FEFFERMAN-STEIN DECOMPOSITION FOR $Q$-SPACES AND MICRO-LOCAL QUANTITIES

Qixiang Yang, Tao Qian and Pengtao Li

Abstract. In this paper, we consider the Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ and give an affirmative answer to an open problem posed by M. Essen, S. Janson, L. Peng and J. Xiao in 2000. One of our main methods is to study the structure of the predual space of $Q_\alpha(\mathbb{R}^n)$ by the micro-local quantities. This result indicates that the norm of the predual space of $Q_\alpha(\mathbb{R}^n)$ depends on the micro-local structure in a self-correlation way.

Contents

1. Introduction 2
2. Preliminaries 4
   2.1. Wavelets and classic function spaces 4
   2.2. $Q$-spaces 6
   2.3. Calderón-Zygmund operators 7
   2.4. Predual spaces of $Q_\alpha(\mathbb{R}^n)$ 8
3. Micro-local quantities for $P_\alpha(\mathbb{R}^n)$ 9
   3.1. Conditional maximum value for non-negative sequence 10
   3.2. Micro-local quantities in $P_\alpha(\mathbb{R}^n)$ 11
4. Wavelet characterization of $P_\alpha(\mathbb{R}^n)$ 14
5. Fefferman-Stein type decomposition of $Q$-spaces 18
   5.1. Adapted $L^1$ and $L^\infty$ spaces 18
   5.2. Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ 22
6. The proof of Theorem 5.8 23
   6.1. A lemma 24
   6.2. The proof of converse part 25
   6.3. The proof of $\tau = 1$ 26
   6.4. The proof of $\tau = 2$ 31
References 33
1. Introduction

In this paper, we give a wavelet characterization of the predual of $Q$-space $Q_\alpha(\mathbb{R}^n)$ without using a family of Borel measures. By this result, we obtain a Fefferman-Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

Let $R_0$ be the unit operator and $R_i, i = 1, \ldots, n$, be the Riesz transforms, respectively. In 1972, in the celebrated paper [4], C. Fefferman and E. M. Stein proved the following result.

**Theorem 1.** ([4], Theorem 3) If $f(x) \in BMO(\mathbb{R}^n)$, then there exist $g_0(x), \ldots, g_n(x) \in L^\infty(\mathbb{R}^n)$ such that, modulo constants, $f = \sum_{j=0}^n R_j g_j$ and $\sum_{j=0}^n \|g_j\|_{L^\infty} \leq C\|f\|_{BMO}$.

The importance of Fefferman-Stein decomposition exists in two aspects. On one hand, there is a close relation between the $\bar{\partial}$-equation and Fefferman-Stein decomposition. On the other hand, this decomposition helps us get a better understanding of the structure of $BMO(\mathbb{R}^n)$ and the distance between $L^\infty(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Due to the above two points, Fefferman-Stein decomposition of $BMO(\mathbb{R}^n)$ has been studied extensively by many mathematicians since 1970s. We refer the readers to Jones [5, 6] and Uchiyama [16] for further information. In latest decades, Fefferman-Stein decomposition is also extended to other function spaces, for example, $BLO$, $C^0$ and $VMO$. We refer the reader to [1], [12] and the reference therein.

As an analogy of $BMO$ space, $Q$-spaces own the similar structure and many common properties (see [2], [3] and [19]). It is natural to seek a Fefferman-Sten type decomposition of $Q$-spaces. For the $Q$-spaces on unit disk, Nicolau-Xiao [10] obtained a decomposition of $Q_\alpha(\bar{\partial}\mathbb{D})$ similar to Fefferman and Stein’s result of $BMO(\partial\mathbb{D})$ (see [10], Theorem 1.2). On Euclidean space $\mathbb{R}^n$, Essen-Janson-Peng-Xiao [3] introduced $Q_\alpha(\mathbb{R}^n)$ as a generalization of $Q_\alpha(\partial\mathbb{D})$. For $\alpha \in (-\infty, \infty)$, $Q_\alpha(\mathbb{R}^n)$ are defined as the spaces of all measurable functions with

\begin{equation}
\sup I |I|^{\frac{n-1}{\alpha}} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy < \infty,
\end{equation}

where the supremum is taken over all cubes $I$ with the edges parallel to the coordinate. They gave a systemic research on $Q_\alpha(\mathbb{R}^n)$ and list Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ as one of several open problems.

**Problem 1.1.** ([3], Problem 8.3) For $n \geq 2$ and $\alpha \in (0, 1)$. Give a Fefferman-Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

In this paper, we give an affirmative answer to this open problem. Generally speaking, there are two methods to obtain the Fefferman-Stein decomposition of $BMO(\mathbb{R}^n)$. In [4], C. Fefferman and E. M. Stein split $BMO$...
functions by an extension theorem, i.e., the Hahn-Banach theorem in functional analysis. In [16], A. Uchiyama gave a constructive proof of Theorem 1. In this paper, using wavelets, we study the micro-local structure of $P^\alpha(\mathbb{R}^n)$, the predual of $Q_\alpha(\mathbb{R}^n)$. By this way, we obtain a Fefferman-Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

For the Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$, the difficulties exist in two aspects.

1. For a function $f$ in $P^\alpha(\mathbb{R}^n)$, $0 < \alpha < \frac{n}{2}$, the higher frequency party and the lower frequency party make different contributions to the norm $\|f\|_{P^\alpha}$. That is to say, $P^\alpha(\mathbb{R}^n)$ have special micro-local structure.

2. For any function $f$, Riesz transforms may cause a perturbation on all the range of its frequencies. To obtain Fefferman-Stein type decomposition, we need to control the range of the perturbation.

To overcome the above two difficulties, on one hand, we analyze the micro-local structure of functions in $P^\alpha(\mathbb{R}^n)$. Such micro-local structure could help us get a wavelet characterization of $P^\alpha(\mathbb{R}^n)$ without involving a group of Borel measures. On the other hand, we use classical Meyer wavelets to control the range of the perturbation. See also Remark 5.9.

In Section 2, we will give the definition of wavelet basis $\{\Phi_{\epsilon jk}(x)\}_{(\epsilon,j,k)\in\Lambda_n}$. It is well-known that a function $g(x)$ can be written as a sum

$$g(x) = \sum_j g_j(x), \text{ where } g_j(x) =: Q_j g(x) = \sum_{\epsilon,k} g_{\epsilon jk} \Phi_{\epsilon jk}(x).$$

Let $g(x)$ be a function in Besov spaces or Triebel-Lizorkin spaces. Roughly speaking, the norms of $g(x)$ can be determined by the $l^p(L^q)$-norm or $L^p(l^q)$-norm of $\{g_j(x)\}$, respectively. See [8] and [15]. For $g(x) \in P^\alpha(\mathbb{R}^n), 0 < \alpha < \frac{n}{2}$, the situation becomes complicated and we can not use the above ideas. In Section 3, we introduce the micro-local quantities with levels to study the structure of functions in $P^\alpha(\mathbb{R}^n)$.

Let $g(x) = \sum_{\epsilon,j,k} g_{\epsilon jk} \Phi_{\epsilon jk}(x) \in P^\alpha(\mathbb{R}^n)$. For any dyadic cube $Q$, we take the function

$$g_Q(x) = \sum_{Q_{j,k} \subset Q} g_{\epsilon jk} \Phi_{\epsilon jk}(x)$$

as the localization of $g(x)$ on $Q$. Then we restrict the range of frequency by limiting the number of $j$. In fact, we consider the function

$$(1.2) \quad g_{t,Q}(x) =: \sum_{Q_{j,k} \subset Q, -\log_2 |Q| \leq j \leq nt-\log_2 |Q|} g_{\epsilon jk} \Phi_{\epsilon jk}(x).$$

We obtain three micro-local quantities about $g_{t,Q}(x)$ by using some basic results in analysis. See Section 3 for details.
In Section 4, applying the above micro-local analysis of functions in $P^\alpha(\mathbb{R}^n)$, we give a new wavelet characterization of this space. As the predual of $Q^\alpha(\mathbb{R}^n)$, $P^\alpha(\mathbb{R}^n)$ has been studied by many authors. One method is to define the predual space $P^\alpha(\mathbb{R}^n)$ by a family of Borel measures. See Kalita [7], Wu-Xie [18] and Yuan-Sickel-Yang [22]. This idea can result in wavelet characterization of the predual space; but the predual space with the induced norm is a pseudo-Banach space. Dafni-Xiao [2] used a method of Hausdorff capacity to study $P^\alpha(\mathbb{R}^n)$. Peng-Yang [11] defined $P^\alpha(\mathbb{R}^n)$ by the atoms (see also [20]). By these methods, $P^\alpha(\mathbb{R}^n)$ are Banach spaces; but these authors did not consider the wavelet characterization of $P^\alpha(\mathbb{R}^n)$.

Compared with the former results of [7], [11], [20] and [22], our result owns the following advantage. Let $f$ be a function in $P^\alpha(\mathbb{R}^n)$. Our wavelet characterization indicates clearly that different frequencies exert different influences to the $P^\alpha$-norm of $f$. See Theorem 3.4 and Theorem 4.2. To obtain the Fefferman-Stein type decomposition of $Q^\alpha(\mathbb{R}^n)$, we need such a wavelet characterization of $P^\alpha(\mathbb{R}^n)$.

In Section 5, by the characterization obtained in Section 4 and the properties of Meyer wavelets and Daubechies wavelets, we characterize $P^\alpha(\mathbb{R}^n)$ associated with Riesz transforms. See Theorem 5.8. Applying this result and the duality between $P^\alpha(\mathbb{R}^n)$ and $Q^\alpha(\mathbb{R}^n)$, we obtain a Fefferman-Stein type decomposition of $Q^\alpha(\mathbb{R}^n)$.

We need to point out that our definition of $Q^\alpha(\mathbb{R}^n)$ is different from the one introduced in [3]. For non-trivial spaces, the scope of $\alpha$ in [3] is restricted to $(0, \min\{1, \frac{2}{n}\})$, while the scope in our definition can be relaxed to $(0, \frac{2}{n})$. More importantly, when $\alpha \in (0, \min\{1, \frac{2}{n}\})$, our definition is equivalent to the one in [3]. So the Fefferman-Stein type decomposition obtained in Section 5 gives a positive answer to Problem 1.1 proposed in [3].

The rest of this paper is organized as follows. In Section 2, we state some preliminary notations and lemmas which will be used in the sequel. In Section 3, we study the micro-local quantities for the functions in $P^\alpha(\mathbb{R}^n)$. In Section 4, we obtain a wavelet characterization of $P^\alpha(\mathbb{R}^n)$ by the micro-local result in Section 3. In Section 5, a Fefferman-Stein type decomposition of $Q^\alpha(\mathbb{R}^n)$ is obtained. For this decomposition, we need a characterization of $P^\alpha(\mathbb{R}^n)$ associated with Riesz transforms. We will give this characterization in Section 6.

2. Preliminaries

In this section, we present some preliminaries on wavelets, functions and operators which will be used in this paper.

2.1. Wavelets and classic function spaces. In this paper, we use real-valued tensor product wavelets; which can be regular Daubechies wavelets
or classical Meyer wavelets. For simplicity, we denote by 0 the zero vector in $\mathbb{R}^n$. Let $E_n = \{0, 1\}^n \setminus \{0\}$. Let $\Phi^0(x)$ and $\Phi^\varepsilon(x), \varepsilon \in E_n$, be the scale function and the wavelet functions, respectively. If $\Phi^\varepsilon(x)$ is a Daubechies wavelet, we assume there exist $m > 8n$ and $M \in \mathbb{N}$ such that

1. $\forall \varepsilon \in \{0, 1\}^n$, $\Phi^\varepsilon(x) \in C_0^m([-2^M, 2^M]^n)$;
2. $\forall \varepsilon \in E_n$, $\Phi^\varepsilon(x)$ has the vanishing moments up to the order $m - 1$.

For further information about wavelets, we refer the reader to [8], [17] and [20].

For $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, we denote by $Q_{j,k}$ the dyadic cube

$$\prod_{s=1}^n [2^{-j}k_s, 2^{-j}(k_s + 1)].$$

Let

$$\Omega = \{Q_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

$$\Lambda_n = \{(\varepsilon, j, k), \varepsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

For $\varepsilon \in \{0, 1\}^n$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we denote $\Phi_{j,k}^\varepsilon(x) = 2^{jn/2} \Phi^\varepsilon(2^j x - k)$.

The following result is well-known.

**Lemma 2.1.** ([8]) \{ $\Phi_{j,k}^\varepsilon(x), (\varepsilon, j, k) \in \Lambda_n$ \} is an orthogonal basis in $L^2(\mathbb{R}^n)$.

Let $f_{j,k}^\varepsilon = \langle f, \Phi_{j,k}^\varepsilon \rangle$, $\forall \varepsilon \in \{0, 1\}^n$ and $k \in \mathbb{Z}^n$. By Lemma 2.1 any $L^2$ function $f(x)$ has a wavelet decomposition

$$f(x) = \sum_{(\varepsilon, j, k) \in \Lambda_n} f_{j,k}^\varepsilon \Phi_{j,k}^\varepsilon(x).$$

We list some knowledge on Sobolev spaces and Hardy space. For $1 < p < \infty$, we denote by $p'$ the conjugate index of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. For a function space $A$, we denote by $(A)'$ the dual space of $A$. Given $1 < p < \infty, r \in \mathbb{R}$. For Sobolev spaces $W^{r,p}(\mathbb{R}^n)$, we know that

$$(W^{r,p}(\mathbb{R}^n))' = W^{-r,p'}(\mathbb{R}^n).$$

See [8], [15] and [20] for details.

