Idempotents of double Burnside algebras, 
$L$-enriched bisets, 
and decomposition of $p$-biset functors

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Abstract: Let $R$ be a (unital) commutative ring, and $G$ be a finite group with order invertible in $R$. We introduce new idempotents $\epsilon^G_{T,S}$ in the double Burnside algebra $RB(G,G)$ of $G$ over $R$, indexed by conjugacy classes of minimal sections $(T, S)$ of $G$ (i.e. sections such that $S \leq \Phi(T)$). These idempotents are orthogonal, and their sum is equal to the identity. It follows that for any biset functor $F$ over $R$, the evaluation $F(G)$ splits as a direct sum of specific $R$-modules indexed by minimal sections of $G$, up to conjugation.

The restriction of these constructions to the biset category of $p$-groups, where $p$ is a prime number invertible in $R$, leads to a decomposition of the category of $p$-biset functors over $R$ as a direct product of categories $F_L$ indexed by atoric $p$-groups $L$ up to isomorphism.

We next introduce the notions of $L$-enriched biset and $L$-enriched biset functor for an arbitrary finite group $L$, and show that for an atoric $p$-group $L$, the category $F_L$ is equivalent to the category of $L$-enriched biset functors defined over elementary abelian $p$-groups.

Finally, the notion of vertex of an indecomposable $p$-biset functor is introduced (when $p \in R^\times$), and when $R$ is a field of characteristic different from $p$, the objects of the category $F_L$ are characterized in terms of vertices of their composition factors.

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1. Introduction

Let $R$ denote throughout a commutative ring (with identity element). For a finite group $G$, we consider the double Burnside algebra $RB(G,G)$ of a $G$ over $R$. In the case where the order of $G$ is invertible in $R$, we introduce idempotents $e^G_{T,S}$ in $RB(G,G)$, indexed by the set $\mathcal{M}(G)$ of minimal sections of $G$, i.e. the set of pairs $(T, S)$ of subgroups of $G$ with $S \leq T$ and $S \leq \Phi(T)$, where $\Phi(T)$ is the Frattini subgroup of $G$ (such sections have been considered in Section 5 of [9]). The idempotent $e^G_{T,S}$ only depends of the conjugacy class of $(T, S)$ in $G$. Moreover, the idempotents $e^G_{T,S}$, where $(T, S)$ runs through a
set $[\mathcal{M}(G)]$ of representatives of orbits of $G$ acting on $\mathcal{M}(G)$ by conjugation, are orthogonal, and their sum is equal to the identity element of $RB(G,G)$.

The idempotents $\epsilon_{G,1}^G$ plays a special role in our construction, and it is denoted by $\varphi_{1}^G$. In particular, when $F$ is a biset functor over $R$ (and the order of $G$ is invertible in $R$), we set $\delta_F^G F(G) = \varphi_1^G F(G)$. We show that $\delta_F^G F(G)$ consists of those elements $u \in F(G)$ such that Res$^G_H u = 0$ whenever $H$ is a proper subgroup of $G$, and Def$^G_{G/N} u = 0$ whenever $N$ is a non-trivial normal subgroup of $G$ contained in $\Phi(G)$. This yields moreover a decomposition

$$F(G) \cong \bigoplus_{(T,S) \in \mathcal{M}(G)} \delta_F^G F(T/S) \cong \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \delta_F^G F(T/S)^{NC(T,S)/T}.$$

Restricting these constructions to the biset category $RC_p$ of $p$-groups with coefficients in $R$, where $p$ is a prime invertible in $R$, we get orthogonal idempotents $b_L$ in the center of $RC_p$, indexed by atoric $p$-groups, i.e. finite $p$-groups which cannot be split as a direct product $C_p \times Q$, for some $p$-group $Q$. We show next that every finite $p$-group $P$ admits a unique largest atoric quotient $P^\alpha$, well defined up to isomorphism, and that there exists an elementary abelian $p$-subgroup of $E$ of $P$ (non unique in general) such that $P \cong E \times P^\alpha$. For a given atoric $p$-group $L$, we introduce a category $RC_p^\alpha$, defined as a quotient of the subcategory of $RC_p$ consisting of $p$-groups $P$ such that $P^\alpha \cong L$. This leads to a decomposition of the category $F_{p,R}$ of $p$-biset functors over $R$ as a direct product

$$F_{p,R} \cong \prod_{L \in [\mathcal{A}_p]} \text{Fun}_R(RC_p^\alpha, R\text{-Mod})$$

of categories of representations of $RC_p^\alpha$ over $R$, where $L$ runs through a set $[\mathcal{A}_p]$ of isomorphism classes of atoric $p$-groups. Similar questions on idempotents in double Burnside algebras and decomposition of biset functors categories have been considered by L. Barker ([1]), R. Boltje and S. Danz ([2], [3]), R. Boltje and B. Külshammer ([4]), and P. Webb ([16]).

In particular, via the above decomposition, to any indecomposable $p$-biset functor $F$ is associated a unique atoric $p$-group, called the vertex of $F$. We show that this vertex is isomorphic to $Q^\alpha$, for any $p$-group $Q$ such that $F(Q) \neq \{0\}$ but $F$ vanishes on any proper subquotient of $Q$.

Going back to arbitrary finite groups, we next introduce the notions of $L$-enriched biset and $L$-enriched biset functor, and show that when $L$ is an atoric $p$-group, the abelian category $\text{Fun}_R(RC_p^\alpha, R\text{-Mod})$ is equivalent to the category of $L$-enriched biset functors from elementary abelian $p$-groups to $R$-modules.

The paper is organized as follows: Section 2 is a review of definitions and basic results on Burnside rings and biset functors. Section 3 is concerned
with the algebra $E(G)$ obtained by “cutting” the double Burnside algebra $RB(G, G)$ of a finite group $G$ by the idempotent $\tilde{e}_G^G$ corresponding to the “top” idempotent $e_G^G$ of the Burnside algebra $RB(G)$. Orthogonal idempotents $\varphi_N^G$ of $E(G)$ are introduced, indexed by normal subgroups $N$ of $G$ contained in $\Phi(G)$. It is shown moreover that if $G$ is nilpotent, then $\varphi_1^G$ is central in $E(G)$. In Section 4, the idempotents $\epsilon_{T,S}^G$ of $RB(G, G)$ are introduced, leading in Section 5 to the corresponding direct sum decomposition of the evaluation at $G$ of any biset functor over $R$. In Section 6, atoric $p$-groups are introduced, and their main properties are stated. In Section 7, the biset category of $p$-groups over $R$ is considered, leading to a splitting of the category $\mathcal{F}_{p,R}$ of $p$-biset functors over $R$ as a direct product of abelian categories $\mathcal{F}_L = \text{Fun}_R(RC_p^L, R\text{-Mod})$ indexed by atoric $p$-groups $L$ up to isomorphism. In Section 8, for an arbitrary finite group $L$, the notions of $L$-enriched biset and $L$-enriched biset functor are introduced, and it is shown that when $L$ is an atoric $p$-group, the category $\mathcal{F}_L$ is equivalent to the category of $L$-enriched biset functors on elementary abelian $p$-groups. Finally, in Section 9, for a given atoric $p$-group $L$, and when $p$ is invertible in $R$, the structure of the category $\mathcal{F}_L$ is considered, and the notion of vertex of an indecomposable $p$-biset functor over $R$ is introduced. In particular, when $R$ is a field of characteristic different from $p$, it is shown that the objects of $\mathcal{F}_L$ are those $p$-biset functors all composition factors of which have vertex $L$.

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2. Review of Burnside rings and biset functors

2.1. Let $G$ be a finite group, let $s_G$ denote the set of subgroups of $G$, let $\overline{s_G}$ denote the set of conjugacy classes of subgroups of $G$, and let $[s_G]$ denote a set of representatives of $\overline{s_G}$.

Let $\mathcal{B}(G)$ denote the Burnside ring of $G$, i.e. the Grothendieck ring of the category of finite $G$-sets. It is a commutative ring, with an identity element, equal to the class of a $G$-set of cardinality 1. The additive group $\mathcal{B}(G)$ is a free abelian group on the set $\{[G/H] \mid H \in [s_G]\}$ of isomorphism classes of transitive $G$-sets.

2.2. When $G$ and $H$ are finite groups, and $L$ is a subgroup of $G \times H$, set

- $p_1(L) = \{ g \in G \mid \exists h \in H, (g, h) \in L \}$,
- $p_2(L) = \{ h \in H \mid \exists g \in G, (g, h) \in L \}$,
- $k_1(L) = \{ g \in G \mid (g, 1) \in L \}$,
- $k_2(L) = \{ h \in H \mid (1, h) \in L \}$.

Recall that $k_i(L) \unlhd p_i(L)$, for $i \in \{1, 2\}$, that $(k_1(L) \times k_2(L)) \unlhd L$, and that there are canonical isomorphisms

$$p_1(L)/k_1(L) \cong L/(k_1(L) \times k_2(L)) \cong p_2(L)/k_2(L) \, .$$

Set moreover $q(L) = L/(k_1(L) \times k_2(L))$.

- When $Z$ is a subgroup of $G$, set

$$\Delta(Z) = \{(z, z) \mid z \in Z\} \leq (G \times G) \, .$$

When $N$ is a normal subgroup of $G$, set

$$\Delta_N(G) = \{(a, b) \in G \times G \mid ab^{-1} \in N\} \, .$$

It is a subgroup of $G \times G$.

- When $G$, $H$, and $K$ are groups, when $L \leq (G \times H)$ and $M \leq (H \times K)$, set

$$L * M = \{(g, k) \in (G \times K) \mid \exists h \in H, (g, h) \in L \text{ and } (h, k) \in K\} \, .$$

It is a subgroup of $(G \times K)$.

2.3. When $G$ and $H$ are finite groups, a $(G, H)$-biset $U$ is a set endowed with
a left action of $G$ and a right action of $H$ which commute. In other words $U$ is a $G \times H^{\text{op}}$-set, where $H^{\text{op}}$ is the opposite group of $H$. The opposite biset $U^{\text{op}}$ is the $(H, G)$-biset equal to $U$ as a set, with actions defined for $h \in H$, $u \in U$ and $g \in G$ by $h \cdot u \cdot g$ (in $U^{\text{op}}$) = $g^{-1}uh^{-1}$ (in $U$).

The Burnside group $B(G, H)$ is the Grothendieck group of the category of finite $(G, H)$-bisets. It is a free abelian group on the set of isomorphism classes $[(G \times H)/L]$, for $L \in [s_{G \times H}]$, where the $(G, H)$-biset structure on $(G \times H)/L$ is given by

$$\forall a, g \in G, \forall b, h \in H, a \cdot (g, h)L \cdot b = (ag, b^{-1}h)L.$$ When $G, H,$ and $K$ are finite groups, there is a unique bilinear product

$$\times_H : B(G, H) \times B(H, K) \to B(G, K)$$

induced by the usual product $(U, V) \mapsto U \times_H V = (U \times V)/H$ of bisets, where the right action of $H$ on $U \times V$ is defined for $u \in U$, $v \in V$ and $h \in H$ by $(u, v) \cdot h = (uh, h^{-1}v)$. This product will also be denoted as a composition $(\alpha, \beta) \mapsto \alpha \circ \beta$ or as a product $(\alpha, \beta) \mapsto \alpha \beta$.

This leads to the following definitions:

2.4. Definition: The biset category of finite groups $\mathcal{C}$ is defined as follows:

- The objects of $\mathcal{C}$ are the finite groups.
- When $G$ and $H$ are finite groups,
  $$\text{Hom}_\mathcal{C}(G, H) = B(H, G).$$
- When $G$, $H$, and $K$ are finite groups, the composition
  $$\circ : \text{Hom}_\mathcal{C}(H, K) \times \text{Hom}_\mathcal{C}(G, H) \to \text{Hom}_\mathcal{C}(G, K)$$
  is the product
  $$\times_H : B(K, H) \times B(H, G) \to B(K, G).$$
- The identity morphism of the group $G$ is the class of the set $G$, viewed as a $(G, G)$-biset by left and right multiplication.

A biset functor is an additive functor from $\mathcal{C}$ to the category of abelian groups.
When $R$ is a commutative (unital) ring, the category $RC$ is defined similarly by extending coefficients to $R$, i.e. by setting

$$\text{Hom}_{RC}(G, H) = R \otimes_{\mathbb{Z}} B(H, G) ,$$

which will be simply denoted by $RB(H, G)$. A biset functor over $R$ is an $R$-linear functor from $RC$ to the category $R\text{-Mod}$ of $R$-modules. The category of biset functors over $R$ (where morphisms are natural transformations of functors) is denoted by $\mathcal{F}_R$.

The correspondence sending a $(G, H)$-biset $U$ to its opposite $U^{op}$ extends to an isomorphism of $R$-modules $RB(G, H) \to RB(H, G)$. These isomorphisms give an equivalence of $R$-linear categories from $RC$ to its opposite category, which is the identity on objects.

2.5. Let $G$ and $H$ be finite groups, and $F$ be a biset functor (with values in $R\text{-Mod}$). For any finite $(H, G)$-biset $U$, the isomorphism class $[U]$ of $U$ belongs to $B(H, G)$, and it yields an $R$-linear map $F([U]) : F(G) \to F(H)$, simply denoted by $F(U)$, or even $f \in F(G) \mapsto U(f) \in F(H)$. In particular:

- When $H$ is a subgroup of $G$, denote by $\text{Ind}_H^G$ the set $G$, viewed as a $(G, H)$-biset for left and right multiplication, and by $\text{Res}_H^G$ the same set, viewed as an $(H, G)$-biset. This gives a map $\text{Ind}_H^G : F(H) \to F(G)$, called induction, and a map $\text{Res}_H^G : F(G) \to F(H)$, called restriction.

- When $N$ is a normal subgroup of $G$, let $\text{Inf}_G^{G/N}$ denote the set $G/N$, viewed as a $(G, G/N)$-biset for the left action of $G$, and right action of $G/N$ by multiplication. Also let $\text{Def}_G^{G/N}$ denote the set $G/N$, viewed as a $(G/N, G)$-biset. This gives a map $\text{Inf}_G^{G/N} : F(G/N) \to F(G)$, called inflation, and a map $\text{Def}_G^{G/N} : F(G) \to F(G/N)$, called deflation.

- Finally, when $f : G \to G'$ is a group isomorphism, denote by $\text{Iso}(f)$ the set $G'$, viewed as a $(G', G)$-biset for left multiplication in $G'$, and right action of $G$ given by multiplication by the image under $f$. This gives a map $\text{Iso}(f) : F(G) \to F(G')$, called transport by isomorphism.

When $G$ and $H$ are finite groups, any $(G, H)$-biset is a disjoint union of transitive ones. It follows that any element of $B(G, H)$ is a linear combination of morphisms of the form $[(G \times H)/L]$, where $L \in s_{G \times H}$. Moreover, any such morphism factors as

$$[(G \times H)/L] = \text{Ind}_G^{p_1(L)} \circ \text{Inf}^{p_1(L)/k_1(L)} \circ \text{Iso}(f) \circ \text{Def}_G^{p_2(L)/k_2(L)} \circ \text{Res}_G^{H/p_2(L)} ,$$

where $f : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$ is the canonical group isomorphism.

In particular, for $N \triangleleft G$,

$$[(G \times G)/\Delta_N(G)] = \text{Inf}_G^{G/N} \circ \text{Def}_G^{G/N} .$$
For finite groups $G, H, K$, for $L \leq (G \times H)$ and $M \leq (H \times K)$, one has that

\[
\frac{(G \times H)}{L} \times_H \frac{(H \times K)}{M} = \sum_{h \in p_2(L) \setminus H/p_1(M)} \frac{(G \times K)}{(L * (h,1)) M}
\]

in $B(G, K)$.

2.9. When $G$ is a finite group, the group $B(G, G)$ is the ring of endomorphisms of $G$ in the category $C$. This ring is called the double Burnside ring of $G$. It is a non-commutative ring (if $G$ is non trivial), with identity element equal to the class of the set $G$, viewed as a $(G, G)$-biset for left and right multiplication.

There is a unitary ring homomorphism $\alpha \mapsto \tilde{\alpha}$ from $B(G)$ to $B(G, G)$, induced by the functor $X \mapsto \tilde{X}$ from $G$-sets to $(G, G)$-bisets, where $\tilde{X} = G \times X$, with $(G, G)$-biset structure given by

\[
\forall a, b, g \in G, \forall x \in X, a(g, x)b = (a g b, a x).
\]

This construction has in particular the following properties ([7], Corollary 2.5.12):

2.10. Lemma: Let $G$ be a finite group.

1. If $H$ is a subgroup of $G$, and $X$ is a finite $G$-set, then there is an isomorphism of $(G, H)$-bisets

\[
\tilde{X} \times_G \text{Ind}^G_H X \cong \text{Ind}^G_H \times_H \text{Res}^G_H X,
\]

and an isomorphism of $(H, G)$-bisets

\[
\text{Res}^G_H \times_G \tilde{X} \cong \text{Res}^G_H X \times_H \text{Res}^G_H.
\]

2. If $H$ is a subgroup of $G$, and $Y$ is a finite $H$-set, then there is an isomorphism of $(G, G)$-bisets

\[
\text{Ind}^G_H \times_H \tilde{Y} \times_H \text{Res}^G_H Y \cong \text{Ind}^G_H Y.
\]

3. If $N$ is a normal subgroup of $G$, and $X$ is a finite $G/N$-set, then there is an isomorphism of $(G/N, G)$-bisets

\[
\tilde{X} \times_{G/N} \text{Def}^G_{G/N} X \cong \text{Def}^G_{G/N} \times_G \text{Inf}^G_{G/N} X.
\]
4. If $N$ is a normal subgroup of $G$, and $X$ is a finite $G$-set, then there is an isomorphism of $(G/N, G/N)$-biset

$$\text{Def}_{G/N}^G \times_G X \times_G \text{Inf}_{G/N}^G \cong \text{Def}_{G/N}^G X.$$ 

2.11. Let $RB(G)$ denote the $R$-algebra $R \otimes \mathbb{Z} B(G)$. Assume moreover that the order of $G$ is invertible in $R$. Then for $H \leq G$, let $e_H^G \in RB(G)$ be defined by

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H) [G/K],$$ 

where $\mu$ is the Möbius function of the poset of subgroups of $G$. The elements $e_H^G$, for $H \in [s_G]$, are orthogonal idempotents of $RB(G)$, and their sum is equal to the identity element of $RB(G)$. It follows that the elements $\tilde{e}_H^G$, for $H \in [s_G]$, are orthogonal idempotents of the $R$-algebra $RB(G, G) = R \otimes \mathbb{Z} B(G, G)$, and the sum of these idempotents is equal to the identity element of $RB(G, G)$. The idempotents $\tilde{e}_G^G$ play a special role, due to the following lemma:

2.13. Lemma: Let $R$ be a commutative ring, and $G$ be a finite group with order invertible in $R$.

1. Let $H$ be a proper subgroup of $G$. Then

$$\text{Res}_H^G \circ \tilde{e}_G^G = 0 \text{ and } \tilde{e}_G^G \circ \text{Ind}_H^G = 0.$$

2. When $N \trianglelefteq G$, let $m_{G,N} \in R$ be defined by

$$m_{G,N} = \frac{1}{|G|} \sum_{X \in [s_G]} |X| \mu(X, G).$$

Then

$$\text{Def}_{G/N}^G \circ \tilde{e}_G^G \circ \text{Inf}_{G/N}^G = m_{G,N} \tilde{e}_{G/N}^G.$$

3. Let $N \trianglelefteq G$, and suppose that $N$ is contained in the Frattini subgroup
\( \Phi(G) \) of \( G \). Then
\[
\widetilde{e}^{G/N}_{G/N} \circ \text{Def}^G_{G/N} = \text{Def}^G_{G/N} \circ \widetilde{e}^G_G \quad \text{and} \quad \text{Inf}^G_{G/N} \circ \widetilde{e}^{G/N}_{G/N} = \widetilde{e}^G_G \circ \text{Inf}^G_{G/N} .
\]

**Proof:** Assertion 1 follows from Lemma 2.10 and Assertion 1 of Theorem 5.2.4. of [7].

Assertion 2 follows from Lemma 2.10 and Assertion 4 of Theorem 5.2.4. of [7].

