Adapted algebras for the Berenstein-Zelevinsky conjecture

Philippe CALDERO

ABSTRACT. Let $G$ be a simply connected semi-simple complex Lie group and fix a maximal unipotent subgroup $U^-$ of $G$. Let $q$ be an indeterminate and let's denote by $\mathcal{B}^*$ the dual canonical basis, [18], of the quantized algebra $\mathbb{C}_q[U^-]$ of regular functions on $U^-$. Following [19], fix a $\mathbb{Z}_{\geq 0}^N$-parametrization of this basis, where $N = \dim U^-$. In [2], A. Berenstein and A. Zelevinsky conjecture that two elements of $\mathcal{B}^*$ q-commute if and only if they are multiplicative, i.e. their product is an element of $\mathcal{B}^*$ up to a power of $q$. For all reduced decomposition $\tilde{w}_0$ of the longest element of the Weyl group of $g$, we associate a subalgebra $A_{\tilde{w}_0}$, called adapted algebra, of $\mathbb{C}_q[U^-]$ such that 1) $A_{\tilde{w}_0}$ is a $q$-polynomial algebra which equals $\mathbb{C}_q[U^-]$ up to localization, 2) $A_{\tilde{w}_0}$ is spanned by a subset of $\mathcal{B}^*$, 3) the Berenstein-Zelevinsky conjecture is true on $A_{\tilde{w}_0}$. Then, we test the conjecture when one element belongs to the $q$-center of $\mathbb{C}_q[U^-]$.

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0. Introduction.

0.1. Let $G$ be a simply connected semi-simple complex group, and let $\mathfrak{g}$ be its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the corresponding triangular decomposition. For all $\lambda$ in the set of integral dominant weights $P^+$, let $V(\lambda)$ be the simple $\mathfrak{g}$-module with highest weight $\lambda$.

For all Lie algebra $\mathfrak{a}$, the enveloping algebra of $\mathfrak{a}$ will be denoted by $U(\mathfrak{a})$. The module $V(\lambda)$ is naturally endowed with a structure of $U(\mathfrak{g})$-module, and it is known that $V(\lambda) = U(\mathfrak{n}^-)v_\lambda$, where $v_\lambda$ is a highest weight vector of $V(\lambda)$. By the work of Lusztig, [19], there exists a base $\mathcal{B}$ of $U(\mathfrak{n}^-)$ with the following property:
Set $B(\lambda) := \{b \in B, bv_\lambda \neq 0\}$. Then, for all $\lambda$ in $P^+$, $B(\lambda).v_\lambda$ is a base of $V(\lambda)$.

Let $U^-$ be the maximal unipotent subgroup of $G$ such that $\text{Lie}(U^-) = n^-$ and let $\mathbb{C}[U^-]$ be the algebra of regular functions on $U^-$. It is well known that $\mathbb{C}[U^-]$ is isomorphic to the symmetric algebra $S(n^-)$, and thus to $S(n)$ via the Killing form. The left multiplication of $U^-$ on itself provides a natural morphism $\phi$ from $U(n^-)$ on the algebra of differential operators on $\mathbb{C}[U^-]$. The pairing on $U(n^-) \times \mathbb{C}[U^-]$ such that $(u, f)$ is the value of $\phi(u)(f)$ on the identity of $U^-$, is non degenerate.

Let $B^*$ be the basis of $\mathbb{C}[U^-]$ dual to $B$ for this pairing.

0.2. Berenstein and Zelevinsky have a conjecture in order to calculate the basis $B^*$. This conjecture involves the standard $q$-deformation of the basis $B^*$. Let’s present some notations. The objects introduced in the previous section have a $q$-deformation. Let $q$ be an indeterminate and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra on $\mathfrak{g}(q)$ as defined by Drinfeld and Jimbo. Let $U_q(n)$ and $U_q(n^-)$ be the quantized subalgebra corresponding to $U(n)$ and $U(n^-)$, see 1.2, and let $\mathbb{C}_q[U^-]$ be the quantized algebra of regular functions on $U^-$. For all $\lambda$ in $P^+$, let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module, 1.3, with highest weight vector $v_\lambda$. Then, there exists

1) A canonical basis, still denoted by $B$, of $U_q(n^-)$, with the following property

Set $B(\lambda) := \{b \in B, bv_\lambda \neq 0\}$. Then, for all $\lambda$ in $P^+$, $B(\lambda).v_\lambda$ is a base of $V_q(\lambda)$.

2) A pairing on $U_q(n^-) \times \mathbb{C}_q[U^-]$ which quantifies the pairing in 0.1 and which permits to define the dual canonical basis $B^*$ in $\mathbb{C}_q[U^-]$.

Remark that we can recover both basis of 0.1 by specializing $q$ on 1. Remark also that $\mathbb{C}_q[U^-]$ is isomorphic to $U_q(n)$.

We say that two elements $u$ and $u'$ in $\mathbb{C}_q[U^-]$ q-commute if there exists an integer $m$ such that $uu' = q^m u'u$. Two elements $b^*$ and $b'^*$ in $B^*$ are said to be multiplicative if there exists an $m$ such that $q^m b^*b'^* \in B^*$. We can formulate the Berenstein-Zelevinsky conjecture.

**Conjecture.** Two elements of the dual canonical basis $B^*$ q-commute if and only if they are multiplicative.

Let $A$ be a subalgebra of $\mathbb{C}_q[G]$ such that $A$ is generated as a space by a subset of $B^*$. In the sequel, we say that the Berenstein-Zelevinsky conjecture is true on $A$ if the assertion is true when both elements are in $A$.

0.3. We present results about the conjecture. Recall that the elements of the dual canonical basis are labelled by elements in $\mathbb{Z}_{\geq 0}^N$, where $N$ is the dimension of $n$. This is the so-called Lusztig parametrization of $B^*$ which depends on the choice of a reduced decomposition $\tilde{w}_0$ of the longest element $w_0$ of the Weyl group.

In [21], the author uses the quiver approach of quantum groups. First of all, he obtains the only if part of the Berenstein-Zelevinsky conjecture, [loc. cit., Corollary 4.5], in the simply laced case. Maybe the most remarkable result of [21] is obtained in the case $A_n$. It can be interpreted as follows : for all quiver orientation $\Omega$ of the diagram of $A_n$, the author proves that there exists a subalgebra $A_\Omega$ of maximal GK-dimension of $\mathbb{C}_q[U^-]$ which is compatible with $B^*$ and on which the Berenstein-Zelevinsky conjecture is true. Moreover, each pair of elements in $B^* \cap A_\Omega$ q-commute (and so are multiplicative).
and the previous parametrization yields a correspondence between $B^* \cap A_\Omega$ and the set $C_\Omega$ of integral points of a simplicial cone in $\mathbb{R}^N$.

Let $g$ of type $A_n$. The non zero classical minors in $\mathbb{C}[U^-]$ belong to the (classical) dual canonical basis. They have a quantum analogue in $\mathbb{C}_q[U^-]$ which belong to the (quantum) dual canonical basis. A natural question comes up : is the assertion of the Conjecture true when both elements are quantum minors. In [13], Leclerc, Nazarov and Thibon give a positive answer to this question.