Let $\chi(x)$ be the characteristic function of the unit cube $[0, 1]^n$. For function $g(x) = \sum_{(\varepsilon, j, k) \in \Lambda_n} g_{j,k}^\varepsilon \Phi_{j,k}^\varepsilon(x)$, we have the following characterization, see [8] and [20]:

**Proposition 2.2.** (i) Given $1 < p < \infty$ and $|r| < m$.

$$g(x) = \sum_{(\varepsilon, j, k) \in \Lambda_n} g_{j,k}^\varepsilon \Phi_{j,k}^\varepsilon(x) \in W^{r,p}(\mathbb{R}^n)$$
if and only if
\[ \left\| \sum_{(r,j,k) \in \Lambda_n} 2^j r^{r+2} |g_{r,j,k}|^2 \chi(2^j x - k) \right\|_{L^p} < \infty. \]

(ii) \( g(x) = \sum_{(r,j,k) \in \Lambda_n} g^r_{r,j,k}(x) \) if and only if
\[ \left\| \sum_{(r,j,k) \in \Lambda_n} 2^n j^n |g_{r,j,k}|^2 \chi(2^j x - k) \right\|_{L^1} < \infty. \]

2.2. \( Q \)-spaces. In [3], the \( Q \)-spaces, \( Q_\alpha(\mathbb{R}^n) \), are defined by (1.1). For \( \alpha \leq 0 \), it is easy to see that \( Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n) \). Further, for \( \alpha \geq 1 \) or \( \alpha > \frac{n}{2} \), there are only constants in \( Q_\alpha(\mathbb{R}^n) \) defined by (1.1).

Now we introduce a new definition of \( Q \)-spaces such that the \( Q \)-spaces are not trivial for \( 1 \leq \alpha \leq \frac{n}{2} \). For \( \alpha \in \mathbb{R} \), denote by
\[ f_{\alpha,Q} = |Q|^{-1} \int_Q (-\Delta)^{\frac{\alpha}{2}} f(x) \] the mean value of the function \((-\Delta)^{\frac{\alpha}{2}} f\) on the cube \( Q \).

For \( \alpha \in \mathbb{R} \), let
\[ B_{\alpha,Q} f = |Q|^\frac{\alpha}{2} \left( |Q|^{-1} \int_Q (-\Delta)^{\frac{\alpha}{2}} f(x) - f_{\alpha,Q}^2 \right)^{\frac{1}{2}}. \]

\( Q \)-spaces \( Q_\alpha(\mathbb{R}^n) \) and \( Q_0^0(\mathbb{R}^n) \) are defined as follows.

**Definition 2.3.** Given \( \alpha \in [0, \frac{n}{2}] \).

(i) \( Q_\alpha(\mathbb{R}^n) \) is a space defined as the set of all measurable functions \( f \) with
\[ \sup_Q B_{\alpha,Q} f < \infty, \]
where the supremum is taken over all cubes \( Q \).

(ii) \( Q_0^0(\mathbb{R}^n) \) is a space defined as the set of all measurable functions \( f \in Q_\alpha(\mathbb{R}^n) \) and satisfying
\[ \lim_{|Q| \to 0} B_{\alpha,Q} f = 0, \]
\[ \lim_{|Q| \to \infty} B_{\alpha,Q} f = 0, \]
where the limit are taken over all cubes \( Q \).

**Remark 2.4.** If \( \alpha = \frac{n}{2} \), \( Q_{\frac{n}{2}}(\mathbb{R}^n) = \dot{B}_2^{\frac{n}{2}}(\mathbb{R}^n) \). For \( 1 \leq \alpha \leq \frac{n}{2} \), the \( Q \)-spaces in Definition 2.3 are not trivial. Further, for other indices \( \alpha \), the corresponding \( Q_\alpha(\mathbb{R}^n) \) coincide with the ones defined in [3]. So \( Q_\alpha(\mathbb{R}^n) \) defined in Definition 2.3 is a generalization of \( Q \)-spaces defined by (1.1).
For $|\alpha| < m$, $Q \in \Omega$ and function $f(x) = \sum_{(e,j,k) \in \Lambda_n} f^e_{jk} \Phi^e_{jk}(x)$, let
\[
C_{\alpha,Q}f = |Q|^{\frac{m}{2} - \frac{|\alpha|}{2}} \left( \sum_{Q,\mu \subset Q} 2^{2j_\mu} |f^e_{jk}|^2 \right)^{\frac{1}{2}}.
\]

Similar to $W^{s,2}(\mathbb{R}^n)$, $Q$-spaces can be also characterized by wavelets.

**Proposition 2.5.** ([22]) Given $0 \leq \alpha \leq \frac{n}{2}$,

(i) $f(x) = \sum_{(e,j,k) \in \Lambda_n} f^e_{jk} \Phi^e_{jk}(x) \in Q_\alpha(\mathbb{R}^n)$ if and only if
\[
\sup_{Q \subset \Omega} C_{\alpha,Q}f < \infty.
\]

(ii) $f(x) = \sum_{(e,j,k) \in \Lambda_n} f^e_{jk} \Phi^e_{jk}(x) \in Q^0_\alpha(\mathbb{R}^n)$ if and only if
\[
\begin{cases}
\sup_{Q \subset \Omega} C_{\alpha,Q}f < \infty, \\
\lim_{Q \subset \Omega, |Q| \to 0} C_{\alpha,Q}f = 0, \\
\lim_{Q \subset \Omega, |Q| \to \infty} C_{\alpha,Q}f = 0.
\end{cases}
\]

By Propositions 2.2 and 2.5 we may identify a function
\[
g(x) = \sum_{(e,j,k) \in \Lambda_n} g^e_{jk} \Phi^e_{jk}(x)
\]
with the wavelet coefficients \{\(g^e_{jk}\)\}_{(e,j,k) \in \Lambda_n}.

### 2.3. Calderón-Zygmund operators

In this subsection, we introduce some preliminaries about Calderón-Zygmund operators, see [8], [9] and [14]. For $x \neq y$, let $K(x,y)$ be a smooth function such that

\[
|\partial^\alpha_x \partial^\beta_y K(x,y)| \leq \frac{C}{|x - y|^{n+|\alpha|+|\beta|}}, \forall |\alpha| + |\beta| \leq N_0,
\]
where $N_0$ is a large enough constant and $N_0 \leq m$.

A linear operator $T$ is said to be a Calderón-Zygmund operator in $CZO(N_0)$ if

1. $T$ is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))'$;
2. there exists a kernel $K(x,y)$ satisfying (2.1) and for $x \notin \text{supp}f(x)$,
\[
Tf(x) = \int K(x,y)f(y)dy;
\]
3. $Tx^\alpha = T^*x^\alpha = 0$, $\forall \alpha \in \mathbb{N}^n$ and $|\alpha| \leq N_0$.

**Remark 2.6.** In (2.1), the values of $K(x,y)$ have not been defined for $x = y$. According to Schwartz kernel theorem, the kernel $K(x,y)$ of a linear continuous operator $T$ is only a distribution in $S'(\mathbb{R}^{2n})$. 

Lemma 2.7. (8) (i) If \( T \in CZO(N_0) \), for all \((\epsilon, j, k)\) and \((\epsilon', j', k')\) \(\in\Lambda_n\), the coefficients \(a_{\epsilon,j,k}^{\epsilon',j',k'}\) satisfy that

\[
|a_{\epsilon,j,k}^{\epsilon',j',k'}| \leq C2^{-|j-j'|(\frac{s}{2}+N_0)}\left(\frac{2^{-j}+2^{-j'}-|k2^{-j}-k'2^{-j'}|}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^{n+N_0}.
\]

(ii) If \(a_{\epsilon,j,k}^{\epsilon',j',k'}\) \((\epsilon, j, k), (\epsilon', j', k')\) \(\in\Lambda_n\) satisfies (2.2), then

\[
K(x, y) = \sum_{(\epsilon, j, k), (\epsilon', j', k')\in\Lambda_n} a_{\epsilon,j,k}^{\epsilon',j',k'}\Phi_{\epsilon,j,k}(x)\Phi_{\epsilon',j',k'}(y)
\]

in the sense of distributions. Further, for any \(0 < \delta < N_0\), we have \(T \in CZO(N_0 - \delta)\).

At the end of this subsection, we list a variant result about the continuity of Calderón-Zygmund operators on Sobolev spaces (see also [12]).

For all \((\epsilon, j, k)\) \(\in\Lambda_n\), denote

\[
g_{\epsilon,j,k}^{\epsilon',j',k'} = \sum_{(\epsilon', j', k')\in\Lambda_n} a_{\epsilon,j,k}^{\epsilon',j',k'}g_{\epsilon',j',k'}^{\epsilon'}.
\]

We have

Lemma 2.8. Given \(|r| < s < m\) and \(1 < p < \infty\). If \(\forall (\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n\),

\[
|a_{\epsilon,j,k}^{\epsilon',j',k'}| \leq C2^{-|j-j'|(\frac{s}{2}+s)}\left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^{n+s},
\]

then

\[
\int \left( \sum_{(\epsilon, j, k)\in\Lambda_n} 2^{j(n+2r)}|g_{\epsilon,j,k}^{\epsilon'}|^2\chi(2^j x - k) \right)^\frac{p}{2} dx
\]

\[
\leq C \int \left( \sum_{(\epsilon, j, k)\in\Lambda_n} 2^{j(n+2r)}|g_{\epsilon,j,k}^{\epsilon'}|^2\chi(2^j x - k) \right)^\frac{p}{2} dx.
\]

2.4. Predual spaces of \(Q_\alpha(\mathbb{R}^n)\). Below we state the standard atoms, the wavelet atoms and the predual spaces of \(Q_\alpha(\mathbb{R}^n)\) which were introduced in [11] and [20].

Definition 2.9. Given \(0 \leq \alpha < \frac{n}{2}\).

(i) A distribution \(g(x)\) is an \((\alpha, 2)\)-atom on a cube \(Q\) if

\[
(1) \|(-\Delta)^{\frac{\alpha}{2}}g\|_{L^2} \leq |Q|^{\frac{1}{2}+\frac{\alpha}{2}},
\]

(2) \(\text{supp} g(x) \subset Q\),

(3) \(\int x^\beta g(x) dx = 0, \forall |\beta| \leq |\alpha|\).
(ii) A distribution \( f(x) \) belongs to \( P^\alpha(\mathbb{R}^n) \) if \[
 f(x) = \sum_{\lambda \in \mathbb{Z}} \lambda u g_u(x),
\] where \( \{\lambda_u\}_{u \in \mathbb{Z}} \in l^1 \) and \( g_u(x) \) are \((\alpha, 2)\)-atoms.

**Definition 2.10.** Given \( 0 \leq \alpha < \frac{n}{2} \).

(i) A distribution \( g(x) = \sum_{e \in E_n, Q_{jj} < Q} g_{f,\ell}^e \Phi_{f,\ell}^e(x) \) is a \((\alpha, 2)\)-wavelet atom on a dyadic cube \( Q \) if
\[
(\sum_{(e, j, k) \in \Lambda_n} 2^{-2j\alpha} |g_{f,\ell}^e|^2)^{\frac{1}{2}} \leq |Q|^{\frac{\alpha}{n} - \frac{1}{2}}.
\]

(ii) A distribution \( f(x) \) belongs to \( P_w^\alpha(\mathbb{R}^n) \) if \[
 f(x) = \sum_{\lambda \in \mathbb{Z}} \lambda u g_u(x),
\] where \( \{\lambda_u\}_{u \in \mathbb{Z}} \in l^1 \) and \( g_u(x) \) are \((\alpha, 2)\)-wavelet atoms.

In fact, \( P^\alpha(\mathbb{R}^n) \) and \( P_w^\alpha(\mathbb{R}^n) \) are the same spaces and Calderón-Zygmund operators are bounded on \( P^\alpha(\mathbb{R}^n) \). The proof of the following lemma can be found in [11] and [20].

**Proposition 2.11.** Given \( 0 \leq \alpha < \frac{n}{2} \).

(i) \( P^\alpha(\mathbb{R}^n) = P_w^\alpha(\mathbb{R}^n) \).

(ii) Any operator \( T \in CZO(N_0) \) is bounded on \( P^\alpha(\mathbb{R}^n) \).

For \( \alpha = \frac{n}{2} \), define \( P_w^{\frac{n}{2}}(\mathbb{R}^n) =: \tilde{B}_2^{\frac{n}{2}, 2}(\mathbb{R}^n) \).

Applying the same ideas in [11], [12], [13] and [22], we have the following duality relation.

**Proposition 2.12.** Given \( 0 \leq \alpha \leq \frac{n}{2} \).

(i) \( (P^\alpha(\mathbb{R}^n))' = Q_\alpha(\mathbb{R}^n) \);

(ii) \( (Q^0_\alpha(\mathbb{R}^n))' = P^\alpha(\mathbb{R}^n) \).

