Rotating Black Holes in Higher Dimensional Einstein-Maxwell Gravity

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Abstract

The strategy of obtaining the familiar Kerr-Newman solution in general relativity is based on either using the metric ansatz in the Kerr-Schild form, or applying the method of complex coordinate transformation to a non-rotating charged black hole. In practice, this amounts to an appropriate re-scaling of the mass parameter in the metric of uncharged black holes. Using a similar approach, we assume a special metric ansatz in $N+1$ dimensions and present a new analytic solution to the Einstein-Maxwell system of equations. It describes rotating charged black holes with a single angular momentum in the limit of slow rotation. We also give the metric for a slowly rotating charged black hole with two independent angular momenta in five dimensions. Finally, we compute the gyromagnetic ratio of these black holes which corresponds to the value $g = N - 1$.

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I. INTRODUCTION

Black holes were originally predicted in the framework of four dimensional general relativity as the endpoint of gravitational collapse of sufficiently massive stars. Subsequently apart from their astrophysical implications, they have also played a profound role in understanding the nature of general relativity itself, resulting in the famous singularity theorems \[1, 2\]. It has been shown that black holes possess a number of remarkable features such as the equilibrium and uniqueness properties and quantum properties of evaporation of microscopic black holes \[3, 4, 5\].

From a pure theoretical point of view one can expect that the properties of black holes might also have played an important role in understanding the nature of gravity in higher dimensions. This expectation has triggered the study of black hole solutions in higher-dimensional gravity theories as well as in string/M-theory \[6, 7\] (see Refs.\[8\] for reviews). Developments have revealed both new possibilities to test the predictions of string/M-theory and new unexpected features of black holes inherent in higher dimensions. For instance, by counting the microstates of certain supersymmetric black holes in five dimensions it has become possible to explain the statistical origin of the Bekenstein-Hawking entropy \[9\]. Furthermore, it turned out that the higher dimensions allow different horizon topologies for black holes, whereas in four dimensions the event horizon is uniquely determined by the topology of a two-sphere \[3, 4\]. Accordingly, some basic properties of black holes change when going from four dimensions to higher dimensions, among them are stability and uniqueness properties. The simplest class of extended black holes (black strings with cylindrical topology of the horizon) exhibits the linear perturbative instability below a certain critical mass \[10\]. There exists a strong evidence that the endpoint of this instability may result in a separate black hole, thereby providing an example of non-uniqueness in the form of a phase transition between black holes and black strings in higher dimensions \[11\].

The first higher dimensional black hole solutions with the spherical topology of the horizon have been found in \[12\]. These solutions generalize the familiar spherically symmetric Schwarzschild and Reissner-Nordstrom solutions of four-dimensional general relativity. It is remarkable that for the static black holes the uniqueness and the stability properties still survive \[13\] in higher dimensions, however the situation is drastically changed for rotating black holes. The rotating black hole solution was found by Myers and Perry \[14\]. This solu-
tion is not unique, unlike its four dimensional counterpart, the Kerr solution. There exists a rotating black ring solution in five dimensions with the horizon topology of $S^2 \times S^1$ [15] which may have the same mass and angular momentum as the Myers-Perry solution. Thus, the black ring solution can be thought of as describing a ”donut-shaped” rotating black hole which is absent in four dimensions. This provides another example of a lack of uniqueness for black holes in higher dimensions. As for the stability of these solutions, it is still not known in the general case, though it has been argued that the Myers-Perry solution becomes unstable at large enough rotation for a fixed mass [16]. The different physical properties of black holes and black rings in higher dimensions have been discussed in a number of papers [17]-[19].

After the advent of Large Extra Dimension Scenarios [20]-[21] the study of black holes in higher dimensions got a new strong impetus. These scenarios consider our observable Universe as a slice, a ”3-brane” in higher dimensional space and give an elegant geometric resolution of the hierarchy problem between the electroweak scale and the fundamental scale of quantum gravity. The large size of the extra dimensions supports the weakness of Newtonian gravity on the brane and makes it possible to lower the scale of quantum gravity down to the same order as the electroweak interaction scale (of the order of a few TeVs). One may naturally assume that in such scenarios black holes would be mainly localized on the brane, however some portion of their event horizon would have a finite extension into the extra dimension as well. On the other hand, it is clear that if the radius of the horizon of a black hole on the brane is much smaller than the size scale of the extra dimensions ($r_+ \ll L$), the black hole would behave as a higher dimensional object. These black holes, to a good enough approximation, can be well described by the exact solutions of higher dimensional Einstein’s equations [12, 14]. Thus, the Large Extra Dimension Scenarios open up new exciting possibilities to relate the properties of higher dimensional black holes to the observable world by direct probing of TeV-size mini black holes at future high energy colliders [22].

In the light of all described above it becomes obvious that the further study of black hole solutions in higher dimensional gravity is of great importance. Of particular interest is the case of charged black holes as after all, black holes produced at colliders may, in general, have an electric charge as well as other type of charges. Though the non-rotating black hole solution to the higher dimensional Einstein-Maxwell equations was found a long time
ago [12], rotating charged black holes have been basically discussed in the framework of
certain supergravity theories and string theory [8, 23, 24]. The rotating black hole solution
in higher dimensional Einstein-Maxwell gravity, that is the counterpart of the usual Kerr-
Newman solution, still remains to be found analytically. Numerical solutions for some special
cases in five dimensions have been found in [25].

