Universal Polar Codes

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Abstract—Polar codes, invented by Arikan in 2009, are known to achieve the capacity of any binary-input memoryless output-symmetric channel. Further, both the encoding and the decoding can be accomplished in $O(N \log(N))$ real operations, where $N$ is the blocklength.

One of the few drawbacks of the original polar code construction is that it is not universal. This means that the code has to be tailored to the channel if we want to transmit close to capacity.

We present two “polar-like” schemes which are capable of achieving the compound capacity of the whole class of binary-input memoryless output-symmetric channels with low complexity.

Roughly speaking, for the first scheme we stack up $N$ polar blocks of length $N$ on top of each other but shift them with respect to each other so that they form a “staircase.” Then by coding across the columns of this staircase with a standard Reed-Solomon code, we can achieve the compound capacity using a standard successive decoder to process the rows (the polar codes) and in addition a standard Reed-Solomon erasure decoder to process the columns. Compared to standard polar codes this scheme has essentially the same complexity per bit but a block length which is larger by a factor $O\left(\frac{\log_2(N)}{\epsilon}\right)$. Here $N$ is the required blocklength for a standard polar code to achieve an acceptable block error probability for a single channel at a distance of at most $\epsilon$ from capacity.

For the second scheme we first show how to construct a true polar code which achieves the compound capacity for a finite number of channels. We achieve this by introducing special “polarization” steps which “align” the good indices for the various channels. We then show how to exploit the compactness of the space of binary-input memoryless output-symmetric channels to reduce the compound capacity problem for this class to a compound capacity problem for a finite set of channels. This scheme is similar in spirit to standard polar codes, but the price for universality is a considerably larger blocklength.

We close with what we consider to be some interesting open problems.

I. INTRODUCTION

Consider a communication scenario where the transmitter does not know the channel over which transmission takes place but only has knowledge of a set that the actual channel belongs to. Hence we require that the coding scheme must be reliable for every channel in this set. The preceding setup is known as the compound channel scenario and the maximum achievable rate is known as the compound capacity. Several variations on this theme are possible and useful. We consider the case where the transmitter only has knowledge of the set but the receiver knows the actual channel that was used. This is not unrealistic. If the channel is constant or changes very slowly then the receiver has ample of time and data to estimate the channel very accurately.

Let $\mathcal{W}$ denote the set of channels. The compound capacity of $\mathcal{W}$, denote it by $C(\mathcal{W})$, is defined as the maximum rate at which we can reliably transmit irrespective of which channel from $\mathcal{W}$ is chosen. It was shown in [1] that

$$C(\mathcal{W}) = \max_{Q} \inf_{a \in \mathcal{W}} I_Q(a),$$

where $I_Q(a)$ denotes the mutual information between the input and the output of $a$, with the input distribution being $Q$. Note that the compound capacity of $\mathcal{W}$ can be strictly smaller than the infimum of the individual capacities. This happens only if the capacity-achieving input distributions for the individual channels are different.

We restrict our attention to the class of binary-input memoryless output-symmetric (BMS) channels. As the capacity-achieving input distribution for all BMS channels is the uniform one (and hence in particular the same), it follows that for any collection $\mathcal{W}$ of BMS channels the compound capacity is equal to the infimum of the individual capacities. Why is this problem of practical relevance? When we design a communications system we typically start with a mathematical model. But in reality no channel is exactly equal to the assumed model. Depending on the conditions of the transmission medium, the channel will show some variations and deviations. Therefore, designing low-complexity universal coding schemes is a natural and important problem for real systems. Spatially coupled codes [2] were the first class of low complexity codes to be shown to be universal.

Consider standard polar codes with the standard successive decoder [3]. For this scheme the question of universality was addressed in [4]. By deriving a sequence of upper and lower bounds, it was shown that in general the compound capacity under successive decoding is strictly smaller than the unrestricted compound capacity described in [1]. In words, standard polar codes under successive decoding are not universal.

One might wonder if this lack of universality is due to the code structure or due to the (suboptimal) successive decoding procedure. To answer this question, let us consider polar codes under MAP decoding. Let $C \in [0, 1]$ and consider the polar code (with the standard kernel $G_2 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$) designed for the binary symmetric channel (BSC) with capacity $C$. It is shown in [5] that under MAP decoding such a code achieves the compound capacity if we take $\mathcal{W}$ to be the class of BMS channels of capacity $C$. Consequently, polar codes, decoded with the optimal MAP decoder, are universal. Hence, it is the suboptimal decoder that is to fault for the lack of universality.

It is therefore interesting to ask whether some suitable modification of the standard polar coding scheme allows us to construct “polar-like” codes which are universal under low-complexity decoding. As we will show, the answer is yes.
In fact, we present two solutions. The first solution combines polar codes with Reed-Solomon (RS) codes which are optimal for the (symbol) erasure channel. The second solution is a slight modification of the standard polar coding scheme and it is itself a polar code where channels are combined in a specific way in order to guarantee universality.

In independent work Şaşoğlu and Wang also consider the problem of constructing universal polar codes. Their solution, see [3], is based on introducing two types of polarization steps. The first one is the usual polarization step and it is used to achieve a low error probability. The second one, which is novel, guarantees that the resulting code is universal.

Before we present our schemes let us agree on notation and let us recall some facts.

Consider a standard polar block of length \( N = 2^n \) generated by the matrix \( G_2 \). Note that we use the word block to denote the structure implied by the \( n \)-fold Kronecker product of \( G_2 \), together with the implied decoding order of the successive decoder.

