Supersymmetric Calogero Models from Superfield Gauging

E. A. Ivanova, *, O. Lechtenfeldb, **, and S. Fedoruka, ***

aBogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow oblast, 141980 Russia
bInstitut für Theoretische Physik and Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Hannover, 30167 Germany

*e-mail: eivanov@theor.jinr.ru
**e-mail: olaf.lechtenfeld@itp.uni-hannover.de
***e-mail: fedoruk@theor.jinr.ru

Received December 20, 2019; revised January 16, 2020; accepted January 29, 2020

Abstract—Using the superfield gauging procedure, we construct new N = 2 and N = 4 superfield systems that generalize Calogero models. In the bosonic limit, these systems yield rational Calogero models and hyperbolic Calogero–Sutherland models in the N = 2 case, and their U(2) spin generalization in the N = 4 case.

DOI: 10.1134/S1063779620040346

1. INTRODUCTION

Calogero models [1–3] are text-book examples of integrable multi-particle one-dimensional (d = 1) systems. The simplest is the so-called rational Calogero model

\[ S_C = \frac{1}{2} \int dt \left( \sum_a x_a \dot{x}_a - \sum_{a < b} \frac{e^2}{4(x_a - x_b)^2} \right), \quad (1.1) \]

which describes the interaction of n identical particles with a potential inversely proportional to the square of the distance and invariant with respect to transformations of the d = 1 conformal group SO(1,2)

\[ \delta t = \alpha, \quad \delta x_a = \frac{1}{2} \alpha x_a, \quad \partial_t^2 \alpha = 0. \quad (1.2) \]

The Calogero–Moser system [1–3] is a generalization of the system (1.1) by adding an oscillator term \( \sum_{a,b} (x_a - x_b)^2 \). Being interesting, in the first turn, as integrable systems, rational Calogero models also bear relationships with superstring theory and M-theory [4, 5].

Besides conformally invariant systems, some other many-particle integrable Calogero-type models are known [6], e.g., Calogero–Sutherland hyperbolic systems [1–3, 7]

\[ S_{CS} = \frac{1}{2} \int dt \left( \sum_a \dot{q}_a \partial_a - \sum_{a < b} \frac{e^2}{4 \sinh^2 \left( \frac{q_a - q_b}{2} \right)} \right), \quad (1.3) \]

and their trigonometric analogues.

A natural generalization of the Calogero and Calogero–Sutherland systems is their supersymmetric variants. An N = 2 superextension was built in [8], where each bosonic coordinate x_a was completed, by two fermionic fields, to the multiplet (1, 2, 1). Thus, the model contains n physical bosons and 2n fermions. The corresponding N = 2, d = 1 superfield action in the limit of zero fermions is reduced to the action of the rational Calogero model. Similarly, one can set up N = 2 extension of the Calogero–Sutherland models [9]. Passing to supersymmetric extensions with higher N encounters certain problems. E.g., when generalizing to the N = 4 case, the coordinate set {x_a} must be enlarged to a set of (1, 4, 3) supermultiplets with n bosonic and 4n fermionic physical fields [10]. However, when constructing the corresponding superfield action, which should yield the potential of the n-particle Calogero system in the bosonic sector, there arise two prepotentials connected by a set of nonlinear differential equations [11] including the WDVV equations [12, 13], explicit solutions to which are known only for small values of n.

There is another type of supersymmetrization, in which the above problems do not arise, although the models constructed in this way are “non-minimal”: they contain Nn^2 fermions for each set of n bosonic coordinates [14–16]. This supersymmetrization is based on the gauging method [17], developed previously in [3, 18, 19] in application to the Calogero bosonic systems. A particular Calogero model arises as a result of eliminating gauge fields in the Lagrangian of some matrix gauge-invariant system. In this talk,
based on the results of [14–16], we expound how this approach can be applied to some $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superfield matrix models in order to obtain new versions of supersymmetric Calogero models.