In this paper we shall present new analytical solutions to the higher dimensional Einstein-
Maxwell equations which describe electrically charged black holes with slow rotation. The
organization of the paper is as follows. In Sec.II we begin with a brief description of the
Myers-Perry metric with a single non-vanishing rotation parameter in $N + 1$ dimensions.
We describe the Killing isometries of the metric, its mass parameter and specific angular
momentum. Assuming that the Myers-Perry black hole may also have a small electric charge
we construct the potential one-form for the electromagnetic field of the charge. In Sec.III we
assume a special ansatz for the metric of charged rotating black holes following the strategy
of obtaining the consistent solution to the Einstein-Maxwell system of equations in four
dimensional space-time. With this ansatz we show that the simultaneous solution of the
Einstein-Maxwell equations exists only when the rotation of the black hole occurs slowly.
Next, in Sec.IV we extend this approach to examine the metric of a slowly rotating charged
black hole with two independent angular momenta associated with two orthogonal 2-planes
of rotation in five dimensions. Finally, in Sec.V we compute the value of the gyromagnetic
ratio for rotating charged black holes in $N + 1$ dimensions which is turned out to be equal to
$g = N - 1$. In Appendix A we present the non-vanishing components of the electromagnetic
source tensor in arbitrary spacetime dimensions. The components of the Ricci tensor for
given metric ansatz are calculated in Appendix B.

II. THE MYERS-PERRY BLACK HOLE WITH A WEAK ELECTRIC CHARGE

A. Properties of the metric

The general stationary and asymptotically flat metric describing rotating black holes with
multiple angular momenta in different orthogonal planes of $N + 1$ dimensional space-time
was found by Myers and Perry [14]. We recall that in $N + 1$ dimensions the rotation group
is $SO(N)$ which possesses $\lfloor N/2 \rfloor$ independent Casimir invariants. (The notation $\lfloor N/2 \rfloor$
denotes the integer part of $N/2$). To be precise, the angular momentum of the system in general is described by an anti-symmetric $N \times N$ tensor. In the center-of-mass frame one can transform this tensor into its block-diagonal form by a suitable rotation of the spatial coordinates. In this case the angular momentum tensor is characterized by $\lfloor N/2 \rfloor$ physical parameters which in accordance with the existence of $\lfloor N/2 \rfloor$ Casimir invariants are associated with rotations in distinct planes. We shall focus on higher-dimensional black holes with a single rotation parameter. The corresponding metric in the Boyer-Lindquist type coordinates can be written in the form

$$ds^2 = -\left(1 - \frac{m}{r^{N-4} \Sigma}\right) dt^2 + \frac{r^{N-2} \Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{ma^2 \sin^2 \theta}{r^{N-4} \Sigma}\right) \sin^2 \theta \, d\phi^2$$

$$-2ma \sin^2 \theta \, \frac{dr}{r^{N-4} \Sigma} \, dt \, d\phi + r^2 \cos^2 \theta \, d\Omega_{N-3}^2 ,$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta , \quad \Delta = r^{N-2}(r^2 + a^2) - mr^2 ,$$

the parameter $m$ is related to the mass of the black hole, while $a$ is a parameter associated with its angular momentum and

$$d\Omega_{N-3}^2 = d\chi_1^2 + \sin^2 \chi_1 (d\chi_2^2 + \sin^2 \chi_2 (...d\chi_{N-3}^2...))$$

is the metric of a unit $(N-3)$-sphere. The square root of the determinant of the metric is given by

$$\sqrt{-g} = \sqrt{\gamma} \, r^{N-3} \sin \theta \, \cos^{N-3} \theta$$

where $\gamma$ is the determinant of the metric. When dropping the last term the metric bears a close resemblance to its counterpart, the Kerr solution of ordinary general relativity, exactly covering it for $N = 3$.

The event horizon is a null surface determined by the equation $\Delta = 0$, which implies that

$$r^2 + a^2 - \frac{m}{r^{N-4}} = 0 .$$

The largest real root of this equation gives the location of the black hole’s outer event horizon. We see that the properties of the horizon essentially depend on the dimension of the space. In particular, from equation it follows that in $N = 3$ and $N = 4$ dimensions
the event horizon exists until its rotation attains the maximum speed bounded by the mass of the black hole. However, for \( N \geq 5 \) the horizon does exist independently of the rotation, that is the black hole with a given mass may have arbitrarily large angular momentum \([14, 16]\).

