Influence Diffusion Dynamics and Influence Maximization in Social Networks with Friend and Foe Relationships

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Abstract

Influence diffusion and influence maximization in large-scale online social networks (OSNs) have been extensively studied, because of their impacts on enabling effective online viral marketing. Existing studies focus on social networks with only friendship relations, whereas the foe or enemy relations that commonly exist in many OSNs, e.g., Epinions and Slashdot, are completely ignored. In this paper, we make the first attempt to investigate the influence diffusion and influence maximization in OSNs with both friend and foe relations, which are modeled using positive and negative edges on signed networks. In particular, we extend the classic voter model to signed networks and analyze the dynamics of influence diffusion of two opposite opinions. We first provide systematic characterization of both short-term and long-term dynamics of influence diffusion in this model, and illustrate that the steady state behaviors of the dynamics depend on three types of graph structures, which we refer to as balanced graphs, anti-balanced graphs, and strictly unbalanced graphs. We then apply our results to solve the influence maximization problem and develop efficient algorithms to select initial seeds of one opinion that maximize either its short-term influence coverage or long-term steady state influence coverage. Extensive simulation results on both synthetic and real-world networks, such as Epinions and Slashdot, confirm our theoretical analysis on influence diffusion dynamics, and demonstrate the efficacy of our influence maximization algorithm over other heuristic algorithms.

Keywords: Signed networks, voter model, influence maximization, social networks

1 Introduction

As the popularity of online social networks (OSNs) such as Facebook and Twitter continuously increases, OSNs have become an important platform for the dissemination of news, ideas, opinions, etc. The openness of the OSN platforms and the richness of contents and user interaction information enable intelligent online recommendation systems and viral marketing techniques. For example, if a company wants to promote a new product, it may identify a set of influential users in the online social network and provide them with free sample products. They hope that these influential users could influence their friends, and friends of friends in the network and so on, generating a large influence cascade so that many users adopt their product as a result of such word-of-mouth effect. The question is how to select the initial users given a limited budget on free samples, so as to influence the largest number of people to purchase the product through this “word-of-mouth” process. Similar situations could apply to the promotion of ideas and opinions, such as political candidates trying to find early supporters for their political proposals and agendas, government authorities or companies trying to win public support by finding and convincing an initial set of early adopters to their ideas.

The above problem is referred to as the influence maximization problem in the literature, which has been extensively studied in recent years [8-10, 16, 18, 20-21, 25, 29, 30]. In these studies, several influence diffusion models are proposed to formulate the underlying influence propagation processes, including linear threshold (LT) model, independent cascade (IC) model, voter model, etc. A number of approximation algorithms and scalable heuristics are designed under these models to solve the influence maximization problem.
However, all existing studies only look at networks with positive (i.e., friend, altruism, or trust) relationships, where in reality, relationships also include negative ones, such as foe, spite or distrust relationships. In Ebay, users develop trust and distrust in agents in the network; In online review and news forums, such as Epinions and Slashdot, readers approve or denounce reviews and articles of each other. Some recent studies \[11,23,24\] already look into the network structures with both positive and negative relationships. As a common sense exploited in many existing social influence studies \[8–10,16,20\], positive relationships carry the influence in a positive manner, i.e., you would more likely trust and adopt your friends’ opinions. In contrast, we consider that negative relationships often carry influence in a reverse direction — if your foe chooses one opinion or votes for one candidate, you would more likely be influenced to do the opposite. This echoes the principles that “the friend of my enemy is my enemy” and “the enemy of my enemy is my friend.” Structural balance theory has been developed based on these assumptions in social science (see Chapter 5 of \[14\] and the references therein). We acknowledge that in real social networks, people’s reactions to the influence from their friends or foes could be complicated, i.e., one could take the opposite opinion of what her foe suggests for one situation or topic, but may adopt the suggestion from the same person for a different topic, because she trusts her foe’s expertise in that particular topic. In this study, we consider the influence diffusion for a single topic, where one always takes the opposite opinion of what her foe suggests. This is our first attempt to model influence diffusion in signed networks, and such topic-dependent simplification is commonly employed in prior influence diffusion studies on unsigned networks \[8–10,16,18,20\]. Our work aims at providing a mathematical analysis on the influence diffusion dynamic incorporated with negative relationship and applying our analysis to the algorithmic problem of influence maximization.

1.1 Our contributions

In this paper, we extend the classic voter model \[13,19\] to incorporate negative relationships for modeling the diffusion of opinions in a social network. Given an unsigned directed graph (digraph), the basic voter model works as follows. At each step, every node in the graph randomly picks one of its outgoing neighbors and adopts the opinion of this neighbor. Thus, the voter model is suitable to interpret and model opinion diffusions where people’s opinions may switch back and forth based on their interactions with other people in the network. To incorporate negative relationships, we consider signed digraphs in which every directed edge is either positive or negative, and we consider the diffusion of two opposite opinions, e.g., black and white colors. We extend the voter model to signed digraphs, such that at each step, every node randomly picks one of its outgoing neighbors, and if the edge to this neighbor is positive, the node adopts the neighbor’s opinion, but if the edge is negative, the node adopts the opposite of the neighbor’s opinion (Section 2).

We provide detailed mathematical analysis on the voter model dynamics for signed networks (Section 3). For short-term dynamics, we derive the exact formula for opinion distribution at each step. For long-term dynamics, we provide closed-form formulas for the steady state distribution of opinions. We show that the steady state distribution depends on the graph structure: we divide signed digraphs into three classes of graph structures — balanced graphs, anti-balanced graphs, and strictly unbalanced graphs, each of which leads to a different type of steady state distributions of opinions. While balanced and unbalanced graphs have been extensively studied by structural balance theory in social science \[14\], the anti-balanced graphs form a new class that has not been covered before, to the best of our knowledge. Moreover, our long-term dynamics not only cover strongly connected and aperiodic digraphs that most of such studies focus on, but also weakly connected and disconnected digraphs, making our study more comprehensive.

We then study the influence maximization problem under the voter model for signed digraphs (Section 4). The problem here is to select at most \(k\) initial white nodes while all others are black, so that either in short term or long term the expected number of white nodes is maximized. This corresponds to the scenario where one opinion is dominating the public and an alternative opinion (e.g. a competing political agenda, or a new innovation) tries to win over supporters as much as possible by selecting some initial seeds to influence on. We provide efficient algorithms that find optimal solutions for both short-term and long-term cases. In particular, for long-term influence maximization, our algorithm provides a comprehensive solution covering weakly connected and disconnected signed digraphs, with nontrivial computations on influence coverage of seed nodes.

Finally, we conduct extensive simulations on both real-world and synthetic networks to verify our analysis and to show the effectiveness of our influence maximization algorithm (Section 5). The simulation results demonstrate that
our influence maximization algorithms perform much better than other heuristic algorithms.

To the best of our knowledge, we are the first to study influence diffusion and influence maximization in signed networks, and the first to apply the voter model to this case and provide efficient algorithms for influence maximization under voter model for signed networks. Our identification of the class of anti-balanced graphs could also be of independent interest in the study of social networks with both positive and negative relationships.

1.2 Related work

To the best of our knowledge, we are the first to study the influence diffusion and influence maximization on signed digraphs. In this subsection, we discuss the topics that are closely related to our problem, such as: (1) influence maximization and voter model, (2) signed networks, and (3) competitive influence diffusion.

Influence maximization and voter model. Influence maximization has been extensively studied in the literature. The initial work [20] proposes several influence diffusion models and provides the greedy approximation algorithm for influence maximization. More recent works [8, 10, 16, 18, 21, 25, 29] study efficient optimizations and scalable heuristics for the influence maximization problem. In particular, voter model is proposed in [13, 19], and is good for modeling opinion diffusions in which people may switch opinions back and forth from time to time due to the interactions with other people in the network. Even-Dar and Shapira [16] study the influence maximization problem in the voter model on simple unsigned and undirected graphs, and they show that the best seeds for long-term influence maximization are simply the highest degree nodes. As a contrast, we show in this paper that seed selection for signed digraphs are more sophisticated, especially for weakly connected or disconnected signed digraphs. More voter model related research are conducted in physics domain, where the voter model, the zero-temperature Glauber dynamics for the Ising model, invasion process, and other related models of population dynamics belong to the class of models with two absorbing states and epidemic spreading dynamics [1, 2, 3, 22–32]. However, none of these works study the influence diffusion and influence maximization of voter model under signed networks.

Signed networks. The signed networks with both positive and negative links have gained attentions recently [8, 22–24]. In [23, 24], the authors empirically study the structure of real-world social networks with negative relationships based on two social science theories, i.e., balance theory and status theory. Kunegis et al. [22] study the spectral properties of the signed undirected graphs, with applications in link predictions, spectral clustering, etc. Borgs et al. [3] proposes a generalized PageRank algorithm for signed networks with application to online recommendations. None of the above work studies influence diffusion and influence maximization in signed networks.

Competitive influence diffusion. A number of recent studies focus on competitive influence diffusion and maximization [2, 4, 6, 7], in which two or more competitive opinions or innovations are diffusing in the network. Although they consider two or more competitive or opposing influence diffusions, they are all on unsigned networks, different from our study here on diffusion with both positive and negative relationships.

2 Voter model for signed networks

We consider a weighted directed graph (digraph) $G = (V, E, A)$, where $V$ is the set of vertices, $E$ is the set of directed edges, and $A$ is the weighted adjacency matrix with $A_{ij} \neq 0$ if and only if $(i, j) \in E$, with $A_{ij}$ as the weight of edge $(i, j)$. The voter model was first introduced for unsigned graphs, with nonnegative adjacency matrices $A$’s. In this model, each node holds one of two opposite opinions, represented by black and white colors. Initially each node has either black or white color. At each step $t \geq 1$, every node $i$ randomly picks one outgoing neighbor $j$ with the probability proportional to the weight of $(i, j)$, namely $A_{ij}/\sum_{\ell} A_{i\ell}$, and changes its color to $j$’s color. The voter model also has a random walk interpretation. If a random walk starts from $i$ and stops at node $j$ at step $t$, then $i$’s color at step $t$ is $j$’s color at step $0$.

In this paper, we extend the voter model to signed digraphs, in which the adjacency matrix $A$ may contain negative entries. A positive entry $A_{ij}$ represents that $i$ considers $j$ as a friend or $i$ trusts $j$, and a negative $A_{ij}$ means that $i$ considers $j$ as a foe or $i$ distrusts $j$. The absolute value $|A_{ij}|$ represents the strength of this trust or distrust relationship. The voter model is thus extended naturally such that one always takes the same opinion from his/her friend, and the opposite opinion of his/her foe. Technically, at each step $t \geq 1$, $i$ randomly picks one outgoing neighbor $j$ with probability $|A_{ij}|/\sum_{\ell} |A_{i\ell}|$, and if $A_{ij} > 0$ (or edge $(i, j)$ is positive) then $i$ changes its color to $j$’s color, but if
A\_i\_j < 0 (or edge (i, j) is negative) then i changes its color to the opposite of j’s color. The random walk interpretation can also be extended for signed networks: if the t-step random walk from i to j passes an even number of negative edges, then i’s color at step t is the same as j’s color at step 0; while if it passes an odd number of negative edges, then i’s color at step t is the opposite of j’s color at step 0.

