SCHELLEKENS’ LIST AND THE VERY STRANGE FORMULA

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Abstract. In \cite{Sch93} (see also \cite{EMS20a}) Schellekens proved that the weight-one space $V_1$ of a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 must be one of 71 Lie algebras. During the following three decades, in a combined effort by many authors, it was proved that each of these Lie algebras is realised by such a vertex operator algebra and that, except for $V_1 = \{0\}$, this vertex operator algebra is uniquely determined by $V_1$. Uniform proofs of these statements are given in \cite{MS19, HM20}.

In this paper we give a fundamentally different, simpler proof of Schellekens’ list of 71 Lie algebras. Using the dimension formula in \cite{MS19} and Kac’s “very strange formula” \cite{Kac90} we show that every strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_1 \neq \{0\}$ can be obtained by an orbifold construction from the Leech lattice vertex operator algebra $V_\Lambda$. This suffices to restrict the possible Lie algebras that can occur as weight-one space of $V$ to the 71 of Schellekens.

Moreover, the fact that each strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 comes from the Leech lattice $\Lambda$ can be used to classify these vertex operator algebras by studying properties of the Leech lattice. We demonstrate this for 43 of the 70 non-zero Lie algebras on Schellekens’ list, omitting those cases that are too computationally expensive.

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1. INTRODUCTION

The weight-one subspace $V_1$ of a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 is a finite-dimensional, reductive Lie algebra \cite{DM04a}. More precisely, $V_1$ is either zero, 24-dimensional abelian or semisimple $g_1 \oplus \ldots \oplus g_r$ of rank at most 24 \cite{DM04a}. In 1993 Schellekens \cite{Sch93} (see also

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operator algebra is uniquely determined by its 

Theorem. Each potential Lie algebra on Schellekens’ list is realised by a strongly rational, holomorphic vertex operator algebra of central charge 24 and this vertex operator algebra is uniquely determined by its $V_1$-structure if $V_1 \neq \{0\}$.

In addition, uniform existence and uniqueness proofs are given in [Hoh17, MS19] (and [HM20]).

Schellekens’ work is based on decomposing the vertex operator algebra $V$ into irreducible modules for the affine vertex operator algebra $(V_1) \cong L_{\mathfrak{g}_i}(k_i,0) \otimes \cdots \otimes L_{\mathfrak{g}_r}(k_r,0)$ generated by $V_1$ (with levels $k_i \in \mathbb{Z}_{>0}$, see [DM06]), assuming $V_1$ to be semisimple, and deriving a set of trace identities (cf. Theorem 6.1 in [EMS20b]) by viewing the character of $V$ as a Jacobi form. The lowest trace identity is

$$\frac{h_\mathfrak{g}^\vee}{k_i} = \frac{\dim(V_1) - 24}{24}$$

for all $i = 1, \ldots, r$ where $h_\mathfrak{g}^\vee$ is the dual Coxeter number of $\mathfrak{g}_i$ (see also [DM04a]). This equation has exactly 221 solutions, listed in Table 3. The higher trace identities translate to a system of linear equations on the multiplicities appearing in the decomposition of $V$ into $\langle V_1 \rangle$-modules. By excluding those Lie algebras for $V_1$ whose corresponding system has no solutions in the non-negative integers one essentially arrives at Schellekens’ list of 71 possible Lie algebras for $V_1$. This approach relies heavily on solving linear programming problems on the computer.

The main goal of this paper is to provide a novel, simpler proof of Schellekens’ list by showing that each strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_1 \neq \{0\}$ can be obtained as orbifold construction $V \cong V_\Lambda^{\text{orb}(g)}$ associated with the Leech lattice vertex operator algebra $V_\Lambda$ (itself one of these vertex operator algebras) and some finite-order automorphism $g \in \text{Aut}(V_\Lambda)$.

The orbifold construction [EMS20b, MG16] is an important method to construct vertex operator algebras. Let $V$ be a strongly rational, holomorphic vertex operator algebra and $g$ an automorphism of $V$ of finite order $n$ and type 0. Then the fixed-point vertex operator subalgebra $V^g$ is strongly rational [Miy15, CM16] and has exactly $n^2$ non-isomorphic irreducible modules, which can be realised as the eigenspaces of $g$ acting on the irreducible twisted modules $V(g^j)$ of $V$. If the twisted modules $V(g^j)$ have positive conformal weight for $j \neq 0 \pmod{n}$, then the sum $V^{\text{orb}(g)} := \oplus_{j \in \mathbb{Z}_n} V(g^j)^g$ is again a strongly rational, holomorphic vertex operator algebra.

Our approach is motivated by the classification of positive-definite, even, unimodular lattices, which bears similarities to the classification of strongly rational, holomorphic vertex operator algebras. In 1973 Niemeier classified the positive-definite, even, unimodular lattices of rank 24 [Nie73] (see also [Ven80, CS99]). He proved that up to isomorphism there are exactly 24 such lattices and that the isomorphism class is uniquely determined by the root system. The Leech lattice $\Lambda$ is the unique one amongst these lattices without roots.

Conway, Parker and Sloane showed that there is a natural bijection, mediated by the “holy construction”, between the 23 classes of deep holes of the Leech lattice $\Lambda$, i.e. points in $\Lambda \otimes \mathbb{Z} \mathbb{R}$ which have maximal distance to $\Lambda$, and the 23 Niemeier lattices different from the Leech lattice [CPS82, CS82] (see also [Bar88]). Let $d$ be a deep hole corresponding to the Niemeier lattice $N$ and let $n$ denote the order of $d + \Lambda \in (\Lambda \otimes \mathbb{Z} \mathbb{R})/\Lambda$. Then $\Lambda^d := \{\alpha \in \Lambda \mid \langle \alpha, d \rangle \in \mathbb{Z}\}$ is an index-$n$ sublattice of $\Lambda$ and of the lattice $\text{span}_\mathbb{Z}\{\Lambda^d, d\}$ generated by $d$ and $\Lambda^d$, which is isomorphic to
the Niemeier lattice $N$. Note that $n$ equals the (dual) Coxeter number $h$ of (any irreducible component of) the root system of $N$.

For the inverse construction, given the Niemeier lattice $N$, let $u := \sum_{i=1}^{r} \rho_i/h$ where the $\rho_i$ denote the Weyl vectors, i.e. the sums of the fundamental weights or the half sums of the positive roots, of the irreducible components of the root system of $N$. Then $N^u$ is an index-$h$ sublattice of $N$ that isomorphic to $\Lambda^d$ where $d$ is the corresponding deep hole.

In [MS19] the deep-hole construction of the Niemeier lattices was generalised to strongly rational, holomorphic vertex operator algebras $V$ of central charge 24. For each of the 71 Lie algebras $g$ on Schellekens’ list there is a generalised deep hole, a certain automorphism of the Leech lattice vertex operator algebra $V_\Lambda$, such that $(V_\Lambda^{\text{orb}(g)})_1 \cong g$. This provides a uniform proof of the existence part of the above theorem.

Complementarily, in this paper we lift the inverse construction to the level of vertex operator algebras (generalising [LS20]). First, combining the dimension formula in [MS19] with the “very strange formula” in [Kac90] we prove:

**Theorem (Theorem 3.3).** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 whose weight-one Lie algebra $V_1 = \bigoplus_{i=1}^{r} g_i$, is semisimple with Cartan subalgebra $\mathfrak{h}$ and $g \neq \text{id}$ an automorphism of $V$ of order $n$ such that $g|_{V_1}$ is inner and characterised by $\delta = \sum_{i=1}^{r} \delta_i \in \mathfrak{h}^*$. Assume furthermore that $g$ is of type 0, that $V^g$ satisfies the positivity condition and that $g|_{V_1}$ is quasirational. Then

$$\dim(V_1^{\text{orb}(g)}) \leq 24 + 12n \sum_{i=1}^{r} h_i^\vee \begin{vmatrix} \delta_i - \frac{\rho_i}{h_i^\vee} \end{vmatrix}^2$$

with Weyl vectors $\rho_i$ and dual Coxeter numbers $h_i^\vee$.

Then, given a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with semisimple weight-one Lie algebra $V_1 = \bigoplus_{i=1}^{r} g_i$, we define the inner automorphism $\sigma_u = e^{(2\pi i)u_0} \in \text{Aut}(V)$ with $u = \sum_{i=1}^{r} \rho_i/h_i^\vee$. Using equation (1) one can show that this automorphism has type 0 so that the orbifold construction exists and the above formula shows that $\dim(V_1^{\text{orb}(\sigma_u)}) \leq 24$, which implies that $V^{\text{orb}(\sigma_u)} \cong V_\Lambda$ by [DM06, DM04b]. The inverse orbifold construction [EMS20a, LS19] implies:

**Theorem (Theorem 4.7).** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_1 \neq \{0\}$. Then $V$ is isomorphic to $V_\Lambda^{\text{orb}(g)}$ for some automorphism $g \in \text{Aut}(V_\Lambda)$.

More precisely, the automorphism $g$ may be taken to be a generalised deep hole as introduced in [MS19]. As an application we reprove Schellekens’ list, only using equation (1) but none of the higher trace identities.

**Theorem (Theorem 6.3).** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_1 \neq \{0\}$. Then $V_1$ is isomorphic to one of the 70 non-zero Lie algebras in Table 1 of [Sch93].

This novel proof is much simpler than the original proof [Sch93, EMS20a] and for the most part does not rely on computer calculations.

Finally, we can also use the result that each strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_1 \neq \{0\}$ comes from the Leech lattice $\Lambda$ to prove the uniqueness of such a vertex operator algebra for a given semisimple Lie algebra $V_1$. One merely has to classify generalised deep holes in $\text{Aut}(V_\Lambda)$. We demonstrate this for 43 of the 70 non-zero Lie algebras on Schellekens’ list.
Outline. In Section 2 we recall some results on the classification of strongly rational, holomorphic vertex operator algebras of central charge 24.

In Section 3 we review the cyclic orbifold construction. Then we combine the dimension formula in [MS19] with Kac’s “very strange formula” to derive the dimension formula in Theorem 3.3.

In Section 4 we use Theorem 3.3 and equation (1) to prove that any strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V \neq \{0\}$ can be obtained from the Leech lattice vertex operator algebra $V_\Lambda$ by a cyclic orbifold construction (see Theorem 4.7).

In Section 5 we review the automorphism group of the Leech lattice vertex operator algebra $V_\Lambda$ and state a classification of the conjugacy classes.

In Section 6 we use Theorem 4.7 and the results in Section 5 to give a simple proof of Schellekens’ list of 71 Lie algebras that can occur as the weight-one space of a strongly rational, holomorphic vertex operator algebra of central charge 24 (see Theorem 6.3).

Finally, in Section 7 we prove the uniqueness of a strongly rational, holomorphic vertex operator algebra of central charge 24 with a given weight-one space for 43 of the 70 non-zero Lie algebras on Schellekens’ list by classifying generalised deep holes in $\text{Aut}(V_\Lambda)$ (see Theorem 7.1).

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2. Holomorphic Vertex Operator Algebras of Central Charge 24

In this section we review some results on strongly rational, holomorphic vertex operator algebras of central charge 24.

A vertex operator algebra $V$ is called strongly rational if it is rational (as defined, e.g., in [DLM97]), C2-cofinite (or lisse), self-contragredient (or self-dual) and of CFT-type. Then $V$ is also simple. Moreover, a rational vertex operator algebra $V$ is said to be holomorphic if $V$ itself is the only irreducible $V$-module. The central charge of a strongly rational, holomorphic vertex operator algebra $V$ is necessarily in $8\mathbb{Z}_{\geq 0}$, a simple consequence of Zhu’s modular invariance result [Zhu96]. Examples of strongly rational vertex operator algebras are those associated with positive-definite, even lattices. If the lattice is unimodular, then the associated vertex operator algebra is holomorphic.

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a vertex operator algebra of CFT-type. Then the zero modes $[a,b] := a_0b$ for $a,b \in V_1$ equip $V_1$ with the structure of a Lie algebra. If $V$ is also self-contragredient, then there exists a non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V$, which is unique up to a non-zero scalar and symmetric [FH93, L94]. We normalise the form such that $\langle 1,1 \rangle = -1$, where $1$ is the vacuum vector of $V$. Then $a_1b = b_0a = \langle a,b \rangle 1$ for all $a,b \in V_1$. 
Moreover, Theorem 3.1 in [DM06] states that for a simple Lie subalgebra 
\[ \rho \]
and an integrable vertex operator subalgebra of a simple Lie algebra \( \hat{g} := g \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C} K \) with central element \( K \) and Lie bracket
\[ [a \otimes t^m, b \otimes t^n] := [a, b] \otimes t^{m+n} + m(a, b)\delta_{m+n,0}K \]
for \( a, b \in g, m, n \in \mathbb{Z} \).

A representation of \( \hat{g} \) is said to have level \( k \in \mathbb{C} \) if \( K \) acts as \( k \) id. For a dominant integral weight \( \lambda \in P_+ \) and \( k \in \mathbb{C} \) let \( L_{\hat{g}}(k, \lambda) \) be the irreducible \( \hat{g} \)-module of level \( k \) obtained by inducing the irreducible highest-weight \( g \)-module \( L_g(\lambda) \) up in a certain way to a \( \hat{g} \)-module and taking its irreducible quotient (see, e.g., [Kac90]).

For \( k \in \mathbb{Z}_{\geq 0} \), \( L_{\hat{g}}(k, 0) \) admits the structure of a rational vertex operator algebra whose irreducible modules are given by the \( L_{\hat{g}}(k, \lambda) \) for \( \lambda \in P^k_+ \), the subset of the dominant integral weights \( P_+ \) of level at most \( k \) \([FZ92]\). The conformal weight of the affine vertex operator algebra \( L_{\hat{g}}(k, \lambda) \) is \( \rho(L_{\hat{g}}(k, \lambda)) = \frac{(\lambda + 2\rho(\lambda))}{2(k + h^\vee)} \) with Weyl vector \( \rho \) and dual Coxeter number \( h^\vee \) (see \([Kac90]\), Corollary 12.8).

For a self-contragredient vertex operator algebra \( V \) of CFT-type the commutator formula implies that the modes satisfy
\[ [a_m, b_n] = (a_0 b)_{m+n} + m(a_1 b)_{m+n-1} = [a, b]_{m+n} + m(a, b)\delta_{m+n,0} \text{id}_V \]
for all \( a, b \in V_1, m, n \in \mathbb{Z} \). Comparing this with the definition above we see that for a simple Lie subalgebra \( g \) of \( V_1 \) the map \( a \otimes t^n \mapsto a_n \) for \( a \in g \) and \( n \in \mathbb{Z} \) defines a representation of \( g \) on \( V \) of some level \( k_g \in \mathbb{C} \) with \( \langle \cdot, \cdot \rangle_g = k_g \langle \cdot, \cdot \rangle \).

