New exact solution of the one-dimensional Dirac equation for the Woods–Saxon potential within the effective mass case

O Panella, S Biondini and A Arda

1 Istituto Nazionale di Fisica Nucleare, Sezione di Perugia, Via A. Pascoli, I-06123 Perugia, Italy
2 Dipartimento di Fisica, Università degli Studi di Perugia, Via A. Pascoli, I-06123 Perugia, Italy
3 Department of Physics Education, Hacettepe University, 06800 Ankara, Turkey

E-mail: orlando.panella@pg.infn.it

Received 12 March 2010, in final form 8 June 2010
Published 9 July 2010
Online at stacks.iop.org/JPhysA/43/325302

Abstract
We study the one-dimensional Dirac equation in the framework of a position-dependent mass under the action of a Woods–Saxon external potential. We find that by constraining appropriately the mass function it is possible to obtain a solution of the problem in terms of the hypergeometric function. The mass function for which this turns out to be possible is continuous. In particular, we study the scattering problem and derive exact expressions for the reflection and transmission coefficients which are compared to those of the constant mass case. For the very same mass function the bound state problem is also solved, providing a transcendental equation for the energy eigenvalues which is solved numerically.

PACS numbers: 03.65.-w, 03.65.Ge, 12.39.Fd

(Some figures in this article are in colour only in the electronic version)

1. Introduction
In recent years, the study of several quantum mechanical systems within the framework of an effective position-dependent mass (PDM) has received increasing attention in the literature. Position-dependent effective masses enter, for example, into the dynamics of electrons in semiconductor hetero-structures [1], and when describing the properties of hetero-junctions and quantum dots [2]. In non-relativistic quantum mechanics, when the mass becomes dependent on the position coordinate, the mass and momentum operators no longer commute, thereby making the generalization of the non-relativistic Hamiltonian (kinetic energy operator) to the PDM case highly non-trivial [3, 4], because of the ambiguities in the choice of a correct ordering of mass and momentum operator [5]. Another important issue is that of a Galilean invariance [6].
The investigation of relativistic effects is of course important in those systems containing heavy atoms or heavy ion doping [7]. Therefore for these types of materials, the investigation of the properties of the Dirac equation in circumstances where the mass becomes a function of the position is certainly of great interest. In addition the problems posed by the ambiguities of the mass and momentum operator ordering are absent in the Dirac equation. An effort in this direction has been reported in some recent literature [7–15]. For example the authors of [7] have reported an interesting numerical investigation of the scattering problem for the three-dimensional Dirac Hamiltonian within a position-dependent mass with a constant asymptotic limit, studying the energy resonance structure. In [16] the scattering problem is solved for a smooth potential and a mass step but in the non-relativistic regime. The authors of [17] reported an approximated solution of the one-dimensional Dirac equation with a position-dependent mass for the generalized Hulthén potential. To the best of our knowledge few attempts have been reported that study the Dirac equation in an external potential with position-dependent effective masses. In [18] the author studies Dirac equation in 3+1 dimensions in the Coulomb field and with a spherically symmetric singular mass distribution. In [10] the author reports an exact solution for the Dirac equation with central potential and mass distribution both inversely proportional to the distance from the center.

It is worth pointing out that graphene (single atomic layer of graphite), a recently discovered material [19, 20] which is receiving a lot of attention, exhibits several properties whose explanation involve the Dirac equation for massless fermions. For a comprehensive review, see for example [21]. Recent reports studying these effects [22, 23] attest the use of the Dirac equation in explaining the properties of single-layer graphene.

This work is an attempt in the same direction as that of [10, 18]. We report an exact solution of the one-dimensional Dirac equation in the position-dependent mass formalism for a particle in the Woods–Saxon (WS) potential. Our approach is based on that of [24] where the author solves the one-dimensional Dirac equation in the WS potential for the ordinary constant mass case. Our method consists in requiring, within the effective-position dependent mass, that the second-order equation still be exactly solvable by the hypergeometric function. This is done by imposing restrictive conditions on the mass function which lead to a first-order differential equation which provides the explicit mass function.

We would also like to stress that our new analytical exact solution of the position-dependent mass Dirac equation in the WS potential will prove to be certainly useful in further studies of effective mass models. Other issues could for example be addressed that go beyond the scope of this work: for example, it would certainly be of interest to study in detail the Klein paradox, as well as the issue of zero momentum resonances that support a bound state at $E = -m$, i.e. super-critical states, in the framework of effective masses. We also note that the study of the transmission coefficient in the case of two-dimensional Dirac equation for massless fermions has already been used to describe the electrical properties of graphene and in particular the possibility of observing the Klein paradox phenomena [25] in this material. In addition the authors of [23] discuss the case of massless electrons that cross a square barrier region where they are instead massive, a situation that can simulate a n–p–n junction in a graphene nanotransistor. We believe that our exact solution derived for the WS potential with an effective position-dependent mass, in the limit $aL \gg 1$ the WS potential barrier reduces to a square barrier, may prove useful to describe such a real physical system. Further investigation in this direction is needed but is beyond the scope of this work.

The plan of the paper is as follows. In section 2, we summarize the basic equations of the problem. In section 3, we solve the effective-mass Dirac equation and provide the mass function for the problem. We study the scattering problem for the potential barrier and deduce the transmission and reflection coefficients by studying the asymptotic behavior of
the wavefunction when $x \to \pm \infty$ and the match at $x = 0$. In section 4 we also address the bound states of the problem by turning the WS potential barrier into a WS potential well. We discuss several numerical examples providing the eigenvalues and wavefunctions corresponding to particular choices of the parameters. Finally we summarize our results and present our conclusions in section 5.

2. The basic equations

We recall the free Dirac equation (using natural units, $\hbar = c = 1$)

$$\left( i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \Psi(x) = 0,$$  \hspace{1cm} (1)

and $\mu = 0, 1, 2, 3$ is a spacetime index. Considering instead the case of one space dimension, it is possible to choose the gamma matrices $\gamma^x$ and $\gamma^0$ of dimension 2 and it is customary to set them respectively to the Pauli matrices $i\sigma_x$ and $\sigma_z$ [9]. Considering a charge particle minimally coupled to an electromagnetic potential, in the absence of the space component of a vector potential, and setting $V(x) = eA_0(x)$ the one-dimensional Dirac equation for a stationary state $\Psi(x, t) = e^{-iEt}\psi(x)$ becomes

$$\left[ \sigma_x \frac{d}{dx} - (E - V(x))\sigma_z + m \right] \psi(x) = 0.$$  \hspace{1cm} (2)