0.5. The main results of this article are

1) a generalization of the result of [21] for all semi-simple Lie algebra $g$, see Theorem 2.2. An adapted subalgebra of $\mathbb{C}_q[U^-]$, see Definition 2.2, is an algebra which is spanned by a maximal subset $P^*$ of $B^*$ such that two elements in $P^*$ are multiplicative. For all reduced decomposition $\tilde{w}_0$ of $w_0$, we provide an adapted algebra $A_{\tilde{w}_0}$ which generalizes the algebra $A_\Omega$ discussed in 0.3. Recall that in [5] and [6], we provide, for all reduced decomposition of $w_0$, an subalgebra $A_{\tilde{w}_0}$ of $\mathbb{C}_q[U^-]$ such that a) the algebra $A_{\tilde{w}_0}$ is generated by a family $S_{\tilde{w}_0}$ of $N$ algebraically independant elements which pairwise $q$-commute b) $A_{\tilde{w}_0}$ and $\mathbb{C}_q[U^-]$ are equal up to localization by the multiplicative part generated by $S_{\tilde{w}_0}$. Using Kashiwara’s crystal basis and some standard monomial theory, we prove that $A_{\tilde{w}_0}$ is spanned by a subset of $B^*$ and that the Conjecture is true on $A_{\tilde{w}_0}$.

Then, we prove that $A_{\tilde{w}_0}$ is an adapted algebra.

2) we test the Berenstein-Zelevinsky conjecture on the $q$-center of $\mathbb{C}_q[U^-]$, i.e. the space generated by elements which $q$-commute with all homogeneous elements of $\mathbb{C}_q[U^-]$. Indeed, as the $q$-center of $\mathbb{C}_q[U^-]$ is spanned by a part of the dual canonical basis, a natural test for the conjecture can be stated as follows : is the conjecture true if one element belongs to the $q$-center? We give a positive answer to this question. Moreover, it reduces the conjecture to a subset of $B^*$, see Corollary 3.2.

3) we study the case when $g$ is of type $B_2$, see 3.4, and we remark that in this case, $\mathbb{C}_q[U^-]$ is a direct sum of adapted algebras.

To conclude, we give a presentation of these results in terms of cones in $\mathbb{Z}_>^N$, which are sets of parametrizations for the part $P_{\tilde{w}_0}^*$ associated to $A_{\tilde{w}_0}$. These cones generalize the cones $C_\Omega$ discussed in 0.4.

1. Notations and recollection on global base.

1.1. Let $g$ be a semi-simple Lie $\mathbb{C}$-algebra of rank $n$. We fix a Cartan subalgebra $h$ of $g$. Let $g = n^- \oplus h \oplus n$ be the triangular decomposition and let $\{\alpha_i\}_i$ be a base of the root system $\Delta$ resulting from this decomposition. Let $\Delta^+$ be the set of positive roots. Let $P$ be the weight lattice generated by the fundamental weights $\varpi_i$, $1 \leq i \leq n$. Let $P^+ := \sum_i \mathbb{Z}_{\geq 0} \varpi_i$ and $P := \sum_i \mathbb{Z} \varpi_i$ be respectively the semigroup of integral dominant weights and the group of integral weights. We endow $P^+$ with the ordering $\leq$ defined by $\lambda \leq \mu \iff \mu - \lambda \in P^+$. Let $W$ be the Weyl group, generated by the reflections corresponding to the simple roots $s_i := s_{\alpha_i}$, with longest element $w_0$. We note $<,>$ the $W$-invariant form on $P$.

1.2. Let $d$ be an integer such that $<P,P> \subset (2/d)\mathbb{Z}$. Let $q$ be a indeterminate and set $\mathbb{K} = \mathbb{C}(q^{1/d})$. Let $U_q(g)$ be the simply connected quantized enveloping algebra on
For all $\beta$, unique form, $[22], [24], (\beta < , > X_1, Y_1)$. In this case, we set $w_t(\alpha U_{\beta}) = \text{positive, resp. negative, weights and the quantum Serre relations.}$ For all $\lambda \in P$, let $K_\lambda$ the corresponding element in the algebra $U_q^0 = \mathbb{K}[P]$ of the torus of $U_q(\mathfrak{g})$. Recall the triangular decomposition $U_q(\mathfrak{g}) = U_q(n^-) \otimes U_q^0 \otimes U_q(n)$. We define the following subalgebras of $U_q(\mathfrak{g})$:

$$U_q(b) = U_q(n) \otimes U_q^0, \quad U_q(b^-) = U_q(n^-) \otimes U_q^0.$$  

$U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra and the comultiplication $\Delta$, the antipode $S$ and the augmentation $\varepsilon$ are given by

$$\Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \quad \Delta F_i = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i, \quad \Delta K_\lambda = K_\lambda \otimes K_\lambda$$

$$S(E_i) = -K_{\alpha_i}^{-2} E_i, \quad S(F_i) = -F_i K_{\alpha_i}^2, \quad S(K_\lambda) = K_{-\lambda}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) = 1.$$  

Let’s denote by $ad$ the adjoint action. Remark that there exists an algebra isomorphism $U_q(n) \simeq U_q(n^-)$ which sends $E_{\alpha_i}$ on $F_{\alpha_i}$.

If $\alpha = \sum m_i \alpha_i$ is in $Q^+ := \sum \mathbb{Z}_{\geq 0} \alpha_i$, resp. $Q^- := \sum \mathbb{Z}_{\leq 0} \alpha_i$, then an element $X$ of the subspace of $U_q(n)$, resp. $U_q(n^-)$, generated by $\{E_i^{n_1} \cdots E_i^{n_k} \mid n_i \alpha_{i_1} + \ldots + n_i \alpha_{i_k} = \alpha\}$, resp. $\{F_i^{n_1} \cdots F_i^{n_k} \mid n_i \alpha_{i_1} + \ldots + n_i \alpha_{i_k} = -\alpha\}$ will be called element of weight $\alpha$. In this case, we set $\text{wt}(X) = \alpha$ and $\text{Tr}(X) = \sum n_i$.

For each $u$ in $U_q(\mathfrak{g})$ we set $u(1) \otimes u(2) = \Delta(u) \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. There exists a unique form, $[22], [24], ( , )$ on $U_q(b) \times U_q(b^-)$ such that:

$$(E_i, F_i) = \delta_{ij}(1-q_i^2)^{-1},$$

$$(u^+ , u^- , u^- -) = (\Delta(u^+), u_1^- \otimes u_2^-), \quad u^+ \in U_q(b); u_1^-, u_2^- \in U_q(b^-)$$

$$(u^+_2 , u^- , u^- ) = (u_2^+ \otimes u_1^-, \Delta(u^-)), \quad u^- \in U_q(b^-); u_1^+, u_2^- \in U_q(b)$$

$$(K_\lambda, K_\mu) = q^{-(\lambda, \mu)}(K_\lambda, F_i) = 0, (E_i, K_\lambda) = 0, \quad \lambda, \mu \in P$$

For all $\beta$ in $Q^+$, let $U_q(n)_{\beta}$, resp. $U_q(n^-)_{-\beta}$, be the subspace of $U_q(n)$, resp. $U_q(n^-)$, with weight $\beta$, resp. $-\beta$. The form $( , )$ is non degenerate on $U_q(n)_{\beta} \times U_q(n^-)_{-\beta}$, $\beta \in Q^+$. We have:

$$(X K_\lambda, Y K_\mu) = q^{-(\lambda, \mu)}(X, Y), \quad X \in U_q(n), Y \in U_q(n^-)$$