As a next component we need to specify the channel over which transmission takes place. All channels we consider are BMS channels. Assume that we are given the channel \( a \), where a might be a binary-erasure channel (BEC), a binary symmetric channel (BSC), a binary additive white-Gaussian noise channel (BAWNC) or any other element of the class of BMS channels. We denote its capacity by \( C(a) \). Once we are given the channel we can compute for the given length \( N \) the set of “good” polar indices. Call this set \( \mathcal{A} \). There are many possible ways of defining this set. To be concrete, we will use the following convention. Fix the rate \( R \) where \( 0 < R < C(a) \). Compute for the given channel the Battacharyya constants associated to all indices \( i, 1 \leq i \leq N \). Sort these numbers from smallest to largest. Include in \( \mathcal{A} \) the smallest \( RN \) such indices. Note that the sum of the Battacharyya constants of the included indices is an upper bound on the block error probability under successive decoding. We denote this error probability by \( P(a) \). Efficient algorithms to determine the set of good indices can be found in [9], [10].

In this respect the following fact, first stated in [11], is important: For any BMS channel \( a \), any \( 0 < R < C(a) \), and \( 0 < \beta < \frac{1}{2} \), we have \( P(a) \leq c(C(a), R, \beta) 2^{-N \alpha} \), where \( c(C(a), R, \beta) \) only depends on \( C(a) \), the chosen rate \( R \), and \( \beta \), but is universal with respect to \( a \). This means in particular that for any fixed \( k > 0 \), by choosing \( N \) sufficiently large, we can make \( N^k P(a) \) as small as desired. Note that this bound not only holds for the block error probability but also for the sum of the Battacharyya constants of the included channels.

In the previous paragraphs we used the word “index” to refer to one of the synthetic channels which are created by the polarization process. The reason for using “index” and not “channel” or “synthetic channel” is that we consider transmission over a set of channels \( \mathcal{W} \) and hence referring to both, the actual transmission channel and the synthetic channels created by the polarization process, as channels might lead to confusion. In the sequel we will always assume that these indices are labeled from 1 to \( N \) and that the processing order of the successive decoder is the one implied by this labeling (i.e., we first process index 1, then 2, and so on).

We will also need a universal upper bound on the block-length which is required if we want to transmit with a standard polar code close to capacity. Such a bound is stated in the next lemma.

**Lemma 1** (Universal Upper Bound on Block Length – [72]): For \( 0 < C < 1 \), \( \Delta > 0 \), and \( P > 0 \) define

\[
  n(C, \Delta, P) = \lceil 7 \log_2 \frac{1}{\Delta} + c(C, P)(\log_2(\log_2 \frac{1}{\Delta}))^2 \rceil.
\]

Then a polar code of length \( N \geq 2^{n(C, \Delta, P)} \) and rate \( R = C - \Delta \) designed for \( a \in \text{BMS}(C) \) has a block error probability under successive decoding of at most \( P/N^2 \). Here, \( c(C, P) \) only depends on \( C \) and \( P \) but is independent of \( R \), and \( a \).

**Discussion:** The scaling of \( P/N^2 \) is somewhat arbitrary. The same result, albeit with a different constant, is true for the more general case where we require \( P/N^k, k > 0 \).

As a final notational convention, we will write \( \text{BMS}(C) \) to denote the set of all BMS channels of capacity at least \( C \).

II. BASE SCHEME FOR TWO CHANNELS

Consider two channels, call them \( a \) and \( b \), both of capacity \( C \). This means that \( W = \{a, b\} \). Assuming that both channels have capacity \( C \) entails no essential loss of generality since for the class of BMS channels the compound capacity of a set of channels is equal to the minimum of the capacities, as was mentioned in the introduction.

Consider two polar blocks of length \( N \) and let \( \mathcal{A}_N \) and \( \mathcal{B}_N \) be the set of “good” indices for channel \( a \) and \( b \), respectively. What we mean with this is that with this chosen set we get “acceptable” block error probabilities, call them \( P(a) \) and \( P(b) \), respectively. As we have discussed in the introduction, one convenient way of defining this set is to fix a rate \( 0 < R < C \) and then to include the \( NR \) indices of the block of length \( N \) that have the smallest Battacharyya parameters.

Since by Lemma 1 polar codes achieve the capacity uniformly over the class of BMS channels, it entails further no essential loss of generality if we assume that \( |\mathcal{A}_N| = |\mathcal{B}_N| \).

The most obvious way of constructing a polar code for this compound case is to place the information in the set \( \mathcal{A}_N \cap \mathcal{B}_N \), i.e., to place information only in the indices which are good for both channels. The block error probability under the standard successive decoder is in this case bounded above by \( \max\{P(a), P(b)\} \), which is good news. Unfortunately, as was mentioned in the introduction, it was shown in [4] that such a scheme in general results in rates which are strictly below the compound capacity even if we let \( N \) tend to infinity. This means that for large \( N \), \( |\mathcal{A}_N \cap \mathcal{B}_N|/N \leq \alpha \min\{|\mathcal{A}_N|, |\mathcal{B}_N|\}/N \), where \( \alpha < 1 \). One notable exception is the case where the channels are ordered by degradation, but this covers only a small range of cases of interest, see [13].

