Information erasure lurking behind measures of complexity

Karoline Wiesner,1,∗ Mile Gu,2 Elisabeth Rieper,2 and Vlatko Vedral3,4,2

1 School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom
2 Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, S15-03-18, Singapore 117543, Singapore
3 Atomic and Laser Physics, Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom
4 Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117543, Singapore

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Complex systems are found in most branches of science. It is still argued how to best quantify their complexity and to what end. One prominent measure of complexity (the statistical complexity) has an operational meaning in terms of the amount of resources needed to forecasting a system’s behaviour. Another one (the effective measure complexity, aka excess entropy) is a measure of mutual information stored in the system proper. We show that for any given system the two measures differ by the amount of information erased during forecasting. We interpret the difference as inefficiency of a given model. We find a bound to the ratio of the two measures defined as information-processing efficiency, in analogy to the second law of thermodynamics. This new link between two prominent measures of complexity provides a quantitative criterion for good models of complex systems, namely those with little information erasure.

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The idea of physics as information has a long history. The concept of entropy, at the heart of information theory, originated in the theory of thermodynamics. It was Maxwell and Boltzmann who, in the beginning of the 19th century, recognized the intricate link between probability distributions over configurations and thermodynamics. This laid the foundation to the field of statistical mechanics. The similarity between the thermodynamic entropy and the information entropy, introduced in 1948 by Shannon, lead to a whole new perspective on physical processes as storing and processing information. It also lead to paradoxes such as Maxwell’s demon which seemed to suggest that work could be generated from heat only with the use of information, which would violate the second law of thermodynamics (for a review, see Refs. [1, 2]). The paradox was solved independently by Penrose and Bennett, in considering the entropy creation caused by erasing information [2, 3]. Since the insight that information is physical and physics is information one has started to regard nature as a grand information processor of both classical and quantum information (see e.g. Refs. [4, 5]). This point of view is especially fruitful in the study of complex systems. Here, the physical laws are often missing. Other means of modelling a system’s behaviour have to be found. Information theory comes in handy. It provides the tools for distinguishing structure from randomness in a given data set [6] – a distinction which is the basis of any reasonable model [7]. Complexity lies between disorder and order. Hence, a good measure of complexity is zero for both completely random objects and trivially ordered objects.

In the following we will concentrate on two computable measures of complexity: The statistical complexity [8, 9] and the effective measure complexity [10], aka excess entropy [6]. The excess entropy measures the internal information of a process which is communicated from the past to the future. The statistical complexity, on the other hand, measures the amount of information required to predict the process. Hence, it is a property more of the model than the process. Any inconsistency between the two is puzzling at first. One should not require more information to model a process than the process itself uses. The central and new result of this paper is that this inconsistency can be explained with information erasure. This allows for a direct computation of the excess entropy from the statistical complexity, which had not been possible before. Until now the excess entropy had to be computed numerically [6] or analytically via a computationally expensive procedure [11]. We also obtain a simple interpretation of the difference between the two measures as the information-processing efficiency of the model used for prediction. We find that this efficiency is bounded in the same way the generation of work is bounded by the second law of thermodynamics.

Many proposals for complexity measures exist, such as effective measure complexity [10], statistical complexity [8], logical depth [12], thermodynamic depth [2], effective complexity [13], to mention a few. Some measure structure as expected [10], some only randomness which renders them unsuitable [2] (see [14]). Some are computable [8], others are not [13]. And most often they are completely unrelated quantities. If one is interested in measures which capture complexity and not randomness and which, in addition, are computable then one is
left with the mutual information and related quantities \[ \mu \] and the statistical complexity \[ C \]. Therefore, these are the measures we focus on in this paper.

