On the smallest poles of Igusa’s p-adic zeta functions

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Abstract

Let $K$ be a $p$-adic field. We explore Igusa’s $p$-adic zeta function, which is associated to a $K$-analytic function on an open and compact subset of $K^n$. First we deduce a formula for an important coefficient in the Laurent series of this meromorphic function at a candidate pole. Afterwards we use this formula to determine all values less than $-1/2$ for $n = 2$ and less than $-1$ for $n = 3$ which occur as the real part of a pole.

1 Introduction

(1.1) Let $K$ be a $p$-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let $R$ be the valuation ring of $K$, $P$ the maximal ideal of $R$ and $q$ the cardinality of the residue field $R/P$. For $z \in K$, let $\text{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of $z$ and $|z| = q^{-\text{ord} z}$ the absolute value of $z$.

(1.2) Let $f$ be a $K$-analytic function on an open and compact subset $X$ of $K^n$ and put $x = (x_1, \ldots, x_n)$. Igusa’s $p$-adic zeta function of $f$ is defined by

$$Z_f(s) = \int_X |f(x)|^s |dx|$$

for $s \in \mathbb{C}$, $\text{Re}(s) \geq 0$, where $|dx|$ denotes the Haar measure on $K^n$, so normalized that $R^n$ has measure 1. Igusa proved that it is a rational function of $q^{-s}$, so that it extends to a meromorphic function $Z_f(s)$ on $\mathbb{C}$ which is also called Igusa’s $p$-adic zeta function of $f$.

(1.3) This zeta function has an interesting connection with number theory. Let $f$ be a $K$-analytic function on $R^n$ defined by a power series over $R$ which is convergent on the whole of $R^n$. Let $M_i$ be the number of solutions of $f(x) \equiv 0 \mod P^i$ in $(R/P^i)^n$. All the $M_i$’s are described by $Z_f(s)$ through the relation

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\[ Z_f(s) = (1 - q^s) P(q^{-s}) + q^s, \] where the Poincaré series \( P(t) \) of \( f \) is defined by

\[ P(t) = \sum_{i=0}^{\infty} M_i(q^{-n} t)^i. \]

Remark that \( P(t) \) is a rational function of \( t \) because \( Z_f(s) \) is a rational function of \( q^{-s} \).

(1.4) The poles of \( Z_f(s) \) are an interesting object of study because they are related to the monodromy conjecture [De2 (2.3.2)] and because they determine the asymptotic behaviour of the \( M_i \). The poles with largest real part give the largest contribution to the \( M_i \). In this paper we are concerned with the smallest real part \( l \) of a pole of \( Z_f(s) \). A non-trivial consequence of the fact that the \( M_i \) are integers is that \( l \) is larger than or equal to \(-n\). Our main results are stated in the next paragraph and sharpen this bound by using a completely different method. This better bound has number theoretic consequences because the knowledge of \( l \) gives us interesting information about the \( M_i \): there exists an \( a \in \mathbb{Z} \) such that \( M_i \) is divisible by \( q^{\lfloor (n+1)i-a \rfloor} \) for all \( i \) (for which \((n+1)i-a \geq 0\)). This is proved in the appendix. Remark that \( a \) is independent of \( i \) and that the number in the exponent is the smallest integer larger than or equal to \((n+1)i-a\).

Let \( F^K_n \) denote the set of all \( K \)-analytic functions defined on an arbitrary open and compact subset of \( K^n \). For \( n \in \mathbb{Z}_{>0} \), we define the set \( \mathcal{P}^K_n \) by

\[ \mathcal{P}^K_n := \{ s_0 \mid \exists f \in F^K_n : Z_f(s) \text{ has a pole with real part } s_0 \}. \]

In this article, we will prove that \( \mathcal{P}^K_n \cap -\infty, -1/2 ] = \{ -1/2 - 1/2 | i \in \mathbb{Z}_{\geq 1} \} = \{ -1, -5/6, -3/4, -7/10, \ldots \} \) and that \( \mathcal{P}^K_n \cap -\infty, -1 ] = \{ -1 - 1/2 | i \in \mathbb{Z}_{>1} \} \). In general, we expect that \( \mathcal{P}^K_n \cap -\infty, -(n-1)/2 ] = \{ -(n-1)/2 - 1/2 | i \in \mathbb{Z}_{>1} \} \).

Remark. One can easily show that \( \mathcal{P}^K_n \cap -\infty, -n + 1 ] = \emptyset \) if \( n \geq 2 \).

(1.5) Let \( f \in K[x_1, x_2] \). Consider \( f \) as a polynomial over \( K^{alg, cl} \). Suppose that the minimal embedded resolution \( g \) of \( f^{-1}(0) \subset (K^{alg, cl})^2 \) is defined over \( K \), i.e., all irreducible components of \( g^{-1}(f^{-1}(0)) \) over \( K^{alg, cl} \) and all points in the intersection of two such components are defined over \( K \). Then it is generally known that an exceptional curve which is intersected once or twice does not contribute to the residues of its candidate poles with candidate order 1. Because \( K^{alg, cl} \iso \mathbb{C} \), we can use the calculations in [SV] to conclude that the real part of a pole of \( Z_f(s) \) is of the form \(-1/2 - 1/2, i \in \mathbb{Z}_{>1} \), if it is smaller than \(-1/2 \).

Let \( f \in K[x_1, x_2, x_3] \). Consider \( f \) again as a polynomial over \( K^{alg, cl} \iso \mathbb{C} \). Suppose that there exists an embedded resolution \( g \) of \( f^{-1}(0) \subset (K^{alg, cl})^3 \iso \mathbb{C}^3 \) for which the induced embedded resolution of the germ at each point \( P \) of \( \mathbb{C}^3 \) satisfies the conditions in [SV (3.1.1)], which is defined over \( K \) and which has good reduction modulo \( P \) (see [De1 section 2]). Then the vanishing results in
and the calculations in [SV] imply that the real part of a pole of $Z_f(s)$ is of the form $-1 - 1/i$, $i \in \mathbb{Z}_{>1}$, if it is smaller than $-1$.

Consequently, starting from [SV], it is rather easy to deal with polynomials which allow an appropriate embedded resolution. However it is very difficult to verify the existence of such an embedded resolution for a concrete function $f$. In a lot of cases there does not exist an embedded resolution which is defined over $K$, and if it exists, the condition of good reduction modulo $P$ is very hard to check. This gives us a strong motivation to study the general case. In this article there are no constraints on $f$: we will not require that $g$ is defined over $K$ and that $g$ has good reduction modulo $P$.

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## 2 The tool for our vanishing results

**2.1** Let $K$ be a $p$-adic field. Let $Y$ be a $n$-dimensional $K$-analytic manifold, $\omega$ a $K$-analytic differential $n$-form on $Y$ and $h$ a $K$-analytic function on $Y$. We say that a chart $(V, y = (y_1, \ldots, y_n))$ on $Y$ is a good chart for $(h, \omega)$ if $h = \varepsilon \prod_{i=1}^{N} y_i^{N_i}$ on $V$ and $\omega = \eta \prod_{i=1}^{K} y_i^{\nu_i-1} dy$ on $V$ for $k \in \{0, \ldots, n\}$, $N_i \in \mathbb{Z}_{>0}$, $\nu_i \in \mathbb{Z}_{>0}$ and non-vanishing $K$-analytic functions $\varepsilon, \eta$ on $V$. We say that $(h, \omega)$ has normal crossings at a point $P \in Y$ if there exists a good chart for $(h, \omega)$ around $P$. So when we say normal crossings, we mean normal crossings over $K$.

Let $x = (x_1, \ldots, x_n)$ be the coordinates of $K^n$. Let $f$ be a $K$-analytic function on an open and compact subset $X$ of $K^n$. Suppose that $f$ does not vanish on an open subset of $X$. An embedded resolution of $(f, dx)$ consists of a $K$-analytic $n$-dimensional manifold $Y$ and a proper $K$-analytic map $g : Y \to X$ such that the restriction $Y \setminus g^{-1}(f^{-1}\{0\}) \to X \setminus f^{-1}\{0\}$ is a $K$-bianalytic map and such that $(f \circ g, g^* dx)$ has normal crossings at every $P \in Y$. We can write $g^{-1}(f^{-1}\{0\})$ as a finite union of closed submanifolds $E_i$, $i \in T$, of codimension one for which there exists a pair of positive integers $(N_i, \nu_i)$, called the numerical data of $E_i$, such that the following condition holds for every point $P \in Y$. If $E_1, \ldots, E_k$ are all the $E_i$ that contain $P$, there exists a chart $(V, y = (y_1, \ldots, y_n))$ around $P$ with $y_i$, $1 \leq i \leq k$, an equation of $E_i$ on $V$ such that

$$f \circ g = \varepsilon \prod_{i=1}^{K} y_i^{N_i} \quad \text{and} \quad g^* dx = \eta \prod_{i=1}^{K} y_i^{\nu_i-1} dy$$

on $V$ for non-vanishing $K$-analytic functions $\varepsilon$ and $\eta$ on $V$. By Hironaka’s theorem [Hi], there always exists an embedded resolution which is a composition of blowing-ups along $K$-analytic closed submanifolds which are contained in the zero locus of the pullback of $f$.

Let $g : Y \to X$ be a $K$-analytic map which is a composition of blowing-ups along $K$-analytic closed submanifolds which are contained in the zero locus of
the pullback of $f$. If $y = (y_1, \ldots, y_n)$ is a system of local parameters at $P \in Y$ such that $f \circ g = \varepsilon \prod_{i=1}^{k} y_i^{N_i}$ for $k \in \{0, \ldots, n\}$, $N_i \in \mathbb{Z}_{>0}$ and $\varepsilon$ a unit in the local ring at $P$, then $g^*dx = \eta \prod_{i=1}^{k} y_i^{\nu_i-1}dy$ for $\nu_i \in \mathbb{Z}_{>0}$ and $\eta$ a unit in the local ring at $P$. Consequently we will talk in this context about an embedded resolution of $f$, about normal crossings of $f \circ g$ at $P$, and about a good chart for $f \circ g$.

(2.2) Fix a uniformizing parameter $\pi$ for $R$. For $z \in K$ let $ac \varepsilon := z\pi^{-ord z}$ be the angular component of $z$. Let $\chi$ be a character of $R^\times$, i.e., a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. Igusa's $p$-adic zeta function of $f$ and $\chi$ is defined by

$$Z_{f,\chi}(s) = \int_X \chi(ac f(x))|f(x)|^s |dx|$$

for $s \in \mathbb{C}$, $\text{Re}(s) \geq 0$. Note that $Z_{f,\chi}(s) = Z_f(s)$ if $\chi$ is the trivial character, which is denoted by 1.

