The BRST quantization of the nonlinear $WB_2$ and $W_4$ algebras

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ABSTRACT

We construct the BRST operator for the non-linear $WB_2$ and $W_4$ algebras. Contrary to the general belief, the nilpotent condition of the BRST operator doesn’t determine all the coefficients. We find a three and seven parameter family of nilpotent BRST operator for $WB_2$ and $W_4$ respectively. These free parameters are related to the canonical transformation of the ghost antighost fields.
1. Introduction

It is well-known that the covariant quantization of the (bosonic) string theory leads to a set of constraints for the physical state $|\text{phys.}\rangle$:

$$L_n |\text{phys.}\rangle = 0, \quad n \geq 0,$$

(1)

where $L_n$’s are the modes of the stress-energy tensor $T(z) = \sum_n L_n z^{-n-2}$ and satisfy the following Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{12} m(m^2 - 1)\delta_{m+n,0},$$

(2)

or in terms of $T(z)$, the following OPE:

$$T(z)T(w) \sim \frac{D}{2(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) T(w).$$

(3)

One way of organizing all these constraints is to introduce the ghost modes $c_m$ and their corresponding antighost modes $b_n$ which satisfy the following anti-commutation relations

$$\{c_m, b_n\} = \delta_{m+n,0},$$

$$\{c_m, c_n\} = \{b_m, b_n\} = 0.$$  

(4)

Then by constructing a BRST operator (firstly introduced in the quantization of gauge fields [2, 3]) [4]

$$Q = \sum_n c_{-n}L_n - \frac{1}{2} \sum_{m,n} :c_{-m}c_{-n}b_{m+n} :,$$

(5)

the physical state conditions can be formulated as the following

$$Q|\text{phys.}\rangle = 0 \quad \text{and} \quad |\text{phys.}\rangle \neq Q|\chi\rangle.$$  

(6)

Here the BRST operator $Q$ is required to be nilpotent, i.e. $Q^2 = 0$, because of the consistency of the formalism. This requirement fixes $D = 26$, the celebrated
critical dimension of the bosonic string theory [4, 1]. If we identify $Q$ as an exterior differential operator, from (6) we see that the physical states are just these closed but not exact differentials, i.e. the cohomology classes. The payoff of using the BRST formalism is that one can use the well-established results in cohomology theory to compute the physical states. In fact a complete understanding of the physical states in minimal model coupled with gravity was reached by computing the cohomology of the corresponding BRST operator [5, 6].

The BRST formalism is quite powerful and has been extended to many algebras. Examples are superconformal algebras which appear in superstring theory [1] and current algebra which appears in Wess-Zumino-Witten model [7]. For all these algebras a (quantum) BRST operator can often easily be constructed. Up to some anomalous terms the Jacobi identities of the algebras guarantee the nilpotence of $Q$. (The vanishing of the anomalous terms gives the critical dimension but this is not always possible.) This is so because all these algebras are (graded) linear algebras. On the other hand, for non-linearly extended algebras the construction of quantum BRST operators is a much more difficult problem. For the simplest non-linear algebra $W_3$ [8], the BRST operator has been constructed in [9] and shown to be nilpotent if the central charge is 100. (See also refs. [10, 11].) By understanding this construction, in [12] a general solution was given for the construction of quantum BRST operator for a sub-class of quadratical non-linear algebras. Nevertheless such restriction on the algebra is so stringent that even the next simplest non-linear $W$-algebras $WB_2$ and $W_4$ are not belong to the class*. The BRST operator for the $WB_2$ algebra was studied in ref. [13] by using an explicit free field realization for part of the stress energy tensor. In this paper we will construct the (quantum) BRST operator for these two simple non-linear algebras from a purely algebraic point of view. The motivation lies behind this study is that we hope to learn something new from these simple cases. Our results hint that no general recipe could exist for the construction of BRST operator for these

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* These two non-linear algebras contain tri-linear term $\Lambda_7 (\sim T^3)$ in the conformal basis, but they are actually quadratic non-linear algebras.
non-linear algebras. Contrary to the general belief we found that there exists not just a unique BRST operator but a family of BRST operators. In the case of $WB_2$ we found a three parameter family of nilpotent BRST operator. For $W_4$ there is a seven parameter family of nilpotent BRST operator. Independently the BRST operator for the $W_4$ algebra was also constructed in ref. [14]. It was pointed out in ref. [14] that these free parameters are related to the canonical transformation of the ghost antighost fields. We will establish this connection explicitly in sect. 5 for the $WB_2$ algebra.

Before presenting the results let me make a few remarks about this work and the writing of this paper. This may be helpful to understand this paper. Based on my study about the cohomology of pure gravity and subsequently some preliminary works on pure $W_3$ gravity [15], I envisage that there should be a general connection between highest states and BRST cohomology in pure $W$-gravity [16, 17]. Central to this conjecture is the assumption that there exists a (unique quantum) BRST operator. So I set to construct the BRST operator for the $W_4$ algebra. In order to fix the notation I rederived the $W_4$ algebra [18, 19] from quantum Miura transformation [20]. The construction of the BRST operator is straightforward although the algebraic calculations are so complicated that the only hope is to use computer symbolic calculation. With the aid of computer, all the coefficients are found and the BRST operator is shown to be nilpotent. Nevertheless the result obtained is quite dirty: the printout (in phyzzx TeX form) of the coefficients is around 20 pages long and a seven parameter family of nilpotent BRST exists*. Next I realized that there is a $WB_2$ algebra which is simpler than $W_4$ algebra because there is no spin-3 fields. With $WB_2$ algebra life becomes much easier. Nevertheless there still exists a three parameter family of nilpotent BRST operator. As we will show in sect. 5, by requiring the total stress energy tensor as $\{Q, b(z)\}$ we can fix some of these free parameters and the expression of $Q$ simplifies a lot. For $WB_2$, all the three free parameters are fixed and we have a unique BRST operator. For $W_4$ there left only

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* In an early version of this paper, I got only a two parameter family by requiring some total derivatives to zero. The solution found is only a subset of the complete solution.
two free parameters and there seems no natural choice for them. The organization of the paper is as follows:

In sect. 2 we gave all the needed OPEs for the $WB_2$ and $W_4$ algebras. The OPEs involving the composite field $\Lambda_1$ are also given explicitly. In sect. 3 we discuss how one can construct a BRST operator for any given algebra. Specifically we gave the ansatz of the BRST currents for the $WB_2$ and $W_4$ algebras. In sect. 4 the solutions of the nilpotent condition $Q^2 = 0$ are given. Nevertheless we gave all the coefficients only for $WB_2$ and dare not to write down all the coefficients for $W_4$. The best we could do is to fix five out of the seven free parameters and gave only the resulting expressions which depend on only two free parameters. In sect. 5 we discuss the results obtained in sect. 4 and explicitly establish the connection between these free parameters and the canonical transformation of the ghost antighost fields which was only briefly touched in ref. [14]. The explicit form of all these canonical transformations are also given.
2. The nonlinear $WB_2$ and $W_4$ algebras

