Transformation media with variable optical axes

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Abstract. We introduce a general framework in performing transformation optics by purely rotating the optical axis of the same anisotropic dielectric material on the subwavelength scale. The transformation medium realizes any area-preserving maps with maximum anisotropy only limited by the original material. By applying different optical-axis profiles on the same medium, a wave expander and a virtual shifter are constructed as examples. The investigations are potentially useful for designing transformation optical devices using only one type of material.

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1. Introduction

Since the first demonstration of an invisibility cloak working at the microwave regime, transformation optics (TO) is becoming an emerging framework in achieving extraordinary optical phenomena based on metamaterials with flexible constitutive parameters [1–3]. These include invisibility cloaking, illusion optics, optical analogue of mechanics at the astronomical scale and various light steering devices [4–9]. For cloaking, several schemes have been developed for working at optical frequencies and also on the macroscopic and the three-dimensional scale. They seek to provide a balance between extreme functionality and fabrication simplicity by choosing a limited class of coordinate maps [10–19]. Although TO generally enjoys its unique functions from a very inhomogeneous medium, it is indeed possible to use only a single type of anisotropic material in constructing a carpet cloak which compresses a two-dimensional region into a flat line. It employs the bilinear mapping first proposed in [20]. The cloak only consists of two regions of the same material with different directions of optical axes. The specific coordinate map ensures a single type of elliptical dispersion and hence the same dielectric material is induced. Similar linear mappings have also been used to construct a field shifter, bender and a rotator in obtaining a single type of dispersion surface [17–19, 21–23]. Although the original intention is to make fabrication much easier, all these examples share the common feature that wave propagation is manipulated by the directions of optical axes instead of the magnitudes of refractive indices. This is complementary to conventional optics and it becomes interesting to ask to what general extent the scheme can work within the theoretical framework of TO. Here, we will establish that the transformation corresponding to such a wave manipulation belongs to a wider class of area-preserving maps. Such mappings can be translated into optical-axis profiles. A wave expander and a virtual shifter are designed and numerically demonstrated using different optical-axis profiles as illustrative examples.

2. Effective medium in mixing optical axes

We are interested in using an optical-axis profile to manipulate wave propagation. However, it does not necessarily mean the coordinate map is restricted to generate only a single type of dispersion surface in previous examples. By mixing the optical axes in a subwavelength scale, the associated effective medium can have a variable dispersion surface. It will make fabrication more difficult due to the subwavelength variation but it can fully unlock the potential of the
Figure 1. Schematic of the checkerboard transformation medium with a single type of anisotropic material. Different colors on the squares represent different directions of the optical axes. A closer view of a unit cell is outlined by the red dashed lines. The two kinds of squares (with the same anisotropic elliptical surface with principal indices \(n_\alpha\) and \(n_\beta\)) have their optical axes \(\hat{\alpha}\) making an angle of \(\theta_A\) and \(\theta_B\) to the \(x\)-axis. (b) Examples of effective anisotropy \(\gamma\) of the transformation medium with constant \(n_\beta/n_\alpha = 2.4\). \(\theta_A = 3\theta_B\) is set as a particular example, numerical results (solid lines) coincide with the approximated results from (1) (dashed lines). The \(\hat{u}/\hat{r}\)-axis is the effective principal axis of the effective medium, \(\theta_u\) is the angle of the \(\hat{u}\)-axis to the \(x\)-axis. The right inset shows the cases of \(\theta_B = 0, 0, -0.7\) the effective elliptical dispersion surfaces with varying anisotropy and direction of principal axis but always with the same area.