Let $g : Y \to X$ be an embedded resolution of $(f, dx)$. We study Igusa's $p$-adic zeta function $Z_{f,\chi}(s)$ by calculating the integral on the resolution $Y$. Because $|\varepsilon|$, $|\eta|$ and $\chi(ac \varepsilon)$ are locally constant functions on each chart and because $Y$ is a compact $K$-analytic manifold, we can choose a finite set $J$ of good charts $(V, y)$ for $(f \circ g, g^*dx)$ such that $|\varepsilon|$, $|\eta|$ and $\chi(ac \varepsilon)$ are constant on each chart, the $V$'s form a partition of $Y$ and for each chart $(V, y)$ we have $y(V) = P^j := P^{j_1} \times \cdots \times P^{j_n}$ for some $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{>0})^n$ depending on $(V, y)$. Remark that we may even require that $j_1 = \cdots = j_n$ and that this value does not depend on the chart, but we will not do this. We obtain

$$Z_{f,\chi}(s) = \int_X \chi(ac f(x))|f(x)|^s |dx|$$

$$= \sum_{(V, y) \in J} \int_V \chi(ac (f \circ g)(y))|(f \circ g)(y)|^s |g^*dx|$$

$$= \sum_{(V, y) \in J} \int_{P^j} \chi(ac \varepsilon \prod_{i=1}^{k} y_i^{N_i})|\varepsilon \prod_{i=1}^{k} y_i^{N_i}|^s |\eta \prod_{i=1}^{k} y_i^{\nu_i-1}dy|$$

$$= \sum_{(V, y) \in J} \chi(ac \varepsilon)|\varepsilon|^s |\eta|q^{-\sum_{i=k+1}^{n} j_i} \prod_{i=1}^{k} \int_{P^{j_i}} \chi^{N_i}(ac y_i)|y_i|^{N_i s + \nu_i - 1} |dy_i|.$$
max\{−ν_i/N_i \mid i \in T\}. We obtain also from this calculation that every pole of 
\(Z_{f,χ}(s)\) is of the form

\(\frac{v_i}{N_i} + \frac{2k\pi \sqrt{-1}}{N_i \log q},\)

with \(k \in \mathbb{Z}\) and \(i \in T\) such that \(χ^{N_i} = 1\). These values are called the candidate poles of \(Z_{f,χ}(s)\). The candidate poles of \(Z_{f,χ}(s)\) associated to \(E_i\), with \(i \in T\) such that \(χ^{N_i} = 1\), are the values \(-\nu_i/N_i + (2k\pi \sqrt{-1})/(N_i \log q), k \in \mathbb{Z}\). Obviously, we do not associate candidate poles to \(E_i\) if \(χ^{N_i} \neq 1\).

Let \(s_0\) be a candidate pole of \(Z_{f,χ}(s)\). Because the poles of \(1/(1 - q^{-N_i s - \nu_i})\) have order one, we define the expected order \(m = m(s_0)\) of \(E_i\)'s with candidate pole \(s_0\) and with non-empty intersection. The order of \(s_0\) is of course less than or equal to \(m\). It is less than \(m\) if and only if \(b_{-m}\), which is defined by the Laurent series

\(\frac{b_{-m}}{(s - s_0)^m} + \frac{b_{-m+1}}{(s - s_0)^{m-1}} + \cdots + b_0 + b_1(s - s_0) + \cdots\)

of \(Z_{f,χ}(s)\) at \(s_0\), is equal to zero. Remark that a candidate pole of expected order one is a pole if and only if \(b_{-1} \neq 0\).

Everything we have done up till now is well known. More details can be found for example in [Ig3].

(2.3) Let \(X\) be an open and compact subset of \(K^n\). Let \(ξ, f, f_1, \ldots, f_l\) be \(K\)-analytic functions on \(X\). Let \(a_i, b_i, 1 \leq i \leq l\), be non-negative integers. We associate to these data the zeta function

\(Z(s_1, \ldots, s_l) = \int_X \chi(a f)|ξ| |f_1|^{|a_1 s_1 + b_1|} \cdots |f_l|^{|a_l s_l + b_l|} dx,\)

which is defined on a set \(U\) that contains all points \((s_1, \ldots, s_l) \in \mathbb{C}^l\) with \(\text{Re}(s_i) \geq -b_i/a_i\) if \(f_i\) vanishes on \(X\) and \(s_i\) arbitrary if \(f_i\) does not vanish on \(X\). Loeser [Lo2] already studied this zeta function. By looking at an embedded resolution of \(ξff_1 \cdots f_l\), one proves in a way that is analogous to the argument in (2.2) that \(Z(s_1, \ldots, s_l)\) is a rational function of \(q^{-s_1}, \ldots, q^{-s_l}\). Consequently, it extends to a meromorphic function on \(\mathbb{C}^l\), which we also denote by \(Z(s_1, \ldots, s_l)\). As before, we can also obtain an explicit description of \(U\), which turns out to be an open subset of \(\mathbb{C}^l\).

The meromorphic continuation of a function \(h\) will be denoted by \([h]^{mc}\) and the evaluation of this meromorphic continuation at the point \(s = s_0\) of the domain will be denoted by \([h]^{mc}_{s=s_0}\).

In our study of Igusa’s \(p\)-adic zeta function, we will have to deal with expressions of the form

\(\left[\int_X \chi(a f)|ξ| |f_1|^{|a_1 s_1 + b_1|} \cdots |f_l|^{|a_l s_l + b_l|} dx\right]^{mc}_{s=s_0}.\)
The zeta function in more complex variables can be used to modify this expression. If \( U \cap \{(s_1, \ldots, s_l) \in \mathbb{C}^l \mid s_1 = s_0 \} \neq \emptyset \), then

\[
\left[ \int_X \chi(ac \, f)|\xi||f_1|^{a_1s+b_1} \cdots |f_1|^{a_ls+b_l} \, dx \right]_{s=s_0}^{mc} = \left[ \int_X \chi(ac \, f)|\xi||f_1|^{a_1s+b_1} \cdots |f_1|^{a_ls+b_l} \, dx \right]_{s_1=\ldots=s_l=s_0}^{mc} = \left[ \int_X \chi(ac \, f)|\xi||f_1|^{a_1s+b_1} \cdots |f_1|^{a_ls+b_l} \, dx \right]_{s=s_0}^{mc}.
\]

We explain the first equality. The composition of the map \( A : \mathbb{C} \to \mathbb{C}^l : s \mapsto (s, \ldots, s) \) with the meromorphic function \( Z(s_1, \ldots, s_l) \) on \( \mathbb{C}^l \) is a meromorphic function on \( \mathbb{C} \) which is equal to the meromorphic function

\[
\left[ \int_X \chi(ac \, f)|\xi||f_1|^{a_1s+b_1} \cdots |f_1|^{a_ls+b_l} \, dx \right]^{mc}
\]

because they agree on an open subset of \( \mathbb{C} \). Consequently, the first equality is nothing more than \( (Z \circ A)(s_0) = Z(A(s_0)) \). For the second equality, we have to use the map \( B : \mathbb{C} \to \mathbb{C}^l : s \mapsto (s_0, s, \ldots, s) \).

(2.4) Let \( f \) be a \( K \)-analytic function on an open and compact subset \( X \) of \( K^n \) and let \( g : Y \to X \) be an embedded resolution of \((f, dx)\). Denote \( E_I = \cap_{i \in I} E_i \) for \( I \subset T \). Let \( \chi \) be a character of \( R^\times \). Let \( s_0 \) be a candidate pole of \( Z_{f, \chi}(s) \) and let \( m \) be its expected order. Let \( E_I, I \in S \), be all the non-empty intersections of \( m \) varieties \( E_i, i \in T \), with candidate pole \( s_0 \) (and thus also with \( \chi^N \)). Fix \( I \in S \) and suppose for the ease of notation that \( I = \{1, \ldots, m\} \). Let \( W_1 \) and \( W_2 \) be open and compact subsets of \( Y \) which satisfy \( E_I \cap W_1 = E_I \cap W_2 \neq \emptyset \) and which do not meet any \( E_K, K \in S \setminus \{I\} \). Then the contribution of \( W_1 \) to \( b_{-m} \) and the contribution of \( W_2 \) to \( b_{-m} \) are the same because they are both equal to the contribution of \( W_1 \cap W_2 \) to \( b_{-m} \). Consequently we can speak of the contribution of \( E_I \cap W_1 = E_I \cap W_2 \) to \( b_{-m} \). In particular, the contribution of \( E_I \) to \( b_{-m} \) is well defined.

Consider a set \( J \) of disjoint compact charts \((V, y)\) that intersect \( E_I \), that cover \( E_I \) and that are disjoint with all \( E_K, K \in S \setminus \{I\} \). This set \( J \) is necessarily finite and the contribution of \( E_I \) to \( b_{-m} \) is the sum over \( J \) of the contributions

\[
\lim_{s \to s_0} (s - s_0)^m \left[ \int_V \chi(ac \, f \circ g)(y)||f \circ g)(y)||^s |g^* dx \right]^{mc}
\]

of \( V \) to \( b_{-m} \).

We introduce some notation. Let \((V, y)\) be a chart. We have that \( \overline{y} = (y_{m+1}, \ldots, y_n) \) determines a chart on the closed submanifold \( \overline{V} \) defined by \( y_1 = \cdots = y_m = 0 \). Denote \( dy_{m+1} \wedge \cdots \wedge dy_n \) by \( d\overline{y} \). It is a volume form on \( \overline{V} \). If \( j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n \), then we denote \( P^{j_{m+1}} \times \cdots \times P^{j_n} \) by \( P^j \).
Suppose that \((V,y)\) is a chart such that \(E_1, \ldots, E_m\) have equations \(y_1 = 0, \ldots, y_m = 0\) respectively, and such that
\[
f \circ g = \alpha \prod_{i=1}^{m} y_i^{N_i} \quad \text{and} \quad g^*dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1}dy
\]
on \(V\), for \(K\)-analytic functions \(\alpha\) and \(\beta\) on \(V\) with \(|\alpha|, |\beta|\) and \(\chi(ac\alpha)\) independent of \(y_1, \ldots, y_m\). Remark that a good chart \((V,y)\) for \((f \circ g, g^*dx)\) in which \(|\varepsilon|, |\eta|\) and \(\chi(ac\varepsilon)\) are constant satisfies this condition for \(\alpha = \varepsilon \prod_{i=m+1}^{k} y_i^{N_i}\) and \(\beta = \eta \prod_{i=m+1}^{k} y_i^{\nu_i - 1}\). Remark also that \(\nabla = V \cap E_I\). Suppose also that \(y(V)\) is of the form \(P^j\) with \(j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n\). Then
\[
\lim_{s \to s_0} (s - s_0)^m \left[ \int_V \chi(ac(f \circ g)(y))(f \circ g)(y)^s |g^*dx| \right]^{mc}
= \lim_{s \to s_0} (s - s_0)^m \left[ \int_{VJ} \chi(ac\alpha)|\alpha|^s|\beta| \prod_{i=1}^{m} \chi^{N_i}(ac\,y_i)|y_i|^{|N_is + \nu_i - 1}|dy| \right]^{mc}
= \left( \prod_{i=1}^{m} \lim_{s \to s_0} (s - s_0) \left[ \int_{\gamma_i} |y_i|^{N_is + \nu_i - 1}|dy_i| \right]^{mc} \right) \left[ \int_{V} \chi(ac\alpha)|\alpha|^s|\beta||dy| \right]^{mc}_{s = s_0}
= \left( \prod_{i=1}^{m} \frac{q - 1}{qN_i \log q} \right) \left[ \int_{V} \chi(ac\alpha)|\alpha|^s|\beta||dy| \right]^{mc}_{s = s_0}
\]
We have shown that the last expression is the contribution of \(V\) to \(b_{-m}\). Consequently, the only aspect of the chart \((V,y)\) that it depends on is \(V\). In the next section we will see that this is still the case if \(|\alpha|, |\beta|\) and \(\chi(ac\alpha)\) depend on \(y_1, \ldots, y_m\) and if we are not in an embedded resolution.

