Mean field games with monotonous interactions through the law of states and controls of the agents

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Abstract. We consider a class of Mean Field Games in which the agents may interact through the statistical distribution of their states and controls. It is supposed that the Hamiltonian behaves like a power of its arguments as they tend to infinity, with an exponent larger than one. A monotonicity assumption is also made. Existence and uniqueness are proved using a priori estimates which stem from the monotonicity assumptions and Leray–Schauder theorem. Applications of the results are given.

Mathematics Subject Classification. 35Q89.

Keywords. Mean Field Games, Partial differential equation.

1. Introduction

The theory of Mean Field Games (MFG for short) aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of agents tends to infinity. It has been introduced in the independent works of Lasry and Lions [34–36], and of Huang, Caines and Malhamé [28, 29]. The agents are supposed to be rational (given a cost to be minimized, they always choose the optimal strategies), and indistinguishable. Furthermore, the agents interact via some empirical averages of quantities which depend on the state variable.

The most common Mean Field Game systems, in which the agents may interact only through their states can often be summarized by a system of two coupled partial differential equations which is named the MFG system. On the one hand, the optimal value of a generic agent at some time $t$ and state $x$ is
denoted by $u(t, x)$ and is defined as the lowest cost that a representative agent can achieve from time $t$ to $T$ if it is at state $x$ at time $t$. The value function satisfies a Hamilton–Jacobi–Bellman equation posed backward in time with a terminal condition involving a terminal cost. On the other hand, there is a Fokker–Planck–Kolmogorov equation describing the evolution of the statistical distribution $m$ of the state variable; this equation is a forward in time parabolic equation, and the initial distribution at time $t = 0$ is given. Here we take a finite horizon time $T > 0$, and we only consider second-order nondegenerate MFG systems. In this case, the MFG system is often written as:

\[
\begin{aligned}
-\partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x)) &= f(t, x, m(t)), \\
\partial_t m(t, x) - \nu \Delta m(t, x) - \text{div}(H_p(t, x, \nabla_x u(t, x))m) &= 0, \\
u \Delta m(T, x) - \text{div}(H_p(T, x, \nabla_x u(T, x))m) &= 0, \\
\end{aligned}
\]

for $(t, x) \in (0, T) \times \mathbb{R}^d$. We refer the reader to [8] for some theoretical results on the convergence of the $N$-agent Nash equilibrium to the solutions of the MFG system. For a thorough study of the well-posedness of the MFG system, see the videos of P.L. Lions’ lecture at the Collège de France [38], and the lecture notes [7]. Let us mention a non exhaustive list of other works studying system (1.1): [2, 9, 11, 18, 25, 27, 30].

In this paper we are considering a class of Mean Field Games in which agents may interact through their states and controls. To underline this, we choose to use the terminology Mean Field Games of Controls (MFGCs); this terminology was introduced in [10].

Since the agents are assumed to be indistinguishable, a representative agent may be described by their state, which is a random process with value in $\mathbb{R}^d$ denoted by $(X_t)_{t \in [0, T]}$ and satisfying the following stochastic differential equation,

\[dX_t = b(t, X_t, \alpha_t, \mu_\alpha(t)) dt + \sqrt{2\nu}dW_t,\]

where $X_0$ is a random process whose law is denoted by $m_0$, $(W_t)_{t \in [0, T]}$ is a Brownian motion on $\mathbb{R}^d$ independent of $X_0$, and $\alpha_t$ is the control chosen by the agent at time $t$. The diffusion coefficient $\nu$ is assumed to be uncontrolled, constant and positive. The drift $b$ naturally depends on the control, and may also depend on the time, the state, and on the mean field interactions of all agents through $\mu_\alpha$ the joint distribution of states and controls. At the equilibrium $\mu_\alpha$ should be the law of the state and the control of the representative agent, i.e. $\mu_\alpha(t) = \mathcal{L}(X_t, \alpha_t)$, for $t \in [0, T]$. The aim of an agent is to minimize the functional given by,

\[\mathbb{E} \left[ \int_0^T L(t, X_t, \alpha_t, \mu_\alpha(t)) + f(t, X_t, m(t)) dt + g(X_T, m(T)) \right],\]

where $m(t)$ is the distribution of agents at time $t$, which should satisfy $m(t) = \mathcal{L}(X_t)$ at the equilibrium. The coupling function $f$ and the terminal cost $g$ depend on $m$ in a nonlocal manner. From the Lagrangian $L$ and the drift $b$,
we define the Hamiltonian $H$ by,

$$H(t, x, p, \mu) = \sup_{\alpha \in \mathbb{R}^d} -p \cdot b(t, x, \alpha, \mu) - L(t, x, \alpha, \mu),$$

(1.2)

for $(t, x) \in [0, T] \times \mathbb{R}^d$, $p \in \mathbb{R}^d$ and $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, where $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$. Under some assumptions on $b$ and $L$ that will be introduced later, there exists a unique $\alpha$ which achieves the supremum in the latter equality and it also satisfies,

$$b(t, x, \alpha, \mu) = -H_p(t, x, p, \mu).$$

In an attempt to keep this paper easy to read, we introduce $\mu_b$ as the joint law of states and drifts defined by

$$\mu_b(t) = \left[ (x, \alpha) \mapsto (x, b(t, x, \alpha, \mu(t))) \right]\#\mu(t).$$

(1.3)

We believe that the fixed point relation satisfied by $\mu_\alpha$ at equilibrium is more clear if we distinguish $\mu_b$ from $\mu_\alpha$. We assume that $b$ is invertible with respect to $\alpha$ in such a way that its inverse map can be expressed in term of $\mu_b$ instead of $\mu_\alpha$, see Assumption B1 below. This allows us to define $\alpha^*: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}^d$ such that

$$\tilde{b} = b\left(t, x, \alpha^*\left(t, x, \tilde{b}, \mu_b\right), \mu_\alpha\right)$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$, $\tilde{b} \in \mathbb{R}^d$ and $\mu_\alpha, \mu_b \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying (1.3). Conversely, for any $\alpha \in \mathbb{R}^d$ we have

$$\alpha = \alpha^*\left(t, x, b\left(t, x, \alpha, \mu_\alpha\right), \mu_b\right),$$

since $b(t, x, \cdot, \mu_\alpha)$ is injective. This implies that the equality (1.3) can be inverted to express $\mu_\alpha$ in term of $\mu_b$ so that we obtain (1.6) below. Within this framework, the usual MFG system (1.1) is replaced by the following Mean Field Game of Controls (MFGC for short) system,

$$-\partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x), \mu_\alpha(t)) = f(t, x, m(t)),\quad (1.4)$$

$$\partial_t m(t, x) - \nu \Delta m(t, x) - \text{div} (H_p(t, x, \nabla_x u(t, x), \mu_\alpha(t)) m) = 0,\quad (1.5)$$

$$\mu_\alpha(t) = \left[ (x, \tilde{b}) \mapsto \left(x, \alpha^*\left(t, x, \tilde{b}, \mu_b\right)\right)\right]\#\mu_b(t),\quad (1.6)$$

$$\mu_b(t) = (I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu_\alpha(t))\#m(t),\quad (1.7)$$

$$u(T, x) = g(x, m(T)),\quad (1.8)$$

$$m(0, x) = m_0(x),\quad (1.9)$$

for $(t, x) \in (0, T) \times \mathbb{R}^d$. The structural assumption under which we prove existence and uniqueness of the solution to (1.4)–(1.9) is that $L$ satisfies the following inequality,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(L(t, x, \alpha, \mu_1) - L(t, x, \alpha, \mu_2)\right) d(\mu_1 - \mu_2)(x, \alpha) \geq 0.$$
for any $t \in [0, T]$, $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. This is the Lasry-Lions monotonicity assumption extended to MFGC that will be referred to as A3. This assumption is particularly adapted to applications in economics or finance.

This work follows naturally the analysis in [32] in which a MFGC system in the $d$-dimensional torus and with $b = \alpha$ is considered. In [32], the monotonicity assumption is replaced by another structural assumption, namely that the optimal control $-H_x$ is a contraction with respect to the second marginal of $\mu$ (when the other arguments and the first marginal are fixed) and that it is bounded by a quantity that depends linearly on the second marginal of $\mu$ with a coefficient smaller than 1.

**Related works**

Monotonicity assumptions for MFGC like A3 have already been discussed in [10,11,19]. In [19], the authors proved uniqueness of the solution to (1.4)–(1.9) with $b = \alpha$ and $\nu = 0$ when it exists. In [11] Section 4.6, existence and uniqueness are proved in the quadratic case with a uniformly convex Lagrangian and under an additional linear growth assumption on $H_x$. In [10], the existence of weak solutions to a MFGC system with a possibly degenerate diffusion operator is proved assuming that the inequalities satisfied by $H$, its derivatives or the optimal control (here defined in B1 as $\alpha^*$), are uniform with respect to the joint law of states and controls $\mu_\alpha$. Here, we prove the existence and uniqueness of the solutions of weak solutions in the time-dependent non-degenerate case, with larger classes of Hamiltonians and drifts than in the above-mentioned works.

A particular application of MFGC satisfying A3, namely the Bertrand and Cournot competition for exhaustible resource, described in paragraph 3.1, has been broadly investigated in the literature. Let us mention a non exhaustive list of such works: [6,13,14,20,22,24,26,32]. Its mean field version has been introduced in [26], and obtained from the $N$-agent game in [13] in the case of a linear supply-demand function. A generalization to the multi-dimensional case is discussed in [6], and an extension to negatively correlated resources is addressed in [32].

The non-monotone case has been studied in [17,32,39]. In [17], an existence result is proved in the stationnary setting and under the assumption that the dependence of $H$ on $\mu$ is small. In [32], the existence of solutions to the MFGC system in the $d$-dimensional torus and with $b = \alpha$ is discussed under similar growth assumptions as here. By and large, existence of solutions to a MFGC system posed on the $d$-dimensional torus and with $b = \alpha$ was proved in [32] in any of the following cases:

- short time horizon,
- small enough parameters,
- weak dependency of $H$ upon $\mu$,
- weak dependency of $H_x$ upon $\mu$,

and uniqueness is proved only for a short time horizon. Indeed without a monotonicity assumption, it is unlikely that uniqueness holds in general, numerical
examples of non-uniqueness of solutions to discrete approximations of (1.4)–(1.9) with \( b = \alpha \) and in a bounded domain are showed in [1]. In [39], a similar model of population dynamics as the one in [1], is analyzed, and an existence result of the solutions of a MFGC system is proved using a Lagrangian setting.

We refer to [4,5,12,21] for other existence and uniqueness results for MFGC systems.

**Organization of the paper**

In Sect. 2, the notations and the assumptions are described, the case when the control is equal to the drift is discussed. The main results of the paper, namely the existence and uniqueness of solution to (1.4)–(1.9), are stated in paragraph 2.3. We give some insights on our strategy for proving the main results in paragraph 2.3. Two applications of the MFGC system (1.4)–(1.9) are presented in Sect. 3. Section 4 is devoted to solving the fixed point relation in the joint law of state and control in the particular case when the drift is equal to the control. Section 5 consists in giving a priori estimates for a MFGC system posed on the \( d \)-dimensional torus. In Sect. 6, we prove existence and uniqueness of the solution to (1.4)–(1.9) and of an intermediate MFGC system.

**2. Assumptions**

**2.1. Notations**

The spaces of probability measures are equipped with the weak* topology. We denote by \( \mathcal{P}_2(\mathbb{R}^d) \) the subset of \( \mathcal{P}(\mathbb{R}^d) \) of probability measures with finite second moments, and \( \mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d) \) the subset of measures \( \mu \) in \( \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) with a second marginal compactly supported. For \( \mu \in \mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d) \) and \( \tilde{q} \in [1,\infty) \), we define the quantities \( \Lambda_{\tilde{q}}(\mu) \) and \( \Lambda_\infty(\mu) \) by,

\[
\Lambda_{\tilde{q}}(\mu) = \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |\alpha|^\tilde{q} d\mu(x,\alpha) \right)^{\frac{1}{\tilde{q}}}, \\
\Lambda_\infty(\mu) = \sup \{|\alpha|, (x,\alpha) \in \text{supp } \mu\}.
\]

For \( R > 0 \), we denote by \( \mathcal{P}_\infty,R(\mathbb{R}^d \times \mathbb{R}^d) \) the subset of measures \( \mu \) in \( \mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d) \) such that \( \Lambda_\infty(\mu) \leq R \). The probability measures \( \mu_\alpha \) and \( \mu_b \) involved in (1.4)–(1.9), have a particular form, since they are the images of a measure \( m \) on \( \mathbb{R}^d \) by \( (I_d,\alpha) \) and \( (I_d,b) \) respectively, where \( \alpha \) and \( b \) are bounded measurable functions from \( \mathbb{R}^d \) to \( \mathbb{R}^d \); in particular they are supported on the graph of \( \alpha \) and \( b \) respectively.

