Hierarchical wave function, Fock cyclic condition and spin-statistics relation in the spin-singlet fractional quantum Hall effect

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ABSTRACT

We construct the hierarchical wave function of the spin-singlet fractional quantum Hall effect, which turns out to satisfy Fock cyclic condition. The spin-statistics relation of the quasi-particles in the spin-singlet fractional quantum Hall effect is also discussed. Then we use particle-hole conjugation to check the wave function.

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1. INTRODUCTION

The hierarchical wave function of the spin-polarized fractional quantum Hall effect (FQHE) has been constructed by Haldane [1] and Halperin [2] and it also has been subjected to intensive studies in the last several years [3,4,5, etc.]. When the magnetic field is not strong enough, the electron spin may not be polarized. Halperin had proposed a class of state with half spins reversed which are spin-singlet states [6] and Haldane and Rezayi had proposed a spin-singlet state at a filling factor $\nu = \frac{1}{2}$ [7]. However the hierarchical wave function for the spin-non-polarized case, for example, spin-singlet FQHE (SFQHE), is not still fully understood as it was pointed out by Girvin [8]. In particular, it is much more difficult to obtain the hierarchical wave function of SFQHE subjected to the requirement of Fock cyclic condition (FCC) (we will discuss it in section 4). So it remains an interesting problem. The task of the present paper is to construct the hierarchical wave function based on Halperin spin-singlet state. However we shall point out that this hierarchical scheme does not include Haldane and Rezayi state [7].

We will use the projective coordinate to construct hierarchical wave function of SFQHE on the sphere. The projective coordinate has been used to construct hierarchical wave function in the spin-polarized FQHE on the sphere [9]. And here the same notation will be used as that in [9].

In the projective coordinates $x, y$, the metric $g$ of the sphere is given by

$$g_{\alpha\beta}(x) = \frac{1}{(1 + r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $r^2 = \frac{x^2 + y^2}{4R^2}$, and $R$ is the radius of the sphere. For simplicity, we assume the radius of the sphere $R = \frac{1}{2}$. The hamiltonian of electron in a magnetic monopole field is (the hamiltonian with Laplace-Beltrami operator ordering differs from this
hamiltonian by a constant) then

\[ H = \frac{2}{M_e} (1 + z\bar{z})^2 (P_z - eA_z)(P_{\bar{z}} - eA_{\bar{z}}) . \]

(1.2)

The magnetic monopole field \( eA_z \) is

\[ eA_z = -\frac{i\phi}{2} \frac{z}{1 + z\bar{z}} , \]

(1.3)

where \( \phi \) is the magnetic flux in the unit of the fundamental flux and it is an integer. \( P_z \) and \( P_{\bar{z}} \) in (1.3) are the operators

\[ P_z = -i\partial z_i , P_{\bar{z}} = -i\partial_{\bar{z}}_i . \]

(1.4)

We have put the Dirac singularity of the monopole field at \( z = \infty \). By solving the equation \( (P_{\bar{z}} - eA_{\bar{z}})\Psi = 0 \), the ground state of the electron is

\[ \psi = z^k (1 + z\bar{z})^{-\frac{\phi}{2}} , \]

(1.5)

where \( 0 \leq k \leq \phi \) with \( k \) being an integer (in order that the ground state is normalizable). The Laughlin wave function [10] shall be

\[ \Psi = \prod_{i<j} (z_i - z_j)^m \prod_i (1 + z_i\bar{z}_i)^{-\frac{\phi}{2}} , \]

(1.6)

where \( m \) is an odd integer. It is known that the state of FQHE on the sphere and plane is non-degenerate due to the reason that, by adding Coulomb interaction, the Landau degeneracy is lifted. Suppose that the system still has rotational symmetry, thus the ground must be rotationally invariant. Under the rotation, the coordinate
is transformed as

\[ z' = \frac{az + b}{cz + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SO(3), \quad (1.7) \]

which is generated by the rotations around three axes

\[
R_x = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 + \cos \alpha)^{1/2} & i(1 - \cos \alpha)^{1/2} \\ i(1 - \cos \alpha)^{1/2} & (1 + \cos \alpha)^{1/2} \end{pmatrix}, \\
R_y = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 - \cos \beta)^{1/2} & (1 - \cos \beta)^{1/2} \\ -(1 - \cos \beta)^{1/2} & (1 - \cos \beta)^{1/2} \end{pmatrix}, \\
R_z = \begin{pmatrix} \exp \frac{i\gamma}{2} & 0 \\ 0 & \exp -\frac{i\gamma}{2} \end{pmatrix}. \quad (1.8)\]

Under the rotation \( z' = \frac{az + b}{cz + d} \), the wave function is transformed into [9]

\[
R(a, b, c, d)\Psi(z_i) = \prod_i \left( \frac{cz_i + \bar{z}_i}{cz_i + d} \right)^{d_{ij}} \Psi\left( \frac{az_i + b}{cz_i + d} \right), \quad (1.9)
\]

where \( R \) is the corresponding quantum operator of the rotation. The transformation of \( z_i - z_j \) and \( 1 + z \bar{z} \) will be:

\[
z_i' - z_j' = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}, \\
1 + z_i' \bar{z}_i' = \frac{1 + z \bar{z}}{(cz + d)(\bar{c}z + d)}, \quad (1.10)
\]

and

\[
d_{ij} = \frac{z_i - z_j}{(1 + z_i \bar{z}_i)^{1/2}(1 + z_j \bar{z}_j)^{1/2}}, \quad (1.11)
\]

will be transformed as

\[
d_{ij}' = \left( \frac{cz_i + d}{cz_i + d} \right)^{1/2} \left( \frac{cz_j + d}{cz_j + d} \right)^{1/2} d_{ij}. \quad (1.12)
\]

Implementing the wave function being rotationally invariant, that is, \( R\Psi = \Psi \), one
gets

\[ \phi = m(N - 1) , \]  

(1.13)

where \( N \) is the number of the electrons. Then the wave function (1.6) is equal to \( \prod_{i<j} d_{ij}^m \) [1].

