AN IDENTITY INVOLVING THE CYCLOTOMIC POLYNOMIALS

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ABSTRACT. We present an elementary identity for the cyclotomic polynomials \( \Phi_n(x) \) which reflects a kind of multiplicative property of \( \Phi_n(x) \) as a function of \( n \), and we explore its connections with the properties of other arithmetical functions.

Important Note: In the first version of this article uploaded to the arXiv, it is said that this result seemed to be new. However, after that, I have learned that this identity in Theorem 1.1 has previously appeared in [CMW95, corollary 2] (with a different proof).

1. INTRODUCTION AND MAIN RESULT

For each natural number \( n \in \mathbb{N} \), let \( \Phi_n(X) \) denote the \( n \)-th cyclotomic polynomial, i.e.: the monic polynomial whose roots are the primitive \( n \)-th roots of unit. Explicitly

\[
\Phi_n(X) = \prod_{1 \leq k \leq n \atop k \perp n} (X - \zeta_k), \quad \zeta_k = e^{2\pi i k/n}.
\]

Here \( k \perp n \) means that \( m \) and \( n \) are coprime or relatively prime (a useful notation introduced in [GKP94, section 4.5]), i.e.

\[ k \perp n \Leftrightarrow \gcd(k,n) = 1 \]

The cyclotomic polynomials are a well-known object in number theory and they also play a key role in field theory, see for instance [DF04, section 13.6]. Nice surveys on the subject of cyclotomic polynomials are [Tha00] and [Ge08]. Also the web page [Wei20] collects some known results and references on them.

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In particular, it is known that the cyclotomic polynomials $\Phi_n(X)$ have integral coefficients, and that many important functions in multiplicative number theory are related to them.

For instance, it is clear from their definition that their degree is given by $\varphi(n)$, Euler’s totient function, which counts the number of integers $k$ in the range $1 \leq k \leq n$ that are coprime with $n$. Also it is easily seen that the cyclotomic polynomials satisfy the following Fundamental identity

$$X^n - 1 = \prod_{d|n} \Phi_d(X)$$  \hfill (1)

(since every $n$-th root of the unit is a $d$-th primitive root for exactly one $d$ dividing $n$), from where we deduce that the cyclotomic polynomials can be computed recursively using the formula

$$\Phi_n(X) = \frac{X^n - 1}{\prod_{d|n, d<n} \Phi_d(X)}.$$  \hfill (2)

Moreover, if we consider the coefficients $a_k(n)$ of $\Phi_n(X)$, i.e. we write

$$\Phi_n(X) = \sum_{k=0}^{\varphi(n)} a_k(n)X^k,$$  \hfill (3)

we have that

$$a_1(n) = a_{\varphi(n)-1}(n) = -\mu(n) \quad \text{for } n > 1$$

where $\mu$ is the Möbius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1p_2\ldots p_k \text{ for distinct primes } p_j. \\ 0 & \text{otherwise} \end{cases}$$

Indeed, it is well known that $\mu(n)$ gives the sum of the $n$-th primitive roots of the unit so that

$$a_{\varphi(n)-1}(n) = -\mu(n)$$

and $a_1(n) = a_{\varphi(n)-1}(n)$ by the symmetry of the cyclotomic polynomial \cite[Lemma 2.1]{Tha00}.

Also from the Fundamental Identity (1) we can derive the expression

$$\Phi_n(X) = \prod_{d|n} (X^{n/d} - 1)^{\mu(d)} = \prod_{d|n} (X^{d} - 1)^{\mu(n/d)}$$

using the multiplicative version of Möbius inversion formula (see Lemma 2.2 below).
Many of the arithmetical functions in multiplicative number theory $f : \mathbb{N} \to R$ (where $R$ is some commutative ring, usually the field $\mathbb{C}$ of complex numbers) are multiplicative in the sense that

$$f(m \cdot n) = f(m) \cdot f(n) \text{ whenever } m \perp n$$

For instance $\varphi$ and $\mu$ have this property (see [HW79] or [Apo76, chapter 2], and section 3 below).

