Non-renormalization of induced charges and constraints on strongly coupled theories.

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Abstract

It is shown that global fermionic charges induced in vacuum by slowly varying, topologically non-trivial background scalar fields are not renormalized provided that expansion in momenta of background fields is valid. This suggests that strongly coupled theories obey induced charge matching conditions which are analogous, but generally not equivalent, to ’t Hooft anomaly matching conditions. We give a few examples of induced charge matching. In particular, the corresponding constraints in softly broken supersymmetric QCD suggest non-trivial low energy mass pattern, in full accord with the results of direct analyses.

1 Introduction

Four dimensional gauge theories strongly coupled at low energies often exhibit interesting content of composite massless fermions. This property is potentially important for constructing composite models of quarks and leptons, which is long being considered as a possible “major step on our way into the nature of matter” [1]. Powerful constraints on the low energy spectrum are provided by t’ Hooft anomaly matching conditions [2]. These are heavily used, in particular, in establishing duality properties of supersymmetric gauge theories (see, e.g., refs. [3, 4] and references therein).

The basis for anomaly matching is provided by the Adler–Bardeen non-renormalization theorem [5]. In non-Abelian theories, the absence of radiative corrections to anomalies is
intimately connected to topology: if one introduces background gauge fields corresponding to the flavor symmetry group, the anomalies in global currents are proportional to the topological charge densities of these background fields. Integer-valuedness of global fermionic charges, on the one hand, and integer-valuedness of topological charges of background gauge fields, on the other, imply that anomaly equations should not get renormalized.

Gauge field backgrounds are not the only ones that may have topological properties. Topology is inherent also in scalar background fields of Skyrmion type. Indeed, one loop calculations show that slowly varying in space, static scalar fields induce, in vacuum, fermionic global charges which are proportional to the topological charges of the background. By analogy to triangle anomalies, this suggests that induced charges do not receive radiative corrections, and hence may serve as constraints on low energy spectrum of strongly coupled theories. Unlike triangle anomalies, however, the one-loop expressions for the induced charges are promoted to full quantum theory only if the expansion in momenta of the background fields is valid in the full theory. The latter property can often be established to all orders of perturbation theory (exceptions are easy to understand); in some models the validity of the expansion in momenta can be also shown non-perturbatively.

We will see that induced charge matching conditions emerging in this way have a certain relation to anomaly matching. However, in some cases the two sets of matching conditions are inequivalent, so the induced charges give additional information on the properties of the low energy theory. This information is particularly interesting in softly broken supersymmetric gauge theories.

Fermionic charges in vacuum are induced due to Yukawa interactions of fermions with background scalar fields. These interactions introduce masses to some of the fermions in the fundamental theory and hence explicitly break a subgroup of the flavor group. As a consequence, some of the fermions of the low energy effective theory acquire masses. Induced charge matching conditions constrain the resulting mass pattern of the effective theory. We will see that these conditions are satisfied automatically (provided the triangle anomalies
match) if all composite fermions charged under explicitly broken flavor subgroup become massive. The latter situation is very appealing intuitively; however, we are not aware of any argument implying that it should be generic.

This paper is organized as follows. In section 2 we show that global charges induced in vacuum by slowly varying background scalar fields do not get renormalized provided that the derivative expansion is valid. In section 3 we discuss exceptional situations by presenting a model where the derivative expansion fails already at the one loop level. In section 4 we give several examples of induced charge matching (ordinary QCD, supersymmetric $N = 1$ QCD exhibiting the Seiberg duality [8], SQCD with softly broken supersymmetry). We conclude in section 5 by discussing the relation between induced charges and triangle anomalies.

2 Non-renormalization of induced charges

To be specific, let us consider QCD with $N_c$ colors and $N_f$ massless fermion flavors. Let $\psi^a$ and $\bar{\psi}^{\tilde{a}}$, $a, \tilde{a} = 1, \ldots, N_f$, denote left-handed quarks and anti-quarks, respectively. To probe this theory, we introduce background scalar fields $m^\tilde{a}_b(x)$ of the following form,

$$m^\tilde{a}_b(x) = m_0 U^\tilde{a}_b(x),$$

where $m_0$ is a constant and $U$ is an $SU(N_f)$ matrix at each point $x$. Let these fields interact with quarks and anti-quarks,

$$\mathcal{L}_{\text{int}} = \bar{\psi}^{\tilde{a}} m^\tilde{a}_b \psi^b + \text{h.c.}$$

(1)

Besides the global $SU(N_f)_L \times SU(N_f)_R$ symmetry, the theory exhibits non-anomalous baryon symmetry, $\psi \to e^{i\alpha} \psi$, $\bar{\psi} \to e^{-i\alpha} \bar{\psi}$, $m \to m$. The baryonic current is conserved and obtains non-vanishing vacuum expectation value in the presence of the background scalar fields.

