Semi-quotient mappings and spaces

1 Introduction

The basic aim of this article is to study properties of topological spaces and mappings between them by weakening the continuity and openness conditions. Semi-continuity [1] and irresolute mappings [2] were a consequence of the study of semi-open sets in topological spaces. In [3] Bohn and Lee defined and investigated the notion of $s$-topological groups and in [4] Siddique et. al. defined the notion of $S$-topological groups. In [5] Siab et. al. defined and studied the notion of irresolute-topological groups by using irresolute mappings. Study of $s$-paratopological groups and irresolute-paratopological groups is a consequence of the study of paratopological groups (see [6]). For the study of semi-topological groups with respect to semi-continuity and irresoluteness we refer the reader to Oner’s papers [7–9].

In this paper we continue the study of properties of $s$-topological and irresolute-topological groups. Keeping in mind the existing concepts, semi-quotient topology on a set is defined as a generalization of the quotient topology for spaces and groups. Various results on semi-quotients of topologized groups are proved. A counter example is given to show that the quotient topology is properly contained in the semi-quotient structure. We define also semi-isomorphisms and $S$-isomorphisms between topologized groups and prove that if certain irresolute-topological groups $G$ and $H$ are semi-isomorphic or $S$-isomorphic, then their semi-quotients are semi-isomorphic. Investigation of $s$-openness and $s$-closedness of mappings on $s$-topological groups is also presented.

2 Definitions and preliminaries

Throughout this paper $X$ and $Y$ are always topological spaces on which no separation axioms are assumed. If $f : X \to Y$ is a mapping between topological spaces $X$ and $Y$ and $B$ is a subset of $Y$, then $f^{-1}(B)$ denotes
the pre-image of $B$. By $\text{Cl}(A)$ and $\text{Int}(A)$ we denote the closure and interior of a set $A$ in a space $X$. Our other topological notation and terminology are standard as in [10]. If $(G, \ast)$ is a group, then $e$ or $e_G$ denotes its identity element, and for a given $x \in G$, $\ell_x : G \rightarrow G$, $y \mapsto x \circ y$, and $r_x : G \rightarrow G$, $y \mapsto y \circ x$, denote the left and the right translation by $x$, respectively. The operation $\ast$ we call the multiplication mapping $m : G \times G \rightarrow G$, and the inverse operation $x \mapsto x^{-1}$ is denoted by $i$.

In 1963, N. Levine [1] defined semi-open sets in topological spaces. Since then, many mathematicians have explored different concepts and generalized them by using semi-open sets (see [2, 11–14]). A subset $A$ be a topological space and $\mathcal{S} \mathcal{O}(X)$ the collection of all semi-open sets in $X$, and $\mathcal{S} \mathcal{O}(X, x)$ is the collection of semi-open sets in $X$ containing the point $x \in X$. The complement of a semi-open set is said to be semi-closed; the semi-closure of $A \subset X$, denoted by $s\text{Cl}(A)$, is the intersection of all semi-closed subsets of $X$ containing $A$ [15, 16]. $x \in s\text{Cl}(A)$ if and only if any $U \in \mathcal{S} \mathcal{O}(X, x)$ meets $A$.

Clearly, every open (resp. closed) set is semi-open (resp. semi-closed). It is known that a union of any collection of semi-open sets is again a semi-open set. The intersection of two semi-open sets need not be semi-open whereas the intersection of an open set and a semi-open set is semi-open. Basic properties of semi-open sets and semi-closed sets are given in [1], and [15, 16].

Recall that a set $U \subset X$ is a semi-neighbourhood of a point $x \in X$ if there exists $A \in \mathcal{S} \mathcal{O}(X, x)$ such that $A \subset U$. If a semi-neighbourhood $U$ of a point $x$ is a semi-open set, we say that $U$ is a semi-open neighbourhood of $x$. A set $A \subset X$ is semi-open in $X$ if and only if $A$ is a semi-open neighbourhood of each of its points. Let $X$ be a topological space and $A \subset X$. Then $x \in X$ is called a semi-interior point of $A$ if there exists a semi-open set $U$ such that $x \in U \subset A$. The set of all semi-interior points of $A$ is called a semi-interior of $A$ and is denoted by $\text{si} \text{nt}(A)$. A nonempty set $A$ is pre-open (or locally dense) [17] if $A \subset \text{Int}(\text{Cl}(A))$. A space $X$ is $s$-compact [18], if every semi-open cover of $X$ has a finite subcover. Every $s$-compact space is compact but converse is not always true. For some applications of semi-open sets see [13].

A mapping $f : X \rightarrow Y$ between topological spaces $X$ and $Y$ is called:

- semi-continuous [1] (resp. irresolute [2]) if for each open (resp. semi-open) set $V \subset Y$ the set $f^{-1}(V)$ is semi-open in $X$. Equivalently, the mapping $f$ is semi-continuous (irresolute) if for each $x \in X$ and for each open (semi-open) neighbourhood $V$ of $f(x)$, there exists a semi-open neighbourhood $U$ of $x$ such that $f(U) \subset V$;
- pre-semi-open [2] if for every semi-open set $A$ of $X$, the set $f(A)$ is semi-open in $Y$;
- $s$-open (s-closed) if for every semi-open (semi-closed) set $A$ of $X$, the set $f(A)$ is open (closed) in $Y$;
- $s$-perfect if it is semi-continuous, $s$-closed, surjective, and $f^{-1}(y)$ is $s$-compact relative to $X$, for each $y$ in $Y$.
- semi-homeomorphism [2, 19] if $f$ is bijective, irresolute and pre-semi-open;
- $S$-homeomorphism [4] if $f$ is bijective, semi-continuous and pre-semi-open.