The hierarchical wave function can be constructed as follows (for the case on the plane, see [3]). The normalized wave function in the presence of quasi-particles at \( z'_\alpha \) and at the filling \( \nu = \frac{1}{m} \) is

\[ \Psi_e = \prod_{i<j} d_{ij}^m \prod_{i\alpha} d_{i\alpha} \prod_{\alpha<\beta} d_{\alpha\beta} \frac{1}{m} , \]  

(1.14)

The Laughlin type wave function of the quasi-particles is

\[ \Psi_q = \prod_{\alpha<\beta} d_{\alpha\beta} m (d_{\alpha\beta})^p , \]  

(1.15)

where \( p \) is a positive even integers. The hierarchical wave function of the electrons is given by

\[ \Psi(z_i) = \int \prod_{\alpha} dv_\alpha \Psi_e(z_i, z'_\alpha) \Psi_q(z'_\alpha) , \]  

(1.16)

where \( dv_\alpha = \frac{d^2_{\alpha}}{(1+z'_\alpha z_\alpha)^2} \) is rotationally invariant measure on the sphere. Requiring \( \Psi(z_i) \) to be rotationally invariant, \( R\Psi(z_i) = R \), we get the relation

\[ m(N_e - 1) + N_q = \phi , \]

\[ N_e - p(N_q - 1) = 0 , \]  

(1.17)

where \( N_e \) is the electron number and \( N_q \) is the quasi-particle number. From (1.17), one can derive the filling

\[ \nu = \frac{1}{m + \frac{1}{p}} . \]  

(1.18)

By considering the excited state of (1.15) and let those quasi-particles of the quasi-particles be condensed, we can get the next hierarchical state. By proceeding in this way, the general hierarchical state can be constructed.
We shall generalize the above method to construct the hierarchical wave function of SFQHE. In the next section we review some basic facts about FQHE with the layered structure, which is useful for the discussions in the following sections. In section 3, we discuss the spin-statistics relation of the quasi-particles in the layered FQHE. In section 4, we construct an hierarchical wave function of SFQHE and prove that the wave function satisfies FCC. In section 5, we try to give a physical explanation of the wave function obtained in section 4. In section 6, we apply the particle-hole conjugation operation to the wave function. Finally in section 7, we make the conclusion of the paper.

2. THE LAYERED FQHE

Halperin had proposed some states with half spin reversed [6],

$$\Psi_{mmn} = \prod_{i<j}^{N} [d(z_i, z_j)]^{m} [d(w_i, w_j)]^{m} [d(z_k, w_l)]^{n}, \quad (2.1)$$

where $m$ is an odd integer, $z_i$ are the coordinates of the up-spin electrons and $w_i$ are the coordinates of the down-spin electrons. We can also interpret $z_i$ as the coordinates of the up-layer electrons and $w_i$ as the coordinates of the down-layer electrons in a double layered FQHE. Now We would like to discuss a more general type wave function [11],

$$\Psi_{m_1, m_2, n} = \prod [d(z_i, z_j)]^{m_1} [d(w_i, w_j)]^{m_2} [d(z_k, w_l)]^{n}. \quad (2.2)$$

Because the up-layer electrons $z_i$ can be distinguished from the down-layer electrons $w_i$, so the wave function does not need to be completely anti-symmetrized. $\Psi_{m_1, m_2, n}$ is rotationally invariant [1,9], and we have

$$m_1(N_1 - 1) + nN_2 = nN_1 + m_2(N_2 - 1) = \phi, \quad (2.3)$$

because all electrons are exposed to the same magnetic field and in the lowest Landau level. $N_1$ ($N_2$) is the number of the up (down)-layer electrons and $\phi$ is the
magnetic flux. According to (2.3), the filling is \((N_1, N_2, \phi \text{ etc. are always assumed to be much larger than } 1)\)

\[
\nu = \frac{N_1 + N_2}{\phi} = \frac{m_1 + m_2 - 2n}{m_1 m_2 - n^2}. \tag{2.4}
\]

When \(m_1 = m_2 = m\), the filling is then \(\frac{2}{m+n}\). Now we introduce a two dimension lattice with bases

\[
e_i \cdot e_j = \Lambda_{i,j} = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix}. \tag{2.5}
\]

The bases of the inverse lattice is defined by \(e_i^* \cdot e_j = \delta_{i,j}\), and thus we have

\[
e_i^* \cdot e_j^* = \Lambda^{-1}_{i,j} = \frac{1}{m_1 m_2 - n^2} \begin{pmatrix} m_2 & -n \\ -n & m_1 \end{pmatrix}. \tag{2.6}
\]

The wave function \(\Psi_{m_1,m_2,n}\) can be written now as

\[
\Psi_{m_1,m_2,n} = \prod [d(z_i, z_j)]^{e_1 \cdot e_1} [d(w_i, w_j)]^{e_2 \cdot e_2} [d(z_k, w_l)]^{e_1 \cdot e_2}. \tag{2.7}
\]

The wave function with quasi-particles at \(z'_\alpha\) and \(w'_\alpha\) is

\[
\Psi_{m_1,m_2,n}(z'_\alpha, w'_\alpha) = \prod d(z_i - z'_\alpha) d(w_i - w'_\alpha) \Psi_{m_1,m_2,n}
\]

\[
= \prod d(z_i - z'_\alpha)^{e_1 \cdot e_1} d(w_i - w'_\alpha)^{e_2 \cdot e_2} \Psi_{m_1,m_2,n}. \tag{2.8}
\]

The equation (2.3)now becomes

\[
m_1(N_1 - 1) + nN_2 + N'_1 = \phi, \\
nN_1 + m_2(N_2 - 1) + N'_2 = \phi, \tag{2.9}
\]

where \(N'_1 (N'_2)\) is the number of the up(down)-layer quasi-particles. The plasma charge of the electron \(z_i (w_i)\) is \(e_1 (e_2)\) and the plasma charge of the quasi-particle
\( z'_\alpha (w'_\alpha) \) is \( e^*_i (e_2^*) \). The normalized wave function can be obtained by using plasma analogue on the sphere,

\[
\Psi_{m_1,m_2,n}(z'_\alpha, w'_\alpha)_{\text{nor}} = \prod_{i,\alpha} d(z_i - z'_\alpha) e^{e_1^* \cdot e_i^*} d(w_i - w'_\alpha) e^{e_2^* \cdot e_i^*} d(z'_\alpha - z'_\beta) e^{e_1^* \cdot e_2^*} d(w'_\alpha - w'_\beta) e^{e_2^* \cdot e_2^*} d(z'_\alpha - z'_\beta) e^{e_1^* \cdot e_2^*} \Psi_{m_1,m_2,n}.
\] (2.10)

The normalization constant of the above wave function will be independent on the coordinates of the quasi-particles in the limit of the quasi-particles being quite far away from each other. The statistics parameter \( \theta_{ij} \) (when exchanging two kinds of particles \( i \) and \( j \), we will get a phase \( e^{i\theta_{ij} \pi} \)) of the quasi-particle can be read from the wave function (2.10),

\[
\theta_{ij} = -e_i^* \cdot e_j^* = -\Lambda^{-1}.
\] (2.11)

The electric charge of the quasi-particle \( z'_\alpha \) is \( \Lambda_{1,1}^{-1} + \Lambda_{1,2}^{-1} = \frac{m_2-n}{m_1m_2-n^2} \) and the charge of the quasi-particle \( w'_\alpha \) is \( \Lambda_{2,1}^{-1} + \Lambda_{2,2}^{-1} = \frac{m_1-n}{m_1m_2-n^2} \), where the electron charge is assumed to be \(-1\) (the above results can be derived by using Berry phase method [12] or the article by D.P. Arovas in [13]. however see also the next section).

