Hypercomplex structures with Hermitian-Norden metrics on four-dimensional Lie algebras

Mancho Manev

Abstract. Integrable hypercomplex structures with Hermitian and Norden metrics on Lie groups of dimension 4 are considered. The corresponding five types of invariant hypercomplex structures with hyper-Hermitian metric, studied by M.L. Barberis, are constructed here. The different cases regarding the signature of the basic pseudo-Riemannian metric are considered.

Mathematics Subject Classification (2010). 53C15, 53C50, 22E30, 53C55, 53C56.

Keywords. hypercomplex structure, 4-dimensional Lie algebra, Hermitian metric, Norden metric, indefinite metric.

Introduction

The present work is inspired by the work of Barberis [2] where invariant hypercomplex structures $H$ on 4-dimensional real Lie groups are classified. In that case the corresponding metric is positive definite and Hermitian with respect to the triplet of complex structures of $H$. Our main goal is to classify 4-dimensional real Lie algebras which admit hypercomplex structures with Hermitian and Norden metrics.

We equip a hypercomplex structure $H$ with a metric structure, generated by a pseudo-Riemannian metric $g$ of neutral signature (see [9,10]). In our case the one (resp., the other two) of the almost complex structures of $H$ acts as an isometry (resp., act as anti-isometries) with respect to $g$ in each tangent fibre. Thus, there exist three $(0,2)$-tensors associated by $H$ except the metric $g$ — a
Kähler form and two metrics of the same type. The metric $g$ is Hermitian with respect to the one almost complex structure of $H$ and $g$ is a Norden (i.e. an anti-Hermitian) metric regarding the other two almost complex structures of $H$. For this reason we call the derived almost hypercomplex structure an almost hypercomplex structure with Hermitian-Norden-metrics or briefly almost hypercomplex HN-metric structure.

Let us remark that in [13] and [12] are classified the invariant complex structures on 4-dimensional solvable simply-connected real Lie groups where the dimension of commutators is less than three and equal three, respectively.

A hypercomplex structure is called Abelian ([4]) if $[J_\alpha x, J_\alpha y] = [x, y]$, for all $x, y \in \mathfrak{g}$ ($\alpha = 1, 2, 3$). Abelian hypercomplex structures are considered in [3], [5] and they can only occur on solvable Lie algebras ([6]). It is clear that the condition $N(x, y) = [Jx, Jy] - J[Jx, y] - J[x, Jy] - [x, y] = 0$ can be rewritten as $[Jx, Jy] - [x, y] = J([Jx, y] + [x, Jy])$ for all $x, y \in \mathfrak{g}$. Thus, Abelian complex structures and therefore Abelian hypercomplex structure are integrable.

If the three almost complex structures of $H$ are parallel with respect to the Levi-Civita connection $\nabla$ of $g$ then such hypercomplex HN-manifolds of Kähler type we call hyper-Kähler HN-manifolds, which are flat according to [10].

The paper is organized as follows. In Sect. 1 we recall some facts about the almost hypercomplex HN-manifolds known from [1, 9, 10, 11]. In Sect. 2 we construct different types of hypercomplex structures on Lie algebras following the Barberis classification.

The basic problem of this work is the existence and the geometric characteristics of hypercomplex HN-structures on 4-dimensional Lie algebras according to the Barberis classification. The main results of this paper is construction of the different types of the considered structures and their characterization.

