TORIC DEGENERATION AND NON-DISPLACEABLE LAGRANGIAN TORI IN $S^2 \times S^2$

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Abstract. In this article, using the idea of toric degeneration and the computation of the full potential function of Hirzebruch surface $F_2$, which is not Fano, we produce a continuum of Lagrangian tori in $S^2 \times S^2$ which are non-displaceable under the Hamiltonian isotopy.

1. Introduction

In [FOOO3, FOOO4], we study Lagrangian Floer theory of Lagrangian torus fibers in toric manifolds, in particular, that of non-displaceable under the Hamiltonian isotopy. In this article, we discuss non-displaceable Lagrangian tori in $(S^2, \omega_{\text{std}}) \times (S^2, \omega_{\text{std}})$, which are not of the type of toric fibers. Here $\omega_{\text{std}}$ denotes the symplectic form on $S^2$ with area $2\pi$. Our main result is the following.

Theorem 1.1. There exist uncountably many Lagrangian tori $T(u)$ in $(S^2, \omega_{\text{std}}) \times (S^2, \omega_{\text{std}})$, parameterized by the real numbers $u \in (0, u_0]$ for some $u_0 > 0$, such that:

1. If $u \neq u'$ then $T(u)$ is not Hamiltonian isotopic to $T(u')$.
2. For any $u$ there exists a pair $(b, b) \in H^2(S^2 \times S^2; \Lambda_+^+) \times H^1(T(u); \Lambda_0)$ such that the Floer cohomology $HF((T(u), (b, b)), (T(u), (b, b)))$ is isomorphic to $H(T(u); \Lambda_0)$. In particular none of them are displaceable.
3. $T(u) \cap T(u') = \emptyset$ if $u \neq u'$.
4. There exists a unique $T(u_0)$ in our family that is monotone.
5. $T(u_0)$ is not symplectically equivalent to $S^1_{\text{eq}} \times S^1_{\text{eq}}$, the direct product of the equators.

The definitions of the Floer cohomology $HF((T(u), (b, b)), (T(u), (b, b)))$ with bulk deformation is given in [FOOO1] section 3.8.

We will give an explicit description of $T(u)$ in later sections. Using either the theory of spectral invariants with bulk deformation or a generation result for Fukaya category of toric manifolds, we can obtain the intersection result $\varphi(T(u)) \cap (S^1_{\text{eq}} \times S^1_{\text{eq}}) \neq \emptyset$ for any Hamiltonian diffeomorphism $\varphi$. (See Remark 6.2.)

In response to Polterovich’s question, Albers and Frauenfelder [AF] proved Hamiltonian non-displaceability of a certain monotone Lagrangian torus in $T^*S^2$ with the standard symplectic structure. Our theorem also implies this non-displaceability result. (See Remark 3.1.)

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We use the universal Novikov ring $\Lambda_{0,\text{nov}}$ and its ideal $\Lambda_{0,\text{nov}}^+$ in this paper. We recall their definitions here. An element of $\Lambda_{0,\text{nov}}$ is a formal sum $\sum a_i T^{\lambda_i} e^{\mu_i}$ with $a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \mu_i \in \mathbb{Z}$ such that $\lambda_i \leq \lambda_{i+1}$ and $\lim_{i \to \infty} \lambda_i = \infty$. $T$ and $e$ are formal parameters. We define a valuation $\nu_T : \Lambda_{0,\text{nov}} \to \mathbb{R}_{\geq 0}$ defined by

$$\nu_T \left( \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \right) = \lambda_1.$$ 

This induces a natural $\mathbb{R}$-filtration on $\Lambda_{0,\text{nov}}$ which in turn induces a non-Archimedean topology thereon. The sum is said to be an element of $\Lambda_{0,\text{nov}}^+$ if $\lambda_i > 0$ for all $i$. If we rearrange the sum $\sum a_i T^{\lambda_i} e^{\mu_i}$ into

$$\sum_{k=1}^{\infty} p_k(e) T^{\lambda_{i_k}}, \quad \lambda_{i_k} < \lambda_{i_{k+1}},$$

then each $p_k$ becomes a complex polynomial of variables $e, e^{-1}$. In particular, we can insert $e = 1$ and the resulting formal sum converges in non-Archimedean topology and satisfies

$$\nu_T \left( \sum_{k=1}^{\infty} p_k(1) T^{\lambda_{i_k}} \right) \geq \nu_T \left( \sum_{k=1}^{\infty} p_k(e) T^{\lambda_{i_k}} \right) = \lambda_1.$$ 

(We note that the value $p_1(1)$ could be zero.) We also use the subring $\Lambda_0$ of $\Lambda_{0,\text{nov}}$ consisting of elements which do not involve $e$: We define

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \left| \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right. \right\}$$

which is isomorphic to $\Lambda_{0,\text{nov}}|_{\{e=1\}}$, the quotient of $\Lambda_{0,\text{nov}}$ by the ideal generated by $e - 1$, and its unique maximal ideal by $\Lambda_+ = \Lambda_0 \cap \Lambda_{0,\text{nov}}^+$. One can unify the definitions of $\Lambda_{0,\text{nov}}$ and of $\Lambda_0$ by introducing a universal Novikov ring $\Lambda_0^{R}$ over a general coefficient ring $R$ defined by

$$\Lambda_0^{R} := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \left| \lambda_i \leq \lambda_{i+1}, a_i \in R, \lim_{i \to \infty} \lambda_i = \infty \right. \right\}$$

Then we can write the above universal Novikov rings as

$$\Lambda_{0,\text{nov}} = \Lambda_0^{C[e,e^{-1}]}, \quad \Lambda_0 = \Lambda_0^{C}$$

for $R = \mathbb{C}[e,e^{-1}], \mathbb{C}$ respectively.

2. Potential function of the Hirzebruch surface $F_2(\alpha)$

We consider toric Hirzebruch surface $F_2(\alpha)$ whose moment polytope is

$$P(\alpha) = \left\{ (u_1, u_2) \in \mathbb{R}^2 \left| u_1 \geq 0, u_2 \leq 1 - \alpha, u_1 + 2u_2 \leq 2 \right. \right\} \quad (1)$$

Recall that the fiber $L(u) = \pi^{-1}(u)$ at $u \in \text{Int} P(\alpha)$ is a Lagrangian torus. We fix an identification of $L(u) \cong T^2$ and an integral basis $\{ e_i^1 \}_{i=1,2}$ of $H^1(L(u); \mathbb{Z}) \cong H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ and its dual basis $\{ e_i \}$ on $H^1(L(u); \mathbb{Z})$. We denote by $\{ x_i \}_{i=1,2}$ the coordinates of $H^1(L(u); \mathbb{R})$ with respect to $\{ e_i \}$, and set $y_i = e^{x_i}$.

$F_2(\alpha)$ is not Fano but nef, i.e. every holomorphic sphere has non-negative Chern number. In fact the toric divisor $D_1 \cong \mathbb{C}P^1$ associated to the facet of $P(\alpha)$,
Let \( \beta_i \in H_2(F_2(\alpha), L(u); \mathbb{Z}) \) (\( i = 1, \ldots, 4 \)) be the classes such that \( \beta_i \cap D_i = \delta_i j \). Then we have \( M_1^{reg}(F_2(\alpha), L(u); \beta_i) \neq \emptyset \) and \( c_1 \cap \beta_i = 1 \). (Here ‘\( \text{reg} \)’ means the moduli space of pseudo-holomorphic disks without bubble.) \([CO]\) Theorem 5.2 implies that there are exactly four homology classes satisfying this condition. \( \beta_1 \) is the same class as before.

We refer to \([FOOO1]\) for the general definition of the potential function \( \mathcal{PO} \). We also provide some description of \( \mathcal{PO} \) specialized to the current circumstance in Appendix of the present paper. The following description of the potential function \( \mathcal{PO} \) of \( F_2(\alpha) \) can be derived from the results from \([FOOO3]\) Example 8.2 and the argument used in the proof of \([FOOO4]\) Proposition 9.4. For readers’ convenience, we give its proof.

**Proposition 2.1.** The potential function \( \mathcal{PO} \) of \( F_2(\alpha) \) has the form

\[
\mathcal{PO} = \mathcal{PO}(y_1, y_2; u_1, u_2) = T^{u_1}y_1 + T^{u_2}y_2 + T^{2-u_1-2u_2}y_1^{-1}y_2^{-2} + (1+c)T^{1-\alpha-u_2}y_2^{-1}
\]

for some element \( c \in \Lambda_+ \) of the form

\[
c = \sum_{k \geq 1} c_k T^{2k\alpha}, \quad c_k \in \mathbb{Q}.
\]

**Proof.** In our current circumstance, we have

\[
\mathcal{PO}(b) = \sum_{\mu(\beta)=2} T^{\omega(\beta)/2\pi} \exp(b \cap \partial \beta) \deg[\text{ev}_{0^*} : M_1(\beta) \rightarrow L(u)].
\]

(See Theorem 7.2 and 40 in Appendix. Here we divide \( \omega(\beta) \) in the exponent of \( T \) by \( 2\pi \) to be consistent with the conventions used in \([FOOO3], [FOOO4]\).) We decompose

\[
\mathcal{PO} = \mathcal{PO}_0 + \text{“higher order part”}
\]

where \( \mathcal{PO}_0 \) is the ‘leading order part’ of \( \mathcal{PO} \) coming from the contribution of the classes \( \beta_i, i = 1, \ldots, 4 \). We have derived in \([FOOO3]\) Example 8.2

\[
\mathcal{PO}_0 = T^{u_1}y_1 + T^{u_2}y_2 + T^{2-u_1-2u_2}y_1^{-1}y_2^{-2} + T^{1-\alpha-u_2}y_2^{-1}.
\]

It remains to identify the “higher order part”. This is the contribution of the singular discs in classes \( \beta \) with \( \mu(\beta) = 2 \).

Let \( \beta \in H_2(X, L(u); \mathbb{Z}) \) with \( \mu(\beta) = 2 \). We assume \( M_{1:}^{\text{main}}(L(u), \beta) \) is nonempty. According to \([FOOO3]\) Theorem 11.1 (5) combined with the fact \( F_2(\alpha) \) being nef, \( \beta \) must be of the form

\[
\beta = \beta_i + k[D_1]
\]

for some \( i = 1, \ldots, 4 \). (See also \([FOOO4]\) Proposition 10.4.) On the other hand, the bordered stable map must have connected image, and that only the holomorphic discs in \( \beta_i \) among \( \beta_i \)'s intersect the divisor \( D_1 \), which follows from the classification theorem \([CO]\) Theorem 5.2 Therefore \( \beta \) must be of the form \( \beta_i + k[D_1] \) for \( k \in \mathbb{Z}_{\geq 0} \).

Writing \( b = x_1e_1 + x_2e_2 \) and noting \( e_1 \cap \partial \beta_i = 0, e_2 \cap \partial \beta_i = -1 \), we obtain

\[
\exp(b \cap \partial \beta_i) = \exp\left((x_1e_1 + x_2e_2) \cap \partial \beta_i\right) = e^{x_1(e_1 \cap \partial \beta_i)}e^{x_2(e_2 \cap \partial \beta_i)} = y_2^{-1}.
\]

Furthermore

\[
\omega(\beta) = \omega(\beta_i) + k\omega([D_1]) = 2\pi((1-\alpha-u_2) + 2k\alpha)
\]
since we have $\omega(\beta_1) = 2\pi(1 - \alpha - u_2)$ (see [CO] Theorem 8.1). Therefore $\beta_1 + k[D_1]$ contributes

$$c_k T^{2k\alpha} T^{1-\alpha-u_2 y_2^{-1}}$$

to $\mathcal{R}_\text{O}^u$ where $c_k$ is given by

$$c_k = \deg \{ ev_0 : \mathcal{M}_1(F_2(\alpha), L(u); \beta_1 + k [D_1]) \to L(u) \}.$$  \hfill (4)

(We remark that the symplectic area of the $(-2)$-curve $D_1$ is $2\pi(2\alpha)$. By summing over $k$’s, we obtain the proposition. \hfill \Box

Appearance of the additional term $c T^{1-\alpha-u_2 y_2^{-1}}$ in $\mathcal{R}_\text{O}$ reflects the fact that $\mathcal{F}_2(\alpha)$ in not Fano.

The following theorem completely determines the full potential $\mathcal{R}_\text{O}$ for this non-Fano toric manifold $\mathcal{F}_2(\alpha)$, which will play an important role in our study of non-displaceable tori later.

**Theorem 2.2.** We have $c = T^{2\alpha}$. In particular $c_k = 0$ for $k \geq 2$ and $c_1 = 1$.

**Remark 2.1.** This result, in the form of convergent power series, is obtained previously by D. Auroux [Au] using a different method. Although Auroux [Au] did not state the version with coefficients in the Novikov ring, he determined all necessary information on the moduli spaces of bordered stable maps in order to determine the potential function by analyzing the wall-crossing phenomenon. Hence Theorem 2.2 follows from his study.

3. **Smoothing of singular toric $\mathcal{F}_2(0)$**

We consider the limit $\alpha \to 0$ of our Hirzebruch surface $\mathcal{F}_2(\alpha)$. At $\alpha = 0$ we obtain an orbifold with a singularity of the form $\mathbb{C}^2/\{ \pm 1 \}$. This singularity is of $A_1$-type: The map

$$(x, y) \mapsto (x^2, y^2, xy)$$

induces an isomorphism between $\mathbb{C}^2/\{ \pm 1 \}$ and $\{ u, v, w \} | uv = w^2 \}$. The link of the singular point is diffeomorphic to $S^3/\{ \pm 1 \} = \mathbb{R}P^3$.

We deform the latter singular surface to a Milnor fiber $\{ (u, v, w) | uv = w^2 + \epsilon^2 \}$. Here we have $\epsilon^2$ in the right hand side in order to obtain the simultaneous resolution of this family. We cut out a neighborhood of the singularity of $\mathcal{F}_2(0)$ and paste the Milnor fiber back into the neighborhood to obtain the desired manifold. We denote it by $\tilde{\mathcal{F}}_2(0)$.

We also have a symplectic description of $\tilde{\mathcal{F}}_2(0)$ which is in order. Consider the preimage $Y(\epsilon) \subset P(\epsilon) \subset P(0)$, $0 < \epsilon < 1$, under the moment map. Then $Y(\epsilon)$ has concave boundary $\partial Y(\epsilon)$ which is diffeomorphic to $S^3/\{ \pm 1 \}$. Moreover, the characteristic foliation of $\partial Y(\epsilon)$ is same as that of the contact manifold $S^3/\{ \pm 1 \}$ equipped with the standard contact form $\theta_{\text{can}}$ whose leaves consist of the fibers of the circle bundle $\mathbb{R}P^3 = S^3/\{ \pm 1 \} \to S^2$.

