Abstract. In this paper, we introduce polytopal $k$-wedge construction and blowdown of a simple polytope and inspect the effect on the retraction sequence of a simple polytope due to $k$-wedge construction and blowdown. In relation to this construction, we introduce the $k$-wedge and blowdown of a quasitoric orbifold. We compare the torsions in the integral cohomologies of $k$-wedges and blowdowns of a quasitoric orbifold with the original one. These two constructions provide infinitely many integrally equivariantly formal quasitoric orbifolds from a given one.

1. Introduction

Simplicial wedge operation is a classical technique in the category of simplicial complexes, see [15] and [23]. The authors of [2] used this idea in the area of toric topology for the first time. Later, several applications have been exploited in [3] and [12]. One of the main objectives of these works is to construct infinite families of toric manifolds from a given one which may simplify the presentation of their integral cohomology rings.

Let $K$ be a simplicial complex with the vertex set \{$v_1, \ldots, v_m$\}. The simplicial wedge of $K$ on $v_i$ is a simplicial complex with the vertices \{$v_{i0}, v_{i1}, \ldots, v_{im} \}$ defined by

$K(v_i) := \{v_{i0}, v_{i1}\} \ast \text{link}_K\{v_i\} \cup \{\{v_{i0}\}, \{v_{i1}\}\}$

where $\ast$ implies the join of simplicial complexes. The dual notion of this construction is called polytopal wedge construction. Precisely, a simple polytope $P$ is called a polytopal wedge of $Q$ if $K_P$ is a simplicial wedge of $K_Q$, where $K_P, K_Q$ are the dual simplicial complexes of $P, Q$ respectively. We note that $K_P$ is a simplicial complex on the set of codimension-1 faces of $P$.

The readers are referred to [2] and [9] for details on these concepts.

On the other hand, Davis and Januszkiewicz introduced toric manifolds and toric orbifolds in the pioneering paper [14]. However, they studied several topological properties of toric manifolds.

Later, toric orbifolds were explicitly defined in [22] with the name ‘quasitoric orbifolds’ to avoid similar terminology in algebraic geometry. Weighted projective spaces and simplicial projective toric varieties are some well-known examples of toric orbifolds. Here, the authors prefer to use the term quasitoric orbifold instead of toric orbifold. A quasitoric orbifold is an even-dimensional effective orbifold equipped with a ‘locally standard’ half-dimensional torus action such that the orbit space has the structure of a simple polytope.

The seminal work [20] computed the integral cohomology ring of weighted projective spaces. This inspired us to study the integral cohomology of quasitoric orbifolds as this may help in classifying quasitoric orbifolds up to diffeomorphisms. Note that a CW-complex structure can be constructed on an effective orbifold following [17]. Several works discussed the de-Rham cohomology, the singular cohomology, the Chen-Ruan cohomology ring, orbifold $K$-theory of orbifolds with rational, real or complex coefficients; see [1] Chapter 2 and 3, [18], [13], [10]. However, the computation of these cohomologies with integral coefficients is considerably difficult.

A quasitoric orbifold is called even if its integral cohomology ring is torsion-free and concentrated in even degrees. The paper [5] initiated the investigation of which (quasi)toric orbifold is even. Subsequently, in [6], they constructed infinitely many even (quasi)toric orbifolds using
the polytopal wedge construction. In this paper, we study several properties of blowdown and $k$-wedge of polytopes and quasitoric orbifolds which generalize the wedge construction of polytopes and $J$-construction of toric orbifolds respectively. Moreover, we extend the discussion on the evenness of quasitoric orbifolds.

The paper is organized as follows. In Section 2, we revisit the concept of the retraction sequence (Definition 2.4) of a simple polytope from [5]. We recall that a retraction sequence of a simple polytope $Q$ induces retraction sequences of $Q \times \Delta$ for any simplex $\Delta$, see Proposition 2.4. Then, following [22], we briefly go through the basic construction of a quasitoric orbifold $X(Q, \lambda)$ from a combinatorial data called an $R$-characteristic pair $(Q, \lambda)$ where

$$\lambda: \mathcal{F}(Q) \to \mathbb{Z}^{\dim Q}$$

is called an $R$-characteristic function on the simple polytope $Q$, see Definition 2.5. We discuss some invariant subspaces of $X(Q, \lambda)$ corresponding to the faces of $Q$ and the orbifold property of these subspaces. We also recollect the computation of the orbifold singularities at the fixed points of $X(Q, \lambda)$ and its invariant subspaces, see (2.7) and (2.8).

In Section 3, we define polytopal $k$-wedge $Q_F(k)$ of a simple polytope $Q$ at a facet $F$ and prove that $Q_F(k)$ is a simple polytope of dimension $(\dim Q + k)$, see Lemma 3.1. We observe that this construction can be carried out at a codimension-$\ell$ face with $2 \leq \ell \leq \dim Q$. However, this may not produce a simple polytope, see Remark 3.3.

In Section 4, we introduce the concept of blowdown of a convex polytope. We show that the blowdown of a simple polytope may not be a simple polytope in general, see Figure 7. We also provide the necessary and sufficient conditions when a blowdown preserves the simplicity of a polytope, see Lemma 4.5.

The main result of this section is that a retraction sequence of $Q$ induces a retraction sequence on its blowdown $Q'$ if $Q'$ is a simple polytope, see Theorem 4.9. Moreover, we construct a retraction sequence of $Q_F(k)$ from a given retraction sequence of $Q$, see Corollary 4.10 and 4.11.

In Section 5, first, we define the blowdown of a quasitoric orbifold, see Definition 5.1. If $(Q, \lambda)$ and $(Q', \lambda')$ are $R$-characteristic pairs such that $Q'$ is a blowdown of $Q$ then we analyze when $(Q', \lambda')$ is a restriction of $(Q, \lambda)$ in the sense (5.1). Then, in Theorem 5.9 we show that if $(Q, \lambda)$ satisfies some combinatorial conditions along with the hypotheses $(A_2)$ and $(A_3)$ then $(Q', \lambda')$ possesses the similar combinatorial conditions. We show that, in general, we may not be able to remove the hypotheses $(A_2)$ and $(A_3)$ from Theorem 5.9; see Example 5.10 and Example 5.11 respectively. We conclude that the integral homology of certain blowdown of a quasitoric orbifold has no $p$-torsion, see Theorem 5.13. If a quasitoric orbifold is obtained by a sequence of blowdown on a quasitoric manifold and each step satisfies the hypotheses of Theorem 5.13 for any prime $p$, then we conclude that the integral cohomology of a blowdown of a quasitoric orbifold is concentrated in even degrees and has no torsion, see Corellary 5.14.

In Section 6, we define $k$-wedge construction on quasitoric orbifolds. We remark that, in general, $k$-wedge on quasitoric orbifold may not be possible to obtain from iterated polytopal wedge construction of $[6]$. Also, the blowdown of a simple polytope may not be possible to construct from the polytopal $k$-wedge constructions of a simple polytope, see Example 6.6. Consequently, we construct infinitely many integrally equivariantly formal quasitoric orbifolds from a given one in more generality.

2. Preliminaries

2.1. Retraction sequences of polytopes. In this subsection, we recall a few basics of retraction sequences on polytopes. The convex hull of a finite set of points in $\mathbb{R}^n$ for some $n$ is called a convex polytope. The vertices, edges, and facets of a convex polytope are faces of dimension 0, 1, and $(n-1)$, respectively. If at each vertex of an $n$-dimensional convex polytope $Q$ exactly $n$ facets intersect, then $Q$ is called a simple polytope. Some well-known examples of simple
polytopes are cubes, simplices and prisms. We denote the set of vertices of a convex polytope
$Q$ by $V(Q)$ and the set of facets of $Q$ by $\mathcal{F}(Q)$ throughout this paper.

**Definition 2.1.** [24 Definition 5.1] A polytopal complex $\mathcal{C}$ is a finite collection of convex polytopes in $\mathbb{R}^n$ such that the following holds:

1. If $E$ is a face of $F$ and $F \in \mathcal{C}$ then $E \in \mathcal{C}$.
2. If $E, F \in \mathcal{C}$ and $E \cap F \neq \emptyset$ then $E \cap F$ is a face of both $E$ and $F$.

The dimension of a polytopal complex is defined to be the maximum dimension of the convex polytope in it. The union of the convex polytopes in $\mathcal{C}$ is called its geometric realization.

Let $Q$ be an $n$-dimensional simple polytope and $\mathcal{L}(Q) := \{ F : F \text{ is a face of } Q \}$. Then $\mathcal{L}(Q)$ is an $n$-dimensional polytopal complex. If $P$ is a subset of $Q$ such that $P$ is the union of some faces of $Q$, then $\mathcal{L}(P)$ is also a polytopal complex. For simplicity in this situation, we call $P$ a subcomplex of $Q$.

**Definition 2.2.** Let $P$ be a subcomplex of $Q$ and $v \in V(P) \subset V(Q)$. The vertex $v$ is called a free vertex of $P$ if $v$ has a neighborhood $U_v$ in $P$ such that $U_v$ is homeomorphic to $\mathbb{R}^d_{>0}$ as a manifold with corners for some $0 \leq d \leq \dim(P)$. The set $U_v$ is called a local neighborhood of the free vertex $v$ in $P$.

**Definition 2.3.** Let $Q$ be a polytope with $m$ vertices and there exists a sequence $\{(B_\ell, E_\ell, b_\ell)\}_{\ell=1}^m$ of triplets such that

1. $B_1 = Q = E_1$ and $b_1$ is a free vertex of $Q$.
2. $B_\ell \subset B_{\ell-1}$ such that $B_\ell = \bigcup\{ F \mid F \text{ is a face in } B_{\ell-1} \text{ and } b_{\ell-1} \notin V(F) \}$.
3. $b_\ell$ is a free vertex in $B_\ell$ and $E_\ell$ is the maximal dimensional face of $B_\ell$ containing the vertex $b_\ell$.
4. $B_m = E_m = b_m$.

Then the sequence $\{(B_\ell, E_\ell, b_\ell)\}_{\ell=1}^m$ is called a retraction sequence of $Q$ starting with the vertex $b_1$ and ending at $b_m$.

Remark that the conditions (2) and (3) of Definition 2.3 imply $B_\ell = B_{\ell+1} \cup E_\ell$ for $\ell = 1, \ldots, m - 1$. Note that a retraction sequence of $Q$ induces an ordering on $V(Q)$. Figure 1 gives an example of a retraction sequence of a prism. In [3], the authors proved that a simple polytope admits at least one retraction sequence. We remark that all convex polytopes may not possess retraction sequences in general. But some convex polytopes admit retraction sequences though they are not simple. For example, there is no retraction sequence of the octahedron; however, we can construct a retraction sequence of a pyramid on a pentagonal base.

**Proposition 2.4.** [7 Proposition 2.5] Let $Q$ be a simple polytope and $\Delta$ be a simplex. Then $Q \times \Delta$ has a retraction sequence induced from the retraction sequences of $Q$ and $\Delta$.