1. Preliminaries

Let $(M, H)$ be a hypercomplex manifold, i.e. $M$ is a 4$n$-dimensional differentiable manifold and $H = (J_1, J_2, J_3)$ is a triple of complex structures on $M$ with the following properties for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I,$$

where $I$ denotes the identity and moreover, it is valid

$$N_\alpha = 0, \quad \alpha \in \{1, 2, 3\}$$

for the Nijenhuis tensors $N_\alpha$ of $J_\alpha$ given by

$$N_\alpha(\cdot, \cdot) = [J_\alpha \cdot, J_\alpha \cdot] - J_\alpha [J_\alpha \cdot, \cdot] - J_\alpha [\cdot, J_\alpha \cdot] - [\cdot, \cdot]$$

on $\mathfrak{X}(M)$. 
A hypercomplex structure on a Lie group $G$ is said to be invariant if left translations by elements of $G$ are holomorphic with respect to $J_\alpha$ for all $\alpha \in \{1,2,3\}$. Obviously, if $\mathfrak{g}$ is the corresponding Lie algebra of the Lie group $G$, a hypercomplex structure on $\mathfrak{g}$ induces an invariant hypercomplex structure on $G$ by left translations.

Let $G$ be a simply connected 4-dimensional real Lie group admitting an invariant hypercomplex structure. A left invariant metric on $G$ is called invariant hyper-Hermitian if it is hyper-Hermitian with respect to some invariant hypercomplex structure on $G$. It is known that all such metrics on given $G$ are equivalent up to homotheties.

If $\mathfrak{g}$ denotes the Lie algebra of $G$ then it is known the following

**Theorem 1.1 ([2]).** The only 4-dimensional Lie algebras admitting an integrable hypercomplex structure are the following types:

- (hc1) $\mathfrak{g}$ is Abelian;
- (hc2) $\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{so}(3)$;
- (hc3) $\mathfrak{g} \cong \text{aff}(\mathbb{C})$;
- (hc4) $\mathfrak{g}$ is the solvable Lie algebra corresponding to $\mathbb{R}H^4$;
- (hc5) $\mathfrak{g}$ is the solvable Lie algebra corresponding to $\mathbb{C}H^2$, where $\mathbb{R} \oplus \mathfrak{so}(3)$ is the Lie algebra of the Lie groups $U(2)$ and $S^3 \times S^1$; $\text{aff}(\mathbb{C})$ is the Lie algebra of the affine motion group of $\mathbb{C}$ – the unique 4-dimensional Lie algebra carrying an Abelian hypercomplex structure; $\mathbb{R}H^4$ is the real hyperbolic space; $\mathbb{C}H^2$ is the complex hyperbolic space.

Let $g$ be a neutral metric on $(M, H)$ with the properties

$$g(\cdot, \cdot) = \varepsilon_\alpha g(J_\alpha \cdot, J_\alpha \cdot),$$

where

$$\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2; 3. \end{cases}$$

Moreover, the associated (Kähler) 2-form $g_1$ and the associated neutral metrics $g_2$ and $g_3$ are determined by

$$g_\alpha(\cdot, \cdot) = g(J_\alpha \cdot, \cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha \cdot).$$

The structure tensors of a such manifold are the following three $(0,3)$-tensors

$$F_\alpha(x,y,z) = g\left((\nabla_x J_\alpha) y, z\right) = \left(\nabla_x g_\alpha\right)(y,z),$$

where $\nabla$ is the Levi-Civita connection generated by $g$. The corresponding Lee 1-forms $\theta_\alpha$ are defined by

$$\theta_\alpha(\cdot) = g^{ij} F_\alpha(e_i, e_j, \cdot)$$

for an arbitrary basis $\{e_1, e_2, \ldots, e_{4n}\}$ of $T_p M$, $p \in M$.

In [10] we study the so-called hyper-Kähler manifolds with HN-metric structure (or pseudo-hyper-Kähler manifolds), i.e. the almost hypercomplex HN-manifold in the class $\mathcal{K}$, where $\nabla J_\alpha = 0$ for all $\alpha = 1,2,3$. A sufficient
condition \((M, H, G)\) be in \(K\) is this manifold be of Kähler-type with respect to two of the three complex structures of \(H\).\(^9\)