We can define a bilinear form $< , >$ on $U_q(\mathfrak{g}) \times U_q(\mathfrak{g})$ by:

$$< X_1 K_\lambda S(Y_1), Y_2 K_\mu S(X_2) > = (X_1, Y_2)(X_2, Y_1)q^{-(\lambda, \mu)/2}$$

where $X_1, X_2 \in U_q(n), Y_1, Y_2 \in U_q(n^-), \lambda, \mu \in P$. Moreover, this form is non degenerate.
We define the ring automorphism \( x \mapsto \eta(x) \) of \( U_q(n) \), resp. \( U_q(n^-) \), such that \( \eta(q) = q^{-1} \) and \( \eta(E_i) = E_i \), resp. \( \eta(F_i) = F_i \) for all \( i \), \( 1 \leq i \leq n \). We also define the antihomomorphism \( \sigma \) of the \( \mathbb{C}(q) \)-algebra \( U_q(n) \), resp. \( U_q(n^-) \), such that \( \sigma(E_i) = E_i \), resp. \( \sigma(F_i) = F_i \) for all \( i \), \( 1 \leq i \leq n \).

### 1.3. For all \( U_q^0 \)-module \( M \), set \( M_\lambda := \{ m \in M, K_\mu.m = q^\langle \mu, \lambda \rangle m \} \). Elements of \( M_\mu \) are called elements of weight \( \mu \). For all \( \lambda \) in \( P^+ \), \( V_q(\lambda) \) denotes the simple \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \), with highest weight vector \( v_\lambda \). It is known, see [10, 4.3.6], that this module verifies the Weyl character formula. In particular, for all \( w \) in \( W \), \( \dim V_q(\lambda)w_\lambda = 1 \), and we can define extremal vectors \( v_\lambda \) in \( V_q(\lambda) \) as in the classical case.

The dual space \( V(\lambda)^* \) is endowed with a structure of left \( U_q(\mathfrak{g}) \)-module by twisting with the antipode.

### 1.4. Fix a reduced decomposition \( w_0 = s_i \ldots s_iN \) of \( w_0 \). Set \( \beta_s := s_i \ldots s_iN(\alpha_i) \), \( 1 \leq s \leq N \). Recall that \( \{ \beta_s, 1 \leq s \leq N \} \) is exactly the set of positive roots. We define a total ordering of the positive roots as follows:

\[
\beta_1 > \beta_2 > \ldots > \beta_N.
\]

For \( i \), \( 1 \leq i \leq n \), let \( T_i \) be the Lusztig automorphism, [19, 37.1.3], [23], associated to \( i \). We introduce the following elements in \( U_q(n) : E_\beta_s = T_i \ldots T_{i-1}(E_{i,1}), 1 \leq s \leq N \). Remark that these elements depend on \( w_0 \). We index the so-called Poincaré-Birkhoff-Witt basis of \( U_q(n) \) by \( m \) in \( \mathbb{Z}_{\geq 0}^N \) as follows

\[
E(m) = E_{\tilde{w}_0}(m) = \prod_{i=N}^{1} \frac{1}{m_i} q_i^{m_i} F_{\beta_i}^{m_i},
\]

where \( [n]_q ! = [n]_q [n-1]_q \ldots [1]_q \), \( [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \). We define in the same way a PBW-basis \( (F(m))_{m \in \mathbb{Z}_{\geq 0}^N} \) of \( U_q(n^-) \) via the isomorphism in 1.2.

Let \( (E(m)^*)_{m \in \mathbb{Z}_{\geq 0}^N} \) be the dual basis of the PBW-basis \( (F(m))_{m \in \mathbb{Z}_{\geq 0}^N} \) for \( (, ) \). By [14], we have

\[
E(m)^* = \sum_{i=1}^{N} \frac{\phi_{m_i}(q_i^2)}{(1 - q_i^2)^{m_i}} E(m),
\]

where \( \phi_m(z) = \prod_{k=1}^{m} (1 - z^k) \). We know by [9, Lemma 1.7] that the lexicographical ordering on the PBW-basis provides a filtration on \( U_q(n) \) whose associated algebra is a \( q \)-polynomial algebra:

**Theorem.** Fix a reduced decomposition \( \tilde{w}_0 \) of \( w_0 \) and set

\[
\mathcal{F}_m(U_q(n)) = \oplus_{n < m} K E_{\tilde{w}_0}(n), \quad m \in \mathbb{Z}_{\geq 0}^N,
\]

where \( \prec \) is the lexicographical ordering of \( \mathbb{Z}_{\geq 0}^N \). Then, the associated graded algebra \( \text{Gr}_{\tilde{w}_0}(U_q(n)) \) is generated by \( \text{Gr}_{\tilde{w}_0}(E_\alpha), \alpha \in \Delta^+ \) and with relations:

\[
\text{Gr}_{\tilde{w}_0}(E_\alpha) \text{Gr}_{\tilde{w}_0}(E_\beta) = q^{\langle \alpha, \beta \rangle} \text{Gr}_{\tilde{w}_0}(E_\beta) \text{Gr}_{\tilde{w}_0}(E_\alpha), \quad \alpha \prec \beta.
\]
In particular $Gr_{\tilde{w}_0}(E^*(m)) Gr_{\tilde{w}_0}(E^*(n)) = Gr_{\tilde{w}_0}(E^*(m+ n))$ up to a power of $q$.  

**1.5.** Let’s define now the following sub-$\mathbb{Z}[q]$-lattices of $U_q(n)$:

$$\mathcal{L} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}^N} \mathbb{Z}[q]E(m), \quad \mathcal{L}^* = \bigoplus_{m \in \mathbb{Z}_{\geq 0}^N} \mathbb{Z}[q]E(m)^*$$

A remarkable result of Lusztig states that these lattices do not depend on a reduced decomposition of $w_0$. The following theorem is due to Lusztig [17]. It introduces the so-called canonical basis $B$ of $U_q(n)$ which will be identified with the canonical basis of $U_q(n^-)$.

**Theorem.** Fix a reduced decomposition $\tilde{w}_0$ of $w_0$. Then, for each $m$ in $\mathbb{Z}_{\geq 0}^N$, there exists a unique element $B(m) = B_{\tilde{w}_0}(m)$ in $U_q(n)$ such that $\eta(B(m)) = B(m)$ and $B(m) \in E(m) + q\mathcal{L}$. The set $B := \{B(m), m \in \mathbb{Z}_{\geq 0}^N\}$ is a basis of $U_q(n)$ which does not depend on $\tilde{w}_0$.  

Hence, for a fixed $\tilde{w}_0$, the bijection $\mathbb{Z}_{\geq 0}^N \rightarrow B$ gives rise to a parametrization of $B$. This will be called the Lusztig’s parametrization of the canonical basis $B$.

Let $B^* = (B(m)^*)_{m \in \mathbb{Z}_{\geq 0}^N}$ be the dual canonical basis of $U_q(n)$. As in [13, Proposition 16], we have

**Proposition.** Fix a reduced decomposition $\tilde{w}_0$ of $w_0$. Then, for each $m$ in $\mathbb{Z}_{\geq 0}^N$, the element $B(m)^*$ is the unique element $X$ of $U_q(n)$ with weight $wt(X) = \sum m_i b_i$ such that

$$\eta(X) = (-1)^{tr(X)} q^{<wt(X), wt(X)>/2} q_X \sigma(X), \quad X = E(m)^* + q\mathcal{L}^*,$$

where $q_X = \prod_i q_i^{m_i}$, $wt(X) = \sum_i m_i \alpha_i$.

