Homogeneous Euler equation: blow-ups, gradient catastrophes and singularity of mappings

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Abstract

The paper is devoted to the analysis of the blow-ups of derivatives, gradient catastrophes and dynamics of mappings of \( \mathbb{R}^n \to \mathbb{R}^n \) associated with the \( n \)-dimensional homogeneous Euler equation. Several characteristic features of the multi-dimensional case (\( n > 1 \)) are described. Existence or nonexistence of blow-ups in different dimensions, boundness of certain linear combinations of blow-up derivatives and the first occurrence of the gradient catastrophe are among of them. It is shown that the potential solutions of the Euler equations exhibit blow-up derivatives in any dimension \( n \). Several concrete examples in two- and three-dimensional cases are analysed. Properties of \( \mathbb{R}^n \to \mathbb{R}^n \) mappings defined by the hodograph equations are studied, including appearance and disappearance of their singularities.

1 Introduction

The homogeneous Euler equation

\[
\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} = 0, \quad i = 1, \ldots, n
\]  

(1.1)

is an important and remarkable representative of the class of multidimensional quasi-linear partial differential equations. It is the basic equation of the hydrodynamics and theory of continuous media, namely the Navier-Stokes equation, in the situation when one can neglect effects of pressure, dissipation, viscosity, dispersion etc (see e.g. [1, 2, 3]). It can be viewed also as the inviscid multidimensional Burgers equation (see e.g. [4, 5]). In spite of such simplification, the equation (1.1) arises in various branches of physics from hydrodynamics to cosmology (see e.g. [1, 2, 3, 4, 6]).

In addition, it has the remarkable property to be solvable by multidimensional version of the classical method of hodograph equations [4, 6, 7, 8]. Namely, any solution of equation (1.1) is obtainable as a solution of the hodograph equations [7, 8]

\[
x_i - u_it - f_i(u) = 0, \quad i = 1, \ldots, n
\]  

(1.2)

where \( f_i(u_1, \ldots, u_n) \) are arbitrary functions associated with the initial data \( u_i(x, 0) \) for equation (1.1).

In virtue of all that the homogeneous Euler equation is an excellent touchstone for the study of various properties of multidimensional quasi-linear equations.

In the present paper we will study singularities associated with the homogeneous Euler equation, namely, blow-ups of the derivatives \( \frac{\partial u_i}{\partial x_k} \), gradient catastrophes (blow-ups at \( t > 0 \)) and the dynamics of singularities of the mappings \( \mathbb{R}^n \to \mathbb{R}^n \) defined by the hodograph equations (1.2). These problems have been already partially addressed in [4, 6, 9, 10] using different techniques. Here the hodograph equations (1.2) will be our principal tool.

It is shown that all above singularities occur on the hypersurface in \( \mathbb{R}^{n+1} \) defined by the equation

\[
t^n + \sum_{k=0}^{n-1} a_k(u)t^k = 0
\]  

(1.3)

where coefficients \( a_k(u) \) depend on the choice of the functions \( f_k(u), k = 1, \ldots, n \). Blow-up (singularity) hypersurface (1.3) has \( m \) branches where \( m \) is the number of real roots of the equation (1.3). Classical property of the roots of

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polynomial equations with real coefficients imply that for odd \( n = 1, 3, 5, \ldots \) there is always at least one branch of the hypersurface (1.3) while for even \( n = 2, 4, \ldots \) the minimal number of possible real branches is zero. This means that for odd dimensions any solution of the equation (1.3) exhibits blow-up of the derivatives. Instead, in the even dimensional case, there are solutions of the Euler equations (corresponding to certain functions \( f_k(u) \)) free of blow-ups for real \( t \). On the other hand, there are subclasses of solutions for which blow-up always happens. It is shown that for the potential solutions (\( u_i = \frac{\partial u}{\partial x_i}, i = 1, \ldots, n \)) of the homogeneous Euler equation in any dimension \( n \), the blow-up hypersurface (1.3) has always \( n \) real branches. Consequently, any potential solution of equation (1.1) at any dimension \( n \) exhibits the blow-up of derivatives.

These multidimensional features of blow-ups are illustrated by explicit examples in the two-dimensional case. It is shown that for the potential flow which correspond to the functions \( f_1 = \frac{\partial u}{\partial x_1} \) and \( f_2 = \frac{\partial u}{\partial x_2} \) there are always two branches of singularity hypersurfaces. Hence, any solution from this subclass exhibits blow-up (for positive or negative \( t \)).

Opposite case can be easily analyzed by rewriting the two-dimensional Euler equations (1.1) in the complex variables \( z = x_1 + i x_2, V = u_1 + i u_2 \), namely, in the form \( V_t + V V_z + \bar{V} \bar{V} = 0 \). Under the reduction \( \bar{V} = \mu(z, \bar{z}, t)V_z \), where \( \mu \) is a certain function, it assumes the form
\[
V_t + (V + \mu(V))V_z = 0, \tag{1.4}
\]
plus the equation for \( \mu \). It is shown that in the case \( \mu \equiv 0 \) (\( V_z = 0 \)), i.e. for the complex Burgers-Hopf equation \( V_t + V V_z = 0 \), considered in [11, 12], the equation (1.3) has no real roots. So, all such analytic solutions are blow-up free. For \( \mu \neq 0 \) blow-up happens for \( |\mu| = 1 \) that exactly coincides with the singularity of the quasi-conformal mapping given by the function \( V(z, \bar{z}) \).

In the paper we consider concrete examples of solutions of the two- and three-dimensional Euler equation (1.1). Blows-ups and gradient catastrophes for them are analysed on emphasis on differences with one-dimensional case. Among the characteristic properties of blow-up and gradient catastrophe (GC) for \( n \)-dimensional Euler equation we note two of them. First, one of the consequences of the equation (1.3) is that GC first happens at the point \((u_1, \ldots, u_n)\) on the blow-up hypersurface (1.3) at the time \( t_c \) and then expands on a whole blow-up hypersurface. Second property is the consequence of the degeneracy of a certain matrix \( M \). Namely, even if all the derivatives \( \frac{\partial u}{\partial x_k} \) blow-up at the hypersurface (1.3), any their linear combinations in certain \((n-r)\)-dimensional subspaces \((r = \text{rank}(M))\) remain finite. This property is manifestly a multi-dimensional feature of the Euler equation.

It is noted that the system (1.2) has an equivalent form, namely,
\[
x_i = x_{0_i} + u_{0_i}(x_0)t, \quad i = 1, \ldots, n, \tag{1.5}
\]
where \( x_i \) and \( x_{0_i} \) are Eulerian and Lagrangian coordinates, respectively, and the functions \( u_{0_i} \) \((i = 1, \ldots, n)\) represent themselves the initial distribution of the components of the velocity \( \mathbf{u} \) (see also [3, 4, 5, 6, 15]). Zeldovich’s theory [26] of the large scale structure of the universe (see also [4]) the system of equations (1.5) describes the motion of the cold, collisionless medium (dust). It has been used also in the other models in physics (see e.g. [3, 6, 9]).

The results obtained in this paper demonstrate that the hodograph equations in the form (1.2) are the simple and effective tool which allow us to perform most of the calculations explicitly up to the values of blow-ups times.

In the paper we also consider the dynamics of the mapping \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by the hodograph equations (1.2). It is shown that these mappings are singular on the hypersurfaces (1.3) with particular degenerations of these mappings around any point on the hypersurface (1.3). Appearance and disappearance of singularities and their alternation are analyzed for some concrete cases. In particular, it is shown the classical stable mappings \( \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) and \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) [13, 14], i.e. folds, cusps and swallow tails remain singular on certain hypersurfaces at any \( t \).

The paper is organized as follows. General properties of blow-ups and gradient catastrophe for \( n \)-dimensional homogeneous Euler equations are studied in Section 2. The mappings \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) associated with the Euler equation are considered in section 3. In Section 4 the blow-ups of the potential solution of the Euler equation are discussed. Section 5 contains the general discussion of the two-dimensional case. Concrete examples of the solutions of the homogeneous Euler equation, their gradient catastrophes and all that are considered in section 6 for 2D cases, in section 7 for dynamics of mappings case and finally in section 8 for 3D cases. In the conclusion (section 9) some possible directions of future investigations are indicated.