As \(g\) is an indefinite metric, there exist isotropic vectors \(x\) on \(M\), i.e. \(g(x, x) = 0, \ x \neq 0\). In \(^9\) we define the invariant square norm
\[
\|\nabla J_\alpha\|^2 = g^{ij} g^{kl} g((\nabla_i J_\alpha) e_k, (\nabla_j J_\alpha) e_l),
\]
where \(\{e_1, e_2, \ldots, e_{4n}\}\) is an arbitrary basis of the tangent space \(T_p M\) at an arbitrary point \(p \in M\) of \(T_p M\). We say that an almost hypercomplex \(HN\)-manifold is an isotropic hyper-Kähler \(HN\)-manifold if \(\|\nabla J_\alpha\|^2 = 0\) for each \(J_\alpha\) of \(H\). Clearly, if the manifold is a hyper-Kähler \(HN\)-manifold, then it is an isotropic hyper-Kähler \(HN\)-manifold. The inverse statement does not hold.

Let us note that according to \(^4\) the manifold \((M, J_1, g)\) is almost Hermitian and the manifolds \((M, J_\alpha, g), \ \alpha = 2, 3\), are almost complex manifolds with Norden metric \(^7\). The basic classes of the mentioned two types of manifolds are given in \(^8\) and \(^7\), respectively. The special class \(W_0(J_\alpha) : F_\alpha = 0 (\alpha = 1, 2, 3)\) of the Kähler-type manifolds belongs to any other class within the corresponding classification. In the 4-dimensional case the four basic classes of the almost Hermitian manifolds are restricted to two: \(W_2(J_1)\), the class of the almost Kähler manifolds and \(W_4(J_1)\), the class of the Hermitian manifolds with respect to \(J_1\). They are determined for \(\dim M = 4\) by:

\[
W_2(J_1) : \mathcal{S} \{F_1(x, y, z)\} = 0;
\]
\[
W_4(J_1) : F_1(x, y, z) = \frac{1}{2} \{g(x, y)\theta_1(z) - g(x, z)\theta_1(y)
\]
\[
- g(x, J_1 y)\theta_1(J_1 z) + g(x, J_1 z)\theta_1(J_1 y)\};
\]

where \(\mathcal{S}\) is the cyclic sum by three arguments \(x, y, z\). The basic classes of the almost Norden manifolds (i.e., for \(\alpha = 2\) or 3) are determined for dimension 4 as follows:

\[
W_1(J_\alpha) : F_\alpha(x, y, z) = \frac{1}{4} \{g(x, y)\theta_\alpha(z) + g(x, z)\theta_\alpha(y)
\]
\[
+ g(x, J_\alpha y)\theta_\alpha(J_\alpha z) + g(x, J_\alpha z)\theta_\alpha(J_\alpha y)\};
\]
\[
W_2(J_\alpha) : \mathcal{S} \{F_\alpha(x, y, J_\alpha z)\} = 0, \ \ \ \ \theta_\alpha = 0;
\]
\[
W_3(J_\alpha) : \mathcal{S} \{F_\alpha(x, y, z)\} = 0.
\]

It is known that the class of the complex manifolds with Norden metric is \(W_1 \oplus W_2\) for \(J_\alpha (\alpha = 2, 3)\).

Then the class of hypercomplex manifolds with Hermitian-Norden metrics is
\[
\mathcal{H} = W_4(J_1) \cap (W_1 \oplus W_2) (J_2) \cap (W_1 \oplus W_2) (J_3).
\]
2. Four-dimensional Lie algebras with such structures

Let \( \{e_1, e_2, e_3, e_4\} \) be a basis of a 4-dimensional real Lie algebra \( g \) with center \( z \) and derived Lie algebra \( g' = [g, g] \). A standard hypercomplex structure on \( g \) is defined as in [14]:

\[
\begin{align*}
J_1 e_1 &= e_2, & J_1 e_2 &= -e_1, & J_1 e_3 &= -e_4, & J_1 e_4 &= e_3; \\
J_2 e_1 &= e_3, & J_2 e_2 &= e_4, & J_2 e_3 &= -e_1, & J_2 e_4 &= -e_2; \\
J_3 e_1 &= -e_4, & J_3 e_2 &= e_3, & J_3 e_3 &= -e_2, & J_3 e_4 &= e_1.
\end{align*}
\]

