On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces

Wadei Al-Omeri and Saeid Jafari

1 Department of Mathematics, Al-Balqa Applied University, Salt 19117, Jordan
2 Department of Mathematics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark
* Correspondence: wadeialomeri@bau.edu.jo; Tel.: +962-77-6690-543
† Current address: Department of Mathematics, Al-Balqa Applied University, Salt 19117, Jordan
‡ These authors contributed equally to this work.

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Abstract: In this paper, the concept of generalized neutrosophic pre-closed sets and generalized neutrosophic pre-open sets are introduced. We also study relations and various properties between the other existing neutrosophic open and closed sets. In addition, we discuss some applications of generalized neutrosophic pre-closed sets, namely neutrosophic $pT_1$ space and neutrosophic $gpT_2$ space. The concepts of generalized neutrosophic connected spaces, generalized neutrosophic compact spaces and generalized neutrosophic extremally disconnected spaces are established. Some interesting properties are investigated in addition to giving some examples.

Keywords: neutrosophic topology; neutrosophic generalized topology; neutrosophic generalized pre-closed sets; neutrosophic generalized pre-open sets; neutrosophic $pT_1$ space; neutrosophic $gpT_2$ space; generalized neutrosophic compact and generalized neutrosophic compact

1. Introduction

Zadeh [1] introduced the notion of fuzzy sets. After that, there have been a number of generalizations of this fundamental concept. The study of fuzzy topological spaces was first initiated by Chang [2,3] in 1968. Atanassov [4] introduced the notion of intuitionistic fuzzy sets (IFs). This notion was extended to intuitionistic $L$-fuzzy setting by Atanassov and Stoeva [5], which currently has the name “intuitionistic $L$-topological spaces”. Coker [6] introduced the notion of intuitionistic fuzzy topological space by using the notion of (IFs). The concept of generalized fuzzy closed set was introduced by Balasubramanian and Sundaram [7]. In various recent papers, Smarandache generalizes intuitionistic fuzzy sets and different types of sets to neutrosophic sets (NSs). On the non-standard interval, Smarandache, Peide and Lupianez defined the notion of neutrosophic topology [8–10]. In addition, Zhang et al. [11] introduced the notion of an interval neutrosophic set, which is a sample of a neutrosophic set and studied various properties.

Recently, Al-Omeri and Smarandache [12,13] introduced and studied a number of the definitions of neutrosophic closed sets, neutrosophic mapping, and obtained several preservation properties and some characterizations about neutrosophic of connectedness and neutrosophic connectedness continuity.

This paper is arranged as follows. In Section 2, we will recall some notions that will be used throughout this paper. In Section 3, we mention some notions in order to present neutrosophic generalized pre-closed sets and investigate its basic properties. In Sections 4 and 5, we study the neutrosophic generalized pre-open sets and present some of their properties. In addition, we provide an application of neutrosophic generalized pre-open sets. Finally, the concepts of generalized neutrosophic
connected space, generalized neutrosophic compact space and generalized neutrosophic extremally disconnected spaces are introduced and established in Section 6 and some of their properties in neutrosophic topological spaces are studied.

This class of sets belongs to the important class of neutrosophic generalized open sets which is very useful not only in the deepening of our understanding of some special features of the already well-known notions of neutrosophic topology but also proves useful in neutrosophic multifunction theory in neutrosophic economy and also in neutrosophic control theory. The applications are vast and the researchers in the field are exploring these realms of research.

2. Preliminaries

**Definition 1.** Let \( \mathcal{S} \) be a non-empty set. A neutrosophic set (NS for short) \( \tilde{S} \) is an object having the form \( \tilde{S} = \{(k, \mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k)) : k \in \mathcal{S}\} \), where \( \gamma_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \mu_{\tilde{S}}(k) \), and the degree of non-membership (namely \( \gamma_{\tilde{S}}(k) \)), the degree of indeterminacy (namely \( \sigma_{\tilde{S}}(k) \)), and the degree of membership function (namely \( \mu_{\tilde{S}}(k) \)), of each element \( k \in \mathcal{S} \) to the set \( \tilde{S} \), see [14].

A neutrosophic set \( \tilde{S} = \{(k, \mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k)) : k \in \mathcal{S}\} \) can be identified as \( (\mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k)) \) in \( 0^-1, 1^+ \) on \( \mathcal{S} \).

**Definition 2.** Let \( \tilde{S} = (\mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k)) \) be an NS on \( \mathcal{S} \). [15] The complement of the set \( \tilde{S} = \tilde{S}(C(\tilde{S}), \text{for short}) \) may be defined as follows:

(i) \( C(\tilde{S}) = \{(k, 1 - \mu_{\tilde{S}}(k), 1 - \gamma_{\tilde{S}}(k)) : k \in \mathcal{S} \} \),
(ii) \( C(\tilde{S}) = \{(k, \sigma_{\tilde{S}}(k), \mu_{\tilde{S}}(k)) : k \in \mathcal{S} \} \),
(iii) \( C(\tilde{S}) = \{(k, \gamma_{\tilde{S}}(k), 1 - \sigma_{\tilde{S}}(k), \mu_{\tilde{S}}(k)) : k \in \mathcal{S} \} \).

Neutrosophic sets (N.Ss) \( 0_N \) and \( 1_N \) [14] in \( \mathcal{S} \) are introduced as follows:

\( 1 - 0_N \) can be defined as four types:

(i) \( 0_N = \{(k, 0, 0, 1) : k \in \mathcal{S}\} \),
(ii) \( 0_N = \{(k, 0, 1, 1) : k \in \mathcal{S}\} \),
(iii) \( 0_N = \{(k, 0, 1, 0) : k \in \mathcal{S}\} \),
(iv) \( 0_N = \{(k, 0, 0, 0) : k \in \mathcal{S}\} \).

2- \( 1_N \) can be defined as four types:

(i) \( 1_N = \{(k, 1, 0, 0) : k \in \mathcal{S}\} \),
(ii) \( 1_N = \{(k, 1, 0, 1) : k \in \mathcal{S}\} \),
(iii) \( 1_N = \{(k, 1, 1, 1) : k \in \mathcal{S}\} \),
(iv) \( 1_N = \{(k, 1, 1, 1) : k \in \mathcal{S}\} \).

**Definition 3.** Let \( k \) be a non-empty set, and generalized neutrosophic sets GNSs \( \tilde{S} \) and \( \tilde{R} \) be in the form \( \tilde{S} = \{k, \mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k)\}, \) \( \tilde{R} = \{k, \mu_{\tilde{R}}(k), \sigma_{\tilde{R}}(k), \gamma_{\tilde{R}}(k)\}. \) Then, we may consider two possible definitions for subsets \( \tilde{S} \subseteq \tilde{R} \) [14]:

(i) \( \tilde{S} \subseteq \tilde{R} \Leftrightarrow \mu_{\tilde{S}}(k) \leq \mu_{\tilde{R}}(k), \sigma_{\tilde{S}}(k) \geq \sigma_{\tilde{R}}(k), \) and \( \gamma_{\tilde{S}}(k) \leq \gamma_{\tilde{R}}(k) \),
(ii) \( \tilde{S} \subseteq \tilde{R} \Leftrightarrow \mu_{\tilde{S}}(k) \leq \mu_{\tilde{R}}(k), \sigma_{\tilde{S}}(k) \geq \sigma_{\tilde{R}}(k), \) and \( \gamma_{\tilde{S}}(k) \geq \gamma_{\tilde{R}}(k) \).

