Yang-Mills-Chern-Simons Supergravity

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ABSTRACT

$N = (1,0)$ supergravity in six dimensions admits $\text{AdS}_3 \times S^3$ as a vacuum solution. We extend our recent results presented in hep-th/0212323, by obtaining the complete $N = 4$ Yang-Mills-Chern-Simons supergravity in $D = 3$, up to quartic fermion terms, by $S^3$ group manifold reduction of the six dimensional theory. The $SU(2)$ gauge fields have Yang-Mills kinetic terms as well as topological Chern-Simons mass terms. There is in addition a triplet of matter vectors. After diagonalisation, these fields describe two triplets of topologically-massive vector fields of opposite helicities. The model also contains six scalars, described by a $GL(3,R)/SO(3)$ sigma model. It provides the first example of a three-dimensional gauged supergravity that can obtained by a consistent reduction of string-theory or M-theory and that admits $\text{AdS}_3$ as a vacuum solution. There are unusual features in the reduction from six-dimensional supergravity, owing to the self-duality condition on the 3-form field. The structure of the full equations of motion in $N = (1,0)$ supergravity in $D = 6$ is also elucidated, and the role of the self-dual field strength as torsion is exhibited.

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1 Introduction

A model of considerable interest in the context of AdS$_3$/CFT$_2$ correspondence is the AdS$_3 \times S^3$ compactification of the $N = (2,0)$ supergravity in $D = 6$ coupled to 21 tensor multiplets [1, 2]. This model arises in Type IIB string on $K3$ and it has 16 real supersymmetries [3]. The AdS$_3 \times S^3$ compactification preserves all supersymmetries, and it was determined in [4] that the propagating massless Kaluza-Klein spectrum consists of 21 hyper-multiplets and a special vector multiplet comprising $SO(4)$ Yang-Mills fields, 6 additional vector fields and 26 scalars. As a first step in finding the AdS$_3$ supergravity with 16 supersymmetries describing the couplings of this system, we recently studied [5] the simpler problem of obtaining a similar AdS$_3$ supergravity with 8 supersymmetries by means of an $S^3$ group manifold reduction [6] of $(1,0)$ supergravity [7, 8, 9, 10, 11, 12]. In particular, in [5] we obtained the bosonic Lagrangian and the supersymmetry transformations of the fermions

\footnote{By a group manifold reduction, we mean a dimensional reduction on a group manifold in which only those fields associated with left-invariant harmonics on the group are retained. Provided that one keeps all such fields, the consistency of the reduction is guaranteed.}
in three dimensions. In this paper, we shall extend these results to obtain the fermionic part of the resulting supergravity Lagrangian and the supersymmetry transformations of the bosons as well. There are subtleties in this process due to the presence of a self-duality equation in the pure \((1,0)\) theory in six dimensions. As we shall see, the use of a \(6D\) Lagrangian for the fermions will require an ansatz for the self-dual field strength that contains special fermionic bilinears terms. While the idea of using the fermionic part of Lagrangian in reduction schemes has been considered before, these schemes apparently have not been carried out fully to find supersymmetric results in lower dimensions. The procedure we have found in this paper fills this gap and it can have applications for the reductions of other chiral supergravity theories with self-dual field strengths.

Some of the salient features of the \(3D\) supergravity we have obtained are:

1. The theory admits AdS\(_3\) as vacuum (this is the first instance of a group manifold type reduction giving rise to such a feature.)

2. The theory has two sets of vector fields; a triplet coming from the metric, which we refer to as A-type vector fields, a triplet coming from the self-dual 3-form, which we refer to as B-type vector fields. The A-type vector fields are \(SU(2)\) gauge fields with Yang-Mills kinetic term and a topological mass term (hence the terminology of Yang-Mills-Chern-Simons supergravity), while the B-type vector fields are \(SU(2)\) matter fields which mix with the \(SU(2)\) gauge fields. After diagonalisation, these fields describe two triplets of topologically-massive vector fields of opposite helicities.

3. Although there exist many gauged supergravities in \(D = 3\) that admit an AdS\(_3\) vacuum, our theory is the only one known so far that has a consistent string or M-theory origin.\(^2\)

In this paper, we also elucidate the structure of the pure \((1,0)\) supergravity theory by providing its full field equations in various forms, and we highlight the occurrence of the self-dual 3-form field strength as bosonic torsion in these equations. This fact was pointed out long ago in [10] up to quartic fermion terms in the action. We extend these results here by the inclusion of all the quartic fermion terms as well.

The first part of this paper, contained in Section 2, deals with the structure of the pure \((1,0)\) supergravity in \(D = 6\). In the second part, contained in Section 3, we describe the \(S^3\)

\(^2\)Recently, a paper has appeared which initiates the direct construction in three dimensions of a broad class of gauged supergravities [13]. This construction, which to date presents the bosonic action only, presumably should encompass our example that arises from dimensional reduction.
group manifold reduction of the theory, including its supersymmetry transformation rules, and we also determine the representation content of the vector fields, showing that they describe two triplets of spin-1 states with opposite helicities. In the concluding Section 4, we comment further on our results.

2 The Complete \((1, 0)\) Supergravity in \(D = 6\)

The pure \((1, 0)\) supergravity in \(D = 6\) was considered in [8] at the level of the purely bosonic field equations, and the lowest order in fermions gravitino field equation and supersymmetry transformation rules. The complete field equations can be obtained by suitable truncations of the matter coupled versions given in [9, 10], or by imposing self-duality condition on the 3-form field strength after the variation of an action (which is not supersymmetric but it does yield supersymmetric field equations by this procedure) [12].

In this section, we begin with the description of the complete field equations with an emphasis on how the self-dual 3-form field strength arises as torsion. This property, noted at lowest order in fermionic contributions in [10], will be shown to hold at the level of the complete field equations. We then proceed to truncate the equations to lowest order in fermionic terms consistent with supersymmetry, which requires that the fermionic bilinear contributions to the bosonic field equations be kept. We use these results in Section 3 to perform the \(S^3\) group manifold reduction.

2.1 The Complete Field Equations and Bosonic Torsion

The \(D = 6, (1, 0)\) supergravity multiplet consists of the vielbein, 2-form potential with self-dual field strength and a gravitino which is symplectic Majorana-Weyl spinor in doublet representation of the R-symmetry group \(Sp(1)\). As is well known, a manifestly covariant action containing these fields alone cannot be written down due to the self-duality condition. However, the coupling of this multiplet to a tensor multiplet, consisting of a two-form potential with anti-self dual field strength, a dilaton and anti-chiral symplectic-Majorana spinor, does admit a Lagrangian formulation. Indeed, the complete Lagrangian, field equations and supersymmetry transformation rules for the coupled system have been constructed in [9]. Starting from these field equations and transformation laws, we can obtain the corresponding ones for the pure supergravity theory by setting the dilaton and the tensor-multiplet spinor to zero, and imposing self-duality condition on the supercovariant
3-form field strength. The resulting supersymmetry transformation rules are

\[ \delta e_\mu^a = \bar{\epsilon} \Gamma^a \psi_\mu , \]  
(2.1)

\[ \delta B_{\mu \nu} = -\bar{\epsilon} \Gamma_{[\mu} \psi_{\nu]} , \]  
(2.2)

\[ \delta \psi_\mu = D_\mu (\omega_+ ) \epsilon , \]  
(2.3)

where \( \Gamma_7 \psi = \psi \), and

\[ D_\mu (\omega_+ ) \epsilon = \left( \partial_\mu + \frac{1}{4} \omega^{ab}_\mu \Gamma^{ab} \right) \epsilon , \]  
(2.4)

\[ \omega^{\pm}_{\mu ab} = \tilde{\omega}_{\mu ab} \pm \tilde{R}^\pm_{\mu ab} . \]  
(2.5)

Here the supercovariant spin connection and 3-form field strengths are defined as

\[ \tilde{\omega}_{\mu ab} = \omega_{\mu ab} (\epsilon) + \kappa_{\mu ab} , \]  
(2.6)

\[ \tilde{H}_{\mu \nu \rho} = H_{\mu \nu \rho} + \frac{3}{2} \bar{\psi}_{[\mu} \Gamma_{\nu} \psi_{\rho]} , \]  
(2.7)

where \( \omega_{\mu ab} (\epsilon) \) is defined in the Appendix B and

\[ \kappa_{\mu ab} = \bar{\psi}_\mu \Gamma_{[a} \psi_{b]} + \frac{1}{2} \bar{\psi}_a \Gamma_\mu \psi_b , \]  
(2.8)

\[ H_{\mu \nu \rho} = 3 \partial_{[\mu} B_{\nu \rho]} . \]  
(2.9)

The (anti)self-dual projections are defined as \( H^{\pm}_{abc} = \frac{1}{2} (H_{abc} \pm \frac{1}{3!} \epsilon_{abcdef} H^{def}) \). Further notations and conventions can be found in Appendix A.