3. Micro-local quantities for \( P^\alpha(\mathbb{R}^n) \)

If \( \alpha = 0 \), \( P^0(\mathbb{R}^n) = H^1(\mathbb{R}^n) \). If \( \alpha = \frac{n}{2} \), \( P_w^{\frac{n}{2}}(\mathbb{R}^n) = \tilde{B}_2^{\frac{n}{2}, 2}(\mathbb{R}^n) \). It is well known that the norms of \( P^0(\mathbb{R}^n) \) and \( P_w^{\frac{n}{2}}(\mathbb{R}^n) \) depend only on the \( L^p(\mathbb{R}^n) \)-norms of function series \( \{f_j = Q_j f\}_{j \in \mathbb{Z}} \) for \( p = 1 \) and \( p = 2 \), respectively.

For the case \( 0 < \alpha < \frac{n}{2} \), the situation is complicated.

In this section, we use wavelets to analyze the micro-local structure of \( P^\alpha(\mathbb{R}^n) \). First, we present a theorem on conditional maximum value in Subsection 3.1. Then we consider the micro-local quantities in Subsection 3.2.
3.1. Conditional maximum value for non-negative sequence. For \( u \in \mathbb{N} \), we denote
\[
\begin{align*}
\Lambda_{u,n} &= \{0, 1, \cdots, 2^u - 1\}; \\
G_{u,n} &= \{(\epsilon, s, v), \epsilon \in E_n, 0 \leq s \leq u, v \in \Lambda_{s,n}\}.
\end{align*}
\]

**Definition 3.1.** Given \( j \in \mathbb{Z}, k \in \mathbb{Z}^n \) and \( t \in \mathbb{N} \),
\[
\tilde{g}^t_{j,k} = \{g^t_{j+s,2^j+k+u}(\epsilon, s, u) \in G_{t,n}\}
\]
is called a non-negative sequence if
\[
g^t_{j+s,2^j+k+u} \geq 0, \quad \forall (\epsilon, s, u) \in G_{t,n}.
\]

For a non-negative sequence \( \tilde{g}^t_{j,k} \), we find the maximum value of the following quantities:
\[
\tau^t_{j,k} = \sum_{(\epsilon, s, u) \in G_{t,n}} f^t_{j,k} \tilde{g}^t_{j,k},
\]
where non-negative sequence \( f^t_{j,k} = \{f^t_{j+s,2^j+k+u}(\epsilon, s, u) \in G_{t,n}\} \) satisfies the following conditions
\[
\begin{align*}
2^{n(j+i)} \sum_{(\epsilon, s, u) \in G_{t,n}} (f^t_{j+s,2^j+k+u})^2 &\leq 1, \quad \forall u \in \Lambda_{t-1}; \\
2^{n(j+i-1)} \sum_{(\epsilon, s, u) \in G_{t,n}} 2^{2s\alpha}(f^t_{j+i-1+s,2^i(2^i-1+k+u)+v})^2 &\leq 1, \quad \forall u \in \Lambda_{t-1}; \\
2^{n(j+i-2)} \sum_{(\epsilon, s, u) \in G_{t,n}} 2^{2s\alpha}(f^t_{j+i-2+s,2^i(2^i-2+k+u)+v})^2 &\leq 1, \quad \forall u \in \Lambda_{t-2}; \\
&\vdots \\
2^{nj} \sum_{(\epsilon, s, u) \in G_{t,n}} 2^{2s\alpha}(f^t_{j+s,2^j+k+u})^2 &\leq 1.
\end{align*}
\]

There exist \( 2^n - 1 \) \( \sum_{0 \leq s \leq t} 2^{ns} \) elements in \( G_{t,n} \). We can see that \( f^t_{j,k} \) is a sequence, where the number of nonnegative terms is at most \( 2^n - 1 \) \( \sum_{0 \leq s \leq t} 2^{ns} \).

**Definition 3.2.** Given \( j \in \mathbb{Z}, k \in \mathbb{Z}^n \) and \( t \in \mathbb{N} \),
\[
f^t_{j,k} = \{f^t_{j+s,2^j+k+u}(\epsilon, s, u) \in G_{t,n}\} \in F^t_{j,k}
\]
if \( f^t_{j,k} \) is a non-negative sequence satisfying the conditions in (3.1).

According to the basic results in analysis, we have:

**Theorem 3.3.** Given \( 0 \leq \alpha < \frac{n}{2} \) and \( t \geq 0 \). For any non-negative sequence \( \tilde{g}^t_{j,k} = \{g^t_{j+s,2^j+k+u}(\epsilon, s, u) \in G_{t,n}\} \), there exists at least a sequence \( f^t_{j,k} = \{f^t_{j+s,2^j+k+u}(\epsilon, s, u) \in G_{t,n}\} \in F^t_{j,k} \) such that
\[
\tau^t_{j,k} \tilde{g}^t_{j,k} = \max_{f^t_{j,k} \in F^t_{j,k}} \tau^t_{j,k} f^t_{j,k} \tilde{g}^t_{j,k}.
\]
Proof. The \( (2^n - 1) \sum_{0 \leq s \leq t} 2^{ns} \) variables \( \{f'_{j+k,2^k n}\}_{(e,s,u) \in G_{i,n}} \) of the sequence \( f'_{j,k} \) are restricted in a closed domain, so the conclusion is obvious. \( \square \)

3.2. Micro-local quantities in \( P^\alpha(\mathbb{R}^n) \). From Proposition \([2.12]\) we know that \( (Q^0(\mathbb{R}^n))^\prime = P^\alpha(\mathbb{R}^n) \). To prove a function \( g \in P^\alpha(\mathbb{R}^n) \), we only need to consider \( \sup_{\|f\|_{Q^0} \leq 1} \langle f, g \rangle \), where the supremum is taken over all \( f \in Q^0(\mathbb{R}^n) \) with \( \|f\|_{Q^0} \leq 1 \). However, by this method, we can not know the micro-local structure of \( g(x) \) in details.

To delete this shortage, we introduce a new method. Let

\[
g(x) = \sum_{(e,j,k) \in \Lambda_n} g_{j,k}^e \Phi_{j,k}(x).
\]

We localize \( g(x) \) by restricting its wavelet coefficients \( g_{j,k}^e \), such that \( Q_{j,k} \subset Q \). Then we limit the range of frequencies of \( g(x) \) and analyze its micro-local information.

For this purpose, we consider the function \( g_{t,Q}(x) \) defined in (1.2). For such a \( g_{t,Q} \), the number of \( (e,j,k) \) such that \( g_{j,k}^e \neq 0 \) is at most \( (2^n - 1) \sum_{0 \leq s \leq t} 2^{ns} \). We study micro-local functions \( g_{t,Q} \) in \( P^\alpha(\mathbb{R}^n) \) and obtain three kinds of micro-local quantities.

For all \( t \geq 0 \), \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \), we consider the series

\[
g'_{j,k}(x) = \sum_{(e,s,u) \in G_{i,n}} g_{j+2^k s+u}^e \Phi_{j+2^k s+u}(x).
\]

Denote

\[
(3.2) \quad g'_{j,k}(x) = \sum_{(e,s,u) \in G_{i,n}} g_{j+2^k s+u}^e \Phi_{j+2^k s+u}(x).
\]

Notice that there is a one-to-one relation between the sequences \( g_{j,k}^e \) and the function \( g'_{j,k}(x) \). Sometimes, we do not distinguish them.

For simplicity, we suppose that our functions are real-valued. Let

\[
\begin{aligned}
f(x) &= \sum_{(e,j,k) \in \Lambda_n} f_{j,k}^e \Phi_{j,k}(x); \\
g(x) &= \sum_{(e,j,k) \in \Lambda_n} g_{j,k}^e \Phi_{j,k}(x).
\end{aligned}
\]

If \( \langle f, g \rangle \) and \( \sum_{(e,j,k) \in \Lambda_n} f_{j,k}^e g_{j,k}^e \) are well defined, we have

\[
(3.3) \quad \tau_{f,g} := \langle f, g \rangle = \sum_{(e,j,k) \in \Lambda_n} f_{j,k}^e g_{j,k}^e.
\]

To compute \( \max_{\|f\|_{Q^0} \leq 1} \tau_{f,g_{t,Q}} \), according to (3.3), we can restrict \( f \) to the function

\[
f'_{j,k}(x) = \sum_{(e,s,u) \in G_{i,n}} f_{j+2^k s+u}^e \Phi_{j+2^k s+u}(x)
\]
with \(\|f'_j\|_{Q^0_\alpha} \leq 1\). The number of \((\epsilon, j, k)\) such that \(f'_j \neq 0\) is at most \((2^n - 1) \sum_{0 \leq s \leq t} 2^{ns}\). In other words, by (3.3), finding the supremum under infinite restricted conditions is transferred to a maximum value problem of the series of quantities \(\{f'_{j+s,2^k+u}(\epsilon, s, u)\}_{(\epsilon, s, u) \in G_{t,n}}\) under the \(\sum 2^{ns}\) restricted conditions.

Based on Theorem 3.3, we begin to consider the micro-local quantities of \(g'_{j,k}\) in \(P^\alpha(\mathbb{R}^n)\).

**Theorem 3.4.** Given \(0 < \alpha < \frac{n}{2}\) and \(t \geq 0\). Let \(g'_{j,k}\) be the function defined by (3.2) and \(\|g'_{j,k}\|_{P^\alpha} > 0\).

(i) There exists a function

\[
S f_{j,k}(x) = \sum_{(\epsilon, s, u) \in G_{t,n}} S f_{j+s,2^k+u} \Phi_{j+s,2^k+u}(x)
\]

with \(\|S f_{j,k}\|_{Q^0_\alpha} \leq 1\) such that

\[
\max_{\|f\|_{Q^0_\alpha} \leq 1} \tau_{f,g'_{j,k}} = \sum_{(\epsilon, s, u) \in G_{t,n}} S f_{j+s,2^k+u} : g'_{j+s,2^k+u}.
\]

(ii) There exists a positive number \(P'_{j,s}g'_{j,k}\) which is defined by the absolute values of wavelet coefficient of \(g'_{j,k}\) such that

\[
P'_{j,s}g'_{j,k} = \|g'_{j,k}\|_{P^\alpha} = \max_{\|f\|_{Q^0_\alpha} \leq 1} \tau_{f,g'_{j,k}} = \tau_{S f_{j,k}, g'_{j,k}}.
\]

(iii) There exists a sequence \(\{Q'_{j,s}g'_{j,k}\}_{\epsilon \in E_n}\) such that \(\sum_{\epsilon \in E_n} Q'_{j,s}g'_{j,k} \Phi_{j,k}(x)\) has the same norm in \(P^\alpha(\mathbb{R}^n)\) as \(g'_{j,k}\) does.

**Proof.** For \(g'_{j,k} = \{g'_{j+s,2^k+u}(\epsilon, s, u)\}_{(\epsilon, s, u) \in G_{t,n}}\), denote \(g'_{j,k} = \{\|g'_{j+s,2^k+u}\|_{(\epsilon, s, u) \in G_{t,n}}\}.\) Let

\[
G_{j,k} = \{(\epsilon, s, u) \in G_{t,n}, g'_{j+s,2^k+u} \neq 0\}.
\]

For \(f_{j,k}(x) = \sum_{(\epsilon, s, u) \in G_{t,n}} f_{j+s,2^k+u} \Phi_{j+s,2^k+u}(x)\), define

\[
f_{j+s,2^k+u} = \left\{ \begin{array}{ll} |f_{j+s,2^k+u}| \cdot |g'_{j+s,2^k+u}|^{-1} g'_{j+s,2^k+u}, & (\epsilon, s, u) \in G_{t,n}; \\ 0, & (\epsilon, s, u) \notin G_{t,n}. \end{array} \right.
\]

We denote by \(F_{j,k}^\epsilon\) the set

\[
\{f_{j,k} : f_{j,k}(x) = \sum_{(\epsilon, s, u) \in G_{t,n}} f_{j+s,2^k+u} \Phi_{j+s,2^k+u}(x)\} \text{ and } \|f_{j,k}\|_{Q^0_\alpha} \leq 1\}.
\]

By (ii) of Proposition 2.5 we have

\[
(3.4) \quad \max_{\|f_{j,k}\|_{Q^0_\alpha} \leq 1} \tau_{f_{j,k}, g'_{j,k}} = \max_{f_{j,k} \in F_{j,k}^\epsilon} \tau_{f_{j,k}, g'_{j,k}} = \max_{f_{j,k} \in F_{j,k}^\epsilon} \tau_{f_{j,k}, g'_{j,k}}.
\]
By the wavelet characterization of $Q_n(\mathbb{R}^n)$, the condition $\|f \|_{Q_n}^2 \leq 1$ is equivalent to the conditions (3.1). Further, for fixed $\tilde{s}'_{jk}$, because of (3.3), if $(\epsilon, s, u) \in G_{t,n}$ and $(\epsilon, s, u) \notin G_{s'_{jk}}$, the coefficients $f_{\epsilon}^{\tau_{j+2^k+u}}$ make no contribution to $\tau_{j+k}^{s'_{jk}}$. We get

$$\max_{j\in \mathcal{F}'} \tau_{j+k}^{s'_{jk}} = \max_{j\in \mathcal{F}'} \tau_{j+k}^{s_{jk}}.$$ 

According to Theorem 3.3, there exists at least a sequence

$$f_{\tilde{s}'_{jk}} = \{f_{j+2^k+u}^{\epsilon}(\epsilon, s, u) \in F'_{j,k}$$

such that

(3.5) $$\tau_{j+k}^{s'_{jk}} = \max_{j\in \mathcal{F}'} \tau_{j+k}^{s_{jk}}.$$ 

Let $S_{j,k}(\epsilon, s, u) = \sum_{(\epsilon, s, u) \in G_{t,n}} S_{j+2^k} f_{j+2^k+u}^{\epsilon}(\epsilon, s, u) \Phi_{j+2^k+u}^{\epsilon}(\epsilon, s, u).$ where

$$S_{j+2^k} f_{j+2^k+u}^{\epsilon}(\epsilon, s, u) = \left\{ \begin{array}{ll} \tilde{f}_{j+2^k+u}^{\epsilon} g_{j+2^k+u}^{\epsilon} & \text{if } \sum_{\epsilon \in E_n} |g_{j,k}^{\epsilon}| = 0; \\ 2^{j+2^k} P' j_{s'_{jk}} \left( \sum_{\epsilon \in E_n} |g_{j,k}^{\epsilon}|^2 \right)^{1/2} g_{j,k}^{\epsilon} & \text{if } \sum_{\epsilon \in E_n} |g_{j,k}^{\epsilon}| 
eq 0. \end{array} \right.$$ 

Applying (ii) of Proposition 2.5 again, we know that $\{Q'_{j,s'_{jk}}\}_{\epsilon \in E_n}$ satisfies the condition (iii). □

Remark 3.5. For $\alpha = 0$ and $\alpha = \frac{1}{2}$, we can deal with $P'_{j'} g_{j,k}$ in a similar way.

