ON THE MULTICOLOR RAMSEY NUMBER OF A GRAPH WITH m EDGES

KATHLEEN JOHST* AND YURY PERSON

Abstract. The multicolor Ramsey number \( r_k(F) \) of a graph \( F \) is the least integer \( n \) such that in every coloring of the edges of \( K_n \) by \( k \) colors there is a monochromatic copy of \( F \). In this short note we prove an upper bound on \( r_k(F) \) for a graph \( F \) with \( m \) edges and no isolated vertices of the form \( k^{6m^{2/3}} \) addressing a question of Sudakov [Adv. Math. 227 (2011), no. 1, 601–609]. Furthermore, the constant in the exponent in the case of bipartite \( F \) and two colors is lowered so that \( r_2(F) \leq 2^{(1+o(1))2^{1/2}m^{2/3}} \) improving the result of Alon, Krivelevich and Sudakov [Combin. Probab. Comput. 12 (2003), no. 5–6, 477–494].

1. Introduction

The by now classical theorem of Ramsey [11] states that no matter how one colors the edges of the large enough complete graph \( K_n \) with two colors, say red and blue, there will always be a monochromatic copy of \( K_t \) in it. The smallest such \( n \) is called the Ramsey number, denoted by \( r(t) \) or \( r(K_t) \). First lower and upper bounds on \( r(t) \) were obtained by Erdős and Szekeres [8]: \( r(t) \leq \left( \frac{2t-2}{t-1} \right) \) and by Erdős [6]: \( r(t) \geq 2^{t/2} \). Despite numerous efforts by various researchers, the best lower and upper bounds remain asymptotically \( 2^{(1+o(1))t/2} \) and \( 2^{(1+o(1))2t} \), for the currently best bounds see Conlon [5] and Spencer [12].

Thus, one turned to the study of Ramsey numbers of graphs other than complete graphs \( K_t \). The multicolor Ramsey number for \( k \) colors of a graph \( F \), denoted \( r_k(F) \), is defined as the smallest number \( n \) such that in any coloring of \( E(K_n) \) by \( k \) colors there is a monochromatic copy of a graph \( F \) in one of the \( k \) colors. Much attention was drawn by the conjectures of Burr and Erdős [2] about Ramsey numbers of graphs \( F \) whose maximum degree is bounded by a constant and which are \( d \)-degenerate for some constant \( d \) stating that these Ramsey numbers are linear in \( r(F) := |V(F)| \). While the first conjecture has been resolved positively by Chvátal, Rödl, Szemerédi and Trotter [4], the latter one is still open and the best bound is due to Fox and Sudakov [10] being \( r_2(F) \leq 2^{c_d \sqrt{m} n} \) for \( c_d \) depending on \( d \) only.

A related conjecture of Erdős and Graham [7] states that among all graphs \( F \) with \( m = \binom{t}{2} \) edges and no isolated vertices the Ramsey number \( r(t) \) of the complete graph \( K_t \) is an upper bound on \( r_2(F) \). A relaxation conjectured by Erdős [3] states that at least \( r(F) \leq 2^{c \sqrt{m}} \) should hold for any graph \( F \) with \( m \) edges and no isolated vertices and some absolute constant \( c \). This was verified by Alon, Krivelevich and Sudakov [1] who showed that if \( F \) is bipartite, has \( m \) edges and no isolated vertices

\* This paper forms part of the first author’s Master’s thesis at Freie Universität Berlin.

Date: November 26, 2013.
then \( r(F) \leq 2^{16\sqrt{m}+1} \), and for nonbipartite \( F \) showing \( r(F) \leq 2^{7\sqrt{m}\log_2 m} \). Finally, the general case was settled by Sudakov [14] who proved \( r(F) \leq 2^{250\sqrt{m}} \).

In his concluding remarks in [14], Sudakov mentions that the methods used to settle the general case are not extendible to more colors and it would be interesting to understand the growth of \( r_k(F) \). It is easy to see that there is an upper bound on \( r_k(F) \) of the form \( k^{4v(F)} \) by finding a monochromatic copy of \( K_{2m} \supset F \) using the classical color focussing argument. In this note we prove to the best of our knowledge a first nontrivial upper bound \( r_k(F) \leq k^{6km^{2/3}} \).

**Theorem 1.** Let \( F \) be a graph with \( m \) edges and no isolated vertices. Then, for \( k \geq 3 \) it holds

\[
r_k(F) \leq k^{3.2^{-1/3}km^{2/3}+k(2m)^{1/3}} 8m.
\]

Further we study the case when \( F \) is bipartite and show an upper bound \( r_k(F) \leq k^{(1+o(1))2\sqrt{mk}} \).