We also note that the Milnor fiber of $\mathbb{C}^2/\{ \pm 1 \}$ is diffeomorphic to $T^*S^2$. Let $S^2$ be the standard round 2-sphere and $D_r(T^*S^2)$ its cotangent disc bundle with radius $r > 0$. $D_r(T^*S^2)$ has convex boundary $\partial D_r(T^*S^2)$ which is diffeomorphic to $S^3/\{ \pm 1 \}$. The characteristic foliation of $\partial D_r(T^*S^2)$ is isomorphic to that of $(S^3/\{ \pm 1 \}, \theta_{\text{can}})$. Hence we can take a suitable radius $r = r(\epsilon) > 0$ so that the symplectic form on a collar neighborhood $N(\epsilon)$ of $\partial Y(\epsilon)$ and the one on a collar
neighborhood of $\partial D_{\epsilon}(T^*S^2)$ can be glued to a symplectic form on $\hat{F}_2(0) = Y(\epsilon) \cup D_{\epsilon}(T^*S^2)$ in a way that the given toric symplectic form on $Y(\epsilon)$ is unchanged on $Y(\epsilon) \setminus N(\epsilon) \subset Y(\epsilon) \setminus \partial Y(\epsilon)$.

Since $H^2(S^3/\{\pm 1\};\mathbb{Q}) = 0$, the glued symplectic form does not depend on the choices of $\epsilon > 0$ or the gluing data up to the symplectic diffeomorphism.

**Remark 3.1.** Note that the projections to $S^2$ of characteristic Reeb orbits in $\partial D_{\epsilon}(T^*S^2)$ are oriented geodesics on $S^2$, all of which are periodic. Hence the space of (unparameterized) oriented geodesics with a minimal period, denoted by $\text{Geo}_{\text{std}}^1(S^2)$, is identified with the Grassmannian of oriented 2-planes in $\mathbb{R}^3$ which is diffeomorphic to $S^2$. The Lagrangian torus $L_N$ studied in [AF] is the union of closed oriented geodesics of unit speed with a minimal period passing through a given point, say the north pole $N$. Denote by $\sigma_\epsilon$ the fiberwise multiplication by $\epsilon > 0$ on $T^*S^2$. We claim that $L_N$ is non-displaceable. Suppose to the contrary that there is a Hamiltonian diffeomorphism $\phi$ of $T^*S^2$ with $\phi(L_N) \cap L_N = \emptyset$. Then we can take a sufficiently small $\epsilon > 0$ such that $\sigma_\epsilon(\text{supp}(\phi)) \subset D_{\epsilon/2}(T^*S^2)$. Therefore $\sigma_\epsilon(L_N)$ can be regarded as a Lagrangian torus contained in $\hat{F}_2(0)$ and $\sigma_\epsilon \circ \phi \circ \sigma_{\epsilon^{-1}}$ is a Hamiltonian diffeomorphism of $\hat{F}_2(0)$, which disjoin $\sigma_\epsilon(L_N)$. However this Lagrangian torus $L$ is the same as the inverse image of a great circle in $S^2$ by the projection $\partial D_{\epsilon}(T^*S^2) \rightarrow S^2 = \text{Geo}_{\text{std}}^1(S^2)$. By taking a sufficiently small constant $\epsilon' > 0$, we find that $L'$ is contained in $Y(\epsilon') \setminus N(\epsilon') \subset \hat{F}_2(0)$ and becomes one of the Lagrangian torus $T(u)$ given by the construction in section 4. This then gives rise to a contradiction to Theorem 1.1 (2) and hence $L_N$ is not displaceable in $T^*S^2$.

**Lemma 3.1.** $\hat{F}_2(0)$ is symplectomorphic to $(S^2,\omega_{\text{std}}) \times (S^2,\omega_{\text{std}})$.

This is a well-known fact. We will prove a stronger fact, Proposition 4.11 in section 4 which contains Lemma 3.1 as a special case. For $u \in \text{Int } P(0)$, we choose $\epsilon > 0$ small enough such that $u \in P(\epsilon)$. Then we find that $\hat{F}_2(0)$ still contains $L(u)$. When $L(u)$ with $u = (u,1-u)$ is considered as a Lagrangian torus in $\hat{F}_2(0)$, we denote it by $T(u)$. We will give more precise description in section 4. The contraction of a vanishing cycle $\cong S^2$ in $\hat{F}_2(0)$ gives a map

$$\hat{\pi} : \hat{F}_2(0) \rightarrow F_2(0).$$

This map induces a canonical commutative diagram in homology

$$
\begin{array}{ccccccccc}
H_2(\hat{F}_2(0);\mathbb{Z}) & \overset{\iota_*}{\longrightarrow} & H_2(\hat{F}_2(0),T(u);\mathbb{Z}) & \overset{\delta}{\longrightarrow} & H_1(T(u);\mathbb{Z}) & \\
\downarrow{\hat{\pi}_*} & & \downarrow{\hat{\pi}_*} & & \downarrow{\hat{\pi}_*} \\
H_2(F_2(0);\mathbb{Z}) & \overset{\iota_*}{\longrightarrow} & H_2(F_2(0),L(u,1-u);\mathbb{Z}) & \overset{\delta}{\longrightarrow} & H_1(L(u,1-u);\mathbb{Z})
\end{array}
$$

Here $\hat{\pi}_* : H_1(T(u);\mathbb{Z}) \rightarrow H_1(L(u,1-u);\mathbb{Z})$ is an isomorphism which becomes the identity map under the above identification. We fix a basis of $H_1(L(u);\mathbb{Z})$, let $\{x_i\}_{i=1,2}$ be the coordinates of $H^1(L(u);\Lambda_0)$ with respect to its dual basis and set $y_i = e^{x_i}$ as before.

**Theorem 3.2.** Let $\mathcal{V}$ be the potential function of $\hat{F}_2(0)$ written in terms of the above mentioned basis. Then we have

$$\mathcal{V} = T^{u_1}y_1 + T^{u_2}y_2 + T^{2-u_1-2u_2}y_1^{-1}y_2^{-2} + 2T^{1-u_2}y_2^{-1}. 
$$

(6)
We note that (3) can be obtained by putting $\alpha = 0$ in (2). However to justify this conclusion, we need some more work to do.

**Remark 3.2.** The idea of using degeneration to a singular toric variety in a calculation of the potential function for a non toric manifold is due to Nishinou-Nohara-Ueda [NNU1, NNU2]. The transition $F_2(\alpha) \to F_2(0) \to \hat{F}_2(0)$ is a baby example of conifold transition in physics literature, the resolution $F_2(0)$ to $\hat{F}_2(0)$ is a baby example of crepant resolution and the degeneration of $\hat{F}_2(0)$ to $F_2(0)$ is an example of toric degeneration of a non-toric $\hat{F}_2(0)$.

Using the formula (6) in Theorem 3.2, we now find critical points of the potential function $\mathcal{P} \mathcal{D}$ at the point $u = (1/2, 1/2)$ for $\hat{F}_2(0)$. Note $L(1/2, 1/2)$ is a monotone Lagrangian torus in $\hat{F}_2(0)$. We have

$$\mathcal{P} \mathcal{D}^u = T^{1/2}(y_1 + y_2 + y_1^{-1}y_2^{-2} + 2y_2^{-1}).$$

The critical point equation of $\mathcal{P} \mathcal{D}^u$ for $(y_1, y_2)$ becomes

$$0 = 1 - y_1^{-2}y_2^{-2},$$

$$0 = 1 - 2y_1^{-1}y_2^{-3} - 2y_2^{-2}. \quad (9)$$

The first equation (8) implies $y_1y_2 = \pm 1$.

*Case 1:* $y_1y_2 = -1$. The second equation (9) becomes $1 = 0$, which is absurd.

*Case 2:* $y_1y_2 = 1$. The equation (9) is equivalent to $y_2 = \pm 2$. Therefore we conclude that there are 2 solutions for the critical point equation at $u = (1/2, 1/2)$.

On the other hand, we remark that for $\alpha > 0$, there is a unique balanced fiber $u = ((1 + \alpha)/2, (1 - \alpha)/2)$ which carries 4 critical points. So the valuation $(v_T(y_1), v_T(y_2))$ of 2 critical points among those 4, which is nothing but the location of the fiber $u = (u_1, u_2)$ (see [FOOO3] section 7 for a detailed explanation), jumps away from $(1/2, 1/2)$ to somewhere else. In fact, they jump to the point $u = (0, 1)$ in the following sense: The valuation point $u = (0, 1)$ corresponds to a singular point of $F_2(0)$ which no longer carries a torus action. However there appears a new Lagrangian sphere $S^2$ in its smoothing $\hat{F}_2(0)$. We may regard that this Lagrangian sphere corresponds to the two missing critical points. The two torus branes are merged and transformed into a sphere brane under a conifold transition!

**Remark 3.3.** For the case of the two-point blow-up of $\mathbb{C}P^2$ we have uncountably many non-displaceable $T^2$ at the very moment when the location of balanced fibers jumps. Their location lies on the line segment joining the positions of balanced fibers before and after the jump. (See [FOOO3] section 5 for such an example.) The same phenomenon occurs here.

We consider the tori $L(u)$ in $F_2(0) \setminus \pi^{-1}((0, 1))$ at $u = (u_1, u_2)$ on the line segment given by

$$u_1 = 2 - u_1 - 2u_2 = 1 - u_2 < u_2.$$ 

It can be considered as a submanifold of $\hat{F}_2(0)$ and we denote it by $T(u) \subset \hat{F}_2(0)$. This is possible by the discussion around (5) if we choose $\varepsilon$ sufficiently small relative to the distance from $u$ to $(0, 1)$. The above equation is equivalent to

$$u_1 = 1 - u_2, \quad u_2 > 1/2. \quad (10)$$

On this line, the leading order term of the potential function is

$$y_1 + y_1^{-1}y_2^{-2} + 2y_2^{-1}.$$
Therefore the leading term equation introduced in [FOOO3] Definition 10.2 is reduced to
\begin{align}
0 & = 1 - y_1^{-2} y_2^{-2}. \\
0 & = -2y_1^{-1} y_2^{-3} - 2y_2^{-2}.
\end{align}
(11) (12)
The equation (11) is equivalent to \( y_1 y_2 = \pm 1 \). In case \( y_1 y_2 = 1 \), (12) has no solution. However in case \( y_1 y_2 = -1 \), (12) becomes vacuous. Therefore the leading term equation has a continuum of solutions which will give rise to the continuum of Lagrangian tori \( T(u) \) mentioned in Theorem [I1].

We would like to note that the Lagrangian torus \( T(u) \) in Theorem [I1] is not a torus fiber of a toric manifold. Because of this, we can not directly apply [FOOO4] Theorem 1.3 to conclude non-vanishing of a Floer cohomology associated to \( T(u) \). However we can still prove the following. (See [FOOO1] section 3.8 for the notations appearing in Theorem [I3].)

**Theorem 3.3.** Let \( M = (S^2, \omega_{\text{std}}) \times (S^2, \omega_{\text{std}}) \). If \((u_1, u_2)\) satisfies (10) then there exist a bounding cochain with bulk \( (b, b) \) of \( b \in H^2(M; \Lambda_+) \) and a bounding cochain \( b \in H^1(T(u); \Lambda_0) \) such that \( HF((T(u), (b, b)), (T(u), (b, b))) \neq 0 \). In addition \( b \) has the property that it carries exactly two elements \( b \in H^1(T(u); \Lambda_0) \) such that \( HF((T(u), (b, b)), (T(u), (b, b))) \neq 0 \).

**Proof.** [Proof of Theorem 3.3 \( \Rightarrow \) Theorem I.1] Statement (2) is Theorem 3.3. (3) is immediate from construction. (1) follows from (2), (3) and the invariance property of Floer cohomology under the action of Hamiltonian diffeomorphism. (4) follows from the fact that only \( T(1/2) \) is monotone among \( T(u) \)'s. Finally we prove (5). The last paragraph of Theorem 3.3 states that there exists \( b \) for which the number of bounding cochain \( b \in H^1(T(u); \Lambda_0) \) with \( HF((T(u), (b, b)), (T(u), (b, b))) \neq 0 \) is exactly 2. On the other hand, we know from [FOOO4] that for any choice of \( b \) there exist exactly 4 different choices of \( b \in H^1(S^1_{\text{eq}} \times S^1_{\text{eq}}) \) such that
\[
HF((S^1_{\text{eq}} \times S^1_{\text{eq}}, (b, b)), (S^1_{\text{eq}} \times S^1_{\text{eq}}, (b, b))) \neq 0.
\]
We recall from section 4.3.3 in [FOOO1] that

1. the potential function \( \mathfrak{P} \mathfrak{O}^b : \tilde{\mathcal{M}}_{\text{weak}}(L; m^b) \rightarrow \Lambda_0 \) is gauge invariant
2. the isomorphism type of Floer cohomology depends only on the gauge equivalence class in the space \( \tilde{\mathcal{M}}_{\text{weak}}(L; m^b) \) of weak bounding cochains.

In particular the set of weak bounding cochains \( b \) with nontrivial Floer cohomology is symplectically invariant. (In this paper, we suppress the formal variable “c” and work with \( \mathbb{Z}/2\mathbb{Z} \)-grading.) In order to extend the coefficients to \( \Lambda_0 \), we use non-unitary flat line bundle on the Lagrangian submanifold as in section 12 of [FOOO3] and adapt the argument in section 4.3.3 in [FOOO1] accordingly. The idea of using non-unitary flat line bundle is originally due to Cho [C].

Hence the proof of (5) is reduced to the following lemma.

**Lemma 3.4.** Let \( L \) be either \( T(u) \) or \( S^1_{\text{eq}} \times S^1_{\text{eq}} \) in \( M = S^2(1) \times S^2(1) \). Then we have a canonical isomorphism
\[
H^1(L; \Lambda_0) \cong \mathcal{M}_{\text{weak}}(L; m^b)
\]
and hence we can naturally identify \( \mathcal{M}_{\text{weak}}(L; m^b) \) with \( H^1(L; \Lambda_0) \) and \( \mathfrak{P} \mathfrak{O}^b \) can be regarded as a function defined on \( H^1(L; \Lambda_0) \) for all \( b \).
Proposition 4.1. from that of Auroux [Au].

By considering the de Rham model. We can also identify and we may assume

\[ H^1(L; \Lambda_0) \hookrightarrow \mathcal{M}_{\text{weak}}(L; m^{\text{can}, b}) \]

and the gauge transformation acts trivially on its image. (See section 7.) Therefore we have an embedding

\[ H^1(L; \Lambda_0) \hookrightarrow \mathcal{M}_{\text{weak}}(L; m^{\text{can}, b}). \]

And we may assume

\[ \mathcal{M}_{\text{weak}}(L; m^{\text{can}, b}) \subset C_{\text{dR}}(L; \Lambda_0) \]

by considering the de Rham model. We can also identify \( \mathcal{M}_{\text{weak}}(L; m^{\text{can}, b}) \) as a sub-variety of \( H^{\text{odd}}(L; \Lambda_0) \) as explained in section 5.4 [FOOO1]. But since both \( L \) are of 2 dimension, we have \( H^1(L; \Lambda_0) = H^{\text{odd}}(L; \Lambda_0) \) and hence \( H^1(L; \Lambda) \cong \mathcal{M}_{\text{weak}}(L; m^{\text{can}, b}) \).

\[ \Box \]

4. PROOF OF THEOREM 2.2

In this section we provide a proof of Theorem 2.2 whose strategy is different from that of Auroux [A].

