THE ZONOID ALGEBRA, GENERALIZED MIXED VOLUMES,
AND RANDOM DETERMINANTS

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ABSTRACT. We show that every multilinear map between Euclidean spaces induces a unique, continuous, Minkowski multilinear map of the corresponding real cones of zonoids. Applied to the wedge product of the exterior algebra of a Euclidean space, this yields a multiplication of zonoids, defining the structure of a commutative, associative, and partially ordered ring, which we call the zonoid algebra. This framework gives a new perspective on classical objects in convex geometry, and it allows to introduce new functionals on zonoids, in particular generalizing the notion of mixed volume. We also analyze a similar construction based on the complex wedge product, which leads to the new notion of mixed $J$–volume. These ideas connect to the theory of random determinants.

1. INTRODUCTION

This paper is at the interface of convex geometry and probability. It links the three topics zonoids, mixed volumes and random determinants under a general new framework.

In convex geometry, sums of line segments are called zonotopes and their limits with respect to the Hausdorff metric are called zonoids. Zonoids form a special, yet fundamental class of convex bodies [Sch14]. In functional analysis, probability, combinatorics, and algebraic geometry, it is sometimes possible to reformulate questions in terms of convex bodies, typically involving the notion of mixed volume. In this paper we show that there is a richer multiplicative structure as soon as we confine ourselves to zonoids.

Let $V$ be a Euclidean space of dimension $m$ and $\mathcal{Z}(V)$ denote the set of its centrally symmetric zonoids, which in the following we simply refer to as zonoids. On $\mathcal{Z}(V)$ we have the Minkowski addition and the multiplication with nonnegative reals, and the Hausdorff distance makes it a complete metric space. We study more closely the tensor products of zonoids. More specifically, if $V_1, \ldots, V_p$ are Euclidean spaces, we define a Minkowski multilinear map

$$Z(V_1) \times \cdots \times Z(V_p) \rightarrow Z(V_1 \otimes \cdots \otimes V_p),$$

which is monotone with respect to inclusion and continuous (this is the same tensor product that was defined earlier by Aubrun and Lancien [AL16]). Based on the tensor product, we prove in Theorem 4.1 that each multilinear map between Euclidean spaces induces a multilinear map of the corresponding spaces of zonoids. The resulting maps are again continuous and monotone with respect to inclusion.

A main contribution of this paper is to define and study the zonoid algebra. For this, we first pass from the cone of zonoids $\mathcal{Z}(V)$ to a vector space $\hat{\mathcal{Z}}(V)$, which we call virtual zonoids. In simple terms, $\hat{\mathcal{Z}}(V)$ consists of formal differences of elements in $\mathcal{Z}(V)$, but this abstract vector space can be naturally embedded into various classical vector spaces, e.g. in the space of measures, or the space of functions on the sphere, see Sections 2.3 and 2.4. In Section 4 we
introduce the zonoid algebra over $V$. As a vector space this is the direct sum of the vector spaces of virtual zonoids in all exterior powers of $V$:

$$\mathcal{A}(V) := \bigoplus_{k=0}^m \mathcal{Z}(\Lambda^k V).$$

The multiplication in $\mathcal{A}(V)$ is the multilinear map on zonoids induced by the *wedge products* $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$, $(v, w) \mapsto v \wedge w$ for $k + l \leq m$. Thus, if $K \in \mathcal{Z}(\Lambda^k(V))$ and $L \in \mathcal{Z}(\Lambda^l(V))$ are zonoids, we can multiply them and get the zonoid $K \wedge L \in \mathcal{Z}(\Lambda^{k+l}(V))$. In particular, the product $K_1 \wedge \cdots \wedge K_p$ of $d$ zonoids $K_1, \ldots, K_d \in \mathcal{Z}(V)$ is a zonoid in $\Lambda^d V$.

The zonoid algebra $\mathcal{A}(V)$ has a rich structure: we prove in Theorem 4.5 that it is a graded, commutative partially ordered algebra. While the Minkowski multilinear map (1.1) is continuous, its extension to virtual zonoids requires careful topological considerations, which we discuss in Section 3.1. We also introduce a subalgebra $\mathcal{G}(V)$ (Definition 4.10), which we call *Grassmann zonoid algebra*, and which plays a crucial role in the probabilistic intersection theory developed in our follow-up work [BBML21].

The connection to the mixed volume enters via the concept of the length $\ell : \mathcal{A}(V) \to \mathbb{R}$, which is a monotone, continuous linear functional; see Definition 2.10 and Theorem 2.22. In fact, the length of a zonoid is its first intrinsic volume. More generally, we show in Theorem 5.2 that if $K \in \mathcal{Z}(V)$ is a zonoid, then $\frac{1}{m!} \ell(K^{\wedge d})$ equals its $d$-th intrinsic volume. Moreover, we show in Theorem 5.1 that, if $K_1, \ldots, K_m$ are zonoids in $V$, then their mixed volume satisfies

$$\text{MV}(K_1, \ldots, K_m) = \frac{1}{m!} \ell(K_1 \wedge \cdots \wedge K_m).$$

For higher degree zonoids, i.e., zonoids in $\Lambda^k(V)$ with $1 < k < m$, the length of their product therefore naturally defines a generalized mixed volume.

For zonoids in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we can repeat the same construction using the *complex wedge product*, seen as a real multilinear map, $\wedge_\mathbb{C} : \Lambda^k(\mathbb{C}^n) \times \Lambda^l(\mathbb{C}^n) \to \Lambda^{k+l}(\mathbb{C}^n)$. In Section 6 we define what we call the *mixed $J$-volume*

$$\text{MV}^J(K_1, \ldots, K_n) := \frac{1}{n!} \ell(K_1 \wedge_\mathbb{C} \cdots \wedge_\mathbb{C} K_n),$$

where $K_1, \ldots, K_n$ are zonoids in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. When the $K_i$ are real, i.e., contained in $\mathbb{R}^n \subset \mathbb{C}^n$, then we retrieve the classical mixed volume MV. Furthermore, we prove in Proposition 6.18 that the $J$-volume $\text{vol}^J_m(K) := \frac{1}{m!} \ell(K^{\wedge m})$ can be extended to polytopes and that it is a weakly continuous, translation invariant valuation. By contrast, using a recent result by Wannerer [Wan20], we prove in Corollary 6.20 that $\text{vol}^J_m$ cannot be extended to a continuous valuation on all convex bodies in $\mathbb{C}^n$. We also relate our definition to Kazarnovskii’s *pseudovolume* [Kaz04]; see Definition 6.21.

We apply our theory to the study of the expected absolute value $|\det[X_1, \ldots, X_m]|$ of the determinant of random matrices, whose column vectors $X_1, \ldots, X_m \in \mathbb{R}^m$ are random vectors. Vitale [Vit91] gave a formula for this expectation in the case where the $X_i$ are i.i.d. random vectors. His formula is in terms of the volume of a zonoid: to each random vector $X$ in $\mathbb{R}^m$ which is *integrable*, i.e., $\mathbb{E} |X| < \infty$, there corresponds an associated zonoid $K(X)$; see Definition 2.3. Although $X$ is a random vector, $K(X)$ is deterministic. Vitale’s result asserts that $\mathbb{E} |\det(M)| = m! \cdot \text{vol}_m(K(X))$, where $M = [x_1 \ldots x_m]$, with i.i.d. random variables $X_i$ having the same distribution as $X$, and where $\text{vol}_m$ denotes the $m$-dimensional volume in $\mathbb{R}^m$.

It is important to emphasize that Vitale’s result makes no assumption on the independence of the entries of the random vector $X$. But the columns must be independent, and identically
determined. If the columns are independent but not identically distributed, we can generalize Vitale’s result as follows:

(1.3) \[ \mathbb{E}_{x_1, \ldots, x_m \text{ independent}} \left| \det(M) \right| = \ell(K(X_1) \wedge \cdots \wedge K(X_m)), \quad \text{where } M = \det[X_1, \ldots, X_m]. \]

If the columns are not independent, the situation is far more complicated. Explicit formulas are known for special cases; see, e.g., [Gir90], but general formulas like (1.3) so far were not available. We fill this gap by showing in Theorem 5.4 that, if \( M = [M_1, \ldots, M_p] \) is a random \( m \times m \) matrix partitioned into independent blocks \( M_j = [v_{j,1}, \ldots, v_{j,d_j}] \) of size \( m \times d_j \), and if \( Z_j = v_{j,1} \wedge \cdots \wedge v_{j,d_j} \in \Lambda^{d_j} \mathbb{R}^m \) are integrable, then

(1.4) \[ \mathbb{E} |\det(M)| = \ell(K(Z_1) \wedge \cdots \wedge K(Z_p)). \]

This formula links the expected determinants of a matrix with independent blocks to the multiplication in the zonoid algebra \( \mathcal{A}(\mathbb{R}^m) \), which can be studied using methods from convex geometry and commutative algebra [Eis13]. Furthermore, using the complex wedge product, we obtain in Theorem 6.8 a new formula for the expected absolute value of the determinant of a random complex matrix: \( \mathbb{E} |\det(M)| = \ell(K(Z_1) \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K(Z_p)) \), where the \( Z_j \in \Lambda_{\mathbb{C}}^{d_j} \mathbb{C}^n \) are complex random vectors. Notice, that if the \( d_j \) are all equal to one, we obtain the mixed \( J \)-volume from (1.2), and if the \( K(Z_j) \) are all real zonoids, this specializes to (1.4).

We remark that (1.3) has interesting consequences when combined with the Alexandrov–Fenchel inequality. For instance, if \( X, Y \in \mathbb{R}^m \) are independent integrable random vectors, then the expected area of the triangle \( \Delta(X,Y) \), whose vertices are the origin and \( X \) and \( Y \), satisfies

\[ (\mathbb{E} \text{vol}_2(\Delta(X,Y)))^2 \geq \mathbb{E} \text{vol}_2(\Delta(X,X')) \cdot \mathbb{E} \text{vol}_2(\Delta(Y,Y')), \]

where \( X \sim X' \) and \( Y \sim Y' \) are independent integrable random vectors. We show this in (5.6). Another remarkable consequence of our result is that we obtain a simple proof for zonoids of the recent reverse Alexandrov–Fenchel inequality by Böröczky and Hug [BH21].

We conclude by pointing out that this work lays out the foundations for the follow-up article [BBML21]. This will aim at developing a probabilistic intersection theory for compact homogeneous spaces, in which the zonoid algebra takes the role played by the Chow ring or the cohomology ring in the classical case. To each subvariety in the space we associate a zonoid in the exterior algebra of the tangent space at a distinguished point. The codimension of the variety equals the degree of the exterior power in which the zonoid lives. Then, the intersection of subvarieties in random position can be identified with the product of their zonoids. This allows to develop further the probabilistic Schubert calculus initiated in [BL20].

Outline. In Section 2 we recall basic facts about zonoids, emphasizing how to represent them by random vectors following an approach by Vitale. We discuss the notion of the length of zonoids and discuss some of its properties. Moreover, we define the vector space of virtual zonoids and study its topology. In Section 3 we give the definition of the tensor product map for zonoids, and we discuss its continuity. In Section 4 we introduce and discuss the zonoid algebra, and we prove that any multilinear map induces a unique multilinear map of virtual zonoids. In Section 5 we explain how to relate our construction to intrinsic volumes, mixed volumes and random determinants. Finally, in Section 6 we introduce and study the new notion of mixed \( J \)-volume.

Acknowledgements. We would like to thank A. Bernig for pointing out the definition of Kazarnovskii’s pseudovolume and its connection to our work. We also thank A. Khovanskii,
G. Aubrun, C. Lancien and Y. Martinez-Maure for insightful discussions. Finally, we thank the anonymous referee for helpful comments.

2. Zonoids

In this section we collect known facts about convex bodies, for which we refer the reader to Schneider’s extensive monograph [Sch14] for more details. We also review and further extend a probabilistic description of zonoids due to Vitale [Vit91], which is a crucial tool in our paper.

Throughout the section, $V$ denotes $m$-dimensional Euclidean space with inner product $\langle \cdot , \cdot \rangle$. The corresponding norm of $v \in V$ is given by $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$. The norm defines a topology on $V$ making it a topological vector space. We denote by $B(V) \subset V$ the closed unit ball centered at zero in $V$ and by $S(V)$ the unit sphere in $V$. We abbreviate $B^m := B(\mathbb{R}^m)$ and $S^{m-1} := S(\mathbb{R}^m)$ for $V = \mathbb{R}^m$ with the standard inner product. We denote by $[x, y] := \{(1 - t)x + ty \mid t \in [0, 1]\}$ the segment joining $x$ and $y \in V$.

2.1. Convex bodies and support functions. A subset $K \subset V$ is convex, if for all $x, y \in K$ also $[x, y] \subset K$. A nonempty compact convex set $K \subset V$ is called convex body and $\mathcal{K}(V)$ denotes the set of convex bodies in $V$. Given two convex bodies $K_1, K_2 \in \mathcal{K}(V)$, we can form their Minkowski sum,

$$K_1 + K_2 := \{x + y \mid x \in K_1, y \in K_2\},$$

and, for $\lambda \geq 0$ the scalar multiple,

$$\lambda K := \{\lambda x \mid x \in K\}.$$

In particular, the Minkowski addition turns $\mathcal{K}(V)$ into a commutative monoid. Note that $\epsilon B(V)$ is the ball of radius $\epsilon > 0$ centered at zero. Moreover, we denote by $\langle K \rangle$ the linear span of $K$. The dimension of $K$ is defined as the dimension of its affine span (since all convex bodies considered in this paper contain the origin, this will be the the same as the dimension of $\langle K \rangle$). We set $\mathcal{K}^m := \mathcal{K}(\mathbb{R}^m)$ for $V = \mathbb{R}^m$ with the standard inner product.

The Hausdorff distance between $K_1, K_2 \in \mathcal{K}(V)$ is defined by

$$d_H(K_1, K_2) := \inf\{\epsilon > 0 \mid K_1 \subset K_2 + \epsilon B(V) \text{ and } K_2 \subset K_1 + \epsilon B(V)\}.$$

This leads to the norm of a convex body:

$$\|K\| := d_H(K, \{0\}) = \max_{x \in K} \|x\|.$$

We also call $\|K\|$ the radius of $K$. The Hausdorff distance (2.2) makes $\mathcal{K}(V)$ a complete metric space, see [Sch14, Theorem 1.8.5].

A convex body $K \in \mathcal{K}(V)$ can be described by its support function $h_K : V \to \mathbb{R}$ defined by:

$$h_K(v) := \max\{\langle v, x \rangle \mid x \in K\}.$$

This function satisfies $h(v + w) \leq h(v) + h(w)$ and $h(\lambda v) = \lambda h(v)$ for $v, w \in V$ and $\lambda \geq 0$. Functions $h : V \to \mathbb{R}$ satisfying these properties are called sublinear.

All properties of convex bodies can be concisely described in terms of their support functions. We summarize the relevant facts in the next well known proposition. Let $C(S(V))$ denote the space of real valued, continuous functions on the unit sphere $S(V)$ of $V$. We think of $C(S(V))$ as a Banach space with respect to either the 1- or the $\infty$-norm. That is, for $f \in C(S(V))$:

$$\|f\|_1 = \frac{1}{\text{vol}_{m-1}(S(V))} \int_{S(V)} |f(x)| \lambda(dx) \quad \text{and} \quad \|f\|_\infty = \max_{x \in S(V)} |f(x)|,$$
Proposition 2.1. The map $h: (K(V), d_H) \to (C(S(V)), \| \cdot \|_\infty)$, $K \mapsto \bar{h}_K$ is a Minkowski linear isometric embedding. That is, for $K, L \in K(V)$ and $\lambda \geq 0$, 
\[
h_{K+L} = h_K + h_L, \quad h_{\lambda K} = \lambda h_K, \quad d_H(K, L) = \|h_K - h_L\|_\infty.
\]
In particular, $\|K\| = \|\bar{h}_K\|_\infty$. Moreover,

1. The image of $h$ consists of the sublinear functions on $V$ restricted to $S(V)$.
2. The support function $h_K$ allows to reconstruct the body $K$ as follows: 
   \[
   K = \{x \in V \mid \forall v \in V \langle x, v \rangle \leq h_K(v)\}.
   \]
3. $K \subseteq L$ if and only if $h_K(u) \leq h_L(u)$ for all $u \in V$.
4. Let $W$ be a Euclidean space and $M: V \to W$ be linear with adjoint $M^T: W \to V$.
   Then $M(K)$ is a convex body with support function $h_{M(K)}(v) = h_K(M^Tv)$.

Proof. The three statements in display are [Sch14, Theorem 1.7.5] and [Sch14, Lemma 1.8.14].

The characterization (1) of support functions as restrictions of sublinear functions is [Sch14, Theorem 1.7.1]. The statement (2) expresses duality and is in the proof of [Sch14, Theorem 1.7.1]. The third statement follows from the second item. The last point is a direct consequence of the definition of the support function in (2.4). \hfill \Box

We will be mainly interested in centrally symmetric convex bodies $K$, characterized by $(-1)K = K$. They form a closed subset of $K(V)$. The fourth item in Proposition 2.1 implies that $K$ is centrally symmetric if and only if $h_K(-v) = h_K(v)$ for all $v \in V$.

2.2. Zonoids and random convex bodies. A zonotope $K$ in $V$ is defined as the Minkowski sum of a finite number of line segments, i.e., it has the form $K = [x_1, y_1] + \cdots + [x_n, y_n]$ with $x_i, y_i \in V$. In general, zonotopes are exactly the polytopes that have a center of symmetry such that all of its 2 dimensional faces also have a center of symmetry; see [Sch14, Theorem 3.5.2]. The centrally symmetric zonotopes (i.e., with the center of symmetry at the origin) are the ones that can be written in the form 
\[
K = \frac{1}{2}[-x_1, x_1] + \cdots + \frac{1}{2}[-x_n, x_n].
\]

We introduce now the main objects of this paper.

Definition 2.2 (Zonoids). A convex body $K$ is called a zonoid if it is the limit of a sequence of zonotopes with respect to the Hausdorff metric.

Our focus will be on the centrally symmetric zonoids. Those are exactly the limits of centrally symmetric zonotopes. We denote by $\mathcal{Z}(V)$ the space of centrally symmetric zonoids in $V$.\footnote{The standard notation for centrally symmetric zonoids would be $\mathcal{Z}_0(V)$, but we omit the subscript in order to avoid unnecessary notation.} By definition, $\mathcal{Z}(V)$ is a closed subspace of $K(V)$. It is known [Sch14, Cor. 3.5.7] that $\mathcal{Z}(V)$ is a proper subset of the set of centrally symmetric convex bodies iff $\dim V > 2$. We also note that if a polytope is a zonoid, then it must be a zonotope (e.g., see [BLM89]). We abbreviate $\mathcal{Z}^m := \mathcal{Z}(\mathbb{R}^m)$.

To deal with zonoids, we shall extensively use a probabilistic viewpoint going back to Vitale [Vit91], which not only is intuitive, but also allows for a great deal of flexibility.
Let $\Omega$ be a probability space. By an integrable random convex body we understand a Borel measurable map $Y: \Omega \to \mathcal{K}(V)$ such that $\mathbb{E}\|Y\| < \infty$; see (2.3) for the definition of the norm. The last condition implies that $\mathbb{E} h_Y(u)$ is a well defined sublinear function of $u$, hence it is the support function of a uniquely defined convex body (see Proposition 2.1(1)). We can thus define the expectation of $Y$ to be the convex body $\mathbb{E}Y \subset V$ with the following support function:

\begin{equation}
(2.6) \quad h_{\mathbb{E}Y}(v) := \mathbb{E} h_Y(v).
\end{equation}

Suppose now that $X$ is a random vector in $V$ satisfying $\mathbb{E}\|X\| < \infty$. We call such a random vector integrable. Then $Y = \frac{1}{2}[-X, X]$ is an integrable, random, centrally symmetric segment. Vitale [Vit91] proved that every zonoid $K \subset V$ (including noncentered zonoids) can be written as $K = \mathbb{E}[0, X] + c$ for some random integrable vector $X \in V$ and a fixed $c \in V$. We can write this as $K = \mathbb{E}\frac{1}{2}[-X, X] + c + \frac{1}{2}\mathbb{E}X$. If $K \in \mathcal{Z}(V)$ is centrally symmetric, then by uniqueness of support functions we have $h_K(u) = h_K(-u)$ for all $u \in V$, which implies $c + \frac{1}{2}\mathbb{E}X = 0$. Thus, we have shown that

\begin{equation}
(2.7) \quad K(X) := \mathbb{E}\frac{1}{2}[-X, X]
\end{equation}

is a centrally symmetric zonoid, and that every centrally symmetric zonoid arises this way.

**Definition 2.3 (Zonoid associated to random variable).** If $X$ is an integrable random vector in $V$, we call $K(X)$ defined by (2.7) the Vitale zonoid associated to the random vector $X$. We also say that $K(X)$ is represented by $X$.

In terms of support functions, we can describe $K(X)$ as follows.

\begin{equation}
(2.8) \quad h_{K(X)}(v) = \mathbb{E} h_{\frac{1}{2}[-X, X]}(v) = \frac{1}{2} \mathbb{E}|\langle v, X \rangle|.
\end{equation}

(This follows from (2.6) using that $h_{[-z, z]}(v) = |\langle v, z \rangle|$.) The observation $K(X) = K(-X)$ shows that the random variable representing a zonoid is not unique. However, it turns out that the random variable representing a zonoid is unique up to sign, if one assumes $X$ to take its values on the unit sphere $S(V)$. This follows from the measure theoretic interpretation of zonoids, we will discuss this point of view in Section 2.4 below (see also [Sch14, Theorem 3.5.3] and [Bol69, Theorem 5.2]).

We found it of great technical advantage to allow the random vector $X$ to take values outside the unit sphere of $V$; the resulting loss in uniqueness is not significant. The next result is a clear indication of the advantage of this viewpoint. It says that the association of a zonoid to an integrable random vector commutes with linear maps. Proving the following proposition using the measure point of view is involved (see Lemma 2.31 below), while the proof using random vectors is almost trivial. This highlights for the first time one of the advantages when working with random vectors.

**Proposition 2.4.** Let $V, W$ be Euclidean spaces, $M : V \to W$ be a linear map and $X$ be an integrable random vector in $V$. Then $M(K(X)) = K(MX)$.