2. CALOGERO AND CALOGERO–SUTHERLAND MODELS AS GAUGE MODELS

To illustrate the method we use, let us first show how one can reproduce the well-known conformal mechanics model [20],

$$ S_0 = \int dt L_0, \quad L_0 = \frac{1}{2}(\dot{x}^2 - c^2x^2), \quad (2.4) $$

from a different $d = 1$ system with gauge symmetry [15]. Consider the model of complex field $\psi(t)$ with the Lagrangian

$$ L_\psi = \frac{1}{2} \dot{\psi} \overline{\dot{\psi}} + \frac{im}{2}(\overline{\dot{\psi}} - \dot{\psi}), \quad (2.5) $$

which is invariant under global transformations $\psi' = e^{-i\theta}\psi, \overline{\psi'} = e^{i\theta}\overline{\psi}$. Now we extend the Lagrangian (2.5) so that it possesses gauge symmetry with a local parameter: $\lambda \rightarrow \lambda(t)$. To accomplish this, we introduce $d = 1$ gauge field $A(t)$ that lengthens the derivatives as $\dot{\psi} \rightarrow \overline{\nabla}{\psi} = \dot{\psi} + iA\psi, \overline{\dot{\psi}} \rightarrow \nabla\overline{\psi} = \ddot{\psi} - iA\overline{\psi}$. The resulting system with the Lagrangian

$$ L^G = \frac{1}{2} \nabla\overline{\nabla}{\psi} + \frac{im}{2}(\overline{\nabla}{\psi} - \nabla\overline{\psi}) + cA \quad (2.6) $$

is invariant, up to a total derivative, under the gauge transformations introduced above, supplemented by the transformation $A' = A + \lambda$. The last term in (2.6), with the constant $c$, is also gauge-invariant up to a total derivative. It is an analogue of the well-known Fayet–Iliopoulos term.

Choosing the gauge $\nu = \overline{\nu} \equiv x(t)$ and eliminating the field $A(t)$ by its equation of motion, we obtain the following expression for the Lagrangian in this gauge

$$ L_{\text{gauge}} = \frac{1}{2} \dot{x}^2 - \frac{1}{2}(mx - cx^{-1})^2. \quad (2.7) $$

For $m = 0$ it coincides with the Lagrangian (2.4). We note that the initial action with the Lagrangian (2.5) for $m = 0$, as well as the gauge-invariant model (2.6), are invariant under the conformal transformations $SO(1,2)$ (1.2), supplemented by the transformations $\delta A(t) = -\dot{\lambda}A(t)$. As a result, the action with the Lagrangian (2.4) also has $d = 1$ conformal symmetry.

In the gauge approach, the Calogero system is described by $U(n)$-invariant matrix system [3, 18, 19], incorporating the $n \times n$ Hermitian matrix field $X^b_a$, $a, b = 1, \ldots, n$; the complex $U(n)$-spinor field $Z_a(t)$, $Z^a = (Z_a)^*$ and $n^2$ Hermitian gauge fields $A^a_b$. The gauge-invariant action has the form

$$ S_c = \frac{1}{2} \int dt \left[ \nabla X \nabla X^{-1} \nabla X \right] + i(\overline{\nabla}Z - \nabla\overline{Z}) + 2c\text{tr}A, \quad (2.8) $$

where the following definitions are used for the covariant derivatives

$$ \nabla X = \dot{X} + i[A, X], \quad \nabla Z = \dot{Z} - iZA. \quad (2.9) $$

The action (2.8) is invariant under local $U(n)$ transformations acting on the spinor indices $a, b$ of all involved quantities, with the matrix field $A^a_b$ as a gauge field. Using $n^2 - n$ local transformations, we can fix the gauge $X^a_a = 0$ with $a \neq b$. The residual gauge transformations generated by the abelian subgroup $\text{U}(1)_n$ are then fixed by the reality conditions $Z^a = Z_a, Z_a$ being subject to the constraints $Z_aZ_a = c$ for each $a$. As a result, after eliminating the auxiliary and gauge fields, the action (2.8) is reduced to the action (1.1) of the Calogero model. Since the original action (2.8) is conformally invariant, the model (1.1) is also conformally invariant.