The time-translation invariance of the metric \([11]\) along with its rotational symmetry in the \( \phi \)-direction imply the existence of the commuting Killing vectors

\[
\xi_{(0)} = \xi_{(t)}^\mu \frac{\partial}{\partial x^\mu}, \quad \xi_{(3)} = \xi_{(\phi)}^\mu \frac{\partial}{\partial x^\mu},
\]

such that their various scalar products are expressed through the metric components as follows

\[
\begin{align*}
\xi_{(0)} \cdot \xi_{(0)} &= g_{00} = -1 + \frac{m}{r^{N-4\Sigma}}, \\
\xi_{(0)} \cdot \xi_{(3)} &= g_{03} = -\frac{ma\sin^2 \theta}{r^{N-4\Sigma}}, \\
\xi_{(3)} \cdot \xi_{(3)} &= g_{33} = \left( r^2 + a^2 + \frac{ma^2\sin^2 \theta}{r^{N-4\Sigma}} \right) \sin^2 \theta.
\end{align*}
\]

The Killing vectors \([6]\) can be used to give a physical interpretation of the parameters \( m, a \) in the metric. Indeed, using the analysis given in \([26]\), we can obtain the following coordinate-independent definitions for these parameters

\[
m = \frac{1}{(N-2)A_{N-1}} \oint \xi_{(t)}^{\mu;\nu} d^{N-1} \Sigma_{\mu \nu},
\]

and

\[
\begin{align*}
\xi_{(3)} \cdot \xi_{(3)} &= g_{33} = \left( r^2 + a^2 + \frac{ma^2\sin^2 \theta}{r^{N-4\Sigma}} \right) \sin^2 \theta.
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\end{align*}
\]

The semicolon denotes covariant differentiation and the integrals are taken over the \((N-1)\)-sphere at spatial infinity with the surface element

\[
d^{N-1} \Sigma_{\mu \nu} = \frac{1}{(N-1)!} \sqrt{-g} \epsilon_{\mu \nu i_1 i_2 \ldots i_{N-1}} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{N-1}},
\]

the quantity

\[
A_{N-1} = \frac{2 \pi^{N/2}}{\Gamma(N/2)}
\]

gives the area of a unit \((N-1)\)-sphere. We have introduced the specific angular momentum parameter \( j \) associated with rotation in the \( \phi \) direction. We note that with these definitions
the relation between the specific angular momentum and the mass parameter looks exactly like the corresponding relation \( J = aM \) of four dimensional Kerr metric.

To justify the definitions \( \xi \) and \( \xi' \) we can calculate the integrands in the asymptotic region \( r \to \infty \). For the dominant terms in the asymptotic expansion we have

\[
\xi_{(t)}^{t; r} = \frac{m (N - 2)}{2 r^{N-1}} + \mathcal{O} \left( \frac{1}{r^{N+1}} \right),
\]

\[
\xi_{(\phi)}^{t; r} = -\frac{j N \sin^2 \theta}{2 r^{N-1}} + \mathcal{O} \left( \frac{1}{r^{N+1}} \right).
\]  

(12)

One can easily show that the substitution of these expressions into the formulas \( \xi \) and \( \xi' \) verifies them. The relation of the above parameters to the total mass \( M \) and the total angular momentum \( J \) of the black hole can be established using the standard Komar integrals in \( N+1 \) dimensions \[14\]. We find that

\[
m = \frac{16 \pi G}{N - 1} \frac{M}{A_{N-1}}, \quad j = \frac{8 \pi G J}{A_{N-1}}.
\]  

(13)

These relations confirm the interpretation of the parameters \( m \) and \( a \) as being related to the physical mass and angular momentum of the black hole.

**B. Electromagnetic potential one-form**

Let us now assume that a rotating black hole in \( N + 1 \) dimensions possesses an electric charge \( Q \). When the charge is small enough compared with the mass of the black hole, \( Q \ll M \), the influence of the electromagnetic field of the charge on the metric of space-time may become negligible, that is the space-time can still be well described by the Myers-Perry metric \[1\]. The potential one-form describing the electromagnetic field is given by the solution of the source-free Maxwell equations. In order to construct this solution we shall use the well-known fact that for a Ricci-flat metric a Killing 1-form field is closed and co-closed, that is, it can serve as a potential one-form for an associated test Maxwell field. Since the Myers-Perry metric is Ricci-flat as well, we can take the potential one-form field as

\[
A = \alpha \hat{\xi}_{(t)},
\]  

(14)

where the Killing one-form field \( \hat{\xi}_{(t)} = \xi_{(t)\mu} \, dx^{\mu} \) is obtained by lowering the index of the temporal Killing vector in \[6\] and \( \alpha \) is an arbitrary constant parameter. To determine this
parameter we examine the Gauss integral for the electric charge of the black hole
\[ Q = \frac{1}{A_{N-1}} \oint \star F , \] (15)
where the \( \star \) operator denotes the Hodge dual, along with expression (8) for the mass parameter. As a result we find that
\[ \alpha = -\frac{Q}{m (N - 2)} . \] (16)
With this in mind and requiring the vanishing behavior of the potential at infinity, we obtain the following expression for the electromagnetic potential one-form
\[ A = -\frac{Q}{(N - 2) r^{N-4} \Sigma} \left( dt - a \sin^2 \theta \, d\phi \right) . \] (17)
Accordingly, the electromagnetic two-form field is given by
\[ F = -\frac{Q}{(N - 2) r^{N-3} \Sigma} \left\{ H \left( dt - a \sin^2 \theta \, d\phi \right) \wedge dr ight. \] (18)
\[ \left. - r a \sin 2\theta \left[ a \, dt - \left( r^2 + a^2 \right) \, d\phi \right] \wedge d\theta \right\} , \]
where
\[ H = (N - 2) \Sigma - 2 a^2 \cos^2 \theta . \] (19)
It is also useful to calculate the contravariant components of the electromagnetic field tensor. They are given by
\[ F^{01} = \frac{Q}{N - 2} \frac{r^2 + a^2}{r^{N-3} \Sigma^3} H = -\frac{r^2 + a^2}{a} F^{13} , \]
\[ F^{23} = \frac{2 a Q}{N - 2} \frac{\cot \theta}{r^{N-4} \Sigma^3} = -\frac{F^{02}}{a \sin^2 \theta} . \] (20)
We recall that equations (17) and (18) describe the electromagnetic field of a higher dimensional rotating black hole carrying a weak (test) electric charge. In the following we shall suppose that the electric charge is no longer weak that its electromagnetic field essentially affects the geometry of space-time around the black hole.