Decomposing the Dirac spinor $\psi(x)$ into upper ($u_1$) and lower ($u_2$) components: $\psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, gives the coupled equations

$$u_1'(x) = -[m + E - V(x)]u_2(x),$$  \hspace{1cm} (3a)

$$u_2'(x) = -[m - E + V(x)]u_1(x).$$  \hspace{1cm} (3b)

It turns out to be convenient to define two auxiliary components $\phi(x)$ and $\chi(x)$ in terms of $u_1(x)$ and $u_2(x)$ as in [9]:

$$\phi(x) = u_1(x) + iu_2(x),$$ \hspace{1cm} (4a)

$$\chi(x) = u_1(x) - iu_2(x).$$ \hspace{1cm} (4b)

Using the above definitions and equations (3a) and (3b), we find the first-order coupled equations for the components $\phi(x)$ and $\chi(x)$:

$$\phi'(x) = -im\chi(x) + i[E - V(x)]\phi(x),$$ \hspace{1cm} (5a)

$$\chi'(x) = +im\phi(x) - i[E - V(x)]\chi(x),$$ \hspace{1cm} (5b)

which give the second-order equations

$$\phi''(x) + [(E - V(x))^2 - m^2 + iV'(x)]\phi(x) = -im'\chi(x),$$ \hspace{1cm} (6a)

$$\chi''(x) + [(E - V(x))^2 - m^2 - iV'(x)]\chi(x) = +im'\phi(x),$$ \hspace{1cm} (6b)

where we have taken into account the fact that the mass may depend on the position coordinate and prime denotes the derivative with respect to $x$. Eliminating $\chi(x)$ using equation (5a), we obtain for $\phi(x)$ the second-order equation

$$\phi''(x) - \frac{m'}{m} \phi'(x) + \left[ (E - V(x))^2 - m^2 + iV'(x) + i \frac{m'}{m} (E - V(x)) \right] \phi(x) = 0.$$ \hspace{1cm} (7)
Figure 1. Left: plot of the Woods–Saxon potential \( (W = 1) \). Right: plot of the position-dependent mass function \( m_0 = 1 \). The parameters are \( L = 2 \) and \( a = 10 \) (solid line), \( a = 5 \) (dashed line) and \( a = 3 \) (dot-dashed line). In the limit \( aL \gg 1 \), the Woods–Saxon potential reduces to a smooth barrier approaching a square barrier.

Solving this second-order differential equation, one can obtain the \( \chi(x) \) component via equation (5a) and then reconstruct the upper \( u_1(x) \) and lower \( u_2(x) \) components of the complete spinor solution \( \psi(x) \). Equation (7) reduces to the one studied in [24] if \( m' = 0 \), i.e. if the mass reduces to a constant. Thus we see that keeping a position dependence in the mass introduces two new terms: one which multiplies \( \phi'(x) \) while the other enters the \( \phi(x) \) term. These new terms must be appropriately constrained in order to be able to solve the equation in terms of the hypergeometric function.

Let us make a final remark before discussing the details of the computations. The attentive reader might wonder what would happen if one were to derive a second-order equation for the \( \chi(x) \) component and then computing it through equation (5b). The second-order equation for the \( \chi(x) \) component turns out to be

\[
\chi''(x) - \frac{m'}{m} \chi'(x) + \left[ (E - V(x))^2 - m^2 - iV'(x) - i \frac{m'}{m} (E - V(x)) \right] \chi(x) = 0.
\]

(8)

It is easily checked that equation (8) can be obtained from equation (7) using the map \( E \rightarrow -E \) and \( V(x) \rightarrow -V(x) \). This can be interpreted as the negative energy solution corresponding to the charge conjugate particle (antiparticle). Since \( V(x) \) is the temporal component of a four-potential, the change \( V(x) \rightarrow -V(x) \) amounts to reversing the charge of the particle. Indeed with \( E \rightarrow -E \) and \( V(x) \rightarrow -V(x) \), we have \( \chi \rightarrow \phi \), and by using equation (5a), \( \phi \rightarrow -\chi \) and by using the inverse of equations (4a), (4b) we have in turn \( u_1 \rightarrow iu_2 \) and \( u_2 \rightarrow iu_1 \) which amounts, up to inessential phase factors, to the charge conjugation symmetry of the Dirac equation [24].

3. Effective-mass Dirac scattering problem

The form of the WS potential (illustrated in figure 1, see left plot) is given by (see also [24])

\[
V(x) = W \left[ \frac{\theta(-x)}{e^{a(x+L)} + 1} + \frac{\theta(x)}{e^{a(x-L)} + 1} \right],
\]

(9)

where \( W \) is a positive parameter in the scattering problem (potential barrier) and negative in the bound state problem (potential well); \( a \) and \( L \) are two real and positive parameters.
3.1. Solution in the negative region \((x < 0)\)

Using the variable \(y = -e^{-a(x+L)}\) and the transformation \(\phi = y^\mu(1 - y)^{-\lambda}f(y)\) used in [24], we will show that it is possible to obtain an exact solution in the form of a hypergeometric function by imposing appropriate constraints on the mass function. With the above transformation equation (7) becomes

\[
y(1 - y) \frac{d^2 f(y)}{dy^2} + \left[ 1 + 2\mu - y(1 + 2\mu - 2\lambda) - \frac{m}{m} y(1 - y) \right] \frac{df(y)}{dy} + \left\{ \frac{\lambda(1 + 2\mu)}{y(1 - y)} \left[ \mu^2 (1 - y)^2 + \lambda(1 + \lambda)y^2 \right] + \frac{1}{a^2} \left[ (E^2 - m^2)(1 - y)^2 + W^2 - 2EW(1 - y) - iayW \right] \right\} f(y)
\]

\[
- \frac{m}{m} y(1 - y) \left[ \frac{\mu}{y} + \frac{\lambda}{1 - y} + \frac{i}{ay} \left( E - \frac{W}{1 - y} \right) \right] f(y) = 0.
\]

(10)

Here and in the following, the dot indicates derivation with respect to the transformed variable \((\dot{m} = dm/dy)\). In order to keep the structure of the hypergeometric differential equation we impose the following condition on this term:

\[
- \frac{m}{m} y(1 - y) = \alpha + \beta y
\]

which has the following solution \((m_0 \text{ integration constant})\):

\[
m(y) = m_0 \left| \frac{y - 1}{y^{\alpha \beta}} \right|^\alpha.
\]

(12)

