Infiniteness of Double Coset Collections in Algebraic Groups

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Abstract

Let $G$ be a linear algebraic group defined over an algebraically closed field. The double coset question addressed in this paper is the following: Given closed subgroups $X$ and $P$, is $X \backslash G/P$ finite or infinite? We limit ourselves to the case where $X$ is maximal rank and reductive and $P$ parabolic. This paper presents a criterion for infiniteness which involves only dimensions of centralizers of semisimple elements. This result is then applied to finish the classification of those $X$ which are spherical. Finally, excluding a case in $F_4$, we show that if $X \backslash G/P$ is finite then $X$ is spherical or the Levi factor of $P$ is spherical. This implies that it is rare for $X \backslash G/P$ to be finite. The primary method is to descend to calculations at the finite group level and then to use elementary character theory.

Keywords: algebraic groups, finite groups of Lie type, double cosets, spherical subgroups, finite orbit modules.

1 Introduction

All algebraic groups in this paper are linear algebraic groups defined over an algebraically closed field and all subgroups are assumed to be closed. Given an algebraic group $G$ we wish to classify those subgroups $X$ and $P$ such that $X \backslash G/P$ is finite. One has that $X \backslash G/P$ is finite if and only if the $G$-orbit $G/P$ splits into finitely many $X$-orbits. This viewpoint makes a complete classification of all finite double coset collections appear unlikely in the near future. For this reason, we will restrict which subgroups we allow. We will
generally assume $G$ is a reductive (or simple) algebraic group, that $P$ is a parabolic subgroup and that $X$ is maximal rank and reductive. This paper is concerned with proving that a double coset collection is infinite. We intend to establish finiteness results in a later paper.

We will state the main results of the paper first with brief indications of how these results relate to earlier work in the field. This is followed by a lengthier description of some of this earlier work.

The first result provides a powerful criterion for establishing that $X \setminus G/P$ is infinite. If $H$ is a group and $g \in H$ we write $H_g$ for the centralizer of $g$ in $H$.

**Theorem 1.1 (Dimension Criterion).** Let $G$ be a reductive algebraic group, $X$ and $P$ subgroups of $G$ with $X$ maximal rank and $P$ parabolic. Let $L$ be a Levi factor of $P$ and let $s \in X \cap L$ be a semisimple element. If $\dim Z(G_s) + \dim G_s > \dim X_s + \dim P_s$ (equivalently, if $\dim Z(G_s) + \frac{1}{2} \dim G_s - \frac{1}{2} \dim X_s - \frac{1}{2} \dim L_s > 0$), then $X_s \setminus G_s/P_s$ and $X \setminus G/P$ are infinite.

**Classification of maximal rank reductive spherical subgroups.** For the next result, suppose $G$ is a simple algebraic group. The first application of the dimension criterion is to finish the classification of maximal rank reductive spherical subgroups. The subgroup $X$ is **spherical** if a Borel subgroup $B$ has a dense orbit upon $X \setminus G$. Brion [5] and Vinberg [28] independently showed that $X$ is spherical if and only if $B$ has a finite number of orbits upon $X \setminus G$, or, equivalently, $X \setminus G/B$ is finite. A maximal rank reductive subgroup is **generic** if a subgroup of the same type exists in all characteristics. The work of Krämer [17], Brundan [6] and Lawther [18] has produced a list of subgroups which are spherical in all characteristics. The generic maximal rank reductive subgroups on this list are given in table 1 where we use the following conventions. We treat $A_0$ and $B_0$ as trivial groups and $D_1$ as a 1-dimensional torus. We list only the Lie type of each group, as the property of being spherical is not affected by which representative of an isogeny class is used. The notation $T_i$ refers to an $i$ dimensional torus, central in $X$. Finally, in the group $A_1 \tilde{A}_1 \leq G_2$ the tilde $\sim$ is used to signify a subgroup with short roots (we don’t use this notation for the other groups as there is no ambiguity). Finally, in attempting to classify the maximal rank reductive spherical subgroups, it suffices to classify only the generic cases as the others arise from isogenies or graph automorphisms, which preserve the property of being spherical (see Lemma 3.2).

To prove that table 1 is complete, we introduce the following root-theoretic property which was inspired by Lawther’s anti-open property (see...
INFINITENESS OF DOUBLE COSET COLLECTIONS

Table 1: Generic Maximal Rank Reductive Spherical Subgroups

| $X \leq G$ | $X \leq G$ |
|------------|------------|
| $A_nA_mT_1 \leq A_{n+m+1}$ | $E_6T_1 \leq E_7$ |
| $B_nD_m \leq B_{n+m}$ | $A_7 \leq E_7$ |
| $A_{n-1}T_1 \leq B_n$ | $A_1D_6 \leq E_7$ |
| $C_nC_m \leq C_{n+m}$ | $A_1E_7 \leq E_8$ |
| $C_{n-1}T_1 \leq C_n$ | $D_8 \leq E_8$ |
| $A_{n-1}T_1 \leq C_n$ | $A_1C_3 \leq F_4$ |
| $D_nD_m \leq D_{n+m}$ | $B_4 \leq F_4$ |
| $A_{n-1}T_1 \leq D_n$ | $A_2 \leq G_2$ |
| $D_5T_1 \leq E_6$ | $A_1A_1 \leq G_2$ |
| $A_1A_5 \leq E_6$ |

Theorem 2.2 below). Let $X$ be a maximal rank reductive subgroup of $G$. We abbreviate the phrase “maximal rank reductive” with MRR. Fix a maximal torus of $X$ and define $\Phi(X)$ and $\Phi(G)$ with respect to this maximal torus.

We say $X$ has an $R$-complement if there exists a closed root subsystem $R \leq \Phi(G) - \Phi(X)$. This is equivalent to the existence of a generic MRR subgroup $K \leq G$ with $\Phi(K) = R$ and $K \cap X$ a maximal torus. The adjective “long” or “short” may be applied if $R$ has only long or short roots.

**Theorem 1.2.** Let $G$ be a simple algebraic group and $X$ a generic MRR subgroup. The following are equivalent:

(i) $X$ is spherical.
(ii) $X$ appears in table 1.
(iii) $X$ has no $A_2$ or $B_2$ complement.

In this paper we show that (i) $\implies$ (iii) $\implies$ (ii) (more precisely, we show $\neg$ (ii) $\implies$ $\neg$ (iii) $\implies$ $\neg$ (i)). The other implications are due to Brundan [6] and Lawther (see Theorem 2.2 below).

**Theorem 1.2** applies to groups acting on the full flag variety $G/B$, where $B$ is a Borel subgroup. Using the dimension criterion, we now obtain more general infiniteness results where $P$ is a parabolic. Since table 1 contains relatively few subgroups, the following theorem places great restrictions upon $X$ and $P$ for $X\backslash G/P$ to be finite. An end node parabolic is conjugate to a standard parabolic obtained by crossing off exactly one of the end nodes in the Dynkin diagram of $G$. 


Theorem 1.3 (Spherical $X$ or Spherical $L$). Let $G$ be a simple algebraic group, $X$ a MRR subgroup, $P$ a parabolic subgroup with Levi factor $L$. If $G = F_4$ suppose $P$ is not an end node parabolic. If $X \backslash G/P$ is finite then $X$ is spherical or $L$ is spherical.

The extra restrictions placed upon $P$ when $G = F_4$ are necessary. In a later paper we will show that $L_1 \backslash F_4/P_1$ and $L_4 \backslash F_4/P_1$ are finite (where $P_i$ is conjugate to the standard parabolic obtained by crossing off the $i^{th}$ node of the Dynkin diagram of $F_4$, and $L_i$ is its Levi factor).

Corollary 1.4. If $X \backslash G/P$ is finite and $P$ is not maximal then $X$ is spherical.

Remark 1.5. The theorem and the corollary give a surprisingly strong dichotomy for MRR subgroups with respect to the double coset problem. Either they are spherical, or they will have an infinite number of orbits on almost all flag varieties. For instance, $A_1A_5$ is spherical in $E_6$, but $X = T_1A_5$ will have an infinite number of orbits on all flag varieties $E_6/P$ except, possibly, if $P$ is an end node parabolic. As another example, suppose one could show that a MRR subgroup $X$ in $GL(V)$ has a finite number of orbits on flags consisting of one and two dimensional subspaces. Then $X$ has a finite number of orbits on full flags, i.e. upon $G/B$ where $B$ is a Borel subgroup.