Given a signed digraph \( G = (V, E, A) \), let \( G^+ = (V, E^+, A^+) \) and \( G^- = (V, E^-, A^-) \) denote the unsigned subgraphs consisting of all positive edges \( E^+ \) and all negative edges \( E^- \), respectively, where \( A^+ \) and \( A^- \) are the corresponding non-negative adjacency matrices. Thus we have \( A = A^+ - A^- \). Similar to unsigned digraphs, \( G \) is aperiodic if the greatest common divisor of the lengths of all cycles in \( G \) is 1, and \( G \) is ergodic if it is strongly connected and aperiodic. A sink component of a signed digraph is a strongly connected component that has no outgoing edges to any nodes outside the component. When studying the long-term dynamics of the voter model, we assume that all signed strongly connected components are ergodic. We first study the case of ergodic graphs, and then extend it to the more general case of weakly connected or disconnected graphs with ergodic sink components. Table 1 provides notations and terminologies used in the paper. Note that one basic fact we often use in studying long-term convergence behavior is: If matrix \( P \) satisfies \( \lim_{t \to \infty} P^t = 0 \), then \( I - P \) is invertible and \( (I - P)^{-1} = \lim_{t \to \infty} \sum_{i=0}^{t} P^i \).

### 3 Analysis of voter model dynamics on signed digraphs

In this section, we study the short-term and long-term dynamics of the voter model on signed digraphs. In particular, we answer the following two questions.

(i) **Short-term dynamics:** Given an initial distribution of black and white nodes, what is the distribution of black and white nodes at step \( t > 0 \)?

(ii) **Convergence of voter model:** Given an initial distribution of black and white nodes, would the distribution converge, and what is the steady state distribution of black and white nodes?

#### 3.1 Short-term dynamics

To study voter model dynamics on signed digraphs, we first define the signed transition matrix as follows.

**Definition 1 (Signed transition matrix).** Given a signed digraph \( G = (V, E, A) \), we define the signed transition matrix of \( G \) as \( P = D^{-1} A \), where \( D = \text{diag}[d_i] \) is the diagonal matrix and \( d_i = \sum_{j \in V} |A_{ij}| \) is the weighted out-degree of node \( i \).
Next proposition characterizes the dynamics of the voter model at each step using the signed transition matrix.

**Proposition 1.** Let $G = (V, E, A)$ be a signed digraph and denote $x_0$ as the initial white color distribution vector, i.e., $x_0(i)$ represents the probability that node $i$ is white initially. Then, the white color distribution at step $t$, denoted by $x_t$, can be computed as

$$x_t = P^t x_0 + \left( \sum_{i=0}^{t-1} P^i \right) g^-,$$

where $g^- = D^{-1} A^{-1}$, i.e. $g^-(i)$ is the weighted fraction of outgoing negative edges of node $i$.

**Proof.** Based on the signed digraph voter model defined in Section 2, $x_t$ can be iteratively computed as

$$x_t(i) = \sum_{j \in V} \frac{A_{ij}^+}{d_i} x_{t-1}(j) + \sum_{j \in V} \frac{A_{ij}^-}{d_i} (1 - x_{t-1}(j)).$$

In matrix form, we have

$$x_t = D^{-1} A x_{t-1} + D^{-1} A^+ I = P x_{t-1} + g^-,$$

which yields Eq. (1) by repeatedly applying Eq. (3).

### 3.2 Convergence of signed transition matrix with relation to structural balance of signed digraphs

Eq. (1) infers that the long-term dynamics, i.e., the vector $x_t$ when $t$ goes to infinity, depends critically on the limit of $P^t$ and $\sum_{i=0}^{t-1} P^i$. We show below that the limiting behavior of the two matrix sequences is fundamentally determined by the structural balance of signed digraph $G$, which connects to the social balance theory well studied in the social science literature (cf. [14]). We now define three types of signed digraphs based on their balance structures.

**Definition 2 (Structural balance of signed digraphs).** Let $G = (V, E, A)$ be a signed digraph.

1. **Balanced digraph.** $G$ is balanced if there exists a partition $S, \bar{S}$ of nodes in $V$, such that all edges within $S$ and $\bar{S}$ are positive and all edges across $S$ and $\bar{S}$ are negative.
2. **Anti-balanced digraph.** $G$ is anti-balanced if there exists a partition $S, \bar{S}$ of nodes in $V$, such that all edges within $S$ and $\bar{S}$ are negative and all edges across $S$ and $\bar{S}$ are positive.
3. **Strictly unbalanced digraph.** $G$ is strictly unbalanced if $G$ is neither balanced nor anti-balanced.

The balanced digraphs defined above correspond to the balanced graphs originally defined in social balance theory. It is known that a balanced graph can be equivalently defined by the condition that all circles in $G$ without considering edge directions contain an even number of negative edges [14]. On the other hand, the concept of anti-balanced digraphs seems not appearing in the social balance theory. Note that balanced digraphs and anti-balanced digraphs are not mutually exclusive. For example, a four node circle with one pair of non-adjacent edges being positive and the other pair being negative is both balanced and anti-balanced. However, for studying long-term dynamics, we only need the above categorization for aperiodic digraphs, for which we show below that balanced digraphs and anti-balanced digraphs are mutually exclusive.

**Proposition 2.** An aperiodic digraph $G$ cannot be both balanced and anti-balanced.

**Proof.** Suppose, for a contradiction, that an aperiodic digraph $G$ is both balanced and anti-balanced. By the equivalent condition of balanced graphs, we know that all cycles of $G$ have an even number of negative edges. Since an anti-balanced graph will become balanced if we negate the signs of all its edges, we know that all cycles of $G$ also have an even number of positive edges. Therefore, all cycles of $G$ must have an even number of edges, which means their lengths have a common divisor $2$, contradicting to the assumption that $G$ is aperiodic. 

\[ \square \]
With the above proposition, we know that balanced graphs, anti-balanced graphs, and strictly unbalanced graphs indeed form a classification of aperiodic digraphs, where anti-balanced graphs and strictly unbalanced graphs together correspond to unbalanced graphs in the social balance theory. We identify anti-balanced graphs as a special category because it has a unique long-term dynamic behavior different from other graphs. An example of anti-balanced graphs is a graph with only negative edges. In general, anti-balanced graphs could be viewed as an extreme in which many hostility exist among individuals, e.g., networks formed by bidders in auctions [5, 28].

**Case of ergodic signed digraphs.** Now, we discuss the limiting behavior of $P^t$ of ergodic signed digraphs with three balance structures. A signed digraph $G = (V, E, A)$ is ergodic if and only if for any node $i$, there always exists a signed path to any other node in $G$ and the common divisor of all cycle path lengths of $i$ is 1. Here, a signed path $R$ in a signed graph $G$ is a sequence of nodes with the edges being directed from each node to the following one, where the length of the path, denoted as $|R|$, is the total number of directed edges in $R$. The sign of a path is positive, if there is an even number of negative edges along the path; otherwise the sign of a path is negative. Below, we first introduce Proposition 3 presenting that the balance structures of ergodic signed digraphs can be interpreted and distinguished in terms of the path lengths and path signs in $G$. As a result, Lemma 1 introduces the various limiting behaviors of $P^t$ of ergodic signed digraphs with respect to three balance structures.

**Proposition 3.** Let $G = (V, E, A)$ be an ergodic strictly unbalanced digraph. There exist two nodes $i$ and $j$, and two directed paths from $i$ to $j$ with the same length but different signs.

**Proof.** Given the following three statements, we prove $\text{Statement 1} \Rightarrow \text{Statement 2} \Rightarrow \text{Statement 3}$, which in turn proves this proposition, i.e., $\neg \text{Statement 3} \Rightarrow \neg \text{Statement 1}$. We assume that $G$ is a signed ergodic digraph.

**Statement 1:** For any two nodes $i$ and $j$, all paths from $i$ to $j$ with the same length have same signs.

**Statement 2:** For any two nodes $i$ and $j$, all paths from $i$ to $j$ with even length have same signs.

**Statement 3:** $G$ is either balanced or anti-balanced.

1) Proof by contradiction for $\text{Statement 1} \Rightarrow \text{Statement 2}$. We assume that in $G$, there exist two even length paths $R_{c1}$ and $R_{c2}$ from $i$ to $j$ with different signs. Since $G$ is ergodic, by Proposition 4 in Appendix A, there must exist a path, denoted by $R_o$, from $j$ to $i$ with odd length (no matter what sign it carries). Denote the length of these three paths as $|R_{c1}|$, $|R_{c2}|$ and $|R_o|$, respectively.

Then, $R_{c1} = R_{c1} + R_o$ forms a cycle at node $i$ with odd length $|R_{c1}| + |R_o|$ and $R_{c2} = R_{c2} + R_o$ forms another cycle at $i$ with odd length $|R_{c2}| + |R_o|$. Clearly, two cycles $R_{c1}$ and $R_{c2}$ carry different signs. Then, let $R'_{c1} = R_{c1}^{R_{c1}}$ denote a cycle of node $i$, by continuing $R_{c1}$ for $|R_{c2}|$ times, which has the same sign with $R_{c1}$ since $|R_{c2}|$ is odd. Similarly, we construct a cycle $R'_{c2} = R_{c2}^{R_{c1}}$ by continuing $R_{c2}$ for $|R_{c1}|$ times, which has the same sign as $R_{c2}$. Thus $R'_{c1}$ and $R'_{c2}$ have the same length of $|R_{c1}|$ but different signs, which contradicts to $\text{Statement 1}$.

2) Proof for $\text{Statement 2} \Rightarrow \text{Statement 3}$. By Proposition 4 in Appendix A, we know that between any two nodes there must exist even-length paths. By $\text{Statement 2}$, we partition $V$ into $S$ and $\bar{S}$, based on the signs of even length paths originated from a particular node $i \in V$. More specifically, $S$ contains the nodes to which all even length paths from $i$ have positive signs, and $\bar{S}$ contains the other set of nodes (note that $i$ may not be in $S$).

We argue that (a) within $S$ and $\bar{S}$, all edges have same signs; and (b) all edges between $S$ and $\bar{S}$ have same signs. Since $G$ contains both negative and positive edges, it must be either balanced or anti-balanced.

For (a), assume to the contrary that there exist two directed edges $R_{ab} = a \rightarrow b$ and $R_{cd} = c \rightarrow d$, which both reside in the same set, e.g., $S$ with different signs. (The case for $\bar{S}$ is similar.)

We construct two even length paths from $i$ to $c$ and $i$ to $d$ as follows.

$$R_e(i, c) = R_e(i, b) + R_e(b, c),$$
$$R_e(i, d) = R_e(i, a) + R_{ab} + R_e(b, c) + R_{cd}$$

where $R_e(x, y)$ represents the constructed even length path from node $x$ to node $y$.

Since both $c, d \in S$, by construction, then $R_e(i, c)$ and $R_e(i, d)$ have same signs

$$\text{sgn}(R_e(i, c)) = \text{sgn}(R_e(i, d)).$$  (4)
On the other hand, since $a$ and $b$ are in the same group as $c$ and $d$, $\text{sgn}(R_e(i, a)) = \text{sgn}(R_e(i, b))$. Then, we have

$$\text{sgn}(R_e(i, c)) = \text{sgn}(R_e(i, b))\text{sgn}(R_e(b, c)), \quad (5)$$

$$\text{sgn}(R_e(i, d)) = \text{sgn}(R_e(i, a))\text{sgn}(R_e(b, c))\text{sgn}(R_{cd})$$

$$= -\text{sgn}(R_e(i, b))\text{sgn}(R_e(b, c)). \quad (6)$$

Eq.(6) comes from the assumption that $R_{ab}$ and $R_{cd}$ have different signs. Eq.(4) contradicts with Eq.(5) and Eq.(6).