For \( m \in \mathcal{P}(\mathbb{R}^d) \), we call \( \mathcal{P}_m(\mathbb{R}^d \times \mathbb{R}^d) \) the set of all measures \( \mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) such that there exists \( \alpha^\mu \in L^\infty(m) \) satisfying \( \mu = (I_d,\alpha^\mu)\#m \). Here, \( \Lambda_{\tilde{q}}(\mu) \) and \( \Lambda_\infty(\mu) \) defined in (2.1) are given by

\[
\Lambda_{\tilde{q}}(\mu) = ||\alpha^\mu||_{L^\tilde{q}(m)}, \quad \Lambda_\infty(\mu) = ||\alpha^\mu||_{L^\infty(m)}.
\]

Let \( C^0([0,T] \times \mathbb{R}^d;\mathbb{R}^n) \) be the set of bounded continuous functions from \([0,T] \times \mathbb{R}^d \) to \( \mathbb{R}^n \), for \( n \) a positive integer. We define \( C^{0,1}([0,T] \times \mathbb{R}^d;\mathbb{R}) \) as the set of the functions \( v \in C^0([0,T] \times \mathbb{R}^d;\mathbb{R}) \) differentiable at any point with
respect to the state variable, and whose gradient $\nabla_x v$ is in $C^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ the set of continuous functions from $[0, T] \times \mathbb{R}^d$ to $\mathbb{R}^d$. We shall have the use of the parabolic spaces of H"older continuous functions $C^{q, \beta}([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$ defined for any $\beta \in (0, 1)$ and $n \geq 1$ by,

$$
C^{q, \beta}([0, T] \times \mathbb{R}^d; \mathbb{R}^n) = \left\{ v \in C^0([0, T] \times \mathbb{R}^d; \mathbb{R}^n), \exists C > 0 \text{ s.t.} \right. \\
|v(t_1, x_1) - v(t_2, x_2)| \leq C \left( |x_1 - x_2|^2 + |t_1 - t_2|^{\frac{\beta}{2}} \right), \\
\forall (t_1, x_1), (t_2, x_2) \in [0, T] \times \mathbb{R}^d
$$

This is a Banach space equipped with the norm,

$$
\|v\|_{C^{q, \beta}} = \|v\|_{\infty} + \sup_{(t_1, x_1) \neq (t_2, x_2)} \frac{|v(t_1, x_1) - v(t_2, x_2)|}{(|x_1 - x_2|^2 + |t_1 - t_2|^{\frac{\beta}{2}})}.
$$

Then we introduce the Banach space $C^{1+\beta, 0+\beta}([0, T] \times \mathbb{R}^d; \mathbb{R})$ for $\beta \in (0, 1)$ as the set of the functions $v \in C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R})$ such that $\nabla_x v \in C^{q, \beta}([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$ and which admits a finite norm defined by,

$$
\|v\|_{C^{1+\beta, 0+\beta}} = \|v\|_{\infty} + \|\nabla_x v\|_{C^{q, \beta}} + \sup_{(t_1, x_1) \neq (t_2, x_2) \in [0, T] \times \mathbb{R}^d} \frac{|v(t_1, x) - v(t_2, x)|}{|t_1 - t_2|^{\frac{\beta}{2}}}.
$$

When the drift $b$ is equal to the control $\alpha$, (1.4)–(1.9) can be simplified in the following system,

$$
- \partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x), \mu(t)) = f(t, x, m(t)), \\
\partial_t m(t, x) - \nu \Delta m(t, x) - \text{div} (H_p(t, x, \nabla_x u(t, x), \mu(t)) m) = 0, \\
\mu(t) = \left( I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t), \\
u(T, x) = g(x, m(T)), \\
m(0, x) = m_0(x),
$$

for $(t, x) \in (0, T) \times \mathbb{R}^d$. Here, making a distinction between $\mu_a$ and $\mu_b$ is pointless since they coincide. Therefore, we simply use the notation $\mu$. For the system (2.2)–(2.6), the Hamiltonian is defined as the Legendre transform of $L$,

$$
H (t, x, \alpha, \mu) = \sup_{\alpha \in \mathbb{R}^d} -p \cdot \alpha - L(t, x, \alpha, \mu).
$$

**Definition 2.1.** We say that $(u, m, \mu_a, \mu_b)$ is a solution to (1.4)–(1.9) if

- $u \in C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R})$ is a solution to the heat equation in the sense of distributions with a right-hand side equal to $(t, x) \mapsto f(t, x, m(t)) - H(t, x, \nabla_x u, \mu(t))$, and satisfies the terminal condition (1.8),

- $m \in C^0([0, T]; P(\mathbb{R}^d))$ is a solution to (1.5) in the sense of distributions, and satisfies the initial condition (1.9),

- $\mu_a(t), \mu_b(t) \in P(\mathbb{R}^d \times \mathbb{R}^d)$ satisfy (1.6) and (1.7) for any $t \in [0, T]$.

We say that $(u, m, \mu)$ is a solution to (2.2)–(2.6) if $u$ and $m$ respectively satisfy the first two points of the latter definition with (2.2)–(2.6) instead of (1.4)–(1.9), and if $\mu(t) \in P(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies (2.4) for any $t \in [0, T]$. 

2.2. Hypothesis
The monotonicity assumption made in this paper concerns the Lagrangian. For this reason and the fact that sometimes it may be hard to obtain an explicit form of the Hamiltonian (like in the example of paragraph 3.2 below), all the assumptions will be formulated in term of the Lagrangian and never in term of the Hamiltonian. In particular, working with the Lagrangian gives more flexibility in the arguments of the proofs.

The constants entering the assumptions are $C_0$ a positive constant, $q \in (1, \infty)$ an exponent, $q' = \frac{q}{q-1}$ its conjugate exponent, and $\beta_0 \in (0, 1)$ a Hölder exponent.

(A1) $L : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P} (\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ is differentiable with respect to $(x, \alpha)$; $L$ and its derivatives are continuous on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_{\infty,R} (\mathbb{R}^d \times \mathbb{R}^d)$ for any $R > 0$; we recall that $\mathcal{P}_{\infty,R} (\mathbb{R}^d \times \mathbb{R}^d)$ is endowed with the weak* topology on measures; we use the notation $L_x, L_\alpha$ and $L_{(x,\alpha)}$ for respectively the first-order derivatives of $L$ with respect to $x, \alpha$ and $(x, \alpha)$.

(A2) The maximum in (1.2) is achieved at a unique $\alpha \in \mathbb{R}^d$.

(A3) $L$ satisfies the following monotonicity condition,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left( L(t,x,\alpha,\mu^1) - L(t,x,\alpha,\mu^2) \right) d(\mu^1 - \mu^2)(x,\alpha) \geq 0,$$

for any $t \in [0,T], \mu^1, \mu^2 \in \mathcal{P} (\mathbb{R}^d \times \mathbb{R}^d)$.

(A4) $L(t,x,\alpha,\mu) \geq C_0^{-1}|\alpha|^{q'} - C_0 \left( 1 + \Lambda q' (\mu)^{q'} \right)$, where $\Lambda q'$ is defined in (2.1),

(A5) $|L(t,x,\alpha,\mu)| + |L_x(t,x,\alpha,\mu)| \leq C_0 \left( 1 + |\alpha|^{q'} + \Lambda q' (\mu)^{q'} \right),$

(A6) $\int_{\mathbb{R}^d} |x|^2 dm^0(x) + \|m^0\|_{C^0} + \|f(t,\cdot,m)\|_{C^1} + \|g(\cdot,m)\|_{C^{2+\beta_0}} \leq C_0$, for any $t \in [0,T]$ and $m \in \mathcal{P} (\mathbb{R}^d)$.

Assumption A3 can be interpreted as a natural extension of the Lasry-Lions monotonicity condition to MFGC. Roughly speaking, the Lasry-Lions monotonicity condition in the case of MFG without interaction through controls, translates the fact that the agents have aversion for crowded regions of the state space. In the case of MFGC, the monotonicity condition implies that the agents favor moving in a direction opposite to the mainstream. Such an assumption is adapted to models of agents trading goods or financial assets. Indeed in most of the models coming from economics or finance, a buyer may prefer to buy when no one else is buying, and conversely a seller may prefer to sell when no one else is selling.

Assumptions A4 and A5 imply that at least asymptotically when $\alpha$ tends to infinity, $L$ behaves like a power of $\alpha$ of exponent $q'$.

Under the monotonicity assumption A3, uniqueness is in general easier to obtain than existence. For uniqueness, we assume that $f$ and $g$ are also monotonous, this is the purpose of the following assumption.

(U) For $m^1, m^2 \in \mathcal{P} (\mathbb{R}^d)$, and $t \in [0,T]$, assume that,

$$\int_{\mathbb{R}^d} \left( f(t,x,m^1) - f(t,x,m^2) \right) d(m^1 - m^2)(x) \geq 0,$$
The two main results in this work are Theorems 2.2 and 2.3 below, which respectively state the uniqueness and existence of the solution to (1.4)–(1.9).

**Theorem 2.2.** Under assumptions A1–A3, U and B1 there is at most one solution to (1.4)–(1.9).

Uniqueness results for MFGC systems with a monotonicity assumption have been proved in [19] and [11]. In [19], uniqueness is proved when the diffusion coefficient is equal to 0 and the drift is equal to the control, i.e. \( \nu = 0 \) and \( b = \alpha \). In [11] Section 4.6, the authors stated uniqueness in the quadratic case. Theorem 2.2 is new in the sense that it yields uniqueness for a large new class of Lagrangians and drift functions. Indeed, beside the monotonicity
assumption A3 and U, we only assume that $L$ satisfies A1 and A2, and that the drift $b$ is invertible in the sense of B1.

**Theorem 2.3.** Under assumptions A1–A6 and B1–B2, there exists a solution to (1.4)–(1.9).

The existence of solutions of the MFGC system is in general much more demanding than for MFG systems without interactions through the controls. Under monotonicity assumptions similar to A3, existence has been proved in [11] Section 4.6, for quadratic and uniform convex Lagrangians with a growth condition on the derivatives of the Hamiltonian. In [10], the existence of weak solutions of the monotonous MFGC system is discussed with a possibly degenerate diffusion operator, under assumptions which are uniform with respect to the joint law of states and controls.

Here, we prove existence of solutions of the monotonous MFGC system for a large class of Lagrangians and drifts. Namely, we assume that the Lagrangians and drifts behave asymptotically like a power of $\alpha$; we allow them to have a growth in the law of the controls of at most the same order as the order of dependency upon $\alpha$.

Before starting the discussion on existence of solutions to the MFGC systems (1.4)–(1.9) and (2.2)–(2.6), we introduce a new MFGC system set in the torus, so that the solutions should have more compactness properties. We define $T_a^d = \mathbb{R}^d / (a\mathbb{Z}^d)$ the $d$-dimensional torus of radius $a > 0$. Namely, we consider:

\begin{align*}
- \partial_t u(t, x) - \nu \Delta u(t, x) + H (t, x, \nabla_x u(t, x), \mu(t)) &= f(t, x, m(t)), \quad (2.8) \\
\partial_t m(t, x) - \nu \Delta m(t, x) - \text{div} \left( H_p (t, x, \nabla_x u(t, x), \mu(t)) m \right) &= 0, \quad (2.9) \\
\mu(t) &= \left( I_d, -H_p (t, \cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t), \quad (2.10) \\
u(T, x) &= g(x, m(T)), \quad (2.11) \\
m(0, x) &= m_0(x), \quad (2.12)
\end{align*}

for $(t, x) \in (0, T) \times T_a^d$. All the assumptions in paragraph 2.2 are stated in $\mathbb{R}^d$. When considering that $L : [0, T] \times T_a^d \times \mathbb{R}^d \times \mathcal{P} (T_a^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$ (like in (2.8)--(2.12)) satisfies one of those assumptions, we shall simply replace $\mathbb{R}^d$ by $T_a^d$ as the state set in the chosen assumption.

The fixed point satisfied by the joint law of states and controls, namely (1.6)–(1.7), (2.4) or (2.10), may be a difficult issue for MFGC systems. Here, we solve this difficulty with the following lemma proved in Sect. 4, using mainly the monotonicity assumption A3 and the compactness of the state space of (2.8)–(2.12).

**Lemma 2.4.** Assume A1–A5. Let $p \in C^0 ([0, T] \times T_a^d; \mathbb{R}^d)$ and $m \in C^0 ([0, T]; \mathcal{P} (T_a^d))$ be such that $t \mapsto p(t, \cdot)$ is continuous with respect to the topology of the local uniform convergence and $m(t)$ admits a finite second moment uniformly bounded with respect to $t \in [0, T]$. For any $t \in [0, T]$, there exists a unique $\mu(t) \in \mathcal{P} (T_a^d \times \mathbb{R}^d)$ such that

$$\mu(t) = \left( I_d, -H_p (t, \cdot, p(t, \cdot), \mu(t)) \right) \# m(t).$$
Moreover, the map \( t \mapsto \mu(t) \) is continuous where \( \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d) \) is equipped with the weak* topology.

The next step in our strategy for proving existence is to look for a priori estimates for the solutions of the MFGC systems and obtain compactness results to use a fixed point theorem. In Sect. 5, we prove the a priori estimates stated in the following lemma for solutions to (2.8)–(2.12).

**Lemma 2.5.** Assume A1–A6. If \((u, m, \mu)\) is a solution to (2.8)–(2.12), then \(\|u\|_\infty, \|\nabla_x u\|_\infty\) and \(\sup_{t \in [0, T]} \Lambda_\infty(\mu(t))\) are uniformly bounded by a constant independent of \(a\).

Let us mention that the a priori estimates of Lemma 2.5 rely on the monotonicity assumption on \(L\) and a Bernstein method introduced in [32]. To our knowledge, these are the first results in the literature of MFGC which use the monotonicity assumption for getting a priori estimates. They are the key ingredients of the proof of the existence of solutions to (2.8)–(2.12) in the following theorem, proved in paragraph 6.1.