The Calogero–Sutherland model can be deduced by a similar gauging procedure from the system involving a nonlinear kinetic term of the sigma-model type for the matrix field $X^a_a$:

$$ S_{CS} = \frac{1}{2} \int dt \left[ \nabla^{-1}\nabla X \nabla^{-1}\nabla X \right] + i(\overline{\nabla}Z - \nabla\overline{Z}) + 2c\text{tr}A. \quad (2.10) $$

Following the same pattern as in the rational case, we arrive at the action

$$ S_{CS} = \frac{1}{2} \int dt \left[ \sum_{a} \delta_{a} x_{a} \frac{\delta_{a} x_{a}}{2} = \sum_{a \neq b} \frac{x_{a} x_{b} c^{2}}{a (x_{a} - x_{b})^2} \right], \quad (2.11) $$

which, in terms of the variables $q_{a} = \ln x_{a}$, coincides with (1.3). Like the initial matrix action, the resulting action does not possess conformal invariance.

3. $\mathcal{N} = 2$ CALOGERO AND CALOGERO–SUTHERLAND MODELS

To construct $\mathcal{N} = 2$ supersymmetric generalization, we will resort to the same strategy, proceeding now from the matrix $\mathcal{N} = 2$ superfields and effecting a superfield gauging procedure. The input superfield set involves $n \times n$ matrix Hermitian superfield with components $F^{ab}_a(t, \theta, \bar{\theta}), a, b = 1, \ldots, n$, describing $n^2$ supermultiplets (1, 2, 1), and a chiral $U(n)$-spinor superfield
SUPERSYMMETRIC CALOGERO MODELS FROM SUPERFIELD GAUGING

The gauge-invariant action has the form

\[ S^{N=2} = \frac{1}{2} \int dt d\theta d\bar{\theta} \left[ \text{tr} \left( \overline{D} \overline{X} D X \right) - \overline{D} \overline{X} \right], \tag{3.12} \]

remains invariant under global U(\(n\))-transformations \( \overline{X} = e^{i\Lambda} \overline{X}, \overline{X}' = e^{-i\Lambda} \overline{X}' \). Gauging these symmetries amounts to passing to the chiral and antichiral superfield parameters \( \lambda \) and \( \overline{\lambda} \). To ensure invariance, the Hermitian gauge superfield \( V \) is introduced, with the transformation law: \( e^{2V} = e^{i\Lambda} e^{2V} e^{-i\Lambda} \). The gauge-invariant action has the form

\[ S_c^{N=2} = \frac{1}{2} \int dt d\theta d\bar{\theta} \left[ \text{tr} \left( \overline{D} \overline{X} e^{2V} D X e^{2V} \right) \right. \]
\[ \left. - \overline{D} e^{2V} \overline{X} + 2ctr V \right], \tag{3.13} \]

where covariant derivatives are defined as

\[ \overline{D} \overline{X} = \overline{D} \overline{X} + e^{-2V} \left( D e^{2V} \right) \overline{X}, \]
\[ \overline{D} \overline{X} = \overline{D} \overline{X} - e^{2V} \left( D e^{-2V} \right). \tag{3.14} \]

It can be shown that the initial matrix action (3.12) and its gauge-invariant analogue (3.13) possess \( N = 2 \) superconformal symmetry SU(1,\( \mathbb{I} \)).