## 2 Blow-ups and gradient catastrophes for \( n \)-dimensional Euler equation

Let us start with the classical textbook case (see e.g. [3]) of the one-dimensional Euler equation (Burgers-Hopf-Riemann equation). Hodograph equation is given by \((u \equiv u_1)\)
\[
x = ut + f(u), \tag{2.1}
\]
where \( f(u) \) is the function locally inverse to the initial data \( u_0(x) = u(t = 0, x) \). Differentiating (2.1) w.r.t. \( x \) and \( t \), and assuming that \( t + \frac{\partial f}{\partial u} \neq 0 \), one gets

\[
\begin{align*}
  u_x &= \frac{1}{t + \frac{\partial f}{\partial u}}, & u_t &= -\frac{u}{t + \frac{\partial f}{\partial u}}.
\end{align*}
\] (2.2)

Consequently, any solution of the equation (2.1) is a solution of the Burgers-Hopf (BH) equation.

For a given function \( f(u) \), i.e. given initial data \( u_0(x) \), the derivatives blow up at \( t = t_0 \) defined by the equation

\[
t + \frac{\partial f}{\partial u} = 0.
\] (2.3)

If \( t_0 > 0 \), then the blow-up of \( u_x \) and \( u_t \) is usually called “gradient catastrophe” (see e.g. [3]). Otherwise will refer to such a situation as “blow-up” of derivatives.

We note that in the one-dimensional case any solutions of the Euler equation (1.1) exhibits a blow-up of derivatives \( u_x, u_t \) (at negative or positive \( t_0 \)) while the gradient catastrophe \( (t_0 > 0) \) occurs only for certain initial data. We will see that at \( n \geq 2 \) the situation is quite different.

In the \( n \)-dimensional case, the hodograph equation (1.2) imply that (see also [7, 8])

\[
\begin{align*}
  \sum_{l=1}^n M_{il} \frac{\partial u_l}{\partial x_k} &= \delta_{ik}, & i, k = 1, \ldots, n, \\
  \sum_{l=1}^n M_{il} \frac{\partial u_l}{\partial t} &= -u_i, & i = 1, \ldots, n,
\end{align*}
\] (2.4)

where

\[
M_{il} = \frac{\partial f_i}{\partial u_l} + t \delta_{il}, & i, l = 1, \ldots, n.
\] (2.5)

Assuming that \( \det M \neq 0 \), one gets

\[
\begin{align*}
  \frac{\partial u_i}{\partial x_k} &= (M^{-1})_{ik}, & i, k = 1, \ldots, n, \\
  \frac{\partial u_i}{\partial t} &= -\sum_k (M^{-1})_{ik} u_k, & i = 1, \ldots, n,
\end{align*}
\] (2.6)

and, hence, any solution \( u_i, i = 1, \ldots, n \) of the hodograph equation (1.2) with \( \det M \neq 0 \) obeys the Euler equation (1.1).

In the hodograph equation (1.2), the functions \( f_i(u) \) are local inverse to the initial values \( u_i(x, t = 0) = u_0_i(x) \), and the solutions of the homogeneous Euler equation (1.1) are given implicitly by the formula [7, 8]

\[
u_i(x, t) = u_0_i(x - u t), & i = 1, \ldots, n.
\] (2.7)

This formula implies that the domain \( D_u \subset \mathbb{R}^n \) of variations of the functions \( u_i(x, t) \) coincides with the domain of variation of the initial data \( u_0_i(x) \).

We note that generically the correspondence between the initial data \( u_0_i(x) \) and the functions \( f_i(u) \) is not, obviously, one-to-one. Similar to the one-dimensional case one may have several functions \( f_i(u) \) for the given initial data \( u_0_i(x) \), one for every open set of invertibility of the initial datum. So, for the given initial data \( u_0_i(x) \) \( (i = 1, \ldots, n) \) one may have several associated matrices \( M \) of the form (2.5).

The matrix \( M \) is the central ingredient in this construction. It is easy to show that it obeys the equation

\[
\frac{dM}{dt} = E,
\] (2.8)

where \( \frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k=1}^n u_k \frac{\partial}{\partial x_k} \) and \( E \) is the identity matrix. For the matrix \( M^{-1} \) one has the equation \( (M^{-1} \equiv U) \)

\[
\frac{dU}{dt} + U^2 = 0,
\] (2.9)

that is the equation used in [9].

The matrix \( M \) is also the key object in the analysis of blow-ups and gradient catastrophes (GC) for the equation (1.1). The formulas (2.4) and (2.6) imply that blow-up of derivatives occurs when

\[
\det M = 0.
\] (2.10)
Due to (2.5) this condition is of the form

$$t^n + a_{n-1}(u)t^{n-1} + \cdots + a_0(u) = 0,$$

where the coefficients $a_k(u)$ are certain real-valued functions of $u = u_1, \ldots, u_n$ and $a_0(u) = \det \frac{\partial f}{\partial u}$. We will refer to the hypersurface in $\mathbb{R}^{n+1}_{(t,u)}$ defined by the equations (2.11) as the blow-up hypersurface. Its structure and properties depend on the solution. For the given initial data $u_0$, one may have several equations of the form (2.11) with different coefficients $a_k(u)$, $k = 0, \ldots, n - 1$. It is important that all of them have the same order $n$. In general the blow-up hypersurface has $m$ branches

$$t = \bigcup_{i \in S} t_i(u), \quad \text{with } t_i(u) = \phi_i(u), \quad i \in S,$$

(2.12)
corresponding to $m$ real roots $t_i$ of the polynomial equations (2.11) where $S$ is the subset of indices $i$ for which $t_i$ is real.

Due to the standard properties of the roots of polynomial with real coefficients the maximal number of branches (2.12) is equal to $n$. Minimal number of branches (2.12) is equal to one for odd $n$ and to zero for even $n$. This means that for $n = 1, 3, 5, \ldots$ each solution of the Euler equation exhibits a blow-up while for $n = 2, 4, 6, \ldots$ there are solutions free of blow-ups. GC appears on the branches $f_i(u)$ for which $t_{0i} > 0$. Several examples of solutions at $n = 2, 3$ with different properties will be presented in subsequent sections.

General properties of the branches (2.12), i.e. their coalescence and intersections, are loosely connected with the well-known problem of the stratifications of the space of matrices [14, 15]. In our case it is the family of $n \times n$ matrices $M$ (2.5) with the parameter $t$. The condition that $s$ branches (2.12) coalesce is equivalent to $s - 1$ partial differential equations

$$\phi_i = \phi_k, \quad i, k \in S,$$

(2.13)

for the functions $f_1, \ldots, f_n$. On the other hand two branches $\phi_a(u), \phi_b(u)$ (2.12) for given $f_1, \ldots, f_n$ generally intersect along $(n - 1)$-dimensional hypersurfaces $\phi_a(u) = \phi_b(u)$.

In the particular case $\frac{\partial f_i}{\partial u_k} = 0$, $i \neq k$, i.e. $f_i = f_i(u_i)$, $i = 1, \ldots, n$ the $n$-dimensional equation (1.1) decomposes into $n$ one-dimensional BH equations for the pairs of variables $(x_m, u_m)$, $m = 1, \ldots, n$. In this case the blow-up hypersurface has $n$ branches

$$t + \frac{\partial f_m(u_m)}{\partial u_m} = 0, \quad m = 1, \ldots, n.$$  

(2.14)

The $m$-th branch (2.14) is associated with the BH equation for the variables $x_m, u_m$ and represent itself a cylindrical hypersurface generated by the curve (2.14).

In physical problems the first time of appearance of GC (minimal value of $t$) is usually of most interest (see e.g. [3]). Let the branch for which $t$ assumes the minimal value among the other is given by

$$t_c = \phi(u).$$  

(2.15)

The minimal value of such $t_c$ is defined by the condition

$$\frac{\partial t_c}{\partial u_i} = \frac{\partial \phi(u)}{\partial u_i} = 0, \quad i = 1, \ldots, n,$$

(2.16)

plus a condition on the second derivatives.

For generic initial data the function $\phi(u)$ is a generic one. Consequently, $n$ equation (2.16) has generically a single solution $u$. Thus, generically, the GC for the homogeneous Euler equation (1.1) first happens at the time

$$t_{c_{\min}} = \phi(u).$$

(2.17)

at the point $u_c$ on the hypersurface (2.15). Then it expands on the whole hypersurface (2.15). It is noted that for the first time such property of the GC for multi-dimensional PDEs has been observed in [9, 16, 17].