**Theorem 2.6.** Under assumptions A1–A6, there exists a solution to system (2.8)–(2.12).

Therefore, for any \(a > 0\) we can construct a solution to (2.8)–(2.12) which satisfies uniform estimates with respect to \(a\). Consequently, we obtain a compact sequence of approximating solutions to (2.2)–(2.6). The limit of a subsequence is a solution to (2.2)–(2.6). This leads to the following theorem proved in paragraph 6.2.

**Theorem 2.7.** Assume A1–A6, there exists a solution to (2.2)–(2.6).

Uniqueness relies on the monotonicity assumptions A3 and U, the following theorem is proved in the arxiv preprint of this work [31] using classical arguments in the MFG literature.

**Theorem 2.8.** Under assumptions A1–A3 and U, there is at most one solution to (2.2)–(2.6) or (2.8)–(2.12).

The idea to pass from (2.2)–(2.6) to (1.4)–(1.9), is to change the optimization problem in \(\alpha\) into a new optimization problem expressed in term of \(b\). In paragraph 6.4, we prove the equivalence between the solutions of these two optimization problems. As a result, a first existence results for (1.4)–(1.9) is stated in Corollary 6.3. Theorem 2.3 is a consequence of Corollary 6.3 with more tractable assumptions. Let us mention that for proving Theorem 2.3, the structure of the Lagrangian should be invariant when passing from one optimization problem to the other. In particular, one may figure out that the assumptions on the Lagrangian behaving asymptotically like a power of \(\alpha\) are preserved under our assumptions on the drift function \(b\).

Finally, Theorem 2.2 is a consequence of Theorem 2.8 and the above-mentioned equivalence between the two optimization problems.
Remark 2.9.

i) If the Lagrangian admits the following form,

\[ L(t, x, \alpha, \mu) = L^0(t, x, \alpha) + L^1(t, \mu), \]

we say that the Lagrangian is separated. Then A3 is automatically satisfied since the left-hand side of the inequality is identically equal to 0. In this case, the assumptions on \( L \) are satisfied if \( L^0 \) behaves asymptotically like a power of \( \alpha \) of exponent \( q' \), and \( L^1 \) at most involves \( \Lambda_{q_0}(\mu)^{q'} \).

ii) All our assumptions are uniform with respect to \( x \). In particular, we restrain from considering more general functions \( f \) and \( g \) since this topic has been investigated in the literature devoted to MFG systems without interaction through controls; we believe that the same tools can be applied to the present case, and that our results may be extended so.

iii) We did not address the case without diffusion, i.e. \( \nu = 0 \). However, all the a priori estimates of Sects. 4 and 5 are uniform with respect to \( \nu \). Here, the diffusion is used to easily obtain compactness results which are central for proving our existence results since the proofs rely on a fixed point theorem and approximating sequences of solutions. Using weaker topological spaces and tools from the literature devoted to weak solutions of systems of MFGs without interaction through controls, we believe that we can extend our results to weak solutions to MFGC systems without diffusion or with possibly degenerate diffusion operators. We plan to address this question in forthcoming works.

General outline

The present work aims at proving Theorems 2.2 and 2.3. We list below the main steps of our analysis to make it easier for the reader to understand the structure of the proofs.

I  We solve the fixed point (2.10) in \( \mu \), which proves Lemma 2.4, in three steps:

I.a in Lemma 4.1 we state a priori estimates for a solution of (2.10);

I.b using the Leray–Schauder fixed point theorem (Theorem 4.2), we solve the fixed point (2.10) at any time \( t \in [0, T] \), in Lemma 4.3;

I.c we prove that the fixed point \( \mu(t) \) defined at any \( t \in [0, T] \) in step I.b, is continuous with respect to time (Lemma 4.4); this implies lemma 2.4.

II  We prove the existence of a solution to (2.8)–(2.12), stated in Theorem 2.6, in two steps:

II.a we obtain a priori estimates for solutions to (2.8)–(2.12) (Lemmas 2.5 and 5.1);

II.b in paragraph 6.1, we use Leray–Schauder fixed point theorem (Theorem 4.2) and the estimates of step II.a to conclude.

III We prove existence and uniqueness of the solution to (2.2)–(2.6) (Theorems 2.7 and 2.8):

III.a the proof of Theorem 2.7 is given in paragraph 6.2;
III.b the proof of Theorem 2.8 is given in the arxiv preprint of this work [31].

IV The proof of existence and uniqueness of the solution to (1.4)–(1.9) (Theorems 2.2 and 2.3) is given in paragraph 6.4.

Contribution
An important novelty in the present work comes from the assumptions we are considering. On the one hand, we consider a general class of monotonous Lagrangians which behave asymptotically like a power of $\alpha$ with any exponent in $(1, \infty)$ (while most of the results in the literature only address the quadratic case with uniformly convex Lagrangian); they may depend on moments of $\mu_\alpha$ at most of the same order as the above-mentioned exponent of $L$ in $\alpha$; we do not require them to depend separately on $(x, \alpha)$ and $\mu_\alpha$. On the other hand, the drift functions are also general since we allow them to behave like power functions and to be not separated too. See the assumptions in paragraph 2.2 for more details.

Moreover, most contributions focus on MFG systems stated on $\mathbb{T}^d$ for simplicity. Here, we introduce a method to extend an existence result for a MFGC system stated on the torus to its counterpart on the whole Euclidean space. In particular, this method holds for MFG system without interaction through controls and the proof becomes easier. See paragraph 6.2. We also introduce a method to extend the well-posedness of MFGC (or MFG) systems to general drift functions, see paragraph 6.4. We would like to insist on the fact that our techniques are designed in order to preserve the structure of the Lagrangian when passing from one setting to another. Here, namely it preserves the monotonicity assumption A3. Furthermore, these methods apply to the conclusions of [32] and consequently generalize them.

2.4. Properties of the Lagrangian and the Hamiltonian
Starting from the assumptions of paragraph 2.2, we state here two Lemmas giving technical results on the Lagrangian $L$ and the Hamiltonian $H$ respectively. The proof can be found in the arxiv preprint of this work [31]. Lemma 2.10 below links assumption A2 with the convexity of $L$.

Lemma 2.10. If $L$ is coercive and differentiable with respect to $\alpha$, and $b = \alpha$, assuming that $L$ is strictly convex is equivalent to A2.

When $b = \alpha$, the conjugacy relation (2.7) between $H$ and $L$ and the assumptions in the paragraph 2.2 imply results on the Hamiltonian given in Lemma 2.11 below. Here, the set $\Omega$ is either $\mathbb{R}^d$ or $\mathbb{T}_a^d$, for some $a > 0$.

Lemma 2.11. Under assumptions A1, A2, A4 and A5, the map $H$, defined in 2.7, is differentiable with respect to $x$ and $p$, $H$ and its derivatives are continuous on $[0, T] \times \Omega \times \mathbb{R}^d \times \mathcal{P}_{\infty, R} (\Omega \times \mathbb{R}^d)$ for any $R > 0$. Moreover there exists $\tilde{C}_0 > 0$ a constant which only depends on $C_0$ and $q$ such that

\[
|H_p (t, x, p, \mu)| \leq \tilde{C}_0 \left(1 + |p|^{q-1} + \Lambda q' (\mu)\right),
\]

\[
|H (t, x, p, \mu)| \leq \tilde{C}_0 \left(1 + |p|^q + \Lambda q' (\mu)^q\right),
\]

(2.13) (2.14)
\[ p \cdot H_p(t, x, p, \mu) - H(t, x, p, \mu) \geq \tilde{C}_0^{-1} |p|^q - \tilde{C}_0 \left( 1 + \Lambda q'(\mu)^q \right), \quad (2.15) \]

\[ |H_x(t, x, p, \mu)| \leq \tilde{C}_0 \left( 1 + |p|^q + \Lambda q'(\mu)^q \right), \quad (2.16) \]

for any \((t, x) \in [0, T] \times \Omega, p \in \mathbb{R}^d\) and \(\mu \in \mathcal{P}(\Omega \times \mathbb{R}^d)\).

Up to replacing \(C_0\) with \(\max(C_0, \tilde{C}_0)\), we can assume that the inequalities in Lemma 2.11 are satisfied with \(C_0\) instead of \(\tilde{C}_0\).

3. Applications

3.1. Exhaustible resource model

Cournot [15] and Bertrand [3] proposed two models for the production of exhaustible resources. Here, we adopt Bertrand’s formulation, where the control of a producer is the price at which they sell one unit of resource (while Cournot’s model considers producers controlling the the quantity of units sold). It is often possible to pass from one formulation to the other, therefore we will use the terminology Bertrand and Cournot model even if it is in fact two different models. The mean field game version of this model in dimension one was introduced in [26] and numerically analyzed in [13]; for theoretical results see [6, 14, 20, 22–24].

Here, we consider a continuum of producers selling exhaustible resources. The production of a representative agent at time \(t \in [0, T]\) is denoted by \(q_t \geq 0\); the agents differ in their production capacity \(X_t \in \mathbb{R}\) (the state variable), that satisfies,

\[ dX_t = -q_t dt + \sqrt{2\nu} dW_t, \]

where \(\nu > 0\) and \(W\) is a \(d\)-dimensional Brownian motion. The selling price \(\alpha\) per unit of resource that a producer can make when they sales \(q\) units of resource, depends naturally on \(q\) and on the quantity produced by the other agents. It satisfies a supply-demand relationship. Using the Bertrand’s formulation [3], we consider that an agent controls their selling price and that their production \(q\) is a function of their selling price and some average quantity of the selling price of all the other agents.

This may lead to a system in the following form,

\[
\begin{aligned}
&-\partial_t u - \nu \Delta u + H(t, x, \nabla_x u + \varphi(x)^T P(t)) = f(t, x, m(t)), \\
&\partial_t m - \nu \Delta m - \text{div} \left( H_p(t, x, \nabla_x u + \varphi(x)^T P(t)) m \right) = 0, \\
P(t) = \Psi \left(t - \int_{\mathbb{R}^d} \varphi(x) H_p(t, x, \nabla_x u + \varphi(x)^T P(t)) \, dm(t, x)\right), \\
u(T, x) = g(x, m(T)), \\
m(0, x) = m_0(x),
\end{aligned}
\]

where \(\varphi : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) and \(\Psi : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) are given functions. The counterpart of the latter system posed on \(\mathbb{T}^d\) has been introduced in [6]. For a simpler system of this model, corresponding to case of linear demand system in one dimension, see the arxiv preprint of this work [31].
Theorems 2.2 and 2.3 provide the uniqueness and existence respectively of the solution to (3.1).

**Proposition 3.1.** Assume $A_1$, $A_2$, U. If the function $\Psi$ is continuous, $\Psi(t, \cdot)$ is monotone, locally Lipschitz continuous, and admits at most a power-like growth of exponent $q' - 1$ with a coefficient uniform in $t \in [0, T]$, there exists at most one solution to (3.1).

**Proposition 3.2.** Assume $A_1$, $A_2$, $A_4$–$A_6$, and that $\Psi$ satisfies the same assumptions as in Proposition 3.1.

There exists a solution to (3.1).

See the arxiv preprint of this work [31] for the details of the proof.

In [6], similar existence and uniqueness results for the counterpart of (3.1) posed on $T^d$ are given in the quadratic setting, with a uniformly convex Lagrangian and $\Psi$ being the gradient of a convex map. Here, we generalize their results to a wider class of Lagrangians and functions $\Psi$.

See [32] for an extension of this model non-monotonic $\Psi$.

Finally, we would like to warn the reader that system (3.1) is hardly representative of exhaustible resource models generally.

### 3.2. A model of crowd motion

This model of crowd motion has been introduced in [32] in the non-monotone setting. It has been numerically studied in [1] in the quadratic non-monotone case. For $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ we define $V(\mu)$ the average drift by,

$$V(\mu) = \frac{1}{Z(\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \alpha k(x) d\mu(x, \alpha),$$

where $Z(\mu)$ is a normalization constant defined by $Z(\mu) = \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} k(x)^{q_1} d\mu(x, \alpha) \right)^{\frac{1}{q_1}}$, for some constant $q_1 \in [q, \infty]$ where $q$ is defined below. To be consistent with the notations used in [32], $k : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a non-negative kernel. By convention, if $Z(\mu) = 0$, we take $V(\mu) = 0$.

The state of a representative agent is given by their position $X_t \in \mathbb{R}^d$ satisfying the following controlled stochastic differential equation,

$$dX_t = \alpha_t dt + \sqrt{2\nu} dW_t.$$ 

Their objective is to minimize the cost functional given by,

$$\mathbb{E} \left[ \int_0^T \frac{\theta}{2} |\alpha_t + \lambda V(\mu(t))|^2 + \frac{1 - \theta}{a'} |\alpha_t|^{a'} + f(t, X_t, m(t)) dt + g(X_T, m(T)) \right],$$

where $\lambda \geq 0$ and $0 \leq \theta \leq 1$ are two constants standing for the intensity of the preference of an individual to have an opposite control as the stream one, and $a' > 1$ is an exponent. In this model we define the Lagrangian $L$ by,

$$L(x, \alpha, \mu) = \frac{\theta}{2} |\alpha + \lambda V(\mu)|^2 + \frac{1 - \theta}{a'} |\alpha|^{a'},$$
52

and the Hamiltonian $H$ as its Legendre transform. The map $H$ does not admit an explicit form for every choice of the parameters $a'$. We take $q' = \max(2, a')$, and $q = \frac{q'}{q' - 1}$ its conjugate exponent.

Here, since the control is equal to the drift, the MFGC system is of the form of (2.2)–(2.6). Therefore, the following proposition is a consequence of Theorems 2.2 and 2.3.