To the leading order in momenta, the one-loop expression for this induced current is

$$\langle j_B^\mu \rangle = \frac{N_c}{24\pi^2} e^{\mu\nu\lambda\rho} \text{Tr} \left( U \partial_\nu U^\dagger U \partial_\lambda U^\dagger U \partial_\rho U^\dagger \right).$$

(2)
This expression can be obtained by considering a configuration which in the vicinity of a given point $x$ has the following form,

$$U(x) = 1 + \epsilon(x) ,$$

where $\epsilon(x)$ is a small and slowly varying anti-Hermitean background field. To the leading order in momenta, one-loop induced baryonic current is then given by the diagram of Fig. 1 with fermions of mass $m_0$ running in the loop. The complete expression (2) is reconstructed by making use of $SU(N_f)_L \times SU(N_f)_R$ global symmetry.

A remarkable property of eq. (2) is that the baryonic charge induced in vacuum by slowly varying, time-independent background field $U(x)$ with

$$U(x) \rightarrow 1 \text{ as } |x| \rightarrow \infty$$

is proportional to the topological number of the background,

$$\langle B \rangle = N_c N[U] ,$$

where

$$N[U] = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{ijk} \text{Tr} \left( U \partial_i U^\dagger U \partial_j U^\dagger U \partial_k U^\dagger \right) ,$$

The higher derivative terms omitted in eq. (2) do not contribute to $\langle B \rangle$. 

Figure 1: Leading order contribution to the induced baryonic current.
Let us see that eq. (4) does not get renormalized in the full quantum theory provided the expansion in momenta of the background field works. Let us consider the same theory with the gauge coupling \( \alpha \) depending on coordinates \( x \). The induced current is now a functional of \( \alpha(y) \) and \( U(y) \),

\[
\langle j^\mu_B(x) \rangle = j^\mu[x; \alpha(y); U(y)] .
\]

At slowly varying \( \alpha(y) \) we expand \( \langle j^\mu_B(x) \rangle \) in derivatives of \( \alpha \) at the point \( x \),

\[
\langle j^\mu_B(x) \rangle = J^\mu [x; \alpha(x); U(y)] + B^{\mu\nu}[x; \alpha(x); U(y)] \partial_\nu \alpha(x) + O \left[(\partial \alpha)^2\right] , \tag{5}
\]

where the coefficients on the right hand side are now functions, rather than functionals, of \( \alpha(x) \) (but still functionals of \( U(y) \)). The structure analogous to the right hand side of eq. (2) appears in the derivative expansion of the first term on the right hand side of eq. (5),

\[
J^\mu = \frac{f(\alpha)}{24\pi^2} \epsilon^{\mu\lambda\rho\sigma} \Tr \left( U \partial_\sigma U^\dagger U \partial_\lambda U^\dagger U \partial_\rho U^\dagger \right) + \ldots
\]

Our purpose is to show that \( f(\alpha) \) is independent of \( \alpha \), and hence \( f(\alpha) = N_c \).

Let us make use of the conservation of \( \langle j^\mu_B(x) \rangle \),

\[
\partial_\mu \langle j^\mu_B(x) \rangle = 0 . \tag{6}
\]

If \( f \) were a non-trivial function of \( \alpha \), the divergence of \( J^\mu \) would contain the term

\[
\frac{1}{24\pi^2} \epsilon^{\mu\lambda\rho\sigma} \Tr \left( U \partial_\sigma U^\dagger U \partial_\lambda U^\dagger U \partial_\rho U^\dagger \right) \frac{\partial f}{\partial \alpha} \partial_\mu \alpha .
\]