Let us turn back to blow-ups. Due to the formulae (2.6) all derivatives of $u_i$ blow-up simultaneously at the blow-up hypersurface (2.15) similar to the case $n = 1$ (2.2). In the multi-dimensional case the blow-ups and GC exhibit additional and novel properties. Indeed, the first of the relations (2.6) can be equivalently rewritten as

$$\frac{\partial u_i}{\partial x_k} = \frac{\tilde{M}_{ik}}{\det \tilde{M}}, \quad i, k = 1, \ldots, n,$$

(2.18)

where $\tilde{M}$ is the adjugate matrix. In vicinity of the blow-up hypersurface (2.11) one has

$$\det \tilde{M}(t = t_0 + \epsilon) = \epsilon^n + A_{k-1}(u)\epsilon^{n-1} + \cdots + A_1 \epsilon,$$

(2.19)
where

Using (2.27) and (2.29), one obtains

and

These imply that

Thus, at $\epsilon \to 0$, all $\frac{\partial u_i}{\partial x_k}$ go to infinity. However, since the matrix $M$ is degenerate one, the matrix $\tilde{M}$ is degenerate too. Let the rank of the matrix $\tilde{M}$ be equal to $r$. So, there are $n - r$ real vectors $\tilde{R}^{(\alpha)} = (\tilde{R}_{1}^{(\alpha)}, \ldots, \tilde{R}_{n}^{(\alpha)})$ and $n - r$ vectors

$L = (L_{1}^{\alpha}, \ldots, L_{n}^{\alpha})$, $\alpha = 1, \ldots, n - r$ such that

and

These imply that

and

where $a_\alpha$ and $b_\beta$ are arbitrary constants. Thus, at $n \geq 2$ derivatives $\frac{\partial u_i}{\partial x_k}$ all blow-up at the blow-up hypersurface. However, there are $(n - r)$-dimensional subspaces where all linear superpositions of derivatives (2.24), (2.25) are finite.

If $A_1 = 0$, then, instead of (2.21), one has

and so on. Thus, the first order blow-up sector for the $n$-dimensional Euler equation has a specific fine structure in contrast to the $n = 1$ case of Burgers-Hopf equation. Higher order blow-up sectors and higher order GCs analogous for those in one-dimensional case [18] will be considered elsewhere.

The matrix $M$ serves also to define the type of behavior of the derivatives $\frac{\partial u_i}{\partial x_k}$ near the blow-up hypersurface. Indeed, considering the infinitesimal variations of $x_i, u_i$ for the fixed $t = t_0$ in the formula (1.2) one gets

Due to the degeneracy of the matrix $M(t_0)$ of rank $r$, there are $n - r$ vectors $\tilde{\mathbf{W}}^{(\alpha)}$ and $\mathbf{w}^{(\alpha)}$ such that

and

Using (2.27) and (2.29), one obtains

where

$$\tilde{\varphi}_{kl}^{\beta} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial u_k \partial u_l} \bigg|_{t_0} L_i^{(\beta)}.$$
In the case of nondegeneracy of the matrices $\tilde{\phi}_{ik}^{\beta}$, the formula (2.27) defines generically the variation $\delta u_k$ as the square roots of the variation $\delta u_k$ as the square roots of the variations

$$\delta \xi^\beta = \sum_{i=1}^n L_i^{(\beta)} \delta x_i, \quad \beta = 1, \ldots, n-r. \quad (2.32)$$

The formula (2.27) has another consequence. If one consider the variation of $u_k$ of the form

$$\delta u_k = \sum_{i=1}^{n-r} L_i^{(\alpha)} \delta a_\alpha, \quad (2.33)$$

where $\delta a_\alpha$ are arbitrary infinitesimals, then the formula (2.27) implies that

$$\delta x_i = \sum_{\alpha, \beta=1}^n \phi_i^{\alpha \beta} \delta a_\alpha \delta a_\beta, \quad i = 1, \ldots, r, \quad (2.34)$$

where

$$\phi_i^{\alpha \beta} = \frac{1}{2} \sum_{k,l=1}^n \partial^2 f_i \left. \right|_{t_0} L_k^{(\alpha)} L_l^{(\beta)}. \quad (2.35)$$

The formula (2.33) together with (2.34) also defines the behavior of $\delta u_i$ as a function of $\delta x_i$.

### 3 Euler equation and mappings $\mathbb{R}^n \to \mathbb{R}^n$

There are at least three ways to treat the mappings associated with the homogeneous Euler equations (1.1).

The first one, most standard, is to consider the family of mappings $\mathbb{R}^n_{(u)} \to \mathbb{R}^n_{(\varphi)}$ given by a solution

$$u_i = u_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \quad (3.1)$$

of equation (1.1). The Jacobian $J_{(\varphi, u)} = |\partial u_i/\partial x_k|$ of these mappings is (see (2.4), (2.9))

$$J = \det M^{-1} = \det U. \quad (3.2)$$

One can show (see [9]) that

$$U = U_0(1 + U_0)^{-1}, \quad (3.3)$$

where $U_{0ik} = \partial u_i/\partial x_k|_{t=0}$. Hence ($J_0 = \det U_0$)

$$J_{(\varphi, u)} = \frac{J_0}{\det(1 + U_0 t)}. \quad (3.4)$$

The mapping (3.4) is singular if $J_{(\varphi, u)} = 0$ that defines the hypersurface

$$J_{(\varphi, u)}(\varphi) = 0, \quad (3.5)$$

in the space $\mathbb{R}^n_{(\varphi)}$ with local coordinates $\varphi = x_1, \ldots, x_n$. The formula (3.4) implies that the mapping (1.1) singular at $t = 0$ ($J_0$) remains singular at all $t$, except (possibly) those values of $t_0$ for which $\det(1 + U_0 t) = 0$. Such $t_0$ corresponds to the blow-ups of the derivatives $\partial u_i/\partial x_k$ for non-singular $J_0 \neq 0$ mappings (1.1) [9].

Second way is to consider the mappings $\mathbb{R}^n_{(\varphi_0)} \to \mathbb{R}^n_{(\varphi)}$ described by the version of the hodograph equation discussed in [4, 6], namely, by equations

$$x_i = x_{i0} + v_i(\varphi_0) t, \quad i = 1, \ldots, n, \quad (3.6)$$

where $v_i(\varphi_0)$ is the Lagrangian velocity of a particle starting from $\varphi_0$. The Jacobian $J_{(\varphi_0, \varphi)}$ of such mapping is

$$J_{(\varphi_0, \varphi)}(\varphi) = \det(\delta \varphi_t + U_{0ik} t), \quad (3.7)$$

The mapping (3.6) is singular when

$$\det(\delta \varphi_t + U_{0ik} t) = 0. \quad (3.8)$$

In this paper we will consider families of mappings $\mathbb{R}^n_{(\varphi_0)} \to \mathbb{R}^n_{(\varphi)}$ defined by the hodograph equations (1.2), namely, by

$$x_i = u_i t + f_i(\varphi), \quad i = 1, \ldots, n. \quad (3.9)$$

For given initial data $u_{0i}(\varphi)$ the formula (3.9) defines the mapping of the domain $\mathcal{D}_{\varphi_0}$ to the corresponding domain $\mathcal{D}_{\varphi}$. 
The Jacobian of these mappings is

$$J_{(x, t)} = \det M,$$

where the matrix $M$ is given by (2.5). The family of mappings (3.9) can be viewed as the deformations of the initial mappings

$$x_i = f_i(u), \quad i = 1, \ldots, n.$$  \hfill (3.11)

with the simple deforming part $u, t$.

The mappings (3.9) are singular $(J_{(x, t)} = 0)$ on the hypersurfaces in the space with the coordinates $(u, t)$ given by equation (2.11). Singularities of the mappings (3.9) are obviously (as in many other cases) in one-to-one correspondence with blow-ups for the derivatives $\frac{\partial u}{\partial a}$.

