Quantum optical gyroscope
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Gyroscope for rotation sensing plays a key role in inertial navigation systems. Developing more precise gyroscopes than the conventional ones bounded by classical shot-noise limit by using quantum resources has attracted much attention. However, the existing quantum gyroscope schemes suffer severe deterioration under the influence of decoherence, which is called the no-go theorem of noisy metrology. Here, by using two quantized optical fields as quantum probe, we propose a quantum-optical-gyroscope scheme breaking through the constraint of the no-go theorem. Our exact analysis on the non-Markovian noise reveals that the ideal sensitivity achieving the super-Heisenberg limit is asymptotically recoverable when each optical field forms a bound state with its environment. The result provides a guideline in realizing high-precision rotation sensing in realistic noisy environments.

**Introduction.**—High-performance gyroscopes for rotation sensing are of pivotal significance for navigation in many types of air, ground, marine, and space applications. Based on the Sagnac effect, i.e., two counter-propagating waves in a rotating loop accumulate a rotation-dependent phase difference, gyroscopes have been realized in optical [1–6] and matter-wave [7–14] systems. The records for precision and stability of commercial gyroscopes are held by optical gyroscopes [15, 16]. However, their precision, which is proportional to the surface area enclosed by the optical path [17], is still limited by the classical shot-noise limit (SNL). It dramatically constrains their practical application and further performance improvement. How to build a gyroscope beating the SNL is highly desired.

Pursuing more precise measurement to physical quantities than the classical SNL by using quantum resources [18–22], such as squeezing [23–25] and entanglement [26–28], quantum metrology supplies a way toward achieving gyroscopes with ultimate sensitivity limits. Based on this idea, many schemes of quantum gyroscopes have been proposed. It was found that the entanglement in N00N states [29, 30], continuous-variable squeezing [31–33], and optical nonlinearity [34] can enhance the sensitivity of optical gyroscopes beyond the SNL. A quantum-enhanced sensitivity can also be achieved in matter-wave gyroscopes [35–38] by using spin squeezing [39–41] or entanglement. However, quantum gyroscopes are still at the stage proof-of-principle study and their superiority over the conventional ones in the absolute value of sensitivity still has not been exhibited [22, 37]. One key obstacle is that the stability of quantum gyroscope is challenged by the decoherence caused by inevitable noise in microscopic world, which generally makes the quantum resources degraded. It was found that the metrology sensitivity using entanglement [42–44] and squeezing [45, 46] exclusively returns to or even becomes worth than the SNL and thus their quantum superiority completely disappears when the photon loss is considered. This is called the no-go theorem of noisy quantum metrology [47, 48] and is one difficulty to achieve a high-precision quantum gyroscope in practice.

In this Letter, we propose a scheme of quantum optical gyroscope (QOG) and discover a mechanism to overcome the constraint of the no-go theorem on our scheme. A super-Heisenberg limit (HL) on the sensitivity is achieved in the ideal case by using two-mode squeezed vacuum state. Our exact analysis on the non-Markovian photon dissipation reveals that the performance of the QOG in the realistic noise intrinsically depends on the energy-spectrum feature of the total system formed by the probe and its environments. The ideal precision is asymptotically recovered and the no-go theorem is avoided when each optical field forms a bound state with its environment. It supplies us a guideline to engineer the optimal working condition of our QOG in the realistic noise.

**Ideal QOG scheme.**—To measure a physical quantity of certain system, three processes, i.e., the initialization of the quantum probe, the quantity encoding via the probe-system coupling, and the measurement, are generally required. In our QOG, we choose two beams of quantized optical fields as the quantum probe. They propa-
gating in opposite directions are input into a 50:50 beam splitter and split into clockwise and counter-clockwise propagating beams [see Fig. 1(a)]. The setup rotates with an angular velocity \( \Omega \) about the axis perpendicular to its plane. Thus the two beams accumulate a phase difference \( \Delta \Omega = N \pi kr^2 \Omega / c \) when they reencounter the beam splitter after \( N \) rounds of propagation in the circular path [49]. Here \( k \) is the wave vector, \( c \) is the speed of light, and \( R \) is the radius of the QOG. Remembering the standing-wave condition \( kR = n (n \in \mathbb{Z}) \) of the optical fields propagating along the circular path and defining \( \Delta \omega \equiv \Delta \theta / \Delta t = 2n\Omega \). Therefore, the QOG can be equivalently treated as two counter-propagating optical fields with a frequency difference \( \Delta \omega \) along the circular path. For concreteness, we choose the basic mode \( n = 1 \). Then the optical fields in the QOG can be quantum mechanically described by \( (h = 1) [50] \)

