\textbf{Abstract}

We present an $S_4$ flavor model to unify quarks and leptons in the framework of the $SU(5)$ GUT. Three generations of 5-plets in $SU(5)$ are assigned 3 of $S_4$ while the first and the second generations of 10-plets in $SU(5)$ are assigned to be 2 of $S_4$, and the third generation of 10-plet is to be 1 of $S_4$. Right-handed neutrinos are also assigned 2 for the first and second generations and 1 for the third generation, respectively. Taking vacuum alignments of relevant gauge singlet scalars, we predict the quark mixing as well as the tri-bimaximal mixing of neutrino flavors. Especially, the Cabbibo angle is predicted to be $15^\circ$ in the limit of the vacuum alignment. We can improve the model to predict observed CKM mixing angles.

\section{Introduction}

Neutrino experimental data provide an important clue for elucidating the origin of the observed hierarchies in mass matrices for quarks and leptons. Recent experiments of the neutrino oscillation go into a new phase of precise determination of mixing angles and mass squared differences \cite{1}, which indicate the tri-bimaximal mixing for three flavors in the lepton sector \cite{2}. These large mixing angles are different from the quark mixing ones. Therefore, it is important to find a natural model that leads to these mixing patterns of quarks and leptons with good accuracy.

Flavor symmetry is expected to explain the mass spectrum and the mixing matrix of quarks and leptons. In particular, some predictive models with non-Abelian discrete flavor symmetries have been explored by many authors. Among them, the tri-bimaximal mixing of leptons has been understood based on the non-Abelian finite group $A_4$, \cite{3}--\cite{24} because these symmetries provide the definite meaning of generations and connects different generations. On the other hand, much attention has been devoted to the question $SU(5)$ whether these models can be extended to describe the observed pattern of quark masses and mixing angles, and whether these can be made compatible with the $SU(5)$ or $SO(10)$ grand unified theory (GUT).

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Recently, group-theoretical arguments indicate that the discrete symmetry $S_4$ is the minimal flavor symmetry compatible with the tri-bimaximal neutrino mixing \[25\]. Actually, the exact tri-bimaximal neutrino mixing is realized in the $S_4$ flavor model \[26\]. Thus, the $S_4$ flavor model is attractive for the lepton sector. \[27\] - \[31\]. Although an attempt to unify quark and lepton sector was presented towards a grand unified theory of flavor \[28\], mixing angles are not predicted clearly. In this work, we present an $S_4$ flavor model to unify the quarks and leptons in the framework of the $SU(5)$ GUT.

## 2 Prototype of $S_4$ flavor model in $SU(5)$ GUT

We present the prototype of the $S_4$ flavor model in the $SU(5)$ GUT to understand the essence of our model clearly. We consider the supersymmetric GUT based on $SU(5)$ The flavor symmetry of quarks and leptons is the discrete group $S_4$ in our model. The group $S_4$ is the permutation group of four distinct objects \[32\] - \[33\]. It is isomorphic to the symmetry group of regular octahedron. It has 24 distinct elements and has five irreducible representations $1$, $2$, $3$, $3_1$, and $3_2$.

| $(T_1, T_2)$ | $T_3$ | $(F_1, F_2, F_3)$ | $(N^c_e, N^c_\mu)$ | $N^c_\tau$ | $H_5$ | $H_5$ |
|-------------|-------|------------------|-------------------|----------|-------|-------|
| $SU(5)$     | 10    | 10              | 5                 | 1        | 1     | 5     | 5     |
| $S_4$       | 2     | $1_1$           | $3_1$             | 2        | 1     | 1     | 1     |
| $Z_4$       | $\omega^3$ | $\omega^2$  | $\omega$         | 1        | 1     | 1     | 1     |

| $SU(5)$ | $\chi_1$ | $(\chi_2, \chi_3)$ | $(\chi_4, \chi_5)$ | $(\chi_6, \chi_7, \chi_8)$ | $(\chi_9, \chi_{10}, \chi_{11})$ | $(\chi_{12}, \chi_{13}, \chi_{14})$ |
|---------|-----------|-------------------|-------------------|---------------------------|---------------------------------|---------------------------------|
| $S_4$   | 1         | 1             | 1                 | 1                         | 1                               | 1                               |
| $Z_4$   | $\omega^2$ | $\omega^2$  | 1                 | $\omega^3$                | 1                               | $\omega$ |

Table 1: Assignments of $SU(5)$, $S_4$, and $Z_4$ representations, where the phase factor $\omega$ is $i$.

