$Q_6$ As The Flavour Symmetry in a Non-minimal SUSY $SU(5)$ Model

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Abstract

We present a non-minimal renormalizable SUSY $SU(5) \times U(1)$ model, with extended Higgs sector and right-handed neutrinos, where the flavour sector exhibits a $Q_6$ flavour symmetry. We analysed the simplest version of this model, in which R-parity is conserved and the right-handed neutrinos in the doublet are mass degenerate. We find the generic form of the mass matrices both in the quark and lepton sectors. We reproduce, according to current data, the mixing in the $CKM$ matrix. At the same time, in the leptonic side, the model predicts a strong inverted hierarchy spectrum and a sum rule among the neutrino masses. As main results, the atmospheric $\theta_{23}^{\text{th}} = 46.18^{+0.66}_{-0.65}$ and solar $\theta_{12}^{\text{th}} = 36.62 \pm 4.06$ mixing angle are calculated consistent with the experimental data. Also, the following value for the reactor mixing angle $\theta_{13}^{\text{th}} = 3.38^{+0.03}_{-0.02}$ is found. Although this value is small compared to the central value from the global fits, it is still fairly large in comparison to the tribimaximal scenario, and would correspond to a lower bound for $\theta_{13}$ in the case of the more general model with non-degenerate neutrino masses, the same as in $S_3$ models.

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1 Introduction

Understanding the flavour sector of the Standard Model (SM) has been a puzzle for a long time, due to the large differences in the Yukawa couplings of the different fermions. The hierarchy between the fundamental particles, the amount of CP violation and the structure of the CKM matrix remain as open [1]. In spite of these subtle facts, the success of the SM is remarkable.

One of the strategies to deal with the flavour problem in the quark and lepton sectors has been to study it in the framework of textures (zeros) in the mass matrices. The textures have been explored for a long time, as an attempt to eliminate the irrelevant free parameters in the Yukawa sector. As it is well known, the Fritzsch textures [2,3] can accommodate the quarks mixing angles, in terms of the quarks masses. This approach seems to work out correctly because the Cabbibo angle is obtained with great accuracy [2]. However, this framework presents some problems with the top mass and the $V_{cb}$ element of the CKM matrix [4,5], as can be seen in [6,7]. Recently, deviations to the Fritzsch textures have appeared in order to overcome these problems. Moreover, the charged lepton and neutrino sector have been included in this kind of ansatz and consistent results on the PMNS matrix [8,9] have been observed in this generic approximation [7]. As alternative textures to the Fritzsch ones, the Nearest Neighbour Interaction ($NNI$) [10,11] textures can also reproduce very well the flavour mixing in the quark and lepton sectors; it is well known that Fritzsch textures could be obtained from the $NNI$ ones as a limiting case [10].

Non-Abelian Flavour symmetries have played an important role in model building to obtain, in an elegant way, $NNI$ textures in the fermion mass matrices. In particular, the flavour symmetry group $Q_6$ [12] has been proposed as responsible of the textures in the quarks as well as in the lepton sector [13–17]. This appealing flavour symmetry allows the appearance of the $NNI$ textures in the quark and lepton sectors so that the mixings are in good agreement with the data [13–17]. The rich phenomenology $Q_6$ provides in the SUSY scenario is remarkable, one of these features is that it prohibits the dangerous terms that mediate fast proton decay rather than invoking the $R$-parity symmetry [14,15]. The immediate question that arises is how the $Q_6$ symmetry can look within a GUT framework, in particular, in the SUSY $SU(5)$ model? The main question that we will address here is if the SUSY $SU(5)$ models are compatible with the $Q_6$ group in order to accommodate masses and mixings for fermions.

From a theoretical point of view, grand unified ideas [18–21] are well motivated for fundamental reasons. In particular, the $SU(5)$ model [21] is considered to be one of the best scenarios to unify the electroweak and strong interactions. However, the model itself faces serious phenomenological and theoretical problems, one of them being that active neutrinos are massless [22,23]. The simplicity of the $SU(5)$ can be retained even if it is promoted to be a supersymmetric model. The SUSY $SU(5)$ [24,26] version cures most of the above problems, for example, proton decay may be solved by introducing the already known discrete symmetry, $R$-parity, which prohibits dangerous terms that can mediate its fast decay. Moreover, recent generic studies about the minimal SUSY $SU(5)$ model have made it clear that it may not be ruled out by the current experimental data on the proton decay rates [27,28]. Taking into account the neutrino mass problem, the SUSY $SU(5)$ version (in general supersymmetric) provides elegant mechanisms to generate massive neutrinos via $R$-parity violation [29,30].
But still, the simplest way to give mass to the left-handed neutrinos in the minimal $SU(5)$ or SUSY scenario is to consider three right-handed neutrinos which are singlets under the gauge group and invoke the type I see-saw mechanism $\cite{31, 35}$.

There are interesting SUSY $SU(5)$ models $\cite{36, 37}$, that can very well reproduce the $CKM$ and $PMNS$ matrices in agreement with the experimental results, but where the simplicity in the matter content has been left aside. Most of these models include a large number of flavons which are required to accommodate correctly the mixings. A different approach consists of extending the Higgs sector of the models. By itself, the SUSY $SU(5)$ matter content provides the tools to accommodate masses and mixings via the extension of the Higgs scalar sector. This philosophy has been worked out with success in non-supersymmetric $\cite{38, 40}$ and supersymmetric $\cite{13, 17}$ scenarios where the concept of flavour has been extended to the Higgs sector.

Therefore, we propose here a renormalizable SUSY $SU(5)$ model where the $Q_6$ group plays an important role in the flavour sector. The need to extend the scalar sector is evident in order to accommodate masses and mixings for quarks and leptons without breaking explicitly the flavour symmetry, so that three families of $H^u$ and $H^d$ 5-plets and two gauge singlet scalars are introduced. The latter scalars provide mass to the right-handed neutrinos and the type I see-saw mechanism is invoked to get small masses for the active neutrinos. At low energies, consistent results are obtained for the $CKM$, and in the leptonic sector, the atmospheric $\theta_{12}^{\ell h} = 46.18^{+0.66}_{-0.65}$ and solar $\theta_{13}^{\ell h} = 36.62 \pm 4.06$ mixing angles are consistent with the experimental data. However, the reactor mixing angle value $\theta_{13}^{\ell h} = 3.38^{+0.03}_{-0.02}$, is not in good agreement with the central values of the global fits but it is still within the error bar of the different experiments, and fairly large in comparison to the tribimaximal scenario. This value corresponds to a lower bound for $\theta_{13}$ for the more general model where the right-handed neutrinos are not mass degenerate, similar to the case of $S_3$ non-supersymmetric models, where relaxing the degeneracy condition in the right-handed neutrino masses gives the right value for $\theta_{13}^{\ell h}$ $\cite{40}$.

The paper is organized as follows: In section 2, we build the extended SUSY $SU(5)$ model which obeys the $Q_6$ flavour symmetry, in addition, we explain the required matter content to get a unified scenario and we do stress the strong assumptions that we will make in the model. The textures in the quark mass matrices, and therefore their consequences in the lepton sector, are analyses in section 3. Also, we describe how to diagonalize the mass matrices for each sector in order to obtain the mixing matrices. In section 4, we present and discuss the results about the mixing angles for each sector. Finally, we give conclusions on this preliminary analysis of the model.