2 Some History of the Problem

Spherical subgroups. The following result is a starting point for classifying reductive spherical subgroups. Let $G$ be an algebraic group and let $C_G(\tau)$ be the centralizer of the involution $\tau \in \text{Aut}(G)$.

Theorem 2.1 (Matsuki [20], Springer [23]). Let $G$ be a reductive group. If the characteristic of the underlying field is not 2, then the centralizer of an involution in $G$ is spherical.

This result was proven by Matsuki in characteristic 0 and extended to all characteristics except $p = 2$ by Springer. Seitz [22] has given an alternative proof and Lawther [18] has given a proof for positive characteristics which also shows that the same subgroups are spherical in characteristic 2. (Note, by “same” subgroups, we mean, by abuse of language, subgroups of the same type.) We will have occasion later to refer to Lawther’s version of this theorem. In the following theorem let $\alpha_1, \ldots, \alpha_n$ be the simple roots of
the simple algebraic group \( G \). Write the high root of \( G \) as \( \sum \lambda_i \alpha_i \). Recall that a generic maximal rank reductive subgroup of \( G \) can be produced via the Borel-de Siebenthal algorithm [4]. This algorithm takes the Dynkin diagram, or the extended Dynkin diagram, of \( G \), crosses off some number of nodes, and recursively applies the same procedure to the new diagrams which have been produced.

**Theorem 2.2 (Lawther [18]):** Let \( G \) be a simple algebraic group with high root \( \sum \lambda_i \alpha_i \) as just described. Let \( X \) be a generic maximal rank reductive subgroup. The following conditions are equivalent and define anti-open. If \( X \) is anti-open then it is spherical.

(i) There do not exist \( \alpha, \beta, \alpha + \beta \in \Phi(G) - \Phi(X) \).

(ii) \( X \) can be produced from the Borel-de Siebenthal algorithm by removing a single node \( \alpha_i \) with \( \lambda_i = 1 \) or by extending the diagram for \( G \) once and removing a single node \( \alpha_i \) with \( \lambda_i = 2 \).

(iii) If the characteristic of the underlying field is not 2, then \( X \) is the centralizer of an involution.

(iv) \((G, X)\) appears in table 1 but \((G, X) \notin \{(B_n, A_{n-1}T_1), (C_n, C_{n-1}T_1), (G_2, A_2)\}\).  

**Remark 2.3.** If \( X \) is anti-open then \( X \) has no \( A_2 \) or \( B_2 \) complement. Theorem 2.2 and Theorem 1.2 show that the converse is often, but not always, true. To be precise, the converse is true unless \((G, X) \in \{(B_n, A_{n-1}T_1), (C_n, C_{n-1}T_1), (G_2, A_2)\}\). For example, let \( G = B_2 \), fix a maximal torus \( T \) and label the Dynkin diagram of \( G \) with \( \alpha_1 \) and \( \alpha_2 \) where \( \alpha_1 \) is long. The group \( X \), generated by \( T \) and the root groups corresponding to the high root and its negative, is of type \( A_1T_1 \). Then \( \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \in \Phi(G) - \Phi(X) \) but \( \alpha_1 + 2\alpha_2 \in \Phi(X) \). So \( X \) does not have an \( A_2 \) or \( B_2 \) complement, but it is not anti-open.

If the characteristic \( p \) is not 2, then \( C_G(\tau) \), the centralizer of an involution, forms a reductive group. The reductive spherical subgroups of the simple algebraic groups have been classified in characteristic 0 by Krämer [17] and in fact most of them are centralizers of involutions. Brundan [6] gave a reduction mod \( p \) argument to show that certain groups in Krämer’s list are spherical in all characteristics, but this argument could not be applied to nine of the cases. These nine exceptions were all groups which would be the centralizer of an involution when \( p \) was not 2. Lawther [18] introduced the concept of anti-open to prove that these remaining cases were spherical when \( p = 2 \). He then proceeded to work at the level of finite groups using character theory. See sections 4 and 5 below for some similar arguments.
In the work just described, the attention is focused on the group $X$. Related work has been done where the attention is focused on the action of a Borel subgroup $B$ on varieties or modules. For example, the homogeneous space $C_G(\tau)\backslash G$ is a symmetric variety, and these have been studied by a wide variety of authors in various contexts of representation theory. See, for example, Helminck [11] for how to compute the orbits of $B$ upon $C_G(\tau)\backslash G$ and for a review of related literature. In other contexts, the spherical subgroups are of interest because they are related to multiplicity free modules (see the discussion below of work by Kac [16] and Arzhantsev [1]).

**Irreducible finite orbit modules.** The next family of examples is of a representation theoretic nature. Let $X$ be a closed, connected subgroup of $G = \text{GL}(V)$ with $V$ a finite dimensional vector space, irreducible under $X$. Then $X$ is a reductive group and we may write $X = X'Z$ where $X'$ and $Z$ are the derived subgroup and the center of $X$ respectively. Then $Z$ acts as scalars upon $V$. To make the following statement easier, we assume that $Z$ equals the full group of scalars. Let $P_1$ be the stabilizer in $\text{GL}(V)$ of a 1-space in $V$. We identify $\text{GL}(V)/P_1$ with the collection of 1-spaces in $V$.

**Theorem 2.4.** Let $X$, $\text{GL}(V)$ and $P_1$ be as described. Then $X$ has a finite number of orbits upon $V$ if and only if $X'\backslash \text{GL}(V)/P_1$ is finite. Moreover, all instances where $X'\backslash \text{GL}(V)/P_1$ is finite have been classified.

The classification was found first by Kac [16] in characteristic 0 and extended to positive characteristics by Guralnick, Liebeck, Macpherson and Seitz [12]. The latter paper also classifies when $X'\backslash \text{GL}(V)/P_i$ is finite where $P_i$ is the stabilizer of an $i$-dimensional subspace. In Kac’s paper, he suggests a broader context for this problem. Essentially, he wants to find those representations which allow a classification of orbits and he lists various properties which such representations might have in common. For instance, one of the properties he studies is that the modules be spherical (i.e. a Borel subgroup has a dense orbit on $V$) which is equivalent to the representation being multiplicity free. A related problem was studied by Arzhantsev [1]. Here the goal was to classify those representations of $G$ on $V$ such that a Borel subgroup has an open orbit in $Gv$ for each $v \in V$. Again, a classification is obtained. This demonstrates Kac’s heuristic that representations with certain nice properties should be amenable to classification. The question remains as to which properties are important. The present paper takes a small step towards making the case that having a finite double coset collection is an important property.
A program for double cosets. The last two families of examples share a theme. In each case we have reductive groups acting on $G/P$ where $P$ is a parabolic subgroup. In an effort to classify all instances where $X \backslash G/Y$ is finite this setting is a natural place to concentrate for reasons we will discuss. Firstly, one should start by classifying finiteness of $X \backslash G/Y$ where both $X$ and $Y$ are maximal. Then the Borel-Tits Theorem (see [14, 30.4]) implies that each of $X$ and $Y$ is reductive or parabolic. If $X$ and $Y$ are both parabolics, then the collection is finite by the Bruhat decomposition. By work of Brundan [7], if $X$ and $Y$ are each reductive and either maximal or the Levi factor of a parabolic subgroup, then finiteness of $X \backslash G/Y$ is equivalent to having a factorization $G = XY$. Such factorizations are rare and have been essentially classified in a paper by Liebeck, Saxl and Seitz [19]. (Note, having such a factorization is equivalent to having $|X \backslash G/Y| = 1$, so this is also an example of a finite double coset problem.) This leaves the case where $X$ is reductive and $Y = P$ is a parabolic. (Further discussion of these matters may be found in the article by Seitz [22].)

Within the setting of $X$ being reductive and $P$ being parabolic, the question becomes which reductive groups should be addressed first. In many problems, the maximal rank reductive subgroups are the most important. Therefore in this paper we will study the case where $X$ is maximal rank. We intend to classify all cases where $X$ is maximal rank and reductive and $X \backslash G/P$ is finite in future papers. Afterwards, it is hoped that a classification may be obtained where the only restriction placed upon $X$ will be that it is reductive.

