ON SOME $p$-ADIC SERIES WITH FACTORIALS

Branko DRAGOVICH
Institute of Physics, P. O. Box 57, 11001 Belgrade, Yugoslavia

1 Introduction

In the last nine years $p$-adic numbers have been successfully applied in various branches of theoretical and mathematical physics (for a review see [1-3]). One of the objects which we often encounter is a $p$-adic series.

My interest in $p$-adic series is mainly motivated by the significant role they might play not only in pure but also in applied mathematics. It was initiated in 1987 [4] by an observation that divergent perturbative series, which we usually face up in theoretical and mathematical physics, are $p$-adically convergent. Namely, such power series have rational coefficients and may be treated in $R$ as well as in any $Q_p$. Loosely speaking, the less convergence in $R$ the more convergence in $Q_p$, and vice versa.

This opportunity induces a question on possible connection between convergence in the one number field and summation of divergent counterpart in the other one. It is natural to expect an answer within the field of rational numbers $Q$, because $Q$ is a subfield of $R$ and all $Q_p$.

Recall that the sum of a divergent series depends on the way how one performs summation. If a series is convergent and has a definite rational sum in $Q_p$ for all but a finite number of $p$ then this sum may be attached to the divergent counterpart. In other words, it seems appropriate to choose the same procedure of summation for divergent as that one for convergent versions of a series.

For often encountered divergent series, the situation is such that only trivial rational points exist (i.e. for argument $x = 0$). However there are divergent power series which are $p$-adically convergent and have non-trivial (usually one) rational points. Such series contain factorials in the numerator of coefficients and they are similar to perturbative expansions in quantum field theory and string theory.
To illustrate divergence in quantum field theory one can consider the simple integral for zero-dimensional scalar theory

\[ Z(m, g) = \int_R d\varphi \exp(-\varphi \frac{m^2}{2} \varphi + g \varphi^3) \]

which leads to a perturbative series

\[ \sum_{n=0}^{\infty} \frac{(6n-1)!!}{(2n)!} \cdot x^n, \]

where \( x = g^2 / m^6 \), \( m \) is a mass and \( g \) is a coupling constant. This series, as well as the partition function in perturbation string theory [5], diverges like \( \sum n!x^n \).

On the other hand, to make a direct connection of \( p \)-adic models with the corresponding real one it is necessary to have convergence of series in \( \mathbb{R} \) and all \( \mathbb{Q}_p \) within the common domain of rational numbers. Namely, all results of measurements belong to \( \mathbb{Q} \), and comparison of theory and experiment performs within \( \mathbb{Q} \). However, the standard power series of theoretical physics do not satisfy the above condition. For example, expansions of functions \( \exp x \sin x \cos x \sinh x \) and \( \cosh x \) are convergent in \( p \)-adic case for \( |x|_p < |2|_p \). As a consequence, there is no \( 0 \neq x \in \mathbb{Q} \) for which these functions are defined for every \( p \). Therefore it is reasonable to consider small modifications of the standard expansions which lead to significant enlargement of the region of convergence in \( p \)-adic counterparts.

This article is based on the author’s papers [6-10]. The wide class of series considered here is given by expression (2.5). Enlargement of \( p \)-adic region of convergence is introduced by a parameter \( q \in \mathbb{Q} \) in the denominator of coefficients. The presence of \( q \) in the form (2.3) makes the series (2.5) convergent everywhere on \( \mathbb{Q}_p \) for every \( p \), and it also does not affect convergence properties in the real case. This \( q \)-modification can be made arbitrarily small in real counterpart leaving \( p \)-adic convergence unchanged. For \( q = 0 \) these series become in a sense the standard ones. The domain of convergence is found for all characteristic cases.

