The analysis of a time delay fractional COVID-19 model via Caputo type fractional derivative

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Novel coronavirus (COVID-19), a global threat whose source is not correctly yet known, was firstly recognised in the city of Wuhan, China, in December 2019. Now, this disease has been spread out to many countries in all over the world. In this paper, we solved a time delay fractional COVID-19 SEIR epidemic model via Caputo fractional derivatives using a predictor–corrector method. We provided numerical simulations to show the nature of the diseases for different classes. We derived existence of unique global solutions to the given time delay fractional differential equations (DFDEs) under a mild Lipschitz condition using properties of a weighted norm, Mittag–Leffler functions and the Banach fixed point theorem. For the graphical simulations, we used real numerical data based on a case study of Wuhan, China, to show the nature of the projected model with respect to time variable. We performed various plots for different values of time delay and fractional order. We observed that the proposed scheme is highly emphatic and easy to implementation for the system of DFDEs.

KEYWORDS
Caputo fractional derivative, COVID-19 epidemic, fixed point theory, predictor–corrector scheme, SEIR model, time delay

MSC CLASSIFICATION
26A33; 34C60; 92C60; 92D30

INTRODUCTION

Novel coronavirus (COVID-19), a global threat whose source is not correctly yet known, was firstly recognised in the city of Wuhan, China, in December 2019.1,2 Now, this disease has been spread out to many countries in all over the world. From the date of its origin, it grows exponentially in mankind and infected more than 20,254,685 with 738,930 deaths and 13,118,618 recoveries on 11 August throughout the globe. For the preventions to this disease, social control measures have been extended up by increased public awareness such as through social or physical distancing measures, good hygiene and not walking out in the open environment. Billions of people have been infected by this virus, and in some particular cases, the virus has also been recognised in animals. The symptoms of the coronavirus are fever, sneeze, a runny nose, fatigue, dry cough, bilateral lung infiltration to severely ill and breathing problems,3 in which fever and dry cough are two most common symptoms.4 So many newly reported cases have been associated to the travel history from an epidemic region or a meeting history to people from the region in the early outbreak phases. Because a person who was infected by COVID-19 can travel to any other region and spread the virus. This coronavirus outbreak has received considerable global attention when, since 31 Dec 2019, the Wuhan Municipal Health Commission reported 27 cases of viral pneumonia, with 7 critically ill cases. Many researchers and doctors have suggested their specialisation to the study of this virus.
COVID-19 has been studied from different points of view, in many fields of study, including epidemiology, microbiology, environmental and occupational health, virology, veterinary science, economics, sociology and media studies. In reaction to early outbreaks of the virus, the United States, Korea and China come out as the leading countries in COVID-19 research. Other research groups focused on the improvement of healthcare systems and virus prevention propounding that the way to restraining virus transmission is the reduction of human contact and social distance by dint of the enclosure of public transport, school closures, the adjournment of common activities, etc.\textsuperscript{5} To date, there is no specific vaccine or clinically established treatment for coronavirus. Nevertheless, according to the suggestions of different prudent infectious disease specialists, several countries have started using chloroquine combined with azithromycin as alternative drugs. But still there are some countries that successfully controlled the pandemic. These include Iceland, New Zealand, Fiji, Vatican City, Tanzania, Montenegro, Seychelles, Papua New Guinea and Mauritius. On 28 February 2020, New Zealand reported its first coronavirus case and then enforced the one of the strictest lockdowns in the world. The United States, Brazil, India and Peru are the countries which are most infected by this virus. Particularly, the United States is one of the most infected country of this virus with 5,251,446 cases, 166,192 deaths and 2,715,934 recoveries on 11 August. In Brazil, 3,057,470 cases with 101,857 deaths and 2,163,812 recoveries have been recorded. In India, there are 2,267,153 cases with 45,353 deaths and 1,581,640 recoveries founded on August 11. Because of this pandemic, lots of social programmes and activities have been cancelled or postponed. T-20 cricket world cup 2020 in Australia, Summer Olympics which was planned to be held in Tokyo, has been postponed. One of the most popular cricket tournament, Indian Premier League (IPL), has been shifted from India to UAE.

