Quasi-hom-Lie Algebras, Central Extensions and 2-cocycle-like Identities

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Abstract

This paper begins by introducing the concept of a quasi-hom-Lie algebra, or simply, a qhl-algebra, which is a natural generalization of hom-Lie algebras introduced in a previous paper [14]. Quasi-hom-Lie algebras include also as special cases (color) Lie algebras and superalgebras, and can be seen as deformations of these by homomorphisms, twisting the Jacobi identity and skew-symmetry. The natural realm for these quasi-hom-Lie algebras is as a generalization-deformation of the Witt algebra \(\mathfrak{d}\) of derivations on the Laurent polynomials \(\mathbb{C}[t, t^{-1}]\).

We also develop a theory of central extensions for qhl-algebras which can be used to deform and generalize the Virasoro algebra by centrally extending the deformed Witt type algebras constructed here. In addition, we give a number of other interesting examples of quasi-hom-Lie algebras, among them a deformation of the loop algebra.

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1 Introduction

The classical Witt and Virasoro algebras are ubiquitous in mathematics and theoretical physics, the latter algebra being the unique one-dimensional central extension of the former \[8\,9\,11\,12\,19\]. Considering the origin of the Witt algebra this is not surprising: the Witt algebra \(\mathfrak{d}\) is the infinite-dimensional Lie algebra of complexified polynomial vector fields on the unit circle \(S^1\). It can also be defined as \(\mathfrak{d} = \mathbb{C} \otimes \text{Vect}(S^1) = \oplus_{n \in \mathbb{Z}} \mathbb{C} \cdot d_n\), where \(d_n = -t^{n+1}d/dt\) is a linear basis for \(\mathfrak{d}\), and the Lie product being defined on the generators \(d_n\) as \(\langle d_n, d_m \rangle = (n - m)d_{n+m}\) and extended linearly to the whole \(\mathfrak{d}\). This means in particular that any \(\hat{f} \in \mathfrak{d}\) can be written as \(\hat{f} = f \cdot d/dt\) with \(f \in \mathbb{C}[t,t^{-1}]\), the algebra of Laurent polynomials, and hence \(\mathfrak{d}\) can be viewed as the (complex) Lie algebra of derivations on \(\mathbb{C}[t,t^{-1}]\). When the usual derivation operator is replaced by its difference discretization or deformation, the underlying algebra is also in general deformed, and the description and understanding of the properties of the new algebra becomes a problem of key importance.

To put the present article into the right perspective and to see where we are coming from we briefly recall the constructions from \[14\]. In that paper we considered deformations of \(\mathfrak{d}\) using \(\sigma\)-derivations, i.e. linear maps \(D\) satisfying a generalized Leibniz rule \(D(ab) = Da \cdot b + \sigma(a) \cdot Db\). As we mentioned above the Witt algebra \(\mathfrak{d}\) can be viewed as the Lie algebra of derivations on \(\mathbb{C}[t,t^{-1}]\). This observation was in fact our starting point.
in [14] in constructing deformations of the Witt algebra. Instead of just considering ordinary derivations on $\mathbb{C}[t, t^{-1}]$ we considered $\sigma$-derivations. In fact, we did something even more general as we considered a commutative, associative $\mathbb{C}$-algebra $\mathcal{A}$ with $1_\mathcal{A}$ and a $\sigma$-derivation $\Delta$ on $\mathcal{A}$. Forming the cyclic left $\mathcal{A}$-module $\mathcal{A} \cdot \Delta$, a left submodule of the $\mathcal{A}$-module $\mathcal{D}_\sigma(\mathcal{A})$ of all $\sigma$-derivations on $\mathcal{A}$, we equipped $\mathcal{A} \cdot \Delta$ with a bracket multiplication $\langle \cdot, \cdot \rangle_\sigma$ such that it satisfied skew-symmetry and a generalized Jacobi identity with six terms
\[
\circ_{x,y,z} \left( \langle \sigma(x), \langle y, z \rangle_\sigma \rangle_\sigma + \delta \cdot \langle x, \langle y, z \rangle_\sigma \rangle_\sigma \right) = 0,
\]
where $\circ_{x,y,z}$ denotes cyclic summation with respect to $x, y, z$ and where $\delta \in \mathcal{A}$. In the case when $\mathcal{A}$ is a unique factorization domain we showed that the whole $\mathcal{A}$-module $\mathcal{D}_\sigma(\mathcal{A})$ is cyclic and can thus be generated by a single element $\Delta$. Since $\mathbb{C}[t, t^{-1}]$ is a UFD this result applies in particular to the $\sigma$-derivations on the Laurent polynomials $\mathbb{C}[t, t^{-1}]$, and so we may regard $\mathcal{D}_\sigma(\mathbb{C}[t, t^{-1}])$ as a deformation of $\mathfrak{d} = \mathcal{D}_\text{id}(\mathbb{C}[t, t^{-1}])$. This means in particular that we have a Jacobi-like identity (1.1) on $\mathcal{D}_\sigma(\mathbb{C}[t, t^{-1}]).$

Furthermore, in [14] we concentrated mainly on the case when $\delta \in \mathbb{C} \setminus \{0\}$ and so the Jacobi-like identity (1.1) simplified to the Jacobi-like identity with three terms
\[
\circ_{x,y,z} \left\langle (\zeta + \text{id})(x), \langle y, z \rangle_\zeta \right\rangle_\zeta = 0,
\]
where $\zeta = \frac{1}{\delta} \bar{\sigma}$ is $1/\delta$-scaled version of $\bar{\sigma} : \mathcal{A} \cdot \Delta \to \mathcal{A} \cdot \Delta$, acting on this left module as $\bar{\sigma}(a \cdot \Delta) = \sigma(a) \cdot \Delta$. Motivated by this we called algebras with a three-term deformed Jacobi identity of this form hom-Lie algebras.

Using that any algebra $\mathbb{C}$-endomorphism $\sigma$ on $\mathbb{C}[t, t^{-1}]$ must be on the form $\sigma(t) = q^s t$ for $s \in \mathbb{Z}$ and $q \in \mathbb{C} \setminus \{0\}$ unless $\sigma(1) = 0$ in which case $\sigma = 0$ identically, we obtained a $\mathbb{Z}$-parametric family of deformations which, when $s = 1$, reduces to a $q$-deformation of the Witt algebra becoming $\mathfrak{d}$ when $q = 1$. This deformation is closely related to the $q$-deformations of the Witt algebra introduced and studied in [1, 3, 4, 5, 6, 7, 20, 25, 26, 27]. However, our defining commutation relations in this case look somewhat different, as we obtained them, not from some conditions aiming to resolve specifically the case of $q$-deformations, but rather by choosing $\mathbb{C}[t, t^{-1}]$ as an example of the underlying coefficient algebra and specifying $\sigma$ to be the automorphism $\sigma_q : f(t) \mapsto f(qt)$ in our general construction for $\sigma$-derivations. By simply choosing a different coefficient algebra or basic $\sigma$-derivation one can construct many other analogues and deformations of the Witt algebra. The important feature of our approach is that, as in the non-deformed case, the deformations and analogues of Witt algebra obtained by various choices of the underlying coefficient algebra, of the endomorphism $\sigma$ and of the basic $\sigma$-derivation,
are precisely the natural algebraic structures for the differential and integral type calculi and geometry based on the corresponding classes of generalized derivation and difference type operators.

We remarked in the beginning that the Witt algebra \( \mathfrak{d} \) has a unique (up to isomorphism) one-dimensional central extension, the Virasoro algebra. In [14] we developed, for the class of hom-Lie algebras, a theory of central extensions, providing cohomological type conditions useful for showing the existence of central extensions and for their construction. For natural reasons we required that the central extension of a hom-Lie algebra is also a hom-Lie algebra. In particular, the standard theory of central extensions of Lie algebras becomes a natural special case of the theory for hom-Lie algebras when no non-identity twisting is present. In particular, this implies that in the specific examples of deformation families of Witt and Virasoro type algebras constructed within the framework of [14], the corresponding non-deformed Witt and Virasoro type Lie algebras are included as the algebras corresponding to those specific values of deformation parameters which remove the non-trivial twisting. We rounded up [14], putting the central extension theory to the test applying it for the construction of a hom-Lie algebra central extension of the \( q \)-deformed Witt algebra producing a \( q \)-deformation of Virasoro Lie algebra. For \( q = 1 \) one indeed recovers the usual Virasoro Lie algebra as is expected from our general approach.

A number of examples of deformed algebras constructed in [14] do not satisfy the three-term Jacobi-like identity of hom-Lie algebras, but obey instead twisted six-term Jacobi-like identities of the form (1.1). These examples are recalled for the convenience of the reader among other examples in Section 3. Moreover, there exists also many examples where skew-symmetry is twisted as well. Taking the Jacobi identity (1.1) as a stepping-stone we introduce in this paper a further generalization of hom-Lie algebras by twisting, not only the Jacobi identity, but also the skew-symmetry and the homomorphism \( \sigma \) itself (\( \sigma \) is denoted by \( \alpha \) in this paper). In addition, we let go of the assumption that \( \delta \) (here denoted \( \beta \)) is an element of \( A \) and assume instead that it is an endomorphism on \( A \). We call these algebras quasi-hom-Lie algebras or in short just qhl-algebras, see Definition 1. In this way we obtain a class of algebras which not only includes hom-Lie algebras but also color Lie algebras and other, more exotic types of algebras, all of which can be viewed as a type of deformation of Lie algebras. Note, however, that with these types of deformations we, by design, leave the category of Lie algebras for a larger category, including the Lie case as a subcategory.

The present paper is organized into two clearly distinguishable parts. The first, consisting of Sections 2 and 3 concerns the definition of qhl-algebras and some more or less elaborated examples of such. The second part, Section 4,
is devoted to the (central) extension theory of qhl-algebras. Let us first comment some on the first part. In Section 3 we give, based on observations and results from [14], examples of qhl-algebras generalizations—deformations or analogues of the classical Witt algebra $\mathfrak{d}$, in addition to showing how the notion of a qhl-algebra also encompasses Lie superalgebras and, more generally, color Lie algebras by introducing gradings on the underlying linear space and by suitable choices of deformation maps. We also remark that we can define generalized color Lie algebras by admitting the twists $\alpha$ and $\beta$. As another, new, example of qhl-algebras we offer in subsection 3.2 a deformed loop algebra. Section 4 is devoted to the development of a central extension theory for qhl-algebras generalizing the theory for (color) Lie algebras and hom-Lie algebras as developed in [14]. We give necessary and sufficient conditions for having a central extension and compare these results to the ones given in the existing literature, for example [14] for hom-Lie algebras and [28, 29] for color Lie algebras. As a last example we consider central extensions of deformed loop qhl-algebras in subsection 4.3.

2 Definitions and notations

Throughout this paper we let $\mathbb{k}$ be a field of characteristic zero and let $\mathcal{L}_k(L)$ denote the linear space of $\mathbb{k}$-linear maps (endomorphisms) of the $\mathbb{k}$-linear space $L$.

**Definition 1.** A quasi-hom-Lie algebra (qhl-algebra, for short) is a tuple $(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega)$ where

- $L$ is a $\mathbb{k}$-linear space,
- $\langle \cdot, \cdot \rangle_L : L \times L \to L$ is a bilinear map called a product or a bracket in $L$,
- $\alpha, \beta : L \to L$, are linear maps,
- $\omega : D_\omega \to \mathcal{L}_k(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$,

such that the following conditions hold:

- ($\beta$-twisting.) The map $\alpha$ is a $\beta$-twisted algebra homomorphism, i.e.
  $$\langle \alpha(x), \alpha(y) \rangle_L = \beta \circ \alpha \langle x, y \rangle_L,$$
  for all $x, y \in L$;

- ($\omega$-symmetry.) The product satisfies a generalized skew-symmetry condition
  $$\langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L,$$
  for all $(x, y) \in D_\omega$;
• (qhl-Jacobi identity.) The bracket satisfies a generalized Jacobi identity
\[ \circ_{x,y,z} \left\{ \omega(z, x) \left( \langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta(x, \langle y, z \rangle_L) \right) \right\} = 0, \]
for all \((z, x), (x, y), (y, z) \in D_\omega.\]

**Remark 1.** Note that if \(\alpha = \text{id}_L\) then, necessarily, \(\beta = \text{id}|_{\langle L, L \rangle}\) restricted to the "commutator ideal" \(\langle L, L \rangle \subseteq L.\)

**Remark 2.** To avoid writing all the maps \(\langle \cdot, \cdot \rangle_L, \alpha_L, \beta_L, \omega_L\) every time we use two abbreviations for a qhl-algebra \((L, \langle \cdot, \cdot \rangle_L, \alpha_L, \beta_L, \omega_L)\):

- \((L, \text{Mor}(L))\), "Mor" for "morphism", or simply
- \(L\), remembering that there are also maps present, implicitly, in the notation.