### 2.2. Some basics of quasitoric orbifolds.

A quasitoric orbifold is an even-dimensional effective orbifold with nice enough half-dimensional torus action. We can realize quasitoric orbifolds as a topological analog of simplicial projective toric varieties. In this subsection, we briefly recall the constructive definition of a quasitoric orbifold, some notion of invariant suborbifolds, and the singularities at some special points following [22]. The authors of [11] and [21] gave a nice exposure to (effective) differentiable orbifolds. Let $Q$ be an $n$-dimensional simple polytope with $V(Q) := \{ b_1, \ldots, b_m \}$ and $\mathcal{F}(Q) := \{ F_1, \ldots, F_r \}$.
Definition 2.5. Let $\lambda: F(Q) \to \mathbb{Z}^n$ be a map such that for $i \in \{1, \ldots, r\}$ each $\lambda(F_i)$ is primitive and
\begin{equation}
\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\} \text{ is linearly independent if } \bigcap_{j=1}^k F_{i_j} \neq \emptyset.
\end{equation}

Then $\lambda$ is called an $R$-characteristic function on $Q$. The vector $\lambda(F_i)$ is denoted by $\lambda_i$ and called the $R$-characteristic vector assigned to the facet $F_i$. The pair $(Q, \lambda)$ is called an $R$-characteristic pair.

Remark 2.6. Let $F$ be a $d$-dimensional face of an $n$-dimensional simple $Q$ with $d < n$. Then
\begin{equation}
F = \bigcap_{j=1}^{n-d} F_{i_j}
\end{equation}
for some unique facets $F_{i_1}, \ldots, F_{i_{n-d}}$ of $Q$. If the set of vectors $\{\lambda_{i_j} | j = 1, \ldots, (n-d)\}$ spans an $(n-d)$-dimensional unimodular subspace of $\mathbb{Z}^n$, then $\lambda$ is called a characteristic function and the pair $(Q, \lambda)$ is called a characteristic pair, see (*) in page 423 of [14]. Note that [24, Definition 3.5] is a generalization of Definition 2.5.

Example 2.7. We give an example of an $R$-characteristic function on a square in Figure 2(a) and on a prism in Figure 2(b).

We recall the basic construction of a quasitoric orbifold from an $R$-characteristic pair $(Q, \lambda)$ following [22]. Let $F$ be a face of dimension $d (0 \leq d < n)$ in $Q$. Then $F = \bigcap_{j=1}^{n-d} F_{i_j}$ for some unique facets $F_{i_1}, \ldots, F_{i_{n-d}}$ of $Q$. Each $\lambda_i \in \mathbb{Z}^n$ determines a line in $\mathbb{R}^n = (\mathbb{Z}^n \otimes \mathbb{Z}) \mathbb{R}$, whose image under the exponential map
\[ \exp: \mathbb{R}^n \to T^n = (\mathbb{Z}^n \otimes \mathbb{Z}) \mathbb{R} / \mathbb{Z}^n \]
is a circle subgroup, denoted by $T_i$. Let $T_F := \langle T_{i_1}, \ldots, T_{i_{n-d}} \rangle$. Then $T_F$ is an $(n-d)$-dimensional subtorus of $T^n$. We define $T_Q = 1 \in T^n$. Consider the equivalence relation $\sim$ on $T^n \times Q$ is defined by
\begin{equation}
(t, x) \sim (s, y) \text{ if and only if } x = y \in \hat{F} \text{ and } t^{-1}s \in T_F,
\end{equation}
where $x$ is in the relative interior of the unique face $F$ of $Q$. The quotient space $X(Q, \lambda) := (T^n \times Q) / \sim$ has an orbifold structure with a natural $T^n$ action. The orbit map
\begin{equation}
\pi: X(Q, \lambda) \to Q
\end{equation}
is defined by $[t, x]_\sim \mapsto x$, where $[t, x]_\sim$ is the equivalence class of $(t, x)$. In [22], the authors discussed the orbifold structure of the space $X(Q, \lambda)$ explicitly. They also show that the axiomatic
The number at the point singularity at the point \( \pi \)

When \( F \) is simple and \( F = \bigcap_{j=1}^{n-d} F_{i_j} \), for some unique facets \( F_{i_1}, \ldots, F_{n-d} \) of \( Q \). Let 

\[
N(F) := \langle \lambda_{i_1}, \ldots, \lambda_{i_{n-d}} \rangle
\]

where \( \lambda_{i_1}, \ldots, \lambda_{i_{n-d}} \) are the \( R \)-characteristic vectors assigned to these facets respectively. Then \( N(F) \) is an \( (n-d) \)-dimensional submodule of \( \mathbb{Z}^n \).

Consider the projection map

\[
\rho_F : \mathbb{Z}^n \to \mathbb{Z}^n/((N(F) \otimes \mathbb{R}) \cap \mathbb{Z}^n) \cong \mathbb{Z}^d.
\]

The facets of \( F \) are the following 

\( F(F) := \{ F \cap F_j \mid F_j \in F(Q) \text{ and } j \neq i_1, \ldots, i_{n-d} \text{ and } F \cap F_j \neq \emptyset \}. \)

Then, one can define a map

\[
\lambda_F : F(F) \to \mathbb{Z}^d
\]

by \( \lambda_F(F \cap F_j) := \text{prim}((\rho_F \circ \lambda)(F_j)) \), where \( \text{prim}((\rho_F \circ \lambda)(F_j)) \) denotes the primitive vector of \( (\rho_F \circ \lambda)(F_j) \). Note that, \( \lambda_F \) is an \( R \)-characteristic function on \( F \). Consequently, it gives a quasitoric orbifold \( X(F, \lambda_F) \) which is an invariant suborbifold of \( X(Q, \lambda) \), see [22, Section 2.3].

Now we recall how the order of singularities associated to each vertex of the face \( F \) is defined. Let \( v \in V(F) \subset V(Q) \) and

\[
\pi_F : X(F, \lambda_F) \to F
\]

be the orbit map. Then \( v = (F \cap F_{j_1}) \cap \cdots \cap (F \cap F_{j_d}) \) for some unique facets \( F_{j_1}, \ldots, F_{j_d} \) of \( Q \). The orbifold singularity at the point \( \pi_F^{-1}(v) \) in \( X(F, \lambda_F) \) is defined by

\[
G_F(v) := \mathbb{Z}^d/\langle \lambda_F(F \cap F_{j_1}), \ldots, \lambda_F(F \cap F_{j_d}) \rangle.
\]

When \( F = Q \), then \( v = F_{i_1} \cap \cdots \cap F_{i_n} \) for some unique facets \( F_{i_j} \) of \( Q \). Then the orbifold singularity at the point \( \pi^{-1}(v) \) in \( X(Q, \lambda) \) is given by

\[
G_Q(v) := \mathbb{Z}^n/\langle \lambda(F_{i_1}), \ldots, \lambda(F_{i_n}) \rangle.
\]

We call the matrices

\[
A_Q^F := (\lambda(F_{i_1})^t \quad \ldots \quad \lambda(F_{i_n})^t)^t, \quad \text{and}
A_C^F := (\lambda_F(F \cap F_{j_1})^t \quad \cdots \quad \lambda_F(F \cap F_{j_d})^t)^t
\]

associated to the vertex \( v \) in \( Q \) and \( F \) respectively. Note the following:

\[
|G_F(v)| = |\det A_C^F| = |\det[\lambda_F(F \cap F_{j_1})^t \cdots \lambda_F(F \cap F_{j_d})^t]|, \quad \text{and}
|G_Q(v)| = |\det A_Q^F| = |\det[\lambda(F_{i_1})^t \cdots \lambda(F_{i_n})^t]|.
\]

The number \( |G_F(v)| \) encodes the order of orbifold singularity of the quasitoric orbifold \( X(F, \lambda_F) \) at the point \( \pi_F^{-1}(v) \).
In this section, we generalize the polytopal wedge construction in a broader sense and call it $k$-wedge construction on a simple polytope $Q$. We further show that this construction produces another simple polytope of dimension $(\dim Q + k)$.

Let $Q$ be an $n$-dimensional simple polytope in $\mathbb{R}^n$ and $F$ a facet of $Q$. We consider the polyhedron $Q \times \mathbb{R}_{\geq 0}^k \subset \mathbb{R}^{n+k}$ and identify $Q \times 0_k$ with $Q$ where $0_k$ is the corner $(0, \ldots, 0)$ in $\mathbb{R}_0^k \subset \mathbb{R}^k$. Let $H$ be a hyperplane in $\mathbb{R}^{n+k}$ such that it intersects the interior of $Q \times \mathbb{R}_{\geq 0}^k$ and divides it into two parts such that one open half space such that one open half space (say $H_{<0}$) of $H$ contains the vertices $V(Q) \setminus V(F)$ as well as $Q \cap H = F$. Let us denote the part containing $Q$ by $Q_F(k)$, that is

$$Q_F(k) := (Q \times \mathbb{R}_{\geq 0}^k) \cap H_{\leq 0}.$$  

When $k = 1$, the construction is called polytopal wedge construction in [11] and [12].

The hyperplane $H$ can be defined as follows. Choose $n$ many vertices $v_1, \ldots, v_n \in V(F)$ which are in ‘general positions’. Now we choose $k$-many points $v_{n+1}, \ldots, v_{n+k}$ from $v \times \mathbb{R}_{\geq 0}^k$ such that $v \in V(Q) \setminus V(F)$ and the line segment joining $v \times 0_k$ and $v_{n+j}$ is a subset of an edge of $Q \times \mathbb{R}_{\geq 0}^k$ for $j = 1, \ldots, k$. Then $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}\}$ are in general positions in $\mathbb{R}^{n+k}$. Take the hyperplane

$$H := \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}\}.$$  

The hyperplane $H$ satisfies the following. Let $p \in F$ be a point and $\overrightarrow{mp}$ a normal on $F$ towards the interior of $Q$. Then for any $x \in (Q \times \mathbb{R}_{\geq 0}^k) \cap H$, the angle between $\overrightarrow{mp}$ and $(x-p)$ is less than $90^\circ$. Therefore $(v' \times \mathbb{R}_{\geq 0}^k) \cap H$ is a $(k-1)$-simplex if $v' \in V(Q) \setminus V(F)$. Thus $H$ is a bounding hyperplane for $(Q \times \mathbb{R}_{\geq 0}^k) \cap H_{\leq 0}$. So $Q_F(k)$ is a convex polytope and $(Q \times \mathbb{R}_{\geq 0}^k) \cap H$ is a facet of $Q_F(k)$. Note that $F$ is a face in $Q_F(k)$ of codimension $(k+1)$, see Figure 3 for an example of this construction.

**Lemma 3.1.** Let $Q$ be an $n$-dimensional simple polytope with a facet $F$. Then $Q_F(k)$ is an $(n+k)$ dimensional simple polytope.