For (b), assume that there exist two edges $R_{ab}$ and $R_{cd}$ with different signs between $S$ and $\bar{S}$. Still consider the two even length paths $R_e(i, c)$ and $R_e(i, d)$ constructed before. Since $c$ and $d$ are not in the same side, $R_e(i, c)$ and $R_e(i, d)$ have opposite signs by the construction, i.e.,

$$\text{sgn}(R_e(i, c)) = -\text{sgn}(R_e(i, d)). \quad (7)$$

On the other hand, since $a$ and $b$ are in the different groups as well, $\text{sgn}(R_e(i, a)) = -\text{sgn}(R_e(i, b))$. Then, we have

$$\text{sgn}(R_e(i, c)) = \text{sgn}(R_e(i, b)) \cdot \text{sgn}(R_e(b, c)), \quad (8)$$

$$\text{sgn}(R_e(i, d)) = \text{sgn}(R_e(i, a))\text{sgn}(R_e(b, c))\text{sgn}(R_{cd})$$

$$= \text{sgn}(R_e(i, b)) \cdot \text{sgn}(R_e(b, c)). \quad (9)$$

However, Eq.(7) contradicts with Eq.(8) and Eq.(9). This completes the proof.

Next lemma characterizes the limiting behavior of $P^t$ of ergodic signed digraphs with all three balance structures. Given a signed digraph $G = (V, E, A)$, let $\bar{G} = (V, E, \bar{A})$ corresponds to its unsigned version ($\bar{A}_{ij} = |A_{ij}|$ for all $i, j \in V$). When $G$ is ergodic, random walk on $\bar{G}$ has a unique stationary distribution, denoted as $\pi$. That is, $\pi^T = \pi^T \bar{P}$, where $\bar{P} = D^{-1} \bar{A}$ is the transition probability matrix for $\bar{G}$. Henceforth, we always use $S, \bar{S}$ to denote the corresponding partition for either balanced graphs or anti-balanced graphs. We define the infinity norm of matrix $M \in \mathbb{R}^{m \times m}$ as: $\|M\|_{\infty} := \max_{1 \leq i \leq m} \sum_{j=1}^{m} |M_{ij}|$.

**Lemma 1.** Given an ergodic signed digraph $G = (V, E, A)$, let $\bar{G} = (V, E, \bar{A})$ be the unsigned digraph. When $G$ is balanced or strictly unbalanced, $P^t$ converges, and when $G$ is anti-balanced, the odd and even subsequences of $P^t$ converge to opposite matrices.

Balanced $G$: \quad $\text{lim}_{t \to \infty} P^t = \hat{I}_S \hat{T}_S$;

Strictly unbalanced $G$: \quad $\text{lim}_{t \to \infty} P^t = 0$;

Anti-balanced $G$: \quad $\text{lim}_{t \to \infty} P^{2t} = \hat{I}_S \hat{T}_S$, \quad $\text{lim}_{t \to \infty} P^{2t+1} = -\hat{I}_S \hat{T}_S$.

**Proof.** (1) When $G$ is balanced, the signed transition matrix $P$ can be written as $P = \hat{I}_S \bar{P} \hat{I}_S$. Since $\bar{G}$ is ergodic, we have $\text{lim}_{t \to \infty} \bar{P}^t = \hat{1} \pi^T$. Thus,

$$\text{lim}_{t \to \infty} P^t = \text{lim}_{t \to \infty} (\hat{I}_S \bar{P} \hat{I}_S)^t = \hat{I}_S \hat{T}_S^t,$$

where we use simple facts $\hat{I}_S^2 = I$, $\hat{I}_S \hat{1} = \hat{1} \hat{S}$, and $\pi^T \hat{1} = \hat{T}_S$.

(2) When $G$ is anti-balanced, we have $P = -\hat{I}_S \bar{P} \hat{I}_S$. Thus,

$$\text{lim}_{t \to \infty} P^{2t} = \text{lim}_{t \to \infty} (-\hat{I}_S \bar{P} \hat{I}_S)^{2t} = \hat{1} \hat{T}_S^T$$

$$\text{lim}_{t \to \infty} P^{2t+1} = \text{lim}_{t \to \infty} (-\hat{I}_S \bar{P} \hat{I}_S)^{2t+1} = -\hat{1} \hat{T}_S^T.$$

(3) By Proposition 3, given a signed strictly unbalanced digraph $G$, there exist a pair of nodes $i$ and $j$, such that two paths $R_1$ and $R_2$ from $i$ to $j$ have the same length $\ell(i)$ and opposite signs. Consider a random walk from $i$. Let $p_1$ (resp. $p_2$) be the probability that the walk exactly follows $R_1$ (resp. $R_2$) in the first $\ell(i)$ steps. Let $R_{i,k}^{\ell(i)}$ be the set of
all paths from \( i \) to \( k \) with length \( \ell(i) \). Then, for a unit vector \( e_i \) with \( i \)-th entry equal to 1 and other entries as 0, we have

\[
||e_i^T P^{(t)}||_1 = \sum_{k \in V} \sum_{R \in R_i^{(t)}} \text{Prob}[R] |\text{sgn}(R)| \leq 1 - \min(p_1, p_2) = \rho_i.
\]

For any other node \( i' \), there must exist a path \( R' \) from \( i' \) to \( i \), due to the ergodicity of \( G \), thus two paths \( R'_1 = R' + R_1 \) and \( R'_2 = R' + R_2 \) from \( i' \) to \( j \) have the same length, but opposite signs. With similar arguments as that for node \( i \), \( ||e_i^T P^{(t)}||_1 \leq \rho_i \) holds for any \( i' \in V \). Let \( \rho = \max \{ \rho_i \} < 1 \) and \( \ell = \max \{ \ell(i) \} \), we conclude for any \( i \in V \), \( ||e_i^T P^{(t)}||_1 \leq \rho \) holds. Hence, when \( t \geq T = 2\ell \), the following inequality holds

\[
||e_i^T P^{(t)}||_1 = ||e_i^T P^\ell||_1 \leq \rho^{[\frac{T}{\ell}]} \leq \rho^\ell.
\]

Hence \( \lim_{t\to\infty} ||P^t||_\infty = 0 \), i.e., \( \lim_{t\to\infty} P^t = 0 \). \( \square \)

The above lemma clearly shows different convergence behaviors of \( P^t \) for three types of graphs. In particular, \( P^t \) of anti-balanced graphs exhibits a bounded oscillating behavior in the long term.

**Case of weakly connected signed digraphs.** Now, we consider a weakly connected signed digraph \( G = (V, E, A) \) with one ergodic sink component \( G_Z \) with node set \( Z \), which only has incoming edges from the rest of the signed digraph \( G_X \) with node set \( X = V \setminus Z \). Then, the signed transition matrix \( P \) has the following block form.

\[
P = \begin{bmatrix}
P_X & P_Y \\
0 & P_Z
\end{bmatrix},
\]

where \( P_X \) and \( P_Z \) are the block matrices for component \( G_X \) and \( G_Z \), and \( P_Y \) represent the one-way connections from \( G_X \) to \( G_Z \). Then, the \( t \)-step transition matrix \( P^t \) can be expressed as

\[
P^t = \begin{bmatrix}
P_X^{(t)} & P_Y^{(t)} \\
0 & P_Z^{(t)}
\end{bmatrix},
\]

where \( P_X^{(t)} = P_X^t \), \( P_Z^{(t)} = P_Z^t \) and \( P_Y^{(t)} = \sum_{i=0}^{t-1} P_X^i P_Y P_Z^{t-1-i} \). When \( G_Z \) is balanced or anti-balanced, we use \( S_Z, \bar{S}_Z \) to denote the partition of \( Z \) defining its balance or anti-balance structure. Then, we denote column vectors

\[
u_b = (I_X - P_X)^{-1} P_Y 1_{Z,S_Z},
\]

and

\[
u_a = (I_X + P_X)^{-1} P_Y 1_{Z,S_Z}.
\]

The reason that \( I_X - P_X \) is invertible is because \( \lim_{t\to\infty} P_X^t = 0 \), which is in turn because there is a path from any node \( i \) in \( G_X \) to nodes in \( Z \) (since \( Z \) is the single sink), and thus informally a random walk from \( i \) eventually reaches and then stays in \( G_Z \). The same reason applies to \( I_X + P_X \). Lemma \( \square \) provides the formal proof of the fact \( \lim_{t\to\infty} P_X^t = 0 \).

Let \( \hat{\pi}_Z \) denote the stationary distribution of nodes in \( G_Z \), and \( \hat{\pi}_{Z,S_Z} \) is signed, with \( \hat{\pi}_{Z,S_Z}(i) = \pi_Z(i) \) for \( i \in S_Z \), and \( \hat{\pi}_{Z,S_Z}(i) = -\pi_Z(i) \) for \( i \in Z \setminus S_Z \). Lemma \( \square \) discloses the convergence of \( P^t \) given various balance structures of \( G_Z \).

**Lemma 2.** For weakly connected signed digraph \( G = (V, E, A) \) with one ergodic sink components, with signed transition matrix given in Eq. \((11)\), we have

- **Balanced \( G_Z \):**
  \[
  \lim_{t\to\infty} P^t = \begin{bmatrix}
  0 & 1_u \\
  0 & 1_{1_{Z,S_Z}, \bar{1}_{Z,S_Z}}
\end{bmatrix}
  \]

- **Strictly unbalanced \( G_Z \):**
  \[
  \lim_{t\to\infty} P^t = \begin{bmatrix}
  0 & -1_u \\
  0 & 1_{1_{Z,S_Z}, \bar{1}_{Z,S_Z}}
\end{bmatrix}
  \]

- **Anti-balanced \( G_Z \):**
  \[
  \lim_{t\to\infty} P^{2t} = \begin{bmatrix}
  0 & -1_u \\
  0 & 1_{1_{Z,S_Z}, \bar{1}_{Z,S_Z}}
\end{bmatrix}, \lim_{t\to\infty} P^{2t+1} = \begin{bmatrix}
  0 & 1_u \\
  0 & 1_{1_{Z,S_Z}, \bar{1}_{Z,S_Z}}
\end{bmatrix}
  \]

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Proof. We discuss the convergence of \( P_X^t, P_Z^t, \) and \( P_Y^t \) in Eq. (11), respectively.

(1) We first prove that \( P_X^t \) converges to 0, i.e., \( \lim_{t \to \infty} P_X^t = 0 \).

Since \( G_X \) does not contain sink components, any node \( i \in X \) has a path to component \( G_Z \). Let \( R_{iZ} \) be the shortest path from \( i \) to some node in \( Z \), and \( \text{Prob}[R_{iZ}] \) denote the probability that a random walk starting from \( i \) takes the path \( R_{iZ} \). Hence we denote

\[ p = \min_{i \in X} \text{Prob}[R_{iZ}], \quad m = \max_{i \in X} |R_{iZ}|, \]

which implies that starting from any node \( i \in X \), after \( m \) steps of random walk, there is at least probability \( p \) that it reaches component \( G_Z \). Hence, we have \( \|P_X^m\|_\infty \leq (1-p) < 1 \). Let \( T = 2m \), then for any \( t > T \), we have

\[ \|P_X^t\|_\infty = \|P_X^t - P_X^m\|_\infty \leq (1-p)^{t-m} \leq (1-p)^t, \]

which implies \( \lim_{t \to \infty} \|P_X^t\|_\infty = 0 \), i.e., \( \lim_{t \to \infty} P_X^t = 0 \).

(2) For subgraph \( G_Z \), Lemma 1 directly yields

\[ \lim_{t \to \infty} P_Z^t = \begin{cases} 
0, & \text{Strictly unbalanced } G_Z; \\
1_{Z,S} \pi_{Z,S}^T, & \text{Balanced } G_Z; \\
1_{Z,S} \pi_{Z,S}^T, & \text{Anti-balanced } G_Z, \text{ even } t; \\
-1_{Z,S} \pi_{Z,S}^T, & \text{Anti-balanced } G_Z, \text{ odd } t.
\end{cases} \tag{14} \]

(3) Below, we focus on proving the results on \( \lim_{t \to \infty} P_Y^t \) using Proposition 6 in Appendix B.

When \( G_Z \) is strictly unbalanced, from Lemma 1 and (1) in this proof, \( \lim_{t \to \infty} P_X^t = 0 \) and \( \lim_{t \to \infty} P_Z^t = 0 \) hold, thus by Proposition 6 in Appendix B \( \lim_{t \to \infty} P_Y^t = 0 \).