Outline of remaining sections. The outline of the rest of this paper is as follows: section 3 includes basic results and preliminaries; section 4 reduces the double coset question of algebraic groups to a related question about finite groups; section 5 applies character theory to the finite groups (roughly following Lawther [18]) and obtains the Dimension Criterion; section 6 proves Theorems 1.2 and 1.3, assuming Proposition 6.4; section 7 proves Proposition 6.4.

3 Preliminaries

In this section we list basic results which will be used later. Many (perhaps all) of the results in this section are known to others. We list them here either for convenience, or because references are difficult to find. For standard facts in algebraic groups we refer to the books by Borel [2], Humphreys [14] and
For a reductive, connected reductive group, $G$ we will use the following notation. Let $T$ be a maximal torus, $\Phi$ be the root system and for each $\alpha \in \Phi$ let $U_\alpha$ be the corresponding root group.

**Lemma 3.1.** Let $G$ be a simple algebraic group. All MRR subgroups of type $A_2$, of the same length, are conjugate. All MRR subgroups of type $B_2$ are conjugate. If the rank of $G$ is at least three then these subgroups are all Levi factors of parabolic subgroups.

**Proof.** The last statement is clear. Let $H$ and $H'$ be two MRR subgroups which are claimed to be conjugate. By conjugation we may assume that $H$ and $H'$ share a common maximal torus $T$. If the rank of $G$ is two then $H$ and $H'$ are equal. Otherwise $H$ and $H'$ are Levi factors and each is generated by $T$ and the root groups (positive and negative) corresponding to a pair of adjacent nodes in the Dynkin diagram of $G$. Then $H$ and $H'$ are conjugate by the action of the Weyl group.

The following lemma allows us to make a variety of easy reductions, or assumptions about $G$.

**Lemma 3.2.** Let $G$ be a group, $X$ and $P$ subgroups. Let $Z$ be the center of $G$, suppose $Z \leq P$ and let $\overline{X}$, $\overline{G}$ and $\overline{P}$ be the images of $X$, $G$ and $P$ under the map $G \to G/Z$. Let $K$ be a finite normal subgroup of $G$ and let $\hat{X}$, $\hat{G}$ and $\hat{P}$ be the images of $X$, $G$ and $P$ under the map $G \to G/K$. Let $g, h \in G$. The following are equivalent:

(i) $|X \backslash G/P| < \infty$.

(ii) $|\hat{X} \backslash \hat{G}/\hat{P}| < \infty$.

(iii) $|\overline{X} \backslash \overline{G}/\overline{P}| < \infty$.

(iv) $|X^g \backslash G/P^h| < \infty$.

**Proof.** These statements can all be proven in an elementary fashion.

Brundan [6] also states most of these. This lemma shows that the question of whether $X \backslash G/P$ is finite depends only upon the Lie type of the groups involved. In particular, it does not depend upon which elements of an isogeny class are chosen, the presence of centers, connectedness etc. By this result, we may work with either $\text{GL}(V)$ or $\text{SL}(V)$ and get essentially the same results. We may also assume that $G$ has simply connected derived subgroup which eases some of the proofs. Finally, if $X$ and $P$ are maximal rank then we may assume that they contain a common maximal torus.
Convention. If \( \tau \) is an endomorphism we denote by \( G_\tau \) the fixed points of \( \tau \) in \( G \). If \( G \) is a group and \( g \in G \) then \( G_g \) denotes the centralizer of \( g \) in \( G \). Finally, \( G_{\tau, g} \) denotes those points in \( G \) fixed by both \( \tau \) and \( g \). The finite groups of Lie type arise as the fixed points in \( G \) of a Frobenius morphism \( \sigma : G \to G \), where \( G \) is defined over the algebraic closure \( \mathbb{F}_p \) of the field \( \mathbb{F}_p \) of \( p \) elements. We refer to [8] and [26] for details.

Lemma 3.3. Let \( G \) be a connected reductive group with simply connected derived subgroup. Let \( T \) be a maximal torus of \( G \).

(i) The center of \( G \) is contained in each maximal torus of \( G \). (This does not require that \( G \) have simply connected derived subgroup.)

(ii) If \( s \in G \) is semisimple then \( G_s \) is reductive and connected.

(iii) For each \( s \in T \) we have \( Z(G_s) \leq T \).

(iv) The set \( \{ G_s \mid s \in T \} \) is finite. Its size may be bounded by a constant depending only upon the root system of \( G \).

(v) Fix \( s \in T \). There exist \( t_1, \ldots, t_r \in T \) such that \( \{ G_t \mid t \in G, G_t > G_s \} = \{ G_{ti} \mid 1 \leq i \leq r \} \). Let \( Z(s) = \{ t \in G \mid G_t = G_s \} \). Then \( Z(s) \) is an open subset of \( Z(G_s) \) and its complement is \( \bigcup_i Z(G_{ti}) \).

(vi) If \( S \) is a torus and \( L = C_G(S) \) then \( \{ s \in S \mid L = G_s \} \) is a dense subset of \( S \).

Proof. Part (i) is [14, 26.2].

Part (ii) is [8, 3.5.4,3.5.6].

Part (iii). Note that \( T \) is a maximal torus of \( G_s \). By part (ii) we may apply part (i) to the group \( G_s \).

Part (iv). By [8, 3.5.3] we have that \( G_s \) is generated by \( T \), the root groups it contains and by certain elements of the Weyl group. Since the Weyl group is finite and the number of root groups is finite, there are finitely many possibilities for \( G_s \).

Part (v). Let \( U \) be the set proposed as the complement of \( Z(s) \). Clearly \( G_t > G_s \iff Z(G_t) < Z(G_s) \leq T \). This, and the previous part, show that \( t_1, \ldots, t_r \) may be chosen in \( T \) as stated and that \( U \subseteq Z(G_s) \). Since the union is finite, \( U \) is a closed set. Given \( t \in Z(G_s) \) we have:

\[
\begin{align*}
t \notin Z(s) & \iff G_t > G_s \\
& \iff G_t = G_{ti} \text{ for some } i \\
& \iff t \in Z(G_{ti}) \text{ for some } i \\
& \iff t \in U.
\end{align*}
\]

This shows that \( U \) is the desired complement.
Part (vi). Note that $S \leq Z(L)$. Then for all $t \in S$ we have $G_t \geq L$. By
an argument similar to that for part (v), one can show that the set of $t \in S$
with $G_t > L$ is a proper, closed subset of $S$. \hfill \qedsymbol

Lemma 3.4. Let $G$ be a connected reductive group and $\sigma : G \to G$ a
Frobenius morphism. Then $Z(G) = Z(G)_{\sigma}$. Moreover, if $G$ has simply
connected derived subgroup and $s, t \in G_{\sigma}$ are semisimple elements, then
$G_s = G_t \iff G_{\sigma,s} = G_{\sigma,t}$.

Proof. Carter [8, 3.6.8] shows that $Z(G)_{\sigma} = Z(G)$. For the second state-
ment note that “$\Rightarrow$” is obvious, for “$\Leftarrow$” suppose $G_{\sigma,s} = G_{\sigma,t}$. By Lemma
3.3(ii) we have that $G_s$ and $G_t$ are connected and reductive. Then the first
statement shows that $Z(G_{\sigma,s}) = Z(G_s)$ whence $t$ is in $Z(G_{\sigma,s}) = Z(G_s)$. This shows that $t$ is in $Z(G_s)$ whence $G_t \geq G_s$. A symmetric argument
shows that $G_s \geq G_t$. \hfill \qedsymbol

Lemma 3.5 (Rational normalizer theorem). Let $G$ be a connected re-
ductive group defined over $\overline{\mathbb{F}_p}$, $\sigma : G \to G$ a Frobenius morphism. Let $P$ be
a $\sigma$-stable parabolic subgroup. Then $N_{G_{\sigma}}(P_{\sigma}) = P_{\sigma} = (N_{G}(P))_{\sigma}$

Proof. It is well known that $P = N_{G}(P)$, which gives the second equality.
For the first equality it is clear that $P_{\sigma} \leq N_{G_{\sigma}}(P_{\sigma})$. The reverse inclusion
follows from the fact that if $\tilde{P}$ is a $\sigma$-stable parabolic subgroup with $\tilde{P}_{\sigma} = P_{\sigma}$
then $\tilde{P} = P$, see [3, 4.20]. \hfill \qedsymbol

Corollary 3.6. Let $G$ be connected and reductive, $P$ a parabolic subgroup
and $x \in G_{\sigma}$. Suppose that $\sigma$ is a Frobenius morphism of $G$ which fixes $P$ and
$x$. Let $(G/P)_x$ be the variety of $G$-conjugates of $P$ which contain $x$. Then
$\sigma$ acts upon $(G/P)_x$ and $1_{G_{\sigma}}(x)$ is equal to the number of $\sigma$-fixed points on
this variety.