We need also some basic information on (topological) groups; for more details see the excellent monograph [20]. If $G$ is a group and $H$ its normal subgroup, then the canonical projection of $G$ onto the quotient group $G/H$ (sending each $g \in G$ to the coset in $G/H$ containing $g$) will be denoted by $p$. A mapping $f : G \rightarrow H$ between two topological groups is called a topological isomorphism if $f$ is an algebraic isomorphism and a topological homeomorphism.

Let $f : X \rightarrow Y$ be a surjection; a subset $C$ of $X$ is called saturated with respect to $f$ (or $f$-saturated) if $f^{-1}(f(C)) = C$ [21].

3 Semi-quotient mappings

**Definition 3.1.** A mapping $f : X \rightarrow Y$ from a space $X$ onto a space $Y$ is said to be semi-quotient provided a subset $V$ of $Y$ is open in $Y$ if and only if $f^{-1}(V)$ is semi-open in $X$.

Evidently, every semi-quotient mapping is semi-continuous and every quotient mapping is semi-quotient. The following simple examples show that semi-quotient mappings are different from semi-continuous mappings and quotient mappings.
Example 3.2. Let $X = \{1, 2, 3\}$ and let $\mathcal{t}_X = \{\emptyset, X, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{t}_Y = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}\}$ be topologies on $X$ and $Y$. Let $f : X \to Y$ be defined by $f(x) = x, x \in X$. Since $\mathcal{t}_Y \subset \mathcal{t}_X$, the mapping $f$ is continuous, hence semi-continuous. On the other hand, this mapping is not semi-quotient because $f^{-1}(\{1, 3\})$ is semi-open in $X$ although $\{1, 3\}$ is not open in $Y$.

Example 3.3. Let $X = \{1, 2, 3, 4\}, Y = \{a, b\}$, $\mathcal{t}_X = \{\emptyset, X, \{1\}, \{3\}, \{1, 3\}\}$, $\mathcal{t}_Y = \{\emptyset, Y, \{a\}\}$. Define $f : X \to Y$ by: $f(1) = f(3) = f(4) = a; f(2) = b$. The mapping $f$ is not a quotient mapping because it is not continuous. On the other hand, $f$ is semi-quotient: the only proper subset of $Y$ whose preimage is semi-open in $X$ is the set $\{a\}$ which is open in $Y$.

The following proposition is obvious.

Proposition 3.4.

(a) Every surjective semi-continuous mapping $f : X \to Y$ which is either $s$-open or $s$-closed is a semi-quotient mapping.

(b) If $f : X \to Y$ is a semi-quotient mapping and $g : Y \to Z$ a quotient mapping, then $g \circ f : X \to Z$ is semi-quotient.

Proof. We prove only (b). A subset $V \subset Z$ is open in $Z$ if and only if $g^{-1}(V)$ is open in $Y$ (because $g$ is a quotient mapping), while the latter set is open in $Y$ if and only if $f^{-1}(g^{-1}(V))$ is semi-open in $X$ (because $f$ is semi-quotient). So, $V$ is open in $Z$ if and only $(g \circ f)^{-1}(V)$ is semi-open in $X$, i.e. $g \circ f$ is a semi-quotient mapping.

The restriction of a semi-quotient mapping to a subspace is not necessarily semi-quotient. Let $X$ and $Y$ be the spaces from Example 3.3, and $A = \{2, 4\}$. Then $\mathcal{t}_A = \{\emptyset, A\}$. The restriction $f_A : A \to Y$ of $f$ to $A$ is not a semi-quotient mapping because $f_A^{-1}(\{a\}) = \{4\}$ is not semi-open in $A$.

To see when the restriction of a semi-quotient mapping is also semi-quotient we will need the following simple but useful lemmas.

Lemma 3.5 ([22, Theorem 1]). Let $X$ be a topological space, $X_0 \in \text{SO}(X)$ and $A \subset X_0$. Then $A \in \text{SO}(X_0)$ if and only if $A \in \text{SO}(X)$.

Lemma 3.6 ([23, Lemma 2.1]). Let $X$ be a topological space, $X_0$ a subspace of $X$. If $A \in \text{SO}(X_0)$, then $A = B \cap X_0$, for some $B \in \text{SO}(X)$.

Lemma 3.7 ([21]). Let $f : X \to Y$ be a mapping, $A$ a subspace of $X$ saturated with respect to $f$, $B$ a subset of $X$. If $g : A \to f(A)$ is the restriction of $f$ to $A$, then:

1. $g^{-1}(C) = f^{-1}(C)$ for any $C \subset f(A)$;
2. $f(A \cap B) = f(A) \cap f(B)$.

Now we have this result.

Theorem 3.8. Let $f : X \to Y$ be a semi-quotient mapping and let $A$ be a subspace of $X$ saturated with respect to $f$, and let $g : A \to f(A)$ be the restriction of $f$ to $A$. Then:

(a) If $A$ is open in $X$, then $g$ is a semi-quotient mapping;
(b) If $f$ is an $s$-open mapping, then $g$ is semi-quotient.

