Independent Distributions on a Multi-Branching AND-OR Tree of Height 2

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Abstract

We investigate an AND-OR tree $T$ and a probability distribution $d$ on the truth assignments to the leaves. Tarsi (1983) showed that if $d$ is an independent and identical distribution (IID) such that probability of a leaf having value 0 is neither 0 nor 1 then, under a certain assumptions, there exists an optimal algorithm that is depth-first. We investigate the case where $d$ is an independent distribution (ID) and probability depends on each leaf. It is known that in this general case, if height is greater than or equal to 3, Tarsi-type result does not hold. It is also known that for a complete binary tree of height 2, Tarsi-type result certainly holds. In this paper, we ask whether Tarsi-type result holds for an AND-OR tree of height 2. Here, a child node of the root is either an OR-gate or a leaf: The number of child nodes of an internal node is arbitrary, and depends on an internal node. We give an affirmative answer. Our strategy of the proof is to reduce the problem to the case of directional algorithms. We perform induction on the number of leaves, and modify Tarsi’s method to suite height 2 trees. We discuss why our proof does not apply to height 3 trees.

Keywords: Depth-first algorithm; Independent distribution; Multi-branching tree; Computational complexity; Analysis of algorithms
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1 Introduction

Depth-first algorithm is a well-known type of tree search algorithm. Algorithm $A$ on a tree $T$ is depth-first if the following holds for each internal node $x$ of $T$:

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Once \( A \) probes a leaf that is a descendant of \( x \), \( A \) does not probe leaves that are not descendant of \( x \) until \( A \) finds value of \( x \).

With respect to analysis of algorithm, the concept of depth-first algorithm has an advantage that it is well-suited for induction on subtrees. An example of such type of induction may be found in our former paper [12].

Thus, given a problem on a tree, it is theoretically interesting question to ask whether there exists an optimal algorithm that is depth-first. Here, cost denotes (expected value of) the number of leaves probed during computation, and an algorithm is optimal if it achieves the minimum cost among algorithms considered in the question.

If an associated evaluation function of a mini-max tree is bi-valued and the label of the root is MIN (MAX, respectively), the tree is equivalent to an AND-OR tree (an OR-AND tree). In other words, the root is labeled by AND (OR), and AND layers and OR layers alternate. Each leaf has a truth value 0 or 1, where we identify 0 with false, and 1 with true.

A fundamental result on optimal algorithms on an AND-OR trees is given by Tarsi. A tree is balanced (in the sense of Tarsi) if (1) any two internal nodes of the same depth (distance from the root) have the same number of child nodes, and (2) all of the leaves have the same depth. A probability distribution \( d \) on the truth assignments to the leaves is an independent distribution (ID) if the probability of each leaf having value 0 depends on the leaf, and values of leaves are determined independently. If, in addition, all the probabilities of the leaves are the same (say, \( p \)), \( d \) is an independent and identical distribution (IID). Algorithm \( A \) is directional [3] if there is a fixed linear order of the leaves and for any truth assignment, the order of probing by \( A \) is consistent with the order.

The result of Tarsi is as follows. Suppose that AND-OR tree \( T \) is balanced and \( d \) is an IID such that \( p \neq 0,1 \). Then there exists an optimal algorithm that is depth-first and directional [13].

This was shown by an elegant induction. For an integer \( n \) such that \( 0 \leq n \leq h = \) (the height of the tree), an algorithm \( A \) is called \( n \)-straight if for each node \( x \) whose distance from the leaf is at most \( n \), once \( A \) probes a leaf that is a descendant of \( x \), \( A \) does not probe leaves that are not descendant of \( x \) until \( A \) finds value of \( x \). The proof by Tarsi is induction on \( n \). Under an induction hypothesis, we take an optimal algorithm \( A \) that is \((n - 1)\)-straight but not \( n \)-straight. Then we modify \( A \) and get two new algorithms. By the assumption, the expected cost by \( A \) is not greater than the costs by the modified algorithms, thus we get inequalities. By means of the inequalities, we can eliminate a non-\( n \)-straight move from \( A \) without increasing cost.

In the above mentioned proof by Tarsi, derivation of the inequalities heavily depends on the hypothesis that the distribution is an IID. In the same paper, Tarsi gave an example of an ID on an AND-OR tree of the following properties: The tree is of height 4, not balanced, and no optimal algorithm is depth-first. Later, we gave another example of such an ID where a tree is balanced, binary and height 3 [10].

On the other hand, as is observed in [10], in the case of a balanced binary AND-OR tree of height 2, Tarsi-type result holds for IDs in place of IID's.
Table 1 summarizes whether Tarsi-type result holds or not. In the table, we assume that an AND-OR tree is balanced, and that the probability of each leaf having value 0 is neither 0 nor 1.

