The perturbation bound of the extended vertical linear complementarity problem∗

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Abstract

In this paper, we discuss the perturbation analysis of the extended vertical linear complementarity problem (EVLCP). Under the assumption of the row \( W \)-property, several absolute and relative perturbation bounds of EVLCP are given, which can be reduced to some existing results. Some numerical examples are given to show the proposed bounds.

Keywords: The extended vertical linear complementarity problem; the row \( W \)-property; the perturbation bound

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1 Introduction

Let \( M_i \in \mathbb{R}^{n \times n} \) and \( q_i \in \mathbb{R}^n \), \( i = 0, 1, \ldots, k \). The following minimization equation is to find \( x \in \mathbb{R}^n \) such that

\[
\min\{M_0x + q_0, M_1x + q_1, \ldots, M_kx + q_k\} = 0,
\]

where \( \min \) is the component minimum operator, which can be reduced to the following two models:

\[
\min\{x, Mx + q\} = 0,
\]

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where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, and

$$\min \{ x, M_1 x + q_1, \ldots, M_k x + q_k \} = 0. \quad (1.3)$$

The model (1.1) is called the extended vertical linear complementarity problem (EVLCP), denoted by EVLCP $(M, q)$ with $M = (M_0, M_1, \ldots, M_k)$ and $q = (q_0, q_1, \ldots, q_k)$, the models (1.2) and (1.3) are called the linear complementarity problem (LCP) (e.g., see [6, 19]) and the vertical linear complementarity problem (VLCP) (e.g., see [5, 25]), respectively.

A very important problem in the computational sciences is that how the solution variation is when the data is perturbed. More specifically, for the model (1.1), when $\Delta M_i$ and $\Delta q_i$ are the perturbation of $M_i$ and $q_i$, respectively, $i = 0, 1, \ldots, k$, how do we characterize the change in the solution of the following perturbed model:

$$\min \{ \tilde{M}_0 y + \tilde{q}_0, \tilde{M}_1 y + \tilde{q}_1, \ldots, \tilde{M}_k y + \tilde{q}_k \} = 0, \quad (1.4)$$

where $\tilde{M}_i = M_i + \Delta M_i$, and $\tilde{q}_i = q_i + \Delta q_i$, $i = 0, 1, \ldots, k$. This problem has been extensively studied for the model (1.2), which is often used to deduce the sensitivity and stability analysis [6] for solving the LCP $(M, q)$. A classical result on the error bound of the LCP $(M, q)$ was given in [17] by Mathias and Pang. In order to introduce the perturbation result for the model (1.2), we first give some definitions and notations.

A matrix $M$ is called a $P$-matrix if all principal minors of $M$ are positive, in this case the model (1.2) has the unique solution (e.g., see [6]). Let

$$c(M) := \min_{\|x\| = 1} \max_{1 \leq i \leq n} x_i (Mx).$$

By the above definition, the perturbation bound for LCP (1.2) was given as follows.

Lemma 1.1 (7.3.10 Lemma, [6]) For $M$ being a $P$-matrix, the following results hold:

(a) For any two vectors $q$ and $\tilde{q}$ in $\mathbb{R}^n$,

$$\|x^* - x^*\| \leq \frac{1}{c(M)} \|q - \tilde{q}\|, \quad (a)$$

where $x^*$ and $x^*$ denote the unique solutions of the LCPs $(M, q)$ and $(M, \tilde{q})$, respectively.

(b) For each vector $q \in \mathbb{R}^n$, there exist a neighborhood $U$ of the pair $(M, q)$ and a constant $c_0 > 0$ such that for any $(\tilde{M}, \tilde{q}), (\hat{M}, \hat{q}) \in U$, $\tilde{M}, \hat{M}$ are $P$-matrices and

$$\|x - y\| \leq c_0 (\|\tilde{q} - \hat{q}\| + \|\tilde{M} - \hat{M}\|), \quad (b)$$

where $x$ and $y$ denote the unique solutions of the LCPs $(\tilde{M}, \tilde{q})$ and $(\hat{M}, \hat{q})$, respectively.
Lemma 1.1 is very important for studying the sensitivity and stability analysis of the LCP model theoretically. Alternatively, Chen and Xiang in [3] provided the following sharper perturbation bounds than the ones in Lemma 1.1 by introduced the following constant:

\[
\beta_p(M) := \max_{d \in [0,1]^n} \| (I - D + DM)^{-1} D \|_p
\]

for a \( P \)-matrix, where \( I \) is the identity matrix, \( D \) is a diagonal matrix whose diagonal entry is in \([0,1]\) and \( \| \cdot \|_p \) denotes the \( p \)-norm with \( p \geq 1 \).

**Lemma 1.2 (Theorem 2.8 of [3])** For \( M \) being a \( P \)-matrix, the following results hold:

(a) For any two vectors \( q \) and \( \tilde{q} \) in \( \mathbb{R}^n \),

\[
\| x^* - x^* \|_p \leq \beta_p(M) \| q - \tilde{q} \|_p,
\]

where \( x^* \) and \( x^* \) denote the unique solutions of the LCPs \((M, q)\) and \((M, \tilde{q})\), respectively.

(b) For \( \hat{M}, \bar{M} \in \mathcal{M} := \{ A \mid \beta_p(M) \| M - A \|_p \leq \eta < 1 \} \) and \( \bar{q}, \hat{q} \in \mathbb{R}^n \),

\[
\| x - y \|_p \leq \frac{\beta_p^2(M)}{(1 - \eta)^2} \| (\tilde{q})_+ \|_p \| \bar{M} - \hat{M} \|_p + \frac{\beta_p(M)}{1 - \eta} \| \bar{q} - \hat{q} \|_p,
\]

where \( x \) and \( y \) denote the unique solutions of the LCPs \((\hat{M}, \bar{q})\) and \((\bar{M}, \tilde{q})\), respectively.

It is difficult to compute \( c(M), c_0 \) and \( \beta_p(M) \). To overcome this drawback, Chen and Xiang in [3] provided some computable bounds for \( M \) being an \( H \)-matrix with positive diagonals, a symmetric positive definite matrix, a positive definite matrix, respectively.

The algorithm, applications and the existence of solutions for the model (1.1) have been given (see, e.g., [8–14, 18, 20, 21, 23, 24, 26]). However, so far, to our knowledge, the perturbation analysis of models (1.1) and (1.3) has not been discussed. In order to fill in this study gap, in this paper, inspired by the success as in LCP model (1.2) as in [3], we focus on discussing the perturbation analysis for EVLCP (1.1) and VLCP (1.3). The contributions are given below:

- The framework of perturbation bound for the EVLCP model is proposed by developing the technique given in [3], from which one may derive the corresponding bound given in [3].
- Some computable bounds are also given for some special block matrices \( M = (M_0, M_1, \ldots, M_k) \).
- Some examples from discretization of Hamilton-Jacobi-Bellman (HJB) equation are given to show the proposed bounds.
The rest of the article is organized as follows. Section 2 is preliminary, in which we give some lemmas. In Section 3, we provide some perturbation bounds for the EVLCP \((M, q)\) by using a general equivalent form of the minimum function under the row \(W\)-property. In Section 4, some relative perturbation bounds are provided. In Section 5, some numerical examples are given to show the feasibility of the perturbation bound. Finally, in Section 6, we give some conclusion remarks to end this paper.

Finally, in this section we give some notations and definitions \([2, 13]\), which will be used in the sequel.