Properties of blow-ups discussed in the previous section have their counterparts for the mappings (3.9). Indeed, considering the infinitesimal variation $\delta u_i$ in vicinity of singular hypersurface, one has

$$\delta x_i = \sum_{k=1}^{n} M_{ik}(0)\delta u_k, \quad i = 1, \ldots, n.$$  \hfill (3.12)

Hence, for the variation of $u_k$ given by

$$\delta u_k = \sum_{\alpha=1}^{n-r} \delta a_{\alpha} R_k^{(\alpha)},$$

where the vectors $R_k^{(\alpha)}$ are defined in (2.28), $\delta a_{\alpha}, \alpha = 1, \ldots, n - r$ are arbitrary infinitesimals, one has

$$\delta x_i = O(1), \quad i = 1, \ldots, n.$$  \hfill (3.14)

Thus, around any point on the singularity hypersurface there is an $(n - r)$-dimensional infinitesimal domain whose image under the mapping (3.9) collapses to zero.

Analogously, due to the existence of $n - r$ vectors $L_i^{(\beta)}$, $\beta = 1, \ldots, n - r$, defined in (2.28), one concludes that

$$\sum_k L_k^{(\beta)} \delta x_i = o(|\delta u|^2).$$  \hfill (3.15)

So, any $n$-dimensional infinitesimal domain around a point on the singularity hypersurface is transformed by the mapping (3.9) into the $r$-dimensional infinitesimal domain in the space $\mathbb{R}^n_r$.

Family of mappings (3.9) can be viewed as that describing dynamics of the initial mapping (3.11) in time. Study of the appearance or disappearance of singularities of mapping or their alternation is, definitely, of interest. Such properties of the mapping (3.11), obviously, are connected with the properties of stable and unstable mappings $\mathbb{R}^n \to \mathbb{R}^n$ studied by Whitney and others [13, 14].

In particular, mappings with stable singularities at $t = 0$ should remain singular at any value of $t$ [13, 14]. To illustrate this point let us consider classical stable mappings at $n = 2$ and $n = 3$, i.e. folds, cusps and swallow tail [13, 14] as the initial ($t = 0$) mapping (3.11). Then for the fold at $n = 2$ the mapping (3.9) is given by

$$x_1 = u_1^2 + u_1 t, \quad x_2 = u_2 + u_2 t,$$

that corresponds to the solution

$$u_1 = \frac{1}{2} \left( -t + \sqrt{t^2 + 4x_1} \right), \quad u_2 = \frac{x_2}{t + 1},$$  \hfill (3.17)

of the Euler equation. The Jacobian of the mapping (3.16) is

$$J = (1 + t)(2u_1 + t).$$  \hfill (3.18)

At $t = 0$ the mapping (3.16) is singular on the line $u_1 = 0$ and for any $t$ it is singular on the line $u_1 = -t/2$. At $t = -1$ the mapping degenerates.

For the cusp at $n = 2$ the mapping (3.9) is

$$x_1 = u_1^3 + u_1 u_2 + u_1 t, \quad x_2 = u_2 + u_2 t,$$

and the corresponding solution of the Euler equation is

$$u_1 = \left( \frac{x_1}{2} + \sqrt{\frac{x_1^2}{4} + \frac{x_2}{11} + \frac{t}{27}} \right)^{1/3} + \left( \frac{x_1}{2} - \sqrt{\frac{x_1^2}{4} + \frac{x_2}{11} + \frac{t}{27}} \right)^{1/3}, \quad u_2 = \frac{x_2}{1 + t}.$$  \hfill (3.20)
The Jacobian for this mapping is

\[ J = (1 + t)(3u_1^2 + u_2 + t). \]  

(3.21)

So, the mapping (3.19) is singular for any \( t \) on the parabolas

\[ 3u_1^2 + u_2 + t = 0. \]  

(3.22)

In the three-dimensional case the corresponding fold and cusp singularities [14] are associated with the mapping

\[
\begin{align*}
x_1 &= u_1^2 + u_1 t, \\
x_2 &= u_2 + u_2 t, \\
x_3 &= u_3 + u_3 t,
\end{align*}
\]

(3.23)

and

\[
\begin{align*}
x_1 &= u_1^3 + u_1 u_2 + u_1 t, \\
x_2 &= u_2 + u_2 t, \\
x_3 &= u_3 + u_3 t,
\end{align*}
\]

(3.24)

Their Jacobian are again given by the formulas (3.18) and (3.21). Hence, the mappings (3.23) and (3.24) are singular for any \( t \) on the hypersurfaces in \( \mathbb{R}^3 \) given by

\[ 2u_1 + t = 0, \]  

(3.25)

and

\[ 3u_1^2 + u_2 + t = 0. \]  

(3.26)

For the swallow tail [14] the mapping (3.9) assumes the form

\[
\begin{align*}
x_1 &= u_4^1 + u_1 u_2^2 + u_3 u_1 + u_1 t, \\
x_2 &= u_2 + u_2 t, \\
x_3 &= u_3 + u_3 t,
\end{align*}
\]

(3.27)

The Jacobian of this mapping is

\[ J = (1 + t)^2(4u_1^3 + 2u_1 u_2 + u_3 + t). \]  

(3.28)

So, the mapping (3.27) is singular for any \( t \) on the hypersurface

\[ 4u_1^2 + 2u_1 u_2 + u_3 + t = 0. \]  

(3.29)

For \( n > 3 \) one has similar situation.

For all these mappings the singularity hypersurface has a single branch defined by the equation

\[ J(u, t) = 0 + t = 0. \]  

(3.30)

Note that \( J = \det M \) where the matrix \( M \) is defined by (2.5). Thus, for above mapping the blow-up hypersurfaces (2.11) are given by

\[ \det M(u, t) = 0 + t = 0, \]  

(3.31)

and corresponding solutions of the homogeneous Euler equation exhibit blow-up at any time \( t \) (for different values of \( u \)).

4 Potential case

Potential flows represent themselves the particular subclass of solutions of \( n \)-dimensional homogeneous Euler equation for which all branches of the blow-up hypersurface are real and consequently any potential solution of (1.1) for any dimension \( n \) exhibits blow-up. Indeed, the existence of a potential \( \phi \) such that \( u_i = \frac{\partial \phi}{\partial x_i}, \ i = 1, \ldots, n \) implies that

\[ \frac{\partial u_i}{\partial x_k} = \frac{\partial u_k}{\partial x_i}, \ i, k = 1, \ldots, n. \]  

(4.1)

Hence, due to the relation (2.6), the matrix \( M^{-1} \) and, consequently, the matrix \( M \) (2.5) are symmetric one. Thus,

\[ \frac{\partial f_i}{\partial u_k} = \frac{\partial f_k}{\partial u_i}, \ i, k = 1, \ldots, n, \]  

(4.2)
\[ f_i = \frac{\partial \widetilde{W}}{\partial u_i}, \quad i = 1, \ldots, n, \] (4.3)

where \( \widetilde{W}(u) \) is some function. Hence, one has

\[ M_{ik} = \frac{\partial^2 \widetilde{W}}{\partial u_i \partial u_k} + t \delta_{ik}, \quad i, k = 1, \ldots, n. \] (4.4)

Thus, in this case the equation (2.10) or (2.11) defining the blow-up hypersurface is the characteristic equation for the symmetric matrix \( \frac{\partial^2 \widetilde{W}}{\partial u_i \partial u_k} \) and values \( t_0 \) in (2.12) coincide (up to a sign) with the eigenvalues of the matrix \( \frac{\partial^2 \widetilde{W}}{\partial u_i \partial u_k} \). Consequently, due to the standard properties of the eigenvalues of a real-valued symmetric matrix all branches (2.12) are real. It is emphasized that this property of the potential flows is valid for any dimension \( n \). Of course, for particular potential flows (particular functions \( \widetilde{W} \)) some branches (2.12) may coalesce. Anyway there is always at least one real branch. This implies that any potential solution of the homogeneous Euler equation (1.1) in any dimension exhibits blow-up (GC occurs only if the critical time is positive).

We note that for potential flows the hodograph equation (1.2) represent themselves the equations for the critical points

\[ \frac{\partial W}{\partial u_i} = 0, \quad i = 1, \ldots, n, \] (4.5)

for the function (see also [19])

\[ W = -\sum_{i=1}^{n} u_i x_i + \frac{t}{2} \sum_{i=1}^{n} u_i^2 + \widetilde{W}(u). \] (4.6)

At the same time the \( \mathbb{R}^n \to \mathbb{R}^n \) mapping (3.9) is of the form

\[ x_i = \frac{\partial W^*}{\partial u_i}, \quad i = 1, \ldots, n, \] (4.7)

where

\[ W^* = \frac{t}{2} \sum_{i=1}^{n} u_i^2 + \widetilde{W}(u). \] (4.8)

So, for the potential solutions of the homogeneous Euler equation (1.1), the associated mappings are the gradient mappings (4.7).