### 3. HIERARCHICAL WAVE FUNCTION AND SPIN-STATISTICS RELATION IN THE LAYERED FQHE

The hierarchical wave function can be obtained when the quasi-particles are condensed. We can analyze the hierarchical wave function to obtain the spin of the quasi-particle [9]. The idea is that, from the hierarchical wave function, we can obtain the hamiltonian of the quasi-particles and then the spin of the quasi-particle by analyzing the hamiltonian. The spin of the quasi-particles can also be obtained by calculating Berry phase when the quasi-particle moves in a closed path [9].

The hamiltonian of the quasi-particles can be obtained by using the fact that the suggested wave function of quasi-particles, which is Laughlin type [2], is the
ground state of the Hamiltonian, or by using Berry phase method [3]. We shall remark that, the Lagrangian of the quasi-particles are described by vortex (center coordinate) dynamics, and the Lagrangian of vortices (quasi-particles) does not contain any mass term [14]. The Hilbert space of the Hamiltonian which we will derive in the following shall be restricted to ground state. The ground state of the following Hamiltonian is the same as the one obtained by analyzing the vortex dynamics theory of the quasi-particles. So to be rigorous, we shall proceed our discussion from the beginning based on vortex dynamics theory.

The problem about the spin of the quasi-particle has also been addressed in [15,16]. The result about the spin of the quasi-particle in [9] agrees with the one in [15]. However there are some differences in the definition of the spin between [9] and [15], which we explain later. $s$ in [9] corresponds to $S_{\text{total}}$ [15]. The reference [15] has also calculated the spin of the quasi-particle in general hierarchical state and multilayered FQHE state (which we can also use the method described in [9] to calculate). The spin-statistics relation of the quasi-particle in general hierarchical state and multilayered FQHE state usually is not standard one [15], even it is found that the quasi-particle in Laughlin state (with filling as $\frac{1}{m}$) has standard spin-statistics relation [9]. We are aware that the spin of the quasi-particle in the Laughlin state calculated in [16] is different from the one in [9,15].

Let us consider the special case $m_1 = m_2$ of the last section for simplicity. Then we have

$$e_i^* \cdot e_j^* = \frac{1}{m^2 - n^2} \begin{pmatrix} m & -n \\ -n & m \end{pmatrix}.$$

The wave function of the condensed quasi-particles in the singular gauge is a Laughlin type wave function, and according to (2.11), it shall be

$$\tilde{\Psi}_p = d(z'_\alpha - z'_\beta)^{e_i^* \cdot e_j^*} d(w'_\alpha - w'_\beta)^{e_i^* \cdot e_j^*} d(z'_\alpha - w'_\beta)^{e_i^* \cdot e_j^*} d(z'_\alpha - w'_\beta) [\ldots] p.$$
Hence the hierarchical wave function is

\[ \Psi_{mmn,p} = \int \prod dv_\alpha \Psi_{mmn}(z_i, w_i, z'_\alpha, w'_\alpha) \Psi^q_{p}(z'_\alpha, w'_\alpha), \quad (3.3) \]

where \( \Psi_{mmn}(z_i, w_i, z'_\alpha, w'_\alpha) \) now is the normalized wave function given by (2.10) and \( dv_\alpha \) are the rotationally invariant volume measures of the quasi-particles. By imposing the rotationally invariant condition on the wave function \( \Psi_{mmn,p} \), we can obtain the relation

\[ \begin{align*}
m(N_1 - 1) + n N_2 + N'_1 &= \phi, \\
n N_1 + m(N_2 - 1) + N'_2 &= \phi, \\
N_1 - p(N'_1 - 1) - p N'_2 &= 0, \\
N_2 - p(N'_2 - 1) - p N'_1 &= 0.
\end{align*} \quad (3.4) \]

The first two equations in (3.4) are equations in (2.9). From (3.4), we get the filling \( \nu \),

\[ \nu = \frac{2}{m + n + \frac{1}{2p}}. \quad (3.5) \]

To discuss the hamiltonian of the quasi-particles, we will use the quasi-particle wave function in non-singular gauge,

\[ \Psi_p' = |d(z'_\alpha - z'_\beta)^{e_1 \cdot e_1} d(w'_\alpha - w'_\beta)^{e_2 \cdot e_2} d(z'_\alpha - w'_\beta)^{e_1 \cdot e_2} | \times [\bar{d}(z'_\alpha - z'_\beta) \bar{d}(w'_\alpha - w'_\beta) \bar{d}(z'_\alpha - w'_\beta)]^p, \quad (3.6) \]

The hamiltonian which has \( \Psi_p' \) as the ground state is

\[ H = \frac{2}{M} \sum (1 + z'_i z'_i)^2 (P_{z'_i} - A_{z'_i}) (P_{z'_i} - A_{z'_i}) + (1 + w'_i w'_i)^2 (P_{w'_i} - A_{w'_i}) (P_{w'_i} - A_{w'_i}), \quad (3.7) \]

where

\[ \begin{align*}
A_{z'_i} &= -\frac{im}{2(m^2 - n^2)} \sum_{j \neq i} \frac{1}{z'_i - z'_j} + \frac{in}{2(m^2 - n^2)} \sum_{i,j} \frac{1}{z'_i - w'_j} + \\
&\quad \frac{i}{2(m - n)(m + n)} \frac{z'_i}{1 + z'_i z'_i} + \frac{i\phi}{2(m + n)} \frac{z'_i}{1 + z'_i z'_i}.
\end{align*} \quad (3.8) \]
and

\[
A_{w'_i} = \frac{-im}{2(m^2 - n^2)} \sum_{j \neq i} \frac{1}{w'_i - w'_j} + \frac{in}{2(m^2 - n^2)} \sum_{i,j} \frac{1}{w'_i - z'_j} + \frac{i}{2} \frac{m(m - n - 1)}{(m - n)(m + n)} \frac{\bar{w}'_i}{1 + w'_i \bar{w}'_i} + \frac{i\phi}{2} \frac{\bar{w}'_i}{1 + w'_i \bar{w}'_i}.
\]