**Proposition 3.3.** Assume A6, there exists a solution to the above MFGC system of crowd motion. Assume U, this solution is unique.

The proof consists in checking that $L$ satisfies A1–A5. See [32] for existence results on this model with $\lambda < 0$.

4. The fixed point (2.10) and the proof of Lemma 2.4

This section is devoted to step I. In paragraph 4.1, we state a priori estimates on a fixed point of (2.10) (Lemma 4.1); then we use these estimates and Leray–Schauder fixed point theorem (Theorem 4.2) and obtain the existence of a fixed point (2.10) at any time $t \in [0, T]$ (Lemma 4.3). We address the continuity in time of the fixed point, i.e. step I.c, in Lemma 4.4.

In this section and the next one, we work on $T^d_a = \mathbb{R}^d / (a\mathbb{Z}^d)$. The $d$-dimensional torus of radius $a > 0$. Here we take $L : [0, T] \times T^d_a \times \mathbb{R}^d \times \mathcal{P}(T^d_a \times \mathbb{R}^d) \to \mathbb{R}$. All the assumptions in paragraph 2.2 are stated in $\mathbb{R}^d$, but, when considering that $L$ satisfies one of those assumptions, we shall simply replace $\mathbb{R}^d$ by $T^d_a$ as the state set in the chosen assumption (note that we keep $\mathbb{R}^d$ as the set of admissible controls). The initial distribution $m_0$ is now in $\mathcal{P}(T^d_a \times \mathbb{R}^d)$. The Hamiltonian $H$ is still defined as the Legendre transform of $L$, i.e. it satisfies (2.7).

4.1. Leray–Schauder Theorem for solving the fixed point in $\mu$

We start by stating a priori estimates for solutions of the fixed point in $\mu$ (2.10), involving $\Lambda_{q'}(\mu)$ and $\Lambda_\infty(\mu)$ defined in (2.1).

**Lemma 4.1.** Assume that $L$ satisfies A1–A5 For any $t \in [0, T]$, $m \in \mathcal{P}(T^d_a)$ and $p \in C^0(T^d_a, \mathbb{R}^d)$, if there exists $\mu \in \mathcal{P}(T^d_a \times \mathbb{R}^d)$ such that

$$\mu = (I_d, -H_p(t, \cdot, p(\cdot), \mu)) \#m, \quad (4.1)$$

then it satisfies

$$\Lambda_{q'}(\mu) q' \leq 4C_0^2 + \frac{(q')^{q-1}(2C_0)^q}{q} \|p\|_{L^q(m)}, \quad (4.2)$$

$$\Lambda_\infty(\mu) \leq C_0 (1 + \|p\|_\infty + \Lambda_{q'}(\mu)). \quad (4.3)$$

**Proof.** We use A3 with $m \times \delta_0$ and $\mu$ satisfying (4.1),

$$\int_{T^d_a \times \mathbb{R}^d} (L(t, x, \alpha, \mu) - L(t, x, \alpha, m \times \delta_0)) \, d\mu(x, \alpha)$$

$$+ \int_{T^d_a} (L(t, x, 0, m \times \delta_0) - L(t, x, 0, \mu)) \, dm(x) \geq 0.$$
From A5, we obtain \( \int_{T_a^d} L(t, x, 0, m \times \delta_0) \, dm(x) \leq C_0 \). The latter two inequalities, A4 and the convexity of \( L \) (stated in Lemma 2.10) yield

\[
C_0^{-1} \int_{T_a^d \times \mathbb{R}^d} |\alpha|^q \, d\mu(x, \alpha) - C_0 \leq C_0 + \int_{T_a^d} (L(t, x, \alpha^\mu(x), \mu) - L(t, x, 0, \mu)) \, dm(x)
\]

\[
\leq C_0 + \int_{T_a^d \times \mathbb{R}^d} \alpha \cdot L_{\mu}(t, x, \alpha, \mu) \, d\mu(x, \alpha),
\]

where \( \alpha^\mu \) is defined in paragraph 2.1. We recall that \( p(x) = -L_{\mu}(t, x, \alpha^\mu(x), \mu) \).

Using the inequality \( yz \leq \frac{y^q}{c^q} q' + \frac{c z^q}{q} \), for \( y, z, c > 0 \), we obtain

\[
\frac{1}{C_0} \int_{T_a^d \times \mathbb{R}^d} |\alpha|^q \, d\mu(x, \alpha) \leq 2C_0 + \left( \frac{2C_0 q'}{q} \right) \int_{T_a^d} |p(x)|^q \, dm(x)
\]

\[
+ \frac{1}{2C_0} \int_{T_a^d \times \mathbb{R}^d} |\alpha|^q \, d\mu(x, \alpha).
\]

This and \( \frac{q}{q'} + 1 = q \) imply (4.2). This and 2.13 implies 4.3, we recall that we assume \( C_0 = \tilde{C}_0 \). \( \square \)

Here, we shall use Leray–Schauder fixed point theorem as stated in [16] Theorem 11.6.

**Theorem 4.2.** (Leray–Schauder fixed point theorem) Let \( B \) be a Banach space and let \( \Psi \) be a compact mapping from \([0, 1] \times B \) into \( B \) such that \( \Psi(0, x) = 0 \) for all \( x \in B \). Suppose that there exists a constant \( C \) such that

\[
\|x\|_B \leq C,
\]

for all \((\theta, x) \in [0, 1] \times B \) satisfying \( x = \Psi(\theta, x) \). Then the mapping \( \Psi(1, \cdot) \) of \( B \) into itself has a fixed point.

From Lemma 4.1 and Theorem 4.2, we obtain the following existence result for a fixed point (2.10).

**Lemma 4.3.** Assume A1–A5. For \( t \in [0, T] \), \( m \in \mathcal{P}(T_a^d) \) and \( p \in C^0(T_a^d; \mathbb{R}^d) \), there exists a unique \( \mu \in \mathcal{P}(T_a^d \times \mathbb{R}^d) \) such that \( \mu = (I_d, -H_p(t, \cdot, p(\cdot), \mu)) \# m \). Moreover, \( \mu \) satisfies the inequality stated in Lemma 4.1.

In the following proof, we will take advantage of the flexibility offered when making all assumptions on the Lagrangian, instead of the Hamiltonian. We will introduce a sequence of new Lagrangians. The associated Hamiltonians may not admit explicit form; therefore it would be difficult to check assumptions on them. Here on the one hand, checking the assumptions on the new Lagrangians is straightforward. On the other hand, we obtain the same conclusions on the new Hamiltonian as stated in Lemma 2.11.

**Proof.** Take \((t, \bar{p}, m)\) satisfying the same assumptions as \((t, p, m)\) in Lemma 4.3. In order to use the Leray–Schauder fixed point theorem later, we introduce the following family of Lagrangians indexed by \( \lambda \in [0, 1] \),

\[
L^{p,\lambda}(x, \alpha, \mu) = \lambda L(t, x, \alpha, \mu) + (1 - \lambda) \left( \frac{|\alpha|^q}{q'} - \alpha \cdot \bar{p}(x) \right),
\]

where \( \bar{p}(x) = -L_{\mu}(t, x, \alpha^\mu(x), \mu) \).
for \((x, \alpha, \mu) \in T^d_a \times \mathbb{R}^d \times \mathcal{P} (T^d_a \times \mathbb{R}^d)\). We denote by \(H^{p, \lambda}\) the Legendre transform of \(L^{p, \lambda}\). For \(\lambda = 0\) it satisfies \(H^{p, 0} (x, p, \mu) = \frac{1}{q} |p - \overline{p}(x)|^q\).

From Young inequality, we obtain that
\[
|\alpha \cdot \overline{p}(x)| \leq \frac{|\alpha|^q'}{2q'} + \frac{2q - 1}{q} \|p\|_\infty.
\]

Therefore, up to changing \(C_0\) into \(\max \left(\frac{1}{2q'}, \frac{2q-1}{q} \|p\|_\infty, C_0\right)\), we may assume that \(L^{p, \lambda}\) satisfies \(A1–A5\), with the same constant \(C_0\) for any \(\lambda \in [0, 1]\).

The map \((\lambda, x, p, \mu) \mapsto -H^{p, \lambda}_p (x, p, \mu)\) is continuous on \([0, 1] \times T^d_a \times \mathbb{R}^d \times \mathcal{P}_\infty (T^d_a \times \mathbb{R}^d)\), for any \(R > 0\), by the same arguments as in the proof of Lemma 2.11.

For \(\alpha \in C^0 (T^d_a; \mathbb{R}^d)\), we set \(\mu = (I_d, \alpha) \# m \in \mathcal{P} (T^d_a \times \mathbb{R}^d)\) and \(\overline{\alpha}(x) = -H^{p, \lambda}_p (x, \overline{p}(x), \mu)\), for \(x \in T^d_a\). We define the map \(\Psi\), from \([0, 1] \times C^0 (T^d_a; \mathbb{R}^d)\) to \(C^0 (T^d_a \times \mathbb{R}^d)\), by \(\Psi(\lambda, \alpha) = \overline{\alpha}\). If \(\alpha\) is a fixed point of \(\Psi(1, \cdot)\), then \(\mu\) satisfies the fixed point in Lemma 4.3. Conversely, if \(\mu\) satisfies the fixed point in Lemma 4.3, then \(\alpha^\mu\) (defined in paragraph 2.1) is a fixed point of \(\Psi(1, \cdot)\).

The map \(\Psi\) is continuous by the continuity of \((\lambda, x, p, \mu) \mapsto -H^{p, \lambda}_p (x, p, \mu)\). For \(R > 0\), the set \(A_R\), defined by \(A_R = [0, 1] \times T^d_a \times B_{\mathbb{R}^d} (0, R) \times \mathcal{P}_\infty, R (T^d_a \times \mathbb{R}^d)\), is compact. By Heine theorem, the map \((\lambda, x, p, \mu) \mapsto -H^{p, \lambda}_p (x, p, \mu)\) is uniformly continuous on \(A_R\). Here, note that we use the fact that \(\mathcal{P}_\infty, R (T^d_a \times \mathbb{R}^d)\) is a metric space since the weak* topology coincides with the topology induced by the 1-Wasserstein distance on \(\mathcal{P}_\infty, R (T^d_a \times \mathbb{R}^d)\). Heine theorem also implies that \(\overline{p}\) is uniformly continuous. Therefore, \(\Psi\) is a compact mapping from \([0, 1] \times C^0 (T^d_a; \mathbb{R}^d)\) to \(C^0 (T^d_a \times \mathbb{R}^d)\), i.e. it maps bounded subsets of \([0, 1] \times C^0 (T^d_a; \mathbb{R}^d)\) into relatively compact subsets of \(C^0 (T^d_a; \mathbb{R}^d)\); this comes from the latter observation and Arzelà–Ascoli theorem.

Take a fixed point of \(\Psi(\lambda, \cdot)\), for \(\lambda \in [0, 1]\), Lemma 4.1 implies that \(||\alpha||_\infty\) is bounded by a constant \(C\) which does not depend on \(\lambda\).

Moreover, it is straightforward to check that \(\Psi(0, \cdot) = 0\). Leray–Schauder Theorem 4.2 implies that there exists a fixed point of the map \(\alpha \mapsto \Psi(1, \alpha)\), which concludes the existence part of the proof.

The proof of uniqueness relies on \(A3\) and the strict convexity of \(L\), see [10] Lemma 5.2 for the detailed proof. \(\Box\)

4.2. The continuity of the fixed point in time

The fixed point result stated in Lemma 4.3 yields the existence of a map \((t, p, m) \mapsto \mu\). The continuity of this map is addressed in the following lemma:

**Lemma 4.4.** Assume \(A1–A5\). Let \((t^n, m^n, p^n)_{n \in \mathbb{N}}\) be a sequence in \([0, T] \times \mathcal{P} (T^d_a) \times C^0 (T^d_a; \mathbb{R}^d)\). Assume that
- \(t^n \rightharpoonup t_{n \to \infty} t \in [0, T]\),
- \((p^n)_{n \in \mathbb{N}}\) is uniformly convergent to \(p \in C^0 (T^d_a; \mathbb{R}^d)\),
- \((m^n)_{n \in \mathbb{N}}\) tends to \(m\) in the weak* topology.
We define $\mu^n$ and $\mu$ as the unique solutions of the fixed point relation of Lemma 4.3 respectively associated to $(t^n, m^n, p^n)$ and $(t, m, p)$, for $n \in \mathbb{N}$. Then the sequence $(\mu^n)_{n \in \mathbb{N}}$ tends to $\mu$ in $\mathcal{P}(\mathbb{T}_d^d \times \mathbb{R}^d)$ equipped with the weak* topology.

**Proof.** The sequence $(p^n)_{n \in \mathbb{N}}$ is uniformly bounded in the norm $\|\cdot\|_\infty$. Therefore $(\mu^n)_{n \in \mathbb{N}}$ is uniformly compactly supported by Lemma 4.1. Thus we can extract a subsequence $(\mu^{\varphi(n)})_{n \in \mathbb{N}}$ convergent to some limit $\tilde{\mu} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ in the weak* topology on measures.