**Proof.** We have for $v \in V$ by (2.8),

\begin{equation}
(2.9) \quad \mathbb{E}h_{K(MX)}(v) = \frac{1}{2}\mathbb{E}|\langle v, MX \rangle| = \frac{1}{2}\mathbb{E}|\langle M^Tv, X \rangle| = h_K(M^Tv) = h_{M(K)}(v),
\end{equation}

where last equality is due to Proposition 2.1(4). \hfill \Box

Proposition 2.4 also implies that linear images of zonoids are again zonoids. Of course, this also follows directly from the definition.

For instance, we obtain for a standard Gaussian vector $X \in \mathbb{R}^m$,

\begin{equation}
(2.9) \quad K(X) = (2\pi)^{-\frac{1}{2}} B^m,
\end{equation}

where $B^m$ denotes the $m$-dimensional unit ball.
where $B^m$ denotes the Euclidean unit ball in $\mathbb{R}^m$. Indeed, $h_{K(X)}(v) = \frac{1}{2} \mathbb{E}|\langle v, X \rangle| = (2\pi)^{-\frac{1}{2}} \|v\|$, by (2.8), since $\sum_{i=1}^m X_i v_i$ is distributed as $\|v\| Z$, where $Z \in \mathbb{R}$ is standard gaussian and has expected value $\mathbb{E}|Z| = \sqrt{2/\pi}$. The function $h_{K(X)}$ equals the support function of $(2\pi)^{-\frac{1}{2}} B^m$ and hence (2.9) follows.

Remark 2.5. Behind Vitale’s result is the following law of large numbers for zonoids [AV75], which provides a geometric way for viewing Vitale zonoids associated to random vectors. Let $X$ be an integrable vector in $V$ and $\{X_k\}_{k \geq 1}$ a sequence of independent samples of $X$. Then, as $n \to \infty$, we have in almost sure convergence $\frac{1}{n} \sum_{k=1}^n \frac{1}{2} [-X_k, X_k] \to K(X)$.

We next show how to realize the Minkowski sum of two zonoids as the Vitale zonoid associated to random vectors. Let $X$ and $Y$ be integrable random vectors in $V$. Then $K(Z) = K(X) + K(Y)$ for the random variable $Z := 2\epsilon X + 2(1-\epsilon)Y$ defined with a fair Bernoulli random variable $\epsilon \in \{0,1\}$ that is independent of $X$ and $Y$.

Proof. By (2.8), we have

$$h_{K(Z)}(v) = \frac{1}{2} \mathbb{E} h_{[-Z,Z]}(v) = \frac{1}{2} \mathbb{E}_{X,Y} \mathbb{E}_{\epsilon} h_{2\epsilon[-X,X] + 2(1-\epsilon)[-Y,Y]}(v).$$

Expanding the expectation over $\epsilon$,

$$\mathbb{E}_{\epsilon} h_{2\epsilon[-X,X] + (1-\epsilon)[-Y,Y]}(v) = \frac{1}{2} h_{2[-X,X]}(v) + \frac{1}{2} h_{2[-Y,Y]}(v) = h_{[-X,X]}(v) + h_{[-Y,Y]}(v)$$

and taking expectations over $X, Y$ finishes the proof.

Lemma 2.6 has a straightforward generalization for adding $n$ random variables. As an application, if $x_1, \ldots, x_n \in V$ are fixed vectors and $Z \in V$ is the random vector taking the value $nx_i$ with probability $1/n$, we see that $K(Z)$ is the zonotope $\frac{1}{n} [-x_1, x_1] + \cdots + \frac{1}{n} [-x_n, x_n]$.

The next observation states that scaling the random variable $X$ with an independent random function only leads to a rescaling of the resulting zonoid.

Lemma 2.7. We have $K(\rho X) = \mathbb{E}[\rho] \cdot K(X)$ if $X \in V$ and $\rho \in \mathbb{R}$ are independent integrable random variables.

Proof. We have $h_{K(\rho X)}(v) = \frac{1}{2} \mathbb{E}|\langle v, \rho X \rangle| = \frac{1}{2} \mathbb{E}\left(|\rho| \cdot |\langle v, X \rangle|\right) = \mathbb{E}[\rho] \cdot h_{K(X)}(v)$, where the first equality is (2.8) and the last equality is due to the independence of $X$ and $\rho$. The assertion follows since convex bodies are determined by their support function.

As an application, we compute the zonoid defined by $\bar{X}$ uniformly distributed on the unit sphere $S^{m-1}$ in $\mathbb{R}^m$. For this, we introduce the following notation:

$$\tau_m := \sqrt{2\pi} \mathbb{E}\|X\|, \quad \text{where } X \in \mathbb{R}^m \text{ is a standard Gaussian random vector.}$$

This number is $\sqrt{2\pi}$ times the expected value of a chi-distributed random variable with $m$ degrees of freedom\(^2\) It has the explicit value $\tau_m := \sqrt{2\pi} \sqrt{2 \Gamma\left(\frac{m+1}{2}\right) / \Gamma\left(\frac{m}{2}\right)}$. We can write $\bar{X} := X / \|X\|$, where $X \in \mathbb{R}^m$ is standard Gaussian. Then, we have

$$K(\bar{X}) = \frac{1}{\tau_m \sqrt{2\pi}} B^m,$$

where, as before, $B^m$ is the unit ball in $\mathbb{R}^m$. This follows from Lemma 2.7 and (2.9), using that $\bar{X}$ is independent of $\|X\|$ and (e.g., see [BC13, §2.2.3])

In the next example we generalize this from spheres to Grassmannians.

\(^2\)In [BL20] the expected value of a chi-distributed random variable with $m$ degrees of freedom is denoted $\rho_m$. So, using their notation we have $\tau_m = \sqrt{2\pi} \rho_m$. The additional factor of $\sqrt{2\pi}$ makes the formulas in this paper easier to grasp.
Example 2.8. Let $\xi_1, \ldots, \xi_k \in \mathbb{R}^m$ be standard, independent Gaussian vectors and consider

$$X := \xi_1 \wedge \cdots \wedge \xi_k \in \Lambda^k(\mathbb{R}^m).$$

Put $\bar{X} := X/\|X\|$. By [Jan54, Theorem 8.1], the random variables $\bar{X}$ and $\|X\|$ are independent and $\bar{X}$ is uniformly distributed on the Grassmannian $\tilde{G}(k, m)$ of oriented $k$-planes in $\mathbb{R}^m$. Moreover, $\|X\|$ is distributed as the square root of the determinant of a Wishart matrix\(^4\). In particular, using [Mui82, Theorem 3.2.15], we get $E\|X\| = 2^k \Gamma_k \left(\frac{m+1}{2}\right) / \Gamma_k \left(\frac{m}{2}\right)$, where $\Gamma_k(x) := \pi^{\frac{k(k-1)}{4}} \prod_{j=1}^k \Gamma(x + \frac{k-j}{2})$ denotes the multivariate Gamma function; see, e.g., [Mui82, Theorem 2.1.12]. Lemma 2.7 tells now that

$$(2.12) \quad K(X) = 2^k \frac{\Gamma_k \left(\frac{m+1}{2}\right)}{\Gamma_k \left(\frac{m}{2}\right)} K(\bar{X}).$$

Notice that, in the case $k = 1$, we get $E\|X\| = \tau_m / \sqrt{2\pi}$ (see (2.10)) and we recover (2.11) as a special case of (2.12).

We now show that the expectation of the norm of an integrable random vector $X$ depends only on its zonoid $K(X)$. Recall that $h_K$ denotes the restriction of $h_K$ to the unit sphere.

**Proposition 2.9.** Let $X$ be an integrable random vector in $\mathbb{R}^m$ and $K = K(X)$. Then

$$E\|X\| = \sqrt{2\pi} \, Eh_K(v) = \tau_m \|h_K\|_1$$

where $v \in \mathbb{R}^m$ is a standard Gaussian vector, and the 1-norm is as in (2.5).

**Proof.** We assume w.l.o.g. that $v$ is independent of $X$. By rotational invariance and homogeneity, we have $\|x\| = \sqrt{\pi/2} E|\langle x, v \rangle|$ for all $x \in \mathbb{R}^m$. This implies, using the independence of $X$ and $v$, that

$$E\|X\| = \sqrt{\frac{\pi}{2}} \, \mathbb{E}_{X,v} \|\langle X, v \rangle\| = \sqrt{\frac{\pi}{2}} \, \mathbb{E}_{X,v} 2 \cdot h_{1/2[-X,X]}(v) = \sqrt{2\pi} \, \mathbb{E}_{v} h_K(v),$$

which shows the first equality. The second equality follows from the factorization $v = \|v\| \cdot \tilde{v}$, noting as before that $\|v\|$ and $\tilde{v}$, are independent and $\tilde{v}$ is uniformly distributed on the unit sphere. We have $E\|v\| = (2\pi)^{-\frac{1}{2}} \tau_m$ and $\mathbb{E}_{v} h_K(v) = \|h_K\|_1$ by (2.5), because $h_K$ is nonnegative. Putting everything together finishes the proof. \(\square\)

Proposition 2.9 shows that the following notion of length is well defined. The length will be investigated more closely in Section 5.1.

**Definition 2.10.** We define the *length* of a zonoid $K \in Z(V)$ by

$$\ell(K) = E\|X\|,$$

where $X$ is an integrable random vector representing $K$.

**Example 2.11.** We see by Proposition 2.9 that if $B^m$ is the unit ball of $\mathbb{R}^m$ then its length is given by $\ell(B^m) = \tau_m$, where $\tau_m$ is given by (2.10).

We next show that the length is additive with respect to Minkowski addition. This might be surprising at first sight, given that Definition 2.10 defines the length as the expected value of a norm.

\(^3\)Recall that $\tilde{G}(k, m)$ can be identified with the set of unit simple vectors in $\Lambda^k(\mathbb{R}^m)$, see [Koz97].

\(^4\)Recall that, if $M = [\xi_1, \ldots, \xi_k]$ is a $\mathbb{R}^{m \times k}$ matrix whose columns are standard independent gaussian vectors in $\mathbb{R}^m$, the matrix $A := M^T M$ is called a Wishart matrix.
Lemma 2.12. Let $K, L \in Z(V)$ and $\lambda \geq 0$. Then, we have
\[ \ell(K + \lambda L) = \ell(K) + \lambda \ell(L). \]

Proof. Let $K = K(X)$ and $L = K(Y)$ for independent integrable vectors $X,Y \in V$. By Lemma 2.7 we have $\lambda L = K(\lambda Y)$. Using this and Lemma 2.6 we can write the Minkowski sum as $K + \lambda L = K(2\epsilon X + 2\lambda(1-\epsilon)Y)$, where $\epsilon$ is a fair Bernoulli random variable $\epsilon \in \{0,1\}$ that is independent of $X$ and $Y$. This implies $\ell(K + \lambda L) = E\|2\epsilon X + 2\lambda(1-\epsilon)Y\|$. Taking first the expectation over $\epsilon$ yields $\ell(K + \lambda L) = \frac{1}{2}E\|2X\| + \frac{1}{2}E\|2\lambda Y\| = \ell(K) + \lambda \ell(L)$. This finishes the proof. \qed

Combining Proposition 2.9 with Proposition 2.1(3) we get the following corollary.

Corollary 2.13 (Monotonicity of the length). Let $K,L \subset V$ are zonoids such that $K \subset L$, then $\ell(K) \leq \ell(L)$.

We recall from Proposition 2.1 and Proposition 2.9 that the radius $\|K\|$ of a zonoid $K$ can be expressed as the $\infty$-norm of its support function $\bar{h}_K$ and that its length $\ell(K)$ can expressed in terms of its 1–norm. The next result compares these two norms.

Corollary 2.14 (Radius and length). We have
\[ 2\|K\| \leq \ell(K) \leq \tau_m \|K\|, \]
with inequality on the left hand side iff $K$ is a (centrally symmetric) segment, and equality holding on the right hand side iff $K$ is rotational invariant.

Proof. Recall $\|K\| = \|\bar{h}_K\|_\infty$ from Proposition 2.1 and $\ell(K) = \tau_m \|\bar{h}_K\|_1$ from Proposition 2.9. By definition of the norms we have $\|\bar{h}_K\|_1 \leq \|\bar{h}_K\|_\infty$ with equality holding iff $h_K$ is constant. The latter means that $K$ is rotationally invariant. This shows the right inequality.

For the left inequality, we write $K = K(X)$. By (2.8) we have $2h_K(u) = E\langle X, u \rangle \leq E\|X\|$, if $\|u\| = 1$. This implies $2\|\bar{h}_K\|_\infty \leq E\|X\| = \ell(K)$ and equality holds if $K$ is a segment. \qed

The next observation will be useful later.

Lemma 2.15. Let $K,L \in Z(V)$ be zonoids and $\langle K \rangle, \langle L \rangle$ denote their linear spans. If $K,L$ are represented by the integrable random vectors $X,Y$, respectively, then
\[ \langle K \rangle \perp \langle L \rangle \iff \langle X, Y \rangle = 0 \text{ almost surely.} \]

Proof. Let $\pi: V \to \langle L \rangle$ denote the orthogonal projection. By Proposition 2.1 and (2.8), the support function of $\pi(K)$, for $u \in \langle L \rangle$, is given by
\[ h_{\pi(K)}(u) = h_K(\pi(u)) = \frac{1}{2}E\langle \pi(u), X \rangle. \]
Thus $\langle K \rangle \perp \langle L \rangle$ iff $\pi(K) = 0$ iff $h_{\pi(K)} = 0$. The latter is equivalent to $\langle \pi(u), X \rangle = 0$ almost surely, for all $u \in \langle L \rangle$. This is easily seen to be equivalent to $\langle X, Y \rangle = 0$ almost surely. \qed

2.3. Virtual zonoids. It is well known that the set of zonoids $Z(V)$ and convex bodies $K(V)$ can be interpreted as cones in vector spaces of “virtual zonoids” $\hat{Z}(V)$ and “virtual convex bodies” $\hat{K}(V)$, respectively (see [BZ88, §25.1] and [Sch14, §3.5]). This allows to investigate them using tools from linear algebra and functional analysis.

We confine ourselves to zonoids, since only for those we can define a satisfying notion of tensor product, see Section 3. The next result summarizes the situation.
Theorem 2.16. \( \mathcal{Z}(V) \) is embedded in an essentially unique way as a subcone of a normed and partially ordered real vector space \( \hat{\mathcal{Z}}(V) \) of virtual zonoids such that any element of \( \hat{\mathcal{Z}}(V) \) can be written as a formal difference \( K_1 - K_2 \) of zonoids. The norm and partial order are defined as follows:

\[
\|K_1 - K_2\| = d_H(K_1, K_2), \quad 0 \leq K_1 - K_2 :\iff h_{K_2} \leq h_{K_1}.
\]

Thus the norm extends the norm of zonoids introduced in (2.3) and the partial order on \( \hat{\mathcal{Z}}(V) \) extends the inclusion of convex bodies (see Proposition 2.1). However, \( \hat{\mathcal{Z}}(V) \) is not complete, so not a Banach space, unless \( \dim V \leq 1 \).

Proof. We can identify convex bodies with their support functions using Proposition 2.1. Then \( \mathcal{K}(V) \) can be seen as a real cone in \( C(S) \), which is closed under addition and multiplication with nonnegative scalars (recall that \( S \) denotes the unit sphere in \( V \)). We define the space of virtual convex bodies \( \hat{\mathcal{K}}(V) \) as the span of \( \mathcal{K}(V) \): it is the subspace of \( C(S) \) consisting of differences of support functions. Similarly, we define the space of virtual zonoids \( \hat{\mathcal{Z}}(V) \) as the span of \( \hat{\mathcal{K}}(V) \), seen as a subset of \( C(S) \). Thus \( \hat{\mathcal{Z}}(V) \subset \hat{\mathcal{K}}(V) \) are endowed with the supremum norm (of functions restricted to the unit sphere), which extends the norm of convex bodies defined in (2.3). Moreover, the pointwise order of functions makes them partially ordered vector spaces. For the uniqueness, we refer to the discussion below. The non-completeness of \( \hat{\mathcal{Z}}(V) \) is shown in Proposition 2.24 below.

In Section 2.4 we will see that virtual zonoids can be identified with the space of signed measures on the sphere. Moreover, virtual zonoids also have a geometric realization as hedgehogs, i.e., envelopes of hyperplanes defined by the difference of the corresponding support functions. For this point of view we refer the reader to [MM01].

Despite the efficiency of support functions, used in the proof of Theorem 2.16, it is useful to think of virtual convex bodies more abstractly as formal differences of convex bodies. The abstract point of view reveals the uniqueness of the construction. Let us therefore formulate the above algebraic features in this language.

Given a commutative monoid \((\mathcal{M}, +)\), its Grothendieck group \((\hat{\mathcal{M}}, +)\) can be defined as the group of equivalence classes of pairs \((m_1, m_2) \in \mathcal{M}\) such that \((m_1, m_2) \sim (n_1, n_2)\) if and only if \(m_1 + n_2 = m_2 + n_1\), and with the addition \([[(m_1, m_2)] + [(n_1, n_2)] := [(m_1 + n_1, m_2 + n_2)]\). The Grothendieck group of \((\mathcal{M}, +)\) is an essentially unique object characterized by a universal property, see [Lan02]. If the cancellation law holds, then \(\mathcal{M} \to \hat{\mathcal{M}}, m \mapsto [(m, 0)]\) is injective so that \(\mathcal{M}\) can be seen as a submonoid of \(\hat{\mathcal{M}}\) and we write the class \([[(m_1, m_2)]\) as \(m_1 - m_2\). If, in addition, there is an action of \(\mathbb{R}_+\) on \(\mathcal{M}\) satisfying the usual axioms of scalar multiplication in vector spaces, then it is immediate to check that the scalar multiplication extends to \(\mathbb{R}\), so that \(\hat{\mathcal{M}}\) becomes a real vector space, which we may call the corresponding Grothendieck vector space.

We apply this construction to the monoids of zonoids and convex bodies in a Euclidean space \(V\), endowed with the Minkowski addition and the scalar multiplication given by (2.1). Using support functions, we see that the cancellation law holds in \((\mathcal{K}(V), +)\), see Proposition 2.1 and [Sch14, §3.1].

Definition 2.17 (The vector space of virtual convex bodies). We denote by \(\hat{\mathcal{K}}(V)\) and \(\hat{\mathcal{Z}}(V)\) the Grothendieck vector spaces of \(\mathcal{K}(V)\) and \(\mathcal{Z}(V)\), respectively. Its elements are called virtual convex bodies and virtual zonoids, respectively.
From the uniqueness of the Grothendieck vector spaces \( \widehat{\mathcal{K}}(V) \) and \( \widehat{\mathcal{Z}}(V) \) up to canonical isomorphy, it follows that they are isomorphic to the subspaces of \( C(S) \) defined via support functions in the proof of Theorem 2.16. We abbreviate \( \widehat{\mathcal{K}}^m := \widehat{\mathcal{K}}(\mathbb{R}^m) \) and \( \widehat{\mathcal{Z}}^m := \widehat{\mathcal{Z}}(\mathbb{R}^m) \).

Let us point out that the formal inverse \(- K\) of a zonoid \( K\), which is a virtual zonoid, should not be confused with the zonoid \((-1)K\), which is equal to \( K\).

**Remark 2.18.** 1. We may define the nonegative cone of \( \widehat{\mathcal{K}}(V) \) corresponding to the above defined order by \( \widehat{\mathcal{K}}(V)_+ := \{ K_2 - K_1 \mid K_1, K_2 \in \mathcal{K}(V), K_1 \leq K_2 \}; \) similarly, we define \( \widehat{\mathcal{Z}}(V)_+ \). Then it is clear that \( \mathcal{K}(V) \subseteq \widehat{\mathcal{K}}(V)_+ \) and \( \mathcal{Z}(V) \subseteq \widehat{\mathcal{Z}}(V)_+ \), however, the inclusions in general are strict [Sch14, §3.2].

**Remark 2.19.** The elements of \( \mathcal{K}(V) \cap \widehat{\mathcal{Z}}(V) \) are the convex bodies that can be written as a difference of zonoids, they are called generalized zonoids [Sch14, p. 195]. They are the convex bodies whose support function has a representation as in (2.13) below, but with an even signed measure \( \mu \). It is known that the generalized zonoids are dense in \( \mathcal{K}(V) \); in particular, there exist centrally symmetric convex bodies, which are not generalized zonoids. This is shown in [Sch14, Note 13, p. 206].

If \( M : V \to W \) is a linear map between Euclidean vector spaces then \( K \mapsto M(K) \) gives a morphism \( \mathcal{Z}(V) \to \mathcal{Z}(W) \) of monoids, which commutes with multiplication with nonnegative scalars. It is immediate that this morphism can be extended to a linear map between the corresponding Grothendieck vector spaces.

**Definition 2.20.** Let \( M : V \to W \) be a linear map between Euclidean vector spaces. We denote by \( \widehat{M} : \widehat{\mathcal{Z}}(V) \to \widehat{\mathcal{Z}}(W) \) the associated linear map defined by
\[
\widehat{M}(K_1 - K_2) := M(K_1) - M(K_2).
\]

We collect some properties of the associated map \( \widehat{M} \).

**Proposition 2.21.** Let \( M : V \to W \) be a linear map between Euclidean vector spaces. Then the associated linear map \( \widehat{M} : \widehat{\mathcal{Z}}(V) \to \widehat{\mathcal{Z}}(W) \) preserves the order, is continuous, and
\[
\| \widehat{M} \|_{\text{op}} = \| M \|_{\text{op}}.
\]
Moreover, \( \widehat{M}(\mathcal{Z}(V)) \subseteq \mathcal{Z}(W) \), i.e., \( \widehat{M} \) maps zonoids to zonoids. Finally, for \( K \in \mathcal{Z}(V) \),
\[
\ell(M(K)) \leq \| M \|_{\text{op}} \ell(K).
\]
In particular, the length of zonoids does not increase under an orthogonal projection.