III. CAPACITY GAP

Consider again the case \( W = \{a, b\} \) and let \( 0 < P < 1 \). Assume that we include in the set \( \mathcal{A}_N \) the maximum number
of indices so that the sum of their Battacharyya constant (with respect to channel \(a\)) does not exceed \(P\) and that we define \(B_N\) in the equivalent manner. Then we know from \([3]\) that the limits \(\lim_{N \to \infty} \frac{|A_N|}{N}\) and \(\lim_{N \to \infty} \frac{|B_N|}{N}\) exist and are equal to \(C(a)\) and \(C(b)\) respectively. Further, it was shown in \([1]\) that

\[
\lim_{N \to \infty} \frac{|A_N \cap B_N|}{N}
\]

exists. Call this limit \(C(a \cap b)\). More generally, we can define the limit \(C(\cap_{a \in \mathcal{W}})\). This is the rate which we can achieve if we only transmit on those indices which are good for all channels in \(\mathcal{W}\). We can now define the gap, call it \(\Delta(\cap_{a \in \mathcal{W}} a)\), as \(\Delta(\cap_{a \in \mathcal{W}} a) = \min_{a \in \mathcal{W}} C(a) - C(\cap_{a \in \mathcal{W}} a)\). For convenience of notation let us define \(C(W) = \min_{a \in \mathcal{W}} C(a)\) as a shorthand for the compound capacity.

\[\text{IV. Scheme I}\]

Let us now describe our first scheme. Represent a polar block of length \(N\) by a row vector as in Figure 1. Take \(N\) such blocks and construct a staircase by stacking these blocks on top of each other as shown in Figure 2. Note that the \(j\)-th such block (counted from the bottom), \(1 \leq j \leq N\), is shifted \((j - 1)\) positions to the right.

\[\text{Fig. 1. A polar block of length } N \text{ as a row vector.}\]

\[\text{Fig. 2. A staircase consisting of } N \text{ basic polar blocks stacked on top of each other, where the } j \text{-th such block (counted from the bottom), } 1 \leq j \leq N, \text{ is shifted } (j - 1) \text{ positions to the right. The columns are labeled from } 1 \text{ to } 2^N - 1 \text{ and the rows are labeled from } 1 \text{ to } N.\]

Next, extend the staircase by placing \(k\) copies of this staircase horizontally next to each other in a consecutive manner, where \(k \in \mathbb{N}\) is a parameter of the construction. Call the result an extended staircase. This is shown in Figure 3 for \(N = 16\) and \(k = 3\).

\[\text{Fig. 3. An extended staircase for } N = 16 \text{ and } k = 3.\]

Finally, take \(\log_2(N) = n\) such extended staircases. Graphically we think of them as being placed in a vertical direction on top of each other. Figure 4 shows the result for \(N = 16\) and \(k = 3\).

\[\text{Fig. 4. The scheme for } N = 16 \text{ and } k = 3, \text{ consisting of } \log_2(16) = 4 \text{ extended staircases.}\]

It remains to explain where to place information and how to recover it. Note that each extended staircase has width \((k + 1)N - 1\), and we assume that the (column) indices run from 1 to \((k + 1)N - 1\). Note further that, except for the boundaries, the columns of each extended staircase have height \(N\). More precisely, all columns in the range \(N \leq i \leq kN\) have height \(N\). We say that such a column has full height. As a final observation, note that in a column of full height, due to the shifts, we “see” exactly the same indices (channels) as in a standard polar block of length \(N\). In other words, we can think of one column of full height as (a cyclic shift of) a standard polar block. This is one key reason why our construction works.

Now recall that according to Lemma 1, regardless of what channel from BMS(\(C\)) is chosen, for sufficiently large \(N\), the number of good indices in one polar block is very close to \(NC\) and the notion of “very close” is uniform with respect to the channel. In words, regardless of what channel is chosen, out of the \(N\) indices in a column about \(NC\) can be (correctly) decoded and they are decoded with high probability. Further, since we know at the receiver what channel has been used, we know which of the indices can be decoded. Therefore, we can treat the undecoded indices as erasures.

The idea is therefore simple. Use in each full-height column an erasure code so that we can reconstruct the whole column if we know roughly \(NC\) components of it. Since we want to do this without loss, we wish to use a maximum distance separable (MDS) code. Since binary MDS codes only exist for very few parameters we take \(\log_2(N) = n\) such staircases. Exploiting this fact we can code over \(\mathbb{GF}(N)\), and over this field there do exist MDS codes of any dimension up to length...
At the boundaries we proceed in a simpler fashion since there we only store information in indices which are good for all channels and all other indices are frozen.

In the above paragraph we have crucially used the following property of polar codes. For BMS channels we can choose the value of frozen bits in any manner we wish as long as these values are known at the decoder.

Let us now clarify why in general we do not use only a single extended staircase but \( n \) of them. This slight modification allows us to deal with the fact that we cannot code over the binary field but need to code at least over a field of size \( N \) in order to construct an MDS code of length \( N \) of dimension roughly \( NC \).

How can we use the \( \log_2(N) \) copies? The crucial observation is that the \( \log_2(N) \) copies behave essentially identical. Therefore, fix a particular full-height column. Assume that for a particular channel we know that lets say index \( i \), \( 1 \leq i \leq N \), is good. Then this index is good for all the \( \log_2(N) \) extended staircases and with high probability we recover all \( \log_2(N) \) of them. So if we think of these \( \log_2(N) \) bits as one symbol of \( GF(N) \) then we can assume that this symbol is known. Conversely, assume that this index is not good for the chosen channel. Then it is not good for any of the \( \log_2(N) \) extended staircases and this fact is known at the receiver. So, if again we combine these \( \log_2(N) \) bits into an element of of \( GF(N) \) then we can think of this element as an erasure and the overall erasure probability is very close to \( 1 - C \), as it should be.