Let \(A = (a_{ij})\), \(B = (b_{ij}) \in \mathbb{R}^{n \times n}\) and \(N = \{1, 2, \ldots, n\}\). Then we denote \(|A| = (|a_{ij}|)\). The order \(A \geq (>) B\) means \(a_{ij} \geq (>) b_{ij}\) for any \(i, j \in N\).

\(A = (a_{ij})\) is called an \(M\)-matrix if \(A^{-1} \geq 0\) and \(a_{ij} \leq 0\) \((i \neq j)\) for \(i, j \in N\); an \(H\)-matrix if its comparison matrix \((A)\) (i.e., \((a)_{ii} = |a_{ii}|, (a)_{ij} = -|a_{ij}| \text{ if } i \neq j\) for \(i, j \in N\)) is an \(M\)-matrix; an \(H_+\)-matrix if \(A\) is an \(H\)-matrix with \(a_{ii} > 0\) for \(i \in N\); a strictly diagonal dominant (sdd) matrix if \(|a_{ii}| > \sum_{j \neq i} |a_{ij}|\), \(i \in N\); an irreducible diagonal dominant (idd) matrix if \(A\) is an irreducible, \(|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|\), \(i \in N\) and \(\{i \in N : |a_{ii}| > \sum_{j \neq i} |a_{ij}|\} \neq \emptyset\).

Let \(e = (1, 1, \ldots, 1)^T\) and by \(\rho(\cdot)\) we denote the spectral radius of a matrix. For a vector \(q \in \mathbb{R}^n\), by \(q_{+}\) and \(q_{-}\) we denote \(q_{+} = \max\{0, q\}\) and \(q_{-} = \max\{0, -q\}\).

In this paper, the norm \(\| \cdot \|_p\) means \(p\)-norm with \(p \geq 1\).

Let

\[
D = \{(D_0, D_1, \ldots, D_k) \mid D_i = \text{diag}(d_i) \text{ with } d_i \in [0, 1]^n \text{ (}i = 0, 1, \ldots, k\text{)} \text{ and } \sum_{i=0}^{k} D_i = I\}.
\]

A block matrix \(M = (M_0, M_1, \ldots, M_k)\) is said to be with \textbf{row \(W\)-property} if

\[
\min(M_0x, M_1x, \ldots, M_kx) \leq 0 \leq \max(M_0x, M_1x, \ldots, M_kx) \Rightarrow x = 0.
\]

It is noted that EVLCP \((M, q)\) has the unique solution for any \(q\) if and only if \(M\) has the row \(W\)-property (see Theorem 17 in \([13]\)).

## 2 Some lemmas

In this section, we give some basic lemmas, which will be used in the sequel. The first one is a general equivalent formula of the minimum function.

### Lemma 2.1

Let all \(a_i, b_i \in \mathbb{R}, i = 1, 2, \ldots, n\). Then there exist \(\lambda_i \in [0, 1]\) with \(\sum_{i=1}^{n} \lambda_i = 1\) such that

\[
\min_{1 \leq i \leq n} \{a_i\} - \min_{1 \leq i \leq n} \{b_i\} = \sum_{i=1}^{n} \lambda_i(a_i - b_i). \tag{2.1}
\]

### Proof.

The result follows immediately from the mean value theorem of Lipschitz functions with the generalized gradient (see 2.3.7 Theorem in \([4]\)). \(\Box\)
Lemma 2.2  The block matrix $M = (M_0, M_1, \ldots, M_k)$ has the row $W$-property if and only if $D_0M_0 + D_1M_1 + \ldots + D_kM_k$ is nonsingular for any $(D_0, D_1, \ldots, D_k) \in D$.

**Proof.** The lemma follows immediately from Lemma 2.1 and Theorem 3 (b) in [22]. □

Lemma 2.3  Let $a_i, b_i, t_i \in \mathbb{R}$ with $a_i > 0$, $t_i \in [0, 1]$ and $\sum_{i=1}^{n} t_i = 1$. Then

$$\frac{\sum_{i=1}^{n} t_i b_i}{\sum_{i=1}^{n} t_i a_i} \leq \max \left\{ \frac{|b_i|}{a_i} \right\}.$$

**Proof.** The result follows immediately from the Cauchy inequality. □

Lemma 2.4  Let $a, b \geq 0$. Then

$$\frac{t}{ta + (1 - t)b} \leq \frac{1}{a}$$

for any $t \in [0, 1]$.

**Proof.** Let

$$f(t) = \frac{t}{ta + (1 - t)b}.$$

Then

$$f'(t) = \frac{b}{(ta + (1 - t)b)^2} > 0,$$

it follows that $f(t)$ is a strictly monotone increasing function of $t$ with $t \in [0, 1]$. Therefore, when $t = 1$, we obtain that $f(t)_{\text{max}} = \frac{1}{a}$. □

## 3 Perturbation bounds

In this section, we always assume that the block matrix $M$ has the row $W$-property without further illustration. In this case, EVLCP $(M, q)$ has the unique solution $x^*$.

### 3.1 Framework of perturbation analysis for EVLCP

In this subsection, we discuss perturbation analysis of EVLCP when both $M$ and $q$ are perturbed to $\tilde{M} = (\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_k)$ and $\tilde{q} = (\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_k)$, respectively. Assume that $\tilde{M}$ has the row $W$-property. Then EVLCP $(\tilde{M}, \tilde{q})$ has the unique solution $y^*$.

Let $r_i = M_i x^* + q_i := (a_1^{(i)}, \ldots, a_n^{(i)})^T$ and $\tilde{r}_i = \tilde{M}_i y^* + \tilde{q}_i := (b_1^{(i)}, \ldots, b_n^{(i)})^T$. It follows from Lemma 2.1 that there exist $d_{\ell}^{(i)} \in [0, 1]$ with $\sum_{i=0}^{k} d_{\ell}^{(i)} = 1$ ($\ell = 1, 2, \ldots, n$) such that

$$\min_{0 \leq i \leq k} \{ a_1^{(i)} \} - \min_{0 \leq i \leq k} \{ b_1^{(i)} \} = \sum_{i=0}^{k} d_{\ell}^{(i)} (a_1^{(i)} - b_1^{(i)}).$$
Let \( \tilde{D}_i = \text{diag}(d_i^{(0)}, d_i^{(1)}, \ldots, d_i^{(k)}) \), \( i = 0, 1, \ldots, k \). Then it is easy to see that for each \( i \), \( \tilde{D}_i \) is a nonnegative diagonal matrix, and \( \sum_{i=0}^{k} \tilde{D}_i = I \) such that
\[
\min\{r_0, r_1, \ldots, r_k\} - \min\{\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_k\} = \sum_{i=0}^{k} \tilde{D}_i (r_i - \tilde{r}_i). \tag{3.1}
\]

Since \( x^* \) and \( y^* \) are the solution of EVLCP (\( M, q \)) and EVLCP (\( \tilde{M}, \tilde{q} \)), respectively, we have
\[
\sum_{i=0}^{k} \tilde{D}_i (r_i - \tilde{r}_i) = 0,
\]
which implies that
\[
(\sum_{i=0}^{k} \tilde{D}_i M_i)x^* = (\sum_{i=0}^{k} \tilde{D}_i \tilde{M}_i)y^* + \sum_{i=0}^{k} \tilde{D}_i (\tilde{q}_i - q_i).
\]
Hence we get
\[
\left( \sum_{i=0}^{k} \tilde{D}_i M_i \right)(x^* - y^*) = \left( \sum_{i=0}^{k} \tilde{D}_i (\tilde{M}_i - M_i) \right)y^* + \sum_{i=0}^{k} \tilde{D}_i (\tilde{q}_i - q_i). \tag{3.2}
\]