We note also the well known fact that for potential flows the homogeneous Euler equation (1.1) is equivalent, when \( u_i \) is asymptotically zero, to a Hamilton-Jacobi equation

\[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial \varphi}{\partial x_i} \right)^2 = 0, \quad \text{with} \quad u \equiv \nabla \varphi. \] (4.9)

It is noted that most solutions of the homogeneous Euler equation discussed in [4, 6] (see also [3]) correspond to the potential flows.

5 Two-dimensional case

In order to simplify notation we denote \( x_1 = x, \ x_2 = y, \ u_1 = u, \ u_2 = v, \ f_1 = f, \ f_2 = g. \) So, the hodograph equations are

\[ x = ut + f(u, v), \quad y = vt + g(u, v). \] (5.1)

The blow-up hypersurface (2.11) is of the form

\[ t^2 + (f_u + g_v)t + f_u g_v - f_v g_u = 0. \] (5.2)

Two roots of the equation (5.2) are given by

\[ t_\pm(u, v) = \frac{1}{2} \left( - (f_u + g_v) \pm \sqrt{(f_u + g_v)^2 - 4(f_u g_v - g_u f_v)} \right) \]
\[ = \frac{1}{2} \left( - (f_u + g_v) \pm \sqrt{(f_u - g_v)^2 + 4g_u f_v} \right). \] (5.3)

So, the blow-up occurs if

\[ \Delta \equiv (f_u + g_v)^2 - 4J_0 = (f_u - g_v)^2 + 4g_u f_v \geq 0, \] (5.4)
in the domain \( D_{\omega} \) where \( J_0 = f_u g_v - g_u f_v \). In the case \( \Delta = 0 \), i.e. when the functions \( f \) and \( g \) obey the PDE

\[
(f_u - g_v)^2 + 4g_u f_v = 0, \tag{5.5}
\]

there is a single branch and at \( f_u + g_v > 0 \) one has blow-up at \( t = -\frac{1}{2}(f_u + g_v) < 0 \). In the opposite case \( f_u + g_v < 0 \), the corresponding solution have GC at \( t = \min u_v - \frac{1}{2}(f_u + g_v) > 0 \).

If \( \Delta > 0 \) in the domain \( D_{\omega} \) there are three different cases.

1st case : \( f_u + g_v > 0 \), \( J_0 > 0 \). One has GCs on the branches \( t_+ \) and \( t_- \).

2nd case : \( f_u + g_v > 0 \), \( J_0 > 0 \). One has GCs on the branches \( t_+ \) and \( t_- \).

3rd case : \( J_0 < 0 \). One has blow-up on the branch \( t_- < 0 \) and GC at times \( t_+ > 0 \).

If \( J_0 = 0 \) one has blow-ups on the branches \( t_+ = 0 \) and \( t_- = -f_u + g_v \).

It is easy to see that for potential flows (\( u = \varphi_x, v = \varphi_y \)) the condition \( \Delta > 0 \) \( (5.4) \) is always satisfied. Indeed, since in such a case (see \( (4.3) \))

\[
f = \frac{\partial \tilde{W}}{\partial u}, \quad g = \frac{\partial \tilde{W}}{\partial v}, \tag{5.6}
\]

then

\[
\Delta = (\tilde{W}_{uu} - \tilde{W}_{uv})^2 + 4(\tilde{W}_{uv})^2 > 0, \tag{5.7}
\]

for any functions \( \tilde{W} \) (except the trivial case with \( \tilde{W} = A(u^2 + v^2) + Bu + Cu + D \) for which \( \Delta = 0 \)).

The case when \( \Delta \) is negative corresponds to the solution of 2D homogeneous Euler equation free of blow-ups. In order to analyse such and other situations in 2D case, it is convenient to rewrite the 2D Euler equation and the corresponding hodograph equations in complex variables. For this purpose we introduce the notations

\[
z = x + iy, \quad V = u + iv, \quad F = f + ig. \tag{5.8}
\]

In these variables the 2D homogeneous Euler equation assumes the form

\[
V_t + V V_z + \nabla V \nabla = 0, \tag{5.9}
\]

and hodograph equations become

\[
z = V_t + F(V, \nabla). \tag{5.10}
\]

Equation \( (5.10) \) implies that

\[
V_z = \frac{\overline{F}_V + t}{\det M}, \quad V_\varphi = -\frac{F_\varphi}{\det M}, \quad V_t = -\frac{V(\overline{F}_V + t) + \nabla F_\varphi}{\det M}, \tag{5.11}
\]

where

\[
\det M = (F_V + t)(\overline{F}_V + t) - F_\varphi \overline{F}_\varphi. \tag{5.12}
\]

So two roots of \( \det M = 0 \) composing the blow-up surface are given by

\[
t_{\pm}(u, v) = \frac{1}{2} \left( -(F_V + \overline{F}_V) \pm \sqrt{(F_V - \overline{F}_V)^2 + 4F_\varphi \overline{F}_\varphi} \right) = -\Re(F_V) \pm \sqrt{-(\Im(F_V))^2 + |F_\varphi|^2}. \tag{5.13}
\]

Now let us consider two particular subclasses of initial data, i.e. functions \( F \). First class is given by analytic functions \( F \), i.e. \( F_\varphi = 0 \). In such case \( V_\varphi = 0 \) and the Euler equation \( (5.9) \) becomes

\[
V_t + V V_z = 0. \tag{5.14}
\]

This complex Burgers-Hopf equation has been considered earlier in the papers \([10][11][12]\) within the study of potential two-dimensional flows in particular hydrodynamical problems.

The reduction \( (5.14) \) of the 2D Euler equation \( (5.9) \) has one particular property. In the domain where \( F_\varphi = 0 \) (and hence \( V_\varphi = 0 \)) the expression in the square root is always negative \( ((F_V - \overline{F}_V)^2 = -(\Im(F_V))^2 < 0) \). Thus, these solutions of the equation \( (5.14) \) are blow-up free. Note that the singularities for equation \( (5.14) \) discussed in \([11][12]\) correspond to the singularities of \( F(V) \) (poles, etc.) in nonphysical domain.

Another interesting reduction of the equation \( (5.9) \) is given by the mapping \( (z, \overline{\varphi}) \rightarrow (V, \nabla) \), defined by the Beltrami equation \([20]\)

\[
V_z = \mu(z, \overline{\varphi}, t)V_\overline{\varphi}, \tag{5.15}
\]

where \( \mu \) is the so-called complex dilation. In the case \( |\mu| < 1 \) the mapping is quasi-conformal \([20]\).

Under this constraint the equation \( (5.9) \) becomes

\[
V_t + (V + \mu \nabla)V_z = 0. \tag{5.16}
\]
The consistency of the constraint (5.15) with the equation (5.16) requires that \( \mu \) obeys the following equation

\[
\mu_t V_z - \mu((V + \mu V_z)_z + ((V + \mu V_z)_z)_z) = 0.
\]

The first two formulae (5.11) imply that the constraint (5.15) is equivalent to the following one

\[
F_V = -\mu(F_V + t),
\]

or

\[
z_V = -\mu z_V.
\]

Equation defining the blow-up surface \( \det M = 0 \) (see 5.12) in this case assumes the form

\[
F_V F_V ((|\mu|^2 - 1)) = 0.
\]

Thus, solutions of the equation (5.16) exhibit blow-ups only in the case \(|\mu| = 1\). It is exactly the case of singularity of quasi conformal mapping defined by the constraint (5.15) (see e.g. [20]).

At \(|\mu| < 1\) the quasi-conformal mapping (5.15) and solutions of the equation (5.9) are free of singularities.

6 Two dimensional case: examples

In this section we will present some illustrative examples.

