Optimal Hedging in Incomplete Markets

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We consider the problem of optimal hedging in an incomplete market with an established pricing kernel. In such a market, prices are uniquely determined, but perfect hedges are usually not available. We work in the rather general setting of a Lévy-Ito market, where assets are driven jointly by an \( n \)-dimensional Brownian motion and an independent Poisson random measure on an \( n \)-dimensional state space. Given a position in need of hedging and the instruments available as hedges, we demonstrate the existence of an optimal hedge portfolio, where optimality is defined by use of an expected least squared-error criterion over a specified time frame, and where the numéraire with respect to which the hedge is optimized is taken to be the benchmark process associated with the designated pricing kernel.

Key words: Incomplete markets, pricing kernels, hedge ratios, Brownian motion, Lévy processes, Lévy measures, Lévy-Ito processes, Poisson random measure, simulations.

I. INTRODUCTION

This paper is concerned with optimal hedging in incomplete markets. Hedging is important, since it lies at the heart of risk management. Historically, hedging in complete markets has played an important role in the foundations of option-pricing theory \[3, 5, 9, 14, 18\]. From a modern perspective, however, hedging arguments need not be invoked in the determination of prices. Instead, pricing is achieved by use of a pricing kernel. The connection between the two approaches is that in a complete market the specification of the price processes of a sufficiently large number of assets is enough to allow one to determine the pricing kernel associated with that market. Nevertheless, in the absence of market frictions, the prices of all of financial assets are determined in an incomplete market, including those of derivatives, once we designate a pricing kernel. In the incomplete market situation, however, one can not in general form a perfect hedge of a given position. This leaves us with a more precise statement of our problem: namely, determination of the optimal strategy for hedging a financial position in an incomplete market, given the set of hedging assets at the hedger's disposal. The optimal hedge corresponds to the maximal possible elimination of risk in a financial position making use of the instruments available for this purpose.

The paper is structured as follows. In Section II we briefly summarize several of the mathematical ideas that we require. In particular, we define what we mean by a Lévy-Ito process and we present a form of Ito’s formula that is applicable to Lévy-Ito processes. Then we present some useful versions of the Ito product and quotient rules for such processes. The Brownian versions of these rules will be familiar, but the corresponding Lévy-Ito rules appear to be less well known, and do not seem previously to have been presented systematically in all their different versions; so we do so here. We also comment on the form that the Ito isometry takes is the Lévy-Ito setting. In Section III we introduce the family of risky assets
that we work with in the hedging problem. In particular, we argue that the most natural approach to hedging arises when the values of the various assets under consideration are expressed in units of the benchmark process associated with the pricing kernel. In Section IV we consider the hedging of a position in a risky asset in a one-dimensional Lévy-Ito market in the situation where the hedging instrument is another risky asset driven by the same one-dimensional Lévy-Ito process. In general, a perfect hedge is not possible in such a market, so one aims for a best possible hedge instead. We take the view that the goal is that of optimal elimination of the risk, which we characterize in a natural way using a quadratic optimization criterion. See [2, 4, 7, 8, 10, 12, 13, 15, 17, 20, 21] for various aspects of quadratic hedging. We obtain a formula for the optimal hedge in the case of a single hedging asset. This is presented in Proposition 4. We refer to the asset being hedged as the contract asset. The terminology is inherited from the language of derivatives pricing, though in the present context the asset need not be a derivative, and indeed the assets involved are essentially on an equal footing. We refer to the second asset as the hedging asset.

We then move on in Section V to consider the case where two hedging assets are available to hedge the contract asset. Again, we are able to work out an explicit formula for the optimal hedge, and this is given in Proposition 5. We illustrate the result in the simplest possible situation: this is the case of a geometric Lévy asset for which the Lévy process is a linear combination of a Brownian motion and a Bernoulli process. We refer to a Lévy process of this type as a Bernoulli jump diffusion. By a Bernoulli process we mean a compound Poisson process for which each jump is characterized by an independent Bernoulli random variable taking one of two possible values depending on the outcome of chance. We consider the situation where the contract asset and the hedging assets are geometric Bernoulli jump diffusions driven by the same Lévy-Ito process. We illustrate the fundamental fact that a better hedge can be obtained by using both of the hedging assets rather than just a single hedging asset, even though a perfect hedge is not obtainable as long as the Brownian component of the driving process is present. On the other hand, if the Brownian volatility is small for the various assets under consideration, then a nearly perfect hedge can be obtained. In Section VI we consider the more general situation where we hedge the contract asset with a position in \( n \) risky assets. In Proposition 6 we work out a general expression for the optimal hedge in such a market, and in Proposition 7 we show that if there is no redundancy among the hedging assets then the optimal hedge obtained with \( n+1 \) hedging instruments is better than the optimal hedge obtained with \( n \) such instruments.

II. MATHEMATICAL PRELIMINARIES

We begin with a brief account of the mathematical context in which we set the hedging problem. The Lévy-Ito market provides a modelling framework of considerable generality. In particular, it contains all of the familiar Brownian motion driven models and Lévy driven models as special cases. The setup is as follows. We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) that supports an \( n \)-dimensional Brownian motion \( \{W_t\}_{t \geq 0} \) alongside an independent Poisson random measure \( \{N(dx, dt)\} \) with mean measure \( \nu(dx)dt \), where \( \nu(dx) \) is taken to be the Lévy measure associated with an \( n \)-dimensional pure-jump Lévy process. We write \( \{\mathcal{F}_t\}_{t \geq 0} \) for the augmented filtration generated by \( \{W_t\} \) and \( \{N(dx, dt)\} \). See [11, 6, 11, 16, 19] for aspects of the theory of Lévy-Ito processes. In the one-dimensional case, by a Lévy-Ito process driven by \( \{W_t\} \) and \( \{N(dx, dt)\} \) we mean a process \( \{X_t\}_{t \geq 0} \) satisfying a stochastic
The product rule for Lévy-Ito processes takes the following form:

\[ dX_t = \alpha_t \, dt + \beta_t \, dW_t + \int_{|x| < 1} \gamma_t(x) \, \tilde{N}(dx, dt) + \int_{|x| \geq 1} \delta_t(x) \, N(dx, dt), \]  

(1)

where

\[ \tilde{N}(dx, dt) = N(dx, dt) - \nu(dx) \, dt. \]  

(2)

We require that \( \{\alpha_t\}_{t \geq 0} \) and \( \{\beta_t\}_{t \geq 0} \) be \( \{\mathcal{F}_t\}\)-adapted, that \( \{\gamma_t(x)\}_{t \geq 0, |x| < 1} \) and \( \{\delta_t(x)\}_{t \geq 0, |x| \geq 1} \) be \( \{\mathcal{F}_t\}\)-predictable, and that

\[ P \left[ \int_0^t \left( |\alpha_s| + \beta_s^2 + \int_{|x| < 1} \gamma_s(x)^2 \nu(dx) \right) \, ds < \infty \right] = 1 \]  

(3)

for \( t \geq 0 \). Then we have the following generalization of Ito’s formula (see, for example, reference [1], Theorem 4.4.7):

