ON SPECIAL RIEMANN XI FUNCTION FORMULAE OF HARDY INVOLVING THE DIGAMMA FUNCTION

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Abstract. We consider some properties of integrals considered by Hardy and Koshliakov, that have have connections to the digamma function. We establish a new general integral formula that provides a connection to the polygamma function. We also obtain lower and upper bounds for Hardy’s integral through properties of the digamma function.

Keywords: Fourier Integrals; Riemann xi function; Digamma function.

2010 Mathematics Subject Classification 11M06, 33C15.

1. Introduction and Main formulas

In a paper written by the well-known G. H. Hardy [10], an interesting integral formula is presented (corrected in [3])

\[
\int_0^\infty \frac{\Xi(t/2)}{1 + t^2 \cosh(\pi t/2)} \frac{\cos(xt)}{\tanh(t/2)} dt = \frac{1}{4} e^{-x} \left( 2x + \frac{1}{2} \gamma + \frac{1}{2} \log(\pi) + \log(2) \right) + \frac{1}{2} e^x \int_0^\infty \psi(t+1)e^{-\pi t^2 e^{4x}} dt,
\]

where \( \psi(x) := \frac{d}{dx} \log(\Gamma(x)) \), \( \Gamma(x) \) being the gamma function [1, pg.1], and \( \Xi(t) := \xi(\frac{1}{2} + it) \), where [11, 15]

\[
\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).
\]

Here we have used the standard notation for the Riemann zeta function \( \zeta(s) := \sum_{n \geq 1} n^{-s} \), for \( \Re(s) > 1 \). Koshliakov [12, eq.(14), eq.(20)] (or [3, eq.(1.15)]) produced this formula as well, but in a slightly different form,

\[
2 \int_0^\infty \frac{\Xi(t/2)}{1 + t^2 \cosh(\pi t/2)} \frac{\cos(xt)}{\tanh(t/2)} dt = e^x \int_0^\infty (\psi(t+1) - \log(t))e^{-\pi t^2 e^{4x}} dt.
\]

Here the trick, which has been exploited in the many studies [3, 4, 5, 6, 7, 12, 14], is to re-write the left side of (1.1) as an inverse Mellin transform by utilizing the classical functional equation \( \xi(s) = \xi(1-s) \) [15]. Titchmarsh has used simpler but similar integrals than the left side of (1.1) to obtain Hardy’s result that \( \Xi(t) \)
has infinitely many real zeros [15]. See Csordas [2] for some more recent work in this direction. A recent paper by Dixit offers a beautiful generalization of formula (1.1), which involves a confluent hypergeometric function [3, Theorem 1.3]. Other refinements can be found in [4, 5].

The purpose of this paper is to offer some further results concerning (1.1) that appear to have been overlooked. A new generalization is offered as well that takes a different route than the ones offered by [3, 4, 5], considering what is essentially the polygamma function [9, pg.904, eq. NH 37(1)]. In the following section, we offer some new inequalities that we noticed upon examining the integral given in (1.1).

Recall the Mellin transform of a suitable function \( f(t) \) is given by [11, pg.90, eq.(4.105)]

\[
\int_0^\infty f(t)t^{s-1}dt = F(s),
\]

and its inverse,

\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)t^{-s}ds,
\]

provided \( c \) is a real number chosen where \( F(s) \) is analytic. We will be applying Parseval’s formula for Mellin transforms throughout the proofs of our theorems [15, pg.34, eq.(2.15.10)]

\[
\int_0^\infty f(t)g(t)dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)G(1-s)ds.
\]

Here the needed condition for validity is to have \( c \) chosen as a real number restricted to the region where \( F(s)G(1-s) \) is analytic.

