Compactification of Extensive Forms and Belief in the Opponents’ Future Rationality

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Abstract

We introduce an operation, called compactification, to reduce an extensive form to a compact one where each decision node in the game tree can be assigned to more than one player. Motivated by Thompson [17]’s interchange of decision nodes, we attempt to capture the notion of a faithful representation of the chronological order of the moves in a dynamic game which plays a vital role in fields like epistemic game theory. The compactification process preserves perfect recall and the unambiguity of the order among information sets. We specify an algorithm, called leaves-to-root process, which compactifies at least as many information sets as any other compactification process. The compact extensive form provides an approach to avoid problems in dynamic game theory due to the vague definition of the chronological order of the moves, for example, belief in the opponents’ future rationality (Perea [12])’s sensitivity to the specific extensive form representation. We show that any strategy which can rationally be chosen under common belief in future rationality in a minimal compact game if and only if it satisfies this property in every extensive form game which is related to it via some compactification process.

Keywords: extensive forms, epistemic game theory, compactification, interchange of decision nodes, belief in the opponents’ future rationality

1. Introduction

The extensive form has been acclaimed from the birth of game theory as a good model of the salient features of dynamic interactive situations. Using the words of Kreps [7] (p.13), it captures “the timing of actions that players may take and the information they will have when they must take those actions.” In many cases, there are several extensive forms to represent the same chronological order of the moves. That is, even if players actually make decisions simultaneously at some stage, there are several orders to describe their moves in extensive forms. Though the orders are arbitrary, two different extensive forms representing different orders can be transformed into each other via Thompson [17]’s interchange of decision nodes. Hence some authors, for example, Thompson [17] and Elmes and Reny [6], claim that the difference among multiple representations of a chronological order is inessential.

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Yet it causes severe problems sometimes, for example, in dynamic epistemic game theory. Epistemic game theory studies how a player may reason about other players before he starts to play (see Perea [11], Dekel and Siniscalchi [3]). A primary criterion of a player’s reasoning in static epistemic games is that he believes in the opponents’ rationality (Tan and Werlang [16]). It is more complicated in dynamic games since it is not always possible for a player to believe that his opponents had behaved rationally in the past (e.g., Binmore [4], Reny [13], [14]); further, his interpretation of the opponents’ irrational behavior determines his beliefs about their future choices (e.g., Battigalli [2], Battigalli and Siniscalchi [3], Perea [12]). Therefore, the order of information sets matters since it specifies the meaning of past and future in a player’s reasoning. An example is the notion of belief in the opponents’ future rationality (Perea [12]) which describes that a player always believes at every stage of the game that her opponents will choose rationally at the stage and in the future. The strategies optimal to the common belief in the opponents’ future rationality is sensitive to the specific extensive form representation. In Section 6.2 of Perea [12], an example shows that representations with different orders of information sets, albeit representing the same dynamic situation, have different optimal strategies to the common belief in the opponents’ future rationality.

Perea [12] claims that “if we insist that the order of the information sets in the dynamic games faithfully represents the actual chronological order of the moves, then there is no problem in using common belief in future rationality as a concept.” Yet the meaning of a faithful representation is not specified there. Simultaneous moves are allowed and play an important role in Perea [12]’s definition of extensive form, in which sense the formulation can be regarded as a “compactification” of the standard one. However, there is no practical method to formulate the compactification process.

Perea [12] also suggests that two extensive forms which can be transformed into each other through Thompson [17]’s interchange of decision nodes can be regarded as representing the same actual chronological order of the moves. Yet if we consider the “compactified” extensive forms, it is not clear whether and how two “equivalent” (standard) extensive forms are related to a compactified form. Neither there is an effective way to connect the distinct sets of strategies selected in “equivalent” extensive forms. Worse, interchange of decision nodes sometimes destroys the unambiguity of the order among information sets (an example is given in Perea [10], p.141) and makes it impossible to select strategies optimal to any belief criterion which relies on the well-defined notions of “past” and “future”.

In this paper, we attempt to solve those problems. We first give a formal definition of the compact extensive form, where each decision node in the game tree can be assigned to more than one player. We then introduce an operation, called compactification, where an information set containing only the immediate successors of a node is “absorbed” into the node. It is motivated by Thompson [17]’s interchange of decision nodes. Yet our compactification operation can be conducted only if the immediate successors form an information set, while interchange of decision nodes allows that the absorbed information set contains other nodes. This difference matters essentially. In Section 5 we will show that the latter causes difficulties in our context.

By the compact extensive form and compactification operation, we attempt to capture the process of constructing a faithful representation of the chronological order of the moves in dynamic games. We show that compactification process preserves perfect recall and the unambiguity of the order among information sets. A problem, however, is that the compactification process is not order-independence. It is caused by the absorption of some information set $h$ may hinder
other information sets which is constituted by immediate successors of some node in \( h \). We specify an algorithm, called leaves-to-root process, which compactifies every information set that can be absorbed.

The compact extensive form provides an approach to alleviate the sensitivity of belief in the opponents’ future rationality (Perea \[12\]) to the specific extensive form representation. We show that any strategy which can rationally be chosen under common belief in future rationality in a minimal compact extensive form game, i.e., a game whose extensive form cannot be compactified any more, if and only if the strategy satisfies this property in every extensive form game which can be related to it via some compactification process.

The rest of the paper is organized as follows. Section 2 defines the compact extensive form. Section 3 specifies the compactification operation. We also give some results on the properties of the compactification process. Section 4 shows the main theorem about the minimal compact extensive form games and the strategies optimal to common belief in the opponents’ future rationality. Section 5 provides an alternative operation which is faithful to Thompson \[17\]’s interchange of decision nodes and compare it with our compactification. Section 6 contains all the proofs.

2. Compact Extensive Forms

A compact extensive form is a tuple \( \Gamma = (T, \geq; I, \tau; \{A\}_{i \in I}, \alpha; \{H_i\}_{i \in I}) \), where

1. \( (T, \geq) \) is a finite tree, that is, \( T \) is a non-empty finite set of nodes and \( \geq \) is a partial order on \( T \) satisfies (i) there is a smallest element \( t_0 \) (called the root or the initial node) in \( T \) with respect to \( \geq \), and (ii) for each \( t \in T, \geq \) is a complete order on the predecessors of \( t \).