The field equations can be obtained either from the closure of the supersymmetry transformations (2.1)-(2.3), or directly from the field equations given in [9] of the (1,0) supergravity coupled to a single tensor multiplet by means of a consistent truncation triggered by setting the dilaton and the tensor-multiplet spinor to zero, and imposing self-duality condition on \( \tilde{H}_{\mu \nu \rho} \). We find that the resulting full field equations of of the pure (1,0) supergravity in six dimensions can be expressed as follows:

\[ \tilde{H}^-_{\mu \nu \rho} = 0 , \]  
(2.10)

\[ \Gamma^\mu \tilde{\psi}_{\mu \nu} = 0 , \]  
(2.11)

\[ \tilde{R}_{\mu \nu} = 0 , \]  
(2.12)

where the supercovariant gravitino curvature is defined as
\[ \hat{\psi}_{\mu\nu} \equiv \mathcal{D}_\mu (\omega_+) \psi_\nu - \mathcal{D}_\nu (\omega_+) \psi_\mu, \]  
(2.13)

and supercovariant generalized Ricci tensor is given by

\[ \hat{R}_{\mu\nu} \equiv R_{\mu\nu}(\omega_+) - \hat{\psi}^a \Gamma_a \hat{\psi}_{\mu\nu} - \hat{\psi}^a \Gamma_m \hat{\psi}_{\nu a} + 2 \hat{\psi}^a \Gamma^b \psi_\nu \hat{H}^+_{\mu ab}. \]  
(2.14)

The definition and properties of the Ricci tensor \( R_{\mu\nu}(\omega_+) \) are given in Appendix B. Note that neither \( \hat{R}_{\mu\nu} \) nor \( R_{\mu\nu}(\omega_+) \) are symmetric. Taking the symmetric and antisymmetric part of (2.12), we find

\[ R(\mu\nu)(\omega_+) - \hat{\psi}^a \Gamma_{(\mu} \hat{\psi}_{\nu)a} + 2 \hat{\psi}^a \Gamma_{(\mu} \hat{H}_{\nu)ab}^+ = 0, \]  
(2.15)

\[ R_{(\mu\nu)}(\omega_+) - \hat{\psi}^a \Gamma_a \hat{\psi}_{\mu\nu} - \hat{\psi}^a \Gamma_{[\mu} \hat{\psi}_{\nu]a} - 2 \hat{\psi}^a \Gamma_{[\mu} \hat{H}_{\nu]ab}^+ = 0. \]  
(2.16)

Computing the antisymmetric part of \( R_{\mu\nu}(\Gamma) \) from its definition yields the result (B.19) given in Appendix B. Using this result in (2.16) we find

\[ \mathcal{D}_\lambda (\Gamma) \hat{H}_{\mu ab}^+ = -\frac{1}{2} \hat{\psi}_a \Gamma_a \hat{\psi}_{\mu\nu} - \hat{\psi}^a \Gamma_{[\mu} \hat{\psi}_{\nu]a} + 2 \hat{\psi}^a \Gamma_{[\mu} \hat{H}_{\nu]ab}^+. \]  
(2.17)

It is gratifying to check that not only this equation is supercovariant but it also follows from the curl of the self-duality equation (2.10), namely from \( \epsilon^{\mu\nu\rho\sigma\lambda\tau} \mathcal{D}_\rho(\Gamma) \hat{H}_{\mu\nu\rho\tau} = 0. \) Therefore, in a formalism where we work with \( \hat{H}_{\mu\nu\rho}^+ \), we can take (2.17) to be the field equation for the 2-form potential. Note also that making use of the formulæ (B.14) and (B.16) given in Appendix B, the supercovariant Einstein equation (2.15) can be written as

\[ R(\mu\nu)(\omega_+) = \hat{H}_{\mu ab}^+ \hat{H}_{\nu}^{+ ab} + \hat{\psi}^a \Gamma_{(\mu} \hat{\psi}_{\nu)a} + 2 \hat{\psi}^a \Gamma^b \psi_{(\mu} \hat{H}_{\nu)ab}^+ . \]  
(2.18)

In checking the super-covariance of various equations we have encountered above, it is useful to note the following results:

\[ \delta \hat{\omega}_{\mu ab} = \epsilon \Gamma_{[a} \hat{\psi}_{b]} \mu - \frac{1}{2} \epsilon \Gamma_\mu \hat{\psi}_{ab} + \epsilon \Gamma^c \psi_\mu \hat{H}_{abc}^+, \]  
(2.19)

\[ \delta \hat{H}_{\mu ab}^+ = \frac{3}{2} \epsilon \Gamma_\mu \hat{\psi}_{ab} + \epsilon \Gamma^c \psi_\mu \hat{H}_{abc}^+. \]  
(2.20)

In deriving the second equation, we have used the self-duality equation (2.10) and the Fierz identity

\[ \Gamma_a \psi_{[\mu} \Gamma^a \psi_{\nu]} = 0. \]  
(2.21)

From (2.20) and the gravitino field equation (2.11), we also find

\[ \delta \hat{H}_{abc}^+ = \frac{3}{2} \epsilon \Gamma_{[a} \hat{\psi}_{bc]} . \]  
(2.22)
The self-duality of the right hand sides is ensured by the gravitino field equation (2.11).

To summarize, the full field equations of the pure \( N = (1, 0) \) supergravity are given by (2.10), (2.11) and (2.12), in which the self-dual and supercovariant 3-form field strength manifestly arises as torsion, or equivalently by (2.11), (2.17) and (2.18).

### 2.2 Truncation of the Quartic Fermions

The bosonic field equations given in the previous section, (2.11), (2.17) and (2.18), have quadratic and quartic in fermion contributions. To simplify the \( SU(2) \) reduction which will be performed in the next section, the quartic fermion terms in these equations and cubic fermion terms in the supersymmetry transformation rules (counting the supersymmetry parameter as one of the fermionic fields) can be truncated such that the field equations are supersymmetric up to quartic fermion terms. While the self-duality equation (2.10) must be implemented fully in this process, the equivalent second order field equation (2.17) can be consistently truncated to keep terms up to quadratic order in fermions.

Implementing the truncation procedure outlined above to the field equations (2.11), (2.17) and (2.18), we find:

\[
\Gamma^\mu \psi^\nu = \frac{1}{4} \Gamma^\mu \Gamma^{ab} \psi^\mu H^+_{ab},
\]

\[
\nabla_\rho H^+_{\mu
u\rho} = \frac{1}{8} \nabla_\rho (\bar{\psi} \gamma^\lambda \mu\nu\rho \psi^\tau - 3! \bar{\psi} \Gamma_{[\mu \nu} \psi^{\rho]} ) + \frac{3}{2} \bar{\psi} \Gamma_{[\mu \psi^{\nu}]} \\
- \bar{\psi} \Gamma_{[\mu \psi^b H^+_{\rho \nu a b}} - \bar{\psi} \Gamma^{b} \psi_{[\mu H^+_{\rho \nu \nu a b]} - \bar{\psi} \Gamma^{c} \psi^a H^+_{\mu \nu a]},
\]

\[
R_{\mu
u} = H^+_{\mu
u} H^+_{\rho \sigma} - \nabla_\rho (\bar{\psi} \gamma^\sigma \psi^a) - \nabla_\sigma (\bar{\psi} \Gamma_{(\mu} \psi_{\nu)}) \\
+ \bar{\psi} \Gamma_{(\mu} \psi_{\nu)} + \frac{1}{4} \bar{\psi} \Gamma^{a b} (\mu \psi^c H^+_{\nu a b} - \frac{1}{4} \bar{\psi} \Gamma^{b c} (\mu \psi_{\nu)} H^+_{a b c}) \\
+ \frac{3}{2} \bar{\psi} \Gamma^{b} \psi_{(\mu} H^+_{\nu a b)} - \bar{\psi} \Gamma_{(\mu} \psi^b H^+_{\nu a b}) - \frac{1}{2} \bar{g} \mu \nu \bar{\psi} \Gamma^{b} \psi^c H^+_{a b c},
\]

where \( R_{\mu
u} \equiv R_{\mu
u}(\omega(e)) \), the covariant derivative \( \nabla_\mu \) is torsion free and

\[
\psi_{\mu\nu} = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{a b} (e) \Gamma_{a b} \right) \psi_\nu - \mu \leftrightarrow \nu.
\]