With this choice of mass function, equation (10) becomes

\[
y(1 - y) \frac{d^2 f(y)}{dy^2} + \left[ 1 + 2\mu - y(1 + 2\mu - 2\lambda) + (\alpha + \beta y) \right] \frac{df(y)}{dy} + \left\{ \frac{\lambda(1 + 2\mu)}{y(1 - y)} \left[ \mu^2 (1 - y)^2 + \lambda(1 + \lambda)y^2 \right] + \frac{1}{a^2} \left[ (E^2 - m^2)(1 - y)^2 + W^2 - 2EW(1 - y) - iayW \right] \right\} f(y) = 0.
\]

(13)

In order that equation (13) keeps the structure of the hypergeometric differential equation as in the \(m = \text{const.} \) case, we may impose the following conditions on the mass function:

\[
\lim_{y \to -\infty} m(y) = m_0 \quad m^2(1 - y)^2 = m_0^2 y^2.
\]

(14)

From these conditions we fix completely the two parameters to \(\alpha = -1\) and \(\beta = 0\) so that the mass function (in the \(y \leq 0 \) region) becomes

\[
m(y) = m_0 \frac{y}{y - 1}.
\]

(15)

The most general condition that we can impose on the term multiplying to \(1/|y(1 - y)|\) in order that the equation be that of the hypergeometric function is that it be equal to a constant
\( \gamma \). Therefore, we get three equations

\[
\begin{align*}
\mu^2 + \frac{E^2}{a^2} + \frac{W^2}{a^2} - 2\frac{EW}{a^2} - \mu - \frac{i}{a}(E - W) &= 0 \\
-2\frac{E^2}{a^2} + 2\frac{EW}{a^2} - i\frac{W - E}{a} &= -2\mu^2 - (\lambda - \mu) = \gamma \\
\mu^2 + \lambda(1 + \lambda) + \frac{E^2 - m_0^2}{a^2} &= -\gamma.
\end{align*}
\]  

(16a) (16b) (16c)

From equation (16a), it is possible to solve for \( \mu \) while \( \lambda \) is found summing equation (16b) and (16c). We finally obtain

\[
\begin{align*}
\lambda &= i\sqrt{\frac{W^2 - m_0^2}{a^2}} \\
\mu &= -i\frac{(E - W)}{a} \\
\gamma &= \nu^2 - \mu^2 - \lambda(\lambda + 1)
\end{align*}
\]  

(17a) (17b) (17c)

having defined \( \nu = ik/a \) where \( \kappa^2 = E^2 - m_0^2 \). Our equation (13) becomes the differential equation of the hypergeometric function

\[
y(1 - y)\frac{d^2 f(y)}{dy^2} + [2\mu - (1 + 2\mu - 2\kappa)y] \frac{df(y)}{dy} - (\mu - \lambda - \nu)(\mu - \lambda + \nu)f = 0,
\]

(18)

and the general solution is (with \( D_1 \) and \( D_2 \) constants)

\[
f(y) = D_1 y^\nu \, _2F_1(\mu - \nu - \lambda, \mu + \nu - \lambda; 2\mu; y) + D_2 y^{1-2\mu} \, _2F_1(1 - \mu - \nu - \lambda, 1 - \mu + \nu - \lambda; 2 - 2\mu; y).
\]

(19)

3.2. Solution in the positive region \( (x > 0) \)

Let us study the other region in which \( x > 0 \); from equation (7) using the variable \( 1/z = 1 + e^{i(x-\delta)} \) and the transformation \( \phi = z^{-\omega}(1 - z)^{-\rho}g(z) \), we obtain

\[
z(1 - z)\frac{dg(z)}{dz} + \left[ 1 - 2\omega - z(2 - 2\rho - 2\omega) - \frac{m}{m} z(1 - z) \right] \frac{dg(z)}{dz}
\]

\[
+ \left[ 2\omega(1 - \rho) - \frac{m}{m} z(1 - z) \left( -\frac{\omega}{z} + \frac{\rho}{1 - z} \right) + \frac{1}{a} \left[ \omega(\omega + 1)(1 - z)^2 \right. \right.
\]

\[
+ \rho(1 + \rho)z^2 - \omega(1 - z) + z\rho - (2z^2)\rho + \frac{1}{a^2} \left[ (E - Wz)^2 - m(z)^2 \right. \right.
\]

\[
\left. \left. - iaWz(1 - z) - ia(E - Wz)\frac{m}{m} z(1 - z) \right] \right] g(z) = 0.
\]

(20)

Following a line of thought similar to the one outlined in the previous subsection, we obtain the mass function

\[
m(z) = m_0(1 - z).
\]

(21)

The final result for the parameters \( \omega \), \( \rho \) and \( \delta \) (the coefficient introduced requiring that the term multiplying the factor \( 1/|z(1 - z)| \) be a constant) is found to be
Let us briefly comment on the change of variables chosen in the negative region convergence (J. Phys. A: Math. Theor. rather on the properties of the differential equation satisfied by two regions as follows in terms of the transformations such as to use adequate analytic continuation identities [26]. We expect in any case that using other use the hypergeometric function outside the radius of convergence of the series, it is necessary would not be convergent as its radius of convergence is the negative region is not appropriate because then the hypergeometric series would in the end not alter our conclusions. The reader might be worried that derivatives of the bound state problem (see section 4, subsections 4.1 and 4.2).

The differential equation of the function \( g(z) \) in equation (20) becomes

\[
\frac{d}{dz}(1-z) \frac{d^2 g(z)}{dz^2} + [1 - 2\omega - z(1 - 2\rho - 2\omega)] \frac{d g(z)}{dz} - (-\rho - \omega - \lambda)(-\rho - \omega + \lambda) g(z) = 0,
\]

with the general solution \((d_1 \text{ and } d_2 \text{ constants})

\[
g(z) = d_1 \, _2F_1(-\rho - v - \lambda, -\rho - v + \lambda; 1 - 2v; z) + d_2 \, z^{2v} \, _2F_1(-\rho - v + \lambda, -\rho + v + \lambda; 1 + 2v; z).
\]

Let us briefly comment on the change of variables chosen in the negative region \( x \in [-\infty, 0] \): \( y = -e^{-a|x|} \) and in the positive region \( x \in [0, +\infty] \): \( 1/z = 1 + e^{a|x|}. \) This implies clearly that \( y \in [-\infty, -e^{-aL}] \) and \( z \in [(1 + e^{-aL})^{-1}, 0] \) (which reduce to \( y \in [-\infty, 0] \) and \( z \in [1, 0] \) with the assumption \( aL \gg 1 \)). The reader might worry that the choice in the negative region is not appropriate because then the hypergeometric series \( _2F_1(a, b, c; y) \) would not be convergent as its radius of convergence is \( |y| < 1 \). However, we would like to stress that (in the negative region) we use the explicit form of the hypergeometric series only in vicinity of \( x \to 0^+ (|y| \ll 1) \) (in the scattering problem) remaining well within the radius of convergence \( |y| < 1 \). When obtaining the asymptotic expression at \( x \to -\infty (y \to -\infty) \) we use the asymptotic expansion of the hypergeometric series given in equation (27). In the positive region when considering the expansion in \( x \to 0^+ (z \to 1^-) \) to avoid problems of convergence we use the continuation identity of the hypergeometric function given in equation (36). We can say that our exact solution does not rely at all on the hypergeometric series but rather on the properties of the differential equation satisfied by \( _2F_1(a, b, c; y) \). If one needs to use the hypergeometric function outside the radius of convergence of the series, it is necessary to use adequate analytic continuation identities [26]. We expect in any case that using other transformations such as \( y = e^{\alpha t} \) and \( z = e^{-\beta t} \) would in the end not alter our conclusions. Similar considerations apply as well to the change of variables chosen in the discussion of the bound state problem (see section 4, subsections 4.1 and 4.2).