Suppose that \( V \) is strongly rational. Then it is shown in \([DM04a]\) that the Lie algebra \( V_1 \) is reductive, i.e., a direct sum of a semisimple and an abelian Lie algebra. Moreover, Theorem 3.1 in \([DM04a]\) states that for a simple Lie subalgebra \( g \) of \( V_1 \) the restriction of \( \langle \cdot, \cdot \rangle \) to \( g \) is non-degenerate, the level \( k_g \) is a positive integer, the vertex operator subalgebra of \( V \) generated by \( g \) is isomorphic to \( L_{\hat{g}}(k_g, 0) \) and \( V \) is an integrable \( \hat{g} \)-module.

Assume in addition that \( V \) is holomorphic and of central charge 24. Then the Lie algebra \( V_1 \) is zero, abelian of dimension 24 or semisimple of rank at most 24 \([DM04a]\). If the Lie algebra \( V_1 \) is semisimple, then it decomposes into a direct sum
\[ V_1 \cong g_1 \oplus \ldots \oplus g_r \]
of simple ideals \( g_i \), and the vertex operator subalgebra \( \langle V_1 \rangle \) of \( V \) generated by \( V_1 \) is isomorphic to the tensor product of affine vertex operator algebras
\[ \langle V_1 \rangle \cong L_{\hat{g}_1}(k_1, 0) \otimes \ldots \otimes L_{\hat{g}_r}(k_r, 0) \]
with levels \( k_i := k_{g_i} \in \mathbb{Z}_{\geq 0} \) and has the same Virasoro vector as \( V \). The decomposition of the vertex operator algebra \( \langle V_1 \rangle \) is called the affine structure of \( V \), denoted by
\[ \mathfrak{g}_{k_1, k_2, \ldots, k_r} \]
with \( k_i \) sometimes omitted if it equals 1.

Since \( \langle V_1 \rangle \cong L_{\hat{g}_1}(k_1, 0) \otimes \ldots \otimes L_{\hat{g}_r}(k_r, 0) \) is rational, \( V \) decomposes into the direct sum of finitely many irreducible \( \langle V_1 \rangle \)-modules
\[ V \cong \bigoplus_{\lambda} m_{\lambda} L_{\hat{g}_1}(k_1, \lambda_1) \otimes \ldots \otimes L_{\hat{g}_r}(k_r, \lambda_r) \]
with \( m_{\lambda} \in \mathbb{Z}_{\geq 0} \) and the sum runs over finitely many \( \lambda = \lambda_1 + \ldots + \lambda_r \) with dominant integral weights \( \lambda_i \in P^k_+(g_i) \), i.e. of level at most \( k_i \).
Let $h_i^\vee$ denote the dual Coxeter number of $g_i$. The modular invariance of the character of $V$ implies that the ratio $h_i^\vee/k_i$ is independent of $g_i$. More precisely,

$$\frac{h_i^\vee}{k_i} = \frac{\dim(V_i) - 24}{24}$$

for all $i = 1, \ldots, r$, which follows from the lowest trace identity in [Sch93] (see also [DM04a]). This equation is the only result from [Sch93] that we shall use in this work in order to reprove Schellekens’ list. Note that equation (1) also implies that the Lie algebra $V_i$ uniquely determines the affine structure, i.e. the levels $k_i$.

Together with the crude inequality $(h_i^\vee)^2 < 4\dim(g_i)$ and $k_i \geq 1$ equation (1) implies that $\dim(V_i)$ is bounded from above, which proves that there are only finitely many affine structures satisfying equation (1). It is then easy to produce a list of the exactly 221 solutions of this equation (see Table 3).

Schellekens then narrowed down this list to 69 possible affine structures (or Lie algebras $V_i$) by solving large integer linear programming problems on the computer that follow from higher trace identities (explicitly written down in [EMS20a, Theorem 6.1]). Together with the zero Lie algebra and the 24-dimensional abelian Lie algebra this gives Schellekens’ list of 71 Lie algebras (see Table 2) that occur as the weight-one space of a strongly rational, holomorphic vertex operator algebra of central charge 24 [Sch93].

In Section 6 we present an alternative approach to proving this result by using the lowest trace identity [1], the dimension formula from [MS9] and Kac’s “very strange formula”.

3. Dimension Formula and “Very Strange Formula”

In this section we recall the cyclic orbifold construction. Then we state the dimension formula for central charge 24 from [MS19] and combine it with Kac’s “very strange formula” to obtain the first main result of this text.

3.1. Orbifold Construction. The cyclic orbifold construction [EMS20a, M6116] is an important tool that allows to construct new vertex operator algebras from known ones.

Let $V$ be a strongly rational, holomorphic vertex operator algebra and $G = \langle g \rangle$ a finite, cyclic group of automorphisms of $V$ of order $n \in \mathbb{Z}_{>0}$.

By [DLLM00] there is an up to isomorphism unique irreducible $g^i$-twisted $V$-module $V(g^i)$ for each $i \in \mathbb{Z}_n$. The uniqueness of $V(g^i)$ implies that there is a representation $\phi_i: G \rightarrow \text{Aut}_C(V(g^i))$ of $G$ on the vector space $V(g^i)$ such that

$$\phi_i(g)Y_{V(v)}(v, x)\phi_i(g)^{-1} = Y_{V(g^i)}(gv, x)$$

for all $i \in \mathbb{Z}_n$, $v \in V$. This representation is unique up to an $n$-th root of unity. Denote the eigenspace of $\phi_i(g)$ in $V(g^i)$ corresponding to the eigenvalue $e^{(2\pi i j)/n}$ by $W^{(i,j)}$. On $V(g^0) = V$ we choose $\phi_0(g) = g$.

By [DM97] and recent results in [Miy15, CM16] the fixed-point vertex operator subalgebra $V^g = W^{(0,0)}$ is again strongly rational. It has exactly $n^2$ irreducible modules, namely the $W^{(i,j)}$, $i, j \in \mathbb{Z}_n$. One can further show that the conformal weight $\rho(V(g))$ of $V(g)$ is in $(1/n^2)\mathbb{Z}$, and we define the type $t \in \mathbb{Z}_n$ of $g$ by $t = n^2 \rho(V(g)) \pmod{n}$.

Assume for simplicity that $g$ has type 0, i.e. that $\rho(V(g)) \in (1/n)\mathbb{Z}$. Then it is possible to choose the representations $\phi_i$ such that the conformal weights satisfy

$$\rho(W^{(i,j)}) = \frac{ij}{n} + \mathbb{Z}$$
and $V^g$ has fusion rules

$$W^{(i,j)} \boxtimes W^{(l,k)} \cong W^{(i+l,j+k)}$$

for all $i,j,l \in \mathbb{Z}_n$ (see [EMS20a, Section 5]), i.e. the fusion ring of $V^g$ is the group ring $\mathbb{C}[\mathbb{Z}_n \times \mathbb{Z}_n]$. In particular, all irreducible $V^g$-modules are simple currents.

In essence, the results in [EMS20a] show that for cyclic $G \cong \mathbb{Z}_n$ and strongly rational, holomorphic $V$ the module category of $V^G$ is the twisted group double $\mathcal{D}_\omega(G)$ where the 3-cocycle $[\omega] \in H^3(G, \mathbb{C}^*) \cong \mathbb{Z}_n$ is determined by the type $t \in \mathbb{Z}_n$. This proves a conjecture by Kirillov [Kir02] who stated it for arbitrary finite $G$ and proved it when $[\omega]$ is trivial.

In general, a simple vertex operator algebra $V$ is said to satisfy the positivity condition if the conformal weight $\rho(W) > 0$ for any irreducible $V$-module $W \ncong V$ and $\rho(V) = 0$.

Now, if $V^g$ satisfies the positivity condition (it is shown in [Mo18] that this condition is almost automatically satisfied if $V$ is strongly rational), then the direct sum of $V^g$-modules

$$V^{\text{orb}(g)} := \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$$

admits the structure of a strongly rational, holomorphic vertex operator algebra of the same central charge as $V$ and is called orbifold construction associated with $V$ and $g$ [EMS20a]. Note that $\bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)}$ gives back the old vertex operator algebra $V$.

We briefly describe the inverse (or reverse) orbifold construction [EMS20a]. Suppose that the strongly rational, holomorphic vertex operator algebra $V^{\text{orb}(g)}$ is obtained in an orbifold construction as described above. Then via $\zeta_v := e^{(2\pi i)/n_v}$ for $v \in W^{(i,0)}$ we define an automorphism $\zeta$ of $V^{\text{orb}(g)}$ of order $n$, and the unique irreducible $\zeta$-twisted $V^{\text{orb}(g)}$-module is given by $V^{\text{orb}(g)}(\zeta^j) = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,j)}$, $j \in \mathbb{Z}_n$. Then

$$V^{\text{orb}(g)}(\zeta^j) \cong \bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)} = V,$$

i.e. orbifolding with $\zeta$ is inverse to orbifolding with $g$.

3.2. Dimension Formula and “Very Strange Formula”. A dimension formula for the weight-one space $V^{\text{orb}(g)}_1$ in the case of central charge 24 was derived in [MS19] by pairing the vector-valued character of $V^g$ with a vector-valued Eisenstein series of weight 2. It implies the following dimension bound:

**Proposition 3.1** (Dimension Formula, [MS19], Corollary 4.9). Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 and $g \neq id$ an automorphism of $V$ of finite order $n$ and type 0 such that $V^g$ satisfies the positivity condition. Then

$$\dim(V^{\text{orb}(g)}_1) \leq 24 + \sum_{d|n} c_n(d) \dim(V^g_d)$$

with the $c_n(d) \in \mathbb{Q}$ determined by $\sum_{d|n}(t,d)c_n(d) = n/t$ for all $t \mid n$.

This generalises earlier results for $n = 2, 3$ in [Mon94] (see also [LS19]), for $n = 2, 3, 5, 7, 13$ in [Mo16] and for all $n$ such that the modular curve $\Gamma_0(n) \backslash \mathbb{H}$ has genus zero in [EMS20a]. An automorphism $g$ such that $\dim(V^{\text{orb}(g)}_1)$ attains this upper bound is called extremal.

We note that the upper bound from the dimension formula depends only on the weight-one Lie algebra $V_1$ and the action of the automorphism $g \in \text{Aut}(V)$ on it. In the following we combine this upper bound with Kac’s “very strange formula” to cast the dimension formula in a more Lie theoretic form.
For simplicity, we only state Kac’s “very strange formula” in the special case of inner automorphisms but the following arguments work equally well in the general case. Recall from Kac’s classification of finite-order automorphisms of simple Lie algebras (see [Kac90], Chapter 8) that an inner automorphism $g$ of a simple Lie algebra $\mathfrak{g}$ of type $X_l$ is characterised by a sequence of non-negative, relatively prime integers $s_0, \ldots, s_l$ associated with the nodes of the untwisted affine Dynkin diagram $X^{(1)}_l$ satisfying $\sum_{i=0}^{l} a_is_i = m$ with Kac labels $a_i$. Equivalently, $g$ is characterised by the vector $\delta$ in the dual Cartan subalgebra $\mathfrak{h}^*$ of $\mathfrak{g}$ defined by $(\alpha_i, \delta) = s_i/m$ for all simple roots $\alpha_i$ of $\mathfrak{g}$. Identifying the Cartan subalgebra $\mathfrak{h}$ with its dual $\mathfrak{h}^*$ via $(\cdot, \cdot)$ we may view $\delta \in \mathfrak{h}$ and then $g$ is conjugate to $e^{ad(2i\pi \delta)}$.

Recall that the Weyl vector $\rho$ is characterized by the half sum of the positive roots of $\mathfrak{g}$ and that $h^\vee$ denotes the dual Coxeter number.

**Proposition 3.2** (“Very Strange Formula”, formula (12.3.6) in [Kac90]). Let $\mathfrak{g}$ be a finite-dimensional, simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and $g$ an inner automorphism of $\mathfrak{g}$ of order $m \in \mathbb{Z}_{>0}$ characterised by $\delta \in \mathfrak{h}^*$ as described above. Then

$$\dim(\mathfrak{g}) = \frac{24}{4m^2} \sum_{j=1}^{m-1} j(m-j) \dim(\mathfrak{g}_{(j)}) = \frac{h^\vee}{2} |\delta - \rho/h^\vee|^2$$

where $\mathfrak{g}_{(j)}$ is the eigenspace $\mathfrak{g}_{(j)} = \{ x \in \mathfrak{g} | \sigma x = e^{2(\pi i)j/n}x \}, j = 0, \ldots, m - 1$.

Note that the squared norm on the right-hand side is formed with respect to bilinear form on $\mathfrak{h}^*$ induced by $(\cdot, \cdot)$.

An automorphism $g$ of $\mathfrak{g}$ of order $m$ is called quasirational (see [Kac90], Section 8.8) if the corresponding eigenspaces satisfy $\dim(\mathfrak{g}_{(i)}) = \dim(\mathfrak{g}_{(j)})$ whenever $(i, m) = (j, m)$ or equivalently if the characteristic polynomial of $g$ has rational coefficients. This means that the characteristic polynomial of $g$ can be written as $\prod_{t|m}(x^t - 1)^{b_t}$ for some $b_t \in \mathbb{Z}$ (see, e.g., Exercise 13.7 in [Kac90]). In this case we say $g$ has cycle shape $\prod_{t|m} b_t$ and it is not difficult to see that the left-hand side of Kac’s “very strange formula” may be rewritten as

$$\frac{\dim(\mathfrak{g})}{24} = \frac{1}{4m^2} \sum_{j=1}^{m-1} j(m-j) \dim(\mathfrak{g}_{(j)}) = \frac{1}{24} \sum_{t|m} b_t.$$ 

We are now in a position to prove the first main result of this text:

**Theorem 3.3.** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 whose weight-one Lie algebra $V_1 = \bigoplus_{i=1}^r \mathfrak{g}_i$ is semisimple with Cartan subalgebra $\mathfrak{h}$ and $g \neq \text{id}$ an automorphism of $V$ of order $n$ such that $g|_{V_1}$ is inner and characterised by $\delta = \sum_{i=1}^r \delta_i \in \mathfrak{h}^*$. Assume furthermore that $g$ is of type 0, that $V^g$ satisfies the positivity condition and that $g|_{V_1}$ is quasirational. Then

$$\dim(V_1^{\text{orb}(g)}) \leq 24 + 12n \sum_{i=1}^r h_i^\vee | \delta_i - \rho_i/h_i^\vee|^2$$

with Weyl vectors $\rho_i$ and dual Coxeter numbers $h_i^\vee$.

