Extra-dimensional models provide a fertile approach to a broad class of problems in the theory of elementary particles and cosmology. In the brane-world model building the question of paramount importance is the localization of the Standard Model particles on the brane. We continue our attempt to put forward the theoretical framework for describing the dynamics of particle escape into extra space. The troublesome aspect of our approach presented in [2] has to do with the time evolution of extra-dimensional part of the scalar field. Namely, it does not preserve the norm of the metastable state and therefore contradicts the wave function interpretation. Let us strive to maintain this interpretation of extra-dimensional part of the field as underpinning for the brane-world model and try to describe the time evolution of the state encoded in the Schrödinger equation. To gain a complete picture it should be noticed that for the Gamow’s method the concrete form of the wave function satisfying the outgoing wave boundary condition. To gain a complete picture it should be noticed that for the Gamow’s method the concrete form of the time evolution of the state encoded in the Schrödinger equation is a crucial moment. So following the Gamow’s method it seems reasonable to take the following ans"{a}tze as a satisfactory point of departure

\[ \psi = e^{i(E_{0}t - \frac{p^{2}}{2m})} g(t, z), \]

where \( g(t, z) \) stands for a free particle with the mass \( m_{0} \) moving in \( x^{\mu} \) space along \( \vec{p} \) while \( g(t, z) \) describes its localization properties across the brane which, in general, may vary in course of time. For this ans"{a}tze one gets the following equation

\[ -\partial_{z}^{2}\psi + 4k \text{sgn}(z) \partial_{z}\psi + \mu^{2}\psi = e^{2k|z|} ((E - E_{0})^{2} - p^{2}) \psi, \]

where \( \mu \) is the bulk mass parameter [4]. Following the paper under assumption \( \mu \ll k \) one finds without much ado

\[ E = 2E_{0} \frac{\sqrt{m_{0}\Gamma}}{\sqrt{\mu^{2} + m_{0}^{2}}} , \quad \Gamma \frac{m_{0}}{m_{0}} = \sqrt{\frac{m_{0}}{\sqrt{\mu^{2} + m_{0}^{2}}}}, \quad m_{0} = \frac{\mu}{\sqrt{2}}, \]

and the corresponding wave function satisfying the outgoing wave boundary condition. In this way one finds that the probability of finding the particle on the brane decays exponentially in time and the corresponding lifetime is given by \( \sim \sqrt{\mu^{2} + m_{0}^{2}}/m_{0}\Gamma}.\]

But, as it is well known the precise quantum mechanical consideration of metastable state decay based on the approach proposed by Fock and Krylov [3] leads to the

\[ \langle g|f \rangle = \int_{-\infty}^{\infty} dz e^{-2k|z|} g^{*}(z)f(z). \]
deviation from exponential decay law for small and large values of time. More precisely in quantum mechanics it is well established that an exponential decay cannot last forever if the Hamiltonian is bounded below and cannot occur for small times if, besides that, the energy expectation value of the initial state is finite \[6\]. Let us consider this problem for the brane-world model following to the general ansatz

\[
\phi = e^{i(E_0 t - \beta z)} g(t, z).
\]

The equation of motion for

\[
f(t, z) = e^{iE_0 t} g(t, z)
\]

takes the form

\[
(\partial_t^2 + p^2) f - e^{2k|z|} \partial_z (e^{-4k|z|} \partial_z f) + e^{-2k|z|} \mu^2 f = 0,
\]

the general solution for which is given by \[5\]

\[
f(t, z) = \int_{-\infty}^{\infty} G_1(t, z, z') f(0, z') dz' + \int_{-\infty}^{\infty} G_2(t, z, z') f(0, z') dz',
\]

where

\[
G_1(t, z, z') = e^{-2k|z'|} \int_{E}^{\infty} dE \cos(E \tau) \varphi_p(E, z) \varphi_p(E, z'),
\]

\[
G_2(t, z, z') = e^{-2k|z'|} \int_{E}^{\infty} dE \frac{\sin(E \tau)}{E} \varphi_p(E, z) \varphi_p(E, z'),
\]

and the functions

\[
\varphi_p(E, z) = \sqrt{\frac{E}{m}} \varphi_m(z),
\]

with \(m = (E^2 - p^2)^{1/2}\) are determined by the normalized eigenfunctions of