Let us present the model of the quark and lepton flavor with the $S_4$ group in $SU(5)$ GUT. In $SU(5)$, matter fields are unified into a $10(q_l, u^c, e^c)_L$ and a $5(d^c, l^c)_L$ dimensional representations. Three generations of $5$, which are denoted by $F_i$, are assigned by $3_1$ of $S_4$. On the other hand, the third generation of the 10-dimensional representation is assigned by $1_1$ of $S_4$, so that the top quark Yukawa coupling is allowed in tree level. While, the first and the second generations are assigned $2$ of $S_4$. These 10-dimensional representations are denoted by $T_3$ and $(T_1, T_2)$, respectively. Right-handed neutrinos, which are $SU(5)$ gauge singlets, are also assigned $1_1$ and $2$ for $N^c_e$ and $(N^c_\mu, N^c_\tau)$, respectively.

We introduce new scalars $\chi_i$ in addition to the 5-dimensional and 5-dimensional Higgs of the $SU(5)$, $H_5$ and $H_5$ which are assigned $1_1$ of $S_4$. These new scalars are supposed to be $SU(5)$ gauge singlets. The $\chi_1$ scalar is assigned $1_1$, $(\chi_2, \chi_3)$, and $(\chi_4, \chi_5)$ are $2$, $(\chi_6, \chi_7, \chi_8)$, $(\chi_9, \chi_{10}, \chi_{11})$, and $(\chi_{12}, \chi_{13}, \chi_{14})$ are $3_1$ of the $S_4$ representations, respectively. The $\chi_1$ and $(\chi_2, \chi_3)$ scalars are coupled with the up-type quark sector, $(\chi_4, \chi_5)$ are coupled with the right-handed Majorana neutrino sector, $(\chi_6, \chi_7, \chi_8)$ are coupled with the Dirac neutrino sector, $(\chi_9, \chi_{10}, \chi_{11})$ and $(\chi_{12}, \chi_{13}, \chi_{14})$ are coupled with the charged lepton and
down-type quark sector, respectively. We also add $Z_4$ symmetry in order to obtain relevant couplings. The particle assignments of $SU(5)$, $S_4$, and $Z_4$ are summarized Table 1.

We can now write down the superpotential at the leading order in terms of the cut off scale $\Lambda$, which is taken to be the Planck scale. The $SU(5)$ invariant superpotential of the Yukawa sector respecting $S_4$ and $Z_4$ symmetries is given as

$$w_{SU(5)}^{(0)} = y_1^u(T_1, T_2) \otimes (T_1, T_2) \otimes \chi_1 \otimes H_5/\Lambda$$

$$+ y_2^u(T_1, T_2) \otimes (T_1, T_2) \otimes (\chi_2, \chi_3) \otimes H_5/\Lambda + y_3^u T_3 \otimes H_5$$

$$+ M_1(N^c_e, N^c_\mu) \otimes (N^c_\mu, N^c_\tau) + M_2 N^c_t \otimes N^c_\tau$$

$$+ y^N(N^c_e, N^c_\mu) \otimes (N^c_\mu, N^c_\tau) \otimes (\chi_4, \chi_5)$$

$$+ y^D(N^c_e, N^c_\mu) \otimes (F_1, F_2, F_3) \otimes (\chi_6, \chi_7, \chi_8) \otimes H_5/\Lambda$$

$$+ y^D N^c_t \otimes (F_1, F_2, F_3) \otimes (\chi_6, \chi_7, \chi_8) \otimes H_5/\Lambda$$

$$+ y_1(F_1, F_2, F_3) \otimes (T_1, T_2) \otimes (\chi_9, \chi_{10}, \chi_{11}) \otimes H_5/\Lambda$$

$$+ y_2(F_1, F_2, F_3) \otimes T_3 \otimes (\chi_{12}, \chi_{13}, \chi_{14}) \otimes H_5/\Lambda,$$

(1)

where $M_1$ and $M_2$ are mass parameters for right-handed Majorana neutrinos, and Yukawa coupling constants $y_i^u$ and $y_i$ are complex in general. By decomposing this superpotential into the quark sector and the lepton sector, we can discuss mass matrices of quarks and leptons in following sections.

## 3 Lepton sector

At first, we begin to discuss the lepton sector of the superpotential $w_{SU(5)}^{(0)}$. Denoting Higgs doublets as $h_u$ and $h_d$, the superpotential of the Yukawa sector respecting the $S_4 \times Z_4$ symmetry is given for charged leptons as

$$w_l = y_1 \left[ \frac{e^c}{\sqrt{2}} (l_\mu \chi_{10} - l_\tau \chi_{11}) + \frac{\mu^c}{\sqrt{6}} (-2l_e \chi_9 + l_\mu \chi_{10} + l_\tau \chi_{11}) \right] h_d/\Lambda$$

$$+ y_2 \tau^c (l_e \chi_{12} + l_\mu \chi_{13} + l_\tau \chi_{14}) h_d/\Lambda.$$  

(2)

For right-handed Majorana neutrinos, the superpotential is given as

$$w_N = M_1(N^c_e N^c_e + N^c_\mu N^c_\mu) + M_2 N^c_\tau N^c_\tau$$

$$+ y^N \left[ (N^c_e N^c_\mu + N^c_\mu N^c_\tau) \chi_4 + (N^c_\mu N^c_\tau - N^c_\mu N^c_\tau) \chi_5 \right],$$

