CYCLIC GROUPS OF AUTOMORPHISMS OF COMPLEX K3 SURFACES

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Abstract. We determine all possible orders of automorphisms of complex K3 surfaces. A positive integer \( N \) is the order of an automorphism of a complex K3 surface if and only if \( \phi(N) \leq 20 \) where \( \phi \) is the Euler function.

We work over the field of complex numbers, and prove the following:

Theorem 0.1. A positive integer \( N \) is the order of an automorphism of a K3 surface if and only if \( \phi(N) \leq 20 \) where \( \phi \) is the Euler function. In other words, the set of all possible orders of automorphisms of K3 surfaces is

\[ \{ N : N \text{ is a positive integer, } \phi(N) \leq 20 \} \cup \{ \infty \}. \]

There is an example of a K3 surface with an automorphism of order \( N \) if \( \phi(N) \leq 20 \). In fact, purely non-symplectic examples of order \( N \) with \( \phi(N) \leq 20 \), \( N \neq 60 \), were given in [7], Section 7 and [12], Proposition 2. Examples of order 60 will be given in Example 3.3.

An elliptic K3 surface with Mordell-Weil rank positive always admits automorphisms of infinite order, e.g., the automorphisms induced by translations by a non-torsion of the Mordell-Weil group of the Jacobian fibration. These are symplectic. Non-symplectic automorphisms of infinite order also exist, e.g., on generic Jacobian Kummer surfaces the composition of odd number of projections or the 192 new automorphisms found in [6].

The proof depends on the results of Piateckii-Shapiro and Shafarevich [13] on the faithfulness of the representation of the automorphism group of a K3 surface on its integral cohomology, of Nikulin [9] on symplectic automorphisms of finite order of K3 surfaces, and uses the methods developed by Kondō [7], by Machida and Oguiso [8] and the topological and holomorphic Lefschetz fixed point formulas. A key idea is that if \( g^n \) is symplectic, then its fixed locus is small (0-dimensional) and contains the fixed locus of \( g^a \) for all \( a \) dividing \( n \). In most cases, this imposes a restriction to the order of \( g \).

For the non-existence of the order 60, we also use the result of Nikulin [10], [11] and of Artebani, Sarti and Taki [1] on non-symplectic automorphisms of prime order.

Notation

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For an automorphism $g$ of a K3 surface $X$, we use the following notation:
- $\omega_X$: a nowhere vanishing holomorphic 2-form on $X$
- $\text{ord}(g) = m.n$: $g$ is of order $mn$ and the representation of the group $\langle g \rangle$ on $H^0(X, \Omega^2_X)$ has kernel of order $m$.
- $[g^*] = [\lambda_1, \ldots, \lambda_{22}]$: the eigenvalues of $g^*|H^2(X, \mathbb{Z})$
- $\zeta_a = \exp(\frac{2\pi \sqrt{-1}}{a})$: a primitive $a$-th root of unity
- $\zeta_a : \phi(a)] \subset [g^*]$: all primitive $a$-th roots of unity appear in $[g^*]$, where $\phi(a)$ indicates the number of them.
- $\lambda.r \subset [g^*]$: $\lambda$ repeats $r$ times in $[g^*]$.
- $\{\zeta_a : \phi(a)).r \subset [g^*]$: the list $\zeta_a : \phi(a)$ repeats $r$ times in $[g^*]$.
- $\text{Fix}(g)$: the fixed locus of $g$

1. Preliminaries

First we recall a result of Piatetski-Shapiro and Shafarevich ([13] Proposition 2) for complex projective K3 surfaces, later extended to Kähler K3 surfaces by Burns and Rapoport [5]. See also [4], Proposition VIII.11.3.

**Proposition 1.1.** [13] Let $X$ be a K3 surface. Then the representation
$$\text{Aut}(X) \to \text{GL}(H^2(X, \mathbb{Z})), \ g \mapsto g^*,$$

is faithful.

For a K3 surface $X$ there is a natural representation of the automorphism group $\text{Aut}(X)$ on the space of global 2-forms $H^{2,0}(X) = \mathbb{C}\omega_X$:
$$\text{Aut}(X) \to \text{GL}(H^{2,0}(X)) \cong \mathbb{C}^*, \ g \mapsto \lambda_g$$
where $g^*\omega_X = \lambda_g\omega_X$. An automorphism $g$ is called symplectic if it is in the kernel of the representation, and non-symplectic otherwise. It is said to be of non-symplectic order $n$ if $\lambda_g$ is a primitive $n$-th root of unity, and of non-symplectic order infinity if $\lambda_g$ is not a root of unity.