The general summation formula (3.1) is derived, which is number field invariant. When a rational sum is obtained for all but a finite number of \( p \) it may be used for summation of divergent counterparts. This method of summation of divergent series is called ”adelic summation” and may be used in some rational points.
In the case $q = 0$ we mainly pay attention to the series which are convergent everywhere on the ring of $p$-adic integers $\mathbb{Z}_p$ for every $p$. One can say, the more simple a series is, the more we go into details. In particular, the series $\sum_{n=0}^{\infty} n! P(n)$, where $P(n)$ is a polynomial in $n$ and with rational coefficients, is investigated rather widely. It is obtained a method to find all $P(n)$ which yield rational sums. The connection between uniqueness of the pair of integers $(u_k, v_k)$ in $\sum n! (n^k + u_k) = v_k$ and possible non-rationality of $\sum n!$ is pointed out. Although there is not yet a proof that $\sum n!$ is not a rational number it seems very likely that it is true. A proof of this conjecture might be very significant.

Since the concept of adeles (Platonov and Rapinchuk [11]) enables to consider properties of $\mathbb{Q}$ simultaneously from real and $p$-adic points of view, in Section 5 one investigates some adelic aspects of our series. It is shown that one can make an adelic sequence of some series at rational points.

In the Appendix we give some examples of characteristic series with a rational sum to illustrate their non-triviality and diversity.

All necessary $p$-adic analysis needed for investigation of these series can be found in an excellent book of Schikhof [12].

2 Convergence

If we have a power series

$$\sum_{n=0}^{\infty} A_n x^n,$$

(2.1)

where $A_n \in \mathbb{Q}$, it can be treated as a $p$-adic series ($x \in \mathbb{Q}_p$) as well as a real one ($x \in \mathbb{R}$). Recall that, in the $p$-adic case, a necessary and sufficient condition for a convergence of (2.1) is

$$|A_n x^n|_p \to 0 \text{ as } n \to \infty.$$

(2.2)

Let

$$I_{\mu\nu}^{(q)} = \frac{(\mu n + \nu)!}{q + ((\mu n + \nu))!^{\mu n + \nu}},$$

(2.3)

where $\mu \in \mathbb{Z}_+ = \{1, 2, \ldots\}$, $\nu \in \mathbb{Z}_0 = \{0, 1, 2, \ldots\}$ and $q$ is a nonnegative rational number ($0 \leq q \in \mathbb{Q}$). Let also

$$P_k(n) = C_k n^k + \cdots + C_0$$

(2.4)
be a polynomial in $n \in \mathbb{Z}_+$ of degree $k$ with the coefficients $C_0, \ldots, C_k \in \mathbb{Q}$.

**Proposition 1**  The power series

$$
\sum_{n=0}^{\infty} \varepsilon^n I_{\mu\nu}^{(q)} \prod_{i=1}^{I} ((\alpha_i n + \beta_i)!)^\lambda_i P_k(n) x^{\mu n + \nu},
$$

(2.5)

where $\varepsilon = +1$ or $-1$, $I_{\mu\nu}^{(q)}$ is defined by (2.3) with $q \neq 0, \alpha_i, I \in \mathbb{Z}_+, \beta_i \in \mathbb{Z}_0, \lambda_i \in \mathbb{Z}$, and $P_k(n)$ is given by (2.4), is $p$-adically convergent for all $x \in \mathbb{Q}_p$ and for any prime number $p$.

**Proof:** The general term of (2.5) has a $p$-adic norm

$$
| I_{\mu\nu}^{(q)} |_p \prod_{i=1}^{I} | (\alpha_i n + \beta_i)! |_p | P_k(n) |_p | x |_p^{\mu n + \nu},
$$

(2.6)

where $| P_k(n) |_p \leq \max_{0 \leq j \leq k} | C_j |_p$, and for a large enough $n$

$$
| I_{\mu\nu}^{(q)} |_p = | q |_p^{-1} | (\mu n + \nu)! |_p^{\mu n + \nu}
$$

(2.7)

as a consequence of the strong triangle inequality for $p$-adic norm. Recall that

$$
| m! |_p = p^{-m - S_m}
$$

(2.8)

where $S_m$ is the sum of digits in the canonical expansion of a positive integer $m$ over $p$. Thus, for a large enough $n$ (2.6) behaves like

$$
\left\{ p^{-\frac{\mu n}{p-1}} p^{-\frac{1}{(p-1)p}} \sum_{i=1}^{I} \alpha_i \lambda_i | x |_p \right\}^{\mu n}.
$$

(2.9)

For any $x \in \mathbb{Q}_p$ and for the above range of parameters the expression (2.9) tends to zero as $n \to \infty$.

