu\nu} \\
\Pi_\delta = 0
\]

which imply that we need to calculate only the two quantities quantities \( \Pi_\mu^{\mu} \) and \( u_{\mu} u_{\nu} \Pi^{\mu\nu} = \Pi^{00} \). These are obtained from Eqs. (20)-(23) by contracting them with \( g^{\mu\nu} \) and \( u_{\mu} u_{\nu} \). This essentially means replacing \( N^{\mu\nu} \) for all the loops with \( N^{00} \) or \( N^{00} \), an explicit list for which has been provided in Appendix C.

Let us now discuss the analytic structure of the self energy functions. We first consider the zero magnetic field case. The imaginary part of the self energy function for \( \pi\pi \) and NN loops as given in Eqs. (22) and (23) each contains four Dirac delta functions. These delta functions represent energy-momentum conservation and they are non-vanishing in certain kinematic domain. They are termed as the Unitary-I, Unitary-II, Landau-II and Landau-I cuts as they appear in those equations. The kinematic regions for the Unitary-I and Unitary-II cuts are given by [35] \( \sqrt{q^2 + 4m_L^2} < q^0 < \infty \) and \( -\infty < q^0 < -\sqrt{q^2 + 4m_L^2} \) whereas the same for the two Landau cuts are \( |q^0| < |q| \) where \( m_L \) is the mass of the loop particle i.e. \( m_L = m_\pi \) or \( m_N \). These cuts correspond to different physical processes such as decay or scattering. For example, Unitary cuts correspond to the decay of \( \rho^0 \) into a \( \pi^+ \pi^- \) or \( N \bar{N} \) pair and the Landau cuts correspond to the scattering of a \( \rho^0 \) with a pion or nucleon producing the same in the final state along with their time reversed processes. If we restrict ourselves to the physical timelike kinematic regions defined in terms of \( q^0 > 0 \) and \( q^2 > 0 \), then only the Unitary-I cut contributes. It is important to note that, a non-trivial Landau cut appears in the physical timelike region only if the loop particles have different masses and lie in the kinematic domain \( |\tilde{q}| < q^0 < \sqrt{q^2 + \Delta m^2} \) where \( \Delta m \) is the mass difference of the loop particles.
Let us now consider the case of both finite temperature and non-zero external magnetic field. In this case the imaginary parts of the self energy as given in Eqs. (52) and (53) also contain four Dirac delta functions corresponding to the Unitary and Landau cuts. It is important to note that the arguments of the delta functions contain only the longitudinal dynamics (because of dimensional reduction) which implies that the analytic structure of the self energy functions will only depend on the longitudinal momentum of $\rho$. On the other hand, the transverse dynamics has appeared as Landau level dependent “dimensionally reduced effective mass” to the loop particles as given in Eq. (35). Therefore, even if the loop particles have the same masses, a non-trivial Landau cut may appear in the physical timelike kinematic domain if the two loop particles reside in different Landau levels. Physically, this means that $\rho$ can get absorbed in a scattering with a pion or a proton in a lower Landau level producing another pion or proton in a higher Landau level as the final state. A detailed discussion on the analytic structure in presence of external magnetic field can be found in Refs. [31, 38]. The Unitary-I and Unitary-II terms for the $\pi \pi$ loop are non-vanishing in the kinematic domains $\sqrt{q_\perp^2 + 4(m_\pi^2 + eB)} < q^0 < \infty$ and $-\infty < q^0 < -\sqrt{q_\perp^2 + 4(m_\pi^2 + eB)}$ whereas the kinematic domain for both the Landau cuts is

$$|q^0| < \sqrt{q_\perp^2 + (\sqrt{m_N^2 + eB} - \sqrt{m_N^2 - 2eB})^2}. \tag{116}$$

The corresponding kinematic domains for the NN loop are $\sqrt{q_\perp^2 + 4m_N^2} < q^0 < \infty$ and $-\infty < q^0 < -\sqrt{q_\perp^2 + 4m_N^2}$ for the Unitary-I and Unitary-II cuts respectively and

$$|q^0| < \sqrt{q_\perp^2 + (m_N - \sqrt{m_N^2 + 2eB})^2} \tag{117}$$

for the Landau cuts. Note that, the threshold of the Landau cuts appears when the “dimensionally reduced effective mass” difference between the loop particles is the maximum. As can be seen from Eqs. (22) and (23), for a particular value of the index $n$, the sum over the index $l$ runs only for three values $(n-1)$, $n$ and $(n+1)$ which implies that, the Landau level difference between the loop particles can be at most one. Thus the maximum difference in their “dimensionally reduced effective mass” appears when one of them is at the lowest Landau level and the other one is at the first Landau level which in turn defines the Landau cut threshold in Eqs. (116) and (117).

We now simplify the expressions of the imaginary parts given in Eqs. (22), (23), (52) and (53) by evaluating one of the integrals using the Dirac delta functions. For the imaginary parts at zero magnetic field, we evaluate the $d(\cos \theta)$ integrals and get (after imposing the kinematic restrictions discussed above),

$$\text{Im} \Pi_{\pi, N}(q^0, \mathbf{q}) = -\text{sign}(q^0) \tan \theta \left[ \frac{1}{2\pi T} \int_{-\omega_-}^{\omega_+} d(\omega_k, \Omega_k) \left( U_{\pi, N}^\mu \right)^\nu (\cos \theta = \cos \theta_{0,N}) \Theta \left( q^0 - \sqrt{q^2 + 4m_{\pi, N}^2} \right) \right]$$

$$+ \int_{-\omega_-}^{\omega_+} d(\omega_k) \left( U_{\pi, N}^\mu \right)^\nu (\cos \theta = \cos \theta_{0,N}) \Theta \left( -q^0 - \sqrt{q^2 + 4m_{\pi, N}^2} \right)$$

$$+ \int_{-\omega_-}^{\omega_+} d(\omega_k) \left( L_{\pi, N}^\mu \right)^\nu (\cos \theta = \cos \theta_{0,N}) \Theta \left( -|q^0| + |\mathbf{q}| \right)$$

$$+ \int_{\omega_-}^{\omega_+} d(\omega_k) \left( L_{\pi, N}^\mu \right)^\nu (\cos \theta = \cos \theta_{0,N}) \Theta \left( -|q^0| + |\mathbf{q}| \right) \tag{118}$$

where,

$$\omega_\pm = \left\{ \begin{array}{cl} \frac{1}{2\pi T} \left[ q^0q^2 \mp |\mathbf{q}| \Lambda^{1/2} \left( q^2, m_{\pi, N}^2, m_{\pi, N}^2 \right) \right] & \text{for } \pi \pi \text{ loop} \\ \frac{1}{2\pi T} \left[ q^0q^2 \mp |\mathbf{q}| \Lambda^{1/2} \left( q^2, m_{\pi, N}^2, m_{\pi, N}^2 \right) \right] & \text{for } \text{NN loop} \end{array} \right\}$$

$$\left( U_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} 1 + f(\omega_k) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\omega_k) f(\omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \right( k^0 = -\omega_k \right) \tag{120}$$

$$\left( U_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} 1 + f(\omega_k) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\omega_k) f(\omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \right( k^0 = \omega_k \right) \tag{121}$$

$$\left( L_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} f(\omega_k) f(\omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\omega_k) f(\omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \right( k^0 = -\omega_k \right) \tag{122}$$

$$\left( L_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} f(\omega_k) f(\omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\omega_k) f(\omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \right( k^0 = \omega_k \right) \tag{123}$$

$$\left( U_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} f(\Omega_k) - f(\Omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\Omega_k) f(\Omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \right( k^0 = -\Omega_k \right) \tag{124}$$

$$\left( U_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} f(\Omega_k) - f(\Omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\Omega_k) f(\Omega_p) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \right( k^0 = \Omega_k \right) \tag{125}$$

$$\left( L_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} -f(\Omega_k) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\Omega_k) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \left( k^0 = \Omega_k \right) \tag{126}$$

$$\left( L_{\pi, N}^\mu \right)^\nu = \left\{ \begin{array}{cl} -f(\Omega_k) & \left( \cos \theta = -\cos \theta_{0,N} \right) \\ f(\Omega_k) & \left( \cos \theta = -\cos \theta_{0,N} \right) \end{array} \right\} N_{\pi, N}^\mu \left( k^0 = -\Omega_k \right) \tag{127}$$
\[
\cos \theta_0^\pi = \left( -\frac{2q^0\omega_k + q^2}{2|q||\vec{k}|} \right),
\]
(128)
\[
\cos \theta_0^{\pi'} = \left( \frac{2q^0\omega_k + q^2}{2|q||\vec{k}|} \right),
\]
(129)
\[
\cos \theta_0^\mathcal{N} = \left( -\frac{2q^0\Omega_k + q^2}{2|q||\vec{k}|} \right) \quad \text{and}
\]
(130)
\[
\cos \theta_0^{\mathcal{N}'} = \left( \frac{2q^0\Omega_k + q^2}{2|q||\vec{k}|} \right),
\]
(131)

with \( \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \) being the Källén function.

For the imaginary parts at finite magnetic field, we evaluate the \( dk_z \) integrals in Eqs. (52) and (53) using the Dirac delta functions. The imaginary part due to \( \pi\pi \) loop simplifies to

\[
\text{Im} \mathcal{\Pi}_{\pi}^{\mu\nu} (q^0, q_z) = -\text{sign} (q^0) \tanh \left( \frac{q^0}{2T} \right) \sum_{n=0}^{(n+1)} \sum_{l=(n-1)}^\infty \frac{1}{4\lambda^{1/2}(q^2_{\parallel}, m_l^2, m_n^2)} \sum_{\tilde{k}_z \in \tilde{k}_z^\pm} \left[ \left( \tilde{U}_{1,nl}^\pi \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( q^0 - \sqrt{q_z^2 + (m_l + m_n)^2} \right) + \left( \tilde{U}_{2,nl}^\pi \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( -q^0 - \sqrt{q_z^2 + (m_l + m_n)^2} \right) + \left( \tilde{L}_{1,nl}^\pi \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( q^0 - \min(q_z, E_\pm) \right) \Theta \left( -q^0 + \max(q_z, E_\pm) \right) + \left( \tilde{L}_{2,nl}^\pi \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( -q^0 - \min(q_z, E_\pm) \right) \Theta \left( q^0 + \max(q_z, E_\pm) \right) \right]
\]
(132)

where,

\[
\left( \tilde{U}_{1,nl}^\pi \right)^{\mu\nu} = \{ 1 + f(\tilde{\omega}_k^\mu) + f(\tilde{\omega}_k^\nu) + 2f(\tilde{\omega}_k^\mu)f(\tilde{\omega}_k^\nu) \} \tilde{\mathcal{N}}_{\pi,nl}^{\mu\nu}(k^0 = -\tilde{\omega}_k),
\]
(133)
\[
\left( \tilde{U}_{2,nl}^\pi \right)^{\mu\nu} = \{ 1 + f(\tilde{\omega}_k^\mu) + f(\tilde{\omega}_k^\nu) + 2f(\tilde{\omega}_k^\mu)f(\tilde{\omega}_k^\nu) \} \tilde{\mathcal{N}}_{\pi,nl}^{\mu\nu}(k^0 = \tilde{\omega}_k),
\]
(134)
\[
\left( \tilde{U}_{1,nl}^\pi \right)^{\mu\nu} = \{ f(\tilde{\omega}_k^\mu) + f(\tilde{\omega}_k^\nu) + 2f(\tilde{\omega}_k^\mu)f(\tilde{\omega}_k^\nu) \} \tilde{\mathcal{N}}_{\pi,nl}^{\mu\nu}(k^0 = \tilde{\omega}_k),
\]
(135)
\[
\left( \tilde{U}_{2,nl}^\pi \right)^{\mu\nu} = \{ f(\tilde{\omega}_k^\mu) + f(\tilde{\omega}_k^\nu) + 2f(\tilde{\omega}_k^\mu)f(\tilde{\omega}_k^\nu) \} \tilde{\mathcal{N}}_{\pi,nl}^{\mu\nu}(k^0 = -\tilde{\omega}_k),
\]
(136)

with \( \tilde{k}_z^\pm = \frac{1}{2|q_z|} \left[ -yq_z \pm |q^0|\lambda^{1/2} \left( q^2_{\parallel}, m_l^2, m_n^2 \right) \right] \), \( y = (q^2_{\parallel} + m_l^2 - m_n^2), \tilde{\omega}_k^\mu = \sqrt{k_z^2 + m_l^2}, \) and \( E_\pm = \frac{m_l \pm m_n}{|m_l - m_n|} \sqrt{q_z^2 + (m_l \pm m_n)^2} \).