**Proof.** The preservation of the order follows from the definition and Proposition 2.1. To prove continuity, let \( Z = K_1 - K_2 \in \widehat{\mathcal{Z}}(V) \). Recall from Theorem 2.16 that the norm defined on \( \widehat{\mathcal{Z}}(V) \) is \( \| Z \| = d_H(K_1, K_2) \). Using the properties from Proposition 2.1, we have
\[
\| \widehat{M}(Z) \| = \| M(K_1) - M(K_2) \|
= \sup \{ |h_{M(K_1)}(u) - h_{M(K_2)}(u)| \mid u \in W, \| u \| = 1 \}
\leq \| M \|_{\text{op}} \sup \{ |h_{K_1}(v) - h_{K_2}(v)| \mid v \in W, \| v \| = 1 \}
\leq \| M \|_{\text{op}} \cdot d_H(K_1, K_2)
= \| M \|_{\text{op}} \cdot \| K_1 - K_2 \| = \| M \|_{\text{op}} \cdot \| Z \|.
\]
This proves $\|\hat{M}\|_{\text{op}} \leq \|M\|_{\text{op}}$ and hence the continuity of the linear map $\hat{M}$. On the other hand, let $v \in V$ be of norm one such that $\|M\|_{\text{op}} = \|Mv\|$ and consider the segment $\frac{1}{2}[-v, v]$, of norm $\frac{1}{2}\|[-v, v]\| = \|v\| = 1$. Then
\[
\|\hat{M}\|_{\text{op}} \geq \|\hat{M}(\frac{1}{2}[-v, v])\| = \|\frac{1}{2}(Mv, Mv)\| = \|Mv\| = \|M\|_{\text{op}},
\]
This proves the stated equality of operator norms.

Since $\hat{M}$ maps segments to segments and $\hat{M}$ is continuous, we have $\hat{M}(Z(V)) \subseteq Z(W)$. For the stated upper bound on the length, let $X$ be a integrable random vector representing $X$. Then $M(X)$ represents the image $M(K)$ and we have by Definition 2.10,
\[
\ell(M(K)) = \mathbb{E}\|M(X)\| \leq \|M\|_{\text{op}} \mathbb{E}\|X\| = \|M\|_{\text{op}} \ell(X),
\]
which completes the proof. \hfill \Box

Our construction may be summarized by saying that we have constructed a covariant functor from the category of Euclidean spaces (with any linear morphisms) to the category of partially ordered and normed vector spaces.

The length of zonoids defined in Definition 2.10 extends to a continuous linear functional.

**Theorem 2.22.** The length extends to a continuous linear functional $\ell : \hat{Z}(V) \to \mathbb{R}$.

**Proof.** Lemma 2.12 shows that the length on $\hat{Z}(V)$ is additive, so we can extend it to a linear functional on $\hat{Z}(V)$ by setting $\ell(K - L) := \ell(K) - \ell(L)$. Using Corollary 2.14 we can characterize this as $\ell(K_1 - K_2) = \tau_m(\|h_{K_1}\|_1 - \|h_{L_1}\|_1)$. Since both $h_{K_1}$ and $h_{L_1}$ are nonnegative we have $\|h_{K_1}\|_1 - \|h_{L_1}\|_1 = \|h_{K_1} - h_{L_1}\|_1$. This shows that the length is continuous. \hfill \Box

We also need the following observation.

**Lemma 2.23.** If $V$ is one-dimensional, the length $\ell : \hat{Z}(V) \to \mathbb{R}$ induces an isomorphism of additive groups, which preserves the standard norm and the standard order. We shall use this to identify $\hat{Z}(V)$ with $\mathbb{R}$ and $\hat{Z}(V)$ with $\mathbb{R}_+$.

**Proof.** The length is a group homomorphism by Theorem 2.22. Let $K = \frac{1}{2}[-x, x]$ be a centrally symmetric zonoid in $Z(V)$; i.e., a segment. Then, $\ell(K) = x$, which shows that $\ell$ is bijective, hence an isomorphism. This also shows that $\ell$ preserves the standard norm and the standard order. \hfill \Box

We finally show that $\hat{Z}(V)$ in general is not a Banach space, as already claimed in Theorem 2.16. Recall that $C(S)$ denotes the Banach space of continuous real functions on the unit sphere $S := S(V)$ endowed with the supremum norm. We denote by $C_{\text{even}}(S)$ its closed subspace of even functions (i.e., $f(-v) = f(v)$). In the proof of Theorem 2.16 we have constructed the normed vector space of virtual convex bodies $\hat{K}(V) \subset C(S)$ and the normed vector space $\hat{Z}(V) \subset C_{\text{even}}(S)$ of virtual zonoids. (Recall that the sup-norm corresponds to the Hausdorff metric (2.2).) Unfortunately, these are not Banach spaces. This follows from known facts that we state in the next proposition for the sake of clarity.

**Proposition 2.24.** The completion of the space of virtual convex bodies $\hat{K}(V)$ equals $C(S)$ and the completion of the space of virtual zonoids $\hat{Z}(V)$ equals $C_{\text{even}}(S)$. Moreover, if $\dim V > 1$, then $\hat{K}(V) \neq C(S)$ and $\hat{Z}(V) \neq C_{\text{even}}(S)$, hence $\hat{K}(V)$ and $\hat{Z}(V)$ are not complete and hence not Banach spaces. In particular, they are real vector spaces of infinite dimension. However, if $\dim V = 1$, then we have $\hat{K}(V) \simeq \mathbb{R}^2$ and $\hat{Z}(V) \simeq \mathbb{R}$. 

Proof. W.l.o.g. we assume \( V = \mathbb{R}^m \). It is known [Sch14, Lemma 1.7.8] that every twice continuously differentiable function \( S^{m-1} \to \mathbb{R} \) can be written as a difference \( h_K - rh_{B^m} \) for some \( K \in \mathcal{K}^m \) and \( r > 0 \). This implies that \( \hat{K}^m \) is dense in \( C(S^{m-1}) \), since continuous function can be uniformly approximated by twice continuously differentiable functions. On the other hand, \( \hat{K}^m \) is strictly contained in \( C(S^{m-1}) \) if \( m > 1 \). One way to see this is that there are continuous, nowhere differentiable functions on \( S^{m-1} \), while every \( h \in \hat{K}^m \) is differentiable at least at some point in \( S^{m-1} \).

The assertion about virtual zonoids is more involved. In [Sch14, Theorem 3.5.4], it is shown that every sufficiently smooth even real function \( f : S^{m-1} \to \mathbb{R} \) has a representation as in Definition 2.25 below, but with a even signed measure \( \mu \), which implies that \( f \in \hat{Z}^m \). This implies that \( \hat{Z}^m \) is dense in \( C_{\text{even}}(S^{m-1}) \). In order to show that \( \hat{Z}^m \) is strictly contained in \( C_{\text{even}}(S^{m-1}) \) one can argue as follows. If we had \( \hat{Z}^m = C_{\text{even}}(S^{m-1}) \), then every centrally symmetric convex body would be a virtual zonoid. This contradicts [Sch14, Note 13, p. 206], which states that there exist centrally symmetric convex bodies, which are not generalized zonoids. Finally, \( \hat{K}^1 \simeq \mathbb{R}^2 \) and \( \hat{Z}^1 \simeq \mathbb{R}^1 \) since \( \mathcal{K}^1 \) consists of the intervals and \( \mathcal{Z}^1 \) consist of the symmetric intervals.

We remark that the fact that the cone \( \hat{Z}(V) \) is a closed subset of the cone \( \mathcal{K}(V) \) does not contradict the fact that, by Proposition 2.24, \( \hat{Z}(V) \) is a dense subset of \( \hat{K}(V) \).

2.4. Virtual zonoids and measures. There is a correspondence between zonoids and even measures on the sphere, that we discuss now. This point of view is classic when dealing with zonoids, so we include here a review of the principal facts of this correspondence. The approach using measures provides a complimentary viewpoint to our approach using random vectors. This alternative viewpoint will become particularly useful in Section 3.1, where we discuss continuity properties of our constructions.

In what follows we identify the space of even measures on the sphere with the space of measures on the projective space. The space of continuous functions on the projective space is \( C(\mathbb{P}^{m-1}) \). The space of (signed) measures on \( \mathbb{P}^{m-1} \) will be denoted by \( \mathcal{M}(\mathbb{P}^{m-1}) \). The cone of positive measures will be denoted by \( \mathcal{M}_+(\mathbb{P}^{m-1}) \). Recall that the weak-\(*\) topology on \( \mathcal{M}(\mathbb{P}^{m-1}) \) is the coarsest topology on \( \mathcal{M}(\mathbb{P}^{m-1}) \) such that for every \( \phi \in C(\mathbb{P}^{m-1}) \) the linear functional \( \mu \mapsto \int_{\mathbb{P}^{m-1}} \phi(x) \mu(dx) \) is continuous.

For practical purposes, an even measure on the sphere \( S^{m-1} \) will correspond to \( \frac{1}{2} \) times its pushforward measure on the projective space. Similarly we identify a function on the projective space with the corresponding even function on the sphere; that is, if \( f : \mathbb{P}^{m-1} \to \mathbb{R} \), then we write \( f(x) := (f \circ \Pi)(x) \), where \( \Pi : \mathbb{R}^m \setminus \{0\} \to \mathbb{P}^{m-1} \) is the projection.

With these identifications, for all \( \mu \in \mathcal{M}(\mathbb{P}^{m-1}) \) and \( f \in C(\mathbb{P}^{m-1}) \), we have

\[
\int_{\mathbb{P}^{m-1}} f(x) \mu(dx) = \frac{1}{2} \int_{S^{m-1}} f(x) \mu(dx).
\]

Passing from zonoids to virtual zonoids corresponds to passing from even measures to even signed measures. The main object of the section is the following map.

Definition 2.25. The cosine transform is the linear map \( H : \mathcal{M}(\mathbb{P}^{m-1}) \to C(\mathbb{P}^{m-1}) \) given for all \( \mu \in \mathcal{M}(\mathbb{P}^{m-1}) \) and \( x \in \mathbb{P}^{m-1} \) by

\[
H(\mu)(x) := \int_{\mathbb{P}^{m-1}} |\langle u, x \rangle| \mu(du).
\]

The image of the cosine transform will be denoted by \( H(\mathbb{P}^{m-1}) \) and the image of \( \mathcal{M}_+(\mathbb{P}^{m-1}) \) by \( H_+(\mathbb{P}^{m-1}) \).
We could not locate any reference for items (4)–(6) of the next theorem. This is why we include our own proof. Recall that we endow \( \mathcal{M}(\mathbb{P}^{m-1}) \) with the weak–* topology and \( C(\mathbb{P}^{m-1}) \) with the topology induced by the \( \infty \)-norm.

**Theorem 2.26.** The cosine transform \( H: \mathcal{M}(\mathbb{P}^{m-1}) \to C(\mathbb{P}^{m-1}) \) satisfies the following properties.

1. \( H \) is injective.
2. \( H(\mathcal{M}(\mathbb{P}^{m-1})) \) is a dense subspace of \( C(\mathbb{P}^{m-1}) \).
3. There is \( c = c(m) > 0 \) such that \( c\|H(\mu)\|_\infty \leq \mu(\mathbb{P}^{m-1}) \leq \|H(\mu)\|_\infty \) for all (nonnegative) measures \( \mu \in \mathcal{M}_+(\mathbb{P}^{m-1}) \).
4. \( H \) is sequentially continuous.
5. The restriction \( H: \mathcal{M}_+(\mathbb{P}^{m-1}) \to H_+(\mathbb{P}^{m-1}) \) is a homeomorphism.
6. The inverse \( H^{-1}: H_+(\mathbb{P}^{m-1}) \to \mathcal{M}(\mathbb{P}^{m-1}) \) is not sequentially continuous for \( m > 1 \).

**Remark 2.27.** Our proof for item (4) does not extend to nets. This is why we only prove sequential continuity of \( H \).

**Proof of Theorem 2.26.** Assertions (1) and (2) are in (the proof of) [Sch14, Theorem 3.5.4]. Assertion (3) follows from Corollary 2.14 stated in our context, and using Remark 2.30 (2) below.

As for assertion (4): because \( H \) is linear it suffices to prove sequential continuity at 0. Recall that for the weak–* topology on \( \mathcal{M}(\mathbb{P}^{m-1}) \), a sequence of measures \( (\mu_i) \) converges to 0, if and only if for every \( \phi \in C(\mathbb{P}^{m-1}) \), the sequence \((\mu_i, \phi)\) goes to zero, where

\[
\langle \mu, \phi \rangle := \int_{\mathbb{P}^{m-1}} \phi(x) \mu(dx).
\]

Suppose that \( (\mu_i) \) converges to 0 in \( \mathcal{M}(\mathbb{P}^{m-1}) \) in the weak–* topology. Let \( h_i := H(\mu_i) \), thus \( h_i(v) := \int_{x \in \mathbb{P}^{m-1}} |\langle v, x \rangle| \mu_i(dx) \), and in particular \( h_i(v) \to 0 \) for all \( v \in \mathbb{P}^{m-1} \). So we have pointwise convergence of the \( h_i \). We are going to show that \( h_i \to 0 \) uniformly. Recall that every measure \( \mu \) has a unique decomposition \( \mu = \mu^+ - \mu^- \) (called the Hahn–Jordan decomposition), where \( \mu^+, \mu^- \in \mathcal{M}_+(\mathbb{P}^{m-1}) \). We define \( |\mu| := \mu^+ + \mu^- \). The Banach–Steinhaus Theorem (e.g., see [Rud73]) implies that \( \kappa := \text{sup}_i (|\mu_i|) < \infty \). Therefore, we have for any \( v \in \mathbb{P}^{m-1} \),

\[
|h_i(v)| \leq \int_{\mathbb{P}^{m-1}} |\langle v, x \rangle| |\mu_i|(dx) \leq |\mu_i|(\mathbb{P}^{m-1}) \leq \kappa,
\]

and hence \( \text{sup}_i \|h_i\|_\infty \leq \kappa \). Moreover, for \( v_1, v_2 \in \mathbb{P}^{m-1} \),

\[
|h_i(v_1) - h_i(v_2)| \leq \int_{\mathbb{P}^{m-1}} |\langle v_1 - v_2, x \rangle| |\mu_i|(dx) \leq \kappa \|v_1 - v_2\|.
\]

The Arzelà-Ascoli Theorem (e.g., see [Rud73]) implies that \( (h_i) \) has a uniformly convergent subsequence \( (h_{i_j}) \). Thus \( h_{i_j} \to 0 \) uniformly since \( h_{i_j} \to 0 \) pointwise. By the same argument we see that any subsequence of \( (h_i) \) has a subsequence that uniformly converges to 0. This implies that \( h_i \to 0 \) uniformly. Therefore, it follows that the map \( H \) is sequentially continuous.

For assertion (5), Bolker [Bol69, Theorem 5.2] showed that \( H: \mathcal{M}_+(\mathbb{P}^{m-1}) \to C(\mathbb{P}^{m-1}) \) is continuous. So we only need to show that the inverse \( H^{-1}: H_+(\mathbb{P}^{m-1}) \to \mathcal{M}_+(\mathbb{P}^{m-1}) \) is continuous. For this it suffices to show that \( H^{-1} \) is sequentially continuous on \( H_+(\mathbb{P}^{m-1}) \), because the norm topology on \( H_+(\mathbb{P}^{m-1}) \) is first countable and for maps whose domain is first countable topological spaces sequential continuity and continuity are equivalent. To show sequential continuity of \((H^{-1})_{|H_+(\mathbb{P}^{m-1})}\) we take a sequence \( (h_i) \subset H_+(\mathbb{P}^{m-1}) \) that converges to \( h \). Let \( (\mu_i) \subset \mathcal{M}_+(\mathbb{P}^{m-1}) \) be the corresponding sequence so that \( H(\mu_i) = h_i \) and let \( \mu \) be a
measure with $H(\mu) = h$. We have to show that $\mu_i$ converges to $\mu$. For this, we fix $\phi \in C(\mathbb{P}^{m-1})$ and show that $\langle \mu_i - \mu, \phi \rangle \to 0$. This would imply that $\mu_i - \mu \to 0$. Let $\varepsilon > 0$. By assertion (2) there are $\xi_1, \ldots, \xi_N \in \mathbb{P}^{m-1}$ and $c_1, \ldots, c_N \in \mathbb{R}$ such that the function $\psi(x) := \sum_{k=1}^{N} c_k |x, \xi_k|$ in $C(\mathbb{P}^{m-1})$ satisfies $\|\psi - \phi\|_{\infty} < \varepsilon/(2c)$. We decompose
\[
\langle \mu_i - \mu, \phi \rangle = \langle \mu_i, \phi - \psi \rangle + \langle \mu_i - \mu, \psi \rangle + \langle \mu, \psi - \phi \rangle.
\]
The sequence of real numbers $\|h_i\|_{\infty}$ converges to $\|h\|_{\infty}$ and is thus bounded so that there is $c > 0$ such that $\sup_i \|h_i\|_{\infty} \leq c$ and $\|h\|_{\infty} \leq c$. An upper bound for the absolute value of third term in (2.15) is $|\langle \mu, \phi - \psi \rangle| = \|\int_{\mathbb{P}^{m-1}} (\phi(x) - \psi(x)) \mu(dx)\| \leq \mu(\mathbb{P}^{m-1}) \|\phi - \psi\|_{\infty}$ (here, we have used that $\mu$ is a measure and not a signed measure). Assertion (4) implies that this is bounded by $c\|\phi - \psi\|_{\infty} < \varepsilon/2$. We get the same bound for the first term. The middle term equals $\sum_{k=1}^{N} c_k (h_i(\xi_k) - h(\xi_k))$ and, by assumption, converges to zero for $i \to \infty$. Therefore, $\limsup_i |\langle \mu_i - \mu, \phi \rangle| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that indeed $\langle \mu_i - \mu, \phi \rangle \to 0$, which proves assertion (6).

Assertion (6) follows from the noncontinuity of the tensor product of zonoids. More precisely, we will prove in Theorem 3.10 below that the map $(K, L) \mapsto h_{K \otimes L}$ is not sequentially continuous. On the other hand, we can write: $h_{K \otimes L} = H(T(H^{-1}(h_K), H^{-1}(h_L)))$, where, for two measures $\mu_1, \mu_2$ the measure $\tilde{T} (\mu_1, \mu_2)$ is defined in (3.4) below. By Theorem 3.10 (3) below, the map $\tilde{T}$ is sequentially continuous and, if $H^{-1}$ were sequentially continuous, then $(K, L) \mapsto h_{K \otimes L}$ would also be sequentially continuous. This contradicts Theorem 3.10.

By Theorem 2.26 (5), the cone $H_+(\mathbb{P}^{m-1})$ coincides with the cone of support functions of zonoids and consequently $H(\mathbb{P}^{m-1})$ coincides with the linear span of support functions of zonoids. Again by Theorem 2.26, since the cone of nonnegative measures can be identified with the cone of zonoids, the vector space of measures can be identified with the vector space of virtual zonoids. In particular, on the space of virtual zonoids $Z(V)$ we can put two different topologies: the topology $T_1$ induced by viewing them (through their support functions) as a subspace of continuous functions on $\mathbb{P}^{m-1}$ with the uniform convergence topology induced by the $\infty$-norm (we denote this topology by $T_\infty$), and the topology $T_2$ induced by viewing them as signed measures with the weak-* topology $T_{\text{weak-*}}$. With this notation, letting $V \simeq \mathbb{R}^m$, we have
\[
(\hat{Z}(V), T_1) \simeq (H(\mathbb{P}^{m-1}), T_\infty) \quad \text{and} \quad (\hat{Z}(V), T_2) \simeq (M(\mathbb{P}^{m-1}), T_{\text{weak-*}}).
\]

With these identifications, the map $H$ can be seen as an inclusion:
\[
H : (\hat{Z}(V), T_2) \xrightarrow{\text{id}} (\hat{Z}(V), T_1) \hookrightarrow (C(\mathbb{P}^{m-1}), T_\infty).
\]
The fact that the inverse of $H$ is not continuous means that the two topologies (2.16) are not the same. What is remarkable, however, is that on the cone of zonoids, they induce the same topology.

Remark 2.28. Next to the weak-* topology, another natural topology on $M(\mathbb{P}^{m-1})$ is the one induced by the total variation norm $\|\mu\| := \|\mu\|_{\text{TV}} := \|\mu^+ - \mu^-\|$ for $\mu = \mu^+ - \mu^-$ being the Hahn-Jordan decomposition of $\mu$. By Remark 2.30 below, for nonnegative measures this coincides with the length. Let us observe that item (3) in Theorem 2.26 does not imply that this topology coincides with either $T_1$ or $T_2$, not even when restricted to $Z(V)$. As an example, one can consider a sequence of pairwise different segments $[v_i, v_i]$ such that all $v_i$ are in the sphere. The corresponding measure is $\mu_i := \frac{1}{2} \delta_{H_i(v_i)}$, where the latter is the Dirac-delta measure. Observe, that for the total variation norm we have $\|\mu_i - \mu_j\| = \frac{1}{2}$. In particular such a sequence cannot converge in the topology induced by the total variation norm, but it can converge in $T_1$ and $T_2$. 

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Next, we discuss the way of passing from the point of view of random vectors to the point of view of positive measures. Recall that for $x \in \mathbb{R}^m \setminus \{0\}$ and $f \in C(\mathbb{P}^{m-1})$ we set $f(x) := (f \circ \Pi)(x)$, where $\Pi : \mathbb{R}^m \setminus \{0\} \to \mathbb{P}^{m-1}$ is the canonical projection.

**Proposition 2.29.** Let $X \in \mathbb{R}^m$ be an integrable random vector with probability measure $\nu$. Let $\nu'$ be the measure such that $\nu'(A) = \int_A \|x\| \nu(dx)$ for all measurable sets $A \subset \mathbb{R}^m$. Then

$$h_K(X) = H(\mu),$$

where $\mu$ is the push-forward measure of $\nu'$ under the projection $\Pi : \mathbb{R}^m \setminus \{0\} \to \mathbb{P}^{m-1}$.

**Proof.** Let $u \in \mathbb{P}^{m-1}$. The support function of $K(X)$ evaluated at $u$ is $h_K(X)(u) = \frac{1}{2} \mathbb{E}\langle u, X \rangle$. We write the integral explicitly as

$$\mathbb{E}\langle u, X \rangle = \int_{\mathbb{R}^m} \langle u, x \rangle \nu(dx) = \int_{\mathbb{R}^m} \langle u, x/\|x\| \rangle \nu'(dx) = \int_{\mathbb{P}^{m-1}} \langle u, y \rangle \mu(dy),$$

the second equality, because $\nu'$ is zero where $\|X\| = 0$ and the third equality, because $\langle u, x/\|x\| \rangle$ is constant on preimages of $\Pi$.

**Remark 2.30.** Let $X$ be an integrable vector, $K = K(X)$, and corresponding measure $\mu$.

1. From Definition 2.10 and Proposition 2.29 it follows that $\ell(K) = 2\mu(\mathbb{P}^{m-1}) = \mu(S^{m-1}).$
2. If $X$ admits an even measurable density $\rho : \mathbb{R}^m \to \mathbb{R}$, then $\mu$ admits the density $\tilde{\rho}(x) := \int_0^{+\infty} t^m \rho(tx_0) dt$, where $x_0 \in S^{m-1}$ is such that $\Pi(x_0) = x$.