**Theorem 1.1.** For \( m \in \mathbb{N}_0 \), we have that

\[
(-1)^m e^x \int_0^\infty t^m \bar{\psi}_m(t)e^{-\pi t^2 e^{ix}} dt
\]

\[
= \frac{1}{2} \int_0^\infty \frac{\Xi(t)}{\cosh(\pi t)(t^2 + \frac{1}{4})} \left( \frac{\Gamma(\frac{1}{2} - it + m)}{\Gamma(\frac{1}{2} - it)} e^{-2\pi t} + \frac{\Gamma(\frac{1}{2} + it + m)}{\Gamma(\frac{1}{2} + it)} e^{2\pi t} \right) dt,
\]

\[
\bar{\psi}_m(t) := \frac{\partial^m}{\partial t^m} (\psi(t + 1) - \log(t)).
\]

**Proof.** This is a special application of Parseval’s theorem for Mellin transforms with one function chosen as \( f(t) = e^{-\pi t^2} \), and the other as \( g(t) = t^m \bar{\psi}_m(t) \). From
Titchmarsh [15, eq.(2.15.7)] we have for \(0 < c < 1\),

\[
\psi(x + 1) - \log(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi \zeta(1-s)}{\sin(\pi s)} x^{-s} ds.
\]

Differentiating (1.3) \(m\) times, gives us

\[
\psi_m(x) = (-1)^{m-1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi \zeta(1-s) \Gamma(s+m)}{\Gamma(s)} x^{-s-m} ds.
\]

This may also be obtained from the integral formula

\[
\psi_m(x) = (-1)^m \frac{1}{t} \int_0^{\infty} \left( 1 - \frac{1}{e^t - 1} \right) t^m e^{-xt} dt,
\]

valid for \(\Re(s) > -m\). Equation (1.4) gives us the Mellin transform for our \(g(t)\).

Now proceeding with our choices for \(f(t)\) and \(g(t)\) in the beginning of the proof, and noting \(\sin((\frac{1}{2} + it)\pi) = \cosh(\pi t)\), the functional equation \(\xi(s) = \xi(1-s)\) with (1.4), the Mellin transform of \(f(t)\), \(\pi^{-s/2}x^{-s} \Gamma(s/2)\), we have

\[
(-1)^m e^x \int_0^{\infty} t^m \bar{\psi}_m(t) e^{-\pi t^2 e^{4x}} dt = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\pi \xi(s) \Gamma(1-s+m)}{\sin(\pi s) s(s-1)\Gamma(1-s)} x^{-s} ds,
\]

for \(0 < c' < 1\). Put \(c' = \frac{1}{2}\) and we find the right side is equal to

\[
\frac{1}{2\sqrt{x}} \int_{-\infty}^{\infty} \frac{\Xi(t)}{(t^2 + \frac{1}{4}) \cosh \pi t} \frac{\Gamma(\frac{1}{2} - it + m)}{\Gamma(\frac{1}{2} - it)} e^{-it} dt.
\]

Splitting this bilateral integral up into two integrals and replacing \(x\) by \(e^{2x}\) leads to the result. \(\square\)

Koshliakov [12, eq.(36), eq.(40)] gives a similar integral to (1.1) but squaring the integrand \(\Xi(t)(t^2 + \frac{1}{4})^{-1}\),

\[
\int_0^{\infty} \frac{\Xi(t)}{(1 + t^2)^2} \frac{\cos(xt)}{\cosh(\pi t/2)} dt = e^{x/2} \int_0^{\infty} K_0(2\pi e^x t) \Lambda(t) dt,
\]

where

\[
\Lambda(t) = \zeta(2) + \gamma^2 - 2\gamma_1 + 2\gamma \log(t) + \frac{1}{2} \log^2(t) + \sum_{n \geq 1} d(n) \left( \frac{1}{t + n} - \frac{1}{n} \right).
\]

Here \(d(n)\) denotes the number of divisors of \(n\), and the Stieltjes constant is

\[
\gamma_1 = \lim_{N \to \infty} \left( \sum_{1 \leq n \leq N} \frac{\log(n)}{n} - \frac{\log(N)^2}{2} \right).
\]

Moll and Dixit state his result in [6, Theorem 4.4], and offer further interesting generalizations. However, we noticed a slightly different form of this integral based on the type of proof we used in the previous theorem.
Theorem 1.2. Let \( K_s(t) \) denote the modified Bessel function. For \( x > 0 \),

\[
\int_0^\infty (\psi(t+1) - \log(t)) \left( \frac{2\pi e^{x/2}}{t} - e^{-x/4} \sum_{n\geq 1} K_0(2\pi nte^{-x}) \right) dt = \int_0^\infty \frac{\Xi^2(t)}{(t^2 + \frac{1}{4})^2 \cosh(\pi t)} dt.
\]