Using the component expansions \( \overline{X} = \theta + \ldots \), the Wess–Zumino gauge \( V = \theta \overline{A} A(t) \) and eliminating auxiliary fields, we obtain the following component action

\[ S_c^{N=2} = \frac{1}{2} \int dt \left[ \text{tr} \nabla^2 \nabla^2 + i \left( \nabla \nabla - \nabla \overline{Z} \right) \right. \]
\[ \left. + 2ctr A + i \text{tr} \left( \overline{\nabla} \Psi - \overline{\nabla} \Psi \right) \right], \tag{3.15} \]

Here \( \nabla \Psi = \Psi + i[A, \Psi], \nabla \overline{\Psi} = \overline{\Psi} + i[\overline{A}, \overline{\Psi}] \), and \( \nabla \overline{\nabla} \) are defined in (2.9). It is easy to show that the bosonic limit of (3.15) coincides with the action of the rational Calogero model in the gauge-invariant formulation (2.8). Thus, we have obtained a new \( N = 2 \) supersymmetric extension of the \( n \)-particle Calogero model with \( n \) physical bosons and \( 2n^2 \) fermions \( \Psi^a, \overline{\Psi}^a \), unlike the standard \( N = 2 \) Calogero system with \( 2n \) fermions proposed in [8].

Note that, after the additional gauge fixing \( Z_a = \overline{Z}^a \), the constraints \( (Z_a)^2 = c - R_a \) contain extra fermionic terms \( R_a \equiv [\Psi^a, \overline{\Psi}^a], (R_a)^{2n-1} \equiv 0 \). At present, it is not clear how to interpret such a proliferation of fermionic fields. Perhaps, their number could be reduced by implementing a new fermionic gauge invariance similar to the well-known \( \kappa \)-symmetry.

To deduce \( N = 2 \) superextension of Calogero–Sutherland model, one proceeds from the gauged superfield sigma-model type action

\[ S_{CS}^{N=2} = \frac{1}{2} \int dt d\theta d\bar{\theta} \left[ \text{tr} \left( \overline{D} \overline{X} e^{2V} D X e^{2V} \right) \right. \]
\[ \left. - \overline{D} e^{2V} \overline{X} + 2ctr V \right], \tag{3.16} \]

Passing over the same steps as in the rational case, we arrive at the component action

\[ S_{CS}^{N=2} = \frac{1}{2} \int dt \left[ \text{tr} \left( X^{-1} \nabla^2 X^{-1} \nabla X \right) \right. \]
\[ \left. + i \left( \overline{Z} \nabla Z - \nabla \overline{Z} \right) + 2ctr A \right. \]
\[ \left. + i \text{tr} \left( X^{-1} \overline{\nabla} X^{-1} \overline{\nabla} \Psi - X^{-1} \overline{\nabla} \Psi X^{-1} \Psi \right) \right] \]
\[ - \frac{1}{2} \text{tr} \left( X^{-1} \overline{\nabla} X^{-1} \overline{\nabla} \Psi X^{-1} \Psi X^{-1} \Psi \right), \tag{3.17} \]

In the bosonic limit, it becomes the gauge–invariant action of the Calogero–Sutherland model (2.10). An alternative superspace formulation of both models has been developed in [21].

4. MANY-PARTICLE \( N = 4 \) SUPERSYMMETRIC SYSTEMS

The universal approach to superfield formulations of \( N = 4 \) mechanics models is the method of \( N = 4 \), \( d = 1 \) harmonic superspace [22], which is \( d = 1 \) version of the \( N = 2 \), \( d = 4 \) harmonic superspace [23]. Unlike the ordinary \( N = 4, d = 1 \) superspace with the coordinates \( (t, \theta^i, \overline{\theta}^i) \), the \( N = 4, d = 1 \) harmonic superspace is parameterized by the coordinates \( (t, \theta^i, \overline{\theta}^i, u^-) \), where \( \theta^i = \theta^i u^- \), \( \overline{\theta}^i = \overline{\theta}^i u^+ \), and \( u^+, u^- \) are \( SU(2) \)-harmonics which parameterize \( 2 \)-sphere \( S^2 \). An important property of harmonic superspace is that it has a harmonic analytic subspace, including only half the original Grassmann variables, \( (\zeta, u) = (t, \theta^i, \overline{\theta}^i, u^+, u^-) \), \( t = t + i(\theta^i \overline{\theta}^i - \overline{\theta}^i \theta^i) \). This analytic superspace is closed under \( N = 4 \) supersymmetry.