III. THE METRIC ANSATZ

For an arbitrary amount of the electric charge of the black hole we must solve the simultaneous system of the Einstein-Maxwell equations. Following the strategy of obtaining the
consistent solution to the Einstein-Maxwell equations in four dimensions which results in
the familiar Kerr-Newman metric, we take the metric ansatz of the form
\[
ds^2 = - \left( 1 - \frac{m}{r^{N-4} \Sigma} + \frac{q^2}{r^{2(N-3)} \Sigma} \right) dt^2 + \frac{r^{N-2} \Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{2a (mr^{N-2} - q^2) \sin^2 \theta}{r^{2(N-3)} \Sigma} dt d\phi
\]
\[
+ \left( r^2 + a^2 + \frac{a^2 (mr^{N-2} - q^2) \sin^2 \theta}{r^{2(N-3)} \Sigma} \right) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\Omega^2_{N-3},
\]
where \( q \) is a parameter related to the physical charge of the black hole and the metric
function \( \Delta \) is now given as
\[
\Delta = r^{N-2} (r^2 + a^2) - m r^2 + q^2 r^{N-4}.
\]
In choosing this metric ansatz we required the potential one-form \([17]\) still to satisfy the
Maxwell equations. Indeed, this idea implicitly lies at the root of using the metric ansatz
in the Kerr-Schild form, as well as the complex coordinate transformation method \([27]\)
to obtain the Kerr-Newman solution in general relativity. Therefore, the metric ansatz
\([21]\) agrees with that suggested in \([28]\) by applying the complex coordinate transformation method of \([27]\) to a non-rotating charged black hole in \( N + 1 \) dimensions. However, in
our case it is obtained from the Myers-Perry metric \([11]\) by a simple re-scaling of the mass
parameter
\[
m \rightarrow m - q^2/r^{N-2}.
\]
Straightforward calculations show that the solution of the source-free Maxwell equations
\[
\partial_\nu (\sqrt{-g} F^{\mu\nu}) = 0
\]
is still given by the potential one-form field \([17]\) and the components of the electromagnetic
field tensor given by \([18]\) and \([20]\) remain unchanged in the metric \([21]\). Therefore, we shall
use them to calculate the energy-momentum source on the right-hand-side of the higher
dimensional Einstein equations
\[
R^\mu_\nu = 8\pi G M^\mu_\nu
\]
with
\[
M^\mu_\nu = \frac{1}{A_{N-1}} \left( F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{2(N-1)} \delta^\mu_\nu F^{\alpha\beta} F_{\alpha\beta} \right).
\]
The non-vanishing components of the energy-momentum tensor as well as the Ricci tensor
are given in Appendix A and B by equations \([A3] - [A8]\) and \([B6] - [B10]\). It is instructive
to start with the case \( N = 3 \). The corresponding energy-momentum tensor has the components

\[
M_0^0 = -M_3^3 = -\frac{Q^2}{8\pi \Sigma^3} \left( r^2 + a^2 + a^2 \sin^2 \theta \right),
\]
\[
M_1^1 = -M_2^2 = -\frac{Q^2}{8\pi \Sigma^2}, \quad M_3^3 = -\frac{M_0^0}{(r^2 + a^2) \sin^2 \theta} = -\frac{aQ^2}{4\pi \Sigma^3}, \tag{27}
\]

while, for the components of the Ricci tensor we have

\[
R_0^0 = -R_3^3 = -\frac{q^2}{\Sigma^3} \left( r^2 + a^2 + a^2 \sin^2 \theta \right),
\]
\[
R_1^1 = -R_2^2 = -\frac{q^2}{\Sigma^2}, \quad R_3^0 = -\frac{R_3^3}{(r^2 + a^2) \sin^2 \theta} = -\frac{2aq^2}{\Sigma^3}. \tag{28}
\]