\[
\hat{H}_S = \omega_0 \sum_{l=1,2} \hat{a}_l^\dagger \hat{a}_l + \Omega (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2). \tag{1}
\]

where \( \hat{a}_l \) is the annihilation operator of the \( l \)th field with frequency \( \omega_0 \). The optical fields couple to the beam splitter twice and output in the state \( |\Psi_{out}\rangle = V\hat{U}_0(\Omega, t)V^\dagger |\Psi_{in}\rangle \), where \( \hat{U}_0(\Omega, t) = \exp(-i\hat{H}_S t) \) is the evolution operator of the fields and \( V = \exp[i\frac{\pi}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)] \) describes the action of the beam splitter. Thus the angular velocity \( \Omega \) is encoded into the state \( |\Psi_{out}\rangle \) of the optical probe via the unitary evolution.

To exhibit the quantum superiority, we employ two-mode squeezed vacuum state as the input state \( |\Psi_{in}\rangle = \mathcal{S}(0,0) \), where \( \mathcal{S} = \exp[r(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2)] \) is the squeeze operator with \( r \) being the squeeze parameter. The total photon number of this input state is \( N = 2 \sinh^2 r \), which is the quantum resource of our scheme. The parity operator \( \hat{P} = \exp(i\pi\hat{a}_1^\dagger \hat{a}_1) \) is measured at the output port [51]. To the output state \( |\Psi_{out}\rangle \), we can calculate \( \mathcal{P} = \langle \Psi_{out}|\hat{P}|\Psi_{out}\rangle = [1 + N (2 + N) \cos^2(2\Omega t)]^{-1/2} \) and \( \delta \Pi = (1 - \hat{P}^2)^{1/2} \), where \( \hat{P}^2 = 1 \) has been used. Then the sensitivity of sensing \( \Omega \) can be evaluated via the error propagation formula \( \delta \Omega = \delta \Pi / \langle \hat{P} \rangle \) as

\[
\min\delta \Omega = \left[ 2t\sqrt{N(2 + N)} \right]^{-1}, \tag{2}
\]

when \( \Omega t = (2n + 1)\pi / 4 \) with \( n \in \mathbb{Z} \). It is remarkable to find that the best sensing error achieved in our scheme is even smaller than the HL \( \Delta \Omega \equiv (tN)^{-1} \), which reflects the absolute superiority of the used squeezing and measured observable in our QOG. It can be verified that this measurement scheme saturates the Cramér–Rao bound governed by quantum Fisher information. We call such metrology sensitivity surpassing the HL the super-\( \mathcal{H} \) [51–53]. The outstanding performance of quantum squeezing has been found in gravitational wave detection [54–56].

Effects of dissipative environments. —The superiority of quantum sensor is challenged by the decoherence of the quantum probe due to the inevitable interactions with its environment. Depending on whether the probe has energy exchange with the environment or not, the decoherence can be classified into dissipation and dephasing. The main decoherence in our QOG is the photon dissipation. The previous works phenomenologically treat the photon dissipation by introducing an imperfect transmission to the beam splitter [42, 45, 46, 57–59], which is equivalent to the Born-Markovian approximate description. It has been found that the system-environment interplay caused by the inherent non-Markovian nature would induce diverse characters absent in the Born-Markovian approximate [60–67]. To reveal the practical performance of our QOG, we, going beyond the Born-Markovian approximation and paying special attention to the non-Markovian effect, investigate the impact of the photon dissipation on the scheme.

We consider that the encoding process is influenced by two independent dissipative environments of the two optical fields. The Hamiltonian of the total system is

\[
\hat{H} = \hat{H}_S + \sum_{l=1,2} \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_{k,l} + g_{k,l} (\hat{a}_l^\dagger \hat{b}_{k,l} + \text{H.c.}), \tag{3}
\]

where \( \hat{b}_{k,l} \) is the annihilation operator of the \( k \)th environmental mode with frequency \( \omega_k \) felt by the \( l \)th optical field and \( g_{k,l} \) is their coupling strength. The coupling is further characterized by the spectral density \( J_l(\omega) = \sum_k g_{k,l}^2 \delta(\omega - \omega_k) \) in the continuous limit of the environmental frequencies. We consider the Ohmic-family spectral density \( J_l(\omega) = J_0(\omega) \equiv J(\omega) = n_\omega \omega^{s-1} e^{-\omega/\omega_c} \) for both environments, where \( n_\omega \) is a dimensionless coupling constant, \( \omega_c \) is a cutoff frequency, and \( s \) is an Ohmcity index [68]. Under the condition that the environments are initially in the vacuum state, we can derive an exact master equation for the encoding process using the Feynman-Vernon influence functional method [60, 69, 70]