Let us summarize. For columns \( 1 \leq i \leq N - 1 \) and \( kN \leq i \leq (k + 1)N - 1 \) we load information only into those polar indices which belong to \( C(\cap_{a \in BMS(C)}a) \). For all other columns, i.e., the columns \( N \leq i \leq kN \) we load into the \( \log_2(N) \) columns of the \( \log_2(N) \) extended staircases at position \( i \) one RS codeword of length \( N \) over the field \( GF(N) \), where the RS code has has rate just a little bit less than \( C \). We then multiply each row by the polar matrix. This specifies the encoding operation.

For the decoding, we run \( N \log_2(N) \) successive decoders in parallel, each working on one of the \( N \log_2(N) \) rows of the scheme. These decoders are synchronized in the sense that they are processing the bits in the same column of the scheme at the same time. Regardless of what channel the transmission takes place, according to Lemma 1 we can decode about a fraction \( NC \) of the \( N \) positions in each extended staircase. Therefore, the RS code which has a rate just a little bit below \( NC \) will be able to recover all symbols.

The subsequent lemma summarizes our observations and gives the precise parameters and the resulting bounds on the error probability as well as the complexity.

**Lemma 2 (Universal Polar Codes):** Let \( BMS(C) \) denote the set of BMS channels of capacity at least \( C \). Let \( \epsilon > 0 \) be the allowed gap to the compound capacity and let \( P > 0 \) be the allowed block error probability. Consider the above construction with the following parameters.

- Pick \( k = 2^\frac{P}{2(1 - C/2)} \).
- Let \( N = 2^{n(P, \epsilon/2)} \), where \( n(P, \epsilon) \) is given in Lemma 1

**Encoding:** Assume that for the columns \( 1 \leq i \leq N \) and \( kN < i \leq (k + 1)N - 1 \) we load only the indices which are good.
for all the channels in BMS(C). For full-height columns, i.e., columns with $N \leq i \leq kN$, we load the columns with RS codewords of length $N$ over the field $GF(N)$ and of dimension $(C - \frac{1}{2})N$. We then multiply each polar block by the polar matrix to accomplish the encoding.

**Decoding:** At the decoder we proceed as follows. For columns $1 \leq i \leq N - 1$ and $kN \leq i \leq (k + 1)N - 1$ we use the standard successive polar decoder to recover those indices which are good for all channels. For the columns $N \leq i < kN$ we first use successive polar decoding for all those rows which the index at the intersection of this row and the current column (the $i$-th column) is a good index for the channel at hand. This knowledge is present at the decoder since we assume that the receiver knows the channel over which transmission takes place and it can hence compute these indices. We then perform a RS erasure decoder along the column to fill in all missing information.

This results in a scheme with the following parameters which hold uniformly over the whole class BMS(C).

- $R \geq C(1 - \epsilon)$
- The blocklength is $N^2 \log_2(N)^2$.
- The block error probability is upper bounded by $P_{\log_2(N)}(N) \leq P$ uniformly over the set BMS(C).
- The encoding complexity per bit is $O((\log_2(N))^{\log_2(3)})$ binary operations.
- The decoding complexity per bit is $O((\log_2(N))$ real operations (for the polar decoder) and $O((\log_2(N))^{1+\log_2(3)})$ binary operations for the decoding of the RS code.

**Proof:** Let us go over each of these claims one by one.

- $[R \geq C(1 - \epsilon),]$ This follows by construction. We loose at most a factor $(1 - \epsilon/2)$ compared to the compound capacity due to boundary effects of the staircase and a further such factor due to the fact that we chose a finite value for $N$ and so we are bounded away from capacity.
- The blocklength is $N^2 \log_2(N)^2$. By construction each extended staircase contains $Nk$ blocks and we have $\log_2(N)$ of those. The claim now follows by our choice $k = 2^2$.
- The block error probability is upper bounded by $P_{\log_2(N)}(N) \leq P$ uniformly over the set. Note that we have in total $N \log_2(N)^2$ polar blocks. By construction, the block error probability for each of them under successive decoding is at most $\frac{P}{2^{N^2}}$ and this bound is uniform over all BMS(C) channels. The claim therefore follows by an application of the union bound.
- The encoding complexity per bit is $O((\log_2(N))^{\log_2(3)})$ binary operations.] The encoding complexity consists of determining the RS codewords. This can be done by computing a Fourier transform of length $N$ which can be accomplished by $O(N \log_2(N))$ operations over the field $GF(N)$. Addition over this field $GF(N)$ can be implemented with $\log_2(N)$ binary operations and multiplication can be implemented in $O((\log_2(N))^{\log_2(3)})$ binary operations. Since one such codeword contains $N \log_2(N)$ bits, the claim follows. A good reference for the spectral view of RS codes is [14].
- The decoding complexity per bit is $O(\log_2(N))$ real operations and $O((\log_2(N))^{1+\log_2(3)})$ binary operations.] There are two components contributing to the decoding complexity. First, we have to decode all $N \log_2(N)^2$ polar blocks. Measured per bit this causes a complexity of $O((\log_2(N))$ real operations. The second operation is the erasure decoding of the RS codes of length $N$ over $GF(N)$. This can be done in $O(N(\log_2(N))^2)$ symbol operations (additions and multiplications) assuming that we preprocess and store $O(N \log_2(N))$ symbols, i.e., $O(N(\log_2(N))^2)$ bits, see [15]. Therefore the bit complexity of the processing is $O(N(\log_2(N))^{2+\log_2(3)})$. The claim follows since this takes care of of $O(N \log_2(N))$ bits.

**Discussion:** At the decoder we need to know what indices belong to the good set for the given BMS channel at hand. As was pointed out in the introduction, this can be computed efficiently as shown in [19], [11]. Note that we only have to do this once every time the channel changes.

