SIMPLE CURRENT EXTENSIONS OF TENSOR PRODUCTS OF VERTEX OPERATOR ALGEBRAS

HIROMICHI YAMADA AND HIROSHI YAMAUCHI

Abstract. We study simple current extensions of tensor products of two vertex operator algebras satisfying certain conditions. We establish the relationship between the fusion rule for the simple current extension and the fusion rule for a tensor factor. In a special case, we construct a chain of simple current extensions. As an example, we obtain a chain of simple current extensions starting from the simple affine vertex operator algebra associated with $\hat{sl}_2$ at level $k \in \mathbb{Z}_{>0}$. The irreducible modules are classified and the fusion rules are determined for those simple current extensions.

1. Introduction

Simple current extensions of vertex operator algebras have been studied for many years, see [6, 7, 18, 33] and the references therein. In this paper, we consider simple current extensions of tensor products of two vertex operator algebras with suitable properties.

We argue under the following setting. Let $W$ and $V$ be simple, rational and $C_2$-cofinite vertex operator algebras of CFT-type such that the conformal weight of any irreducible module is positive except the vertex operator algebras themselves. Assume that all irreducible $V$-modules are simple currents. This means that $\text{Irr}(V) = \{V^\alpha | \alpha \in C\}$ is a $C$-graded set of simple current $V$-modules for a finite abelian group $C$ with $V^0 = V$. Let $D$ be a subgroup of $C$ and $\{W^\beta | \beta \in D\}$ a $D$-graded set of simple current $W$-modules with $W^0 = W$. Assume further that the conformal weight of $W^\beta \otimes V^\beta$ is an integer for all $\beta \in D$. Then a direct sum $U = \bigoplus_{\beta \in D} W^\beta \otimes V^\beta$ has a unique vertex operator algebra structure as an extension of $W \otimes V$. The vertex operator algebra $U$ is simple, rational, $C_2$-cofinite and of CFT type. Moreover, the conformal weight of any irreducible $U$-module is positive except $U$.

Such a triple $(U, V, W)$ appears when we consider the commutant of a subalgebra in a vertex operator algebra. In fact, the commutant of $V$ in $U$ is $W$ and the commutant of $W$ in $U$ is $V$ in our setting.

One of the important examples is the parafermion vertex operator algebra $K(g, k)$ associated with a finite dimensional simple Lie algebra $g$ and a positive integer $k$. Let $L_{\tilde{g}}(k; 0)$ be a simple affine vertex operator algebra associated with the affine Kac-Moody Lie algebra $\tilde{g}$ at level $k$ [4, 23]. Let $Q_L$ be the sublattice of the root lattice of $g$ spanned by the long roots. Then $L_{\tilde{g}}(k; 0)$ contains a lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ associated with $\sqrt{k}$ times the lattice $Q_L$ and $K(g, k)$ is the commutant of $V_{\sqrt{k}Q_L}$ in $L_{\tilde{g}}(k; 0)$. Thus $L_{\tilde{g}}(k; 0)$ contains a tensor product $K(g, k) \otimes V_{\sqrt{k}Q_L}$ as a subalgebra. Moreover, $L_{\tilde{g}}(k; 0)$ is

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a simple current extension of $K(\mathfrak{g},k) \otimes V_{\sqrt{k}QL}$. The parafermion vertex operator algebras have been studied extensively in the last decade, see for example [2, 4, 10, 11, 15, 16, 17].

Another important example of such a triple $(U, V, W)$ is known in a certain W-algebras, see for example [3, 8].

The representation theory of simple current extensions was developed in [33]. Since any irreducible $U$-module is a direct sum of inequivalent irreducible $W \otimes V$-modules by our assumption on the vertex operator algebras $W$ and $V$, the argument for the representation theory of $U$ is much simpler than that for a general one.

As our first result, we classify the irreducible $U$-modules and establish the relationship between the fusion rules for $U$ and $W$ under the above setting. The relationship between the fusion algebras of $U$ and $W$ is discussed as well, see Section 3 for details. The argument for the classification of irreducible $U$-modules is standard [33]. As to the relationship between the fusion rules for $U$ and $W$, our argument is motivated by the proofs of [2, Theorem 5.2] and [17, Theorem 4.2]. The quantum dimension plays a role.

One of the features of a $D$-graded simple current extension is that an irreducible twisted $U$-module but also those irreducible twisted $U$-modules, see Theorem 3.7.

In Section 3.3, we discuss a duality between the fusion algebras of $U$ and $W$. Since $\text{Irr}(V) = \{V^{\alpha} \mid \alpha \in C\}$ is a $C$-graded set of simple current $V$-modules, the group $C$ has a structure of a quadratic space with a quadratic form $q_V$ and its associated bilinear form $b_V$ [18, Theorem 3.4]. The bilinear form $b_V$ is non-degenerate [18, Proposition 3.5]. It turns out that $U$ has a $D^\perp$-graded set $\{U^{\gamma} \mid \gamma \in D^\perp\}$ of simple current $U$-modules, where $D^\perp$ is the orthogonal complement of $D$ in $C$ with respect to $b_V$. We consider an action of $D$ on $\text{Irr}(W)$ (resp. $D^\perp$ on $\text{Irr}(U)$) defined by $X \mapsto W^\beta \boxtimes W X$ for $\beta \in D$, $X \in \text{Irr}(W)$ (resp. $X \mapsto U^{\gamma} \boxtimes U X$ for $\gamma \in D^\perp$, $X \in \text{Irr}(U)$). We show that the set of $D$-orbits in $\text{Irr}(W)$ and the set of $D^\perp$-orbits in $\text{Irr}(U)$ are in one to one correspondence. The fusion algebras of $U$ and $W$ are related to each other through the correspondence.

In the case $U = L_\mathfrak{g}(k,0)$ and $W = K(\mathfrak{g},k)$, Corollary 3.10 and Theorem 3.19 correspond to [2, Theorem 5.1] and [2, Theorem 5.2], respectively. Furthermore, Remark 3.21 is related to [2, Theorem 4.7].

Our second result is a construction of a chain of simple current extensions starting from $U$ in the case where $V$ is a rank one lattice vertex operator algebra, see Theorem 2.11. In particular, simple, rational and $C_2$-cofinite vertex operator algebras of CFT-type with the same central charge are obtained.

Our third result is an example of such a chain of simple current extensions. We study a simple current extension $U^0 = \bigoplus_{j=0}^{k-1} M_{(k)}^j \otimes V_{zd - \frac{j}{k} d}$ of $K(\mathfrak{sl}_2, k) \otimes V_{zd}$ with $\langle d, d \rangle = 2k(sk + 1)$ for any $s \in \mathbb{Z}_{\geq 0}$. We describe the irreducible modules explicitly and determine the fusion rules for the vertex operator algebra $U^0$, see Section 4 for details. If $s = 0$, then $U^0$ coincides with the simple affine vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)$.

The paper is organized as follows. Section 2 is devoted to preliminaries. We recall basic properties of fusion rules and quantum dimensions of modules for vertex operator algebras. We also review simple current extensions of vertex operator algebras. We consider a chain of simple current extensions of special type. In Section 3, we study a simple current extension $U$ of a tensor product $W \otimes V$ of vertex operator algebras $W$.
and V with suitable properties. We discuss irreducible U-modules as well as irreducible twisted U-modules. We establish the relationship between the fusion rules for U and W and discuss the fusion algebras of U and W. In Section 4, we introduce a vertex operator algebra $U^0$, which is a simple current extension of a tensor product of $K(\mathfrak{sl}_2, k)$ and a rank one lattice vertex operator algebra. We classify the irreducible modules and determine the fusion rule for $U^0$.

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2. Preliminaries

In this section, we recall some basic properties of fusion rules and quantum dimensions of modules for vertex operator algebras. Moreover, we discuss simple current extensions of vertex operator algebras and their irreducible twisted modules. We mention a chain of simple current extensions as well. Our notations for vertex operator algebras and their modules are standard [19, 20, 27].

2.1. Fusion rules. Let $V$ be a simple, rational and $C_2$-cofinite vertex operator algebra of CFT-type. Then for $V$-modules $M_i$, $i = 1, 2, 3$, the set $I_V\left(\prod_i M_i\right)$ of all intertwining operators of type $\prod_i M_i$ is a finite dimensional vector space. The dimension $\dim I_V\left(\prod_i M_i\right)$ is called the fusion rule among $M_1$, $M_2$ and $M_3$. Furthermore, a fusion product $M \boxtimes V N$ over $V$ of any $V$-modules $M$ and $N$ exists [24, 28]. The fusion product is commutative and associative [22].

Denote by $\text{Irr}(V)$ the set of equivalence classes of irreducible $V$-modules. Then

$$M_1 \boxtimes V M_2 = \sum_{M_3 \in \text{Irr}(V)} \dim I_V\left(M_3\left(M_1 M_2\right)\right)M_3$$

(2.1)

for $M_1, M_2 \in \text{Irr}(V)$. A vector space $R(V)$ with basis $\text{Irr}(V)$ equipped with multiplication $\boxtimes V$ is called the fusion algebra of $V$.

An irreducible $V$-module $M$ is called a simple current if $M \boxtimes V X$ is an irreducible $V$-module for any $X \in \text{Irr}(V)$. We denote by $\text{Irr}(V)_{sc}$ the set of equivalence classes of simple current $V$-modules.

A set $M^\alpha$, $\alpha \in D$ of simple current $V$-modules indexed by an abelian group $D$ is said to be $D$-graded if $M^\alpha$, $\alpha \in D$ are inequivalent to each other with $M^0 = V$ and

$$M^\alpha \boxtimes V M^\beta = M^{\alpha + \beta}, \quad \alpha, \beta \in D.$$

It is shown in [26, Corollary 1] that $\text{Irr}(V)_{sc}$ forms a finite abelian group in the fusion algebra $R(V)$ when $V$ is self-dual. That is, $\text{Irr}(V)_{sc}$ is a set of simple current $V$-modules graded by a finite abelian group if $V$ is self-dual. The inverse of $M \in \text{Irr}(V)_{sc}$ with respect to the fusion product is its contragredient module $M'$.

The fusion product by a simple current changes the fusion rule slightly. In fact, the following proposition holds.
Proposition 2.1. Let V be a simple, rational, $C_2$-cofinite and self-dual vertex operator algebra of CFT-type. Let $\text{Irr}(V) = \{ M^i \mid i \in I \}$ with $|\text{Irr}(V)| = |I|$. Then for $A \in \text{Irr}(V)_\text{sc}$ and $i, j, k \in I$, the fusion rules satisfy
\[
\dim I_V \left( \frac{A \boxtimes_V M^k}{A \boxtimes_V M^i} \right) = \dim I_V \left( \frac{A \boxtimes_V M^k}{M^i \boxtimes_V M^j} \right) = \dim I_V \left( \frac{M^k}{M^i \boxtimes_V M^j} \right). \tag{2.2}
\]

Proof. Suppose $A \boxtimes_V M^i \cong M^{\sigma(i)}$ with $\sigma(i) \in I$. Then $\sigma$ induces a permutation on $I$ such that $A' \boxtimes_V M^i \cong M^{\sigma^{-1}(i)}$. Denote the fusion product by
\[
M^i \boxtimes V M^j = \sum_{k \in I} n^k_{ij} M^k, \quad n^k_{ij} = \dim I_V \left( \frac{M^k}{M^i \boxtimes V M^j} \right).
\]
Since the fusion product is commutative and associative, we have
\[
(A \boxtimes_V M^i) \boxtimes V M^j = M^i \boxtimes V (A \boxtimes_V M^j) = \sum_{k \in I} n^k_{ij} A \boxtimes V M^k. \tag{2.3}
\]
Therefore, we obtain $n^\sigma(k)_{\sigma(i)j} = n^\sigma(k)_{i\sigma(j)} = n^k_{ij}$ and the assertion follows. \hfill \Box

The following two propositions \cite[Proposition 2.9, Theorem 2.10]{[1]} will be used later.

Proposition 2.2. Let $M^1$, $M^2$, $M^3$ be modules for a vertex operator algebra $V$ among which $M^1$ and $M^2$ are irreducible. Let $U$ be a vertex operator subalgebra of $V$ with the same conformal vector. Suppose $N^1$ and $N^2$ are irreducible $U$-submodules of $M^1$ and $M^2$, respectively. Then the restriction map from $I_V \left( \frac{M^3}{M^1 \boxtimes V M^2} \right)$ to $I_U \left( \frac{M^3}{N^1 \boxtimes N^2} \right)$ is injective and
\[
\dim I_V \left( \frac{M^3}{M^1 \boxtimes V M^2} \right) \leq \dim I_U \left( \frac{M^3}{N^1 \boxtimes N^2} \right). \tag{2.4}
\]

Proposition 2.3. Let $V^1$ and $V^2$ be rational and $C_2$-cofinite vertex operator algebras and let $M^i$ and $N^i$, $i = 1, 2, 3$ be modules for $V^1$ and $V^2$, respectively. Then
\[
I_{V^1 \boxtimes V^2} \left( \frac{M^3 \boxtimes N^3}{M^1 \boxtimes N^1 \boxtimes N^2} \right) \cong I_{V^1} \left( \frac{M^3}{M^1 \boxtimes M^2} \right) \boxtimes I_{V^2} \left( \frac{N^3}{N^1 \boxtimes N^2} \right) \tag{2.5}
\]
as vector spaces.

2.2. Quantum dimensions. We assume that the vertex operator algebras discussed in this section satisfy the following hypothesis.

Hypothesis 2.4. $V$ is a simple, rational and $C_2$-cofinite vertex operator algebra of CFT-type and the conformal weight of any irreducible $V$-module except $V$ itself is positive.

Let $V$ be a vertex operator algebra satisfying Hypothesis 2.4. Since the contragredient module $V'$ of the adjoint module $V$ has conformal weight 0, it follows from Hypothesis 2.4 that $V$ is self-dual.

Let $M = \bigoplus_{n \in \mathbb{Z}_{\geq 2}} M_{\lambda + n}$ be an irreducible $V$-module with conformal weight $\lambda$. For $a \in V$ and $\tau$ in the upper half plane, we define the 1-point function by
\[
Z_M(a, \tau) = \text{tr}_M o(a) q^{L(0) - c/24} = q^{\lambda - c/24} \sum_{n=0}^{\infty} \text{tr}_{M_{\lambda+n}} o(a) q^n,
\]
where \( q = e^{2\pi \sqrt{-1} \tau} \), \( o(a) = a_{(\text{wt}(a) - 1)} \) is the zero-mode operator and \( c \) is the central charge of \( V \). Note that \( \text{ch}_q(M) = Z_M(1, \tau) \) is the conformal character of \( M \). It is shown in [34] that there exist unique \( S_{M,N} \in \mathbb{C} \) such that

\[
Z_M(a, -1/\tau) = \tau^{\text{wt}[a]} \sum_{N \in \text{Irr}(V)} S_{M,N} Z_N(a, \tau), \tag{2.6}
\]

where \( \text{wt}[a] \) is the \( L[0] \)-weight of \( a \in V \) in (loc. cit.) and \( S_{M,N} \) are independent of \( a \) and \( \tau \). The matrix \( S = [S_{M,N}]_{M,N \in \text{Irr}(V)} \) is called the \( S \)-matrix of \( V \). It is shown in [23] that \( S \) is symmetric and satisfies \( (S^2)_{M,N} = \delta_{M,N} \). Moreover, the fusion rules of \( V \)-modules are described by the Verlinde formula associated with \( S \).

Under Hypothesis [2,4], the limit

\[
\text{qdim}_V M = \lim_{y \to +0} \frac{Z_M(1, \sqrt{-1} y)}{Z_V(1, \sqrt{-1} y)} \tag{2.7}
\]

exists [4, Lemma 4.2], where \( y \) is positive real. The limit \( \text{qdim}_V M \) is called the quantum dimension of \( M \) over \( V \) [4, Section 3.1]. The notion of quantum dimensions amounts to a linear character of the fusion algebra \( R(V) \) of \( V \). The following properties of the quantum dimensions can be found in [4].