Proof. Using the Lang-Steinberg Theorem [25] it is easy to show that $\varphi :$
$G_{\sigma}/P_{\sigma} \to (G/P)_{\sigma}$ taking $gP_{\sigma}$ to $gP$ is an $x$-equivariant bijection. Together
with the rational normalizer theorem this shows that we have bijections between $(G/P)_{\sigma,x}$, $(G_{\sigma}/P_{\sigma})_x$ and \{ $gP_{\sigma} \mid g \in G_{\sigma}$, $x \in gP_{\sigma}$ \}. Elementary character theory shows that $1_{G_{\sigma}}(x)$ equals the size of the last collection. \hfill \qedsymbol

Lemma 3.7 ([21, 3.5]). Let $H$ be a connected algebraic group of dimension
d and $\sigma : H \to H$ a standard $q^{th}$ power Frobenius map. Then $|H_{\sigma}| \leq \dfrac{|H|}{(q - 1)^d} \leq (q + 1)^d$. 

Lemma 3.8. Let $G$ be a connected reductive group with simply connected derived subgroup, $\sigma : G \to G$ a standard $q^{th}$ power Frobenius map and $T$ a $\sigma$-stable maximal torus. Fix $s \in T_\sigma$ and let $Z(s)$ and $t_1, \ldots, t_r$ be as in Lemma 3.3. Let $c_1$ and $d_2$ be the number of components and dimension of $Z(G_s)$ respectively. Let $I \subseteq \{1, \ldots, r\}$ such that $\dim Z(G_{t_i}) < \dim Z(G_s) \iff i \in I$. Let $m = |I|$, let $c_2$ and $d_2$ be the maximal number of components and the greatest dimension of the $Z(G_{t_i})$ with $i \in I$. Note that if $m \neq 0$ then $d_2 < d_1$.

Then $Z(s)$ is $\sigma$-stable and

$$(q - 1)^{d_1} - m c_2 (q + 1)^{d_2} \leq |Z(s)_\sigma| \leq c_1 (q + 1)^{d_1}.$$  

Proof. If $t \in Z(s)$ then $G_{\sigma(t)} = \sigma(G_t) = \sigma G_s = G_s$ which shows that $Z(s)$ is $\sigma$-stable. We will be using Lemma 3.7 throughout the proof without further mention.

Since $Z(s) \subseteq Z(G_s)$ we see that $|Z(s)_\sigma| \leq |Z(G_s)_\sigma| \leq c_1 (q + 1)^{d_1}$ where the second inequality is found by calculating $|Z(G_s)_\sigma|$ under the assumption that $\sigma$ stabilizes each component of $Z(G_s)$.

From Lemma 3.3 we have that $Z(G_s) = Z(s) \sqcup \bigcup_{i \geq 1} Z(G_{t_i})$, where “$\sqcup$” indicates a disjoint union. Intersecting with $s Z(G_s)^o$ and taking fixed points we have

$$|s Z(G_s)^o \cap Z(s)_\sigma| = |(s Z(G_s)^o)_\sigma| - \left|\left(s Z(G_s)^o \cap \bigcup_{i \geq 1} Z(G_{t_i})\right)_\sigma\right|.$$  

It is easy to check that:

$$|s Z(G_s)^o \cap Z(s)_\sigma| \leq |Z(s)_\sigma|,$$

$$(q - 1)^{d_1} \leq |Z(G_s)^o| = |(s Z(G_s)^o)_\sigma|,$$

$$\left|\left(s Z(G_s)^o \cap \bigcup_{i \in I} Z(G_{t_i})\right)_\sigma\right| \leq m c_2 (q + 1)^{d_2}.$$  

To finish we will prove that $s Z(G_s)^o \cap \bigcup_{i \geq 1} Z(G_{t_i}) = s Z(G_s)^o \cap (\bigcup_{i \in I} Z(G_{t_i}))$.

It suffices to show that $s Z(G_s)^o \cap Z(G_{t_i})$ is empty if $\dim Z(G_{t_i}) = \dim Z(G_s)$. Let $\dim Z(G_{t_i}) = \dim Z(G_s)$. Then $Z(G_{t_i}) = Z(G_s)^o$ and $s Z(G_s)^o \cap Z(G_{t_i})$ is empty or all of $s Z(G_s)^o$. However, by definition of the $t_i$, we have $s \not\in Z(G_{t_i})$ so we are done.  

4 Reduction to finite groups

In this section we prove the fundamental results which relate double cosets in algebraic groups to double cosets in finite groups. These results seem
intuitive, but use material surprisingly far from group theory. By a reduced algebraic group scheme over \( \mathbb{Z} \), we mean (naively) that the group \( G \) is defined, as a subgroup of \( \text{GL}_n(\mathbb{Z}) \), with a finite number of polynomials over \( \mathbb{Z} \) and that \( \mathbb{Z}[G] \) has no nilpotents except 0. This is the case for the simple algebraic groups, as well as their parabolic subgroups and generic MRR subgroups (see [9] or [15]). Such a group scheme has a group of points over every field. For an algebraically closed field \( k \) one may identify the group of points (of the group scheme) over \( k \) with the algebraic group (in the naive sense) over \( k \). The field \( \overline{\mathbb{F}}_p \) is the algebraic closure of the field of \( p \) elements for the prime \( p \). Lawther [18] also uses some of these results.

**Proposition 4.1.** Let \( G \) be a simple algebraic group scheme and \( X \) and \( P \) closed algebraic subgroup schemes of \( G \), all of which are reduced over \( \mathbb{Z} \). For a field \( \mathbb{F} \) we denote by \( G(\mathbb{F}) \), \( X(\mathbb{F}) \) and \( P(\mathbb{F}) \) the group of points over \( \mathbb{F} \) of \( G \), \( X \) and \( P \). Let \( k \) be an algebraically closed field.

(i) If \( \text{char} \ k = 0 \) then

\[ |X(k) \setminus G(k) / P(k)| < \infty \iff \lim_{p \to \infty} |X(\overline{\mathbb{F}}_p) \setminus G(\overline{\mathbb{F}}_p) / P(\overline{\mathbb{F}}_p)| < \infty. \]

(ii) If \( \text{char} \ k = p > 0 \) then

\[ |X(k) \setminus G(k) / P(k)| < \infty \iff |X(\overline{\mathbb{F}}_p) \setminus G(\overline{\mathbb{F}}_p) / P(\overline{\mathbb{F}}_p)| < \infty. \]

**Proof.** Part (ii) is proven in [12]. (We view the group \( X(k) \times P(k) \) as acting on the affine space \( G(k) \). The assumption in [12] that \( X(k) \times P(k) \) should be reductive is not used.) It may also be proven using a model theoretic argument similar in nature to the one we give now for part (i). For basic facts about model theory we refer to the textbooks by Fried-Jarden [10] or Hodges [13].

For \( p \) equal to 0 or a prime, let \( ACF_p \) be the theory of algebraically closed fields of characteristic \( p \). Then \( ACF_p \) is a complete theory.

For a field \( \mathbb{F} \) we identify \( G(\mathbb{F}) \) as a set of matrices in \( \text{GL}_n(\mathbb{F}) \) using the defining polynomials over \( \mathbb{Z} \). We make similar identifications for \( X \) and \( P \). Since \( G \), \( X \) and \( P \) are defined over \( \mathbb{Z} \) we can express membership in \( G(\mathbb{F}) \), \( X(\mathbb{F}) \) and \( P(\mathbb{F}) \) with first order sentences. Let \( \varphi \) be the sentence which, applied to the model \( \mathbb{F} \), gives \( \exists g_1, \ldots, g_n \in G(\mathbb{F}), \forall g \in G(\mathbb{F}), \exists x \in X(\mathbb{F}), \exists y \in P(\mathbb{F}), \exists i \in \{1, \ldots, n\} \) such that \( xgy = g_i \). In other words, \( \varphi \) applied to \( \mathbb{F} \) states that \( |X(\mathbb{F}) \setminus G(\mathbb{F}) / P(\mathbb{F})| \leq n \).