6.1 First example

We begin with the simple example corresponding to the initial data

\[
u_0 = \tanh(x + 2y), \quad v_0 = \tanh(x + y).
\]

So, the hodograph equations are

\[
x - ut = - \operatorname{atanh}(u) + 2\operatorname{atanh}(v), \quad y - vt = \operatorname{atanh}(u) - \operatorname{atanh}(v),
\]

where the domain \( D_\mathbb{R} \) is the square \(-1 \leq u \leq 1, -1 \leq v \leq 1\). The corresponding Jacobian is

\[
\begin{pmatrix}
  f_u & g_v \\
  g_u & f_v
\end{pmatrix} = \begin{pmatrix}
  -\frac{1}{1-u^2} & \frac{2}{1-v^2} \\
  \frac{2}{1-v^2} & -\frac{1}{1-u^2}
\end{pmatrix}.
\]

and the blow-up surface is given by the equation

\[
t^2 = \frac{2 - u^2 - v^2}{(1-u^2)(1-v^2)} t - \frac{1}{(1-u^2)(1-v^2)} = 0.
\]

It has two branches \( t = t_+ \cup t_- \)

\[
t_{\pm}(u, v) = \frac{2 - u^2 - v^2 \pm \sqrt{(2-u^2-v^2)^2 + 4(1-u^2)(1-v^2)}}{2(1-u^2)(1-v^2)}.
\]

It is easy to see that for the (+) branch \( t > 0 \) always and for the (−) branch \( t < 0 \). We report in figure 1 the plot of (+) branch. Thus, one has GC on the (+) branch at \( t_c = 1 + \sqrt{2} \) at \( u_c = v_c = 0 \) (and a blow-up on the (−) branch for \( t < 1 - \sqrt{2} \) starting from \( u_c = v_c = 0 \)). Finally from the hodograph solution (6.2) we get the catastrophe position \( x_c = y_c = 0 \).

In the case (6.2) and at the point \( t_c = 1 + \sqrt{2} \), the matrices \( M(0) \) and \( \tilde{M}(0) \) are of the form

\[
M(0) = \begin{pmatrix}
\sqrt{2} & 2 \\
1 & \sqrt{2}
\end{pmatrix}, \quad \tilde{M}(0) = \begin{pmatrix}
\sqrt{2} & -2 \\
-1 & \sqrt{2}
\end{pmatrix}.
\]

Hence, the vectors \( \tilde{R}, \tilde{L}, \tilde{R}, \) and \( \tilde{L} \), defined by (2.22), (2.23), (2.28), and (2.29) are

\[
\tilde{R} = (\sqrt{2}, 1)a, \quad \tilde{L} = (1, \sqrt{2})b, \quad \tilde{R} = (-\sqrt{2}, 1)c, \quad \tilde{L} = (1, -\sqrt{2})d.
\]
where \(a, b, c, d\) are arbitrary real constants. So, at the of GC described above, the following combinations of blow-up derivatives remain finite at time \(t_c = 1 + \sqrt{2}\)

\[
\sqrt{2} \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial y} \sim O(1), \quad i = 1, 2,
\]

\[
\frac{\partial u_1}{\partial x} + \sqrt{2} \frac{\partial u_2}{\partial x} \sim O(1), \quad \frac{\partial u_1}{\partial y} + \sqrt{2} \frac{\partial u_2}{\partial y} \sim O(1).
\]

(6.8)

On the other hand, the matrices \(\frac{\partial^2 f}{\partial u_k \partial u_l} i = 1, 2\) are identically zero in the catastrophe point \((u, v) = (u_c, v_c) = (0, 0)\). So, the second order term in the r.h.s. of the formula (2.27) vanishes and consequently the variations of \(x_i\) are cubic polynomials in \(\delta u_i\). This is a consequence of the fact that the GC point \(t_c = 1 + \sqrt{2}\), being the first blow-up point, is non-generic.

Figure 1: Evolution of the multivalued region with initial data given in (6.1). The black dot identifies the catastrophe time. The folding region is the region where the velocity \(u\) is multivalued: this corresponds to the region enclosed by the line of blow-up.

6.2 1 + \(\epsilon\) dimensional case

As the second example we consider the situation when initial data (and so functions \(f\) and \(g\)) depend on parameter. Namely, let

\[
u(x, 0) = \tanh(x + \epsilon y), \quad v(x, 0) = \tanh(\epsilon x + y), \quad \epsilon \geq 0.
\]

(6.9)

At \(\epsilon = 0\) one has the decomposition into two BH equation with no GC for both fields \(u\) and \(v\). At \(\epsilon = 1\) the problem degenerates to the reduction \(u = v\). For the initial data (6.9) the hodograph equations are

\[
x - ut = \frac{\epsilon \tanh(v) - \tanh(u)}{\epsilon^2 - 1}, \quad y - vt = \frac{\epsilon \tanh(u) - \tanh(v)}{\epsilon^2 - 1}.
\]

(6.10)

and the domain \(D_u\) is the square \(-1 \leq u \leq 1, -1 \leq v \leq 1\). The blow-up surface is defined by the equation

\[
t^2 = \frac{2 - u^2 - v^2}{(1 - u^2)(1 - v^2)(\epsilon^2 - 1)} t - \frac{1}{(1 - u^2)(1 - v^2)(\epsilon^2 - 1)} = 0
\]

(6.11)

Two branches \(t_{+}\) of the surface \(t = t_{+} \cup t_{-}\) defined by (6.11) are given by

\[
t_{\pm} = \frac{2 - u^2 - v^2 \pm \sqrt{(2 - u^2 - v^2)^2 + 4(1 - u^2)(1 - v^2)(\epsilon^2 - 1)}}{2(1 - u^2)(1 - v^2)(\epsilon^2 - 1)}.
\]

(6.12)

It is easy to see that for the \((-)\) branch \(t\) is negative for all values of \(\epsilon\). Instead, for the \((+)\) branch \(t > 0\) at \(\epsilon > 1\) and \(t < 0\) at \(0 < \epsilon < 1\).

The catastrophe time \(t_c\) coincides with the minimum \(t_{min}\) of the \((+)\) branch. Evaluating the \(t_{min}\) for the \((+)\) branch one finds that it corresponds to \(u_c = v_c = 0\) and

\[
t_{min} = t_c = \frac{1}{\epsilon - 1}.
\]

(6.13)

Thus, the solutions of the 2d-homogeneous Euler equation with the initial data (6.9) and “small” coupling of dimensions \((0 < \epsilon < 1)\) do not exhibit GC as in the decoupled case \((\epsilon = 0)\). For \(\epsilon > 1\) the time of the first appearance of GC decreases with increasing of the coupling \(\epsilon\).
Evaluating \( t \) for the \((-\) branch at \( u = v = 0 \) one gets

\[
t_{-\text{max}} = -\frac{1}{\epsilon + 1}.
\] (6.14)

So, for \( 0 < \epsilon < 1 \)

\[
t_{-\text{max}} > t_{\text{min}}. \tag{6.15}
\]

Thus, at \( 0 < \epsilon < 1 \) the solution (6.10) exhibit blow-ups at negative \( t \) with maximum given by \( t_{-\text{max}} \).

### 6.3 Generic 2d case

Let us consider the case which initial data which resemble standard initial data in the one-dimensional case, namely,

\[
u(x,0) = u_0(1 - \tanh(\alpha x + \beta y)), \quad v(x,0) = v_0(1 - \tanh(\gamma x + \delta y)),
\] (6.16)

where \( u_0, v_0, \alpha, \beta, \gamma, \delta \) are constants, \( u_0, v_0 > 0 \) and the domain \( D_u \) is the rectangle \( 0 \leq u \leq 2u_0, 0 \leq v \leq 2v_0 \). The corresponding hodograph equations are

\[
x - ut = \frac{1}{\Delta} \left( \delta \tanh \left( 1 - \frac{u}{u_0} \right) - \beta \tanh \left( 1 - \frac{u}{u_0} \right) \right),
\]
\[
y - vt = \frac{1}{\Delta} \left( -\gamma \tanh \left( 1 - \frac{u}{u_0} \right) + \alpha \tanh \left( 1 - \frac{u}{u_0} \right) \right),
\] (6.17)

where \( \Delta = \alpha \delta - \beta \gamma \). One has

\[
J = \frac{1}{\Delta} \left( \frac{-\delta u_0 v(2v_0 - v) + \alpha v_0 u(2u_0 - u)}{u(2u_0 - u) - v(2v_0 - v)} \right),
\] (6.18)

and

\[
T \equiv \text{tr} J = -\frac{1}{\Delta} \frac{\delta u_0 v(2v_0 - v) + \alpha v_0 u(2u_0 - u)}{u(2u_0 - u) - v(2v_0 - v)},
\]
\[
D \equiv \det J = \frac{1}{\Delta} \frac{u_0 v_0}{u(2u_0 - u) - v(2v_0 - v)}.
\] (6.19)

So the blow-up surface is \( t = t_+ \cup t_- \) with the two branches given by

\[
t_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}.
\] (6.20)

Critical values of \( t \), given by \( t_{\pm u} = t_{\pm v} = 0 \), are reached where \( u = u_0 \) and \( v = v_0 \) corresponding to

\[
t_{\pm} = \frac{\alpha u_0 + \delta v_0 \pm \sqrt{(\alpha u_0 + \delta v_0)^2 - 4\Delta u_0 v_0}}{2\Delta u_0 v_0}.
\] (6.21)

The value \( t_{\pm c} \) is real when

\[
\Delta < \frac{(\alpha u_0 + \delta v_0)^2}{4u_0 v_0}.
\] (6.22)

Let us now consider some particular cases.