### 3.3. Mass function

From conditions given in equations (15) and (21), we have obtained the mass function in the two regions as follows in terms of the \( x \) variable (illustrated in figure 1, see right plot):

\[
m(x) \equiv m_0 \left[ \frac{e^{-a|x|} + 1}{e^{-a|x|} + 1} \Theta(-x) + \frac{e^{a|x|} + 1}{e^{a|x|} + 1} \Theta(x) \right].
\]

Even if the conditions on the parameters \( \alpha \) and \( \beta \) are different in the two regions we obtain a continuous function, in fact the limit for \( x \to 0^+ \) and \( x \to 0^- \) is the same; the value in \( x = 0 \) is \( m(0) = m_0 e^{-aL}/(e^{-aL} + 1) < m_0 \) as given in figure 1. This mass function has the desired asymptotic behavior at \( x \to \pm \infty \) where it assumes the desired constant value \( m_0 \). The reader might be worried that derivatives of the \( \Theta \)-functions might introduce singularities in the problem, since in equation (7) there appear terms with first derivatives of the mass function.
However, it is quite straightforward to check that, due to the continuity of the mass function at \( x = 0 \), such \( \delta \)-function contributions do cancel exactly.

An interesting and relevant point to address comes from the fact that the mass function is of the same type of the vector potential (the WS potential). Indeed the mass function that we have derived (equation (25)) can be written in the following form:

\[
m(x) = m_0 - \gamma V(x),
\]

where \( \gamma = m_0 / W \). Therefore, the position-dependent mass problem could be looked at as that of a particle of constant mass \( m_0 \) coupling to both a vector potential \( (V(x)) \) and a scalar potential \( S(x) = -\gamma V(x) \). When discussing the barrier \( (W > 0) \) it turns out that \( \gamma > 0 \) and therefore since \( S(x) + V(x) = (1 - \gamma)V(x) \) there might be regions of the parameter space, \( m_0 = W \) (or \( m_0 \approx W \)) where the system is endowed with an exact (or approximate) pseudo-spin symmetry which is defined by the condition \( S(x) + V(x) = \text{constant} \). Such a symmetry, the near equality of an attractive scalar potential with a repulsive vector potential, is well known in the literature \[27, 28\] of the Dirac equation and has been proved very useful in describing the motion of nucleons in the relativistic mean fields resulting form nucleon–meson interactions, nucleon–nucleon Skyrme-type interactions and QCD sum rules.

Further investigation of the possible consequences of this symmetry for the system under consideration goes however beyond the scope of this work.

### 3.4. Asymptotic expressions and boundary conditions of the scattering problem

In the negative region from equation (19), we have

\[
\phi_L(y) = D_1 y^\mu (1 - y)^{-\lambda} \, _2F_1(\mu - v - \lambda, \mu + v - \lambda; 2\mu; y) + D_2 y^{1-\mu} (1 - y)^{-\lambda} \, _2F_1(1 - \mu - v - \lambda, 1 - \mu + v - \lambda; 2 - 2\mu; y). \tag{26}
\]

We can derive the asymptotic expression as \( x \to -\infty \) (\( y \to -\infty \)) by using the following formula for the asymptotic behavior of the hypergeometric function \[26\]

\[
_2F_1(a, b, c; y) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-y)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-y)^{-b}, \tag{27}
\]

and obtain

\[
\phi_L(x) \sim G e^{-ik(x+L)} + H e^{ik(x+L)}, \tag{28}
\]

where

\[
G = D_1 A e^{i\pi \mu} - D_2 C e^{-i\pi \mu} \tag{29a}
\]

\[
H = D_1 B e^{i\pi \mu} - D_2 D e^{-i\pi \mu} \tag{29b}
\]

and \( A, B, C, D \) are given by

\[
A = \frac{\Gamma(2\mu)\Gamma(2v)}{\Gamma(\mu + v - \lambda)\Gamma(\mu + v + \lambda)}, \tag{30a}
\]

\[
B = \frac{\Gamma(2\mu)\Gamma(-2v)}{\Gamma(\mu - v - \lambda)\Gamma(\mu - v + \lambda)}, \tag{30b}
\]

\[
C = \frac{\Gamma(2 - 2\mu)\Gamma(2v)}{\Gamma(1 - \mu + v - \lambda)\Gamma(1 - \mu + v + \lambda)}, \tag{30c}
\]

\[
D = \frac{\Gamma(2 - 2\mu)\Gamma(-2v)}{\Gamma(1 - \mu - v - \lambda)\Gamma(1 - \mu - v + \lambda)}. \tag{30d}
\]
Similarly we can derive the asymptotic form of the lower component \( \chi(x) \) from equation (5a):

\[
\lim_{x \to \pm \infty} \chi(x)_L = G \frac{(E + k)}{m_0} e^{-ik(x+L)} + H \frac{(E - k)}{m_0} e^{ik(x+L)}. \tag{31}
\]

Similarly for the solution in the positive region we have from equation (24)

\[
\phi_R(z) = d_1 z^{\nu} (1-z)^{-\rho} 2F_1(-\rho - \nu - \lambda, -\rho + \nu + \lambda; 1 - 2\nu; z)
+ d_2 z^{\nu} (1-z)^{-\rho} 2F_1(-\rho + \nu - \lambda, -\rho + v + \lambda; 1 + 2\nu; z). \tag{32}
\]

Now we recall that \( z \to 0 \) when \( x \to \infty \) and imposing the boundary condition of the scattering problem that in the \( (x > 0) \) region we only have a wave traveling to the right (only the transmitted wave), we find

\[
\lim_{x \to +\infty} \phi_R(x) = d_1 e^{ik(x-L)} \tag{33}
\]

and \( \chi_R(x) \) is found again through equation (5a) in terms of \( \phi_R \):

\[
\lim_{x \to +\infty} \chi_R(x) = d_1 \frac{(E - k)}{m_0} e^{ik(x-L)}. \tag{34}
\]

### 3.5. Match of solution at \( x = 0 \)

So far we have derived asymptotic expressions at \( x \to \pm \infty \) for the wavefunction of the scattering problem in the negative region, \( x < 0, (\phi_L) \), and in the positive region, \( x > 0, (\phi_R) \), which depend respectively on two, \( D_1 \) and \( D_2 \), and one, \( d_1 \), unknown constants. In order to have a physical solution of the scattering problem, one needs to match the two solutions \( \phi_L \) and \( \phi_R \) at \( x = 0 \). This is done by imposing the continuity of the wavefunction and of its derivative at \( x = 0 \) which gives two conditions and two of the three unknown constants can be expressed in terms of the one left out as the ordinary normalization constant.