\[
\dot{\rho}(t) = \sum_{l=1,2} \left\{ -i [\omega_l (\hat{a}_l^\dagger \hat{a}_l + 1)] \rho(t) + \gamma_l(t) \hat{D}_l \rho(t) \right\}, \tag{4}
\]

where \( \hat{D}_l = 2 \hat{a}_l^\dagger \hat{a}_l^\dagger - \hat{a}_l^\dagger \hat{a}_l - \hat{a}_l \hat{a}_l^\dagger \) is the Lindblad superoperator, \( \omega_l(t) = -i \text{Im}[\hat{u}_l(t)/\hat{u}_l(t)] \) is the renormalized frequency, and \( \gamma_l(t) = -\text{Re}[\hat{u}_l(t)/\hat{u}_l(t)] \) is the dissipation rate. The time-dependent functions \( u_l(t) \) satisfy

\[
\dot{u}_l(t) + i \omega_l u_l(t) + \int_0^t f(t - \tau) u_l(\tau) d\tau = 0, \tag{5}
\]

under \( u_l(0) = 1 \), where \( \omega_{1,2} = \omega_0 \pm \Omega \) and \( f(x) = \int_0^\infty J_0(\omega) e^{-i\omega x} d\omega \) is the environmental correlation function. Equation (4) indicates that all the non-Markovian effects induced by the environmental backactions have been incorporated into these time-dependent coefficients self-consistently. Solving Eq. (4), we obtain [71]

\[
\bar{\Pi}(t) = \langle x^4 m_1^2 \delta^2 + 4 \delta_p (m_1^2 - \delta p_1^2) + (1 - p_1 p_2)^2 + 16 |m_1 m_2|^2 \rangle \tag{6}
\]
where \( x = (\sqrt{A_1A_2} \cosh^2 r)^{-1} \), \( m_l = \frac{-i u(t)^2 \tanh r}{Z_{l}} \), and \( p_l = |u(t)|^2 (1-A_l^{-1}) \), with \( A_l = 1-(|u(t)|^2-1)^2 \tanh^2 r \). The analytical form of \( \delta \Omega \) can then be calculated in the similar manner as the ideal case.

In the special case when the probe-environment coupling is weak and the time scale of \( f(t-\tau) \) is smaller than the typical time scale of the probe, we can apply the Born-Markovian approximation in Eqs. (5) [42, 58]. Their approximate solutions read \( u_{BA}(t) = e^{-[\kappa_l + i(\omega_l + \Delta(\omega_l))t]} \), with \( \kappa_l = \pi J(\omega_l) \) and \( \Delta(\omega_l) = \mathcal{P} \int_0^\infty \frac{J(\omega)}{\omega^2 - \omega_I^2} d\omega \) [72]. Substituting them into Eq. (6) and using the error propagation formula, we obtain [71]

\[
\delta \Omega_{BA}(t) = \frac{(2e^{2\kappa_l} + nC e^{-2\kappa_l})\sqrt{C}}{\sqrt{8N(N+2)^2|\sin(4\Omega t)|}},
\]

where \( C = 4e^{2\kappa_l} + N - 2 + (N + 2) \cos(\Omega t) \). Here we have chosen \( \kappa_1 = \kappa_2 \equiv \kappa \). We plot in Fig. 1(b) the evolution of \( \delta \Omega_{BA}(t) \). It can be found that \( \delta \Omega_{BA}(t) \) experiences an obvious oscillation with time. However, the best sensitivity manifested by the profile of its local minima tends to be divergent with time. Thus, being in sharp contrast to the ideal case in Eq. (2), the superiority of time as a resource in enhancing the precision of the QOG disappears. After optimizing the encoding time, we obtain the global minimum \( \Omega \) [see the red dot in Fig. 1(b)]. The numerical fitting reveals \( \min \delta \Omega = 5.44N^{-0.23} \) [see Fig. 1(c)], which is even worse than the SNL. Therefore, being consistent with the previous quantum sensing schemes [42, 45, 46, 57–59], the photon dissipation under the Born-Markovian approximation makes the quantum advantages of our scheme completely vanish. It is called the no-go theorem of noisy quantum metrology [47, 48] and is the main obstacle to achieve a high-precision quantum sensing in practice.

