Boxicity and Treewidth

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Abstract

An axis-parallel $b$-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$ where $R_i$ (for $1 \leq i \leq b$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph $G$, its boxicity $\text{box}(G)$ is the minimum dimension $b$, such that $G$ is representable as the intersection graph of (axis-parallel) boxes in $b$-dimensional space. The concept of boxicity finds applications in various areas such as ecology, operation research etc. Though many authors have investigated this concept, not much is known about the boxicity of many well-known graph classes (except for a couple of cases) perhaps due to lack of effective approaches. Also, little is known about the structure imposed on a graph by its high boxicity.

The concepts of tree decomposition and treewidth play a very important role in modern graph theory and has many applications to computer science. In this paper, we relate the seemingly unrelated concepts of treewidth and boxicity. Our main result is that, for any graph $G$, $\text{box}(G) \leq \text{tw}(G) + 2$, where $\text{box}(G)$ and $\text{tw}(G)$ denote the boxicity and treewidth of $G$ respectively. We also show that this upper bound is (almost) tight. Since treewidth and tree decompositions are extensively studied concepts, our result leads to various interesting consequences, like bounding the boxicity of many well known graph classes, such as chordal graphs, circular arc graphs, AT-free graphs, co-comparability graphs etc. For all these graph classes, no bounds on their boxicity were known previously. All our bounds are shown to be tight up to small constant factors. An algorithmic consequence of our result is a linear time algorithm to construct a box representation for graphs of bounded treewidth in a space of constant dimension.

Another consequence of our main result is that, if the boxicity of a graph is $b \geq 3$, then there exists a simple cycle of length at least $b - 3$ as well as an induced cycle of length at least $\lfloor \log_\Delta (b - 2) \rfloor + 2$, where $\Delta$ is its maximum degree. We also relate boxicity with the cardinality of minimum vertex cover, minimum feedback vertex cover etc. Another structural consequence is that, for any fixed planar graph $H$, there is a constant $c(H)$ such that, if $\text{box}(G) \geq c(H)$ then $H$ is a minor of $G$. 

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1 Introduction

1.1 Boxicity

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe $U$, where $V$ is an index set. The intersection graph $\Omega(\mathcal{F})$ of $\mathcal{F}$ has $V$ as node set, and two distinct nodes $x$ and $y$ are adjacent if and only if $S_x \cap S_y \neq \emptyset$.

Representations of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. A prime example of a graph class defined in this way is the class of interval graphs: A graph $G$ is an interval graph if and only if $G$ has an interval realization: i.e., each node of $G$ can be associated to an interval on the real line such that two intervals intersect if and only if the corresponding nodes are adjacent. Motivated by theoretical as well as practical considerations, graph theorists have tried to generalize the concept of interval graphs in various ways. One such generalization is the concept of boxicity defined as follows.

An axis-parallel $b$–dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$ where $R_i$ (for $1 \leq i \leq b$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph $G$, its boxicity $\text{box}(G)$ is the minimum dimension $b$, such that $G$ is representable as the intersection graph of (axis–parallel) boxes in $b$–dimensional space. It is easy to see that the class of graphs with $b \leq 1$ is exactly the class of interval graphs. A $b$–dimensional box representation of a graph $G = (V, E)$ is a mapping that maps each each $u \in V$ to an axis-parallel $b$–dimensional box $B_u$ such that $G$ is the intersection graph of the family $\{B_u : u \in V\}$.

The concept of boxicity was introduced by F. S. Roberts [26]. It finds applications in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research. (See [15].) It was shown by Cozzens [14] that computing the boxicity of a graph is NP–hard. This was later strengthened by Yannakakis [33], and finally by Kratochvil [23] who showed that deciding whether boxicity of a graph is at most 2 itself is NP–complete. The complexity of finding the maximum independent set in bounded boxicity graphs was considered by [21,18].

There have been many attempts to estimate or bound the boxicity of graph classes with special structure. In his pioneering work, F. S. Roberts proved that the boxicity of complete $k$–partite graphs are $k$. Scheinerman [28] showed that the boxicity of outer planar graphs is at most 2. Thomassen [30] proved that the boxicity of planar graphs is bounded above by 3. The boxicity of split graphs is investigated by Cozzens and Roberts [15]. Apart from these results, not much is known about the boxicity of most of the well-known graph classes. Also, little is known about the structure imposed on a graph by its high boxicity.

Researchers have also tried to generalize or extend the concept of boxicity in vari-
ous ways. The poset boxicity [31], the rectangular number [12], grid dimension [3], circular dimension [17, 29] and the boxicity of digraphs [11] are some examples.

1.2 Treewidth

The notions of \textit{tree–decomposition} and \textit{treewidth} were first introduced (under different names) by R. Halin and later rediscovered independently by Robertson and Seymour. (See [16], Chapter 12 for historical details.) Roughly speaking, the treewidth of a graph $G$ is the minimum $k$ such that $G$ can be decomposed into pieces forming a tree structure with at most $k + 1$ nodes per piece. Such a decomposition is called a tree decomposition. See section 3 for the formal definition of tree decomposition and treewidth.

These notions underly several important and sometimes very deep results in graph theory and graph algorithms and are very useful for the analysis of several practical problems. Recent research has shown that many NP–complete problems become polynomial or even linear time solvable, or belong to NC, when restricted to graphs with small treewidth (See [1, 2, 5]). The concepts of treewidth and pathwidth have applications in many practically important fields like VLSI layouts, Cholesky factorization, Expert systems, Evolution theory, and natural language processing. (See [5] for references).

The decision problem of checking whether $\text{tw}(G)$ is at most $k$, given $G$ and $k$ is known to be NP-complete. Hence the problem of determining the treewidth of an arbitrary graph is NP-hard and the research on determining the treewidth and pathwidth has been focused on special classes. Linear or polynomial time or NC algorithms for producing optimal tree decompositions have been proposed for several special classes of graphs like graphs of bounded treewidth [6, 7], chordal graphs, cographs, circular arc graphs, chordal bipartite graphs, permutation graphs, circle graphs, and distance hereditary graphs. For an extensive bibliography on treewidth, see [5].

2 Our Results

Our main result is the following theorem which connects the boxicity of a graph to its treewidth.

\textbf{Theorem.} For any graph $G$, $\text{box}(G) \leq \text{tw}(G) + 2$. Moreover, we construct a family of graphs such that, for any $t \geq 1$, there is a graph $G$ of treewidth at most $t + \sqrt{t}$ and boxicity at least $t - \sqrt{t}$ in this family. In other words, for this family of graphs, $\text{tw}(G)(1 - o(1)) \leq \text{box}(G) \leq \text{tw}(G) + 2$, and thus the upper bound is
almost sharp.

As far as we know, the only known general upper bound for boxicity of $G$ is given by Roberts [26], who showed that $box(G) \leq \frac{n}{2}$, where $n$ is the number of vertices in $G$. It was also shown that this bound is tight. Thus it is interesting to look for upper bounds on boxicity which can provide better structural insight about graphs with respect to their boxicity. We believe that our upper bound in terms of treewidth is a progress in that direction. Since treewidth is an extensively studied concept, our upper bound also leads to many nonintuitive results about boxicity, which appear difficult to prove using direct approaches.

Consequences on special graphs. Except for the few classes mentioned in the introduction, not much progress is made in bounding the boxicity of various well-known graph classes, perhaps due to the lack of effective approaches. As consequences of the above theorem, we are able to derive tight (up to constant factors) upper bounds for the boxicity of various graph classes, in terms of their maximum degree and clique number.

For a graph $G$, let $\Delta(G)$ denote its maximum degree and let $\omega(G)$ denote its clique number i.e., the size (number of nodes) of the maximum clique in $G$. We summarize our results for the boxicity of different graph classes in the following table.

| Graph class                  | Upper bounds on $box(G)$ |
|------------------------------|--------------------------|
| Chordal Graphs               | $\omega(G) + 1$         |
|                              | $\Delta(G) + 2$         |
| Circular Arc Graphs          | $2\omega(G) + 1$        |
|                              | $2\Delta(G) + 3$        |
| AT-free Graphs               | $3\Delta(G)$            |
| Co-comparability Graphs      | $2\Delta(G) + 1$        |
| Permutation Graphs           | $2\Delta(G) + 1$        |
| Any minor closed family      | constant                 |
| which excludes at least one planar graph |

Each of the above upper bounds is shown to be tight up to small constant factors.