We recall that $\alpha^\mu$ is defined in paragraph 2.1. Here, since $\mu^{\varphi(n)}$ and $\mu$ are fixed points like in Lemma 4.3, they satisfy:

$$\alpha^{\mu^{\varphi(n)}}(x) = -H_p\left(t^{\varphi(n)}, x, p^{\varphi(n)}(x), \mu^{\varphi(n)}\right),$$

$$\alpha^\mu(x) = -H_p(t, x, p(x), \mu),$$

for $x \in \mathbb{T}_d^d$ and $n \in \mathbb{N}$. From the continuity of $H_p$ stated in Lemma 2.11, $(\alpha^{\varphi(n)})_{n \in \mathbb{N}}$ tends uniformly to the function $\tilde{\alpha} : x \mapsto -H_p(t, x, p, \tilde{\mu})$. Then $\left((I_d, \alpha^{\varphi(n)}) \# m^n\right)_{n \in \mathbb{N}}$ tends to $(I_d, \tilde{\alpha}) \# m$ in the weak* topology. Hence $\tilde{\mu}$ satisfies the same fixed point relation as $\mu$; by uniqueness we deduce that $\tilde{\mu} = \mu$. This implies that all the convergent subsequences of $(\mu^n)_{n \in \mathbb{N}}$ have the same limit $\mu$, thus the whole sequence converges to $\mu$. $\square$

Lemma 4.3 states that for all time the fixed point (2.10) has a unique solution. Then Lemma 4.4 yields the continuity of the map defined by the fixed point under suitable assumptions. Therefore, the conclusion of step I.c and the Lemma 2.4 are straightforward consequences of these two lemmas.

**Remark 4.5.** All the conclusions of this section hold when we relax Assumption A3, assuming that the inequality holds only when $\mu^1$ and $\mu^2$ have the same first marginal. Some applications of MFGC do not satisfy A3, but satisfy the above-mentioned relaxed monotonicity assumption. This is the case of the MFG version of the Almgren and Chriss’ model for price impact and high-frequency trading, discussed in [10–12,32].

However, the a priori estimates in the next section do not hold under this relaxed monotonicity assumption. We refer to [32] for estimates which do not rely on A3 (Assumptions FP1 and FP2 in [32] are unnecessary if $L$ satisfies the relaxed monotonicity assumption).

## 5. A priori estimates for the solutions to (2.8)–(2.12)

In order to use the Leray–Schauder fixed point theorem later, we introduce the following family of Lagrangians indexed by $\theta \in (0, 1]$,

$$L^\theta\left(t, x, \alpha, \mu\right) = \theta L\left(t, x, \theta^{-1}\alpha, \Theta(\mu)\right),$$

where the map $\Theta : \mathcal{P}(\mathbb{T}_d^d \times \mathbb{R}^d) \to \mathcal{P}(\mathbb{T}_d^d \times \mathbb{R}^d)$ is defined by $\Theta(\mu) = (I_d \times \theta^{-1}I_d) \# \mu$. Then the Hamiltonian defined as the Legendre transform of $L^\theta$ is given by

$$H^\theta\left(t, x, p, \mu\right) = \theta H\left(t, x, p, \Theta(\mu)\right).$$
The definition of the Hamiltonian can naturally be extended to $\theta = 0$ by $H^0 = 0$, the associated Lagrangian is $L^0 = 0$ if $\alpha = 0$ and $L^0 = \infty$ otherwise. We introduce the following system of MFGC,

$$\begin{align*}
- \partial_t u(t, x) - \nu \Delta u(t, x) + H^\theta(t, x, \nabla_x u(t, x), \mu(t)) &= \theta f(t, x, m(t)), \\
\partial_t m(t, x) - \nu \Delta m(t, x) - \text{div} \left( H^\theta_p(t, x, \nabla_x u(t, x), \mu(t)) m \right) &= 0,
\end{align*}$$

for $(t, x) \in (0, T) \times \mathbb{R}^d$. When $\theta = 1$, the latter system coincides with (2.8)–(2.12). When $\theta = 0$, (5.3)–(5.7) consists in a situation in which the state of a representative agent satisfies a non-controlled stochastic differential equation. Alternatively it can be interpreted as a game in which the agents pay an infinite price as soon as they try to use a control different than 0. In particular the case $\theta = 0$ is specific and easier than the case when $\theta > 0$. Therefore, in the rest of this section, we only consider $\theta \in (0, 1]$.

Let us mention that assumptions $A1$–$A3$ are preserved when replacing $L$ and $H$ by $L^\theta$ and $H^\theta$ respectively. Moreover the inequalities from $A4$, $A5$, become respectively

$$\begin{align*}
L^\theta(t, x, \alpha, \mu) &\geq C^\theta(t) |\alpha| q - C_0 \theta - C_0 \theta^{1-q'} \Lambda_{q'}(\mu) q', \\
|L^\theta(t, x, \alpha, \mu)| &\leq C_0 \theta + C_0 \theta^{1-q'} \left( |\alpha| q' + \Lambda_{q'}(\mu) q' \right),
\end{align*}$$

since $\Lambda_{q'}(\Theta(\mu)) = \theta^{-1} \Lambda_{q'}(\mu)$. Furthermore, the conclusions of Lemma 2.11 hold and the inequalities become respectively

$$\begin{align*}
|H^\theta_p(t, x, p, \mu)| &\leq C_0 \theta \left( 1 + |p| q - 1 \right) + C_0 \Lambda_{q'}(\mu), \\
|H^\theta(t, x, p, \mu)| &\leq C_0 \theta \left( 1 + |p| q - 1 \right) + C_0 \theta^{1-q'} \Lambda_{q'}(\mu) q', \\
p \cdot H^\theta_p(t, x, p, \mu) - H^\theta(t, x, p, \mu) &\geq C_0 \theta |p| q - C_0 \theta - C_0 \theta^{1-q'} \Lambda_{q'}(\mu) q', \\
|H^\theta_z(t, x, p, \mu)| &\leq C_0 \theta \left( 1 + |p| q - 1 \right) + C_0 \theta^{1-q'} \Lambda_{q'}(\mu) q'.
\end{align*}$$

We recall that without loss of generality, we assumed $\tilde{C}_0 = C_0$ where $\tilde{C}_0$ is defined in Lemma 2.11.

Instead of proving Lemma 2.5 and step II.a, we address the more general following lemma which provides an a priori estimates not only for solutions to (2.8)–(2.12) but also for solutions to (5.3)–(5.7). This will help to use the Leray–Schauder theorem in the next section.

**Lemma 5.1.** Under assumptions $A1$–$A6$, there exists a positive constant $C$ which only depends on the constants in the assumptions and not on $a$ or $\theta$, such that the solution to (5.3)–(5.7) satisfies: $\|u\|_\infty \leq C \theta$, $\|\nabla_x u\|_\infty \leq C \theta^{\frac{1}{2}}$ and $\sup_{t \in [0, T]} \Lambda_{\infty}(\mu(t)) \leq C \theta$. 

Proof. First step: controlling \( \int_0^T \Lambda q' (\mu(t)) q' \, dt \)

Let us take \((X, \alpha)\) defined by
\[
\begin{cases} 
\alpha_t = \alpha^{\mu(t)}(t, X_t) = -H^\theta_p(t, X_t, \nabla_x u(t, X_t), \mu(t)), \\
dX_t = \alpha_t dt + \sqrt{2\nu} dB_t, \\
X_0 = \xi \sim m_0,
\end{cases}
\]
where \((B_t)_{t \in [0, T]}\) is a Brownian motion independent of \(\xi\).

The function \(u\) is the value function of an optimization problem, i.e. the lowest cost that a representative agent can achieve from time \(t\) to \(T\) if \(X_t = x\), when the probability measures \(m\) and \(\mu\) are fixed, i.e.
\[
\alpha_{|s \in [t, T]} = \arg\min_{\alpha'} E \left[ \int_t^T L^\theta(s, X_s^{\alpha'}, \alpha_s, \mu(s)) + \theta f(s, X_s^{\alpha'}, m(s)) \, ds \right. \\
\left. + \theta g(\hat{X}_T^{\alpha'}, m(T)) \right],
\]
where for a control \(\alpha'\), we define
\[
dX_t^{\alpha'} = \alpha' dt + \sqrt{2\nu} dB_t, \quad X_0^{\alpha'} = \xi \sim m_0,
\]
and \((B_t')_{t \in [0, T]}\) is a Brownian motion independent of \(\xi'\). Let us recall that for any \(t \in [0, T]\), \(m(t)\) is the law of \(X_t\), and \(\mu(t)\) is the law of \((X_t, \alpha_t)\). We introduce \(\hat{X}\) the stochastic process defined by
\[
d\hat{X}_t = \sqrt{2\nu} dB_t, \quad \hat{X}_0 = \xi \sim m_0.
\]
We set \(\tilde{m}(t) = \mathcal{L}(\hat{X}_t)\) and \(\tilde{\mu}(t) = \mathcal{L}(\hat{X}_t) \times \delta_0\), for \(t \in [0, T]\). For \(\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\) we define \(I(\mu^1, \mu^2)\) by,
\[
I(\mu^1, \mu^2) = \int_0^T \int_{\mathbb{T}_d^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu^1(t)) \, d\mu^2(t, x, \alpha) dt.
\]
For the strategy consisting in taking \(\alpha' = 0\), (5.14) yields the inequality:
\[
I(\mu, \mu) + \int_0^T \int_{\mathbb{T}_d^d} \theta f(t, x, m(t)) \, dm(t, x) dt + \int_{\mathbb{T}_d^d} \theta g(x, m(T)) \, dm(T, x)
\leq I(\mu, \tilde{\mu}) + \int_0^T \int_{\mathbb{T}_d^d} \theta f(t, x, m(t)) \, d\tilde{m}(t, x) dt + \int_{\mathbb{T}_d^d} \theta g(x, m(T)) \, d\tilde{m}(T, x).
\]
This and A6 imply that,
\[
I(\mu, \mu) - I(\mu, \tilde{\mu}) \leq 2C_0 \theta (1 + T).
\]
Integrating over \([0, T]\) the inequality in Assumption A3 with \((\mu(t), \tilde{\mu}(t))\), yields
\[
I(\tilde{\mu}, \mu) \leq I(\mu, \mu) - I(\mu, \tilde{\mu}) + I(\tilde{\mu}, \tilde{\mu}) \leq 2C_0 \theta (1 + T) + I(\tilde{\mu}, \tilde{\mu}),
\]
where we used (5.15) to get the second line. Then, from A5 we obtain that,
\[
I(\tilde{\mu}, \tilde{\mu}) = \int_{\mathbb{T}_d^d} \theta L(t, x, 0, \tilde{\mu}(t)) \, d\tilde{m}(t, dx) \leq C_0 \theta.
\]
Finally, using the latter two inequalities to get an upper bound on $I(\tilde{\mu}, \mu)$, and using Assumption (A4) to get a lower bound on $I(\tilde{\mu}, \mu)$, we obtain,

$$
\int_0^T \int_{\mathcal{T}_d^a} \left(C_0^{-1}\theta^{1-q'} |\alpha|^{q'} - C_0\theta\right) \, d\mu(t, x, \alpha) dt \leq I(\tilde{\mu}, \mu) \leq C_0\theta(2 + 3T).
$$

This implies

$$
\int_0^T \Lambda_{q'}(\mu(t))^{q'} \, dt \leq 2C_0^2\theta^{q'}(1 + 2T).
$$

(5.16)

Second step: the uniform estimate on $\|u\|_\infty$

Let us rewrite (5.3) in the following way,

$$
-\partial_t u - \nu \Delta u + \left[ \int_0^1 H_p^\theta(t, x, s\nabla_x u, \mu(t)) \, ds \right] \cdot \nabla_x u = H^\theta(t, x, 0, \mu(t)) + \theta f(t, x, m(t)),
$$

for $(t, x) \in (0, T) \times \mathcal{T}_d^a$. The maximum principle for second-order parabolic equation, A6, and (2.14) yield that

$$
\|u\|_\infty \leq C_0\theta(1 + 2T) + C_0\theta^{1-q'} \int_0^T \Lambda_{q'}(\mu(t))^{q'} \, dt,
$$

which implies that $u$ is uniformly bounded using the conclusion of the previous step.

Third step: the uniform estimate on $\|\nabla_x u\|_\infty$.

The proof of this step relies on the same Bernstein-like method introduced in [32] Lemma 6.5. We refer to the proof of the latter results for more details in the derivation of the equations below.

Let us introduce $\rho \in C^\infty \left([-\frac{a}{2}, \frac{a}{2}]^d\right)$ a nonnegative mollifier such that $\rho(x) = 0$ if $|x| \geq \frac{a}{4}$ and $\int_{-\frac{a}{2}}^{\frac{a}{2}} \rho(x) \, dx = 1$.

For any $0 < \delta < 1$ and $t \in [0, T]$, we introduce $\rho^\delta = \delta^{-d} \rho \left(\frac{x}{\delta}\right)$ and $u^\delta(t) = \rho^\delta * u(t)$ with * being the convolution operator with respect to the state variable.