In the general non-Markovian case, the analytical solution of Eq. (5) can be found by the method of Laplace transform, which converts Eq. (5) into \( \tilde{u}_l(z_l) = \int_0^\infty \frac{J(\omega) d\omega}{\omega^2 - \omega_I^2} \). Then \( u(t) \) is obtained by making the inverse Laplace transform to \( \tilde{u}_l(z_l) \), which can be done by finding its poles from

\[
Y_l(E_l) \equiv -\omega_l - \int_0^\infty \frac{J(\omega)}{\omega - E_l} d\omega = E_l, (E_l = iz_l).
\]

Here, \( E_l \) are also the eigenenergies in the single-excitation subspace of the total systems formed by each optical field and its environment. To see this, we expand the eigenstate as \( |\Phi_l\rangle = (x |\alpha_1\rangle + \sum_k y_k |\beta_k\rangle) |0, \{0_{k,l}\} \rangle \). From the stationary Schrödinger equation, we have \( |E_t - (\omega_l \pm \Omega)| x_l = \sum_k y_k x_l k_l \) and \( y_{k,l} = h_{k,l} x_l / (E_l - h_{k,l}) \) with \( E_l \) being the eigenenergies. The two equations readily result in Eq. (8) in the continuous limit of the environmental frequencies. It implies that the dissipation of the optical probe is intrinsically determined by the energy-spectrum character of the probe-environment system in the single-excitation subspace, even though the subspaces with any excitation numbers are involved. Due to \( Y_l(E_l) \) are decreasing functions in the regime \( E_l < 0 \), each of Eqs. (8) has one isolated root \( E_{b,l} \) in this regime provided \( Y_l(0) < 0 \). While \( Y_l(E_l) \) are not well analytic when \( E_l > 0 \), thus they have infinite roots in this regime, which form a continuous energy band. We call the eigenstates of the isolated eigenenergies \( E_{b,l} \) bound states [61]. Making the inverse Laplace transform, we obtain \( u_l(t) = Z_l e^{-iE_{b,l} t} + \int_0^\infty \Theta(E) e^{-iE t} dE \), where \( Z_l = \int_0^\infty \frac{J(\omega) d\omega}{(E_l - \omega)^2} \). The integral in \( u_l(t) \) is from the energy band and tends to zero in the long-time limit due to the out-of-phase interference. Thus, when the bound state is formed, we have \( \lim_{t \to \infty} u_t = Z_l e^{-iE_{b,l} t} \), characterizing the suppressed dissipation, otherwise, we have \( \lim_{t \to \infty} u_t = 0 \), meaning a complete dissipation. It can be determined that the bound state is formed for the Ohmic-family spectral density when \( \omega_c < \eta \omega_c \Gamma(s) \), where \( \Gamma(s) \) is the Euler’s \( \Gamma \) function.

We have three parameter regimes where zero, one, and two bound states are formed, respectively. It is natural to expect that \( \delta \Omega \) in the former two regimes is qualitatively consistent with the Born-Markovian approximate result (7) due to the complete dissipation in either two or one optical fields. Focusing on the case in the presence of two bound states and substituting the asymptotic solution \( Z_l e^{-iE_{b,l} t} \) into Eq. (6), we obtain [71]

\[
\lim_{t \to \infty} \delta \Omega(t) = \frac{F \sqrt{2F - 4}}{2N(2 + N)} \left[ \frac{\partial_3 Z_1^2 - Z_2^2 - 1}{4 + 2N} + Z_2^2 Z_1^2 \left[ (Z_1 + Z_2) \sin(2G t) - 2 \partial_3 \ln(Z_1 Z_2) \cos^2(G t) \right] \right]^{-1},
\]

where \( F = 2 + N \sum_i Z_i^2 (2 - Z_i^2) + N Z_2^2 Z_1^2 [N + (2 + N) \cos(2G t)] + G = E_{b_1} - E_{b_2} \). We have used \( \partial_3 E_{b,l} = (-1)^{l-1} Z_l \) derived from Eq. (8). It is observed from Eq. (9) that \( \lim_{t \to \infty} \delta \Omega(t) \) is dominated by the second term, which matches as perfectly as its ideal case \( t^{-1} \) of Eq. (2). Therefore, the formation of two bound states overcomes the problem of no-go theorem and retrieve the encoding time as a resource to enhance the sensitivity.

