Quantum State Tomography and Quantum Games

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A technique is developed for single qubit quantum state tomography using the mathematical setup of generalized quantization scheme for games. In this technique, Alice sends an unknown pure quantum state to Bob who appends it with $|0\rangle\langle0|$ and then applies the unitary operators on the appended quantum state and finds the payoffs for Alice and himself. It is shown that for a particular set of unitary operators, these payoffs are equal to Stokes parameters for an unknown quantum state. In this way an unknown quantum state can be measured and reconstructed. Strictly speaking, this technique is not a game as no strategic competitions are involved.

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All information about a quantum system is encoded in the state of the system but it is one of the great challenges for experimentalists to measure the state of the quantum system perfectly.\[1\] This is due to the fact that the state of a quantum system is not an observable in quantum mechanics\[2\] that makes it impossible to perform all measurements on the single state to extract the whole information about the system. On the other hand, the no-cloning theorem does not allow creation of a perfect copy of the system without prior knowledge about its state.\[3\] Hence, there remains no way, even in principle, to infer the unknown quantum state of a single quantum system.\[4\] However it is possible to estimate the unknown quantum state of a system when many identical copies of the system are available. This procedure of reconstructing an unknown quantum state through a series of measurements on a number of identical copies of the system is called the quantum state tomography. In this process, each measurement gives a new dimension of the system and therefore, an infinite number of copies are required to reconstruct the exact state of a quantum system. The problem of quantum state tomography was first addressed by Fano\[5\] who recognized the need to measure two non commuting observables. However it remained mere speculation until an original proposal for quantum tomography and its experimental verification.\[14,6,7\] Since then it has been applied for the measurement of photon statistics of a semiconductor laser\[8\] reconstruction of the density matrix of a squeezed vacuum\[9\] and probing the entangled states of light and ions.\[10\] The other main tasks where the role of quantum state tomography is necessary for the complete characterization of the state of the system are: to study the effects of decoherence\[11\] to optimize the performance of quantum gates,\[12\] to quantify the amount of information that various parties can obtain by quantum communication protocols\[13\] and utilization of quantum error correction protocols in real world situations effectively.\[14\]

In this Letter, by making use of the mathematical setup of the generalized quantization scheme for games\[15\] a technique for quantum state tomography is developed. Strictly speaking, this arrangement is not a game but only the mathematical setup of quantum games is used as a tool. It works as follows: Alice sends an unknown pure quantum state $\rho$ to Bob who appends it with $|0\rangle\langle0|$ resulting in the initial state of the game as $\rho_{in} = |0\rangle\langle0| \otimes \rho$. On this appended quantum state, Bob applies unitary operator $U = U_A \otimes U_B$ and finds the the payoffs ($S_A, S_B$) using the predefined payoff operators $P^A$ and $P^B$. For a particular set of unitary operators (strategies) and payoff operators these payoffs become Stokes parameters of the given quantum state $\rho$. In this way an unknown quantum state can be measured and reconstructed. It is common to use the entangled state as an input for quantum games when one is interested in finding the solution of a game such as the resolution of dilemma in prisoner dilemma game.\[16\] However, the payoffs cannot be independent of the initial quantum state whether the state is product or entangled and hence the information about the input quantum state is reflected as a function of payoffs at the output. This makes it possible to estimate an unknown quantum state. Furthermore this technique does not improve the standard technique of quantum state tomography but it is a step forward for strengthening the established link between quantum games and quantum information theory.\[17\] Before presenting the main result of this study, we give a brief introduction to a single qubit tomography following Refs.\[14,19\].