Let \( \tilde{S}_M = \sum_{i=0}^{k} \tilde{D}_i M_i \). It follows from Lemma 2.2 that \( \tilde{S}_M \) is nonsingular, and then by Eq. \( 3.2 \), we have
\[
x^* - y^* = \tilde{S}_M^{-1} \left[ \left( \sum_{i=0}^{k} \tilde{D}_i (\tilde{M}_i - M_i) \right)y^* + \sum_{i=0}^{k} \tilde{D}_i (\tilde{q}_i - q_i) \right] = \sum_{i=0}^{k} \tilde{S}_M^{-1} \tilde{D}_i [(\tilde{M}_i - M_i)y^* + (\tilde{q}_i - q_i)]. \tag{3.3}
\]

For any \((D_0, D_1, \ldots, D_k) \in \mathcal{D}\), setting
\[
S_M = \sum_{i=0}^{k} D_i M_i
\]
and
\[
\alpha_i(M) = \max_{(D_0, D_1, \ldots, D_k) \in \mathcal{D}} \|S_M^{-1} D_i\|, i = 0, 1, \ldots, k. \tag{3.4}
\]
Then
\[
\alpha_i(M) \geq \|\tilde{S}_M^{-1} \tilde{D}_i\|, i = 0, 1, \ldots, k.
\]
So, by (3.3) and (3.4) we have
\[ \| x^* - y^* \| \leq \sum_{i=0}^{k} \alpha_i(M) \| (\tilde{M}_i - M_i)y^* + (\tilde{q}_i - q_i) \|, \] (3.5)
which gives a perturbation bound for EVLCP.

**Theorem 3.1** Let \( M = (M_0, M_1, \ldots, M_k) \) have the row \( W \)-property and \( \alpha_i(M) \) be defined as in (3.4). Then for both block vectors \( q = (q_0, q_1, \ldots, q_k) \) and \( \tilde{q} = (\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_k) \), EVLCP \( (M, q) \) and EVLCP \( (M, \tilde{q}) \) have the unique solution \( x^* \) and \( y^* \), respectively. Furthermore,
\[ \| x^* - y^* \| \leq \sum_{i=0}^{k} \alpha_i(M) \| \tilde{q}_i - q_i \|. \] (3.6)

**Remark 3.1** Taking \( k = 1 \) and \( M_0 = I \), it is easy to see that \( \beta_p(M) = \alpha_1(M) \). Furthermore, taking \( q_0 = 0, \tilde{q}_0 = 0 \), the bound (3.6) reduces to the corresponding one in [3].

Let \( M = (M_0, M_1, \ldots, M_k) \) have the row \( W \)-property. Then we define \( \mathfrak{M}_\eta \) as follows:
\[ \mathfrak{M}_\eta := \{ A = (A_0, A_1, \ldots, A_k) \mid \sum_{i=0}^{k} \alpha_i(M) \| A_i - M_i \| \leq \eta < 1 \}, \]
where \( \alpha_i(M) \) is defined as in (3.4).

In order to get a general perturbation bound, it needs the following lemma:

**Lemma 3.1** The following statements hold:

1. A block matrix \( A = (A_0, A_1, \ldots, A_k) \in \mathfrak{M}_\eta \) is with the row \( W \)-property.
2. For any \( A \in \mathfrak{M}_\eta \), \( \alpha_i(A) \leq \tau_i(M) \), where
\[ \tau_i(M) = \frac{1}{1 - \eta} \alpha_i(M). \]

**Proof.** First, we show the assertion (1) holds. For any \( A = (A_0, A_1, \ldots, A_k) \in \mathfrak{M}_\eta \), since
\[
\| S^{-1}_M \sum_{i=0}^{k} D_i(A_i - M_i) \| = \| S^{-1}_M D_0(A_0 - M_0) + S^{-1}_M D_1(A_1 - M_1) \\
+ \ldots + S^{-1}_M D_k(A_k - M_k) \| \\
\leq \| S^{-1}_M D_0 \| \| A_0 - M_0 \| + \| S^{-1}_M D_1 \| \| A_1 - M_1 \| \\
+ \ldots + \| S^{-1}_M D_k \| \| A_k - M_k \| \\
\leq \sum_{i=0}^{k} \alpha_i(M) \| A_i - M_i \| \leq \eta < 1
\]
and
\[
\sum_{i=0}^{k} D_iA_i = S_M[I + S_M^{-1} \sum_{i=0}^{k} D_i(A_i - M_i)],
\]
(3.7)
it is known that the matrix \( \sum_{i=0}^{k} D_iA_i \) is nonsingular for any \((D_0, D_1, \ldots, D_k) \in \mathcal{D}. \) By Lemma 2.2, \( A \) has the row \( \mathcal{W} \)-property. This proves the assertion (1).

Next, we will prove the assertion (2). From (3.7) we have
\[
(\sum_{i=0}^{k} D_iA_i)^{-1} = [I + S_M^{-1} \sum_{i=0}^{k} D_i(A_i - M_i)]^{-1} S_M^{-1} D_i,
\]
which together with the definition of \( S_A \) gives
\[
\|S_A^{-1} D_i\| \leq \|[I + S_M^{-1} \sum_{i=0}^{k} D_i(A_i - M_i)]^{-1}\| S_M^{-1} D_i \|.
\]
It is easy to see that
\[
\|[I + S_M^{-1} \sum_{i=0}^{k} D_i(A_i - M_i)]^{-1}\| \leq \frac{1}{1 - \sum_{i=0}^{k} \alpha_i(M)} \|A_i - M_i\| \leq \frac{1}{1 - \eta}.
\]
Hence we have
\[
\|S_A^{-1} D_i\| \leq \frac{1}{1 - \eta} \|S_M^{-1} D_i\|.
\]
(3.8)
The desired bound follows from (3.8). This proves (2).

\( \square \)

**Lemma 3.2** Let \( M = (M_0, M_1, \ldots, M_k) \) have the row \( \mathcal{W} \)-property, and \( q = (q_0, q_1, \ldots, q_k) \). Then there exists \((\tilde{D}_0, \tilde{D}_1, \ldots, \tilde{D}_k) \in \mathcal{D} \) such that the unique solution \( x^* \) of EVLCP \((M, q)\) is given by
\[
x^* = -\sum_{i=0}^{k} \tilde{S}_M^{-1} \tilde{D}_i q_i,
\]
(3.9)
where \( \tilde{S}_M = \sum_{i=0}^{k} \tilde{D}_i M_i \).