Theoretical scheme in manipulating waves purely using optical axes. Here, we focus on wave propagation in two dimensions in the H-polarization for simplicity (while a similar analysis for the E-polarization can be done by swapping \(\varepsilon\) and \(\mu\)). Figure 1(a) shows a particular example in mixing optical axes in a checkerboard pattern. The blue/orange region represents the A/B type of squares. Each square comes from the same kind of anisotropic material (showed by the same shape of elliptical dispersion surface in black color) with fixed principal refractive indices \(n_\alpha = \sqrt{\varepsilon_\beta\mu_z}\) and \(n_\beta = \sqrt{\varepsilon_\alpha\mu_z}\). Each square can thus be represented only by the optical axis \(\hat{\alpha}\) with \(n_\alpha < n_\beta\), making an angle \(\theta_A/B\) to the \(x\)-axis (the other optical axis \(\hat{\beta}\) is defined as normal to \(\hat{\alpha}\)). As an example, we set a single kind of anisotropic material with \(n_\beta/n_\alpha = 2.4\), and keep
\( \theta_A = 3 \theta_B \) where \( \theta_B \in [-\pi/4, \pi/4] \). Then we can obtain the effective medium (tensor \(\varepsilon_{\text{eff}}\)) of the checkerboard with a new set of principal axes \(\hat{u}\) and \(\hat{v}\), where the \(\hat{u}\)-axis makes an angle \(\theta_u\) to the \(x\)-axis, and \(\hat{v}\) is normal to \(\hat{u}\). The effective anisotropy \(\gamma = \sqrt{\varepsilon_u/\varepsilon_v}\) varies with \(\theta_u\), where \(\varepsilon_u\) and \(\varepsilon_v\) are two eigenvalues of the effective permittivity \(\varepsilon_{\text{eff}}\). In figure 1(b), we obtain the index ellipsoid by solving the eigenfrequency problem of the unit cell (by using COMSOL Multiphysics), the result converges well when the wavelength is larger than three periods. The anisotropy showed has a range of \(1 \leq \gamma \leq n_\beta/n_\alpha\) and \(\theta_u \in [-\pi/2, \pi/2]\). Moreover, \(n_\alpha n_\gamma = n_\alpha n_\beta\) is a constant, where \(n_\alpha = \sqrt{\varepsilon_1\mu_3}\) and \(n_\gamma = \sqrt{\varepsilon_3\mu_1}\) are the principal indices of the effective medium. We are showing a very particular situation \((\theta_A = 3 \theta_B)\) just for demonstration but actually approximated expressions for the effective anisotropy and the principal axes direction (when \(n_\beta/n_\alpha\) is not much larger than unity) can be found as

\[
\gamma = (n_\beta/n_\alpha)^\cos(\theta_A-\theta_B),
\]

\[
\theta_u = (\theta_A + \theta_B)/2.
\]  

Such an approximation fits the numerical results well, as shown in figure 1(b). With (1), both \(\gamma\) and \(\theta_u\) can be controlled by choosing \(\theta_A\) and \(\theta_B\). For more details, we pick the cases of \(\theta_B = 0.5, 0, -0.7\); the elliptical dispersion surfaces of the effective medium are showed in the inset of figure 1(b). The size and anisotropy factor of the corresponding effective dispersion surface can be represented by the geometric mean \((n)\) and ratio \((\gamma)\) among the major and minor principal refractive indices \([10]\). First, this result indicates that any dispersion surfaces of the same size \((n = \sqrt{n_\alpha n_\beta}\) can be proved invoking \([24]\) or \([25]\)) that are less anisotropic than an original anisotropic dielectric \((1 \leq \gamma \leq n_\beta/n_\alpha)\), which are available in our scheme. Second, the constant \(n\) in the whole transformation medium means a constant determinant of metric \([10]\); this corresponds to a coordinate map (in TO) with an area-preserving property. Any regions before and after transformation have the same area ratio; such a map has been previously used for designing non-magnetic TO devices \([23, 26]\). We have chosen the checkerboard configuration as a constructive example to demonstrate the above statement. In fact, even if the optical axes are mixed in any other geometries (e.g. stratified layers, a cylinder array, etc), one can still prove the product of the eigenvalues of the permittivity tensor remains the same before and after mixing. Therefore, we have established that the usage of an optical-axis profile (coming from a single type of anisotropic dielectric material) as a transformation medium corresponds to a general area-preserving map. It only requires the anisotropic factor of this dielectric material to be larger than the maximum one implied from the coordinate map.