**Proof.** By definition, $Q_F(k)$ is a convex polytope. Thus it is enough to show that, at every vertex, exactly $n+k$ facets of $Q_F(k)$ intersect. Note that, the polyhedron

$$L_s^{k-1} := \{(x_1, \ldots, x_k) \in \mathbb{R}_0^k | x_s = 0\}$$  

is a facet of $\mathbb{R}_0^k$ for $s = 1, \ldots, k$. Let

$$V(Q) = \{v_1^Q, \ldots, v_n^Q\} \quad \text{and} \quad F(Q) = \{F_1^Q, \ldots, F_r^Q\}.$$  

be the vertex and facet set of $Q$, respectively. Then the facets of $Q \times \mathbb{R}_{\geq 0}^k$ are given by

$$\{Q \times L_s^{k-1} | s = 1, \ldots, k\} \cup \{F_j^Q \times \mathbb{R}_{\geq 0}^k | j = 1, \ldots, r\}.$$  

**Figure 3.** Example of a polytopal $k$-wedge construction.
Without loss of generality, let $F_i^Q = F$. Then the facet set of $Q_F(k)$ is given by

\[
F(Q_F(k)) = \{F_1, \ldots, F_r, F_{r+1}, \ldots, F_{r+k}\}
\]

where

\[
F_i := \begin{cases} 
(F_i^Q \cap \mathbb{R}_{\geq 0}) \cap H & \text{for } i = 1, \ldots, r - 1 \\
(Q \times \mathbb{R}_{\geq 0}) \cap H & \text{for } i = r \\
(Q \times L_{s-1}) \cap H_{\leq 0} & \text{for } i = r + s, s = 1, \ldots, k.
\end{cases}
\]

Note that there exists a projection $\rho: F_i = (Q \times \mathbb{R}^k) \cap H \to Q$. So $\rho$ is face preserving, and it takes facets to facets. Also, naturally, $Q$ is identified with the face $Q \times 0_k$ of $Q_F(k)$.

Let $v \in V(Q_F(k)) \setminus V(F)$. Then $\rho(v) = v^Q$ for some $v^Q = \bigcap_{j=1}^n F_{i_j}^Q \in V(Q)$ where $F_{i_j}^Q$'s are some unique facets in $F(Q) \setminus \{F_r^Q\}$. Then we have

\[
v = \left( \bigcap_{j=1}^n F_{i_j} \right) \cap F_r \cap \left( \bigcap_{s=1}^k F_{r+s} \right)
\]

for some $t \in \{1, \ldots, k\}$. Thus, exactly $(n + k)$ facets intersect at $v$ in $Q_F(k)$ for this case.

Let $v \in V(Q) \setminus V(F)$. Considering $v$ as a vertex of the simple polytope $Q$ and denote it by $v^Q$, we have $v^Q = \bigcap_{j=1}^n F_{i_j}^Q$ for some unique facets $F_{i_1}^Q, \ldots, F_{i_n}^Q$ in $F(Q) \setminus \{F_r^Q\}$. Thus, for this case, we have

\[
v = \left( \bigcap_{j=1}^n F_{i_j} \right) \cap \left( \bigcap_{s=1}^k F_{r+s} \right).
\]

Therefore, $v$ is the intersection of $(n + k)$ many facets in $Q_F(k)$.

Let $v \in V(F)$. Then $v = v^Q \in V(F) \subset V(Q) \subset V(Q_F(k))$. So $v^Q = \bigcap_{j=1}^{n-1} F_{i_j}^Q \cap F_r^Q$ for some unique facets $F_{i_1}^Q, \ldots, F_{i_{n-1}}^Q$ of $Q$. Thus, considering $v$ as a vertex of $Q_F(k)$, we get

\[
v = \left( \bigcap_{j=1}^{n-1} F_{i_j} \right) \cap F_r \cap \left( \bigcap_{s=1}^k F_{r+s} \right).
\]

Therefore, in this case also, $v$ is the intersection of $(n + k)$ many facets of $Q_F(k)$. Thus we get the result.

We call the simple polytope $Q_F(k)$ the \textit{polytopal $k$-wedge} of $Q$ at $F$.

\textbf{Example 3.2.} Let $Q$ be an interval $I = [0, 1]$ with two facets $\{0\}$ and $\{1\}$. If we take $k = 2$ and $F = \{1\}$ then the polytopal 2-wedge of $Q$ at $F$ is a tetrahedron. In Figure 3 we provide another example.
Remark 3.3. If $F$ is not a facet in $Q$, then the construction $Q_F(k)$ may not give a simple polytope in general. Consider a square $Q$ and a vertex $v_1 \in V(Q)$ as in Figure 4. Then $Q \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^3$. Next we take $v_2$ and $v_3$ in general positions of $Q \times \mathbb{R}_{\geq 0}$ to construct the hyperplane $H$ and eventually $Q_{v_1}(1)$. Note that at $v_1 \in Q_{v_1}(1)$, four facets intersect while $Q_{v_1}(1)$ is 3-dimensional, see Figure 4. Thus $Q_{v_1}(1)$ is not a simple polytope.

4. Blowdowns of polytopes

The concept of blowdown of a simple polytope was discussed in [16, Section 4] as follows. If $Q_1, Q_2$ are simple polytopes and $Q_1$ is a blowup (see Definition 4.1) of $Q_2$ then $Q_2$ is called a blowdown of $Q_1$. But it is not a precise definition, as $Q_1$ may be a blowup of another simple polytope $Q_3$, see Figure 5. In this section, we give the precise definition of blowdown of convex polytope which enriches its beauty. We provide the necessary and sufficient conditions for which a blowdown of a simple polytope is again a simple polytope. We also show that the new one possesses an ‘induced retraction sequence’ in the sense of Definition 2.3. We prove that blowdown of a polytope is a generalization of the polytopal $k$-wedge construction which is a generalization of the polytopal wedge construction of [11].

**Definition 4.1** (Blowup of a convex polytope). Let $Q$ be an $n$-dimensional convex polytope in $\mathbb{R}^n$ and $F$ be a face of $Q$. Take an $(n-1)$ dimensional hyperplane $H$ in $\mathbb{R}^n$ such that one open half space (say $H_{<0}$) contains $V(F)$ and $V(Q) \setminus V(F)$ is a subset of the open half space $H_{>0}$. Then $\bar{Q} := Q \cap H_{>0}$ is called a blowup of $Q$ along the face $F$.

Note that if $F_1$ is a facet of $Q$, then $\bar{F}_1 := F_1 \cap \bar{Q}$ is the facet of $\bar{Q}$ corresponding to $F_1$. The new facet $\bar{Q} \cap H$ is called the facet corresponding to the face $F$ and denoted by $\bar{F}$. We refer the reader to [19] for several properties of manifold with corners and maps between them.

**Definition 4.2** (Blowdown of a convex polytope). Let $F$ and $\bar{F}$ be two faces of an $n$-dimensional convex polytope $Q$ such that $F \subset \bar{F}$ and $\bar{F}$ is a facet. Let $F'$ be a convex polytope with a face $F'$ such that $F'$ is homeomorphic to $F$ as a manifold with corners. If the blowup $\bar{Q}'$ of $Q'$ along the face $F'$ is homeomorphic to $Q$ as a manifold with corners and the restriction on the facet $\bar{F}'$ is homeomorphic to $\bar{F}$ as a manifold with corners where $\bar{F}'$ is corresponding to the face $F'$, then $Q'$ is called a blowdown of $Q$ of the facet $\bar{F}$ on $F$. 

Figure 5. Both $Q_2$ and $Q_3$ are blowdowns as in [16].

Figure 6. Blowdown of a pentagonal prism of the face $\bar{F}$ on $F$. 

Remark 3.3. If $F$ is not a facet in $Q$, then the construction $Q_F(k)$ may not give a simple polytope in general. Consider a square $Q$ and a vertex $v_1 \in V(Q)$ as in Figure 4. Then $Q \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^3$. Next we take $v_2$ and $v_3$ in general positions of $Q \times \mathbb{R}_{\geq 0}$ to construct the hyperplane $H$ and eventually $Q_{v_1}(1)$. Note that at $v_1 \in Q_{v_1}(1)$, four facets intersect while $Q_{v_1}(1)$ is 3-dimensional, see Figure 4. Thus $Q_{v_1}(1)$ is not a simple polytope.
Remark 4.3. If $f : Q \to \overline{Q}'$ is the homeomorphism as a manifold with corners in Definition 4.2 then $f|_{\tilde{F}} : \tilde{F} \to \tilde{F}'$ is a homeomorphism as a manifold with corners. Also let $F(Q) = \{\tilde{F}, F_1, \ldots, F_r\}$ be the facets of $Q$ and $F$ a proper face of $\tilde{F}$. Then the facets $F_1', \ldots, F_r'$ of $\overline{Q}'$ are such that the facet $\tilde{F}_i$ in $\overline{Q}'$ is homeomorphic to $F_i$ as a manifold with corners through $f|_{F_i}$ for $1 \leq i \leq r$.

Example 4.4. Let $Q$ be a pentagonal prism with $\tilde{F}$ and $F$ as shown in Figure 6(a). Also, $Q'$ and $F'$ be as in Figure 6(b) where $F \cong F'$ is a manifold with corners. The blowup of $Q'$ along $F'$ is $\overline{Q}'$ in Figure 6(c), which is homeomorphic to $Q$ as a manifold with corners. So $Q'$ is a blowdown of $Q$ of the facet $\tilde{F}$ on $F$.

We note that a blowdown of a simple polytope may not be simple in general, see Figure 7. However, the following lemma gives a criterion when a blowdown of a simple polytope is simple.

Lemma 4.5. Let $Q$ be an $n$-dimensional simple polytope having a facet $\tilde{F}$ homeomorphic to $F \times \Delta^{n-d-1}$ as a manifold with corners where $F$ is a face of $\tilde{F}$ and $\Delta^{n-d-1}$ is a simplex for $0 \leq \dim(F) = d \leq (n-1)$. Let $Q'$ be the blowdown of $Q$ of the facet $\tilde{F}$ on $F$. Then $Q'$ is an $n$-dimensional simple polytope. Conversely if $Q'$, the blowdown of $Q$ of the facet $\tilde{F}$ on $F$, is simple then $\tilde{F}$ is homeomorphic to $F \times \Delta^{n-d-1}$ as a manifold with corners.

Proof. Let $V(F) = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ and $V(\Delta^{n-d-1}) = \{v_1, \ldots, v_{n-d}\}$. Then we may write

\[ V(\tilde{F}) := \{(b_{i_1}, v_q) : 1 \leq i \leq k \text{ and } 1 \leq q \leq (n-d)\} \subset V(Q). \]

By the definition of blowdown, $Q$ is homeomorphic to $\overline{Q}'$ as a manifold with corners. We denote this homeomorphism by $f : Q \to \overline{Q}'$ as in Remark 4.3. This induces a bijection between $V(Q)$ and $V(\overline{Q}')$.