**Proposition 1.** Let \( F : \mathbb{R} \to \mathbb{R} \) admit a continuous second derivative and let \( \{X_t\} \) be a Lévy-Ito process for which the dynamics are given as in [1]. Then for \( t \geq 0 \) it holds that

\[ dF(X_t) = \left[ \alpha_t F'(X_{t-}) + \frac{1}{2} \beta_t^2 F''(X_{t-}) \right] dt + \beta_t F'(X_{t-}) \, dW_t \]
\[ + \int_{|x| < 1} [F(X_{t-} + \gamma_t(x)) - F(X_{t-}) - \gamma_t(x)F'(X_{t-})] \, \nu(dx) \, dt \]
\[ + \int_{|x| < 1} [F(X_{t-} + \gamma_t(x)) - F(X_{t-})] \, \tilde{N}(dx, dt) \]
\[ + \int_{|x| \geq 1} [F(X_{t-} + \delta_t(x)) - F(X_{t-})] \, N(dx, dt). \]  

(4)

We can use the generalized Ito formula to work out the Ito product and quotient rules for such processes. The results are useful, but do not seem to have been systematically recorded in the literature, so we set them down in full below. Let \( \{X^1_t\}_{t \geq 0} \) and \( \{X^2_t\}_{t \geq 0} \) be Lévy-Ito processes, each satisfying stochastic differential equations of the form [1], such that

\[ dX^1_t = \alpha^1_t \, dt + \beta^1_t \, dW_t + \int_{|x| < 1} \gamma^1_t(x) \, \tilde{N}(dx, dt) + \int_{|x| \geq 1} \delta^1_t(x) \, N(dx, dt) \]  

(5)

and

\[ dX^2_t = \alpha^2_t \, dt + \beta^2_t \, dW_t + \int_{|x| < 1} \gamma^2_t(x) \, \tilde{N}(dx, dt) + \int_{|x| \geq 1} \delta^2_t(x) \, N(dx, dt). \]  

(6)

**Lemma 1.** The product rule for Lévy-Ito processes takes the following form:

\[ d(X^1_t \, X^2_t) = [\alpha^1_t X^2_{t-} + \alpha^2_t X^1_{t-} + \beta^1_t \beta^2_t] \, dt + (\beta^1_t X^2_{t-} + \beta^2_t X^1_{t-}) \, dW_t + \int_{|x| < 1} \gamma^1_t(x) \gamma^2_t(x) \, \nu(dx) \, dt \]
\[ + \int_{|x| < 1} (\gamma^1_t(x) \gamma^2_t(x) + \gamma^1_t(x) X^2_{t-} + \gamma^2_t(x) X^1_{t-}) \, \tilde{N}(dx, dt) \]
\[ + \int_{|x| \geq 1} (\delta^1_t(x) \delta^2_t(x) + \delta^1_t(x) X^2_{t-} + \delta^2_t(x) X^1_{t-}) \, N(dx, dt). \]  

(7)
In the proportional case, the product rule takes the form

$$X_t^1 X_t^2 = \frac{1}{4} (X_t^1 + X_t^2)^2 - \frac{1}{4} (X_t^1 - X_t^2)^2. \quad (8)$$

A calculation then gives the result claimed.

Proof. This is similar to the proof of the corresponding result for Ito processes, and is obtained by applying Ito’s formula to each side of the identity

Now let \( \{X_t^1\} \) and \( \{X_t^2\} \) be Lévy-Ito processes such that \( \{X_t^1\}, \{X_t^2\} \) are strictly positive. Then we obtain the following.

**Lemma 2.** The quotient rule for Lévy-Ito processes is given by

\[
\begin{align*}
\frac{dX_t^1}{X_t^2} &= \left[ \frac{\alpha_t^1 X_t^2 - \alpha_t^2 X_t^1}{(X_t^2)^2} + \frac{(\beta_t^2)^2 X_t^1 - \beta_t^1 \beta_t^2 X_t^2}{(X_t^2)^3} \right] dt + \frac{\beta_t^1 X_t^2 - \beta_t^2 X_t^1}{(X_t^2)^2} dW_t + \int_{|x|<1} (\gamma_t^1(x) X_t^2 - \gamma_t^2(x) X_t^1) \tilde{N}(dx, dt) + \int_{|x|\geq 1} \frac{\delta_t^1(x) X_t^2 - \delta_t^2(x) X_t^1}{X_t^2(X_t^2 + \delta_t^2(x))} N(dx, dt) \, .
\end{align*}
\]

\[
(9)
\]

Proof. First one uses Proposition 1 to work out the dynamics of the process \( \{1/X_t^2\} \). Then one uses Lemma 1 to work out the dynamics of the product \( \{X_t^1 \times 1/X_t^2\} \).

For applications in finance, one often makes use of the “proportional” versions of the Lévy-Ito product and quotient rules, which are applicable if we assume that \( \{X_t^1\}, \{X_t^-\}, \{X_t^-\} \) are strictly positive. The stochastic differential equations for \( \{X_t^1\} \) and \( \{X_t^2\} \) will be assumed in Lemmas 3 and 4 to take the proportional form

\[
\begin{align*}
\frac{dX_t^1}{X_t^-} &= \alpha_t^1 \, dt + \beta_t^1 \, dW_t + \int_{|x|<1} \gamma_t^1(x) \tilde{N}(dx, dt) + \int_{|x|\geq 1} \delta_t^1(x) \, N(dx, dt) \quad (10)
\end{align*}
\]

and

\[
\begin{align*}
\frac{dX_t^2}{X_t^-} &= \alpha_t^2 \, dt + \beta_t^2 \, dW_t + \int_{|x|<1} \gamma_t^2(x) \tilde{N}(dx, dt) + \int_{|x|\geq 1} \delta_t^2(x) \, N(dx, dt) \quad . \quad (11)
\end{align*}
\]

Then we have the following formulae, which arise as consequences of Lemmas 1 and 2.

**Lemma 3.** In the proportional case, the product rule takes the form

\[
\begin{align*}
\frac{d(X_t^1 X_t^2)}{X_t^- X_t^-} &= \left[ \alpha_t^1 + \alpha_t^2 + \beta_t^1 \beta_t^2 + \int_{|x|<1} \gamma_t^1(x) \gamma_t^2(x) \nu(dx) \right] dt + (\beta_t^1 + \beta_t^2) dW_t + \int_{|x|<1} (\gamma_t^1(x) \gamma_t^2(x) + \gamma_t^1(x) + \gamma_t^2(x)) \tilde{N}(dx, dt) + \int_{|x|\geq 1} (\delta_t^1(x) \delta_t^2(x) + \delta_t^1(x) + \delta_t^2(x)) \, N(dx, dt) \right].
\end{align*}
\]

\[
(12)
\]
Lemma 4. In the proportional case, the quotient rule takes the form
\[
\begin{align*}
\frac{d}{dt} \left( \frac{X_t^1}{X_t^2} \right) &= \frac{X_t^1}{X_t^2} \left[ \frac{\alpha_t^1 - \alpha_t^2 - \beta_t^2 (\beta_t^1 - \beta_t^2)}{1 + \gamma_t^2(x)} - \int_{|x| < 1} \frac{\gamma_t^1(x) - \gamma_t^2(x)}{1 + \gamma_t^2(x)} \nu(dx) \right] dt \\
&+ (\beta_t^1 - \beta_t^2) dW_t + \int_{|x| < 1} \frac{\gamma_t^1(x) - \gamma_t^2(x)}{1 + \gamma_t^2(x)} \tilde{N}(dx, dt) + \int_{|x| \geq 1} \frac{\delta_t^1(x) - \delta_t^2(x)}{1 + \delta_t^2(x)} N(dx, dt).
\end{align*}
\] (13)