Proof. (a) Let $V$ be an open subset of $f(A)$. Then $V = W \cap f(A)$ for some open subset $W$ of $X$, so that $g^{-1}(V) = f^{-1}(W \cap f(A)) = f^{-1}(W) \cap A$ is a semi-open set in $A$.

Let now $V$ be a subset of $f(A)$ such that $g^{-1}(V)$ is semi-open in $A$. We have to prove that $V$ is open in $f(A)$. Since $g^{-1}(V)$ is semi-open in $A$ and $A$ is open in $X$ we have that $g^{-1}(V)$ is semi-open in $X$. By Lemma 3.7, $g^{-1}(V) = f^{-1}(V)$; the set $f^{-1}(V)$ is semi-open in $X$ since $f$ is semi-quotient, hence $g^{-1}(V)$ is semi-open in $f(A)$. This means that $V$ is open in $Y$ and thus in $f(A)$. This completes the proof that $g$ is a semi-quotient mapping.
(b) Let now \( f \) be \( s \)-open and \( V \) a subset of \( f(A) \) such that \( g^{-1}(V) \) is semi-open in \( A \). Again we must prove that \( V \) is open in \( f(A) \). Since \( g^{-1}(V) = f^{-1}(V) \) and \( g^{-1}(V) \) is semi-open in \( A \), by Lemma 3.6 we have \( f^{-1}(V) = U \cap A \), for some \( U \) semi-open in \( X \). As \( f \) is surjective, it holds \( f(f^{-1}(V)) = V \). By Lemma 3.7, then \( V = f(f^{-1}(V)) = f(U \cap A) = f(U) \cap f(A) \). The set \( f(U) \) is open in \( Y \) because \( f \) is \( s \)-open, so that \( V \) is open in \( f(A) \). Other part is the same as in (a). \( \square \)

As a complement to Proposition 3.4 we have the following two theorems.

**Theorem 3.9.** Let \( X, Y \) and \( Z \) be topological spaces, \( f : X \to Y \) a semi-quotient mapping, \( g : Y \to Z \) a mapping. Then the mapping \( g \circ f : X \to Z \) is semi-quotient if and only if \( g \) is a quotient mapping.

*Proof.* If \( g \) is a quotient mapping, then \( g \circ f \) is semi-quotient as the composition of a semi-quotient and a quotient mapping (Proposition 3.4).

Conversely, let \( g \circ f \) be semi-quotient. We have to prove that a subset \( V \) of \( Z \) is open if and only if \( g^{-1}(V) \) is open. Since \( f^{-1}(V) \) is open in \( Y \) for \( f \) is semi-quotient as the composition of a semi-quotient and a quotient mapping (Proposition 3.4).

**Theorem 3.10.** Let \( f : X \to Y \) be a mapping and \( g : X \to Z \) a mapping which is constant on each set \( f^{-1}([y]) \), \( y \in Y \). Then \( g \) induces a mapping \( h : Y \to Z \) such that \( g = h \circ f \).

1. If \( f \) is pre-semi-open and irresolute, then \( h \) is a semi-continuous mapping if and only if \( g \) is semi-continuous;
2. If \( f \) is semi-quotient, then \( h \) is continuous if and only if \( g \) is semi-continuous.

*Proof.* Since \( g \) is constant on the set \( f^{-1}([y]) \), \( y \in Y \), then for each \( y \in Y \), the set \( g(f^{-1}([y])) \) is a one-point set in \( Z \), say \( h(y) \). Define now \( h : Y \to Z \) by the rule

\[
h(y) = g(f^{-1}(y)) \quad (y \in Y).
\]

Then for each \( x \in X \) we have

\[
g(x) = g(f^{-1}(f(x))) = h(f(x)),
\]

i.e. \( g = h \circ f \).

1. Suppose \( g \) is a semi-continuous mapping. If \( V \) is an open set in \( Z \), then \( h^{-1}(V) = f(g^{-1}(V)) \in SO(Y) \) because \( f \) is pre-semi-open and \( g^{-1}(V) \) is semi-open in \( X \). Thus \( h \) is a semi-continuous mapping.

Conversely, suppose \( h \) is a semi-continuous. Let \( V \) be an open set in \( Z \). The set \( g^{-1}(V) = f^{-1}(h^{-1}(V)) \) is semi-open in \( X \) because \( f \) is irresolute and \( h^{-1}(V) \in SO(Y) \). So, \( g \) is semi-continuous.

2. If \( g \) is semi-continuous, then for any open set \( V \) in \( Z \) we have \( g^{-1}(V) \) is a semi-open set in \( X \). But, \( g^{-1}(V) = f^{-1}(h^{-1}(V)) \). Since \( f \) is semi-quotient it follows that \( h^{-1}(V) \) is open in \( Y \). So, \( h \) is continuous.

Conversely, suppose \( h \) is continuous. For a given open set \( V \) in \( Z \), \( h^{-1}(V) \) is an open set in \( Y \). We have then \( g^{-1}(V) = f^{-1}(h^{-1}(V)) \) is semi-open in \( X \) because \( f \) is semi-quotient. Hence \( g \) is semi-continuous. \( \square \)

At the end of this section we describe now a typical construction which shows how the notion of semi-quotient mappings may be used to get a topology or a topology-like structure on a set.