Another arithmetical function closely related to the cyclotomic polynomials is the Ramanujan sum $c_n(q)$ (introduced in [Ram00]), defined as the sum of the $q$-powers of the $n$-th primitive roots of the unit

$$c_n(q) = \sum_{1 \leq k \leq n \atop k \perp n} \zeta_k^q, \quad \zeta_k = e^{\frac{2\pi ik}{n}}.$$

As we have mentioned before

$$\mu(n) = c_n(1) \text{ for all } n \in \mathbb{N}.$$

The Ramanujan sums are multiplicative as a function of $n$ (see [HW79, Theorem 67])

$$c_{mn}(q) = c_m(q) \cdot c_n(q) \text{ whenever } m \perp n$$

and also satisfy the following more complex multiplicative property [Apo76, Theorem 8.7]

$$c_{mn}(ab) = c_m(a) \cdot c_n(b) \text{ whenever } a \perp n \text{ and } b \perp m.$$

Likewise, other arithmetical functions defined by sums involving the roots of the unit, like Gauss quadratic sums and Kloosterman sums, enjoy similar multiplicative properties [HW79, section 5.6].

A question that naturally arises is whether $\Phi_n(X)$, considered as an arithmetical function of $n \in \mathbb{N}$ into the ring $\mathbb{Z}[X]$ of polynomials with integral coefficients, has some property of this kind.

In this note, we present an elementary identity involving the cyclotomic polynomials, answering this question.

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**Theorem 1.1.** Let $m$ and $n$ be coprime. Then,

$$\Phi_n(X^m) = \prod_{d|\text{lcm}(m,n)} \Phi_{d\cdot n}(X)$$
Proof. The proof uses (complete) induction on \( n \). We will show that for each \( n \in \mathbb{N} \), \((6)\) holds for every \( m \in \mathbb{N} \) such that \( m \perp n \).

Indeed for \( n = 1 \), \( \Phi_1(X) = X - 1 \) and \((6)\) reduces to the Fundamental Identity \((1)\). Next, we assume then that \((6)\) holds for any \( n' < n \) in place of \( n \), and we will show that it holds for \( n \).

From \((2)\) (substituting \( X^m \) for \( X \)), we have that

\[
\Phi_n(X^m) = \frac{(X^m)^n - 1}{\prod_{d \mid m, d < n} \Phi_d(X^m)},
\]

and using the inductive hypothesis (with \( n' = d_2 \)), this can be written as

\[
(7) \quad \Phi_n(X^m) = \frac{(X^m)^n - 1}{\prod_{d \mid n} \prod_{d_1 \mid m, d_2 < n} \Phi_{d_1 \cdot d_2}(X)}.
\]

Here we have used the fact that since \( d_2 \mid n \), \( d_2 \) is also coprime with \( m \).

On the other hand, from the Fundamental Identity \((1)\),

\[
(X^m)^n - 1 = X^{mn} - 1 = \prod_{d \mid mn} \Phi_d(X).
\]

Now we observe that the Fundamental Theorem of Arithmetic implies that, since \( m \) and \( n \) are coprime, each divisor \( d \) of \( mn \) can be uniquely decomposed as \( d = d_1 \cdot d_2 \) where \( d_1 \mid m \) and \( d_2 \mid n \).

Hence, we can write

\[
(X^m)^n - 1 = \prod_{d_1 \mid m} \prod_{d_2 \mid n} \Phi_{d_1 \cdot d_2}(X)
\]

\[
= \left[ \prod_{d_1 \mid m} \prod_{d_2 \mid n, d_2 < n} \Phi_{d_1 \cdot d_2}(X) \right] \cdot \left[ \prod_{d_1 \mid m} \Phi_{d_1 \cdot n}(X) \right]
\]

(splitting the factor with \( d_2 = n \)). Replacing in \((7)\), it follows that

\[
\Phi_n(X^m) = \prod_{d_1 \mid m} \Phi_{d_1 \cdot n}(X)
\]

as claimed.

By the principle of (complete) mathematical induction it follows that the theorem holds for every \( n, m \in \mathbb{N} \). \(\square\)
Remark 1.2. It is easily seen that (6) fails if \( n \) and \( m \) are not coprime. For instance if \( m = 2 \) and \( n = 4 \)

\[
\Phi_4(X) = X^2 + 1 \Rightarrow \Phi_n(X^m) = \Phi_4(X^2) = X^4 + 1,
\]

whereas

\[
\prod_{d|n} \phi_{d,n}(X) = \Phi_4(X)\Phi_8(X) = (X^2 + 1)(X^4 + 1) = X^6 + X^4 + X^2 + 1
\]

It is my pleasure to acknowledge that the software Sagemath [The20] was used to find this counterexample and to check many of the identities in this work, and to thank their developers for this wonderful tool.

2. A dual form of the main identity

In this section, we prove a dual form of our main identity.

Theorem 2.1. If \( n \) and \( m \) are coprime,

\[
\Phi_{nm}(X) = \prod_{d|m} \Phi_n(X^d)^{\mu(m/d)} = \prod_{c|m} \phi_n(X^{m/c})^{\mu(c)}
\]

For the proof we need a the M"obius inversion formula that we state as a lemma (see [HW79, theorems 266 and 267] for a proof).