Recall that $Q' \cap H_{\geq 0} = \overline{Q}'$, see Definition 4.1 Then $M_g = Q' \cap H_{\geq 0}$ is a mapping cylinder for the projection map

\[ g : \tilde{F}' \cong F' \times \Delta^{n-d-1} \to F'. \]

So there is a face preserving homotopy of $M_g$ on $F'$. Let us consider a tubular neighborhood $N_{\tilde{F}'}$ of $\tilde{F}'$ in $\overline{Q}'$ such that it does not contain any vertices in $V(\overline{Q}') \setminus V(\tilde{F}')$. We define

\[ f' : \overline{Q}' \to \overline{Q}' \]

by $f'(N_{\tilde{F}'}) \hookrightarrow N_{\tilde{F}'} \cup M_g$ preserving the face structure and $f'(x) = x$ if $x \in \overline{Q}' \setminus N_{\tilde{F}'}$. Now let

\[ \tilde{f} = f' \circ f : Q \to \overline{Q}'. \]

Since $f$ is a homeomorphism and face preserving, the map $\tilde{f}$ is a face preserving map. Then

\[ V(Q') = \{\tilde{f}(v) : v \in V(Q)\} \]

where $\tilde{f}(b_{i_1}, v_1) = \tilde{f}(b_{i_1}, v_2) = \cdots = \tilde{f}(b_{i_1}, v_{n-d})$ for all $i = 1, \ldots, k$, and $\tilde{f}$ is one-one otherwise.
Let $\tilde{f}(b) \in V(Q')$ such that $b \in V(\tilde{F})$. Then $b = (b_{t_i}, v_q)$ for some $b_{t_i} \in V(F)$ and $v_q \in V(\Delta^{n-d-1})$ and $\tilde{f}(b) = \tilde{f}(b_{t_i})$. Then, considering $b_{t_i}$ as a vertex of $Q$, we have

$$\tilde{f}(b) = (\bigcap_{j=1}^{n-1} F_{i_j}) \cap \tilde{F}$$

(4.5)

for some unique facets $F_{i_1}, ..., F_{i_{n-1}}$ of $Q$. Also, $b_{t_i} \times \Delta^{n-d-1}$ is homeomorphic to $\Delta^{n-d-1}$ as a manifold with corners. Precisely, since $F(\Delta^{n-d-1}) := \{F_{\Delta, 1}, ..., F_{\Delta, n-d}\}$ be the set of facets of $\Delta^{n-d-1}$.

Exactly one of these facets does not contain $b_{t_i}$, say $F_{\Delta, j}$ without loss of generality. If we consider $F \times F_{\Delta, j}$ for $1 \leq s \leq (n-d)$ then they are facets of $\tilde{F}$ and codimension 2 faces in $Q$. Note that these may not be the total collection of facets of $\tilde{F}$. Thus $F \times F_{\Delta, j} = \tilde{F} \cap P_{s_i}$ for a unique facet $P_{s_i}$ in $Q$ for all $1 \leq s \leq (n-d)$. Except $P_{s_i}$ all other facets in $\{P_{s_i} : 2 \leq s \leq (n-d)\}$ contain $b_{t_i}$. So

$$\{P_{s_i} : 2 \leq s \leq (n-d)\} \subseteq \{F_{i_j} : 1 \leq j \leq n-1\}.$$

(4.6)

Since $\tilde{f}$ is face preserving and $\tilde{f}(\tilde{F}) = F$, from Remark 4.3 and (4.5), we have

$$\tilde{f}(b) = (\bigcap_{j=1}^{n-1} \tilde{f}(F_{i_j})) \cap F = (\bigcap_{j=1}^{n-1} \tilde{f}(F_{i_j})) \cap \tilde{f}(P_{s_i})$$

(4.7)

where $P_{s_i}$ is described in the previous paragraph. So at the vertex $\tilde{f}(b)$ in $Q'$ exactly $n$ facets intersect. A similar construction can be done for any vertex of $\tilde{F}$.

Now let $b \in Q \setminus \tilde{F}$ be any vertex. Then $b = \bigcap_{i=1}^{n} F_{i_i}$ for some unique facets $F_{i_i}$ of $Q$ and

$$\tilde{f}(b) = \bigcap_{i=1}^{n} \tilde{f}(F_{i_i}).$$

(4.8)

This concludes at every vertex of $Q'$ exactly $n$ facets meet. So, $Q'$ is a simple polytope.

The converse part follows from Definition 1.1 and 1.2 as $Q$ and $Q'$ are homeomorphic as manifold with corners. Precisely, since $Q'$ is simple and $F(\cong F'$ as manifold with corners) is a face then the facet $\tilde{F}'$ of $Q'$ corresponding to face $F'$ is homeomorphic to $F \times \Delta$ as a manifold with corners where $\Delta$ is a simplex of dimension $(\text{dim}(Q) - \text{dim}(F) - 1)$. \hfill $\Box$

**Remark 4.6.** Let $Q'$ be a blowdown of $Q$ of the facet $\tilde{F}$ on $F$ such that $Q'$ is simple. If $E \cap \tilde{F} = \emptyset$, for a face $E$ of $Q$ then $\tilde{f}(E)$ is homeomorphic to $E$ as manifold with corners.

The following lemma investigates how a face $E$ of $Q$ is changed due to blowdown when $E \cap \tilde{F} \neq \emptyset$. Note that $E \cap \tilde{F}$ is a face of $\tilde{F} \cong F \times \Delta^{n-d-1}$. Thus $E \cap \tilde{F} = E^F \times E^\Delta$ as manifold with corners for some faces $E^F$ and $E^\Delta$ of $F$ and $\Delta^{n-d-1}$ respectively. Now $E^\Delta$ is again a simplex as it is a facet of the simplex $\Delta^{n-d-1}$. Thus

$$E \cap \tilde{F} = E^F \times \Delta^q$$

for some $0 \leq q \leq (n-d-1)$. The face $E \cap \tilde{F}$ shrinks to $E^F$ due to blowdown.

**Lemma 4.7.** Let $Q'$ be a blowdown of $Q$ of the facet $\tilde{F}$ on $F$ such that $Q'$ is simple. If $E \cap \tilde{F} \neq \emptyset$, then $\tilde{f}(E)$ is either a blowdown of $E$ of the facet $E \cap \tilde{F}$ on $E^F$ or homeomorphic to a face of $E$ as a manifold with corners.

**Proof.** Let $\dim(E \cap \tilde{F}) = \beta$. If $\beta = 0$, i.e., $E \cap \tilde{F}$ is a vertex, then $\tilde{f}(E)$ is homeomorphic to $E$ as manifold with corners. Now we consider the cases where $0 < \beta \leq (n-1)$. It is evident from $\dim(E \cap \tilde{F}) = \beta$ that $\dim(E) \geq \beta$. If $\dim(E) = \beta$, then $E \subset \tilde{F}$ and $\tilde{f}(E)$ is homeomorphic to a face of $E$. If $\dim(E) = \beta + 1$, then $E \cap \tilde{F}$ is a facet of $E$. We have $E \cap \tilde{F} = E^F \times \Delta^q$. If $q = 0$, then $\tilde{f}(E)$ is homeomorphic to $E$ as a manifold with corners. If $q > 0$ then $\tilde{f}(E)$ is a blowdown of $E$ of the face $E \cap \tilde{F}$ on $E^F$. 

Now we show $\dim(E) \neq \beta + j$ for $j \geq 2$. First, let $\dim(E) = \beta + 2$ and $v \in E \cap \tilde{F}$ be a vertex. Exactly $\beta + 2$ edges meet at $v$ in $E$, out of which $\beta$ edges are also edges of $\tilde{F}$ and two are not. Also, $v$ being a vertex of the facet $\tilde{F}$, exactly $n - 1$ edges meet at $v$ in $\tilde{F}$. This implies that exactly $n + 1$ edges meet at $v$ in $Q$, which is a contradiction to $Q$ is an $n$-dimensional simple polytope. Therefore, $\dim(E) \neq \beta + 2$, and by similar observation, $\dim(E) \neq \beta + j$ for $j > 2$. Thus, the claim of the lemma follows.

**Corollary 4.8.** Polytopal $k$-wedge of a simple polytope $Q$ at a facet $F$ is a blowdown of $Q \times \Delta^k$ of the facet $F \times \Delta^k$ on $F$.

**Proof.** If we blowup $Q_F(k)$ along the face $F$, then $Q_F(k)$ is homeomorphic to $Q \times \Delta^k$ as manifold with corners where $\Delta^k$ is a $k$-simplex. Also the facet $F \times \Delta^k$ arises in $Q_F(k)$ corresponding to the face $F$ of $Q$. Thus the polytopal $k$-wedge construction of $Q$ at $F$ is nothing but a blowdown of $Q \times \Delta^k$ of the face $F \times \Delta^k$ on $F$.

Now we investigate how the blowdown of a simple polytope affects its retraction sequences if the polytope remains simple after the blowdown.

**Theorem 4.9.** Let $Q'$ be the blowdown of an $n$-dimensional simple polytope $Q$ of the facet $\tilde{F}$ on $F$, and $Q'$ is simple. For a retraction sequence $\{(B_t, E_t, b_t)\}_{t=1}^m$ of $Q$ where $m = |V(Q)|$, there exists a retraction sequence $\{(B'_t, E'_t, b'_t)\}_{t=1}^{m-k}$ of $Q'$ which preserves the ordering on vertices.

**Proof.** We adhere to the notations from the proof of Lemma 4.5. Also, recall $F$ is a face of dimension $d$ in $Q$. Then from the converse part of Lemma 4.5, $\tilde{F}$ is homeomorphic to $F \times \Delta^{n-d-1}$ as a manifold with corners. Let $V(F) := \{b_{\ell_1}, \ldots, b_{\ell_k}\} \subset V(Q)$ be the vertices of $F$ such that $\ell_1 < \cdots < \ell_k$. We construct a retraction sequence $\{(B'_t, E'_t, b'_t)\}_{t=1}^{m-k}$ of $Q'$ inductively. First we define $\{(B'_1, E'_1, b'_1) := (Q', Q', b'_1)\}$ where $b'_1 := \tilde{f}(b_1)$ since $Q'$ is a simple polytope. Now we may encounter the following 3 cases to construct the second triple for a retraction sequence of $Q'$.

**Case 1 of the 2-nd step:** Let $b_2$ be neither in $V(\tilde{F})$ nor adjacent to any vertex in $V(\tilde{F})$ and $C'_1 := \cup\{E : E$ is a face of $B'_1$ containing the vertex $b'_1\}$. Then we take $b'_2 := \tilde{f}(b_2)$ and define

$$B'_2 := B'_1 \setminus C'_1, \quad \text{and} \quad E'_2 := \tilde{f}(E_2).$$

The definition of blowdown implies that $Q'$ does not change locally at the points which are not in $V(\tilde{F})$ or adjacent to a vertex in $V(\tilde{F})$, see Remark 4.6. This implies $E'_2 \cong E_2$ as a manifold with corners, in which $b'_2$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1}_{\geq 0}$ as a manifold with corners. So we can construct the next triple $(B'_2, E'_2, b'_2)$.

**Case 2 of the 2-nd step:** Let $b_2 \in V(\tilde{F})$. If $\tilde{f}(b_2) = \tilde{f}(b_1)$, then $\tilde{f}(b_2)$ is already retracted. Then to define $b'_2$ we need to go to the next vertex $\tilde{f}(b_3)$. Otherwise, we take $b'_2 := \tilde{f}(b_2)$. So we can get $(B'_2, E'_2, b'_2)$ where $B'_2$ and $E'_2$ is defined as in (4.9). As $b'_2$ is connected to $b'_1$ through an edge, $b'_2$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1}_{\geq 0}$ in $B'_2$ as a manifold with corners. Thus we get the second entry $(B'_2, E'_2, b'_2)$ of a retraction sequence for $Q'$.