All \( N = 4, d = 1 \) multiplets can be described by harmonic supersfiels. In particular, the \( N = 4 \) multiplet \((1, 4, 3)\) can be represented as a real harmonic superfield \( \mathcal{X}(t, \theta^i, \overline{\theta}^i, u^-) \) subjected to certain constraints (see details in [22]), or as an analytic prepotential \( \mathcal{V}(\zeta, u) \) defined through the integral representation

\[ \mathcal{X}(t, \theta^i, \overline{\theta}^i) = \int du^- \mathcal{V}(t, \theta^i, \overline{\theta}^i, u^-) |u^- = \theta^i \overline{\theta}^i - \overline{\theta}^i \theta^i \], \tag{4.18} \]

up to gauge transformations \( \delta \mathcal{V} = D^+ \lambda^- \), with the local analytic parameter \( \lambda^- (\zeta, u) \). In this section, we also use the \( N = 4 \) hypermultiplet described by complex analytical superfields \( \mathcal{X}^+, \overline{\mathcal{X}}^+ \) subjected to the constraint
$D^+\mathcal{F}^+ = 0$, where $D^+ = u^+\partial/\partial u^- + 2i\theta\overline{\theta}\partial \mathcal{A}_j$ is the analyticity-preserving harmonic derivative (in the analytical basis). Gauge fields are accommodated by the unconstrained analytic gauge prepotential $V^{++}$. Gauge transformation are realized on this superfield as

$$V^{++} = e^{i\mathcal{A}}V^{++}e^{-i\mathcal{A}} - ie^{i\mathcal{A}}(D^+ e^{-i\mathcal{A}}),$$  \hspace{1cm} (4.19)

where $\lambda_{ab}(\xi, u^\pm) \in u(n)$ is an Hermitian analytic matrix parameter. Using this gauge freedom, we can choose the Wess–Zumino gauge $V^{++} = 2i\theta\overline{\theta}\mathcal{A}(t_a)$. 

### 4.1. $\mathcal{N} = 4$ Supersymmetric Calogero Model

The matrix superfield action

$$S^{N=4} = \sum_{\nu} S_{\nu}^{N=4} + S_{\nu}^{N=4} + S_{\nu}^{N=4}$$  \hspace{1cm} (4.20)

possesses the most general $\mathcal{N} = 4$, $d = 1$ superconformal symmetry $D(2,1;\alpha)$ provided that the items in (4.20) are of the form

$$S_{\nu}^{N=4} = \frac{1}{4(1 + \alpha)} \int \mu_A(\text{tr} \mathcal{Z}^2) - \frac{1}{2} \int \mu_A(\text{tr} V^2),$$

$$S_{\nu}^{N=4} = \frac{1}{2} \mu_A(\text{tr} V^2),$$

$$S_{\nu}^{N=4} = -i \int \mu_A(-2) \text{tr} V^{++},$$  \hspace{1cm} (4.21)

where $\mu_A$ and $\mu_A(-2)$ are integration measures in the full and analytic harmonic superspaces. All superfields in (4.21) are defined by the constraints employing derivatives which are covariant with respect to local $u(n)$-transformations,

$$\mathcal{X} = e^{i\mathcal{A}}e^{-i\mathcal{A}}, \mathcal{X}^{++} = e^{i\mathcal{A}}\mathcal{X}^{++},$$

$$e^{i\mathcal{A}} = e^{i\mathcal{A}}e^{-i\mathcal{A}},$$  \hspace{1cm} (4.22)

e.g., $D^+\mathcal{F}^+ \rightarrow D^+\mathcal{F}^+ + D^+\mathcal{F}^+ + iV^{++}\mathcal{F}^+$. In addition, the superfield $V$ is a real analytic prepotential for the $u(n)$-singlet superfield $\mathcal{X}_0 = \text{tr}(\mathcal{X})$. They are related by the integral transform (4.18).