(3.9)

we can check that, by using the relation (3.4), \( P_{z'_i} - A_{z'_i} \) or \( P_{w'_i} - A_{w'_i} \) acting on the wave function \( \Psi'_p \) is zero. The lagrangian of the quasi-particles is (for the case of disc geometry, see [14]),

\[
L = \sum A_{z'_i} \frac{dz'_i}{dt} + A_{\bar{z}'_i} \frac{d\bar{z}'_i}{dt} + A_{w'_i} \frac{dw'_i}{dt} + A_{\bar{w}'_i} \frac{d\bar{w}'_i}{dt}.
\]

(3.10)

From the lagrangian, we use Noether theorem to derive the angular momenta of the quasi-particle. Then from the angular momenta, we can get the spin of the quasi-particle. The first and second terms in the right of the equations (3.8) and (3.9) tell us that the quasi-particles satisfy fractional statistics. The last terms in (3.8) and (3.9) represent the interaction between the quasi-particles and magnetic field (so the electric charge of the quasi-particle is \( \frac{1}{m+n} \)). The statistics parameters and the electric charge of the quasi-particle given by (3.8) and (3.9) are consistent with the ones given by the last section.

The spin of the particle will be changed by the presence of the magnetic monopole field or other particles with monopole charges [17]. For example, if an electron interacts with a magnetic monopole of odd integer flux, the spin of the electron will be an integer instead of \( \frac{1}{2} \). The spin we would like to discuss is the intrinsic spin which shall not depend on the presence of the applied magnetic monopole field or other particles with monopole charges.

Let us consider the spin of quasi-particle \( z'_i \). By calculating Noether currents of rotational invariance of the lagrangian ([9] or the chapter 3 in [18]), we find that the terms in (3.10), for example, \( \frac{-im}{2(m^2 - n^2)} \frac{1}{z'_i - z'_j} \frac{dz'_i}{dt} \) and \( \frac{in}{2(m^2 - n^2)} \frac{1}{z'_i - w'_j} \) (and their complex conjugate) will contribute to spin of the quasi-particle. Also the interaction between the quasi-particle and magnetic field, which is described by the
term in (3.10) as \( \frac{i\phi}{2(m+n)} \frac{z_i'}{1+z_i'\bar{z}_i'} \frac{dz_i'}{dt} \) (and its complex conjugate) will contribute to spin of the quasi-particle. However there is another term in (3.10), \( \frac{i}{2} \frac{m(m-n-1)}{(m-n)(m+n)} \frac{z_i'}{1+z_i'\bar{z}_i'} \frac{dz_i'}{dt} \) (and its complex conjugate), which represents the interaction between the quasi-particle and a monopole field with flux \( \frac{m(m-n-1)}{(m-n)(m+n)} \). This term is independent on the presence of the applied magnetic monopole field or the presence of other quasi-particles. So its contribution to the spin of the quasi-particle is intrinsic. The contribution to the spin is \( \frac{1}{2} \frac{m(m-n-1)}{(m-n)(m+n)} \). Thus we identify the intrinsic spin (from this time on, we will just simply call intrinsic spin as spin) of the quasi-particle as

\[
s = \frac{m(m-n-1)}{2(m-n)(m+n)}. \tag{3.11}
\]

This result can also be obtained by using the formula for \( S_{\text{total}} \) in [15].

When \( m = 1, n = 0 \), up(down)-layer FQHE is an integer quantum Hall effect with the filling as 1. The quasi-particles are electrons or the holes of the electrons. From (3.11), \( s \) will be equal to zero when \( m = 1, n = 0 \). However one should expect that the electron spin is not 0, but \( \frac{1}{2} \). What is the reason for the deficit of \( \frac{1}{2} \)? In the layered FQHE, the Pauli spin of the up(down)-layer electron is polarized. So only one component of the electron is taken into account and the Pauli spin \( \frac{1}{2} \) usually is forgotten for this reason. The wave function of the electron in the up-lower is \( \psi_{ui}(e) \) where \( i \) is Pauli spin index (\( i = 1, 2 \)). In the polarized case, one component is zero, for example, \( \psi_{u2}(e) = 0 \). The same reasoning shall also be applied to the quasi-particle wave function. The up-layer quasi-particle wave function is \( \psi_{ui}(q) \). In the polarized case, \( \psi_{u2}(q) = 0 \). Thus We shall include the Pauli spin to the (intrinsic) spin of the quasi-particle and now the spin will be equal to \( s^t = s + \frac{1}{2} \). This definition of the spin of the quasi-particle is consistent with the fact that the electron spin is \( \frac{1}{2} \).

The generalized standard spin-statistics relation shall be

\[
s^t = \frac{\theta}{2} + \text{integer}. \tag{3.12}
\]

In the present problem, \( \theta \) equals to \( \frac{-m}{m^2-n^2} \) (\( \theta \) corresponds to \( \theta_{11} \) in the last section).
and $s^t = s + \frac{1}{2}$ is given by the equation (3.11). However the relation (3.12) is not satisfied in this case.

When $m = n + 1$, the wave function $\Psi_{mmn}$ satisfies Fock cyclic condition (we will discuss it in the next section). Thus $\Psi_{n+1,n+1,n}$ can be used to describe the un-layered spin-singlet FQHE (SFQHE) [19]. When $m = n + 1$, $s^t$ equals to $\frac{1}{2}$. However it is not clear to us that the definition of the quasi-particle spin by the equation (3.12) is suitable for the quasi-particle spin in the spin-singlet FQHE or not.