In the decoupled case when \( \beta = \gamma = 0 \) then \( \Delta = \alpha \delta \) and

\[
t_{+ c} = \frac{1}{\delta v_0}, \quad t_{- c} = \frac{1}{\alpha u_0}.
\] (6.23)

So one gets, for the \( t_{\text{min}} \) for the GC

\[
t_{-\text{min}} = \frac{1}{\alpha u_0}, \quad \text{for the field } u,
\]
\[
t_{+\text{min}} = \frac{1}{\delta v_0}, \quad \text{for the field } v.
\] (6.24)

Therefore the appearance of GC is resumed in the following table

| \( \alpha \) | \( \delta \) | fields with GC |
|-------------|-------------|----------------|
| > 0         | > 0         | \( u, v \)     |
| > 0         | < 0         | \( u \)        |
| < 0         | > 0         | \( v \)        |
| < 0         | < 0         | none           |
This table reproduces well known situation for GC of BH equation. The same results are obtained in the semi-decomposed case $\beta \neq 0, \gamma = 0$ or $\beta = 0, \gamma \neq 0$.

The quasi-decomposed case corresponds to $\beta = \gamma = \epsilon > 0$ where $\epsilon = o(\alpha) = o(\beta)$. The behavior for small $\epsilon$ of the blow-up time (6.21) depends critically on values of $\alpha u_0 - \delta v_0$: If $\alpha u_0 - \delta v_0 \neq 0$ the correction to the decouple case (6.24) is quadratic in $\epsilon$

$$t_{+c} = \frac{1}{\delta v_0} + \frac{u_0}{\delta^2 v_0 (\alpha u_0 - \delta v_0)} \epsilon^2 + O(\epsilon^3),$$

$$t_{-c} = \frac{1}{\alpha u_0} - \frac{v_0}{\alpha^2 u_0 (\alpha u_0 - \delta v_0)} \epsilon^2 + O(\epsilon^3).$$

(6.26)

Otherwise, if $\alpha u_0 - \delta v_0 = 0$ the corrective term is linear in $\epsilon$

$$t_{+c} = \frac{1}{\delta v_0} + \frac{1}{\alpha \delta u_0 v_0} \epsilon + O(\epsilon^2),$$

$$t_{-c} = \frac{1}{\alpha u_0} - \frac{1}{\alpha \delta u_0 v_0} \epsilon + O(\epsilon^2).$$

(6.27)

6.4 Direct case example and qualitative comparison with numerics

When the local inversion of initial data is computationally complicated, the mathematical characterization of the blow-ups and catastrophes could be done using a direct characteristic-like method depicted in appendix A.

Let us consider the initial data

$$u(x, 0) = e^{-x^2 - y^2}, \quad v(x, 0) = e^{-x^2 - 2y^2}. \tag{6.28}$$

In this case the local inversion of (6.28) is possible but it would require the study of many local inverses: we will use the direct method. The Jacobian of the initial data (6.28) is

$$
\begin{pmatrix}
-2xe^{-x^2 - y^2} & -2ye^{-x^2 - y^2} \\
-2xe^{-x^2 - 2y^2} & -4ye^{-x^2 - 2y^2}
\end{pmatrix}.
$$

(6.29)

The opposite inverse eigenvalues of such matrix give the blow-up region $t = t_+ \cup t_-$ where

$$t_{\pm}(x_0, y_0) = \frac{e^{x_0^2 + 2y_0^2}}{x_0e^{y_0^2} + 2y_0 \pm \sqrt{x_0^2 e^{2y_0^2} + 4y_0^2}}.$$ 

(6.30)

where $x_0$ is a suitable parameter related to characteristics (see appendix A). Using Mathematica one can compute the minimum of the functions (6.30) and the other catastrophe parameters obtaining

$$t_c = 0.7281359, \quad u_c = 0.705897, \quad v_c = 0.572292, \quad x_c = 0.886099, \quad y_c = 0.874767. \tag{6.31}$$

In figure 2 we compare the numerical evolution of the initial data (6.28) with the catastrophe point computed by the theory.

Figure 2: Evolution for initial data given in (6.28) at catastrophe time ($t = 0.728$): the black dot is the catastrophe point.
7 Dynamics of mappings. Examples in two-dimensional case

Let us consider the hodograph equation from the point of view of the mappings. The first example is

\[ x = ut - 2u^3 - 2v - v^3, \quad y = vt - u - 5v^3 - 3u^3. \]  

(7.1)

The Jacobian of the map (7.1) looks like

\[ M = \begin{pmatrix} t - 6u^2 & -3v^2 - 2 \\ -9u^2 - 1 & t - 15v^2 \end{pmatrix} \]  

(7.2)

and its determinant, giving the blow-up region, is

\[ J = \det(M) = t^2 + (-6u^2 - 15v^2) t + 63u^2 v^2 - 18u^2 - 3v^2 - 2. \]  

(7.3)

At \( t = 0 \) the mapping (7.1) is singular on the curve

\[ v = \pm \sqrt{\frac{18u^2 + 2}{63u^2 - 3}}. \]  

(7.4)

In the figure 3, both the branches of the blow-up region (7.3) are plotted. The positive minimum identifies also a catastrophe point.

Figure 3: Plot of the the blow-up region \( t = t(u, v) \) implicitly defined by (7.3).

Now let us consider the family of mappings on the whole plane \((u, v) \rightarrow (x, y)\)

\[ x = tu - \frac{1}{3}u^3 + \frac{2}{3}u^3 v - u + 2v, \]

\[ y = tv + \frac{1}{3}u^3 - \frac{1}{3}v^3 + u - v. \]  

(7.5)

It is singular on the curve given by the equation

\[ t^2 - t \left( 2 + u^2 + v^2 \right) - \left( 1 + u^2 \right) \left( 1 + v^2 \right) = 0 \]  

(7.6)

So, at \( t = 0 \), the mapping (7.5) is free of singularities \( \left( (1 + u^2) \left( 1 + v^2 \right) \right) \neq 0 \). The curve (7.6) defines a surface \( t = t_+ \cup t_- \) where

\[ t_{\pm}(u, v) = \frac{1}{2} \left( 2 + u^2 + v^2 \pm \sqrt{u^4 + v^4 + 6u^2 v^2 + 8u^2 + 8v^2 + 8} \right). \]  

(7.7)

The (+)-branch is the surface in \( \mathbb{R}^3 \) with the coordinates \((t, u, v)\) with the minimum \( t_{\text{min}} = 1 + \sqrt{2} \) at the point \( u_c = v_c = 0 \). The (-)-branch is the surface in \( \mathbb{R}^3 \) with the coordinates \((t, u, v)\) with the maximum \( t_{\text{min}} = 1 - \sqrt{2} \) at the point \( u_c = v_c = 0 \). So, the dynamics of the mapping (7.5) is rather specific: it has a singularity at \( t < 1 - \sqrt{2} \) on the (-)-branch. It has no singularity on the interval \( 1 - \sqrt{2} < t < 1 + \sqrt{2} \) and it becomes again singular for \( t > 1 + \sqrt{2} \) on the (+)-branch.