For the boundaries by design we only use indices which are good for all channels in BMS(C). There is currently no efficient algorithm to compute this set. But we can efficiently compute subsets, e.g., the set of indices which is good for the channel which is the least degraded with respect to the whole family BMS(C), [16].

It is easy to improve the error probability substantially by using the RS code not only for erasure decoding but also for error correction. We leave the details to the reader.

As a final remark. In the above complexity computation we list the complexity of the polar decoding as the number of real operations. In [17] it is shown how to accomplish the decoding in binary operations if we allow a small gap in capacity.

A. Variations on the Theme – Two Channels

Many variations on the basic construction are possible. Let us briefly discuss one of them. Assume that we only have two channels, i.e., $\mathcal{W} = \{a, b\}$ and that rather than achieving the compound capacity $C(\mathcal{W})$ we just want to improve the achievable rate.

In this case we can construct a much shorter code. Let us quickly explain. Rather than stacking up $N$ polar blocks on top of each other, we only stack up $2^l$ for some $1 \leq l \leq n$. Further, we only use a single extended staircase (rather than $n$). Let us describe this scheme in some more detail.

Consider a basic polar block. Each position $i$ in the basic block of length $N$, $1 \leq i \leq N$, can have one of four possible types, namely it can be in $A$ or not and it can be in $B$ or not. Let us indicate this by shades of gray as shown in Figure 5.

Our construction is best seen visually. Take $2^l$ polar codes of length $N$. Visualize each such code as a row vector of height $N$ as in Figure 5. Place the $2^l$ row vectors on top of each other but shift each copy one position further to the right so that they visually form a “staircase.” To be concrete, assume that the top-most copy is the one that is shifted the furthest to the right. Further, take $k$ such basic units (staircases) and

\footnote{In the sequel we write $A$ instead of $A_N$ in order to simplify our notation.}
place them next to each other in a contiguous manner. The total blocklength of this construction is hence $k2^l N$.

Consider this construction for $N = 16$, $l = 2$, and $k = 3$. This is shown in Figure 6.

![Figure 6](image)

To complete the construction we will now match pairs of indices which appear in the same column. For the following explanation we will refer to Figure 6 for a concrete case.

Number the columns from left to right from 1 to $kN + 2^l − 1$. Note that for each column in the range $2^l ≤ i ≤ kN$ we see $2^l$ distinct polar indices and that towards the boundary we still see distinct polar indices but fewer.

For each column $i$ find a suitable matching. For our example this means the following. White boxes correspond to polar indices which are bad for both channels. Freeze such boxes. Black boxes correspond to polar indices which are good for both channels. To each such box we associate an independent bit to be transmitted. Finally, try to match each light gray box (which corresponds to a polar index which is good for the first but not good for the second channel) with a dark gray box (which has the converse property). Any index that cannot be matched is set to frozen.

For position 1 we see a single light gray box. Since we cannot match it, we freeze it. For position 2 we see a white box and a light gray box. Again, freeze both. For position 3 we can associate one bit to the black box and have to freeze the remaining two. The first interesting position is position 7 where we can in fact match a light and dark gray box.

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The rate of this construction is

$$\frac{|A \cap B| + \frac{k-1}{N} |A \setminus B|}{N}.$$

The scheme is visualized in Figure 8 for the case $k = 3$.

Why is this a reasonable guess? Assume that the various types were distributed over the columns uniformly at random. In this case, if we look at a particular column then the distribution of the types $(1,0)$ and $(0,1)$ are both Bernoulli with equal mean. It follows that the expected number of such types which cannot be matched decreases like one over the square root of the number of copies. Our guess therefore stems from assuming that for large blocklengths the distribution effectively looks like this random case.
Discussion: Recall that if we were to use a standard polar code for the compound scenario involving the channels a and b we could only transmit within the set \([A \cap B]\). This results in a rate-loss of \(\frac{|A \cap B|}{N}\) compared to what we can achieve when transmitting over a single channel. For the chaining construction the achievable rate on the other hand can be made as close to \(|A|/N\) as we want by choosing \(k\) sufficiently large.

Example 4: Let us go through the case with \(k = 3\) shown in Figure 8 in more detail.

In block one (the left-most block in the figure) we put information in the positions indexed by \(A \cap B\) and \(A \setminus B\). The positions indexed by \(B \setminus A\) as well as \((A \cup B)^c\) are frozen and can be set to 0.

In block two (the middle block in the figure) we put information in the positions indexed by \(A \cap B\) and \(A \setminus B\). In the positions indexed by \(B \setminus A\) we repeat the information which is in the positions indexed by \(A \setminus B\) in block one. The positions indexed by \((A \cup B)^c\) are again frozen and can be set to 0.

Finally, in block three (the right-most block in the figure) we put information in the positions indexed by \(A \cap B\). In the positions indexed by \(B \setminus A\) we repeat the information which is in the positions indexed by \(A \setminus B\) in block two. The positions indexed by \((A \cup B)^c\) are again frozen and can be set to 0.

Let us now discuss how to decode this code. The decoder sees the received word and is aware of the channel which was used. Since the construction is symmetric we can assume without loss of generality that it is the channel \(a\). In this case the decoder can decode block one (the left-most block in the figure) reliably. This is true since we only placed information in the sets \(A \cap B\) and \(A \setminus B\), both of which are good for channel \(a\). All the other positions were frozen. Once block one has been decoded, we copy the information which was contained in the set \(A \setminus B\) to the position indexed by \(B \setminus A\) in block two. Now we can reliably decode block two. Note that we have crucially used the fact that frozen positions can contain any value as long as the value is known to the receiver.

