GOOD CODES FROM DIHEDRAL GROUPS

SAMIR ASSUENA AND CÉSAR POLCINO MILIES

Abstract. We give a simple construction of codes from left ideals in group algebras of certain dihedral groups and give an example to show that they can produce codes with weights equal to those of the best known codes of the same length.

1. Introduction

Throughout this paper, $G$ will always denote a finite group and $F$ a field such that $\text{char}(F) \nmid |G|$.

We recall that a group $G$ is metacyclic if $G$ contains a cyclic normal subgroup $H = \langle a \rangle$ such that the factor group $G/H = \langle bH \rangle$ is also cyclic.

Then:

$$G = \langle a, b \mid a^m = 1, b^n = a^s, bab^{-1} = a^i \rangle$$

and the integers $m, n, s, i$ are such that

$$s \mid m, \quad m \mid s(i - 1), \quad i < m, \quad \gcd(i, m) = 1.$$

In the special case when $i = -1$ and $n = 2$ we obtain the well-known family of Dihedral groups.

A group code over a field $F$ is any ideal $I$ of the group algebra $FG$ of a finite group $G$. A code is said to be metacyclic, abelian, or dihedral in case the given group $G$ is of that kind of groups.

If $I$ is two-sided, then it is called a central code. A minimal central code is an ideal $I$ which is minimal in the set of all (two-sided) ideals of $FG$.

The Hamming distance between two elements $\alpha = \sum_{g \in G} \alpha_g g$ and $\beta = \sum_{g \in G} \beta_g g$ in $FG$ is

$$d(\alpha, \beta) = | \{ g \mid \alpha_g \neq \beta_g, \ g \in G \} |$$

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that is, the number of elements of the support or $\alpha - \beta$. The weight of an ideal $I$ is the number

$$w(I) = \min\{d(\alpha, \beta) \mid \alpha \neq \beta, \alpha, \beta \in I\}.$$ 

The following result was proved in [1, Proposition 2.1] when $H \triangleleft G$ but, actually, the proof does not require normality.

**Lemma 1.1.** Let $G$ be a finite group and let $\mathbb{F}$ a field. Let $H, K$ be subgroups of $G$ with $H \subset K$. Write $e = \hat{H} - \hat{K}$. Then:

1. $\dim_{\mathbb{F}}(\mathbb{F}e) = (G : H) - (G : K)$;
2. $w(\mathbb{F}e) = 2|H|$;
3. If $A$ is a transversal of $K$ in $G$ and $\tau$ is a transversal of $H$ in $K$ containing $1$, then the set
   \[\{r(1-t)\hat{H} \mid r \in A, t \in \tau \setminus \{1\}\}\]
   is a basis of $(\mathbb{F}G)e$ over $\mathbb{F}$.

Sabin and Lomonaco showed in [6] that central metacyclic codes are not better than cyclic codes, in a sense that, for each central metacyclic code, there exists a cyclic code that is equivalent to it. We shall show that, in the particular case of Dihedral codes, there exist non central codes that are not equivalent to any abelian code and, in the last section, we exhibit examples of Dihedral codes that have weights equal to the weights of the best known codes of the same dimension.

## 2. Dihedral Codes of Length $2p^m$

Throughout this section, $D$ will denote the dihedral group of order $2p^m$ with presentation

$$D = \langle a, b \mid a^{p^m} = 1 = b^2, bab = a^{-1} \rangle.$$ 

and $\mathbb{F}_q$ will denote a finite field with $q$ elements such that $\gcd(2p^m, q) = 1$. Also, in what follows we shall always assume that $\mathcal{U}(\mathbb{Z}_{p^m}) = (\overline{q})$; i.e. that the multiplicative order of $q$ modulo $p^m$ is $\varphi(p^m) = p^{m-1}(p-1)$.