By a strong quasi-hom-Lie algebra morphism (or simply a quasi-hom-Lie algebra morphism)
\[ \phi : (L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega) \to (L', \langle \cdot, \cdot \rangle_{L'}, \alpha', \beta', \omega') \]
we mean a linear map from \(L\) to \(L'\) such that the following conditions hold for \(x, y \in L:\)

**M1.** \(\phi(\langle x, y \rangle_L) = \langle \phi(x), \phi(y) \rangle_{L'}\),

**M2.** \(\phi \circ \alpha = \alpha' \circ \phi\),

**M3.** \(\phi \circ \beta = \beta' \circ \phi\).

Conditions M2 and M3 are often referred to as "intertwining conditions".

**Lemma 1.** Let \(\phi : L \to L'\) be a map satisfying condition M1. Then
\[ \omega_{L'}(\phi(x), \phi(y)) \circ \phi = \phi \circ \omega_L(x, y) \]
on \(\langle L, L \rangle_L\) if \((x, y) \in D_\omega\) and \((\phi(x), \phi(y)) \in D_{\omega_{L'}}.\)

**Proof.** On the one hand
\[ \phi(x, y)_L = \langle \phi(x), \phi(y) \rangle_L = \omega(\phi(x), \phi(y)) \langle \phi(y), \phi(x) \rangle_L = \omega(\phi(x), \phi(y)) \circ \phi \langle y, x \rangle_L, \]
and on the other,
\[ \phi(x, y)_L = \phi \circ \omega(x, y) \langle y, x \rangle_L. \]

Comparison proves the lemma.
A weak quasi-hom-Lie algebra morphism is a linear map $L \to L'$ such that just condition M1 holds. If the weak (strong) morphism $\varphi$ is a monomorphism and epimorphism it is a weak (strong) isomorphism.

**Definition 2.** Two qhl-algebras $L$ and $L'$ are said to be of $\varphi$-related $\omega$-type if there is a weak morphism $\varphi : L \to L'$ such that

$$\omega_{L'}(\varphi(x), \varphi(y)) \circ \varphi = \varphi \circ \omega_L(x, y)$$

for all $x, y \in L$.

**Definition 3.** By an exact sequence of qhl-algebras $\{(L_i, \text{Mor}(L_i))\}_{i \in \mathbb{Z}}$, with all $L_{i-1}$ and $L_i$ of the same $\varphi$-related $\omega$-type for each $i$, we mean a commutative diagram with strong morphisms $\varphi_i$

$$\cdots \longrightarrow L_{i-1} \xrightarrow{\varphi_i} L_i \xrightarrow{\varphi_{i+1}} L_{i+1} \longrightarrow \cdots$$

such that the rows are exact, that is, $\text{im}(\varphi_i) = \ker(\varphi_{i+1})$.

**Definition 4.** A short exact sequence of qhl-algebras $L_i$ is a commutative diagram as (2.1) with all algebras but three consecutive ones zero, i.e. a diagram

$$\cdots \longrightarrow a \xrightarrow{\iota} E \xrightarrow{\text{pr}} L \longrightarrow 0$$

with commutative "boxes" and exact rows where $\iota$ and $\text{pr}$ are strong morphisms.

**Remark 3.** Note that this means, in particular, that

- $\iota \circ \omega_a(a, b) = \omega_E(\iota(a), \iota(b)) \circ \iota$.
- $\text{pr} \circ \omega_E(e, e') = \omega_L(\text{pr}(e), \text{pr}(e')) \circ \text{pr}$.

**Definition 5.** A short exact sequence as (2.2) is a quasi-hom-Lie algebra extension of $L$ by $a$, or by a slight abuse of language, we say that $E$ is an extension of $L$ by $a$. 

\[\begin{array}{c}
\end{array}\]
3 Examples

Example 1. By taking \( \beta \) to be the identity \( \text{id}_L \) and \( \omega = -\text{id}_L \) we get the hom-Lie algebras discussed in a previous paper \([14]\). We recall the definition for the reader’s convenience. A hom-Lie algebra is a non-associative algebra \( L \) with bracket multiplication \( \langle \cdot, \cdot \rangle_\alpha \) with \( \alpha : L \to L \) an algebra endomorphism, such that

- \( \langle x, y \rangle_\alpha = -\langle y, x \rangle_\alpha \), (Skew-symmetry)
- \( \bigcirc_{x,y,z} \left( \langle \text{id}_L + \alpha)(x), \langle y, z \rangle_\alpha \right) = 0 \) (\( \alpha \)-deformed Jacobi identity).

Restricting further we get a Lie algebra by taking \( \alpha \) also equal to the identity \( \text{id}_L \).

\[
\Gamma \text{ be an abelian group. Then a } \Gamma\text{-graded linear space is a linear space } V \text{ with a direct sum decomposition into subspaces labeled by } \Gamma, \text{ i.e.}
\]
\[
V = \bigoplus_{\gamma \in \Gamma} V_\gamma.
\]

This means that any element in \( V \) can be written uniquely as a finite sum \( v = \sum_{\gamma \in \Gamma} v_\gamma \). The elements \( v_\gamma \in V_\gamma \) are called homogeneous of degree \( \gamma \).

A \( \Gamma \)-graded algebra is a \( \Gamma \)-graded linear space with bilinear multiplication \( \ast \) respecting the grading in the following sense

\[
V_{\gamma_1} \ast V_{\gamma_2} \subseteq V_{\gamma_1 + \gamma_2}.
\]

Example 2. A Lie superalgebra is a \( \mathbb{Z}_2 \)-graded (\( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \)) algebra \( L \) decomposing as \( L = L_0 \oplus L_1 \) with a bilinear product \( \langle \cdot, \cdot \rangle \) satisfying

- \( \langle L_{\gamma_1}, L_{\gamma_2} \rangle \subseteq L_{\gamma_1 + \gamma_2} \)
- \( \langle x, y \rangle = -(-1)^{\gamma_x \gamma_y} \langle y, x \rangle \) for \( x \in L_{\gamma_x} \) and \( y \in L_{\gamma_y} \), and
- \( -(-1)^{\gamma_x \gamma_z} \langle x, \langle y, z \rangle \rangle + (-1)^{\gamma_y \gamma_z} \langle y, \langle z, x \rangle \rangle + (-1)^{\gamma_x \gamma_y} \langle z, \langle x, y \rangle \rangle = 0 \), for \( x \in L_{\gamma_x}, y \in L_{\gamma_y} \) and \( z \in L_{\gamma_z} \).

Note that \( L_0 \) is a subalgebra of \( L \) but that \( L_1 \) is not. \( L_0 \) is the even component of \( L \) and \( L_1 \) the odd. Lie superalgebras are qhl-algebras. This can be easily seen letting \( L \) be \( \mathbb{Z}_2 \)-decomposed in the definition of the qhl-algebras as \( L = L_0 \oplus L_1 \), \( \alpha = \beta = \text{id}_L \) and \( \omega(x, y)v = \omega(\gamma_x, \gamma_y)v = -(-1)^{\gamma_x \gamma_y}v \) for \( v \in L \) and where \( (x, y) \in D_\omega = L_0 \cup L_1 \) and \( \gamma_x, \gamma_y \in \mathbb{Z}_2 \) are the graded degrees (or parity) of \( x \) and \( y \) respectively. \(\square\)
Example 3. The super Lie algebra example is covered by the more general notion of a color Lie algebra or Γ-graded ε-Lie algebra. In this case Γ is any abelian group and the color Lie algebra $L$ with bracket $\langle \cdot, \cdot \rangle$ decomposes as $L = \bigoplus_{\gamma \in \Gamma} L_{\gamma}$ where $\langle L_{\gamma_1}, L_{\gamma_2} \rangle \subseteq L_{\gamma_1 + \gamma_2}$, for $\gamma_1, \gamma_2 \in \Gamma$. In addition, the "color structure" includes a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{k}$, called a commutation factor, satisfying

- $\varepsilon(\gamma x, \gamma y) \varepsilon(\gamma y, \gamma x) = 1$,
- $\varepsilon(\gamma x + \gamma y, \gamma z) = \varepsilon(\gamma x, \gamma z) \varepsilon(\gamma y, \gamma z)$,
- $\varepsilon(\gamma x, \gamma y + \gamma z) = \varepsilon(\gamma x, \gamma y) \varepsilon(\gamma x, \gamma z)$,

for $\gamma_x, \gamma_y, \gamma_z \in \Gamma$. The color skew-symmetry and Jacobi condition are now stated, with the aid of $\varepsilon$, as

- $\langle x, y \rangle = -\varepsilon(\gamma x, \gamma y) \langle y, x \rangle$
- $\varepsilon(\gamma x, \gamma z) \langle x, \langle y, z \rangle \rangle + \varepsilon(\gamma x, \gamma y) \langle y, \langle z, x \rangle \rangle + \varepsilon(\gamma y, \gamma z) \langle z, \langle x, y \rangle \rangle = 0$

for $x \in L_{\gamma x}, y \in L_{\gamma y}$ and $z \in L_{\gamma z}$. Color Lie algebras are examples of qhl-algebras. This can be seen by putting a grading on $L$ in the definition of qhl-algebras $L = \bigoplus_{\gamma \in \Gamma} L_{\gamma}$, $\alpha = \beta = \text{id}_L$ and $\omega(x, y)v = -\varepsilon(\gamma x, \gamma y)v$, for $v \in L$ and where $(x, y) \in D_\omega = \bigcup_{\gamma \in \Gamma} L_{\gamma}$ and $\gamma_x, \gamma_y \in \Gamma$ are the graded degrees of $x$ and $y$. The $\omega$-symmetry and the qhl-Jacobi identity gives us the respective identities in the definition of a color Lie algebra.

Remark 4. Since $\alpha_L = \beta_L = \text{id}_L$ for (color) Lie algebras, there is only one notion of morphism in this case, namely the usual (color) Lie algebra homomorphism.

Remark 5. By not restricting $\alpha_L$ to be the identity in Example 3 we can define color hom-Lie algebras, and similarly, with $\beta_L \neq \text{id}_L$, color qhl-algebras.

In the next section we summarize several results from [14] providing non-trivial examples of qhl-algebras. For proofs and more details we refer to [14].

3.1 $\sigma$-derivations

In this section, we let $\mathcal{A}$ denote a commutative, associative $\mathbb{k}$-algebra with unity, and let $\mathcal{D}_\sigma(\mathcal{A})$ denote the set of $\sigma$-derivations on $\mathcal{A}$, that is the set of all $\mathbb{k}$-linear maps $D : \mathcal{A} \to \mathcal{A}$ satisfying the $\sigma$-Leibniz rule

$$D(ab) = D(a)b + \sigma(a)D(b).$$
We now fix a homomorphism $\sigma : \mathcal{A} \to \mathcal{A}$, an element $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$, and an element $\delta \in \mathcal{A}$, and we assume that these objects satisfy the following two conditions:

\[(3.1) \quad \sigma(\text{Ann}(\Delta)) \subseteq \text{Ann}(\Delta),\]

\[(3.2) \quad \Delta(\sigma(a)) = \delta \sigma(\Delta(a)), \quad \text{for } a \in \mathcal{A},\]

where $\text{Ann}(\Delta) = \{a \in \mathcal{A} \mid a \cdot \Delta = 0\}$. Let

$$\mathcal{A} \cdot \Delta = \{a \cdot \Delta \mid a \in \mathcal{A}\}$$

denote the cyclic $\mathcal{A}$-submodule of $\mathcal{D}_\sigma(\mathcal{A})$ generated by $\Delta$ and extend $\sigma$ to $\mathcal{A} \cdot \Delta$ by $\sigma(a \cdot \Delta) = \sigma(a) \cdot \Delta$. We have the following theorem, which introduces a $k$-algebra structure on $\mathcal{A} \cdot \Delta$.

**Theorem 2.** (cf. [14]) If (3.1) holds then the map

$$\langle \cdot, \cdot \rangle_\sigma : \mathcal{A} \cdot \Delta \times \mathcal{A} \cdot \Delta \to \mathcal{A} \cdot \Delta$$

defined by setting

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a) \cdot \Delta) \circ (b \cdot \Delta) - (\sigma(b) \cdot \Delta) \circ (a \cdot \Delta), \quad \text{for } a, b \in \mathcal{A},$$

(3.3)

where $\circ$ denotes elementwise composition, is a well-defined $k$-algebra product on the $k$-linear space $\mathcal{A} \cdot \Delta$, and it satisfies the following identities:

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a)) \cdot \Delta,$$

(3.4)

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = -\langle b \cdot \Delta, a \cdot \Delta \rangle_\sigma,$$

(3.5)

for $a, b, c \in \mathcal{A}$. In addition, if (3.2) holds, then

$$\odot_{a,b,c} \left( \langle \sigma(a) \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \rangle_\sigma + \delta \cdot \langle a \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \rangle_\sigma \right) = 0.$$  

(3.6)

The algebra $\mathcal{A} \cdot \Delta$ from the theorem is then a qhl-algebra with $\alpha = \sigma$, $\beta = \delta$ and $\omega = -1$.