When $X$ is projective, the image of $\text{Aut}(X)$ on $\text{GL}(H^{2,0}(X))$ is a finite cyclic group [9]. Thus projective K3 surfaces do not admit an automorphism of non-symplectic order infinity.

When $X$ is not projective, the image of $\text{Aut}(X)$ on $\text{GL}(H^{2,0}(X))$ is either trivial or an infinite cyclic group. Thus non-projective K3 surfaces do not admit an automorphism of non-symplectic order infinity.

Given an automorphism $g$ of a K3 surface $X$, we may regard the transcendental lattice $T_X$ as a $\mathbb{Z}[\langle g \rangle]$-module via the natural action of $g^*$ on $T_X$. If $g$ is of non-symplectic order $n$, then $T_X$ is a $\mathbb{Z}[\langle g \rangle]/\langle \Phi_n(g) \rangle$-module where $\Phi_n(x) \in \mathbb{Z}[x]$ is the $n$-th cyclotomic polynomial, thus $T_X$ can be viewed as a $\mathbb{Z}[\zeta_n]$-module via the isomorphism $\mathbb{Z}[\langle g \rangle]/\langle \Phi_n(g) \rangle \cong \mathbb{Z}[\zeta_n]$.

**Proposition 1.2.** [9] Let $X$ be a K3 surface.
(1) A symplectic automorphism \( g \) of finite order of \( X \) has finitely many fixed points, the number of fixed points \( \#\text{Fix}(g) \) depends only on the order of \( g \) and the possible pairs \((\text{ord}(g), \#\text{Fix}(g))\) are as follows:
\[
(\text{ord}(g), \#\text{Fix}(g)) = (2, 8), (3, 6), (4, 4), (5, 4), (6, 2), (7, 3), (8, 2).
\]

(2) If \( n \) is the non-symplectic order of an automorphism of \( X \), then the transcendental lattice \( T_X \) is a free module over \( \mathbb{Z}[\zeta_n] \). In particular \( \phi(n) \) divides \( \text{rank } T_X \) where \( \phi \) is the Euler function.

Denote by \( \epsilon(m) \) the number of fixed points of a symplectic automorphism of order \( m \). That is,
\[
\epsilon(m) = 8, 6, 4, 4, 2, 3, 2
\]
if \( m = 2, 3, 4, 5, 6, 7, 8 \), respectively.

**Lemma 1.3.** For a symplectic automorphism \( h \) of order \( m \)
\[
\text{Tr}(h^*|H^2(X, \mathbb{Z})) = \epsilon(m) - 2.
\]

**Proof.** Apply the topological Lefschetz formula and Proposition 1.2. \[ \square \]

**Lemma 1.4.** Let \( X \) be a K3 surface.

(1) If \( h \) is a purely non-symplectic automorphism of order 4 of a K3 surface, then the Euler characteristic \( \epsilon(\text{Fix}(h)) \) is divisible by 4.

(2) If \( h \) is a non-symplectic automorphism of order 4 of \( X \) such that \( h^2 \) is symplectic, then \( \text{Fix}(h) = \phi \).

**Proof.** This is an easy consequence of the holomorphic Lefschetz fixed point formula (\[2\], p542 and \[3\], p567).

(1) Assume that \( h^*\omega_X = \zeta_4 \omega_X \). Assume that \( \text{Fix}(h) \) consist of \( k \) isolated points and \( l \) curves \( R_1, \ldots, R_l \). The local action of \( h \) at a fixed point can be diagonalized as
\[
\begin{pmatrix}
-\zeta_4 & 0 \\
0 & -1
\end{pmatrix},
\begin{pmatrix}
\zeta_4 & 0 \\
0 & 1
\end{pmatrix},
\]
respectively if the fixed point is isolated and otherwise. Since on a K3 surface \( R_j^2 = 2g(R_j) - 2 \), the holomorphic Lefschetz fixed point formula yields
\[
1 - \zeta_4 = \sum_{j=0}^{2} \text{Tr}(h^*|H^j(X, \mathcal{O}_X)) = \frac{k}{2(1 + \zeta_4)} + \frac{\zeta_4 - 1}{2} \sum_{j=1}^{l} (1 - g(R_j)),
\]
hence \( k = 4 + 2 \sum_{j=1}^{l} (1 - g(R_j)) \). Now
\[
\epsilon(\text{Fix}(h)) = k + 2 \sum_{j=1}^{l} (1 - g(R_j)) = 4 + 4 \sum_{j=1}^{l} (1 - g(R_j)).
\]

(2) Suppose \( \text{Fix}(h) \) is not empty. The local action of \( h \) at a fixed point can be diagonalized as
\[
\begin{pmatrix}
\zeta_4 & 0 \\
0 & \zeta_4
\end{pmatrix}.
\]
It follows that $\text{Fix}(h)$ is finite. Applying the holomorphic Lefschetz formula, we see that the number of fixed points is 0. □

**Lemma 1.5.** Let $g$ be an automorphism of finite order of a projective variety $X$. Then $g$ has an invariant ample divisor, which corresponds to a non-zero $g^*$-invariant cohomology class in $H^2(X, \mathbb{Z})$. In particular 1 appears as an eigenvalue of $g^*$ acting on $H^2(X, \mathbb{Z})$.