Mathematical models estimate the number of cases in best and worst case scenarios when disease dynamics are unclear and can also be used to estimate the effects of preventive (healthcare) measures. A lot of research articles have been come in literature to analyse the effects of COVID-19 via mathematical modelling.\textsuperscript{6-10} In Khan et al,\textsuperscript{11} authors formulated a new mathematical model for the dynamics of COVID-19 with quarantine and isolation. They studied the dynamics of the COVID-19 using fractal–fractional Atangana–Baleanu derivative. A study on the mathematical modelling of the impact of nonpharmaceutical interventions on the dynamics of COVID-19 with optimal control analysis is given in Ullah and Khan.\textsuperscript{12} A study on solution of a COVID-19 model via new generalised fractional derivative (FD) is done by Erturk and Kumar.\textsuperscript{13} They used a modified predictor–corrector scheme to perform the numerical simulations with the real data of Wuhan, China. The applications of fractional calculus have been received in various branches of Science and Engineering.\textsuperscript{14-18} Unlike an ordinary derivative (OD) operator, an FD operator is non-local in nature. Due to this non-local behaviour of the FD operator, it can formulate processes having memory and hereditary properties. Recently, an attention to the delay fractional differential equations (DFDEs) has considered cause of their applications in the mathematical modelling of real-world problems. The delay differential equation is a differential equation in which the derivative of the function at any time depends on the solution at previous time. It is well known that an ordinary delay differential equation has a unique local solution under some Lipschitz conditions; furthermore, using continuation property, one can derive global solutions as well, but in the noninteger, the existence of unique solutions (local and global) is more intricate because of the noninteger order feature of the equation which implies history dependence of the solutions; hence, among others, the continuation property is not applicable.

The aim of this article is to study the time delay fractional COVID-19 epidemic model using a real numerical data of a case study of Wuhan, China, from the literature to show the nature of the given model. We also performed some analysis to show the role of time delay parameter $\tau$ by the help of Caputo FD with predictor–corrector algorithm. The paper is formulated as follows. In Section 2, we recall some important definitions of the FDs. Section 3 is devoted for the description of the ordinary differential equations (ODE) model following the fractional order model. Existence and uniqueness analysis of the problem are performed in Section 4. Solution of the projected model is done in Section 5. Simulation results are performed in Section 6. A conclusion completes the paper.

## 2 | PRELIMINARIES

Here, we remind some basic definitions and properties.

**Definition 1** (Podlubny\textsuperscript{17}). The Riemann–Liouville (R-L) definition of noninteger order integral of order $\zeta > 0$ of a function $G : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$I_{t}^{\zeta}G(t) = \frac{1}{\Gamma(\zeta)} \int_{0}^{t} (t-\theta)^{\zeta-1}G(\theta)d\theta. \quad (1)$$
**Definition 2** (Podlubny\(^1\)) The R-L definition of noninteger order derivative of order \(\zeta > 0\) of a function \(G : (0, \infty) \to \mathbb{R}\) is defined by

\[
D^\zeta_L G(t) = \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n - \zeta)} \int_0^t (t - \theta)^{n-\zeta-1} G(\theta) d\theta, \tag{2}
\]

where \(n = [\zeta] + 1\) and \([\zeta]\) is the integer part of \(\zeta\).

**Definition 3** (Podlubny\(^1\)). The Caputo definition of noninteger order derivative of order \(\zeta > 0\) of a function \(G : (0, \infty) \to \mathbb{R}\) is defined by

\[
D^\zeta_C G(t) = \frac{1}{\Gamma(n - \zeta)} \int_0^t (t - \theta)^{n-\zeta-1} G^n(\theta) d\theta, \tag{3}
\]

where \(n = [\zeta] + 1\) and \([\zeta]\) is the integer part of \(\zeta\).

**Definition 4** (Podlubny\(^1\)). The Mittag–Leffler type function with one parameter is defined as follows \(E_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)}, \mu > 0, z \in \mathbb{C}\).

## 3 | MODEL DESCRIPTION

### 3.1 | Description of the ODE model

There are so many mathematical models have been introduced in the literature to study the outbreaks of COVID-19. So many researchers have analysed various type of models to study the dynamics of COVID-19 with the case study of different particular countries. In this section, we analyse the time delay ordinary model studied by Cakan\(^{19}\) to introduce the COVID-19 epidemic. The considered model consists of four compartments with individuals of susceptible \(S(t)\), exposed \(E(t)\), infectious \(I(t)\) and recovered \(R(t)\). The author presented and derived the projected model in the sense of ordinary time delay differential equations as follows:

\[
\begin{aligned}
S'(t) & = b - \beta S(t) I(t) - dS(t), \\
E'(t) & = \beta S(t) I(t) - \gamma \beta S(t - \tau) I(t - \tau) e^{-\delta \tau} - dE(t) - \delta E(t), \\
I'(t) & = \gamma \beta S(t - \tau) I(t - \tau) e^{-\delta \tau} - \left[ \vartheta_1 + \vartheta_2 \left(1 - c(t) \right) \right] I(t) - \left[ \theta_1 + \theta_2 c(t) \right] I(t) - dI(t), \\
R'(t) & = \left[ \theta_1 + \theta_2 c(t) \right] I(t) + \delta E(t) - dR(t).
\end{aligned} \tag{4}
\]

In the given ordinary model (4), \(b\) is the birth rate, \(\beta\) is the contact rate of susceptible to infected persons and natural death rate denoted by \(d\), \(\gamma\) and \(\delta\) are the progression rate of exposed humans into infectious population and rate of exposed to removed, respectively. The brief description of all other parameters with the numerical values is given in Table 1. The function \(c(t)\) presents the available opportunities level by healthcare system to public who are infected at time \(t\). It can be observed that all hospital facilities are almost consumed away when \(c(t)\) tends to zero and all hospital facilities (opportunities) can be used fully when \(c(t) = 1\) with respect to the time variable \(t\). \(\tau\) is a time delay corresponding the latent period of the COVID-19.