When \( G_Z \) is balanced, Lemma 1 and Proposition 8 in Appendix A directly yield \( P_Z - 1_{Z,S} \pi_{Z,S}^T = P_Z - 1_{Z,S} \pi_{Z,S}^T \) for any integer \( t > 0 \), and \( \lim_{t \to \infty} (P_Z - 1_{Z,S} \pi_{Z,S}^T)^t = 0 \), thus

\[ \lim_{t \to \infty} P_Y^t = \lim_{t \to \infty} \sum_{i=0}^{t-1} P_Y^t P_Y (P_Z - 1_{Z,S} \pi_{Z,S}^T)^{t-1-i} + \lim_{t \to \infty} \sum_{i=0}^{t-2} P_X^t P_Y 1_{Z,S} \pi_{Z,S}^T = (I_X - P_X)^{-1} P_Y 1_{Z,S} \pi_{Z,S}^T \]

where the first term in the second line being 0 is due to Proposition 6(ii) in Appendix B.

When \( G_Z \) is anti-balanced, applying Lemma 1 and Proposition 8 in Appendix A we have for any integer \( t > 0 \), \( (P_Z + 1_{Z,S} \pi_{Z,S}^T)^t = P_Z - (1)^t 1_{Z,S} \pi_{Z,S}^T \), and \( \lim_{t \to \infty} (P_Z + 1_{Z,S} \pi_{Z,S}^T)^t = 0 \) hold true, thus

\[ \lim_{t \to \infty} P_Y^t = \lim_{t \to \infty} \sum_{i=0}^{t-1} P_Y^t P_Y (P_Z + 1_{Z,S} \pi_{Z,S}^T)^{t-1-i} + \lim_{t \to \infty} \sum_{i=0}^{t-2} P_X^t P_Y 1_{Z,S} \pi_{Z,S}^T = (I_X - P_X)^{-1} P_Y 1_{Z,S} \pi_{Z,S}^T \]

Hence, we have for anti-balanced \( G_Z \): \( \lim_{t \to \infty} P_Y^{2t} = -u_T \pi_{Z,S}^T \), and \( \lim_{t \to \infty} P_Y^{2t+1} = u_T \pi_{Z,S}^T \). \( \square \)
**Multiple sink components and disconnected signed digraphs.** When there exist \( m > 1 \) ergodic sink components, i.e., \( G_{Z_1}, G_{Z_2}, \ldots, G_{Z_m} \), the rest of the graph \( G \) is considered as \( G_X \). Then the signed transition matrix \( P \) and \( P^t \) can be written as

\[
P = \begin{bmatrix}
P_X & P_{Y_1} & \cdots & P_{Y_m} \\
0 & P_{Z_1} & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & P_{Z_m}
\end{bmatrix},
P^t = \begin{bmatrix}
P_X & (1) \cdots & (t) \cdots & P_{Y_m} \\
0 & (1) \cdots & (t) \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & (t)
\end{bmatrix},
\]

where \( P_{Y_i}^{(t)} = \sum_{j=0}^{t-1} P_X^{(j)} P_{Y_1}^{(t-1-j)} \), \( 1 \leq i \leq m \). Hence, each sink ergodic component \( P_{Z_i} \) along with \( P_X \) and \( P_{Y_1} \) independently follows Lemma 2. For disconnected signed digraph, with \( m \geq 1 \) ergodic or weakly connected components, each of which satisfies Lemma 1 or Lemma 2 respectively. For brevity, we omit the details here.

### 3.3 Long-term dynamics

Based on the structural balance classification and the convergence of signed transition matrix discussed above, we are ready now to analyze the long-term dynamics of the voter model on signed digraphs. Formally, we are interested in characterizing \( x_t \) with \( t \to \infty \), i.e.,

\[
x = \lim_{t \to \infty} x_t = \lim_{t \to \infty} (P^t x_0 + \sum_{i=0}^{t-1} P^i g^-).
\]

If the even and odd subsequences of \( x_t \) converge separately, we denote \( x_e = \lim_{t \to \infty} x_{2t} \), \( x_o = \lim_{t \to \infty} x_{2t+1} \).

Before presenting the results on long-term dynamics of voter model, we first introduce the following useful lemma connecting a signed digraph \( G \) with another graph \( G' \) where all edge signs in \( G \) are negated.

**Lemma 3.** Given a signed digraph \( G = (V, E, A) \), let \( G' = (V, E, -A) \) be a signed digraph with all edge signs negated from \( G \). Then, for any initial color distribution \( x_0 \), at any \( 2t \) steps \( (t > 0) \), the color distributions \( x_{2t}(G) \) on \( G \) and \( x_{2t}(G') \) on \( G' \) are identical.

**Proof.** Let \( P' = -P \) denote the signed transition matrix of \( G' \), and denote the vector \( g^- = D^{-1} A^- 1 \) and \( g^- = D^{-1} (-A)^- 1 = D^{-1} A^+ 1 \). Thus \( g^- = 1 - g^- \). By Eq. (1), after two steps, we have

\[
x_2(G') = P'^2 x_0 + P' g^- + g^- = P'^2 x_0 - P' (1 - g^-) + 1 - g^- = P'^2 x_0 + Py^- + g^- = x_2(G),
\]

where the last equality uses facts \( 1 = D^{-1} A 1 \) and \( P = D^{-1} A \). Since the lemma holds for two steps, then clearly it holds for all even steps.

Next theorem discusses the case of ergodic signed digraphs.

**Theorem 1.** Let \( G = (V, E, A) \) be an ergodic signed digraph, we have

**Balanced G:** \( x = \hat{I} S \hat{\pi}_z^T (x_0 - \frac{1}{2} I) + \frac{1}{2} I \)

**Strictly unbalanced G:** \( x = \frac{1}{2} I \)

**Anti-balanced G:** \( x_e = \hat{I} S \hat{\pi}_z^T (x_0 - \frac{1}{4} I) + \frac{1}{2} I \)
\( x_o = -\hat{I} S \hat{\pi}_z^T (x_0 - \frac{1}{2} I) + \frac{1}{4} I \)

**Proof.** We discuss the limit in Eq. (16) for three possible balance structures of \( G \).

**Balanced digraphs.** From Lemma 1 and Proposition in Appendix A it is easy to prove \( P^m - \hat{I} S \hat{\pi}_z^T = (P - \hat{I} S \hat{\pi}_z^T)^m \) for any integer \( m > 0 \), which yields the following result on the second part in Eq. (16).

\[
\lim_{t \to \infty} \sum_{i=0}^{t-1} P^i g^- = (I - P + \hat{I} S \hat{\pi}_z^T)^{-1} g^- + \lim_{t \to \infty} \sum_{i=1}^{t-1} \hat{I} S \hat{\pi}_z^T g^- = (I - P + \hat{I} S \hat{\pi}_z^T)^{-1} g^- = \frac{1}{2} \frac{1}{2} \hat{I} S \hat{\pi}_z^T 1,
\]

\[
\lim_{t \to \infty} \sum_{i=0}^{t-1} P^i g^- = (I - P + \hat{I} S \hat{\pi}_z^T)^{-1} g^- = \frac{1}{2} \frac{1}{2} \hat{I} S \hat{\pi}_z^T 1,
\]

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where the last term of Eq. (21) is canceled out due to the digraph flow circulation law \cite{12, 26}, i.e.,

\[
\hat{\pi}_S g^- = \hat{\pi}_S D^{-1} A^- 1 = \sum_{i \in S} \pi(i) \sum_{j \in S} \hat{P}_{ij} - \sum_{i \in S} \pi(i) \sum_{j \in S} \hat{P}_{ij} = 0.
\]

The last equality in Eq. (22) holds because

\[
\frac{1}{2} (I - P + \mathbf{1} S \hat{\pi}_S^T) (1 - \mathbf{1} S \hat{\pi}_S^T) - g^- = 0.
\]

Eq. (17) is obtained by combining Eq. (22) with Lemma [1].

**Anti-balanced Digraphs.** Lemma [3] directly yields Eq. (19). The odd step influence distribution sequence is obtained by

\[
x_o = P x_e + g^- = -\mathbf{1} S \hat{\pi}_S^T (x_0 - \frac{1}{2} \mathbf{1}) + \frac{1}{2} \mathbf{1}.
\]

**Strictly unbalanced digraphs.** From Theorem [11] \(\lim_{t \to \infty} P^t = \mathbf{0}\) holds and thus we have

\[
\lim_{t \to \infty} \sum_{i=0}^{t-1} P^i g^- = (I - P)^{-1} g^- = (D - A)^{-1} A^- \mathbf{1} = \frac{1}{2} \mathbf{1}.
\]  

(23)

The last equality comes from the facts \((D - A)I = 2A^-\mathbf{1}\).

Theorem [11] has several implications. First of all, for strictly unbalanced digraphs, each node has equal steady state probability of being black or white, and it is not determined by the initial distribution \(x_0\). Secondly, anti-balanced digraphs has the same steady state distribution as the corresponding balanced graph for even steps, and for odd steps, the distribution oscillates to the opposite \((x_o = 1 - x_e)\). Moreover, Eq. (17) can also be intuitively explained from the random walk interpretation of the voter model. In particular, starting from node \(i\), if we perform a random walk for an infinite number of steps, the probability that the random walk stops at \(j\) is given by the stationary distribution \(\pi(j)\). For balanced graphs, if \(i\) and \(j\) are from the same component (either \(S\) or \(\bar{S}\)), then the random walk must pass an even number of negative edges, so \(i\) takes the same color as \(j\); if \(i\) and \(j\) are from opposite components, then the walk passes an odd number of negative edges and \(i\) takes the opposite of \(j\)'s color. Thus, the steady distribution of \(i \in S\) being white is given by \(\hat{\pi}_S x_0 S + \hat{\pi}_S^T (\mathbf{1} \bar{S} - x_0 \bar{S})\), and the case of \(i \in \bar{S}\) is symmetric. Some algebra manipulations can lead us to Eq. (17).

For a balanced ergodic digraph \(G\) with partition \(S, \bar{S}\), it is easy to check that it has the following two equilibrium states: in one state all nodes in \(S\) are white while all nodes in \(\bar{S}\) are black; and in the other state all nodes in \(S\) are black while all nodes in \(\bar{S}\) are white. We call these two states the polarized states. Using random walk interpretation, we show in the following theorem that with probability 1, the voter model dynamic converges to one of the above two equilibrium states.

**Theorem 2.** Given an ergodic signed digraph \(G = (V, E, A)\), if \(G\) is balanced with partition \(S, \bar{S}\), the voter model dynamic converges to one of the polarized states with probability 1, and the probability of nodes in \(S\) being white is \(\hat{\pi}_S^T (x_0 - \frac{1}{2} I) + \frac{1}{2}\). Similarly, if \(G\) is anti-balanced, with probability 1 the voter model dynamic oscillates between the two polarized states eventually, and the probability of nodes in \(S\) being white at even steps is \(\hat{\pi}_S^T (x_0 - \frac{1}{2} I) + \frac{1}{2}\).

**Proof.** Consider a balanced ergodic digraph \(G\) with partition \(S, \bar{S}\). By ergodicity, given any two nodes \(i\) and \(j\), with probability 1 the random walks starting from \(i\) and \(j\) will meet eventually. If \(i\) and \(j\) are both in \(S\), when the two walks meet at some node \(u\), they both pass either an even number of negative edges (if \(u \in S\)) or an odd number of negative edges (if \(u \in \bar{S}\)). Therefore, \(i\) and \(j\) must be in the same color with probability 1. If \(i\) and \(j\) are from different components \(S\) and \(\bar{S}\), a similar argument shows that they will have the opposite color with probability 1. Therefore the final state is one of the two polarized states. The probability of nodes in \(S\) being white is simply given by Theorem [11] Eq. (17). The case of anti-balanced ergodic digraphs can be argued in a similar way.