(11)

Let us introduced a pseudo-Euclidian metric \( g \) with neutral signature as follows

\[
g(x, y) = x^1 y^1 + x^2 y^2 - x^3 y^3 - x^4 y^4,
\]

(12)

where \( x(x^1, x^2, x^3, x^4), y(y^1, y^2, y^3, y^4) \in g \). This metric satisfies (4) and (5). Then the metric \( g \) generates an almost hypercomplex HN-metric structure on \( g \).

Let us consider the different cases of Theorem 1.1.

2.1. Hypercomplex HN-metric structure of type (hc1)

Obviously, in this case the considered manifold belongs to the class \( K \).

2.2. Hypercomplex HN-metric structure of type (hc2)

Let \( g \) be not solvable and let us determine it by

\[
\begin{align*}
[e_2, e_4] &= e_3, & [e_4, e_3] &= e_2, & [e_3, e_2] &= e_4.
\end{align*}
\]

(13)

In this consideration the (+)-unit \( e_1 \in \mathbb{R} \), i.e. \( g(e_1, e_1) = 1 \), is orthogonal to \( g' \) with respect to \( g \).

Then we compute covariant derivatives in the basis and the nontrivial ones are

\[
\begin{align*}
\nabla_{e_2} e_3 &= -\frac{3}{2} e_4, & \nabla_{e_3} e_2 &= \frac{1}{2} e_4, & \nabla_{e_4} e_2 &= \frac{1}{2} e_3, \\
\nabla_{e_4} e_3 &= \frac{3}{2} e_3, & \nabla_{e_3} e_4 &= -\frac{1}{2} e_2, & \nabla_{e_4} e_3 &= -\frac{1}{2} e_2.
\end{align*}
\]

(14)
By virtue of (14), (11) and (6), we obtain components \((F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k)\), \(i, j, k \in \{1, 2, 3, 4\}\), as follows:

\[
(F_1)_{314} = -(F_1)_{323} = (F_1)_{332} = -(F_1)_{341} = - (F_1)_{413} = -(F_1)_{424} = (F_1)_{431} = (F_1)_{442} = \frac{1}{2};
\]

\[
(F_2)_{214} = -(F_2)_{232} = -(F_2)_{232} = (F_2)_{241} = \frac{3}{2}, \quad (F_2)_{322} = -1,
\]

\[
(F_2)_{412} = (F_2)_{421} = (F_2)_{434} = (F_2)_{443} = \frac{1}{2}, \quad (F_2)_{344} = -1;
\]

\[
(F_3)_{213} = (F_3)_{224} = (F_3)_{231} = (F_3)_{242} = \frac{3}{2}, \quad (F_3)_{422} = 1,
\]

\[
(F_3)_{312} = (F_3)_{321} = -(F_3)_{334} = -(F_3)_{343} = \frac{1}{2}, \quad (F_3)_{433} = 1.
\]

The only non-zero components \((\theta_\alpha)_i = (\theta_\alpha)(e_i), i = 1, 2, 3, 4\), of the corresponding Lee forms are

\[
(\theta_1)_2 = -1, \quad (\theta_2)_3 = -2, \quad (\theta_3)_4 = 2.
\]

Using the results in (15), (16) and the classification conditions (9), (10), we obtain

**Proposition 2.1.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (13), belongs to the largest class of the considered manifolds, i.e. \(\mathcal{H}\), as well as this manifold does not belong to neither \(\mathcal{W}_1\) nor \(\mathcal{W}_2\) for \(J_2\) and \(J_3\).

The other possibility is the signature of \(g\) on \(\mathbb{R}\) to be \((-\ viability, e.g. \(e_3 \in \mathbb{R}\), where \(g(e_3, e_3) = -1\). By similar computations we establish the same class in the statement of Proposition 2.1.