**Case 3 of the 2-nd step:** Let $b_2$ be adjacent to a vertex in $V(\tilde{F})$. We define $b'_2 := \tilde{f}(b_2)$ along with $B'_2$ and $E'_2$ as in (4.9). As $b'_2$ is connected to $b'_1$ by an edge, $b'_2$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1}_{\geq 0}$ in $B'_2$ as a manifold with corners. Thus we get the second triple $(B'_2, E'_2, b'_2)$ for the retraction sequence of $Q'$.

Continuing a similar way, suppose that we are at $t$-th step of a retraction of $Q'$. In the meantime, we are at $i_t$-th step of retraction of $Q$ where $t \leq i_t$. At this step, three cases may arise.

**Case 1 of the $t$-th step:** Let $b_{i_t}$ is neither in $V(\tilde{F})$ nor adjacent to any vertex in $V(\tilde{F})$ and $C'_{i_t-1} := \cup\{E : E$ is a face of $B'_{i_t-1}$ containing the vertex $b'_{i_t-1}\}$. Then define

$$b'_t := \tilde{f}(b_{i_t}), \quad B'_t := B'_{i_t-1} \setminus C'_{i_t-1}, \quad \text{and} \quad E'_t := \tilde{f}(E_{i_t}) (\cong E_{i_t}$ as a manifold with corners).

This gives $t$-th triple $(B'_t, E'_t, b'_t)$ of a retraction sequence for $Q'$. 


Proof. There always exists a retraction sequence \((B_t, E_t, b_t)\) for a polytopal \(k\)-wedge \(Q_F(k)\) of \(Q\) at \(F\), where \(u = (k + 1)m - k\alpha\) and \(\alpha = |V(F)|\).

Corollary 4.10. Let \(\{(B_t, E_t, b_t)\}_{t=1}^{m-1}\) be a retraction sequence of \(Q\), where \(m = |V(Q)|\). There always exists a retraction sequence \(\{(B_t, E_t, b_t)\}_{t=1}^{i}\) for a polytopal \(k\)-wedge \(Q_F(k)\) of \(Q\) at \(F\), where \(u = (k + 1)m - k\alpha\) and \(\alpha = |V(F)|\).

Proof. Proposition 2.4 gives an induced retraction sequence \(\{(\tilde{B}_t, \tilde{E}_t, \tilde{b}_t)\}_{t=1}^{m}\) on \(Q \times \Delta^k\) such that \(\tilde{E}_{(k+1)t-(k+1-s)} = E_t \times \Delta^{k+1-s}\) for \(t = 1, \ldots, m\) and \(s = 1, \ldots, k + 1\). Then the claim follows from Corollary 4.8 and Theorem 4.9.

\[\]

\[\]
Moreover, we get the following if \(\{(\Delta^{k+1-s}, \Delta^{k+1-s}, e_s)\}_{s=1}^{k+1}\) is a retraction sequence of \(\Delta^k\).

**Corollary 4.11.** Let \(Q\) be a simple polytope with a facet \(F\) such that \(|V(F)| = \alpha\) and there exists a retraction sequence \(\{(B_t, E_t, v_t)\}_{t=1}^u\) such that the vertices of \(F\) to be retracted at the end. For \(u = (k+1)(m-\alpha) + \alpha\) there exists a retraction sequence \(\{(B'_t, E'_t, b'_t)\}_{t=1}^u\) for \(Q_F(k)\) such that

\[
\begin{align*}
(1) &\quad E'_{(k+1)\ell-(k+1)-s} = E_{\ell} \times \Delta^{k+1-s} \text{ for } \ell = 1, \ldots, m - \alpha \text{ and } s = 1, \ldots, k + 1, \text{ and } E'_t = E_{m-\alpha+\ell} \text{ for } t = (k+1)(m-\alpha) + \ell \text{ and } \ell = 1, \ldots, \alpha. \\
(2) &\quad b'_{(k+1)\ell-(k+1)-s} = (v_{\ell}, e_s) \text{ for } \ell = 1, \ldots, m - \alpha \text{ and } s = 1, \ldots, k + 1, \text{ and } b'_t = v_{m-\alpha+\ell} \text{ for } t = (k+1)(m-\alpha) + \ell \text{ and } \ell = 1, \ldots, \alpha.
\end{align*}
\]

5. **BLOWDOWNS OF QUASITORIC ORBIFOLDS AND TORSIONS IN THEIR INTEGRAL COHOMOLOGIES**

In this section, we study the effects of blowdowns of simple polytopes on their corresponding quasitoric orbifolds. Note that the blowdown of a quasitoric orbifold is discussed in [18]. However, we study blowdowns of quasitoric orbifolds in more generality. We also investigate the torsions in the integral cohomology of quasitoric orbifolds after blowdowns and prove no new torsion arises in certain blowdowns. We adhere to the notation of previous sections.

Let \(Q\) be an \(n\)-dimensional simple polytope with \(\mathcal{F}(Q) = \{\tilde{F}, F_1, \ldots, F_r\}\). Consider a blowdown \(Q'\) of \(Q\) on a facet \(\tilde{F}\) on a facet \(F\) such that \(Q'\) is a simple polytope. Let \(\mathcal{F}(Q') = \{F'_1, \ldots, F'_r\}\) as in Remark [5.1]. Let \(\lambda: \mathcal{F}(Q) \to \mathbb{Z}^n\) and \(\lambda': \mathcal{F}(Q') \to \mathbb{Z}^n\) be two \(\mathcal{R}\)-characteristic functions on \(Q\) and \(Q'\) respectively such that

\[
\lambda'(F'_i) = \lambda'(\tilde{f}(F_i)) = \lambda(F_i)
\]

for \(1 \leq i \leq r\) where \(\tilde{f}\) is defined in [4.3]. Then we call the pair \((Q', \lambda')\) a restriction of \((Q, \lambda)\).

**Definition 5.1** (Blowdown of a quasitoric orbifold). Let \((Q, \lambda)\) and \((Q', \lambda')\) be \(\mathcal{R}\)-characteristic pairs such that \((Q', \lambda')\) is a restriction of \((Q, \lambda)\). Then the quasitoric orbifold \(X(Q', \lambda')\) is called a blowdown of \(X(Q, \lambda)\).

**Example 5.2.** Let \((Q, \lambda)\) be an \(\mathcal{R}\)-characteristic function and \(Q'\) a blowdown of \(Q\) such that \(Q'\) is simple. Then the natural restriction of \(\lambda\) on \(\mathcal{F}(Q')\) using Remark [4.3] and [5.1] may not be an \(\mathcal{R}\)-characteristic function. For example, consider a blowdown of a cube as in Figure 9 and the \(\mathcal{R}\)-characteristic function \(\lambda\) on the facets of \(Q\) by

\[
\begin{align*}
\lambda(F_0) &= (1, 0, 0), & \lambda(F_1) &= (1, 0, 0), & \lambda(F_2) &= (2, 3, 5), \\
\lambda(F_3) &= (1, 3, 2), & \lambda(F_4) &= (4, 1, 0), & \lambda(\tilde{F}) &= (1, 0, 1).
\end{align*}
\]

If we define \(\lambda': \mathcal{F}(Q') \to \mathbb{Z}^3\) by natural restriction following [5.1], then \(\lambda'\) is not an \(\mathcal{R}\)-characteristic function on \(Q'\) since

\[
\det[\lambda'(F'_0), \lambda'(F'_1), \lambda'(F'_4)] = 0
\]

where \(F'_0 \cap F'_1 \cap F'_4\) is a vertex in \(Q'\). Thus the pair \((Q', \lambda')\) does not determine any quasitoric orbifold.

This justifies our definition of restriction of an \(\mathcal{R}\)-characteristic pair as well as the blowdown of quasitoric orbifolds. Next, we give a sufficient condition when the natural restriction is an \(\mathcal{R}\)-characteristic function.

Let \(Q'\) be a blowdown of \(Q\) of the facet \(\tilde{F}\) on \(F\). Then by Lemma [4.3] \(Q'\) is simple if and only if \(\tilde{F} \cong F \times \Delta^{n-d-1}\) as a manifold with corners. If \(b \in Q\) be any vertex such that \(b \notin \tilde{F}\), then the facets adjacent to \(\tilde{f}(b)\) remain the same, see [4.3]. Now let us fix a vertex \(b \in \tilde{F}\). So
Proposition 5.3. Let \( b = \bigcap_{j=1}^{n-1} F_{ij} \cap \tilde{F} \) as in (1.3). Similar construction as in the proof of Lemma 4.3 leads us to

\[
\tilde{f}(b) = \bigcap_{j=1}^{n-1} \tilde{f}(F_{ij}) \cap \tilde{f}(P_{b}),
\]

for a unique facet \( P_{b} \) of \( Q \) such that \( b \notin P_{b} \), see (4.7). We define a set

\[
S_{b} := \{ \lambda(F_{i1}), \ldots, \lambda(F_{i_{n-1}}), \lambda(P_{b}) \},
\]

for each vertex \( b \in \tilde{F} \). As a vertex of \( \tilde{F} \), \( b \) can be considered as \( (b_{F}, v_{q}) \) for some \( b_{F} \in V(F) \) and \( v_{q} \in V(\Delta^{n-d-1}) \). Notice that for any \( v, v' \in V(\Delta^{n-d-1}) \) we have

\[
S_{(b_{F}, v)} = S_{(b_{F}, v')}. \]

So we denote \( S_{b_{F}} := S_{(b_{F}, v)} \). Then, we can conclude the following.

**Proposition 5.3.** Let \( Q' \) be the blowdown of \( Q \) of the facet \( \tilde{F} \) on \( F \). If \( S_{b_{F}} \) is linearly independent for each \( b_{F} \in V(F) \), then the pair \((Q', \lambda')\) is an \( \mathcal{R} \)-characteristic pair as well as a restriction of \((Q, \lambda)\), where \( \lambda' \) is defined in (5.1). 

**Example 5.4.** Let \( Q \) be a cube as in Figure 9. We define \( \lambda \) on \( F(Q) \) by

\[
\lambda(F_0) = (0, 2, 1), \quad \lambda(F_1) = (1, 1, 2), \quad \lambda(F_2) = (0, 1, 1), \\
\lambda(F_3) = (1, 0, 1), \quad \lambda(F_4) = (1, 0, 0), \quad \lambda(\tilde{F}) = (1, 3, 3).
\]

This gives an \( \mathcal{R} \)-characteristic pair \((Q, \lambda)\) and consequently a quasitoric orbifold \( X(Q, \lambda) \). In Figure 9 \( Q' \) is the blowdown of \( Q \) of the face \( \tilde{F} \) on the face \( F \). We define \( \lambda' \) on the facets of \( Q' \) by (5.1). This gives a restriction \((Q', \lambda')\) of \((Q, \lambda)\). So \( X(Q', \lambda') \) is a blowdown of \( X(Q, \lambda) \).

**Remark 5.5.** Let \( Q' \) be a blowdown of an \( n \)-dimensional simple polytope \( Q \) of the facet \( \tilde{F} = F \times \Delta^{n-d-1} \) on a \( d \)-dimensional face \( F \). If \( F(\Delta^{n-d-1}) = \{ F^1_{\Delta}, \ldots, F_{\Delta_{n-d}} \} \), then \( F \times F_{\Delta} \) are some facets of \( \tilde{F} \) and \( F \times F_{\Delta} = \tilde{F} \cap P_{s} \) for a unique \( P_{s} \in F(Q) \) for \( 1 \leq s \leq (n-d) \).

