A diagrammatic approach to map-state and channel-state dualities

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Diagrammatic representation and manipulation of tensor networks has proven to be a useful tool in mathematics, physics, and computer science. Here we present several important and mostly well-known theorems regarding the dualities between linear maps and bipartite pure quantum states, and the dualities between quantum channels and bipartite mixed quantum states, in diagrammatic form. The graphical presentation makes the proofs very compact and in some cases even intuitive.

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I. INTRODUCTION

Diagrammatic methods for describing tensors and their connections have a long history in physics, mathematics and computer science. Their importance stems from the fact that they enable one to perform mathematical reasoning and even actual calculations using intuitive graphical objects instead of abstract mathematical entities. In the early 1970s Penrose introduced a somewhat informal but expressive graphical notation for representing tensor network expressions such as the ones one encounters in general relativity [1]. Perhaps the first one to note the importance of diagrammatic methods in quantum information science was David Deutsch. Today quantum circuit diagrams (QCDs), a well-defined subclass of tensor network diagrams, are a standard tool for describing quantum algorithms and protocols.

The channel-state duality, or Choi-Jamiołkowski isomorphism, is a central result in quantum information science. It establishes a direct one-to-one correspondence between quantum channels (processes that map valid quantum states to valid quantum states), and mixed quantum states on a larger Hilbert space. In this work we aim to provide a complementary viewpoint to the usual abstract algebraic derivation and presentation of these results. The graphical presentation tends to make the proofs very compact, easy to follow, and in some cases even intuitive. Our presentation occasionally follows [2] and [3]. Related results can be also found in [4].

II. NOTATION

Here we briefly explain the basic string diagram notation for tensor networks. For a more complete treatment together with proofs, see [6].

Horizontal wires represent Hilbert spaces. We assume that every Hilbert space $A$ comes with a preferred computational basis $\{|k\rangle_A\}_k$. Stacking the wires vertically corresponds to a system comprised of several subsystems, as shown in Fig. 1a. Unless the types are clear from the context, each wire should be explicitly labeled. Unlike in standard QCDs, the wires are allowed to deviate from a straight horizontal line as long as they do not cross each other, or reverse direction. Soon, however, we will relax both of these rules by introducing some additional wire-like diagram elements.

Linear maps between Hilbert spaces are represented by labeled boxes, with the domain wire(s) connecting to the left side of the box, and the codomain wire(s) to the right side. Maps are concatenated simply by placing the boxes next to each other on the wire. Similarly, two boxes stacked atop each other represent the tensor product of the corresponding maps,
Figure 1. Basic wire structures. (a) Identity map $\mathbb{1}_{A\otimes B\otimes C}$, or the Hilbert space $A \otimes B \otimes C$. (b) Cup $|\supset\rangle_{A\otimes A}$. (c) Cap $\langle\subset|_{A\otimes A}$. (d) SWAP$_{A\otimes B}$ gate.

Figure 2. Linear maps and vectors. (a) Map $f : A \rightarrow B$. (b) Composition. (c) Tensor product. (d) Vector $|\psi\rangle \in A$. (e) Map acting on a vector. (f) Inner product $\langle \phi, \psi \rangle$.

as shown in Fig. 2a–c. Consequently, a bare wire can also be understood as the identity map.

Vectors (or kets) are represented by labeled left-pointing triangles, and the corresponding covectors (one-forms/functionals/bras, via Riesz’s theorem) as labeled right-pointing triangles. Maps operating on vectors, inner products, and tensor products are represented in the obvious way, as shown in Fig. 2d–f.

Taking the Hermitian conjugate of an expression is accomplished by mirroring the corresponding diagram in the left-right direction, and adding a dagger symbol to each box label. Now we will complete our notation by introducing three additional wire-like structures.

**Definition 1 (Cup and cap).** Given the computational basis, a *cup* is the diagram element
that corresponds to the bipartite vector

$$|\triangleright\rangle_{A\otimes A} := \sum_i |i\rangle_A \otimes |i\rangle_A.$$  \hspace{1cm} (1)

Likewise, a cap is the diagram element that corresponds to the corresponding covector:

$$\langle\subset|_{A\otimes A} := \sum_i \langle i|_A \otimes \langle i|_A.$$  \hspace{1cm} (2)

The corresponding diagrams are presented in Fig. 1bc. Note that normally the symbol inside the ket or bra does not change when taking the dagger: $|\psi\rangle^\dagger = \langle\psi|$. Here it does, $|\triangleright\rangle_{A\otimes A}^\dagger = \langle\subset|_{A\otimes A}$, but only for the sake of aesthetics. We essentially define two redundant symbols for the same thing: $|\triangleright\rangle_{A\otimes A} = |\subset\rangle_{A\otimes A}$.

It is easy to notice that interpreted as a pure quantum state, $|\triangleright\rangle_{A\otimes A}$ is proportional to the maximally entangled generalized Bell state $B_{0,0}$,

$$|B_{0,0}\rangle_{A\otimes A} = \frac{1}{\sqrt{\text{dim} A}} |\triangleright\rangle_{A\otimes A},$$  \hspace{1cm} (3)

and that the other generalized Bell states are locally equivalent to it.

**Definition 2** (SWAP gate). $\text{SWAP}_{A\otimes B}: A \otimes B \rightarrow B \otimes A$ exchanges the order of two subsystems and is consequently represented by crossing wires, see Fig. 1c.

$$\text{SWAP}_{A\otimes B} = \sum_{ab} |ba\rangle_B \otimes \langle ab|_A.$$  \hspace{1cm} (4)

In the case where $A = B$, we can instead interpret it as swapping the states of the two subsystems, thus recovering the usual definition.

Now we will list (without proof) some fundamental properties of cups, caps and SWAPs, corresponding to the diagram identities in Fig. 3.

