COMPLEXITY OF MODULES OVER CLASSICAL LIE SUPERALGEBRAS

HOUSSEIN EL TURKEY

Abstract. The complexity of the simple and the Kac modules over the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ of type $A$ was computed by Boe, Kujawa, and Nakano in [2]. A natural continuation to their work is computing the complexity of the same family of modules over the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2|2n)$ of type $C$. The two Lie superalgebras are both of Type I which will result in similar computations. In fact, our geometric interpretation of the complexity agrees with theirs. We also compute a categorical invariant, $z$-complexity, introduced in [2], and we interpret this invariant geometrically in terms of a specific detecting subsuperalgebra. In addition, we compute the complexity and the $z$-complexity of the simple modules over the Type II Lie superalgebras $\mathfrak{osp}(3|2), D(2, 1; \alpha), G(3)$, and $F(4)$.

1. Introduction

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a classical Lie superalgebra (hence $\mathfrak{g}_0$ is a reductive Lie algebra) over the complex numbers, $\mathbb{C}$. Let $\mathcal{F} := \mathcal{F}_{(\mathfrak{g}_0 \oplus \mathfrak{g}_0)}$ be the category of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over $\mathfrak{g}_0$. The authors in [3] showed that $\mathcal{F}$ has enough projectives and it satisfies: (i) it is a self-injective category and (ii) every module in this category admits a projective resolution which has a polynomial rate of growth. For a module $M \in \mathcal{F}$, the complexity $c_{\mathcal{F}}(M)$ is the rate of growth of the minimal projective resolution of $M$.

In this paper we compute the complexity of the simple and the Kac modules for the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2n)$. Let $K(\lambda)$ (resp. $L(\lambda)$) be the Kac (resp. simple) module of highest weight $\lambda$. Let $\text{atyp}(\lambda)$ denote the atypicality of $\lambda$ (see Subsection 2.2). For $\mathfrak{osp}(2|2n)$, atyp($\lambda$) is either zero or one. For typical $\lambda$ (i.e. atyp($\lambda$) = 0), the simple and the Kac modules are projective and hence they have a zero complexity. For atypical $\lambda$ (i.e. atyp($\lambda$) = 1), the complexity is computed in Theorems 3.2.1 and 3.5.1

\[ c_{\mathcal{F}}(L(\lambda)) = 2n + 1, \quad c_{\mathcal{F}}(K(\lambda)) = 2n. \]

These computations can be interpreted geometrically as follows. For a module $M$, let $\mathcal{X}_M$ denotes the associated variety defined by Duflo and Serganova [7], and $\mathcal{V}_{(\mathfrak{g}_0 \oplus \mathfrak{g}_0)}(M)$ the support variety as defined in [5]. Then, if $X(\lambda)$ is a Kac or a simple module, we have the geometric interpretation of the complexity in Theorem 4.2.2

\[ c_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{X}_{X(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g}_0 \oplus \mathfrak{g}_0)}(X(\lambda)). \quad (1.0.1) \]

The authors in [2] introduced a categorical invariant called the $z$-complexity of modules and denoted it by $z_{\mathcal{F}}(-)$ (see [2, Section 9]). They computed the $z$-complexity of the simple
and the Kac modules over \( \mathfrak{gl}(m|n) \) and then used a detecting subsuperalgebra \( \mathfrak{f} \) to interpret their computations geometrically. We carry these computations over \( \mathfrak{osp}(2|2n) \) and conclude in Theorems 5.1.1 and 5.2.1 that for an atypical \( \lambda \), we have
\[
 z_F(L(\lambda)) = 2, \quad z_F(K(\lambda)) = 1. \tag{1.0.2}
\]
Moreover, we show in Theorem 5.3.1 that if \( X(\lambda) \) is a Kac or a simple module, we have
\[
z_F(X(\lambda)) = \dim V_{(f,f_0)}(X(\lambda)). \tag{1.0.3}
\]

The fact that our geometric interpretations of the complexity and the \( z \)-complexity agree with the results obtained in [2] was expected since both types \( A \) and \( C \) are Type I Lie superalgebras (Subsection 3.1). It was interesting to know if these interpretations would hold over Type II Lie superalgebras, hence we computed the complexity and the \( z \)-complexity of the simple (finite-dimensional) modules over \( \mathfrak{osp}(3|2) \), and the three exceptional Lie superalgebras \( D(2,1;\alpha) \), \( G(3) \), and \( F(4) \). Our results show that equations (1.0.1) and (1.0.3) hold for the simple modules over these Lie superalgebras. The results in this paper raises the question whether these geometric interpretations will hold over other classical Lie superalgebras, in particular types \( B \) and \( D \).