Theorem [3] below introduces the long-term dynamics of the weakly connected signed digraphs. We consider weakly connected \(G\) with a single sink ergodic component \(G_{S}\), and use the same notations as in Section 3.2.

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Theorem 3. Let \( G = (V, E, A) \) be a weakly connected signed digraph with a single sink component \( G_Z \) and the rest subgraph \( G_X \). The long-term white color distribution vector \( x \) is expressed in two parts:

\[
x^T = \lim_{t \to \infty} x_t^T = [x_{XY}^T, x_Z^T],
\]

where \( x_Z \) is the limit of \( x_{1Z} \) on \( G_Z \) with initial distribution \( x_{0Z} \) and is given as in Theorem 1 and vector \( x_{XY} \) is given below with respect to the balance structure of \( G_Z \):

- **Balanced** \( G_Z \):
  \[ x_{XY} = \frac{1}{\pi} x_{0X}, \]

- **Strictly unbalanced** \( G_Z \):
  \[ x_{XY} = \frac{1}{\pi} x_{0X} + u_b \tilde{\pi}_{Z,S} (x_{0Z} - \frac{1}{\pi} I_Z) \]

- **Anti-balanced** \( G_Z \), even \( t \):
  \[ x_{XY,e} = \frac{1}{\pi} x_{0X} - u_u \tilde{\pi}_{Z,S} (x_{0Z} - \frac{1}{\pi} I_Z) \]

- **Anti-balanced** \( G_Z \), odd \( t \):
  \[ x_{XY,o} = \frac{1}{\pi} x_{0X} + u_u \tilde{\pi}_{Z,S} (x_{0Z} - \frac{1}{\pi} I_Z) , \]

where \( u_b \) and \( u_u \) are defined in Eq. (12) and Eq. (13).

**Proof.** Let initial distribution \( x_t^T = [x_{0X}^T, x_{0Z}^T] \) and \( g_t^T = [g_X^T, g_Z^T] \). When \( t \to \infty \), Eq. (1) can be written as

\[
x_t^T = \lim_{t \to \infty} (P^t x_0)^T = [x_{XY}^T, x_Z^T] = [x_X^T + x_Y^T, x_Z^T],
\]

where \( x_X = \lim_{t \to \infty} \sum_{i=0}^{t-1} P_i g_X, x_Y = \lim_{t \to \infty} \sum_{i=0}^{t-1} P_i g_Y \), and \( x_Z = \lim_{t \to \infty} \sum_{i=0}^{t-1} P_i g_Z \).

From Lemma 2 \( \lim_{t \to \infty} P_X = 0 \), thus \( x_X = (I_X - P_X)^{-1} g_X \) holds for any ergodic \( G_Z \). Since \( G_Z \) is ergodic, \( x_Z \) follows Theorem 1. Below we will focus on deriving \( x_Y \), where the first part of \( x_Y \) satisfies Lemma 2 i.e.,

\[
\lim_{t \to \infty} P_Y(t) x_{0Z} = \begin{cases} 0 & G_Z \text{ is strictly unbalanced} \\ u_b \tilde{\pi}_{Z,S} x_{0Z} & G_Z \text{ is balanced} \\ -u_u \tilde{\pi}_{Z,S} x_{0Z} & G_Z \text{ is anti-balanced, even } t \\ u_u \tilde{\pi}_{Z,S} x_{0Z} & G_Z \text{ is anti-balanced, odd } t. \end{cases}
\]

The second part of \( x_Y \) can be further written down as

\[
\lim_{m \to \infty} \sum_{t=1}^{m} P_Y(t) g_Z = \lim_{m \to \infty} \sum_{t=0}^{m-1} \sum_{i=0}^{t-1} (P_X^t P_Y P_Z) g_Z = \sum_{i=0}^{\infty} P_X P_Y P_Z g_Z = (I_X - P_X - P_Y)^{-1} g_Z (24)
\]

Now we discuss Eq. (24) under different balance structures of \( G_Z \).

1) **\( G_Z \) is strictly unbalanced.** From Lemma 2 \( \lim_{t \to \infty} P^t = 0 \). Then by Eq. (23) we directly obtain that \( x_{XY} = \frac{1}{2} 1_X \). Applying Eq. (23) to \( \sum_{i=0}^{\infty} P_i g_Z \) in Eq. (24), we have

\[
\lim_{m \to \infty} \sum_{t=1}^{m} P_Y(t) g_Z = \frac{1}{2} (I_X - P_X)^{-1} P_Y 1_Z = \frac{1}{2} 1_X.
\]

Thus, we obtain the following equation:

\[
x_{XY} = x_X + x_Y = (I_X - P_X)^{-1} (g_X + \frac{1}{2} P_Y 1_Z) = \frac{1}{2} 1_X.
\]

2) **\( G_Z \) is balanced.** Using Eq. (22), we have

\[
\lim_{m \to \infty} \sum_{t=1}^{m} P_Y(t) g_Z = \frac{1}{2} (I_X - P_X)^{-1} P_Y (1_Z - \pi_{Z,S} \tilde{\pi}_{Z,S} 1_Z)
\]
Hence, we have

\[ x_{XY} = (I_X - P_X)^{-1} \left( g \tilde{x} + \frac{1}{2} P_Y \mathbf{1}_Z \right) + u_b \tilde{\pi}_{Z,S_Z}^T (x_{0Z} - \frac{1}{2} \mathbf{1}_Z) = \frac{1}{2} \mathbf{1}_X + u_b \tilde{\pi}_{Z,S_Z}^T (x_{0Z} - \frac{1}{2} \mathbf{1}_Z) \]  

(25)

(3) \( G_Z \) is anti-balanced. Using Lemma[3] we can negate the signs of all edges in \( G \) so that the sink becomes balanced. Hence, we know that at even steps in long term,

\[ x_{XY,e} = \frac{1}{2} \mathbf{1}_X - u_b \tilde{\pi}_{Z,S_Z}^T (x_{0Z} - \frac{1}{2} \mathbf{1}_Z), \]  

(26)

where Eq. (26) and Eq. (25) are identical in the sense that \( P_X \)'s and \( P_Y \)'s in Eq. (26) and Eq. (25) have opposite signs. Moreover, the odd step influence distribution sequence is obtained

\[ x_{XY,o} = P_X x_{XY,e} + P_Y x_{Z,e} + g \tilde{x} = \frac{1}{2} \mathbf{1}_X + u_b \tilde{\pi}_{Z,S_Z}^T (x_{0Z} - \frac{1}{2} \mathbf{1}_Z). \]  

(27)

Theorem 3 characterizes the long-term dynamics when the underlying graph is a weakly connected signed digraph with one ergodic sink component. We can see that the results for balanced and anti-balanced sink components are more complicated than the ergodic digraph case, since how non-sink components are connected to the sink subtly affects the final outcome of the steady state behavior. In steady state, while the sink component is still in one of the two polarized states as stated in Theorem 2, the non-sink components exhibit more complicated color distribution, for which we provide probability characterizations in Theorem 3. Using Eq. (15), Theorem 1 and Theorem 3 can be readily extended to the case with more than one ergodic sink components and disconnected digraphs.

4 Influence maximization

With the detailed analysis on voter model dynamics for signed digraphs, we are ready now to solve the influence maximization problem. Intuitively, we want to address the following question: If only at most \( k \) nodes could be selected initially and be turned white while all other nodes are black, how should we choose seed nodes so as to maximize the expected number of white nodes in short term and in long term, respectively?

4.1 Influence maximization objectives

Influence maximization objectives. We consider two types of short-term influence objectives, one is the instant influence, which counts the total number of influenced nodes at a step \( t > 0 \); the other is the average influence, which takes the average number of influenced nodes within the first \( t \) steps. These two objectives have different implications and applications. For example, political campaigns try to convince voters who may change their minds back and forth, but only the voters’ opinions on the voting day are counted, which matches the instant influence. On the other hand, a credit card company would like to have customers keep using their credit card service as much as possible, which is better interpreted by the average influence. When \( t \) is sufficiently large, it becomes the long-term objective, and long-term average influence coincides with long-term instant influence when the dynamic converges.

Formally, we define the short-term instant influence \( f_t(x_0) \) and the short-term average influence \( \bar{f}_t(x_0) \) as follows:

\[ f_t(x_0) := \mathbf{1}_X^T x_t(x_0) \]  

and

\[ \bar{f}_t(x_0) := \frac{\sum_{i=0}^t f_i(x_0)}{t+1}. \]  

(28)

Moreover, we define long term influence as

\[ f(x_0) := \lim_{t \to \infty} \frac{\sum_{i=0}^t f_i(x_0)}{t+1}. \]  

(29)
Note that when the dynamic converges (e.g., ergodic balanced or ergodic strictly unbalanced graphs), \( f(x_0) = \lim_{t \to \infty} f_t(x_0) \). For ergodic anti-balanced graphs (or sink components), it is essentially the average of even- and odd-step limit influence.

Given a set \( W \subseteq V \). Let \( e_W \) be the vector in which \( c_W(j) = 1 \) if \( j \in W \) and \( c_W(j) = 0 \) if \( j \notin W \), which represents the initial seed distribution with only nodes in \( W \) as white seeds. Let \( e_i \) be the shorthand of \( c_{\{i\}} \). Unlike unsigned graphs, if initially no white seeds are selected on a signed digraph \( G \), i.e., \( x_0 = \emptyset \), the instant influence \( f_t(\emptyset) \) at step \( t \) is in general non-zero, which is referred to as the ground influence of the graph \( G \) at \( t \). The influence contribution of a seed set \( W \) do not count such ground influence, as shown in Definition 5.

**Definition 3** (Influence contribution). The instant influence contribution of a seed set \( W \) to the \( t \)-th step instant influence objective, denoted by \( c_t(W) \), is the difference between the instant influence at step \( t \) with only nodes in \( W \) selected as seeds and the ground influence at step \( t \): \( c_t(W) = f_t(e_W) - f_t(\emptyset) \). The average influence contribution \( \bar{c}_t(W) \) and long-term influence contribution \( c(W) \) are defined in the same way: \( \bar{c}_t(W) = f_t(e_W) - f_t(\emptyset) \) and \( c(W) = f(e_W) - f(\emptyset) \).

We are now ready to formally define the influence maximization problem.

**Definition 4** (Influence maximization). The influence maximization problem for short-term instant influence is finding a seed set \( W \) of at most \( k \) seeds that maximizes \( W \)'s instant influence contribution at step \( t \), i.e., finding \( W^* = \arg\max_{|W| \leq k} c_t(W) \). Similarly, the problem for average influence and long-term influence is finding \( W^*_t = \arg\max_{|W| \leq k} \bar{c}_t(W) \) and \( W^* = \arg\max_{|W| \leq k} c(W) \), respectively.

We now provide some properties of influence contribution, which lead to the optimal seed selection rule. By Eq. (30), we have

\[
(c_t(W) = f_t(e_W) - f_t(\emptyset) = 1^T x_t(e_W) - 1^T x_t(\emptyset) = 1^T P^t e_W.
\]

Let \( c_t(i) \) be the shorthand of \( c_t(\{i\}) \), and let \( c_t = [c_t(i)] \) denote the vector of influence contribution of individual nodes. Then \( c^T_t = [c_t(i)]^T = 1^T P^t \). When \( t \to \infty \), the long term influence contributions of individual nodes are obtained as a vector \( c^* \):

\[
c^* = \lim_{t \to \infty} \frac{1^T}{t+1} \sum_{i=0}^t P^i.
\]

When \( P^t \) converges, we simply have

\[
c^* = 1^T \lim_{t \to \infty} P^t.
\]

Lemma 4 below discloses the important property that the influence contribution is a linear function.