**Proposition 2**  The power series (2.5) for the same parameters as in Proposition 1, but $q = 0$, is $p$-adically convergent in the domain

$$
| x |_p < p^{\frac{1}{(p-1)p} \sum_{i=1}^{I} \alpha_i \lambda_i}.
$$

(2.10)

**Proof:** Since $I_{\mu\nu}^{(0)} = 1$, instead of (2.9) one obtains

$$
\left\{ p^{-\frac{1}{(p-1)p} \sum_{i=1}^{I} \alpha_i \lambda_i | x |_p } \right\}^{\mu n}
$$

(2.11)
which tends to zero as $n \to \infty$ iff $|x|_p$ satisfies (2.10).

In order to have $p$-adic convergence for any $x \in Q$ at all but a finite number of $p$, we will be mainly interested in the cases when

$$\sum_{i=1}^{I} \alpha_i \lambda_i \geq \mu$$

(2.12)
in (2.11).

While in the $p$-adic case factors $I^{(q)}_{\mu_n+\nu}$, for $q \neq 0$, serve to extend the domain of convergence, in the real one they do not play a such role. Namely, since in the real case $I^{(q)}_{\mu_n+\nu} \to 1$ as $n \to \infty$, these factors do not influence a change of the domain of convergence of series (2. 5) for various values of the parameter $q$. However, other parameters are more or less important and convergence in detail may be determined using the d’Alembert criterion.

In particular, the following two simple classes of (2. 5) deserve to be noted:

$$E_{\varepsilon,q}^{\mu,\nu}(x) = \sum_{n=0}^{\infty} \varepsilon^n I^{(q)}_{\mu_n+\nu} \frac{x^{\mu n+\nu}}{(\mu n + \nu)!} ,$$

(2.13)
which is everywhere convergent in $R$ and all $Q_p$ if $q \neq 0$, and for $|x|_p < |2|_p$ if $q = 0$;

$$F_{\varepsilon,q}^{\mu,\nu}(x) = \sum_{n=0}^{\infty} \varepsilon^n I^{(q)}_{\mu_n+\nu} (\mu n + \nu)! x^{\mu n+\nu} ,$$

(2.14)
which is everywhere divergent in $R$, everywhere $p$-adic convergent if $q \neq 0$, and $p$-adic convergent for $|x|_p \leq 1$ if $q = 0$. Series of the form (2.13) contain $q$-modification of the expansions for the well-known functions like exponential ($\varepsilon = 1, \mu = 1, \nu = 0$), cosine ($\varepsilon = -1, \mu = 2, \nu = 0$) and sine ($\varepsilon = -1, \mu = 2, \nu = 1$). Expansion (2.14) has factors $(\mu n + \nu)!$ which are inverse to (2.13).

### 3 Summation

Starting from the series (2. 3) and owing to the factorization of expressions with factorials one can obtain a summation formula for a wide class of series.

Let $(m+1)_\mu = (m+1)(m+2) \cdots (m+\mu)$.
Proposition 3  The summation formula

\[ \sum_{n=0}^{\infty} \varepsilon^n \left( (\mu n + \nu)! \right)^{\mu n + \nu} \prod_{i=1}^{l} \left( (x_i n + \beta_i)! \right)^{\lambda_i} \]

\[ \times \left\{ \frac{((\mu n + \nu)!)^{\mu}}{q + \left[ ((\mu(n+1) + \nu)!)^{\mu(n+1)+\nu} \prod_{i=1}^{l} (\alpha_i n + \beta_i + 1)^{\lambda_i} P_k(n+1) x^\mu \right]} \right\} x^{\mu n + \nu} \]

\[ = -\varepsilon \left( \nu! \right)^{\nu} \prod_{i=1}^{l} (\beta_i)!^{\lambda_i} P_k(0) x^\nu \]  \hspace{1cm} (3.1)

has a place in the region of parameters and variable \( x \) which are determined by convergence of the series (2.5).