Planar Graph Minors and Bboxicity. Study of graph minors is one of the most important areas in modern graph theory (see [16] for the definition of graph minors). Combing our upper bound result with a result of Robertson and Seymour [27], we obtain the following.
Theorem. For every planar graph $H$, there is a constant $c(H)$ such that every graph with boxicity $\geq c(H)$ has a minor isomorphic to $H$.

Cycles and Boxicity. The properties that imposes various kinds of long cycles in a graph is an extensively explored topic in graph theory. (See Chapter 1 of [9] for an introductory survey, or the book by Voss [32].) A structural consequence of our main result is that, high boxicity imposes a long simple cycle as well as a long induced cycle (chordless cycle) in the graph.

Theorem. In any graph $G$ of boxicity $b$, there exists a simple cycle of length at least $b - 3$. Moreover, there exists a graph $G$ whose boxicity is $b$ but the length of any simple cycle in it is at most $2b$.

Theorem. Let $G$ be a graph with maximum degree $\Delta$ and boxicity $b \geq 3$. Then there exists a induced cycle (chordless cycle) of length at least $\lceil \log_\Delta (b - 2) \rceil + 2$.

Boxicity and Vertex Cover. The subset $S \subseteq V(G)$ is called a vertex cover of $G$ if every edge of $G$ is incident on at least one vertex from $S$. A vertex cover of minimum cardinality is called a minimum vertex cover. We denote the cardinality of a minimum vertex cover of $G$ by $MVC(G)$. Our upper bound theorem yields the following.

Theorem. For any graph $G$, $box(G) \leq MVC(G) + 2$.

The reader may note that for a graph $G$ on $n$ nodes, $MVC(G) = n - \alpha(G)$, where $\alpha(G)$ is the cardinality of a maximum independent set in $G$. Thus $box(G) \leq n - \alpha(G) + 2$. Similarly, 2 times the cardinality of any maximal matching in $G$ is an upper bound for $MVC(G)$ and thus, we obtain an upper bound for the boxicity of $G$ in terms of the cardinality of maximal matchings also.

We also connect boxicity to a parameter, which is a variant of minimum vertex cover, namely the cardinality of a minimum feedback vertex cover of $G$.

Algorithmic Consequences. From an algorithmic point of view, it is interesting to efficiently construct a box representation (see introduction for the definition of box representation) of the given graph in low dimensional space. For example, note that if a dense graph has such a representation in constant dimensional space, then the memory required to store this graph (nodes and edges) is only linear in the number of nodes. The proof of our main upper bound also yields an efficient construction of the box representation of the given graph $G$ from its tree decomposition. Since, for bounded treewidth graphs, the tree decompositions can be constructed in linear time [6], we have the following.

Theorem. For a bounded treewidth graphs, box representation in constant dimension can be constructed in linear (in the number of vertices) time.
Efficient polynomial time algorithms (exact or approximation) are known for constructing the tree decompositions of many special graph classes such as chordal graphs, cographs, circular arc graphs, chordal bipartite graphs, permutation graphs, circle graphs, distance hereditary graphs etc. An immediate consequence of this in conjunction with the constructive proof of our main upper bound is that the corresponding box representations can also be computed in polynomial time.

Complexity theoretic consequences: By our main upper bound result, the class of bounded treewidth graphs is a subset of bounded boxicity graphs. Hence if a problem is NP-hard for bounded treewidth graphs, it is also NP-hard for bounded boxicity graphs. For example, it is shown in [25] that the channel assignment problem is NP-complete for graphs of treewidth at least 3. It follows from our result that this problem is NP-complete for graphs of boxicity at least 5.

3 Tree Decompositions and the Treewidth

**Definition 1** A tree decomposition of \( G = (V, E) \) is a pair \((\{X_i : i \in I\}, T)\), where \( I \) is an index set, \( \{X_i : i \in I\} \) is a collection of subsets of \( V \) and \( T \) is a tree (connected) whose node set is \( I \), such that the following conditions are satisfied:

1. \( \bigcup_{i \in I} X_i = V \).
2. \( \forall (u, v) \in E, \exists i \in I \) such that \( u, v \in X_i \).
3. \( \forall i, j, k \in I: \text{if } j \text{ is on a path in } T \text{ from } i \text{ to } k, \text{ then } X_i \cap X_k \subseteq X_j \).

The width of a tree decomposition \((\{X_i : i \in I\}, T)\) is \( \max_{i \in I} |X_i| - 1 \). The treewidth of \( G \) is the minimum width over all tree decompositions of \( G \) and is denoted by \( tw(G) \). Node \( i \) of a tree decomposition \((\{X_i : i \in I\}, T)\) refers to the node \( i \) of the tree \( T \).

**Rooted Tree.** A tree with a fixed root is called a rooted tree. The height(i) of a node \( i \) in a rooted tree \( T \) with root \( r \) is defined as usual: height(r) of the root \( r \) is 0, and height(x) for any other node \( x \) is exactly one more than the height of its parent. A node \( i \neq j \) is the ancestor of node \( j \) if \( i \) is in the path from \( j \) to \( r \). A node \( j \) is a descendant of \( i \) if either \( i = j \) or \( i \) is the ancestor of \( j \).

**Definition 2** A normalized tree decomposition of a graph \( G = (V, E) \) is a triple \((\{X_i : i \in I\}, r \in I, T)\) where \((\{X_i : i \in I\}, T)\) is a tree decomposition of \( G \) that additionally satisfies the following two properties.

4. It is a rooted tree where the subset \( X_r \) that corresponds to the root node \( r \) contains exactly one vertex.
5. For any node \( i \), if \( i' \) is a child of \( i \), then \( |X_{i'} - X_i| = 1 \).
Lemma 3 For any graph $G$ there is a normalized tree decomposition with width equal to $tw(G)$.

**PROOF.** Consider a tree decomposition $(\{X_i : i \in I\}, T)$ of $G = (V, E)$ with width $tw(G)$. We convert it into a normalized tree decomposition $(\{X_i : i \in I'\}, r, T')$ as follows.

As the first step, we convert $T$ into a rooted tree $T_1$ as follows. Let $i$ be an arbitrary node of $T$ such that $X_i$ is non–empty. Let $u \in X_i$. Create a new node $r$ (where $r \notin I$), and define $X_r = \{u\}$. Now connect node $r$ to $i$. Let the resulting tree on the node set $I \cup \{r\}$ be $T_1$. It is easy to verify that $(\{X_i : i \in I \cup \{r\}\}, T_1)$ is a tree decomposition of $G$. From here on, we view $T_1$ as a rooted tree, with root $r$.

Consider any edge $(j, j')$ of $T_1$ where $j'$ is a child of $j$. Without loss of generality, we can assume that $X_{j'} \subseteq X_j$. (If $X_{j'} \subseteq X_j$ then the following operations do not violate the defining properties of tree decomposition: (a) Remove $j'$ from $I \cup \{r\}$ and hence from $T_1$ (b) make each child of $j'$ a child of $j$.) Let $X_{j'} - X_j = \{u_1, \ldots, u_h\}$. If $h = 1$ then we retain this edge as such. If $h > 1$ then we replace the edge $(j, j')$ by a path $j, k_1, k_2, \ldots, k_{h-1}, j'$, where $k_1, k_2, \ldots, k_{h-1}$, are new nodes, and define the subset $X_{k_i} = (X_j \cap X_{j'}) \cup \{u_1, u_2, \ldots, u_i\}$ for $1 \leq i \leq h - 1$. Note that $|X_{k_i}| \leq |X_{j'}|$ for $1 \leq i \leq h - 1$ and thus by introducing these new nodes we have not increased the width of the tree decomposition. We repeat this process for each edge of $T_1$. Let $T'$ be the new rooted tree (rooted at $r$) obtained after these operations. Let $I'$ be the node set of $T'$. Note that the root $r$ of $T'$ still corresponds to the singleton set $X_r$. Now, it is straightforward to verify that $(\{X_i : i \in I'\}, r, T')$ is a normalized tree decomposition.