**Proposition 1** (Snake equation (Fig. 3a)). A cup and a cap can combine to cancel each other. In other words, a double bend in a wire can be pulled straight.

**Proposition 2** (Inverse of SWAP (Fig. 3d)). The inverse of $\text{SWAP}_{A\otimes B}$ is simply $\text{SWAP}_{B\otimes A}$. Thus two stacked wires can be pushed through each other.

**Proposition 3** (Cup and cap symmetry (Fig. 3c)). Since the cup corresponds to a symmetric state, it immediately follows that the relative order of the two subsystems is irrelevant. Diagrammatically this means the order of the outgoing wires can be swapped, or a bend “twisted” into a loop.

**Proposition 4** (Cup crossing a wire (Fig. 3e)). $\text{SWAP}$ interacts with a cup in the obvious way. Hence a bend can be moved across a wire.

With these propositions, the interpretation of Fig. 1a becomes clear. One can manipulate the wires almost as if they were rubber bands confined in a two-dimensional plane without changing the meaning of the diagram. Mathematically the purpose of the various wire structures is to reorder and entangle subsystems in various ways. When interpreting the diagrams physically, one can think of time flowing from left to right, much like in ordinary QCDs, in which case the different reshapes of a diagram correspond to different ways of obtaining the same physical effect.
Figure 3. Wire identities. (a) Snake equation. (c) \( \text{SWAP}^{-1}_{A \otimes B} = \text{SWAP}_{B \otimes A} \). (c) Cup symmetry. (d) Cup crossing a wire. (e) “Sliding” an operator \( f \) around a cup (or cap) transposes it in the computational basis. Definition of the operator state \( |f \rangle \). (f) Trace. (g) Conjugate state.

**Proposition 5** (Sliding operators around cups and caps; operator states (Fig. 3e)). An operator \( f : A \rightarrow B \) can be moved (“slid”) around a cup or a cap by transposing it in the computational basis. Alternatively, there is a trivial isomorphism between a cup followed by the operator \( f \) on the second subsystem, the vector \( |f \rangle \), and a cup followed by the operator \( f^T \) on the first subsystem.

Note that \( |f \rangle \) is obtained by taking the matrix representation of \( f \) in the computational basis and rearranging it column by column, left to right, into a column vector, much like the vec operation used in numerical software.

**Proposition 6** (Trace (Fig. 3f)). Looping a wire from an output to an input of an operator \( f \) corresponds to taking a partial trace of \( f \). Consequently, looping all the outputs into corresponding inputs corresponds to the full trace.

**Proposition 7** (Conjugate states (Fig. 3g)). Cups and caps, together with the dagger function, induce an isomorphism between states \( |\psi \rangle \) and their conjugate states \( |\psi^\dagger \rangle \), obtained by complex conjugating the coefficients of the state in the computational basis.

At first it might seem strange that we should encounter basis-dependent operations such as transposition and complex conjugation. However, this is a direct result of us having chosen a preferred computational basis and defined the cup and cap structures in terms of it.
III. MAP-STATE DUALITY

We will now explore the cup-induced isomorphism (introduced in Proposition 5) between linear operators \( f : A \to B \) and bipartite vectors \( |f\rangle_{AB} \), which may be interpreted as pure quantum states. This is equivalent to the common \( \text{vec} \)-mapping, or columnwise vectorization of a matrix. One immediate result is that all matrix/operator decompositions yield a corresponding bipartite state decomposition, and vice versa.

**Proposition 8 (SVD \iff Schmidt decomposition).** The singular value decomposition (SVD) of an operator \( f : A \to B \) is equivalent to the Schmidt decomposition of \( |f\rangle \), with the singular values \( \{\sigma_i\}_{i=0}^{d-1} \) of \( f \) corresponding to the Schmidt coefficients of \( |f\rangle \), as shown in Fig. 4. \( d := \min(\dim A, \dim B) \).

*Proof.* The SVD of \( f \) is
\[
  f = U \Sigma V,
\]
where \( U : B \to B \) and \( V : A \to A \) are unitary operators and \( \Sigma : A \to B \) is diagonal in the computational basis, with the (real, nonnegative) singular values of \( f \) on the diagonal. \( \Sigma \) can be represented diagrammatically as an order \((1,1)\) diagonal dot (ddot), preceded or followed by a dimension-changing ones-on-diagonal operator \( Q \) if \( \dim A \neq \dim B \). (Without loss of generality we may assume that \( \dim A \leq \dim B \).)

The ddot can further be split into a \( \text{COPY}^{2\to1} \) dot and a unique ket \( |\sigma\rangle \) with nonnegative coefficients:
\[
  \Sigma = \sum_{j=0}^{d-1} \sigma_j |j\rangle_B \langle j|_A = \sum_{i=0}^{d-1} |i\rangle_B \langle i|_A \sum_j |j\rangle_A \langle jj|_A \sum_k \sigma_k |k\rangle_A
\]
(\text{where } \sigma_k \geq 0)

The diagrammatic SVD is presented in Fig. 4a. By applying a cup to the SVD diagram we obtain \( |f\rangle \). We then slide \( V \) around the bend and bend the ddot leg forward, arriving at a diagrammatic representation for the Schmidt decomposition of \( |f\rangle \),
\[
  |f\rangle = \sum_{k=0}^{d-1} \sigma_k V_k^i U_k^j |ij\rangle,
\]
shown in Fig. 4b.

Conversely, given a bipartite state \( |\psi\rangle \) we may apply the snake equation, apply the diagrammatic SVD and then perform the steps above to obtain the diagrammatic Schmidt decomposition of \( |\psi\rangle \). See also [7] for some applications.