We close this section by proving that linear maps between spaces of measures are continuous with respect to the weak-$*$ topology.

**Lemma 2.31.** Let $M : V \to W$ be a linear map between Euclidean vector spaces, $m := \dim V$ and $n := \dim W$. Consider the induced linear map $\tilde{M} : \mathcal{M}(\mathbb{P}^{m-1}) \to \mathcal{M}(\mathbb{P}^{n-1})$ that sends the measure associated to the zonoid $K$ to the measure associated to the zonoid $\tilde{M}(K)$; that is, if $H(\mu) = \tilde{h}_K$, then $H(\tilde{M}(\mu)) = \tilde{h}_{\tilde{M}(K)}$. Then, $\tilde{M}$ is sequentially continuous with respect to the weak-$*$ topology.

**Proof.** Since $\tilde{M}$ is linear, it is enough to show continuity at $0$. So, let $\mu_i$ be a sequence of measures converging to $0$, and let $\nu_i := \tilde{M}(\mu_i)$. We have to show that $\nu_i$ converges to $0$. Let $K_i$ be the zonoid associated to $\mu_i$. Moreover, let us denote the pairing as in (2.14):

$$\langle \mu, \phi \rangle = \int_{\mathbb{P}^{m-1}} \phi(x) \mu(dx)$$

for $\phi \in C(\mathbb{P}^{m-1})$. From Proposition 2.1 (4) we know that, if $K$ is a zonoid, then $h_{\tilde{M}(K)}(v) = h_K(M^T v)$. Take $v \in S(W)$. Then, we have:

$$h_{\tilde{M}(K)}(v) = \begin{cases} \|M^T v\| \cdot \tilde{h}_K(M^T v/\|M^T v\|), & \text{if } M^T v \neq 0, \\ 0, & \text{else.} \end{cases}$$

This implies that for any $v$ such that $M^T v \neq 0$ and for every $i$ we have:

$$\int_{\mathbb{P}^{m-1}} |\langle v, y \rangle| \nu_i(dy) = \|M^T v\| \cdot \int_{\mathbb{P}^{m-1}} |\langle M^T v/\|M^T v\|, x \rangle| \mu_i(dx) = \int_{\mathbb{P}^{m-1}} |\langle M^T v, x \rangle| \mu_i(dx),$$

and $\int_{\mathbb{P}^{m-1}} |\langle v, y \rangle| \nu_i(dy) = 0$ otherwise. Since $(\mu_i)$ converges to $0$ we see that $\int_{\mathbb{P}^{m-1}} |\langle v, y \rangle| \nu_i(dy)$ converges to $0$ for every $v \in S(W)$. Now, we can proceed as we did in (2.15), approximating any continuous function $\phi \in C(\mathbb{P}^{m-1})$ with a linear combination of functions of the form $y \mapsto |\langle v, y \rangle|$. This concludes the proof.
3. Tensor product of zonoids

In this section we introduce and study the notion of tensor product of zonoids. The only previous appearance of this notion we are aware of is [AL16, Definition 3.2].

In the whole section, $V$ and $W$ denote Euclidean spaces. We start with the following central definition.

**Definition 3.1 (Tensor product of zonoids).** Let $K$ be a zonoid in $V$, $L$ be a zonoid in $W$ and $X \in V$ and $Y \in W$ be integrable random vectors representing $K$ and $L$, respectively. We define the tensor product of $K$ and $L$ as

$$K \otimes L := K(X \otimes Y).$$

Of course, we need to check that this tensor product does not depend on the choice of random vectors representing the zonoids. This is guaranteed by Lemma 3.2 below. For stating it, we introduce the following notation: for $x \in V$ and $y \in W$ we define the following linear operators

$$T_x := \langle \cdot, x \rangle \otimes \text{id}_W : V \otimes W \to W \quad \text{and} \quad T_y := \text{id}_V \otimes \langle \cdot, y \rangle : V \otimes W \to V.$$

Notice that their operator norms satisfy

$$\|T_x\|_{\text{op}} = \|x\| \quad \text{and} \quad \|T_y\|_{\text{op}} = \|y\|.$$

**Lemma 3.2.** Let $K \in Z(V)$ and $L \in Z(W)$ be zonoids represented by independent random vectors $X \in V$ and $Y \in W$, i.e., $K = K(X)$ and $L = K(Y)$. Then

$$h_{K(X \otimes Y)}(u) = \mathbb{E}_Y h_{K(X)}(T_Y(u)) = \mathbb{E}_X h_{K(Y)}(T_X(u))$$

and $K(X \otimes Y) \in Z(V \otimes W)$ depends only on $K$ and $L$, and not on the choice of the random vectors $X$ and $Y$.

**Proof of Lemma 3.2.** We first show that

$$\forall u \in V \otimes W \quad \langle u, X \otimes Y \rangle = \langle T_Y(u), X \rangle.$$ 

It suffices to check this for simple tensors of the form $u = v \otimes w$ where $v \in V$ and $w \in W$, because the simple vectors generate $V \otimes W$. For such $u$, the definition of the scalar product in $V \otimes W$ indeed implies

$$\langle v \otimes w, X \otimes Y \rangle = \langle v, X \rangle \langle w, Y \rangle = \langle \langle w, Y \rangle v, X \rangle = \langle T_Y(v \otimes w), X \rangle.$$ 

By \((2.8)\) the support function of the zonoid $K(X \otimes Y)$ is given by

$$h_{K(X \otimes Y)}(u) = \frac{1}{2} \mathbb{E} |\langle u, X \otimes Y \rangle|,$$

where the expectation is over the joint distribution of $X$ and $Y$. Taking first the expectation over $X$ and using the independence of $X$ and $Y$, we get

$$h_{K(X \otimes Y)}(u) = \mathbb{E}_Y h_{K(X)}(T_Y(u)).$$

This shows that the dependence of $K(X \otimes Y)$ on $X$ is only through $K = K(X)$. A symmetric argument for $Y$ completes the proof. \(\square\)

**Example 3.3 (Tensor product of balls).** Let $X \in \mathbb{R}^k$ and $Y \in \mathbb{R}^m$ be independent, standard Gaussian vectors. Then $K(\sqrt{2\pi}X) = B^k$ and $K(\sqrt{2\pi}Y) = B^m$ by \((2.9)\). Therefore,

$$B^k \otimes B^m = \sqrt{2\pi}K(X) \otimes \sqrt{2\pi}K(Y) = 2\pi K(X \otimes Y).$$

The tensor product $K(X \otimes Y)$ was called Segre zonoid in [BL20]. It is a convex body in the space $\mathbb{R}^k \otimes \mathbb{R}^m \simeq \mathbb{R}^{k \times m}$, and its support function depends on singular values\(^5\) in the

\(^5\)In a recent preprint [SS20], Sanyal and Saunderson have introduced the notion of spectral convex bodies, i.e., convex bodies in the space of symmetric operators whose support function depends on eigenvalues only.
following sense. Assuming \( k \leq m \) and denoting by \( sv(M) \in \mathbb{R}^k \) the list of singular values of a matrix \( M \in \mathbb{R}^{k \times m} \), we have \( h_{B^k \otimes B^m}(M) = (2\pi)^{\frac{1}{2}} g_k(sv(M)) \), by [BL20, Lemma 5.5], where the function \( g_k : \mathbb{R}^k \to \mathbb{R} \) is defined by \( g_k(\sigma_1, \ldots, \sigma_k) := \mathbb{E} (\sigma_1^2 \xi_1^2 + \cdots + \sigma_k^2 \xi_k^2)^{\frac{1}{2}} \), and \( \xi_1, \ldots, \xi_k \) are independent standard gaussians. Recall from (2.10) the definition of \( \tau_m \). We obtain \( \|B^k \otimes B^m\| = \tau_k/\sqrt{k} =: r_k \) from [BL20, Lemma 5.5], which remarkably only depends on \( k \). Thus \( r_k B^k m \) is the smallest centered ball containing \( B^k \otimes B^m \). In [BL20, Theorem 6.3], it was shown that for fixed \( k \) and \( m \to \infty \), \( B^k \otimes B^m \) is not much smaller in volume than \( r_k B^km \); we have log \( \text{vol}_{km}(B^k \otimes B^m) = \log \text{vol}_{km}(r_k B^km) - O(\log m) \) for \( m \to \infty \).

The next result shows that the tensor product of zonoids behaves well with respect to Minkowski addition, scalar multiplication, norm, and inclusion.

**Proposition 3.4.** The tensor product of zonoids is componentwise Minkowski additive and positively homogenous. Moreover, the tensor product is monotonically increasing in each variable; that is, \( K_1 \subset K_2 \) and \( L_1 \subset L_2 \) implies \( K_1 \otimes L_1 \subset K_2 \otimes L_2 \). Finally, the tensor product of zonoids is associative.

**Proof.** Given \( K_1, K_2 \in Z(V) \), \( \lambda_1, \lambda_2 \geq 0 \), and a random vector \( Y \in W \) representing \( L \), we use (3.2) to write the support function of \( (\lambda_1 K_1 + \lambda_2 K_2) \otimes L \) as

\[
\begin{align*}
h_{(\lambda_1 K_1 + \lambda_2 K_2) \otimes L}(u) &= \mathbb{E}_Y h_{\lambda_1 K_1 + \lambda_2 K_2}(T_Y(u)) \\
&= \mathbb{E}_Y \lambda_1 h_{K_1}(T_Y(u)) + \mathbb{E}_Y \lambda_2 h_{K_2}(T_Y(u)) \\
&= \lambda_1 h_{K_1 \otimes L}(u) + \lambda_2 h_{K_2 \otimes L}(u).
\end{align*}
\]

For the second factor we argue analogously. Therefore Minkowski additivity and positive homogeneity in each factor follows from Proposition 2.1.

For the monotonicity we assume in addition that \( K_1 \subset K_2 \). Proposition 2.1 implies \( h_{K_1} \leq h_{K_2} \). Again, (3.2) gives \( h_{K_1 \otimes L}(u) = \mathbb{E}_Y h_{K_1}(T_Y(u)) \) and \( h_{K_2 \otimes L}(u) = \mathbb{E}_Y h_{K_2}(T_Y(u)) \). Therefore, \( h_{K_1 \otimes L} \leq h_{K_2 \otimes L} \). Again using Proposition 2.1 shows \( K_1 \otimes L \subset K_2 \otimes L \). For the second factor we argue analogously. Finally, the associativity immediately follows from the one of the usual tensor product. \( \square \)

**Example 3.5** (Tensor product of zonotopes). The tensor product of symmetric segments is given by

\[
\begin{align*}
\frac{1}{2}[-v_1, v_1] \otimes \cdots \otimes \frac{1}{2}[-v_p, v_p] &= \frac{1}{2}[-v_1 \otimes \cdots \otimes v_p, v_1 \otimes \cdots \otimes v_p],
\end{align*}
\]

where \( v_1 \in V_1, \ldots, v_p \in V_p \). (Indeed, just take for \( X_j \in V_j \) a random variable taking the constant value \( v_j \) and note that \( K(X_j) = \frac{1}{2}[-v_j, v_j] \).) Together with the biadditivity of the tensor product (Proposition 3.4), we conclude that the tensor product of two zonotopes is

\[
\left( \sum_{i=1}^n \frac{1}{2}[-x_i, x_i] \right) \otimes \left( \sum_{j=1}^m \frac{1}{2}[-y_j, y_j] \right) = \sum_{i=1}^n \sum_{j=1}^m \frac{1}{2}[-x_i \otimes y_j, x_i \otimes y_j].
\]

We now show that the length is multiplicative with respect to the tensor product and prove an upper bound on the norm of a tensor product of zonoids.

**Proposition 3.6.** For \( K \in K(V) \) and \( L \in K(W) \) such that \( m = \dim V \leq \dim W \), we have

\[
\ell(K \otimes L) = \ell(K) \ell(L), \quad \|K \otimes L\| \leq 2\sqrt{m}\|K\|\|L\|.
\]

**Proof.** Suppose that \( K = K(X) \) and \( L = K(Y) \) with independent random vector \( X \in V \) and \( Y \in W \). Then, by Definition 2.10, \( \ell(K \otimes L) = \mathbb{E}\|X \otimes Y\| = \mathbb{E}\|X\| \cdot \mathbb{E}\|Y\| = \ell(K) \ell(L) \), showing the first assertion.
For the norm inequality, assume $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$ and w.l.o.g. $m \leq n$. Recall that 
the nuclear norm of a matrix $M \in \mathbb{R}^{m \times n}$ is defined as the sum of its singular values.
The corresponding unit ball $B_{\text{nuc}}$ equals the convex hull of the rank one matrices $v \otimes w$ such that $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ have norm one, e.g., see [Der16]. If we denote by $B$ the unit ball with respect to the Frobenius norm, we get $B \subseteq \sqrt{m} B_{\text{nuc}}$, where we used that $m \leq n$. We obtain
\[
\|K \otimes L\| = \max_{u \in B} h_{K \otimes L}(u) \leq \sqrt{m} \max_{u \in B_{\text{nuc}}} h_{K \otimes L}(u) = \frac{1}{2} \sqrt{m} \max_u \mathbb{E} |\langle u, X \otimes Y \rangle|.
\]
But for $u = v \otimes w$ with unit vectors $v, w$, we have
\[
\mathbb{E} |\langle v \otimes w, X \otimes Y \rangle| = \mathbb{E} |\langle v, X \rangle| \cdot \mathbb{E} |\langle w, Y \rangle| = 4h_K(v)h_L(w) \leq 4\|K\| \cdot \|L\|.
\]
Using the convexity of $h_{K \otimes L}$, implies the second assertion. \hfill \Box

It is straightforward to extend the tensor product of zonoids to a bilinear map between spaces of virtual zonoids.

**Proposition 3.7** (Tensor product of virtual zonoids). The tensor product of zonoids from Definition 3.1 uniquely extends to a bilinear map $\hat{T} : \hat{Z}(V) \times \hat{Z}(W) \to \hat{Z}(V \otimes W)$. The resulting tensor product of virtual zonoids is associative.

**Proof.** The only possible way to define the map $\hat{T}$ is by setting

\[(K_1 - K_2) \otimes (L_1 - L_2) := K_1 \otimes L_1 + K_2 \otimes L_2 - K_1 \otimes L_2 - K_2 \otimes L_1.
\]

Using the biadditivity of the tensor product of zonoids (Proposition 3.4) it is straightforward to check that this is well defined and defines a bilinear map. The associativity follows from the associativity of the tensor product of zonoids. \hfill \Box

### 3.1. Continuity of the tensor product

Here we discuss the continuity of the tensor product map. The main result is that the tensor product is continuous on zonoids, but not on virtual zonoids with the norm topology. It is only separately continuous in each variable, meaning that it is continuous in each component. However, viewing virtual zonoids as measures via the correspondence described in Section 2.4 and endowed with the weak-$*$ topology, the tensor product turns out to be sequentially continuous; see Theorem 3.10 below.

As above we denote by \[\hat{T} : \hat{Z}(V) \times \hat{Z}(W) \to \hat{Z}(V \otimes W)\]
the tensor product map, $\hat{T}(K, L) = K \otimes L$, and its restriction to zonoids is denoted $T : Z(V) \times Z(W) \to Z(V \otimes W)$.

Viewing virtual zonoids as measures, we obtain a bilinear map \[\tilde{T} : \mathcal{M}(\mathbb{P}(V)) \times \mathcal{M}(\mathbb{P}(W)) \to \mathcal{M}(\mathbb{P}(V \otimes W)).\]

Concretely, this map can be described as follows. Let $\mu \in \mathcal{M}(\mathbb{P}(V))$ and $\nu \in \mathcal{M}(\mathbb{P}(W))$, and let $K_\mu \in \hat{Z}(V)$, $K_\nu \in \hat{Z}(W)$ be such that $h_{K_\mu} = H(\mu)$ and $h_{K_\nu} = H(\nu)$, where $H$ is the cosine transform from Definition 2.25. Then $\tilde{T}(\mu, \nu)$ is the measure on $\mathbb{P}(V \otimes W)$ characterized by $H(\tilde{T}(\mu, \nu)) = h_{K_\mu \otimes K_\nu}$.

We now show that the map $\tilde{T}$ has a direct natural characterization. For this, we first recall that the Segre embedding $\mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$, $[v] \otimes [w] \mapsto [v \otimes w]$, is an isomorphism onto its image, which allows to view $\mathbb{P}(V) \times \mathbb{P}(W)$ as a subspace of $\mathbb{P}(V \otimes W)$. Taking the pushforward gives

\[(3.4) \quad \overline{T} : \mathcal{M}(\mathbb{P}(V)) \times \mathcal{M}(\mathbb{P}(W)) \to \mathcal{M}(\mathbb{P}(V \otimes W))\]
which is a bilinear map.
Lemma 3.8. The map $\tilde{T} : M(\mathcal{P}(V)) \times M(\mathcal{P}(W)) \to M(\mathcal{P}(V \otimes W))$ equals the tensor product of measures composed with the pushforward of the Segre map.

Proof. Since the map $\tilde{T}$ is bilinear we can assume without loss of generality that $\mu \in M(\mathcal{P}(V))$ and $\nu \in M(\mathcal{P}(W))$ are probability measures; the general case then follows by homogeneity and linearity. In that case, by Proposition 2.29, $K_\mu = K(X)$ where $X \in S(V)$ is a random vector of law $\mu$ (recall that we identify a measure on $\mathbb{P}(V)$ with the corresponding even measure on the sphere). Similarly, $K_\nu = K(Y)$, where $Y \in S(W)$ is a random vector of law $\nu$, that we can assume to be independent of $X$. By definition, we have $K_\mu \otimes K_\nu = K(X \otimes Y)$. The law of $X \otimes Y$ is the pushforward by the Segre map of the tensor product of measures $\mu \otimes \nu \in \mathcal{M}_+(\mathbb{P}(V) \times \mathbb{P}(W))$, and this concludes the proof. \qed

One could take Lemma 3.8 as the definition of the tensor product of zonoids, as it may appear simpler. This simplicity however entirely relies on the fact that the tensor product on vectors (the Segre map) sends the product of spheres to the sphere. When later in Section 4, we will deal with multilinear maps that do not have this property, the point of view of random vectors is easier to handle. 

To investigate the continuity of $T$ and $\tilde{T}$, let us prove an inequality.

Lemma 3.9. For $K_1, K_2 \in Z(V)$ and $L \in Z(W)$ we have with $m := \dim W$

$$\|K_1 \otimes L - K_2 \otimes L\| \leq \tau_m \|L\| \|K_1 - K_2\|,$$

where $\tau_m$ is defined as in (2.10).

Proof. Let $u \in V \otimes W$ be such that $\|K_1 \otimes L - K_2 \otimes L\| = |h_{K_1 \otimes L}(u) - h_{K_2 \otimes L}(u)|$. Let $L = K(Y)$. From (3.2) we get

$$h_{K_1 \otimes L}(u) - h_{K_2 \otimes L}(u) = \mathbb{E}_Y (h_{K_1} - h_{K_2})(T_Y(u)),$$

hence,

$$|h_{K_1 \otimes L}(u) - h_{K_2 \otimes L}(u)| \leq \mathbb{E}_Y |(h_{K_1} - h_{K_2})(T_Y(u))| \leq \mathbb{E}_Y \|h_{K_1} - h_{K_2}\|_\infty \|T_Y(u)\| \leq \mathbb{E}_Y \|Y\| \cdot \|K_1 - K_2\| = \ell(L) \cdot \|K_1 - K_2\|$$

where we used (3.1) for the third inequality. Applying Corollary 2.14 completes the proof. \qed

The main continuity properties are summarized in the following result.

Theorem 3.10. Suppose that $\dim V, \dim W \geq 2$. Then, the tensor product map satisfies the following.

1. $T : Z(V) \times Z(W) \to Z(V \otimes W)$ is continuous. More specifically, for $K_1, K_2 \in Z(V)$ and $L_1, L_2 \in Z(W)$, we have $d_H (K_1 \otimes L_1, K_2 \otimes L_2) \leq (\tau_m \|L_1\| + \tau_n \|K_2\|) \left(d_H (K_1, K_2) + d_H (L_1, L_2)\right)$, where $n := \dim V$ and $m := \dim W$;

2. $\tilde{T} : \tilde{Z}(V) \times \tilde{Z}(W) \to \tilde{Z}(V \otimes W)$ with the norm topology on both sides is not sequentially continuous, but separately (i.e., componentwise) continuous;

3. $\tilde{T} : M(\mathbb{P}(V)) \times M(\mathbb{P}(W)) \to M(\mathbb{P}(V \otimes W))$ with the weak-* topology on both sides is sequentially continuous.
Proof. For proving (1), recall that \( d_H(K, L) = \|K - L\| \). From the multiadditivity of the tensor product and the triangle inequality of the norm, we get
\[
\|K_1 \otimes L_1 - K_2 \otimes L_2\| \leq \|K_1 \otimes L_1 - K_2 \otimes L_1\| + \|K_2 \otimes L_1 - K_2 \otimes L_2\|.
\]
Combined with Lemma 3.9, this yields
\[
\|K_1 \otimes L_1 - K_2 \otimes L_2\| \leq \tau_n\|L_1\||\|K_1 - K_2\| + \tau_n\|K_2\||\|L_1 - L_2\|,
\]
which proves the first assertion.

As for (2), the separate continuity follows directly from Lemma 3.9. To prove that \( \hat{T} \) is not (sequential) continuous, we begin with a general observation. Let \( \varphi : E \times F \to G \) be a bilinear map of real normed vector spaces. Then \( \varphi \) is (sequential) continuous if and only if it has finite operator norm:
\[
\|\varphi\|_{\text{op}} := \sup_{\|x\| \leq 1, \|y\| \leq 1} \|\varphi(x, y)\| < \infty.
\]
We show now that \( \hat{T} \) has infinite operator norm. It suffices to prove this for \( V = W = \mathbb{R}^2 \). Consider the sequence of vectors \( a_n := (n, 1) \), \( b_n := (n, 0) \) and the corresponding sequence of segments \( A_n := \frac{1}{2}[-a_n, a_n] \), \( B_n := \frac{1}{2}[-b_n, b_n] \) in \( \mathbb{R}^2 \). This defines the sequence of virtual zonoids \( A_n - B_n \in \hat{Z}(\mathbb{R}^2) \). It is immediate to check that
\[
\|A_n - B_n\| = d_H(A_n, B_n) = \frac{1}{2}.
\]
Consider \( P_n := (A_n - B_n) \otimes (A_n - B_n) \in \hat{Z}(\mathbb{R}^2 \otimes \mathbb{R}^2) \). It suffices to show that \( \lim_{n \to \infty} \|P_n\| = \infty \).

