SIMPLE RESTRICTED MODULES OVER THE HEISENBERG-VIRASORO ALGEBRA AS VOA MODULES

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ABSTRACT. In this paper, we determine all simple restricted modules over the mirror Heisenberg-Virasoro algebra $\mathcal{D}$, and the twisted Heisenberg-Virasoro algebra $\bar{\mathcal{D}}$ with nonzero level. As applications, we characterize simple Whittaker modules and simple highest weight modules over $\mathcal{D}$. A vertex-algebraic interpretation of our result is the classification of simple weak twisted and untwisted modules over the Heisenberg-Virasoro vertex operator algebras $\mathcal{V} \cong \mathcal{V}_{\mathfrak{c}} \otimes M(1)$. We also present a few examples of simple restricted $\mathcal{D}$-modules and $\bar{\mathcal{D}}$-modules induced from simple modules over finite dimensional solvable Lie algebras, that are not tensor product modules of Virasoro modules and Heisenberg modules. This is very different from the case of simple highest weight modules over $\mathcal{D}$ and $\bar{\mathcal{D}}$ which are always tensor products of simple Virasoro modules and simple Heisenberg modules.

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1. Introduction

Throughout the paper we denote by \(\mathbb{Z}, \mathbb{Z}^*, \mathbb{N}, \mathbb{Z}_+, \mathbb{Z}_{\leq 0}, \mathbb{R}, \mathbb{C},\) and \(\mathbb{C}^*\) the sets of integers, nonzero integers, non-negative integers, positive integers, non-positive integers, real numbers, complex numbers, and nonzero complex numbers, respectively. All vector spaces and Lie algebras are assumed to be over \(\mathbb{C}\). For a Lie algebra \(\mathfrak{g}\), the universal algebra of \(\mathfrak{g}\) is denoted by \(U(\mathfrak{g})\).

The Virasoro algebra \(\mathfrak{Vir}\) and the Heisenberg algebra \(\mathfrak{H}\) are infinite-dimensional Lie algebras with bases \(\{c, d_n : n \in \mathbb{Z}\}\) and \(\{l, h_n : n \in \mathbb{Z}\}\), respectively. Their Lie brackets are given by

\[
[\mathfrak{Vir}, c] = 0, \quad [d_m, d_n] = (m - n)d_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \quad m, n \in \mathbb{Z},
\]

and

\[
[\mathfrak{H}, l] = 0, \quad [h_m, h_n] = m\delta_{m+n,0}l, \quad m, n \in \mathbb{Z},
\]

respectively. The twisted Heisenberg-Virasoro algebra \(\mathfrak{D}\) is the universal central extension of the Lie algebra \(\{f(t)\frac{d}{dt} + g(t) : f, g \in \mathbb{C}[t, t^{-1}]\}\) of differential operators of order at most one on the Laurent polynomial algebra \(\mathbb{C}[t, t^{-1}]\). Since the Lie algebra \(\mathfrak{D}\) contains the Virasoro algebra \(\mathfrak{Vir}\) and the Heisenberg algebra \(\mathfrak{H}\) as subalgebras (but not the semi-direct product of the two subalgebras), many properties of \(\mathfrak{D}\) are closely related to the algebras \(\mathfrak{Vir}\) and \(\mathfrak{H}\).

The Virasoro algebra \(\mathfrak{Vir}\), the Heisenberg algebra \(\mathfrak{H}\) and the twisted Heisenberg-Virasoro algebra \(\mathfrak{D}\) are very important infinite-dimensional Lie algebras in mathematics and in mathematical physics because of its beautiful representation theory (see \([32, 33]\)), and its widespread applications to vertex operator algebras (see \([19, 23]\)), quantum physics (see \([26]\)), conformal field theory (see \([18]\)), and so on. Many other interesting and important algebras contain the Virasoro algebra as a subalgebra, such as the Schrödinger-Virasoro algebra (see \([30, 31]\)), the mirror Heisenberg-Virasoro algebra \(\mathfrak{D}\) (see \([7, 25, 38]\)) and so on. These Lie algebras have nice structures and perfect theory on simple Harish-Chandra modules. The mirror Heisenberg-Virasoro algebra \(\mathfrak{D}\) is the even part of the mirror \(N = 2\) superconformal algebra (see \([7]\)), and is the semi-direct product of the Virasoro algebra and the twisted Heisenberg algebra (see Definition \([2, 1]\)).

1.1. Connection with representation theory of Lie algebras. Representation theory of Lie algebras has attracted a lot of attentions of mathematicians and physicists. For a Lie algebra \(\mathfrak{g}\) with a triangular decomposition \(\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-\) in the sense of \([49]\), one can study its weight and non-weight representation theory. For weight representation approach, to some extent, Harish-Chandra modules are well understood for many infinite-dimensional Lie algebras, for example, the affine Kac-Moody algebras in \([12, 49]\), the Virasoro algebra in \([20, 33, 44]\), the twisted Heisenberg-Virasoro algebra in \([4, 41]\), the Schrödinger-Virasoro algebra (partial results) in \([30, 31, 37]\), and the mirror Heisenberg-Virasoro algebra in \([38]\). There are also some researches about weight modules with infinite-dimensional weight spaces (see \([9, 16, 43]\)).

Recently, non-weight module theory over Lie algebras \(\mathfrak{g}\) attracts more attentions from mathematicians. In particular, \(\mathcal{U}(\mathfrak{h})\)-free \(\mathfrak{g}\)-modules, Whittaker modules, and restricted modules have
been widely studied for many Lie algebras. The notation of $\mathfrak{u}(\mathfrak{b})$-free modules was first introduced by Nilsson [50] for the simple Lie algebra $\mathfrak{sl}_{n+1}$. At the same time these modules were introduced in a very different approach in the paper [53]. Later, $\mathfrak{u}(\mathfrak{b})$-free modules for many infinite-dimensional Lie algebras are determined, for example, the Kac-Moody algebras in [17, 11, 28], the Virasoro algebra in [39, 42, 46], the Witt algebra in [53], the twisted Heisenberg-Virasoro algebra and $W(2, 2)$ algebra in [13, 15, 43], and so on.

Whittaker modules for $\mathfrak{sl}_2(\mathbb{C})$ were first constructed by Arnal and Pinzcon (see [5]). Whittaker modules for arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ were introduced and systematically studied by Kostant in [34], where he proved that these modules with a fixed regular Whittaker function (Lie homomorphism) on a nilpotent radical are (up to isomorphism) in bijective correspondence with central characters of $\mathfrak{u}(\mathfrak{g})$. In recent years, Whittaker modules for many other Lie algebras have been investigated (see [1, 2, 8, 10, 17, 47, 48]).

1.2. Restricted modules. The restricted modules for a $\mathbb{Z}$-graded Lie algebra are the modules in which any vector can be annihilated by sufficiently large positive part of the Lie algebra. Whittaker modules and highest weight modules are restricted modules, and, in some sense, restricted modules can be seen as generalization of Whittaker modules and highest weight modules. Understanding restricted modules for an infinite-dimensional Lie algebra with a $\mathbb{Z}$-gradation is one of core topics in Lie theory, for this class of modules are closely connected with the modules for corresponding vertex operator algebras. The first step of studying restricted modules is to classify all restricted modules for a Lie algebra. But this is a difficult challenge. Up to now all simple restricted modules for the Virasoro algebra are classified in [46]. There are some partial results of simple restricted modules for other Lie algebras. Some simple restricted modules for twisted Heisenberg-Virasoro algebra and mirror Heisenberg-Virasoro algebra with level 0 were constructed in [14, 24, 38]. Rudakov investigated a class of simple modules over Lie algebras of Cartan type $W, S, H$ in [51, 52], and these modules are restricted modules over the Cartan type Lie algebras of the formal power series.

1.3. Vertex algebraic approach. For many infinite-dimensional $\mathbb{Z}$-graded Lie algebras and superalgebras $\mathfrak{g}$, one can construct the associated (universal) vertex algebra $\mathcal{V}_\mathfrak{g}$ with the property:

- Any restricted $\mathfrak{g}$-module is a weak $\mathcal{V}_\mathfrak{g}$-module;
- Any weak module for the vertex algebra $\mathcal{V}_\mathfrak{g}$ has the structure of a restricted $\mathfrak{g}$-module.

This approach is very prominent for the following cases:

- Affine Kac-Moody algebra of type $X^{(1)}_n$, when the associated vertex algebra is the universal affine vertex algebra $\mathcal{V}^\Lambda(\mathfrak{g})$ for certain simple Lie algebra $\mathfrak{g}$. This approach was used in [2] for studying Whittaker modules.
- Virasoro Lie algebra, when the associated vertex algebra is the universal Virasoro vertex algebra $\mathcal{V}_{Vir}$ (cf. [35]).
- Heisenberg vertex algebra, when the associated vertex algebra is $M(1)$ (cf. [35]).
- Heisenberg-Virasoro algebra; super conformal algebras, etc.

From the vertex-algebraic point of view, the twisted Heisenberg-Virasoro algebra and its untwisted modules were investigated in [3, 27].
The restricted representations of nonzero level for the twisted Heisenberg-Virasoro algebra corresponds to representations of the Heisenberg-Virasoro vertex operator algebra $\mathcal{V}^c = V_{Vir}^c \otimes M(1)$, where $V_{Vir}^c$ is the universal Virasoro vertex operator algebra of central charge $c = \ell_1 - 1$, and $M(1)$ is the Heisenberg vertex operator algebra of level 1. (Since $M(\ell_2) \cong M(1)$ (cf. [35]), we usually assume that the level $\ell_2 = 1$.)

Moreover, the restricted representations of the mirror Heisenberg-Virasoro algebra $\mathcal{D}$ can be treated as twisted modules for the Heisenberg-Virasoro vertex operator algebra $\mathcal{V}^c = V_{Vir}^c \otimes M(1)$.

We summarize:

- The category of restricted $\mathcal{D}$-modules of level 1 is equivalent to the category of weak (untwisted) modules for the vertex operator algebra $\mathcal{V}^c$;
- The category of restricted $\mathcal{D}$-modules of level 1 is equivalent to the category of weak twisted modules for the vertex operator algebra $\mathcal{V}^c$.

1.4. Main results. In this paper, our main goal is to classify all simple restricted modules for mirror Heisenberg-Virasoro algebra $\mathcal{D}$, and classify simple restricted modules with non-zero level for the twisted Heisenberg-Virasoro algebra $\mathcal{D}$. As applications, we describe the simple untwisted and twisted modules for Heisenberg-Virasoro vertex operator algebras $\mathcal{V}^c$. The main results are the following theorems:

**Main theorem A** (Theorem 4.13) Let $S$ be a simple restricted module over the mirror Heisenberg-Virasoro algebra $\mathcal{D}$ with level $\ell \neq 0$. Then

(i) $S \cong H^\mathcal{D}$ where $H$ is a simple restricted module over the Heisenberg algebra $\mathcal{H}$, or

(ii) $S$ is an induced $\mathcal{D}$-module from a simple restricted $\mathcal{D}^{(0,-n)}$-module, or

(iii) $S \cong U^\mathcal{D} \otimes H^\mathcal{D}$ where $U$ is a simple restricted $\mathcal{Vir}$-module, and $H$ is a simple restricted module over the Heisenberg algebra $\mathcal{H}$.

**Main theorem B** (Theorem 5.8) Let $M$ be a simple restricted module over the twisted Heisenberg-Virasoro algebra $\mathcal{D}$ with level $\ell \neq 0$. Then

(i) $M \cong K(z)^\mathcal{D}$ where $K$ is a simple restricted $\mathcal{H}$-module and $z \in \mathbb{C}$, or

(ii) $M$ is an induced $\mathcal{D}$-module from a simple restricted $\mathcal{D}^{(0,-n)}$-module for some $n \in \mathbb{Z}_+$, or

(iii) $M \cong K(z)^\mathcal{D} \otimes U^\mathcal{D}$ where $z \in \mathbb{C}$, $K$ is a simple restricted $\mathcal{H}$-module and $U$ is a simple restricted $\mathcal{Vir}$-module.

These simple restricted modules over the (mirror) Heisenberg-Virasoro algebra are actually all simple weak ($\theta$-twisted) modules over Heisenberg-Virasoro vertex operator algebras $\mathcal{V}^c$ (where the involution $\theta$ is defined in Section A, see Theorem A.2, Theorem A.3). As a consequence, we obtain the classification of twisted and untwisted simple modules for the Heisenberg-Virasoro vertex operator algebra $\mathcal{V}^c$, i.e., we obtain all weak simple $\mathcal{V}^c$-modules and all weak simple $\theta$-twisted $\mathcal{V}^c$-modules.

It is important to notice that certain weak modules induced from simple restricted $\mathcal{D}^{(0,-n)}$, as a (twisted) modules for $V_{Vir}^c \otimes M(\ell_2)$, do not have the form $M_1 \otimes M_2$ (see Section 7). This is interesting, since in the category of ordinary (twisted) modules for the vertex operator algebra, such modules don’t exist (see [21] Theorem 4.7.4] and its twisted analogs).
1.5. **Organization of the paper.** The present paper is organized as follows. In Section 2, we recall notations related to the algebras $D$ and $\bar{D}$, collect some known results, and establish a general result for a simple module to be a tensor product module over a class of Lie algebras (Theorem 2.13). In Section 3, we construct a class of induced simple $D$-modules (Theorem 3.1). In Section 4, we completely determine all simple restricted modules over the mirror Heisenberg-Virasoro algebra $\mathbb{D}$ (Theorems 4.13, 2.11). In Section 5, we use a similar method as in Section 4 to classify the simple restricted modules of level nonzero over the twisted Heisenberg-Virasoro algebra $\bar{D}$ (see Theorem 5.8). In Section 6, we apply Theorem 4.13 to generalize the result in [45] to the algebra $D$, i.e., we give a new characterization of simple highest weight modules over $D$ (Theorem 6.1). We also characterize simple Whittaker modules over $D$ (Theorem 6.3). In Section 7, we present a few examples of simple restricted $D$-modules and $\bar{D}$-modules induced from simple modules over finite dimensional solvable Lie algebras, that are not tensor product modules of Virasoro modules and Heisenberg modules. This is very different from the case of simple highest weight modules over $D$ and $\bar{D}$ which are always tensor products of simple Virasoro modules and simple Heisenberg modules. In Appendix A, we apply Theorems 4.13 and 5.8 to classify simple weak twisted and untwisted modules over the Heisenberg-Virasoro vertex operator algebras $V^c \cong V^c_{Vir} \otimes M(1)$ (Theorems A.2, A.3). The main method in this paper is the (twisted) weak module structure of the Heisenberg vertex operator algebra $M(1)$ on simple restricted $D$-modules.

2. **Notations and preliminaries**

In this section we recall some notations and known results related to the algebras $D$ and $\bar{D}$.

**Definition 2.1.** The twisted Heisenberg-Virasoro algebra $\mathfrak{D}$ is a Lie algebra with a basis

$$\{d_m, h_r, \bar{c}_1, \bar{c}_2, \bar{c}_3 : m, r \in \mathbb{Z}\}$$

and subject to the commutation relations

$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} \bar{c}_1,$$

$$[d_m, h_r] = -rh_{m+r} + \delta_{m+r,0}(m^2 + m)\bar{c}_2,$$

$$[h_r, h_s] = r\delta_{r+s,0} \bar{c}_3,$$

$$[\bar{c}_1, \mathfrak{D}] = [\bar{c}_2, \mathfrak{D}] = [\bar{c}_3, \mathfrak{D}] = 0,$$

for $m, n, r, s \in \mathbb{Z}$.

It is clear that $\mathfrak{D}$ contains a copy of the Virasoro subalgebra $\mathfrak{Vir} = \text{span}\{\bar{c}_1, d_i : i \in \mathbb{Z}\}$ and the Heisenberg algebra $\mathfrak{H} = \bigoplus_{r \in \mathbb{Z}} \mathbb{C} h_r \oplus \mathbb{C} \bar{c}_3$. So $\mathfrak{D}$ has a quotient algebra that is isomorphic to a copy of Heisenberg-Virasoro algebra

$$\widetilde{\mathfrak{D}} = \text{span}_\mathbb{C} \{d_m, h_r, \bar{c}_1, \bar{c}_3 : m, r \in \mathbb{Z}\}$$

whose relations are defined by (2.1) (but the second and fourth equalities are replaced by $[d_m, h_r] = -rh_{m+r}$ and $[\bar{c}_1, \mathfrak{D}] = [\bar{c}_3, \mathfrak{D}] = 0$).
Note that $\mathcal{D}$ is $\mathbb{Z}$-graded and equipped with a triangular decomposition: $\mathcal{D} = \mathcal{D}^+ \oplus \mathfrak{h} \oplus \mathcal{D}^-$, where

$$\mathcal{D}^\pm = \bigoplus_{n,r \in \mathbb{Z}_c} (\mathbb{C}d_{zn} \oplus \mathbb{C}h_{zr}), \quad \mathfrak{h} = \mathbb{C}d_0 \oplus \mathbb{C}h_0 \oplus \mathbb{C}c_1 + \mathbb{C}c_2 + \mathbb{C}c_3.$$ 

Moreover, $\mathcal{D} = \oplus_{i \in \mathbb{Z}} \mathcal{D}_i$ is $\mathbb{Z}$-graded with $\mathcal{D}_i = \mathbb{C}d_i \oplus \mathbb{C}h_i$ for $i \in \mathbb{Z}^+$, $\mathcal{D}_0 = \mathfrak{h}$.

**Definition 2.2.** The mirror Heisenberg-Virasoro algebra $\mathcal{D}$ is a Lie algebra with a basis

$$\left\{ d_m, h_r, c_1, c_2 \mid m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z} \right\}$$

and subject to the commutation relations

$$[d_m, d_n] = (m - n)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12}c_1,$$

$$[d_m, h_r] = -rh_{m+r},$$

$$[h_r, h_s] = r\delta_{r+s,0}c_2,$$

$$[c_1, \mathcal{D}] = [c_2, \mathcal{D}] = 0,$$

for $m, n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z}$.

It is clear that $\mathcal{D}$ is the semi-direct product of the Virasoro subalgebra $\mathfrak{Vir} = \text{span}\{c_1, d_i \mid i \in \mathbb{Z}\}$ and the twisted Heisenberg algebra $\mathcal{H} = \bigoplus_{r \in \frac{1}{2} + \mathbb{Z}} \mathbb{C}h_r \oplus \mathbb{C}c_2$. Note that $\mathcal{D}$ is $\frac{1}{2}\mathbb{Z}$-graded and equipped with triangular decomposition: $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^0 \oplus \mathcal{D}^-$, where

$$\mathcal{D}^\pm = \bigoplus_{n \in \mathbb{Z}_c} \mathbb{C}d_{zn} \oplus \bigoplus_{r \in \frac{1}{2} + \mathbb{N}} \mathbb{C}h_{zr}, \quad \mathcal{D}^0 = \mathbb{C}d_0 \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2.$$ 

Moreover, $\mathcal{D} = \oplus_{i \in \mathbb{Z}} \mathcal{D}_i$ is $\mathbb{Z}$-graded with $\mathcal{D}_i = \mathbb{C}d_i \oplus \mathbb{C}h_{i+\frac{1}{2}}$ for $i \in \mathbb{Z}^+ \setminus \{-1\}$, $\mathcal{D}_0 = \mathbb{C}d_0 \oplus \mathbb{C}h_{\frac{1}{2}} \oplus \mathbb{C}c_1$ and $\mathcal{D}_{-1} = \mathbb{C}d_{-1} \oplus \mathbb{C}h_{-\frac{1}{2}} \oplus \mathbb{C}c_2$.

**Definition 2.3.** Let $\mathcal{G} = \oplus_{i \in \mathbb{Z}} \mathcal{G}_i$ be a $\mathbb{Z}$-graded Lie algebra. A $\mathcal{G}$-module $V$ is called the **restricted module** if for any $v \in V$ there exists $n \in \mathbb{N}$ such that $\mathcal{G}_n v = 0$, for $i > n$. The category of restricted modules over $\mathcal{G}$ will be denoted as $\mathcal{R}_\mathcal{G}$.

**Definition 2.4.** Let $a$ be a subalgebra of a Lie algebra $\mathcal{G}$, and $V$ be a $\mathcal{G}$-module. We denote

$$\text{Ann}_V(a) = \{ v \in V : av = 0 \}.$$ 

**Definition 2.5.** Let $\mathcal{G}$ be a Lie algebra and $V$ a $\mathcal{G}$-module and $x \in \mathcal{G}$.

1. If for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $x^n v = 0$, then we say that the action of $x$ on $V$ is **locally nilpotent**.

2. If for any $v \in V$ we have $\dim(\sum_{n \leq N} \mathbb{C}x^n v) < +\infty$, then the action of $x$ on $V$ is said to be **locally finite**.

3. The action of $\mathcal{G}$ on $V$ is said to be **locally nilpotent** if for any $v \in V$ there exists an $n \in \mathbb{Z}_+$ (depending on $v$) such that $x_1 x_2 \cdots x_n v = 0$ for any $x_1, x_2, \cdots, x_n \in L$.

4. The action of $\mathcal{G}$ on $V$ is said to be **locally finite** if for any $v \in V$ there is a finite-dimensional $L$-submodule of $V$ containing $v$.

**Definition 2.6.** If $W$ is a $\mathcal{D}$-module (resp. $\mathcal{D}$-module) on which $c_1$ (resp. $\bar{c}_1$) acts as complex scalar $c$, we say that $W$ is of **central charge** $c$. If $W$ is a $\mathcal{D}$-module (resp. $\mathcal{D}$-module) on which $c_2$ (resp. $\bar{c}_3$) acts as complex scalar $\ell$, we say that $W$ is of **level** $\ell$. 
Note that if \( V \) is a \( \mathfrak{vir} \)-module, then \( V \) can be easily viewed as a \( \mathfrak{d} \)-module (resp. \( \tilde{\mathfrak{d}} \)-module) by defining \( \mathcal{H}V = 0 \) (resp. \((\mathcal{H} + \mathbb{C}\mathfrak{e}_2)V = 0\)), the resulting module is denoted by \( V_{\mathcal{H}} \) (resp. \( V_{\tilde{\mathcal{H}}} \)).

Thanks to [38], for any \( H \in \mathcal{R}_{\mathcal{H}} \) with the action of \( \mathfrak{e}_2 \) as a nonzero scalar \( \ell \), we can give \( H \) a \( \mathfrak{d} \)-module structure denoted by \( H_{\mathcal{H}} \) via the following map
\[
d_n \mapsto L_n = \frac{1}{2\ell} \sum_{k \in \mathbb{Z} + \frac{1}{2}} h_{n-k}h_k, \quad \forall n \in \mathbb{Z}, n \neq 0, \tag{2.2}
\]
\[
d_0 \mapsto L_0 = \frac{1}{2\ell} \sum_{k \in \mathbb{Z} + \frac{1}{2}} h_{-\ell}|h| + \frac{1}{16}, \tag{2.3}
\]
\[
h_r \mapsto h_r, \quad \forall r \in \frac{1}{2} + \mathbb{Z}, \quad c_1 \mapsto 1, \quad c_2 \mapsto \ell. \tag{2.4}
\]

**Remark 2.7.** The vertex operator algebra interpretation of the formula \((2.2)-(2.3)\) will be given in Section A. The operators \( L_n, n \in \mathbb{Z}, \) will be represented as components of the field \( L_{\mathcal{Heis}}(z) \) in \((A.2)\).