\[- \partial_z^2 \varphi + 4k \text{sgn}(z) \partial_z \varphi + \mu^2 \varphi = e^{2k|z|} m^2 \varphi, \tag{7}\]

satisfying the junction condition across the brane

\[
\varphi = \sqrt{\frac{2k}{2}} e^{2k|z|} \left[ a(m) J_\nu \left( \frac{m}{k} e^{k|z|} \right) + b(m) Y_\nu \left( \frac{m}{k} e^{k|z|} \right) \right],
\]

with the index \(\nu = \sqrt{1 + \mu^2 / k^2}\) and coefficients

\[
a(m) = - \frac{A(m)}{\sqrt{1 + A^2(m)}}, \quad b(m) = \frac{1}{\sqrt{1 + A^2(m)}},
\]

\[
A(m) = \frac{Y_{\nu-1}(m/k) - (\nu - 2)(k/m) Y_{\nu}(m/k)}{J_{\nu-1}(m/k) - (\nu - 2)(k/m) J_{\nu}(m/k)}.
\]

To compute \(f(t, z)\) one needs to know the initial data \(f(0, z), \dot{f}(0, z)\). In general, arbitrarily taking the initial velocity \(\dot{f}(0, z)\) we face the problem that the norm of \(f(t, z)\) may not be preserved \[1\].

With the above comment in mind, to keep as close as possible to the time dependence of Schrödinger equation, the further insight into the evaluation of escape dynamics can be gained by considering the following solution

\[
g(t, z) = e^{-iE_0 t} \int_{p}^{\infty} dE e^{-iEt} \varphi_p(E, z) \langle \varphi_p(E)|g(0)\rangle, \tag{8}\]

where \(g(0, z)\) denotes the brane localized initial state. So that the resulting prescription for evaluating the probability of particle to be confined at instant \(t\) on the brane

\[|\langle g(0)|g(t)\rangle|^2,\]

is similar to the quantum mechanical one considered in \[3\]. Correspondingly the time evolution of the decay can be divided into three domains as in the quantum mechanical case indicated above. Under assumption \(\mu / k \ll 1\) one finds that the function \(|C(E)|^2\) where

\[
C(E) = \langle \varphi_p(E)|g(0)\rangle,
\]

has a simple pole in the fourth quadrant of a complex \(E\) plane \[4\]. \[4\]

\[
\tilde{E} = \sqrt{\beta^2 + m_0^2} - \frac{im_0 \Gamma}{\sqrt{\beta^2 + m_0^2}}.
\]

Since, when \(t > 0\) and \(|E| \to \infty\), \(e^{-iEt} \to 0\) in the fourth quadrant, for evaluating of the transition amplitude

\[
\langle g(0)|g(t)\rangle = e^{-iE_0 t} \int_{p}^{\infty} dE e^{-iEt} |C(E)|^2,
\]

is similar to the quantum mechanical one considered in \[3\]. Correspondingly the time evolution of the decay can be divided into three domains as in the quantum mechanical case indicated above. Under assumption \(\mu / k \ll 1\) one finds that the function \(|C(E)|^2\) where

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Since, when \(t > 0\) and \(|E| \to \infty\), \(e^{-iEt} \to 0\) in the fourth quadrant, for evaluating of the transition amplitude

\[
\langle g(0)|g(t)\rangle = e^{-iE_0 t} \int_{p}^{\infty} dE e^{-iEt} |C(E)|^2,
\]
Let us make a brief clarification of the above discussion. Usually in constructing the relativistic quantum mechanics the square root from the Hamiltonian of a relativistic free particle

\[ i\partial_t\psi = \sqrt{H}\psi , \quad (11) \]

is removed. The resulting Klein-Gordon equation

\[ -\partial_t^2\psi = H\psi , \quad (12) \]

is altered from the Schrödinger equation in that it contains the second order time derivative that in general proves impossible the probability interpretation of the solution. Knowing the eigenfunction spectrum

\[ H\varphi = E^2\varphi , \]

loosely speaking one can construct the solution of Eq. (11)

\[ \psi(t) = \int dE e^{-iEt}\langle \varphi(E)|\psi(0)\rangle , \]

satisfying the Eq. (12) as well. This solution admits a straightforward probability interpretation as in the Schrödinger case and is thereby unique. This is essentially what we have done in the present paper for describing the time evolution of the quasilocalized scalar particle on the brane. This brief paper corrects and complements our previous consideration [1].

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