The only possible source of cancellation of this term in eq. (6) is the second term on the right hand side of eq. (5). The cancellation would occur iff \( B^{\mu\nu} \) contained the term of the following structure

\[
\beta^{\mu\nu}[U] \frac{\partial f}{\partial \alpha}
\]

with

\[
\partial_\mu \beta^{\mu\nu}[U] = -\frac{1}{24\pi^2} \epsilon^{\sigma\lambda\rho\nu} \Tr \left( U \partial_\sigma U^\dagger U \partial_\lambda U^\dagger U \partial_\rho U^\dagger \right) . \tag{7}
\]
However, the right hand side of eq. (7) is not a complete divergence of any tensor that is
invariant under the flavor group (recall that the right hand side of eq. (7) is a topological
current). Hence, the conservation of the baryonic current requires that $\partial f/\partial \alpha = 0$.

This argument is straightforward to generalize to other conserved currents and to the-
ories other than QCD. As discussed in section [I], it implies that induced charges should
match in fundamental and low energy theories. Examples of such a matching are given in
section [I].

### 3 Failure of derivative expansion: an example

An important ingredient in the above argument is the derivative expansion. While in QCD
and many other models the derivative expansion works, at least to all orders of perturbation
theory, one can design models where the derivative expansion fails, and the induced charges
cannot be reliably calculated even within perturbation theory. As an example, let us consider
a model of free left-handed fermions $\psi^a$, $\tilde{\psi}_a$ and $\chi_\tilde{a}$, $a, \tilde{a} = 1, \ldots, N_f$, with the mass term

$$L_{\mu_0} = \mu_0 \tilde{\psi}_a \chi^{\tilde{a}} + \text{h.c.} \quad (8)$$

The model is invariant under the global $SU(N_f)_L \times SU(N_f)_R$ symmetry under which $\psi$, $\tilde{\psi}$, and $\chi$ transform as $(N_f, 1)$, $(1, \bar{N}_f)$, and $(1, N_f)$, respectively. The “baryon numbers” of
fermions $\psi$, $\tilde{\psi}$ and $\chi$ are $+1$, $-1$ and $+1$, respectively. Let us introduce background fields $m_\xi(x)$ and their interaction with fermions $\psi$ and $\tilde{\psi}$ in the same way as in eq. (7).

To see that the derivative expansion is not reliable in this model, let us again consider
the background field of the form (8). At $\epsilon = 0$, fermions $\tilde{\psi}$ and $\xi = \text{const} \cdot (\mu_0 \chi + m_0 \psi)$
form massive Dirac field, while $\eta = \text{const} \cdot (m_0 \chi - \mu_0 \psi)$ remains massless Weyl field. Both
types of fermions interact with the background field $\epsilon(x)$. In an attempt to calculate the
induced baryonic current, one faces diagrams with massless internal fermion lines like one
shown in Fig. 2. It is straightforward to see that the derivative expansion of these diagrams
The fact that the derivative expansion fails in this model manifests itself in different values of induced charges in various limits. Namely, at $\mu_0 \gg m_0$ one can ignore the background fields, and $\langle B \rangle = 0$. On the other hand, at $\mu_0 \ll m_0$, the mass term (8) becomes irrelevant, so $\langle B \rangle = N[U]$. As outlined above, this phenomenon is due to the fact that not all fermions charged under $SU(N_f)_L \times SU(N_f)_R$ obtain masses upon introducing the background fields $m(x)$.

This example shows that the validity of the derivative expansion requires that the background scalar fields provide masses to all relevant fermions. This will be the case in all examples presented in the next section.

4 Examples of induced charge matching
4.1 QCD

We again consider conventional $SU(N_c)$ QCD with $N_f$ massless flavors. Let us generalize slightly the discussion of section 2 by introducing background fields

$$m_q^\tilde{p}(x) = m_0 U_q^\tilde{p}(x)$$  \hfill (9)

which give $x$-dependent masses to $N_0$ quark flavors only, $\mathcal{L}_{\text{int}} = m_q^\tilde{p}(x) \bar{\psi}_q \psi^\tilde{q} + \text{h.c.}$. The fields $m(x)$ are $N_0 \times N_0$ matrices; hereafter the indices $p, q, r, \tilde{p}, \tilde{q}, \tilde{r}$ run from 1 to $N_0$ with $N_0 < N_f$. $(N_f - N_0)$ flavors remain massless. In all examples of this section we consider background fields that are constant at spatial infinity; by a global $SU(N_f)_L$ rotation we set $U(x) \to 1$ as $|x| \to \infty$. Besides the baryon number symmetry $U(1)_B$, we will be interested in a vector subgroup $U(1)_8^f$ of the original $SU(N_f)_L \times SU(N_f)_R$ flavor group, whose (unnormalized) generator is

$$T_8^f = \text{diag} \left( 1, \ldots, 1, -\frac{N_0}{N_f - N_0}, \ldots, -\frac{N_0}{N_f - N_0} \right).$$

The background fields are charged under neither $U(1)_B$ nor $U(1)_8^f$.