We follow the auxiliary notations defined in Kreps and Wilson \[8\]. For each node \( x \), we denote by \( (T_x, \geq) \) (sometimes only \( T_x \) for simplicity when no confusion is caused) the subtree of \( (T, \geq) \) with \( x \) as its root. We use \( Z \) to denote the set of terminal nodes in \( T \) and \( X \) the set of non-terminal nodes (called the decision nodes), i.e., \( X = T \setminus Z \). For each \( x \in X \), we denote by \( S(x) \) the set of immediate successors of \( t \), \( P(x) \) the set of predecessors of \( x \), and \( p_n(x) \) the \( n \)-th predecessor of \( x \) (we stipulate that \( p_0(x) = x \)).

2. \( I \) is the finite set of players. The function \( i : X \to 2^I \setminus \{\emptyset\} \) assigns to each decision node a set of players who have to make decision there. We say that a player \( i \) is active at a decision node \( x \in X \) iff \( i \in i(x) \). For each \( i \in I \), we define \( X_i = \{ x \in X : i \in i(x) \} \).

3. For each \( i \in I \), \( A_i \) is a non-empty set of actions of \( i \). We define \( \overline{A}_i = A_i \cup \{\epsilon\} \), where \( \epsilon \) is a symbol not belonging to any \( A_i \). By \( \epsilon \) we mean “no actions”. The function \( \alpha : T \setminus \{t_0\} \to \prod_{i \in I} \overline{A}_i \) assigning to each non-initial node the last profile of actions taken to reach it. It satisfies the following conditions:

   (3.1) \( \alpha_i(t) \in \overline{A}_i \) (i.e., \( \alpha_i(t) \neq \epsilon \)) if and only if \( i \in i(p_1(t)) \). Here \( \alpha_i \) is the projection of \( \alpha \) on the \( i \)-th dimension.

   (3.2) For each \( x \in X \) and \( y, z \in S(x) \), \( \alpha(y) \neq \alpha(z) \). Combined with (3.1), it means that for some \( i \in i(x) \), \( \alpha_i(y) \neq \alpha_i(z) \).

   (3.3) For each \( x \in X \) with \( i \in i(x) \) and each \( y, z \in S(x) \), \( \|y\|_{\alpha_i} = \|z\|_{\alpha_i} \), where \( \|y\|_{\alpha_i} := \{ y' \in S(x) : \alpha_i(y') = \alpha_i(y) \} \). Combined with (3.1) and (3.2), it implies that for each \( x \in X \), \( S(x) = \prod_{i \in i(x)} \alpha_i(S(x)) \).

4. For each \( i \in I \), \( H_i \) is a partition of \( X_i \). For each \( x \in X_i \), we use \( H_i(x) \) to denote equivalent class in \( H_i \) containing \( x \). We require that \( H_i(x) = H_i(y) \) implies \( \alpha_i(S(x)) = \alpha_i(S(y)) \). In the following, we denote by \( \alpha_i(h_i) \) the set of available actions of player \( i \) at information set \( h_i \).
Extensive forms which allow simultaneous moves appear and play an essential role in Perea [11]. Here we give it a formal formulation with some small modification. We call the model the compact extensive form since it seems like that we eliminate redundant nodes and consolidate simultaneous moves together in a standard extensive form (Selten [15], Kreps and Wilson[8]). To accommodate to the multiple-players-one-node situation, the action assignment function \( \alpha \) assigns to each non-initial node (or, equivalently, an edge) a profile of actions where for each player active at the immediate predecessor node there is a concrete action, and for those inactive a dummy symbol \( \epsilon \) meaning that he does not need to act. Instead of one edge representing one action for one player and different edges represent distinct actions of her in the standard extensive game, here different edges may represent the same action(s) for some player(s). In (3.2) we require that two edges radiating from one decision node should represent distinct profiles, i.e., at least one player active at that decision node should have different actions. Condition (3.3) describes that the edges radiating from one decision node represent the different combinations of independent actions of players active at that node. Finally, since there might be multiple players active at each node, information sets could overlap.

It is easy to see that the standard extensive form is a special case of the compact one, i.e., \( |\alpha(x)| = 1 \) for every \( x \in X \). In this sense, the notion of compact extensive form is a generalization of the standard one.

Like the extensive form game, a compact extensive form game is obtained by specifying each player \( i \)'s von Neumann-Morgenstern utility function \( u_i : Z \to \mathbb{R} \) for terminal nodes.

In Figure 1, we show how a compact extensive form (game) looks like and compare it with the standard one.

The graph in the left-hand side is adopted from Perea [12]'s Figure 1. The situation is, as Perea [12] put it, “at the beginning of the game, \( \emptyset \), player 1 chooses between \( a \) and \( b \), and player 2 simultaneously chooses between \( c \) and \( d \). So, \( \emptyset \) is an information set that belongs to both players 1 and 2. If player 1 chooses \( b \), the game ends, and the utilities are as depicted. If he chooses \( a \), then the game moves to information set \( h_{21} \) or information set \( h_{22} \), depending on whether player 2 has chosen \( c \) or \( d \). Player 1, however, does not know whether player 2 has chosen \( c \) or \( d \), so player 1
faces information set \( h_1 \) after choosing \( a \). Hence \( h_{21} \) and \( h_{22} \) are information sets that belong only to player 2, whereas \( h_1 \) is an information set that belongs only to player 1.

The graph in the middle is a representation of the situation by a (standard) extensive form game. There, since simultaneous moves are not allowed, information set \( \emptyset \) is decomposed into \( \emptyset_1 \) and \( \emptyset_2 \) which correspond to player 1 and 2’s information set at the beginning of the game (to differentiate players, we use continuous-line circles to represent player 1’s information sets and dashed-line circles to represent player 2’s information sets). Also, \( h_1 \) and \( h_{21}, h_{22} \) are displayed separately.

The graph in the right-hand side represents the situation by a compact extensive form game. Since simultaneous moves are allowed, each decision node represents a move for both players, and informations sets overlap. Indeed, \( \emptyset_1 \) and \( \emptyset_2 \) coincide with each other, and \( h_1 = h_{21} \cup h_{22} \). Note that each edge (i.e., each non-initial node) represents a vector of the active players’ choices. It can be seen that this form is indeed more compact than the one in the middle, with the terminal nodes unchanged.