Consider the choice $\alpha = -1/2$, for which $D(2,1;\alpha) \sim \text{osp}(4|2)$. In Wess–Zumino gauge and after eliminating a part of the auxiliary fields, the action (4.20) takes the form

$$S^{N=4}_N = -\frac{1}{4} \int d\tau \left[ A^\tau X \nabla X + 2cA \right]$$

$$+ \nu_0 (Z^i Z^i) + i X_0 (Z^i \nabla Z^i - \nabla Z^i Z^i)$$

$$+ \frac{1}{2} \int d\tau \left[ (\overline{\Psi}_k \nabla \Psi_k - \nabla \overline{\Psi}_k \Psi_k) - i X_0 (\overline{\Psi}_k \Psi_k) \right],$$  \hspace{1cm} (4.23)

where $X_0 := \text{tr}(X)$, $\Psi_0^i := \text{tr}(\Psi_i)$, $\overline{\Psi}_0^i := \text{tr}(\overline{\Psi}_i)$. After gauge-fixing of the residual gauge symmetry, eliminating the fields $A^\alpha, a \neq b$, and a proper field redefinition, the bosonic part of the action can be written as

$$S^{N=4}_N = -\frac{1}{4} \int d\tau \left[ \sum_a (\dot{\xi}_a + i \sum_c (Z^c \dot{Z}^c - \dot{Z}^c Z^c)$$

$$- \sum_{a \neq b} (\nabla X_0) (\dot{S}_{a,b}) - \frac{1}{2} \text{tr} (\dot{S}_S) \right],$$  \hspace{1cm} (4.24)

where $S_{a,b} := Z^a_0 Z^b_0$, $\dot{S}_S := \sum (S_{a,b}) - \frac{1}{2} \delta_{ab}(S_{a,b})$ and the fields $Z^a_0$ obey the constraints $Z^a_0 Z^a_0 = c$ (for any $a$). The Wess–Zumino term for $Z$-variables in (4.24) generates Dirac brackets $[Z^a_0, Z^b_0] = i\delta^{ab}\delta_{0i}$, which as a consequence imply the relation

$$[[S_{a,b}, S_{a,b}]]_\mu = i\delta_{ab} \left[ \delta(S_{a,b}) - \delta(S_{a,b}) \right].$$  \hspace{1cm} (4.25)

In other words, for each value of the index $a$, the quantities $S_{a,b}$ form mutually commuting algebras $u(2)$, and $(\dot{S}_S)$ is the conserved Noether SU(2)-charge of this system.

Unlike the $\mathcal{N} = 2$ cases, not all out of the $d = 1$ fields $Z_0^a$ turn out to be auxiliary: after quantization, they become U(2)-spin degrees of freedom (i.e. harmonics in the target space). In addition, the quantity $\text{tr} S S$ is an integral of motion that generates in the $\mathcal{N} = 4$ case a conformal potential in the center-of-mass sector. Modulo this extra conformal potential, the bosonic limit of the $\mathcal{N} = 4$ system constructed coincides with the integrable U(2)-spin Calogero model [3].

There exists other types of superextensions of the $n$-particle Calogero model with $su(n)$ or $so(n)$ spin variables [24, 25]. Here, the $su(n)$ spin variables can be removed by a Hamiltonian reduction, keeping only the $\mathcal{N}n^2$ fermions for any number $\mathcal{N}$ of supersymmetries.