**Proof.** Let \( r_i = M_i x^* + q_i := (a_1^{(i)}, \ldots, a_n^{(i)})^T, \ i = 0, 1, \ldots, k. \) Since \( x^* \) is the solution of the EVLCP \((M, q)\), we have \( \min\{r_0, r_1, \ldots, r_k\} = 0. \) This implies that for any \( s \) there is a vector \( r_i \) whose the \( s \)-th component is equal to zero, \( s = 0, 1, \ldots, k. \) Assume that the number of vectors in \( \{r_0, r_1, \ldots, r_k\} \) whose the \( s \)-th component is 0 is \( t_s, \) say \( a_s^{(i_1)} = a_s^{(i_2)} = \ldots = a_s^{(i_{t_s})} = 0. \) Taking non-negative diagonal matrices \( \tilde{D}_i = \text{diag}(d_1^{(i)}, \ldots, d_n^{(i)}) \) such that \( d_s^{(i_1)} = \ldots = d_s^{(i_{t_s})} = 1/t_s \) and \( d_s^{(i)} = 0, \ i \neq i_1, \ldots, i_{t_s}. \) Then it is easy to see that \((\tilde{D}_0, \tilde{D}_1, \ldots, \tilde{D}_k) \in \mathcal{D} \) and
\[
\sum_{i=0}^{k} (\tilde{D}_i(M_i x^* + q_i)) = \sum_{i=0}^{k} \tilde{D}_i r_i = 0.
\]
This implies
\[
\left( \sum_{i=0}^{k} \tilde{D}_i M_i \right) x^* = - \sum_{i=0}^{k} \tilde{D}_i q_i,
\]
from which it follows that
\[
x^* = - \left( \sum_{i=0}^{k} \tilde{D}_i M_i \right)^{-1} \sum_{i=0}^{k} \tilde{D}_i q_i.
\]
This proves the lemma.  

The following theorem is the framework of EVLCP perturbation.

**Theorem 3.2** Let both \( A = (A_0, A_1, \ldots, A_k) \) and \( B = (B_0, B_1, \ldots, B_k) \) be in \( \mathcal{M}_\eta \), and let \( \bar{q} = (\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_k) \) and \( \bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k) \). Then EVLCP \((A, \bar{q})\) and EVLCP \((B, \bar{p})\) have the unique solutions \( x^* \) and \( y^* \), respectively. Moreover, we have
\[
\|x^* - y^*\| \leq \left( \sum_{i=0}^{k} \tau_i(M) \|A_i - B_i\| \right) \left( \sum_{i=0}^{k} \tau_i(M) \|\bar{p}_i\| \right) + \sum_{i=0}^{k} \tau_i \|\bar{q}_i - \bar{p}_i\|,
\]
where \( \tau_i(M) \) is given by Lemma 3.1.

**Proof.** The first assertion follows from Lemma 3.1. Next we show the perturbation bound. By (3.5) we have
\[
\|x^* - y^*\| \leq \sum_{i=0}^{k} \alpha_i(A)(\|B_i - A_i\| \|y^*\| + \|\bar{p}_i - \bar{q}_i\|). \quad (3.11)
\]
It follows from Lemma 3.2 that
\[
y^* = - \sum_{i=0}^{k} \tilde{S}_B^{-1} \tilde{D}_i \bar{p}_i,
\]
where \( \tilde{S}_B = \sum_{i=0}^{k} \tilde{D}_i B_i \). Clearly we have
\[
\alpha_i(B) \geq \|\tilde{S}_B^{-1} \tilde{D}_i\|, \quad i = 0, 1, \ldots, k.
\]
Hence,
\[
\|y^*\| \leq \sum_{i=0}^{k} \alpha_i(B) \|\bar{p}_i\|. \quad (3.12)
\]
By (3.11), (3.12) and Lemma 3.1, we obtain the desired bound (3.10). \( \square \)
Next, we discuss perturbation analysis of VLCP \((M, q)\), i.e., \(M_0 = I\) and \(q_0 = 0\) for Eq. (1.1).

Let
\[
\tilde{S}_v^M = \tilde{D}_0 + k \sum_{i=1}^k \tilde{D}_i M_i, \quad S_v^M = D_0 + \sum_{i=1}^k D_i M_i
\]
and
\[
\tilde{S}_v^B = \tilde{D}_0 + \sum_{i=1}^k \tilde{D}_i B_i, \quad S_v^B = D_0 + \sum_{i=1}^k D_i B_i,
\]
where matrices \(\tilde{D}_i = \text{diag}(\tilde{d}_i)\) with \(\tilde{d}_i \in [0, 1]^n\) \((i = 0, 1, \ldots, k)\) are nonnegative diagonal and \(\sum_{i=0}^k \tilde{D}_i = I\), matrices \(D_i = \text{diag}(d_i)\) with \(d_i \in [0, 1]^n\) \((i = 0, 1, \ldots, k)\) are arbitrary nonnegative diagonal and \(\sum_{i=0}^k D_i = I\). Then we take
\[
\tilde{\alpha}_i(M) = \max_{(D_0, D_1, \ldots, D_k) \in D} \| (S_v^M)^{-1} D_i \|, \quad \tilde{\alpha}_i(B) = \max_{(D_0, D_1, \ldots, D_k) \in D} \| (S_v^B)^{-1} D_i \|, \quad i = 1, \ldots, k.
\]

(3.13)

Clearly,
\[
\tilde{\alpha}_i(M) \geq \| (S_v^M)^{-1} \tilde{D}_i \|, \quad \tilde{\alpha}_i(B) \geq \| (S_v^B)^{-1} \tilde{D}_i \|, \quad i = 1, \ldots, k.
\]

Notice that 0 is the solution of the VLCP \((B, p)\) and \(y^*\) is the unique solution of the VLCP \((B, p)\). Then we have
\[
\| y^* - 0 \| \leq \sum_{i=0}^k \tilde{\alpha}_i(B)\| (-p_i)_+ \|.
\]

By analogous proof to Theorem 3.1 and Theorem 3.2, we can obtain the following theorem.

**Theorem 3.3** Let \(M = (I, M_1, \ldots, M_k)\) have the row \(W\)-property and \(\tilde{\alpha}_i(M)\) be defined as in (3.13). Then the following statements hold:

(i) For any two block vectors \(q = (0, q_1, \cdots, q_k), \tilde{q} = (0, \tilde{q}_1, \cdots, \tilde{q}_k)\) with \(q_i, \tilde{q}_i \in \mathbb{R}^n\),
\[
\| x^* - y^* \| \leq \sum_{i=1}^k \tilde{\alpha}_i(M)\| \tilde{q}_i - q_i \|,
\]
where \(x^*\) and \(y^*\) are the unique solutions of VLCP \((M, q)\) and VLCP \((M, \tilde{q})\), respectively.

(ii) Every block matrix \(\tilde{A} = (I, A_1, \cdots, A_k) \in \tilde{M} := \{ \tilde{A} | \sum_{i=1}^k \tilde{\alpha}_i(M)\| A_i - M_i \| \leq \eta < 1 \}\) has the row \(W\)-property. Let
\[
\tilde{\tau}_i(M) = \frac{1}{1 - \eta} \tilde{\alpha}_i(M).
\]
Remark 3.2 Here we consider a special case. It is noted that \( \alpha \) in Theorems 3.2 and 3.3, respectively. However, it is difficult to compute quantities \( \alpha \).

Here we consider the case (1), i.e., all the diagonal entries of matrices \( M \) are positive. If \( M \) has the row \( W \)-property if and only if \( M \) is a \( P \)-matrix (see page 696 of [22]). By Remark 3.1, \( \beta_p(M) = \alpha_1(M) \), the bounds in Theorem 3.3 reduces to the corresponding ones in [3] (also see Lemma 1.2).