4. HIERARCHICAL WAVE FUNCTION
WITH FOCK CYCLIC CONSTRAINT

For FQHE with half spins reversed and without layered structure, all electrons are identical and the wave function needs to be completely anti-symmetrized. The wave function is [19]

$$\Phi_{mmn} = \sum_{P} \frac{(-1)^p}{(2N)!} \Psi_{mmn}(z_{P(1)}, \cdots, z_{P(N)}; z_{P(1+N)}, \cdots, z_{P(2N)}) \Psi_s(P),$$

(4.1)

where $z_{i+N} = w_i$, $P$ is permutation operator, $p$ is the parity of the permutation $P$ and $\Psi_s(P)$ is the spin function

$$\Psi_s(P) = (\alpha_{P(1)}, \cdots, \alpha_{P(N)}; \beta_{P(N+1)}, \cdots, \beta_{P(2N)}),$$

(4.2)

where $\alpha$ and $\beta$ represent the spin-up and spin-down states. $\Phi_{mmn}$ is the eigenstate of $S^z = \sum_{i=1}^{2N} S^z_i = 0$. However it may not be the eigenstate of $S^2 = 0$ (if $S^2 = 0$, then $S^z$ must be zero). Now the operator of the rotation is given by

$$R^t(a, b, c, d) = R^s(a, b, c, d)R(a, b, c, d),$$

(4.3)

where $R(a, b, c, d)$ is defined as that in (1.9) and $R^s(a, b, c, d)$ is the rotational operator on spin function. Hence if the wave function (4.1) is rotationally invariant,
that is \( R^s \Phi_{mnm} = \Phi_{mnm} \), we may require

\[
R \Phi_{mnm} = \Phi_{mnm}, \quad R^s \Phi_{mnm} = \Phi_{mnm}.
\]  

(4.4)

The condition \( R^s \Phi_{mnm} = \Phi_{mnm} \) is equal to the condition \( S^2 = 0 \). If we require \( S^2 \Psi_{mnm} = 0 \), then we obtain Fock cyclic condition (FCC) on the wave function \( \Psi_{mnm} \) [19]. FCC is a condition given by

\[
E_{zi} \Psi = \sum_j e(z_i, w_j) \Psi = \Psi,
\]  

(4.5)

where \( e(z_i, w_j) \) is the operator which exchanges the coordinates \( z_i \) and \( w_j \) of the function. If \( \Psi = \Psi_{mnm} \), then \( m = n + 1 \) is the only solution of FCC. It is well-known fact that \( \prod_{i<j} (z_i - z_j)(w_i - w_j) \) satisfies FCC (4.5). Then we can easily show that \( \prod_{i<j} d(z_i, z_j)d(w_i, w_j) \) satisfies FCC and so does \( \Psi_{n+1,n+1,n} \). One interesting problem is how to construct the hierarchical wave function \( \Phi \) on which \( S^2 \) is 0.

We may first construct the hierarchical wave function \( \Psi \) based on the parent state \( \Psi_{n+1,n+1,n} \) by using the construction discussed in the last section (now we must have \( N_1 = N_1' \) in order to have a rotationally invariant state), then we wish that it fulfills FCC. We will show that it is indeed so! Following the last section, the general hierarchical wave function of \( \Psi_{n+1,n+1,n} \) is

\[
\Psi(z_1(1), w_1(1)) = \int \prod dv_q \Psi_1(z_1(1), w_1(1); z_2(2), w_2(2)) \times \\
\Psi_2(z_2(2), w_2(2); z_3(3), w_3(3)) \times \cdots \\
\Psi_{l-1}(z_l(l-1), w_l(l-1); z_l(l), w_l(l)) \Psi_1(z_l(l), w_l(l)).
\]  

(4.6)

\( z_i(1) = z_i \) (\( w_i(1) = w_i \)) are up(down)-spin electron coordinates and \( z_i(k) \) (\( w_i(k) \)) are up(down)-spin quasi-particle coordinates of \( k^{th} \) hierarchy. \( \Psi_1(z_1(1), w_1(1); z_2(2), w_2(2)) \) is the normalized electron wave functions in the presence of quasi-particles at \( z_2(2), w_2(2) \) and \( \Psi_k(z_k(k), w_k(k); z_k(k+1), w_k(k+1)) \) is the normalized wave function of quasi-particles in \( k^{th} \) hierarchy in the presence of
the next \(((k + 1)^{th})\) hierarchical quasi-particles. We mean that the quasi-particles in the second hierarchy is the quasi-particles of the electrons, etc. The index in \(z_i(k)\) \((w_i(k))\) ranges over \(1 \leq i \leq N_k\) \((1 \leq i \leq N'_k)\). \(N_1\) \((N'_1)\) is the number of the up(down)-spin electrons and \(N_k\) \((N'_k)\) is the number of the up(down)-spin quasi-particles in \(k^{th}\) hierarchy. The integration in (4.6) is over all quasi-particles coordinates (excluding the electron coordinates) and the integral measure over every quasi-particle coordinate is rotationally invariant measure on the sphere (see the first section).

Let us define

\[
F_k = \prod_{i<j} d(z_i(k), z_j(k))d(w_i(k), w_j(k)) \prod_{m,n} d(z_m(k), w_n(k)),
\]

\[
G_k = \prod_{i<j} d(z_i(k), z_j(k))d(w_i(k), w_j(k)),
\]

\[
S_k = \prod_{i,j} d(z_i(k), z_j(k+1))d(w_i(k), w_j(k+1)).
\]

(4.7)

Then the general hierarchical wave function \(\Psi\) in (4.6) is given by

\[
\Psi_1 = F_1^{p_1}G_1S_1(F_2)^{-p_1}G_2,
\]

\[
\cdots
\]

\[
\tilde{\Psi}_k = (F_k)^{s_k+p_k}G_kS_k(F_{k+1})^{-p_k}G_{k+1},
\]

\[
\cdots
\]

\[
\tilde{\Psi}_l = (F_l)^{s_l+p_l}G_l.
\]

(4.8)

\(p_1 = n, p_l\) with \(l > 1\) are even integers and

\[
\tilde{\Psi}_k = \begin{cases} \Psi_k, & \text{if } k = \text{odd integer;} \\ \Psi_k, & \text{otherwise.} \end{cases}
\]

(4.9)

\(s_k\) is given by the recursion relation

\[
s_{k+1} = -\frac{s_k + p_k}{2(s_k+p_k)+1},
\]

(4.10)

with \(s_1 = 0\). The statistics parameter of the condensed quasi-particle in \(k^{th}\) hier-
archy is

\[ \theta_k = (-1)^{k-1}(s_k + 1), \quad (4.11) \]

The charge of the condensed quasi-particle in \( k^{th} \) hierarchy is given by the recursion relation

\[ e_{k+1} = -\frac{e_k}{2(s_k + p_k) + 1}, \quad (4.12) \]

with the electron charge \( e_1 = -1 \).