(3)

and for neutrino Yukawa couplings, the superpotential is

$$w_D = y_1^D \left[ \frac{N^c_e}{\sqrt{2}} (l_\mu \chi_7 - l_\tau \chi_8) + \frac{N^c_\mu}{\sqrt{6}} (-2l_e \chi_6 + l_\mu \chi_7 + l_\tau \chi_8) \right] h_u/\Lambda$$

$$+ y_2^D N^c_\tau (l_e \chi_6 + l_\mu \chi_7 + l_\tau \chi_8) h_u/\Lambda.$$  

(4)

Higgs doublets $h_u, h_d$ and gauge singlet scalars $\chi_i$, are assumed to develop their vacuum expectation values (VEVs) as follows:

$$\langle h_u \rangle = v_u, \quad \langle h_d \rangle = v_d, \quad \langle (\chi_4, \chi_5) \rangle = (u_4, u_5), \quad \langle (\chi_6, \chi_7, \chi_8) \rangle = (u_6, u_7, u_8),$$

$$\langle (\chi_9, \chi_{10}, \chi_{11}) \rangle = (u_9, u_{10}, u_{11}), \quad \langle (\chi_{12}, \chi_{13}, \chi_{14}) \rangle = (u_{12}, u_{13}, u_{14}),$$

(5)
which are supposed to be real. Then, we obtain the mass matrix for charged leptons as

$$M_l = y_1 v_d \begin{pmatrix} 0 & \alpha_{10}/\sqrt{2} & -\alpha_{11}/\sqrt{2} \\ -2\alpha_9/\sqrt{6} & \alpha_{10}/\sqrt{6} & \alpha_{11}/\sqrt{6} \\ 0 & 0 & 0 \end{pmatrix} + y_2 v_d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_{12} & \alpha_{13} & \alpha_{14} \end{pmatrix},$$

(6)

while the right-handed Majorana neutrino mass matrix is given as

$$M_N = \begin{pmatrix} M_1 + y^N_5 \alpha_5 \Lambda & y^N_4 \alpha_4 \Lambda & 0 \\ y^N_4 \alpha_4 \Lambda & M_1 - y^N_5 \alpha_5 \Lambda & 0 \\ 0 & 0 & M_2 \end{pmatrix},$$

(7)

and the Dirac mass matrix of neutrinos is

$$M_D = y^D_1 v_u \begin{pmatrix} 0 & \alpha_7/\sqrt{2} & -\alpha_8/\sqrt{2} \\ -2\alpha_6/\sqrt{6} & \alpha_7/\sqrt{6} & \alpha_8/\sqrt{6} \\ 0 & 0 & 0 \end{pmatrix} + y^D_2 v_u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_6 & \alpha_7 & \alpha_8 \end{pmatrix},$$

(8)

where we denote $\alpha_i \equiv u_i/\Lambda$.

Let us discuss lepton masses and mixing angles by considering mass matrices in Eqs.(6), (7) and (8). In order to get the left-handed mixing of charged leptons, we investigate $M_l^\dagger M_l$.

If we can take vacuum alignments $(u_9, u_{10}, u_{11}) = (u_9, u_{10}, 0)$ and $(u_{12}, u_{13}, u_{14}) = (0, 0, u_{14})$, that is $\alpha_{11} = \alpha_{12} = \alpha_{13} = 0$, we obtain

$$M_l^\dagger M_l = v_d^2 \begin{pmatrix} \frac{2}{3} |y_1|^2 \alpha_9^2 & -\frac{1}{3} |y_1|^2 \alpha_9 \alpha_{10} & 0 \\ -\frac{1}{3} |y_1|^2 \alpha_9 \alpha_{10} & \frac{2}{3} |y_1|^2 \alpha_{10}^2 & 0 \\ 0 & 0 & |y_2|^2 \alpha_{14}^2 \end{pmatrix},$$

(9)

which gives $\theta_{13}^l = \theta_{23}^l = 0$, where $\theta_{ij}^l$ denote left-handed mixing angles to diagonalize the charged lepton mass matrix. Since the electron mass is tiny compared with the muon mass, we expect $\alpha_9 \ll \alpha_{10}$ and then we get the mixing angle $\theta_{12}^l$ as,

$$\tan \theta_{12}^l \approx -\frac{\alpha_9}{2\alpha_{10}},$$

(10)

and charged lepton masses,

$$m_e^2 \approx \frac{1}{2} |y_1|^2 \alpha_9^2 v_d^2, \quad m_\mu^2 \approx \frac{2}{3} |y_1|^2 \alpha_{10}^2 v_d^2 + \frac{1}{6} |y_1|^2 \alpha_9^2 v_d^2 \approx \frac{2}{3} |y_1|^2 \alpha_{10}^2 v_d^2, \quad m_\tau^2 = |y_2|^2 \alpha_{14}^2 v_d^2.$$  