Finally the first part of Assertion 3 follows from Lemma 2.10 and Assertion 3 of Theorem 5.2.4. of [7]: indeed \( \text{Inf}^G_{G/N}\widetilde{e}^{G/N}_{G/N} \) is equal to the sum of the different idempotents \( e^G_X \) of \( \text{RB}(G, G) \) indexed by subgroups \( X \) such that \( XN = G \). If \( N \leq \Phi(G) \), then \( XN = G \) implies \( X\Phi(G) = G \), hence \( X = G \). The second part of Assertion 3 follows by taking opposite bisets, since \( \widetilde{e}^G_G \) and \( \widetilde{e}^{G/N}_{G/N} \) are equal to their opposite bisets, and since \( (\text{Def}^G_{G/N})^{\text{op}} \cong \text{Inf}^G_{G/N} \).

**2.14. Remark:** For the same reason, if \( N \leq \Phi(G) \), then \( m_{G,N} = 1 \).

**2.15. Remark:** It follows from Assertion 1 and Remark 2.6 that if \( G \) and \( H \) are finite groups and if \( L \leq (G \times H) \), then \( e^G_G[(G \times H)/L] = 0 \) if \( p_1(L) \neq G \), and \( [(G \times H)/L]e^H_H = 0 \) if \( p_2(L) \neq H \).

## 3. Idempotents in \( \mathcal{E}(G) \)

**3.1. Notation:** When \( G \) is a finite group with order invertible in \( R \), denote by \( \mathcal{E}(G) \) the \( R \)-algebra defined by
\[
\mathcal{E}(G) = \widetilde{e}^G_G\text{RB}(G, G)\widetilde{e}^G_G .
\]
Set
\[
\Sigma(G, G) = \{ M \in s_{G \times G} \mid p_1(L) = p_2(L) = G \} ,
\]
and for \( L \in s_{G \times G} \), set
\[
Y_L = \widetilde{e}^G_G[(G \times G)/L]\widetilde{e}^G_G \in \mathcal{E}(G) .
\]

The \( R \)-algebra \( \mathcal{E}(G) \) has been considered in [5], Section 9, in the case \( R \) is a field of characteristic 0. The extension of the results proved there to the
case where $R$ is a commutative ring in which the order of $G$ is invertible is straightforward. In particular:

**3.2. Proposition:** Let $G$ be a finite group with order invertible in $R$.

1. If $L \in s_{G \times G} - \Sigma(G, G)$, then $Y_L = 0$.
2. The elements $Y_L$, for $L$ in a set of representatives of $(G \times G)$-conjugacy classes on $\Sigma(G, G)$, form a $R$-basis of $E(G)$.
3. For $L, M \in \Sigma(G, G)$

$$Y_L Y_M = \frac{m_{G,k_2(L) \cap k_1(M)}}{|G|} \sum_{Z \leq G} |Z| \mu(Z, G) Y_{L \Delta(Z) \ast M}$$

in $E(G)$.

**3.3. Corollary:** Let $L, M \in \Sigma(G, G)$. If one of the groups $k_2(L)$ or $k_1(M)$ is contained in $\Phi(G)$, then

$$Y_L Y_M = Y_{L \ast M} .$$

**Proof:** Indeed if $k_2(L) \leq \Phi(G)$, then $Z k_2(L) = G$ implies $Z \Phi(G) = G$, hence $Z = G$. Similarly, if $k_1(M) \leq \Phi(G)$, then $Z k_1(M) = G$ implies $Z = G$. In each case, Proposition [3.2] then gives

$$Y_L Y_M = m_{G,k_2(L) \cap k_1(M)} Y_{L \ast M} ;$$

and moreover $m_{G,k_2(L) \cap k_1(M)} = 1$ since $k_2(L) \cap k_1(M) \leq \Phi(G)$, by Remark [2.14].

**3.4. Notation:** For a normal subgroup $N$ of $G$ such that $N \leq \Phi(G)$, set

$$\varphi^G_N = \sum_{M \leq G} \mu_{G}(N, M) Y_{M \ast \Delta(G)} ;$$

where $\mu_{G}$ is the Möbius function of the poset of normal subgroups of $G$. 

3.5. Proposition: Let $N \unlhd G$ with $N \leq \Phi(G)$. Then

$$\varphi^G_N = \text{Inf}_{G/N}^G \varphi_1^{G/N} \text{Def}_{G/N}^G.$$ 

Proof: Indeed if $N \leq M \unlhd G$, then $\mu_{\leq G}(N, M) = \mu_{\leq G/N}(1, M/N)$. Since moreover $N \leq \Phi(G)$, setting $\overline{G} = G/N$ and $\overline{M} = M/N$, we have by Lemma 2.13

$$\text{Inf}_{G/N}^G Y_{\Delta_G(M)} \text{Def}_{G/N}^G = \text{Inf}_{G/N}^G \circ \widetilde{e}_G^G((\overline{G} \times \overline{G})/\Delta_G(\overline{M})) \widetilde{e}_G^G \circ \text{Def}_{G/N}^G$$

$$= \widetilde{e}_G^G \circ \text{Inf}_{G/N}^G((\overline{G} \times \overline{G})/\Delta_G(\overline{M})) \text{Def}_{G/N}^G \circ \widetilde{e}_G^G$$

$$= \widetilde{e}_G^G((G \times G)/\Delta_M(G)) \widetilde{e}_G^G$$

$$= Y_{\Delta_M(G)},$$

since $\text{Inf}_{G/N}^G((\overline{G} \times \overline{G})/\Delta_G(\overline{M})) \text{Def}_{G/N}^G = (G \times G)/\Delta_M(G).$ 

3.6. Proposition:

1. Let $N \unlhd G$, with $N \leq \Phi(G)$. Then

$$\varphi^G_N = \widetilde{e}_G^G \times_G \left( \sum_{M \leq G \atop N \leq M \leq \Phi(G)} \mu_{\leq G}(N, M)((G \times G)/\Delta_M(G)) \right)$$

$$= \left( \sum_{M \leq G \atop N \leq M \leq \Phi(G)} \mu_{\leq G}(N, M)((G \times G)/\Delta_M(G)) \right) \times_G \widetilde{e}_G^G.$$

2. In particular

$$\varphi^G_1 = \frac{1}{|G|} \sum_{X \leq G, M \leq G \atop M \leq \Phi(G) \leq X \leq G} |X| \mu(X, G) \mu_{\leq G}(1, M) \text{Indinf}_{X/M}^G \circ \text{Defres}_{X/M}^G.$$

Proof: For Assertion 1, by definition

$$\varphi^G_N = \sum_{M \leq G \atop N \leq M \leq \Phi(G)} \mu_{\leq G}(N, M) \widetilde{e}_G^G((G \times G)/\Delta_M(G)) \times_G \sum_{X \leq G} |X| \mu(X, G)((G \times G)/\Delta(X)).$$

Moreover $[(G \times G)/\Delta_M(G)] \times_G [(G \times G)/\Delta(X)] = [(G \times G)/((\Delta_M(G) \ast \Delta(X)))]$, by (2.8), and $\Delta_M(G) \ast \Delta(X) = \{(xm, x) \mid x \in X, m \in M\}$. The first

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projection of this group is equal to $XM$, hence it is equal to $G$ if and only if $X = G$, since $M \leq \Phi(G)$. The first equality of Assertion 1 follows, by Remark \[2.15\]. The second one follows by taking opposite bisets, since $\tilde{e}_G^G$ and $[(G \times G)/\Delta_M(G)]$ are equal to their opposite.

Assertion 2 follows in the special case where $N = 1$, expanding $\tilde{e}_G^G$ as

\[
\tilde{e}_G^G = \frac{1}{|G|} \sum_{X \leq G} |X| \mu(X, G)[(G \times G)/\Delta(X)] ,
\]

observing that $\mu(X, G) = 0$ unless $X \geq \Phi(G)$, and that if $X \geq \Phi(G) \geq M$, then

\[
[(G \times G)/\Delta(X)] \circ [(G \times G)/\Delta_M(G)] = [(G \times G)/\Delta_M(X)] ,
\]

which is equal to $\text{Indinf}^G_{X/M} \circ \text{Defres}^G_{X/M}$.  

3.7. Corollary:

1. Let $H < G$. Then $\text{Res}^G_H \varphi_N^G = 0$ and $\varphi_N^G \text{Ind}^G_H = 0$.

2. Let $M \leq G$. If $M \cap \Phi(G) \nless N$, then $\text{Def}^G_{G/M} \varphi_N^G = 0$ and $\varphi_N^G \text{Inf}^G_{G/M} = 0$.

Proof: The first part of Assertion 1 follows from Lemma \[2.13\], since

\[
\text{Res}^G_H \varphi_N^G = \text{Res}^G_H \tilde{e}_G^G \varphi_N^G = 0 .
\]

The second part follows by taking opposite bisets.

For Assertion 2, let $P = M \cap \Phi(G)$. Since $\text{Def}^G_{G/M} = \text{Def}^G_{G/M} \circ \text{Def}^G_{G/P}$, it suffices to consider the case $M = P$, i.e. the case where $M \leq \Phi(G)$. Then, since $[(G \times G)/\Delta_M(G)] = \text{Inf}^G_{G/M} \text{Def}^G_{G/M}$ for any $M \leq G$, by \[2.1\] and since $\text{Def}^G_{G/M} \text{Inf}^G_{G/Q} = \text{Inf}^G_{G/MQ} \text{Def}^G_{G/MQ}$ for any $M, Q \leq G$,

\[
\text{Def}^G_{G/M} \varphi_N^G = \text{Def}^G_{G/M} \sum_{Q \leq G} \mu_{G}(N, Q) \text{Inf}^G_{G/Q} \text{Def}^G_{G/Q} \tilde{e}_G^G = \sum_{Q \leq G} \mu_{G}(N, Q) \text{Inf}^G_{G/MQ} \text{Def}^G_{G/MQ} \tilde{e}_G^G = \sum_{P \leq G} \left( \sum_{Q \leq G} \mu_{G}(N, Q) \right) \text{Inf}^G_{G/P} \text{Def}^G_{G/P} \tilde{e}_G^G .
\]
Now for a given \( P \leq G \) with \( P \subseteq \Phi(G) \), the sum \( \sum_{Q \leq G, N \leq Q \leq \Phi(G)} \mu_{\leq G}(N, Q) \) is equal to zero unless \( NM = N \), that is \( M \leq N \), by classical properties of the Möbius function (\cite{15} Corollary 3.9.3). This proves the first part of Assertion 2, and the second part follows by taking opposite bisets. 

3.8. Theorem: Let \( G \) be a finite group with order invertible in \( R \).

1. The elements \( \varphi^G_N \), for \( N \triangleleft G \) with \( N \leq \Phi(G) \), form a set of orthogonal idempotents in the algebra \( \mathcal{E}(G) \), and their sum is equal to the identity element \( e_G^G \) of \( \mathcal{E}(G) \).

2. Let \( N \leq G \) with \( N \leq \Phi(G) \), and let \( H \) be a finite group.

   (a) If \( L \leq (G \times H) \), then \( \varphi^G_N \times_G [(G \times H)/L] = 0 \) unless \( p_1(L) = G \) and \( k_1(L) \cap \Phi(G) \leq N \).

   (b) If \( L' \leq (H \times G) \), then \( [(H \times G)/L'] \times_G \varphi^G_N = 0 \) unless \( p_2(L') = G \) and \( k_2(L') \cap \Phi(G) \leq N \).

Proof: For \( N \triangleleft G \), set \( u_N^G = Y_{\Delta_N(G)} \). Since \( \Delta_N(G) * \Delta_M(G) = \Delta_{NM}(G) \) for any normal subgroups \( N \) and \( M \) of \( G \), it follows from Corollary 3.3 that if either \( N \) or \( M \) is contained in \( \Phi(G) \), then \( u_N^G u_M^G = u_{NM}^G \).

Now Assertion 1 follows from the following general procedure for building orthogonal idempotents (see \cite{13} Theorem 10.1 for details): we have a finite lattice \( P \) (here \( P \) is the lattice of normal subgroups of \( G \) contained in \( \Phi(G) \)), and a set of elements \( g_x \) of a ring \( A \), for \( x \in P \) (here \( A = \mathcal{E}(G) \) and \( g_N = u_N^G \)), with the property that \( g_x g_y = g_{x \lor y} \) for any \( x, y \in P \), and \( g_0 = 1 \), where 0 is the smallest element of \( P \) (here this element is the trivial subgroup of \( G \), and \( u_1^G = Y_{\Delta_1(G)} = e_G^G \)). The elements \( f_x \) defined for \( x \in P \) by

\[
f_x = \sum_{y \leq P, x \leq y} \mu(x, y) g_y,
\]

where \( \mu \) is the Möbius function of \( P \), are orthogonal idempotents of \( A \), and their sum is equal to the identity element of \( A \). This is exactly Assertion 1 (since \( f_x = \varphi^G_N \) here, for \( x = N \in P \)).

Let \( L \leq (G \times H) \), then by 2.6

\[
\varphi^G_N \times_G [(G \times H)/L] = \varphi^G_N \circ \text{Ind}_{p_1(L)}^G \circ [(p_1(L) \times H)/L] = 0
\]

unless \( p_1(L) = G \), by Corollary 3.7. And if \( p_1(L) = G \), then by 2.6

\[
\varphi^G_N \times_G [(G \times H)/L] = \varphi^G_N \circ \text{Ind}_{G/k_1(L)}^G \circ [(G/k_1(L) \times H)/L].
\]
for some subgroup $L_1$ of $(G/k_1(L) \times H)$. Again, by Corollary 3.7 this is equal to 0 unless $k_1(L) \cap \Phi(G) \leq N$. The proof of Assertion (b) is similar. Alternatively, one can take opposite bisets in (a).

3.9. Proposition: Let $G$ be a finite group with order invertible in $R$.

1. Let $L \in \Sigma(G, G)$. Then

$$
\varphi_1^G Y_L = \sum_{N \leq G \atop N \leq \Phi(G)} \mu_{\leq G}(1, N) Y_{(N \times 1)L}.
$$

This is non zero if and only if $k_1(L) \cap \Phi(G) = 1$. Similarly

$$
Y_L \varphi_1^G = \sum_{N \leq G \atop N \leq \Phi(G)} \mu_{\leq G}(1, N) Y_L(1 \times N),
$$

and $Y_L \varphi_1^G \neq 0$ if and only if $k_2(L) \cap \Phi(G) = 1$.

2. The elements $\varphi_1^G Y_L$ (resp. $Y_L \varphi_1^G$), when $L$ runs through a set of representatives of conjugacy classes of elements of $\Sigma(G, G)$ such that $k_1(L) \cap \Phi(G) = 1$ (resp $k_2(L) \cap \Phi(G) = 1$), form an $R$-basis of the right ideal $\varphi_1^G \mathcal{E}(G)$ (resp. the left ideal $\mathcal{E}(G) \varphi_1^G$) of $\mathcal{E}(G)$.

Proof: Let $L \in \Sigma(G, G)$. By Proposition 3.8 we have

$$
\varphi_1^G Y_L = \widetilde{e}_G^G \times_G \left( \sum_{N \leq G \atop N \leq \Phi(G)} \mu_{\leq G}(1, N) \left[ (G \times G) / \Delta_N(G) \right] \right) \times_G \left[ (G \times G) / L \right] \times_G \widetilde{e}_G^G
$$

$$
= \widetilde{e}_G^G \times_G \left( \sum_{N \leq G \atop N \leq \Phi(G)} \mu_{\leq G}(1, N) \left[ (G \times G) / (\Delta_N(G) \ast L) \right] \right) \times_G \widetilde{e}_G^G
$$

$$
= \widetilde{e}_G^G \times_G \left( \sum_{N \leq G \atop N \leq \Phi(G)} \mu_{\leq G}(1, N) \left[ (G \times G) / (N \times 1) L \right] \right) \times_G \widetilde{e}_G^G.
$$

$$
= \sum_{N \leq G \atop N \leq \Phi(G)} \mu_{\leq G}(1, N) Y_{(N \times 1)L}.
$$

Set $M = k_1(L) \cap \Phi(G)$. Then $M \leq G$, and $(N \times 1)L = (NM \times 1)L$ for any
normal subgroup $N$ of $G$ contained in $\varphi(G)$. Thus
\begin{equation}
\varphi^G Y_L = \sum_{P \leq G \atop M \leq P \leq \Phi(G)} \left( \sum_{N \leq G \atop N M = P} \mu_{\leq G}(1, N) \right) Y_{(P \times 1)L}.
\end{equation}

If $M \neq 1$, then $\left( \sum_{N \leq G \atop N M = P} \mu_{\leq G}(1, N) \right) = 0$ for any $P \leq G$ with $M \leq P \leq \Phi(G)$. Hence $\varphi^G Y_L = 0$ in this case. And if $M = 1$, Equation (3.10) reads
\begin{equation}
\varphi^G Y_L = \sum_{P \leq G \atop P \leq \Phi(G)} \mu_{\leq G}(1, P) Y_{(P \times 1)L}.
\end{equation}

The element $Y_{(P \times 1)L}$ is equal to $Y_L$ if and only if $(P \times 1)L$ is conjugate to $L$. This implies that $k_1((P \times 1)L)$ is conjugate to (hence equal to) $k_1(L)$. Thus $P \leq k_1((P \times 1)L) \leq k_1(L) \cap \Phi(G)$, hence $P = 1$. So the coefficient of $Y_L$ in $\varphi^G Y_L$ is equal to 1, hence $\varphi^G Y_L \neq 0$. The remaining statements of Assertion 1 follow by taking opposite bisets.

Assertion 2 follows from Proposition 3.2, and from the fact that the coefficient of $Y_L$ in $\varphi^G Y_L$ is equal to 1 when $k_1(L) \cap \Phi(G) = 1$.

3.11. Corollary: Let $G$ be a finite group of order invertible in $R$. If every minimal (non-trivial) normal subgroup of $G$ is contained in $\Phi(G)$, then $\varphi^G_1$ is central in $\mathcal{E}(G)$, and the algebra $\varphi^G_1 \mathcal{E}(G)$ is isomorphic to $R\text{Out}(G)$.