**Theorem 2.5.** Let \( V \) be a vertex operator algebra satisfying Hypothesis [2,4]. Then the following assertions hold.

1. \( \text{qdim}_V M = S_{V,M}/S_{V,V} \) for \( M \in \text{Irr}(V) \).
2. \( \text{qdim}_V(M^1 \boxtimes_V M^2) = \text{qdim}_V M^1 \cdot \text{qdim}_V M^2 \) for \( M^1, M^2 \in \text{Irr}(V) \).
3. \( 1 \leq \text{qdim}_V M < \infty \) for \( M \in \text{Irr}(V) \).
4. \( M \in \text{Irr}(V) \) is a simple current if and only if \( \text{qdim}_V M = 1 \).

Let \( V^1 \) and \( V^2 \) be vertex operator algebras and \( M^1 \) and \( M^2 \) modules for \( V^1 \) and \( V^2 \), respectively. It follows from the definition [23] of the quantum dimension that [4, Lemma 2.10]

\[
\text{qdim}_{V^1 \boxtimes V^2} M^1 \otimes M^2 = \text{qdim}_{V^1} M^1 \cdot \text{qdim}_{V^2} M^2. \tag{2.8}
\]

**2.3. Simple currents.** Let \( V \) be a vertex operator algebra satisfying Hypothesis [2,4]. For a \( V \)-module \( M \), we denote its conformal weight by \( \rho(M) \). It is shown in [13, Theorem 11.3] that \( \rho(M) \) and the central charge \( c \) are rational. Recall that the set \( \text{Irr}(V)_{\text{sc}} \) of equivalence classes of simple current \( V \)-modules forms a finite abelian group inside \( R(V) \).

We define two \( \mathbb{Q}/\mathbb{Z} \)-valued maps on \( \text{Irr}(V) \) and \( \text{Irr}(V)_{\text{sc}} \times \text{Irr}(V) \) by

\[
q_V(M) = \rho(M) + \mathbb{Z}, \quad b_V(A,M) = \rho(A \boxtimes_V M) - \rho(A) - \rho(M) + \mathbb{Z} \tag{2.9}
\]

for \( A \in \text{Irr}(V)_{\text{sc}} \) and \( M \in \text{Irr}(V) \). The maps \( q_V \) and \( b_V \) are introduced in [18] in the case where \( \text{Irr}(V)_{\text{sc}} = \text{Irr}(V) \). The same argument as in [18] Propositions 3.3, 3.5, Theorem 3.4 shows the following relations (see also [30, Propositions 2.1.4, 2.1.6]).

**Proposition 2.6.** Let \( V \) be a vertex operator algebra satisfying Hypothesis [2,4]. \( A \in \text{Irr}(V)_{\text{sc}} \) and \( M, N \in \text{Irr}(V) \). Then the following assertions hold.

1. \( S_{A&M,N} S_{V,N} = S_{A,N} S_{M,N} \).
2. \( S_{A&M} = e^{-2\pi \sqrt{-1} b_V(A,M)} S_{V,A&M} \).
(3) The restriction of \( b_V \) to \( \text{Irr}(V)^{sc} \times \text{Irr}(V)^{sc} \) is a symmetric bilinear form on \( \text{Irr}(V)^{sc} \).

(4) \( b_V \) is a non-degenerate symmetric bilinear form on \( \text{Irr}(V) \) if \( \text{Irr}(V)^{sc} = \text{Irr}(V) \).

(5) \( \kappa_V(A^{n\mathbb{Z}}) = n^2 \kappa_V(A) \) for \( n \in \mathbb{Z} \).

By this proposition, we see that \( \text{Irr}(V)^{sc} \) carries a structure of a finite quadratic space which is not necessarily non-degenerate when \( \text{Irr}(V)^{sc} \neq \text{Irr}(V) \).

**Corollary 2.7.** The map \( \kappa_V \) defines a quadratic form on \( \text{Irr}(V)^{sc} \).

The map \( b_V \) enjoys the following partial bilinearity on \( \text{Irr}(V)^{sc} \times \text{Irr}(V) \).

**Proposition 2.8.** Let \( V \) be a vertex operator algebra satisfying Hypothesis 2.4. \( A, B \in \text{Irr}(V)^{sc} \) and \( M \in \text{Irr}(V) \). Then the following assertions hold.

1. \( b_V(A \otimes B, M) = b_V(A, M) + b_V(B, M) \).
2. \( b_V(A, B \otimes M) = b_V(A, B) + b_V(A, M) \).

**Proof.** It follows from (2) of Proposition 2.6 and Theorem 2.5 that

\[
\frac{S_{A,M}}{S_{V,V}} = e^{-2\pi \sqrt{-1}b_V(A, M)} \kappa_V M
\]

for \( A \in \text{Irr}(V)^{sc} \) and \( M \in \text{Irr}(V) \).

We compute \( S_{A \otimes B, M}/S_{V,V} \) in two ways. On the one hand, we have

\[
\frac{S_{A \otimes B, M}}{S_{V,V}} = \frac{S_{A,M}}{S_{V,V}} \frac{S_{B,M}}{S_{V,V}} = \frac{S_{A,M}}{S_{V,V}} \frac{S_{B,M}}{S_{V,V}} \frac{S_{V,V}}{S_{V,M}} = e^{-2\pi \sqrt{-1}(b_V(A, M) + b_V(B, M))} \frac{(\kappa_V M)^2}{\kappa_V M}
\]

by (2.10). On the other hand, we have

\[
\frac{S_{A \otimes B, M}}{S_{V,V}} = e^{-2\pi \sqrt{-1}(b_V(A, B \otimes M))} \cdot \kappa_V B \otimes M = e^{-2\pi \sqrt{-1}(b_V(A, B \otimes M))} \cdot \kappa_V M
\]

by (2.10). Thus we obtain the assertion (1).

We consider \( S_{A, B \otimes M}/S_{V,V} \) in two ways. On the one hand, we have

\[
\frac{S_{A, B \otimes M}}{S_{V,V}} = \frac{S_{A, B \otimes M}}{S_{V,V}} \frac{S_{B, M}}{S_{V,V}} = \frac{S_{A, B}}{S_{V,V}} \frac{S_{A, M}}{S_{V,V}} \frac{S_{V,V}}{S_{V, A}} = e^{-2\pi \sqrt{-1}(b_V(A, B) + b_V(A, M))} \cdot \kappa_V B \cdot \kappa_V M
\]

by (2.10). On the other hand, since the \( S \)-matrix is symmetric, we have

\[
\frac{S_{A, B \otimes M}}{S_{V,V}} = \frac{S_{B, M}}{S_{V,V}} = \frac{S_{B, A} S_{M, A}}{S_{V,V}} = \frac{S_{A, B}}{S_{V,V}} \frac{S_{A, M}}{S_{V,V}} \frac{S_{V,V}}{S_{V, A}}
\]

Since \( \kappa_V A = \kappa_V B = 1 \), we have

\[
\frac{S_{A, B}}{S_{V,V}} \frac{S_{A, M}}{S_{V,V}} \frac{S_{V,V}}{S_{V, A}} = e^{-2\pi \sqrt{-1}(b_V(A, B) + b_V(A, M))} \cdot \kappa_V B \cdot \kappa_V M
\]

by (2.10). Comparing the equalities above, we obtain the assertion (2). \( \square \)

Suppose \( \text{Irr}(V)^{sc} = \{ V^{\alpha} \mid \alpha \in C \} \) is graded by a finite abelian group \( C \) with \( V = V^0 \) and \( V^{\alpha} \otimes_V V^{\beta} \cong V^{\alpha + \beta} \) for \( \alpha, \beta \in C \). We regard \( \kappa_V \) as a quadratic form on \( C \) by setting
abelian group $C$. 

$q_V(\alpha) = q_V(V\alpha)$. Likewise, we set $b_V(\alpha, \beta) = b_V(V\alpha, V\beta)$. For a subgroup $D$ of $C$, we obtain a $D$-graded set $\{V^\alpha \mid \alpha \in D\}$ of simple current $V$-modules. Set

$$D^\perp = \{\alpha \in C \mid b_V(\alpha, D) = 0\}. \quad (2.11)$$

Then $D^\perp$ is a subgroup of $C$. We say $\alpha \in C$ is isotropic if $q_V(\alpha) = 0$ and $D$ is totally isotropic if $q_V(D) = 0$. Note that $D \subset D^\perp$ if $D$ is totally isotropic, but the converse is not generally true since $b_V(\alpha, \alpha) = 2q_V(\alpha)$.

As discussed in [13], the set $\text{Irr}(V)_{sc}$ of equivalence classes of simple currents canonically defines a normalized abelian 3-cocycle $(F_V, \Omega_V) \in Z^3_{ab}(C, \mathbb{C}^\times)$, which is unique up to 3-coboundary, where $F_V : C^3 \to \mathbb{C}^\times$ and $\Omega_V : C^2 \to \mathbb{C}^\times$ are determined by the associativity and the skew-symmetry of the intertwining operators, respectively.

It is known in [12, 29] that the normalized cohomology class of $(F_V, \Omega_V)$ is uniquely determined by the associated quadratic form defined by $q_{\Omega}(\alpha) = \Omega_V(\alpha, \alpha)$ for $\alpha \in C$. Only the case $\text{Irr}(V)_{sc} = \text{Irr}(V)$ is considered in [13]. Based on Proposition 2.6, we can slightly generalize [18, Theorem 4.1] as follows.

**Theorem 2.9.** Let $V$ be a vertex operator algebra satisfying Hypothesis 2.4 and $\text{Irr}(V)_{sc} = \{V^\alpha \mid \alpha \in C\}$ the set of equivalence classes of simple current $V$-modules graded by a finite abelian group $C$. Then the direct sum $V_C = \bigoplus_{\alpha \in C} V^\alpha$ has a unique structure of a simple abelian intertwining algebra, which extends the $V$-module structure on $V_C$, with normalized abelian 3-cocycle $(F_V, \Omega_V)$ whose associated quadratic form is given by $q_{\Omega}(\alpha) = e^{-2\pi \sqrt{-1} q_V(\alpha)}$ for $\alpha \in C$.

As an application of Theorem 2.9, we obtain the following [18, Theorem 4.2] (see also [3, Theorem 2.4]).

**Theorem 2.10.** Let $V$ be a vertex operator algebra satisfying Hypothesis 2.4 and $\text{Irr}(V)_{sc} = \{V^\alpha \mid \alpha \in C\}$ the set of equivalence classes of simple current $V$-modules graded by a finite abelian group $C$. Let $D$ be a subgroup of $C$. Then the direct sum $V_D = \bigoplus_{\alpha \in D} V^\alpha$ has a unique structure of a simple vertex operator algebra which extends the $V$-module structure on $V_D$ if and only if $D$ is a totally isotropic subgroup of $C$ with respect to the quadratic form $q_V$.

The extension $V_D$ associated with a totally isotropic subgroup $D$ of $C$ is called a $D$-graded simple current extension of $V = V^0$. A general theory of representations of simple current extensions is well developed and it is known that $V_D$ also satisfies Hypothesis 2.4, see [13].

Let $D$ be a totally isotropic subgroup of $C$. We consider $V_D$-modules. Since $V_D = \bigoplus_{\alpha \in D} V^\alpha$ is a $D$-graded simple current extension, the character group $D^*$ of $D$ naturally acts on $V_D$. In fact, $\chi \in D^*$ acts by a scalar multiplication $\chi(\alpha)$ on $V^\alpha$. We regard $D^*$ as a subgroup of $\text{Aut}(V_D)$. Then the fixed point vertex operator subalgebra of $V_D$ by $D^*$ is $(V_D)^{D^*} = V$. If an automorphism $g$ of $V_D$ acts trivially on $V$, then there is $\chi \in D^*$ such that $\chi$ agrees with $g$.

By the definition of simple currents, the group $D$ acts on the set $\text{Irr}(V)$ by

$$M \mapsto V^\alpha \boxtimes_V M$$
for $\alpha \in D$ and $M \in \text{Irr}(V)$. Let $\mathcal{O} \subset \text{Irr}(V)$ be a $D$-orbit and take $M \in \mathcal{O}$. It follows from (1) of Proposition 2.8 that the map
\[ \xi_M : D \to \mathbb{Q}/\mathbb{Z}; \quad \alpha \mapsto b_V(V^\alpha, M) \] (2.12)
is a group homomorphism. Moreover, since $D \subset D^\perp$, it follows from (2) of Proposition 2.8 that this map is independent of the choice of $M \in \mathcal{O}$ and we can denote $\xi_M$ by $\xi_\mathcal{O}$.

By exponentiation we obtain a linear character
\[ \tilde{\xi}_\mathcal{O}(\alpha) = e^{2\pi i \xi_\mathcal{O}(\alpha)} \in \mathbb{C}^\times \] (2.13)
of $D$. Therefore, each $M \in \text{Irr}(V)$, as well as a $D$-orbit $\mathcal{O}$ in $\text{Irr}(V)$, defines an automorphism $\tilde{\xi}_\mathcal{O}$ of $V_D$. If $D$ acts on $\mathcal{O}$ freely, it is known that the $D$-orbit $\mathcal{O}$ uniquely defines an irreducible twisted $V_D$-module by [31, Proposition 3.8] and [33, Theorem 3.3].

**Theorem 2.11.** Let $V$ be a vertex operator algebra satisfying Hypothesis 2.4 and $\text{Irr}(V)_{\text{sc}} = \{V^\alpha \mid \alpha \in C\}$ the set of equivalence classes of simple current $V$-modules graded by $C$. Let $V_D$ be a simple current extension of $V$ associated with a totally isotropic subgroup $D$ of $C$. Let $M$ be an irreducible $V$-module and $\mathcal{O} = \{V^\alpha \boxtimes_V M \mid \alpha \in D\}$ its $D$-orbit in $\text{Irr}(V)$. Suppose $D$ acts on $\mathcal{O}$ freely. Then there exists a unique structure of an irreducible $\tilde{\xi}_\mathcal{O}$-twisted $V_D$-module on the direct sum $V_D \boxtimes_V M = \bigoplus_{\alpha \in D} V^\alpha \boxtimes_V M$ of $V$-modules which extends the $V$-module structure on $M$.

As to the notion of a $g$-twisted module for a vertex operator algebra with respect to its automorphism $g$, we adopt the definition in [33]. Thus a $g$-twisted module in [33] means a $g^{-1}$-twisted module in this paper.

For untwisted $V_D$-modules described in the theorem above, their fusion rules can be determined as follows [33, Lemma 2.16] (see also [31, Lemma 3.12]).

**Theorem 2.12.** Let $V$ be a vertex operator algebra satisfying Hypothesis 2.4 and $\text{Irr}(V)_{\text{sc}} = \{V^\alpha \mid \alpha \in C\}$ the set of equivalence classes of simple current $V$-modules graded by $C$. Let $V_D$ be a simple current extension of $V$ associated with a totally isotropic subgroup $D$ of $C$. Let $\mathcal{O}_i$, $i = 1, 2, 3$, be $D$-orbits in $\text{Irr}(V)$ on which $D$ acts freely. Suppose the associated characters $\tilde{\xi}_{\mathcal{O}_i}$ are trivial for $i = 1, 2, 3$. Take $M_i \in \mathcal{O}_i$ for $i = 1, 2, 3$ and consider $V_D$-modules $P_i = V_D \boxtimes_V M_i$. Then we have the following linear isomorphism
\[ I_{V_D}(P_3; P_1, P_2) \cong \bigoplus_{\alpha \in D} I_V(V^\alpha \boxtimes_V M_3; M_1, M_2). \]

In fact, the restriction map from $I_{V_D}(P_3; P_1, P_2)$ to $I_V(M_3; P_1, P_2)$ is bijective.