The corresponding expression of the imaginary part due to NN loop reads

\[
\text{Im} \mathcal{\Pi}_{\mathcal{N}}^{\mu\nu} (q^0, q_z) = \frac{1}{2} \text{Im} \mathcal{\Pi}_{\mathcal{N}}^{\mu\nu} (q^0, q_z) - \text{sign} (q^0) \tanh \left( \frac{q^0}{2T} \right) \sum_{n=0}^{(n+1)} \sum_{l=(n-1)}^\infty \frac{1}{4\lambda^{1/2}(q^2_{\parallel}, M_l^2, M_n^2)} \sum_{\tilde{k}_z \in \tilde{k}_z^\pm} \left[ \left( \tilde{U}_{1,nl}^\mathcal{N} \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( q^0 - \sqrt{q_z^2 + (M_l + M_n)^2} \right) + \left( \tilde{U}_{2,nl}^\mathcal{N} \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( -q^0 - \sqrt{q_z^2 + (M_l + M_n)^2} \right) + \left( \tilde{L}_{1,nl}^\mathcal{N} \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( q^0 - \min(q_z, E'_\pm) \right) \Theta \left( -q^0 + \max(q_z, E'_\pm) \right) + \left( \tilde{L}_{2,nl}^\mathcal{N} \right)^{\mu\nu} (k_z = \tilde{k}_z) \Theta \left( -q^0 - \min(q_z, E'_\pm) \right) \Theta \left( q^0 + \max(q_z, E'_\pm) \right) \right]
\]
(137)
where,

\[
\begin{align*}
\left( \bar{U}_{1,\mu}^p \right)^{\mu\nu} &= \left\{ 1 - f^- (\Omega_k^0) - f^+ (\Omega_p^0) + 2 f^- (\Omega_k^l) f^+ (\Omega_p^l) \right\} \bar{N}_{p,\mu\nu}^{\mu\nu} (k^0 = -\Omega_k^0), \\
\left( \bar{U}_{2,\mu}^p \right)^{\mu\nu} &= \left\{ 1 - f^+ (\Omega_k^0) - f^- (\Omega_p^0) + 2 f^+ (\Omega_k^l) f^- (\Omega_p^l) \right\} \bar{N}_{p,\mu\nu}^{\mu\nu} (k^0 = -\Omega_k^0), \\
\left( \bar{U}_{3,\mu}^p \right)^{\mu\nu} &= \left\{ - f^+ (\Omega_k^0) - f^- (\Omega_p^0) + 2 f^+ (\Omega_k^l) f^- (\Omega_p^l) \right\} \bar{N}_{p,\mu\nu}^{\mu\nu} (k^0 = -\Omega_k^0), \\
\left( \bar{U}_{4,\mu}^p \right)^{\mu\nu} &= \left\{ - f^- (\Omega_k^0) - f^+ (\Omega_p^0) + 2 f^- (\Omega_k^l) f^+ (\Omega_p^l) \right\} \bar{N}_{p,\mu\nu}^{\mu\nu} (k^0 = -\Omega_k^0),
\end{align*}
\]

with, \( \tilde{K}_z = \frac{1}{2\pi \eta} \left[ -Y q_z \pm \eta^{1/2} \left( \sqrt{q^2} - M_{\pi}^2 + M_{\pi}^2 \right) \right] \), \( Y = (q_0^2 + M_{\pi}^2 - M_{\pi}^2) \), \( \Omega_k^0 = \sqrt{\Omega_z^2 + M_k^2} \), and \( E_{\pm} = \frac{M_{\pi} - M_{\pi}}{M_{\pi} + M_{\pi}} \sqrt{q_z^2 + (M_1 + M_\pi)^2} \). The first term on the RHS of Eq. \((137)\) is the contribution from the neutron-neutron loop (which is not affected by the external field) whose simplified form is given in Eq. \((118)\).

VIII. NUMERICAL RESULTS

We begin this section by presenting the real and imaginary parts of the in-medium self energy functions of \( \rho^0 \). As can be seen from Eqs. \((89)\)-\((115)\), we have only two non-zero form factors for the self energy which are \( \Pi_{\alpha} \) and \( \Pi_{\beta} \) for \( q_L = 0 \). Let us first consider the zero magnetic field case for which the imaginary and real parts of \( \Pi_{\alpha} \) and \( \Pi_{\beta} \) are depicted in Figs. \(2\) and \(3\) respectively. In Fig. \(2\)(a), \( \text{Im}\Pi_{\alpha} \) and \( \text{Im}\Pi_{\beta} \) due to \( \pi\pi \) loop are plotted as a function of invariant mass (\( \sqrt{q^2} \)) of \( \rho^0 \) for vacuum as well as for medium \( (T = 160 \text{ MeV} \text{ and } \mu_B = 400 \text{ MeV}) \) with \( q_z = 250 \text{ MeV} \). The vacuum self energy for \( T = \mu_B = 0 \) is compared with the in-medium one obtained at temperature \( T = 160 \text{ MeV} \) and baryon chemical potential \( \mu_B = 400 \text{ MeV} \) for the (a) \( \pi\pi \) loop and (b) NN Loop.

![Imaginary part of the self energy of \( \rho^0 \) as a function of invariant mass at zero magnetic field and at \( \rho^0 \) three momentum \( |q| = 250 \text{ MeV} \). The vacuum self energy for \( T = \mu_B = 0 \) is compared with the in-medium one obtained at temperature \( T = 160 \text{ MeV} \) and baryon chemical potential \( \mu_B = 400 \text{ MeV} \) for the (a) \( \pi\pi \) loop and (b) NN Loop.]( Vacuum T = 160 MeV - α --- Vacuum T = 160 MeV - β --- \( q = 250 \text{ MeV} \) , \( \text{eB} = 0 \) \( \pi\pi \) - Loop \( NN \) - Loop \( q = 250 \text{ MeV} \) , \( \text{eB} = 0 \))

It is to be understood that in the case of vacuum the two form factors are equal. In this case, the only contribution comes from the Unitary-I cut which starts at \( 2m_{\pi} \) in the invariant mass axis. With the increase in temperature, the degeneracy between the form factor get lifted as well as they are enhanced with respect to the vacuum. This is due to the enhancement of the thermal factor in Eq. \((120)\) which increases the available phase space with the increase in temperature. The corresponding results for the NN loop is shown in Fig. \(2\)(b) for which the threshold of the Unitary-I cut is \( 2m_N \). In this case, with the increase in temperature and density, the imaginary part decreases slightly with respect to the vacuum which can be understood from Eq. \((124)\) where, because of the negative signs in front of the thermal distribution functions of the nucleons, the thermal factor reduces with the increase in temperature thus showing opposite behaviour as compared to the \( \pi\pi \) loop.

In Fig. \(3\) \( \text{Re}\Pi_{\alpha} \) and \( \text{Re}\Pi_{\beta} \) are shown as a function of \( \rho^0 \) invariant mass at zero external magnetic field with \( \rho^0 \) longitudinal momentum \( q_z = 250 \text{ MeV} \) at temperature \( T = 130 \text{ MeV} \). For the \( \pi\pi \) loop, the real part is positive at low invariant mass and becomes negative in the high invariant mass region in contrast to the NN loop for which the contribution to the real part is always negative. The real part due to NN loop is shown for two different values of
baryon chemical potential $\mu_B = 200$ and 400 MeV respectively. For low values of $\mu_B$, the contribution of the NN loop is almost of the same order as $\pi\pi$ loop, however at high $\mu_B$, the contribution from NN loop dominates over the $\pi\pi$ loop.

We now turn on the external magnetic field. For the check of consistency of the calculation at non-zero magnetic field, it is essential that $eB \to 0$ limit of non-zero magnetic field results reproduces the $eB = 0$ one. In order to take the $eB \to 0$ limit numerically, we have considered upto 500 Landau levels for a convergent result. We have shown the imaginary part of the self energy as a function of invariant mass of $\rho^0$ with longitudinal momentum $q_z = 250$ MeV at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV for the two cases: $eB = 0$ and $eB \to 0$ in Fig. 4 separately for the $\pi\pi$ and NN loops. Fig. 4(a) shows Im$\Pi_{\alpha}$ for the $\pi\pi$ loop in which the $eB \to 0$ graph has a series of spikes infinitesimally separated from each other all over the whole invariant mass region whereas the $eB = 0$ graph is finite and well behaved. Interestingly, the $eB \to 0$ graph does not miss the $eB = 0$ curve which implies that when average is done, the $eB = 0$ line will be exactly reproduced. The appearance of these spikes are due to the “threshold singularities” at each Landau level as can be understood from Eq. (132) where the Källén function goes to zero at each threshold of the Unitary and Landau cuts defined in terms of the unit step functions therein, which is a consequence of the dimensional reduction. In order to extract physical and finite results out of these spikes, we have used Ehrenfest’s coarse-graining (CG) in this method, the whole invariant mass region has been discretized in small bins followed by bin averages. In other words, the self energy at a given $\sqrt{q^2}$ is approximated by its average over the neighbourhood around that point. This in turn smears out the spike like structures. As can be seen in the figure, after CG, Im$\Pi_{\alpha}$ exactly matches with the analytic $eB = 0$ graph. The corresponding comparison of $eB \to 0$ and $eB = 0$ result for Im$\Pi_{\beta}$ due to $\pi\pi$ loop is shown in Fig. 4(b). In this case, $eB \to 0$ graph is finite and free from the threshold singularities and it matches exactly with the $eB = 0$ graph. The absence of the threshold singularities in this case is due to an overall factor of Källén functions coming from $\hat{N}_{\pi,00}$ in Eq. (133) which cancels the Källén functions in the denominator of Eq. (132). Thus the Im$\Pi_{\beta}$ due to the $\pi\pi$ loop does not require to be coarse grained.

The corresponding results for the NN loop is depicted in Figs. 4(c) and 4(d). In this case, both the Im$\Pi_{\alpha}$ and Im$\Pi_{\beta}$ suffer threshold singularities as there is no overall Källén functions coming from $\hat{N}_{p,nl}$. So both the form factors have to be coarse grained after which they exactly reproduce the $eB = 0$ graphs.