**Numerical results.**—We now numerically verify our general result by choosing the Ohmic spectral density. Figure 2(a) shows the energy spectrum of the total system consisting of the optical fields and their environment. It can be seen that the two branches of bound states divide the energy spectrum into three regimes: without bound state when \( \omega_c < 19.8 \omega_0 \), one bound state when \( \omega_c \in (19.8, 20.2) \omega_0 \), and two bound states when \( \omega_c > 20.2 \omega_0 \). The result confirms our analytical criterion that the bound states are formed when \( \omega_c > \omega_c / \eta \Gamma (s) \). Numerically solving Eq. (5) and using Eq. (6), we obtain the exact evolution of \( \delta \Omega(t) \) in the three regimes. When no or one bound state is formed, the local-minima profile of \( \delta \Omega(t) \) tends to diverge in the long-time limit and
the quantum superiority of the scheme completely disappears [see Figs. 2(b) and 2(c)], which is qualitatively similar to the Markovian result. However, as long as two bound states are formed, the profile of the local minima becomes a decreasing function of the encoding time. The matching of the numerical result with the long-time behavior (9) verifies the validity of the result in (9). Thus, the encoding time as a resource in sensing $\Omega$ is recovered as perfectly as the ideal case by the formation of the two bound states.

Figure 3(a) shows the evolution of the local minima of $\delta \Omega(t)$ in Eq. (9) in different $\omega_c$ when two bound states are formed. The formation of the bound state causes the abrupt increase of the corresponding $|u(t)\rangle$ from zero to a finite value exactly matching with $Z_t$ [see the inset of Fig. 3(b)]. It is interesting to find that not only the encoding time as a resource is retrieved, but also the ideal precision is asymptotically recovered. This is double confirmed by the long-time behavior of min $\delta \Omega(t)$ as a function of the photon number $N$ in Fig. 3(b). The similar performance is found by changing the coupling constant $\eta$ (see Fig. 4). All the results demonstrate the constructive role played by the two bound states and the non-Markovian effect in retrieving the quantum superiority of our QOG. It offers us a guideline to achieve a noise-tolerant QOG by manipulating the formation of the bound states. It is noted that, according to the condition of forming the bound states, we see that what really matters is the relative value $\omega_c/\omega_0$. The equivalent result is achievable by tuning $\omega_0$ for given $\omega_c$ and $\eta$.

Discussion and conclusions.—Our scheme is independent of the form of the spectral density. Although only the Ohmic spectral density is considered, our scheme can be generalized to other spectra. Given the rich way in controlling the spectral density in the setting of the technique of quantum reservoir engineering [73–77], we deem that our scheme is realizable in the state-of-the-art quantum-optical experiments. Actually, the non-Markovian effect has been observed in the linear optical systems [78, 79]. The bound state and its dynamical effects have been observed in circuit QED [80] and ultracold atom [81] systems. These experimental advances provide a support for the feasibility of our scheme.

In summary, we have proposed a QOG scheme by using two quantized fields as quantum probe, which achieves a super-HL sensitivity in measuring the angular velocity. However, the photon dissipation under the convention...
tional Born-Markovian approximation forces this sensitivity even being worse than the classical SNL. To overcome this problem, we have presented a mechanism to retrieve the ideal sensitivity by relaxing this approximation. It is found that the ideal sensitivity is asymptotically recoverable when each optical field forms a bound state with its environment, which can be realized by the technique of quantum reservoir engineering. Exhibiting the optimal working condition of QOG, our mechanism breaks through the constraint of the no-go theorem of noisy quantum metrology and supplies a guideline in developing high-precision rotation sensing for next-generation inertial navigation systems.

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Supplemental material for “Quantum optical gyroscope”

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**EXPECTATION VALUE OF PARITY OPERATOR**

In this section, we give the derivation of Eq. (6) in the main text. The influence-functional theory of Feynman and Vernon enables us to derive the evolution of the reduced density matrix of the quantum probe formed by two quantized optical fields exactly [1]. By expressing the forward and backward evolution operators of the density matrix of the probe and the environments as a double path integral in the coherent-state representation, and performing the integration over the environmental degrees of freedom, we incorporate all the environmental effects on the probe in a functional integral named influence functional. The reduced density matrix fully describing the encoding dynamics of the probe is given by [2, 3]

$$
\rho(\alpha_f, \alpha'_f; t) = \int d\mu(\alpha_i) d\mu(\alpha'_i) J(\alpha_f, \alpha'_f; t|\alpha_i, \alpha'_i; 0) \times \rho(\alpha_i, \alpha'_i; 0),
$$