Lemma 2.2 (Möbius inversion formula). Let \( f, g : \mathbb{N} \to R \) be two functions, where \( R \) is a commutative ring.

i) (Additive form) The relation

\[
g(m) = \sum_{d|m} f(d) \text{ for every } m \in \mathbb{N}
\]

is equivalent to

\[
f(m) = \sum_{d|m} g(d) \mu\left(\frac{m}{d}\right) = \sum_{c|m} g\left(\frac{m}{c}\right) \mu(c) \text{ for every } m \in \mathbb{N}
\]

ii) (Multiplicative form) Assume that \( R \) is a field. Then, the relation

\[
g(m) = \prod_{d|m} f(d) \text{ for every } m \in \mathbb{N}
\]

is equivalent to

\[
f(m) = \prod_{d|m} g(d)^{\mu(m/d)} = \prod_{c|m} g(m/c)^{\mu(c)} \text{ for every } m \in \mathbb{N}
\]

Here we make the convention that \( x^0 = 1 \) even if \( x = 0 \).
In our application of the multiplicative form of Möbius inversion formula, $R = \mathbb{Q}(x)$ is the field of rational functions with rational coefficients. Now we see that using the lemma, Theorem 2.1 follows from Theorem 1.1 by fixing $n$ and considering

$$f(d) = \begin{cases} 
\Phi_{dn}(x) & \text{if } d \perp n \\
0 & \text{otherwise}
\end{cases}$$

$$g(m) = \begin{cases} 
\Phi_n(X^m) & \text{if } m \perp n \\
0 & \text{otherwise}
\end{cases}$$

The relation (9) is just (6) if $n \perp m$. Likewise (10) reduces to (8) when $m \perp n$ as $d \mid m$ implies that $d \perp n$. If not, both sides of (9) vanish as $d = m$ is one of the divisors in the right hand side.

Some known properties of the cyclotomic polynomial follow easily from our identity.

**Corollary 2.3.** [Ge08, Corollary 2.3] If $p$ is a prime and $k \geq 1$ then,

$$\Phi_{p^k \cdot n}(X) = \begin{cases} 
\Phi_n(X^{p^k}) & \text{if } p \text{ divides } n \\
\frac{\Phi_n(X^{p^k})}{\Phi_n(X^{p^k-1})} & \text{if } p \text{ does not divide } n.
\end{cases}$$

**Proof.** We first consider the case in which $p$ does not divide $n$. We use theorem 2.1 with $m = p^k$.

$$\Phi_{p^k \cdot n}(X) = \prod_{c \mid p^k} \Phi_n(X^{m/c})^{\mu(c)} = \prod_{j=0}^{k} \Phi_n(X^{m/p^j})^{\mu(p^j)} = \frac{\Phi_n(X^{p^k})}{\Phi_n(X^{p^k-1})}$$

since by the definition of the Möbius function

$$\mu(p^j) = \begin{cases} 
1 & \text{for } j = 0 \\
-1 & \text{for } j = 1 \\
0 & \text{for } j \geq 2
\end{cases}$$

this proves the corollary in this case.

If $p$ divides $n$, we write $n = p^j \cdot n'$ where $p$ does not divide $n'$. Then, using what we have already proved,

$$\Phi_{p^k \cdot n}(X) = \Phi_{p^k+j \cdot n'}(X) = \frac{\Phi_{n'}(X^{p^k+j})}{\Phi_{n'}(X^{p^k+j-1})}.$$

Likewise

$$\Phi_n(X) = \Phi_{p^k \cdot n'}(X) = \frac{\Phi_{n'}(X^{p^k})}{\Phi_{n'}(X^{p^k-1})}.$$
Then, substituting $X^{p^j}$ for $X$,

$$
\Phi_n(X^{p^j}) = \Phi_n\left((X^{p^{k+1}})^{p^{k-1}}\right) = \Phi_{p^k-n}(X)
$$

as we have claimed. □

3. The multiplicative property of Euler’s totient function

In this section, we show how identity (6) is related to the multiplicative property of $\varphi$.

We remark that comparing the degree of both sides in the Fundamental Identity (1) gives a well-known property of Euler’s totient function

$$
\sum_{d|m} \varphi(d) = n.
$$

Likewise if we compare the degree of both sides in (6), we get that

$$
\sum_{d|m} \varphi(dn) = m\varphi(n) \text{ when } m \perp n.
$$

**Theorem 3.1.** The identity (12) is equivalent to the multiplicative property of $\varphi$

$$
\varphi(mn) = \varphi(m)\varphi(n) \text{ when } m \perp n
$$

in the sense that each property can be deduced from the other using (11).