Under this hypothesis, it has been shown in [2] that, setting $A = \langle a \rangle$, the structure of the cyclic group algebra $\mathbb{F}A$ depends only on the subgroup structure of $A$. As an application, one can describe the set of primitive idempotents of $\mathbb{F}D$ as follows. Let

$$A = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_m = \{1\}$$

be the descending chain of all subgroups of $A$, i.e., $H_j = \langle a^{p^j} \rangle, 0 \leq i \leq m$. Consider the idempotents

$$e_0 = \hat{A}, \quad e_j = \hat{H}_j - \hat{H}_{j-1}, \quad 1 \leq j \leq m$$

and write $e_0 = \langle a \rangle$ as the following sum of idempotents

$$e_{11} = \left(\frac{1+b}{2}\right)e_0, \quad e \quad e_{22} = \left(\frac{1-b}{2}\right)e_0.$$
Theorem 2.1. [1] Theorem 3.3] Under the hypotheses above, the set of primitive central idempotents of $F_qD$ is

$$\{e_{11}, e_{22}\} \cup \{e_j, \ 1 \leq j \leq m\}.$$ 

Sabin and Lomonaco, [6] introduced the following.

Definition 2.2. Let $G_1$ and $G_2$ be finite groups of the same order and let $F$ be a field. Let $FG_1$ and $FG_2$ be the corresponding group algebras. A combinatorial equivalence is a $F$ vector space isomorphism $\phi : FG_1 \rightarrow FG_2$ induced by a bijection $\phi : G_1 \rightarrow G_2$. Codes $C_1 \subset FG_1$ and $C_2 \subset FG_2$ are combinatorially equivalent if there exists a combinatorial equivalence $\phi : FG_1 \rightarrow FG_2$ such that $\phi(C_1) = C_2$.

Clearly, combinatorially equivalences are Hamming isometries and, thus, combinatorially equivalent codes have the same weights and dimensions.

Let $G$ be a metacyclic group with presentation

$$G = \langle a, b \mid a^m = 1 = b^n, \ bab^{-1} = a^i \rangle.$$ 

The result we quote below shows that central metacyclic codes are no better that abelian codes.

Theorem 2.3. [6 Theorem 1] Let $C_m$ and $C_n$ be cyclic groups of order $m$ and $n$, with generators $\bar{a}$ and $\bar{b}$ respectively and let $\phi : G \rightarrow C_m \times C_n$ be the bijection given by $\phi(a^i b^j) = \bar{a}^i \bar{b}^j$. If $C$ is a code generated by a central idempotent $e$ in $FG$, then $\phi(C)$ is the ideal of $F[C_m \times C_n]$ generated by $\phi(e)$, an idempotent of $F[C_m \times C_n]$. Consequently, every central metacyclic code is combinatorially equivalent to an abelian code.

We shall construct some non-central ideals in a rather simple way and prove that these are not equivalent to any abelian code.

Recall, from Theorem 2.1, that the set of primitive central idempotents of $F D$ is

$$\{e_{11}, e_{22}\} \cup \{e_j, \ 1 \leq j \leq m\}.$$ 

Let us denote by $e$, for short, one idempotent chosen from the set $\{e_j, \ 1 \leq j \leq m\}$ which we fix for the rest of this section. It was shown in [1 Proposition 2.1] that $(a - a^{-1})e$ is invertible in $FD$ and that the elements
\[ e_{11} = \left( \frac{1 + b}{2} \right) e, \quad e_{12} = \left( \frac{1 + b}{2} \right) a \left( \frac{1 - b}{2} \right) e, \]

\[ e_{21} = 4((a - a^{-1})e)^{-2} \cdot \left( \frac{1 - b}{2} \right) a \left( \frac{1 + b}{2} \right) e, \quad e_{22} = \left( \frac{1 - b}{2} \right) e. \]

form a set of matrix units for \((FD)e\); i.e., they verify the identities
\[ e_{11} + e_{22} = e \quad \text{and} \quad e_{ij} e_{hk} = \delta_{jh} e_{1k}. \]
Hence, by [8, Lemma VI.3.11], \((FD)e\) is the full ring of \(2 \times 2\) matrices over the centralizer of \(e_{12}\) in \((FD)e\) and thus, \(e_{11}\) and \(e_{22}\) are primitive (non-central) idempotents.