We need to find the behavior of the function \( \phi_L(x) \) and \( \phi_R(x) \) in the vicinity of \( x = 0 \).

As \( x \to 0 \) we have \( |y| \approx e^{-aL} \ll 1 \) as our only assumption throughout the paper is that \( aL \gg 1 \). Thus, \( (1 - y)^{-\lambda} \approx 1 \) and from equation (26) we obtain for \( \phi_L(y) \)

\[
\phi_L(x) \sim D_1 (-e^{-a(x+L)})^\mu + D_2 (-e^{-a(x+L)})^{1-\mu}
\]

having evaluated the two hypergeometric functions to unity as their argument vanishes; the above equation can be put as

\[
\phi_L(x) \sim D_1 e^{i\pi \mu} e^{-a(x+L))} - D_2 e^{-i\pi \mu} e^{-a(x+L)} e^{i\mu(x+L)}. \tag{35}
\]

In order to extract the behavior of \( \phi_R(x) \) as \( x \to 0 \) we use the continuation identity of the hypergeometric function [26]

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} 2F_1(a, b; a+b-c+1; 1-z)
+ (1-z)^{-(a+b)} \frac{\Gamma(c)\Gamma(a+b-c)\Gamma(a)\Gamma(b)}{\Gamma(a+b)\Gamma(c-a-b)} 2F_1(c-a, c-b; c-a-b+1; 1-z). \tag{36}
\]

Proceeding similarly to the case \( x < 0 \) we find (for \( x > 0 \)) that as \( x \to 0^+ \), \( z \to 1 \) and \( 1 - z \approx e^{i(x-L)} \ll 1 \) (we assume \( aL \gg 1 \)) and recalling that \( \mu = \rho \), from equation (32) with \( d_2 = 0 \),

\[
\phi(x)_R \sim d_1 M e^{-a\mu(x-L)} + \overline{d_1} \overline{N} e^{i\mu(x-L)}, \tag{37}
\]

where
\[
M = \frac{\Gamma(1 - 2\nu)\Gamma(1 + 2\mu)}{\Gamma(1 + \mu - \nu + \lambda)\Gamma(1 + \mu - \nu - \lambda)}
\]

\[
N = \frac{\Gamma(1 - 2\nu)\Gamma(-1 - 2\mu)}{\Gamma(-\mu - \nu + \lambda)\Gamma(-\mu - \nu - \lambda)}
\]

(38a)

(38b)

The match of the two solutions \(\phi_L(x)\) and \(\phi_R(x)\) is done by imposing the continuity of the wavefunction and of its derivative at \(x = 0\) which gives

\[
D_1 e^{i\pi\mu} e^{-\mu aL} - D_2 e^{-i\pi\mu} e^{-(1-\mu)aL} = d_1 [M e^{\mu aL} + N e^{-(\mu+1)aL}]
\]

\[
- \mu a D_1 e^{i\pi\mu} e^{-a aL} + (1 - \mu) a D_2 e^{-i\pi\mu} e^{-(1-\mu)aL} = d_1 [-\mu a M e^{a aL} + N(\mu + 1) a e^{-(\mu+1)aL}]
\]

and solving

\[
D_1 = \frac{d_1 e^{-i\pi\mu}}{1 - 2\mu} [M(1 - 2\mu) e^{2\mu aL} + 2N e^{-a aL}]
\]

\[
D_2 = \frac{d_1 e^{i\pi\mu}}{1 - 2\mu} [N(2\mu + 1) e^{-2a aL}].
\]

(39a)

(39b)

We would like to remark that when solving the Dirac equation the continuity condition at a given boundary \((x = 0\) in our case\) should be imposed by requiring the match of both the upper and lower spinor components \((u_1(x)\) and \(u_2(x)\)). In our second-order approach based on the introduction of the auxiliary components \(\phi(x)\) and \(\chi(x)\), the derivative of one of the two \((\phi')\) is connected to the other \((\chi)\) because of equation \((5a)\). In turn both the initial upper \(u_1\) and lower \(u_2\) components can be expressed in terms of \(\phi\) and \(\phi'\). Indeed solving equations \((4a)\) and \((4b)\) and using equation \((5a)\), one finds for \(u_1\) and \(u_2\)

\[
u_1(x) = \frac{1}{2} \left[ \left(1 + \frac{E - V(\chi)}{m}\right) \phi(x) + \frac{i}{m} \phi'(x) \right]
\]

\[
u_2(x) = -\frac{i}{2} \left[ \left(1 - \frac{E - V(\chi)}{m}\right) \phi(x) - \frac{i}{m} \phi'(x) \right],
\]

which shows how the matching of the wavefunction \(\phi(x)\) and of its derivative \(\phi'(x)\) is totally equivalent to requiring the continuity of \(u_1(x)\) and \(u_2(x)\). We also recall the reader that this method of matching the solution in \(x = 0\) has also been used in [24]. The above consideration also applies to subsection 4.3.