**Definition 4.** Let \( \{\tilde{S}_j : j \in J\} \) be an arbitrary family of NSs in \( \mathcal{S} \). Then,

(i) \( \cap \tilde{S}_j \) can be defined as two types:

\( \cap \tilde{S}_j = \{(k, \mu_{\tilde{S}_j}(k), \sigma_{\tilde{S}_j}(k), \gamma_{\tilde{S}_j}(k))\}, \)
\( \cap \tilde{S}_j = \{(k, \mu_{\tilde{S}_j}(k), \sigma_{\tilde{S}_j}(k), \gamma_{\tilde{S}_j}(k))\}. \)
(ii) \( \bigcup \tilde{S}_j \) can be defined as two types: 
\[
\bigcup \tilde{S}_j = (k, \vee_{j \in J} \mu_{\tilde{S}_j}(k), \vee_{j \in J} \sigma_{\tilde{S}_j}(k), \wedge_{j \in J} \gamma_{\tilde{S}_j}(k)),
\]
\[
\bigcup \tilde{S}_j = (k, \vee_{j \in J} \mu_{\tilde{S}_j}(k), \wedge_{j \in J} \sigma_{\tilde{S}_j}(k), \wedge_{j \in J} \gamma_{\tilde{S}_j}(k)),
\]
see [14].

**Definition 5.** A neutrosophic topology (NT for short) [16] and a non empty set \( \mathcal{X} \) is a family \( \Gamma \) of neutrosophic subsets of \( \mathcal{X} \) satisfying the following axioms:

(i) \( 0_N, 1_N \in \Gamma \),
(ii) \( \tilde{S}_1 \cap \tilde{S}_2 \in \Gamma \) for any \( \tilde{S}_1, \tilde{S}_2 \in \Gamma \),
(iii) \( \bigcup \tilde{S}_i \in \Gamma, \forall \{ \tilde{S}_i | j \in J \} \subseteq \Gamma \).

In this case, the pair \( (\mathcal{X}, \Gamma) \) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in \( \Gamma \) is known as neutrosophic open set NOS \( \in \mathcal{X} \). The elements of \( \Gamma \) are called neutrosophic open sets. A closed neutrosophic set \( \tilde{R} \) if and only if its \( \tilde{C}(\tilde{R}) \) is neutrosophic open.

Note that, for any NTS \( \tilde{S} \) in \( (\mathcal{X}, \Gamma) \), we have \( \tilde{NCl}(\tilde{S}) = [\tilde{NInt}(\tilde{S})]^c \) and \( \tilde{NInt}(\tilde{S})^c = [\tilde{NCl}(\tilde{S})]^c \).

**Definition 6.** Let \( \tilde{S} = \{ \mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k) \} \) be a neutrosophic open set and \( B = \{ \mu_B(k), \sigma_B(k), \gamma_B(k) \} \) a neutrosophic set on a neutrosophic topological space \( (\mathcal{X}, \Gamma) \). Then,

(i) \( \tilde{S} \) is called neutrosophic regular open [14] iff \( \tilde{S} = \tilde{NInt}(\tilde{NCl}(\tilde{S})) \).
(ii) \( \tilde{S} \) is called neutrosophic regular closed [14] iff \( \tilde{S} = \tilde{NCl}(\tilde{NInt}(\tilde{S})) \).

**Definition 7.** Let \( (k, \Gamma) \) be NT and \( \tilde{S} = \{ k, \mu_{\tilde{S}}(k), \sigma_{\tilde{S}}(k), \gamma_{\tilde{S}}(k) \} \) an NS in \( \mathcal{X} \). Then,

(i) \( \tilde{NCL}(\tilde{S}) = \cap \{ U : U \text{ is an NCS in } \mathcal{X}, \tilde{S} \subseteq U \} \),
(ii) \( \tilde{NInt}(\tilde{S}) = \cup \{ V : V \text{ is an NOS in } \mathcal{X}, \tilde{S} \subseteq V \} \), see [14].

It can be also shown that \( \tilde{NCl}(\tilde{S}) \) is an NCS and \( \tilde{NInt}(\tilde{S}) \) is an NOS in \( \mathcal{X} \). We have

(i) \( \tilde{S} \) is in \( \mathcal{X} \) iff \( \tilde{NCl}(\tilde{S}) \).
(ii) \( \tilde{S} \) is an NCS in \( \mathcal{X} \) iff \( \tilde{NInt}(\tilde{S}) = \tilde{S} \).

**Definition 8.** Let \( \tilde{S} \) be an NS and \( (\mathcal{X}, \Gamma) \) an NT,

(i) Neutrosophic semiopen set (NSOS) [12] if \( \tilde{S} \subseteq \tilde{NCl}(\tilde{NInt}(\tilde{S})) \),
(ii) Neutrosophic preopen set (NPPOS) [12] if \( \tilde{S} \subseteq \tilde{NInt}(\tilde{NCl}(\tilde{S})) \),
(iii) Neutrosophic \( \alpha \)-open set (NaOS) [12] if \( \tilde{S} \subseteq \tilde{NInt}(\tilde{NCl}(\tilde{NInt}(\tilde{S}))) \),
(iv) Neutrosophic \( \beta \)-open set (NBOS) [12] if \( \tilde{S} \subseteq \tilde{NCl}(\tilde{NInt}(\tilde{NCl}(\tilde{S}))) \).

The complement of \( \tilde{S} \) is an NSOS, NaOS, NPOS, and NROS, which is called NSCS, NaCS, NPCS, and NRCS, resp.

**Definition 9.** Let \( \tilde{S} = \{ \tilde{S}_1, \tilde{S}_2, \tilde{S}_3 \} \) be an NS and \( (\mathcal{X}, \Gamma) \) an NT. Then, the \( \ast \)-neutrosophic closure of \( \tilde{S} \) \((\ast = \tilde{NCl}(\tilde{S}) \text{ for short [12]) and \( \ast \)-neutrosophic interior \((\ast = \tilde{NInt}(\tilde{S}) \text{ for short [12]) \) of \( \tilde{S} \) are defined by