Proof. First, since \([9, pg.920, eq. WA 231, 245(9)] K_0(t) = O(t^{-1/2}e^{-t})\), for \( t \to \infty \) in \( |\arg(t)| < 3\pi/2 \) where as usual \( K_s(t) \) is the modified Bessel function \([9, pg.928]\), it follows that the series on the right side of the formula (by \([9, pg. 676, eq. EH II 9(27)]\))

\[
\sum_{n\geq 1} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma^2\left(\frac{s}{2}\right) \left( \frac{x}{2n} \right)^{-s} ds = 4 \sum_{n\geq 1} K_0(n\pi x),
\]

converges absolutely. Summing through, we find for \( d' > 0 \),

\[
\frac{1}{2\pi i} \int_{d'-i\infty}^{d'+i\infty} \Gamma^2\left(\frac{s}{2}\right) \left( \frac{x}{2} \right)^{-s} \zeta(s) ds = 4 \sum_{n\geq 1} K_0(n\pi x),
\]

which has a simple pole at \( s = 1 \). Note that \( \lim_{s\to 1} (s-1)\Gamma^2\left(\frac{s}{2}\right) \left( \frac{x}{2} \right)^{-s} \zeta(s) = \frac{2\pi}{x} \), since \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) and \( \lim_{s\to 1} (s-1)\zeta(s) = 1 \). If we calculate this residue and move the line of integration to \( \Re(s) = b, 0 < b < 1 \), we find

\[
\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma^2\left(\frac{s}{2}\right) \left( \frac{x}{2} \right)^{-s} \zeta(s) ds + \frac{2\pi}{x} = 4 \sum_{n\geq 1} K_0(n\pi x).
\]

Now noting (1.3) and (1.6) in conjunction with Parseval’s theorem for Mellin transforms, we obtain the result after applying the functional equation for \( \xi(s) \). □

Other associated integral formulae related to Fourier cosine transforms may be obtained from using known evaluations.

Theorem 1.3. We have for \( |\beta'| > \frac{\pi}{2} > |\Re(\beta')|\),

\[
\int_0^\infty \frac{\cosh(t)}{\cosh(2t) + \cosh(2\beta')} \left( e^{t/2} - 2e^{-t/2} \sum_{n\geq 1} e^{-\pi n^2e^{-2t}} \right) dt
\]

\[
= \frac{\pi e^{\beta'}}{4 \cosh(\beta')} \int_0^\infty (\psi(t+1) - \log(t))e^{-\pi t^2 e^{4\beta'}} dt.
\]

Proof. We apply the classical [15, eq.(2.16.2)]

\[
\int_0^\infty \frac{\Xi(t)}{(t^2 + \frac{1}{4})} \cos(xt) dt = \frac{\pi}{2} \left( e^{x/2} - 2e^{-x/2} \sum_{n\geq 1} e^{-\pi n^2e^{-2x}} \right),
\]
with an integral from [9, pg.511, eq. ET I 31(16)], valid for \( \pi \Re(\alpha) > \Im(\alpha \beta) \), and
\[
\int_0^\infty \cos(ty) \cosh(t\alpha/2) dt = \frac{\pi}{2\alpha} \frac{\cos(\beta y/\alpha)}{\cosh(\beta/2) \cosh(\pi y/\alpha)}.
\]
We need only let \( \beta \to 2\beta' \), and \( \alpha = 2 \), and then we have \( \frac{\pi}{2} > |\Im(\beta')| \). This coupled with Parseval’s theorem gives the result. \[\Box\]

The integral formula in Theorem 1.3 may be useful for fast numerical approximations, since the integrand of the integral on the left side is comprised entirely of exponentials. A possible avenue would be to truncate the series at the first term, since it is a double exponential. Note that the integral on the left side of (1.7) hints to us about the non-negativity of the integral on the right side. We will investigate this further in the next section.