**Proof.** For any ample divisor $D$ the finite sum $\sum g^i(D)$ is ample and $g$-invariant. □

**Lemma 1.6.** Let $\eta_1, \ldots, \eta_k$ be a collection of $m$-th roots of unity, at least one of them is primitive. Assume that the sum $\sum_{j=1}^{k} \eta_j$ is an integer.

1. If $m$ is a prime, then $\sum \eta_j \geq s - r$ where $k = (m-1)r + s$, $0 \leq s \leq m - 2$. In particular, if $m = 2$, then $\sum \eta_j \geq -k$.
2. If $m = 4$, then $\sum \eta_j \geq -k + 2$.
3. If $m = 6$, then $\sum \eta_j \geq -k + 3$.
4. If $m = 8$, then $\sum \eta_j \geq -k + 4$.

**Proof.** Since the sum $\sum_{j=1}^{k} \eta_j$ is an integer, the collection is a disjoint union of sub-collections, each a collection of all primitive $m'$-th roots of unity for some $m'$ dividing $m$.

1. When $m$ is a prime, the sum of all primitive $m$-th roots of unity $\zeta_m + \zeta_m^2 + \cdots + \zeta_m^{m-1} = -1$.
2. The sum takes its minimum when the collection is $\zeta_4, -\zeta_4, -1, \ldots, -1$.
3. The sum takes its minimum when the collection is $\zeta_6, \zeta_6^5, -1, \ldots, -1$.
4. The sum takes its minimum when the collection is $\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7, -1, \ldots, -1$. □

The following easy lemma also will be used frequently.

**Lemma 1.7.** Let $S$ be a set and $\text{Aut}(S)$ be the group of bijections of $S$. For any $g \in \text{Aut}(S)$ and positive integers $a$ and $b$,

1. $\text{Fix}(g) \subset \text{Fix}(g^a)$;
2. $\text{Fix}(g^a) \cap \text{Fix}(g^b) = \text{Fix}(g^d)$ where $d = \gcd(a, b)$;
3. $\text{Fix}(g) = \text{Fix}(g^a)$ if $\text{ord}(g)$ is finite and $a$ is prime to it.

2. PROOF OF THE BOUND $\phi(N) \leq 20$

It is well known ([14], [9]) that non-projective K3 surfaces do not admit a non-symplectic automorphism of finite order.

From now on we assume that $X$ is a projective K3 surface and $g$ is an automorphism of $X$ of finite order $N$. Assume that

$$\text{ord}(g) = N = m n$$

i.e., $g$ is of order $N = mn$ and of non-symplectic order $n$. From Proposition 1.2 we know that

$$m \leq 8.$$
Since rank $T_X \leq 21$, $\phi(n) \leq 20$. For convenience, we list all possible $n$ with $\phi(n) \leq 20$ as follows:

| $\phi(n)$ | $n$            |
|----------|---------------|
| 20       | 66, 50, 44, 33, 25 |
| 18       | 54, 38, 27, 19   |
| 16       | 60, 48, 40, 34, 32, 17 |
| 12       | 42, 36, 28, 26, 21, 13 |
| 10       | 22, 11          |
| 8        | 30, 24, 20, 16, 15 |
| 6        | 18, 14, 9, 7    |
| 4        | 12, 10, 8, 5    |
| 2        | 6, 4, 3         |
| 1        | 2, 1            |

The bound $\phi(N) \leq 20$ will be proved in the following lemmas.

**Lemma 2.1.** If $\phi(n) > 13$, then $m = 1$.

**Proof.** The primitive $n$-th root $\zeta_n = \exp\left(\frac{2\pi \sqrt{-1}}{n}\right)$ is an eigenvalue of $g^*|H^2(X, \mathbb{Z})$, hence

$$[g^*] = [1, \zeta_n : \phi(n), \lambda_1, \ldots, \lambda_{21-\phi(n)}]$$

where $\zeta_n : \phi(n)$ means all primitive $n$-th roots of unity and $\phi(n)$ indicates the number of them. Here we use that $1 \in [g^*]$ by Lemma 1.5. Note that $g^n$ is symplectic of order $m$ and

$$[g^{n*}] = [1, 1, \phi(n), \eta_1, \ldots, \eta_{21-\phi(n)}]$$

where $1, \phi(n)$ means that 1 appears with multiplicity $\phi(n)$ and $\eta_j$’s are $m$-th roots of unity, not necessarily primitive.