**Proof.** Since $g|_{V_1}$ is quasirational (and of some order $m \mid n$), Kac’s “very strange formula”, generalised to the semisimple Lie algebra $V_1$, becomes

$$\frac{1}{24} \sum_{t|m} b_t = \frac{r}{2} \sum_{i=1}^r h_i^\vee | \delta_i - \rho_i/h_i^\vee|^2$$
where $\prod_{t|m} t^{b_t}$ is the cycle shape of $g_{|V_1}$. Also note that
\[
\dim(V_1^{g_{|V_1}}) = \sum_{t|m} b_t(t,d)
\]
for any $d \in \mathbb{Z}$. The dimension formula then yields
\[
\dim(V_1^{\text{orb}(g)}) \leq 24 + \sum_{d|n} c_n(d) \dim(V_1^{g_{|V_1}}) = 24 + \sum_{d|n} c_n(d) \sum_{t|m} b_t(t,d)
\]
\[
= 24 + \sum_{t|m} b_t \sum_{d|n} c_n(d)(t,d) = 24 + n \sum_{t|m} b_t
\]
\[
= 24 + 12n \sum_{i=1}^{r} h_i^\vee \left| \delta_i - \frac{\rho_i}{h_i^\vee} \right|^2
\]
where we used the defining relations of the $c_n(d)$.

4. THE NEIGHBOURS OF THE LEECH LATTICE VERTEX OPERATOR ALGEBRA

In this section, given a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_1 = \bigoplus_{i=1}^r \mathfrak{g}_i$ semisimple, we define a certain inner automorphism $\sigma_u \in \text{Aut}(V)$ and show using Theorem 3.3 and equation (1) that the corresponding orbifold construction $V_{\text{orb}(\sigma_u)}$ is isomorphic to the Leech lattice vertex operator algebra $V_\Lambda$. By the inverse orbifold construction this then proves that every strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_1 \neq \{0\}$ can be obtained by an orbifold construction from the Leech lattice vertex operator algebra $V_\Lambda$.

The upper bound in Theorem 3.3 suggests a canonical choice for the automorphism $\sigma_u \in \text{Aut}(V)$, namely
\[
\sigma_u = e^{(2\pi i)u_0} \quad \text{with} \quad u := \sum_{i=1}^{r} \rho_i / h_i^\vee
\]
with Weyl vectors $\rho_i$ and dual Coxeter numbers $h_i^\vee$, where we view $u$ in the Cartan subalgebra $\mathfrak{h}$ of $V_1$ by means of the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ via the invariant bilinear form $(\cdot, \cdot)$, which we normalised on each simple ideal $\mathfrak{g}_i$ such that long roots have norm 2.

First, we study some properties of this automorphism. We denote by $l_i \in \{1, 2, 3\}$ the lacing number of $\mathfrak{g}_i$.

Proposition 4.1. Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_1 = \bigoplus_{i=1}^r \mathfrak{g}_i$ semisimple. Then the inner automorphism $\sigma_u = e^{(2\pi i)u_0}$ with $u = \sum_{i=1}^{r} \rho_i / h_i^\vee$ has order $n = \text{lcm}(l_i h_i^\vee)_{i=1}^{r}$ (as does the restriction $\sigma_u|_{V_1}$ to $V_1$) and type 0, and $\sigma_u|_{V_1}$ is quasirational.

Proof. First, we describe the Lie algebra automorphism $\sigma_u|_{V_1} = e^{ad(2\pi i)u_0}$ in the language of [Kac90], Chapter 8. Since the Weyl vector $\rho_i$ is the sum of the fundamental weights (dual to the simple coroots) of $\mathfrak{g}_i$, it is easy to determine the sequence of integers $s_0, \ldots, s_l$ (or equivalently the vector $\delta_i$ in the dual Cartan subalgebra $\mathfrak{h}^*_i$ of $\mathfrak{g}_i$) characterising the action of $\sigma_u|_{\mathfrak{g}_i} = e^{ad(2\pi i\rho_i / h_i^\vee)}$ on the simple roots of $\mathfrak{g}_i$. Indeed, if $\mathfrak{g}_i$ is simply-laced, i.e. of type $A_l$, $D_l$ or $E_l$, then $\sigma_u|_{\mathfrak{g}_i}$ has type $(1, \ldots, 1; 1)$ (and this is the up to conjugation unique minimal-order regular automorphism of $\mathfrak{g}_i$, see Exercise 8.11 in [Kac90]). For types $B_l$, $C_l$, $F_4$ or $G_2$ the automorphism $\sigma_u|_{\mathfrak{g}_i}$ has type $(2, \ldots, 2, 1; 1)$, $(2, 1, \ldots, 1, 2, 1)$, $(2, 2, 2, 1, 1; 1)$ or $(3, 3, 1, 1)$, respectively. Not surprisingly, this shows that $\delta_i = \rho_i / h_i^\vee$. 
The quasirationality of \( \sigma_u \mid V \) now follows from the quasirationality on each simple ideal \( \mathfrak{g}_i \). If \( \mathfrak{g}_i \) is simply-laced, this is Exercise 8.12 in [Kac90]. For the non simply-laced cases, it is not difficult to show that an automorphism of the above-mentioned types is quasirational.

The order of \( \sigma_u \mid \mathfrak{g}_i \) is \( l_i h_i^\vee \) with dual Coxeter number \( h_i^\vee \) and lacing number \( l_i \) so that the Lie algebra automorphism \( \sigma_u \mid \mathfrak{g}_i \) has order \( n = \text{lcm}(\{l_i h_i^\vee\}_{i=1}^r) \).

In principle, the order of \( \sigma_u \) on the vertex operator algebra \( V \) is a multiple of \( n \). In the following we show that in fact \( |\sigma_u| = n = \text{lcm}(\{l_i h_i^\vee\}_{i=1}^r) \). Recall that \( V \) decomposes into a direct sum of irreducible \( (V_i) \)-modules \( V \cong \bigoplus_{i=1}^r \bigotimes_{\lambda} L_{\vartheta_i}(k_i, \lambda_i) \) where the sum runs over weights \( \lambda = \sum_{i=1}^r \lambda_i \) with dominant integral weights \( \lambda_i \in P_+^{\vartheta_i}(\mathfrak{g}_i) \). In order to prove that \( \sigma_u \) has order \( n \), it suffices to show that \( n(u, \lambda) \in \mathbb{Z} \) for all weights \( \lambda \) appearing in this decomposition. (It is easy to see that \( 2n(u, \lambda) \in \mathbb{Z} \) for any weight \( \lambda \) in the weight lattice \( P_\vartheta \).)

The conformal weight of a module \( \bigotimes_{i=1}^r L_{\vartheta_i}(k_i, \lambda_i) \) (see Section 2) appearing in the decomposition of \( V \) must be an integer, i.e.

\[
\sum_{i=1}^r \frac{2\rho_i + (\lambda_i, \lambda_i)}{2(k_i + h_i^\vee)} = \frac{h_i^\vee}{2(h_i^\vee + k_i)} \sum_{i=1}^r \left( \frac{2\rho_i}{h_i^\vee} + \frac{1}{h_i^\vee} (\lambda_i, \lambda_i) \right) = \frac{1}{2n(1 + k_i/h_i^\vee)} \left( 2n(u, \lambda) + \sum_{i=1}^r \frac{n}{h_i^\vee} (\lambda_i, \lambda_i) \right) \in \mathbb{Z}
\]

where we used that \( h_i^\vee/(h_i^\vee + k_i) \) is independent of \( i \) by equation (1). Then, since \( n(1 + k_i/h_i^\vee) \in \mathbb{Z} \), this implies that \( 2n(u, \lambda) + \sum_{i=1}^r \frac{n}{h_i^\vee} (\lambda_i, \lambda_i) \in \mathbb{Z} \) and, recalling that \( 2n(u, \lambda) \in \mathbb{Z} \), we obtain

\[
\sum_{i=1}^r \frac{n}{h_i^\vee} (\lambda_i, \lambda_i) \in \mathbb{Z}.
\]

Now, for any irreducible Lie algebra \( \mathfrak{g} \) (with Cartan subalgebra \( \mathfrak{h} \), root system \( \Phi \subseteq \mathfrak{h}^* \), positive roots \( \Phi^+ \subseteq \Phi \) and dual Coxeter number \( h^\vee \)) the identity

\[
\sum_{\alpha \in \Phi^+} (\alpha, y)^2 = \frac{1}{2} \sum_{\alpha \in \Phi} (\alpha, y)^2 = h^\vee(y, y)
\]

holds for all \( y \in \mathfrak{h}^* \). For the simply-laced root systems this is Proposition 1.6 in [Be13] and it is easy to verify the statement for the other cases.

Then, since the Weyl vector \( \rho_i = \frac{1}{2} \sum_{\alpha \in \Phi^+_i} \alpha \) is the half sum of positive roots,

\[
(2n(u, \lambda))^2 = \left( \sum_{i=1}^r \frac{n}{h_i^\vee} (2\rho_i, \lambda_i) \right)^2 = \left( \sum_{i=1}^r \sum_{\alpha \in \Phi^+_i} \frac{n}{h_i^\vee} (\alpha, \lambda_i) \right)^2
\]

\[
\equiv \sum_{i=1}^r \sum_{\alpha \in \Phi^+_i} \left( \frac{n}{h_i^\vee} \right)^2 (\alpha, \lambda_i)^2 \pmod{2}
\]

\[
= \sum_{i=1}^r \left( \frac{n}{h_i^\vee} \right)^2 h_i^\vee (\lambda_i, \lambda_i) = n \sum_{i=1}^r \frac{n}{h_i^\vee} (\lambda_i, \lambda_i).
\]

Note that \( \frac{n}{h_i^\vee} (\lambda_i, \lambda_i) \in \mathbb{Z} \) for any root \( \alpha \in \Phi^+_i \).

If \( n \) is even, then \( n \sum_{i=1}^r \frac{n}{h_i^\vee} (\lambda_i, \lambda_i) \) is also even and so is \( 2n(u, \lambda)^2 \). Thus, \( 2n(u, \lambda) \) is even and \( n(u, \lambda) \) is an integer. If \( n \) is odd, then all lacing numbers \( l_i \) and dual Coxeter numbers \( h_i^\vee \) are odd. This means that all irreducible components of \( V_l \) are of type \( A_l \) with \( l \) even. However, \( \rho_i \) is in the root lattice in this case. Therefore, \( n(u, \lambda) \in \mathbb{Z} \) for any \( \lambda \) in the weight lattice \( P_\vartheta \).
To prove that \( \sigma_u \) has type 0 we use the “strange formula” of Freudenthal–de Vries \( |\rho_i|^2/(2h_i^+) = \dim(g_i)/24 \) (see, e.g., equation (12.1.8) in [Kac90]), which is a special case of the “very strange formula”, and equation (1). The \( L_0 \)-operator of the twisted module \( V(\sigma_u) \) is shifted in comparison to \( V \) by \(-u_0 + (1/2)(u,u)\) id (see [Li90], Section 5). Since \( \sigma_u \) has order \( n, u_0 \) has eigenvalues in \((1/n)\mathbb{Z}\), and hence \( \sigma_u \) is of type 0 if and only if \((u,u)/2 \in (1/n)\mathbb{Z}\). We compute
\[
\frac{1}{2} \langle u, u \rangle = \sum_{i=1}^{r} \frac{\langle \rho_{\gamma_i}, \rho_{\gamma_i} \rangle}{2(h_{\gamma_i}^+)^2} = \sum_{i=1}^{r} \frac{k_i \langle \rho_{\gamma_i}, \rho_{\gamma_i} \rangle}{2(h_{\gamma_i}^+)^2} = \sum_{i=1}^{r} \frac{k_i \dim(g_i)}{24h_{\gamma_i}^+}
\]
where we used the “strange formula” in the last step. With equation [1] this becomes
\[
\frac{1}{2} \langle u, u \rangle = \sum_{i=1}^{r} \frac{\dim(g_i)}{\dim(V_i) - 24} = \frac{\dim(V_1)}{\dim(V_1) - 24} = \frac{h_{\gamma_i}^+ + k_i}{h_{\gamma_i}^+} = \frac{l_i (h_{\gamma_i}^+ + k_i)}{l_i h_{\gamma_i}^+}
\]
for all \( i = 1, \ldots, r \). Since \( k_i \in \mathbb{Z} \) by [DM06] and \( h_{\gamma_i}^+ \in \mathbb{Z} \), this implies that \( \langle u, u \rangle/2 \in (1/n)\mathbb{Z} \). Hence \( \sigma_u \) is of type 0.

The following result is probably known:

**Lemma 4.2.** Let \( V \) be a simple vertex operator algebra and \( U \) a full vertex operator subalgebra of \( V \) that is simple and rational. If \( U \) satisfies the positivity condition, then so does \( V \).

**Proof.** By the rationality of \( U \), every \( V \)-module decomposes into a direct sum of irreducible \( U \)-modules. Because the conformal weights \( \rho(M) \) are non-negative for all irreducible \( U \)-modules \( M \), the same is true for all irreducible \( V \)-modules.

Now, let \( M \) be an irreducible \( V \)-module with \( \rho(M) = 0 \). Then, as \( U \)-module, \( M \) contains \( U \) since \( U \) is rational and \( U \) is the only irreducible \( U \)-module with non-positive conformal weight. The vacuum vector \( \nu := 1 \in U \) is a vacuum-like vector of \( U \) when we view \( U \) as a \( U \)-module, i.e., \( L_{-1}\nu = 0 \). Since \( U \) and \( V \) have the same Virasoro vector by assumption, \( \nu \) is also a vacuum-like vector of the \( V \)-module \( M \). Then, by Proposition 3.4 in [Li90] there is a non-zero \( V \)-module homomorphism from \( V \) to \( M \) and since both \( V \) and \( M \) are irreducible, \( V \cong M \) as \( V \)-modules by Schur’s lemma. Hence, \( V \) satisfies the positivity condition. \( \square \)

**Proposition 4.3.** Let \( V \) be a strongly rational, holomorphic vertex operator algebra of central charge 24 with \( V_1 := \bigotimes_{i=1}^{r} g_i \) semisimple and consider the inner automorphism \( \sigma_u = e^{(2\pi i)u} \) with \( u = \sum_{i=1}^{r} \rho_{\gamma_i}/h_{\gamma_i}^+ \). Then the fixed-point vertex operator subalgebra \( V_{\sigma_u} \) satisfies the positivity condition.