**Remark.** Suppose that $\tilde{w}_0$ corresponds to a quiver orientation. Then, by [21, Lemma 4.1], see also [20, par. 7] for the non simply laced case,

\begin{equation}
Gr_{\tilde{w}_0}(B^*(m)) = Gr_{\tilde{w}_0}(E^*(m)).
\end{equation}

Recall that two elements of the dual canonical basis are called multiplicative if their product is an element of the dual canonical basis, up to a power of $q$. The following corollary of the proposition is due to Reineke, [21, Corollary 4.5]. It is an implication in the Berenstein-Zelevinsky conjecture. His proof is also available for the non simply laced case.

**Corollary.** If two elements of the dual canonical basis are multiplicative, then they $q$-commute.  

**1.6.** In this section we consider the canonical basis $B$ in $U_q(n^-)$.

Let $\tilde{E}_i, \tilde{F}_i : U_q(n^-) \rightarrow U_q(n^-)$ be the Kashiwara operators, [19]. For $b \in B$, $\tilde{E}_i(b)$, resp. $\tilde{F}_i(b)$, equals some $b'$ in $B \cup \{0\}$ modulo $q\mathbb{Z}[q]B$. The rule $b \mapsto b'$ defines maps $\tilde{e}_i$ and $\tilde{f}_i$ from $B$ to $B \cup \{0\}$. For $b \in B$, $1 \leq i \leq n$, set $\varepsilon_i(b) = \text{Max}\{r, \tilde{e}_i^r(b) \neq 0\}$, and $\mathcal{E}(b) = \sum_{i=1}^n \varepsilon_i(b) \omega_i$. 

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Fix $\lambda$ in $P^+$ and define the following subset of $B$

$$B(\lambda) = \{ b \in B, \mathcal{E}(\sigma(b)) \leq \lambda \}.$$ 

Let $L(\lambda)$ be the $\mathbb{C}[q]$-sublattice of $V_q(\lambda)$ generated by $\{bv_\lambda, b \in B(\lambda)\}$ and define the following subset of $L(\lambda)/qL(\lambda)$

$$B(\lambda) = \{ b \mod qL(\lambda), b \in B(\lambda) \}.$$ 

In the sequel, the element $b \mod qL(\lambda)$ will be denoted $[b]_\lambda$ or $[b]$ if no confusion occurs. The following is well known, [11, 12.3] :

**Theorem.** For all $\lambda \in P^+$, we have :

(i) the set $B(\lambda)$ is the set of elements $b$ of $B$ such that $bv_\lambda$ is non zero in $V_q(\lambda)$, moreover $B(\lambda)v_\lambda$ is a basis of $V_q(\lambda)$,

(ii) the pair $(L(\lambda), B(\lambda))$ is a crystal basis of $V_q(\lambda)$, see [11, par 4].

**1.7.** Recall, [10, 6.4.27] that for all $\lambda$ in $P^+$, the Kashiwara crystal $B(\lambda)$ is isomorphic to the Littelmann path crystal $C(\lambda)$. Moreover, in this isomorphism, the tensor product can be defined in terms of concatenation of paths [15]. Through this isomorphism, the following propositions are direct consequences of [16, Theorem 10.1].

**Proposition A.** Let $\lambda_i$ in $P^+$, $y_i$ in $W$, $b_i$ in $B(\lambda_i)$, with weight $y_i\lambda_i$, $1 \leq i \leq r$. Suppose $y_1 \leq y_2 \leq \ldots \leq y_r$ for the Bruhat ordering. Then, $b_r \otimes b_{r-1} \otimes \ldots \otimes b_1$ belongs to the component of type $B(\lambda_r + \lambda_{r-1} + \ldots + \lambda_1)$ in the crystal $B(\lambda_r) \otimes B(\lambda_{r-1}) \otimes \ldots \otimes B(\lambda_1)$.

**Proposition B.** Let $\lambda$, $\mu$ in $P^+$, $b$ in $B(\lambda)$, $c$ in $B(\mu)$ with weight $w_0\mu$. Then, $c \otimes b$ belongs to the component of type $B(\mu + \lambda)$ in the crystal $B(\mu) \otimes B(\lambda)$.

For all $\lambda$ in $P^+$ and $w$ in $W$, let $B_w(\lambda)$ the subset of $B(\lambda)$ corresponding to the Demazure module associated to $w$, see [11, 12.4].

**Proposition C.** Let $\lambda$, $\mu$ in $P^+$, $w$ in $W$, $b$ in $B_w(\lambda)$, $c$ in $B_w(\mu)$ with weight $w\mu$. Then, $c \otimes b$ belongs to the component of type $B(\mu + \lambda)$ in the crystal $B(\mu) \otimes B(\lambda)$.

**1.8.** In this section, we define the string parametrization of the canonical basis. For $b$ in $B$ and $i$, $1 \leq i \leq n$, set $\tilde{c}_i^{max}(b) = \tilde{c}_i^{n_i}(b)$, where $n_i(b) := \max\{k, \tilde{c}_i^k(b) \neq 0\}$. Let $\tilde{w}_0 = s_{i_1} \ldots s_{i_N}$ be a reduced decomposition of $w_0$. To an element $b$ in $B$, we associate the element $A_{\tilde{w}_0}(b)$ in $\mathbb{Z}^N_{\geq 0}$ such that the $k$-th component of $A_{\tilde{w}_0}(b)$ is $n_{i_k}(\tilde{c}_{i_{k-1}} \ldots (\tilde{c}_{i_1}^{max}(b)))$. It is known, see [1], that $A_{\tilde{w}_0}$ is injective and that $A_{\tilde{w}_0}(B)$ is the set of integral points of a cone in $\mathbb{R}^N$. Hence, the map $A_{\tilde{w}_0}$ defines a parametrization of the canonical basis. It is called the string parametrization of $B$.

**Proposition.** We have

(i) Fix a reduced decomposition $\tilde{w}_0$ of the longest element of the Weyl group. Let $b, b', b''$ be in $B$ with $A_{\tilde{w}_0}(b) + A_{\tilde{w}_0}(b') = A_{\tilde{w}_0}(b'')$. Then, the $b''$-component of $b'\ast b''$ in the dual canonical basis $B^*$ is a power of $q$. 

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(ii) In the framework of Propositions 1.7, there exists a reduced decomposition of $w_0$ for which the string parametrization of the tensor product is the sum of the string parametrizations of the factors.