Proof: Indeed if $L \in \Sigma(L, L)$ and $\varphi^G_1 Y_L \neq 0$, then $k_1(L) \cap \Phi(G) = 1$. It follows that $k_1(L)$ contains no minimal normal subgroup of $G$, and then $k_1(L) = 1$. Equivalently $q(L) \cong p_1(L)/k_1(L) \cong G \cong p_2(L)/k_2(L)$, i.e. $k_2(L) = G$ also, or equivalently $k_2(L) \cap \Phi(G) = 1$. Hence $\varphi^G_1 Y_L \neq 0$ if and only if $Y_L \varphi^G_1 \neq 0$, and in this case, there exists an automorphism $\theta$ of $G$ such that
\[ L = \Delta_\theta(G) = \{(\theta(x), x) \mid x \in G \}. \]

In this case for any normal subgroup $N$ of $G$ contained in $\Phi(G)$
\begin{align*}
(N \times 1)L &= \{(a, b) \in G \times G \mid a \theta(b)^{-1} \in N\} \\
&= \{(a, b) \in G \times G \mid a^{-1} \theta(b) \in N\} \\
&= L(1 \times \theta^{-1}(N)).
\end{align*}

Now $N \mapsto \theta^{-1}(N)$ is a permutation of the set of normal subgroups of $G$ contained in $\Phi(G)$. Moreover $\mu_{\leq G}(1, N) = \mu_{\leq G}(1, \theta^{-1}(N))$. 

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It follows that $\varphi^G_Y = Y_L \varphi^G_1$, so $\varphi^G_1$ is central in $E(G)$. Moreover the map $\theta \in \text{Aut}(G) \mapsto \varphi^G_1 \Delta_{\theta}(G)$ clearly induces an algebra isomorphism $R\text{Out}(G) \rightarrow \varphi^G_1 E(G)$. 

\[ \text{3.12. Theorem:} \quad \text{Let } G \text{ be a finite group with order invertible in } R. \text{ If } G \text{ is nilpotent, then } \varphi^G_1 \text{ is a central idempotent of } E(G). \]

\textbf{Proof:} Let $L \in \Sigma(G, G)$. Setting $Q = q(L)$, there are two surjective group homomorphisms $s, t : G \rightarrow Q$ such that $L = \{(x, y) \in G \times G \mid s(x) = t(y)\}$. Then $k_1(L) = \text{Ker} s$ and $k_2(L) = \text{Ker} t$. Now by Proposition 3.3,

\[ \varphi^G_1 Y_L = \sum_{N \subseteq G} \mu \in G(1, N) Y_{(N \times 1) L}, \]

and this is non zero if and only if $\text{Ker} s \cap \Phi(G) = 1$. Now $s(\Phi(G))$ is equal to $\Phi(Q)$ since $G$ is nilpotent: indeed $G = \prod_p G_p$ (resp. $Q = \prod_p Q_p$) is the direct product of its $p$-Sylow subgroups $G_p$ (resp. $Q_p$), and $s$ induces a surjective group homomorphism $G_p \rightarrow Q_p$, for any prime $p$. Moreover $\Phi(G) = \prod_p \Phi(G_p)$ (resp. $\Phi(Q) = \prod_p \Phi(Q_p)$). Finally $\Phi(G_p)$ is the subgroup of $G_p$ generated by commutators and $p$-powers of elements of $G_p$, hence it maps by $s$ onto the subgroup of $Q_p$ generated by commutators and $p$-powers of elements of $Q_p$, that is $\Phi(Q_p)$. Similarly $t(\Phi(G)) = \Phi(Q)$.

If $\text{Ker} s \cap \Phi(G) = 1$, it follows that $s$ induces an isomorphism from $\Phi(G)$ to $\Phi(Q)$. Then the surjective homomorphism $\Phi(G) \rightarrow \Phi(Q)$ induced by $t$ is also an isomorphism, and in particular $\text{Ker} t \cap \Phi(G) = 1$.

Let $D = L \cap (\Phi(G) \times \Phi(G))$. Then $k_1(D) \subseteq k_1(L) \cap \Phi(G) = \text{Ker} s \cap \Phi(G)$, hence $k_1(D) = 1$. Similarly $k_2(D) \subseteq k_2(L) \cap \Phi(G) = \text{Ker} t \cap \Phi(G) = 1$, hence $k_2(D) = 1$. Since $s(\Phi(G)) = \Phi(Q) = t(\Phi(G))$, we have moreover $p_1(D) = \Phi(G) = p_2(D)$. It follows that there is an automorphism $\alpha$ of $\Phi(G)$ such that $D = \{(x, \alpha(x)) \mid x \in \Phi(G)\}$.

Moreover for any $y \in G$, there exists $z \in G$ such that $(y, z) \in L$. It follows that $(x^y, \alpha(x)^z) \in D$ for any $x \in \Phi(G)$, that is $\alpha(x^y) = \alpha(x)^z$. In particular if $N$ is a normal subgroup of $G$ contained in $\Phi(G)$, then so is $\alpha(N)$. Hence $\alpha$ induces an automorphism of the poset of normal subgroups of $G$ contained in $\Phi(G)$. In particular $\mu_{\in G}(1, N) = \mu_{\in G}(1, \alpha(N))$.

Moreover for $n \in N$ and $(y, z) \in L$, we have

\[ (n, 1)(y, z) = (y, z)(n^y, 1) = (y, z)(n^y, \alpha(n^y))(1, \alpha(n^y)^{-1}). \]

Since $(n^y, \alpha(n^y)) \in D \leq L$, we have $(N \times 1)L = L(1 \times \alpha(N))$. It follows
that
\[
\varphi^G Y_L = \sum_{N \leq \Phi(G)} \mu_{\leq G}(1, N) Y_{L(1 \times 1)} = \sum_{N \leq \Phi(G)} \mu_{\leq G}(1, N) Y_{L(1 \times 1, N)}
\]
\[
= \sum_{N \leq \Phi(G)} \mu_{\leq G}(1, 1) Y_{L(1 \times 1, N)} = \sum_{N \leq \Phi(G)} \mu_{\leq G}(1, N) Y_{L(1 \times 1, N)}
\]
\[
= Y_L \varphi^G_Y ,
\]
as was to be shown.

\[\square\]

3.13. Remark: When \( G \) is not nilpotent, it is not true in general that \( \varphi^G Y_L \) is central in \( E(G) \). This is because \( t(\Phi(G)) \) need not be equal to \( \Phi(Q) \) for a surjective group homomorphism \( t : G \to Q \). For example, there is a surjection \( t \) from the group \( G = C_4 \times (C_5 \rtimes C_4) \) to \( Q = C_4 \) with kernel \( C_4 \times C_5 \), containing \( \Phi(G) = C_2 \times 1 \), and another surjection \( s : G \to Q \) with kernel \( 1 \times (C_5 \rtimes C_4) \), intersecting trivially \( \Phi(G) \). In this case, the group \( L = \{(x, y) \in G \times G \mid s(x) = t(y)\} \) is in \( \Sigma(G, G) \), and \( k_1(L) \cap \Phi(G) = 1 \), but \( k_2(L) \cap \Phi(G) = \Phi(G) \neq 1 \). By Proposition 3.9 we have \( \varphi^G Y_L \neq 0 \) and \( Y_L \varphi^G_Y = 0 \), so \( \varphi^G \) is not central in \( E(G) \).

4. Idempotents in \( RB(G, G) \)

4.1. Definition: When \( G \) is a finite group, a section \((T, S)\) of \( G \) is a pair of subgroups of \( G \) such that \( S \subseteq T \).

A section \((T, S)\) is called minimal (cf. [9]) if \( S \leq \Phi(T) \). Let \( \mathcal{M}(G) \) denote the set of minimal sections of \( G \).

A group \( H \) is called a subquotient of \( G \) (notation \( H \sqsubseteq G \)) if there exists a section \((T, S)\) of \( G \) such that \( T/\Sigma \cong H \).

A section \((T, S)\) is minimal if and only if the only subgroup \( H \) of \( T \) such that \( H/(H \cap S) \cong T/\Sigma \) is \( T \) itself.

4.2. Notation: Let \( G \) be a finite group, and let \((T, S)\) be a section of \( G \).

1. Let \( \text{Indinf}_{T/S}^G \in B(G, T/S) \) denote (the isomorphism class of) the \((G, T/S)\)-biset \( G/S \), and let \( \text{Defres}_{T/S}^G \in B(T/S, G) \) denote (the isomorphism class of) the \((T/S, G)\)-biset \( S\backslash G \).

2. Let \( R \) be a commutative ring in which the order of \( G \) is invertible. Let
\[ u_{T,S}^G \in RB(G, T/S) \text{ be defined by} \]
\[ u_{T,S}^G = \text{Ind}_{T/S}^G \varphi_{1}^{T/S} , \]

and let \[ v_{T,S}^G \in RB(T/S, G) \text{ be defined by} \]
\[ v_{T,S}^G = \varphi_{1}^{T/S} \text{Defres}_{T/S}^G . \]

4.3. Remark: Observe that \[ v_{T,S}^G = (u_{T,S}^G)^{op} : \text{ indeed } (G/S)^{op} \cong S\backslash G, \text{ and } (\varphi_{1}^{T/S})_{op} = \varphi_{1}^{T/S}. \]

4.4. Theorem: Let \( G \) be a finite group with order invertible in \( R \).

1. If \((T, S)\) and \((T', S')\) are minimal sections of \( G \), then

\[ v_{T',S'}^G u_{T,S}^G = 0 \]

unless \((T, S)\) and \((T', S')\) are conjugate in \( G \).

2. If \((T, S)\) is a minimal section of \( G \), then

\[ v_{T,S}^G u_{T,S}^G = \varphi_{1}^{T/S} \left( \sum_{g \in N_G(T,S)/T} \text{Iso}(c_g) \right) , \]

where \( N_G(T, S) = N_G(T) \cap N_G(S) \), and \( c_g \) is the automorphism of \( T/S \) induced by conjugation by \( g \).

**Proof:** Indeed \((S'\backslash G) \times_{G} (G/S) \cong S'\backslash G/S \) as a \((T'/S', T/S)\)-biset. Hence

\[ v_{T',S'}^G u_{T,S}^G = \varphi_{1}^{T'/S'} \left( \sum_{g \in (T'/S') \times (T/S) \backslash (T'/S')} S'\backslash T'gT/S \right) \varphi_{1}^{T/S} . \]

For any \( g \in G \), the \((T'/S', T/S)\)-biset \( S'\backslash T'gT/S \) is transitive, isomorphic to \(( (T'/S') \times (T/S) ) / L_g \), where

\[ L_g = \{ (t'S', tS) \in (T'/S') \times (T/S) \mid t'gt^{-1} \in S'gS \} . \]

Then \( t'S' \in p_1(L_g) \) if and only if \( t' \in S' \cdot gTg^{-1} \cap T' \). Hence

\[ p_1(L_g) = (aT \cap T')S'/S' . \]

Similarly \( p_2(L_g) = (Tg \cap T)S/S' \). In particular \( p_1(L_g) = T'/S' \) if and only if \((aT \cap T')S' = T'\), i.e. \( aT \cap T' = T' \), since \( S' \leq \Phi(T') \). Thus \( p_1(L_g) = T'/S' \).
if and only if $T' \leq gT$. Similarly $p_2(L_g) = T/S$ if and only if $T \leq T^g$. By Theorem 3.8, it follows that $\varphi_1^{T'/S'}(S' \setminus T'gT/S)\varphi_1^{T/S} = 0$ unless $T' = gT$.

Assume now that $T' = gT$. Then $t'S' \in k_1(L_G)$ if and only if $t'$ lies in $S' \cdot gSg^{-1} \cap T'$. Hence

$$k_1(L_g) = (gS \cap T')S'/S' ,$$

and similarly $k_2(L_g) = (S^g \cap T)S/S$. But since $S \leq \Phi(T)$ and $S \leq T$, it follows that $gS \leq gT = T'$ and $gS \leq g\Phi(T) = \Phi(T')$. Hence $gS \cdot S'/S'$ is contained in $k_1(L_g) \cap \Phi(T')/S'$. Moreover $\Phi(T')/S' = \Phi(T'/S')$, as

$$\Phi(T'/S') = \bigcap_{S' \leq M' < T'} (M'/S') = \bigcap_{M' < T'} (M'/S') = ( \bigcap_{M' < T'} M')/S' = \Phi(T'/S') ,$$

where $M'$ runs through maximal subgroups of $T'$, which all contain $S'$ since $S' \leq \Phi(T')$.

It follows that if $k_1(L_g) \cap \Phi(T'/S') = 1$, then $gS \cdot S' = S'$, that is $gS \leq S'$. Similarly if $k_2(L_g) \cap \Phi(T/S) = 1$, then $S^g \leq S$. By Theorem 3.8, it follows that $\varphi_1^{T'/S'}(S' \setminus T'gT/S)\varphi_1^{T/S} = 0$ unless $T' = gT$ and $S' = gS$. This proves Assertion 1.

For Assertion 2, the same computation shows that

$$\psi_{T,S}^G g_{T,S}^G = \sum_{g \in N_G(T,S)/T} \varphi_1^{T/S}(S \setminus TgT/S)\varphi_1^{T/S} .$$

But $S \setminus TgT = SgT/S$ if $g \in N_G(T,S)$, and this $(T/S, T/S)$-biset is isomorphic to $\text{Iso}(c_g)$. Assertion 2 follows, since moreover $\varphi_1^{T/S}$ commutes with any biset of the form $\text{Iso}(\theta)$, where $\theta$ is an automorphism of $T/S$. \hfill $\Box$

### 4.5. Notation:

*For a minimal section $(T, S)$ of the group $G$, set*

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S)/T|} \varphi_{T,S}^G \psi_{T,S}^G = \frac{1}{|N_G(T,S)/T|} \text{Indinf}^G_{T/S} \varphi_{T/S}^G \text{Defres}^G_{T/S} \in RB(G, G) .$$

Note that $\epsilon_{T,S}^G = \epsilon_{T,S}^T$ for any $g \in G$, and that $\epsilon_{G,N}^G = \varphi_{N}^G$ when $N \leq G$ and $N \leq \Phi(G)$, by Proposition 3.5

### 4.6. Proposition:

*Let $(T, S)$ be a minimal section of $G$. Then*

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S)|} \sum_{X \leq T, M \leq T} |X| \mu(X, T)\mu_{2T}(S, M) \text{Indinf}^G_{X/M} \circ \text{Defres}^G_{X/M} .$$

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Proof: This is a straightforward consequence of the above definition of $\epsilon_{T,S}^G$, and from Assertion 2 of Proposition 3.6.

4.7. Theorem: Let $G$ be a finite group with order invertible in $R$, let $[\mathcal{M}(G)]$ be a set of representatives of conjugacy classes of minimal sections of $G$. Then the elements $\epsilon_{T,S}^G$, for $(T, S) \in [\mathcal{M}(G)]$, are orthogonal idempotents of $RB(G, G)$, and their sum is equal to the identity element of $RB(G, G)$.

Proof: Let $(T, S)$ and $(T', S')$ be distinct elements of $[\mathcal{M}(G)]$. Then

$$\epsilon_{T',S'}^G \epsilon_{T,S}^G = \frac{1}{|N_G(T',S'):T'| |N_G(T,S):T|} u_{T',S'}^G v_{T',S'}^G u_{T,S}^G v_{T,S}^G = 0,$$

since $v_{T',S'}^G v_{T,S}^G = 0$ by Theorem 4.4. Moreover:

$$\sum_{(T, S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G = \sum_{(T, S) \in [\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G = \sum_{(T, S) \in [\mathcal{M}(G)]} \frac{1}{|G:T|} \text{Ind}_{T/S}^G \varphi_{T/S} \text{Res}_{T/S}^G.$$

Now $\varphi_{T/S} = \epsilon_{T/S}^G f_{T/S}$ by Proposition 3.6, where

$$f_{T/S} = \sum_{N/S \subseteq (T/S)} \mu_{\Phi}(1, N/S) [(T/S) \times (T/S)]/[\Delta_{N/S}(T/S)] .$$

Hence $\varphi_{T/S} = \epsilon_{T/S}^G \text{Def}_{T/S}^T \text{Inf}_{T/S}^T f_{T/S}$, and

$$\sum_{(T, S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G = \sum_{(T, S) \in [\mathcal{M}(G)]} \frac{1}{|G:T|} \text{Ind}_{T/S}^G \text{Inf}_{T/S}^T \text{Def}_{T/S}^T \text{Inf}_{T/S}^T f_{T/S} \text{Def}_{T/S}^T \text{Res}_{T/S}^G .$$

Now $\text{Inf}_{T/S}^T \text{Def}_{T/S}^T = \text{Inf}_{T/S}^T e_{T/S}^T$, and $\text{Inf}_{T/S}^T e_{T/S}^T$ is equal to the sum over subgroups $X$ or $T$ such that $XS = T$, up to conjugation, of the idempotents $e_X^T$. Since $S \leq \Phi(T)$, the only subgroup $X$ of $T$ such that $XS = T$ is $T$ itself. Hence

$$\text{Inf}_{T/S}^T e_{T/S}^T \text{Def}_{T/S}^T = e_T^T .$$
On the other hand

\[
\text{Inf}_{T/S}^T \left( [(T/S) \times (T/S)] / \Delta_N(T/S) \right) \text{Def}_{T/S}^T = [(T \times T) / \Delta_N(T)].
\]

It follows that the sum \( \Sigma = \sum_{(T, S) \in \mathcal{M}(G)} e_{T, S}^G \) is equal to

\[
\Sigma = \sum_{(T, S) \in \mathcal{M}(G)} \frac{1}{|G:T|} \text{Ind}_{T}^{G} e_{T}^{\sim} \sum_{S \leq T} \mu_{S \leq T, N} \left( \frac{(T \times T)}{\Delta_N(T)} \right) \text{Res}_{T}^{G}
\]

\[
= \sum_{(T, S) \in \mathcal{M}(G)} \frac{1}{|G:T|} \text{Ind}_{T}^{G} e_{T}^{\sim} \varphi_{S}^{T} \text{Res}_{T}^{G} \quad \text{[by definition of } \varphi_{S}^{T}] 
\]

\[
= \sum_{T \leq G} \frac{1}{|G:T|} \text{Ind}_{T}^{G} e_{T}^{\sim} \sum_{S \leq T} \varphi_{S}^{T} \text{Res}_{T}^{G}
\]

\[
= \sum_{T \leq G} \frac{1}{|G:T|} \text{Ind}_{T}^{G} e_{T}^{\sim} \text{Res}_{T}^{G} \quad \text{[by Theorem 3.8]}
\]

\[
= \sum_{T \leq G} \frac{1}{|G:T|} \text{Ind}_{T}^{G} e_{T}^{\sim} \quad \text{[by Lemma 2.10]}
\]

\[
= \sum_{T \leq G} \frac{1}{|G:NG(T)|} \text{Ind}_{T}^{G} e_{T}^{\sim} \quad \text{[by (2.12)]}
\]

\[
= \sum_{T \in \mathcal{S}(G)} \text{Res}_{T}^{G} = \bar{G}/G = \left( (G \times G) / \Delta(G) \right).
\]

So the sum \( \Sigma \) is equal to the identity of \( RB(G, G) \). Since \( e_{T, S}^{G} e_{T, S'}^{G} = 0 \) if \((T, S)\) and \((T', S')\) are distinct elements of \( \mathcal{M}(G) \), it follows that for any \((T, S) \in \mathcal{M}(G)\)

\[
e_{T, S}^{G} = e_{T, S}^{G} \Sigma = (e_{T, S}^{G})^2,
\]

which completes the proof of the theorem. \( \square \)

5. Application to biset functors

5.1. Notation: Let \( F \) be a biset functor over \( R \). When \( G \) is a finite group with order invertible in \( R \), we set

\[
\delta_{\Phi} F(G) = \varphi_{1}^{G} F(G)
\]
5.2. Proposition: Let $F$ be a biset functor over $R$. Then for any finite group $G$ with order invertible in $R$, the $R$-submodule $\delta_{\Phi}F(G)$ of $F(G)$ is the set of elements $u \in F(G)$ such that

$$
\begin{align*}
\text{Res}_H^G u &= 0 & \forall H < G \\
\text{Def}_N^G u &= 0 & \forall N \leq G, N \cap \Phi(G) \neq 1
\end{align*}
$$

Proof: If $u \in \delta_{\Phi}F(G) = \varphi_1^GF(G)$, then $\text{Res}_H^G u = 0$ for any proper subgroup $H$ of $G$, and $\text{Def}_N^G u = 0$ for any $N \leq G$ such that $N \cap \Phi(G) \neq 1$, by Corollary 3.7.