According to (9.4.13) and (9.4.15) in [22], we know that for all \( m, n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}, \) we have
\[
[L_n, h_r] = -rh_{n+r},
\]
\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}. \tag{2.5}
\]

Moreover, since
\[
[d_m, h_{n-k}h_k] = [d_m, h_{n-k}]h_k + h_{n-k}[d_m, h_k] = [L_m, h_{n-k}]h_k + h_{n-k}[L_m, h_k] = [L_m, h_{n-k}h_k],
\]
we see that
\[
[d_m, L_n] = [L_m, L_n] \tag{2.6}
\]

By [43], for any \( z \in \mathbb{C} \) and \( H \in \mathcal{R}_{\tilde{\mathcal{H}}} \) with the action of \( \mathfrak{e}_3 \) as a nonzero scalar \( \ell \), we can give \( H \) a \( \tilde{\mathfrak{d}} \)-module structure (denoted by \( H(z)_{\tilde{\mathcal{H}}} \)) via the following map
\[
d_n \mapsto \tilde{L}_n = \frac{1}{2\ell} \sum_{k \in \mathbb{Z}} :h_{n-k}h_k:\ + \frac{(n+1)z}{\ell}h_n, \quad \forall n \in \mathbb{Z}, \tag{2.7}
\]
\[
h_r \mapsto h_r, \quad \forall r \in \mathbb{Z}, \quad \tilde{c}_1 \mapsto 1 - \frac{12z^2}{\ell}, \quad \tilde{c}_2 \mapsto z, \quad \tilde{c}_3 \mapsto \ell, \tag{2.8}
\]
where the normal order is define as
\[
:h_1h_s: = :h_sh_r: = h_sh_r, \text{ if } r \leq s.
\]

According to (8.7.9), (8.7.13) in [22] and by some simple computation, we deduce that for all \( m, n, r \in \mathbb{Z}, \)
\[
[\tilde{L}_m, h_r] = -rh_{m+r} + \delta_{m+r,0}(m^2 + m)z,
\]
\[
[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}(1 - \frac{12z^2}{\ell}). \tag{2.9}
\]

**Remark 2.8.** The vertex operator algebra interpretation of the operators \((2.7)\) will be given later in \((A.1)\). Then relations \((2.9)\) can be obtained using commutator formula, similarly as in [3].
Moreover, since
\[ [d_m, h_{n-k}] = [d_m, h_{n-k}] h_k + h_{n-k} [d_m, h_k] = [\tilde{L}_m, h_{n-k}] h_k + h_{n-k} [\tilde{L}_m, h_k] = [\tilde{L}_m, h_{n-k} h_k], \]
we see that
\[ [d_m, \tilde{L}_n] = [\tilde{L}_m, \tilde{L}_n]. \tag{2.10} \]

For convenience, we define the following subalgebras of \( \mathcal{D} \). For any \( m \in \mathbb{N}, n \in \mathbb{Z} \), set
\[
\mathcal{D}^{(m,n)} = \sum_{i \in \mathbb{N}} C_d d_{m+i} \oplus C_h h_{n+i} + \frac{1}{2} \oplus C_c c_1 \oplus C_c c_2, \\
\mathcal{D}^{(m,-\infty)} = \sum_{i \in \mathbb{N}} C_d d_{m+i} \oplus \sum_{i \in \mathbb{Z}} C_h h_i + C_c c_1 + C_c c_2, \\
\mathcal{D}^m = \sum_{i \in \mathbb{N}} C_d d_{m+i}, \\
\mathcal{D}_{\leq 0} = \sum_{i \in \mathbb{N}} C_d d_i, \\
\mathcal{D}^{\geq 0} = \mathcal{D}^{(m,0)} = \sum_{i \in \mathbb{N}} C_d d_{m+i} \oplus C_h h_{n+i} + \frac{1}{2} \oplus C_c c_1 \oplus C_c c_2, \\
\mathcal{D}^\geq n = \sum_{i \in \mathbb{N}} C_h h_{n+i} + \frac{1}{2}. 
\]

Similarly, we define the subalgebras of \( \tilde{\mathcal{D}} \) as following: for \( m \in \mathbb{N}, n \in \mathbb{Z} \), set
\[
\tilde{\mathcal{D}}^{(m,n)} = \sum_{i \in \mathbb{N}} C_d d_{m+i} \oplus C_h h_{n+i} \oplus C_{\tilde{c}} \tilde{c}_1 \oplus C_{\tilde{c}} \tilde{c}_2 + C_{\tilde{c}} c_3, \\
\tilde{\mathcal{D}}^{(m,-\infty)} = \sum_{i \in \mathbb{N}} C_d d_{m+i} \oplus \sum_{i \in \mathbb{Z}} C_h h_i + C_{\tilde{c}} \tilde{c}_1 \oplus C_{\tilde{c}} \tilde{c}_2 + C_{\tilde{c}} c_3, \\
\tilde{\mathcal{D}}^m = \sum_{i \in \mathbb{N}} C_d d_{m+i}, \\
\tilde{\mathcal{D}}_{\leq 0} = \sum_{i \in \mathbb{N}} C_d d_i, \\
\tilde{\mathcal{D}}^{\geq 0} = \tilde{\mathcal{D}}^{(m,0)} = \sum_{i \in \mathbb{N}} C_d d_{m+i} \oplus C_h h_{n+i} + \frac{1}{2} \oplus C_{\tilde{c}} \tilde{c}_1 \oplus C_{\tilde{c}} \tilde{c}_2, \\
\tilde{\mathcal{D}}^{\geq n} = \sum_{i \in \mathbb{N}} C_h h_{n+i}. 
\]

Note that we use the same notations \( \mathcal{D}^{(m,n)}, \mathcal{D}^{m}, \mathcal{D}_{\leq 0}, \mathcal{D}^{\geq 0}, \mathcal{D}^\geq n, \) to denote the subalgebras of \( \mathcal{D} \) and of \( \tilde{\mathcal{D}} \) since there will be no ambiguities in later contexts.
Denote by $\mathbb{M}$ the set of all infinite vectors of the form $i := (\ldots, i_2, i_1)$ with entries in $\mathbb{N}$, satisfying the condition that the number of nonzero entries is finite. We can make $(\mathbb{M}, +)$ a monoid by

$$(\ldots, i_2, i_1) + (\ldots, j_2, j_1) = (\ldots, i_2 + j_2, i_1 + j_1).$$

Let $\mathbf{0}$ denote the element $(\ldots, 0, 0) \in \mathbb{M}$ and for $i \in \mathbb{Z}$, let $e_i = (\ldots, 0, 1, 0, \ldots, 0) \in \mathbb{M}$, where $1$ is in the $i$'th position from right. For any $i \in \mathbb{M}$ we define

$$w(i) = \sum_{n \in \mathbb{Z}_+} i_n \cdot n,$$

(2.13)

Let $<$ be the reverse lexicographic total order on $\mathbb{M}$, that is, for any $i, j \in \mathbb{M}$,

$$j < i \iff \text{there exists } r \in \mathbb{N} \text{ such that } j_r < i_r \text{ and } j_s = i_s, \forall 1 \leq s < r.$$

We extend the above total order on $\mathbb{M} \times \mathbb{M}$, that is, for all $i, j, k, l \in \mathbb{M}$,

$$(i, j) < (k, l) \iff (j, w(j), w(i) + w(j)) < (l, w(l), w(k) + w(l)), \text{ or } (j, w(j), w(i) + w(j)) < (l, w(l), w(k) + w(l)), \text{ and } i < k.$$

Now we define another total order $\prec'$ on $\mathbb{M} \times \mathbb{M}$: for all $i, j, k, l \in \mathbb{M}$,

$$(i, j) <' (k, l) \iff (j, i) < (l, k)$$

The symbols $\leq$ and $\leq'$ have the obvious meanings.

It is not hard to verify that

$$(a, b) \leq (c, d) \& (c', d') < (a', b') \implies (a - a', b - b') < (c - c', d - d'),$$

provided $(a, b), (c, d), (c', d'), (a', b'), (a - a', b - b'), (c - c', d - d') \in \mathbb{M} \times \mathbb{M}$, where the difference is the corresponding entry difference.

For $n \in \mathbb{Z}$, let $V$ be an simple $\mathbb{Z}^{(0,-n)}$-module. According to the PBW Theorem, every nonzero element $v \in \text{Ind}^{\mathbb{Z}^{(0,-n)}}(V)$ can be uniquely written in the following form

$$v = \sum_{i, k \in \mathbb{M}} h^i d^k v_{i,k},$$

(2.14)

where

$$h^i d^k = \ldots h^i_{-2-n+2} h^i_{-1-n+1} \ldots d^k_{-2} d^k_{-1} \in U(\mathbb{Z}^-), v_{i,k} \in V,$$

and only finitely many $v_{i,k}$ are nonzero. For any nonzero $v \in \text{Ind}(V)$ as in (2.14), we will use the following notations for later use:

1. Denote by $\text{supp}(v)$ the set of all $(i, k) \in \mathbb{M} \times \mathbb{M}$ such that $v_{i,k} \neq 0$.
2. Denote by $w(v) = \max\{w(i) + w(k) : (i, k) \in \text{supp}(v)\}$, called the length of $v$.
3. Denote by $\text{deg}(v)$ to be the largest element in $\text{supp}(v)$ with respect to the total order $\prec$.
4. Denote by $\text{deg}'(v)$ to be the largest element in $\text{supp}(v)$ with respect to the total order $\prec'$.

We first recall from [46] the classification for simple restricted $\mathfrak{vir}$-modules.

**Theorem 2.9.** Any simple restricted $\mathfrak{vir}$-module is a highest weight module, or isomorphic to $\text{Ind}_{\mathfrak{vir}^+}^{\mathfrak{vir}_+} V$ for an simple $\mathfrak{vir}_+$-module $V$ such that for some $k \in \mathbb{Z}_+$,
(a) $d_k$ acts injectively on $V$;
(b) $d_iV = 0$ for all $i > k$.

Simple restricted $\mathfrak{D}$-modules with level 0 are classified in [33] by the following two theorems.

**Theorem 2.10.** Let $V$ be a simple $\mathfrak{D}^{(0,-n)}$-module for some $n \in \mathbb{Z}_+$ and $c \in \mathbb{C}$ such that $c_1v = cv, c_2v = 0$ for any $v \in V$. Assume that there exists an integer $k \geq -n$ satisfying the following two conditions:

(a) the action of $h_{k+\frac{1}{2}}$ on $V$ is bijective;
(b) $h_{m+\frac{1}{2}}V = 0 = d_{m+n}V$ for all $m > k$.

Then the induced $\mathfrak{D}$-module $\text{Ind}_{\mathfrak{D}^{(0,-n)}}(V)$ is simple.

**Theorem 2.11.** Every simple restricted $\mathfrak{D}$-module of level 0 is isomorphic to a restricted $\mathfrak{Vir}$-module with $\mathfrak{HS} = 0$, or $S \cong \text{Ind}_{\mathfrak{D}^{(0,0)}}(V)$ for some $n \in \mathbb{N}$ and a simple $\mathfrak{D}^{(0,-n)}$-module $V$ as in Theorem 2.10.

Actually the simple $\mathfrak{D}^{(0,-n)}$-module $V$ can be considered as a simple module over a finite dimensional solvable Lie algebra $\mathfrak{D}^{(0,-n)}/\mathfrak{D}^{(t+n+1,\ell-n)}$ for some $t \in \mathbb{Z}_+$ and injective action of $h_{k+\frac{1}{2}}$ on $V$.

For simple restricted $\mathfrak{D}$-modules with level 0, we know the following results from [14].

**Theorem 2.12.** Let $n \in \mathbb{N}$ and $V$ be a simple module over $\mathfrak{D}^{(0,-n)}$ or over $\mathfrak{D}^{(0,-\infty)}$ with $\ell = 0$, $h_0 = \mu, c_2 = z$. If there exists $k \in \mathbb{N}$ such that

(a) \[
\begin{cases}
h_k \text{ acts injectively on } V, & \text{if } k \neq 0, \\
\mu + (1 - r)z \neq 0, \forall r \in \mathbb{Z} \setminus \{0\}, & \text{if } k = 0;
\end{cases}
\]

(b) $h_iV = d_jV = 0$ for all $i > k$ and $j > k + n$.

then

(1) $\text{Ind}(V)$ is a simple $\mathfrak{D}$-module;
(2) $h_i, d_j$ act locally nilpotently on $\text{Ind}(V)$ for all $i > k$ and $j > k + n$.

Now we generalize Theorem 12 in [43] as follows.

Let $\mathfrak{g} = a \ltimes b$ be a Lie algebra where $a$ is a Lie subalgebra of $\mathfrak{g}$ and $b$ is an ideal of $\mathfrak{g}$. Let $M$ be a $\mathfrak{g}$-module with a $b$-submodule $H$ so that the $b$-submodule structure on $H$ can be extended to a $\mathfrak{g}$-module structure on $H$. We denote this $\mathfrak{g}$-module by $H^b$. For any $\alpha$-module $U$, we can make it into a $\mathfrak{g}$-module by $bU = 0$. We denote this $\mathfrak{g}$-module by $U^b$.

**Theorem 2.13.** Let $\mathfrak{g} = a \ltimes b$ be a countable dimensional Lie algebra where $a$ is a Lie subalgebra of $\mathfrak{g}$ and $b$ is an ideal of $\mathfrak{g}$. Let $M$ be a simple $\mathfrak{g}$-module with a simple $b$-submodule $H$ so that an $H^b$ exists. Then $M \cong H^b \otimes U^b$ as $\mathfrak{g}$-modules for some simple $\alpha$-module $U$.

**Proof.** Define the one-dimensional $b$-module $\mathbb{C}v_0$ by $bv_0 = 0$. Then $H \cong H \otimes \mathbb{C}v_0$ as $b$-modules. Now from Lemma 8 in [43], we have

$$\text{Ind}_b^H \cong \text{Ind}_b^H (H \otimes \mathbb{C}v_0) \cong H^b \otimes \text{Ind}_b^H C v_0.$$
Note that $\text{Ind}_\mathfrak{b}^\mathfrak{b} \mathfrak{c}_V \equiv \mathcal{W}^0$ for the universal $\mathfrak{a}$-module $\mathcal{W}$. Since $\mathcal{M}$ is a simple quotient of $\text{Ind}_\mathfrak{b}^\mathfrak{b} \mathcal{H}$, from Theorem 7 in [43] we know that there is a simple quotient $\mathfrak{a}$-module $\mathcal{U}$ of $\mathcal{W}$ such that $\mathcal{M} \cong \mathcal{H}_{\mathfrak{b}} \otimes \mathcal{U}^0$ as $\mathfrak{g}$-modules. Now the theorem follows. \hfill \Box

**Remark.** This theorem has particular meaning for $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ since $\mathcal{H}^0$ automatically exists (see for example [36]). Also, Theorem 2.13 holds for associative algebras.

Applying the above theorem to our mirror Heisenberg-Virasoro algebra $\mathcal{D} = \mathfrak{Vir} \ltimes \mathcal{H}$ and twisted Heisenberg-Virasoro algebra $\mathcal{D}_\mathfrak{g} = \mathfrak{Vir} \ltimes (\mathcal{H} + \mathcal{C}_2)$, we have the following results.

**Corollary 2.14.** Let $\mathcal{V}$ be a simple $\mathcal{D}$-module with nonzero action of $\mathfrak{c}_2$. Then $\mathcal{V} \cong \mathcal{H}_{\mathfrak{b}} \otimes \mathcal{U}_{\mathfrak{b}}$ as a $\mathcal{D}$-module for some simple module $\mathcal{H} \in \mathcal{R}_\mathcal{H}$ and some simple $\mathfrak{Vir}$-module $\mathcal{U}$ if and only if $\mathcal{V}$ contains a simple $\mathcal{H}$-submodule $\mathcal{H} \in \mathcal{R}_\mathcal{H}$.

**Proof.** The sufficiency follows from Theorem 2.13 and the necessity follows from that $\mathcal{H} \otimes u$ is a simple $\mathcal{H}$-submodule of $\mathcal{H}_{\mathfrak{b}} \otimes \mathcal{U}_{\mathfrak{b}}$ for any nonzero $u \in \mathcal{U}$. \hfill \Box

## 3. Induced modules over the mirror Heisenberg-Virasoro algebra $\mathcal{D}$

In this section, we construct some simple restricted $\mathcal{D}$-modules induced from some simple ones over some subalgebras $\mathcal{D}^{(0,-n)}$ for $n \in \mathbb{Z}_+$. For that, we need the following formulas in $\mathcal{U}(\mathcal{D})$ which can be shown by induction on $t$: let $i, j, s \in \mathbb{Z}$, $1 \leq s \leq t$ with $j_1 \leq j_2 \leq \cdots \leq j_s$,

\begin{equation}
[h_{-\frac{i}{2}}, h_{j_1 + \frac{i}{2}} h_{j_2 + \frac{i}{2}} \cdots h_{j_s + \frac{i}{2}}] = \sum_{1 \leq s \leq t} \delta_{i+j,s,0} (i - \frac{1}{2}) \mathfrak{c}_2 h_{j_1 + \frac{i}{2}} \cdots h_{j_s + \frac{i}{2}},
\end{equation}

\begin{equation}
[d_{i}, h_{j_1 + \frac{i}{2}} h_{j_2 + \frac{i}{2}} \cdots h_{j_s + \frac{i}{2}}] = \sum_{1 \leq s \leq t} (-j_s - \frac{1}{2}) h_{j_1 + \frac{i}{2}} \cdots h_{j_s + \frac{i}{2}} h_{j_s + \frac{i}{2}},
\end{equation}

\begin{equation}
[h_{-\frac{i}{2}}, d_{j_1} d_{j_2} \cdots d_{j_s}] = \sum_{1 \leq s \leq t} (i - \frac{1}{2}) d_{j_1} \cdots d_{j_s},
\end{equation}

\begin{equation}
[d_{i}, d_{j_1} d_{j_2} \cdots d_{j_s}] = \sum_{1 \leq s \leq t} (i - j_s) d_{j_1} \cdots d_{j_{s-1}} d_{j_s} d_{j_s},
\end{equation}

where $\hat{h}_{j_s + \frac{i}{2}}, \hat{d}_{j_s}$ mean that $h_{j_s + \frac{i}{2}}, d_{j_s}$ are deleted in the corresponding products, $a_{s_1, s_2, \cdots, a_{1,2,\cdots,t}}, b_{s_1, s_2, \cdots b_{1,2,\cdots,t}} \in \mathbb{C}$, and $d_{i+j_1+\cdots+j_s} = d_{i+j_1+\cdots+j_s} + \frac{\ell-1}{2} \delta_{i+j_1+\cdots+j_s,0} \mathfrak{c}_1$, $1 \leq s \leq t$.

We are now in the position to present the following main result in this section.

**Theorem 3.1.** Let $k \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$ with $k \geq n$. Let $V$ be a simple $\mathcal{D}^{(0,-n)}$-module with level $\ell \neq 0$ such that there exists $l \in \mathbb{N}$ satisfying both conditions:

(a) $h_{k-\frac{i}{2}}$ acts injectively on $V$;


\[
\]
(b) \( h_{-\frac{1}{2}} V = d_j V = 0 \) for all \( i > k \) and \( j > l \).

Then \( \text{Ind}^{\mathcal{D}(0\to n)}_ {\mathcal{D}(0\to m)}(V) \) is a simple \( \mathcal{D} \)-module if one of the following conditions holds:

1. \( k = n, l \geq 2n, \) and \( d_l \) acts injectively on \( V \);
2. \( k > n, k + n \geq 2, \) and \( l = n + k - 1 \).

Theorem 3.1 follows from Lemmas 3.2, 3.5 directly.

Lemma 3.2. Let \( n \in \mathbb{Z}_+ \) and \( V \) be a \( \mathcal{D}(0\to n) \)-module such that \( h_{-\frac{1}{2}} \) acts injectively on \( V \), and \( h_{-\frac{1}{2}} V = 0 \) for all \( i > n \). For any \( v \in \text{Ind}(V \setminus V) \), let \( \text{deg}(v) = (i, j) \). If \( i \neq 0 \), then \( \text{deg}(h_{p+n-\frac{1}{2}} v) = (i - \epsilon_p, j) \) where \( p = \min\{i : i \neq 0\} \).

\textbf{Proof.} Write \( v \) in the form of (2.14) and let \( (k, l) \in \text{supp}(v) \).

Noticing that \( h_{p+n-\frac{1}{2}} V = 0 \), we have
\[
h_{p+n-\frac{1}{2}} h^k d^i v_{k,l} = [h_{p+n-\frac{1}{2}}, h^k] d^i v_{k,l} + h^k [h_{p+n-\frac{1}{2}}, d^i] v_{k,l}.
\]

First we consider the term \([h_{p+n-\frac{1}{2}}, h^k] d^i v_{k,l}\) which is zero if \( k_p = 0 \). In the case that \( k_p > 0 \), since the level \( k \neq 0 \), it follows from (3.1) that \([h_{p+n-\frac{1}{2}}, h^k] = \lambda h^{k-\epsilon_p} \) for some \( \lambda \in \mathbb{C}^* \).

\[
\text{deg}([h_{p+n-\frac{1}{2}}, h^k] d^i v_{k,l}) = (k - \epsilon_p, l) \leq (i - \epsilon_p, j),
\]

where the equality holds if and only if \( (k, l) = (i, j) \).

Now we consider the term \( h^k [h_{p+n-\frac{1}{2}}, d^i] v_{k,l} \) which is by (3.3) a linear combination of some vectors in the form \( h^k d^i h_{p+n-\frac{1}{2}} v_{k,l} \) with \( j \in \mathbb{Z}_+ \) and \( w(l_j) = w(l) - j \). If \( h^k d^i h_{p+n-\frac{1}{2}} v_{k,l} \neq 0 \), we denote \( \text{deg}(h^k d^i h_{p+n-\frac{1}{2}} v_{k,l}) = (k^*, l^*) \). We will show that
\[
(k^*, l^*) < (i - \epsilon_p, j).
\]

We have four different cases to consider.

(a) \( j < p \). Then \( p + n - j > n \) and \( h_{p+n-\frac{1}{2}} v_{k,l} = 0 \). Hence \( h^k d^i h_{p+n-\frac{1}{2}} v_{k,l} = 0 \).

