Inverse scattering problem with data at fixed energy and fixed incident direction

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Abstract

Let $A_q(\alpha', \alpha, k)$ be the scattering amplitude, corresponding to a local potential $q(x)$, $x \in \mathbb{R}^3$, $q(x) = 0$ for $|x| > a$, where $a > 0$ is a fixed number, $\alpha', \alpha \in S^2$ are unit vectors, $S^2$ is the unit sphere in $\mathbb{R}^3$, $\alpha$ is the direction of the incident wave, $k^2 > 0$ is the energy. We prove that given an arbitrary function $f(\alpha') \in L^2(S^2)$, an arbitrary fixed $\alpha_0 \in S^2$, an arbitrary fixed $k > 0$, and an arbitrary small $\varepsilon > 0$, there exists a potential $q(x) \in L^2(D)$, where $D \subset \mathbb{R}^3$ is a bounded domain such that

$$\|A_q(\alpha', \alpha_0, k) - f(\alpha')\|_{L^2(S^2)} < \varepsilon.$$  \hspace{1cm} (*)

The potential $q$, for which (*) holds, is nonunique. We give a method for finding $q$, and a formula for such a $q$.

1 Introduction

If $q(x) = 0$ for $|x| > a$, where $a > 0$ is some number, $q(x) \in L^2(B_a)$, $B_a = \{x : |x| \leq a\}$, then the corresponding scattering amplitude $A_q(\alpha', \alpha, k)$ is defined as follows. Let $\alpha \in S^2$ be a given unit vector, where $S^2$ is the unit sphere. The scattering problem consists in finding the scattering solution $u(x, \alpha, k)$, which solves the equation

$$[\nabla^2 + k^2 - q(x)]u = 0 \text{ in } \mathbb{R}^3,$$ \hspace{1cm} (1)

MSC: 35J05, 35J10, 35R30, 74J25, 81U40, 81V05
PACS: 03.04.Kf
key words: inverse scattering, properties of scattering amplitudes, quantum mechanics
and satisfies the following asymptotics:

\[ u = e^{ik\alpha \cdot x} + A_q(\alpha', \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha' := \frac{x}{r}. \]  \hspace{1cm} (2)

Vector \( \alpha \) is called the incident direction, the direction of the incident plane wave \( e^{ik\alpha \cdot x} \), vector \( \alpha' \) is the direction in which the incident wave is scattered, and the coefficient \( A_q(\alpha', \alpha, k) \) is called the scattering amplitude. The properties of the scattering amplitude, corresponding to a real-valued, rapidly decaying at infinity \( q \), has been studied in detail in the literature. (See, e. g., [2]).

The inverse scattering problem with fixed-energy data consists in finding a potential \( q(x) \) from the knowledge of \( A_q(\alpha', \alpha, k) \) at a fixed \( k > 0 \) and all \( \alpha', \alpha \in S^2 \). The uniqueness of the solution to this problem in the class of real-valued, compactly supported, square-integrable potentials was first proved by the author ([4]), who also gave a method for recovery of \( q \) from the exact fixed-energy data, an error estimate for this method, and stability estimates for the recovery problem ([7]) as well as a method for a stable recovery of \( q \) from noisy fixed-energy data, and an error estimate for this method [3], [6], [7]. Until now this is the only known rigorously justified method for recovery of \( q \) from noisy fixed-energy data.

In this paper a new problem is studied. Let us assume that the incident direction \( \alpha \) is fixed, \( \alpha = \alpha_0 \), and \( k > 0 \) is also fixed, \( k = k_0 > 0 \). Denote by \( A_q(\alpha') := A_q(\alpha', \alpha_0, k_0) \) the corresponding scattering amplitude, and let \( \beta := \alpha' \). Let \( f(\beta) \in L^2(S^2) \) be an arbitrary function. In part of our arguments, in Lemma 3, specifically, it is convenient to assume first that the norm of \( f \) is sufficiently small norm \( \|f(\beta)\|_{L^2(S^2)} \). The role of this "smallness" assumption will be clear from our arguments. Under this "smallness" assumption we derive an analytical explicit formula for the potential which generates the scattering amplitude at a fixed \( k > 0 \) and a fixed incident direction \( \alpha \) with any desired accuracy. The "smallness" assumptions will be dropped in Lemma 4.

Fix an arbitrary small number \( \varepsilon > 0 \). Let \( D \subset \mathbb{R}^3 \) be a bounded domain.