**Construction:** Let \( X \) be a topological space and \( Y \) a set. Let \( f : X \to Y \) be a mapping. Define

\[
s_{\tau_Q} := \{ V \subset Y : f^{-1}(V) \in SO(X) \}.
\]

It is easy to see that the family \( s_{\tau_Q} \) ia a generalized topology on \( Y \) (i.e. \( \emptyset \in s_{\tau_Q} \) and union of any collection of sets in \( s_{\tau_Q} \) is again in \( s_{\tau_Q} \)) generated by \( f \); we call it the *semi-quotient generalized topology*. But \( s_{\tau_Q} \) need not be a topology on \( Y \). It happens if \( X \) is an extremally disconnected space, because in this case the intersection of two semi-open sets in \( X \) is semi-open \([24]\). It is trivial fact that in the latter case \( s_{\tau_Q} \) is the finest topology \( \sigma \) on \( Y \) such that \( f : X \to (Y, \sigma) \) is semi-continuous. In fact, \( f : X \to (Y, s_{\tau_Q}) \) is a quotient mapping in this case.

In particular, let \( \rho \) be an equivalence relation on \( X \). Let \( p : X \to X/\rho \) be the natural (or canonical) projection from \( X \) onto the quotient set \( X/\rho \); for each \( x \in X \), \( p \) sends \( x \) to the equivalence class \( \rho(x) \). Then the family \( s_{\tau_Q} \)
generated by $p$ is a generalized topology on the quotient set $Y/\rho$, and a topology when $X$ is extremally disconnected. This topology will be called the semi-quotient topology on $X/\rho$. Observe, that we forced the mapping $p$ to be semi-continuous, that is semi-quotient.

This kind of construction will be applied here to topologized groups: to $s$-topological groups and irresolute-topological groups.

The following example shows that a quotient topology on a set generated by a mapping and the semi-quotient topological groups.

Example 3.11. Let the set $X = \{1, 2, 3, 4\}$ be endowed with the topology

$$\tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$ 

Then the set $SO(X)$ is

$$\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$ 

Define the relation $R$ on $X$ by $xRy$ if and only if $x + y$ is even. Therefore,

$$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$$

is an equivalence relation, and $X/R = \{\{R(1), R(2)\} = \{\{1, 3\}, \{2, 4\}\}$. Let $p : X \rightarrow X/R$ be the canonical projection. Then, $p^{-1}(R(1)) = \{1, 3\} \in SO(X)$, and $p^{-1}(R(2)) = \{2, 4\} \in SO(X)$, so that

$$s\tau_\Omega = \{\emptyset, X/R, \{R(1)\}, \{R(2)\}\}$$

is the semi-quotient topology on $X/R$. On the other hand, $p^{-1}(\{R(1)\}) = \{1, 3\} \in \tau$, but $p^{-1}(\{R(2)\}) = \{2, 4\} \notin \tau$. Therefore, the quotient topology on $X/R$ is

$$\tau_\Omega = \{\emptyset, X/R, \{R(1)\}\}.$$

4 Topologized groups

In this section we give some information on $s$-topological groups and irresolute-topological groups introduced and studied first in [4] and [5], respectively.

Definition 4.1 ([3]). An $s$-topological group is a group $(G, \ast)$ with a topology $\tau$ such that for each $x, y \in G$ and each neighbourhood $W$ of $xy^{-1}$ there are semi-open neighbourhoods $U$ of $x$ and $V$ of $y$ such that $U \ast V^{-1} \subset W$.

Definition 4.2 ([5]). A triple $(G, \ast, \tau)$ is an irresolute-topological group if $(G, \ast)$ is a group, and $\tau$ a topology on $G$ such that for each $x, y \in G$ and each semi-open neighbourhood $W$ of $xy^{-1}$ there are semi-open neighbourhoods $U$ of $x$ and $V$ of $y$ such that $U \ast V^{-1} \subset W$.

Lemma 4.3 ([25]). If $(G, \ast, \tau)$ is an $s$-topological group, $y \in G$, and $K$ an $s$-compact subset of $G$, then $y \ast K^{-1}$ is $s$-compact in $G$. In particular, $K^{-1}$ is $s$-compact.

Theorem 4.4. If a mapping $f : X \rightarrow Y$ between topological spaces $X$ and $Y$ is $s$-perfect, then for any compact subset $K$ of $Y$, the pre-image $f^{-1}(K)$ is an $s$-compact subset of $X$.

Proof. Let $\{U_i : i \in \Lambda\}$ be a semi-open cover of $f^{-1}(K)$. Then for each $x \in K$ the set $f^{-1}(x)$ can be covered by finitely many $U_i$; let $U(x)$ denote their union. Then $O(x) = Y \setminus f(X \setminus U(x))$ is an open neighbourhood of $x$ in $Y$ because $f$ is an $s$-closed map. So, $K \subset \bigcup_{x \in K} O(x)$, and because $K$ is assumed to be compact, there are finitely many points $x_1, x_2, \ldots, x_n$ in $K$ such that $K \subset \bigcup_{i=1}^n O(x_i)$. It follows that $f^{-1}(K) \subset \bigcup_{i=1}^n f^{-1}(O(x_i)) \subset \bigcup_{i=1}^n U(x_i)$, hence $f^{-1}(K)$ is $s$-compact in $X$. \qed
The following results are related to $s$-topological groups, and they are generalizations of some results for topological groups.

