Computing the $A_\alpha -$ eigenvalues of a bug

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Abstract

Let $G$ be a simple undirected graph. For $\alpha \in [0,1]$, let

$$A_\alpha (G) = \alpha D(G) + (1-\alpha) A(G),$$

where $A(G)$ is the adjacency matrix of $G$ and $D(G)$ is the diagonal matrix of the degrees of $G$. In particular, $A_0(G) = A(G)$ and $A_1(G) = \frac{1}{2} Q(G)$ where $Q(G)$ is the signless Laplacian matrix of $G$. A bug $B_{p,q,r}$ is a graph obtained from a complete graph $K_p$ by deleting an edge and attaching paths $P_q$ and $P_r$ to its ends. In [4], Hansen and Stevanović proved that, among the graphs $G$ of order $n$ and diameter $d$, the largest spectral radius of $A(G)$ is attained by the bug $B_{n-d+2,\lfloor d/2 \rfloor,\lfloor d/2 \rfloor}$. In [5], Liu and Lu proved the same result for the spectral radius of $Q(G)$. Let $\rho_\alpha(G)$ be the spectral radius of $A_\alpha(G)$. In this note, for a bug $B$ of order $n$ and diameter $d$, it is shown that $(n-d+2)\alpha - 1$ is an eigenvalue of $A_\alpha(B)$ with multiplicity $n-d-1$ and that the other eigenvalues, among them $\rho_\alpha(B)$, can be computed as the eigenvalues of a symmetric tridiagonal matrix of order $d + 1$. It is also shown that $\rho_\alpha(B_{n-d+2,\lfloor d/2 \rfloor,\lfloor d/2 \rfloor})$ can be computed as the spectral radius of a symmetric tridiagonal matrix of order $\frac{d}{2} + 1$ whenever $d$ is even.

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1 Introduction

Let $G = (V(G),E(G))$ be a simple undirected graph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ be the diagonal matrix of the degrees of $G$. Let $A(G)$ be the adjacency matrix of $G$. In [6], the family of matrices $A_\alpha(G)$,

$$A_\alpha (G) = \alpha D(G) + (1-\alpha) A(G)$$

with $\alpha \in [0,1]$, is introduced together with a number of some basic results and several open problems.

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Observe that $A_0(G) = A(G)$ and $A_{1/2}(G) = \frac{1}{2}Q(G)$.  
Since $A_1(G) = D(G)$, from now on, we take $\alpha \in [0,1)$.

A bug $B_{p,q,r}$ is a graph obtained from a complete graph $K_p$ by deleting an edge $uv$ and attaching the paths $P_q$ and $P_r$ by one of their end vertices at $u$ and $v$, respectively. Observe that $B_{p,q,r}$ is a graph of order $p + q + r - 2$ and diameter $q + r$.

**Example 1** For instance $B_{5,3,4}$ is the graph

![Graph](image)

of 10 vertices and diameter 7 which is obtained from $K_5$ deleting the edge $uv$ and attaching the paths $P_3$ and $P_4$ at $u$ and $v$ respectively.

Let $\rho(M)$ be the spectral radius of the matrix $M$.

In [4], Hansen and Stevanović proved the following result:

**Theorem 1** Let $G$ be a graph of order $n$ and diameter $d$. If $d = 1$, then $\rho(A(G)) = \rho(A(K_n))$. If $d \geq 2$, then

$$\rho(A(G)) \leq \rho(A(B_{n-d+2,\lfloor d/2 \rfloor,\lceil d/2 \rceil})).$$

The equality holds if and only if $G = B_{n-d+2,\lfloor d/2 \rfloor,\lceil d/2 \rceil}$.

In [5], Liu and Lu proved the same result for the spectral radius of $Q(G)$:

**Theorem 2** Let $G$ be a graph of order $n$ and diameter $d$. If $d = 1$, then $\rho(Q(G)) = \rho(Q(K_n))$. If $d \geq 2$, then

$$\rho(Q(G)) \leq \rho(Q(B_{n-d+2,\lfloor d/2 \rfloor,\lceil k/2 \rceil})).$$

The equality holds if and only if $G = B_{n-d+2,\lfloor d/2 \rfloor,\lceil d/2 \rceil}$.

Let $\rho_\alpha(G)$ be the spectral radius of $A_\alpha(G)$. From the Perron - Frobenius Theory for non-negative matrices, for a connected graph $G$, $\rho_\alpha(G)$ is a simple eigenvalue of $A_\alpha(G)$ having a positive eigenvector.