Proof:  Taking into account the identity

\[ \left[ (\mu(n+1) + \nu)! \right]^{\mu(n+1)+\nu} = (\mu n + \nu)!^{\mu(n+1)+\nu} (\mu n + \nu + 1)^{\mu(n+1)+\nu} \]  \hspace{1cm} (3.2)

the LHS of expression (3.1) may be rewritten as

\[ \sum_{n=1}^{\infty} \varepsilon^{n-1} I_{\mu n + \nu}^{(q)} \prod_{i=1}^{l} \left( (\alpha_i n + \beta_i)! \right)^{\lambda_i} P_k(n) x^{\mu n + \nu} \]

\[ - \sum_{n=0}^{\infty} \varepsilon^{n-1} I_{\mu n + \nu}^{(q)} \prod_{i=1}^{l} \left( (\alpha_i n + \beta_i)! \right)^{\lambda_i} P_k(n) x^{\mu n + \nu} \]

\[ = \varepsilon I_{\nu}^{(q)} \prod_{i=1}^{l} (\beta_i)!^{\lambda_i} P_k(0) x^\nu. \]  \hspace{1cm} (3.3)

Although the formula (3.1) is based on mutual cancellation in pairs of terms in the LHS of (3.3), leaving only the first one, it is very useful and yields highly non-trivial results.
The summation in (3.1) does not depend on the number field. It is worth noting that for any \( x \in Q \) the sum is a definite rational number given by the RHS and this result holds in all \( Q_p \) if \( q \neq 0 \).

In the real case the LHS of (3.1) can be divergent. If so, a sum depends on the way of summation. In such case, among all reasonable ways of summation it seems that the number field invariant one is the most natural.

**Definition** (Adelic summation) Let a series be divergent in the real case and convergent in \( Q_p \) for all but a finite number of \( p \). Let such series allows a number field invariant summation with rational sum for some variable \( x \in Q \) in the domain of convergence. Extrapolation of the number field invariant summation to the divergent counterparts we will call adelic summation.

### 4 Case \( q = 0 \)

The results obtained in the previous sections are mainly related to the series (2.5) with \( q \neq 0 \), where two particular cases are noted: (2.13) and (2.14). Here we will consider the case \( q = 0 \), i.e. we will investigate some aspects of

\[
\sum_{n=0}^{\infty} \varepsilon^n \prod_{i=1}^{I} \left( (\alpha_i n + \beta_i)! \right)^{\lambda_i} P_k(n) x^{\mu n + \nu}. \tag{4.1}
\]

Recall that the domain of convergence of (4.1) is already derived and given by (2.10). The corresponding summation formula follows from (3.1) and it reads:

\[
\sum_{n=0}^{\infty} \varepsilon^n \prod_{i=1}^{I} \left( (\alpha_i n + \beta_i)! \right)^{\lambda_i} \left\{ \prod_{i=1}^{I} (\alpha_i n + \beta_i + 1)^{\lambda_i} P_k(n + 1) x^{\mu} \right\} - \varepsilon P_k(n) \right\} x^{\mu n + \nu} = -\varepsilon \prod_{n=1}^{I} (\beta_i!)^{\lambda_i} P_k(0) x^{\nu}. \tag{4.2}
\]

In some pure theoretical, as well as practical problems a knowledge on existence of non-trivial rational points may be very important.

**Proposition 4** The series (4.1) has a rational sum for some \( x \in Q \) which satisfies (2.10), if there exists a polynomial \( A_\eta(n) \) such that

\[
P_k(n) = \prod_{i=1}^{I} (\alpha_i n + \beta_i + 1)^{\lambda_i} x^{\mu} A_\eta(n + 1) - \varepsilon A_\eta(n). \tag{4.3}
\]
Proof: Let there exist a polynomial \( A_\eta(n) \) with rational coefficients which satisfies equation (4.3), then according to formula (4.2) the rational sum of (4.1) does exist and the sum is
\[
S = -\varepsilon \prod_{i=1}^{I} (\beta_i!)^{\lambda_i} A_\eta(0) x^\nu .
\] (4.4)

Note that Proposition 4 defines a sufficient condition. So far, on a necessary condition one can only conjecture. It is clear that all series (4.1) may not have a non-trivial rational point (i.e. for \( x \neq 0 \)).