**Theorem 2.** Let \( F \) be a bipartite graph with \( m \) edges and no isolated vertices. Then, for \( k \geq 2 \) it holds

\[
r_k(F) \leq 2^{6m^{8/3}k^{2\sqrt{km}+1/2}}.
\]

Note that in the case \( k = 2 \), Theorem 2 is an improvement of the above mentioned result of Alon, Krivelevich and Sudakov [1] to \( r(F) \leq 2^{(1+o(1))2\sqrt{km}} \). Remarkably, this upper bound is asymptotically the “same” as the upper bound \( 2^{(1+o(1))2k} \) for \( r(k) \) with \( m = \binom{k}{2} \).

The methods we use are slight modifications of the arguments of Fox and Sudakov from [9] and of Alon, Krivelevich and Sudakov [1]. The paper is organized as follows. In the next section, Section 2 we collect some results and observations used in our proofs, in Section 3 we prove Theorem 2 and in Section 4 we show Theorem 1.

2. Some auxiliary results

Here we collect several results from [9] and one small graph theoretic estimate. The first prominent lemma we use is the so-called dependent random choice lemma, stating that in a bipartite dense graph one finds a large vertex subset in one class, most of whose \( d \)-tuples have many common neighbours on the other side.

**Lemma 3** (Dependent Random Choice, Lemma 2.1[9]). If \( \varepsilon > 0 \) and \( G = (V_1, V_2; E) \) is a bipartite graph with \( |V_1| = |V_2| = N \) and at least \( \varepsilon N^2 \) edges, then for all positive integers \( a, d, t, x \), there is a subset \( A \subset V_2 \) with \( |A| \geq 2^{-\frac{t}{2}} x \) \( N \), such that for all but at most \( 2\varepsilon^{-ta} \left( \frac{N}{a} \right)^t \binom{N}{d} \left( \frac{a}{N} \right)^a \) \( d \)-sets \( S \) in \( A \), we have \( |N(S)| \geq x \).

The following lemma allows one to embed a graph \( H \) with bounded degree and bounded chromatic number into a dense graph \( G \) given along with a nested sequence of sets, where the parts of \( H \) are supposed to be embedded into.

**Lemma 4** (Lemma 4.2 in [9]). Suppose \( G \) is a graph with vertex set \( V_1 \), and let \( V_1 \supset \ldots \supset V_q \) be a family of nested subsets of \( V_1 \) such that \( |V_q| \geq x \geq 4n \), and for \( 1 \leq i < q \), all but less than \( (2d)^{-d} \left( \frac{x}{4} \right)^n \) \( d \)-sets \( U \subset V_{i+1} \) satisfy \( |N(U) \cap V_i| \geq x \). Then, for every \( q \)-partite graph \( H \) with \( n \) vertices and maximum degree at most \( \Delta(H) \leq d \), there are at least \( \left( \frac{q}{4} \right)^n \) labeled copies of \( H \) in \( G \).
We will also need the following Turán-type result, from which the currently best known upper bound on the Ramsey number of a bounded degree bipartite graph follows.

**Theorem 5** (Theorem 1.1 from [9]). Let $H$ be a bipartite graph with $n$ vertices and maximum degree $\Delta \geq 1$. If $\varepsilon > 0$ and $G$ is a graph with $N \geq 32\varepsilon^{-\Delta} n$ vertices and at least $\varepsilon \binom{N}{2}$ edges, then $H$ is a subgraph of $G$.

Finally we need one simple observation, whose proof we provide here for completeness.

**Proposition 6.** Let $F = (V, E)$ be a graph with $m$ edges. Then there exists a subset $U \subseteq V$ with $|U| < \frac{m}{d}$ such that $\Delta(F \setminus U) \leq d$.

**Proof.** Let $v_1$ be a vertex of maximum degree in $F_1 := F$ and set $d_1 := \Delta(F)$. We delete $v_1$ from $F$ denoting the new graph by $F_2$. We proceed inductively, deleting from $F_i$ a vertex of maximum degree $v_i$, setting $d_i := \Delta(F_i)$ and defining the new graph $F_{i+1} := F_i - v_i$ and stop with $F_{|V(F)|+1} = \emptyset$. Let $j$ be the smallest integer with $\Delta(F_{j+1}) \leq d$. Thus, till we obtained $F_{j+1}$ we must have deleted $j$ vertices, each of degree larger than $d$. Moreover, by the construction of the sequence of $v_i$s, we have $m = |E(F)| = \sum_{i=1}^{|V(F)|} \Delta(F_i)$. Therefore, $jd < m$ and the claim follows with $U := \{v_1, \ldots, v_j\}$. \qed

Often we try to avoid using floor and ceiling signs as they will not affect our calculations.