(S1)

where \(\rho(\alpha_f, \alpha'_f; t) = \langle \alpha_f | \rho(t) | \alpha'_f \rangle\) is the reduced density matrix expressed in coherent-state representation and \(J(\alpha_f, \alpha'_f; t|\alpha_i, \alpha'_i; 0)\) is the propagating function. In the derivation of Eq. (S1), we have used the coherent-state representation \(|\alpha\rangle = \prod_{i=1}^{2} |\alpha_i \rangle\) with \(|\alpha_i \rangle = \exp(\alpha_i a_i^\dagger)|0\rangle\), which are the eigenstates of annihilation operators, i.e. \(\hat{a}_i^\dagger |\alpha_i \rangle = \alpha_i |\alpha_i \rangle\) and obey the resolution of identity,

$$
\int d\mu(\alpha) |\alpha\rangle \langle \alpha| = 1 \quad \text{with} \quad d\mu(\alpha) = \prod_{i=1}^{2} e^{-\alpha_i^2} \frac{d\alpha_i d\alpha_i^\dagger}{2\pi r}, \quad \alpha \text{ denotes the complex conjugate of } \alpha.
$$

The time evolution of the reduced density matrix is determined by the propagating function \(J(\alpha_f, \alpha'_f; t|\alpha_i, \alpha'_i; 0)\). The propagating function is expressed as the path integral governed by an effective action which consists of the free actions of the forward and backward propagators of the optical probe and the influence functional obtained from the integration of environmental degrees of freedom. After evaluation of the path integral, the final form of the propagating function is obtained as follows

$$
J(\alpha_f, \alpha'_f; t|\alpha_i, \alpha'_i; 0) = \exp \left\{ \sum_{i=1}^{2} \left[ u_i(t) \alpha_i^\dagger \alpha_i + \bar{u}_i(t) \alpha_i^\dagger \alpha_i^\dagger + 1 - |u_i(t)|^2 |\alpha_i^\dagger \alpha_i| \right] \right\},
$$

(S2)

where \(u_i(t)\) satisfies

$$
\dot{u}_i(t) + i\omega u_i(t) + \int_{0}^{t} f(t-\tau) u_i(\tau) = 0
$$

(S3)

with \(f(x) = \int_{0}^{\infty} J(\omega) e^{-i\omega x} d\omega\) and \(u_i(0) = 1\). We have assumed that the spectral density of the two environments are identical.

The input state of the probe is a two-mode squeezed vacuum state \(|\Psi_{in}\rangle = \exp[r(\alpha_1^\dagger \alpha_2^\dagger - \alpha_1^\dagger \alpha_2)]|0\rangle\), where \(r\) is the squeezing parameter. After passing the first beam splitter of the quantum optical gyroscope, the state changes into \(|\Psi(0)\rangle \equiv |V\rangle |\Psi_{in}\rangle\), with \(V = \exp[i \frac{r}{2} (\alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2)]\), which acts as the initial state of the encoding dynamics. In the coherent-state representation, this initial state is given by

$$
\rho(\alpha_i, \alpha'_i; 0) = \frac{1}{\cosh^2 r} \exp[-i \frac{\tanh r}{2} \sum_l (\alpha_i^2 - \alpha_i^4)],
$$

(S4)

The time-dependent reduced density matrix is obtained by integrating the propagating function over the initial state of Eq. (S1). It reads

$$
\rho(\alpha_f, \alpha'_f; t) = x \exp \left\{ \sum_l \left( m_l \alpha_i^2 + \bar{m}_l \alpha_i^4 + p_l \alpha_i^2 \alpha'_i \right) \right\}
$$

(S5)

where \(x = (\sqrt{A_1 A_2} \cosh r)^{-1}\), \(m_l = -\frac{i u_l(t)^2 \tanh r}{2 A_l}\), and \(p_l = |u_l(t)|^2 (1 - A_l^{-1})\), with \(A_l = 1 - (|u_l(t)|^2 - 1)^2 \tanh^2 r\). Remembering \(\rho(t) = \int d\mu(\alpha_f) d\mu(\alpha'_f) \rho(\alpha_f, \alpha'_f; t) |\alpha_{i1}, \alpha_{i2}\rangle \langle \alpha_{i1}^\dagger, \alpha_{i2}^\dagger|\alpha_f^\dagger, \alpha_f^\dagger\rangle\) and \(\rho_{out} = \hat{V} \rho(t) \hat{V}^\dagger\), we obtain

$$
\rho_{out} = \int d\mu(\alpha_f) d\mu(\alpha'_f) \rho(\alpha_f, \alpha'_f; t) \times |\alpha_{i1} + i\alpha_{i2} \rangle \langle \alpha_{i1} + i\alpha_{i2}| \langle \alpha_{i1}^\dagger - i\alpha_{i2}^\dagger, \alpha_{i1}^\dagger - i\alpha_{i2}^\dagger | \langle \alpha_{i1}^\dagger - i\alpha_{i2}^\dagger, \alpha_{i1}^\dagger - i\alpha_{i2}^\dagger|\alpha_f^\dagger, \alpha_f^\dagger\rangle.
$$