In this section we gave the complete OPEs of the nonlinear $WB_2$ and $W_4$ algebras. We will use the same notation for both algebras (but different normalization for the spin-4 field $U(z)$). Firstly the $WB_2$ algebra is generated by the stress-energy tensor $T(z)$ and a spin-4 primary field $U(w)$. The basic OPEs were discussed in a number of papers [21-23, 18, 19]. They are

\[
T(z)T(w) \sim \left(\frac{2}{(z-w)^2} + \frac{1}{z-w} \partial_w + \frac{3}{10} \partial_w^2 + \frac{1}{15} (z-w) \partial_w^3 + \frac{1}{84} (z-w)^2 \partial_w^4\right)T(w)
+ \frac{c/2}{(z-w)^4} + \left(1 + \frac{1}{2} (z-w) \partial_w + \frac{5}{36} (z-w)^2 \partial_w^2\right)\Lambda_1(w) + (z-w)^2 \Lambda_2(w),
\]

\[
T(z)U(w) \sim \left(\frac{4}{(z-w)^2} + \frac{1}{z-w} \partial_w + \frac{1}{6} \partial_w^2\right)U(w) + \Lambda_5(w),
\]

\[
U(z)U(w) \sim \frac{c/4}{(z-w)^8} + \left(\frac{2}{(z-w)^6} + \frac{1}{(z-w)^3} \partial_w + \frac{3/10}{(z-w)^4} \partial_w^2 + \frac{1/15}{(z-w)^3} \partial_w^3\right)T(w)
+ \frac{1/84}{(z-w)^2} \partial_w^4\right)T(w) + \left(\frac{1}{(z-w)^4} + \frac{1/2}{(z-w)^3} \partial_w + \frac{5/36}{(z-w)^2} \partial_w^2\right)
\times \left(c_0 U(w) + \frac{42}{(22+5c)} \Lambda_1(w)\right) + \frac{28c_0}{3(c+24)} \Lambda_5(w)
\]

\[
+ \frac{3(19c - 524)}{5(68 + 7c)(2c - 1)} \frac{\Lambda_2(w)}{(z-w)^2} + \frac{24(72c + 13)}{(22+5c)(68+7c)(2c-1)} \frac{\Lambda_7(w)}{(z-w)^2},
\]

where $c_0 = \sqrt{\frac{54(24+c)(c^2-172c+196)}{(22+5c)(68+7c)(2c-1)}}$. Notice that in (7) we didn’t give the simple pole terms in $U(z)U(w)$ because these terms can easily be obtained from the symmetric property of this OPE and they are not needed explicitly in the following. Also some regular terms are included in $T(z)T(w)$ and $T(z)U(w)$ explicitly in order to define the quasi primary fields $\Lambda_i(w)$ ($i = 1, 2, 5$). The other quasi primary field $\Lambda_7(w)$ appears in the regular term of the following OPE:

\[
T(z)\Lambda_1(w) \sim \frac{(22+5c)}{5} \left(\frac{4}{(z-w)^2} + \frac{1}{z-w} \partial_w + \frac{1}{6} \partial_w^2\right)\Lambda_1(w) + \Lambda_7(w).
\]

(8)
In the computation of $Q^2$ we will also need the OPEs between $U(z)$ and $\Lambda_1(w)$, and $\Lambda_1(z)$ with itself because the composite field $\Lambda_1(z)$ also enters the construction of $Q$ due to the presence of a tri-linear term $\Lambda_7(w)$ in $U(z)U(w)$. These OPEs can be computed from the Wick theorem involving the contraction of composite fields [24]. Explicitly we have

\[ U(z)\Lambda_1(w) \sim \frac{84}{5} \frac{1}{(z-w)^4} + \frac{1/2}{(z-w)^3} \partial_w + \frac{5/36}{(z-w)^2} \partial_w^2 U(w) + \frac{8}{(z-w)^2} \Lambda_3(w), \]

\[ \Lambda_1(z)\Lambda_1(w) \sim \frac{c(22 + 5c)}{10(z-w)^8} + \frac{4(22 + 5c)}{5} \left( \frac{1}{(z-w)^6} + \frac{1/2}{(z-w)^5} \partial_w \right) + \frac{3/20}{(z-w)} \partial_w^2 + \frac{1/15}{(z-w)^3} \partial_w^3 + \frac{1/68}{(z-w)^2} \partial_w^4 T(w) + \frac{2(64 + 5c)}{5} \left( \frac{1}{(z-w)^4} + \frac{1/2}{(z-w)^3} \partial_w + \frac{5/36}{(z-w)^2} \partial_w^2 \right) \Lambda_1(w) + \frac{2(22 + 5c)}{5} \frac{\Lambda_2(w)}{(z-w)^2} + \frac{8}{(z-w)^2} \Lambda_7(w). \]

(9)

For $W_4$ algebra there is a spin-3 primary field $W(z)$ besides the stress-energy tensor $T(z)$ and the spin-4 primary field $U(z)$ appearing in $WB_2$ algebra. However the $WB_2$ algebra is not a sub-algebra of the $W_4$ algebra. The OPE $U(z)U(w)$ in $W_4$ algebra is different from the one in $WB_2$ algebra because the presence of an additional spin-6 composite field $\Lambda_6 \sim W^2$. The complete structure of the $W_4$ algebra was determined by using Jacobi identities in [18, 19]. Recently we have rederived it from quantum Miura transformation [20]. For later convenience we will write the OPEs in a non-standard (but natural from quantum Miura transformation) normalization. The OPEs $T(z)T(w)$, $T(z)U(w)$, $T(z)\Lambda_1(w)$, $U(z)\Lambda_1(w)$ and $\Lambda_1(z)\Lambda_1(w)$ are the same as those given above for $WB_2$. The OPE $U(z)U(w)$ changed to the following

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\[
\frac{U(z)U(w)}{C_4} \sim \frac{c/4}{(z-w)^8} + \left(\frac{2}{(z-w)^6} + \frac{1}{(z-w)^5}\partial_w + \frac{3/10}{(z-w)^4}\partial_w^2 + \frac{1/15}{(z-w)^3}\partial_w^3 \right) + \frac{1/84}{(z-w)^2}\partial_w^4 \right) T(w) + \left(\frac{1}{(z-w)^4} + \frac{1/2}{(z-w)^2}\partial_w + \frac{5/36}{(z-w)^2}\partial_w^2 \right) \Lambda_1(w) \times \frac{42}{(22+5c)} \Lambda_1(w) - \frac{90(c^2 + c + 218)}{(2 + c)(7 + c)(7c + 114)} U(w)
\]
\[
- \frac{120}{(2 + c)(7 + c)(z-w)^2} \Lambda_5(w) + \frac{225(22 + 5c)}{(2 + c)(7 + c)(7c + 114)(z-w)^2} \Lambda_6(w) + \frac{3(19c - 582)}{10(2 + c)(7c + 114)(z-w)^2} \Lambda_2(w) + \frac{96(9c - 2)}{(2 + c)(22 + 5c)(7c + 114)(z-w)^2} \Lambda_7(w)
\]

where \( C_4 = \frac{(2+c)(7+c)(7c+114)}{300(22+5c)} \). The additional OPEs \( T(z)W(w), W(z)W(w), W(z)U(w) \) and \( W(z)\Lambda_1(w) \) are given as in the following:

\[
T(z)W(w) \sim \left(\frac{3}{(z-w)^2} + \frac{1}{z-w}\partial_w + \frac{3}{14}\partial_w^2 + \frac{3}{84}(z-w)\partial_w^3 \right) W(w)
\]
\[
+ (1 + \frac{2}{5}(z-w)\partial_w)\Lambda_3(w) + (z-w)\Lambda_4(w),
\]

\[
W(z)W(w) \sim \left(\frac{c/3}{(z-w)^4} + \left(\frac{2}{(z-w)^4} + \frac{1}{(z-w)^3}\partial_w + \frac{3/10}{(z-w)^2}\partial_w^2 + \frac{1/15}{z-w}\partial_w^3 \right) + \frac{1/84}{(z-w)^2}\partial_w^4 \right) T(w) + \left(\frac{1}{(z-w)^2} + \frac{1/2}{z-w}\partial_w + \frac{5/36}{z-w}\partial_w^2 \right) \Lambda_1(w)
\]
\[
+ \frac{40}{(7 + c)} \left(\frac{1}{(z-w)^2} + \frac{1/2}{z-w}\partial_w + \frac{5/36}{z-w}\partial_w^2 \right) U(w) + \left(\frac{10}{(7 + c)} \Lambda_6(w),
\]

\[
W(z)U(w) \sim \left(\frac{(c+2)(7c+114)}{10(22 + 5c)} \right) \left(\frac{1}{(z-w)^4} + \frac{1/3}{(z-w)^3}\partial_w + \frac{1/14}{(z-w)^2}\partial_w^2 + \frac{1/84}{z-w}\partial_w^3 \right) W(w)
\]
\[
+ \frac{26(c+2)}{5(22 + 5c)} \left(\frac{1}{(z-w)^2} + \frac{2/5}{z-w}\partial_w \right) \Lambda_3(w) + \left(\frac{7c+114}{10(22 + 5c)} \Lambda_4(w),
\]

\[
W(z)\Lambda_1(w) \sim \frac{48}{5} \left(\frac{1}{(z-w)^4} + \frac{1/3}{(z-w)^3}\partial_w + \frac{1/14}{(z-w)^2}\partial_w^2 + \frac{1/84}{z-w}\partial_w^3 \right) W(w)
\]
\[
+ 6 \left(\frac{1}{(z-w)^2} + \frac{2/5}{z-w}\partial_w \right) \Lambda_3(w) - \left(\frac{4}{(z-w)} \Lambda_4(w).
\]
3. The Ansatz