(i) For $\alpha = 0$, according to the wavelet characterization of Hardy space in [3], $P'_{j'} g_{j,k}$ can be equivalent to

$$\left\| \left( \sum_{(\epsilon, s, u) \in G_{t,n}} 2^{n(j+s)} |g_{j+2^k+u}^{\epsilon}|^2 \chi(2^{j+2^k} - 2^k u - u) \right)^{1/2} \right\|_{L^1},$$

(ii) For $\alpha = \frac{1}{2}$, $P'_{j'} g_{j,k}$ can be written as

$$\left( \sum_{(\epsilon, s, u) \in G_{t,n}} 2^{-n} |g_{j,k}^{\epsilon}|^2 \right)^{1/2}.$$
But for $0 < \alpha < \frac{n}{2}$, $P^s g_{j,k}^t$ can not be expressed in an explicit way. Luckily, the three parts
\[
\begin{align*}
\{Q^t|_{S_{j,k}}\}_{t \in E_n}, \\
S_{j,k}^f = \sum_{(e,s,a) \in G_{j,n}} S_{j,s+2^{k+u}}^f \Phi_{j,s,2^{k+u}}^t(x), \\
P_{j}^s g_{j,k}^t
\end{align*}
\]
indicate the micro-local characteristics in both the frequency structure and the local structure.

In the rest of this paper, the above three kinds of quantities will be used in the following sections repeatedly. These micro-local quantities tell us the global information of functions in $P^s(\mathbb{R}^n)$. In Section 4, this idea will be used to get a characterization of $P^s(\mathbb{R}^n)$ by a group of $L^1$ functions defined by the absolute values of wavelet coefficients. Such wavelet characterization does not involve the action of a group of Borel measures.

4. Wavelet characterization of $P^s(\mathbb{R}^n)$

In recent years, as the predual of $Q_\alpha(\mathbb{R}^n)$, $P^s(\mathbb{R}^n)$ was studied by many authors. We refer the reader to [2], [11], [20], [21] and [22] for details. In this section, we give a new wavelet characterization of $P^s(\mathbb{R}^n)$ by using the micro-local results obtained in Section 3.

In the famous book [8], Y. Meyer proved that the Hardy space $H^1(\mathbb{R}^n)$ can be characterized by some $L^1$ functions defined by wavelet coefficients without using a family of Borel measures. Based on the micro-local results in Section 3, we give a similar result. We show that each element in $P^s(\mathbb{R}^n)$ can be characterized by a group of $L^1$ functions $P_{s,t,N} g(x)$. In Section 5, this result will be used to get a characterization of $P^s(\mathbb{R}^n)$ associated with Riesz transformations.

For $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, let
\[
\Omega^{s,N} = \{Q \in \Omega : 2^{-sn} \leq |Q| \leq 2^{(N-s)n}\}.
\]
For $0 \leq t \leq N$, $m \in \mathbb{Z}^n$ and $Q = Q_{s-N,m}$, define
\[
\Omega_{s,t,Q}^{L,N} = \{Q' \in \Omega : 2^{-sn} \leq |Q'| \leq 2^{(N-s)n}, Q' \subset Q_{s-N,m}\}.
\]
We can see that $\Omega^{s,N} = \bigcup_{m \in \mathbb{Z}^n} \Omega_{s,N}^{L,N}$.

For any $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, we define
\[
(4.1) 
\sum_{Q_{j,k} \in \Omega_{s,N}^{L,N}} g_{j,k}^t \Phi_{j,k}^t(x) 
\]
and

\[(4.2) \quad g_{s,N}(x) = \sum_{m \in \mathbb{Z}^n} g_{s-N,m}^N(x). \]

For \( g(x) = \sum_{(\epsilon, j, k) \in L_n} g_{\epsilon, j, k}^\epsilon \Phi_{\epsilon, j, k}(x) \in P^\alpha(\mathbb{R}^n) \), let

\[\Lambda_n^g = \{(\epsilon, j, k) \in \Lambda_n : g_{\epsilon, j, k}^\epsilon \neq 0\}.\]

Let \( \{f_{\epsilon, j, k}^\epsilon, g\} \) be the sequence such that

\[f_{\epsilon, j, k}^\epsilon, g = \left\{ \begin{array}{ll}
|f_{\epsilon, j, k}^\epsilon| \|g_{\epsilon, j, k}^\epsilon\|^{-1} g_{\epsilon, j, k}^\epsilon, & (\epsilon, j, k) \in \Lambda_n^g; \\
0, & (\epsilon, j, k) \notin \Lambda_n^g.
\end{array} \right.\]

We denote by \( Q_0^{\alpha, g} \) the set

\[\{f : f(x) = \sum_{(\epsilon, j, k) \in L_n} f_{\epsilon, j, k}^\epsilon \Phi_{\epsilon, j, k}(x) \text{ and } \|f\|_{Q_0^{\alpha}} \leq 1\}.\]

By (4.3), we have

\[(4.3) \quad \sup_{\|f\|_{Q_0^{\alpha}} \leq 1} \tau_{f, g} = \sup_{f \in Q_0^{\alpha, g}} \tau_{f, g}.\]

We prove first an approximation lemma on \( P^\alpha(\mathbb{R}^n) \).

Lemma 4.1. For \( g(x) = \sum_{(\epsilon, j, k) \in L_n} g_{\epsilon, j, k}^\epsilon \Phi_{\epsilon, j, k}(x) \in P^\alpha(\mathbb{R}^n) \), let

\[\tilde{g}_{s,N}(x) = \sum_{|m| \leq 2^n} g_{s-N,m}^N(x).\]

For arbitrary \( \delta > 0 \), there exist \( s \) and \( N \) such that \( \|g - \tilde{g}_{s,N}\|_{P^\alpha} \leq \delta. \)

Proof. For any \( 0 < \delta < \frac{\|f\|_{P^\alpha}}{8} \), according to \((\alpha, 2)-wavelet atom decomposition, there exist \( \{\lambda_u\}_{u \in \mathbb{N}_0} \in l^1 \) and a group of \((\alpha, 2)-wavelet\) atoms \( a_u(x) \) such that \( g = \sum_{u \in \mathbb{N}_0} \lambda_u a_u(x) \) and

\[\sum_{u \in \mathbb{N}_0} |\lambda_u| - \|g\|_{P^\alpha} \leq \frac{\delta}{8}.\]

Further, there exists an integer \( N_0 > 0 \) such that

\[(4.4) \quad \sum_{u > N_0} |\lambda_u| \leq \frac{\delta}{8}.\]

Now, for \( u = 1, \cdots, N_0 \), we consider the atoms

\[a_u(x) = \sum_{(\epsilon, j, k) \in L_n, Q_j \subset Q_u} a_{\epsilon, j, k}^u \Phi_{\epsilon, j, k}(x).\]
Since
\[
\left( \sum_{(\epsilon, j, k) \in \Lambda_n, Q_{j,k} \subset Q_u} 2^{-2 ju} |a_{j,k}^{\epsilon,n}|^2 \right)^{1/2} \leq |Q_u|^{\frac{\alpha}{2}},
\]
then there exists an integer $N_\delta > 0$ such that
\[
(4.5) \quad \left( \sum_{(\epsilon, j, k) \in \Lambda_n, Q_{j,k} \subset Q_u, j \leq N_\delta} 2^{-2 ju} |a_{j,k}^{\epsilon,n}|^2 \right)^{1/2} \leq \frac{\delta}{16g\|p_u\|} |Q_u|^{\frac{\alpha}{2}}.
\]
Since $1 \leq u \leq N_\delta$, there exists an integer $j_\delta \in \mathbb{Z}$ such that
\[
(4.6) \quad \bigcup_{1 \leq u \leq N_\delta} Q_u \subset \bigcup_{|m| \leq 2^n} Q_{j_\delta,m}.
\]
For $u = 1, \cdots, N_\delta$, let
\[
b_u(x) = \sum_{(\epsilon, j, k) \in \Lambda_n, Q_{j,k} \subset Q_u, j \leq N_\delta} a_{j,k}^{\epsilon,n} \Phi_{j,k}^\epsilon(x).
\]
By (4.4) and (4.5), we know that
\[
(4.7) \quad \begin{cases}
\left\| \sum_{u \geq N_\delta} \lambda_u a_u \right\|_{p_u} \leq \frac{\delta}{8} \\
\left\| \sum_{1 \leq u \leq N_\delta} \lambda_u (a_u - b_u) \right\|_{p_u} \leq \sum_{1 \leq u \leq N_\delta} |\lambda_u| \frac{\delta}{16g\|p_u\|} \leq \frac{\delta}{8}.
\end{cases}
\]
Let
\[
g_\delta(x) = \sum_{u > N_\delta} \lambda_u a_u + \sum_{1 \leq u \leq N_\delta} \lambda_u (a_u - b_u) = \sum_{(\epsilon, j, k) \in \Lambda_n} g_{j,k}^{\epsilon,\delta} \Phi_{j,k}^\epsilon(x).
\]
By (4.7), we get $\|g_\delta(x)\|_{p_u} \leq \frac{\delta}{4}$.

Let
\[
g_{1,\delta}(x) = \sum_{(\epsilon, j, k) \in \Lambda_n, j \geq N_\delta, Q_{j,k} \subset \bigcup_{|m| \leq 2^n} Q_{j_\delta,m}} g_{j,k}^{\epsilon,\delta} \Phi_{j,k}^\epsilon(x)
\]
and
\[
g_{2,\delta}(x) = g_\delta(x) - g_{1,\delta}(x).
\]
By (4.3), we have $\|g_{1,\delta}(x)\|_{p_u} \leq \frac{\delta}{4}$ and $\|g_{2,\delta}(x)\|_{p_u} \leq \frac{\delta}{4}$.

Take $s = N_\delta$ and $N = s - j_\delta$. Let
\[
\tilde{g}_{s,N}(x) = g_{1,\delta}(x) + \sum_{1 \leq u \leq N_\delta} \lambda_u b_u.
\]
According to (4.6) and the above construction process, $\tilde{g}_{s,N}(x)$ satisfies the condition of Lemma 4.1. \qed

Given $0 \leq t \leq N$, $m \in \mathbb{Z}^n$ and $Q = Q_{s-N,m}$. If $t = 0$, we denote
\[
g_{j,k}^{\epsilon,s,t,N} = \begin{cases}
0, & j > s; \\
g_{j,k}^\epsilon, & j = s.
\end{cases}
\]
For $t \geq 1$ and $Q_j g_{j,k}$ defined in Theorem 4.4 we denote
\[
 g_{j,k}^{s,t,N} = \left\{ \begin{array}{ll}
 0, & j > s - t; \\
 Q_j g_{j,k}, & j = s - t; \\
 g_{j,k}, & j < s - t.
\end{array} \right.
\]

Denote $g_{s,t,N}(x) = \sum_{j,k} g_{j,k}^{s,t,N} \Phi_{j,k}(x)$. We define
\[
P_{s,t,N}g(x) = \left( \sum_{j,k \in \mathbb{Z}^2, j \leq t} 2^{jn} g_{j,k}^{s,t,N} |\chi(2^j x - k)|^2 \right)^{1/2},
\]
\[
Q_{s,t,N}g = \left\| \left( \sum_{j,k \in \mathbb{Z}^2, j = t} 2^{jn} g_{j,k}^{s,t,N} |\chi(2^j x - k)|^2 \right)^{1/2} \right\|_{L^1}.
\]

For $t = N$, we have
\[
(4.8) \quad Q_{s,N,N}g_{s,N,m} = \|P_{s,N,N}g_{s-N,m}\|_{L^1}.
\]

Now we give a wavelet characterization of $P^\alpha(\mathbb{R}^n)$ without using Borel measures.