We continue in this fashion. E.g., in the next step, copy the information which was contained in block two in the positions indexed by \(A \setminus B\) to block three to the positions indexed by \(B \setminus A\). Now we can reliably decode block three. We go on with this scheme until we have reached block \(k\). If, on the other hand, the information was transmitted on channel \(b\) we proceed in an equivalent fashion but start the decoding from the right-most block.

What is the overall probability of error? If we have \(k\) blocks then by a simple union bound the error probability is at most \(k \max\{P(a), P(b)\}\). Recall that if \(k\) is large, then the common rate in 2 tends to \(|A|/N\), which we know can be made as close to capacity as we desire by picking a sufficiently large blocklength \(N\) and a properly chosen index set \(A\). Let us summarize this discussion by formulating these observations as lemmas.

Lemma 5 (Chaining Construction is Good): Consider two BMS channels of capacity \(C\), \(0 < C < 1\). Call the two channels \(a\) and \(b\). Assume that under the standard successive decoding the good indices under channel \(a\) are \(A\) and that the good indices under channel \(b\) are \(B\) and let \(P(a)\) and \(P(b)\) denote the respective single-channel block error probabilities.

Then for each \(k \geq 2\), the \(k\)-chain described in Definition 3 has an error probability of at most \(k \max\{P(a), P(b)\}\) for transmission over channel \(a\) as well for transmission over channel \(b\) and a rate given in 2.

If for a fixed \(k\) we let \(N\) tend to infinity then we will achieve the rate \(C(\{a, b\}) = \frac{1}{2} \Delta(a \cap b)\) and an arbitrarily low error probability. If in addition we let \(k\) tend to infinity then we achieve the compound capacity \(C(\{a, b\})\).

B. A Polar View of Chains for Two Channels

Let us now give a slightly different interpretation of the previous construction. Rather than thinking of the previous construction as a chain which is decoded either from the left to the right or vice versa depending on which channel we use, let us observe that we can construct from it a “real” polar block where we have a fixed decoding order and use a standard successive decoder according to this order.

Consider once again the situation depicted in Figure 1 i.e., we have two channels called \(a\) and \(b\) and their respective good sets are \(A\) and \(B\). Instead of constructing from this a 2-chain by matching up the indices of \(A \setminus B\) in one block with the indices of \(B \setminus A\) in the other block and by repeating the same information in these two sets of indices, let us combine the two blocks via a special polar step where we match channels in a particular way.

The scheme is shown in Figure 9. Whereas in the 2-chain the exact matching of the bits was immaterial, we will now make a very specific choice. Recall that by our convention the labels of the polar indices are ordered according to the processing order of the successive decoder. This means, if the indices \(i\) go from 1 to \(N\), then we first process index 1, then 2, and so on. Let \(|A \setminus B| = |B \setminus A| = S\) and let \(\{a_1, \ldots, a_S\}\) denote the subset of \([1, N]\) which corresponds to the set \(A \setminus B\) and let \(\{b_1, \ldots, b_S\}\) denote the equivalent for the set \(B \setminus A\). Further, we assume that these indices are ordered in a strictly increasing order.
The idea is to polarize the index \( a_1 \) of the first block with the index \( b_1 \) of the second block. None of the other indices are polarized but they are simply "passed through" unchanged. Associated to this scheme we use the following processing order. We start with index 1 from block one. We go down the list of indices of block one in the natural order until and including position \( a_1 - 1 \). We then process all the bits in block two just until and including position \( b_1 - 1 \). We then process position \( a_1 \) of block one, immediately followed by position \( b_1 \) from block two. At this point we have performed our first "matching" by polarizing the first element of \( A \setminus B \) of block one with the first element of \( B \setminus A \) of block two. We now continue in exactly the same fashion, processing elements of block one until and including index \( a_2 - 1 \). The we process the positions of block two until and including index \( b_2 - 1 \). We then process index \( a_2 \) of block one followed immediately by processing index \( b_2 \) in block two and so on.

Note that in this way we have created a new polar block of length \( 2N \) and which has a specified processing order. Let us introduce the following notation. We say that an index is of type \((1,0)\) if the index is contained in the set \( A \) but not in the set \( B \). This means, this is an index which is good for the \( a \) channel but bad for the \( b \) channel. The equivalent definitions apply for the three remaining types, namely \((0,0)\), \((0,1)\) and \((1,1)\).

The key observation is that under successive decoding the various indices have the following type: An index which used to have type \((0,0)\) or \((1,1)\) in either of the original blocks is simply passed through by this construction and still has this type. Let us clarify what we mean that it has this type. What we mean is that if we define it to have this type then, under successive decoding, all indices which are declared to be of type \((1,1)\) will have a small error probability. So in this sense these indices are "good."

An index which used to have type \((0,1)\) in the original block one or type \((1,0)\) in the original block two is also simply passed through and maintains its type.

But an index which used to be of type \((1,0)\) in block one is now polarized with an index of type \((0,1)\) in block two and this creates two new indices. The first one is of type \((0,0)\) and the second is of type \((1,1)\). This means that we have converted two indices, one which was good for \( b \) but bad for \( a \) and one which was good for \( b \) but bad for \( a \) into two new indices where one is bad for both and one is good for both.

The preceding paragraph encapsulates the main idea of the construction. It is worth going over the two possible cases in more detail since this is the main building block of our scheme. So consider a structure like a standard polar transform but assume that you have two different boxes as shown in Figure 10. The top box represents a perfect channel in case the actual channel is \( a \) and a completely useless channel in case it is \( b \). The box on the bottom branch has the same property except that the roles of \( a \) and \( b \) are exchanged. Now regardless which channel is used, the top polar index (assuming uniform random information along the bottom index) is a useless index, but once we have processed and hence decided upon the top index the bottom index is good for either of the two channels \( a \) and \( b \) since either the information will flow along the top branch or along the bottom branch.