Assume that $m \geq 2$. By Lemma 1.6

$$\sum \eta_j \geq \phi(n) - 21.$$ Since $\phi(n) > 13$, $\sum \eta_j \geq 21$. Hence

$$\text{Tr}(g^{n*}|H^2(X, \mathbb{Z})) = 1 + \phi(n) + \sum \eta_j \geq 1 + \phi(n) + \phi(n) - 21 > 6 \geq \epsilon(m) - 2.$$ This contradicts Lemma 1.3, so we have proved that $m = 1$.

**Lemma 2.2.** Assume that $\phi(n) = 12$. Then $m = 1$ if $n = 42, 36, 28, 26$, and $m \leq 2$ if $n = 21, 13$.

**Proof.** By the same proof as in the previous lemma, we see that

$$[g^{n*}] = [1, 1, 12, \eta_1, \ldots, \eta_9]$$

where $\eta_j$’s are $m$-th roots of unity. Note that $g^n$ is symplectic of order $m$.

If $m \geq 3$, then we see from Lemma 1.6 that

$$\text{Tr}(g^{n*}|H^2(X, \mathbb{Z})) = 13 + \sum \eta_j > \epsilon(m) - 2,$$ a contradiction to Lemma 1.3. So we have proved that $m \leq 2$.

Assume $m = 2$. Since $g^n$ is symplectic of order 2,

$$\text{Tr}(g^{n*}|H^2(X, \mathbb{Z})) = 13 + \sum \eta_j = \epsilon(2) - 2 = 6,$$
hence \( \sum \eta_j = -7 \). This occurs only when \([\eta_1, \ldots, \eta_9] = [1, -1.8] \). Assume that \( n = 2n' \). Then \( g^{n'} \) is a non-symplectic automorphism of order 4 such that \( h^2 \) is symplectic. We infer that
\[
[g^{n'}] = [1, -1.12, \pm 1, (\zeta_4 : 2).4],
\]
Thus by topological Lefschetz formula the Euler characteristic \( e(\text{Fix}(g^{n'})) \neq 0 \), contradicting Lemma 1.4.

**Lemma 2.3.** Assume that \( \phi(n) = 10 \). Then \( m = 1 \) if \( n = 22 \) and \( m \leq 2 \) if \( n = 11 \).

*Proof.* As in the previous proof, we see that \( g^n \) is symplectic of order \( m \) and
\[
[g^n] = [1, 1.10, \eta_1, \ldots, \eta_{11}]
\]
where \( \eta_j \)'s are \( m \)-th roots of unity.

Assume that \( m \geq 5 \) or \( m = 3 \). From Lemma 1.6 we see that
\[
\text{Tr}(g^n|H^2(X, \mathbb{Z})) = 11 + \sum \eta_j > \epsilon(m) - 2,
\]
a contradiction.

Assume that \( m = 4 \). Since \( g^n \) is symplectic of order 4, we see that
\[
\sum \eta_j = -9,
\]
hence
\[
[\eta_1, \ldots, \eta_{11}] = [\zeta_4 : 2, -1.9].
\]
Then \( g^{2n} \) is a symplectic involution with
\[
[g^{2n}] = [1.11, -1.2, 1.9],
\]
whose trace is too big.

Assume that \( m = 2 \) and \( n = 22 \). Then \( g^{11} \) is non-symplectic of order 4 with a symplectic square. Since
\[
[g^{22}] = [1, 1.10, -1.8, \ldots, 1.3],
\]
we infer that
\[
[g^{11}] = [1, -1.10, (\zeta_4 : 2).4, \pm 1, \pm 1, \pm 1].
\]
In any case, \( \text{Tr}(g^{11}|H^2(X, \mathbb{Z})) \neq -2 \), a contradiction to Lemma 1.4. \( \square \)

**Lemma 2.4.** If \( \phi(n) = 8 \), then \( m \leq 3 \).

*Proof.* As in the previous proof, we see that \( g^n \) is symplectic of order \( m \) and
\[
[g^n] = [1, 1.8, \eta_1, \ldots, \eta_{13}]
\]
where \( \eta_j \)'s are \( m \)-th roots of unity, not necessarily primitive.

Assume that \( m = 8 \). By Lemma 1.3
\[
\text{Tr}(g^n|H^2(X, \mathbb{Z})) = \epsilon(8) - 2 = 0,
\]
so
\[
[\eta_1, \ldots, \eta_{13}] = [\zeta_8 : 4, -1.9].
\]
Then \( \text{Tr}(g^{2n}|H^2(X, \mathbb{Z})) = 18 \). But, \( g^{2n} \) is symplectic of order 4, so has
\[
\text{Tr}(g^{2n}|H^2(X, \mathbb{Z})) = \epsilon(4) - 2 = 2,
\]
a contradiction.