The supersymmetry transformation up to cubic fermions, on the other hand, take the form

\[
\delta e^a_\mu = \bar{e} \Gamma^a \psi_\mu,
\]

\[
\delta B_{\mu\nu} = -\bar{e} \Gamma_{[\mu \psi_\nu]},
\]

\[
\delta \psi_\mu = \left[ \partial_\mu + \frac{1}{4} (\omega_\mu^{a b} (e) + H^+_{\mu\nu} a b) \Gamma_{a b} \right] \epsilon,
\]
The above field equations are rather complicated. The strategy now is to determine if they can be derived from the variation of an action by a consistent procedure. A natural candidate is to start from the Lagrangian \([12]\)

\[
e^{-1} \mathcal{L} = \frac{1}{4} R - \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} - \frac{1}{4} \bar{\psi}_{\mu} \Gamma^{\mu \nu \rho} \psi_{\nu \rho} - \frac{1}{24} \psi^{\lambda} \Gamma_{[\lambda} \Gamma^{\mu \nu \rho} \Gamma_{\tau]} \bar{\psi}_{\tau} H_{\mu \nu \rho} ,
\]

which is the up to quartic fermion Lagrangian of \((1,0)\) supergravity coupled to a tensor multiplet and where the dilaton and spinor fields of that multiplet are set to zero. Varying with respect to the graviton, gravitino and the 2-form potential, and after the variation making use of the full supersymmetric self-duality condition \((2.10)\) which can be expressed as

\[
H_{\mu \nu \rho} = H_{\mu \nu \rho}^+ - \frac{1}{8} \psi^{\lambda} \Gamma_{[\lambda} \Gamma^{\mu \nu \rho} \Gamma_{\tau]} \bar{\psi}_{\tau}
\]

we find the field equations

\[
\Gamma^{\mu \nu \rho} \psi_{\nu \rho} = -2 H_{\mu \nu \rho}^+ \gamma_\nu \psi_{\rho} ,
\]

\[
\nabla_\mu H_{\mu \nu \rho}^+ = -\frac{1}{8} \nabla_\mu \left( \psi^{\lambda} \Gamma_{[\lambda} \Gamma^{\mu \nu \rho} \Gamma_{\tau]} \bar{\psi}_{\tau} \right)
\]

\[
R_{\mu \nu} = H_{\mu \nu \rho}^+ H_{\nu \rho}^+ - \nabla_{(\mu} \left( \bar{\psi}_{\nu)} \Gamma^{\mu \alpha} \psi_{\alpha} \right) - \nabla_{\rho} \left( \bar{\psi}^{\rho} \Gamma_{(\mu} \psi_{\nu)} \right) - \bar{\psi}_{\alpha} \Gamma^{\alpha \beta (\mu} \psi_{\nu)} + \frac{1}{2} \bar{\psi}_{(\mu} \psi_{\nu)} \Gamma^{\alpha \beta} \psi_{\alpha} H_{\mu \nu \rho}^+ .
\]

In particular, the coefficient \(-1/8\) in \((2.33)\) results from two contributions of the same type; one from the variation of the Pauli coupling term in the Lagrangian and another from the substitution \((2.31)\).

The question now is whether these equations agree with \((2.23)\), \((2.24)\) and \((2.25)\) which were derived from first principles. A lengthy computation in which the field equations are used repeatedly shows that this is indeed the case.

Note that the self-dual part of \(H\) is automatically picked up in the Pauli coupling in the Lagrangian. Yet, the above Lagrangian is not supersymmetric because the kinetic term contains the anti-selfdual part as well, and indeed the self-duality equation does not follow from this Lagrangian. Nonetheless, we have proven above that the field equations following from this Lagrangian followed by the use of the self-duality equation \((2.31)\) gives the correct equation of motion derived from first principles. The above result will simplify considerably the \(SU(2)\) reduction described in the next section.
3 The $S^3$ Group Manifold Reduction

We begin by explaining our strategy for handling the self-duality condition in performing the $S^3$ group manifold reduction. The resulting 3D supergravity Lagrangian is given by (3.1), (3.22) and (3.33), and the supersymmetry transformations in (3.39), (3.40), (3.43), (3.44), (3.46), (3.47) and (3.52).

3.1 The Strategy for the Reduction of the Self-Duality Condition

The most straightforward, though certainly not the most economical, way to perform Scherk-Schwarz reduction of the model described above on $S^3$ group manifold down to $D = 3$ is to first reduce the field equations, and then find a $D = 3$ Lagrangian from which they can be derived. In the case of bosonic field equations, this was done in [5]. To obtain the supergravity up to quartic fermions, however, a considerably simpler way to proceed is to make use of the Lagrangian (2.30). To do so, we first observe that:

1) The gravitino field equation obtained from the Lagrangian (2.30) is evidently the same regardless of whether the self-duality equation (2.31) is used before or after the variation of the Lagrangian. This is due to the fact that the anti-self-dual part of the 3-form field strength is automatically dropped out of the Pauli coupling, while the H-kinetic term, of course, does not effect the gravitino field equation.

2) The Einstein equation obtained from the variation of the Lagrangian (2.30) is also the same regardless of whether (2.31) is used before or after the variation. This is not obvious but we have checked that the substitution (2.31) into the Einstein’s equation does not yield fermionic contributions over and above those which arise from the last term in (2.30).

It follows from the above observations that a candidate for the $S^3$ reduced action in $D = 3$ is

$$S^{3D} = \int d^3x L_B^{3D} + \int d^6x L_F , \quad \text{(3.1)}$$

where $L_B^{3D}$ is the 3Dbosonic Lagrangian obtained in [5] (see (3.22)) and

$$L_F = -\frac{1}{4} \hat{e} \hat{\psi}_A \Gamma^{ABC} \hat{\psi}_{BC} - \frac{1}{2} \hat{e} \hat{\psi}_A \Gamma_B \hat{\psi}_C \hat{R}^{ABC} , \quad \text{(3.2)}$$

where

$$\hat{H}_{ABC} = 3 \hat{\partial}_{[A} \hat{B}_{BC]} , \quad \hat{\partial}_A = \hat{e}^M_A \partial_M , \quad A, M = 0, 1, ..., 5 ,$$

$$\hat{\psi}_{AB} = \left[ \left( \hat{\partial}_A + \frac{i}{2} \hat{\omega}_A^{BC}(e) \Gamma_{BC} \right) \hat{\psi}_B + \hat{\omega}_{AB}^C(e) \hat{\psi}_C - A \leftrightarrow B \right] , \quad \text{(3.3)}$$
Note that in this section \((M, A)\) to denote the world and tangent space indices in 6D and hats only refer to six dimensional quantities and \textit{not} supercovariantizations as they did in the previous section. There should be no confusion in this notation because we shall never use supercovariantized objects in this section but rather explicitly write the terms required for supercovarianizations if need be.

It will prove to be useful to express the Lagrangian (3.2) as

\[
\mathcal{L}_F = -\frac{1}{4} \hat{e} \hat{\psi}_A \Gamma^{ABC} \hat{\psi}_B - \frac{1}{24} \hat{e} \hat{X}^{ABC} \hat{H}^+_A ,
\]

where

\[
\hat{X}^{ABC} \equiv \hat{\psi}^D \Gamma_{[D} \Gamma^{ABC} \Gamma_{E]} \hat{\psi}^E \equiv e^{-\phi} \hat{X}^{ABC} .
\]

Note that \(\hat{X}^{ABC}\) is ant-self-dual. In view of the arguments given above, this Lagrangian yields the gravitino and Einstein’s field equations straightforwardly. The crucial check remaining is to establish that reduction of the field equation for the self-dual potential (2.33), which in the notation of this section we write as

\[
\hat{\nabla}_A \hat{H}^{ABC} = -\frac{1}{8} \hat{\nabla}_A \hat{X}^{ABC} ,
\]

(here \(\hat{\nabla}\) is covariant derivative with respect to the ordinary spin connection \(\hat{\omega}(e)\)) agrees with that obtained from our proposed action (2.30). That check will ensure that the ansatz for the self-dual field strength has the correct fermionic bilinear terms. This we shall do in the remainder of this section.