Following the standard procedure we introduce ghost anti-ghost pairs \((c(z), b(z))\), \((\gamma(z), \beta(z))\) and \((\delta(z), \alpha(z))\) for \(T(z), W(z)\) and \(U(z)\) respectively. These ghost anti-ghost fields have spins \((-1, 2), (-2, 3)\) and \((-3, 4)\) and their mode expansions are as follows

\[
c(z) = \sum_n c_n z^{-n+1}, \quad b(z) = \sum_n b_n z^{-n-2},
\]
\[
\gamma(z) = \sum_n \gamma_n z^{-n+2}, \quad \beta(z) = \sum_n \beta_n z^{-n-3},
\]
\[
\delta(z) = \sum_n \delta_n z^{-n+3}, \quad \alpha(z) = \sum_n \alpha_n z^{-n-4}.
\]  

These modes satisfy the usual anti-commutation relations which can be derived from the following OPEs

\[
c(z) b(w) \sim \frac{1}{z - w},
\]
\[
\gamma(z) \beta(w) \sim \frac{1}{z - w},
\]
\[
\delta(z) \alpha(w) \sim \frac{1}{z - w}.
\]  

The other OPEs are all 0.

Because of the complexity with normal ordering we will not use mode expansions. All our calculation are done with (the holomorphic) fields. The normal ordering for the ghost anti-ghost fields are such that the following equations are true

\[
c(z) b(w) = \frac{1}{z - w} + : c(z) b(w) :,
\]
\[
\gamma(z) \beta(w) = \frac{1}{z - w} + : \gamma(z) \beta(w) :,
\]
\[
\delta(z) \alpha(w) = \frac{1}{z - w} + : \delta(z) \alpha(w) :.
\]  

This is possible because all these fields are free fields.
With all the above knowledge, we now construct the quantum BRST operator. One way to start is to construct the corresponding classical BRST operator [25, 26]. The quantum BRST operator is then assumed to be the same form as the classical one with possible renormalization of some coefficients and addition of some zero mode terms due to normal ordering. By imposing the nilpotence condition, one could determine all these coefficients. For linear algebras this route is quite successful. The same strategy has been applied to \(W_3\) [9] and in [12] to a class of quadratic non-linear algebra. But the simplicity of this construction doesn’t apply to \(WB_2\) and other non-linear algebras. As we will show explicitly a moment later, there are new terms which are not present in the classical BRST operator. So we should look for other method.

From our experience with BRST operator in known examples we know that the (quantum) BRST operator is the contour integration of a spin-1 current \(j(z)\) with ghost number 1. \(j(z)\) is the summation of various terms. Each individual term is the (normal ordered) product of the basic fields and their derivatives. For example, the BRST operator in (5) is the integration of \(j(z) = c(z)T(z) + c(z)\partial z c(z) b(z)\): \(Q = \oint [dz]j(z)^*\). (\(j(z)\) is called the BRST current.) We can simply assume that this is true in general. As the BRST operator is presumably coming from the quantization of a constrained system, we will also assume that the single ghost terms in \(j(z)\) is given by \(\sum c_i(z)T_i(z)\), where \(T_i(z)\)’s generate the constraints and \(c_i(z)\)’s are the corresponding ghost fields. For \(WB_2\) algebra this is \(c(z)T(z) + \delta(z)U(z)\). There is no terms like \(\delta(z)\partial^2 z T(z)\) which also has spin 1. (However see discussion in sect. 5.) Then we can just write down the most general spin-1 current with ghost number 1 (this is possible because these terms are finite in number, see below). Presumably the nilpotence of the BRST operator constructed from this current by contour integration should fix all the unknown coefficients. Let us now apply these rules to the construction of BRST currents for the \(WB_2\) and \(W_4\) algebras.

* \([dz] \equiv dz/2\pi i.\)
First let us see how we can generate all the possible terms for $WB_2$. The ghost number condition put the constraint that an individual term should consist of $n+1$ ghost fields and $n$ anti-ghost fields with $n = 1, 2, \ldots$. Define a generating function

$$P(x) = \sum_{n=1}^{\infty} \left( \frac{c(w + x)}{x} + \frac{\delta(w + x)}{x^3} \right)^{n+1} (x^2b(w + x) + x^4\alpha(w + x))^n \equiv \sum_l P_l x^l. \quad (15)$$

Here we should first think these ghost anti-ghost fields as commutating. Only at the end of the expansion we consider them as anti-commutating and put $(c(w))^2$, $(\partial_w b(w))^3$, etc. to 0. It is not quite difficult to convince oneself that $P_l$ is a spin-$l$ current with ghost number 1. Because all the bosonic fields have positive spins, only the currents $P_l$ with $l \leq 1$ could possibly be included in BRST current $j(z)$. Due to the anti-commutativity of the ghost fields, these currents are finite in number. For $l < 1$, one can construct spin-1 fields from $P_l$ by multiplying with bosonic fields. We claim that these are all the possible terms. Of course every $P_l$ consists of several terms and their coefficients should be set free in $j(z)$.

After describing the general principle let us see what is the maximum number of ghost fields a term can have. Because all the non-positive spin ghost fields are $(\delta(z), \delta'(z), \delta''(z), c(z), \delta^{(3)}(z), c'(z))$, the lowest spin (first consideration) fields with lowest ghost number (second consideration) is $\delta(z)\delta'(z)\delta''(z)c(z)$ and has spin $-7$ and ghost number 4. Nevertheless the lowest spin field with ghost number $-3$ are $b(z)b'(z)b''(z)$ and $b(z)b'(z)\alpha(z)$ and have spin 9. So there is no way to construct a spin-1 field by taking the product. The next possibility is to have 3 ghost fields and 2 anti-ghost field. Here there are a lot of terms. Some of them are

$$b(z)b'(z)\delta(z)\delta'(z)\delta''(z)T(z), \quad c(z)b(z)b'(z)\delta(z)\delta'(z)T(z),$$
$$c(z)c'(z)b(z)b'(z)\delta(z), \quad b(z)b'(z)\delta(z)\delta'(z)\delta^{(4)}. \quad (16)$$

One subtle point in the above construction of the BRST current is that there is a redundancy of terms because the integration of a total derivative term identically gives zero. So we should set some terms to zero in order to fix this redundancy.
Bearing this in mind, the most general expression of the BRST current \( j(z) \) for \( WB_2 \) can be easily constructed. Splitting it as the sum of various term of fixing total number of ghost antighost fields, we have

\[
j(z) = j_0(z) + j_1(z) + j_2(z), \tag{17}
\]

where the single ghost term is

\[
j_0(z) = c(z)T(z) + \delta(z)U(z). \tag{18}
\]

The three ghost antighost term \( j_1 \) is

\[
j_1 = b\delta'(a_0\Lambda_1 + a_1U) + (a_2b\delta'(3) + a_3b\delta' + a_4b\delta' + a_5b\delta' + a_6cb\delta' + a_7cb\delta

+ a_8c'b\delta + a_9\delta'\alpha)T + (m_1\delta(5) + m_2\delta(4) + m_3\delta''\delta(3))b + c(m_4\delta'\alpha' + m_5\delta\alpha')

