On Investigation of Crisp Bi-Objective Formulations for Fuzzy Traveling Salesman Problem and Fuzzy $p$-Median Problem

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Abstract. We consider two classic problems: the Traveling Salesman Problem and the $p$-Median Problem. In both problems the total traveling cost (in some sense) is minimized, assuming that all traveling times are known. In most real-word applications the input data are uncertain, and this fact should be taken into account when a solution is constructed. In this paper we investigate bi-objective versions of the problems, where the total cost and fuzzy uncertainty associated with the traveling times (membership function) are optimized. A greedy approximation algorithm is proposed and experimentally tested on instances from TSPLIB library.

1. Introduction
We consider two classic problems: the Traveling Salesman Problem and the $p$-median Problem. The problems are subproblems in such practically important problems as Vehicle Routing problems, Facility Location problems and their aggregations [1, 2]. In both problems the total traveling cost (in some sense) is minimized, assuming that all traveling times are known. However, usually input data have uncertain nature, in particular we could not certain evaluate traveling times for the considered problems. There are various approaches to deal with uncertain data, e.g. stochastic programming [3], threshold robust optimization [4], fuzzy uncertainty [5] and others. In the stochastic programming a distribution of random parameters is given and the mean, the median, the probability of some quality criterion depending on random parameters or other quantiles of the distribution are optimized. In the cases when there is no fully reliable information about possible changes of parameters (probability distribution is unknown), a threshold robust model is often used. Here estimates of parameters and a threshold value for some quality measure are given. The robustness of a solution is calculated as the smallest error in the parameter estimates which makes the quality measure exceeds the threshold. Various robust optimization methods were also successfully used for practical applications (see, e.g., [6–8]). In the proposed approaches, characteristics of the input data and relations between the considered objects are taken into account. Fuzzy optimization problems operates with input data such as fuzzy numbers, where the grade of membership of elements to the set of real numbers is given. The worst case scenarios, crisp versions and multiobjective formulations are used in approaches to combinatorial optimization problems with fuzzy uncertainty [9–13]. In [14], a transportation
problem with fuzzy costs and demands is solved by a simple iterative procedure. A two stage algorithm was proposed for a vehicle routing problem with fuzzy time windows formulated as a two-criteria problem [15]. A general approach for optimization problems with fuzzy coefficients in objectives and constraints was developed in [16]. Here a new model with only fuzzy coefficients in objectives is formulated and solved.

In this paper we investigate crisp bi-objective versions of the fuzzy Traveling Salesman Problem and fuzzy p-Median Problem, where the total cost and the membership function of fuzzy uncertainty are optimized. We consider traveling costs as triangular fuzzy numbers. A greedy algorithm is proposed to construct approximation of the Pareto set. The results of the experimental evaluation on instances from TSPLIB library [17] are presented.

2. Problem Statement

2.1. Classic Problem Statement

In the Traveling Salesman Problem (TSP) we are given a set of cities \( I = \{1, \ldots, n\} \) and traveling times \( c_{ij} \) between cities \( i \) and \( j \), where \( i, j \in I \). The goal is to find a Hamiltonian circuit (tour) of minimum total traveling time. The problem can be formulated as integer linear program in the classic way [1]. Introduce Boolean variables

\[ x_{ij} = \begin{cases} 
1 & \text{if salesman visits city } i \text{ and travels to city } j, \\
0 & \text{otherwise},
\end{cases} \]

and auxiliary real variables \( u_i, i \in I \), for subtour elimination.

\[ f = \sum_{i=1}^{n} \sum_{j \in I} c_{ij} x_{ij} \rightarrow \min, \]

\[ \sum_{i=1}^{n} x_{ij} = 1, \ j \in I, \] (2)

\[ \sum_{j \in I} x_{ij} = 1, \ i = 1, \ldots, n, \] (3)

\[ u_i - u_j + nx_{ij} \leq n - 1, \ i, j \in I, i \neq j, i, j > 1, \] (4)

\[ x_{ij} \in \{0, 1\}, \ u_i \geq 0, \ i, j \in I. \] (5)