Possibly after modifying the constant $C$ appearing in the first step, we can assume that $\|u\|_\infty + (1 + C_0) \theta^{1-q'} \int_0^T \Lambda_{q'}(\mu(s))^{q'} \, ds \leq C$ using the first two steps in such a way that $C$ depends only on the constants in the assumptions, and not on $\theta$. Then we introduce $\varphi : [-C, C] \rightarrow (0, \infty)$ and $w^\delta$ defined by

$$
\varphi(v) = \exp(\exp(-v)),
$$

$$
w^\delta(t, x) = \varphi \left(u^\delta(T - t, x) + (1 + C_0) \theta^{1-q'} \int_{T-t}^T \Lambda_{q'}(\mu(s))^{q'} \, ds \right) |\nabla_x u^\delta|^{2}(T - t, x),
$$

(5.17)

for $(t, x) \in [0, T] \times \mathcal{T}_d^a$, $v \in B_{R^d}(0, C)$. In particular $\varphi' < 0$, and $\varphi, 1/\varphi, -\varphi'$ and $-1/\varphi'$ are uniformly bounded. We refer to the proof of Lemma 6.5 in [32] for the derivation of the following partial differential equation satisfied by $w^\delta$,

$$
\partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p^\theta(x, \nabla_x u^\delta, \mu) - 2\nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta + 2\nu \varphi |D^2_{x,x} u^\delta|^2
$$
\[ \begin{align*}
\n= \frac{\varphi'}{\varphi} w^\delta \left[ \nabla_x u^\delta \cdot H_p^\theta (x, \nabla_x u^\delta, \mu) - H^\theta (x, \nabla_x u^\delta, \mu) + (1 + C_0) \theta^{1-q'} \Lambda_{q'} (\mu)^q' \right] \\
- \nu \frac{\varphi''}{\varphi^3} - 2 (\varphi')^2 (w^\delta)^2 - 2 \varphi \nabla_x u^\delta \cdot H^\theta_p (x, \nabla_x u^\delta, \mu) \\
+ 2 \theta \varphi \nabla_x u^\delta \cdot f^\delta (x, m) + R^\delta (t, x)
\end{align*} \tag{5.18} \]

in which \( H^\theta, f, f^\delta, u, u^\delta \) and \( \mu \) are taken at time \( T - t \) and \( w^\delta \) at time \( t \), and where \( f^\delta \) and \( R^\delta \), are defined by,

\[
\begin{align*}
&f^\delta (x, m) = \rho^\delta \ast (f(\cdot, m)) (x), \\
&R^\delta (t, x) = - \varphi' |\nabla_x u^\delta|^2 \left[ \rho^\delta \ast (H^\theta (\cdot, \nabla_x u, \mu)) (x) - H^\theta (x, \nabla_x u^\delta, \mu) \right] \\
&\quad - 2 \varphi \nabla_x u^\delta \cdot \left[ (\rho^\delta \ast H^\theta_p (\cdot, \nabla_x u, \mu)) (x) - H^\theta_p (x, \nabla_x u^\delta, \mu) \right], \\
&\quad + 2 \varphi \nabla_x u^\delta \cdot D^2_{x,x} u^\delta H^\theta_p (x, \nabla_x u^\delta, \mu) - \rho^\delta \ast (D^2_{x,x} u^\delta H^\theta_p (\cdot, \nabla_x u, \mu)).
\end{align*}
\]

From (5.12), we obtain that

\[
\nabla_x u^\delta \cdot H^\theta_p (x, \nabla_x u^\delta, \mu) - H^\theta (x, \nabla_x u^\delta, \mu) + (1 + C_0) \theta^{1-q'} \Lambda_{q'} (\mu)^q' \geq C_0^{-1} \theta |\nabla_x u^\delta|^q + \theta^{1-q'} \Lambda_{q'} (\mu)^q' - C_0 \theta.
\]

Therefore, using A6, (5.18), (5.13), the facts that \( \varphi' < 0 \), that \( \varphi'' - 2 (\varphi')^2 \geq 0 \), that \( \varphi, \varphi^{-1}, \varphi' \), \( (\varphi')^{-1} \) are bounded, and the latter inequality, we get

\[
\begin{align*}
\partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H^\theta_p (x, \nabla_x u^\delta, \mu) - 2 \nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta \\
\leq &- C_0^{-1} \left( \theta (w^\delta)^{\frac{3}{2}} + \theta^{1-q'} \Lambda_{q'} (\mu)^q' \right) w^\delta \\
&+ C (w^\delta)^{\frac{1}{2}} \left[ \theta + \theta (w^\delta)^{\frac{1}{2}} + \theta (w^\delta)^{\frac{3}{2}} + \theta^{1-q'} \Lambda_{q'} (\mu)^q' \right] + \|R^\delta\|_\infty, \tag{5.19}
\end{align*}
\]

up to updating \( C \). We notice that the terms with the highest exponents in \( w^\delta \) and \( \Lambda_{q'} (\mu)^q' \) in the right-hand side of the latter inequality is non-positive. Let us use Young inequalities and obtain

\[
\begin{align*}
(w^\delta)^{\frac{3}{2}} \Lambda_{q'} (\mu)^q' \leq &\varepsilon w^\delta \Lambda_{q'} (\mu)^q' + \frac{1}{4 \varepsilon} \Lambda_{q'} (\mu)^q', \\
(w^\delta)^{\frac{3}{2}} \Lambda_{q'} (\mu)^q' \leq &\varepsilon (w^\delta)^{1+\frac{3}{2}} + \frac{q + 2 - 2 \tilde{q}}{q + 2} \left( \varepsilon (q + 2) \right)^{\frac{2}{2 \tilde{q}}} \\
&- \frac{2}{q + 2 - 2 \tilde{q}},
\end{align*}
\]

for any \( \tilde{q} < 1 + \frac{2}{q} \) and \( \varepsilon > 0 \). Using systematically these two inequalities in (5.19) and taking \( \varepsilon \) small enough we finally obtain,

\[
\begin{align*}
\partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H^\theta_p (x, \nabla_x u^\delta, \mu) - 2 \nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta \\
\leq &C_\varepsilon \left( \theta + \theta^{1-q'} \Lambda_{q'} (\mu)^q' \right) + \|R^\delta\|_\infty,
\end{align*}
\]

where \( C_\varepsilon \) is a constant which depends on \( \varepsilon \) and the constants in the assumptions. From A6, the initial condition of \( w^\delta \) is bounded. Therefore the maximum
principle for second-order parabolic equations implies that
\[ \|w^\delta\|_\infty \leq C_\varepsilon \left( \theta + \theta T + \theta^{1-q'} \int_0^T \Lambda_{q'}(\mu(t))^{q'} \, dt \right) + T \|R^\delta\|_\infty. \] (5.20)

Let us point out that \( \nabla_x u \) is the solution of the following backward \( d \)-dimensional parabolic equation,
\[-\partial_t \nabla_x u - \nu \Delta \nabla_x u + D_{x,x}^2 u H_p(x, \nabla_x u, \mu) = \nabla_x f(x, m) - H_x(x, \nabla_x u, \mu),\]
which has bounded coefficients and right-hand side, and a terminal condition in \( C^{1+\beta_0}(\mathbb{T}_d^d) \). Theorem 6.48 in [37] states that \( \nabla_x u \) and \( D_{x,x}^2 u \) are continuous. This and the continuity of \( H_{\theta} \) and \( H_{\theta}^\infty \) stated in Lemma 2.11 imply that \( R^\delta \) is uniformly convergent to 0 when \( \delta \) tends to 0. We conclude this step of the proof by passing to the limit in (5.20) as \( \delta \) tends to 0, using the estimate on \( \int_0^T \Lambda_{q'}(\mu(t))^{q'} \, dt \) computed in the first step. We obtain that \( \nabla_x u \) is uniformly bounded by a constant which depends on the constants in the assumptions, and depends linearly on \( \theta_1^2 \).

Fourth step: obtaining uniform estimates on \( \Lambda_{q'}(\mu) \) and \( \Lambda_{\infty}(\mu) \).
Repeating the calculation in the proof of Lemma 4.1 with \( L \) satisfying (5.8) and (5.9), we obtain:
\[ \Lambda_{q'}(\mu(t))^{q'} \leq 4C_0^2 \theta q' + \frac{(q')^{q-1}(2C_0)^q}{q} \theta q' \|\nabla_x u(t)\|_{L^q(m(t))}^q. \] (5.21)

This and the third step of this proof yield that \( \sup_{t \in [0,T]} \Lambda_{q'}(\mu(t)) \leq C \theta \) for some \( C \) depending only on the constants of the assumptions. We conclude that \( \sup_{t \in [0,T]} \Lambda_{\infty}(\mu(t)) \) satisfies a similar inequality using (5.10).

\[ \square \]

6. Existence and uniqueness results
Paragraph 6.1 is devoted to proving the existence of solutions to (2.8)–(2.12), which is step II.b. In paragraph 6.2, we propose a method to extend the existence result to system (2.2)–(2.6) which is stated on \( \mathbb{R}^d \); this concludes step III.a. This method relies on compactness results using the uniform estimates of \( \nabla_x u \) that we obtained in Lemma 5.1. Then the main results of the paper and step IV are addressed in paragraph 6.4. We introduce a one-to-one correspondence between solutions to (1.4)–(1.9) and (2.2)–(2.6), which allows us to obtain directly the existence and the uniqueness of the solution to (1.4)–(1.9) from the ones to (2.2)–(2.6).

6.1. Proof of Theorem 2.6: existence of solutions to (2.8)–(2.12)
We will use the a priori estimates stated in Sect. 5 and the latter fixed point theorem, in order to achieve step II.b and prove the existence of solutions to (2.8)–(2.12).

Proof of Theorem 2.6. We would like to use the Leray–Schauder theorem 4.2 on a map which takes a flow of measures \( (\tilde{m}_t)_{t \in [0,T]} \in \{ \mathcal{P}(\mathbb{T}_d^d) \}_{[0,T]} \) as an argument. However, \( \mathcal{P}(\mathbb{T}_d^d) \) is not a Banach space. A way to go through this
difficulty is to compose the latter map with a continuous map from a convenient Banach space to the set of such flows of measures. Here, we consider the map introduced in [6], namely \( \rho: C^0([0,T] \times \mathbb{T}^d, \mathbb{R}) \rightarrow C^0([0,T] \times \mathbb{T}^d, \mathbb{R}) \) defined by

\[
\rho(\tilde{m})(t,x) = \frac{\tilde{m}_+(t,x) - a^{-d} \int \tilde{m}_+(t,y)dy}{\max(1, \int \tilde{m}_+(t,y)dy)} + a^{-d},
\]

where \( \tilde{m}_+(t,x) = \max(0, \tilde{m}(t,x)) \). We will also have the use of \( \tilde{m}^0 \) defined as the unique weak solution of

\[
\partial_t \tilde{m}^0 - \nu \Delta \tilde{m}^0 = 0 \text{ on } (0,T) \times \mathbb{T}^d,
\]

and \( \tilde{m}^0(0,\cdot) = m^0 \). \hspace{1cm} (6.1)

We are now ready to construct the map \( \Psi \) on which we will use the Leray–Schauder theorem 4.2. Take \( \theta \in [0,1], u \in C^{0,1}([0,T] \times \mathbb{T}^d; \mathbb{R}) \) and \( \tilde{m} \in C^0([0,T] \times \mathbb{T}^d; \mathbb{R}) \). We define \( m = \rho(\tilde{m} + \tilde{m}^0) \) and \( (\mu, \alpha) \in C^0([0,T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)) \times C^0([0,T] \times \mathbb{T}^d; \mathbb{R}) \) by,

\[
\alpha(t,x) = -H^\theta_p(t,x,\nabla_x u(t,x), \mu(t)),
\]

\[
\mu(t) = (I_d, \alpha(t,\cdot)) \# m(t).
\]

This definition comes from the conclusions of Lemma 2.4 when \( \theta > 0 \). For \( \theta = 0 \), it simply consists in taking \( \alpha = 0 \) and \( \mu(t) = m(t) \times \delta_0 \). Here we can repeat the calculation and obtain inequality (5.21). This and (5.10) implies that \( ||\alpha||_\infty \) is bounded by \( C\theta \) for some constant \( C > 0 \) which depends on \( ||\nabla_x u||_\infty \) and is independent of \( \theta \) and \( a \).

Then we define \( \overline{m} \) the solution in the sense of distributions of

\[
\partial_t \overline{m} - \nu \Delta \overline{m} + \text{div}(\alpha \overline{m}) = 0,
\]

supplemented with the initial condition \( \overline{m}(0,\cdot) = m^0 \), with \( m^0 \) being \( \beta_0 \)-Hölder continuous. Theorem 2.1 section V.2 in [33] states that \( \overline{m} \) is uniformly bounded by a constant which depends on \( ||m^0||_\infty \) and \( ||\alpha||_\infty \). Theorem 6.29 in [37] yields that \( m \in C^{\frac{1}{2}+\beta}_+([0,T] \times \mathbb{T}^d) \) for \( \beta \in (0, \beta_0) \), and that its associated norm can be estimated from above by a constant which depends on \( ||\nabla_x u||_\infty \), \( \beta \), \( a \) and the constants in the assumptions. The same arguments applied to \( \tilde{m}^0 \) defined in (6.1) imply that \( \tilde{m}^0 \) is in \( C^{\frac{1}{2}+\beta}_+([0,T] \times \mathbb{T}^d) \) and its associated norm is bounded.

Then we take \( \overline{u}(t) = (I_d, \alpha(t,\cdot)) \# \overline{m}(t) \) for any \( t \in [0,T] \), and \( \overline{u} \in C^{0,1}([0,T] \times \mathbb{T}^d; \mathbb{R}) \) the unique solution in the sense of distributions of the following heat equation with bounded right-hand side,

\[
-\partial_t \overline{u} - \nu \Delta \overline{u} = -H^\theta(t,x,\nabla_x u, \overline{u}(t)) + \theta f(x, \overline{m}(t)),
\]

supplemented with the terminal condition \( \overline{u}(T,\cdot) = \theta g(\cdot, \overline{m}(T)) \) which is in \( C^{1+\beta_0}([0,T]) \). Classical results (see for example Theorem 6.48 in [37]) state that \( u \) is in \( C^{\frac{1}{4}+\frac{1}{2},1+\beta}_+([0,T] \times \mathbb{T}^d) \) and its associated norm is bounded by a constant which depends on \( ||\nabla_x u||_\infty \), \( \beta \), \( a \) and the constants in the assumptions.