\textbf{(2.5)} Suppose that \(g : Y = Y_t \to X = Y_0\) is a composition \(g_1 \circ \cdots \circ g_t\) of blowing-ups \(g_i : Y_i \to Y_{i-1}\). Suppose that each \(g_i\) is a blowing-up along a \(K\)-analytic closed submanifold of codimension bigger than one which has only normal crossings with the union of the exceptional varieties of \(g_1 \circ \cdots \circ g_{i-1}\). Let \(I = \{1, \ldots, m\} \subset S\) as in (2.4). Let \(r \in \{0, \ldots, t\}\). Suppose that \(E_t\) already exists in \(Y_r\) and that the \(E_i, i \in I\), intersect transversally in \(Y_r\). Remark that the last condition is satisfied if all the \(E_i, i \in I\), are exceptional. We will write \(E_I \subset Y_r\) if we want to stress that we consider \(E_I\) as a subset of \(Y_r\).

We call a chart \((V,y)\) a good chart for \(E_I \subset Y_r\) if \((V,y)\) is a chart on \(Y_r\) such that \(V\) intersects \(E_I\) and such that \(y_1 = 0, \ldots, y_m = 0\) are the equations of respectively \(E_1, \ldots, E_m\) on \(V\).

Let \((V,y)\) be a good chart for \(E_I \subset Y_r\). Then we have
\[
f \circ g_1 \circ \cdots \circ g_r = \alpha \prod_{i=1}^{m} y_i^{N_i} \quad \text{and} \quad (g_1 \circ \cdots \circ g_r)^*dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1}dy
\]
on $V$, for $K$-analytic functions $\alpha$ and $\beta$ on $V$.

We will now prove that the only aspect of the chart $(V, y)$ that

$$\left[ \int_V \chi(ac \alpha)|\alpha|^s|\beta||d\overline{y}| \right]_{s=s_0}^{mc}$$

(1)

depends on is $\overline{V}$.

Let $(W, z)$ be another chart on $Y$, such that $\overline{V} = \overline{W}$ and such that $z_1 = 0, \ldots, z_m = 0$ are the equations of respectively $E_1, \ldots, E_m$ on $W$. We may suppose that $V = W$ because we can restrict them both to $V \cap W$. For every $i \in \{1, \ldots, m\}$ there exists a non-vanishing $K$-analytic function $f_i$ on $V$ such that $y_i = f_i z_i$ because $y_i$ and $z_i$ are equations of the same $E_i$. Thus

$$f \circ g_1 \circ \cdots \circ g_r = \alpha \left( \prod_{i=1}^{m} f_i^{N_i} \right) \prod_{i=1}^{m} z_i^{N_i}$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx = \beta \left( \prod_{i=1}^{m} f_i^{\nu_i-1} \right) \det \left( \frac{\partial y}{\partial z} \right) \prod_{i=1}^{m} z_i^{\nu_i-1} dz.$$

We have to prove that (1) is equal to

$$\left[ \int_V \chi(ac \alpha)|\alpha|^s|\beta| \left( \prod_{i=1}^{m} \chi^{N_i}(ac f_i)|f_i|^{N_i s + \nu_i - 1} \right) \det \left( \frac{\partial y}{\partial z} \right) |d\overline{z}| \right]_{s=s_0}^{mc}.$$ 

(2)

In (1) we have that $|d\overline{y}| = |\det(\partial \overline{y}/\partial \overline{z})||d\overline{z}|$. Recall that $\chi^{N_i} = 1$ for $i \in \{1, \ldots, m\}$. Because $f_i$, $i \in I$, is a non-vanishing function, we may replace each $|f_i|^{N_i s + \nu_i - 1}$ in (2) by $|f_i|^{N_i s_0 + \nu_i - 1}$ according to (2.3), and this is equal to $|f_i|^{-1}$ because $N_is_0 + \nu_i = 0$ for $i \in I$. Consequently, we have to prove that

$$\left[ \int_V \chi(ac \alpha)|\alpha|^s|\beta| \left( \prod_{i=1}^{m} \chi^{N_i}(ac f_i)|f_i|^{-1} \right) \det \left( \frac{\partial y}{\partial z} \right) |d\overline{z}| \right]_{s=s_0}^{mc}.$$ 

Because $(\partial y/\partial z)$ is equal to

$$\left( \begin{array}{cccc}
\frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_m} & \frac{\partial f_1}{\partial z_{m+1}} \\
\frac{\partial f_2}{\partial z_1} & \cdots & \frac{\partial f_2}{\partial z_m} & \frac{\partial f_2}{\partial z_{m+1}} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_m} & \frac{\partial f_m}{\partial z_{m+1}} \\
\frac{\partial y_1}{\partial z_1} & \cdots & \frac{\partial y_1}{\partial z_m} & \frac{\partial y_1}{\partial z_{m+1}} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial y_m}{\partial z_1} & \cdots & \frac{\partial y_m}{\partial z_m} & \frac{\partial y_m}{\partial z_{m+1}}
\end{array} \right),$$
we obtain that

\[
\left[ \det \left( \frac{\partial y}{\partial z} \right) \right]_{z_1=\cdots=z_m=0} = \left[ \left( \prod_{i=1}^{m} f_i \right) \det \left( \frac{\partial y}{\partial z} \right) \right]_{z_1=\cdots=z_m=0}.
\]

Consequently we have proved our statement.

(2.6) In order to formulate the next important proposition, we recall the setting.

**Data and notations.** Let $f$ be a $K$-analytic function on an open and compact subset $X$ of $K^n$. Let $\chi$ be a character of $R^\times$. Let $g : Y = Y_I \to X = Y_0$ be an embedded resolution of $(f, dx)$ which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_i - 1$. Suppose that each $g_i$ is a blowing-up along a $K$-analytic closed submanifold of codimension larger than one which has only normal crossings with the union of the exceptional varieties of $g_1 \circ \cdots \circ g_{i-1}$. Let $s_0$ be a candidate pole of $Z_{f,\chi}(s)$ and let $m$ be its expected order. Let $b_{-m}$ be the coefficient of $1/(s-s_0)^m$ in the Laurent series of $Z_{f,\chi}(s)$ at $s_0$. For $I \subset T$, denote the intersection in $Y$ of the $E_i \subset Y$, $i \in I$, by $E_I$. Let $E_I$, $I \in S$, be all the non-empty intersections in $Y$ of $m$ varieties $E_i$, $i \in T$, with candidate pole $s_0$ (and thus also with $\chi^{N_i} = 1$). Fix $I \in S$ and suppose for the ease of notation that $I = \{1, \ldots, m\}$. Let $r \in \{0, \ldots, t\}$. Suppose that $E_I$ already exists in $Y_r$ and that the $E_i$, $i \in I$, intersect transversally in $Y_r$.

**Definition.** The contribution of an open and compact subset $U$ of $E_I \subset Y_r$ to $b_{-m}$ is the contribution of the strict transform of $U$ in $Y$ to $b_{-m}$.

**Remark.** The contribution of $E_I$ to $b_{-m}$ is not necessarily equal to the contribution of a ‘very small’ neighbourhood of $E_I \subset Y_r$ to $b_{-m}$, because it can happen that an $E_K$, $K \in S \setminus \{I\}$, lies above $E_I \subset Y_r$.

**Data and notations.** Let $(V, y)$ be a good compact chart for $E_I \subset Y_r$, i.e., $(V, y)$ is a good chart on $Y_r$ such that $V$ is compact. Write

\[
f \circ g_1 \circ \cdots \circ g_r = \alpha \prod_{i=1}^{m} y_i^{N_i} \quad \text{and} \quad (g_1 \circ \cdots \circ g_r)^* dx = \beta \prod_{i=1}^{m} y_i^{\nu_i-1} dy
\]
on $V$, for $K$-analytic functions $\alpha$ and $\beta$ on $V$. We have that \( \overline{y} = (y_{m+1}, \ldots, y_n) \) determines coordinates on the closed submanifold $\overline{V} = V \cap E_I$ which is defined by $y_1 = \cdots = y_m = 0$. Consider the volume form $d\overline{y} = dy_{m+1} \wedge \cdots \wedge dy_n$ on $\overline{V}$.

**Proposition.** The contribution of $\overline{V}$ to $b_{-m}$ is equal to

\[
\left( \prod_{i=1}^{m} \frac{q-1}{qN_i \log q} \right) \left[ \int_{\overline{V}} \chi(\alpha) |\alpha|^s |\beta||d\overline{y}| \right]_{s=s_0}^{mc}.
\]

**Notation.** For the ease of notation, we denote $\kappa = \prod_{i=1}^{m} (q-1)/(qN_i \log q)$.
Remark. One can also prove that the proposition is true on \((2.7)\) and that for every neighbourhood \(O\) of \(F\) (respectively \(\pi(F)\)), there exists a positive integer \(n\) such that \(B_n \subset O\) (respectively \(\pi(B_n) \subset O\)).

Let \(J\) be a set of good compact charts \((W, z)\) for \(E_t \subset Y_t\) such that the \(\bar{W}\)'s form a partition of \(\pi^{-1}(\bar{V})\). Let \(s \in \mathbb{C}\) with \(\text{Re}(s) > 0\). Then

\[
\int_{\bar{V}} \chi(ac) |\alpha|^s |\beta||d\bar{g}| = \lim_{n \to \infty} \int_{\bar{V} \setminus \pi(B_n)} \chi(ac) |\alpha|^s |\beta||d\bar{g}|
\]

\[
= \lim_{n \to \infty} \sum_{(W, z) \in J} \int_{\bar{W} \setminus B_n} \chi(ac) |\alpha|^s |\beta||d\bar{z}|
\]

\[
= \sum_{(W, z) \in J} \int_{\bar{W}} \chi(ac) |\alpha|^s |\beta||d\bar{z}|
\]

The first equality holds because

\[
\lim_{n \to \infty} \int_{\bar{V} \cap \pi(B_n)} \chi(ac) |\alpha|^s |\beta||d\bar{g}| = 0.
\]

Indeed, the measure of \(\bar{V} \cap \pi(B_n)\) for \(|d\bar{g}|\) decreases to zero if \(n \to \infty\) and the real and the complex part of the integrand are bounded for complex numbers \(s\) satisfying \(\text{Re}(s) > 0\). The last equality is obtained by using the same argument.