**Remark 6.** Let $\Delta$ be a non-empty family of commuting $\sigma$-derivations on $\mathcal{A}$ closed under composition of maps. Then $\Delta$ generates a left $\mathcal{A}$-module $\mathcal{A} \otimes \Delta$.
via the rule \(b(a \otimes d) = (ba) \otimes d\), where \(a, b \in \mathcal{A}\) and \(d \in \Delta\). We extend any \(d \in \Delta\) from \(\mathcal{A}\) to

\[
\mathcal{A} \otimes \Delta = \left\{ \sum_{\text{finite}} a \otimes d \mid a \in \mathcal{A}, d \in \Delta \right\}
\]

by the rule

\[
d(a \otimes d') = d(a) \otimes d' + \sigma(a) \otimes dd',
\]

where \(dd'\) denotes (associative) composition \(dd'(a) = d(d'(a))\). For \(a \in \mathcal{A}\) and \(d \in \Delta\) we can identify \(a \otimes d\) and \(ad\) as operators on \(\mathcal{A}\) by \(a \otimes d(r) = a(d(r))\) for \(r \in \mathcal{A}\). Define a product on monomials of \(\mathcal{A} \otimes \Delta\) by

\[
\langle a \otimes d_1, b \otimes d_2 \rangle = \sigma(a) \otimes d_1(b \otimes d_2) - \sigma(b) \otimes d_2(a \otimes d_1),
\]

and extend linearly to the whole \(\mathcal{A} \otimes \Delta\). Then a simple calculation using the commutativity of \(\mathcal{A}\) and \(\Delta\) shows that

\[
\langle a \otimes d_1, b \otimes d_2 \rangle = (\sigma(a)d_1(b)) \otimes d_2 - (\sigma(b)d_2(a)) \otimes d_1.
\]

Skew-symmetry also follows from this. Note also that if \(d_1, d_2 \in \Delta\) then \(d_1 - d_2 \in \mathcal{D}_\sigma(\mathcal{A})\). If \(\Delta\) is maximal with respect to being commutative then \(d_1 - d_2 \in \Delta\). We see that part of the above theorem generalizes to a setting with multiple \(\sigma\)-derivations. However, if there is a nice Jacobi-like identity as in the theorem is uncertain at this moment. The above construction parallels the one given in \([24]\) with the difference that \([24]\) considers the construction in a color Lie algebra setting.

Under the assumption that \(\mathcal{A}\) is a unique factorization domain there exists \(\Delta \in \mathcal{D}_\sigma(\mathcal{A})\) such that \(\mathcal{A} \cdot \Delta = \mathcal{D}_\sigma(\mathcal{A})\). This means that \(\mathcal{D}_\sigma(\mathcal{A})\) affords a qhl-algebra structure under the above assumptions on \(\mathcal{A}\). For more details see \([14]\).

We now apply these ideas for the construction of some explicit examples.

### 3.1.1 A q-deformed Witt algebra

Let \(\mathcal{A}\) be the unique factorization domain \(\mathbb{k}[t, t^{-1}]\), the algebra of Laurent polynomials in \(t\) over the field \(\mathbb{k}\). Then \(\mathcal{D}_\sigma(\mathcal{A})\) can be generated by a single element \(D\) as a left \(\mathcal{A}\)-module, that is \(\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A} \cdot D\). When \(\sigma(t) = qt\) and \(q \neq 1, 0\), one can take

\[
D = \frac{id - \sigma}{1 - q} : f(t) \mapsto \frac{f(t) - f(qt)}{1 - q} = \frac{f(qt) - f(t)}{q - 1}.
\]

Then \([3.1]\) and \([3.2]\) are satisfied with \(\delta = 1\), and Theorem \([2]\) yields the following qhl-algebra structure on \(\mathcal{D}_\sigma(\mathcal{A})\). This is, in fact, an example of a qhl-algebra which is also a hom-Lie algebra (see Example 1).
Theorem 3. (cf. [14]) Let $\mathcal{A} = \mathbb{k}[t, t^{-1}]$. Then the $\mathbb{k}$-linear space

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{k} \cdot d_n,$$

with $d_n = -t^n D$ can be equipped with the bracket multiplication

$$\langle \cdot, \cdot \rangle : \mathfrak{D}_\sigma(\mathcal{A}) \times \mathfrak{D}_\sigma(\mathcal{A}) \rightarrow \mathfrak{D}_\sigma(\mathcal{A})$$

defined on generators by (3.3) as

$$\langle d_n, d_m \rangle_\sigma = q^n d_n d_m - q^m d_m d_n$$

with commutation relations

$$\langle d_n, d_m \rangle_\sigma = (\{n\} - \{m\}) d_{n+m}.$$

This bracket satisfies skew-symmetry

$$\langle d_n, d_m \rangle_\sigma = -\langle d_m, d_n \rangle_\sigma,$$

and a $\sigma$-deformed Jacobi-identity

$$(q^n + 1)\langle d_n, \langle d_l, d_m \rangle \rangle_\sigma + (q^l + 1)\langle d_l, \langle d_m, d_n \rangle \rangle_\sigma +$$

$$+ (q^m + 1)\langle d_m, \langle d_n, d_l \rangle \rangle_\sigma = 0.$$

Remark 7. The linear space $\mathfrak{D}_\sigma(\mathcal{A})$ is a hom-Lie algebra with bilinear bracket defined in Theorem 3 and $\alpha : \mathfrak{D}_\sigma(\mathcal{A}) \rightarrow \mathfrak{D}_\sigma(\mathcal{A})$ given by

$$\alpha(d_n) = \alpha(-t^n D) = \sigma(-t^n) D = -q^n t^n D = q^n d_n.$$

3.1.2 Non-linearly deformed Witt algebras

The most general non-zero endomorphism $\sigma$ on $\mathbb{k}[t, t^{-1}]$ is one on the form $\sigma(t) = q t^s$ for $s \in \mathbb{Z}$ and $q \in \mathbb{k} \setminus \{0\}$. With this $\sigma$, the left $\mathcal{A}$-module $\mathfrak{D}(\mathcal{A})$ is generated by a single element

$$D = \eta t^{-k+1} \frac{id - \sigma}{t - qt^s}, \quad \eta \in \mathbb{k}.$$

The element $\delta$ for this $D$ such that (3.2) holds is

$$\delta = q^k t^{k(1-s)} \sum_{r=0}^{s-1} (qt^{s-1})^r.$$

Equation (3.1) is clearly still valid. We then get, using Theorem 3, the following qhl-algebra structure on $\mathfrak{D}_\sigma(\mathcal{A})$. 

Theorem 4. (cf. [11]) Let $\mathcal{A} = \mathbb{k}[t, t^{-1}]$. Then the $\mathbb{k}$-linear space

$$\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{k} \cdot d_n,$$

with $d_n = -t^n D$, can be equipped with the bracket product

$$\langle \cdot, \cdot \rangle_\sigma : \mathcal{D}_\sigma(\mathcal{A}) \times \mathcal{D}_\sigma(\mathcal{A}) \to \mathcal{D}_\sigma(\mathcal{A})$$

defined on generators by (3.3) as

$$\langle d_n, d_m \rangle_\sigma = q^n d_n s d_m - q^m d_m s d_n$$

and satisfying defining commutation relations

$$\langle d_n, d_m \rangle_\sigma = \eta \text{sign}(n - m) \sum_{l = \min(n,m)}^{\max(n,m) - 1} q^{n+m-l} d_{(s(s-1)) l + n + m - k}$$

for $n, m \geq 0$;

$$\langle d_n, d_m \rangle_\sigma = \eta \left( \sum_{l=0}^{-m-1} q^{n+m+l} d_{(s(s-1)) l + n + m - k} + \sum_{l=0}^{-n-1} q^{m+n+l} d_{(s(s-1)) l + n + m - k} \right)$$

for $n \geq 0, m < 0$;

$$\langle d_n, d_m \rangle_\sigma = \eta \left( \sum_{l_1=0}^{m-1} q^{n+l_1} d_{(s(s-1)) l_1 + n + m - k} + \sum_{l_2=0}^{-n-1} q^{m+n+l_2} d_{(s(s-1)) l_2 + n + m - k} \right)$$

for $m \geq 0, n < 0$;

$$\langle d_n, d_m \rangle_\sigma = \eta \text{sign}(n - m) \sum_{l = \min(-n,-m)}^{\max(-n,-m) - 1} q^{n+m+l} d_{(s(s-1)) l + n + m - k}$$

for $n, m < 0$.

Furthermore, this bracket satisfies skew-symmetry

$$\langle d_n, d_m \rangle_\sigma = -\langle d_m, d_n \rangle_\sigma,$$

and a $\sigma$-deformed Jacobi-identity

$$\bigtriangleup_{n,m,l} \left( q^n \langle d_n s, \langle d_m, d_l \rangle_\sigma \rangle_\sigma + q^k t^k(s-1) \sum_{r=0}^{s-1} (q^r t^r)^r \langle d_n, \langle d_m, d_l \rangle_\sigma \rangle_\sigma \right) = 0.$$
3.1.3 A generalization to several variables

As shown in [14], by not restricting the denominator in $D \in D_\sigma(A)$ to greatest common divisors (on a certain subset of $A$), but instead considering only common divisors on this subset, we get, not the whole $A$-module $D_\sigma(A)$, but a proper submodule $\mathcal{M}$. Theorem 2 can now be used to endow this submodule with the structure of a $qhl$-algebra. For this, we need not restrict our attention to one-variable Laurent polynomials.

Theorem 5. (cf. [14]) Let $A = \mathbb{k}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. The $\mathbb{k}$-linear space

$$\mathcal{M} = \bigoplus_{l \in \mathbb{Z}^n} \mathbb{k} \cdot d_l$$

spanned by $d_l = -z_1^{l_1} \cdots z_n^{l_n} D$, where $D$ is given by

$$D = Q \frac{id - \sigma}{z_1^{G_1} \cdots z_n^{G_n}},$$

with $Q \in \mathbb{k} \setminus \{0\}$ and $(G_1, \ldots, G_n) \in \mathbb{Z}^n$, can be endowed with a bracket defined on generators (by (3.3)) as

$$\langle d_k, d_l \rangle_\sigma = q_{z_1}^{k_1} \cdots q_{z_n}^{k_n} d_{\alpha_1(k) + \cdots + \alpha_n(k)} d_l - q_{z_1}^{l_1} \cdots q_{z_n}^{l_n} d_{\alpha_1(l) + \cdots + \alpha_n(l)} d_k$$

and satisfying relations

$$\langle d_k, d_l \rangle_\sigma = Q q_{z_1}^{l_1} \cdots q_{z_n}^{l_n} d_{\alpha_1(l) + \cdots + \alpha_n(l) + k_1 - G_1 - \cdots - k_n - G_n} - Q q_{z_1}^{k_1} \cdots q_{z_n}^{k_n} d_{\alpha_1(k) + \cdots + \alpha_n(k) + l_1 - G_1 - \cdots - l_n - G_n}.$$

The bracket satisfies the $\sigma$-deformed Jacobi-identity

$$\bigcirc_{k,l,h} \left( q_{z_1}^{k_1} \cdots q_{z_n}^{k_n} \langle d_{\alpha_1(k) + \cdots + \alpha_n(k)}, d_l, d_h \rangle_\sigma \bigg)_{\sigma} + q_{z_1}^{G_1} \cdots q_{z_n}^{G_n} \langle d_k, d_l, d_h \rangle_\sigma = 0.$$

3.2 A $(\alpha, \beta, \omega)$-deformed loop algebra

Loop algebras are special cases of Lie algebras known as current algebras. Given a topological space $X$ and a Lie algebra $\mathfrak{g}$ the current algebra with respect to $X$ and $\mathfrak{g}$ consists of maps $X \to \mathfrak{g}$ with (Lie) bracket multiplication

$$[f, g](x) = [f(x), g(x)]_\mathfrak{g}, \quad \text{for } x \in X.$$  

In the classical definition of current algebras it is usually required that $X$ is a smooth manifold, $\mathfrak{g}$ is a finite-dimensional Lie algebra, and all maps
belonging to the current algebra are smooth as well [13]. These smoothness assumptions and even the assumption that \(X\) is a topological space and not just a set, are not important for the algebraic constructions in this article, since we do not discuss the topologies on the current algebra. It is enough for us that the set of maps comprising the algebra is a linear space closed under the commutator bracket multiplication (3.7). Taking \(X\) to be the unit circle \(S^1\), and restricting to (trigonometric) polynomials we get the loop algebra of \(g\), oftentimes denoted by \(\hat{g}\). It is not difficult to see that \(\hat{g} = g \otimes \mathbb{C}[t, t^{-1}]\) with bilinear multiplication determined by

\[
[f \otimes t^n, g \otimes t^m]_{\hat{g}} = [f, g]_g \otimes t^{n+m}.
\]

Loop algebras are important in physics, especially in conformal field theories and superstring theory [8, 9, 11, 12].