Inspecting equation (26) with expressions (27) and (28) we see that it is satisfied if \( q^2 = GQ^2 \). This is the case of a rotating charged black hole in four dimensions and the metric (21), with dropped last term in it, reduces to the Kerr-Newman metric. However, for the case \( N \geq 4 \) equation (25) is not satisfied with the expressions (A3) - (A8) and (B6) - (B10). It is only satisfied when we restrict ourselves to a slow rotation of the black hole. Indeed, to first order in the rotation parameter we obtain that

\[
M_0^0 = M_1^1 = -\frac{N - 2}{(N - 1) A_{N-1}} \frac{Q^2}{r^{2(N-1)}},
\]
\[
M_2^2 = M_3^3 = M_4^4 = \frac{1}{(N - 1) A_{N-1}} \frac{Q^2}{r^{2(N-1)}}, \tag{29}
\]
\[
M_3^0 = -r^2 \sin^2 \theta M_0^0 = \frac{a \sin^2 \theta}{A_{N-1}} \frac{Q^2}{r^{2(N-1)}}.
\]

We note that all the components \( M_i^i \) with \( i \geq 4 \) are equal to each other. We also have the Ricci components

\[
R_0^0 = R_1^1 = -\frac{q^2}{r^{2(N-1)}} (N - 2)^2,
\]
\[
R_2^2 = R_3^3 = R_4^4 = \frac{q^2}{r^{2(N-1)}} (N - 2), \tag{30}
\]
\[
R_3^0 = -r^2 \sin^2 \theta R_0^0 = \frac{q^2 a \sin^2 \theta}{r^{2(N-1)}} (N - 1)(N - 2)
\]

along with the components \( R_i^i \) identical to each other for all \( i \geq 4 \). Inspecting now equation (26) we see that it is satisfied provided that the charge parameter \( q \) is related to the physical
charge of the black hole as
\[ q = \pm Q \left[ \frac{8\pi G}{(N - 2)(N - 1) A_{N-1}} \right]^{1/2}. \] (31)

With this in mind, the metric of a slowly rotating and charged black hole in \( N+1 \) dimensions can be obtained from (21) by ignoring all terms involving \( a^2 \) and higher powers in \( a \). We arrive at the metric
\[
\begin{align*}
    ds^2 &= -\left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2(N-2)}} \right) dt^2 + \left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2(N-2)}} \right)^{-1} dr^2 \\
    &+ \frac{2a \sin^2 \theta}{r^{N-2}} \left( \frac{m}{r^{N-2}} - \frac{q^2}{r^{N-2}} \right) dt \, d\phi + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \theta \, d\Omega_{N-3}^2 \right). \quad (32)
\end{align*}
\]

The potential one-form for associated electromagnetic field must have the form
\[ A = -\frac{Q}{(N - 2) r^{N-2}} \left( dt - a \sin^2 \theta \, d\phi \right). \] (33)

This metric generalizes the higher dimensional Schwarzschild-Tangherlini solution for non-rotating charged black holes in the Einstein-Maxwell gravity to include an arbitrarily small angular momentum of the black holes. For \( N = 4 \) this solution reduces to that given in [29].

IV. TWO INDEPENDENT ANGULAR MOMENTA

As we have mentioned above, in the general case, the metric describing rotating black holes in higher dimensions involves multiple angular momenta in different orthogonal planes of rotation. For slow enough rotation one can also include an electric charge into this metric, keeping only linear in rotation parameter terms in it and using the re-scaling procedure determined by equation (23). As an instructive example, we shall focus on the simplest case with two independent angular momenta associated with two orthogonal 2-planes of rotation in five dimensions. We start with the metric ansatz
\[
\begin{align*}
    ds^2 &= -dt^2 + \Xi \left( \frac{r^2}{\Pi} d\theta^2 + d\phi^2 \right) + (r^2 + a^2) \sin^2 \theta \, d\phi^2 + (r^2 + b^2) \cos^2 \theta \, d\psi^2 \\
    &+ \frac{m r^2 - q^2}{\Xi r^2} \left( dt - a \sin^2 \theta \, d\phi - b \cos^2 \theta \, d\psi \right)^2, \quad (34)
\end{align*}
\]

where the metric functions \( \Xi \) and \( \Pi \) are given as
\[ \Xi = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Pi = (r^2 + a^2)(r^2 + b^2) - m r^2 + q^2. \] (35)
One can show that in this metric the electromagnetic field of an electric charge $Q$ is described by the potential one-form

$$A = -\frac{Q}{2r^3} \left( dt - a \sin^2 \theta \, d\phi - b \cos^2 \theta \, d\psi \right).$$

(36)

This is exactly the same potential as that found in [18] for the unperturbed Myers-Perry metric with a small enough amount of the electric charge. However, the system of Einstein-Maxwell equations (25) and (24) becomes consistent provided that we discard all terms with higher powers of rotation parameters, keeping only linear in $a$ and $b$ terms. As a result, we have the metric

$$ds^2 = -\left(1 - \frac{m}{r^2} + \frac{q^2}{r^4}\right) dt^2 + \left(1 - \frac{m}{r^2} + \frac{q^2}{r^4}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \theta \, d\psi^2\right)$$

$$-\frac{2}{r^2} \left(m - \frac{q^2}{r^2}\right) \left( a \sin^2 \theta \, dt \, d\phi + b \cos^2 \theta \, dt \, d\psi \right),$$