(i) \( \ast \tilde{NCl}(\tilde{S}) = \cap \{ V : V \text{ is an NRC in } \mathcal{X}, \tilde{S} \subseteq V \} \),
(ii) \( \ast \tilde{NInt}(\tilde{S}) = \cup \{ U : U \text{ is an NRO in } \mathcal{X}, \tilde{S} \subseteq U \} \),
(iii) \( \ast \tilde{NCl}(\tilde{S}) = \cap \{ V : V \text{ is an NPC in } \mathcal{X}, \tilde{S} \subseteq V \} \),
(iv) \( \ast \tilde{NInt}(\tilde{S}) = \cup \{ U : U \text{ is an NPO in } \mathcal{X}, \tilde{S} \subseteq U \} \),
(v) \( \ast \tilde{NCl}(\tilde{S}) = \cap \{ V : V \text{ is an NSC in } \mathcal{X}, \tilde{S} \subseteq V \} \),
(vi) \( \ast \tilde{NInt}(\tilde{S}) = \cup \{ U : U \text{ is an NSO in } \mathcal{X}, \tilde{S} \subseteq U \} \),
(vii) \( \ast \tilde{NCl}(\tilde{S}) = \cap \{ V : V \text{ is an NCBO in } \mathcal{X}, \tilde{S} \subseteq V \} \),
(viii) \( \ast \tilde{NInt}(\tilde{S}) = \cup \{ U : U \text{ is a NBO in } \mathcal{X}, \tilde{S} \subseteq U \} \),
(ix) \( \ast \tilde{NCl}(\tilde{S}) = \cap \{ V : V \text{ is an NRC in } \mathcal{X}, \tilde{S} \subseteq V \} \),
(x) \( \ast \tilde{NInt}(\tilde{S}) = \cup \{ U : U \text{ is an NRO in } \mathcal{X}, \tilde{S} \subseteq U \} \).
**Definition 10.** An \((NS) S\) of an \(NT (\mathcal{X}, \Gamma)\) is called a generalized neutrosophic closed set [17] \((\text{GNC in short)}\) if \(\text{NCI}(S) \subseteq B\) wherever \(S \subseteq B\) and \(B\) is a neutrosophic closed set in \(\mathcal{X}\).

**Definition 11.** An \(NS S\) in an \(NT \mathcal{X}\) is said to be a neutrosophic a generalized closed set \((\text{NagCS \cite{18}})\) if \(\text{NaNCI}(S) \subseteq B\) whenever \(S \subseteq B\) and \(B\) is an \(\text{NOS in} \ \mathcal{X}\). The complement \(C(S)\) of an \(\text{NagCS} S\) is an \(\text{NagOS in} \ \mathcal{X}\).

### 3. Neutrosophic Generalized Connected Spaces, Neutrosophic Generalized Compact Spaces and Generalized Neutrosophic Extremally Disconnected Spaces

**Definition 12.** Let \((\mathcal{X}, \Gamma)\) and \((\mathcal{X}', \Gamma_1)\) be any two neutrosophic topological spaces.

(i) A function \(g : (\mathcal{X}, \Gamma) \rightarrow (\mathcal{X}', \Gamma_1)\) is called generalized neutrosophic continuous \((\text{GN-continuous}) g^{-1}\) of every closed set in \((\mathcal{X}, \Gamma_1)\) is \(\text{GN-closed in} \ (\mathcal{X}, \Gamma)\).

Equivalently, if the inverse image of every \(\text{open set in} \ (\mathcal{X}, \Gamma_1)\) is \(\text{GN-open in} \ (\mathcal{X}, \Gamma)\):

(ii) A function \(g : (\mathcal{X}, \Gamma) \rightarrow (\mathcal{X}', \Gamma_1)\) is called generalized neutrosophic irresolute \((\text{GN-irresolute}) g^{-1}\) of every \(\text{GN-closed set in} \ (\mathcal{X}, \Gamma)\) is \(\text{GN-closed in} \ (\mathcal{X}, \Gamma)\).

Equivalently \(g^{-1}\) of every \(\text{GN-open set in} \ (\mathcal{X}, \Gamma)\) is \(\text{GN-open in} \ (\mathcal{X}', \Gamma_1)\):

(iii) A function \(g : (\mathcal{X}, \Gamma) \rightarrow (\mathcal{X}', \Gamma_1)\) is said to be strongly neutrosophic continuous if \(g^{-1}(S)\) is both \(\text{neutrosophic open and neutrosophic closed in} \ (\mathcal{X}, \Gamma)\) for each neutrosophic set \(S\) in \((\mathcal{X}, \Gamma_1)\).

(iv) A function \(g : (\mathcal{X}, \Gamma) \rightarrow (\mathcal{X}', \Gamma_1)\) is said to be strongly \(\text{GN-continuous if the inverse image of every} \ \text{GN-open set in} \ (\mathcal{X}, \Gamma)\), see \((\text{17 for more details)}\).

**Definition 13.** An \(\text{NTS} (\mathcal{X}, \Gamma)\) is said to be neutrosophic-\(T_1\) \((\text{NT}_1\ \text{in short})\) if every \(\text{GNC in} \ \mathcal{X}\) is an \(\text{NC in} \ \mathcal{X}\).

**Definition 14.** Let \((\mathcal{X}, \Gamma)\) be any neutrosophic topological space. \((\mathcal{X}, \Gamma)\) is said to be generalized neutrosophic disconnected (in shortly \(\text{GN-disconnected})\) if there exists a generalized neutrosophic open and generalized neutrosophic closed set \(\bar{R}\) such that \(\bar{R} \neq 0_N\) and \(\bar{R} \neq 1_N\). \((\mathcal{X}, \Gamma)\) is said to be generalized neutrosophic connected if it is not generalized neutrosophic disconnected.

**Proposition 1.** Every \(\text{GN-connected space is neutrosophic connected. However, the converse is not true.}\)

**Proof.** For a \(\text{GN-connected} \ \ (\mathcal{X}, \Gamma)\) space and let \((\mathcal{X}, \Gamma)\) not be neutrosophic connected. Hence, there exists a proper neutrosophic set, \(\bar{S} = \{\mu_3(x), \sigma_3(x), \gamma_3(x)\} \neq 0_N\), \(\bar{S} \neq 1_N\), such that \(\bar{S}\) is both neutrosophic open and neutrosophic closed in \((\mathcal{X}, \Gamma)\). Since every neutrosophic open set is \(\text{GN-open and neutrosophic closed set is} \ \text{GN-closed,} \ \mathcal{X}\) is not \(\text{GN-connected. Therefore,} \ (\mathcal{X}, \Gamma)\) is neutrosophic connected.

**Example 1.** Let \(\mathcal{X} = \{u, v, w\}\). Define the neutrosophic sets \(\bar{S}, \bar{R}\) and \(\mathcal{Z}\) in \(\mathcal{X}\) as follows:

\[S = \langle x_1, \langle a_1, b_1, c_1 \rangle, \langle a_2, b_2, c_2 \rangle, \langle a_3, b_3, c_3 \rangle \rangle, \quad \bar{S} = \langle x_1, \langle a_1, b_1, c_1 \rangle, \langle a_2, b_2, c_2 \rangle, \langle a_3, b_3, c_3 \rangle \rangle.\]

Then, the family \(\Gamma = \{0_N, 1_N, \bar{S}, \bar{R}\}\) is neutrosophic topology on \(\mathcal{X}\). It is obvious that \((\mathcal{X}, \Gamma)\) is \(\text{NTS}. \ \text{Now,} \ \ (\mathcal{X}, \Gamma)\) is neutrosophic connected. However, it is not a \(\text{GN-connected} \ \mathcal{Z} = \langle x_1, \langle a_1, b_1, c_1 \rangle, \langle a_2, b_2, c_2 \rangle, \langle a_3, b_3, c_3 \rangle \rangle\) is \(\text{GN open and GN closed in} \ (\mathcal{X}, \Gamma)\).