Let \(g\) be a qhl-algebra \((g, \langle \cdot, \cdot \rangle_g, \alpha_g, \beta_g, \omega_g)\). Assume \(\omega_g\) is linear over the tensor product \(\otimes := \otimes_k\) and form the vector space

\[
\hat{g} \overset{\text{def}}{=} g \otimes k[t, t^{-1}].
\]

This means that \(\hat{g}\) can be considered as the algebra of Laurent polynomials with coefficients in the qhl-algebra \(g\). Define further

\[
\begin{align*}
\alpha_{\hat{g}} &\overset{\text{def}}{=} \alpha_g \otimes \text{id} \\
\beta_{\hat{g}} &\overset{\text{def}}{=} \beta_g \otimes \text{id} \\
\omega_{\hat{g}} &\overset{\text{def}}{=} \omega_g \otimes \text{id}.
\end{align*}
\]

Furthermore, define a product on \(\hat{g}\) as

\[
\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{g}} = \langle x, y \rangle_g \otimes t^{n+m}.
\]

**Theorem 6.** The vector space \(\hat{g}\) admits the structure of a qhl-algebra \((\hat{g}, \langle \cdot, \cdot \rangle_{\hat{g}}, \alpha_{\hat{g}}, \beta_{\hat{g}}, \omega_{\hat{g}})\).

**Proof.** The proof consists of a checking the axioms from Definition 1 of qhl-algebras.

1. The \(\omega_{\hat{g}}\)-skew symmetry is checked as follows:

\[
\begin{align*}
\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{g}} &= \langle x, y \rangle_g \otimes t^{n+m} = (\omega_g(x, y) \langle y, x \rangle_g) \otimes t^{n+m} = \\
&= (\omega_g(x, y) \otimes \text{id})(\langle y, x \rangle_g \otimes t^{n+m}) = \\
&= \omega_{\hat{g}}(x, y)\langle y \otimes t^m, x \otimes t^n \rangle_{\hat{g}}.
\end{align*}
\]
2. Next we prove the $\beta_\mathfrak{g}$-twisting of $\alpha_\mathfrak{g}$. First $\alpha_\mathfrak{g}(x \otimes t^n) = \alpha_\mathfrak{g}(x) \otimes t^n$, and so

$$\langle \alpha_\mathfrak{g}(x \otimes t^n), \alpha_\mathfrak{g}(y \otimes t^m) \rangle_\mathfrak{g} = \langle \alpha_\mathfrak{g}(x) \otimes t^n, \alpha_\mathfrak{g}(y) \otimes t^m \rangle_\mathfrak{g} =$$

$$= \langle \alpha_\mathfrak{g}(x), \alpha_\mathfrak{g}(y) \rangle_\mathfrak{g} \otimes t^{n+m} =$$

$$= (\beta_\mathfrak{g} \circ \alpha_\mathfrak{g}(x, y)) \otimes t^{n+m} =$$

$$= \beta_\mathfrak{g} \circ (\alpha_\mathfrak{g}(x, y) \otimes t^{n+m}) =$$

$$= \beta_\mathfrak{g} \circ \alpha_\mathfrak{g}((x, y) \otimes t^{n+m}).$$

3. The qhl-Jacobi identity lastly, is as follows. The left hand side is

$$\circ \omega_\mathfrak{g}(z \otimes t^l, x \otimes t^n) \left( \langle \alpha_\mathfrak{g}(x \otimes t^n), \langle y \otimes t^m, z \otimes t^l \rangle_\mathfrak{g} \rangle_\mathfrak{g} +$$

$$+ \beta_\mathfrak{g} \langle x \otimes t^n, \langle y \otimes t^m, z \otimes t^l \rangle_\mathfrak{g} \rangle_\mathfrak{g} \right),$$

where the notation $\circ$ here is used for cyclic summation with respect to $x \otimes t^n, y \otimes t^m, z \otimes t^l$. The first term in the parantheses is

$$\langle \alpha_\mathfrak{g}(x \otimes t^n), \langle y \otimes t^m, z \otimes t^l \rangle_\mathfrak{g} \rangle_\mathfrak{g} = \langle \alpha_\mathfrak{g}(x) \otimes t^n, \langle y \otimes t^m, z \otimes t^l \rangle_\mathfrak{g} \rangle_\mathfrak{g} =$$

$$= \langle \alpha_\mathfrak{g}(x), \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} \otimes t^{n+m+l}$$

and the second

$$\beta_\mathfrak{g} \langle x \otimes t^n, \langle y \otimes t^m, z \otimes t^l \rangle_\mathfrak{g} \rangle_\mathfrak{g} = \beta_\mathfrak{g} \langle \langle x, \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} \rangle_\mathfrak{g} \otimes t^{n+m+l} =$$

$$= (\beta_\mathfrak{g} \langle \langle x, \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} \rangle_\mathfrak{g} \otimes t^{n+m+l}).$$

Adding up these and using that $\omega_\mathfrak{g}(z \otimes t^l, x \otimes t^n) = \omega_\mathfrak{g}(z, x) \otimes \text{id}$, we get

$$\circ (\omega_\mathfrak{g}(z, x) \otimes \text{id}) \left( \langle \alpha_\mathfrak{g}(x), \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} + \beta_\mathfrak{g} \langle x, \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} \right) \otimes t^{n+m+l} =$$

$$= \left( \circ_{x, y, z} \omega_\mathfrak{g}(z, x) \left( \langle \alpha_\mathfrak{g}(x), \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} + \beta_\mathfrak{g} \langle x, \langle y, z \rangle_\mathfrak{g} \rangle_\mathfrak{g} \right) \otimes t^{n+m+l}$$

and the parantheses is zero since $\mathfrak{g}$ is a qhl-algebra.

$$\square$$

4 Extensions

Throughout this section we use that exact sequences of linear spaces

$$0 \longrightarrow a \overset{i}{\longrightarrow} E \overset{\text{pr}}{\longrightarrow} L \longrightarrow 0 \quad (4.1)$$
$split$ in the sense that there is a $k$-linear map $s : L \to E$ called a $section$ such that $\text{pr} \circ s = \text{id}_L$. Note that the condition $\text{pr} \circ s = \text{id}_L$ means that the $k$-linear section $s$ is one-to-one and so $L \cong s(L)$ (as linear spaces). This, together with the exactness, lets us deduce that $E \cong s(L) \oplus \iota(a)$ as linear spaces. Hence a basis of $E$ can be chosen such that any $e \in E$ can be decomposed as $e = s(l) + \iota(a)$ for $a \in a$ and $l \in L$, that is, we consider $\iota(a)$ and $s(L)$ as subspaces of $E$.

To return to extensions, let us from now on assume that $L, a$ and $E$ from (4.1) are qhl-algebras, and that we have a section $s : L \to E$ such that $\omega_E$ and $\omega_L$ are intertwined with $s$, that is

\[ \omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ s = s \circ \omega_L(x, y), \quad (4.2) \]

if $(x, y) \in D_{\omega_L}$ and $(s(x) + \iota(a), s(y) + \iota(b)) \in D_{\omega_E}$. In particular,

\[ \omega_E(s(x), s(y)) \circ s = s \circ \omega_L(x, y) \]

if $a$ and $b$ are taken to be zero. By definition of a section $\text{pr} \circ s = \text{id}_L$, and since $\text{pr}$ is a homomorphism of algebras we have

\[ 0 = \text{pr} \circ (\langle s(x), s(y) \rangle_E - s(x, y)_L) \]

which gives that

\[ \langle s(x), s(y) \rangle_E = s(x, y)_L + \iota \circ g(x, y), \quad (4.3) \]

where $g : L \times L \to a$ is a 2-cocycle-like $k$-bilinear map which depends in a crucial way on the section $s$. Thus $g$ is a measure of the deviation of $s$ from being a weak qhl-algebra morphism. Furthermore, $g$ has to satisfy a generalized skew-symmetry condition on $D_{\omega_L}$

\[ \iota \circ g(x, y) = \langle s(x), s(y) \rangle_E - s(x, y)_L = \]

\[ = \omega_E(s(x), s(y)) \circ \langle s(y), s(x) \rangle_E - s \circ \omega_L(x, y) \circ \langle y, x \rangle_L = \]

\[ = \omega_E(s(x), s(y)) \circ (\langle s(y), s(x) \rangle_E - s(y, x)_L) = \]

\[ = \omega_E(s(x), s(y)) \circ \iota \circ g(y, x), \quad (4.4) \]

for $(x, y) \in D_{\omega_L}$ such that $(s(x), s(y)) \in D_{\omega_E}$. Here we have used that $s$ intertwines $\omega_E$ and $\omega_L$.

**Definition 6.** Denote the set of all maps $L \times L \to a$ satisfying (4.4) by $\text{Alt}^2_{\omega}(L, a; \mathcal{U})$, the $\omega$-alternating $k$-bilinear mappings associated with the extension (4.1), which we denote by $\mathcal{U}$ to keep notation short.
Remark 8. For Lie algebras $\omega_E(s(x), s(y))$ is just multiplication by $-1$ and thus by linearity and injectivity of the map $\iota$, the condition (4.4) reduces to

$$g(x, y) = -g(y, x),$$

which is the classical skew-symmetry, independent on the extension.

By the commutativity of the boxes in (2.2) we have

$$\alpha_L \circ \text{pr} = \text{pr} \circ \alpha_E$$

which means that

$$\text{pr} \circ (\alpha_E - s \circ \alpha_L \circ \text{pr}) = 0$$

and so

$$\alpha_E = s \circ \alpha_L \circ \text{pr} + \iota \circ f,$$  \hspace{1cm} (4.5)

where $f : E \to a$ is a $k$-linear map. By a similar argument we get

$$\beta_E = s \circ \beta_L \circ \text{pr} + \iota \circ h,$$  \hspace{1cm} (4.6)

for a $k$-linear $h : E \to a$. Obviously, both $f$ and $h$ depends on the section chosen. To simplify notation we do not indicate explicitly this dependence in what follows.

Since any $e \in E$ and $e' \in E$ can be decomposed as $e = s(x) + \iota(a)$ and $e' = s(y) + \iota(b)$ with $x, y \in L$ and $a, b \in a$, we have

$$\langle e, e' \rangle_E = \langle s(x) + \iota(a), s(y) + \iota(b) \rangle_E =$$

$$= \langle \iota(a), \iota(a) \rangle_E + \langle s(x), s(y) \rangle_E + \langle \iota(a), s(y) \rangle_E + \langle s(x), s(y) \rangle_E.$$

With $\langle s(x), s(y) \rangle_E = s\langle x, y \rangle_L + \iota \circ g(x, y)$ we can re-write this noting that by definition $\iota$ is a morphism of algebras:

$$\langle e, e' \rangle_E = \iota(\langle a, b \rangle_a) + \langle s(x), \iota(b) \rangle_E + \langle \iota(a), s(y) \rangle_E + \langle s(x), y \rangle_L + \iota \circ g(x, y) =$$

$$= s\langle x, y \rangle_L + \left( \iota(\langle a, b \rangle_a) + \langle s(x), \iota(b) \rangle_E + \langle \iota(a), s(y) \rangle_E + \iota \circ g(x, y) \right)$$

where the expression in parentheses is in $\iota(a)$ since $\iota(a)$ is an ideal in $E$ by the exactness. The extension is called inessential if $g \equiv 0$, which is equivalent to viewing $L$ as a subalgebra of $E$.