2.3. Hypercomplex HN-metric structure of type (hc3)

We analyze separately the cases of signature (1,1), (0,2) and (2,0) of \(g\) on \(g'\).

2.3.1. Firstly, we consider \(g\) of signature (1,1) on \(g'\).

Let us determine \(g\) by

\[
[e_2, e_3] = [e_1, e_4] = e_2, \quad [e_2, e_1] = [e_4, e_3] = e_4.
\]

Then we compute covariant derivatives and the nontrivial ones are

\[
\nabla_{e_2} e_1 = \nabla_{e_4} e_3 = e_4, \quad \nabla_{e_2} e_2 = -\nabla_{e_4} e_4 = e_3,
\]

\[
\nabla_{e_2} e_3 = -\nabla_{e_4} e_1 = e_2, \quad \nabla_{e_2} e_4 = \nabla_{e_4} e_2 = e_1.
\]
By virtue of (17), (11) and (6), we obtain that $F_1 = 0$ and the other components $(F_\alpha)_{ijk}$, $\alpha = 2, 3$, are as follows
\[
(F_2)_{212} = (F_2)_{221} = (F_2)_{234} = (F_2)_{243} =
\]
\[
= -(F_2)_{414} = (F_2)_{423} = (F_2)_{432} = -(F_2)_{441} = 2;
\]
\[
(F_3)_{211} = -(F_3)_{222} = -(F_3)_{233} = (F_3)_{244} =
\]
\[
= (F_3)_{413} = (F_3)_{424} = (F_3)_{431} = (F_3)_{442} = -2.
\]
The only non-zero components of the corresponding Lee forms are
\[
(\theta_2)_1 = (\theta_3)_2 = 4.
\]
Using that $F_1 = 0$, the results in (10), (20) and the classification conditions (9), (10), we obtain

**Proposition 2.2.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (17), belongs to the subclass of the Kähler manifold with respect to $J_1$ of the largest class of the considered manifolds, i.e.
\[
W_0(J_1) \cap (W_1 \oplus W_2) (J_2) \cap (W_1 \oplus W_2) (J_3),
\]
as well as this manifold does not belong to neither $W_1$ nor $W_2$ for $J_2$ and $J_3$.

2.3.2. Secondly, we consider $g$ of signature $(2,0)$ on $g'$. The case for signature $(0,2)$ is similar.

Let us determine $g$ by
\[
[e_1, e_3] = [e_4, e_2] = e_1, \quad [e_1, e_4] = [e_2, e_3] = e_2.
\]
Then we compute covariant derivatives and the nontrivial ones are
\[
\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -\nabla_{e_4} e_1 = e_2,
\]
\[
\nabla_{e_1} e_3 = \nabla_{e_4} e_2 = e_1.
\]
By virtue of (21), (11) and (6), we obtain the following components of $(F_\alpha)$:
\[
(F_1)_{114} = -(F_1)_{123} = (F_1)_{132} = -(F_1)_{141} =
\]
\[
= (F_1)_{213} = (F_1)_{224} = -(F_1)_{231} = -(F_1)_{242} = -1;
\]
\[
(F_2)_{111} = (F_2)_{133} = 2,
\]
\[
(F_2)_{212} = (F_2)_{221} = (F_2)_{234} = (F_2)_{243} =
\]
\[
= -(F_2)_{414} = (F_2)_{423} = (F_2)_{432} = -(F_2)_{441} = 1;
\]
\[
(F_3)_{222} = (F_3)_{233} = 2,
\]
\[
-(F_3)_{112} = -(F_3)_{121} = (F_3)_{134} = (F_3)_{143} =
\]
\[
= (F_3)_{413} = (F_3)_{424} = (F_3)_{431} = (F_3)_{442} = -1.
\]
The only non-zero components of the corresponding Lee forms are

\[(\theta_1)_4 = -2, \quad (\theta_2)_1 = (\theta_3)_2 = 4.\]  \hfill (24)