Definition 4 With respect to the normalized tree decomposition $(\{X_i : i \in I\}, r, T)$ of a graph $G = (V, E)$, we define the following two functions.

a) $b : V \rightarrow I$ is defined as follows. For $v \in V$, $b(v) = i$, where $i$ is the (unique) node in $I$ such that $\text{height}(i)$ is minimum subject to the condition that $v \in X_i$.

b) $h : V \rightarrow N$ is defined by $h(v) = \text{height}(b(v))$.

Observe that the function $b(v)$ is well-defined. That is, there is exactly one node $i$ of $T$, such that $v \in X_i$ and $\text{height}(i)$ is the minimum possible. To see this, assume that there is one more node $j$ such that $v \in X_j$ and $\text{height}(j) = \text{height}(i)$. Then, by Property 3 of Definition 1, there should be a node $k$ with height less than $\text{height}(i)$ and $v \in X_k$. This contradicts the assumption that node $i = b(v)$ has the minimum possible height.

Lemma 5 The function $b : V \rightarrow I$ is a bijection.
PROOF. First we show that \( b : V \to I \) is injective. That is, for any two distinct vertices \( u, v \in V \), \( b(u) \neq b(v) \). If not, let \( b(u) = b(v) = i \). Since \( X_i \) contains at least two vertices (namely \( u \) and \( v \)), \( i \) is not the root node of the normalized tree decomposition. Let \( j \) be the parent of \( i \). Since \( b(u) = b(v) = i \), \( X_j \) does not contain \( u \) and \( v \) by the definition of \( b(\cdot) \). That is, \( |X_i - X_j| \geq 2 \). This contradicts Property 5 of Definition 2.

Now assume that \( b : V \to I \) is not surjective i.e., there exists a node \( i \in I \) that do not have a pre-image in \( V \). Let \( X_i = \{ u_1, \ldots, u_r \} \). Consider any vertex \( u_j \in X_i \). Let \( b(u_j) = k \neq i \). By the definition of \( b(u_j) \), \( \text{height}(k) \leq \text{height}(i) \). Let \( j \) be the parent of \( i \). Clearly, \( j \) is on the path between \( k \) and \( i \) in \( T \), and thus by Property 3 of the Definition 1, \( u_j \in X_j \). This implies that \( X_i \subseteq X_j \), which contradicts Property 5 of Definition 2. Thus \( b : V \to I \) injective and surjective i.e., bijective.

Lemma 6 For any \( i \in I \) such that \( u \in X_i \), node \( i \) is a descendant of \( b(u) \).

PROOF. Otherwise, since \( u \in X_i \) as well as \( X_{b(u)} \), by Property 3 of Definition 1, \( u \in X_j \) also where \( j \) is the parent of \( b(u) \). This contradicts the definition of \( b(u) \).

4 Box Representation and Interval Graph Representation

Let \( G = (V, E(G)) \) be a graph and let \( I_1, \ldots, I_k \) be \( k \) interval graphs such that each \( I_j = (V, E(I_j)) \) is defined on the same set of vertices \( V \). If

\[
E(G) = E(I_1) \cap \cdots \cap E(I_k),
\]

then we say that \( I_1, \ldots, I_k \) is an interval graph representation of \( G \). The following equivalence is well-known.

Theorem 7 (Roberts [26]) The minimum \( k \) such that there exists an interval graph representation of \( G \) using \( k \) interval graphs \( I_1, \ldots, I_k \) is the same as \( \text{box}(G) \).

Recall that a \( b \)-dimensional box representation of \( G \) is a mapping of each node \( u \in V \) to \( R_1(u) \times \cdots \times R_b(u) \), where each \( R_i(u) \) is a closed interval of the form \([\ell_i(u), r_i(u)]\) on the real line. It is straightforward to see that an interval graph representation of \( G \) using \( b \) interval graphs \( I_1, \ldots, I_b \), is equivalent to a \( b \)-dimensional box representation in the following sense. Let \( R_i(u) = [\ell_i(u), r_i(u)] \) denote the closed interval corresponding to node \( u \) in an interval realization of \( I_i \). Then the \( b \)-dimensional box corresponding to \( u \) is simply \( R_1(u) \times \cdots \times R_b(u) \). Conversely, given a \( b \)-dimensional box representation of \( G \), the set of intervals \( \{ R_i(u) : u \in V \} \) forms the \( i \)th interval graph \( I_i \) in the corresponding interval graph representation.
5 Treewidth vs Boxicity: The Upper Bound

Let $G = (V, E)$ be a graph. In this section, we assume that $(\{X_i : i \in I\}, r, T)$ is a normalized tree decomposition of $G$ with width $tw(G)$.

**Lemma 8** Let $G = (V, E)$ be a graph and $(\{X_i : i \in I\}, r, T)$ be its normalized tree decomposition of width $tw(G)$. Then, there exists a function $\theta : V \to \{0, \ldots, tw(G)\}$, such that for any $i \in I$ and for any two distinct nodes $u, v \in X_i$, $\theta(u) \neq \theta(v)$.

**PROOF.** Sort the nodes in $I$ in the increasing order of their height (breaking ties arbitrarily). Let the order be $i_1, \ldots, i_n$. Let $u_j = b^{-1}(i_j)$. We inductively define $\theta(u_j)$ in the order $u_1, \ldots, u_n$. Define $\theta(u_1) = 0$. Assume inductively that for $k < j$, $\theta(u_k)$ is defined, and for any $u, v \in X_{i_k}, \theta(u) \neq \theta(v)$. Observe that node $i_1$ is the root of $T$ and thus $X_{i_1}$ is the singleton set $\{u_1\}$, and thus the inductive assumption is trivially true for $i_1$. Let $i_h$ be the parent of $i_j$ in $T$. First we observe that $u_j \in X_{i_j} - X_{i_h}$, by the definition of $b(u_j)$. Hence $X_{i_j} - \{u_j\} \subseteq X_{i_h}$ by Property 5 of Definition 2. Consider a vertex $v \in X_{i_j} - \{u_j\}$. Observe that $b(v) = i_r$, for some $r < j$. (This is because, $v \in X_{i_1} - \{u_j\} \subseteq X_{i_h}$ and $height(i_h) < height(i_j)$.) So $\theta(v)$ is already defined at this point by the induction assumption. Now define $\theta(u_j) = t$, where $t \neq \theta(v)$ for any $v \in X_{i_j} - \{u_j\}$. There is such a $t$ because $|X_{i_j} - \{u_j\}| \leq tw(G)$ but there exists $tw(G) + 1$ distinct possible values for $t$. Now we claim that for any $u, v \in X_{i_j} - \{u_j\}$, $\theta(u) \neq \theta(v)$. This is because, $u, v \in X_{i_j} - \{u_j\} \subseteq X_{i_h}$, and since $h < j$, the inductive assumption is valid for $i_h$.

**Lemma 9** If $(u, v) \in E(G)$ then $\theta(u) \neq \theta(v)$.

**PROOF.** Since $(u, v) \in E(G)$ then there exists an $X_i$ such that $u, v \in X_i$ by Property 2 of Definition 1. Now, by Lemma 8, $\theta(u) \neq \theta(v)$.

**Lemma 10** If $(u, v) \in E(G)$, then either $b(u)$ is an ancestor of $b(v)$ or $b(v)$ is an ancestor of $b(u)$ in $T$.

**PROOF.** Due to Property 2 of Definition 1, there is a node $i \in I$ such that $u, v \in X_i$. Because of Lemma 6, node $i$ is the descendant of $b(u)$ and also $b(v)$. Thus both $b(u)$ and $b(v)$ are in the path from $i$ to the root $r$ in $T$. Moreover, $b(u) \neq b(v)$ since $b(\cdot)$ is a bijection by Lemma 5. Thus the result follows.

**Lemma 11** Let $(u, v) \in E(G)$ and let $b(u)$ be the ancestor of $b(v)$. For any vertex $w \in V - \{u\}$, $\theta(w) \neq \theta(u)$ if $b(w)$ is in the path from $b(v)$ to $b(u)$ in $T$. 