**3. Multicolor Ramsey number of bipartite graphs with $m$ edges**

The idea of the proof of Theorem 2 is quite simple. Given a coloring of $E(K_N)$, we will perform a color focussing argument by considering the densest color class and taking a vertex with maximum degree in it. Then we iterate on the colored neighborhood of that vertex. After less than $km/d$ steps we arrive at the situation, where we can embed all $m/d$ vertices from $U$ (of high degree in $F$) onto the vertices specified in the focussing process, the remaining graph $F - U$ has maximum degree at most $d$ (by Proposition 6) and is bipartite, and thus can be embedded in one round, by Theorem 5.

**Proof of Theorem 2.** Given a bipartite graph $F$ with $m$ edges and no isolated vertices. We choose with foresight $d = \sqrt{km}$. By Proposition 6, let $U$ be a set of $t = \lfloor m/d \rfloor = \lfloor \sqrt{m/k} \rfloor$ vertices such that $\Delta(F \setminus U) \leq d$. Further observe that $|V(H)| \leq 2m$.

Let us be given an arbitrary but fixed $k$-edge coloring of the graph $G := K_N$ with the colors $1, \ldots, k$, where $N = 32dk^{d+kt}m$.

We will construct a sequence of sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_t$ and a sequence of colors $c(1), \ldots, c(s)$ as follows. First we set $A_1 = [N]$ and let $c(1)$ be a densest color in $G[A_1]$. Since we used $k$ colors there exists a vertex $v_1 \in A_1$ connected to at least $\frac{|A_1|-1}{k}$ vertices in color $c(1)$. We denote the set of these vertices by $A_2$. Then we proceed inductively as follows. Given a sequence of sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_{i+1}$ and the corresponding sequences of vertices $v_1, v_i$ and colors $c(1), \ldots, c(i)$, we do the following. Let $c(i + 1)$ be the densest color in $G[A_{i+1}]$. If $c(i + 1)$ occurs at most $t$ times in the sequence $c(1), \ldots, c(i + 1)$ of colors constructed so far, then we choose $v_{i+1} \in A_{i+1}$ such that $v_{i+1}$ is connected to at least $\frac{|A_{i+1}|-1}{k}$ vertices in color
c(i + 1) and we denote these vertices by $A_{i+2}$. Otherwise we stop. It is clear that we stop after at most $kt + 1$ steps, that is with $i + 1 \leq kt + 1$.

Next we identify $t$ vertices $v_{j_1}, \ldots, v_{j_t}$ such that $c(j_1) = \ldots = c(j_t) = c(i + 1)$. Observe that all vertices $v_{j_t}$ are connected in color $c(i + 1)$ to each other and also to all vertices in $A_{i+1}$. Moreover, at least $\frac{1}{k} \binom{|A_{i+1}|}{2}$ edges of $G[A_{i+1}]$ are colored in $c(i + 1)$. Therefore, we can embed the vertices from $U$ in $F$ onto $v_{j_1}, \ldots, v_{j_t}$, and then one needs to find an embedding $F \setminus U$ into $G[A_{i+1}]$ in color $c(i + 1)$. But this is asserted to us by Theorem 5, as long as

$$|A_{i+1}| \geq 32dk^d |V(F) \setminus U|.$$ 

Since $i + 1 \leq kt + 1$ we obtain $|A_{i+1}| \geq \frac{N}{km} - 1$, and since we can assume that $F$ is not a matching (otherwise Theorem 5 implies the result immediately), we have $|V(F)| < 2m$ and therefore

$$|A_{i+1}| \geq 32dk^d 2m - 1 \geq 32dk^d |V(F) \setminus U|.$$ 

Thus, $r_k(F) \leq N = 32dk^d+2 2m \leq 2^6 \sqrt{km}^3 k^2 \sqrt{km} = k^{(1+o(1))2\sqrt{km}}$. 

As an immediate consequence we get.

**Corollary 7.** Let $F$ be a bipartite graph with $m$ edges and without isolated vertices, then $r_2(F) \leq 2^{(1+o(1))2\sqrt{km}}$.

4. **Multicolor Ramsey number of general graphs with $m$ edges**

In this section we heavily rely on the tools developed by Fox and Sudakov in [9]. There they showed that $r_k(F) \leq k^{2k\Delta q}n$ for a graph $F$ with $n$ vertices, $\Delta(F) = \Delta$ and $\chi(F) = q$ (more generally, it holds for $k$ not necessarily isomorphic graphs $F_1, \ldots, F_k$ with the same properties). Their proof combines Lemmas 3 and 4.