2.4. A chain of simple current extensions. Let $k, m \in \mathbb{Z}_{>0}$ and $Zd$ a rank one lattice spanned by an element $d$ with square norm $\langle d, d \rangle = 2km$. Let $W$ be a vertex operator algebra satisfying Hypothesis 2.4 and $\{W_j \mid j \in \mathbb{Z}_k\}$ a $\mathbb{Z}_k$-graded set of simple current $W$-modules with $W^0 = W$. We also recall the fusion rule for $V_{2d}$ [14, Chapter 12]. Let $U = \bigoplus_{j=0}^{k-1} W_j \otimes V_{Zd-\frac{j}{k}d}$ be a vertex operator algebra which is a $\mathbb{Z}_k$-graded simple current extension of $W \otimes V_{Zd}$. This in particular means that $W_j \otimes V_{Zd-\frac{j}{k}d}, 0 \leq j < k$ are of integral weight.

Let $s \in \mathbb{Z}$ with $m + sk > 0$ and $d'$ an element with square norm $\langle d', d' \rangle = 2k(m + sk)$. Then both the conformal weight of $V_{Zd-\frac{1}{k}d}$ and the conformal weight of $V_{Zd'-\frac{1}{k}d'}$ are
congruent to $\frac{\beta m}{k}$ modulo $\mathbb{Z}$. Thus $W^j \otimes V_{\mathbb{Z}d' - \frac{k}{d}d}$, $0 \leq j < k$ are of integral weight. These irreducible $W \otimes V_{\mathbb{Z}d'}$-modules form a $\mathbb{Z}_k$-graded set of simple current modules. Since the vertex operator algebra $W \otimes V_{\mathbb{Z}d'}$ satisfies Hypothesis 2.4, the direct sum $\tilde{U} = \bigoplus_{j=0}^{k-1} W^j \otimes V_{\mathbb{Z}d' - \frac{j}{d}}$ has a unique vertex operator algebra structure as an extension of $W \otimes V_{\mathbb{Z}d'}$ by Theorem 2.10. Hence the following theorem holds.

**Theorem 2.13.** Let $k, m \in \mathbb{Z}_{>0}$ and $d$ an element with $(d, d) = 2km$. Let $W$ be a vertex operator algebra satisfying Hypothesis 2.4. Let $\{W^j \mid j \in \mathbb{Z}_k\}$ be a $\mathbb{Z}_k$-graded set of simple current $W$-modules with $W^0 = W$. Assume that $U = \bigoplus_{j=0}^{k-1} W^j \otimes V_{\mathbb{Z}d' - \frac{j}{d}}$ is a $\mathbb{Z}_k$-graded simple current extension of $W \otimes V_{\mathbb{Z}d'}$. Then for $s \in \mathbb{Z}$ with $m + sk > 0$ and an element $d'$ with $(d', d') = 2k(m + sk)$, we have a $\mathbb{Z}_k$-graded simple current extension $\tilde{U} = \bigoplus_{j=0}^{k-1} W^j \otimes V_{\mathbb{Z}d' - \frac{j}{d}d'}$ of $W \otimes V_{\mathbb{Z}d'}$. The vertex operator algebra $\tilde{U}$ satisfies Hypothesis 2.4.

In this way, we obtain a chain of simple current extensions. The central charge of $\tilde{U}$ coincides with the central charge of $U$ and it is the central charge of $W$ plus 1.

In the case where $W$ is the parafermion vertex operator algebra $K(sl_2, k)$ and $m = 1$, we have $U = L_{sl_2}(k, 0)$. The vertex operator algebra $\tilde{U}$ for $U = L_{sl_2}(k, 0)$ will be discussed in Section 4.

### 3. Irreducible $U$-modules and Fusion Rules for $W$ and $U$

In this section, we assume the following hypothesis.

**Hypothesis 3.1.** $W$ and $V$ are vertex operator algebras satisfying the following conditions.

1. $W$ and $V$ satisfy Hypothesis 2.4.
2. All the irreducible $V$-modules are simple currents and $\text{Irr}(V) = \{V^\alpha \mid \alpha \in C\}$ is $C$-graded by a finite abelian group $C$ with $V^0 = V$ and $V^\alpha \otimes_V V^\beta = V^{\alpha + \beta}$ for $\alpha, \beta \in C$.
3. $D$ is a subgroup of $C$ and $\{V^\alpha \mid \alpha \in D\}$ is a $D$-graded set of simple current $W$-modules with $W^0 = W$ and $W^\alpha \otimes_V W^\beta = W^{\alpha + \beta}$ for $\alpha, \beta \in D$.
4. $\{W^\beta \otimes V^\beta \mid \beta \in D\}$ is a totally isotropic subgroup of $(\text{Irr}(W \otimes V)_{sc}, q_{W \otimes V})$. That is, the conformal weight of $W^\beta \otimes V^\beta$ is an integer for any $\beta \in D$.

Since $\text{Irr}(W \otimes V) = \{M \otimes V^\alpha \mid M \in \text{Irr}(W), \alpha \in C\}$, the tensor product $W \otimes V$ also satisfies Hypothesis 2.4. Then it follows from Theorem 2.10 that

$$U = \bigoplus_{\beta \in D} W^\beta \otimes V^\beta \quad (3.1)$$

is a $D$-graded simple current extension of $W \otimes V$ which also satisfies Hypothesis 2.4. In this section, we consider representations of $U$.

Recall that $b_V$ is a non-degenerate symmetric bilinear form on $C$ by Proposition 2.6 with $b_V(\alpha, \beta) = b_V(V^\alpha, V^\beta)$, $\alpha, \beta \in C$. We denote the orthogonal complement of $D$ in $C$ with respect to $b_V$ by

$$D^\perp = \{\alpha \in C \mid b_V(\alpha, D) = 0\}. \quad (3.2)$$

The following lemma is a direct consequence of Hypothesis 3.1.

**Lemma 3.2.** The commutant of $W$ in $U$ is $V$ and the commutant of $V$ in $U$ is $W$. 
Proof. Since \( W \) satisfies Hypothesis 2.3, the conformal weight of \( W^\beta \) is positive unless \( \beta = 0 \). Hence \( \{ u \in U \mid (\omega_W)_1 u = 0 \} = V \), where \( \omega_W \) is the conformal vector of \( W \). Thus the commutant of \( W \) in \( U \) is \( V \). Likewise, we have that the commutant of \( V \) in \( U \) is \( W \). \( \square 
\)

3.1. \( \text{Irr}(W) \) and \( \text{Irr}(U) \). Since \( \{ W^\alpha \mid \alpha \in D \} \) is a \( D \)-graded set of simple current \( W \)-modules, the group \( D \) acts on the set \( \text{Irr}(W) \) by

\[
X \mapsto W^\beta \otimes W X
\]

for \( \beta \in D \) and \( X \in \text{Irr}(W) \). Let

\[
\text{Irr}(W) = \bigcup_{i \in I} O^i_W
\]

be the orbit decomposition. For \( i \in I \) let

\[
D_i = \{ \beta \in D \mid V^\beta \otimes V M \cong M \text{ for any } M \in O^i_W \}.
\]

Since \( D \) is abelian, \( D_i \) coincides with the stabilizer of \( M \) for any \( M \in O^i_W \). Therefore, the length of \( O^i_W \) is \( [D : D_i] \).

Lemma 3.3. \( D_i \subset D^\perp \) for any \( i \in I \).

Proof. Let \( \gamma \in D_i, \beta \in D \) and \( M \in O^i_W \). Then by (2) of Proposition 2.8 we have

\[
b_W(W^\beta, M) = b_W(W^\beta, W^\gamma \otimes W M) = b_W(W^\beta, W^\gamma) + b_W(W^\beta, M)
\]

and \( b_W(W^\beta, W^\gamma) = 0 \). Since \( b_W(W^\beta, W^\gamma) = -b_V(\beta, \gamma) \) by (4) of Hypothesis 3.1, it follows that \( \gamma \in D^\perp \). \( \square 
\)

For each \( i \in I \), we pick \( W^{i,0} \in O^i_W \) and fix it. Here we assume that \( 0 \in I \) and assign \( W^{0,0} = W \) so that \( O^0_W = \{ W^\beta \mid \beta \in D \} \). Then we set

\[
W^{i,\beta} = W^\beta \otimes_V W^{i,0}
\]

for \( i \in I \) and \( \beta \in D \). Clearly we have \( O^i_W = \{ W^{i,\beta} \mid \beta \in D \} \).

Lemma 3.4. Let \( i, i' \in I \) and \( \beta, \beta' \in D \). Then \( W^{i,\beta} \cong W^{i',\beta'} \) as \( W \)-modules if and only if \( i = i' \) and \( \beta \equiv \beta' \mod D_i \).

Proof. Since the decomposition in (3.4) is disjoint, it is clear that \( W^{i,\beta} \cong W^{i',\beta'} \) only if \( i = i' \). By (3.9), \( W^{i,\beta} \cong W^{i,\beta+(\beta-\beta')} \) so that \( W^{i,\beta} \cong W^{i,\beta'} \) if and only if \( \beta - \beta' \in D_i \). \( \square 
\)

For \( i, i' \in I \) and \( \beta, \beta' \in D \), we write

\[
(i, \beta) \sim (i', \beta')
\]

if \( W^{i,\beta} \cong W^{i',\beta'} \) as \( W \)-modules. It is clear that this is an equivalence relation on \( I \times D \) and every element of \( \text{Irr}(W) \) is uniquely indexed by an element of \( I \times D/\sim \) as \( W^{i,\beta} \).

\[
\text{Irr}(W) = \{ W^{i,\beta} \mid (i, \beta) \in (I \times D)/\sim \}.
\]

(3.8)

Here and further we identify \( (i, \beta) \in I \times D \) with its equivalence class in \( (I \times D)/\sim \) by abuse of notation.

We can consider another action of \( D \) on \( \text{Irr}(W \otimes V) \) by

\[
X \mapsto (W^\beta \otimes V^\beta) \otimes W \otimes V X
\]

for \( \beta \in D \) and \( X \in \text{Irr}(W \otimes V) \).
Lemma 3.5. $D$ acts on $\text{Irr}(W \otimes V)$ freely by $(3.9)$.  

Proof. Let $\beta \in D$ and $X \in \text{Irr}(W \otimes V)$. By $(3.8)$, $X$ is isomorphic to $W^{i,\gamma} \otimes V^\alpha$ for some $(i, \gamma) \in I \times D$ and $\alpha \in C$. Then

$$
(W^\beta \otimes V^\beta) \boxtimes_{U \otimes V} X \cong (W^\beta \boxtimes_{W} W^{i,\gamma}) \otimes (V^\beta \boxtimes_{W} V^\alpha) \cong W^{i,\gamma + \beta} \otimes V^{\alpha + \beta}.
$$

Since $V^\alpha \cong V^{\alpha + \beta}$ as $V$-modules if and only if $\beta = 0$, the group action in $(3.9)$ is free. \hfill \square

We will call a free action of $D$ on $\text{Irr}(W \otimes V)$ given in $(3.9)$ the diagonal action. By Theorem 2.11, every irreducible $W \otimes V$-module can be uniquely extended to an irreducible $\chi$-twisted $U$-module for some $\chi \in D^*$. To describe irreducible twisted $U$-modules precisely, we introduce two $\mathbb{Q}/\mathbb{Z}$-valued maps on $D$ by

\begin{align*}
\eta_\alpha(\beta) &= b_V(V^\beta, V^\alpha), \\
\xi_{i,\alpha}(\beta) &= b_{W \otimes V}(W^\beta \otimes V^\beta, W^{i,0} \otimes V^\alpha) = b_W(W^\beta, W^{i,0}) + \eta_\alpha(\beta)
\end{align*}

for $i \in I$, $\alpha \in C$ and $\beta \in D$, where $b_W$ and $b_V$ are defined as in $(2.9)$. It follows from Proposition 2.8 that $\eta_\alpha$ and $\xi_{i,\alpha}$ are linear maps on $D$ so that we can define linear characters $\hat{\eta}_\alpha$ and $\hat{\xi}_{i,\alpha}$ of $D$ by

\begin{align*}
\hat{\eta}_\alpha(\beta) &= \exp(2\pi \sqrt{-1} \eta_\alpha(\beta)), \\
\hat{\xi}_{i,\alpha}(\beta) &= \exp(2\pi \sqrt{-1} \xi_{i,\alpha}(\beta)) \quad (3.10)
\end{align*}

for $\beta \in D$.

Lemma 3.6. Let $i \in I$. The following assertions hold.

1. The map $\hat{\eta}_C : C \to D^*$; $\alpha \mapsto \hat{\eta}_\alpha$ is an epimorphism with kernel $D^\perp$.
2. $\hat{\xi}_{i,\alpha + \delta} = \xi_{i,\alpha} + \eta_\delta$ and $\xi_{i,\alpha + \delta} = \xi_{i,\alpha} \delta$ for $\alpha, \delta \in C$.
3. $\hat{\xi}_{i,\alpha} = \xi_{i,\alpha'}$ if and only if $\alpha \equiv \alpha' \mod D^\perp$.

Proof. The assertion (1) follows from (4) of Proposition 2.6, and the assertion (2) is clear by the definitions in $(3.10)$ and $(3.11)$. The assertion (3) follows from (1) and (2). \hfill \square

Let

$$
U^{i,\alpha} = U \boxtimes_{W \otimes V} (W^{i,0} \otimes V^\alpha) = \bigoplus_{\beta \in D} W^{i,\beta} \otimes V^{\alpha + \beta} \quad (3.12)
$$

for $i \in I$ and $\alpha \in C$. The index $\alpha$ of $U^{i,\alpha}$ depends on the choice of a representative $W^{i,0}$ of the $D$-orbit $O_{W}^{i,0}$ in $\text{Irr}(W)$. In fact, $U \boxtimes_{W \otimes V} (W^{i,\beta} \otimes V^\alpha) = U^{i,\alpha - \beta}$.

Theorem 3.7. The following assertions hold.

1. $U^{i,\alpha}$ is an irreducible $\hat{\xi}_{i,\alpha}$-twisted $U$-module for $(i, \alpha) \in I \times C$.
2. For $(i, \alpha) = (i', \alpha') \in I \times C$, we have $U^{i,\alpha} \cong U^{i',\alpha'}$ as twisted $U$-modules if and only if $i = i'$ and $\alpha \equiv \alpha' \mod D_i$.
3. For $\chi \in D^*$ and $i \in I$, there exists $\alpha \in C$ such that $U^{i,\alpha}$ is an irreducible $\chi$-twisted $U$-module.
4. Let $\chi \in D^*$. Then any irreducible $\chi$-twisted $U$-module is isomorphic to $U^{i,\alpha}$ for some $(i, \alpha) \in I \times C$.

Proof. Since $U$ is a $D$-graded simple current extension of $W \otimes V$, the assertion (1) follows from Lemma 3.3 and Theorem 2.11.

It follows from the uniqueness in Theorem 2.11 that $U^{i,\alpha} \cong U^{i',\alpha'}$ as $U$-modules if and only if $U^{i,\alpha} \cong U^{i',\alpha'}$ as $W \otimes V$-modules. Since the summands of $U^{i,\alpha}$ and $U^{i',\alpha'}$ are those in $D$-orbits of $W^{i,0} \otimes V^\alpha$ and $W^{i',0} \otimes V^{\alpha'}$, respectively, $U^{i,\alpha}$ and $U^{i',\alpha'}$ are isomorphic.
$U$-modules if and only if $W^{i,0} \otimes V^\alpha$ and $W^{i',0} \otimes V^{\alpha'}$ are in the same $D$-orbit with respect to the diagonal action (3.9), which is equivalent to $i = i'$ and $\alpha \equiv \alpha' \mod D_i$. Thus we obtain the assertion (2).

As for (3), let $\chi \in D^*$ and $i \in I$. By Lemma 3.6, there exists $\alpha \in C$ such that $\chi = \hat{\xi}_{i,\alpha}$ as characters of $D$, and $U^{i,\alpha}$ gives an irreducible $\chi$-twisted $U$-module by (1).