3.2 Estimations of \( \alpha_i(M) \) and \( \tilde{\alpha}_i(M) \)

Under assumption of the row \( W \)-property, we have given perturbation bounds of EVLCP and VLCP in Theorems 3.2 and 3.3, respectively. However, it is difficult to compute quantities \( \alpha_i(M) \) and \( \tilde{\alpha}_i(M) \). In this subsection, we explore computability estimations for \( \alpha_i(M) \) and \( \tilde{\alpha}_i(M) \) for the special block matrix \( M \).

To calculate \( \alpha_i(M) \) and \( \tilde{\alpha}_i(M) \), we consider two types of special matrices: (1) all the diagonal entries of the matrices \( M_i \) in \( M \) are positive; (2) all the matrices \( M_i \) in \( M \) are a sdd matrix.

3.2.1 Case 1

Here we consider the case (1), i.e., all the diagonal entries of matrices \( M_i \) in \( M \) are positive. Let \( A_i \in \mathbb{R}^{n \times n}, i = 0, 1, \ldots, k \). We denote by \( \max_{0 \leq i \leq k} \{ A_i \} \) a matrix whose the \((i, j)\)-entry is the largest \((i, j)\)-entry among all matrices \( A_i \in \mathbb{R}^{n \times n}, i = 0, 1, \ldots, k \).

Let \( \wedge_i \) be the diagonal part of \( M_i \), and let \( M_i = \wedge_i - C_i, i = 0, 1, \ldots, k \). First we give a lemma.

Lemma 3.3 Let all the diagonal elements of the matrices \( M_i \) in \( M \) be positive. If

\[
\rho(\max_{0 \leq i \leq k} \{ \wedge_i^{-1} |C_i| \}) < 1.
\]

Then \( M = (M_0, M_1, \ldots, M_k) \) has the row \( W \)-property.

Proof. Let \( V = \sum_{i=0}^k D_i \wedge_i \) and \( U = \sum_{i=0}^k D_i C_i \). Then it is easy to see that the matrix \( V \) is nonsingular, and thus

\[
S_M = V - U = V(I - V^{-1}U).
\]

From Lemma 2.3, we have

\[
V^{-1}U \leq V^{-1}|U| \leq \max_{0 \leq i \leq k} \{ \wedge_i^{-1} |C_i| \}.
\]
By Theorem 8.1.18 of [15], we get

\[ \rho(V^{-1}U) \leq \rho(V^{-1}|U|) \leq \rho(\max_{0 \leq i \leq k}\{\wedge_i^{-1}|C_i|\}), \]

which implies that \( S_M \) is nonsingular. Hence, \( M = (M_0, M_1, \ldots, M_k) \) has the row \( W \)-property.

**Remark 3.3** It is known that all the diagonal entries are positive for some special matrices, e.g., each \( M_i \) in \( M \) is a symmetric positive definite matrix, an \( M \)-matrix or an \( H_+ \)-matrix. However the condition (3.14) in Lemma 3.3 cannot be omitted, a counter-example is given below. Taking \( k = 1 \), and

\[
M_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.
\]

Obviously, both \( M_0 \) and \( M_1 \) are symmetric positive definite matrices, and also are \( H_+ \)-matrices or idd matrices. If we take

\[
\hat{D}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

then the matrix

\[
\hat{D}_0 M_0 + \hat{D}_1 M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

is singular.

**Remark 3.4** Remark 3.3 implies that the results in Theorem 2.5 and Theorem 2.7 in [3] do not extend to EVLCP \((M, q)\). That is to say, the general forms of Theorem 2.5 and Theorem 2.7 in [3] are no longer valid, e.g., taking \( M_0 \) and \( M_1 \) in Remark 3.3,

\[
\beta_p(M) = \max_{d \in [0,1]^{n}} \|((I - D)M_0 + DM_1)^{-1}D\|_p
\]

does not exist.

Next theorem shows the computability of \( \alpha_\ell(M) \).

**Theorem 3.4** Under the assumption of Lemma 3.3 we have

\[
\alpha_\ell(M) \leq \|\gamma_\ell\|, \quad \ell = 0, 1, \ldots, k,
\]

where

\[
\gamma_\ell = (I - \max_{0 \leq i \leq k}\{\wedge_i^{-1}|C_i|\})^{-1} \wedge^{-1}_\ell.
\]
**Proof.** For the sake of convenience, we only prove the case when $\ell = 0$, for $\ell = 1, 2, \ldots, k$, the proof is analogical. Now we show that

$$\alpha_0(M) \leq \| (I - \max_{0 \leq i \leq k} \{ \wedge_i^{-1}|C_i| \} )^{-1} \wedge_0^{-1} \|.$$

(3.15)

By the proof of Lemma 3.3, we have

$$S_M^{-1}D_0 = (V - U)^{-1}D_0 = (I - V^{-1}U)^{-1}V^{-1}D_0$$

and

$$|(I - V^{-1}U)^{-1}| \leq [I - \max_{0 \leq i \leq k} \{ \wedge_i^{-1}|C_i| \}]^{-1}.$$

From Lemma 2.4 and

$$D_1 \wedge_1 + \ldots + D_k \wedge_k \geq (I - D_0) \min\{\wedge_1, \wedge_2, \ldots, \wedge_k\},$$

we have

$$V^{-1}D_0 = [D_0 \wedge_0 + D_1 \wedge_1 + \ldots + D_k \wedge_k]^{-1}D_0$$

$$\leq [D_0 \wedge_0 + (I - D_0) \min\{\wedge_1, \wedge_2, \ldots, \wedge_k\}]^{-1}D_0$$

$$\leq \wedge_0^{-1}.$$

Hence, we have

$$\| S_M^{-1}D_0 \| \leq \| (I - V^{-1}U)^{-1}V^{-1}D_0 \|$$

$$\leq \| [I - \max_{0 \leq i \leq k} \{ \wedge_i^{-1}|C_i| \}]^{-1} \wedge_0^{-1} \|,$$

which implies that the desired bound (3.15) holds.

For the VLCP $(M, q)$, we have the following corollary.

**Corollary 3.1** Let all the diagonal entries of the matrices $M_i$ in $M = (I, M_1, \ldots, M_k)$ be positive. If

$$\rho(\max_{1 \leq i \leq k} \{ \wedge_i^{-1}|C_i| \}) < 1,$$

then we have

(i) $M = (I, M_1, \ldots, M_k)$ has the row $W$-property.

(ii) The following inequality holds:

$$\tilde{\alpha}_\ell(M) \leq \tilde{\gamma}_\ell, \ell = 1, 2, \ldots, k,$$

where

$$\tilde{\gamma}_\ell = \| (I - \max_{1 \leq i \leq k} \{ \wedge_i^{-1}|C_i| \})^{-1} \wedge_\ell^{-1} \|.$$

**Remark 3.5** It is noted that for $k = 1$ $M_1$ is an $H_+$-matrix in Corollary 3.1, Corollary 3.1(ii) directly reduces to Theorem 2.5 of [3].
3.2.2 Case 2

Let \( M = (M_0, M_1, \ldots, M_k) \). In the subsection, we consider the case that \( M_i \) in \( M \) is an sdd matrix for \( i = 0, 1, \ldots, k \). First, we have

**Lemma 3.4** Let \( M_i \) in \( M \), \( i = 0, 1, \ldots, k \), be an sdd matrix with the \( s \)-th diagonal entry having the same sign, \( s = 1, \ldots, n \). Then matrix \( S_M \) for any \((D_0, D_1, \ldots, D_k) \in \mathcal{D}\) is also an sdd matrix.