Consider a complex system which is studied in a sequence of observations. We call the time-independent probability distribution \( \Pr(S) \) over such a (infinite) sequence of observations \( S \) a stationary process. The framework of computational mechanics provides the tools to infer a computation-theoretic model of the process, which is provably the most compact description while reproducing the statistics exactly \[ C, \mu \]. This computation-theoretic model, called \( \epsilon \)-machine, consists of an output alphabet \( A \), a set of causal states \( S \) and stochastic transitions between them. For every pair of states \( SS' \), \( \Pr(S'|S,a) \) gives the probability of going from state \( S \) to state \( S' \) while outputting symbol \( a \in A \). The statistical complexity of a process is the Shannon entropy over the causal-state distribution of its \( \epsilon \)-machine \[ C_{\epsilon} \]:

\[
C_{\epsilon} = H(S). \tag{1}
\]

\( C_{\epsilon} \) measures the minimum number of bits that need to be stored to optimally predict a given process. The \( \epsilon \)-machine is a minimal and optimal predictor of the process. Fig. 1 shows an example of an \( \epsilon \)-machine. At any given point in time the machine is in one of its states. It chooses the next state according to the transition probabilities. Once the transition is taken the label of the corresponding edge will be put out as symbol. This procedure repeats indefinitely. Successful applications of computational mechanics range from dynamical systems \[ \mu \], spin systems \[ C \], and crystallographic data \[ C \] to molecular dynamics \[ C \], atmospheric turbulence \[ C \], and self-organisation \[ C \].

The second measure of complexity under consideration is the effective-measure complexity EMC introduced by Grassberger \[ C \]. It is based on the entropy rate \( h \) of a process, defined as the increase in Shannon entropy over increasing block sizes \( s^L \): \( h_L = \lim_{L \to \infty} H(s^L)/L \). The limit \( h = \lim_{L \to \infty} h_L \) is a measure of information production or unpredictability. Completely periodic processes, for example, have zero entropy rate \[ C \]. EMC is defined as \( EMC = \sum_{L=0}^{\infty} (h_L - h) \). Equivalent to it is the excess entropy \( E \) which, for our purposes, is more conveniently defined \[ C \]. The excess entropy of a process is the average mutual information between its semi-infinite past and semi-infinite future:

\[
E = I[S; S']. \tag{2}
\]

Note, that \( E = EMC \). It can easily be shown that the excess entropy is a lower bound of the statistical complexity \[ C \][Theorem 5]: \( E \leq C_{\epsilon} \). The excess entropy is the average number of bits a process stores about its past and transmits into the future. The effective measure complexity and the excess entropy can be considered as special cases of a complexity measure introduced by Tononi et al. \[ C \]. It has recently been shown that the excess entropy can be interpreted as the mutual information between a process’s predictive and retrodictive causal states \[ C \]. This result also led to a first closed-form expression of the excess entropy. The computation of that expression, however, requires the inference of both the predictive and retrodictive causal states, which is computationally very costly. We now derive a simple expression for calculating the excess entropy of a process from the predictive causal states only.

Once an \( \epsilon \)-machine is given, there are several quantities of interest. The average amount of information stored is given by \( C_{\epsilon} \). The average amount of information generated is given by the entropy rate \( h \). For finite-state machines like the \( \epsilon \)-machine it can be calculated in closed form as the uncertainty of the next symbol given the last symbol \( h = H(a|S) \). The quantity which had not been considered in this context is the amount of information which is erased at each time step. We now show how to calculate it and its significance for characterising a model of a complex system.

Landauer defines an operation to be logically irreversible if the output of the operation does not uniquely define the inputs \[ C \]. In other words, logically irreversible operations erase information about the computer’s preceding logical state. How can we apply this logical irreversibility to the \( \epsilon \)-machine of a process? \( \epsilon \)-machines are deterministic in the sense that the current state and the next symbol determine the next state uniquely. The reverse, however, is not necessarily true. Given the next state and the last symbol, the previous state is not always uniquely determined. If it is not the \( \epsilon \)-machine is logically irreversible. The information erased at each step \( I_{\text{erased}} \) is given by the entropy over the last state given the last symbol. \( I_{\text{erased}} \) can be calculated from the \( \epsilon \)-machine as follows. Given the triplet ’last state, symbol, next state’ \( SaS' \), the amount of information erased is as follows:

\[
I_{\text{erased}} = H(Sa) - H(aS') \tag{3}
\]

This uncertainty of the previous state given that the symbol and next state are known quantifies the irreversibility of the \( \epsilon \)-machine. We can now derive an exact relationship between statistical complexity and excess entropy of a process.