+ m_6cc'b + c(m_7b\delta'' + m_8b\delta' + m_9\delta' + m_{10}b(3)\delta) + \delta(m_{11}\delta'\alpha'' + m_{12}\delta(3)\alpha), \tag{19}
\]

and the five ghost antighost term \( j_2 \) is

\[
j_2 = (a_{10}bb'\delta'\delta'' + a_{11}cbb'\delta\delta')T + m_{13}cc'bb'\delta + b\delta(m_{14}\delta''\alpha' + m_{15}\delta'(3)\alpha)

+ b(m_{16}\delta'(4) + m_{17}b\delta'(3) + m_{18}b(3)\delta'\delta')\delta + c(m_{19}b'\delta'\alpha + m_{20}b\delta'\alpha' + m_{21}b\delta''\alpha)\delta

+ c(m_{22}bb'\delta\delta(3) + m_{23}bb'\delta'' + m_{24}bb''\delta\delta' + m_{25}bb'\delta'\delta + m_{26}bb'\delta'\delta'). \tag{20}
\]

There is no 7 or higher ghost antighost terms. Notice that in \( j_1(z) \) there also appears a term with \( \Lambda_1(z) \) (\( \sim T^2(z) \)). This is necessary because the OPE of two \( U(z) \)'s gives \( \Lambda_7(w) \) which can only be cancelled by the terms \( \Lambda_7(w) \) coming from \( T(z)\Lambda_1(w) \) and \( \Lambda_1(z)\Lambda_1(w) \). As we will show later, the coefficient \( a_0 \) of this term is determined to be a pure number and so can never be tuned to zero we have some freedom to adjust other coefficients. (In a different basis where the \( WB_2 \) algebra is quadratic, there is no need for \( \Lambda_1 \) in the BRST operator. Similar remarks also apply to \( W_4 \).)

\* All the fields \( b, c, \delta, \alpha \) and their derivatives are holomorphic function of \( z \).
For the $W_4$ algebra one can perform similar analysis. As we noted in ref. [20], there is a selection rule for $W_4$ algebra. All the fields are classified into even and odd sets. The even set consists of $T(z)$, $U(z)$, $\Lambda_1(z)$, $\Lambda_2(z)$, $\Lambda_5(z)$ to $\Lambda_7(z)$ and all their derivatives. The odd set consists of $W(z)$, $\Lambda_3(z)$, $\Lambda_4(z)$ and all their derivatives. The OPEs of (even) $\times$ (even) and (odd) $\times$ (odd) give only even fields and the OPEs of (even) $\times$ (odd) give only odd fields as one can see from eqs. (7) to (11). If we also assign a parity to the ghost antighost fields and their derivatives ($(\gamma, \beta)$ are odd, $(c, b)$ and $(\delta, \alpha)$ are even), the BRST operator or the current is then an even object. Bearing this in mind we arrived at the following ansatz for the BRST current $j(z)$:

$$j = j_0 + j_1 + j_2 + j_3,$$

(21)

where the single ghost term $j_0$ is

$$j_0 = cT + \gamma W + \delta U,$$

(22)

and the three ghost antighost term $j_1$ is

$$j_1 = (a_1 \gamma \delta' \beta + a_2 \gamma' \delta \beta + a_3 \gamma \delta \beta' + a_4 \delta \delta' (3)b + a_5 \delta' \delta'' b + a_6 \delta'' b')
+ a_7 \delta \delta' b'' + a_8 \delta \delta' \alpha + a_9 \gamma \gamma' b + a_{10} \delta' b + a_{11} \delta'' b + a_{12} \delta b') T
+ (a_{22} \gamma \delta' b + a_{23} \gamma' \delta b + a_{24} \gamma \delta b' + a_{25} \delta' \beta) W + (a_{26} U + a_{27} \Lambda_1) \delta \delta' b
+ c \delta' b + (c \gamma' - 2c' \gamma) \beta + (c \delta' - 3c' \delta) \alpha + (2c_1 \gamma \gamma (3) - 3c_2 \gamma' \gamma'') b
+ c_3 \gamma' \alpha + (c_4 \gamma \delta (3) - 3c_5 \gamma' \delta'' + 5c_6 \gamma'' \delta' - 5c_7 \gamma (3) \delta) \beta
+ (3c_8 \delta (5) - 5c_9 \delta (4) + 6c_{10} \delta'' \delta (3) b + (c_{11} \delta \delta (3) - 2c_{12} \delta' \delta'') \alpha
+ (c_{13} \delta (3) + c_{14} \delta'' + c_{15} \delta'' b + c_{16} \delta (3) \delta) b),$$

(23)
The five ghost antighost term $j_2$ contains more than 60 terms:

$$j_2 = (a_{13}c\gamma\delta b\beta + a_{14}c\delta'\delta b' + a_{15}c\delta''\delta b'' + a_{16}c\gamma'\delta b' + a_{17}c\gamma'\delta b\beta$$

$$+ a_{18}c\gamma\delta'' b\beta + a_{19}c\gamma\delta' b' b + a_{20}c\gamma\delta' b\beta')T + a_{21}c\gamma\delta b\beta W$$

$$+ (m_{8c}\delta + m_{9c}\gamma')c b b' + (m_{10c}\delta' b' + m_{11c}\delta(3) + m_{12c}\delta'' b + m_{13c}\delta b')b b'$$

$$+ (m_{14c}\gamma(3)\delta + m_{15c}\gamma'\delta + m_{16c}\gamma''\delta') b + (m_{17c}\gamma''\delta) b b'$$

$$+ (m_{18c}\delta'\delta(4) + m_{19c}\delta''(3) + m_{20c}\delta' b b' + (m_{21c}\gamma'\delta + m_{22c}\gamma'\delta')(b + \delta')(m_{23c}\gamma(3) + m_{24c}\delta'' b + m_{25c}\delta'' b)$$

$$+ cd' + m_{26c}' b b'' + m_{27c}b + m_{28c}\delta b' + \gamma\delta(b + m_{29c}\gamma'\delta b' b'' + \delta'(m_{30c}\beta + m_{31c}\beta')$$

$$+ \delta' + (m_{32c}' b b'' + m_{33c}\beta'(3) + (m_{34c}\gamma' b'' + m_{35c}\gamma' b + m_{36c}\gamma' b' + m_{37c}\gamma'' b + m_{38c}\gamma' b$$

$$+ m_{39c}'(3) + m_{40c}\gamma' b' + m_{41c}\gamma' b + m_{42c}\gamma b + m_{43c}\gamma b b' + m_{44c}\gamma b\alpha$$

$$+ (m_{45c}\gamma\delta(4) + m_{46c}\gamma'\delta(3) + m_{47c}\gamma\delta(3) + m_{48c}\gamma b + m_{49c}\gamma'\delta' b b'$$

$$+ m_{50c}\gamma(3)\delta' b b + (m_{51c}\gamma\delta(3) + m_{52c}\gamma'\delta'' b + m_{53c}\gamma'\delta'' b + m_{54c}\gamma'\delta' b b' + m_{55c}\gamma\delta b b'$$

$$+ (m_{56c}\gamma\delta'' + m_{57c}\gamma'\delta b b' + (m_{58c}\gamma\delta'' + m_{59c}\gamma'\delta' b + m_{60c}\gamma\delta' b\alpha)$$

$$+ (24)$$

and there also exist some 7 ghost antighost terms

$$j_3 = (m_{61c}\gamma\delta b + m_{62c}\gamma'\delta b + m_{63c}\gamma\delta b + m_{64c}\gamma\delta b b'$$

$$+ m_{65c}\gamma\delta b b\beta + m_{66c}\gamma\delta b b' b + m_{67c}\gamma b b' + m_{68c}\gamma\delta\delta b b'$$

$$+ (m_{69c}\gamma'\delta' b b + m_{70c}\gamma b b b b + m_{71c}\gamma'\delta' b b' b + m_{72c}\gamma\delta\delta b b' b'$$

$$+ (25)$$

Fortunately no higher than 7 ghost antighost terms exist. In the next section we will solve the nilpotent condition $Q^2 = 0$ of such constructed BRST operators.
4. The Solutions

The BRST operator is the contour integration of the current $j(z)$:

$$Q = \oint [dz] j(z).$$

(26)

The square of $Q$ is

$$Q^2 = \frac{1}{2} \{Q, Q\} = \int_0 [dw] \int_w [dz] j(z) j(w).$$

(27)

To compute $Q^2$ one must do these two integration over $z$ and $w$. The integration over $z$ is straightforward. Firstly one expands $j(z)j(w)$ and compute the OPEs (or equivalently arranging them into normal ordered products by doing various contractions). For the $WB_2$ algebra the needed OPEs for the bosonic fields $T(z)$, $U(z)$ and $\Lambda_1(z)$ are given in (7), (8) and (9). Notice that we have purposely expanded these OPEs up to the fields with highest spin 6. This is the necessary and sufficient degree of accuracy. Because of the anti-commutativity of ghost antighost fields, terms containing higher spin bosonic fields are automatically zero. (One can easily convince oneself by constructing the lowest spin product of ghost antighost fields with ghost number 2.) For the contraction of ghost antighost fields we use exactly Wick theorem theorem in quantum field theory while taking into account the fermionic property of these (free) fields. After doing all these, the integration over $z$ amounts to evaluate the residue of the resulting expression at $w$.