In the p-median Problem we are given a set \( I \) of \( n \) location points and matrix \( c \) with the traveling costs \( c_{ij} \) for satisfying the demand required at point \( i \) from the facility located at point \( j \), \( i, j \in I \). It is required to select \( p \) facilities among \( n \) points and allocate the demand points to the facilities. The objective is to minimize the total traveling cost between the demand points and the facilities. Introduce the Boolean variables:

\[ x_{ij} = \begin{cases} 
1 & \text{if demand point } j \text{ is assigned to facility located at point } i, \\
0 & \text{otherwise},
\end{cases} \]

\[ z_i = \begin{cases} 
1 & \text{if a facility is located at point } i, \\
0 & \text{otherwise.}
\end{cases} \]

The classic integer linear formulation has the following form [2]:

\[ f = \sum_{i=1}^{n} \sum_{j \in I} c_{ij} x_{ij} \rightarrow \min, \] (6)
solution form \((f, \mu_i)\) of the TSP or \(p_i\) membership function where parameters each value a real number in the interval \([0, 1]\)

We model the uncertainty of travelling times by Triangular Fuzzy Numbers (TFN), namely fuzzy traveling time is denoted by triplet \((c_{ij} - \delta_{ij}, c_{ij}, c_{ij} + \varepsilon_{ij})\) with membership function (assigns to each value a real number in the interval \([0, 1]\])

\[
\mu_{c_{ij}}(c) = \begin{cases} 
0 & \text{if } c < c_{ij} - \delta_{ij}, \\
\frac{c - c_{ij} + \delta_{ij}}{\delta_{ij}} & \text{if } c_{ij} - \delta_{ij} \leq c < c_{ij}, \\
\frac{c_{ij} + \varepsilon_{ij} - c}{\varepsilon_{ij}} & \text{if } c_{ij} \leq c < c_{ij} + \varepsilon_{ij}, \\
0 & \text{if } c \geq c_{ij} + \varepsilon_{ij},
\end{cases}
\]

where parameters \(\varepsilon_{ij}, \delta_{ij} > 0\) correspond to deviations from estimated traveling time \(c_{ij}\) and membership function \(\mu_{c_{ij}}(c)\) indicates the grade of membership of value \(c\) to the set of possible traveling times between \(i\) and \(j\).

According to [13] the sum of TFNs also form a TFN. Thus each feasible solution \(x = (x_{ij})\) of the TSP or \(p\)-median problem gives fuzzy value of the objective function having TFN form \((f_x - \delta_{f_x}, f_x, f_x + \varepsilon_{f_x})\), where \(f_x = \sum^n_{i=1} \sum_{j \in I} c_{ij} x_{ij}\), \(\delta_{f_x} = \sum^n_{i=1} \sum_{j \in I} \delta_{ij} x_{ij}\), \(\varepsilon_{f_x} = \sum^n_{i=1} \sum_{j \in I} \varepsilon_{ij} x_{ij}\). The membership function of the objective function \(f\) corresponded to the solution \(x\) we denote by

\[
\mu_{f_x}(y) = \begin{cases} 
0 & \text{if } y < f_x - \delta_{f_x}, \\
\frac{y - f_x + \delta_{f_x}}{\delta_{f_x}} & \text{if } f_x - \delta_{f_x} \leq y < f_x, \\
\frac{f_x + \varepsilon_{f_x} - y}{\varepsilon_{f_x}} & \text{if } f_x \leq y < f_x + \varepsilon_{f_x}, \\
0 & \text{if } y \geq f_x + \varepsilon_{f_x},
\end{cases}
\]

The membership function \(\mu_f\) of objective function \(f\) is the upper bound of union of TFNs corresponding to all feasible solutions \(X\): \(\mu_f(y) = \sup_{x \in X} \{\alpha \in [0, 1] \mid \mu_x(y) = \alpha\}\).

According to the approach from [18] we modify our optimization problems with fuzzy objective functions to crisp bi-criteria problems, where objective functions and their membership function represent the criteria of crisp problem, i.e. we have vector criterion \(g = (f, \mu_f)\), for which \(g(y) = (y, \mu_f(y))\) for \(y\) from interval \([\min_{x \in X} (f_x - \delta_{f_x}), \max_{x \in X} (f_x + \varepsilon_{f_x})]\).