We have written \(\alpha_z\) and \(\beta_z\) to stress that these functions depend on the chart. For the second equality, we have to use that the \(\bar{W}\)'s form a partition of \(\pi^{-1}(\bar{V})\), that \(\pi : Y_t \to Y_r\) is a \(K\)-bianalytic map on a neighbourhood of \(\bar{W} \setminus B_n\) in \(Y_t\) for every \((W, z) \in J\) and that the contribution is independent of the chosen coordinates, a fact we explained in \((2.5)\).

Finally, we evaluate the meromorphic continuation in \(s = s_0\) and we obtain:

\[
\left[ \int_{\bar{V}} \chi(ac) |\alpha|^s |\beta||d\bar{g}| \right]_{s=s_0}^{mc} = \sum_{(W, z) \in J} \left[ \int_{\bar{W}} \chi(ac) |\alpha|^s |\beta||d\bar{z}| \right]_{s=s_0}^{mc}.
\]

The right hand side multiplied by \(\kappa\) is the contribution of \(\bar{V}\) to \(b_m\) because the proposition is true in the case \(r = t\). Consequently, the proof is done. \(\square\)

Remark. One can also prove that the proposition is true on \(Y_r\) if it is true on \(Y_{r+1}\) by using coordinate transformations of the blowing-up \(g_{r+1}\). This alternative proof can be found in [Se, page 58].

\((2.7)\) Let \(T_t\) be the set of all \(j \in T \setminus I\) for which \(E_j\) intersects \(E_t\) in \(Y_t\). Let \(F_j, j \in T_t\), be the intersection of \(E_j\) and \(E_t\) in \(Y_t\). We have that \(F_j\) has codimension
one in $E_I \subset Y_I$. The set of all $j, j \in T_I$, for which $(g_{r+1} \circ \cdots \circ g_I)(F_j)$ has also codimension one in $E_I \subset Y_I$ will be denoted by $T_I$. For $j \in T_I$ we denote $(g_{r+1} \circ \cdots \circ g_I)(F_j)$ also by $F_j$ and we put $\alpha_j = N_j s_0 + \nu_j$.

Let $(V, y)$ be a good chart for $E_I$ on $Y_r$ on which $F_j, j \in T_r$, is given by $y_1 = \cdots = y_m = y_{m+1} = 0$. Write

$$\alpha(0, \ldots, 0, y_{m+1}, \ldots, y_n) = y_{m+1}^{N_{j,r}} h_1 \quad \text{and} \quad \beta(0, \ldots, 0, y_{m+1}, \ldots, y_n) = y_{m+1}^{\nu_{j,r} - 1} h_2$$

with $h_1$ and $h_2$ not divisible by $y_{m+1}$. Then we denote $N_{j,r} s_0 + \nu_{j,r}$ by $\alpha_{j,r}$.

We deduce now the relations that will be used later. In this paragraph we suppose that $m = 1$ and that $E_I = E_r$ is created by the blowing-up at a point $P$ of $Y_{r-1}$. Suppose that there exists a chart $(V, y)$ centred at $P$ on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part a homogeneous polynomial for which every irreducible factor over $K^{\text{alg}}$ is defined over $K$ and for which the zero locus in $\mathbb{P}^{n-1}$ of every irreducible factor (over $K^{\text{alg}}$) contains a non-singular point defined over $K$. Remark that these conditions are satisfied if the lowest degree part is a product of linear factors defined over $K$. Write $f \circ g_1 \circ \cdots \circ g_{r-1} = e \left( \prod_{j \in T_r} f_j^{N_{j,r}} \right) + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho \left( \prod_{j \in T_r} f_j^{\nu_{j,r} - 1} \right) dy$, where $f_j$ is the equation of $F_j \subset E_r$ in the homogeneous coordinates $(y_1: \cdots: y_n)$ on $E_r \subset Y_r$, $e \in K^\times$, $\theta$ is a power series with multiplicity larger than the degree of the homogeneous polynomial $\prod_{j \in T_r} f_j^{N_{j,r}}$ and $\rho$ is a $K$-analytic function which does not vanish at $P$. Because the multiplicity of $f \circ g_1 \circ \cdots \circ g_{r-1}$ at $P$ is equal to $N_r$, we obtain the first relation:

$$\sum_{j \in T_r} (\deg F_j) N_{j,r} = N_r. \quad \text{(Relation 1)}$$

Our second relation will involve the $\alpha_{j,r}, j \in T_r$. There will appear differential forms with rational exponents in the calculations. One can make sense to this by considering them as an element of a tensor power of the module of rational differential forms (see [Ja]), but we will not give details here. Let $i \in \{1, \ldots, n\}$. We look at the chart $(O, z = (z_1, \ldots, z_n))$ on $Y_r$ for which $g_r(z_1, \ldots, z_n) = (z_1 z_i, \ldots, z_{i-1} z_i, z_i, z_{i+1} z_i, \ldots, z_n z_i)$. Then

$$f \circ g_1 \circ \cdots \circ g_r = z_i^{N_r} \left( e \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{N_{j,r}} + z_i \theta \circ g_r \right)$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx = z_i^{\nu_r - 1} (\rho \circ g_r) \left( \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\nu_{j,r} - 1} \right) dz.$$
Consequently the Poincaré residue of \((f \circ g_1 \circ \cdots \circ g_r)^{-\nu_r/N_r}(g_1 \circ \cdots g_r)^* dx\) on \(E_r \subset Y_r\) (see [Ja]) is equal to
\[
e^{-\nu_r/N_r} \rho(P) \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\alpha_{j,r}-1} dz,
\]
so that the canonical divisor of \(E_r\) is \(\sum_{j \in T_r} \deg F_j (\alpha_{j,r} - 1) F_j\). Because we know that the degree of the canonical divisor on \(E_r \cong \mathbb{P}^{n-1}\) is \(-n\), we obtain the second relation:
\[
\sum_{j \in T_r} (\deg F_j)(\alpha_{j,r} - 1) = -n. \tag{Relation 2}
\]
Remark that the condition on the lowest degree part of \(f \circ g_1 \circ \cdots \circ g_{r-1}\) has to be satisfied because otherwise some terms on the left hand side are missing. We need the two relations which we just derived in section 3. In the next paragraph we will deduce that \(\alpha_{j,r} = \alpha_j\) and that \(N_{j,r} \equiv N_j \pmod{N_r}\) so that we obtain
\[
\sum_{j \in T_r} (\deg F_j)N_j \equiv 0 \pmod{N_r} \quad \text{and}
\]
\[
\sum_{j \in T_r} (\deg F_j)(\alpha_j - 1) = -n.
\]
One can find these relations in a more general form in [Ve1], [Ve2] and [Ve4].

We prove that \(\alpha_{j,r} = \alpha_j\) for \(j \in T_r\). Because \(g_{r+1} \circ \cdots \circ g_i\) is a composition of a finite number of blowing-ups, it is enough to prove that \(\alpha_{j,r} = \alpha_{j,r+1}\). If the centre of \(g_{r+1}\) does not contain \(F_j\), then \(N_{j,r} = N_{j,r+1}\) and \(\nu_{j,r} = \nu_{j,r+1}\) so that we are done. If the centre of \(g_{r+1}\) contains \(F_j\), we may suppose that \(g_{r+1}\) is the blowing-up along \(y_1 = \cdots = y_a = y_{m+1} = 0\), where \(0 < a \leq m\). The relevant chart is determined by the transformation
\[
(z_1, \ldots, z_n) \mapsto (z_1 z_{m+1}, \ldots, z_a z_{m+1}, z_{a+1}, \ldots, z_m, z_{m+1}, \ldots, z_n).
\]
Because
\[
f \circ g_1 \circ \cdots \circ g_{r+1} = g_{r+1}^* \left( \prod_{i=1}^m z_i^{N_i} \right) \sum_{i=1}^a N_i z_i^{m+1}
\]
and
\[
(g_1 \circ \cdots \circ g_{r+1})^* dx = g_{r+1}^* \left( \prod_{i=1}^m z_i^{\nu_i-1} \right) \sum_{i=1}^a \nu_i d z,
\]
we have to prove that \(N_{j,s_0} + \nu_{j,r} = (N_{j,s} + \sum_{i=1}^a N_i) s_0 + (\nu_{j,s} + \sum_{i=1}^a \nu_i)\). This follows from the fact that \(N_i s_0 + \nu_i = 0\) for \(i \in \{1, \ldots, a\}\). Remark that it follows also from these calculations that
\[
N_{j,r} \equiv N_j \pmod{\gcd(N_1, \ldots, N_m)}.
\]
Example. We give an illustration which is easy and well known. Let $f = x_1^2 + x_2^3$. Let $X = \mathbb{Z}_p \times \mathbb{Z}_p$. We want to determine the poles of Igusa’s $p$-adic zeta function associated to $f$. Notice that $-1$ is a square in $\mathbb{Q}_p$ if and only if $-1$ is a square $\mathbb{Z}/(p)$ and $p \neq 2$.

If $-1$ is a square in $\mathbb{Q}_p$, then $(f, dx)$ has already normal crossings. We obtain a good chart for $(f, dx)$ by applying the coordinate transformation $(y_1, y_2) \mapsto ((y_1 + y_2)/2, (y_1 - y_2)/(2a))$, where $a$ denotes a square root of $-1$. Because $|a| = 1$ we obtain

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} |x_1^2 + x_2^3|^s dx_1 \wedge dx_2 = \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |y_1y_2|^s dy_1 \wedge dy_2.$$ 

Consequently, the only candidate poles of $Z_f(s)$ are $-1 + (2k\pi \sqrt{-1})/(\log p)$, $k \in \mathbb{Z}$. They are all poles because $b_{-2} = ((p - 1)/(p \log p))^2$ for each candidate pole.

If $-1$ is not a square in $\mathbb{Q}_p$, then $(f, dx)$ does not have normal crossings at the origin. We obtain an embedded resolution after one blowing-up $g$. Remark that the zero locus of $f$ contains only the origin and that the zero locus of $f \circ g$ is equal to the exceptional curve $E$ of $g$. We will use the two charts on the blowing-up determined by $(y_1, y_2) \mapsto (y_1 y_2, y_2)$ and $(z_1, z_2) \mapsto (z_1, z_1 z_2)$. The sets \{(y_1, y_2) \mid y_1 \in \mathbb{Z}_p, y_2 = 0\} and \{(z_1, z_2) \mid z_1 = 0, z_2 \in p \mathbb{Z}_p\} form a partition of $E$. The candidate poles of $Z_f(s)$ are $s_k = -1 + (2k\pi \sqrt{-1})/(2 \log p)$, $k \in \mathbb{Z}$, and each $b_{-1}$ is equal to

$$\left(\frac{p - 1}{2p \log p}\right) \left(\int_{\mathbb{Z}_p} |y_1^2 + 1|^s dy_1\right)_{s=s_k}^{mc} + \left(\int_{p \mathbb{Z}_p} |1 + z_2^2|^s dz_2\right)_{s=s_k}^{mc}.$$ 

If $p \neq 2$, we have that $|1 + x^2| = 1$ for every $x \in \mathbb{Z}_p$, so that $b_{-1} = (p^2 - 1)/(2p^2 \log p)$. If $p = 2$, we have that $|1 + x^2| = 1$ for every $x \in 2 \mathbb{Z}_p$ and $|1 + x^2| = 1/2$ for every $x \in 1 + 2 \mathbb{Z}_2$, so that $b_{-1} = 1/(2 \log 2)$ if $k$ is even and $b_{-1} = 0$ if $k$ is odd.