**Proposition 5** For a given \( x \in \mathbb{Q} \) in the series (4.1) one can always find a polynomial \( P_k(n) \) so that the sum is a rational number. The degree of \( P_k(n) \) is \( k = \max\{\sum_{i=1}^{I} \alpha_i \lambda_i + \eta, \eta\} \).

Proof: According to (4.3) for a given \( x \in \mathbb{Q} \) there is a polynomial \( P_k(n) \) which depends on a particular choice of a polynomial \( A_\eta(n) \). The degree of \( P_k(n) \) also follows from (4.3).

If \( \sum_{i=1}^{I} \alpha_i \lambda_i \geq 1 \), then \( k = \sum_{i=1}^{I} \alpha_i \lambda_i + \eta \). Condition (2.12) belongs to this case.

The simplest factorial form of (4.1) with \( P_k(n) \) is
\[
\sum_{n=0}^{\infty} n! P_k(n) ,
\] (4.5)

which is divergent in \( \mathbb{R} \) and convergent in all \( \mathbb{Q}_p \). The corresponding summation formula is
\[
\sum_{n=0}^{\infty} n![(n + 1) A_\eta(n + 1) - A_\eta(n)] = -A_\eta(0) ,
\] (4.6)

where \( A_\eta(n) = a_\eta n^n + a_{\eta-1} n^{\eta-1} + \cdots + a_0 \), with \( a_\eta, \cdots, a_0 \in \mathbb{Q} \).

It is obvious that all possible polynomials \( A_\eta(n) \) generate the corresponding polynomials \( P_k(n) \), \( k = \eta + 1 \geq 1 \), which allow rational sums \(-A_\eta(0)\). Searching of all possible \( P_k(n) \) may be reduced to
\[
\sum_{n=0}^{\infty} n!(n^k + u_k) = v_k , (k \geq 1) ,
\] (4.7)

8
where $u_k, v_k \in Q$ have to be determined, i.e. one has to solve equation

$$(n+1)A_{k-1}(n+1) - A_{k-1}(n) = n^k + u_k.$$  \hspace{1cm} (4.8)$$

To Eq. (4.7) corresponds a system of $k+1$ linear equations with $k+1$ unknowns $(a_0, a_1, \cdots, a_{k-1}, u_k)$, which has always a solution. Note that $v_k = -A_{k-1}(0)$. The first five of pairs $(u_k, v_k)$ are: $(u_1, v_1) = (0, -1)$, $(u_2, v_2) = (1, 1)$, $(u_3, v_3) = (-1, 1)$, $(u_4, v_4) = (-2, -5)$, $(u_5, v_5) = (9, 5)$.

**Proposition 6** If pairs of rational numbers $(u_k, v_k)$ are not unique for a given $k \geq 1$, then $\sum_{n=0}^{\infty} n!$ is a rational number in $\mathbb{Q}_p$.

**Proof:** Suppose that in addition to $(u_k, v_k)$ exists $(u_k', v_k') \neq (u_k, v_k)$. Then $\sum_{n=0}^{\infty} n! = (v_k - v_k')/(u_k - u_k')$.

**Proposition 7** If $\sum_{n=0}^{\infty} n!$ is a rational number then $\sum_{n=0}^{\infty} n!n^k$, for any $k \in \mathbb{Z}_+$, are also rational numbers.

**Proof:** It follows from (4.7).

There is not yet an exact proof that $\sum_{n=0}^{\infty} n!$ is not a rational number (Schikhof [12]). But it seems reasonable to suppose that it is not a rational number. If so, then pairs $(u_k, v_k)$ are unique, and existence of a polynomial $A_\eta(n)$ is not only sufficient but also a necessary condition for rational summation of (4.5). Thus the general form of (4.5) with rational sums is

$$\sum_{n=0}^{\infty} n!(C_k n^k + C_{k-1} n^{k-1} + \cdots + C_0) = D_k,$$

where $C_0 = \sum_{j=1}^{k} C_j u_j$ , $D_k = \sum_{j=1}^{k} C_j v_j$ , and $C_1 \cdots, C_k \in Q$.