### 3.2 The Reduction Ansatz

We begin with the introduction of the left-invariant \(SU(2)\) 1-forms \(\sigma^\alpha\), which satisfy the Maurer-Cartan algebra

\[
d\sigma^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma ,
\]

where \(f^\alpha_{\beta\gamma}\) are the \(SU(2)\) structure constants. The Kaluza-Klein metric reduction ansatz will then be given by

\[
d\hat{s}^2 = e^{2\alpha \phi} ds^2 + \frac{4}{g^2} e^{2\beta \phi} h_{\alpha\beta} \nu^\alpha \nu^\beta ,
\]

where \(\phi\) is the “breathing-mode” scalar, \(h_{\alpha\beta}\) denotes the remaining \(n\)-dimensional scalar fields (with the symmetric tensor \(h_{\alpha\beta}\) being unimodular), and \(\nu^\alpha\) is given by

\[
\nu^\alpha \equiv \sigma^\alpha - g A^\alpha .
\]
The constants $\alpha$ and $b$ are chosen to be
\[ \alpha = -3\beta = \sqrt{3/2}, \] (3.10)
such that one obtains the standard Hilbert-Einstein term in $D = 3$. In (3.9), $A^\alpha$ denotes the $SU(2)$ Yang-Mills potentials corresponding to the right-acting $SU(2)$ isometry of the 3-sphere.

It is convenient to work in a vielbein basis, which we take to be
\[ \hat{e}^a = e^\alpha \phi e^a, \quad \hat{e}^i = 2g^{-1} e^\beta \phi L^i_\alpha \nu^\alpha, \quad a, i = 1, 2, 3, \] (3.11)
where $L^i_\alpha$ is the vielbein on $SL(3,R)/SO(3)$. Here $e^a$ is a vielbein basis for the 3-dimensional metric $ds^2$, and $L^i_\alpha$ is a “square root” of $h_{\alpha\beta}$, and so
\[ h_{\alpha\beta} = L^i_\alpha L^i_\beta, \quad \det(L^i_\alpha) = 1. \] (3.12)

More explicitly,
\[ \hat{e}_M^A = \begin{pmatrix} e^{\alpha\phi} e^a_\mu & -\frac{2}{g} e^{\beta\phi} A^a_\mu L^i_\alpha \\ 0 & \frac{2}{g} e^{\beta\phi} L^i_\alpha \end{pmatrix}, \quad \hat{e}_M^a = \begin{pmatrix} e^{-\alpha\phi} e^\mu_\alpha & 0 \\ e^{-\alpha\phi} e^\mu_\alpha A^\alpha_\mu & \frac{2}{g} e^{-\beta\phi} L^i_\alpha \end{pmatrix} \] (3.13)

Note that $\hat{e}^a = 0$ and $\hat{e}^\mu = 0$. Defining the Yang-Mills field strengths $F^\alpha = dA^\alpha + \frac{1}{2} g f^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma$, we have:
\[ D F^\alpha \equiv dF^\alpha + g f^\alpha_{\beta\gamma} A^\beta \wedge F^\gamma = 0, \]
\[ D \nu^\alpha \equiv d\nu^\alpha + g f^\alpha_{\beta\gamma} A^\beta \wedge \nu^\gamma = -g F^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} \nu^\beta \wedge \nu^\gamma. \] (3.14)

It is also useful to define the Yang-Mills covariant exterior derivative acting on the scalars $L^i_\alpha$:
\[ D L^i_\alpha \equiv dL^i_\alpha - g f^\beta_{\gamma\alpha} A^\gamma L^i_\beta. \] (3.15)

The torsion-free spin connection $\hat{\omega}^A_B$, defined by $d\hat{e}^A = -\hat{\omega}^A_B \wedge \hat{e}^B$ and $\hat{\omega}_{AB} = -\hat{\omega}_{BA}$, where $A = 1, ..., 6$ and the hats denote six dimensional quantities, are found to be [5]
\begin{align*}
\hat{\omega}_{cab} &= e^{-\alpha\phi} [\omega_{cab} + \alpha (\partial_c \phi \eta_{ac} - \partial_a \phi \eta_{bc})], \\
\hat{\omega}_{iab} &= e^{-\phi} f_{abi}, \\
\hat{\omega}_{abi} &= -e^{-\phi} f_{abi}, \\
\hat{\omega}_{ija} &= e^{-\alpha\phi} (P_{a ij} - \frac{1}{3} \alpha \partial_a \phi \delta_{ij}), \\
\hat{\omega}_{aij} &= e^{-\alpha\phi} Q_{aij}, \\
\hat{\omega}_{kij} &= \frac{1}{4} g e^{\phi} (C_{k,ij} - C_{i,jk} - C_{j,ki}). \quad (3.16)
\end{align*}
where we have defined

\[ L^\alpha D_a L^j_\alpha = P_{ij}^a + Q_{ij}^a, \quad P_{ij} = P_{ji}, \quad Q_{ij} = -Q_{ji}, \quad (3.17) \]

\[ T^{ij} = L^i_\alpha L^j_\alpha, \quad C_{k,ij} = T_k^\ell \epsilon_{\ell ij} \]

\[ F_{ab}^i = L^i_\alpha F^\alpha, \quad L^i_\alpha L^j_\alpha = \delta^j_i. \quad (3.18) \]

It is also useful to define

\[ D_a L^i_\alpha = D_a L^i_\alpha + Q_{a ij} L^j_\beta, \quad (3.19) \]

from which it follows that

\[ L^\alpha D_a L^j_\alpha = P_{ij}^a. \quad (3.20) \]

Finally, we make following ansatz for the self-dual 3-form field strength

\[ \hat{H}^+_{abc} = m e^{\alpha \phi} \epsilon_{abc}(1 + X), \quad \hat{H}^+_{ab} = -e^{-\frac{1}{2} \alpha \phi} \epsilon_{ab} c (B^i_c + Y^i_c), \]

\[ \hat{H}^+_{ijk} = m e^{\alpha \phi} \epsilon_{ijk}(1 + X), \quad \hat{H}^+_{aij} = e^{-\frac{1}{4} \alpha \phi} \epsilon_{ijk} (B^k_i + Y^k_i), \quad (3.21) \]

where we have defined \( B^i \equiv L^i_\alpha B^\alpha \), and \( X, Y^i_a \) are bilinear in fermions which are to be determined from the requirement of the 3-form field equation following from the proposed action (3.1) agrees with the \( SU(2) \) reduction of the field equation (2.33). The ansatz above with \( X \) and \( Y^i_a \) set to zero was used in [5] in obtaining the bosonic Lagrangian in \( D = 3 \).

These fermionic modifications do not affect the reduction of \( \mathcal{L}_F \) in (3.4) but they are crucial in the reduction of the field equation (2.33) and comparing the result with that obtain from the variation of (3.1) with respect to the field \( B^i_a \). Finally, a word of caution with our notation: the hats in this section, for example, in (3.16) and in (3.21) do not refer to supercovariantization but merely to the six dimensional nature of the quantities.

### 3.3 The 3D Supergravity Lagrangian

The total 3D supergravity Lagrangian is the sum (3.1), where the bosonic Lagrangian is given by [5]

\[
e^{-1} \mathcal{L}^{3D}_B = \frac{1}{4} R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{4} P_{ij}^\mu P_{ij}^{\mu} - \frac{1}{2} e^{-\frac{\alpha}{3} \phi} F^\mu_{ij} F^{\mu ij} - m^2 e^{\alpha \phi} - \frac{1}{6} g^2 e^{\frac{2}{3} \alpha \phi} \left( T_{ij} T^{ij} - \frac{1}{2} T^2 \right) - e^{\frac{4}{3} \alpha \phi} B^\mu_{ij} B^{\mu ij} - 2 g^{-1} e^{\mu \nu \rho} \left( D_{\mu} B_{\nu}^\alpha - 2 m F_{\mu \rho}^\alpha \right) B_{\rho}^\alpha - 8 g^{-1} m^2 e^{\mu \nu \rho} \left( A_{\mu}^\alpha \partial_{\nu} A_{\rho}^\alpha + \frac{1}{4} \epsilon_{\alpha \beta \gamma} A_{\mu}^\alpha A_{\nu}^\beta A_{\rho}^\gamma \right). \quad (3.22)
\]
This is the result of reducing the bosonic Einstein equation and the bosonic self-duality equation in 6D on \( S^3 \) group manifold and constructing a Lagrangian which yields the resulting 3D bosonic field equations [5]. Our task is now to perform the \( SU(2) \) reduction of \( \mathcal{L}_F \) given in (3.4).

We begin by making an ansatz for the reduction of the gravitino field. In doing so, we shall make use of the original treatment of this problem in [6], and [14] where it has been studied further in the context of \( S^3 \) reduction of \( D = 11 \) supergravity. One technical aspect of the reduction is the diagonalization of the lower dimensional gravitino and spinor kinetic terms. It is convenient to treat the diagonalization problem after performing the \( S^3 \) reduction. Thus, we begin with the ansatz

\[
\hat{\psi}_a(x, y) = e^{\frac{1}{2} \alpha \phi(x)} \psi_a(x) , \quad \hat{\psi}_i(x, y) = e^{\frac{1}{2} \alpha \phi(x)} \chi_i(x) , \tag{3.23}
\]

where the exponential factors are chosen such that the gravitino kinetic term is canonical, i.e. with no dilaton prefactor.

