On Generalized $q$-Poly-Bernoulli Numbers and Polynomials

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Abstract. Many mathematicians in ([1], [2], [5], [14], [20]) introduced and investigated the generalized $q$-Bernoulli numbers and polynomials and the generalized $q$-Euler numbers and polynomials and the generalized $q$-Genocchi numbers and polynomials.

Mahmudov ([15], [16]) considered and investigated the $q$-Bernoulli polynomials $B_{n,q} (x, y)$ in $x, y$ of order $\alpha$ and the $q$-Euler polynomials $E_{n,q} (x, y)$ in $x, y$ of order $\alpha$. In this work, we define generalized $q$-poly-Bernoulli polynomials $B_{n,q}^{[k,\alpha]} (x, y)$ in $x, y$ of order $\alpha$. We give new relations between the generalized $q$-poly-Bernoulli polynomials $B_{n,q}^{[k,\alpha]} (x, y)$ in $x, y$ of order $\alpha$ and the generalized $q$-poly-Euler polynomials $E_{n,q}^{[k,\alpha]} (x, y)$ in $x, y$ of order $\alpha$ and the $q$-Stirling numbers of the second kind $S_{2,k} (n, k)$.

1. Introduction, Definitions and Notations

As usual, throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

In the usual notations, let $B_n (x)$ and $E_n (x)$ denote respectively, the classical Bernoulli polynomials and the classical Euler polynomials in $x$ defined by the generating functions, respectively

$$\sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}, \ |t| < 2\pi. \ (1)$$

and

$$\sum_{n=0}^{\infty} E_n (x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt}, \ |t| < \pi. \ (2)$$

Also, let

$$B_n (0) := B_n \text{ and } E_n (0) := E_n$$

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where $B_n$ and $E_n$ are respectively, the Bernoulli numbers and the Euler numbers. If $k \in \mathbb{Z}$ and $k \geq 1$, then $k$-th polylogarithm is defined by ([3], [7], [13]) as

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$  

(3)

This function is convergent for $|z| < 1$, when $k = 1$,

$$Li_1(z) = -\log(1-z)$$  

(4)

[15]. The $q$-numbers and $q$-factorial are defined by

$$[n]_q = \frac{1 - q^n}{1 - q} , q \neq 1$$

$$[n]_q! = [n]_q [n-1]_q [n-2]_q \ldots [1]_q , n \in \mathbb{N}, q \in \mathbb{Z}$$

respectively, where $[0]_q! = 1$. The $q$-binomial coefficients are defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} , 0 \leq k \leq n$$

The $q$-power basis is defined by

$$(x+y)^n_q = \left\{ \begin{array}{ll} (x+y)(x+qy)\ldots(x+q^{n-1}y), & n = 1,2,\ldots \\ 1, & n = 0 \end{array} \right.$$  

From above equality, we get

$$(x+y)_q^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q x^k y^{n-k}.$$  

The $q$-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, 0 < |q| < 1, |z| < \frac{1}{|1-q|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{(n)} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1+(1-q)q^k z \right), 0 < |q| < 1, z \in \mathbb{C}.$$  

From here, we easily see that $e_q(z) E_q(-z) = 1$. The above $q$-notation can be found in ([8], [13]). Mahmudov in ([15], [16]) defined the $q$-Bernoulli polynomials $B^{(\alpha)}_{n,q}(x,y)$ in $x, y$ of order $\alpha$ and the $q$-Euler polynomials $E^{(\alpha)}_{n,q}(x,y)$ in $x, y$ of order $\alpha$, respectively

$$\sum_{n=0}^{\infty} B^{(\alpha)}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1} \right)^\alpha e_q(tx) E_q(ty) , |t| < 2\pi$$  

(5)

and

$$\sum_{n=0}^{\infty} E^{(\alpha)}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)+1} \right)^\alpha e_q(tx) E_q(ty) , |t| < \pi$$  