Any single qubit density matrix $\rho$ can uniquely be represented with the help of three parameters $\{S_1, S_2, S_3\}$ and Pauli matrices $\sigma_i$’s by the expression

$$\rho = \frac{1}{2} \sum_{i=0}^{3} S_i \sigma_i, \quad (1)$$

where $S_0 = 1$, and the other parameters obey the re-
The parameters $S_i$’s are called the Stokes parameters and for a quantum state $\rho$ these can be calculated by
\[ S_i = \text{Tr}(\sigma_i \rho). \tag{2} \]
Physically these parameters give the outcome of projective measurements as
\[
\begin{align*}
S_0 &= P_{\langle 0 \rangle} + P_{\langle 1 \rangle}, \\
S_1 &= P_{\langle 0 \rangle(0)+\langle 1 \rangle) - P_{\langle 0 \rangle(1)-\langle 1 \rangle)}, \\
S_2 &= P_{\langle 0 \rangle(0)+\langle 1 \rangle) - P_{\langle 0 \rangle(1)-\langle 1 \rangle)}, \\
S_3 &= P_{\langle 0 \rangle} - P_{\langle 1 \rangle},
\end{align*}
\tag{3}
\]
where $P_{\langle i \rangle}$ is the probability to measure state $|i\rangle$ given by
\[
P_{\langle i \rangle} = \langle i | \rho | i \rangle = \text{Tr}(|i\rangle\langle i| \rho), \tag{4}
\]
If we are provided with many copies of a quantum state, then with the help of orthogonal set of matrices $\sigma_0/\sqrt{2}, \sigma_1/\sqrt{2}, \sigma_2/\sqrt{2}, \sigma_3/\sqrt{2}$, the density matrix (1) can be written as
\[
\rho = \frac{\text{Tr}(\rho \sigma_0 + \text{Tr}(\rho \sigma_1) \sigma_1 + \text{Tr}(\rho \sigma_2) \sigma_2 + \text{Tr}(\rho \sigma_3) \sigma_3}{2}, \tag{5}
\]

where the expression like $\text{Tr}(\rho \sigma_i)$ represents the expectation value of the observable. To estimate $\text{Tr}(\rho \sigma_3)$, for example, we measure $\sigma_3$ for $m$ numbers of time giving the values $z_1, z_2, \ldots, z_m$, all equal to +1 or -1. The average $\sum z_i/m$ is an estimate of the true value of the quantity $\text{Tr}(\rho \sigma_3)$. By central limit theorem this estimate has a standard deviation $\Delta \sigma_3/\sqrt{m}$, where $\Delta \sigma_3$ is the standard deviation for a single measurement of $\sigma_3$ that is upper bounded by 1. Therefore, the standard deviation for estimating $\sum z_i/m$ is at most $1/\sqrt{m}$. The standard deviation for each of the measurements in Eq. (5) is the same. In this way with the help of Eq. (5), tomography can be performed for an unknown single qubit state.

A single qubit state can very conveniently be represented by a vector in three-dimensional vector space spanned by Pauli matrices. This representation provides a very helpful way for geometrical visualization of a single qubit state, where all the legal states fall within a unit sphere (Bloch sphere). In this representation, all the pure states lie on the surface of the sphere and mixed states fall inside the sphere. The pure states can be written as
\[
|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i \phi} \sin \frac{\theta}{2} |1\rangle, \tag{6}
\]
where $\theta$ and $\phi$ map them on the surface of the sphere. Any state $|\psi\rangle$ and its orthogonal component $|\psi^\perp\rangle$ fall on two opposite points on the surface of the sphere such that the line connecting these points form the axis of the sphere.

![Fig. 1](image1.png)

**Fig. 1.** Measurement in the $\sigma_3$ basis that confines the unknown quantum state to a plane of $z = \cos \theta$.

![Fig. 2](image2.png)

**Fig. 2.** Measurement in the $\sigma_2$ basis that confines the state to the $y = \sin \theta \sin \phi$ plane. When this measurement is combined with the first measurement the unknown state reduces to a line parallel to the $x$-axis.

![Fig. 3](image3.png)

**Fig. 3.** The last measurement in the $\sigma_1$ basis that pinpoints the state as a point that results from the intersection of three orthogonal planes.