For this, we calculate
\[
\begin{aligned}
a_n \otimes a_n &= \begin{bmatrix} n^2 & n \\ n & 1 \end{bmatrix}, 
& b_n \otimes b_n = \begin{bmatrix} n^2 & 0 \\ 0 & 0 \end{bmatrix},
& a_n \otimes b_n = \begin{bmatrix} n^2 & 0 \\ n & 0 \end{bmatrix},
& b_n \otimes a_n = \begin{bmatrix} n^2 & n \\ 0 & 0 \end{bmatrix}.
\end{aligned}
\]
Their inner product with the matrix \( w_n := \begin{bmatrix} 1 & -n \\ -n & 0 \end{bmatrix} \) is given by
\[
\begin{aligned}
\langle a_n \otimes a_n, w_n \rangle &= -n^2,
& \langle b_n \otimes b_n, w_n \rangle = n^2,
& \langle a_n \otimes b_n, w_n \rangle = \langle b_n \otimes a_n, w_n \rangle = 0.
\end{aligned}
\]
Using (2.8), we obtain
\[
\begin{aligned}
h_{A_n \otimes A_n}(w_n) &= \frac{1}{2} \langle a_n \otimes a_n, w_n \rangle = \frac{n^2}{2},
& h_{B_n \otimes B_n}(w_n) = \frac{n^2}{2},
& h_{A_n \otimes B_n}(w_n) = h_{B_n \otimes A_n}(w_n) = 0.
\end{aligned}
\]
Therefore,
\[
\frac{h_{P_n}(w_n)}{\|w_n\|} = \frac{n^2}{\|w_n\|} = \Omega(n),
\]
which completes the proof of the second item.

For item (3) we recall from Lemma 3.8 that \( \hat{T} \) equals the tensor product of measures composed with the pushforward of the Segre map. The pushforward of a measure under a continuous map is weak–* continuous. Mapping two measures to their product measure is sequentially continuous by [Bil99, Theorem 2.8]. This finishes the proof for the third assertion. \( \square \)

By Proposition 2.24, the completion of the normed vector space \( \hat{Z}(V) \) is isomorphic to the Banach space of even real valued continuous function defined on the unit sphere of \( V \), or equivalently to \( C(\mathbb{P}(V)) \). One may hope to extend the tensor product map to the completions to obtain a separate continuous bilinear map. We show now that, unfortunately, this is impossible. For proving this, we will rely on Theorem 5.1 below and on [Sch14, Theorem 5.2.2].

**Proposition 3.11.** There is no bilinear map \( C(\mathbb{P}(V)) \times C(\mathbb{P}(W)) \to C(\mathbb{P}(V \otimes W)) \) that is separate continuous and extends the tensor product map \( \hat{T} : \hat{Z}(V) \times \hat{Z}(W) \to \hat{Z}(V \otimes W) \), provided \( \dim V, \dim W \geq 2 \).
Proof. Suppose by way of contradiction that there is a bilinear map as in the proposition. From this it is straightforward to construct such map \( \varphi: C(\mathbb{P}^1) \times C(\mathbb{P}^1) \to C(\mathbb{P}^3) \) for \( V = W = \mathbb{R}^2 \). Thus it is enough to prove it for this case. The determinant map \( \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) is a bilinear map and hence it factors via a linear map \( \hat{M}: \mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{R} \). By Theorem 5.1 below, the associated linear map \( \hat{M}: \hat{Z}(\mathbb{R}^2) \to \hat{Z}(\mathbb{R}) \simeq \mathbb{R} \) gives the mixed volume of the tensor product in the sense that, for \( K_1, K_2 \in \mathcal{Z}(\mathbb{R}^2) \),

\[
\hat{M}(K_1 \otimes K_2) = \ell(K_1 \wedge K_2) = 2 \text{MV}(K_1, K_2).
\]

Since \( \hat{M} \) is continuous (Proposition 2.21), we can extend it to a continuous linear map \( \lambda: C(\mathbb{P}(\mathbb{R}^2 \otimes \mathbb{R}^2)) \to C(\mathbb{P}^0) \simeq \mathbb{R} \) by Hahn-Banach. Then, \( \lambda \circ \varphi: C(\mathbb{P}^1) \times C(\mathbb{P}^1) \to \mathbb{R} \) is a componentwise continuous bilinear map extending \( 2 \cdot \det: \mathcal{Z}(\mathbb{R}^2) \times \mathcal{Z}(\mathbb{R}^2) \to \mathbb{R} \). Now recall that \( C(\mathbb{P}^1) \) is identified with the even functions in \( C(S^1) \).

The existence of \( \lambda \circ \varphi \) contradicts the fact that there is no separate continuous bilinear function \( V: C(S^1) \times C(S^1) \to \mathbb{R} \) that satisfies \( V(h_K, h_L) = \text{MV}(K, L) \) for \( K, L \in \mathcal{Z}(\mathbb{R}^2) \). The latter follows by inspecting the proof of [Sch14, Theorem 5.2.2], where the analogous statement is made about bilinear functions \( V \) satisfying the above condition for all \( K, L \in \mathcal{K}(\mathbb{R}^2) \). \( \square \)

4. Multilinear maps induced on zonoids

In this section we show how to associate with any multilinear map between Euclidean spaces a corresponding multilinear map of the corresponding vector spaces of zonoids. This will allow us to construct the zonoid algebra. The zonoid algebra inherits a duality notion from the Hodge star operator.

4.1. Zonoids and multilinear maps. We learned how to associate with a linear map \( M: V \to W \) of Euclidean spaces a continuous, order preserving linear map \( \hat{M}: \hat{Z}(V) \to \hat{Z}(W) \) of the corresponding vector spaces of zonoids (Definition 2.20). We now transfer this construction to multilinear maps via the tensor product. Note that if \( V_1, \ldots, V_p \) are Euclidean vector spaces, then each \( \hat{Z}(V_i) \) is an partially ordered and normed real vector space. The product space \( \hat{Z}(V_1) \times \cdots \times \hat{Z}(V_p) \) carries a componentwise order and the product topology, where \( \hat{Z}(V_i) \) carries the topology defined by the norm of virtual zonoids from Theorem 2.16.

**Theorem 4.1** (Induced multilinear zonoid maps). Let \( V_1, \ldots, V_p \) and \( W \) be Euclidean vector spaces and \( M: V_1 \times \cdots \times V_p \to W \) be a multilinear map. There exists a unique multilinear, separate continuous map

\[
\hat{M}: \hat{Z}(V_1) \times \cdots \times \hat{Z}(V_p) \to \hat{Z}(W),
\]

such that for every \( v_1 \in V_1, \ldots, v_p \in V_p \)

\[
\hat{M}(\frac{1}{2}[-v_1, v_1], \ldots, \frac{1}{2}[-v_p, v_p]) = \frac{1}{2}[-M(v_1, \ldots, v_p), M(v_1, \ldots, v_p)].
\]

Restricting to zonoids, we get a continuous map \( \mathcal{Z}(V_1) \times \cdots \times \mathcal{Z}(V_p) \to \mathcal{Z}(W) \). The map \( \hat{M} \) preserves the componentwise inclusion order of zonoids. If we interpret this map on the level of signed measures, then we obtain a multilinear map \( \hat{M} \), which is sequentially continuous with respect to the weak-* topology.

**Proof.** To show existence, we rely on the universal property of tensor product: there is a unique linear map \( L: V_1 \otimes \cdots \otimes V_p \to W \) such that \( L(v_1 \otimes \cdots \otimes v_p) = M(v_1, \ldots, v_p) \). Consider the linear continuous map \( \hat{L}: \hat{Z}(V_1 \otimes \cdots \otimes V_p) \to \hat{Z}(W) \) given by Definition 2.20. For \( (K_1, \ldots, K_p) \in \hat{Z}(V_1) \times \cdots \times \hat{Z}(V_p) \), we define the map \( \hat{M} \) by

\[
(4.1) \quad \hat{M}(K_1, \ldots, K_p) := \hat{L}(K_1 \otimes \cdots \otimes K_p).
\]
This is the composition of the linear map \( \tilde{L} \) with the multilinear tensor product map from Proposition 3.7, therefore it is multilinear.

Restricting to zonoids and using Proposition 3.4, we see \( \hat{M}(\mathcal{Z}(V_1) \times \cdots \times \mathcal{Z}(V_p)) \subseteq \mathcal{Z}(W) \). The asserted formula for the image of \( \hat{M} \) on tuples of segments is a direct consequence of (3.3) and the definition of the map \( \hat{M} \).

Since \( \tilde{L} \) is continuous and the tensor product map from Proposition 3.7 is separate continuous, \( \hat{M} \) is separate continuous. Similarly, \( \hat{M}|_{\mathcal{Z}(V_1) \times \cdots \times \mathcal{Z}(V_p)} \) is continuous since the tensor product map on zonoids is continuous (Proposition 3.4).

For the uniqueness of the map \( \hat{M} \) we argue as follows. Since by definition, any zonoid in \( V_i \) can be approximated by symmetric segments and the values of \( \hat{M} \) are determined on tuples of segments, the componentwise continuity of \( \hat{M} \) determines \( \hat{M} \) on \( \mathcal{Z}(V_1) \times \cdots \times \mathcal{Z}(V_p) \). In turn, this determines \( \hat{M} \) by multilinearity.

By Proposition 3.4, the tensor product map of zonoids preserves the componentwise order. Moreover, \( \tilde{L} \) preserves the order by Proposition 2.21. This implies that \( \hat{M} \) preserves the componentwise order.

The last statement about the sequential continuity with respect to the weak–\( \ast \) topology follows from Lemma 2.31 and Theorem 3.10 (3). \( \square \)

Remark 4.2. The map \( \hat{M} : \widehat{\mathcal{Z}}(V_1) \times \cdots \times \widehat{\mathcal{Z}}(V_p) \rightarrow \widehat{\mathcal{Z}}(W) \) is in general not sequentially continuous for the norm topology. Indeed Theorem 3.10 shows that the tensor product map for two factors is not (sequentially) continuous, and this immediately extends to any number of factors. By contrast, we showed that \( \hat{M} \) is sequentially continuous for the weak–\( \ast \) topology.

Our construction is nicely compatible with the description of zonoids by random vectors.

Corollary 4.3. Let \( X_1 \in V_1, \ldots, X_p \in V_p \) be integrable and independent random vectors. For a multilinear map \( M : V_1 \times \cdots \times V_p \rightarrow W \) we have

\[
\hat{M}(K(X_1), \ldots, K(X_p)) = K(M(X_1, \ldots, X_p)).
\]

Proof. This follows from the construction (4.1) of the map \( \hat{M} \) and Proposition 2.4. \( \square \)

A recent work that can be interpreted from the point of view of multilinear functions of zonoids is the paper [MM21] by Meroni and Mathis, in which they study so-called fiber bodies.

4.2. Zonoid algebra. We assign to the exterior powers of a Euclidean space the corresponding zonoid vector spaces and apply Theorem 4.1 to the wedge product of the exterior algebra to arrive at a commutative and associative graded algebra, which we call the zonoid algebra of \( V \).

Let \( V \) denote a Euclidean vector space of dimension \( m \) throughout this section. Consider the \( d \)-th exterior power \( \Lambda^d V \) of \( V \), defined for \( d \in \mathbb{N} \), and note that \( \Lambda^d V = 0 \) if \( d > m \). This space inherits a Euclidean structure from \( V \), which can be described as follows: if \( \{e_1, \ldots, e_m\} \) is an orthonormal basis for \( V \), an orthonormal basis for \( \Lambda^d V \) is given by \( \{e_{i_1} \wedge \cdots \wedge e_{i_d}\}_{1 \leq i_1 < \cdots < i_d \leq m} \).

We form the direct sum of zonoid vector spaces

\[
\mathcal{A}(V) := \bigoplus_{d=0}^{m} \widehat{\mathcal{Z}}(\Lambda^d V).
\]

The elements of \( \widehat{\mathcal{Z}}(\Lambda^k V) \) will be said to have degree \( d \). By Lemma 2.23 we shall identify \( \widehat{\mathcal{Z}}(\Lambda^0 V) = \widehat{\mathcal{Z}}(\mathbb{R}) \cong \mathbb{R} \).
Remark 4.4. Proposition 2.24 implies that $\mathcal{A}(V)$ is an infinite dimensional real vector space, unless $\dim(V) \leq 1$.

Via Theorem 4.1, we associate with each wedge product $\wedge: \Lambda^d V \times \Lambda^e V \to \Lambda^{d+e} V$ a componentwise continuous bilinear map

$$\wedge: \hat{\mathcal{Z}}(\Lambda^d V) \times \hat{\mathcal{Z}}(\Lambda^e V) \to \hat{\mathcal{Z}}(\Lambda^{d+e} V),$$

and, by extension, a componentwise continuous bilinear map $\mathcal{A}(V) \times \mathcal{A}(V) \to \mathcal{A}(V)$, all denoted by the same symbol. We call this map the wedge product of virtual zonoids. Theorem 4.1 implies that the wedge product of zonoids is a zonoid. We write $K^\wedge d$ for the wedge of $K$ with itself $d$ many times.

In an analogous way, we define the wedge product of several (virtual) zonoids. We can describe this product explicitly as follows. Assume that $X_1 \in \Lambda^d V, \ldots, X_p \in \Lambda^e V$ are independent random vectors representing the zonoids $A_j \in \mathcal{Z}(\Lambda^d V)$. Then Corollary 4.3 implies that

$$A_1 \wedge \cdots \wedge A_p = K(X_1) \wedge \cdots \wedge K(X_p) = K(X_1 \wedge \cdots \wedge X_p) \in \mathcal{Z}(\Lambda^{d+\cdots+d_p} V).$$

We call $\mathcal{A}(V)$ the zonoid algebra associated with $V$. This naming is justified by the following theorem.

**Theorem 4.5.** The wedge product turns $\mathcal{A}(V)$ into a graded, associative and commutative real algebra. The wedge maps of zonoids

$$\mathcal{Z}(\Lambda^d V) \times \cdots \times \mathcal{Z}(\Lambda^e V) \to \mathcal{Z}(\Lambda^{d+\cdots+d_p} V), \ (A_1, \ldots, A_p) \mapsto A_1 \wedge \cdots \wedge A_p$$

are continuous. These maps preserve the inclusion order of zonoids: if we have $A_j' \subset A_j$ for zonoids in $\mathcal{Z}(\Lambda^d V)$, then

$$A_1' \wedge \cdots \wedge A_p' \subset A_1 \wedge \cdots \wedge A_p.$$

Moreover, the wedge product of zonoids does not increase the length:

$$\ell(A_1 \wedge \cdots \wedge A_p) \leq \ell(A_1) \cdots \ell(A_p).$$

**Proof.** The associativity follows from the associativity of the wedge product and (4.2). The distributivity is a consequence of Proposition 3.7. The gradedness follows from the definition of $\mathcal{A}(V)$. The multiplicative unit lies in $\hat{\mathcal{Z}}(\Lambda^0 V) = \hat{\mathcal{Z}}(\mathbb{R}) \sim \mathbb{R}$. The commutativity of the wedge follows with (4.2) from the known relation $X \wedge Y = \pm Y \wedge X$ and the fact that $K(-Z) = K(Z)$.

The wedge map of zonoids is continuous since it is obtained by composing a continuous linear map with the continuous tensor product of zonoids (see Theorem 3.10). The preservation of the inclusion order follows from Theorem 4.1.

For the length inequality, we use that the antisymmetrization map $\otimes_j \Lambda^d V \to \Lambda^{d_1+\cdots+d_p} V$ is an orthogonal projection. Hence $\ell(A_1 \wedge \cdots \wedge A_p) \leq \ell(A_1 \otimes \cdots \otimes A_p) = \ell(A_1) \cdots \ell(A_p)$ by Proposition 2.21 and Proposition 3.6. \hfill $\square$

Here is an immediate yet important observation about wedge products of zonoids.

**Lemma 4.6.** Let $K \in \mathcal{Z}(V)$ be a zonoid. Recall that we defined the dimension of $K$ as the dimension of its linear span $\langle K \rangle$. Then, $K^\wedge d = 0$ for all $d > \dim(K)$.

**Proof.** Let us write $K = K(X)$. By (4.2) we have $K^\wedge d = K(X_1 \wedge \cdots \wedge X_d)$, where $X_1, \ldots, X_d$ are independent copies of $X$. With probability one we have $X \in \langle K \rangle$, and so with probability one the $X_i$ are linearly dependent. Hence, $X_1 \wedge \cdots \wedge X_d = 0$ almost surely, so that $K^\wedge d = 0$. \hfill $\square$
In the next section, we will link the length to the mixed and intrinsic volumes. More specifically, Theorem 5.2 show that the jth intrinsic volume $V_j(K)$ can be expressed as \( \frac{1}{j!} \ell(K^\wedge j) \). This allows to immediately derive Corollary 4.7, which is a reverse Alexandrov-Fenchel inequality, that was independently found very recently by Böröczky and Hug [BH21].

It is useful to denote by $K[d]$ the zonoid $K$ repeated $d$ times.

**Corollary 4.7.** Let $K_1, \ldots, K_p \in \mathcal{Z}(V)$ be zonoids and $\langle K_1 \rangle, \ldots, \langle K_p \rangle$ their spans. Assume $d_1, \ldots, d_p \in \mathbb{N}$ satisfy $d_1 + \ldots + d_p = m$. Then

\[
\frac{m!}{d_1! \cdots d_p!} \text{MV}(K_1[d_1], \ldots, K_p[d_p]) \leq V_{d_1}(K_1) \cdots V_{d_p}(K_p).
\]

Equality holds if and only if $\langle K_1 \rangle, \ldots, \langle K_p \rangle$ are pairwise orthogonal or if $K_i \wedge d_i = 0$ for at least one $i$.

By Lemma 4.6 the condition that $K^\wedge d = 0$ is equivalent to either $K = 0$ or $d > \dim(K)$.

**Proof of Corollary 4.7.** By Theorem 5.2, the stated inequality can be rephrased as

\[
\ell(K_1^{\wedge d_1} \wedge \cdots \wedge K_p^{\wedge d_p}) \leq \ell(K_1^{\wedge d_1}) \cdots \ell(K_p^{\wedge d_p}),
\]

which is a consequence of the submultiplicativity of the length, see Theorem 4.5.

For analyzing when equality holds, we assume $p = 2$ to simplify notation. We write $K_1 = K(X)$ and $K_2 = K(Y)$. Let $X_1, \ldots, X_{d_1}$ be independent copies of $X$ and $Y_1, \ldots, Y_{d_2}$ be independent copies of $Y$. The above inequality (4.3) can be written as an inequality of expectations $\mathbb{E}\|X_1 \wedge \cdots \wedge X_{d_1} \wedge Y_1 \wedge \cdots \wedge Y_{d_2}\| \leq \mathbb{E}\|X_1 \wedge \cdots \wedge X_{d_1}\| \cdot \mathbb{E}\|Y_1 \wedge \cdots \wedge Y_{d_2}\|$, and equality holds if and only if $\|X_1 \wedge \cdots \wedge X_{d_1} \wedge Y_1 \wedge \cdots \wedge Y_{d_2}\| = \|X_1 \wedge \cdots \wedge X_{d_1}\| \cdot \|Y_1 \wedge \cdots \wedge Y_{d_2}\|$ almost surely. We have equality if and only if $\langle X_i, Y_j \rangle = 0$ almost surely for all $i, j$, or if one of the wedge products is almost surely zero. By Lemma 2.15, the first case is equivalent to $\langle K_1 \rangle$ and $\langle K_2 \rangle$ being orthogonal, the second case means that $K_1 \wedge d_1 = 0$ or $K_2 \wedge d_2 = 0$. \( \square \)

### 4.3. Grassmannian zonoids

Here we describe a particular subalgebra of $\mathcal{A}(V)$, which will play a role in our forthcoming work [BBML21].

**Definition 4.8.** Let $K \in \mathcal{Z}(\Lambda^k V)$. We say that $K$ is a Grassmannian zonoid if there is a random vector $X \in \Lambda^k V$ such that $K = K(X)$ and such that $X$ is almost surely simple, i.e. $X$ almost surely takes values in the cone of simple vectors $\{v_1 \wedge \cdots \wedge v_k \mid v_1, \ldots, v_k \in V\}$. The set of Grassmannian zonoids will be denoted $\mathcal{G}(k, V) \subset \mathcal{Z}(\Lambda^k V)$ and its linear span in $\mathcal{Z}(\Lambda^k V)$ will be denoted by $\hat{\mathcal{G}}(k, V)$

In the correspondence between zonoids and measures on the projective space described in Section 2.4, Grassmannian zonoids have a simple description. Indeed using the definition above and Proposition 2.29 we see that a zonoid $K \in \mathcal{Z}(\Lambda^k V)$ is Grassmannian if and only if its corresponding measure is supported on the Grassmannian, considered as the simple vectors in $\mathbb{P}(\Lambda^k V)$ via the Plücker embedding $\text{span}(v_1, \ldots, v_k) \mapsto [v_1 \wedge \cdots \wedge v_k]$. Hence the name Grassmannian zonoids.

**Proposition 4.9.** The wedge product of two Grassmannian zonoids is a Grassmannian zonoid. Moreover, if they are of the same degree, the sum of two Grassmannian zonoids is a Grassmannian zonoid. Hence $\hat{\mathcal{G}}(k, V) \subset \mathcal{Z}(\Lambda^k V)$ consists only of differences of Grassmannian zonoids and $\mathcal{G}(k, V)$ is a convex cone in $\hat{\mathcal{G}}(k, V)$. 
Proposition 4.11. Finite sums of zonoids of the form \( K_1 \wedge \cdots \wedge K_k \in \mathcal{G}(k,V) \) are dense in \( \mathcal{G}(k,V) \). Hence the set \( \{ K_1 \wedge \cdots \wedge K_k \mid K_1, \ldots, K_k \in Z(V) \} \) spans a dense subspace in the virtual Grassmann zonoids \( \mathcal{G}(k,V) \).