(6)
where $q \in \mathbb{C}$, $\alpha \in \mathbb{N}_0$, $0 < |q| < 1$. It is obvious that
\[
\mathcal{B}^{(\alpha)}_{n,q} : = \mathcal{B}^{(\alpha)}_{n,q}(0,0), \quad \lim_{q \to 1} \mathcal{B}^{(\alpha)}_{n,q}(x, y) = B^{(\alpha)}_n(x + y), \quad \lim_{q \to 1} \mathcal{B}^{(\alpha)}_{n,q} = B^{(\alpha)}_n
\]
\[
\mathcal{E}^{(\alpha)}_{n,q} : = \mathcal{E}^{(\alpha)}_{n,q}(0,0), \quad \lim_{q \to 1} \mathcal{E}^{(\alpha)}_{n,q}(x, y) = E^{(\alpha)}_n(x + y), \quad \lim_{q \to 1} \mathcal{E}^{(\alpha)}_{n,q} = E^{(\alpha)}_n
\]
Carlitz defined in [6] the $q$-Stirling numbers of the second kind $S_{2,q}(n, k)$ as
\[
\sum_{m=0}^{\infty} S_{2,q}(m, k) \frac{t^n}{[m]_q^k} = \left(\frac{e_q(t) - 1}{[k]_q^k}\right)^k
\]
For $k = 1$, from (4). We get $B^{(\alpha)}_n(x) = B_n(x)$ and $E^{(\alpha)}_n(x) = E_n(x)$.

By this motivation, we define the generalized $q$-poly-Bernoulli polynomials $\mathcal{B}^{[\alpha]q}_{n,q}(x, y)$ in $x$, $y$ of order $\alpha$ and the generalized $q$-poly-Euler polynomials $\mathcal{E}^{[\alpha]q}_{n,q}(x, y)$ in $x$, $y$ of order $\alpha$ as the following generating functions, respectively
\[
\sum_{n=0}^{\infty} \mathcal{B}^{[\alpha]q}_{n,q}(x, y) \frac{t^n}{[n]_q^k} = \left(\frac{L_i (1 - e^{-t})}{t (e^t - 1)}\right)^k e_q(xt) E_q(ty)
\]
and
\[
\sum_{n=0}^{\infty} \mathcal{E}^{[\alpha]q}_{n,q}(x, y) \frac{t^n}{[n]_q^k} = \left(\frac{2L_i (1 - e^{-t})}{t (e^t - 1)}\right)^k e_q(xt) E_q(ty)
\]
For $k = 1$, from $L_i (x) = -\log (1 - x)$. The equations (10) and (11) reduces to (5) and (6) respectively.

Srivastava in [20] and Srivastava et al. in [21] gave basic knowledge the Bernoulli polynomials, the Euler polynomials and $q$-Bernoulli polynomials and $q$-Euler polynomials.

Kim et al. in [11] introduced the poly-Bernoulli polynomials, Luo in [14] and Sadjang in [17] and Simsek in [18] considered and gave some relations the $q$-Bernoulli polynomials and the Stirling numbers of the second kind.

Carlitz in [5] gave some properties of $q$-Bernoulli polynomials. Mahmudov in ([15], [16]) considered and investigated some recurrences relations between $q$-Bernoulli polynomials $\mathcal{B}^{(\alpha)}_{n,q}(x, y)$ in $x$, $y$ of order $\alpha$ and $q$-Euler polynomials $\mathcal{E}^{(\alpha)}_{n,q}(x, y)$ in $x$, $y$ of order $\alpha$.

Firstly, Kaneko in [9] defined poly-Bernoulli numbers. Bayat et al. in [3] and Hamahata in [7] gave some identities for the poly-Bernoulli polynomials and the poly-Euler polynomials. Kim et al. in [10] and Kurt in [12] gave some relations and identities for the $q$-Bernoulli polynomials $\mathcal{B}^{(\alpha)}_{n,q}(x, y)$ in $x$, $y$ of order $\alpha$. 
2. Explicit Relations for The Generalized q-Poly-Bernoulli Polynomials $B_{n,q}^{[k,\alpha]}(x, y)$ in $x, y$ of Order $\alpha$

In this section, we give some identities and relations for the generalized q-poly-Bernoulli polynomials $B_{n,q}^{[k,\alpha]}(x, y)$ in $x, y$ of order $\alpha$. Also, we prove the closed theorem between the generalized q-poly-Bernoulli polynomials $B_{n,q}^{[k,\alpha]}(x, y)$ and the q-Stirling numbers of the second kind $S_{2,q}(n, k)$.

Theorem 2.1. The generalized q-poly-Bernoulli polynomials $B_{n,q}^{[k,\alpha]}(x, y)$ in $x, y$ of order $\alpha$ satisfy the following relations.

$$B_{n,q}^{[k,\alpha]}(x, y) = \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} (x + y)^{l} B_{n-l,q}^{[k,\alpha]}.$$  

$$B_{n,q}^{[k,\alpha]}(x, y) = \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} B_{n-l,q}^{[k,\alpha]}(x, 0) q^{(l)} y^{l}.$$  

$$B_{n,q}^{[k,\alpha]}(x, y) = \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} B_{n-l,q}^{[k,\alpha]}(0, y) x^{l}.$$  