The paper is organized as follows. In Section 2, we introduce the preliminaries for classical Lie superalgebras and their representations. We define the atypicality, complexity, and \( z \)-complexity of modules. In Section 3, we compute the complexity and the \( z \)-complexity of simple and Kac modules over \( \mathfrak{osp}(2|2n) \). We construct an explicit minimal projective resolution of the trivial module and use the fact that simple modules of the same atypicality have the same complexity ([14, Theorem 4.1.1]). We then use the equivalence of blocks defined in [9] to compute the complexity of the Kac modules. In Section 4, we relate the support and associated variety of these modules to complexity as mentioned earlier. At the end of Section 5, we show that \( z \)-complexity will be the same as the dimension of the support variety of a detecting subalgebra, \( \mathfrak{f} \), of \( \mathfrak{osp}(2|2n) \). In Section 6 we compute the complexity and the \( z \)-complexity of the simple modules over \( \mathfrak{osp}(3|2) \), \( D(2,1;\alpha) \), \( G(3) \), and \( F(4) \). We also show that the same geometric interpretation holds in these cases.

I would like to acknowledge my Ph.D advisor, Jonathan Kujawa, for many insightful discussions. His comments were very helpful throughout this work. I would also like to thank him for proof-reading this paper.

2. Preliminaries

2.1. Lie superalgebras and representations. We will use the notations and conventions developed in [2, 3, 4, 5]. We will work over the complex numbers \( \mathbb{C} \) throughout this paper.

A superspace \( V \) is a \( \mathbb{Z}_2 \)-graded vector space (i.e. \( V = V_0 \oplus V_1 \)). Given a homogeneous vector \( v \in V \), we write \( \bar{v} \in \mathbb{Z}_2 \) for the parity (or degree) of \( v \). Elements of \( V_0 \) (resp. \( V_1 \)) are called even (resp. odd). Note that if \( M \) and \( M' \) are two superspaces, then the space \( \text{Hom}_{\mathbb{C}}(M,M') \) is naturally \( \mathbb{Z}_2 \)-graded by declaring \( f \in \text{Hom}_{\mathbb{C}}(M,M') \), \( (r \in \mathbb{Z}_2) \) if \( f(M_s) \subseteq M'_{r+s} \) for all \( s \in \mathbb{Z}_2 \).

Let \( \mathfrak{g} \) be a Lie superalgebra; that is, a \( \mathbb{Z}_2 \)-graded vector space \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) with a bracket operation \([,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\) which preserves the \( \mathbb{Z}_2 \)-grading and satisfies graded versions of the usual Lie bracket axioms. The subspace \( \mathfrak{g}_0 \) is a Lie algebra under the bracket and
g_1 is a g_0-module. A finite dimensional Lie superalgebra g is called classical if there is a
corresponding reductive algebraic group G_0 such that \text{Lie}(G_0) = g_0, and an action of G_0 on g_0
which differentiates to the adjoint action of g_0 on g_1. In particular, if g is classical, then
\text{ad}(g) is a reductive Lie algebra and g_1 is semisimple as a g_0-module. A basic classical Lie
superalgebra is a classical Lie superalgebra with a nondegenerate invariant supersymmetric
even bilinear form. The Lie superalgebras considered in this paper are basic classical Lie
superalgebras.

Let U(g) be the universal enveloping superalgebra. Let us describe the category of
g-superalgebras. The objects are all left U(g)-modules which are \mathbb{Z}_2-graded; that is, super-
spaces \mathcal{M} = M_0 \oplus M_1 satisfying U(g), M_s \subseteq M_{r+s} for all r, s \in \mathbb{Z}_2. If M is a g-superalgebra,
then \mathcal{N} \subseteq \mathcal{M} is a subsupermodule if it is a supermodule which inherits its grading from \mathcal{M}
in the sense that \mathcal{M}_r \cap \mathcal{N} = \mathcal{N}_r for r \in \mathbb{Z}_2. Given g-superalgebras \mathcal{M} and \mathcal{N}
one can define a g-superalgebra structure on the contragradient dual \mathcal{M}^* and the tensor product \mathcal{M} \otimes \mathcal{N}
using the antipode and coproduct of U(g). A morphism of U(g)-superalgebras is an element
of Hom_C(\mathcal{M}, \mathcal{M}^*) satisfying \text{f}(xm) = (-1)^{\beta(z)f}(m) for all m \in \mathcal{M} and x \in U(g). In this
definition, f and x are assumed to be homogeneous. The general case can be obtained by
linearity.

We write \mathcal{F} = \mathcal{F}(g, g_0) for the full subcategory of all finite dimensional g-superalgebras
which are completely reducible over g_0. As only superalgebras will be considered in this
paper, we will from now on use the term “module” with the understanding that the prefix
“super” is implicit.

2.2. Atypicality. Let h be a Cartan subalgebra of g and let h^* be the dual Cartan. Then
h^* is equipped with a bilinear form (\cdot, \cdot). We then have a set of roots \Phi with a corresponding
root decomposition of g. Assuming a choice of the Borel b has been made, the root vectors
that lie in b will be called positive root vectors and their corresponding roots \mathcal{L} \in \Phi will
be called the positive roots. The set of positive roots will be denoted by \Phi^+. The negative
roots are \Phi^- := \Phi \setminus \Phi^+. The set of simple roots will be denoted by \Delta.