Perfect recall can be defined similarly for compact extensive form games. Formally, each player in \( I \) has perfect recall iff the following condition is satisfied. For each \( x,y,y' \in X \) with \( i \in i(x) \cap i(y) \cap i(y') \), if \( x < y \) and \( H_i(y) = H_i(y') \), then there is \( x' \in X \) such that \( P(x') \cap H_i(x) = \{x'\} \), and if \( x = p_n(y) \) and \( x' = p_m(y') \), \( \alpha_i(p_{n-1}(y)) = \alpha_i(p_{m-1}(y')) \).

3. Compactifying Extensive Forms

We have defined the compact extensive form. In this section, we specify how to “compactify” such a form. Also, we show some important features of the compactification process.

3.1. The compactification operation

Consider two compact extensive forms \( \Gamma = \langle T, \geq ; I, i; \{A\}_{i \in I}, \alpha; \{H_i\}_{i \in I} \rangle \) and \( \Gamma' = \langle T', \geq' ; I, i'; \{A'\}_{i \in I}, \alpha'; \{H'_i\}_{i \in I} \rangle \). For simplicity, we assume that for each \( x \in X \), if \( i \in i(x) \), then \( S(x) \notin H_i \). We say that \( \Gamma' \) is a compactification of \( \Gamma \), denoted by \( \Gamma \rightarrow_{\text{COMP}} \Gamma' \), iff there exist \( x,y_1,...,y_k,z_{11},...,z_{1\ell_1},...,z_{k\ell_k} \in Z \) satisfying

1. \( S(x) = \{y_1,...,y_k\} \) and \( S(y_r) = \{z_{r1},...,z_{r\ell_r}\} \) for each \( r = 1,...,k \).
2. There is some \( i \in I \) such that \( i \in i(y_r) \) for each \( r = 1,...,k \), and \( H_i(y_1) = ... = H_i(y_k) = \{y_1,...,y_k\} \). We let \( b = |\alpha_i(S(y_1))| = |\alpha_i(S(y_r))| \) for each \( r = 1,...,k \).
3. Without loss of generality, we assume that for each \( r = 1,...,k \), \( \alpha_i(z_{r1}) = ... = \alpha_i(z_{r\ell_r}/b) = \alpha_i(z_{11}/b), \alpha_i(z_{r(\ell_r)/b}+1) = ... = \alpha_i(z_{r2\ell_r}/b) = \alpha_i(z_{12}/b),..., \alpha_i(z_{r,\ell_r-(\ell_r/b)+1}) = ... = \alpha_i(z_{1,\ell_1}/b) = \alpha_i(2\ell_1) \).
4. \( T' \) is obtained from \( T \) by replacing \( T_x \) with the following subtree (i.e., \( T' - T'_x = T - T_x \)).
5. (3.1) We preserve \( x \), and let \( S'(x) = \{t_{11},...,t_{1b}; t_{21},...,t_{2b};...,t_{k1},...,t_{kb}\} \) where these \( t_{rs} (r = 1,...,k, s = 1,...,b) \) are symbols which do not belong to \( T \).
6. (3.2) For each \( r = 1,...,k \), if \( |\nu(y_r)| = 1 \) (i.e., \( \nu(y_r) = \{i\} \)), then \( \ell_r = b \). In this case, we regard \( t_{r1},...,t_{rb} \) as the same with \( z_{r1},...,z_{r\ell_r} \), and each subtree \( T_{z_{rs}} \) can be directly “grafted” to \( t_{rs} (s = 1,...,b) \). If \( |\nu(y_r)| > 1 \), then \( \ell_r > b \) and \( b \) is a divisor of \( \ell_r \). In this case, we let \( S(t_{r1}) = \{z_{r1},...,z_{r\ell_r/b}\} \), \( S(t_{r2}) = \{z_{r,\ell_r-(\ell_r/b)+1},...,z_{r,2\ell_r/b}\} \), ... \( S(t_{rb}) = \{z_{r,\ell_r-(\ell_r/b)+1},...,z_{r,\ell_r}\} \), and the subtrees \( T_{z_{rn}} \) are all preserved (\( n = 1,...,\ell_r \)).

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1. This condition is satisfied when each player has perfect recall (and perhaps with some consolidation of redundant actions like coalescing of information sets in Thompson [17]).
4. For the player assignment function \( f', f'(x) = j(x) \cup \{i\} \). For each \( t_{rs} \) \((r = 1, \ldots, k, s = 1, \ldots, b)\), if \(|j(y_r)| = 1\), then since we have stipulated that \( t_{rs} = z_{rs} \), it follows that \( f'(t_{rs}) = j(z_{rs}) \); if \(|j(y_r)| > 1\), then \( f'(t_{rs}) = j(y_r) \cup \{i\} \).

5. For the set of actions \( \{A'_i\}_{i \in I} \) and action assignment function \( \beta' \), since we have assumed that for each \( x \in X \), if \( i \in j(x) \), then there is no \( h_i \in H_i \) such that \( S(y) \subseteq h_i \), it follows that \( i \notin j(x) \), i.e., for each \( y_r \) \((r = 1, \ldots, k)\), \( \beta_i(y_r) = \epsilon \). Let \( A'_j = A_j \) for each \( j \in I \). For each \( r = 1, \ldots, k \) and \( s = 1, \ldots, b \), let \( \alpha'_i(t_{rs}) = \alpha_i(z_{1,s}, \ldots, b) \) and \( \alpha'_j(t_{rs}) = \alpha_j(y_r) \) for each \( j \neq i \). Let \( \alpha'_j(t) = \alpha_j(t) \) for all other nodes \( t \) and all \( j \in I \).

6. For the information sets, let \( H'_i = (H_i \setminus H_i(y_1)) \cup \{x\} \), and in the following sometimes we say that \( H_i(y_1) \) is absorbed by/into \( x \). For \( j \neq i \), let

\[
H'_j = (H_j \setminus \bigcup_{r=1}^k H_j(y_r)) \cup \bigcup_{r=1}^k \left( H_j(y_r) \setminus \{y_1, \ldots, y_k\} \right) \cup \bigcup_{s:y_r \in H_j(y_r)} \{t_{s1}, \ldots, t_{sb}\}
\]

(1)

In the above formula, we manipulate that \( H_j(y_r) = \emptyset \) if \( j \neq i \) for each \( r = 1, \ldots, k \).