Conversely, if $u \in F(G)$ fulfills the two conditions of the proposition, then $\widetilde{e}_G u = u$, because $\widetilde{e}_G$ is equal to the identity element $[(G \times G)/\Delta(G)]$ of $RB(G, G)$, plus a linear combination of elements of the form $[(G \times G)/\Delta(H)] = \text{Inf}_H \circ \text{Res}_H^G$, for proper subgroups $H$ of $G$. Similarly $\text{Inf}_{N}^{G/M} \text{Def}_{N}^{G/M} u = 0$ for any non-trivial normal subgroup of $G$ contained in $\Phi(G)$, thus $\varphi_1^G u = u$. $
$

5.3. Remark: Since $\text{Def}_{N}^{G/N} = \text{Def}_{N}^{G/M} \circ \text{Def}_{N}^{G/M}$, where $M = N \cap \Phi(G)$, saying that $\text{Def}_{N}^{G/N} u = 0$ for any $N \leq G$ with $N \cap \Phi(G) \neq 1$ is equivalent to saying that $\text{Def}_{N}^{G/N} u = 0$ for any non-trivial normal subgroup $N$ of $G$ contained in $\Phi(G)$.

5.4. Theorem: Let $F$ be a biset functor over $R$. Then for any finite group $G$ with order invertible in $R$, the maps

$$
\begin{align*}
F(G) &\xrightarrow{(T,S) \in [M(G)]} \bigoplus_{(T,S) \in [M(G)]} \left(\delta_{\Phi}F(T/S)\right)^{N_G(T,S)/T} \\
\sum_{(T,S)} w_T^G w_{T,S} &\xrightarrow{U} \bigoplus_{(T,S)} w_{T,S} \\
U &\xrightarrow{\varphi_1^G} \bigoplus_{(T,S)} w_{T,S}
\end{align*}
$$

are well defined isomorphisms of $R$-modules, inverse to one other.

Proof: We have first to check that if $w \in F(G)$, then the element $v_{T,S}^G w$ of $\varphi_1^{T/S} F(T/S) = \delta_{\Phi} F(T/S)$ is invariant under the action of $N_G(T, S)/T$. But for any $g \in N_G(T, S)$

$$
\text{Iso}(c_g) v_{T,S}^G = v_{N_G(T,S):T}^G \text{Iso}(c_g) = v_{T,S}^G \text{Iso}(c_g),
$$

where $\text{Iso}(c_g) : F(G) \to F(G)$ on the right hand side is conjugation by $g$, that is an inner automorphism, hence the identity map, for $g \in G$. 

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Now for \( w \in F(G) \)
\[
UV(w) = \sum_{(T,S) \in |M(G)|} \frac{1}{|N_G(T,S) : T|} u^G_{T,S} v^G_{T,S} w
\]
\[
= \sum_{(T,S) \in |M(G)|} e^G_{T,S} w = w,
\]
so \( UV \) is the identity map of \( F(G) \).

Conversely, if \( w_{T,S} \in (\delta_F (T/S))^{N_G(T,S)/T} \), for \( (T, S) \in |M(G)| \), then
\[
VU \left( \bigoplus_{(T,S) \in |M(G)|} w_{T,S} \right) = \bigoplus_{(T,S) \in |M(G)|} \sum_{(T',S') \in |M(G)|} \frac{1}{|N_G(T,S) : T'|} v^G_{T,S} u^G_{T',S'} w_{T',S'}
\]
\[
= \bigoplus_{(T,S) \in |M(G)|} \frac{1}{|N_G(T,S) : T|} v^G_{T,S} u^G_{T,S} w_{T,S}
\]
\[
= \bigoplus_{(T,S) \in |M(G)|} \frac{1}{|N_G(T,S) : T|} \sum_{g \in N_G(T,S)/T} \text{Iso}(c_g) w_{T,S}
\]
\[
= \bigoplus_{(T,S) \in |M(G)|} w_{T,S},
\]
so \( VU \) is also equal to the identity map.

\[ \blacksquare \]

6. Atoric \( p \)-groups

For the remainder of the paper, we denote by \( p \) a (fixed) prime number.

6.1. Notation and Definition:

- If \( P \) is a finite \( p \)-group, let \( \Omega_1 P \) denote the subgroup of \( P \) generated by the elements of order \( p \).

- A finite \( p \)-group \( P \) is called atoric if it does not admit any decomposition \( P = E \times Q \), where \( E \) is a non-trivial elementary abelian \( p \)-group. Let \( \text{At}_p \) denote the class of atoric \( p \)-groups, and let \( [\text{At}_p] \) denote a set of representatives of isomorphism classes in \( \text{At}_p \).

The terminology “atoric” is inspired by [14], where elementary abelian \( p \)-groups are called \( p \)-tori. Atoric \( p \)-groups have been considered (without naming them) in [6], Example 5.8.

6.2. Lemma: Let \( P \) be a finite \( p \)-group, and \( N \) be a normal subgroup of \( P \). The following conditions are equivalent:
1. \( N \cap \Phi(P) = 1 \)

2. \( N \) is elementary abelian and central in \( P \), and admits a complement in \( P \).

3. \( N \) is elementary abelian and there exists a subgroup \( Q \) of \( P \) such that \( P = N \times Q \).

**Proof:**

1 \( \Rightarrow \) 3

Let \( N \trianglelefteq P \) with \( N \cap \Phi(P) = 1 \). Then \( N \) maps injectively in the elementary abelian \( p \)-group \( P/\Phi(P) \), so \( N \) is elementary abelian. Let \( Q/\Phi(P) \) be a complement of \( N\Phi(P)/\Phi(P) \) in \( P/\Phi(P) \). Then \( Q \geq \Phi(P) \geq [P, P] \), so \( Q \) is normal in \( P \). Moreover \( Q \cdot N = P \) and \( Q \cap N\Phi(P) = (Q \cap N)\Phi(P) = \Phi(P) \), thus \( Q \cap N \leq \Phi(P) \cap N = 1 \). Now \( N \) and \( Q \) are normal subgroups of \( P \) which intersect trivially, hence they centralize each other. It follows that \( P = N \times Q \).

3 \( \Rightarrow \) 2

This is clear.

2 \( \Rightarrow \) 1

If \( P = N \cdot Q \) for some subgroup \( Q \) of \( P \), and if \( N \) is central in \( P \), then \( P = N \times Q \). Thus \( \Phi(P) = 1 \times \Phi(Q) \), as \( N \) is elementary abelian. Then \( N \cap \Phi(P) \leq N \cap Q = 1 \).

6.3. Lemma: Let \( P \) be a finite \( p \)-group. The following conditions are equivalent:

1. \( P \) is atoric.

2. If \( N \trianglelefteq P \) and \( N \cap \Phi(P) = 1 \), then \( N = 1 \).

3. \( \Omega_1 Z(P) \leq \Phi(P) \).

**Proof:**

1 \( \Rightarrow \) 2

Suppose that \( P \) is atoric. Let \( N \trianglelefteq P \) with \( N \cap \Phi(P) = 1 \). Then by Lemma 6.2, the group \( N \) is elementary abelian and there exists a subgroup \( Q \) of \( P \) such that \( P = N \times Q \). Hence \( N = 1 \).

2 \( \Rightarrow \) 3

Suppose now that Assertion 2 holds. If \( x \) is a central element of order \( p \) of \( P \), then the subgroup \( N \) of \( P \) generated by \( x \) is normal in \( P \), and non trivial. Then \( N \cap \Phi(P) = 1 \), hence \( N \leq \Phi(P) \) since \( N \) has order \( p \), thus \( x \in \Phi(P) \).

3 \( \Rightarrow \) 1

Finally, if Assertion 3 holds, and if \( P = E \times Q \) for some subgroups \( E \) and \( Q \) of \( P \) with \( E \) elementary abelian, then \( \Phi(P) = 1 \times \Phi(Q) \). Moreover \( E \leq \Omega_1 Z(P) \leq \Phi(P) \leq Q \), so \( E = E \cap Q = 1 \), and \( P \) is atoric.
6.4. Proposition: Let $P$ be a finite $p$-group, and $N$ be a maximal normal subgroup of $P$ such that $N \cap \Phi(P) = 1$. Then:

1. The group $N$ is elementary abelian and there exists a subgroup $T$ of $P$ such that $P = N \times T$.
2. The group $P/N \cong T$ is atoric.
3. If $Q$ is an atoric $p$-group and $s : P \to Q$ is a surjective group homomorphism, then $s(T) = Q$. In particular $Q$ is isomorphic to a quotient of $T$.

Proof: (1) This follows from Lemma \ref{lem:6.2}.

(2) By (1), there exists $T \leq P$ such that $P = N \times T$. In particular $P/N \cong T$. Now if $T = E \times S$, for some subgroups $E$ and $S$ of $T$ with $E$ elementary abelian, then $P \cong P_1 = (N \times E) \times S$, and $N \times E$ is an elementary abelian normal subgroup of $P_1$ which intersects trivially $\Phi(P_1) = \Phi(S)$. By maximality of $N$, it follows that $E = 1$, so $T \cong P/N$ is atoric.

(3) Let $s : P \to Q$ be a surjective group homomorphism, where $Q$ is atoric. By (1), the group $N$ is elementary abelian, and there exists a subgroup $T$ of $P$ such that $P = N \times T$. Then $T \cong P^{\tilde{}}$, and $\Phi(P) = \Phi(T)$. Moreover $s(\Phi(P)) = \Phi(Q)$ as $P$ is a $p$-group, and $s(Z(P)) \leq Z(Q)$ as $s$ is surjective. It follows that $s(N)$ is an elementary abelian central subgroup of $Q$, so $s(N) \leq \Phi(Q)$ since $Q$ is atoric, by Lemma \ref{lem:6.3}. Now $s(P) = Q = s(N)s(T)$, thus $Q = \Phi(Q)s(T)$, and $s(T) = Q$, as was to be shown.

6.5. Notation: When $P$ is a finite $p$-group, and $N$ is a maximal normal subgroup of $P$ such that $N \cap \Phi(P) = 1$, we set $P^{\tilde{}} = P/N$.

By Proposition \ref{prop:6.4} the group $P^{\tilde{}}$ does not depend on the choice of $N$, up to isomorphism: it is the greatest atoric quotient of $P$, in the sense that any atoric quotient of $P$ is isomorphic to a quotient of $P^{\tilde{}}$. In particular $P^{\tilde{}}$ is trivial if and only if $P$ is elementary abelian.

6.6. Proposition: Let $s : P \to Q$ be a surjective group homomorphism. Then $P^{\tilde{}} \cong Q^{\tilde{}}$ if and only if $\text{Ker}(s) \cap \Phi(P) = 1$.

Proof: Let $E$ be a maximal normal subgroup of $P$ such that $E \cap \Phi(P) = 1$, and $T$ be a subgroup of $P$ such that $P = E \times T$. Then $E$ is elementary abelian, and $\Phi(P) = \Phi(T)$. Let $\pi : Q \to Q^{\tilde{}}$ be the canonical projection. By definition, we have $T \cong P^{\tilde{}}$, and by Proposition \ref{prop:6.4} we have $\pi \circ s(T) = Q^{\tilde{}}$. 25
Hence $Q^\alpha$ is a quotient of $P^\alpha$, and $P^\alpha \cong Q^\alpha$ if and only if the map $\pi \circ s$ induces an isomorphism from $T$ to $Q^\alpha$, that is if $\text{Ker}(\pi \circ s) \cap T = 1$. This implies $\text{Ker}(s) \cap T = 1$, hence $\text{Ker}(s) \cap \Phi(P) = 1$.

Conversely, if $\text{Ker}(s) \cap \Phi(P) = 1$, then $\text{Ker}(s) \cap \Phi(T) = 1$. Now the group $M = \text{Ker}(s) \cap T$ is a normal subgroup of $T$ such that $M \cap \Phi(T) = 1$. Since $T$ is atoric, it follows from Lemma [6.3] that $M = 1$, hence $s(T) \cong T$.

Now $Q = s(E)s(T)$, and $s(E)$ is a central elementary abelian subgroup of $Q$, since $s$ is surjective. Let $F$ be a complement of $G = s(E) \cap s(T)$ in $s(E)$. Then $Q = (F \cdot G)s(T) = F \cdot s(T)$, thus $Q = F \times s(T)$ since $F$ is central in $Q$. It follows that $s(T)$ is a quotient of $Q$. Since $s(T) \cong T \cong P^\alpha$ is atoric, the group $P^\alpha$ is isomorphic to a quotient of $Q^\alpha$, thus $P^\alpha \cong Q^\alpha$.

\[\square\]

6.7. Proposition: Let $P$ be a finite $p$-group, and $Q$ be a subquotient of $P$. Then $Q^\alpha$ is a subquotient of $P^\alpha$.

Proof: Let $(V, U)$ be a section of $P$ such that $V/U \cong Q$. Then $Q^\alpha$ is isomorphic to a quotient of $V^\alpha$, by Lemma [6.3]. Hence it suffices to prove that $V^\alpha$ is a subquotient of $P^\alpha$.

Let $E$ be a maximal normal subgroup of $P$ such that $E \cap \Phi(P) = 1$, and $T$ be a subgroup of $P$ such that $P = E \times T$. Then $V \leq E \times T$, so there exist a subgroup $F$ of $E$, a subgroup $X$ of $T$, a group $Y$, and surjective group homomorphisms $\alpha : F \to Y$ and $\beta : X \to Y$ such that

$$V = \{ (f, x) \in F \times X \mid \alpha(f) = \beta(x) \}.$$ Now $F \leq E$ is elementary abelian. If $(f, x), (f', x') \in V$, then $[(f, x), (f', x')] = (1, [x, x'])$, so $[V, V] \leq 1 \times [X, X]$. Conversely if $x, x' \in X$, then there exist $f, f' \in F$ such that $\alpha(f) = \beta(x)$ and $\alpha(f') = \beta(x')$, i.e. $(f, x), (f', x') \in V$. Then $[(f, x), (f', x')] = (1, [x, x'])$, and it follows that $[V, V] = 1 \times [X, X]$. Similarly, if $(f, x) \in V$, then $(f, x)^p = (1, x^p)$. Conversely, if $x \in X$, then there exists $f \in F$ such that $\alpha(f) = \beta(x)$, i.e. $(f, x) \in V$, and $(1, x^p) = (f, x)^p$. It follows that $\Phi(V) = 1 \times \Phi(X)$.

Now $N = \text{Ker}(\alpha) \times 1$ is a normal subgroup of $V$, and $N \cap \Phi(V) = 1$. By Proposition [6.6] it follows that $V^\alpha \cong (V/N)^\alpha$. Moreover the group homomorphism $(f, x) \in V \mapsto x \in X$ is surjective with kernel $N$, hence $V/N \cong X$. It follows that $V^\alpha \cong X^\alpha$ is isomorphic to a quotient of the subgroup $X$ of $T \cong P^\alpha$. Hence $V^\alpha$ is a subquotient of $P^\alpha$, as was to be shown.

\[\square\]

6.8. Proposition: Let $P$ be a finite $p$-group, let $N$ be a normal subgroup of $P$ such that $P/N \cong P^\alpha$, and let $Q$ be a subgroup of $P$. The following are
equivalent:

1. \( Q^\alpha \cong P^\alpha \).

2. \( QN = P \).

3. There exists a central elementary abelian subgroup \( E \) of \( P \) such that \( P = EQ \).

4. There exists an elementary abelian subgroup \( E \) of \( P \) such that \( P = E \times Q \).

**Proof:**

1 \( \Rightarrow \) 2 Suppose \( Q^\alpha \cong P^\alpha \). We have \( N \cap \Phi(T) = 1 \), by Proposition 6.6. Moreover \( \Phi(Q) \leq \Phi(P) \), as \( P \) is a \( p \)-group. Setting \( M = N \cap Q \), we have \( M \cap \Phi(Q) = 1 \), so \( (Q/M)^\alpha \cong Q^\alpha \cong P^\alpha \). But \( \overline{Q} = Q/M \cong QN/N \) is a subgroup of \( P/N \cong P^\alpha \), and moreover there exists an elementary abelian subgroup \( E \) of \( \overline{Q} \) such that \( \overline{Q} \cong E \times \overline{Q}^\alpha \cong E \times P^\alpha \). Hence \( E = 1 \) and \( \overline{Q} \cong QN/N \cong P/N \), so \(QN = P\), as was to be shown.

2 \( \Rightarrow \) 3 We have \( N \cap \Phi(P) = 1 \), by Proposition 6.6. Hence \( N \) is elementary abelian, and central in \( P \), and 2 implies 3.

2 \( \Rightarrow \) 3 Let \( E \) be an elementary abelian central subgroup of \( P \) such that \( P = EQ \). Let \( F \) be a complement of \( E \cap Q \) in \( E \). Then \( F \) is elementary abelian and central in \( P \). Moreover \( QF = QE = P \), and \( Q \cap F = 1 \). Hence \( P = F \times Q \).

4 \( \Rightarrow \) 1 If \( P = E \times Q \) and \( E \) is elementary abelian, then \( \Phi(P) = 1 \times \Phi(Q) \). Thus \( E \cap \Phi(P) = 1 \), so \( (P/E)^\alpha \cong P^\alpha \) by Proposition 6.6 and \( Q^\alpha \cong P^\alpha \).

6.9. Proposition:

1. Let \( L \) be an atoric \( p \)-group, let \( P = E \times L \) and \( Q = F \times L \), where \( E \) and \( F \) are elementary abelian \( p \)-groups, and let \( s : P \to Q \) be a group homomorphism. Then \( s \) is surjective if and only if there exist a surjective group homomorphism \( a : E \to F \), group homomorphisms \( b : L \to F \) and \( c : E \to \Omega_1 Z(L) \), and an automorphism \( d \) of \( L \) such that

\[
\forall (e, l) \in E \times L, \quad s(e, l) = (a(e)b(l), c(e)d(l)).
\]

Moreover in this case \( b \circ c(e) = 1 \) for any \( e \in E \), and \( s \) is an isomorphism if and only if \( a \) is an isomorphism.