As all fundamental fermions that interact with the background scalar fields acquire masses due to this interaction, the derivative expansion is justified, at least order by order in perturbation theory. Hence, for slowly varying $m(x)$ one has

$$\langle B \rangle = N_c N[U],$$  \hfill (10)

$$\langle T_8^f \rangle = N_c N[U].$$  \hfill (11)

Let us see that the low energy effective theory of QCD — the non-linear sigma-model — indeed reproduces eqs. (10) and (11). In the absence of the background fields, the non-linear sigma-model action contains only derivative terms for the $SU(N_f)$ matrix valued dynamical sigma-model field $V(x)$, including the usual kinetic term and the Wess–Zumino term. The background field $m(x)$ introduces a potential term into the low energy effective Lagrangian,

$$\Delta \mathcal{L}_{\text{eff}} = \text{Tr} \left( m^\dagger V + V^\dagger m \right).$$

8
For slowly varying \( m \), the effective potential is minimized at
\[
V(x) = \begin{pmatrix} U(x) & 0 \\ 0 & 1 \end{pmatrix}.
\] (12)

Hence, the induced baryonic charge appears at the classical level \([9]\); as the baryonic charge of \( V(x) \) is equal to its topological number \( N[V] \times N_c \), the induced baryonic charge is indeed given by eq. \((11)\). Likewise, it follows from the structure of the Wess–Zumino term that the \( T^f_8 \) current of the configuration of the form \((12)\) is (cf. ref. \([10]\))
\[
j^f_{8,\mu} = \frac{N_c}{24\pi^2} \varepsilon_{\mu\nu\lambda\rho} \text{Tr} \left( U \partial_\nu U^\dagger U \partial_\lambda U^\dagger U \partial_\rho U^\dagger \right),
\]
so the \( T^f_8 \) charge of the configuration \((12)\) is given by eq. \((11)\).

We see that the induced charges in QCD and its low energy effective theory match rather trivially. The way the induced charges match becomes more interesting when low energy theories (in the absence of background scalar fields) contain massless fermions.

### 4.2 Supersymmetric QCD

Let us now consider supersymmetric QCD with \( N_c \) colors and \( N_f \) flavors. To be specific, we discuss the case \( 3N_c > N_f > N_c + 3 \). This theory exhibits the Seiberg duality \([8]\): the fundamental theory contains the superfields of quarks \( Q^i \) and anti-quarks \( \tilde{Q}_i \), while its effective low energy counterpart at the origin of moduli space is an \( SU(N_f - N_c) \) magnetic gauge theory with magnetic quarks \( q_i \), magnetic anti-quarks \( \tilde{q}_i \) and mesons \( M^i_j \) with the superpotential \( qM\tilde{q} \).

Let us probe this theory by adding the scalar background fields \( m_{\mu}^{\tilde{q}}(x) \) with the same properties as above, i.e., by introducing the term
\[
m_{\mu}^{\tilde{q}}(x) \tilde{Q}_q Q^p
\] (13)
into the superpotential of the fundamental theory. Let us take for definiteness \(2 \leq N_0 < N_f - N_c - 1 \). The calculation of the induced baryon and \( T^f_8 \) charges in the fundamental theory proceeds as above, and we again obtain eqs. \((10)\) and \((11)\).
We now turn to the effective low energy theory. For slowly varying \( m(x) \), the term (13) translates into \( \text{Tr}(mM) \), so the total superpotential of the magnetic theory is

\[
qM\tilde{q} + \mu_0 \text{Tr}(mM) ,
\]

where \( \mu_0 \) is the dimensionful parameter inherent in the magnetic theory. The ground state near the origin of the moduli space has the following non-vanishing \( x \)-dependent expectation values\(^1\) of the magnetic quarks and anti-quarks,

\[
\langle \hat{q}^p_q \rangle = \mu^p_q , \quad p = 1, \ldots, N_0 , \quad q = 1, \ldots, N_0 \tag{15}
\]