**Proof.** Assume first that that $\varphi$ is multiplicative. Then (12) follows easily from (11) since $d|m \Rightarrow d \perp n$. Therefore,

$$
\sum_{d|m} \varphi(dn) = \sum_{d|m} \varphi(d)\varphi(n) = \varphi(n) \sum_{d|m} \varphi(d) = m\varphi(n).
$$

On the other hand, assume that (12) holds. We will show (13) holds by induction on $m$ (for every $n$ coprime with $m$). For $m = 1$, it holds trivially since $\varphi(1) = 1$. Assume then (13) holds for any $m' < m$. Then using (12)

$$
\sum_{d|m,d<m} \varphi(dn) + \varphi(nm) = m\varphi(n)
$$

Since $d|m \Rightarrow d \perp n$ and since $d < m$, we deduce using the induction hypothesis that

$$
\sum_{d|m,d<m} \varphi(d)\varphi(n) + \varphi(nm) = m\varphi(n)
$$
or

\[ \varphi(n) \sum_{d|m, d<m} \varphi(d) + \varphi(nm) = m\varphi(n) \]

But (11) gives

\[ \sum_{d|m, d<m} \varphi(d) = m - \varphi(m) \]

Therefore

\[ \varphi(n)[m - \varphi(m)] + \varphi(nm) = m\varphi(n) \Rightarrow \varphi(n)\varphi(m) = \varphi(nm) \]

By the principle of (complete) mathematical induction it follows that (13) holds for every \(m, n \in \mathbb{N} \).

\[ \square \]

4. Ramanujan sums

In this section, we will apply (6) to the Ramanujan sums (4), and deduce a formula for computing the coefficients of the cyclotomic polynomials.

We will make use of the logarithmic derivative operator

\[ L[P] = \frac{P'}{P} \]

on polynomials. We observe that it has the fundamental property

(14) \[ L[P \cdot Q] = L[P] + L[Q] \]

We will also use the method of generating functions. We need the following lemma (taken from [RPT60, appendix III to chapter X]):

**Lemma 4.1.** Let \( P \in \mathbb{C}[X] \) be a polynomial of degree \( N \) with complex coefficients,

\[ P(z) = \sum_{j=0}^{N} a_j z^j \quad \text{with } a_N \neq 0. \]

Let \( \rho_1, \rho_2, \ldots, \rho_N \) be the roots of \( P \) (repeated according to their multiplicity) and let

\[ S_q = S_q[P] := \rho_1^q + \rho_2^q + \ldots + \rho_N^q \]

be the sum of its \( q \)-powers. Then \( L[P] \) has the following Laurent expansion

(15) \[ L[P](z) = \frac{P'(z)}{P(z)} = \sum_{q=0}^{\infty} \frac{S_q}{z^{q+1}} \]

for \(|z| > M = \max_{1 \leq j \leq N} |\rho_j| \).
Proof. We have that
\[ P = a_n (z - \rho_1)(z - \rho_2) \ldots (z - \rho_N) \]
Using (14), we have that
\[ L[P](z) = \sum_{j=1}^{N} \frac{1}{z - \rho_j} \]
The lemma follows by expanding each term in a geometric series
\[ \frac{1}{z - \rho_j} = \frac{1}{z} \cdot \frac{1}{1 - (\rho_j/z)} = \frac{1}{z} \sum_{q=0}^{\infty} \left( \frac{\rho_j}{z} \right)^q = \sum_{q=0}^{\infty} \frac{\rho_j^q}{z^{q+1}} \text{ for } |z| > |\rho_j| \]
and adding the results (which is legitimate for |z| > M by the absolute convergence of the series).

By applying this lemma to the cyclotomic polynomial \( \Phi_n(z) \) we immediately get

Corollary 4.2. We have the following Laurent expansion for the logarithmic derivative of the cyclotomic polynomials:
\[ L[\Phi_n](z) = \frac{\Phi'_n(z)}{\Phi_n(z)} = \sum_{q=0}^{\infty} \frac{c_n(q)}{z^{q+1}} \text{ for } |z| > 1. \]

Remark 4.3. Let \( P \) be a polynomial and let \( Q(z) = P(z^m) \). Then
\[ L[Q](z) = m z^{m-1} L[P](z^m). \]
We are ready to see how property (6) applies to the Ramanujan sums:

Proposition 4.4. (6) implies that if \( n \perp m \),
\[ \sum_{d|m} c_{dn}(q) = \begin{cases} m \cdot c_n(q/m) & \text{if } m \mid q \\ 0 & \text{otherwise} \end{cases} \]