Next, it is convenient to work out the \( S^3 \) reduction of the gravitino curvature (3.3). Using the vielbein basis (3.11) and the spin connection (3.16), we find:

\[
\hat{\psi}_{ab} = e^{-\frac{3}{2} \alpha \phi} \left[ \psi_{ab} + \alpha \Gamma_a \Gamma^c \psi_b \partial_c \phi + e^{-\frac{4}{3} \alpha \phi} F_{ac} \left( \Gamma^c \Gamma^{ab} - 2 \delta_b^a \Gamma^c \chi_i \right) \right] \tag{3.24}
\]

\[
\hat{\psi}_{ij} = e^{-\frac{3}{2} \alpha \phi} \left[ \left( P_{cik} - \frac{1}{3} \alpha \delta_{ik} \partial_c \phi \right) \Gamma^k \Gamma^c \chi_j + \frac{1}{2} e^{-\frac{4}{3} \alpha \phi} F_{ab} \Gamma^{ab} \chi_j 
\right.
\]

\[
+ \frac{1}{8} g e^{\frac{4}{3} \alpha \phi} \left( \left( C_{i, k \ell} - 2 C_{k, i \ell} \right) \Gamma^{k \ell} \chi_j - 4 C_{k, ij} \chi^k \right) \right] \tag{3.25}
\]

\[
\hat{\psi}_{ai} = e^{-\frac{3}{2} \alpha \phi} \left[ D_a \chi_i + \frac{1}{2} P_{cij} \left( \Gamma^c \Gamma^j \psi_a + 2 \delta_a^c \chi_j \right) 
\right.
\]

\[
- \frac{1}{6} \alpha \left[ \Gamma^c \Gamma_i \psi_a + (5 \delta_a^c - 3 \Gamma^a \Gamma^c) \chi_i \right] \partial_c \phi 
\]

\[
- \frac{1}{4} e^{-\frac{4}{3} \alpha \phi} F_{cd}^k \left( \delta_{k t} \Gamma^{cd} \psi_a - 2 \Gamma^c \delta^d \Gamma_k \chi_i \right) 
\]

\[
- \frac{1}{16} g e^{\frac{4}{3} \alpha \phi} \left( C_{i, j k} - 2 C_{j, i k} \right) \Gamma^{j k} \psi_a \right] , \tag{3.26}
\]

where the antisymmetrizations in \([ab]\) and \([ij]\) on the right hand sides are understood, and

\[
D_a \psi_b = \left( \partial_a + \frac{1}{2} \omega_{acd} \Gamma^{cd} + \frac{1}{4} Q_{akl} \Gamma^{kl} \right) \psi_b + \omega_{abc} \psi_c , \tag{3.27}
\]

\[
D_a \chi_i = \left( \partial_a + \frac{1}{2} \omega_{acd} \Gamma^{cd} + \frac{1}{4} Q_{akl} \Gamma^{kl} \right) \chi_i + Q_{aij} \chi_j . \tag{3.28}
\]

Using the above results, we find the \( S^3 \) reduction of the Lagrangian (3.2) to be
\[ e^{-1} L_F = -\frac{1}{4} \tilde{\psi}_\mu \Gamma^{\rho \mu \nu} D_\nu \psi_\rho - \bar{\chi}_j \Gamma^i \Gamma^{\mu \nu} D_\mu \psi_\nu + \frac{1}{2} \bar{\chi}_i \Gamma^{ij} \Gamma^{\mu} D_\mu \chi_j \]

\[ + \frac{1}{2} \tilde{\psi}_\mu \Gamma^{\mu \nu} \Gamma^i \chi^j + \bar{\chi}_k \Gamma^{\mu \nu} \Gamma^i \chi^j \right) P_{\mu \nu} - \frac{2}{3} \alpha \left( \tilde{\psi}_\mu \Gamma^{\mu \nu} \Gamma^i \chi_i \right) \partial_\nu \phi \]

\[ - \frac{1}{4} e^{-4 \alpha \phi / 3} \Gamma^{\mu \nu} \left[ \tilde{\psi}^j \Gamma_i \psi^\nu - \tilde{\psi}_\rho \left( \Gamma^{\mu \nu} \Gamma^j + \Gamma^{i \mu \nu} \Gamma_\rho \delta^j_\nu \right) \right] \chi_k \]

\[ + \frac{1}{2} \bar{\chi}_j \left( \Gamma^{ij} - 4 \Gamma^{jik} \right) \Gamma^{\mu \nu} \chi_k \]

\[ - \frac{1}{16} g \epsilon^{\alpha \phi / 3} C_{i,j,k} \left( \tilde{\psi}_\mu \Gamma^{\mu \nu} \Gamma^{ij} \psi^\nu - 4 \tilde{\psi}_\mu \Gamma^{\mu \nu} \Gamma^{ij} \chi^k - 4 \chi^j \Gamma^{kij} \chi^\ell \right) \]

\[ + 2 \bar{\chi} \chi^j \Gamma^{ij} + 4 \bar{\chi} \chi^j \Gamma^{ij} \chi^k \right) - \frac{1}{24} m e^{2 \alpha \phi / 3} \left( \epsilon_{abc} X^{abc} + \epsilon_{ijk} X^{ijk} \right) \]

\[ + \frac{1}{8} \epsilon^{2 \alpha \phi / 3} \left( \epsilon_{abc} X^{abi} - \epsilon_{ijk} X^{cij} \right) B^c_i , \quad (3.29) \]

where \( \psi_\mu = \epsilon_\mu^n \psi_n \), \( \Gamma_\mu = \epsilon_\mu^n \Gamma_n \) and \( X^{ABC} \) are defined in (3.5). The explicit form of the last four terms will be given shortly. It is convenient to leave them in this form in comparing the B-field equation that follows from (3.1) with \( L_F \) as given above, with the \( S^3 \) reduction of (3.6). The former is

\[ \epsilon^{\mu \nu \rho} \left( D_\mu B_\rho^{\alpha} L^i_\alpha - m F_\mu^{ij} + \frac{1}{4} \epsilon^{\alpha \phi / 3} \epsilon_{\mu \nu \alpha} T^{ij} B_j^\alpha \right) = \frac{1}{8} \epsilon^{2 \alpha \phi / 3} T^{ij} \left( \epsilon^{\mu \nu \rho} \epsilon_{\rho \nu \alpha} \right) \epsilon_{ijk} X^{ijk} . \quad (3.30) \]

Comparing this result with the \( SU(2) \) reduction of the B-field equation (3.6), we find a perfect agreement by choosing

\[ Y^i_\alpha = -\frac{1}{16} e^{-2 \alpha \phi / 3} \epsilon_{abc} X^{bc} \]

\[ X = \frac{1}{48} m e^{-2 \alpha \phi} \epsilon_{abc} X^{abc} . \quad (3.31) \]

The ansatz (3.21) then takes the form

\[ \hat{H}^+_{abc} = m e^{2 \alpha \phi} \epsilon_{abc} - \frac{1}{8} e^{-2 \alpha \phi} X^{abc} , \quad \hat{H}^+_{abi} = -e^{-2 \alpha \phi} \epsilon_{abc} B_\epsilon^i - \frac{1}{8} e^{-2 \alpha \phi} X_{abi} , \]

\[ \hat{H}^+_{ijk} = m e^{2 \alpha \phi} \epsilon_{ijk} + \frac{1}{8} e^{-2 \alpha \phi} X_{ijk} , \quad \hat{H}^+_{a\ell j} = e^{-2 \alpha \phi} \epsilon_{ijk} B_\ell^k + \frac{1}{8} e^{-2 \alpha \phi} X_{ija} . \quad (3.32) \]

The sign differences in the \( X \)-terms are essential for the self duality of \( H \). These terms are responsible for the cancellation of the differentiated fermion terms that arise on the right hand side of (3.6).