**Lemma 4.** Given a white seed set \( W \), \( c_t(W) = \sum_{i \in W} c_t(i) \), \( \bar{c}_t(W) = \sum_{i \in W} \bar{c}_t(i) \), and \( c(W) = \sum_{i \in W} c(i) \).

**Proof.** From Eq. (30), we have

\[
c_t(W) = 1^T P^t e_W = 1^T P^t \sum_{i \in W} e_i = \sum_{i \in W} 1^T P^t e_i = \sum_{i \in W} c_t(i).
\]

The linearity of \( \bar{c}_t \) and \( c \) can be derived from that of \( c_t \).

Given a vector \( v \), let \( n^+(v) \) denote the number of positive entries in \( v \). By applying Lemma 4, we have the optimal seed selection rule for instant influence maximization as follows.

**Optimal seed selection rule for instant influence maximization.** Given a signed digraph and a limited budget \( k \), selecting top \( \min\{k, n^+(c_t)\} \) seeds with the highest \( c_t(i) \)'s, \( i \in V \), leads to the maximized instant influence at step \( t \).

Note that the influence contributions of some nodes may be negative and these nodes should not be selected as white seeds, and thus the optimal solution may have less than \( k \) seeds. The rules for average influence maximization and long-term influence maximization are patterned in the same way. Therefore, the central task now becomes the computation of the influence contributions of individual nodes. Below, we will introduce our SVM algorithm, for Signed Voter model Influence Maximization.
4.2 Short-term influence maximization

By applying Definition 3 and Lemma 4, we develop SVIM-S algorithm to solve the short-term instant and average influence maximization problem, as shown in Algorithm 1.

Algorithm 1 Short-term influence maximization SVIM-S

1: INPUT: Signed transition matrix $P$, short-term period $t$, budget $k$;
2: OUTPUT: White seed set $W$.
3: $c_t = 1$; $\bar{c}_t = 1$;
4: for $i = 1 : t$ do
5: $c^T c_t = c^T c_t P_t$ (for instant influence maximization.)
6: $\bar{c}_t = \bar{c}_t + c_t$; (for average influence maximization.)
7: $W = \text{top } \min\{k, n^+(c_t)\}$ (resp. $\min\{k, n^+(\bar{c}_t)\}$) nodes with the highest $c_t(i)$ (resp. $\bar{c}_t(i)$) values, for instant (resp. average) influence maximization.

SVIM-S algorithm requires $t$ vector-matrix multiplications, each of which takes $|E|$ times entry-wise multiplication operations. Hence the total time complexity of SVIM-S is $O(t \cdot |E|)$.

4.3 Long-term influence maximization

We now study the long-term influence contribution $c$ and introduce the corresponding influence maximization algorithm SVIM-L. We will see that the computation of influence contribution $c$ and seed selection schemes depends on the structural balance and connectedness of the graph. While seed selection for balanced ergodic digraphs still has intuitive explanations, the computation for weakly connected and disconnected digraphs is more involved and less intuitive.

4.3.1 Case of ergodic signed digraphs

When the signed digraph $G = (V, E, A)$ is ergodic, Lemma 5 below characterizes the long-term influence contributions of nodes, with respect to various balance structures.

Lemma 5. Consider an ergodic signed digraph $G = (V, E, A)$. If $G$ is balanced, with bipartition $S$ and $\bar{S}$, the influence contribution vector $c = (|S| - |\bar{S}|)\pi_S$. If $G$ is anti-balanced or strictly unbalanced, $c = 0$.

Proof. (1) When $G$ is balanced, by Lemma 1 and Eq. (32),

$$c^T = 1^T \lim_{t \to \infty} P^t = 1^T \hat{1}_S \pi^T_S = (|S| - |\bar{S}|)\pi_S.$$  

(2) When $G$ is strictly unbalanced, again by Lemma 1 and Eq. (32), we have $c^T = 1^T \lim_{t \to \infty} P^t = 0$.

(3) When $G$ is anti-balanced, by Lemma 1 and Eq. (31), we have

$$c^T = 1^T \lim_{t \to \infty} P^{2t} + \lim_{t \to \infty} P^{2t+1} = 0.$$  

Based on Lemma 5, Algorithm 2 summarizes how to compute the long-term influence contribution $c$ on ergodic signed digraphs.

Lemma 5 suggests that for ergodic balanced digraphs, we should pick the larger component, e.g., $S$, if $|S| > |\bar{S}|$, and select the top $\min\{k, |S|\}$ nodes from $S$ with the largest stationary distributions as white seeds. Selecting these nodes will make the probability of the larger component being white the largest.

Theorem 1 indicates that given an anti-balanced digraph $G$, with bipartition $S$ and $\bar{S}$, the long-term dynamic $x_t$ oscillates on odd and even steps, and their long-term influence contribution is 0. However, we can still maximize the strength of the oscillation of the voter model on an anti-balanced ergodic digraph by properly choosing the initial white seeds (See Remark 1.)
Algorithm 2 $c = \text{ergodic}(G)$

1: **INPUT:** Signed transition matrix $P$.
2: **OUTPUT:** Long term influence contribution vector $c$
3: Detect the structure of ergodic signed digraph $G$;
4: if $G$ is balanced, with bipartition $S$ and $\bar{S}$ then
5: Compute stationary distribution $\pi$ of $P$;
6: $c = (|S| - |\bar{S}|)\pi_S$;
7: else
8: $c = 0$;

**Remark 1.** In an anti-balanced ergodic digraph $G = (V, E, A)$ with the bipartition $S$ and $\bar{S}$ and a budget $k$. Let $W'$ (resp. $W''$) denote two initial sets, where $\min\{k, |S|\}$ (resp. $\min\{k, |\bar{S}|\}$) nodes, with highest stationary distribution $\pi(i)$’s in $S$ (resp. $\bar{S}$), are selected. Then, the optimal $W^*$ that maximizes the strength of oscillation is

$$W^* := \arg\max_{W \in \{W', W''\}} |\pi_S^T(e_W - \frac{1}{2}I)|. \quad (33)$$

**Proof.** From Theorem 1 when $t$ becomes sufficiently large, the vector $x$ oscillates at two vectors on odd and even steps, respectively. The strength of the oscillation is

$$\frac{|f_o(x_0) - f_e(x_0)|}{2} = \frac{|1^T x_o(x_0) - x_e(x_0)|}{2} = \frac{|1^T \hat{1}_S \hat{\pi}_S^T(x_0 - \frac{1}{2}1)|}{2} = ||S| - |\bar{S}|| \cdot |\pi_S^T(x_0 - \frac{1}{2}1)|.
$$

Let $W$ be the initial seed set, then the oscillation strength maximization is formulated as

$$\max_{|W| \leq k} ||S| - |\bar{S}|| \cdot |\pi_S^T(e_W - \frac{1}{2}1)| = ||S| - |\bar{S}|| \cdot \max_{|W| \leq k} \{\max_{|W| \leq k} \{\pi_S^T e_W\} - \frac{1}{2} \pi_S^T 1, \max_{|W| \leq k} \{-\pi_S^T e_W\} + \frac{1}{2} \pi_S^T 1\}, \quad (34)$$

which contains two sub-problems, i.e., $\max_{|W| \leq k} \{\pi_S^T e_W\}$ and $\max_{|W| \leq k} \{-\pi_S^T e_W\}$. The first maximization problem can be rewritten as

$$\max_{|W| \leq k} \{\pi_S^T e_W\} = \max_{|W| \leq k} \left(\sum_{i \in S} \pi(i) e_W(i) - \sum_{j \in \bar{S}} \pi(i) e_W(j)\right). \quad (35)$$

Thus, let $W'$ denote the optimal solution to the problem in Eq. (35), which is obtained by choosing $\min\{k, |S|\}$ seeds with highest $\pi(i)$’s from $S$. Similarly, choosing $\min\{k, |\bar{S}|\}$ nodes with the highest $\pi(i)$’s from $\bar{S}$ yields the optimal solution, denoted by $W''$, to the second maximization problem $\max_{|W| \leq k} \{-\pi_S^T e_W\}$. The optimal $W$ to the problem in eq. (34) that maximizes the oscillation strength is the one in $\{W', W''\}$, with higher $|\pi_S^T(e_W - \frac{1}{2}1)|$, which completes the proof of eq. (33).

\[\square\]

4.3.2 Case of weakly connected signed digraphs

We first consider a weakly connected signed $G$ which has a single ergodic sink component $G_Z$ with only incoming edges from the remaining nodes $X = V \setminus Z$.

**Lemma 6.** Consider a weakly connected digraph $G = (V, E, A)$ with a single ergodic sink component $G_Z$. If $G_Z$ is balanced, with partition $S_Z$ and $\bar{S}_Z$, the long term influence contribution vector $c^T = [c_X^T, c_Z^T]$, where $c_X = 0_X$ and $c_Z = (1\bar{x}_u + |S_Z| - |\bar{S}_Z|)\pi_{Z,S_Z}$. If $G$ is anti-balanced or strictly unbalanced, $c = 0$. 

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Lemma 6 indicates that influence contribution of the balanced ergodic sink component is more complicated than that of the balanced ergodic digraph. This is because the sink component affects the colors of the non-sink component in a complicated way depending on how non-sink and sink components are connected. Therefore, the optimal seed selection depends on the calculation of the influence contributions of each sink node, and is not as intuitive as that for the ergodic digraph case.

Theorem 3 shows that in a weakly connected signed digraph $G$, with single anti-balanced sink component $G_Z$, the long term influence $f(x_0)$ oscillates on odd and even steps, and the average is $|V|/2$, which is invariant to the initial seed selection. Similar to Remark 1, we can maximize the oscillation strength by properly selecting initial seeds, i.e.,

$$W^* = \arg\max_{|W| \leq k} \frac{|f_e(ew) - f_o(ew)|}{2}$$

$$= \arg\max_{|W| \leq k} \left| (1^T_X u_b + 1^T_Z \hat{\pi}_{Z,S_Z}) P_\pi \left( (1^T_X u_b + 1^T_Z \hat{\pi}_{Z,S_Z}) (ew_Z - \frac{1}{2} 1_Z) \right) \right|$$

$$= \left| 1^T_X u_b + |S_Z| - |\hat{\pi}_{Z,S_Z}| \cdot \arg\max_{|W| \leq k} \left| \pi_Z S_1 \right| \right|$$

where the maximization objective is independent from $x_0$, thus oscillation strength maximization problem objective in Eq. (36) for $G$ is identical to that in Remark 1. Hence, Remark 1 also applies here.

Using Eq. (15), Lemma 5 and Lemma 6 can be readily extended to the case with more than one ergodic sink components and disconnected digraphs. Algorithm 3 below summarizes how to compute the node influence contributions of weakly connected signed digraphs. Note that by our assumption, we consider all sink components to be ergodic.

Algorithm 3 $c =$ weakly connected signed graph $G$

1. **INPUT:** Signed transition matrix $P$.
2. **OUTPUT:** Influence contribution vector $c$.
3. Detect the structure of the weakly connected signed digraph $G$, and find its $m \geq 1$ signed ergodic sink components $G_{Z_1}, \ldots, G_{Z_m}$.
4. for $i = 1 : m$
5. if $G_{Z_i}$ is balanced with partition $S_{Z_i}, \bar{S}_{Z_i}$ then
6. Compute stationary distribution $\bar{\pi}_{Z_i}$ of $P_{Z_i}$;
7. $u_{b_i} = (I_X - P_X)^{-1} P_{Y_i} 1_{Z_i,S_{Z_i}}$;
8. $c_{Z_i} = (1^T_X u_{b_i} + |S_{Z_i}| - |\bar{\pi}_{Z_i,S_{Z_i}}|) \hat{\pi}_{Z_i,S_{Z_i}}$;
9. $c = [0_X; c_{Z_1}; \ldots; c_{Z_m}]$

### 4.3.3 General case and SVIM-L algorithm

Given the above systematic analysis, we are now in a position to summarize and introduce our SVIM-L algorithm which solves the long-term voter model influence maximization problem for general aperiodic signed digraphs.