We consider only central extensions, i.e. extensions satisfying

$$\iota(a) \subseteq Z(E) = \{ e \in E \mid \langle e, E \rangle_E = 0 \},$$
where \( a \) is abelian, that is \( \langle a, a \rangle_a = 0 \). This means in particular that

\[
\langle e, e' \rangle_E = s\langle x, y \rangle_L + (t(\langle a, b \rangle_a) + \langle s(x), t(b) \rangle_E + \langle t(a), s(y) \rangle_E + \iota \circ g(x, y)) = \\
= s\langle x, y \rangle_L + \iota \circ g(x, y).
\]

To study the necessary conditions on \( g \), when having a Jacobi-like identity on \( E \), under the centrality assumption, we calculate

\[
\omega_E(s(z) + \iota(c), s(x) + \iota(a)) \left( \langle \alpha_E(s(x)), \langle s(y), s(z) \rangle_E \right) + \\
+ \beta_E \langle s(z), \langle s(y), s(z) \rangle_E \right) \right)
\]

and sum up this cyclically. The reason for not considering products of elements on the form \( s(x) + \iota(a) \) in the brackets is that, since the extension is assumed to be central, the terms involving elements from \( \iota(a) \) within the brackets vanish automatically. So, we have

\[
\langle \alpha_E(s(x)), \langle s(y), s(z) \rangle_E \right) = \langle \alpha_E(s(x)), s(y, z)_L + \iota \circ g(y, z) \right) = \\
= \langle s(\alpha_L(x)), s(y, z)_L \right) = \\
= s(\alpha_L(x), \langle y, z \rangle_L_L + \iota \circ g(\alpha_L(x), \langle y, z \rangle_L).
\]

In a similar fashion we see that

\[
\beta_E \left( s(x), \langle s(y), s(z) \right) = \beta_E \left( s(x), s(y, z)_L + \iota \circ g(y, z) \right) = \\
= \beta_E \circ s(\langle x, \langle y, z \rangle_L \right)_L + \beta_E \circ \iota \circ g(x, \langle y, z \rangle_L).
\]

Noting that by exactness

\[
\beta_E \circ \iota \circ g = \iota \circ h \circ \iota \circ g,
\]

we can re-write the above as

\[
\beta_E \circ s(\langle x, \langle y, z \rangle_L \right)_L + \beta_E \circ \iota \circ g(x, \langle y, z \rangle_L) = \\
= s \circ \beta_L(\langle x, \langle y, z \rangle_L \right)_L + \iota \circ h \circ s(\langle x, \langle y, z \rangle_L \right)_L + \iota \circ h \circ \iota \circ g(x, \langle y, z \rangle_L) = \\
= s \circ \beta_L(\langle x, \langle y, z \rangle_L \right)_L + \iota \circ h \circ (s(\langle x, \langle y, z \rangle_L \right)_L + \iota \circ g(x, \langle y, z \rangle_L))
\]
and so the qhl-Jacobi identity becomes

$$0 = \bigcirc_{(x,a), (y,a), (z, c)} \omega_E \big( \langle s(z) + \iota(c), s(x) + \iota(a) \rangle \circ \\
\circ \big( s(\alpha_L(x), \langle y, z \rangle_L) + \iota \circ g(\alpha_L(x), \langle y, z \rangle_L) + \\
+ s \circ \beta_L \langle x, \langle y, z \rangle_L \rangle + \iota \circ h \circ \big( s(x, \langle y, z \rangle_L) + \iota \circ g(x, \langle y, z \rangle_L) \big) \big) \big) = \\
= \bigcirc_{(x,a), (y,a), (z, c)} \omega_E \big( s(z) + \iota(c), s(x) + \iota(a) \rangle \circ \big( \iota \circ g(\alpha_L(x), \langle y, z \rangle_L) + \\
+ \iota \circ h \circ \big( s(x, \langle y, z \rangle_L) + \iota \circ g(x, \langle y, z \rangle_L) \big) \big) = 0. \quad (4.7)$$

Or, after re-writing the second term,

$$\bigcirc_{(x,a), (y,a), (z, c)} \omega_E \big( s(z) + \iota(c), s(x) + \iota(a) \rangle \circ \\
\circ \big( \iota \circ g(\alpha_L(x), \langle y, z \rangle_L) + \iota \circ h \circ \langle s(x, s(y) \rangle_E \big) = 0. \quad (4.8)$$

We denote the subset of all $\omega$-alternating $g \in \text{Alt}_2^\omega(L, a; U)$, $g : L \times L \to a$ satisfying the condition (4.7) by $Z_{\omega,h}^2(L, a; U)$, the set of 2-cocycle-like maps, by analogy with the Lie algebra case. The subscript $h$ is to remind us that the 2-cocycle-like maps may depend on $h$. We will now show that (4.7), or equivalently (4.8), is independent of the choice of section $s$ and the $h$.

Taking another section $\tilde{s}$ with $\text{pr} \circ \tilde{s} = \text{id}_L$ satisfying the intertwining condition $\omega_E \big( \tilde{s}(x) + \iota(a), \tilde{s}(y) + \iota(b) \big) \circ \tilde{s} = \tilde{s} \circ \omega_L(x, y)$, we see that

$$(\tilde{s} - s)(x) = \iota \circ k(x),$$

for some linear $k : L \to a$, and so

$$\tilde{s} = s + \iota \circ k. \quad (4.9)$$

Hence

$$\iota \circ \tilde{g}(x, y) = \langle \tilde{s}(x), \tilde{s}(y) \rangle_E - \tilde{s}(x, y)_L = \langle s(x) + \iota \circ k(x), s(y) + \iota \circ k(y) \rangle_E - \\
- s(x, y)_L - \iota \circ k(\langle x, y \rangle) = \iota \circ g(x, y) - \iota \circ k(\langle x, y \rangle_L)$$
since the extension is central. By the injectivity of \( \iota \) we get

\[
\tilde{g}(x, y) = g(x, y) - k(\langle x, y \rangle_L).
\]

Further

\[
\iota \circ (\tilde{h} - h)(x) = (\beta_E - \tilde{s} \circ \beta_L \circ \text{pr} - \beta_E + s \circ \beta_L \circ \text{pr})(x) = (s - \tilde{s}) \circ \beta_L \circ \text{pr}(x) = -\iota \circ k \circ \beta_L \circ \text{pr}(x)
\]
giving since \( \iota \) is an injection

\[
\tilde{h} = h - k \circ \beta_L \circ \text{pr}.
\]  \hspace{1cm} (4.10)

Therefore

\[
\circ_{(x,a),(y,a),(z,c)} \left( \tilde{g}(\alpha_L(x), \langle y, z \rangle_L) + \tilde{h}(\tilde{s}(x), \langle \tilde{s}(y), \tilde{s}(z) \rangle_E) \right) = \\
= \circ_{(x,a),(y,a),(z,c)} \left( g(\alpha_L(x), \langle y, z \rangle_L) - k\langle \alpha_L(x), \langle y, z \rangle_L \rangle + \\
+ h(\langle s(x), \langle s(y), s(z) \rangle_E \rangle_E - k \circ \beta_L \langle x, \langle y, z \rangle_L \rangle \right) = \\
= \circ_{(x,a),(y,a),(z,c)} \left( g(\alpha_L(x), \langle y, z \rangle_L) + h(\langle s(x), \langle s(y), s(z) \rangle_E \rangle_E \right)
\]
since \( L \) is a qhl-algebra. Knowing that

\[
\langle \alpha_E(x), \alpha_E(y) \rangle_E = \beta_E \circ \alpha_E \langle x, y \rangle_E
\]
by the definition of a qhl-algebra, we can deduce another relation that necessarily must hold for \( E \) to be a qhl-algebra. The left hand side of the above equality for \( x = s(x) \) and \( y = s(y) \) can be written as follows:

\[
\langle \alpha_E(s(x)), \alpha_E(s(y)) \rangle_E = (s \circ \alpha_L \circ \text{pr} + \iota \circ f) \circ s(x), (s \circ \alpha_L \circ \text{pr} + \iota \circ f) \circ s(y) \rangle_E = \\
= \langle s \circ \alpha_L(x), s \circ \alpha_L(y) \rangle_E = s\langle \alpha_L(x), \alpha_L(y) \rangle_L + \\
+ \iota \circ g(\alpha_L(x), \alpha_L(y)),
\]
where we have used the centrality in the second equality. The right hand side is

\[
\beta_E \circ \alpha_E(\langle s(x), s(y) \rangle_E) = \beta_E \circ s \circ \alpha_L(\langle x, y \rangle_L + \beta_E \circ \iota \circ f(\langle s(x), s(y) \rangle_E = \\
= (s \circ \beta_L \circ \text{pr} + \iota \circ h) \circ s \circ \alpha_L(\langle x, y \rangle_L + \\
+ (s \circ \beta_L \circ \text{pr} + \iota \circ h) \circ \iota \circ f(\langle s(x), s(y) \rangle_E = \\
= s \circ \beta_L \circ \alpha_L(\langle x, y \rangle_L + \iota \circ h \circ s \circ \alpha_L(\langle x, y \rangle_L + \\
+ \iota \circ h \circ \iota \circ f(\langle s(x), s(y) \rangle_E
\]
and after comparing and using injectivity of $\iota$, we get that
\[ g(\alpha_L(x), \alpha_L(y)) = h \circ (s \circ \alpha_L(x, y)_L + \iota \circ f(s(x), s(y))_E). \]

Hence, we get the following result.

**Theorem 7.** Suppose $(L, \alpha_L, \beta_L, \omega_L)$ and $(a, \alpha_a, \beta_a, \omega_a)$ are qhl-algebras with $a$ abelian. If there exists a central extension $(E, \alpha_E, \beta_E, \omega_E)$ of $(L, \alpha_L, \beta_L, \omega_L)$ by $(a, \alpha_a, \beta_a, \omega_a)$, then for a section $s : L \to E$, satisfying (4.2) there is an $\omega$-alternating bilinear $g : L \times L \to a$ and linear maps $f : E \to a$ and $h : E \to a$ such that
\[ f \circ \iota = \alpha_a, \quad (4.11) \]
\[ h \circ \iota = \beta_a \quad (4.12) \]

and also
\[ g(\alpha_L(x), \alpha_L(y)) = h \circ (s \circ \alpha_L(x, y)_L + \iota \circ f(s(x), s(y))_E) \quad (4.13) \]

and
\[ \varnothing_{(x,a),(y,a),(z,c)} \omega_E (s(z) + \iota(c), s(x) + \iota(a)) \circ \left( \iota \circ g(\alpha_L(x), y, z)_L + \iota \circ g(x, y, z)_L \right) = 0 \quad (4.14) \]

for all pairs $(x, a), (y, b), (z, c) \in L \times a$ such that $(s(z) + \iota(c), s(x) + \iota(a)), (s(x) + \iota(a), s(y) + \iota(b)), (s(y) + \iota(b), s(z) + \iota(c)) \in D_{\omega_E}$. Moreover, equation (4.14) is independent of the choice of section $s$ and function $h$, if only sections $s$ satisfying (4.2) are considered.

**Proof.** It remains to prove (4.11) and (4.12). However these follows directly from (4.5), (4.6), the commutativity of (2.2) and injectivity of $\iota$. \[ \square \]

**Example 4.** By taking $\beta_L = \text{id}_L$, $\beta_E = \text{id}_E$, $\beta_a = \text{id}_a$ and $\omega_L(x,y)v_L = -1 \cdot v_L$ for all $x, y, v_L \in L$, $\omega_E(e, e')v_E = -1 \cdot v_E$ for all $e, e', v_E \in E$ (we have $D_{\omega_L} = L \times L$ and $D_{\omega_E} = E \times E$ here), that is if we consider only hom-Lie algebras, we recover the results from the previous paper [14]. To see this consider first (4.14). The assumption that $\beta_E = \text{id}_E$ and $\beta_L = \text{id}_L$ implies that
\[ \iota \circ h = \text{id}_E - s \circ \text{pr}, \quad (4.15) \]
and hence by exactness
\[ ι \circ h \circ \left( s\langle x, ⟨y, z⟩_L \rangle_L + ι \circ g(x, ⟨y, z⟩_L) \right) = \]
\[ = (id_E - s \circ pr) \circ \left( s\langle x, ⟨y, z⟩_L \rangle_L + ι \circ g(x, ⟨y, z⟩_L) \right) = \]
\[ = s\langle x, ⟨y, z⟩_L \rangle_L - s\langle x, ⟨y, z⟩_L \rangle_L + ι \circ g(x, ⟨y, z⟩_L) = \]
\[ = ι \circ g(x, ⟨y, z⟩_L). \]

This means that (4.14) can be re-written using that $ι$ is an injective qhl-algebra morphism as
\[ \bigcirc_{x,y,z} g(\left(id_L + α_L\right)(x), ⟨y, z⟩_L) = 0, \]
which is the condition obtained in the previous work [14]. In the same manner one can see, using (4.15) and injectivity of $ι$ that (4.13) reduces to
\[ g(α_L(x), α_L(y)) = f(s(x), s(y))_E \]
which also is a formula from our previous article.

Note that (4.15) can be written as $h \circ ι\big|_a = id_a$ and $h \circ s\big|_L = 0$. Indeed, we can decompose any $e ∈ E$ as $e = s(x) + ι(a)$ and so
\[ ι \circ h(s(x) + ι(a)) = (id_E - s \circ pr)(s(x) + ι(a)) = \]
\[ = 0 + ι(a) = ι(a). \]

Since $ι$ is an injection this gives $h(s(x) + ι(a)) = ι(a)$. Restricting even further to (color) Lie algebras and thus having $α_L = α_E = id$, we have $f$ satisfies a similar condition $f \circ ι\big|_a = id_a$ and $f \circ s\big|_L = 0$. □

**Example 5.** Consider the following short exact sequence of color Lie algebras $L, E$ and $a$ with the same $Γ$-grading and commutation factor $ε$
\[ 0 \rightarrow a \rightarrow E \rightarrow L \rightarrow 0 \]
with $ι(a)$ central in $E$. This setup is a special case of the construction of Scheunert and Zhang [29] and Scheunert [28], special in the sense that we consider central extensions and not just abelian. We shall show that our construction encompasses the ones in [28, 29] for central extensions.