One property of the compactification operation is that it has no influence on the “grandsons” of \( x \), i.e., the nodes \( z \) with \( p_2(z) = x \) (with respect to isomorphism). Hence it implies that a compactification operation does not alter the terminal nodes and, consequently, any utility functions.

Here we use an example to illustrate the compactification operation.

| Figure 2: Compactifying an extensive form |

Consider the (standard) extensive form on the left-hand side of Figure 2. There are three players. Player 1 moves at first, choosing \( a \) or \( b \). Player 2 moves without knowing player 1’s choice, and he can choose either \( c \) or \( d \). It turns out that if player 1 chooses \( b \), then the game terminates after player 2 makes a choice, while if player 1 chooses \( a \), then after player 2’s move it is player 3’s turn. Player 3 knows what player 1 has chosen yet does not know player 2’s choice. The game terminates after player 3 makes up his choice.

Here, player 3’s two decision nodes in \( h_3 \) can be absorbed by their immediate predecessor, i.e., the node on the left-hand side in player 2’s information set \( h_2 \). By doing this, the number of edges
radiating from that node increases from two to four since now each edge represents a pair of player 2 and 3’s actions. Now we obtain the compact extensive form in the middle.

Further, we can combine player 2’s information set $h_2$ with player 1’s $h_1$. Now four edges radiate from the root which is occupied by both players 1 and 2. Note that $h_3$ cannot be absorbed by the root since $h_3$ does not contain all immediate predecessors of the root node. In the reduced form on the right-hand side, $h_3$ now contains two nodes which represent player 1’s action $a$ and player 2’s two actions. This faithfully describes player 3’s knowledge at $h_3$: he knows that player 1’s action is $a$, yet he has no idea on player 2’s choice.

The following proposition states that perfect recall is preserved in the compactification operation.

**Proposition 1. (Preservation of perfect recall)** Consider two compact extensive forms $\Gamma$ and $\Gamma'$ with $\Gamma \rightarrow_{\text{COM}} \Gamma'$. If every player has perfect recall in $\Gamma$, then so does everyone in $\Gamma'$.

Another problem concerns epistemic game theorists is the unambiguity of the order among information sets. Consider a compact extensive form $\Gamma = \langle T, \succeq; I, i; \{A\}_{i \in I}, \alpha; \{H_i\}_{i \in I}\rangle$ and two information sets $h$ and $h'$ (they may belong to different players). We say that $h$ is followed by $h'$ (or $h'$ follows $h$), denoted by $h \succ h'$, iff there is $x \in h$ and $y \in h'$ such that $x$ is on the unique path from the root to $y$. We say that $h$ and $h'$ are simultaneous, denoted by $h \sim h'$, iff $h \cap h' \neq \emptyset$. An information set $h$ is weakly followed by $h'$ (or $h'$ weakly follows $h$), denoted by $h \succeq h'$, iff $h$ is followed by $h'$ or they are simultaneous. We say the order among the information sets is unambiguous iff for every information sets $h, h'$, if $h \preceq h'$, then it does not hold that $h \succ h'$.

The unambiguity of the order among information sets is vital in dynamic epistemic game theory. The algorithm which screens out strategies that is optimal to common belief in the opponents’ future rationality, called backward dominance procedure, works if and only if the information sets are ordered unambiguously. Hence it is relevant to consider whether the compactification process preserves the unambiguity of the order among information sets. The following statement gives a positive answer.

**Proposition 2. (Preservation of unambiguity of the order among information sets)** Consider two compact extensive forms $\Gamma$ and $\Gamma'$ with $\Gamma \rightarrow_{\text{COM}} \Gamma'$. If $\Gamma$ has an unambiguous order among information sets, then so does $\Gamma'$.

### 3.2. The compactification process and its properties

Figure 2 actually shows a compactification process. Indeed, the operation can be applied repeatedly, and we can obtain a compactification sequence. Formally, let $\Gamma$ be a compact extensive form. A compactification process from $\Gamma$ is a sequence $\langle \Gamma_0, \Gamma_1, ..., \Gamma_m \rangle$ where $\Gamma_0 = \Gamma$, for each $r = 0, ..., m - 1$, $\Gamma_r \rightarrow_{\text{COM}} \Gamma_{r+1}$, and $\Gamma_m$ can not be compactified anymore. A compact extensive form is said to be minimal iff it cannot be compactified any more.

The problem of the compactification process is that it is not order-independent. In Figure 3 we give an example. Consider the extensive form on the left-hand side. The symbols $A, ..., N$ represent the terminal nodes. We give two compactification processes from it. In the process on the top, first the four nodes in $h'_1$ are absorbed by the node on the left-hand side of $h_2$ (as well as $h_3$). Then the three nodes in $h_2$ are absorbed by the root. In the process on the button, in contrast, the absorption of $h_2$ is conducted first. However, this makes the compactification process unable to continue, since now no node has its immediate successors in one information set. It is clear that the two processes have distinct terminal terms.
Though order-independence does not hold in general, by peering closely at Figure 3, one may notice that the process on the top makes the original extensive form more compact, i.e., one more information set is absorbed. Here we define a compactification process which absorbs at least as many informations sets as any other process. We start from the nodes next to the terminal nodes, i.e., \( \{ y \in X : S(y) \subseteq Z \} \) and check whether some of them can be absorbed by their immdiate predecessor, if so, then conduct the compactification operation on them. When all compactification is done for those nodes, we go to the nodes next to them, etc. We call it a leaves-to-root (LTR) compactification process. We have the following statement.

**Proposition 3. (LTR process absorbs more information sets)** Consider the LTR compactification process and an arbitrary compactification processes from a compact extensive form \( \Gamma \). If an information set in \( \Gamma \) is absorbed in the latter, then so it is in the former.

For each compact extensive form \( \Gamma \), we use \( COM(\Gamma) \) to denote the set of all compact extensive forms (not necessarily minimal) which can be reached through a compactification process from \( \Gamma \). Let \( \Gamma^* \) be a minimal compact extensive form. We define \( [\Gamma^*] = \bigcup \{ COM(\Gamma) : \Gamma^* \text{ is the terminal term of the LTR compactification process from } \Gamma \} \). Each \( \Gamma' \in [\Gamma^*] \) can be said as related to \( \Gamma^* \) via some compactification process; either it is a compact extensive form which has \( \Gamma^* \) as the final term of the LTR process, like the one on the left-hand side of Figure 3, or it can be reached via a compactification process from another compact extensive form which reaches \( \Gamma^* \) through the LTR process (though itself may not be able to reach \( \Gamma^* \)), like the terminal term on the bottom process in Figure 3.