Having determined the quantities \( X \), \( Y^i_\alpha \) and \( Y^k_\alpha \) in the ansatz (3.21) for \( \hat{H}^{ABC} \), we can now express the fermionic Lagrangian (3.29) as follows:
$$e^{-1} \mathcal{L}_F = -\frac{1}{2} \bar{\psi} \mu \Gamma_{\mu \nu} \partial_\nu \psi_\nu - \bar{\chi}_i \Gamma^i \Gamma_{\mu \nu} \partial_\mu \psi_\nu + \frac{1}{2} \bar{\chi}_i \Gamma^{ij} \partial_\mu \chi_j$$

$$+ \frac{1}{2} (\bar{\psi} \nu \Gamma^\nu \Gamma^i \chi^j + \bar{\chi}_k \Gamma^k \Gamma_{\mu \nu} \chi_i) P_{\mu ij} - \frac{2}{3} \alpha (\bar{\psi} \nu \Gamma^\nu \partial_\mu \chi_i) \partial_\mu \phi$$

$$+ \frac{1}{2} e^{-4 \alpha \phi / 3} F^i_{\mu \nu} (\bar{\psi} \mu \Gamma_{\mu \nu} \chi_i + \bar{\chi}_k \Gamma^k \Gamma_{\mu \nu} \chi_i)$$

$$- \frac{1}{8} e^{-4 \alpha \phi / 3} G^+_{\mu \nu i} (2 \bar{\psi}^\mu \Gamma^i \psi_\nu - 2 \bar{\psi}^\mu \Gamma^\rho \partial_\nu \chi_\rho + \bar{\chi}_j \Gamma^{ij} \chi_k + \bar{\chi}_j \Gamma^{ij} \Gamma_k \psi_\nu)$$

$$- \frac{1}{32} g e^{4 \alpha \phi / 3} C_{i, j k} (\bar{\psi} \mu \Gamma^{ij} \Gamma^{jk} \psi_\nu - 4 \bar{\psi} \mu \Gamma^{ij} \chi_k - 4 \bar{\chi}_j \Gamma^{ij} \chi_\ell + 2 \bar{\chi}_j \Gamma^{ij} \chi_\ell)$$

$$+ 2 \bar{\chi}_j \Gamma^{ij} \chi_k + 4 \bar{\chi}_j \Gamma^{ij} \chi_k + \frac{1}{2} m e^{2 \alpha \phi / 3} (\bar{\psi} \mu \Gamma_{\mu \nu} \sigma_1 \psi_\nu - \bar{\chi}_i \Gamma^{ij} \sigma_1 \chi_j),$$

where

$$G^+_{\mu \nu i} = F_{\mu \nu i} + 2 e e^{2 \alpha \phi / 3} \epsilon_{\mu \nu \rho} B_\rho^i.$$

Thus, the total 3D supergravity Lagrangian is given by (3.1), (3.22) and (3.33). As expected, the gravitino and spinor kinetic terms are mixed. We have verified that they can be simultaneously be diagonalized for any dimension $n$. In particular, the field redefinitions which do the job for $n = 3$ are given by

$$\psi_\mu = \psi_\mu' - \frac{1}{2} \Gamma_\mu \Gamma^k \chi'_k,$$

$$\chi_i = -\frac{1}{2} \Gamma^k \Gamma_i \chi'_k.$$ 

(3.35)

The inverse transformation is

$$\psi_\mu' = \psi_\mu + \Gamma_\mu \Gamma^k \chi_k,$$

$$\chi'_i = \Gamma_{ij} \chi^j.$$ 

(3.36)

The kinetic terms become diagonal in terms of the primed fields and in particular the spinor kinetic term becomes $-\frac{1}{2} \bar{\chi}^\mu \Gamma^{\mu \nu} \partial_\nu \chi_i$.

### 3.4 The 3D Supersymmetry Transformations

The reduction of the gravitino transformation rule has been carried out already in [5]. Here, we shall also perform the $S^3$ reduction of the supersymmetry transformation rules of the graviton and the self-dual field strength, thereby obtaining all the transformation rules up to cubic fermion terms.

---

3The redefinition $\psi_\mu = \psi_\mu' - \Gamma_\mu \Gamma^k \chi_k$ suffices to eliminate the mixing between the fermion kinetic terms and puts the gravitino its canonical form, while the kinetic term for $\chi_i$ remains non-diagonal in the $SU(2)$ space. This kind of procedure was adopted in [14].
Let us begin with the combined supersymmetry and Lorentz transformations of the vielbein $N = (1, 0)$ supergravity in six dimensions:

\[ \hat{e}_A^M \delta \hat{e}_{MB} = \hat{e} \Gamma_B \hat{\psi}_A + \hat{\lambda}_{AB} , \]  

(3.37)

In order that the 3D gravitino transformation takes the form $\delta \hat{\psi}_a(x) = D_a \epsilon(x) + \cdots$, we need to make the ansatz

\[ \hat{\epsilon}(x, y) = e^{\alpha \phi/2} \epsilon(x) . \]  

(3.38)

The $ij$ projection of (3.37), and its trace, then yield

\[ L^i_\alpha \delta L^i_{\alpha j} = \hat{\epsilon} \Gamma_{[i} \chi_{j]} - \frac{1}{3} \delta_{ij} \left( \hat{\epsilon} \Gamma^k \chi_k \right) + \lambda_{ij} , \]  

(3.39)

\[ \delta \phi = -\frac{1}{\alpha} \hat{\epsilon} \Gamma^i \chi_i , \]  

(3.40)

where the composite $SO(3)$ transformation parameter $\lambda_{ij}$ in 3D theory is defined as

\[ \lambda_{ij} = \hat{\lambda}_{ij} - \hat{\epsilon} \Gamma_{[i} \chi_{j]} . \]  

(3.41)

The $(ia)$ projection of (3.37) gives

\[ \hat{\lambda}_{ia} = \hat{\epsilon} \Gamma_a \chi_i , \]  

(3.42)

which is the required Lorentz transformation to maintain the triangular gauge implied by (3.11). Next, the $(ai)$ projection of (3.37) gives

\[ (\delta A^a_\mu) L^i_\alpha = -\frac{1}{2} g e^{Aa\phi/3} \left( \hat{\epsilon} \Gamma^i \hat{\psi}_\mu + \hat{\epsilon} \Gamma_\mu \chi^i \right) , \]  

(3.43)

where we have used (3.41). Finally, the $ab$ projection of (3.37) yields

\[ \delta e^a_\mu = \hat{\epsilon} \Gamma^a \psi_\mu + \left( \hat{\epsilon} \Gamma^i \chi_i \right) e^a_\mu - \lambda^{ab} e^a_\mu , \]  

(3.44)

where the second term disappears if we were to make the field redefinition (3.35) which diagonalizes the fermionic kinetic terms as discussed earlier, and the Lorentz transformation parameter in 3D theory is $\lambda_{ab} \equiv \hat{\lambda}_{ab}$.

The reduction of the gravitino transformation rule

\[ \delta \hat{\psi}_A = \hat{\nabla}_A \hat{\epsilon} + \frac{1}{2} \hat{R}^A_{CD} \Gamma^C_D \hat{\epsilon} \]  

(3.45)

is straightforward, given the expressions (3.16) for the spin connection. The Lorentz transformations are not written down as they do not give rise to induced supersymmetry transformations to lowest order in fermions. The reduction of the above transformation rule
gives [5]

\[ \delta \psi_\mu = D_\mu \epsilon + \frac{1}{2} \alpha \Gamma_\mu \Gamma^\nu \partial_\nu \phi - \frac{1}{2} e^{-4\alpha \phi/3} F^i_{\mu \nu} \Gamma_i \epsilon \]
\[ + \frac{1}{2} e^{2\alpha \phi/3} B_\mu \Gamma^\nu \Gamma_\mu \Gamma_i \epsilon - \frac{1}{2} m e^{2\alpha \phi} \Gamma_\mu \epsilon, \] (3.46)

\[ \delta \chi_i = \frac{1}{2} \left( P_{\mu ij} - \frac{1}{16} \alpha \delta_{ij} \partial_\mu \phi \right) \Gamma^j \Gamma^i \epsilon + \frac{1}{4} g e^{-4\alpha \phi/3} \left( T_{ij} - \frac{1}{2} \delta_{ij} T \right) \Gamma^j \Gamma^i \epsilon \]
\[ + \frac{1}{2} e^{-4\alpha \phi/3} F^i_{\mu \nu} \Gamma_\mu \Gamma_\nu \epsilon + \frac{1}{2} e^{2\alpha \phi/3} B^i_\mu \Gamma_\mu \Gamma_i \Gamma^i \epsilon. \] (3.47)

There remains the transformation rule for \( B_\mu \). To this end, it is convenient to first define

\[ h_{ABC} \equiv \hat{H}^{+\text{cov}}_{ABC} - \frac{1}{2} \epsilon_{ABCDEF} \hat{\psi}^D T^E \hat{\psi}^F, \] (3.48)

where \( \hat{H}^{+\text{cov}}_{ABC} \) is the supercovariant self-dual field strength defined as

\[ \hat{H}^{+\text{cov}}_{ABC} = \hat{H}^{+}_{ABC} - \frac{1}{2} \hat{\psi}^D \Gamma^{DE} \hat{\psi}^E. \] (3.49)