**Theorem 1.** Let \((\mathcal{X}, \Gamma)\) be a neutrosophic \(T_1\) space; then, \((\mathcal{X}, \Gamma)\) is neutrosophic connected iff \((\mathcal{X}, \Gamma)\) is \(\text{GN-connected}\).

**Proof.** Suppose that \((\mathcal{X}, \Gamma)\) is not \(\text{GN-connected, and there exists a neutrosophic set} \ \bar{S}\) which is both \(\text{GN-open and GN-closed. Since} \ \ (\mathcal{X}, \Gamma)\) is neutrosophic \(T_1\), \(\bar{S}\) is both neutrosophic open and neutrosophic closed. Hence, \((\mathcal{X}, \Gamma)\) is \(\text{GN-connected. Conversely, let} \ (\mathcal{X}, \Gamma)\) is \(\text{GN-connected. Suppose that} \ (\mathcal{X}, \Gamma)\) is not neutrosophic connected, and there exists a neutrosophic set \(\bar{S}\) such that \(\bar{S}\) is both \(\text{NCs and NOs in} \ (\mathcal{X}, \Gamma)\).
Theorem 2. Suppose that $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$. Then, $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$. Moreover, if $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$, then $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$. Therefore, $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$. Hence, if $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$, then $\mathcal{U}$ is a neutrosophic open cover of $\mathcal{X}$.

Proposition 2. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are any two neutrosophic open sets in $\mathcal{X}$. Then, $\mathcal{U} \cup \mathcal{V}$ is a neutrosophic open cover of $\mathcal{X}$. Hence, if $\mathcal{U}$ and $\mathcal{V}$ are neutrosophic open sets in $\mathcal{X}$, then $\mathcal{U} \cup \mathcal{V}$ is a neutrosophic open cover of $\mathcal{X}$. Therefore, if $\mathcal{U}$ and $\mathcal{V}$ are neutrosophic open sets in $\mathcal{X}$, then $\mathcal{U} \cup \mathcal{V}$ is a neutrosophic open cover of $\mathcal{X}$.

Proof. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are neutrosophic open sets in $\mathcal{X}$. Then, $\mathcal{U} \cup \mathcal{V}$ is a neutrosophic open cover of $\mathcal{X}$. Hence, if $\mathcal{U}$ and $\mathcal{V}$ are neutrosophic open sets in $\mathcal{X}$, then $\mathcal{U} \cup \mathcal{V}$ is a neutrosophic open cover of $\mathcal{X}$. Therefore, if $\mathcal{U}$ and $\mathcal{V}$ are neutrosophic open sets in $\mathcal{X}$, then $\mathcal{U} \cup \mathcal{V}$ is a neutrosophic open cover of $\mathcal{X}$.

Definition 15. Let $(\mathcal{X}, \Gamma)$ be an NT. If a family $\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\}$ of GN open sets in $(\mathcal{X}, \Gamma)$ satisfies the condition $\bigcup\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\} = 1_N$, then it is called a GN open cover of $(\mathcal{X}, \Gamma)$. A finite subfamily of a GN open cover $\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\}$ of $(\mathcal{X}, \Gamma)$, which is also a GN open cover of $(\mathcal{X}, \Gamma)$, is called a finite subcover of

$$\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\}.$$ 

Definition 16. An NT $(\mathcal{X}, \Gamma)$ is called GN compact if every GN open cover of $(\mathcal{X}, \Gamma)$ has a finite subcover.

Theorem 2. Let $(\mathcal{X}, \Gamma)$ and $(\mathcal{X}, \Gamma_1)$ be any two NTs, and $g : (\mathcal{X}, \Gamma) \to (\mathcal{X}, \Gamma_1)$ be GN continuous surjection. Suppose that $g$ is GN continuous, $g^{-1}(S)$ is GN-open and GN-closed in $(\mathcal{X}, \Gamma)$. Thus, $(\mathcal{X}, \Gamma)$ is not GN connected. Hence, $(\mathcal{X}, \Gamma_1)$ is neutrosophic connected.

Proof. Let $G_i = \{y, \mu_G(x), \sigma_G(x), \gamma_G(y) : i \in I\}$ be a neutrosophic open cover in $(\mathcal{X}, \Gamma_1)$ with

$$\bigcup_{i \in I} G_i = 1_N.$$

Since $g$ is GN continuous, $g^{-1}(G_i) = G_i = \{y, \mu_{g^{-1}(G_i)}(x), \sigma_{g^{-1}(G_i)}(x), \gamma_{g^{-1}(G_i)}(x) : i \in I\}$ is GN open cover of $(\mathcal{X}, \Gamma)$. Now,

$$\bigcup_{i \in I} g^{-1}(G_i) = g^{-1}\left(\bigcup_{i \in I} G_i\right) = 1_N.$$

Since $(\mathcal{X}, \Gamma)$ is GN compact, there exists a finite subcover $J_0 \subset I$, such that

$$\bigcup_{i \in J_0} g^{-1}(G_i) = 1_N.$$

Hence,

$$g\left(\bigcup_{i \in J_0} g^{-1}(G_i) = 1_N\right) = g^{-1}\left(\bigcup_{i \in J_0} (G_i) = 1_N\right).$$

That is,

$$\bigcup_{i \in J_0} (G_i) = 1_N.$$

Therefore, $(\mathcal{X}, \Gamma_1)$ is neutrosophic compact.

Definition 17. Let $(\mathcal{X}, \Gamma)$ be an NT and $K$ be a neutrosophic set in $(\mathcal{X}, \Gamma)$. If a family $\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\}$ of GN open sets in $(\mathcal{X}, \Gamma)$ satisfies the condition $K \subseteq \bigcup\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\} = 1_N$, then it is called a GN open cover of $K$. A finite subfamily of a GN open cover $\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\}$ of $K$, which is also a GN open cover of $K$ is called a finite subcover of $\{k, \mu_G(k), \sigma_G(k), \gamma_G(k) : i \in I\}$.
**Definition 18.** An NT $(\mathcal{X}, \Gamma)$ is called GN compact iff every GN open cover of $K$ has a finite subcover.

**Theorem 3.** Let $(\mathcal{X}, \Gamma)$ and $(\mathcal{X}, \Gamma_1)$ be any two NTs, and $g : (\mathcal{X}, \Gamma) \rightarrow (\mathcal{X}, \Gamma_1)$ be an GN continuous function. If $K$ is GN-compact, then so is $g(K)$ in $(\mathcal{X}, \Gamma_1)$.