Remark that Igusa’s $p$-adic zeta function of $x_1^2 + x_2^3$ can be calculated completely elementarily in all the cases.

3 The vanishing results

3.1 Curves

Let $X$ be an open and compact subset of $K^2$. Let $f$ be a $K$-analytic function on $X$. Let $g : Y \to X$ be an embedded resolution of $f$. Write $g = g_1 \circ \cdots \circ g_t : Y = Y_t \to X = Y_0$ as a composition of blowing-ups $g_i : Y_i \to Y_{i-1}$, $i \in \{1, \ldots, t\}$. The exceptional curve of $g_i$ and also the strict transforms of this curve are denoted by $E_i$. Let $\chi$ be a character of $R^x$. 

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Proposition. Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up $g_r$. Suppose that the expected order of a candidate pole $s_0$ associated to $E_r$ is one. Suppose that there exists a chart $(V, y = (y_1, y_2))$ centred at $P$ on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part a (non-constant) monomial. Then the contribution of $E_r$ to the residue $b_{-1}$ of $Z_{f, \chi}(s)$ at $s_0$ is zero.

Remark. This proposition is essentially well known. Our proof differs slightly from the ones in [Ig2] and [Lo1] because we will calculate the contribution of $E_r$ to $b_{-1}$ just after the creation of $E_r$ instead of on the embedded resolution. We incorporate this proof here because the same technique will be used in the proof of the more difficult result of section 3.2.

Proof. We may suppose that $(V, y)$ is a chart centred at $P$ such that $f \circ g_1 \circ \cdots \circ g_{r-1} = ey_1^k y_2^l + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{c-1} y_2^d dy$ with $k, l \in \mathbb{Z}_{\geq 0}$, $c, d \in \mathbb{Z}_{> 0}$, $e \in K^*$ and $\rho, \theta$ $K$-analytic functions satisfying $\rho(0, 0) \neq 0$ and $\text{mult}(\theta) < k + l$.

We consider here that $k$ and $l$ are both not zero. The case that $k$ or $l$ is zero can be treated analogously.

We look at the chart $(O, z = (z_1, z_2))$ on $Y_r$ for which $g_r(z_1, z_2) = (z_1, z_1 z_2)$. Then

$$f \circ g_1 \circ \cdots \circ g_r = z_1^{k+l} \left( e z_2^l + z_1 \frac{\theta(z_1, z_1 z_2)}{z_1^{k+l+1}} \right).$$

Remark that the equation of $E_r$ is $z_1 = 0$, that $N_r = k + l$ and that $\nu_r = c + d$.

Using the notation of (2.7), let $T_r = \{1, 2\}$ and let $F_1$ be the origin of this chart. The contribution to $b_{-1}$ of an open and compact subset $A$ of $E_r$ which is contained in $O$ is equal to

$$\left( \frac{q-1}{q N_r \log q} \right) \left[ \int_A \chi(ac e) \chi(a z_2) |e| |\rho(0, 0)||z_2|^{l s + d - 1} |dz_2| \right]_{s=s_0}^{mc}.$$

Let $(O', z' = (z_1', z_2'))$ be the chart on $Y_r$ for which $g_r(z_1', z_2') = (z_1' z_2', z_2')$. The origin of this chart is the point $F_2$. Analogously as before, we obtain that the contribution to $b_{-1}$ of an open and compact subset $B$ of $E_r$ which is contained in $O'$ is equal to

$$\left( \frac{q-1}{q N_r \log q} \right) \left[ \int_B \chi(ac e) \chi^k(ac z_1') |e| |\rho(0, 0)||z_1'|^{k s + c - 1} |dz_1'| \right]_{s=s_0}^{mc}.$$

Because $\chi^N = 1$ (otherwise there are no candidate poles associated to $E_r$) and because $k + l = N_r$, we have that $\chi^k = 1$ if and only if $\chi^l = 1$.

Case 1: $\chi^k = \chi^l = 1$. Then the contribution of $E_r$ to $b_{-1}$ is equal to

$$\left( \frac{\chi(ac e)|e| s_0 |\rho(0, 0)||q-1|}{q N_r \log q} \right) \left[ \int_R |z_2|^{l s + d - 1} |dz_2| \right]_{s=s_0}^{mc} + \left[ \int_P |z_1'|^{k s + c - 1} |dz_1'| \right]_{s=s_0}^{mc}.$$
The last equality follows from $\alpha_1 + \alpha_2 = 0$, which is relation 2 of (2.7).

Case 2: $\chi^k \neq 1$ and $\chi^l \neq 1$. Then the contribution of $E_r$ to $b_{-1}$ is equal to zero because both terms in the sum

$$\left[ \int_{R} \chi^l(ac z_2)|z_2|^{s+d-1}|dz_2| \right]_{s=s_0}^{mc} + \left[ \int_{P} \chi^k(ac z'_1)|z'_1|^{k+c-1}|dz'_1| \right]_{s=s_0}^{mc}$$

are equal to zero. \(\square\)

### 3.2 Surfaces

Let $X$ be an open and compact subset of $K^3$. Let $f$ be a $K$-analytic function on $X$. Let $g : Y = Y_t \rightarrow X = Y_0$ be an embedded resolution of $f$ which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \rightarrow Y_{i-1}$ with centre a $K$-analytic closed submanifold which has only normal crossings with the union of the exceptional surfaces in $Y_{i-1}$ and with exceptional surface $E_i$.

**Proposition.** Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up $g_r$. Suppose that the expected order of a candidate pole $s_0$ associated to $E_r$ is one. Suppose that there exists a chart $(V, y = (y_1, y_2, y_3))$ centred at $P$ on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part of the form $e y_1^a y_2^b y_3^c (y_1 + y_2)^n$, with $e \in K^*$ and $k,l,m,n \in \mathbb{Z}_{\geq 0}$. Then the contribution of $E_r$ to the residue $b_{-1}$ of $Z_f(s)$ at $s_0$ is zero.

**Proof.** We may suppose that $f \circ g_1 \circ \cdots \circ g_{r-1} = e y_1^a y_2^b y_3^c (y_1 + y_2)^n + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{a-1} y_2^{b-1} y_3^{c-1} (y_1 + y_2)^{d-1} dy$ with $a, b, c, d \in \mathbb{Z}_{>0}$ and $\rho, \theta$ $K$-analytic functions satisfying $\rho(0,0) \neq 0$ and $\text{mult}(\theta) > k+l+m+n$. Remark that at least one of the numbers $a, b, d$ is equal to 1. We consider here the case that $k,l,m$ and $n$ are all different from zero. The other cases are treated analogously. Let $T_r = \{1, 2, 3, 4\}$ and suppose that $F_i$, $i \in \{1, 2, 3\}$, is given by $y_i = 0$ and that $F_4$ is given by $y_1 + y_2 = 0$ in the homogeneous coordinates $(y_1 : y_2 : y_3)$ on $E_r \subset Y_r$. Analogously as in Section 3.1, we can calculate $f \circ g_1 \circ \cdots \circ g_r$ and $(g_1 \circ \cdots \circ g_r)^* dx$ in the three charts on $Y_r$ for which respectively $g_r(z_1, z_2, z_3) = (z_1 z_3, z_2 z_3, z_3)$, $g_r(z'_1, z'_2, z'_3) = (z'_1 z'_2 z'_3, z'_2 z'_3)$ and $g_r(z''_1, z''_2, z''_3) = (z''_1 z''_2 z''_3, z''_1 z''_3)$. The contribution of $E_r$ to the residue $b_{-1}$ of $Z_f(s)$ at $s_0$ turns out to be $\kappa |e|^{s_0} |\rho(0,0,0)|$ times

$$\left[ \int_{P \times P} |z_1|^{k+s+a-1} |z_2|^{l+s+b-1} |z_1 + z_2|^{n+s+d-1} |dz_1 \wedge dz_2| \right]_{s=s_0}^{mc}$$

\((*)\)
Analogously, we obtain that the contribution of

$$\sum_{i=1}^{\infty} \left[ \sum_{i=1}^{p+1} \left( \int_{P \setminus P^{i+1}} \left| z_1 \right|^{k_s+a-1} \left| z_2 \right|^{s+b-1} \left| z_1 + z_2 \right|^{n_s+d-1} \right) \right]^{mc} \Bigg|_{s=s_0} = \left( \frac{q-1}{q} \right)^2 \frac{1}{q^{k_s+a-1}} \sum_{i=1}^{\infty} q^{-i(k_s+a+s+b+n_s+d-1)} \Bigg|_{s=s_0} = \left( \frac{q-1}{q} \right)^2 \frac{1}{q^{k_s+a-1}(q^{k_s+a+s+b+n_s+d-1} - 1)} \Bigg|_{s=s_0} = 1.$$  

Consequently, we have to prove that this expression is equal to zero.

To calculate the first term in (*), we partition $P \times P$ into

$$A_1 = \{(z_1, z_2) \in P \times P \mid \text{ord } z_1 > \text{ord } z_2 \} = \bigcup_{i \in \mathbb{Z}_{>0}} \{(z_1, z_2) \mid \text{ord } z_1 > \text{ord } z_2 = i\}$$

$$A_2 = \{(z_1, z_2) \in P \times P \mid \text{ord } z_1 < \text{ord } z_2 \} = \bigcup_{i \in \mathbb{Z}_{>0}} \{(z_1, z_2) \mid i = \text{ord } z_1 < \text{ord } z_2\}$$

$$A_3 = \{(z_1, z_2) \in P \times P \mid \text{ord } z_1 = \text{ord } z_2 \} = \bigcup_{i \in \mathbb{Z}_{>0}} \{(z_1, z_2) \mid \text{ord } z_1 = \text{ord } z_2 = i\}$$

The contribution of $A_1$ to the first term in (*) is equal to

$$\left[ \sum_{i=1}^{\infty} \left( \int_{P^{i+1}} \left| z_1 \right|^{k_s+a-1} \left| z_2 \right|^{s+b-1} \left| z_1 + z_2 \right|^{n_s+d-1} \right) \right]^{mc} \Bigg|_{s=s_0} = \left( \frac{q-1}{q} \right)^2 \frac{1}{q^{k_s+a-1}} \sum_{i=1}^{\infty} q^{-i(k_s+a+s+b+n_s+d-1)} \Bigg|_{s=s_0} = \left( \frac{q-1}{q} \right)^2 \frac{1}{q^{k_s+a-1}(q^{k_s+a+s+b+n_s+d-1} - 1)} \Bigg|_{s=s_0} = \left( \frac{q-1}{q} \right)^2 \frac{1}{q^{k_s+a-1}(q^{k_s+a+s+b+n_s+d-1} - 1)}.$$  