Note that the above consideration performed for (4.5) can be done for

$$\sum_{n=0}^{\infty} (-1)^n n!P_k(n)$$

with analogous conclusions.
5 Adelic Aspects

Recall (Platonov and Rapinchuk [11]) that an adele is an infinite sequence

\[ x = (x_\infty, x_2, \cdots, x_p, \cdots), \]  

(5.1)

where \( x_\infty \in \mathbb{R} \), \( a_p \in \mathbb{Q}_p \) with a restriction that \( a_p \in \mathbb{Z}_p = \{ t : | t |_p \leq 1 \} \) for all but a finite number of \( p \). The set of all adeles is a ring under componentwise addition and componentwise multiplication. The space of adeles \( \mathcal{A} \) may be presented as

\[ \mathcal{A} = \bigcup_S A(S), \quad A(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p, \]  

(5.2)

where \( S \) is a set of finite number of primes \( p \). \( \mathcal{A} \), as a topological space, has a basis of open sets which are of the form \( W_\infty \times \prod_{p \in S} W_p \times \prod_{p \notin S} \mathbb{Z}_p \), where \( W_\infty \) and \( W_p \) are open sets in \( \mathbb{R} \) and \( \mathbb{Q}_p \), respectively.

Note that \( \mathcal{A} \) is an instrument which enables a simultaneous treatment of all completions of \( \mathbb{Q} \). The field \( \mathbb{Q} \) can be embedded into \( \mathcal{A} \) by mapping \( x \rightarrow (x, x, \cdots, x, \cdots) \), where \( x \in \mathbb{Q} \).

**Proposition 8**  Let one has a sequence

\[ E_{\mu,\nu}^0(x_\infty) = \left( E_{\mu,\nu}^{0,0}(x_\infty), E_{\mu,\nu}^{0,2-s}(x_2), \cdots, E_{\mu,\nu}^{0,p-s}(x_p), \cdots \right), \]  

(5.3)

where \( E_{\mu,\nu}^{0,0}(x_\infty) \) is a real and \( E_{\mu,\nu}^{0,p-s}(x_p) \) is a \( p \)-adic series defined by (2.13), and \( s \in \mathbb{Z}_+ \). When \( x = (x_\infty, x_2, \cdots, x_p, \cdots) \) is an adele, \( E_{\mu,\nu}^0(x) \) is also an adele.

**Proof:** For any \( x_\infty \in \mathbb{R} \), the series \( E_{\mu,\nu}^{0,0}(x_\infty) \) is well defined in real case. The general term of the \( p \)-adic series (2.13) for \( q = p^{-s} \) is

\[ \varepsilon^n \frac{p^s((\mu n + \nu)!)^{\mu n+\nu-1}}{1 + p^s((\mu n + \nu)!)^{\mu n+\nu} x_p^{\mu n+\nu}} \]  

(5.4)

Since \( 1 + p^s((\mu n + \nu)!)^{\mu n+\nu} \big|_p = 1 \), the \( p \)-adic norm of (5.4) is

\[ \frac{1}{p^s} | ((\mu n + \nu)!)^{\mu n+\nu-1} \big|_p x_p^{\mu n+\nu}, \]  

(5.5)

which can be larger than 1 only for a finite number of \( p \).
Thus (5.3) is an adelic sequence of series. It is clear that in (5.3) one can take 
\[ x_\infty = x_2 = \cdots = x_p = \cdots = x, \] where \( x \in Q \).

**Proposition 9**  The sequence of series

\[
\left( H_{\mu,\nu}^{\varepsilon,q}(x), H_{\mu,\nu}^{\varepsilon,q}(x), \cdots, H_{\mu,\nu}^{\varepsilon,q}(x), \cdots \right),
\]

where

\[
H_{\mu,\nu}^{\varepsilon,q}(x) = \sum_{n=0}^{\infty} (((\mu n + \nu)!)^{\mu n + \nu - 1} \mu n + \nu + 1)^{\mu (n+1) + \nu - 1} \mu n + \nu + \nu - 1 \mu n + \nu + 1 \right) \frac{1}{q + ((\mu n + \nu)!)^{\mu n + \nu}} \right) x^{\mu n + \nu},
\]

(5.7)
is an adele of series if \( x \in Q \).