Proof. We consider the identity in Theorem 1.1. By taking the logarithmic derivative on both sides and using the previous remark, we get for \( |z| > 1, \)
\[ m z^{m-1} L[\Phi_n](z^m) = \sum_{d|m} L[\Phi_{d,n}](z). \]

We expand each side in a Laurent series
\[ \sum_{r=0}^{\infty} \frac{m c_n(r)}{z^{(r+1)m-(m-1)}} = \sum_{d|m} \sum_{q=0}^{\infty} \frac{c_{dn}(q)}{z^{q+1}} = \sum_{q=0}^{\infty} \left( \sum_{d|m} c_{dn}(q) \right) \frac{1}{z^{q+1}}. \]
By the uniqueness of the Laurent expansion, 
\[ \sum_{d|\gcd(m,q)} c_{dn}(q) = m \cdot c_n(r) \]
when \((r + 1)m - (m - 1) = q + 1 \Leftrightarrow rm = q\), and that the sum is zero otherwise. 

\[ \square \]

**Remark 4.5.** When \(q = 0\) this property reduces to (12), since \(c_n(0) = \varphi(n)\). 

As before, using the additive version of Möbius inversion formula, we get

**Corollary 4.6.** If \(n \perp m\),
\[ c_{mn}(q) = \sum_{d|\gcd(m,q)} d c_n\left(\frac{q}{d}\right) \mu\left(\frac{m}{d}\right). \]

In particular, if we choose \(n = 1\), \(c_1(q/d) = 1\) and we get the following known explicit formula for the Ramanujan sums due to Kluyver [Klu06]

See also [Apo76, Theorem 8.6]

\[ \text{(16)} \quad c_m(q) = \sum_{d|\gcd(m,q)} d \mu\left(\frac{m}{d}\right) \quad \forall m \in \mathbb{N}. \]

Another explicit formula for the Ramanujan sums is

\[ \text{(17)} \quad c_m(q) = \frac{\mu\left(\frac{m}{\gcd(m,q)}\right) \varphi(m)}{\varphi\left(\frac{m}{\gcd(m,q)}\right)} \]

The function on the right hand side was initially studied by von Sternneck [Ste02]. Later, Kluyver [Klu06] and also Hölder [Hölo36] proved that it coincides with the Ramanujan sums. See [FGK14] for more information on the Ramanujan sums and their history.

We conclude this note by explaining how the coefficients of the cyclotomic polynomial \(\Phi_n(X)\) can be recursively computed using the Ramanujan sums, without the need of factoring polynomials.

**Lemma 4.7 (Newton Relations).** Let \(P \in \mathbb{C}[X]\) of degree \(N\) and consider the sums \(S_q\) of the \(q\)-powers of its roots as in lemma 4.1. Then the coefficients \(a_j\) of \(P\) are related to the sums \(S_q\) by:
\[ a_{N-\ell} = -\frac{1}{\ell} \sum_{j=0}^{\ell-1} a_{N-j} \cdot S_{\ell-j} \quad \text{for} \quad j = 1, 2, \ldots, N - 1 \]
This result follows from Lemma 4.1 by writing (15) as
\[
\sum_{r=1}^{N} r a_r z^{r-1} = \left( \sum_{j=0}^{N} a_j z^j \right) \cdot \left( \sum_{q=0}^{\infty} S_q z^{q+1} \right)
\]
and equating the coefficients on both sides. See [RPT60, appendix III to chapter X] for details\footnote{Beware that in this book the notation for the coefficient of $X^j$ in $P$ is $a_{N-j}$ instead of $a_j$.}

**Corollary 4.8.** Let $\Phi_n(X)$ be the cyclotomic polynomial. Its coefficients $a_j(n)$ (for $0 \leq j \leq N = \varphi(n)$) can be recursively computed in terms of the Ramanujan sums using the relation
\[
a_{N-\ell}(n) = -\frac{1}{\ell} \sum_{j=0}^{\ell-1} a_{N-j}(n) \cdot c_n(\ell - j) \quad \text{for } \ell = 1, 2, \ldots, N - 1,
\]
starting from
\[
a_N = 1
\]
Together with (16) or (17) these formulas provide an algorithm for computing $\Phi_n(X)$ without the need of dividing polynomials.

More information on the coefficients of cyclotomic polynomials and their relations to other arithmetical functions can be found in [HM20]. We also refer those readers who are interested in efficient algorithms for the computation of cyclotomic polynomials to [AM11] and [Bre93].

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