The wave function with \( l \)-hierarchies is characterized by the \( 2l \times 2l \) matrix \( \Lambda \)

\[
\Lambda = \begin{pmatrix}
I + p_1 C & I & 0 & \ldots & 0 & 0 \\
I & -p_2 C & -I & 0 & \ldots & 0 \\
0 & -I & p_3 C & I & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & (-1)^{l-1}I & (-1)^{l-2}p_{l-1}C & (-1)^{l}I \\
0 & 0 & \ldots & 0 & (-1)^{l}I & (-1)^{(l-1)}p_{l}C
\end{pmatrix},
\]

where \( p_i \) positive even integers (except \( p_1 \) can be zero) and \( I, C \) are matrices,

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (4.14) \]

In order that the wave function \( \Phi \) be rotationally invariant, we apply the first condition in (4.4) and thus obtain the relation,

\[ \sum_j \Lambda_{i,j}(H_j - \delta_{i,j}) = \begin{cases} \phi, & \text{if } i = 1, 2; \\ 0, & \text{otherwise}, \end{cases} \quad (4.15) \]

where

\[ H_{2i-1} = N_i, \quad H_{2i} = N'_i. \quad (4.16) \]
It is clear that $N_k = N'_k$ in this case. From (4.15) we can derive filling as

$$\nu = \frac{2}{2p_1 + 1 + \frac{1}{2p_2 + 1 + \cdots + \frac{1}{2p_l}}}.$$  \hfill (4.17)

The second condition in (4.4) is equal to Fock cyclic condition (FCC) on $\Psi$. So we need to prove that the wave function (4.6) satisfies FCC. First we introduce the operators which are needed in the proving,

$$O_k = \prod_i^{N_k} \left(1 + e(z_i(k), w_i(k))\right) \frac{1}{2},$$

$$A_k^1 f(z_i(k)) = \sum_p \frac{(-1)^p}{N_k!} f(z_{P_i}(k)),$$

$$A_k^2 f(w_i(k)) = \sum_p \frac{(-1)^p}{N_k!} f(w_{P_i}(k)),$$

where $P$ are the permutations on 1, 2, …, $N_k$. Let us consider the simplest hierarchical wave function ($l = 2$). The wave function is now

$$\Psi = \int \prod_i^{N_k} dv_i F_1^{p_1} G_1 S_1(F_2) G_2(F_2)^{p_2} + p_2 G_2.$$

$G_1$ satisfies FCC (4.5). As an important fact, we can prove that $G_1 O_2(S_1 G_2)$ also satisfies FCC or the equation (4.5)

$$E_{z_i}(G_1 O_2(S_1 G_2)) = G_1 O_2(S_1 G_2).$$  \hfill (4.20)

The formula (4.20) is valid because

$$\prod_{i < j} (z_i(1) - z_j(1))(w_i(1) - w_j(1)) \times$$

$$O_2 \left[ \prod_{i, j} (z_i(1) - z_j(2))(w_i(1) - w_j(2)) \times \right.$$

$$\left. \prod_{i < j} (z_i(2) - z_j(2))(w_i(2) - w_j(2)) \right]$$?
satisfies FCC. Now we shall show that

\[
\int \prod dv_q F_1^{p_1}|(F_2)|^{\frac{-2p_1}{p_1+1}}(\bar{F}_2)^{p_2}G_1O_2(S_1G_2)\bar{G}_2
\]

(4.21)
is proportional to the wave function Ψ in (4.19). Because

\[
F_1^{p_1}|(F_2)|^{\frac{-2p_1}{p_1+1}}(\bar{F}_2)^{p_2}
\]
is the completely symmetric function of the coordinates \(z_i(2), w_i(2)\), the operator \(O_2\) acting on the function \(S_1G_2\) can be removed to act on the function \(\bar{G}_2\) inside the integration. So (4.21) is equal to

\[
\int \prod dv_q F_1^{p_1}|(F_2)|^{\frac{-2p_1}{p_1+1}}(\bar{F}_2)^{p_2}G_1S_1G_2(O_2\bar{G}_2).
\]

(4.22)

But \(G_2\) is the anti-symmetric function of the coordinates \(z_i(2)\) and the anti-symmetric function of the coordinates \(w_i(2)\), hence (4.22) will be equal to

\[
\int \prod dv_q F_1^{p_1}|(F_2)|^{\frac{-2p_1}{p_1+1}}(\bar{F}_2)^{p_2}G_1S_1G_2(A_1^1A_2^2O_2)\bar{G}_2.
\]

(4.23)

It can be shown that

\[(A_1^1A_2^2O_2)\bar{G}_2\]
is proportional to \(\bar{G}_2\). Hence we can conclude that (4.21) is proportional to Ψ in (4.19). But the formula (4.21) satisfies FCC due to the identity (4.20), so the wave function (4.19) also satisfies FCC. For the case of the general hierarchical wave function, we leave the proving to the next section. We shall mention that, Moore and Read had discussed the Halperin spin-singlet state from the point of view of Conformal Field Theory [5]. They had also discussed ordinary spin polarized hierarchical wave function by using Conformal Field Theory. It will be interesting to see how to obtain the above spin-singlet hierarchical wave function by using the method developed in [5].
5. OBTAINING THE HIERARCHICAL WAVE FUNCTION OF SFQHE FROM MORE PHYSICAL POINT OF VIEW?

The last section gives us an impression that we obtain the hierarchical wave function (4.6) satisfying FCC only by guess. In fact, we did not get the wave function (4.6) directly during this work. In this section, we shall show our original reasoning (based on the physical intuition) which we used to obtain the hierarchical wave function. The picture presented in the following may be not right, but the final wave function obtained in this picture is the same as that in the last section and we think that it is worthwhile to include it here. The picture presented in this section shall be called as pairing picture. The pairing picture had firstly and extensively been used by the authors in [5] to construct spin-singlet state in FQHE, for example, Halperin spin-singlet state, Haldane and Rezayi spin-singlet state. Pfaffian state at a filling factor $\frac{1}{q}$ with $q$ as even integer was also obtained based on pairing picture [5].