**Proof:** Note that \( H_{\mu,\nu}^{\varepsilon,q}(x) \) is a particular case of the LHS in (3.1) induced by (2.13), which is convergent for every \( x \in R \) and every \( x \in Q_p \). It has a sum

\[
S = - \frac{1}{q + (\nu!)^{\mu - 1} x^{\nu}},
\]

(5.8)

which for \( x \in Q \) is a \( p \)-adic integer for all but a finite number of \( p \).

In virtue of (4.6) one can construct adeles from \( \sum_{n=0}^{\infty} n! P_k(n) \) with \( P_k(n) = (n + 1)A_{k-1}(n + 1) - A_{k-1}(n) \) where \( A_{k-1}(n), (k \geq 1) \), are arbitrary polynomials with rational coefficients. The corresponding sum \(-A_{k-1}(0)\), which is valid for all \( p \)-adic cases, may be also attributed to the real case by method of adelic summation of divergent series.

6 **Appendix**

Here we give some particular examples which illustrate various forms of \( p \)-adic series with rational sums contained in the preceding sections of this article. These series are convergent in all \( Q_p \) and have number field independent rational sum. The first series is also convergent in \( R \), but all other are not. Some other examples can be found in the author’s paper [7].
\[
\sum_{n=0}^{\infty} (-1)^n \left[ \frac{((n+1)!)^n}{q + ((n+1)!)^{n+1}} + \frac{(n!)^{n-1}}{q + (n!)^n} \right] = \frac{1}{q + 1} \quad 0 \leq q \in Q . \quad (A1)
\]

\[
\sum_{n=0}^{\infty} n! [C_5 n^5 + C_4 n^4 + C_3 n^3 + C_2 n^2 + C_1 n + (9C_5 - 2C_4 - C_3 + C_2)]
\]

\[
= 5C_5 - 5C_4 + C_3 + C_2 - C_1 . \quad C_1, \cdots, C_5 \in Q . \quad (A2)
\]

\[
\sum_{n=0}^{\infty} (n + \beta)! [C_2 n^2 + C_1 n - C_2 \beta^2 + C_1 \beta + C_2]
\]

\[
= \beta! [C_2 (\beta + 1) - C_1] . \quad C_1, C_2 \in Q . \quad (A3)
\]

\[
\sum_{n=0}^{\infty} (2n + \beta)! [4n^2 + 2(2\beta + 3)n + \beta^2 + 3\beta + 1] = -\beta! . \quad (A4)
\]

\[
\sum_{n=0}^{\infty} (2n + \beta)! [8n^3 - 2(3\beta^2 + 9\beta + 8)n - 2\beta^3 - 9\beta^2 - 11\beta - 1]
\]

\[
= \beta!(2\beta + 5) . \quad (A5)
\]

\[
\sum_{n=0}^{\infty} (2n + \beta)! \frac{4n^2 + 2(2\beta + 3)n + \beta(\beta + 3)}{2^n} = -\beta!2 . \quad (A6)
\]

\[
\sum_{n=0}^{\infty} (2n + \beta)! \frac{8n^3 - 6(\beta^2 + 3\beta + 3)n - 2\beta^3 - 9\beta^2 - 9\beta + 4}{2^n}
\]

\[
= \beta!2(2\beta + 5) . \quad (A7)
\]

\[
\sum_{n=0}^{\infty} ((n + \beta)!)^2 [n^2 + 2(\beta + 1)n + \beta^2 + 2\beta] = -(\beta!)^2 . \quad (A8)
\]

\[
\sum_{n=0}^{\infty} ((n + \beta)!)^2 [n^3 - (3\beta^2 + 6\beta + 4)n - 2(\beta + 1)^3 + 2\beta + 3]
\]
\[(A9)\]
\[
\sum_{n=0}^{\infty} ((n + \beta)!)^2 n^2 \frac{2(n^2 + 2(\beta + 1)n + \beta^2 + 2\beta - 1)}{2^n} = -2(\beta!)^2 .
\]