**Proof.** \( V \) contains the full vertex operator subalgebra \( \langle V_1 \rangle \cong \bigotimes_{i=1}^{r} L_{\hat{g}_i}(k_i, 0) \), which in turn contains the full vertex operator subalgebra \( V_Q \otimes W \) where \( V_Q \) is the lattice vertex operator algebra associated with the lattice \( Q := \bigoplus_{i=1}^{r} \sqrt{r_i} Q_i^0, Q_i^0 \) is the lattice spanned by the long roots of \( g_i \) normalised to have squared norm 2 (see, e.g., Corollary 5.7 in [DM06]) and \( W = \text{Com}(K_i(V_Q)) \) is the commutant (or centraliser) of \( V_Q \) in \( \langle V_1 \rangle \). By definition, \( W \cong \bigotimes_{i=1}^{r} K(g_i, k_i) \) where \( K(g_i, k_i) \) denotes the parafermion vertex operator algebra associated with \( g_i \) at level \( k_i \).

Since \( \sigma_u = e^{(2\pi i)u} \) is an inner automorphism associated with an element \( u \) in the Cartan subalgebra of \( V_1 \), the fixed-point vertex operator subalgebra \( V_{\sigma_u} \) contains the full vertex operator subalgebra \( V_Q \otimes W \) with lattice \( Q^0 = \{ \alpha \in Q \mid (\alpha, u) \in \mathbb{Z} \} \). It is well-known that lattice vertex operator algebras are rational and satisfy the positivity condition. For parafermion vertex operator algebras this is shown in [DR17] (using the rationality results in [CM16]). Therefore, \( V_Q \otimes W \) satisfies the positivity condition, and by Lemma 4.2 so does \( V_{\sigma_u} \). \( \square \)
Remark 4.4. We sketch another proof of Proposition 4.3 based on the geometry of affine Weyl groups. Let \( g \) be a simple Lie algebra and \( \lambda \) a dominant integral weight in \( P^+_k, k \in \mathbb{Z}_{>0} \). We identify the Cartan subalgebra \( h \) with its dual \( h^* \) via the form \( (\cdot, \cdot) \). For any non-zero vector in the integrable \( \hat{g} \)-module \( M = L_g(k, \lambda) \), homogeneous of weight \( \mu \in h^* \) and \( L_0 \)-eigenvalue \( \Delta \), the bound
\[
\Delta \geq \frac{|\mu|^2}{2k} + \left( \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)} - \frac{|\lambda|^2}{2k} \right)
\]
must be satisfied, with equality only possible for \( \mu \in \lambda + kQ^\vee \) where \( Q^\vee \) denotes the coroot lattice. The weights of the twisted module \( M(\sigma_u), u = \rho/h^\vee \), satisfy a similar inequality with \( \mu + ku \) in place of \( \mu \). The term in parentheses is non-negative for all \( \lambda \in P^+_k \), vanishing only if \( \lambda = k \Lambda_i \) where \( \Lambda_i \) is a fundamental weight with Kac label \( a_i = 1 \). (The Kac labels are the coefficients of the highest root \( \theta \) in terms of the simple roots \( \alpha_i \) of \( g \).) It is well known that, modulo \( Q^\vee \), this set of fundamental weights forms a complete set of representatives of \( Q^*/Q^\vee \) where \( Q^* \) is the dual of the root lattice [FKW92].

These remarks, together with the easy fact that the smallest positive integer \( p \) for which \( pp/\rho/h^\vee \in Q^* \) is \( p = lh^\vee \), imply that \( V(\sigma_{pu}) \) has positive conformal weight whenever \( 0 < p < lh^\vee = |\sigma_u| \). The same statement for semisimple \( g \) follows, and thus too Proposition 4.3.

Having studied the properties of the automorphism \( \sigma_u \) we are now in a position to prove:

**Proposition 4.5.** Let \( V \) be a strongly rational, holomorphic vertex operator algebra of central charge 24 with \( V_1 = \bigoplus_{i=1}^r g_i \), semisimple and consider the inner automorphism \( \sigma_u = e^{(2\pi i)\cdot u_0} \) with \( u = \sum_{i=1}^r \rho_i/h_i^\vee \). Then the corresponding orbifold construction \( V_{\text{orb}}(\sigma_u) \) is isomorphic to the Leech lattice vertex operator algebra \( \Lambda \).

Note that \( \sigma_u \) is exactly the automorphism chosen in Section 2.3 of [LS20] if \( V_1 \) is simply-laced and reduces to the "holy construction" in [CS82] if \( V \) is one of the 23 vertex operator algebras associated with the Niemeier lattices other than the Leech lattice \( \Lambda \).

**Proof.** We checked in Proposition 4.1 and Proposition 4.3 that \( \sigma_u \in \text{Aut}(V) \) admits an orbifold construction and satisfies the assumptions of Theorem 3.3. It follows from the definition of \( \sigma_u \) restricted to \( V_1 \) (recall that we showed \( \delta_i = \rho_i/h_i^\vee \) ) that
\[
\dim(V_{\text{orb}}(\sigma_u)) \leq 24.
\]
On the other hand, \( \dim(V_{\text{orb}}(\sigma_u)) \geq \dim(V^{\sigma_u}) > 0 \). Then equation \([1]\) implies that \( V_{\text{orb}}(\sigma_u) \) cannot be semisimple so that \( V_{\text{orb}}(\sigma_u) \) must be 24-dimensional abelian [DM04b]. By Theorem 3 in [DM04b], \( V_{\text{orb}}(\sigma_u) \cong \Lambda \). \( \square \)

With the inverse orbifold construction, Proposition 4.5 immediately implies that every strongly rational, holomorphic vertex operator algebra \( V \) of central charge 24 with \( V_1 \) semisimple can be obtained by an orbifold construction \( V \cong V_{\text{orb}}(g) \) from the Leech lattice vertex operator algebra \( \Lambda \). More precisely, the properties of the inner automorphism \( \sigma_u \) imply that the inverse-orbifold automorphism \( g \in \text{Aut}(\Lambda) \) is a generalised deep hole. Generalised deep holes were introduced in [MS19] and arise naturally as generalisations of the deep holes of the Leech lattice:

**Definition 4.6** (Generalised Deep Hole, [MS19]). Let \( V \) be a strongly rational, holomorphic vertex operator algebra of central charge 24 and \( g \in \text{Aut}(V) \) of finite order \( n > 1 \). Suppose \( g \) has type 0 and \( V^g \) satisfies the positivity condition. Then \( g \) is called a generalised deep hole of \( V \) if
(1) \( g \) is extremal, i.e. \( \dim((V_{\text{orb}}^g)_L) \) attains the upper dimension bound in Proposition 3.1.

(2) \( \text{rk}(V_{\text{orb}}^g) = \text{rk}(V_{\text{orb}}^g) \).

Note that both \( V^g \) and \( V_{\text{orb}}^g \) are reductive by [DM04]. By convention we also call the identity a generalised deep hole so that the Leech lattice vertex operator algebra \( V_\Lambda = V_{\text{orb}(\text{id})} \) may be included in the following theorem.

**Theorem 4.7.** Let \( V \) be a strongly rational, holomorphic vertex operator algebra of central charge 24 with \( V_1 \neq \{0\} \). Then \( V \) is isomorphic to \( V_{\text{orb}}^g \) for some generalised deep hole \( g \) in \( \text{Aut}(V_\Lambda) \).

**Proof.** We may assume that \( V_1 \) is semisimple. We showed in Proposition 4.5 that \( V_{\text{orb}(\sigma_u)} \) is isomorphic to the Leech lattice vertex operator algebra \( V_\Lambda \). Since the upper bound of 24 in the dimension formula is attained, \( \sigma_u \) is extremal. We then consider the inverse-orbifold automorphism \( g \in \text{Aut}(V_\Lambda) \) with \( V_{\text{orb}(g)} \cong V \). In particular, \( g \) is of type 0 and satisfies the positivity condition. Moreover, \( g \) is extremal since \( \sigma_u \) is by Proposition 4.11 in [MS19] and \( \text{rk}((V_{\text{orb}}^g)_L) = \text{rk}(V_{\text{orb}}) = \text{rk}(V_1) = \text{rk}((V_{\text{orb}}^g)_1) \) since \( \sigma_u | V_1 \) is inner. Hence, \( g \) is a generalised deep hole. \( \square \)

We remark that Theorem 4.7 is similar to Theorem 5.6 in [MS19] with the crucial difference that here we assume the existence of the vertex operator algebra \( V \) with a certain weight-one Lie algebra \( g \) while in [MS19] the existence is proved for each of the 71 Lie algebras on Schellekens’ list by explicitly specifying a generalised deep hole in \( \text{Aut}(V_\Lambda) \) such that \( (V_{\text{orb}}^g)_1 \cong g \).

5. **Automorphisms of the Leech Lattice Vertex Operator Algebra**

In this section we describe automorphisms of the Leech lattice vertex operator algebra \( V_\Lambda \) [Bor86, FLM88] and in particular their conjugacy classes, which were classified in [MS19].

For any positive-definite, even lattice \( L \) (with bilinear form \( \langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z} \)) the associated vertex operator algebra is given by

\[ V_L = M(1) \otimes \mathbb{C}_{[L]} \]

with the Heisenberg vertex operator algebra \( M(1) \) of rank \( \text{rk}(L) \) associated with \( L_{C} = L \otimes \mathbb{C} \) and the twisted group algebra \( \mathbb{C}_{[L]} \), the algebra with basis \( \{ e_\alpha | \alpha \in L \} \) and product \( e_\alpha e_\beta = \varepsilon(\alpha, \beta) e_{\alpha + \beta} \) for all \( \alpha, \beta \in L \) where \( \varepsilon : L \times L \to \{ \pm 1 \} \) is a choice of 2-cocycle satisfying \( \varepsilon(\alpha, \beta) / \varepsilon(\alpha, \alpha) = (-1)^{\langle \alpha, \beta \rangle} \).

Let \( O(L) \) denote the orthogonal group (or automorphism group) of the lattice \( L \). For \( \nu \in O(L) \) and a function \( \eta : L \to \{ \pm 1 \} \) the map \( \phi_\eta(\nu) \) acting on \( \mathbb{C}_{[L]} \) as \( \phi_\eta(\nu)(e_\alpha) = \eta(\alpha) e_{\nu \alpha} \) for \( \alpha \in L \) and as \( \nu \) on \( M(1) \) defines an automorphism of \( V_L \) if and only if

\[ \frac{\eta(\alpha) \eta(\beta)}{\eta(\alpha + \beta)} = \frac{\varepsilon(\alpha, \beta)}{\varepsilon(\nu \alpha, \nu \beta)} \]

for all \( \alpha, \beta \in L \). In this case \( \phi_\eta(\nu) \) is called a lift of \( \nu \) and all such automorphisms form the subgroup \( O(\hat{L}) \) of \( \text{Aut}(V_L) \). There is a short exact sequence

\[ 1 \to \text{Hom}(L, \{ \pm 1 \}) \to O(\hat{L}) \to O(L) \to 1 \]

with the surjection \( O(\hat{L}) \to O(L) \) given by \( \phi_\eta(\nu) \mapsto \nu \). The image of \( \text{Hom}(L, \{ \pm 1 \}) \) in \( O(\hat{L}) \) are exactly the lifts of \( \text{id} \in O(L) \). For later use let \( \nu \mapsto \hat{\nu} \) denote a fixed section \( O(L) \to O(\hat{L}) \).

If the restriction of \( \eta \) to the fixed-point lattice \( L' \) is trivial, we call \( \phi_\eta(\nu) \) a standard lift of \( \nu \). It is always possible to choose \( \eta \) in this way (see [Lep85], Section 5). It was proved in [EMS20a], Proposition 7.1, that all standard lifts of a
given \( \nu \in O(L) \) are conjugate in \( \text{Aut}(V_L) \). For convenience, let us assume that the section \( O(L) \to O(\hat{L}) \) maps \( \nu \mapsto \hat{\nu} \) only to standard lifts. This is not essential but simplifies the presentation.

For any vertex operator algebra \( V \) of CFT-type \( K := (e^{2\pi i} | \nu \in V_1) \) defines a subgroup of \( \text{Aut}(V) \), called the inner automorphism group of \( V \). By [DN99, Theorem 2.1], the automorphism group of \( V_L \) is of the form

\[
\text{Aut}(V_L) = O(\hat{L}) \cdot K,
\]

\( K \) is a normal subgroup of \( \text{Aut}(V_L), \text{Hom}(L, \{ \pm 1 \}) \) a subgroup of \( K \cap O(\hat{L}) \) and \( \text{Aut}(V_L)/K \) is isomorphic to some quotient group of \( O(L) \).

In the following we specialise to the Leech lattice \( \Lambda \), the up to isomorphism unique unimodular, positive-definite, even lattice of rank 24 without roots, i.e. vectors of norm 2. The automorphism group \( O(\Lambda) \) is the Conway group \( Co_0 \). Since \( (V_\Lambda)_1 = \{ h(-1) \otimes e_0 | h \in \Lambda_C \} \cong \Lambda_C \), the inner automorphism group satisfies

\[
K = \{ e^{ha} | h \in \Lambda_C \}
\]

and is abelian. Moreover, \( \text{Aut}(V_\Lambda)/K \cong O(\Lambda) \), i.e. there is a short exact sequence

\[
1 \to K \to \text{Aut}(V_\Lambda) \to O(\Lambda) \to 1.
\]

This is because \( K \cap O(\hat{\Lambda}) = \text{Hom}(L, \{ \pm 1 \}) \) in the special case of the Leech lattice. By the above, every automorphism of \( V_\Lambda \) is of the form

\[
\phi_\eta(\nu)\sigma_h
\]

for a lift \( \phi_\eta(\nu) \) of some \( \nu \in O(\Lambda) \) with \( \sigma_h := e^{-2\pi i h a} \) for some \( h \in \Lambda_C \). The surjection \( \text{Aut}(V_\Lambda) \to O(\Lambda) \) in the short exact sequence is given by \( \phi_\eta(\nu)\sigma_h \mapsto \nu \).

It suffices to take \( \phi_\eta(\nu) = \hat{\nu} \) from the section \( O(\Lambda) \to O(\hat{\Lambda}) \) since any two lifts only differ by a homomorphism \( \Lambda \to \{ \pm 1 \} \), which can be absorbed into \( \sigma_h \). Moreover, since \( \sigma_h = \text{id} \) if and only if \( h \in \Lambda' = \Lambda \), it is enough to take \( h \in \Lambda_C/\Lambda \).