In general, a signed digraph consists $m \geq 1$ disconnected components, within each of which the node influence contribution follows Lemma 6. The long-term signed voter model influence maximization (SVIM-L) algorithm is constructed in Algorithm 3. The complexity analysis. We consider $G = (V, E, A)$ to be weakly connected, since disconnected graph case can be treated independently for each connected component for the time complexity. SVIM-L algorithm consists of two parts. The first part extracts the connectivity and balance structure of the graph, which can be done using depth-first
Algorithm 4 Long-term influence maximization SVIM-L

1: **INPUT:** Signed transition matrix \( P \), budget \( k \).
2: **OUTPUT:** White seed set \( W \).
3: Detect the structure of a general aperiodic signed digraph \( G \), and find the \( m \geq 1 \) disconnected components \( G_1, \cdots, G_m \).
4: **for** \( i = 1 : m \) **do**
5: \( c_{G_i} = \text{weakly}_{G_i}(G_i) \);
6: \( c = [c_{G_1}; \cdots; c_{G_m}] \);
7: \( W = \text{top} \min \{k, n^x(c)\} \) nodes with the highest \( c(i) \) values.

Algorithm 4 Long-term influence maximization SVIM-L

1: **INPUT:** Signed transition matrix \( P \), budget \( k \).
2: **OUTPUT:** White seed set \( W \).
3: Detect the structure of a general aperiodic signed digraph \( G \), and find the \( m \geq 1 \) disconnected components \( G_1, \cdots, G_m \).
4: **for** \( i = 1 : m \) **do**
5: \( c_{G_i} = \text{weakly}_{G_i}(G_i) \);
6: \( c = [c_{G_1}; \cdots; c_{G_m}] \);
7: \( W = \text{top} \min \{k, n^x(c)\} \) nodes with the highest \( c(i) \) values.

search with complexity \( O(|E|) \). The second part uses Algorithm 3 to compute influence contributions of balanced ergodic sink components. The dominant computations are on the stationary distribution \( \pi_Z \)'s and \( (I_X - P_X)^{-1} \), which can be done by solving a linear equation system \([33]\) and matrix inverse in \( O(|Z|^3) \) and \( O(n_X^3) \), respectively, where \( n_X = |X| \). Let \( b \) be the number of balanced sink components in \( G \), \( n_Z \) be the number of nodes in the largest balanced sink component. Thus SVIM-L can be done in \( O(bn_Z^2 + n_X^3) \) time. Alternatively, we can use iterative method for computing both \( \pi_Z \)'s and \( 1_X(I_X - P_X)^{-1} \), if the largest convergence time \( t_C \) of \( P_Z \)'s and \( P_X \) is small (note that the convergence time of ergodic digraphs could be exponentially large in general, as illustrated by an example in Appendix C). In this case, each iteration step involves vector-matrix multiplication and can be done in \( O(m_B) \) time, where \( m_B \) is the number of edges of the induced subgraph \( G_B \) consisting of all nodes in the balanced sink components and \( X \). Note that \( m_B \) and \( t_C \) are only related to subgraph \( G_B \), which could be significantly smaller than \( G \), and thus \( O(t_C m_B) \) could be much smaller than the time of naive iterations on the entire graph. Overall SVIM-L can be done in \( O(|E| + \min(bn_Z^2 + n_X^3, t_C m_B)) \) time.

5 Evaluation

In this section, we use both synthetic datasets and real social network datasets to evaluate the efficacy of our short-term and long-term seed selection schemes. For different scenarios, we compare our SVIM-L and SVIM-S algorithms with three heuristics, i.e., (1) selecting seed nodes with the highest weighted outgoing degrees (denoted by \( d^+ + d^- \) in the figures), (2) highest weighted outgoing positive degrees (denoted by \( d^+ \)), (3) highest differences between weighted outgoing positive and negative degrees (denoted by \( d^+ - d^- \)). Our evaluation results demonstrate that our seed selection scheme can increase up to 72% long-term influence, and 51% short-term influence over other heuristics.

5.1 Synthetic datasets

In this part, we generate synthetic datasets with different structures to validate our results.

Dataset generation model. In our tests, we generate six types of signed digraphs, including balanced ergodic digraphs, anti-balanced ergodic digraphs, strictly unbalanced ergodic digraphs, weakly connected signed digraphs, disconnected signed digraphs with ergodic components, and disconnected signed digraph with weakly connected components (WCCs). All edges have unit weights. The following are graph configuration details.

We first create an unsigned ergodic digraph \( G \) with 9500 nodes, which has two ergodic components \( G_A \) and \( G_B \), with \([3000, 6500] \) nodes and \([3000, 6500] \times 8 \) random directed edges, respectively. Moreover, there are \( 3000 \times 8 \) random directed edges across \( G_A \) and \( G_B \). Ergodicity is checked through a simple connectivity and aperiodicity check. Given \( G \), a balanced digraph is obtained by assigning all edges within \( G_A \) and \( G_B \) with positive signs, and those across them with negative signs. Then, an anti-balanced digraph is generated by negating all edge signs of the balanced ergodic digraph. To generate a strictly unbalanced digraph, we randomly assign edge signs to all edges in \( G \) and make sure that there does not exist a balanced or anti-balanced bipartition.

Moreover, we generated a disconnected signed digraph and a weakly connected signed digraph for our study. We first generate 5 ergodic unsigned digraphs, \( \tilde{G}_1, \cdots, \tilde{G}_5 \) with \([500, 200, 800, 300, 2700] \) nodes and \([500, 200, 800, 300, 2700] \times 8 \) edges, respectively. Then, we group \( G_{23} = (G_2, G_3) \) and \( G_{45} = (G_4, G_5) \) to form two ergodic balanced digraphs, and generate a strictly unbalanced ergodic digraph \( G_1 \) by randomly assigning signs.
to edges in $G_1$. Three disconnected components $G_1, G_{23}, G_{45}$ together form a disconnected signed digraph. To form a weakly connected signed digraph, we place in total 3000 random direct edges from $G_1$ to the balanced ergodic components $G_{23}$ and $G_{45}$, where the nodes in subgraph $G_1$ only have outgoing edges to $G_{23}$ and $G_{45}$. Moreover, we combine the above generated balanced ergodic digraph and the weakly connected signed digraph together forming a larger disconnected signed digraph, with the weakly connected signed digraph as a component. Fig. 1(a)-Fig. 1(f) present the evaluation results for one set of digraphs, where we observe that all digraphs we randomly generated exhibit consistent results. Our tests are conducted using Matlab on standard PC server.

![Graphs](image)

Figure 1: Long-term influence maximization on various signed digraphs

### 5.1.1 Long-term influence maximization

In the evaluations, we set the influence budget as $k = 500$. Fig. 1(a) shows that in the balanced ergodic digraph, SVIM-L algorithm achieves the highest long-term influence over other heuristics with up to 14% influence increase. Fig. 1(b) shows the clear oscillating behavior on the anti-balanced ergodic digraph, and the average influence is the same for all algorithms. The inset shows that our algorithm (denoted as “Max. Osc.”) indeed provides the largest oscillation, which confirms Remark 1. Fig. 1(c) shows the results in strictly unbalanced case, where the long-term influences of all algorithms converge to $4750 = |V|/2$, which matches Theorem 1. Fig. 1(d) and Fig. 1(e) show that SVIM-L algorithm increases up to 72% long-term influence over other heuristics in the weakly connected signed digraph and the disconnected signed digraph. Fig. 1(f) shows that in a more general signed digraph, which consists of a weakly connected signed component and a balanced ergodic component, SVIM-L algorithm outperforms all other heuristics with up to 17% more long term influence. In general, we see that for weakly connected and disconnected digraphs, SVIM-L has larger winning margins over other heuristics than the case of balanced ergodic digraphs (Fig. 1(d) vs. Fig. 1(a)). We attribute this to our accurate computation of influence contribution in the more involved weakly connected and disconnected digraph cases. Moreover, in all cases, the dynamics converge very fast, i.e., in only a few steps, which indicates that the convergence time of voter model on these random graphs are very small.
5.2 Real datasets

In this section, we use real datasets, namely, Epinions and Slashdot datasets, to validate our theoretical results and our SVIM algorithm.

Table 2: Statistics of Epinions and Slashdot datasets

| Statistics          | Epinions | Slashdot |
|---------------------|----------|----------|
| # of nodes          | 131580   | 77350    |
| # of edges          | 840799   | 516575   |
| # of positive edges | 717129   | 396378   |
| # of negative edges | 123670   | 120197   |

5.2.1 Epinions Dataset

Epinions.com [15] is a consumer review online social site, where users can write reviews to various items and vote for or against other users. The signed digraph is formed with positive or negative directed edge \((u, v)\) meaning that \(u\) trusts or distrusts \(v\). The statistics are shown in Table 2. We apply our short-term SVIM-S algorithm to the entire Epinions digraph as well as the largest strongly connected component (SCC), and compare it with three heuristics, i.e., \(d^+ + d^-\), \(d^+\), and \(d^+ - d^-\).

Our first batch of tests are on the entire digraph. We first look at the seed selection schemes for maximizing the instant influence at step \(t\). Fig. 2(a) shows the expected maximum instant influence at each step by different methods. Note that since the initial seeds selected by SVIM-S algorithm hinge on \(t\), the values on the curve of our selection scheme are associated with different optimal initial seed sets. On the other hand, the seed selection of other heuristics are independent to \(t\), thus the corresponding curves represent the same initial seed sets. We can see from Fig. 2(a)
that with different budgets, i.e., 500 and 6k seeds, our SVIM-S algorithm outperforms other heuristics. For example, SVIM-S algorithm has up to 47% more influence over other heuristics for $t = 2$. Fig. 2(b) shows how the influence evolves over steps, given that the goal is to maximize the influence at step 3, where SVIM-S algorithm achieves up to 27% more influence at step 3.

Next we compare the seed selection scheme for maximizing the average influence within the first $t$ steps. Fig. 2(c) shows the expected maximum average influence within the first $t$-th steps by different methods. Again, the values on the curve of SVIM-S algorithm are associated with different initial seed sets. Fig. 2(c) shows that with different budgets, i.e., 500 and 6k seeds, SVIM-S algorithm outperforms other heuristics with up to 26% more influence when $t = 6$. Fig. 2(d) shows how the expected average influence evolves over steps for $t = 3$, where SVIM-S algorithm achieves up to 36% more average influence over the first 3 steps. Moreover, in all these figures, we observe that our seed selection scheme results in the highest long-term influence over other heuristics.

In the second set of simulation results, we compare the influences obtained by different schemes on the largest strongly connected component (SCC) of Epinions dataset, which is ergodic and strictly unbalanced. In Fig. 3(a)-3(d), we show similar results as that in the entire Epinions dataset, where SVIM-S algorithm always outperforms other heuristics. The slight difference is that the performance of algorithms converges faster than on the entire graph, because SCC has better connectivity than the entire graph.

5.2.2 Slashdot Dataset

Slashdot.org [31] provides a discussion forum on various technology-related topics, where members can submit their stories, and comment on other members’ stories. Its Slashdot Zoo feature allows members to tag each other as friends or foes, which in turn forms a signed online social network. The network was collected on 6-th November 2008 [24] and the statistics are shown in Table 2.