Analogously, we obtain that the contribution of $A_2$ to the first term in (*) is equal to

$$\left( \frac{q-1}{q} \right)^2 \frac{1}{q^{k_s+a-1}(q^{k_s+a+s+b+n_s+d-1} - 1)}.$$  

The contribution of $A_3$ to the first term in (*) is

$$\left[ \sum_{i=1}^{\infty} \left( \int_{P^{i+1}} \left| z_1 \right|^{k_s+a-1} \left| z_2 \right|^{s+b-1} \left| z_1 + z_2 \right|^{n_s+d-1} \right) \right]^{mc} \Bigg|_{s=s_0}.$$  

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One can verify (see [Se, Section 3.3.2]) that this is equal to

\[
\left( \frac{q - 1}{q} \right)^2 \frac{1}{(q^{a_4} - 1)(q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1)} + \left( \frac{q - 1}{q} \right) \left( \frac{q - 2}{q} \right) \frac{1}{q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1}.
\]

(5)

(6)

The second term of (\(*\)) is equal to

\[
\left[ \int_P |z_1|^{ks+a_1-1} |dz_1| \int_R |z_3|^{ms+c-1} |dz_3| \right]_{s=s_0}^{mc} = \left( \frac{q - 1}{q} \right)^2 \frac{1}{(q^{\alpha_1} - 1)(1 - q^{-\alpha_3})}.
\]

(7)

The third term of (\(*\)) is equal to

\[
\left[ \int_R |z_2|^{ls+b-1} (1 + z_2)^{ns+d-1} |dz_2| \int_R |z_3|^{ms+c-1} |dz_3| \right]_{s=s_0}^{mc} = \left( \frac{q - 1}{q} \right) \left( \frac{q - 2}{q} \right) \frac{1}{1 - q^{-\alpha_3}} + \left( \frac{q - 1}{q} \right)^2 \frac{1}{(q^{\alpha_2} - 1)(1 - q^{-\alpha_3})} + \left( \frac{q - 1}{q} \right)^2 \frac{1}{(q^{\alpha_4} - 1)(1 - q^{-\alpha_3})}.
\]

(8)

(9)

(10)

Relation 2 of (2.7) is \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 1 = 0\), so that we obtain

\[
\frac{1}{q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1} + \frac{1}{1 - q^{-\alpha_3}} = \frac{1 - q^{-\alpha_3} + q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1}{(q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1)(1 - q^{-\alpha_3})} = \frac{q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1}{(q^{\alpha_1 + \alpha_2 + \alpha_4 - 1} - 1)(q^{\alpha_3} - 1)} = 0,
\]

and consequently (3) + (7) = 0. Analogously, we obtain that (4) + (9) = (5) + (10) = (6) + (8) = 0. Consequently, the contribution of \(E_r\) to \(b_{-1}\) is equal to zero. □
Remark. Let $\chi$ be a character of $R^\times$. Suppose that we are in the analogous situation to this proposition with $Z_{f,\chi}(s)$. Then one can show that the contribution of $E_r$ to the residue $b_{-1}$ of $Z_{f,\chi}(s)$ at $s_0$ is zero. The proof consists of very long calculations involving character sums and is written down in [Se, Chapter 4].

4 Determination of the smallest poles

The main ideas and results of this section have the same flavour as those in [SV], where the local topological zeta function is studied. However here the situation is more complicated because the field $K$ is not algebraically closed.

4.1 Curves

(4.1.1) In this section we will determine $P_{\mathbb{K}}^{\mathbb{K}} \cap ]-\infty, -1/2[$. Let $f$ be a $K$-analytic function on an open and compact subset of $K^2$ and let $g$ be the minimal embedded resolution of $f$. The poles of $Z_f(s)$ with real part less than $-1/2$ and different from $-1$ are only associated to exceptional curves. Consequently, these poles are completely determined by the germs of $f$ at the points where $f$ does not have normal crossings. It is thus sufficient to study the germs of $K$-analytic functions at the origin, which will be identified with the convergent power series. The set of all convergent power series in the variables $x$ and $y$ is classically denoted by $K<\!<x, y>\!>$. 

(4.1.2) Let $f \in K<\!<x, y>\!>$. Let $g: Y \to X$ be the minimal embedded resolution of a representative of $f$. Write $g = g_1 \circ \cdots \circ g_t : Y = Y_t \to X = Y_0$ as a composition of blowing-ups $g_i : Y_i \to Y_{i-1}$, $i \in \{1, \ldots, t\}$. The exceptional curve of $g_i$ and also the strict transforms of this curve are denoted by $E_i$. Let $T$ be as in (2.1) and obviously we suppose that $\{1, \ldots, t\} \subset T$.

Let $k \in \{1, \ldots, t\}$. Let $P \in Y_k$ be a point on an exceptional curve, i.e., a point which is mapped to the origin under the map $g_1 \circ \cdots \circ g_k$. The strict transform of $f$ around $P$ is defined as the germ at $P$ of the $K$-analytic function $f \circ g_1 \circ \cdots \circ g_k$ divided by the highest possible powers of local equations of exceptional curves through $P$. Remark that the strict transform of $f$ around $P$ is defined modulo the germ of a $K$-analytic function which does not vanish at $P$ as a factor.

We call a complex number ‘a candidate pole of $Z_f(s)$’ if it is a candidate pole associated to an $E_i$, $i \in T$, satisfying $0 \in g(E_i)$. A candidate pole of $Z_f(s)$ is called a pole of $Z_f(s)$ if there exists an arbitrarily small neighbourhood of 0 for which it is a pole.

The following lemma is trivial.

(4.1.3) Lemma. Suppose that we have blown up $k$ times but we do not yet have an embedded resolution. Let $P$ be a point at which $f \circ g_1 \circ \cdots \circ g_k$ does not
have normal crossings. Let $\mu$ be the multiplicity at $P$ of the strict transform of $f$ around $P$ and let $g_{k+1}$ be the blowing-up at $P$.

(a) Suppose that two exceptional curves $E_i$ and $E_j$ contain $P$. Then $-\nu_{k+1}/N_{k+1}$ is equal to $-(\nu_i + \nu_j)/(N_i + N_j + \mu)$ and this value is larger than $\min\{-\nu_i/N_i, -\nu_j/N_j\}$.

(b) Suppose that exactly one exceptional curve $E_i$ contains $P$ and that $\mu \geq 2$. Then $E_{k+1}$ has numerical data $(N_i + \mu, \nu_i + 1)$ and $-(\nu_i + 1)/(N_i + \mu)$ lies between $-1/\mu$ and $-\nu_i/N_i$.

(c) Suppose that exactly one exceptional curve $E_i$ contains $P$ and that $\mu = 1$. Remark that the two curves are tangent at $P$ because we do not have normal crossings at $P$. Let $g_{k+2}$ be the blowing-up at $E_i \cap E_{k+1}$. Remark that we do not have to blow up at a point of $E_{k+1}$ anymore. The numerical data of $E_{k+2}$ are $(2N_i + 2, 2\nu_i + 1)$, and $-(2\nu_i + 1)/(2N_i + 2)$ lies between $-1/2$ and $-\nu_i/N_i$. Let $s_0$ be a candidate pole associated to $E_{k+1}$. Because $s_0$ is not a candidate pole associated to $E_{k+2}$, which is a consequence of $-\nu_{k+1}/N_{k+1} \neq -\nu_{k+2}/N_{k+2}$, the contribution of $E_{k+1}$ to the coefficient $b_{-2}$ in the Laurent series of $Z_f(s)$ at $s_0$ is zero. It follows from the proposition in 3.1 that $E_{k+1}$ does not give a contribution to the residue $b_{-1}$ of $Z_f(s)$ at $s_0$.

(4.1.4) Suppose that after some blowing-ups, the pullback of $f$ does not have normal crossings at a point $P$. Suppose also that the real parts of the candidate poles associated to the exceptional curves through $P$ are all larger than or equal to $-1/2$. Then it follows from the above lemma that the components above $P$ in the final resolution do not give a contribution to a candidate pole with real part less than $-1/2$.

**Corollary.** Zeta functions of convergent power series of multiplicity at least $4$ do not have a pole with real part in $]-\infty, -1/2]\setminus\{-1\}$.

Indeed, every exceptional curve in the minimal embedded resolution of $f$ lies above a point of $E_1$ (considered in the stage when it is created), which has a candidate pole with real part larger than or equal to $-1/2$.

(4.1.5) To deal with multiplicity $2$ and $3$, we will study an ‘easier’ element of $K<x, y>$. We will use the following theorem (see [Ig3, Theorem 2.3.1]).

**Weierstrass Preparation Theorem.**

If $f(z_1, \ldots, z_{n-1}, w) = f(z, w) \in K<x, w>$ is not identically zero on the $w$-axis, then $f$ can be written uniquely as $f = (w^e + a_1(z)w^{e-1} + \cdots + a_e(z))h$, where $a_i(z) \in K<x,z>$ satisfies $a_i(0) = 0$ and $h \in K<x, w>$ satisfies $h(0) \neq 0$.

Because $h(0) \neq 0$ implies that $|h|$ is constant on a neighbourhood of $0$, we have that Igusa’s $p$-adic zeta functions of $f$ and $w^e + a_1(z)w^{e-1} + \cdots + a_e(z)$ have the same poles. After an appropriate coordinate transformation, the desired form will appear. For example, the coordinate transformation $(z, w) \mapsto (z, w - a_1(z)/e)$ cancels the term $a_1(z)w^{e-1}$.
(4.1.6) Example. Let \( f \in K \ll x, y \gg \) have multiplicity 3 and let \( f_3 = y^3 + xy^2 = y^2(y + x) \) be the homogeneous part of \( f \) of degree 3. By the Weierstrass preparation theorem, we may work with a function of the form \( y^3 + a_1(x)y^2 + a_2(x)y + a_3(x) \), with \( \text{mult}(a_1(x)) = 1 \), \( \text{mult}(a_2(x)) \geq 3 \) and \( \text{mult}(a_3(x)) \geq 4 \). One can check that there exists a coordinate transformation \((x, y) \mapsto (x, y - k(x))\) such that the function becomes of the form \( y^3 + b_1(x)y^2 + b_3(x) \), with \( \text{mult}(b_1(x)) = 1 \) and \( \text{mult}(b_3(x)) \geq 4 \). After another coordinate transformation, we get the form \( y^3 + xy^2 + g(x) \), with \( \text{mult}(g(x)) \geq 4 \).

(4.1.7) Theorem. We have

\[
\mathcal{P}_2^K \cap \left[-\infty, -\frac{1}{2}\right] = \left\{ -\frac{1}{2} - \frac{1}{i} \right\}_{i \in \mathbb{Z}_{>1}}
\]

and at most one value in \( ]-1, -1/2[ \) is the real part of a pole of a fixed Igusa’s \( p \)-adic zeta function. Moreover, if \( f \in K \ll x, y \gg \) has multiplicity at least 4, then \( Z_f(s) \) has no pole with real part in \( ]-\infty, -1/2[ \setminus \{-1\} \).

Proof. Because the calculations are analogous to the calculations in [SV] for the local topological zeta function, we do not treat all the cases in this paper.