Our proof strategy is in fact a slight modification of their argument intertwined with the process of first embedding high degree vertices. The idea of embedding high degree vertices already occurs in [1]. More precisely, since we are given a general graph $F$ with $m$ edges, we first seek to embed vertices of high degree (which will be done in a similar way as in the proof of Theorem 2). However, this time we are going to use Lemma 3 instead of Theorem 5 repeatedly. The authors in [9] show $r_k(F) \leq k^{2k\Delta q}n$ by applying iteratively Lemma 3 roughly $qk$ times, “loosing” each time roughly a factor of $k^{-k\Delta}$. Afterwards one identifies a long enough nested sequence to perform embedding (Lemma 4). In our case however, we first need to reduce the maximum degree of $F$, and only then we will apply Lemma 3. However, its applications intertwine with the focussing argument similar to the previous section, as each color might get filled up differently quickly.

**Proof of Theorem 1.** We choose with foresight $d = (m/4)^{1/3}$ and $\ell = |m/d| = \lfloor (2m)^{2/3} \rfloor$. Furthermore, we set $x = k^{-(2d+2)kd-k\Delta} N$ and $N = k^{(2d+2)kd+kd} 8m$.

Take a given graph $F$ with $m$ edges and no isolated vertices. By Proposition 6, let $U$ be a set of at most $\ell$ vertices such that $\Delta(F \setminus U) \leq d$. Further observe that $|V(H)| \leq 2m$ and $\chi(F \setminus U) \leq d + 1$.

Let an arbitrary but fixed coloring of the edges of the graph $G := K_N$ by $k$ colors be given.

We set $A_1 = [N]$ and construct a sequence of sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_s$ and a sequence of colors $c(1), \ldots, c(s - 1)$ inductively as follows.
Given a sequence of sets $A_1 \supset A_2 \supset \ldots \supset A_\ell$ and the sequence of colors $c(1), \ldots, c(\ell-1)$, we do the following. Let $c(i)$ be the densest color in $G[A_i]$. If $c(i)$ occurs at most $\ell$ times in the sequence $c(1), \ldots, c(\ell)$ of colors constructed so far, then we choose $v_i \in A_i$ such that $v_i$ is connected to at least $\frac{|A_i|-1}{k}$ vertices in color $c(i)$ and we denote these vertices by $A_{i+1}$ (and we refer to this step as focusing). If, however, the color $c(i)$ occurs more than $\ell$ times then we call $c(i)$ saturated. As long as the saturated color $c(i)$ occurs at most $t + d$ times among $c(1), \ldots, c(i)$, we consider a balanced bipartition of $A_i = A_{i,1} \cup A_{i,2}$ (assume $|A_{i,1}| \leq |A_{i,2}|$) such that at least $\frac{1}{2}|A_{i,1}| |A_{i,2}|$ edges are colored by the color $c(i)$ (simply take a random balanced bipartition). Furthermore, we apply now Lemma 3 with $\varepsilon = \frac{1}{k}$, $a = 1$ and $t = 2d$ and thus we find a subset $A_{i+1} \subset A_{i,2} \subset A_i$ with $|A_{i+1}| \geq 2^{-1}k^{-2d} |A_{i,2}| \geq k^{-2d-2}|A_i|$ (use $|A_{i,1}| \leq |A_{i,2}|$), such that all but at most

$$2 \cdot k^{2d} \left( \frac{x}{|A_{i,2}|} \right)^{2d} \left( \frac{|A_{i+1}|}{|A_{i,2}|} \right)^{d} \left( \frac{|A_{i,2}|}{d} \right)$$

(1)

d-sets $S$ in $A_{i+1}$ have at least $x$ common neighbors in $G[A_i]$ in color $c(i)$ (actually in $A_{i,1}$). We refer to such a step as nesting. Moreover, we can use $|A_{i,2}| \geq \frac{1}{2} k^{-k\ell-2d+2k(d-1)}N$ to simplify and bound (1) as:

$$\frac{x}{|A_{i,2}|} \left( \frac{x}{d} \right)^d \leq \left( \frac{2k}{d} \right)^d \frac{x}{|A_{i,2}|} \left( \frac{x}{d} \right)^d \leq (2d)^{-d} \frac{x}{d}.$$

We stop constructing a sequence once we end up with colors $c(1), \ldots, c(s)$ and sets $A_1 \supset \ldots \supset A_{s+1}$ and there is one color $c$ which occurs $\ell + d$ times. Clearly, $s \leq \ell + k(d-1) + 1$, since we first focus in one color $t$ times before it gets saturated and then we need to nest $d$ times in some color, before we stop the sequence construction. By the choice of $N$, $x$, $\ell$ and $d$ we can clearly proceed for $kt + k(d-1) + 1$ steps if necessary.