![Fig. 10. How to combine two polar blocks to improve the compound capacity for two channels.](image)

If we use the so constructed polar block in a compound setting we see that we have halved the gap \( \Delta(a \cap b) \). Hence, the gap of the compound capacity of this polar block compared to true compound capacity \( C(\{a,b\}) \) has been halved (assuming that we have sufficiently long blocks so that for individual channels we essentially achieve capacity).

Even better. The block so constructed has exactly the same structural properties as a standard polar block. It follows that we can recurse this construction. If we perform \( \kappa \) recursion steps then we have created a block of length \( 2^\kappa N \) which has a block error probability of at most \( 2^\kappa \max\{P(a),P(b)\} \) and whose gap to the real compound capacity \( C(\{a,b\}) \) has decreased by a factor \( 2^{-\kappa} \). In other words, this construction is exactly as efficient as the chain construction which we presented in the previous section. But as we will discuss in the next section, from this point of view the generalization to more than two channels is immediate.

C. Polar Codes Which are Good for a Finite Set of Channels

Very little is needed to lift the previous construction for two channels to a finite number of channels. Here is the general recipe. We start with two channels and a basic polar block of length \( N \), we then recurse \( \kappa \) times until we have "aligned" essentially all good indices for these two channels. Those that are still not aligned are thrown away. We hence have a polar block of length \( 2^\kappa N \) which is simultaneously good for two channels. We now recurse on this block but with the aim of also aligning the good indices for a third channel. We proceed in this fashion until we have aligned the indices for all channels.

Let us go over these steps a little bit more in detail. Assume that we have a finite set of channels, call them \( \mathcal{W} = \{a_1, \ldots, a_t\} \), all of capacity \( C_i, 0 < C < 1 \).

The construction is as follows. We proceed recursively. We are given a target gap of \( \epsilon > 0 \) to the compound capacity \( C(\mathcal{W}) \) (here we have \( C(\mathcal{W}) = C \)). Further, we are given a target block error probability \( P > 0 \).

We start with a standard polar block of length \( N = 2^n \), where \( N \) will be chosen later on sufficiently large to fulfill
the various requirements. Let \( a = a_1 \) and \( b = a_2 \). Further, let \( A = A_1 \) and \( B = A_2 \). Construct a polar block which is good for channels \( a \) and \( b \) as described in the previous section in the sense that the fraction of indices which are good for channel \( a \) but not good for channel \( b \) is at most \( \epsilon/t \). Let us assume that this takes \( k_2 \) recursions of the basic scheme.

Now let \( A \) be the set of indices which are jointly good for channel \( a \) and \( b \) (this means in particular that we “throw away” any indices which at this point are good for only one channel; but we are assured that there are only few of them so the rate-loss is minor). Further, set \( b = a_3 \) and \( B = A_3 \). Since our construction resulted in a polar block we can recurse the construction, taking now as our building block the block we previously constructed and which has length \( 2^{\epsilon_2}N \). In the second stage we recurse as many times as are needed so that we incur an additional gap of at most \( \epsilon/t \). Assume that this takes \( k_3 \) steps.

We proceed in this manner, adding always one channel at a time. In this case we will have created a block which has the property that the fraction of indices which are good for all the \( t \) channels simultaneously is at least \( C - \epsilon \). How many iterations do we need? Consider one step of this process. A priori the gap to the compound capacity is at most \( \Delta = t \). Extra polarization step we decrease this gap by a factor of \( 1 - \epsilon/C \). Therefore, we need at most \( \log_2 \frac{\Delta}{\epsilon} \) steps so that the gap is no larger than \( \epsilon/t \). If we have \( t \) channels, then we have \( t - 1 \) such steps. We conclude that the sum of the required steps is at most

\[
(t - 1) \log_2 \frac{\Delta}{\epsilon} \leq (t - 1) \log_2 \frac{1}{\epsilon}
\]

so that the total blocklength is at most \( 2^{t \log_2 \frac{1}{\epsilon}} t \). Note that since we have a fixed upper bound on the total number of recursions (and this upper bound does not depend on the block length \( N \)), it is possible to fix \( N \) sufficiently large so that the final block error probability is sufficiently small.

The above bounds are quite pessimistic and an actual construction is likely to have significantly better parameters. In particular, once we have constructed a polar code which is good for several channels, it is likely that for any further channel we add we only need a few recursions. One would therefore assume that the pre-factor which is currently \( \log_2 \frac{\Delta}{\epsilon} \) is in reality much smaller.

**D. Compactness of the Space of BMS Channels**

It remains to transition from a finite set of BMS channels to the whole set of BMS channels, let’s say of capacity \( C \), call this set \( \text{BMS}(C) \). The crucial observation in this respect is that (i) \( \text{BMS}(C) \) is compact, and that (ii) we can modify the finite set of representatives implied by the compactness so that all channels in \( \text{BMS}(C) \) are upgraded with respect to these representatives.

**Lemma 6 (Construction of a Dominating Set):** Let \( \text{BMS}(C) \) denote the set of BMS channels of capacity at least \( C \). Let \( \epsilon > 0 \). Then we can explicitly construct a set of channels of cardinality \( K(\epsilon) \), denote this set by \( \{ c_i \}_{i=1}^{K(\epsilon)} \), with the following two properties:

(i) \( C(c_i) \geq C - \epsilon \) for all \( 1 \leq i \leq K(\epsilon) \).
(ii) For any \( c \in \text{BMS}(C) \), there exists at least one \( i, 1 \leq i \leq K(\epsilon) \), so that \( c_i \prec c \).