In some situations it can be useful to consider processes for which the dynamical equation takes the form
\[
\begin{align*}
dX_t &= \alpha_t dt + \beta_t dW_t + \int_{|x| < 1} \gamma_t(x) \tilde{N}(dx, dt) + \int_{|x| \geq 1} \delta_t(x) N(dx, dt),
\end{align*}
\] (14)

where the integral involving the large jumps is taken with respect to the compensated Poisson random measure. In order for this to be possible, \(\{\delta_t(x)\}\) must satisfy
\[
\mathbb{P} \left[ \int_{|x| \geq 1} |\delta_t(x)| \nu(dx) < \infty \right] = 1,
\] (15)

which is sufficient to ensure that the integral with respect to the compensated Poisson random measure exists for large jumps. If we impose the stronger condition
\[
\mathbb{P} \left[ \int_{|x| \geq 1} \delta_t(x)^2 \nu(dx) < \infty \right] = 1,
\] (16)

we can simplify and unify the notation by using a common symbol \(\{\gamma_t(x)\}_{t \geq 0, x \in \mathbb{R}}\) for the coefficients of the compensated Poisson random measures for small jumps and large jumps. Then we write
\[
\begin{align*}
dX_t &= \alpha_t dt + \beta_t dW_t + \int_x \gamma_t(x) \tilde{N}(dx, dt),
\end{align*}
\] (17)

and the associated condition on the coefficients takes the form
\[
\mathbb{P} \left[ \int_0^t \left( |\alpha_s| + \beta_s^2 + \int_x \gamma_s(x)^2 \nu(dx) \right) ds < \infty \right] = 1,
\] (18)

in place of (3). We shall refer to processes satisfying (17) and (18) as being “symmetric” since large and small jumps are treated similarly. In the symmetric case Ito’s formula takes the following form:

Proposition 2. Let \(F : \mathbb{R} \rightarrow \mathbb{R}\) admit a continuous second derivative and let \(\{X_t\}\) be a symmetric Lévy-Ito process for which the dynamics are as in (17). Then for \(t \geq 0\) we have
\[
\begin{align*}
dF(X_t) &= \left[ \alpha_t F'(X_t^-) + \frac{1}{2} \beta_t^2 F''(X_t^-) \right] dt + \beta_t \frac{dF'}{dx}(X_t^-) dW_t \\
&+ \int_x \left[ F(X_t^- + \gamma_t(x)) - F(X_t^-) - \gamma_t(x) F'(X_t^-) \right] \nu(dx) dt \\
&+ \int_x \left[ F(X_t^- + \gamma_t(x)) - F(X_t^-) \right] \tilde{N}(dx, dt).
\end{align*}
\] (19)
The various forms of the Ito product and quotient rules simplify for symmetric proportional processes. Let \( \{X^1_t\}, \{X^1_{-t}\}, \{X^2_t\}, \{X^2_{-t}\} \) be strictly positive. Then we can write

\[
dX^1_t = X^1_t \left[ \alpha^1_t \, dt + \beta^1_t \, dW_t + \int_x \gamma^1_t(x) \tilde{N}(dx, dt) \right]
\]

and

\[
dX^2_t = X^2_t \left[ \alpha^2_t \, dt + \beta^2_t \, dW_t + \int_x \gamma^2_t(x) \tilde{N}(dx, dt) \right],
\]

and we obtain the following.

**Lemma 5.** In the symmetric proportional case the product rule takes the form

\[
d(X^1_t X^2_t) = X^1_t X^2_t \left[ \left( \alpha^1_t + \alpha^2_t + 2 \beta^1_t \beta^2_t + \int_x \gamma^1_t(x) \gamma^2_t(x) \nu(dx) \right) dt + (\beta^1_t + \beta^2_t) dW_t \right.
\]

\[
+ \left( \int_x (\gamma^1_t(x) + \gamma^2_t(x)) \quad \tilde{N}(dx, dt) \right].
\]

**Lemma 6.** In the symmetric proportional case the quotient rule takes the form

\[
d \left( \frac{X^1_t}{X^2_t} \right) = \frac{X^1_t}{X^2_t} \left[ \left( \alpha^1_t - \alpha^2_t - 2 \beta^1_t \beta^2_t - 2 \int_x \gamma^1_t(x) \gamma^2_t(x) \nu(dx) \right) dt \right.
\]

\[
+ \left( \beta^1_t - \beta^2_t \right) dW_t + \left( \int_x \frac{\gamma^1_t(x) - \gamma^2_t(x)}{1 + \gamma^2_t(x)} \tilde{N}(dx, dt) \right].
\]

The corresponding results for \( n \)-dimensional Lévy-Ito process are straightforward. Finally, we note that the Ito isometry can be generalized in a useful way in the present context. So far, we have not imposed any integrability conditions on the processes that we have considered. For the Ito isometry we require that the process should satisfy an \( L^2 \) condition.

**Proposition 3.** Let \( \{X_t\}_{t \geq 0} \) be a Lévy-Ito process such that

\[
X_t = X_0 + \int_0^t \beta_s \, dW_s + \int_0^t \int_x \gamma_s(x) \tilde{N}(dx, ds),
\]

where \( X_0 \) is a constant and

\[
\mathbb{P} \left[ \int_0^t \left( \beta_s^2 + \int_x \gamma_s(x)^2 \nu(dx) \right) ds < \infty \right] = 1.
\]

If \( \mathbb{E} [X^2_t] < \infty \) for \( t \geq 0 \), then \( \{X_t\}_{t \geq 0} \) is a martingale and for \( t \geq 0 \) it holds that

\[
\mathbb{E} [(X_t - X_0)^2] = \mathbb{E} \left[ \int_0^t \left( \beta_s^2 + \int_x \gamma_s(x)^2 \nu(dx) \right) ds \right].
\]

Again, the corresponding result for an \( n \)-dimensional Lévy-Ito process is straightforward.
III. RISKY ASSETS

We proceed to consider the problem of optimal hedging. It should be emphasized from the outset that we are not concerned here with the problem of derivative pricing via hedging arguments. We assume that prices are known and we look instead at the problem of hedging a position in one asset by use of a self-financing portfolio of other assets. In a complete market we know that an exact hedge can be obtained in such a situation; but we work in an incomplete market, where exact hedges are generally not available, so we look for an optimal hedge instead. We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) where \(\mathbb{P}\) is the real-world measure. The market filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) is taken to be the augmented filtration generated by a one-dimensional Brownian motion \(\{W_t\}\) and an independent one-dimensional Poisson random measure \(\{N(dx, dt)\}\), where the Poisson random measure is that associated with a one-dimensional pure-jump Lévy process in the sense discussed in Section II.