Suppose \( X(k) \setminus G(k) / P(k) \) is infinite in characteristic zero. Then \( \varphi \) is false in \( k \). Then \( ACF_0 \models \neg \varphi \) by completeness. This means that we have
a finite number of steps each of which only uses a finite number of axioms to derive \( \neg \varphi \). In particular, only finitely many axioms which assert that \( m \cdot 1 \neq 0 \) are used and so there exists a prime \( p_0 \) which is greater than every \( m \) which is used in this manner. For all primes \( p \geq p_0 \) the axioms and steps which are used in the proof of \( ACF_0 \vdash \neg \varphi \) may also be used to conclude \( ACF_p \vdash \neg \varphi \). Therefore, for all such \( p \) we have \( |X(\mathbb{F}_p) \backslash G(\mathbb{F}_p)/P(\mathbb{F}_p)| > n \) whence \( \limsup_{p \to \infty} |X(\mathbb{F}_p) \backslash G(\mathbb{F}_p)/P(\mathbb{F}_p)| > n \).

Conversely, a similar argument shows that \( ACF_0 \vdash \varphi \Rightarrow ACF_p \vdash \varphi \) for all \( p \) sufficiently large. Therefore finiteness in characteristic 0 implies boundedness of \( |X(\mathbb{F}_p) \backslash G(\mathbb{F}_p)/P(\mathbb{F}_p)| \) as \( p \to \infty \).

**Lemma 4.2.** Let \( G \) be a connected algebraic group defined over \( k = \mathbb{F}_p \), \( \sigma : G \to G \) a Frobenius morphism, \( X \) and \( P \) closed subgroups which are \( \sigma \)-stable. If \( X \backslash G/P \) is infinite let \( C = 1 \). If \( X \backslash G/P \) is finite let \( C \) be an upper bound on the number of components of stabilizers of \( X \times P \) acting on \( G \). Then

\[
\frac{1}{C} \limsup_{n \to \infty} |X_{\sigma^n} \backslash G_{\sigma^n}/P_{\sigma^n}| \leq |X \backslash G/P| \leq \limsup_{n \to \infty} |X_{\sigma^n} \backslash G_{\sigma^n}/P_{\sigma^n}|.
\]

**Proof.** Suppose \( \limsup_{n \to \infty} |X_{\sigma^n} \backslash G_{\sigma^n}/P_{\sigma^n}| \) is finite and less than \( m \). We will show that \( |X \backslash G/P| < m \). Let \( g_1, \ldots, g_m \in G \). There is a number \( n \in \mathbb{N} \) such that \( g_1, \ldots, g_m \in G_{\sigma^n} \) and \( m > |X_{\sigma^n} \backslash G_{\sigma^n}/P_{\sigma^n}| \). Then at least two of \( g_1, \ldots, g_m \) are in the same \( X_{\sigma^n} \times P_{\sigma^n} \)-orbit, whence they are in the same \( X \times P \)-orbit. Since this holds for every \( g_1, \ldots, g_m \in G \) we see that \( |X \backslash G/P| < m \).

Suppose now that \( X \backslash G/P \) is finite, let \( n \) be given and let \( (X \backslash G/P)_{\sigma^n} \) be the collection of \( \sigma^n \)-stable \( X \times P \)-orbits. Then the Lang-Steinberg Theorem [25] shows that \( C |X \backslash G/P| \geq C |(X \backslash G/P)_{\sigma^n}| \geq |X_{\sigma^n} \backslash G_{\sigma^n}/P_{\sigma^n}| \). \( \square \)

5 Character Theory and the Dimension Criterion

**Strategy and conventions.** By Lemma 3.2 we may assume that \( G \) has simply connected derived subgroup when convenient and use Lemma 3.3, Lemma 3.4 etc. We wish to establish a criterion for infiniteness which is independent of characteristic. By section 4 it suffices to work over the algebraic closures of finite fields and establish infiniteness independently of the field. Let \( G \) be defined over the algebraic closure of a field of positive characteristic. Let \( \sigma : G \to G \) be a \( q^{th} \) power Frobenius morphism. We assume that \( X \) and \( P \) are \( \sigma \)-stable. Then to prove infiniteness it suffices to
show that $|X_{\sigma} \setminus G_{\sigma}/P_{\sigma}|$ is unbounded as $n$ approaches infinity. For fixed points we will use the notation $G_{\sigma}$, $P_{\sigma}$, etc as described in section 3. If $H$ is a group, the notation $[h] \subseteq H$ means $h$ is an element of $H$ and $[h]$ is its $H$-conjugacy class. An element denoted by $s$ will be semisimple, and an element denoted by $u$ will be unipotent. This development roughly follows Lawther [18]. Essentially the following lemma regroups the terms in the inner product of characters.

**Lemma 5.1.** Let $X$ and $P$ be subgroups of $G$ with $P$ parabolic. Define an equivalence relation on semisimple elements in $X_{\sigma}$ as follows: $s$ and $t$ are equivalent if $G_{\sigma,s}$ and $G_{\sigma,t}$ are $X_{\sigma}$-conjugate. Denote the equivalence class of $s$ by $E(s, \sigma)$. Choose a set $S_{\sigma}$ of representatives of these equivalence classes. Then

$$|X_{\sigma} \setminus G_{\sigma}/P_{\sigma}| = \sum_{s \in S_{\sigma}} \sum_{[u] \subseteq X_{\sigma,s}} \frac{|E(s, \sigma)|}{|X_{\sigma}|} \frac{|X_{\sigma,s}|}{|X_{\sigma,s,u}|} 1_{G_{\sigma}}(su).$$

**Proof.** Basic character theory gives

$$|X_{\sigma} \setminus G_{\sigma}/P_{\sigma}| = (1_{X_{\sigma}}, 1_{P_{\sigma}})_{G_{\sigma}} = (1_{X_{\sigma}}, 1_{P_{\sigma}})_{X_{\sigma}} = \frac{1}{|X_{\sigma}|} \sum_{x \in X_{\sigma}} 1_{P_{\sigma}}(x).$$

Applying the Jordan-Chevalley decomposition within the finite group $X_{\sigma}$ we get that this last sum is equal to

$$\frac{1}{|X_{\sigma}|} \sum_{s \in X_{\sigma}} \sum_{u \in X_{\sigma,s}} 1_{P_{\sigma}}(su).$$

Now we claim that $t \in E(s, \sigma)$ implies that

$$\sum_{u \in X_{\sigma,t}} 1_{P_{\sigma}}(tu) = \sum_{u \in X_{\sigma,s}} 1_{P_{\sigma}}(su).$$

Let $x \in X_{\sigma}$ with $(G_{\sigma}, x) = G_{\sigma,s}$. The crucial step is to show that for all $u \in X_{\sigma,t}$ we have $1_{P_{\sigma}}(tu) = 1_{P_{\sigma}}(su^x)$. Once this is done, conjugation by $x$ shows that the sums are equal. We work at the level of algebraic groups. Given $u \in X_{\sigma,t}$, let $(G/P)_tu$ and $(G/P)_su^x$ be the varieties of conjugates of $P$ which contain $tu$ and $su^x$ respectively. Then, by Lemma 3.6 $1_{P_{\sigma}}(tu)$ and $1_{P_{\sigma}}(su^x)$ are the numbers of $\sigma$-rational points on these varieties. Let $g \in G$ such that $t \in P^g$, let $T$ be a maximal torus of $P^g$ which contains $t$. We apply Lemma 3.4 to see that $(G_{t^x})_x = G_s$. We have the following:

$$T \leq G_t \implies T^x \leq G_s \implies s \in T^x \implies s \in P^g x.$$
It is now easy to see that $tu \in P^g \implies su^x \in P^{gx}$. Therefore, conjugation by $x$ gives a $\sigma$-equivariant bijection $(G/P)_{tu} \rightarrow (G/P)_{su^x}$. Taking $\sigma$-fixed points and applying Lemma 3.6 finishes the claim.