**Proposition 9 (Spectral decomposition \iff conjugate state decomposition).** If the operator \( f : A \to A \) is normal, i.e. \( f^\dagger f = ff^\dagger \), we may write its spectral (eigenvalue) decomposition as
\[
  f = U \Lambda U^\dagger = \sum_{k=0}^{d-1} \lambda_k |\lambda_k\rangle \langle \lambda_k|,
\]
Figure 4. Correspondence between SVD and Schmidt decomposition. (a) SVD of $f : A \to B$. (b) Schmidt decomposition of $|f\rangle$.

where $U : A \to A$ is unitary and $\Lambda : A \to A$ is diagonal, and $U = |\lambda_k\rangle\langle k|$. Much like in Proposition 8, we may write this in diagram form, with $|\sigma\rangle$ replaced by $|\lambda\rangle$, holding the spectrum of $f$. Likewise, we may then find the corresponding decomposition for $|f\rangle$:

$$|f\rangle = \sum_{k=0}^{d-1} \lambda_k U_k^U U_k^J |ij\rangle = \sum_{k=0}^{d-1} \lambda_k |\overline{\lambda_k}\rangle |\lambda_k\rangle.$$

In the special case where $f$ is hermitian, the spectrum is real. Furthermore, if $f \geq 0$, the spectrum is nonnegative as well.

**Proposition 10** (Purification of positive operators). Any positive operator $\rho : B \to B$ can be purified, i.e., represented as the partial trace of a bipartite positive rank-1 operator $|f\rangle\langle f|$ in $\text{End}(A \otimes B)$:

$$\rho = f f^\dagger = \text{Tr}_A(|f\rangle\langle f|),$$

where $f : A \to B$, for any $\text{dim} A \geq \text{rank} \rho$. Conversely, for every nonvanishing bipartite vector $|f\rangle \in A \otimes B$ the expression $\text{Tr}_A(|f\rangle\langle f|)$ gives a positive operator of rank $\leq \min(\text{dim} A, \text{dim} B)$.

**Proof.** See the diagram identity below:

$$\begin{align*}
B \rho B &= B U^B Q^B A U B = B f^\dagger A f B = \begin{array}{c}
\infty\end{array}.
\end{align*}$$

The first equality is the spectral decomposition of an arbitrary positive operator $\rho$ of rank $\leq \text{dim} A$. The second introduces an arbitrary unitary matrix $V$ to construct the SVD $f = U \Sigma V$. Finally, the $f$ operators are converted to the corresponding bipartite states $|f\rangle$ using cups and caps. Conversely, any vector $|f\rangle$ inserted into the diagram on the right yields a positive $\rho$.

These results can be used to derive correspondence rules between the properties of operators and the bipartite pure states dual to them, as shown in Table II.
Table I. Cup-induced isomorphism between operators and bipartite pure quantum states.

| operator state vector | state vector |
|-----------------------|--------------|
| $f : A \to B$         | $|f\rangle \in A \otimes B$ |
| $\mathbf{1}$         | $|\supset\rangle$ |
| $\text{Tr}(f)$       | $\langle \subset |f\rangle$ |
| real                  | real |
| symmetric             | symmetric ($|f\rangle = \text{SWAP}|f\rangle = |f^T\rangle$) |
| antisymmetric         | antisymmetric |
| hermitian             | $\text{SWAP}|f\rangle = |f\rangle$ |
| unitary               | locally equivalent to a Bell state |
| rank                  | Schmidt rank |
| invertible (full-rank)| full Schmidt rank |
| rank-1 ($f = |x\rangle\langle y|$) | factorizable |
| diagonal              | $\text{ddot}^{0 \to 2}$ |
| SVD                   | Schmidt decomposition |
| spectral decomposition| conjugate state decomposition |

IV. CHANNEL-STATE DUALITY

Much like in the previous section, we now use cups and caps to construct an isomorphism between linear operator maps (channels) and bipartite operators, and use it to prove several correspondence rules.

A. Definitions

Given two Hilbert spaces, $A$ and $B$, a linear operator

$$\Omega : \text{End}(A) \to \text{End}(B)$$

mapping linear operators on $A$ to linear operators on $B$ is called a channel between $A$ and $B$. Compatible channels can be concatenated, and all channels of the form $\Omega : \text{End}(A) \to \text{End}(A)$ form a monoid, i.e. the concatenation is associative and there is an identity element. Inverses are not guaranteed, hence not a group.

Definition 3 (Choi-Jamiołkowski isomorphism). Using cup- and cap-induced dualities it follows that

$$\text{Hom}(\text{End}(A), \text{End}(B)) \cong \text{Hom}(A \otimes A, B \otimes B) \cong \text{End}(A \otimes B).$$

Hence any channel $\Omega : \text{End}(A) \to \text{End}(B)$ may be represented equally well using the related linear maps

$$\tilde{\Omega} : A \otimes A \to B \otimes B \quad \text{and}$$
$$\hat{\Omega} : A \otimes B \to A \otimes B,$$
presented in Fig. 5a. $\tilde{\Omega}$ is the usual “vec-superoperator” representation of $\Omega$, i.e. an operator operating on vectorized operators: $|\Omega(\rho)\rangle = \tilde{\Omega}|\rho\rangle$. It is commonly used in numerical implementations. The second one, $\hat{\Omega}$, is often called the Choi matrix of $\Omega$ and corresponds to $\tilde{\Omega}$ with the inputs and outputs permuted as shown in Fig. 5b. In these representations, the identity channel is given by $\tilde{1} = 1$ and $\hat{1} = |\rangle\langle\|$. respectively.

![Figure 5. Channel-operator identities. (a) Cup-induced isomorphisms between $\Omega \in \text{Hom}(\text{End}(A), \text{End}(B)), \tilde{\Omega} \in \text{Hom}(A \otimes A, B \otimes B)$ and $\hat{\Omega} \in \text{End}(A \otimes B)$. (b) Choi matrix $\hat{\Omega}$ in terms of the superoperator $\tilde{\Omega}$.](image)

We shall now classify different types of channels based on the properties they conserve.