**Theorem.** The difference between statistical complexity and excess entropy is given by the logical irreversibility of the \( \epsilon \)-machine:

\[
C_{\epsilon} - E = I_{\text{erased}} \tag{4}
\]

**Proof.** From the definition of the excess entropy in terms of mutual information we arrive at the following expression for \( E \):

\[
I[S; S'] = H(S) + H(S') - H(SS') = H(S) + H(S') - H(S', S) = H(S) - H(S') + I[S; S'] = H(S) - H(S') + I[S; S']
\]

Since \( H(S) = C_{\epsilon} \) and \( H(S') = E \), we get

\[
C_{\epsilon} - E = I_{\text{erased}}
\]
\[ E = I[\overline{S} | \overline{S}] = H(\overline{S}) - H(\overline{S} | \overline{S}) = H(\overline{S}) - H(\overline{S} | S) = H(S) - H(S | \overline{S}) = C_\mu - H(S | \overline{S}) . \]

To prove the Theorem we have to show \( H(S | \overline{S}) = I_{\text{erased}} \). Since knowledge of the entire future starting with symbol \( a \) eventually uniquely determines the next state following on \( a \), we have \( H(S'| \overline{S}) = 0 \) (note the difference between uncertainty in last state, which can be non-vanishing, and next state).

We can write
\[
H(S | \overline{S}) = H(S | \overline{S}) - H(S' | \overline{S}) = H(S | \overline{S}) - H(S' | S) = H(S) - H(S | \overline{S}) = H(S) - H(S | a) - H(S | a') + H(S | a) - H(S | a') = H(S) - H(S | a) - H(S | a') .
\]

The superscript \(-1\) indicates the removal of the first or last symbol, respectively. In the next to last step, we used the fact that \( \epsilon \)-machines are deterministic, which means \( H(S'|Sa) = 0 \).

This leads to the corollary that for reversible \( \epsilon \)-machines \( E = C_\mu \). In analogy to thermodynamical efficiency we define the information-processing efficiency \( \eta \) of an \( \epsilon \)-machine as the ratio between excess entropy and statistical complexity:
\[
\eta = \frac{E}{C_\mu} = 1 - \frac{I_{\text{erased}}}{C_\mu} . \tag{5}
\]

From Eq. 8 we find an upper bound on the amount of information which can be erased at each step. We can rewrite Eq. 8 in the following way:
\[
I_{\text{erased}} = H(a|S) - H(a|S') = h - H(a|S') , \tag{6}
\]
where we used the fact that the entropy over last state \( S \) and next state \( S' \) are the same. \( H(a|S') \) is the uncertainty of the last symbol given the next state. Hence, we obtain an upper bound for the cost of erasure as
\[
I_{\text{erased}} \leq h . \tag{7}
\]

Thus, the amount of information which can be erased in the causal-state model is upper bounded by the amount of information which is created, as one would expect.

Let us illustrate the Theorem with two example processes. The first one generates single 1’s surrounded by 0’s. No consecutive ones are allowed. The process is called Golden Mean Process, and its \( \epsilon \)-machine is shown in Fig. 1. The amount of irreversibility of this \( \epsilon \)-machine is illustrated in Table I. The information erased at each time step can be calculated from the transitions from ‘last state , symbol’ to ‘symbol , next state’.