**Proof.** Let $G_i = \{ \langle y, \mu_{G_i}(x), \sigma_{G_i}(x), \gamma_{G_i}(x) : i \in I \rangle \}$ be a neutrosophic open cover of $g(K)$ in $(\mathcal{X}, \Gamma_1)$. That is,

$$g(K) \subseteq \bigcup_{i \in I} G_i.$$

Since $g$ is GN continuous, $g^{-1}(G_i) = \{ \langle x, \mu_{g^{-1}(G_i)}(x), \sigma_{g^{-1}(G_i)}(x), \gamma_{g^{-1}(G_i)}(x) : i \in I \rangle \}$ is GN open cover of $K$ in $(\mathcal{Z}, \Gamma)$. Now,

$$K \subseteq \bigcup_{i \in I} g^{-1}(G_i) \subseteq \bigcup_{i \in I} g^{-1}(G_i).$$

Since $K$ is $(\mathcal{Z}, \Gamma)$ is GN compact, there exists a finite subcover $J_0 \subseteq J$, such that

$$K \subseteq \bigcup_{i \in J_0} g^{-1}(G_i) = 1_N.$$

Hence,

$$g(K) \subseteq g\left( \bigcup_{i \in J_0} g^{-1}(G_i) \right) \bigcup_{i \in J_0} (G_i).$$

Therefore, $g(K)$ is neutrosophic compact. □

**Proposition 3.** Let $(\mathcal{Z}, \Gamma)$ be a neutrosophic compact space and suppose that $K$ is a GN-closed set of $(\mathcal{Z}, \Gamma)$. Then, $K$ is a neutrosophic compact set.

**Proof.** Let $K_j = \{ \langle y, \mu_{K_j}(x), \sigma_{K_j}(x), \gamma_{K_j}(x) : i \in I \rangle \}$ be a family of neutrosophic open set in $(\mathcal{Z}, \Gamma)$ such that

$$K \subseteq \bigcup_{i \in J} K_j.$$

Since $K$ is GN-closed, $NCl(K) \subseteq \bigcup_{i \in J} K_j$. Since $(\mathcal{Z}, \Gamma)$ is a neutrosophic compact space, there exists a finite subcover $J_0 \subseteq J$. Now, $NCl(K) \subseteq \bigcup_{i \in J_0} K_j$. Hence, $K \subseteq NCl(K) \subseteq \bigcup_{i \in J_0} K_j$. Therefore, $K$ is a neutrosophic compact set. □

**Definition 19.** Let $(\mathcal{Z}, \Gamma)$ be any neutrosophic topological space. $(\mathcal{Z}, \Gamma)$ is said to be GN extremally disconnected if $NCl(K)$ neutrosophic open and $K$ is GN open.

**Proposition 4.** For any neutrosophic topological space $(\mathcal{Z}, \Gamma)$, the following are equivalent:

(i) $(\mathcal{Z}, \Gamma)$ is GN extremally disconnected.
(ii) For each GN closed set $K$, $NCl(N\Gamma)(\mathcal{S})$ is a GN closed set.
(iii) For each GN open set $K$, we have $NCl(K) + NCl(\mathcal{S} - NCl(\mathcal{S})) = 1$.
(iv) For each pair of GN open sets $K$ and $M$ in $(\mathcal{Z}, \Gamma)$, $NCl(K) + M = 1$, we have $NCl(K) + NCl(B) = 1$.

4. Generalized Neutrosophic Pre-Closed Set

**Definition 20.** An NS $\mathcal{S}$ is said to be a neutrosophic generalized pre-closed set (GNPCS in short) in $(\mathcal{Z}, \Gamma)$ if $pNCl(\mathcal{S}) \subseteq B$ whenever $\mathcal{S} \subseteq B$ and $B$ is an NO in $\mathcal{Z}$. The family of all GNPCSs of an NT $(\mathcal{Z}, \Gamma)$ is defined by $GNPC(\mathcal{Z})$. 
Example 2. Let \( \mathcal{Z} = \{a, b\} \) and \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, T\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( T = \langle(0.2,0.3,0.5),(0.8,0.7,0.7)\rangle \). Then, the NS \( \hat{S} = \langle(0.2,0.2,0.2),(0.8,0.7,0.7)\rangle \) is GNPCs in \( \mathcal{Z} \).

Theorem 4. Every NC is a GNPC, but the converse is not true.

Proof. Let \( \hat{S} \) be an NC in \( \mathcal{Z} \), \( \hat{S} \subseteq \hat{B} \) and \( \hat{B} \) is an NOS in \( (\mathcal{Z}, \Gamma) \). Since \( pNCI(\hat{S}) \subseteq NCI(\hat{S}) \) and \( \hat{S} \) is NCS in \( \mathcal{Z} \), \( pNCI(\hat{S}) \subseteq NCI(\hat{S}) = \hat{S} \subseteq \hat{B} \). Therefore, \( \hat{S} \) is GNPCs in \( \mathcal{Z} \). \( \Box \)

Example 3. Let \( \mathcal{Z} = \{u, v\} \) and \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, H\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( H = \langle(0.2,0.3,0.5),(0.8,0.7,0.7)\rangle \). Then, the NS \( \hat{S} = \langle(0.2,0.2,0.2),(0.8,0.7,0.7)\rangle \) is a GNPC in \( \mathcal{Z} \) but not an NCS in \( \mathcal{Z} \).

Theorem 5. Every NaCS is GNPC, but the converse is not true.

Proof. Let \( \hat{S} \) be an NaCS in \( \mathcal{Z} \) and let \( \hat{S} \subseteq \hat{B} \) and \( \hat{B} \) is an NOS in \( (\mathcal{Z}, \Gamma) \). Now, \( NCI(NInt(\hat{S})) \subseteq \hat{S} \). Since \( \hat{S} \subseteq NCI(\hat{S}) \), \( NCI(NInt(\hat{S})) \subseteq NCI(NInt(\hat{S})) \subseteq \hat{S} \). Hence, \( pNCI(\hat{S}) \subseteq \hat{S} \subseteq \hat{B} \). Therefore, \( \hat{S} \) is GNPCs in \( \mathcal{Z} \). \( \Box \)

Example 4. Let \( \mathcal{Z} = \{u, v\} \) and \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, H\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( H = \langle(0.4,0.2,0.5),(0.6,0.7,0.6)\rangle \). Then, the NS \( \hat{S} = \langle(0.3,0.1,0.4),(0.7,0.8,0.7)\rangle \) is a GNPC in \( \mathcal{Z} \) but not NaCSs in \( \mathcal{Z} \) since \( NCI(NInt(\hat{S})) = \langle(0.5,0.6,0.5),(0.5,0.3,0.6)\rangle \not\subseteq \hat{S} \).

Theorem 6. Every GNaC is a GNPC, but the converse is not true.

Proof. Let \( \hat{S} \) be GNaCs in \( \mathcal{Z} \), \( \hat{S} \subseteq \hat{B} \) and \( \hat{B} \) be an NOSs in \( (\mathcal{Z}, \Gamma) \). By Definition 6, \( \hat{S} \cup NCI(NInt(\hat{S})) \subseteq \hat{B} \). This implies \( NCI(NInt(\hat{S})) \subseteq \hat{B} \) and \( NCI(NInt(\hat{S})) \subseteq \hat{B} \). Therefore, \( pNCI(\hat{S}) = \hat{S} \cup NCI(NInt(\hat{S})) \subseteq \hat{B} \). Hence, \( \hat{S} \) is GNPCs in \( \mathcal{Z} \). \( \Box \)

Example 5. Let \( \mathcal{Z} = \{u, v\} \) and \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, H\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( H = \langle(0.5,0.6,0.6),(0.5,0.4,0.4)\rangle \). Then, the NS \( \hat{S} = \langle(0.4,0.50.5),(0.6,0.5,0.5)\rangle \) is GNPC in \( \mathcal{Z} \) but not GNaC in \( \mathcal{Z} \) since \( aNCI(\hat{S}) = 1_{\mathcal{N}} \not\subseteq \hat{H} \).