As we know [19], the set of pareto-optimal solutions with respect to criteria \(g\) and feasible set \(Y\) is \(P_g(Y) = \{y \in Y \mid \exists g^* \in Y : g_1(y^*) \leq g_1(y), g_2(y^*) \geq g_2(y), g(y^*) \neq g(y)\}\). The Pareto set is defined as \(P(G) = g(P_g(Y))\), or \(P(G) = \{g' \in G \mid \exists g^* \in G : g'_1 \leq g'_1, g'_2 \geq g'_2, g^* \neq g'\}\), where \(G = g(Y)\).

3. Greedy Algorithm for Finding an Approximation of the Pareto Set
We suppose that all input data are integer. The Pareto set of the considered bi-criteria problems corresponds to the north-west border of the set of feasible outcomes \(G\). The membership function of \(f(\cdot)\) corresponds to the upper border of combination of TFNs, so the Pareto set \(P(G)\) is
computed as the most left broken line. As shown in [18] this set may be formed by the left parts of TFNs representing by the minimum and close-to-minimum values of the objective $f(\cdot)$. In the proposed here new greedy algorithm an approximation of the Pareto set is formed by taking into account only parts of all possible TFNs. The main idea is to iteratively find the optimal solution of the classical problem with a lower bound on the main criterion, which is 1 more than the optimum at the previous iteration. Then we form TFNs, corresponding to the obtained solutions, and build by them an approximation of the Pareto set using the constructive algorithm from [18]. The detailed description of the greedy algorithm is presented in Algorithm 1.

We note, that from the obtained approximation of the Pareto set $\tilde{P}(G)$, we can easily came to the corresponding solutions of initial fuzzy problem $x$ (call them “near-optimal”). It will be such solution, that gives the upper bound for vector $y = g^{-1}(\tilde{P}(G))$ in definition of membership function $\mu_f(y)$.

**Algorithm 1 Greedy Approximation Algorithm**

1. Find optimal solution of model (1)-(5) for TSP problem or model (6)-(10) for $p$-median problem. Let $f^*_1$ denote the optimal objective value, and triplet $(f^*_1 - \delta^*_1, f^*_1 + \varepsilon^*_1)$ is the corresponding TFN. Put the current iteration number $iter = 1$.

2. Repeat until $iter \leq m_{\text{max}}$.

   2.1 Put $iter = iter + 1$.

   2.2 Find optimal solution of model (1)-(5) for TSP problem or model (6)-(10) for $p$-median problem with additional constraint $f \geq f^*_1 - \delta^*_1 + 1$.

   2.3 Let $f^*_1$ denote the obtained optimal objective value, and triplet $(f^*_1 - \delta^*_1, f^*_1 + \varepsilon^*_1)$ is the corresponding TFN.

3. Find an approximation of the Pareto set from $\{(f^*_1 - \delta^*_1, f^*_1 + \varepsilon^*_1), \ldots, (f^*_m - \delta^*_m, f^*_m + \varepsilon^*_m)\}$ by means of the algorithm from [18].

4. **Computational Experiment**

The experiments were done on the following test problems from TSPLIB library: fri26, gr17, gr21, dantzig42, hk48, swiss42 (symmetric instances), br17, ftv33, ftv35, p43, ry48p, ft70, kro124p (asymmetric instances). We set the number of facilities $p = 3$ for $p$-median problems. Integer programming models were coded in GAMS and solved by Gurobi-solver (version 9.0.3).

Parameters $\delta_{ij} = \varepsilon_{ij}$ are generated as follows:

$$\delta_{ij} = \begin{cases} 6 \cdot \frac{c_{ij}}{\bar{c}} + \frac{r_{ij}}{\bar{c}} & \text{if } c_{ij} > c_{\text{aver}}, \\ A_{ij} \cdot \frac{c_{ij}}{\bar{c}} + \frac{r_{ij}}{\bar{c}} & \text{otherwise}, \end{cases}$$

where $c_{\text{aver}}$ is the average traveling time, $k = n$ for TSP instances and $k = n - p$ for $p$-median instances, random factor $r_{ij}$ is generated randomly with uniform distribution from interval $[\frac{c_{\text{aver}}}{2}, c_{\text{aver}}]$ for $c_{ij} > c_{\text{aver}}$ and from interval $[\frac{c_{\text{aver}}}{8}, \frac{c_{\text{aver}}}{4}]$ for $c_{ij} \leq c_{\text{aver}}$, parameter $A_{ij}$ is generated randomly from $[1, 3]$. Here we consider values $\delta_{ij} = \varepsilon_{ij}$ proportional to traveling costs with additional random factors, and this is consistent with practical applications. Note that if the ordering of the traveling costs $c_{ij}$ in increasing order corresponds to the ordering of the deviations $\delta_{ij}$ in descending order, then we have only one solution of fuzzy problem corresponding the Pareto set $P(G)$.