(b) \( j = p \). Noting that \( h_{-\frac{1}{2}} \) acts injectively on \( V \), we see \( (k^*, l^*) = (k, l_p) \) and \( w(k^*) + w(l^*) = (k) + w(l) - p \) with \( w(l_p) = w(l) - p < w(l) \).

If \( w(k) + w(l) < w(i) + w(j) \), then \( (k^*, l^*) < (i - \epsilon_p, j) \).

If \( w(k) + w(l) = w(i) + w(j) \), then there is \( \tau \in \mathcal{M} \) such that \( w(\tau) = p \) and \( l_p = l - \tau \). Since \( (\epsilon_p, 0) < (0, \tau) \) and \( (k, l) \leq (i, j) \), we see that
\[
(k^*, l^*) = (k, l) - (0, \tau) < (i, j) - (\epsilon_p, 0) = (i - \epsilon_p, j).
\]

In both cases, (3.5) holds.

(c) \( p < j < 2n + p \). Then \( h_{p+n-\frac{1}{2}} \notin \mathcal{D}(0\to n) \) and \( h_{p+n-\frac{1}{2}} v_{k,l} \in V \). So
\[
w(k^*) + w(l^*) = w(k) + w(l) - j < w(k) + w(l) - p
\]
and (3.5) holds.
(d) \( j \geq 2n + p \). Then \( p + n - \frac{1}{2} - j < -n + \frac{1}{2} \). Assume \( p + n - \frac{1}{2} - j = -s - n + \frac{1}{2} \) for some \( s \in \mathbb{Z}_+ \), that is, \( -j + s = -2n - p + 1 < -p \). Clearly, the corresponding vector \( h^k d^l h_{p+n-\frac{1}{2}} v_{k,1} \) can be written in the form
\[
h^k h_{-s-n+\frac{1}{2}} d^l v_{k,1} + \text{lower terms},
\]
which means
\[
w(k') + w(l') = w(k) + w(l) - j + s < w(k) + w(l) - p,
\]
and hence (3.5) holds.

In conclusion, \( \deg(h_{p+n-\frac{1}{2}} h^k d^l v_{k,1}) \leq (i - \epsilon, j) \), where the equality holds if and only if \((k, l) = (i, j)\), that is, \( \deg(h_{p+n-\frac{1}{2}} v_{k,1}) = (i - \epsilon, j) \).

**Lemma 3.3.** Let \( n \in \mathbb{Z}_+ \) and \( V \) be a \( \mathcal{D}^{(0,-n)} \)-module satisfying Conditions (a), (b) and (i) in Theorem 3.1. If \( v \in \text{Ind}(V) \setminus V \) with \( \deg(v) = (0, j) \), then \( \deg(d_{q+l} v) = (0, j - \epsilon_q) \) where \( q = \min\{s : j_s \neq 0\} \).

**Proof.** Write \( v \) in the form of (2.14) and let \((k, l) \in \text{supp}(v)\).

Since \( d_{q+l} V = 0 \), we have
\[
d_{q+l} h^k d^l v_{k,1} = [d_{q+l}, h^k d^l v_{k,1} + h^k [d_{q+l}, d^l] v_{k,1}].
\]

We first consider the degree of \( h^k [d_{q+l}, d^l] v_{k,1} \) with \( d_{q+l} h^k d^l v_{k,1} = 0 \). Clearly, by (3.4) we see that \( h^k [d_{q+l}, d^l] v_{k,1} \) is a linear combination of some vectors of the forms \( h^k d^l d_{q+l-j} v_{k,1} \), \( j \in \mathbb{Z}_+ \), and \( h^k d_{q+l} v_{k,1} \) where \( w(l) = w(l) - j \). Clearly, \( \deg(h^k d_{q+l} v_{k,1}) = (k, l_{q+l}) \) has weight
\[
w(k) + w(l) - q - l < w(k) + w(l) - q \leq (w(l) - q),
\]
so \( \deg(h^k d_{q+l} v_{k,1}) < (0, j - \epsilon_q) \). Then we need only to consider \( h^k d^l d_{q+l-j} v_{k,1} \). Denote \( \deg(h^k d^l d_{q+l-j} v_{k,1}) \) by \((k, l')\). We will show that
\[
(k, l') \leq (0, j - \epsilon_q),
\]
where the equality holds if and only if \((k, l) = (0, j)\). We have four different cases to consider.

(i) \( j < q \). Then \( q + l - j > l \) and \( h^k d^l d_{q+l-j} v_{k,1} = 0 \).

(ii) \( j = q \). Then \( q + l - j = l \). Since \( d_l \) acts injectively on \( V \), we see \((k, l') = (k, l_q)\) and \( w(k) + w(l') = w(k) + w(l) - q \). If \( w(k) + w(l) < w(0) + w(j) \), then \((k, l') < (0, j - \epsilon_q)\). If \( w(k) + w(l) = w(0) + w(j) \), there is \( \tau \in \mathbb{M} \) such that \( w(\tau) = q \) and \( l_q = l - \tau \). Then \((0, \epsilon_q) \leq (0, \tau)\). Since \((k, l) \leq (0, j)\), we see that
\[
(k, l') = (k, l) - (0, \tau) \leq (0, j) - (0, \epsilon_q) = (0, j - \epsilon_q).
\]
In both cases we have that
\[
(k, l') \leq (0, j - \epsilon_q),
\]
where the equality holds if and only if \((k, l) = (0, j)\).

(iii) \( q + 1 \leq j \leq q + l \). Then \( 0 \leq q + l - j \leq l - 1 \) and \( d_{q+l-j} v_{k,1} \in V \). So if \( h^k d^l d_{q+l-j} v_{k,1} \neq 0 \), then \( w(k) + w(l') = w(k) + w(l) - j < w(k) + w(l) - q \).

(iv) \( j > q + l \). Then \( q + l - j < 0 \). Clearly, \( w(l') = w(l_j) + (j - q - l) = w(l) - q - l \), and hence
\[
w(k) + w(l') = w(k) + w(l) - q - l < w(k) + w(l) - q.
\]
Therefore, we conclude that (3.6) holds, i.e., \( \sum_{(k,l)} h^k d^l v_{k,1} \) has degree \((0, j - \epsilon_q)\).
Next we consider the degree of the nonzero vector \([d_{q+l}, h^k]d^l v_{k,l}\). By (3.2) we can see that \([d_{q+l}, h^k]d^l v_{k,l}\) is a linear combination of some vectors of the forms \(h^k h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l}\), \(s \in \mathbb{Z}_+\) and \(h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l}\), where \(w(k_s) = w(k) - s\). Noting that \(l \geq 2n\), the degree of \(h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l}\) has weight
\[
w(k) - (q + l + 1 - 2n) + w(l) < w(k) + w(l) - q.
\]
So
\[
deg(h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l}) < (0, j - \epsilon_q).
\]
Next we will show that
\[
deg(h^k h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l}) < (0, j - \epsilon_q),
\] (3.7)
We have two different cases to consider.

(a) \(s > q + l\). The degree of \(h^k h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l}\) has weight
\[
w(k_s) + (s - q - l) + w(l) = w(k) + w(l) - q - l < w(k) + w(l) - q.
\]
So, (3.7) holds in this case.

(b) \(1 \leq s \leq q + l\). We have
\[
h^k h^{q+l-s-n+\frac{1}{2}} d^l v_{k,l} = h^k [h^{q+l-s-n+\frac{1}{2}}, d^l] v_{k,l} + h^k d^l h^{q+l-s-n+\frac{1}{2}} v_{k,l}.
\]
Noting that \(h^{q+l-s-n+\frac{1}{2}} v_{k,l} \in V\) (in particular, \(h^{q+l-s-n+\frac{1}{2}} v_{k,l} = 0\) for \(1 \leq s \leq q + l - 2n\)), we see that if \(h^k d^l h^{q+l-s-n+\frac{1}{2}} v_{k,l} \neq 0\) for \(q + l - 2n + 1 \leq s \leq q + l\), its degree has weight
\[
w(k_s) + w(l) = w(k) + w(l) - s < w(k) + w(l) - q.
\]
Now we consider \(deg(h^k [h^{q+l-s-n+\frac{1}{2}}, d^l] v_{k,l})\) which is denoted by \((\tilde{k}, \tilde{l})\).

(b1) \(1 \leq s \leq q\), that is, \(q + l - s - n \geq n\). Then \(q + l - s - n + \frac{1}{2} = n + p - \frac{1}{2}\) for some \(p \in \mathbb{Z}_+\) and hence \(s + p = q + l - 2n + 1 \geq q + 1\). Thus, by the same arguments in proof of Lemma 3.2, we see
\[
w(\tilde{k}) + w(\tilde{l}) \leq w(k_s) + w(l) - p = w(k) - s + w(l) - p \leq w(k) + w(l) - q - 1 < w(k) + w(l) - q.
\]
So, (3.7) holds in this case.

(b2) \(q + 1 \leq s \leq q + l\). Then by (3.3) and the same arguments in proof of Lemma 3.2, we see
\[
w(\tilde{k}) + w(\tilde{l}) \leq w(k_s) + w(l) = w(k) + w(l) - s \leq w(k) + w(l) - q - 1 < w(k) + w(l) - q.
\]
So, (3.7) holds in this case as well.

Therefore, \(deg(d_{q+l}v) = (0, j - \epsilon_q)\), as desired.

\[\square\]

\textbf{Lemma 3.4.} Let \(k \in \mathbb{Z}_+, n \in \mathbb{Z}\) with \(k \geq n\) and \(k + n \geq 2\), and let \(V\) be a \(\mathcal{D}^{(0, -n)}\)-module such that \(h_{\frac{1}{2}}\) acts injectively on \(V\), and \(h_{\frac{1}{2}} V = 0\) for all \(i > k\). If \(v \in \text{Ind}(V) \setminus V\) with \(deg(v) = (i, j)\) and \(j \neq 0\), then \(deg'(h_{p+k-\frac{1}{2}} v) = (i, j - \epsilon_p)\) where \(p = \min\{s : j_s \neq 0\}\).

\textbf{Proof.} As in (2.14), write \(v = \sum_{\{k,l\}} h^k d^l v_{k,l}\). Consider \(deg'(h_{p+k-\frac{1}{2}} h^k d^l v_{k,l})\) if \(h_{p+k-\frac{1}{2}} h^k d^l v_{k,l} \neq 0\). Noting that \(h_{p+k-\frac{1}{2}} V = 0\), we see
\[
h_{p+k-\frac{1}{2}} h^k d^l v_{k,l} = [h_{p+k-\frac{1}{2}}, h^k] d^l v_{k,l} + h^k [h_{p+k-\frac{1}{2}}, d^l] v_{k,l}.
\]
First we consider the term \([h_{p+k-\frac{1}{2}}, h^k]d^iv_{k,1}\) which is zero if \(k_p' = 0\) for \(p' := p + k - n\). In the case that \(k_p' > 0\), since the level \(\ell \neq 0\), it follows from \((3.3)\) that \([h_{p+k-\frac{1}{2}}, h^k] = \lambda h^{k-k_p'}\) for some \(\lambda \in \mathbb{C}\). Note that \((k, l) \preceq (i, j), (0, \epsilon_p) \preceq (\epsilon_p, 0)\). So

\[
\text{deg}'([h_{p+k-\frac{1}{2}}, h^k]d^iv_{k,1}) = (k - \epsilon_p, l) - (\epsilon_p, 0) \preceq (i, j) - (0, \epsilon_p) = (i, j - \epsilon_p).
\]

Now we consider the term \(h^k[h_{p+k-\frac{1}{2}}, d^iv_{k,1}]\) which is by \((3.3)\) a linear combination of some vectors in the form \(h^kd^ih_{p+k-\frac{1}{2}}v_{k,1}\) with \(j \in \mathbb{Z}_+\) and \(w(l_i) = w(l) - j\). We will show that

\[
\text{deg}'(h^kd^ih_{p+k-\frac{1}{2}}v_{k,1}) = (k', l') \preceq (i, j - \epsilon_p),
\]

where the equality holds if and only if \((k, l) = (i, j)\). We have four different cases to consider.

(a) \(j < p\). Then \(p + k - j > n\) and \(h_{p+k-\frac{1}{2}}v_{k,1} = 0\). Hence \(h^kd^ih_{p+k-\frac{1}{2}}v_{k,1} = 0\).

(b) \(j = p\). Noting that \(h_{\frac{1}{2}}\) acts injectively on \(V\), we see \((k', l') = (k, l_p)\) and \(w(k') + w(l') = w(k) + w(l) - p\).

If \(w(k) + w(l) < w(i) + w(j)\), then \((k', l') \preceq (i, j - \epsilon_p)\).

If \(w(k) + w(l) = w(i) + w(j)\), then there is \(\tau \in \mathbb{M}\) such that \(w(\tau) = p\) and \(l_p = 1 - \tau\). Since \((0, \epsilon_p) \preceq (0, \tau)\) and \((k, l) \preceq (i, j)\), we see that

\[
(k', l') = (k, l) - (0, \tau) \preceq (i, j) - (0, \epsilon_p) = (i, j - \epsilon_p),
\]

where the equality holds if and only if \((k, l) = (i, j)\).

(c) \(p < j < n + k + p\). Then \(h_{p+k-\frac{1}{2}}v_{k,1} \in \mathbb{D}(0,-n)\) and \(h_{p+k-\frac{1}{2}}v_{k,1} \in V\). So

\[
w(k') + w(l') = w(k) + w(l) - j < w(k) + w(l) - p
\]

and \((k', l') \preceq (i, j - \epsilon_p)\).

(d) \(n \geq n + k + p\). Then \(p + k - \frac{1}{2} - j < -n + \frac{1}{2}\). Assume \(p + k - \frac{1}{2} - j = -s - n + \frac{1}{2}\) for some \(s \in \mathbb{Z}_+\), that is, \(-j + s = -n - k + p + 1 < -p\) since \(k + n \geq 2\). Since the corresponding vector \(h^kd^ih_{p+k-\frac{1}{2}}v_{k,1} = h^kh_{-s-n+\frac{1}{2}}d^iv_{k,1} = h^kh_{-s-n+\frac{1}{2}}d^iv_{k,1}\), by \((3.3)\) and simple computations we see \(h^kd^ih_{p+k-\frac{1}{2}}v_{k,1}\) can written as a linear combination of vectors in the form \(h^kh_{-s'-n+\frac{1}{2}}d^iv_{k,1}\) where \(s' \in \mathbb{N}\) and \(\text{deg}'(h^kh_{-s'-n+\frac{1}{2}}d^iv_{k,1})\) has weight

\[
w(k) + s' + s + w(l_{s'+j}) = w(k) + w(l) + s - j.
\]

So

\[
w(k') + w(l') = w(k) + w(l) - j + s < w(k) + w(l) - p \leq w(i) + w(j) - p
\]

and hence \((k', l') \preceq (i, j - \epsilon_p)\).

In conclusion, \(\text{deg}'(h_{p+k-\frac{1}{2}}h^kd^iv_{k,1}) \preceq (i, j - \epsilon_p)\), where the equality holds if and only if \((k, l) = (i, j)\), that is, \(\text{deg}'(h_{p+k-\frac{1}{2}}v) = (i, j - \epsilon_p)\). \(\Box\)

**Lemma 3.5.** Let \(k \in \mathbb{Z}_+, n \in \mathbb{Z}\) such that \(k > n\) and \(k + n \geq 2\), and \(V\) be a \(\mathbb{D}(0,-n)\)-module such that \(h_{\frac{1}{2}}\) acts injectively on \(V\), and \(h_{\frac{1}{2}}V = d^iV = 0\) for all \(i > k\), \(j > k + n - 1\). Assume that \(v = \sum_{(k,l)}h^kd^iv_{k,1} \in \text{Ind}(V) \setminus V\) with \(\text{deg}'(v) = (i,0)\). Set \(q = \min\{s : i_s \neq 0\}\).

(1) If the sum \(\sum_{(k,l)}h^kd^iv_{k,1}\) does not contain terms \(h^kd^iv_{k,1}\) satisfying

\[
w(k) + w(l) = w(i), w(i) - q < w(k) < w(i),
\]

(3.9)
then \( \deg'(d_{q+k+n-1}) = (i - \epsilon_q, 0) \);
(2) Assume that the sum \( \sum_{(k,l)} h^k d^l v_{k,l} \) contains terms \( h^k d^l v_{k,l} \) satisfying (3.9). Let \( \nu' = \nu - \sum_{w(k,l) = w(i)} h^k \nu_{k,0} \) and \( \deg'(\nu') = (k^*, l^*) \) with \( t = \min(s : l^r_s \neq 0) \). Then \( \deg'(h_{k+t'}^{v^*}) = (k^*, l^* - \epsilon_j) \).

**Proof.** Consider \( \deg'(d_{q+k+n-1} h^k d^l v_{k,l}) \) with \( d_{q+k+n-1} h^k d^l v_{k,l} \neq 0 \). Noting that \( d_{q+k+n-1} V = 0 \), we see that

\[
d_{q+k+n-1} h^k d^l v_{k,l} = [d_{q+k+n-1}, h^k] d^l v_{k,l} + h^k [d_{q+k+n-1}, d^l] v_{k,l}.
\]

First we consider the term \([d_{q+k+n-1}, h^k] d^l v_{k,l}\). It follows from (3.2) that \([d_{q+k+n-1}, h^k] d^l v_{k,l}\) is a linear combination of vectors in the forms \( h^k [h_{(q-j)+k-\frac{1}{2}}, d^l] v_{k,l} \) and \( h^k d^l v_{k,l} \) where \( k_j = k - \epsilon_j \), \( w(s) = w(k) - (k + q - n) \). If \( l = 0 \), it is not hard to see that \( \deg'(d_{q+k+n-1} h^k d^l v_{k,l}) \leq (i - \epsilon_q, 0) \) where the equality holds if and only if \((k, l) = (i, 0)\).

Next we assume that \( l \neq 0 \), and continue to consider the term \([d_{q+k+n-1}, h^k] d^l v_{k,l}\). We first consider the term \( h^k [h_{(q-j)+k-\frac{1}{2}}, d^l] v_{k,l} \). We break the arguments into four different cases next.

(a) \( j < q \). In this case, we have \( h^k [h_{(q-j)+k-\frac{1}{2}}, d^l] v_{k,l} = h^k [h_{(q-j)+k-\frac{1}{2}}, d^l] v_{k,l} \). Then it follows from (3.3) that \( h^k [h_{(q-j)+k-\frac{1}{2}}, d^l] v_{k,l} \) is a linear combination of vectors in the form \( h^k d^l h_{(q-j-s)+k-\frac{1}{2}} v_{k,l} \) where \( w(l_s) = w(l) - s \).

(a1) If \( s < q - j \), then \( h^k d^l h_{(q-j-s)+k-\frac{1}{2}} v_{k,l} = 0 \).

(a2) If \( s = q - j \), then \( \deg'(h^k d^l h_{k-s} v_{k,l}) \) has weight

\[
w(k) + w(l) - w(k) + w(l) - j - s = w(k) + w(l) - q.
\]

If \( w(k) + w(l) < w(i) \), or \( w(k) + w(l) = w(i) \) and \( w(k) < w(i) - q \), then \( \deg'(h^k d^l h_{k-s} v_{k,l}) \leq (i - \epsilon_q, 0) \). We will discuss the remaining cases that \((k, l)\) satisfies (3.9) in Case (2). later.

(a3) If \( q - j < s \leq q + k + n - 1 - j \), then \( h_{(q-j-s)+k-\frac{1}{2}} v_{k,l} \in V \) and \( \deg'(h^k d^l h_{k-s} v_{k,l}) \) has weight

\[
w(k) + w(l) = w(k) + w(l) - j - s < w(k) + w(l) - q \leq w(i) - q.
\]

So \( \deg'(h^k d^l h_{(q-j-s)+k-\frac{1}{2}} v_{k,l}) \sim (i - \epsilon_q, 0) \).

(a4) If \( s > q + k + n - 1 - j \), then \( q - j - s + k - \frac{1}{2} = -s' + n + \frac{1}{2} \) for some \( s' \in \mathbb{Z}_+ \). It is easy to see \( h^k d^l h_{(q-j-s')+n+\frac{1}{2}} v_{k,l} \) can be written as a linear combination of vectors of the form \( h^k h_{s'-s'-s'+s'+n+\frac{1}{2}} v_{k,l}, 0 \leq s' \leq w(l) \). Note that both \( \deg'(h^k d^l h_{s'-s'-s'+s'+n+\frac{1}{2}} v_{k,l}) \) and \( \deg'(h^k d^l h_{s'-s'-s'+s'+n+\frac{1}{2}} v_{k,l}) \) have the same weight and \( j - s + s' = -q - k - n + 1 < -q \), we see \( \deg'(h^k d^l h_{(q-j-s)+k-\frac{1}{2}} v_{k,l}) \) has weight

\[
w(k) + w(l) + s' = w(k) + w(l) - j - s + s' < w(k) + w(l) - q \leq w(i) - q.
\]

So \( \deg'(h^k d^l h_{(q-j-s)+k-\frac{1}{2}} v_{k,l}) \sim (i - \epsilon_q, 0) \).

(b) \( j = q \). In this case, we have \( h^k h_{k-s} v_{k,l} = h^k d^l h_{k-s} v_{k,l} + h^k [h_{k-s}, d^l] v_{k,l} \). Clearly, \( \deg'(h^k d^l h_{k-s} v_{k,l}) \sim (k_q, l) \sim (i - \epsilon_q, 0) \) since \( l \neq 0 \). By (3.3) and the similar arguments in Cases (a3) and (a4) we can deduce that \( \deg'(h^k [h_{k-s}, d^l] v_{k,l}) \sim (i - \epsilon_q, 0) \). Hence \( \deg'(h^k h_{k-s} d^l v_{k,l}) \sim (i - \epsilon_q, 0) \).
(c) \( q < j \leq q + k + n - 1 \). In this case, we have \( h^k h_{(q-j)+k-\frac{j}{2}} d^1 v_{k,1} = h^k d^1 h_{(q-j)+k-\frac{j}{2}} v_{k,1} + h^k [h_{(q-j)+k-\frac{j}{2}}, d^1] v_{k,1} \). Clearly, \( \deg'(h^k d^1 h_{(q-j)+k-\frac{j}{2}} v_{k,1}) = w(k) + w(l) - j < w(i) - q \). Then by (3.3) and the similar arguments in Cases (a3) and (a4) we can deduce that \( \deg'(h^k [h_{(q-j)+k-\frac{j}{2}}, d^1] v_{k,1}) <' (i - \epsilon_q, 0) \). Hence, \( \deg'(h^k h_{(q-j)+k-\frac{j}{2}} d^1 v_{k,1}) <' (i - \epsilon_q, 0) \).