### Table 1 | Parameter values for simulations

| Parameter | Description | Value/range | Reference |
|-----------|-------------|-------------|-----------|
| \(b\)     | Birth rate  | 3.210       | Estimated |
| \(\beta\) | Contact rate susceptible to infected | \(0.62 \times 10^{-8}\) | Yang and Wang\(^{24}\) |
| \(d\)     | Natural death rate | \(3.57 \times 10^{-5}\) | Khan et al\(^{11}\) |
| \(\gamma\) | Rate of exposed to infected | 0.143 | Yang and Wang\(^{24}\) |
| \(\delta\) | Rate of exposed to removed | 0.006 | Fitted |
| \(c\)     | Level of available opportunities by health care systems | \([0, 1]\) | Yang and Wang\(^{24}\) |
| \(\alpha_1\) | Natural recovery rate of the infectious class | 0.0005 | Fitted |
| \(\alpha_2\) | Recovery rate of the infectious class | 0.0667 | Yang and Wang\(^{24}\) |
| \(\mu_1\) | Minimum disease-induced death rate | 0.01 | Yang and Wang\(^{24}\) |
| \(\mu_1 + \mu_2\) | Maximum disease-induced death rate | 0.02 | Fitted |
3.2 Description of the Caputo fractional model

The theory of the time DFDEs is a well known phenomena of fractional calculus. Now, we generalise the above ODE model in the Caputo FD sense. Here, we are including the death equation also. In this generalisation, we replace the OD operator by the Caputo fractional operator \( {}^C\!D_t^\gamma \). So the generalisation of the given ordinary time delay differential equation system into the time DFDEs system is as follows:

\[
\begin{align*}
{}^C\!D_t^\gamma S(t) &= b - \beta S(t) I(t) - dS(t), \\
{}^C\!D_t^\gamma E(t) &= \beta S(t) I(t) - \gamma \beta S(t - \tau) I(t - \tau) e^{-\tau \gamma} - dE(t) - \delta E(t), \\
{}^C\!D_t^\gamma I(t) &= \gamma \beta S(t - \tau) I(t - \tau) e^{-\tau \gamma} - [v_1 + v_2 (1 - c(t))] I(t) - [\theta_1 + \theta_2 c(t)] I(t) - dI(t), \\
{}^C\!D_t^\gamma R(t) &= [\theta_1 + \theta_2 c(t)] I(t) + \delta E(t) - dR(t), \\
{}^C\!D_t^\gamma D(t) &= [v_1 + v_2 (1 - c(t))] I(t),
\end{align*}
\]

where \( {}^C\!D_t^\gamma \) denotes the Caputo FD operator.

By means of the next generation matrix method, here, we find the basic reproduction number \( R_0 \) for the fractional model (5) as follows:

Let us assume \( Y = (I, S)^T \). Then, the model (5) can be written as

\[ {}^C\!D_t^\gamma Y = F(Y) - V(Y), \]

where

\[ F(Y) = \begin{bmatrix} \gamma \beta S(t - \tau) I(t - \tau) e^{-\tau \gamma} \\ 0 \end{bmatrix} \]

and

\[ V(Y) = \begin{bmatrix} [v_1 + v_2 (1 - c(t)) + \theta_1 + \theta_2 c(t) + d] I(t) \\ \beta S(t) I(t) + dS(t) - b, \end{bmatrix}. \]

\( Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is the unique disease-free equilibrium point of the model (6).

Then, Jacobian matrices at the disease-free equilibrium point \( P^* = (S^*, I^*) = \begin{pmatrix} b \\ d \end{pmatrix} \) of \( F(Y) \) and \( V(Y) \) by looked to the FDs with respect to \( I \) and \( S \) are obtained as

\[ JF(P^*) = \begin{bmatrix} \gamma \beta S^* e^{-\gamma} & 0 \\ 0 & \gamma \beta I^* e^{-\gamma} \end{bmatrix}, \]

and

\[ JV(P^*) = \begin{bmatrix} [v_1 + v_2 (1 - c(t)) + \theta_1 + \theta_2 c(t) + d] & 0 \\ \beta S^* & \beta I^* + d \end{bmatrix}, \]

respectively. So \( F \) and \( V \), which are the new infection terms and the remaining transfer terms of the model, respectively, are determined as

\[ F = JF_{1x1} = \begin{bmatrix} b \beta \gamma e^{-\gamma} \\ 0 \end{bmatrix} \]

and

\[ V = JV_{1x1} = [v_1 + v_2 (1 - c(t)) + \theta_1 + \theta_2 c(t) + d]. \]