**Proposition 4.1.** For \( 0 < \alpha < 1 \), \( (S^2, (1 - \alpha)\omega_{\text{std}}) \times (S^2, (1 + \alpha)\omega_{\text{std}}) \) is symplectomorphic to \( F_2(\alpha) \). For \( \alpha = 0 \), \( (S^2, \omega_{\text{std}}) \times (S^2, \omega_{\text{std}}) \) is symplectomorphic to \( \tilde{F}_2(0) \).

From now on, we write

\[ S^2(1 + \alpha) \times S^2(1 - \alpha) = (S^2, (1 + \alpha)\omega_{\text{std}}) \times (S^2, (1 - \alpha)\omega_{\text{std}}). \]

Although this proposition is well-known, we give a proof for the later purpose of computing the full potential of the resolution \( \tilde{F}_2(0) \) of \( F_2(0) \). (See [Mc1], [MK].)

**Proof.** We define a holomorphic vector bundle \( \mathcal{V} \rightarrow \mathbb{C} \times \mathbb{C}P^1 \) as follows. Write \( U_0 = \mathbb{C}P^1 \setminus \{ \infty \} \) and \( U_\infty = \mathbb{C}P^1 \setminus \{ 0 \} \) and take the transition function

\[ f(a, z) : (v_1, v_2) \in \mathbb{C}^2 \mapsto (zv_1 + av_2, z^{-1}v_2) \in \mathbb{C}^2, \quad (13) \]

for \((a, z) \in \mathbb{C} \times (U_0 \cap U_\infty)\). The restriction \( \mathcal{V}_a := \mathcal{V}|_{\{a\} \times \mathbb{C}P^1} \) is isomorphic to the bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \) over \( \mathbb{C}P^1 \) for \( a = 0 \) while it is holomorphically trivial for \( a \neq 0 \).

Take its projectivization

\[ \pi : \mathcal{X} = \mathbb{P}(\mathcal{V}) \rightarrow \mathbb{C} \times \mathbb{C}P^1. \]

Let \( p_i \) be the projection from \( \mathbb{C} \times \mathbb{C}P^1 \) to the \( i \)-th factor. We denote

\[ X_a = \mathcal{X}|_{\{a\} \times \mathbb{C}P^1} \]

as a complex manifold and denote its complex structure by \( J^a \).

By the construction and the definition of Hirzebruch surfaces (see e.g., [MK]), we have

**Lemma 4.2.** \( X_a \) is biholomorphic to \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) for \( a \neq 0 \) but \( X_0 \) is biholomorphic to \( F_2 \).
Recall that any two cohomologous Kähler forms on a compact manifold with a fixed complex structure are isotopic by Moser’s theorem. We will define Kähler forms on $X_a$ in suitable cohomology classes. We would like to note that our argument is basically gluing $F_0(0) \setminus \{O\}$ with a local model of simultaneous resolution of $\mathbb{C}^2/(\pm 1)$ and does not require the following specific construction.

Note that $X_0$ contains a $(−2)$-curve $C_{−2}$ and its normal bundle in $X$ is $O(−1) \oplus O(−1)$ [LO]. Therefore we have a contraction of $C_{−2}$. Here we give such a map explicitly. Firstly we define a holomorphic map from $X$ to $\mathbb{C}P^4$ as follows. By the definition of $\mathcal{V}$, we have

$$P(\mathcal{V}) = \mathbb{C} \times U_0 \times (\mathbb{C} \oplus \mathbb{C}) \cup \mathbb{C} \times U_\infty \times (\mathbb{C} \oplus \mathbb{C}),$$

where $(a, [1, z], [v_1, v_2]) \in \mathbb{C} \times U_0 \times (\mathbb{C} \oplus \mathbb{C})$ and $(a, [z, 1], (zv_1 + av_2, z^{−1}v_2) \in \mathbb{C} \times U_\infty \times (\mathbb{C} \oplus \mathbb{C})$ are identified. Define $\Phi_0 : \mathbb{C} \times U_0 \times \mathbb{P}^1 \to \mathbb{P}^4$ by

$$(a, [1, z], [v_1, v_2]) \mapsto [av_2 + zv_1, v_1, zv_2 + z^2v_1, z_1v_1, v_2]$$

and $\Phi_\infty : \mathbb{C} \times U_\infty \times \mathbb{P}^1 \to \mathbb{P}^4$ by

$$(a, [z, 1], [u_1, u_2]) \mapsto [z_0u_1, z_0^2u_1 - az_0u_2, u_1, z_0u_1 - au_2, u_2].$$

Then we find that $\Phi_0$ and $\Phi_\infty$ are glued to a map $\Phi : P(\mathcal{V}) \to \mathbb{P}^4$.

Let $[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ be the homogenous coordinates on $\mathbb{P}^4$. Then the image of $\Phi$ is described by the equation $\xi_1\xi_4 = \xi_2\xi_3$. Note that $\xi_1 - \xi_4 = a\xi_5$. By changing the coordinates

$$\eta_1 = \xi_1 + \xi_4, \quad \eta_2 = \xi_2 - \xi_3, \quad \eta_3 = i(\xi_2 + \xi_3), \quad \eta_4 = i(\xi_1 - \xi_4), \quad \eta_5 = \xi_5,$$

the equation becomes $\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 0$. From the relation $\eta_4 = i\alpha \eta_5$, we find that the image of $X_a \subset P(\mathcal{V})$ is described by

$$\eta_1^2 = \eta_2^2 + \eta_3^2 + \eta_4^2 = a^2 \eta_5^2, \quad \eta_5 = i\alpha \eta_5.$$

Note that $\Phi|_{X_a}$ becomes $({\mathbb{C}^*})^2$-equivariant with respect to a suitable $({\mathbb{C}^*})^2$-action on $\mathbb{C}P^4$. For a later convenience, we modify the Fubini-Study form on $\mathbb{C}P^4$ to a Kähler form $\Omega_{\mathbb{C}P^4}$, so that the corresponding Kähler metric is flat in a small neighborhood of $\Phi(O)$. We may also assume that $\Omega_{\mathbb{C}P^4}$ is invariant under the action of the maximal torus $S^1 \times S^1 \subset ({\mathbb{C}^*})^2$. Then we define

$$\Omega = \Phi^* \Omega_{\mathbb{C}P^4}.$$

For $a = 0$, the restriction of $\Omega$ is Kähler away from the $(−2)$-section $C_{−2}$, along which $\Omega$ degenerates, and semi-positive with respect to the complex structure on $F_2$.

For $\epsilon > 0$, we set

$$\Omega_\epsilon = \Omega + \epsilon(p_2 \circ \pi)^* \omega_{\mathbb{C}P^1},$$

where $\omega_{\mathbb{C}P^1}$ is the Fubini-Study form on $\mathbb{C}P^1$. Then the restriction of $\Omega_\epsilon$ to any $X_a$ is Kähler, and hence it follows from Moser’s theorem that for each given $\epsilon > 0$ ($X_a, \Omega_\epsilon|_{X_a}$) are symplectomorphic for all $a \in \mathbb{C}P^1$.

Set $\epsilon = 2a/(1 - a)$ and take a suitable scale change,

$$\Omega^\alpha = (1 - \alpha)\Omega^\frac{\alpha}{1-a}. \quad (14)$$

We define a symplectic form on $X_a$

$$\omega^\alpha_a = \Omega^\alpha|_{X_a}. \quad (15)$$
Then it follows that \((X_0, \omega_0^a)\) is symplectomorphic to \((X_a, \omega_0^a)\) by the above discussion. On the other hand, the above discussion together with Lemma 4.2 implies that \((X_0, \omega_0^a)\) is symplectomorphic to \(F_2(\alpha)\) and \((X_a, \omega_0^a)\) is symplectomorphic to \(S^2(1 - \alpha) \times S^2(1 + \alpha)\) for \(a \neq 0\). Combination of these now proves the statement of the proposition for \(0 < \alpha < 1\).

Now we consider the case \(\alpha = 0\). We write \(\bar{\rho} := p_1 \circ \pi : X' \to C\). Denote by \(\bar{X}\) the image of \(X\) by \(\bar{\rho} \times \Phi\). The projection \(\bar{\rho}\) descends to \(p : \bar{X} \to C\). The \((-2)\)-section \(C_{-2}\) of \(X_0\) is contracted to the \(A_1\)-singularity by \(\Phi\). Other \(X_a\) are mapped to their images biholomorphically. Therefore, \(p : \bar{X} \to C\) gives a smoothing of the \(A_1\)-singularity. Hence \(X_a\) is isomorphic to \(\bar{F}_2(0)\). For \(a \neq 0\), the restriction \(\Omega^a = 0|_{X_a} = \omega_0^a\) is a Kähler form, which represents a cohomology class proportional to \(PD[\mathbb{C}P^1 \times \{p\}] + PD[\{pt\} \times \mathbb{C}P^1]\).

Hence we obtain the statement for \(\alpha = 0\). \(\square\)

Using Proposition 4.1, we describe our family of Lagrangian submanifolds as follows.

Equip \(X\) with \(\Omega^a, \alpha > 0\). Then the central fiber \(X_0\) is \(F_2(\alpha)\) as a Kähler manifold. We now describe its toric structure. Consider the Kähler \(S^1\)-action on \(\mathbb{C}P^1\). Denote by \(\{D_1\}\) the domain bounded by an \(S^1\)-orbit \(C_0\) such that the orientation as the boundary of \(D_1\) is the same as the direction by the \(S^1\)-action and the area of \(D_1\) is \(s/2\) of the area of \(\mathbb{C}P^1\). In particular, \(C_1\) bounds two discs in \(\mathbb{C}P^1\) of equal area.

We lift this action holomorphically to

\[
y_0 \cong \mathcal{O}(-1) \oplus \mathcal{O}(1) \to \{0\} \times \mathbb{C}P^1.
\]

Such a lift is not unique, but unique up to the \(S^1\)-action obtained by the multiplication of \(e^{ik\theta}\) for some integer \(k\). The lifted \(S^1\)-action and the action of \(S^1 \times S^1\) by fiberwise multiplication commute and give a holomorphic \(T^2\)-action, which induces the \(T^2\)-action on \(X_0\). Here \(T^2\) is the quotient of \(T^3\) by the diagonal subgroup in the second and third factors. This \(T^2\)-action on \((X_0, J^0)\) is isomorphic to the \(T^2\)-action on \(F_2(\alpha)\) with the standard complex structure as a toric manifold. In particular, \(J^0\) is isomorphic to \(T^2\)-invariant complex structure on the toric Hirzebruch surface \(F_2\). In this description, the torus fibers \(L(u_1, u_2) \subset F_2(\alpha)\) for \((u_1, u_2) \in \text{Int} P(\alpha)\) are identified with the \(T^2\)-orbits. We find that \(L(u_1, u_2) \subset (p_2 \circ \pi|_{X_0})^{-1} C_{u_1/(1-u_2)} \subset X_0\). In particular, when \(u_1 + u_2 = 1\), it is contained in \((p_2 \circ \pi|_{X_0})^{-1} C_1\).

Next, we consider a hypersurface \(S = \pi^{-1}(\mathbb{R} \times \mathbb{C}P^1)\) and the characteristic foliation \(\text{Char}_{\Omega^0}(\mathcal{S})\).

**Definition 4.1.** For \(a \in \mathbb{R}\) and \(\alpha > 0\), the map \(\psi_0^a : X_0 \to X_a\) is the symplectomorphism induced by integration of the characteristic foliation \(\text{Char}_{\Omega^0}(\mathcal{S})\). When \(\alpha = 0\), we obtain an embedding \(\psi_0^a : X_0 \setminus C_{-2} \to X_a\) in a similar way. We define the tori

\[
L_0^a(u_1, u_2) = \psi_0^a(L(u_1, u_2)) \subset (X_a, \omega_0^a)
\]

which are Lagrangian with respect to \(\omega_0^a\). In particular, for \(a \neq 0\), we obtain Lagrangian tori \(L_0^a(u, 1-u) = \psi_0^a(L(u, 1-u))\) in \((X_a, \omega_0^a)\).

Here we explain how the construction above is related to the one of \(\bar{F}_2(0)\) given in section 3. Let \(N\) be a tubular neighborhood of \(C_{-2}\) in \(X_0\) such that the characteristic foliation on \(\partial N\) is isomorphic to the fibers of \(S^2/\pm 1 \to \mathbb{C}P^1\). Denote
by $W$ a tubular neighborhood of the vanishing cycle in $X_a$, which is bounded by $\psi^0_a(\partial N)$. Then we have $X_a = \psi^0_a(X_0 \setminus N) \cup W$. Applying the symplectic cutting construction along the boundary of $W$, we obtain a closed symplectic 4-manifold $N$ containing $+2$-curve $S$ and the symplectic form is exact on the complement of $S$. Then the work of McDuff [Mc2] implies that $N$ is symplectomorphic to the product of two copies of $S^2$ of the same area. It implies that $W$ is symplectomorphic to a disc bundle of $T^*S^2$. Therefore we find that $(X_a, \omega^0_a, a \neq 0$, is symplectomorphic to $F_2(0)$.

Pick a symplectomorphism $\varphi_a : (X_a, \omega^0_a) \to S^2(1) \times S^2(1)$. Then the Lagrangian tori $T(u)$ in Theorem 1.1 are given by

$$T(u) = \varphi_a(L^0_a(u, 1 - u)).$$

We use the next result for the proof of Theorem 2.2.

**Theorem 4.3.** The critical values of the potential function of $S^2(1 - \alpha) \times S^2(1 + \alpha)$ are equal to those of $F_2(\alpha)$. In fact, they are equal to the eigenvalues of the quantum multiplication of the first Chern class.

**Proof.** The fact that the critical values of the potential function are equal to the eigenvalues of the quantum multiplication of the first Chern class is proved for the Fano toric manifolds in [FOOO3] (See Remark 5.3 and Theorem 1.9 therein. We note $S^2(1 - \alpha) \times S^2(1 + \alpha)$ is Fano.)

Since $F_2(\alpha)$ is not Fano, we need to prove the corresponding fact separately. We prove this by combining the following facts: There exists an isomorphism $\varphi$ from the small quantum cohomology ring of $F_2(\alpha)$ to the Jacobian ring $\text{Jac}(\mathfrak{D})$ of the potential function $\mathfrak{D}$ of $F_2(\alpha)$. Moreover $\varphi$ sends $c_1(F_2(\alpha))$ to the element $\mathfrak{D} \in \text{Jac}(\mathfrak{D})$. We can prove it by the argument of [FOOO3] Remark 6.15. (See [FOOO5] for detail.)