**Definition 4 (Basic properties of channels).** A channel $\Omega$ is

(i) hermitianness-preserving (HP) iff $\rho$ is hermitian $\implies$ $\Omega(\rho)$ is hermitian

(ii) positivity-preserving (PP) iff $\rho \geq 0 \implies \Omega(\rho) \geq 0$

(iii) completely positivity-preserving (CPP) iff $\mathbb{1}_C \otimes \Omega$ is PP for all $C$

(iv) trace-preserving (TP) iff $\text{Tr}(\Omega(\rho)) = \text{Tr}(\rho)$ for all $\rho$

(v) unital

iff $\Omega(\mathbb{1}_A) = \mathbb{1}_B$

If $\Omega$ is HP, this implies that $\mathbb{1}_C \otimes \Omega$ is HP for all $C$. This can be shown by expanding a bipartite hermitian $\rho$ in a factorizable hermitian operator basis, and then using Lemma 12. However, $\Omega$ is PP does not imply that $\mathbb{1}_C \otimes \Omega$ is PP. This is why we need to introduce the stronger property, CPP. Clearly CPP $\implies$ PP $\implies$ HP. Note that the order of the factors in a tensor product carries no fundamental importance, hence the identity $\mathbb{1}_C$ could be on the right as well.

Since $\text{Tr}(\hat{\Omega}) = \text{Tr}(\Omega(\mathbb{1}_A))$, we find that for a TP channel $\text{Tr}(\hat{\Omega}) = \text{dim} A$, and for a unital channel $\text{Tr}(\hat{\Omega}) = \text{dim} B$. Therefore if $\Omega$ is both TP and unital (doubly stochastic), this immediately implies that $\text{dim} A = \text{dim} B$.

**Definition 5 (Quantum channel).** In quantum mechanics, any state can be described using a state operator $\rho$, also called a density operator, that is positive and has unit trace. A
quantum channel is any linear map that maps state operators to state operators, also when applied only to a part of a larger system. Therefore it has to be both CPP and TP.

We will now show an important property of the Choi matrix: Both concatenation and tensor product of channels correspond to a tensor product of the corresponding Choi matrices, conjugated by some additional wire structure.

Definition 6 (Concatenated channels). Given channels \( \Omega_1 : \text{End}(A) \to \text{End}(B) \) and \( \Omega_2 : \text{End}(B) \to \text{End}(C) \) we can concatenate them: \( \Omega = \Omega_2 \circ \Omega_1 \). The Choi matrix \( \hat{\Omega} \) is obtained by tensoring the Choi matrices of the concatenated channels together and then conjugating with \( Q = 1_A \otimes \langle \subset | \otimes B \otimes 1_C \):

\[
\hat{\Omega}_2 \circ \hat{\Omega}_1 =
\]

\[
\hat{\Omega}_1 \hat{\Omega}_2
\]

Definition 7 (Factorizable channel). A channel is factorizable iff it is of the form \( \Omega = \Omega_1 \otimes \Omega_2 \), where \( \Omega_k : \text{End}(A_k) \to \text{End}(B_k) \). In this case \( \hat{\Omega} \) is obtained by tensoring the subchannel Choi matrices and then conjugating with \( Q = 1_{A_1} \otimes \text{SWAP}_{B_1 \otimes A_2} \otimes 1_{B_2} \):

\[
\hat{\Omega}_1 \otimes \hat{\Omega}_2 =
\]

\[
\hat{\Omega}_1 \hat{\Omega}_2
\]

B. Correspondence rules

Here we present proofs for several well-known theorems connecting the properties of channels \( \Omega \) to the properties of the corresponding Choi operators \( \hat{\Omega} \). As we shall soon see, for every quantum channel the corresponding Choi operator can be interpreted as the supernormalized state operator of a bipartite quantum state, and vice versa. This channel-state duality leads to many interesting and useful results. The diagrammatic approach makes the proofs shorter, more intuitive and easier to follow.

We shall start by presenting three simple diagrammatic lemmas related to tensoring a channel \( \Omega \) with an identity channel. Remembering that \( \hat{1} = \hat{\mathbb{I}} = | \supset \rangle \langle \subset | \) and inserting it into the diagram in Def. 7 immediately yields Lemma \( \square \).
Lemma 11. Choi matrix diagram for the channel $1_C \otimes \Omega$.

Lemma 12. Applying Lemma 11 to an operator $\rho$.

Lemma 13. Applying Lemma 12 in the case where $C = A$ and $\rho = |\supset\rangle\langle\subset|$, we obtain $(1_A \otimes \Omega)(|\supset\rangle\langle\subset|) = \hat{\Omega}$:

As an immediate consequence of this last lemma, we obtain an operational interpretation for the Choi matrix $\hat{\Omega}$. It can be understood, up to scaling by dim $A$, as the state operator that results when Alice prepares a cup state and then sends one half of it through the quantum channel $\Omega$ to Bob.

Next, we will derive three fundamental correspondence rules between the properties of channels and the corresponding Choi matrices.

Proposition 14 (Hermitianness preservation). $\Omega$ is HP iff $\hat{\Omega}$ is hermitian.

Proof.

$\Leftarrow$: Follows immediately by taking the dagger of the diagram on the right in Fig. 5a.

$\Rightarrow$: $|\supset\rangle\langle\subset|$ is hermitian, which means we may use Lemma 13:

$\Omega$ is HP $\Rightarrow$ $1_A \otimes \Omega$ is HP $\Rightarrow$ $(1_A \otimes \Omega)(|\supset\rangle\langle\subset|) = \hat{\Omega}$ is hermitian.
Proposition 15 (Positivity preservation). $\Omega$ is PP iff for all separable $\sigma \geq 0$ we have $\langle \sigma, \hat{\Omega} \rangle \geq 0$. This also implies that $\hat{\Omega}$ is hermitian.