(11)

Therefore, the mixing of $\theta_{12}^l$ is estimated as

$$|\tan \theta_{12}^l| \approx \frac{1}{\sqrt{3}} \frac{m_\mu}{m_e} \approx 2.8 \times 10^{-3},$$

(12)

which is negligibly small. Hereafter, we do not consider the mixing from the charged lepton sector.
Taking vacuum alignments \((u_4, u_5) = (0, u_5)\) and \((u_6, u_7, u_8) = (u_6, u_6, u_6)\) in Eq. (17). By using the seesaw mechanism \(M_\nu = M_D^TM_NM_D^{-1}\), the left-handed Majorana neutrino mass matrix is written as

\[
M_\nu = \begin{pmatrix}
a + \frac{2}{3}b & a - \frac{1}{3}b & a - \frac{1}{3}b \\
a - \frac{1}{3}b & a + \frac{1}{3}b + \frac{1}{2}c & a + \frac{1}{3}b - \frac{1}{2}c \\
a - \frac{1}{3}b & a + \frac{1}{3}b - \frac{1}{2}c & a + \frac{1}{3}b + \frac{1}{2}c
\end{pmatrix},
\]

where

\[
a = \frac{(y_D^2 D_\nu u_a)^2}{M_2}, \quad b = \frac{(y_D^1 D_\nu u_a)^2}{M_1 - y_\alpha N_5}, \quad c = \frac{(y_D^0 D_\nu u_a)^2}{M_1 + y_\alpha N_5}.
\]

The neutrino mass matrix is decomposed as

\[
M_\nu = \frac{b + c}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} + \frac{3a - b}{3} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} + \frac{b - c}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

which gives the tri-bimaximal mixing matrix \(U_{\text{tri-bi}}\) and mass eigenvalues as follows:

\[
U_{\text{tri-bi}} = \begin{pmatrix}
\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}, \quad m_1 = b, \quad m_2 = 3a, \quad m_3 = c.
\]

### 4 Quark sector

In this section, we discuss quark sector of the superpotential \(w^{(0)}_{SU(5)}\). For up-type quarks, the superpotential of the Yukawa sector with \(S_4 \times Z_4\) is given as

\[
w_u = y_1^u (u^c q_1 + c^c q_2) \chi_1 h_u / \Lambda \\
+ y_2^u [u^c q_2 + c^c q_1] \chi_2 + (u^c q_1 - c^c q_2) \chi_3] h_u / \Lambda + y_3^u t^c q_3 h_u.
\]

For down-type quarks, we can write the superpotential as follows:

\[
w_d = y_1 \left[ \frac{1}{\sqrt{2}} (s^c \chi_{10} - b^c \chi_{11}) q_1 + \frac{1}{\sqrt{6}} (-2d^c \chi_9 + s^c \chi_{10} + b^c \chi_{11}) q_2 \right] h_d / \Lambda \\
+ y_2 (d^c \chi_{12} + s^c \chi_{13} + b^c \chi_{14}) q_3 h_d / \Lambda.
\]

We assume that scalar fields, \(\chi_i\), develop their VEVs as \(u_i\). Then, the mass matrix for up-type quarks is given as

\[
M_u = v_u \begin{pmatrix}
y_1^u \alpha_1 + y_2^u \alpha_3 & y_2^u \alpha_2 & 0 \\
y_2^u \alpha_2 & y_1^u \alpha_1 - y_2^u \alpha_3 & 0 \\
0 & 0 & y_3^u
\end{pmatrix},
\]

and the down-type quark mass matrix is given as

\[
M_d = y_1 v_d \begin{pmatrix}
0 & -2 \alpha_9 / \sqrt{6} & 0 \\
\alpha_{10} / \sqrt{2} & \alpha_{10} / \sqrt{6} & 0 \\
-\alpha_{11} / \sqrt{2} & \alpha_{11} / \sqrt{6} & 0
\end{pmatrix} + y_2 v_d \begin{pmatrix}
0 & 0 & \alpha_{12} \\
0 & 0 & \alpha_{13} \\
0 & 0 & \alpha_{14}
\end{pmatrix}.
\]
Let us discuss mixing of the quark sector. For up-type quarks, if we take
\[ \alpha_3 = 0, \quad y_1^u \alpha_1 = y_2^u \alpha_2, \]
which will be reexamined to get observed CKM mixing angles in section 5.2, then we have
\[ M_u = v_u \begin{pmatrix} y_1^u \alpha_1 & y_1^u \alpha_1 & 0 \\ y_1^u \alpha_1 & y_1^u \alpha_1 & 0 \\ 0 & 0 & y_3^u \end{pmatrix}, \]
which is diagonalized by the orthogonal matrix \( U_u \)
\[ U_u = \begin{pmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