Also, the characteristic polynomial of

\[ FV^{-1} = \frac{b \beta \gamma e^{-\gamma}}{d (v_1 + v_2 (1 - c(t)) + \theta_1 + \theta_2 c(t) + d)} \]
is
\[ \det \left( \lambda I_1 - FV^{-1} \right) = \lambda - \frac{b\beta \gamma e^{-\delta t}}{d(v_1 + v_2(1 - c(t)) + \theta_1 + \theta_2 c(t) + d)}. \]

It follows from Van den Driessche and Watmough\(^{20}\) that the basic reproduction number of the model (5) is the spectral radius of the next generation matrix \(FV^{-1}\). In that case,
\[ R_0 = \frac{b\beta \gamma e^{-\delta t}}{d(v_1 + v_2(1 - c(t)) + \theta_1 + \theta_2 c(t) + d)}. \] (7)

The equivalent compact form of the above system (5) is as follows:
\[
\begin{aligned}
    CD_{\tau}^{\xi} S(t) &= G_1(t, S(t), S(t - \tau)), \\
    CD_{\tau}^{\xi} E(t) &= G_2(t, E(t), E(t - \tau)), \\
    CD_{\tau}^{\xi} I(t) &= G_3(t, I(t), I(t - \tau)), \\
    CD_{\tau}^{\xi} R(t) &= G_4(t, R(t), R(t - \tau)), \\
    CD_{\tau}^{\xi} D(t) &= G_5(t, D(t), D(t - \tau)),
\end{aligned}
\] (8)

with the initial conditions taken as \(S(0) = k_1, E(0) = k_2, I(0) = k_3, R(0) = k_4\) and \(D(0) = k_5\).

4 | MATHEMATICAL ANALYSIS OF THE FRACTIONAL MODEL

4.1 | Existence and uniqueness analysis

In this section, we give the existence of unique solution for the projected fractional time delay COVID-19 model by the help of the consequences of fixed point theory. In this regard, many results have been given in the literature, and here, we are following the procedure proposed by Cong and Tuan.\(^{21}\) We show the analysis for \(I(t)\), and for other equations of the system (8), it will be similar. Let us consider the fractional time delay equations
\[ CD_{\tau}^{\xi} I(t) = G_3(t, I(t), I(t - \tau)), \quad t \in [0, T], \quad 0 < \zeta \leq 1, \] (9)

with the initial condition
\[ I(t) = k_3, \quad t \in [-\tau, 0], \] (10)

where \(I \in \mathbb{R}^n, T > 0, \& G_3 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous.
(\(\mathbb{R}^n\) be the \(n\)-dimensional Euclidean space defined with a norm \(\| \cdot \|\))

**Lemma 1** (Cong and Tuan\(^{21}\)). The function \(\phi \in C([-\tau, T]; \mathbb{R}^n)\) (space of continuous functions \(\phi : [-\tau, T] \to \mathbb{R}^n\) with the sup norm \(\| . \|_\infty\) is a solution of the initial value problem (IVP) (Equations 9–10) on the interval \([-\tau, T]\) if it is a solution of the delay integral equation
\[ I(t) = I(0) + \frac{1}{\Gamma(\zeta)} \int_{0}^{t} (t - \xi)^{\zeta - 1} G_3(\xi, I(\xi), I(\xi - \tau))d\xi, \quad \forall t \in [0, T] \] (11)

with the initial condition
\[ I(t) = k_3, \quad t \in [-\tau, 0]. \] (12)

**Note:**\(^{21}\) In this particular way to show the existence of unique global solutions, we don’t need to require Lipschitz property of \(G_3\) with respect to the time delay variable of \(G_3\), but only the Lipschitz property of \(G_3\) with respect to the nondelay (second) variable.
Theorem 1 (Existence and uniqueness of global solutions). Assume that \( G_3 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and agree with the following Lipschitz condition with respect to the nondelay variable: there exists a non-negative continuous function \( L : [0, T] \times \mathbb{R}^n \to \mathbb{R}_{\geq 0} \), such that

\[
\|G_3(t, I, z) - G_3(t, I_1, z)\| \leq L(t, z)\|I - I_1\|
\]

\( \forall t \in [0, T], I, z, I_1 \in \mathbb{R}^n \). Then, the IVP (Equations 9–10) has a unique global solution \( \phi \) on the interval \([- \tau, T]\).