**Proof.** Recall the notations given in Section 1. Then we set
\[
\langle M_i \rangle e = (r_1^{(i)}, r_2^{(i)}, \ldots, r_n^{(i)})^T.
\]
Since \( M_i \) is an sdd matrix, we have \( r_i = \min_j \{ r_j^{(i)} \} > 0, j = 1, 2, \ldots, n \). By the simple computations we have
\[
\langle S_M \rangle e \geq \sum_{i=0}^{k} D_i \langle M_i \rangle e \geq \sum_{i=0}^{k} D_i r_i e \geq \min \{ r_i \} e \sum_{i=0}^{k} D_i = \min \{ r_i \} e > 0,
\]
which shows that \( S_M \) is an sdd matrix. \(\square\)

**Theorem 3.5** Under the assumption of Lemma 3.4 we have
\[
(\alpha_\ell(M))_\infty \leq \delta_\ell, \ell = 0, 1, \ldots, k,
\]
where \( (\alpha_\ell(M))_\infty = \max_{\mathcal{D}} \| S_M^{-1} D_\ell \|_\infty \) and
\[
\delta_\ell = \frac{1}{\min_{i \in \mathbb{N}} \{ (\langle M_\ell \rangle e)_i \}}.
\]

**Proof.** We only prove that
\[
(\alpha_0(M))_\infty \leq \frac{1}{\min_{i \in \mathbb{N}} \{ (\langle M_0 \rangle e)_i \}}. \tag{3.16}
\]

By Lemma 3.4, \( S_M \) is an sdd matrix. Let \( D_0 = \text{diag}(d_1^{(0)}, \ldots, d_n^{(0)}) \). It follows from Lemma 4 of [16] that
\[
\| S_M^{-1} D_0 \|_\infty \leq \max_{i} \frac{d_i^{(0)}}{(\langle S_M \rangle e)_i}. \tag{3.17}
\]

Let \( r = \min_{1 \leq i \leq k; 1 \leq j \leq n} (\langle M_i \rangle e)_j \). Since \( M_i \) is an sdd matrix, we get \( r > 0 \) and \( \langle M_i \rangle e \geq re \). By the proof of Lemma 3.4 we have
\[
\langle S_M \rangle e \geq D_0 \langle M_0 \rangle e + \sum_{i=1}^{k} D_i \langle M_i \rangle e \geq D_0 \langle M_0 \rangle e + r \sum_{i=1}^{k} D_i e = D_0 \langle M_0 \rangle e + r(I - D_0) e.
\]
Hence,
\[
\frac{d_i^{(0)}}{(S_M e)_i} \leq \frac{d_i^{(0)}}{d_i^{(0)}(M_0 e)_i + r(1 - d_i^{(0)})}.
\]
(3.18)

Then the desired bound follows from (3.17), (3.18) and Lemma 2.4. This completes the proof of the theorem.

The following result is for the VLCP case.

**Corollary 3.2** Let \( M_i \) in \( M = (I, M_1, \ldots, M_k) \) be an sdd matrix with positive diagonals. Then the following statements hold:

(i) \( S^r_M \) for any \((D_0, D_1, \ldots, D_k) \in D\) is an sdd matrix with positive diagonals, and \( M = (I, M_1, \ldots, M_k) \) has the row \( W \)-property.

(ii) The following bound holds:
\[
(\tilde{\alpha}_\ell(M))_\infty \leq \tilde{\delta}_\ell, \quad \ell = 1, 2, \ldots, k,
\]
where \((\tilde{\alpha}_\ell(M))_\infty = \max \{\|S^{-1}_M D_\ell\|_\infty\} \) and
\[
\tilde{\delta}_\ell = \frac{1}{\min_{i \in N\{((M_\ell e)_i}\}}).
\]

**Remark 3.6** The conditions in Lemmas 3.3 and 3.4 are not included each other, e.g., taking the block matrix \( M = (M_0, M_1) \) as follows:
\[
M_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}.
\]

By the simple computation, we get
\[
\rho(\max\{\Lambda_0^{-1}|C_0|, \Lambda_1^{-1}|C_1|\}) = 0 < 1.
\]
Then \( M \) satisfies the condition in Lemma 3.3. But \( M_0 \) and \( M_1 \) are not sdd, i.e., \( M \) does not satisfy the condition in Lemma 3.4. Now we take \( M = (M_0, M_1) \) as follows:
\[
M_0 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}.
\]
Then both \( M_0 \) and \( M_1 \) are sdd. However, one may get
\[
\rho(\max\{\Lambda_0^{-1}|C_0|, \Lambda_1^{-1}|C_1|\}) = 1,
\]
which shows that \( M \) does not satisfy the condition in Lemma 3.3.
4 Relative perturbation bounds

In this section, we discuss the relative perturbation bounds for EVLCP \((M, q)\) and the VLCP \((M, q)\).

**Theorem 4.1** Let \(M = (M_0, M_1, \ldots, M_k)\) have the row \(W\)-property, and let the perturbation \(\Delta M_i \in \mathbb{R}^{n \times n}\) and \(\Delta q_i \in \mathbb{R}^n\) satisfy \(\|\Delta M_i\| \leq \epsilon_i \|M_i\|\) and \(\|\Delta q_i\| \leq \epsilon_i \|(-q_i)_+\|\), respectively. If the perturbation \(\epsilon_i\) is small enough such that \(\eta := \sum_{i=0}^{k} \epsilon_i \alpha_i(M)\|M_i\| < 1\), then we have:

(i) \(\tilde{M} = (M_0 + \Delta M_0, M_1 + \Delta M_1, \ldots, M_k + \Delta M_k)\) has the row \(W\)-property.

(ii) Let \(x\) and \(y\) be the solution of EVLCP \((M, q)\) and EVLCP \((\tilde{M}, \tilde{q})\), respectively. Then

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{2\eta}{1 - \eta}.
\]  

**Proof.** (i) follows immediately from Lemma 3.1.

(ii) By (3.8) and (3.11) it is easy to show that

\[
\alpha_i(\tilde{M}) \leq \frac{1}{1 - \eta} \alpha_i(M)
\]

and

\[
\|y - x\| \leq \frac{1}{1 - \eta} \left( \left( \sum_{i=0}^{k} \alpha_i(M)\|\Delta M_i\| \right) \|x\| \right) + \sum_{i=0}^{k} \alpha_i(M)\|\Delta q_i\|.
\]  

From \(M_i x + q_i \geq 0\), we deduce that

\((-q_i)_+ \leq (M_i x)_+ \leq |M_i x| \leq |M_i| |x|,\)

which together with the assumption gives

\[
\|\Delta q_i\| \leq \epsilon_i \|(-q_i)_+\| \leq \epsilon_i \|(M_i x)_+\| \leq \epsilon_i \|M_i\| \|x\|.
\]  

Combining (4.2) and (4.3) together gives

\[
\|y - x\| \leq \frac{1}{1 - \eta} \left( \sum_{i=0}^{k} \alpha_i(M)\|\Delta M_i\| \right) \|x\| + \sum_{i=0}^{k} \alpha_i(M)\|\Delta q_i\| \|x\| \leq \frac{2\eta}{1 - \eta} \left( \sum_{i=0}^{k} \epsilon_i \alpha_i(M)\|M_i\| \right) \|x\|,
\]

from which one may deduce the desired bound (4.1).