For down-type quarks, putting \( \alpha_{11} = \alpha_{12} = \alpha_{13} = 0 \), which is the condition in the charged lepton sector, we have
\[ M_d^\dagger M_d = v_d^2 \begin{pmatrix} |y_1|^2 \alpha_{10}^2 & \frac{1}{2\sqrt{3}} |y_1|^2 \alpha_{10}^2 & 0 \\ \frac{1}{2\sqrt{3}} |y_1|^2 \alpha_{10}^2 & \frac{1}{3} |y_1|^2 (4\alpha_9^2 + \alpha_{10}^2) & 0 \\ 0 & 0 & |y_2|^2 \alpha_{14}^2 \end{pmatrix}. \]
Then, the mass matrix is diagonalized by the orthogonal matrix \( U_d \) as
\[ U_d = \begin{pmatrix} \cos 60^\circ & \sin 60^\circ & 0 \\ -\sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
where the small \( \alpha_9 \) is neglected.

Now, we get the CKM matrix as follows:
\[ V_{CKM} = U_u^\dagger U_d = \begin{pmatrix} \cos 15^\circ & \sin 15^\circ & 0 \\ -\sin 15^\circ & \cos 15^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Therefore, in our prototype model of SU(5) GUT with the \( S_4 \) flavor symmetry, the quark sector has a non-vanishing mixing angle 15\(^\circ\) only between the first and second generations while the lepton flavor mixing is tri-bimaximal. In order to get the non-vanishing but small mixing angles \( V_{cb}^{CKM} \) and \( V_{ub}^{CKM} \), we improve the prototype model in the next section.

## 5 Improved \( S_4 \) flavor model in \( SU(5) \) GUT

We improve the prototype model to get the observed quark and lepton mass spectra and the CKM mixing matrix. We introduce the \( SU(5) \) 45-dimensional Higgs \( h_{45} \), which is required to get the difference between the charged lepton mass spectrum and the down-type quark mass spectrum. Moreover, we add an \( S_4 \) doublet \((\chi_2', \chi_3')\) and an \( S_4 \) triplet \((\chi_9', \chi_{10}', \chi_{11}')\), which
are $SU(5)$ gauge singlet scalars. These assignments of $SU(5)$, $S_4$, and $Z_4$ are summarized Table 2. Since the additional scalars do not contribute to the neutrino sector, the result of the neutrino sector in the prototype model is not changed. Therefore, we discuss only the charged lepton sector and the quark sector in this section.

The superpotential of the Yukawa sector respecting the $SU(5)$, $S_4$ and $Z_4$ symmetries is given as

$$w_{SU(5)} = w_{SU(5)}^{(0)} + w_{SU(5)}^{(1)},$$

where we denote

$$w_{SU(5)}^{(1)} = y_1^u(T_1, T_2) \otimes T_3 \otimes (\chi_2', \chi_3') \otimes H_5/\Lambda$$

$$+ y_1^d(F_1, F_2, F_3) \otimes (T_1, T_2) \otimes (\chi_9', \chi_{10}', \chi_{11}') \otimes h_{45}/\Lambda.$$  

5.1 Improved lepton sector

Masses and mixing angles of the charged lepton sector are similar to those of the prototype model in Eqs. (41) and (42). If we can take the vacuum alignments $(u_9, u_{10}, u_{11}) = (u_9, u_{10}, 0)$, $(u_9', u_{10}', u_{11}') = (u_9', u_{10}', 0)$ and $(u_{12}, u_{13}, u_{14}) = (0, 0, u_{14})$, that is $\alpha_{11} = \alpha_{11}' = \alpha_{12} = \alpha_{13} = 0$, we obtain charged lepton mass matrix as follow:

$$M_l = v_d \begin{pmatrix} 0 & (y_1\alpha_{10} - 3\bar{y}_1\alpha_{10}')/\sqrt{6} & 0 \\ -2(y_1\alpha_9 - 3\bar{y}_1\alpha_9')/\sqrt{5} & (y_1\alpha_{10} - 3\bar{y}_1\alpha_{10}')/\sqrt{6} & 0 \\ 0 & 0 & y_2\alpha_{14} \end{pmatrix},$$

where we replace $y_1' v_45$ with $\bar{y}_1 v_d$. Masses and mixing angles of the charged leptons as follows:

$$m^2_e \approx \frac{1}{2} |y_1\alpha_9 - 3\bar{y}_1\alpha_9'|^2 v_d^2,$$

$$m^2_\mu \approx \frac{2}{3} |y_1\alpha_{10} - 3\bar{y}_1\alpha_{10}'|^2 v_d^2,$$

$$m^2_\tau \approx |y_2|^2 \alpha_{14}^2 v_d^2,$$

$$|\theta_{12}^l| = \left| - \frac{y_1\alpha_9 - 3\bar{y}_1\alpha_9'}{2(y_1\alpha_{10} - 3\bar{y}_1\alpha_{10}')} \right| \approx \frac{1}{\sqrt{3}} \frac{m_e}{m_\mu} \approx 2.8 \times 10^{-3}, \quad \theta_{23}^l = 0, \quad \theta_{13}^l = 0.$$  

Thus, the charged lepton mass matrix is almost diagonal, and so the tri-bimaximal mixing of neutrino flavors is also realized in this improved model.