Let $\chi \in D^*$ and $M$ an irreducible $\chi$-twisted $U$-module. Since $W \otimes V$ is rational, we can take an irreducible $W \otimes V$-submodule $X \cong W^{i,\gamma} \otimes V^\alpha$ for some $i \in I$, $\gamma \in D$ and $\alpha \in C$ by (3.8). Since the diagonal action (3.9) is free, the $W \otimes V$-modules $(W^{\beta} \otimes V^{\beta}) \mathbb{Z}_{W \otimes V} X$, $\beta \in D$, are all inequivalent. Therefore, by the irreducibility, $M$ is isomorphic to

$$U \otimes_{W \otimes V} X = \bigoplus_{\beta \in D} W^{i,\beta+\gamma} \otimes V^{\alpha+\beta} = U^{i,\alpha-\gamma}$$

as $W \otimes V$-modules. Since $W \otimes V$-module structure uniquely determines the $U$-module structure by Theorem 2.11, $M \cong U^{i,\alpha-\gamma}$ as $U$-modules. This completes the proof of the assertion (4).

For $i \in I$ and $\chi \in D^*$, we define

$$C(i, \chi) = \{ \gamma \in C \mid \hat{\xi}_{i,\gamma} = \chi \}. \quad (3.13)$$

**Lemma 3.8.** The following assertions hold.

1. $C(i, \chi)$ is a coset of $D^\perp$ in $C$ for $i \in I$ and $\chi \in D^*$.
2. $C(i, \eta_\alpha) = \alpha + C(i, \hat{1})$ for $\alpha \in C$, where $\hat{1}$ is the principal character of $D$.

**Proof.** It is shown in (3) of Theorem 3.7 that $C(i, \chi) \neq \emptyset$. By (1) and (2) of Lemma 3.6, we see that $D^* = \{ \xi_{i,\alpha} \mid \alpha \in C \}$, and the assertion (1) follows from (3) of Lemma 3.6.

The assertion (2) follows from (2) of Lemma 3.6. \qed

**Proposition 3.9.** For $\chi \in D^*$, the number of inequivalent irreducible $\chi$-twisted module is $|C||\text{Irr}(W)|/|D|^2$.

**Proof.** By Theorem 3.4, the set of irreducible $\chi$-twisted $U$-modules is given by

$$\{ U^{i,\alpha} \mid i \in I, \alpha \in C(i, \chi) \},$$

where $U^{i,\alpha} \cong U^{i',\alpha'}$ if and only if $i = i'$ and $\alpha \equiv \alpha' \mod D_i$. It follows from Lemmas 3.3 and 3.8 that among $U^{i,\alpha}$ with $\alpha \in C(i, \chi)$ there are exactly $[D^\perp : D_i]$ inequivalent irreducible $\chi$-twisted $U$-modules. Therefore, the number of inequivalent irreducible $\chi$-twisted $U$-modules is

$$\sum_{i \in I} [D^\perp : D_i] = \frac{|D^\perp|}{|D|} \sum_{i \in I} [D : D_i] = \frac{|C : D|}{|D|} \sum_{i \in I} |\mathcal{O}_W^i| = \frac{|C|}{|D|^2}|\text{Irr}(W)|$$

by (3.4). This completes the proof. \qed

**Corollary 3.10.** $|\text{Irr}(U)| = |C||\text{Irr}(W)|/|D|^2$.

**Remark 3.11.** Since every irreducible $U$-module is $\chi$-stable for $\chi \in D^*$, one can also show Corollary 3.10 by using the modular invariance of twisted modules [13, Theorem 10.2].
We have shown that the set of irreducible untwisted \( U \)-modules is given by
\[
\{ U^{i,\alpha} \mid i \in I, \alpha \in C(i, \hat{1}) \},
\]  
where \( C(i, \hat{1}) \) is a coset of \( D/i \) in \( C \) and \( U^{i,\alpha} \cong U^{i',\alpha'} \) as \( U \)-modules if and only if \( i = i' \) and \( \alpha \equiv \alpha' \mod D_i \).

3.2. Fusion rules of \( U \)-modules. We discuss the relationship between the fusion rules of \( U \)-modules and \( W \)-modules. First, we consider the quantum dimensions of irreducible \( U \)-modules. By a similar argument as in the proof of \([2, \text{Theorem 4.5}]\), we have the following theorem.

**Theorem 3.12.** Let \( i \in I, \alpha \in C(i, \hat{1}) \) and \( \beta \in D \). Then
\[
q\text{dim}_U U^{i,\alpha} = q\text{dim}_W W^{i,\beta}.
\]

**Proof.** It follows from \((3.6)\) and Theorem 2.5 that
\[
q\text{dim}_W W^{i,\beta} = q\text{dim}_W W^{\beta} \cdot q\text{dim}_W W^{i,0} = q\text{dim}_W W^{i,0}.
\]
Since any irreducible \( V \)-module is a simple current, we also have
\[
q\text{dim}_{W \otimes V} W^{i,\beta} \otimes V^{\alpha+\beta} = q\text{dim}_W W^{i,\beta} \cdot q\text{dim}_V V^{\alpha+\beta} = q\text{dim}_W W^{i,0}
\]
by \((2.8)\). Thus \( q\text{dim}_{W \otimes V} U^{i,\alpha} = |D| q\text{dim}_W W^{i,0} \) by the definition of \( U^{i,\alpha} \) in \((3.12)\). In particular, \( q\text{dim}_{W \otimes V} U = |D| q\text{dim}_W W^{0,0} = |D| \). Thus we have
\[
q\text{dim}_U U^{i,\alpha} = \frac{q\text{dim}_{W \otimes V} U^{i,\alpha}}{q\text{dim}_{W \otimes V} U} = q\text{dim}_W W^{i,0}.
\]
This completes the proof. \( \square \)

As an immediate corollary, we obtain

**Corollary 3.13.** Let \( I_{\text{sc}} = \{ i \in I \mid W^{i,0} \in \text{Irr}(W)_{\text{sc}} \} \). Then
\[
\text{Irr}(U)_{\text{sc}} = \{ U^{i,\alpha} \mid i \in I_{\text{sc}}, \alpha \in C(i, \hat{1}) \}
\]
and \( |\text{Irr}(U)_{\text{sc}}| = |C||\text{Irr}(W)_{\text{sc}}|/|D|^2 \). In particular, \( \text{Irr}(U)_{\text{sc}} = \text{Irr}(U) \) if and only if \( \text{Irr}(W)_{\text{sc}} = \text{Irr}(W) \).

**Proof.** An irreducible \( U \)-module is a simple current if and only if its quantum dimension is one so that \( \text{Irr}(U)_{\text{sc}} \) is as in the assertion by Theorem 3.12. Since \( \text{Irr}(W)_{\text{sc}} \) forms a multiplicative group in \( R(V) \), we have \( D_i = 0 \) for \( i \in I_{\text{sc}} \). Therefore, \(|\text{Irr}(W)_{\text{sc}}| = |I_{\text{sc}}|D| \) and \(|\text{Irr}(U)_{\text{sc}}| = |I_{\text{sc}}|D/|D^i| \). Thus \(|\text{Irr}(U)_{\text{sc}}| \) and \(|\text{Irr}(W)_{\text{sc}}| \) are related as in the assertion. \( \square \)

The global dimensions of \( W \) and \( U \) are related as follows (cf. \([2, \text{Theorem 2.9}]\)).

**Proposition 3.14.** The global dimensions of \( W \) and \( U \) are as follows.

1. \( \text{glob}(W) = \sum_{i \in I} [D : D_i](q\text{dim}_W W^{i,0})^2 \).
2. \( \text{glob}(U) = \sum_{i \in I} [D^i : D_i](q\text{dim}_W W^{i,0})^2 \).
3. \( |C| \text{glob}(W) = |D|^2 \text{glob}(U) \).

**Proof.** The assertion (1) follows from Lemma 3.4, and the assertion (2) follows from \((3.14)\) and Theorem 3.12. The assertion (3) follows from (1) and (2) as \(|C| = |D|D^i| \) (see also \([2, \text{Lemma 2.10}]\)). \( \square \)
Theorem 3.15. Let \( i, i', i'' \in I, \alpha \in C(i, 1), \alpha' \in C(i', 1), \alpha'' \in C(i'', 1) \) and \( \beta, \beta', \beta'' \in D \). Then the fusion rules of \( U \)-modules and those of \( W \)-modules are related as follows.

1. If \( \alpha + \alpha' - \alpha'' \in D \) then
   \[
   \dim I_U \left( \frac{U^{i''}, \alpha''}{U^{i', \alpha}} \right) = \dim I_W \left( \frac{W^{i''}, \alpha + \alpha' + \beta - \alpha''}{W^{i', \beta}} \right),
   \]
   and otherwise
   \[
   \dim I_U \left( \frac{U^{i''}, \alpha''}{U^{i', \alpha}} \right) = 0.
   \]

2. If \( \alpha + \beta + \alpha' + \beta' - \alpha'' - \beta'' \in D^\perp \) then
   \[
   \dim I_W \left( \frac{W^{i''}, \alpha''}{W^{i', \beta}} \right) = \dim I_U \left( \frac{U^{i''}, \alpha + \alpha' + \beta + \beta' - \beta''}{U^{i', \alpha'}} \right),
   \]
   and otherwise
   \[
   \dim I_W \left( \frac{W^{i''}, \alpha''}{W^{i', \beta}} \right) = 0.
   \]

Proof. By Proposition 2.3, Theorem 2.12 and (3.12), we have
\[
I_U \left( \frac{U^{i''}, \alpha''}{U^{i', \alpha}} \right) \cong \bigoplus_{\beta'' \in D} I_W \left( \frac{W^{i''}, \alpha''}{W^{i', \beta}} \right) \odot V^{\alpha + \beta''} \otimes V^{\alpha' + \beta''} = \bigoplus_{\beta'' \in D} I_W \left( \frac{W^{i''}, \alpha''}{W^{i', \beta}} \right) \otimes \left( \frac{V^{\alpha + \beta''}}{V^{\alpha' + \beta''}} \right).
\]
Since \( V^{\alpha + \beta} \boxtimes V^{\alpha' + \beta'} = V^{\alpha + \beta + \alpha' + \beta'} \), the assertion (1) holds.

For the proof of the assertion (2), we consider the fusion product
\[
W^{i, \beta} \boxtimes_W W^{i', \beta'} = \bigoplus_{X \in \text{Irr}(W)} \dim I_W \left( \frac{X}{W^{i, \beta}} \right) X.
\]
(3.15)
Since \( \text{Irr}(W) = \{W^{i'', \beta''} \mid (i'', \beta'') \in (I \times D)/\sim \} \) by (3.8), (3.13) can be written as
\[
W^{i, \beta} \boxtimes_W W^{i', \beta'} = \bigoplus_{(i'', \beta'') \in (I \times D)/\sim} n(i'', \beta'') W^{i'', \beta''},
\]
(3.16)
where
\[
n(i'', \beta'') = \dim I_W \left( \frac{W^{i'', \beta''}}{W^{i', \beta'}} \right).
\]

We fix \( \alpha, \alpha', \beta \) and \( \beta' \) and set
\[
P = \{ (i'', \beta'') \in I \times D \mid \alpha + \beta + \alpha' + \beta' - \beta'' \in C(i'', 1) \}, \quad P_\sim = (I \times D) \setminus P.
\]
Since \( C(i'', 1) = \alpha'' + D^\perp \) by Lemma 3.3, the condition \( \alpha + \beta + \alpha' + \beta' - \beta'' \in C(i'', 1) \) is equivalent to the condition \( \alpha + \beta + \alpha' + \beta' - \alpha'' - \beta'' \in D^\perp \). Since \( D_{i''} \subset D^\perp \) by Lemma 3.3, \( P \) is a union of equivalence classes with respect to the equivalence relation \( \sim \) in (3.7).
Therefore we can divide \((I \times D)/\sim\) into disjoint subsets as \((I \times D)/\sim = (P/\sim) \cup (P^c/\sim)\). Then (3.16) can be expressed as
\[
W^{i,\beta} \otimes_W W^{i',\beta'} = \bigoplus_{(i'',\beta'') \in P/\sim} n(i'',\beta'') W^{i'',\beta''} + \bigoplus_{(i'',\beta'') \in P^c/\sim} n(i'',\beta'') W^{i'',\beta''}. \tag{3.17}
\]
Since \(V^{\alpha+\beta} \otimes V^{\alpha'+\beta'} = V^{\alpha+\alpha'+\beta'+\beta''}\), we have
\[
n(i'',\beta'') W^{i'',\beta''} \otimes V^{\alpha+\alpha'+\beta'+\beta''} \subset (W^{i,\beta} \otimes V^{\alpha+\beta}) \otimes_W (W^{i',\beta'} \otimes V^{\alpha'+\beta'}). \]
Taking fusion products with \(U\) over \(W \otimes V\) we obtain
\[
U \otimes_W \otimes_V (W^{i,\beta} \otimes V^{\alpha+\beta}) = \bigoplus_{\gamma \in D} W^{i,\beta+\gamma} \otimes V^{\alpha+\beta+\gamma} = U^{i,\alpha},
\]
\[
U \otimes_W \otimes_V (W^{i',\beta'} \otimes V^{\alpha'+\beta'}) = \bigoplus_{\gamma \in D} W^{i',\beta'+\gamma} \otimes V^{\alpha'+\beta'+\gamma} = U^{i',\alpha'},
\]
\[
U \otimes_W \otimes_V (W^{i'',\beta''} \otimes V^{\alpha+\alpha'+\beta'+\beta''}) = \bigoplus_{\gamma \in D} W^{i'',\beta''+\gamma} \otimes V^{\alpha+\alpha'+\beta'+\beta''+\gamma} = U^{i'',\alpha+\alpha'+\beta'+\beta''}. \tag{3.18}
\]
This shows
\[
n(i'',\beta'') = \dim_I U \begin{pmatrix} U^{i'',\alpha+\alpha'+\beta'+\beta''} \\ U^{i,\alpha} \\ U^{i',\alpha'} \end{pmatrix}
\]
if \((i'',\beta'') \in P\). Now taking the quantum dimensions of both sides of (3.17) over \(W\) we obtain
\[
\text{qdim}_W W^{i,\beta} \cdot \text{qdim}_W W^{i',\beta'} = \sum_{(i'',\beta'') \in P/\sim} n(i'',\beta'') \text{qdim}_W W^{i'',\beta''} + \sum_{(i'',\beta'') \in P^c/\sim} n(i'',\beta'') \text{qdim}_W W^{i'',\beta''}. \tag{3.19}
\]
On the other hand, by Theorem 3.12, the quantum dimensions of both sides of (3.18) over \(U\) leads to
\[
\text{qdim}_U W^{i,\beta} \cdot \text{qdim}_U W^{i',\beta'} = \sum_{(i'',\beta'') \in P/\sim} n(i'',\beta'') \text{qdim}_U W^{i'',\beta''}. \tag{3.20}
\]
By (3.19) and (3.20) we obtain
\[
\sum_{(i'',\beta'') \in P^c/\sim} n(i'',\beta'') \text{qdim}_W W^{i'',\beta''} = 0
\]
and \(n(i'',\beta'') = 0\) for \((i'',\beta'') \in P^c\) as quantum dimensions are positive. This completes the proof of the assertion (2). \(\square\)
Remark 3.16. Let $i, i', i'' \in I$, $\alpha \in C(i, \hat{1})$, $\alpha' \in C(i', \hat{1})$, $\alpha'' \in C(i'', \hat{1})$ and $\beta, \beta', \beta'' \in D$ be as in Theorem 3.14. If $\alpha + \beta + \alpha' + \beta' - \alpha'' - \beta'' \in D^\perp$, then we can replace $\alpha''$ with $\alpha + \beta + \alpha' + \beta' - \beta''$ in the assertion (1) and obtain the first part of the assertion (2). Thus (1) and (2) are related under the correspondence

$$\beta'' = \alpha + \beta + \alpha' + \beta' - \alpha'' \quad \iff \quad \alpha'' = \alpha + \beta + \alpha' + \beta' - \beta''$$

(3.21)

provided that $\alpha + \alpha' - \alpha'' \in D$ and $\alpha + \beta + \alpha' + \beta' - \alpha'' - \beta'' \in D^\perp$.