We first briefly recall Scheunert and Zhang’s construction (as given in [28]). Thus we consider our algebras graded as
\[ a = \bigoplus_{γ ∈ Γ} a_γ, \quad L = \bigoplus_{γ ∈ Γ} L_γ, \quad E = \bigoplus_{γ ∈ Γ} E_γ \]
and so the above sequence becomes
\[ 0 \longrightarrow \bigoplus_{\gamma \in \Gamma} a_{\gamma} \overset{\iota}{\longrightarrow} \bigoplus_{\gamma \in \Gamma} E_{\gamma} \overset{\text{pr}}{\longrightarrow} \bigoplus_{\gamma \in \Gamma} L_{\gamma} \longrightarrow 0. \]

Note that this means that \( \iota \) and \( \text{pr} \) are color Lie algebra homomorphisms and this in turn implies that they are homogeneous of degree zero. Take a section \( s : L \to E \) which is homogeneous of degree zero, that is, \( s(L_\gamma) \subseteq E_\gamma \) for all \( \gamma \in \Gamma \). With this data the Scheunert 2-cocycle condition can be expressed as
\[ \circlearrowleft_{x,y,z} \varepsilon(\gamma_z, \gamma_x) g(x, \langle y, z \rangle) = 0, \quad (4.16) \]
for homogenous elements \( x, y, z \) and where \( \gamma_x, \gamma_y, \gamma_z \) are the graded degrees of \( x, y, z \) respectively.

Putting the above in a qhl-algebras setting means letting \( \omega \) play the role of the commutation factor \( \varepsilon \), where \( \omega \) is then defined on homogeneous elements, \( D_\omega = D_x = \bigcup_{\gamma \in \Gamma} L_\gamma \), and dependent only on the graded degree of these elements. The set \( \text{Alt}^2_\omega(L, a; U) \) includes all \( g \) coming from the ”defect”-relation \( \iota \circ g(x, y) = \langle s(x), s(y) \rangle_E - s(x, y)_L \), for \( s \) a homogeneous section of degree zero. Hence all such \( g \)'s are also homogeneous of degree zero. Noting that \( h \circ |_a = \text{id}_a \) and \( h \circ s|_L = 0 \) from the Example 4, the relation (4.14) now becomes,
\[ \circlearrowleft_{(x,a),(y,a),(z,c)} \varepsilon_L(s(z) + \iota(c), s(x) + \iota(a)) \circ \\
\circ \left( \iota \circ g(\alpha_L(x), \langle y, z \rangle_L) + \iota \circ h \circ \left( s(x, \langle y, z \rangle_L) + \iota \circ g(x, \langle y, z \rangle_L) \right) \right) = \\
= 2 \circlearrowleft_{(x,a),(y,a),(z,c)} \varepsilon(s(z) + \iota(c), s(x) + \iota(a)) \circ \iota \circ g(x, \langle y, z \rangle_L) = 0 \]
implying that
\[ \circlearrowleft_{x,y,z} \varepsilon(z, x) g(x, \langle y, z \rangle_L) = 0, \]
for homogeneous elements, which is Scheunert’s 2-cocycle condition for central extensions [28].

### 4.1 Equivalence between extensions

Let \( \varphi : E \to E' \) be a weak qhl-algebra morphism such that \( E \) and \( E' \) are of the same \( \varphi \)-related \( \omega \)-type. We call two extensions
\[ 0 \longrightarrow (a, \text{Mor}(a)) \overset{\iota}{\longrightarrow} (E, \text{Mor}(E)) \overset{\text{pr}}{\longrightarrow} (L, \text{Mor}(L)) \longrightarrow 0 \]
and
\[ 0 \longrightarrow (a, \text{Mor}(a)) \overset{\iota'}{\longrightarrow} (E', \text{Mor}(E')) \overset{\text{pr}'}{\longrightarrow} (L, \text{Mor}(L)) \longrightarrow 0 \]
weakly equivalent or a weak equivalence if the diagram

\[
\begin{array}{ccccccc}
0 & \to & (\mathfrak{a}, \text{Mor}(\mathfrak{a})) & \xrightarrow{\iota} & (E, \text{Mor}(E)) & \xrightarrow{\text{pr}} & (L, \text{Mor}(L)) & \to & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \to & (\mathfrak{a}, \text{Mor}(\mathfrak{a})) & \xrightarrow{\iota'} & (E', \text{Mor}(E')) & \xrightarrow{\text{pr}'} & (L, \text{Mor}(L)) & \to & 0 \\
\end{array}
\]  

commutes. Similarly one defines strong equivalence as a diagram with the map \( E \to E' \) being a strong morphism. That \( \varphi \) is automatically an isomorphism of linear spaces follows from the 5-lemma.

**Definition 7.** The set of weak equivalence classes of extensions of \( L \) by \( \mathfrak{a} \) is denoted by \( E^-(L, \mathfrak{a}) \), and the set of strong equivalence classes by \( E^+(L, \mathfrak{a}) \).

**Remark 9.** In the case of Lie algebras, or generally, color Lie algebras, weak and strong extensions coincide since weak and strong morphisms do.

We pick sections \( s : L \to E \) and \( s' : L \to E' \) satisfying \( \text{pr} \circ s = \text{id}_L = \text{pr}' \circ s' \) such that

\[
\omega_{E'}(s'(x) + \iota'(a), s'(y) + \iota'(b)) \circ s' = s' \circ \omega_L(x, y)
\]

and

\[
\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ s = s \circ \omega_L(x, y).
\]

Then there is a \( g' \in \text{Alt}_L^2(L, \mathfrak{a}; \mathcal{U}') \) associated with the extension \( \mathcal{U}' \) of \( L \) by \( \mathfrak{a} \) such that \( \langle s'(x), s'(y) \rangle_{\mathcal{U}'} = s'(s, y)_L + \iota' \circ g'(x, y) \).

Given a map \( \varphi : E \to E' \) such that the diagram (4.17) commutes means in particular that

\[
\text{pr}' \circ \varphi = \text{id}_L \circ \text{pr}
\]

and so

\[
\text{pr}' \circ \varphi(s(x)) = x,
\]

which gives

\[
0 = \text{pr}' \circ s'(x) - \text{pr}' \circ \varphi(s(x)) = \text{pr}'(s'(x) - \varphi(s(x))).
\]

Hence

\[
s'(x) = \varphi \circ s(x) + \iota' \circ \xi(x)
\]  

(4.18)
for some $k$-linear $\xi : L \to a$. Taking $x, y \in L$ we have, using the centrality,

\[
\iota' \circ g'(x, y) = (s'(x), s'(y))_{E'} - s'(x, y)_{E'} = \\
= (\varphi(s(x)) + \iota' \circ \xi(x), \varphi(s(y)) + \iota' \circ \xi(y))_{E'} - \varphi(s(x, y)_L) - \iota' \circ \xi((x, y)_L) = \\
= \varphi \circ (s(x, s(y))_{E} - s(x, y)_L) - \iota' \circ \xi((x, y)_L) = \varphi \circ \iota g(x, y) - \iota' \circ \xi((x, y)_L)
\]

and since $\varphi \circ \iota = \iota'$ by \eqref{eq:4.17} we get

\[
\iota' \circ g'(x, y) = \iota' \circ g(x, y) - \iota' \circ \xi((x, y)_L)
\]

or

\[
g'(x, y) = g(x, y) - \xi((x, y)_L), \tag{4.19}
\]

by the injectivity of $\iota'$.

**Remark 10.** Note that \eqref{eq:4.19} implies

\[
(\omega_{E'}(s'(y), s'(x)) - \text{id}_{E'}) \circ \iota' \circ g'(x, y) = \\
= (\omega_E(\varphi \circ s(y), \varphi \circ s(x)) - \text{id}_{E'}) \circ \iota' \circ g(x, y) - \iota' \circ \xi_{L}(\omega_{L}(y, x) - \text{id}_{L})(x, y)_L)
\]

by exchanging $x$ and $y$ and subtracting equalities. For Lie algebras we get

\[
(- \text{id}_{E'} - \text{id}_{E'}) \circ \iota' \circ g'(x, y) = (- \text{id}_{E'} - \text{id}_{E'}) \circ \iota' \circ g(x, y) - \\
- \iota' \circ \xi_{L}(- \text{id}_{L} - \text{id}_{L})(x, y)_L).
\]

This is obviously equivalent to

\[
\iota' \circ g'(x, y) = \iota' \circ g(x, y) - \iota' \circ \xi((x, y)_L)
\]

or, by the injectivity of $\iota'$

\[
g'(x, y) = g(x, y) - \xi((x, y)_L)
\]

which is nothing but the standard equivalence 2-cocycle-condition for the second cohomology group $H^2$ for Lie algebras.

For color Lie algebra extensions we have the same commutation formula and $\Gamma$-grading for both $E, E'$ and $L, a$. Hence

\[
\omega_{E'}(s'(y), s'(x))_{v_{E'}} = \varepsilon(\deg(s'(y)), \deg(s'(x)))_{v_{E'}}, \\
\omega_{E'}(\varphi \circ s(y), \varphi \circ s(x))_{v_{E'}} = \varepsilon(\deg(\varphi \circ s(y)), \deg(\varphi \circ s(x)))_{v_{E'}}, \\
\omega_{L}(y, x)_{v_L} = \varepsilon(\deg(y), \deg(x))_{v_L},
\]
on homogeneous elements, where \( \deg(x) \) denotes the graded degree of \( x \) and \( v_L \in L, v_{E'} \in E' \). This means that

\[
\left( \varepsilon \left( \deg(s'(y)), \deg(s'(x)) \right) - \text{id}_{E'} \right) \circ \iota' \circ g'(x, y) = \\
= \left( \varepsilon \left( \deg(\varphi \circ s(y)), \deg(\varphi \circ s(x)) \right) - \text{id}_{E'} \right) \circ \iota' \circ g(x, y) - \\
- \iota' \circ (\varepsilon(\deg(y), \deg(x)) - \text{id}_L)(\langle x, y \rangle_L)
\]

for homogeneous elements \( x, y \) and \( z \). Since \( s \) and \( \varphi \) is of homogeneous degree zero this implies that

\[
\varepsilon(x, y) = \varepsilon \left( \deg(s'(y)), \deg(s'(x)) \right) = \varepsilon \left( \deg(\varphi \circ s(y)), \deg(\varphi \circ s(x)) \right) = \\
= \varepsilon(\deg(y), \deg(x))
\]

and so

\[
g'(x, y) = g(x, y) - \xi(\langle x, y \rangle_L)
\]

for color Lie algebras also.

We can view \( \xi(\langle x, y \rangle_L) \) as a "2-coboundary" thus motivating the following definition.

**Definition 8.** The set of all 2-cocycle-like maps modulo 2-coboundary-like with respect to a weak isomorphism is denoted by \( H^{2}_{\omega,-}(L, a; U) \) and with respect to a strong isomorphism \( H^{2}_{\omega,+}(L, a; U) \).

Now, given the relation \( g'(x, y) = g(x, y) - \xi(\langle x, y \rangle_L) \) and two extensions \((E, \text{Mor}(E))\) and \((E', \text{Mor}(E'))\) of \( L \) by \( a \), can we construct a weak (strong) equivalence, that is, a weak (strong) isomorphism making (4.17) commute? We can view \( E \) and \( E' \) as

\[
E = s(L) \oplus \iota(a) \quad \text{(as linear spaces)}
\]

\[
E' = s'(L) \oplus \iota'(a) \quad \text{(as linear spaces)}
\]

since the sequences on the form (4.1) is a split sequence. This means that any element \( e \in E \) can be decomposed as

\[
e = s(l) + \iota(a)
\]

for \( a \in a \) and \( l \in L \). We define a map \( \varphi : E \to E' \) by

\[
\varphi \left( s(l) + \iota(a) \right) \overset{\text{def}}{=} s'(l) + \iota'(a - \xi(l)).
\]

(4.20)
We will show that this is a weak isomorphism of qhl-algebras. That it is surjective is clear. Suppose that

\[ s'(l) + l'(a - \xi(l)) = s'(\tilde{l}) + l'(\tilde{a} - \xi(\tilde{l})). \]

This is equivalent to

\[ s'(l - \tilde{l}) + l'(a - \tilde{a} + \xi(l - \tilde{l})) = 0 \]

and so injectivity follows from the injectivity of \( l' \) and \( s' \). To have a weak equivalence we must check

\[ \langle \varphi(x), \varphi(y) \rangle_{E'} = \varphi(x, y)_E. \] (4.21)

On the one hand,

\[ \langle \varphi(s(l) + \iota(a)), \varphi(s(\tilde{l}) + \iota(\tilde{a})) \rangle_{E'} = \langle s'(l) + l'(a - \xi(l)), s'(\tilde{l}) + l'(\tilde{a} - \xi(\tilde{l})) \rangle_{E'} = \\
= \langle s'(l), s'(\tilde{l}) \rangle_{E} = s'(l, \tilde{l})_L + l' \circ g'(l, \tilde{l}) = \\
= s'(l, \tilde{l})_L + l' \circ (g(l, \tilde{l}) - \xi(l, \tilde{l})_L). \]

On the other hand,

\[ \varphi \circ \langle s(l) + \iota(a), s(\tilde{l}) + \iota(\tilde{a}) \rangle_E = \varphi\left( \langle s(l), s(\tilde{l}) \rangle_E \right) = \varphi\left( s(l, \tilde{l})_L + \iota \circ g(l, \tilde{l}) \right) = \\
= s'(l, \tilde{l})_L + l' \circ (g(l, \tilde{l}) - \xi(l, \tilde{l})_L) \]

from which we see that (4.21) is proved. Hence,

**Theorem 8.** With definitions and notations as above, there is a one-to-one correspondence between elements of \( E^-(L, a) \) and elements of \( H^2_{\omega, -}(L, a) \).