**PROOF.** Because of Property 2 of Definition 1, there is an \( X_i \) such that \( u, v \in X_i \). By Lemma 6, we know that \( i \) is a descendant of \( b(u) \) and also \( b(v) \). This in conjunction with the assumption that \( b(u) \) is the ancestor of \( b(v) \), implies that \( b(v) \) is in the path from \( b(u) \) to node \( i \). Thus, for any node \( k \) in the path from \( b(v) \) to \( b(u) \), \( u \in X_k \), by Property 3 of Definition 1. Now, for any vertex \( x \in X_k - \{u\} \), \( \theta(x) \neq \theta(u) \) by Lemma 8. In particular, this is true for \( k = b(w) \) and \( x = w \). ■

Using the function \( \theta : V \to \{0, \ldots, tw(G)\} \) (see Lemma 8) and function \( h : V \to N \) (see Definition 4), we construct \( tw(G) + 2 \) different interval super graphs of \( G \) as follows. Let \( i \) be such that \( 0 \leq i \leq tw(G) \). The interval graph \( I_i \) is defined as follows.

**Definition of interval graph** \( I_i \), for \( 0 \leq i \leq tw(G) \): We define the interval \([\ell_i(v), r_i(v)]\) for each \( v \in V \) as follows.

1. If \( \theta(v) = i \) then \( \ell_i(v) = 2h(v) \) and \( r_i(v) = 2h(v) + 1 \).
2. If \( \theta(v) \neq i \) then let \( S = \theta^{-1}(i) \cap N(v) \), where \( \theta^{-1}(i) = \{u \in V \mid \theta(u) = i\} \) and \( N(v) = \{u \in V - \{v\} \mid (u, v) \text{ is an edge of } G\} \).
   a. If \( S = \emptyset \) then \( \ell_i(v) = 3n \) and \( r_i(v) = 3n \).
   b. If \( S \neq \emptyset \) then \( \ell_i(v) = \min_{u \in S} r_i(u) \) and \( r_i(v) = 3n \).

**Definition of interval graph** \( I_{tw(G) + 1} \): Consider a depth-first ordering of the nodes of \( T \). The depth-first ordering of rooted tree \( T \) rooted at \( r \) is an ordered list of the nodes of \( T \) denoted as \( df(T, r) \). If \( T \) has only one node, namely its root \( r \), then \( df(T, r) = \langle r, r \rangle \). Otherwise, let \( r_1, \ldots, r_k \) be the children of \( r \) and let \( T_i \) be the rooted sub-tree rooted at \( r_i \). Then, \( df(T, r) \) is the concatenation of the lists \( \langle r \rangle, df(T_1, r_1), \ldots, df(T_k, r_k), \langle r \rangle \) in that order. Observe that each node of \( T \) appears exactly two times in \( df(T, r) \). Thus we can associate with each node \( i \), two numbers \( \text{first}(i) \) and \( \text{last}(i) \) that denote its sequence number in the ordered list \( df(T, r) \) corresponding to its first occurrence and last occurrence respectively. Now, for each vertex \( v \in V \), \( \ell_{tw(G) + 1}(v) = \text{first}(b(v)) \) and \( r_{tw(G) + 1}(v) = \text{last}(b(v)) \).

The resulting interval graph is \( I_{tw(G) + 1} \).

**Lemma 12** Each \( I_i \), \( 0 \leq i \leq tw(G) + 1 \), \( E(G) \subseteq E(I_i) \).

**PROOF.** Let \( (x, y) \in E(G) \). First, assume that \( 0 \leq i \leq tw(G) \). By Lemma 9, we have \( \theta(x) \neq \theta(y) \). Without loss of generality, assume that \( \theta(y) \neq i \). Hence \( r_i(y) = 3n \) because of Case 2 of the definition of \( I_i \). If \( \theta(x) \neq i \) then \( r_i(x) \) is also \( 3n \), and thus \( (x, y) \in E(I_i) \). Now assume that \( \theta(x) = i \). Hence, \( x \in S = \theta^{-1}(i) \cap N(y) \). Hence \( r_i(y) = 3n \geq r_i(x) \geq \min_{z \in S} r_i(z) = \ell_i(y) \), and thus \( r_i(x) \in [\ell_i(y), r_i(y)] \). It follows that \( (x, y) \in E(I_i) \).
It remains to show that \((x, y) \in E(I_{tw(G)+1})\). Because of Lemma 10, we can assume without loss of generality that \(b(x)\) is an ancestor of \(b(y)\). Consider the depth-first order \(df(T, r)\) of the nodes of \(T\). It is straightforward to verify that 
\[ \text{first}(b(x)) \leq \text{first}(b(y)) \leq \text{last}(b(y)) \leq \text{last}(b(x)), \]
and thus the intervals in \(I_{tw(G)+1}\) corresponding to \(x\) and \(y\) intersect. It follows that \((x, y) \in E(I_{tw(G)+1})\).

**Lemma 13** For any \((x, y) \notin E(G)\), there exists some \(i, 0 \leq i \leq tw(G) + 1\), such that \((x, y) \notin E(I_i)\).

**PROOF.** Let \((x, y) \notin E(G)\). Suppose that neither \(b(x)\) is an ancestor of \(b(y)\) nor \(b(y)\) an ancestor of \(b(x)\) in \(T\). Then, we claim that \((x, y) \notin E(I_{tw(G)+1})\). This is because, if \(b(x)\) is not an ancestor of \(b(y)\) or vice versa, then in the depth-first order \(df(T, r)\), either \(\text{last}(b(x)) < \text{first}(b(y))\) or \(\text{last}(b(y)) < \text{first}(b(x))\), and thus their corresponding intervals do not intersect. From now on, we assume without loss of generality, that \(b(x)\) is an ancestor of \(b(y)\).

Let \(t = \theta(x)\). Note that by the definition of \(\theta\), \(0 \leq t \leq tw(G)\). We claim that \((x, y) \notin E(I_t)\). Since function \(b : V \to I\) is bijective (Lemma 5), \(b(x) \neq b(y)\), and thus, since \(b(x)\) is an ancestor of \(b(y)\), we have \(h(x) < h(y)\).

Now, if \(\theta(y)\) also equals \(t\), then the intervals corresponding to \(x\) and \(y\) do not intersect since \(h(x) \neq h(y)\) (see definition of \(I_t\)). From now on, we assume that \(\theta(y) \neq \theta(x)\).

Let \(S = \theta^{-1}(t) \cap N(y)\). If \(S = \emptyset\) then the interval corresponding to \(y\) in \(I_t\) is \([3n, 3n]\) by definition. Since \(r_t(x) = 2h(x) + 1 < 3n\), \((x, y) \notin E(I_t)\) as required. If \(S \neq \emptyset\) then let \(z\) be the node such that \(r_t(z) = \min_{w \in S} r_t(w)\). Note that \(x \neq z\) since \(x \notin N(y)\). Since \(z \in N(y)\), there is an ancestorial relation between \(b(z)\) and \(b(y)\) by Lemma 10 i.e., \(b(z)\) is an ancestor of \(b(y)\) or vice versa. Recalling that \(b(x)\) is an ancestor of \(b(y)\), it follows that \(there is a pair-wise ancestorial relation between b(x), b(y) and b(z)\). Noting that \(x \neq y \neq z\), by Lemma 5 \(b(x) \neq b(y) \neq b(z)\). It follows that \(h(x) \neq h(y) \neq h(z)\).

Let \(h(x) > h(z)\). Recalling that \(h(y) > h(x),\) we have \(h(y) > h(x) > h(z)\) Hence \(b(x)\) is in the path in \(T\) from \(b(y)\) to \(b(z)\), and thus by Lemma 11, it follows that \(\theta(x) \neq \theta(z)\). But recall that, \(z \in S \subseteq \theta^{-1}(t)\), by definition, and therefore \(\theta(z) = t = \theta(x)\), which is a contradiction. Hence the only possibility is that \(h(x) < h(z)\), and thus \(r_t(x) < r_t(z)\) by case (1) of definition of \(I_t\). Recall that \(r_t(z) = \ell_t(y)\) by case 2(b) of definition of \(I_t\). Hence we have \(r_t(x) < \ell_t(y)\), and thus \((x, y) \notin E(I_t)\).