Proof.

$\Omega$ is PP $\iff$ $\langle \tau, \Omega(\varsigma) \rangle \geq 0 \quad \forall \varsigma, \tau \geq 0$ (by Lemma 28)

$\iff$ $\langle \tau^T \otimes \tau, \hat{\Omega} \rangle \geq 0 \quad \forall \varsigma, \tau \geq 0$

$\iff$ $\langle \sigma, \hat{\Omega} \rangle \geq 0 \quad \forall$ separable $\sigma \geq 0$.

□

Proposition 16 (Complete positivity preservation (Choi’s theorem on CPP maps \[10, 11\])). $\Omega$ is CPP iff $\hat{\Omega}$ is positive.

Proof.

$\Leftarrow$ : Choose an arbitrary $C$, $\sigma \in \text{End}(C \otimes A) \geq 0$, $\tau \in \text{End}(C \otimes B) \geq 0$, and define

$\omega = \begin{array}{c}
A \\
\sigma \\
C \\
B \\
\tau \\
A \\
C
\end{array}$

Now $\omega \geq 0$, since it is the tensor product of two positive operators conjugated by some wire structure. Thus we have

$\hat{\Omega} \geq 0$ $\implies$ $\langle \hat{\Omega}, \omega \rangle \geq 0 \quad \forall \sigma, \tau \geq 0$ (by Lemma 28)

$\iff$ $\langle \sigma \otimes \tau, \mathbb{1}_C \otimes \Omega \rangle \geq 0 \quad \forall \sigma, \tau \geq 0$ (by Lemma 11)

$\iff$ $\mathbb{1}_C \otimes \Omega$ is PP $\quad \forall C$ (by Proposition 15)

$\iff$ $\Omega$ is CPP.

$\Rightarrow$ : $\langle \rangle | \langle \rangle \|$ is positive, which means we may use Lemma 13

$\Omega$ is CPP $\implies$ $\mathbb{1}_A \otimes \Omega$ is PP $\implies$ $(\mathbb{1}_A \otimes \Omega)(| \rangle \langle \|) = \hat{\Omega}$ is positive.

□

Combining the three propositions above, we obtain

$\Omega$ is CPP $\implies$ $\Omega$ is PP $\implies$ $\Omega$ is HP

$\hat{\Omega} \geq 0$ $\implies$ $\langle \sigma, \hat{\Omega} \rangle \geq 0$ $\implies$ $\hat{\Omega}^\dagger = \hat{\Omega}$

where $\sigma$ is any separable state operator.
Proposition 17 (Trace preservation). \( \Omega \) is TP iff \( \text{Tr}_B(\hat{\Omega}) = 1_A \).

 Proof. 

\[
\text{Tr}(\Omega(\rho)) = \sum_A \rho_A \hat{\Omega} = \sum_A \rho_A = \text{Tr}(\rho)
\]

The above equation presents the TP condition in diagram form. Denoting \( \text{Tr}_B(\hat{\Omega}) \) by \( \omega \), we can see that

\[
\Omega \text{ is TP} \iff \text{Tr}(\Omega(\rho)) = \text{Tr}(\omega^T \rho) = \text{Tr}(\rho) \quad \forall \rho
\]

\[
\iff \langle \omega, \rho \rangle = \langle 1_A, \rho \rangle \quad \forall \rho
\]

\[
\iff \langle \omega - 1_A, \rho \rangle = 0 \quad \forall \rho
\]

\[
\iff \omega = \text{Tr}_B(\hat{\Omega}) = 1_A.
\]

\( \square \)

By using Lemma 26, we can see that restricting the domain of \( \Omega \) to \( \rho \geq 0 \) does not change anything.

Proposition 18 (Unitality). \( \Omega \) is unital iff \( \text{Tr}_A(\hat{\Omega}) = 1_B \).

 Proof. Evident by examining the corresponding diagram:

\[
\Omega(1_A) = \sum_B \rho_B \hat{\Omega} = \sum_B \rho_B = 1_B
\]

\( \square \)

Let us now take a look at a few illustrative examples of channels, quantum and otherwise.

Example 19 (Information-erasing channel). The quantum channel given by \( \hat{\Omega} = 1_A \otimes \rho_{\text{out}} \) where \( \rho_{\text{out}} \) is a valid quantum state, is clearly both CPP and TP. It maps any quantum state to \( \rho_{\text{out}} \), thereby erasing all the information in the input state. As a special case, for \( \rho_{\text{out}} = 1_B / \dim B \) we obtain the channel \( \hat{\Omega} = 1_{AB} / \dim B \) which maps any quantum state on \( A \) to the maximum-entropy state on \( B \).
**Example 20** (Unitary channel). A channel of the form \( \hat{\Omega} = (1 \otimes U) |\ldots\rangle\langle\ldots| (1 \otimes U^\dagger) \) is easily seen to be CPP, TP and unital. It corresponds to a quantum evolution by the unitary propagator \( U \).