(a) Suppose that \( f \) is an element of \( K \ll x_1, x_2 \gg \) with multiplicity 2. When we apply the ideas of (4.1.5), we see that it is enough to consider \( x_1^3 + ax_1^2 \), with \( l \in \mathbb{Z}_{>1} \) and \( a \in K^\times \). If \( f = x_1^2 \), the candidate poles of \( Z_f(s) \) are \(-1/2 + (k \pi \sqrt{-1})/(\log q) \), \( k \in \mathbb{Z} \). If \( l = 2 \), the calculations are analogous as in (2.8). If \( l \) is odd, write \( l = 2r + 1 \). After \( r \) blowing-ups, the strict transform of \( f^{-1}{0} \) is non-singular and tangent to \( E_r \). The numerical data of \( E_i \), \( i = 1, \ldots, r \), are \((2i, i + 1)\). To get the minimal embedded resolution, we now blow up twice. Let \( E_0 \) be the strict transform of \( f^{-1}{0} \). Remark that \( T = \{0, 1, \ldots, r + 2\} \).

The dual resolution graph and the numerical data are given below.

\[
E_1 \quad E_2 \quad E_3 \quad \ldots \quad E_r \quad E_{r+2} \quad E_{r+1} \quad E_1(2, 2) \quad E_r(2r, r + 1) \\
E_2(4, 3) \quad E_{r+1}(2r + 1, r + 2) \\
E_3(6, 4) \quad E_{r+2}(4r + 2, 2r + 3)
\]

It follows from section 3.1 that the candidate poles associated to \( E_1, \ldots, E_{r+1} \) are not poles. The other candidate poles have real part \(-1 \) or \(-2r + 3/(4r + 2) = -1/2 - 1/(2r + 1) \). We calculate the residue of \( Z_f(s) \) at the candidate pole \( s_0 = -1/2 - 1/(2r + 1) \). Because

\[
\left[ \int_{aR} |y_1|^{|2r+1)s+r+1|y_1 + a|^s|dy_1| \right]_{s=s_0}^{mc} = |a|^{-1/(2r+1)} \left[ \int_{R} |y|^{|2r+1)s+r+1|y + 1|^s|dy| \right]_{s=s_0}^{mc} = |a|^{-1/(2r+1)} \left[ \int_{R}\mathbb{A}_{-1+P} |y|^{|2r+1)s+r+1|dy| + \int_{-1+P} |y + 1|^s|dy| \right]_{s=s_0}^{mc}
\]

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\[
= |a|^{-1/(2r+1)} \left( \frac{q - 2}{q} + \frac{q - 1}{q} \frac{1}{q^{\alpha_{r+1}} - 1} + \frac{q - 1}{q} \frac{1}{q^{\alpha_0} - 1} \right)
\]
and
\[
\left[ \int_{t P^r} |y|^{2rs+r}|1 + a y^s|dy_2 \right]_{s=s_0}^{mc} = |a|^{-1/(2r+1)} \left[ \int_{P^r} |y|^{2rs+r}|1 + y^s|dy \right]_{s=s_0}^{mc} = |a|^{-1/(2r+1)} \left[ \int_{P^r} |y|^{2rs+r}|dy \right]_{s=s_0}^{mc} = |a|^{-1/(2r+1)} \frac{q - 1}{q} \frac{1}{q^{\alpha_r} - 1}
\]
the residue of \(Z_f(s)\) at the candidate pole \(s_0 = -1/2 - 1/(2r + 1)\) is
\[
|a|^{-1/(2r+1)} \left( \frac{q - 2}{q} + \frac{q - 1}{q} \frac{1}{q^{\alpha_{r+1}} - 1} + \frac{q - 1}{q} \frac{1}{q^{\alpha_0} - 1} + \frac{q - 1}{q} \frac{1}{q^{\alpha_r} - 1} \right)
\]
multiplied by the factor \(\kappa\) which is different from zero (see (2.6)). Because \(\alpha_{r+1} = (2r + 1)s_0 + r + 2 = 1/2 > 0, \alpha_0 = s_0 + 1 = 1/2 - 1/(2r + 1) > 0\) and \(\alpha_r = 2rs_0 + r + 1 = 1/(2r + 1) > 0\), we have that the last three terms of this expression are strictly positive. Consequently the whole expression is strictly positive and thus different from zero, so that \(-1/2 - 1/(2r + 1)\) is a pole of \(Z_f(s)\).

If \(l\) is even and larger than 2, write \(l = 2r\). We have to blow up \(r\) times to obtain an embedded resolution. We have \(E_1(2, 2), E_2(4, 3), E_3(6, 4), \ldots, E_{r-1}(2r - 2, r), E_r(2r, r + 1)\). We obtain the first dual resolution graph if \(-a\) is a square in \(K\). Otherwise, we obtain the second dual resolution graph.

\[E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \rightarrow E_{r-1} \rightarrow E_r\]

It follows from section 3.1 that the candidate poles associated to \(E_1, \ldots, E_{r-1}\) are not poles. The other candidate poles have real part \(-1\) or \(- (r + 1)/(2r) = -1/2 - 1/(2r)\) in the first case and \(- (r + 1)/(2r) = -1/2 - 1/(2r)\) in the second case. Now we prove that \(-1/2 - 1/(2r)\) is an element of \(\mathcal{P}_2^K\). Suppose first that \(p \neq 2\). Then there exists an element \(a\) of \(K\) with norm 1 for which \(-a\) is not a square in \(K\). For such an \(a\), the residue of \(Z_f(s)\) at \(s_0 = -1/2 - 1/(2r)\) is the non-zero factor \(\kappa\) times
\[
\frac{q - 1}{q} \frac{1}{q^{\alpha_r} - 1} + 1.
\]
Suppose now that \(p = 2\). Remark that every element of the residue field is a square in this case. Let \(b \in R^\times\). If \(b' \in b + P\), then \(b'^2 - b^2 \in P^2\). Consequently, there exists an \(a \in -b^2 + P\) such that \(|a + x^2| = 1/q\) for all \(x \in b + P\). For such an \(a\), the residue of \(Z_f(s)\) at \(s_0 = -1/2 - 1/(2r)\) is the non-zero factor \(\kappa\) times
\[
\frac{q - 1}{q} \frac{1}{q^{\alpha_r} - 1} + \frac{q - 1}{q} + \frac{1}{q} \left( \frac{1}{q} \right)^{s_0}.
\]
Because $\alpha_{r-1} = (2r - 2)s_0 + r = 1/r > 0$, we obtain in the two cases that this residue is strictly positive, which implies that $-1/2 - 1/(2r)$ is a pole.

Our conclusion of part (a) is thus
\[
\{s_0 \mid \exists f \in K<<x_1, x_2>> : \text{mult}(f) = 2 \text{ and } Z_f(s) \text{ has a pole with real part } s_0\} = \left\{-\frac{1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\right\} \cup \left\{-\frac{1}{2}\right\}.
\]

Remark that Newton polyhedra could also be used to deal with (a), see [DH].

(b) Suppose that $f$ is an element of $K<<x_1, x_2>>$ with multiplicity 3. Up to an affine coordinate transformation, there are three cases for $f_3$.

We consider the case that $f_3$ is a product of three different linear factors over $K^\text{alg cl}$. Then we obtain an embedded resolution after one blowing-up. There are three possibilities for the dual resolution graph, depending on whether $f_3$ splits into linear factors over $K$, $f_3$ is a product of a linear factor and an irreducible factor of degree 2 over $K$ or $f_3$ is irreducible over $K$. The dual resolution graphs are respectively

\[\text{\begin{table}[
\begin{array}{c}
\ast & \ast & \ast \\
\ast & & \\
& & \\
\end{array}\end{table}}\]

The equations of $f_3 \circ g$ in the charts determined by $(y_1, y_2) \mapsto (y_1, y_1 y_2)$ and $(z_1, z_2) \mapsto (z_1 z_2, z_2)$ are respectively of the form $y_1^3 h_1$ and $z_3^3 h_2$. In the last case for example, we have that $h_1$ and $h_2$ are non-vanishing on the exceptional curve. The real parts of the candidate poles of $Z_f(s)$ are $-1$ and $-2/3 = -1/2 - 1/6$ in the first two cases and $-2/3 = -1/2 - 1/6$ in the last case.

The other cases are treated in [SV] for the topological zeta function and are very similar for Igusa’s $p$-adic zeta function.

(c) Suppose that $f$ is an element of $K<<x_1, x_2>>$ with multiplicity at least 4. We explained in (4.1.4) that $Z_f(s)$ has no pole with real part in $]-\infty, -1/2\} \cup \{-1\}$. □

(4.1.8) Let $\chi$ be a character of $R^\times$. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}_{n,\chi}^K$ by
\[
\mathcal{P}_{n,\chi}^K := \{s_0 \mid \exists f \in F_n^K : Z_{f,\chi}(s) \text{ has a pole with real part } s_0\}.
\]

**Theorem.** We have
\[
\mathcal{P}_{2,\chi}^K \cap ]-\infty, -1/2[ \subseteq \left\{-\frac{1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\right\}
\]

and at most one value in $]-1, -1/2]$ is the real part of a pole of a fixed Igusa’s $p$-adic zeta function.
Proof. This inclusion is proved in the same way as the analogous inclusion of the previous theorem. Again we need the proposition in 3.1. □

4.2 Surfaces

In this section, we prove the following theorem.

(4.2.0) Theorem. We have
\[ P_K^3 \cap -\infty, -1] = \left\{ -1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>1} \right\}. \]

Moreover, if \( f \in K<<x, y, z>> \) has multiplicity at least 3, then \( Z_f(s) \) has no pole with real part less than -1.

Remark. (i) It is a priori not obvious that the smallest value of \( P_K^3 \) is \(-\frac{3}{2}\).
This is in contrast with the fact that it easily follows from lemma 4.1.3 that the smallest value of \( P_K^2 \) is \(-1\).

(ii) Let \( \chi \) be a character of \( R^\times \). Then one proves analogously as below that an element of \( P_{3, \chi}^K \) less than \(-1\) is of the form \(-\frac{1}{i} - 1\), \( i \in \mathbb{Z}_{>1} \). Using the remark in section 3.2, the arguments below will also imply that \( Z_{f, \chi}(s) \) has no pole with real part less than \(-1\) if \( f \in K<<x, y, z>> \) has multiplicity at least 3.

4.2.1 Multiplicity 2

(4.2.1.1) Let \( f(x), x = (x_1, \ldots, x_n) \), be a \( K \)-analytic function on an open and compact subset \( X \) of \( K^n \). Let \( g(y), y = (y_1, \ldots, y_m) \), be a \( K \)-analytic function on an open and compact subset \( Y \) of \( K^m \). Then \( f(x) + g(y) \) is a \( K \)-analytic function on the open and compact subset \( X \times Y \) of \( K^{n+m} \). Put \( A(s, \rho) := q^{s+1} - 1 \) if \( \rho \) is the trivial character of \( R^\times \) and \( A(s, \rho) := 1 \) if \( \rho \) is another character of \( R^\times \).