Definition 21. An NS \( \hat{S} \) is said to be a neutrosophic generalized pre-closed set (GNCSs ) in \( (\mathcal{Z}, \Gamma) \) if \( SNCl(\hat{S}) \subseteq \hat{B} \) whenever \( \hat{S} \subseteq \hat{B} \) and \( \hat{B} \) is an NO in \( \mathcal{Z} \). The family of all GNSCSs of an NT \( (\mathcal{Z}, \Gamma) \) is defined by \( GNCS(\mathcal{Z}) \).

Proposition 5. Let \( \hat{S}, \hat{B} \) be a two GNPCs of an NT \( (\mathcal{Z}, \Gamma) \). NGSC and NGPC are independent.

Example 6. Let \( \mathcal{Z} = \{u, v\} \), \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, H\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( H = \langle(0.5,0.4,0.4),(0.5,0.6,0.5)\rangle \). Then, the NS \( \hat{S} = H \) is GNSC but not GNPC in \( \mathcal{Z} \) since \( \hat{S} \subseteq H \) but \( pNCI(\hat{S}) = \langle(0.5,0.6,0.4),(0.5,0.4,0.5)\rangle \not\subseteq H \)

Example 7. Let \( \mathcal{Z} = \{u, v\} \), \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, H\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( H = \langle(0.7,0.9,0.7),(0.3,0.1,0.1)\rangle \). Then, the NS \( \hat{S} = \langle(0.6,0.7,0.6),(0.4,0.3,0.4)\rangle \) is GNPC but not GNSsC in \( \mathcal{Z} \) since \( sNCI(\hat{S}) = 1_{\mathcal{N}} \not\subseteq H \).

Proposition 6. NSC and GNPC are independent.

Example 8. Let \( \mathcal{Z} = \{a,b\} \), \( \Gamma = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, T\} \) be a neutrosophic topology on \( \mathcal{Z} \), where \( T = \langle(0.5,0.2,0.3),(0.5,0.6,0.5)\rangle \). Then, the NS \( \hat{S} = T \) is an NSC but not GNPC in \( \mathcal{Z} \) since \( \hat{S} \subseteq T \) but \( pNCI(\hat{S}) = 1_{\mathcal{N}} \not\subseteq T \).
Example 9. Let \( \mathcal{X} = \{u, v\}, \Gamma = \{0_N, 1_N, H\} \) be a neutrosophic topology on \( \mathcal{X} \), where \( H = ((0.8, 0.8, 0.8), (0.2, 0.2, 0.2)) \). Then, the NS \( \tilde{S} = ((0.8, 0.8, 0.8), (0.2, 0.2, 0.2)) \) is GNPC but not an NSC in \( \mathcal{X} \) since \( \mathrm{NInt}(\mathrm{NCl}(\tilde{S})) \not\subset \tilde{S} \).

The following Figure 1 shows the implication relations between GNPC set and the other existed ones.

![Figure 1. Relation between GNPC and others exists set.](image)

**Remark 1.** Let \( \tilde{S}, B \) be a two GNPCs of an NT \((\mathcal{X}, \Gamma)\). Then, the union of any two GNPCs is not a GNPC in general—see the following example.

**Example 10.** Let \((\mathcal{X}, \Gamma)\) be a neutrosophic topology set on \( \mathcal{X} \), where \( \mathcal{X} = \{u, v\} \), \( T = ((0.6, 0.8, 0.6), (0.4, 0.2, 0.2)) \). Then, \( \Gamma = \{0_N, 1_N, T\} \) is neutrosophic topology on \( \mathcal{X} \) and the NS \( \tilde{S} = ((0.2, 0.9, 0.3), (0.8, 0.2, 0.6)) \), \( B = ((0.6, 0.7, 0.6), (0.4, 0.3, 0.4)) \) are GNPCs but \( \tilde{S} \cup B \) is not a GNPC in \( \mathcal{X} \).

5. Generalized Neutrosophic Pre-Open Sets

In this section, we present generalized neutrosophic pre-open sets and investigate some of their properties.

**Definition 22.** An NS \( \tilde{S} \) is said to be a generalized neutrosophic pre-open set (GNPOS) in \((\mathcal{X}, \Gamma)\) if the complement \( \tilde{S}^c \) is a GNPCS in \( \mathcal{X} \). The family of all GNPOSs of NTS \((\mathcal{X}, \Gamma)\) is denoted by GNPO(\( \mathcal{X} \)).
Example 11. Let $\mathcal{Z} = \{u,v\}$ and $\Gamma = \{0_N,1_N,H\}$ be a neutrosophic topology on $\mathcal{Z}$, where $H = \{(0.8,0.7,0.8),(0.3,0.4,0.3)\}$. Then, the NS $\tilde{S} = \{(0.9,0.8,0.8),(0.3,0.3,0.3)\}$ is GNPO in $\mathcal{Z}$.

Theorem 7. Let $(\mathcal{Z},\Gamma)$ be an NT. Then, for every $\tilde{S} \in GNPO(\mathcal{Z})$ and for every $\tilde{R} \in NS(\mathcal{Z})$, $pNInt(\tilde{S}) \subseteq \tilde{R} \subseteq \tilde{S}$ implies $\tilde{R} \in GNPO(\mathcal{Z})$.

Proof. By Theorem 7. Let $\tilde{S}^c \subseteq R^c \subseteq (pNInt(\tilde{S}))^c$. Let $R^c \subseteq R$ and $R$ be NOs. Since $\tilde{S}^c \subseteq \tilde{B}^c$, $\tilde{S}^c \subseteq R$. However, $\tilde{S}^c$ is a GNPC, $\tilde{pNCl}(\tilde{S}^c) \subseteq \tilde{R}$. In addition, $\tilde{R}^c \subseteq (pNInt(\tilde{S}))^c = \tilde{pNCl}(\tilde{S}^c)$ (by theorem). Therefore, $\tilde{pNCl}(\tilde{R}^c) \subseteq \tilde{pNCl}(\tilde{S}^c) \subseteq \tilde{R}$. Hence, $\tilde{B}^c$ is GNPC. This implies that $\tilde{R}$ is a GNPO of $\mathcal{Z}$. \qed

Remark 2. Let $\tilde{S}, \tilde{R}$ be two GNPOs of an NT $(\mathcal{Z},\Gamma)$. The intersection of any two GNPOs is not a GNPO in general.

Example 12. Let $\mathcal{Z} = \{u,v\}$ and $\Gamma = \{0_N,1_N,H\}$ be a neutrosophic topology on $\mathcal{Z}$, where $H = \{(0.6,0.8,0.6),(0.4,0.2,0.4)\}$. Then, the NSs, $\tilde{S} = \{(0.9,0.2,0.1),(0.1,0.8,0.2)\}$ and $\tilde{R} = \{(0.4,0.3,0.4),(0.6,0.7,0.6)\}$ is GNPO, but $\tilde{S} \cap \tilde{R}$ is not GNPO in $\mathcal{Z}$.