Rephrased, the theorem says that there is a one-to-one correspondence between weak equivalence classes of central extensions of \( L \) by an abelian \( a \) and bilinear maps \( g \) transforming according to (4.19) under weak isomorphisms \( E \to E' \).

If we are seeking strong equivalence we also have to condition and check the intertwining conditions

**M2a.** \( \varphi \circ \alpha_E = \alpha_{E'} \circ \varphi \), and

**M3a.** \( \varphi \circ \beta_E = \beta_{E'} \circ \varphi \).
The check of M3a is completely analogous to M2a so we can restrict our attention to M2a. To find conditions for M2a we first note that

\[ f \circ \iota = \alpha_a = f' \circ \iota' \]

which can be seen from the commutativity of (4.17). Hence,

\[
\varphi \circ \alpha_E(s(l) + \iota(a)) = \varphi(s \circ \alpha_L(l) + \iota \circ f(\iota(a) + s(l))) = \\
= s' \circ \alpha_L(l) + \iota' \circ (f(\iota(a) + s(l)) - \xi(\alpha_L(l))) = \\
= s' \circ \alpha_L(l) + \iota' \circ (\alpha_a(a) + f \circ s(l) - \xi(\alpha_L(l))), \quad (4.22)
\]

and

\[
\alpha_{E'} \circ \varphi(s(l) + \iota(a)) = \alpha_{E'} \circ (s'(l) + \iota'(a - \xi(l))) = \\
= s' \circ \alpha_L(l) + \iota' \circ f' \circ (s'(l) + \iota'(a - \xi(l))) = \\
= s' \circ \alpha_L(l) + \iota' \circ (\alpha_a(a) - \alpha_a(\xi(l)) + f' \circ s'(l)). \quad (4.23)
\]

Combining (4.22) and (4.23) shows that a sufficient condition for M2a to hold is that

**S1α.** \( \alpha_a \circ \xi = \xi \circ \alpha_L \) and

**S2α.** \( f \circ s = f' \circ s' \)

or in commutative diagram form

\[
\begin{array}{ccc}
L & \xrightarrow{\xi} & a \\
\alpha_L \downarrow & & \downarrow \alpha_a \\
L & \xrightarrow{\xi} & a
\end{array}
\]

and

\[
\begin{array}{ccc}
L & \xrightarrow{s} & E & \xrightarrow{f} & a \\
& \parallel & \parallel & & \\
L & \xrightarrow{s'} & E' & \xrightarrow{f'} & a.
\end{array}
\]

We also have the analogous condition for M3a

**S1β.** \( \beta_a \circ \xi = \xi \circ \beta_L \) and

**S2β.** \( h \circ s = h' \circ s' \)
or, as above, in commutative diagram form

\[
\begin{array}{ccc}
L & \xrightarrow{\xi} & a \\
\beta_L & \downarrow & \beta_a \\
L & \xrightarrow{\xi} & a
\end{array}
\]

and

\[
\begin{array}{ccc}
L & \xrightarrow{s} & E & \xrightarrow{h} & a \\
\| & & \| & & \\
L & \xrightarrow{s'} & E' & \xrightarrow{k'} & a.
\end{array}
\]

The morphism \( \varphi \) has to be subjected to these four conditions, S1\( \alpha, \beta \) and S2\( \alpha, \beta \), for it to be a strong equivalence.

### 4.2 Existence of extensions

So far we have shown how the 2-cocycle-like bilinear maps \( g \) (that is, elements \( g \in \text{Alt}_2^\omega(L, a; \mathcal{U}) \) such that \( [4.17] \) holds) satisfying \( [4.19] \) corresponds in a one-to-one fashion to weak equivalence classes of extensions. We now show the existence of an extension \( E \) of \( L \) by \( a \) given the following set of data.

**Data A.** A function \( f : L \oplus a \to a \) such that \( f(0, a) = \alpha_a(a) \).

**Data B.** A function \( h : L \oplus a \to a \) such that \( h(0, a) = \beta_a(a) \).

**Data C.** An alternating bilinear form \( g \) satisfying \( [4.7] \) and also

\[
g(\alpha_L(x), \alpha_L(y)) = h(\alpha_L(\langle x, y \rangle_L), f(\langle x, y \rangle_L, g(x, y))).
\]

Now we put \( E := L \oplus a \) and

\[
\langle (l, a), (l', a') \rangle_E \overset{\text{def}}{=} \langle (l, l')_L, g(l, l') \rangle.
\]

In addition we choose the canonical section \( s(x) = (x, 0) \) and define \( \text{pr} \) and \( \iota \) to be the natural projection and inclusion respectively:

\[
\text{pr} : E \to L, \quad \text{pr}(x, a) = x;
\]

\[
\iota : a \to E, \quad \iota(a) = (0, a).
\]

We also define \( \omega_E \) by

\[
\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ s = s \circ \omega_L(x, y),
\]

\[
\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ \iota = \iota \circ \omega_a(a, b)
\]
4.2 Existence of extensions

for $(x, y) \in D_{\omega_L}$, $(a, b) \in D_{\alpha}$ and $(s(x) + \iota(a), s(y) + \iota(b)) \in D_{\omega_E}$. First note that the definition of the bracket can be written in the usual form

$$\langle s(x), s(y) \rangle_E = s(x, y)_L + \iota \circ g(x, y).$$

This gives

$$\langle (l, a), (l', a') \rangle_E = \langle (l, l')_L, g(l, l') \rangle =$$

$$= s(l, l')_L + \iota \circ g(l, l') =$$

$$= s \circ \omega_L(l, l')(l', l)_L + \omega_E(s(l), s(l')) \circ \iota \circ g(l', l) =$$

$$= \omega_E(s(l), s(l')) \circ s(l', l)_L + \omega_E(s(l), s(l')) \circ \iota \circ g(l', l) =$$

$$= \omega_E(s(l), s(l')) \circ (s(l', l)_L + \iota \circ g(l', l))$$

which amounts to

$$\langle (l, a), (l', a') \rangle_E = \omega_E(s(l), s(l')) \langle (l', a'), (l, a) \rangle_E.$$

Furthermore we define

$$\alpha_E(x, a) \overset{\text{def}}{=} (\alpha_L(x), f(x, a))$$

$$\beta_E(x, a) \overset{\text{def}}{=} (\beta_L(x), h(x, a)).$$

That $\alpha_E$ satisfies the $\beta$-twisting condition follows from the following calculation. First,

$$\langle \alpha_E(x, a), \alpha_E(y, b) \rangle_E = \langle (\alpha_L(x), f(x, a)), (\alpha_L(y), f(y, b)) \rangle_E =$$

$$= \langle (\alpha_L(x), \alpha_L(y))_L, g(\alpha_L(x), \alpha_L(y)) \rangle.$$

Secondly,

$$\beta_E \circ \alpha_E((x, a), (y, b)) = \beta_E \circ \alpha_E((x, y)_L, g(x, y)) =$$

$$= \beta_E(\alpha_L(x, y)_L, f((x, y)_L, g(x, y))) =$$

$$= (\beta_L \circ \alpha_L(x, y)_L, h(\alpha_L(x, y)_L, f((x, y)_L, g(x, y))))$$

and so comparing and using Data C gives us the result. Remember that $\beta_E$ is only supposed to be a linear map by the definition of a qhl-algebra. The qhl-Jacobi identity is checked as

$$\langle \alpha_E(x, a), (y, b), (z, c) \rangle_E = \langle (\alpha_L(x), f(x, a)), (y, z)_L \circ g(y, z) \rangle_E =$$

$$= \langle (\alpha_L(x), (y, z)_L)_L, g(\alpha_L(x), (y, z)_L) \rangle.$$
and
\[
\beta_E((x, a), (y, b), (z, c))_L = \beta_E((x, a), (y, z)_L, g(y, z))_E = \\
= \beta_E((x, y, z)_L, g(x, y, z)_L) = \\
= \left(\beta_L((x, y, z)_L, h((x, y, z)_L, g(x, y, z)_L))\right)
\]
so we end up with
\[
g(\alpha_L(x), (y, z)_L) + h((x, y, z)_L, g(x, y, z)_L)
\]
since \(L\) is a qhl-algebra. We know that \(s\) is the canonical section \(x \mapsto (x, 0)\) and so by (4.7) the above expression is zero upon cyclic summation.

That the diagram (2.2) has exact rows is obvious from the definition of \(\iota\) and \(pr\). Next we show that these maps are homomorphisms.

\[
pr((x, a), (y, b))_E = pr((x, y)_L, g(x, y)) = (x, y)_L = \\
= pr(x, a), pr(y, b))_L,
\]
\[
\langle \iota(a), \iota(b)\rangle_E = \langle (0, a), (0, b)\rangle_E = (0, 0) = \iota(0) = \iota(a, b)_a
\]
since \(a\) was abelian. This shows that \(pr\) and \(\iota\) are homomorphisms. In fact they are also qhl-algebra morphisms, because
\[
pr \circ \alpha_E(x, a) = pr(\alpha_L(x), f(x, a)) = \alpha_L(x) = \alpha_E \circ pr(x, a)
\]
and
\[
\alpha_E \circ \iota(a) = \alpha_E(0, a) = (\alpha_L(0), f(0, a)) = (0, \alpha_a(a)) = \iota \circ \alpha_a(a)
\]
and similary with \(\beta_E\). We have proved

**Theorem 9.** Suppose \((L, \text{Mor}(L))\) and \((a, \text{Mor}(a))\) are qhl-algebras with \(a\) abelian and put \(E = L \oplus a\). Then for every \(g \in \text{Alt}_{\omega}^2(L, a; U)\) and every linear map \(f : L \oplus a \to a\) and \(h : L \oplus a \to a\) such that
\[
f(0, a) = \alpha_a(a) \quad \text{for } a \in a, \quad (4.24)
\]
\[
h(0, a) = \beta_a(a) \quad \text{for } a \in a, \quad (4.25)
\]
\[
g(\alpha_L(x), \alpha_L(y)) = h(\alpha_L((x, y)_L), f((x, y)_L, g(x, y))) \tag{4.26}
\]
and
\[
\big((x, a), (y, a), (z, c)\big) \bigwedge_{E}((z, c), (x, a)) \bigcirc \big(\iota \circ g(\alpha_L(x), (y, z)_L) + \\
+ \iota \circ h((x, y, z)_L, g(x, (y, z)_L))\big) = 0, \quad (4.27)
\]
4.2 Existence of extensions

for \( x, y, z \in L \) and \( ((z, c), (x, a)), ((x, a), (y, b)), ((y, b), (z, c)) \in D_{\omega_E}, \) the linear direct sum \((E, \text{Mor}(E))\) with morphisms \( \alpha_E, \beta_E, \omega_E \) is a qhl-algebra central extension of \((L, \text{Mor}(L))\) by \((a, \text{Mor}(a))\).

Remark 11. Let \( L \) be a qhl-algebra and \( a \) abelian. We seek central extensions of \( L \) by \( a \). Put \( E = L \oplus a \) and let

\[
\langle \cdot, \cdot \rangle_E = \left( \langle \cdot, \cdot \rangle_L, g(\cdot, \cdot) \right)
\]

for some bilinear \( g : L \times L \to a \). In addition to this pick the canonical section \( s : L \to E, x \mapsto (x, 0) \). Note that the definition of \( \langle \cdot, \cdot \rangle_E \) is compatible with the choice of \( s \). We want to endow \( E \) with a qhl-algebra structure, that is, we want to find appropriate maps \( \alpha_E, \beta_E \) and \( \omega_E \) satisfying the necessary conditions. This means finding \( f \) and \( h \) such that \( f(0, a) = \alpha_a(a) \) and \( h(0, a) = \beta_a(a) \). Define \( \omega_E \) by the relations

\[
\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ s = s \circ \omega_L(x, y)
\]

and

\[
\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ \iota = \iota \circ \omega_a(a, b).
\]

We let \( \iota \) denote the canonical injection \( \iota : a \hookrightarrow E, a \mapsto (0, a) \) and make the general ansatz

\[
\iota \circ f(l, a) = (0, \alpha_a(a) + F(l))
\]

\[
\iota \circ h(l, a) = (0, \beta_a(a) + H(l)),
\]

for \( F, H : L \to a \) linear. A simple calculation shows that

\[
\alpha_E(l, a) = (\alpha_L(l), \alpha_a(a) + F(l))
\]

\[
\beta_E(l, a) = (\beta_L(l), \beta_a(a) + H(l)).
\]

Furthermore, note that

\[
\langle s(x), s(y, z) \rangle_E = \left( \langle x, \langle y, z \rangle_L \rangle_L, g(x, \langle y, z \rangle_L) \right),
\]

and

\[
\iota \circ h\langle s(x), s(y, z) \rangle_E = (0, \beta_a(g(x, \langle y, z \rangle_L)) + H(\langle x, \langle y, z \rangle_L \rangle_L)).
\]