3.6. Probability current density, reflection and transmission coefficients

The reader might wonder whether the Dirac equation with a position-dependent mass still has a conserved current. It is well known that a continuity equation for the current is related to the conservation of probability, or unitarity. It is quite straightforward to show that a coordinate dependence of the mass does not bring in any change in the derivation of the conserved current. This is related to the fact that the mass multiplies the spinor wavefunction and in deriving the conserved current such terms simply drop out as in the constant mass case. The probability current density is given by

\[
J(x) = \bar{\psi}(x) \gamma_5 \psi(x) = i[u_1^*(x)u_2(x) - u_2^*(x)u_1(x)],
\]

which can also be given in terms of the auxiliary functions \(\phi\) and \(\chi\):

\[
J(x) = \frac{1}{2} [ |\phi(x)|^2 - |\chi(x)|^2].
\]
With the asymptotic form of the wavefunction we can compute the left \((x < 0)\) \(J_L\) and the right \((x > 0)\) \(J_R\) current density as

\[
J_L = J_{\text{inc}} - J_{\text{refl}} = |H|^2 \frac{k(E - k)}{m_0^2} - |G|^2 \frac{k(E + k)}{m_0^2},
\]

\[
J_R = J_{\text{trans}} = |d_1|^2 \frac{k(E - k)}{m_0^2},
\]

and we can define the transmission and reflection coefficients as

\[
T = \frac{J_{\text{trans}}}{J_{\text{inc}}} = \frac{|d_1|^2}{|H|^2} \tag{40}
\]

\[
R = \frac{J_{\text{refl}}}{J_{\text{inc}}} = \frac{(E + k)}{(E - k)} \frac{|G|^2}{|H|^2} \tag{41}
\]

and from the current conservation \(\partial_x J(x) = 0\) it follows that \(\int_{-\infty}^{+\infty} \partial_x J(x) = J_R - J_L = 0\), and therefore \(J_L = J_R\) from which we have the unitarity condition \(R + T = 1\).

\[T = \frac{|1 - 2\mu|^2}{|MB(1 - 2\mu)e^{2iaL} + N[B(2 - 2\mu)e^{-2iaL} - D(2\mu + 1)e^{-2iaL}|]^2} \tag{44a}
\]

\[R = \frac{E + k}{E - k} \frac{|A[M(1 - 2\mu)e^{2iaL} + N(2 - 2\mu)e^{-aL}] - C[N(1 + 2\mu)e^{-2iaL}]|^2}{|MB(1 - 2\mu)e^{2iaL} + N(2 - 2\mu)e^{-aL} - D[N(1 + 2\mu)e^{-2iaL}]|^2} \tag{44b}
\]

Figure 2 shows the transmission coefficient in the constant mass case (left plots) and for the position-dependent mass case (right plot) for two choices of the parameters \((a, L)\) and in the so-called Klein range \(m < E < W - m\). We note that in the PDM case we still observe the transmission resonances found for constant mass. We observe that, while for \(m = m_0\) when \(E \to W - m, T \to 0\), in the PDM case \(T \to 1\). This is an important fact worthwhile to be pointed out. In figure 3 we plot the transmission coefficient as a function of the barrier height \(W\). We note that as opposed to the case of constant mass where \(T = 0\) for \(E - m < W < E + m\), in the position-dependent mass case, \(T(W)\) always oscillates and does not go to zero in the interval \(E - m < W < E + m\). We also note that we have verified our numerical calculations of the constant mass case with those of [24] finding complete agreement. Finally, we have numerically checked the validity of the unitarity condition \(R + T = 1\).

4. Effective-mass Dirac equation, bound states

Let us study the bound states for the particle with position-dependent mass. In order to do this, we take the \(W\)-\(S\) potential with \(W \to -W\).

4.1. Negative region

In the study of the discrete spectrum it is convenient to use a different variable. Now we choose \(y^{-1} = 1 + e^{-a(x + L)}\), with \(d/dx = ay(1 - y) d/dy, V(y) = -Wy\) and \(m(y) = m_0(1 - y)\) and
using the parametric transformation $\phi = y^\sigma (1 - y)^\epsilon h(y)$, we obtain from equation (7)

$$y(1 - y) \frac{d^2 h(y)}{dy^2} + [1 + 2\sigma - y(1 + 2\sigma + 2\epsilon)] \frac{dh(y)}{dy} \frac{1}{a^2 y(1 - y)} (E + Wy)^2 - m_0^2 (1 - y)^2 - i\alpha Wy(1 - y) - i\alpha y(E + Wy) h(y) + \frac{1}{y(1 - y)} \{\sigma(\sigma - 1)(1 - y)^2 + \epsilon(\epsilon - 1)y^2 + \sigma(1 - y)^2\} h(y) + (-2\epsilon\sigma - \epsilon) h(y) = 0.$$  (45)
The most general condition that we can impose on the term multiplying $1/[y(1-y)]$ in order that the equation be that of the hypergeometric function is that it be equal to a constant $\zeta$. Therefore, we get three equations

\begin{align}
\sigma^2 + E^2 \frac{a^2}{a^2} - m_0^2 \frac{a^2}{a^2} &= 0 \quad (46a) \\
2m_0^2 \frac{a^2}{a^2} + 2 \frac{EW}{a^2} - i \frac{W + E}{a} - 2\sigma^2 &= \zeta \quad (46b) \\
\sigma^2 + \epsilon^2 - \epsilon + \frac{W^2}{a^2} - m_0^2 \frac{a^2}{a^2} &= -\zeta. \quad (46c)
\end{align}

From equation (46a) we can solve for $\sigma$, while summing equation (46b) and (46c) we obtain the equation for $\epsilon$:

\begin{align}
\sigma &= \sqrt{m_0^2 - E^2} \frac{a}{a} \quad (47a) \\
\frac{(E + W)^2}{a^2} - i \frac{W + E}{a} - \epsilon + \epsilon^2 &= 0. \quad (47b)
\end{align}

So $\sigma = -\nu$ ($\nu$ as defined in equation (22b)). A solution of the second one is $\epsilon = -i(E + W)/a$, which is the same as that of $\mu$ in the scattering problem but with the replacement $W \to -W$. In determining the parameters $a$ and $b$ ($c = 1 + 2\sigma$) of the hypergeometric equation, we make use of the relation in equation (46c), where we define $\lambda = i\sqrt{(W^2 - m_0^2)/a^2}$ and finally obtain

\begin{align}
y(1 - y) \frac{d^2 f(y)}{dy^2} + [1 + 2\sigma - (1 + 2\epsilon + 2\sigma)y] \frac{df(y)}{dy^2} - (\epsilon + \sigma + \lambda)(\epsilon + \sigma - \lambda)f(y) = 0. \quad (48)
\end{align}

4.2. Positive region

We consider the same substitution of scattering states for the variable $x$. So we use the variable $1/z = 1 + e^{(x-L)}$, and the transformation $\phi = z^{-\tau}[(1-z)^{-\eta}g(z)]:$

\begin{align}
z(1 - z) \frac{d^2 g(z)}{dz^2} + [1 - 2\tau - z(1 - 2\eta - 2\tau)] \frac{dg(z)}{dz} + \frac{1}{a^2 z(1 - z)} [(E + Wz)^2 - m_0^2 (1 - z)^2 & \\
+ i a Wz(1 - z) + i a z(E + Wz)] g(z) + \frac{1}{z(1 - z)} [\tau (\tau + 1)(1 - z)^2 & \\
+ \eta (\eta + 1) z^2 - \tau (1 - z) + \eta z - 2\eta z^2 - \tau z(1 - z) + \eta z^2] g(z) & \\
+ (2\tau - 2\tau \eta) g(z) = 0. \quad (49)
\end{align}