2. Let \( P \) be a finite \( p \)-group. For a group homomorphism

\[
\lambda : P \to \Omega_1 Z(P) \cap \Phi(P),
\]

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Let \( \alpha_\lambda : P \to P \) be defined by \( \alpha(x) = x\lambda(x) \), for \( x \in P \). Then \( \alpha_\lambda \) is an automorphism of \( P \).

3. Let \( P \) be a finite \( p \)-group, and let \( P = E \times Q \), where \( Q \) is atomic and \( E \) is elementary abelian. Then the correspondence \( \lambda \mapsto \alpha_\lambda(E) \) is a bijection from the set of group homomorphisms \( \lambda : P \to \Omega_1Z(P) \cap \Phi(P) \) such that \( Q \leq \ker \lambda \) to the set of subgroups \( N \) of \( P \) such that \( P = N \times Q \).

**Proof:** (1) If \( s \) is surjective, then \( s(E) \) is central in \( Q \), so \( s(E) \leq \Omega_1Z(Q) = F \times \Omega_1Z(L) \). Hence there exists group homomorphisms \( a : E \to F \) and \( c : E \to \Omega_1Z(L) \) such that \( s(e, 1) = (a(e), c(e)) \), for any \( e \in E \). Let \( b : L \to F \) and \( d : L \to L \) be the group homomorphisms defined by \( s(1, l) = (b(l), d(l)) \), for \( l \in L \). Then \( s(e, l) = s(e, 1)s(1, l) = (a(e)b(l), c(e)d(l)) \) for all \( (e, l) \in P \). Moreover \( b \circ c(e) = 1 \) for any \( e \in E \), since \( c(E) \leq \Omega_1Z(L) \leq \Phi(L) \), as \( L \) is atomic, and \( \Phi(L) \leq \ker b \), as \( F \) is elementary abelian.

Now the composition of \( s \) with the projection \( F \times L \to L \) is surjective, hence \( s(1 \times L) = L \) by Proposition 6.4. In other words \( d \) is surjective, hence it is an automorphism of \( L \).

Since \( s \) is surjective, for any \((f, y) \in Q\), there exists \((e, x) \in P\) such that \( a(e)b(x) = f \) and \( c(e)d(x) = y \). The latter gives \( x = d^{-1}(c(e)^{-1}y) \). Then \( b(x) = bd^{-1}(c(e)^{-1})bd^{-1}(y) \), and \( bd^{-1}(c(e)^{-1}) = 1 \) since \( d^{-1}(c(e)^{-1}) \in d^{-1}\Omega_1Z(L) = \Omega_1Z(L) \), and \( \Omega_1Z(L) \leq \Phi(L) \leq \ker b \). Then \( b(x) = bd^{-1}(y) \), and \( f = a(e)bd^{-1}(y) \). In particular, taking \( y = 1 \), we get that for any \( f \in L \), there exists \( e \in E \) such that \( f = a(e) \). In other words \( a \) is surjective.

Conversely, given a surjective group homomorphism \( a : E \to F \), a group homomorphism \( b : L \to F \), a group homomorphism \( c : E \to \Omega_1Z(L) \), and an automorphism \( d \) of \( L \), we can define \( s : P \to Q \) by \( s(e, x) = (a(e)b(x), c(e)d(x)) \), for \( (e, x) \in P \). This is clearly a group homomorphism, as \( F \) is abelian, and the image of \( c \) is central in \( L \). We have again \( \Omega_1Z(L) \leq \Phi(L) \leq \ker b \), since \( F \) is elementary abelian. If \((f, y) \in Q\), we can choose an element \( e \in E \) such that \( f = a(e)bd^{-1}(y) \), and then set \( x = d^{-1}(c(e)^{-1})y \), i.e. \( c(e)d(x) = y \). We also have \( b(x) = bd^{-1}(y) \), since \( d^{-1}(c(e)) \in \Omega_1Z(L) \), so \( f = a(e)b(x) \). Hence \( s(e, x) = (f, y) \), and \( s \) is surjective.

Finally, if \( s \) is an isomorphism, then \( E \cong F \), and then the surjection \( a \) is an isomorphism. Conversely, if \( a \) is an isomorphism, then \( E \cong F \), so \( P \cong Q \), and the surjection \( s \) is an isomorphism.

(2) Clearly \( \alpha_\lambda \) is a group homomorphism, since \( \lambda(P) \leq Z(P) \). Moreover if \( x \in \ker \alpha_\lambda \), then \( \lambda(x) = x \), so \( x \in \Omega_1Z(P) \cap \Phi(P) \leq \Phi(P) \leq \ker \lambda \), since \( \Omega_1Z(P) \cap \Phi(P) \) is elementary abelian. Thus \( x = 1 \), and \( \alpha_\lambda \) is injective. Hence it is an automorphism.

(3) Since \( P = E \times Q \), we have \( \Omega_1Z(P) = E \times \Omega_1Z(Q) \), and \( \Phi(P) = 1 \times \Phi(Q) \). 28
So if $\lambda$ is a group homomorphism from $P$ to $\Omega_1 Z(P) \cap \Phi(P)$ with $Q \leq \ker \lambda$, we have $\lambda(e, l) = (1, \beta(e))$ for some group homomorphism $\beta : E \to \Omega_1 Z(Q)$. Then the group $N = \alpha_\lambda(E) = \{(e, \beta(e)) \mid e \in E\}$ is central in $P$. Moreover $N \cap Q = 1$, and $NQ = P$, so $P = N \times Q$. Note that $N$ determines the homomorphism $\beta$, hence also the homomorphism $\lambda$, so the map $\lambda \mapsto \alpha_\lambda(E)$ is injective.

It is moreover surjective: indeed, if $N$ is a subgroup of $P = E \times Q$ such that $P = N \times Q$, then $N \cong P/Q \cong E$ is elementary abelian, hence central in $P$. Since $NQ = P$, for any $e \in E$, there exists $(a, b) \in N$ and $q \in Q$ such that $(e, 1) = (a, b)(1, q)$, that is $e = a$ and $q = b^{-1}$. In other words $p_1(N) = E$. Moreover $N \cap Q = 1$, so $k_2(N) = 1$. So for $e \in E$, there exists a unique $x \in Q$ such that $(e, x) \in N$. Setting $x = \beta(e)$, we get a group homomorphism $\beta : E \to Q$, such that $N = \{(e, \beta(e)) \mid e \in E\}$. Since $N$ is central in $P$, the image of $\beta$ is contained in $\Omega_1 Z(Q) \leq \Phi(Q)$. Moreover $\Omega_1 Z(P) = E \times \Omega_1 Z(Q)$, and $\Phi(P) = 1 \times \Phi(Q)$, so $(1 \times \beta(E)) \leq \Omega_1 Z(P) \cap \Phi(P)$. Setting $\lambda(e, l) = (1, \beta(e))$, we get a group homomorphism from $P$ to $\Omega_1 Z(P) \cap \Phi(P)$, such that $Q \leq \ker \lambda$, and $N = \alpha_\lambda(E)$.

7. Splitting the biset category of $p$-groups, when $p \in R^X$

7.1. Notation and Definition: Let $R\mathcal{C}_p$ denote the full subcategory of the biset category $R\mathcal{C}$ consisting of finite $p$-groups. A $p$-biset functor over $R$ is an $R$-linear functor from $R\mathcal{C}_p$ to the category of $R$-modules. Let $F_{p,R}$ denote the full subcategory of $F_R$ consisting of $p$-biset functors over $R$.

In the statements below, we indicate by $[p \in R^X]$ the assumption that $p$ is invertible in $R$.

7.2. Theorem: $[p \in R^X]$ Let $P$ and $Q$ be finite $p$-groups, let $(T, S)$ be a minimal section of $P$, and $(V, U)$ be a minimal section of $Q$. Then

$$
eq_{V,U} Q \, RB(Q, P) \, e_{T,S}^P \neq \{0\} \implies (V/U)^\alpha \cong (T/S)^\alpha.$$ 

Proof: If $e_{V,U}^Q RB(Q, P) e_{T,S}^P \neq \{0\}$, there exists $a \in RB(Q, P)$ such that $e_{V,U}^Q a e_{T,S}^P = \operatorname{Indinf}_{V/U}^Q \varphi_{V/U}^T \operatorname{Defres}_{V/U}^Q a \operatorname{Indinf}_{T/S}^P \varphi_{T/S}^T \operatorname{Defres}_{T/S}^P \neq 0$, and in particular the element $b = \operatorname{Defres}_{V/U}^Q a \operatorname{Indinf}_{T/S}^P$ of $RB(V/U, T/S)$ is such that $\varphi_{V/U}^T b \varphi_{T/S}^T \neq 0$. It follows that there is a subgroup $L$ of the
product \((V/U) \times (T/S)\) such that
\[
\varphi^{-1} \left[ \frac{(V/U) \times (T/S)}{L} \right] \varphi^{-1} \neq 0.
\]
Then Theorem 3.8 implies that \(p_1(L) = V/U\), \(k_1(L) \cap \Phi(V/U) = 1\), \(p_2(L) = T/S\), and \(k_2(L) \cap \Phi(T/S) = 1\). By Proposition 6.6 it follows that
\[
(V/U)^\alpha \cong (p_1(L)/k_1(L))^\alpha \cong (p_2(L)/k_2(L))^\alpha \cong (T/S)^\alpha,
\]
as was to be shown.

\[\text{7.3. Notation: } [ p \in R^\times ] \quad \text{Let } L \text{ be an atoric } p\text{-group. If } P \text{ is a finite } p\text{-group, we set}
\]
\[
b^P_L = \sum_{(T,S) \in [\mathcal{M}(G)]} \epsilon^P_{T,S}.
\]

\[\text{7.4. Theorem: } [ p \in R^\times ]
\]

1. Let \(L\) be an atoric \(p\)-group, and \(P\) be a finite \(p\)-group. Then \(b^P_L \neq 0\) if and only if \(L \trianglelefteq P^\alpha\).

2. Let \(L\) and \(M\) be atoric \(p\)-groups, and let \(P\) and \(Q\) be finite \(p\)-groups. If \(b_M^Q RB(Q, P) b^P_L \neq \{0\}\), then \(M \cong L\).

3. Let \(L\) be an atoric \(p\)-group, and let \(P\) and \(Q\) be finite \(p\)-groups. Then for any \(a \in RB(Q, P)\)
\[
b^Q_L a = a b^P_L.
\]

4. The family of elements \(b^P_L \in RB(P, P)\), for finite \(p\)-groups \(P\), is an idempotent endomorphism \(b^P_L\) of the identity functor of the category \(RC_p\) (i.e. an idempotent of the center of \(RC_p\)). The idempotents \(b^P_L\) for \(L \in [\mathcal{A}_p]\), are orthogonal, and their sum is equal to the identity element of the center of \(RC_p\).

5. For a given finite \(p\)-group \(P\), the elements \(b^P_L\), for \(L \in [\mathcal{A}_p]\) such that \(L \trianglelefteq P^\alpha\), are non zero orthogonal central idempotents of \(RB(P, P)\), and their sum is equal to the identity of \(RB(P, P)\).

\[\text{Proof:} (1)\text{ The idempotent } b^P_L \text{ is non zero if and only if there exists a minimal section } (T, S) \text{ of } P \text{ such that } (T/S)^\alpha \cong L. \text{ Then } L \trianglelefteq P^\alpha, \text{ by Proposition 6.7.}
\]

Conversely, if \(L \subseteq P^\alpha\), then \(L \subseteq P\), and there exists a minimal section \((T, S)\) of \(P\) such that \(T/S \cong L\). Then \((T/S)^\alpha \cong L^\alpha \cong L\), so \(\epsilon^P_{T,S}\) appears in the sum defining \(b^P_L\), thus \(b^P_L \neq 0\).
(2) If \( b^Q_M \cdot RB(Q, P) \cdot b^P_L \neq \{0\} \), then there exist a minimal section \((V, U)\) of \(Q\) with \((V/U)^\# \cong M\) and a minimal section \((T, S)\) of \(P\) with \((T/S)^\# \cong L\) such that \( e^Q_{V,U} \cdot RB(Q, P) \cdot e^P_{T,S} \neq 0\). Then \((V/U)^\# \cong (T/S)^\#\) by Theorem 7.2, that is \(M \cong L\).

(3) The identity element of \(RB(P, P)\) is equal to the sum of the idempotents \(e^P_{T,S}\), for \((T, S) \in [\mathcal{M}(P)]\). Grouping those idempotents \(e^P_{T,S}\) for which \((T/S)^\#\) is isomorphic to a given \(L \in [\mathcal{A}_p]\) shows that the identity element of \(RB(P, P)\) is equal to the sum of the idempotents \(b^P_L\), for \(L \in [\mathcal{A}_p]\) (and there are finitely many non zero \(b^P_L\); by (1)). It follows that

\[
b^Q_M a = b^Q_M a \sum_{L \in [\mathcal{A}_p]} b^P_L = \sum_{L \in [\mathcal{A}_p]} b^Q_M a b^P_L
\]

\[
= b^Q_M a b^P_M \quad \text{[by (2)]}
\]

\[
= \sum_{L \in [\mathcal{A}_p]} b^Q_L a b^P_M \quad \text{[by (2)]}
\]

\[
= a b^P_M ,
\]

since \(\sum_{L \in [\mathcal{A}_p]} b^Q_L\) is the identity element of \(RB(Q, Q)\).

It follows that the family \(b^P_L\), where \(P\) is a finite \(p\)-group, is an element \(b_L\) of the center of \(RC_p\). Clearly \(b^2_L = b_L\), and if \(L\) and \(M\) are non isomorphic atoric \(p\)-groups, then \(b_L b_M = 0\), by (2). Moreover the infinite sum \(\sum_{L \in [\mathcal{A}_p]} b_L\) is actually locally finite, i.e. for each finite \(p\)-group \(P\), the sum \(\sum_{L \in [\mathcal{A}_p]} b^P_L\) has only finitely many non zero terms. The sum \(\sum_{L \in [\mathcal{A}_p]} b_L\) is clearly equal to the identity endomorphism of the identity functor of \(RC_p\).

(4) This is a straightforward consequence of (1) and (3).

\[\blacksquare\]

7.5. \textbf{Corollary:} [ \(p \in R^\times\)]

1. Let \(L\) be an atoric \(p\)-group. For a \(p\)-biset functor \(F\), the family of maps \(F(b^P_L) : F(P) \rightarrow F(P)\), for finite \(p\)-groups \(P\), is an endomorphism of \(F\), denoted by \(F(b_L)\).

2. If \(\theta : F \rightarrow G\) is a natural transformation of \(p\)-biset functors, the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{F(b_L)} & F \\
\downarrow{\theta} & & \downarrow{\theta} \\
G & \xrightarrow{G(b_L)} & G
\end{array}
\]
is commutative. Hence the family of endomorphisms $F(b_L)$, for $p$-biset functors $F$, is an idempotent of the center of the category $\mathcal{F}_{p,R}$, denoted by $\hat{b}_L$.

3. The idempotents $\hat{b}_L$, for $L \in [At_p]$, are orthogonal idempotents of the center of $\mathcal{F}_{p,R}$, and their sum is the identity.

4. If $F$ is a $p$-biset functor over $R$, let $\hat{b}_LF$ denote the image of the endomorphism $F(b_L)$ of $F$. Then $F = \bigoplus_{L \in [At_p]} \hat{b}_LF$.

5. Let $\hat{b}_L\mathcal{F}_{p,R}$ denote the full subcategory of $\mathcal{F}_{p,R}$ consisting of functors $F$ such that $F = \hat{b}_LF$. Then $\hat{b}_L\mathcal{F}_{p,R}$ is an abelian subcategory of $\mathcal{F}_{p,R}$. Moreover the functor

$$F \in \mathcal{F}_{p,R} \mapsto (\hat{b}_LF)_{L \in [At_p]} \in \prod_{L \in [At_p]} \hat{b}_L\mathcal{F}_{p,R}$$

is an equivalence of categories.

**Proof**: All assertions are straightforward consequences of Theorem 7.4.

### 7.6. Notation:
For an atoric $p$-group $L$, let $RC_p^L$ denote the full subcategory of $RC_p$ consisting of the class $\mathcal{Y}_L$ of finite $p$-groups $P$ such that $P^{\text{ab}} \subseteq L$. When $p \in R^\times$, Let moreover

$$b_L^+ = \sum_{H \in [At_p]} b_H$$

be the sum of the idempotents $b_H$ corresponding to atoric subquotients of $L$, up to isomorphism.

The class $\mathcal{Y}_L$ is closed under taking subquotients, by Proposition 6.7. It follows that we can apply the results of Section 6 (Appendix) of [12]: if $F$ is a $p$-biset functor over $R$, we can restrict $F$ to an $R$-linear functor from $RC_p^L$ to $R$-Mod. This yields a forgetful functor $\mathcal{O}_{\mathcal{Y}_L} : \mathcal{F}_{p,R} \rightarrow \text{Fun}_R(\mathcal{Y}_L, \text{R-Mod})$. The right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ of this functor is described in full detail in Section 6 of [12], as follows: if $G$ is an $R$-linear functor from $RC_p^L$ to $R$-Mod, and $P$ is a finite $p$-group, set

$$(7.7) \quad \mathcal{R}_{\mathcal{Y}_L}(G)(P) = \lim_{\leftarrow} G(X/M)$$

the inverse limit of modules $G(X/M)$ on the set $\Sigma_L(P)$ of sections $(X, M)$.
of \( P \) such that \((X/M)^a \subseteq L\), i.e. the set of sequences \((l_{X,M})_{(X,M) \in \Sigma_L(P)}\) with the following properties:

1. if \((X, M) \in \Sigma_L(P)\), then \(l_{X,M} \in G(X/M)\).
2. if \((X, M), (Y, N) \in \Sigma_L(P)\) and \(M \leq N \leq Y \leq X\), then
   \[
   \text{Defres}_{Y/N}^{X/M} l_{X,M} = l_{Y,N} .
   \]
3. if \(x \in P\) and \((X, M) \in \Sigma_L(P)\), then
   \[
   x l_{X,M} = l_{x,X,M} .
   \]

Recall now that for finite groups \( P \) and \( Q \), and for a finite \((Q, P)\)-biset \( U \), for a subgroup \( T \) of \( Q \) and an element \( u \) of \( U \), the subgroup \( T u \) of \( P \) is defined by
\[
T u = \{ x \in P \mid \exists t \in T \; tu = ux \}. \]
By Lemma 6.4 of [12], if \((T, S)\) is a section of \( Q \), then \((T^u, S^u)\) is a section of \( P \), and \( T^u / S^u \) is a subquotient of \( T / S \).