We introduce a fiat currency, which we call the domestic currency, in units of which prices are conventionally expressed. The market is assumed to be endowed with a pricing kernel \(\{\pi_t\}_{t \geq 0}\) for which the dynamics take the form

\[
d\pi_t = -\pi_t \left[ r_t dt + \lambda_t dW_t + \int_x \Lambda_t(x) \dot{N}(dx, dt) \right].
\]

We assume that the domestic short rate \(\{r_t\}_{t \geq 0}\) and the Brownian market price of risk \(\{\lambda_t\}_{t \geq 0}\) are adapted and that the jump market price of risk \(\{\Lambda_t(x)\}_{t \geq 0, x \in \mathbb{R}}\) is predictable and such that \(\Lambda_t(x) < 1\) for \(t \geq 0\) and \(x \in \mathbb{R}\). The solution for the pricing kernel is then

\[
\pi_t = \exp \left[ -\int_0^t r_s ds - \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right. \\
\left. - \int_0^t \int_x \kappa_s(x) \dot{N}(dx, ds) - \int_0^t \int_x (e^{-\kappa_s(x)} - 1 + \kappa_s(x)) \nu(dx) ds \right],
\]

where \(\{\kappa_t(x)\}_{t \geq 0, x \in \mathbb{R}}\) is defined by

\[
\kappa_t(x) = \log \left[ \frac{1}{1 - \Lambda_t(x)} \right].
\]

We assume that the market includes a money market asset \(\{B_t\}_{t \geq 0}\) satisfying \(dB_t = r_t B_t dt\), along with one or more risky assets. For a typical risky asset we let \(\{S_t\}_{t \geq 0}\) denote the price process, and we assume that the associated dynamics are of the form

\[
\frac{dS_t}{S_t} = \left[ r_t + \lambda_t \sigma_t + \int_x \Lambda_t(x) \Sigma_t(x) \nu(dx) \right] dt + \sigma_t dW_t + \int_x \Sigma_t(x) \dot{N}(dx, dt),
\]

where \(\{\sigma_t\}_{t \geq 0}\) is adapted, \(\{\Sigma_t(x)\}_{t \geq 0, x \in \mathbb{R}}\) is predictable, and \(\Sigma_t(x) > -1\) for \(t \geq 0\) and \(x \in \mathbb{R}\). For simplicity we assume that \(\{S_t\}\) pays no dividend.

We require that for any such asset the process determined by the product of the pricing kernel and the asset price should be a \(\mathbb{P}\)-martingale. Thus we have

\[
S_t = \frac{1}{\pi_t} \mathbb{E}_t [\pi_u S_u]
\]
for $0 \leq t \leq u < \infty$, where $\mathbb{E}_t$ denotes conditional expectation with respect to $\mathcal{F}_t$. There is another way of expressing this condition which turns out to be useful for our purposes. It is well known that the process \( \{ \xi_t \}_{t \geq 0} \) defined by $\xi_t = 1/\pi_t$ for $t \geq 0$ can be interpreted as a “natural numeraire” or “benchmark”. By the definition of the pricing kernel, we see that for any asset \( \{ S_t \} \) that pays no dividend the process \( \{ \bar{S}_t \} \) defined by $\bar{S}_t = S_t / \xi_t$ represents the price of the original asset expressed in units of the natural numeraire. It follows that the “natural” price of any such asset is a martingale. Then we have

$$S_t = \xi_t \mathbb{E}_t \left[ \frac{S_u}{\xi_u} \right]$$

(32)

for $0 \leq t \leq u < \infty$, or equivalently

$$\bar{S}_t = \mathbb{E}_t \left[ \bar{S}_u \right].$$

(33)

Equation (32) shows that the domestic value of the asset at time $t$ can be represented as the product of the natural numeraire (which can be interpreted as a dividend-adjusted proxy for the market as a whole) and a fluctuating term, given by the conditional expectation of the natural value of the asset at some later time $u$. A form of (33) is used in the theory of derivatives, for instance, when we make use of the pricing formula

$$H_t = \frac{1}{\pi_t} \mathbb{E}_t [\pi_T H_T],$$

(34)

valid for $0 \leq t < T < \infty$, which shows that the natural value $\bar{H}_t = \pi_t H_t$ of the derivative at $t$ is given by the conditional expectation of the natural value of the payoff $\bar{H}_T = \pi_T H_T$.

A calculation making use of (27), (30) and Lemma 5 shows that the stochastic differential equation satisfied by the natural value of the risky asset takes the form

$$d \bar{S}_t = \bar{\sigma}_t dW_t + \int_x \hat{\Sigma}_t(x) \tilde{N}(dx, dt),$$

(35)

where $\bar{\sigma}_t = \sigma_t - \lambda_t$ and $\hat{\Sigma}_t(x) = \Sigma_t(x)(1 - \Lambda_t(x)) - \Lambda_t(x)$, or equivalently

$$\sigma_t = \bar{\sigma}_t + \lambda_t, \quad \Sigma_t(x) = \frac{\hat{\Sigma}_t(x) + \Lambda_t(x)}{1 - \Lambda_t(x)}.$$

(36)

The relations given in (36) show that the Brownian and jump volatilities of the asset with domestic price process \( \{ S_t \} \) can each be decomposed into terms involving only the intrinsic “natural” volatility of the asset and terms associated with the volatility of the domestic pricing kernel but not associated with any particular asset.

One can check that as a consequence of (27) and Proposition 2, the stochastic differential equation satisfied by \( \{ \xi_t \} \) takes the form

$$d \xi_t = \left[ r_t + \lambda_t^2 + \int_x \Lambda_t(x)^2 \nu(dx) \right] dt + \lambda_t dW_t + \int_x \Lambda_t(x) \tilde{N}(dx, dt),$$

(37)

which is indeed of the type appropriate to an asset that pays no dividend, as one sees by comparing (30) with (37). The benchmark process has the property that its Brownian proportional volatility coincides with the Brownian market price of risk and its jump proportional volatility coincides with the jump market price of risk.
The significance of the benchmark asset in the present investigation is as follows. We are concerned with the problem of hedging a position in a risky asset with a position in a portfolio consisting of one or more other risky assets. Now, when such a hedge is carried out, this involves a choice of base currency with respect to which the hedge is optimized. Clearly, the choice of base currency is largely arbitrary, and it does not make sense to insist on minimizing exclusively the magnitude of the residual value of the hedge portfolio in units of the domestic currency. Sometimes it is argued that there may be a favoured choice of base currency – for example the currency in which a household has to meet its daily obligations, or in which a business has to accommodate a series of cashflows in connection with its activities. But such considerations bring additional elements of structure into the argument, and the fact remains that there is no a priori reason why one fiat currency should be favoured over another in the absence of a more detailed specification of the problem. Of all the choices of hedging currencies there is, however, a “preferred” numeraire involving no additional elements of structure, and this is the benchmark. So we take the view that the optimization problem takes the form of minimizing a function of the magnitude of the value of the hedge portfolio when that value is expressed in units of the benchmark.