As we mentioned in Section 3.1, compactification operations have no influence on the terminal nodes. Hence any utility function applied to one \( \Gamma' \in [\Gamma^*] \) can be applied to any compact extensive form in \( [\Gamma^*] \) without any alteration.

### 4. Compactification and Belief in the Opponents’ Future Rationality

We have developed the compact extensive form as a faithful representation of the actual chronological order of the moves in Perea [12], a notion essential for the application of backward dominance...
procedure which screens out strategies optimal to types with common belief in future rationality. This section discusses this relationship between compactification process and belief in the opponents’ future rationality in detail.

Consider a compact extensive form game \( G = (\Gamma, \{u_i\}_{i \in I}) \), where \( \Gamma = \{ T_i \geq; I, j; \{A\}_{i \in I}, \beta; \{H_i\}_{i \in I} \) is a compact extensive form and for each \( i \in I \), \( u_i \) is player \( i \)'s von Neumann-Morgenstern utility function for terminal nodes. An information set \( h_i \in H_i \) is called initial for player \( i \) iff for any \( x \in h_i \) and any \( y < x \), there is no \( h'_i \in H_i \) such that \( y \in h'_i \). A strategy for player \( i \) is a function \( s_i : \hat{H}_i \rightarrow \bigcup_{h_i \in \hat{H}_i} \beta_i(h_i) \) which assigns each \( h_i \in \hat{H}_i \subseteq H_i \) a choice \( s_i(h_i) \in \beta_i(h_i) \) satisfying

1. Every initial information set \( i \) belongs to \( \hat{H}_i \), and
2. \( \hat{H}_i = H_i(s_i) \), where \( H_i(s_i) \) is the information sets that strategy \( s_i \) allows for.

Consider an information set \( h \). We adopt the following symbols from Perea [12]:

- \( S(h) := \{ (s_i)_{i \in I} \in \prod_{i \in I} S_i : (s_i)_{i \in I} \) reaches some node in \( h \} \)
- \( S_i(h) := \{ s_i \in S_i : (s_i, s_{-i}) \in S(h) \) for some \( s_{-i} \in S_{-i} \}
- \( S_{-i}(h) := \{ s_{-i} \in S_{-i} : (s_i, s_{-i}) \in S(h) \) for some \( s_i \in S_i \} \)

Consider a compact extensive form game \( G = (\Gamma, \{u_i\}_{i \in I}) \). An epistemic model for \( G \) is a tuple \( M = (T_i, b_i)_{i \in I} \) where for each \( i \in I \),

1. \( T_i \) is a finite set of types for player \( i \),
2. \( b_i \) is a function that assigns to every type \( t_i \in T_i \), and every information set \( h_i \in H_i \), a probability distribution \( b_i(t_i, h_i) \in \Delta(S_{-i}(h_i) \times T_{-i}) \).

Consider a type \( t_i \), a strategy \( s_i \), and an information set \( h_i \in H_i(s_i) \). By \( u_i(s_i, t_i|h_i) \) we denote the expected utility from choosing \( s_i \) under the conditional belief that \( t_i \) holds at \( h_i \). Strategy \( s_i \) is optimal for type \( t_i \) iff \( u_i(s_i, t_i|h_i) \geq u_i(s'_i, t_i|h_i) \) for all \( s'_i \in S_i(h_i) \). Strategy \( s_i \) is rational for type \( t_i \) iff \( s_i \) is optimal for \( t_i \) at every \( h_i \in H_i(s_i) \).

Consider a type \( t_i \), an information set \( h_i \in H_i \), and an opponent \( j \neq i \). Type \( i \) believes at \( h_i \) in \( j \)'s future rationality iff \( b_i(t_i, h_i) \) only assigns positive probability to \( j \)'s strategy-type pairs \( (s_j, t_j) \) where \( s_j \) is optimal for \( t_j \) at every \( h_j \in H_j(s_j) \) that weakly follows \( h_i \). Type \( t_i \) believes in the opponents’ future rationality iff at every \( h_i \in H_i \), type \( t_i \) believes in every opponents’ rationality.

Consider a compact extensive form game \( G \) and an epistemic model \( M = (T_i, b_i)_{i \in I} \) for \( G \).

**Initial step.** Define for every player \( i \) the set of types
\[
T_i^0 = \{ t_i \in T_i : t_i \text{ believes in the opponents’ future rationality} \}
\]

**Inductive step.** Let \( k \geq 2 \), and suppose that \( T_i^{k-1} \) has been defined for all player \( i \). Then, we define
\[
T_i^k = \{ t_i \in T_i^{k-1} : b_i(t_i, h_i)(S_{-i} \times T_{-i}^{k-1}) = 1 \text{ for all } h_i \in H_i \}.
\]
A type \( t_i \) expresses common belief in future rationality iff \( t_i \in T_i^k \) for every \( k \).

A strategy can **rationally be chosen under common belief in future rationality** iff there is some epistemic model \( M = (T_j, b_j)_{j \in I} \) and some type \( t_i \in T_i \) such that \( t_i \) expresses common belief in future rationality and \( s_i \) is rational for \( t_i \).

We have the following statement

**Theorem 4.** (Minimal form and the smallest set of rational strategies) Let \( \Gamma^* \) be a minimal compact extensive form and \( (u_i)_{i \in I} \) a vector of utility functions for each players on the terminal nodes of \( \Gamma^* \). A strategy can **rationally be chosen under common belief in future rationality** in \( (\Gamma^*, (u_i)_{i \in I}) \) if and only if for each \( \Gamma \in [\Gamma^*] \), it can **rationally be chosen under common belief in future rationality** in \( (\Gamma, (u_i)_{i \in I}) \).
5. Concluding Remarks: An Alternative Compactification

In the definition of compactification in Section 3.1, we require that the immediate successors of \( x \) form an information set for some player \( i \), i.e., \( H_i(y_1) = \ldots = H_i(y_k) = \{y_1, \ldots, y_k\} \). It might be wondered why we did not follow Thompson [17]'s interchange of decision nodes faithfully and require only \( H_i(y_1) = \ldots = H_i(y_k) \). Technically, it is possible, and the definition of compactification can be preserved except that, in term 6, we have to define

\[
H'_i = (H_i \setminus H_i(y_1)) \cup \{(H_i(y_1) \setminus \{y_1, \ldots, y_k\}) \cup \{x\}\}.
\]

(2)

We call it a Thompson compactification (T-compactification), denoted by \( \Gamma \rightarrow_{TCOM} \Gamma' \). This definition allows that only some nodes in an information set, instead of the whole set, are absorbed. The advantage of T-compactification is that the process is order-independence. Formally, we have the following statement.