(S6)

Then the expectation value \(\bar{\Pi} = Tr[\hat{\Pi} \rho_{out}]\) of the parity operator \(\hat{\Pi} = \exp(i \pi \alpha_{i1}^\dagger \alpha_{i1})\) can be calculated as

$$
\bar{\Pi} = x [4m_1 (m_2^2 - m_1 p_2^2) + 4m_2 (m_1^2 - m_2 p_1^2)] + (1 - p_2) + 16|m_1 m_2|^2 - 1/2,
$$

(S7)

where \(\bar{\Pi}(\alpha, \beta) = |e^{i\pi \alpha, \beta}|\) has been used. The sensing sensitivity of \(\Omega\) is calculated by \(\delta \Omega = \frac{1}{|\bar{\Pi}|} \frac{\partial |\bar{\Pi}|}{\partial \Omega}\).

In the ideal limit, the solution of Eq. (S3) reads \(u_i(t) = \exp(-i\omega t)\) and thus \(A_l = 1, p_l = 0,\) and \(m_l = -\frac{i \omega^2}{2} \tanh r\). Then Eq. (S7) reduces to \(\bar{\Pi} = \frac{1}{2} + \frac{N(2 + N) \cosh^2(2\Omega t)}{2} - (N - 1)\) with \(N = 2 \sinh^2 r\).
SENSITIVITY UNDER THE BORN-MARKOVIAN APPROXIMATION

Defining \( u_l(t) = e^{-i\omega_l t}u_l'(t) \), we can rewritten Eq. (3) as

\[
\dot{u}_l(t) + \int_0^t d\tau \int_0^\infty d\omega J(\omega)e^{-i(\omega-\omega_l)(t-\tau)}u_l'(-\tau) = 0. \quad (S8)
\]

When the probe-environment coupling is weak and the time scale of the environmental correlation function is much smaller than the one of the probe, we can apply the Born-Markovian approximation to Eq. (S8) by neglecting the memory effect, i.e., \( u_l'(t) \approx u_l(t) \), and extending the upper limit of the integral to infinity, i.e. \( \int_0^\infty d\tau \approx \int_0^\infty d\tau \). The utilization of the identity \( \lim_{t \to \infty} \int_0^t d\tau e^{-i(\omega-\omega_0)(t-\tau)} = \pi \delta(\omega-\omega_0) + i\mathcal{P} \frac{1}{\omega-\omega_0} \), with \( \mathcal{P} \) being the Cauchy principal value, results in \( u_{l,MA}'(t) = e^{-i\kappa t}\Delta(\omega_l)l(t) \), where \( \kappa_l = \pi J(\omega_l) \) and \( \Delta(\omega_l) = \mathcal{P} \int_0^\infty J(\omega)l(\omega-\omega_0)dw \). We thus have the Born-Markovian approximate solution of \( u_l(t) \) as \( u_{l,MA}(t) = e^{-i\kappa t}\Delta(\omega_l)l(t) \) [4].

Substituting \( u_{l,MA}(t) \) into Eq. (S7) and using the error propagation formula, we analytically obtain the sensitivity under the Born-Markovian approximation as

\[
\delta\Omega_{BA}(t) = \frac{(2e^{2\kappa c} + nC e^{-2\kappa c})\sqrt{C}}{\sqrt{8N(N+2)t} \sin(4\Omega t)}, \quad (S9)
\]

where \( C = 4e^{2\kappa c} + N - 2 + (N + 2)\cos(4\Omega t) \). Here we have chosen \( \kappa_1 = \kappa_2 \equiv \kappa \) and neglected the constant \( \Delta(\omega_l) \), which is generally renormalized into \( \omega_0 \) [5]. We readily see from Eq. (S9) that the sensitivity under the Born-Markovian approximation tend to be divergent in the long-time limit.