5.2 Improved quark sector

Let us discuss the quark sector of the superpotential $w_{SU(5)}$. For up-type quarks, the mass matrix is given as

$$M_u = v_u \begin{pmatrix} y_1^u\alpha_1 + y_2^u\alpha_3 & y_2^u\alpha_2 & y_4^u\alpha_2' \\ y_2^u\alpha_2 & y_1^u\alpha_1 - y_2^u\alpha_3 & y_4^u\alpha_3' \\ y_4^u\alpha_2' & y_4^u\alpha_3' & y_3^u \end{pmatrix}.$$
while the down-type quark mass matrix is given as

\[
M_d = y_1 v_d \begin{pmatrix}
0 & -2\alpha_9/\sqrt{6} & 0 \\
\alpha_{10}/\sqrt{2} & \alpha_{10}/\sqrt{6} & 0 \\
-\alpha_{11}/\sqrt{2} & \alpha_{11}/\sqrt{6} & 0
\end{pmatrix}
+ y_2 v_d \begin{pmatrix}
0 & 0 & \alpha_{12} \\
0 & 0 & \alpha_{13} \\
0 & 0 & \alpha_{14}
\end{pmatrix}
+ y_1' v_{45} \begin{pmatrix}
0 & -2\alpha'_9/\sqrt{6} & 0 \\
\alpha'_{10}/\sqrt{2} & \alpha'_{10}/\sqrt{6} & 0 \\
-\alpha'_{11}/\sqrt{2} & \alpha'_{11}/\sqrt{6} & 0
\end{pmatrix}.
\]

We consider the quark mixing. The up-type quark mass matrix (31) turns to the following one after rotating by \(\theta_{12}^u = 45^\circ\):

\[
\hat{M}_u = v_u \begin{pmatrix}
y_1^u \alpha_1 - y_2^u \alpha_2 & y_2^u \alpha_3 & \frac{y_3^u}{\sqrt{2}} (\alpha'_2 - \alpha'_3) \\
y_2^u \alpha_3 & y_1^u \alpha_1 + y_2^u \alpha_2 & \frac{y_3^u}{\sqrt{2}} (\alpha'_2 + \alpha'_3) \\
\frac{y_3^u}{\sqrt{2}} (\alpha'_2 - \alpha'_3) & \frac{y_3^u}{\sqrt{2}} (\alpha'_2 + \alpha'_3) & y_3^u
\end{pmatrix}.
\]

In order to obtain the non-vanishing quark mixing of \(V_{CB}^{CKM}\) and \(V_{ub}^{CKM}\), we take

\[y_2^u \alpha_3 \gg y_1^u \alpha_1, \ y_2^u \alpha_2, \ \alpha'_2 = \alpha'_3,\]

which are realized by vacuum alignments \(u_1 = 0\), \((u_2, u_3) = (0, u_3)\) and \((u'_2, u'_3) = (u'_2, u'_3)\). This situation of VEVs is completely different from that of the prototype model as seen in Eq. (21), in which \(V_{CB}^{CKM}\) and \(V_{ub}^{CKM}\) vanish. Then, we obtain the so-called Fritzsch-type mass matrix [34]

\[
\hat{M}_u \simeq v_u \begin{pmatrix}
0 & y_2^u \alpha_3 & 0 \\
y_2^u \alpha_3 & 0 & \sqrt{2} y_3^u \alpha'_2 \\
0 & \sqrt{2} y_3^u \alpha'_2 & y_3^u
\end{pmatrix}.
\]

As well known, the complex phases in this \(3 \times 3\) matrix can be removed by the phase matrix \(P\) as \(P^\dagger \hat{M}_u P\):

\[
P = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-i\rho} & 0 \\
0 & 0 & e^{-i\sigma}
\end{pmatrix}.
\]

Therefore, up-type quark masses are

\[
m_u = \left| \frac{y_3^u y_2^u \alpha_2^2}{2 y_4^u \alpha_2^2} \right| v_u, \quad m_c = \left| \frac{2 y_4^u \alpha_2^2}{y_3^u} \alpha'_2 \right| v_u, \quad m_t = |y_3^u| v_u,
\]

and the mixing matrix to diagonalize \(\hat{M}_u\) in Eq. (35), \(V_F (M_u^{\text{diagonal}} = V_F^\dagger \hat{M}_u V_F)\), is

\[
V_F \approx \begin{pmatrix}
1 & \sqrt{\frac{m_c}{m_t}} & -\sqrt{\frac{m_c}{m_t}} \\
-\sqrt{\frac{m_c}{m_t}} & 1 & \sqrt{\frac{m_c}{m_t}} \\
\sqrt{\frac{m_c}{m_t}} & -\sqrt{\frac{m_c}{m_t}} & 1
\end{pmatrix}.
\]