The integration over $w$ is actually not needed because our purpose is to set $Q^2$ to zero and determine all the unknown coefficients. Evidently the integrand should be a total derivative if the contour integration is zero. This is necessary and sufficient. By setting the integrand to total derivatives one obtains a set of equations among the unknown coefficients and the central charge $c$. All these equation can have a solution only if the central charge is $c = 172$. This is in
agreement with the simple counting that the total central charge of the matter
and ghost antighost system is zero. (The \((b, c)\) system has central charge \(-26\) and
the \((\alpha, \delta)\) system has central charge \(-146\).) By setting \(c = 172\) and solving these
equations we found that some of the coefficients are pure numbers

\[ a_0 = \frac{253}{654444}, \quad m_4 = 4, \quad m_5 = 3, \quad m_6 = 1. \]  \hspace{1cm} (28)

The rest coefficients depend on three arbitrary parameters which we denoted as \(c_i\)
\((i = 1, 2, 3)\). We have (in order of increasing complexity)
\[ a_6 = c_1 \quad a_7 = 2c_1 \quad a_8 = c_1 \]

\[ a_9 = c_2 \quad m_{13} = c_1 \quad m_{19} = 4c_1 \]

\[ m_{20} = -7c_1 \quad m_{21} = -4c_1 \quad m_{11} = -\frac{7c_2}{2} \]

\[ a_1 = \frac{1}{21\sqrt{742}} - c_2 \quad m_7 = \frac{13}{2}c_1 + 12c_2 \quad m_{12} = \frac{7c_2}{2} \]

\[ m_9 = \frac{11c_1}{2} + \frac{17c_2}{2} \quad m_{10} = \frac{5c_1}{2} + \frac{5c_2}{3} \quad m_8 = -\frac{2c_1}{3} + \frac{31c_2}{6} \]

\[ a_2 = \frac{337}{311640} + \frac{7c_1^2}{60} + \frac{71c_1c_2}{60} - \frac{41c_2^2}{60} - \frac{c_3}{5} \]

\[ a_3 = -\frac{239}{155820} + \frac{c_1^2}{5} + \frac{71c_1c_2}{60} + \frac{2c_2^2}{5} - \frac{c_3}{5} \]

\[ a_4 = \frac{1}{1060} - \frac{17c_1^2}{20} + \frac{71c_1c_2}{30} - \frac{29c_2^2}{20} - \frac{2c_3}{5} \]

\[ a_5 = \frac{1}{1060} - \frac{7c_1^2}{4} - \frac{3c_2^2}{4} \quad a_{11} = c_1^2 + c_1c_2 \]

\[ m_1 = \frac{43}{63600} - \frac{127c_1^2}{100} + \frac{3037c_1c_2}{1800} - \frac{29c_2^2}{600} + \frac{14c_3}{75} \]

\[ m_2 = -\frac{43}{38160} + \frac{149c_1^2}{120} + \frac{217c_1c_2}{720} + \frac{81c_2^2}{40} + \frac{13c_3}{60} \]

\[ m_3 = \frac{43}{44520} + \frac{77c_1^2}{30} + \frac{497c_1c_2}{360} - \frac{17c_2^2}{60} - \frac{7c_3}{30} \]

\[ m_{14} = \frac{43}{51940} - \frac{14c_1^2}{5} + \frac{781c_1c_2}{60} - \frac{151c_2^2}{10} - \frac{11c_3}{5} \]

\[ m_{15} = \frac{43}{77910} - \frac{58c_1^2}{15} + \frac{71c_1c_2}{15} - \frac{151c_2^2}{15} - \frac{4c_3}{5} \]

\[ m_{22} = \frac{5c_1^2}{2} + \frac{155c_1c_2}{12} + c_3 \quad m_{23} = -\frac{5c_1^2}{2} - \frac{155c_1c_2}{12} + c_3 \]

\[ m_{24} = 3c_1^2 + \frac{5c_1c_2}{3} + 2c_3 \quad m_{25} = \frac{17c_1^2}{6} + \frac{3c_1c_2}{4} + c_3 \quad m_{26} = -\frac{215c_1c_2}{12} + c_3 \]

\[ a_{10} = -\frac{1}{2646\sqrt{742}} + \frac{239c_1}{155820} + \frac{c_1^3}{30} + \frac{163c_2}{20776} + \frac{13c_1^2c_2}{30} - \frac{11c_1c_2^2}{10} - \frac{13c_2^3}{12} - \frac{c_1c_3}{5} \]
\[
m_{16} = \frac{1}{3969 \sqrt{742}} + \frac{1849 c_1}{1869840} - \frac{899 c_1^3}{360} - \frac{473 c_2}{93492} - \frac{1907 c_1^2 c_2}{360} + \frac{217 c_1 c_2^2}{18} + \frac{199 c_2^3}{30} + c_2 c_3
\]

\[
m_{17} = \frac{5}{5292 \sqrt{742}} - \frac{43 c_1}{38955} - \frac{103 c_1^3}{30} - \frac{258 c_2}{12985} - \frac{799 c_1^2 c_2}{90} + \frac{2659 c_1 c_2^2}{15} + \frac{391 c_2^3}{15} + 82 c_2 c_3
\]

\[
m_{18} = -\frac{11}{15876 \sqrt{742}} + \frac{145 c_1^3}{36} + \frac{13373 c_2}{934920} + \frac{1063 c_1^2 c_2}{120} + \frac{994 c_1 c_2^2}{45} - \frac{3581 c_2^3}{180} - \frac{7 c_1 c_3}{2} - \frac{56 c_2 c_3}{15}
\]

With the above explicit expressions for all these coefficients, the BRST operator is shown to be nilpotent for arbitrary \( c_i \)'s. We will discuss this solution in the next section.

Quite similarly but becoming more complicated, the above procedure can be carried out for the \( W_4 \) algebra. The central charge must be 246 in order to have a nilpotent BRST operator. This is the number we would expect for the ghost antighost system \((b, c), (\beta, \gamma)\) and \((\alpha, \delta)\). However the explicit expression of all these coefficients are quite long. Let us first state the results in words. All the coefficients are determined as algebraic functions of seven coefficients which we choose (quite arbitrarily) to be \( c_{15}, c_{16}, m_8, m_{14}, m_{22}, m_{29} \) and \( m_{32} \). The explicit expression of all the coefficients in terms of these seven (free) coefficients are quite long. Nevertheless, with these explicit expressions we do succeed in proving that the BRST operator is indeed nilpotent.