Assume that \( m = 5 \) or \( 7 \). From Lemma 1.6 we see that
\[
\text{Tr}(g^n|H^2(X, \mathbb{Z})) = 9 + \sum \eta_j > \epsilon(m) - 2.
\]
Assume that $m = 6$. Since $g^{3n}$ is symplectic of order 2,

$$[g^{3n}] = [1, 1.8, -1.8, 1.5].$$

From this we infer that

$$[\eta_1, \ldots, \eta_{13}] = [(\zeta_6 : 2).4, 1.5], \ [(\zeta_6 : 2).4, \zeta_3 : 2, 1.3] \text{ or } \ [(\zeta_6 : 2).4, (\zeta_3 : 2).2, 1].$$

In any case, $\text{Tr}(g^{n*}|H^2(X, \mathbb{Z})) > \epsilon(6) - 2 = 0$, a contradiction.

Assume that $m = 4$. Since $g^{2n}$ is symplectic of order 2,

$$[g^{2n}] = [1, 1.8, -1.8, 1.5].$$

From this we infer that

$$[\eta_1, \ldots, \eta_{13}] = [(\zeta_4 : 2).4, \pm 1, \pm 1, \pm 1, \pm 1].$$

In any case, $\text{Tr}(g^{n*}|H^2(X, \mathbb{Z})) > \epsilon(4) - 2 = 2$, a contradiction. \hfill \Box

**Lemma 2.5.** (1) If $n = 16$, then $m = 1$ or 3.

(2) If $n = 30, 24, 15$, then $m \leq 2$.

**Proof.** By Lemma 2.4, $m \leq 3$.

(1) We know that $\zeta_{16} \in [g^*]$. If $m = 2$, then $\zeta_{32} \in [g^*]$ by Proposition 2.1, but then $\phi(16) + \phi(32) > 22$.

(2) Claim: $\text{ord}(g) \neq 3.15$.

On the contrary, suppose that $\text{ord}(g) = 3.15$. Since $g^{15}$ is symplectic of order 3, $\text{Tr}(g^{15*}|H^2(X, \mathbb{Z})) = 4$. As in the previous proof, we see that

$$[g^{15}] = [1, 1.8, (\zeta_3 : 2).6, 1].$$

Since $\phi(45) > 13$, $\zeta_{45} \notin [g^*]$ and we infer that

$$[g^*] = [1, \zeta_{15} : 8, (\zeta_9 : 6).2, 1].$$

Thus

$$[g^{3*}] = [1, (\zeta_5 : 4).2, (\zeta_3 : 2).6, 1],$$

hence the trace

$$\text{Tr}(g^{3*}|H^2(X, \mathbb{Z})) = -6.$$

On the other hand, $\text{Fix}(g^3)$ is contained in $\text{Fix}(g^{15})$, so is finite and by the topological Lefschetz

$$\text{Tr}(g^3|H^2(X, \mathbb{Z})) = c(\text{Fix}(g^3)) - 2 \geq -2,$$

a contradiction.

Claim: $\text{ord}(g) \neq 3.30$.

Suppose that $\text{ord}(g) = 3.30$. Then $g^2$ is of order 3.15. But such an order cannot occur by the previous claim.

Claim: $\text{ord}(g) \neq 3.24$.

Suppose that $\text{ord}(g) = 3.24$. Since $g^{24}$ is symplectic of order 3, we see that

$$[g^{24}] = [1, 1.8, (\zeta_3 : 2).6, 1].$$
The eigenvalue $\zeta_3 \in [g^{24*}]$ must come from $\zeta_9, \zeta_{18}, \zeta_{36}$ or $\zeta_{72} \not\in [g^*]$. Since $\phi(72) > 12, \zeta_{72} \not\in [g^*]$. Since the 4th power of $\zeta_9, \zeta_{18}, \zeta_{36}$ is a 9th root of unity, we infer that

$$[g^{14*}] = [1, (\zeta_6 : 2).4, (\zeta_9 : 6).2, 1]$$

where $\zeta_6 : 2).4$ comes from $\zeta_{24} : 8$ in $[g^*]$. Thus

$$\text{Tr}(g^{14*}|H^2(X, \mathbb{Z})) = 6 > \text{Tr}(g^{24*}|H^2(X, \mathbb{Z})) = 4.$$  

But $\text{Fix}(g^4) \subset \text{Fix}(g^{24})$ and $\text{Fix}(g^{24})$ is finite, so the inequality is impossible. \hfill \Box

**Lemma 2.6.** Assume that $\phi(n) = 6$.

1. If $n = 18$, then $m \leq 2$.
2. If $n = 9$, then $m \leq 2$ or 4.
3. If $n = 14$, then $m \leq 3$.
4. If $n = 7$, then $m \leq 4$.