(here the upper and lower indices refer to magnetic color and flavor, respectively)

\[
\langle \tilde{\hat{q}}^\tilde{p}_{\tilde{q}} \rangle = \tilde{\mu}^\tilde{p}_{\tilde{q}} , \quad \tilde{p} = 1, \ldots, N_0 , \quad \tilde{q} = 1, \ldots, N_0 \tag{16}
\]

(here the lower index refers to magnetic color). The expectation values obey

\[
\tilde{\mu}^{\tilde{p}}_{\tilde{q}}(x)\mu^{p}_{q}(x) = -\mu_0 m^p_{q}(x) .
\]

They also satisfy the \( D \)-flatness condition at each point in space, \( \mu^{\dagger q}_{p} \mu^{r}_{q} = \tilde{\mu}^{\dagger \tilde{q}}_{\tilde{p}} \tilde{\mu}^{\dagger \tilde{r}}_{\tilde{q}} \). With our choice of background fields, eq.(9), one has

\[
\mu = \pm \sqrt{\mu_0 m_0} \ W(x) , \quad \tilde{\mu} = \mp \sqrt{\mu_0 m_0} \tilde{W}(x) ,
\]

where \( W \) and \( \tilde{W} \) are \( N_0 \times N_0 \) unitary matrices obeying

\[
(\tilde{W}W)(x) = U(x) .
\]

Indeed, at \( m = m_0 \cdot 1 \), the matrices \( \mu \) and \( \tilde{\mu} \) are proportional to \( N_0 \times N_0 \) unit matrix, up to magnetic color rotation. At \( m = m_0 U(x) \) one has \( \mu = \pm \sqrt{\mu_0 m_0} \ U_c U , \ \tilde{\mu} = \mp \sqrt{\mu_0 m_0} \ U_c^{\dagger} \)

where \( U_c(x) \) is a slowly varying matrix belonging to \( SU(N_0) \) subgroup of the magnetic color group. The explicit form of \( U_c(x) \) is to be found from the minimization of the gradient energy, and it is not important for our purposes.

\(^1\)Hereafter we use the same notations for superfields and their scalar components.
Since the gradient energy has to vanish at spatial infinity, $W(x)$ and $\tilde{W}(x)$ are constant at $|x| \to \infty$, so they can be characterized by their winding numbers $N[W]$ and $N[\tilde{W}]$. Because of eq. (17) one has

$$N[W] + N[\tilde{W}] = N[U].$$

In this ground state, the magnetic color is broken down to $SU(N_f - N_c - N_0)$. At small $m_0$, the ground state (15), (16) is close to the origin of the moduli space, so the magnetic description is reliable.

Both the baryon number and $T^f_8$ are broken in this vacuum. However, there exist combinations of these generators and magnetic color generators that remain unbroken. Recalling [8] that the baryon number of magnetic quarks equals $N_c/(N_f - N_c)$ and that the magnetic quarks and anti-quarks transform as $(\bar{N}_f, 1)$ and $(1, N_f)$, respectively, under the global $SU(N_f)_L \times SU(N_f)_R$ group, the unbroken generators are

$$B' = B - \frac{N_c}{N_f - N_c} T^{mc}_8,$$  \hspace{1cm} (18)

$$T'_8 = T^f_8 + T^{mc}_8,$$  \hspace{1cm} (19)

where $T^{mc}_8$ is the following generator of the magnetic color group,

$$T^{mc}_8 = \text{diag} \left( 1, \ldots, 1, -\frac{N_0}{N_f - N_c - N_0}, \ldots, -\frac{N_0}{N_f - N_c - N_0} \right).$$

As the fundamental quarks and gluons are singlets under magnetic color, the induced charges $\langle B' \rangle$ and $\langle T'_8 \rangle$ calculated in the magnetic theory should match eqs. (10) and (11). Let us check that this is indeed the case.