We have thus proved the second half of Theorem 4.3. The first half follows from the second half and Proposition 4.1. □

A key idea of the proof of Theorem 2.2 is our usage of Theorem 4.3 in our computation of the coefficient $c$. Both $F_2(\alpha)$ and $S^2(1 - \alpha) \times S^2(1 + \alpha)$ are toric manifolds. Hence the potential functions are defined for them. Although $F_2(\alpha)$ and $S^2(1 - \alpha) \times S^2(1 + \alpha)$ are not isomorphic as toric Kähler manifolds, they are symplectomorphic. Therefore their quantum cohomologies are isomorphic. In particular, the eigenvalues of the quantum multiplication by the first Chern class are the same. By comparing the critical values of potential functions in the two pictures, we will determine the coefficient $c$. The detail is now in order.

The moment polytope of $S^2(1 - \alpha) \times S^2(1 + \alpha)$ is

$$P^s(\alpha) = \{(u_1, u_2) \mid 0 \leq u_1 \leq 1 - \alpha, \ 0 \leq u_2 \leq 1 + \alpha\}.$$ 

There is a unique balanced fiber over $u = ((1 - \alpha)/2, (1 + \alpha)/2)$, where the potential function is

$$T^{(1 - \alpha)/2}(y_1 + y_1^{-1}) + T^{(1 + \alpha)/2}(y_2 + y_2^{-1}) .$$ 

It has 4 critical points, $(y_1, y_2) = (\pm 1, \pm 1)$. Their associated critical values are

$$\pm 2T^{(1 - \alpha)/2}(1 \pm T^\alpha).$$


On the other hand the balanced fiber of $F_2(\alpha)$ is located at $((1+\alpha)/2, (1-\alpha)/2)$, where the corresponding potential function is

$$\mathfrak{PO} = T^{(1-\alpha)/2}(y_2 + (1 + c)y_2^{-1}) + T^{(1+\alpha)/2}(y_1 + y_1^{-1}y_2^{-2}).$$

(19)

The condition for $(y_1, y_2)$ being critical is

$$0 = 1 - y_1^{-2}y_2^{-2}. \quad (20)$$

$$0 = 1 - 2T^\alpha y_1^{-1}y_2^{-3} - (1 + c)y_2^{-2}. \quad (21)$$

The equation (21) implies $y_1 y_2 = \pm 1$.

Case 1: $y_1 y_2 = -1$. (21) implies $y_2 = 1 + c - 2T^\alpha$. Then the critical values are

$$\pm 2T^{(1-\alpha)/2}\sqrt{1 + c - 2T^\alpha}. \quad (22)$$

Case 2: $y_1 y_2 = 1$. (21) implies $y_2 = 1 + c + 2T^\alpha$. Then the critical values are

$$\pm 2T^{(1-\alpha)/2}\sqrt{1 + c + 2T^\alpha}. \quad (23)$$

We have already found four different solutions, while the number (counted with multiplicity) of critical points is the sum of Betti numbers, which is four in this case. Therefore we must have

$$1 \pm T^\alpha = \sqrt{1 + c \pm 2T^\alpha}$$

from which $c = T^{2\alpha}$ follows immediately. This finishes the proof of Theorem 3.2. □

5. PROOF OF THEOREM 3.2

We first insert $c = T^{2\alpha}$ obtained in Theorem 2.2 into the formula (2) and obtain the potential function

$$\mathfrak{PO} = \mathfrak{PO}(y_1, y_2; u_1, u_2) = T^{u_1}y_1 + T^{u_2}y_2 + T^{2-u_1-2u_2}y_1^{-1}y_2^{-2} + (1 + T^{2\alpha})T^{1-\alpha-u_2}y_2^{-1}$$

(24)

for $F_2(\alpha)$ for any $0 < \alpha < 1$. As we pointed out before, Theorem 3.2 is obtained by formally setting $\alpha = 0$ in (24).

In this section, we give two different justifications of this ‘formal insertion’ to obtain the potential function of $\tilde{F}_2(0)$. Both proofs use Theorem 2.2. The first proof is based on the deformation family $\tilde{p} = p_1 \circ \pi : \mathcal{X} \to \mathbb{C}$ constructed in the proof of Proposition 4.4. The second proof uses the standard gluing argument. The latter may be applied to more general cases, while the former uses a special feature of Hirzebruch surfaces.

We remark that potential function $\mathfrak{PO}$ of Lagrangian submanifold as a $\Lambda_0$-valued function on $H^1(L; \Lambda_0)$ is well-defined up to a change of coordinate that is congruent to identity. (See Remark 7.3) In the case of toric fiber, we can use a $T^n$-equivariant perturbation so that $\mathfrak{PO}$ is strictly well-defined as a function on $H^1(L; \Lambda_0)$. We emphasize that we are studying the case of Lagrangian submanifolds which is not of toric fiber.

Precisely speaking, the first proof implies the following: For each $(u_1, u_2)$, there exists a tame almost complex structure $J_{u_1, u_2}$ on $\tilde{F}_2(0)$ such that we have strict equality (13) for the potential function with respect to this $J_{u_1, u_2}$.

The second proof implies the following: For each $(u_1, u_2)$ and $E$ there exists a compatible almost complex structure $J_E$ on $\tilde{F}_2(0)$ such that

$$\mathfrak{PO}^{JE}(y_1, y_2; u_1, u_2) \equiv T^{u_1}y_1 + T^{u_2}y_2 + T^{2-u_1-2u_2}y_1^{-1}y_2^{-2} + 2T^{1-u_2}y_2^{-1} \mod T^E.$$

(25)
holds for the potential function $\mathcal{P} \mathcal{D}^{JE}$ of $L(u_1,u_2)$ with respect to the almost complex structure $J_E$.

Both the first and the second proofs imply the equality (6) modulo a coordinate change of $(y_1, y_2)$ congruent to the identity modulo $\Lambda_\pm$. To prove Theorem 1.1 this statement is sufficient.

5.1. Proof I: Deformation method. Let $\text{Int } P = \bigcup_{\alpha > 0} \text{Int } P(\alpha) = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 > 0, u_2 < 1, u_1 + 2u_2 < 2\}$. We put $X_\alpha = (\mathcal{X}, \Omega_\alpha)$, where $\alpha \in [0, 1]$ and $\mathcal{X} = \mathbb{P}(V)$ is as in section 4. As a family of $C^\infty$-manifolds, we have a trivialization $\bigcup_{\alpha \in \mathbb{C}} X_\alpha \cong \mathbb{C} \times (S^2 \times S^2)$. When we specify the symplectic form $\omega_\alpha$ on $X_\alpha$, we write $X_\alpha = (X_\alpha, \omega_\alpha)$. Since $\Phi|_{X_\alpha}$ is equivariant under the torus action, we may assume that $\omega_\alpha^0$, $\alpha > 0$ is a toric Kähler form on $X_0$.

The complement of the $(+2)$-section $C_2$ in $p : X_0 \to \mathbb{C}P^1$ is biholomorphic to the total space of $H^{\otimes -2}$, where $H$ is the hyperplane section bundle of $\mathbb{C}P^1$. We equip $H^{\otimes -2}$ with a hermitian metric such that the corresponding hermitian connection satisfies the following condition. Denote by $\omega_{\mathbb{C}P^1}$ the Fubini-Study form on $\mathbb{C}P^1$ with total area 1 and by $\theta$ the hermitian connection form of $H^{\otimes -2}$. Then the curvature of the connection $\theta$ is a multiple of $\omega_{\mathbb{C}P^1}$.

Recall that we modified the Fubini-Study form on $\mathbb{C}P^4$ such that it becomes flat in a neighborhood of $\Phi(O)$ and invariant under $S^1 \times S^1$-action. Then we can express

$$\omega_0^\alpha = (1 - \alpha) \left( \frac{1}{2} d(\rho(r)^2 \theta) + \frac{2\alpha}{1 - \alpha} p^* \omega_{\mathbb{C}P^1} \right),$$

where $r$ is the fiberwise norm and $\rho : [0, \infty) \to [0, 1)$ is a strictly increasing smooth function with $\rho(r) = r$ around $r = 0$. We will modify the function $\rho$ according to $(u_1, u_2)$ later.

Recall from (14), (15) that we have

$$\omega_0^\alpha = (1 - \alpha) \left( \omega_0^0 + \frac{2\alpha}{1 - \alpha} (p_2 \circ \pi)^* \omega_{\mathbb{C}P^1} \right).$$

Therefore we obtain

$$\omega_0^0 = \frac{1}{2} d(\rho(r)^2 \theta).$$

When we collapse the $(−2)$-curve $C_{−2}$ in $X_0$, we obtain $F_2(0)$ with an isolated singular point $O$. Note that $X_0 \setminus (C_2 \cup C_{−2}) \cong F_2(0) \setminus (C_2 \cup \{O\})$ is biholomorphic to $(-\infty, \infty) \times S^3/\{\pm 1\}$ with a complex structure, which is invariant under the translation in $(-\infty, \infty)$. In the cylindrical coordinates, we have

$$\omega_0^0 = \frac{1}{2} d(e^{2\sigma(s)} \theta),$$

where $s$ is the coordinate on $(-\infty, \infty)$ and $\sigma(s) = \log \rho(e^s)$. In particular, $\sigma(s) = s$ for sufficiently small $s$.

**Lemma 5.1.** For $(u_1, u_2) \in \text{Int } P$, we consider the Lagrangian submanifold $L_\alpha^0(u_1, u_2)$ given in (19). Then there is a constant $\delta(u_1, u_2) > 0$ with the following properties:

1. If $|a| < \delta(u_1, u_2)$, $\alpha < \delta(u_1, u_2)$, we have a diffeomorphism $\phi_{a,(u_1,u_2),\alpha} : X_\alpha \to X_\alpha$ such that

$$L_\alpha^a(u_1, u_2) = \phi_{a,(u_1,u_2),\alpha}(L_\alpha^0(1/2, 1/2)).$$

2. There is a neighborhood $U(u_1, u_2)$ of $C_{−2}$ in $\bigcup_a X_\alpha$ such that $\phi_{a,(u_1,u_2),\alpha}$ is the identity map on $U(u_1, u_2) \cap X_\alpha$. Here we regard $C_{−2} \subset X_0 \subset X_\alpha.$
(3) There exists a smooth family of almost complex structures $J_{a,u_1,u_2}^{(1)}$ on $X_a$, which is tamed both by $\omega^a_1$ and $\phi^*_{a,(u_1,u_2)} \omega^a_1$ for $|a|, \alpha < \delta(u_1,u_2)$.

Proof. Firstly, we consider the case that $a = \alpha = 0$. Since the $(\mathbb{C}^*)^2$-action induced by the toric structure is transitive on the set of all torus fibers $L(u_1, u_2)$, we have a biholomorphic map $g_{u_1,u_2}$ such that $g_{u_1,u_2}(L(1/2,1/2)) = L(u_1, u_2) = L^0_{0}(u_1, u_2)$. We modify $g_{u_1,u_2}$ to obtain $\phi_{0,(u_1,u_2),0}$ and find $J_{0,u_1,u_2}^{(1)}$ so that items 1-3 of Lemma 5.1 are also satisfied. Namely we have:

Sublemma 5.2. There exists a neighborhood $U \subset X_0$ of $C_{-2}$ depending only on $(u_1, u_2)$, and a map $\phi_{0,(u_1,u_2),0} : X_0 \to X_0$, and an almost complex structure $J_{0,u_1,u_2}^{(1)}$ such that the following holds:

1. $L^0_{0}(u_1, u_2) = \phi_{0,(u_1,u_2),0}(L^0_{0}(1/2,1/2))$.
2. $\phi_{0,(u_1,u_2),0}$ is an identity map on $U$.
3. $J_{0,u_1,u_2}^{(1)}$ is tamed both by $\omega^0_1$ and $\phi^*_{0,(u_1,u_2),0} \omega^0_1$.

Proof. We identify $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ with $(-\infty, \infty) \times S^3/\{\pm 1\}$, which is $(\mathbb{C}^*)^2$-equivariantly biholomorphic to $X_0 \setminus (C_2 \cup C_{-2})$. From now on, we fix this identification. Here $O \in \mathbb{C}^2/\{\pm 1\}$ corresponds to the limit of $\{s\} \times S^3/\{\pm 1\}$ as $s$ tends to $-\infty$. The complex torus $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2/\{\pm 1\}$ by

$$(w_1, w_2) \cdot [z_1, z_2] = [w_1z_1, w_2z_2],$$

where $(w_1, w_2) \in (\mathbb{C}^*)^2$ and $(z_1, z_2) \in \mathbb{C}^2$. Note that this map extends to a biholomorphic automorphism of $X_0$. (The action by $(\mathbb{C}^*)^2/\{\pm 1\}$ is the torus action as a toric manifold.)

For $(u_1, u_2)$, we pick $a < b \in \mathbb{R}$ so that

$L(u_1, u_2), L(1/2,1/2) \subset (a, b) \times S^3/\{\pm 1\} \subset X_0$.

We choose the function $\rho$ such that $\rho(s) = s$ on $(-\infty, b]$. We set

$$i(c_1, c_2) = (c_1c_2, c_1^{-1}c_2).$$

Since $(\mathbb{C}^*)^2$ acts on the set of torus fibers $L(u_1, u_2)$ transitively, there is $(c_1, c_2) \in (\mathbb{R}_+)^2$ such that

$$i(c_1, c_2) \cdot L(1/2,1/2) = L(u_1, u_2).$$

Thus we may identify $g_{u_1,u_2}$ with the multiplication by $i(c_1, c_2) \in (\mathbb{C}^*)^2$.

Let $S$ be a sufficiently large positive number to be chosen later. We will modify the action by $i(c_1, 1)$ in the region $(-\infty, -S) \times S^3/\{\pm 1\}$ so that it becomes the identity on a neighborhood of $C_{-2}$.

Identifying $S^3/\{\pm 1\} = (\mathbb{C}^2 \setminus \{0\})/\mathbb{R}^*$, we denote by $[[x_1, x_2]] \in S^3/\{\pm 1\}$ the equivalence class of $(x_1, x_2) \in \mathbb{C}^2 \setminus \{0\}$. We define $\phi_1 : (-\infty, \infty) \times S^3/\{\pm 1\} \to (-\infty, \infty) \times S^3/\{\pm 1\}$ by

$$\phi_1(s, [[x_1, x_2]]) = (s + h_S(s, [[x_1, x_2]]), [[x_1^{-f_S(s)} x_1, c_1^{-f_S(s)} x_2]])$$

where $f_S$ is a nondecreasing function such that

$$f_S(s) = \begin{cases} 0 & s < -3S + 1, \\ 1 & s > -2S \end{cases}$$
Then if we take $X_{a,\alpha}$, a taming condition holds on $h$. Large $S$ depends only on $c_1$, $c_2$, $\phi_1$ is a diffeomorphism of $X_0$ and $J_0$ is tamed by $\phi_1^* \omega_0$. We also choose $S$ large enough such that $-S < a - \max |h_S| - 1$.