Proof. We can see easily from (10). \(\Box\)

Theorem 2.2. There is the following relation between the q-poly-Bernoulli polynomials $B_{n,q}^{[k,\alpha]}(x, y)$ and the q-Stirling numbers of the second kind $S_{2,q}(n, k)$

$$\sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} B_{n-l,q}^{[k,\alpha]}(x + y) - B_{n,q}^{[k,\alpha]}(x + y) = \sum_{m=0}^{\infty} (-1)^{m+n-l} \frac{(m + 1)!}{(m + 1)^{k}} S_{2,q}(n - l, m + 1).$$  \(12\)

Proof. By (7) and (10), for $\alpha = 1$ and $q \to 1^{-}$, we have (12). \(\Box\)

Theorem 2.3. The following relation holds true

$$n B_{n-1}^{[1]}(x + y) = \sum_{m=0}^{\infty} \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} B_{l}(x + y) \frac{(-1)^{m+n-l}}{(m + 1)^{k}} (m + 1)! S_{2,q}(n - l, m + 1).$$  \(13\)

Proof. By (10) for $\alpha = 1$, by using (7), we write as

$$\sum_{n=0}^{\infty} \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} B_{n,q}^{[l,1]}(x, y) \frac{p^{n}}{[n]_{q}!} = \frac{1}{[l]_{q}!} \left[ \frac{1}{e_{q}(t)} - 1 \right]^{-l} L_{k}(1 - e^{-t})$$

$$= \sum_{m=0}^{\infty} \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q} B_{l,q}(x, y) \frac{p^{l}}{[l]_{q}!} \frac{(-1)^{m+n-l}}{(m + 1)^{k}} (m + 1)! S_{2,q}(p, m + 1) \frac{(-1)^{p+m}}{p!}.$$  

We take to limit $q \to 1^{-}$ both sides and by the Cauchy product, we have (13). \(\Box\)

Theorem 2.4 (Closed Formula). The following relation holds true

$$B_{n,q}^{[l,1]}(x + y) = \sum_{j=0}^{\min(n,k)} (j!)^{2} S_{2}(n, j, x + y) S_{2}(k, j, 1).$$  \(14\)
Proof. By replacing $k$ by $-k$ in (10), for $\alpha = 1$, we get

$$\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} = \sum_{m=0}^{\infty} (m+1)^k \left(1 - e^{-t}\right)^{m+1} \left(e_q(t) E_q(ty) e_q(t) - 1\right)$$

we take to limit $q \to 1^-$ in both sides, we have

$$\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} = \sum_{m=0}^{\infty} (m+1)^k \left(1 - e^{-t}\right)^{m+1} \frac{e^{xt+ty}}{e^t - 1},$$

From here, we write as

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_{n,q}^{(-k, 1)}(x, y) \frac{t^n u^k}{n! k!} \sum_{k=0}^{\infty} \frac{1}{e^t - 1} \sum_{m=0}^{\infty} (m+1)^k \left(1 - e^{-t}\right)^{m+1} e^{xt+ty} u^k k!$$

$$= \frac{1}{e^t - 1} \sum_{m=0}^{\infty} \left(1 - e^{-t}\right)^{m+1} e^{xt+ty} e^{(m+1)u}$$

$$= \frac{e^{xt+ty} (1 - e^{-t}) e^u}{e^t - 1} \sum_{m=0}^{\infty} \left(1 - e^{-t}\right) e^u \right)^m$$

(15)

Carlitz et al [6] defined the weighted Stirling numbers of the second is defined kind as

$$\frac{e^x}{k!} (e^t - 1)^k = \sum_{n=0}^{\infty} S_2 (n, k, x) \frac{t^n}{n!} \frac{u^k}{k!}$$

(16)

[18]. By using (15) and (16), we get

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_{n,q}^{(-k, 1)}(x, y) \frac{t^n u^k}{n! k!} = \frac{e^{(x+y)} e^u}{1 - (e^t - 1) (e^u - 1)}$$

$$= \sum_{j=0}^{\infty} e^{(x+y)t} (e^t - 1)^j e^u (e^u - 1)^j$$

$$= \sum_{j=0}^{\infty} \frac{j! e^{(x+y)t} (e^t - 1)^j}{j!} \frac{e^u (e^u - 1)^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{j! e^{(x+y)t} (e^t - 1)^j}{j!} \frac{e^u (e^u - 1)^j}{j!} \sum_{n=0}^{\infty} S_2 (n, j, x + y) \frac{t^n}{n!} \frac{u^k}{k!}$$

By using Cauchy product and comparing the coefficients of both sides, we have (14). □

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