\(^1\)One may consider to remove \(\chi_1\), which is \(S_4\) singlet, in our scheme.
The conditions from the lepton sector \(\alpha_{11} = \alpha_{12} = \alpha_{13} = 0\) give the down-type quark mass matrix:

\[
M_d = v_d \begin{pmatrix}
0 & -2(y_1\alpha_{10} + \bar{y}_1\alpha'_{10})/\sqrt{6} & 0 \\
(y_1\alpha_{10} + \bar{y}_1\alpha'_{10})/\sqrt{2} & 0 & 0 \\
0 & 0 & y_2\alpha_{14}
\end{pmatrix},
\]

where we denote \(\bar{y}_1 v_d = y'_1 v_{45}\). Then, down-type quark masses are given as

\[
m_d^2 \approx \frac{1}{2}|y_1\alpha_{10} + \bar{y}_1\alpha'_{10}|^2 v_d^2, \quad m_s^2 \approx \frac{2}{3}|y_1\alpha_{10} + \bar{y}_1\alpha'_{10}|^2 v_d^2, \quad m_b^2 \approx |y_2|^2 \alpha_{14}^2 v_d^2,
\]

and the mixing angle \(\theta_{12}^d\) is \(60^\circ + \delta\theta_{12}^d\), where

\[
\delta\theta_{12}^d = -\frac{\sqrt{3}|y_1\alpha_{10} + \bar{y}_1\alpha'_{10}|^2}{4|y_1\alpha_{10} + \bar{y}_1\alpha'_{10}|^2} = -\frac{m_d^2}{\sqrt{3}m_s^2} \approx -1.5 \times 10^{-3}.
\]

Therefore, \(\theta_{12}^d\) is almost \(60^\circ\).

Let us discuss the CKM matrix. The unitary matrices diagonalizing the up-type quark mass matrix and the down-type quark one, \(U_u\) and \(U_d\) are given, respectively,

\[
U_u = \begin{pmatrix}
\cos 45^\circ & \sin 45^\circ & 0 \\
-\sin 45^\circ & \cos 45^\circ & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad U_{d1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-i\rho} & 0 \\
0 & 0 & e^{-i\sigma}
\end{pmatrix}, \quad U_{d2} = \begin{pmatrix}
\cos 60^\circ & \sin 60^\circ & 0 \\
-\sin 60^\circ & \cos 60^\circ & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Since the CKM matrix is given as \(U_u^\dagger U_d\), we obtain the relevant CKM mixing as

\[
|V_{us}^{\text{CKM}}| = |\sin 15^\circ - \cos 15^\circ \sqrt{\frac{m_u}{m_c}} e^{i\rho}|, \quad |V_{cb}^{\text{CKM}}| = \sqrt{\frac{m_c}{m_t}}, \quad |V_{ub}^{\text{CKM}}| = \sqrt{\frac{m_u}{m_t}}.
\]

In the limit of neglecting the CP violating phase, \(\rho = 0\), putting typical values at the GUT scale \(m_u = 1.04 \times 10^{-3}\) GeV, \(m_c = 302 \times 10^{-3}\) GeV, \(m_t = 129\) GeV, which are derived in Ref. [35], we predict

\[
|V_{us}^{\text{CKM}}| = 0.202, \quad |V_{cb}^{\text{CKM}}| = 0.048, \quad |V_{ub}^{\text{CKM}}| = 0.003.
\]

By adjusting the non-zero phase \(\rho = 50^\circ\), we can get the central value of the observed Cabbibo angle 0.226. Another phase \(\sigma\) is still a free parameter.

### 6 Summary

We have presented a flavor model with the \(S_4\) symmetry to unify quarks and leptons in the framework of the \(SU(5)\) GUT. Three generations of \(5\)-plets in \(SU(5)\) are assigned \(3_1\) of \(S_4\)
while the first and the second generations of 10-plets in $SU(5)$ are assigned to be 2 of $S_4$, and the third generation of 10-plet is to be $1_1$ of $S_4$. These assignments of $S_4$ for 5 and 10 lead to the different structure of quark and lepton mass matrices. Right-handed neutrinos, which are $SU(5)$ gauge singlets, are also assigned 2 for the first and second generations and 1 for the third generation, respectively. These assignments are essential to realize the tri-bimaximal mixing of neutrino flavors. Vacuum alignments of scalars are also required to realize the tri-bimaximal mixing of neutrino flavors. Our model predicts the quark mixing as well as the tri-bimaximal mixing of leptons. Especially, the Cabbibo angle is predicted to be $15^\circ$ in the limit of the vacuum alignment. We improve the model to predict observed CKM mixing angles. The deviation from $15^\circ$ in $|V_{us}^{\text{CKM}}|$, the non-vanishing $|V_{cb}^{\text{CKM}}|$, and $|V_{ub}^{\text{CKM}}|$ are given by up-type quark masses.