**Proposition 5.** (Order-independence of T-compactification process) Consider a compact extensive form \( \Gamma \) and two T-compactification processes \( \langle \Gamma_0, \Gamma_1, \ldots, \Gamma_m \rangle \) and \( \langle \Gamma'_0, \Gamma'_1, \ldots, \Gamma'_n \rangle \) from \( \Gamma \). Then \( \Gamma_m = \Gamma'_n \).

However, on the other hand, T-compactification has a severe disadvantage: the partial absorption may reverse the order of two information sets, sometimes even destroy the unambiguity of the order. Figure 4 gives an example.

![Figure 4: Change of the unambiguity of the order between information sets](image)

On the left-hand side of Figure 4 is a fragment of some game. The symbols \( A, \ldots, M \) represent the continuation of the game. Only player 1’s information set \( h_1 \) and player 2’s \( h_2 \) are depicted with \( h_2 \) unambiguously follows \( h_1 \), and all other nodes belong to players other than 1 and 2. It can be seen that four nodes in \( h_2 \) can be absorbed by their immediate predecessors respectively, and we obtain the fragment in the middle. Now, \( h_2 \) is simultaneous with \( h_1 \) and follows \( h_1 \). Further, the two nodes on the left-hand side in \( h_2 \) in the middle can be absorbed by their immediate predecessor, and we obtained the fragment on the right-hand side. Here, the unambiguity of the order between \( h_1 \) and \( h_2 \) is destroyed; indeed, \( h_2 \) follows \( h_1 \) and is also followed by \( h_1 \), which obfuscates the meaning.
of “future” for moves in $h_1$ and makes it impossible to apply Perea \cite{Perea2012}'s backward dominance procedure.

The requirement of absorption of the whole information set can avoid this difficulty. Yet it is difficult to deny the validity of combining only some nodes in an information set since the strategic features of the original game seems invariant in this process. Perhaps we can consider some restrictions on the original game which guarantee the preservation of the unambiguity of the order of the information sets, for example, every two nodes in an information set should have experienced the same history in the sense of information sets, i.e., if the path to one node passes an information set $h$, so does the other. More research is expected in this direction.

6. Proofs

6.1. Preservation of perfect recall

In this subsection, we will prove Proposition 1 in Section 3.1. First, it can be seen that a compactification operation can be characterized by a pair $(x, i)$ where $x$ is the node which absorbs an information set constitute by its immediate successors and $i$ is the player who possesses that information set. Hence for each compact form $\Gamma = (T, \geq; I, \psi; \{A\}_{i \in I}, \alpha; \{H_i\}_{i \in I})$, each $x \in X$, and each $i \in I$ with $S(x) \in H_i$, we can define $\psi(\Gamma; (x, i))$ to be the compact extensive form obtained from $\Gamma$ via the compactification operation where player $i$’s information set $S(x) \in H_i$ is absorbed by $x$. Conversely, for each $\Gamma, \Gamma'$ with $\Gamma \rightarrow_{\text{COM}} \Gamma'$, there exist uniquely $x \in X$ and $i \in I$ such that $\Gamma' = \psi(\Gamma; (x, i))$.

Consider a compact extensive form $\Gamma$ which satisfies perfect recall, and $\Gamma' = \psi(\Gamma; (x, i))$. Let $y, z, z' \in X$ with $j \in \psi(y, i) \cap \psi(z, i) \cap \psi(z', i)$, $y < z$, and $H_j(z) = H_j(z')$. It can be seen straightforwardly that condition for perfect recall is satisfied in $\Gamma'$ if one of the following conditions is satisfied: (i) $j \neq i$; (ii) $y, z, z'$ and $x$ are on different subtrees; (iii) $H_i(z) > x$, (iv) $p_n(y) = x$ with $n \geq 2$. The reason is, in each case, the compactification has no essential effect on the configuration of the tree between $y$ and $z, z'$.

Hence we only need to consider the case where $j = i$, $y, z, z'$ and $x$ are on the same sub-tree, and $x = p_1(y)$ or $y < x$.\footnote{Suppose that $x = p_1(y)$. Since the original $\Gamma$ satisfies perfect recall, there is some $y' \in H_i(y)$ leading to $z'$ via the same action as $y$ for $z$. After the compactification, $y'$ and $y$ are both absorbed into $x$, it can be seen that in $\Gamma'$, now $z$ and $z'$ are reached from $x$ (now $H_i(y)$ is replaced by $\{x\}$) through two edges sharing the same action of $i$. For the case of $y < x$, it is easy to see that the existence of $y'$ and the action from it leading to $z'$ is invariant in the compactification. Here we have shown that perfect recall is satisfied in $\Gamma'$.}

6.2. Preservation of unambiguity of the order among information sets

In this subsection, we will show Proposition 2 in Section 3.1. Consider two compact extensive form $\Gamma$ and $\Gamma'$ with $\Gamma' = \psi(\Gamma; (x, i))$. It can be seen that for each $j \in I$, there is a bijection $\varrho_j$ from $H_j$ to $H'_j$. Indeed, using the symbols in the definition of the compactification in Section 3.1, $\varrho_i(H_i(y_1)) = \{x\}$, and for each $j \neq i$ with $j \in \psi(y_r)$ for some $y_r \in S(x)$, $\varrho_j(H_j(y_r)) = (H_j(y_r)\{y_1, \ldots, y_k\}) \cup \bigcup_{s \in y_r \in H_j(y_r)} \{t_{s1}, \ldots, t_{sh}\}$, and for every other information set, since it is preserved in the compactification, $\varrho_i$ maps it to itself.

\[ \varrho_i(H_1(z)) \text{ and } x \text{ is not essential here.} \]
Suppose that the information sets in \( \Gamma \) are ordered unambiguously. To see whether the unambiguity is preserved in \( \Gamma' \), we only need to consider those information sets \( h_j \) with \( \varrho_j(h_j) \neq h_j \). It is easy to see that \( H_i(y_1) \) does not cause any trouble, because in \( \Gamma' \) it is degenerated into a singleton. For every \( j \neq i \) with \( j \in i(y_r) \) for some \( y_r \in S(x) \), since the only difference between \( H_j(y_r) \) and \( \varrho_j(H_j(y_r)) \) is that every \( y_s \in H_j(y_r) \) with \( y_s \in S(x) \) is now replaced by a group of doppelgangers \( t_{s1}, ..., t_{sb} \), the order between \( H_j(y_r) \) and any information set other than \( H_i(y_1) \) in \( \Gamma \) is preserved in \( \Gamma' \) (we only need to replace \( H_j(y_r) \) by \( \varrho_j(H_j(y_r)) \)). Also, it is clear that \( \{x\} \succ \varrho_j(H_j(y_r)) \). Here, we have shown that the order of information sets in \( \Gamma' \) are ordered unambiguously.