For the tomography of an unknown single qubit state, three consecutive measurements are required. Each measurement gives one dimension of the system until one becomes aware of all dimensions after the complete set of measurement. For example, a single qubit state $\rho = |\psi\rangle \langle \psi|$, where $|\psi\rangle$ is defined by Eq. (6), can be expressed as
\[
\rho = \frac{1}{2}(\sigma_0 + \sin \theta \cos \phi \sigma_1 + \sin \theta \sin \phi \sigma_2 + \cos \theta \sigma_3). \tag{7}
\]
Comparing Eqs. (1) and (7) the Stokes parameters for this state become
\[
S_1 = \sin \theta \cos \phi, \quad S_2 = \sin \theta \sin \phi, \quad S_3 = \cos \theta. \tag{8}
\]
For an unknown state in the form of Eq. (7), when a measurement is performed in \( \sigma_3 \) basis, it confines the state to a plane \( z = \cos \theta \); as shown in Fig. 1.

Then a measurement in the \( \sigma_2 \) basis is performed, which further confines it to the plane \( y = \sin \theta \sin \phi \). The combined effect of both these measurements restricts the unknown quantum state to a line parallel to the \( x \)-axis as shown in Fig. 2.

Lastly the measurement in the \( \sigma_1 \) basis pinspoints the state as a point lying on this line (resulting from the intersection of \( y \) and \( z \) planes) at distance \( z = \sin \theta \cos \phi \); as illustrated in Fig. 3. Since the resultant state is due to the intersection of three orthogonal planes, the order of these measurements is immaterial in the whole process.

Next we show how a single qubit quantum state tomography can be performed using a mathematical setup of the generalized quantization scheme for games. \(^{[16]} \) Let Alice forward an unknown quantum state of the form of Eq. (6) to Bob who appends it with \(|0 \rangle |0 \rangle \) resulting from the initial state of the game as

\[
\rho_{\alpha} = \cos^2 \frac{\theta}{2} |00 \rangle \langle 00 | + \sin^2 \frac{\theta}{2} |01 \rangle \langle 01 | + \frac{\sin \theta e^{i \phi}}{2} (|01 \rangle \langle 00 | + e^{-i 2 \phi} |00 \rangle \langle 01 |),
\]

and then applies the unitary operator

\[
U = U_A \otimes U_B
\]

on the appended state (9). Here for \( k = A, B \), we have

\[
U_k = \cos \frac{\beta_k}{2} R_k + \sin \frac{\beta_k}{2} P_k
\]

with \( 0 \leq \beta_k \leq \pi \). The operations of \( R_k \) and \( P_k \) on \(|0 \rangle \) and \(|1 \rangle \) are defined as

\[
R_k |0 \rangle = e^{i \alpha_k} |0 \rangle, \quad R_k |1 \rangle = e^{-i \alpha_k} |1 \rangle,
\]

\[
P_k |0 \rangle = -|1 \rangle, \quad P_k |1 \rangle = |0 \rangle.
\]

The unitary operators (10) transform the initial state (9) to

\[
\rho_f = (U_A \otimes U_B) \rho_{\alpha} (U_A \otimes U_B)^{\dagger}.
\]

and then Bob finds the payoffs by using the formula

\[
\hat{S}^k(U_A, U_B, \theta, \phi) = \text{Tr}(P^k \rho_f),
\]

where \( P^k \) the payoff operators given as

\[
P^k = \hat{S}^k_{00} |00 \rangle \langle 00 | + \hat{S}^k_{01} |01 \rangle \langle 01 | + \hat{S}^k_{10} |10 \rangle \langle 10 | + \hat{S}^k_{11} |11 \rangle \langle 11 |,
\]

with \( \hat{S}^k_{ij} \) being the entries of payoff matrix in the \( i \)-th row and \( j \)-th column for player \( k \). With the help of Eqs. (6), (14) and (15), the payoffs come out to be