**Proof.** As in the previous proof, we see that $g^n$ is symplectic of order $m$ and

$$[g^{9*}] = [1, 1.6, \eta_1, \ldots, \eta_{15}]$$

where $\eta_j$’s are $m$-th roots of unity, not necessarily primitive.

Assume that $m = 8$. By Lemma [1.3], $\text{Tr}(g^{9*}|H^2(X, \mathbb{Z})) = \epsilon(8) - 2 = 0$. Since $g^{4n}$ is symplectic of order 2, $[g^{9*}] = [1, 1.6, -1.8, 1.7]$. Thus

$$[\eta_1, \ldots, \eta_{15}] = [\zeta_8 : 4, \zeta_8 : 4, -1.7].$$

Then $\text{Tr}(g^{2n*}|H^2(X, \mathbb{Z})) = 14$. But, $g^{2n}$ is symplectic of order 4, so has $\text{Tr}(g^{2n*}|H^2(X, \mathbb{Z})) = \epsilon(4) - 2 = 2$, a contradiction.

Assume that $m = 5$ or 7. From Lemma [1.6] we see that

$$\text{Tr}(g^{9*}|H^2(X, \mathbb{Z})) = 7 + \sum \eta_j > \epsilon(m) - 2.$$  

Assume that $m = 6$. Since $g^{3n}$ is symplectic of order 2,

$$[g^{9*}] = [1, 1.6, -1.8, 1.7].$$

From this we infer that

$$[\eta_1, \ldots, \eta_{15}] = [(\zeta_6 : 2).4, 1.7], [(\zeta_6 : 2).4, \zeta_3 : 2, 1.5],$$

$$[(\zeta_6 : 2).4, (\zeta_3 : 2).2, 1.3] \text{ or } [(\zeta_6 : 2).4, (\zeta_3 : 2).3, 1].$$

In any case, $\text{Tr}(g^{9*}|H^2(X, \mathbb{Z})) > 6 - 2 = 0$, a contradiction.

We have proved that $m \leq 4$.

Assume that $m = 3$. Then $n$ cannot be divisible by 9, since $\phi(n) + \phi(3^3) > 22$.

It remains to show that 4.18 and 4.14 do not occur.

Claim: $\text{ord}(g) \neq 4.18$.

On the contrary, suppose that $\text{ord}(g) = 4.18$. Since $g^{18}$ is symplectic of order 4,

$$[g^{18*}] = [1, 1.6, (\zeta_4 : 2).4, -1.6, 1].$$
Note that \( \zeta_{72} \notin [g^*] \). Thus the eigenvalue \( \zeta_4 \in [g^{18*}] \) must come from \( \zeta_8 \) or \( \zeta_{24} \in [g^*] \), but not from \( \zeta_{72} \). Similarly, \(-1 \in [g^{18*}] \) must come from \( \zeta_4 \) or \( \zeta_{12} \in [g^*] \), but not from \( \zeta_{36} \), since \( \phi(36) > 6 \). From this we infer that

\[ [g^{8*}] = [1, -1.6, (\zeta_8 : 4).2, (\zeta_4 : 2).3, \pm 1]. \]

Thus

\[ \text{Tr}(g^{8*}|H^2(X, \mathbb{Z})) \leq -4. \]

But \( \text{Fix}(g^9) \) being contained in the finite set \( \text{Fix}(g^{18}) \) has non-negative Euler number, so \( \text{Tr}(g^{9*}|H^2(X, \mathbb{Z})) \geq -2. \)

Claim: \( \text{ord}(g) \neq 4.14. \)

Suppose that \( \text{ord}(g) = 4.14. \) Since \( g^{14} \) is symplectic of order 4,

\[ [g^{14*}] = [1, 1.6, (\zeta_4 : 2).4, -1.6, 1]. \]

The eigenvalue \( \zeta_4 \in [g^{14*}] \) must come from \( \zeta_8 \in [g^*] \), but not from \( \zeta_{56} \), since \( \phi(56) > 8 \). Similarly, \(-1 \in [g^{14*}] \) must come from \( \zeta_4 \in [g^*] \), but not from \( \zeta_{28} \), since \( \phi(28) > 6 \). Thus we see that

\[ [g^7] = [1, \zeta_{14} : 6, (\zeta_8 : 4).2, (\zeta_4 : 2).3, \pm 1]. \]