**Example 21** (Transposing channel). Taking the transpose of a state is a classic example of an operation that is TP, unital and PP but not CPP, and thus not a valid quantum evolution. This can be shown diagrammatically by forming the Choi matrix \( \hat{\Omega}_T = \text{SWAP}_{A \otimes A} \) (see Fig. 6) and showing that it is not positive by presenting a state \( |\psi\rangle \) that corresponds to a strictly negative eigenvalue. In this case \( |\psi\rangle \) can be chosen to be any antisymmetric state in \( A \otimes A \) (which always exist whenever \( A \) is nontrivial). However, \( \hat{\Omega}_T \) is easily seen to be both TP and unital by taking the appropriate partial traces of \( \hat{\Omega}_T \), and PP by \( \langle \sigma, \hat{\Omega}_T \rangle = \sum_k p_k \text{Tr}(\varsigma_k \tau_k) \geq 0 \), where the inequality is obtained using Lemma 27.

\[
\begin{array}{c}
A \\
A
\end{array}
\begin{array}{c}
A \\
A
\end{array}
\hat{\Omega}_T
\begin{array}{c}
A \\
A
\end{array}
\begin{array}{c}
A \\
A
\end{array} =
\begin{array}{c}
A \\
A
\end{array}
\begin{array}{c}
A \\
A
\end{array}
\]

Figure 6. Transposing channel.

![Diagram of transposing channel](image)

**Figure 7.** For any pair of Hilbert spaces \( A \) and \( B \), the linear space of Choi matrices \( \hat{\Omega} \in \text{End}(A \otimes B) \) is isomorphic to the linear space of channels \( \Omega : \text{End}(A) \to \text{End}(B) \). The cone of CPP channels (positive Choi matrices) is self-dual. The cone of separable channels is dual to the cone of PP channels.

Fig. 7 illustrates the relationships between some of the most important subsets of Choi matrices.

**C. Concatenated and factorizable channels**

Using the Choi matrix diagrams in Defs. 6 and 7 together with the correspondence rules in Sec. IVB we may quickly derive the following properties for concatenated and factorizable
channels. In both cases the Choi matrix has the structure \( \hat{\Omega} = Q(\hat{\Omega}_1 \otimes \hat{\Omega}_2)Q^\dagger \), which means that we may often use similar proofs.

**Theorem 22** (Properties of factorizable channels).

(i) \( \Omega_1, \Omega_2 \) are HP \( \implies \hat{\Omega}^\dagger = Q(\hat{\Omega}_1^\dagger \otimes \hat{\Omega}_2^\dagger)Q^\dagger = \hat{\Omega} \) \( \implies \Omega_2 \circ \Omega_1 \) and \( \Omega_1 \otimes \Omega_2 \) are HP.

The converse is not true since \( i \otimes i = -1 \).

(ii) \( \Omega_1, \Omega_2 \) are PP \( \implies \Omega_1(\rho) \geq 0 \ \forall \rho \geq 0 \) \( \implies (\Omega_2 \circ \Omega_1)(\rho) \geq 0 \ \forall \rho \geq 0 \) \( \implies \Omega_2 \circ \Omega_1 \) is PP.

The converse is not true since \( -1 \times -1 = 1 \).

Not true for factorizable channels since \( 1 \) is PP, yet \( 1 \otimes \Omega \) is not always PP. Again, this is the reason we need to introduce the concept of CPP.

(iii) \( \Omega_1, \Omega_2 \) are CPP \( \implies \hat{\Omega} = Q(\hat{\Omega}_1 \otimes \hat{\Omega}_2)Q^\dagger \geq 0 \) \( \implies \Omega_2 \circ \Omega_1 \) and \( \Omega_1 \otimes \Omega_2 \) are CPP.

The converse is not true since \(-1 \otimes -1 = 1\).

(iv) Concatenated channel:

\( \Omega_1, \Omega_2 \) are TP \( \implies \text{Tr}_C(\hat{\Omega}) = \text{Tr}_B(\hat{\Omega}_1) = 1_A \) \( \implies \Omega_2 \circ \Omega_1 \) is TP.

Factorizable channel:

\( \Omega_1, \Omega_2 \) are TP \( \implies \text{Tr}_B(\hat{\Omega}) = \text{Tr}_{B_1}(\hat{\Omega}_1) \otimes \text{Tr}_{B_2}(\hat{\Omega}_2) = 1_A \) \( \implies \Omega_1 \otimes \Omega_2 \) is TP.

The converse is not true since \( x1 \otimes x^{-1}1 = 1 \).

(v) Concatenated channel:

\( \Omega_1, \Omega_2 \) are unital \( \implies \text{Tr}_A(\hat{\Omega}) = \text{Tr}_B(\hat{\Omega}_2) = 1_C \) \( \implies \Omega_2 \circ \Omega_1 \) is unital.

Factorizable channel:

\( \Omega_1, \Omega_2 \) are unital \( \implies \text{Tr}_A(\hat{\Omega}) = \text{Tr}_{A_1}(\hat{\Omega}_1) \otimes \text{Tr}_{A_2}(\hat{\Omega}_2) = 1_B \) \( \implies \Omega_1 \otimes \Omega_2 \) is unital.

The converse is not true (like above).
The identity channel \( \mathbb{1} = | \oplus \rangle \langle \ominus | \) is CPP, TP and unital. This means that we can use these results to obtain the properties of \( \mathbb{1} \otimes \Omega \).

**Example 23** (Partial transpose). Partial transpose of a bipartite state amounts to transposing one of the subsystems while leaving the other one invariant. It thus corresponds to the channel \( \Omega_{\text{PT}} = \mathbb{1} \otimes \Omega_T \), where \( \Omega_T \) is the transposing channel from Example 21. Using the results above, we immediately see that \( \Omega_{\text{PT}} \) is HP, TP and unital, but not PP.

These properties can also be shown directly by forming the Choi matrix \( \hat{\Omega}_{\text{PT}} \) (see Fig. 8).

**Figure 8. Partial transpose w.r.t. subsystem \( B \).**

**D. Decompositions**

Using the results of Sec. [IVB](#) we may now use standard matrix decompositions to derive corresponding decompositions for channels.