**Theorem 4.2.** If $0 < \alpha < \frac{n}{2}$, then
\[
P^\alpha(\mathbb{R}^n) = \left\{ g : \sup_{s \in \mathbb{Z}, N \in \mathbb{N}} \min_{0 \leq j \leq N} \|P_{s,N}g\|_{L^1} < \infty \right\}.
\]

**Proof.** By (4.3) and Lemma 4.1 for any $\delta > 0$, there exists $\tau_\delta > 0$ such that for $s > \tau_\delta$ and $N \geq 2s$,
\[
(4.9) \quad \|g_{s,N} - g\|_{P^\alpha} + \sum_{|m| \leq 2^n} \|g_{s-N,m}^N\|_{P^\alpha} \leq \delta
\]
and
\[
(4.10) \quad 8^{-n} \max_{|m| \leq 2^n} \|g_{s-N,m}^N\|_{P^\alpha} - \delta \leq \|g_{s,N}\|_{P^\alpha} \leq \sum_{|m| \leq 2^n} \|g_{s-N,m}^N\|_{P^\alpha} + \delta,
\]
where $g_{s,N}$ and $g_{s-N,m}^N$ are defined by (4.1) and (4.2).

By (4.8) and Theorem 3.4 we have
\[
(4.11) \quad \|g_{s,N}\|_{P^\alpha} = Q_{s,N,N}g_{s-N,m} = \|P_{s,N,N}g_{s-N,m}\|_{L^1}.
\]

Furthermore, we have
\[
(4.12) \quad \|g_{s,t,N}\|_{P^\alpha} \leq \|g_{s,t,N}\|_{H^1} = \|P_{s,t,N}g\|_{L^1}.
\]

According to (4.9)-(4.12), the proof of Theorem 4.2 is completed. \(\square\)
5. Fefferman–Stein type decomposition of $Q$-spaces

In this section, by the wavelet characterization obtained in Theorem 4.2, we give a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

In [3], M. Essen, S. Janson, L. Peng and J. Xiao proved that $Q_\alpha(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. We can see that the Hardy space $H^1(\mathbb{R}^n)$ is contained in $P^\alpha(\mathbb{R}^n)$.

**Proposition 5.1.** If $0 < \alpha < \frac{n}{2}$, then $H^1(\mathbb{R}^n) \subset P^\alpha(\mathbb{R}^n)$.

Further, in the proof of Theorem 5.7, we need some special properties of Daubechies wavelets. Except for Theorem 5.7, we use the classical Meyer wavelets in [8]. The supports of classical Meyer wavelets in [8] are $\Phi_0$ and $\Phi^1$ satisfy the following conditions

\[
\begin{align*}
\text{supp } \hat{\Phi}_0(\xi) &\subset \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]; \\
\text{supp } \hat{\Phi}^1(\xi) &\subset \left[-\frac{8\pi}{3}, \frac{8\pi}{3}\right]\setminus\left(\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)\right).
\end{align*}
\]

For tensor product Meyer wavelet satisfying (5.1), $\forall (\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n$ and $|j - j'| \geq 2$, we have

\[
\langle R_i \Phi^\epsilon_{j,k}, \Phi^{\epsilon'}_{j',k'} \rangle = 0, \forall i = 1, \cdots, n.
\]

5.1. Adapted $L^1$ and $L^\infty$ spaces. For $g(x) = \sum_{(\epsilon, j, k) \in \Lambda_n} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$ and $j \in \mathbb{Z}$, denote

\[
Q_j g(x) = \sum_{\epsilon, j, k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).
\]

For any $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, we set

\[
P_{s, N} g(x) = \sum_{\epsilon, s - N \leq j \leq s, k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).
\]

For all integers $t = 0, 1, \cdots, N$, denote

\[
T_{s, t, N}^1 g(x) = \sum_{\epsilon, s - t \leq j \leq s, k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)
\]

and

\[
T_{s, t, N}^2 g(x) = \sum_{\epsilon, s - N \leq j < s - 1, k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).
\]

By Theorem 4.2, we introduce a space $\hat{P}^\alpha(\mathbb{R}^n)$.

**Definition 5.2.** Given $\alpha \in [0, \frac{n}{2})$. $g(x) \in \hat{P}^\alpha(\mathbb{R}^n)$ if

\[
\sup_{s \in \mathbb{Z}, N \in \mathbb{N}} \inf_{0 \leq t \leq N} (\|T_{s, t, N}^1 g\|_{P^\alpha} + \|T_{s, t, N}^2 g\|_{H^1}) < \infty.
\]
This space is not really new. In fact,

**Theorem 5.3.** (i) If $\alpha = 0$, then $P^0(\mathbb{R}^n) = \tilde{P}^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

(ii) If $0 < \alpha < \frac{n}{2}$, then $P^\alpha(\mathbb{R}^n) = \tilde{P}^\alpha(\mathbb{R}^n)$.

**Proof.** $P^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ is known, so (i) is evident. Now we consider the cases $0 < \alpha < \frac{n}{2}$. If $g(x) \in P^\alpha(\mathbb{R}^n)$, then $\|P_s,N g\|_{P^\alpha} \leq \|g\|_{P^\alpha}$. Further

$$\inf_{0 \leq t \leq N} (\|T^1_{s,t,N} g\|_{P^\alpha} + \|T^2_{s,t,N} g\|_{H^1}) \leq \|T^1_{s,N,N} g\|_{P^\alpha} = \|P_s,N g\|_{P^\alpha}.$$  

Hence

$$\sup_{s \in \mathbb{Z}} \inf_{0 \leq t \leq N} (\|T^1_{s,t,N} g\|_{P^\alpha} + \|T^2_{s,t,N} g\|_{H^1}) \leq \|g\|_{P^\alpha}.$$

Conversely, we have

$$\|P_s,N g\|_{P^\alpha} \leq \|T^1_{s,t,N} g\|_{P^\alpha} + \|T^2_{s,t,N} g\|_{H^1}.$$  

Hence

$$\|P_s,N g\|_{P^\alpha} \leq \inf_{0 \leq t \leq N} (\|T^1_{s,t,N} g\|_{P^\alpha} + \|T^2_{s,t,N} g\|_{H^1}).$$

By (4.9), if $g(x) \in \tilde{P}^\alpha(\mathbb{R}^n)$, then $g(x) \in P^\alpha(\mathbb{R}^n)$. \hfill \Box

**Theorem 5.3** tells us that if $g \in P^\alpha(\mathbb{R}^n)$, the $P^\alpha$-norm of $g$ is equivalent to

$$\sup_{s \in \mathbb{Z}} \inf_{0 \leq t \leq N} (\|T^1_{s,t,N} g\|_{P^\alpha} + \|T^2_{s,t,N} g\|_{H^1}).$$

In other words, for $0 < \alpha < \frac{n}{2}$, the higher frequency part $T^1_{s,t,N} g(x)$ and the lower frequency part $T^2_{s,t,N} g(x)$ make different contributions to the norm.

Now we use this result to construct $L^{1,\alpha}(\mathbb{R}^n)$ and $L^{\infty,\alpha}(\mathbb{R}^n)$ which will be adapted to Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$.

Let $f(x) = \sum_{(e,j,k) \in \Lambda_n} f^e_{jk} \Phi^e_{jk}(x)$. For $s, t, N \in \mathbb{Z}$ and $0 \leq t \leq N$, we denote

$$P_{s,N} f(x) = \sum_{e, s-N \leq s, t < s} f^e_{jk} \Phi^e_{jk}(x),$$  

$$S^1_{s,t,N} f(x) = \sum_{e, s-t \leq j < s} f^e_{jk} \Phi^e_{jk}(x),$$  

$$S^2_{s,t,N} f(x) = \sum_{e, s-N \leq j \leq s-t} f^e_{jk} \Phi^e_{jk}(x).$$

The spaces $L^{1,\alpha}(\mathbb{R}^n)$ and $L^{\infty,\alpha}(\mathbb{R}^n)$ are defined as follows.

**Definition 5.4.** Given

$$f(x) = \sum_{(e,j,k) \in \Lambda_n} f^e_{jk} \Phi^e_{jk}(x).$$
and
\[ g(x) = \sum_{(e,j,k) \in \Lambda_n} g_{e,j,k}^e \Phi_{e,j,k}^e(x). \]

(i) \( g(x) \in L^{1,\alpha}(\mathbb{R}^n) \) if
\[ \sup_{s \in \mathbb{Z}, N \in \mathbb{N}} \min_{0 \leq t \leq N} \left( \|T^{1}_{s,t,N}g\|_{L^\alpha} + \|T^{2}_{s,t,N}g\|_{L^1} \right) < \infty. \]

(ii) \( f(x) \in L^{\infty,\alpha}(\mathbb{R}^n) \) if
\[ \sup_{s \in \mathbb{Z}, N \in \mathbb{N}} \sup_{0 \leq t \leq N} \left( \|S^{1}_{s,t,N}f\|_{Q_0} + \|S^{2}_{s,t,N}f\|_{L^\infty} \right) < \infty. \]

The following theorem can be deduced from Proposition 5.1 and Theorem 5.3.

**Theorem 5.5.** Given \( 0 \leq \alpha < \frac{n}{2} \),
(i) \( P^\alpha(\mathbb{R}^n) \subset L^{1,\alpha}(\mathbb{R}^n) \),
(ii) \( L^{\infty,\alpha}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \),
(iii) \( (L^{1,\alpha}(\mathbb{R}^n))^' = L^{\infty,\alpha}(\mathbb{R}^n) \).

**Remark 5.6.** For \( \alpha = 0 \), we have:
(i) \( P^0(\mathbb{R}^n) = H^1(\mathbb{R}^n) \) and \( Q_0(\mathbb{R}^n) = BMO(\mathbb{R}^n) \);
(ii) \( L^{1,0}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \) and \( L^{\infty,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \).

Now we use Daubechies wavelets to prove that
\[ L^{\infty,\alpha}(\mathbb{R}^n) \not\subset Q_\alpha(\mathbb{R}^n). \]

We know that there exist some integer \( M \) and Daubechies scale function \( \Phi^0(x) \in C_0^{2+2}([-2^{M}, 2^{M}]^n) \) satisfying
\[ C_D = \int_{\mathbb{R}^n} \frac{\Phi^0(y - 2^{M+1}e)}{|y|^{n+1}} dy < 0, \]
where \( e = (1, 1, \cdots, 1) \).

**Theorem 5.7.** Let \( \Phi(x) = \Phi^0(x - 2^{M+1}e) \) and let
\[ f(x) = \sum_{j \in \mathbb{Z}} \Phi(2^j x). \]

If \( 0 \leq \alpha < \frac{n}{2} \), then \( f(x) \in L^{\infty,\alpha}(\mathbb{R}^n) \) and \( R_{f(x)} \not\in L^\infty(\mathbb{R}^n) \). That is to say, \( L^{\infty,\alpha}(\mathbb{R}^n) \not\subset Q_\alpha(\mathbb{R}^n) \).

**Proof.** For \( j, j' \in 2\mathbb{N} \) and \( j \neq j' \), the supports of \( \Phi(2^j x) \) and \( \Phi(2^{j'} x) \) are disjoint. Hence the above \( f(x) \) in (5.6) belongs to \( L^\infty(\mathbb{R}^n) \). The same reasoning gives, for any \( j' \in \mathbb{N} \),
\[ \sum_{j \in \mathbb{Z}, 2^j > j'} \Phi(2^{2j}/x) \in L^\infty(\mathbb{R}^n). \]
Now we compute the wavelet coefficients of $f(x)$ in (5.6). For $(\epsilon', j', k') \in \Lambda_n$, let $f_{j', k'}^{\epsilon'} = \langle f, \Phi_{j', k'}^{\epsilon'} \rangle$. We distinguish two cases: $j' < 0$ and $j' \geq 0$.

For $j' < 0$, since the support of $f$ is contained in $[-3 \cdot 2^M, 3 \cdot 2^M]^n$, we know that if $|k'| > 2^{2M+5}$, then $f_{j', k'}^{\epsilon'} = 0$. If $|k'| \leq 2^{2M+5}$, we have

$$|f_{j', k'}^{\epsilon'}| \leq C 2^{-\frac{2M}{j'}} \int |f(x)| dx \leq C 2^{-\frac{2M}{j'}}.$$ 

For $j' \geq 0$, by orthogonality of the wavelets, we have

$$f_{j', k'}^{\epsilon'} = \langle f, \Phi_{j', k'}^{\epsilon'} \rangle = \langle \sum_{\epsilon \in \mathbb{N}, 2 \cdot j > j'} \Phi(2^{j'} x), \Phi_{j', k'}^{\epsilon'} \rangle.$$ 

Similarly, for the case $j' \geq 0$, we know that if $|k'| > 2^{2M+5}$, then $f_{j', k'}^{\epsilon'} = 0$. Since $\sum_{\epsilon \in \mathbb{N}, 2 \cdot j > j'} \Phi(2^{j'} x) \in L^\infty$, if $|k'| \leq 2^{2M+5}$, we have

$$|f_{j', k'}^{\epsilon'}| \leq C \int |\Phi_{j', k'}^{\epsilon'}(x)| dx \leq C 2^{-\frac{2M}{j'}}.$$ 

By the above estimates and the wavelet characterization of $Q$-spaces, we conclude that $f(x) \in Q_\alpha(\mathbb{R}^n)$, that is,

$$f(x) \in Q_\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Since $\Phi^0(x) \in C^{n+2}_0([-2^M, 2^M]^n)$, we know that

$$\Phi(x) = \Phi^0(x - 2^M e) \in C^{n+2}_0([2^M, 3 \cdot 2^M]^n).$$

Further, if $|x| \leq 2^{M-1}$ and $y \in [2^M, 3 \cdot 2^M]^n$, then $|x-y| > 2^{M-1}$. Hence $R_1 \Phi(x)$ is smooth in the ball $\{x : |x| \leq 2^{M-1}\}$.