We can now construct the map \( \Psi : (\theta, u, \tilde{m}) \mapsto (\overline{u}, \overline{m} - \tilde{m}^0) \), from \( C^{0,1}([0,T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0,T] \times \mathbb{T}^d; \mathbb{R}^d) \) into itself. This map is continuous and compact, it satisfies \( \Psi(0,u, \tilde{m}) = 0 \) for any \( (u, \tilde{m}) \). In particular, the fact
that \( \| \alpha \|_{\infty} \leq C \theta \) in the previous paragraph, implies that \( \bar{m} \) tends to \( \bar{m}^0 \) and \( \bar{\pi} \) tends to 0 as \( \theta \) tends to 0. This gives the continuity of \( \Psi \) at \( \theta = 0 \). Moreover the fixed points of \( \Psi(\theta) \) are exactly the solutions to (5.3)–(5.7), which are uniformly bounded by Lemma 5.1. Therefore, by the Leray–Schauder fixed point theorem 4.2, there exists a solution to (2.8)–(2.12).

\( \square \)

6.2. Proof of Theorem 2.7: passing from the torus to \( \mathbb{R}^d \)

The purpose of this paragraph is to extend the existence result to the system (2.2)–(2.6) and achieve step III.a.

**Proof of Theorem 2.7. First step: constructing a sequence of approximate solutions.**

For \( a > 0 \) we define \( \tilde{m}^{0,a} = \pi^a \# m^0 \), where \( \pi^a : \mathbb{R}^d \to T_a^d \) is the quotient map. Let \( \chi^a : T_a^d \to \mathbb{R} \) be the canonical injection from the one-dimensional torus of radius \( a \) to \( \mathbb{R} \), which image is \( \left[ \frac{-a}{2}, \frac{a}{2} \right] \). Take \( \tilde{\psi} \in C^2(\mathbb{R}; \mathbb{R}) \) periodic with a period equal to 1 and such that,

\[
\tilde{\psi}(x) = x, \quad \text{if } |x| \leq \frac{1}{4},
\]

\[
|\tilde{\psi}(x)| \leq |x|, \quad \text{for any } x \in \left[ -\frac{1}{2}, \frac{1}{2} \right],
\]

(6.2)

We define \( \psi^a : T_a^d \to \mathbb{R}^d \) by \( \psi^a(x)_i = a \tilde{\psi}(a^{-1} \chi^a(x)_i) \) for \( i = 1, \ldots, d \), this is a \( C^2 \) function. Since \( \tilde{\psi} \left( \frac{\cdot}{a} \right) \) has a period of \( a \), the function \( \psi^a \circ \pi^a : \mathbb{R}^d \to \mathbb{R}^d \) satisfies

\[
\psi^a \circ \pi^a(x)_i = a \tilde{\psi} \left( \frac{x_i}{a} \right),
\]

(6.3)

for \( i = 1, \ldots, d \) and \( x \in \mathbb{R}^d \), and is a \( C^2 \) function.

We define the periodic approximations of \( L \), \( f \) and \( g \) by,

\[
L^a(t,x,\alpha,\mu) = L(t,\psi^a(x),\alpha,(\psi^a \times I_d) \# \mu),
\]

\[
f^a(t,x,m) = f(t,\psi^a(x),\psi^a \# m),
\]

\[
g^a(x,m) = g(\psi^a(x),\psi^a \# m),
\]

for \( (t,x) \in [0,T] \times T_a^d \), \( \alpha \in \mathbb{R}^d \), \( \mu \in \mathcal{P}(T_a^d \times \mathbb{R}^d) \). Let \( H^a \) be the periodic Hamiltonian associated with \( L^a \) by the Legendre transform:

\[
H^a(t,x,p,\mu) = H(t,\psi^a(x),p,(\psi^a \times I_d) \# \mu).
\]

Let us point out that the fact that \( L, H, f \) and \( g \) satisfy A1–A6, implies that \( L^a, H^a, f^a \) and \( g^a \) satisfy these assumptions too with \( C_0 \| \tilde{\psi}' \|_{\infty} \) instead of \( C_0 \).

So we can define \( (\tilde{u}^a, \tilde{m}^a, \tilde{\mu}^a) \) a solution to (2.8)–(2.12) with \( H^a \), \( f^a \), \( g^a \) and \( \tilde{m}^{0,a} \) instead of \( H, f, g \) and \( m^0 \). We define \( u^a \in C^0\left( [0,T] \times \mathbb{R}^d ; \mathbb{R} \right) \), \( m^a \in C^0\left( [0,T] ; \mathcal{P}(\mathbb{R}^d) \right) \) and \( \mu^a \in C^0\left( [0,T] ; \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \right) \) respectively by

\[
u^a(t,x) = \tilde{u}^a \left( t, \psi^a(x) \right), \quad m^a(t) = \psi^a \# \tilde{m}^a(t), \quad \text{and} \quad \mu^a(t) = (\psi^a \times I_d) \# \tilde{\mu}^a(t),
\]

for \( (t,x) \in [0,T] \times \mathbb{R}^d \).

Second step: Proving that \( m^a \) is compact.
We are going to use the Arzelà–Ascoli Theorem on $C^0\left([0, T]; (\mathcal{P}(\mathbb{R}^d), W_1)\right)$ ($\mathcal{P}(\mathbb{R}^d)$ is endowed with the $1$-Wassertein distance). First we prove that for any $t \in [0, T]$, the sequence $(m^a(t))_{a>1}$ is compact with the $1$-Wassertein distance, by proving that $\int_{\mathbb{R}^d} |x|^2 dm^a(t, x)$ is uniformly bounded in $a$. At time $t = 0$, we have

\[
\int_{\mathbb{R}^d} |x|^2 dm^a(0, x) = \int_{\mathbb{T}^d_a} |\psi^a(x)|^2 d\tilde{m}^{a,0}(x) = \int_{\mathbb{R}^d} |\psi^a \circ \pi^a(x)|^2 dm^0(x)
\]

\[
\leq \int_{\mathbb{R}^d} |x|^2 dm^0(x) \leq C_0,
\]

using (6.2), (6.3) and A6. Let us differentiate $\int_{\mathbb{R}^d} |x|^2 dm^a(t, x)$ with respect to time, perform some integrations by part and obtain that

\[
\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 dm^a = \frac{d}{dt} \int_{\mathbb{T}^d_a} |\psi^a(x)|^2 d\tilde{m}^a
\]

\[
= \int_{\mathbb{T}^d_a} |\psi^a(x)|^2 \left( \nu \Delta \tilde{m}^a(t, x) - \text{div} \left( \alpha \tilde{\alpha}^a(t, \tilde{m}^a(t, x)) \right) \right) dx
\]

\[
= 2 \int_{\mathbb{T}^d_a} \sum_{i=1}^d \left[ \nu \tilde{\psi}^{(i)} \left( \frac{\psi^a(x^i)}{a} \right) \tilde{\psi}^{(i)} \left( \frac{\psi^a(x^i)}{a} \right) \right] \nu \tilde{\psi}^{(i)} \left( \frac{\psi^a(x^i)}{a} \right) dx
\]

\[
+ |\psi^a(x)|^2 \tilde{\psi}^{(i)} \left( \frac{\psi^a(x^i)}{a} \right) \alpha \tilde{\alpha}^a(t, i(x)) d\tilde{m}^a
\]

\[
\leq 2\nu d \left\| \tilde{\psi}^{(i)} \right\|_{\infty} \left\| \tilde{\psi} \right\|_{\infty} + 2\nu d \left\| \tilde{\psi} \right\|_{\infty}^2 + \left\| \psi^a \right\|_{\infty}^2 \left\| \alpha \tilde{\alpha}^a(t) \right\|_{\infty}^2 + \int_{\mathbb{T}^d_a} |\psi^a(x)|^2 d\tilde{m}^a
\]

\[
\leq 2\nu d \left\| \tilde{\psi}^{(i)} \right\|_{\infty} \left\| \tilde{\psi} \right\|_{\infty} + 2\nu d \left\| \tilde{\psi} \right\|_{\infty}^2 + \left\| \tilde{\psi} \right\|_{\infty}^2 \left\| \alpha \tilde{\alpha}^a(t) \right\|_{\infty}^2 + \int_{\mathbb{R}^d} |x|^2 dm^a.
\]

We recall that $(t, x) \mapsto \alpha \tilde{\alpha}^a(t, x)$ is uniformly bounded with respect to $t$ and $a$ by Lemma 5.1. Therefore, the latter two inequalities and a comparison principle for ordinary differential equation imply that $\int_{\mathbb{R}^d} |x|^2 dm^a(t, x)$ is uniformly bounded with respect to $a$ and $t$.

We define $X^a$ a random process on $\mathbb{R}^d$ by

\[
dX^a_t = \alpha \mu^a(t) \left( \pi^a \left( X^a_t \right) \right) dt + \sqrt{2\nu d} dB_t, \quad \mathcal{L}(X^a_0) = m^0,
\]

where $B$ is a Brownian motion on $\mathbb{R}^d$ independent of $X^a_0$. For $t, s \in [0, T]$, we have that,

\[
\mathbb{E} \left[ |X^a_t - X^a_s| \right] \leq \mathbb{E} \left[ |X^a_t - X^a_s|^2 \right]^{\frac{1}{2}}
\]

\[
\leq \mathbb{E} \left[ \left| \int_s^t \sqrt{2\nu d} dW_r \right|^2 \right]^{\frac{1}{2}} + \mathbb{E} \left[ \left| \int_s^t \alpha \mu^a(r) \right|^2 \right]^{\frac{1}{2}}
\]

\[
\leq \sqrt{2\nu d} |t - s|^{\frac{1}{2}} + |t - s| \sup_{r \in [0, T]} \left\| \alpha \mu^a(r) \right\|_{\infty}.
\]
We define $\tilde{X}^a_t = \pi^a \left( \chi_t^a \right) \in \mathbb{T}_a$ and $X^a_t = \psi^a \left( \tilde{X}^a_t \right) \in \mathbb{R}^d$, for $t \in [0, T]$. One may check that the law of $\tilde{X}^a_t$ satisfies the same Fokker-Planck equation in the sense of distributions as $\tilde{m}^a(t)$ by testing it with $C^\infty((0, T) \times \mathbb{T}^d)$ test functions. Therefore, the law of $\tilde{X}^a_t$ is $\tilde{m}^a(t)$ and the law of $X^a_t$ is $m^a(t)$. By definition of the 1-Wasserstein distance, we obtain

$$W_1 (m^a(t), m^a(s)) \leq \mathbb{E} \left[ |X^a_t - X^a_s| \right] \leq \mathbb{E} \left[ |\psi^a \circ \pi^a (\tilde{X}^a_t) - \psi^a \circ \pi^a (\tilde{X}^a_s)| \right] \leq \|\psi\|_{\infty} \mathbb{E} \left[ |\tilde{X}^a_t - \tilde{X}^a_s| \right] \leq \|\psi\|_{\infty} \left( \sqrt{2\nu d|t-s|^\frac{1}{2}} + |t-s| \sup_{r \in [0, T]} \left\| \alpha^{a\circ r} \right\|_{\infty} \right),$$

where we used (6.3) and the mean value theorem to pass from the second to the third line in the latter chain of inequalities. Therefore by the Arzelà–Ascoli theorem, $(m^a)_a \geq 1$ is relatively compact in $C^0([0, T]; (\mathcal{P} (\mathbb{R}^d), W_1))$.