Through the proof above, it can be seen that the order of information sets are weakly preserved in the compactification operation. Formally, we have the following statement.

**Corollary 6.** (Weak preservation of the order of information sets) Consider two compact extensive forms \( \Gamma, \Gamma' \) with \( \Gamma \rightarrow \text{COM} \) \( \Gamma' \) and two information sets \( h_i, h'_j \) in \( \Gamma \). If \( h_i \prec h'_j \), then \( \varrho_i(h_i) \preceq \varrho_j(h'_j) \).

6.3. The efficiency of the LTR process

In this subsection we show Proposition 3 in Section 3. Suppose that some information set \( h \) is absorbed in some compactification process but not in the LTR process. The reason can only be that some information set smaller than \( h \) (i.e., nearer to the root) is absorbed before. Yet it is contradictory to the definition of “leaf to root”. Hence LTR process absorbs at least as many information sets as any other compactification process.

Also, it will be shown in Section 6.5 that, even for those processes with which LTR shares the same terminal term, LTR is at least as fast as any of them. In summary, we can say that LTR is the most efficient compactification process.

6.4. Minimal compact extensive form and backward dominance procedure

In this subsection we show Theorem 4 in Section 4. We need an algorithm called backward dominance procedure defined in Perea [12].

For a given information set \( h \), a subset \( \Lambda(h) \subseteq S(h) \) is called a decision problem at \( h \) iff for every active player \( i \) at \( h \) there are some \( D_i \subseteq S_i(h) \) and \( D_{-i} \subseteq S_{-i}(h) \) such that \( \Lambda(h) = D_i \times D_{-i} \). Consider an information set \( h \), a player \( i \) who is active at \( h \), and a decision problem \( \Lambda(h) = D_i \times D_{-i} \) at \( h \). We say that a strategy \( s_i \in D_i \) is strictly dominated within the decision problem \( \Lambda(h) \) iff there is some randomized strategy \( \mu_i \in \Delta(D_i) \) such that \( u_i(\mu_i, s_{-i}) > u_i(s_i, s_{-i}) \) for all \( s_{-i} \in S_{-i} \). By \( sd_i(\Lambda(h)) \) we denote the set of strategies in \( D_i \) that are strictly dominated within \( \Lambda(h) \) for the active player \( i \). Similarly, we define

\[
sd(\Lambda(h)) = \{(s_j)_{j \in I} \in \Lambda(h) : s_i \in sd_i(\Lambda(h)) \text{ for some } i \text{ that is active at } h\}
\]  

(3)

The backward dominance procedure is defined as follows:

**Initial step.** For every information set \( h \), let \( \Lambda^0(h) = S(h) \);

**Inductive step.** Let \( k \geq 1 \), and suppose that \( \Lambda^{k-1}(h) \) has been defined for every information set \( h \). Then, at every information set \( h \) we define

\[
\Lambda^k(h) = \Lambda^{k-1}(h) \setminus \bigcup_{h' \geq h} sd(\Lambda^{k-1}(h'))
\]

(4)
A strategy $s_i$ survives the backward dominance procedure iff there is some $s_{-i} \in S_{-i}$ such that $(s_i, s_{-i}) \in \Lambda^k(h^o)$ for each $k \in N$. Here $h^o$ is the information set which contains (only) the root.

Perea [12] showed the following result (also see Perea [11]).

**Lemma 7.** (Strategies surviving the backward dominance procedure) Player $i$ can rationally choose $s_i$ under common belief in future rationality if and only if $s_i$ survives the backward dominance procedure.

Based on Corollary 6, we can show the following statement.

**Lemma 8.** (Decrease of strategies surviving the backward dominance procedure in a compactification) Consider compact extensive forms $\Gamma, \Gamma'$ with $\Gamma \rightarrow_{\text{COM}} \Gamma'$, and a vector of utility functions $(u_i)_{i \in I}$. If $s_i$ survives the backward dominance procedure in $(\Gamma', (u_i)_{i \in I})$, so it does in $(\Gamma, (u_i)_{i \in I})$.

Indeed, through a compactification operation, an information set $h$ which used to follow another information set $h'$ is now simultaneous with $h'$, which may lead to that more strategies in some $S_i(h)$ as well as strategies reaching some information sets before $h$ and $h'$ to be eliminated. Based on this lemma, it is easy to see by mathematical induction that, if $\Gamma$ is the final term of some LTR process from $\Gamma'$, then every $s_i$ survives the backward dominance procedure in $\Gamma$, so it does in $\Gamma'$. In other words, if a strategy $s_i$ does not survive the procedure for some compact extensive form, then $s_i$ does not survive in the terminal of the LTR process from it.

Consider a minimal compact extensive form $\Gamma^*$, some $\Gamma$ from which the LTR process leads to $\Gamma^*$, and some $\Gamma'$ which is the terminal term of a compactification process from $\Gamma$. Remember that we have shown in Proposition 3 that for every information set, if it is absorbed in $\Gamma'$, then so it does in $\Gamma^*$. For simplicity, we assume that at each node, each player has more than two actions. It can be seen that (1) every information set in $\Gamma$ can be absorbed for at most one time, and it is determined that to where they are absorbed. (2) If $\Gamma' \neq \Gamma^*$, it is because in the compactification process from $\Gamma$ to $\Gamma'$, some information set $h$ is absorbed while some other information set $h'$ following it is still not absorbed and is not able to absorbed due to the absorption of $h$. Therefore, $\Gamma'$ can be informally said as a “incomplete” version of $\Gamma^*$. Hence, similarly to the argument in the previous paragraph, it can be seen that every strategy which does not survive the backward dominance procedure in $\Gamma'$ does not survive the procedure in $\Gamma^*$. Combined with Lemma 8, it follows that the statement holds for any middle term in any compactification process from $\Gamma$. Hence, we have shown the only if part of Theorem 4.