**Proposition 5.6.** Let \( Q' \) be a blowdown of an \( n \)-dimensional simple polytope \( Q \) of the facet \( \tilde{F} = F \times \Delta^{n-d-1} \) on a \( d \)-dimensional face \( F \). Let \( \lambda \) be an \( \mathcal{R} \)-characteristic function on \( Q \) such that

\[
\lambda(\tilde{F}) = \sum_{s=1}^{n-d} c_{s} \lambda(P_{s}),
\]

for some \( c_{s} \in \mathbb{Q} \setminus \{0\} \), where \( P_{s} \) are described in Remark 5.5 for \( 1 \leq s \leq (n-d) \). Then \((Q', \lambda')\) is a restriction of \((Q, \lambda)\) where \( \lambda' \) is defined as (5.1).
Proof. Let \( b \in V(F) \). So \( \tilde{f}(b) \in V(F') \subset V(Q') \). The arguments in the proof of Lemma 4.5 and (4.6) give us

\[
(5.5) \quad b = \left( \bigcap_{j=1}^{n-1} F_{ij} \right) \cap \tilde{F} \quad \text{and} \quad \tilde{f}(b) = \left( \bigcap_{j=1}^{n-1} \tilde{f}(F_{ij}) \right) \cap \tilde{f}(P_b),
\]

for some unique facets \( F_{i1}, \ldots, F_{in-1}, \tilde{F} \) and \( P_b \) of \( Q' \). Note that \( P_b \) is the unique facet in \( \{ P_s : 1 \leq s \leq (n-d) \} \) such that it does not contain the vertex \( b \). If we define \( \lambda' : F(Q') \to \mathbb{Z}^n \) by using (5.1) from \( \lambda \), (4.5) give us

Thus (5.1) becomes an \( \mathbb{R} \)-linear space, that is the vectors assigned to facet \( s \) adjacent to the vertex \( \lambda \) are \( \mathbb{R} \)-linearly independent, that is the vectors assigned to facets adjacent to the vertex \( \tilde{f}(b) \) in \( Q' \) are \( \mathbb{R} \)-linearly independent. We can do the above construction for each vertex in \( V(F) \). Thus at each vertex of \( F' \) in \( Q' \) the vectors assigned to the adjacent facets are linearly independent.

Now let \( b' \in V(Q') \setminus V(F') \). Then there exists unique \( b \in V(Q) \setminus V(\tilde{F}) \) such that \( \tilde{f}(b) = b' \). The vectors assigned to the adjacent facets of \( b \) in \( Q \) and \( b' \) in \( Q' \) are the same. So the induced \( \lambda' \) using (5.1) becomes an \( \mathbb{R} \)-characteristic function on \( Q' \). Thus \( (Q', \lambda') \) is a restriction of \( (Q, \lambda) \). \( \square \)

Let \( E \) be a face of \( Q \) such that \( E \cap \tilde{F} = E^F \times \Delta^q \) for some \( q > 0 \) and \( E \cap \tilde{F} \) is a facet of \( E \). Then using Lemma 4.3, \( \tilde{f}(E) \) is a blowdown of \( E \) of the facet \( E \cap \tilde{F} \) to \( E^F \). The next lemma deduces that if the \( \mathbb{R} \)-characteristic function \( \lambda \) on \( Q \) satisfies (5.4), then a similar relation also holds for \( \lambda_E \). Recall the facet set \( F(\Delta^{n-d-1}) = \{ F_{1}^{\Delta}, F_{2}^{\Delta}, \ldots, F_{n-d}^{\Delta} \} \) from Remark 5.5 Let \( \{ F_{1}^{\Delta}, F_{2}^{\Delta}, \ldots, F_{q+1}^{\Delta} \} \subset \{ F_{1}^{\Delta}, F_{2}^{\Delta}, \ldots, F_{n-d}^{\Delta} \} \) such that \( F_{ij} \cap \Delta^q \) is a facet of \( \Delta^q \) for all \( j = 1, 2, \ldots, q + 1 \) and \( \bigcap_{j=1}^{n-d} F_{ij} = \Delta^q \). Thus \( \{ P_{i1}, P_{i2}, \ldots, P_{q+1} \} \subset \{ P_1, P_2, \ldots, P_{n-d} \} \) such that \( \{ P_{i1} \cap E, \ldots, P_{q+1} \cap E \} \subset \tilde{f}(E) \).

Lemma 5.7. If \( \lambda \) be an \( \mathbb{R} \)-characteristic function on \( Q \) satisfying (5.4) then

\[
\lambda_E(E \cap \tilde{F}) = \left( \sum_{j=1}^{q+1} c_j d_j \lambda(E \cap P_{ij}) \right) / d_E,
\]

for some positive integers \( d_E \) and \( d_j \)'s where \( j = 1, 2, \ldots, q + 1 \).

Proof. The projection map is defined by

\[
\rho_E : \mathbb{Z}^n \to \mathbb{Z}^{\dim(E)}.
\]

This map is \( \mathbb{Z} \)-linear and any \( \mathbb{Z} \)-linear map is \( \mathbb{Q} \)-linear. From the definition of the induced \( \mathbb{R} \)-characteristic function on \( E \)

\[
(5.7) \quad \lambda_E : F(E) \to \mathbb{Z}^{\dim(E)},
\]

we have

\[
(5.8) \quad \lambda_E(E \cap P_{ij}) = \text{prim}\{\rho_E(\lambda(P_{ij}))\} = \frac{\rho_E(\lambda(P_{ij}))}{d_j}
\]

Thus

\[
(5.9) \quad \lambda_E(E \cap \tilde{F}) = \text{prim}\{\rho_E(\lambda(\tilde{F}))\} = \text{prim}\{\rho_E\left( \sum_{i=1}^{n-d} c_i \lambda(P_i) \right) \} = \text{prim}\{\sum_{i=1}^{n-d} c_i \rho_E(\lambda(P_i)) \}
\]

\[
= \text{prim}\{\sum_{j=1}^{q+1} c_j d_j \lambda_E(P_{ij} \cap E) \} = \left( \sum_{j=1}^{q+1} c_j d_j \lambda_E(P_{ij} \cap E) / d_E \right),
\]

for some unique positive integer \( d_E \). \( \square \)
Let \((Q, \lambda)\) be an \(\mathcal{R}\)-characteristic pair and \(\{(B_\ell, E_\ell, b_\ell)\}_{\ell=1}^m\) be a retraction sequence of \(Q\). Then we denote \(|G_{B_\ell}(b_\ell)| := |G_{E_\ell}(b_\ell)|\) for all \(\ell = 1, \ldots, m\).

**Proposition 5.8** (Proposition 4.5, [2]). Let \((Q, \lambda)\) be an \(\mathcal{R}\)-characteristic pair and \((F, \lambda_F)\) the induced \(\mathcal{R}\)-characteristic pair on a face \(F\) with \(V(F) = \{b_{\ell_1}, \ldots, b_{\ell_k}\}\). If \(\{(B^F_\ell, E^F_\ell, b^F_\ell)\}_{\ell=1}^k\) is an induced retraction sequence of \(F\) from \(\{(B_\ell, E_\ell, b_\ell)\}_{\ell=1}^m\) of \(Q\) such that \(B^F_\ell = B_\ell \cap F\), \(b^F_\ell = b_\ell\), and \(E^F_\ell\) is the maximal dimensional face of \(B^F_\ell\) containing \(b^F_\ell\); then \(|G_{E^F_\ell}(b^F_\ell)|\) divides \(|G_{E_\ell}(b_\ell)|\).

Next, we discuss how the singularities are affected after certain blowdowns of quasitoric orbifolds.

**Theorem 5.9.** Let \((Q, \lambda)\) be an \(\mathcal{R}\)-characteristic pair satisfying the hypothesis in Proposition 4.7 and \(Q'\) a blowdown of \(Q\) of the facet \(\hat{F}\) on \(F\) with \(|V(F)| = k, \dim F = d\). Let \(p\) be a prime such that the following holds:

\[
\begin{align*}
(A_1) & \text{ There exists a retraction sequence } \{(B_\ell, E_\ell, b_\ell)\}_{\ell=1}^m \text{ of } Q \text{ such that } \gcd(|G_{E_\ell}(b_\ell)|, p) = 1 \text{ for all } \ell = 1, \ldots, m. \\
(A_2) & \text{ The map } \lambda: \mathcal{F}(Q) \to \mathbb{Z}^n \text{ satisfies } (5.4) \text{ such that } \gcd(\text{denominator of } c_s, p) = 1 \text{ for } s = 1, \ldots, (n-d) \\
(A_3) & \text{ If } E_\ell \cap \hat{F} \text{ is a facet of } E_\ell \text{ such that } \tilde{f}(E_\ell) \text{ is a blowdown of } E_\ell \text{ of the facet } E_\ell \cap \hat{F} \text{ to some face } E^F_\ell \text{ then } \gcd(d_{E_\ell}, p) = 1, \text{ where } \lambda_{E_\ell}(E_\ell \cap \hat{F}) = d_{E_\ell} \cdot \rho_{E_\ell}(\lambda(\hat{F})).
\end{align*}
\]

Then \(X(Q', \lambda')\) is a blowdown of \(X(Q, \lambda)\) where \(\lambda'\) is defined in (5.1) and \(Q'\) has a retraction sequence \(\{B'_\ell, E'_\ell, b'_\ell\}_{\ell=1}^{m-k}\) such that \(|G_{E'_\ell}(b'_\ell)|, p) = 1 \text{ for all } t = 1, \ldots, m - k.

**Proof.** Let \((Q, \lambda)\) be a quasitoric orbifold over an \(n\)-dimensional simple polytope \(Q\) having a retraction sequence \(\{(B_\ell, E_\ell, b_\ell)\}_{\ell=1}^m\).

Let \(Q'\) be a blowdown of \(Q\) of the facet \(\hat{F}\) on the face \(F\) such that \(Q'\) is simple. Then there is a retraction sequence \(\{B'_\ell, E'_\ell, b'_\ell\}_{\ell=1}^{m-k}\) of \(Q'\) where \(|V(F)| = k\), see the proof of Theorem 4.9.

Suppose that \(\lambda\) satisfies (5.4) and \(p\) is a prime number such that

\[
\gcd(\text{denominator of } c_s, p) = 1 \text{ for } s = 1, \ldots, (n-d).
\]

Then \(X(Q', \lambda')\) is a blowdown of \(X(Q, \lambda)\) by Proposition 5.13.

For an arbitrary vertex \(b \in V(F)\), there exists \(b' = \tilde{f}(b) \in V(F')\) during the induced retraction as in the proof of Theorem 4.9. Then from (5.6), we have

\[
(5.10) \quad |G_Q(b)| = c_s |G_{Q'}(b')|
\]

for some \(s \in \{1, \ldots, (n-d)\}\). For the quasitoric orbifold \(X(Q, \lambda)\), let us assume

\[
\gcd(|G_{E_\ell}(b_\ell)|, p) = 1 \text{ for } \ell = 1, \ldots, m.
\]

Now we want to see how the orders of the singularities \(G_{E'_\ell}(b'_\ell)\) behave due to blowdown where \(b'_\ell = \tilde{f}(b_\ell)\) for some \(b_\ell \in V(Q)\). Depending on \(b_\ell \in V(Q)\) three cases may arise during the induced retraction of \(Q'\).