It follows from (3.32) that

\[ h_{abi} = -e^{-\alpha \phi/3} \epsilon_{abc} B^c_i. \] (3.50)

The advantage of working with \( h_{ABC} \) is that its transformation rule is somewhat simpler to compute. Using (2.22), and recalling that we work up to cubic fermions, we find that the combined supersymmetry and Lorentz transformation of \( h_{ABC} \) is given by

\[ \delta h_{ABC} = \frac{3}{2} \epsilon \Gamma_{[ABC]} (\omega (\epsilon)) - \frac{3}{2} \epsilon \Gamma^D \hat{\psi}_{[A} \hat{H}^{+}_{BC]D} - \frac{1}{2} \epsilon_{ABCDEF} \hat{\psi}^D T^E \delta \hat{\psi}^F \]
\[ + 3 \hat{\lambda}_{A} h_{BCD}. \] (3.51)

From this equation, using (3.50) and the formulae provided above for various quantities occurring in this equation, including the compensating \( SO(3) \) transformation (3.41), we find the supersymmetry transformation rule

\[ (\delta B_\alpha^i)^ L_i = e^{-2\alpha \phi/3} \left( 2 \chi_1 \chi^j \delta \psi_\mu + \epsilon^{ijk} \chi_j \Gamma_k \delta \chi_k \right) \]
\[ - \frac{1}{2} m e^{4\alpha \phi/3} \epsilon \Gamma_\mu \chi^i + \frac{1}{4} e^{2\alpha \phi/3} \left( \epsilon_{\mu \nu \rho} \Gamma^i \psi^{\nu \rho} - 2 \epsilon \Gamma_\mu \psi^{\nu i} \right) \]
\[ + \epsilon \left( g_{\mu \nu} (- \Gamma_i \chi^j + \frac{1}{2} \delta_{ij} \Gamma^k \chi_k) - \Gamma_{\mu \nu} \Gamma^{ij} \chi_k + \Gamma_{(\mu} \psi_{\nu)} \delta_{ij} + \frac{1}{2} \Gamma_{\mu \nu \rho} \Gamma^{ij} \psi^{\rho} \right) B^j_i. \] (3.52)

To summarize, the supersymmetry transformation rules of the 3D supergravity are given by (3.39), (3.40), (3.43), (3.44), (3.46), (3.47) and (3.52). It is straightforward, though considerably tedious, to perform the redefinitions (3.35) if one wishes to work with diagonalized fermionic kinetic terms. We also note that the transformation rule for \( B_\mu \) is rather complicated and it has an unusual form. This is due to the fact that this field originates directly from the 3-form field strength, instead of its potential. There are many ways to express this transformation rule, and a better understanding of how to do so in a manner that would be natural from a direct three-dimensional construction would be desirable.
3.5 Yang-Mills and Matter Vector Fields With Opposite Helicity

In this section we take a closer look at the coupled field equations for the $SU(2)$ Yang-Mills vector fields and the triplet of matter vector fields. The bosonic part of their field equations are given by

$$
D_b (e^{-\frac{2}{3} \alpha \phi} F_{ab}^i) = -e^{-\frac{2}{3} \alpha \phi} P^{bij} P_{abj} + \frac{1}{4} g^2 \epsilon^{ijk} T_k \ P_{aj\ell} - 4m e^{\frac{4}{3} \alpha \phi} B_a^i + 2 \epsilon^{ijk} \epsilon_{abc} B^b_j \ B^c_k ,
$$

$$
\epsilon^{abc} (D_b B^a_c - m F_{bc}^a) L^i_\alpha = g e^{\frac{4}{3} \alpha \phi} T^{ij} B^a_j .
$$

To analyze the representation content of the vector fields, it suffices to examine their linearized equations of motion. In doing so, we set $g = 4m$ which implies that the AdS$_3$ vacuum arises with vanishing scalar fields. (Note that $R_{ab} = -2m^2 \eta_{ab}$, and hence, $m$ is the inverse AdS$_3$ radius). The linearized equations of motion then take the form

$$
d{*} F^i = -4m F^i + \frac{1}{2} G^i , \quad B^i = \frac{1}{2m} * G^i - * F^i ,
$$

where $G^i = dB^i$. We now define the one-forms $A^i_1$ and $A^i_2$ by

$$
A^i = A^i_1 + A^i_2 , \quad B^i = m (4A^i_1 + A^i_2) .
$$

With these definitions, it follows from (3.54) that

$$
d{*} F^i_1 = 4m F^i_1 , \quad d{*} F^i_2 = -2m F^i_2 ,
$$

where $F^i_1 = dA^i_1$ and $F^i_2 = dA^i_2$. These equations can be derived from an action which is the sum of two actions each one containing a Yang-Mills kinetic term and Chern-Simons mass terms with opposite signs:

$$
e^{-1}\mathcal{L} = -\frac{1}{4} F_{1ab} F^{ab}_1 + m \epsilon^{abc} F_{1ab} A_{1c} - \frac{1}{4} F_{2ab} F^{ab}_2 - \frac{1}{2} m \epsilon^{abc} F_{2ab} A_{2c} .
$$

The sign of the topological mass term dictates the sign of the spin-1 helicity, and thus we see that the vector fields $A_1$ and $A_2$ describe spin-1 fields with opposite helicity [4]. This result is in exact agreement with that of [4] (with AdS$_3$ radius set to 1), where these spin-1 fields arise as a subset of those which come from an $SO(4)$ gauge invariant KK reduction from six dimensions on $S^3$. As explained in [4], the Lagrangian (3.57) is only meant to be interpreted at the free level (which is sufficient for deducing the representation content) and that the interacting Lagrangian cannot be written in this way. Indeed, that would have implied an $SO(4)$ gauge symmetry, while we know that the full theory is only $SU(2)$ gauge invariant.
4 Conclusions

In this paper, we have completed the construction of the three-dimensional supergravity obtained by an $S^3$ group-manifold reduction of pure $(1,0)$ chiral supergravity in six dimensions. The bosonic action, and the fermionic transformation rules, had been obtained previously in [5]. The resulting three-dimensional supergravity has $SU(2)$ Yang-Mills fields with topological mass terms, and a triplet of massive vector fields in the adjoint representation of $SU(2)$, together with six scalars described by the coset $GL(3,R)/SO(3)$.

In our reduction we neglected the quartic fermion terms in the six-dimensional Lagrangian, and the associated cubic-fermion terms in the supersymmetry transformation rules. However, we did keep all contributions following from the quadratic fermion terms in the Lagrangian, meaning in particular that we kept all the bilinear fermion contributions in the bosonic equations of motion. It was therefore necessary to work with the full self-duality condition for the six-dimensional 3-form field, including the fermion bilinear terms. Thus although the procedure for performing a group-manifold reduction is a mechanical one, there were considerable subtleties in this case arising from the reduction of the full self-duality condition.

Although the six-dimensional theory cannot have a Lagrangian formulation, owing to the self-duality of the 3-form, the reduced three-dimensional theory does have a Lagrangian formulation. Furthermore, it is unnecessary to introduce potentials for the three $B$-type matter vector fields defined in (3.32) which arose directly from the self-dual 3-form field strength (as opposed to its potential) in six dimensions. This is because the six-dimensional self-duality is transformed into an “odd-dimensional self-duality” in $D = 3$, which admits a Lagrangian formulation even though the equations of motion are of first order.

The matter-coupled AdS$_3$ supergravity that we have obtained here has a specific field content and rather elaborate interactions which would be difficult to guess from a direct construction in three dimensions. It is of interest, however, to find a systematic way of directly constructing not only this model but its generalizations with arbitrary field content, as well as higher supersymmetries. Significant progress in this direction has been made in [13]. In particular, the model we have constructed has been suggested to correspond to a Chern-Simons type of supergravity with CS gauge group a semidirect product of $SO(3)$ with a 6-parameter group generated by 3 abelian and 3 nilpotent generators. This is coupled to 12 scalars that parametrize $SO(4,3)/SO(4) \times SO(3)$. In [13], a procedure is described in which 3 of the scalars are dualised to what we call $B$-type vector fields, and 3 scalars are gauged away, thereby leaving 6 scalars that parametrize $GL(3,R)/SO(3)$. So far, the
bosonic action has been provided in [13], and while it seems to have a structure similar to our bosonic action, the exact equivalence remains to be shown.