Following a line of thought similar to that outlined in the previous subsection ($x < 0$), we obtain $\tau^2 = -(E^2 - m_0^2)/a^2 = \sigma^2 = \nu^2$ and $\eta = -i(E + W)/a = \epsilon$ so that equation (49) reduces to

\begin{align}
z(1 - z) \frac{d^2 f(z)}{dz^2} + [1 - 2\sigma - (1 - 2\epsilon - 2\sigma) z] \frac{df(z)}{dz} & \\
- (-\epsilon - \sigma + \lambda)(-\epsilon - \sigma - \lambda) f(z) = 0. \quad (50)
\end{align}
4.3. Bound state wavefunction and match at \( x = 0 \)

We note that the wavefunction in the \( x > 0 \) region can be obtained from that of the \( x < 0 \) region simply letting \( \sigma \to -\sigma \) and \( \epsilon \to -\epsilon \). The general solutions to equations (48) and (50) are

\[
h(y) = A' y^\lambda \frac{\Gamma (1 \sigma + \lambda)}{\Gamma (1 \sigma + \lambda)} F_1 (1 \epsilon + \sigma + \lambda, 1 \epsilon + \sigma \lambda, 1 + 2 \sigma; y)
\]

\[
g(z) = C' y^\lambda \frac{\Gamma (1 - \sigma + \lambda)}{\Gamma (1 - \sigma + \lambda)} F_1 (1 - \epsilon - \sigma + \lambda, 1 - \epsilon - \sigma \lambda, 1 + 2 \sigma; z)
\]

Recall the parametric transformation for \( \phi_{L,R} \):

\[
\phi_L = y^\sigma (1 - z)^{-\sigma} g(z) \quad \text{and} \quad \phi_R = y^\sigma (1 - y)^{\sigma} h(y)
\]

Therefore, imposing the boundary condition of a bound state (vanishing wavefunction at infinity), we obtain \( B' = C' = 0 \) and we are left with

\[
\phi_L (y) = A' y^\sigma (1 - y)^{\sigma} F_1 (1 \epsilon + \sigma + \lambda, 1 - \sigma + \lambda, 1 + 2 \sigma; y)
\]

\[
\phi_R (z) = D' z^\sigma (1 - z)^{-\sigma} F_1 (1 - \epsilon + \sigma + \lambda, 1 + \sigma - \lambda, 1 + 2 \sigma; z)
\]

With the help of the continuation formula of the hypergeometric function [26] we can extract the behavior of the solution in the vicinity of \( x = 0 \) (recall that for \( x \to 0, y, z \to 1 \) and \( 1 - y \approx e^{-a(x-L)} \) while \( 1 - z \approx e^{a(x-L)} \):

\[
\phi_L (x) \approx A' \left\{ \frac{\Gamma (1 + 2 \sigma)}{\Gamma (1 - \epsilon + \sigma - \lambda) \Gamma (1 - \epsilon + \sigma + \lambda)} e^{a(x-L)} + \frac{\Gamma (1 - \epsilon + \sigma + \lambda)}{\Gamma (1 + 2 \sigma) \Gamma (1 - \epsilon + \sigma - \lambda)} e^{-a(x-L)} \right\}
\]

\[
\phi_R (x) \approx D' \left\{ \frac{\Gamma (1 + 2 \sigma)}{\Gamma (1 + \epsilon + \sigma + \lambda) \Gamma (1 + \epsilon + \sigma - \lambda)} e^{a(x-L)} + \frac{\Gamma (1 + \epsilon + \sigma - \lambda)}{\Gamma (1 + 2 \sigma) \Gamma (1 - \epsilon + \sigma + \lambda)} e^{-a(x-L)} \right\}
\]

Upon defining as \( S, T, U, V \) respectively the various combinations of gamma functions appearing in the above expressions, the wavefunctions are written as

\[
\phi_L (x) \approx A' \left[ S e^{-a(x-L)} + T e^{-(1-\epsilon)a(x-L)} \right]
\]

\[
\phi_R (x) \approx D' \left[ U e^{-a(x-L)} + V e^{(1+\epsilon)a(x-L)} \right].
\]

Now we have to match the two solutions in \( x = 0 \) requiring continuity of the wavefunction \( \phi_L (0) = \phi_R (0) \) and of its first derivative \( \phi'_L (0) = \phi'_R (0) \). This gives the homogeneous system

\[
A' \left[ S e^{-aL} + T e^{-(1-\epsilon)aL} \right] = 0
\]

\[
A' \left[ -\epsilon S e^{-aL} - (1-\epsilon)T e^{-(1-\epsilon)aL} \right] = 0
\]

\[
A' \left[ -\epsilon S e^{-aL} - (1-\epsilon)T e^{-(1-\epsilon)aL} \right] = D' \left[ U e^{aL} + (1+\epsilon)V e^{aL} \right] = 0,
\]

which admits a solution only if its determinant is zero. This provides a condition for extracting the energy eigenvalue:

\[
\mathcal{F}(E) = \frac{SV}{TU} e^{2aL} \frac{2\epsilon - 1}{2\epsilon + 1} = 0.
\]

When equation (52) is satisfied the relation between \( A' \) and \( D' \) is found to be

\[
D' = \frac{T}{V} \frac{2\epsilon - 1}{2\epsilon + 1} e^{2aL} A' = \frac{S}{U} e^{-2aL} A',
\]

and \( A' \) is the usual normalization constant. The condition in equation (52) is a transcendental equation which can be solved numerically. We provide numerical examples of the bound
Figure 4. The energy spectrum is derived by solving numerically with respect to the real variable \( E \), the transcendental equations \( \text{Re}[\mathcal{F}(E)] = 0, \text{Im}[\mathcal{F}(E)] = 0 \). In the figure, we plot \( \text{Re}[\mathcal{F}(E)] \) (solid line) and \( \text{Im}[\mathcal{F}(E)] \) (dotted line) as functions of the energy \( E \) for the position-dependent mass case (left panel) and for the constant mass case (right panel). The eigenvalues (full disks) are given by the points on the \( E \)-axis where the two curves cross. The corresponding numerical values are given in table 1.