In this paper, we ask whether Tarsi-type result holds for the case where a tree is height 2 and the number of child nodes is arbitrary. We give an affirmative answer.

We show a slightly stronger result. We are going to investigate a tree of the following properties. The root is an AND-gate, and a child node of the root is either an OR-gate or a leaf. The number of child nodes of an internal node is arbitrary, and depends on an internal node. Figure 1 is an example of such a tree. Now, suppose that an ID on the tree is given and that at each leaf, the probability of having value 0 is neither 0 nor 1. Under these mild assumptions, we show that there exists an optimal algorithm that is depth-first and directional.

Our strategy of the proof is to reduce the problem to the case of directional algorithms. We perform induction on the number of leaves, and modify Tarsi’s method to go along with properties particular to height 2 trees. The first author and the third author showed, in their talk [8], a restricted version of the present.
result in which only directional (possibly non-depth-first) algorithms are taken into consideration. In the talk, the core of the strategy is suggested by the first author. The second author reduced the general case, in which non-directional algorithms are taken into consideration, to the case of directional algorithms.

The paper [9] gives an exposition of the background. It is a short survey on the work of Liu and Tanaka [2] and its subsequent developments [11, 12, 6]. The paper [5] is also in this line. Some classical important results by 1980’s may be found in the papers [1, 3, 4, 13] and [7].

We introduce notation in section 2. We show our result in section 3. In section 4, we discuss why our proof does not apply to the case of height 3, and discuss future directions.

2 Preliminaries

If $T$ is a balanced tree (in the sense of Tarsi, see Introduction) and there exists a positive integer $n$ such that all of the internal nodes have exactly $n$ child nodes, we say $T$ is a complete $n$-ary tree.

For algorithm $A$ and distribution $d$, we denote (expected value of) the cost by $\text{cost}(A, d)$.

We are interested in a multi-branching AND-OR tree of height 2, where a child node of the root is either a leaf or an OR-gate, and the number of leaves depend on each OR-gate. For simplicity of notation, we investigate a slightly larger class of trees.

Hereafter, by “a multi-branching AND-OR tree of height at most 2”, we denote an AND-OR tree $T$ of the following properties.

- The root is an AND-gate.
- We allow an internal node to have only one child, provided that the tree has at least two leaves.
- All of the child nodes of the root are OR-gates.

The concept of “multi-branching AND-OR tree of height at most 2” include the multi-branching AND-OR trees of height 2 in the original sense (because an OR-gate of one leaf is equivalent to a leaf), the AND-trees of height 1 (this case is achieved when all of the OR-gates have one leaf) and the OR-trees of height 1 (this case is achieved when the root has one child). The simplest case is a tree of just two leaves, and this case is achieved exactly in either of the following two. (1) The tree is equivalent to a binary AND-tree of height 1: (2) The tree is equivalent to a binary OR-tree of height 1.

Suppose that $T$ is a multi-branching AND-OR tree of height at most 2.

- We let $x_{\lambda}$ denote the root.
- By $r$ we denote the number of child nodes of the root. $x_0, \ldots, x_{r-1}$ are the child nodes of the root.
For each $i \ (0 \leq i < r)$, we let $a(i)$ denote the number of child leaves of $x_i$. $x_{i,0}, \ldots, x_{i,a(i)-1}$ are the child leaves of $x_i$.

Figure 2 is an example of such a tree, where $r = 5$, $a(0) = 1$ and $a(4) = 3$. The tree is equivalent to the tree in Figure 1.

Figure 2: Example of an AND-OR tree of height at most 2

Suppose that $d$ is an ID on $T$. For each $i \ (0 \leq i < r)$ and $j \ (0 \leq j < a(i))$, we use the following symbols.

- $p(i,j)$ is the probability of $x_{i,j}$ having value 0.
- $p(i)$ is the probability of $x_i$ having value 0.
- Since $d$ is an ID, its restriction to the child nodes of $x_i$ is an ID on the subtree whose root is $x_i$. Here, we denote it by the same symbol $d$. Then we define $c(i)$ as follows.

$$c(i) = \min_A \text{cost}(A, d),$$

where $A$ runs over algorithms finding value of $x_i$, and $\text{cost}(A, d)$ is expected cost.

Thus, $p(i)$ is the product $p(i,0) \cdots p(i,a(i)-1)$. If $a(i) = 1$ then $c(i) = 1$. If $a(i) \geq 2$ and we have $p(i,0) \leq \cdots \leq p(i,a(i)-1)$, then $c(i) = 1 + p(i,0) + p(i,0)p(i,1) + \cdots + p(i,0) \cdots p(i,a(i)-1)$.