Remark 3.17. We can also prove Theorem 3.13 from Propositions 2.2 and 2.3. Indeed, we have the following inequalities from Propositions 2.2 and 2.3:

$$\dim I_U(U_{i, \alpha}^\prime U_{i', \alpha'}^\prime) \leq \sum_{\beta'' \in D} \dim I_W(W_{i, \beta} \otimes V_{\alpha'' + \beta''})$$

$$\dim I_U(U_{i, \alpha}^\prime U_{i', \alpha'}^\prime) \leq \dim I_W(W_{i, \beta} \otimes V_{\alpha'' + \beta''}) \cdot \dim I_Y(V_{\alpha'' + \beta''})$$

These inequalities are in fact sufficient to prove all the the assertions of Theorem 3.15 more directly without the use of Theorem 3.13.

We express the fusion products of $U$-modules and $W$-modules explicitly. We fix $i, i' \in I$, $\alpha \in C(i, \hat{1})$, $\alpha' \in C(i', \hat{1})$ and $\beta, \beta' \in D$. Set

$$P = \{(i'', \beta'') \in I \times D \mid \alpha + \beta + \alpha' + \beta' - \beta'' \in C(i'', \hat{1})\},$$

$$Q = \{(i'', \alpha'') \in I \times C \mid \alpha'' \in C(i'', \hat{1}), \alpha + \alpha' - \alpha'' \in D\}.$$ (3.22)

The set $P$ has already appeared in the proof of Theorem 3.13. Define a map $\psi$ by

$$\psi : P \rightarrow Q; \quad (i'', \beta'') \mapsto (i'', \alpha + \beta + \alpha' + \beta' - \beta'').$$ (3.23)

Then $\psi$ is a bijection as we discussed in Remark 3.10. The inverse $\psi^{-1}$ is given by

$$\psi^{-1} : Q \rightarrow P; \quad (i'', \alpha'') \mapsto (i'', \alpha + \beta + \alpha' + \beta' - \alpha'').$$ (3.24)

Since $\psi$ is a bijection, we can consider an equivalence relation $\sim$ on $Q$ by pullback. That is, we define $x \sim y$ for $x, y \in Q$ if and only if $\psi^{-1}(x) \sim \psi^{-1}(y)$ in $P$. By abuse of notation, we identify an element in $P$ (resp. $Q$) with its equivalence classes in $P/\sim$ (resp. $Q/\sim$).

Lemma 3.18. All $U_{i', \alpha''}$ are inequivalent irreducible $U$-modules for $(i', \alpha'') \in Q/\sim$.

Proof. Since $D_{i'} \subset D$, the assertion follows from (2) of Theorem 3.4. □

By Theorem 3.13, the fusion products of $U$-modules and $W$-modules are mutually related as follows.
Theorem 3.19. Let \( i, i' \in I, \alpha \in C(i, \widehat{1}), \alpha' \in C(i', \widehat{1}) \) and \( \beta, \beta' \in D \). Then the fusion product \( W^{i,\beta} \boxtimes_W W^{i',\beta'} \) and the fusion product \( U^{i,\alpha} \boxtimes_U U^{i',\alpha'} \) are given as follows.

\[
W^{i,\beta} \boxtimes_W W^{i',\beta'} = \sum_{(i'',\alpha'') \in Q/\sim} \dim_U \left( \frac{U^{i'',\alpha''}}{U^{i,\alpha}} \frac{U^{i',\alpha'}}{U^{i',\alpha'}} \right) W^{\psi^{-1}(i'',\alpha'')},
\]

\[
U^{i,\alpha} \boxtimes_U U^{i',\alpha'} = \sum_{(i'',\alpha'') \in P/\sim} \dim_W \left( \frac{W^{i'',\beta''}}{W^{i,\beta}} \frac{W^{i',\beta'}}{W^{i',\beta'}} \right) U^{\psi(i'',\alpha'')},
\]

where \( P, Q, \psi \) and \( \psi^{-1} \) are defined as in (3.22), (3.23) and (3.24), respectively.

Next, we describe some simple current \( U \)-modules. Let

\[
U^\gamma = U^0,\gamma = \bigoplus_{\beta \in D} W^\beta \otimes V^{\beta + \gamma}
\]

for \( \gamma \in D^\perp \).

Corollary 3.20. \( \{U^\gamma \mid \gamma \in D^\perp\} \) is a \( D^\perp \)-graded set of simple current \( U \)-modules. Furthermore, \( U^\gamma \boxtimes_U U^i,\alpha = U^{i,\alpha + \gamma} \) for \( U^i,\alpha \in \text{Irr}(U) \).

Proof. By Theorem 3.19 and Corollary 3.13, all \( U^\gamma, \gamma \in D^\perp \) are inequivalent simple current \( U \)-modules. Let \( U^i,\alpha \in \text{Irr}(U) \). Then \( U^{i,\alpha + \gamma} \in \text{Irr}(U) \) by Lemma 3.8 and we obtain the desired fusion product \( U^\gamma \boxtimes_U U^i,\alpha = U^{i,\alpha + \gamma} \) by Theorem 3.19 as \( W^\beta \boxtimes_W W^{i',\beta'} = W^{i',\beta + \beta'} \).

In particular, we have \( U^\gamma \boxtimes_U U^\gamma = U^{\gamma + \gamma} \) for \( \gamma, \gamma' \in D^\perp \). \( \square \)

Remark 3.21. Let \( X \in \text{Irr}(W) \) and \( M, N \in \text{Irr}(U) \). Suppose \( X \subset M \). Then it follows from Corollary 3.20 that \( N \) contains an irreducible \( W \)-submodule isomorphic to \( X \) if and only if \( N \cong U^\gamma \boxtimes_U M \) for some \( \gamma \in D^\perp \).

The following proposition follows from Proposition 2.1 and Corollary 3.20.

Proposition 3.22. Let \( i, i', i'' \in I \). Then the following assertions hold.

1. For \( \beta, \beta', \delta, \delta' \in D \), we have

\[
\dim I_W \left( \frac{W^{i'',\beta'' + \delta''}}{W^{i,\beta + \delta}} \frac{W^{i',\beta' + \delta'}}{W^{i',\beta'}} \right) = \dim I_W \left( \frac{W^{i'',\beta'' - \delta' - \delta + \delta''}}{W^{i,\beta}} \frac{W^{i',\beta' - \delta' - \delta + \delta''}}{W^{i',\beta'}} \right)
\]

2. For \( \alpha \in C(i, \widehat{1}), \alpha' \in C(i', \widehat{1}) \), \( \alpha'' \in C(i'', \widehat{1}) \) and \( \gamma, \gamma', \gamma'' \in D^\perp \), we have

\[
\dim I_U \left( \frac{U^{i'',\alpha'' + \gamma''}}{U^{i,\alpha + \gamma}} \frac{U^{i',\alpha' + \gamma'}}{U^{i',\alpha'}} \right) = \dim I_U \left( \frac{U^{i'',\alpha'' - \gamma - \gamma' + \gamma''}}{U^{i,\alpha}} \frac{U^{i',\alpha' - \gamma - \gamma' + \gamma''}}{U^{i',\alpha'}} \right)
\]

For our later purpose, we restate Theorem 3.13 in the following form.

Theorem 3.23. Let \( X^p \in \text{Irr}(W) \) and \( M^p \in \text{Irr}(U) \) for \( p = 1,2,3 \) and suppose \( X^1 \otimes V^{\alpha} \subset M^1, X^2 \otimes V^{\alpha'} \subset M^2 \) and \( X^3 \otimes V^{\alpha''} \subset M^3 \) for some \( \alpha, \alpha', \alpha'' \in C \). Let \( \gamma = \alpha + \alpha' - \alpha'' \).

1. If \( \gamma \in D \) then

\[
\dim I_U \left( \begin{array}{c} M^3 \\ M^1 \\ M^2 \end{array} \right) = \dim I_W \left( \begin{array}{c} W^\gamma \boxtimes_W X^3 \\ X^1 \\ X^2 \end{array} \right)
\]

and otherwise

\[
\dim I_U \left( \begin{array}{c} M^3 \\ M^1 \\ M^2 \end{array} \right) = 0.
\]
(2) If $\gamma \in D^\perp$ then
\[
\dim I_W\left(\begin{array}{c}
X^3 \\
X^1 \\
X^2
\end{array}\right) = \dim I_U\left(\begin{array}{c}
U^\gamma \boxtimes_U M^3 \\
M^1 \\
M^2
\end{array}\right),
\]
and otherwise
\[
\dim I_W\left(\begin{array}{c}
X^3 \\
X^1 \\
X^2
\end{array}\right) = 0.
\]

Indeed, since $W_{i,0}$ is an arbitrary chosen representative of a $D$-orbit $O_W$ in $\text{Irr}(W)$, we may take $X^1 = W_{i,0}$, $X^2 = W_{i',0}$ and $X^3 = W_{i'',0}$ for some $i, i', i'' \in I$. Then $M^1 = U^i, M^2 = U^{i', \alpha'}$ and $M^3 = U^{i'', \alpha''}$. Apply Theorem 3.15 with $\beta = \beta' = \beta'' = 0$. Then Theorem 3.23 follows from Corollary 3.20 and Proposition 3.22.

3.3. $R(U)$ versus $R(W)$. In the previous subsections, we have described the structure of the fusion algebra $R(U)$ based on $R(V)$ and $R(W)$. In this subsection, we discuss representations of $W$ based on $V$ and $U$. In fact, we will show that the structure of $R(W)$ can be described in terms of $R(U)$ and $R(V)$. Suppose $U$ is a $D$-graded simple current extension of $W \otimes V$ as in (3.1) where $W$ and $V$ satisfy Hypothesis 3.1. As shown in Lemma 3.2, $W$ is the commutant of $V$ in $U$ and $V$ is the commutant of $W$ in $U$.

Recall that $\{U^\gamma | \gamma \in D^\perp\}$ is a $D^\perp$-graded set of simple current $U$-modules by Corollary 3.20, where $U^\gamma, \gamma \in D^\perp$ are defined as in (3.28). We define an action of $D^\perp$ on $\text{Irr}(U)$ by
\[
X \mapsto U^\gamma \boxtimes_U X
\]
for $\gamma \in D^\perp$ and $X \in \text{Irr}(U)$. It also follows from Lemma 3.3 and Corollary 3.20 that for each $i \in I$ the set
\[
O^i_U = \{U^{i, \alpha} | \alpha \in C(i, \tilde{1})\}
\]
forms a $D^\perp$-orbit and we obtain the $D^\perp$-orbit decomposition
\[
\text{Irr}(U) = \bigcup_{i \in I} O^i_U.
\]
That is, we can use the same index set $I$ as in (3.4). Note that $O^0_U = \{U^\gamma | \gamma \in D^\perp\}$.

We forget the description (3.28) and regard $O^i_U$ as a $D^\perp$-orbit in $\text{Irr}(U)$. For each $i \in I$, we pick $M^{i,0} \in O^i_U$ and fix it, where we choose $M^{0,0} = U$. Then we set
\[
M^{i, \gamma} = U^\gamma \boxtimes_U M^{i,0}
\]
for $\gamma \in D^\perp$. Clearly we have
\[
O^i_U = \{M^{i, \gamma} | \gamma \in D^\perp\}
\]
and $\text{Irr}(U) = \{M^{i, \gamma} | i \in I, \gamma \in D^\perp\}$. We define
\[
(D^\perp)_i = \{\gamma \in D^\perp | U^\gamma \boxtimes_U M \cong M \text{ for any } M \in O^i_U\}.
\]
Since $D^\perp$ is abelian, $(D^\perp)_i$ coincides with the stabilizer of $M$ for any $M \in O^i_U$. We can describe the stabilizers $D_i$ in (3.4) in terms of $R(U)$ and $R(V)$ as follows.

Lemma 3.24. $(D^\perp)_i = D_i$ for any $i \in I$. 
Proof. If we regard \( \mathcal{O}_i^U \) as in (3.28), the equivalence classes of irreducible \( U \)-modules in \( \mathcal{O}_i^U \) are determined by (2) of Theorem 3.7. It follows from Lemma 3.3 and Corollary 3.20 that \( (D^\perp)_i = D_i \) for all \( i \in I \).

Next, we describe \( \text{Irr}(W) \). By Theorem 3.7, any irreducible \( W \)-module appears as a submodule of an irreducible \( U \)-module. Let \( M^{i,\gamma} \in \mathcal{O}_i^U \). Then \( M^{i,\gamma} \cong U^{i,\alpha} \) for some \( \alpha \in C(i, \mathbf{1}) \), and this \( \alpha \) is uniquely determined modulo \( D_i = (D^\perp)_i \) by Lemma 3.24. By the structure of irreducible \( U \)-modules, there exists a coset \( \lambda(i, \gamma) + D \in C/D \) such that \( M^{i,\gamma} \) has a decomposition of the form

\[
M^{i,\gamma} = \bigoplus_{\delta \in \lambda(i, \gamma) + D} X^{i,\gamma,\delta} \otimes V^\delta
\]

as a \( W \otimes V \)-module, where \( X^{i,\gamma,\delta} \in \text{Irr}(W) \) is the multiplicity of \( V^\delta \) in \( M^{i,\gamma} \). The set \( \{X^{i,\gamma,\delta} \mid \delta \in \lambda(i, \gamma) + D\} \) is independent of \( \gamma \) and coincides with a \( D \)-orbit \( \mathcal{O}_W = \{W^{i,\beta} \mid \beta \in D\} \) in \( \text{Irr}(W) \) in (3.4). In other words, the \( D \)-orbit \( \mathcal{O}_W \) is uniquely determined by the \( D^\perp \)-orbit \( \mathcal{O}_i^U \) as we mentioned in Remark 3.21. Moreover, since \( D_i = (D^\perp)_i \), we have \( X^{i,\gamma,\delta} \cong X^{i,\gamma,\delta'} \) as \( W \)-modules if and only if \( \delta \equiv \delta' \mod (D^\perp)_i \) by (3.12) and (2) of Theorem 3.7. Therefore, we obtain a decomposition of \( \text{Irr}(W) \) in terms of \( R(U) \) and \( R(V) \) as follows.

**Theorem 3.25.** Define a \( D^\perp \)-graded set of simple currents \( \{U^\gamma \mid \gamma \in D^\perp\} \subset \text{Irr}(U)_{sc} \) as in (3.26) and decompose \( \text{Irr}(U) \) into a disjoint union of \( D^\perp \)-orbits as in (3.29). Pick an irreducible \( U \)-module \( M^{i,0} \in \mathcal{O}_i^U \) for each \( i \in I \) and collect \( X^{i,0,\delta} \in \text{Irr}(W) \) in the decomposition (3.32) of \( M^{i,0} \). Then \( \text{Irr}(W) = \{X^{i,0,\delta} \mid i \in I, \delta \in \lambda(i,0) + D\} \) where \( X^{i,0,\delta} \cong X^{i',0,\delta'} \) if and only if \( i = i' \) and \( \delta \equiv \delta' \mod (D^\perp)_i \).

Now the fusion rules of irreducible \( W \)-modules are described as in (2) of Theorem 3.23 in terms of \( R(U) \) and \( R(V) \). Therefore, the structure of \( R(W) \) is completely determined by those of \( R(U) \) and \( R(V) \).

We close this section by summarizing outcomes of Hypothesis 3.3 as follows.

Let \( W \) and \( V \) be vertex operator algebras satisfying Hypothesis 3.1 and let \( U \) be a \( D \)-graded simple current extension of \( W \otimes V \) as in (3.1). Then \( U \) satisfies Hypothesis 2.7. We have a \( D \)-graded set of simple current \( W \)-modules \( \{W^\beta \mid \beta \in D\} \) as in Hypothesis 3.1 and a \( D^\perp \)-graded set of simple current \( U \)-modules \( \{U^\gamma \mid \gamma \in D^\perp\} \) which is defined as in (3.26). Consider the actions of \( D \) on \( \text{Irr}(W) \) and of \( D^\perp \) on \( \text{Irr}(U) \) as in (3.3) and (3.27), respectively. Let \( \mathcal{O}_W \) be the set of \( D \)-orbits in \( \text{Irr}(W) \) and \( \mathcal{O}_U \) the set of \( D^\perp \)-orbits in \( \text{Irr}(U) \), respectively. Then \( |\mathcal{O}_W| = |\mathcal{O}_U| = |I| \). Define \( \Phi : \mathcal{O}_U \rightarrow \mathcal{O}_W \) as follows. Let \( \mathcal{O} \in \mathcal{O}_U \) and \( M \in \mathcal{O} \). We define \( \Phi(\mathcal{O}) \) to be the set of equivalence classes of irreducible \( W \)-submodules of \( M \). Then \( \Phi(\mathcal{O}) \) is independent of the choice of \( M \) and well-defined. Moreover, \( \mathcal{O} \rightarrow \Phi(\mathcal{O}) \) is a bijection between \( \mathcal{O}_U \) and \( \mathcal{O}_W \).