Proceeding with our investigation of optimal hedging, let us write \( \{C_t\}_{t \geq 0} \) for the domestic price process of another risky asset, which we call the contract asset. We shall assume that \( \{C_t\}_{t \geq 0} \) is strictly positive and that
\[
\frac{dC_t}{C_t} = \left[ r_t + \lambda_t \sigma_t^c + \int_x \Lambda_t(x) \Sigma_t^c(x) \nu(dx) \right] dt + \sigma_t^c dW_t + \int_x \Sigma_t^c(x) \tilde{N}(dx, dt),
\]
where \( \{\sigma_t^c\}_{t \geq 0} \) is adapted, \( \{\Sigma_t^c(x)\}_{t \geq 0, x \in \mathbb{R}} \) is predictable, and \( \Sigma_t^c(x) > -1 \) for \( t \geq 0 \) and \( x \in \mathbb{R} \). We can think of \( \{C_t\} \) as representing the domestic value process of the position that we wish to hedge, and \( \{S_t\} \) as being the domestic value process of the hedging asset.

For applications, one usually needs to impose stronger conditions on the price processes under consideration. For example, in the case of a derivative, with payoff \( H_T \) at time \( T \), it is reasonable to assume not merely that the payoff should satisfy \( \mathbb{E}[\bar{H}_T] < \infty \), but also that it should satisfy \( \mathbb{E}[\bar{H}_T^2] < \infty \). In other words, for derivative risk management, we typically desire that some measure of the uncertainty of the payoff can be worked out, such as its variance. Indeed, in financial markets, one does not really wish to be working with instruments that are so volatile or ill-behaved that it is not possible to assign a meaningful value to the variance of the payoff. Since, in international markets, there is no particular reason to prefer one currency to another, it makes sense to introduce a minimalist assumption to the effect that the variance of the natural value of the payoff should be quantifiable. Thus, we shall assume at the very least that \( \text{Var}\ \bar{H}_T < \infty \). One could consider other choices for a measure of the riskiness of the payoff, and one could work this out in other units, but the choice that we have indicated is convenient from a mathematical perspective since the category of square-integrable random variables is well understood, and the use of natural units is well defined already under the assumption that we have made. Indeed, financial markets, one does not really wish to be working with instruments that are so volatile or ill-behaved that it is not possible to assign a meaningful value to the variance of the payoff. Since, in international markets, there is no particular reason to prefer one currency to another, it makes sense to introduce a minimalist assumption to the effect that the variance of the natural value of the payoff should be quantifiable. Thus, we shall assume at the very least that \( \text{Var}\ \bar{H}_T < \infty \). One could consider other choices for a measure of the riskiness of the payoff, and one could work this out in other units, but the choice that we have indicated is convenient from a mathematical perspective since the category of square-integrable random variables is well understood, and the use of natural units is well defined already under the assumption that we have made. One might object that insisting on a finite variance is too strong an assumption; but the reply can be put in normative terms – namely, that for a financial instrument to be considered as a legitimate object of commerce, it needs in principle to be capable of being risk-managed in a reasonably conventional manner; and the requirement that the value of the instrument can be modelled as having a finite variance is a step in this direction, an embodiment of this idea.
IV. OPTIMAL HEDGING IN A LÉVY-ITO MARKET

We consider setting up a trading strategy to hedge the natural value of a position in a given asset. Going forward, we shall for this purpose assume that all values are given in natural units – that is, in units of the natural benchmark numeraire. Thus, we henceforth drop the use of the “bar” notation, and let \( \{S_t\} \) and \( \{C_t\} \) denote the natural prices of the hedging asset and the contract asset, respectively. For the associated price dynamics we write

\[
\frac{dS_t}{S_t} = \sigma_t \, dW_t + \int_x \Sigma_t(x) \tilde{N}(dx, dt) \tag{39}
\]

and

\[
\frac{dC_t}{C_t} = \sigma^c_t \, dW_t + \int_x \Sigma^c_t(x) \tilde{N}(dx, dt). \tag{40}
\]

The corresponding price processes are given more explicitly by

\[
S_t = S_0 \exp \left( \int_0^t \sigma_u \, dW_u - \frac{1}{2} \int_0^t \sigma^2_u \, du \right) \times \exp \left( \int_0^t \int_x \sigma_u(x) \tilde{N}(dx, du) - \int_0^t \int_x (e^{\sigma_u(x)} - \sigma_u(x) - 1) \nu(dx) \, du \right) \tag{41}
\]

and

\[
C_t = C_0 \exp \left( \int_0^t \sigma^c_u \, dW_u - \frac{1}{2} \int_0^t (\sigma^c_u)^2 \, du \right) \times \exp \left( \int_0^t \int_x \sigma^c_u(x) \tilde{N}(dx, du) - \int_0^t \int_x (e^{\sigma^c_u(x)} - \sigma^c_u(x) - 1) \nu(dx) \, du \right). \tag{42}
\]

The hedging problem can be formulated as follows. The hedger is holding a position in one unit of the contract asset. The value process of this asset is \( \{C_t\} \) in natural units. The value process of the hedging asset is \( \{S_t\} \) in natural units. We assume that the hedging asset can be borrowed in any quantity at no cost, and that a short position in the hedging asset can be maintained and adjusted on a continuous basis at no cost. The value of the hedge portfolio at time \( t \) is

\[
V_t = C_t - \phi_t S_t + \theta_t, \tag{43}
\]

where the predictable process \( \{\phi_t\} \) denotes the number of units of the hedging asset being shorted, and the predictable process \( \{\theta_t\} \) denotes the number of benchmark units held in the hedge portfolio. Initially, we have \( \theta_0 = \phi_0 S_0 \). That is to say, the proceeds of the initial short sale of the hedging asset are deposited in the benchmark account. Thereafter, the portfolio is managed on a self-financing basis: thus, the change in the value of the portfolio over a small interval of time is given by

\[
dV_t = dC_t - \phi_t \, dS_t. \tag{44}
\]

It follows from (43) and (44) that the position in the benchmark account at time \( t \) is

\[
\theta_t = \phi_t \, S_t - \int_0^t \phi_u \, dS_u, \tag{45}
\]
or equivalently
\[ \theta_t = \phi_t S_t - \int_0^t \phi_u \, dS_u, \quad (46) \]
where the integrals on the right-hand sides of (45) and (46) are understood as being over the intervals \([0, t]\) and \([0, t]\), respectively. Then for the dynamics of the hedge portfolio we have
\[ dV_t = (\sigma_t^c C_t^- - \phi_t \sigma_t S_t^-) \, dW_t + \int_x \left( \Sigma_t^c(x) C_t^- - \phi_t \Sigma_t(x) S_t^- \right) \tilde{N}(dx, dt). \quad (47) \]