Proof. By Remark 2.5, any zonoid in \( \mathcal{G}(k,V) \) is the limit of finite sums of segments that are of the form \( \frac{1}{2}[-w,w] \) with \( w = x_1 \wedge \cdots \wedge x_k \). It is then enough to see that such segments are decomposable. Indeed we have \( \frac{1}{2}[-w,w] = \frac{1}{2}[-x_1,x_1] \wedge \cdots \wedge \frac{1}{2}[-x_k, x_k] \). \( \square \)

This last fact gives good hope to extend properties of decomposable zonoids to the Grassmannian ones. For example we conjecture the following.

Conjecture. For any \( K, L \in Z(\mathbb{R}^m) \) and any \( C \in \mathcal{G}(m-2,\mathbb{R}^m) \), we have

\[
\ell(K \wedge L \wedge C)^2 \geq \ell(K \wedge K \wedge C) \ell(L \wedge L \wedge C).
\]

This would generalize Alexandrov–Fenchel, which corresponds to the case where \( C \) is decomposable; i.e., of the form \( C = K_1 \wedge \cdots \wedge K_{m-2} \) with \( K_1, \ldots, K_{m-2} \in Z(\mathbb{R}^m) \), see (5.2) below.

4.4. Hodge duality. The exterior algebra of an oriented Euclidean vector space \( V \) comes with the duality given by the Hodge star operation. Let us briefly recall this notion. Upon choosing an orientation of \( V \), \( \Lambda^m V \) becomes an oriented one dimensional Euclidean vector space that can be identified with \( \mathbb{R} \). More specifically, if \( e_1, \ldots, e_m \) is an oriented orthonormal basis of \( V \), then \( e_1 \wedge \cdots \wedge e_m \) is the distinguished generator of \( \Lambda^m V \). When \( d_1 + \cdots + d_p = m \), the wedge product thus induces a multilinear map

\[
\hat{\Lambda}^p(\Lambda^d V) \times \cdots \times \hat{\Lambda}^p(\Lambda^d V) \to \hat{\Lambda}^p(\Lambda^m V) \cong \mathbb{R}.
\]

Therefore the wedge product \( K_1 \wedge \cdots \wedge K_p \) of zonoids, which is a segment in \( \Lambda^m V \), can be identified with the real number giving the length of this segment using Lemma 2.23. This remark will play a role in Section 5.1.

The Hodge star operation is the isometric linear map \( \Lambda^d V \to \Lambda^{m-d} V, v \mapsto *v \) characterized by \( \langle u, v \rangle = u \wedge *v \) for all \( u \in V \). This defines an involution (up to sign) of the exterior algebra \( \Lambda V \), see [Fla89, p.16]. Via Definition 2.20 we associate with the Hodge star operation the linear isomorphism

\[
\hat{\Lambda}(\Lambda^d V) \to \hat{\Lambda}(\Lambda^{m-d} V), \ K \mapsto *K
\]
that we conveniently denote with the symbol $\star$ as well. If $X \in \Lambda^k V$ is an integrable random variable, then by Proposition 2.4
\[ \star(K(X)) = K(\star X). \]
This shows that, if $K \in Z(\Lambda^k V)$ is a zonoid, then $\star K$ is a zonoid as well. The support function of $\star K$ satisfies
\[ h_{\star K}(\star u) = \frac{1}{2} \mathbb{E}|\langle u, \star X \rangle| = \frac{1}{2} \mathbb{E}|\langle u, X \rangle| = h_K(u), \]
where we used that $\star$ is isometric for the second equality.

**Proposition 4.12.** The Hodge star operation on zonoids $\tilde{Z}(\Lambda^d V) \rightarrow \tilde{Z}(\Lambda^{d-d} V)$ is a norm and order preserving linear isomorphism. It also preserves the length of zonoids, so that for all zonoids $K \in Z(\Lambda^d V)$
\[ \|K\| = \|\star K\|, \quad \ell(K) = \ell(\star K). \]
The Hodge star operation defines a linear involution of the zonoid algebra $\mathcal{A}(V)$.

**Proof.** Using (4.4) and Proposition 2.1, we obtain
\[ \|\star K\| = \|\tilde{h}_{\star K}\|_{\infty} = \|\tilde{h}_K\|_{\infty} = \|K\|. \]
Moreover, using Definition 2.10, we get
\[ \ell(K) = \mathbb{E}\|X\| = \mathbb{E}\|\star X\| = \ell(\star K). \]
The fact that it is an involution on the space of zonoids follows from the fact that the Hodge star is an involution up to sign and that the zonoids are centrally symmetric. □

**Remark 4.13.** $\star K$ should not be confused with the polar dual of $K \in Z(\Lambda^d V)$, which is a convex body living in the same space as $K$, and in general not a zonoid, see [Bol69].

**Remark 4.14.** Note that since the Hodge star operation preserves simple vectors, if $K \in G(k, V)$ is a Grassmannian zonoid then $\star K \in G(m - k, V)$ is also Grassmannian. In other words, the map $\star$ preserves the subalgebra $G(V)$ of Grassmannian zonoids.

Let $K \in Z(V)$. The Hodge dual of $K^{\wedge(m-1)}$ is a zonoid in $\Lambda^1 V = V$. We show now that it is the so called projection body $\Pi K$ of $K$. According to [Sch14, Section 10.9], $\Pi K$ is the convex body whose support function is given by $u \mapsto \|u\| \text{vol}_{m-1}(\pi_u(K))$, where $\pi_u$ denotes the orthogonal projection $V \rightarrow u^\perp$.

**Proposition 4.15.** We have $\star(K^{\wedge(m-1)}) = \frac{(m-1)!}{2} \Pi K$.

**Proof.** We show that these zonoids have the same support function. Let $X \in V$ be a random vector representing $K = K(X)$. By definition,
\[ \star(K^{\wedge(m-1)}) = K(Y), \quad \text{where} \quad Y := \star(X_1 \wedge \cdots \wedge X_{m-1}) \]
and $X_1, \ldots, X_{m-1}$ are i.i.d. copies of $X$. The definition of the Hodge dual yields
\[ |\langle \star(X_1 \wedge \cdots \wedge X_{m-1}), u \rangle| = |X_1 \wedge \cdots \wedge X_{m-1} \wedge u| \]
and so
\[ |\langle Y, u \rangle| = |X_1 \wedge \cdots \wedge X_{m-1} \wedge u| = \|\pi_u(X_1) \wedge \cdots \wedge \pi_u(X_{m-1})\| \cdot \|u\|. \]
By (2.8), we get
\[ h_K(Y)(u) = \frac{1}{2} \mathbb{E}|\langle Y, u \rangle| = \frac{1}{2} \mathbb{E}\|\pi_u(X_1) \wedge \cdots \wedge \pi_u(X_{m-1})\| \cdot \|u\|. \]
Theorem 5.1 from the next section, applied to the space $u^\perp \simeq \mathbb{R}^{m-1}$, yields
\[ h_K(Y)(u) = \frac{(m-1)!}{2} \text{vol}_{m-1}(\pi_u(K)) \cdot \|u\|. \]
This shows that \( \ast(K^{\wedge(m-1)}) \) has the same support function as \( \frac{(m-1)!}{2} \Pi K \). \( \square \)

We close this section by giving an interesting property of Hodge duals. It concerns orthogonal zonoids and will be of relevance in our upcoming work [BBML21].

**Corollary 4.16.** Let \( K, L \subset \Lambda^d V \) be zonoids and denote by \( \langle K \rangle \) and \( \langle L \rangle \) their linear spans. Then

\[
K \wedge \ast L = 0 \iff \langle K \rangle \perp \langle L \rangle.
\]

**Proof.** Let \( X \) and \( Y \) be integrable random vectors with \( K = K(X) \) and \( L = K(Y) \). Then \( K \wedge \ast L = K(X \wedge \ast Y) \). By the definition of the Hodge dual, we have \( X \wedge \ast Y = \langle X,Y \rangle \). Therefore, \( K \wedge \ast L = 0 \) if and only if \( \langle X,Y \rangle = 0 \) almost surely. By Lemma 2.15, this is equivalent to \( \langle K \rangle \perp \langle L \rangle \). \( \square \)

5. **Mixed volumes and random determinants**

In this section we show how the classical notion of mixed volume fits into our framework. We then apply our theory to study expected absolute determinants of random matrices. Again we denote by \( V \) a Euclidean vector space of dimension \( m \).

We recall that the mixed volume is the Minkowski multilinear, translation invariant and continuous map \( \text{MV}: K(V)^m \to \mathbb{R} \), defined on an \( m \)-tuple \( K_1 \ldots, K_m \) of convex bodies by

\[
\text{MV}(K_1, \ldots, K_m) = \frac{1}{m!} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m} \text{vol}_m(t_1 K_1 + \cdots + t_m K_m) \bigg|_{t_1=\cdots=t_m=0},
\]

see [Sch14, Theorem 5.1.7]. For instance, if \( K_i = \frac{1}{2}[-v_i, v_i] \) are segments, then

\[
\text{MV}(\frac{1}{2}[-v_1, v_1], \ldots, \frac{1}{2}[-v_m, v_m])
\]
equals the volume of the parallelootope spanned by \( v_1, \ldots, v_m \), divided by \( m! \). The volume of \( K \) is \( \text{vol}_m(K) = \text{MV}(K, \ldots, K) \), which means that the mixed volume is obtained from the volume function of a single body by polarization.

It is a key insight that the mixed volume of zonoids equals the length of their wedge product, up to a constant factor.

**Theorem 5.1** (The wedge product of zonoids). Let \( \hat{\det}: \hat{Z}(V)^m \to \hat{Z}^1 \) denote the multilinear map associated to the multilinear determinant map \( \det: V^m \to \mathbb{R} \) via Theorem 4.1. Then, for every \( K_1, \ldots, K_m \in \mathcal{Z}(V) \), we have

\[
\hat{\det}(K_1, \ldots, K_m) = K_1 \wedge \cdots \wedge K_m
\]

and

\[
\text{MV}(K_1, \ldots, K_m) = \frac{1}{m!} \ell(K_1 \wedge \cdots \wedge K_m).
\]

**Proof.** The first identity is immediate from the definition of associated multilinear map \( \hat{\det} \).

For the second identity, we observe that both sides are Minkowski multilinear and continuous maps \( \hat{Z}(V)^m \to \mathbb{R} \) (for the right hand side, use Theorem 2.22). It therefore suffices to verify the identity for segments. So let \( v_1, \ldots, v_m \in V \). Note that by Theorem 4.1,

\[
L := \frac{1}{2}[-v_1, v_1] \wedge \cdots \wedge \frac{1}{2}[-v_m, v_m] = \frac{1}{2}[-\det(v_1, \ldots, v_m), \det(v_1, \ldots, v_m)],
\]

This is an interval in \( \Lambda^m V \simeq \mathbb{R} \) and hence \( \ell(L) \) equals the volume of the parallelootope spanned by \( v_1, \ldots, v_m \). On the other hand, this equals \( \text{MV}(\frac{1}{2}[-v_1, v_1], \ldots, \frac{1}{2}[-v_m, v_m]) \), divided by \( m! \). This verifies the identity for segments and completes the proof. \( \square \)
The previous theorem implies that for $K \in \mathcal{Z}(V)$

\begin{equation}
\vol_m(K) = \frac{1}{m!} \ell(K^\wedge m),
\end{equation}

where, again, $K^\wedge m = K \wedge \cdots \wedge K$ with $m$ factors.

5.1. **Length functional and intrinsic volumes.** Recall [KR97, Sch14] that the $d$-th intrinsic volume $V_d(K)$ of a zonoid $K$ is defined as

$$V_d(K) := \frac{\binom{m}{d}}{\vol_{m-d}(B_{m-d})} \MV(K[d], B_m[m - d]).$$

In the following, we show that the length of a zonoid, introduced in Definition 2.10, is nothing but its first intrinsic volume (see [KR97, Sch14]), and that the higher intrinsic volumes can also be expressed using the length. As before, we denote by $B = B(V)$ the unit ball in $V$ and $B^m := B(\mathbb{R}^m)$.

**Theorem 5.2.** The $d$th intrinsic volume of a zonoid $K \in \mathcal{Z}(V)$ is given by

$$V_d(K) = \frac{1}{d!} \ell(K^{\wedge d}),$$

where, as before, $K^{\wedge d}$ is the wedge product of $d$ copies of $K$. In particular,

$$V_1(K) = \ell(K).$$

**Proof.** Suppose $X_1, \ldots, X_d, Y_1, \ldots, Y_{m-d}$ are independent random vectors with values in $V$ such that the $X_i$ represent $K$ and the $Y_j$ are standard Gaussian. Recall that $B_m = \sqrt{2\pi} K(Y_j)$. By Theorem 5.1, we can write

$$\MV(K[d], B_m[m - d]) = \frac{(2\pi)^{m-d}}{m!} \mathbb{E}|X_1 \wedge \cdots \wedge X_d \wedge Y_1 \wedge \cdots \wedge Y_{m-d}|.$$

We first integrate over the $Y_j$ while leaving the $X_i$ fixed, thus

$$\MV(K[d], B_m[m - d]) = \frac{(2\pi)^{m-d}}{m!} \mathbb{E} \mathbb{E}_{X_i} |X_1 \wedge \cdots \wedge X_d \wedge Y_1 \wedge \cdots \wedge Y_{m-d}|.$$

By orthogonal invariance of $Y := Y_1 \wedge \cdots \wedge Y_{m-d}$, we can assume that in the inner expectation the space spanned by the $X_i$ is the span of a fixed orthonormal frame $e_1, \ldots, e_d$. Then, $X_1 \wedge \cdots \wedge X_d = \|X_1 \wedge \cdots \wedge X_d\| e_1 \wedge \cdots \wedge e_d$ and so, using that $Y$ is independent of the $X_i$:

$$\MV(K[d], B_m[m - d]) = \frac{c}{m!} \mathbb{E}_{X_i} \|X_1 \wedge \cdots \wedge X_d\| = \frac{c}{m!} \ell(K),$$

with the constant $c := (2\pi)^{m-d} \mathbb{E}_Y \|e_1 \wedge \cdots \wedge e_d \wedge Y\|$. In order to determine this constant, we use that $\|e_1 \wedge \cdots \wedge e_d \wedge Y\| = \|\tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_d\|$, where $\tilde{Y}_j$ denotes the orthogonal projection of $Y_j$ onto the orthogonal complement $\mathbb{R}^{m-d}$ of $\mathbb{R}^d = \text{span}\{e_1, \ldots, e_d\}$. Since the unit ball $B_{m-d}$ is represented by $\sqrt{2\pi} \tilde{Y}_j$, we obtain with (5.1),

$$(2\pi)^{m-d} \mathbb{E}|\tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_{m-d}| = \ell(B^{\wedge(m-d)}) = (m - d)! \vol_{m-d}(B_{m-d}).$$

We therefore conclude that

$$\MV(K[d], B_m[m - d]) = \frac{1}{m!} \vol_{m-d}(B_{m-d}) \ell(K^{\wedge d}),$$

which finishes the proof. \qed

An inspection of the proof of Theorem 5.2 reveals the following general insight.
Corollary 5.3. Let $L$ be a zonoid in $\Lambda^e V$, which is invariant under the action of the orthogonal group $O(V)$ (for instance, $L = B^{\Lambda^e}$). Further, let $d + e \leq m$. Then there exists a constant $c_{d,m}(L)$, only depending on $d$, $m$ and $L$, such that

$$\ell(K \wedge L) = c_{d,m}(L) \ell(K)$$

for any zonoid $K = K_1 \wedge \cdots \wedge K_d$ with $K_1, \ldots, K_d \in \mathcal{Z}(V)$.

The Alexandrov–Fenchel inequality for convex bodies is one of deepest results of the Brunn-Minkowski theory. Via Theorem 5.1, we can express this inequality for zonoids in terms of the length as follows: if $K_1, K_2, \ldots, K_m \in \mathcal{Z}(V)$ are zonoids in $V$, then

$$\ell(K_1 \wedge K_2 \wedge \cdots \wedge K_m)^2 \geq \ell(K_1 \wedge K_2) \cdot \ell(K_2 \wedge K_3 \wedge \cdots \wedge K_m)$$

where $K = K_1 \wedge \cdots \wedge K_m$; see, e.g., [Sch14, Theorems 6.3.1]. More generally, the general Brunn-Minkowski theorem [Sch14, Theorem 6.4.3] implies that for any $1 \leq d \leq m$:

$$t \mapsto \ell(K_t^d \wedge K_{d+1} \wedge \cdots \wedge K_m)^2$$

is concave for $t \in [0,1]$, where $K_t := tK_1 + (1-t)K_2$.

Using Corollary 5.3, we deduce from (5.2) the following special case:

$$\ell(K \wedge L)^2 \geq \ell(K \wedge K) \cdot \ell(L \wedge L)$$

for all zonoids $K, L \in \mathcal{Z}(V)$. Moreover, (5.3) means that

$$t \mapsto \ell((tK + (1-t)L)^{\wedge d})^2$$

is concave for $t \in [0,1]$. By Corollary 5.3, we can generalize (5.4) by replacing $K \wedge L$ with the wedge product of $K \wedge L$ with any orthogonally invariant zonoid in $M \in \mathcal{Z}(\Lambda^k(V))$ such that $k + d \leq m$, and get

$$\ell(K \wedge L \wedge M)^2 \geq \ell(K \wedge K \wedge M) \cdot \ell(L \wedge L \wedge M).$$

Similarly, in (5.5) we may also take the product of the $d$-th wedge power with $M$ and obtain that the function $\ell((tK + (1-t)L)^{\wedge d} \wedge M)^2$ is concave for $t \in [0,1]$.

5.2. Random determinants. The purpose of this section is to generalize a result due to Vitale. In [Vit91] Vitale showed that if $X \in \mathbb{R}^m$ is an integrable random vector and $M_X$ is the $m \times m$ matrix whose columns are i.i.d. copies of $X$, then $E|\det(M_X)| = m! \text{vol}_m(K(X))$. We generalize this result to independent blocks that can give different distributions. This is to be compared with Theorem 6.8 below, in which we prove a similar result, but for complex random matrices.

**Theorem 5.4** (Expected absolute determinant of independent blocks). Let $M = (M_1, \ldots, M_p)$ be a random $m \times m$ matrix partitioned into blocks $M_j$ of size $d_j \times d_j$, with $d_1 + \cdots + d_p = m$. We denote by $v_{j,1}, \ldots, v_{j,d_j}$ the columns of $M_j$ and assume that $Z_j := v_{j,1} \wedge \cdots \wedge v_{j,d_j} \in \Lambda^{d_j} \mathbb{R}^m$ is integrable. If the random vectors $Z_1 \in \Lambda^{d_1}(\mathbb{R}^m), \ldots, Z_p \in \Lambda^{d_p}(\mathbb{R}^m)$ are independent, then:

$$E|\det(M)| = \ell((K(Z_1) \wedge \cdots \wedge K(Z_p))).$$

In particular, if $p = m$ and $d_1 = \cdots = d_p = 1$, then

$$E|\det(M)| = m! \text{MV}(K(Z_1), \ldots, K(Z_m)).$$

The formula for independent columns, i.e., where $p = m$, was already proved by Weil in [Wei76, Theorem 4.2]. In fact, Weil proved a more general version in the case $p = m$ for convex bodies, not just zonoids.

Vitale’s theorem corresponds to the special case $d_1 = \cdots = d_m = 1$ and where the matrices $M_1, \ldots, M_m$ (which are now column vectors) all have the same distribution: $M_j \sim X$. In this case, $E|\det(M)| = m! \text{MV}(K(X), \ldots, K(X)) = m! \text{vol}_m(K(X))$. 
Proof of Theorem 5.4. Let us first observe that \( \det[v_1, \ldots, v_m] = \|v_1 \wedge \cdots \wedge v_m\| \) for vectors \( v_1, \ldots, v_m \in \mathbb{R}^m \). Applying this in our case, we get for the absolute determinant of \( M \) that \( \det(M) = \|v_{1,1} \wedge \cdots \wedge v_{p,p}\| = \|Z_1 \wedge \cdots \wedge Z_p\| \). Taking expectations on both sides, we obtain \( \mathbb{E} \det(M) = \ell(K(Z_1 \wedge \cdots \wedge Z_p)) \). By the definition of the wedge product, we have \( K(Z_1 \wedge \cdots \wedge Z_p) = K(Z_1) \wedge \cdots \wedge K(Z_p) \), which shows the first assertion. The second claim follows from Theorem 5.1. This concludes the proof.

We give two additional examples in which Theorem 5.4 is applied.

Example 5.5. Let \( Z_1, \ldots, Z_n \in \mathbb{C}^n \) be integrable random vectors and \( L \in \mathbb{C}^{n \times n} \) be the random matrix \( L = (Z_1, \ldots, Z_n) \). We show how to compute \( \mathbb{E} |\det(L)|^2 \) with Theorem 5.4 (in the next section we will explain how to compute \( \mathbb{E} |\det(L)| \); see Theorem 6.8). To this end, we decompose \( Z_j = X_j + \sqrt{-1}Y_j \) with real random vectors \( X_j, Y_j \in \mathbb{R}^n \) (possibly dependent), we put \( m := 2n \) and consider the random matrix \( M = (M_1, \ldots, M_n) \), where \( M_j = \left( \begin{array}{c} X_j - Y_j \\ Y_j \end{array} \right) \), which satisfies the hypothesis of Theorem 5.4. Observe that \( \det(L) = \det(M) \). If we define the integrable random vector \( Q_j := \left( \begin{array}{c} X_j \\ Y_j \end{array} \right) \), we get by Theorem 5.4 that \( \mathbb{E} |\det(L)|^2 = \mathbb{E} |\det(M)| = \ell(K(Q_1) \wedge \cdots \wedge K(Q_n)) \).

Example 5.6. We interpret here Theorem 5.4 geometrically as follows: we consider a random parallelootope \( P \subset \mathbb{R}^m \) spanned by \( k \leq m \) random vectors, and ask for its expected \( k \)-dimensional volume. Suppose that the random vectors are \( v_{i,j} \) for \( 1 \leq i \leq p \) and \( 1 \leq j \leq d \) with \( \sum_{j=1}^d d_j = k \) and such that \( v_{i_1,j_1} \) and \( v_{i_2,j_2} \) are independent, if \( i_1 \neq i_2 \). Setting \( Z_i := v_{i,1} \wedge \cdots \wedge v_{i,d_i} \), we therefore have \( Z_1 \wedge \cdots \wedge Z_p \in \Lambda^k(\mathbb{R}^m) \) and

\[
\mathbb{E} (\text{vol}_k(P)) = \ell(K(Z_1) \wedge \cdots \wedge K(Z_p)).
\]

This shows that the length functional can be used to compute the expected volume of \( P \).