Notice that, setting \(H_j^* = \langle b \rangle H_j, \ 1 \leq j \leq m\), we have that
\[ e_{11} = \hat{H}_j^* - \hat{H}_{j-1}^* \]
so Lemma 1.1 readily gives
\[ \dim([FG]e_{11}) = \varphi(p^j) \quad \text{and} \quad w([FG]e_{11}) = 2|H_j| = 4p^{m-j}. \]

It is not difficult to show that \((FG)e_{22}\) has the same dimension and weight.

It turns out that these particular idempotents generate codes \(I_{11} = (FD)e_{11}\) and \(I_{22} = (FD)e_{22}\) that are equivalent to cyclic codes. However, it will be easy to use them, under certain hypotheses, to produce better codes.

To see this, consider first a cyclic group \(G\) of order \(2p^m\), which we write as \(G = \hat{A} \times C\), where \(\hat{A}\) is the cyclic subgroup of \(G\) of order \(p^m\) and \(C = \{1, t\}\) its subgroup of order 2. Denote by
\[ \hat{A} = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_m = \{1\} \]
the descending chain of subgroups of \(\hat{A}\) and, as before, write \(\hat{e}_j = \hat{K}_j = \hat{K}_{j-1}\).

**Proposition 2.4.** With the notations above, the map
\[ \gamma : D \longrightarrow G = A \times C_2 \]
\[ \gamma(a^i b^j) \longmapsto \hat{a}^i \hat{t}^j \]
extended linearly to \(D\) is such that \(\gamma(e_{11}) = \left( \frac{1 + t}{2} \right) \hat{e}_j\) and \(\gamma(e_{22}) = \left( \frac{1 - t}{2} \right) \hat{e}_j\) so

\[ \gamma(I_{j_1}) = \mathbb{F}_q G \left( \frac{1 + t}{2} \right) \hat{e}_j \quad \text{and} \quad \gamma(I_{j_2}) = \mathbb{F}_q G \left( \frac{1 - t}{2} \right) \hat{e}_j. \]

Consequently, \(I_{11}\) and \(I_{22}\) are combinatorially equivalent to cyclic codes.
Proof. It follows directly, from the definition of $\gamma$ that $\gamma(\hat{H}_i) = \hat{K}_i$ and $\gamma(\hat{bH}_i) = \hat{tK}_i$. We claim that $\gamma(ge_{11}) = \gamma(g) \left( \frac{1 + t}{2} \right) \hat{e}_j$, for all $g \in D$.

In fact, $g$ is of the form $g = a^h b$, for some positive integer $h$ and, since $b(1 + b)/2 = (1 + b)/2$, we have

$$
\gamma(ge_{11}) = \gamma \left( a^h b \left( \frac{1 + b}{2} \right) e \right) = \gamma \left( \left( \frac{a^h + ab}{2} \right) (\hat{H}_j - \hat{H}_{j-1}) \right)
= \left( \frac{\hat{a}^h + \hat{a}^h t}{2} \right) (\hat{K}_j - \hat{K}_{j-1}) = \hat{a}^h \hat{b} \hat{e}_j = \gamma(g) \hat{e}_j.
$$

Also, since $b(1 - b)/2 = -(1 - b)/2$, in a similar way we obtain that $\gamma(ge_{22}) = \gamma(g) \hat{e}_{22}$.

Since $\gamma$ is linear, the results follow. $\square$

Set $\alpha = e_{11} + e_{12} + e_{22}$. Then, as $\alpha(e_{11} - e_{12} + e_{22}) = e$ it follows that $\alpha$ is invertible in the component $(\mathbb{F}G)e$ with $\alpha^{-1} = e_{11} - e_{12} + e_{22}$. Then $\alpha e_{11} \alpha^{-1} = e_{11} - e_{12}$, a non central idempotent.

Since conjugation is an $\mathbb{F}$-automorphism of $(\mathbb{F}D)e$, the dimension of the left ideal $I = \mathbb{F}_q D(e_{11} - e_{12})$ is also equal to $\varphi(p^\ell)$.