The inverse \( \Psi : \mathcal{O}_W \rightarrow \mathcal{O}_U \) of \( \Phi \) is described as follows. Let \( \mathcal{O}' \in \mathcal{O}_W \) and \( X \in \mathcal{O}' \). Then the \( \mathbb{Q}/\mathbb{Z} \)-valued map \( \beta \mapsto b_W(W^\beta, X) \) is linear on \( D \) by Proposition 2.8. Since \( b_W \) is non-degenerate on \( C \) by (4) of Proposition 2.6, we can find \( \lambda \in C \) such that \( b_W(W^\beta, X) + b_V(V^\beta, V^\lambda) = 0 \) for all \( \beta \in D \). Such a \( \lambda \) is unique modulo \( D^\perp \) and uniquely determines a coset \( \lambda + D^\perp \) of \( D^\perp \) in \( C \). By (1) of Theorem 3.7, \( U \boxtimes_{W \otimes V} (X \otimes V^\alpha) \) is an irreducible untwisted \( U \)-module for \( \alpha \in \lambda + D^\perp \). We define

\[
\Psi(\mathcal{O}') = \{U \boxtimes_{W \otimes V} (X \otimes V^\alpha) \mid \alpha \in \lambda + D^\perp\},
\]
which in turn recovers \( \mathcal{O} = \Psi(\mathcal{O}') \) if \( \mathcal{O}' = \Phi(\mathcal{O}) \). Although \( \lambda \) depends on the choice of \( X \), the resulting \( D^\perp \)-orbit \( \Psi(\mathcal{O}') \) is independent of the choice of \( X \) and uniquely determined by \( \mathcal{O}' \), see the comment just after (3.12) and Remark 3.21.

It is shown in Theorem 3.12 that \( q\dim_U M = q\dim_W X \) if \( M \in \mathcal{O} = \Psi(\mathcal{O}') \) and \( X \in \mathcal{O}' = \Phi(\mathcal{O}) \) for \( \mathcal{O} \in \mathcal{O}_U \) and \( \mathcal{O}' \in \mathcal{O}_W \).

For \( \mathcal{O} \in \mathcal{O}_U \) and \( \mathcal{O}' \in \mathcal{O}_W \), define their stabilizers by
\[
(D^\perp)_\mathcal{O} = \{ \gamma \in D^\perp \mid U^\gamma \otimes_U M \cong M \text{ for any } M \in \mathcal{O} \},
\]
\[
D_{\mathcal{O}'} = \{ \beta \in D \mid W^\beta \otimes_W X \cong X \text{ for any } X \in \mathcal{O}' \}.
\]

Then \( D_{\Phi(\mathcal{O})} = (D^\perp)_\mathcal{O} \) and \( (D^\perp)_{\Phi(\mathcal{O}')} = D_{\mathcal{O}'} \) by Lemma 3.24 and we have
\[
\frac{|\mathcal{O}|}{|\Phi(\mathcal{O})|} = \frac{|\Psi(\mathcal{O}')|}{|\mathcal{O}'|} = \frac{|D^\perp|}{|D|}.
\]

This equation essentially explains Corollary 3.10 and (3) of Proposition 3.14.

The equivalence classes \( \mathrm{Irr}(U) \) and \( \mathrm{Irr}(W) \) are mutually described by Theorems 3.7 and 3.23, and their fusion rules are mutually described by Theorem 3.23. Thus, we can completely describe the structures of \( R(U) \) and \( R(W) \) each other based on the duality between \( D \) and \( D^\perp \) in the quadratic space \((C, q_V)\) associated with \( R(V) \).

\textbf{Remark 3.26.} Let \( \mathrm{Irr}(U; \chi) \) be the set of equivalence classes of irreducible \( \chi \)-twisted \( U \)-modules for \( \chi \in D^\times \). If \( \chi = \eta_\alpha \) for \( \alpha \in C \), then by Theorem 3.7 and (2) of Lemma 3.8 there is a bijection between \( \mathrm{Irr}(U) \) and \( \mathrm{Irr}(U; \chi) \) given by
\[
\Theta_\alpha : \mathrm{Irr}(U) \to \mathrm{Irr}(U; \chi); \quad \Theta_\alpha(M) = M \boxtimes_{W \otimes V} (W \otimes V^\alpha)
\]
for \( M \in \mathrm{Irr}(U) \). We can similarly consider the action of \( D^\perp \) on \( \mathrm{Irr}(U; \chi) \) and obtain the \( D^\perp \)-orbit decomposition of \( \mathrm{Irr}(U; \chi) \). Then the map \( \Theta_\alpha \) induces a bijective correspondence between \( \mathcal{O}_W \) and the set of \( D^\perp \)-orbits in \( \mathrm{Irr}(U; \chi) \) as well.

4. Simple current extension of \( K(\mathfrak{sl}_2, k) \otimes V_{Z^d} \)

In this section, we discuss a simple current extension of a tensor product of the parafermion vertex operator algebra \( K(\mathfrak{sl}_2, k) \) and a certain rank one lattice vertex operator algebra. The simple current extension is an example of the vertex operator algebra \( \gamma \)-current algebra \( U \) studied in Section 3. We provide an explicit description of the irreducible modules and argue as in Section 3 to determine the fusion rule.

4.1. Parafermion vertex operator algebra \( K(\mathfrak{sl}_2, k) \). We recall basic properties of the parafermion vertex operator algebra \( k(\mathfrak{sl}_2, k) \) associated with \( \mathfrak{sl}_2 \) and a positive integer \( k \).

The parafermion vertex operator algebra \( K(\mathfrak{sl}_2, k) \) is realized in a vertex operator algebra \( V_L \) associated with a lattice \( L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k \), where \( \langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j} \) [11], Section 4]. In fact, \( V_L \) contains a subalgebra isomorphic to \( L^-_{\mathfrak{sl}_2}(k, 0) \). Let \( \gamma = \alpha_1 + \cdots + \alpha_k \). Then \( \langle \gamma, \gamma \rangle = 2k \) and \( L^-_{\mathfrak{sl}_2}(k, 0) \) contains a subalgebra isomorphic to \( V_{Z^d} \). The commutant of \( V_{Z^d \gamma} \) in \( L^-_{\mathfrak{sl}_2}(k, 0) \) is \( K(\mathfrak{sl}_2, k) \). In the case \( k = 1 \), \( K(\mathfrak{sl}_2, k) \) reduces to the trivial vertex operator algebra \( \mathbb{C} \). Let
\[
M^1_{(k)} = \{ v \in L^-_{\mathfrak{sl}_2}(k, 0) \mid \gamma(n)v = -2j\delta_{n,0}v \text{ for } n \geq 0 \}.
\]
for $0 \leq j < k$. Then $M^0_{(k)} = K(\mathfrak{sl}_2, k)$ and

$$L_{\mathfrak{sl}_2}(k, 0) = \bigoplus_{j=0}^{k-1} M^j_{(k)} \otimes V_{Z_\gamma - \frac{j}{k}\gamma} \quad (4.1)$$

as $M^0_{(k)} \otimes V_{Z_\gamma}$-modules [11, Lemma 4.2].

An irreducible $L_{\mathfrak{sl}_2}(k, 0)$-module $L_{\mathfrak{sl}_2}(k, i)$ with $i + 1$ dimensional top level can be constructed in $V_L$ for $0 \leq i \leq k$, where $L^0 = \frac{1}{2}L$ is the dual lattice of $L$. Let

$$M^{i,j}_{(k)} = \{ v \in L_{\mathfrak{sl}_2}(k, i) \mid \gamma(n)v = (i - 2j)\delta_{n,0}v \text{ for } n \geq 0 \}$$

for $0 \leq j < k$. Then $M^0_{(k)} = M^j_{(k)}$ and

$$L_{\mathfrak{sl}_2}(k, i) = \bigoplus_{j=0}^{k-1} M^{i,j}_{(k)} \otimes V_{Z_\gamma + \frac{i-2j}{2k}\gamma} \quad (4.2)$$

as $M^0_{(k)} \otimes V_{Z_\gamma}$-modules [11, Lemma 4.3]. The index $j$ of $M^j_{(k)}$ and $M^{i,j}_{(k)}$ can be considered to be modulo $k$.

The following properties of $K(\mathfrak{sl}_2, k)$ can be found in [11 Theorems 7.3, 8.2], [11 Corollary 6.2], [11, Theorem 4.1] and [11, Theorem 4.4, Proposition 4.5].

1. $M^0_{(k)} = K(\mathfrak{sl}_2, k)$ is a simple, rational, $C_2$-cofinite vertex operator algebra of CFT-type with central charge $\frac{2(k-1)}{k+2}$.

2. $M^{i,j}_{(k)}, 0 \leq i \leq k, 0 \leq j < k$ are irreducible $M^0_{(k)}$-modules. They are inequivalent to each other except the isomorphism

$$M^{i,j}_{(k)} \cong M^{k-i,j-i}_{(k)} \quad (4.3)$$

The $\frac{1}{2}k(k + 1)$ irreducible modules $M^{i,j}_{(k)}, 0 \leq j < i \leq k$ is a complete set of representatives of equivalence classes of irreducible $M^0_{(k)}$-modules.

3. The top level of $M^{i,j}_{(k)}$ is one dimensional and its conformal weight is

$$\frac{1}{2k(k+2)} \left(2k(i-j) - (i - 2j)^2 + 2k(i + j + 1)j \right) \quad (4.4)$$

for $0 \leq j \leq i \leq k$. Eq. (4.4) is valid even in the case $j = i$.

The conformal weight (4.4) of $M^{i,j}_{(k)}$ is positive except $M^{i,j}_{(k)} = M^{k,0}_{(k)} \cong M^0_{(k)}$. Hence $M^0_{(k)}$ satisfies Hypothesis 2.3. The conformal weight of $M^{k,j}_{(k)} \cong M^{j}_{(k)}$ is $\frac{j(k-1)}{k}$.

The fusion product of irreducible modules for $K(\mathfrak{sl}_2, k)$ are known [17, Theorem 4.2] (see also [5, Section 6]). Among other things, $M^j_{(k)}, 0 \leq j < k$ are the simple currents and

$$M^j_{(k)} \boxtimes M^0_{(k)} = M^{j,j}_{(k)}, \quad M^j_{(k)} \boxtimes M^{i,j}_{(k)} = M^{i,j}_{(k)} \quad (4.5)$$

for $0 \leq j, j' < k, 0 \leq i \leq k$.

4. The automorphism group Aut($M^0_{(k)}$) of $M^0_{(k)}$ is trivial if $k = 1, 2$ and Aut($M^0_{(k)}$) = $\langle \theta \rangle$ is of order 2 if $k \geq 3$. Moreover, $M^{i,j}_{(k)} \circ \theta \cong M^{i,j}_{(k)}$. Actually, $\theta$ is the restriction of an automorphism of $V_L$ which is a lift of the $-1$ isometry of the lattice $L$. 
4.2. Irreducible modules for \( U^0 \). Let \( k \in \mathbb{Z}_{>0} \) and \( s \in \mathbb{Z}_{>0} \). Let \( Zd \) be a rank one lattice spanned by an element \( d \) with square norm
\[
\langle d, d \rangle = 2k(sk + 1).
\]

We consider a tensor product \( M^0_d(k) \otimes V_{Zd} \) of the parafermion vertex operator algebra \( M^0_d(k) = K(\mathfrak{s}\mathfrak{l}_2, k) \) and a vertex operator algebra \( V_{Zd} \) associated with the lattice \( Zd \). The vertex operator algebra \( M^0_d(k) \otimes V_{Zd} \) satisfies Hypothesis 2.4.

Let
\[
U^0 = \bigoplus_{j=0}^{k-1} M^j_d(k) \otimes V_{Zd-\frac{j}{k}d}.
\]

Then \( U^0 \) has a unique vertex operator algebra structure as a simple current extension of \( M^0_d(k) \otimes V_{Zd} \) by (4.1) and Theorem 2.13. The vertex operator algebra \( U^0 \) satisfies Hypothesis 2.4 and its central charge is \( \frac{3k}{k+2} \). Note that \( U^0 = L_{\hat{\mathfrak{sl}}_2}(k, 0) \) if \( s = 0 \).

The notation here corresponds to the notation in Section 3 as \( U = U^0, V = V_{Zd}, W = M^0_d(k), C = \mathbb{Z}_{2k(sk+1)} \) and \( D = \mathbb{Z}_k \).

We explicitly describe the irreducible modules for \( U^0 \). It turns out that there are exactly \((k+1)(sk+1)\) inequivalent irreducible \( U^0 \)-modules, which agrees with Corollary 3.10.

Recall from Section 4.1 that \( M^j_d(k), 0 \leq j < i \leq k \) is a complete set of representatives of equivalence classes of irreducible \( M^0_d(k) \)-modules. There is another choice of representatives. In fact, (4.5) implies that the group \( \mathbb{Z}_k \) acts on \( \text{Irr}(M^0_d(k)) \) via \( M^j_d(k) \mathbb{Z}_k M^0_d(k) \) for \( X \mapsto M^{ij}_d(k) \mathbb{Z}_k M^0_d(k) \). For \( X \in \text{Irr}(M^0_d(k)), 0 \leq j < k \).

If \( k \) is odd, then it follows from (4.3) that \( M^{0,0}_d(k), 0 \leq i \leq \frac{k-1}{2} \) is a complete set of representatives of the \( \mathbb{Z}_k \)-orbits in \( \text{Irr}(M^0_d(k)) \). Moreover, \( M^j_d(k) \mathbb{Z}_k M^0_d(k) \mathbb{Z}_k M^j_d(k), 0 \leq i \leq \frac{k-1}{2}, 0 \leq j < k \) is a complete set of representatives of equivalence classes of irreducible \( M^0_d(k) \)-modules. In this case, \( D_i = 0, 0 \leq i \leq \frac{k-1}{2} \).

If \( k \) is even, then \( M^{\frac{1}{2},0}_d(k) \cong M^j_d(k) \mathbb{Z}_k \frac{1}{2} \) as \( M^0_d(k) \)-modules by (4.3) and \( M^{0,i}_d(k), 0 \leq i \leq \frac{k}{2} \) is a complete set of representatives of the \( \mathbb{Z}_k \)-orbits in \( \text{Irr}(M^0_d(k)) \). Moreover, \( M^{ij}_d(k), 0 \leq i \leq \frac{k}{2}-1, 0 \leq j < k, M^{\frac{1}{2},j}_d(k), 0 \leq j < \frac{k}{2} \) is a complete set of representatives of equivalence classes of irreducible \( M^0_d(k) \)-modules. In this case, \( D_i = 0, 0 \leq i \leq \frac{k}{2} - 1 \) and \( D_{\frac{k}{2}} = \mathbb{Z}_2 \).

This choice of representatives of equivalence classes of irreducible \( M^0_d(k) \)-modules agrees with the notation in (3.8). In fact, \( I = \{0, 1, \ldots, \frac{k-1}{2}\} \) if \( k \) is odd and \( I = \{0, 1, \ldots, \frac{k}{2}\} \) if \( k \) is even. Moreover, \( (\frac{k}{2}, j) \sim (\frac{k}{2}, j - \frac{k}{2}) \) if \( k \) is even. Nevertheless, the representatives in this notation depend on the case \( k \) is odd or even. Therefore, in this section we choose \( M^{ij}_d(k), 0 \leq j < i \leq k \) as representatives of equivalence classes of irreducible \( M^0_d(k) \)-modules. The notation for the irreducible \( U^0 \)-modules here are also different from the notation for the irreducible \( U \)-modules used in Section 3, see (3.12) and (4.3) below. Thus the assertions in this section are slightly different in appearance from those in Section 3.