2 SUSY $SU(5) \otimes U(1) \otimes Q_6$ model

We will consider the SUSY $SU(5)$ model where three right-handed neutrinos have been included in the matter content. In addition, the scalar sector has to be necessarily extended as we will show. The assigned matter content under $Q_6$ flavour symmetry and $U(1)$ charges are displayed on Table 1. Let us comment on our notation and the matter content: $\phi$ and $\tilde{\phi}$ are singlet scalars under the gauge group, the former gives mass to the right-handed neutrinos and the latter is introduced in order to cancel anomalies in the $U(1)$ Abelian group.
In addition, $\phi$ breaks the SUSY $SU(5) \otimes U(1) \otimes Q_6$ gauge group into SUSY $SU(5) \otimes Q_6$; $\Phi^a_b$ stands for the 24 adjoint scalar representation which breaks the SUSY $SU(5)$ gauge group to the MSSM; there are three families of Higgs type $H_u^a$ and $H_d^b$. On the other hand, we do need to include $H_{45}$ and $\bar{H}_{45}$ scalar representations so as to fix the incorrect relation $M_d = M_{\nu e}^T$, although there is another way to achieve that see [27]. Regarding the fermion sector, $N_i$ denotes the right-handed neutrino, which is a singlet under the $SU(5)$ gauge group; $F_{i}$ and $T_{j}$ stand for the 5-plets and the 10 antisymmetric-plet, respectively. Here, $a,b,c$ are $SU(5)$ indices, and $i,j$ are family indices. More explicitly,

$$
F_{ia} = (d^c, L); \quad T_{j}^{ab} = \frac{1}{\sqrt{2}} (u^c, q, \ell^c); \quad L = \left( \begin{array} {c} \nu \ell \\ \ell \end{array} \right); \quad q = \left( \begin{array} {c} u \\ d \end{array} \right).
$$

$$
H^{ua} = \left( \begin{array} {c} H^u \\ H^a \end{array} \right); \quad H^d_b = \left( \begin{array} {c} H^d \\ H^d \end{array} \right); \quad H^u = \left( \begin{array} {c} h^{+u} \\ h^0_{ua} \end{array} \right); \quad H^d = \left( \begin{array} {c} h^{0d} \\ h^{-d} \end{array} \right).
$$

Here, $H^u$ and $H^d$ are the coloured triplet scalars that mediate proton decay. For the time being, it will be assumed that these are heavy enough to keep proton lifetime bounded and under control. In addition, we have to point out that this subtle issue will be left aside, since we are only interested in studying the masses and mixings when implementing $Q_6$ as a flavour symmetry in this model. Also, we do not dwell on the details of full gauge symmetry breaking from SUSY $SU(5) \otimes U(1) \otimes Q_6$ to the MSSM group. Such issues are part of a wider study of this particular model, that is still in progress. However, there are some ideas involving this line of thought, see for example [41–43]. On the other hand, $H^d$ and $H^u$ are identified as the weak doublets of the MSSM group. In consequence, we employ the usual pattern to break the SUSY $SU(5)$ model via the vacuum expectation values (vev’s) for every

| (5) | $Q_6$ | $U(1)$ |
|-----|--------|--------|
| $(H_u^1, H_d^1)$ | 5 | $2_1$ | $-x$ |
| $H_d^1$ | 5 | $1_{+2}$ | $-x$ |
| $(H_u^1, H_d^2)$ | 5 | $2_1$ | $x$ |
| $H_u^1$ | 5 | $1_{+2}$ | $x$ |
| $(F_1, F_2)$ | 5 | $2_2$ | $\frac{3x}{2}$ |
| $F_3$ | 5 | $1_{-3}$ | $\frac{3x}{2}$ |
| $(T_1, T_2)$ | 10 | $2_2$ | $-\frac{x}{2}$ |
| $T_3$ | 10 | $1_{-3}$ | $-\frac{x}{2}$ |
| $(N^c_1, N^c_2)$ | 1 | $2_2$ | $-\frac{5x}{2}$ |
| $N^c_3$ | 1 | $1_{-1}$ | $-\frac{5x}{2}$ |
| $\phi$ | 1 | $1_{+2}$ | $5x$ |
| $\bar{\phi}$ | 1 | $1_{+2}$ | $-5x$ |
| $H_{45}$ | 45 | $1_{+2}$ | $-x$ |
| $H_{45}$ | 45 | $1_{+2}$ | $x$ |
| $\Phi$ | 24 | $1_{+0}$ | 0 |
scalar representation.

\[ \langle \Phi \rangle = \text{diag} \left( b, b, -\frac{3b}{2}, -\frac{3b}{2} \right), \quad \langle H^u \rangle = \left( \begin{array}{c} 0 \\ \langle H^u \rangle \end{array} \right), \quad \langle H^d \rangle = \left( \begin{array}{c} 0 \\ \langle H^d \rangle \end{array} \right), \]

\[ \langle H_{45} \rangle^5 = v_{45}, \quad \langle H_{45} \rangle^4_{i5} = -3v_{45}, \quad \langle H_{45} \rangle^4_{ii5} = v_{45}, \quad \langle H_{45} \rangle^4_{i5} = -3v_{45}; \]

\[ \langle \phi \rangle = v_s, \quad \langle \phi \rangle = \bar{v}_s, \quad \alpha, \beta = 1, 2, 3. \] (2)

For simplicity's sake, we have deliberately neglected the scalar triplets, $H^u = 0$ and $H^d = 0$, in the $H^u$ and $H^d$ 5-plets, assuming they are extremely heavy.

Having presented the assigned matter fields under $Q_6$ flavour symmetry, we will now introduce the superpotential, gauge invariant under the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ group. All necessary details for building the flavour symmetry singlets are given in the appendix, where a brief review of this dihedral group is offered. The trilinear terms in the superpotential are given by

\[ W = \sqrt{2}y_1^d (F_1 T_2 - F_2 T_1) H_3^d + \sqrt{2}y_2^d (F_1 T_3 H_2^d - F_2 T_3 H_1^d) + \sqrt{2}y_3^d F_3 (T_1 H_2^d - T_2 H_1^d) \]

\[ + \sqrt{2}y_4^d F_4 T_3 H_3^d + \frac{y_5^d}{4} (T_1 T_2 - T_2 T_1) H_3^d + \frac{y_6^d}{4} (T_1 T_3 H_2^d - T_2 T_3 H_1^d) \]

\[ + \frac{y_7^d}{4} F_3 T_2 H_2^u - T_2 H_1^u \right) + \frac{y_8^d}{4} T_3 T_3 H_3^d + \sqrt{2}Y_1 (F_1 T_2 - F_2 T_1) H_{45} + \sqrt{2}Y_2 F_3 T_3 H_{45} \]

\[ + \frac{Y_3}{4} (T_1 T_2 - T_2 T_1) H_{45} + \frac{Y_4}{4} T_3 T_3 H_{45} + y_5^m (N_1^c F_2 - N_2^c F_1) H_3^u \]

\[ + y_6^m (N_1^c F_3 H_2^d - N_2^c F_3 H_1^d) + y_7^m N_3^c (F_1 H_2^d + F_2 H_1^d) + y_8^m (N_1^c \psi N_1^c + N_2^c \phi N_2^c) \]

\[ + y_9^m N_3^c \phi N_3^c. \] (3)