References

[1] T. Schwetz, M. Tortola, and J.W.F. Valle, arXiv:0808.2016; G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A.M. Rotunno, Phys. Rev. Lett. 101 141801 (2008); arXiv:0806.2649

[2] P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. B 530, 167 (2002); P.F. Harrison and W.G. Scott, Phys. Lett. B 535, 163 (2002).

[3] E. Ma and G. Rajasekaran, Phys. Rev. D 64, 113012 (2001); E. Ma, Mod. Phys. Lett. A 17, 2361 (2002).

[4] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B 552, 207 (2003), arXiv:hep-ph/0206292.

[5] M. Hirsch, J.C. Romao, S. Skadhauge, J.W.F. Valle, and A. Villanova del Moral, Phys. Rev. D 69, 093006 (2004), arXiv:hep-ph/0312265.

[6] E. Ma, Phys. Rev. D 70, 031901 (2004), arXiv:hep-ph/0404199.

[7] G. Altarelli and F. Feruglio, Nucl. Phys. B 720, 64 (2005), arXiv:hep-ph/0504165; Nucl. Phys. B 741, 215 (2006), arXiv:hep-ph/0512103.

[8] S.-L. Chen, M. Frigerio, and E. Ma, Nucl. Phys. B 724, 423 (2005), arXiv:hep-ph/0504181.

[9] A. Zee, Phys. Lett. B 630, 58 (2005), arXiv:hep-ph/0508278.

[10] B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma, and M. K. Parida, Phys. Lett. B 638, 345 (2006), arXiv:hep-ph/0603059.

[11] J. W. F. Valle, J. Phys. Conf. Ser. 53, 473 (2006), arXiv:hep-ph/0608101.

[12] X.-G. He, Y.-Y. Keum, and R.R. Volkas, JHEP 0604, 039 (2006).

[13] E. Ma, H. Sawanaka, and M. Tanimoto, Phys. Lett. B 641, 301 (2006).
[14] B. Adhikary and A. Ghosal, Phys. Rev. D 75, 073020 (2007), [arXiv:hep-ph/0609193]

[15] E. Ma, Phys. Rev. D 70, 031901 (2004), Phys. Rev. D 72, 037301 (2005), Mod. Phys. Lett. A 22, 101 (2007), [arXiv:hep-ph/0610342]

[16] G. Altarelli, F. Feruglio, and Y. Lin, Nucl. Phys. B 775, 31 (2007), [arXiv:hep-ph/0610165]

[17] S. F. King and M. Malinský, Phys. Lett. B 645, 351 (2007).

[18] M. Hirsch, A. S. Joshipura, S. Kaneko, and J.W.F. Valle, Phys. Rev. Lett. 99, 151802 (2007), [arXiv:hep-ph/0703046]

[19] L. Lavoura and H. Kühböck, Mod. Phys. Lett. A 22, 181 (2007), [arXiv:0711.0670]

[20] M. Honda and M. Tanimoto, Prog. Theor. Phys. 119, 585 (2008), [arXiv:0801.0181]

[21] F. Bazzocchi, S. Kaneko, and S. Morisi, JHEP 03, 063 (2008), [arXiv:0707.3032]

[22] F. Bazzocchi, M. Frigerio, and S. Morisi, [arXiv:0809.3573]

[23] Y. Lin, [arXiv:0804.2867]

[24] M. Hirsch, S. Morisi, and J.W.F. Valle, [arXiv:0810.0121]

[25] C. S. Lam, [arXiv:0809.1185]

[26] F. Bazzocchi and S. Morisi, [arXiv:0811.0345]

[27] E. Ma, Phys. Lett. B 632, 352 (2006), [arXiv:hep-ph/0508231]

[28] C. Hagedorn, M. Lindner, and R. N. Mohapatra, JHEP 06, 042 (2006), [arXiv:hep-ph/0602244]

[29] Y. Cai and H.-B. Yu, Phys. Rev. D 74, 115005 (2006), [arXiv:hep-ph/0608022]

[30] H. Zhang, Phys. Lett. B 655, 132 (2007), [arXiv:hep-ph/0612214]

[31] Y. Koide, JHEP 08, 086 (2007), [arXiv:0705.2275]

[32] J. S. Lomont, Applications of Finite Groups, Acad. Press (1959) 346 p.

[33] S. Califano, Vibrational States, Wiley, London (1976) 365 p.

[34] H. Fritzsch, Phys. Lett. B 73, 317 (1978); Nucl. Phys. B 115, 189 (1979).

[35] H. Fusaoka and Y. Koide, Phys. Rev. D 57, 3986 (1998), [hep-ph/9712201]

[36] T. Kobayashi, Y. Omura, and K. Yoshioka, [arXiv:0809.3064], G. Seidl, [arXiv:0811.3775].