**Lemma 4.1.** Let \( 0 \leq i \leq k, 0 \leq j < k \) and \( 0 \leq p < 2k(sk+1) \). Then the difference of the conformal weight of the irreducible \( M^0_d(k) \otimes V_{Zd} \)-modules \( M^{ij}_d(k) \otimes V_{Zd-\frac{p}{2k(sk+1)}d-\frac{q}{k}d} \) for

\[
\langle d, d \rangle = 2k(sk + 1).
\]
$q \equiv 0, j$ is congruent to $\frac{j(i-p)}{k}$ modulo $\mathbb{Z}$.

$$\rho(M_{i,j}^{0}) \otimes V_{Zd+\frac{p}{2k(sk+1)d-kd}} - \rho(M_{i,j}^{0}) \otimes V_{Zd+\frac{p}{2k(sk+1)d}} \equiv \frac{j(i-p)}{k} \pmod{\mathbb{Z}}. \quad (4.7)$$

In particular, it is an integer for all $0 \leq j < k$ if and only if $i \equiv p \pmod{k}$.

Proof. Let $v^{i,j}$ be a nonzero element of the top level of $M_{i,j}^{0}$ for $0 \leq j \leq i \leq k$. Then the conformal weight of $M_{i,j}^{0}$ is

$$\text{wt } v^{i,j} = \frac{1}{2k(k+2)}(k(i-2j) - (i-2j)^2 + 2k(i-j+1)j)$$

for $0 \leq j \leq i \leq k$ by (4.4). Since $M_{i,j}^{0} \cong M_{i,j}^{k-i,-j-i}$ by (4.3), the conformal weight of $M_{i,j}^{0}$ is

$$\text{wt } v^{k-i,j-i} = \frac{1}{2k(k+2)}(k(k+i-2j) - (k+i-2j)^2 + 2k(k-j+1)(j-i))$$

for $0 \leq i \leq j \leq k$. Thus the difference of the conformal weight of $M_{i,j}^{0}$ and the conformal weight of $M_{i,j}^{0}$ is

$$\text{wt } v^{i,j} - \text{wt } v^{i,0} = \frac{j(i-j)}{k}$$

if $j \leq i$ and

$$\text{wt } v^{k-i,j-i} - \text{wt } v^{i,0} = \frac{(k-j)(j-i)}{k} \equiv \frac{j(i-j)}{k} \pmod{\mathbb{Z}}$$

if $j \geq i$.

The conformal weight of $V_{Zd+\frac{p}{2k(sk+1)d-kd}}$ is congruent to

$$\frac{1}{2}(p - 2(sk+1)q)^2 \langle d, d \rangle = \frac{(p - 2(sk+1)q)^2}{4k(sk+1)}$$

modulo $\mathbb{Z}$ for $0 \leq q < k$ and $0 \leq p < 2k(sk+1)$ as $\langle d, d \rangle = 2k(sk+1)$. Thus the difference of the conformal weight for $q = 0$, $j$ is congruent to $\frac{j(i-p)}{k}$ modulo $\mathbb{Z}$. Hence the left hand side of (4.7) is congruent to $\frac{j(i-j)}{k} + \frac{j(i-p)}{k} = \frac{j(i-p)}{k}$ modulo $\mathbb{Z}$. The proof is complete. \qed

Let $N$ be an irreducible $U^0$-module. Then $N$ is a direct sum of irreducible $M_{i,j}^{0} \otimes V_{Zd}$ modules. A direct summand of $N$ is of the form

$$X = M_{i,j}^{0} \otimes V_{Zd+\frac{p}{2k(sk+1)d}}$$

for some $0 \leq i \leq k$, $0 \leq q < k$, $0 \leq p < 2k(sk+1)$.

Now, $U^0 \cdot X = \text{span}_{\mathbb{C}} \{ a_{q}v \mid a \in U^0, v \in X, n \in \mathbb{Z} \}$ is the $U^0$-submodule of $N$ generated by $X$. Since $N$ is irreducible, we have $N = U^0 \cdot X$ and

$$N = \bigoplus_{j=0}^{k-1} M_{i,j}^{0} \otimes V_{Zd+\frac{p}{2k(sk+1)d-kd}}$$

by the fusion product

$$(M_{i,j}^{0} \otimes V_{Zd-\frac{p}{d-kd}} \boxtimes M_{i,j}^{0} \otimes V_{Zd+\frac{p}{2k(sk+1)d}}) = M_{i,j}^{0} \otimes V_{Zd+\frac{p}{2k(sk+1)d-kd}}$$
for $M^0_{(k)} \otimes V_{zd}$. Here $M^{i,l+j}_{(k)} \otimes V_{zd+\frac{p}{2k(k+1)}d-\frac{l}{k}d}$, $0 \leq j < k$ are inequivalent irreducible $M^0_{(k)} \otimes V_{zd}$-modules. Therefore, we may assume that

$$X = M^{i,0}_{(k)} \otimes V_{zd+\frac{p}{2k(k+1)}d}.$$ 

Then

$$(M^i_{(k)} \otimes V_{zd-\frac{l}{k}d}) \cdot X = M^{i,j}_{(k)} \otimes V_{zd+\frac{p}{2k(k+1)}d-\frac{l}{k}d}$$

is contained in $N$ as a direct summand. In particular, the difference of the conformal weight of those direct summands is an integer. Thus $i \equiv p \pmod{k}$ by Lemma [4.1]. Hence $p \equiv i - \ell k \pmod{2k(k+1)}$ for some $0 \leq \ell < 2(sk+1)$. Define $0 \leq l < 2(sk+1)$ by $l \equiv si + \ell \pmod{2(sk+1)}$. Then $\mathbb{Z}d + \frac{p}{2k(sk+1)}d = \mathbb{Z}d - \frac{l}{2(sk+1)}d + \frac{1}{2k}d$. Therefore,

$$N = \bigoplus_{j=0}^{k-1} M^{i,j}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{j-2i}{2k}d}$$

as $M^0_{(k)} \otimes V_{zd}$-modules.

Let

$$U^{i,l} = U^0 \boxtimes M^0_{(k)} \otimes V_{zd} (M^{i,0}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{1}{2k}d})$$

$$= \bigoplus_{j=0}^{k-1} M^{i,j}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{j-2i}{2k}d} \quad (4.8)$$

for $0 \leq i \leq k, 0 \leq l < 2(sk+1)$. The index $l$ can be considered to be modulo $2(sk+1)$.

Since $M^{i,j}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{j-2i}{2k}d}$, $0 \leq j < k$ are inequivalent irreducible $M^0_{(k)} \otimes V_{zd}$-modules and since the difference of the conformal weight of these irreducible modules is an integer, $U^{i,l}$ has a unique irreducible $U^0$-module structure as an extension of the $M^0_{(k)} \otimes V_{zd}$-module $M^{i,0}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{1}{2k}d}$. Any irreducible $U^0$-module is isomorphic to one of these $U^{i,l}, 0 \leq i \leq k, 0 \leq l < 2(sk+1)$.

Note that the definition (4.8) of $U^{i,l}$ for $0 \leq i \leq k$ and $0 \leq l < 2(sk+1)$ is slightly different from the definition (3.12) of $U^{i,\alpha}$ for $i \in I$ and $\alpha \in C$.

As we will show below, not all of those irreducible $U^0$-modules $U^{i,l}, 0 \leq i \leq k, 0 \leq l < 2(sk+1)$ are inequivalent to each other.

**Lemma 4.2.** (1) Let $0 \leq i, i' \leq k, 0 \leq j, j' < k$ and $0 \leq l, l' < 2(sk+1)$. Then

$$M^{i,j}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{j-2i}{2k}d} \cong M^{i',j'}_{(k)} \otimes V_{zd-\frac{l'}{2(sk+1)}d+\frac{j'-2i'}{2k}d} \quad (4.9)$$

as $M^0_{(k)} \otimes V_{zd}$-modules if and only if one of the following conditions holds.

(i) $i = i', j = j'$ and $l = l'$.

(ii) $i' = k - i, j' \equiv j - i \pmod{k}$ and $l' \equiv sk + 1 + l \pmod{2(sk+1)}$.

(2) Among $M^{i,j}_{(k)} \otimes V_{zd-\frac{l}{2(sk+1)}d+\frac{j-2i}{2k}d}$, $0 \leq i \leq k, 0 \leq j < k, 0 \leq l < 2(sk+1)$, exactly $k(k+1)(sk+1)$ are inequivalent to each other.

**Proof.** We fix $i, j, l$ and consider $i', j', l'$ for which (1.9) holds. Note that (1.8) holds if and only if

(a) $M^{i,j}_{(k)} \cong M^{i',j'}_{(k)}$ as $M^0_{(k)}$-modules, and
Proof. We fix Lemma 4.3. modules if and only if one of the following conditions holds. 

The condition (a) implies that \( i = i' \) and \( j = j' \), or \( i = k - i \) and \( j' \equiv j - i \pmod{k} \) by (4.3). The condition (b) is equivalent to 

\[(i - i') - 2(j - j')(sk + 1) - (l - l')k \equiv 0 \pmod{2k(sk + 1)}.

Suppose \( i = i' \) and \( j = j' \). Then \( l - l' \equiv 0 \pmod{2k(sk + 1)} \) by (b). Hence (i) holds. Next, suppose \( i' = k - i \) and \( j' \equiv j - i \pmod{k} \). Then \( i - i' = 2i - k \) and \( 2(j - j') \equiv 2i \pmod{2k} \). Hence (b) implies \( k(skr + 1) + (l - l')k \equiv 0 \pmod{2k(skr + 1)} \) and so \( sk + 1 + l - l' \equiv 0 \pmod{2(skr + 1)} \). This completes the proof of the assertion (1).

The assertion (2) follows directly from (1). □

Lemma 4.3. Let \( 0 \leq i, i' \leq k \) and \( 0 \leq l, l' < 2(skr + 1) \). Then \( U^{i,l} \cong U^{i',l'} \) as \( M^0_{(k)} \otimes V_{Zd} \)-modules if and only if one of the following conditions holds.

(i) \( i = i' \) and \( l = l' \).

(ii) \( i' = k - i \) and \( l' \equiv sk + 1 + l \pmod{2(skr + 1)} \).

Proof. We fix \( i \) and \( l \). Let \( i' = k - i \) and \( l' \equiv sk + 1 + l \pmod{2(skr + 1)} \). Moreover, for \( 0 \leq j < k \), define \( 0 \leq j' < k \) by \( j' \equiv j - i \pmod{k} \). Then by Lemma 4.2, we have

\[ U^{i,l} \cong \bigoplus_{j=0}^{k-1} M^{i,j'}_{(k)} \otimes V_{Zd- \frac{i}{2(skr+1)} d+ \frac{i'}{2k} d} \]

as \( M^0_{(k)} \otimes V_{Zd} \)-modules. Since the map \( j \mapsto j' \) is a permutation on the set \( \{0, 1, \ldots, k-1\} \), we arrange the summation so that the right hand side of the above equation is

\[ \bigoplus_{j'=0}^{k-1} M^{i,j'}_{(k)} \otimes V_{Zd- \frac{i}{2(skr+1)} d+ \frac{i'}{2k} d} = U^{i',l'} . \]

Hence \( U^{i,l} \cong U^{i',l'} \) as \( M^0_{(k)} \otimes V_{Zd} \)-modules.

Conversely, assume that \( U^{i,l} \cong U^{i',l'} \) as \( M^0_{(k)} \otimes V_{Zd} \)-modules. Then

\[ M^{i,j}_{(k)} \otimes V_{Zd- \frac{i}{2(skr+1)} d+ \frac{i'}{2k} d} \cong M^{i',j'}_{(k)} \otimes V_{Zd- \frac{i'}{2(skr+1)} d+ \frac{i'}{2k} d} \]

for some \( 0 \leq j, j' < k \) by (4.8). Hence (i) or (ii) holds by Lemma 4.2. The proof is complete. □

It follows from (4.8) that

\[ U^{k-i, sk+1+l} = U^0 \boxtimes M^0_{(k)} \otimes V_{Zd} \left( M^{k-i,0}_{(k)} \otimes V_{Zd- \frac{k+1}{2(skr+1)} d+ \frac{k-i}{2k} d} \right) \]

as \( M^0_{(k)} \otimes V_{Zd} \)-modules. Since

\[ M^{k-i,0}_{(k)} \otimes V_{Zd- \frac{k+1}{2(skr+1)} d+ \frac{k-i}{2k} d} \cong M^{i,0}_{(k)} \otimes V_{Zd- \frac{i}{2(skr+1)} d+ \frac{i}{2k} d} \]

as \( M^0_{(k)} \otimes V_{Zd} \)-modules by Lemma 4.2 and since

\[(M^{i}_{(k)} \otimes V_{Zd+ \frac{i}{2k} d}) \boxtimes M^0_{(k)} \otimes V_{Zd} \left( M^{i}_{(k)} \otimes V_{Zd- \frac{i}{2(skr+1)} d+ \frac{i}{2k} d} \right) = M^{i,0}_{(k)} \otimes V_{Zd- \frac{i}{2(skr+1)} d+ \frac{i}{2k} d} \]

we see that \( U^{k-i, sk+1+l} \cong U^{i,l} \) as modules for the \( \mathbb{Z}_k \)-graded simple current extension \( U^0 \) of \( M^0_{(k)} \otimes V_{Zd} \).

We summarize the above arguments as in the following theorem.
Theorem 4.4. (1) Any irreducible $U^0$-module is isomorphic to $U^{i,l}$ for some $0 \leq i \leq k$, $0 \leq l < 2(sk + 1)$.

(2) The irreducible $U^0$-modules $U^{i,l}$, $0 \leq i \leq k$, $0 \leq l < 2(sk + 1)$ are inequivalent to each other except the isomorphism \[ U^{i,l} \cong U^{k-i,sk+1+1}. \] (4.10)

(3) There are exactly $(k + 1)(sk + 1)$ inequivalent irreducible $U^0$-modules.

(4) The conformal weight of any irreducible $U^0$-module except $U^0$ itself is positive.

There are $\frac{1}{2}k(k + 1) \cdot 2k(sk + 1) = k^2(k + 1)(sk + 1)$ inequivalent irreducible $M^0_k \otimes V_{zd}$-modules

\[ M^{i,j}_k \otimes V_{zd + \frac{p}{2(sk + 1)}}, \quad 0 \leq j < i \leq k, \quad 0 \leq p < 2k(sk + 1). \]

Only $k(k + 1)(sk + 1)$ of them can be direct summands of an irreducible $U^0$-module by Lemma 4.1 and each irreducible $U^0$-module $U^{i,l}$ is a direct sum of $k$ inequivalent irreducible $M^0_k \otimes V_{zd}$-modules of those $k(k + 1)(sk + 1)$ irreducible modules.