We next define $(-\infty, \infty) \times S^3/\{\pm 1\} \to (-\infty, \infty) \times S^3/\{\pm 1\}$ by

$$\phi_2(s, [[x_1, x_2]]) = (\chi_S(s), [[x_1, x_2]])$$

Here $\chi_S : (0, \infty) \to (-\infty, \infty)$ is a strictly increasing smooth function such that $\chi_S(s) = s$ for $s < a - S$ and $\chi_S(s) = s + \log c_2$ for $s \geq a$. Using the fact that $d\chi_S/ds > 0$, we find that $J_0$ is tamed by $\phi_2^* \omega_0$. Note also that $\phi_2$ coincides with the action by $i(1, c_2)$ in the region $[a, \infty) \times /\{\pm 1\}$. In particular, $\phi_2$ extends to $C_2$ smoothly.

We remark that $\phi_2$ is the identity map on the image of $(-\infty, \infty) \times S^3/\{\pm 1\}$ by $\phi_1$ and that $\phi_1$ is holomorphic on $(-2S, \infty) \times S^3/\{\pm 1\}$. Therefore $J_0$ is tamed by $(\phi_2 \phi_1)^* \omega_0$.

Since $\phi = \phi_2 \phi_1$ is the action of $(c_1, c_2)$ on $X_0 \setminus (C_{-2} \cup (-\infty, -2S) \times S^3/\{\pm 1\}$, it follows that $J_{0;U_1, U_2}$ is tamed by $\phi^* \omega_0$.

By definition $\phi(L(1/2, 1/2)) = L(u_1, u_2)$. Since $\phi$ is the identity map near $C_{-2}$ and biholomorphic near $C_2$, $\phi_1$ is extend to a diffeomorphism $\phi_{0;u_1,0} u_2$ of $X$, resp. a complex structure $J_{0;u_1,2}$, which have the required properties.

We next consider the case $(a, \alpha) \neq (0, 0)$. We set

$$\phi_{a, u_1, u_2} = \psi_a^\alpha \circ \phi_{0;u_1, u_2} \circ (\psi_a^\alpha)^{-1}. \quad (26)$$

(See Definition 3.11.) Note that $\phi_{a, u_1, u_2}, \alpha$ is the identity map on $\psi_a^\alpha(U)$, which is a neighborhood of the vanishing cycle. Therefore $\phi_{a, u_1, u_2}, \alpha$ is a diffeomorphism of $X_a$. We have $L_{a}^{(1)}(u_1, u_2) = \phi_{a, u_1, u_2}, \alpha(L_{a}^0(1/2, 1/2, 2))$.

Fix $(u_1, u_2) \in \text{Int } P$. We extend $J_{0;u_1, u_2}$ to a smooth family of almost complex structures $J_{a;u_1, u_2}$ on $X_a$ such that $J_{a;u_1, u_2} = J^a$ on $\psi_a^\alpha(U)$ for $|a|, \alpha < \delta$. Here we pick and fix a positive constant $\delta$.

Pick an open subset $U'$ such that $C_{-2} \subset U' \subset U$. The condition that $J_{a;u_1, u_2}$ is tamed both by $\omega_a^\alpha$ and $\psi_a^\alpha \circ \phi_{0;u_1, u_2}, \alpha \circ (\psi_a^\alpha)^{-1}$ on $X_0 \setminus U'$ is an open condition for $a, \alpha$. (The degenerates along $C_{-2}$.) Note also that the above taming condition holds on $X_0 \setminus U$ at $a = \alpha = 0$ and $J^a$ is compatible with $\omega_a^\alpha$ for all $\alpha \geq 0$ when $a \neq 0$. Therefore we find $\delta(u_1, u_2)$ enjoying the required two properties.

We identify $(X_0, \omega_a^\alpha)$ with the toric $F_2(\alpha)$ by a fixed symplectomorphism. Consider $L(1/2, 1/2) = L_{0}^{0}(1/2, 1/2) \subset F_2(\alpha) = X_0$. For $a \in \mathbb{R} \setminus \{0\}$,

$$L_{a}^{0}(1/2, 1/2) = \psi_{a}^{0}(L(1/2, 1/2)) \subset X_{a,0}.$$
is a Lagrangian submanifold with respect to $\omega_a^0$. In particular, it is a monotone Lagrangian submanifold with respect to $\omega_a^0$. The ambient space is also monotone. Thus we find that the Maslov index of any non constant $J$-holomorphic disc with boundary on the Lagrangian torus $L_a^0(1/2,1/2)$ is at least 2. Here $J$ is any almost complex structure tamed by $\omega_a^0$. Namely Assumption 4.1 is satisfied for such an almost complex structure $J$.

Using Theorem 7.3 in Appendix and the above remark, we find that the conclusion of Theorem 2.2 holds for any such $J$ and $L(1/2,1/2)$. (Theorem 2.2 itself is the case of toric complex structure.) In particular it holds for $J_{a;u_1,u_2}^{(1)}$.

We put

$$J_{a,\alpha;u_1,u_2}^{(2)} = \langle \phi_{a,(u_1,u_2),\alpha} \rangle_{a;u_1,u_2}.$$ \(J_{a,\alpha;u_1,u_2}^{(2)}\) is tamed by $\omega_a^0$ for sufficiently small $\alpha$, $a$.

Clearly the moduli space of $J_{a,\alpha;u_1,u_2}^{(2)}$-holomorphic discs with boundary on the torus $L_a^0(u_1,u_2)$ is identified with the moduli space of $J_{a;u_1,u_2}^{(1)}$-holomorphic discs which bound $L_a^0(1/2,1/2)$. (Lemma 5.4.)

Hence, for any $\beta$ with $\mu(\beta) = 2$ and $\alpha$, $a_1$, $a_2$ with $|a_1|, |a_2| < \delta(u_1,u_2)$, we have a cobordism between

$$\mathcal{M}(L_{a_1}^0(u_1,u_2); J_{a_1,\alpha;u_1,u_2}^{(2)}; \beta) \cong \mathcal{M}(L_{a_2}^0(1/2,1/2); J_{a_2;u_1,u_2}^{(1)}; \beta)$$

and

$$\mathcal{M}(L_{a_2}^0(u_1,u_2), J_{a_2,\alpha;u_1,u_2}^{(2)}; \beta) \cong \mathcal{M}(L_{a_1}^0(1/2,1/2); J_{a_1;u_1,u_2}^{(1)}; \beta).$$

On the other hand, Gromov’s compactness implies that when $\alpha > 0$, for any given $E > 0$ there is some $a_0 > 0$ such that for $a \in (0,a_0)$, the potential functions of $L_a^0(u_1,u_2) \subset (X,\omega^0_a, J_{0,\alpha;u_1,u_2}^{(2)})$ and $L_a^0(u_1,u_2) \subset (X,\omega^0_a, J_{a,\alpha;u_1,u_2}^{(2)})$ coincide up to the order of $T^E$.

Combining these facts, we conclude that the potential functions of $L_a^0(u_1,u_2) \subset (X,\omega^0_a, J_{0,\alpha;u_1,u_2}^{(2)})$ and of $L_a^0(u_1,u_2) \subset (X,\omega^0_a, J_{a,\alpha;u_1,u_2}^{(2)})$ coincide for $a \leq \delta(u_1,u_2)$ and any $\alpha$ with $|a| < \delta(u_1,u_2)$. Recall that Theorem 2.2 gives the potential function of $L_a^0(u_1,u_2)$, hence we obtain the potential function of $L_a^0(u_1,u_2)$. We put

$$J_{a;u_1,u_2}^a = \lim_{\alpha \to 0} J_{a,\alpha;u_1,u_2}^{(2)} = \lim_{\alpha \to 0} \langle \phi_{a,(u_1,u_2),\alpha} \rangle_{a;u_1,u_2}.$$\(J_{a;u_1,u_2}^a\) is the identity map on $\psi_a^0(U)$, $J_{a,\alpha;u_1,u_2}^{(2)}$ is independent of $\alpha$ around the vanishing cycle. Therefore $J_{a;u_1,u_2}^a$ is well-defined. Then we find that $c_k = \deg(\pi_0; \mathcal{M}_1(X, L(u_1,u_2); J_{a,\beta;u_1,u_2}; \beta_1 + k[D_1]) \to L(u_1,u_2)).$\n
The potential function of $L_a^0(u_1,u_2)$ in $(X,\omega_a^0)$ depends on $\alpha$ only through the exponents of $T$, i.e., $\omega_a^0$-areas of bordered stable maps. Namely, $c_k$’s, which appear as coefficients in the potential function, does not depend on $\alpha$.

When $a \neq 0$, we have a family of Kähler forms $\omega_a^0$, $\alpha \leq \delta(u_1,u_2)$. (Namely the family $\omega_a^0$ extends to $\alpha = 0$.) Letting $\alpha \to 0$, we find that the potential function of $L_a^0(u_1,u_2) \subset (X,\omega_a^0, J_{a;u_1,u_2}^a)$ becomes

$$\mathcal{PO} = T^{u_1}y_1 + T^{u_2}y_2 + T^{2-u_1-2u_2}y_1^{-2}y_2^{-2} + 2T^{1-u_2}y_2^{-1}.$$ \(\square\)

This is the conclusion of Theorem 3.2.
5.2. Proof II: Gluing method. Now we present the second proof. Let $F_2(0)_0$ be the $F_2(0)$ minus the singular point. We symplectically identify the end of $F_0(0)_0$ with the semi-infinite cylinder

$$((-\infty, 0] \times S^3/\{\pm 1\}, d(e^s\lambda))$$

over the real projective space $\mathbb{RP}^3 = S^3/\{\pm 1\}$. Here $\lambda$ is the standard contact form on $S^3/\{\pm 1\}$ and $S_0 \in \mathbb{R}$ is sufficiently large.

For each given $u \in \text{IntP}$, we consider the moduli space

$$\widetilde{\mathcal{M}}^2_{(1;1)}(L(u); F_2(0)) = \{(v, z; z_0) \mid z_0 \in \text{Int } D^2, \ z_0 \in \partial D^2, $$

$$v : (D^2 \setminus \{z_0\}, \partial D^2) \to (-\infty, 0] \times S^1 \to F_2(0)_0$$

is a map with the following properties:

1. $v$ is proper and pseudo-holomorphic.
2. There exists a loop $\gamma : S^1 \to S^3/\{\pm 1\}$ which is a simple closed Reeb orbit and such that there exists $\tau_0 \in \mathbb{R}$ and $e^{it_0} \in S^1$ with

$$\lim_{t \to -\infty} d(\tau, \gamma(t), v_\gamma(t + \tau_0)) = 0$$

where $z = e^{i\tau + it}$ is an analytic coordinate such that $z = 0$ at $z_0$, $d$ is the cylindrical metric on $(-\infty, 0] \times S^3/\{\pm 1\}$ and $\gamma$ is a trivial cylinder in $\mathbb{R} \times S^3/\{\pm 1\}$ defined by $\gamma(t) = (\tau, \gamma(t))$.
3. $v(z) \in L(u)$ for $z \in \partial D^2$.
4. We have

$$\int v^* \omega \leq 1 - u_2 + \delta$$

where the constant $\delta$ is a fixed constant so small that the corresponding homotopy class of such map $v$ is unique.

We note that $\text{PSL}(2, \mathbb{R})$ acts on $\widetilde{\mathcal{M}}^2_{(1;1)}(L(u); F_2(0))$ by

$$g \cdot (v, z_-, z_0) = (v \circ g^{-1}, g(z_-), g(z_0)).$$

Then we define the moduli space $\mathcal{M}^2_{(1;1)}(L(u); F_2(0))$ to be the quotient

$$\mathcal{M}^2_{(1;1)}(L(u); F_2(0)) = \mathcal{M}^2_{(1;1)}(L(u); F_2(0))/\text{PSL}(2, \mathbb{R}).$$

We remark that the set $\tilde{\mathcal{R}}_1(\lambda) := \text{Reeb}_1(S^3/\{\pm 1\})$ of (parameterized) Reeb orbits $\gamma$ given in (2) above can be identified with the set of parameterized closed geodesic of $S^2$ with minimal length and so it is diffeomorphic to $S^1/\{\pm 1\}$. Here the subscript ‘1’ stands for the ‘minimal period’. We take its quotient by $S^1$-action defined by changing the base point of the parametrization. Let $\mathcal{R}_1(\lambda)$ be the corresponding quotient space, which is diffeomorphic to $S^2$.

On the cylinder $\mathbb{R} \times S^3/\{\pm 1\}$, we denote by $s$, resp. $\Theta$ the projection to $\mathbb{R}$, resp. $S^3/\{\pm 1\}$. By abuse of notation, we also use $s$ and $\Theta$ on a subcylinder contained in $\mathbb{R} \times S^3/\{\pm 1\}$. There is an obvious asymptotic evaluation map

$$ev^\dagger : \mathcal{M}^2_{(1;1)}(L(u); F_2(0)) \to \mathcal{R}_1(\lambda) \cong S^2$$

which assigns to $(v, z_-, z_0) \in \mathcal{M}^2_{(1;1)}(L(u); F_2(0))$ its asymptotic Reeb orbit $\gamma$ defined by

$$\gamma(t) = \lim_{t \to -\infty} \Theta \circ v(\tau/T, t/T)$$

where $\tau = s$ is a fixed constant so small that the corresponding
where $z = e^{\gamma + it}$ is the analytic coordinates adapted to $z_-$ and $T$ is the period of the Reeb orbit $\gamma$. We note that the period $T$ is determined by the asymptotic behavior of $v$ by the formula
\[ T = \lim_{\tau \to -\infty} \int (\Theta \circ v_\tau)^* \lambda \]
where $v_\tau$ is the loop defined by $v_\tau(t) = v(\tau, t)$. We have another evaluation map
\[ ev_0 : \mathcal{M}_{(1,1)}^t (L(u); F_2(0)) \to L(u) \quad (27) \]
which assigns to $(v, z_-, z_0)$ the point $v(z_0) \in L(u)$.