Now, if both of the assets are driven purely by the Brownian motion, and there are no jumps, then a perfect hedge can be carried out in such a way that the value of the hedge portfolio is constant. In that case a short calculation shows that
\[ \phi_t = \frac{\sigma_t^c}{\sigma_t} C_t \quad \text{and} \quad \theta_t = C_0 + \left( \frac{\sigma_t^c}{\sigma_t} - 1 \right) C_t. \quad (48) \]
The expression for the hedge ratio will look familiar, of course, but one should keep in mind that the hedge here is for the natural value of the contract asset, not its value in units of the fiat currency. In the general situation, when jumps are allowed, it is not possible to find a perfect hedge in the sense of completely erasing the riskiness of the position. Instead, we proceed as follows. We assume that the natural values of the assets under consideration are square-integrable in the sense that
\[ \mathbb{E} \left[ S_t^2 \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ C_t^2 \right] < \infty \quad (49) \]
for \( t \geq 0 \), and that the self-financing hedging strategy \( \{\phi_t, \theta_t\}_{t \geq 0} \) is such that the portfolio value at any time \( t \geq 0 \) over which the hedge is maintained satisfies
\[ \mathbb{E} \left[ V_t^2 \right] < \infty. \quad (50) \]

We fix a time interval \([0, T]\). Our goal is to choose the hedging strategy in such a way as to minimize the expected squared deviation of the value of the hedge portfolio at time \( T \) from its value at time 0. Thus if for any admissible choice of \( \{\phi_t\}_{0 \leq t \leq T} \) we write
\[ \Delta_T(\phi) = \mathbb{E} \left[ (V_T - V_0)^2 \right], \quad (51) \]
then we have
\[ \Delta_T(\phi) = \mathbb{E} \left[ \left( \int_0^T (\sigma_u^c C_u^- - \phi_u \sigma_u S_u^-) \, dW_u + \int_0^T \int_x \left( \Sigma_u^c(x) C_u^- - \phi_u \Sigma_u(x) S_u^- \right) \tilde{N}(dx, du) \right)^2 \right]. \]
Therefore, by use of Proposition 3 we obtain
\[ \Delta_T(\phi) = \mathbb{E} \left[ \int_0^T (\sigma_u^c C_u^- - \phi_u \sigma_u S_u^-)^2 \, du + \int_0^T \int_x \left( \Sigma_u^c(x) C_u^- - \phi_u \Sigma_u(x) S_u^- \right)^2 \nu(dx) \, du \right]. \]

It follows that the error takes the form
\[ \Delta_T(\phi) = \mathbb{E} \left[ \int_0^T \left( K_u C_u^2 - 2\phi_u L_u S_u - C_u^- + \phi_u^2 M_u S_u^2 \right) \, du \right], \quad (52) \]
where
\[ K_t = \sigma_t^c \sigma_u + \int_x \Sigma_t^c(x)^2 \nu(dx), \quad L_t = \sigma_t \sigma_u^c + \int_x \Sigma_t(x) \Sigma_t^c(x) \nu(dx), \quad M_t = \sigma_t^2 + \int_x \Sigma_t(x)^2 \nu(dx). \]
Thus we are led to the following.

**Proposition 4.** Let the contract asset \( \{C_t\} \) be hedged with \( \{\phi_t\} \) units of the asset \( \{S_t\} \) and \( \{\theta_t\} \) units of the benchmark. Then the optimal hedge \( \hat{\phi}_t \) for \( t \) is given by

\[
\hat{\phi}_t = \frac{\sigma_t \sigma_t^c + \int_x \Sigma_t(x) \Sigma_t^c(x) \nu(dx)}{\sigma_t^2 + \int_x \Sigma_t(x)^2 \nu(dx)} \frac{C_{t-}}{S_{t-}}.
\]

**Proof.** A standard argument using the calculus of variations establishes (53) as a candidate for the optimal hedge. To prove that the candidate is indeed optimal, we need to show that the mean squared error in any alternative hedge is no less than the mean squared error in the candidate. Let \( \{\psi_t\}_{0 \leq t \leq T} \) denote an alternative admissible hedge. We shall say that two such hedging strategies \( \{\psi_t^1\} \) and \( \{\psi_t^2\} \) are distinct if there exists an interval of time \( A \subset [0, T] \) of length greater than zero such that

\[
P \left[ (\psi_t^1 - \psi_t^2)^2 > 0 \right] = 1
\]

for almost all \( t \in A \). A calculation gives

\[
\Delta_T(\psi) - \Delta_T(\hat{\phi}) = \mathbb{E} \left[ \int_0^T (\psi_u - \hat{\phi}_u)^2 S_u^2 M_u du \right],
\]

and one sees that the right-hand side is nonnegative for any choice of the alternative hedge. In fact, the optimal hedge dominates any alternative hedge that is distinct from it. \( \square \)

V. HEDGING WITH TWO ASSETS

Let us now consider the problem of setting up a trading strategy to hedge the natural value of a position in a given contract asset by use of two risky hedging assets. The problem will be framed in the case where all three of the assets are driven by a one-dimensional Brownian motion \( \{W_t\} \) and a one-dimensional Poisson random measure \( \{N(dx, dt)\} \). The hedging assets each have dynamics of the form (39). We write \( \{S_t^i\}_{i=1,2} \) for the hedging assets, and we write \( \{\phi_t^i\}_{i=1,2} \) for the holdings in these assets. Then we obtain the following.

**Proposition 5.** Let \( \{C_t\} \) be hedged over \([0, T]\) with \( \{\phi_t^1\} \) units of \( \{S_t^1\} \), \( \{\phi_t^2\} \) units of \( \{S_t^2\} \), and \( \{\theta_t\} \) units of the benchmark. The optimal hedge \( \hat{\phi}_t^1 \), \( \hat{\phi}_t^2 \) is given by

\[
\hat{\phi}_t^i = \frac{P_{t}^{i2} - Q_{t}^{i2}}{R_{t}^{i2}} \frac{C_{t-}}{S_{t-}^i}, \quad \hat{\phi}_t = \frac{\sigma_t^2 + \int_x \Sigma_t^i(x)^2 \nu(dx)}{\sigma_t^2 + \int_x \Sigma_t^i(x)^2 \nu(dx)} \frac{C_{t-}}{S_{t-}}.
\]

for \( t \in [0, T] \), where we set

\[
P_{t}^{ij} = \left( \sigma_i \sigma_j + \int_x \Sigma_i(x) \Sigma_j(x) \nu(dx) \right) \left( \sigma_i^2 + \int_x \Sigma_i^2(x) \nu(dx) \right),
\]

\[
Q_{t}^{ij} = \left( \sigma_i \sigma_j + \int_x \Sigma_i(x) \Sigma_j(x) \nu(dx) \right) \left( \sigma_i \sigma_j + \int_x \Sigma_i^2(x) \nu(dx) \right),
\]

\[
R_{t}^{ij} = \left( \sigma_i^2 + \int_x \Sigma_i(x) \Sigma_j(x) \nu(dx) \right) \left( \sigma_i^2 + \int_x \Sigma_i(x)^2 \nu(dx) \right) - \left( \sigma_i^2 \right)^2.
\]
It will be shown in Section VI that one can generalize this expression to the case where there are \( n \) hedging assets. The rather general proof given there leads back to the formulae given above in Proposition 5 for \( n = 2 \). We mention these formulae here since they form the basis of the simulations that we shall present shortly. One can also prove that as the number of hedging assets increases an overall better hedge can generally be achieved.