Applying (5.5), there exists a positive $\delta > 0$ such that for $|x| < \delta$,

$$R_1 \Phi(x) < \frac{C_D}{2} < 0.$$ 

We can see that if $2^j |x| < \delta$, then

$$R_1 \Phi(2^j x) < \frac{C_D}{2} < 0.$$ 

Hence

$$R_1 f(x) \not\in L^\infty(\mathbb{R}^n).$$

□
5.2. Fefferman-Stein decomposition of \( Q_\alpha(\mathbb{R}^n) \). In [4], C. Fefferman and E. M. Stein used Riesz transformations and \( L^1 \)-norm to characterize Hardy space \( H^1(\mathbb{R}^n) \).

**Theorem 2.** \( g(x) \in H^1(\mathbb{R}^n) \) if and only if \( g(x) \in L^1(\mathbb{R}^n) \) and \( R_i g(x) \in L^1(\mathbb{R}^n) \) for \( i = 1, \cdots, n \).

Theorem 2 results in the solving of Fefferman-Stein decomposition of \( BMO(\mathbb{R}^n) \). The following theorem is a similar result for \( P_\alpha(\mathbb{R}^n) \) and it extends Theorem 2. If \( \alpha = 0 \), Theorem 5.8 becomes Theorem 2. We omit the proof for this case. The proof for the cases \( 0 < \alpha < \frac{n}{2} \) is very long. So we only state this result here and postpone the proof to Section 6.

**Theorem 5.8.** If \( 0 \leq \alpha < \frac{n}{2} \), then \( g(x) \in P_\alpha(\mathbb{R}^n) \) if and only if \( R_i g(x) \in L^{1,\alpha}(\mathbb{R}^n) \), \( i = 0, 1, \cdots, n \).

**Remark 5.9.** For the cases \( 0 < \alpha < \frac{n}{2} \), in [3], the authors list several open problems about \( Q_\alpha(\mathbb{R}^n) \). Their open problems attract a lot of attention and have been studied extensively by many authors. See [2], [11], [18], [20], [22] and the references therein. In the open problem 8.3 of [3], the authors ask what would be the suitable subspaces of \( Q_\alpha(\mathbb{R}^n) \) in which Fefferman-Stein decomposition is valid.

Unlike the situation of \( BMO(\mathbb{R}^n) \), \( L^\infty \) does not belong to \( Q_\alpha(\mathbb{R}^n) \). By duality, \( P_\alpha(\mathbb{R}^n) \) is not a subspace of \( L^1(\mathbb{R}^n) \) either. If we want to solve the problem 8.3 by Fefferman-Stein’s idea, we need to establish a relation between \( P_\alpha(\mathbb{R}^n) \) and the functions in \( L^1(\mathbb{R}^n) \). For this purpose, we should apply much more skills. For example, in Section 3, we consider the micro-local property of \( P_\alpha(\mathbb{R}^n) \). In Subsection 5.1, we construct the space \( L^{1,\alpha}(\mathbb{R}^n) \) by Meyer wavelets.

If Theorem 5.8 holds, by Theorem 5.7, we could obtain a Fefferman-Stein type decomposition of \( Q_\alpha(\mathbb{R}^n) \) using Fefferman-Stein’s skill in [4]. This result solves Problem 1.1 (Problem 8.3 in [3]).

**Theorem 5.10.** If \( 0 \leq \alpha < \frac{n}{2} \), then \( f(x) \in Q_\alpha(\mathbb{R}^n) \) if and only if \( f(x) = \sum_{0 \leq i \leq n} R_i f_i(x), \) where \( f_i(x) \in Q_\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), i = 1, 2, \cdots, n \).

**Proof.** By the continuity of the Calderón-Zygmund operators on \( Q \)-spaces, we know that if \( f_i(x) \in Q_\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), then

\[
\sum_{0 \leq i \leq n} R_i f_i(x) \in Q_\alpha(\mathbb{R}^n).
\]

Now we prove the converse result. Let

\[
B = \{(g_0, g_1, \cdots, g_n) : g_i \in L^{1,\alpha}(\mathbb{R}^n), i = 0, \cdots, n\}.
\]
The norm of \( B \) is defined as
\[
\| (g_0, g_1, \cdots, g_n) \|_B = \sum_{i=0}^{n} \|g_i\|_{L^1,\alpha}.
\]
We define
\[
S = \{(g_0, g_1, \cdots, g_n) \in B : g_i = R_i g_0, i = 0, 1, \cdots, n\}.
\]

\( S \) is a closed subset of \( B \) and \( g_0 \to (g_0, R_1 g_0, \cdots, R_n g_0) \) define a norm preserving map from \( P^\alpha(\mathbb{R}^n) \) to \( S \). Hence the set of continuous linear functionals on \( P^\alpha(\mathbb{R}^n) \) is equivalent to the set of continuous linear functionals on \( S \). We know that the dual space of \( L^1,\alpha(\mathbb{R}^n) \oplus \cdots \oplus L^1,\alpha(\mathbb{R}^n) \) is \( L^\infty,\alpha(\mathbb{R}^n) \oplus \cdots \oplus L^\infty,\alpha(\mathbb{R}^n) \).

For any \( f \in Q^\alpha(\mathbb{R}^n) \), \( f \) defines a continuous linear functional \( l \) on \( P^\alpha(\mathbb{R}^n) \) and also on \( S \). Hence there exist \( \tilde{f}_i \in L^\infty,\alpha(\mathbb{R}^n) \), \( i = 0, 1, \cdots, n \), such that for any \( g_0 \in P^\alpha(\mathbb{R}^n) \),
\[
l(f) = \int_{\mathbb{R}^n} f(x) g_0(x) dx = \int_{\mathbb{R}^n} \tilde{f}_0(x) g_0(x) dx + \sum_{i=1}^{n} \int_{\mathbb{R}^n} \tilde{f}_i(x) R_i g_0(x) dx
\]
\[
= \int_{\mathbb{R}^n} \tilde{f}_0(x) g_0(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^n} R_i(\tilde{f}_i)(x) g_0(x) dx.
\]

Hence \( f(x) = \tilde{f}_0(x) - \sum_{i=1}^{n} \int_{\mathbb{R}^n} R_i(\tilde{f}_i)(x) \). \( \square \)

6. The proof of Theorem 5.8

In [4], C. Fefferman and E. M. Stein used the Riesz transformations to characterize Hardy space \( H^1(\mathbb{R}^n) \) in terms of the \( L^1 \) norm. In this section, we prove a similar result for the cases \( 0 < \alpha < \frac{n}{2} \).

First, we prove that
\[
(6.1) \quad g(x) \in P^\alpha(\mathbb{R}^n) \implies R_i g(x) \in L^{1,\alpha}(\mathbb{R}^n), \ i = 0, 1, \cdots, n.
\]

By (ii) of Proposition 2.11, Riesz transforms are bounded on \( P^\alpha(\mathbb{R}^n) \). The fact \( g(x) \in P^\alpha(\mathbb{R}^n) \) implies that
\[
R_i g(x) \in P^\alpha(\mathbb{R}^n), \ i = 1, \cdots, n.
\]

By (i) of Theorem 5.5, we have
\[
R_i g(x) \in L^{1,\alpha}(\mathbb{R}^n), \ i = 0, 1, \cdots, n.
\]

The proof of the converse of (6.1) is cumbersome. We divide the proof into four parts which are given in Subsections 6.1-6.4, respectively.
6.1. A lemma. We first prove the following lemma.

**Lemma 6.1.** For \( g(x) = \sum_{(\epsilon, jk) \in \Lambda_n} g^\epsilon_{j,k}(x) \) and arbitrary \( j \in \mathbb{Z} \), denote \( g_j(x) = \sum_{\epsilon \in \mathbb{E}_n, k \in \mathbb{Z}^n} g^\epsilon_{j,k}(x) \) and denote \( \tilde{g}_j(x) = \sum_{f \leq j} g_f(x) \). For \( 0 < \alpha < \frac{n}{2} \), we have

(i) \( \|g_j\|_{H^1} \leq C\|g\|_{L^1} \).

(ii) \( \max \{\|\tilde{g}_j\|_{P^1}, \|g - \tilde{g}_j\|_{P^1}\} \leq \|g\|_{P^1} \leq \|\tilde{g}_j\|_{P^1} + \|g - \tilde{g}_j\|_{P^1} \).

(iii) \( \|g_j\|_{P^1} \leq C\|g\|_{L^1,\alpha} \).

**Proof.** (i) By applying wavelet characterization of Hardy spaces and the orthogonality properties of the Meyer wavelets, we have

\[
\|g_j\|_{H^1} \leq C\left\| \left( \sum_{\epsilon \in \mathbb{E}_n, k \in \mathbb{Z}^n} 2^{nj/2} |\langle g, \Phi^\epsilon_{j,k} \rangle|^2 \chi(2^j \cdot -k) \right)^{1/2} \right\|_{L^1} \leq C \sum_{\epsilon \in \mathbb{E}_n} \left\| \sum_{k \in \mathbb{Z}^n} 2^{nj/2} |\langle g, \Phi^\epsilon_{j,k} \rangle| \chi(2^j \cdot -k) \right\|_{L^1} \leq C\|g\|_{L^1}.
\]

(ii) \( P^1(\mathbb{R}^n) \) is a Banach space, hence we have

\[
\|g\|_{P^1} \leq \|\tilde{g}_j\|_{P^1} + \|g - \tilde{g}_j\|_{P^1}.
\]

To prove the inequality on the left side, denote \( G_g = \{(\epsilon, j, k) \in \Lambda_n, g^\epsilon_{j,k} \neq 0 \} \).

For \( f(x) = \sum_{(\epsilon, jk) \in \Lambda_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x) \), define

\[
f^\epsilon_{j,k} = \begin{cases} |f^\epsilon_{j,k}| \cdot |g^\epsilon_{j,k}|^{-1} \tilde{g}^\epsilon_{j,k}, & (\epsilon, j, k) \in G_g; \\ 0, & (\epsilon, j, k) \notin G_g.\end{cases}
\]

We denote by \( F_g \) the set

\[
\{f : f(x) = \sum_{(\epsilon, jk) \in G_g} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x) \text{ and } \|f\|_{Q_g^0} \leq 1\}.
\]

Define

\[
\tilde{f}(x) = \sum_{(\epsilon, jk) \in \Lambda_n} |f^\epsilon_{j,k}| \Phi^\epsilon_{j,k}(x)
\]

and

\[
g(x) = \sum_{(\epsilon, jk) \in \Lambda_n} |g^\epsilon_{j,k}| \Phi^\epsilon_{j,k}(x).
\]

By (ii) of Proposition 2.5, we have

\[
(6.2) \quad \sup_{\|f\|_{Q_g^0} \leq 1} \tau_{f,g} = \sup_{f \in F_g} \tau_{f,g} = \sup_{f \in F_g} \tau_{f,g} = \sup_{\|f\|_{Q_g^0} \leq 1} \tau_{f,g}.
\]
Hence, by (3.3) and (6.2), we can get
\[ \max \left\{ \| g \|_{L^p}, \| g - \tilde{g} \|_{L^p} \right\} \leq \| g \|_{L^p}. \]

(iii) By the definition of the norm of \( g(x) \) in \( L^{1,\alpha}(\mathbb{R}^n) \), for any \( s \in \mathbb{Z}, N \in \mathbb{N} \) and \( s - N \leq j \leq s \), there exists \( j_0 \) such that \( 0 \leq j_0 \leq N \) and
\[ \left\| \sum_{s - j_0 \leq f \leq s} g_f \right\|_{L^p} + \left\| \sum_{s - N \leq f < s - j_0} g_f \right\|_{L^1} \leq \| g \|_{L^{1,\alpha}}. \]

If \( j < s - j_0 \), we apply (i) of this lemma to get the desired assertion. If \( j \geq s - j_0 \), we apply (ii) of this lemma to get the desired assertion. \( \square \)

6.2. The proof of converse part. For the converse of (6.1), it is sufficient to prove that \( \forall s_1 \in \mathbb{Z}, N_1 \geq 1 \) and \( g_{s_1,N_1}(x) = P_{s_1,N_1}g(x) \) defined in (5.4), we have
\[ \| g_{s_1,N_1} \|_{L^p} \leq C \sum_{i = 0}^{\ldots,n} \| R_i g_{s_1,N_1} \|_{L^{1,\alpha}}. \]

Owing to (5.2), there exists \( \{ g^i_{j,k}(\epsilon,j,k) \} \subset \mathbb{A}_n \) such that for \( i = 1, 2, \ldots, n \),
\[ R_i g_{s_1,N_1}(x) = \sum_{\epsilon,j,k} g^i_{j,k}(\epsilon,j,k)(x). \]