The if part of Theorem 4 holds straightforwardly since $\Gamma^*$ itself belongs to $[\Gamma^*]$. Here we have shown Theorem 4.

### 6.5. Order-independence of the T-compactification process

In this subsection we show Proposition 5 in Section 5, i.e., the order independence of the T-compactification process. We use a lemma proved in Newman [9] (also see Apt [1]) which gives a sufficient condition for order-independence in an abstract reduction system.

An abstract reduction system is a pair $(\Theta, \rightarrow)$, where $\Theta$ is a non-empty set and $\rightarrow$ is a binary relation on $\Theta$. An element $\theta \in \Theta$ is called an endpoint in $(\Theta, \rightarrow)$ if there is no $\theta' \in \Theta$ such that $\theta \rightarrow \theta'$. We say that $\{\theta_n : n = 0, 1, ...\}$ (which can be finite or infinite) is a $\rightarrow$-sequence in $(\Theta, \rightarrow)$ iff $\theta_n \in \Theta$ for each $n$ and $\theta_n \rightarrow \theta_{n+1}$ (as far as $\theta_{n+1}$ is defined). We use $\rightarrow^*$ to denote the reflexive and transitive closure of $\rightarrow$. We say that $(X, \rightarrow)$ is weakly confluent iff for each $\eta, \xi \in \Theta$, if $\theta \rightarrow \eta$ and $\theta \rightarrow \zeta$, then there is some $\theta' \in \Theta$ such that $\eta \rightarrow^* \theta'$ and $\zeta \rightarrow^* \theta'$.
Lemma 9. (Newman’s lemma) If an abstract reduction system \((X, \to)\) satisfies the following two conditions: \((N1)\) each \(\to\)-sequence is finite, and \((N2)\) \((X, \to)\) is weakly confluent, then for each \(\theta \in \Theta\) there is a unique endpoint \(\theta' \in \Theta\) such that \(\theta \to^* \theta'\).

The set of all compact extensive forms and \(\to_{TCOM}\) form an abstract reduction system. Since a T-compactification operation does not generate new information set which can be absorbed by its immediate predecessor, is straightforward to see that each \(\to_{TCOM}\)-sequence is finite. Hence, by Newman’s lemma, to show that the T-compactification process is order-independent, we only need to show that weak confluence is satisfied.

First, it can be seen that a T-ompactification operation can still be characterized by a pair \((x, i)\) where \(x\) is the node which absorbs an information set constitute by its immediate successors and \(i\) is the player who possesses that information set. So here we abuse the symbol and still use \(\psi(\Gamma; (x,i))\) to show what is changed in a T-compactification on \(\Gamma\). Consider three compact extensive forms \(\Gamma, \Gamma', \Gamma''\) with \(\Gamma' = \psi(\Gamma; (x, i))\) and \(\Gamma'' = \psi(\Gamma; (y, j))\). Intuitively, it can be seen easily that if \(x\) and \(y\) are “far” from each other, then for each form we just repeat the operation that leads to the other and we can obtain the same outcome. Formally, if \(x\) and \(y\) are on different subtrees (i.e., neither \(x \leq y\) nor \(y \leq x\)) or \(x = p_k(y)\) with \(k \geq 2\) (or symmetrically \(y = p_k(x)\) with \(k \geq 2\)), then since the two T-compactification operations do not influence each other and both \(x, S(x)\) and \(y, S(y)\) are preserved in \(\Gamma''\), \(\Gamma\) respectively, we have \(\psi(\Gamma'; (y,j)) = \psi(\Gamma''; (x,i))\).

Suppose that \(x = y\) (i.e., \(x = p_0(y)\)). If \(i = j\), then \(\Gamma' = \Gamma''\). Consider the case that \(i \neq j\). It means that \(S(x) \in H_i\) and \(S(x) \in H_j\). By the equation (1) in the definition of T-compactification in Section 5, \(x\) is preserved in \(\Gamma'\) and \(\Gamma''\), and \(S'(x) \in H_j\) and \(S''(x) \in H_i\). Hence we can further combine \(S'(x)\) with \(x\) in \(\Gamma'\) and \(S''(x)\) with \(x\) in \(\Gamma''\), and it can be seen that \(\psi(\Gamma'; (y,j)) = \psi(\Gamma''; (x,i))\).

Finally, suppose that \(x = p_1(y)\). By our assumption, in this case \(i \neq j\). When \(|i(y)| = 1\) (i.e., \(i(y) = \{i\}\)), it can be seen that \(\psi(\Gamma''; (x, i)) = \Gamma'\). Suppose that \(|i(y)| > 1\). We still use the symbols used in Section 3 and let \(b = |i(S(y))|\). In \(\Gamma'\), now \(y\) is “replaced” by dummies \(t_1, \ldots, t_b\), and \(\bigcup_{r=1}^b S'(t_r) \subseteq h_j\) for some \(h_j\) in \(H_j\). Hence we can make each \(t_r\) to absorb \(S(t_r)\). On the other hand, in \(\Gamma''\), \(x\) still can absorb \(S''(x) = S(x)\). It can be seen that \(\psi(...\psi(\psi(\Gamma'; (t_1, j)); (t_2, j))\ldots; (t_b, j)) = \psi(\Gamma''; (x, i))\). Here we have shown that condition \((N2)\) in Newman’s lemma is satisfied, that is, we have shown that T-composition process is order-independent.

Though the terminal term is invariant among different T-compactification processes, it can be seen from the above proof that the LTR process is the fastest one. Indeed, the outcome that LTR process can achieve in two steps if we start from the node which is “nearer” to the terminal nodes (i.e., \(y\)). If we start from the farther one (i.e., \(x\)), then it costs \(b\) times of operation. Therefore, LTR process still has its distinct value from the viewpoint of algorithmic complexity for T-compactification.

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\[\text{It should be noted that in this proof, all the equality holds in the sense of isomorphism, i.e., strictly speaking there may need some re-naming of the nodes.}\]
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