Proof. According to Lemma (1), Equation (9) with Equation (10) is equivalent to the initial value problem (11)–(12). First, we take the case \( 0 < T \leq \tau \). In this case, Equation (11) has the form:

\[
I(t) = I(0) + \frac{1}{\Gamma(\xi)} \int_0^t (t - \xi)^{\xi-1} G_3(\xi, I(\xi), k_3) d\xi, \quad \forall t \in [0, T].
\]

For this integral equation, by Tisdell [22 Theorem 6.4, p. 310], there exists a unique solution on the interval \([0, T]\). Identify that solution by \( \kappa_\tau \) and put

\[
\phi_\tau(t, k_3) := \begin{cases} 
k_3, & t \in [-\tau, 0], 
k_\tau(t), & t \in [0, T]. \end{cases}
\]

Then, \( \phi_\tau(t, k_3) \) is the unique solution of the problem (11)–(12) on \([- \tau, T]\).

Now, in another case when \( T > \tau \), we break the interval \([0, T]\) into \([0, r]\) \( \cup \ldots \cup ([m_0 - 1]r, m_0r) \cup [m_0r, T] \), where \( m_0 \in \mathbb{N} \) and \( 0 \leq T - m_0r < \tau \). On the interval \([- \tau, r]\), using the same arguments as above, we can derive a unique solution of the IVP (Equations 11–12), which is identified by \( \phi_\tau \). We will prove the existence of the unique solution on the interval \([- \tau, m_0r]\) by induction. Let us assume that the problem (11)–(12) has a unique solution on the interval \([- \tau, mr]\) for some \( 1 \leq m < m_0 \). We denote that solution by \( \phi_{mr}(\cdot, k_3) \). On \([mr, (m + 1)r]\), we define an operator \( F_{(m+1)r,k_3} : C([mr, (m+1)r]; \mathbb{R}^n) \to C([mr, (m+1)r]; \mathbb{R}^n) \) as follows:

\[
(F_{(m+1)r,k_3})(t) := I(0) + \frac{1}{\Gamma(\xi)} \int_0^{mr} (t - \xi)^{\xi-1} G_3(\xi, \phi_{mr}(\xi, k_3), \phi_{mr}(\xi - r, k_3)) d\xi
\]

\[
+ \frac{1}{\Gamma(\xi)} \int_{mr}^t (t - \xi)^{\xi-1} G_3(\xi, \kappa(\xi), \phi_{mr}(\xi - r, k_3)) d\xi, \quad \forall t \in [mr, (m+1)r].
\]

Let \( \beta_m \) be a positive constant satisfying \( \beta_m > 2 \max_{\xi \in [mr, (m+1)r]} L(t, \phi_{mr}(t - r, k_3)) \). On the space \( C([mr, (m+1)r]; \mathbb{R}^n) \), we define a new metric \( d_{\beta_m} \) by

\[
d_{\beta_m}(\kappa, \kappa_1) := \sup_{t \in [mr, (m+1)r]} \frac{\|\kappa(t) - \kappa_1(t)\|}{E_\zeta(\beta_m t^\zeta)}, \quad \forall \kappa, \kappa_1 \in C([mr, (m+1)r]; \mathbb{R}^n),
\]

where \( E_\zeta : \mathbb{R} \to \mathbb{R} \) is the Mittag–Leffler function which is defined in Definition 4. Then, the space \( C([mr, (m+1)r]; \mathbb{R}^n) \) equipped the metric \( d_{\beta_m} \) is complete. We will show that the operator \( F_{(m+1)r,k_3} \) is contractive on \( C([mr, (m+1)r]; \mathbb{R}^n), d_{\beta_m} \). Indeed, for any \( \kappa, \kappa_1 \in C([mr, (m+1)r]; \mathbb{R}^n) \) and any \( t \in [mr, (m+1)r] \), we have

\[
\|(F_{(m+1)r,k_3})(t) - (F_{(m+1)r,k_3})(t)\|
\]

\[
\leq \frac{\max_{\xi \in [mr, (m+1)r]} L(t, \phi_{mr}(t - r, k_3))}{\Gamma(\xi)} \int_{mr}^t (t - \xi)^{\xi-1} \|\kappa(\xi) - \kappa_1(\xi)\| d\xi
\]

\[
\leq \frac{\max_{\xi \in [mr, (m+1)r]} L(t, \phi_{mr}(t - r, k_3))}{\Gamma(\xi)} E_\zeta(\beta_m t^\zeta) \int_{mr}^t (t - \xi)^{\xi-1} E_\zeta(\beta_m \xi^\zeta) d\xi,
\]