Then

\[ [g^{7*}] = [1, -1 : 6, (\zeta_8 : 4).2, (\zeta_4 : 2).3, \pm 1], \]

hence

\[ \text{Tr}(g^{7*}|H^2(X, \mathbb{Z})) \leq -4. \]

But \( \text{Fix}(g^{7}) \) is contained in the finite set \( \text{Fix}(g^{18}) \), so has non-negative Euler number. \( \square \)

**Lemma 2.7.** Assume that \( \phi(n) = 4. \)

1. If \( n = 12 \), then \( m \leq 5. \)
2. If \( n = 10 \) or \( 5 \), then \( m \leq 4. \)
3. If \( n = 8 \), then \( m \leq 3 \) or \( 5. \)

**Proof.** As in the previous proof, we see that \( g^n \) is symplectic of order \( m \) and

\[ [g^{m*}] = [1, 1.4, \eta_1, \ldots, \eta_{17}] \]

where \( \eta_j \)'s are \( m \)-th roots of unity.

Assume that \( m = 7. \) From Lemma 1.6 we see that

\[ \text{Tr}(g^{m*}|H^2(X, \mathbb{Z})) = 5 + \sum \eta_j > \epsilon(7) - 2 = 1. \]

Assume that \( m = 6. \) Since \( g^{3n} \) is symplectic of order 2,

\[ [g^{3n*}] = [1, 1.4, -1.8, 1.9]. \]

From this we infer that

\[ [\eta_1, \ldots, \eta_{17}] = [(\zeta_6 : 2).4, 1.9], [(\zeta_6 : 2).4, \zeta_3 : 2, 1.7], [(\zeta_6 : 2).4, (\zeta_3 : 2).2, 1.5], 
[(\zeta_6 : 2).4, (\zeta_3 : 2).3, 1.3] \text{ or } [(\zeta_6 : 2).4, (\zeta_3 : 2).4, 1]. \]

In any case, \( \text{Tr}(g^{n*}|H^2(X, \mathbb{Z})) > \epsilon(6) - 2 = 0, \) a contradiction.
(1) Assume that $m = 8$. Then $[\zeta_{32} : 16] \subset [g^*]$, so $[-1 : 16] \subset [g^{18*}]$, but $g^{48}$ is symplectic of order 2.

(2) Assume that $m = 8$. Since $g^{2n}$ is symplectic of order 4,

$$[g^{2n*}] = [1, 1.4, (\zeta_4 : 2), -1.6, 1.3].$$

The eigenvalue $-1 \in [g^{2n*}]$ must come from $\zeta_8 \in [g^*]$, so its multiplicity must be divisible by 4. But 6 is not.

Assume that $m = 5$. Then $\zeta_{25}$ or $\zeta_{50} \in [g^*]$, but $\phi(10) + \phi(25) > 22$.

(3) Assume that $m = 8$. Then $\zeta_{64} \in [g^*]$, but $\phi(64) > 22$.

Assume that $m = 4$. Then $[\zeta_{32} : 16] \subset [g^*]$, so $[-1 : 16] \subset [g^{16*}]$, but $g^{16}$ is symplectic of order 2.

\[ \square \]

3. The order 60

Assume that there is an automorphism $g$ of order 60 of a projective K3 surface. It cannot be purely non-symplectic (Main Theorem 3) and cannot be of order 4.15 or 6.10 (Lemma 2.4 and 2.7), hence must be of order 2.30, 3.20 or 5.12. We will rule out the first two possibilities.

**Lemma 3.1.** $\text{ord}(g) \neq 2.30$.

**Proof.** Suppose that $\text{ord}(g) = 2.30$. Since $g^{30}$ is symplectic of order 2,

$$[g^{30*}] = [1, 1.8, -1.8, 1.5]$$

where 1.8 come from $\zeta_{30} : 8$ in $[g^*]$. The eigenvalues $-1.8$ in $[g^{30*}]$ must come from either $\zeta_{20} : 8$ or a combination of $\zeta_4 : 2$ and $\zeta_{12} : 4$ in $[g^*]$, but not from $\zeta_{60}$, since $\phi(60) > 8$. The eigenvalues 1.5 in $[g^{30*}]$ come, as the 30-th power, from a combination of 1, $-1$, $\zeta_3 : 2$, $\zeta_6 : 2$, $\zeta_5 : 4$, $\zeta_{10} : 4$ in $[g^*]$. Note that 1 \in $[g^*]$, corresponding to a $g$-invariant ample divisor class.

Claim: $\zeta_{20} : 8$ do not appear in $[g^*]$.