SENSITIVITY IN THE NON-MARKOVIAN DYNAMICS

In the non-Markovian case, Eq. (3) can be analytically solvable by the method of Laplace transform, which converts Eq. (3) into \( \dot{u}_l(z_l) = [z_l + i\omega_l + \int_0^\infty J(\omega)l(\omega-\omega_l)dw]^{-1} \). Then \( u_l(t) \) is obtained by applying the inverse Laplace transform on \( \dot{u}_l(z_l) \), we obtain

\[
u_l(t) = \frac{1}{2\pi i} \int_{\sigma+i\infty}^{\sigma-i\infty} \frac{e^{-E_l t}dE}{E_l - \omega_l + \int_0^\infty J(\omega)l(\omega-\omega_l)dw} \]

where \( E_l = iz_l \) and \( \sigma \) is chosen to be larger than all the poles of the integrand. Finding the pole of Eq. (S10) from

\[
Y_l(E_l) \equiv \omega_l - \int_0^\infty \frac{J(\omega)}{\omega - E_l}d\omega = E_l. \quad (S11)
\]

It is noted that \( E_l \) are also the eigenenergy in the single-excitation subspace of the total systems formed by each optical field and its environment. To see this, we expand the eigenstate of \( |\Phi_i\rangle = (x_i \partial_i^a + \sum_k y_{ik} \partial_k^b)l(0, \{q_{kl}\}) \).

From the stationary Schrödinger equation, we have \( |E_l - (\omega_l \pm \Omega)|x_l = \sum_k y_{ik} \partial_k^b\partial_k^b |E_l - \omega_{kl}\rangle \) with \( E_l \) being the eigenenergy. These two equations readily result in Eq. (S11) in the continuous limit of the environmental frequencies. According to the residue theorem, we have

\[
u_l(t) = Z_l e^{-iE_l t} + \int_0^\infty \Theta(E)e^{-iE_l t}dE. \quad (S12)
\]

where \( Z_l = [1 + \int_0^\infty \frac{J(\omega)}{(E_l - \omega)^2}d\omega]^{-1} \) and \( \Theta(E) = \frac{J(E)}{|E_l - \omega - \Delta(E)|^2 + |\pi J(E)|^2} \). The first and the second terms of Eq. (S12) are the residues contributed from the poles of Eq. (S11) in the regime \( E_l < 0 \) and \( E_l > 0 \), respectively. It can be found that \( Y_l(E_l) \) are decreasing functions in the regime \( E_l < 0 \), each of Eqs. (S11) has one isolated root \( E_{kl,l} \) in this regime provided \( Y_l(0) < 0 \). While \( Y_l(E_l) \) are not well analytic in the regime \( E_l > 0 \), they have infinite roots in this regime, which form a continuous energy band. We call the eigenstates of the isolated eigenenergies \( E_{kl,l} \) bound states.

Due to the out-of-phase interference, the second term in Eq. (S12) tends to vanish in the long-time limit. Therefore, we have the asymptotical solution of Eq. (S12) as

\[
\lim_{t \to \infty} \nu_l(t) = \begin{cases} 0, & Y_l(0) \geq 0 \\ Z_l e^{-iE_l t}, & Y_l(0) < 0 \end{cases}. \quad (S13)
\]

Since the dominated role played by \( u_l(t) \) in the encoding dynamics of the quantum probe, the two qualitatively different asymptotical behaviors in Eq. (S13) manifest the significance of the bound states in determining the photon dissipation. The former case of Eq. (S12) characterizes a complete dissipation, while the latter case denotes a suppressed dissipation.

Choosing on the case in the presence of two bound states and substituting the asymptotic solution \( \lim_{t \to \infty} \nu_l(t) = Z_l e^{-iE_l t} \) into Eq. (S7), we obtain

\[
\lim_{t \to \infty} \delta\Omega(t) = \frac{F \sqrt{2F - 4} \left| \partial_\Omega[Z_1^2 + Z_2^2 - 1] \right|^2}{2N(2 + N)} + Z_1^2 Z_2^2 [t(Z_1 + Z_2) \sin(2Gt) - 2\partial_\Omega \ln(Z_1 Z_2) \cos^2(Gt)] \bigg|^{-1}, \quad (S14)
\]

where \( F = 2 + N \sum Z_k^2 (2 - Z_k^2) + N Z_1 Z_2 [N + (2 + N) \cos(2Gt)] \) with \( G = E_{b1} - E_{b2} \). We have used \( \partial_\Omega Z_{1/2} = Z_1 \) and \( \partial_\Omega Z_{3/2} = -Z_2 \) derived from Eq. (S11). It can be observed that Eq. (S14) is dominated by the second term, which behaves as \( t^{-1} \), in the large-time limit due to the time-independence of its first and the third terms. It implies that the problem that the sensing error asymptotically tends to be divergent in the Born-Markovian approximation is overcome.
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