**Theorem 4.5.** Let $G$, $H$ and $K$ be $s$-topological groups, $\varphi : G \to H$ a semi-continuous homomorphism, $\psi : G \to K$ an irresolute endomorphisms, such that $\ker \psi \subset \ker \varphi$. Assume also that for each open neighbourhood $U$ of $e_H$ there is a semi-open neighbourhood $V$ of $e_K$ with $\psi^{-1}(V) \subset \varphi^{-1}(U)$. Then there is a semi-continuous homomorphism $f : K \to H$ such that $\varphi = f \circ \psi$.

**Proof.** The existence of a homomorphism $f$ such that $\varphi = f \circ \psi$ is well-known fact in group theory. We verify the semi-continuity of $f$. Suppose $U$ is an open neighbourhood of $e_H$ in $H$. By our assumption, there is a semi-open neighbourhood $V$ of $e_K$ in $K$ such that $\psi^{-1}(V) \subset \varphi^{-1}(U)$. Then $f = f \circ \psi$ implies $f(V) = \psi(\psi^{-1}(V)) \subset \varphi(\varphi^{-1}(U)) = U$, which means that $f$ is semi-continuous at the identity element $e_K$ of $K$. By [4, Theorem 3.5], $f$ is semi-continuous on $K$.

**Theorem 4.6.** Suppose that $G$, $H$ and $K$ are $s$-topological groups. Let $\varphi : G \to H$ be a semi-continuous homomorphism, $\psi : G \to K$ an irresolute endomorphisms such that $\ker \psi \subset \ker \varphi$. If $\psi$ is pre-semi-open, then there is a semi-continuous homomorphism $f : K \to H$ such that $\varphi = f \circ \psi$.

**Proof.** By Theorem 4.5, there exists a homomorphism $f : K \to H$ satisfying $\varphi = f \circ \psi$. We prove that $f$ is semi-continuous. Let $V$ be an open set in $H$. From $\varphi = f \circ \psi$ it follows $f^{-1}(V) = \psi^{-1}(\varphi^{-1}(V))$. Since, $\psi$ is semi-continuous, the set $\varphi^{-1}(V)$ is semi-open in $G$, and pre-semi-openness of $\psi$ implies that $\psi(\varphi^{-1}(V))$ is semi-open, i.e. $f^{-1}(V)$ is semi-open in $K$. This means that $f$ is semi-continuous.

**Theorem 4.7.** Let $(G, \ast, \tau_G)$ and $(H, \cdot, \tau_H)$ be $s$-topological groups, and $f : G \to H$ a homomorphism of $G$ onto $H$ such that for some non-empty open set $U \subset G$, the set $f(U)$ is semi-open in $H$ and the restriction $f \mid_U : U \to f(U)$ is a pre-semi-open mapping. Then $f$ is pre-semi-open.

**Proof.** We have to prove that if $x \in G$ and $W \in \text{SO}(G, x)$, then $f(W) \in \text{SO}(H, f(x))$. Pick a fixed point $y \in U$ and consider the mapping $\ell_{y \ast x^{-1}} : G \to G$. Evidently, $\ell_{y \ast x^{-1}}(x) = y$, and by [4, Theorem 3.1], $\ell_{y \ast x^{-1}}$ is an $S$-homeomorphism of $G$ onto itself. Thus the set $V = U \cap \ell_{y \ast x^{-1}}(W)$ is semi-open as the intersection of an open set $U$ and a semi-open set $\ell_{y \ast x^{-1}}(W)$, i.e. $V$ is a semi-open neighbourhood of $y$ in $U$. By assumption on $f$, the set $f(V)$ is semi-open in $f(U)$ and also in $H$ by Lemma 3.5. Set $z = f(x \ast y^{-1})$ and consider the mapping $\ell_z : H \to H$. We have $\ell_z(f(y)) = z \ast f(y) = f(x)$. Clearly, $\ell_z \circ f \circ \ell_{y \ast x^{-1}} = f$, hence $(\ell_z \circ f \circ \ell_{y \ast x^{-1}})(W) = f(W)$. However, $\ell_z$ is an $S$-homeomorphism of $H$ onto itself. As $f(W)$ is semi-open in $H$, the set $\ell_z(f(V))$ is also semi-open in $H$. Therefore, the set $f(W)$ contains a semi-open neighbourhood $\ell_z(f(V))$ of $f(x)$ in $H$, so that $f(W) \in \text{SO}(H, f(x))$ as required.

## 5 Semi-quotients of topologized groups

In this section we apply the construction of $s \tau_Q$ described in Section 3 to topologized groups and establish some properties of their semi-quotients.

If $G$ is a topological group and $H$ a subgroup of $G$, we can look at the collection $G/H$ of left cosets of $H$ in $G$ (or the collection $H \backslash G$ of right cosets of $H$ in $G$), and endow $G/H$ (or $G \backslash H$) with the semi-quotient structure induced by the natural projection $p : G \to G/H$. Recall that $G/H$ is not a group under coset multiplication unless $H$ is a normal subgroup of $G$.

The following simple lemmas may be quite useful in what follows.