In this paper, we have focussed on the $S^3$ group-manifold reduction of pure $(1,0)$ six-dimensional supergravity, which is, of course, anomalous. Since one of the principal motivations for studying such reductions is to obtain lower-dimensional AdS supergravities that are embedded in string theory, it is natural to extend our work by taking an anomaly-free six-dimensional theory as the starting point for the $S^3$ reduction. There exists a variety of such six-dimensional anomaly-free theories, including $(1,0)$ examples with appropriate matter multiplets, which can be obtained from K3 reductions of the heterotic string, and the $(2,0)$ theory coupled to 21 tensor multiplets, which comes from the type IIB string reduced on K3. All the resulting three-dimensional supergravities will presumably be encompassed within the general class that has been proposed in [13]. While it is useful to have a direct construction of the most general matter-coupled AdS supergravities in three dimensions, it is also of importance to establish a connection with those which follow from $S^3$ reduction, in order to see how they are embedded in string theory. This would be of particular importance, for example, for studying the AdS$_3$/CFT$_2$ correspondence.
A Conventions

We use the signatures $\eta_{AB} = (- + \cdots +)$ and $\eta_{ab} = (- +)$, and use the $\Gamma$-matrices

$$\Gamma^a = \gamma^a \times 1 \times \sigma^1, \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad \gamma^{abc} = \epsilon^{abc},$$

(A.1)

$$\Gamma^i = 1 \times \gamma^i \times \sigma^2, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \gamma^{ijk} = -i\epsilon^{ijk},$$

$$\Gamma_7 = 1 \times 1 \times \sigma^3, \quad \Gamma_{A_1A_2\cdots A_6} = \epsilon_{A_1A_2\cdots A_6} \Gamma_7,$$

(A.2)

$$C = (i\sigma^2) \times (i\sigma^2) \times \sigma^1.$$  

The 6D spinors are symplectic Majorana-Weyl and $\Gamma^7 \epsilon = \epsilon$. The $Sp(1)$ doublet index $r = 1, 2$ is suppressed. In all the fermionic bilinears the north-west south-east contraction rule is understood, and $\epsilon_{rs} \epsilon^{rt} = \delta^t_s$. With the $Sp(1)$ indices suppressed, we have the symmetry property

$$\bar{\psi}_1 \Gamma^{A_1 \cdots A_n} \psi_2 = (-1)^n \bar{\psi}_2 \Gamma^{A_n \cdots A_1} \psi_1,$$

(A.3)

for any two anticommuting symplectic Majorana spinors $\psi_1$ and $\psi_2$.

B Identities for Curvature with Torsion

We begin by noting the usual vielbein postulate

$$\partial_{\mu} e^a_{\nu} + \omega_{\mu ab}(e) e_{\nu b} - \frac{\rho}{\mu \nu} e^a_{\rho} = 0,$$

(B.1)

from which one solves for the spin connection

$$\omega_{\mu ab}(e) = \left( e^\nu_a \partial_{[\mu} e_{\nu] b} + \frac{1}{2} e^\rho_a e^\nu_b e^\nu_c \partial_{\nu} e_{\rho c} - a \leftrightarrow b \right)$$

(B.2)

We also require the condition $D_{\mu} (\Gamma, \omega_+) e^a_{\nu} = 0$, where $\omega_{\pm \mu}^{ab}$ is given by (2.4). This condition explicitly takes the form

$$\partial_{\mu} e^a_{\nu} + \omega_{\mu}^{ab} e_{\nu b} - \Gamma^{\lambda}_{\mu \nu} e^a_{\lambda} = 0.$$  

(B.3)

Taking the antisymmetric part and using (B.1) we learn that

$$\Gamma^{\lambda}_{[\nu \mu]} = S_{\mu \nu}^{\lambda} + \hat{H}_{\mu \nu}^{\lambda},$$

(B.4)

$$S_{\mu \nu}^{\lambda} = -\frac{1}{2} \bar{\psi}_1 \Gamma^{\lambda} \psi_2.$$  

(B.5)

The antisymmetric part of $\Gamma^{\lambda}_{\mu \nu}$ has the interpretation of torsion. In addition to the usual fermionic bilinear contribution to torsion, here the supercovariant self-dual field strength $\hat{H}_{\mu \nu \rho}^{+}$ also appears as a contribution to torsion.
Subtracting (B.1) from (B.3), we also learn that
\[ \Gamma^\lambda_{\mu\nu} = \left\{ \lambda \atop \mu\nu \right\} - K_{\mu\nu}^\lambda, \tag{B.6} \]
\[ K_{\mu\nu}^\lambda = \kappa_{\mu\nu}^\lambda + \hat{H}_{\mu\nu}^\lambda. \tag{B.7} \]
Note that \( K_{\lambda(\mu, \nu)} = 0 \), where \( K_{\mu\nu, \rho} \equiv K_{\mu\nu}^\lambda g_{\lambda\rho} \). Observe also that we can write (B.6) as
\[ \Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} - \hat{H}_{\mu\nu}^\lambda, \tag{B.8} \]
\[ \hat{\Gamma}^\lambda_{\mu\nu} \equiv \left\{ \lambda \atop \mu\nu \right\} - \kappa_{\mu\nu}^\lambda. \tag{B.9} \]
Using these equations, and recalling the definition of \( \omega_{\pm \mu}^{ab} \) given in (2.4), we find that the vielbein postulate (B.3) can also be written as
\[ \partial_{\mu} e_a^\nu + \hat{\omega}_{\mu}^{ab} e_b^\nu - \hat{\Gamma}_{\mu\nu}^\lambda e_a^\lambda = 0. \tag{B.10} \]
Next, we define the Riemann tensors
\[ R^\sigma_{\rho, \mu\nu}(\Gamma) = \left( \partial_{\mu} \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \mu \leftrightarrow \nu \right), \tag{B.11} \]
\[ R_{\mu\nu}^{ab}(\omega) = \left( \partial_{\mu} \omega_{\nu}^{ab} + \omega_{\mu}^{ac} \omega_{\nu}^{cb} - \mu \leftrightarrow \nu \right). \tag{B.12} \]
Using the vierbein postulates (B.3) and (B.10), we find that
\[ R^\sigma_{\rho, \mu\nu}(\Gamma) = R_{\mu\nu}^{ab}(\hat{\omega}) e_a^\sigma e_b^\rho, \tag{B.13} \]
\[ R^\sigma_{\rho, \mu\nu}(\hat{\Gamma}) = R_{\mu\nu}^{ab}(\hat{\omega}) e_a^\sigma e_b^\rho. \tag{B.14} \]
Next, we define \( R_{\rho\sigma, \mu\nu}(\Gamma) = g_{\rho\lambda} R_{\lambda\sigma, \mu\nu}(\Gamma) \) and recalling (B.6) we find:
\[ R_{\rho\sigma, \mu\nu}(\Gamma) = R_{\rho\sigma, \mu\nu}(\hat{\Gamma}) + 2K_{[\mu\nu]}^\lambda K_{\lambda\sigma, \rho} \]
\[ + \left( K_{\mu\nu}^\lambda K_{\lambda\sigma, \rho} + D_{\mu}(\Gamma) K_{\nu\sigma, \lambda, \rho - \mu \leftrightarrow \nu} \right). \tag{B.15} \]
It follows that \( R_{(\rho\sigma), \mu\nu}(\Gamma) = 0 \). The following form of the above equation is also useful:
\[ R_{\rho\sigma, \mu\nu}(\Gamma) = R_{\rho\sigma, \mu\nu}(\hat{\Gamma}) + 2D_{[\mu}(\Gamma) \hat{H}_{\nu] \rho\sigma}^+ - \hat{H}_{\mu\nu}^+ \hat{\Gamma}_{\rho\sigma \lambda}^+ \]. \tag{B.16} \]
In obtaining this result, we have used the identity
\[ \hat{H}_{[\mu\nu}^\lambda \hat{\Gamma}_{\rho\sigma \lambda}^+ = 0. \tag{B.17} \]
Next, we define the Ricci tensor as
\[ R_{\mu\nu}(\Gamma) \equiv g^{\rho\sigma} R_{\mu\rho, \nu\sigma}(\Gamma). \tag{B.18} \]
By direct computation we find from (B.15) that
\[ R_{[\mu\nu]}(\Gamma) = D_{\lambda}(\Gamma) \hat{H}_{\mu\nu}^+ \lambda + \frac{1}{2} \bar{\psi}_a^\lambda \Gamma^\lambda_{\mu\nu} + 2 \bar{\psi}^a \Gamma_{[\mu} \psi_{\nu]} \hat{H}_{\mu\nu}^+ \]. \tag{B.19} \]
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