It is noticed that VLCP is a special case of EVLCP. The relative perturbation bound for VLCP \((M, q)\) can be deduced from Theorem 4.1. Here we omit it.
Remark 4.1 If we take \( k = 1, q_0 = 0, M_0 = I, \) and \( M_1 \) is a \( P \)-matrix in Theorem 4.1, then Theorem 3.1 of \([3]\) can be derived from Theorem 4.1.

In the following, we consider the special cases where \( M_i, i = 0, 1, \ldots, k, \) has positive diagonals or an sdd matrix as in the subsections 3.2.1 and 3.2.2, see Corollary 4.1. Its proof is similar to Theorem 4.1. We omit it.

**Corollary 4.1** Under the same assumption as in Theorem 4.1, if \( M_i \) satisfies the same assumption as in Lemma 3.3 or Lemma 3.4 for any \( i = 0, 1, \ldots, k, \) respectively, then \( \tilde{M} = (M_0 + \Delta M_0, M_1 + \Delta M_1, \ldots, M_k + \Delta M_k) \) has the row \( \mathcal{W} \)-property and

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{2\eta_i}{1 - \eta_i}, \tag{4.4}
\]

provided that the perturbation \( \epsilon_i \) is small enough such that \( \sum_{i=0}^{k} \epsilon_i \| \xi_i \| \| M_i \| = \eta_i < 1, \) where \( \xi = \gamma \) or \( \delta \) is given by Theorem 3.4 or Theorem 3.5, respectively.

It is noted that \( \| \delta_i \| = \delta_i \) in the bound (4.4) (see Theorem 3.5). Next, we consider a special norm for the relative perturbation bound. The proof is tedious, we omit it.

**Theorem 4.2** Let \( \bar{M} = (M_0, M_1, \ldots, M_k) \) have the row \( \mathcal{W} \)-property, the perturbation \( \Delta M_i \in \mathbb{R}^{n \times n} \) and \( \Delta q_i \in \mathbb{R}^n \) satisfy \( |\Delta M_i| \leq \epsilon_i \| M_i \| \) and \( |\Delta q_i| \leq \epsilon_i (-(q_i)_+), \) respectively. Let \( M_i = \Lambda_i - C_i \) be an \( H_+ \)-matrix and \( \rho(\max_{0 \leq i \leq k} \{|\Lambda_i^{-1}|C_i|\}) < 1, \) where \( \Lambda_i \) is the diagonal part of \( M_i, i = 0, 1, \ldots, k. \) If the perturbation \( \epsilon_i \) is small enough such that

\[
\sum_{i=0}^{k} \epsilon_i \| \gamma_i \| \| M_i \| = \hat{\eta} < 1,
\]

then the following statements hold:

(i) \( \bar{M} = (M_0 + \Delta M_0, M_1 + \Delta M_1, \ldots, M_k + \Delta M_k) \) has the row \( \mathcal{W} \)-property.

(ii) Let \( x \) and \( y \) be the solution of EVLCP \( (M, q) \) and EVLCP \( (\bar{M}, \bar{q}), \) respectively. Then

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{2(\epsilon_0 \| \gamma_0 \| M_0 \| + \cdots + \epsilon_k \| \gamma_k \| M_k \|)}{1 - \hat{\eta}}. \tag{4.5}
\]

**Remark 4.2** Taking \( k = 1, M_0 = I \) and \( q_0 = 0, \) the bound (4.5) reduces to the corresponding one in Theorem 3.3 of \([3]\).
5 Numerical examples

In this section, some numerical examples are given to show the feasibility of the relative perturbation bound. For the sake of convenience, we only use the infinity norm in all numerical experiments. Let \( x \) and \( y \), respectively, be the solution of EVLCP (1.1) and the perturbed EVLCP (1.4), which can be obtained by directly using Lemke’s complementarity pivoting method [5] for the following examples, and \( r \) be the real relative error given by

\[
r = \frac{\|x - y\|_{\infty}}{\|x\|_{\infty}}.
\]

The perturbation for EVLCP (1.1) can be set as:

\[
\Delta M_i = \frac{\epsilon \|M_i\|_{\infty}}{\|S_i\|_{\infty}} S_i, \quad \Delta q_i = \frac{\epsilon \|q_i\|_{\infty}}{\|t_i\|_{\infty}} t_i,
\]

with \( S_i \) and \( t_i \), respectively, being an arbitrary random matrix and vector. In this case we have

\[
\|\Delta M_i\|_{\infty} \leq \epsilon \|M_i\|_{\infty}, \quad \|\Delta q_i\|_{\infty} \leq \epsilon \|q_i\|_{\infty}.
\]

All the computations are done in Matlab R2021b.

**Example 5.1** ([7, 21]) Let \( k = 1 \) in (1.1), \( M = (M_0, M_1) \) and \( q = (q_1, q_2) \), respectively, be form

\[
M_0 = \begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 4 & -4 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad q_0 = q_1 = (-1, -1)^T.
\]

It is easy to check that \( M \) has the row \( \mathcal{W} \)-property. This implies that the corresponding EVLCP \((M, q)\) has a unique solution. In fact, its unique solution \( x = (2.25, 2)^T \).

| \( \epsilon \)   | \( y \)            | \( r \)   | \( \bar{\tau}_\gamma \) | \( \tau \)   | \( \nu \)   |
|----------------|-------------------|----------|-------------------------|-----------|-----------|
| 0.01           | (1.9277, 1.7120)^T| 0.1444   | 1.2359                  | 2.3478    | 1.5145    |
| 0.001          | (2.2132, 1.9671)^T| 0.0155   | 0.0518                  | 0.1142    | 0.0736    |
| 0.0001         | (2.2463, 1.9967)^T| 0.0017   | 0.0057                  | 0.0109    | 0.0070    |

Table 1: Relative perturbation bounds of Example 5.1.

By the simple computations, we obtain that

\[
\rho(\max\{\wedge_0^{-1}|C_0|, \wedge_1^{-1}|C_1|\}) = 0.8944 < 1.
\]

This shows that the condition in Theorem 4.2 and Corollary 4.1 with Lemma 3.3 are satisfied. Since \( M_1 \) is not a strictly row diagonally dominant matrix, the conditions of Corollary 4.1 with Lemma 3.4 are not satisfied. Based on (4.2), Theorem 4.2 and Corollary 4.1 with Lemma 3.3, we set \( \eta_\gamma = \epsilon \sum_{i=0}^{1} \|\gamma_i\|_{\infty} \|M_i\|_{\infty} \|

\[
\bar{\tau}_\gamma = \frac{1}{1 - \eta_\gamma} \left( \sum_{i=0}^{1} \|\gamma_i\|_{\infty} \|\Delta M_i\|_{\infty} + \sum_{i=0}^{1} \|\gamma_i\|_{\infty} \|\Delta q_i\|_{\infty} \right)/\|x\|_{\infty}.
\]
\[
\tau = \frac{2\eta}{1 - \eta}, \nu = \frac{2\epsilon(\|\gamma_0| M_0 \|_\infty + \|\gamma_1| M_1 \|_\infty)}{1 - \eta}.
\]

In our computations, we choose some values of \(\epsilon\) such that \(\eta < 1\), see Table 1.

We report the numerical result for three relative perturbation bounds in Table 1, from which we find that \(\bar{\tau} < \tau\) and \(\bar{\nu} < \nu\), and also illustrates that the proposed bounds are very close to the real relative value when the perturbation is very small. This show that the relative perturbation bounds given by Corollary 4.1 and Theorem 4.2 are feasible and effective under some suitable condition. The following example is given by [25].