It is by now well-known that there is an appropriate Fredholm theory for the study of $\mathcal{M}_{(1,1)}^t (L(u); F_2(0))$. We omit explanation of this Fredholm theory here just referring to \[\text{FOOO6}\] for detailed exposition given in a similar context. See also \[\text{H1}\]. We can check that the moduli space $M$ of such maps is classified by its intersection number with the zero section $C_s$ where $\lambda$ is the coordinate on $[0, \infty)$. The end of $X(\alpha)$, $0 < \alpha < 1$, is symplectomorphic to the cylinder
\[ ([0, \infty) \times S^3 / \{ \pm 1 \}, d(e^s \lambda)), \]
where $s$ is the coordinate on $[0, \infty)$. We consider the set of smooth maps $u : \mathbb{C} \to X(\alpha)$. For the purpose of describing the gluing process precisely, we identify $\mathbb{C}$ with $\mathbb{C} P^1 \setminus \{ pt \}$, and so consider the moduli space of maps
\[ w : \mathbb{C} \to X(\alpha), \]
with the following properties:

1. $w$ is proper and pseudo-holomorphic.
2. There exist $(\tau_0, t_0) \in \mathbb{R} \times S^1$ and $\gamma$ where $\gamma$ is as in (2) in the definition of $\mathcal{M}_{(1,1)}^t (L(u); F_2(0))$ such that
\[ \lim_{\tau \to +\infty} d(w(e^{\tau + \tau_0 + (t + t_0)i}), v_\gamma(\tau, t)) = 0. \]

where $d$ is the cylindrical metric on the cylinder $([0, \infty) \times S^3 / \{ \pm 1 \}, d(e^s \lambda))$. We need to assume finiteness of an appropriate energy, more specifically the Hofer energy \[\text{H1}\]. In addition, we omit the precise formulation thereof because it is by now standard.

We denote the asymptotic boundary of $X(\alpha)$ by $\partial_{\infty} X(\alpha)$ and the relative (Moore) homology class
\[ [w] \in H_2(X(\alpha), \partial_{\infty} X(\alpha)) \]
of such $w$ is classified by its intersection number with the zero section $\mathbb{C} P^1 \subset X(\alpha)$. Let $\bar{\beta}$ be the class with intersection number 1. Other classes are $\bar{\beta} + k [\mathbb{C} P^1]$. Let $\tilde{\mathcal{M}}(X(\alpha); k)$ be the moduli space of such $w$ in homology class $[w] = \bar{\beta} + k [\mathbb{C} P^1]$. We take the quotient of the space $\tilde{\mathcal{M}}(X(\alpha); k)$ by the $\text{Aut}(\mathbb{C})$-action given by
\[ g \cdot w = w \circ g^{-1} \]
and take its stable-map compactification. Denote the resulting compactified moduli space by $\mathcal{M}^2(X(\alpha); k)$. There is an obvious asymptotic evaluation map 

$$ev^\tau : \mathcal{M}^2(X(\alpha); k) \to R_1(\lambda)$$

which assigns to $w$ an asymptotic Reeb orbit $\gamma$ given by 

$$\gamma(t) = \lim_{\tau \to \infty} \Theta \circ w(\epsilon^{\tau+\tau}/T), \quad T = \lim_{\tau \to \infty} \int (\Theta \circ w_\tau)^* \lambda$$

where $w_\tau(t) = w(\epsilon^{\tau+\tau})$.

We next describe a family of almost complex structures we use.

Let $\alpha$ and $S_0$ be positive numbers satisfying $e^{2S_0}\alpha < 1$. We glue $F_2(0) \setminus (-\infty, -S_0) \times S^3/\{\pm 1\}$ with $X(e^{2S_0}\alpha) \setminus [S_0, \infty) \times S^3/\{\pm 1\}$ along $[-S_0] \times S^3/\{\pm 1\}$ and $\{S_0\} \times S^3/\{\pm 1\}$. (We put the symplectic form $e^{-2S_0}d(e^\lambda)$ on $X(e^{2S_0}\alpha)$.) We then obtain $F_2(\alpha)$.

This space contains the cylindrical region $[-S_0, S_0] \times S^3/\{\pm 1\}$. Using this identification, we define a compatible almost complex structure $J_{S_0}$ for each $S_0$, on $F_2(\alpha)$ so that its restriction to $[-S_0, S_0] \times S^3/\{\pm 1\}$ is of product type and its restriction to the other part is independent of $S_0$. Then the zero section $\mathbb{C}P^1$ of $X(\alpha)$ becomes the $(-2)$-curve $D_1$ in $F_2(\alpha)$.

Now we consider the homology class of maps $(D^2, \partial D^2) \to (F_2(\alpha), L(u))$. Such a homology class is determined by the intersection numbers with irreducible components of the toric divisor. We denote by the class $\beta$ the one that has intersection number 1 with the $(-2)$-curve $D_1$ and 0 with all other irreducible components of toric divisors. We consider the class $\beta = \beta_1 + k [D_1]$ and let $\mathcal{M}_1(F_2(\alpha), L(u); J_{S_0}; \beta_1 + k [D_1])$ be the compactified moduli space of stable maps $(\Sigma, \partial \Sigma) \to (F_2(\alpha), L(u))$ of genus zero, in homology class $\beta_1 + k [D_1]$ and with one boundary marked point.

We have the natural fiber product

$$\mathcal{M}^2(X(\alpha); k) \times_{ev^\tau} \mathcal{M}^2_{(1,1)}(L(u); F_2(0))$$

and the evaluation map

$$ev_0 : \mathcal{M}^2(X(\alpha); k) \times_{ev^\tau} \mathcal{M}^2_{(1,1)}(L(u); F_2(0)) \to L(u)$$

such that the following diagram

$$\begin{array}{ccc}
\mathcal{M}^2(X(\alpha); k) \times_{ev^\tau} \mathcal{M}^2_{(1,1)}(L(u); F_2(0)) & \xrightarrow{\pi_2} & \mathcal{M}^2_{(1,1)}(L(u); F_2(0)) \\
\downarrow ev_0 & & \downarrow ev_0 \\
L(u) & & L(u)
\end{array}$$

commutes.

**Lemma 5.3.** For each $k$, $u$ and a constant $C > 0$ there exists $S_0(k, u, C)$ such that if $S_0 > S_0(k, u, C)$ and $C^{-1} \leq e^{2S_0}\alpha \leq C$ then the virtual fundamental cycle of the evaluation map

$$ev_0 : \mathcal{M}^2(X(e^{2S_0}\alpha); k) \times ev^\tau \mathcal{M}^2_{(1,1)}(L(u); F_2(0)) \to L(u)$$

defines the same homology class in $L(u)$ as that of

$$ev_0 : \mathcal{M}_1(F_2(\alpha), L(u); J_{S_0}; \beta_1 + k [D_1]) \to L(u).$$
Using the fact that \( \gamma \) is the Reeb orbit of smallest period, the lemma follows from a standard gluing result. (See [LR] [FOOO6].)

**Remark 5.1.** We can prove the equality

\[
\dim \mathcal{M}^\#(X(\alpha), k) = \dim \mathcal{M}^\#_{(1,1)}(L(u); F_2(0)) = 2
\]
as follows. (Here \( \dim \) is the virtual dimension that is the dimension as the space with Kuranishi structure.)

Since \( c_1 = 0 \) in \( X(\alpha) \), it follows that \( \dim \mathcal{M}^\#(X(\alpha), k) \) is independent of \( k \). In case \( k = 0 \), our moduli space \( \mathcal{M}^\#(X(\alpha), 0) \) consists of the fibers of the \( \mathbb{C} \) vector bundle \( X(\alpha) \to \mathbb{C}P^1 \). We also find that the linearization operator is surjective. Therefore \( \dim \mathcal{M}^\#(X(\alpha), k) = 2 \).

On the other hand we have \( \dim \mathcal{M}^\#_1(F_2(\alpha), L(u); \beta_1) = 2 \) (see [CO] Theorem 5.1). Therefore we obtain \( \dim \mathcal{M}^\#_{(1,1)}(L(u); F_2(0)) = 2 \) from the (analytic) index sum formula which is a part of the (standard) gluing result, Lemma 5.1.

**Lemma 5.4.** For each \( k, u, C \), there exists \( S_0(k, u, C) \) such that if \( S_0 > S_0(k, u, C) \),

\[
C^{-1} \leq e^{2S_0} \alpha \leq C, \text{ and } ke^{2S_0} \alpha \leq \delta_0(u)
\]

then the mapping degree of the map

\[
ev_0: \mathcal{M}_1(F_2(\alpha), L(u); J_{S_0}; \beta_1 + k[D_1]) \to L(u)
\]
is equal to \( c_k \). Here \( \delta_0(u) \) is a sufficiently small positive constant depending only on \( u \), and the integer \( c_k \) is as in Theorem 5.4 for \( k \neq 0 \) and \( c_0 = 1 \).

**Proof.** Let \( J \) be the complex structure of \( F_2(\alpha) \) as a toric manifold. It follows from [CO] section 7 and Theorem 5.2 that the mapping degree of the map

\[
ev_0: \mathcal{M}_1(F_2(\alpha), L(u); J; \beta_1 + k[D_1]) \to L(u)
\]
is \( c_k \). By our choice of \( J_{S_0} \), the difference between \( J \) and \( J_{S_0} \) in \( C^k \) norm converges to zero as \( S_0 \to \infty \). (Here we equip the neck region \( \Xi \equiv [-S_0, S_0] \times S^3/\{\pm 1\} \) with a cylindrical metric and the other part with a metric independent of \( S_0 \) but depending only on \( e^{2S_0} \alpha \).) We join \( J \) with \( J_{S_0} \) by a short path \( \{J_{S_0,t}\}_{0 \leq t \leq 1} \).

It suffices to show (see the proof of Theorem 5.2) that if \( S_0 \) is sufficiently large then all the \( J_{S_0,t} \)-holomorphic discs bounding \( L(u) \) that have energy \( \leq \beta_1 \cap [\omega] + \delta_0(u) \) have positive Maslov index. We prove this by contradiction.

Suppose to the contrary that we have \( \alpha_i, S_i \) and \( v_i : (\Sigma_i, \partial \Sigma_i) \to (X(\alpha_i), L(u)) \) such that \( S_i \to \infty \), \( C^{-1} \leq e^{2S_0} \alpha_i \leq C \), \( v_i \) is \( J_{S_i,t} \)-holomorphic with energy \( \leq \beta_1 \cap [\omega] + \delta_0(u) \), \( v_i \) is non-constant and the Maslov index of \( [v_i] \) is non-positive. By choosing a subsequence, we may assume \( \lim_{t \to \infty} e^{2S_0} \alpha_i = \alpha' \) converges.

We can use a compactness result such as the one in [BEHWZ] (see also [FOOO6]), and can take a subsequence with the following properties: Consider \( [-S_i, S_i] \times S^3/(\{\pm 1\}) \subset (F_2(\alpha_i), J_{S_i,t}) \) and cut \( \Sigma_i \) along the dividing curve \( v_i^{-1}(c_i \times S^3/(\{\pm 1\})) \) for a regular value \( c_i \in (0, 1) \) of \( s \circ v_i : \Sigma_i \to \mathbb{R} \). \( s : [-S_i, S_i] \times S^3/(\{\pm 1\}) \to \mathbb{R} \) is the projection to the first factor.) Let \( \Sigma^0_i \) be the part which is mapped to \( F_2(\alpha_i) \setminus \{(-\infty, c_i - S_i) \times S^3/(\{\pm 1\}) \cup C_-\} \). Then \( v_i|_{\Sigma^0_i} \) converges in appropriate compact \( C^\infty \) topology to a map \( v_\infty : \Sigma_\infty \to F_2(0) \) where \( \Sigma_\infty \) conformally a disc with punctures.

**Remark 5.2.** We note that \( \{c_i\} \times S^3/(\{\pm 1\}) \subset [-S_i, S_i] \times S^3/(\{\pm 1\}) \subset F_2(\alpha_i) \) is identified with \( \{e^{-S_i}\} \times S^3/(\{\pm 1\}) \subset F_2(0)_0 \) in the conical coordinate \( r = e^s \). In fact we identified \( \{e^{-S_i}\} \times S^3/(\{\pm 1\}) \subset F_2(0)_0 \) with \( \{S_i\} \times S^3/(\{\pm 1\}) \subset X(\alpha_i) \), which corresponds to \( c_i = 0 \), i.e., to \( \{0\} \times S^3/(\{\pm 1\}) \subset F_2(\alpha_i) \).
We remark that image of the limit map $v_\infty$ can not be entirely contained in the compact subset of $F_2(0)_0$. This is because $F_0(0)_0$ does not contain $J$-holomorphic disc with boundary on $L(u)$ and of Maslov index non-positive.

Then by choosing $\delta_0(u)$ sufficiently small, this implies that $\Sigma_\infty$ consists of one component such that $\int_{\Sigma_\infty} v_\infty^* \omega = \beta \cap [\omega] = 1 - u_2$. Namely it defines an element of $\mathcal{M}_{(1,1)}^d(L(u); F_2(0))$.

On the other hand, the first Chern classes of $\mathbb{R} \times S^3/\{\pm 1\}$ and of $X(\alpha')$ are trivial. Therefore the Maslov index of $v_i$ is 2 for sufficiently large $i$, a contradiction to the hypothesis that the Maslov index is non-positive. \hfill \Box

**Lemma 5.5.** The mapping degree of (27) is 1.

**Proof.** We consider $\mathcal{M}^d(X(e^{2S_0}\alpha); 0)$. It is easy to see that they consist of fibers of the $\mathbb{C}$ vector bundle $X(e^{2S_0}\alpha) \to \mathbb{C}P^1$. Therefore $ev^d : \mathcal{M}^d(X(e^{2S_0}\alpha); 0) \to \mathcal{R}_1(\lambda)$ is a diffeomorphism. It follows from Lemma 5.4 that the degree $d_0$ of $ev_0 : \mathcal{M}_1(F_2(\alpha), L(u); J_{\beta_1}) \to L(u)$ is 1 if we choose $S_0$ sufficiently large. It then follows from Lemma 5.3 that

$$\deg \left[ ev_0 : \mathcal{M}^d(X(e^{2S_0}\alpha); 0) \to \mathcal{R}_1(\lambda) \right] = 1.$$ (28)

This proves Lemma 5.5. \hfill \Box

**Lemma 5.6.** The degree of $ev^d : \mathcal{M}^d(X(\alpha); k) \to \mathcal{R}_1(\lambda)$ is 1 if $k = 0, 1$ and is zero otherwise.

**Proof.** We choose an almost complex structure of $X(\alpha)$ so that it is independent of $\alpha \neq 0$.

We also remark that for given $k, u$ there exist $\alpha, C$ and $S_0$ such that the assumption of Lemma 5.4 is satisfied. In fact we first choose $C$ such that $2kC^{-1} \leq \delta_0(u)$, next $S_0 > S_0(k,u,C)$ and then finally we choose $\alpha$ so that $C^{-1} \leq e^{2S_0}\alpha \leq 2C^{-1} \leq C$. Then $ke^{2S_0}\alpha \leq 2C^{-1}k \leq \delta_0(u)$ as required.