The problem (P) is:

Given \( \varepsilon > 0, f(\beta) \in L^2(S^2), \|f(\beta)\|_{L^2(S^2)}, \alpha_0 \in S^2, \) and \( k_0 > 0, \) does there exist a potential \( q \in L^2(D) \) such that

\[ \|f(\beta) - A_q(\beta)\|_{L^2(S^2)} < \varepsilon, \]  \hspace{1cm} (3)

and how does one calculate such a \( q \)?

Our basic result consists of an answer to these questions. In [8] the approximation problem problem similar to the above was considered for the first time, but the argument in [8] does require the "smallness" assumption.

**Theorem 1.** For any \( f \in L^2(S^2) \), an arbitrary small \( \varepsilon > 0 \), any fixed \( \alpha = \alpha_0 \in S^2 \), any fixed \( k = k_0 > 0 \), and any bounded domain \( D \subset \mathbb{R}^3 \), there exists a (non-unique) potential \( q(x) \in L^2(D) \), such that [3] holds.
To calculate such a potential, we need some auxiliary results.

**Lemma 1.** Given an arbitrary $f(\beta) \in L^2(S^2)$ and an arbitrary small $\varepsilon > 0$, there exists an $h \in L^2(D)$, such that

$$\left\| f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx \right\| < \varepsilon. \quad (4)$$

In Lemma 1, and in Lemma 2 below, we do not need the ”smallness” assumption. In what follows we always assume that $\alpha = \alpha_0 \in S^2$ and $k = k_0 > 0$ are fixed, and write $\alpha, k$ in place of $\alpha_0$ and $k_0$.

The following formula for the scattering amplitude is well known:

$$A_q(\beta, \alpha, k) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} q(x) u(x) dx, \quad (5)$$

where $u(x) := u(x, \alpha, k)$ is the scattering solution and the dependence on $\alpha$ and $k$ is not shown because $\alpha$ and $k$ are fixed.

If

$$h(x) = q(x) u(x), \quad (6)$$

then (4) is identical to (3).

We now explain how one calculates $q$ from equation (6) if $h$ is given, and how one calculates $h$, satisfying (4), if $f$ is given.

**Lemma 2.** Given $f \in L^2(S^2)$, one calculates $h$, satisfying (4), by the formula:

$$h_{\ell,m} = \begin{cases} 
-(-i)^\ell \frac{f_{\ell,m}}{\sqrt{\pi^2 g_1, \ell+1, \ell+1}(k)}, & \ell \leq L, \\
0, & \ell > L
\end{cases} \quad (7)$$

where $g_{\mu, \nu}(k) := \int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx) dx$. This integral can be calculated analytically (1, formula 8.5.8), and we have assumed that $h(x) = 0$ for $|x| > 1$.

If $h$ is found such that (4) holds, then $q$ can be calculated from equation (6), which is a nonlinear equation for $q$, because the scattering solution $u$ depends on $q$. Under the ”smallness” assumption we derive an analytical explicit formula for $q$, namely, formula (9) below. This formula holds not necessarily under the ”smallness” assumption, as follows from Lemma 4.

In Lemma 3 we use the ”smallness” assumption for the first time. This assumption guarantees that inequality (8) holds. This inequality is sufficient for the formula (9) to produce an $L^2(D)$ potential $q$.

In Lemma 4 we prove that there exists an $h_\delta$, a small perturbation of $h$, $\|h - h_\delta\|_{L^2(D)} < \delta$, which leads by formula (9), with $h$ replaced by $h_\delta$, to a potential $q_\delta$, generating the radiation pattern $A(\beta)$ as close to the desired $f$ as one wishes.
Lemma 3. Assume that \( \sup_{x \in D} \left| \int_D gh \, dy \right| < 1 \), or, more generally, that
\[
\sup_{x \in D} \left| u_0(x) - \int_D g(x, y) h(y) \, dy \right| > 0, \quad g = g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}.
\] (8)

Then equation (6) has a unique solution:
\[
q(x) = \frac{h(x)}{u_0(x) - \int_D g(x, y) h(y) \, dy}, \quad u_0(x) = e^{ik\alpha \cdot x}, \quad q \in L^2(D).
\] (9)

Formula (9) holds also if condition (8) holds.