**Proposition 24** (Kraus operator sum representation \( \iff \) spectral decomposition). *Given the Choi matrix \( \hat{\Omega} \geq 0 \), we may write its spectral decomposition as*

\[
\hat{\Omega} = \sum_k \omega_k |\psi_k \rangle \langle \psi_k| = \sum_k |f_k \rangle \langle f_k|,
\]

*where the eigenvalues \( \omega_k \geq 0 \) can be absorbed into the rescaled eigenvectors \( |f_k \rangle = \sqrt{\omega_k} |\psi_k \rangle \).*

*By elementary diagram manipulation as shown in Fig. 9, we can see that this yields the Kraus operator sum representation for the CPP channel \( \Omega \), and vice versa. The number of Kraus operators \( f_k \) in the representation is equal to rank \( \hat{\Omega} \). Likewise, \( \Omega \) is TP if and only if \( \mathbb{1}_A = \text{Tr}_B(\hat{\Omega}) = \sum_k f_k^\dagger f_k \), the familiar criterion on the completeness of a set of Kraus operators.*

**Figure 9. Spectral decomposition of \( \hat{\Omega} \) is equivalent to the Kraus operator sum representation of \( \Omega \).**
The eigenvalues are nonnegative and have been absorbed into \( f_k \) and \( f_k^\dagger \). The dashed line denotes summation over the shared index.
E. Dual channels

Definition 8 (Dual channel). Given a channel $\Omega$, we define its dual channel $\Omega^* : \text{End}(B) \to \text{End}(A)$ with respect to the Hilbert-Schmidt inner product $\langle \cdot, \cdot \rangle$ to be the channel that fulfills

$$\langle x, \Omega(\rho) \rangle = \langle \Omega^*(x), \rho \rangle \quad \forall \rho, x. \quad (10)$$

From the definitions in Fig. 5, it immediately follows that

$$\tilde{\Omega}^* = \tilde{\Omega}^\dagger \quad \text{and} \quad \hat{\Omega}^* = \text{SWAP}_{A \otimes B} \tilde{\Omega} \text{SWAP}_{B \otimes A}. \quad (11)$$

$$\hat{\Omega}^* = \text{SWAP}_{A \otimes B} \tilde{\Omega} \text{SWAP}_{B \otimes A}. \quad (12)$$

When $\Omega$ is a quantum channel and $\rho$ a quantum state, $x$ can be interpreted as a hermitian observable or a POVM element, and the duality transformation itself as the change between the Schrödinger and Heisenberg pictures.

Using Eq. (12) and the correspondence results in Sec. IV B, we may derive a set of equivalences between the properties of channels and their duals, presented in Table II.

Table II. Equivalences between the properties of channels and their duals.

| channel | dual channel |
|---------|--------------|
| $\Omega : \text{End}(A) \to \text{End}(B)$ | $\Omega^* : \text{End}(B) \to \text{End}(A)$ |
| HP | HP |
| PP | PP |
| CPP | CPP |
| TP | unital |
| unital | TP |
| unitary, $U$ | unitary, $U^\dagger$ |

V. DISCUSSION

We summarize the results we have presented on channel-operator correspondence in Table III.

Using tensor network diagrams to describe quantum mechanical systems is a broad topic with has attracted considerable interest recently. Reference [4] gives detailed diagrammatic proofs of several results not elaborated here, e.g. the Stinespring theorem.

ACKNOWLEDGMENTS

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Table III. Isomorphism between channels and bipartite operators.

| channel | Choi operator |
|---------|---------------|
| $\Omega : \text{End}(A) \to \text{End}(B)$ | $\hat{\Omega} : A \otimes B \to A \otimes B$ |

- **HP** hermitian
- **PP** in the dual cone to separable states
- **CPP** positive
- **TP** $\text{Tr}_B(\hat{\Omega}) = I_A$
- **unital** $\text{Tr}_A(\hat{\Omega}) = I_B$
- **Tr($\Omega(1_A)$)** $\text{Tr}(\hat{\Omega})$

- pure, single Kraus term pure
- unitary locally equivalent to a Bell state
- $\langle X^T, \rho \rangle Y$ factorizable, $X \otimes Y$
- separable separable
- Kraus decomposition spectral decomposition
- Kraus rank rank

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[10] E. C. G. Sudarshan, P. M. Mathews, and Jayaseetha Rau, “Stochastic dynamics of quantum-mechanical systems,” *Phys. Rev.* 121, 920–924 (1961).

[11] Man-Duen Choi, “Completely positive linear maps on complex matrices,” *Linear Algebra and its Applications* 10, 285–290 (1975).
Appendix A: Some linear algebra background

Throughout this work we explicitly use the Hilbert-Schmidt inner product and the Frobenius norm.

**Definition 9** (Hilbert-Schmidt inner product). Let \( f, g : A \to B \) be linear operators between the Hilbert spaces \( A \) and \( B \). The Hilbert-Schmidt inner product on the space \( \text{Hom}(A, B) \) is defined as
\[
\langle f, g \rangle := \text{Tr}(f^\dagger g).
\]
It induces the Frobenius norm:
\[
\|f\|_F := \sqrt{\langle f, f \rangle}.
\]
For kets and bras interpreted as operators in \( \text{Hom}(1, A) \) and \( \text{Hom}(A, 1) \), respectively, the Hilbert-Schmidt inner product reduces to the usual inner product of vectors on a Hilbert space.

**Definition 10** (Dual cone). Given a linear space \( A \) and the Hilbert-Schmidt inner product on it, we define the dual cone of a set \( Q \subset A \) to be
\[
Q^* = \{ y \in A | \langle y, x \rangle \geq 0 \; \forall x \in Q \}.
\]

\( Q^* \) is clearly a convex cone, since any linear combination of its elements with nonnegative scalar multipliers yields another element of \( Q^* \). We say that \( Q \) is self-dual iff \( Q = Q^* \).