(d) \( j > q + k + n - 1 \). In this case, we have \( h^k h_{(q-j)+k-\frac{j}{2}} d^1 v_{k,1} = h^k h_{(q-j)+(k+n+1)-\frac{j}{2}} d^1 v_{k,1} \). Then \( \deg'(h^k h_{(q-j)+k-\frac{j}{2}} d^1 v_{k,1}) = (k^*, 1) \) with weight \( w(k^*) + w(l) = w(k) + w(l) - (q + k + n - 1) < w(i) - q \). Hence, \( \deg'(h^k h_{(q-j)+k-\frac{j}{2}} d^1 v_{k,1}) <' (i - \epsilon_q, 0) \).

Next consider the term \( h^k d^1 v_{k,1} \). Since \( w(\deg'(h^k d^1 v_{k,1})) = w(s) + w(l) < w(k) + w(l) - q \leq w(i) - q \), it follows that \( \deg'(h^k d^1 v_{k,1}) <' (i - \epsilon_q, 0) \).

Thus, if \( h^k d^1 v_{k,1} \) does not satisfy (3.9) we have

\[ \deg'( [d_{q+k+n-1}, h^k] d^1 v_{k,1}) \leq' (i - \epsilon_q, 0) \]

where the equality holds if and only if \( (k, l) = (i, 0) \).

Now, consider the term \( h^k [d_{q+k+n-1}, d^1] v_{k,1} \) where we still assume that \( l \neq 0 \). By (3.4) we see \( h^k [d_{q+k+n-1}, d^1] v_{k,1} \) is a linear combination of vectors \( h^k d^1 d_{q+k+n-1} v_{k,1} \) and \( h^k d^1 v_{k+n+1} v_{k,1} \) where \( w(l_j) = w(l) - j, j \in \mathbb{N} \). Since \( \deg'(h^k d^1 v_{k+n+1} v_{k,1}) \) has weight

\[ w(k) + w(l) - (q + k + n - 1) < w(k) + w(l) - q \leq w(i) - q, \]

we see \( \deg'(h^k d^1 v_{k+n+1} v_{k,1}) <' (i - \epsilon_q, 0) \). So we need only to consider the vector \( h^k d^1 d_{q+k+n-1} v_{k,1} \).

There are four different cases.

(i) \( j < q \). Then \( q + k + n - 1 - j > k + n - 1 \) and \( h^k d^1 d_{q+k+n-1} v_{k,1} = 0 \). In particular, for \( w(l) < q \) we have \( h^k d^1 d_{q+k+n-1} v_{k,1} = 0 \).

(ii) \( j = q \). Then \( q + k + n - 1 - q = k + n - 1 \) and hence \( \deg'(h^k d^1 d_{k+n-1} v_{k,1}) = (k, l_q) \) (in the case \( d_{k+n-1} v_{k,1} \neq 0 \) with \( w(k) + w(l_q) = w(k) + w(l) - q \).

If \( w(k) + w(l) < w(i) \), or \( w(k) + w(l) = w(i) \) and \( w(k) < w(i) - q \), then \( (k, l_q) <' (i - \epsilon_q, 0) \). We will discuss the remaining cases that \( (k, l) \) satisfies (3.9) in Case (2) later.

(iii) \( q < j \leq q + k + n - 1 \). Then \( d_{q+k+n-1} v_{k,1} \in V \) and \( h^k d^1 d_{q+k+n-1} v_{k,1} = 0 \) or \( \deg'(h^k d^1 d_{q+k+n-1} v_{k,1}) \) has weight

\[ w(k) + w(l_j) = w(k) + w(l) - j < w(k) + w(l) - q \leq w(i) - q, \]

so \( \deg'(h^k d^1 d_{q+k+n-1} v_{k,1}) <' (i - \epsilon_q, 0) \).

(iv) \( j > q + k + n - 1 \). Then \( q + k + n - 1 - j < 0 \). Assume \( q + k + n - 1 - j = -j', j' \in \mathbb{Z}_+ \).

Then \( -j + j' = -(q + k + n - 1) < -q \). So \( \deg'(h^k d^1 d_{q+k+n-1} v_{k,1}) \) has weight

\[ w(k) + w(l_j) + j' = w(k) + w(l) - j + j' = w(k) + w(l) - (q + k + n - 1) < w(i) - q, \]

which means \( \deg'(h^k d^1 d_{q+k+n-1} v_{k,1}) <' (i - \epsilon_q, 0) \).

(1) If \( v = \sum_{(k,l)} h^k d^1 v_{k,1} \) does not contain a term \( h^k d^1 v_{k,1} \) satisfying (3.9), then by the above arguments we see \( \deg'(d_{q+k+n-1} v) = (i - \epsilon_q, 0) \).

(2) If \( v = \sum_{(k,l)} h^k d^1 v_{k,1} \) contains \( h^k d^1 v_{k,1} \) satisfying (3.9), then we see \( \deg'(v') = (k^*, l') \) with

\[ w(k^*) + w(l^*) = w(i), w(k^*) \geq w(i) - q, 1 \leq w(l^*) \leq q. \]

Then by Lemma (3.4) we see \( \deg'(h_{i+k-\frac{j}{2}} v') = (k^*, l' - \epsilon_i) \).
Noticing that \( k > n \), by (3.1) we see \( h_{t+k-rac{1}{2}}h^{k}v_{k,0} = 0 \) or \( \lambda h^{k'}v_{k,0}, \lambda \in \mathbb{C} \) with \( t' = t+k-n > t \) and \( w(k') = w(k) - t' \), so \( \deg'(h_{t+k-rac{1}{2}}(h^{k}v_{k,0})) = (k', 0) \) has weight \( w(k') = w(k) - t' < w(k^*) + w(l^*) - t = w(k^*) + w(l^* - \epsilon) \). Hence
\[
\deg'(h_{t+k-rac{1}{2}}v) = \deg'(h_{t+k-rac{1}{2}}v - \sum_{w(k) = w(l)} h^{k}v_{k,0}) = (k^*, l^* - \epsilon).
\]
\( \square \)

4. Simple restricted \( \mathcal{D} \)-modules

In this section we will determine all simple restricted \( \mathcal{D} \)-modules. Based on Theorem 2.11 we only need to determine all simple restricted \( \mathcal{D} \)-modules \( S \) of level \( \ell \neq 0 \).

For a given simple restricted \( \mathcal{D} \)-module \( S \) with level \( \ell \neq 0 \), we define the following invariants of \( S \) as follows:
\[
S(r) = \text{Ann}_{S}(\mathcal{J}r), n_{S} = \min\{r \in \mathbb{Z} : S(r) \neq 0\}, W_{0} = S(n_{S}),
\]
and
\[
U(r) = \text{Ann}_{W_{0}}(\mathcal{J}r), m_{S} = \min\{r \in \mathbb{Z} : U(r) \neq 0\}, U_{0} = U(m_{S}).
\]

Lemma 4.1. Let \( S \) be a simple restricted \( \mathcal{D} \)-module with level \( \ell \neq 0 \).

(i) \( h_{n_{S}-\frac{1}{2}} \) acts injectively on \( W_{0} \), \( d_{m_{S}-1} \) acts injectively on \( U_{0} \).
(ii) \( n_{S}, m_{S} \in \mathbb{N} \).
(iii) \( W_{0} \) is a nonzero \( \mathcal{D}^{(0, -n_{S})} \)-module, and is invariant under the action of the operators \( L_{n} \) defined in (2.2)-(2.4) for \( n \in \mathbb{N} \).
(iv) If \( m_{S} \geq 2n_{S} \), then \( U_{0} \) is a nonzero \( \mathcal{D}^{(0, -n_{S})} \)-submodule of \( W_{0} \), and is invariant under the action of the operators \( L_{n} \) defined in (2.2)-(2.4) for \( n \in \mathbb{N} \).

Proof. (i) follows from the definitions of \( n_{S} \) and \( m_{S} \).

(ii) Suppose \( n_{S} < 0 \), take any nonzero \( v \in W_{0} \), we then have
\[
h_{\frac{1}{2}}v = 0 = h_{-\frac{1}{2}}v.
\]
This implies that \( \frac{1}{2}\ell v = [h_{\frac{1}{2}}, h_{-\frac{1}{2}}]v = 0 \), a contradiction. Hence, \( n_{S} \in \mathbb{N} \).

Suppose \( m_{S} < 0 \). Take any nonzero \( v \in U_{0} \), we then have \( d_{-1}v = 0 = h_{m_{S}+\frac{1}{2}}v \). Then
\[
-(n_{S} + \frac{1}{2})h_{n_{S}-\frac{1}{2}}v = [d_{-1}, h_{n_{S}+\frac{1}{2}}]v = 0,
\]
a contradiction with (1). Hence, \( m_{S} \in \mathbb{N} \).

(iii) It is obvious that \( W_{0} \neq 0 \) by definition. For any \( w \in W_{0}, i, j, k \in \mathbb{N} \), we have
\[
h_{k+n_{S}+\frac{1}{2}}d_{i}w = d_{i}h_{k+n_{S}+\frac{1}{2}}w + (k + n_{S} + \frac{1}{2})h_{i+k+n_{S}+\frac{1}{2}}w = 0,
\]
and
\[
h_{k+n_{S}+\frac{1}{2}}h_{j-n_{S}+\frac{1}{2}}w = h_{j-n_{S}+\frac{1}{2}}h_{k+n_{S}+\frac{1}{2}}w = 0.
\]
Hence, \( d_{i}u \in W_{0} \) and \( h_{j-n_{S}+\frac{1}{2}}u \in W_{0} \), i.e., \( W_{0} \) is a nonzero \( \mathcal{D}^{(0, -n_{S})} \)-module.
For $n \in \mathbb{N}, i \in \mathbb{N}, w \in W_0$, by (2.5) we have
\[ h_{i+nS} L_n w = (L_n h_{i+nS} - (i + nS + \frac{1}{2}) h_{n+i+nS} - \frac{1}{2}) w = 0. \]
This implies that $L_n w \in W_0$ for $i \in \mathbb{N}$, that is, $W_0$ is invariant under the action of the operators $L_i$ for $i \in \mathbb{N}$.

(iv) It is obvious that $0 \neq U_0 \subseteq W_0$. Suppose that $mS \geq 2nS$. For any $u \in U_0$, $i, j, k \in \mathbb{N}$, it follows from (iii) that $d_i u \in W_0$ and $h_{j-nS + \frac{1}{2}} u \in W_0$. Furthermore,
\[ d_{k+mS} d_i u = d_i d_{k+mS} u + (k - i - mS) d_{k+i+mS} u = 0, \]
and
\[ d_{k+mS} h_{j-nS + \frac{1}{2}} u = h_{j-nS + \frac{1}{2}} d_{k+mS} u - (j - nS + \frac{1}{2}) h_{k+j+mS - nS + \frac{1}{2}} u = 0. \]
Hence, $d_i u \in U_0$ and $h_{j-nS + \frac{1}{2}} u \in U_0$, i.e., $U_0$ is a nonzero $\mathfrak{D}(0, -nS)$-submodule of $W_0$.

Furthermore, if in addition $mS > 0$, then for $n, i \in \mathbb{N}, u \in U_0$, it follows from (iii) that $L_n u \in W_0$. Moreover, for $n \in \mathbb{N}$, using (2.2) we have
\[ d_{i+mS} L_n u = L_n d_{i+mS} u + [d_{i+mS}, L_n] u = (n - i - mS) L_{i+n+mS} u = 0. \]
This implies that $L_i u \in U_0$ for $i \in \mathbb{N}$, that is, $U_0$ is invariant under the action of the operators $L_i$ for $i \in \mathbb{N}$. 

\[ \Box \]

**Proposition 4.2.** Let $S$ be a simple restricted $\mathfrak{D}$-module with level $\ell \neq 0$.

(i) If $nS = 0$, then $S \cong H^\mathfrak{D} \otimes U^\mathfrak{D}$ as $\mathfrak{D}$-modules for some simple modules $H \in \mathfrak{R}_\mathfrak{D}$ and $U \in \mathfrak{R}_{\mathfrak{D}_{\text{triv}}}$.

(ii) If $mS > 2nS > 0$, then $S \cong \text{Ind}_{\mathfrak{D}(0, -nS)}(U_0)$ and $U_0$ is a simple $\mathfrak{D}(0, -nS)$-module.

(iii) If $mS < 2nS$, then $U_0$ is a nonzero $\mathfrak{D}(0, -(mS - nS))$-submodule of $W_0$. Moreover,

(iii-1) If $mS \geq 2$, then $S \cong \text{Ind}_{\mathfrak{D}(0, -(mS - nS))}(U_0)$ and $U_0$ is a simple $\mathfrak{D}(0, -(mS - nS))$-module.

(iii-2) If $mS = 0$ or 1, and $nS > 1$, then $U(2)$ is a simple $\mathfrak{D}(0, -(2nS))$-module, and $S \cong \text{Ind}_{\mathfrak{D}(0, -(2nS))}(U(2))$.

**Proof.** (i) Since $nS = 0$, we take any nonzero $\nu \in W_0$. Then $\mathbb{C} \nu$ is a trivial $\mathfrak{H}(0)$-module. Let $H = U(\mathfrak{H}) \nu$, the $\mathfrak{H}$-submodule of $S$ generated by $\nu$. It follows from representation theory of Heisenberg algebras (or from the same arguments as in the proof of Lemma 3.2) that $\text{Ind}_{\mathfrak{D}(0)}(\mathbb{C} \nu)$ is a simple $\mathfrak{H}$-module. Consequently, the following surjective $\mathfrak{H}$-module homomorphism
\[ \varphi : \text{Ind}_{\mathfrak{D}(0)}(\mathbb{C} \nu) \longrightarrow H \]
\[ \sum_{i \in \mathbb{M}} a_i h^i \otimes \nu \mapsto \sum_{i \in \mathbb{M}} a_i h^i \nu \]
is an isomorphism, that is, $H$ is a simple $\mathfrak{H}$-module, which is certainly restricted. Then the desired assertion follows directly from Corollary 2.14.

(ii) By taking $V = U_0, k = n = nS$ and $l = mS - 1$ in Theorem 3.1, we see that any nonzero $\mathfrak{D}$-submodule of $\text{Ind}_{\mathfrak{D}(0, -nS)}(U_0)$ has a nonzero intersection with $U_0$. Consequently, the surjective $\mathfrak{D}$-module homomorphism
\[ \varphi : \text{Ind}_{\mathfrak{D}(0, -nS)}(U_0) \longrightarrow S \]
\[ \sum_{i \in \mathbb{M}} h^i d^k \otimes \nu_{i,k} \mapsto \sum_{i \in \mathbb{M}} h^i d^k \nu_{i,k} \]
is an isomorphism, i.e., \( S \cong \text{Ind}^\mathbb{D}_{(0,-n_S)}(U_0) \). Since \( S \) is simple, we see \( U_0 \) is a simple \( \mathbb{D}^{(0,-n_S)} \)-module.

(iii) Suppose that \( m_S < 2n_S \). For any \( u \in U_0, i, j, k \in \mathbb{N} \), it follows from Lemma 4.1(iii) that \( d_i u \in W_0 \) and \( h_{j-(m_S-n_S)+\frac{1}{2}} u \in W_0 \). Furthermore,

\[
d_{k+m_S} d_i u = d_i d_{k+m_S} u + (k - i + m_S) d_{k+i+m_S} u = 0,
\]

and

\[
d_{k+m_S} h_{j-(m_S-n_S)+\frac{1}{2}} u = h_{j-(m_S-n_S)+\frac{1}{2}} d_{k+m_S} u - (j - (m_S - n_S) + \frac{1}{2}) h_{k+j+n_S+\frac{1}{2}} u = 0.
\]

Hence, \( d_i u \in U_0 \) and \( h_{j-(m_S-n_S)+\frac{1}{2}} u \in U_0 \), i.e., \( U_0 \) is a nonzero \( \mathbb{D}^{(0,-(m_S-n_S))} \)-module of \( W_0 \).

Now suppose \( m_S \geq 2 \). Then it follows from Theorem 3.1(2) that any nonzero \( \mathbb{D} \)-submodule of \( \text{Ind}^\mathbb{D}_{(0, -(m_S-n_S))}(U_0) \) has a nonzero intersection with \( U_0 \) by taking \( k = n_S, n = m_S - n_S \) and \( l = m_S - 1 \) therein. Consequently, \( S \cong \text{Ind}^\mathbb{D}_{(0, -(m_S-n_S))}(U_0) \) by similar arguments as in (ii). Since \( S \) is simple, we see \( U_0 \) is a simple \( \mathbb{D}^{(0,-n_S)} \)-module.

Suppose that \( m_S = 0 \) or 1, and \( n_S > 1 \). Then \( \mathbb{D}^{(0, -(2-n_S))} \subseteq \mathbb{D}^{(0,-n_S)} \). Hence, \( W_0 \) is a \( \mathbb{D}^{(0,-(2-n_S))} \)-module. Moreover, for any \( u \in U(2), i, j \in \mathbb{N} \), we have

\[
d_{i+2} d_i u = d_{i+2} d_i u = 0,
\]

and

\[
d_{i+2} h_{i-(2-n_S)+\frac{1}{2}} u = h_{i-(2-n_S)+\frac{1}{2}} d_{i+2} u + (2 - n_S - i + \frac{1}{2}) h_{i+j+n_S+\frac{1}{2}} u = 0.
\]

Therefore, \( U(2) \) is a \( \mathbb{D}^{(0, -(2-n_S))} \)-module. Then it follows from Theorem 3.1(2) that any nonzero \( \mathbb{D} \)-submodule of \( \text{Ind}^\mathbb{D}_{(0, -(2-n_S))}(U(2)) \) has a nonzero intersection with \( U(2) \) by taking \( V = U(2), k = n_S, n = 2 - n_S \) and \( l = 1 \) therein. Consequently, \( S \cong \text{Ind}^\mathbb{D}_{(0, -(2-n_S))}(U(2)) \) by similar arguments as in (ii). In particular, \( U(2) \) is a simple \( \mathbb{D}^{(0, -(2-n_S))} \)-module.

From Proposition 4.2, what remains to consider are the following two cases: (1) \( m_S = 2n_S > 0 \), (2) \( m_S = 0 \) or 1, and \( n_S = 1 \).

Now we first consider Case (1): \( m_S = 2n_S > 0 \). For that, we define the operators \( d'_n = d_n - L_n \) on \( S \) for \( n \in \mathbb{Z} \). Since \( S \) is a restricted \( \mathbb{D} \)-module, then \( d'_n \) is well-defined for any \( n \in \mathbb{Z} \). By (2.5) and (2.6), we have

\[
[d'_n, c_1] = 0, [d'_n, d'_m] = (m - n) d'_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} (c - 1), m, n \in \mathbb{Z},
\]

where \( c_1 = c_1 - \text{id}_S \) and \( c \) is the central charge of \( S \). So the operator algebra

\[
\mathbb{Vir}' = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d'_n \oplus \mathbb{C} c_1
\]

is isomorphic to the Virasoro algebra \( \mathbb{Vir} \). Since \( [d_n, h_{k+\frac{1}{2}}] = [L_n, h_{k+\frac{1}{2}}] = -(k + \frac{1}{2}) h_{n+k+\frac{1}{2}} \), we have \( [d'_n, h_{k+\frac{1}{2}}] = 0, n, k \in \mathbb{Z} \) and hence \( [\mathbb{Vir}', \mathcal{H}] = 0 \). Clearly, the operator algebra \( \mathbb{Vir}' = \mathbb{Vir}' \oplus \mathcal{H} \) is a direct sum, and \( S = \mathfrak{U}(\mathbb{D}) v = \mathfrak{U}(\mathbb{Vir}') v, 0 \neq v \in S \). Similar to (2.11) we can define its subalgebras, \( \mathbb{D}^{(m,n)} \) and the likes.

Let

\[
Y_n = \bigcap_{p \geq n} \text{Ann}_{U(0)}(d'_p), r_S = \min \{ n \in \mathbb{Z} : Y_n \neq 0 \}, K_0 = Y_{r_S}.
\]
If $Y_n \neq 0$ for any $n \in \mathbb{Z}$, we define $r_S = -\infty$. Denote by $K = U(\mathcal{H})K_0$.

**Lemma 4.3.** Let $S$ be a simple restricted $\mathcal{D}$-module with level $\ell \neq 0$. Assume that $m_S = 2n_S > 0$. Then the following statements hold.

(i) $-1 \leq r_S \leq m_S$ or $r_S = -\infty$.
(ii) $K_0$ is a $\mathcal{D}(0,-m_S)$-module and $h_{n_S - \frac{1}{2}}$ acts injectively on $K_0$.
(iii) $K$ is a $\mathcal{D}(0,-\infty)$-module and $K^\mathcal{D}$ has a $\mathcal{D}$-module structure by (2.2)-(2.4).
(iv) $K_0$ and $K$ are invariant under the action of $d_n'$ for $n \in \mathbb{N}$.
(v) If $r_S \neq -\infty$, then $d_{r_S-1}$ acts injectively on $K_0$ and $K$.

**Proof.** (i) Since $m_S = 2n_S > 0$, the operators $d_n$ and $L_m = \frac{1}{2m} \sum_{k \in \mathbb{Z}+\frac{1}{2}} h_{m-k}h_k$ act trivially on $U_0$ for any $m \geq m_S$. This implies that $Y_{m_S} = U_0 \neq 0$. Consequently, $r_S \leq m_S$ by the definition of $r_S$.

If $Y_{-2} \neq 0$, then $d_{-2}'K_0 = d_{-1}'K_0 = 0$. We deduce that $\mathcal{H}K_0 = 0$ and hence $r_S = -\infty$.

If $Y_{-2} = 0$, then $r_S \geq -1$ and hence $-1 \leq r_S \leq m_S$.

(ii) For any $0 \neq v \in K_0$ and $x \in \mathcal{D}(0,-m_S)$, it follows from Lemma 4.1(iv) that $xv \in U_0$. We need to show that $d_{n}'xv = 0$, $p \geq r_S$. Indeed, $d_{n}'h_{k+\frac{1}{2}}v = h_{k+\frac{1}{2}}d_{n}'v = 0$ by (2.5) for any $k \geq -n_S$. Moreover, it follows from (2.6) and (4.1) that

$$d_{n}'dv = d_{n}d_{n}'v + [d_{n}',d_n]v = (p-n)d_{p+n}'v = 0.$$  

Hence, $d_{n}'xv = 0$, $p \geq r_S$, that is, $xv \in K_0$, as desired.

Since $0 \neq K_0 \subseteq U_0 \subseteq W_0$, we see that $h_{n_S - \frac{1}{2}}$ acts injectively on $K_0$ by Lemma 4.1(i).

(iii) follows from (ii).

(iv) follows from Lemma 4.1(iv) that $U_0$ is invariant under the action of $d_n'$ for $n \in \mathbb{N}$, so is $K_0$ by (4.1). Moreover, since $[\mathcal{H}',\mathcal{H}] = 0$, $K$ is also is invariant under the action of $d_n'$ for $n \in \mathbb{N}$.