Suppose that for a given \( h \in L^2(D) \) condition (8) is not satisfied. Let us approximate \( h \) by an analytic function \( h_1 \) in \( D \), for example, by a polynomial, so that
\[
||f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h_1(x) \, dx|| < \varepsilon.
\]

Denoting \( h_1 \) by \( h \) again, we may assume that \( h \) is analytic in \( D \) and in a domain which contains \( D \). In the following lemma we prove that it is possible to perturb \( h \) slightly so that for the perturbed \( h \), denoted \( h_\delta \), condition (8) is satisfied, and formula (9) yields a potential \( q_\delta \in L^2(D) \), for which inequality (3) holds.

Lemma 4. Assume that \( h \) is analytic in \( D \) and bounded in the closure of \( D \). There exists a small perturbation \( h_\delta \) of \( h \), \( ||h - h_\delta||_{L^2(D)} < \delta \), such that the function
\[
q_\delta := \frac{h_\delta(x)}{u_0(x) - \int_D g(x, y) h_\delta(y) \, dy}
\]
is bounded.

In Section 2 proofs are given.

2 Proofs

Proof of Lemma 7 Assume the contrary. Then
\[
\int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta \cdot x} h(x) \, dx = 0 \quad \forall h \in L^2(D).
\]
This implies
\[
\int_{S^2} f(\beta) e^{-ik\beta \cdot x} d\beta = 0 \quad x \in D.
\] (10)

The left-hand side of (10) is the Fourier transform of a distribution \( f(\beta) \frac{\delta(\lambda - k)}{\lambda} \), where \( \delta(\lambda - k) \) is the delta-function and \( \lambda \) is the Fourier transform variable in spherical coordinates. By the injectivity of the Fourier transform, (10) implies that \( f(\beta) = 0 \). \qed
Remark 1. In this proof and in the proof of Lemma \(2\) below, the domain \(D\) can be taken such that \(\text{diam}D\) is arbitrarily small.

Proof of Lemma 2. Note that Lemma 2 implies Lemma 1. It is known that

\[
e^{-ik\beta \cdot x} = \sum_{\ell=0}^{\infty} 4\pi (-i)^{\ell} j_{\ell}(kr) Y_{\ell,m}(\alpha') Y_{\ell,m}(\beta),
\]

where \(\sum_{\ell} = \sum_{\ell} \sum_{m=-\ell}^{\ell} j_{\ell}(r) = \sqrt{\frac{2}{\ell}} J_{\ell+\frac{1}{2}}(r)\), \(J_{\ell}(r)\) is the Bessel function, \(Y_{\ell,m}\) are orthonormal in \(L^2(S^2)\) spherical harmonics, \(Y_{\ell,m}(-\beta) = (-1)^{\ell} Y_{\ell,m}(\beta), \alpha' = \frac{\pi}{r}, r = |x|\), and the overbar stands for complex conjugate. Expand \(f\) in the Fourier series:

\[
f(\beta) = \sum_{\ell=0}^{\infty} f_{\ell,m} Y_{\ell,m}(\beta).
\]

Choose \(L\), such that

\[
\sum_{\ell > L} |f_{\ell,m}|^2 < \varepsilon^2.
\]

Let \(\ell > L, h(x) = \sum_{\ell=0}^{\infty} h_{\ell,m}(r) Y_{\ell,m}(\alpha'), \alpha' = \frac{\pi}{r}, r = |x|\).

Define \(h_{\ell,m}\) by (7). Then, by Parseval’s equation, inequality (4) holds. Lemma 2 is proved.

Remark 2. Alternatively, one may look for \(h\) in the form \(h = \sum_{j=1}^{J} c_j \varphi_j(x)\), where \((\varphi_j, \varphi_i) = \delta_{ij}\) and \(c_j = \text{const}\), and minimize the left-hand side of (4) with respect to \(c_j\), \(1 \leq j \leq J\). If \(J\) is sufficiently large, the minimum will be \(\leq \varepsilon\). One may take not an orthonormal system of \(\varphi_j\), but just a linearly independent, complete (total) in \(L^2(D)\), system of functions \(\{\varphi_j\}\).