Let $c \in [k]$ be the color which occurs $\ell + d$ times and let $v_{i_1}, \ldots, v_{i_k}$ be the vertices which got selected in the first $\ell$ steps when the color $c$ was chosen (these were focussing steps) and let $A_{i_{t+1}}, \ldots, A_{i_{t+d}}$ be the sets with majority color $c$ during the nesting steps. Next we show how to find a $c$-colored copy of $F$, whose vertices are embedded onto $v_{i_1}, \ldots, v_{i_k}$ and into the sets $A_{i_{t+1}}, A_{i_{t+1}+1}, \ldots, A_{i_{t+d+1}}$ (where $i_{t+d} = s$).

By Proposition 6, we have a set $U$ of $\ell$ vertices of high degree which get embedded onto $v_{i_1}, \ldots, v_{i_k}$ and then all we need to do is to embed a copy of $F \setminus U$ into the nested sequence $A_{i_{t+1}}, A_{i_{t+1}+1}, \ldots, A_{i_{t+d+1}}$ in color $c$. But this can be done by Lemma 4. This shows that $r_k(F) \leq N$. \hfill \square

**Corollary 8.** Let $F$ be a graph with $m$ edges and without isolated vertices. Then $r_k(F) \leq k^{6k^2m/3}$. \hfill \square

### 5. Concluding Remarks

In this note we showed a first nontrivial upper bound on $r_k(F)$ to be $k^{6k^2e(F)^{2/3}}$. Certainly, there should be a room for improvement, maybe even to $k^{O(\sqrt{e(F)})}$, thus generalizing the result of Sudakov. Another interesting direction would be to improve the result of Sudakov to $r_2(F) \leq 2^{(1+o(1))2\sqrt{2m}}$ by obtaining asymptotically the same upper bound as the best one known for $r_2(K_{\ell})$ with $\binom{\ell}{2} = m$. This was noted by us to hold if $F$ is bipartite.
After the completion of this paper we learned that Conlon, Fox and Sudakov obtained a result similar to our Theorem 1 independently [13].

References

[1] N. Alon, M. Krivelevich, and B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, Combin. Probab. Comput. 12 (2003), no. 5-6, 477–494, Special issue on Ramsey theory. 1, 1, 4

[2] S. A. Burr and P. Erdős, On the magnitude of generalized Ramsey numbers for graphs, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. 1, North-Holland, Amsterdam, 1975, pp. 215–240. Colloq. Math. Soc. János Bolyai, Vol. 10. 1

[3] F. Chung and R. Graham, Erdős on graphs, A K Peters Ltd., Wellesley, MA, 1998, His legacy of unsolved problems. 1

[4] C. Chvátal, V. Rödl, E. Szemerédi, and W. T. Trotter, Jr., The Ramsey number of a graph with bounded maximum degree, J. Combin. Theory Ser. B 34 (1983), no. 3, 239–243. 1

[5] D. Conlon, A new upper bound for diagonal Ramsey numbers, Ann. of Math. (2) 170 (2009), no. 2, 941–960. 1

[6] P. Erdős, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947), 292–294. 1

[7] P. Erdős and R. L. Graham, On partition theorems for finite graphs, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, North-Holland, Amsterdam, 1975, pp. 515–527. Colloq. Math. Soc. János Bolyai, Vol. 10. 1

[8] P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935), 463–470. 1

[9] J. Fox and B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results, Combinatorica 29 (2009), no. 2, 153–196. 1, 2, 3, 4, 5, 4

[10] , Two remarks on the Burr-Erdős conjecture, European J. Combin. 30 (2009), no. 7, 1630–1645. 1

[11] F. P. Ramsey, On a problem in formal logic, Proc. Lond. Math. Soc. 30 (1930), 264–286. 1

[12] J. Spencer, Ramsey’s theorem—a new lower bound, J. Combinatorial Theory Ser. A 18 (1975), 108–115. 1

[13] B. Sudakov, personal communication. 5

[14] B. Sudakov, A conjecture of Erdős on graph Ramsey numbers, Adv. Math. 227 (2011), no. 1, 601–609. 1

Freie Universität Berlin, Institut für Mathematik, Berlin, Germany
E-mail address: kathleen.johst@fu-berlin.de

Goethe-Universität, Institut für Mathematik, Robert-Mayer-Str. 10, 60325 Frankfurt am Main, Germany
E-mail address: person@math.uni-frankfurt.de