As a numerical illustration of the general methodology let us consider the situation where each of the assets follows a geometric Lévy process for which the Lévy process takes the form of a jump diffusion consisting of a standard Brownian motion superposed on a compound Poisson Process. It should be recalled that even if the driving process in the exponent of the asset price is a Lévy process, the asset price itself follows a Lévy-Ito process. We consider the simplest possible case, namely, that for which the pure-jump component of the Lévy process is a Bernoulli process. Let

\[
\{ \delta_{g}(dx) \} \quad \text{and} \quad \{ \delta_{h}(dx) \}
\]

where \( \delta_{g}(dx) \) is the Dirac measure concentrated at \( g \) and \( \delta_{h}(dx) \) is the Dirac measure concentrated at \( h \). Then the price processes of the assets under consideration have dynamics of the form \((39)-(40)\), with deterministic time-independent volatilities. Since we are working with a geometric Lévy process, the jump volatility is of the form \( \Sigma(x) = \exp(\beta x) - 1 \), for some \( \beta \in \mathbb{R}^{+} \). The price of a typical non-dividend paying risky asset in a Bernoulli jump diffusion market with this set up is thus of the form

\[
S_{t} = S_{0} \exp \left( \sigma W_{t} - \frac{1}{2} \sigma^{2} t + \beta X_{t} - mt \left( p (e^{\beta g} - 1) + (1 - p) (e^{\beta h} - 1) \right) \right),
\]

where \( \sigma \) is a constant. For our simulations we consider a contract asset \( \{ C_{t} \} \) and a pair of hedging assets \( \{ S^{1}_{t} \} \) and \( \{ S^{2}_{t} \} \), each of the form \((58)\), with a view to forming an optimal hedge of the contract asset with positions in one or both of the hedging assets.

In Figure 1, we show on the left-hand side a random sample path for the Lévy process \( \{ X_{t} \} \) alongside the underlying Poisson process \( \{ N_{t} \} \). On the right-hand side one finds the corresponding paths for the contract asset \( \{ C_{t} \} \) and the two hedging assets \( \{ S^{1}_{t} \} \) and \( \{ S^{2}_{t} \} \). The inputs for this numerical example are as follows: \( S_{0}^{1} = 100, S_{0}^{2} = 100, C_{0} = 100, \sigma^{1} = 0.20, \sigma^{2} = 0.10, \sigma^{c} = 0.15, \beta^{1} = 0.30, \beta^{2} = 0.20, \beta^{c} = 0.25, m = 15, p = 0.5, g = 1, h = -1, \) and \( T = 1 \). The number of time steps is one thousand.

Now, we know from general theory that if the Brownian motion is non-vanishing then the hedge can never be perfect; but if the Brownian component is small for all three assets, then a reasonably good hedge should be obtainable using just two assets in the case of a Bernoulli jump diffusion. In Figure 2, we show the effect of using either \( \{ S^{1}_{t} \} \) or \( \{ S^{2}_{t} \} \) alone as a hedge and we plot the residual movements in the values of the hedged portfolios.

In Figure 3, we show the effect of using both hedging assets together to hedge the contract asset, and we note in particular the significant drop in the variance of the hedged portfolio. If we reduce the volatilities of the Brownian components still further, then we get a near perfect hedge, as illustrated in Figure 4. The Brownian volatilities for Figure 4 are given by \( \sigma^{1} = 0.003, \sigma^{2} = 0.001 \) and \( \sigma^{c} = 0.002 \).
Figure 1: Bernoulli jump-diffusion market. The chart on the left above shows an outcome of chance for the Lévy process in blue, with the underlying Poisson process in red. The chart on the right above plots the value process of the contract asset in green. The high volatility hedging asset 1 is shown in red, and the low volatility hedging asset 2 is shown in blue.

Figure 2: Single-asset hedges. The chart on the left plots at each step the change in the value of the hedge portfolio, when asset 1 alone is used as the hedge. The lengthy downward spikes correspond to jumps, whereas the shorter spikes are due to Brownian volatility. In the chart on the right, asset 2 alone is used as the hedge. The lengthy upward spikes correspond to jumps.

One sometimes hears that Lévy markets are incomplete, except in the Brownian case when the number of available assets is no less than the number of Brownian motions. But this of course is not quite true, since a pure Poisson market is also complete. If a pair of geometric Lévy assets are driven by a common Poisson process, then either can be hedged by use of the other. A pure Bernoulli market is also complete, in the sense that if three geometric Lévy assets are driven by a common Bernoulli process, then any one can be hedged by use of the other two. Similarly, a compound Poisson process market with \( k \) possible outcomes at each jump is complete if \( k \) hedging assets are available. If a Brownian component is introduced into any of these scenarios, then the resulting market is incomplete. But if the Brownian volatilities are small, then near perfect hedges can be achieved, as we see in Figure 4.
VI. MULTIPLE HEDGING ASSETS

We turn finally to consider hedging a contract asset \( \{ C_t \} \) with a collection of \( n \) hedging assets \( \{ S^i_t \}_{i=1}^n \) each with dynamics of the form (39). Thus we have

\[
\frac{dS^i_t}{S^i_t} = \sigma^i_t \, dW_t + \int_x \Sigma^i_t(x) \, \tilde{N}(dx, dt).
\]  

(59)

We assume the set of hedging assets is non-degenerate in the sense that no one of the assets can be replicated by holding a portfolio in the remaining \( n-1 \) assets. That is, no strategy \( \{ \eta^i_t \}_{i=1}^n \) exists such that for some interval \( A \in [0, T] \) of length greater than zero we have

\[
\sum_{i=1}^n \eta^i_t \, \sigma^i_t \, S^i_t = 0 \quad \text{and} \quad \sum_{i=1}^n \eta^i_t \, \Sigma^i_t(x) \, S^i_t = 0
\]

(60)
for almost all \( t \in A \), for all \( x \in \mathbb{R} \). The hedge portfolio then takes the form
\[
V_t = C_t - \sum_{i=1}^{n} \phi_i t S_t^i + \theta_t,
\]
(61)
and we impose the self-financing condition
\[
dV_t = dC_t - \sum_{i=1}^{n} \phi_i t dS_t^i.
\]
(62)
We would like to choose the hedging strategy \( \{\phi_i t, \theta_t\} \) in such a way that the mean squared error in the portfolio value
\[
\Delta_T(\phi_i) = \mathbb{E} [(V_T - V_0)^2]
\]
is minimized. Then by (40), (59), (62) and (63) we have
\[
\Delta_T(\phi_i) = \\
\mathbb{E} \left[ \int_0^T \left( \sigma_t^c C_{t^-} - \sum_{i=1}^{n} \phi_i t^c S_t^i \right) dW_u + \int_0^T \int_x \left( \Sigma_t^c(x) C_{t^-} - \sum_{i=1}^{n} \phi_i t^c \Sigma_t^i(x) S_t^i \right) \tilde{N}(dx, du) \right]^2,
\]
and by use of the Ito isometry we obtain
\[
\Delta_T(\phi_i) = \\
\mathbb{E} \left[ \int_0^T \left( \sigma_t^c C_{t^-} - \sum_{i=1}^{n} \phi_i t^c S_t^i \right)^2 du + \int_0^T \int_x \left( \Sigma_t^c(x) C_{t^-} - \sum_{i=1}^{n} \phi_i t^c \Sigma_t^i(x) S_t^i \right)^2 \nu(dx)du \right].
\]
Expanding the squares and gathering together the various terms we get
\[
\Delta_T(\phi_i) = \mathbb{E} \left[ \int_0^T \left( G_u + \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_i u \phi_j u M_{ij}^u - 2 \sum_{i=1}^{n} \phi_i u F_i u \right) du \right],
\]
(64)
where
\[
M_{ij}^u = S_{u^-} S_{u^-}^j \left[ \sigma_u^i \sigma_u^j + \int_x \Sigma_u^i(x) \Sigma_u^j(x) \nu(dx) \right],
\]
(65)
\[
F_i^u = S_{u^-} C_{u^-} \left[ \sigma_u^i \sigma_u^c + \int_x \Sigma_u^i(x) \Sigma_u^c(x) \nu(dx) \right],
\]
(66)
\[
G_u = C_{u^-}^2 \left[ \sigma_u^c \sigma_u^c + \int_x \Sigma_u^c(x) \nu(dx) \right].
\]
(67)
Applying a small perturbation $\{\phi^i_t\} \rightarrow \{\phi^i_t + \eta^i_t\}$ to $\Delta_T(\phi^i)$ we find that to first order in $\{\eta^i_t\}$ it holds that