Tarsi investigated a depth-first algorithm that probes the leaves from left to right, skipping a leaf whenever there is sufficient information to evaluate one of its ancestors, and he called it SOLVE. We investigate a similar algorithm depending on a given independent distribution.
Definition 1. Suppose that $T$ is a multi-branching AND-OR tree of height at most 2.

1. Suppose that $d$ is an ID on $T$ and for each $i$ ($0 \leq i < r$) and $j$ ($0 \leq j < a(i)$), we have $p(i, j) \neq 0, 1$.

By $\text{SOLVE}_d$, we denote the unique depth-first directional algorithm such that the following hold for all $i, s, j, k$ ($0 \leq i < r$, $0 \leq s < r$, $0 \leq j < a(i)$, $0 \leq k < a(i)$).

(a) If $c(i)/p(i) < c(s)/p(s)$ then priority of (probing the leaves under) $x_i$ is higher than that of $x_s$.
(b) If $c(i)/p(i) = c(s)/p(s)$ and $i < s$ then priority of $x_i$ is higher than that of $x_s$.
(c) If $p(i, j) < p(i, k)$ then priority of (probing) $x_{i, j}$ is higher than $x_{i, k}$.
(d) If $p(i, j) = p(i, k)$ and $j < k$ then priority of $x_{i, j}$ is higher than $x_{i, k}$.

2. Suppose that we remove some nodes (except for the root of $T$) from $T$, and if a removed node has descendants, we remove them too. Let $T'$ be the resulting tree. Suppose that $\delta$ is an ID on $T'$ and for each $i$ ($0 \leq i < r$) and $j$ ($0 \leq j < a(i)$) such that $x(i, j)$ is a leaf of $T'$, we have $p(i, j) \neq 0, 1$.

By $\text{SOLVE}^{T'}_d$, we denote the unique depth-first directional algorithm of the following properties. For all $i, s, j, k$ ($0 \leq i < r$, $0 \leq s < r$, $0 \leq j < a(i)$, $0 \leq k < a(i)$), if $x_i$ and $x_s$ are nodes of $T'$, the above-mentioned assertions (a) and (b) hold; and if $x_{i, j}$ and $x_{i, k}$ are leaves of $T'$, the above-mentioned assertions (c) and (d) hold.

Lemma 1. Suppose that $T$ is a multi-branching AND-OR tree of height at most 2. Suppose that $d$ is an ID on $T$ and for each $i$ ($0 \leq i < r$) and $j$ ($0 \leq j < a(i)$), we have $p(i, j) \neq 0, 1$. Then $\text{SOLVE}_d$ achieves the minimum cost among all of the algorithms (depth-first or non-depth-first, directional or non-directional). To be more precise, if $A$ is a depth-first directional algorithm then $\text{cost}(\text{SOLVE}_d, d) \leq \text{cost}(A, d)$.

Proof. It is straightforward.

3 Result

Theorem 2. Suppose that $T$ is a multi-branching AND-OR tree of height at most 2. Suppose that $d$ is an ID on $T$ and for each $i$ ($0 \leq i < r$) and $j$ ($0 \leq j < a(i)$), we have $p(i, j) \neq 0, 1$. Then $\text{SOLVE}_d$ achieves the minimum cost among all of the algorithms (depth-first or non-depth-first, directional or non-directional). Therefore, there exists a depth-first directional algorithm that is optimal among all of the algorithms.

Proof. We perform induction on the number of leaves. The base cases are the binary AND-trees of height 1 and the binary OR-trees of height 1. In general, if $T$ is equivalent to a tree of height 1, the assertion of the theorem clearly holds.
To investigate induction step, we assume that $T$ has at least three leaves. Our induction hypothesis is that for any multi-branching AND-OR tree $T'$ of height at most 2, if the number of leaves of $T'$ is less than that of $T$ then the assertion of theorem holds for $T'$.

We fix an algorithm $A$ that minimizes $\text{cost}(A, d)$ among all of the algorithms (depth-first or non-depth-first, directional or non-directional).

Case 1: At the first move of $A$, $A$ makes a query to a leaf $x_{i,0}$ such that $a(i) = 1$. In this case, if $A$ finds that $x_{i,0}$ has value 0 then $A$ returns 0 and finish. Otherwise, $A$ calls a certain algorithm (say, $A'$) on $T - x_i$, that is, the tree given by removing $x_i$ and $x_{i,0}$ from $T$. The probability distribution given by restricting $d$ to $T - x_i$ is an ID, and a probability of any leaf is neither 0 nor 1.