We now describe the conjugacy classes of \( \text{Aut}(V_\Lambda) \). For \( \nu \in O(\Lambda) \) let \( \pi_\nu = \frac{1}{|\nu|} \sum_{i=0}^{\nu-1} \nu^i \) denote the projection of \( \Lambda_C \) onto the elements of \( \Lambda_C \) fixed by \( \nu \). By Lemma 8.3 in [EMS20a] (see also Lemma 3.4 in [LS19]) the automorphism \( \phi_\eta(\nu)\sigma_h \) is conjugate to \( \phi_\eta(\nu)\sigma_{\pi_\nu(h)} \) for any \( h \in \Lambda_C \) and \( \phi_\eta(\nu) \) and \( \sigma_{\pi_\nu(h)} \) commute.

In [MS19] all automorphisms in \( \text{Aut}(V_\Lambda) \) were classified up to conjugacy. A similar result for arbitrary lattice vertex operator algebras is proved in [HM20].

**Proposition 5.1** (MS19, Proposition 2.2). Let \( N \) denote a set of representatives for the conjugacy classes of \( O(\Lambda) \cong Co_0 \). For \( \nu \in N \) let \( H_\nu \) denote a set of representatives for the orbits of the action of the centraliser \( C_{O(\Lambda)}(\nu) \) on \( \pi_\nu(\Lambda_C) / \pi_\nu(\Lambda) \).

Then \( (\nu, h) \mapsto \hat{\nu}\sigma_h \) (with a fixed section \( O(\Lambda) \to O(\hat{\Lambda}), \nu \mapsto \hat{\nu} \)) defines a bijection from the set \( Q := \{ (\nu, h) | \nu \in N, h \in H_\nu \} \) to the conjugacy classes of \( \text{Aut}(V_\Lambda) \).

We also describe the conjugacy classes in \( \text{Aut}(V_\Lambda) \) of a given finite order \( n \). First note that a standard lift \( \phi_\eta(\nu) \) of \( \nu \) has order \( m := |\nu| \) if \( m \) is odd or if \( m \) is even and \( (\alpha, \nu^{m/2} \alpha) \in 2\mathbb{Z} \) for all \( \alpha \in \Lambda \) and order \( 2m \) otherwise. In the latter case we say that \( \nu \) exhibits order doubling. Then \( \phi_\eta(\nu)\epsilon_\alpha = (-1)^m (\pi_\nu(\alpha), \pi_\nu(\alpha)) \epsilon_\alpha = (-1)^{(\alpha, \nu^{m/2} \alpha)} \epsilon_\alpha \) for all \( \alpha \in \Lambda \) (Note that \( \alpha \mapsto m (\pi_\nu(\alpha), \pi_\nu(\alpha)) + 2\mathbb{Z} = (\alpha, \nu^{m/2} \alpha) + 2\mathbb{Z} \) defines a homomorphism \( \Lambda \to 2\mathbb{Z} \)).

Given a standard lift \( \phi_\eta(\nu) \) that exhibits order doubling there exists a vector \( s_\nu \in (1/2m)\Lambda' \) defining an inner automorphism \( \sigma_{s_\nu} = e^{-2\pi i (\nu, s_\nu)} \) of order \( 2m \) such that \( \phi_\eta(\nu)\sigma_{s_\nu} \) has order \( m \). (If \( \nu \) does not exhibit order-doubling, then we set \( s_\nu = 0 \).) Then the order of an automorphism \( \phi_\eta(\nu)\sigma_{s_\nu + f} \) for \( f \in \Lambda \otimes \mathbb{Q} \) is given by \( \text{lcm}(m, k) \) where \( k \) is minimal in \( \mathbb{Z}_{>0} \) such that \( kf \) is in \( \Lambda \) or equivalently in the fixed-point lattice \( \Lambda' \).
For convenience, we define the $s_\nu$-shifted action of $C_{O(\Lambda)}(\nu)$ on $\pi_\nu(\Lambda_C)$ by
$$\tau.f = \tau f + (\tau - \text{id})s_\nu$$
for all $\tau \in C_{O(\Lambda)}(\nu)$ and $f \in \pi_\nu(\Lambda_C)$. The following result is immediate:

**Proposition 5.2.** A complete system of representatives for the conjugacy classes of automorphisms in $\text{Aut}(V_\Lambda)$ of order $n \in \mathbb{Z}_{>0}$ consists of the $\nu\sigma_\nu + f$ where

1. $\nu$ is from the representatives in $N \subset O(\Lambda)$ of order $n$ dividing $n$,
2. $f$ is from the orbit representatives of the $s_\nu$-shifted action of $C_{O(\Lambda)}(\nu)$ on $(\Lambda^\nu/n)/\pi_\nu(\Lambda)$ such that $\text{lcm}(m, |\sigma_f|) = n$.

We conclude this section by recalling some results on the twisted modules of lattice vertex operator algebras. For a standard lift $\phi_\nu(\nu)$ the irreducible $\nu$-twisted modules of a lattice vertex operator algebra $V_L$ are described in [DL96, BK04]. Together with the results in Section 5 of [Li96] this allows us to describe the irreducible $g$-twisted $V_L$-modules for all finite-order automorphisms $g \in \text{Aut}(V_L)$.

For simplicity, let $L$ be unimodular. Then $V_L$ is holomorphic and there is a unique irreducible $g$-twisted $V_L$-module $V_L(g)$ for each $g \in \text{Aut}(V_L)$ of finite order. Let $g = \phi_\nu(\nu)\sigma_h$ for some standard lift $\phi_\nu(\nu)$ and $\sigma_h = e^{-2\pi i/h}$ for some $h \in \pi_\nu(L \otimes_{\mathbb{Z}} \mathbb{Q})$. Then
$$V_L(g) = M(1)[\nu] \otimes e_h \mathbb{C}[\pi_\nu(L)] \otimes \mathbb{C}^{d(\nu)}$$
with twisted Heisenberg module $M(1)[\nu]$, grading by the lattice coset $h + \pi_\nu(L)$ and defect $d(\nu) \in \mathbb{Z}_{>0}$.

Assume that $\nu$ has order $m$ and cycle shape $\sum_{t|\nu} b_t \nu$ with $b_t \in \mathbb{Z}$, i.e. the extension of $\nu$ to $L_C$ has characteristic polynomial $\prod_{t|\nu} (x^t - 1)^{b_t}$. Then the conformal weight of $V_L(g)$ is given by
$$\rho(V_L(g)) = \frac{1}{24} \sum_{t|\nu} b_t \left( t - \frac{1}{2} \right) + \min_{\alpha \in h + \pi_\nu(L)} \langle \alpha, \alpha \rangle / 2 \geq 0,$$
where $\rho_\nu := \frac{1}{24} \sum_{t|\nu} b_t \left( t - \frac{1}{2} \right)$ is called the *vacuum anomaly* of $V_L(g)$ [DL96]. Note that $\rho_\nu$ is positive for $\nu \neq \text{id}$.

6. **A New Proof of Schellekens’ List**

In this section we give a simpler proof of Schellekens’ list of possible weight-one Lie algebras of strongly rational, holomorphic vertex operator algebras of central charge 24. The proof uses [Theorem 4.7] namely that all strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with non-zero weight-one space $V_1$ can be obtained as orbifold constructions from the Leech lattice vertex operator algebra $V_\Lambda$ associated with generalised deep holes in $\text{Aut}(V_\Lambda)$.

First, based on this result, we derive a couple of simple identities that must be satisfied by any such weight-one Lie algebra. For convenience we include equation (1), which was already used in the proof of [Theorem 4.7].

**Lemma 6.1.** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_1$ semisimple and affine structure $\mathfrak{g}_1,k_1, \ldots, \mathfrak{g}_r,k_r$. Then

1. $h_\gamma^\Lambda/k_i = (\dim(V_1) - 24)/24$
2. $\text{rk}(\Lambda^\nu) = \text{rk}(V_1)$,
3. $|\nu| \mid \text{lcm}(\{l_ih_\gamma^\Lambda\}_{i=1}^r)$,
4. $1/(1 - \rho_\nu) = \text{lcm}(\{l_i\})_{i=1}^r$.

**Proof.**
Recall that $\rho_\nu$ denotes the vacuum anomaly of $\nu$ and only depends on the cycle shape of $\nu$. Also recall the lacing numbers $l_i$ and dual Coxeter numbers $h_i^\vee$.

**Proof.** By Proposition 4.3 and Theorem 4.7 there must be a generalised deep hole $g \in \text{Aut}(V_\Lambda)$ of order $n = |\sigma_\nu|$ such that $V_{\Lambda_1}^{\text{arb}(g)} \cong V$. As explained in Section 5 $g$ projects to an automorphism $\nu \in O(\Lambda)$ of order dividing $n$. The order $n = \text{lcm}(\{l_i h_i^\vee\}_{i=1}^r)$ of $\sigma_\nu$ was determined in Proposition 4.1. This implies (6.1).

Since the Leech lattice $\Lambda$ has no roots, $(V_{\Lambda_1}^\vee)_1 = \langle h(-1) \otimes e_0 \mid h \in \pi_\nu(\Lambda_C) \rangle$, which is an abelian Lie algebra of rank equal to $\dim(\pi_\nu(\Lambda_C)) = \text{rk}(\Lambda^\vee)$, the rank of the fixed-point lattice. The rank condition in the definition of generalised deep hole states that $\text{rk}(V_\nu^\vee) = \text{rk}(V_{\Lambda_1}^{\text{arb}(g)})$ so that (6.1) holds.

A generalised deep hole $g$ is also extremal, i.e. the upper bound in the dimension formula is attained. This bound simplifies for the Leech lattice vertex operator $V_\nu$ because $(V_{\Lambda_1}^\vee)_1$ is isomorphic to $\Lambda_C$, on which $g$ acts as $\nu$. Suppose $\nu$ has order $m$ and cycle shape $\prod_{i\in \mathbb{N}} l_i^{b_i}$. Then

$$\dim((V_{\Lambda_1}^{\text{arb}(g)})_1) - 24 = \sum_{d|n} c_n(d) \dim(V_\nu^{g^d}) = n \sum_{i|n} \frac{h_i^\vee}{l_i} = 24 n (1 - \rho_\nu)$$

with vacuum anomaly $\rho_\nu$. In the second step we used exactly the same argument as in the proof of Theorem 3.3. With equation (1) this implies

$$\frac{h_i^\vee}{l_i} = \frac{\dim((V_{\Lambda_1}^{\text{arb}(g)})_1) - 24}{24} = n (1 - \rho_\nu)$$

for all $i = 1, \ldots, r$, which entails item (6.1).

It is straightforward to list all solutions, i.e. pairs of affine structures and automorphisms of the Leech lattice $\Lambda$, to the equations in Lemma 6.1.

**Proposition 6.2.** There are exactly 82 pairs of affine structures and conjugacy classes in $O(\Lambda)$ satisfying the four equations in Lemma 6.1. These are the 69 semisimple cases on Schellekens’ list [Sch93], each with one of 11 conjugacy classes in $O(\Lambda)$ (see Table 2), plus the 13 spurious cases listed in Table 1.

| $\nu \in O(\Lambda)$ | $\rho_\nu$ | $n$ | Aff. Struc. |
|----------------------|-------------|-----|-------------|
| $6^4$                | $35/36$     | 6   | $D_{4,36}$  |
| $4^6$                | $15/16$     | 4   | $A_{3,16}^5$|
|                     | $8$         |     | $C_{3,8} A_{3,8}$ |
| $3^8$                | $8/9$       | 6   | $D_{4,9} A_{4,3}$|
|                     | $12$        |     | $G_{2,3}$   |
| $2^{4,4}$            | $7/8$       | 4   | $A_{3,8}^2 A_{2,4}^2$ |
|                     | $8$         |     | $C_{3,4} A_{2,2}$ |
| $1^{2} 2^{2} 3^{2} 6^2$ | $5/6$      | 6   | $D_{4,6} B_{2,3}^2$|
|                     | $4$         |     | $A_{3,4} A_{1,2}^6$ |
| $2^{12}$             | $3/4$       | 8   | $D_{5,4} A_{3,2} A_{4,1}$|
|                     | $8$         |     | $C_{3,2} A_{1,1}^3$ |
|                     | $8$         |     | $C_{2,2} A_{3,2}$ |

Table 1. 13 spurious cases in Proposition 6.2
Note that there is no affine structure that appears in more than one pair.

**Proof.** The 221 affine structures satisfying equation (1) are listed in Table 3. The 167 conjugacy classes of $O(\Lambda) = \text{Co}_0$ are well-known (see, e.g., \[CCN+88\]). It is now a purely combinatorial problem to find the pairs consisting of an affine structure $g_1 \cdot \Lambda \ldots g_\nu \cdot \Lambda$, and a conjugacy class $\nu \in O(\Lambda)$ satisfying items (6.1) to (6.1).

For example, since $\text{rk}(V_1) > 0$, item (6.1) implies that $\nu$ has non-trivial fixed-point lattice, leaving only 72 conjugacy classes in $O(\Lambda)$, and item (6.1) implies that $\rho(\nu) < 1$, which further reduces the possible conjugacy classes to 50. Also, since $\text{rk}(\Lambda^\nu)$ is always even, which is a special property of the Leech lattice $\Lambda$, all affine structures with $\text{rk}(V_1)$ odd are eliminated. Moreover, as we mentioned in the proof of Proposition 4.1 if the order $n$ is odd, then $V_1$ can only contain simple ideals $A_t$ with $t$ even.

In the following we rule out the 13 spurious cases. So far, we have only considered the compatibility of the affine structure with the projection of the generalised deep hole to $O(\Lambda)$. To use the full strength of Theorem 4.7 we classify (see Proposition 5.1 and Proposition 5.2) all possible generalised deep holes in $\text{Aut}(V_\Lambda)$ with given projection to $O(\Lambda)$ and order $n$ and show that none exist whose corresponding orbifold constructions have the affine structures in Table 1.

**Theorem 6.3.** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_1 \neq \{0\}$. Then $V_1$ is isomorphic to one of the 70 non-zero Lie algebras in Table 1 of \[Sch93\], each uniquely specifying the affine structure of $(\Lambda)$, and $V$ is isomorphic to $V_\Lambda^{\text{orb}(g)}$ for a generalised deep hole $g \in \text{Aut}(V_\Lambda)$, projecting to one of 11 conjugacy classes in $O(\Lambda) = \text{Co}_0$.