From this superpotential, there is a missing term which is invariant under SUSY $SU(5) \otimes Q_6$ such as $y_{ij} N_i^c \bar{\phi} N_j^c$, however, it is prohibited by the $U(1)$ group. It is important to remember that SUSY must be broken via soft breaking terms, so these soft breaking terms should be included in a complete study of the full scalar potential, but we do not include them in this preliminary analysis. The scalar superpotential is

\[ W_s = \mu_b Tr (\Phi^2) + \lambda_b Tr (\Phi^3) + m_{45} H_{45} H_{45} + m_b \bar{\phi} \phi + \mu_1 (H^u H^d - H^d H^u) \]

\[ + \mu_2 H^u H^d + \lambda_1 (H^u H^d - H^d H^u) + \lambda_2 H^u H^d + a_1 H_{45} H_{45} \]

\[ + a_2 H_{45} H_{45} + a_3 H_{45} H_{45}. \] (4)

From the superpotential given in Eq. (3), one must obtain the MSSM effective superpotential that contains the Yukawa mass term after spontaneous symmetry breaking via the vev's of the $H^u$ and $H^d$ weak doublets scalar superfields. We will work in the following basis

\[ \mathcal{L} = \mathcal{L}^u + \mathcal{L}^d \] (5)

with

\[ \mathcal{L}^u = -\bar{d}_{iR} (\mathbf{M}_u)_{ij} d_{jL} - \bar{u}_{iR} (\mathbf{M}_u)_{ij} u_{jL} + h.c., \]

\[ \mathcal{L}^d = \bar{\ell}_{iR} (\mathbf{M}_d)_{ij} \ell_{jL} - \bar{N}_{iR} (\mathbf{M}_d)_{ij} v_{jL} - \frac{1}{2} \bar{N}_{iR} (\mathbf{M}_R)_{ij} N_{jR}^c + h.c., \] (6)

In general, the up, down, and charged lepton mass matrices are given by (see [44])

\[ \mathbf{M}_u = \frac{(Y^u + Y^u T)}{2} \langle H^u \rangle - \langle \bar{Y}^T - Y \rangle v_{45}, \quad \mathbf{M}_d = Y^d \langle H^d \rangle + 2Y v_{45}, \quad \mathbf{M}_l = Y^{dT} \langle H^d \rangle - 6Y^T v_{45}. \] (7)
In this particular model, from Eqs. (3) and (7) we have that the up, down and charged lepton mass matrices have respectively the following structures:

\[
M_u = \begin{pmatrix}
0 & -2\tilde{Y}_1 v_{45} & \tilde{y}_u h_{0u}^d \\
2Y_1 v_{45} & 0 & -\tilde{y}_d h_{0u}^d \\
\tilde{y}_u h_{0u}^d & -\tilde{y}_d h_{0u}^d & 0
\end{pmatrix},
M_d = \begin{pmatrix}
0 & y_1^d h_{0d}^d + 2Y_1 v_{45} & y_2^d h_{0d}^d \\
y_1^d h_{0d}^d - 2Y_1 v_{45} & 0 & -y_3^d h_{0d}^d \\
y_2^d h_{0d}^d & y_3^d h_{0d}^d & 0
\end{pmatrix};
\]

\[
M_{\ell} = \begin{pmatrix}
y_1^d h_{0d}^d - 6Y_1 v_{45} & 0 & -y_2^d h_{0d}^d \\
y_1^d h_{0d}^d - 6Y_1 v_{45} & 0 & -y_2^d h_{0d}^d \\
y_1^d h_{0d}^d - 6Y_1 v_{45} & 0 & -y_2^d h_{0d}^d
\end{pmatrix}.
\]

where $\tilde{y}^u \equiv (y_2^u + y_3^u)/2$. As can be seen, the $M_u$ mass matrix turns out almost symmetric due to the flavour structure. At the same time, we were able to correct the wrong relationship between the down quarks and the charged leptons; this was achieved including the $H_{45}$ scalar representation. The Dirac and right-handed mass matrices are given by

\[
M_D = \begin{pmatrix}
-\tilde{y}_d h_{0u}^d & y_1^d h_{0u}^d & y_2^d h_{0u}^d \\
y_1^d h_{0u}^d & 0 & -y_2^d h_{0u}^d \\
y_2^d h_{0u}^d & y_2^d h_{0u}^d & 0
\end{pmatrix}
\quad \text{and} \quad
M_R = \begin{pmatrix}
M_{R_1} & 0 & 0 \\
0 & M_{R_1} & 0 \\
0 & 0 & M_{R_2}
\end{pmatrix}.
\]

Therefore, after the type I see-saw mechanism, the neutrino mass term is given by

\[
\mathcal{L}_\nu = -\tilde{N}_{IR} (M_D)_{ij} \nu_{jL} - \frac{1}{2} \tilde{N}_{IR} (M_R)_{ij} N_{jR}^c + h.c. = -\frac{1}{2} \bar{\nu}^c L M_\nu \nu_L - \frac{1}{2} \bar{N}_{IR} (M_R)_{ij} N_{jR}^c.
\]

where the $M_\nu = M_D^T M_R^{-1} M_D$ effective neutrino mass matrix has the following structure:

\[
M_\nu = \begin{pmatrix}
(x (y_1^h h_{0u}^0)^2 + y (y_2^h h_{0u}^0)^2) & x (y_1^h h_{0u}^0)^2 h_3^2 + y (y_1^h h_{0u}^0)^2 h_2^2 & x y_1^h y_2^h h_0^0 h_3^0 h_2^u \\
y(y_2^h h_{0u}^0)^2 h_1^2 + y (y_1^h h_{0u}^0)^2 h_2^2 & x (y_1^h h_{0u}^0)^2 + y (y_2^h h_{0u}^0)^2 & x y_2^h y_1^h h_0^0 h_1^u h_3^u \\
x y_1^h y_2^h h_0^0 h_3^0 h_1^u & x y_2^h y_1^h h_0^0 h_3^0 h_1^u & x (y_2^h h_{0u}^0)^2 + (y_1^h h_{0u}^0)^2
\end{pmatrix},
\]

where $x = M_{R_1}^{-1}$ and $y = M_{R_2}^{-1}$.

Interesting phenomenology can be extracted from $M_\nu$ because the free parameters might be reduced substantially as a result of imposing conditions on the vev’s that take the down quark and charged lepton mass matrices to the correct NNI form as in ref. [13–17].