**Case 1:** Let the vertex \(b_\ell \in V(Q)\) be neither in \(V(\hat{F})\) nor adjacent to any vertex in \(V(\hat{F})\). Then \(E'_\ell\) is homeomorphic to \(E_\ell\) as a manifold with corners, see the proof of Theorem 4.9. Thus

\[
|G_{E'_\ell}(b'_\ell)| = |G_{E_\ell}(b_\ell)|.
\]

This implies \(\gcd(|G_{E'_\ell}(b'_\ell)|, p) = 1\) for the vertices considered in this case.

**Case 2:** Let \(b_\ell \in V(\hat{F})\) and \(b'_\ell = \tilde{f}(b_\ell)\). Then from the proof of Theorem 4.9 either \(E'_\ell = \tilde{f}(E_\ell)\) or \(E'_\ell\) is a face of \(\tilde{f}(E_\ell)\). Now \(\tilde{f}(E_\ell)\) is either homeomorphic to \(E_\ell\) as a manifold with corners or homeomorphic to a face of \(E_\ell\) as a manifold with corners or a blowdown of \(E_\ell\), see Remark 4.6 and Lemma 4.7.
Subcase 1: If $E'_t$ is homeomorphic to $E_t$, then
\[
|G_{E'_t}(b'_t)| = |G_{E_t}(b_t)|.
\]

Subcase 2: If $E'_t$ is homeomorphic to a face of $E_t$, then, from Proposition 5.8
\[
|G_{E'_t}(b'_t)| \text{ divides } |G_{E_t}(b_t)|.
\]

Subcase 3: If $E'_t$ is a blowdown of $E_t$, then from Lemma 5.7 and 5.10
\[
(5.11) \quad \frac{c_s d_s}{d_{E_t}} |G_{E'_t}(b'_t)| = |G_{E_t}(b_t)| \text{ for some } s \in \{1, \ldots, (n - d)\}
\]
where $d_{E_t}$ comes from 5.9 while computing the determinant of the corresponding matrices
given by 2.10. Thus $|G_{E'_t}(b'_t)| = \frac{d_{E_t}}{c_s d_s} |G_{E_t}(b_t)|$ for some $s \in \{1, \ldots, (n - d)\}$. Since we have
d$s \in \mathbb{Z}$ from 5.8, then $d_s$ is a factor of $|G_{E_t}(b_t)|$. Therefore, if we assume $\gcd(d_{E_t}, p) = 1$
then for the above three subcases we have $\gcd(|G_{E'_t}(b'_t)|, p) = 1$ where $b'_t = \hat{f}(b_t)$.

Case 3: Let $b_t$ be adjacent to a vertex of $V(\hat{F})$ and $b'_t = \hat{f}(b_t)$ in the blowdown. Then either
$E'_t = \hat{f}(E_t)$ or $E'_t$ is a face of $\hat{f}(E_t)$. Here also three subcases arise as in Case 2 and deduction
follows in a similar way. Thus
\[
\gcd(|G_{E'_t}(b'_t)|, p) = 1 \text{ for all } 1 \leq t \leq m - k.
\]
The claim $X(Q', \lambda)$ is a blowdown of $X(Q, \lambda)$ follows directly from Proposition 5.6 □

The next two examples show that, in general, we may not relax the hypotheses $(A_2)$ and $(A_3)$
in Theorem 5.9

Example 5.10. Let $Q$ be a 3-dimensional cube and $Q'$ a blowdown of $Q$ as in Figure 9. Define
an $\mathcal{R}$-characteristic function $\lambda$ on $Q$ by
\[
(5.12) \quad \lambda(F_0) = (2, 1, 4), \quad \lambda(F_1) = (6, 3, 5), \quad \lambda(F_2) = (3, 1, 7),
\]
\[
\lambda(F_3) = (1, 2, 6), \quad \lambda(F_4) = (4, 1, 3), \quad \lambda(\hat{F}) = (2, 3, 5).
\]

Then $|G_Q(b_1)| = 5$. Consider the retraction sequence of $Q$ as in Figure 10.
Now we calculate the order of $G_{E_t}(b_t)$. As $E_3$ is the face $F_1$, we extend $\lambda(F_1)$ to a basis
$\{(6, 3, 5), (1, 0, 0), (0, 2, 3)\}$ of $\mathbb{Z}^3$. Thus the projection map $\rho_{F_1}$ defined in 2.31
becomes
\[
\rho_{F_1} : \mathbb{Z}^3 \to \mathbb{Z}^3 / \langle(6, 3, 5)\rangle \cong \mathbb{Z}^2.
\]
The facets of $F_1$ are $\{F_1 \cap F_2, F_1 \cap F_3, F_1 \cap F_4, F_1 \cap \hat{F}\}$. Therefore the map $\lambda_{F_1} : \mathcal{F}(F_1) \to \mathbb{Z}^2$
as in 2.5 is defined by
\[
\lambda_{F_1}(F_1 \cap F_2) = \rho_{F_1}(\lambda(F_2)) = (-63, -16),
\]
\[
\lambda_{F_1}(F_1 \cap F_3) = \rho_{F_1}(\lambda(F_3)) = (-35, -8),
\]
\[
\lambda_{F_1}(F_1 \cap F_4) = \rho_{F_1}(\lambda(F_4)) = (-14, -4),
\]
\[
\lambda_{F_1}(F_1 \cap \hat{F}) = \rho_{F_1}(\lambda(\hat{F})) = (4, 0)
\]
Thus $|G_{E_t}(b_3)| = 64$. Note that 5.12 induces an $\mathcal{R}$-characteristic function on $Q'$ using 5.1
though $(2, 3, 5)$ is not a $\mathcal{Q}$-linear combination of $(2, 1, 4)$ and $(6, 3, 5)$. Then $|G_Q(b'_1)| = 7$. Here,
new prime factor 7 arises in the order of singularity at $b'_1$ after blowdown, which was neither
in $|G_Q(b_1)|$ nor in $|G_{E_t}(b_3)|$. Therefore the hypothesis $(A_2)$ may not be possible to relax in
Theorem 5.9 □

Example 5.11. Let $(Q, \lambda)$ and $(Q', \lambda')$ be $\mathcal{R}$-characteristic pairs as in Example 5.4. We consider
the retraction sequences of $Q$ and $Q'$ as in Figure 10. In the induced retraction sequence of $Q'$
from $Q$, $E'_2$ is a blowdown of $E_2$. Similar calculation to Example 5.10 gives $|G_{E'_2}(b_2)| = 1$ but
$|G_{E'_2}(b'_2)| = 3$. Here $d_{E_2} = 3$ comes while taking determinant, see 5.11. Thus we cannot relax
the hypothesis $(A_3)$ in Theorem 5.9 in general. □
Then for a $k$-dimensional simplex $\Delta^k$ we get a simple polytope $Q \times \Delta^k$ with

$$\mathcal{F}(Q \times \Delta^k) = \{Q \times F_0^\Delta, \ldots, Q \times F_k^\Delta, F_1^Q \times \Delta^k, \ldots, F_r^Q \times \Delta^k\}$$

where $\mathcal{F}(\Delta^k) = \{F_0^\Delta, \ldots, F_k^\Delta\}$. Let us consider a blowdown of $Q \times \Delta^k$ of the face $F_s^Q \times \Delta$ on $F_s^Q$ for some $s \in \{1, \ldots, r\}$ and denote it by $(Q \times \Delta^k)’$. By Corollary 4.8 this is a polytopal $k$-wedge of $Q$ at $F_s^Q$. Without loss of generality let $s = r$ and denote the polytopal $k$-wedge by $Q_F(k)$.

Let $\{e_1, \ldots, e_k\}$ be the standard basis of $\mathbb{Z}^k$. Now we define a map

$$\tilde{\lambda} : \mathcal{F}(Q \times \Delta^k) \to \mathbb{Z}^{n+k}.$$
induced from the characteristic function $\lambda$ by the following way

$$
\tilde{\lambda}(F) = \begin{cases} 
(0_k, \lambda(F_j^Q)) & \text{if } F = F_j^Q \times \Delta^k \text{ for } j = 1, \ldots, r \\
(\sum_{j=1}^k e_j, \lambda(F_j^Q)) & \text{if } F = Q \times F_0^\Delta \\
(1, a, 0_{n+k-2}) & \text{if } F = Q \times F_1^\Delta, \ a \in \mathbb{Z}\setminus\{1\} \\
(e_j, 0_n) & \text{if } F = Q \times F_j^\Delta \text{ for } j = 2, \ldots, k
\end{cases}
$$

where $0_j$ represents the zero vector in $j$-dimension, depending on the condition on the facet $F$.

**Lemma 6.1.** Let $(Q, \lambda)$ be an $\mathcal{R}$-characteristic pair and $\Delta^k$ a $k$-simplex. Then the map $\tilde{\lambda}$ defined in (6.1) is an $\mathcal{R}$-characteristic map over $Q \times \Delta^k$.

**Proof.** We investigate the order of singularities defined in (2.10) at the vertices of $Q \times \Delta^k$ and show they are non-zero. Let $b_i \in V(Q \times \Delta^k)$ with $b_i = v_\ell \times v^\Delta$ for $v_\ell \in V(Q)$ and $v^\Delta \in V(\Delta^k)$. If $a = 0$ then clearly $\tilde{\lambda}$ is an $\mathcal{R}$-characteristic function and $|G_{Q \times \Delta^k}(b_i)| = |G_Q(v_\ell)|$.

Now let $a \neq 0, 1$. If $b_i \in V(Q \times \Delta^k) \setminus V(Q \times F_0^\Delta)$ then $|G_{Q \times \Delta^k}(b_i)| = |G_Q(v_\ell)|$

where $b_i = v_\ell \times v^\Delta$. Let $b_i \in V(Q \times F_\ell^\Delta)$ with $b_i = v_\ell \times v^\Delta$ and $v_\ell = \bigcap_{t=1}^n F_j^Q$. Then

$$b_i = \left(\bigcap_{t=1}^n (F_j^Q \times \Delta^k)\right) \cap \left(\bigcap_{\substack{j=0 \\text{ for } \alpha \neq a}}^k (Q \times F_j^\Delta)\right)$$

for $\alpha = 0, 2, \ldots, k$. To calculate the order of $G_{Q \times \Delta^k}(b_i)$, we can visualise the matrix associated to the vertex $b_i$ in $Q \times \Delta^k$ as the following block matrix

$$A_{b_i}^{Q \times \Delta^k} = \begin{pmatrix} 0 & B \\ A_{b_i}^Q & C \end{pmatrix}$$

where $A_{b_i}^Q$ is defined as in (2.5) and $B$ and $C$ are determined by the vectors assigned to the facets $Q \times F_j^\Delta$ of $Q \times \Delta^k$ for $j = 0, 1, \ldots, k$. Thus

$$|G_{Q \times \Delta^k}(b_i)| = |\det A_{b_i}^{Q \times \Delta^k}| = |\det A_{b_i}^Q| \times |\det B|.$$

If $\alpha = 0$ or 2 then $|\det B| = 1$ and $|G_{Q \times \Delta^k}(b_i)| = |G_Q(v_\ell)| \neq 0$. If $\alpha \neq 0, 2$ then $|\det B| = |(1-a)|$ and $|G_{Q \times \Delta^k}(b_i)| = |(1-a)| |G_Q(v_\ell)| \neq 0$. This concludes the proof of the lemma.