Fix a character \( \chi \) of \( R^\times \). Suppose that the only critical value of \( f \) and \( g \) is zero. Then the poles of \( A(s, \chi)Z_{f+g, \chi}(s) \) are of the form \( s_1 + s_2 \) with \( s_1 \) a pole of \( A(s, \chi')Z_{f, \chi'}(s) \) and \( s_2 \) a pole of \( A(s, \chi'')Z_{g, \chi''}(s) \) for some characters \( \chi' \) and \( \chi'' \) of \( R^\times \) satisfying \( \chi' \chi'' = \chi \) (see [Ig1] or [De2 (5.1)]).

(4.2.1.2) Proposition. The set \( \{ s_0 \mid \exists f \in K<<x, y, z>>: \text{mult}(f) = 2 \text{ and } Z_f(s) \text{ has a pole with real part } s_0 \} \cap ]-\infty, -1[ \) is equal to
\[ \left\{ -1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>1} \right\}. \]

Proof. Let \( f \) be an element of \( K<<x, y, z>> \) with multiplicity 2. Up to an affine coordinate transformation, the part of degree 2 of \( f \) is equal to \( ax^2 + by^2 + cz^2 \), with \( a, b, c \in K \) and \( a \neq 0 \). Using (4.1.5), we may suppose that \( f \) is of the form \( x^2 + g(y, z) \) with \( g(y, z) \in K<<y, z>> \). The statement in (4.2.1.1) and the result
for curves imply that the real part of a pole of $Z_f(s)$ is of the form $-1 - 1/i$, $i \in \mathbb{Z}_{>1}$, if it is less than $-1$.

Now we prove the other inclusion. Using the $p$-adic stationary phase formula \[ \text{Ig3, Theorem 10.2.1}, \] we obtain that Igusa’s $p$-adic zeta function of $xy + z^i$, $i \geq 2$, is equal to

$$
\left( \frac{q-1}{q} \right) \left( \frac{1 - q^{-s-3} + (q-1)(q^{-2s-4} + q^{-3s-5} + \cdots + q^{-(i-1)s-(i+1)})}{(1 - q^{-s-1})(1 - q^{-is-(i+1)})} \right).
$$

The real poles of this zeta function are $-1$ and $-1 - 1/i$. \qed

### 4.2.2 Multiplicity larger than 2

#### (4.2.2.1) Let $f$ be an element of $K \ll x, y, z \gg$. Fix a (small enough) neighbourhood $X$ of $0 \in K^3$ on which $f$ is convergent and an embedded resolution $g : Y \to X$ of $f$ which is a $K$-bianalytic map at the points where $f$ has normal crossings and which is a composition of blowing-ups $g_{ij} : X_i \to X_j$ with centre a $K$-analytic closed submanifold $D_j$ and with exceptional surface $E_i$ satisfying:

(a) the codimension of $D_j$ in $X_j$ is at least 2;
(b) $D_j$ is a subset of the zero locus of the strict transform of $f$ on each chart (the strict transform of $f$ is not defined globally);
(c) the union of the exceptional varieties in $X_j$ has only normal crossings with $D_j$, i.e., for all $P \in D_j$, there are three surface germs through $P$ which are in normal crossings such that each exceptional surface germ through $P$ is one of them and such that the germ of $D_j$ at $P$ is the intersection of some of them;
(d) the image of $D_j$ in $X \subset K^3$ contains the origin of $K^3$; and
(e) $D_j$ contains a point in which the pullback of $f$ has not normal crossings.

Remark that such a resolution always exists by Hironaka’s theorem \[ \text{[H]} \].

#### (4.2.2.2) The following table gives the numerical data of $E_i$. In the columns, the dimension of $D_j$ is kept fixed. In the rows, the number of exceptional surfaces through $D_j$ is kept fixed. So $E_k$, $E_l$ and $E_m$ represent exceptional surfaces that contain $D_j$. The multiplicity of the strict transform of $f$ at $D_j$ is denoted by $\mu_{D_j}$.

| $D_j$ is a point $P$ | $D_j$ is a curve $L$ |
|---------------------|---------------------|
| $E_k$               | $(\mu_P, 3)$        |
| $E_k$ and $E_l$     | $(N_k + \mu_P, \nu_k + 2)$ |
| $E_k$, $E_l$ and $E_m$ | $(N_k + N_l + \mu_P, \nu_k + \nu_l + 1)$ |
|                     | $(\mu_L, 2)$        |
|                     | $(N_k + \mu_L, \nu_k + 1)$ |
|                     | $(N_k + N_l + \mu_L, \nu_k + \nu_l)$ |
|                     | $(N_k + N_l + N_m + \mu_P, \nu_k + \nu_l + \nu_m)$ |
|                     | /                   |

#### (4.2.2.3) Lemma. Suppose that $\text{mult}(f) \geq 3$. If there is no exceptional surface through $D_j$, then $-\nu_i/N_i \geq -1$. 

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Proof. The analogous statement for the local topological zeta function is treated in [SV] (3.3.3). The proof of the lemma is a trivial adaptation of the proof there. □

(4.2.2.4) Suppose that $D_j$ is contained in at least one exceptional surface and that the real parts of the candidate poles associated to the exceptional surfaces that pass through $D_j$ are larger than or equal to $-1$. Then the table in (4.2.2.2) implies that also $-\nu_i/N_i \geq -1$, unless $D_j$ is a regular point $P$ of the strict transform of $f$ around $P$ through which only one exceptional surface $E_0$ passes and $-\nu_0/N_0 = -1$. Suppose that we are in this situation. Let $Z_0$ be a (small enough) neighbourhood of $P$ such that, if we restrict the blowing-ups $g_{ij}$ to the inverse image of $Z_0$, we get an embedded resolution $h = h_1 \circ \cdots \circ h_s$ of the pullback of $f$ which is a composition of blowing-ups $h_i : Z_i \to Z_{i-1}$, $i \in \{1, \ldots, s\}$, with centre $D'_{i-1} := D_{i-1} \cap Z_{i-1}$ and exceptional surface $E'_i := E_i \cap Z_i$ for which $P$ is in the image of $D'_{i-1}$ under $h_1 \circ \cdots \circ h_{i-1}$.

Remark that it can happen that $g_{ij}$ is a $K$-bianalytic map on the inverse image of $Z_0$. Because we did not specify the indices in (4.2.2.1), we were able to get a nice notation here. From now on, we study the resolution $h : Z_s \to Z_0$ of the pullback of $f$.

**Lemma.** (a) If $D_i = D'_i$, then $D_i$ is a subset of $E'_0 := E_0 \cap Z_0$.

(b) Suppose that $\text{mult}(f) \geq 3$. Then we have $\nu_i \leq N_i + 1$ for every exceptional surface $E_i$, $i \in \{1, \ldots, s\}$. Moreover, $\nu_i = N_i + 1$ if and only if $D_{i-1}$ is a point and the numerical data of every exceptional surface $E_j$ different from $E_0$ and through $D_{i-1}$ satisfy $\nu_j = N_j + 1$.

(c) If $\text{mult}(f) \geq 3$ and if the numerical data of $E_i$ satisfy $\nu_i = N_i + 1$, then $-\nu_i/N_i \neq -\nu_j/N_j$ for every exceptional surface $E_j$ that intersects $E_i$ at some stage of the resolution process.

**Proof.** See [SV] (3.3.5),(3.3.6) and (3.3.7)]. □

**Proposition.** If $\text{mult}(f) \geq 3$, then $Z_f(s)$ has no pole with real part less than $-1$.

**Proof.** The proof is analogous to the one in [SV] (3.3.8)], except that we have to use the proposition in 3.2. □

**Appendix. Poles and divisibility of the $M_i$**

Suppose that $f$ is a $K$-analytic function on $R^n$ defined by a power series over $R$ which is convergent on the whole of $R^n$. Let $l$ be the smallest real part of a pole of $Z_f(s)$ and let $M_i$ be the number of solutions of $f(x) \equiv 0 \mod P^i$ in $(R/P^i)^n$.

**Proposition.** There exists an integer $a$ which is independent of $i$ such that $M_i$...
is an integer multiple of $q^{(n+l)i-a}$ for all $i \in \mathbb{Z}_{\geq 0}$.

**Remark.** (i) The number $\lceil (n+l)i-a \rceil$ is the smallest integer larger than or equal to $(n+l)i-a$, which rises $(n+l > 0)$ linearly as a function of $i$ with a slope depending on $l$.

(ii) The statement is trivial if $(n+l)i-a \leq 0$ because the $M_i$ are integers. If $(n+l)i-a > 0$, which is the case for $i$ large enough, it claims that $M_i$ is divisible by $q^{(n+l)i-a}$.

**Proof.** Put $t = q^{-s}$. It follows from (2.2) that we can write

$$Z_f(t) = \frac{A(t)}{\prod_{j \in T} (1 - q^{-\nu_j tN_j})},$$

where $A(t)$ is a polynomial with coefficients in the set $S := \{z/q^i \mid z \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 0}\}$. By using the division algorithm for polynomials we can write

$$Z_f(t) = \frac{B(t)}{\prod_{j \in K} (1 - q^{-\nu_j tN_j})},$$

where $B(t)$ is a polynomial with coefficients in $S$ and where $K := \{j \in T \mid -\nu_j/N_j \geq l\}$.

The Poincaré series $P(t)$ of $f$ is defined by

$$P(t) = \sum_{i=0}^{\infty} M_i \frac{t^i}{q^{ni}},$$

and can be obtained from $Z_f(t)$ by the relation

$$P(t) = \frac{1 - tZ_f(t)}{1 - t}.$$

It easily follows from the defining integral of Igusa’s $p$-adic zeta function that $Z_f(t = 1) = 1$. Consequently, $1 - tZ_f(t)$ is divisible by $1 - t$ and $P(t)$ can be written as

$$P(t) = \frac{C(t)}{\prod_{j \in K} (1 - q^{-\nu_j tN_j})},$$

where $C(t)$ is a polynomial with coefficients in $S$.

We will say that a formal power series in $t$ has the divisibility property if the coefficient of $t^i/q^{ni}$ is an integer multiple of $q^{(n+l)i}$ for every $i$.

For $j \in K$, the series

$$\frac{1}{1 - q^{-\nu_j tN_j}} = \sum_{i=0}^{\infty} q^{-i\nu_j tN_j} = \sum_{i=0}^{\infty} q^{i(nN_j-\nu_j)} \frac{t^{iN_j}}{q^{niN_j}}$$

has the divisibility property because $nN_j - \nu_j$ is an integer larger than or equal to $N_j(n+l)$. Let $a$ be an integer such that the polynomial $D(t) := q^aC(t)$ has the divisibility property. Remark that $C(t) = q^{-a}D(t)$.
One can easily check that the product of a finite number of power series with the divisibility property also has the divisibility property. This implies that \( P(t) \) is a power series with the divisibility property, multiplied by \( q^{-a} \). Hence \( M_i \) is an integer multiple of \( q^{\langle n+l \rangle i^2-a} = q^{\langle n+l \rangle i-i^3} \) for all \( i \). \( \square \)

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