### 3 Mesas and mixings in the NNI scenario

There are two ways to obtain the NNI textures in the down quark and charged lepton sector: The first scenario consists in taking the condition $h_{22}^u = h_{11}^u \equiv h_{0u}^u$ and $h_{22}^d = h_{11}^d \equiv h_{0d}^d$, we obtain the following mass matrices:

\[
M_{(d,\ell)} = \begin{pmatrix}
0 & \pm A_{(d,\ell)} & B_{(d,\ell)} \\
\mp A_{(d,\ell)} & 0 & -B_{(d,\ell)} \\
C_{(d,\ell)} & -C_{(d,\ell)} & D_{(d,\ell)}
\end{pmatrix},
\quad \text{and} \quad
M_u = \begin{pmatrix}
0 & -A_u & B_u \\
A_u & 0 & -B_u \\
B_u & -B_u & D_u
\end{pmatrix}.
\]

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Here, the upper (lower) sign corresponds to the $\mathbf{M}_d$ ($\mathbf{M}_e$) mass matrix. In addition, the coefficients can read of Eq. (8). The $\mathbf{M}_{d,\ell}$ and $\mathbf{M}_u$ mass matrices contain implicitly the NNI and Fritzsch textures respectively which appear explicitly as follows: the above mass matrices are diagonalized by unitary matrices, $\mathbf{U}_{f(R,L)}$. According to Eq. (6) one obtains $\tilde{\mathbf{M}}_f = \mathbf{U}_{f(R,L)}^\dagger \mathbf{M}_f \mathbf{U}_{f(L)}$, in general. Here, $\tilde{\mathbf{M}}_f = \text{Diag}(\tilde{m}_{f_1}, \tilde{m}_{f_2}, 1)$. For simplicity, we have normalized the above expressions so that $\tilde{m}_{f_1} = m_{f_1}/m_{f_3}$, $\tilde{m}_{f_2} = m_{f_2}/m_{f_3}$ and $\tilde{\mathbf{M}}_f = \mathbf{M}_f/m_{f_3}$ are dimensionless parameters. Then, taking $\mathbf{U}_{f(R,L)} = \mathbf{U}_{\pi/4} \mathbf{u}_{f(R,L)}$, one can get easily $\tilde{\mathbf{M}}_f = \mathbf{u}_{fR}^\dagger \mathbf{M}_f \mathbf{u}_{fL}$ where

$$m_f = \mathbf{U}_{\pi/4}^\dagger \tilde{\mathbf{M}}_f \mathbf{U}_{\pi/4} = \begin{pmatrix} 0 & \pm \tilde{A}_f & 0 \\ \mp \tilde{A}_f & 0 & -\sqrt{2}\tilde{B}_f \\ 0 & -\sqrt{2}\tilde{C}_f & \tilde{D}_f \end{pmatrix}, \text{and} \quad \mathbf{U}_{\pi/4} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{2}}{\sqrt{2}} & -\frac{\sqrt{2}}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{2}} & 0 \end{pmatrix} \tag{13}$$

We should point out that the $\sqrt{2}$ factor will be absorbed in the $\tilde{B}$ and $\tilde{C}$ dimensionless free parameters for the quark and lepton sectors, respectively. In addition, notice that for the $\mathbf{m}_u$ mass matrix given in Eq. (13), $\tilde{C}_u = \tilde{B}_u$, according to Eq. (12). We will now show how the degeneracy in the vacuum expectation values for two scalar fields modifies the effective neutrino mass matrix which is now given as

$$\mathbf{M}_\nu = \begin{pmatrix} A_\nu^2 + B_\nu^2 & B_\nu^2 & A_\nu C_\nu \\ B_\nu^2 & A_\nu^2 + B_\nu^2 & A_\nu C_\nu \\ A_\nu C_\nu & A_\nu C_\nu & 2C_\nu^2 \end{pmatrix}, \tag{14}$$

where we have defined $A_\nu = \sqrt{3}y_h^0 h_3^0$, $B_\nu = \sqrt{3}y_3^0 h_u^0$ and $C_\nu = \sqrt{3}y_3^0 h_u^0$. Let us add a comment on the degeneracy on the vacuum expectation values. As we have remarked, the conditions $h_2^0 = h_3^0 \equiv h_{2d}^0$ and $h_2^{0d} = h_1^{0d} \equiv h_{2d}^{0d}$ have been assumed so far, it is not clear yet that these relations will arise in a natural way upon minimization of the scalar potential. We expect that it will be the case and the study of the full scalar potential may be done along the lines as those given in [13][16].

### 3.2 Quark and lepton mixings

We will describe briefly how to diagonalize the mass matrices, $\mathbf{m}_{(d,\ell)}$ and $\mathbf{m}_u$, respectively. Let us first start with the down quark and charged lepton mass matrices which have the NNI textures, we will not enter in great detail since these kind of matrices have been well studied in [10][11]. The above mentioned description is applied to the $\mathbf{m}_u$ mass matrix where the Fritzsch texture [2][3][7] is present. For a pedagogical method to diagonalize these mass matrices see [38].

Going back to the expression $\tilde{\mathbf{M}}_f = \mathbf{u}_{fR}^\dagger \mathbf{m}_f \mathbf{u}_{fL}$, we are interested in obtaining the $\mathbf{u}_{fL}$ left-handed matrices that appear in the CKM matrix, and for this we must build the bilineal form: $\tilde{\mathbf{M}}_f \mathbf{u}_{fL} = \mathbf{U}_{f(L,R)} \mathbf{m}_f \mathbf{U}_{f(L,R)}$. From this relation, we can factorize the CP phases that comes from $\mathbf{m}_f = Q_f (\mathbf{m}_f^\dagger \mathbf{m}_f) Q_f^\dagger$, see [45], such that, $Q_f = \text{Diag} (1, \exp(-i\eta_{f_2}), \exp(-i\eta_{f_3}))$ and $\mathbf{m}_f^\dagger \mathbf{m}_f$ have the following form

$$(\mathbf{m}_f^\dagger \mathbf{m}_f) = \begin{pmatrix} |\tilde{A}_f|^2 & 0 & |\tilde{A}_f||\tilde{B}_f| \\ 0 & |\tilde{A}_f|^2 + |\tilde{C}_f|^2 & |\tilde{C}_f||\tilde{D}_f| \\ |\tilde{A}_f||\tilde{B}_f| & |\tilde{C}_f||\tilde{D}_f| & |\tilde{B}_f|^2 + |\tilde{D}_f|^2 \end{pmatrix}, \tag{15}$$
we should keep in mind that the $|\tilde{A}_f|$, $|\tilde{B}_f|$, $|\tilde{C}_f|$ and the $|\tilde{D}_f|$ free parameters are real and dimensionless, and that for the up mass matrix, $|\tilde{B}_u| = |\tilde{C}_u|$.

Having factorized out the phases associated with CP violation in the bilinear form, we choose $u_{f\ell} = Q_f O_{f\ell}$, where $O_{f\ell}$ is the orthogonal matrix that diagonalizes to the $(m^f_j m^f_k)$ matrix. The matrix $O_{f\ell}$ is given by

$$O_{f\ell} = \begin{pmatrix} |f_1|, |f_2|, |f_3| \end{pmatrix},$$

where the three eigenvectors have the following form

$$|f_i\rangle = N_{f_i} \left( \begin{pmatrix} \hat{m}_f^2 - |\tilde{A}_f|^2 - |\tilde{C}_f|^2 |\tilde{A}_f| |\tilde{B}_f| \\ \hat{m}_f^2 - |\tilde{A}_f|^2 |\tilde{C}_f| |\tilde{D}_f| \\ (\hat{m}_f^2 - |\tilde{A}_f|^2)(\hat{m}_f^2 - |\tilde{A}_f|^2 - |\tilde{C}_f|^2) \end{pmatrix} \right),$$

$$|f_3\rangle = N_{f_3} \left( \begin{pmatrix} (1 - |\tilde{A}_f|^2 - |\tilde{C}_f|^2)|\tilde{A}_f||\tilde{B}_f| \\ (1 - |\tilde{A}_f|^2)|\tilde{C}_f||\tilde{D}_f| \\ (1 - |\tilde{A}_f|^2)(1 - |\tilde{A}_f|^2 - |\tilde{C}_f|^2) \end{pmatrix} \right).$$