Note that $|\mathcal{F}(Q_F(k))| = |\mathcal{F}(Q \times \Delta^k)| - 1$, since the facet $F_{\ell}^Q \times \Delta^k$ is identified with $F_{\ell}^Q$ after blowdown. We recall that the facet set of $Q_F(k)$ is defined in (3.1). Now we restrict $\lambda$ in (6.1) to obtain

$$\lambda^F_k : \mathcal{F}(Q_F(k)) \rightarrow \mathbb{Z}^{n+k}$$

by the following way

$$\lambda^F_k(F_i) = \begin{cases} 
(0_k, \lambda_i) & \text{for } i = 1, \ldots, r - 1 \\
(-\sum_{j=1}^k e_j, \lambda(F_j^Q)) & \text{for } i = r \\
(1, a, 0_{n+k-2}) & \text{for } i = r + 1, \ a \in \mathbb{Z}\setminus\{1\} \\
(e_s, 0_n) & \text{for } i = r + s \text{ and } s = 2, \ldots, k
\end{cases}$$

where $0_j$ represents the zero vector of dimension $j$, depending on the condition on the facet $F_i$ of $Q_F(k)$.

**Lemma 6.2.** Let $(Q, \lambda)$ be an $\mathcal{R}$-characteristic pair over an $n$-dimensional simple polytope $Q$ with $\mathcal{F}(Q) = \{F_1, \ldots, F_r\}$. If $Q_F(k)$ is $k$-wedge of $Q$ at $F = F_r$ and $\lambda^F_k : \mathcal{F}(Q_F(k)) \rightarrow \mathbb{Z}^{n+k}$ is defined as in (6.3), then $\lambda^F_k$ is an $\mathcal{R}$-characteristic function on $Q_F(k)$. 

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The above text is a representation of the content from a mathematical document, specifically focusing on the definition and proofs related to characteristic functions and polytopes. The text involves complex mathematical expressions and concepts, relating to characteristic maps and singularities. The document aims to provide a rigorous mathematical framework for understanding and analyzing these concepts within the context of polytopes and their associated functions.
Proof. This proof is similar to the proof of Lemma 6.1. □

Definition 6.3. Let \( X(Q, \lambda) \) be a quasitoric orbifold over a simple polytope \( Q \) with a facet \( F \) and \( Q_F(k) \) a polytopal \( k \)-wedge of \( Q \) at \( F \). Let \( \lambda_F^k \) be defined as in (6.3) induced from \( \lambda \). Then we call the quasitoric orbifold \( X(Q_F(k), \lambda_F^k) \) a \( k \)-wedge of the quasitoric orbifold \( X(Q, \lambda) \).

Remark 6.4. Observe that \( (Q_F(k), \lambda_F^k) \) is a restriction of the characteristic pair \( (Q \times \Delta^k, \tilde{\lambda}) \). Thus the quasitoric orbifold \( X(Q_F(k), \lambda_F^k) \) is a blowdown of the quasitoric orbifold \( X(Q \times \Delta^k, \lambda) \). Moreover, if \( a = 0 \) in (6.3) we can use Theorem 5.9 to the quasitoric orbifold \( X(Q_F(k), \lambda_F^k) \) as a blowdown of \( X(Q \times \Delta^k, \tilde{\lambda}) \) and get similar conclusion as in Theorem 5.13. Some results for the cases for \( a \neq 0 \) are discussed further in the following.

Theorem 6.5. Let \( (Q, \lambda) \) be a quasitoric orbifold over a simple polytope \( Q \) with a facet \( F \) and for a prime \( p \) there exists a retraction sequence \( \{(B_t, E_t, v_t)\}_{t=1}^m \) such that \( \{V(F) = \{v_{m-\alpha+1}, \ldots, m\} \) and satisfying \( \text{gcd}(|G_{E_t}(v_t)|, p) = 1 \) for \( t = 1, \ldots, m \). If \( X(Q_F(k), \lambda_F^k) \) is a \( k \)-wedge of \( X(Q, \lambda) \) where \( \lambda_F^k \) is defined as in (6.3) such that \( \text{gcd}(|1 - a|, p) = 1 \), then there is no p-torsion in \( H_a(X(Q_F(k), \lambda_F^k); \mathbb{Z}) \) and \( H_{odd}(X(Q_F(k), \lambda_F^k); \mathbb{Z}_p) = 0 \).

Proof. Recall the facets of \( Q_F(k) \) from (3.1), the induced retraction sequence \( \{(B_t^f, E_t^f, b_t^f)\}_{t=1}^u \) of \( Q_F(k) \) from Corollary 4.11 and the R-characteristic vector is defined in (6.3). If we prove \( \text{gcd}(|G_{E_t^f}(b_t^f)|, p) = 1 \) for \( t = 1, \ldots, u \), we can conclude the result using [4, Theorem 1.1]. For that, we have to deal with the following cases.

Case 1: Let \( 1 \leq t \leq (k + 1)(m - \alpha) \). In this case \( E_t^f = E_t \times \Delta^{k+1-s} \) and \( b_t^f = (v_t, e_s) \) for \( t = (k + 1)\ell - (k + s) \) where \( \ell = 1, \ldots, m - \alpha \) and \( s = 1, \ldots, k + 1 \). Let \( \text{dim}(E_t^f) = d \). Then \( \text{dim}(E_t) = d + q \) and

\[
E_t^f = \left( \bigcap_{t=1}^{n-d} (F_t^{Q \times \Delta^k}) \right) \bigcap \left( \bigcap_{j=1}^{k-q} (Q \times F_s^{\Delta}) \right)
\]

where \( 0 \leq q \leq k \). From the discussion in Subsection 2.2 we obtain a \((d + q) \times (d + q)\) matrix \( A_{b_t^f}^{E_t^f} \) associated to the vertex \( b_t^f \) in \( E_t^f \) by projecting \( \lambda_F^k \) on the face \( E_t^f \) as follows. First we extend the set of \( d + q \) vectors

\[
S(E_t^f) = S(E_t) \cup S(\Delta^q) := \{ \lambda_F^k(E_t^{Q \times \Delta^k}) \mid t = 1, \ldots, n - d \} \cup \{ \lambda_F^k(Q \times F_s^{\Delta}) \mid j = 1, \ldots, k - q \}
\]

to a basis of \( \mathbb{Z}^{n+k} \). Since the first \( k \) entries of the vectors in \( S(E_t^f) \) are zero, we extend them to \( n \) linearly independent vectors in \( \mathbb{Z}^{n+k} \) similar to the extension of \( \{ \lambda(F_t^{Q \times \Delta^k}) \mid t = 1, \ldots, n - d \} \) to a basis of \( \mathbb{Z}^n \) in \( Q \). We denote this linearly independent set of \( n \) vectors by \( S(Q_t^f) \). Also along with \( S(\Delta^q) \), we add \( q \) many vectors from the standard basis vectors \( \{e_1, \ldots, e_k\} \) of \( \mathbb{Z}^{n+k} \) to extend \( S(Q_t^f) \) to a basis \( S(n + k) \) of \( \mathbb{Z}^{n+k} \).

Now if we visualize the matrix \( A_{b_t^f}^{E_t^f} \) as block matrix of the form

\[
A_{b_t^f}^{E_t^f} = \begin{pmatrix}
M_1^{(q \times d)} & M_2^{(q \times q)} \\
M_3^{(d \times d)} & M_4^{(d \times q)}
\end{pmatrix}
\]

then we have \( M_1 = 0_{(q \times d)} \) and \( M_3 = \left( \lambda(F_t \cap F_i)^t \right)_{d \times d} = A_{v_t}^{E_t} \) from the above discussion where \( v_t = \cap_{j=1}^d (E_t \cap F_j) \). Thus

\[
|G_{E_t^f}(b_t^f)| = |\det A_{b_t^f}^{E_t^f}| = |\det A_{v_t}^{E_t^f}| \times |\det M_2| = |G_{E_t^f}(v_t)| \times |\det M_2|.
\]

If \( E_t \cap F_{r+1} \neq \emptyset \), then there exists two subcases. If \( e_2 \in S(n + k) \), then \( |\det M_2| = 1 \). Otherwise, \( |\det M_2| = |(1 - a)| \). This implies \( |G_{E_t^f}(b_t^f)| \) divides \( |(1 - a)||G_{E_t^f}(v_t)| \).

If \( E_t \cap F_{r+1} = \emptyset \), then \( |G_{E_t^f}(b_t^f)| = |G_{E_t^f}(v_t)| \). For \( \text{gcd}(|1 - a|, p) = 1 \), we can conclude

\[
\text{gcd}(|G_{E_t^f}(b_t^f)|, p) = 1 \quad \text{for} \quad t = 1, \ldots, (k + 1)(m - \alpha).
\]
Figure 11. A blowdown that cannot be obtained by polytopal $k$-wedge construction.

**Case 2:** Let $(k + 1)(m - \alpha) + 1 \leq t \leq u = (k + 1)(m - \alpha) + \alpha$. In this case $E'_t = E_{m-\alpha+\ell}$ and $b'_t = v_{m-\alpha+\ell}$ for $t = (k + 1)(m - \alpha) + \ell$ and $\ell = 1, \ldots, \alpha$. Then $|G_{E'_t}(b'_t)| = |G_{E_{m-\alpha+\ell}}(v_{m-\alpha+\ell})|$. Thus, we show

$$\gcd(|G_{E'_t}(b'_t)|, p) = 1 \text{ for } t = 1, \ldots, (k + 1)(m - \alpha) + \alpha$$

and eventually conclude the result. \qed

**Example 6.6.** In Figure 11, we show a blowdown $Q'$ of a simple polytope $Q$ that cannot be obtained by a polytopal wedge construction. Define an $\mathcal{R}$-characteristic function on $Q$ by

$$\lambda(F_0) = (0, 2, 1), \quad \lambda(F_1) = (1, 1, 2), \quad \lambda(F_2) = (1, 3, 3),$$

$$\lambda(F_3) = (0, 1, 1), \quad \lambda(F_4) = (1, 0, 1), \quad \lambda(F_5) = (1, 0, 0), \quad \lambda(F_6) = (3, 2, 7).$$

Then $(Q, \lambda)$ is an $\mathcal{R}$-characteristic pair and provides us a quasitoric orbifold $X(Q, \lambda)$.

We define the $\mathcal{R}$-characteristic function $\lambda'$ on $Q'$ using (6.1). Then $(Q', \lambda')$ is a restriction of $(Q, \lambda)$. Therefore $X(Q', \lambda')$ is a blowdown of $X(Q, \lambda)$. Note that $X(Q', \lambda')$ cannot be obtained by a $J$-construction of [8] on a 4-dimensional quasitoric orbifold as its orbit space is a polygon. \qed

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