Table 1. Numerical values of the energy eigenvalues (discrete spectrum) for the bound states. The model parameters are \( m_0 = 1, W = 2, a = 10 \) and \( L = 2 \). In the upper row we have the spectrum of the constant mass case, while in lower row we have the position-dependent mass case.

|          | \( E_1 \)       | \( E_2 \)       | \( E_3 \)       | \( E_4 \)       | \( E_5 \)       |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( m = \text{const} \) | -1              | -0.759 003      | -0.273 555      | +0.271 144      | +0.788 942      |
| \( m(x) \)    | -0.633 251      | -0.008 067 37   | +0.605 869      | -               | -               |

states. As we are studying bound states, we seek numerical solutions of equation (52) in the interval \( -W \leq E \leq m \). Since \( \mathcal{F}(E) = 0 \) is complex, the energy eigenvalues \( E \) are found by solving numerically, for real solutions, the two (independent) equations \( \text{Re}[\mathcal{F}(E)] = 0 \) and \( \text{Im}[\mathcal{F}(E)] = 0 \). Figure 4 shows graphically the details of the numerical computations both for the position-dependent mass and the constant mass cases. In table 1 we give the numerical results for the spectra of the position-dependent mass case and that of the constant mass case, with the same values of the parameters \( (m_0 = 1; L = 2; a = 10; W = 2) \) in order to make a meaningful comparison between the two cases. For the case of the constant mass we have used the results of [24]. We observe that in the position-dependent mass case the number of bound states decreases relative to the constant mass. In figures 5 and 6 we provide some example of the (normalized) wavefunctions and probability densities both for constant mass case and position-dependent mass. Also, comparing figure 5 with figure 6, one can infer that in the position-dependent mass case the probability density is almost flat in the region inside the potential well as opposed to the constant mass case where for the highest excited states it oscillates strongly. We note, in the constant mass case (right panel of figure 4), an eigenvalue corresponding to \( E = -m = -1 \) which merges with the negative continuum. This situation has previously been considered in the literature [24, 29, 30] and has been referred to as super-criticality. Such super-critical states are also called half-bound states and are characterized by the fact that one of the spinor components (the upper \( u_1 \) or the lower \( u_2 \)) is not strictly normalizable. We show in table 1 the eigenvalue \( E = -1 \) only because it is a solution of the equation that in the constant mass case corresponds to our equation (52) (see reference [24]).
Figure 5. The case of position-dependent mass. Upper panel: plot of the normalized wavefunction $\text{Re}[u_1]$ and of the probability density for $E = -0.633\,251$ (ground state); lower panel: the same but for $E = +0.605\,869$ (highest excited state). The model parameters are $m_0 = 1$, $W = 2$, $a = 10$ and $L = 2$.

We do not address further the issue of super-criticality within the position-dependent mass case as it goes beyond the scope of this work.

Finally in figure 7 we give another example in the case of the position-dependent mass when the potential well is deeper, $W = 3$, with the other parameters as in figure 5. We note that in this case the highest level is close to the continuum ($E = m_0 = 1$) and indeed the wavefunction converges less rapidly and the probability density as well. We have computed for example that in this case the probability of the particle to be outside the potential well ($|x| > 2$) is $P_{\text{outside}} \approx 0.57$ which is even greater than the probability to be inside ($-L \leq x \leq L$): $P_{\text{inside}} \approx 0.43$. The less-rapid convergence of the wavefunction is due to the fact that the coefficient that controls such behavior (in this case as $x \to \infty$, $\phi_R(x) \approx e^{-\sigma|x|}$) is $\sigma = \sqrt{m_0^2 - E^2}/a$, and therefore very small giving a wavefunction that vanishes much slower than those corresponding to the lowest lying bound states.

5. Discussion and conclusions

We have solved the scattering problem for the one-dimensional Dirac equation with the WS potential in the position-dependent mass formalism. We have set some conditions on the equation in order to keep the structure of the hypergeometric equation which gives a suitable mass function. These conditions provide a first-order differential equation which can be solved exactly. The physical requirement that the mass function at infinity goes to a constant mass $m_0$ completely specifies the mass function. Once the mass function has been found, cf
The case of constant mass. Upper panel: plot of the normalized wavefunction $\text{Re}[u_1]$ and probability density for $E = -0.759003$ (lowest energy bound state); lower panel: the same but for $E = +0.7888942$ (highest excited state). The model parameters are the same as those of figure 5: $m_0 = 1$, $W = 2$, $a = 10$ and $L = 2$.

Position-dependent mass case. Plot of the normalized wavefunction $\text{Re}[u_2]$ (left panel) and of the probability density (right panel) for $E = 0.97248$ (highest excited state). The model parameters are the same as those of figure 5 except for $W$ which now is $W = 3$.

Equation (25), we have followed the same technique employed in [24] solving the equation in the negative and positive region separately and giving the solution in terms of the hypergeometric function $\text{F}_1$. For the scattering problem, ordinary boundary conditions at infinity are imposed and then the match at $x = 0$ allows us to specify all unknowns up to a normalization constant.
We note that our method of solving the Dirac equation for a particular case of effective position-dependent mass function has the drawback of providing a mass function which does not interpolate smoothly with the constant mass case. In other words our mass function does not contain a parameter such that when set to zero reduces the mass function to the constant case ($m_0$). Further studies in this direction should be pursued in order to overcome such difficulties.

We have obtained analytical expressions for the transmission and reflection coefficients, and we explicitly verified that unitarity ($R + T = 1$) is preserved in the PDM case. We have also studied the bound states, i.e. the discrete spectrum of the WS potential well with the effective position-dependent mass, finding an exact analytical condition for the energy eigenvalues (in the form of a transcendental equation which needs to be solved numerically). We have provided an explicit numerical example finding the eigenvalues and the wavefunction for a specific choice of the parameters.

Our approach offers one of the few examples where the Dirac equation is solved exactly in the position-dependent mass case and in an external potential. To the best of our knowledge the only other example is reported in [18] where the three-dimensional Dirac equation is solved for the PDM case in the Coulomb external field. We note a similarity between the present work and that reported in [18]. In both cases, the mass function for which an exact solution is found shares similarities with the external potential. In [18] the spherically symmetric mass function for which the problem is solved is $m(r) = 1 + \mu \lambda^2 / r$ where, in atomic units, $m_0 = \hbar = 1$ and $\lambda$ is the Compton wavelength. Our mass function, cf equation (25), is also certainly related to the shape of the Woods–Saxon external potential.

Acknowledgments

This work is an outcome of the diploma thesis of SB presented at the University of Perugia in September 2009. AA acknowledges warm hospitality from the Physics Department of the University of Perugia and INFN—Istituto Nazionale di Fisica Nucleare—Sezione di Perugia.

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