Having stated the result in general terms let us be more specific. As we will discuss in more details for the BRST operator of the \( WB_2 \) algebra in the next section, a quite natural procedure to fix these free coefficients is to require \( \{Q, b(z)\} \) to be the total stress-energy tensor. By imposing this condition, five out of the seven free coefficients are fixed. Quite surprisingly the resulting expression for all the coefficients become quite simple. 36 coefficients are identically equal to zero. These are: \( a_{10} \) to \( a_{14} \), \( c_{13} \) to \( c_{16} \), \( m_8 \) to \( m_{13} \), \( m_{26} \) and \( m_{21} \), \( m_{25} \) to \( m_{27} \), \( m_{34} \) to
Some coefficients are pure numbers:

\[ c_3 = 2 \quad a_9 = \frac{506}{1565} \quad a_{25} = \frac{3}{8} \]

\[ a_{27} = \frac{139909}{2449225} \quad c_{11} = -\frac{1703}{6260} \quad m_{33} = -\frac{69}{64} \]

Setting \( m_{22} = cv \) and trading the other free parameters as \( vc \) (which is identified with the free parameter \( \beta \) in ref. [14]) the rest coefficients are

\[
\begin{align*}
a_1 &= \frac{37 \, cv}{26} \\
a_7 &= \frac{258819}{1252000} - \frac{143301 \, cv^2}{66248} \\
a_{22} &= \frac{1565}{1565} - \frac{106 \, cv}{91} \\
a_{26} &= -\frac{459}{1565} + \frac{185 \, cv}{182} \\
c_4 &= \frac{12520}{12520} + \frac{6 \, cv}{91} \\
c_7 &= \frac{9891}{62600} + \frac{20 \, cv}{91} \\
m_{30} &= \frac{1518}{1565} - \frac{459 \, cv}{91} \\
m_{58} &= -\frac{918}{1565} + \frac{279 \, cv}{91}
\end{align*}
\]

\[
\begin{align*}
a_2 &= -\frac{159 \, cv}{91} \\
a_8 &= -\frac{185 \, cv}{182} \\
a_{23} &= -\frac{1059}{1565} + \frac{365 \, cv}{182} \\
c_1 &= \frac{18780}{1547} - \frac{1271 \, cv}{675} \\
c_5 &= -\frac{12520}{1513} + \frac{364 \, cv}{185} \\
c_{12} &= \frac{25040}{104} \\
m_{31} &= -\frac{1941}{6260} + \frac{495 \, cv}{182} \\
m_{59} &= -\frac{3777}{3130} + \frac{1275 \, cv}{182}
\end{align*}
\]

\[
\begin{align*}
a_{16} &= \frac{512072}{2449225} - \frac{116127 \, cv}{142415} \\
a_{24} &= -\frac{459}{3130} + \frac{47 \, cv}{182} \\
c_2 &= \frac{12397}{18780} - \frac{159 \, cv}{91} \\
c_6 &= \frac{832}{7825} + \frac{35 \, cv}{52} \\
m_{23} &= \frac{2024}{1565} - \frac{703 \, cv}{91} \\
m_{32} &= -\frac{15931409}{1773553152} - 11 \, vc \\
m_{52} &= -\frac{1773553152}{8367} + \frac{1335 \, cv}{182} \\
m_{60} &= -\frac{6260}{6260} + \frac{1335 \, cv}{182}
\end{align*}
\]

and
\[ a_4 = \frac{145947672356389}{763292937792000} + vc + \frac{621693 \cdot cv}{2506504} - \frac{1060309 \cdot cv^2}{1093092} \]

\[ a_5 = -\frac{355891353306971}{763292937792000} + vc + \frac{621693 \cdot cv}{2506504} + \frac{1698633 \cdot cv^2}{728728} \]

\[ a_6 = \frac{48206198862109}{381646468896000} + 2 \cdot vc + \frac{621693 \cdot cv}{1253252} - \frac{222333 \cdot cv^2}{104104} \]

\[ a_{17} = \frac{80986927589317}{381646468896000} + 2 \cdot vc + \frac{621693 \cdot cv}{1253252} - \frac{2697385 \cdot cv^2}{364364} \]

\[ a_{18} = -\frac{381646468896000}{76962564246203} + 2 \cdot vc + \frac{621693 \cdot cv}{1253252} + \frac{793465 \cdot cv^2}{364364} \]

\[ a_{19} = -\frac{30689454039803}{190823234448000} + 4 \cdot vc + \frac{621693 \cdot cv}{626626} + \frac{4995 \cdot cv^2}{91091} \]

\[ a_{20} = -\frac{33360383297723}{381646468896000} + 2 \cdot vc + \frac{621693 \cdot cv}{1253252} - \frac{489135 \cdot cv^2}{364364} \]

\[ a_{21} = \frac{4209291047557}{381646468896000} + 2 \cdot vc - \frac{32547 \cdot cv}{89518} + \frac{123789 \cdot cv^2}{91091} \]

\[ c_8 = \frac{49068317152000}{38110049205397} - 14 \cdot vc - \frac{1537379 \cdot cv}{10742160} - \frac{534983 \cdot cv^2}{1093092} \]

\[ c_9 = \frac{52763531459717}{366380610140160} + 13 \cdot vc - \frac{371291 \cdot cv}{2148432} - \frac{6567019 \cdot cv^2}{8744736} \]