The induced charges appear in the magnetic theory through $x$-dependent mass terms of fermions. These are generated by the expectation values (15), (16). The mass terms coming from the superpotential (14) are

$$\tilde{\mu}_q^p(x) \psi^i_p \psi^q_i + \mu_q^p(x) \bar{\psi}^j_p \psi^q_j,$$  \hspace{1cm} (20)
where $\Psi$, $\psi$ and $\tilde{\psi}$ are fermionic components of mesons, magnetic quarks and magnetic anti-quarks, respectively. The gauge interactions give rise to other mass terms,

$$
\mu_{ij}^{\star}(x) \psi_i^a \chi_a^j - \mu_{ij}^a(\mathbf{x}) \lambda_a^j \psi_i^a ,
$$

where $\lambda_a^i$ is the gluino field, $a, b = 1, \ldots, (N_f - N_c)$ are magnetic color indices.

To calculate the induced baryon number $\langle B' \rangle$ we observe that the only fermions carrying non-zero $B'$ are magnetic quarks $\psi_\alpha^i$ with $i = 1, \ldots, N_f$, $\alpha = (N_f - N_c - N_0 + 1), \ldots, (N_f - N_c)$, magnetic anti-quarks $\tilde{\psi}_\alpha^i$ and gluinos $\lambda_p^\alpha$, $\lambda_p^\alpha$. Their $B'$-charges are

$$
\psi_\alpha^i : \frac{N_c}{N_f - N_c} - \frac{N_c}{N_f - N_c} \left( -\frac{N_0}{N_f - N_c - N_0} \right) = \frac{N_c}{N_f - N_c - N_0} ,
$$

$$
\lambda_p^\alpha : \frac{N_c}{N_f - N_c - N_0} ,
$$

$$
\tilde{\psi}_\alpha^i , \lambda_p^\alpha : -\frac{N_c}{N_f - N_c - N_0} .
$$

Hence, the induced $B'$ is due to the $x$-dependent mass term (21) and is equal to

$$
\langle B' \rangle = -\frac{N_c}{N_f - N_c - N_0} \cdot (N_f - N_c - N_0) \left( N[W] + N[\tilde{W}] \right) .
$$

It follows from eq. (17) that $\langle B' \rangle$ indeed coincides with $N_c N[U]$, the induced baryon number calculated in the fundamental theory.

The induced charge $\langle T'_8 \rangle$ is calculated in a similar way. The relevant $T'_8$ charges of magnetic quarks are

$$
\psi_p^\alpha : -\frac{N_f - N_c}{N_f - N_c - N_0} ,
$$

$$
\psi_p^u : \frac{N_f}{N_f - N_0} , \ u = (N_0 + 1), \ldots, N_f ,
$$

and similarly for magnetic anti-quarks, gluinos and mesons. We find that both $x$-dependent mass terms, (20) and (21), contribute to $\langle T'_8 \rangle$, and obtain

$$
\langle T'_8 \rangle = \frac{N_f}{N_f - N_0} \cdot (N_f - N_0) \left( N[W] + N[\tilde{W}] \right)
+ \frac{N_f - N_c}{N_f - N_c - N_0} \cdot (N_f - N_c - N_0) \left( N[W] + N[\tilde{W}] \right) .
$$
This is equal to $N_c N[U]$, so the induced $T'_{8}$ charges also match in the fundamental and low energy theories.

### 4.3 Softly broken SQCD

As our last example, let us consider supersymmetric QCD with small soft masses of scalar quarks, $m^2_{Q}$, that explicitly break supersymmetry \[1\]. We again probe this theory by introducing the term \[13\] into the superpotential. The restrictions on $N_f$, $N_c$ and $N_0$ are the same as in the previous example.

The induced charges, as calculated in the fundamental theory, are still given by eqs. \[10\] and \[11\]. The low energy theory near the origin is still the magnetic theory, but now with soft mass terms of scalar mesons and scalar magnetic quarks \[14\]. The scalar potential of the magnetic theory near the origin at small $m^2_{Q}$ is determined both by the superpotential \[14\] and these soft terms,

\[
V(M, q, \tilde{q}) = |\tilde{q}q + \mu_0 m|^2 + |qM|^2 + |M\tilde{q}|^2 + m^2_M M^\dagger M + m^2_{\tilde{q}}(q^\dagger q + \tilde{q}^\dagger \tilde{q}) + D\text{-terms}, \tag{22}
\]