We now turn our attention to the real part of the self energy at non-zero magnetic field and show how a numerical limit of $eB \to 0$ agrees with the $eB = 0$ results. This has been shown in Fig. 5 where the real part of the form factors is shown as a function of $\rho^0$ invariant mass with longitudinal momentum $q_z = 250$ MeV at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV for the two cases $eB \to 0$ and $eB = 0$. The contributions from the $\pi\pi$ and NN loops are shown separately. Fig. 5(a) depicts Re$\Pi_{\alpha}$ whereas Fig. 5(b) shows Re$\Pi_{\beta}$. As can be seen from the figure, the $eB \to 0$ graphs exactly reproduce the $eB = 0$ for the case of NN loop. Whereas, for the $\pi\pi$ loop, $eB \to 0$ is slightly deviated from the $eB = 0$ graph but with an excellent qualitative agreement in their behaviour with respect to the variation of invariant mass of $\rho^0$. This small disagreement between the $eB \to 0$ and $eB = 0$ graph
is due to the inaccuracy in the numerical principal value integration of Eqs. (20) and (50) for which the two particle bound state threshold \( \sqrt{q_z^2} > 2m_\pi = 280 \text{ MeV} \) is less than the \( \rho^0 \) mass pole \( m_\rho = 0.770 \) (in contrast, for the NN loop, the two particle bound state threshold is at \( \sqrt{q_z^2} > 2m_N = 1.878 \text{ GeV} \) much higher than the range of the plot).

Having checked the consistency of the non-zero magnetic field calculations, we now proceed to present the imaginary part of the self energy for nonzero values of the magnetic field. In Fig. 6, the variation of \( \text{Im}\Pi_\alpha \) is shown as a function of \( \rho_0 \) invariant mass with longitudinal momentum \( q_z = 250 \text{ MeV} \) at temperature \( T = 130 \text{ MeV} \) and at baryon chemical potential \( \mu_B = 300 \text{ MeV} \). We have plotted the self energy up to \( \sqrt{q_z^2} = 1.5 \text{ GeV} \) for which the Unitary cut of the NN loop does not contribute. Fig. 6(a) depicts \( \text{Im}\Pi_\alpha \) at magnetic field \( eB = 0 \).

This is more clearly visible in the CG points which are used to obtain a coarse-grained interpolated (CGI) graph. Fig. 6(b) shows the CGI imaginary part at two different values of the magnetic field (\( eB = 0 \), 0.05 and 0.10 GeV respectively); both of them are found to oscillate about the \( eB = 0 \) graph. Moreover, with the increase in magnetic field, the oscillation frequency decreases with an increase in the oscillation amplitude. This behavior of the imaginary part with increasing magnetic field is consistent with Fig. 4 where for the \( eB \to 0 \) case, the oscillation frequency becomes infinite and amplitude becomes zero, thus reproducing the \( eB = 0 \) graph. Also with the increase in magnetic field, the threshold of the unitary cut moves towards the higher invariant mass value as discussed in Sec. VII. This has been shown clearly in the inset plot.
The real part of the form factors as a function of the invariant mass at $eB = 0$ have been compared with the real part at non zero magnetic field in the numerical limit $eB \to 0$ at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ with $\rho^0$ longitudinal momentum $q_z = 250$ MeV. The contribution from the form factors (a) $\Pi_\alpha$ and (b) $\Pi_\beta$ are shown separately due to $\pi\pi$ and NN loop.

The corresponding results for the $\text{Im}\Pi_\beta$ due to $\pi\pi$ loop as a function of $\rho^0$ invariant mass with longitudinal momentum $q_z = 250$ MeV at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV are shown in Fig. 7 for the two different values of the magnetic field $eB = 0.10$ and 0.20 GeV$^2$. Analogous to $\text{Im}\Pi_\alpha$, $\text{Im}\Pi_\beta$ also oscillates about $eB = 0$ curve, but in this case the oscillation frequency is much smaller as compared to $\text{Im}\Pi_\alpha$. The threshold of the Unitary cut moves towards higher invariant mass with the increase in magnetic field as clearly depicted in the inset plot.

As discussed in Sec. VII, a non trivial Landau cut contribution in presence of external magnetic field may appear even if the loop particles have the same mass. In this case, we have observed Landau cut contribution only in $\text{Im}\Pi_\alpha$, whereas the Landau cut does not appear in $\text{Im}\Pi_\beta$. This can be understood from the expressions of trace and 00 component of $\tilde{N}_{\pi,nl}$ and $\tilde{N}_{p,nl}$ as given in Appendix F. It can be noticed that, for both the $\pi\pi$ and proton-proton loops, the expression for the trace (i.e $\tilde{N}_{\mu\mu}^0$) contains two additional Kronecker delta functions $\delta_l^{n+1}$ along with $\delta_l^n$. 

FIG. 5. The real part of the form factors as a function of the invariant mass at $eB = 0$ have been compared with the real part at non zero magnetic field in the numerical limit $eB \to 0$ at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ with $\rho^0$ longitudinal momentum $q_z = 250$ MeV. The contribution from the form factors (a) $\Pi_\alpha$ and (b) $\Pi_\beta$ are shown separately due to $\pi\pi$ and NN loop.

FIG. 6. The contribution from the form factor $\text{Im}\Pi_\alpha$ to the imaginary part of the $\rho^0$ self energy is shown as a function of invariant mass at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ with $\rho^0$ longitudinal momentum $q_z = 250$ MeV for (a) two different values of magnetic field ($eB = 0$ and 0.05 GeV$^2$ respectively) and (b) three different values of magnetic field ($eB = 0$, 0.05 and 0.10 GeV$^2$ respectively). The coarse-grained (CG) as well as coarse-grained interpolated (CGI) results are shown in (a) whereas (b) shows only the CGI results. The inset plot in (b) shows the movement of the Unitary cut threshold by focusing in smaller range of invariant mass.
FIG. 7. The contribution from the form factor Im\(\Pi_\beta\) to the imaginary part of the \(\rho^0\) self energy is shown as a function of invariant mass at temperature \(T = 130\) MeV and at baryon chemical potential \(\mu_B = 300\) with \(\rho^0\) longitudinal momentum \(q_z = 250\) MeV for three different values of magnetic field (\(eB = 0, 0.05\) and 0.10 GeV\(^2\) respectively). The inset plot shows the movement of the Unitary cut threshold by focusing in smaller range of invariant mass.

FIG. 8. The contribution from the form factor \(\Pi_\alpha\) to the Landau cut of the coarse grained (CG) imaginary part of the \(\rho^0\) self energy is shown as a function of invariant mass with \(\rho^0\) longitudinal momentum \(q_z = 250\) MeV (a) at temperature \(T = 130\) MeV and at baryon chemical potential \(\mu_B = 300\) for three different values of magnetic field (\(eB = 0.05, 0.07\) and 0.10 GeV\(^2\) respectively) and (b) at magnetic field \(eB = 0.10\) GeV\(^2\) for two different values of temperature (\(T = 100\) and 130 MeV respectively) and at baryon chemical potential (\(\mu_B = 200\) and 300 MeV respectively). The contribution from the \(\pi\pi\) and NN loops are shown separately in which the later is scaled with different factors for the sake of presentation.

which is absent in the expressions for the 00 component (i.e \(\tilde{\Sigma}^{00}\)) (see Eqs. (F1)-(F10)). This implies that, for Im\(\Pi_\alpha\), the loop particles can be in different Landau levels whereas for Im\(\Pi_\beta\) the loop particles will always stay in the same Landau levels. Thus, as discussed in Sec. VII, the non-trivial Landau cuts will appear only in Im\(\Pi_\alpha\) and not in Im\(\Pi_\beta\). The contribution of the CGI Landau cuts to Im\(\Pi_\alpha\) as a function of \(\rho^0\) invariant mass with longitudinal momentum \(q_z = 250\) MeV is shown in Fig. 8. It is to be noted that, the Landau cuts also contain the threshold singularities and thus have to be coarse grained. Fig. 8(a) shows the variation of Im\(\Pi_\alpha\) at a temperature \(T = 130\) MeV and at baryon chemical potential \(\mu_B = 300\) MeV for three different values of the magnetic field (\(eB = 0.05, 0.07\) and 0.10 GeV\(^2\) respectively), whereas Fig. 8(b) shows the corresponding variation at magnetic field (\(eB = 0.10\) GeV\(^2\)) for two different values of temperature (\(T = 100\) and 130 MeV respectively). The contributions due to \(\pi\pi\) loop and proton-proton loops are shown separately and in Fig. 8(b); the contribution due to proton-proton loop is shown for two different values of baryon chemical potential (\(\mu_B = 200\) and 300 MeV). As can be seen from the figures, the threshold of the Landau cuts due to \(\pi\pi\) loop is different (greater) than that of proton-proton loop which can be understood.
from the discussions of Sec. VII. The threshold for $\pi \pi$ loop is $\sqrt{q^2_\parallel} < \left( \sqrt{m^2_\pi + eB} - \sqrt{m^2_\pi + 3eB} \right)$, whereas the same for proton-proton loop is $\sqrt{q^2_\parallel} < \left( m_N - \sqrt{m^2_N + 2eB} \right)$. The shift of the Landau cut threshold towards the higher invariant mass values with the increase in magnetic field can be clearly seen in Fig. 8(a). It is observed that the magnitude of the Landau cut contribution due to proton-proton loop is much less than that of $\pi \pi$ loop at lower values of the magnetic field and they become comparable to each other only at $eB \gtrsim 0.10$ GeV$^2$. In Fig. 8(a), we observe that with the increase in temperature and density, the Landau cut contribution increases without changing its threshold in the invariant mass axis.

![Figure 9](image1.png)  
**FIG. 9.** The real part of the thermal self energy of $\rho^0$ as a function of invariant mass at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV with $\rho^0$ longitudinal momentum $q_z = 250$ MeV is shown for three different values of magnetic field ($0, 0.05$ and $0.10$ GeV$^2$ respectively).

![Figure 10](image2.png)  
**FIG. 10.** The $eB$-dependent vacuum contribution to the real part of the self energy of $\rho^0$ as a function of invariant mass with $\rho^0$ longitudinal momentum $q_z = 250$ MeV is shown at two different values of magnetic field ($0, 0.05$ and $0.10$ GeV$^2$ respectively) for the form factors (a) $\Pi_\alpha$ and (b) $\Pi_\beta$. The contribution due to $\pi \pi$ and proton-proton loops are shown separately.

We now turn our attention to the real part of the self energy at finite temperature under external magnetic field. In Fig. 11 we have shown the thermal contribution to the real part of the self energy as a function of invariant mass with $\rho^0$ longitudinal momentum $q_z = 250$ MeV at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV for two different values of the magnetic field ($eB = 0.05$ and $0.10$ GeV$^2$ respectively). The contributions from the $\pi \pi$ and NN loops are summed up in this figure. We notice that, with the increase in magnetic field, the thermal contribution to the real part of the self energy oscillates about the $eB = 0$ curve. The oscillation frequency and the
oscillation amplitude respectively decreases and increases with the magnetic field.

Next in Fig. 11, the “eB-dependent vacuum” contribution to the real part of the self energy is shown as a function of \( \rho^0 \) invariant mass with longitudinal momentum \( q_z = 250 \text{ MeV} \) for two different values of magnetic field \( (eB = 0.10 \text{ and } 0.20 \text{ GeV}^2 \text{ respectively}).\) Figs. (a) and (b) show the contributions from \( \Pi_\alpha \) and \( \Pi_\beta \) respectively. The contributions due to \( \pi \pi \) and proton-proton loops are shown separately. First of all, we note that at \( eB = 0 \), these term will vanish. With the increase of the magnetic field, the eB-dependent vacuum term also increases and the contribution of \( \Pi_\beta \) is more than \( \Pi_\alpha \).