In this note, for a bug $B$ of order $n$ and diameter $d$, we prove that $(n - d + 2)\alpha - 1$ is an eigenvalue of $A_\alpha(B)$ with multiplicity $n - d - 1$ and that the other eigenvalues, among them $\rho_\alpha(B)$, can be computed as the eigenvalues of a symmetric tridiagonal matrix of order $d + 1$. We also prove that $\rho_\alpha(B_{n-d+2,d/2,d/2})$ can be computed as the spectral radius of a symmetric tridiagonal matrix of order $\frac{d}{2} + 1$ whenever $d$ is even. This note extends the results obtained in [1] for the signless Laplacian matrix of bugs of order $n$ and diameter $d$. 

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2 On the $\alpha$–spectrum of a bug

We recall the notion of the $H$–join of graphs [2, 3]. Let $H$ be a graph of order $k$ with $V(H) = \{1, \ldots, k\}$. Let $\{G_1, \ldots, G_k\}$ be a set of pairwise vertex disjoint graphs. For $1 \leq j \leq k$, the vertex $j \in V(H)$ is assigned to the graph $G_j$. The $H$–join of the graphs $G_1, \ldots, G_k$, denoted by

$$G = \bigvee_{H} \{G_j : 1 \leq j \leq k\},$$

is the graph $G$ obtained from the graphs $G_1, \ldots, G_k$ and the edges connecting each vertex of $G_i$ with all the vertices of $G_j$ if and only if $ij \in E(H)$. That is, $G$ is the graph with vertex set

$$V(G) = \bigcup_{i=1}^{k} V(G_i)$$

and edge set

$$E(G) = \left( \bigcup_{i=1}^{k} E(G_i) \right) \cup \left( \bigcup_{ij \in E(H)} \{uv : u \in V(G_i), v \in V(G_j)\} \right).$$

Clearly if each $G_i$ is a graph of order of $n_i$, then $H$–join of $G_1, \ldots, G_k$ is a graph of order $n_1 + n_2 + \ldots + n_k$.

In particular, for $i = 1, \ldots, d - 1$, $B_{n-d+2,i,d-i}$ is the $P_{d+1} – join$ of the regular graphs $G_1 = \ldots = G_i = K_1, G_{i+1} = K_{n-d}, G_{i+2} = \ldots = G_{d+1} = K_1$. Since $B_{n-d+2,i,d-i}$ and $B_{n-d+2,i,d-i}$ are isomorphic graphs, we may take $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Example 2 Below are displayed the non-isomorphic bugs of order 10 and diameter 7:

$B_{5,1,6}$ is the $P_8$–join of $G_1 = K_1, G_2 = K_3$ and $G_i = K_1$ for $i = 3, \ldots, 8$:

![Diagram](image1)

$B_{5,2,5}$ is the $P_8$–join of $G_1 = G_2 = K_1, G_3 = K_3$ and $G_i = K_1$ for $i = 4, \ldots, 8$:

![Diagram](image2)
$B_{5,3,4}$ is the $P_8 - \text{join}$ of $G_1 = G_2 = G_3 = K_1, G_4 = K_3$ and $G_i = K_1$ for $i = 5, \ldots, 8$:

![Graph](image)

It is immediate that if $G$ is a $r-$regular graph of order $n$ then $A_\alpha(G) = arI_n + (1 - \alpha)A(G)$ and $\rho(A(G)) = r$. Hence $\rho_\alpha(G) = ar + (1 - \alpha)r = r$ for any $r$-regular graph $G$.

Let $\sigma(M)$ be the spectrum of a matrix $M$. In [2], Theorem 5, the spectrum of the adjacency matrix of the $H$-join of regular graphs is obtained. The version of the corresponding result for the spectrum of $A_\alpha$ is given below and its proof is similar.