\( \forall t \in [mr, (m+1)r] \).
This implies that

\[
\frac{\| (F_{(m_0 \tau, k)}(t) - (F_{(m_0 \tau, k_1)}(t)) \|}{E_\zeta(\beta_m t^\zeta)} \leq \max_{t \in [m \tau, (m+1) \tau]} L(t, \phi_{m \tau}(t - \tau, k_3)) \frac{1}{E_\zeta(\beta_m t^\zeta)} d_{\beta_m}(\kappa_1) \frac{\Gamma(\zeta)}{t^\zeta} \int_0^t (t - \xi)^{\zeta - 1} E_\zeta(\beta_m \xi^\zeta) d\xi
\]

for all \(t \in [m \tau, (m+1) \tau]\). Therefore,

\[
d_{\beta_m}(F_{(m+1) \tau, k}, F_{(m_0 \tau, k_1)}(t)) \leq \max_{t \in [m \tau, (m+1) \tau]} L(t, \phi_{m \tau}(t - \tau, k_3)) \frac{1}{\beta_m} d_{\beta_m}(\kappa_1) \leq \frac{1}{2} d_{\beta_m}(\kappa_1),
\]

for all \(\kappa, \kappa_1 \in C([m \tau, (m+1) \tau]; \mathbb{R}^n)\). By a statement of the Banach fixed point theorem, there exists a unique fixed point \(k^*_r \) of \(F_{(m+1) \tau, k}\) in \(C([m \tau, (m+1) \tau]; \mathbb{R}^n)\). Put

\[
\phi_{(m+1) \tau}(t, k_3) := \begin{cases} 
\phi_{m \tau}(t), & \forall t \in [-\tau, m \tau], \\
\kappa^*_r(t), & \forall t \in [m \tau, (m+1) \tau]. 
\end{cases}
\]

Then, \(\phi(m+1) \tau(t, k_3)\) is the unique solution of the problem (11)–(12) on \([-\tau, (m+1) \tau]\).

Finally, on the interval \([m_0 \tau, T]\), we derive an operator \(F_k : C([m_0 \tau, T]; \mathbb{R}^n) \to C([m_0 \tau, T]; \mathbb{R}^n)\) by

\[
(F_k)(t) := I(t) + \frac{1}{\Gamma(\zeta)} \int_0^{m_0 \tau} (t - \xi)^{\zeta - 1} G_3(\xi, \phi_{m_0 \tau}(\xi, k_3, t), \phi_{m_0 \tau}(\xi - \tau, k_3)) d\xi
\]

\[
+ \frac{1}{\Gamma(\zeta)} \int_{m_0 \tau}^t (t - \xi)^{\zeta - 1} G_3(\xi, \phi_{m_0 \tau}(\xi, \kappa(\xi), t), \phi_{m_0 \tau}(\xi - \tau, k_3)) d\xi \quad \forall t \in [m_0 \tau, T].
\]

Let \(\beta_{m_0}\) be a positive constant satisfying \(\beta_{m_0} > 2 \max_{t \in [m_0 \tau, T]} L(t, \phi_{m_0 \tau}(t - \tau, k_3))\). On the space \(C([m_0 \tau, T]; \mathbb{R}^n)\), we define a new metric \(d_{\beta_{m_0}}\) by

\[
d_{\beta_{m_0}}(\kappa, \kappa_1) := \sup_{t \in [m_0 \tau, T]} \frac{||k(t) - \kappa_1(t)||}{E_\zeta(\beta_{m_0} t^\zeta)},
\]
and repeating arguments as above, we can show that the operator $F_k$ has a unique fixed point $\kappa^*$ on $[m_0r, T]$. Define a function

$$\phi_T(t, k_3) := \begin{cases} \phi_{m, t}^*(t, k_3), & \forall t \in [-r, m_0r], \\ \kappa^*(t), & \forall t \in [m_0r, T]. \end{cases} \quad (19)$$

It is evident that $\phi_T$ is the unique solution of the problem (11)–(12) on the interval $[-r, T]$. □

5 | SOLUTION OF THE PROJECTED MODEL USING PREDICTOR–CORRECTOR ALGORITHM

We know that the numerical methods or techniques used for solving ordinary differential equations cannot be used directly to solve noninteger order differential equations. Lots of numerical methods have been used to solve DFDEs arising in biology. In this section, we find the solution of the projected model by the help of Adams–Bashforth–Moulton predictor–corrector scheme described in Bhalekar and Daftardar-Gejji for solving DFDEs.