In Table 2 we list the 69 semisimple cases together with some properties of the corresponding generalised deep holes in $\text{Aut}(V_\Lambda)$. The 11 conjugacy classes in $O(\Lambda)$ are exactly those given in Table 4 (and Tables 5 to 15) in \[Höh17\] where they arise in a very different approach.

In [MS19] a generalised deep hole for each weight-one Lie algebra is explicitly listed, thus providing a uniform proof of the existence of a strongly rational, holomorphic vertex operator algebra of central charge 24 for each of the 71 Lie algebras on Schellekens' list.

**Proof.** Given Lemma 6.1 and Proposition 6.2 it remains to eliminate the 13 potential spurious cases in Table 1. Using Proposition 5.1 and Proposition 5.2 we classify the conjugacy classes of automorphisms $g \in \text{Aut}(V_\Lambda)$ of order $n$ projecting to the listed conjugacy class $\nu \in O(\Lambda)$. Then we show that either none of these classes $g$ is a generalised deep hole or that it is a generalised deep hole corresponding to one of the 69 semisimple Lie algebras on Schellekens' list.

The computationally most challenging part is the determination of the orbits of the action of $\text{Co}_0(\nu)$ on $(\Lambda^\nu/n)/\pi(\Lambda)$, which has $\text{rk}(\nu)/((\Lambda^\nu)/\Lambda^\nu)$ elements. This is performed on the computer using Magma [BCP97]. For the cases considered, the computations take between seconds and a couple of minutes on a standard desktop computer.

In order to prove that an automorphism in $\text{Aut}(V_\Lambda)$ is not a generalised deep hole (see Definition 4.6) we show that it does not have type 0 (by computing the conformal weight $\rho(V_\Lambda(g))$, see Section 5), is not extremal or does not satisfy the rank condition (2).

For the latter, we assume that the rank condition is fulfilled, which is equivalent to $(V_\Lambda^g)_1$ being a Cartan subalgebra of the semisimple Lie algebra $(V_\Lambda^{\text{orb}(g)})_1$, so that the action of the zero modes of $(V_\Lambda^g)_1$ on $(V_\Lambda^{\text{orb}(g)})_1$ yields the root space.
decomposition of \((V^\text{orb}(g))_1\). The action of \((V^g)_1\) on the irreducible \(g\)-twisted \(V\)-
module \(V(g)\), \(i \in \mathbb{Z}_n\), follows directly from the explicit construction in [DL96, BKKH] (see Section 5). We then consider the inclusions

\[
(V^g)_1 \oplus \bigoplus_{i \in \mathbb{Z}_n} (V(g))_1 \subseteq (V^\text{orb}(g))_1 \subseteq (V^g)_1 \oplus \bigoplus_{i \in \mathbb{Z}_n} (V(g))_1,
\]

which allow us to compute a sub- and superset of the root system of \((V^\text{orb}(g))_1\), including the lengths with respect to the unique non-degenerate, invariant bilinear form \(\langle \cdot, \cdot \rangle\) on \(V^\text{orb}(g)\) normalised such that \(\langle 1, 1 \rangle = -1\). This leads to a contradiction if the sub- or superset is not compatible with the affine structure, for example, if the root spaces would have to have dimension greater than 1.

Sufficient and necessary criteria for extremality can be derived from the dimension formula (see, e.g., Proposition 4.9 in MIS99). For instance, if \(g\) is extremal, then \(\rho(V(g)) \geq 1\) for all \(i \in \mathbb{Z}_n\) with \((i, n) = 1\).

We compute that in \(\text{Aut}(V)\):

1. There are exactly one conjugacy class of order 6 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(6^4\) but its type is not 0.
2. There is exactly one conjugacy class of order 4 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(4^8\) but its type is not 0.
3. There are exactly six conjugacy classes of order 8 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(4^4\). Five of these do not have type 0, and one has type 0 but does not satisfy the rank condition.
4. There is exactly one conjugacy class of order 3 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(3^8\) but its type is not 0.
5. There are exactly two conjugacy classes of order 6 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(3^8\) but their types are not 0.
6. There are exactly eight conjugacy classes of order 12 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(3^8\) but their types are not 0.
7. There are exactly three conjugacy classes of order 4 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(2^4 4^4\). Two of these do not have type 0, and one has type 0 but does not satisfy the rank condition.
8. There are exactly 15 conjugacy classes of order 4 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(2^4 4^4\). 13 of these do not have type 0, and two have type 0 but do not satisfy the rank condition.
9. There are exactly 13 conjugacy classes of order 6 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(1^2 2^2 3^2 6^2\). Nine of these do not have type 0, two have type 0 but do not satisfy the rank condition, and the two remaining ones are excluded below.
10. There are exactly seven conjugacy classes of order 4 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(2^2 12\). Five of these do not have type 0, and two have type 0 but do not satisfy the rank condition.
11. There are exactly 56 conjugacy classes of order 8 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(2^2 12\). 50 of these do not have type 0, five have type 0 but do not satisfy the rank condition, and one further case has type 0 but is not extremal since \(\rho(V(g)) = 7/8 < 1\).

It remains to study the two remaining conjugacy classes of order 6 and type 0 projecting to the conjugacy class in \(O(\Lambda)\) with cycle shape \(1^2 2^2 3^2 6^2\) from item 9.

The first conjugacy class is extremal. However, it is not difficult to see that \(V^\text{orb}(g)\) cannot have affine structure \(D_{1,6}B_{2,3}^2\) (nor can it have affine structure \(A_{5,6}B_{2,3}A_{1,2}\), and so it cannot satisfy the rank condition).
The second conjugacy class satisfies the rank condition and is extremal. Hence, it is a generalised deep hole. However, it is again easy to see that $V_{\Lambda}$ cannot have affine structure $D_4, B_2^2, 3$. In fact, this conjugacy class can only yield $(V_{\hat{\Lambda}})^1 \cong A_5, B_2, A_{1,2}$, corresponding to case 8 on Schellekens’ list.

We remark that in this work we only determine the 69 possible affine structures, i.e. the affine vertex operator subalgebras $(V_{\hat{\Lambda}}) \cong \bigotimes_{i=1}^k L_{k_i}$, that can occur in a strongly rational, holomorphic vertex operator algebra of central charge 24. Schellekens, in contrast, also determines the possible decompositions of $V$ as a $(V_1)$-module (see Table 1 in [Sch93]). He obtains that this decomposition is unique up to an outer automorphism.

7. Towards a Uniform Uniqueness Proof

So far, the proof of the uniqueness of a strongly rational, holomorphic vertex operator algebra of central charge 24 with a given non-zero weight-one space is scattered over many publications [DM04a, LS19, LS15, LL20, EMS20b, LS20, KLL18].

A second potential application of Theorem 4.7 is a more uniform proof of this uniqueness statement. In this section we demonstrate this approach for 43 of the 70 non-zero weight-one Lie algebras on Schellekens’ list. While we believe that this method works in principle for all cases, we are currently restricted by computational limitations.

Another uniform statement of the uniqueness is given in [HL20] based on orbifold constructions not just associated with the Leech lattice but with all Niemeier lattices.

It was conjectured in [MS19], Conjecture 5.10, that the cyclic orbifold construction $g \mapsto V_{\Lambda}^{\text{orb}(g)}$ defines a bijection between the algebraic conjugacy classes of generalised deep holes $g \in \text{Aut}(V_{\Lambda})$ with $\text{rk}(V_{\Lambda}^{\text{orb}(g)}) > 0$ and the isomorphism classes of strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_1 \neq \{0\}$.

The surjectivity of this map follows directly from Theorem 4.7 while the injectivity is still open.

In the following we try to classify the generalised deep holes in $\text{Aut}(V_{\Lambda})$. If we can show that there is only one conjugacy class $g$ of generalised deep holes in $\text{Aut}(V_{\Lambda})$ whose corresponding orbifold construction has a certain weight-one Lie algebra $g$ on Schellekens’ list, then any strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_1 \neq \{0\}$ must be isomorphic to $V_{\Lambda}^{\text{orb}(g)}$, proving the uniqueness for this $g$. Moreover, we will have also added evidence to the injectivity in the conjecture of [MS19].

**Theorem 7.1.** Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with weight-one space $V_1 \neq \{0\}$. Then the Lie algebra structure of $V_1$ uniquely determines the vertex operator algebra $V$ up to isomorphism if $V_1$ is one of the 24 Lie algebras of rank 24 or corresponding to cases $2, \ldots, 14$ and $16, \ldots, 21$ on Schellekens’ list. Moreover, there is a unique conjugacy class of generalised deep holes in $\text{Aut}(V_{\Lambda})$ such that $V_{\Lambda}^{\text{orb}(g)} \cong V$ in these cases.

**Proof.** We begin with the case of $\text{rk}(V_1) = 24$. Then $V \cong V_{\Lambda}^{\text{orb}(g)}$ for some generalised deep hole $g \in \text{Aut}(V_{\Lambda})$ projecting to $\nu \in O(\Lambda)$ with $\text{rk}(\Lambda') = 24$. This means that $\nu = \text{id}$, i.e. $g$ is inner and conjugate to $\sigma_h = e^{-(2\pi i)h}$ for some $h \in \Lambda_C/\Lambda$. Recall that $g = \text{id}$ is a generalised deep hole by convention and this is the unique generalised deep hole $g$ such that $V_{\Lambda}^{\text{orb}(g)} \cong V_{\Lambda}$. In the following we assume that $g \neq \text{id}$ or equivalently that $V_1$ is semisimple.
By the dimension formula in [MS19], the extremality of $g$ implies $\rho(V_\Lambda(g)) \geq 1$. On the other hand, for $d \in \Lambda_C$, $\rho(V_\Lambda(\sigma_d)) = \min_{\alpha \in \Lambda + d} \langle \alpha, \alpha \rangle / 2$. Since the covering radius of the Leech lattice is $\sqrt{2}$ [CPSS2], the conformal weight can be at most 1 and it is 1 by definition if and only if $d$ is a deep hole of the Leech lattice.

Hence, $h$ must be a deep hole. The deep holes of the Leech lattice $\Lambda$ were classified in [CPSS2] and it is shown that there are exactly 23 orbits of deep holes in $\Lambda_C/\Lambda$ under the action of $O(\Lambda) = C_{O(\Lambda)}(id)$. This shows that there are at most 24 conjugacy classes of generalised deep holes with $\text{rk}(\Lambda') = 24$ in $\text{Aut}(V_\Lambda)$, each one except for the identity represented by $\sigma_h$ where $h$ is a representative of a deep hole of $\Lambda$. It is not difficult to see that each of the 23 deep holes in fact defines a generalised deep hole in this way and based on [CSS2] it is proved in [MS19] that the orbifold constructions associated with these 23 generalised deep holes give exactly the vertex operator algebras associated with the 23 Niemeier lattices other than the Leech lattice. This proves the uniqueness of the 24 lattice cases on Schellekens’ list, i.e. the cases with $\text{rk}(V_1) = 24$.

For the weight-one Lie algebras of rank less than 24 we proceed by classifying the corresponding generalised deep holes on the computer using Magma [BCP97] similar to the proof of Theorem 6.3. Recall that it was shown in [MS19] that there is at least one generalised deep hole corresponding to each case on Schellekens’ list.

We compute that in $\text{Aut}(V_\Lambda)$:

1. There are exactly six conjugacy classes of order 10 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $2^410^2$. Four of these do not have type 0, and one has type 0 but does not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 4 on Schellekens’ list.

2. There are exactly four conjugacy classes of order 6 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $2^36^3$. Two of these do not have type 0, and one has type 0 but does not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 3 on Schellekens’ list.

3. There are exactly 156 conjugacy classes of order 18 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $2^63^6$. 135 of these do not have type 0, and four have type 0 but do not satisfy the rank condition. Of the remaining 17 classes 16 are not extremal. The remaining class must be a generalised deep hole corresponding to case 14 on Schellekens’ list.

4. There are exactly 25 conjugacy classes of order 8 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^22^41^28^2$. 21 of these do not have type 0, and three have type 0 but do not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 10 on Schellekens’ list.

5. There are exactly eight conjugacy classes of order 7 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^37^3$. Six of these do not have type 0, and one has type 0 but does not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 11 on Schellekens’ list.

6. As was already discussed in the proof of Theorem 6.3, there are exactly 13 conjugacy classes of order 6 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^22^33^26^2$. Nine of these do not have type 0, and three have type 0 but do not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 8 on Schellekens’ list.

7. There are exactly 194 conjugacy classes of order 12 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^22^33^26^2$. 172 of these do not have type 0, and eight have type 0 but do not satisfy the rank condition. Of the remaining 14 classes 13 are not extremal. The remaining class must be a generalised deep hole corresponding to case 21 on Schellekens’ list.
(8) There are exactly six conjugacy classes of order 5 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^45^4$. Four of these do not have type 0, and one has type 0 but does not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 9 on Schellekens’ list.

(9) There are exactly 54 conjugacy classes of order 10 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^45^4$. 47 of these do not have type 0, and one has type 0 but does not satisfy the rank condition. Of the remaining six classes five are not extremal. The remaining class must be a generalised deep hole corresponding to case 20 on Schellekens’ list.

(10) There are exactly nine conjugacy classes of order 4 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^42^4$. Six of these do not have type 0, and two have type 0 but do not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 7 on Schellekens’ list.

(11) There are exactly 94 conjugacy classes of order 8 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^42^4$. 82 of these do not have type 0, and four have type 0 but do not satisfy the rank condition. Of the remaining eight classes six are not extremal. The two remaining classes must be generalised deep holes corresponding to cases 18 and 19 on Schellekens’ list.

(12) There is exactly one conjugacy class of order 2 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $2^2$. This class must be a generalised deep hole corresponding to case 2 on Schellekens’ list.

(13) There are exactly 14 conjugacy classes of order 6 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $2^2$. Eight of these do not have type 0, four have type 0 and satisfy the rank condition but are not extremal. The two remaining classes must be generalised deep holes corresponding to cases 12 and 13 on Schellekens’ list.

(14) There are exactly four conjugacy classes of order 3 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^63^6$. Two of these do not have type 0, and one has type 0 but does not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 5 on Schellekens’ list.

(15) There are exactly 24 conjugacy classes of order 6 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^63^6$. 18 of these do not have type 0, and one has type 0 but does not satisfy the rank condition. Of the remaining five classes four are not extremal. The remaining class must be a generalised deep hole corresponding to case 17 on Schellekens’ list.