We set the number of iterations in the greedy algorithm to 100. At each iteration of Algorithm 1 the optimal solution was found in less then half an hour. The results of the
Table 1. Experimental Results

| Instance | n | Approximation |
|----------|---|---------------|
| fri26    | 26| 3             |
| gr17     | 17| 3             |
| gr21     | 21| 4             |
| dantzig42| 42| 4             |
| hk48     | 48| 3             |
| swiss42  | 48| 4             |
| br17     | 17| 3             |
| ftv33    | 34| 4             |
| ftv35    | 36| 5             |
| p43      | 43| 1             |
| ry48p    | 48| 3             |
| ft70     | 48| 4             |
| kro124p  | 100| 4            |

Table 2. Average deviations $\delta_{i}^{\text{aver}}$ for facilities in solutions for fri26 ($p$-median problem)

| Solution | facility 1 | facility 2 | facility 3 |
|----------|------------|------------|------------|
|          | $i_{3}$    | $i_{5}$    | $i_{9}$    |
| 1        | 9.5        | 3.7        | 4.75       |
| 2        | 2.7        | 6.0        | 13.5       |
| 3        | 4.3        | 6.0        | 13.5       |

Table 3. Average deviations $\delta_{i}^{\text{aver}}$ for facilities in solutions for ftv35 ($p$-median problem)

| Solution | facility 1 | facility 2 | facility 3 |
|----------|------------|------------|------------|
|          | $i_{3}$    | $i_{8}$    | $i_{10}$   |
| 1        | 11.8       | 4.7        | 3.5        |
| 2        | 5.0        | 6.0        | 30.2       |

computational experiment are presented in Table 1, where |Approximation| is the number of “near-optimal” solutions of fuzzy problem in the approximation. As we can see it contains at most 5 elements in all tested instances. The obtained left borders of the TFNs giving the approximations of the Pareto set for instances fri26 and ftv35 are presented in Fig. 1, 2. Tables 2 and 3 present the average deviations $\delta_{i}^{\text{aver}} = \frac{\sum_{j=1}^{n} \delta_{ij} x_{ij}}{\sum_{j=1}^{n} x_{ij}}$ for facilities $i$ located in “near-optimal” solutions of $p$-median problems generated from fri26 and ftv35. These solutions are characterized by different sets of facilities and different deviations, and this allows to have more than one solution in approximation. Similar properties take place for the rest instances.

In particular, for fri26 ($p$-median problem) solution 1 has the degree of confidence in interval $[0, 0.6]$, the degree of confidence of solution 2 is in interval $[0.6, 0.61]$, and solution 3 has the degree in interval $[0.61, 1]$. For ftv35 ($p$-median problem) we have the following correspondings: solution 1 has interval $[0, 0.38]$, solution 2 is assigned to interval $[0.38, 1]$.

We also tested the case, when parameters $\delta_{ij}$, $\varepsilon_{ij}$ are generated randomly from interval $\left[\frac{c_{\text{aver}}}{2}, \frac{3c_{\text{aver}}}{2}\right]$ with uniform distribution. For all considered instances, |Approximation| = 1 in
Figure 1. Left borders of TFNs giving the Pareto set approximation of fri26 (left is for TSP problem, right is for $p$-median problem)

Figure 2. Left borders of TFNs giving the Pareto set approximation of ftv35 (left is for TSP problem, right is for $p$-median problem)

this experiment.

Conclusion
We investigated fuzzy versions of two classic optimization problems with summed objective function: TSP problem and $p$-median problem. The problems were formulated as bi-objective ones, where the total traveling cost (main objective) and its membership function are optimized. An approximation of the Pareto set was constructed by the greedy algorithm using MIP-solver. Experimental evaluation on instances from TSPLIB library shows that the approximate set of the non-dominated solutions has a very low cardinality, which is appealing from the practical point of view. The possibilities of MIP-solvers are limited by small dimensions in solving NP-hard problems, so further research can be undertaken to develop fast metaheuristics for instances with large number of cities and location points.

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