**Theorem 3** Let $H$ be a graph of order $k$. Let $G = \bigvee_H \{G_j : 1 \leq j \leq k\}$. If each $G_j$ is a $r_j$-regular graph of order $n_j$ then

$$\sigma(A_\alpha(G)) = \bigcup_{G_j \neq K_1} \{as_j + \lambda : \lambda \in \sigma(A_\alpha(G_j)) \setminus \{r_j\}\} \cup \sigma(M(G))$$

where $M(G)$ is a matrix of order $k \times k$ given by

$$M(G) = \begin{bmatrix} as_1 + r_1 & \beta \delta_{12} \sqrt{n_1 n_2} & \cdots & \beta \delta_{1k} \sqrt{n_1 n_k} \\ \beta \delta_{12} \sqrt{n_1 n_2} & as_2 + r_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \beta \delta_{(k-1)k} \sqrt{n_{k-1} n_k} \\ \beta \delta_{1k} \sqrt{n_1 n_k} & \cdots & \beta \delta_{(k-1)k} \sqrt{n_{k-1} n_k} & as_k + r_k \end{bmatrix} \quad (1)$$

with

$$\beta = 1 - \alpha,$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } ij \in E(H) \\ 0 & \text{otherwise} \end{cases}$$

and, for $j = 1, 2, \ldots, k$,

$$s_j = \sum_{j \in V(H)} n_i.$$  

For brevity, let $B(i) = B_{n-d+i,d-i}$ with $1 \leq i \leq \left[\frac{d}{2}\right]$.

Since, for each $1 \leq i \leq k$, $B(i)$ is the $P_{d+1} - \text{join}$ of the regular graphs $G_1 = \ldots = G_i = K_1, G_{i+1} = K_{n-d}, G_{i+2} = \ldots = G_{d+1} = K_1$, Theorem 3 can be applied to determine the spectrum of $A_\alpha(B(i))$ with the advantage that the matrix $M(B(i))$ in (1), of order $d + 1$, is a symmetric
A tridiagonal matrix. For instance, the matrices $M(B(1))$, $M(B(2))$ and $M(B(3))$ are

$$
M(B(1)) = \begin{bmatrix}
\alpha(n-d) & \beta\sqrt{n-d} & & & \\
\beta\sqrt{n-d} & 2\alpha + n - d - 1 & \beta\sqrt{n-d} & & \\
& \beta\sqrt{n-d} & 2\alpha + n - d - 1 & \beta\sqrt{n-d} & \\
& & \beta\sqrt{n-d} & 2\alpha + n - d - 1 & \beta\sqrt{n-d} \\
& & & \beta\sqrt{n-d} & \alpha(n-d+1) \ \\
& & & & \beta
\end{bmatrix},
$$

$$
M(B(2)) = \begin{bmatrix}
\alpha & \beta & & & \\
\beta & \alpha(n-d+1) & \beta\sqrt{n-d} & & \\
& \beta\sqrt{n-d} & 2\alpha + n - d - 1 & \beta\sqrt{n-d} & \\
& & \beta\sqrt{n-d} & 2\alpha + n - d - 1 & \beta\sqrt{n-d} \\
& & & \beta\sqrt{n-d} & \alpha(n-d+1) \ \\
& & & & \beta
\end{bmatrix},
$$

and

$$
M(B(3)) = \begin{bmatrix}
\alpha & \beta & & & \\
\beta & 2\alpha & \beta & & \\
& \beta & \alpha(n-d+1) & \beta\sqrt{n-d} & \\
& & \beta\sqrt{n-d} & 2\alpha + n - d - 1 & \beta\sqrt{n-d} \\
& & & \beta\sqrt{n-d} & \alpha(n-d+1) \\
& & & & \beta
\end{bmatrix}.
$$

**Definition 1** Let $\alpha \in [0,1)$ and $\beta = 1 - \alpha$. Let

$$
R = \begin{bmatrix}
\alpha(n-d+1) & \beta\sqrt{n-d} & 0 \\
\beta\sqrt{n-d} & 2\alpha + (n-d-1) & \beta\sqrt{n-d} \\
0 & \beta\sqrt{n-d} & \alpha(n-d+1)
\end{bmatrix},
$$

$$
T_1 = [\alpha]
$$
and, for $s \geq 2$, let

$$T_s = \begin{bmatrix}
\alpha & \beta \\
\beta & 2\alpha & \beta \\
\beta & \ddots & \ddots \\
\ddots & 2\alpha & \beta \\
\beta & 2\alpha
\end{bmatrix}$$

of order $s \times s$.

Let $I$ be the identity matrix, $0$ the zero matrix, $J$ the exchange matrix (the matrix with ones in the secondary diagonal and zeros elsewhere) and $F$ the matrix whose entries are zeros except for the entry in the last row and first column which is equal to 1. All of them of the appropriate order.