Using the results in (23), (24) and the classification conditions (9), (10), we obtain that the considered manifold belongs to the class \(W_4(J_1) \cap W_1(J_2) \cap W_1(J_3)\). Remark that, according to [10], necessary and sufficient conditions a 4-dimensional almost hypercomplex HN-manifold to be in the class \(W = W_4(J_1) \cap W_1(J_2) \cap W_1(J_3)\) are:

\[\theta_2 \circ J_2 = \theta_3 \circ J_3 = -2 (\theta_1 \circ J_1).\]  \hfill (25)

These conditions are satisfied bearing in mind (24).

Let us consider the class \(W^0 = \{W \mid d((\theta \circ J_1) = 0)\}\), which is the class of the (locally) conformally equivalent \(K\)-manifolds, where a conformal transformation of the metric is given by \(\bar{g} = e^{2u}g\) for a differential function \(u\) on the manifold.

Using (24) and (25), we establish that the considered manifold belongs to the subclass \(W^0\).

**Proposition 2.3.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (21), belongs to the class \(W^0\) of the (locally) conformally equivalent \(K\)-manifolds.

### 2.4. Hypercomplex HN-metric structure of type (hc4)

In this case, \(g\) is solvable and the derived Lie algebra \(g'\) is 3-dimensional and Abelian.

#### 2.4.1. Firstly, we fix \(e_1 \in g\), for which \(g(e_1, e_1) = 1\), as an element orthogonal to \(g'\) with respect to \(g\). Therefore \(g\) is determined by

\[[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4.\]  \hfill (26)

Then we compute covariant derivatives and the nontrivial ones are

\[\nabla_{e_2} e_1 = -e_2, \quad \nabla_{e_3} e_1 = -e_3, \quad \nabla_{e_4} e_1 = -e_4,\]

\[\nabla_{e_2} e_2 = -\nabla_{e_3} e_3 = -\nabla_{e_4} e_4 = e_1.\]  \hfill (27)
By similar computation as in the previous cases, the components \((F_\alpha)_{ijk}\), \(\alpha = 1, 2, 3\), are as follows:

\[
(F_1)_{314} = -(F_1)_{323} = (F_1)_{332} = -(F_1)_{341} = \\
= -(F_1)_{413} = -(F_1)_{424} = (F_1)_{431} = (F_1)_{442} = 1; \\
(F_2)_{311} = (F_2)_{333} = -2,
(F_2)_{214} = -(F_2)_{223} = -(F_2)_{232} = (F_2)_{241} \\
= (F_2)_{412} = (F_2)_{421} = (F_2)_{434} = (F_2)_{443} = -1;
(F_3)_{411} = (F_3)_{444} = 2,
(F_3)_{213} = (F_3)_{224} = (F_3)_{231} = (F_3)_{242} \\
= (F_3)_{312} = (F_3)_{321} = -(F_3)_{334} = -(F_3)_{343} = -1.
\]

The only non-zero components of the corresponding Lee forms are

\[
(\theta_1)_2 = -(\theta_2)_3 = (\theta_3)_4 = -2. 
\]

The results in (28), (29) and the classification conditions (9), (10) imply

**Proposition 2.4.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (26), belongs to the largest class of the considered manifolds, i.e. \(\mathcal{H}\), as well as this manifold does not belong to neither \(\mathcal{W}_1\) nor \(\mathcal{W}_2\) for \(J_2\) and \(J_3\).