Here, $N_{f_i} (i = 1, 2)$ and $N_{f_3}$ stand for the normalization factors whose definition must be read directly from the above expression. On the other hand, three free parameters can be fixed in terms of the physical masses and $|\tilde{D}_f| \equiv y_f$ remains as the only free parameters $^{[10][11]}$. Explicitly, these are given by

$$|\tilde{A}_f| = \frac{q_f}{y_f}, \quad |\tilde{B}_f| = \sqrt{\frac{1 + P_f - y_f^2 - R_f}{2} - \left( \frac{q_f}{y_f} \right)^2}, \quad |\tilde{C}_f| = \sqrt{\frac{1 + P_f - y_f^2 + R_f}{2} - \left( \frac{q_f}{y_f} \right)^2},$$

where

$$P_f = \hat{m}_f^2 + \hat{m}_f^2, \quad q_f = \sqrt{\hat{m}_f^2 \hat{m}_f^2}, \quad R_f = \sqrt{\left(1 + P_f - y_f^2\right)^2 - 4\left(P_f + q_f^2\right) + 8q_f^2y_f^2}.$$  

Here, there are two free parameters, $y_f$, for $f = d, \ell$. These parameters should be tuned in order to get reliable mixing matrices as we will see later. So far, we have found the $U_{(d,\ell)L}$ left-handed matrices that diagonalize the $M_{(d,\ell)}$ mass matrices which have the NNI textures. Let us now focus on $m_u$. Going back to Eq. (15), we must remember that $|\tilde{B}_u| = |\tilde{C}_u|$, then one can determine the three free parameters in terms of the physical masses. Explicitly, we obtain

$$|\tilde{A}_u| = \frac{\hat{m}_u \hat{m}_c}{1 - \hat{m}_c + \hat{m}_u}, \quad |\tilde{B}_u| = \sqrt{\frac{(1 - \hat{m}_c)(1 + \hat{m}_u)(\hat{m}_c - \hat{m}_u)}{1 - \hat{m}_c + \hat{m}_u}}, \quad |\tilde{D}_u| = 1 - \hat{m}_c + \hat{m}_u. \quad (20)$$

Following the same procedure, the $O_{uL}$ orthogonal matrix that diagonalizes $(m^u_j m^u_k)$ is fixed in terms of above parameters. Thus, using the expression given in Eq. (17), we get

$$O_{uL} = \begin{vmatrix} \frac{m_u(1 - m_u)}{m_u(1 - m_u)} & \frac{m_u(1 - m_u)}{m_u(1 - m_u)} & \frac{m_u(1 - m_u)}{m_u(1 - m_u)} \\ \frac{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)}{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)} & \frac{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)}{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)} & \frac{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)}{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)} \\ \frac{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)}{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)} & \frac{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)}{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)} & \frac{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)}{(1 - m_u)(m_c + m_u)(1 - m_c + m_u)} \end{vmatrix} \quad (21)$$

Therefore, the full left-handed unitary matrices are given, in general, by $U_{fL} = U_{\pi/4} u_{fL}$ where $u_{fL} = Q_f O_{fL}$. Then, the $CKM$ matrix may be completely determined and given by $V_{CKM} = U_{uL}^T U_{dL} = O_{uL}^T Q_q O_{dL}$, where we have defined $Q_q = Q_u^T Q_d = \text{Diag}(1, \exp i\alpha, \exp i\beta)$. In this way, the $CKM$ matrix can be obtained analytically or numerically, however, we are now just interested in getting a numerical expression for it which will be done in the next section.
In the leptonic sector, the PMNS matrix is defined by $V_{PMNS} = U_{LL}^T U_{vL}^\dagger K$ where $U_{LL} = U_{\pi/4} Q_{\nu} O_{LL}$ and the $U_{vL}$ neutrino contribution will be obtained next. We should point out that we will neglect the $K$ Majorana CP phases, which are currently unobservable.

Before diagonalizing the effective neutrino mass matrix, let us show the set of neutrino observables which is considered along the analytic and numerical analysis [46, 47]. This is given below

\[
\Delta m^2_\odot [10^{-5} eV^2] = m^2_{\nu_2} - m^2_{\nu_1} = 7.59^{+0.20}_{-0.18}
\]

\[
\Delta m^2_{ATM} [10^{-3} eV^2] = |m^2_{\nu_3} - m^2_{\nu_1}| = 2.50^{+0.09}_{-0.16} (-2.40^{+0.08}_{-0.09})
\]

\[
\sin^2 \theta_{12} = 0.312^{+0.017}_{-0.015}
\]

\[
\sin^2 \theta_{23} = 0.52^{+0.06}_{-0.07} (0.52 \pm 0.06)
\]

\[
\sin^2 \theta_{13} = 0.013^{+0.007}_{-0.005} (0.016^{+0.008}_{-0.006})
\]

\[
\sin^2 2\theta_{13} = (0.076 \pm 0.068) \quad (MINOS).
\]

Here, the data appearing in parentheses stand for the inverted case. Leaving aside the experimental results, we consider the $M_\nu$ mass matrix given in Eq. (14). Due to the form of $M_\nu$ we can rotate the left-handed neutrino field as follows: $\nu_L = U_{v}\nu_L$, where $U_{v} = u_{\pi/4} u_{v}$ so that one gets: $\tilde{M}_\nu = \text{Diag} (m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) = u_{v}^T \tilde{m}_\nu u_{v}$ where

\[
\tilde{m}_\nu = u_{\pi/4}^T M_\nu u_{\pi/4} = \begin{pmatrix} A^2_\nu + 2 B^2_\nu & \sqrt{2} A_\nu C_\nu & 0 \\ \sqrt{2} A_\nu C_\nu & 2 C^2_\nu & 0 \\ 0 & 0 & A^2_\nu \end{pmatrix} \quad \text{and} \quad u_{\pi/4} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}.
\]

(22)

The $m_\nu$ block matrix can be easily diagonalized. First, let us factorize the CP phases of $m_\nu$ [45]. So that, $\tilde{m}_\nu = P_\nu \hat{m}_\nu P_\nu$, where $P_\nu$ and $\hat{m}_\nu$ are given in Eq. (23). We can associate immediately $|A_\nu|^2 = m_{\nu_3}$. Thus, we just have to diagonalize the left upper block of $m_\nu$

\[
P_\nu = \begin{pmatrix} \exp i \eta_{\nu_1} & 0 & 0 \\ 0 & \exp i \eta_{\nu_2} & 0 \\ 0 & 0 & \exp i \eta_{\nu_3} \end{pmatrix} \quad \text{and} \quad \hat{m}_\nu = \begin{pmatrix} |A_\nu|^2 + 2 |B_\nu|^2 & \sqrt{2} |A_\nu| |C_\nu| & 0 \\ \sqrt{2} |A_\nu| |C_\nu| & 2 |C_\nu|^2 & 0 \\ 0 & 0 & |A_\nu|^2 \end{pmatrix}.
\]

(23)

