Asymptotic solutions of a nonlinear diffusive equation in the framework of $\kappa$-generalized statistical mechanics

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Abstract. The asymptotic behavior of a nonlinear diffusive equation obtained in the framework of the $\kappa$-generalized statistical mechanics is studied. The analysis based on the classical Lie symmetry shows that the $\kappa$-Gaussian function is not a scale invariant solution of the generalized diffusive equation. Notwithstanding, several numerical simulations, with different initial conditions, show that the solutions asymptotically approach to the $\kappa$-Gaussian function. Simple argument based on a time-dependent transformation performed on the related $\kappa$-generalized Fokker-Planck equation, supports this conclusion.

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1 Introduction

In the framework of non-equilibrium thermostatistics, irreversible processes can be often described by means of Fokker-Planck equations (FPE’s) [1] whose time evolutions are characterized by monotonically non-increasing Lyapunov functionals. In a previous work [2], we derived a non-linear FPE in the picture of a generalized statistical mechanics based on the $\kappa$-entropy [3,4], and discussed its relation with the associate Lyapunov functional or Bregman type divergence [5]. Based on the monotonic behavior of the Lyapunov functional, one can show that any initial localized state, which evolves according to the $\kappa$-generalized FPE, converges to a stationary solution which minimizes the Lyapunov functional. In addition, in the linear drift case, the corresponding stationary solution is nothing but the $\kappa$-Gaussian, which is a generalization of Gaussian by replacing the standard exponential with its $\kappa$-generalized version [6].

As well known, the diffusive equations play a fundamental role in the description of several physical phenomena. In particular, the linear diffusive equation arising in the Boltzmann-Gibbs (BG) statistical mechanics has been widely studied jointly with its Gaussian self-similar solution [7]. Similarly, the $q$-generalized diffusive equation arises, in a natural way, in the framework of the Tsallis statistical physics. This is equivalent to the porous medium equation (PME), which is a transport equation of gases or fluids in a porous medium. The main properties of PME are now well known [8] and its solutions have been widely investigated in literature [9,10], in particular its self-similar solution is given by the so-called Barenblatt solution [11].

In this work we study a different nonlinear diffusive equation obtained in the picture of the $\kappa$-generalized statistical mechanics. As known, a useful method to obtain the solutions of a partial differential equation is based on the study of its Lie symmetries and the related group invariant solutions [12,13]. We accomplished such analysis for the $\kappa$-diffusive equation considering only the classical Lie symmetries whose generators are functions of the independent and the dependent variables. We will leave the study of the generalized Lie symmetries, whose generators are functions also of the derivatives of the depending variables, to a future work.

In particular, we show that the only group-invariant solutions for the $\kappa$-diffusive equation are the kink-like solutions (a non-normalizable family of self-similar solutions) and the traveling wave solutions (physically irrelevant because they are divergent). Next, we explore numerically the evolutions of a localized initial state, whose spreading is governed by the $\kappa$-diffusive equation. It is shown that, independently from the initial states, any numerical solution approaches to a shape which is asymptotically well fitted by the $\kappa$-Gaussian distribution, although this one is not a group invariant solution of the equation under investigation.

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The paper is organized as follows. In the next Section 2 we briefly review the \( \kappa \)-generalized thermostatistics and its associated non-linear FPE. In Section 3, after presenting an alternative derivation of the \( \kappa \)-diffusive equation, we classify its classical Lie symmetries and their related group invariant solutions. Section 4 deals with the numerical analysis and the final Section 5 is devoted to summary.

### 2 \( \kappa \)-generalized thermostatistics

We briefly review the generalized thermostatistics based on the \( \kappa \)-entropy \([3,4]\) given by

\[
S_\kappa[p] \equiv - \int_{-\infty}^{\infty} p(v) \ln_{\kappa} p(v) \, dv,
\]

where \( \ln_{\kappa}(x) \) is the \( \kappa \)-logarithm defined us

\[
\ln_{\kappa}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}.
\]

The \( \kappa \)-entropy \( S_\kappa[p] = S_{-\kappa}[p] \) is an extension of the BG entropy by means of a real parameter \( |\kappa| \in [0,1] \).

The inverse function of \( \ln_{\kappa}(x) \), namely \( \exp_{\kappa}(x) \), given by

\[
\exp_{\kappa}(x) = \left( \kappa x + \sqrt{1 + \kappa^2 x^2} \right)^{\frac{1}{\kappa}},
\]

is called \( \kappa \)-exponential.