Due to (6.4), to estimate the \( L^{1,\alpha} \)-norm of \( R_i g_{s_1,N_1}(x), i = 0, 1, \ldots, n \),
\[ \| R_i g_{s_1,N_1} \|_{L^{1,\alpha}} \]
it is sufficient to consider \( s = s_1 + 1 \) and \( N = N_1 + 2 \). For such \( s \) and \( N \), there exist \( \tilde{g}^0_{s,N} \) and \( u^i_{s,N} \) such that
\[ \left\| T_{s}^{1}_{s,N} \tilde{g}^0_{s,N} g_{s_1,N_1} \right\|_{L^p} + \left\| T_{s}^{2}_{s,N} \tilde{g}^0_{s,N} g_{s_1,N_1} \right\|_{L^1} \]
\[ = \min_{0 \leq i \leq n} \left( \left\| T_{s}^{1}_{s,N} R_i g_{s_1,N_1} \right\|_{L^p} + \left\| T_{s}^{2}_{s,N} R_i g_{s_1,N_1} \right\|_{L^1} \right); \]
and for \( 1 \leq i \leq n \),
\[ \left\| T_{s}^{1}_{s,N} R_i g_{s_1,N_1} \right\|_{L^p} + \left\| T_{s}^{2}_{s,N} R_i g_{s_1,N_1} \right\|_{L^1} \]
\[ = \min_{0 \leq i \leq n} \left( \left\| T_{s}^{1}_{s,N} R_i g_{s_1,N_1} \right\|_{L^p} + \left\| T_{s}^{2}_{s,N} R_i g_{s_1,N_1} \right\|_{L^1} \right). \]
There exist integers \( 1 \leq \tau \leq n, 1 \leq p_1 < p_2 < \cdots < p_\tau = n \) and a set
\[ \{ i_1, \ldots, i_{p_1}, i_{1+p_1}, \ldots, i_{p_2}, \ldots, i_{1+p_{p_1-1}}, \ldots, i_{p_\tau} \} \]
which is a rearrangement of the set \( \{ 1, \ldots, n \} \) such that
\[ u^i_{s,N} = \cdots = u^{i_{p_1}}_{s,N} < u^{i_{1+p_1}}_{s,N} = \cdots = u^{i_{p_2}}_{s,N} < \cdots < u^{i_{1+p_{p_1-1}}}_{s,N} = \cdots = u^{i_{p_\tau}}_{s,N}. \]

If the dimension \( n = 1 \), then \( \tau = 1 \). If the dimension \( n > 1 \), \( \tau \) can be any number between 1 and \( n \). That is to say, according to (6.7), there exist \( n \)
possibilities for $u^i_{s,N}$. In fact, by induction, we can transfer the cases $\tau > 2$ to the case $\tau = 2$. Hence we only give the proofs for $\tau = 1$ and $\tau = 2$.

6.3. The proof of $\tau = 1$. For $\tau = 1$, we denote

$$I^1_{s,N} = u^i_{s,N}, \ 1 \leq i \leq n.$$ 

We divide the proof into three subcases.

**Subcase 6.3.1: $I^0_{s,N} = t^1_{s,N}$.** Let $Q_j$ be the projection operators defined by (5.3). We divide the function $g_{s,1,N}$ into two functions

$$g_{s,1,N}(x) = g^1_{s,N}(x) + g^2_{s,N}(x),$$

where

$$g^1_{s,N}(x) = \sum_{j \geq s - \rho_{s,N}} Q_j g_{s,1,N}(x)$$

and

$$g^2_{s,N}(x) = \sum_{j < s - \rho_{s,N}} Q_j g_{s,1,N}(x).$$

By (6.5), we have $g^2_{s,N}(x) \in L^1(\mathbb{R}^n)$. This fact implies

$$Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x) \in H^1(\mathbb{R}^n)$$

and

$$g^2_{s,N}(x) - (Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x)) \in L^1(\mathbb{R}^n).$$

Further, for $i = 1, \cdots, n$, we have

$$T^2_{s,\lambda,N} R_i g_{s,1,N}(x) = T^2_{s,\lambda,N} R_i \left[ g^2_{s,N}(x) + Q_{s - \rho_{s,N}} g_{s,1,N}(x) \right]$$

$$= T^2_{s,\lambda,N} R_i \left[ g^2_{s,N}(x) - \left( Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x) \right) + \left( Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x) \right) \right]$$

$$= R_i \left[ g^2_{s,N}(x) - \left( Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x) \right) \right]$$

$$+ T^2_{s,\lambda,N} R_i \left( Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x) \right)$$

$$+ T^2_{s,\lambda,N} R_i Q_{s - \rho_{s,N}} g_{s,1,N}(x).$$

Hence, by the equation (6.8), for $i = 1, \cdots, n$,

$$II^1(x) = R_i \left[ g^2_{s,N}(x) - \left( Q_{s - \rho_{s,N} - 1} g_{s,1,N}(x) + Q_{s - \rho_{s,N} - 2} g_{s,1,N}(x) \right) \right]$$

$$+ T^2_{s,\lambda,N} R_i Q_{s - \rho_{s,N}} g_{s,1,N}(x) \in L^1(\mathbb{R}^n).$$
By equation (5.2), there exists \( \{ \tau_{j,k}^{\epsilon,i} \}_{(\epsilon,j) \in \Lambda_n} \) such that

\[
I_i(x) = R_1 \left[ g_{s,N}^2(x) - \left( Q_{s-\rho_N} g_{s_1,N_1}(x) + Q_{s-\rho_N} g_{s_1,N_1}(x) \right) \right]
\]

\[
= \sum_{(\epsilon,j,k) \in \Lambda_n, j \leq s} \tau_{j,k}^{\epsilon,i} \Phi_{j,k}(x)
\]

and

\[
R_1 Q_{s-\rho_N} g_{s_1,N_1}(x) = \sum_{(\epsilon,j,k) \in \Lambda_n, s-\rho_N \leq j \leq s-\rho_N + 1} \tau_{j,k}^{\epsilon,i} \Phi_{j,k}(x).
\]

For arbitrary \( L^\infty \) function

\[
h(x) = \sum_{(\epsilon,j,k) \in \Lambda_n} h_{j,k}^{\epsilon,i} \Phi_{j,k}(x)
\]

and \( j_0 \in \mathbb{Z} \), denote the operator

\[
P_{j_0} h(x) = \sum_{k \in \mathbb{Z}^n} \langle h(x), \Phi_{j_0,k}(x) \rangle \Phi_{j_0,k}(x).
\]

We can see that \( P_{j_0} h(x) \in L^{\infty}(\mathbb{R}^n) \). In fact, by the fact

\[
|\langle h(x), \Phi_{j_0,k}(x) \rangle| \leq C 2^{-n j_0},
\]

we can get

\[
|P_{j_0} h(x)| \leq C \sum_{k \in \mathbb{Z}^n} 2^{-n j_0} |\Phi_{j_0,k}(x)|
\]

\[
\leq C \sum_{k \in \mathbb{Z}^n} |\Phi_{j_0}(2^{j_0} x - k)|
\]

\[
\leq C.
\]

Let

\[
h_0(x) = P_{s-\rho_N} h(x) = \sum_{(\epsilon,j,k) \in \Lambda_n, j \leq s-\rho_N - 2} h_{j,k}^{\epsilon,i} \Phi_{j,k}(x).
\]

Hence \( h_0(x) \in L^{\infty}(\mathbb{R}^n) \). Further, because \( II_i(x) \in L^1(\mathbb{R}^n) \),

\[
|\langle I_i, h \rangle| = |\langle I_i, h_0 \rangle|
\]

\[
= |\langle II_i, h_0 \rangle|
\]

\[
\leq ||II_i||_{L^1} ||h_0||_{\infty}.
\]

The last estimate implies that for \( i = 1, \ldots, n \), the functions

\[
I_i(x) = R_1 \left[ g_{s,N}^2(x) - \left( Q_{s-\rho_N} g_{s_1,N_1}(x) + Q_{s-\rho_N} g_{s_1,N_1}(x) \right) \right] \in L^1(\mathbb{R}^n).
\]
This fact and (6.9) imply that
\[ g_{s,N}^2(x) - \left( Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) + Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) \right) \in H^1(\mathbb{R}^n). \]

By (6.8), we get
\[ g_{s,N}^2(x) \in H^1(\mathbb{R}^n). \]

Further, we have \( g_{s,N}^1(x) \in P^s(\mathbb{R}^n) \). Applying (6.5), we get
\[ g_{s_1,N_1}(x) \in P^s(\mathbb{R}^n). \]

**Subcase 6.3.2:** \( r_{s,N}^0 > r_{s,N}^1 \). For this case, we decompose \( g_{s_1,N_1}(x) \) into three functions
\[ g_{s_1,N_1}(x) = g_{s,N}^1(x) + g_{s,N}^2(x) + g_{s,N}^3(x), \]
where
\[ g_{s,N}^1(x) = \sum_{j \geq r_{s,N}^1} Q_j g_{s_1,N_1}(x), \]
\[ g_{s,N}^2(x) = \sum_{s-r_{s,N}^0 \leq j < s-r_{s,N}^1} Q_j g_{s_1,N_1}(x) \]
and
\[ g_{s,N}^3(x) = \sum_{j < s-r_{s,N}^1} Q_j g_{s_1,N_1}(x). \]

We know that
\[ T_{s_1,N_1}^2 R_i g_{s_1,N_1}(x) = T_{s_1,N_1}^2 R_i \left[ g_{s,N}^3(x) + g_{s,N}^2(x) + Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) \right]. \]

Then \( \forall h(x) = \sum_{\epsilon, s-N \leq j < s-r_{s,N}^0} h^{\epsilon}_j \Phi^{\epsilon}_j(x) \) and \( ||h||_L^1 \leq 1 \), we know that
\[ \langle T_{s_1,N_1}^2 R_i g_{s_1,N_1}, h \rangle = \langle R_i g_{s,N}^3, h \rangle. \]

By (6.5), \( g_{s,N}^3(x) \in L^1(\mathbb{R}^n) \). This fact implies that
\[ Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) + Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) \in H^1(\mathbb{R}^n). \]

Owing to (6.10) and (6.11), for \( i = 0, \cdots, n \), we have
\[ R_i \left[ g_{s,N}^3(x) - \left( Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) + Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) \right) \right] \in L^1(\mathbb{R}^n). \]

Hence we obtain
\[ g_{s,N}^3(x) - \left( Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) + Q_{s-r_{s,N}^0} g_{s_1,N_1}(x) \right) \in H^1(\mathbb{R}^n). \]

So we have \( g_{s,N}^3(x) \in H^1(\mathbb{R}^n) \). Since
\[ g_{s,N}^1(x) + g_{s,N}^2(x) \in P^s(\mathbb{R}^n), \]
we have \( g_{s,N_1}(x) \in P^r(\mathbb{R}^n) \).

**Subcase 6.3.3:** \( t^0_{s,N} < t^1_{s,N}\). We decompose \( g_{s,N_1}(x) \) into three functions

\[
\begin{align*}
g_{s,N_1}(x) &= g^1_{s,N}(x) + g^2_{s,N}(x) + g^3_{s,N}(x),
\end{align*}
\]

where

\[
\begin{align*}
g^1_{s,N}(x) &= \sum_{j \geq s - t^0_{s,N}} Q_j g_{s,N_1}(x),
\end{align*}
\]

\[
\begin{align*}
g^2_{s,N}(x) &= \sum_{s - t^1_{s,N} \leq j < s - t^0_{s,N}} Q_j g_{s,N_1}(x)
\end{align*}
\]

and

\[
\begin{align*}
g^3_{s,N}(x) &= \sum_{j < s - t^1_{s,N}} Q_j g_{s,N_1}(x).
\end{align*}
\]

For \( i = 1, \ldots, n \), we know that

\[
\begin{align*}
T^2_{s,t^1_{s,N},N} R_i g_{s,N_1}(x)
&= T^2_{s,t^1_{s,N},N} R_i \left[ g_{s,N}(x) + Q_{s-t^0_{s,N}} g_{s,N_1}(x) \right]
\end{align*}
\]

\[
\begin{align*}
&= R_i \left[ g_{s,N}(x) - \left( Q_{s-t^1_{s,N}} - 1 g_{s,N_1}(x) + Q_{s-t^0_{s,N} - 2} g_{s,N_1}(x) \right) \right]
\end{align*}
\]

\[
\begin{align*}
&+ T^2_{s,t^1_{s,N},N} R_i \left[ Q_{s-t^1_{s,N} - 1} g_{s,N_1}(x) + Q_{s-t^0_{s,N} - 2} g_{s,N_1}(x) \right]
\end{align*}
\]

\[
\begin{align*}
&+ T^2_{s,t^1_{s,N},N} R_i Q_{s-t^1_{s,N}} g_{s,N_1}(x).
\end{align*}
\]

Define \( h_i(x), i = 1, 2, 3, 4 \), as

\[
\begin{align*}
h_1(x) &= \sum_{\varepsilon, s-N \leq j < s - t^1_{s,N} - 2} h_{j,k} \Phi_{j,k}(x),
\end{align*}
\]

\[
\begin{align*}
h_2(x) &= \sum_{\varepsilon, j = s - t^1_{s,N} - 2} h_{j,k} \Phi_{j,k}(x),
\end{align*}
\]

\[
\begin{align*}
h_3(x) &= \sum_{\varepsilon, j = s - t^1_{s,N} - 1} h_{j,k} \Phi_{j,k}(x),
\end{align*}
\]

\[
\begin{align*}
h_4(x) &= \sum_{\varepsilon, j = s - t^1_{s,N} + k} h_{j,k} \Phi_{j,k}(x),
\end{align*}
\]

where the sequences \( \{ h_{j,k}(x) \} \), \( i = 1, 2, 3, 4 \), are four arbitrary sequences satisfying the condition \( ||h_i||_{L^\infty} \leq 1 \).