\[ c_{10} = \frac{26716946702321}{785101307443200} + 7 \cdot vc + \frac{52087 \cdot cv}{661056} - \frac{712139 \cdot cv^2}{8744736} \]
\[
m_{14} = \frac{206930376169691}{163562772384000} - \frac{14}{3} + \frac{113136277}{18798780} + \frac{938965}{182182} \]  
\[
m_{15} = -\frac{87631707259531}{25443097926400} - \frac{2}{vc} + \frac{156384181}{7519512} - \frac{5820277}{182182} \]  
\[
m_{16} = \frac{660084599304911}{38164646889600} + \frac{6}{vc} - \frac{3226031}{289212} + \frac{3572671}{182182} \]  
\[
m_{17} = -\frac{86745615934751}{76329293779200} + \frac{2}{vc} + \frac{80929421}{9399390} - \frac{3112537}{182182} \]  
\[
m_{24} = \frac{2368726689487}{38164646889600} - \frac{4}{vc} - \frac{4433323}{5013008} - \frac{271025}{56056} \]  
\[
m_{29} = -\frac{125960351}{117562800} + \frac{16960361}{3417960} + \frac{12301}{728728} \]  
\[
m_{45} = \frac{1823102103976543}{4579757626752000} + \frac{19}{vc} - \frac{79769}{447590} + \frac{175935}{364364} \]  
\[
m_{46} = -\frac{143117425836000}{39187600} + \frac{3}{vc} + \frac{1790360}{2278640} + \frac{728728}{66248} \]  
\[
m_{47} = \frac{523287237}{39187600} - \frac{10763643}{2278640} - \frac{264207}{66248} \]  
\[
m_{48} = \frac{11520872866549}{109041848256000} + \frac{7}{vc} - \frac{7651269}{12532520} + \frac{4241739}{364364} \]  
\[
m_{49} = -\frac{40592014272799}{152658587558400} + \frac{vc}{vc} - \frac{8187009}{2566504} + \frac{4962}{1001} \]  
\[
m_{50} = \frac{136142989370639}{114993940668800} - \frac{20}{vc} + \frac{13780917}{2566504} + \frac{136075}{728728} \]  
\[
m_{51} = -\frac{1001912494309861}{763292937792000} - \frac{vc}{vc} + \frac{9178583}{6266260} + \frac{635931}{364364} \]  
\[
m_{52} = -\frac{373185538120107}{127215489632000} + \frac{6}{vc} + \frac{132646001}{12532520} - \frac{295651}{91091} \]  
\[
m_{53} = \frac{23956380445407}{95411617224000} + \frac{8}{vc} - \frac{260289693}{2566504} + \frac{1570815}{182182} \]  
\[
m_{54} = \frac{113975631706187}{152658587558400} - \frac{5}{vc} - \frac{30290837}{5013008} + \frac{6156635}{728728} \]  
\[
m_{55} = -\frac{130048022434427}{457975762675200} + \frac{5}{vc} - \frac{1336781}{1253252} + \frac{16525}{4004} \]  
\[
m_{56} = -\frac{63541384149247}{38164646889600} + \frac{4}{vc} + \frac{17699379}{6266260} + \frac{467931}{182182} \]  
\[
m_{57} = \frac{12043053}{4898450} - \frac{3067341}{227864} + \frac{607545}{33124} \]  
\[
m_{66} = -\frac{7058543840763}{95411617224000} + \frac{8}{vc} + \frac{8732908}{1566565} - \frac{962718}{91091} \]
\[
\begin{align*}
m_{67} &= \frac{73757449136699}{76329293779200} - 10 \, \text{vc} \left( -\frac{12120831}{1253252} \right) + \frac{8735145\, \text{cv}^2}{364364} \\
m_{68} &= -\frac{1558365166597}{95411617224000} - 8 \, \text{vc} \left( -\frac{13299969}{6266260} \right) - \frac{162615\, \text{cv}^2}{14014} \\
a_{15} &= \frac{209663806}{3833037125} - \frac{14389709725980989\, \text{cv}}{46306438226048000} + \frac{789\, \text{vc}\, \text{cv}}{182} \\
&\quad + \frac{490515777\, \text{cv}^2}{395882425\, \text{cv}^3} + \frac{456183728}{33157124} \\
m_{18} &= \frac{11126358008106501049}{11467713097387008000} + \frac{101333\, \text{vc}}{15024} - \frac{230025922125632783\, \text{cv}}{333406355227545600} \\
&\quad + \frac{145\, \text{vc}\, \text{cv}}{168} + \frac{206953330853\, \text{cv}^2}{18247349120} + \frac{507905105\, \text{cv}^3}{122426304} \\
m_{32} &= \frac{4320751301911}{2943772512000} + \frac{14751898953\, \text{cv}}{1426428640} - \frac{133780861\, \text{cv}^2}{7405580} - \frac{6840745\, \text{cv}^3}{10334688} \\
m_{19} &= \frac{21227957302577}{273040788033024000} + \frac{2205\, \text{vc}}{2504} + \frac{224455082507749573\, \text{cv}}{46306438226048000} \\
&\quad - \frac{933\, \text{vc}\, \text{cv}}{182} - \frac{28655403533\, \text{cv}^2}{795770976} - \frac{3964530205\, \text{cv}^3}{395882425\, \text{cv}^3} \\
m_{69} &= \frac{2934212453988242167}{159273930192640000} + \frac{12513\, \text{vc}}{6260} - \frac{191792451947131327\, \text{cv}}{11576609556512000} \\
&\quad - \frac{618\, \text{vc}\, \text{cv}}{91} + \frac{253409589117\, \text{cv}^2}{4561837280} + \frac{1364553837\, \text{cv}^3}{18946928} \\
m_{70} &= \frac{95702179714575577}{2389106895288960000} + \frac{21179\, \text{vc}}{3130} + \frac{39163300881728239\, \text{cv}}{27783862935628800} \\
&\quad - \frac{1489\, \text{vc}\, \text{cv}}{182} + \frac{186796629\, \text{cv}^2}{912367456} - \frac{177977111\, \text{cv}^3}{33157124} \\
m_{71} &= \frac{3725038049030721}{72397178645120000} + \frac{52371\, \text{vc}}{3130} + \frac{790319105345479\, \text{cv}}{485731869504000} \\
&\quad - \frac{3883\, \text{vc}\, \text{cv}}{91} + \frac{20276331\, \text{cv}^2}{3190096} - \frac{59350407\, \text{cv}^3}{3014284} \\
m_{72} &= \frac{8572847559656227}{20076528531840000} + \frac{46529\, \text{vc}}{3130} + \frac{51941209398171283\, \text{cv}}{6314514303552000} \\
&\quad - \frac{1901\, \text{vc}\, \text{cv}}{91} + \frac{10675480773\, \text{cv}^2}{285114830} + \frac{3255147397\, \text{cv}^3}{66314248}
\end{align*}
\]

In the next section after we have a better understanding of the connection between these free parameters and the canonical transformation of the ghost antighost fields, we will show how one can obtain the complete solution from the above...
special solution.
5. Discussions

In this section we will analyse the solutions obtained in the last section in details and explicitly show that the free parameters in the BRST operator are related to the canonical transformation of the ghost antighost fields as first discussed in ref. [14] for the BRST operator of the $W_4$ algebra.

First let us look at the BRST operator of the $W_{B_2}$ algebra. As in other extended conformal algebras, we define a generalized (total) stress-energy tensor

$$T_{\text{tot.}}(z) \equiv \{Q, b(z)\} = T(z) + 2c^\prime(z)b(z) + c(z)b^\prime(z) + 4\delta^\prime(z)\alpha(z) + 3\delta(z)\alpha^\prime(z) + (a_6b\delta^\prime + a_7b\delta)T$$

$$- a_8(b\delta^\prime)^\prime + a_{11}bb^\prime\delta^\prime T + m_8b\delta^{(3)} + m_9b\delta^\prime + m_{10}b\delta^\prime^\prime$$

$$+ a_{11}bb\delta^\prime T + m_{13}(c'b^\prime + (cbb')^\prime) + (m_{19}b\delta'\alpha + m_{20}b\delta'\alpha' + m_{21}b\delta''\alpha)\delta$$

$$+ (m_{22}bb\delta^{(3)} + m_{23}bb\delta'\delta'' + m_{24}bb'\delta\delta'' + m_{25}bb^{(3)}\delta^\prime + m_{26}bb''\delta\delta').$$

(29)

As one can see from the above, this stress-energy tensor contains more terms than needed (we need only the terms on the second line of the above equation). So a natural choice for the free parameters $c_i$’s is to require these extra terms to be zero. This fixes all the $c_i$’s to zero. With these zero $c_i$, all the coefficients become pure numbers and we have a unique BRST operator. This BRST operator is the contour integration of the following BRST current

$$j = c(z)T(z) + \delta(z)U(z) + b\delta^\prime \left( \frac{253}{65444} \Lambda_1 + \frac{1}{21\sqrt{742}} U \right)$$

$$+ \frac{1}{1060} \left( b'\delta'' + b''\delta' - \frac{1}{6}b\delta^{(4)} + \frac{1}{7}b\delta^{(3)} \right) T$$

$$+ \frac{43}{636} \left( \left( \frac{1}{6}b\delta^{(5)} - \frac{1}{10}b\delta^{(4)} + \frac{1}{7}b\delta^{(3)} \right) + c(4\delta'\alpha + 3\delta\alpha') + cc'b \right) (30)$$

$$- \frac{1}{2646\sqrt{742}} cbb'\delta^\prime T + \frac{43}{25970} \left( \frac{1}{2}b\delta^\prime\delta''\alpha' + \frac{1}{3}b\delta^\prime\delta^{(3)}\alpha \right)$$

$$+ \frac{1}{1323\sqrt{742}} \left( \frac{1}{3}b\delta^\prime\delta''\delta^{(4)} + \frac{5}{4}b\delta^\prime\delta''\delta^{(3)} - \frac{11}{12}b\delta^{(3)}\delta^\prime\delta'' \right).$$

Having obtained such simple expression for the BRST operator of the nonlinear
$WB_2$ algebra, we would like to ask why there appear so many free parameters in $Q$ and they could all be fixed by requiring that the total stress-energy tensor is the BRST commutator of the antighost $b(z)$. As it was first discussed in ref. [14] for the $W_4$ algebra, these free parameters are related to the canonical transformation of the ghost antighost fields. Now we show that this is also true for the $WB_2$ algebra.