The use of Theorem 3 yields to:

**Theorem 4** Let $\alpha \in [0, 1)$ and $\beta = 1 - \alpha$. The eigenvalues of $A_\alpha(B(i))$ are $(n - d + 2)\alpha - 1$ with multiplicity $n - d - 1$ and the other eigenvalues, among them $\rho_\alpha(B(i))$, can be computed as the eigenvalues of the $(d + 1) \times (d + 1)$ symmetric tridiagonal matrix

$$M(B(i)) = \begin{bmatrix}
X_i & \beta F \\
\beta F^T & JT_{d-i-1}J
\end{bmatrix}$$

where

$$X_i = \begin{bmatrix}
\alpha(n - d) & \beta \sqrt{n - d} & 0 \\
\beta \sqrt{n - d} & 2\alpha + n - d - 1 & \beta \sqrt{n - d} \\
0 & \beta \sqrt{n - d} & \alpha(n - d + 1)
\end{bmatrix}$$

whenever $i = 1$, and

$$X_i = \begin{bmatrix}
T_{i-1} & \beta F \\
\beta F^T & R
\end{bmatrix}$$

whenever $2 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$.

Since each $M(B(i))$ is a tridiagonal matrix with nonzero codiagonal entries, its eigenvalues are simple.

**Example 3** Consider the bug $B_{8,2,3}$
This bug is the $P_6$−join of $G_1 = G_2 = K_1$, $G_3 = K_6$ and $G_i = K_1$ for $i = 4, 5, 6$. From Theorem 4, to four decimal places and for $\alpha = 0.6$, the eigenvalues of this bug are $(11 - 5 + 2) \cdot 0.6 - 1 = 3.8$ with multiplicity $11 - 5 - 1 = 5$ and the eigenvalues of the $6 \times 6$ matrix

\[
M(B(2)) = M(B_{8,2,3}) = \begin{bmatrix}
0.6 & 0.4 & 0.9798 \\
0.4 & 4.2 & 0.9798 \\
0.9798 & 6.2 & 0.9798 \\
0.9798 & 4.2 & 0.4 \\
0.4 & 1.2 & 0.4 \\
0.4 & 0.6 \\
\end{bmatrix}.
\]

The eigenvalues of $M(B_{8,2,3})$ are 0.3909, 0.5539, 1.3521, 3.5403, 4.2486, 6.9144.

**Theorem 5** Let $\alpha \in [0, 1)$ and $\beta = 1 - \alpha$. Let $d \geq 4$ be an even integer. Then $\rho_\alpha(B_{n-d+2, \frac{d}{2}, \frac{d}{2}})$ can be computed as the largest eigenvalue of the symmetric tridiagonal matrix

\[
\begin{bmatrix}
\alpha & \beta \\
\beta & 2\alpha & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
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& & & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & & & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

of order $\frac{d}{2} + 1$.

**Proof** From Theorem 4, the eigenvalues of $B_{n-d+2, \frac{d}{2}, \frac{d}{2}}$ are $(n - d + 2)\alpha + 1$ with multiplicity $n - d - 1$ and the eigenvalues of the matrix

\[
M(B(\frac{d}{2})) = \begin{bmatrix}
S & b & 0 \\
b^T & 2\alpha + n - d + 1 & b^T J \\
0 & J b & J S J \\
\end{bmatrix}
\]

of order $d + 1$ where

\[
S = \begin{bmatrix}
\alpha & \beta \\
\beta & 2\alpha & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
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& & & & & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & & & & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

of order $\frac{d}{2}$, $b^T = [0 \cdots \cdots 0 \beta \sqrt{n - d}]$, $J$ is the reverse matrix and $0$ is the zero matrix, all of them of the appropriate sizes. Consider the orthogonal matrix

\[
Q = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & 0 & J \\
0^T & \sqrt{2} & 0^T \\
-J & 0 & I \\
\end{bmatrix}.
\]

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An easy calculation shows that

$$Q M(B(d))Q^T = \begin{bmatrix}
S & \sqrt{2}b & 0 \\
\sqrt{2}b^T & 2x + n - d - 1 & 0 \\
0 & 0 & JSJ
\end{bmatrix}.$$ 

Then the eigenvalues of $M(B(d))$ are the eigenvalues of $\begin{bmatrix}
S & \sqrt{2}b \\
\sqrt{2}b^T & 2x + n - d - 1
\end{bmatrix}$ and the eigenvalues of $S$. Since the eigenvalues of $S$ strictly interlace the eigenvalues of $\begin{bmatrix}
S & \sqrt{2}b \\
\sqrt{2}b^T & 2x + n - d - 1
\end{bmatrix}$, the result follows.

\[\square\]

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