We consider

\[
\begin{align*}
\int T^2_{s,t^1_{s,N},N} R_i g_{s,N_1}(x) h_i(x) dx.
\end{align*}
\]
By (6.6) and the definition of $t_{s,N}^1$, we have

$$g_{s,N}^3(x) + g_{s,N}^2(x) \in L^1(\mathbb{R}^n).$$

Hence

(6.12) $$g_{s,N}^2(x) - \left( Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) \right) \in L^1$$

and

(6.13) $$Q_{s-t^i_{s,N}-i} g_{s_1,N_1}(x) \in H^1(\mathbb{R}^n), i = 0, 1, 2.$$

Similar to Subcase 6.3.1, by (5.2), the fact that

$$T^2_{s,t^i_{s,N}} R_i g_{s_1,N_1}(x) \in L^1(\mathbb{R}^n), i = 1, \cdots, n,$$

implies that for $i = 1, \cdots, n$,

$$R_i \left[ g_{s,N}^3(x) - \left( Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) \right) \right] \in L^1(\mathbb{R}^n).$$

Therefore we have

$$g_{s,N}^3(x) - \left( Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) + Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) \right) \in H^1(\mathbb{R}^n).$$

Hence $g_{s,N}^3(x) \in H^1(\mathbb{R}^n)$.

For $i = 1, \cdots, n$, we have

$$T^1_{s,t^i_{s,N}} R_i g_{s_1,N_1}(x) = T^1_{s,t^i_{s,N}} R_i \left[ g_{s,N}^1(x) + g_{s,N}^2(x) + Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) \right].$$

So the conditions

$$T^1_{s,t^i_{s,N}} R_i \left[ g_{s,N}^1(x) + g_{s,N}^2(x) + Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) \right] \in P_0(\mathbb{R}^n), i = 1, \cdots, n$$

and $g_{s,N}^1(x) \in P_0(\mathbb{R}^n)$ implies

$$T^1_{s,t^i_{s,N}} R_i \left[ g_{s,N}^2(x) + Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) \right] \in P_0(\mathbb{R}^n).$$

For $i = 1, \cdots, n$, we have

$$T^1_{s,t^i_{s,N}} R_i \left[ g_{s,N}^2(x) + Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) \right] = R_i \left[ g_{s,N}^2(x) - Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) \right] + T^1_{s,t^i_{s,N}} R_i \left[ Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) + Q_{s-t^i_{s,N}-1} g_{s_1,N_1}(x) \right].$$

Applying (6.13), we obtain

$$R_i \left[ g_{s,N}^2(x) - Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) \right] \in P_0(\mathbb{R}^n).$$

Hence

$$g_{s,N}^2(x) - Q_{s-t^i_{s,N}-2} g_{s_1,N_1}(x) \in P_0(\mathbb{R}^n).$$
That is to say, $g_{s_1,N_1}(x)$ satisfies the conditions (6.12) and (6.13). By (6.5) and (6.6),

$$g_{s,N}^1(x) \in \mathcal{P}^\rho(\mathbb{R}^n)$$

and

$$g_{s,N}^3(x) \in \mathcal{H}^1(\mathbb{R}^n).$$

Putting the four parts together, we complete the proof of (6.3) for $\tau = 1$.

6.4. **The proof of** $\tau = 2$. For $\tau = 2$, we denote

$$
\begin{aligned}
& t_{s,N}^1 = u_{s,N}^{i_1} = \cdots = u_{s,N}^{i_p}, \\
& t_{s,N}^2 = u_{s,N}^{i_{p+1}} = \cdots = u_{s,N}^{i_{p+2}}.
\end{aligned}
$$

We deal with the five subcases for $t_{s,N}^i, 0 \leq i \leq 2$, separately:

- **Subcase 6.4.1:** $t_{s,N}^0 < t_{s,N}^1 < t_{s,N}^2$;
- **Subcase 6.4.2:** $t_{s,N}^0 = t_{s,N}^1 < t_{s,N}^2$;
- **Subcase 6.4.3:** $t_{s,N}^1 < t_{s,N}^0 < t_{s,N}^2$;
- **Subcase 6.4.4:** $t_{s,N}^1 < t_{s,N}^0 = t_{s,N}^2$;
- **Subcase 6.4.5:** $t_{s,N}^1 < t_{s,N}^0 < t_{s,N}^2$.

**Step 1.** At first, we consider Subcase 6.4.1. Let

$$
\begin{aligned}
\bar{s}_1 &= s_1 - t_{s,N}^0 - 1, \\
\bar{N}_1 &= N_1 - t_{s,N}^0 - 1.
\end{aligned}
$$

Since $\|g_{s_1,N_1}\|_{L^\rho} \leq \|g_{s_1,N_1}\|_{L^{1,\alpha}}$, by the continuity of the Calderón-Zygmund operators, we have

$$
(6.14) \quad \|R_{i}g_{s_1,N_1}\|_{L^\rho} \leq C\|g_{s_1,N_1}\|_{L^{1,\alpha}}, \quad i = 1, \cdots, n.
$$

To prove (6.3), it is sufficient to prove that the function

$$
\tilde{g}_{\bar{s}_1,\bar{N}_1}(x) = g_{s_1,N_1}(x) - g_{s_1,N_1}(x)
$$

satisfies the following condition:

$$
(6.15) \quad \|\tilde{g}_{\bar{s}_1,\bar{N}_1}\|_{L^\rho} \leq C\sum_{i=0,\cdots, N} \|R_{i}g_{s_1,N_1}\|_{L^{1,\alpha}}.
$$

According to (5.2), for $i = 1, \cdots, n$, there exists $\{g_{j,k}^{\epsilon,i}(\epsilon,j,k)\in \Lambda_\alpha$ such that

$$
(6.16) \quad R_{i}g_{\bar{s}_1,\bar{N}_1}(x) = \sum_{(\epsilon,j,k)\in \Lambda_\alpha, \bar{s}_1 = \bar{s}_1, \bar{N}_1 = \bar{N}_1 - 1 \leq \bar{s}_1 + 1} g_{j,k}^{\epsilon,i} \Phi_{j,k}(x).
$$

Similar to the case $\tau = 1$, we denote

$$
\begin{aligned}
\tilde{s}_1 &= \bar{s}_1; \\
\tilde{N}_1 &= \bar{N}_1 + 2; \\
\rho_{\tilde{s},\tilde{N}} &= t_{s,N}^1 - t_{s,N}^0; \\
\tilde{t}_{\tilde{s},\tilde{N}} &= t_{s,N}^2 - t_{s,N}^0.
\end{aligned}
$$
Since \( t_{s,N}^0 < t_{s,N}^1 < t_{s,N}^2 \), by (6.15), we have the following inequality.

\[
(6.17) \quad \| g_{s_1,n_1} \|_{L^1} \leq \| g_{s_1,n_1,1} \|_{L^{1,\alpha}}.
\]

For \( i = i_1, \ldots, i_{p_1} \), by the fact \( t_{s,N}^0 < t_{s,N}^1 < t_{s,N}^2 \), we have

\[
(6.18) \quad T_{s,1}^{2} \| g_{s_1,n_1} \|_{L^1} = T_{s,1}^{2} (R_g g_{s_1,n_1}(x)).
\]

Further, by (6.14) and Lemma 6.1 for \( i = i_1, \ldots, i_{p_1} \), we have

\[
(6.19) \quad \| T_{s,1}^{1} R g_{s_1,n_1} \|_{L^1} \leq C (\| g_{s_1,n_1} \|_{L^{1,\alpha}} + \| R_g g_{s_1,n_1} \|_{L^{1,\alpha}}).
\]

By (5.2), (5.6) and (6.16), for \( i = i_1, \ldots, i_{p_1} \), we have

\[
(6.20) \quad \| T_{s,1}^{1} R g_{s_1,n_1} \|_{L^1} \leq C (\| g_{s_1,n_1} \|_{L^{1,\alpha}} + \| R_g g_{s_1,n_1} \|_{L^{1,\alpha}}).
\]

Similarly, for \( i = i_1 + p_1, \ldots, i_{p_2} \),

\[
(6.21) \quad \| T_{s,1}^{2} R g_{s_1,n_1} \|_{L^1} \leq C (\| g_{s_1,n_1} \|_{L^{1,\alpha}} + \| R_g g_{s_1,n_1} \|_{L^{1,\alpha}}).
\]

Then we divide \( g_{s_1,n_1}(x) \) into three functions as Subcases 6.3.2 and 6.3.3. Because \( g_{s_1,n_1}(x) \) satisfies (6.17), (6.18) and (6.19), we obtain the estimate (6.15) by a similar proof.

Step 2. For Subcases 6.4.2 and 6.4.4, we decompose \( g_{s_1,n_1}(x) \) into three functions like what we did for Subcases 6.3.2 and 6.3.3. By a similar proof, we get the estimate (6.3).

Step 3. For Subcase 6.4.3, we take \( \tilde{g} = g_{s_1,n_1} \). Then we have

\[
(6.22) \quad \| \tilde{g} \|_{L^{1,\alpha}} \leq \| g_{s_1,n_1} \|_{L^{1,\alpha}}.
\]

For Subcase 6.4.5, we take \( \tilde{g} = g_{s_1,n_1} \). According to Lemma 6.1 we know that

\[
(6.23) \quad \| \tilde{g} \|_{L^1} \leq \| g_{s_1,n_1} \|_{L^{1,\alpha}}.
\]

Similar to Step 1, we consider the function \( g_{s_1,n_1} - \tilde{g} \). According to (6.20) and (6.21), we can deal with \( g_{s_1,n_1} - \tilde{g} \) by the same ideas as in Step 1.
Precisely, we decompose $g_{s_i, N_i} - \bar{g}$ into three functions as Subcases 6.3.2 and 6.3.3. By a similar proof, we can obtain the estimate (6.3).

By the above three steps, we can obtain the estimate (6.3) for the case $\tau = 2$. The proof of Theorem 5.8 is thus completed.

References

[1] R. Coifman, R. Rochberg, Another characterization of BMO, Proc. A. M. S. 79 (1980), 249-254.
[2] G. Dafni, J. Xiao, Some new tent spaces and duality theorem for fractional Carleson measures and $Q_\alpha(\mathbb{R}^n)$, J. Funct. Anal. 208 (2004), 377-422.
[3] M. Essen, S. Janson, L. Peng, J. Xiao, $Q$ spaces of several real variables, Indiana Univ. Math. J. 49 (2000), 575-615.
[4] C. Fefferman, E. M. Stein, $H^p$ spaces of several variables, Acta Math. 129 (1972), 137-193.
[5] P. Jones, Carleson measures and the Fefferman-Stein decomposition of $BMO(\mathbb{R})$, Ann. Math. 111 (1980), 197-208.
[6] P. Jones, $L^\infty$-estimates for the $\bar{\partial}$ problem in a half-plane, Acta Math. 150 (1983), 137-152.
[7] E. Kalita, Dual Morrey spaces, Dokl. Math. 361 (1998), 447-449.
[8] Y. Meyer, Ondelettes et Opérateurs, I et II, Hermann, Paris, 1991-1992.
[9] Y. Meyer, Q. Yang, Continuity of Calderon-Zygmund operators on Besov or Triebel-Lizorkin spaces, Anal. Appl. (Singap.) 6 (2008), 51-81.
[10] A. Nicolau, J. Xiao, Bounded functions in Möbius invariant Dirichlet spaces, J. Funct. Anal. 150 (1997), 383-425.
[11] L. Peng, Q. Yang, Predual spaces for $Q$ spaces, Acta Math. Sci. Ser. B 29 (2009), 243-250.
[12] D. Sarason, Functions of vanishing mean oscillation, Trans. A. M. S. 207 (1975), 391-405.
[13] A. Stegenga, Bounded Toeplitz operators on $H^1$ and Applications of the duality between $H^1$ and the functions of bounded mean oscillation, Amer. J. Math. 98 (1976), 573-589.
[14] E. M. Stein, Harmonic analysis: real variable methods, orthogonality, and oscillatory integrals, Princeton Univ. Press, 1993.
[15] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.
[16] A. Uchiyama, A constructive proof of the Fefferman-Stein decomposition of $BMO(\mathbb{R}^n)$, Acta Math. 148 (1982), 215-241.
[17] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Mathematical Society Student Texts 37, Cambridge University Press, 1997.
[18] Z. Wu, C. Xie, $Q$ spaces and Morrey spaces, J. Funct. Anal. 201 (2003), 282-297.
[19] J. Xiao, The $Q_p$ Carleson measure problem, Adv. Math. 217 (2008), no. 5, 2075-2088.
[20] Q. Yang, Wavelet and Distribution, Beijing Science and Technology Press, 2002.
[21] Q. Yang, Y. Zhu, Characterization of multiplier spaces by wavelets and logarithmic Morrey spaces, Nonlinear Anal. TMA 75 (2012), 4920-4935.
[22] W. Yuan, W. Sickel, D. Yang, Morrey and Campanato Meet Besov, Lizorkin and Triebel, Lecture Notes in Mathematics 2005, Editors: J.-M. Morel, Cachan F. Takens, Groningen B. Teissier, Paris, 2010.