(16) There are exactly three conjugacy classes of order 2 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^2$. One of these does not have type 0, and one has type 0 but does not satisfy the rank condition. The remaining class must be a generalised deep hole corresponding to case 1 on Schellekens’ list.

(17) There are exactly 14 conjugacy classes of order 4 projecting to the conjugacy class in $O(\Lambda)$ with cycle shape $1^2$. Nine of these do not have type 0, and one has type 0 but does not satisfy the rank condition. Of the remaining four classes three are not extremal. The remaining class must be a generalised deep hole corresponding to case 16 on Schellekens’ list. □

For the other cases on Schellekens’ list determining the conjugacy classes in $\text{Aut}(V_\Lambda)$ and specifically the orbit representatives of the action of $C_{O(\Lambda)}(\nu)$ on $(\Lambda^\nu/n)/\pi_{\nu}(\Lambda)$ takes prohibitively long on a standard desktop computer.
### Appendix A. Tables

**Table 2.** The 69 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_1$ semisimple and the corresponding generalised deep holes in $\nu \in \text{Aut}(V_{\Lambda})$ (continued on next page).

| $\nu \in O(\Lambda)$ | Rk. | $n$ | Dim. | Aff. Struct. | [Höh17] | [Sch93] |
|----------------------|-----|-----|------|--------------|--------|--------|
| $1^{24}$             | 24  |     |      |              |        |        |
|                       | 46  | 1128| 24   | $D_{24,1}$   | $A1$   | [70]  |
|                       | 30  | 744 | 24   | $D_{16,1}E_{8,1}$ | $A2$ | [69]  |
|                       | 30  | 744 |      | $E_{8,1}^3$   | $A3$   | [68]  |
|                       | 25  | 624 | 24   | $A_{24,1}$    | $A4$   | [67]  |
|                       | 22  | 552 | 24   | $D_{12,1}^2$  | $A5$   | [66]  |
|                       | 18  | 456 |      | $A_{17,1}E_{7,1}$ | $A6$ | [65]  |
|                       | 18  | 456 |      | $D_{10,1}E_{7,1}^2$ | $A7$ | [64]  |
|                       | 16  | 408 | 24   | $A_{15,1}D_{9,1}$ | $A8$ | [63]  |
|                       | 14  | 360 | 16   | $D_{8,1}^3$   | $A9$   | [61]  |
|                       | 13  | 336 | 16   | $A_{12,1}^3$  | $A10$  | [60]  |
|                       | 12  | 312 | 16   | $A_{11,1}D_{7,1}E_{6,1}$ | $A11$ | [59]  |
|                       | 12  | 312 | 16   | $E_{6,1}^4$   | $A12$  | [58]  |
|                       | 10  | 264 | 16   | $A_{9,1}^2D_{6,1}$ | $A13$ | [55]  |
|                       | 10  | 264 | 16   | $D_{6,1}^4$   | $A14$  | [54]  |
|                       | 9   | 240 | 16   | $A_{4,1}^2$   | $A15$  | [51]  |
|                       | 8   | 216 | 16   | $A_{8,1}^2D_{4,1}^2$ | $A16$ | [49]  |
|                       | 7   | 192 | 8    | $A_{6,1}^2$   | $A17$  | [46]  |
|                       | 6   | 168 | 8    | $A_{4,1}^2D_{4,1}$ | $A18$ | [43]  |
|                       | 6   | 168 | 8    | $D_{4,1}^6$   | $A19$  | [42]  |
|                       | 5   | 144 | 8    | $A_{2,1}^4$   | $A20$  | [37]  |
|                       | 4   | 120 | 8    | $A_{3,1}^2$   | $A21$  | [30]  |
|                       | 3   | 96  | 8    | $A_{1,1}^2$   | $A22$  | [24]  |
|                       | 2   | 72  | 4    | $A_{4,1}^{24}$ | $A23$ | [15]  |
|                       | 1   | 24  | 4    | $C_{24}^{24}$ | $A24$ | [1]   |
| $1^{8}2^8$           | 16  |     |      |              |        |        |
|                       | 30  | 384 |      | $E_{8,2}B_{8,1}$ | $B1$ | [62]  |
|                       | 22  | 288 |      | $C_{10,1}B_{6,1}$ | $B2$ | [56]  |
|                       | 18  | 240 |      | $C_{8,1}F_{4,1}^2$ | $B3$ | [52]  |
|                       | 18  | 240 |      | $E_{7,2}B_{5,1}F_{4,1}$ | $B4$ | [53]  |
|                       | 16  | 216 |      | $D_{9,2}A_{7,1}$ | $B5$ | [50]  |
|                       | 14  | 192 |      | $D_{8,2}B_{2,1}^2$ | $B6$ | [47]  |
|                       | 14  | 192 |      | $C_{8,1}^2B_{4,1}$ | $B7$ | [48]  |
|                       | 12  | 168 |      | $E_{5,2}C_{5,1}A_{5,1}$ | $B8$ | [44]  |
| $\nu \in \mathcal{O}(\Lambda)$ | Rk. | $n$ | Dim. | Aff. Struct. | Hohl7 | Sch93 |
|---|---|---|---|---|---|---|
| $1^82^8$ | 16 | 10 | 144 | $A_{9,2}A_{4,1}B_{5,1}$ | $B9$ | [40] |
| 10 | 144 | $D_{6,2}C_{4,1}B_{6,1}^2$ | $B10$ | [39] |
| 10 | 144 | $C_{4,1}^4$ | $B11$ | [38] |
| 8 | 120 | $A_{7,2}C_{4,1}^2A_{3,1}$ | $B12$ | [33] |
| 8 | 120 | $D_{6,2}^2A_{3,1}^2$ | $B13$ | [31] |
| 6 | 96 | $A_{4,2}^2B_{2,1}A_{2,1}^2$ | $B14$ | [26] |
| 6 | 96 | $D_{4,2}^2B_{2,1}^2$ | $B15$ | [25] |
| 4 | 72 | $A_{3,2}^2A_{1,1}^2$ | $B16$ | [16] |
| 2 | 48 | $A_{1,2}^{16}$ | $B17$ | [5] |
| 18 | 168 | $E_{7,3}A_{3,1}$ | $C1$ | [45] |
| 12 | 120 | $D_{7,3}A_{3,1}G_{2,1}$ | $C2$ | [34] |
| 12 | 120 | $E_{6,3}G_{2,1}^2$ | $C3$ | [32] |
| 9 | 96 | $A_{8,3}A_{2,1}^2$ | $C4$ | [27] |
| 6 | 72 | $A_{5,3}D_{4,3}A_{1,1}^2$ | $C5$ | [17] |
| 3 | 48 | $A_{6,3}^2$ | $C6$ | [6] |
| 46 | 300 | $B_{1,2}^{12}$ | $D1a$ | [57] |
| 22 | 156 | $B_{6,2}^2$ | $D1b$ | [41] |
| 14 | 108 | $B_{4,2}^{3,2}$ | $D1c$ | [29] |
| 10 | 84 | $B_{3,2}^4$ | $D1d$ | [23] |
| 6 | 60 | $B_{6,2}^6$ | $D1e$ | [12] |
| 2 | 36 | $A_{1,4}^{12}$ | $D1f$ | [2] |
| 18 | 132 | $A_{8,2}F_{4,2}$ | $D2a$ | [36] |
| 10 | 84 | $C_{4,2}^2A_{2,2}^2$ | $D2b$ | [22] |
| 6 | 60 | $D_{4,4}A_{2,2}^2$ | $D2c$ | [13] |
| 16 | 120 | $C_{7,2}A_{3,1}$ | $E1$ | [35] |
| 12 | 96 | $E_{6,4}B_{2,1}A_{2,1}$ | $E2$ | [28] |
| 8 | 72 | $A_{7,4}A_{1,1}^2$ | $E3$ | [18] |
| 8 | 72 | $D_{5,4}C_{3,2}A_{2,1}^2$ | $E4$ | [19] |
| 4 | 48 | $A_{4,4}^4A_{1,2}$ | $E5$ | [7] |
| 10 | 72 | $D_{6,5}A_{1,1}^2$ | $F1$ | [20] |
| 5 | 48 | $A_{4,5}^2$ | $F2$ | [9] |
| 12 | 72 | $C_{5,3}G_{2,2}A_{1,1}$ | $G1$ | [21] |
| 6 | 48 | $A_{5,6}B_{2,3}A_{1,2}$ | $G2$ | [8] |
| 6 | 48 | $A_{6,7}$ | $H1$ | [11] |
| 8 | 48 | $D_{5,8}A_{1,2}$ | $I1$ | [10] |
| 18 | 60 | $F_{6,4}A_{2,2}$ | $J1a$ | [14] |
| 6 | 36 | $D_{4,12}A_{2,6}$ | $J1b$ | [3] |
| 4 | 36 | $C_{4,10}$ | $K1$ | [4] |

Table 2. (continued)
| Dim. | Aff. Struct. | Dim. | Aff. Struct. | Dim. | Aff. Struct. |
|------|-------------|------|-------------|------|-------------|
| 25   | $A_{3.96}B_{2.72}$ | 32   | $A_{3.12}A_{2.9}A_{1.6}^3$ | 48   | $A_{3.5}C_{3.4}A_{1.2}$ |
| 25   | $G_{2.96}A_{2.72}A_{1.48}$ | 32   | $G_{2.12}B_{2.9}A_{1.6}^2$ | 48   | $A_{3.5}A_{1.6}^4$ |
| 25   | $B_{2.72}A_{1.48}^5$ | 32   | $G_{2.12}A_{1.6}^6$ | 48   | $A_{3.5}A_{1.2}$ |
| 25   | $A_{2.72}A_{1.48}^5$ | 32   | $B_{2.9}A_{1.6}^6$ | 48   | $B_{3.5}A_{1.2}$ |
| 26   | $A_{3.38}A_{2.30}A_{1.24}$ | 32   | $B_{2.9}A_{2.9}A_{1.6}^2$ | 48   | $B_{3.3}A_{1.2}$ |
| 26   | $G_{2.48}A_{1.24}^4$ | 32   | $A_{2.9}A_{1.6}^6$ | 48   | $B_{3.3}A_{1.2}$ |
| 26   | $B_{2.36}A_{1.24}^7$ | 32   | $A_{2.9}A_{1.6}^6$ | 48   | $B_{3.3}A_{1.2}$ |
| 26   | $B_{2.36}A_{1.36}^2$ | 36   | $B_{4.14}$ | 48   | $B_{3.3}A_{1.2}$ |
| 26   | $A_{2.36}A_{1.24}^6$ | 36   | $C_{4.10}$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $A_{4.40}A_{1.16}$ | 36   | $D_{4.12}A_{2.6}$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $B_{3.40}A_{1.16}^5$ | 36   | $A_{4.10}A_{1.4}^1$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $C_{3.32}A_{1.16}^2$ | 36   | $B_{3.10}A_{3.8}$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $A_{3.32}A_{1.16}^4$ | 36   | $C_{3.8}A_{3.8}$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $G_{2.32}A_{2.24}A_{1.16}$ | 36   | $B_{3.10}A_{1.4}^1$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $G_{2.24}A_{2.24}A_{1.16}^3$ | 36   | $A_{3.8}A_{1.4}^1$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $A_{1.24}^1A_{1.16}$ | 36   | $A_{3.8}A_{1.4}^1$ | 48   | $B_{3.3}A_{1.2}$ |
| 27   | $A_{1.16}^{1,16}$ | 36   | $A_{3.8}A_{2.6}A_{2.6}A_{1.14}$ | 48   | $B_{3.3}A_{1.2}$ |
| 28   | $D_{4.36}$ | 36   | $A_{3.8}A_{1.4}^1$ | 48   | $B_{3.3}A_{1.2}$ |
| 28   | $A_{3.24}B_{2.18}A_{1.12}$ | 36   | $G_{2.6}A_{1.6}$ | 48   | $A_{3.3}A_{1.2}$ |
| 28   | $G_{2.24}$ | 36   | $G_{2.6}B_{2.6}A_{1.4}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 28   | $G_{2.24}A_{2.18}A_{1.12}^2$ | 36   | $G_{2.6}B_{2.6}A_{1.4}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 28   | $B_{2.36}A_{2.18}A_{1.12}$ | 36   | $B_{2.6}A_{1.4}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 28   | $B_{2.18}A_{6.12}$ | 36   | $B_{2.6}A_{6.6}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 28   | $A_{2.18}A_{1.12}^4$ | 36   | $B_{2.6}A_{2.6}A_{1.6}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $A_{4.20}A_{1.18}^3$ | 36   | $A_{2.6}A_{1.4}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $B_{3.20}A_{1.18}^3$ | 40   | $D_{4.9}A_{1.3}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $C_{3.16}A_{1.8}^3$ | 40   | $G_{2.6}A_{1.3}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $A_{3.16}A_{1.8}^3$ | 40   | $B_{2.4}A_{2.4}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $A_{1.8}^1A_{1.8}$ | 48   | $A_{6.7}$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $G_{2.18}B_{2.12}A_{1.8}^2$ | 48   | $D_{5.8}A_{1.2}$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $G_{2.18}A_{2.12}^2$ | 48   | $B_{4.7}A_{4.2}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $B_{2.12}A_{1.8}^2$ | 48   | $C_{4.5}A_{1.2}$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $B_{2.12}A_{2.12}A_{1.8}^2$ | 48   | $A_{5.6}B_{2.3}A_{1.2}$ | 48   | $A_{3.3}A_{1.2}$ |
| 30   | $A_{1.8}^3$ | 48   | $D_{4.6}A_{2.4}A_{2.2}^4$ | 48   | $A_{3.3}A_{1.2}$ |
| 32   | $A_{4.15}A_{2.9}$ | 48   | $D_{4.9}B_{2.3}^3$ | 48   | $A_{3.3}A_{1.2}$ |
| 32   | $B_{3.15}A_{2.9}A_{1.6}$ | 48   | $B_{2.9}A_{4.2}A_{1.2}$ | 48   | $A_{3.3}A_{1.2}$ |
| 32   | $B_{3.12}A_{2.9}A_{1.6}$ | 48   | $A_{2.5}$ | 48   | $A_{3.3}A_{1.2}$ |
| 32   | $A_{3.12}G_{2.12}A_{1.6}$ | 48   | $A_{4.5}B_{3.5}A_{1.2}$ | 48   | $A_{3.3}A_{1.2}$ |