**Lemma 5.1 ([20]).** Let $p : G \to G/H$ be a canonical projection map. Then for any subset $U$ of $G$, $p^{-1}(p(U)) = U \ast H$.
Lemma 5.2 ([25]). Let \((G, *, \tau)\) be an s-topological group, \(K\) an s-compact subset of \(G\), and \(F\) a semi-closed subset of \(G\). Then \(F * K\) and \(K * F\) are semi-closed subsets of \(G\).

Lemma 5.3 ([4]). Let \((G, *, \tau)\) be an s-topological group. Then each left (right) translation in \(G\) is an \(S\)-homeomorphism. Moreover they and symmetry mappings are actually semi-homeomorphism (see [1, Remark 1]).

Lemma 5.4 ([26]). If \(f : X \to Y\) is a semi-continuous mapping and \(X_0\) is an open set in \(X\), then the restriction \(f |_{X_0} : X_0 \to Y\) is semi-continuous.

Theorem 5.5. Let \((G, *, \tau)\) be an extremally disconnected irresolute-topological group and \(H\) its invariant subgroup. Then \(p : (G, *, \tau) \to (G/H, *, st_{Q})\) is semi-open.

**Proof.** Let \(V \subset G\) be semi-open. By the definition of semi-quotient topology, \(p(V) \subset G/H\) is open if and only if \(p^{-1}(p(V)) \subset G\) is open. By Lemma 5.1 \(p^{-1}(p(V)) = V * H\). Since \(V\) is semi-open, \(V * H\) is semi-open and so \(p(V)\) is semi-open. Hence \(p\) is semi-open. \(\square\)

Theorem 5.6. Let \((G, *, \tau)\) be an extremally disconnected irresolute-topological group, \(H\) its invariant subgroup. Then \((G/H, *, st_{Q})\) is an irresolute-topological group.

**Proof.** First, we observe that \(st_{Q}\) is a topology on \(G/H\). Let \(x \in G, y \in G/H\) and let \(W \subset G/H\) be a semi-open neighbourhood of \((x * H) \backslash (y * H)^{-1}\). By the definition of \(st_{Q}\) (induced by \(p\)), the set \(p^{-1}(W)\) is a semi-open neighbourhood of \(x \backslash (y^{-1})\) in \(G\), and since \(G\) is an irresolute-topological group, there are semi-open sets \(U \subset SO(G, x)\) and \(V \subset SO(G, y)\) such that \(U \backslash V \subset p^{-1}(W)\). By Theorem 5.5, the sets \(p(U) = U \cap H\) and \(p(V) = V \cap H\) are semi-open in \(G/H\), contain \(x \cap H\) and \(y \cap H\), respectively, and satisfy

\[
(U \cap H) \backslash (V \cap H)^{-1} = (U \cap V^{-1}) \cap H = p(U \cap V^{-1}) \subset p^{-1}(W) \subset W.
\]

This just means that \((G/H, *, st_{Q})\) is an irresolute-topological group. \(\square\)

Theorem 5.7. Let \((G, *, \tau)\) be an s-topological group and \(H\) a subgroup of \(G\). Then for every semi-open set \(U \subset G\), the set \(p(U)\) belongs to \(st_{Q}\). In particular, if \(G\) is extremally disconnected, then \(p\) is an s-open mapping from \(G\) to \((G/H, st_{Q})\).

**Proof.** Let \(V \subset G\) be semi-open. By definition of \(st_{Q}\), \(p(V) \in st_{Q}\) if and only if \(p^{-1}(p(V)) \subset G\) is semi-open, i.e. \(V \cap H\) is semi-open in \(G\). But \(V \cap H\) is semi-open in \(G\) because \(V \subset SO(G)\) and \((G, *, \tau)\) is an s-topological group. Clearly, if \(st_{Q}\) is a topology, the last condition actually says that \(p\) is an s-open mapping. \(\square\)

The following theorem is similar to Theorem 5.7.

Theorem 5.8. If \(H\) is an s-compact subgroup of an s-topological group \((G, *, \tau)\), then for every semi-closed set \(F \subset G\), the set \(p(G \setminus F)\) belongs to \(st_{Q}\). If \(st_{Q}\) is a topology, then \(p\) is an s-perfect mapping.

**Proof.** Let \(F \subset G\) be semi-closed. By Lemma 5.2 the set \(p^{-1}(p(F)) = F \cap H\) is semi-closed. By definition of \(st_{Q}\), \((G/H) \setminus (F \cap H) \in st_{Q}\).

Let now \(st_{Q}\) be a topology on \(G/H\). Take any semi-closed subset \(F\) of \(G\). The set \(F \cap H\) is semi-closed in \(G\) and \(F \cap H = p^{-1}(p(F))\). This implies, \(p(F)\) is closed in the semi-quotient space \(G/H\). Thus \(p\) is an s-closed mapping. On the other hand, if \(z \in H\) and \(p(z) = z \cap H\) for some \(z \in G\), then \(p^{-1}(z \cap H) = p^{-1}(p(z)) = z \cap H\), and by Lemmas 4.3 and 5.3 this set is s-compact in \(G\). Therefore, \(p\) is s-perfect. \(\square\)

Corollary 5.9. Let \((G, *, \tau)\) be an extremally disconnected s-topological group and \(H\) its s-compact subgroup. If the semi-quotient space \((G/H, st_{Q})\) is compact, then \(G\) is s-compact.