\[(A10)\]
\[
\sum_{n=0}^{\infty} ((n + \beta)!)^2 n^3 \frac{-(3\beta^2 + 6\beta + 5)n - 2(\beta + 1)^3 + 2(2\beta + 3)}{2^{n+1}} = (\beta!)^2(2\beta + 3) .
\]

\[(A11)\]
\[
\sum_{n=0}^{\infty} (n + \beta_1)!(n + \beta_2)! [n^2 + (\beta_1 + \beta_2 + 2)n + (\beta_1 + 1)(\beta_2 + 1) - 1] = -(\beta_1)!(\beta_2)! .
\]

\[(A12)\]
\[
\sum_{n=0}^{\infty} (n + \beta_1)!(n + \beta_2)! [n^3 - (\beta_1^2 + \beta_2^2 + \beta_1\beta_2 + 3\beta_1 + 3\beta_2 + 4)n - (\beta_1 + \beta_2 + 2)(\beta_1\beta_2 + \beta_1 + \beta_2) + 1] = (\beta_1)!(\beta_2)!(\beta_1 + \beta_2 + 3) .
\]

\[(A13)\]
\[
\sum_{n=0}^{\infty} (-1)^n (n + \beta_1)!(n + \beta_2)! [n^2 + (\beta_1 + \beta_2 + 2)n + (\beta_1 + 1)(\beta_2 + 1) + 1] = (\beta_1)!(\beta_2)! .
\]

\[(A14)\]
\[
\sum_{n=0}^{\infty} (-1)^n (n + \beta_1)!(n + \beta_2)! [n^3 - (\beta_1^2 + \beta_2^2 + \beta_1\beta_2 + 3\beta_1 + 3\beta_2 + 2)n - (\beta_1 + \beta_2 + 2)(\beta_1\beta_2 + \beta_1 + \beta_2 + 2) - 1] = -(\beta_1)!(\beta_2)!(\beta_1 + \beta_2 + 3) .
\]

\[(A15)\]
\[
\sum_{n=0}^{\infty} \varepsilon^n \prod_{i=1}^{l} (\alpha_i n + \beta_i)! [\prod_{i=1}^{l} (\alpha_i n + \beta_i + 1)\alpha_i(n + 1)^k - \varepsilon n^k] = 0 .
\]
References

[1] L. Brekke and P. G. O. Freund, $p$-Adic Numbers in Physics, *Phys. Rep.*, 233: 1-66 (1993).

[2] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *$p$-Adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.

[3] A. Khrennikov, *$p$-Adic Valued Distributions in Mathematical Physics*, Kluwer Academic Publishers, Dordrecht, 1994.

[4] I. Ya. Aref’eva, B. Dragovich and I. V. Volovich, On the $p$-Adic Summability of the Anharmonic Oscillator, *Phys. Lett.*, B 200: 512-514 (1988).

[5] D. J. Gross and V. Periwall, String Perturbation Theory Diverges, *Phys. Rev. Lett.*, 60: 2105-2108 (1988).

[6] B. Dragovich, $p$-Adic Perturbation Series and Adelic Summability, *Phys. Lett.*, B 256: 392-396 (1991).

[7] B. Dragovich, On $p$-Adic Aspects of Some Perturbation Series, *Teor. Mat. Fiz.*, 93: 211-218 (1992).

[8] B. Dragovich, Power Series Everywhere Convergent on $R$ and All $Q_p$, *J. Math. Phys.*, 34: 1143-1148 (1993); [math-ph/0402037](http://arxiv.org/abs/math-ph/0402037).

[9] B. Dragovich, Rational Summation of $p$-Adic Series, *Teor. Mat. Fiz.*, 100: 342-353 (1994).

[10] B. Dragovich, On $p$-Adic Series in Mathematical Physics, *Proc. V. A. Steklov Inst. Math.*, 203: 255-270 (1994).

[11] V. P. Platonov and A. S. Rapinchuk, *Algebraic Groups and Number Theory*, Nauka, Moscow, 1990.

[12] W. H. Schikhof, *Ultrametric Calculus: An Introduction to $p$-Adic Analysis*, Cambridge University Press, Cambridge, 1984.