Theorem 8. For any an NTS $(\mathcal{Z},\Gamma)$, the following hold:

(i) Every NO is GNPO.
(ii) Every NSO is GNPO.
(iii) Every NaO is GNPO.
(iv) Every NPO is GNPO.

Proof. The proof is clear, so it has been omitted. \qed

The converses are not true in general.

Example 13. Let $\mathcal{Z} = \{u,v\}$ and $\Gamma = \{0_N,1_N,H\}$ is a neutrosophic topology on $\mathcal{Z}$, an NS $\tilde{S} = \{(0.8,0.7,0.7),(0.2,0.2,0.2)\}$ is an NSO in $(\mathcal{Z},\Gamma)$ but not an NO $\in \mathcal{Z}$.

Example 14. Let $\mathcal{Z} = \{u,v\}$ and $\Gamma = \{0_N,1_N,H\}$ be neutrosophic topology on $\mathcal{Z}$, where $H = \{(0.6,0.4,0.7),(0.7,0.4,0.6)\}$. Then, an NS $\tilde{S} = \{(0.2,0.7,0.7),(0.8,0.3,0.8)\}$ is GNPO but not an NSO $\in \mathcal{Z}$.

Example 15. Let $\mathcal{Z} = \{u,v\}$ and $\Gamma = \{0_N,1_N,H\}$ be neutrosophic topology on $\mathcal{Z}$, where $H = \{(0.4,0.2,0.4),(0.6,0.7,0.6)\}$. Then, an NS $\tilde{S} = \{(0.8,0.9,0.8),(0.4,0.2,0.3)\}$ is GNPO but not an NaO $\in \mathcal{Z}$.

Example 16. Let $\mathcal{Z} = \{u,v\}$ and $\Gamma = \{0_N,1_N,H\}$ be neutrosophic topology on $\mathcal{Z}$, where $H = \{(0.6,0.5,0.6),(0.5,0.6,0.5)\}$. Then, an NS $\tilde{S} = \{(0.8,0.7,0.8),(0.4,0.5,0.3)\}$ is GNPO but not an NPO $\in \mathcal{Z}$.

Theorem 9. Let $(\mathcal{Z},\Gamma)$ be an NT. If $\tilde{S} \in GNPO(\mathcal{Z})$, then $\tilde{R} \subseteq NInt(NCl(\tilde{S}))$ whenever $\tilde{R} \subseteq \tilde{S}$ and $\tilde{R}$ is an NC in $\mathcal{Z}$.

Proof. Let $\tilde{S} \in GNPO(\mathcal{Z})$. Then, $\tilde{S}^c$ is GnPCs in $\mathcal{Z}$. Therefore, $pNCl(\tilde{S}^c) \subseteq \tilde{B}$ whenever $\tilde{S}^c \subseteq \tilde{B}$ and $\tilde{B}$ is an NO in $\mathcal{Z}$. That is, $NCl(NInt(\tilde{S}^c)) \subseteq \tilde{B}$. This implies $\tilde{B}^c \subseteq NInt(NCl(\tilde{S}))$ whenever $\tilde{B}^c \subseteq \tilde{S}$ and $\tilde{B}^c$ is NCs in $\mathcal{Z}$. Replacing $\tilde{B}^c$, by $\tilde{R}$, we get $\tilde{R} \subseteq NInt(NCl(\tilde{S}))$ whenever $\tilde{R} \subseteq \tilde{S}$ and $\tilde{R}$ is an NC in $\mathcal{Z}$. \qed

Theorem 10. For NS $\tilde{S}$, $\tilde{S}$ is an NO and GNPC in $\mathcal{Z}$ if and only if $\tilde{S}$ is an NRO in $\mathcal{Z}$.
Theorem 13. Let $\tilde{S}$ be an NO and a GNPCS in $\mathcal{X}$. Then, $pNCl(\tilde{S}) \subseteq \tilde{S}$. This implies $NCI(NInt(\tilde{S})) \subseteq \tilde{S}$. Since $\tilde{S}$ is an NO, it is an NPO. Hence, $\tilde{S} \subseteq NInt(NCl(\tilde{S}))$. Therefore, $\tilde{S} = NInt(NCl(\tilde{S}))$. Hence, $\tilde{S}$ is an NRO in $\mathcal{X}$.

$\iff$ Let $\tilde{S}$ be an NRO in $\mathcal{X}$. Therefore, $\tilde{S} = NInt(NCl(\tilde{S}))$. Let $\tilde{S} \subseteq \tilde{B}$ and $\tilde{B}$ be an NO in $\mathcal{X}$. This implies $pNCl(\tilde{S}) \subseteq \tilde{S}$. Hence, $\tilde{S}$ is GNPC in $\mathcal{X}$. $\square$

Theorem 11. An NS $\tilde{S}$ of an NT $(\mathcal{X}, \Gamma)$ is a GNPO iff $H \subseteq pNInt(\tilde{S})$, whensoever $H$ is an NC and $H \subseteq \tilde{S}$.

Proof. $\implies$ Let $\tilde{S}$ be the NO in $\mathcal{X}$. Let $H$ be an NCs and $H \subseteq \tilde{S}$. Then, $H^c$ is an NOS in $\mathcal{X}$ such that $\tilde{S}^c \subseteq H^c$. Since $\tilde{S}^c$ is GNPC, we have $pNCl(\tilde{S}^c) \subseteq H^c$. Hence, $(pNInt(\tilde{S}))^c \subseteq H^c$. Therefore, $H \subseteq pNInt(\tilde{S})$.

$\iff$ Suppose $\tilde{S}$ is an NO of $\mathcal{X}$ and let $H \subseteq pNInt(\tilde{S})$ whensoever $H$ is an NC and $H \subseteq \tilde{S}$. Then, $\tilde{S}^c \subseteq H^c$ and $H^c$ is an NO. By assumption, $(pNInt(\tilde{S}))^c \subseteq H^c$, which implies $pNCl(\tilde{S}^c) \subseteq H^c$. Therefore, $\tilde{S}^c$ is a GNPOS of $\mathcal{X}$. Hence, $\tilde{S}$ is a GNPOS of $\mathcal{X}$.

Corollary 1. An NS $\tilde{S}$ of an NTS $(\mathcal{X}, \Gamma)$ is a GNPO iff $H \subseteq NInt(NCl(\tilde{S}))$, whensoever $H$ is an NC and $H \subseteq \tilde{S}$.

Proof. $\implies$ Let $\tilde{S}$ be a GNPOS in $\mathcal{X}$. Let $H$ be an NCs and $H \subseteq \tilde{S}$. Then, $H^c$ is an NOS in $\mathcal{X}$ such that $\tilde{S}^c \subseteq H^c$. Since $\tilde{S}^c$ is GNPC, we have $pNCl(\tilde{S}^c) \subseteq H^c$. Therefore, $NInt(NCl(\tilde{S}^c)) \subseteq H^c$. Hence, $(NInt(NCl(\tilde{S}))^c \subseteq H^c$. This implies $H \subseteq NInt(\tilde{S})$.