By combining Lemma 12 and Lemma 13, we can infer that \(E(G) = E(I_1) \cap \cdots \cap E(I_{tw(G)+1})\). Thus, by Theorem 7, we obtain the following.
Theorem 14 For any graph $G$, $\text{box}(G) \leq \text{tw}(G) + 2$.

6 Tightness Result

When we consider various simple examples, it is tempting to conjecture that the tight upper bound on the boxicity is $\frac{\text{tw}(G)}{2}$. (For example, consider the Roberts graph explained in Section 7.2). But we show that the above upper bound is asymptotically tight. More precisely,

Theorem 15 For any integer $k \geq 1$, there exists a graph $G$ with $\text{tw}(G) \leq k$ and $\text{box}(G) \geq k \left(1 - \frac{2}{\sqrt{k}}\right) = k \left(1 - o(1)\right)$.

PROOF.

We show the following. Fix any $t \geq 1$. We construct a graph $G$ such that $\text{tw}(G) \leq t + \sqrt{t}$ and $\text{box}(G) \geq t - \sqrt{t}$. For any fixed $k$, we get the result by choosing $t$ to be the largest integer such that $t + \sqrt{t} \leq k$.

The graph $G$ is as follows. The node set of $G$ is the disjoint union of $\alpha + 1$ sets $P_0, P_1, \ldots, P_\alpha$, where

$$\alpha = \sum_{j=1}^{\lfloor \sqrt{t} \rfloor} \binom{t}{j}.$$ 

(Note that $\alpha$ corresponds to the total number of non-empty subsets of at most $\sqrt{t}$ elements from a collection of $t$ distinct elements.) Let $|P_0| = t$. The cardinality of $P_i$ for $1 \leq i \leq \alpha$ is defined as follows. Let $S = \{A \subseteq P_0 \mid 1 \leq |A| \leq \sqrt{t}\}$. Note that $|S| = \alpha$. Let $\Pi : S \to \{P_1, \ldots, P_\alpha\}$ be a bijective map. We define $|\Pi(A)| = |A|$. In other words, $|P_i| = |\Pi^{-1}(P_i)|$.

For $i \in \{1, \ldots, \alpha\}$, let $c_i : P_i \to \Pi^{-1}(P_i)$ be a bijection.

Edge $(u, v) \in E(G)$ if and only if any one of the following conditions holds

a) $u, v \in P_i$ for some $i$.

b) $u \in P_0$ and $v \in P_i$ for some $i \geq 1$ and $u \neq c_i(v)$.

Claim 16 $\text{tw}(G) \leq t + \lfloor \sqrt{t} \rfloor$.

The above claim can be seen as follows. Define a tree decomposition $(\{X_i : i \in I\}, T)$ of $G$ where $I = \{0, \ldots, \alpha\}$. Define $X_0 = P_0$ and $X_i = P_0 \cup P_i$ for $i \in \{1, \ldots, \alpha\}$. The edge set of $T$ is $\{(0, i) : 1 \leq i \leq \alpha\}$.

(Note that $T$ is a star with node 0 at the center). It is straightforward to verify that this is a valid tree-decomposition of $G$. Recalling that each $P_i, 1 \leq i \leq \alpha$, has at
most $\sqrt{t}$ nodes, it follows that the width of this decomposition is $t + \lfloor \sqrt{t} \rfloor$.

Claim 17 $\text{box}(G) \geq t - \lfloor \sqrt{t} \rfloor$.

Proof: Assume by contradiction that $\text{box}(G) < t - \lfloor \sqrt{t} \rfloor$. Then, consider an interval graph representation of $G$ using $\gamma = t - \lfloor \sqrt{t} \rfloor$ interval graphs. That is, let

$$E(G) = E(I_1) \cap \ldots \cap E(I_\gamma),$$

where $I_1, \ldots, I_\gamma$ are the $\gamma$ interval graphs. Fix any arbitrary interval realizations for $I_1, \ldots, I_\gamma$. From now on (abusing the terminology), we refer to this interval realization of $I_i$ also as $I_i$.

Let $P : \{I_1, \ldots, I_\gamma\} \rightarrow P_0 \times P_0$ be a function, where $P(I_j)$ is defined as follows. For node $w \in V$, let $[\ell_j(w), r_j(w)]$ denote its corresponding interval in $I_j$. Let $u, v \in V$ be such that $\ell_j(u) = \max_{w \in P_0} \ell_j(w)$ and $r_j(v) = \min_{w \in P_0} r_j(w)$ (resolving ties arbitrarily). Define $P(I_j) = (u, v)$. Recalling that $P_0$ induces a complete graph in $G$, it follows that for any node $w \in P_0$, $r_j(w) \geq \ell_j(u)$ (otherwise intervals corresponding to nodes $u$ and $w$ will not intersect). Thus $r_j(v) \geq \ell_j(u)$ and therefore $[\ell_j(u), r_j(v)]$ is a valid interval. Now it is straightforward to see that for any node $w \in P_0$, $[\ell_j(u), r_j(v)] \subseteq [\ell_j(w), r_j(w)]$. Define $P(I_j) = (u, v)$. Now it is easy to see that there cannot be a node $x \in P_k$, for any $k \geq 1$, that is adjacent to both $u$ and $v$ but not to some $y \in P_0$, in this interval graph. This is because, if $x$ is adjacent to both $u$ and $v$, then the interval corresponding to $x$ has a non-empty intersection with $[\ell_j(u), r_j(v)]$, and thus it has a non-empty intersection with the interval for any node $y \in P_0$. This is summarized as follows.

Claim 18 Consider any interval graph $I \in \{I_1, \ldots, I_\gamma\}$. Let $P(I) = (u, v)$. Let $x \in P_k$ for any $k \in \{1, \ldots, \alpha\}$. If $c_k(x) \notin \{u, v\}$ then edge $(x, c_k(x)) \in E(I)$.

Define a multi-graph $H = (V_H, E_H)$, where

$$V_H = P_0 \quad \text{and the multi-set} \quad E_H = \{P(I_1), P(I_2), \ldots, P(I_\gamma)\}.$$

Note that $H$ has $t$ nodes and $\gamma$ edges.

Applying Lemma 20 to $H$ by fixing $r = \lfloor \sqrt{t} \rfloor$, we infer the following. The multi-graph $H$ has a connected component $K = (V_K, E_K)$ on $k$ nodes and exactly $k - 1$ edges, where $1 \leq k \leq \lfloor \sqrt{t} \rfloor$. Let

$$S = \{I \mid P(I) \in E_K\}. \quad \text{Clearly} \quad |S| = k - 1.$$

Let $\Pi(V_K) = P_k$. Define a function $f : P_k \rightarrow \{I_1, \ldots, I_\gamma\}$ such that for $x \in P_k$, $f(x) = I_j$ where $I_j$ is an interval graph such that $(x, c_k(x)) \notin E(I_j)$. (Reader may
note that there exists one such interval graph because \((x, c_k(x)) \notin E(G)\). On the other hand, there can be more than one interval graph where the edge \((x, c_k(x))\) is not present. The function \(f(x)\) maps \(x\) to one such interval graph.)

**Claim 19** For any \(x \in P_k = \Pi(V_K), \ f(x) \in S\).

Proof: Let \(I \in \{I_1, \ldots, I_g\} - S\) and let \(P(I) = (u, v)\). Since \((u, v) \in E(H)\), both \(u\) and \(v\) belong to the same connected component of \(H\). We claim that \(u, v \notin V_K\). Otherwise, \((u, v) = P(I) \in E_K\), and hence by definition of \(S\), \(I \in S\), a contradiction. Recall that \(P_k = \Pi(V_K)\) and thus for any \(x \in P_k\), \(c_k(x) \in V_K\) by definition of the function \(c_k(\cdot)\). Therefore \(c_k(x) \notin \{u, v\}\). It follows by Claim 18, that \((x, c_k(x)) \in E(I)\) and therefore \(f(x) \neq I\) by the definition of \(f(x)\). The claim follows.