3. Transformation optics using an optical-axis profile

3.1. A wave expander

Next, we use a wave expander to illustrate how a TO function is translated into a series of effective dispersion surfaces realizable by an optical-axis profile of the same anisotropic dielectric material. The device, bounded by \(x = 0\) to \(b\) and \(y = \pm f(x)\), is outlined by black solid lines in figure 2(a). It has a height \(2(w + h)\) at boundary \(x = 0, b\) with \(w = b/2 = 3h = 20\); the details are shown in the caption. Being transformed from a rectangle \(b \times 2w\) in a homogeneous virtual space of coordinates \((x', y')\), it can expand the width of a Gaussian beam travelling from left to right by a factor of \((w + h)/(w - h)\).
Figure 2. (a) Wave expander bounded by $x = 0$ to $b$ and $y = \pm f(x)$ where $f(x) = w + h \sin((\pi(x/b - 1/2))$ with $w = b/2 = 3h = 20$. The purple grid lines at regular $x'$ and $y'$ in the virtual space divide the device into equal areas. The color pattern/white arrows show the profile of the anisotropy factor $\gamma/\hat{u}$ of the dispersion surfaces. (b) $H_z$-field pattern from a full-wave simulation of the wave expander constructed with checkerboard squares of optical axes of the same anisotropic dielectric. A Gaussian beam at wavelength 5 with width 16 impinging from the left is expanded in width.

Here, a ‘slicing’ technique is introduced to generate an area-preserving map with prescribed boundaries in two steps. Firstly, the device is cut into slices of equal areas using the purple vertical lines in the $x$-direction. Secondly, each slice is cut into pieces of equal areas in the $y$-direction using the purple curved lines. These two sets of lines denote the map from a regular grid in $(x', y')$ and is equivalent to solving

$$\frac{dx(x')}{dx'} = \frac{w}{f(x)} = \frac{y'}{y},$$

with $0 \leq x' \leq b$. The map induces, from TO, the effective permittivity tensor where $\varepsilon_{xx} = w^2/f^2(x)$, $\varepsilon_{xy} = w^2yf'(x)/f^3(x)$, and $\varepsilon_{yy} = (f^6(x) + w^4y^2(f'(x))^2)/(w^2f^4(x))$. From the effective permittivity, we can obtain the effective dispersion surfaces of varying the anisotropy factor $\gamma$ and varying the direction of the principal axes $\hat{u}$ (taking the one with the smaller principal index in this example), as shown by the color pattern and the white arrows in figure 2(a) respectively. Now, we use a dielectric material with $n_\beta/n_\alpha = 2.4$ (this has to be larger than the maximum $\gamma$ of 2.2 implied from the map) to construct the transformation medium. For comparison, some liquid crystals have $\gamma$ around 1.4 and also layered silicon mixing with air, which can obtain $\gamma$ around 2 [19, 27]. By further optimizing the shape of the wave expander instead of a simple sine function used here, we expect that $\gamma$ can be further lowered. The constant index $n$ in the whole space is always $\sqrt{n_\alpha n_\beta}$ (we set it equal to one for simplicity, it should be scaled by the index of the background medium in practice), while a unit $\gamma$ is chosen outside the device. To construct the prescribed $\gamma$ and $\hat{u}$, the optical axis at each square is solved from (1) which is found to have good accuracy to design devices working in the effective medium regime when the wavelength is larger than around five periods of the checkerboard. The resultant $\theta_{A/B}$ is slowly varying in the subwavelength.

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Even outside the TO device, the background medium with unit γ can also be achieved by the small squares similarly. Full-wave simulations (COMSOL Multiphysics) are then performed with such an optical-axis profile. The operational wavelength is chosen as 5 while the sizes of the squares are set to be 0.5, which is much smaller than the wavelength to validate the effective medium description. The field pattern is plotted using a color pattern on figure 2(b). A Gaussian beam of width 16 impinging from the left is expanded by around 2 times in width after passing through the device as expected. Actually, we have also performed a simulation when the device is only modelled using the effective medium. The two field patterns are found to be almost identical. Therefore, the medium with the designed optical-axis profile accurately realizes the wave expander.

3.2. A virtual shifter

The same single anisotropic dielectric medium can be used to perform other functionalities by tuning to another optical-axis profile. Here, we consider a virtual shifter. It makes a scatterer look as if it is shifted by a distance \[28\]. Figure 3(a) shows the shifter that has an outer radius \(R_1 = 10\) coating on a scatterer of radius \(R_0 = 0.5\) centering at a distance \(s_0 = -2.5\) from the origin on the \(x\)-axis. The corresponding shape in the virtual space is the area between two concentric circles of radii \(R_1\) and \(R_0\) centered at the origin. The device makes the scattering field look the same as those coming from a bare scatterer at the origin with the same radius \(R_0\). The coordinate map from the virtual space (with cylindrical coordinates radius \(r'\) and angle \(\tau'\)) to the physical space is shown by the black solid lines for different \(r'\) and \(\tau'\) values which are chosen to divide the device into equal areas in both the radial and the azimuthal directions. We have used the same ‘slicing’ technique to generate the map in two steps. Here, we first use a series of ellipses to divide the device into rings of equal areas. Second, every ring is then cut into pieces.
Figure 4. The $H_z$-field pattern of (a) a virtual shifter of radius 10 constructed with checkerboard squares of optical axes in shifting a scatterer of radius 0.5 from $(-2.5, 0)$ to origin, (b) a bare scatterer at origin (perceived situation), and (c) the same bare scatterer at $(-2.5, 0)$ without shifter.