As the third example we consider the mapping

\[ x = tu + \frac{1}{3}u^3 + \frac{2}{3}u^2 v^2 - 2v, \]

\[ y = tv + \frac{1}{3}v^3 - \frac{1}{3}u^2 v + u. \]  

(7.8)
The Jacobian of this mapping is

\[ J = t^2 + \frac{1}{3} t \left( 4u^2 + 5v^2 \right) + \frac{1}{3} (u^2v^2 + u^4 + 2v^4 + 6). \]  

(7.9)

So, the mapping (7.8) is obviously nonsingular at \( t = 0 \). Two branches of the singular surface \( t = t_+ \cup t_- \) are given by

\[ t_\pm(u, v) = \frac{1}{6} \left( -4u^2 - 5v^2 \pm \sqrt{(4u^2 + 5v^2)^2 - 12(u^2v^2 + u^4 + 2v^4 + 6)} \right). \]  

(7.10)

So, for both branches \( t_\pm < 0 \). Thus, the mapping (7.8) has no singularities for \( t \geq 0 \). Hence, the possible singularities of such mapping for \( t < 0 \) disappear at \( t \geq 0 \).

8 Three-dimensional examples

Let us now consider a simple, yet not trivial, example of a 3D case. We rename \( y \) components as \( y = (u, v, w) \). The initial data

\[ u(x, 0) = \frac{1 - \tanh(y)}{2}, \quad v(x, 0) = \frac{1 - \tanh(z)}{2}, \quad w(x, 0) = \frac{1 - \tanh(x)}{2}, \]  

(8.1)

lead to the hodograph equations

\[ x - ut = \text{atanh}(1 - 2w), \quad y - vt = \text{atanh}(1 - 2v), \quad z - wt = \text{atanh}(1 - 2v). \]  

(8.2)

The associated matrix \( M \) (2.5) is

\[
M = \begin{pmatrix}
-\frac{2}{t - (1 - 2w)^2} & 0 & -\frac{2}{t - (1 - 2w)^2} \\
0 & t & 0 \\
-\frac{2}{t - (1 - 2w)^2} & 0 & t
\end{pmatrix}.
\]  

(8.3)

The real solution of the equation \( \det(M) = 0 \) is

\[ t_1 = \left( \frac{2}{(1 - (1 - 2u)^2)(1 - (1 - 2v)^2)(1 - (1 - 2w)^2)} \right)^{1/3}. \]  

(8.4)

The branch \( t_1(u, v, w) \) admits a positive minimum \( t_c \equiv t_1(1/2, 1/2, 1/2) = 2 \). Finally, the catastrophe values for the initial data (8.1) are

\[ t_c = 2, \quad u_c = v_c = w_c = 1/2, \quad x_c = y_c = z_c = 1. \]  

(8.5)

As our last example we consider solution of the 3-dimensional Euler equation with initial data

\[ u(x, 0) = \tanh(x + \epsilon y), \quad v(x, 0) = \tanh(y + \epsilon z), \quad w(x, 0) = \tanh(z + \epsilon x), \]  

(8.6)

where \( \epsilon \) is a parameter (\( \epsilon \neq 1 \)). The corresponding hodograph equations are

\[ x - ut = \frac{1}{1 + \epsilon^3} \left( \text{atanh}(u) - \epsilon \text{atanh}(v) + \epsilon^2 \text{atanh}(w) \right) \]

\[ y - vt = \frac{1}{1 + \epsilon^3} \left( \text{atanh}(v) - \epsilon \text{atanh}(w) + \epsilon^2 \text{atanh}(u) \right) \]  

(8.7)

\[ z - wt = \frac{1}{1 + \epsilon^3} \left( \text{atanh}(w) - \epsilon \text{atanh}(u) + \epsilon^2 \text{atanh}(v) \right). \]

The domain \( D_3 \) is given by the cube \(-1 \leq u \leq 1, -1 \leq v \leq 1, -1 \leq w \leq 1 \).

The blow-up hypersurface is given by the equation

\[ \left( t + \frac{1}{1 - u^2} \right) \left( t + \frac{1}{1 - v^2} \right) \left( t + \frac{1}{1 - w^2} \right) + \epsilon^3 t^3 = 0. \]  

(8.8)

which manifestly shows the “coupling” of three one-dimensional BH equations at \( \epsilon \neq 0 \). Solutions of each of these three BH equations does not exhibit the GC. In the 3-dimensional case (\( \epsilon \neq 0 \)) situation depends in the value of \( \epsilon \). The symmetry by permutation on \( u, v, w \) of the relation (8.8) implies that the minimum \( t_c \) of \( t \) is reached in \( u_c = v_c = w_c \) and by direct computation one obtains \( u_c = v_c = w_c = 0 \). The corresponding value of \( t_c \) can be easily calculated using equation (8.8). Indeed one has \( (t_c + 1)^3 + \epsilon^3 t_c^3 = 0 \), and hence

\[ t_c = -\frac{1}{1 + \epsilon}. \]  

(8.9)

Thus the solution (8.7) exhibits GC for the parameter \( \epsilon < -1 \).
9 Conclusion

In this paper we have addressed the problem of the blow-ups and catastrophes for homogeneous Euler equation (1.1).

The approach is based on the extension of the hodograph method for Burgers-Hopf equation to many dimensions and it allows to associate a mapping on $\mathbb{R}^n$ to every solution. A complete classification of the singularities appearing on these mappings has been performed in [18] for the 1D case: in the multi-dimensional case, the existence of a hierarchy of singularities (depending on different classes of initial data) is a natural question.

Another problem to be addressed is the regularization of the gradient catastrophes described above. In contrast to the multidimensional analogs of the Rankine- Hugoniot condition, shock waves etc discussed in [21, 22] one can adopt an approach proposed in the one-dimensional case in [23] and develop its multidimensional version.

The $n$-dimensional generalization of the Jordan system proposed in [19] and other $n$-dimensional reductions of the $(n + 1)$-dimensional homogeneous Euler equation considered there can be appropriate regularizing systems. The comparison of such regularizing systems and the Navier-Stokes equation for the same initial data may indicate the regimes for which the homogeneous case approximate the full case.

Applications of the results obtained in the present paper to the concrete problems in physics and comparison with some previously known results (in particular, the numerical ones see [24]) will be discussed elsewhere.

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A On the method of characteristics

Here, for convenience, we briefly recall a widely used approach for the calculation of the gradient catastrophe times, which is based on the equation (1.5) (see e.g. [1, 3, 6, 9, 5]), i.e.

$$x - x_0 = u_0(x_0)t. \quad (A.1)$$

The crossing condition of two characteristics passing through $x_0$ and $x_0 + h_0$

$$x - x_0 = u_0(x_0)t, \quad x - (x_0 + h_0) = u_0(x_0 + h_0)t. \quad (A.2)$$

is given by

$$h_0 = u_0(x_0 + h_0)t. \quad (A.3)$$

For two initially close characteristics (small $|h_0|$) the previous condition is

$$x - x_1 - h_1 = u_0(x_0)t + h_0 \cdot \nabla u_0(x_0) + O(|h_0|^2) \quad (A.4)$$

and the intersection point satisfies

$$0 = h_0 + h_0 \cdot \nabla u_0(x_0)t + O(|h_0|^2). \quad (A.5)$$

In the limit of small $|h_0|$ the intersection point tends to the solution of

$$\sum_{s=1}^{n} h_{0,s} \left( \delta_s + \frac{\partial}{\partial x_s} u_0_s(x_0) t \right) = 0. \quad (A.6)$$

Therefore, for two infinitely close characteristic the blow-up surface is a function of $x_0$ defined by the equation

$$t_{\alpha} = -\frac{1}{\lambda_{\alpha}(x_0)}, \quad (A.7)$$

where $\lambda_{\alpha}(x_0)$ are the eigenvalues of the initial data Jacobian and the vector $h_{0,s}$ defined in (A.2) satisfies (A.6). Consequently, the first moment of the gradient catastrophe is

$$t_c = \min_{\alpha, x_0} \left( -\frac{1}{\lambda_{\alpha}(x_0)} \right) \equiv -\frac{1}{\lambda_{c}(x_{0,c})}. \quad (A.8)$$

The value of $u$ and position of the catastrophe are given by

$$u_c \equiv u_0(x_{0,c}), \quad x_c \equiv x_{0,c} + u_c t_c. \quad (A.9)$$
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