If the wave function $\Psi_{p_1+1,p_1+1,p_1}$ in the presence of quasi-particles is

$$\Psi_1^P = F_1^{p_1}(F_2)^{p_{p_1+1}} G_1 O_2(S_1 G_2), \quad (5.1)$$

then it will satisfy FCC (4.5) because $G_1 O_2(S_1 G_2)$ satisfies FCC. From the wave function $\Psi_1^P$, we shall suppose that the Laughlin wave function of the quasi-particles is

$$\Psi_2^P = (F_2)^{p_{p_1+1} + p_2} O_2(G_2). \quad (5.2)$$

It is reasonable to assume that the spin function of the quasi-particles is

$$\prod \frac{(\alpha_{z_i}(2)\beta_{w_i}(2) - \alpha_{w_i}(2)\beta_{z_i}(2))}{\sqrt{2}}. \quad (5.3)$$

The spin function of the quasi-particles given by (5.3) will insure that the excited state is the eigenstate of $S^2$ with the eigenvalue being 0. The excitations of the
Laughlin states look like Skymion excitations. The Skymion excitation is specified by the coordinates $z_i(2)$ and $w_i(2)$, and it is a bound state of the Laughlin quasi-particles at $z_i(2)$ and $w_i(2)$. This bound state looks like Cooper pair in superconductivity. We can demonstrate this point more clearly if we write the wave function of the quasi-particles as

$$\Psi'_2 = |F|_{2}^{\frac{\pi i}{2}} \bar{F}^{p_2} O_2(\bar{G}_2), \quad (5.4)$$

which is related to the wave function in (5.2) by a singular gauge transformation. In the new wave function of the quasi-particles (the wave function now is the spin function (5.3) multiplied by the wave function (5.4)), when $z_i(2)$ exchanges with $w_i(2)$, we shall get a minus sign and when exchange the coordinates $z_i(2)$ and $w_i(2)$ with the coordinates $z_j(2)$ and $w_j(2)$, the sign of the wave function remains unchanged. So in this gauge, the quasi-particle is fermion and the bound state is boson. Thus it exactly looks like the case of Cooper pairs in superconductivity.

We can proceed to construct the next hierarchy in a similar way. The quasi-particles in any hierarchy all are bounded to pairs. Then the general hierarchical wave function is

$$\Psi^p = \int \prod dv_q \Psi_1^p \Psi_2^p \cdots \Psi_l^p,$$

with

$$\Psi_1^p = F_1^{p_1} G_1(F_2) \bar{F}_1^{p_1} O_2(S_1 G_2),$$

$$\cdots,$$

$$\bar{\Psi}_k^p = (F_k)^{s_k+p_k} (F_{k+1})_{\bar{F}_k^{p_k}}^{-(s_k+p_k)} (O_k O_{k+1})(G_k S_k G_{k+1}), \quad (5.5)$$

$$\cdots,$$

$$\bar{\Psi}_l^p = (F_l)^{s_l+p_l} O_l(G_l).$$

Because $\Psi^1_1$ satisfies FCC, it is clear that $\Psi^p$ shall satisfy FCC.

We surprisingly find that $\Psi^p$ is proportional to $\Psi$ in (4.6). Take the simplest case, $l = 2$, the proving of the above statement is rather easy. The wave function
\( \Psi^p \) is then

\[
\Psi^p = \int \prod dv_q (\text{symmetric function of } z_i(2), w_i(2)) O_2(S_1G_2)O_2(\bar{G}_2), \quad (5.6)
\]

where \( \text{symmetric function of } z_i(2), w_i(2) = F_1^p |(F_2)| \bar{F}_2^{p+1} \). Inside the integration, \( O_2(S_1G_2)O_2(G_2) \) can be changed to \( (S_1G_2)(O_2O_2)(G_2) = (S_1G_2)O_2(G_2) \) due to \( O_kO_k = O_k \). The remaining proving can be found in the last section. For the general case, we take \( l = 3 \) as an example. The wave function is now

\[
\Psi^p = \int \prod dv_q (\text{symmetric function of } z_i(2), w_i(2))

(\text{symmetric function of } z_i(3), w_i(3))

O_2(S_1G_2)(O_2O_3)(\bar{G}_2 \bar{S}_2 \bar{G}_3)O_3(G_3).
\]

Inside the integration, we can change

\[
O_2(S_1G_2)(O_2O_3)(\bar{G}_2 \bar{S}_2 \bar{G}_3)O_3(G_3)
\]

to

\[
(S_1G_2)(O_2(O_2O_3))(\bar{G}_2 \bar{S}_2 \bar{G}_3)O_3(G_3)
\]

which is equal to

\[
S_1G_2(O_2O_3)(\bar{G}_2 \bar{S}_2 \bar{G}_3)O_3(G_3).
\]  (5.8)

However because \( S_1G_2 \) is the anti-symmetric function of the coordinates \( z_i(1) \) and the anti-symmetric function of the coordinates \( w_i(1) \), so (5.8) is equal to

\[
S_1G_2(A_1^1A_2^2O_2O_3)(\bar{G}_2 \bar{S}_2 \bar{G}_3)O_3(G_3).
\]  (5.9)

It can be shown that \( A_1^1A_2^2O_2O_3(\bar{G}_2 \bar{S}_2 \bar{G}_3) \) is proportional to \( \bar{G}_2O_3(\bar{S}_2 \bar{G}_3) \). So (5.9) is proportional to

\[
S_1G_2 \bar{G}_2O_3(\bar{S}_2 \bar{G}_3)O_3(G_3).
\]  (5.10)

Using the same reasoning as the one between the formula (5.7) and the formula (5.8), one can show that the formula (5.10) inside the integration can be replaced...
by
\[ S_1 G_2 \bar{G}_2 \bar{S}_2 \bar{G}_3 ((O_3 O_3)(G_3)) = S_1 G_2 \bar{G}_2 \bar{S}_2 \bar{G}_3 O_3 (G_3). \] (5.11)

Because \( \bar{S}_2 \bar{G}_3 \) is the anti-symmetric function of the coordinates \( z_i(3) \) and the anti-symmetric function of the coordinates \( w_i(3) \), (5.11) turns out to be equal to
\[ S_1 G_2 G_2 \bar{G}_2 \bar{S}_2 \bar{G}_3 = S_1 G_2 G_2 \bar{S}_2 \bar{G}_3 A_3 A_3^2 O_3 (G_3). \] (5.12)

Due to \( A_3 A_3^2 O_3 (G_3) \) being proportional to \( G_3 \), so (5.12) is proportional to
\[ S_1 G_2 \bar{G}_2 \bar{S}_2 \bar{G}_3 G_3. \] (5.13)

Thus we finally conclude that \( \Psi^p \) is proportional to the corresponding \( \Psi \) in the last section when \( l = 3 \). Actually they are the same wave functions. So \( \Psi \) given by (4.6) also satisfies FCC. If \( l > 3 \), the proving can follow the same way as that we did in the case of \( l = 3 \).

Let us now give a summarization of the current section. We have constructed the wave function \( \Psi^p \) which satisfies FCC. We have also proven that the wave function \( \Psi \) and \( \Psi^p \) are actually the same wave functions. Thus automatically we show that the hierarchical wave function \( \Psi \) in the last section satisfies FCC.