Theorem 5.4 has an interesting consequence when combined with the Alexandrov–Fenchel inequality (5.2) and the general Brunn-Minkowski theorem (5.3).

Corollary 5.7 (Brunn–Minkowski theorem for expected determinants). Let \( X_1, \ldots, X_m \) be independent integrable random vectors in \( \mathbb{R}^m \), and let \( X_1' \sim X_1 \) and \( X_2' \sim X_2 \) be independent of \( X_1, X_2 \), respectively. Then:

\[
(\mathbb{E} |\det[x_1 x_2 x_3 \ldots x_m]|)^2 \geq \mathbb{E} |\det[x_1 x_1' x_3 \ldots x_m]| \cdot \mathbb{E} |\det[x_2 x_2' x_3 \ldots x_m]|.
\]

More generally, for any \( 1 \leq d \leq m \), the following function is concave for \( t \in [0, 1] \):

\[
t \mapsto \left( \mathbb{E} |\det[x_1^{(t)} \ldots x_d^{(t)} x_{d+1} \ldots x_n]| \right)^{\frac{1}{d}},
\]

where \( X_i^{(t)}, \ldots, X_d^{(t)} \) are independent copies of \( \epsilon \mathbb{E} X_1 + (1-t)(1-\epsilon)2X_2 \) and \( \epsilon \) is a Bernoulli random variable with success probability \( \frac{1}{2} \), which is independent of \( X_1, X_2, X_{d+1}, \ldots, X_n \).

Proof. We set \( K_i := K(X_i) \) for \( 1 \leq i \leq m \). The first inequality is (5.2) combined with Theorem 5.4. For proving the second statement we let \( K_i := iK_1 + (1-t)K_2 \). Then, by Lemma 2.6, \( K_i = K(X^{(t)}) \), where \( X^{(t)} = t\epsilon X_1 + (1-t)(1-\epsilon)2X_2 \). We combine Theorem 5.4 with (5.3) to conclude.
Remark 5.8. The equality case of Alexandrov–Fenchel for zonoids was described in [Sch88]. From this, one can deduce the equality case for random determinants in Corollary 5.7.

Example 5.9. We can combine (5.4) and (5.5) with Theorem 5.4 to obtain a result about expected volumes of random triangles (this is the case $k = 2$ in Example 5.6):

\[(E \text{vol}_2(\Delta(X, Y)))^2 \geq E \text{vol}_2(\Delta(X, X')) \cdot E \text{vol}_2(\Delta(Y, Y'))\]

where $X, Y, X', Y'$ are independent vectors with finite expected norm, $X \sim X'$ and $Y \sim Y'$, and $\Delta(X, Y)$ is the triangle, whose vertices are the origin and $X$ and $Y$. We also get that the function

\[t \mapsto (E \text{vol}_d(P(X_1^{(t)}, \ldots, X_d^{(t)})))^{\frac{1}{d}}\]

is concave in $t \in [0, 1]$, where the $X_i^{(t)}$ are defined as in Corollary 5.7 and $P(X_1^{(t)}, \ldots, X_d^{(t)})$ is the parallelootope spanned by the $X_i^{(t)}$.

6. Mixed $J$–volume and random complex determinants

In this section $V$ will be a complex vector space of complex dimension $n$, that is a real vector space of real dimension $2n$ together with a real linear endomorphism $J : V \rightarrow V$ such that $J^2 = -I$. $J$ will be called the complex structure of $V$. We recall that such a complex structure induces an isomorphism $V \cong \mathbb{C}^n$ under which the automorphism $J$ corresponds to multiplication by $i$.

Here we will introduce a notion similar to mixed volume for zonoids in $V$, which is adapted to the complex structure. We call it the mixed $J$–volume and denote it by $\text{MV}^J$. It takes $n$ zonoids in a $2n$–dimensional real vector space, while the ordinary mixed volume $\text{MV} : Z(V)^{2n} \rightarrow \mathbb{R}$ is instead a function of $2n$ arguments. Furthermore, $\text{MV}^J$ is Minkowski additive and positively homogeneous in each argument; see Definition 6.2 below. We have already seen an application of our theory to zonoids in a complex vector space in Example 5.5. This example, however, used real multilinear maps. In this section, we consider complex multilinear maps, which leads to a different notion.

There are some key properties that make the mixed $J$–volume interesting: (1) it is compatible with the complex structure (see Proposition 6.6 (3)); it allows to formulate a complex version of Vitale’s theorem, for computing the expectation of the modulus – and not the modulus squared, as it is done in Example 5.5 – of the determinant of a random $n \times n$ matrix with rows which are independent random variables in $\mathbb{C}^n$ (Theorem 6.8); (3) it can be defined on all polytopes (but does not continuously extends all convex bodies, see Corollary 6.20); and finally (4) it equals the classical mixed volumes when restricted to polytopes in $\mathbb{R}^n \subset \mathbb{C}^n$; see Proposition 6.6 (2).

The extension of the $J$–volume to polytopes in $V \cong \mathbb{C}^n$ is a so called valuation; see Definition 6.17. The proof of the extension of the $J$–volume to polytopes is especially interesting and uses some particular combinatorial properties of zonotopes, and it connects to similar notions, such as Kazarnovskii’s pseudovolume (see Definition 6.21 below).

6.1. The mixed $J$–volume of zonoids. We endow our complex vector space $(V, J)$ with a hermitian structure $\phi : V \times V \rightarrow \mathbb{C}$. The associated scalar product is the real part of $\phi$. When $V = \mathbb{C}^n$ we consider the standard complex structure where $J$ is the multiplication by $i = \sqrt{-1}$ and the standard hermitian structure. Moreover, for $0 \leq k \leq n$ we denote by $\Lambda^k_C(V)$ the complex exterior algebra and, given vectors $v_1, \ldots, v_k \in V$, we denote by $v_1 \wedge_C \cdots \wedge_C v_k \in \Lambda^k_C(V)$ their complex exterior product. Note that this construction depends on the choice of the complex structure $J$, however we prefer the notation with “$\mathbb{C}$” that we find easier to read.
The hermitian structure on $V$ induces an hermitian structure on all the complex exterior powers and, in particular, taking its real part, a real scalar product on each of them. This implies that we have a Euclidean norm on each $\Lambda^K_n(V)$ and consequently we have a length functional $\ell : \mathcal{Z}(\Lambda^K_n(V)) \rightarrow \mathbb{R}$; see Definition 2.10.

Moreover the complex wedge product is, in particular, a real multilinear map. Therefore we can apply Theorem 4.1 to obtain a well-defined notion of complex product of virtual zonoids.

**Definition 6.1.** Consider the (real) multilinear map $F: V^n \rightarrow \Lambda^n_k(V)$ defined by the complex wedge $F(v_1, \ldots, v_n) := v_1 \wedge \cdots \wedge v_n$. For any $K_1, \ldots, K_n \in \mathcal{Z}(V)$, we define:

$$K_1 \wedge \cdots \wedge K_n := \hat{F}(K_1, \ldots, K_n).$$

The next definition uses this construction to define the mixed $J$–volume.

**Definition 6.2** (Mixed $J$–volume). We define the mixed $J$–volume $MV^J: \mathcal{Z}(V)^n \rightarrow \mathbb{R}$ to be the $\mathbb{R}$–multilinear map given, for all $K_1, \ldots, K_n \in \mathcal{Z}(V)$, by:

$$MV^J(K_1, \ldots, K_n) := \frac{1}{n!} \ell(K_1 \wedge \cdots \wedge K_n).$$

The $J$–volume of a zonoid $K \in \mathcal{Z}(V)$ is defined to be:

$$\text{vol}^J_n(K) := MV^J(K, \ldots, K).$$

**Remark 6.3.** Notice that, since $\Lambda^{2n}(V) \simeq \mathbb{R}$ is of real dimension one, zonoids in $\Lambda^{2n}(V)$ are just segments. By contrast, the top complex exterior power $\Lambda^n_c(V) \simeq \mathbb{C}$ is of real dimension two and centered zonoids in this space are more than segments (in fact they are precisely the centrally symmetric convex bodies; see [Sch14, Theorem 3.5.2]). Thus $K_1 \wedge \cdots \wedge K_n$ is a zonoid in $\Lambda^n_c(V) \simeq \mathbb{C}$. Then taking its length loses some information. However, it is easy to see using Definition 6.1, that if one of the $K_i$ is invariant under the $U(1)$ action on $V$, then $K_1 \wedge \cdots \wedge K_n$ is also $U(1)$ invariant and hence must be a disc. We compute the length of a disc in Lemma 6.5 below.

Let us study some of the properties of the mixed $J$–volume. On some classes of zonoids of the complex space $V$ it behaves particularly well. The first case is when $V = \mathbb{C}^n$ and all the zonoids are contained in the real $n$–plane $\mathbb{R}^n \subset \mathbb{C}^n$. In that case, we will show that the mixed $J$–volume is equal to the classical mixed volume (see Proposition 6.6 (2)).

Next, we consider complex discs.

**Definition 6.4.** Let $z \in V$. We define $D_z$ to be the closed centered disc of radius $|z|$ in the complex line $\mathbb{C}z \cong \mathbb{R}^2$.

In order to describe a random vector representing $D_z$, let us introduce the following notation. For $\theta \in \mathbb{R}$ we denote by $e^{\theta J}: V \rightarrow V$ the linear operator $e^{\theta J} := \cos(\theta)\text{Id} + \sin(\theta)J$ where $\text{Id}$ denotes the identity on $V$. We then have the following lemma.

**Lemma 6.5.** Let $\theta \in [0, 2\pi]$ be a uniformly distributed random variable and $z \in V$ nonzero. Consider the random vector $X_z \in V$ defined by $X_z := \pi e^{\theta J}z$. Then:

$$K(X_z) = D_z \quad \text{and} \quad \ell(D_z) = \pi|z|.$$

**Proof.** Since for every $\theta \in [0, 2\pi]$ the vector $e^{\theta J}z$ belongs to $\mathbb{C}z$, we have $h_{K(X_z)}(u) = 0$ for every $u \in (\mathbb{C}z)^\perp$. This implies that $K(X_z)$ is contained in $\mathbb{C}z$. It is straightforward to verify that $E|\langle e^{\theta J}z, z \rangle| = |z|^2 E|\cos \theta| = \frac{2}{\pi}|z|^2$. This implies for $\lambda \in \mathbb{C}$ that

$$h_{K(X_z)}(\lambda z) = \frac{1}{2} E|\langle X_z, \lambda z \rangle| = \frac{1}{2\pi} |\lambda| E|\langle e^{\theta J}z, z \rangle| = |\lambda||z|^2 = ||z|| \cdot |\lambda z|. $$
On the other hand, $h_{D_2}(\lambda z) = \|z\| |\lambda z|$, hence the first assertion follows. The second statement follows immediately from the fact that $\|X_z\| = \pi \|z\|$ almost surely. \hfill \square

**Proposition 6.6** (Properties of the mixed $J$–volume). The following properties hold:

1. The mixed $J$–volume of zonoids $\text{MV}^J: Z(V)^n \to \mathbb{R}$ is symmetric, multilinear, and monotonically increasing in each variable.

2. Suppose $V = \mathbb{C}^n$ and let $K_1, \ldots, K_n \in Z(\mathbb{R}^n) \subset Z(\mathbb{C}^n)$. Then:

   $$\text{MV}^J(K_1, \ldots, K_n) = \text{MV}(K_1, \ldots, K_n).$$

3. Let $T: V \to V$ be a $C$–linear transformation (i.e., such that $TJ = JT$), and denote by $\det_C(T)$ its complex determinant. Then, for all $K_1, \ldots, K_n \in Z(V)$,

   $$\text{MV}^J(TK_1, \ldots, TK_n) = |\det_C(T)| \text{MV}^J(K_1, \ldots, K_n).$$

4. For every $z_1, \ldots, z_n \in V$ we have $\text{MV}^J(Dz_1, \ldots, Dz_n) = \frac{n!}{n!} |z_1 \wedge_C \cdots \wedge_C z_n|$

5. For every $\theta \in \mathbb{R}$ and every $K_1, \ldots, K_n \in Z(V)$ we have

   $$\text{MV}^J(e^{i\theta} K_1, K_2, \ldots, K_n) = \text{MV}^J(K_1, \ldots, K_n).$$

**Proof.** Multilinearity of $\text{MV}^J$ follows from the definition and Theorem 4.1. To see that $\text{MV}^J$ is symmetric, given zonoids $K_1, \ldots, K_n$ in $V$, let $X_1, \ldots, X_n \in V$ be independent integrable random vectors such that $K_j = K(X_j)$. We have

$$K_1 \wedge_C K_2 \wedge_C \cdots \wedge_C K_n = K(X_1 \wedge_C X_2 \wedge_C \cdots \wedge_C X_n)$$

$$= K(-X_2 \wedge_X X_1 \wedge_C \cdots \wedge_C X_n)$$

$$= K(X_2 \wedge_C X_1 \wedge_C \cdots \wedge_C X_n)$$

(by Lemma 2.7)

$$= K_2 \wedge_C K_1 \wedge_C \cdots \wedge_C K_n.$$

The same argument gives symmetry in each pairs of variables. The fact that the mixed $J$–volume is monotonically increasing in each variable is a direct consequence of the definition, Theorem 4.1 and the monotonicity of the length (Corollary 2.13).

Let us prove point (2). Let $K_1, \ldots, K_n \in Z(\mathbb{R}^n) \subset Z(\mathbb{C}^n)$ and let $X_1, \ldots, X_n \in \mathbb{R}^n$ be independent random (real) vectors such that $K_j = K(X_j)$, $1 \leq j \leq n$. By Definition 6.2 the mixed $J$–volume is $\text{MV}^J(K_1, \ldots, K_n) = \frac{1}{n!} \ell \langle K_1 \wedge_C \cdots \wedge_C K_n \rangle = \frac{1}{n!} \mathbb{E} \|X_1 \wedge_C \cdots \wedge_C X_n\|$. Because the $X_j$ are real vectors, the span of the $X_j$ defines a *Lagrangian plane* (a plane $E$ that is orthogonal to $JE$), and so we have $\|X_1 \wedge \cdots \wedge X_n\| = \|X_1 \wedge \cdots \wedge X_n\|$, by Lemma 6.13. We get $\text{MV}^J(K_1, \ldots, K_n) = \frac{1}{n!} \mathbb{E} \|X_1 \wedge \cdots \wedge X_n\|$. We conclude from Theorem 5.1 that the latter is equal to $\text{MV}(K_1, \ldots, K_n)$. This finishes the proof of point (2).

In order to prove point (3), let $K_1, \ldots, K_n \subset V$ be zonoids and let again $X_j \in V$ be random vectors such that $K_j = K(X_j)$. Then $TK_j = K(TX_j)$ and we have

$$\text{MV}^J(TK_1, \ldots, TK_n) = \frac{1}{n!} \mathbb{E} |TX_1 \wedge \cdots \wedge TX_n|$$

$$= \frac{1}{n!} \mathbb{E} |(\det_C(T))X_1 \wedge \cdots \wedge X_n|$$

$$= \frac{1}{n!} |\det_C(T)| \mathbb{E} |X_1 \wedge \cdots \wedge X_n|$$

$$= |\det_C(T)| \text{MV}^J(K_1, \ldots, K_n).$$
To show point (4) we use Lemma 6.5 and write, for \( \theta_1, \ldots, \theta_n \in [0,2\pi] \) independent and uniformly distributed:

\[
\text{MV}^J(D_{z_1}, \ldots, D_{z_n}) = \frac{1}{n!} \ell(D_{z_1} \wedge_C \cdots \wedge_C D_{z_n})
\]

\[
= \frac{1}{n!} \ell \left( K(\pi e^{j\theta_1} z_1) \wedge_C \cdots \wedge_C K(\pi e^{j\theta_n} z_n) \right)
\]

\[
= \frac{1}{n!} \mathbb{E} \left| (\pi e^{j\theta_1} z_1) \wedge_C \cdots \wedge_C (\pi e^{j\theta_n} z_n) \right| = \frac{\pi^n}{n!} |z_1 \wedge_C \cdots \wedge_C z_n|,
\]

which is what we wanted.

Finally, to show the last item, let again \( X_j \in V \) be random vectors such that \( K_j = K(X_j) \). Then \( e^{\theta_j} K_j = K(e^{\theta_j}X_j) \) and we have

\[
\text{MV}^J(e^{\theta_j} K_1, K_2, \ldots, K_n) = \frac{1}{n!} \mathbb{E}[e^{\theta_j} X_1 \wedge_C X_2 \wedge_C \cdots \wedge_C X_n] 
\]

\[
= \frac{1}{n!} \mathbb{E}|X_1 \wedge_C \cdots \wedge_C X_n| = \text{MV}^J(K_1, \ldots, K_n).
\]

This concludes the proof. \( \Box \)

**Remark 6.7.** It is unknown to us if the mixed \( J \)-volume satisfies an *Alexandrov–Fenchel inequality* and we leave this for future work.

### 6.2. Random complex determinants.

Here we state and prove a complex version of Theorem 5.4 which gives a way to describe the expectation of the modulus of the determinant of a random complex matrix with independent blocks (in Example 5.5 we used Theorem 5.4 to compute instead the expectation of the *square* of the modulus of the determinant). To do that, mimicking the definition of Section 4.2, we associate with each complex wedge product \( \wedge_C : \Lambda^d_C V \times \Lambda^e_C V \to \Lambda^{d+e}_C V \) the componentwise continuous bilinear map

\[
\wedge_C : \mathbb{Z}(\Lambda^d_C V) \times \mathbb{Z}(\Lambda^e_C V) \to \mathbb{Z}(\Lambda^{d+e}_C V).
\]

induced from it by Theorem 4.1. We then have the following.

**Theorem 6.8.** Let \( M = (M_1, \ldots, M_p) \in \mathbb{C}^{n \times n} \) be a random complex \( n \times n \) matrix partitioned into blocks \( M_j \) of size \( n \times d_j \), with \( d_1 + \cdots + d_p = n \). For \( j = 1, \ldots, n \), denote by \( v_{j,1}, \ldots, v_{j,d_j} \) the columns of \( M_j \) and assume that \( Z_j = v_{j,1} \wedge_C \cdots \wedge_C v_{j,d_j} \in \Lambda^{d_j}_C \mathbb{C}^n \) is integrable. If the random vectors \( Z_1 \in \Lambda^{d_1}_C (\mathbb{C}^n), \ldots, Z_p \in \Lambda^{d_p}_C (\mathbb{C}^n) \) are independent, then:

\[
\mathbb{E}|\det(M)| = \ell(K(Z_1) \wedge_C \cdots \wedge_C K(Z_p)).
\]

In particular, if \( p = n \) and \( a_1 = \cdots = a_p = 1 \), then

\[
\mathbb{E}|\det(M)| = n! \text{MV}^J(K(Z_1), \ldots, K(Z_n)).
\]

**Proof.** Recall from Definition 6.2 that

\[
n! \text{MV}^J(K(Z_1), \ldots, K(Z_n)) = \ell(K(Z_1) \wedge_C \cdots \wedge_C K(Z_n)).
\]

Moreover we have \( \ell(K(Z_1) \wedge_C \cdots \wedge_C K(Z_n)) = \mathbb{E}|Z_1 \wedge_C \cdots \wedge_C Z_n| \). The last term is equal to \( \mathbb{E}|\det(M)| \). This concludes the proof. \( \Box \)

**Remark 6.9.** Notice that, in the case where \( p = n \) and \( d_1 = \cdots = d_p = 1 \) and if the random matrix is almost surely real, Theorem 6.8 agrees with Theorem 5.4, since \( \text{MV} \) and \( \text{MV}^J \) coincide on *real* zonoids; see Proposition 6.6 (2).

As an application of Theorem 6.8, we compute the \( J \)-volume of balls.
Corollary 6.10. The $J$–volume of the unit ball $B^{2n} \subset \mathbb{C}^n$ equals:

$$\nu^n(J)(B^{2n}) = \frac{(4\pi)^{n/2}}{n!} \prod_{j=1}^{n} \frac{\Gamma(j + \frac{1}{2})}{\Gamma(j)}.$$ 

Notice that applying Corollary 6.10 when $n = 1$ we get $\nu^n(J)(B^2) = \pi = \nu_2(B^2)$, but already when $n = 2$ we get $\nu^n(J)(B^4) = \frac{3\pi^2}{4} \neq \pi^2 = \nu_4(B^4)$. In general $\nu^n(J)$ and $\nu_2n$ are different, starting by the fact that the first is homogeneous of degree $n$ while the other is of degree $2n$.

Proof of Corollary 6.10. Let $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a random vector filled with independent standard complex Gaussians $z_j = \frac{1}{\sqrt{2}}(\xi_{j,1} + i\xi_{j,2})$, that is, $\xi_{j,1}, \xi_{j,2}, \ldots, \xi_{n,1}, \xi_{n,2}$ are independent standard real Gaussians. We claim that

$$K(Z) = \frac{1}{2\sqrt{\pi}} B^{2n}. \quad (6.1)$$

To see this, we compute the support function of $K(Z)$. Let $u \in \mathbb{C}^n$. By definition, we have $h_{K(Z)}(u) = \frac{1}{2} \mathbb{E}|\langle Z, u \rangle|$. Using the $U(n)$–invariance of $Z$, we can then assume that $u = ||u||e_1$ where $e_1$ is the first vector of the standard basis of $\mathbb{C}^n$. We obtain

$$h_{K(Z)}(u) = \frac{1}{2} \mathbb{E}|\text{Re}(Z^T \pi)| = \frac{1}{2\sqrt{2}} ||u|| \mathbb{E} |\xi_{1,1}| = \frac{1}{2\sqrt{2}} ||u|| \sqrt{\frac{2}{\pi}} = \frac{1}{2\sqrt{\pi}} ||u|| = h_{\frac{1}{2\sqrt{\pi}} B^{2n}}(u).$$

Proposition 2.1 gives (6.1).