Example 5.2 Consider the EVLCP \((\mathbf{M}, \mathbf{q})\), in which is given \(\mathbf{M} = (M_0, M_1)\), where

\[
M_0 = \begin{bmatrix} 1 & 3/4 & 0 \\ 3/4 & 1 & 0 \\ 0 & 3/4 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 3/4 \\ 3/4 & 0 & 1 \end{bmatrix},
\]

Clearly, \(M_0\) and \(M_1\) are two strictly row diagonally dominant matrices. Hence, EVLCP has a unique solution for any \(\mathbf{q}\) (see Lemma 3.4). Let \(\mathbf{q}_1 = \mathbf{q}_2 = (-1.75, -1.75, -1.75)^T\). Then it is easy to see that \(x = (1, 1, 1)^T\) is its unique solution.

| \(\epsilon\) | \(y\) | \(r\) | \(\bar{\tau}_\delta\) | \(\nu\) |
|----------|------|------|----------------|------|
| 0.01     | \((1.0510, 1.0545, 0.9615)^T\) | 0.0545 | 0.3256 | 0.3256 |
| 0.001    | \((1.0078, 1.0076, 0.9930)^T\) | 0.0078 | 0.0284 | 0.0284 |
| 0.0001   | \((1.0007, 1.0007, 0.9994)^T\) | 7.0368e-04 | 0.0028 | 0.0028 |

Table 2: Relative perturbation bounds of Example 5.2.

Clearly, the condition of Corollary 4.1 with Lemma 3.4 is satisfied. However, \(\mathbf{M}\) does not satisfy the condition of Theorem 4.2 and Corollary 4.1 with Lemma 3.3 because

\[
\rho(\max \{\land_0^{-1}|C_0|, \land_1^{-1}|C_1|\}) = 1.5 > 1.
\]

Based on (4.2) and Corollary 4.1 with Lemma 3.4, we set \(\eta_\delta = \epsilon \sum_{i=0}^{1} \delta_i \|M_i\|_\infty\),

\[
\bar{\tau}_\delta = \frac{1}{1 - \eta_\delta} \left( \sum_{i=0}^{1} \delta_i \|\Delta M_i\|_\infty + \frac{\sum_{i=0}^{1} \delta_i \|\Delta q_i\|_\infty}{\|x\|_\infty} \right), \text{ and } \nu = \frac{2\eta_\delta}{1 - \eta_\delta}.
\]

We take some values of \(\epsilon\) such that \(\eta_\delta < 1\).

The numerical bounds are reported in Table 2, which show that \(\bar{\tau}_\delta = \nu\). It also illustrates that the numerical bounds in Example 5.2 show the same perturbing behavior as in Example 5.1 although conditions are different.

Next, we give an example from the discretization of Hamilton-Jacobi-Bellman (HJB) equation, in which the conditions in both Corollary 4.1 and Theorem 4.3 hold.
Example 5.3 Consider the following EVLCP $(M, q)$
\[
\min \{M_0 x + q_0, M_1 x + q_1\} = 0,
\]
where $M_i$ and $q_i$ ($i = 0, 1$) comes from the discretization of Hamilton-Jacobi-Bellman (HJB) equation
\[
\left\{ \begin{array}{l}
\max_{0 \leq i \leq 1} \{L_i + f_i\} = 0 \text{ in } \Gamma, \\
u = 0 \text{ on } \partial \Gamma,
\end{array} \right.
\]
with $\Gamma = \{(x, y) | 0 < x < 2, 0 < y < 1\}$,
\[
\begin{align*}
L_0 &= 0.002u_{xx} + 0.001u_{yy} - 20u, f_0 = 1, \\
L_1 &= 0.001u_{xx} + 0.001u_{yy} - 10u, f_1 = 1,
\end{align*}
\]
see [1] for more details. Here, by making use of the central difference scheme to discretize the above HJB equation, Example 5.3 can be obtained and $q_0 = q_1 = -e$.

Here, $M_0$ and $M_1$ obtained from the above HJB equation are two strictly row diagonally dominant matrices. What’s more,
\[
\rho(\max\{\lambda_0^{-1}|C_0|, \lambda_1^{-1}|C_1|\}) \leq \|W\|_\infty < 1,
\]
where $W = \max\{\lambda_0^{-1}|C_0|, \lambda_1^{-1}|C_1|\}$. This means that the conditions in Corollary 4.1 are satisfied, so do the conditions in Theorem 4.2. This implies that Example 5.3 has a unique solution because the block matrix $M$ has the row $W$-property.

Tables 3-6 list some relative perturbation bounds for Example 5.3 with the different dimension and $\epsilon$, in which $r$, $\tau$, $\nu$, $\nu$, $\nu$, and $\nu$ are the above defined in Example 5.1 and Example 5.2. Again, we chose some values of $\epsilon$ such that $\eta_\nu < 1$ and $\eta_\nu < 1$. (a), (b), (c) and (d) in Figure 1 are in line with Table 3, Table 4, Table 5 and Table 6, respectively. In Figure 1, ‘RPB’ denotes the value of the relative perturbation bound and ‘$n$’ denotes the order of the system matrix.

| $\epsilon$ | $r$  | $\tau$  | $\nu$  | $\nu$  |
|-----|-----|-----|-----|-----|
| 0.01 | 0.0175 | 0.0500 | 0.0454 | 0.0499 |
| 0.015 | 0.0273 | 0.0760 | 0.0688 | 0.0759 |
| 0.02 | 0.0356 | 0.1026 | 0.0929 | 0.1025 |
| 0.025 | 0.0433 | 0.1300 | 0.1174 | 0.1297 |
| 0.03 | 0.0547 | 0.1580 | 0.1426 | 0.1577 |

Table 3: Relative perturbation bounds of Example 5.3 with $n = 16$.

Similar to what happens in Example 5.1 and Example 5.2, from Tables 3-6, we can draw the same conclusion. In other word, for the same dimension, with $\epsilon$ decreasing, $r$, $\nu$, $\nu$, $\nu$, $\nu$, and $\nu$ are decreasing, also see Figure 1. The reason is the same as Example 5.1 and Example 5.2. In addition, we find that for the same $\epsilon$, with the dimension increasing, $\nu$, $\nu$, $\nu$, $\nu$, $\nu$, and $\nu$ are
Figure 1: The value and number of solutions of Example 5.3.
increasing (the reason is similar to the same dimension with $\epsilon$ decreasing), and the values of $\tau$ and $\nu$ are fairly close, not much different in size.

No matter what, from the above numerical results in Tables 3-6, we still verify that under some suitable condition, Corollary 4.1 and Theorem 4.2 indeed provide some valid relative perturbation bounds.

## 6 Conclusion

In this paper, we discuss the perturbation analysis of the EVLCP ($\mathbf{M}, q$). By making use of a general equivalent form of the minimum function, under the assumption of row $W$-property, some perturbation bounds for the EVLCP ($\mathbf{M}, q$) are presented, which cover some existing results in [3]. Particularly, for all diagonal elements of the matrices $M_i$ in $\mathbf{M}$ being positive and all the matrices $M_i$ in $\mathbf{M}$ being a strictly row diagonally dominant matrix, some computable perturbation bounds are provided as well. Some numerical examples are given to show the proposed bounds.
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