A necessary condition to factorize the phases in the above way is that the $A^2_\nu$ and $B^2_\nu$ phases must be aligned, although they may be different in magnitude. On the other hand, we appropriately choose $u_\nu = P_{\nu}^\dagger O_\nu$. Here, $O_\nu$ is a real orthogonal matrix that diagonalizes the $\hat{m}_\nu$ matrix. Using the left upper block of $m_\nu$, we fix the $|B_\nu|^2$ and $|C_\nu|^2$ free parameters through the following equations

\[
m_{\nu_2} + m_{\nu_1} = |A_\nu|^2 + 2 |B_\nu|^2 + 2 |C_\nu|^2 \quad \text{and} \quad m_{\nu_2} m_{\nu_1} = 4 |B_\nu|^2 |C_\nu|^2.
\]

(24)

Solving for the rest of the free parameters we find that

\[
|B_\nu|^2 = \frac{1}{4} (m_{\nu_2} + m_{\nu_1} - m_{\nu_3} \mp R_\nu) \quad \text{and} \quad |C_\nu|^2 = \frac{1}{4} (m_{\nu_2} + m_{\nu_1} - m_{\nu_3} \pm R_\nu),
\]

(25)
where \( R_\nu \equiv \sqrt{(m_{\nu_2} + m_{\nu_1} - m_{\nu_3})^2 - 4m_{\nu_2}m_{\nu_1}} \). As we can observe, there are two solutions for \(|B_\nu|^2\) and \(|C_\nu|^2\), respectively. However, following a straightforward analysis, it is clear that one solution is discarded by demanding that two free parameters (for the normal and for the inverted hierarchy) should be real and positive definite since they come from a real symmetric matrix. As a result of this, we realize that the normal spectrum is ruled out. For the inverted case \((m_{\nu_2} > m_{\nu_1} > m_{\nu_3})\), \(|B_\nu|^2\) and \(|C_\nu|^2\) turn out being real and positive, if and only if, the \(m_{\nu_3}\) lightest neutrino mass is very small. Actually, from the definition of \(R_\nu\) we obtain the following sum rule

\[
m_{\nu_3} \leq (\sqrt{m_{\nu_2}} - \sqrt{m_{\nu_1}})^2, \tag{26}\]

where the equality in the above expression means an upper bound for the \(m_{\nu_3}\) lightest mass. Having fixed \(|B_\nu|^2\) and \(|C_\nu|^2\) in terms of the physical neutrino masses, the \(O_\nu\) matrix is well determined by them. Explicitly, \(O_\nu\) is given by

\[
O_\nu = \begin{pmatrix}
\sqrt{m_{\nu_3} (m_{\nu_2} + m_{\nu_1} - m_{\nu_3} + R_\nu)} & \sqrt{m_{\nu_3} (m_{\nu_2} + m_{\nu_1} - m_{\nu_3} + R_\nu)} & 0 \\
\frac{m_{\nu_2} - m_{\nu_1}}{2 (m_{\nu_2} - m_{\nu_1})} & \frac{m_{\nu_2} - m_{\nu_1}}{2 (m_{\nu_2} - m_{\nu_1})} & 1
\end{pmatrix}. \tag{27}\]

Therefore, the \(M_\nu\) neutrino mass matrix is diagonalized by \(U_\nu = u_{\pi/4} P^T_\nu O_\nu\). Then, one obtains

\[
V_{PMNS} = U^T_\ell U_\nu = O^T_\nu \mathbf{S}_{23} P_\nu^T \mathbf{O}_{\nu L}. \tag{28}\]

Here, \(\mathbf{S}_{23} = U^T_{\pi/4 u_{\pi/4}}\) is the permutation matrix which is an element of the \(S_3\) family group, at the same time, it is an element of the \(Q_6\) family since \(S_3\) is a subgroup of it. Therefore, the \(PMNS\) mixing matrix has the following form

Comparing this matrix with the standard parametrization given in [48], we find that the reactor, the atmospheric and the solar mixing angles are well determined as follows

\[
|\sin \theta_{13}| = |O_{21\ell}|, \quad |\sin \theta_{23}| = \frac{|O_{22\ell}|}{\sqrt{1 - |O_{21\ell}|^2}}, \quad |\tan \theta_{12}|^2 = \frac{|O_{11\ell}O_{12\bar{\nu} + O_{31\ell}O_{22\bar{\nu}} \exp (i \bar{\eta}_{3e}) |^2}{|O_{11\ell}O_{11\bar{\nu} + O_{31\ell}O_{21\bar{\nu}} \exp (i \bar{\eta}_{3e}) |^2}. \tag{29}\]

Let us point out a remarkable coincidence between the above formulas and those showed in [38,39], their functional behaviour seems to be the same, at least. As we already commented briefly, it is not a surprise since the \(Q_6\) family group is the double covering of the \(S_3\) one, so that in this particular model the \(S_{23}\) presence in the leptonic sector is not simply a coincidence. Of course, we expect that our results turn out being different to the \(S_3\) case, since the charged lepton and neutrino contributions are different in both models, as we will see next.
4 Numerical analysis for the mixing matrices

4.1 CKM mixing matrix

The CKM matrix is defined as $V_{CKM} = U_{uL}^\dagger U_{dL} = O_{uL}^T Q_d O_{dL}$, with the form of the mass matrices found in the previous sections. So far, there are three free parameters ($y_d$ and two CP-violating phases in $Q_d$) if the quark mass ratios are taken as inputs. As it is well known, the physical masses depend on the scale at which they are measured, in this model the CKM matrix may be obtained numerically with masses at the GUT scale. However, the mass ratios do not change drastically at different energy scales as one can verify directly from [49]. Therefore, we will assume that the form of the mass matrices will remain the same from the GUT scale to the electroweak scale. Of course, in the more detailed analysis that is currently in progress, the effects of the extra Higgs fields in the model and the running of the renormalization group will be taken into account. Thus, for the rest of the analysis we will assume we are already at the electroweak scale.

At low energies, we have used the following values for the quark mass ratios given in [50,51], $m_u/m_t = (1.73 \pm 0.75) \times 10^{-5}$, $m_c/m_t = (3.46 \pm 0.43) \times 10^{-3}$; $m_d/m_b = (1.12 \pm 0.007) \times 10^{-3}$ and $m_s/m_b = (2.32 \pm 0.84) \times 10^{-2}$.