Proof. (i) is a part of [8, Theorem 2.3]. Let’s prove (ii). It is enough to prove it in the framework of Proposition 1.7. C. Indeed, the case A is obtained by induction from the case C and the case B is obtained from the case C by setting $w = w_0$. First suppose that $c$ and $b$ are respectively in $B(\mu)$ and $B(\lambda)$ and that for some $i$, $f_i(c) = 0$. Then, by [11, 4.3],

$$\varepsilon_i(c \otimes b) = \varepsilon_i(c) + \varepsilon_i(b)$$

and

$$e_i^{\text{max}}(c \otimes b) = e_i^{\text{max}}(c) \otimes e_i^{\text{max}}(b).$$

Let $\tilde{w} = s_{i_1} \ldots s_{i_k}$ be a reduced decomposition of $w$ and let $\tilde{w}_0 = s_{i_1} \ldots s_{i_k} s_{i_{k+1}} \ldots s_{i_N}$ be the reduced decomposition of $w_0$ which completes $\tilde{w}$ on the right hand. For all $w'$ in $W$, let $\tilde{c}_{w'}$ be the extremal element of $B(\mu)$ corresponding to $w'$. By [11, 12.4], we know that $f_{i_1}(c_w) = 0$ and $e_i^{\text{max}}(c_w) = c_{s_{i_2} \ldots s_{i_k}}$. Using (**) by induction, we obtain that $e_i^{\text{max}}(\ldots(e_i^{\text{max}}(c_w \otimes b))) = e_i^{\text{max}}(\ldots(e_i^{\text{max}}(c_w))) \otimes e_i^{\text{max}}(\ldots(e_i^{\text{max}}(b))).$ By [11, Theorem 12.4], the hypothesis $b \in B_w(\lambda)$ implies that $e_i^{\text{max}}(\ldots(e_i^{\text{max}}(b)))$ is the highest weight vector $b_\lambda$. In particular, $e_i^{\text{max}}(\ldots(e_i^{\text{max}}(c_w)))$ is the highest weight vector $b_\mu$. Using (*) by induction, we obtain the proposition. \(\square\)

1.9. The following proposition is a given in [13, Proposition 33]. In [loc. cit.], the authors study the $A_n$ case, but the generalization of their proof for all $\mathfrak{g}$ is straightforward.

Proposition. Fix $r$ in $\mathbb{Z}_{\geq 0}$ and $\lambda_i$, $1 \leq i \leq r$, in $P^+$. Let $b_i$, $1 \leq i \leq r$, and $b$ respectively in $B(\lambda_1)$, $B(\sum \lambda_i)$, such that

(i) $[b_1] \otimes [b_2] \ldots \otimes [b_r]$ is in the connected component of the crystal graph of $B(\lambda_1) \otimes B(\lambda_2) \otimes \ldots \otimes B(\lambda_r)$ of type $B(\sum \lambda_i)$,

(ii) $[b]$ identifies with $[b_1] \otimes [b_2] \otimes \ldots \otimes [b_r]$ in $B(\sum \lambda_i)$,

then there exists an integer $m$ such that the multiplication rule holds in $U_q(\mathfrak{n})$

$$q^m b_1^* b_2^* \ldots b_r^* = b^* \mod q\mathfrak{L}^*.$$

Corollary. Let $b_i$, $1 \leq i \leq r$, $b$, $m$ as in the previous proposition. Moreover, suppose that

(iii) the $b_i$ pairwise $q$-commute

(iv) for some $\tilde{w}_0$, the string parametrization of $[b]$ is the sum of the string parametrizations of the $[b_i]$.

Then, $q^m b_1^* b_2^* \ldots b_r^* = b^*$.

Proof. From the previous proposition, we obtain

$$q^m b_1^* b_2^* \ldots b_r^* = b^* + q \sum_{b^* \in B^*} c_{b^*}(q)b^*,$$

where $c_{b^*}(q)$ is the coefficient of $[b_1] \otimes [b_2] \otimes \ldots \otimes [b_r]$ in $[b]$. Since $[b_1] \otimes [b_2] \otimes \ldots \otimes [b_r]$ is in the connected component of the crystal graph of $B(\lambda_1) \otimes B(\lambda_2) \otimes \ldots \otimes B(\lambda_r)$, we have $c_{b^*}(q) = 0$ for all $b^* \not\in B^*$. Therefore,

$$q^m b_1^* b_2^* \ldots b_r^* = b^* + q \sum_{b^* \in B^*} c_{b^*}(q)b^* = b^*.$$
where $c_{b^*}(q) \in \mathbb{Z}[q]$. Clearly, the elements of the dual canonical basis of the right hand term of (1.9.1) have the same weight equal to $\sum \text{wt}(b_i^*)$. Applying the automorphism $\eta$ and the antiautomorphism $\sigma$ on (1.9.1), Proposition 1.5 and (iii) give:

$$q^m b_1^* \ldots b_{r-1}^* b_r^* = q^k b^* + q^{k-1} \sum_{b^* \in B^*} c_{b^*}(q^{-1}) b'^*,$$

for some integer $k$. Now, using (iv) and Proposition 1.8 (i), we obtain that the sum in the right hand side of (1.9.1) does not contain any term in $b^*$. Hence, by comparison, $k = 0$ and then we obtain $c_{b^*}(q) = 0$. This gives the corollary.  

Remark that, by a weight argument, the condition (iv) of the corollary implies (but is not equivalent to) the condition (i) of the proposition.

2. Adapted algebras

2.1. For each reduced decomposition of $w_0$, we define a set of pairwise $q$-commuting elements of the dual canonical basis.

**Lemma.** Fix a reduced decomposition $\tilde{w}_0 = s_{i_1} \ldots s_{i_N}$ of $w_0$. For all $r$, $1 \leq r \leq N$, set $y_r = s_{i_1} \ldots s_{i_r}$. Then,

(i) there exists a unique $X^r_{\tilde{w}_0}$ in $U_q(n)$ such that $v^*_{y_r \varpi_r}(uv\varpi_{r'}) = (K_{-\varpi_r}X^r_{\tilde{w}_0}, u)$, for all $u$ in $U_q(b^-)$,

(ii) up to a multiplicative scalar $X^r_{\tilde{w}_0}$ is in $B^*$,

(iii) the $X^r_{\tilde{w}_0}$, $1 \leq r \leq N$, pairwise $q$-commute.

**Proof.** The existence of the $q$-commuting family of elements $\{X^r_{\tilde{w}_0}, 1 \leq r \leq N\}$ is proved in [6, Proposition 3.2]. By [loc. cit., Theorem 1.6], $K_{-2\varpi_r}X^r_{\tilde{w}_0}$ is in $\text{ad}U_q(n)K_{-2\varpi_r}$. Hence, by [7, Theorem 1.6], $X^r_{\tilde{w}_0}$ is in the space generated by $B(\varpi_{r'})^*$. By the Weyl character formula, see 1.3, $X^r_{\tilde{w}_0}$ is equal to an element of $B(\varpi_{r'})^*$ up to a multiplicative scalar. This gives the lemma.

**Remark and Definition.** We can choose the extremal vectors $v_{y_r \varpi_r}$, such that $X^r_{\tilde{w}_0} \in B^*$. Let $m^r$, such that $B(m^r)^* = X^r_{\tilde{w}_0}$, $1 \leq r \leq N$. The elements $m^r$ can be easily calculated at least when the reduced decomposition $\tilde{w}_0$ corresponds to some quiver orientation of the Dynkin diagram of $g$. Let $m^r_k$ be the $k$-th entry of $m^r$. By [6, (3.2.2)] and (1.5.1):

$$m^r_k = \begin{cases} 
1 & \text{if } i_k = i_r \text{ and } k \leq r \\
0 & \text{if not}
\end{cases}$$

It is likely that this formula generalizes for all decomposition $\tilde{w}_0$.