If there are only $(b, c)$ and $(\beta, \gamma)$ ghost antighost fields, there doesn’t exist any canonical transformations (except the parity violating linear transformation). However for the ghost antighost fields $(b, c)$ and $(\alpha, \delta)$ appearing in the BRST quantization of the $WB_2$ algebra, there exists a three parameter family of canonical transformation. These transformations form an Abelian group. By explicit calculation we found that these transformations are generated from the following three independent transformations:

\[
\begin{align*}
\{b, c, \alpha, \delta\} &\longrightarrow \\
\left\{ b + c_1 b'b' \delta, c - c_1 (cb\delta' + 2 cb'\delta + c'b\delta) + \frac{c_1^3}{6} bb' \delta' \delta'' \right. \\
&\quad + \frac{c_1^2}{12} \left( 12 cb'\delta'\delta'' - 3(b\delta + 9 b'\delta' - 15 bb'\delta'' - 7 bb'(3)\delta) + \frac{13 c_1^3}{12} bb'b' \delta' \delta'' \right), \delta \\
&\left. + \frac{c_1^2}{12} \left( 6 b'b'' \delta - 9 bb' \delta'' - 15 bb'' \delta' - 7 bb'(3) \delta + \frac{13 c_1^3}{12} bb'b'' \delta' \delta'' \right) \right\}, \\
\{b, c - c_2 \delta' \alpha - \frac{c_2^2}{12} \left( 13 b\delta \delta''(3) + 27 b' \delta' \delta'' + 9 b'' \delta' \delta'' \right) + \frac{13 c_2^3}{12} bb' \delta' \delta'' \right\}, \\
\left\{ b, c - c_3 \left( b\delta(3) + 2 b\delta'' + 2 b' \delta'' \right), \delta + c_3 \left( 2 bb' \delta' + bb'(3) \delta + b'b' \delta \right) \right\},
\end{align*}
\]

(31)

where $c_1$, $c_2$ and $c_3$ are free parameters of these transformation. As one can show by explicit calculation (see below), non of the above transformations or any com-
A combination of them leaves the standard ghost antighost stress-energy tensor $T_{\text{gh.}}$, which is given by

$$T_{\text{gh.}}(z) = 2c'(z)b(z) + c(z)b'(z) + 4\delta'(z)\alpha(z) + 3\delta(z)\alpha'(z),$$  \hspace{1cm} (32)

invariant. This is why we get a unique BRST operator after imposing the condition: 
\(\{Q, b(z)\} = T(z) + T_{\text{gh.}}(z)\). The general solution can be obtained from this unique one just by doing a general canonical transformation of these ghost antighost fields.

Of course the reverse is also true. In this way we established the connection between the free parameters in the BRST operator and the canonical transformation of the ghost antighost fields for $W_{B2}$.

Quite similarly but becoming more complicated, the above connection also applies to the $W_4$ algebra, as first discussed in ref. [14]. Apart from the three parameter transformations given in eq. (31), we have four more independent canonical transformations by adding $(\beta, \gamma)$ to the existing ghost antighost fields in $W_{B2}$. Explicitly these four transformations are ($b$ and $\delta$ are unchanged)
\{c, \beta, \gamma, \alpha\} \rightarrow \\
\left\{c + c_4 \gamma' \beta + \frac{c_4^2}{12} \left(12b\gamma\beta\delta'' - b\delta\delta^{(3)} - 3b\delta\delta'' - 3b''\delta\delta' + 12b'\gamma\beta\delta' + 24b'\gamma\beta\delta\right) - \frac{2c_3^3}{3}bb\delta\delta''\delta', \beta - c_4b\beta\delta - c_4^2bb\delta\delta', \gamma - c_4 (b\gamma\delta + b\gamma'\delta + b'\gamma\delta) - c_4^2bb\gamma\delta\delta', \alpha - c_4b\gamma\beta' + \frac{2c_4^3}{3}bb'\delta\delta' + \frac{c_4^2}{12} \left(24bb'\gamma\beta\delta' - 3bb'\delta'' - 3bb'\delta' - bb^{(3)}\delta + 12bb'\gamma'\beta\delta + 12bb''\gamma\beta\right) \right\}, \\
\left\{c - c_5 \gamma' \beta\delta - \frac{c_5^3}{6} bb\delta\delta''\delta' - \frac{c_5^2}{12} \left(b\delta\delta^{(3)} + 3b'\delta\delta'' + 3b''\delta\delta' - 12b'\gamma'\beta\delta' \right), \beta - c_5 (b\beta\delta' + b\beta'\delta + b'\beta\delta'), \gamma - c_5 b\gamma'\delta, \\
\alpha + c_5 b\gamma'\beta - \frac{c_5^2}{12} \left(3bb''\delta'' + 3bb''\delta' + bb^{(3)}\delta - 12bb'\gamma'\beta\delta \right) + \frac{c_5^3}{6} bb'bb''\delta' \right\}, \\
\left\{b, c + c_6 \gamma' \beta\delta - \frac{c_6^2}{2} \left(b\delta\delta'' + 3b'\delta\delta'' \right), \beta - c_6 b\beta\delta', \gamma + c_6 b\gamma'\delta, \\
\alpha + c_6 (b\gamma'\beta + b\gamma'\beta + b'\gamma'\beta) + \frac{c_6^2}{2} \left(bb''\delta' + bb^{(3)}\delta + b'b''\delta - bb''\delta'' \right) \right\}, \\
\left\{b, c - c_7 (b\gamma'\beta\delta'' + b\gamma'\beta'\delta' + b\gamma'\beta\delta' + 2b'\gamma'\beta\delta') , \beta - c_7 bb'\beta\delta', \\
\gamma + c_7 bb'\gamma'\delta', \alpha - c_7 (2bb'\gamma'\beta\delta' + bb'\gamma'\beta'\delta + bb'\gamma'\beta\delta + bb''\gamma\beta \right) \right\}.

(33)

where $c_4$ to $c_7$ are free parameters of these transformations. In total we have a seven parameter canonical transformation of the ghost antighost field in $W_4$. So we should expect at least seven free parameters in the BRST operator if there exists one BRST operator. This is indeed the case as we have found precisely a seven parameter family of nilpotent BRST operator by explicit construction. As one can show by calculation there exist three independent canonical transformations, parametrized by $c_4$, $c_5$ and $c_7$ with the other $c_i$'s depending on them as follows

$$
\{c_1, c_2, c_3, c_6\} = \left\{ \frac{53}{64}a_4 - \frac{47}{384}a_5 - \frac{23}{64}a_4 - \frac{203}{384}a_5, \\
\frac{435}{4096}a_4^2 - \frac{21145}{12288}a_4a_5 - \frac{101045}{147456}a_5^2 - \frac{a_7}{2}, a_4 + \frac{2a_5}{3} \right\},
$$
which leave the ghost antighost stress-energy tensor invariant. Nevertheless only two of them also leave $b(z)$ unchanged. This is why we found only a two (instead of three) parameter family of BRST charge after imposing the condition:

$$T_{\text{tot.}} \equiv \{ Q, b(z) \} = T(z) + T_{\text{gh.}}(z),$$

(34)

where

$$T_{\text{gh.}}(z) = 2c'(z)b(z) + c(z)b'(z) + 3\gamma'(z)\beta(z) + 2\gamma(z)\beta'(z) + 4\delta'(z)\alpha(z) + 3\delta(z)\alpha'(z),$$

(35)

is the stress-energy tensor of the ghost antighost fields in $W_4$. With eqs. (31) and (33) and the explicit solution given in the last section, one can easily obtain the most general solution with seven free parameters.

One point worth noticing is that contrary to the general belief the nilpotent condition doesn’t fix all the coefficients. What is not quite interesting is the fact that these free parameters are all related the freedom of canonically changing the ghost antighost fields. We effectively proved a no-go theorem. As discussed in several papers [27, 13, 28], for higher spin extended $W$-algebra, it is often possible to split the BRST operator into a sum of mutually anticommuting and separately nilpotent operators. If this is the case it is quite easy to show that the BRST operator must depend on some free parameters. If $Q = Q_0 + \sum_{i=1}^{n} Q_i$ satisfy

$$\{ Q_i, Q_j \} = 0, \quad i, j = 0, 1, \ldots, n,$$

(36)

the operator $\tilde{Q} = Q_0 + \sum_{i=1}^{n} a_i Q_i$ is also nilpotent and depends on $n - 1$ free parameters. Because our ansatz for the BRST charge is the most general form and we found no free parameters in $Q$, this proves that in order to write $Q$ as the above form, one must use more inputs besides the algebra, i.e. using an explicit realization of the algebra*.

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