### 4.2. $\mathcal{N} = 4$ Calogero–Sutherland Models

The main distinguishing feature of this system is the choice of the non-linear sigma-model type action for $\mathcal{X}$ in (4.20),

$$S^{N=4}_N = -\frac{1}{4} \int \mu_A(\text{tr} \mathcal{X}),$$  \hspace{1cm} (4.26)

with preserving the form of two other terms in (4.20), (4.21). The full structure of the component action is restored by the same procedure as in the case of rational Calogero. The number of physical fermions is again $4n^2$. The action (4.26) has only “flat” $\mathcal{N} = 4, d = 1$ supersymmetry and SU(2) $R$-symmetry.
Introducing new variables $q_a$ through the replacement $x_a = e^{2\phi_a}$ brings the bosonic part of the action to the form

$$S_{CS,b}^{N=4} = \frac{1}{2} \int dt \sum_{a,b} (\mathcal{S}_a^{(ik)}(\mathcal{S}_b)_{(jk)}) \text{tr}(X^2) - \mathcal{Z}_a^2 \mathcal{Z}_b^2 - \sum_{a,b} \left( (\mathcal{S}_a)^k_b (\mathcal{S}_b)^l_a \right) \right), \quad \text{(4.27)}$$

where $\text{Tr}(X^2) = \sum a e^{2\phi_a}$, $X_a = \sum_c e^{\phi_c}$, the constraints $\mathcal{Z}_a^2 = c$ are satisfied for each $a$ and

$$S_{CS,b}^{N=4} = \frac{1}{2} \int dt \left[ \sum (\mathcal{S}_a)^k_b (\mathcal{S}_b)^l_a - (\mathcal{Z}_a^2)^k_b (\mathcal{Z}_b^2)^l_a \right]$$

Therefore, modulo the last term, the action (4.27) describes the hyperbolic $U(2)$-spin Calogero–Sutherland system [3].

The choice of the action $S_{WZ}$ in (4.21) for $N=4$ rational Calogero model was mainly motivated by superconformal invariance. In the hyperbolic case, such symmetry is absent from the very beginning. In particular, the action (4.26) for $\mathcal{F}$ has no longer this invariance, and there is no reason to insist on it in other parts of the total action. Therefore, in the action (4.20) it is natural to choose, instead of (4.21), the simplest action for the multiplets (4, 4, 4)

$$S_{WZ}^{N=4} = -\frac{1}{2} \int dt \mu^{(2)}_{ A} \mathcal{F}^{2} \mathcal{Z}_a^2 \quad \text{(4.29)}$$

The new total action in its bosonic sector yields the “pure” hyperbolic $U(2)$-spin Calogero–Sutherland system for any $n$, without any additional interaction. The coordinate of the center of mass is completely separated and is described by a free action in this model.

Also for the Calogero–Sutherland models, there exists an $N=4$ supersymmetric extension free of spin variables and still containing $4n^2$ fermions [26].

5. CONCLUSIONS

We have described a universal method of constructing supersymmetric extensions of Calogero-type models based on the superfield gauging procedure. This method leads to a non-standard supersymmetrization with $N n^2$ physical fermionic fields. Using it, we constructed new $N=2$ and $N=4$ superfield systems containing rational Calogero models and hyperbolic Calogero–Sutherland systems as bosonic limits for $N=2$ case and their $U(2)$-spin analogs for $N=4$ case.

We finish by listing some further possible tasks in the framework of the approach proposed:

— Studying the classical and quantum integrability of new supersymmetric Calogero models;

— Considering the possibilities of representing spin variables in various $N=4$ Calogero systems by other $\mathcal{N} = 4, d = 1$ multiplets, for example, multiplets $(2, 4, 2)$ or $(3, 4, 1)$;

— Generalization of the gauge approach to the case of $N=4$ “weak” supersymmetries $SU(2|1)$ [27–29] and similar deformed versions of $N=8$ supersymmetry [30], with some additional oscillator-type terms;

— Quantization of all these models like it was recently done in [31] for Calogero–Moser systems with $SU(2|1)$ supersymmetry;

— Reproducing, by the superfield gauging method, the multiparticle systems constructed in [24, 25] in the Hamiltonian on-shell approach for arbitrary $\mathcal{N}$;

— Supersymmetrizing other integrable many-particle models from the list of [6], e.g., trigonometric Calogero–Sutherland models, elliptic models, etc.