(v) follows directly from the definition of $r_S$ and $K$. □

**Proposition 4.4.** Let $S$ be a simple restricted $\mathcal{D}$-module with central charge $c$ and level $\ell \neq 0$. Assume that $m_S = 2n_S > 0$. If $r_S = -\infty$, then $c = 1$. Moreover, $S = K^\mathcal{D}$ and $K$ is a simple $\mathcal{H}$-module.

**Proof.** Since $r_S = -\infty$, we see that $\mathcal{H}'K_0 = 0$. This together with (4.1) implies that $c = 1$. Noting that $[\mathcal{H}',\mathcal{H}] = 0$, we further obtain that $\mathcal{H}'K = 0$, that is, $d_nv = L_nv \in K$ for any $v \in K$ and $n \in \mathbb{Z}$. Hence $K^\mathcal{D}$ is a $\mathcal{D}$-submodule of $S$, yielding that $S = K^\mathcal{D}$. In particular, $K$ is a simple $\mathcal{H}$-module. □

**Proposition 4.5.** Let $S$ be a simple restricted $\mathcal{D}$-module with level $\ell \neq 0$. If $r_S \geq 2$, then $K_0$ is a simple $\mathcal{D}(0,-m_S)$-module and $S \cong \text{Ind}_{\mathcal{D}(0,-m_S)}^{\mathcal{D}}K_0$.

**Proof.** We first show that $\text{Ind}_{\mathcal{D}(0,-m_S)}^{\mathcal{D}}K_0 \cong K$ as $\mathcal{D}(0,-\infty)$ modules. For that, let

$$\phi : \text{Ind}_{\mathcal{D}(0,-m_S)}^{\mathcal{D}}K_0 \longrightarrow K$$

$$\sum_{k \in \mathbb{Z}} h_k \otimes v_k \mapsto \sum_{k \in \mathbb{Z}} h_kv_k.$$
where \( h^k = \cdots h^{k_{2}}_{-2r_S+1} h^{k_{1}}_{-1-r_S+1} \). Then \( \phi \) is a \( \mathcal{D}(0, -\infty) \)-module epimorphism and \( \phi|_{K_0} \) is one-to-one. By similar arguments in the proof of Lemma 3.2 we see that any nonzero submodule of \( \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K_0 \) contains nonzero vectors of \( K_0 \), which forces that the kernel of \( \phi \) must be zero and hence \( \phi \) is an isomorphism.

By Lemma 4.3(v), we see that \( d'_{r_S-1} \) acts injectively on \( K \).

As \( \mathcal{D} \)-modules,
\[
\text{Ind}^{\mathcal{D}(0, -n_S)}_{\mathcal{D}(0, -\infty)} K_0 \cong \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} (\text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K_0) \cong \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -\infty)} K.
\]

And we further have \( \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K \cong \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K \) as vector spaces. Moreover, we have the following \( \mathcal{D} \)-module epimorphism
\[
\pi : \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K = \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K \rightarrow S,
\]
\[
\sum_{i \in M} d^i \otimes v_i \mapsto \sum_{i \in M} d^i v_i,
\]
where \( d^i = \cdots (d'_{2})^i_1 (d'_{-1})^i_1 \). We see that \( \pi \) is also a \( \text{Vir} \)-module epimorphism. By the proof of Theorem 2.1 in [46] we know that any nonzero \( \text{Vir} \)-submodule of \( \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K \) contain nonzero vectors of \( K \). Note that \( \pi|_{K} \) is one-to-one, we see that the image of any nonzero \( \mathcal{D} \)-submodule (and hence \( \text{Vir} \)-submodule ) of \( \text{Ind}^{\mathcal{D}(0, -\infty)}_{\mathcal{D}(0, -n_S)} K \) must be a nonzero \( \mathcal{D} \)-submodule of \( S \) and hence be the whole module \( S \), which forces that the kernel of \( \pi \) must be 0. Therefore, \( \pi \) is an isomorphism. Since \( S \) is simple, we see \( K_0 \) is a simple \( \mathcal{D}(0, -n_S) \)-module. \( \square \)

As a direct consequence of Proposition 4.5 we have

**Corollary 4.6.** Let \( S \) be a simple restricted \( \mathcal{D} \)-module with level \( \ell \neq 0 \). If \( m_S \leq 1 \) and \( n_S = 1 \), then \( K_0 \) is a simple \( \mathcal{D}(0, -1) \)-module and \( S \cong \text{Ind}^{\mathcal{D}(0, -1)}_{\mathcal{D}(0, -1)} K_0 \).

**Proof.** For any nonzero \( u \in U_0 \), since \( m_S \leq 1 \) and \( n_S = 1 \), it follows from the definitions of \( m_S, n_S \) and Lemma 4.1(i) that
\[
d_1 u = 0, \quad L_1 u = \frac{1}{2\ell} \sum_{k \in \mathbb{Z}+1} h_{1-k} h_k u = \frac{1}{2\ell} (h^2)^u u \neq 0.
\]
This implies that \( d'_{1} u \neq 0 \), i.e., \( d'_1 \) acts injectively on \( U_0 \). Hence \( r_S \geq 2 \). More precisely, since
\[
d_{2+i} v = L_{2+i} v = 0, \quad \forall \quad i \in \mathbb{N}, v \in U_0,
\]
we see that \( r_S = 2 \). Now the desired assertion follows directly from Proposition 4.5 \( \square \)

**Remark 4.7.** From Corollary 4.6 we have dealt with the Case (2).

What remains to consider for Case (1) is that \( m_S = 2n_S \geq 2 \) and \( r_S \leq 1 \). In this case we will show that \( K \) is a simple \( \mathcal{H} \)-module.

For the Verma module \( M_{\text{Vir}}(c, h) \) over \( \text{Vir} \), it is well-known from [6, 20] that there exist two homogeneous elements \( P_1, P_2 \in \text{Vir}^{-} \) such that \( \text{Vir}^{-} P_1 w_1 + \text{Vir}^{-} P_2 w_1 = \text{Vir}^{-} \), where \( P_1, P_2 \) are allowed to be zero and \( w_1 \) is the highest weight vector in \( M_{\text{Vir}}(c, h) \).

**Lemma 4.8.** Let \( d = 0, -1 \). Suppose \( M \) is a \( \text{Vir}^{(d)} \)-module on which \( d_0 \) acts as multiplication by a given scalar \( \lambda \). Then there exists a unique maximal submodule \( N \) of \( \text{Ind}^{\text{Vir}^{(d)}} M \) with \( N \cap M = 0 \). More precisely, \( N \) is generated by \( P_1 M \) and \( P_2 M \), i.e., \( N = \text{Vir}^{-}(P_1 M + P_2 M) \).
Proof. Note that $d_0$ acts diagonally on $\text{Ind}_{\text{Vir}}^{\text{Vir}} M$ and its submodules, and
\[ M = \{ u \in \text{Ind}_{\text{Vir}}^{\text{Vir}} M \mid d_0 u = \lambda u \}, \]
i.e., $M$ is the highest weight space of $\text{Ind}_{\text{Vir}}^{\text{Vir}} M$. Let $N$ be the sum of all $\text{Vir}$-submodules of $\text{Ind}_{\text{Vir}}^{\text{Vir}} M$ which intersect with $M$ trivially. Then $N$ is the desired unique maximal $\text{Vir}$-submodule of $\text{Ind}_{\text{Vir}}^{\text{Vir}} M$ with $N \cap M = 0$.

Let $N'$ be the $\text{Vir}$-submodule generated by $P_1 M$ and $P_2 M$, i.e., $N' = \bigcup (\text{Vir})(P_1 M + P_2 M)$. Then $N' \cap M = 0$. Hence, $N' \subseteq N$. Suppose there is a proper submodule $U$ of $\text{Ind}_{\text{Vir}}^{\text{Vir}} M$ that is not contained in $N'$. There is a nonzero homogeneous $v = \sum_{i=1}^r u_i v_i \in U \setminus N'$ where $u_i \in \bigcup (\text{Vir})$ and $v_1, \ldots, v_r \in M$ are linearly independent. Note that all $u_i$ have the same weight. Then some $u_i v_i \notin N'$, say $u_1 v_1 \notin N'$. There is a homogeneous $u \in \bigcup (\text{Vir})$ such that $u u_1 v_1 = v_1$. Noting that all $uu_i$ has weight 0, so $uu_i v_i \in \mathbb{C} v_i$. Thus $uv \in M \setminus \{0\}$. This implies that $N \subseteq N'$. Hence, $N = N'$, as desired. □

**Proposition 4.9.** Let $S$ be a simple restricted $\mathcal{D}$-module with level $\ell \neq 0$. If $m_S = 2n_s \geq 2$, and $r_S = 0$ or $-1$, then $K$ is a simple $\mathcal{H}$-module and $S \cong U \mathcal{D} \otimes K \mathcal{D}$ for some simple $U \in \mathcal{R}_{\text{Vir}}$.

Proof. By Lemma 4.3 (iii), we see that $K \mathcal{D}$ is a $\mathcal{D}$-module, and hence $K \mathcal{D}'$ is a $\mathcal{D}'$-module with $d_n' K \mathcal{D} = 0$ for any $n \in \mathbb{Z}$. Let $C_{v_0}$ be a one-dimensional $\mathcal{D}^{r_s, -\infty}$-module with module structure defining by $d_n' v_0 = h_{k+2} v_0 = c_2 v_0 = 0, n \geq r_S, k \in \mathbb{Z}, c_1 v_0 = (c-2)v_0$. Then $C_{v_0} \otimes K \mathcal{D}'$ is a $\mathcal{D}^{r_s, -\infty}$-module with central charge $c - 1$ and level $\ell$. It is easy to see that we have the following $\mathcal{D}^{r_s, -\infty}$-module homomorphism
\[ \tau_K : C_{v_0} \otimes K \mathcal{D}' \longrightarrow S, \]
\[ v_0 \otimes u \mapsto u, \forall u \in K. \]
Clearly, $\tau_K$ is an injective map and can be extended to a $\mathcal{D}'$-module epimorphism
\[ \tau : \text{Ind}_{\mathcal{D}^{r_s, -\infty}}^\mathcal{D} (C_{v_0} \otimes K \mathcal{D}') \longrightarrow S, \]
\[ x(v_0 \otimes u) \mapsto xu, x \in \bigcup (\mathcal{D}'), u \in K. \]

By Lemma 8 in [43] we know that
\[ \text{Ind}_{\mathcal{D}^{r_s, -\infty}}^\mathcal{D} (C_{v_0} \otimes K \mathcal{D}') \equiv (\text{Ind}_{\mathcal{D}^{r_s, -\infty}}^\mathcal{D} C_{v_0}) \otimes K \mathcal{D}' = (\text{Ind}_{\text{Vir}}^{\text{Vir}} C_{v_0}) \mathcal{D}' \otimes K \mathcal{D}'. \]

Then we have the following $\mathcal{D}'$-module epimorphism
\[ \tau' : (\text{Ind}_{\text{Vir}}^{\text{Vir}} C_{v_0}) \mathcal{D}' \otimes K \mathcal{D}' \longrightarrow S, \]
\[ x_{v_0} \otimes u \mapsto xu, x \in \bigcup (\text{Vir}'), u \in K. \]

Note that $(\text{Ind}_{\text{Vir}}^{\text{Vir}} C_{v_0}) \mathcal{D}' \otimes K \mathcal{D}' \cong \text{Ind}_{\text{Vir}}^{\text{Vir}} (C_{v_0} \otimes K \mathcal{D}')$ as $\text{Vir}'$-modules, and $\tau'$ is also a $\text{Vir}'$-module epimorphism, $\tau'|_{C_{v_0} \otimes K \mathcal{D}'}$ is one-to-one, and $(\text{Ind}_{\text{Vir}}^{\text{Vir}} C_{v_0}) \mathcal{D}' \otimes K \mathcal{D}'$ is a highest weight $\text{Vir}'$-module.

Let $V = \text{Ind}_{\text{Vir}}^{\text{Vir}} C_{v_0}$ and $R = \ker(\tau')$. It should be noted that
\[ C_{v_0} \otimes K \mathcal{D}' = \{ u \in V \mathcal{D}' \otimes K \mathcal{D}' \mid d_0 u = 0 \}. \]

We see that $(C_{v_0} \otimes K \mathcal{D}') \cap R = 0$. Let $R'$ be the sum of all $\text{Vir}'$-submodules $W$ of $V \mathcal{D}' \otimes K \mathcal{D}'$ with $(C_{v_0} \otimes K \mathcal{D}') \cap W = 0$, that is, the unique maximal $\text{Vir}'$-submodule of $V \mathcal{D}' \otimes K \mathcal{D}'$ with trivial intersection with $(C_{v_0} \otimes K \mathcal{D}')$. It is obvious that $R \subseteq R'$. Next we further show that $R = R'$. For that, take any $\text{Vir}'$-submodule $W$ of $V \mathcal{D}' \otimes K \mathcal{D}'$ such that $(C_{v_0} \otimes K \mathcal{D}') \cap W = 0$. Then for any weight
vector \( w = \sum_{i \in M} d_i^0 v_i \otimes u_i \in W \), where \( u_i \in K^{V_i} \), \( d_i^0 = \cdots (d_{i-2}^0)^2(d_{i-1}^0)^1 \) if \( r_S = 0 \), or \( d_i^0 = \cdots (d_{i-2}^0)^2 \) if \( r_S = -1 \), and all \( w(i) \geq 1 \) are equal. Note that \( h_{k+1} w = \sum_{i \in M} d_i^0 v_i \otimes h_{k+1} u_i \) either equals to 0 or has the same weight as \( w \) under the action of \( d_i^0 \). So \( U(\mathfrak{C}') \mathfrak{K} \cap (\mathcal{C}_0 \otimes K^{V_i}) = 0 \). The maximality of \( \mathfrak{K}' \) forces that \( \mathfrak{K}' = U(\mathfrak{C}') \mathfrak{K} \) is a proper \( \mathfrak{C}' \)-submodule of \( V^{V_i} \otimes K^{V_i} \). Since \( \mathfrak{K} \) is a maximal proper \( \mathfrak{C}' \)-submodule of \( V^{V_i} \otimes K^{V_i} \), it follows that \( \mathfrak{K} = \mathfrak{K}' \).

By Lemma [4.3] we know that \( \mathfrak{K} \) is generated by \( P_1(\mathcal{C}_0 \otimes K^{V_i}) = \mathcal{C} P_1 v_0 \otimes K^{V_i} \) and \( P_2(\mathcal{C}_0 \otimes K^{V_i}) = \mathcal{C} P_2 v_0 \otimes K^{V_i} \). Let \( V' \) be the maximal submodule of \( V \) generated by \( P_1 v_0 \) and \( P_2 v_0 \), then \( \mathfrak{K} = V^{V_i} \otimes K^{V_i} \). Therefore,

\[
S \cong (V^{V_i} \otimes K^{V_i})/(V^{V_i} \otimes K^{V_i}) \cong (V/V')^{V_i} \otimes K^{V_i},
\]

which forces that \( K^{V_i} \) is a simple \( \mathfrak{C}' \)-module and hence a simple \( \mathcal{C} \)-module. So \( S \) contains a simple \( \mathcal{C} \)-module \( K \). By Corollary [2.4] we know there exists a simple \( \mathfrak{W} \)-module \( U \in \mathcal{R}^{\mathfrak{W}} \) such that \( S \cong U^{\mathfrak{W}} \otimes K^{\mathfrak{W}} \), as desired.

**Lemma 4.10.** Let \( M \) be a \( \mathfrak{W}^{(0)} \)-module on which \( \mathfrak{W}^{(1)} \) acts trivially. If any finitely generated \( \mathcal{C}[d_0] \)-submodule of \( M \) is a free \( \mathcal{C}[d_0] \)-module, then any nonzero submodule of \( \text{Ind}_{\mathfrak{W}^{(0)}}^{\mathfrak{W}} M \) intersects with \( M \) non-trivially.

**Proof.** Let \( V \) be a nonzero submodule of \( \text{Ind}_{\mathfrak{W}^{(0)}}^{\mathfrak{W}} M \). Take a nonzero \( u \in V \). If \( u \in M \), there is nothing to do. Now assume \( u \in V \setminus M \). Write \( u = \sum_{i=1}^n a_i u_i \) where \( a_i \in \mathcal{U}(\mathfrak{W}^{(0)}) \), \( u_i \in M \). Since \( M_1 = \sum_{1 \leq i \leq n} \mathcal{C}[d_0] u_i \) (a \( \mathfrak{W}^{(0)} \)-submodule of \( M \)) is a finitely generated \( \mathcal{C}[d_0] \)-module, we see \( M_1 \) is a free module over \( \mathcal{C}[d_0] \) by the assumption. Without loss of generality, we may assume that \( M_1 = \bigoplus_{1 \leq i \leq n} \mathcal{C}[d_0] u_i \) with bases \( u_1, \ldots, u_n \) over \( \mathcal{C}[d_0] \). Note that each \( a_i \) can be expressed as a sum of eigenvalue subspaces of \( \text{ad} d_0 \) for \( 1 \leq i \leq n \). Assume that \( a_i \) has a maximal eigenvalue among all \( a_i \) for \( 1 \leq i \leq n \). Then \( a_i u_i \notin M \). For any \( \lambda \in \mathcal{C} \), let \( M_1(\lambda) \) be the \( \mathcal{C}[d_0] \)-submodule of \( M_1 \) generated by \( u_2, u_3, \ldots, u_n, d_0 u_1 - \lambda u_1 \). Then \( M_1(\lambda) \) is a one-dimensional \( \mathfrak{W}^{(0)} \)-module with \( d_0(u_1 + M_1(\lambda)) = \lambda u_1 + M_1(\lambda) \). By the Verma module theory for Virasoro algebra we know that there exists some \( 0 \neq \lambda_0 \in \mathcal{C} \) such that the corresponding Verma module \( \mathfrak{V} = \text{Ind}_{\mathfrak{W}^{(0)}}^{\mathfrak{W}} (M_1/M_1(\lambda_0)) \) is irreducible. We know that \( u = a_1 u_1 \neq 0 \) in \( \mathfrak{V} \). Hence we can find a homogeneous \( w \in \mathcal{U}(\mathfrak{V}^{(0)}) \) such that \( w a_1 u_1 = f_1(d_0) u_1 \) in \( \text{Ind}_{\mathfrak{W}^{(0)}}^{\mathfrak{W}} M \), where \( 0 \neq f_1(d_0) \in \mathcal{C}[d_0] \). So \( w u = \sum_{i=1}^n w a_i u_i = \sum_{i=1}^n f_1(d_0) u_i \) for \( f_1(d_0) \in \mathcal{C}[d_0] \), \( 1 \leq i \leq n \). Therefore, \( 0 \neq w u \in W \cap M_1 \subset M \), as desired.

**Proposition 4.11.** Let \( S \) be a simple restricted \( \mathfrak{C} \)-module with level \( \ell \neq 0 \). If \( m_S = 2n_S \geq 2 \), \( r_S = 1 \), then \( d_0' \) has an eigenvector in \( K \).

**Proof.** Suppose first that any finitely generated \( \mathcal{C}[d_0'] \)-submodule of \( K = \text{Ind}_{\mathfrak{C}(-\infty)}^{\mathfrak{C}} K_0 \) is a free \( \mathcal{C}[d_0'] \)-module. By Lemma [4.10] we see that the following \( \mathfrak{C}' \)-module homomorphism

\[
\tau : \text{Ind}_{\mathfrak{C}^{(0)}}^{\mathfrak{C}} K = \text{Ind}_{\mathfrak{W}^{(0)}}^{\mathfrak{W}} K \longrightarrow S, \\
x \otimes u \mapsto xu, x \in \mathcal{U}(\mathfrak{W}), u \in K.
\]

is an isomorphism. So \( S = \text{Ind}_{\mathfrak{W}^{(0)}}^{\mathfrak{W}} K \), and consequently, \( K \) is an irreducible \( \mathfrak{C}'^{(0, -\infty)} \)-module. Since \( \mathfrak{W}^{(1)} K = 0 \), we consider \( K \) as an irreducible module over the Lie algebra \( \mathcal{C} \otimes d_0' \). Since \( d_0' \) is the center of the Lie algebra \( \mathcal{C} \otimes d_0' \), we see that the action of \( d_0' \) on \( K \) is a scalar, a contradiction. So this case does not occur.

Now there exists some finitely generated \( \mathcal{C}[d_0'] \)-submodule \( M \) of \( K \) that is not a free \( \mathcal{C}[d_0'] \)-module. Since \( \mathcal{C}[d_0'] \) is a principal ideal domain, by the structure theorem of finitely generated
modules over a principal ideal domain, there exists a monic polynomial \( f(d'_0) \in \mathbb{C}[d'_0] \) with positive degree and nonzero element \( u \in M \) such that \( f(d'_0)u = 0 \). Furthermore, we can write \( f(d'_0) = (d'_0 - \lambda_1)(d'_0 - \lambda_2) \cdots (d'_0 - \lambda_p) \) for some \( \lambda_1, \ldots, \lambda_p \in \mathbb{C} \). Then there exists some \( s \leq p \) such that \( w := \prod_{j=s+1}^{p}(d'_0 - \lambda_j)u \neq 0 \) and \( d'_0w = \lambda_sw \), where we make convention that \( w = u \) if \( s = p \). Then \( w \) is a desired eigenvector of \( d'_0 \).

**Proposition 4.12.** Let \( S \) be a simple restricted \( \mathcal{D} \)-module with level \( \ell \neq 0 \). If \( m_S = 2n_s \geq 2 \), \( r_S = 1 \), then \( K \) is a simple \( \mathcal{H} \)-module and \( S \cong U^\mathcal{D} \otimes K^\mathcal{D} \) for some simple \( U \in \mathcal{R}_{\text{Vir}} \).