Having obtained the real and imaginary parts of the self energy, we now proceed to evaluate the in-medium spectral functions of \( \rho^0 \) under external magnetic field. We have from Eq. (106), the complete in-medium interacting propagator is given by

\[
\mathcal{D}^{\mu\nu}(q^0, q_z) = A_\alpha P_1^{\mu\nu} + A_\beta P_2^{\mu\nu} + A_\gamma P_3^{\mu\nu} + A_\delta Q^{\mu\nu} + \xi q^\mu q^\nu
\]  

(142)

where the coefficients are given in Eq. (107)-(111) and the basis tensors are provided in Eqs. (72)-(75). Since we will be considering the special case \( q_\perp = 0 \) for which \( \Pi_\alpha = \Pi_\gamma \) and \( \Pi_\beta = 0 \) as given in Eq. (112), the coefficients in the above equation become

\[
A_\alpha = \left( \frac{1}{q^2 - m_\rho^2 + \Pi_\alpha} \right)
\]  

(143)

\[
A_\beta = \left( \frac{1}{q^2 - m_\rho^2 + \Pi_\beta} \right)
\]  

(144)

\[
A_\gamma = \left( \frac{1}{q^2 - m_\rho^2 + \Pi_\gamma} \right)
\]  

(145)

\[A_\delta = 0
\]  

(146)

\[
\xi = -\frac{1}{q^2 m_\rho^2}
\]  

(147)

so that the complete in-medium interacting propagator is given by

\[
\mathcal{D}^{\mu\nu}(q^0, q_z) = \frac{P_1^{\mu\nu}}{\left(q^2 - m_\rho^2 + \Pi_\alpha\right)} + \frac{P_2^{\mu\nu}}{\left(q^2 - m_\rho^2 + \Pi_\beta\right)} + \frac{P_3^{\mu\nu}}{\left(q^2 - m_\rho^2 + \Pi_\gamma\right)} - \frac{q^\mu q^\nu}{q^2 m_\rho^2}.
\]  

(148)

It is clear from the above equation, that there will be three modes for the propagation of \( \rho^0 \) meson in magnetized medium for vanishing transverse momentum of \( \rho^0 \). Of the three modes, two are found to be degenerate (the first and third in the RHS of above equation) leaving two distinct modes for the propagation of \( \rho^0 \) which we denote as Mode-A and Mode-B respectively.

We now define the spectral function \( S_\rho \) of \( \rho^0 \) for the two distinct modes as the imaginary part of the complete propagator which is obtained from Eq. (148) as

\[
S_\rho^{(A)} = \text{Im} \left[ \frac{-1}{q^2 - m_\rho^2 + \Pi_\alpha} \right] = \frac{\text{Im}\Pi_\alpha}{(q^2 - m_\rho^2 + \text{Re}\Pi_\alpha)^2 + (\text{Im}\Pi_\alpha)^2}
\]  

(149)

and

\[
S_\rho^{(B)} = \text{Im} \left[ \frac{-1}{q^2 - m_\rho^2 + \Pi_\beta} \right] = \frac{\text{Im}\Pi_\beta}{(q^2 - m_\rho^2 + \text{Re}\Pi_\beta)^2 + (\text{Im}\Pi_\beta)^2}.
\]  

(150)

In Fig. 11, the spectral function for the two modes at zero magnetic field is shown as a function of \( \rho^0 \) invariant mass with \( \rho^0 \) longitudinal momentum \( q_z = 250 \text{ MeV} \) at baryon chemical potential \( \mu_B = 300 \text{ MeV} \) for three different values of temperature \( (T = 100, 130 \text{ and } 160 \text{ MeV} \text{ respectively}).\) The vacuum spectral function (which is same for the two modes) is also shown for comparison. We find that, the spectral functions have a nice Breit-Wigner shape around the \( \rho^0 \) mass pole with a width \( \mathcal{O}(150 \text{ MeV}) \) corresponding to the decay of \( \rho^0 \to \pi^+ \pi^- \). With the increase in temperature, the width of the spectral function increases and the peak decreases. Physically, it corresponds to the enhancement of the decay process in the medium implying that the \( \rho^0 \) become more unstable at a high temperature. It is important to note that, for the invariant mass region shown in the plot, the imaginary part of the self energy that enters in the calculation of spectral function is completely due to the Unitary-I cut of \( \pi \pi \) loop. On the other hand, the real part of the self energy that enters in the spectral function calculation has contributions from both the \( \pi \pi \) and NN loops.
We now turn on the external magnetic field and show the spectral function of $\rho^0$ as a function of its invariant mass for the two modes in Fig. 12. The range of the invariant mass axis is taken as 0.5-1.2 GeV which is dominated by the Unitary cut contributions from the $\pi\pi$ loop. In Fig. 12(a), the spectral function with $\rho^0$ longitudinal momentum $q_z = 250$ MeV at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV is shown for three different values of the magnetic field ($eB = 0.10, 0.15$ and $0.20$ GeV$^2$ respectively). It is observed that, with the increase in the magnetic field, the two modes get well separated from each other and the threshold of the spectral function moves towards higher values of invariant mass corresponding to the magnetic field dependent Unitary cut threshold of the imaginary part of the self energy. At sufficiently high values of the magnetic field, the spectral function misses the $\rho^0$ mass pole (770 MeV) so that it looses its Breit-Wigner shape which may be termed as $\rho^0$ “melting” in presence of magnetic field. The critical value of the magnetic field for a given temperature and baryon chemical potential for which the $\rho^0$ will melt is discussed later.

In Fig. 12(b), the spectral function with $\rho^0$ longitudinal momentum $q_z = 250$ MeV at a magnetic field $eB = 0.10$ GeV$^2$ and at a baryon chemical potential $\mu_B = 300$ MeV is shown for three different values of temperature ($T = 100, 130$ and $160$ MeV respectively). In this case, the threshold of the spectral function remains fixed and for both the modes, the spectral function becomes shorter and wider with the increase in temperature with a marginal shift of its peak. The shift of the peak is due to the modification in the real part of the self energy with the change in temperature.

Fig. 12(c) depicts the spectral function with $\rho^0$ longitudinal momentum $q_z = 250$ MeV at a magnetic field $eB = 0.10$ GeV$^2$ and at a temperature $T = 160$ MeV for three different values of the baryon chemical potential ($\mu_B = 200, 300$ and $400$ MeV respectively). Analogous to the previous case, the threshold of the spectral function remains fixed for both the modes. Since the baryon chemical potential only affects the real part of the self energy in the given kinematic region, the peak of the spectral function changes its position (keeping the width almost same) with the change in baryon chemical potential. It can be noticed, that in contrast to Fig. 12(b), the peak position of the spectral function is more sensitive to $\mu_B$ as compared to the temperature which is due to the dominant contribution coming from NN loop.

In Fig. 12(d), the spectral function at a magnetic field $eB = 0.10$ GeV$^2$ and at a temperature $T = 130$ MeV with baryon chemical potential $\mu_B = 300$ MeV is shown for two different values of $\rho^0$ longitudinal momentum ($q_z = 0$ and $500$ MeV). In this case, the threshold of the spectral function remains same and the height of the spectral function increases with the increase of the longitudinal momentum.

We have already mentioned that, a non-trivial Landau cut in the physical kinematic region would appear in presence of the external magnetic field. In our case, the non-zero contribution to the Landau cut comes only from the form factor Im$\Pi_q$ which is reflected in the the spectral function of Mode-(A). In Fig. 13 the spectral function as a function of $\rho^0$ invariant mass with $\rho^0$ longitudinal momentum $q_z = 250$ MeV is shown in the low invariant mass region which is dominated by the Landau cut contribution. It can be observed that the magnitude of the spectral function in this region is much lower as compared to the Unitary cut regions. Fig. 13(a) shows the spectral function at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV for three different values of magnetic field ($eB = 0.10$, etc.)
FIG. 12. The in-medium spectral functions of $\rho^0$ as a function of invariant mass is shown for different modes (a) at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV with $\rho^0$ longitudinal momentum $q_z = 250$ MeV for three different values of magnetic field ($eB = 0.10, 0.15$ and $0.20$ GeV$^2$) respectively (b) at magnetic field $eB = 0.10$ GeV$^2$ and at baryon chemical potential $\mu_B = 300$ MeV with $\rho^0$ longitudinal momentum $q_z = 250$ MeV for three different values of temperature ($T = 100, 130$ and $160$ MeV respectively) (c) at magnetic field $eB = 0.10$ GeV$^2$ and at temperature $T = 160$ MeV with $\rho^0$ longitudinal momentum $q_z = 250$ MeV for three different values of baryon chemical potential ($\mu_B = 200, 300$ and $400$ MeV respectively) and (d) at magnetic field $eB = 0.10$ GeV$^2$ and at temperature $T = 130$ MeV with baryon chemical potential $\mu_B = 300$ MeV for two different values of $\rho^0$ longitudinal momentum ($q_z = 0$ and $500$ MeV respectively). The vacuum spectral function is also shown for comparison.

0.15 and 0.20 GeV$^2$ respectively). As can be seen in the graph, the threshold of the Landau cut moves towards the higher values of invariant mass with the increase in magnetic field as a consequence of similar behaviour of the Landau cut contribution to the imaginary part as shown in Fig. 8. Also the height of the spectral function is enhanced with the increase in $eB$. Fig. 12(b) shows the corresponding plots of spectral function at magnetic field $eB = 0.10$ GeV$^2$ for four different combinations of temperature and baryon chemical potential ($(T = 100$ MeV, $\mu_B = 300$ MeV), $(T = 130$ MeV, $\mu_B = 300$ MeV), $(T = 160$ MeV, $\mu_B = 300$ MeV) and $(T = 160$ MeV, $\mu_B = 400$ MeV) respectively). As can be seen in the graph, the height of the spectral function increases with the increase in temperature and density owing to an enhancement of the corresponding scattering processes in presence of external magnetic field.

We now proceed to obtain the effective mass and dispersion relation of the $\rho^0$ in a magnetized medium. They follow from the pole of the complete $\rho^0$ propagator given in Eq. (148) which are obtained by solving the following transcendental equations

$$\omega^2 - q_z^2 - m^2_\rho + \text{Re}\Pi_\omega(q^0 = \omega, q_z, eB, T, \mu_B) = 0$$

$$\omega^2 - q_z^2 - m^2_\rho + \text{Re}\Pi_\beta(q^0 = \omega, q_z, eB, T, \mu_B) = 0$$

whose numerical solutions $\omega = \omega(q_z, eB, T, \mu_B)$ represent the dispersion relations for the Mode-(A) and (B) corre-
sponding to $\rho^0$ propagation in the magnetized medium. The effective mass $m^*_{\rho}$ of $\rho^0$ is obtained from the dispersion relation by setting $q_z = 0$ i.e. $m^*_{\rho}(eB,T,\mu_B) = \omega(q_z = 0, eB, T, \mu_B)$.