Let us consider the fractional delay differential equation

$$^CD^\zeta I(t) = G_3(t, I(t), I(t - \tau)), \quad t \in [0, T], \ 0 < \zeta \leq 1, \quad (20a)$$

$$I(t) = k_3, \quad t \in [-r, 0]. \quad (20b)$$

Consider a uniform grid $\{t_m = mh : m = -n, -n + 1, \ldots, -1, 0, 1, \ldots, N\}$, where $n$ and $N$ are integers such that $h = T/N$ and $h = \tau/n$. Let

$$I_h(t_i) = k_3, i = -n, -n + 1, \ldots, -1, 0, \quad (21)$$

and note that

$$I_h(t_i - \tau) = I_h(ih - nh) = I_h(t_{i-n}), i = 0, 1, \ldots, N. \quad (22)$$

Suppose we have already calculated the approximations $I_h(t_i) \approx I(t_i), (i = -n, -n + 1, \ldots, -1, 0, 1, \ldots, m)$ and we want to calculate $I_h(t_{m+1})$ using the volterra integral equation equivalent to the Equations (20a) and (20b)

$$I(t_{m+1}) = I(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{m+1}} (t_{m+1} - \xi)^{-1} G_3(\xi, I(\xi), I(\xi - \tau))d\xi. \quad (23)$$

We use approximations $I_h(t_i)$ for $I(t_i)$ in Equation (23). Further, the integral in Equation (23) is evaluated using product trapezoidal quadrature formula. The corrector formula is thus

$$I_h(t_{m+1}) = I(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_3(t_{m+1}, I_h(t_{m+1}), I_h(t_{m+1} - \tau))$$

$$+ \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^m a_{i,m+1} G_3(t_i, I_h(t_i), I_h(t_i - \tau))$$

$$= I(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_3(t_{m+1}, I_h(t_{m+1}), I_h(t_{m+1} - \tau))$$

$$+ \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^m a_{i,m+1} G_3(t_i, I_h(t_i), I_h(t_i - \tau)). \quad (24)$$
where

\[
a_{i,m+1} = \begin{cases} 
  m^{i+1} - (m - \zeta)(m + 1)^\zeta, & i = 0 \\
  (m - i + 2)^{i+1} - 2(m - i + 1)^\zeta + (m - i)^{i+1}, & 1 \leq i \leq m \\
  1, & i = m + 1.
\end{cases}
\]

The unknown term \( I_h(t_{m+1}) \) present on both sides of Equation (24) and due to non-linearity of \( G_3 \) Equation (24) cannot be solved explicitly for \( I_h(t_{m+1}) \). So we replace the term \( I_h(t_{m+1}) \) on the right hand side by an approximation \( I^p_h(t_{m+1}) \), called predictor. Product rectangle rule is used in Equation (24) to evaluate predictor term

\[
I^p_h(t_{m+1}) = I(0) + \frac{1}{\Gamma(\zeta)} \sum_{i=0}^{m} b_{i,m+1} G_3(t_i, I_h(t_i), I_h(t_i - \tau))
\]

(25)

where

\[
b_{i,m+1} = \frac{h^\zeta}{\zeta} ((m + 1 - i)^\zeta - (m - i)^\zeta).
\]

So from the above calculations, the corrector formulas for all five equations of system (8) are

\[
S_h(t_{m+1}) = S(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_1(t_{m+1}, S_h(t_{m+1}), S_h(t_{m+1-n})) + \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^{m} a_{i,m+1} G_1(t_i, S_h(t_i), S_h(t_{i-n})),
\]

\[
E_h(t_{m+1}) = E(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_2(t_{m+1}, E_h(t_{m+1}), E_h(t_{m+1-n})) + \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^{m} a_{i,m+1} G_2(t_i, E_h(t_i), E_h(t_{i-n})),
\]

\[
I_h(t_{m+1}) = I(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_3(t_{m+1}, I_h(t_{m+1}), I_h(t_{m+1-n})) + \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^{m} a_{i,m+1} G_3(t_i, I_h(t_i), I_h(t_{i-n})),
\]

\[
R_h(t_{m+1}) = R(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_4(t_{m+1}, R_h(t_{m+1}), R_h(t_{m+1-n})) + \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^{m} a_{i,m+1} G_4(t_i, R_h(t_i), R_h(t_{i-n})),
\]

\[
D_h(t_{m+1}) = D(0) + \frac{h^\zeta}{\Gamma(\zeta + 2)} G_5(t_{m+1}, D_h(t_{m+1}), D_h(t_{m+1-n})) + \frac{h^\zeta}{\Gamma(\zeta + 2)} \sum_{i=0}^{m} a_{i,m+1} G_5(t_i, D_h(t_i), D_h(t_{i-n})),
\]

where

\[
a_{i,m+1} = \begin{cases} 
  m^{i+1} - (m - \zeta)(m + 1)^\zeta, & i = 0 \\
  (m - i + 2)^{i+1} - 2(m - i + 1)^\zeta + (m - i)^{i+1}, & 1 \leq i \leq m \\
  1, & i = m + 1.
\end{cases}
\]