Table 3. The 221 solutions of equation (1) (continued on next page).
| Dim. | Aff. Struct. | Dim. | Aff. Struct. | Dim. | Aff. Struct. |
|------|-------------|------|-------------|------|-------------|
| 48   | $B_{2,3}A_{2,3}A_{1,1}^{10}$ | 84   | $C_{1,2}A_{2,2}^{4}$ | 192  | $D_{8,2}B_{4,1}^{2}$ |
| 48   | $A_{2,3}^{2}$   | 84   | $B_{1,2}^{4}$    | 192  | $C_{6,1}B_{4,1}^{4}$ |
| 48   | $A_{2,3}^{2}A_{1,1}^{4}$ | 96   | $A_{8,3}A_{2,1}^{2}$ | 192  | $A_{6,1}B_{1,1}^{2}$ |
| 48   | $A_{1,1}^{10}$  | 96   | $E_{6,4}A_{2,1}^{2}$ | 216  | $D_{8,2}A_{7,1}$ |
| 56   | $C_{3,3}G_{2,3}$ | 96   | $F_{1,3}A_{4,2}^{2}$ | 216  | $A_{7,1}^{2}D_{6,1}^{2}$ |
| 56   | $G_{3,3}^{2}$   | 96   | $A_{3,2}A_{1,1}^{2}$ | 240  | $C_{8,1}F_{1,1}^{2}$ |
| 60   | $F_{1,6}A_{2,2}$ | 96   | $D_{2,2}B_{1,2}^{3}$ | 240  | $E_{7,2}B_{5,1}F_{4,1}$ |
| 60   | $D_{3,4}A_{1,1}^{2}$ | 96   | $D_{2,4}B_{1,2}^{5}$ | 240  | $A_{3,1}^{3}$ |
| 60   | $B_{2,2}^{6}$   | 96   | $D_{2,4}B_{2,1}^{2}A_{1,1}$ | 264  | $A_{3,1}^{2}D_{6,1}$ |
| 60   | $B_{2,2}^{2}A_{1,1}^{4}$ | 108  | $B_{2,2}^{3}$   | 264  | $D_{4,1}^{4}$ |
| 72   | $D_{6,5}A_{2,1}^{2}$ | 120  | $C_{7,2}A_{3,1}$ | 288  | $C_{10,1}B_{6,1}$ |
| 72   | $A_{7,2}A_{1,1}^{4}$ | 120  | $D_{7,2}A_{1,1}G_{2,1}$ | 300  | $B_{12,2}$ |
| 72   | $A_{5,3}G_{2,2}A_{1,1}$ | 120  | $E_{6,3}C_{2,1}$ | 312  | $A_{11,1}D_{7,1}E_{6,1}$ |
| 72   | $D_{5,4}A_{2,1}^{2}$ | 120  | $E_{6,3}G_{2,1}$ | 312  | $E_{6,1}^{4}$ |
| 72   | $A_{5,3}G_{2,2}^{2}A_{1,1}$ | 120  | $E_{6,3}G_{2,1}$ | 336  | $A_{12,1}^{2}$ |
| 72   | $D_{4,3}A_{1,1}^{2}$ | 120  | $E_{6,3}G_{2,1}$ | 360  | $D_{8,1}$ |
| 72   | $D_{4,3}G_{2,2}^{2}A_{1,1}$ | 120  | $E_{6,3}G_{2,1}$ | 384  | $E_{8,2}B_{8,1}$ |
| 72   | $D_{3,4}A_{2,2}G_{2,2}A_{1,1}$ | 120  | $E_{6,3}G_{2,1}$ | 408  | $A_{15,1}D_{9,1}$ |
| 72   | $D_{3,4}A_{2,2}A_{1,1}^{10}$ | 120  | $A_{7,2}A_{1,1}^{2}$ | 456  | $A_{17,1}E_{7,1}$ |
| 72   | $C_{3,2}A_{3,1}^{4}$ | 120  | $A_{7,2}A_{3,1}G_{2,1}$ | 456  | $D_{10,1}E_{7,1}^{2}$ |
| 72   | $C_{3,2}A_{3,1}^{2}$ | 120  | $A_{7,2}A_{3,1}G_{2,1}$ | 552  | $D_{12,1}^{2}$ |
| 72   | $C_{3,2}A_{3,1}^{2}$ | 120  | $A_{7,2}A_{3,1}G_{2,1}$ | 624  | $A_{24,1}$ |
| 72   | $C_{3,2}A_{3,1}^{2}$ | 120  | $A_{7,2}A_{3,1}G_{2,1}$ | 744  | $D_{16,1}E_{8,1}$ |
| 72   | $C_{3,2}A_{3,1}^{2}$ | 120  | $A_{7,2}A_{3,1}G_{2,1}$ | 744  | $E_{8,1}^{3}$ |
| 72   | $C_{3,2}A_{3,1}^{2}$ | 132  | $A_{8,2}F_{1,2}$ | 1128 | $D_{24,1}$ |
| 72   | $C_{3,2}A_{3,2}A_{1,1}^{12}$ | 144  | $A_{9,2}A_{1,1}B_{3,1}$ |  | |
| 72   | $C_{3,2}A_{3,2}A_{1,1}^{12}$ | 144  | $D_{6,2}A_{4,1}B_{3,1}$ |  | |
| 72   | $C_{3,2}A_{3,2}A_{1,1}^{12}$ | 144  | $C_{4,1}^{1}$ |  | |
| 72   | $A_{3,2}A_{1,1}^{10}$ | 144  | $A_{3,1}^{4}$ |  | |
| 72   | $A_{3,2}A_{1,1}^{10}$ | 144  | $A_{3,1}^{4}$ |  | |
| 72   | $A_{2,3}A_{1,1}^{10}$ | 156  | $B_{6,2}$ |  | |
| 72   | $A_{3,2}A_{2,2}A_{1,1}$ | 168  | $E_{7,3}A_{5,1}$ |  | |
| 72   | $A_{3,2}A_{2,2}A_{1,1}$ | 168  | $E_{6,2}C_{5,1}A_{5,1}$ |  | |
| 72   | $G_{3,2}A_{2,1}^{10}$ | 168  | $A_{4,1}^{4}D_{4,1}$ |  | |
| 72   | $A_{2,1}^{10}$ | 168  | $D_{8,1}^{2}$ |  | |

Table 3. (continued)
References

[BCP97] Wieb Bosma, John Cannon and Catherine Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4):235–265, 1997. [http://magma.maths.usyd.edu.au]

[BK04] Bojko N. Bakalov and Victor G. Kac. Twisted modules over lattice vertex algebras. In Heinz-Dietrich Doebner and Vladimir K. Dobrev, editors, Lie theory and its applications in physics V, pages 3–26. World Scientific, 2004. [arXiv:math/0402315v3 [math.QA]].

[Bor85] Richard E. Borcherds. The Leech lattice and other lattices. Ph.D. thesis, University of Cambridge, 1985. [arXiv:math/9911195v1 [math.NT]].

[Bor86] Richard E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster. Proc. Nat. Acad. Sci. U.S.A., 83(10):3068–3071, 1986. [http://math.berkeley.edu/~reb/papers/va/va.pdf].

[CCN+85] John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker and Robert A. Wilson. Atlas of finite groups: Maximal subgroups and ordinary characters for simple groups. Oxford University Press, 1985. With computational assistance from J. G. Thackray.

[CM16] Scott Carnahan and Masahiko Miyamoto. Regularity of fixed-point vertex operator subalgebras. [arXiv:1603.05645v4 [math.RT]], 2016.

[CPS82] John H. Conway, Richard A. Parker and Neil J. A. Sloane. The covering radius of the Leech lattice. Proc. Roy. Soc. London Ser. A, 380(1779):261–290, 1982.

[CS82] John H. Conway and Neil J. A. Sloane. Twenty-three constructions for the Leech lattice. Proc. Roy. Soc. London Ser. A, 381(1781):275–283, 1982.

[CS99] John H. Conway and Neil J. A. Sloane. Sphere packings, lattices and groups, volume 290 of Grundlehren Math. Wiss. Springer, 3rd edition, 1999.

[DL96] Chongying Dong and James I. Lepowsky. The algebraic structure of relative vertex operators. J. Pure Appl. Algebra, 110(3):259–295, 1996. [arXiv:q-alg/9604022v1].

[DL97] Chongying Dong, Haisheng Li and Geoffrey Mason. Regularity of rational vertex operator algebras. Adv. Math., 132(1):148–166, 1997. [arXiv:alg/9508018v1].

[DL00] Chongying Dong, Haisheng Li and Geoffrey Mason. Modular-invariance of trace functions in orbifold theory and generalized moonshine. Commun. Math. Phys., 214:1–56, 2000. [arXiv:q-alg/9703016v2].

[DM97] Chongying Dong and Geoffrey Mason. On quantum Galois theory. Duke Math. J., 86(2):305–321, 1997. [arXiv:hep-th/9412037v1].

[DM04a] Chongying Dong and Geoffrey Mason. Holomorphic vertex operator algebras of small central charge. Pacific J. Math., 213(2):253–266, 2004. [arXiv:math/0203005v1 [math.QA]].

[DM04b] Chongying Dong and Geoffrey Mason. Rational vertex operator algebras and the effective central charge. Int. Math. Res. Not., 2004(56):2989–3008, 2004. [arXiv:math/0201318v1 [math.QA]].

[DM06] Chongying Dong and Geoffrey Mason. Integrability of $C_2$-cofinite vertex operator algebras. Int. Math. Res. Not., 2006:1–15, Art. ID 80468, 2006. [arXiv:math/0601569v1 [math.QA]].

[DN99] Chongying Dong and Kiyokazu Nagatomo. Automorphism groups and twisted modules for lattice vertex operator algebras. In Naihuan Jing and Kailash C. Misra, editors, Recent developments in quantum affine algebras and related topics, volume 248 of Contemp. Math., pages 117–133. Amer. Math. Soc., 1999. [arXiv:math/9808088v1 [math.QA]].

[DR17] Chongying Dong and Li Ren. Representations of the parafermion vertex operator algebras. Adv. Math., 315:88–101, 2017. [arXiv:1411.6058v5 [math.QA]].

[Ebe13] Wolfgang Ebeling. Lattices and Codes. Adv. Lectures Math. Springer, 3rd edition, 2013. A course partially based on lectures by Friedrich Hirzebruch.

[EMS20a] Jethro van Ekeren, Sven Möller and Nils R. Scheithauer. Construction and classification of holomorphic vertex operator algebras. J. Reine Angew. Math., 759:61–99, 2020. [arXiv:1507.08114v3 [math.RT]].

[EMS20b] Jethro van Ekeren, Sven Möller and Nils R. Scheithauer. Dimension formulae in genus zero and uniqueness of vertex operator algebras. Int. Math. Res. Not., 2020(7):2145–2204, 2020. [arXiv:1704.00478v3 [math.QA]].

[FHL93] Igor B. Frenkel, Yi-Zhi Huang and James I. Lepowsky. On Axiomatic Approaches to Vertex Operator Algebras and Modules, volume 104 of Mem. Amer. Math. Soc. Amer. Math. Soc., 1993.
J. Pure Appl. Math. 94(1996), 123–168.

Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. Duke Math. J., 66(1):123–168, 1992.

Gerald Höhn and Sven Möller. Systematic orbifold constructions of Schellekens’ vertex operator algebras from Niemeier lattices. In preparation, 2020.

Gerald Höhn. On the genus of the moonshine module. [math.QA], 2017.

Victor G. Kac. Infinite-dimensional Lie algebras. Cambridge University Press, 3rd edition, 1990.

Alexander Kirillov, Jr. Modular categories and orbifold models. Comm. Math. Phys., 153(1):159–185, 1993. [arXiv:hep-th/9403088v1].

Masahiko Miyamoto and Kenichiro Tanabe. Uniform product of Lie algebras associated with \( \mathfrak{A}_{1,6,22} \), J. Pure Appl. Algebra, 224(3):1241–1279, 2020. [arXiv:1612.08123v2 [math.QA]].

Ching Hung Lam and Hiroki Shimakura. Classification of holomorphic framed vertex operator algebras of central charge 24. Amer. J. Math., 137(1):111–137, 2015. [arXiv:1209.4677v1 [math.QA]].

Ching Hung Lam and Hiroki Shimakura. Reverse orbifold construction and uniqueness of holomorphic vertex operator algebras. Trans. Amer. Math. Soc., 372(10):7001–7024, 2019. [arXiv:1606.08979v3 [math.QA]].

Ching Hung Lam and Hiroki Shimakura. On orbifold constructions associated with the Leech lattice vertex operator algebra. Math. Proc. Cambridge Philos. Soc., 168(2):261–285, 2020. [arXiv:1705.01281v2 [math.QA]].

Masahiko Miyamoto. \( C_2 \)-cofiniteness of cyclic-orbifold models. Comm. Math. Phys., 335(3):1279–1286, 2015. [arXiv:1306.5031v1 [math.QA]].

Sven Möller. A Cyclic Orbifold Theory for Holomorphic Vertex Operator Algebras and Applications. Ph.D. thesis, Technische Universität Darmstadt, 2016. [arXiv:1611.09843v1 [math.QA]].

Sven Möller. Orbifold vertex operator algebras and the positivity condition. In Toshiyuki Abe, editor, Research on algebraic combinatorics and representation theory of finite groups and vertex operator algebras, number 2086 in RIMS Kôkyûroku, pages 163–171. Research Institute for Mathematical Sciences, 2018. [arXiv:1803.03702v1 [math.QA]].

Paul S. Montague. Orbifold constructions and the classification of self-dual \( c = 24 \) conformal field theories. Nucl. Phys. B, 428(1–2):233–258, 1994. [arXiv:hep-th/9403088v1].

Sven Möller and Nils R. Scheithauer. Dimension formulae and generalised deep holes of the Leech lattice vertex operator algebra. [arXiv:1910.04947v1 [math.QA]], 2019.

Masahiko Miyamoto and Kenichiro Tanabe. Uniform product of \( \mathfrak{A}_{9,n} \) for an orbifold model \( V \) and \( G \)-twisted Zhu algebra. J. Algebra, 274(1):80–96, 2004. [arXiv:math/0112054v3 [math.QA]].

Hans-Volkamer Niemeier. Definite quadratische Formen der Dimension 24 und Diskriminante 1. J. Number Theory, 5:142–178, 1973.

A. N. Schellekens. Meromorphic \( c = 24 \) conformal field theories. Comm. Math. Phys., 153(1):159–185, 1993. [arXiv:hep-th/9205072v1].

Boris B. Venkov. On the classification of integral even unimodular 24-dimensional quadratic forms. Proc. Steklov Inst. Math., 148:63–74, 1980.

Yongchang Zhu. Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc., 9(1):237–302, 1996.