Fig. 13(a) depicts the variation of $m^*_{\rho}/m_{\rho}$ as a function of magnetic field at a temperature $T = 130$ MeV and at a baryon chemical potential $\mu_B = 300$ MeV. The effective mass for the two modes starts from the same value around $eB = 0$ and with the increase in magnetic field, they get separated. For both the modes, the effective $\rho^0$ mass decreases with the increase in the magnetic field which is due to the strong positive contribution coming from the dominating $eB$-dependent vacuum part. The effect of magnetic field is found to be more in Mode-(B) as compared to Mode-(A). At a magnetic field value $eB = 0.20$ GeV$^2$, the effective $\rho^0$ mass in Mode-(A) decreases by about 2% whereas for the Mode-(B) it decreases by about 10%. Fig. 13(b) depicts the corresponding variation of effective mass with temperature at a magnetic field $eB = 0.10$ GeV$^2$ and at a baryon chemical potential $\mu_B = 300$ MeV. We find that, for both the modes effective mass of $\rho^0$ get enhanced by a small amount with the increase in temperature. Even at $T = 160$ MeV the change in effective mass is less than 2%. In Fig. 13(c), the variation of effective $\rho^0$ mass is shown as a function of baryon chemical potential at a magnetic field $eB = 0.10$ GeV$^2$ and at a temperature $T = 130$ MeV. In this case also, we observe an enhancement of the effective mass for both the modes with the increase in baryon density. Though the effect of $\mu_B$ on effective mass is more at a higher value of $\mu_B$ the change in the effective mass remains less than 2% even at $\mu_B = 500$ MeV.

Next, we present the dispersion curves of $\rho^0$ propagation in magnetized medium for both the modes in Fig. 15. We have plotted the energy $\omega$ of the $\rho^0$ scaled with the inverse of the vacuum rho mass $m_{\rho} = 770$ MeV as a function of the longitudinal momentum of $\rho^0$. Fig. 15(a) depicts the dispersion curves at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV for two different values of magnetic field ($eB = 0.10$ and 0.20 GeV$^2$ respectively). Fig. 15(b) shows the same at a magnetic field $eB = 0.10$ GeV$^2$ and baryon chemical potential $\mu_B = 300$ MeV for two different temperatures ($T = 100$ and 160 MeV respectively). Finally, Fig. 15(c) shows the corresponding graphs at a magnetic field $eB = 0.10$ GeV$^2$ and at a temperature $T = 130$ MeV for two different values of baryon chemical potential ($\mu_B = 200$ and 400 MeV respectively). In all the cases, the dispersion curves are well separated from each other at lower transverse momentum. With the increase in $q_z$, the loop correction becomes subleading with respect to the kinetic energy of $\rho^0$ and thus, it approaches to a light-like dispersion.

Finally we calculate the decay width of $\rho^0$ for the decay into charged pions which is defined for the two modes as

$$\Gamma^{(A)}(eB,T,\mu_B) = \frac{\text{Im}\Pi_\omega(q^0 = m^*_{\rho}, q_z = 0, eB, T, \mu_B)}{m^*_{\rho}(eB,T,\mu_B)}$$

(153)

$$\Gamma^{(B)}(eB,T,\mu_B) = \frac{\text{Im}\Pi_\omega(q^0 = m^*_{\rho}, q_z = 0, eB, T, \mu_B)}{m^*_{\rho}(eB,T,\mu_B)}$$

(154)

In Fig. 16 the variation of the decay width $\Gamma$ of $\rho^0$ scaled with inverse of its vacuum width ($\Gamma_0 = 156$ MeV) for the
FIG. 14. The ratio of effective mass of $\rho^0$ to its vacuum mass for different modes (a) as a function of magnetic field at temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV (b) as a function of temperature at magnetic field $eB = 0.10$ GeV$^2$ and at baryon chemical potential $\mu_B = 300$ MeV and (c) as a function of baryon chemical potential at temperature $T = 130$ MeV and at magnetic field $eB = 0.10$ GeV$^2$. The green dash-dotted curve in (a) corresponds to the Unitary cut threshold for decay of $\rho^0 \rightarrow \pi^+ \pi^-$. Here $m_\rho = 770$ MeV.

two modes is shown as a function of magnetic field. Note that the vacuum decay width is obtained from the imaginary part of the vacuum self energy as

$$
\Gamma_0 = \frac{\text{Im}\Pi_{\text{pure-vac}}(q^0 = m_\rho, \vec{q} = \vec{0})}{m_\rho} = 156 \text{ MeV} .
$$

Results are presented for two different combinations of temperature and baryon chemical potential ($(T = 130$ MeV, $\mu_B = 300$ MeV) and $(T = 160$ MeV, $\mu_B = 400$ MeV) respectively). Because of the presence of threshold singularity in Im$\Pi_\alpha$, $\Gamma^{(A)}$ also suffers from the presence of threshold singularity for which it needs to be coarse grained. However, Im$\Pi_{\beta}$ and hence $\Gamma^{(B)}$ is finite and free from the singularities. As can be seen from the figure, the ratio $\Gamma/\Gamma_0$ starts from a value greater than unity near $eB = 0$ which is due to the enhancement of the decay width over its vacuum value due to the effect of finite temperature and density. Also for a particular value of magnetic field, larger decay width is observed at higher temperature and density. Near $eB = 0$, the two modes have almost the same decay widths which begin to differ from each other with the increase in the magnetic field. An oscillatory behaviour of the decay width can be clearly seen throughout the magnetic field range. One should also notice that, for both the modes, the oscillation amplitude increases whereas oscillation frequency decreases with $eB$. Finally at a critical value of the magnetic field, the decay width becomes zero. This is because of fact that, the $eB$-dependent Unitary cut threshold for the $\pi\pi$ loop has to satisfy

$$
m^*_\rho(eB) > 2\sqrt{m_\pi^2 + eB}
$$

(156)
FIG. 15. The dispersion relations of $\rho_0$ for different modes: (a) At temperature $T = 130$ MeV and at baryon chemical potential $\mu_B = 300$ MeV for two different values of magnetic field ($eB = 0.10$ and 0.20 GeV$^2$ respectively), (b) at magnetic field $eB = 0.10$ GeV$^2$ and at baryon chemical potential $\mu_B = 300$ MeV for two different temperatures ($T = 100$ and 160 MeV respectively) and (c) at magnetic field $eB = 0.10$ GeV$^2$ and at temperature $T = 130$ MeV for two different values of baryon chemical potential ($\mu_B = 200$ and 400 MeV respectively).

for a kinematically favorable decay of $\rho^0 \to \pi^+\pi^-$. But, with the increase in magnetic field, the RHS of the above equation increases, whereas $m^*_\rho$ in the LHS decreases so that at some critical value of magnetic field, the above inequality is violated and the decay width becomes zero. Physically it means, that $\rho^0$ becomes stable against the decay into $\pi^+\pi^-$ pair. This critical value of the field may be considered as the critical value of the magnetic field required for the “melting” of the spectral function of $\rho^0$.

In order to calculate the critical value of the magnetic field $eB_c$ for a given temperature $T$ and baryon chemical potential $\mu_B$, we need to solve the transcendental equation

$$m^*_\rho(eB_c, T, \mu_B) = 2\sqrt{m^2_\pi + eB_c}.$$ (157)

The green dash-dotted curve in Fig. 14(a) corresponds to $m^*_\rho/m_\rho = 2\sqrt{m^2_\pi + eB}$ so that, the intersection of this curve with the $m^*_\rho = m^*_\rho(eB)$ represents the solution of the above equation. In Fig. 17 we show the variation of the critical magnetic field $eB_c$ for the two decay modes. Fig. 17(a) depicts $eB_c$ as a function of temperature for two different values of baryon chemical potential ($\mu_B = 50$ and 200 MeV) whereas Fig. 17(b) shows the corresponding variation with baryon chemical potential at two different values of temperature ($T = 100$ and 160 MeV). Although, with fixed temperature, the variation with respect to $\mu_B$ shows monotonically increasing trend, both the plots suggests non-monotonic variations of the critical magnetic field with respect to the temperature. More specifically, there exists a maximum value of chemical potential (see Fig. 17(b)) below which the critical field decreases with the temperature there by requiring relatively weaker magnetic field to completely stop the particular decay channel. However, for even larger values of $\mu_B$, a significant increase with temperature can be observed for both of the decay modes.
FIG. 16. The ratio of the decay width of $\rho^0$ to its vacuum width as a function of magnetic field for different modes with two different combinations of temperature and baryon chemical potential (($T = 130$ MeV, $\mu_B = 300$ MeV) and ($T = 160$ MeV, $\mu_B = 400$ MeV) respectively). Here $\Gamma_0 = 156$ MeV.

FIG. 17. The variation of the critical value of magnetic field for stopping the decay of $\rho^0$ into $\pi^+\pi^-$ pair for different modes as a function of (a) temperature at two different values of baryon chemical potential ($\mu_B = 50$ and 200 MeV respectively) and (b) baryon chemical potential at two different values of temperature ($T = 100$ and 160 MeV respectively).

IX. SUMMARY AND CONCLUSIONS

In this work, the spectral properties of the neutral rho meson is studied at finite temperature and density in a constant external magnetic field using the real time formalism of finite temperature field theory. The effective $\rho\pi\pi$ and $\rho NN$ interactions are considered for the evaluation of the one loop self energy of $\rho^0$. Accordingly, the magnetically modified in-medium propagators for pions and protons are used which contain infinite sum over the Landau levels implying no constraint on the strength of the external magnetic field. From the self-energy, the $eB$-dependent vacuum part is extracted by means of dimensional regularization in which the ultraviolet divergence corresponding to the pure vacuum self energy is isolated as the pole of gamma and Hurwitz zeta functions. It is shown that the external magnetic field does not create additional divergences so that the vacuum counter terms required in absence of the background field remain sufficient to renormalize the theory at non zero magnetic field.

The general Lorentz structure for the in-medium massive vector boson self energy in presence of external magnetic field has been constructed with four linearly independent basis tensors out of which three form a mutually orthogonal set. Thus, the extraction of the form factors from the self energy becomes considerably simple. Moreover, it is shown that with vanishing perpendicular momentum of the external particle, one can arrive at new set of constraint
relations among the form factors which essentially leave only two form factors to be determined from the self energy. As a consistency check, the numerical $B \to 0$ limit of the real as well as imaginary parts of the form factors are shown to reproduce the zero field results. Solving the Dyson-Schwinger equation with the one loop self energy, the complete interacting $\rho^0$ propagator is obtained. Consequently, two distinct modes are observed in the study of the effective mass, dispersion relations and the spectral function of $\rho^0$ where one of the modes (Mode-A) possesses two fold degeneracy. It is known [31, 38] that non trivial Landau cuts appear in presence of external magnetic field along with finite temperature even if the loop particles are of equal mass which is completely a magnetic field effect. However, in contrast to Mode-A, the non-trivial Landau cut is found to be absent in case of Mode-B. Also, sharper decrease in the effective mass is observed for the later which essentially stems from the dominant $eB$-dependent vacuum contribution in the real part of the corresponding form factor.