With this notation, when \( P \) and \( Q \) are finite \( p \)-groups, when \( U \) is a finite \((Q, P)\)-biset, and \( l = (l_{X,M})_{(X,M) \in \Sigma_L(P)}\) is an element of \( \mathcal{R}_Y L(G)(P) \), we denote by \( Ul \) the sequence indexed by \( \Sigma_L(Q) \) defined by
\[
(Ul)_{Y,N} = \sum_{u \in [Y \setminus U/P]_L} (N \setminus Yu)(l_{Y^u,N^u})
\]
where \([Y \setminus U/P]_L\) is a set of representatives of \((Y \times P)\)-orbits on \( U \), and \( N \setminus Yu \) is viewed as a \((Y/N, Y^u/N^u)\)-biset. It shown in Section 6 of [12] that \( Ul \in \mathcal{R}_Y L(G)(Q) \), and that \( \mathcal{R}_Y L(G) \) becomes a \( p \)-biset functor in this way. Moreover:

\[\textbf{7.8. Theorem:} \quad [\text{[12] Theorem 6.15}] \quad \text{The assignment } G \mapsto \mathcal{R}_Y L(G) \text{ is an } R\text{-linear functor } \mathcal{R}_Y L \text{ from } \text{Fun}_R(\mathcal{R}C^L_P, R\text{-Mod}) \text{ to } \mathcal{F}_P, \text{ which is right adjoint to the forgetful functor } \mathcal{O}_Y L. \text{ Moreover the composition } \mathcal{O}_Y L \circ \mathcal{R}_Y L \text{ is isomorphic to the identity functor of } \text{Fun}_R(\mathcal{R}C^L_P, R\text{-Mod}).\]

\[\textbf{7.9. Theorem:} \quad [p \in R^\times] \quad \text{For an atoric } p\text{-group } L, \text{ let } \hat{b}_L^+ F_{p,R} \text{ be the full subcategory of } F_{p,R} \text{ consisting of functors } F \text{ such that } \hat{b}_L^+ F = F. \text{ Then the forgetful functor } \mathcal{O}_Y L \text{ and its right adjoint } \mathcal{R}_Y L \text{ restrict to quasi-inverse equivalences of categories}
\]
\[
\hat{b}_L^+ F_{p,R} \xrightarrow{\mathcal{O}_Y L} \text{Fun}_R(\mathcal{R}C^L_P, R\text{-Mod}) .
\]

\[1\text{In Theorem 6.15 of [12], only the case } R = \mathbb{Z} \text{ is considered, but the proofs extend trivially to the case of an arbitrary commutative ring } R.\]
Proof: First step: The first thing to check is that the image of the functor \( \mathcal{R}_L \) is contained in \( \hat{b}_L^+ F_{p,R} \). We first prove that if \( H \) is an atoric \( p \)-group, if \( F \in F_{p,R} \), and if \( \mathcal{O}_{Y_L}(\hat{b}_H F) \neq 0 \), then \( H \subseteq L \); indeed in that case, there exists \( P \in Y_L \) such that \( b_H^P F(P) \neq 0 \). In particular \( b_H^P \neq 0 \) by Theorem 7.4, hence \( H \subseteq P^\alpha \). Since \( P^\alpha \subseteq L \) as \( P \in Y_L \), it follows that \( H \subseteq L \), as claimed.

In particular

\[
\mathcal{O}_{Y_L}(F) = \mathcal{O}_{Y_L} \left( \sum_{H \in \left[ A_t^p \right]} \hat{b}_H F \right) = \mathcal{O}_{Y_L}(\hat{b}_L^+ F) .
\]

Set \( G_p^L = \text{Fun}_R(RC_p^L, R\text{-Mod}) \), and let \( G \in G_p^L \). Let \( H \) be an atoric \( p \)-group such that \( H \nsubseteq L \). If \( F \in F_{p,R} \), then

\[
\text{Hom}_{F_{p,R}}(F, \hat{b}_H \mathcal{R}_L(G)) = \text{Hom}_{F_{p,R}}(\hat{b}_H F, \hat{b}_H \mathcal{R}_L(G)) = \text{Hom}_{F_{p,R}}(\hat{b}_H F, \mathcal{R}_L(G)) \cong \text{Hom}_{G_p^L}(\mathcal{O}_{Y_L}(\hat{b}_H F), G) = \{0\} .
\]

So the functor \( F \mapsto \text{Hom}_{F_{p,R}}(F, \hat{b}_H \mathcal{R}_L(G)) \) is the zero functor, and it follows from Yoneda’s lemma that \( \hat{b}_H \mathcal{R}_L(G) = 0 \) if \( H \nsubseteq L \). In other words \( \mathcal{R}_L(G) = \hat{b}_L^+ \mathcal{R}_L(G) \), as was to be shown.

Second step: The first step shows that we have adjoint functors

\[
\hat{b}_L^+ F_{p,R} \xrightarrow{\mathcal{O}_{Y_L}} \text{Fun}_R(RC_p^L, R\text{-Mod}) = G_p^L .
\]

Moreover, the composition \( \mathcal{O}_{Y_L} \circ \mathcal{R}_L \) is isomorphic to the identity functor, by Theorem 7.4. All we have to show is that the unit of the adjunction is also an isomorphism, in other words, that for any \( F \in \hat{b}_L^+ F_{p,R} \) and any finite \( p \)-group \( P \), the natural map

\[
(7.10) \quad \eta_P : F(P) \to \mathcal{R}_L \mathcal{O}_{Y_L}(F)(P) = \lim_{(X,M) \in \Sigma_L(P)} F(X/M)
\]

sending \( u \in F(P) \) to the sequence \( (\text{Defres}_{X/M}^P u)_{(X,M) \in \Sigma_L(P)} \), is an isomorphism.

The map \( \eta_P \) is injective: indeed, if \( u \in F(P) \), then \( u = \sum_{H \in \left[ A_t^p \right]} b_H^P u \), as \( F = \hat{b}_L^+ F \). If \( \text{Defres}_{X/M}^P u = 0 \) for any section \( (X, M) \) of \( P \) with \( (X/M)^\alpha \subseteq L \),
then \(F(\epsilon_{T,S}^P)(u) = 0\) for any section \((T, S)\) of \(P\) such that \((T/S)^{\alpha} \subseteq L\), by Proposition 4.6 and Proposition 6.7. In particular \(b_H^P u = 0\) for any atoric subquotient \(H\) of \(L\), hence \(u = 0\).

To prove that \(\eta_P\) is also surjective, we generalize the construction of Theorem A.2 of [11] (which is the case \(L = 1\)), and we define, for an element \(v = (v_{X,M})_{(X, M) \in \Sigma_L(P)}\) in \(\mathcal{R}_Y \mathcal{O}_{Y_2}(F)(P)\), an element \(u = \iota_P(v)\) of \(F(P)\) by

\[
u = \frac{1}{|P|} \sum_{(T, S) \in M(P)} \sum_{X \leq T, M \leq T} \sum_{S \leq M, \Phi(T) \leq X \leq T} \frac{|X|}{|P|} \mu(X, T) \mu_{\leq T}(S, M) \text{Ind}_{X/M} \nu_{X,M}.
\]

This yields an \(R\)-linear map \(\iota_P : \mathcal{R}_Y \mathcal{O}_{Y_2}(F)(P) \to F(P)\).

For \((Y, N) \in \Sigma_L(P)\), set \(u_{Y,N} = \text{Defres}_{Y/N}^P u\). Then:

\[
u_{Y,N} = \sum_{(T, S) \in M(P)} \sum_{X \leq T, M \leq T} \sum_{S \leq M, \Phi(T) \leq X \leq T} \frac{|X|}{|P|} \mu(X, T) \mu_{\leq T}(S, M) \text{Defres}_{Y/N}^P \text{Ind}_{X/M} \nu_{X,M}.
\]

Moreover

\[
\text{Defres}_{Y/N}^P \text{Ind}_{X/M} \nu_{X,M} = \sum_{g \in [Y \setminus P/X]} \text{Ind}_{I_g/I'_g}^{Y/N} \text{Iso}(\phi_g) \text{Defres}_{I_g/I'_g}^{g(X)/gM} \nu_{X,N},
\]

where \(J_g = N(Y \cap gX), J'_g = N(Y \cap gM, I_g = gM(Y \cap gX), I'_g = gM(N \cap gX),\) and \(\phi_g\) is the isomorphism \(I_g/I'_g \to J_g/J'_g\) sending \(xI_g\) to \(xJ_g\), for \(x \in Y \cap gX\). Hence

\[
\text{Defres}_{Y/N}^P \text{Ind}_{X/M} \nu_{X,M} = \sum_{g \in [Y \setminus P/X]} \text{Ind}_{I_g/I'_g}^{Y/N} \text{Iso}(\phi_g) \nu_{I_g/I'_g}.
\]

Thus

\[
u_{Y,N} = \sum_{(T, S) \in M(P)} \sum_{X \leq T, M \leq T} \sum_{S \leq M, \Phi(T) \leq X \leq T} \frac{|Y \cap gX|}{|P||X|} \mu(X, T) \mu_{\leq T}(S, M) \text{Ind}_{I_g/I'_g}^{Y/N} \text{Iso}(\phi_g) \nu_{I_g/I'_g}.
\]

Now \(\mu(X, T) = \mu(gX, gT)\) and \(\mu_{\leq T}(S, M) = \mu_{\leq gT}(gS, gM)\), so summing over \((gT, gS, gX, gM)\) instead of \((T, S, X, M)\) we get

\[
u_{Y,N} = \sum_{(T, S) \in M(P)} \sum_{X \leq T, M \leq T} \sum_{S \leq M, \Phi(T) \leq X \leq T} \frac{|Y \cap X|}{|Y|} \mu(X, T) \mu_{\leq T}(S, M) \text{Ind}_{I_g/I'_g}^{Y/N} \text{Iso}(\phi_1) \nu_{I_g/I'_g}.
\]

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Setting $W = Y \cap X$, we have $J_1 = NW$, $J'_1 = N(W \cap M)$, $I_1 = MW$, $I'_1 = M(N \cap W)$, and these four groups only depend on $W$, once $M$ and $N$ are given. Hence, for given $T, S$ and $M$, we can group together the terms of the above summation for which $Y \cap X$ is a given subgroup $W$ of $Y \cap T$. This gives

$$u_{Y, N} = \sum_{(T, S) \in M(P)} \left( \sum_{\substack{\Phi(T) \leq X \leq T \\ X \cap Y = W}} \mu(X, T) \frac{|W|}{|Y|} \mu_{\leq T}(S, M) \text{Ind}_{J_1/J'_1} \text{Iso}(\phi_1)v_{I_1, I'_1}. \right.$$  

Moreover

$$\sum_{\Phi(T) \leq X \leq T} \mu(X, T) = \sum_{X \leq T} \mu(X, T), \text{ since } \mu(X, T) = 0 \text{ unless } X \geq \Phi(T), \text{ and the latter summation vanishes unless } Y \cap T = T, \text{ by classical combinatorial lemmas (15 Corollary 3.9.3). This gives:}$$

$$u_{Y, N} = \sum_{(T, S) \in M(P)} \frac{|W|}{|Y|} \mu(W, T) \mu_{\leq T}(S, M) \text{Ind}_{J_1/J'_1} \text{Iso}(\phi_1)v_{I_1, I'_1}. \right.$$  

Moreover in this summation $J_1 = NW$, $J'_1 = N(W \cap M) = NM$, $I_1 = MW = W$, $I'_1 = M(N \cap W) = MN \cap W$. All these groups remain unchanged if we replace $M$ by $M(N \cap \Phi(T))$, so for given $T, S$ and $W$, we can group together those terms for which $M(N \cap \Phi(T))$ is a given normal subgroup $U$ of $T$ with $U \leq \Phi(T)$. The sum

$$\sum_{S \leq M \leq T} \mu_{\leq T}(S, M) = 0 \text{ (by the same above-mentioned classical combinatorial lemmas) unless } N \cap \Phi(T) \leq S.$$  

Hence

$$u_{Y, N} = \sum_{(T, S) \in M(P)} \frac{|W|}{|Y|} \mu(W, T) \mu_{\leq T}(S, U) \text{Ind}_{J_1/J'_1} \text{Iso}(\phi_1)v_{I_1, I'_1},$$  

where $J_1 = NW$, $J'_1 = NU$, $I_1 = W$, $I'_1 = UN \cap W$.

Now if $N \cap \Phi(T) \leq S \leq \Phi(T) \leq T \leq Y$, then $(TN/N)^{\alpha} \subseteq (Y/N)^{\alpha}$. Moreover the normal subgroup $(N \cap T)/(N \cap \Phi(T))$ of $T/(N \cap \Phi(T))$ intersects trivially the Frattini subgroup

$$\Phi\left(T/(N \cap \Phi(T))\right) = \Phi(T)(N \cap \Phi(T))/(N \cap \Phi(T)),$$  

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so \( (T/(N \cap \Phi(T)))^\alpha \cong (T/(T \cap N))^\alpha \cong (TN/N)^\alpha \) by Proposition 6.6. Then 
\((T/S)^\alpha \subseteq (T/(N \cap \Phi(T)))^\alpha \subseteq (TN/N)^\alpha \subseteq (Y/N)^\alpha \). As \((Y/N)^\alpha \subseteq L\) by assumption, it follows that

\[
\mathcal{H} = (T/S)^\alpha \cap (N \cap \Phi(T)) = 0.
\]

Then \(u_{Y,N} = \sum_{S \leq T \leq Y} \frac{|W|}{|Y|} \mu(W,T) \mu_{\leq T} (S,U) \text{Indinf}_{J_1/I_1'} \text{Iso}(\phi_1) v_{I_1,I_1'}\).

Now the sum \(\sum_{S \leq T \leq Y} \mu_{\leq T} (S,U)\) is equal to zero unless \(U = N \cap \Phi(T)\). Hence

\[
u_{Y,N} = \sum_{\Phi(T) \leq W \leq Y} \frac{|W|}{|Y|} \mu(W,T) \text{Indinf}_{J_1/I_1'} \text{Iso}(\phi_1) v_{I_1,I_1'}\].

For a given subgroup \(W\) of \(Y\), the sum \(\sum_{\Phi(T) \leq W \leq Y} \mu(W,T)\) is equal to \(\sum_{W \leq T \leq Y} \mu(W,T)\) since \(\mu(W,T) = 0\) unless \(W \geq \Phi(T)\), and the latter is equal to zero if \(W \neq Y\), and to 1 if \(W = Y\). Thus

\[
u_{Y,N} = \frac{|Y|}{|Y|} \text{Indinf}_{J_1/I_1'} \text{Iso}(\phi_1) v_{I_1,I_1'},\]

where \(J_1 = NY = Y\), \(J_1' = N(Y \cap U) = N\), \(I_1 = Y\), \(I_1' = UN \cap Y = N\). Hence \(I_1 = J_1 = Y\) and \(I_1' = J_1' = N\), so \(\phi_1\) is equal to the identity. It follows that \(u_{Y,N} = v_{Y,N}\) for any \((Y,N) \in \Sigma_L(P)\), so \(\eta_P(u) = v\). This proves that the map \(\eta_P\) is surjective, hence an isomorphism, with inverse \(\iota_P\). This completes the proof of Theorem 7.9.

7.11. Definition: Let \(RC_p^{LL}\) be the following category:

- The objects of \(RC_p^{LL}\) are the finite \(p\)-groups \(P\) such that \(P^\alpha \cong L\).
- If \(P\) and \(Q\) are finite \(p\)-groups such that \(P^\alpha \cong Q^\alpha \cong L\), then

\[
\text{Hom}_{RC_p^{LL}}(P,Q) = RB(Q,P)/\sum_{S \leq L} RB(Q,S)B(S,P)
\]

is the quotient of \(RB(Q,P)\) by the \(R\)-submodule generated by all morphisms from \(P\) to \(Q\) in \(RC_p\) which factor through a \(p\)-group \(S\) which do not admit \(L\) as a subquotient.

- The composition of morphisms in \(RC_p^{LL}\) is induced by the composition of morphisms in \(RC_p\).
7.12. **Remark:** Morphisms in $RC_p$ which factor through a $p$-group $S$ such that $L \nsubseteq S$ clearly generate a two-sided ideal, so the composition in $RC_p^{tL}$ is well defined. Moreover the category $RC_p^{tL}$ is $R$-linear. Let $\text{Fun}_R(RC_p^{tL}, R\text{-Mod})$ denote the category of $R$-linear functors from $RC_p^{tL}$ to the category $R\text{-Mod}$ of $R$-modules.

7.13. **Lemma:** Let $p$ be a prime, and $L$ be an atoric $p$-group. Let $P$ and $Q$ be finite $p$-groups.

1. If $P^@ \cong L$ or $Q^@ \cong L$, and if $M \leq (Q \times P)$, then $q(M)^@ \subseteq L$. Moreover $q(M)^@ \cong L$ if and only if $L \subseteq q(M)$.

2. If $P^@ \cong Q^@ \cong L$, then

$$\text{Hom}_{RC_p^{tL}}(P, Q) = RB(Q, P)/ \sum_{S^@ \subseteq L} RB(Q, S)B(S, P)$$

is also the quotient of $RB(Q, P)$ by the $R$-submodule generated by all morphisms from $P$ to $Q$ in $RC_p$ which factor through a $p$-group $S$ such that $S^@$ is a proper subquotient of $L$.

3. If $P^@ \cong Q^@ \cong L$, then $\text{Hom}_{RC_p^{tL}}(P, Q)$ has an $R$-basis consisting of the (images of the) transitive $(Q, P)$-bisets $(Q \times P)/M$, where $M$ is a subquotient of $(Q \times P)$ such that $q(M)^@ \cong L$ (up to conjugation).

**Proof:** (1) Indeed $q(M)$ is a subquotient of $P$, and a subquotient of $Q$. Hence $q(M)^@$ is a subquotient of $P^@$ and a subquotient of $Q^@$, thus $q(M) \subseteq L^@ \cong L$. Now suppose that $q(M)^@ \cong L$. Then $L$ is a quotient of $q(M)$, so $L \subseteq q(M)$. Conversely, if $L \subseteq q(M)$, then $L \cong L^@$ is a subquotient of $q(M)^@$, which is a subquotient of $L$. So $q(M)^@ \cong L$.

(2) Let $S$ be a finite $p$-group such that $L \nsubseteq S$, or equivalently, $L \nsubseteq S^@$. Any element of $RB(Q, S)B(S, P)$ is a linear combination of $(Q, P)$-bisets of the form $(Q \times P)/(M \ast N)$, for $M \leq (Q \times S)$ and $N \leq (S \times P)$. This biset $(Q \times P)/(M \ast N)$ also factors through $T = q(M \ast N)$, by 2.6. Moreover $T$ is a subquotient of $q(M)$ and $q(N)$, hence a subquotient of $Q$, $S$, and $P$. Hence $T^@ \subseteq Q^@ \cong L$, and $T^@ \cong L$, since $L \nsubseteq S^@$. Hence $T \subseteq L$.

(3) The (images of the) elements $(Q \times P)/M$, where $M$ is a subgroup of $(Q \times P)$ such that $q(M)^@ \cong L$ (up to conjugation), clearly generate $\text{Hom}_{RC_p^{tL}}(P, Q)$. Moreover, the proof of (2) shows that they are linearly independent, since any transitive $(Q, P)$-biset $(Q \times P)/N$ appearing in an
element of the sum $\sum_{S^\alpha \subset L} RB(Q,S)B(S,P)$ is such that $q(N)^\alpha \subset L$. 