**Proof.** By Theorem 5.8, the projection \(p : G \to G/H\) is s-perfect. Then by Theorem 4.4 we obtain that \(p^{-1}(p(G)) = G \cap H = G\) is s-compact. \(\square\)
Theorem 5.10. Suppose that \((G, \ast, \tau)\) is an extremally disconnected s-topological group, \(H\) an invariant subgroup of \(G\), \(p : G \to (G/H, \ast_Q)\) the canonical projection. Let \(U\) and \(V\) be semi-open neighbourhoods of \(e\) in \(G\) such that \(V^{-1} \ast V \subseteq U\). Then \(Cl(p(V)) \subseteq p(U)\).

Proof. Let \(p(x) \in Cl(p(V))\). Since \(V \ast x\) is a semi-open neighbourhood of \(x \in G\) and, by Theorem 5.7, \(p\) is \(s\)-open, we have that \(p(V \ast x)\) is an open neighbourhood of \(p(x)\). Therefore, \(p(V \ast x) \cap p(V) \neq \emptyset\). It follows that for some \(a, b \in V\) we have \(p(a \ast x) = p(b)\), that is \(a \ast x \ast h_1 = b \ast h_2\) for some \(h_1, h_2 \in H\). Hence,

\[
x = a^{-1} \ast b \ast h_2 \ast h_1^{-1} = (a^{-1} \ast b) \ast (h_2 \ast h_1^{-1}) \in U \ast H
\]

since \(a^{-1} \ast b \in V^{-1} \ast V \subseteq U\) and \(H\) is a subgroup of \(G\). Therefore, \(p(x) \in p(U \ast H) = U \ast H \ast H = U \ast H = p(U)\).

Theorem 5.11. Let \((G, \ast, \tau)\) be an extremally disconnected irresolute-topological group and \(H\) an invariant subgroup of \(G\). Then the semi-quotient space \((G/H, \ast_Q)\) is regular.

Proof. Let \(W\) be an open neighbourhood of \(p(e_G) = H\) in \(G/H\). By semi-continuity of \(p\), we can find a semi-open neighbourhood \(U\) of \(e_G\) such that \(p(U) \subseteq W\). As \(G\) is extremally disconnected and irresolute-topological group, it follows from \(e_G \ast e_G^{-1} = e_G\) that there is a semi-open neighbourhood \(V\) of \(e_G\) such that \(V \ast V^{-1} \subseteq U\). By Theorem 5.10 we have \(Cl(p(V)) \subseteq p(U) \subseteq W\). By Theorem 5.7, \(p(V)\) is an open neighbourhood of \(p(e_G)\). This proves that \((G/H, \ast_Q)\) is a regular space.

If \((G, \ast)\) is a group, \(H\) its subgroup, and \(a \in G\), then we define the mapping \(\lambda_a : G/H \to G/H\) by \(\lambda_a(x \ast H) = a \ast (x \ast H)\). This mapping is called a left translation of \(G/H\) by \(a\) [20].

Theorem 5.12. If \((G, \ast, \tau)\) is an extremally disconnected irresolute-topological group, \(H\) a subgroup of \(G\), and \(a \in G\), then the mapping \(\lambda_a : G/H \to G/H\) by \(\lambda_a(x \ast H) = a \ast (x \ast H)\) is a semi-homeomorphism and \(p \circ \ell_a = \lambda_a \circ p\) holds.

Proof. Since \(G\) is a group, it is easy to see that \(\lambda_a\) is a (well defined) bijection on \(G/H\). We prove that \(\lambda_a \circ p = p \circ \ell_a\). Indeed, for each \(x \in G\) we have \((p \circ \ell_a)(x) = p(a \ast x) = (a \ast x) \ast H = a \ast (x \ast H) = \lambda_a(p(x)) = (\lambda_a \circ p)(x)\). This is required. It remains to prove that \(\lambda_a\) is irresolute and pre-semi-open.

This follows from the following facts. Let \(x \ast H \in G/H\). For any semi-open neighbourhood \(U\) of \(e_G\), \(p(x \ast U \ast H)\) is a semi-open neighbourhood of \(x \ast H\) in \(G/H\). Similarly, the set \(p(a \ast x \ast U \ast H)\) is a semi-open neighbourhood of \(a \ast x \ast H\) in \(G/H\). Since

\[
\lambda_a(p(x \ast U \ast H)) = p(\ell_a(x \ast U \ast H)) = p(a \ast x \ast U \ast H),
\]

it follows that \(\lambda_a\) is a semi-homeomorphism.

Definition 5.13. A mapping \(f : X \to Y\) is:

- an S-isomorphism if it is an algebraic isomorphism and (topologically) an S-homeomorphism;
- a semi-isomorphism if it is an algebraic isomorphism and a semi-homeomorphism.

Theorem 5.14. Let \((G, \ast, \tau_G)\) and \((H, \cdot, \tau_H)\) be extremally disconnected irresolute-topological groups and \(f : G \to H\) a semi-isomorphism. If \(G_0\) is an invariant subgroup of \(G\) and \(H_0 = f(G_0)\), then the semi-quotient irresolute-topological groups \((G_0/G_0, \ast_Q)\) and \((H/H_0, \ast_Q)\) are semi-isomorphic.