**Proof.** We see that \( S \) is a weight \( \mathcal{D}' \)-module since \( S \) is a simple \( \mathcal{D}' \)-module and \( d'_0 \) has an eigenvector. From Lemma 4.3(iii), \( K \) and \( K_0 \) are weight \( \mathcal{D}' \)-modules as well. We can take some \( 0 \neq u_0 \in K \) such that \( d'_0u_0 = \lambda u_0 \) for some \( \lambda \neq 0 \) by Proposition 4.11. Set \( K' = U(H)u_0 \), which is an \( \mathcal{H} \) submodule of \( K \). Then we have the \( \mathcal{D}' \)-module \( K'^{\mathcal{D}'} \), on which \( \text{Vir} \) acts trivially by definition for any \( n \in \mathbb{Z} \). Let \( \mathbb{C}v_0 \) be the one-dimensional \( \mathcal{D}'^{(0,-\infty)} \)-module defined by \( d'_0v_0 = \lambda v_0, d'_nv_0 = h_{k+1}v_0 = c_kv_0 = 0, n \in \mathbb{Z}^+, k \in \mathbb{Z}, c'_1v_0 = (c - 2)v_0 \). Then \( \mathbb{C}v_0 \otimes K'^{\mathcal{D}'} \) is a \( \mathcal{D}'^{(0,-\infty)} \)-module with central charge \( c - 1 \) and level \( \ell \). There is a \( \mathcal{D}'^{(0,-\infty)} \)-module homomorphism

\[
\tau_{K'} : \mathbb{C}v_0 \otimes K'^{\mathcal{D}'} \rightarrow S, \\
v_0 \otimes u \mapsto u, \forall u \in K',
\]

which is injective and can be extended to be the following \( \mathcal{D}' \)-module homomorphism

\[
\tau : \text{Ind}_{\mathcal{D}'^{(0,-\infty)}}^{\mathcal{D}'}(\mathbb{C}v_0 \otimes K'^{\mathcal{D}'})(U_0) \rightarrow S, \\
x(v_0 \otimes u) \mapsto xu, x \in \text{Vir}(\mathcal{D}'), u \in K'.
\]

Since \( S \) is a simple \( \mathcal{D} \)-module and \( \tau \neq 0 \), we see that \( \tau \) is surjective. By similar arguments in the proof of Proposition 4.9 we can obtain that \( K' \) is a simple \( \mathcal{H} \)-module. By Corollary 3.14 we know there exists a simple \( \text{Vir} \)-module \( U \in \mathcal{R}_{\text{Vir}} \) such that \( S \cong U^\mathcal{D} \otimes K'^{\mathcal{D}} \), as desired. Now it is clear that \( K = K' \).

We are now in a position to present the following main result on a classification of simple restricted \( \mathcal{D} \)-modules with nonzero level.

**Theorem 4.13.** Let \( S \) be a simple restricted \( \mathcal{D} \)-module with level \( \ell \neq 0 \). The invariants \( m_S, n_s, r_S \) of \( S, U_0, U(2), K_0, K \) are defined as before. Then one of the following cases occurs.

Case 1: \( n_s = 0 \).

In this case, \( S \cong H^\mathcal{D} \otimes U^\mathcal{D} \) as \( \mathcal{D} \)-modules for some simple modules \( H \in \mathcal{R}_{\text{Vir}} \) and \( U \in \mathcal{R}_{\text{Vir}} \).

Case 2: \( n_s > 0 \).

In this case, we further have the following three subcases.

Subcase 2.1: \( m_S > 2n_s \).

In this subcase, \( S \cong \text{Ind}_{\mathcal{D}'^{(0,-n_s)}}(U_0) \).

Subcase 2.2: \( m_S = 2n_s \).

In this subcase, we have

\[
S \cong \begin{cases} 
K^\mathcal{D}, & \text{if } r_S = -\infty, \\
U^\mathcal{D} \otimes K^\mathcal{D}, & \text{if } -1 \leq r_S \leq 1, \\
\text{Ind}_{\mathcal{D}'^{(0,-n_s)}}K_0, & \text{otherwise}, 
\end{cases}
\]

Subcase 2.3: \( m_S < 2n_s \).

In this subcase, we have

\[
S \cong \begin{cases} 
K^\mathcal{D}, & \text{if } r_S = -\infty, \\
U^\mathcal{D} \otimes K^\mathcal{D}, & \text{if } -1 \leq r_S \leq 1, \\
\text{Ind}_{\mathcal{D}'^{(0,-n_s)}}K_0, & \text{otherwise}, 
\end{cases}
\]
where $U \in \mathcal{R}_{\text{vir}}$.

Subcase 2.3: $m_S < 2n_S$.

In this subcase, we have

$$S \cong \begin{cases} \text{Ind}_{\mathbb{Z}(0,-(m_S - n_S))}^\mathcal{D}(U_0), & \text{if } m_S \geq 2, \\ \text{Ind}_{\mathbb{Z}(0,2-n_S)}^\mathcal{D}(U(2)), & \text{if } m_S < 2, n_S > 1, \\ \text{Ind}_{\mathbb{Z}(0,-1)}^\mathcal{D}K_0, & \text{otherwise}. \end{cases}$$

**Proof.** The assertion follows directly from Proposition 4.2, Proposition 4.4, Proposition 4.9, Corollary 4.6, and Proposition 4.12. □

**Remark 4.14.** By Theorems 2.11 and 4.13, we know that any simple restricted module $S$ is a highest weight $\mathfrak{vir}$-module with trivial action of $\mathfrak{H}$, or a tensor product of a simple restricted $\mathfrak{vir}$-module and a simple restricted $\mathfrak{H}$-module, or an induced module from some simple module $M$ over certain subalgebra of $\mathfrak{D}$. Moreover, $M$ can be viewed as a simple module over some finite-dimensional solvable Lie algebra. This reduces the study of such $\mathfrak{D}$-modules to the study of simple modules over the corresponding finite-dimensional solvable Lie algebras.

5. **Simple restricted $\mathfrak{D}$-modules with nonzero level**

In this section we will determine all simple restricted $\mathfrak{D}$-modules $M$ of level $\ell \neq 0$. The main method we will use is similar to the one used in Section 4.

For a given simple restricted $\mathfrak{D}$-module $M$ with level $\ell \neq 0$, we define the following invariants of $M$ as follows:

$$M(r) = \text{Ann}_M(\mathfrak{D}^r), n_M = \min\{r \in \mathbb{Z} : M(r) \neq 0\}, M_0 = M(n_M).$$

**Lemma 5.1.** Let $M$ be an irreducible restricted $\mathfrak{D}$-module with level $\ell \neq 0$.

(i) $n_M \in \mathbb{N}$, and $h_{n_M-1}$ acts injectively on $M_0$.

(ii) $M_0$ is a nonzero $\mathfrak{D}^{(0,-(n_M-1))}$-module, and is invariant under the action of the operators $\bar{L}_n$ defined in (2.2) for $n \in \mathbb{N}$.

**Proof.** (i) Assume that $n_M < 0$. Take any nonzero $v \in M_0$, we then have

$$h_1v = 0 = h_{-1}v.$$ 

This implies that $v = \frac{1}{4}[h_1, h_{-1}]v = 0$, a contradiction. Hence, $n_M \in \mathbb{N}$.

The definition of $n_M$ means that $h_{n_M-1}$ acts injectively on $M_0$.

(ii) It is obvious that $M_0 \neq 0$ by definition. For any $w \in M_0$, $i, j, k \in \mathbb{N}$, we have

$$h_{k+n_M}d_iw = d_jh_{k+n_M}w + (k + n_M)h_{i+k+n_M}w = 0,$$

and

$$h_{k+n_M}h_{j-n_M+1}w = h_{j-n_M+1}h_{k+n_M}w = 0.$$ 

Hence, $d_iw, h_{j-n_M+1}w \in M_0$, i.e., $M_0$ is a nonzero $\mathfrak{D}^{(0,-(n_M-1))}$-module.

For $i, n \in \mathbb{N}$, $w \in M_0$, noticing $n_M > 0$ by (i), it follows from (2.4) that

$$h_{i+n_M}\bar{L}_n w = (\bar{L}_n h_{i+n_M} + (i + n_M)h_{n+i+n_M})w = 0.$$
Proposition 5.2. Let $M$ be a simple restricted $\mathfrak{H}$-module with level $\ell \neq 0$. If $n_M = 0, 1$, then $M \cong H^\delta \otimes U^\delta$ as $\mathfrak{H}$-modules for some simple modules $H \in \mathcal{R}_\mathfrak{H}$ and $U \in \mathcal{R}_\mathfrak{Vir}$.

Proof. Since $n_M = 0, 1$, we take any nonzero $v \in M$. Then $\mathcal{C}v$ is a $\mathfrak{H}(0)$-module. Let $H = \mathcal{U}(\mathfrak{H})v$, the $\mathfrak{H}$-submodule of $M$ generated by $v$. It follows from representation theory of Heisenberg algebras that $\text{Ind}_{\mathfrak{H}(0)}^{\mathfrak{H}}(\mathcal{C}v)$ is a simple $\mathfrak{H}$-module. Consequently, the following surjective $\mathfrak{H}$-module homomorphism

$$
\varphi : \text{Ind}_{\mathfrak{H}(0)}^{\mathfrak{H}}(\mathcal{C}v) \longrightarrow H \\
\sum_{i \in H} a_i h_i \otimes v \mapsto \sum_{i \in H} a_i h_i v
$$

is an isomorphism, that is, $H$ is a simple $\mathfrak{H}$-module, which is certainly restricted. Then the desired assertion follows directly from [43, Theorem 12]. □

Next we assume that $n_M \geq 2$.

We define the operators $d'_n = d_n - \bar{L}_n$ on $M$ for $n \in \mathbb{Z}$. Since $M$ is a restricted $\mathfrak{H}$-module, then $d'_n$ is well-defined for any $n \in \mathbb{Z}$. By (2.4) and (2.10), we have

$$
[d'_m, \bar{c}'] = 0, [d'_m, d'_n] = (m-n)d'_{m+n} + \frac{m^3 - m}{12}\bar{c}'_1, m, n \in \mathbb{Z},
$$

(5.1)

where $\bar{c}'_1 = c - (1 - \frac{12\bar{c}^2}{7})\text{id}_M$ and $c$ is the central charge of $M$. So the operator algebra

$$
\mathfrak{Vir}' = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}d'_n \oplus \mathbb{C}\bar{c}'_1
$$

is isomorphic to the Virasoro algebra $\mathfrak{Vir}$. Since $[d'_n, h_k] = [\bar{L}_n, h_k] = -kh_{n+k} + \delta_{n+k,0}(n^2 + n)\bar{c}_2$, we have

$$
[d'_n, h_k] = 0, n, k \in \mathbb{Z}
$$

(5.2)

and hence $[\mathfrak{Vir}', \mathfrak{H} + \mathbb{C}\bar{c}_2] = 0$. Clearly, the operator algebra $\mathfrak{H}' = \mathfrak{Vir}' \oplus (\mathfrak{H} + \mathbb{C}\bar{c}_2)$ is a direct sum, and $M = \mathcal{U}(\mathfrak{H}')v = \mathcal{U}(\mathfrak{H}')v$ for any $v \in M \setminus \{0\}$. Let

$$
Y_n = \bigcap_{p \geq n} \text{Ann}_{M_0}(d'_p), r_M = \min\{n \in \mathbb{Z} : Y_n \neq 0\}, K_0 = Y_{r_M}.
$$

Noting that $M$ is a restricted $\mathfrak{H}$-module, we know that $r_M < +\infty$. If $Y_n \neq 0$ for any $n \in \mathbb{Z}$, we define $r_M = -\infty$. Denote by $K = \mathcal{U}(\mathfrak{H})K_0$.

Lemma 5.3. Let $M$ be a simple restricted $\mathfrak{H}$-module with level $\ell \neq 0$. Then the following statements hold.

(i) $r_M \geq -1$ or $r_M = -\infty$.
(ii) If $r_M \geq -1$, then $K_0$ is a $\mathfrak{H}^{(0, -(n_M-1))}$-module and $h_{n_M-1}$ acts injectively on $K_0$.
(iii) $K$ is a $\mathfrak{H}^{(0, -\infty)}$-module and $K(z)^\mathfrak{H}$ has a $\mathfrak{H}$-module structure by (2.2)-(2.3).
(iv) $K_0$ and $K$ are invariant under the actions of $L_n$ and $d'_n$ for $n \in \mathbb{N}$.
(v) If $r_M \neq -\infty$, then $d'_{r_M-1}$ acts injectively on $K_0$ and $K$. 
Proof. (i) If $Y_{-2} \neq 0$, then $d'_p K_0 = 0$, $p \geq -2$. We deduce that $\mathfrak{S}it' K_0 = 0$ and hence $r_M = -\infty$.

If $Y_{-2} = 0$, then $r_M \geq -1$.

(ii) For any $0 \neq v \in K_0$ and $x \in \mathfrak{S}(0, -(m_1 - 1))$, it follows from Lemma 5.1(ii) that $xv \in M_0$. We need to show that $d'_p xv = 0, p \geq r_M$. Indeed, $d'_p h_k v = h_k d'_p v = 0$ by (5.2) for any $k \geq -(n - 1)$. Moreover, it follows from (2.10) and (5.1) that

$$d'_p d_n v = d_n d'_p v + [d'_p, d_n]v = (n - p)d'_{p+n}v = 0, \forall n \in \mathbb{N}.$$ 

Hence, $d'_p xv = 0, p \geq r_M$, that is, $xv \in K_0$, as desired.

Since $0 \neq K_0 \subseteq M_0$, we see that $h_{n_{r_{-1}}} \text{acts injectively on } K_0$ by Lemma 5.1(i).

(iii) follows from (ii).

(iv) Note that if $n_M = 0$, then $\hat{L}_n K_0 = 0$ for any $n \in \mathbb{N}$. For $n_M > 0$ we compute that

$$\hat{L}_n = \frac{1}{2\ell} \sum_{k \in \mathbb{Z}} : h_{n-k} h_k : + \frac{n+1}{\ell} h_n = \frac{1}{2\ell} \sum_{(n_{\ell-1}) \subseteq k \leq n_{\ell} - 1} : h_{n-k} h_k : + \frac{n+1}{\ell} h_n, n \in \mathbb{N}.$$ 

We see $\hat{L}_n K_0 \subset K_0$ and $\hat{L}_n K \subset K$ by (ii), and hence $d'_n K_0 \subset K_0$ and $d'_n K \subset K$.

(v) follows directly from the definitions of $r_M$ and $K$. \qed

We first consider the case $r_M = -\infty$.

**Proposition 5.4.** Let $M$ be a simple restricted $\mathfrak{S}$-module with central charge $c$ and level $\ell \neq 0$. If $r_M = -\infty$, then $M = K(z)^\mathfrak{S}$ for some $z \in \mathbb{C}$. Hence $c = 1 - \frac{12z^2}{\ell}$ and $K$ is a simple $\mathfrak{S}$-module.

Proof. Since $r_M = -\infty$, we see that $\mathfrak{S}it' K_0 = 0$. This together with (5.1) implies that $c = 1 - \frac{12z^2}{\ell}$. Noting that $[\mathfrak{S}it', \mathfrak{F} + \mathfrak{C} \mathfrak{e}_2] = 0$, we further obtain that $\mathfrak{S}it' K = 0$, that is, $d_n v = \hat{L}_n v \in K$ for any $v \in K$ and $n \in \mathbb{Z}$. Hence $K(z)^\mathfrak{S}$ is a $\mathfrak{S}$-submodule of $M$, yielding that $M = K(z)^\mathfrak{S}$. In particular, $K$ is a simple $\mathfrak{F}$-module. \qed

**Proposition 5.5.** Let $M$ be a simple restricted $\mathfrak{S}$-module with level $\ell \neq 0$. If $r_M \geq 2$ and $n_M \geq 2$, then $K_0$ is a simple $\mathfrak{S}(0, -(n_{\ell-1}))$-module and $M \cong \text{Ind}_{\mathfrak{S}(0, -(n_{\ell-1}))}^{\mathfrak{S}(0, -(n_{\ell-1}))} K_0$.

Proof. We first show that $\text{Ind}_{\mathfrak{S}(0, -(n_{\ell-1}))}^{\mathfrak{S}(0, -(n_{\ell-1}))} K_0 \cong K$ as $\mathfrak{S}(0, -\infty)$ modules. For that, let

$$\phi : \text{Ind}_{\mathfrak{S}(0, -(n_{\ell-1}))}^{\mathfrak{S}(0, -(n_{\ell-1}))} K_0 \longrightarrow K$$

$$\sum_{k \in \mathbb{M}} h^k \otimes v_k \mapsto \sum_{k \in \mathbb{M}} h^k v_k,$$

where $h^k = \cdots h^k_{l_2 -(n_{\ell-1}) - 1} h^k_{l_1 - 1} \in \mathfrak{U}(\mathfrak{S})$. Then $\phi$ is a $\mathfrak{S}(0, -\infty)$-module epimorphism and $\phi|_{K_0}$ is one-to-one.

Claim. Any nonzero submodule $V$ of $\text{Ind}_{\mathfrak{S}(0, -(n_{\ell-1}))}^{\mathfrak{S}(0, -(n_{\ell-1}))} K_0$ does not intersect with $K_0$ trivially.

Assume $V \cap K_0 = 0$. Let $v = \sum_{k \in \mathbb{M}} h^k \otimes v_k \in V \setminus K_0$ with minimal degree $i$. Then $0 < i$.

Let $p = \min \{s : i_s \neq 0\}$. Since $h^p v + h_{n_I - 1} v_k = 0$, we have $h^p v = [h^p v + h_{n_I - 1} v_k] v_k$. The following equality

$$[h_i, h_j h_{j_1} \cdots h_{j_i}] = \sum_{1 \leq s \leq t} \delta_{i + j_s, 0} i \mathfrak{C}_j h_{i_s} \cdots h_{j_s}, i, j_1 \leq j_2 \leq \cdots \leq j_t, i, j \in \mathbb{Z}.$$
implies that if $k_p = 0$ then $h_{p+nM-1}h^kv_k = 0$; and if $k_p \neq 0$, noticing the level $\ell \neq 0$, then
\[ [h_{p+n}, h^k] = \lambda h^{k-\ell} \text{ for some } \lambda \in \mathbb{C}^* \text{ and hence} \]
\[ \text{deg}((h_{p+nM-1}, h^k)v_k) = k - \epsilon_p \leq i - \epsilon_p, \]
where the equality holds if and only if $k = i$. Hence $\text{deg}(h_{p+nM-1}v) = i - \epsilon_p < i$ and $h_{p+nM-1}v \in V$, contrary to the choice of $v$. Thus, the claim holds.

From the Claim we know that the kernel of $\phi$ must be zero and hence $\phi$ is an isomorphism.

By Lemma\[ 5.3 \text{(v)}, \]
we see that $d'_{rM-1}$ acts injectively on $K$.

As $\tilde{\mathcal{D}}$-modules,
\[ \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K_0 \cong \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} \left( \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)}(\tilde{\mathcal{E}}(\infty,0), K_0) \right) \cong \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K. \]

And we further have $\text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K \cong \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K$ as vector spaces. Moreover, we have the following $\tilde{\mathcal{D}}$-module epimorphism
\[ \pi : \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K = \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K \to M, \]
\[ \sum_{l \in \mathbb{M}} d'^l \otimes v_l \mapsto \sum_{l \in \mathbb{M}} d'^l v_l, \]
where $d'^1 = \cdots (d'_\infty)^{12} (d'_{-\infty})^{11}$. We see that $\pi$ is also a $\mathcal{D}'$-module epimorphism. By the proof of Theorem 2.1 in [46] we know that any nonzero $\mathcal{D}'$-submodule of $\text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K$ contain nonzero vectors of $K$. Note that $\pi|_K$ is one-to-one, we see that the image of any nonzero $\tilde{\mathcal{D}}$-submodule (and hence $\mathcal{D}'$-submodule) of $\text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{E}}(0,\infty)} K$ must be a nonzero $\tilde{\mathcal{D}}$-submodule of $M$ and hence be the whole module $M$, which forces that the kernel of $\pi$ must be 0. Therefore, $\pi$ is an isomorphism. Since $M$ is simple, we see $K_0$ is a simple $\tilde{\mathcal{E}}(0,\infty)$-module. \hfill $\square$

**Proposition 5.6.** Let $M$ be a simple restricted $\tilde{\mathcal{D}}$-module with level $\ell \neq 0$. If $r_M = 1$, then $d'_0$ has an eigenvector in $K$.

**Proof.** Lemma\[ 5.3 \text{(iv)} \]means that $K$ is a $\tilde{\mathcal{E}}(0,\infty)$-module. Assume that any finitely generated $\mathbb{C}[d'_0]$-submodule of $K$ is a free $\mathbb{C}[d'_0]$-module. By Lemma\[ 4.10 \]we see that the following $\tilde{\mathcal{D}}'$-module homomorphism
\[ \tau : \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\tilde{\mathcal{D}}'} K = \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\mathcal{D}' R \tilde{\mathcal{E}}(0,\infty)} K \to M, \]
\[ x \otimes u \mapsto xu, x \in \mathbb{C}(\mathcal{D}'), u \in K. \]
is an isomorphism. So $M = \text{Ind}_{\tilde{\mathcal{E}}(0,\infty)}^{\mathcal{D}' R \tilde{\mathcal{E}}(0,\infty)} K$, and consequently, $K$ is a simple $\tilde{\mathcal{E}}(0,\infty)$-module. Since $r_M = 1$ and $\mathcal{D}' R \tilde{\mathcal{E}}(0,\infty) K = 0$, $K$ can be seen as a simple module over the Lie algebra $\mathcal{H} \oplus \mathbb{C} \mathbb{C} \oplus \mathbb{C} d'_0$, where $\mathbb{C} d'_0$ lies in the center of the Lie algebra. Schur’s lemma tells us that $d'_0$ acts as a scalar on $K$, a contradiction. So this case will not occur.

Therefore, there exists some finitely generated $\mathbb{C}[d'_0]$-submodule $W$ of $K$ that is not a free $\mathbb{C}[d'_0]$-module. Since $\mathbb{C}[d'_0]$ is a principal ideal domain, by the structure theorem of finitely generated modules over a principal ideal domain, there exists a monic polynomial $f(d'_0) \in \mathbb{C}[d'_0]$ with minimal positive degree and nonzero element $u \in W$ such that $f(d'_0)u = 0$. Write $f(d'_0) = \Pi_{1 \leq i \leq s}(d'_0 - \lambda_i), \lambda_1, \cdots, \lambda_s \in \mathbb{C}$. Denote $w := \prod_{i=1}^s (d'_0 - \lambda_i)u \neq 0$, we see $(d'_0 - \lambda_s)w = 0$ where we make convention that $w = u$ if $s = 1$. Then $w$ is a desired eigenvector of $d'_0$. \hfill $\square$

**Proposition 5.7.** Let $M$ be a simple restricted $\tilde{\mathcal{D}}$-module with level $\ell \neq 0$. If $r_M = 0, \pm 1$, then $K$ is a simple $\mathcal{H}$-module and $M \cong K(z) \tilde{\mathcal{D}} \otimes U \tilde{\mathcal{D}}$ for some simple module $U \in \mathcal{R}_{\mathcal{D}'}$ and some $z \in \mathbb{C}$.
Proof. If $r_M = 1$, then by Proposition 5.6 we know that there exists $0 \neq u \in K$ such that $d'_0 u = \lambda u$ for some $\lambda \neq 0$; if $r_M = 0, -1$, then $d'_0 K = 0$. In summary, for all the three cases, $d'_0$ has an eigenvector in $K$. Since $M$ is a simple $\check{\mathcal{D}}'$-module, Schur’s lemma implies that $h_0, \check{e}_1, \check{e}_2, \check{e}_3$ act as scalars on $M$. So $M$ is a weight $\check{\mathcal{D}}'$-module, and $K$ is a weight module for $\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}$. Take a weight vector $u_0 \in K$ with $d'_0 u_0 = \lambda_0 u_0$ for some $\lambda_0 \in \mathbb{C}$.