In order to obtain the numerical values for the three free parameters, we perform a $\chi^2$ analysis on the parameter space, to find their best fit points. It is built as in \cite{50,51}

$$\chi^2 = \frac{(|V_{ud}| - |V_{ud}^{ex}|)^2}{\sigma_{V_{ud}}^2} + \frac{(|V_{us}| - |V_{us}^{ex}|)^2}{\sigma_{V_{us}}^2} + \frac{(|V_{ub}| - |V_{ub}^{ex}|)^2}{\sigma_{V_{ub}}^2} + \frac{(\mathcal{J}_q^{th} - \mathcal{J}_q^{ex})^2}{\sigma_{\mathcal{J}_q}^2}$$

(30)

where we have taken the following experimental values for the $V_{CKM}$ elements we used to construct the $\chi^2$ function:

$|V_{ud}| = 0.97427 \pm 0.00015$, $|V_{us}| = 0.2253 \pm 0.007$, $|V_{ub}| = 0.00351 \pm 0.00015$, $\mathcal{J}_q^{ex} = (2.96 \pm 0.18) \times 10^{-5}$ \cite{48}. Notice that using the Jarlskog invariant in the $\chi^2$ function implies unitarity as a constraint. The best values for the free parameters are thus found to be

$$y_d = 0.981977^{+0.002843}_{-0.002117},$$
$$\alpha = (168.78^{+2.10}_{-1.11})^\circ,$$
$$\beta = (132^{+105}_{-25})^\circ.$$  

(31)

at 70 % C. L with $\chi^2 = 0.0515$ as the minimal value. These correspond to the following values for the $V_{CKM}$ elements

$$|V_{ud}| = 0.97428^{+0.00016}_{-0.00014}, \quad |V_{us}| = 0.2252^{+0.00062}_{-0.00067},$$
$$|V_{ub}| = 0.00351^{+0.00017}_{-0.00016}, \quad \mathcal{J}_q = 2.95^{+0.20}_{-0.19} \times 10^{-5}.$$

(32)

4.2 PMNS mixing matrix

As we observe from Eq. (29), the reactor and atmospheric angles turn out to be independent of the neutrino masses. These observables only depend explicitly on the $y_e$ free parameter, the charged lepton masses and the $\eta_{2e}$ Dirac phase; the latter may be ignored since we are
just interested in the absolute values of the two mixing angles. On the other hand, the solar mixing angle depends on the $y_e$ free parameter, the $\bar{\eta}_{3e}$ phase, as well as the charged lepton and neutrino masses.

In order to show that in this model we can describe the masses and mixing of flavour of the leptons we will also make a $\chi^2$ fit using the theoretical expressions for the atmospheric and reactor angles given in Eq. (29) and compare them with the current experimental data for these mixing angles. Thus, for the atmospheric mixing angle we consider the following experimental value $\sin^2 \theta_{\ell 23} = 0.52 \pm 0.06$. On other hand, within this theoretical framework we considered the particular case when the first and second right-handed neutrinos are mass degenerate. So we can only determine a lower bound for the value of the reactor angle [40]. Therefore, to perform the $\chi^2$ fit we considered the values for the reactor angle reported by the MINOS experiment; $\sin^2 2\theta_{\ell 13} = 0.076 \pm 0.068$. We considered also the following charged lepton masses values: $m_e = 0.51099$ MeV, $m_\mu = 105.6583$ MeV and $m_\tau = 1776.82$ MeV [52]. As a result of this $\chi^2$ analysis, we obtained that for the best fit with $\chi^2 = 0.85$ as the minimum value, the free parameter $y_e$ at 1σ has the following range of values: $y_e = 0.8478^{+0.0045}_{-0.0046}$. Moreover, the best theoretical values for the atmospheric and reactor angles that come from this analysis, at 1σ, are:

$$\theta_{\ell 23}^{th} = 46.18^{+0.66}_{-0.65} \quad \text{and} \quad \theta_{\ell 13}^{th} = 3.38^{+0.03}_{-0.02}. \quad (33)$$

As can be observed, the atmospheric angle value is in good agreement with the experimental values, however, the obtained reactor mixing value is smaller than the central value obtained in the global fits. In fact, this theoretical value of the reactor angle is more than 3σ away from the values reported in ref. [53, 54]. But it is worth noting that our model predicts that the reactor angle must be different from zero and that is large in comparison to the one obtained from the tribimaximal mixing matrix. Moreover, as already stated above, it is a lower bound for the value obtained in a more general model where the right-handed neutrino masses are not degenerate. In figure 2 we show the dependence of the atmospheric and reactor angles on the parameter $y_e$.

Having determined the allowed values for the $y_e$ free parameter, then, the solar mixing angle just depends on the neutrino masses and one Dirac phase. We can reduce further the free parameters, noticing that the neutrino masses can be determined using the sum rule.
Figure 2: Atmospheric and reactor mixing angle values at 65% (orange) and 95% (yellow) of C.L.

given in Eq. (26). This is written in terms of the observables $\Delta m_{\odot}^2$ and $\Delta m_{ATM}^2$ as

$$m_{\nu_3} \leq \left( \sqrt{m_{\nu_3}^2 + \Delta m_{\odot}^2 + \Delta m_{ATM}^2} - \sqrt{m_{\nu_3}^2 + \Delta m_{ATM}^2} \right)^2. \quad (34)$$

From the above expression and using the experimental results of $\Delta m_{\odot}^2$ and $\Delta m_{ATM}^2$, we can get an upper bound for the $m_{\nu_3}$ lightest neutrino mass. Therefore, the allowed values for the lightest neutrino mass are: $0 \leq m_{\nu_3} \leq 4 \times 10^{-6}$ eV. As a result, the $m_{\nu_2}$ and $m_{\nu_1}$ neutrino masses are easily calculated in the following way

$$m_{\nu_2} = \sqrt{\Delta m_{\odot}^2 + \Delta m_{ATM}^2} \sqrt{1 + \frac{m_{\nu_3}^2}{\Delta m_{\odot}^2 + \Delta m_{ATM}^2}} \approx 5.0 \pm 0.087 \times 10^{-2} \text{ eV},$$

$$m_{\nu_1} = \sqrt{\Delta m_{ATM}^2} \sqrt{1 + \frac{m_{\nu_3}^2}{\Delta m_{ATM}^2}} \approx 4.90 \pm 0.089 \times 10^{-2} \text{ eV}. \quad (35)$$

As we already commented, our model predicts an inverted ordering among the neutrino masses and the above values clarify explicitly our statement. Since we have calculated the neutrino masses, these cease to be considered as free parameters in the solar mixing angle expression given in Eq. (29). Furthermore, we obtain easily the solar mixing value that is allowed by the fixed free parameters, which are $y_e$ and the neutrino masses. Actually, with the particular value for the $\bar{\eta}_{\beta e} = \pi$ Dirac phase, and using the $y_e$ value at 90% at C.L, we obtain for the solar mixing angle

$$\theta_{12}^{\text{sin}^2} = 36.62 \pm 4.06 \quad (36)$$

where we have used $m_{\nu_2} = 0.05080$ eV, $m_{\nu_1} = 0.04987$ eV and $m_{\nu_3} = 3.9 \times 10^{-6}$ eV. As we observe the solar mixing angle value in Eq. (36) is in good agreement with the experimental data. In figure 3 we show explicitly the dependence of the solar angle on the $y_e$ parameter and the $m_{\nu_3}$ lightest neutrino mass, respectively.

It is clear from these results that the solar angle has a strong dependence on the $m_{\nu_2}$ mass, and that the sum rule for the neutrino masses does play an important role to determine the above mixing angle in good agreement with the experimental data.
Figure 3: Solar mixing angle values as function of the $y_e$ parameter and the $m_{\nu_3}$ neutrino mass considering $m_{\nu_2} = 0.05080$ eV, $m_{\nu_1} = 0.04987$ eV and $\bar{\eta}_e = \pi$.