Finally, the decay width for $\rho^0 \to \pi^+ \pi^-$ channel is obtained for the two distinct modes and is found to become zero at certain critical values of magnetic field depending upon the temperature and baryon chemical potential. The corresponding variation of the critical field with these external parameters shows increasing trend for large baryonic chemical potential. However, it is observed that, both the distinct modes possess a maximum value of $\mu_B$ below which the temperature dependence gets reversed. Especially, at a given temperature (say $T = 160$ MeV), $eB_c$ attains the lowest values (123 MeV$^2$ for Mode-A and 116 MeV$^2$ for Mode-B) in case of zero chemical potential. In Ref.[27], charged rho meson condensation has been studied at finite temperature and density. For charged rho mesons, the critical field for which the vector meson mass vanishes is observed to lie in the range of 0.2-0.6 GeV$^2$ at zero density with temperature in the range 0.2-0.5GeV. However, in case of $\rho^0$, the absence of the trivial Landau shift in the energy eigenvalue results in much slower decrease in the effective mass. As a consequence, unrealistically high magnetic field values are required to observe neutral rho condensation in presence of temperature and medium (see Fig.14). In this scenario, the suppression in the $\rho^0 \to \pi^+ \pi^-$ channel can serve as an important observable. It should be mentioned here that magnetic modification of rho meson properties studied in this work deals with effective hadronic interactions. Thus, the observable modification can only occur if the initial burst of magnetic field survives up to hadronization retaining an appreciable field strength. The requirements can be ensured only after a comprehensive knowledge of the space time dependence of the produced magnetic field is obtained which definitely demands a significant amount of research work in future.

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Appendix A: Useful Identities

We have the following list of $d$-dimensional integrals in Minkowski space $\mathbb{R}^d$:

\[
\begin{align*}
\int \frac{d^d k}{(2\pi)^d (k^2 - \Delta)^n} & = \frac{i \, (-1)^n \, \Gamma(n - d/2)}{\Gamma(n)} \left( \frac{1}{\Delta} \right)^{n-d/2} \\
\int \frac{d^d k}{(2\pi)^d (k^2 - \Delta)^n} & = \frac{i \, (-1)^n \, \Gamma(n - 1 - d/2)}{\Gamma(n)} \left( \frac{1}{\Delta} \right)^{n-1-d/2} \\
\int \frac{d^d k}{(2\pi)^d (k^2 - \Delta)^n} & = \frac{i \, (-1)^n \, \Gamma(n - d/2)}{\Gamma(n)} \left( \frac{1}{\Delta} \right)^{n-d/2}.
\end{align*}
\]

Using the orthogonality properties of the generalized Laguerre polynomials, one can derive the following identities

\[
\begin{align*}
\int \frac{d^d k}{(2\pi)^d} e^{-2\alpha_k} L_i(2\alpha_k) L_n(2\alpha_k) & = -g_{\mu\nu} (eb)^2 / 32\pi \left[ (2n+1)\delta_i^n - n\delta_i^{n+1} - n\delta_i^{n-1} \right] \\
\int \frac{d^2 k}{(2\pi)^2} e^{-2\alpha_k} L_i(2\alpha_k) L_n(2\alpha_k) & = eB / 8\pi \delta_i^n \\
\int \frac{d^2 k}{(2\pi)^2} e^{-2\alpha_k} L_i^1(2\alpha_k) L_n(2\alpha_k) k^{\mu} k^{\nu} & = -g_{\mu\nu} (eb)^2 / 32\pi n\delta_i^{n-1} \\
\int \frac{d^2 k}{(2\pi)^2} e^{-2\alpha_k} L_i^1(2\alpha_k) L_n^1(2\alpha_k) k^{\mu} k^{\nu} & = -(eb)^2 / 16\pi n\delta_i^{n-1}
\end{align*}
\]

where, $\alpha_k = -k^2 / eb$.

Appendix B: Calculation of Vacuum Self Energy

In order to evaluate the momentum integrals in Eqs. (3) and (4), they are rewritten as

\[
\begin{align*}
(\Pi_\pi^{\mu\nu})_{\text{pure-vac}}(q) & = i \int \frac{d^d k}{(2\pi)^d} \frac{N_\pi^{\mu\nu}(q, k)}{(k^2 - m_\pi^2 + i\epsilon)(\epsilon^\mu k^\nu + \epsilon^\nu k^\mu)} \\
(\Pi_N^{\mu\nu})_{\text{pure-vac}}(q) & = i \int \frac{d^d k}{(2\pi)^d} \frac{N_N^{\mu\nu}(q, k)}{(k^2 - m_N^2 + i\epsilon)(\epsilon^\mu k^\nu + \epsilon^\nu k^\mu)}
\end{align*}
\]

where, $N_N^{\mu\nu}(q,k)$ contains the trace over Dirac matrices:

\[
N_N^{\mu\nu}(q,k) = -2g_{\rho\sigma N} \text{Tr} \left[ \Gamma^\nu(q)(\not{\xi} + m_N)\Gamma^\mu(-q)(\not{\xi} + m_N) \right] \\
= -8g_{\rho\sigma N} \left[ (m_N^2 - k^2 - k \cdot q)g^{\mu\nu} + 2k^\mu k^\nu + (q^\mu k^\nu + q^\nu k^\mu) + \kappa_\rho (q^2 g^{\mu\nu} - q^\mu q^\nu) \right. \\
+ \left. \frac{\kappa_\rho^2}{4m_N} \left( (m_N^2 + k^2 - k \cdot q)(q^2 g^{\mu\nu} - q^\mu q^\nu) - 2q^\nu k^\mu + 2q^\mu k^\nu - 2(k \cdot q)^2 g^{\mu\nu} + 2(k \cdot q)(q^\mu k^\nu + q^\nu k^\mu) \right) \right]
\]

Applying standard Feynman paramORIZATION, the denominators of Eqs. (31) and (32) are combined to get,

\[
\begin{align*}
(\Pi_\pi^{\mu\nu})_{\text{pure-vac}}(q) & = i \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \Delta^{2-d/2}_\pi \frac{N_\pi^{\mu\nu}(q, k)}{([k + x q]^2 - \Delta_\pi^2)^2} \\
(\Pi_N^{\mu\nu})_{\text{pure-vac}}(q) & = i \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \Delta^{2-d/2}_N \frac{N_N^{\mu\nu}(q, k)}{([k + x q]^2 - \Delta_N^2)^2}
\end{align*}
\]

where,

\[
\begin{align*}
\Delta_\pi^2 & = m_\pi^2 - x(1-x)q^2 - i\epsilon \\
\Delta_N^2 & = m_N^2 - x(1-x)q^2 - i\epsilon
\end{align*}
\]
and the space-time dimension has been changed from 4 to $d$ in order to work with the dimensional regularization so that the additional scale parameters $\Lambda_\pi$ and $\Lambda_N$ of dimension GeV$^2$ have been introduced to keep the overall dimension of the self energy same. It is now straightforward to perform the momentum integrals of the above equations after a momentum shift $k \rightarrow (k - xq)$ using the identities provided in Appendix A so that, the vacuum self energies becomes

$$
(\Pi^\mu\nu_{\pi})_{\text{pure-vac}} (q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left( \frac{g_{\rho\pi}^2 q^2}{32\pi^2} \right) \int_0^1 dx \Gamma(\varepsilon - 1) \left( \frac{\Delta_\pi}{4\pi\Lambda_\pi} \right)^{-\varepsilon} \bigg|_{\varepsilon \to 0}
$$

(B8)

$$
(\Pi^\mu\nu_N)_{\text{pure-vac}} (q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left( \frac{g_{\rho N N}^2}{2\pi^2} \right) \int_0^1 dx \left\{ 2x(1 - x) + \kappa_\pi + \frac{\kappa_\pi^2}{2} \right\} \Gamma(\varepsilon) + \left( \frac{\Delta_N}{4\pi\Lambda_N} \right)^{-\varepsilon} \bigg|_{\varepsilon \to 0}
$$

(B9)

where $\varepsilon = (2 - d)/2$.

Expanding the above equations about $\varepsilon = 0$, we get

$$
(\Pi^\mu\nu_\pi)_{\text{pure-vac}} (q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left( \frac{-g_{\rho\pi}^2 q^2}{32\pi^2} \right) \int_0^1 dx \Delta_\pi \left[ \frac{1}{\varepsilon} - \gamma_E + 1 - \ln \left( \frac{\Delta_\pi}{4\pi\Lambda_\pi} \right) \right] \bigg|_{\varepsilon \to 0}
$$

(B10)

$$
(\Pi^\mu\nu_N)_{\text{pure-vac}} (q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left( \frac{g_{\rho N N}^2}{2\pi^2} \right) \int_0^1 dx \left\{ 2x(1 - x) + \kappa_\pi + \frac{\kappa_\pi^2}{2} - \frac{\kappa_\pi^2}{2} \right\} \Delta_N \bigg|_{\varepsilon \to 0}
$$

(B11)

where, $\gamma_E$ is the Euler-Mascheroni constant.

**Appendix C: Calculation of eB-dependent Vacuum Contribution for $\pi\pi$ Loop**

In this appendix, we sketch how to obtain Eqs. (46) and (48). We rewrite Eq. (44) as

$$
(\Pi^\mu\nu_\pi)_{\text{vac}} (q, eB) = i \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \int \frac{d^2k_l}{(2\pi)^2} \int \frac{d^2k_{\perp}}{(2\pi)^2} \frac{N^\mu\nu_{\pi, nl}(q, k)}{(k^2 - m_l^2 + i\epsilon)(q + k)^2 - m_n^2 + i\epsilon)}
$$

(C1)

For the simplicity in analytic calculations, we take the transverse momentum of the $e^0$ to be zero i.e. $q_{\perp} = 0$. This implies that the $d^2k_{\perp}$ integration can be performed analytically using the orthogonality of the Laguerre polynomial details of which can be obtained from Appendix B so that the self energy becomes

$$
(\Pi^\mu\nu_\pi)_{\text{vac}} (q, eB) = i \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \int \frac{d^2k_l}{(2\pi)^2} \frac{N^\mu\nu_{\pi, nl}(q, k)}{(k^2 - m_l^2 + i\epsilon)(q + k)^2 - m_n^2 + i\epsilon)}
$$

(C2)

where, $N^\mu\nu_{\pi, nl}(q, k)$ is given in Eq. (55). Next, we use the standard Feynman parametization technique to combine the denominators of Eq. (52) and change the reduced space-time dimension from 2 to $d$ in order to apply the dimensional regularization for which a scale parameter $\Lambda_\pi$ of dimension GeV$^2$ has to be introduced in order to keep the overall dimension of the self energy same. This leads to

$$
(\Pi^\mu\nu_\pi)_{\text{vac}} (q, eB) = i \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \int dx \int \frac{d^d k_{\parallel}}{(2\pi)^d} \frac{N^\mu\nu_{\pi, nl}(q, k)}{[k_{\parallel} + xq_{\parallel}]^2 - \Delta_{nl}^2)}
$$

(C3)

where,

$$
\Delta_{nl}^\pi = \Delta_\pi (q_{\parallel} = 0) + 2eB \{ l + 1 - x(l - n) \}
$$

(C4)

with $\Delta_\pi$ is defined in Eq. (30). It is now trivial to perform the $d^d k_{\parallel}$ integration after a shift of momentum $k_{\parallel} \rightarrow (k_{\parallel} - xq_{\parallel})$ using the identities provided in Appendix A so that the self energy becomes

$$
(\Pi^\mu\nu_\pi)_{\text{vac}} (q, eB) = \frac{-g_{\rho\pi}^2 q^2}{16\pi^2} eB \int_0^1 dx \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{n+1} (4\pi\Lambda_\pi)^{\varepsilon} \left[ -(q^2 g^{\mu\nu} - q^\mu q^\nu) \delta_l^{nl} \Gamma(\varepsilon) (\Delta_{nl}^\pi)^{-\varepsilon} \right. \left. \left( q^2 g^{\mu\nu} - q^\mu q^\nu \right)^{\varepsilon} \right]
$$