Putting this together to form the qhl-Jacobi identity we see

\[
\bigcirc_{(x, a), (y, a), (z, c)} \omega_a(c, a) \left( g(\alpha_L(x), \langle y, z \rangle_L) + \beta_a(g(x, \langle y, z \rangle_L)) + H(\langle x, \langle y, z \rangle_L \rangle_L) \right) = 0,
\]
or, assuming $\beta_a = \text{id}_a$,

$$\omega_a(c, a) \left( g((\alpha_L + \text{id}_L)(x), \langle y, z \rangle_L) + H((\langle x, \langle y, z \rangle_L \rangle_L) \right) = 0.$$  

Notice the similarity with the corresponding hom-Lie algebra identity. In addition we must also have

$$\iota \circ g(\alpha_L(x), \alpha_L(y)) = \iota \circ h(\alpha_L(x, y), f(s(x), s(y))_E) =$$

$$= (0, \beta_a \circ f(s(x), s(y))_E + H(\alpha_L(x, y)_L))$$

and so, if $\beta_a = \text{id}_a$,

$$g(\alpha_L(x), \alpha_L(y)) = f(s(x), s(y))_E + H(\alpha_L(x, y)_L) =$$

$$= f((x, y)_L, g(x, y)) + H(\alpha_L(x, y)_L) =$$

$$= \alpha_a(g(x, y)) + F((x, y)_L) + H(\alpha_L(x, y)_L).$$

**Example 6 (Example 4 continued).** Taking $L$ and $a$ to be hom-Lie algebras, meaning that $h \circ \iota|_a = \text{id}_a$ and $h \circ s|_L = 0$, we see that we recover Theorem 7 from our previous paper [14]. □

**Example 7 (Example 5 continued).** Consider two color Lie algebras $L$ and $a$ with the same grading group $\Gamma$ and the same commutation factor $\varepsilon$. This means that (see Example 5)

$$a = \bigoplus_{\gamma \in \Gamma} a_{\gamma} \quad \text{and} \quad L = \bigoplus_{\gamma \in \Gamma} L_{\gamma}.$$  

Forming the vector space

$$E = \bigoplus_{\gamma \in \Gamma} E_{\gamma} = \bigoplus_{\gamma \in \Gamma} (L_{\gamma} \oplus a_{\gamma}) = L \oplus a,$$

we see immediately that $E$ is $\Gamma$-graded. We know from Theorem 9 and the deduction preceding it that we can endow this with a color structure as follows.

From Examples 4 and 5 we see that $f \circ \iota|_a = \text{id}_a$, $f \circ s|_L = 0$ and $h \circ \iota|_a = \text{id}_a$, $h \circ s|_L = 0$ and so (4.24) and (4.25) are true. Taking $s : x \mapsto (x, 0)$ and defining the product on $E$ by

$$\langle (x, a), (y, b) \rangle_E = \langle (x, y)_L, g(x, y) \rangle$$

for some $g \in \text{Alt}_2^2(L, a; U)$. That equation (4.27) is satisfied we saw already in Example 5. Note that (4.26) becomes tautological. Hence we have a color central extension of $L$ by $a$. 
4.3 Central extensions of the \((\alpha, \beta, \omega)\)-deformed loop algebra

Now \(E\) is a color Lie algebra central extension of \(L\) by \(a\). Note, however, that we have not constructed an explicit extension. What we have done is constructing an extension \(\text{given} a \in \text{Alt}_2^\epsilon(L, a; \mathcal{U})\) satisfying (4.7) or its colored restriction. The existence of such a \(g\) is \textit{not} guaranteed in general. See Scheunert [28] Proposition 5.1 for a result that emphasizes this. In our setting this proposition implies that \(H_2^\epsilon(L, a; \mathcal{U}) = \{0\}\) and so there are no non-trivial central extensions. The actual construction of extensions, qualifying to finding 2-cocycles, is a highly non-trivial task.

We now specialize the above slightly to one-dimensional central extensions with \(a = k\). The \(k\)-space \(k\) comes with a natural \(\Gamma\)-grading as

\[
k = \bigoplus_{\gamma \in \Gamma} K_\gamma, \quad \text{where } K_0 = k, \quad K_\gamma = \{0\}, \quad \text{for } \gamma \neq 0.
\]

Then we have a product on

\[
E = \bigoplus_{\gamma \in \Gamma} E_\gamma = \bigoplus_{\gamma \in \Gamma} (L_\gamma \oplus K_\gamma)
\]

defined by

\[
\langle (x, a), (y, b) \rangle_E = \langle x, y \rangle_L g(x, y),
\]

where \(g : L \times L \to k\) is the \(k\)-valued 2-cocycle.

\[\square\]

**Remark 12.** It would be of interest to develop a theory for quasi-hom-Lie algebra extensions of one qhl-algebra by another qhl-algebra, and apply it to get qhl-algebra extensions of the Virasoro algebra by a Heisenberg algebra [16].

4.3 Central extensions of the \((\alpha, \beta, \omega)\)-deformed loop algebra

Now form the vector space

\[
\hat{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus k \cdot c
\]

for a "central element" \(c\) and pick the section \(s : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}, x \otimes t^n \mapsto (x \otimes t^n, 0)\). Define a bilinear product \(\langle \cdot, \cdot \rangle_{\hat{\mathfrak{g}}}\) on \(\hat{\mathfrak{g}}\) as

- \(\langle x \otimes t^n, c \rangle_{\hat{\mathfrak{g}}} = \langle c, x \otimes t^n \rangle_{\hat{\mathfrak{g}}} = 0\), and

- \(\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{\mathfrak{g}}} = \langle x, y \rangle_{\hat{\mathfrak{g}}} \otimes t^{n+m} + g(x \otimes t^n, y \otimes t^m) \cdot c\), for a 2-cocycle-like bilinear map \(g : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to k\).
It is easy to check that this definition is compatible with the section $s$. Define, furthermore,
\[
\alpha_\hat{\mathfrak{g}}(x \otimes t^n + a \cdot c) = \alpha_\hat{\mathfrak{g}}(x \otimes t^n) + a \cdot c
\]
\[
\beta_\hat{\mathfrak{g}}(x \otimes t^n + a \cdot c) = \beta_\hat{\mathfrak{g}}(x \otimes t^n) + a \cdot c,
\] and
\[
\omega_\hat{\mathfrak{g}}(x \otimes t^n + a \cdot c, y \otimes t^m + b \cdot c) = \omega_\hat{\mathfrak{g}}(x \otimes t^n, y \otimes t^m) + \text{id}.
\]

We are now going to investigate when this 1-dimensional ”central extension” $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ can be given the structure of a qhl-algebra.

The $\omega$-skew symmetry of $\langle \cdot, \cdot \rangle_\hat{\mathfrak{g}}$ and $\beta$-twisting of $\alpha_\hat{\mathfrak{g}}$ is an easy check. To check the $\omega$-skew-symmetry condition observe that the product can be written as, where we have put $\omega$ and the section $s \tilde{\omega}$ written as, where we have put $\omega$
\[
\langle \hat{\mathfrak{g}}(s(u), s(v)) \rangle = \langle s(u), s(v) \rangle \hat{\omega} = \hat{\omega}(s(u), s(v)) \circ \iota \circ g(v, u) =\]
\[
\langle s(u), s(v) \rangle \hat{\omega} = \omega(\hat{\mathfrak{g}}(s(u), s(v))) \langle s(v, u) \hat{\omega} \rangle = \omega(s(u), s(v)) \langle s(v, u) \rangle =\]
\[
\omega(s(u), s(v)) \langle s(v, u) \rangle = \omega(s(u), s(v))(v, u) \hat{\omega},
\]

and the $\beta$-twisting:
\[
\langle \alpha_\hat{\mathfrak{g}}(u + a \cdot c), \alpha_\hat{\mathfrak{g}}(v + b \cdot c) \rangle = \langle \alpha_\hat{\mathfrak{g}}(u) + a \cdot c, \alpha_\hat{\mathfrak{g}}(v) + b \cdot c \rangle =\]
\[
= \langle \alpha_\hat{\mathfrak{g}}(u), \alpha_\hat{\mathfrak{g}}(v) \rangle + \beta_\hat{\mathfrak{g}} \circ \alpha_\hat{\mathfrak{g}}(u, v) = \beta_\hat{\mathfrak{g}}(u + a \cdot c, v + b \cdot c)\).
\]

For the qhl-Jacobi identity we have first
\[
\langle \alpha_\hat{\mathfrak{g}}(x \otimes t^n), (y \otimes t^m, z \otimes t^l) \rangle =\]
\[
= \langle \alpha_\hat{\mathfrak{g}}(x) \otimes t^n, (y, z) \otimes t^{m+l} + g(y \otimes t^m, z \otimes t^l) \cdot c \rangle =\]
\[
= \langle \alpha_\hat{\mathfrak{g}}(x) \otimes t^n, (y, z) \otimes t^{m+l} \rangle =\]
\[
= \langle \alpha_\hat{\mathfrak{g}}(x), (y, z) \rangle \otimes t^{n+m+l} + g(\alpha_\hat{\mathfrak{g}}(x) \otimes t^n, (y, z) \otimes t^{m+l}) \cdot c.
\]

Secondly,
\[
\beta_\hat{\mathfrak{g}}(x \otimes t^n, (y \otimes t^m, z \otimes t^l) \hat{\omega}) =\]
\[
= \beta_\hat{\mathfrak{g}}((x, (y, z) \hat{\omega}) \otimes t^{n+m+l} + g(x \otimes t^n, (y, z) \hat{\omega} \otimes t^{m+l}) \cdot c) =\]
\[
= \beta_\hat{\mathfrak{g}}((x, (y, z) \hat{\omega}) \hat{\omega} \otimes t^{n+m+l} + g(x \otimes t^n, (y, z) \hat{\omega} \otimes t^{m+l}) \cdot c
\]

and so combining and summing up cyclically, using that $\mathfrak{g}$ is a qhl-algebra, we end up with
\[
\circ_{(x, n), (y, m), (z, l)} g((\alpha_\mathfrak{g} + \text{id}_\mathfrak{g})(x) \otimes t^n, (y, z) \otimes t^{m+l}) = 0. \quad (4.28)
\]
Now to do this a little more explicit and more in tune with the classical Lie algebra case we construct the product on \( \mathfrak{g} \) a bit differently. We assume \( \omega_\mathfrak{g} = \omega_\mathfrak{g}^\ast = \omega_\mathfrak{g}^\sim = -1 \), that is, that the product is skew-symmetric, and take a homogeneous \( \sigma \)-derivation \( D \) on \( k[t, t^{-1}] \), homogeneous in the sense that \( \sigma \) is a homogeneous mapping \( t \mapsto qt \), for \( q \in k^* \), the multiplicative group of non-zero elements of \( k \).

Explicitly we have

\[
D = \eta t^{-k} \frac{\id - \sigma}{1 - q}
\]

and leading to

\[
D(t^n) = \eta \{n\}_q t^{n-k}.
\]

Take a bilinear form \( B(\cdot, \cdot) \) on \( \mathfrak{g} \) and factor the 2-cocycle-like bilinear map \( g \) as

\[
g(x \otimes t^n, y \otimes t^m) = B(x, y) \cdot (D(t^n) \cdot t^m)_0,
\]

where the notation \((f)_0\) is the zeroth term in the Laurent polynomial \( f \) or, put differently, \( t \) times the residue \( \text{Res}(f) \). The above trick to factor the 2-cocycle (in the Lie algebra case) as \( B \) times a ”residue” is apparently due to Kac and Moody from their seminal papers where they introduced what is now known as Kac-Moody algebras, [17] and [23], respectively. This means that

\[
(D(t^n) \cdot t^m)_0 = \eta \{n\}_q \delta_{n+m,k}.
\]

Calculating the 2-cocycle-like condition now leads to

\[
\bigtriangleup(x,n),(y,m),(z,l) \ (\eta \cdot \{n\}_q \cdot \delta_{n+m+l,k}) \cdot B((\alpha_\mathfrak{g} + \id)(x), \langle y, z \rangle_\mathfrak{g}) = 0,
\]

and for \( \alpha_\mathfrak{g} = \id, \eta = 1, q = 1 \) and \( k = 0 \) we retrieve the classical 2-cocycle discovered by Kac and Moody. Notice, however, that in the Lie algebra case it is assumed that \( B \) is symmetric and \( \mathfrak{g} \)-invariant, this leading to a nice 2-cocycle identity unlike the one we have here. What we thus obtained by the preceding factorization is a \( (\alpha, \beta, -1)q \)-deformed, one-dimensional central extension of the (Lie) loop algebra, where the \( q \)-subscript is meant to indicate that we have \( q \)-deformed the derivation on the Laurent polynomial as well as the underlying algebra. Note that the 2-cocycle-like condition only depends on the ”base algebra” \( \mathfrak{g} \).

\[\text{1} \]One would be tempted to try a more general \( \sigma \)-derivation with \( \sigma(t) = qt^s \) as in section (3.1.2) but the following construction seems only to work with \( s = 1 \).
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