$$\Delta_T(\phi^i + \eta^i) - \Delta_T(\phi^i) = 2 \mathbb{E} \left[ \int_0^T \left( \sum_{i=1}^n \sum_{j=1}^n \eta^i_u \phi^i_u M^{ij}_u - \sum_{i=1}^n \eta^i_u F^i_u \right) du \right]. \quad (68)$$

A sufficient condition for the right-hand side of (68) to vanish for any choice of $\{\eta^i_t\}$ is that the $\{\phi^i_t\}$ should satisfy the first-order condition

$$\sum_{j=1}^n M^{ij}_t \phi^j_t = F^i_t. \quad (69)$$

We arrive at the same condition if we minimize the average of $\{\Delta_t(\phi^i)\}_{0 \leq t \leq T}$ over the time interval $[0, T]$. We are thus led to the following.

**Proposition 6.** Let $\{C_t\}$ be hedged with $\{\phi_t^i\}$ units of $\{S_t^i\}$ for $i = 1, \ldots, n$ and $\{\theta_t\}$ units of the benchmark. Then the optimal hedge $\{\hat{\phi}_t^i\}_{0 \leq t \leq T}$ takes the form

$$\hat{\phi}_t^i = \sum_{j=1}^n N^{ij}_t F^j_t, \quad (70)$$

where $\{N^{ij}_t\}$ is the inverse of $\{M^{ij}_t\}$.

**Proof.** The inverse of $\{M^{ij}_t\}$ exists because of the non-degeneracy condition. Equation (69) then gives a candidate optimal hedge. Putting (70) back into (64), we get

$$\Delta_T(\hat{\phi}^i) = \mathbb{E} \left[ \int_0^T \left( G_u - \sum_{i=1}^n \sum_{j=1}^n N^{ij}_u F^j_u \right) du \right]. \quad (71)$$

As in the case of a single hedging asset, we need to show that the error in any alternative hedge is no less than the error in the candidate solution. Letting $\{\psi_t^i\}_{0 \leq t \leq T}$ be any alternative hedge that is distinct from the candidate, one finds that

$$\Delta_T(\psi^i) - \Delta_T(\hat{\phi}^i) = \mathbb{E} \left[ \int_0^T \sum_{i=1}^n \sum_{j=1}^n M^{ij}_u (\psi_u^i - \hat{\phi}_u^i)(\psi_u^j - \hat{\phi}_u^j) du \right]. \quad (72)$$

It follows as a consequence of (65) and the non-degeneracy condition that the matrix process $\{M^{ij}_t\}$ is positive definite. Therefore the right side of (71) is strictly positive, and we deduce that $\{\hat{\phi}_t^i\}$ is optimal and indeed that it dominates any strategy distinct from it. \qed

Next, we would like to show that if we add an further non-redundant hedging asset to an existing collection of $n$ hedging assets, the hedge will be improved by using all $n + 1$ of the hedging assets. This is a characteristic feature of incomplete markets.

Given $\{C_t\}$ and $\{S_t^i\}_{i=1, \ldots, n}$, let $\{\hat{\phi}_t^i\}_{i=1, \ldots, n}$ denote the optimal hedge determined in Proposition 6. Let $\{S_0^j\}$ be another hedging asset, distinct from the contract asset and the original $n$ hedging assets in the sense that it cannot be realized as a portfolio formed from the other assets. Then we have the following.
\textbf{Proposition 7.} For any contract asset \( \{ C_t \} \), the optimal hedge \( \{ \hat{\Gamma}^i_t \}_{i=0,1,...,n} \) obtained by use of the \( n+1 \) hedging assets \( \{ S^i_t \}_{i=0,1,...,n} \) is better than the optimal hedge \( \{ \hat{\phi}^i_t \}_{i=1,...,n} \) obtained \( \frac{n}{2} \) by use of the \( n \) hedging assets \( \{ S^0_t \}_{i=1,...,n} \).

\textit{Proof.} Let \( \{ U_t \}_{0 \leq t \leq T} \) denote the value process of the hedge portfolio defined by

\[ U_t = \sum_{i=1}^{n} \hat{\phi}^i_t S^i_t + \theta_t. \]  

(73)

It follows by the self-financing condition that \( \{ U_t \} \) itself can be treated as an asset. Now consider a hedging strategy of the form \( \{ \gamma_t, \delta_t \} \) where \( \{ \gamma_t \} \) denotes the holdings in \( \{ U_t \} \) and \( \{ \delta_t \} \) denotes the holdings in \( \{ S^0_t \} \). It is easy to see that an optimal portfolio of two distinct hedging instruments will perform better than the optimal hedge obtained by use of just one of the two instruments. This is because the optimal hedge involving a single instrument is an example of a hedge involving two instruments. It follows that the portfolio \( \{ \gamma_t U_t + \delta_t S^0_t \} \) will perform better than \( \{ U_t \} \) alone as a hedge for \( \{ C_t \} \). That is to say,

\[ \Delta_T (\gamma \hat{\phi}^i, \delta) < \Delta_T (\hat{\phi}^i). \]  

(74)

On the other hand, we observe that \( \{ \hat{\Gamma}^i_t \}_{i=1,...,n} \) is the \textit{optimal} hedge involving the \( n+1 \) assets available for hedging, whereas \( \{ \gamma_t \hat{\phi}^i, \delta_t \}_{i=1,...,n} \) is merely an example of a hedge involving the \( n+1 \) hedging assets. Therefore

\[ \Delta_T (\hat{\Gamma}^i) < \Delta_T (\gamma \hat{\phi}^i, \delta), \]  

(75)

and thus

\[ \Delta_T (\hat{\Gamma}^i) < \Delta_T (\hat{\phi}^i). \]  

(76)

Hence, we deduce that the optimal hedge involving \( n+1 \) hedging instruments will perform better than the optimal hedge formed from any \( n \) of them.

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