Proposition 2.5. Write $f = e_{11} - e_{12}$. Then the set

$$
\mathcal{B} = \{ f, af, a^2 f, \ldots, a^{\varphi(p^\ell)}^{-1} f \}
$$

is a basis for $I$ over $\mathbb{F}_q$.

Proof. We first prove that the set $\mathcal{B}$ is linearly independent.

Notice that

$$
f = e_{11} - e_{12} = \left( \frac{1 + b}{2} \right) e - \left( \frac{1 + b}{2} \right) a \left( \frac{1 - b}{2} \right) e
= \left( \frac{1 + b}{2} \right) \left( 1 - a \left( \frac{1 - b}{2} \right) \right) e
= \left( \frac{1 + b}{2} \right) \left( \frac{2 - a + ab}{2} \right) e
= \frac{1}{4} \left[ (2 - a + a^{-1}) + (2 + a - a^{-1})b \right].
$$

Assume that

$$
\alpha_0 f + \alpha_1 af + \ldots + \alpha_{\varphi(p^\ell)}^{-1} a^{\varphi(p^\ell)}^{-1} f = 0.
$$

Then
\[
\left( \sum_{k=0}^{\varphi(p^j)-1} \alpha_k a^k \right) (2 - a + a^{-1}) + \left( \sum_{k=0}^{\varphi(p^j)-1} \alpha_k a^k \right) (2 + a - a^{-1})b e = 0.
\]

and thus, in particular,
\[
\left[ \left( \sum_{k=0}^{\varphi(p^j)-1} \alpha_k a^k \right) (2 - a + a^{-1}) \right] e = 0,
\]

in \( \mathbb{F}_q \langle a \rangle \). This implies that
\[
\left( \sum_{k=0}^{\varphi(p^j)-1} \alpha_k a^k \right) (2-a+a^{-1})e = \left[ \left( \sum_{k=0}^{\varphi(p^j)-1} \alpha_k a^k \right) e \right] [(2 - a + a^{-1})e] = 0.
\]

Since \( e \) is a primitive idempotent of \( \mathbb{F}_q \langle a \rangle \), the ideal \( \mathbb{F}_q \langle a \rangle e \) is a field.

If \( j \geq 2 \), the elements \( 2e, ae \) and \( a^{-1}e \) have disjoint support so the element \( (2 - a + a^{-1})e \) cannot be zero.

If \( j = 1 \), then \( (2 - a + a^{-1})e_1 = 2\widehat{H}_1 - a\widehat{H}_1 + a^{-1}\widehat{H}_1 - \widehat{H}_0 \). If \( (2 - a + a^{-1})e_1 = 0 \) so \( 2\widehat{H}_1 - a\widehat{H}_1 + a^{-1}\widehat{H}_1 = \widehat{H}_0 \) a contradiction.

So \( \sum_{k=0}^{\varphi(p^j)-1} \alpha_k a^k e_j = 0 \) and, by [7 Theorem 4.9], \( \alpha_k = 0 \), for all \( k \).

Since the number of elements of \( \mathcal{B} \) is \( \varphi(p^j) \) and the dimension of \( I \) is also \( \varphi(p^j) \), the result follows.

\[\square\]

3. An Example

Let \( D_9 \) be dihedral group of order 18, set \( e = e_1 = H_1 - h_0 \), \( f = e_{11} - e_{22} \) and let us consider the ideal \( I = \mathbb{F}_q D_9 f \).

By Proposition 2.3, the set \( \{ f, af \} \) is a basis of this ideal. We have that
\[
ae^{(1)} = (2a - a^2 + 1)e_1 + (2a + a^2 - 1)e_1b
\]

where
\[
(2 - a + a^{-1})(\widehat{H}_1 - \widehat{H}_0) = 2\widehat{H}_1 - a\widehat{H}_1 + a^{-1}\widehat{H}_1 - 2\widehat{H}_0
\]
\[
a(2 - a + a^{-1})(\widehat{H}_1 - \widehat{H}_0) = 2a\widehat{H}_1 - a^2\widehat{H}_1 + \widehat{H}_1 - 2\widehat{H}_0;
\]