We evaluate our SVIM-S algorithm on the entire slashdot dataset (Fig. 4(a)-4(d)) and its largest strongly connected component (Fig. 5(a), Fig. 5(d)), respectively, and our results show that our SVIM-S algorithm performs the best among
all methods tested, especially in the early steps.
Moreover, the convergence times for both real-world datasets are fast, in a few tens of steps, indicating good
connectivity and fast mixing property of real-world networks. In summary, our evaluation results on both synthetic
and real-world networks validate our theoretical results and demonstrate that our SVIM algorithms for both short term
and long term are indeed the best, and often with significant winning margins.

![Figure 4: Short-term influence maximization on the entire Slashdot dataset.](image)

6 Conclusion

In this paper, we propose and study voter model dynamics on signed digraphs, and apply it to solve the influence
maximization problem. We provide rigorous mathematical analysis to completely characterize the short-term and
long-term dynamics, and provide efficient algorithms to solve both short-term and long-term influence maximization
problems. Extensive simulation results on both synthetic and real-world graphs demonstrate the efficacy of our signed
color voter model influence maximization (SVIM) algorithms. We also identify a class of anti-balanced digraphs, which is
not covered in the social balance theory before, and exhibits oscillating steady state behavior.

There exist several open problems and future directions. One open problem is the convergence time of voter
model dynamics on signed digraphs. For balanced and anti-balanced ergodic digraphs, our results show that their
convergence times are the same as the corresponding unsigned digraphs. For strictly unbalanced ergodic digraphs and
more general weakly connected signed digraphs, the problem is quite open. A future direction is to study influence
diffusion in signed networks under other models, such as the voter model with a background color, the independent
cascade model, and the linear threshold model.
Figure 5: Short-term influence maximization on the largest strongly connected component of Slashdot dataset.

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A Properties of ergodic digraphs

Proposition 4. Let $G = (V, E, A)$ be an ergodic digraph. For any nodes $i, j \in V$, there exist two paths from $i$ to $j$ with even and odd length, respectively.

Proof. Suppose, for a contradiction, that all paths from $i$ to $j$ have even lengths. This implies that all cycles passing through $i$ must be even length, since otherwise we could follow node $i$’s odd-length cycle followed by the even length path from $i$ to $j$, making the entire path from $i$ to $j$ odd. Now we can consider any cycle $C_r$ in $G$, not necessarily passing $i$. We claim that $C_r$ must have even length. In fact, we can pick any node $u$ on $C_r$, and construct a path from $i$ to $j$ with the following segments: $R_1$ from $i$ to $u$, $C_r$, $R_2$ from $u$ back to $i$, and $R_3$ from $i$ to $j$. Since we know that $R_1 + R_2$ has even length and $R_3$ has even length, it must be the case that $C_r$ has even length by our assumption. However, this means that all cycles in $C$ has even lengths, contradicting to the aperiodicity of $G$.

The case of odd length paths can be proved in the same way. □

Proposition 5. Let $\bar{G} = (V, E, \bar{A})$ be an ergodic unsigned digraph, with transition probability matrix $\bar{P}$ and stationary distribution vector $\pi$. $\bar{P}^t - I\pi^T = (\bar{P} - I\pi^T)^t$ holds for any integer $t > 0$.

Proof. Using the facts that $\bar{P}1 = 1$ and $\pi^T \bar{P} = \pi^T$, it is easy to prove by induction that for any integer $t > 0$ $\bar{P}^t - I\pi^T = (\bar{P} - I\pi^T)^t$ holds. □

B Special matrix power series

Proposition 6. Let $X \in \mathbb{R}^{m \times m}$, $Y \in \mathbb{R}^{m \times n}$ and $Z \in \mathbb{R}^{n \times n}$. If $\lim_{t \to \infty} X^t = \lim_{t \to \infty} Z^t = 0$, the following equalities hold:

\[(i) \lim_{t \to \infty} \sum_{i=0}^{t-1} X^i = (I-X)^{-1}, \quad (37)\]

\[(ii) \lim_{t \to \infty} \sum_{i=0}^{t-1} X^iYZ^{-1-i} = 0, \quad (38)\]

Proof. (i) Let $\rho(X)$ be the spectral radius of matrix $X$, i.e., the largest absolute value of the eigenvalues of $X$. Notice that $\lim_{t \to \infty} X^t = 0$ if and only if $\rho(X) < 1$.

We first claim that, $I - X$ and $I - Z$ are invertible. Suppose $I - X$ is not invertible, there is a non-zero vector $p$ such that $(I - X)p = 0$. Therefore, $p$ is the eigenvector of $X$ with eigenvalue 1, which contradicts $\lim_{t \to \infty} X^t = 0$. Same argument can be applied to $I - Z$. Hence, the left hand side of Eq.\((37)\) equals to

\[
\lim_{t \to \infty} \sum_{i=0}^{t} X^i = \lim_{t \to \infty} (I-X)^{-1}(I-X^{t+1}) = (I-X)^{-1}.
\]
(ii) The max-norm of $X$ is given by $\|X\|_{\text{max}} = \max_{i,j \leq m} \{X_{ij}\}$. Let $X = QXJQ^{-1}$ be the standard Jordan form of $X$, where $QX$ is an invertible matrix. Denote $J = 11^T$ as the all-one matrix. Hence, we have

$$\|X^t\|_{\text{max}} = \|QXJ^tQ^{-1}\|_{\text{max}} \leq \|QX\|_{\text{max}}\|Q^{-1}\|_{\text{max}}\|J\|_{\text{max}}J^t$$

$$\leq \|QX\|_{\text{max}}\|Q^{-1}\|_{\text{max}}m^2\|J^t\|_{\text{max}}$$

$J^t$ is in form as

$$J^t = \begin{bmatrix}
\lambda_1^t & C_1^1\lambda_1^{t-2} & 0 & 0 \\
0 & \lambda_i^t & C_i^i\lambda_i^{t-1} & 0 \\
0 & 0 & \lambda_i^t & 0 \\
0 & 0 & 0 & \lambda_m^t \\
0 & 0 & 0 & \lambda_m^t
\end{bmatrix}, \quad (39)$$

where $C_i^t = \frac{i^t}{(i-1)!} \leq i^m$ and each non-zero entry in $J^t$ can be expressed as $C_i^t\lambda_i^{t-\ell}$, $1 \leq k \leq m_0$, $1 \leq \ell \leq \ell_0(k)$, with $m_0$ as the number of different eigenvalues of $X$ and $\ell_0(k)$ as the multiplicity of the $k$-th eigenvalue of $X$. Hence, the absolute value of each non-zero entry in $J^t$ is upper bounded as $|C_i^t\lambda_i^{t-\ell}| \leq i^m \rho(X)^t\rho(X)^{t-m}$, which implies that

$$\|X^t\|_{\text{max}} \leq \|QX\|_{\text{max}}\|Q^{-1}\|_{\text{max}}m^2i^m \rho(X)^{t-m}$$

Let $\rho = \max(\rho(X), \rho(Z))$, we have

$$\lim_{t \to \infty} \|X^t\|_{\text{max}} \leq \lim_{t \to \infty} t m n \|X^t\|_{\text{max}}\|Y\|_{\text{max}}\|Z\|_{\text{max}} \leq \lim_{t \to \infty} T_{\text{max}}(m^2i^m \rho^{t-m})(m^3n^3 \rho^{t-1-n}) \leq m^3n^3 T_{\text{max}} i^m n^3 \rho^{t-1-n} = 0$$

where $T_{\text{max}} = \|Y\|_{\text{max}}\|QX\|_{\text{max}}\|Q^{-1}\|_{\text{max}}\|QZ\|_{\text{max}}\|Q^{-1}\|_{\text{max}}$.

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C Illustration of exponential convergence time of $P^t$ on ergodic digraph.

![Diagram](image)

Figure 6: An example digraph with exponential convergence time. All edges are with unit weights.

Given an unsigned ergodic digraph $G = (V, E, A)$, with transition probability matrix $P$, it has fixed stationary distribution $\pi$, i.e., $\pi^T = \pi^T P$.

The convergence time (or mixing time) of a random walk Markov chain on $G$ is the time until the Markov chain is “close” to its stationary distribution $\pi$. To be precise, for an initial distribution $x_0$, let $x_T = x_0^T P^t$ be the distribution at step $t$. The variation distance mixing time is defined as the smallest $t$ such that for any subset $W \subseteq V$,

$$|(x_T^t - \pi^T)e_W| \leq \frac{1}{4},$$

where $e_W$ is the vector such that $e_W(i) = 1$ if $i \in W$, and $e_W(i) = 0$ if $i \in V \setminus W$.

The convergence time is said to be exponentially large if there exists $x_0$ such that the convergence time of the random walk starting from $x_0$ is $2^O(n)$, where $n = |V|$. Lemma 7 below illustrates that the convergence time of random walk on ergodic digraphs could be exponentially large.
Lemma 7. There exist ergodic digraphs, such that the convergence time of the random walks on these digraphs are exponentially large.

Proof. We prove this by construction. Fig. 6 shows an example digraph \( G \), with \( |V| = 2m \) nodes. On the left hand side, there are \( m \geq 3 \) nodes \( L_1, L_2, \ldots, L_m \) connected by \( m - 1 \) directed edges from \( L_1 \) to \( L_m \), and every node \( L_i \) with \( i > 1 \) has a directed connection to the leftmost node \( L_1 \). The right hand side nodes have symmetric connections as the left hand side. Moreover, node \( L_m \) and \( R_m \) also have one more connection to \( R_1 \) and \( L_1 \), respectively, which connect two components together. It is clear that the graph is strongly connected and aperiodic (there exist cycles of length 2 and 3), and thus ergodic.

Let \( x_t(L_i) \) denote the probability that the random walk is at node \( L_i \) at step \( t \), and \( x(L_i) \) be its stationary distribution. Similarly define \( x_t(R_i) \) and \( x(R_i) \) for node \( R_i \). The graph is symmetric, thus we have \( x(L_i) = x(R_i) \) for \( 1 \leq i \leq m \). Let \( x(L_1) = x(R_1) = \rho/4 \), we have \( x(L_i) = x(R_i) = \rho/2^i \) for \( i = 2, 3, \ldots, m \). Then, by solving \( \sum_{i=1}^{m} (x(L_i) + x(R_i)) = 1 \), we obtain \( \rho = \frac{2^{m-1}}{2^m - 1} \). It is easy to verify that indeed the obtained \( \rho \) is the stationary distribution of the random walks on the digraph.

Then, we consider the initial distribution as \( x_0 = [1, 0, 0, \ldots, 0] \), and the subset \( W = \{ R_1, \ldots, R_m \} \) including all \( m \) nodes on the right-hand side. Let \( x_t(W) = x_t^W \cdot e_W \) denote the total probability that the random walk is in some node in \( W \) at step \( t \). The only edge from the left half to the right half is the edge from \( L_m \) to \( R_1 \). Thus all additions to \( x_{t+1}(W) \) from \( x_t(W) \) comes from this edge, namely \( x_{t+1}(W) - x_t(W) \leq x_t(L_m)/2 \). We now bound \( x_t(L_m) \). For \( t \leq m - 1 \), we know that \( x_t(L_m) = 0 \). For \( t \geq m \), we have

\[
x_t(L_m) = x_{t-1}(L_{m-1})/2 = x_{t-2}(L_{m-2})/2^2 = \cdots = x_{t-m+2}(L_2)/2^{m-2} \leq 1/2^{m-2}.
\]

Hence, we have

\[
x_t(W) = \sum_{i=1}^{t} (x_i(W) - x_{i-1}(W)) \leq t \cdot x_t(L_m)/2 \leq t/2^{m-1}.
\]

Therefore, the smallest \( t \) that satisfies \( |(x_t^W - \pi^W)e_W| = |x_t(W) - 1/2| \leq 1/4 \) is such that \( x_t(W) \geq 1/4 \), which implies that \( t/2^{m-1} \geq 1/4 \) and \( t \geq 2^{m-3} \). This completes the proof. \( \Box \)