Recall that \(|V_K| = k\), where \(1 \leq k \leq \lfloor \sqrt{t} \rfloor\). It follows that \(|P_k| = k\) since \(P_k = \Pi(V_K)\) and recalling that \(|\Pi(V_K)| = |V_K|\) by definition. But recall that \(|S| = k - 1\). By Claim 19, for any \(x \in P_k\), \(f(x) \in S\). It follows (by pigeon hole principle) that there exists \(x, y \in P_k\) such that \(f(x) = f(y) = I_z \in S\). By definition of graph \(G\), it contains the four cycle \((x, y, c_k(x), c_k(y), x)\). Since \(E(I_z) \supseteq E(G)\), the same four cycle is present in \(I_z\) also. But by the definition of \(f(\cdot), (x, c_k(x)), (y, c_k(y)) \notin I_z\). Thus it follows that the above four cycle is chordless in \(I_z\), which is a contradiction since \(I_z\) is an interval graph. It follows that \(box(G) \geq t - \lfloor \sqrt{t} \rfloor\). 

**Lemma 20** If a multi-graph \(M\) has \(n\) nodes and at most \(n - \frac{n}{r}\) edges, for some \(r \geq 1\), then there is a connected component \(C\) in \(M\) that has \(k\) nodes and exactly \(k - 1\) edges, for some \(k\) with \(1 \leq k \leq r\).

**PROOF.** Consider those connected components in \(M\) where each of them have at least \(r+1\) nodes. Call them ‘large’ connected components. A connected component which is not large is called ’small’ component. Let \(g\) and \(h\) respectively denote the number of large and small connected components. Let \(n_1, \ldots, n_g \) respectively be the number of nodes in each of these \(g\) large connected components. Let \(n_L = n_1 + \cdots + n_g\) denote the total number of nodes in the \(g\) large components. Let the total number of edges in these \(g\) connected components together be denoted as \(m_L\). Observe that \(m_L \geq n_L - g\). Let \(n_S\) and \(m_S\) respectively be the total number of nodes and edges in the \(h\) small connected components. We have,

\[n_S = n - n_L \quad \text{and} \quad m_S \leq n - \frac{n}{r} - m_L\]

Recalling that \(m_L \geq n_L - g\), we get \(n_S \geq m_S + \frac{n}{r} - g\). Since each large component contains at least \(r+1\) nodes, we have \(g < \frac{n}{r}\). It follows that \(n_S > m_S\). Thus we can infer that there exists at least one small connected component \(C\), on \(k\) nodes and exactly \(k - 1\) edges, as required. 

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7 Consequences on Special Classes of Graphs

7.1 Chordal Graphs

Let $C$ be a cycle in a graph $G$. A chord of $C$ is an edge of $G$ joining two nodes of $C$ which are not consecutive. A graph $G$ is called a chordal (rigid circuit or triangulated) graph if and only if every cycle in $G$, of length 4 or more has a chord. Chordal graphs arise in many applications (see [19]). Chordal graphs constitute one of the most important subclasses of perfect graphs [19]. It is easy to see that the class of interval graphs do not have chordless cycles of length more than 3, and thus is a subclass of chordal graphs. Thus it is natural to study the boxicity of chordal graphs. In fact, the question of bounding the boxicity of a subclass of chordal graphs, namely the split graphs, was already addressed by Cozzens and Roberts [15]. A graph $G$ is a split graph if and only if $G$ and its complement $\overline{G}$ is chordal. Every split graph has the following special structure: Its node set can be partitioned into two sets $V_1$ and $V_2$ such that $V_1$ induces a complete graph in $G$, and $V_2$ induces an independent set. (See [19], Chapter 6 for more information on split graphs.) Cozzens and Roberts prove the following Theorem.

**Theorem 21 (Cozzens and Roberts [15])** If $G$ is a split graph with clique number $\omega(G)$, then $\text{box}(G) \leq \left\lceil \frac{1}{2} \omega(G) \right\rceil$.

The class of split graphs indeed has a very special structure, and is hardly representative of the much wider class of chordal graphs. Using our upper bound (Theorem 14), we derive an upper bound for the boxicity of chordal graphs.

The following result is well-known. (See [16], Chapter 12.)

**Lemma 22** For a chordal graph $G$, $\text{tw}(G) = \omega(G) - 1$.

Combining Lemma 22 with Theorem 14, and noting that $\omega(G) - 1 \leq \Delta(G)$ for any graph $G$, where $\Delta(G)$ is its maximum degree, we get the following.

**Theorem 23** For a chordal graph $G$, $\text{box}(G) \leq \omega(G) + 1 \leq \Delta(G) + 2$.

**Sharpness of Theorem 23** In [15], Cozzens and Roberts show that for any $k \geq 1$, there exists a split graph $G$ such that $\omega(G) = k$ and $\text{box}(G) = \left\lceil \frac{\omega(G)}{2} \right\rceil$. Since the class of split graphs is subclass of chordal graphs, it follows that the upper bound of Theorem 23, is tight up to a factor of 2.
7.2 Circular Arc graphs

A graph $G$ is a circular arc graph if and only if there exists a one-to-one correspondence between its nodes and a set of arcs of a circle, such that two nodes are adjacent if and only if the corresponding arcs intersect. Since the definition of circular arc graphs look very similar to that of interval graphs, it is natural to think that their boxicity will be small (possibly bounded above by a constant), since interval graphs are boxicity 1 graphs. In his pioneering paper [26], F.S. Roberts considers the following graph.

**Definition 24 (Roberts Graph)** The Roberts graph on $2n$ nodes is obtained by removing the edges of a perfect matching from a complete graph on $2n$ nodes.

Roberts showed that the boxicity of Roberts graph on $2n$ nodes is $n$. A little inspection will convince the reader that Roberts graph is indeed a circular arc graph. Thus, there exists a circular arc graph of $2n$ nodes, whose boxicity is $n$.

But still, it is possible to get an upper bound for the boxicity of circular arc graphs in terms of clique number $\omega(G)$ and thus in terms of its maximum degree $\Delta(G)$ as follows.

We claim that the pathwidth (and hence treewidth) of a circular arc graph $G$ is at most $2\omega(G) - 1$. Consider a representation of $G$ as the intersection graph of arcs of a circle. Let $p_0, \ldots, p_k$, be the end points (left or right) of the arcs on this circle, as we traverse the circle in the clock-wise direction, starting from an arbitrarily fixed position. Let $X_i$ denote the set of nodes of $G$, whose arcs contain $p_i$. Clearly, $X_i$ for any $i$ induces a complete graph in $G$ and thus $|X_i| \leq \omega(G)$. It is straightforward to verify that $(X_1 \cup X_0), \ldots, (X_k \cup X_0)$ constitutes a valid path decomposition of $G$ and thus the pathwidth of $G$ is at most $2\omega(G) - 1$.

It follows from Theorem 14 that

**Theorem 25** For a circular arc graph $G$,

$$box(G) \leq 2\omega(G) + 1 \leq 2\Delta(G) + 3.$$  

**Tightness of Theorem 25:** Recall that Roberts graph $G$ on $2n$ nodes (Definition 24) is a circular arc graph and its boxicity is $n$. It is easy to see that $\omega(G) = n$. Thus the upper bounds given by Theorem 25, in terms of $\omega(G)$ and $\Delta(G)$, is tight up to a factor 2 and 4 respectively.
7.3 Asteroidal Triple-free Graphs, Co–comparability Graphs and Permutation Graphs

An independent set of three nodes in $G$, such that each pair is joined by a path that avoids the neighborhood of the third is called an Asteroidal Triple (AT). A graph is AT-free if and only if it contains no asteroidal triples. The concept of Asteroidal triples and AT-free graphs was introduced by Lekkerkerker and Boland [24], to characterize the chordal graphs which are not interval graphs. They showed that a graph $G$ is an interval graph if and only if, it is simultaneously a chordal graph and an AT-free graph.

AT-free graphs generalize (in addition to the class of interval graphs) some very important and practically useful classes of graphs- for example co-comparability graphs, trapezoidal graphs, and the permutation graphs. (See [13] for a discussion of how AT-free graphs are in some sense a unifying generalization of these graph classes.)

In this section we give an upper bound for the boxicity of AT-free graphs in terms of their maximum degree.