Let $A_{\tilde{w}_0}$ be the algebra generated by the $X^r_{\tilde{w}_0}$, $1 \leq r \leq N$. Then, by [6, Corollary 3.2]

**Proposition.** Fix a reduced decomposition $\tilde{w}_0$ of $w_0$. The algebra $A_{\tilde{w}_0}$ is an algebra of regular functions on a quantum space. Moreover, let $S_{\tilde{w}_0}$ be the multiplicative part generated by the $X^r_{\tilde{w}_0}$, $1 \leq r \leq N$. Then $S_{\tilde{w}_0}$ is an Ore set in $A_{\tilde{w}_0}$ and $S_{\tilde{w}_0}^{-1}A_{\tilde{w}_0} = S_{\tilde{w}_0}^{-1}U_q(n)$.  

2.2. Let’s start with a definition.
**Definition.** Fix a reduced decomposition $\tilde{w}_0$ of $w_0$. A subalgebra $A$ of $\mathbb{C}_q[U^-]$ will be called adapted if it is spanned by a subset $P^*$ of $B^*$ which verifies the following properties:

(i) (multiplicativity) For all $p_i^*$ in $P^*$, $1 \leq i \leq k$, and all $(n_i) \in \mathbb{Z}_{\geq 0}^k$, we have $\prod_i (p_i^*)^{n_i} \in B^*$ up to a power of $q$.

(ii) (maximality) for all $q^*$ in $B^* \setminus P^*$, there exists $p^*$ in $P^*$ such that $p^*$ and $q^*$ are not multiplicative.

We now prove the main theorem of this article. It is a generalization of [21, Theorem 6.1] for all semi-simple Lie algebra $\mathfrak{g}$ and for all reduced decomposition of $w_0$.

**Theorem.** Fix a reduced decompositions $\tilde{w}_0$ of $w_0$. The algebra $A_{\tilde{w}_0}$ is adapted. Moreover, the Ore set $S_{\tilde{w}_0}$, see Proposition 2.1, is a subset of $B^*$ which spans $A_{\tilde{w}_0}$.

**Proof.** Let’s prove that the algebra $A_{\tilde{w}_0}$ is adapted. We know from Lemma 2.1 (iii) that the $X_{\tilde{w}_0}$ $q$-commute. Hence, the multiplicativity part is a consequence of Corollary 1.9. Indeed, the conditions of the corollary are satisfied by Proposition A in 1.7 and Proposition 1.8 (ii). The last assertion of the theorem holds.

We now prove the maximality property. Fix a reduced decomposition $\tilde{w}_0'$ which is compatible with a quiver orientation. Let $C_{\tilde{w}_0}'$ be the set of parametrizations of elements of $S_{\tilde{w}_0}$ for the Lusztig’s parametrization associated to $\tilde{w}_0'$. By Theorem 1.4, (1.5.1) and Proposition 2.1. $C_{\tilde{w}_0}'$ is the set of integral points of a simplicial cone in $\mathbb{R}^N$.

Suppose $q^*$ in $B^* \setminus S_{\tilde{w}_0}$. Then, by Proposition 2.1, there exists an element $p^*$ in $S_{\tilde{w}_0}$ such that $u := p^* q^* \in A_{\tilde{w}_0}$. Suppose that $u$ is a monomial, i.e. belongs to $S_{\tilde{w}_0}$ up to a multiplicative scalar. Then, Theorem 1.4 and (1.5.1) imply $m + n \in C_{\tilde{w}_0}'$, where $m$ and $n$ are respectively the $\tilde{w}_0$-parametrizations of $p^*$ and $q^*$. As $m$ is in the simplicial cone $C_{\tilde{w}_0}'$, we have $n \in C_{\tilde{w}_0}'$, which contradicts the hypothesis $q^*$ in $B^* \setminus S_{\tilde{w}_0}$. This ends the proof of the Theorem. $\Diamond$

In view of Proposition 2.1 and the previous theorem, we can see that the Berenstein-Zelevinsky conjecture is true “up to localization”.

**Remark.** By [19, 14.2.5 and Lemma 1.2.8 (b)], the dual canonical basis is stable by $\sigma$. Hence, for all adapted algebra $A$, the algebra $\sigma(A)$ is adapted. In the section 4.2, we give some examples for $\mathfrak{g}$ of type $A_2$ and $B_2$.

3. The multiplicativity property of the $q$-center.

3.1. Let $Z_q$ be the $q$-center of $U_q(\mathfrak{n})$, i.e. the subspace of $U_q(\mathfrak{n})$ generated by elements which $q$-commute with all homogeneous elements of $U_q(\mathfrak{n})$. Let’s describe $Z_q$ as a space and as an algebra.

**Lemma.** We have:

(i) For all $\lambda$ in $P^+$, there exists an element $z_\lambda$ in $U_q(\mathfrak{n})$ such that $v_{w_0}\lambda (uv_\lambda) = (K_{-\lambda}z_\lambda, u)$, for all $u$ in $U_q(\mathfrak{b}^-)$

(ii) $z_\lambda$ $q$-commutes with all homogeneous elements of $U_q(\mathfrak{n})$ and $Z_q$ is generated as a space by $z_\lambda$, $\lambda \in P^+$,
(iii) \( z_\lambda z_\mu = z_{\lambda + \mu} \) and so, the algebra \( \mathbb{Z}_q \) is generated by the \( z_{\omega_k} \), \( 1 \leq k \leq n \).

(iv) up to a multiplicative scalar \( z_\lambda \) belongs to the dual canonical basis.

Proof. (i), (ii), and (iii) are proved in [4] and the proof of (iv) is similar to the proof of Lemma 2.1 by remarking that \( z_\lambda \) corresponds to the extremal vector \( v_{w_0}^* \).

In the sequel, we choose \( v_{w_0} \) such that \( z_\lambda \) is in \( \mathbb{B}^* \). Fix a reduced decomposition \( \tilde{w}_0 = s_{i_1} \ldots s_{i_N} \) of \( w_0 \) and for all \( k \), \( 1 \leq k \leq n \), let \( r(k) \) be the greatest integer such that \( i_{r(k)} = k \). Then, it is easily seen that \( z_{\omega_k} = X_{\tilde{w}_0}^{r(k)} \). Set \( \mathbf{n}_k = \mathbf{m}_{r(k)} \). Then, when \( \tilde{w}_0 \) corresponds to a quiver orientation, Remark 2.1 gives:

\begin{equation}
z_\lambda = B_{\tilde{w}_0}^* \left( \sum_k \lambda_k \mathbf{n}_k \right), \text{ where } \lambda = \sum \lambda_k \omega_k \in P^+.
\end{equation}

3.2. As in the proof of Theorem 2.2, the following Proposition is a consequence of Corollary 1.9, Proposition B in 1.7 and Proposition 1.8 (ii).

**Proposition.** Let \( b^* \) be in \( \mathbb{B}^* \) and let \( c^* \) be in \( \mathbb{Z}_q \cap \mathbb{B}^* \), then \( b^* \) and \( c^* \) are multiplicative. ◇

Let \( J_q \) be the ideal of \( U_q(\mathfrak{n}) \) generated by the \( z_{\omega_k} \), \( 1 \leq k \leq n \). The previous proposition implies that \( J_q \) is generated as a space by a part \( \mathbb{B}^*(J_q) \) of \( \mathbb{B}^* \). Let \( H_q \) be the subspace generated by the complementary set \( \mathbb{B}^*(H_q) = \mathbb{B}^* \setminus \mathbb{B}^*(J_q) \).