$\iff$ Suppose $\tilde{S}$ is an NS of $\mathcal{X}$ and let $H \subseteq NInt(\tilde{S})$ whensoever $H$ is an NC and $H \subseteq \tilde{S}$. Then, $\tilde{S}^c \subseteq H^c$ and $H^c$ is an NO. By assumption, $(NInt(NCl(\tilde{S}))^c \subseteq H^c$. Hence, $NInt(NCl(\tilde{S})) \subseteq H^c$. This implies $pNCl(\tilde{S}^c) \subseteq H^c$. Hence, $\tilde{S}$ is a GNPOS of $\mathcal{X}$. $\square$

6. Applications of Generalized Neutrosophic Pre-Closed Sets

Definition 23. An NTS $(\mathcal{X}, \Gamma)$ is said to be neutrosophic-\textit{pT}$_1$ ($\textit{NpT}_1$) in short space if every GNPC in $\mathcal{X}$ is an NCs in $\mathcal{X}$.

Definition 24. An NTS $(\mathcal{X}, \Gamma)$ is said to be neutrosophic-\textit{gpT}$_1$ ($\textit{NgpT}_1$) in short space if every GNPC in $\mathcal{X}$ is an NPOS in $\mathcal{X}$.

Theorem 12. Every $\textit{NpT}_1$ space is an $\textit{NgpT}_1$ space.

Proof. Let $\mathcal{X}$ be an $\textit{NpT}_1$ space and $\tilde{S}$ be GNPC in $\mathcal{X}$. By assumption, $\tilde{S}$ is NC in $\mathcal{X}$. Since every NC is an NPC, $\tilde{S}$ is NPC in $\mathcal{X}$. Hence, $\mathcal{X}$ is an $\textit{NgpT}_1$ space. $\square$

The converse is not true.

Example 17. Let $\mathcal{X} = \{u, v\}$, $H = \{(0.9, 0.9, 0.9), (0.1, 0.1, 0.1)\}$ and $\Gamma = \{0_N, 1_N, H\}$. Then, $(\mathcal{X}, \Gamma)$ is an $\textit{NgpT}_1$ space, but it is not $\textit{NpT}_1$ since an NS $H = \{(0.2, 0.3, 0.3), (0.8, 0.7, 0.7)\}$ is GNPC but not an NCS in $\mathcal{X}$.

Theorem 13. Let $(\mathcal{X}, \Gamma)$ be an NT and $\mathcal{X}$ is an $\textit{NpT}_1$ space; then,

(i) the union of GNPCs is GNPC,
(ii) the intersection of GNPOs is GNPO.

Proof. (i) Let $\{\tilde{S}_i\}_{i \in I}$ be a collection of GNPCs in an $\textit{NpT}_1$ space $(\mathcal{X}, \Gamma)$. Thus, every GNPCs is an NCS. However, the union of an NC is an NCS. Therefore, the Union of GNPCs is GNPCs in $\mathcal{X}$.

(ii) Proved by taking complement in (i). $\square$
Theorem 14. An NT $\mathcal{X}$ is an $NgpT_2$ space iff $GNPO(\mathcal{X}) = NPO(\mathcal{X})$.

Proof. $\implies$ Let $\tilde{\mathcal{S}}$ be a $GNPO$ in $\mathcal{X}$; then, $\tilde{\mathcal{S}}^c$ is $GNPCs$ in $\mathcal{X}$. By assumption, $\tilde{\mathcal{S}}^c$ is an $NPCs$ in $\mathcal{X}$. Thus, $\tilde{\mathcal{S}}$ is $NPOs$ in $\mathcal{X}$. Hence, $GNPO(\mathcal{X}) = NPO(\mathcal{X})$.

$\impliedby$ Let $\tilde{\mathcal{S}}$ be $GNPC \in \mathcal{X}$. Then, $\tilde{\mathcal{S}}^c$ is $GNPO$ in $\mathcal{X}$. By assumption, $\tilde{\mathcal{S}}^c$ is an $NPO$ in $\mathcal{X}$. Thus, $\tilde{\mathcal{S}}$ is an $NPC \in \mathcal{X}$. Therefore, $\mathcal{X}$ is an $NgpT_2$ space.

Theorem 15. For an NTS $(\mathcal{X}, \Gamma)$, the following are equivalent:

(i) $(\mathcal{X}, \Gamma)$ is a neutrosophic pre-$T_2$ space.

(ii) Every non-empty set of $\mathcal{X}$ is either an NPCS or NPOS.

Proof. (i) $\implies$ (ii). Suppose that $(\mathcal{X}, \Gamma)$ is a neutrosophic pre-$T_2$ space. Suppose that $\{x\}$ is not an NPCS for some $x \in \mathcal{X}$. Then, $\mathcal{X} - \{x\}$ is not an NPOS and hence $\mathcal{X}$ is the only an NPOS containing $\mathcal{X} - \{x\}$. Hence, $\mathcal{X} - \{x\}$ is an NPGCS in $(\mathcal{X}, \Gamma)$. Since $(\mathcal{X}, \Gamma)$ is a neutrosophic pre-$T_2$ space, then $\mathcal{X} - \{x\}$ is an NPCS or equivalently $\{x\}$ is an NPOS. (ii) $\implies$ (i). Let every singleton set of $\mathcal{X}$ be either NPCS or NPOS. Let $\tilde{\mathcal{S}}$ be an NPGCS of $(\mathcal{X}, \Gamma)$. Let $x \in \mathcal{X}$. We show that $x \in \mathcal{X}$ in two cases.

Case (i): Suppose that $\{x\}$ is NPCS. If $x \notin \tilde{\mathcal{S}}$, then $x \in pNCl(\tilde{\mathcal{S}}) - \tilde{\mathcal{S}}$. Now, $pNCl(\tilde{\mathcal{S}}) - \tilde{\mathcal{S}}$ contains a non-empty NPCS. Since $\tilde{\mathcal{S}}$ is NPGCS, by Theorem 7, we arrived to a contradiction. Hence, $x \in \mathcal{X}$.

Case (ii): Let $\{x\}$ be NPOS. Since $x \in pNCl(\tilde{\mathcal{S}})$, then $\{x\} \cap \tilde{\mathcal{S}} \neq \emptyset$. Thus, $x \in \mathcal{X}$. Thus, in any case $x \in \mathcal{X}$. Thus, $PNCI(\tilde{\mathcal{S}}) \subseteq \tilde{\mathcal{S}}$. Hence, $\tilde{\mathcal{S}} = pNCl(\tilde{\mathcal{S}})$ or equivalently $\tilde{\mathcal{S}}$ is an NPCS. Thus, every NPGCS is an NCS. Therefore, $(\mathcal{X}, \Gamma)$ is neutrosophic pre-$T_2$ space.

7. Conclusions

We have introduced generalized neutrosophic pre-closed sets and generalized neutrosophic pre-open sets over neutrosophic topology space. Many results have been established to show how far topological structures are preserved by these neutrosophic pre-closed. We also have provided examples where such properties fail to be preserved. In this paper, we have studied a few ideas only; it will be necessary to carry out more theoretical research to establish a general framework for decision-making and to define patterns for complex network conceiving and practical application.

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