Proof of Lemma 3 The scattering solution, corresponding to a potential \(q\), solves the equation

\[
u = u_0 - \int_D g(x,y)q(y)u(y)dy, \quad u_0 := e^{ik\alpha \cdot x}.
\]

If (4) holds, i.e., if \(h\) corresponds to a \(q \in L^2(D)\), then \(\nu = u_0 - \int_D ghdy\). Multiply this equation by \(q\) and get

\[
q(x)u(x) = q(x)u_0(x) - q(x) \int_D g(x,y)h(y)dy.
\]

Using (4) and solving for \(q\), one gets (9), provided that (8) holds. Condition (8) holds if \(\|h\|_{L^2(D)}\) is sufficiently small. One has

\[
\left| \int_D g(x,y)hd\gamma \right| \leq \frac{1}{4\pi} \sup_x \left\| \frac{1}{|x-y|} \right\|_{L^2(D)} \|h\|_{L^2(D)},
\]
and \( \frac{1}{4\pi} \left( \int_D \frac{dy}{|x-y|^2} \right)^{\frac{1}{2}} \leq \frac{a}{\sqrt{4\pi}} \), where \( a = \frac{1}{3} \text{diam} D \). If, for example, \( \frac{\sqrt{\pi}}{\sqrt{4\pi}} \|h\|_{L^2(D)} < 1 \), then condition (8) holds, and formula (9) yields the corresponding potential. This explains the role of the "smallness" assumption.

If \( h \in L^2(D) \) is an arbitrary function, such that condition (8) holds, then formula (9) defines a potential \( q \in L^2(D) \), formula (10) defines the function \( u = \frac{h}{q} \), and this \( u \) solves equation (13), which is the equation for the scattering solution. Thus, the function \( u = \frac{h}{q} \) is the scattering solution, and the potential \( q \), defined by formula (9), corresponds to the given \( h \). The above argument is valid in the case of complex-valued potential \( q \).

By Lemma 4, proved at the end of the paper, if \( h \) is such that condition (8) fails, then a small perturbation \( h_\delta \) of \( h \) leads by formula (9), with \( h_\delta \) in place of \( h \), to a potential \( q_\delta \) which generates the radiation pattern close to the desired \( f \).

Lemma 3 is proved. \( \square \)

Remark 3. Formula (9) yields, possibly, a complex-valued potential. If \( k > 0 \), then \( k^2 > 0 \) is not an eigenvalue of the Schrödinger operator \( -\nabla^2 + q(x) \) with a complex-valued compactly supported \( q \in L^2(D) \). This follows, e.g., from the results in [3] for \( q = o \left( \frac{1}{|x|} \right) \) as \( |x| \to \infty \), and can be proved for a compactly supported \( q \) by a simple argument.

In the proof of Lemma 2 one may take \( h(x) = \sum_{\ell=0}^\infty h_{\ell,m} Y_{\ell,m} \left( \frac{x}{b} \right) \), \( |x| \leq b < 1 \), where \( h_{\ell,m} \) do not depend on \( r = |x| \), \( h(x) = 0 \) for \( |x| > b \), and \( b > 0 \) can be an arbitrary small number.

Proof of Theorem 1. Given an arbitrary \( f \in L^2(S^2) \) and an arbitrary small \( \varepsilon > 0 \), choose \( h \), such that (14) holds. This is possible by Lemma 1 and is done analytically in Lemma 2. If such an \( h \) is found, then one calculates \( q \) by formula (9), provided that \( h \) is sufficiently small, i.e., \( f(\beta) \) is sufficiently small. If \( f(\beta) \) is arbitrary, so that conditions (8) or (14) are not satisfied, then \( q \) is found numerically.

Let us explain the role of the "smallness" assumption on \( f \). From Lemma 2 it follows that \( f \) and \( h \) are proportional, so that if \( ||f||_{L^2(S^2)} \) is sufficiently small, then \( ||h||_{L^2(D)} \) is small, and then condition (8) is satisfied. In this case the potential is given by formula (9). Theorem 1 is proved. \( \square \)

Remark 4. If (8) fail, then formula (9) may yield a \( q \not\in L^2(D) \). As long as formula (9) yields a potential \( q \in L^p(D) \), \( p \geq 1 \), our arguments essentially remain valid. In our presentation we have used \( p = 2 \) because the numerical minimization in \( L^2 \)-norm is simpler.

The difficulty arises when formula (9) yields a potential which is not locally integrable. Numerical experiments showed that this case did not occur in practice in several test examples in which the "smallness" condition was not satisfied.