**Corollary** The \( \mathbb{Z}_q \)-module \( U_q(\mathfrak{n}) \) is free with (canonical) basis \( \mathbb{B}^*(H_q) \), i.e. \( U_q(\mathfrak{n}) = \mathbb{Z}_q \otimes H_q \). The Berenstein-Zelevinsky conjecture is true on \( \mathbb{B}^* \) if and only if it is true on \( \mathbb{B}^*(H_q) \).

**Proof.** Fix a reduced decomposition of \( \tilde{w}_0 = s_{i_1} \ldots s_{i_N} \) which is compatible with a quiver orientation. For all \( k \), \( 1 \leq k \leq n \), set \( I_k := \{ r \mid i_r = k \} \). remark that \( \{1, \ldots, N\} \) is the disjoint union of the \( I_k \), \( 1 \leq i \leq n \). Let \( P_{\tilde{w}_0}(\mathbb{Z}_q) \), resp. \( P_{\tilde{w}_0}(H_q) \), be the set of parametrizations of \( \mathbb{Z}_q \cap \mathbb{B}^* \), resp. \( \mathbb{B}^*(H_q) \), for \( \tilde{w}_0 \). By Proposition 3.2, (3.1.1) and Remark 1.5, it is easy to describe \( P_{\tilde{w}_0}(\mathbb{Z}_q) \) and \( P_{\tilde{w}_0}(H_q) \):

The semigroup \( P_{\tilde{w}_0}(\mathbb{Z}_q) \) is generated by \( \mathbf{n}_k \), \( 1 \leq k \leq n \), whose \( r \)-th entry is one if \( r \in I_k \) and 0 if not.

The set \( P_{\tilde{w}_0}(H_q) \) is the set of elements \( \mathbf{m} \) of \( \mathbb{Z}^N_{\geq 0} \) such that for all \( k \), \( 1 \leq k \leq n \), there exists \( r \) in \( I_k \) such that the \( r \)-th entry of \( \mathbf{m} \) is 0.

Now, for all \( \mathbf{m} \) in \( \mathbb{Z}^N_{\geq 0} \), and \( 1 \leq k \leq n \), let \( \mathbf{m}(k) \) be the minimal \( r \)-th entry of \( \mathbf{m} \) when \( r \) runs over \( I_k \). It is clear that \( \mathbf{m} = \sum_k \mathbf{m}(k) \mathbf{n}_k + (\mathbf{m} - \sum_k \mathbf{m}(k) \mathbf{n}_k) \) is the unique decomposition of \( \mathbf{m} \) into a sum of an element of \( P_{\tilde{w}_0}(\mathbb{Z}_q) \) and an element of \( P_{\tilde{w}_0}(H_q) \). This fact and the proposition implies the freeness assertion.

Now, suppose that the Berenstein-Zelevinsky conjecture is true on \( \mathbb{B}^*(H_q) \). Suppose that two elements \( b^* \) and \( b^*_z \) of \( \mathbb{B}^* \) \( q \)-commute. Let’s prove that they are multiplicative. By the previous proposition and the previous assertion, we can decompose \( b^* = b^*_z b^*_h \), \( b^*_z = b^*_z b^*_h \), up to a power of \( q \), where \( b^*_z, b^*_z, b^*_h, b^*_h \in \mathbb{B}^*(H_q) \). By Theorem 1.4, two elements in \( U_q(\mathfrak{n}) \) commute up to a scalar imply that they \( q \)-commute. If we suppose that \( b^*_h \) and \( b^*_h \) do not \( q \)-commute, then we obtain that there exists \( u \notin \mathbb{Z} b^*_h b^*_h \) such that \( b^*_h b^*_h = q^m b^*_h b^*_h + u \). This implies that \( b^* b^*_z = q^m b^*_z b^*_z + q^m u b^*_z b^*_z \), and
$ub^*_zb^*_z \not\in \mathbb{K}b^*b'^*$. This is in contradiction with the fact that $b^*$ and $b'^*$ $q$-commute. Hence, $b^*_h$ and $b'^*_h$ $q$-commute. By the hypothesis, this implies that $b^*_h b'^*_h$ belongs to $B^*$ up to a power of $q$. Hence, this also holds for $b^*_h b'^*_h b^*_z b'^*_z$ by the previous proposition. This implies that $b^*$ and $b'^*$ are multiplicative. \hfill \Box

3.3. In this section, we prove that the intersection of all adapted algebras is $Z_q$. To be more precise:

**Proposition.** Each adapted algebra contains $Z_q$. Moreover, $\bigcap_{\tilde{w}_0} A_{\tilde{w}_0} = Z_q$, where $\tilde{w}_0$ runs over the set of reduced decomposition of $w_0$.

**Proof.** By Proposition 3.2 and by the maximality condition, each adapted algebra contains $Z_q$.

Remark that $\bigcap_{\tilde{w}_0} A_{\tilde{w}_0}$ is spanned by its intersection with $B^*$. Now, let $b^* \in \bigcap_{\tilde{w}_0} A_{\tilde{w}_0} \cap B^*$. Then, by construction, $b^*$ $q$-commutes with the $X_{\tilde{w}_0}$’s. In particular, $b^*$ $q$-commutes with $X^1_{\tilde{w}_0} = E_{i_1}$, where $\tilde{w}_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$. Since each $s_k$, $1 \leq k \leq n$ can be the first factor of a reduced decomposition of $w_0$, it implies that $b^*$ is in $Z_q$. So, the proposition holds. \hfill \Box

Remark that in the intersection formula, we can choose a subset of cardinal $n$ of the set $\tilde{w}_0$ of reduced decomposition of $w_0$.

4. Adapted semigroups and examples.

4.1. Fix a reduced decomposition $\tilde{w}_0$ of $w_0$ which corresponds to a quiver orientation. In this section, we study the set of parametrizations of $A \cap B^*$, where $A$ is an adapted algebra, for $\tilde{w}_0$.

**Proposition.** Let $A$ be a adapted algebra, then the set $C^A_{\tilde{w}_0}$ of parametrizations of $A \cap B^*$ for $\tilde{w}_0$ is a subsemigroup of $\mathbb{Z}^N_{\geq 0}$.

**Proof.** This is a direct consequence of Theorem 1.4 and (1.5.1). \hfill \Box

In the sequel, $C^A_{\tilde{w}_0}$ will be called adapted semigroup for $\tilde{w}_0$. For any reduced decomposition $\tilde{w}'_0$ of $w_0$, set $C^{\tilde{w}'_0}_{\tilde{w}_0} := C^A_{\tilde{w}'_0 \tilde{w}_0}$. Proposition 2.1 and Theorem 2.2 give:

**Proposition.** The subset $C^{\tilde{w}'_0}_{\tilde{w}_0}$ of $\mathbb{Z}^N_{\geq 0}$ is the set of integral points of a simplicial cone of $\mathbb{R}^N$. Moreover, $C^{\tilde{w}'_0}_{\tilde{w}_0}$ generates $\mathbb{Z}^N$ as a group. \hfill \Box

We do not know if an adapted semigroup is in general the set of integral points of a real cone. If this case occurs, it will be called adapted cone.