Using the claim we have

$$\frac{1}{|X_\sigma|} \sum_{s \in X_\sigma} \sum_{u \in X_{\sigma,s}} |G^x_{P_\sigma}(su)| = \frac{1}{|X_\sigma|} \sum_{s \in S_{\sigma}} \sum_{u \in X_{\sigma,s}} |E(s,\sigma)| 1^{G_{P_\sigma}}_{P_\sigma}(su).$$

To finish the proof note that if $u$ and $u'$ are conjugate in $X_{\sigma,s}$ then $1^{G_{P_\sigma}}_{P_\sigma}(su) = 1^{G_{P_\sigma}}_{P_\sigma}(su')$. Therefore the sum over unipotent elements in $X_{\sigma,s}$ may be replaced by representatives of unipotent classes. The size of such a unipotent class is

$$|u^{X_{\sigma,s}}| = \frac{|X_{\sigma,s}|}{|X_{\sigma,s,u}|}.$$

**Lemma 5.2.** Let $X$ and $P$ be subgroups of $G$ with $X$ maximal rank. Let $T$ be a $\sigma$-stable maximal torus in $X$, $W = N_G(T)$ the Weyl group, and $s \in T_\sigma$. Let $Z(s,\sigma) = \{ t \in G_\sigma \mid G_{\sigma,t} = G_{\sigma,s} \}$, $Z(s) = \{ t \in G \mid G_t = G_s \}$ (as in Lemma 3.3) and let $E(s,\sigma)$ be as in the previous lemma. Then we have:

(i) $Z(s,\sigma) = Z(s)_\sigma$.

(ii) $\frac{1}{|W|} \left| s^{X_\sigma} \times Z(s,\sigma) \right| \leq |E(s,\sigma)| \leq \left| s^{X_\sigma} \times Z(s,\sigma) \right|.$

**Proof.** Part (i). If $t \in Z(s)_\sigma$ then $G_s = G_t$ which implies $G_{\sigma,s} = G_{\sigma,t}$, whence $Z(s)_\sigma \subseteq Z(s,\sigma)$. If $t \in Z(s,\sigma)$ then $G_{\sigma,s} = G_{\sigma,t}$. By Lemma 3.4 this implies $G_s = G_t$ whence $t \in Z(s)$.

Part (ii). It suffices to show that the following is a surjective map, with fibers bounded in size by $|W|$:

$$\varphi : s^{X_\sigma} \times Z(s,\sigma) \longrightarrow E(s,\sigma).$$

$$(s^x, t) \longrightarrow t^x$$

Note that this map is well-defined as every element in $X_\sigma$ which centralizes $s$ also centralizes $t$. To see that the map is surjective, let $t \in E(s,\sigma)$ and let $x \in X_\sigma$ with $(G_{\sigma,t})^x = G_{\sigma,s}$. Then $(s^{x^{-1}}, t^x)$ is in the domain of $\varphi$ and $\varphi(s^{x^{-1}}, t^x) = t$.

The remainder of the proof bounds the size of the fiber. Let $(s^x, t_1)$ be an element of the domain. We claim that

$$\varphi^{-1}(t_1) = \left\{ (s^{w^{-1}x}, t_1^w) \mid w \in W \right\} \cap \left( s^{X_\sigma} \times Z(s,\sigma) \right).$$
It is easy to see that the set on the right is contained in \( \varphi^{-1}(t_1^y) \). Conversely, let \((s^y, t_2) \in \varphi^{-1}(t_1^y)\) and note \( t_2 = t_1^{xy^{-1}} \).

Claim: \( T \) contains \( t_1, t_2 = t_1^{xy^{-1}}, s, \) and \( s^{yx^{-1}} \). By assumption \( s \in T \). Note that Lemmas 3.3 and 3.4 show \( Z(G_{\sigma,s}) \leq T \). By definition \( t_1, t_2 \in Z(G_{\sigma,s}) \). The following calculation shows that \( s^{yx^{-1}} \in Z(G_{\sigma,s}) \) (which finishes the claim):

\[
G_{\sigma,s^{yx^{-1}}} = (G_{\sigma,s})^{yx^{-1}} = (G_{\sigma,t_2})^{yx^{-1}} = G_{\sigma,t_2}^{xy^{-1}} = G_{\sigma,t_1} = G_{\sigma,s}.
\]

We will use \([8, 3.7.1]\), and the (standard) notation which appears there.

Write \( xy^{-1} = utwu' \) in the Bruhat canonical form. Since \( t_2 = t_1^{xy^{-1}} \) we have \( t_2 = t_1^w \). Since \( xy^{-1} \) conjugates \( s^{yx^{-1}} \) to \( s \) we have \( (s^{yx^{-1}})^w = s \) and \( s^y = s^{w^{-1}x} \). Therefore \( (s^y, t_2) = (s^{w^{-1}x}, t_1^w) \).

**Corollary 5.3.** Let \( X \) be a maximal rank subgroup of \( G \) and \( P \) a parabolic subgroup of \( G \). Let \( S_\sigma \) and \( Z(s, \sigma) \) be as in Lemmas 5.1 and 5.2 respectively. We have

\[
\frac{1}{|W|} \sum_{[u]} \left| \frac{Z(s, \sigma)}{|X_{\sigma,s,u}|} \right| 1_{P_\sigma}^G(su) \leq \left| \frac{X_\sigma \setminus G_\sigma / P_\sigma}{|X_{\sigma,s,u}|} \right| \leq \sum_{[u]} \left| \frac{Z(s, \sigma)}{|X_{\sigma,s,u}|} \right| 1_{P_\sigma}^G(su),
\]

where each sum is taken over \( s \in S_\sigma, [u] \subseteq X_{\sigma,s} \).

**Proof.** Combine Lemma 5.1 and the bounds for \( E(s, \sigma) \) just obtained in Lemma 5.2. Note that

\[
\left| \frac{s^X \times Z(s, \sigma)}{|X_\sigma|} \right| \left| \frac{X_{\sigma,s}}{|X_{\sigma,s,u}|} \right| = \left| \frac{Z(s, \sigma)}{|X_{\sigma,s,u}|} \right|.
\]

**Proof 5.4 (Proof of the Dimension Criterion).** We have that \( X \) is maximal rank, \( P \) parabolic and \( s \in X \cap P \). We assume \( \dim Z(G_s) + \dim G_s > \dim X_s + \dim P_s \). We also assume that \( X \) and \( P \) are \( \sigma \) stable where \( \sigma \) is a standard \( q \)-th power Frobenius map.

Let \( \overline{G_s}, \overline{X_s} \) and \( \overline{P_s} \) denote the quotients of \( G_s, X_s \) and \( P_s \) by \( Z(G_s) \). The dimension inequality we have assumed implies that \( \dim G_s - \dim Z(G_s) > \dim X_s - \dim Z(G_s) + \dim P_s - \dim Z(G_s) \). This implies \( |X_s \setminus G_s / P_s| = \infty \) whence \( |X_s \setminus G_s / P_s| = \infty \) by Lemma 3.2.

It remains to show that \( |X \setminus G/P| = \infty \). Using Corollary 5.3 and Theorem 4.2, it suffices to show that the term in Corollary 5.3 corresponding
to \( s \in S_{\sigma}, 1 = [u] \subseteq X_{\sigma,s} \) is unbounded as we replace \( \sigma \) with \( \sigma^n \) and let \( n \to \infty \). This term is
\[
\frac{1}{|W|} \frac{|Z(s, \sigma^n)|}{|X_{\sigma^n,s}|} G_{\sigma^n}^s(s).
\]
It is easy to show that \( \frac{1}{|W|} \frac{|Z(s, \sigma^n)|}{|X_{\sigma^n,s}|} G_{\sigma^n}^s(s) \geq \frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|} \) whence this term is bounded below by
\[
\frac{1}{|W|} \frac{|Z(s, \sigma^n)|}{|X_{\sigma^n,s}|} \frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|}.
\]
Therefore it suffices to show that
\[
\limsup_{n \to \infty} \frac{1}{|W|} \frac{|Z(s, \sigma^n)|}{|X_{\sigma^n,s}|} \frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|} = \infty.
\]
By Lemmas 3.7 and 3.8 we have
\[
\limsup_{n \to \infty} \frac{1}{|W|} \frac{|Z(s, \sigma^n)|}{|X_{\sigma^n,s}|} \frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|} = \lim_{n \to \infty} C \frac{(q^n)^\dim Z(G_s) + \dim G_s}{(q^n)^\dim X_s + \dim P_s},
\]
for some constant \( C \). It is now easy to see that this limit is infinite. \( \square \)

6 Proof of Theorems 1.2 and 1.3

Throughout this section \( G \) is a simple algebraic group, \( X \) a generic MRR subgroup, \( P \) a parabolic subgroup with Levi factor \( L \). Starting with Proposition 6.4 we will use \( H \) for arguments which apply to both \( X \) and \( L \).