The \( \kappa \)-logarithm and \( \kappa \)-exponential functions are the building blocks of the statistical mechanics based on the \( \kappa \)-entropy. In particular, the \( \kappa \)-exponential is a monotonic increasing and convex function, fulfilling the relationship \( \exp_{\kappa}(-x) = 1/\exp_{\kappa}(x) \). Moreover, \( \exp_{\kappa}(-\infty) = 0 \), \( \exp_{\kappa}(0) = 1 \) and \( \exp_{\kappa}(\infty) = \infty \), as the ordinary exponential function does. For \( |x| \ll 1 \), the \( \kappa \)-exponential is well approximated by the standard exponential, whereas for \( |x| \to \infty \), it asymptotically approaches to a power-law: \( \exp_{\kappa}(x) \sim |2\kappa x|^{1/\kappa} \). In the \( \kappa \to 0 \) limit, both \( \ln_{\kappa}(x) \) and \( \exp_{\kappa}(x) \) reduce to the standard logarithm and exponential functions, respectively. Accordingly, the \( \kappa \)-entropy reduces to the BG entropy.

Maximizing \( S_\kappa[p] \) under the constraints of the linear kinetic energy average and the normalization

\[
\frac{\delta}{\delta p(v)} \left( S_\kappa[p] - \beta \int_{-\infty}^{\infty} \frac{v^2}{2} p(v) \, dv - \gamma \int_{-\infty}^{\infty} p(v) \, dv \right) = 0,
\]

leads to the \( \kappa \)-Gaussian function

\[
p_{\mathrm{MF}}^\kappa(v) = \alpha \exp_{\kappa} \left( \frac{1}{\kappa} \left( \gamma + \beta \frac{v^2}{2} \right) \right).
\]

Here \( \gamma \) is the constant fixing the normalization of the distribution and depends on the Lagrange multiplier \( \beta \) which controls the dispersion of the distribution. Finally, the parameters \( \alpha \) and \( \lambda \) are \( \kappa \)-dependent constants given by

\[
\alpha = \left( \frac{1 - \kappa}{1 + \kappa} \right)^{\frac{1}{2\kappa}}, \quad \lambda = \sqrt{1 - \kappa^2},
\]

respectively, and are related each to the other through the relation

\[
\alpha = \exp_{\kappa}(-1/\lambda).
\]

The \( \kappa \)-entropy and the corresponding statistical mechanics preserve several properties of the BG theory \([3]\). In particular, the Legendre structure of the thermon-statistics theory has been investigated in \([14]\) through the introduction of several thermodynamic potentials.

For instance, it is found that the generalized partition function, as a function of \( \beta \), given by

\[
\ln_{\kappa}(Z_\kappa) = T_\kappa[p] + \gamma,
\]

satisfies the Legendre relation

\[
\frac{d}{d\beta} \ln_{\kappa}(Z_\kappa) = -U,
\]

and that the \( \kappa \)-entropy and the generalized partition function are related through the relationship

\[
S_\kappa = \ln_{\kappa}(Z_\kappa) + \beta U,
\]

where

\[
U = \int_{-\infty}^{\infty} \frac{1}{2} v^2 p(v) \, dv.
\]

The function \( T_\kappa[p] \) in equation (8) is defined by

\[
T_\kappa[p] \equiv \int_{-\infty}^{\infty} p(v) u_{\kappa}(p(v)) \, dv,
\]

where the quantity \( u_{\kappa}(x) = u_{\kappa}(-\kappa)(x) \), defined as

\[
u_{\kappa}(x) = \frac{x^\kappa + x^{-\kappa}}{2},
\]

fulfills the following properties \( u_{\kappa}(x) = u_{\kappa}(1/x) \) and \( u_{\kappa}(\alpha) = 1/\lambda \). In the \( \kappa \to 0 \) limit, it reduces to the unity: \( u_{\kappa}(x) = 1 \).

Both functions \( u_{\kappa}(x) \) and \( T_\kappa[p] \) play an important role in the development of a theory based on the \( \kappa \)-entropy.

Starting from the definition of the partition function \( Z_\kappa \), we can introduce a \( \kappa \)-generalization of the free-energy, according to

\[
F_\kappa[p] \equiv -\frac{1}{\beta} \ln_{\kappa}(Z_\kappa),
\]

In this way, it was shown that the \( \kappa \)-free-energy satisfies the Legendre transformation structures \([14]\) summarized by the following relationships

\[
F_\kappa[p] = U[p] - \frac{1}{\beta} S_\kappa[p], \quad \frac{d}{d\beta} \left( \beta F_\kappa \right) = U.
\]