FUNDING

E.I. and S.F. thank the Russian Science Foundation, grant no. 16-12-10306, for a partial support.

REFERENCES

1. F. Calogero, J. Math. Phys. 10, 2191 (1969); F. Calogero, J. Math. Phys. 12, 419 (1971).
2. J. Moser, Adv. Math. 16, 197 (1975).
3. A. P. Polychronakos, J. Phys. A 39, 12793 (2006).
4. G. W. Gibbons and P. K. Townsend, Phys. Lett. B 454, 187 (1999).
5. E. Ivanov, S. Krivonos, and J. Niederle, Nucl. Phys. B 677, 485 (2004).
6. M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71, 313 (1981); M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 94, 313 (1983).
7. B. Sutherland, J. Math. Phys. 12, 246 (1971); B. Sutherland, Phys. Rev. A 5, 1372 (1972).
8. D. Z. Freedman and P. F. Mende, Nucl. Phys. B 344, 317 (1990).
9. P. Desrosiers, L. Lapointe, and P. Mathieu, Nucl. Phys. B 606, 547 (2001).
10. N. Wyllard, J. Math. Phys. 41, 2826 (2000).
11. S. Bellucci, A. Galajinsky, and E. Latini, Phys. Rev. D: Part. Fields 71, 044023 (2005).
12. E. Witten, Nucl. Phys. B 340, 281 (1990).
13. R. Dijkgraaf, H. L. Verlinde, and E. P. Verlinde, Nucl. Phys. B 352, 59 (1991).
14. S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Phys. Rev. D: Part. Fields 79, 105015 (2009).
15. S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Phys. Rev. A 45, 173001 (2012).
16. S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Nucl. Phys. B 944, 11463 (2019).
17. F. Delduc and E. Ivanov, Nucl. Phys. B 753, 211 (2006).
18. A. P. Polychronakos, Phys. Lett. B 266, 29 (1991).
19. A. Gorsky and N. Nekrasov, Teor. Mat. Fiz. 100, 97 (1994).
20. V. De Alfaro, S. Fubini, and G. Furlan, Nuovo Cim. A 34, 569 (1976).
21. S. Krivonos, O. Lechtenfeld, and A. Sutulin, (2019). arXiv:1912.05989 [hep-th].
22. E. Ivanov and O. Lechtenfeld, J. High Energy Phys. 2003, 73 (2003).
23. A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, and E. Sokatchev, Classical Quantum Gravity 1, 469 (1984).
24. S. Krivonos, O. Lechtenfeld, and A. Sutulin, Phys. Lett. B 784, 137 (2018); S. Krivonos, O. Lechtenfeld, and A. Sutulin, Phys. Lett. B 790, 191 (2019).
25. S. Krivonos, O. Lechtenfeld, A. Provorov, and A. Sutulin, Phys. Lett. B 791, 385 (2019).
26. S. Krivonos and O. Lechtenfeld, Phys. Rev. D 101, 086010 (2020).
27. S. Bellucci and A. Nersessian, Phys. Rev. D: Part. Fields 67, 065013 (2003).
28. A. V. Smilga, Phys. Lett. B 585, 173 (2004).
29. E. Ivanov and S. Sidorov, Classical Quantum Gravity 31, 075013 (2014).
30. E. Ivanov, O. Lechtenfeld, and S. Sidorov, J. High Energy Phys., No. 11, 31 (2016); E. Ivanov, O. Lechtenfeld, and S. Sidorov, J. High Energy Phys., No 8, 193 (2018).
31. S. Fedoruk, E. Ivanov, O. Lechtenfeld, and S. Sidorov, J. High Energy Phys., No. 4, 43 (2018).