We prove that a suitable small perturbation \( h_\delta \) of \( h \) in \( L^2(D) \)-norm yields by formula (9) a bounded potential \( q_\delta \). This means that the "smallness" restriction on the norm of \( f \) can be dropped. The proof is given below.
Proof of Lemma 4. Let

\[ N := \{ x : \psi(x) = 0, x \in D \} , \]

where

\[ \psi := u_0(x) - \int_D g(x, y) h_\delta(y) dy. \]

This set is generically a line, defined by two simultaneous equations \( \psi_j = 0, j = 1, 2 \), where \( \psi_1 := \text{Re}\psi \) and \( \psi_2 := \text{Im}\psi \). Let

\[ N_\delta := \{ x : |\psi(x)| < \delta, x \in D \} \]

and \( D_\delta := D \setminus N_\delta \). Generically, \( |\nabla \psi| \geq c > 0 \) on \( N \), and, therefore, by continuity, in \( N_\delta \). A small perturbation of \( h \) will lead to these generic assumptions. Consider the new coordinates

\[ s_1 = \psi_1, \quad s_2 = \psi_2, \quad s_3 = x_3. \]

Choose the origin on \( N \). The Jacobian \( \frac{\partial(s_1, s_2, s_3)}{\partial(x_1, x_2, x_3)} \) is non-singular in \( N_\delta \). The vectors \( \nabla \psi_j, j = 1, 2 \), are linearly independent in \( N_\delta \). Define \( h_\delta = h \) in \( D_\delta \) and \( h_\delta = 0 \) in \( N_\delta \). Let

\[ q_\delta := \frac{h_\delta}{\psi_\delta}, \]

where

\[ \psi_\delta := u_0(x) - \int_D g(x, y) h_\delta(y) dy. \]

We wish to prove that the function \( q_\delta \) is bounded. It is sufficient to check that \( |\psi_\delta| > c > 0 \) in \( N_\delta \). By \( c \) we denote various positive constants independent of \( \delta \). One has

\[ |\psi_\delta| \geq |\psi| - I(\delta) \geq \delta - I(\delta), \]

where

\[ I(\delta) := \frac{M}{4\pi} \int_{N_\delta} \frac{dy}{|x - y|}, \quad x \in D_\delta, \quad M = \max_{x \in N_\delta} |\psi|. \]

The proof will be completed if the estimate

\[ I(\delta) = O(\delta^2 |\ln(\delta)|) \]

is established. Let us derive this estimate. It is sufficient to check this estimate for the integral

\[ I := \int_{N_\delta} \frac{dy}{|y|} = 2\pi \int_{\rho \leq \delta} d\rho \int_0^1 \frac{ds_3}{\sqrt{s_3^2 + \rho^2}}, \]

where \( \rho^2 = s_1^2 + s_2^2 \). A direct calculation yields the desired estimate: \( I = O(\delta^2 |\ln(\delta)|) \). Lemma is proved.
Finally we make some remarks about ill-posedness of our algorithm for finding $q$ given $f$. This problem is ill-posed because an arbitrary $f \in L^2(S^2)$ cannot be the scattering amplitude $A_q(\beta)$ corresponding to a compactly supported potential $q$. Indeed, it is proved in [5], [7], that $A(\beta)$ is infinitely differentiable on $S^2$ and is a restriction to $S^2$ of a function analytic on the algebraic variety in $\mathbb{C}^3$, defined by the equation $\beta \cdot \beta = k^2$. Finding $h$ satisfying (4) is an ill-posed problem if $\varepsilon$ is small. It is similar to solving the first-kind Fredholm integral equation

$$\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx = -f(\beta),$$

whose kernel is infinitely smooth. Our solution (7) shows the ill-posedness of the problem because the denominator in (7) tends to zero as $\ell$ grows. Methods for stable solutions of ill-posed problems (see [7]) should be applied to finding $h$. If $h$ is found, then $q$ is found by formula (7), provided that (8) holds. If (8) does not hold, one perturbs slightly $h$ according to Lemma 4, and get a potential $q_\delta$ by formula (9) with $h_\delta$ in place of $h$.

References

[1] Bateman, H., Erdelyi, A., *Tables of integral transforms*, McGraw-Hill, New York, 1954.

[2] Cycon, H., Froese, R., Kirsch, W., Simon, B., *Schrödinger operators*, Springer, Berlin, 1986.

[3] Kato, T., “Growth properties of solutions of the reduced wave equation with a variable coefficient”, Comm. Pure and Appl. Math., 12, (1959), 403-425.

[4] Ramm, A. G. , “Recovery of the potential from fixed energy scattering data”. Inverse Problems, 4, (1988), 877-886.

[5] ———, “Stability estimates in inverse scattering”, Acta Appl. Math., 28, N1, (1992), 1-42.

[6] ———, “Stability of solutions to inverse scattering problems with fixed-energy data”, Milan Journ of Math., 70, (2002), 97-161.

[7] ———, *Inverse problems*, Springer, New York, 2005.

[8] Completeness of the set of scattering amplitudes, Phys. Lett. A, (to appear)