Let now $M \in \mathbb{C}^{n \times n}$ be a random complex matrix whose columns are i.i.d. copies of $Z$. Then, Theorem 6.8 gives

$$\mathbb{E} |\det(M)| = n! \nu^n(J)(\frac{1}{2\sqrt{\pi}} B^{2n}, \ldots, \frac{1}{2\sqrt{\pi}} B^{2n}) = \frac{n!}{(2\sqrt{\pi})^n} \nu^n(J)(B^{2n}).$$

To conclude the proof, it suffices to verify that $\mathbb{E} |\det(M)| = \prod_{j=1}^{n} \Gamma(j + \frac{1}{2})/\Gamma(j)$. Note that $|\det M| = |\det(W)|^2$ and $W = MM^*$ is a complex Wishart matrix. Following [BC13, p. 83-84], we see that $\det(W)$ is distributed as $\frac{1}{2^n} \chi_{2n} \cdot \chi_{2n-2} \cdot \chi_2 \cdot \cdots$, where each $\chi_{2j}$ denotes a chi–square distribution with $2j$ degrees of freedom and the $\chi_2$ are independent. Therefore, $|\det(M)|$ has the distribution $\frac{1}{2^n} \chi_{2n} \cdot \chi_{2n-2} \cdot \chi_2$. Recall from (2.10) that $\mathbb{E} \chi_{2j} = \frac{\sqrt{2} \Gamma(j + \frac{1}{2})}{\Gamma(j)}$. Using independence, the assertion follows.

6.3. The extension of the $J$–volume to polytopes. In this section we show that it is possible to extend the notion of $J$–volume to polytopes in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. To do so, we develop an alternative formula for the $J$–volume of zonotopes that makes sense for any polytope (Theorem 6.15). The functional we obtain is a weakly continuous, translation invariant and $U(n)$–invariant valuation, see Proposition 6.18. However, as we will see, it is not possible to continuously extend the $J$–volume from polytopes to all convex bodies (see Corollary 6.20). We start by introducing some terminology.

As a first step, we will give in Proposition 6.14 an alternative way of writing the $J$–volume of zonotopes. This involves the following quantity.

Definition 6.11. For every $E \in G(n, 2n)$, we define

$$\sigma^J(E) := |e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n| \in [0, 1]$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $E$. 

One can check that this definition does not depend on the choice of an orthonormal basis. Moreover, $\sigma^J$ is invariant under the action of $U(n)$ on $G(n, 2n)$. Note that $\sigma^J(E) = 1$ if and only if $E$ is Lagrangian, i.e., if $E$ and $JE$ are orthogonal. Moreover $\sigma^J(E) = 0$ if and only if $E$ contains a complex line.

Remark 6.12. Denoting by $\theta_1(E) \leq \cdots \leq \theta_{\lfloor \frac{n}{2} \rfloor}(E)$ the Kähler angles of $E$, introduced in [Tas01], one can easily verify that

$$\sigma^J(E) = \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (\sin \theta_j(E))^2.$$

In general, $\sigma^J(E)$ can be computed using the following lemma.

Lemma 6.13. Let $z_1, \ldots, z_n \in \mathbb{C}^n$ be $\mathbb{R}$-linearly independent and denote by $E \in G(n, 2n)$ its real span. Then, writing $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}^n$, we have

$$|z_1 \land_C \cdots \land_C z_n|^2 = |\det(X + iY)|^2 = |\det \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \cdots |\det \begin{bmatrix} x_n & -y_n \\ y_n & x_n \end{bmatrix}|^2.$$
Proof. We write \( P = \sum_{j=1}^{p} \frac{1}{2}[-z_j, z_j] \). By definition, we have \( \text{vol}_n^j(P) = \frac{1}{n!} \ell(P \wedge n) \). Using multilinearity and Theorem 4.1, we can write
\[
\left( \sum_{j=1}^{p} \frac{1}{2}[-z_j, z_j] \right) \wedge \cdots \wedge \left( \sum_{j=1}^{p} \frac{1}{2}[-z_j, z_j] \right) = \sum_{1 \leq j_1, \ldots, j_n \leq p} \frac{1}{2}[-w_{j_1, \ldots, j_n}, w_{j_1, \ldots, j_n}],
\]
where \( w_{j_1, \ldots, j_n} := z_{j_1} \wedge \cdots \wedge z_{j_n} \). Therefore, using the linearity of the length,
\[
\text{vol}_n^j(P) = \frac{1}{n!} \sum_{1 \leq j_1, \ldots, j_n \leq p} |w_{j_1, \ldots, j_n}| = \sum_{j_1 < \ldots < j_n} |w_{j_1, \ldots, j_n}|.
\]
We may assume the sum runs only over the \( j_1 < \ldots < j_n \) such that the real span \( E_{j_1, \ldots, j_n} \) of \( z_{j_1}, \ldots, z_{j_n} \) has dimension \( n \). We write \( z_j = x_j + iy_j \) with \( x_j, y_j \in \mathbb{R}^n \) and use Lemma 4.13 to obtain
\[
|w_{j_1, \ldots, j_n}| = \left\| \begin{bmatrix} x_{j_1} \\ y_{j_1} \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_{j_n} \\ y_{j_n} \end{bmatrix} \right\| \cdot \sigma^j(E_{j_1, \ldots, j_n})^{\frac{1}{2}}.
\]
Combining this and exchanging the order of summation, we arrive at
\[
(6.3) \quad \text{vol}_n^j(P) = \sum_{E \in G_n(P)} \sigma^j(E)^{\frac{1}{2}} \cdot \sum_{E_{j_1, \ldots, j_n}=E} \left\| \begin{bmatrix} x_{j_1} \\ y_{j_1} \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_{j_n} \\ y_{j_n} \end{bmatrix} \right\|,
\]
where for fixed \( E \in G_n(P) \), the second sum runs over all \( j_1 < \ldots < j_n \) such that \( E = E_{j_1, \ldots, j_n} \). Shephard’s formula \([\text{She74, Equation (57)}]\) applied to the zonoid \( F_P(E) \) (see (6.2)) tells us that
\[
\sum_{E_{j_1, \ldots, j_n}=E} \left\| \begin{bmatrix} x_{j_1} \\ y_{j_1} \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_{j_n} \\ y_{j_n} \end{bmatrix} \right\| = \text{vol}_n(F_P(E)).
\]
Substituting this into (6.3) gives the statement. \( \square \)

We now turn to a key result of this section. For this, we need to introduce more notation. Let \( F \) be a \( k \)-dimensional face of a polytope \( P \subset \mathbb{R}^n \) and \( N_P(F) \) denote its normal cone. Note that \( N_P(F) \) is contained in the orthogonal complement \( E_F^P \cong \mathbb{R}^{m-k} \), where we recall that \( E_F \) denotes the vector space parallel to \( F \). We define the normal angle of \( P \) at \( F \) as
\[
\Theta_P(F) := \frac{\text{vol}_{m-k-1}(N_P(F) \cap S^{m-1})}{\text{vol}_{m-k-1}(S^{m-k-1})}.
\]
The \( J \)-volume of zonotopes can now be expressed as follows.

**Theorem 6.15.** Let \( P \subset \mathbb{C}^n \) be a zonotope. Then
\[
\text{vol}_n^j(P) = \sum_{F \in \mathcal{F}_n(P)} \text{vol}_n(F) \cdot \Theta_P(F) \cdot \sigma^j(E_F)^{\frac{1}{2}},
\]
where \( \mathcal{F}_n(P) \) denotes the set of \( n \)-dimensional faces of \( P \).

**Proof.** We will prove that the right hand side in this theorem is equal to the right hand side in Proposition 6.14. Let \( z_1, \ldots, z_p \in \mathbb{C}^n \) be such that \( P = \sum_{j=1}^{p} \frac{1}{2}[-z_j, z_j] \) and let \( E \in G_n(P) \).
As we discussed at the beginning of this subsection (see (6.2)), all the faces \( F \) of \( P \) such that \( E_F = E \) are translates of the vectorial face \( F_P(E) = \sum_{z_j \in E} \frac{1}{2}[-z_j, z_j] \), we can thus write:
\[
\sum_{F \in \mathcal{F}_n(P)} \text{vol}_n(F) \Theta_P(F) \sigma^j(E_F)^{\frac{1}{2}} = \sum_{E \in G_n(P)} \text{vol}_n(F_P(E)) \sigma^j(E_F)^{\frac{1}{2}} \sum_{E_F=E} \Theta_P(F).
\]
It remains to prove that for every \( E \in G_n(P) \) we have \( \sum_{E_F=E} \Theta_P(F) = 1 \).

To this end, given \( E \in G_n(P) \), pick a nonzero \( u \in E^\perp \). Then, the set
\[
P^u := \{ x \in K \mid h_P(u) = \langle u, x \rangle \}
\]
is a face of \( K \) and therefore it equals a translate of a vectorial face (see (6.2)). More precisely, there is \( v_u \in \mathbb{C}^n \) such that:

\[
P^u = v_u + \sum_{z_j \in u^\perp} \frac{1}{2}[-z_j, z_j],
\]

In addition, if \( F \) is a face of \( P \) such that \( E_F = E \), \( F \) is a translate of \( \sum_{v_j \in E} \frac{1}{2}[-v_j, v_j] \). Since \( E \subset u^\perp \), the face \( P^u \) contains a translate of \( F \). Moreover, \( \dim(F) = n \) and it follows that \( \dim(P^u) \geq n \) which implies \( \dim(N_K(K^u)) \leq n \). In other words we proved that if \( E \in G_n(K) \) and \( u \in E^\perp \), then \( \dim(N_P(P^u)) \leq n \).

We now show that for almost all \( u \in E^\perp \) we have \( \dim(N_P(P^u)) = n \). Indeed, for this it is enough to write \( E^\perp \subseteq \bigcup_{u \in E^\perp} N_K(K^u) \), thus the set \( \{u \in E^\perp \mid \dim(N_P(P^u)) < n \} \) is contained in a finite union of cones of dimension at most \( n - 1 \).

Let now \( S_{E^\perp}^{n-1} \) be the unit sphere in \( E^\perp \). Denote by \( \mathcal{F} \) the set of faces \( F \) of \( P \) such that \( E_F \supseteq E \) and \( \dim(N_P(F)) < n \). Then, by the above reasoning,

\[
\{u \in S_{E^\perp}^{n-1} \mid \dim(P^u) < n \} \subseteq \bigcup_{F \in \mathcal{F}} N_P(F) \cap S_{E^\perp}^{n-1}.
\]

Each set \( N_P(F) \cap S_{E^\perp}^{n-1} \) with \( F \in \mathcal{F} \) has dimension at most \( n - 2 \). Since the set \( \mathcal{F} \) is finite, it implies, as above, that \( \{u \in S_{E^\perp}^{n-1} \mid \dim(P^u) < n \} \) is contained in a finite union of sets of dimension at most \( n - 2 \), and in particular it has measure zero in \( S_{E^\perp}^{n-1} \). It follows that \( \{u \in S_{E^\perp}^{n-1} \mid \dim(P^u) = n \} \subset S_{E^\perp}^{n-1} \) has full measure. Letting \( u \) vary in \( S_{E^\perp}^{n-1} \) the set \( \{P^u\} \) exhausts all \( n \)-dimensional faces \( F \) with \( E_F = E \) and therefore:

\[
\sum_{E_F = E} \Theta_F(F) = \sum_{E_F = E} \frac{\text{vol}_{n-1}(N_P(F) \cap S_{E^\perp}^{2n-1})}{\text{vol}_{n-1}(S_{E^\perp}^{n-1})} = \frac{\text{vol}_{n-1}(S_{E^\perp}^{n-1})}{\text{vol}_{n-1}(S_{E^\perp}^{n-1})} = 1.
\]

This concludes the proof. \( \square \)

We now note that the formula in Theorem 6.15 still makes sense for polytopes that are not zonotopes. We use this to define the \( J \)-volume on polytopes.

**Definition 6.16.** Let \( P \) be polytope in \( \mathbb{C}^n \). We define its \( J \)-volume to be

\[
\text{vol}_n^J(P) := \sum_{F \in \mathcal{F}_n(P)} \text{vol}_n(F) \cdot \Theta_F(F) \cdot \sigma^J(E_F)^{\frac{1}{2}}
\]

where \( \mathcal{F}_n(P) \) denotes the set of \( n \)-dimensional faces of \( P \).

We next study the \( J \)-volume in the framework of the theory of valuations on polytopes. Let us first recall the notion of a valuation. We denote by \( \mathcal{P}(V) \) the set of polytopes in a finite dimensional real vector space \( V \).

**Definition 6.17** (Valuation). A function \( \nu : \mathcal{P}(V) \to \mathbb{R} \) is called a valuation on \( \mathcal{P}(V) \) if

\[
\nu(K \cup L) + \nu(K \cap L) = \nu(K) + \nu(L)
\]

for every pair of convex polytopes \( K, L \in \mathcal{P}(V) \) such that \( K \cup L \) is still a polytope (an analogous definition applies for valuations on \( \mathcal{K}(V) \)).

We call \( \nu \) continuous if it is continuous with respect to the Hausdorff metric. We say that the valuation \( \nu \) is \( k \)-homogeneous if \( \nu(\lambda K) = \lambda^k \nu(K) \) for every \( K \in \mathcal{P}(V) \) and \( \lambda > 0 \). If a group \( G \) acts on \( \mathcal{P}(V) \), the valuation \( \nu \) is said to be \( G \)-invariant if \( \nu(gK) = \nu(K) \) for all \( K \in \mathcal{P}(V) \) and \( g \in G \).
The map \( \nu^J : \mathcal{P}(\mathbb{C}^n) \to \mathbb{R} \) is a weakly continuous, translation invariant valuation.

(2) The valuation \( \nu_n^J \) is \( n \)-homogeneous and \( U(n) \)-invariant.

(3) Let \( P \subset \mathbb{R}^n \subset \mathbb{C}^n \) be a polytope. Then \( \nu_n^J(P) = \nu_n(P) \).

Proof. The first item follows from [McM83, Theorem 1]. The second item follows from the \( U(n) \)-invariance of \( \sigma^J \) and Definition 6.16. For the third item, if \( P \) is of dimension less than \( n \), both volumes are zero and there is nothing to prove. If \( \dim(P) = n \), its only face of dimension \( n \) is \( P \) itself and \( E_P = \mathbb{R}^n \). Moreover \( \sigma^J(\mathbb{R}^n) = 1 \). Finally since \( N_P(P) = (\mathbb{R}^n)^\perp \) we have \( \Theta_P(P) = 1 \). The claim follows with Definition 6.16.

The valuation \( \nu_n^J \) is a special case of a so called angular valuation, see [Wan20]. Let \( V \) be a Euclidean space of (real) dimension \( m \) and let \( \varphi : G(k, m) \to \mathbb{R} \) be a measurable function. The associated angular valuation \( \nu_\varphi \) on \( \mathcal{P}(V) \) is defined by

\[
\nu_\varphi(P) := \sum_{F \in \mathcal{F}_k(P)} \operatorname{vol}_k(F) \cdot \Theta_P(F) \cdot \varphi(E_F),
\]

where \( \mathcal{F}_k(P) \) denotes the set of \( k \)-dimensional faces of a polytope \( P \in \mathcal{P}(V) \). It is known [McM83] that \( \nu_\varphi : \mathcal{P}(V) \to \mathbb{R} \) is a weakly continuous valuation.

The possibility of continuously extending an angular valuation from polytopes to convex bodies was studied by Wannerer. The following is [Wan20, Theorem 1.2]. For its statement, we recall that \( G(k, m) \) can be seen as a subset of \( \mathbb{P}(\Lambda^k V) \) via the Plücker embedding.

Proposition 6.19. The angular valuation \( \nu_\varphi : \mathcal{P}(V) \to \mathbb{R} \) can be extended to a continuous valuation on \( \mathcal{K}(V) \), if and only if \( \varphi \) is the restriction to \( G(k, m) \) of a homogeneous quadratic polynomial on \( \Lambda^k V \).

If \( n = 1 \), then \( \sigma^J \) is the function which is constant and equal to 1. The previous proposition implies that in this case we can extend \( \nu_n^J \) to a continuous valuation on \( \mathcal{K}(V) \). If \( n \geq 2 \), however, this is not possible as we will show next.

Corollary 6.20. If \( n \geq 2 \), there is no continuous valuation on \( \mathcal{K}(\mathbb{C}^n) \) that is equal to \( \nu_n^J \) on \( \mathcal{P}(\mathbb{C}^n) \).

Proof. Using the notation of Proposition 6.19, we have \( \nu_n^J = \nu_{(\sigma^J)^{1/2}} \). We identify \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) and let \( J \) be the standard complex structure on it. Consider the homogeneous quadratic polynomial \( p : \Lambda^2 \mathbb{R}^{2n} \to \Lambda^2 \mathbb{R}^{2n}, w \mapsto w \wedge Jw \). From Definition 6.11, \( \sigma^J(w) = |p(w)| \) for \( w \in G(n, 2n) \) (in the Plücker embedding). Suppose there were a homogeneous quadratic polynomial \( q : \Lambda^2 \mathbb{R}^{2n} \to \mathbb{R} \) such that we have \( |p(w)| = q(w) \) for all \( w \in G(n, 2n) \). Let us show that this leads to a contradiction, which will complete the proof by Proposition 6.19. First
of all, we notice that \( q(w) \) must be a nonnegative polynomial and that we have \( |p(w)| = q(w)^2 \) on \( G(n, 2n) \).

Let \( e_1, \ldots, e_n \in \mathbb{C}^n \) be the standard basis, so that \( |e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n| = 1 \). We define the curve \( w(\theta) := (\cos(\theta)e_1 + \sin(\theta)Je_2) \wedge e_2 \cdots \wedge e_n \) in \( G(n, 2n) \) for \( \theta \in [0, \pi] \).

This curve interpolates between a Lagrangian plane (for \( \theta = 0 \)) and a plane, which contains a complex line (for \( \theta = \pi \)). We have that
\[
p(w(\theta)) = (\cos(\theta)e_1 + \sin(\theta)Je_2) \wedge e_2 \cdots \wedge e_n \wedge (\cos(\theta)Je_1 - \sin(\theta)e_2) \wedge e_2 \cdots \wedge e_n
= \cos(\theta)^2(e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n),
\]
and so \( |p(w(\theta))| = \cos(\theta)^2 \). If we have \( |p(w(\theta))| = q(w(\theta))^2 \), then \( q(w(\theta)) = \cos(\theta) \), because \( q \) is nonnegative. Since \( q \) is a quadratic form and by the definition of \( w(\theta) \), there are \( a, b, c \in \mathbb{R} \) such that \( q(w(\theta)) = a \cos(\theta)^2 + b \cos(\theta) \sin(\theta) + c \sin(\theta)^2 \) for all \( \theta \). Thus, we have an equality of functions \( a \cos(\theta)^2 + b \cos(\theta) \sin(\theta) + c \sin(\theta)^2 = \cos(\theta) \). It can be checked that such an equality is not possible, so our assumption was wrong and \( (\sigma^T)^{1/2} \) cannot be the restriction of the square of a quadratic form to \( G(n, 2n) \). Proposition 6.19 implies the assertion.

From the proof, we see that if we remove the square root in Definition 6.16, we could extend the valuation continuously to convex bodies. This leads to the notion of Kazarnovskii’s pseudovolume \([\text{Kaz04}]\). We use the expression found in \([\text{Ale03}]\) in the proof of his Proposition 3.3.1. The normalization constant can be determined using the fact that it agrees with the classical volume on \( \mathbb{R}^n \subset \mathbb{C}^n \) just like the \( J \)-volume.

Definition 6.21. The Kazarnovskii’s pseudovolume \( \text{vol}_n^K \) is given for any polytope \( P \subset \mathbb{C}^n \) by the formula
\[
\text{vol}_n^K(P) = \sum_{F \in \mathcal{F}_n(P)} \text{vol}_n(F) \cdot \Theta_K(F) \cdot \frac{1}{(\omega_n)^2} \text{vol}_{2n}(B(E_F) + JB(E_F)),
\]
where \( \mathcal{F}_n(P) \) denotes the set of \( n \)-dimensional faces of \( P \), and as before \( B(E_F) \) denotes the unit ball of \( E_F \), and \( \omega_n := \text{vol}_n(B(\mathbb{R}^n)) \).

In our setting we prove the following, to be compared to Definition 6.16.

Proposition 6.22. For any polytope \( P \in \mathcal{P}(\mathbb{C}^n) \) the Kazarnovskii pseudovolume is given by
\[
\text{vol}_n^K(P) = \sum_{F \in \mathcal{F}_n(P)} \text{vol}_n(F) \cdot \Theta_K(F) \cdot \sigma^J(E_F),
\]
where \( \mathcal{F}_n(P) \) denotes the set of \( n \)-dimensional faces of \( P \).

Proof. We need to prove that for any \( E \in G(n, 2n) \) we have
\[
(6.4) \quad \text{vol}_{2n}(B(E) + JB(E)) = (\omega_n)^2 \sigma^J(E).
\]
Using Theorem 5.1 we write
\[
\text{vol}_{2n}(B(E) + JB(E)) = \frac{1}{(2n)!} \ell \left( (B(E) + JB(E))^\wedge 2n \right)
= \frac{1}{(2n)!} \sum_{j=0}^{2n} \binom{2n}{j} \ell \left( (B(E))^{\wedge j} \wedge (JB(E))^{\wedge (2n-j)} \right).
\]
where we used from Theorem 2.22 that $\ell$ is linear. Since $\dim(B(E)) = \dim(JB(E)) = n$, we see from Lemma 4.6 that $(B(E))^{\wedge j} = 0$ whenever $j > n$ and that $(JB(E))^{\wedge (2n-j)} = 0$ whenever $j < n$. In other words, only the index $j = n$ contributes to the sum and we get

$$\text{vol}_{2n}(B(E) + JB(E)) = \frac{1}{(n!)^2} \ell \left( (B(E))^{\wedge n} \right).$$

Next let $X \in E$ be a random vector such that $K(X) = B(E)$ (for instance, (2.9) shows that we can take $X$ to be $\sqrt{2\pi}$ times a standard Gaussian vector in $E$) and let $X_1, \ldots, X_n$ be i.i.d. copies of $X$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $E$. Note that we have

$$X_1 \wedge \cdots \wedge X_n = \pm \|X_1 \wedge \cdots \wedge X_n\| \ e_1 \wedge \cdots \wedge e_n.$$

With this in mind, (6.5) gives

$$\text{vol}_{2n}(B(E) + JB(E)) = \frac{1}{(n!)^2} \mathbb{E}\|X_1 \wedge \cdots \wedge X_n \wedge JX_{n+1} \wedge \cdots \wedge X_{2n}\|

= \frac{1}{(n!)^2} (\mathbb{E}\|X_1 \wedge \cdots \wedge X_n\|)^2 \sigma^J(E)$$

Again, by Theorem 5.2, we have $\mathbb{E}\|X_1 \wedge \cdots \wedge X_n\| = n! \omega_n$ and this gives (6.4), which concludes the proof.

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