A caterpillar is a tree such that a path (called the spine) is obtained by when all its leaves are deleted. In the proof of Theorem 3.16 of [22], Kloks et al. show that every connected AT-free graph $G$ has a spanning caterpillar subgraph $T$, such that adjacent nodes in $G$ are at distance at most four in $T$. Moreover, for any edge $(u, v) \in E(G)$ with $u$ and $v$ at distance exactly four in $T$, both $u$ and $v$ are leaves of $T$. Let $p_0, \ldots, p_k$ be the nodes along the spine of $G$. Let $X_i$ be the union of $p_i$ and the leaf nodes attached to $p_i$ in the caterpillar. Now it is easy to check that $(X_0 \cup X_1 \cup X_2), \ldots, (X_i \cup X_{i+1} \cup X_{i+2}), \ldots, (X_{k-2} \cup X_{k-1} \cup X_k)$ constitute a path decomposition (and thus a tree decomposition) of $G$.

**Lemma 26** Let $G$ be an AT-free graph. Then $tw(G) \leq 3\Delta(G) - 2$, where $\Delta(G)$ is the maximum degree of $G$.

**Theorem 27** For an AT-free graph $G$, $box(G) \leq 3\Delta(G)$.

We get better upper bounds when we restrict ourselves to sub classes of AT-free graphs. Consider the class of co–comparability graphs: A graph $G$ is a co-comparability graph if and only if its complement is a comparability graph. (See Chapter 5 of [19], for more information on comparability graphs.) An interesting characterization of co–comparability graphs is that they are exactly the class of intersection graphs of function diagrams [20]. (A function diagram is a set of curves $C$, where each $c_i \in C$ is a curve $\{(x, f_i(x)) : 0 \leq x \leq 1\}$ for some $f_i : [0, 1] \to \mathbb{R}$.)

It is known that co-comparability graphs are properly contained in the class of AT-free graphs, but in turn is a strict super class of permutation graphs, trapezoidal
Lemma 28  For a co-comparability graph $G$, $tw(G) \leq 2\Delta(G) - 1$.

PROOF. Let $E(G)$ and $E(\overline{G})$ denote the edge set of $G$ and its complement $\overline{G}$ respectively and let $V$ be the node set. Let $|V| = n$. Since $\overline{G}$ is a comparability graph, there exists a partial order $\prec$ in $\overline{G}$ on the node set $V$ such that $(u, v) \in E(\overline{G})$ if and only if $u$ and $v$ are comparable (that is either $u \prec v$ or $v \prec u$). This partial order gives an orientation to the edge set $E(\overline{G})$, namely, if $u \prec v$, then the edge $(u, v)$ is directed from $u$ to $v$ and we denote this directed edge as $[u, v]$. Define an ordering (i.e., a bijection) $f : V \rightarrow \{1, \ldots, n\}$ for $V$ such that if $(u, v) \in E(\overline{G})$ then $u \prec v$ if and only if $f(u) < f(v)$. Clearly such an ordering exists for $\overline{G}$; for instance, a topological sort on $\overline{G}$ after orienting its edges as described above, gives such an ordering. Let $(u, v) \in E(G)$ and $w$ be such that $f(u) < f(w) < f(v)$. We claim that $w$ is adjacent to either $u$ or $v$ or both in $G$. Assume otherwise. That is, $(u, w), (w, v) \in E(\overline{G})$. Since it is given that $f(u) < f(w) < f(v)$, it follows that $u \prec w \prec v$ in $\overline{G}$ by the definition of $f(\cdot)$. Thus by transitivity of $\prec$, $u \prec v$ and $(u, v) \in E(\overline{G})$, which is a contradiction. Having shown that if $f(u) < f(w) < f(v)$ then $w$ is adjacent to either $u$ or $v$ or both in $G$, it is easy to infer that if edge $(u, v) \in E(G)$, then there can be at most $2\Delta(G) - 2$ nodes whose $f(\cdot)$ values are between $f(u)$ and $f(v)$. Therefore, $|f(u) - f(v)| \leq 2\Delta(G) - 1$. Now it is easy to verify that there is a path decomposition for $G$ (and hence a tree decomposition) $\{\{X_i : i \in I\}, T\}$, where $I = \{1, \ldots, n\}$ and $T$ is a simple path $(1, 2, \ldots, n)$, such that $X_i = \{u \mid i \leq f(u) \leq i + 2\Delta(G) - 1\}$. It is straightforward to verify that the pathwidth of this path decomposition is $2\Delta(G) - 1$.

Now it follows from our upper bound (Theorem 14) that

Theorem 29  For a co–comparability graph $G$, $box(G) \leq 2\Delta(G) + 1$.

Permutation graphs are defined as follows. Let $\pi$ be a permutation of the numbers $1, 2, \ldots, n$, and let graph $G[\pi] = (V, E)$ be defined as follows: $V = \{1, \ldots, n\}$ and $(i, j) \in E(G)$ if and only if $(i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$. A graph $G$ is a permutation graph if there exists a permutation $\pi$ such that $\overline{G}$ is isomorphic to $G[\pi]$. It is well-known that the complement of a permutation graph is also a permutation graph (See Chapter 7, [19]).

Permutation graphs are sub-classes of co-comparability graphs (See Chapter 7, [19]). Therefore the above upper bound on boxicity holds also for permutation graphs.

Theorem 30  For a permutation graph $G$, $box(G) \leq 2\Delta(G) + 1$.

Tightness of Theorems 27, 29 and 30: It is not difficult to see that Roberts graph on $2n$ nodes (Definition 24) is a permutation graph (because its complement is
trivially a permutation graph), and hence it is both AT-free and co-comparability. This graph has maximum degree $2n - 1$ and boxicity $n$. It follows that Theorem 29 and Theorem 30 are tight up to a factor of 4 and Theorem 27 is tight up to a factor of 6.

8 Planar Graph Minors and Boxicity

The following theorem is well-known.

**Theorem 31 (Robertson and Seymour [27])** For every planar graph $H$, there is a constant $c(H)$ such that every graph with treewidth $\geq c(H)$ has a minor isomorphic to $H$.

Combining the upper bound theorem (Theorem 14) with Theorem 31, we obtain the following.

**Theorem 32** For every planar graph $H$, there is a constant $c(H)$ such that every graph with boxicity $\geq c(H)$ has a minor isomorphic to $H$.

For instance, consider the cycle graph. Note that if a cycle graph on $k$ nodes is a minor of $G$, then it is also a subgraph of $G$. In other words, by Theorem 32, there exists a constant $c(k)$ such that, if $\text{box}(G) \geq c(k)$ then $G$ contains a cycle on $k$ nodes as a subgraph. In the next section, we show that $c(k)$ is not more than $2k$ by means of a direct approach.

A restatement of Theorem 32 is as follows.

**Theorem 33** For every planar graph $H$, there is a constant $c(H)$ such that any minor closed family of graphs which excludes $H$ has boxicity at most $c(H)$.

9 Cycles and Boxicity

The study of various kinds of circuits in graphs is a well-established area in graph theory. There is indeed an extensive literature on this topic. (Chapter 1 of [9] gives an introductory survey. See the book by Voss [32] for an extensive treatment.)

A recent result of Birmele [4] relates the treewidth of a graph with the length of the longest simple cycle in $G$. The length of the longest simple cycle in a graph is also known as its circumference.

**Lemma 34 (E. Birmele [4])** For any graph $G$, its circumference is at least $\text{tw}(G) - 1$. 

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Combining Lemma 34 with our upper bound Theorem 14, we get

**Theorem 35** In any graph $G$ of boxicity $b$, there exists a simple cycle of length at least $b - 3$.

**Sharpness of Theorem 35:** It is not difficult to see that Roberts graph on $2n$ nodes is Hamiltonian. Since its boxicity is known to be $n$, the upper bound for Theorem 35 is tight up to a factor of 2.

Now we consider induced cycles in a graph. A cycle in a graph is called an induced cycle or a chordless cycle if there are no chords for that cycle. An induced cycle of length 4 or more is sometimes referred to as a hole, especially in the perfect graph literature: For example, the strong perfect graph theorem states that a graph is perfect if and only if it does not contain any odd holes. The length of the largest induced cycle in a graph $G$ is called the chordality of $G$. It may be noted that the chordality of a chordal graph is 3, that of co-comparability graphs is at most 4 and that of AT-free graphs is at most 5. See [10] for a survey of graph classes with low chordality. A great deal of research is done also with respect to the existence of cycles in a graph with a given number of chords (diagonals) [32].