**Proposition 25** (Leg-bending is a HS isometry). Let \( f, g : A \to B \) be linear maps between the Hilbert spaces \( A \) and \( B \). Since bending of tensor legs using cups and caps amounts to merely reshaping the corresponding matrix, any leg-bending operation \( \Xi \) preserves the Hilbert-Schmidt inner product:
\[
\langle \Xi(f), \Xi(g) \rangle = \text{Tr}\left( (\Xi(f))^\dagger \Xi(g) \right) = \text{Tr}(f^\dagger g) = \langle f, g \rangle.
\]
This is easy to verify using diagrams. Hence any such \( \Xi \) is an isometry and preserves orthogonality between sets of tensors. The induced Frobenius norm is also preserved.

**Lemma 26.** On a complex Hilbert space \( A \), positive operators span all linear operators:
\[
\text{span}\{ \rho \in \text{End}(A) | \rho \geq 0 \} = \text{End}(A).
\]

**Proof.** Any operator \( x \in \text{End}(A) \) can be decomposed to its hermitian and antihermian parts: \( x = h_1 + ih_2 \), where \( h_1, h_2 \) are hermitian. Furthermore, any hermitian operator can be expressed as the difference of two positive operators: \( h = \rho_1 - \rho_2 \).

**Corollary:** Assume we have an arbitrary linear functional \( \phi : \text{End}(A) \to \mathbb{C} \). Now
\[
\phi(\rho) = 0 \; \forall \rho \geq 0 \iff \phi(x) = 0 \; \forall x \iff \phi = 0,
\]
and thus
\[
\langle x, \rho \rangle = 0 \; \forall \rho \geq 0 \iff x = 0,
\]
since \( \langle x, \cdot \rangle \) is a linear functional.
Lemma 27. The product of two positive operators $\sigma$, $\tau$ is not necessarily positive itself, but it has a nonnegative spectrum of eigenvalues.

Proof. Given $\sigma, \tau \geq 0$, assume $\lambda$ is an eigenvalue of $\sigma \tau$:

$$\sigma \tau |x\rangle = \lambda |x\rangle \quad \text{for some } |x\rangle \neq 0.$$ 

This implies

$$\sqrt{\tau} \sigma \sqrt{\tau} (\sqrt{\tau} |x\rangle) = \lambda (\sqrt{\tau} |x\rangle).$$

If $\sqrt{\tau} |x\rangle \neq 0$ this means that $\lambda$ is an eigenvalue of $\sqrt{\tau} \sigma \sqrt{\tau} \geq 0$, and thus nonnegative. If $\sqrt{\tau} |x\rangle = 0$, we have $|x\rangle \in \text{Ker}(\tau)$ and thus $\lambda = 0$.

Corollary: For any two $\sigma, \tau \geq 0$ we have $\langle \sigma, \tau \rangle = \text{Tr}(\sigma \tau) \geq 0$.

Lemma 28 shows that the set of positive operators is its own dual cone.

Lemma 28. Positive operators form a self-dual convex cone:

$$\langle \sigma, \tau \rangle \geq 0 \quad \forall \sigma \geq 0 \iff \tau \geq 0. \quad (A8)$$

Proof.

$\Leftarrow$: By Lemma 27

$\Rightarrow$: Choosing $\sigma = |\psi\rangle \langle \psi|$ we obtain $\langle \sigma, \tau \rangle = \text{Tr}(|\psi\rangle \langle \psi| \tau) = \langle \psi | \tau | \psi \rangle \geq 0$. Since this holds for an arbitrary $|\psi\rangle$, we find that $\tau \geq 0$.

Appendix B: Quantum states, separability and entanglement

In quantum mechanics, every physical system is associated with a complex Hilbert space $\mathcal{H}$, called a state space. Every state of the system can be described using a state operator $\rho \in \text{End}(\mathcal{H})$, also called a density operator, which is positive semidefinite and has unit trace, and vice versa. Given $\mathcal{H}$, the set of all state operators of this system is denoted as $S(\mathcal{H})$, and is seen to be convex and closed. It is also bounded due to the unit trace requirement, without which we would obtain the convex cone of positive operators instead.

The extremal points of $S(\mathcal{H})$ constitute the pure states $P(\mathcal{H})$, i.e., states of the form $\rho = |\psi\rangle \langle \psi|$ where $|\psi\rangle \in \mathcal{H}$. However, all the boundary points need not be pure states. This can be seen using a dimensional argument: using $d := \dim_{\mathbb{C}} \mathcal{H}$, clearly $\dim_{\mathbb{R}} P(\mathcal{H}) = 2d - 2$ whereas $\dim_{\mathbb{R}} \partial S(\mathcal{H}) = d^2 - 2$. In the single-qubit case ($d = 2$) $\partial S(\mathcal{H})$ coincides with $P(\mathcal{H})$, but in higher dimensions $\dim_{\mathbb{R}} P(\mathcal{H})$ is strictly smaller than $\dim_{\mathbb{R}} \partial S(\mathcal{H})$.

A system is bipartite if it has a state space of the form $\mathcal{H} = A \otimes B$, where $A$ and $B$ are the state spaces of its subsystems (neither of which has to be elementary in the physical sense). This idea of dividing a system into two parts enables us to introduce the concept of entanglement.

Definition 11 (Factorizable, separable and entangled states). A bipartite state operator $\rho \in S(A \otimes B)$ is
• factorizable iff \( \rho = \zeta \otimes \tau \), where \( \zeta \in S(A) \) and \( \tau \in S(B) \),

• separable iff it can be expressed as a convex combination of factorizable state operators:

\[
\rho = \sum_k p_k \zeta_k \otimes \tau_k, \quad \text{where } p_k \geq 0 \text{ and } \sum_k p_k = 1,
\]

(B1)

• entangled iff it is not separable.

Due to its construction, the set of separable states is also convex, closed and bounded. Again, if one relaxes the unit trace requirement, one obtains the convex cone of separable positive operators.