**Lemma 6.1.** Let \( s \in X \cap L \). If either of the following holds then \( X \setminus G/P \) is infinite:

(i) \( G_s \) is of type \( A_2 \) and \( X_s \) and \( L_s \) are tori.
(ii) \( G_s \) is of type \( B_2 \), \( X_s \) is a torus and \( L_s \) is of type \( A_1 \) or a torus.

**Proof.** Using the dimension criterion it suffices to show that
\[
\dim Z(G_s) + \frac{1}{2} \dim G_s - \dim X_s - \frac{1}{2} \dim L_s > 0.
\]
It is easy to check in each case that the quantity on the left is at least 1. \( \square \)

**Corollary 6.2.** If either of the following hold then \( X \setminus G/P \) is infinite:

(i) \( X \) and \( L \) have conjugate \( A_2 \) complements.
(ii) \( X \) has a \( B_2 \) complement \( K \), and for some conjugate \( \tilde{K} = K^g \) we have that \( \tilde{K} \cap L \) is a MRR subgroup which is a torus or of type \( A_1 \).

**Proof.** If \( G \) has rank 2 and (i) or (ii) holds then it is easy to show that \( X \setminus G/P \) is infinite by dimension.

Assume now that the rank of \( G \) is at least 3. If (i) holds let \( K \) be the \( A_2 \) complement of \( X \). Using Lemma 3.2 we replace \( P \), if necessary, by a conjugate so that in (i) the \( A_2 \) complements of \( X \) and \( L \) coincide, or so that in (ii), we may take \( \tilde{K} = K \). Since the rank of \( G \) is at least 3, we have that \( K \) is a Levi factor of a parabolic (see Lemma 3.1), whence is of the form \( C_G(S) \) for some torus \( S \). Apply Lemma 3.3, to see that there exists \( s \in S \) with the centralizer of \( s \) in \( G \) equal to \( K \). We are done by the previous lemma.

**Corollary 6.3.** If \( X \) has an \( A_2 \) or \( B_2 \) complement then \( X \) is not spherical.

**Proof.** Apply Lemma 6.2, noting that the Levi factor for a Borel subgroup is a torus, which has every type of complement possible.

**Proposition 6.4.** Let \( H \) be a generic MRR subgroup of \( G \) which does not appear in table 1. The following hold, and, in particular, \( H \) has an \( A_2 \) or \( B_2 \) complement in all cases.

(i) If \( G \) has single root length, then \( H \) has an \( A_2 \) complement.

(ii) If \( H \) is the Levi factor of a parabolic with non-abelian unipotent radical then \( H \) has an \( A_2 \), \( B_2 \) or \( G_2 \) complement.

(iii) Let \( G \) equal \( B_n \) or \( C_n \).

\( a \) If \( n = 2 \) then \( H \) has a \( B_2 \) complement.

\( b \) If \( G = B_n \) and \( H \) equals \( D_{n-1}T_1 \) or \( D_{n_1}D_{n_2} \) then \( H \) has a \( B_2 \) complement.

\( c \) Otherwise \( H \) has an \( A_2 \) complement.

(iv) Let \( G = B_n \). If \( H \) is a Levi factor then there exists a MRR subgroup \( K \) of type \( B_2 \) with \( H \cap K \) a MRR subgroup which is either a torus or of type \( A_1 \).

(v) If \( G = F_4 \) the maximal possibilities for \( H \) are \( C_3T_1, A_2A_2, B_3T_1, A_1A_1B_2, A_1A_3, D_4 \). The first possibility has a long \( A_2 \) complement, the next has both long and short \( A_2 \) complements, and the rest have short \( A_2 \) complements. In particular, if \( L \) is a Levi factor for a parabolic subgroup which is not an end node parabolic, then \( L \) has both long and short \( A_2 \) complements.

The proof of this proposition is delayed until the next section.
Proof 6.5 (Proof of Theorem 1.2). The work of Brundan [6] and Theorem 2.2 show that (ii) $\implies$ (i). Corollary 6.3 shows that (i) $\implies$ (iii). Proposition 6.4 shows that (iii) $\implies$ (ii).

Proof 6.6 (Proof of Theorem 1.3). We assume that $X$ and $L$ are not spherical we will show that $X \setminus G/P$ is infinite.

If $G = G_2$ then by dimension one finds that $X$ non-spherical implies $X \setminus G/P$ is infinite. For the remainder of the proof assume $G \neq G_2$.

Recall our convention that $D_1$ is a 1-dimensional torus. If $(G, X) \neq (B_n, D_n_1 D_n_2)$ then let $H_X$ be an $A_2$ complement for $X$ and let $H_L$ be an $A_2$ complement of $L$, of the same length as $H_X$ (length is only an issue for $F_4$). If $G = B_2$, or $(G, X) = (B_n, D_n_1 D_n_2)$ then let $H_X$ be a $B_2$ complement for $X$ and let $H_L$ be a MRR subgroup of type $B_2$ with $L \cap H_L$ a MRR subgroup of type $A_1$ or a maximal torus. Apply Lemma 3.1 to see that $H_X$ and $H_L$ are conjugate. Apply Lemma 6.2 to see that $X \setminus G/P$ is infinite.

We offer some insight into this result. Suppose the rank of $G$ is at least $3$ and that $(G, X) \neq (B_n, D_n_1 D_n_2)$. If $X$ and $L$ are both not spherical, then both have $A_2$ complements. Assume these $A_2$ complements are conjugate. Let $T_2$ and $B$ be a maximal torus and Borel subgroup of $A_2$ respectively. Then $T_2 \setminus A_2/B$ is infinite, and, in a sense, it may be “embedded” in $X \setminus G/P$.

The embedding comes from the expansion of terms in the inner product of finite group characters corresponding to $X \setminus G/P$. Take $s$ such that its centralizer (modulo a central torus) gives $T_2$, $A_2$ and $B$ in place of $X$, $G$ and $P$. The terms which arise by applying Corollary 5.3 to $X$, $G$ and $P$ and $s$ are the same terms one would obtain by applying Corollary 5.3 to the groups $\bar{X} = T_2$, $\bar{G} = A_2$ and $\bar{P} = B$.

7 Proof of Proposition 6.4

We prove parts (i) and (ii) immediately. Parts (iii)-(v) follow after Corollary 7.3.

Proposition 6.4(i). Recall $G$ has single root length and $H$ is a MRR subgroup which fails to appear in table 1. Then, by Theorem 2.2, $H$ is not anti-open, whence there exist $\alpha, \beta, \alpha + \beta \in \Phi(G) - \Phi(H)$. Let $T$ be the maximal torus used to define these roots and denote the corresponding root groups by $U_\alpha, U_\beta$ etc. Let $K$ be the group generated by $T$ and $U_{\pm \alpha}, U_{\pm \beta}$. Then $K$ is an $A_2$ complement for $G$.

Proposition 6.4(ii). Recall that $H$ is the Levi factor of a parabolic with non-abelian unipotent radical. Let $Q$ be the unipotent radical of the
parabolic associated with \( H \). Fix a maximal torus \( T \) in \( H \) and let \( U_\alpha \) and \( U_\beta \) be root groups contained in \( Q \) which do not commute. Let \( K \) be the group generated by \( T \) and \( U_{\pm \alpha}, U_{\pm \beta} \). Then \( K \) is of type \( A_2, B_2 \) or \( G_2 \) and it remains to show that \( K \cap H = T \). Let \( Q^- \) be the opposite unipotent radical of \( Q \). Note that \( \Phi(K) \subseteq \Phi(Q) \cup \Phi(Q^-) \) whence \( \Phi(K) \cap \Phi(H) = \emptyset \) whence \( K \cap H = T \). Thus \( K \) is an \( A_2, B_2 \) or \( G_2 \) complement for \( H \).

**Lemma 7.1.** Let \( \varphi \) be an irreducible root system in a Euclidean space \( \mathbb{E} \) with inner product \( (\ , \ ) \). Let \( \widehat{\varphi} \) be a proper, closed subsystem of \( \varphi \). Then \( \varphi - \widehat{\varphi} \) spans \( \mathbb{E} \) and for each \( \beta \in \widehat{\varphi} \) there exists \( \alpha \in \varphi - \widehat{\varphi} \) with \( (\alpha, \beta) \neq 0 \).