### 2. Inequalities

We first start with some preliminaries about some special functions which can be found in [1]. First we recall the error function [9, pg.887],
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,
\]
and its complement, \( \text{erfc}(x) = 1 - \text{erf}(x) \). Also, \( \text{erfi}(x) := -i \text{erf}(ix) \). We also use the exponential integral [9, pg.883]
\[
\text{Ei}(x) = -\int_{-x}^\infty \frac{e^{-t}}{t} dt.
\]

**Theorem 2.1.** Define the Hardy integral for \( y > 0 \) as
\[
I(y) := \int_0^\infty (\psi(t + 1) - \log(t))e^{-yt^2} dt.
\]
Then we have,
\[
\frac{(2c_1)^2\sqrt{y}}{\sqrt{\pi \text{erfi}(\sqrt{y})}} - \frac{\text{Ei}(-y)}{4} - \frac{e^{-y}}{12} + \frac{\sqrt{y \pi}}{12} \text{erfc}(\sqrt{y}) \leq I(y).
\]
Furthermore,
\[
I(y) \leq \sqrt{c_2} \sqrt{\frac{\pi}{2y}} \text{erf}(\sqrt{2y}) - \frac{\text{Ei}(-y)}{4},
\]
where \( c_1 = 0.952894 \), and \( c_2 = 1.56624 \).

**Proof.** We first need a result from the paper [13, Corollary 1], for \( x > 0 \),
\[
\frac{1}{2x} - \frac{1}{12x^2} < \psi(x + 1) - \log(x) < \frac{1}{2x}.
\]
For the upper bound we first write,

\[ I(y) = \int_0^1 (\psi(t + 1) - \log(t)) e^{-yt^2} dt + \int_1^{\infty} (\psi(t + 1) - \log(t)) e^{-yt^2} dt. \]

Applying (2.1) we have that this is

\[ \leq \int_0^1 (\psi(t + 1) - \log(t)) e^{-yt^2} dt - \frac{\text{Ei}(-y)}{4}. \]

By Schwarz’s inequality and the fact that

\[ \int_0^1 (\psi(t + 1) - \log(t))^2 dt = 1.56624 \]

we have the upper bound. For the lower bound, we write the Schwarz’s inequality in the form

\[ \left( \int_0^1 (\psi(t + 1) - \log(t))^{1/2} dt \right)^2 \leq \left( \int_0^1 (\psi(t + 1) - \log(t)) e^{-yt^2} dt \right) \left( \int_0^1 e^{yt^2} dt \right), \]

and also note,

\[ \int_0^1 \sqrt{\psi(t + 1) - \log(t)} dt = 0.952894. \]

This calculation, coupled with similar computations derived from the left side of (2.1) and the definitions of the special integral functions, gives our result. □

Now it can be seen that the lower bound actually implies the Hardy integral \( I(x) \) is non-negative for \( x > 0 \), and also the squeeze theorem shows that \( I(x) \to 0 \) when \( x \to \infty \).

### 3. Other observations and further comments

While studies such as [6] show there is a plethora of integral formulae like Hardy’s (1.1) to be found, it would be interesting to see if these integrals could be incorporated into similar work like Csordas [2]. We were interested in seeing if (1.1) could be adapted to imply information beyond Hardy’s theorem, that \( \Xi(t) \) has infinitely many real zeros. The classical ideas are included in Titchmarsh’s text [15, pg. 256-260], and are based around the idea of first assuming \( \Xi(t) \) is of one sign, and then arriving at a contradiction from properties of an integral of the form \( \int_0^{\infty} \Xi(t)k(t) dt, k(t) \) non-negative for \( t > 0 \). The idea in [15, pg. 260] considers the
moments \( k(t) = t^{2n} \). Apply the operator \( \bar{\partial}_x = -\partial_x^2 + \frac{1}{4} \) of Glasser \([8]\) to (1.1) (after replacing \( t \) by \( 2t \)) and set \( x = x' + i\pi \) to get

\[
\lim_{x \to x' + i\pi} \bar{\partial}_x(e^{x/2}I(\pi e^{2x})) = \int_0^\infty \Xi(t) \cos(x't)dt + i \int_0^\infty \Xi(t) \frac{\sin(x't) \sinh(\pi t)}{\cosh(\pi t)}dt.
\]

Differentiating \( 2n \) times and setting \( x' = 0 \) gives us the right side as the value \((-1)^n \int_0^\infty t^{2n} \Xi(t)dt \). Note that the integral \( I(x) \) clearly converges only for \( \Re(x) > 0 \), and hence \( I(\pi e^{2z}) \) converges when \(-\frac{\pi}{4} < \Im(z) < \frac{\pi}{4} \). It would be interesting if new information could be obtained about \( \Xi(t) \) from alternative forms of these moments like those found here.

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