5 Outlook and Remarks

We have studied a non-minimal supersymmetric $SU(5)$ model where the $Q_6$ flavour symmetry plays an important role in accommodating the masses and mixings for quarks and leptons. For the former sector, the $CKM$ mixing matrix has been obtained with great accuracy and it is consistent with the experimental results. For leptons, the flavour symmetry implies an appealing sum rule for the neutrino masses, which leads to an inverted hierarchy and is crucial to determine the solar mixing angle. As a main result, we have that the atmospheric $\theta_{23}^{\text{exp}} = 46.18^{+0.66}_{-0.65}$ and solar angle $\theta_{12}^{\text{exp}} = 36.62 \pm 4.06$ are in good agreement with the experimental data, however, the reactor angle value, $\theta_{13}^{\text{exp}} = 3.38^{+0.03}_{-0.02}$, is more than $3\sigma$ away from the central value from the global fits. It is worth pointing out that the model predicts a non-zero value, unlike the tri-bimaximal case, and this value would constitute the lower bound for $\theta_{13}$ in the more general model, where the right-handed neutrinos in the $Q_6$ doublet are not mass degenerate. It is expected then in a more general study of the model it will be possible to obtain a value for the reactor mixing angle within one or two $\sigma$ of the central value, as is the case in the non-supersymmetric $S_3$ models [40].

Although in this preliminary analysis we have found the form of the mass and mixing matrices for quarks and leptons and we have shown that they lead to realistic values, we have left aside subtle issues as the full analysis of the scalar superpotential, the details of the proton decay and all the phenomenology that the model provides by itself. These topics will be taken into account in a complete study of the model, as we already commented. In general, the model seems to work out very well and it may be considered as a realistic one.

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A  $Q_6$ flavour symmetry

The $Q_6$ group has twelve elements which are contained in six conjugacy classes, therefore, it contains six irreducible representations. We will use the notation given in [15], there are various notations and extensive studies for this group, see for example [12,55]. The $Q_6$ family symmetry has 2 two-dimensional irreducible representations denoted by $2_1$ and $2_2$, 4 one-dimensional ones which are denoted by $1_{+,0}$, $1_{+,2}$, $1_{-,1}$ and $1_{-,3}$. As it is well known, $2_1$ is a pseudo real and $2_2$ is a real representation. In addition, for $1_{+,n}$ we have that $n = 0, 1, 2, 3$ is the factor $\exp(in\pi/2)$ that appears in the matrix given by $B$. The $\pm$ stands for the change of sign under the transformation given by the $A$ matrix. So that the first two one-dimensional representations are real and the two latter ones are complex conjugate to each other.

$$Q_6 = \{1, A, A^2, A^3, A^4, A^5, B, AB, A^2B, A^3B, A^4B, A^5B\},$$

where the $A$ and $B$ are two-dimensional matrices whose explicit forms are given by

$$A = \begin{pmatrix} \cos(2\pi/6) & \sin(2\pi/6) \\ -\sin(2\pi/6) & \cos(2\pi/6) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$  

Let us write the multiplication rules among the six irreducible representations which will be useful to build a phenomenological model:

$$1_{+,2} \otimes 1_{+,2} = 1_{+,0}, \quad 1_{+,3} \otimes 1_{-,3} = 1_{+,2}, \quad 1_{-,1} \otimes 1_{-,1} = 1_{+,2}, \quad 1_{-,1} \otimes 1_{-,3} = 1_{+,0};$$

$$1_{+,2} \otimes 1_{-,1} = 1_{-,3}, \quad 1_{+,2} \otimes 1_{-,3} = 1_{-,1}, \quad 2_1 \otimes 1_{+,2} = 2_1, \quad 2_1 \otimes 1_{-,3} = 2_2;$$

$$2_1 \otimes 1_{-,1} = 2_2, \quad 2_2 \otimes 1_{+,2} = 2_2, \quad 2_2 \otimes 1_{-,3} = 2_1, \quad 2_2 \otimes 1_{-,1} = 2_1;$$

$$1_{+,0} + 1_{+,2} + 2_2,$$
B Superpotential

According to the assignment table given in Section 1, we have that the renormalisable gauge and $Q_6$ flavour invariant superpotential is built in the following way,

$$W = y_1^d \left[ (2 \otimes 2)_{1+2} \otimes 1_{\pm2} \right]_{1+0} + y_2^d \left[ (2 \otimes 1_{-3})_{21} \otimes 2 \right]_{1+0} + y_3^d \left[ (1_{-3} \otimes 2)_{21} \otimes 2 \right]_{1+0}$$

$$+ y_1^n \left[ (1_{-3} \otimes 1_{-3})_{1+2} \otimes 1_{\pm2} \right]_{1+0} + y_2^n \left[ (2 \otimes 2)_{1+2} \otimes 1_{\pm2} \right]_{1+0}$$

$$+ y_3^n \left[ (1_{-3} \otimes 2)_{21} \otimes 2 \right]_{1+0} + y_1^n \left[ (2 \otimes 2)_{1+2} \otimes 1_{-3} \right]_{1+0}$$

$$+ y_2^n \left[ (2 \otimes 1_{-3})_{21} \otimes 2 \right]_{1+0} + y_3^n \left[ (1_{-3} \otimes 1_{-3})_{1+2} \otimes 1_{\pm2} \right]_{1+0}$$

$$+ y_1^m \left[ (2 \otimes 1_{+2})_{22} \otimes 2 \right]_{1+0} + y_2^m \left[ (1_{+2} \otimes 1_{+2})_{1-3} \right]_{1+0}.$$  \(\text{(43)}\)

Let us explain our notation to construct the singlets invariant under the $Q_6$ flavour symmetry. Considering the first term in the superpotential, we know from the multiplication rules given above that

$$\left(2 \otimes 2\right) = 1_{+0} + 1_{+2} + 2 \Rightarrow \left(2 \otimes 2\right)_{1+2} = 1_{+2},$$  \(\text{(44)}\)

therefore, the complete expression means that

$$\left[ (2 \otimes 2)_{1+2} \otimes 1_{\pm2} \right]_{1+0} = [1_{+2} \otimes 1_{\pm2}] = 1_{+0}.$$  \(\text{(45)}\)

The rest of the terms should be understood in the same way. In the scalar superpotential we have the same nomenclature, so that we will just write it in terms of the assignment of $Q_6$,

$$W_s = \mu_\Phi \left( 1_{+0} \otimes 1_{+0} \right)_{1+0} + m_{45} \left( 1_{+2} \otimes 1_{+2} \right)_{1+0} + m_\Phi \left( 1_{+2} \otimes 1_{+2} \right)_{1+0} + \mu_1 \left( 2_1 \otimes 2_1 \right)_{1+0}$$

$$+ \mu_2 \left( 1_{+2} \otimes 1_{+2} \right)_{1+0} + \lambda_\Phi \left( 1_{+0} \otimes 1_{+0} \otimes 1_{+0} \right)_{1+0} + \lambda_1 \left[ (2_1 \otimes 1_{+0})_{21} \otimes 2_1 \right]_{1+0}$$

$$+ \lambda_2 \left[ (1_{+2} \otimes 1_{+0})_{1+2} \otimes 1_{+2} \right]_{1+0} + a_1 \left[ (1_{+2} \otimes 1_{+2}) \otimes 1_{+0} \right]_{1+0}$$

$$+ a_2 \left[ (1_{+2} \otimes 1_{+0})_{1+2} \otimes 1_{+2} \right]_{1+0} + a_3 \left[ (1_{+2} \otimes 1_{+0})_{1+2} \otimes 1_{+2} \right]_{1+0}.$$  \(\text{(46)}\)

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