(37)

where the parameter $q$ can be related to the physical charge of the black hole by inspecting the Einstein equations (25) with the associated components of the source tensor

$$M_0^0 = M_1^1 = -\frac{1}{3\pi^2} \frac{Q^2}{r^6}, \quad M_2^2 = M_3^3 = M_4^4 = \frac{1}{6\pi^2} \frac{Q^2}{r^6},$$

$$M_3^0 = -r^2 \sin^2 \theta \, M_0^3 = \frac{a \sin^2 \theta \, Q^2}{2\pi^2 \, r^6}, \quad M_4^0 = -r^2 \cos^2 \theta \, M_0^4 = \frac{b \cos^2 \theta \, Q^2}{2\pi^2 \, r^6},$$

and those of the Ricci tensor

$$R_0^0 = R_1^1 = -\frac{4q^2}{r^6}, \quad R_2^2 = R_3^3 = R_4^4 = \frac{2q^2}{r^6},$$

$$R_3^0 = -r^2 \sin^2 \theta \, R_0^3 = \frac{6q^2a \sin^2 \theta}{r^6}, \quad R_4^0 = -r^2 \cos^2 \theta \, R_0^4 = \frac{6q^2b \cos^2 \theta}{r^6}.$$ 

We find the relation

$$q = \pm Q \sqrt{\frac{2G}{3\pi}}$$

(40)

which, of course, agrees with (31).

V. GYROMAGNETIC RATIO

It is clear that many of the interesting physical properties of generic rotating black holes in higher dimensional Einstein-Maxwell gravity, such as the surface gravity and the geometry of
the event horizon, must crucially depend on $a^2$ values of the rotation parameter. Fortunately
the gyromagnetic ratio of the black hole can be learned from the limit of slow rotation
described by the metric we discussed above. We recall that the gyromagnetic ratio is the ratio
of the magnetic dipole moment of a rotating charged black hole to its angular momentum.
One of the remarkable facts about the black hole in the Einstein-Maxwell theory in four
dimensions is that it can be assigned a gyromagnetic ratio $g = 2$ just like the electron
in Dirac theory [30], rather than the usual charged matter in classical electrodynamics, for
which $g = 1$. Here we wish to know how does the value of the gyromagnetic ratio change
for higher dimensional black holes.

From the asymptotic behavior of the metric (32) we find that
\[ g_{03} = - \frac{j \sin^2 \theta}{r^{N-2}} + O \left( \frac{1}{r^{2(N-2)}} \right) , \]
which gives the specific angular momentum $j$ defined in equation (9). As for the magnetic
dipole moment, it can also be determined from the far distant behavior of the magnetic field
generated by a rotating and charged black hole around itself. To describe the magnetic field
in $N + 1$ dimensions it is useful to introduce the magnetic $(N-2)$-form defined as
\[ \hat{B}_{N-2} = i \xi(t) \star F = \left( \xi(t) \wedge F \right) , \]
which can also be written in the alternative form as follows
\[ \hat{B}_{N-2} = \frac{Qa}{r^2} \sqrt{\gamma} \cos^{N-3} \theta \left( \frac{2 \cos \theta}{N-2} \frac{dr}{r} + \sin \theta d\theta \right) \wedge d\chi_1 \wedge \cdots \wedge d\chi_{N-3} . \]
Substituting into this equation the non-vanishing components of the electromagnetic field
which is given by (20) taken in the limit of slow rotation, we obtain that
\[ \hat{B}_{N-2} = \frac{Qa}{r^2} \sqrt{\gamma} \cos^{N-3} \theta \left( \frac{2 \cos \theta}{N-2} \frac{dr}{r} + \sin \theta d\theta \right) \wedge d\chi_1 \wedge \cdots \wedge d\chi_{N-3} . \]
In the asymptotic rest frame of the black hole the magnetic $(N-2)$-form has the following
orthonormal components
\[ B_{\hat{r} \hat{x}_1 \hat{x}_2 \cdots \hat{x}_{N-3}} = \frac{2 Qa}{N-2} \frac{\cos \theta}{r^N} , \quad B_{\hat{\theta} \hat{x}_1 \hat{x}_2 \cdots \hat{x}_{N-3}} = \frac{Qa \sin \theta}{r^N} , \]
which is obtained by projecting (44) on the basis (B3). These expressions describe the
dominant behavior of the magnetic field far from the black hole and show that the black
hole can be assigned a magnetic dipole moment given by
\[ \mu = Qa . \]
We see that the coupling of the rotation parameter of the black hole to its charge to give the magnetic dipole moment looks exactly the same as its coupling to the mass parameter to determine the specific angular momenta \( j = am \). Thus, we can write

\[
\mu = \frac{Qj}{m} = (N - 1) \frac{QJ}{2M}, \tag{47}
\]

where we have used the relations \(\ref{13}\).