**Third Step: passing to the limit for a subsequence.**

We recall that $\tilde{u}^a$ and $\nabla_x \tilde{u}^a$ are uniformly bounded with respect to $a$, so are $u^a$ and $\nabla_x u^a$. Moreover $u^a$ satisfies the following PDE,

$$-\partial_t u^a - \nu \Delta u^a + H(t, \psi^a \circ \pi^a (x), \nabla_x u(t, x), \mu^a(t)) = f(t, \psi^a \circ \pi^a (x), m^a(t)),$$

for $(t, x) \in (0, T) \times B_{\mathbb{R}^d}(0, a)$, we recall that $\psi^a \circ \pi^a (x) = x$ if $|x| \leq \frac{a}{4}$. For $a_0 > 0$, we choose $a$ such that $a > 4(a_0 + 1)$, this implies that $u^a$ satisfies a backward heat equation on $B_{\mathbb{R}^d}(0, a_0 + 1)$ with a bounded right-hand side, a bounded terminal condition, and bounded boundary conditions. Classical results on the heat equation (see for example Theorem 6.48 in [37]) state that $u^a$ is in $C^{\frac{\beta}{2} + \gamma, 1+\beta}([0, T] \times B_{\mathbb{R}^d}(0, a_0); \mathbb{R})$ and that its associated norm is bounded by a constant which depends on the constants in the assumptions and $a_0$, but not on $a$. Therefore $(u^a_{\tilde{B}_{\mathbb{R}^d}(0, a_0)})_{a \geq 1}$ is a compact sequence in $C^{0,1}([0, T] \times B_{\mathbb{R}^d}(0, a_0); \mathbb{R})$ for any $a_0 > 0$. Then by a diagonal extraction method, there exists $a_n$ an increasing sequence tending to $\infty$ such that

$$m^{a_n} \to m \quad \text{in } C^0([0, T], (\mathcal{P} (\mathbb{R}^d), W_1)),$$

$$u^{a_n} \to u \quad \text{locally in } C^{0,1},$$

for some $(u, m) \in C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R}) \times C^0([0, T]; (\mathcal{P} (\mathbb{R}^d), W_1))$. Let us prove that for $t \in [0, T]$, $\mu^{a_n}(t)$ converges to a fixed point of (2.4) when $n$ tends to infinity; indeed we notice that

$$\mu^{a_n}(t) = (\psi^{a_n} \times I_d) \# \tilde{\mu}^{a_n}(t) = (\psi^{a_n} \times I_d) \# [ (I_d, -H^{a_n}_p (t, \cdot, \nabla_x \tilde{u}^{a_n} (t, \pi^{a_n} \circ \psi^{a_n} (\cdot)), \tilde{\mu}^{a_n} (t))) \# \tilde{m}^{a_n} ] = (\psi^{a_n}, -H_p (t, \psi^{a_n} (\cdot), \nabla_x u^{a_n} (t, \psi^{a_n} (\cdot)), \mu^{a_n} (t))) \# \tilde{m}^{a_n} = (I_d, -H_p (t, \cdot, \nabla_x u^{a_n} (t, \cdot), \mu^{a_n} (t))) \# m^{a_n}.$$
In particular, \( \alpha \tilde{\mu}^{a_n}(t) = \alpha \mu^{a_n}(t) \circ \psi^{a_n} \) so \( \| \alpha \tilde{\mu}^{a_n}(t) \|_{L^\infty(m)} \) is not larger than \( \| \alpha \mu^{a_n}(t) \|_{L^\infty(m)} \) since the support of \( m^{a_n} \) is contained in the image of the support of \( \tilde{m}^{a_n} \) by \( \psi^{a_n} \). We proved in the previous step that \((m^a(t))_{a \geq 1}\) is compact in \( (P(\mathbb{R}^d), W_1) \), and so is \((\mu^{a_n}(t))_{n \geq 1}\) in \( (P(\mathbb{R}^d \times \mathbb{R}^d), W_1) \), since they are the pushforward measures of \((m^{a_n}(t))_{n \geq 1}\) by \((I_d, \alpha \mu^{a_n}(t))\). Let \( \mu(t) \in P(\mathbb{R}^d \times \mathbb{R}^d) \) be the limit of a convergent subsequence of \((\mu^{a_n}(t))_{n \geq 1}\). Passing to the limit in the weak* topology in the latter chain of equalities implies that

\[
\mu(t) = (I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu(t))) \# m(t).
\]

Moreover, the uniqueness of the fixed point \(2.4\) holds here, see [10] Lemma 5.2 for the proof. We obtained that there exists a unique fixed point satisfying \(2.4\), and that it is the limit of any convergent subsequence of \((\mu^{a_n}(t))\). This implies that the whole sequence \((\mu^{a_n}(t))_{n \geq 1}\) tends to \(\mu(t)\) in \((P(\mathbb{R}^d \times \mathbb{R}^d), W_1)\).

Let us point out that \( m^{a_n} \) satisfies

\[
\frac{\partial}{\partial t} m^{a_n} - \nu \Delta m^{a_n} - \text{div} \left( H_p(t, x, \nabla_x u^{a_n}, \mu^{a_n}) m^{a_n} \right) = 0
\]

in the sense of distributions on \((0, T) \times B \left( 0, \frac{a_n}{4} \right) \), by the definitions of \( \psi^a \) and \( \tilde{\psi} \). Furthermore, at time \( t = 0 \) we know that \( m^{a_n}(0) = (\psi^a \circ \pi^{a_n}) \# m^0 \). We recall that \( \psi^a \circ \pi^{a_n}(x) = x \) for \( x \in B_{\mathbb{R}^d} \left( 0, \frac{a_n}{4} \right) \). This implies that \( m^{a_n}(0) \) tends to \( m^0 \) in the weak* topology of \( P(\mathbb{R}^d) \).

Finally we obtain that \((u, m, \mu)\) is a solution to \((2.2)–(2.6)\), by passing to the limit as \( n \) tends to infinity in the equations satisfied by \((u^{a_n}, m^{a_n}, \mu^{a_n})\).

\[\square\]

**Remark 6.1.** In the above proof, we obtain that there exists a unique fixed point satisfying \(2.4\). We have thereby extended the conclusions of Lemma 4.3 to system \((2.2)–(2.6)\). Similarly, one may extend the conclusions of Lemma 2.4 to system \((2.2)–(2.6)\).

### 6.3. Proof of Theorem 2.8: uniqueness of the solutions to \((2.2)–(2.6)\) and \((2.8)–(2.12)\)

**Step III.b**, namely the uniqueness of the solution to \((2.2)–(2.6)\), is obtained from the monotonicity assumptions \(A3\) and \(U\), and the same arguments as in the case of MFG without interaction through controls.

**Proof of Theorem 2.8.** Here, we write the proof for the system \((2.2)–(2.6)\). However, none of the arguments below is specific to the domain \(\mathbb{R}^d\), therefore this proof can be repeated for \((2.8)–(2.12)\).

We suppose that \((u^1, m^1, \mu^1)\) and \((u^2, m^2, \mu^2)\) are two solutions to \((2.2)–(2.6)\). Throughout the proof, we will omit the dependence on \( t \) and \( x \) in the functions \( H, L, f, g, u^i, m^i \) and \( \mu^i \) for \( i = 1, 2 \), to simplify notations.

On the one hand, we subtract the HJB equation satisfied by \( u^2 \) and the one satisfied by \( u^1 \), we multiply the resulting quantity by \((m^1 - m^2)\); on the other hand, we subtract the FPK equation satisfied by \( m^2 \) and the one satisfied by \( m^1 \), we multiply the resulting quantity by \((u^2 - u^1)\). Then, we add
the latter two quantities and integrate over \([0, T] \times \mathbb{R}^d\), after performing some integration by parts (see \cite{35} for more details), we obtain,

\[
0 = \int_0^T \int_{\mathbb{R}^d} \left[ \nabla_x (u^1 - u^2) \cdot H_p \left( \nabla_x u^1, \mu^1 \right) - H \left( \nabla_x u^1, \mu^1 \right) + H \left( \nabla_x u^2, \mu^2 \right) \right] \, dm^1 \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left[ \nabla_x (u^2 - u^1) \cdot H_p \left( \nabla_x u^2, \mu^2 \right) - H \left( \nabla_x u^2, \mu^2 \right) + H \left( \nabla_x u^1, \mu^1 \right) \right] \, dm^2 \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left( f(m^1) - f(m^2) \right) \, dm^1 \, dt
\]

\[
+ \int_{\mathbb{R}^d} \left( g(m^1(T)) - g(m^2(T)) \right) \, dm^1 \, dt
\]

(6.4)

Recall that

\[
L \left( \alpha^{i^1}, \mu^1 \right) = \nabla_x u^i \cdot H_p \left( \nabla_x u^i, \mu^i \right) - H \left( \nabla_x u^i, \mu^i \right),
\]

\[
\nabla_x u^i = - L_{\alpha} \left( \alpha^{i^1}, \mu^1 \right),
\]

(6.5)

because \( L \) is the Legendre transform of \( H \). From \( U \), (6.4) and (6.5), we obtain that,

\[
0 \geq \int_0^T \int_{\mathbb{R}^d} \left[ L \left( \alpha^{i^1}, \mu^1 \right) - L \left( \alpha^{i^2}, \mu^2 \right) - \left( \alpha^{i^1} - \alpha^{i^2} \right) \cdot L_{\alpha} \left( \alpha^{i^2}, \mu^2 \right) \right] \, dm^1 \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left[ L \left( \alpha^{i^2}, \mu^2 \right) - L \left( \alpha^{i^1}, \mu^1 \right) - \left( \alpha^{i^2} - \alpha^{i^1} \right) \cdot L_{\alpha} \left( \alpha^{i^1}, \mu^1 \right) \right] \, dm^2 \, dt
\]

(6.6)

The function \( L \) is strictly convex in \( \alpha \) by Lemma 2.10, which implies that,

\[
L \left( \alpha^{i^1}, \mu^2 \right) - L \left( \alpha^{i^2}, \mu^2 \right) - \left( \alpha^{i^1} - \alpha^{i^2} \right) \cdot L_{\alpha} \left( \alpha^{i^2}, \mu^2 \right) \geq 0,
\]

\[
L \left( \alpha^{i^2}, \mu^1 \right) - L \left( \alpha^{i^1}, \mu^1 \right) - \left( \alpha^{i^2} - \alpha^{i^1} \right) \cdot L_{\alpha} \left( \alpha^{i^1}, \mu^1 \right) \geq 0,
\]

(6.7)

and (6.7) turn to identities if and only if \( \alpha^{i^1} = \alpha^{i^2} \). The latter inequalities and (6.6) yield

\[
0 \geq \int_0^T \int_{\mathbb{R}^d} \left[ L \left( \alpha^{i^1}, \mu^1 \right) - L \left( \alpha^{i^1}, \mu^2 \right) \right] \, dm^1 \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left[ L \left( \alpha^{i^2}, \mu^2 \right) - L \left( \alpha^{i^2}, \mu^1 \right) \right] \, dm^2 \, dt
\]

\[
= \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[ L \left( \alpha, \mu^1 \right) - L \left( \alpha, \mu^2 \right) \right] \, d(\mu^1 - \mu^2) \, (\alpha) \, dt.
\]

Assumption A3 turns the latter inequality into an equality. This, the case of equality in (6.7) and the continuity of \( \alpha^{i^1} \) and \( \alpha^{i^2} \) yield that \( \alpha^{i^1} = \alpha^{i^2} \).

This implies that \( m^1 = m^2 \) by the uniqueness of the solution to (2.3), (2.6). Therefore, we obtain \( \mu^1 = \mu^2 \), and then \( u^1 = u^2 \) by the uniqueness of the solution to (2.2),2.5. \( \square \)
6.4. Theorems 2.2 and 2.3: existence and uniqueness of (1.4)–(1.9)

So far, no distinction has been made between $\mu_b$ and $\mu_\alpha$, because they coincide for (2.2)–(2.6) and (2.8)–(2.12). Now they may differ since the drift function and the control may be different. In this case $\mu_b$ defined by $\mu_b(t) = [(x, \alpha) \mapsto (x, b(t, x, \alpha, \mu_\alpha(t)))] \# \mu_\alpha(t)$ is naturally the joint law of the states and the drifts. The idea here to pass from (2.2)–(2.6) to (1.4)–(1.9), is to assume that $b$ is invertible with respect to $\alpha$, which changes the optimization problem in $\alpha$ into a new optimization problem expressed in term of $b$. This consists in changing the Lagrangian from $L(t, x, \alpha, \mu_\alpha)$ into

$$L^b(t, x, b, \mu_b) = L \left( t, x, \alpha^* (t, x, b, \mu_b), \left[ \left( x, \tilde{b} \right) \mapsto \left( x, \alpha^* \left( t, x, \tilde{b}, \mu_b \right) \right) \right] \# \mu_b \right).$$

The Hamiltonian $H^b$ defined as the Legendre transform of $L^b$ is given by

$$H^b(t, x, p, \mu_b) = H \left( t, x, p, \left[ \left( x, \tilde{b} \right) \mapsto \left( x, \alpha^* \left( t, x, \tilde{b}, \mu_b \right) \right) \right] \# \mu_b \right). \quad (6.8)$$

Conversely, we can obtain $L$ and $H$ from $L^b$ and $H^b$ with the following relations,

$$L(t, x, \alpha, \mu_\alpha) = L^b \left( t, x, b(t, x, \alpha, \mu_\alpha), \left[ \left( x, \alpha \right) \mapsto \left( x, b(t, x, \alpha, \mu_\alpha) \right) \right] \# \mu_\alpha \right),$$

$$H(t, x, p, \mu_\alpha) = H^b \left( t, x, p, \left[ \left( x, \alpha \right) \mapsto \left( x, b(t, x, \alpha, \mu_\alpha) \right) \right] \# \mu_\alpha \right).$$

Now we can state the following lemma which allows us to pass from (2.2)–(2.6) to (1.4)–(1.9), or vice versa.

**Lemma 6.2.** Under assumption B1, $(u, m, \mu_\alpha, \mu_b)$ is a solution to (1.4)–(1.9) if and only if $(u, m, \mu_b)$ is a solution to (2.2)–(2.6) with $H^b$ instead of $H$.

The proof is straightforward and only consists in checking on the one hand, that (1.4) and (1.5) are respectively equivalent to (2.2) and (2.3) with $H^b$ instead of $H$; on the other hand, that (1.6) and (1.7) are equivalent to (2.4) with $H^b$, where we take $\mu_b = \mu$ and $\mu_\alpha$ defined by (1.6).

The following existence theorem is a direct consequence of Lemma 6.2, and Theorem 2.7.

**Corollary 6.3.** If $L^b$ satisfies A1–A6, and $b$ satisfies B1, there exists a solution to (1.4)–(1.9).

Theorem 2.3, i.e. the existence part of step IV, is a consequence of the latter existence result in which the assumptions on $L^b$ are stated on $L$ instead, which makes them more tractable. However, we have to make the additional assumption B2.

If $L$ and $b$ satisfy the assumptions of Theorem 2.3, it is straightforward to check that $L^b$ satisfies A1–A5. Therefore, Theorem 2.3 is a consequence of Corollary 6.3. Finally, Theorem 2.2 and the uniqueness part of step IV are direct consequences of Theorem 2.8 and Lemma 6.2.

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Received: 16 July 2021.
Accepted: 13 May 2022.