7.14. Remark: If $G$ is an $R$-linear functor from $RC_p^\mu L$ to the category $R\text{-Mod}$ of $R$-modules, we can extend $G$ to an $R$-linear functor from $RC_p^{\mu L}$ to $R\text{-Mod}$ by setting $G(P) = \{0\}$ if $P$ is a finite $p$-group such that $P^\alpha$ is a proper subquotient of $L$. Conversely, an $R$-linear functor from $RC_p^{\mu L}$ to $R\text{-Mod}$ which vanishes on $p$-groups $P$ such that $P^\alpha \not\subset L$ can be viewed as an $R$-linear functor from $RC_p^{\mu L}$ to $R\text{-Mod}$. In the sequel, we will freely identify those two types of functors, and consider $\text{Fun}_R(RC_p^{\mu L}, R\text{-Mod})$ as the full subcategory of $\text{Fun}_R(RC_p^L, R\text{-Mod})$ consisting of functors which vanish on $p$-groups $P$ such that $P^\alpha \not\subset L$.

7.15. Theorem: $[\ p \in R^x\ ]$ Let $L$ be an atoric $p$-group.

1. If $F$ is a $p$-biset functor over $R$ such that $F = \hat{b}_L F$, and $P$ is a finite $p$-group such that $L \not\subset P$, then $F(P) = \{0\}$.

2. If $G$ is an $R$-linear functor from $RC_p^{\mu L}$ to $R\text{-Mod}$, then $\hat{b}_L\mathcal{R}_L(G) = \mathcal{R}_{L}(G)$.

3. The forgetful functor $\mathcal{O}_{L}$ and its right adjoint $\mathcal{R}_{L}$ restrict to quasi-inverse equivalences of categories

$$\hat{b}_L\mathcal{F}_p R \xrightarrow{\mathcal{O}_{L}} \mathcal{R}_{L} \xleftarrow{\text{Fun}_R(RC_p^{\mu L}, R\text{-Mod})}$$

Proof: (1) Since $\hat{b}_L F = F$, then in particular $F(b_L^P) F(P) = F(P)$. If $L \not\subset P$, then there is no minimal section $(T, S)$ of $P$ with $(T/S)^\alpha \cong L$, thus $b_L^P = 0$, and $F(P) = \{0\}$.

(2) Let $G$ be an $R$-linear functor from $RC_p^{\mu L}$ to $R\text{-Mod}$, in other words an $R$-linear functor from $L\mathcal{F}RC_p^{\mu L}$ to $R\text{-Mod}$ which vanishes on $p$-groups $P$ such that $P^\alpha$ is a proper subquotient of $L$. By Theorem 7.13 we have $\hat{b}_L^P \mathcal{R}_L(G) = \mathcal{R}_L(G)$. If $H$ is an atoric $p$-group which is a proper subquotient of $L$, then $G$ vanishes over any subquotient $Q$ of $H$, since $Q^\alpha \subset H \subset L$ if $Q \subset H$. In particular $b_H^P$ acts by 0 on $\mathcal{R}_L(G)(P)$, for any finite $p$-group $P$; indeed $b_H^P$ is a linear combination of terms of the form $\text{Ind}_{X/M}^{P/M} \Phi_{X/M}$, where $(X, M)$ is a section of $P$ such that $S \leq M \leq \Phi(T) \leq X \leq T$, for some section $(T, S)$ of $P$ with $(T/S)^\alpha \cong H$. For such a section $(X, M)$ of $P$, we have $(X/M)^\alpha \subset (T/S)^\alpha \subset H$, thus $G$ vanishes on any subquotient of $X/M$, so $\mathcal{R}_L(G)(X/M) = \{0\}$, hence $b_H^P = 0$ on $\mathcal{R}_L(G)(P)$, as claimed. It follows
that \( \hat{b}_L R \gamma_L(G) = 0 \), hence \( \hat{b}_L^* R \gamma_L(G) = R \gamma_L(G) = \hat{b}_L R \gamma_L(G) \).

(3) This is a straightforward consequence of (1) and (2), by Theorem 7.9 □

The following proposition gives some detail on the structure of the category \( RC_L^p \):

7.16. Proposition: Let \( p \) be a prime, and \( L \) be an atoric \( p \)-group.

1. Let \( P \) be a finite \( p \)-group. Then \( P^\alpha \cong L \) if and only if there exists an elementary abelian \( p \)-group \( E \) such that \( P \cong E \times L \).

2. Let \( P = E \times L \) and \( Q = F \times L \), where \( E \) and \( F \) are elementary abelian \( p \)-groups. If \( M \leq (Q \times P) \), then \( q(M)^\alpha \cong L \) if and only if
   \[
   p_{1,2}(M) = p_{2,2}(M) = L \quad \text{and} \quad k_{1,2}(M) = k_{2,2}(M) = 1 ,
   \]
   where \( p_{1,2} \) and \( p_{2,2} \) are the morphisms from \( ((H \times L) \times (G \times L)) \) to \( L \)
   defined by \( p_{1,2}((h, x), (g, y)) = x \) and \( p_{2,2}((h, x), (g, y)) = y \), and
   \[
   k_{1,2}(M) = \{ x \in L \mid ((1, x), (1, 1)) \in M \} ,
   \]
   \[
   k_{2,2}(M) = \{ x \in L \mid ((1, 1), (1, x)) \in M \} .
   \]

Proof: (1) This follows from Proposition 6.8.

(2) By Lemma 7.13 the \( R \)-module \( RB(Q, P) \) has a basis consisting of the isomorphism classes of \((Q, P)\)-biset of the form \((Q \times P)/M\), where \( M \) is a subgroup of \((Q \times P)\), up to conjugation, and \( q(M)^\alpha \cong L \). If \( M \) is such a subgroup, then \( L \cong (p_1(M)/k_1(M))^\alpha \subset (p_1(M))^\alpha \subset Q^\alpha \cong L \), so \( p_1(M)^\alpha \cong L \), and similarly \( p_2(M)^\alpha \cong L \). By Proposition 6.8 \( p_1(M)^\alpha \cong L \) if and only if \( E p_1(M) = P \), which in turn is equivalent to \( p_{1,2}(M) = L \). Similarly \( p_2(M)^\alpha \cong L \) if and only if \( p_{2,2}(M) = L \).

Then \( (p_1(M)/k_1(M))^\alpha \cong L \) if and only if \( k_1(M) \cap \Phi(p_1(M)) = 1 \), by Proposition 6.6. Moreover \( \Phi(p_1(M)) = \Phi(P) \), as there exists an elementary abelian subgroup \( E' \) of \( P \) such that \( P = E' \times p_1(M) \), by Proposition 6.8 again. Since \( \Phi(P) = 1 \times \Phi(L) \), it follows that \( k_1(M) \cap (1 \times \Phi(L)) = 1 \). Now \( N = k_1(L) \cap (1 \times L) \) is a normal subgroup of \((1 \times L)\) (since \( p_{1,2}(M) = L \)), which intersect trivially \((1 \times \Phi(L))\). Since \( L \) is atoric, by Lemma 6.3 any central element of order \( p \) of \((1 \times L)\) is contained in \((1 \times \Phi(L))\), so \( N \) contains no non trivial central element of \((1 \times L)\), hence \( N = 1 \). Thus \( k_1(L) \cap (1 \times L) = 1 \), or equivalently \( k_{1,2}(M) = 1 \). Similarly \( k_{2,2}(M) = 1 \). Hence \( q(M)^\alpha \cong L \) if and only if \( p_{1,2}(M) = p_{2,2}(M) = L \) and \( k_{1,2}(M) = k_{2,2}(M) = 1 \). □
8. L-enriched bisets

8.1. Notation: Let $G$ and $H$ be finite groups. If $U$ is an $(H,G)$-biset, and $u \in U$, let $(H,G)_u$ denote the stabilizer of $u$ in $(H \times G)$, i.e.

$$(H,G)_u = \{(h,g) \in (H \times G) \mid hu = ug\}.$$ 

Let $H_u = k_1((H,G)_u)$ denote the stabilizer of $u$ in $H$, and $uG = k_2((H,G)_u)$ denote the stabilizer of $u$ in $G$. Set moreover

$q(u) = q((H,G)_u) = (H,G)_u/(H_u \times uG).$ 

8.2. Definition: Let $L$ be a finite group. For two finite groups $G$ and $H$, an $L$-enriched $(H,G)$-biset is a $(H \times L, G \times L)$-biset $U$ such that $L \subseteq q(u)$, for any $u \in U$. A morphism of $L$-enriched $(H,G)$-bisets is a morphism of $(H \times L, G \times L)$-bisets.

The disjoint union of two $L$-enriched $(H,G)$-bisets is again an $L$-enriched $(H,G)$-biset. Let $B[L](H,G)$ denote the Grothendieck group of finite $L$-enriched $(H,G)$-bisets for relations given by disjoint union decompositions. The group $B[L](H,G)$ is called the Burnside group of $L$-enriched $(H,G)$-bisets.

8.3. Lemma: Let $G, H, L$ be finite groups, and $U$ be an $(H \times L, G \times L)$-biset. Let $U^{\#L}$ denote the set of elements $u \in U$ such that $L \subseteq q(u)$. Then $U^{\#L}$ is the largest sub-$L$-enriched $(H,G)$-biset of $U$.

Proof: It suffices to show that $U^{\#L}$ is a sub-$(H \times L, G \times L)$-biset of $U$, for then it is clearly the largest sub-$L$-enriched $(H,G)$-biset of $U$. And this is straightforward, since for any $(u,g,h,x,y) \in (U \times G \times H \times L \times L)$, if $v = (h,y)u(g,x)^{-1}$, then

$$(H \times L, G \times L)_v = ((h,y),(g,x))(H \times L, G \times L)_u,$$

and this conjugation induces a group isomorphism $q(v) \cong q(u)$. ☐

8.4. Lemma: Let $G, H, L$ be finite groups.

1. Let $U$ be an $L$-enriched $(H,G)$-biset. If $V$ is a sub-$(H \times L, G \times L)$-biset of $U$, then $V$ is an $L$-enriched $(H,G)$-biset.
2. The group $B[L](H, G)$ has a $\mathbb{Z}$-basis consisting of the transitive bisets $\left((H \times L) \times (G \times L)\right)/M$, where $M$ is a subgroup of $\left((H \times L) \times (G \times L)\right)$ (up to conjugation) such that $L \subseteq q(M)$.

**Proof:** (1) This is straightforward.

(2) It follows from (1) that $B[L](H, G)$ has a basis consisting of the isomorphism classes of $L$-enriched $(H, G)$-bisets which are transitive $(H \times L, G \times L)$-bisets. These are of the form $U = \left((H \times L) \times (G \times L)\right)/M$, for some subgroup $M$ of $\left((H \times L) \times (G \times L)\right)$. Now if $u$ is the element $\left((1, 1), (1, 1)\right)\:M$, the group $(H \times L, G \times L)_u$ is equal to $M$, hence $q(u) \cong q(M)$. □

### 8.5. Lemma:
Let $G, H, K, L$ be finite groups.

1. For an $(H, G)$-biset $U$, endow $U \times L$ with the $(H \times L, G \times L)$-biset structure defined by

   $$\forall h \in H, \forall g \in G, \forall x, y, z \in L, \forall u \in U, \quad (h, x)(u, y)(g, z) = (hug, xyz).$$

   Then $U \times L$ is an $L$-enriched $(H, G)$-biset.

2. In particular, for any finite group $G$, the identity biset of $G \times L$ is an $L$-enriched $(G, G)$-biset.

3. If $U$ is an $(H, G)$-biset and $V$ is a $(K, H)$-biset, then there is an isomorphism

   $$(V \times L) \times_{(H \times L)} (U \times L) \cong (V \times_{H} U) \times L$$

   of $L$-enriched $(H, G)$-bisets.

**Proof:** (1) For $u \in U$ and $l \in L$,

$$\left((H \times L, G \times L)\right),(u, l) = \{(h^t, x), (g, x) \mid hug = u, l \in L\} \cong (H, G)u \times L.$$

In particular $(H \times L)\,(u, l) = H_u \times 1$ and $(u, l)(G \times L) = G \times 1$, and $q((u, l)) \cong q(u) \times L$ has a (sub)quotient isomorphic to $L$.

(2) In particular, if $H = G$ and $U$ is the identity $(G, G)$-biset, then $U \times L$ is the identity biset of $(G \times L)$.

(3) It is straightforward to check that the maps

$$[(v, x), (u, y)] \in (V \times L) \times_{(H \times L)} (U \times L) \longrightarrow ([v, u], xy) \in (V \times_{H} U) \times L$$

$$[(v, 1), (u, l)] \in (V \times L) \times_{(H \times L)} (U \times L) \longrightarrow ([v, u], l) \in (V \times_{H} U) \times L$$

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are well defined isomorphisms of \((K \times L, G \times L)\)-bisets, inverse to one another.

8.6. Notation: Let \(G, H, K, L\) be finite groups. If \(U\) is an \(L\)-enriched \((H, G)\)-biset and \(V\) is an \(L\)-enriched \((K, H)\)-biset, let \(V \times^L_H U\) denote the \(L\)-enriched \((K, G)\)-biset defined by

\[
V \times^L_H U = (V \times_{(H \times L)} U)^L.
\]

8.7. Lemma: Let \(G, H, J, K, L\) be finite groups.

1. If \(V\) is a \((K \times L, H \times L)\)-biset and \(U\) is an \((H \times L, G \times L)\)-biset, then

\[
(V \times_{(H \times L)} U)^L = V^L \times^L_H U^L.
\]

In particular, if \(V\) and \(U\) are \(L\)-enriched bisets, so is \(V \times^L_H U\).

2. If \(U\) and \(U'\) are \(L\)-enriched \((H, G)\)-bisets, if \(V, V'\) are \(L\)-enriched \((K, H)\)-bisets, then there are isomorphisms

\[
V \times^L_H (U \sqcup U') \cong (V \times^L_H U) \sqcup (V \times^L_H U')
\]

\[
(V \sqcup V') \times^L_H U \cong (V \times^L_H U) \sqcup (V' \times^L_H U)
\]

of \(L\)-enriched \((K, G)\)-bisets.

3. If moreover \(W\) is an \(L\)-enriched \((J, K)\)-biset, then there is a canonical isomorphism

\[
(W \times^L_K V) \times^L_H U \cong W \times^L_K (V \times^L_H U)
\]

of \(L\)-enriched \((J, G)\)-bisets.

Proof: (1) Denote by \([v, u]\) the image in \(V \times_{(H \times L)} U\) of a pair \((v, u)\) \(\in (V \times U)\). By Lemma 2.3.20 of [7],

\[
(K \times L, G \times L)_{[v, u]} = (K \times L, H \times L)_v \ast (H \times L, G \times L)_u,
\]

so by Lemma 2.3.22 of [7], the group \(q([v, u])\) is a subquotient of \(q(v)\) and \(q(u)\). So if \([v, u] \in (V \times_{(H \times L)} U)^L\), then \(L\) is a subquotient of \(q([v, u])\), hence it is a subquotient of \(q(v)\) and \(q(u)\), that is \(v \in V^L\) and \(u \in U^L\). Hence

\[
(V \times_{(H \times L)} U)^L \subseteq (V^L \times_{(H \times L)} U^L)^L = V^L \times^L_H U^L,
\]

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and the reverse inclusion \((V^\mathbb{L} \times (H \times L)) U^\mathbb{L} \subseteq (V \times (H \times L)) U^\mathbb{L}\) is obvious. Hence \((V \times (H \times L)) U^\mathbb{L} = V^\mathbb{L} \times_H U^\mathbb{L}\). If \(V\) and \(U\) are \(L\)-enriched bisets, i.e. if \(V = V^\mathbb{L}\) and \(U = U^\mathbb{L}\), this gives \((V \times (H \times L)) U^\mathbb{L} = V^\mathbb{L} \times_H U\), so \(V^\mathbb{L} \times_H U\) is an \(L\)-enriched biset.

(2) This is straightforward.

(3) With the above notation, there is a canonical isomorphism
\[
\alpha : (W \times (K \times L)) V \times (H \times L) U \rightarrow W \times (K \times L) (V \times (H \times L) U)
\]
sending \([w, v], u\) to \([w, [v, u]]\). Hence
\[
(W^\mathbb{L} \times_{K} V)^\mathbb{L} \times_H U = ((W^\mathbb{L} \times_{K} V) \times (H \times L) U)^\mathbb{L}
\]
\[
= ((W \times (K \times L) V)^\mathbb{L} \times (H \times L) U)^\mathbb{L}
\]
\[
= ((W \times (K \times L) V) \times (H \times L) U)^\mathbb{L} \quad \text{[by (1)]}
\]
Similarly
\[
W^\mathbb{L} \times_{K} (V^\mathbb{L} \times_H U) = (W \times (K \times L) (V^\mathbb{L} \times_H U))^\mathbb{L}
\]
\[
= (W \times (K \times L) (V \times (H \times L) U)^\mathbb{L})^\mathbb{L}
\]
\[
= (W \times (K \times L) (V \times (H \times L) U))^\mathbb{L} \quad \text{[by (1)]}.
\]

Hence \(\alpha\) induces an isomorphism \((W^\mathbb{L} \times_{K} V)^\mathbb{L} \times_H U \cong W^\mathbb{L} \times_{K} (V^\mathbb{L} \times_H U)\).

8.8. Definition: Let \(L\) be a finite group, and \(R\) be a commutative ring. The \(L\)-enriched biset category \(RC[L]\) of finite groups over \(R\) is defined as follows:

- The objects of \(RC[L]\) are the finite groups.
- For finite groups \(G\) and \(H\),
  \[
  \text{Hom}_{RC[L]}(G, H) = R \otimes_{\mathbb{Z}} B[L](H, G) = RB[L](H, G)
  \]
  is the \(R\)-linear extension of the Burnside group of \(L\)-enriched \((H, G)\)-bisets.
- The composition in \(RC[L]\) is the \(R\)-linear extension of the product \((V, U) \mapsto V^\mathbb{L} \times_H U\) defined in 8.6.
The identity morphism of the group $G$ is (image in $RB[L](G, G)$ of) the identity $\text{bi}-\text{set}$ of $G \times L$, viewed as an $L$-enriched $(G, G)$-$\text{bi}-\text{set}$.

The category $RC[L]$ is $R$-linear. An $L$-enriched $\text{bi}-\text{set}$ functor over $R$ is an $R$-linear functor from $RC[L]$ to $R\text{-}\text{Mod}$. The category of $L$-enriched $\text{bi}-\text{set}$ functors over $R$ is denoted by $\mathcal{F}_R[L]$. It is an abelian $R$-linear category.

**8.9. Theorem:** Let $p$ be a prime number, and $R$ be a commutative ring.

1. If $L$ is an atoric $p$-group, the category $RC_p^{NL}$ of Definition $\mathcal{D}$ is equivalent to the full subcategory $REl_p[L]$ of $RC[L]$ consisting of elementary abelian $p$-groups.

2. If $p \in \mathcal{F}^\times$, the category $\mathcal{F}_{p, R}$ of $p$-$\text{bi}-\text{set}$ functors over $R$ is equivalent to the direct product of the categories $\text{Fun}_R(REl_p[L], R\text{-}\text{Mod})$ of $R$-linear functors from $REl_p[L]$ to $R\text{-}\text{Mod}$, for $L \in [\mathcal{A}t_p]$.