Defining now the gyromagnetic parameter \( g \) in the usual way, as a constant of proportionality in the equation

\[
\mu = g \frac{QJ}{2M}, \tag{48}
\]

and comparing this equation with \(\ref{47}\) we infer that a rotating charged black hole in \( N + 1 \) dimensions possesses the value of the gyromagnetic ratio

\[
g = N - 1 . \tag{49}
\]

We recall that the surface of the event horizon of the black hole is topologically equivalent to a \((N - 1)\)-sphere. It is interesting to note that the value of the gyromagnetic ratio coincides with the dimension of this sphere. In the same way, one can show that a five-dimensional black hole which is described by the metric \(\ref{37}\) can be assigned a gyromagnetic ratio of value \( g = 3 \). In \[18\] we proved that a five dimensional weakly charged Myers-Perry black holes must have the value of the gyromagnetic ratio \( g = 3 \). We see that in our case this value remains unchanged for an arbitrary amount of the electric charge. Moreover, as the specific angular momentum and the magnetic dipole moment first appear at linear order in rotation, on the grounds of all described above (see also Ref.\[18\]), it is straightforward to verify that the relation

\[
\mu_{(i)} = \frac{Qj_{(i)}}{m} \tag{50}
\]

holds for rotating black holes with multiple independent rotation parameters \( a_{(i)} \) and accordingly, with multiple magnetic dipole moments \( \mu_{(i)} \). Since for each orthogonal plane of rotation the specific angular momentum is given by \(\ref{13}\), from equation \(\ref{50}\) we conclude that the value of the gyromagnetic ratio \(\ref{49}\) is also true in the general case of multiple rotations in \( N + 1 \) dimensions.
VI. CONCLUSION

We have presented a new analytical solution subject to the higher dimensional Einstein-Maxwell equations which describes rotating charged black holes with a single angular momentum in the limit of slow rotation. Earlier, black hole solutions in the Einstein and Einstein-Maxwell gravity have been discussed by Tangherlini, who found the exact counterparts of the Schwarzschild and Reissner-Nordstrom metrics in arbitrary dimensions. In further developments Myers and Perry have given the exact metric for a rotating black hole using for it the familiar Kerr-Schild ansatz in higher dimensions. We have examined the case of a rotating black hole carrying a Maxwell electric charge. Following the strategy of obtaining the Kerr-Newman solution in general relativity we have assumed a special ansatz for the metric of the rotating charged black hole in arbitrary dimensions. We have shown that this approach enables one to write down the consistent solution of the Einstein-Maxwell equations only in the case of slow rotation. We have also extend this approach to give the metric for a slowly rotating charged black hole with two independent angular momenta associated with two orthogonal 2-planes of rotation in five dimensions.

Although we could not write down the desired solution for all values of rotation parameter, the solutions presented in the limit of slow rotation are sufficient to compute the gyromagnetic ratio for black holes in higher dimensional Einstein-Maxwell gravity. Since the angular momenta and the magnetic dipole momenta of these black holes first appear at the linear order in rotation parameter we have led to the conclusion that the value of the gyromagnetic ratio \( g = N - 1 \) remains the same for multi-rotating and charged black holes with the spherical topology of the horizon.

APPENDIX A: THE ELECTROMAGNETIC SOURCE TENSOR

Since in higher dimensions the energy-momentum tensor of the electromagnetic field

\[
T^\mu_\nu = \frac{1}{A_{N-1}} \left( F^\mu_\alpha F^\nu_\alpha - \frac{1}{4} \delta^\mu_\nu F^\alpha_\beta F^\alpha_\beta \right) \tag{A1}
\]

has the non-vanishing trace, the source on the right-hand-side of the Einstein equation is given by \( T^\mu_\nu \) and it is related to the canonical form \( T^\mu_\nu \) as

\[
M^\mu_\nu = T^\mu_\nu - \frac{1}{N-1} \delta^\mu_\nu T. \tag{A2}
\]
Using equations (18) and (20) in (26) we find that the non-vanishing components of the electromagnetic source tensor have the form

\[ M_{00}^0 = \frac{\lambda^2}{\Sigma^5} \left[ \Sigma - (N - 1)(r^2 + a^2) \right] \frac{H^2 - 4a^2r^2\cos^2\theta \left[ \Sigma + (N - 1)a^2\sin^2\theta \right]}{r^{2(N-3)}} , \quad (A3) \]

\[ M_{33}^3 = \frac{\lambda^2}{\Sigma^5} \left[ \Sigma + (N - 1)a^2\sin^2\theta \right] \frac{H^2 - 4a^2r^2\cos^2\theta \left[ \Sigma - (N - 1)(r^2 + a^2) \right]}{r^{2(N-3)}} , \quad (A4) \]

\[ M_{00}^3 = -\frac{M_3^0}{(r^2 + a^2)\sin^2\theta} = -\frac{\lambda^2 a}{\Sigma^4} \frac{(N - 1) \left[ (N - 2)\Sigma - 4a^2(N - 3)\cos^2\theta \right]}{r^{2(N-3)}} , \quad (A5) \]

\[ M_1^1 = -\frac{\lambda^2}{\Sigma^4} \frac{(N - 2) H^2 + 4 a^2 r^2 \cos^2\theta}{r^{2(N-3)}} , \quad (A6) \]

\[ M_2^2 = \frac{\lambda^2}{\Sigma^4} \frac{H^2 + 4 a^2 r^2(N - 2)\cos^2\theta}{r^{2(N-3)}} , \quad (A7) \]

\[ M_4^4 = \frac{\lambda^2}{\Sigma^4} \frac{H^2 - 4 a^2 r^2 \cos^2\theta}{r^{2(N-3)}} , \quad (A8) \]

and all \( M_i^i \) with \( i \geq 4 \) become equal to each other. In these expressions the function \( H \) is given by (19) and we have also introduced the notation

\[ \lambda^2 = \frac{Q^2}{(N - 2)^2 (N - 1) A_{N-1}} . \quad (A9) \]

We note that the relation

\[ M_0^0 + M_3^3 = M_1^1 + M_2^2 = -(N - 3) M_4^4 \quad (A10) \]

holds between the components of the energy-momentum tensor.