Therefore, by induction hypothesis, without loss of generality, we may assume that $A'$ is a depth-first directional algorithm on $T - x_i$. Therefore, $A$ is a depth-first directional algorithm. Hence, by Lemma 1, the same cost as $A$ is achieved by $\text{SOLVE}_d$.

Case 2: Otherwise. At the first move of $A$, $A$ makes a query to a leaf $x_{i,j}$ such that $a(i) \geq 2$.

Let $T_0 := T - x_{i,j}$, the tree given by removing $x_{i,j}$ from $T$. In addition, let $T_1 := T - x_i$, the tree given by removing $x_i$ and all of the leaves under $x_i$ from $T$. Here, $T_0$ and $T_1$ inherit all of the indices (for example, “3,1” of $x_{3,1}$) from $T$.

If $T_1$ is empty then $T$ is equivalent to a tree of height 1, and this case reduces to our observation in the base case. Thus, throughout rest of Case 2, we assume that $T_1$ is non-empty.

If $A$ finds that $x_{i,j}$ has value 0 then $A$ calls a certain algorithm (say, $A_0$) on $T_0$.

If $A$ finds that $x_{i,j}$ has value 1 then $A$ calls a certain algorithm (say, $A_1$) on $T_1$.

For each $s = 0, 1$, let $d[s]$ be the restriction of $d$ to $T_s$. In the same way as Case 1, without loss of generality, we may assume that $A_s$ is $\text{SOLVE}_{d[s]}^T$ on $T_s$.

Hence, there is a permutation $X = \langle x_{i,s(0)}, \ldots, x_{i,s(a(i) - 2)} \rangle$ of the leaves under $x_i$ except $x_{i,j}$, and there are possibly empty sequences of leaves, $Y = \langle y_0, \ldots, y_{k-1} \rangle$ and $Z = \langle z_0, \ldots, z_{m-1} \rangle$, with the following properties.

- The three sets $\{x_i\}$, $Y^* = \{y : y$ is a parent of a leaf in $Y\}$, and $Z^* = \{z : z$ is a parent of a leaf in $Z\}$ are mutually disjoint, and their union equals $\{x_0, \ldots, x_{r-1}\}$, the set of all child nodes of $x_\lambda$.
- The search priority of $A_0$ is in accordance with $YXZ$ (thus, $y_0$ is the first).
- The search priority of $A_1$ is in accordance with $YZ$.

Case 2.1: $Z$ is non-empty.

We are going to show that $Y$ is empty. Assume not. Let $B$ be the depth-first directional algorithm on $T$ whose search priority is $Yx_{i,j}XZ$. 

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Let $A_Y (A_X, A_Z)$ denote the depth-first directional algorithm on the subtree above by $Y (X, Z)$, where search priority is in accordance with $Y (X, Z)$. Thus, we may write $A_0$ as “$A_Y; A_X; A_Z$”, $A_1$ as “$A_Y; A_Z$”, and $B$ as “$A_Y; \text{Probe } x_{i,j}; A_X; A_Z$”.

We look at the following events. Recall that $Y^*$ is the set of all parent nodes of leaves in $Y$.

- $E_Y$: “At least one element of $Y^*$ has value 0.”
- $E_X$: “All of the elements of $X$ have value 0.”

Since the tree is height 2 and $Z$ is non-empty, in each of $A_0, A_1$ and $B$, the following holds: “$A_Y$ finds value of $x_\lambda$ if and only if $E_Y$ happens.”

In the same way, in $A_0$ and in $B$, under assumption that $A_X$ is called, $A_X$ finds value of the root if and only if $E_X$ happens.

Thus, flowcharts of $A$ and $B$ as Boolean decision trees are as described in Figure 3 and Figure 4, respectively.

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**Figure 3:** Flowchart of $A$ (Case 2.1, in the presence of $Y$)

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**Figure 4:** Flowchart of $B$ (in the presence of $Y$)

Therefore, letting $p_Y = \text{prob}[\neg E_Y]$ (that is, probability of the negation) and $p_X = \text{prob}[\neg E_X]$, the cost of $A$ and $B$ are as follows. In the following formulas, $C_Y$ denotes $\text{cost}(A_Y, d_Y)$, where $d_Y$ denotes the probability distribution given by restricting $d$ to $Y$. $C_X$ and $C_Z$ are similarly defined.
\[ \text{cost}(A, d) = 1 + p(i, j)\{C_Y + p_Y(C_X + p_X C_Z)\} + (1 - p(i, j))(C_Y + p_Y C_Z) \]
\[ = 1 + C_Y + p(i, j)p_Y(C_X + p_X C_Z) + (1 - p(i, j))p_Y C_Z \] 

\[ \text{cost}(B, d) = C_Y + p_Y\{1 + \{p(i, j)(C_X + p_X C_Z) + (1 - p(i, j))C_Z\}\} \]
\[ = C_Y + p_Y + p_Y p(i, j)(C_X + p_X C_Z) + p_Y(1 - p(i, j))C_Z \] 