In words, every channel in \( \text{BMS}(C) \) is upgraded with respect to at least one channel in \( \{ c_i \}_{i=1}^{K(\epsilon)} \), and every \( c_i \) has capacity at least equal to \( C - \epsilon \). We have the bound \( K(\epsilon) \leq \frac{2A(\epsilon)}{\sqrt{A(\epsilon)}} \) where

\[
A(\epsilon) = \frac{9}{8h_2^{-1}(\epsilon)^2}.
\]

**Proof:** The proof uses the machinery developed in [2]. In brief, every BMS channel can be represented by a probability density on the unit interval \([0,1]\) (this is sometimes called the \([D]\)-representation, see [18]). Consider a BMS channel \( c \). By some abuse of notation we let \( c(x) \) denote the density on \([0,1]\) which represents \( c \). The capacity of \( c \) can be computed from \( c(x) \) via the integral

\[
C(c) = 1 - \int_0^1 h_2 \left( \frac{1-x}{2} \right) c(x) dx,
\]

where \( h_2(x) \) denotes the binary entropy function, \( h_2(x) = -x \log_2(x) - (1-x) \log_2(1-x) \). Further, as in [18], we equip the space \([0,1]\) with the Wasserstein metric which we denote by \( d(\cdot,\cdot) \).

Let us first show how to find a set of BMS channels, denote them by \( \{ e_i \}_{i=1}^{K(\epsilon)} \), such that for any \( c \in \text{BMS}(C) \), there exists at least one \( i, 1 \leq i \leq K(\epsilon) \), so that

\[
d(c, e_i) < \frac{4}{9} h_2^{-1}(\epsilon)^2.
\]

Define the discrete alphabet \( \mathcal{X} = \{ x_0, x_1, \cdots, x_T \} \), where \( x_i = \frac{i}{T}, \quad 0 \leq i \leq T \), and where \( T = \left\lceil \frac{9}{2h_2^{-1}(\epsilon)^2} \right\rceil \).

In other words, \( \mathcal{X} \) consists of \( T + 1 \) points equally spaced in the unit interval \([0,1]\).

Consider the space of all densities that have the following form

\[
T \sum_{i=0}^{T} p_i \delta(x - x_i),
\]

where \( \sum_i p_i = 1 \) and \( p_i \in \mathcal{X} \). Let us denote this space by \( \text{BMS}_T \). A computation shows that this set has cardinality \( \left( \frac{T}{T} \right)^{T} \). From the properties of the Wasserstein distance we know that for any BMS channel \( c \), there exists an \( e \in \text{BMS}_T \) such that

\[
d(c, e) \leq \frac{2}{T} \leq \frac{4}{9} h_2^{-1}(\epsilon)^2.
\]

We construct now from each \( e_i \), \( 1 \leq i \leq K(\epsilon) \), another density \( c_i \) such that

(i) \( C(c_i) \geq C(\epsilon) - \epsilon \).
(ii) For any \( c \in \text{BMS}(C) \) there exists and \( i \) so that \( c_i \prec c \).

It is shown in Corollary 43 in [2] how to modify the set \( \{ e_i \}_{i=1}^{K(\epsilon)} \) into the set \( \{ c_i \}_{i=1}^{K(\epsilon)} \) to ensure the degradedness condition and so that the Wasserstein distance, assuming it was \( \delta \) beforehand, is at most \( 3\sqrt{\delta} \) afterwards. But a Wasserstein distance of \( 3\sqrt{\delta} \) implies a loss in capacity of at most \( h_2 \left( \frac{3}{2}\sqrt{\delta} \right) \).
VI. DISCUSSION AND OPEN QUESTIONS

Let us quickly discuss some interesting open questions. Although we have proposed two solutions which solve the compound capacity problem at low computational cost, both solutions require a significant increase in blocklength compared to standard polar codes. It is therefore interesting to see if we can find variations of the proposed solutions, or perhaps different solutions that do not suffer from the same problem.

Let us first consider scheme I. Recall that we payed in blocklength a factor $N \log_2(N)$ compared to standard polar codes. The factor $N$ was due to the fact that we stacked up $N$ polar blocks on top of each other. This simplified the analysis considerably since in this case we know that the number of good indices within one column is essentially independent of the channel and very close to $NC$. It is tempting to conjecture that a much smaller number would suffice. The crucial property which one needs is that the number of good indices within this set does not vary significantly as a function of the channel and very close to $NC$. It is tempting to conjecture that a much smaller number would suffice. The crucial property which one needs is that the number of good indices within this set does not vary significantly as a function of the channel and very close to $NC$. It is tempting to conjecture that a much smaller number would suffice.

Let us now look at scheme II. In this case one significant source of inefficiency of the scheme stems from the fact that we first performed polarization steps to achieve a good polarization for single channels and then in a second step performed polarization steps to achieve universality. It is tempting to conjecture in this case that the two operations can be performed together and that in fact for the second stage we do not need to explicitly polarize for each single channel but that we can achieve universality in a more direct and natural way by modifying the basic construction. I.e., one could imagine that for the basic construction we decide at each step not to polarize all indices but only a fraction and that the mix the indices which are polarized. The price we might have to pay is a somewhat slower polarization but we might be able to achieve universality directly. The investigation of such ideas is slated for future research.

Finally, let us point out that such universal polar schemes might also be useful in other information theoretic scenarios (see for example \[19\textit{–}22\]).

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