Recall from Section 4.1 that $\text{Aut}(M^0_k) = \langle \theta \rangle$, which is trivial if $k = 1, 2$ and of order 2 if $k \geq 3$. The automorphism $\theta$ and an automorphism $V_{zd}$ of order 2 lifted from the $\overline{-1}$ isometry of $zd$ naturally induce an automorphism of $M^0_k \otimes V_{zd}$ of order 2, which can be extended to an automorphism of the vertex operator algebra $U^0$ of order 2. We denote the automorphism of $U^0$ by the same symbol $\theta$. Since

\[ M^{i,j}_k \circ \theta \cong M^{i,j}_k, \quad V_{zd + \frac{p}{2(k+1)}} \circ \theta \cong V_{zd - \frac{p}{2(k+1)}}, \]

we see from (4.8) that $\theta$ induces a permutation on the irreducible $U^0$-modules as

\[ U^{i,l} \circ \theta \cong U^{i,l}. \] (4.11)

Remark 4.5. In the case $s = 0$, we have $\langle d, d \rangle = 2k$ and $U^0 = L_{sl_2}(k, 0)$. In this case Theorem 4.2 implies that the irreducible $U^0$-modules are $U^{i,l}$, $0 \leq i \leq k$, $0 \leq l < 2$ with $U^{i,0} \cong U^{k-i,1}$. In fact, $U^{i,0} = \bigoplus_{j=0}^{k-1} M^{i,j}_k \otimes V_{zd + \frac{2j}{2k}d}$ coincides with $L_{sl_2}(k, i)$ by (4.2) and (4.8). Moreover, since $zd - \frac{d}{2} + \frac{i-2j}{2k}d = zd + \frac{k-i-2(j-i)}{2k}d$, it follows from (4.3) and (4.8) that

\[ U^{i,1} = \bigoplus_{j=0}^{k-1} M^{i,j}_k \otimes V_{zd - \frac{d}{2} + \frac{i-2j}{2k}d} \]

\[ = \bigoplus_{j=0}^{k-1} M^{k-i,j-i}_k \otimes V_{zd + \frac{k-i-2(j-i)}{2k}d} \]

\[ = \bigoplus_{j=0}^{k-1} M^{k-i,j}_k \otimes V_{zd + \frac{k-i-2(j-i)}{2k}d} = U^{k-i,0}, \]

which coincides with $L_{sl_2}(k, k-i)$.

4.3. Fusion rule for $U^0$. We discuss the fusion rule of the irreducible modules for the vertex operator algebra $U^0$.

For $0 \leq i, i' \leq k$, let $R(i, i')$ be the set of integers $p$ satisfying

\[ |i - i'| \leq p \leq \min\{i + i', 2k - (i + i')\}, \quad i + i' + p \in 2Z. \] (4.12)
It is known [17, Theorem 4.2] (see also [3, Section 6]) that the fusion product of irreducible modules for $M^0_{(k)} = K(\mathfrak{sl}_2, k)$ is
\[ M^{i,j}_{(k)} \boxtimes M'^{i',j'}_{(k)} = \sum_{p \in R(i,i')} M^{p, \frac{1}{2}(2j-i+2j'-i'+p)}_{(k)} \tag{4.13} \]
for $0 \leq i, i' \leq k$, $0 \leq j, j' < k$, where the irreducible $M^0_{(k)}$-modules $M^{p, \frac{1}{2}(2j-i+2j'-i'+p)}_{(k)}$, $p \in R(i,i')$ are inequivalent to each other. The fusion product for $M^0_{(k)}$ and the fusion product for $L_{\mathfrak{sl}_2}(k,0)$ [21, 22]
\[ L_{\mathfrak{sl}_2}(k,i) \boxtimes L_{\mathfrak{sl}_2}(k,i') L_{\mathfrak{sl}_2}(k,p) = \sum_{p \in R(i,i')} L_{\mathfrak{sl}_2}(k,p) \tag{4.14} \]
for $0 \leq i, i' \leq k$ are related to each other.

By a similar argument as in the proof of Theorem 3.12, we have the following theorem.

Theorem 4.6. Let $0 \leq i \leq k$, $0 \leq j < k$ and $0 \leq l < 2(sk+1)$. Then
\[ \text{qdim}_U U^{i,l} = \text{qdim}_M^0 M^{i,j}_{(k)} . \]

We obtain the fusion product for $U^0$ from the fusion product for $M^0_{(k)}$ as in Theorem 3.13.

Theorem 4.7. (1) Let $0 \leq i, i' \leq k$ and $0 \leq l, l' < 2(sk+1)$. Then
\[ U^{i,l} \boxtimes U^0 U^{i',l'} = \sum_{p \in R(i,i')} U^{p,l+l'}, \tag{4.15} \]
where $R(i,i')$ is the set of integers $p$ satisfying (1.12) and $l$ and $l'$ are considered to be modulo $2(sk+1)$. The irreducible $U^0$-modules $U^{p,l+l'}$, $p \in R(i,i')$ on the right hand side of (1.13) are inequivalent to each other.

(2) The fusion product (4.14) is compatible with the isomorphism $U^{i,l} \cong U^{k-i,sk+1+l}$ (1.10) of $U^0$-modules.

Proof. We first prove the assertion (1). For simplicity of notation, we write $M^j$ and $M^{i,j}$ for $M^j_{(k)}$ and $M^{i,j}_{(k)}$, respectively.

Let $0 \leq i'' \leq k$ and $0 \leq l'' < 2(sk+1)$. Since $U^0$ is a $\mathbb{Z}_k$-graded simple current extension of $M^0_{(k)} \otimes V_{Zd}$ and since each of the irreducible $U^0$-modules $U^{i,l}$, $U^{i',l'}$ and $U^{i'',l''}$ is a direct sum of $k$ inequivalent irreducible $M^0_{(k)} \otimes V_{Zd}$-modules, it follows from Theorem 2.12 and Proposition 2.3 that
\[ \dim I_{U^0} \left( U^{i'',l''} \mid U^{i,l} \right) \]
\[ = \sum_{j''=0}^{k-1} \dim I_{M^0 \otimes V_{Zd}} \left( M^{i',0} \otimes V_{Zd-\frac{l''}{2(sk+1)}d+\frac{l''+2j''}{2k}d} \mid M^{i'',0} \otimes V_{Zd-\frac{l''}{2(sk+1)}d+\frac{l''+2j''}{2k}d} \right) \]
\[ = \sum_{j''=0}^{k-1} \dim I_{M^0} \left( M^{i'',0} \mid M^{i,0} \right) \cdot \dim I_{V_{Zd}} \left( V_{Zd-\frac{l''}{2(sk+1)}d+\frac{l''+2j''}{2k}d} \mid V_{Zd-\frac{l''}{2(sk+1)}d+\frac{l''+2j''}{2k}d} \right). \tag{4.16} \]
The term
\[
\dim I_{V^d}(V^{d_0} - \frac{m}{2(x-k)}d + \frac{2m-i''}{2k}d, V^{d_0} - \frac{m}{2(x-k)}d + \frac{m}{2k}d, V^{d_0} - \frac{m}{2(x-k)}d + \frac{i''}{2k}d, V^{d_0} - \frac{m}{2(x-k)}d + \frac{m}{2k}d)
\]
on the right hand side of (4.16) is 0 or 1 for \(0 \leq j'' < k\) and it is 1 if and only if
\[-k(l+l') + (sk+1)(i+i') \equiv -kl'' + (sk+1)(i''-2j'' \pmod {2k(sk+1)})\] (4.17)
by the fusion rule for \(V^d\). Moreover, the term \(\dim I_{M^0}(M_{l''}^{i''}, M_{i''}^{j''})\) is 0 or 1 and it is 1 if and only if \(i'' \in R(i, i')\) and \(i''-2j'' \equiv i+i' \pmod {2k}\) by (4.13).

Hence it follows from (4.16) that \(\dim I_{U^0}(U_i^{i''}, U_{i'}^{j''}, U_i^{j''}, U_{i'}^{j''})\) is 0 or 1 and it is 1 if and only if \(i'' \in R(i, i'), i''-2j'' \equiv i+i' \pmod {2k}\) and the condition (4.17) is satisfied. If \(i'' \in R(i, i')\), then there is a unique \(0 \leq j'' < k\) such that \(i''-2j'' \equiv i+i' \pmod {2k}\). If \(i''-2j'' \equiv i+i' \pmod {2k}\), then the condition (4.17) is equivalent to the condition that \(l+l' \equiv l'' \pmod {2k(sk+1)}\). Therefore, (1.13) holds.

For a fixed \(l''\), (2) of Theorem 4.4 implies that \(U^{p,l''}, 0 \leq p \leq k\) are inequivalent to each other. The proof of the assertion (1) is complete.

Next, we prove the assertion (2). Fix \(0 \leq i \leq k\) and \(0 \leq l < 2(sk+1)\). We define
\[\hat{i} = k-i, \quad \hat{l} = sk+1+l \pmod {2(sk+1)}\]
Then \(U^{i,l} \cong \hat{U}^{i,\hat{l}}\) as \(U^0\)-modules. The range \(R(i, i')\) of \(p\) can be written as
\[R(i, i') = \{p \mid \max\{i-i', -i+i'\} \leq p \leq \min\{i+i', 2k-(i+i')\}, i+i'+p \in 2\mathbb{Z}\} \]
The fusion product of \(\hat{U}^{i,\hat{l}}\) and \(U^{i',\hat{l}'}\) is
\[
\hat{U}^{i,\hat{l}} \boxtimes_{U^0} U^{i',\hat{l}'} = \sum_{p \in R(\hat{i}, \hat{l}')} U^{p,\hat{i}+\hat{l}'}, \tag{4.18}
\]
where
\[R(\hat{i}, \hat{l}') = \{p \mid \max\{\hat{i}-\hat{l}', -\hat{i}+\hat{l}'\} \leq p \leq \min\{\hat{i}+\hat{l}', 2k-(\hat{i}+\hat{l}')\}, \hat{i}+\hat{l'}+p \in 2\mathbb{Z}\} \]
For \(0 \leq p \leq k\), we have \(k-p \in R(\hat{i}, \hat{l}')\) if and only if \(p \in R(i, i')\). Let \(p \in R(\hat{i}, \hat{l}')\). Then \(p = k-p\) for some \(p \in R(i, i')\) and
\[U^{p,\hat{i}+\hat{l}'} = U^{k-p,sk+1+l'+l'} \cong U^{p,l'+l'}\]
as \(U^0\)-modules. Hence
\[\sum_{p \in R(\hat{i}, \hat{l}')} U^{p,\hat{i}+\hat{l}'} \cong \sum_{p \in R(i, i')} U^{p,l'+l'}\]
as \(U^0\)-modules. Thus \(\hat{U}^{i,\hat{l}} \boxtimes_{U^0} U^{i',\hat{l}'} \cong U^{i,l} \boxtimes_{U^0} U^{i',l'}\) by (1.13) and (4.18).

Since the fusion product is commutative, we have that any of the \(U^0\)-modules \(\hat{U}^{i,\hat{l}} \boxtimes_{U^0} U^{i',\hat{l}'}\), \(U^{i,l} \boxtimes_{U^0} U^{i',l'}\), \(\hat{U}^{i,\hat{l}} \boxtimes_{U^0} U^{\hat{i},\hat{l}'}\), \(U^{i,l} \boxtimes_{U^0} U^{\hat{i},\hat{l}'}\) is isomorphic to \(U^{i,l} \boxtimes_{U^0} U^{i',l'}\). Thus the assertion (2) holds.

Corollary 4.8. Let \(\zeta = \exp(2\pi \sqrt{-1}/(sk+1))\) and define a map
\[\lambda : U^{i,l} \mapsto \zeta^l U^{i,l}\]
for \(0 \leq i \leq k\), \(0 \leq l < 2(sk+1)\). Then the following assertions hold.
(1) The map $\lambda$ is compatible with the isomorphism $U^{i,l} \cong U^{k-i,sk+1+l}$ of $U^0$-modules so that $\lambda$ can be defined on the set $\text{Irr}(U^0)$ of equivalence classes of irreducible $U^0$-modules.

(2) Extend $\lambda$ linearly. Then
$$\lambda(U^{i,l} \boxtimes_{U^0} U^{i',l'}) = \lambda(U^{i,l}) \boxtimes_{U^0} \lambda(U^{i',l'})$$
for $0 \leq i, i' \leq k$, $0 \leq l, l' < 2(sk + 1)$. That is, the map $\lambda$ gives an automorphism of the fusion algebra of $U^0$ of order $sk + 1$.

Proof. Since $\zeta^{sk+1} = 1$, we have $\zeta^{sk+1+l} = \zeta^l$. Thus the assertion (1) holds. The assertion (2) is clear from (4.15). \qed

As a special case of Theorem 4.1, we have the following corollary.

**Corollary 4.9.** There are exactly $2(sk+1)$ inequivalent simple current $U^0$-modules, which are represented by $U^{0,l} \cong U^{k,sk+1+l}$, $0 \leq l < 2(sk + 1)$.

**Lemma 4.10.** Let $0 \leq i \leq k$ and $0 \leq p, l < 2(sk + 1)$. Then $U^{0,p} \boxtimes_{U^0} U^{i,l} = U^{i,l}$ if and only if one of the following conditions holds.

(i) $p = 0$.

(ii) $k$ is even, $i = k/2$ and $p = sk + 1$.

Proof. We see from Theorem 4.4 that $U^{i,l} \cong U^{i,l+p}$ if and only if $l \equiv l+p$ (mod $2(sk+1)$) or $k-i = i$ and $sk + 1 + l \equiv l + p$ (mod $2(sk + 1)$). Since $U^{0,p} \boxtimes_{U^0} U^{i,l} = U^{i,l+p}$, the assertion holds. \qed

The fusion product of the simple current $U^0$-modules $U^{0,l}$, $0 \leq l < 2(sk + 1)$ is
$$U^{0,l} \boxtimes_{U^0} U^{0,l'} = U^{0,l+l'}.$$ Thus the set $\{U^{0,l} \mid 0 \leq l < 2(sk + 1)\}$ of simple currents is $\mathbb{Z}_{2(sk+1)}$-graded.

While the fusion product for $U^0$ is compatible with the isomorphism $U^{i,l} \cong U^{k-i,sk+1+l}$ of $U^0$-modules, the assignment $U^{0,l} \mapsto l$ (mod $2(sk + 1)$) is not compatible with the isomorphisms $U^{0,l} \cong U^{k,sk+1+l}$. Therefore, we need to denote the simple currents as $U^{0,l}$ and not as $U^{k,l}$ when we refer to the $\mathbb{Z}_{2(sk+1)}$-grading.

The irreducible modules for $M^0_{(k)} = K(\mathfrak{sl}_2, k)$ are denoted as $M^{i,j}_{(k)}$ by using $0 \leq i \leq k$ and $0 \leq j < k$ with isomorphism $M^{i,j}_{(k)} \cong M^{k-i,j-i}_{(k)}$, where $j$ is considered to be modulo $k$. There is another description of the irreducible modules for $M^0_{(k)}$. Take $0 \leq q < 2k$ such that
$$q \equiv i - 2j \pmod{2k}.$$ Then $q \equiv i$ (mod 2) and $j \equiv \frac{1}{2}(i - q)$ (mod $k$). Thus there is a one to one correspondence between $(i,j)$ with $0 \leq i \leq k$, $0 \leq j < k$ and $(i,q)$ with $0 \leq i \leq k$, $0 \leq q < 2k$, $q \equiv i$ (mod 2). Let
$$\tilde{M}^{i,q}_{(k)} = M^{i,j}_{(k)}.$$ Then the fusion product (4.13) for $M^0_{(k)}$ can be written as
$$\tilde{M}^{i,q}_{(k)} \boxtimes_{M^0_{(k)}} \tilde{M}^{i',q'}_{(k)} = \sum_{p \in R(i,i')} \tilde{M}^{p,q+q'}_{(k)}.$$ (4.19)
The relationship between the fusion product (4.15) for $U^0$ and the fusion product (4.19) for $M_0^0(k)$ is clear. Moreover, the isomorphisms $M_{i,j}^i(k) \cong M_{k-i,j}^{k-i,i}$ and $M_{i,j}^i(k) \circ \theta \cong M_{i,j}^{i-j}(k)$ can be written as

$$\tilde{M}_{i,q}^i(k) \cong \tilde{M}_{k-i,k}^{k-i,q}(k), \quad \tilde{M}_{i,q}^i(k) \circ \theta \cong \tilde{M}_{i,j}^{i-j}(k).$$

The fusion product is compatible with the isomorphism $M_{i,j}^i(k) \cong M_{k-i,j}^{k-i,i}(k)$. It is well known that a map defined by $\tilde{M}_{i,q}^i(k) \mapsto \eta^q \tilde{M}_{i,q}^i(k)$ with $\eta = \exp(2\pi\sqrt{-1}/k)$ is compatible with the isomorphism $\tilde{M}_{i,q}^i(k) \cong \tilde{M}_{k-i,k+q}^i(k)$ and it induces an automorphism of the fusion algebra of $K(\mathfrak{sl}_2, k)$ of order $k$.

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