The following Lemma follows from Theorem 14 of [8].

**Lemma 36 (Bodlaender and Thilikos [8])** Let $G$ be a graph with maximum degree $\Delta$ and chordality $c$. Then $tw(G) \leq \Delta^{c-2}$.

By combining Lemma 36 with our upper bound (Theorem 14), we get the following result

**Theorem 37** Let $G$ be a graph with maximum degree $\Delta$ and boxicity $b \geq 3$. Then there exists a induced cycle (chordless cycle) of length at least $\lfloor \log_\Delta(b - 2) \rfloor + 2$.

### 10 Boxicity, Vertex Cover and Related Parameters

The subset $S \subseteq V(G)$ is called a *vertex cover* of $G$ if every edge of $G$ is incident on at least one vertex from $S$. A vertex cover of minimum cardinality is called a *minimum vertex cover*. We denote the cardinality of a minimum vertex cover of $G$ by $MVC(G)$. It is easy to observe that if $S$ is a minimum vertex cover of $G$, then $V(G) - S$ forms an independent set of $G$. Thus, $MVC(G) = n - \alpha(G)$, where $\alpha(G)$ is the independence number (the cardinality of the maximum independent set) of $G$. It is easy to prove the following Lemma.

**Lemma 38** For any graph $G$, $tw(G) \leq MVC(G)$.

Now, applying Theorem 14 we get:
Theorem 39  For any graph $G$, $box(G) \leq MVC(G) + 2 = n - \alpha(G) + 2$.

It is interesting to investigate whether the above bound in terms of $MVC(G)$ can be further tightened. For instance, we can show that if a graph $G$ has a vertex cover which induces a complete graph, then $box(G) \leq \lceil (MVC(G) + 1)/2 \rceil$. To see this, first we recall from Section 7.1 that if the node set $V$ of a graph $G$ can be partitioned into $V_1$ and $V_2$ such that $V_1$ induces a complete graph and $V_2$ induces an independent set, then $G$ is a split graph. It follows that if a vertex cover in $G$ induces a complete graph, then $G$ is a split graph. Now the above bound follows from Theorem 21, because $\omega(G) - 1 \leq MVC(G) \leq \omega(G)$ for a split graph $G$.

A set of dominating edges $D$ is a collection of edges of $G$ such that any edge in $E(G) - D$ is adjacent to at least one edge in $D$. For example the reader may notice that any maximal matching in $G$ constitutes a dominating edge set. A dominating edge set of minimum cardinality is called a minimum dominating edge set. We denote the cardinality of the minimum dominating edge set of $G$ by $MED(G)$. It is easy to see that $MVC(G) \leq 2 MED(G)$. Combining this with Theorem 39, we have

Theorem 40  For any graph $G$, $box(G) \leq 2 MED(G) + 2$.

In this connection, we note that Cozzens and Roberts [15] had proved the following: for any graph $G$, $box(G) \leq MED(G)$. Clearly our result (Theorem 40) complements their result, by showing that $MED(G)$ itself can control the boxicity of $G$. Thus,

Theorem 41  For any graph $G$, $box(G) \leq \min\{2 MED(G) + 2, MED(G)\}$.

Now, let us consider a variant of minimum vertex cover, namely the minimum feedback vertex cover. A feedback vertex cover $S$ is a subset of $V(G)$ such that the induced subgraph on $V - S$ is a forest. A feedback vertex cover of minimum cardinality is called a minimum feedback vertex cover, and we denote its cardinality by $MFVC(G)$. Clearly every vertex cover of $G$ is a feedback vertex cover also, and thus $MFVC(G) \leq MVC(G)$. The reader may also note that in general $MFVC(G)$ can be much smaller than $MVC(G)$: For example for a cycle on $n$ nodes, $\frac{MVC(G)}{MFVC(G)} = \Omega(n)$.

Theorem 42  $box(G) \leq MFVC(G) + 3$.

PROOF. Let $S$ be a minimum feedback vertex cover of $G$. Since the induced subgraph on $V - S$ is a forest, there exists a tree decomposition $({\{X_i : i \in I}\}, T)$, whose width is 1. Clearly, $({\{X_i \cup S : i \in I\}, T})$ is a valid tree decomposition of $G$ whose width is $|S| + 1 = MFVC(G) + 1$. Now, applying the upper bound theorem (Theorem 14), the result follows.
**Sharpness of Theorems 40 and 42:** It is easy to check that for Roberts’ graph \( G \) on \( 2n \) nodes (see Definition 24), \( MVC(G) = 2n - 2 \), whereas its boxicity is \( n \). Thus, the upper bound of Theorem 40 is tight up to a factor of 2. Similarly, it is easy to see that \( MFVC(G) \geq 2n - 4 \), and thus the upper bound of Theorem 42 is also tight up to a factor of 2.

11 Algorithmic Consequence

**Theorem 43** *For a bounded treewidth graph \( G = (V, E) \) on \( n \) nodes, a box representation of \( G \) in constant dimension can be constructed in \( O(n) \) time.*

**PROOF.** We construct the interval graph representation of \( G \) using \( tw(G) + 2 \) interval graphs \( I_0, \ldots, I_{tw(G)+1} \), as described in the proof of Theorem 14. It is not difficult to observe that the proof of Theorem 14 is constructive. It remains to show that this construction can be implemented in linear time when \( tw(G) \) is bounded. (Recall from Section 4 that the interval graph representation of \( G \) is equivalent to its box representation.)

It is well-known (see for instance [6]) that if \( tw(G) \leq k \) then \( |E(G)| \leq kn - \frac{1}{2}k(k + 1) \).

We convert the constructive proof of Theorem 14 into a linear time algorithm consisting of the following steps. We show that each of these steps can be implemented in linear time.

1. Given a bounded treewidth graph \( G \), Bodlaender [6] gives an \( O(n) \) algorithm to construct the optimum tree decomposition \( (\{X_i : i \in I\}, T) \) of \( G \). (In this tree decomposition, \( |I| = O(n) \).)
2. Convert this tree decomposition into a normalized tree decomposition \( (\{X_i : i \in I\}, r, T) \) as described in the proof of Lemma 3. It is straightforward to verify that this conversion takes \( O(n) \) time. It is also easy to see that, while doing this conversion, we can additionally obtain the following, without increasing the time complexity:
   a. An ordering of \( I \), sorted in the non-increasing order of their heights in the rooted tree \( T \).
   b. The \( b(u) \) and \( h(u) \) values for each \( u \in V \).
3. Compute \( \theta(u) \) for each \( u \in V \) as described in the proof of Lemma 8. The sorted order of \( I \) as required in this proof is already computed in Step 2. The remaining steps in this proof can be implemented in a straightforward way such that it takes only constant time for computing \( \theta(u) \) for each node \( u \in V \). Thus the total time is \( O(n) \).
(4) Now we construct the interval graphs $I_0, \ldots, I_{tw(G)}$. To construct $I_i$, $0 \leq i \leq tw(G)$, we need to compute $\ell_i(v)$ and $r_i(v)$ for each $v \in V$, as described in Section 5. If $\theta(v) = i$ then computing $\ell_i(v)$ and $r_i(v)$ is trivial. If $\theta(v) \neq i$, then first we have to compute the set $S = \theta^{-1}(i) \cap N(v)$. It is easy to see that this can be done in $O(|N(v)|)$ time. After this, we have to compute $\min_{u \in S} r_i(u)$. This takes additional $O(|S|)$ time. Thus the total time taken for node $v$ is $O(|N(v)| + |S|) = O(|N(v)|)$. Hence the overall time taken to construct $I_i$ is $O(|E(G)|) = O(n)$. (Recall that for a bounded treewidth graph $G$, $O(|E(G)|) = O(n)$.)

(5) It remains to construct the interval graph $I_{tw(G)+1}$ as described in Section 5. For this, it is required to compute the depth first traversal order of $T$. It is trivial to see that such an ordering can be computed in $O(|I|) = O(n)$ time. Assigning for each $u \in V$, its corresponding sequence numbers in the traversal order to $\ell_i(u)$ and $r_i(u)$ can be done without increasing the time complexity.

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