2.4.2. Secondly, we choose \(e_4 \in g\), for which \(g(e_4, e_4) = -1\), as an element orthogonal to \(g'\) with respect to \(g\). Therefore, in this case \(g\) is determined by

\[
[e_4, e_1] = e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = e_3.
\]

Therefore, the nontrivial covariant derivatives are

\[
\nabla e_1 e_1 = \nabla e_2 e_2 = -\nabla e_3 e_3 = -e_4,
\n\nabla e_1 e_4 = -e_1, \quad \nabla e_2 e_4 = -e_2, \quad \nabla e_3 e_4 = -e_3.
\]

In a similar way we obtain:

\[
(F_1)_{113} = (F_1)_{124} = -(F_1)_{131} = -(F_1)_{142} = \\
= -(F_1)_{214} = (F_1)_{223} = -(F_1)_{232} = (F_1)_{241} = -1; \\
(F_2)_{222} = (F_2)_{244} = -2,
(F_2)_{112} = (F_2)_{121} = (F_2)_{134} = (F_2)_{143} \\
= (F_2)_{314} = -(F_2)_{323} = -(F_2)_{332} = (F_2)_{341} = -1; \\
(F_3)_{111} = (F_3)_{144} = 2,
-(F_3)_{212} = -(F_3)_{221} = (F_3)_{234} = (F_3)_{243} \\
= (F_3)_{313} = (F_3)_{324} = (F_3)_{331} = (F_3)_{342} = -1.
\]
The only non-zero components of the corresponding Lee forms are

\[(\theta_1)_3 = -2, \quad (\theta_2)_2 = -(\theta_3)_1 = -4. \quad (33)\]

Then, analogously of Case 2.3.2 we obtain the following

**Proposition 2.5.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (30), belongs to the class \(W^0\) of the (locally) conformally equivalent K-manifolds.

### 2.5. Hypercomplex HN-metric structure of type (hc5)

In this case, \(g\) is solvable and \(g'\) is a 3-dimensional Heisenberg algebra.

#### 2.5.1.

Firstly, we fix \(e_1 \in g\), for which \(g(e_1, e_1) = 1\), as an element orthogonal to \(g'\) with respect to \(g\). Then \(g\) is determined by

\[ [e_1, e_2] = e_2, \quad [e_1, e_3] = \frac{1}{2} e_3, \quad [e_1, e_4] = \frac{1}{2} e_4, \quad [e_3, e_4] = \frac{1}{2} e_2. \quad (34) \]

Then we compute covariant derivatives and the nontrivial ones are

\[
\begin{align*}
\nabla_{e_2} e_2 &= -2 \nabla_{e_3} e_3 = -2 \nabla_{e_4} e_4 = e_1, \\
-\nabla_{e_2} e_1 &= 4 \nabla_{e_3} e_4 = -4 \nabla_{e_4} e_3 = e_2, \\
-4 \nabla_{e_2} e_4 &= -2 \nabla_{e_3} e_1 = -4 \nabla_{e_4} e_2 = e_3, \\
4 \nabla_{e_2} e_3 &= 4 \nabla_{e_3} e_2 = -2 \nabla_{e_4} e_1 = e_4. \\
\end{align*}
\]

Analogously of the previous cases we obtain the non-zero components \((F_\alpha)_{ijk}\), \(\alpha = 1, 2, 3\), as follows:

\[
(F_1)_{314} = -(F_1)_{323} = (F_1)_{332} = -(F_1)_{341} = \\
= -(F_1)_{413} = -(F_1)_{424} = (F_1)_{431} = (F_1)_{442} = \frac{1}{4};
\]

\[
(F_2)_{214} = -(F_2)_{223} = -(F_2)_{232} = (F_2)_{241} = -\frac{5}{4},
\]

\[
(F_2)_{311} = -2(F_2)_{322} = -(F_2)_{333} = -2(F_2)_{344} = -1,
\]

\[
(F_2)_{412} = (F_2)_{421} = (F_2)_{434} = (F_2)_{443} = \frac{3}{4},
\]

\[
(F_3)_{213} = (F_3)_{224} = (F_3)_{231} = (F_3)_{242} = -\frac{5}{4},
\]

\[
(F_3)_{312} = (F_3)_{321} = -(F_3)_{334} = -(F_3)_{343} = -\frac{3}{4},
\]

\[
(F_3)_{411} = -2(F_3)_{422} = -2(F_3)_{433} = (F_3)_{444} = 1.
\]