Set $K' = \mathcal{U}(\check{\mathcal{D}}') u_0$, which is an $\check{\mathcal{D}}'$ submodule of $K$. Now we define the $\check{\mathcal{D}}'$-module $K^{\check{\mathcal{D}}'}$ with trivial action of $\mathcal{Vir}'$. Let $\mathbb{C} V_0$ be the one-dimensional $\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}$-module defined by

$$\check{e}_1 v_0 = (c - 1 + \frac{12\pi^2}{l}) v_0, \quad d'_0 v_0 = \lambda_0 v_0, \quad d'_1 v_0 = h_1 v_0 = \check{e}_2 v_0 = \check{e}_3 v_0 = 0, \quad 0 \neq n \geq r_M, k \in \mathbb{Z}.$$ 

Then $\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}$ is a $\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}$-module with central charge $c - 1 + \frac{12\pi^2}{l}$ and level $l$. There is a $\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}$-module homomorphism

$$\tau_{K'} : \mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'} \longrightarrow M, \quad v_0 \otimes u \mapsto u, \forall u \in K',$$ 

which is injective and can be extended to be the following $\check{\mathcal{D}}'$-module epimorphism

$$\tau : \text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) \longrightarrow M, \quad x(v_0 \otimes u) \mapsto xu, x \in \mathcal{U}(\check{\mathcal{D}}'), u \in K'.$$

By Lemma 8 in [43] we know that

$$\text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) \cong (\text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0) \otimes K^{\check{\mathcal{D}}'}) \cong (\text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0) \otimes K^{\check{\mathcal{D}}'}) \otimes K^{\check{\mathcal{D}}'}.$$

Then we have the following $\check{\mathcal{D}}'$-module epimorphism

$$\tau' : (\text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0) \otimes K^{\check{\mathcal{D}}'}) \otimes K^{\check{\mathcal{D}}'} \longrightarrow M, \quad x v_0 \otimes u \mapsto xu, x \in \mathcal{U}(\mathcal{Vir}', u \in K').$$

Note that $(\text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0) \otimes K^{\check{\mathcal{D}}'})$ as $\mathcal{Vir}'$-modules, and $\tau'$ is also a $\mathcal{Vir}'$-module epimorphism, $\tau'|_{\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}}$ is one-to-one, and $(\text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0) \otimes K^{\check{\mathcal{D}}'})$ is a highest weight $\mathcal{Vir}'$-module. Let $V = \text{Ind}_{\check{\mathcal{D}}^{(r_M - \delta_M, 1, -\infty)}}^{\check{\mathcal{D}}}(\mathbb{C} V_0)$ and $\mathcal{R} = \text{Ker}(\tau')$. It should be noted that

$$\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'} = \{ u \in V^{\check{\mathcal{D}}'} \otimes K^{\check{\mathcal{D}}'} \mid d'_0 u = \lambda_0 u \}.$$ 

We see that $(\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) \cap \mathcal{R} = 0$. Let $\mathcal{R}'$ be the sum of all $\mathcal{Vir}'$-submodules $W$ of $V^{\check{\mathcal{D}}'} \otimes K^{\check{\mathcal{D}}'}$ with $(\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) \cap W = 0$, that is, the unique maximal (weight) $\mathcal{Vir}'$-submodule of $V^{\check{\mathcal{D}}'} \otimes K^{\check{\mathcal{D}}'}$ with trivial intersection with $(\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'})$. It is obvious that $\mathcal{R} \subseteq \mathcal{R}'$. Next we further show that $\mathcal{R} = \mathcal{R}'$. For that, take any $\mathcal{Vir}'$-submodule $W$ of $V^{\check{\mathcal{D}}'} \otimes K^{\check{\mathcal{D}}'}$ such that $(\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) \cap W = 0$. Then for any weight vector $w = \sum_{l \in \mathbb{N}} d^l v_0 \otimes u_l \in W$, where $u_l \in K^{\check{\mathcal{D}}'}$, $d^1 = \cdots (d_{-2})^3 (d_{-1})^3$ if $r_M = 1, 0$, or $d^1 = \cdots (d_{-2})^3$ if $r_M = -1$, and all $w(l) \geq 1$ are equal. Note that $h_1 w = \sum_{l \in \mathbb{N}} d^l v_0 \otimes h_1 u_l$ either equals to 0 or has the same weight as $w$ under the action of $d'_0$. So $\mathcal{U}(\check{\mathcal{D}}') W \cap (\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) = 0$, i.e., $\mathcal{U}(\check{\mathcal{D}}') W \subset \mathcal{R}'$. Hence $\mathcal{U}(\check{\mathcal{D}}') \mathcal{R}' \cap (\mathbb{C} V_0 \otimes K^{\check{\mathcal{D}}'}) = 0$. The maximality of $\mathcal{R}'$ forces that $\mathcal{R}' = \mathcal{U}(\check{\mathcal{D}}') \mathcal{R}'$ is a proper $\check{\mathcal{D}}'$-submodule of $V^{\check{\mathcal{D}}'} \otimes K^{\check{\mathcal{D}}'}$. Since $\mathcal{R}$ is a maximal proper $\check{\mathcal{D}}'$-submodule of $V^{\check{\mathcal{D}}'} \otimes K^{\check{\mathcal{D}}'}$, it follows that $\mathcal{R} = \mathcal{R}'$. 


By Lemma 4.8, we know that $\mathfrak{H}$ is generated by $P_1(\mathbb{C}v_0 \otimes K^{\mathbb{C}}) = \mathbb{C}P_1v_0 \otimes K^{\mathbb{C}'}$ and $P_2(\mathbb{C}v_0 \otimes K^{\mathbb{C}''}) = \mathbb{C}P_2v_0 \otimes K^{\mathbb{C}''}$. Let $V'$ be the maximal submodule of $V$ generated by $P_1v_0$ and $P_2v_0$, then $\mathfrak{H} = V^{\mathbb{C}'} \otimes K^{\mathbb{C}''}$. Therefore,

$$M \cong (V^{\mathbb{C}'} \otimes K^{\mathbb{C}''})/(V^{\mathbb{C}'} \otimes K^{\mathbb{C}''}) \cong (V/V')^{\mathbb{C}'} \otimes K^{\mathbb{C}''},$$

which forces that $K^{\mathbb{C}''}$ is a simple $\mathfrak{H}$-module and hence a simple $\mathfrak{H}$-module. So $K'$ is a simple $\mathfrak{H}$-module. By [43 Theorem 12] we know there exists a simple $\mathfrak{Vir}$-module $U \in \mathcal{R}_{\mathfrak{Vir}}$ such that $M \cong K^{\mathbb{C}''} \otimes U^{\mathbb{C}'}$. From this isomorphism and some computations we see that $K_0 \subseteq K^{\mathbb{C}''} \otimes v_0$ where $v_0$ is a highest weight vector. So $K = K'$.

We are now in a position to present the following main result on characterization of simple restricted $\mathfrak{H}$-modules with nonzero level.

**Theorem 5.8.** Let $M$ be a simple restricted $\mathfrak{H}$-module with level $\ell \neq 0$. The invariants $n_M, r_M$ of $M, K_0, K$ are defined as before. Then

$$M \cong \begin{cases} K(z)^{\mathbb{C}}, & \text{if } r_M = -\infty, \\ K(z)^{\mathbb{C}} \otimes U^{\mathbb{C}}, & \text{if } -1 \leq r_M \leq 1 \text{ or } n_M = 0, 1, \\ \text{Ind}_{\mathfrak{Vir}_1}^{\mathfrak{Vir}}(\mathfrak{H}(0, -m_1, 1))K_0, & \text{otherwise}, \end{cases}$$

for some $U \in \mathcal{R}_{\mathfrak{Vir}}$ and some $z \in \mathbb{C}$.

**Proof.** The assertion follows directly from Proposition 5.2, Proposition 5.4, Proposition 5.5, Proposition 5.7.

The following result characterizes simple Whittaker modules over the twisted Heisenberg-Virasoro algebra $\tilde{\mathfrak{H}}$.

**Theorem 5.9.** Let $M$ be a $\tilde{\mathfrak{H}}$-module (not necessarily weight) on which the algebra $\tilde{\mathfrak{H}}^+$ acts locally finitely. Then the following statements hold.

(i) The module $M$ contains a nonzero vector $v$ such that $\tilde{\mathfrak{H}}^+v \subseteq \mathcal{C}v$.

(ii) If $M$ is simple, then $M$ is a Whittaker module or a highest weight module.

**Proof.** (i) Let $(M_1, \rho)$ be a finite dimensional $\tilde{\mathfrak{H}}^+$-submodule of $M$. Then $M_1$ is also a finite dimensional $\mathfrak{Vir}_{\geq 1}$-module. Let $a := \ker(\rho|_{\mathfrak{Vir}_{\geq 1}})$ be the kernel of the representation map of $\mathfrak{Vir}_{\geq 1}$ on $M_1$. Then $a$ is an ideal of $\mathfrak{Vir}_{\geq 1}$ of finite codimension. We claim that $d_n \in a$ for some $n \in \mathbb{Z}_+$. If this is not true, then there exists a minimal $m \in \mathbb{Z}_+$ such that $a$ contains an element of the form $a_1d_{i_1} + a_2d_{i_2} + \cdots + a_{m+1}d_{i_{m+1}}$ for positive integers $i_1 < i_2 < \cdots < i_{m+1}$ and nonzero complex numbers $a_1, a_2, \cdots, a_{m+1}$. We further see that $a$ contains

$$[d_{i_1}, a_i d_{i_2} + \cdots + a_{m+1}d_{i_{m+1}}] = a_i(i_2 - i_1)d_{i_1+i_2} + a_i(i_3 - i_1)d_{i_1+i_3} + \cdots + a_{m+1}(i_{m+1} - i_1)d_{i_1+i_{m+1}},$$

which contradicts with the minimality of $m$. Hence the claim follows. Consequently,

$$\mathfrak{Vir}_{\geq 2n} := \sum_{i 

Then

$$\mathfrak{Vir}_{\geq 2n} + \mathfrak{H}_{\geq 2n+1} = \mathfrak{Vir}_{\geq 2n} + [\mathfrak{H}_{\geq 2n}, \mathfrak{Vir}_{\geq 2n}] \subseteq \ker(\rho).$$

This implies that $M_1$ is a finite dimensional module over a finite dimensional solvable Lie algebra $\tilde{\mathfrak{H}}^+/(\mathfrak{Vir}_{\geq 2n} + \mathfrak{H}_{\geq 2n+1})$. The desired assertion follows directly from Lie Theorem.
(ii) follows directly from (i) and \[46\].

\[\textbf{Remark 5.10.}\] From Theorem \[5.9\] we know that if \( M \) is a simple Whittaker module over \( \hat{D} \) with nonzero level, and \( \hat{D}^+ v \subset C v \) for some nonzero vector \( v \in M \), then \( K = \mathcal{U}(\hat{\mathcal{F}})v = \mathcal{U}(\oplus_{r \in \mathbb{Z}_+} \mathbb{C} h_r)v \) is a simple Whittaker module over \( \hat{\mathcal{F}} \). Therefore, \[43\], Theorem 12] implies that \( M \cong U^\mathbb{C} \otimes K(z)^\mathbb{C} \) for some \( U \in \mathcal{R}_\text{Wir} \). Clearly, \( U \) is a simple Whittaker module or a simple highest weight module over \( \mathcal{R}_\text{Wir} \).

6. Application one: characterization of simple highest weight modules and Whittaker modules over the mirror Heisenberg-Virasoro algebra

Based on the results on structure of simple restricted modules over the mirror Heisenberg-Virasoro algebra \( \mathcal{D} \) given in Theorem 2.11 and Theorem 4.13, we give characterization of simple highest weight \( \mathcal{D} \)-modules and simple Whittaker \( \mathcal{D} \)-modules in this section.

We first have the following result characterizing simple highest weight modules over the mirror Heisenberg-Virasoro algebra.

\[\textbf{Theorem 6.1.}\] Let \( \mathcal{D} \) be the mirror Heisenberg-Virasoro algebra with the triangular decomposition \( \mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^0 \oplus \mathcal{D}^- \). Let \( S \) be a \( \mathcal{D} \)-module (not necessarily weight) on which every element in the algebra \( \mathcal{D}^+ \) acts locally nilpotently. Then the following statements hold.

(i) The module \( S \) contains a nonzero vector \( v \) such that \( \mathcal{D}^+ v = 0 \).

(ii) If \( S \) is simple, then \( S \) is a highest weight module.

\[\textbf{Proof.}\] (i) It follows from \[45\], Theorem 1] that there exists a nonzero vector \( v \in S \) such that \( d_i v = 0 \) for any \( i \in \mathbb{Z}_+ \). If \( h_{\frac{1}{2}} v = 0 \), then \( \mathcal{D}^+ v = 0 = d_1, d_2 \) and \( h_{\frac{1}{2}} \) generate \( \mathcal{D}^+ \). Assume that \( w := h_{\frac{1}{2}} v \neq 0 \). Then

\[ d_i w = d_i h_{\frac{1}{2}} v = h_{\frac{1}{2}} d_i v + [d_i, h_{\frac{1}{2}}] v = -\frac{1}{2} h_{\frac{1}{2}} v. \]

Similar arguments yield that the element \( d'_i w = \lambda h_{j+\frac{1}{2}} v \) for some \( \lambda \in \mathbb{C}^* \) and \( j \in \mathbb{Z}_+ \). As \( d_i \) acts locally nilpotently on \( S \), it follow that there exists some \( n \in \mathbb{Z}_+ \) such that \( h_{j+\frac{1}{2}} v = 0 \) for \( j \geq n \).

We now show that for every \( m \in \mathbb{N} \) there exists some nonzero element \( u \in S \) such that \( d_i u = h_{k+\frac{1}{2}} u = 0 \) for \( i \in \mathbb{Z}_+ \) and \( k \geq m \) by a backward induction on \( m \). The above arguments imply that the assertion is true for \( m \geq n \). Assume that \( 0 \neq u \in S \) satisfies that \( d_i u = h_{k+\frac{1}{2}} u = 0 \) for \( i \in \mathbb{Z}_+ \) and \( k \geq m > 0 \). If \( h_{m-\frac{1}{2}} u = 0 \), then the induction step is proved. Otherwise, \( h_{m-\frac{1}{2}} u \neq 0 \), and there exists some \( l \in \mathbb{N} \) such that \( u := h_{l-\frac{1}{2}} u \neq 0 \) and \( h_{m-\frac{1}{2}} u' = h_{l+\frac{1}{2}} u = 0 \). Moreover, \( d_i u' = h_{k+\frac{1}{2}} u' = 0 \) for \( i \in \mathbb{Z}_+ \) and \( k \geq m - 1 \). The induction step follows.

(ii) By (i), we know that \( S \) is a simple restricted \( \mathcal{D} \)-module with \( n_S = 0 \) and \( m_S \leq 1 \). From Theorem 2.11 and Case 1 of Theorem 4.13, we know that \( S \cong H^\mathbb{C} \otimes U^\mathbb{C} \) as \( \mathcal{D} \)-modules for some simple modules \( H \in \mathcal{R}_\text{Vir} \) and \( U \in \mathcal{R}_\text{Wir} \). Moreover, \( H = \text{Ind}^\mathbb{C} \mathcal{D}_\mathbb{C} \) is a simple highest weight module over \( \mathcal{D} \). Note that every element in the algebra \( \mathcal{R}_\text{Wir} \) acts locally nilpotently on \( \mathcal{D}_\mathbb{C} \otimes U \) by the assumption. This implies that the same property also holds on \( U \). From \[45\], Theorem 1] we know that \( U \) is a simple highest weight \( \mathcal{R}_\text{Wir} \)-module. This completes the proof. \( \square \)

As a direct consequence of Theorem 6.1, we have
Corollary 6.2. Let $S$ be an simple restricted $\mathcal{D}$-module with $m_S \leq 1$ and $n_S = 0$. Then $S$ is a highest weight module.

Proof. The assumption that $m_S \leq 1$ and $n_S = 0$ implies that there exists a nonzero vector $v \in M$ such that $\mathcal{D}^+ v = 0$. Then $M = \mathcal{U}(\mathcal{D}^- + \mathcal{D}^0) v$. It follows that each element in $\mathcal{D}^+$ acts locally nilpotently on $M$. Consequently, the desired assertion follows directly from Theorem 6.1. □

The following result characterizes simple Whittaker modules over the mirror Heisenberg-Virasoro algebra.

Theorem 6.3. Let $M$ be a $\mathcal{D}$-module (not necessarily weight) on which the algebra $\mathcal{D}^+$ acts locally finitely. Then the following statements hold.

(i) The module $M$ contains a nonzero vector $v$ such that $\mathcal{D}^+ v \subseteq \mathbb{C} v$.

(ii) If $M$ is simple, then $M$ is a Whittaker module or a highest weight module.

Proof. (i) Let $(M_1, \rho)$ be a finite dimensional $\mathcal{D}^+$-submodule of $M$. Then $M_1$ is also a finite dimensional $\mathfrak{Vir}_{\geq 1}$-module. Let $\alpha := \ker(\rho|_{\mathfrak{Vir}_{\geq 1}})$ be the kernel of the representation map of $\mathfrak{Vir}_{\geq 1}$ on $M_1$. Then $\alpha$ is an ideal of $\mathfrak{Vir}_{\geq 1}$ of finite codimension. We claim that $d_n \in \alpha$ for some $n \in \mathbb{Z}_+$. If this is not true, then there exists a minimal $m \in \mathbb{Z}_+$ such that $\alpha$ contains an element of the form $a_1 d_1 + a_2 d_2 + \cdots + a_{m+1} d_{m+1}$ for positive integers $i_1 < i_2 < \cdots < i_{m+1}$ and nonzero complex numbers $a_1, a_2, \ldots, a_{m+1}$. We further see that $\alpha$ contains

$$[d_i, a_1 d_1 + a_2 d_2 + \cdots + a_{m+1} d_{m+1}] = a_2 (i_1 - i_2) d_{i_1 + i_2} + a_3 (i_1 - i_3) d_{i_1 + i_3} + \cdots + a_{m+1} (i_1 - i_{m+1}) d_{i_1 + i_{m+1}},$$

which contradicts with the minimality of $m$. Hence the claim follows. Consequently,

$$\mathfrak{Vir}_{\geq n} := \sum_{i \geq n, i \neq 2n} \mathbb{C} d_i = \mathbb{C} d_n + [d_n, \mathfrak{Vir}_{\geq 1}] \subseteq \alpha.$$

Then

$$\mathfrak{Vir}_{\geq n} + \mathcal{H}_{\geq n} = \mathfrak{Vir}_{\geq n} + [\mathcal{H}^{1/2}_o + \mathcal{H}^{1/2}, \mathfrak{Vir}_{\geq n}] \subseteq \ker(\rho).$$

This implies that $M_1$ is a finite dimensional module over a finite dimensional solvable Lie algebra $\mathcal{D}^+ / ([\mathfrak{Vir}_{\geq n} + \mathcal{H}_{\geq n})$. The desired assertion follows directly from Lie Theorem.

(ii) follows directly from (i). □

7. Examples

In this section, we will give a few examples of simple restricted $\mathcal{\widehat{D}}$- and $\mathcal{D}$-modules, which are also weak (simple) untwisted and twisted $\mathcal{V}^e$-modules.

Example 7.1. For any $n \in \mathbb{Z}_+$, let $\mathcal{W}_0 = \mathbb{C}[x_1, \cdots, x_n]$ be the polynomial algebra in indeterminates $x_1, \cdots, x_n$. Define the $\mathcal{H}^{(-n)}$-module structure on $\mathcal{W}_0$ by

$$h_{-i^{1/2}} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = \lambda_i f(x_1, \cdots, x_i - 1, \cdots, x_n),$$

$$h_{-i^{1/2}} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = -\frac{\ell(i - \frac{1}{2})}{\lambda_i} (x_i + a_i) f(x_1, \cdots, x_i + 1, \cdots, x_n),$$

$$h_{i^{1/2}} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = 0,$$  \hspace{1cm} (7.1)

$$c_2 \cdot f(x_1, \cdots, x_n) = \ell f(x_1, \cdots, x_n)$$
where $\ell, \lambda_i \in \mathbb{C}^*, a_i \in \mathbb{C}, j \in \mathbb{N}, 1 \leq i \leq n$. It is not hard to check that $\mathcal{W}_0$ is a simple $\mathfrak{H}^{(-n)}$-module. Then the induced $\mathfrak{H}$-module $K = \text{Ind}_{\mathfrak{h}^{(-n)}}^{\mathfrak{H}}\mathcal{W}_0$ is a simple restricted $\mathfrak{H}$-module. So $K^{\mathbb{C}}$ is a simple restricted $\mathfrak{D}$-module with central charge 1 and level $\ell$. We may denote $K^{\mathbb{C}} = K^{\mathbb{C}}(\ell, \Lambda_n, a_0)$ for any $\ell \in \mathbb{C}^*$, $\Lambda_n = (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^*)^n$, $a_0 = (a_1, \cdots, a_n) \in \mathbb{C}^n$.

Let $U$ be a simple restricted $\mathfrak{g}^{\mathbb{C}}$-module (Theorem 2.9 classified all simple restricted $\mathfrak{g}^{\mathbb{C}}$-modules). From Corollary 2.14 then $S = U^{\mathbb{C}} \otimes K^{\mathbb{C}}(\ell, \Lambda_n, a_0)$ is a simple restricted $\mathfrak{D}$-module.

If we replace (7.7) by

$$
\begin{align*}
& h_i \cdot f(x_1, \cdots, x_i, \cdots, x_n) = \lambda_i f(x_1, \cdots, x_i - 1, \cdots, x_n), \\
& h_{-i} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = -\frac{\ell_i}{\lambda_i} (x_i + a_i) f(x_1, \cdots, x_i + 1, \cdots, x_n), \\
& h_{n+j+1} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = 0, \\
& \bar{c}_3 \cdot f(x_1, \cdots, x_n) = \ell f(x_1, \cdots, x_n)
\end{align*}
$$

for $\ell, \lambda_i \in \mathbb{C}^*, a_i \in \mathbb{C}, j \in \mathbb{N}, 1 \leq i \leq n$, then $\mathcal{W}_0$ is a simple $\mathfrak{H}^{(-n)}$-module, and the induced $\mathfrak{H}$-module $\bar{K} = \text{Ind}_{\mathfrak{h}^{(-n)}}^{\mathfrak{H}}\mathcal{W}_0$ is a simple restricted $\bar{\mathfrak{H}}$-module. Hence, for any $z \in \mathbb{C}$, we have the simple $\mathfrak{D}$-module $\bar{K}(z)^{\mathbb{C}} = \bar{K}(z)^{\mathbb{C}}(\ell, \Lambda_n, a_0)$ for any $\ell \in \mathbb{C}^*$, $\Lambda_n = (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^*)^n$, $a_0 = (a_1, \cdots, a_n) \in \mathbb{C}^n$. For any simple $\mathfrak{g}^{\mathbb{C}}$-module $U \in \mathcal{R}_{\mathfrak{g}^{\mathbb{C}}}$, the tensor product $M = U^{\mathbb{C}} \otimes \bar{K}(z)^{\mathbb{C}}(\ell, \Lambda_n, a_0)$ is a simple restricted $\mathfrak{D}$-module.