of equal area in the azimuthal direction. The described map can be written mathematically as

$$x(r', \tau') = a(r') r' \cos (\psi(r', \tau')) + s(r'),$$

$$y(r', \tau') = \frac{r'}{a(r')} \sin (\psi(r', \tau')).$$

(3)

Every ellipse is transformed from a circle centered at the origin in the virtual space with the same area and it is characterized by $a^2(r')$, the ratio of the lengths of the major and minor axes of ellipse, and $s(r')$, the center of ellipse. They are chosen to have polynomial forms for simplicity:

$$a(r') = 1 + \frac{a_0}{r'} \left( \frac{R_1 - r'}{R_1 - R_0} \right)^2 \frac{r' - R_0}{R_1 - R_0},$$

$$s(r') = \left( \frac{R_1 - r'}{R_1 - R_0} \right)^2 \left( s_0 \frac{R_1 - r'}{R_1 - R_0} + s_1 \frac{r' - R_0}{R_1 - R_0} \right),$$

(4)

with coefficients $a_0 = 1.13$ and $s_1 = 3.84$. We have also chosen $a(R_0) = a(R_1) = 1$, $s(R_0) = s_0$, $s(R_1) = 0$ and $a'(R_1) = s'(R_1) = 0$ to ensure that the inner boundary of the device is mapped to the boundary of the perceived scatterer at origin, while the outermost region of the device approaches smoothly into the background medium of unit $\gamma$. The function $\psi(r', \tau')$ can be determined by requiring that the map cuts each ring into pieces of equal area; it is obtained (through the unit determinants of metric) by solving

$$\tau' = \psi + \frac{1}{a} \frac{ds}{dr'} \sin \psi + \frac{r'}{2a} \frac{da}{dr'} \sin 2\psi.$$

(5)

Similar to the wave expander, $\gamma$ and $\hat{u}$ can be obtained from TO and are shown by the color pattern and white arrows in figure 3(a). $\gamma$ ranges from 1 to 2.4, achievable by the previous single anisotropic material. By applying (1), the global optical axes direction profile of the A/B squares (with size 0.5) is found and shown in figure 3(b). Again, full-wave simulations are performed with the optical-axis profile. Figure 4(a) shows the $H_z$-field pattern when a Gaussian

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beam of width 10 and wavelength 5 impinges from the upper side. The beam is scattered by the scatterer (assumed to be a perfect magnetic conductor for simplicity) that is coated by the shifter. Compared with the case of a bare scatterer centered at the origin as shown in figure 4(b), the field patterns, outside the shifter region (outlined by dashed lines in figure 4), are almost identical. For comparison, the field pattern for a bare scatterer at (−2.5, 0) is shown in figure 4(c) as a contrast before shifting.

4. Conclusion

We have shown the usage of a single kind of anisotropic dielectric material with variable optical-axis profile arranged in a checkerboard fashion for TP, extending the usage of checkerboards with isotropic materials [29–31]. The resultant medium can realize dispersion surfaces of constant size but variable anisotropy to construct an arbitrary area-preserving map. By switching to different optical-axis profiles, the medium can be used to construct various devices, including a wave expander and a virtual shifter as illustrative examples. Moreover, one is free to choose the geometry the optical axes are mixed in; the transformation medium can be more functional when it uses a material with larger anisotropy. These investigations will be useful for designing future TO devices with both spatial and temporal functionalities.