Now Lemmas 5.4 and 5.5 imply the degree of $ev_0 : \mathcal{M}_1(F_2(\alpha), L(u); J_{\beta_1} + kD_1) \to L(u)$ is $c_k$. Lemma 5.6 now follows from Lemma 5.3. \hfill \Box

We next consider a simultaneous resolution of the family $F_2(0; \epsilon)$. Here $F_2(0; \epsilon)$ is obtained from $F_2(0)$ by deforming singularity as in section 3. Note it is independent of $\epsilon$ as a symplectic manifold. Here we consider both symplectic and almost complex structures. The latter depends on $\epsilon$. Existence of a simultaneous resolution implies the existence of the family $F_2(\alpha; \epsilon)$ parameterized by $\alpha$ and $\epsilon$ such that for $(\alpha, \epsilon) \neq (0, 0)$ it is smooth, $F_2(0; \epsilon)$ is as above and $F_2(\alpha; 0) = F_2(\alpha)$. We note that $F_2(0; \epsilon)$ is symplectomorphic to the smoothing $\tilde{F}_2(0)$ of $F_2(0)$ introduced in section 3. We denote by

$$S^2_{\text{van}}$$

a vanishing cycle of $F_2(0; \epsilon)$, which is isotoped to the $(-2)$-curve in $X_0$ and shrinks to the singular point of $F_2(0)$ as $\epsilon \to 0$.

Similarly we construct 2-parameter family of local models $X(\alpha; \epsilon)$. We may assume that $X(\alpha; \epsilon)$ coincides with $X(\alpha; 0) = X(\alpha)$ outside compact set. We use this fact to define $\mathcal{M}^d(X(\alpha; \epsilon); k)$ in the same way as $\mathcal{M}^d(X(\alpha); k)$.

**Lemma 5.7.** The degree of $ev^d : \mathcal{M}^d(X(\alpha; \epsilon); k) \to \mathcal{R}_1(\lambda)$ is independent of $(\alpha, \epsilon) \neq (0, 0)$. 


Proof. This can be proved by a standard cobordism argument using the fact that element of $\mathcal{R}_1(\lambda) \cong S^2$ represents a Reeb orbit with smallest action. \hfill \Box

Corollary 5.8. The degree of $ev^2 : \mathcal{M}^2(X(0;\epsilon); k) \to \mathcal{R}_1(\lambda)$ is 1 if $k = 0, 1$ and is 0 otherwise.

This is immediate from Lemmas 5.7 and 5.8.

Remark 5.3. We remark that for $\epsilon = 0$, $\alpha \neq 0$, the elements of $\mathcal{M}^2(X(\alpha;0);0)$ are the fibers of the $\mathcal{C}$ bundle $X(\alpha;0) \to \mathbb{C}P^1$. For $k \neq 0$ the moduli space $\mathcal{M}^2(X(\alpha;0);k)$ consists of those fibers together with sphere bubbles, which are (multiple cover of) the zero section $\mathbb{C}P^1$. So it is nonempty for all $k \geq 0$. A direct counting of them is rather cumbersome. We use Theorem 2.2 and a gluing argument to count them in Lemma 5.7 and Corollary 5.8. If $\epsilon \neq 0$ then there is no pseudo-holomorphic sphere in $X(0;\epsilon)$ since the symplectic form on $X(0;\epsilon)$ is exact. Therefore $\mathcal{M}^2(X(0;\epsilon);k)$, if non-empty, necessarily consists of (proper) holomorphic maps from $\mathbb{C}$ without bubble.

We now need a slight modification of Lemma 5.4 to deal with the moduli spaces associated with $(F_2(0;\epsilon),T(u))$ instead of $(F_2(\alpha),L(u,1-u))$. By gluing $X(0;\epsilon)$ with $F_2(0)_{0}$ in a similar way as before we obtain $F_2(0;\epsilon)$ equipped with an almost complex structures $J_{S_0,\epsilon}$ parameterized by $S_0$.

$F_2(0;\epsilon)$ contains a cylindrical region $\cong [-S_0, S_0] \times S^3/\{\pm 1\}$ on which $J_{S_0,\epsilon}$ is translation invariant. It also contains $X(0;\epsilon) \setminus [S_0,\infty) \times S^3/\{\pm 1\}$ equipped with the symplectic form $e^{-2S_0 \omega_{X(0,\epsilon)}}$.

Recall that when $L(u,1-u) \subset F_2(0)$ is considered as a submanifold in $\hat{F}_2(0) \cong F_2(0,\epsilon)$, we denote it by $T(u)$. We define $\mathcal{M}_1(F_2(0;\epsilon),T(u); J_{S_0,\epsilon}; \beta_1 + k S^2_{\text{van}})$, and $\mathcal{M}^\#(X(0;\epsilon);k)$ in the same way as $\mathcal{M}_1(F_2(\alpha),L(u,1-u); \beta_1 + k \mathbb{C}P^1)$ and $\mathcal{M}^\#(X(\alpha);k)$, respectively.

Lemma 5.9. For each $k$, $u$ and $\epsilon$, there exists a constant $S_0(k,u,\epsilon)$ such that if $S_0 > S_0(k,u,\epsilon)$ the virtual fundamental cycle of the evaluation map

$$ev_0 : \mathcal{M}^2(X(0;\epsilon); k) \cong \mathcal{M}^2_{(1,1)}(L(u,1-u); F_2(0)) \to L(u,1-u)$$

defines the same homology class in $L(u,1-u) \cong T(u)$ as that of

$$ev_0 : \mathcal{M}_1((F_2(0;\epsilon),T(u)); J_{S_0,\epsilon}; \beta_1 + k S^2_{\text{van}}) \to T(u).$$

Proof. We remark that in Lemma 5.3 we need an assumption $C^{-1} \leq e^{2S_0 \alpha}$ since the 'symplectic structure' of $X(0)$ is degenerate. On the other hand, the symplectic structure of $X(0;\epsilon)$ is non-degenerate. Once this fact is understood the proof is the same as Lemma 5.4 and is now standard. \hfill \Box

Using Lemmas 5.5, 5.9 and Corollary 5.8 we finally prove the following:

Proposition 5.10. There exists $S_1(u,\epsilon)$ depending only on $u$ and $\epsilon$ such that if $S_0 > S_1(u,\epsilon)$, then the degree of $ev_0 : \mathcal{M}_1(F_2(0;\epsilon),T(u); J_{S_0,\epsilon}; \beta_1 + k S^2_{\text{van}}) \to T(u)$ is 1 if $k = 0, 1$ and is 0 otherwise.

Proof. We first show the following:

Lemma 5.11. There exists $k(\epsilon)$ such that $\mathcal{M}^2(X(0,\epsilon);k) = \emptyset$ if $|k| > k(\epsilon)$.
Proof. We remark $S^2_{van} \cap \{\omega\} = \emptyset$ in our case. Therefore it follows that the Hofer energy of elements of

$$\bigcup_k \mathcal{M}^2(X(0, \epsilon); k)$$

are independent of $k$ and constant (and so are bounded). The lemma now follows from the Gromov-Hofer compactness.

By Lemma 5.9 the conclusion of Proposition 6.10 follows if $|k| \leq k(\epsilon)$ and

$$S_0 \geq \max_{k, |k| \leq k(\epsilon)} S_0(k, u, \epsilon).$$

Therefore it is enough to prove that there exists a sufficiently large $S_0(u, \epsilon) > 0$ such that

$$\mathcal{M}_1(F(2; \epsilon), T(u); J_{S_0, \epsilon}; \beta_1 + k S^2_{van}) = \emptyset$$

for all $|k| > k(\epsilon)$ if $S_0 > S_0(u, \epsilon)$. We prove this statement by contradiction.

Suppose to the contrary that there exists a sequence $S_i \to \infty$, $k_i$ with $|k_i| > k(\epsilon)$ and $(\Sigma_i, v_i, z_0) \in \mathcal{M}_1(F(2; \epsilon), T(u); J_{S_i, \epsilon}; \beta_1 + k_i S^2_{van})$.

We first remark that $(\beta_1 + k_i S^2_{van}) \cap \{\omega\} = \beta_1 \cap \{\omega\}$ is independent of $k$. Therefore the symplectic energy of $v_i$ is uniformly bounded.

We next consider $[-S_i, S_i] \times (S^3/\{\pm 1\}) \subset F(2; \epsilon)$. We divide the domain of $v_i$ along $s = c_i$ for a regular value $c_i \in [S_i - 1, S_i]$ in the same way as in the proof of Lemma 5.3. We consider the part that goes to $X(0; \epsilon) \setminus (c_i, \infty) \times S^3/\{\pm 1\}$, which we denote by $v_i^0 : \Sigma_i \to X(0; \epsilon)$. We claim that $(\Sigma_i^0, v_i^0)$ converges to an element of $\mathcal{M}^2(X(0, \epsilon); k)$ for some $k$ in compact $C^\infty$ topology, after taking a subsequence.

In fact we can use energy bound to find a subsequence such that the restriction of $v_i$ to $v_i^{-1}([S_i - 1, S_i] \times S^3/\{\pm 1\})$ converges in $C^\infty$ topology. Therefore the action

$$\int_{v_i^{-1}([c_0] \times S^3/\{\pm 1\})} (\Theta \circ v_i)^* \lambda$$

is uniformly bounded. On the other hand, since $\alpha = 0$, the symplectic form $\omega$ is exact on $X(0; \epsilon)$.

Existence of a convergent subsequence of $(\Sigma_i^0, v_i^0)$ is then again a consequence of Gromov-Hofer compactness.

Now by Lemma 5.11 we have $|k_i| \leq k(\epsilon)$ for sufficiently large $i$. This contradicts to our assumption that $|k_i| > k(\epsilon)$.

We next prove that only the moduli spaces for $k = 0, 1$ in Proposition 5.10 contribute to the potential function $\Phi_{\Sigma^\infty_{0, \epsilon}}$. Let $\beta_i \in H_2(F(2; \alpha), L(u_1, u_2); \mathbb{Z})$ ($i = 1, \ldots, 4$) be the classes such that $\mathcal{M}^1_{\text{reg}}(F(2; \alpha), L(u_1, u_2); \beta_i) \neq \emptyset$ and $c_1 \beta_i = 2$.

(Here ‘reg’ means the moduli space of pseudo-holomorphic disks without bubble.)

Theorem 5.2 implies that there are exactly four homology classes satisfying this condition. $\beta_1$ is the same class as before. We denote by $\beta_i \in H_2(F(2; \epsilon), T(u); \mathbb{Z})$ the class corresponding to them by the obvious diffeomorphism $(F(2; \epsilon), T(u)) \cong (F(2; \alpha), L(u, 1 - u))$.

Lemma 5.12. For each $E, \epsilon$ and $(u, 1 - u) \in \text{Int} P(0)$ there exists $S_0(E, \epsilon, u)$ such that the following holds for $S_0 > S_0(E, \epsilon, u)$: Suppose $\beta \in H_2(F(2; \epsilon), T(u); \mathbb{Z})$ satisfies

$$\beta \neq \beta_i, (i = 1, 2, 3, 4), \quad \beta \neq \beta_1 + k S^2_{van}, k \in \mathbb{Z},$$

(30a)
and
\[ \beta \cap \omega \leq E, \quad \mu(\beta) = 2, \quad (30b) \]
where \( \mu \) is the Maslov index. Then we have
\[ \mathcal{M}_1((F_2(0; \varepsilon), T(u)); J_{S_{j\varepsilon}}; \beta_j) = \emptyset. \]

Proof. Suppose to the contrary that there exists a sequence \( \beta_j \in \mathcal{H}_2(F_2(0; \varepsilon), T(u); \mathbb{Z}) \) and
\[ (\Sigma_j, v_j, z_j) \in \mathcal{M}_1((F_2(0; \varepsilon), T(u)); J_{S_{j\varepsilon}}; \beta_j) \]
such that \( \beta = \beta_j \) satisfies (30) and \( S_j \to \infty \). By the same reason as in the proof of Proposition 5.10 we have a constant \( C > 0 \) such that
\[ \int_{v_j^{-1}((c_j) \times S^3/\{\pm 1\})} (\Theta \circ v_j)^* \lambda \leq C \]
for some regular value \( c_j \in [S_j - 1, S_j] \) of \( s \circ v_j \). (Here \([0; S_j, S_j] \times S^3/\{\pm 1\} \subset F_2(0; \varepsilon)\).) This implies that
\[ \int_{v_j^{-1}((d_j) \times S^3/\{\pm 1\})} (\Theta \circ v_j)^* \lambda \leq C \]
where \( d_j \in [-S_j, -S_j + 1] \) is a regular value of \( s \circ v_j \).

We then consider the subsets \( \Sigma_j^{\text{int}} \) and \( \Sigma_j^{\text{out}} \) of \( \Sigma_j \) which are mapped to \( X(0; \varepsilon) \setminus (c_j, \infty) \times S^3/\{\pm 1\} \) and \( F_2(0; \varepsilon) \setminus (-\infty, d_j) \times S^3/\{\pm 1\} \) respectively. We put \( \Sigma_j^{\text{neck}} = \Sigma_j^{\text{int}} \cap \Sigma_j^{\text{out}} \) which is mapped into \([d_j, c_j] \times S^3/\{\pm 1\}\).

The action bound (31) implies that \((\Sigma_j, v_j)\) converge to a pseudo-holomorphic curve \((\Sigma_\infty, v_\infty)\) in \( X(0; \varepsilon) \) in compact \( C^\infty \) topology.

The bound (32) implies that \((\Sigma_j^{\text{out}}, v_j, z_j)\) converge to a pseudo-holomorphic curve \((\Sigma_\infty^{\text{out}}, v_\infty, z_\infty)\) in \( F_2(0; 0) \) in compact \( C^\infty \) topology.

Moreover \((\Sigma_\infty^{\text{neck}}, v_\infty)\) converges to a union of pseudo-holomorphic maps \((\Sigma_\infty^{\text{neck}, c}, v_{\infty, c})\) in \( \mathbb{R} \times S^3/\{\pm 1\}, \) such that \( (\Sigma_\infty^{\text{int}, c}, v_{\infty, c}), (\Sigma_\infty^{\text{out}, c}, v_{\infty, c}, z_{\infty}), \) and \( (\Sigma_\infty^{\text{neck}, c}, v_{\infty, c}, c = 1, \ldots, K) \) comprise the limit of \((\Sigma_j, v_j, z_j)\) in an appropriate stable map compactification.

Remark 5.4. Here we do not discuss the full details of the proof of the above stable map convergence result which, for example, follows from the argument used in [HL, BEHWZ]. We only need this stable map convergence to calculate the index of the linearized operator of Cauchy-Riemann equation of \((\Sigma_j, v_j, z_j)\) by summing over the indices of the components of the limit and to analyze the indices of each component.

Now we provide details of dimension counting arguments of various moduli spaces we have considered.

The virtual dimension of the moduli space containing \((\Sigma_\infty^{\text{int}, c}, v_{\infty, c})\) is 2 or larger and is 2 only if this moduli space is \( M_1(X(0; \varepsilon), k) \). The virtual dimension of the moduli space containing \((\Sigma_\infty^{\text{out}, c}, v_{\infty, c})\) is 2 or larger and is 2 only if it is \( M^2_{1, (1, 1)}(T(u); F_2(0)) \).