**TABLE I:** Transition diagram: ‘last state , symbol’ to ‘symbol , next state’

| Last State | Symbol | New Symbol |
|------------|--------|------------|
| Sa         | aS'    | A0 → 0A    |
| A0         | 0A     | B0 → 0A    |
| A1         | 1B     |            |

Whenever state \( A \) is entered (on symbol 0) the last state is maximally uncertain. The amount of irreversibility can now be calculated from the transition probabilities shown in Fig. 1 and using Eq. 4. The statistical complexity of the Golden Mean Process is \( C_\mu = H(2/3, 1/3) = 0.9183 \) bits, which yields an excess entropy \( E = 0.2516 \) bits. The known numerical value for the excess entropy of this process was 0.252 bits \( [6] \). \( h \) in this example is 2/3 bits. The Golden Mean Process reaches the upper bound of erased information, Eq. 6, since the uncertainty of the last symbol given the next state \( H(a|S') \) is zero.

The second example process generates even sequences of 1’s surrounded by 0’s. This process is interesting because it has an infinite list of forbidden words. The previous process had but one forbidden word, two consecutive 1’s (and all words containing it). We can see from the graph in Fig. 2 that the corresponding \( \epsilon \)-machine is completely reversible. This is confirmed in Table II listing the possible transitions. Whenever the state and the last symbol are known, the last state can be uniquely traced. Hence, according to the Theorem, the statistical complexity is equal to the excess entropy and we obtain for the Even Process: \( C_\mu = E = 0.9183 \) bits. The numerical value known for \( E \) was 0.902 bits and was known to converge very slowly \( [6] \).

**TABLE II:** Transition diagram: ‘last state , symbol’ to ‘symbol , next state’

| Last State | Symbol | New Symbol |
|------------|--------|------------|
| Sa         | aS'    | A0 → 0A    |
| A0         | 0A     | B0 → 0A    |
| A1         | 1B     |            |

Theorem allows us to see the fact that \( E \) is a lower bound of \( C_\mu \) in new light. If there existed a process with \( C_\mu < E \) it would generate more negative entropy than...
it uses. We consider this an information-theoretic analogue to the second law of thermodynamics for models of complex systems. No optimally predictive model can store less than $E$ bits. On the other hand, any predictive model of a complex system needs to store at least $E$ bits of information. Anything beyond that must be considered inefficient. This point of view corroborates the original interpretation of the effective-measure complexity given by Grassberger who noted that EMC is the amount of memory required to optimally predict the future of a process if one could use it to 100% efficiency.

This leaves us with a new puzzle: How can the $\epsilon$-machine – provably minimal and optimal given that one operates on discrete states and variables – be inefficient? There is the possibility that an irreversible $\epsilon$-machine indicates that the process itself is information-theoretically inefficient. If, on the other hand, it implies a model-inefficiency, the model can only be improved by assuming different kinds of resources, such as a different architecture, continuous states or other. Indeed, there are cases where the efficiency of a model is improved when the states are allowed to have quantum mechanical properties. It is also possible that one has to consider a trade-off between the size of the model and its irreversibility. We conclude that an $\epsilon$-machine’s logical irreversibility can potentially be used as a criterion for model discrimination, both in terms of size and in terms of architecture. Measuring the complexity of a process can now be motivated by the search for an information-theoretically efficient model.

To summarise, we have applied Landauer’s logical irreversibility to models of complex systems. We showed that the difference between the statistical complexity and the excess entropy of a complex system is given by the logical irreversibility of its $\epsilon$-machine. Our result provides a means to quantitatively discriminate between models of complex systems. The here presented information-theoretic analogue to the second law of thermodynamics requires that any model requires at least $E$ bits of stored information. Furthermore, if one takes the $\epsilon$-machine as the starting point for a physical model of the system, the nature of resources which minimise the information erased becomes physically meaningful. The here presented results motivate further studies of computation-theoretic representations of complex systems like the $\epsilon$-machine.

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\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{even_process.png}
\caption{Graphical representation of the Even Process. Edges are labelled $\Pr(S'|S,a)$. The Even Process is reversible and thus maximally efficient: $\iota = 1$.}
\end{figure}