The only non-zero components of the corresponding Lee forms are

\[
(\theta_1)_2 = -\frac{1}{2}, \quad (\theta_2)_3 = -(\theta_3)_4 = 3. \quad (37)
\]
The results in (36), (37) and the classification conditions (9), (10) imply

**Proposition 2.6.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (34), belongs to the largest class of the considered manifolds, i.e. $\mathcal{H}$, as well as this manifold does not belong to neither $\mathcal{W}_1$ nor $\mathcal{W}_2$ for $J_2$ and $J_3$.

2.5.2. The other possibility is to choose $e_4 \in g$, for which $g(e_4, e_4) = -1$, as an element orthogonal to $g'$ with respect to $g$. We rearrange the basis in (34) and then $g$ is determined by

\[
[e_1, e_2] = -\frac{1}{2}e_3, \quad [e_1, e_4] = -\frac{1}{2}e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2, \quad [e_3, e_4] = -e_3. \tag{38}
\]

By similar computations we establish the same statement as of Proposition 2.6 for the Heisenberg algebra introduced by (35).

**References**

[1] Alekseevsky, D.V., Marchiafava, S.: Quaternionic structures on a manifold and subordinated structures. Ann. Mat. Pura Appl. **CLXXI** (IV), 205–273 (1996)

[2] Barberis, M.L.: Hypercomplex structures on four-dimensional Lie groups. Proc. AMS **128** (4), 1043–1054 (1997)

[3] Barberis, M.L., Dotti, I.: Abelian complex structures on solvable Lie algebras. J. Lie Theory **14** (1), 25–34 (2004)

[4] Barberis, M.L., Dotti, I., Miatello, R.: On certain locally homogeneous Clifford manifolds, Ann. Glob. Anal. Geom. **13**, 289–301 (1995)

[5] Dotti, I., Fino, A.: Hyper-Kähler with torsion structures invariant by nilpotent Lie groups, Class. Quantum Grav. **19**, 1–12 (2002)

[6] Fino, A., Grantcharov, G.: Properties of manifolds with skew-symmetric torsion and special holonomy, Adv. Math. **189**, 439–450 (2004)

[7] Ganchev, G., Borisov, A.: Note on the almost complex manifolds with a Norden metric. C. R. Acad. Bulgare Sci. **39**, 31–34 (1986)

[8] Gray, A., Hervella, L.M.: The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann. Mat. Pura Appl. **CXXIII** (IV), 35–58 (1980)

[9] Gribachev, K., Manev, M.: Almost hypercomplex pseudo-Hermitian manifolds and a 4-dimensional Lie group with such structure. J. Geom. **88** (1-2), 41–52 (2008)

[10] Gribachev, K., Manev, M., Dimiev, S.: On the almost hypercomplex pseudo-Hermitian manifolds. In: Dimiev, S., Sekigawa, K. (eds.) Trends of Complex Analysis, Differential Geometry and Mathematical Physics, pp. 51–62. World Sci. Publ., Singapore (2003)

[11] Manev, M.: A connection with parallel torsion on almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics. J. Geom. Phys. **61** (1), 248–259 (2011)

[12] Ovando, G.: Invariant complex structures on solvable real Lie groups. Manuscripta Math. **103**, 19–30 (2000)
[13] Snow, J. E.: *Invariant complex structures on four-dimensional solvable real Lie groups*. Manuscripta Math. **66**, 397–412 (1990)

[14] Sommese, A.: *Quaternionic manifolds*, Math. Ann. **212**, 191–214 (1975)

Mancho Manev  
Paisii Hilendarski University of Plovdiv  
Faculty of Mathematics, Informatics and IT  
Department of Algebra and Geometry  
236 Bulgaria blvd  
Plovdiv 4027  
Bulgaria  

e-mail: mmanev@uni-plovdiv.bg