Moreover for the case of \( \Sigma_\infty^{\text{neck}, c} \cong \mathbb{R} \times S^1 \), the virtual dimension of the moduli space containing \((\Sigma_\infty^{\text{neck}, c}, v_{\infty, c})\) is greater than or equal to 2, and is 2 only if the component \((\Sigma_\infty^{\text{neck}, c}, v_{\infty, c})\) is a trivial cylinder.

The assumption \( \mu(\beta_j) = 2 \) is equivalent to \( \dim M_1(F_2(0; \varepsilon), T(u)); J_{S_{j\varepsilon}}; \beta_j = 2 \) which is possible only when \((\Sigma_\infty^{\text{neck}, c}, v_{\infty, c})\) is a trivial cylinder. For otherwise the
translation along the \([-S_j, S_j]\)-direction of \([-S_j, S_j] \times S^3/\{\pm 1\} \subset (F_2(0; \epsilon), J_{S_j, \epsilon})\) would provide an additional parameter.

Therefore, \(\mathcal{M}_1(F_2(0; \epsilon), L(u, 1 - u)); J_{S_j, \epsilon}; \beta_j\) for a sufficiently large \(j\) has the same dimension as that of the fiber product over \(S^2\) of the moduli spaces containing \((\Sigma^\text{int}_\infty, v_\infty)\) and \((\Sigma^\text{out}_\infty, v_\infty)\). This implies that both moduli spaces have dimension 2.

Hence \(\beta_j = \beta_1 + k_j S^2\text{van}\) for some \(k_j\) if \(j\) is sufficiently large. This is a contradiction. \(\Box\)

Theorem 3.2 now follows from Proposition 5.10 and Lemma 5.12. (See Theorem 7.2 and (10) for the definition of \(\mathfrak{D}\)). \(\square\)

6. PROOF OF THEOREM 3.3

In this section, we prove Theorem 3.3. Note that the trivialization of the family \(\bigcup_{\alpha \in \mathbb{C}} X_\alpha\) as smooth manifold identifies the cohomology class \([D_1]\) in \(X_0\) and \([S^2\text{van}]\) in \(X_\alpha\). Using the proof of Theorem 3.2 in the last section, each of the degree 1 in Lemma 5.8 contributes 1 to the coefficient 2 in front of the last term of (6). The homology classes \(\beta_1\) and \(\beta_1 + [S^2\text{van}]\) satisfy the relations

\[
\beta_1 \cap [S^2\text{van}] = 1, \quad (\beta_1 + [S^2\text{van}]) \cap [S^2\text{van}] = -1. \tag{33}
\]

Now we consider the cohomology class \(b = T^\rho PD[S^2\text{van}] \in H^2(\mathcal{F}_2(0), \Lambda_+)\).

Then (33) and Theorem 3.2 imply that the potential function with bulk, \(\mathfrak{D}^b\), becomes

\[
\mathfrak{D}^b = T^{u_1} y_1 + T^{u_2} y_2 + T^{2 - u_1 - 2 u_2} y_1^{-1} y_2^{-2} + (e^{T^\rho} + e^{-T^\rho}) T^{1 - u_2} y_2^{-1}. \tag{34}
\]

(See FOOO4 Theorem 3.4.) Now let \(u_1, u_2\) be as in Theorem 3.3. We put

\[
2 \rho = u_2 - u_1 = u_2 - (1 - u_2) = 2 u_2 - 1 \tag{35}
\]

and consider (33) at \(u = (u_1, u_2)\) for some \(\rho\). It becomes

\[
T^{u_1}(y_1 + y_1^{-1} y_2^{-2} + 2 y_2^{-1}) + T^{u_1 + 2 \rho}(y_2 + y_2^{-1}) + (\text{higher order}) y_2^{-1}. \tag{36}
\]

The condition for \((y_1, y_2)\) being its critical point is

\[
0 = 1 - y_1^{-2} y_2^{-2} \tag{37}
\]

\[
0 = -2 y_1^{-1} y_2^{-3} - 2 y_2^{-2} + T^{2 \rho}(1 - y_2^{-2}) + (\text{higher order}) y_2^{-2}. \tag{38}
\]

We consider the solution \(y_1 y_2 = -1\) of (37). Then (38) has a solution \(y_2 = \pm 1 + (\text{higher order})\). The proof of Theorem 3.3 is complete. \(\Box\)

Remark 6.1. The result of this paper, combining with the study of spectral invariants, also implies that among the 4 quasi-morphisms

\[
\text{Ham}(S^2(1) \times S^2(1)) \to \mathbb{R}
\]

obtained by EP using quantum-cohomology (without bulk deformations), two of them are different from the other two. This fact and the following theorem will be proven in a forthcoming paper FOOO7. (We use spectral invariants with bulk deformation to prove Theorem 6.1.)
Theorem 6.1. Let $\widetilde{\text{Ham}}(S^2(1) \times S^2(1))$ be the universal cover of the group of Hamiltonian diffeomorphisms of $S^2(1) \times S^2(1)$. Then there exists an infinitely many Calabi quasi-morphisms $\widetilde{\text{Ham}}(S^2(1) \times S^2(1)) \to \mathbb{R}$ such that any finite subset thereof is linearly independent.

Remark 6.2. The statement that Fukaya category of a toric manifold is split-generated by (strongly) balanced torus fibers, which we will prove in a future article with M. Abouzaid, implies that $\varphi(T(u)) \cap (S^1_{\text{eq}} \times S^1_{\text{eq}}) \neq \emptyset$ for any Hamiltonian diffeomorphism. This intersection statement also follows from the theory of spectral invariant with bulk deformation. We will discuss these points in a future article [FOOO7].

7. Appendix: $m_0(1)$ in the canonical model.

In section 5.4 of [FOOO1], we gave a construction of the canonical model for the filtered $A_{\infty}$-algebra (see [FOOO2] for a concise exposition). In [FOOO1], we first constructed a filtered $A_{\infty}$-algebra structure $\{m_k\}$ on a certain subcomplex $C(L; \Lambda_0)$ of the smooth singular chain complex $S(L; \Lambda_0)$ with $\Lambda_0$-coefficients. Then using the argument of summation over certain rooted decorated trees, we obtained its canonical model, i.e., a filtered $A_{\infty}$-algebra structure $\{m_k^{\text{can}}\}$ on the classical cohomology $H(L; \Lambda_0)$ with $\Lambda_0$-coefficients.

Here we only give the definition of the set $G_k^+$ of rooted decorated trees used in the construction of $m_k^{\text{can}}$. (See (5.4.31) in [FOOO1] or section 3 of [FOOO2] for details.) The set $G_k^+$ consists of quintets $(T, i, v_0, V_{\text{tad}}, \eta)$:

1. $T$ is a tree with an embedding $i$ to the unit disc.
2. The set of vertices of valency 1 is the disjoint union of the set $C^0_{\text{ext}}(T)$ of exterior vertices and $V_{\text{tad}}$, tad poles. We set $k = \#C^0_{\text{ext}}(T)$.
3. A root vertex $v_0$ is an element of $C^0_{\text{ext}}(T)$.
4. Denote by $C^0_{\text{int}}(T)$ the union of the set of vertices of valency at least 2 and $V_{\text{tad}}$. The set $C^0_{\text{int}}$ is equipped with $\eta$, which assign a class in $H_2(M, L; \mathbb{Z})$ to each element of $C^0_{\text{int}}$.
5. For each vertex $v \in C^0_{\text{int}}(T)$ with $\eta(v) \neq 0$, the valency at $v$ is at least 3.

In this appendix, we summarize basic properties of the potential function for the canonical model under Assumption 7.1 below. Namely, we prove the following Theorems 7.1 and 7.2. Let $M$ be a symplectic manifold and $L$ be a relatively spin Lagrangian submanifold.

Assumption 7.1. If $\beta \in \pi_2(M, L)$ is nonzero and $\mathcal{M}_{k+1}(\beta) \neq \emptyset$, then $\mu_L(\beta) \geq 2$.

Typical examples of Lagrangian submanifold satisfying this assumption are

(a) Lagrangian torus fibers of Fano toric manifolds [CO], [FOOO3],
(b) monotone Lagrangian submanifolds with minimal Maslov number 2,
(c) $\dim L = 2$ and the almost complex structure is generic.

Let $L$ be a relatively spin Lagrangian submanifold such that it satisfies Assumption 7.1 and $\beta \in \pi_2(M, L)$ with $\mu_L(\beta) = 2$. In this case it is easy to see that the moduli space $\mathcal{M}_1(\beta)$ has no boundary (in the sense of Kuranishi structure) and $\dim \mathcal{M}_1(\beta) = n$. So the virtual fundamental cycle

$$ev_{0*}(\left[\mathcal{M}_1(\beta)\right]) \in H_n(L; \mathbb{Z}) \cong \mathbb{Z}$$

(39)
is well-defined. (It is an integral class since \( \mathcal{M}_1(\beta) \) has no sphere components and so has trivial automorphism group.)

**Theorem 7.1.** Let \((M, L)\) satisfy Assumption 7.1 and consider the canonical model \((H(L; \Lambda_{0,\text{nov}}), \mathbf{m}^{\text{can}})\) of the filtered \(A_\infty\) algebra in Theorem A of [FOOO1]. Then we have the deformed

\[
\mathbf{m}^{\text{can}}_0(1) = \sum_{\mu_L(\beta)=2} e T^{\omega(\beta)} ev_{0*}([\mathcal{M}_1(\beta)]).
\]

Here right hand side is an element of \(H_n(L; \Lambda_{0,\text{nov}}) \cong H^0(L; \Lambda_{0,\text{nov}})\).

**Remark 7.2.** We remark that Assumption 7.1 in general depends on the almost complex structure \(J\) of \(M\). If we fix \(J\) satisfying Assumption 7.1, then \(\mathbf{m}^{\text{can},J}_0(1)\) is well-defined and is independent of the other choices. If we have \(J\) and \(J'\) both of which satisfy Assumption 7.1, then \(\mathbf{m}^{\text{can},J}_0(1) = \mathbf{m}^{\text{can},J'}_0(1)\) if \(J\) can be joined to \(J'\) by a path of compatible almost complex structures satisfying Assumption 7.1.

This follows from Theorem 7.1 and the fact that the homology class of the virtual fundamental cycle (39) is well-defined in the above sense.

**Proof.** We consider the filtered \(A_\infty\) structure

\[
\mathbf{m}_k : B_k C(L; \Lambda_{0,\text{nov}})[1] \to C(L; \Lambda_{0,\text{nov}})[1]
\]
on the countably generated subcomplex \(C(L; \Lambda_{0,\text{nov}})\) of smooth singular chain complex. By definition we have

\[
\mathbf{m}_{k,\beta} = 0
\]

unless \(\beta = \beta_0\) or \(\mu_L(\beta) \geq 2\).

Now starting from the filtered \(A_\infty\)-structure \(\{\mathbf{m}_k\}\), we construct the filtered \(A_\infty\) structure \(\{\mathbf{m}^{\text{can}}_k\}\) on \(H(L; \Lambda_{0,\text{nov}})\) by the method of §5.4.4 in [FOOO1]. Then we have

\[
\mathbf{m}^{\text{can}}_k(1) = \sum_{\Gamma} \mathbf{m}_{\Gamma}(1)
\]

where \(\Gamma\) runs over the set \(G^+_1\).

By definition, each element of \(G^+_1\) has no exterior edge other than the root vertex \(v_0\). Hence all vertices other than \(v_0\) with one edge must be a tad pole. There must be at least one tad pole. If the sum of Maslov indices of the class assigned to the vertices is greater than 2, then \(\mathbf{m}_{\Gamma}(1)\) for such a \(\Gamma\) is zero, since it will be an element of \(H_{n+\mu_L(\beta)-2}\). (Here \(\beta\) is the sum of the all homotopy class assigned to the vertices.)

Therefore if \(\mathbf{m}_{\Gamma}(1)\) is non-zero then there can be only one vertex (that is the tad pole) and no other vertices. Namely we have obtained

\[
\mathbf{m}^{\text{can}}_0(1) = \sum_{\mu_L(\beta)=2} e T^{\omega(\beta)} ev_{0*}([\mathcal{M}_1(\beta)]).
\]

Under Assumption 7.1 any \(b = \sum x_i e_i\), where \(x_i \in \Lambda_0\) of degree 0 and \(\{e_i\}\) is the basis of \(H^1(L; \mathbb{Z})\) chosen before, is a solution of the (weak) Maurer-Cartan equation.

Then the following theorem can be proved by the same strategy of [FOOO3] section 11.
Theorem 7.2. Let $(M, L)$ satisfy Assumption 7.1 we have
\[ m_{0}^{\text{can}, b}(1) = \sum_{\mu_{L}(\beta) = 2} e^{T(\beta)} \exp (b \cap \partial \beta) e_{\omega}(M_{1}(\beta)). \]

Here $b \cap \partial \beta = \sum x_{i} (e_{i} \cap \partial \beta) \in \Lambda_{0}$.

We remark
\[ \mathfrak{PD}^{L}(b) = m_{0}^{\text{can}, b}(1) \cap [L]/e. \quad (40) \]

We note by definition that
\[ ev_{0}(M_{1}(\beta)) \cap [L] = \deg [ev_{0} : M_{1}(\beta) \to L]. \]

Therefore, using the same argument in Remark 7.2, we have the following

Theorem 7.3. Let $\{J_{t}\}_{0 \leq t \leq 1}$ be a family of tame almost complex structures such that Assumption 7.1 holds for all $J_{t}$. Then we have $\mathfrak{PD}^{L, J_{0}} = \mathfrak{PD}^{L, J_{1}}$.

Remark 7.3. In Theorem 7.3 we put a rather strong hypothesis that Assumption 7.1 holds for all $J_{t}$. This assumption is satisfied for the family of almost complex structures used in the first proof of Theorem 7.2, in section 5.

In our two dimensional case, Assumption 7.1 holds for generic $J$ but there may exist a codimension one set of $J$ for which Assumption 7.1 is not satisfied. As is shown in [FOOO1] the potential function $\mathfrak{PD} : M_{\text{weak}}(L) \to \Lambda_{0}$ is well-defined. In our two dimensional case, dimension counting implies that $\sum_{k=0}^{\infty} m_{k}(b \otimes k) \in H^{0}(L; \Lambda_{0})$ for any $b \in H^{1}(L; \Lambda_{0})$. (This is a consequence of Assumption 7.1.) Therefore we have an isomorphism $H^{1}(L; \Lambda_{0}) \to M_{\text{weak}}(L)$. However this isomorphism depends on the choice of almost complex structure. Therefore if we regard $\mathfrak{PD}$ as a function $H^{1}(L; \Lambda_{0}) \to \Lambda_{0}$ it is well-defined up to coordinate change congruent to the identity modulo $\Lambda_{0, \text{nov}}$. We however remark that for the proof of Theorem 7.1 and other applications in this paper, it is enough to calculate $\mathfrak{PD}$ modulo coordinate change.

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