\[
\begin{align*}
\hat{S}^A(U_A, U_B, \theta, \phi) &= (\rho_{00} \chi + \rho_{11} \Omega + \rho_{01} \xi + \rho_{00} \eta) \cos^2 \frac{\theta}{2} + (\rho_{00} \xi + \rho_{11} \eta + \rho_{01} \chi + \rho_{00} \Omega) \sin^2 \frac{\theta}{2} \\
&+ \{(\rho_{00} - \rho_{01}) \Phi + (\rho_{10} - \rho_{11}) \Theta \} \cos \alpha_B \sin \theta \cos \phi \\
&+ \{(\rho_{00} - \rho_{01}) \Phi + (\rho_{10} - \rho_{11}) \Theta \} \sin \alpha_B \sin \theta \sin \phi,
\end{align*}
\]

where

\[
\begin{align*}
\chi &= \cos^2 \frac{\beta_A}{2} \cos^2 \frac{\beta_B}{2}, \quad \xi = \cos \frac{\beta_A}{2} \sin \frac{\beta_B}{2}, \\
\Omega &= \sin^2 \frac{\beta_A}{2} \cos^2 \frac{\beta_B}{2}, \quad \eta = \sin \frac{\beta_A}{2} \cos \frac{\beta_B}{2}, \\
\Phi &= \frac{1}{2} \cos^2 \frac{\beta_A}{2} \sin \beta_B, \quad \Theta = \frac{1}{2} \sin^2 \frac{\beta_A}{2} \sin \beta_B.
\end{align*}
\]

It is evident from Eq. (16) that the payoffs contains the information about the initial quantum state in terms of Stokes parameters defined in Eq. (8) that can be extracted by using a suitable set of strategies. For \( S^{A}_{10} = S^{B}_{10} = S^{B}_{11} = 1 \) and \( S^{A}_{11} = S^{A}_{01} = S^{B}_{00} = S^{B}_{01} = -1 \), Bob performs the following steps for a single qubit quantum state tomography:

**Step 1:** When \( \beta_A = \beta_B = \alpha_B = \frac{\pi}{2} \), with the help of Eq. (16) we obtain

\[
\begin{align*}
\hat{S}^A &= \sin \theta \sin \phi, \\
\hat{S}^B &= -\sin \theta \sin \phi.
\end{align*}
\]

Comparing the result (18) with Eq. (8) we see that the payoff of Alice is one of the Stokes parameters.

**Step 2:** When \( \beta_A = \beta_B = \frac{\pi}{2} \) and \( \alpha_B = 0 \), Eq. (16) reduces to

\[
\begin{align*}
\hat{S}^A &= \sin \theta \cos \phi, \\
\hat{S}^B &= -\sin \theta \cos \phi.
\end{align*}
\]

Comparing Eqs. (19) and (8) it is evident that it is also one of the Stokes parameters.

**Step 3:** When \( \beta_A = \beta_B = 0 \), Eq. (16) gives

\[
\begin{align*}
\hat{S}^A &= \cos \theta, \\
\hat{S}^B &= -\cos \theta.
\end{align*}
\]

Comparison of the result (20) with Eq. (8) shows that the payoff of Alice is the third Stokes parameter.

From Eqs. (18) (19) and (20) we see that the payoffs are equal to the Stokes parameters of quantum state that helps us to reconstruct the quantum state. Furthermore the standard deviation for all of the above cases is bounded above by 1. This technique is simple and not beyond the reach of recent technology. \(^{[20],[21]} \)
The state of the quantum system contains all the information about the system. In classical mechanics it is possible in principle, to devise a set of measurements that can fully recover the state of the system. In quantum mechanics, two fundamental theorems, Heisenberg uncertainty principle and no cloning theorem forbid to recover the state of a quantum system without having some prior knowledge. This problem, however, can be solved with the help of quantum state tomography, where an unknown quantum state is estimated through a series of measurements on a number of identical copies of a system. Here we show how an unknown quantum state can be reconstructed by making use of a mathematical setup of the generalized quantization scheme of games. In our technique, Alice sends an unknown pure quantum state to Bob who appends it with $|0\rangle \langle 0|$ and then applies the unitary operators on the appended quantum state and finds the payoffs for Alice and Bob. It is shown that for a particular set of unitary operators, the payoffs become equal to Stokes parameters for the unknown quantum state. In this way an unknown quantum state can be measured and reconstructed.

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