where $m^2_M$ and $m^2_{\tilde{q}}$ are proportional to $m^2_{Q}$. Were the soft terms in eq. \[22\] positive, the ground state of this theory at $m^2_q > \mu_0 m_0$ would be at the origin, $\langle q \rangle = \langle \tilde{q} \rangle = \langle M \rangle = 0$. The masses of fermions in the magnetic theory would vanish, the induced charges $\langle B \rangle$ and $\langle T^f_8 \rangle$ would be zero, so the induced charge matching would not occur. Hence, the induced charge matching requires that either $m^2_q$ and/or $m^2_M$ are negative, so that the ground state even at $m_0 = 0$ is far away from the origin, or $m^2_q = 0$, $m^2_M \geq 0$ with the ground state being the same as in the previous example. This is in accord with explicit calculations. Indeed, it has been found in ref. \[12\] (see also ref. \[13\]) that $m^2_q < 0$ at $N_c + 1 < N_f < 3N_c/2$, i.e., when the magnetic theory is weakly coupled. As regards the conformal window $3N_c/2 \leq N_f < 3N_c$, the complete analysis of the infrared soft masses is still lacking. However, the existing results \[14\] (see also ref. \[12\]) suggest that either the origin $q = \tilde{q} = M = 0$ is again unstable, or $m^2_q = m^2_M = 0$. We conclude that the induced charge matching provides qualitative
understanding of the behavior of soft masses in low energy description of softly broken SQCD.

One can show that the same phenomenon occurs in softly broken supersymmetric theories with $SO(N_c)$ and $Sp(2k)$ gauge groups and fundamental quarks at $N_f$, $N_c$ and $k$ such that the Seiberg duality holds. In particular, in theories with weakly coupled magnetic description, the soft masses of magnetic squarks are negative. This is again in full accord with induced charge matching conditions.

It is worth noting that there exists an example [12] where soft masses of fundamental scalar quarks single out the vacuum at the origin of the moduli space (in the absence of the background fields $m(x)$). This is the theory with $Sp(2k)$ gauge group and $2k + 4 = 2N_f$ quarks $Q_i$, $i = 1, \ldots, 2N_f$, in the fundamental representation. The low energy effective theory [15] contains antisymmetric mesons $M_{ij}$ and has superpotential $\text{Pf} M$. One can probe this theory by adding $x$-dependent mass terms $m^{p\tilde{q}}(x)Q_{\tilde{q}}Q_p$ where $p = 1, \ldots, N_f$, $\tilde{q} = (N_f + 1), \ldots, 2N_f$. In the theory without soft supersymmetry breaking, the induced charges match in a similar way as in the previous example: scalar mesons obtain the expectation values $\langle M_{\tilde{q}p}(x) \rangle \propto m^{\dagger}_{\tilde{q}p}(x)$ which give $x$-dependent masses to fermionic mesons. After the soft scalar quark masses are introduced, the scalar potential of the low energy theory contains soft meson masses, $m^2_M M^{\dagger}M$ where $m^2_M > 0$ at $k > 1$ [12]. At first sight, this ruins the induced charge matching at small $m_0$, as the ground state appears to be at $M = 0$ and no $x$-dependent masses of fermionic mesons seem to be generated. However, the symmetries of the theory allow for a linear supersymmetry breaking term in the scalar potential, $m^2_Q f(m^2_Q, m_0) mM$, which shifts the ground state to $\langle M \rangle \propto m^{\dagger}$ and in this way restores induced charge matching. Hence, we argue that this linear term is indeed generated in the low energy theory.
Let us discuss the relation between induced charges and triangle anomalies; we consider induced baryon number in QCD as an example. We use the notations of section 4.1. The \( x \)-dependence of the background field \( m(x) \) can be removed at the expense of modification of the gradient term in the quark Lagrangian. Namely, after the \( SU(N_0)_L \) rotation of the left-handed quark fields, \( \psi(x) \rightarrow U^{-1}(x)\psi(x) \), \( \bar{\psi}(x) \rightarrow \bar{\psi}(x) \), first \( N_0 \) quarks and antiquarks have \( x \)-independent masses \( m_0 \), and the gradient term of these quarks becomes

\[
\bar{\psi} i\gamma \cdot \left(D + \frac{1-\gamma^5}{2}A^L\right)\psi
\]

where \( A^L_0 = 0 \), \( A^L_i = U\partial_i U^{-1} \), the covariant derivative \( D \) contains dynamical gluon fields, and we switched to four-component notations. The addition to the gradient term may be viewed as the interaction of massive quarks with the background pure gauge vector fields corresponding to \( SU(N_0)_L \) subgroup of the flavor group; these background fields are small, time-independent, and slowly varying in space.