Therefore, \( \text{cost}(A, d) - \text{cost}(B, d) = 1 - p_Y \). However, by our assumption on \( d \) that probability of each leaf (having value 0) is neither 0 nor 1, \( E_Y \) has positive probability, thus \( p_Y < 1 \). Thus \( \text{cost}(A, d) - \text{cost}(B, d) \) is positive, and this contradicts to the assumption that \( A \) achieves the minimum cost.

Hence, we have shown that \( Y \) is empty. Therefore, \( A \) is the following algorithm (Figure 5): "Probe \( x_{i,j} \). If \( x_{i,j} = 0 \) then perform depth-first directional search on \( T_0 = T - x_{i,j} \), where search priority is in accordance with \( XZ \). Otherwise (that is, \( x_{i,j} = 1 \)), perform depth-first directional search on \( T_1 = T - x_i \), where search priority is in accordance with \( Z \)."

![Figure 5: Flowchart of A (in the absence of Y)](image)

Thus, \( A \) is a depth-first directional algorithm. Hence, by Lemma 1 the same cost as \( A \) is achieved by \( \text{SOLVE}_d \).

Case 2.2: Otherwise. In this case, \( Z \) is empty. The proof is similar to Case 2.1. \( \square \)

4 Concluding remarks

4.1 Difference between height 2 case and height 3 case

As is mentioned in Introduction, the counterpart to Theorem 2 does not hold for the case of height 3. We are going to discuss why the proof of Theorem 2
does not work for the case of height 3.

Figure 6 is a complete binary OR-AND tree of height 3.

Figure 6: A binary OR-AND tree of height 3

Suppose that algorithm \( A \) is as follows. Let \( Y = \langle x_{100}, x_{101} \rangle \), \( X = \langle x_{001} \rangle \), and \( Z = \langle x_{010}, x_{011}, x_{110}, x_{111} \rangle \). At the first move, \( A \) probes \( x_{000} \). If \( x_{000} = 0 \) then \( A \) probes in accordance with order \( YXZ \). If \( x_{000} = 1 \) then \( A \) probes in accordance with order \( YZ \).

Let \( A'_Y \) be the algorithm on \( Y \) such that \( x_{100} \) has higher priority of probing than \( x_{101} \).

Suppose that an ID \( d \) on the tree is given, and that, at each leaf, probability of having value 0 is neither 0 nor 1. We investigate the following event.

\( E'_Y \): “\( x_{10} \) has value 0.”

On the one hand, by our assumption on ID \( d \), \( E'_Y \) has positive probability. On the other hand, whether \( E'_Y \) happens or not, \( A'_Y \) does not find value of \( x_\lambda \). In other words, probability of “\( A'_Y \) finds value of \( x_\lambda \)” is 0. Therefore, \( E'_Y \) is not equivalent to the assertion “\( A'_Y \) finds value of \( x_\lambda \)”.

Hence, counterpart to our observation in Case 2.1 of Theorem 2 does not work for the present setting.

4.2 Summary and future directions

Given a tree \( T \), let \( \text{IID}^+_{T} \) (\( \text{ID}^+_{T} \), respectively) denote the set of all IIDs on \( T \) (IDs on \( T \)) such that, at each leaf, probability having value 0 is neither 0 nor 1. Now we know the following.

1. (Tarsi \([13]\)) Suppose that \( T \) is a balanced AND-OR tree of any height, and that \( d \in \text{IID}^+_{T} \). Then there exists an optimal algorithm that is depth-first and directional.

2. (S. \([10]\)) Suppose that \( T \) is a complete binary OR-AND tree of height 3. Then there exists \( d \in \text{ID}^+_{T} \) such that no optimal algorithm is depth-first.
3. (Theorem 2) Suppose that $T$ is an AND-OR tree of height 2, and that $d \in ID_T^+$. Then there exists an optimal algorithm that is depth-first and directional.

Suppose that $T$ is a complete binary AND-OR tree of height $h \geq 3$. There is yet some hope to find a subset $D$ of $ID_T^+$ of the following properties.

- $IID_T^+ \subseteq D \subseteq ID_T^+$
- For each $d \in D$, there exists an optimal algorithm that is depth-first and directional.

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