(C5)
where \( \varepsilon = (1 - d/2) \) and the presence of Kronecker delta functions in Eq. (125) has made the double sum into a single one or in other words the sum over index \( l \) runs only from \( (n-1) \) to \( (n+1) \). The infinite sum in the above equations can be expressed in terms of Hurwitz zeta function so that we get after some simplifications

\[
(\Pi_{\pi}^{\mu \nu})_{\text{vac}}(q_\parallel, eB) = -\frac{g_{\mu \nu}}{16 \pi^2} eB \int_0^1 dx \left( \frac{4 \pi \Lambda_\pi}{2 eB} \right)^\varepsilon \left[ -(q_\parallel^{\mu} g_\parallel^{\nu} - q_\parallel^{\nu} g_\parallel^{\mu}) \Gamma(\varepsilon, \varepsilon, \pi + \frac{1}{2}) - \frac{q_\parallel^2}{2} g_\parallel^{\mu \nu} \Gamma(\varepsilon + 1) \times \left\{ \zeta(\varepsilon, \pi + \frac{1}{2}) + \zeta(\varepsilon, \pi + x + \frac{1}{2}) - \zeta(\varepsilon + 1, \pi + \frac{1}{2}) - \zeta(\varepsilon + 1, \pi + x + \frac{1}{2}) \right\} \right] \right|_{\varepsilon \to 0}
\]

where, \( \zeta = \frac{\Delta_\pi(q_\perp=0)}{2eB} \). Expanding the above equation about \( \varepsilon = 0 \), we get,

\[
(\Pi_{\pi}^{\mu \nu})_{\text{vac}}(q_\parallel, eB) = -\frac{g_{\mu \nu}}{32 \pi^2} \int_0^1 dx \left\{ \left( \frac{1}{\varepsilon} - \gamma_\pi + \ln \left( \frac{4 \pi \Lambda_\pi}{2 eB} \right) \right) \Delta_\pi(q_\parallel = 0) (q_\parallel^2 g_\mu^{\nu} - q_\parallel^2 g_\nu^{\mu}) 
- (q_\parallel^2 g_\mu^{\nu} - q_\parallel^2 g_\nu^{\mu}) 2 eB \left\{ \ln \Gamma(\pi, \pi + \frac{1}{2}) - \ln \sqrt{2 \pi} \right\} 
+ q_\parallel^2 g_\mu^{\nu} \left\{ \Delta_\pi(q_\parallel = 0) + \frac{eB}{2} - \frac{1}{2} \Delta_\pi(q_\parallel = 0) \left\{ \psi \left( z_\parallel + \frac{1}{2} \right) + \psi \left( z_\parallel + x + \frac{1}{2} \right) \right\} \right\} \right\} \right|_{\varepsilon \to 0}
\]

which is finite and independent of scale.

Appendix D: Calculation of eB-dependent Vacuum Contribution for proton-proton Loop

In this appendix, we sketch how to obtain Eqs. (47) and (49). We rewrite Eq. (45) as

\[
(\Pi_\pi^{\mu \nu})_{\text{vac}}(q, eB) = i \sum_{l=0}^\infty \sum_{n=0}^\infty \int \frac{d^2 k_\parallel}{(2\pi)^2} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{N_{p,nl}^{\mu \nu}(q, k)}{(k_\parallel^2 - M_\pi^2 + i\epsilon)(q_\parallel^2 + k_\parallel^2 - M_\pi^2 + i\epsilon)}
\]

where, \( N_{p,nl}^{\mu \nu}(q, k) \) is given in Eq. (122). For the simplicity in analytic calculations, we take the transverse momentum of the \( p^0 \) to be zero i.e. \( q_\perp = 0 \). This implies that the \( d^2 k_\perp \) integration can be performed analytically using the orthogonality of the Laguerre polynomial details of which can be obtained from Appendix [4] so that the self energy becomes

\[
(\Pi_\pi^{\mu \nu})_{\text{vac}}(q_\parallel, eB) = i \sum_{l=0}^\infty \sum_{n=0}^\infty \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{\Delta_\pi^{\mu \nu}(q_\parallel, k_\parallel)}{(k_\parallel^2 - M_\pi^2 + i\epsilon)(q_\parallel^2 + k_\parallel^2 - M_\pi^2 + i\epsilon)}
\]

where, \( \Delta_\pi^{\mu \nu}(q_\parallel, k_\parallel) \) can be read off from Eq. (28). Next, we use the standard Feynman parametran technique to combine the denominators of Eq. (122) and change the reduced space-time dimension from 2 to \( d \) in order to apply the dimensional regularization for which a scale parameter \( \Lambda_N \) of dimension GeV\(^2\) has to be introduced in order to keep the overall dimension of the self energy same. This leads to

\[
(\Pi_\pi^{\mu \nu})_{\text{vac}}(q_\parallel, eB) = i \sum_{l=0}^\infty \sum_{n=0}^\infty \int_0^1 dx \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{\Delta_\pi^{\mu \nu}(q_\parallel, k_\parallel)}{[(k_\parallel + xq_\parallel)^2 - \Delta_\pi^{\mu \nu}(k_\parallel)]^{d-2}}
\]
where,

$$\Delta^p_{nl} = \Delta_N(q_{\perp} = 0) + 2eB \{l - x(l - n)\} \quad (D4)$$

with $\Delta_N$ is defined in Eq. (B7). It is now trivial to perform the $d^d k_{\parallel}$ integration after a shift of momentum $k_{\parallel} \rightarrow (k_{\parallel} - xq_{\parallel})$ using the identities provided in Appendix A so that the self energy becomes

$$(\Pi^\mu_\nu)_{\text{vac}} (q_{\parallel}, eB) = \frac{g^2_{\pi N}}{4\pi^2} eB \int_0^1 dx \sum_{n=0}^\infty \sum_{l=(n-1)}^{(n+1)} (1)^{n+l} \left(4\pi\Lambda_\pi\right) \varepsilon \left[4eBg_{\parallel\nu} n\delta_{l-1}^{n-1} + \left\{(m_N^2 + x(1-x)q_{\parallel}^2)g_{\parallel\mu}^\nu - 2x(1-x)q_{\parallel}^\mu q_{\parallel}^\nu \right\} \left(\delta_{l-1}^{n-1} + \delta_{l}^{n} \right) - (m_N^2 + x(1-x)q_{\parallel}^2)g_{\parallel\mu}^\nu \left(\delta_{l-1}^{n-1} + \delta_{l}^{n} \right) \right] \times \Gamma(\varepsilon + 1) (\Delta^p_{nl})^{-\varepsilon} - 1 - g_{\parallel\mu}^\nu (\delta_{l-1}^{n-1} + \delta_{l}^{n}) \varepsilon + g_{\parallel\mu}^\nu \left(\delta_{l-1}^{n-1} + \delta_{l}^{n} \right) (-\varepsilon + 1) \Gamma(\varepsilon) (\Delta^p_{nl})^{-\varepsilon} + \kappa_\rho \left\{(q_{\parallel}^2 g_{\parallel\mu}^\nu - q_{\parallel}^\mu q_{\parallel}^\nu) \left(\delta_{l-1}^{n-1} + \delta_{l}^{n} \right) - q_{\parallel}^2 g_{\parallel\mu}^\nu \left(\delta_{l-1}^{n-1} + \delta_{l}^{n} \right) \right\} \Gamma(\varepsilon + 1) (\Delta^p_{nl})^{-\varepsilon} - 1 \left. \varepsilon \rightarrow 0 \right\} \varepsilon = (1 - d/2) \text{ and the presence of Kronecker delta functions in Eq. (E8) has made the double sum into a single one or in other words the sum over index } l \text{ runs only from } (n-1) \text{ to } (n+1). \text{ The infinite sum in the above equations can be expressed in terms of Hurwitz zeta function so that we get after some simplifications}

$$(\Pi^\mu_\nu)_{\text{vac}} (q_{\parallel}, eB) = \frac{g^2_{\pi N}}{4\pi^2} eB \int_0^1 \left(\frac{4\pi\Lambda_N}{2eB}\right)^\varepsilon \left[2eBg_{\parallel\mu}^\nu \{\zeta(\varepsilon, z_N) - zN\zeta(\varepsilon + 1, z_N)\} \right. + \left\{(m_N^2 + x(1-x)q_{\parallel}^2)g_{\parallel\mu}^\nu - 2x(1-x)q_{\parallel}^\mu q_{\parallel}^\nu \right\} \left(\zeta(\varepsilon + 1, z_N) - \frac{1}{2}z_N^{-\varepsilon - 1} \right) \right] \left. \varepsilon \rightarrow 0 \right\} \varepsilon \rightarrow 0.$$
where, \( z_N = \frac{\Delta_N(q_z=0)}{2eB} \). Expanding the above equation about \( \varepsilon = 0 \), we get,

\[
(\Pi^\mu_\parallel)_\text{vac}(q_\parallel, eB) = \frac{g^2_{2N}}{4\pi^2} \int_0^1 dx \left[ \left\{ \frac{1}{\varepsilon} - \gamma_B + \ln \left( \frac{4\pi\Lambda_N}{2eB} \right) \right\} \left\{ 2x(1-x) + \kappa_\rho + \frac{\kappa_\rho^2}{2} - \frac{\kappa_\rho^2}{4m_N^2} \Delta_N(q_\perp = 0) \right\} (q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu) - 2x(1-x) \left( \psi(z_N) + \frac{1}{2z_N} \right) (q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu) + 2eBq_\parallel^\mu q_\parallel^\nu \left\{ \left( z_N - \frac{m^2_N}{eB} \right) \psi(z_N + x) + z_N \right\} + \ln \Gamma(z + x) - \ln \sqrt{2\pi} \right] - \kappa_\rho \left\{ (q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu) \left( \psi(z_N) + 1 \right) + q_\parallel^2 g^\mu\nu \psi(z + x) \right\} + \left[ \frac{m^2_N}{eB} \left( \psi(z_N) + \frac{1}{2z_N} \right) + \frac{1}{2} \ln(z_N) + \ln \Gamma(z_N) - \ln \sqrt{2\pi} \right] \frac{\kappa_\rho^2}{4m_N^2} (2eB) \left( q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu \right) \psi(z_N + x) + \Delta_N(q_\perp = 0) \bigg|_{\varepsilon \to 0}.
\] (D7)

It is now trivial to check that, in the limit \( eB \to 0 \), the above equation exactly boils down to the \( \frac{1}{2} \) times pure vacuum contribution given in Eq. (10). Thus extracting the pure vacuum contribution from the above equation we get,

\[
(\Pi^\mu_\parallel)_\text{vac}(q_\parallel, eB) = \frac{1}{2} (\Pi^\mu_\parallel_\text{pure-vac}(q_\parallel)) + (\Pi^\mu_\parallel)_\text{eB-vac}(q_\parallel, eB)
\] (D8)

where,

\[
(\Pi^\mu_\parallel)_\text{eB-vac}(q_\parallel, eB) = \frac{g^2_{2N}}{4\pi^2} \int_0^1 dx \left[ \ln \left( \frac{\Delta_N(q_\perp = 0)}{2eB} \right) \left\{ 2x(1-x) + \kappa_\rho + \frac{\kappa_\rho^2}{2} - \frac{\kappa_\rho^2}{4m_N^2} \Delta_N(q_\perp = 0) \right\} (q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu) - 2x(1-x) \left( \psi(z_N) + \frac{1}{2z_N} \right) (q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu) + 2eBq_\parallel^\mu q_\parallel^\nu \left\{ \left( z_N - \frac{m^2_N}{eB} \right) \psi(z_N + x) + z_N \right\} + \ln \Gamma(z + x) - \ln \sqrt{2\pi} \right] - \kappa_\rho \left\{ (q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu) \left( \psi(z_N) + \frac{1}{2z_N} \right) + q_\parallel^2 g^\mu\nu \psi(z + x) \right\} + \left[ \frac{m^2_N}{eB} \left( \psi(z_N) + \frac{1}{2z_N} \right) + \frac{1}{2} \ln(z_N) + \ln \Gamma(z_N) - \ln \sqrt{2\pi} \right] \frac{\kappa_\rho^2}{4m_N^2} (2eB) \left( q_\parallel^2 g^\mu\nu - q_\parallel^2 q_\parallel^\nu \right) \psi(z_N + x) + \Delta_N(q_\perp = 0) \bigg|_{\varepsilon \to 0}.
\] (D9)

which is finite and independent of scale.