Now, consider an adiabatic process (either in Minkowskian or in Euclidean space-time) in which the background vector fields \( A^L(x) \) (in the gauge \( A^L_0 = 0 \)) change in time from \( A^L_i = 0 \) to \( A^L_i = U\partial_i U^{-1} \), always varying slowly in space and vanishing at spatial infinity (an example of such a process is an instanton of large size). Suppose that this process begins with the system in the ground state which has zero induced charges because of the triviality of the background. As the background vector fields interact with massive degrees of freedom only, the system remains in its ground state in the entire process, at least order by order in perturbation theory. The induced baryon number in the final state — the quantity we are interested in — is equal to \( \langle B \rangle = \int d^4x \partial_\mu j_B^\mu \) which in turn is determined by the anomaly in the gauge-invariant baryonic current \( j_B^\mu \). Hence, we recover eq. (10):

\[
\langle B \rangle = -\frac{N_c}{16\pi^2} \int d^4x \text{ Tr} F_{\mu\nu}^L F_{\mu\nu}^L = N_c N[U] .
\]

This observation relates induced charges and anomalies.

Let us note in passing that the phenomenon discussed in section 3 may be understood in this language as follows. After the background field \( m(x) \) is introduced, and its phase
$U(x)$ is rotated away, there remain *massless* fermions interacting with the background field $A^L(x)$. In this situation the adiabatic process does not necessarily end up in the ground state, because some energy levels of massless fermions may cross zero. In that case the baryon number induced in the ground state by the background field $A^L_i(x) = U \partial_i U^{-1}(x)$ would be different from eq. (23), as the anomaly determines the total change in the baryon number. This is the reason for the dependence of the induced charges on the parameters of the theory ($m_0/\mu_0$ in the example of section 3).

One may wonder whether similar phenomenon (fermion level crossing) might occur even if all relevant fundamental fermions obtain masses upon introducing the background scalar fields, i.e., whether the final state might actually contain excitations carrying non-zero net baryon number. To argue that this does not happen, let us consider QCD again. The appearance, in the final state, of excitations with non-zero net baryon number would show up as a non-vanishing index of the four-dimensional Euclidean Dirac operator $\mathcal{D}[A^L] = \gamma \cdot \left( D + \frac{1-\gamma^5}{2} A^L(x) \right) + m_0$, so that the vacuum-to-vacuum amplitude would vanish while matrix elements of baryon number violating operators between the initial and final vacua would not. However, for arbitrary gluon fields, the eigenvalues $\omega$ of the operator $\mathcal{D}[A^L = 0] = \gamma \cdot D + m_0$ obey $|\omega| > m_0$ (the Euclidean operator $\gamma \cdot D$ is anti-Hermitean) so the operator $\mathcal{D}[A^L]$ has no zero modes when the background fields $A^L(x)$ are small ($A^L(x) \ll m_0$ at all $x$) and slowly vary in space-time. This argument implies that eq. (23) is valid in full quantum theory even at $m_0 < \Lambda_{QCD}$. Although the situation in theories with fundamental scalars is more complicated, it is likely that analogous arguments may be designed in those theories as well.

Finally, the same adiabatic process may be considered within the low energy effective theory. The induced baryon number is now related to the anomaly in the effective theory, *provided* all low energy degrees of freedom interacting with $SU(N_0)_L$ gauge fields become massive upon introducing the mass $m_0$ to $N_0$ flavors of fundamental quarks. As the $U(1)_B \times SU(N_0)_L \times SU(N_0)_R$ anomalies are the same in the fundamental and low energy theories,
the induced baryon numbers match automatically in that case. This observation has an obvious generalization: a sufficient condition for induced charge matching is that no low energy degrees of freedom transforming non-trivially under a subgroup of the flavor group remain massless when this subgroup is explicitly broken by masses of some fermions of the fundamental theory. This property is certainly valid in supersymmetric theories where no phase transition is expected to occur as the masses of some of the flavors flow from small to large values, i.e., where massive flavors smoothly decouple. On the other hand, this property does not seem to be guaranteed in non-supersymmetric models, though it is intuitively appealing and may well be quite generic.

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