consequently we can write an arbitrary element \( \alpha \) of \( \mathbb{F}_q D_9 f \) as
\[\]
\[ \alpha_0 f + \alpha_1 af = (2\alpha_0 + \alpha_1)\hat{H}_1 + (-\alpha_0 + 2\alpha_1)a\hat{H}_1 + (\alpha_0 - \alpha_1)a^2\hat{H}_1 + (-2\alpha_0 - 2\alpha_1)\hat{H}_0 + (2\alpha_0 - \alpha_1)\hat{H}_1 b + (\alpha_0 + 2\alpha_1)a\hat{H}_1 b + (-\alpha_0 + \alpha_1)a^2\hat{H}_1 b + (-2\alpha_0 - 2\alpha_1)\hat{H}_0 b. \]

A direct computation shows that, if the characteristic of \( \mathbb{F}_q \) is different from 2, 3, 5 and 7, then at most just one of the coefficients of \( \alpha \) can be equal to zero. So, the weight of \( \alpha \) is \( w(\alpha) \geq 15 \).

As it is easy to exhibit elements of \( I \) of weight 15, we have:

1. The dimension of \( I = \mathbb{F}_q D_9 f \) is \( \varphi(3) = 2 \);
2. The weight of \( I \) is \( w(I) = 15 \).

We remark that the weight of this code is the same as that of the best known code of same dimension (see [www.codetables.de]), for example in the case when the field is \( \mathbb{F}_{11} \), which satisfies our conditions.

Finally, we shall show that, when the multiplicative order of \( q \) modulo 9 is \( \varphi(9) = 6 \), this code is not equivalent to any abelian code. Let \( A \) be an abelian group of order 18; then it can be written as \( A = C_2 \times B \), where \( C_2 \) is a cyclic group of order 2 generated by an element \( t \), and \( B \) is an abelian 3-group.

Supose, by way of contradiction, that \( I = \mathbb{F}_q D_9 f \) is combinatorially equivalent to a code \( J \) of \( \mathbb{F}_q A \). Since \( U(\mathbb{Z}_p^r) = \langle 7 \rangle \), by [2, Lemma 5 and Theorem 4.1], the primitive idempotents of \( \mathbb{F}_q A \) are of the form \( \left( \frac{1+t}{2} \right) e_H \), or \( \left( \frac{1-t}{2} \right) e_H \) where \( H \) is a subgroup of \( B \) such that \( B/H \) is cyclic of order 9 or 3. We have two possibilities:

1. \( J = I_{e_H} \).
   Since the dimension of \( I \) is 2, there exists a subgroup \( H \) of \( B \) such that \( B/H \) is cyclic of order 3 and \( J = (\mathbb{F}_q A)(\frac{1+t}{2})e_H \).
   By [2, Theorem 4.2], the weight of \( J \) is 12, but the weight of \( \mathbb{F}_q D_9 f \) is 15.
2. \( J = I_{e_{H_1}} \oplus I_{e_{H_2}} \).
   Since the dimension of \( \mathbb{F}_q D_9 f \) is 2, the dimension of \( I_{e_{H_1}} \) and \( I_{e_{H_2}} \) must be equal to 1. In this case, the order of \( B/H_1 \) and \( B/H_2 \) should be 2, which cannot happen since \( |B| = 9 \).

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Centro Universitário da FEI, Av. Humberto de Alencar Castelo Branco 3972, São Bernardo do Campo, CEP:09850-901, Brazil and Centro Universitário do Instituto Maurá de Tecnologia-I.M.T., Praça Maurá 1, São Caetano do Sul, SP, CEP: 09580-900, Brazil

E-mail address: samir.assuena@fei.edu.br and samir.assuena@maua.br

Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66.281, CEP 05314-970, Brazil and CMCC, Universidade Federal do ABC, Av. dos Estados, 5001, Santo André, SP, CEP 09210-580, Brazil

E-mail address: polcino@ime.usp.br and polcino@ufabc.edu.br