**APPENDIX B: CALCULATING THE COMPONENTS OF THE RICCI TENSOR**

In order to calculate the non-vanishing components of the Ricci tensor it is convenient to use the method of orthonormal basis forms. The basis one-forms for the metric (21) can be chosen as

\[ \omega^\alpha = \sqrt{|g_{\alpha\beta} - \omega^2 g_{33}|}^{1/2} d t \]
$$\omega^1 = \left( g_{11} \right)^{1/2} dr$$
$$\omega^2 = \left( g_{22} \right)^{1/2} d\theta$$,
$$\omega^3 = \left( g_{33} \right)^{1/2} (d\phi - \omega dt)$$
$$\omega^4 = r \cos \theta d\chi_1$$,
$$\omega^5 = r \cos \theta \sin \chi_1 d\chi_2$$,
\[\vdots = :\] (B1)

where we have introduced the ”angular velocity” of rotation

$$\omega = -\frac{g_{03}}{g_{33}}$$ (B2)

The corresponding dual basis is given by

$$e_0 = \left| g_{00} - \omega^2 g_{33} \right|^{-1/2} \left( \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi} \right)$$
$$e_1 = \frac{1}{\left( g_{11} \right)^{1/2}} \frac{\partial}{\partial r}$$
$$e_2 = \frac{1}{\left( g_{22} \right)^{1/2}} \frac{\partial}{\partial \theta}$$,
$$e_3 = \frac{1}{\left( g_{33} \right)^{1/2}} \frac{\partial}{\partial \phi}$$,
$$e_4 = \frac{1}{r \cos \theta} \frac{\partial}{\partial \chi_1}$$,
$$e_5 = \frac{1}{r \cos \theta \sin \chi_1} \frac{\partial}{\partial \chi_2}$$,
\[\vdots = :\] (B3)

We calculate the anti-symmetric connection one-forms and the components of the Riemann tensor through the Cartan’s structure equations

$$d\omega^{\hat{\mu}} + \omega^{\hat{\mu}}_{\hat{\nu}} \wedge \omega^{\hat{\nu}} = 0$$, \[\mathcal{R}^{\hat{\mu}}_{\hat{\nu}} = d\omega^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}}_{\hat{\alpha}} \wedge \omega^{\hat{\alpha}}_{\hat{\nu}} ,\] (B4)

where the curvature 2-form is defined as

$$\mathcal{R}^{\hat{\mu}}_{\hat{\nu}} = \frac{1}{2} R^{\hat{\mu}}_{\hat{\alpha} \hat{\beta}} \omega^{\hat{\alpha}} \wedge \omega^{\hat{\beta}}.$$ (B5)
Next, after straightforward calculations we find that the non-vanishing components of the Ricci tensor in the coordinate basis are given by

\[ R^0_0 = - \frac{q^2 (N-2)}{\Sigma^3} \left\{ \frac{(N-2)(r^2 + a^2) - a^2}{r^2(N-2)} \right\} + 2 a^2 r^2 \sin^2 \theta, \]  
\( (B6) \)

\[ R^3_3 = \frac{q^2 (N-2)}{4 \Sigma^3} \left\{ \frac{(N-3) a^4 \sin^2 2 \theta + 4 r^2 \left[ \Sigma + (N-1) a^2 \sin^2 \theta \right]}{r^2(N-2)} \right\}, \]  
\( (B7) \)

\[ R^0_0 = - \frac{R^3_3}{(r^2 + a^2) \sin^2 \theta} = - \frac{q^2 a(N-2)}{\Sigma^3} \left\{ \frac{(N-1) \Sigma - 2 a^2 \cos^2 \theta}{r^2(N-2)} \right\}, \]  
\( (B8) \)

\[ R^1_1 = - \frac{q^2(N-2)}{\Sigma^2} \left\{ \frac{(N-2) \Sigma - a^2 \cos^2 \theta}{r^2(N-2)} \right\}, \quad R^2_2 = R^4_4 \frac{r^2}{\Sigma}, \]  
\( (B9) \)

\[ R^4_4 = \frac{q^2}{\Sigma} \frac{N-2}{r^2(N-2)}. \]  
\( (B10) \)

together with the components \( R^i_i \) which are the same for all \( i \geq 4 \). Furthermore, one can show that similar to \( (A10) \), the components of the Ricci tensor obey the relation

\[ R^0_0 + R^3_3 = R^1_1 + R^2_2 = -(N-3) R^4_4. \]  
\( (B11) \)

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