**Proof:** (1) Let $E$ be an elementary abelian $p$-group. Then $(E \times L)^{\mathcal{A}} \cong L$, so $E \times L$ is an object of $RC_p^{NL}$. Set $\mathcal{I}(E) = E \times L$. If $E$ and $F$ are elementary abelian $p$-groups, and if $U$ is a finite $L$-enriched $(F, E)$-$\text{bi}-\text{set}$, then $U$ is in particular an $(F \times L, E \times L)$-$\text{bi}-\text{set}$, an we can consider its image $\mathcal{I}(U)$ in the quotient $\text{Hom}_{RC_p^{NL}}(E \times L, F \times L)$ of $RB(F \times L, E \times L)$. This yields a unique $R$-linear map $RB[L](F, E) \rightarrow \text{Hom}_{RC_p^{NL}}(E \times L, F \times L)$, still denoted by $\mathcal{I}$.

We claim that these assignments define a functor $\mathcal{I}$ from $REl_p[L]$ to $RC_p^{NL}$: indeed, the identity $(E \times L, E \times L)$-$\text{bi}-\text{set}$ is clearly mapped to the identity morphism of $\mathcal{I}(E)$. Moreover, if $G$ is an elementary abelian $p$-group, if $V$ is an $L$-enriched $(G, F)$-$\text{bi}-\text{set}$ and $U$ is an $L$-enriched $(F, E)$-$\text{bi}-\text{set}$, it is clear that

$$\mathcal{I}(V \times_{p} U) = \mathcal{I}(V) \circ \mathcal{I}(U),$$

where the right hand side composition is in the category $RC_p^{NL}$: indeed, the transitive $\text{bi}-\text{sets}$ $(Q \times P)/M$ with $q(M)^{\mathcal{A}} \subseteq L$ appearing in the product $V \times_{(F \times L)} U$ are exactly those vanishing in $\text{Hom}_{RC_p^{NL}}(\mathcal{I}(E), \mathcal{I}(F))$, by Lemma $\mathcal{L}$. Hence $\mathcal{I}$ is an isomorphism

$$\mathcal{I} : RB[L](F, E) \rightarrow \text{Hom}_{RC_p^{NL}}(\mathcal{I}(E), \mathcal{I}(F)) .$$

In other words $\mathcal{I}$ is a fully faithful functor from $REl_p[L]$ to $RC_p^{NL}$. Moreover, by Proposition $\mathcal{G}$ if $P$ is a finite $p$-group with $P^{\mathcal{A}} \cong L$, there exists an elementary abelian $p$-group $E$ such that $P$ is isomorphic to $E \times L$, hence $P$ is isomorphic to $E \times L$ in the category $RC_p^{NL}$.

It follows that the functor $\mathcal{I}$ is fully faithful and essentially surjective, so it is an equivalence of categories.

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(2) This is a straightforward consequence of (1), Assertion 5 of Corollary 7.5, and Assertion 3 of Theorem 7.15.

9. The category $\widehat{b}_L F_{p,R}$, for an atoric $p$-group $L$ ($p \in R^\times$)

Let $L$ be a fixed atoric $p$-group. In this section, we give some detail on the structure of the category $\widehat{b}_L F_{p,R}$ of $p$-biset functors invariant by the idempotent $\widehat{b}_L$.

We start by straightforward consequences of Theorem 7.15. For a finite $p$-group $P$, we denote by $\Sigma_{\#L}(P)$ the subset of $\Sigma_{L}(P)$ consisting of sections $(X, M)$ of $P$ such that $(X/M)^\# \cong L$. When $G$ is an $R$-linear functor from $R_{\#L}^p$ to $R$-Mod, we can compute $R_Y L(F)$ at $P$ by restricting the inverse limit of 7.7 to the subset $\Sigma_{\#L}(P)$, i.e. by

$$R_Y L(F)(P) = \lim_{\leftarrow (X, M) \in \Sigma_{\#L}(P)} G(X/M).$$

9.1. Proposition: \([ p \in R^\times \] Let $L$ be an atoric $p$-group. If $F$ is a $p$-biset functor in $\widehat{b}_L F_{p,R}$, and $P$ is a finite $p$-group, then

$$F(P) \cong \lim_{(X, M) \in \Sigma_{\#L}(P)} F(X/M),$$

$$\cong \bigoplus_{(T, S) \in [M(P)]} \delta_{\Phi(F)} F(T/S)^{N_P(T,S)/T}.$$  

Proof: The isomorphism $F(P) \cong \lim_{(X, M) \in \Sigma_{\#L}(P)} F(X/M)$ is Assertion 3 of Theorem 7.15. The second isomorphism follows from Theorem 5.4 which implies that for $(T, S) \in \mathcal{M}(P)$

$$\delta_{\Phi(F)} F(T/S)^{N_P(T,S)/T} \cong F(\epsilon_{T,S}^P)(F(P)).$$

Moreover $F(b_L^p) F(P) = F(P)$ since $F \in \widehat{b}_L F_{p,R}$, and

$$F(\epsilon_{T,S}^P F(b_L^p)) = F(\epsilon_{T,S}^P b_L^p) = 0$$

unless $(T/S)^\# \cong L$. Thus $\delta_{\Phi(F)} F(T/S)^{N_P(T,S)/T} = \{0\}$ unless $(T/S)^\# \cong L$, which completes the proof.
The decomposition of the category $\mathcal{F}_{p,R}$ of $p$-biset functors stated in Corollary 7.5 leads to the following natural definition:

**9.2. Definition:** [ $p \in R^\times$ ] Let $F$ be an indecomposable $p$-biset functor over $R$. There exists a unique atoric $p$-group $L$ (up to isomorphism) such that $F = \hat{b}_L F$. The group $L$ is called the vertex of $F$.

**9.3. Remark:** It follows in particular from this definition that if $F$ and $F'$ are indecomposable $p$-biset functors over $R$ with non-isomorphic vertices, then $\text{Ext}_{\mathcal{F}_{p,R}}^*(F, F') = \{0\}$.

**9.4. Theorem:** [ $p \in R^\times$ ] Let $F$ be an indecomposable $p$-biset functor over $R$ and let $L$ be a vertex of $F$. If $Q$ is a finite $p$-group such that $F(Q) \neq \{0\}$, but $F$ vanishes on any proper subquotient of $Q$, then $L \sim \hat{b}_Q L$.

**Proof:** Let $Q$ be a finite $p$-group such that $F(Q) \neq \{0\}$ and $F(Q') = \{0\}$ for any proper subquotient $Q'$ of $Q$. By Proposition 4.6 if $(T, S)$ is a minimal section of $Q$, then

$$e_{T,S}^Q = \frac{1}{|N_Q(T, S)|} \sum_{X \leq T, M \leq T, S \leq M \leq \Phi(T) \leq X \leq T} |X| \mu(X, T) \mu \leq T(S, M) \text{Ind}_{X/M}^Q \circ \text{Defres}_{X/M}^Q .$$

Now if $X/M$ is a proper subquotient of $Q$, i.e. if $X \neq Q$ or $M \neq 1$, then $F(X/M) = \{0\}$, and $F(\text{Ind}_{X/M}^Q \circ \text{Defres}_{X/M}^Q) = 0$. Hence $F(e_{T,S}^Q) = 0$ unless $T = Q$ and $S = 1$, and moreover

$$F(e_{Q,1}^Q) = \frac{1}{|Q|} |Q| \mu(Q, Q) \mu \leq Q(1, Q) F(\text{Ind}_{Q/1}^Q \circ \text{Defres}_{Q/1}^Q) = \text{Id}_{F(Q)} .$$

If $\hat{b}_L F = F$, then in particular $F(b_Q^L)$ is equal to the identity map of $F(Q)$. This can only occur if the idempotent $e_{Q,1}^Q$ appears in the sum defining $b_Q^L$, in other words if $(Q/1)^{\alpha} \cong L$, i.e. $Q^\alpha \cong L$. Conversely, if $Q^\alpha \cong L$, then $F(b_Q^L) = F(e_{Q,1}^Q) = \text{Id}_{F(Q)} \neq 0$. It follows that $\hat{b}_L F \neq 0$, hence $\hat{b}_L F = F$, since $F$ is indecomposable. Hence $Q^\alpha$ is (isomorphic to) the vertex of $F$, as was to be shown.

We assume from now on that $R = k$ is a field. Recall (7] Chapter 4) that the simple $p$-biset functors over $k$ are indexed by pairs $(Q, V)$ consisting of a $p$-group $Q$ and a simple $k\text{Out}(Q)$-module $V$. 

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9.5. Corollary: Let $k$ be a field of characteristic different from $p$.

1. If $Q$ is a finite $p$-group, and $V$ is a simple $k\text{Out}(Q)$-module, then the vertex of the simple $p$-biset functor $S_{Q,V}$ is isomorphic to $Q^0$.

2. Let $Q$ (resp. $Q'$) be a finite $p$-group, and $V$ (resp. $V'$) be a simple $k\text{Out}(Q)$-module (resp. a simple $k\text{Out}(Q')$-module). If $Q^0 \not\cong Q'^0$, then $\text{Ext}_F^1(S_{Q,V}, S_{Q',V'}) = \{0\}$.

Proof: (1) Indeed $Q$ is a minimal group for $S_{Q,V}$, so $S_{Q,V}(Q) \neq \{0\}$, but $S_{Q,V}$ vanishes on any proper subquotient of $Q$.

(2) Follows from (1) and Remark 9.3.

9.6. Definition: Let $F$ be a $p$-biset functor. A functor $S$ is a subquotient of $F$ (notation $S \subset F$) if there exist subfunctors $F_2 < F_1 \leq F$ such that $F_1/F_2 \cong S$. A composition factor of $F$ is a simple subquotient of $F$.

9.7. Lemma: Let $k$ be a field, and $F$ be a $p$-biset functor over $k$.

1. If $F$ is a non zero, then $F$ admits a composition factor.

2. If $S$ is a family of simple $p$-biset functors over $k$, there exists a greatest subfunctor of $F$ all composition factors of which belong to $S$.

Proof: (1) Let $P$ be a finite $p$-group such that $F(P) \neq \{0\}$. Then $F(P)$ is a $kB(P,P)$-module. Choose $m \in F(P) - \{0\}$, and consider the $kB(P,P)$-submodule $M$ of $F(P)$ generated by $m$. Since $kB(P,P)$ is finite dimensional over $k$, the module $M$ is also finite dimensional over $k$, hence it contains a simple submodule $V$. By Proposition 3.1 of [8], there exists a simple $p$-biset functor $S$ such that $S(P) \cong V$ as $kB(P,P)$-module. Then $S(P)$ is a subquotient of $F(P)$, so by Proposition 3.5 of [8], there exists a subquotient of $F$ isomorphic to $S$.

(2) Observe first that if $M, N$ are subfunctors of $F$, then any composition factor of $M + N$ is a composition factor of $M$ or a composition factor of $N$: indeed, if $S$ is a composition factor of $M + N$, let $F_2 < F_1 \leq M + N$ with $S \cong F_2/F_1$, and consider the images $F'_1$ and $F'_2$ of $F_1$ and $F_2$, respectively, in the quotient $(M + N)/N \cong M/(M \cap N)$. If $F'_1 \neq F'_2$, that is if $F_1 + N \neq F_2 + N$, then $F'_1/F'_2 \cong (F_1 + N)/(F_2 + N) \cong F_1/F_2 \cong S$ is a subquotient of $(M + N)/N \cong M/(M \cap N)$, hence $S$ is a subquotient of $M$. Otherwise $F_1 + N = F_2 + N$, so $F_1 = F_2 + (F_1 \cap N)$, hence $S \cong F_1/F_2 \cong (F_1 \cap N)/(F_2 \cap N)$.
is a subquotient of $N$. It follows by induction that any subquotient $S$ of a finite sum $\sum_{M \in \mathcal{I}} M$ of subfunctors of $F$ is a subquotient of some $M \in \mathcal{I}$.

The latter also holds when $\mathcal{I}$ is infinite: let $\Sigma = \sum_{M \in \mathcal{I}} M$ be an arbitrary sum of subfunctors of $F$, and $S$ be a composition factor of $\Sigma$. Let $F_2 < F_1$ be subfunctors of $\Sigma$ such that $S \cong F_1/F_2$. If $P$ is a $p$-group such that $S(P) \cong F_1(P)/F_2(P) \neq 0$, let $U$ be a finite subset of $F_1(P)$ such that $F_1(P)/F_2(P)$ is generated as a $kB(P, P)$-module by the images of the elements of $U$ (such a set exists because $S(P)$ is finite dimensional over $k$, for any $P$). If $V$ is the $kB(P, P)$-submodule of $F_1(P)$ generated by $U$, then $V$ maps surjectively on the module $F_1(P)/F_2(P)$, so there is a $kB(P, P)$-submodule $W$ of $V$ such that $V/W \cong S(P)$. Now since $U$ is finite, there exists a finite subset $\mathcal{J}$ of $\mathcal{I}$ such that $U \subseteq \sum_{M \in \mathcal{J}} M(P)$. Setting $\Sigma_1 = \sum_{M \in \mathcal{J}} M$, it follows that $V/W \cong S(P)$ is a subquotient of $\Sigma_1(P)$, so by Proposition 3.5 of [8], there exists a subquotient of $\Sigma_1$ isomorphic to $S$. By the observation above $S$ is a subquotient of some $M \in \mathcal{J} \subseteq \mathcal{I}$.

Now let $\mathcal{I}$ the set of subfunctors $M$ of $F$ such that all the composition factors of $M$ belong to $\mathcal{S}$, and $N = \sum_{M \in \mathcal{I}} M$. The above discussion shows that $N \in \mathcal{I}$, so $N$ is the greatest element of $\mathcal{I}$.

\[ 9.8. \text{Theorem:} \quad \text{Let } k \text{ be a field of characteristic different from } p, \text{ and } L \text{ be an atoric } p\text{-group. Let } \mathcal{F}_{p,k}[L] \text{ the full subcategory of } \mathcal{F}_{p,k} \text{ consisting of functors whose composition factors all have vertex } L, \text{ i.e. are all isomorphic to } S_{P,V}, \text{ for some } p\text{-group } P \text{ such that } P^{\alpha} \cong L, \text{ and some simple } k\text{Out}(P)\text{-module } V. \]

1. If $F$ is a $p$-biset functor, then $\hat{b}_L \mathcal{F}_{p,k}$ is the greatest subfunctor of $F$ which belongs to $\mathcal{F}_{p,k}[L]$.

2. In particular $\hat{b}_L \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[L]$.

\[ \text{Proof:} \] (1) Let $F$ be a $p$-biset functor over $k$, and let $F_1 = \hat{b}_L F$. If $S$ is a composition factor of $F_1$, then $S = \hat{b}_L S$, as $S$ is a subquotient of $F_1$. Hence $S$ has vertex $L$, by Definition 9.2. It follows that $F_1$ is contained in the greatest subfunctor $F_2$ of $F$ which belongs to $\mathcal{F}_{p,k}[L]$ (such a subfunctor exists by Lemma 9.7).

Conversely, we know that $F_2 = \bigoplus_{Q \in [At_p]} \hat{b}_Q F_2$. For $Q \in [At_p]$, any composition factor $S$ of $\hat{b}_Q F_2$ has vertex $Q$, by Definition 9.2. But $S$ is also a direct summand of $F_2$, so $Q \cong L$. It follows that if $Q \not\cong L$, then $\hat{b}_Q F_2$ has no com-
position factor, so \( \hat{b}_Q F_2 = \{0\} \), by Lemma 9.7. In other words \( F_2 = \hat{b}_L F_2 \), hence \( F_2 \leq F_1 \), and \( F_2 = F_1 \), as was to be shown.

(2) Let \( F \) be a \( p \)-biset functor. Then \( F \in \hat{b}_L \mathcal{F}_{p,k} \) if and only if \( F = \hat{b}_L F \), i.e. by (1) if and only if all the composition factors of \( F \) have vertex \( L \).

9.9. Example: the Burnside functor. Let \( k \) be a field of characteristic \( q \neq p \) (\( q \geq 0 \)). It was shown in [10] Theorem 8.2 (see also [7] 5.6.9) that the Burnside functor \( kB \) is uniserial, hence indecomposable. As \( kB(1) \neq 0 \), the vertex of \( kB \) is the trivial group, by Theorem 9.4, thus \( kB \) is an object of \( \hat{b}_1 \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[1] \). It means that all the composition factors of \( kB \) have to be of form \( S_{Q,V} \), where \( Q^a = 1 \), i.e. \( Q \) is elementary abelian. And indeed by [10] Theorem 8.2, the composition factors of \( kB \) are all of the form \( S_{Q,K} \), where \( Q \) runs through a specific set of elementary abelian \( p \)-groups which depends on the order of \( p \) modulo \( q \) (suitably interpreted when \( q = 0 \)).

References

[1] L. Barker. Blocks of Mackey categories. *J. Algebra*, 446:34–57, 2016.

[2] R. Boltje and S. Danz. A ghost ring for the left-free double Burnside ring and an application to fusion systems. *Adv. Math.*, 229(3):1688–1733, 2012.

[3] R. Boltje and S. Danz. A ghost algebra of the double Burnside algebra in characteristic zero. *J. Pure Appl. Algebra*, 217(4):608–635, 2013.

[4] R. Boltje and B. Külshammer. Central idempotents of the bifree and left-free double Burnside ring. *Israel J. Math.*, 202(1):161–193, 2014.

[5] S. Bouc. Foncteurs d’ensembles munis d’une double action. *J. of Algebra*, 183(0238):664–736, 1996.

[6] S. Bouc. Polynomial ideals and classes of finite groups. *J. of Algebra*, 229:153–174, 2000.

[7] S. Bouc. *Biset functors for finite groups*, volume 1990 of *Lecture Notes in Mathematics*. Springer, 2010.

[8] S. Bouc, R. Stancu, and J. Thévenaz. Simple biset functors and double Burnside ring. *Journal of Pure and Applied Algebra*, 217:546–566, 2013.

[9] S. Bouc, R. Stancu, and J. Thévenaz. Vanishing evaluations of simple functors. *J.P.A.A.*, 218:218–227, 2014.
[10] S. Bouc and J. Thévenaz. The group of endo-permutation modules. \textit{Invent. Math.}, 139:275–349, 2000.

[11] S. Bouc and J. Thévenaz. Gluing torsion endo-permutation modules. \textit{J. London Math. Soc.}, 78(2):477–501, 2008.

[12] S. Bouc and J. Thévenaz. A sectional characterization of the Dade group. \textit{Journal of Group Theory}, 11(2):155–298, 2008.

[13] S. Bouc and J. Thévenaz. The algebra of essential relations on a finite set. \textit{J. reine angew. Math.}, 712:225–250, 2016.

[14] D. Quillen. Homotopy properties of the poset of non-trivial $p$-subgroups. \textit{Adv. in Maths}, 28(2):101–128, 1978.

[15] R. P. Stanley. \textit{Enumerative combinatorics}, volume 1. Wadsworth & Brooks/Cole, Monterey, 1986.

[16] P. Webb. Stratifications and Mackey functors II: globally defined Mackey functors. \textit{J. K-Theory}, 6(1):99–170, 2010.

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