Suppose they do. Since $\zeta_{15}^{15} = -1$ and $\zeta_{20}^{15} = \zeta_4^3$, we see that

$$[g^{15*}] = [1, -1.8, (\zeta_4 : 2), -1.8, \eta_1, \ldots, \eta_5].$$

Since $[\eta_1, \ldots, \eta_5]$ comes from a combination of 1, $-1$, $\zeta_3 : 2$, $\zeta_6 : 2$, $\zeta_5 : 4$, $\zeta_{10} : 4$ in $[g^*]$, we see that $\sum \eta_j \leq 5$ and

$$\text{Tr}(g^{15*} | H^2(X, \mathbb{Z})) = 1 - 8 + 0 + \sum \eta_j \leq -2.$$ 

On the hand, since $\text{Fix}(g^{15}) \subset \text{Fix}(g^{30})$ and $\text{Fix}(g^{30})$ consists of 8 points, we see that

$$-2 \leq \text{Tr}(g^{15*} | H^2(X, \mathbb{Z})) \leq \text{Tr}(g^{30*} | H^2(X, \mathbb{Z})) = 6.$$ 

It follows that $[\eta_1, \ldots, \eta_5] = [1, \ldots, 1]$, hence none of $-1$, $\zeta_6$, $\zeta_{10}$ can appear in $[g^*]$, as their 15-th power is $-1$. If $[\eta_1, \ldots, \eta_5]$ comes from a combination of 1, $\zeta_3 : 2$ in $[g^*]$, then $[g^{3*}] = [1, (\zeta_{10} : 4), 2, \zeta_{20} : 8, 1.5]$, hence

$$\text{Tr}(g^{3*} | H^2(X, \mathbb{Z})) = 1 + 2 + 0 + 5 > \text{Tr}(g^{30*} | H^2(X, \mathbb{Z})).$$
Suppose that \( \text{ord}(g) \neq 3.20 \). This case is similar to the previous case. Since \( g^{20} \) is symplectic of order 3,
\[
[g^{20}] = [1, 1.8, (\zeta_3:2), 6, 1]
\]
where 1.8 come from \( \zeta_{20} : 8 \) in \([g^*]\). The eigenvalues \((\zeta_3:2), 6\) in \([g^{20}]\) come from a combination of \( \zeta_3 : 2, \zeta_6 : 2, \zeta_{12} : 4, \zeta_{15} : 8, \zeta_{30} : 8 \) in \([g^*]\), but not from
\(\zeta_{60}\), since \(\phi(60) > 12\). Note that \(1 \in [g^*]\), corresponding to a \(g\)-invariant ample divisor class.

Claim: Neither \([\zeta_{15}:8]\) nor \([\zeta_{30}:8]\) appears in \([g^*]\).
Suppose that one of the two does. Since \(\zeta_{15}^{10} = \zeta_2^3\) and \(\zeta_{30}^{10} = \zeta_3\), we see that \([g^{10*}] = [1, -1.8, (\zeta_3:2), 4, \eta_1, \ldots, \eta_4, 1]\).
Since \([\eta_1, \ldots, \eta_4]\) comes from a combination of \(\zeta_3:2\) and \(\zeta_6:2\) in \([g^*]\), we see that \(\sum \eta_j \leq 2\) and
\[
\text{Tr}(g^{10*}|H^2(X, \mathbb{Z})) = 1 - 8 - 4 + \sum \eta_j + 1 \leq -8 < -2.
\]
On the hand, since \(\text{Fix}(g^{10}) \subset \text{Fix}(g^{20})\) and \(\text{Fix}(g^{20})\) consists of 6 points, we see that
\[
-2 \leq \text{Tr}(g^{10*}|H^2(X, \mathbb{Z})) \leq \text{Tr}(g^{20*}|H^2(X, \mathbb{Z})) = 4.
\]
This proves the claim.

By the claim, the eigenvalues \((\zeta_3:2), 6\) in \([g^{20*}]\) come from a combination of \(\zeta_3:2\), \(\zeta_6:2\), \(\zeta_{12}:4\) in \([g^*]\). In any case,
\([g^{4*}] = [1, (\zeta_5:4), 2, (\zeta_5:2), 6, 1]\),
thus \(\text{Tr}(g^{4*}|H^2(X, \mathbb{Z})) = -6 < -2\), a contradiction to \(\text{Fix}(g^4) \subset \text{Fix}(g^{20})\). \(\square\)

Example 3.3. In char \(p \neq 2, 3, 5\), the K3 surface
\[X: y^2 + x^3 + t^{11} - t = 0\]
admits an automorphism of order 60 = 5.12
\[g(t, x, y) = (\zeta_{10}t, \zeta_{30}x, \zeta_{20}y).\]
The surface \(X\) has 12 type \(II\)-fibers at \(t = \infty\) and \(t^{11} - t = 0\).

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