For characterizing simple induced restricted $\mathfrak{D}$-and $\bar{\mathfrak{D}}$-module which are not tensor product modules, we need the following

**Lemma 7.2.** Let $S = U^{\mathbb{C}} \otimes V^{\mathbb{C}}$ be a simple restricted $\mathfrak{D}$-module with $n_S > 0$ and nonzero level, where $U \in \mathcal{R}_{\mathfrak{g}^{\mathbb{C}}}$ and $V \in \mathcal{R}_{\mathfrak{H}}$. Let $V_0 = \text{Ann}_{\mathfrak{g}}(\mathfrak{H}^{(-n_S)})$ and $W_0 = \text{Ann}_{\mathfrak{g}}(\mathfrak{H}^{(-n_S)})$. Then $V_0$ is a simple $\mathfrak{D}^{(0,-n_S)}$-module, and $W_0 = U \otimes V_0$. Hence $W_0$ contains a simple $\mathfrak{H}^{(-n_S)}$ submodule.

**Proof.** This is clear. \qed

We also have the $\bar{\mathfrak{D}}$-module version of Lemma 7.2.

**Lemma 7.3.** Let $M = H(z)^{\mathbb{C}} \otimes U^{\mathbb{C}}$ be a simple restricted $\mathfrak{D}$-module with $n_M > 1$ and nonzero level, where $z \in \mathbb{C}$, $H \in \mathcal{R}_{\mathfrak{H}}$ and $U \in \mathcal{R}_{\mathfrak{g}^{\mathbb{C}}}$. Let $H_0 = \text{Ann}_{\mathfrak{g}}(\mathfrak{H}^{(-n_M)})$ and $M_0 = \text{Ann}_{\mathfrak{g}}(\mathfrak{H}^{(-n_M)})$. Then $H_0$ is a simple $\mathfrak{D}^{(0,-n_M+1)}$-module, and $M_0 = H_0 \otimes U$. Hence $M_0$ contains a simple $\mathfrak{H}^{(-n_M+1)}$ submodule.

Lemma 7.2 (resp. Lemma 7.3) means that if $S \in \mathcal{R}_\mathfrak{D}$ (resp. $M \in \mathcal{R}_\bar{\mathfrak{D}}$) is not a tensor product module, then $W_0$ (resp. $M_0$) contains no simple $\mathfrak{H}^{(-n)}$-submodule (resp. $\mathfrak{H}^{(-n+1)}$-submodules).

Here we will first consider the case $n_S = 1$ (resp. $n_M = 2$). Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{C} e$ be the 2-dimensional solvable Lie algebra with basis $h, e$ and subject to Lie bracket $[h, e] = e$. The following concrete example using [40], Example 13] tells us how to construct induced restricted $\mathfrak{D}$-module (resp. $\bar{\mathfrak{D}}$-module) from a $\mathbb{C}[e]$-torsion-free simple $\mathfrak{b}$-module.

**Example 7.4** (Simple induced restricted module, $n_S = 1/n_M = 2$). Let $c_1, c_2 \in \mathbb{C}$ with $c_2 \neq 0$. Let $W' = (t - 1)^{-1}\mathbb{C}[t, t^{-1}]$. From [40], Example 13] we know that $W'$ is a simple $\mathfrak{b}$-module whose structure is given by

$$
h \cdot f(t) = \frac{d}{dt}(f(t)) + \frac{f(t)}{t^2(t-1)}, \quad e \cdot f(t) = tf(t), \quad \forall f(t) \in W'.
$$
We can make $W'$ into a $\mathcal{D}^{(0,0)}$-module by
\[
\begin{align*}
    c_1 \cdot f(t) &= c_1 f(t), & c_2 \cdot f(t) &= c_2 f(t), \\
    d_0 \cdot f(t) &= -\frac{1}{2} h \cdot f(t), & h_1 \cdot f(t) &= e \cdot f(t), & d_i \cdot f(t) &= h_{i+1} \cdot f(t) = 0, & i & \in \mathbb{Z}_+.
\end{align*}
\]

Then $W'$ is a simple $\mathcal{D}^{(0,0)}$-module. Clearly, the action of $h^\pm$ on $W'$ implies that $W'$ contains no simple $\mathcal{H}^{(0)}$-module. Then $W_0 = \text{Ind}_{\mathcal{D}^{(0,0)}}^{\mathcal{D}^{(0,-1)}} W'$ is a simple $\mathcal{D}^{(0,-1)}$-module and contains no simple $\mathcal{H}^{(-1)}$-module. So $W_0$ is not a tensor product $\mathcal{D}^{(0,-1)}$-module. Let $S = \text{Ind}_{\mathcal{D}^{(0,0)}}^{\mathcal{D}^{(0,-1)}} W_0$. It is easy to see $n_S = 1, m_S = 2 = r_S$, and $W_0 = U_0 = K_0$. The proof of Proposition \[4.5\] implies that $S$ is a simple restricted $\mathcal{D}$-module. And Lemma \[7.2\] means that $S$ is not a tensor product $\mathcal{D}$-module.

For $c, z, z' \in \mathbb{C}, \ell \in \mathbb{C}^\ast$, we also can make $W'$ into a $\tilde{\mathcal{D}}^{(0,0)}$-module by
\[
\begin{align*}
    d_0 \cdot f(t) &= h \cdot f(t), & h_1 \cdot f(t) &= e \cdot f(t), \\
    h_0 \cdot f(t) &= z' f(t), & h_{i+1} \cdot f(t) &= d_i \cdot f(t) = 0, & i & \in \mathbb{Z}_+, \\
    c_1 \cdot f(t) &= cf(t), & c_2 \cdot f(t) &= zf(t), & c_3 \cdot f(t) &= \ell f(t),
\end{align*}
\]
where $f(t) \in W'$. Then $W'$ is a simple $\tilde{\mathcal{D}}^{(0,0)}$-module. Clearly, the action of $h_1$ on $W'$ implies that $W'$ contains no simple $\tilde{\mathcal{H}}^{(0)}$-module. Then $M_0 = \text{Ind}_{\tilde{\mathcal{D}}^{(0,0)}}^{\tilde{\mathcal{D}}^{(0,-1)}} W'$ is a simple $\tilde{\mathcal{D}}^{(0,-1)}$-module and contains no simple $\tilde{\mathcal{H}}^{(-1)}$-module. Let $M = \text{Ind}_{\tilde{\mathcal{D}}^{(0,0)}}^{\tilde{\mathcal{D}}^{(0,-1)}} M_0$. It is easy to see $n_M = 2, r_M = 3$. The proof of Proposition \[5.3\] implies that $M$ is a simple restricted $\mathcal{D}$-module. And Lemma \[7.2\] means that $M$ is not a tensor product $\mathcal{D}$-module.

**Example 7.5** (Simple induced modules of semi-Whittaker type, $n_S \geq 2, n_M \geq 3$). \[1\] Take $p, q \in \mathbb{Z}_+, a = (a_1, \ldots, a_p) \in (\mathbb{C}^\ast)^p, b = (b_1, \ldots, b_p) \in (\mathbb{C}^\ast)^p, c, \ell \in \mathbb{C}$ with $\ell \neq 0$. Define the 1-dimensional $\mathcal{D}^{(p,q)}$-module $\mathcal{C}_{a,b} = \mathbb{C}v_0$ with
\[
\begin{align*}
    c_1 \cdot v_0 &= cv_0, & c_2 \cdot v_0 &= \ell v_0, \\
    d_0 v_0 &= a_1 v_0, & d_1 v_0 &= a_2 v_0, & d_i v_0 &= 0 \text{ for } i > p + q - 1, \\
    h_{q+\frac{1}{2}} v_0 &= b_1 v_0, & h_{p+q-\frac{1}{2}} v_0 &= b_p v_0, & h_{i+\frac{1}{2}} v_0 &= 0 \text{ for } i > p + q.
\end{align*}
\]
(7.2)

It is not hard to show that $U(a, b) := \text{Ind}_{\mathcal{D}^{(p,q)}}^{\mathcal{D}^{(0,0)}} \mathcal{C}_{a,b}$ is a simple $\mathcal{D}^{(0,-1)}$-module. Then in Theorem \[3.3\] we have $V = U(a, b), n = 1, k = p + q = l$, and so $S = \mathcal{U}(a, b) := \text{Ind}_{\mathcal{D}^{(0,0)}}^{\mathcal{D}^{(0,-1)}} U(a, b)$ is a simple restricted $\mathcal{D}$-module. In Lemma \[7.2\] we know that $S$ is a simple $\mathcal{H}^{(-p,q)}$-module. Then $\mathcal{U}(a, b)$ does not contain any simple $\mathcal{H}^{(-p,q)}$-module (for $h_{\pm 1/2}$ acts freely on $W_0$). Hence, by Lemma \[7.2\] $\tilde{U}(a, b)$ is not a tensor product $\mathcal{D}$-module.

If we, in the above example, replace (7.2) by
\[
\begin{align*}
    \check{c}_1 \cdot v_0 &= cv_0, & \check{c}_2 \cdot v_0 &= z v_0, & \check{c}_3 v_0 &= \ell v_0, \\
    d_0 v_0 &= a_1 v_0, & \cdots, & d_{p+q-1} v_0 &= a_q v_0, & d_i v_0 &= 0 \text{ for } i > p + q - 1, \\
    h_{q+\frac{1}{2}} v_0 &= b_1 v_0, & \cdots, & h_{p+q-\frac{1}{2}} v_0 &= b_p v_0, & h_{i+\frac{1}{2}} v_0 &= 0 \text{ for } i > p + q,
\end{align*}
\]
where $z \in \mathbb{C}$ and leave other parts invariant, then for any $z' \in \mathbb{C}$, the induced $\tilde{\mathcal{D}}^{(0, -(p+q))}$-module
\[
\check{V} = \text{Ind}_{\tilde{\mathcal{D}}^{(p,q)}}^{\tilde{\mathcal{D}}^{(0, -(p+q))}} \mathcal{C}_{a,b} / \left( \mathcal{U}(\tilde{\mathcal{D}}^{(0, -(p+q))})(h_0 - z')(1 \otimes v_0) \right)
\]

---

1This example is a modified version of the one provided by Drazen Adamovic.
is a simple $\mathcal{E}^{(0)-(p+q)}$-module. Let $M = \text{Ind}_{\mathcal{E}^{(p)}_{0,0}}^{\mathcal{E}^{(p+q)}} \mathcal{V}$. The proof of Theorem 5.5 implies that $M$ is a simple restricted $\mathcal{E}$-module where $n_M = p + q + 1, r_M = 2(p + q) + 1$, and $K_0 = \mathcal{V} = M_0$. Since $\mathcal{V}$ contains no simple $\mathcal{E}^{(-nm+1)}$-module, we see, by Lemma 7.3, that $M$ is not a tensor product $\mathcal{E}$-module.

Remark 7.6. From Theorem 4.13 (resp. Theorem 5.2) we know that if $n_S = 0$ (resp. $n_M = 0, 1$), then simple restricted $\mathcal{E}$-modules (resp. $\mathcal{E}$-modules) must be tensor product modules. And Examples 7.4-7.5 mean that for any $n_S > 0$ (resp. $n_M > 1$), there do exist simple restricted $\mathcal{E}$-modules (resp. $\mathcal{E}$-modules) which are not tensor product modules. Clearly, the $\mathcal{E}$-modules here are simple restricted $\mathcal{E}$-modules for $z = 0$.

Appendix A. Application two: simple modules for Heisenberg-Virasoro vertex operator algebras $\mathcal{V}^{c}$ (by Drazen Adamovic)

A connection between restricted modules over the Heisenberg-Virasoro algebra and VOA modules in untwisted cases was considered by Guo and Wang in [27]. In this appendix we extend this connection for restricted modules for the mirror Heisenberg-Virasoro algebra. Restricted modules of nonzero level for the mirror Heisenberg-Virasoro algebra can be treated as weak modules for the Heisenberg-Virasoro vertex algebras, and restricted modules of nonzero level for the twisted Heisenberg-Virasoro algebra can be treated as weak modules for the Heisenberg-Virasoro vertex algebras. For convenience, in the Heisenberg-Virasoro algebra $\mathcal{E}$, we denote $c_1 = \tilde{c}_1, c_2 = \tilde{c}_2, c_3 = \tilde{c}_3$.

Let $\mathcal{P}$ be the subalgebra of $\mathcal{E}$ spanned by $c_1, c_2, d_m, h_m, (m \in \mathbb{Z}_{\geq 0})$.

Let $(\ell_1, \ell_2) \in \mathbb{C}^2$. Consider the 1-dimensional $\mathcal{P}$-module $\mathbb{C}v_{\ell_1, \ell_2}$ such that $c_1 v_{\ell_1, \ell_2} = \ell_1 v_{\ell_1, \ell_2}, c_2 v_{\ell_1, \ell_2} = \ell_2 v_{\ell_1, \ell_2}, d_m v_{\ell_1, \ell_2} = h_m v_{\ell_1, \ell_2} = 0 \ (m \in \mathbb{Z}_{\geq 0})$.

Let $\mathcal{V}^{\ell_1, \ell_2}$ be the following induced $\mathcal{E}$-module:

$$\mathcal{V}^{\ell_1, \ell_2} = U(\mathcal{E}) \otimes_{U(\mathcal{P})} \mathbb{C}v_{\ell_1, \ell_2}.$$ 

Then $\mathcal{V}^{\ell_1, \ell_2}$ is a highest weight $\mathcal{E}$-module, with the highest weight vector $\mathbf{1}_{\ell_1, \ell_2} = 1 \otimes v_{\ell_1, \ell_2}$.

Define the following fields acting on $\mathcal{V}^{\ell_1, \ell_2}$:

$$h(z) = \sum_{m \in \mathbb{Z}} h_m z^{-m-1}, \quad d(z) = \sum_{m \in \mathbb{Z}} d_m z^{-m-2}.$$ 

- Let $V^{c}_{V^{c}_{\ell}}$ denotes the universal Virasoro vertex algebra of central charge $c$ generated by the Virasoro field $L^{(1)}(z) = \sum_{m \in \mathbb{Z}} L^{(1)}_m z^{-m-2}$ (cf. [35], [23]).
- Let $M(\ell)$ denotes the Heisenberg vertex algebra of level $\ell$, with the vertex operator $Y_{M(\ell)}(\cdot, z)$ which is uniquely generated with the Heisenberg field

$$Y_{M(\ell)}(h(-1)\mathbf{1}, z) = h(z) = \sum_{m \in \mathbb{Z}} h_m z^{-m-1}.$$ 

If $\ell \neq 0$, then $M(\ell)$ is simple and always isomorphic to $M(1)$ (cf. [35]).
• $M(\ell)$ contains a Virasoro vertex subalgebra $V_{\text{Vir}}^{c-1}$ generated by the Virasoro field

$$L^{\text{Heis}}(z) := \frac{1}{2\ell} : h(z)h(z) : = \sum_{n \in \mathbb{Z}} L_n^{\text{Heis}} z^{-n-2}.$$ 

Moreover, we have

$$L_n^{\text{Heis}} = \frac{1}{2\ell} \sum_{k \in \mathbb{Z}} : h_k h_{n-k} :$$

For details see [22], [35].

• $M(\ell)$ contains the Virasoro vector

$$\omega_{\lambda} = \frac{1}{2\ell} (h_{-1}^2 - 1) - \frac{\lambda}{\ell} h_{-2} 1$$

of the central charge $c = 1 - 12\lambda^2 / \ell$. The components of the vertex operator

$$Y_{M(\ell)}(\omega_{\lambda}) = \bar{L}(z) = L^{\text{Heis}}(z) - \frac{\lambda}{\ell} \partial_z h(z) = \sum_{n \in \mathbb{Z}} \bar{L}_n z^{-n-2}$$ (A.1)

represent the operators $\bar{L}_n$ defined directly by (2.2).

• Note that the field $\bar{L}(z)$ and operators $\bar{L}_n$ acts on any weak $M(\ell)$-module, and in particular on any restricted module for the Heisenberg algebra.

Note that $M(\ell_2)$ is naturally a vertex subalgebra of $\mathcal{V}^{\ell_1,\ell_2}$.

**Proposition A.1.** Assume that $\ell_2 \neq 0$. We have the following isomorphism of vertex algebras

$$V_{\text{Vir}}^c \otimes M(\ell_2) \cong \mathcal{V}^{\ell_1,\ell_2},$$

such that $c = \ell_1 - 1$ and

$L^{(1)}(z) \mapsto d(z) - L^{\text{Heis}}(z), \quad h(z) \mapsto h(z).$ In particular, $\mathcal{V}^{\ell_1,\ell_2} \cong V_{\text{Vir}}^c \otimes M(1)$.

Since $M(\ell) \cong M(1)$, without loss of generality we can assume that $\ell = 1$. In what follows we set $\mathcal{V} := V_{\text{Vir}}^c \otimes M(1)$.

The following theorem relates restricted $\bar{D}$-modules as (untwisted) modules for the vertex operator algebra $\mathcal{V}$.

**Theorem A.2.** (cf. [27]) The following statements holds.

(1) Assume that $W$ is a (simple) restricted $\bar{D}$-module of central charge $\ell_1$ and level one. Then $W$ has the unique structure of a weak (simple) $\mathcal{V}^{\ell_1-1}$-module generated by the fields:

$$h(z) = \sum_{r \in \mathbb{Z}} h_r z^{-r-1}, \quad L^{(1)}(z) = d(z) - L^{\text{Heis}}(z) = \sum_{n \in \mathbb{Z}} L_n^{(1)} z^{-n-2}.$$ (2) Assume that $W$ is a weak (simple) $\mathcal{V}^c$-module. Then $W$ has the structure of a (simple) restricted $\bar{D}$-module of level one such that

$$h_n \mapsto h_n, \quad d_n = L_n^{(1)} + L_n^{\text{Heis}}$$
The vertex–algebraic interpretation of the restricted $\mathcal{D}$–modules is via the twisted $\mathcal{V}^c$-modules. There is an automorphism $\theta_1$ of order two of $M(1)$ such that $\theta_1(h) = -h$ (cf. [22]). We extend this automorphism to the automorphism $\theta = \text{Id} \otimes \theta_1$ of $\mathcal{V}^c$.

We have the following theorem.

**Theorem A.3.** The following statements holds.

1. Assume that $W$ is a (simple) restricted $\mathcal{D}$-module of central charge $\ell_1$ and level one. Then $W$ has the unique structure of a weak (simple) $\theta$-twisted $\mathcal{V}^c=\mathcal{V}^c_{\ell_1-1}$-module generated by the fields:

$$h^{tw}(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} h_r z^{-r-1}, L^{(1)}(z) = d(z) - L^{Heis}_{tw}(z),$$

where

$$L^{Heis}_{tw}(z) := \frac{1}{2} : h^{tw}(z)^2 : + \frac{1}{16} z^{-2} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \tag{A.2}$$

2. Assume that $W$ is a weak (simple) $\theta$-twisted $\mathcal{V}^c$-module. Then $W$ has the structure of a (simple) restricted $\mathcal{D}$-module of level one such that $h_r \mapsto h_r$, $d_n = L^{(1)}_n + L_n$.

**Proof.** As in [22] (see also [54], [29]) we see that the field $h^{tw}(z)$ defines on $W$ the unique structure of a $\theta_1$-twisted $M(1)$-module with the Virasoro field (A.2). Then we define $L^{(1)}(z) = d(z) - L^{Heis}_{tw}(z) = \sum_{n \in \mathbb{Z}} L^{(1)}_n z^{-n-2}$. The field $L^{(1)}(z)$ defines on $W$ the structure of a restricted module for the Virasoro algebra of central charge $c = \ell_1 - 1$. Since

$$[L^{(1)}_n, h_r] = 0, \quad \forall n \in \mathbb{Z}, \quad r \in \frac{1}{2} + \mathbb{Z},$$

we have that the action of $L^{(1)}(z)$ commutes with $h^{tw}(z)$. Therefore $W$ is a $\theta$-twisted $\mathcal{V}^c_{Vir}$-module. Since all components of the vertex operators are obtained from the action of $d_n, h_r, n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}$, we see that $W$ is an irreducible $\mathcal{D}$-module if and only if $W$ is an irreducible module for the vertex algebra $\mathcal{V}^c$. This proves the assertion (1).

Assume that $W$ is $\theta$-twisted $\mathcal{V}^c$-module with the vertex operator $Y^{tw}_{W}(v, z), v \in \mathcal{V}^c$. Then the twisted Jacobi identity (cf. [22]) shows that:

- The components of the field $L^{(1)}(z) = Y^{tw}_{W}(L^{(1)}_{-2} 1, z) = \sum_{n \in \mathbb{Z}} L^{(1)}_n z^{-n-2}$ define on $W$ structure of a restricted module for the Virasoro algebra with central charge $c$;

- The components of the field $h^{tw}(z) = Y^{tw}_{W}(h_{-1} 1, z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} h_r z^{-r-1}$ define on $W$ the structure of a restricted module for the Heisenberg algebra of level one.

- The fields $h^{tw}(z)$ and $L^{(1)}(z)$ commute.
The field
\[ L_{tw}^{\text{Heis}}(z) = Y(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \]
is a Virasoro field commuting with \( L^{(1)}(z) \) such that
\[ [L_n, h_r] = -rh_{n+r}, \quad n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}. \]

Define \( d(z) = L^{(1)}(z) + L_{tw}^{\text{Heis}}(z) = \sum_{n \in \mathbb{Z}} d_n z^{-n-2} \). Then the components of the fields \( d(z) \) and \( h^{tw}(z) \) define on \( W \) the structure of a restricted \( \mathfrak{D} \)-module of central charge \( \ell_1 = c + 1 \) and level one.

Arguments for the irreducibility are the same as in (1). This proves the assertion (2). \( \square \)

**Remark A.4.** The simple modules in Examples 7.4-7.5 show that the vertex operator algebra \( V_{\text{Vir}}^c \otimes M(1) \) has simple weak (untwisted and twisted) modules which are not isomorphic to any tensor product \( S_1 \otimes S_2 \) for any simple weak \( V_{\text{Vir}}^c \)-module \( S_1 \) and any simple weak (untwisted and twisted) \( M(1) \)-module \( S_2 \). It would be interesting to find analogs of these non-tensor product modules for other types of vertex operator algebras.

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