4.2. In this section, we study the cases where $\mathfrak{g}$ is of type $A_2$ and $B_2$. For each case, we fix a reduced decomposition $\tilde{w}_0$ for the parametrization of the dual canonical basis and then, we give explicitly the generators of the cones $C^{\tilde{w}'_0}_{\tilde{w}_0}$ for all $\tilde{w}'_0$. The generators of $C^{\tilde{w}'_0}_{\tilde{w}_0}$ are calculated from Remark 2.1 and the "reparametrization" $C^{\tilde{w}'_0}_{\tilde{w}_0}$ is calculated with the help of [1, Proposition 7.1 (3)]. The images $\sigma(C^{\tilde{w}'_0}_{\tilde{w}_0})$ can be easily guessed by using [11, Proposition 8.2].
**Example 1.** Let $\mathfrak{g}$ of type $A_2$. Fix $\tilde{w}_0 = s_1s_2s_1$, $\tilde{w}_0' = s_2s_1s_2$. Then, $\mathbb{C}_{\tilde{w}_0}$ is generated by $\{(1, 0, 0), (0, 1, 0), (1, 0, 1)\}$. $\mathbb{C}_{\tilde{w}_0'}$ is generated by $\{(0, 0, 1), (0, 1, 0), (1, 0, 1)\}$. Both cones are stable by $\sigma$ and the union $\mathbb{C}_{\tilde{w}_0} \cup \mathbb{C}_{\tilde{w}_0'}$ is the whole semigroup $\mathbb{Z}_{\geq 0}$.

**Example 2.** Let $\mathfrak{g}$ of type $B_2$ with minuscule weight $\varpi_2$. Fix $\tilde{w}_0 = s_2s_1s_2s_1$, $\tilde{w}_0' = s_1s_2s_1s_2$. The previous construction gives four cones, with intersection $P_{\tilde{w}_0}(Z_q)$ generated by $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. In the following, we present the four cones $\mathbb{C}_{\tilde{w}_0}, \mathbb{C}_{\tilde{w}_0'}, \sigma(\mathbb{C}_{\tilde{w}_0}), \sigma(\mathbb{C}_{\tilde{w}_0'})$ and their canonical set of generators:

$$
\begin{align*}
\mathbb{C}_{\tilde{w}_0} & = \{(1, 0, 0, 0), (0, 1, 0, 0), (1, 0, 1, 0), (0, 1, 0, 1)\}, \\
\mathbb{C}_{\tilde{w}_0'} & = \{(0, 0, 0, 1), (1, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}, \\
\sigma(\mathbb{C}_{\tilde{w}_0}) & = \{(1, 0, 0, 0), (2, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}, \\
\sigma(\mathbb{C}_{\tilde{w}_0'}) & = \{(0, 0, 0, 1), (0, 0, 1, 0), (1, 0, 1, 0), (0, 1, 0, 1)\},
\end{align*}
$$

where $\sigma$ is the map which commutes with the parametrization corresponding with $\tilde{w}_0$. With the same methods used in 2.1 and 2.2, we obtain that the semigroup $\mathcal{D}$ generated by

$$
\mathcal{D} = \{(0, 1, 0, 0), (0, 0, 1, 0), (1, 0, 1, 0), (0, 1, 0, 1)\}
$$

is an adapted cone. Hence, $\sigma(\mathcal{D})$, which is generated by

$$
\sigma(\mathcal{D}) = \{(2, 0, 0, 1), (1, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\},
$$

is also an adapted cone. Let’s give a few details.

**Lemma.** The cones $\mathcal{D}$ and $\sigma(\mathcal{D})$ are adapted.

**Sketch of the proof.** It is enough to prove that $\mathcal{D}$ is an adapted cone, see Remark 2.2. Set $a = (0, 0, 1, 0)$, $b = (0, 1, 0, 0)$, $c = (1, 0, 1, 0)$, $d = (0, 1, 0, 1)$. By (3.1.1), $B^*_{\tilde{w}_0}(c)$ $B^*_{\tilde{w}_0}(d)$ generate the $q$-center $Z_q$. By 3.2, in order to prove the multiplicativity part, it is enough to prove that, up to a power of $q$, $B^*_{\tilde{w}_0}(a)^a B^*_{\tilde{w}_0}(b)^b$ is in $B^*$ for all non-negative integer $a$ and $b$. Through Corollary 1.9 and Proposition 1.7. C, this is implied by the following assertions:

1) $B^*_{\tilde{w}_0}(a)$ and $B^*_{\tilde{w}_0}(b)$ q-commute,
2) $B^*_{\tilde{w}_0}(a)^a$ is in $B^*$ up to a power of $q$,
3) In the $W$-stratification of $B^*$, see [11, 12.4], $B^*_{\tilde{w}_0}(a)^a$ belongs to the $s_2s_1$ component of $B^*$ and $B^*_{\tilde{w}_0}(b)$ corresponds to the extremal vector $v^*_{s_2s_1\varpi_1}$ of $B(\varpi^*_1)$.

Let sketch the proof for 1) 2) and 3). By Remark 1.5, we can describe the parametrizations of $B(\varpi^*_i)$ for $i = 1, 2$. We then obtain that $B^*_{\tilde{w}_0}(a)$ and $B^*_{\tilde{w}_0}(b)$ are in $B(\varpi^*_1)$. Let $v^*$ and $w^*$ be weight vectors in $V_q(\varpi^*_1)$ corresponding respectively to $B^*_{\tilde{w}_0}(a)$ and $B^*_{\tilde{w}_0}(b)$. It is easy to prove that for all positive root $\alpha$, then $E_\alpha v^*$ is non zero implies that $F_\alpha w^*$ is zero. By [5, 1.5], this implies 1). Now, by [11, Proposition 8.2],
σ(B_{\tilde{w}_0}^*(a)) is in B(\varpi_2)^*, so it corresponds to an extremal vector. By Proposition 1.7, A, Proposition 1.8 (ii) and Corollary 1.9, it gives that σ(B_{\tilde{w}_0}^*(a))^a is in B^*. This implies 2). Moreover, we can obtain by considering the crystal graph of V_q(\varpi_1) that B_{\tilde{w}_0}^*(a) belongs to the s_2s_1 component of B^*. This implies 3).

The maximality property is proved as in Theorem 2.2 by remarking that D generates \mathbb{Z}^N as a group.

Let \mathcal{G}^* be the finite subset of B^* parametrized via \tilde{w}_0 by the union of the canonical generators of the cones \mathcal{C}_{\tilde{w}_0}, \mathcal{C}_{\tilde{w}_0}', \sigma(\mathcal{C}_{\tilde{w}_0}), \sigma(\mathcal{C}_{\tilde{w}_0}'), D, \sigma(D), then it is easily verified that

**Proposition.** We have the decomposition \mathbb{Z}^N_{\geq 0} = \mathcal{C}_{\tilde{w}_0} \cup \mathcal{C}_{\tilde{w}_0}' \cup \sigma(\mathcal{C}_{\tilde{w}_0}) \cup \sigma(\mathcal{C}_{\tilde{w}_0}') \cup D \cup \sigma(D).

Hence, each element of B^* can be decomposed into a product of elements of \mathcal{G}^*.  

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