**Proof.** Let \( n \) be the dimension of \( \mathbb{E} \) and fix a Dynkin diagram \( \Delta \) of \( \varphi \). Given \( \alpha, \beta \in \Delta \) the path connecting \( \alpha \) to \( \beta \) is the shortest such path and includes \( \alpha \) and \( \beta \). The sum over this path means the sum of each element of \( \Delta \) which is contained in the path. It is easy to check that such a sum is itself a root.

For the first conclusion it suffices to show that we have \( n \) independent vectors in \( \varphi - \widehat{\varphi} \). Since \( \widehat{\varphi} \) is a proper, closed subsystem we have that \( \Delta - \widehat{\varphi} \) is non-empty. For each element \( \alpha \in \Delta \) let \( \gamma_\alpha \) be the path connecting \( \alpha \) to some element of \( \Delta - \widehat{\varphi} \). We re-index these paths so that for \( i \in \{1, \ldots, n\} \) the path \( \gamma_i \) contains a node which does not appear in \( \gamma_1, \ldots, \gamma_{i-1} \). Let \( \beta_1, \ldots, \beta_n \) be the sums over the paths just constructed. By the manner in which the paths \( \gamma_i \) were indexed, it is easy to see that \( \beta_1, \ldots, \beta_n \) are linearly independent. By the manner in which the paths were chosen, we may write each \( \beta_i \) as the sum of a root in \( \widehat{\varphi} \) and a root outside of \( \widehat{\varphi} \). This shows that \( \beta_i \) is not in \( \widehat{\varphi} \).

For the final conclusion note that \( \beta \) is not orthogonal to \( \mathbb{E} \), whence it is not orthogonal to \( \varphi - \widehat{\varphi} \). \( \square \)

**Corollary 7.2.** Let \( G \) be a reductive algebraic group, \( H \) a MRR subgroup. Let \( \varphi \leq \Phi(G) \) be an irreducible root system and let \( \varphi(H) = \varphi \cap \Phi(H) \). Suppose \( \widehat{\varphi} \) is a closed subsystem of \( \varphi \) with \( \varphi > \widehat{\varphi} > \varphi(H) \).

(i) If \( \varphi \) has single root length then \( H \) has an \( A_2 \) complement (whose length is the same as \( \varphi \)).

(ii) If \( \varphi \) is closed, \( G = B_n \) and \( \widehat{\varphi} - \varphi(H) \) contains a short root then \( H \) has a \( B_2 \) complement.

**Proof.** Fix \( \beta \in \widehat{\varphi} - \varphi(H) \) and assume \( \beta \) is short if (ii) holds. By the previous lemma there exists \( \alpha \in \varphi - \widehat{\varphi} \) with \( (\alpha, \beta) \neq 0 \). Note that \( \alpha \neq \pm \beta \) and that \( i\alpha + j\beta \in \Phi(G) \implies i\alpha + j\beta \in \varphi \). If \( (\alpha, \beta) > 0 \) we may replace one root with its negative and assume that \( (\alpha, \beta) < 0 \), whence \( \alpha + \beta \in \varphi \). Since \( \alpha \notin \widehat{\varphi} \) and \( \beta \in \widehat{\varphi} \) we see that \( \alpha + \beta \notin \widehat{\varphi} \). Similarly, we see that \( \alpha + 2\beta \notin \widehat{\varphi} \) (of course it may not even be a root) and that \( 2\alpha + \beta \) is not a root. Let \( T \)
be a maximal torus in \( H \) used to define these roots. Set \( K \) to be the group generated by \( T \) and all root groups \( U_{i\alpha+j\beta} \) where \( i\alpha+j\beta \) is a root with \( i \) and \( j \) integers. Then \( K \cap H = T \) and \( K \) is of type \( A_2, B_2 \) or \( G_2 \). If (i) holds then \( K \) is a group of type \( A_2 \). If (ii) holds then \( K \) is a group of type \( B_2 \) since \( \beta \) is short and \( B_n \) has no MRR subgroups of type short \( A_2 \) or \( G_2 \).

**Corollary 7.3.** Let \( G \) be a reductive algebraic group, \( H \) a MRR subgroup, \( \varphi \leq \Phi(G) \) an irreducible root system with single root length. If \( \varphi(H) \) is submaximal in \( \varphi \) then there exists \( \hat{\varphi} \) as in the previous corollary.

**Proof.** In a root system with single root length, every root subsystem is closed. \( \square \)

**Proof 7.4 (Proof of Proposition 6.4 (iii),(iv),(v)).** Part (iii). Recall \( G \) equals \( B_n \) or \( C_n \) and \( H \) is a generic MRR subgroup which does not appear in table 1.

Part (a). Since \( n = 2 \) and \( H \) does not appear in table 1 we see that \( H \) is just a torus and \( G \) itself is a \( B_2 \) complement.

Part (b). We have \( n \geq 3 \), \( G = B_n \) and \( H \in \{ D_{n-1}T_1, D_nD_n \} \). If \( H = D_{n_1}D_{n_2} \) then \( n_1 \geq 2 \). Let \( \varphi = \Phi(G) \) so \( \varphi(H) = \Phi(H) \). If \( H = D_{n-1}T_1 \) let \( \hat{\varphi} = \Phi(B_{n-1}T_1) \). If \( H = D_nD_n \) let \( \hat{\varphi} = \Phi(B_nD_n) \). In each case \( \hat{\varphi} \) contains a short root and \( \varphi(H) \) does not. Then \( \hat{\varphi} - \varphi(H) \) contains a short root and we are done by part Corollary 7.2(ii).

Part (c). Note that \( n \geq 3 \) and if \( G = B_n \) that \( H \notin \{ D_{n-1}T_1, D_nD_n \} \). If \( G = B_n \) let \( \varphi \) and \( \varphi(H) \) equal the long roots in \( \Phi(G) \) and \( \Phi(H) \) respectively. If \( G = C_n \) let \( \varphi \) and \( \varphi(H) \) equal the short roots in \( \Phi(G) \) and \( \Phi(H) \) respectively. In both cases \( \varphi \) is of type \( D_n \), and is irreducible since \( n \geq 3 \). The maximal subsystems of \( \varphi \) are \( A_{n-1}, D_{n-1} \) and \( D_nD_n \). The subsystem \( \varphi(H) \) cannot equal \( D_n, A_{n-1}, D_{n-1} \) or \( D_nD_n \) as this would contradict either the assumption that \( H \) is not in table 1 or the extra restrictions on \( H \) when \( G = B_n \). Therefore \( \varphi(H) \) is a submaximal subsystem of \( \varphi \) and we are done by Corollary 7.3.

Part (iv). We have that \( G = B_n \) and that \( H \) is a Levi factor of \( G \). Let \( T \) be a maximal torus in \( H \), let \( \alpha_1, \ldots, \alpha_n \) be the nodes in the Dynkin diagram of \( G \) in the usual order and suppose that \( H \) is described by “crossing off” certain nodes. Let \( \beta_1 = \alpha_1 \) and \( \beta_2 = \alpha_2 + \cdots + \alpha_n \). Let \( K \) be the group generated by \( T, U_{\pm \beta_1}, \) and \( U_{\pm \beta_2} \). Then \( K \) is of type \( B_2 \). The subsystem \( \Phi(H) \) may contain either \( \beta_1 \) or \( \beta_2 \), but not both. Similarly \( \Phi(H) \) may not contain \( i\beta_1 + j\beta_2 \) with \( i \) and \( j \) positive integers. This shows that \( K \cap H \) is either \( T \) or of type \( A_1 \).
Part (v). We have that $G = F_4$. To construct all short $A_2$ complements, take $\varphi$ equal to all the short roots in $\Phi(F_4)$, so $\varphi(H)$ equals all the short roots in $\Phi(H)$. Observe that $\varphi$ is of type $D_4$. By examining each possibility for $H$ it is easy to verify that $\varphi(H)$ is submaximal in a $D_4$ root system and we are done by Corollary 7.3. To construct the long $A_2$ complements, one proceeds similarly with $\varphi$ equal to all the long roots in $\Phi(F_4)$.

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