6. PARTICLE-HOLE CONJUGATION

Let us recall the particle-hole conjugation in the spin-polarized (un-layered) FQHE. If there is a state of FQHE with the filling \( \nu = \frac{1}{m} \), then the filling of the conjugated state is \( \nu^c = 1 - \frac{1}{m} \). The vacuum state of FQHE is defined as there are no electrons in the lowest Landau level. So the conjugate vacuum \( \Omega \) is the state filled with every orbital in the lowest Landau level occupied [20],
\[ \Omega(z_1, z_2, \cdots, z_{\phi+1}) = \prod_{i<j} d(z_i, z_j), \] (6.1)
where \( \phi \) is the magnetic flux and \( \phi + 1 \) is the number of the orbital in the lowest Landau level. If there is a state \( \Psi(z_1, z_2, \cdots, z_N) \), then the corresponding conjugate
state is
\[ \Psi_C = \int dv_1 \cdots dv_N \Omega(z_1, z_2, \cdots, z_{\phi+1}) \Psi^*(z_1, z_2, \cdots, z_N). \tag{6.2} \]

The Laughlin wave function of FQHE state with the filling \( \frac{1}{m} \) is \( \Psi(z_1, z_2, \cdots, z_N) = \prod_{i<j}^N d^{m}(z_i, z_j) \), where we have the relation \( m(N-1) = \phi \). Thus the conjugate state is
\[
\Psi_C = \int dv_1 \cdots dv_N \prod_{i<j}^N d(z_i, z_j) \prod_{i<j} d^{m}(z_i, z_j) \\
= \int dv_1 \cdots dv_N \prod_{N+1 \leq i < j \leq \phi+1} d(z_i, z_j) \\
\prod_{N+1 \leq i \leq \phi+1, 1 \leq j \leq N} \prod_{1 \leq i \leq N} |d(z_i, z_j)|^{2d^{m-1}(z_i, z_j)}, \tag{6.3}
\]

The filling of this state is
\[ \nu = 1 - \frac{1}{m} = \frac{1}{1 + \frac{1}{m-1}}, \tag{6.4} \]

and the wave function (6.3) actually belongs to the hierarchical wave function constructed by Blok and Wen [3] (see also [2] and [9]). We can show that, by using the conjugation operation, the conjugate state of the hierarchical wave function, of which the filling is
\[ \nu = \frac{1}{p_1 + \frac{1}{1 + \frac{1}{p_2 + \frac{1}{\cdots + \frac{1}{p_l}}}}}, \tag{6.5} \]
is given by another hierarchical state with the filling as
\[ \nu_c = 1 - \nu = \frac{1}{1 + \frac{1}{p_1 - 1 + \frac{1}{p_2 + \frac{1}{\cdots + \frac{1}{p_l}}}}}. \tag{6.6} \]
The conjugate vacuum state of the spin-singlet state $\Omega_s$ now is

$$\Omega_s = \Phi_{1,1,0}(z_1, z_2, \cdots, z_{\phi+1}; w_1, w_2, \cdots, w_{\phi+1}).$$

(6.7)

Now we consider the conjugate state of $\Phi_{n+1,n+1,n}(z_1, z_2, \cdots, z_N; w_1, w_2, \cdots, w_N)$,

$$\Phi_c = \int \prod_{1 \leq i \leq N} dv_z i dv_w_i \Phi_{n+1,n+1,n}^\dagger \Omega_s$$

(6.8)

The filling of the state $\Phi_c$ is $2 - \frac{2}{2n+1} = \frac{2}{1+\frac{1}{2n}}$, and by explicit calculation, $\Phi_c$ turns out to be the hierarchical state constructed in section 3 or 4 with $l = 2, p_1 = 0, p_2 = n$. Generally, the conjugate wave function of the hierarchical wave function specified by the parameters $(p_1, p_2, \cdots, p_l)$ is the hierarchical wave function specified by the parameters $(p'_1, p'_2, \cdots, p'_{l+1}) = (0, p_1, p_2, \cdots, p_l)$. The summation of the fillings of two states, which are conjugate with each other, is always equal to 2. The above discussion offers some kinds of checking to the hierarchical wave function constructed in section 3 or 4. We finally remark that when $p_1 = 0, p_2 = 2, l = 2$, the filling is $\frac{8}{9}$, and this state is conjugate to the state specified by the parameters $p_1 = 2, l = 1$ with the filling as $\frac{2}{3}$.

7. CONCLUSION

We have constructed the hierarchical wave function of SFQHE which satisfies FCC. The particle-hole conjugation has been used to check the wave function. We have also discussed the spin of the quasi-particles and the spin-statistics relation in some cases. The hierarchical state of SFQHE has also been discussed in [21]. The relation between [21] and this work is not clear. There are some other approaches to non-polarized FQHE, for example, the effective Ginzburg-Landau theory approach [22,23]. Although the Ginzburg-Landau theory of the spin-polarized FQHE has been well developed ([3,24,25,26,27,etc.] and also the review articles [18,28]), the Ginzburg-Landau theory of SFQHE has not been much studied in the literature.
While we know how to implement the rotational invariance condition on the wave function of SFQHE (on the sphere), we do not know how to fully implement the rotational invariance condition in the effective Ginzburg-Landau theory. In the microscopic approach to SFQHE, the rotational invariance condition on the spin sector of the wave function turns out to be Fock cyclic condition on the wave function. The rotational invariance condition on the space sector will give us a set of relations from which we can derive the filling of the state. But how can we apply the rotational invariance correctly and sufficiently on the corresponding Ginzburg-Landau theory?

We can also obtain the wave function of SFQHE on the torus (the wave function of the spin-polarized FQHE on the torus, which has the filling as $\frac{1}{m}$ with $m$ being an odd integer, has been constructed in [29] and the hierarchical wave function on the torus has been constructed in [30]). Although the rotational invariance is broken for the space part of the wave function (we have another important invariance on the torus, that is translational invariance), we can still require that the wave function is the eigenstate of $S^2$ with the eigenvalue as 0. $\theta_a(\sum_i z_i)\theta_a(\sum_i w_i) \prod_{i<j} \theta_3(z_i - z_j)\theta_3(w_i - w_j)$ is a solution to FCC and so it can be used to construct the wave function on the torus. We need to use Fay’ trisecant identity [31] to prove this Fock cyclic identity on the torus. $\theta_a$ with $a = 1, 2, 3, 4$ are the $\theta$ functions on the torus, in which $\theta_3$ is the odd function [31].

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