Proof. Let \(p : G \to G/G_0\), \(x \mapsto x \ast G_0\), and \(\pi : H \to H/H_0\), \(f(x_0) \mapsto f(x_0) \cdot H_0\) (\(x_0 \in G_0\)) be the canonical projections. Consider the mapping \(\varphi : G/G_0 \to H/H_0\) defined by

\[
\varphi(x \ast G_0) = f(x) \cdot f(G_0), \quad x \in G, \ y = f(x).
\]

Then for \(x_1 \ast G_0, x_2 \ast G_0 \in G/G_0\) we have

\[
\varphi(x_1 \ast G_0 \ast x_2 \ast G_0) = \varphi(x_1 \ast x_2 \ast G_0) = f(x_1 \ast x_2) \cdot f(G_0) = y_1 \cdot y_2 \cdot H_0 = \varphi(x_1 \ast G_0) \cdot \varphi(x_2 \ast G_0).
\]
i.e. is a homomorphism. Let us prove that is one-to-one. Let be an arbitrary element of . Set . If , then implies and , so is one-to-one.

Next, we have where is a semi-homeomorphism, and and are semi-open, semi-continuous homomorphisms, we conclude that is open and continuous. Hence is a semi-homeomorphism and a semi-isomorphism.

**Theorem 5.15.** Let be an extremally disconnected irresolute-topological group, an invariant subgroup of , an open subgroup of , and the canonical projection. Then the semi-quotient group is semi-isomorphic to the subgroup of .

*Proof.* It is clear that . As is a homomorphism it follows that is a subgroup of . Let be a homomorphism. We have , i.e. is . It is easy to conclude that and are semi-isomorphic.

References

[1] Levine N., Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 1963, 70, 36-41
[2] Crossley S.G., Hildebrand S.K., Semi-topological properties, Fund. Math. 1972, 74, 233-254
[3] Bohn E., Lee J., Semi-topological groups, Amer. Math. Monthly, 1965, 72, 996-998
[4] Siddique Bosan M., Moiz ud Din Khan, Kočinac Lj. D.R., On s-topological groups, Math. Moravica, 2014, 19:2, 35-44
[5] Siab A., Moiz ud Din Khan, Kočinac Lj. D.R., Irresolute-topological groups, Math. moravica, 2015, 19:1, 73-80
[6] Moiz ud Din Khan and Rafaqat Noreen, On Paratopologized Groups, Analele Universitatii din Oradea - Fasc. Matematica, 2016, Tom XXIII:2, 147-157
[7] Oner T., Kandemir M.B., Tanay B., Semi-topological groups with respect to semi-continuity and irresoluteness, J. Adv. stud. topology, 2013, 4:3, 23-28
[8] Oner T., Ozek A., A note on quasi irresolute topological groups, JLTAA, 2016, 5:1, 41-46
[9] Oner T., Ozek A., On semi-topological groups with respect to irresoluteness, Int. J. Recent Sci. Res., 2015, 6:12, 7914-7916
[10] R. Engelking, General Topology, 2nd edition, Heldermann-Verlag, Berlin, 1989
[11] D.R. Anderson, J.A. Jensen, Semi-continuity on topological spaces, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat., 1967, 42, 782-783
[12] T.R. Hamlett, A correction to the paper “Semi-open sets and semi-continuity in topological spaces” by Norman Levine, Proc. Amer. Math. Soc., 1975, 49, 458-460
[13] Nour T.M., A note on some applications of semi-open sets, Internat. J. Math. Math. Sci., 1998, 21, 205-207
[14] Piotrowski Z., On semi-homeomorphisms, Boll. U.M.I., 1979, 5:16-A, 501–509
[15] Crossley S.G., Hildebrand S.K., Semi-closure, Texas J. Sci., 1971, 22, 99-112
[16] Crossley S.G., Hildebrand S.K., Semi-closed sets and semi-continuity in topological spaces, Texas J. Sci., 1971, 22, 123-126
[17] Corson H.H., Michael E., Metrizability of certain countable unions, Illinois J. Math., 1964, 8, 351-360
[18] Carnahan D.A., Some properties related to compactness in topological spaces, Ph.D. Thesis, University of Arkansas, 1973
[19] Lee J.P , On semihomeomorphisms, Internat. J. Math. Math. Sci., 1990, 13, 129-134
[20] Arhangel’ski M.V., Tkachenko M., Topological Groups and Related Structures, Atlantis Studies in Mathematics, Vol. 1, Atlantis Press/World Scientific, Amsterdam-Paris, 2008
[21] Munkres J.R., Topology (Second edition), Prentice Hall, Inc., 2000
[22] Noiri T., On semi-continuous mappings, Atti Accad. Naz. Lincei, Ser., 1973, VIII 54:2, 210-214
[23] Pipitone V., Russo G., Spazi semiaperti e spazi semicompatti, Rend. Circ. Mat. Palermo, 1975, 24:3, 273-285
[24] Njastad O., On some classes of nearly open sets, Pacific J. Math., 1965, 15, 961-970
[25] Siddique Bosan M., Moiz ud Din Khan, A note on s-topological groups, Life Science Journal, 2014, 11:7s, 370-374
[26] Noiri T., On semi-continuous mappings, Atti.Lin.Rend.Sc.Fis.Mat.Natur., 1973, 8:54, 210-215.