**Appendix E: Analytic Evaluation of \( d^2k_\perp \) Integral for \( q_\perp = 0 \)**

In this appendix we will calculate the quantities

\[
\hat{N}^\nu_{\pi,\mu}(q_\parallel, k_\parallel) = \int \frac{d^2k_\perp}{(2\pi)^2} N^\nu_{\pi,\mu}(q_\parallel, q_\perp = 0, k_\perp)
\] (E1)

\[
\hat{N}^\mu_{\pi,\nu}(q_\parallel, k_\parallel) = \int \frac{d^2k_\perp}{(2\pi)^2} N^\mu_{\pi,\nu}(q_\parallel, q_\perp = 0, k_\perp).
\] (E2)

We have the expression for \( N^\nu_{\pi,\mu}(q, k) \) from Eqs. (10) and (17) as

\[
N^\nu_{\pi,\mu}(q, k) = 4g^2_{\rho_{\pi,\nu}}(-1)^{n+l}e^{-\alpha_k - \alpha_{\rho}}L_n(2\alpha_k)L_n(2\alpha_{\rho}) \left[ q^4 k^\mu k^\nu + (q \cdot k)^2 q^\mu q^\nu - q^2 (q \cdot k) (q^\nu k^\mu + q^\mu k^\nu) \right].
\] (E3)

which for \( q_\perp = 0 \) becomes

\[
N^\nu_{\pi,\mu}(q_\parallel, k) = 4g^2_{\rho_{\pi,\nu}}(-1)^{n+l}e^{-2\alpha_k}L_n(2\alpha_k)L_n(2\alpha_{\rho}) \left[ q^4 k^\mu k^\nu + (q_\parallel \cdot k_\parallel)^2 q^\mu q^\nu - q^2 (q_\parallel \cdot k_\parallel) (q^\nu k^\mu + q^\mu k^\nu) \right].
\] (E4)
We now perform the $d^2k_\perp$ integration using the orthogonality of the Laguerre polynomial (identities provided in Appendix A) to obtain

$$N^\mu_\pi_{\nu}(q_\parallel, k_\parallel) = 4g^2 e^{-\alpha k} \frac{eB}{\pi} \left\{ \left( \frac{m^2_q - k_\parallel^2 - k_\parallel \cdot q_\parallel}{(m^2_q - k_\parallel^2 - k_\parallel \cdot q_\parallel)^2} + \left( q_\parallel \cdot q_\parallel \right) g_{\mu\nu} - q_\parallel^2 \right) \delta^{\nu}_{\nu} \right\}$$

(E5)

Similarly, $N^{\mu\nu}_{p,nl}(q, k)$ is obtained from Eq. (12) as

-Evaluating the trace over the Dirac matrices in the above equation, we get for $q_\perp = 0$ (considering the Lorentz symmetric part since the self energy should be symmetric in the two Lorentz indices)

$$N^{\mu\nu}_{p,nl}(q, k) = -g^2 e^{-\alpha k} \frac{eB}{\pi} \left\{ \left( \frac{m^2_q - k_\parallel^2 - q_\parallel^2}{(m^2_q - k_\parallel^2 - q_\parallel^2)^2} + \left( q_\parallel \cdot q_\parallel \right) g_{\mu\nu} - q_\parallel^2 \right) \delta^{\nu}_{\nu} \right\}$$

(E6)

We now perform the $d^2k_\perp$ integration using the orthogonality of the Laguerre polynomial (identities provided in Appendix A) to obtain,

$$N^{\mu\nu}_{p,nl}(q_\parallel, k_\parallel) = -g^2 e^{-\alpha k} \frac{eB}{\pi} \left\{ \left( \frac{m^2_q - k_\parallel^2 - q_\parallel^2}{(m^2_q - k_\parallel^2 - q_\parallel^2)^2} + \left( q_\parallel \cdot q_\parallel \right) g_{\mu\nu} - q_\parallel^2 \right) \delta^{\nu}_{\nu} \right\}$$

(E7)

It is to be noted that, a Kronecker delta with -ve index is zero which comes from our constraint on the Laguerre polynomials $L^\perp_{l+1} = 0$. 

\[
\delta_{l-1}^{n-1} + \delta_{l-1}^{n} + \delta_{l}^{n-1} + \delta_{l}^{n} = 0
\]

\[
\delta_{l}^{n} \equiv \int d^2k_\perp L^\perp_{l+1} L^\perp_{l} = 0
\]

\[
\delta_{l}^{n} \equiv \int d^2k_\perp L^\perp_{l+1} L^\perp_{l} = 0
\]
Appendix F: Details of $\mathcal{N}^\mu_\mu$ and $\mathcal{N}^{00}$ for different loop

In this appendix, we list the explicit forms of $\mathcal{N}^\mu_\mu$ and $\mathcal{N}^{00}$ for all the different loops. For the zero magnetic field case, we have for the $\pi\pi$ Loop

$$g_{\pi\pi} \mathcal{N}^\mu_\mu (q, k) = g^2_{\pi\pi} \left[ q^4 k^\mu k'^\nu + (q \cdot k)^2 q^2 - q^2 (q \cdot k) 2q \cdot k \right]$$  \hspace{1cm} \text{(F1)}$$

$$\mathcal{N}^{00}_\pi (q, k) = g^2_{\pi\pi} \left[ q^4 k^0_0 + (q \cdot k)^2 q^2 - q^2 (q \cdot k) 2q^0 k^0 \right]$$  \hspace{1cm} \text{(F2)}$$

and for the NN-Loop,

$$g_{\rho N} \mathcal{N}^\mu_\mu (q, k) = -8g^2_{\rho N} \left[ (m_N^2 - k^2 - k \cdot q) 4 + 2k^2 + q \cdot k + \kappa_\rho q^2 \right]$$

$$+ \frac{\kappa_\rho^2}{4m_N^2} \left\{ (m_N^2 + k^2 - k \cdot q) 3q^2 - 2q^2 k^2 - 2(k \cdot q)^2 4 + 4(k \cdot q)^2 \right\} .$$  \hspace{1cm} \text{(F4)}$$

$$\mathcal{N}^{00}_N (q, k) = -8g^2_{\rho N} \left[ (m_N^2 - k^2 - k \cdot q) + 2k^2_0 + 2q^0 k^0 - \kappa_\rho q^2 \right]$$

$$+ \frac{\kappa_\rho^2}{4m_N^2} \left\{ -(m_N^2 + k^2 - k \cdot q) q^2 - 2q^2 k^2_0 - 2(k \cdot q)^2 + 4(k \cdot q) 2q^0 k^0 \right\} .$$  \hspace{1cm} \text{(F5)}$$

The corresponding expressions for $\pi\pi$ loop for finite magnetic field case are given by

$$g_{\pi\pi} \mathcal{N}^\mu_\mu (q || k ||, k) = 4g^2_{\pi\pi} \left( -1 \right)^{n+l} \frac{eB}{8\pi} \left\{ \frac{q^4 k^2 || + (q || \cdot k ||)^2 q^2 - q^2 (q || \cdot k ||) 2q || \cdot k ||}{2} \right\} \delta_i^n$$

$$- q^2 || \frac{1}{2} ( (2n + 1) \delta_i^n - n \delta_i^{n+1} - n \delta_i^{n+1} - 1 )$$  \hspace{1cm} \text{(F6)}$$

$$\mathcal{N}^{00}_{\pi, n} (q || k ||) = 4g^2_{\pi\pi} \left( -1 \right)^{n+l} \frac{eB}{8\pi} \left\{ \frac{q^4 k^2 || + (q || \cdot k ||)^2 q^2 - q^2 (q || \cdot k ||) 2q^0 k^0}{2} \right\} \delta_i^n$$  \hspace{1cm} \text{(F7)}$$

whereas the same for proton-proton loop are

$$g_{\rho N} \mathcal{N}^\mu_\mu (q || k ||, k) = -g^2_{\rho N} \left( -1 \right)^{n+l} \frac{eB}{\pi} \left\{ 8eB \delta_i^{n+1} || + \left\{ (m_N^2 - k^2 - k || \cdot q ||) 2 + 2k^2 || + 2q || \cdot k || \right\} \right\} \left( \delta_i^{n+1} - \delta_i^n \right)$$

$$- (m_N^2 - k^2 - k || \cdot q ||) 2 ( \delta_i^{n+1} + \delta_i^n - 1 ) + \kappa_\rho \left\{ q^2 || ( \delta_i^{n+1} + \delta_i^n - 1 ) - q^2 || ( \delta_i^{n+1} + \delta_i^n - 1 ) \right\}$$

$$+ \frac{\kappa_\rho^2}{4m_N^2} \left[ -4eB q^2 n \delta_i^{n+1} - \left\{ 2 (k || \cdot q ||)^2 + 2q || k^2 || - 2 (k || \cdot q ||) 2q || k || \right\} \right]$$

$$- (m_N^2 + k^2 - k || \cdot q ||) q^2 || \left\{ \delta_i^{n+1} + \delta_i^n \right\} - \left\{ q^2 || (m_N^2 + k^2 - k || \cdot q ||) - 2 (k || \cdot q ||)^2 \right\} \right\} 2 ( \delta_i^{n+1} + \delta_i^n - 1 ) \right\}$$  \hspace{1cm} \text{(F9)}$$

$$\mathcal{N}^{00}_{\rho, n} (q || k ||, k) = -g^2_{\rho N} \left( -1 \right)^{n+l} \frac{eB}{\pi} \left\{ 4eB \delta_i^{n+1} || + \left\{ (m_N^2 - k^2 - k || \cdot q ||) + 2k^2_0 + 2q^0 k^0 \right\} \right\} \left( \delta_i^{n+1} + \delta_i^n \right)$$

$$+ \kappa_\rho \left\{ q^2 || ( \delta_i^{n+1} + \delta_i^n - 1 ) \right\} + \frac{\kappa_\rho^2}{4m_N^2} \left[ 4eB q^2 n \delta_i^{n+1} - \left\{ 2 (k || \cdot q ||)^2 + 2q || k^2 || - 2 (k || \cdot q ||) 2q^0 k^0 \right\} \right]$$

$$+ (m_N^2 + k^2 - k || \cdot q ||) q^2 || \left\{ \delta_i^{n+1} + \delta_i^n \right\} \right\} .$$  \hspace{1cm} \text{(F10)}$$

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