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References

[1] Pendry J B, Schurig D and Smith D R 2006 Controlling electromagnetic fields Science 312 1780
[2] Leonhardt U 2006 Optical conformal mapping Science 312 1777
[3] Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Metamaterial electromagnetic cloak at microwave frequencies Science 314 977–80
[4] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Electromagnetic wormholes and virtual magnetic monopoles from metamaterials Phys. Rev. Lett. 99 183901
[5] Cai W, Chettiar U K, Kildishev A V and Shalaev V M 2007 Optical cloaking with metamaterials Nature Photon. 1 224
[6] Lai Y, Ng J, Chen H Y, Han D Z, Xiao J J, Zhang Z-Q and Chan C T 2009 Illusion optics: the optical transformation of an object into another object Phys. Rev. Lett. 102 253902
[7] Genov D A, Zhang S and Zhang X 2009 Mimicking celestial mechanics in metamaterials Nature Phys. 5 687
[8] Narimanov E E and Kildishev A V 2009 Optical black hole: broadband omnidirectional light absorber Appl. Phys. Lett. 95 041106
[9] Cheng Q, Cui T J, Jiang W X and Cai B G 2010 An omnidirectional electromagnetic absorber made of metamaterials New J. Phys. 12 063006
[10] Li J and Pendry J B 2008 Hiding under the carpet: a new strategy for cloaking Phys. Rev. Lett. 101 203901
[11] Valentine J, Li J, Zentgraf T, Bartal G and Zhang X 2009 An optical cloak made of dielectrics Nature Mater. 8 568
[12] Gabrielli L H, Cardenas J, Poitras C B and Lipson M 2009 Silicon nanostructure cloak operating at optical frequencies Nature Photon. 3 461
[13] Ergin T, Stenger N, Brenner P, Pendry J B and Wegener M 2010 Three-dimensional invisibility cloak at optical wavelengths Science 328 337
[14] Renger J, Kadic M, Dupont G, Acimovic S S, Guenneau S, Quidant R and Enoch S 2010 Hidden progress: broadband plasmonic invisibility Opt. Express 18 15757–68
[15] Fischer J, Ergin T and Wegener M 2011 Three-dimensional polarization-independent visible-frequency carpet invisibility cloak Opt. Lett. 36 2059
[16] Urzhumov Y A and Smith D R 2010 Transformation optics with photonic band gap media Phys. Rev. Lett. 105 163901
[17] Chen X, Luo Y, Zhang J, Jiang K, Pendry J B and Zhang S 2011 Macroscopic invisibility cloaking of visible light Nature Commun. 2 176
[18] Zhang B, Luo Y, Liu X and Barbastathis G 2011 Macroscopic invisibility cloak for visible light Phys. Rev. Lett. 106 033901
[19] Zhang J, Liu L, Luo Y, Zhang S and Mortensen N A 2011 Homogeneous optical cloak constructed with uniform layered structures Opt. Express 19 8625
[20] Luo Y, Zhang J, Chen H, Ran L, Wu B-I and Kong J A 2009 A rigorous analysis of plane-transformed invisibility cloaks IEEE Trans. Antennas Propag. 57 3926–33
[21] Chen H and Chan C T 2008 Electromagnetic wave manipulation by layered systems using the transformation media concept Phys. Rev. B 78 054204
[22] Chen H, Hou B, Chen S, Ao X, Wen W and Chan C T 2009 Design and experimental realization of a broadband transformation media field rotator at microwave frequencies Phys. Rev. Lett. 102 183903
[23] Han T, Qiu C-W, Dong J-W, Tang X and Zouhdi S 2011 Homogeneous and isotropic bends to tunnel waves through multiple different/equal waveguides along arbitrary directions Opt. Express 19 13020
[24] Nevard J and Keller J B 1985 Reciprocal relations for effective conductivities of anisotropic media J. Math. Phys. 26 2761–65
[25] Guenneau S and Zolla F 2003 Duality relation for the Maxwell system Phys. Rev. E 67 026610
[26] Vasic B, Isic G, Gajic R and Hingerl K 2009 Coordinate transformation based design of confined metamaterial structures Phys. Rev. B 79 085103
[27] Werner D H, Kwon D-H, Khoo I-C, Kildishev A V and Shalaev V M 2007 Liquid crystal clad near-infrared metamaterials with tunable negative-zero-positive refractive indices Opt. Express 15 3342
[28] Jiang W X and Cui T J 2010 Moving targets virtually via composite optical transformation Opt. Express 18 5161
[29] Mortola S and Steffe S 1985 A two-dimensional homogenization problem Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. 78 77–82
[30] Craster R V and Obnosov Y V 2001 Four-phase checkerboard composites SIAM J. Appl. Math. 61 1839–56
[31] Milton G W 2001 Proof of a conjecture on the conductivity of checkerboards J. Math. Phys. 42 4873–82