Similarly, the predictor terms are

\[
S^p_h(t_{m+1}) = S(0) + \frac{1}{\Gamma(\zeta)} \sum_{i=0}^{m} b_{i,m+1} G_1(t_i, S_h(t_i), S_h(t_{i-n})),
\]

\[
E^p_h(t_{m+1}) = E(0) + \frac{1}{\Gamma(\zeta)} \sum_{i=0}^{m} b_{i,m+1} G_2(t_i, E_h(t_i), E_h(t_{i-n})),
\]

\[
I^p_h(t_{m+1}) = I(0) + \frac{1}{\Gamma(\zeta)} \sum_{i=0}^{m} b_{i,m+1} G_3(t_i, I_h(t_i), I_h(t_{i-n})),
\]

\[
R^p_h(t_{m+1}) = R(0) + \frac{1}{\Gamma(\zeta)} \sum_{i=0}^{m} b_{i,m+1} G_4(t_i, R_h(t_i), R_h(t_{i-n})),
\]

\[
D^p_h(t_{m+1}) = D(0) + \frac{1}{\Gamma(\zeta)} \sum_{i=0}^{m} b_{i,m+1} G_5(t_i, D_h(t_i), D_h(t_{i-n})),
\]

\[
(27)
\]
where
\[ b_{i,m+1} = \frac{h^\zeta}{\zeta} (m + 1 - i)^\zeta - (m - i)^\zeta. \]

6 | SIMULATION RESULTS

To perform numerical simulations, we use parameter values based on a case study of Wuhan, China, cited from literature and with some assumptions, summarized in Table 1. We also use the following initial conditions \( S(0) = 89,985,050, E(0) = 10,050, I(0) = 475, R(0) = 10 \) and \( D(0) = 2 \) for Table 1. The function \( c(t) \) is taken as \( c(t) = \frac{321,000}{321,000 + R(t)} \). Initial population \( N(0) \) is considered as 89,995,587, and birth rate is \( b = d \times N(0) \).
Figure 1A exemplifies the behaviour of achieved results by given solution procedure for $S(t)$ at different values of fractional order $\zeta$. In Figure 1B, we analyse the exposed individuals for different fractional order $\zeta$ with respect to time $t$. Figure 1C, D and E analyses the nature of infectious, recovered and deaths classes at $\zeta = 0.85, 0.90, 0.98$ and 1, respectively. Figure 1A–E examines the nature of all given classes at time delay $\tau = 2$. Similarly, the graphical simulations from Figure 2A–E are for time delay $\tau = 6$ for parameter values given in Table 1. In Figure 2A, we show the nature of susceptible individuals, in Figure 2B, exposed classes; in Figure 2C, infectious individuals; in Figure 2D, recovered and in Figure 2E, the deaths, with respect to time variable $t$ for parameter values used from Table 1. In this same manner, we observed the plots for time delay $\tau = 10$ in Figure 3A–E. All graphs are computed using Mathematica software. For the all above simulations, we used time delay $\tau = 2, 6$ and 10, respectively, and time step was 0.05. From the cited figures, we can observe that the given model exceedingly depends on the order and gestures more degree of flexibility. Moreover, the fractional

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Nature of the given classes for time delay = 6 at different fractional order values}
\end{figure}
method gives more interesting results than the integer order model and permit to better examine the obtained results. We observed that time delay variable $\tau$ plays an important role in this COVID-19 dynamics. The biological meaning of time delay in COVID-19 is cleared from the above simulations.

In Figure 4A–C, we study the nature of basic reproductive number $R_0$ at different fractional order values and for different time delay $\tau$. We can see that the maximum value of basic reproductive number for the given time range is near about 1.038. Numerically, we have calculated that when $c(t)$ tends to one, $R_0$ tends to 1.032, and when $c(t)$ tends to zero, then $R_0$ tends to 3.880. It is clear that the recovery rate and death rate are directly related to whether healthcare capacity is exceeded or not. For the all above graphical simulations, we can observe that the given model works well to study the dynamics of reproductive number respect to the time $t$. We have clearly observed that the parameter $c$ plays a very important role in the simulations and time delay can show the effect of healthcare opportunities.
7 | CONCLUSIONS

In this paper, we solved a time delay fractional COVID-19 compartmental model via Caputo FDs applying predictor–corrector algorithm. After preliminaries and the model description, the existence and uniqueness analysis of the given time delay fractional system are established by the applications of fixed point theory (particularly, a mild Lipschitz condition using properties of a weighted norm, Mittag–Leffler functions and the Banach fixed point theorem). All necessary graphical simulations are done to specified the nature of the achieved solutions in Caputo noninteger order derivative sense. We analysed the role of time delay in the coronavirus epidemic by the graphical simulations for different values. We also presented the plotting of the of basic reproductive number at different fractional order values and various values of time delay. The projected scheme is strong and highly credible in finding the solution to delay fractional models of biological, physical and medical importance.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

AUTHORS CONTRIBUTION

Conceptualization, formal analysis, investigation, methodology, resources, visualization and writing—original draft: Pushpendra Kumar. Conceptualization, investigation, software and writing—review & editing: Vedat Suat Erturk.

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