Let \rho be half the sum of the positive even roots minus half the sum of the positive
odd roots. For \mathcal{L} \in h^*, Define the atypicality of \mathcal{L} to be the maximal number of pairwise
orthogonal positive isotropic roots which are also orthogonal to \mathcal{L} + \rho with respect to
the bilinear form on h^*. We will write atyp(\mathcal{L}) for the atypicality of \mathcal{L}. A weight is called
atypical if its atypicality is not zero, and is called typical otherwise. In the Lie superalgebras
considered in this paper, the atypicality is either zero or one.

If \mathcal{L}(\mathcal{L}) is a simple g-module of highest weight \mathcal{L}, then we define atyp(\mathcal{L}(\mathcal{L})) := atyp(\mathcal{L}).
It is known that the atypicality of a simple module is independent of the choice of Cartan
and Borel subalgebras and, furthermore, is the same for all simple modules in a given block.
Hence it makes sense to refer to the atypicality of a block. The Kac modules over osp(2|2n)
will be defined in Subsection 3.1. They are finite-dimensional in this case and are indexed
by highest weights \mathcal{L}. They will be denoted by K(\mathcal{L}). Let P(\mathcal{L}) be the projective cover of
\mathcal{L}(\mathcal{L}). By [12] Theorem 1 we know that if atyp(\mathcal{L}) = 0, then P(\mathcal{L}) = \mathcal{L}(\mathcal{L}) = K(\mathcal{L}) hence
\mathcal{L}(\mathcal{L}) and K(\mathcal{L}) are projective.

2.3. Complexity. Let \{V_t | t \in \mathbb{N}\} = \{V_\bullet\} be a sequence of finite dimensional \mathbb{C}-vector
spaces. The rate of growth of \text{r}(V_\bullet), is the smallest nonnegative integer c such that
there exists a constant $C > 0$ with $\dim V_t \leq Ct^{c-1}$ for all $t$. If no such integer exists then $V_\bullet$ is said to have infinite rate of growth.

Let $M \in \mathcal{F}$ and $P_\bullet \to M$ be a minimal projective resolution. Define the complexity of $M$ to be $c_\mathcal{F}(M) := r(P_\bullet)$. As shown in [3, Theorem 2.5.1] the complexity is always finite, in particular if $M$ is an object of $\mathcal{F}$, then $c_\mathcal{F}(M) \leq \dim \mathfrak{g}_1$. Moreover, [3, Proposition 2.8.1] provides a characterization of the complexity via rates of growth of extension groups in $\mathcal{F}$:

$$c_\mathcal{F}(M) = r \left( \operatorname{Ext}_{(\mathfrak{g}, \mathfrak{g}_0)}(M, \bigoplus \dim P(S)) \right),$$

where the sum is over all the simple modules in $\mathcal{F}$, and $P(S)$ is the projective cover of $S$. Here and elsewhere, we write $\operatorname{Ext}_{(\mathfrak{g}, \mathfrak{g}_0)}(M, N)$ for the relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_0)$ as introduced in [4, Section 2.3]. In some sense, the complexity of a module measures how far the module is from being projective. For example, by [3, Corollary 2.7.1], $c_\mathcal{F}(M) = 0$ if and only if $M$ is projective.

**Proposition 2.3.1.** If $\lambda$ is typical (i.e. $\operatorname{atyp}(\lambda) = 0$), then $c_\mathcal{F}(L(\lambda)) = c_\mathcal{F}(K(\lambda)) = 0$.

The complexity will be interpreted geometrically using support and associated varieties. The definition of support varieties can be found in [4, Section 6]. For the definition of the associated variety, we refer the reader to [7, Section 2].

2.4. **Support variety.** Let $R = H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ be the cohomology ring of $\mathfrak{g}$ and let $M \in \mathcal{F}$. According to [4, Theorem 2.7], $\operatorname{Ext}_\mathcal{F}(M, M)$ is a finitely generated $R$-module. Set

$$J := \operatorname{Ann}_R(\operatorname{Ext}_\mathcal{F}(M, M)).$$

The *support variety* of the $M$ is defined by

$$\mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M) := \operatorname{MaxSpec}(R/J).$$

2.5. **Associated variety.** Let $\mathcal{X} = \{ x \in \mathfrak{g}_1 \mid [x, x] = 0 \}$. If $M \in \mathcal{F}$, then Duflo and Serganova [7] define an associated variety of $M$ which is equivalent to:

$$\mathcal{X}_M = \{ x \in \mathcal{X} \mid M \text{ is not projective as a } U(\langle x \rangle)-\text{module} \} \cup \{0\},$$

where $U(\langle x \rangle)$ denotes the enveloping algebra of the Lie superalgebra generated by $x$.

2.6. **$z$-complexity.** Let $M$ be a module in $\mathcal{F}$. The $z$-complexity of $M$ is

$$z_\mathcal{F}(M) := r \left( \operatorname{Ext}_{(\mathfrak{g}, \mathfrak{g}_0)}(M, \bigoplus S) \right),$$

where the direct sum runs over all simple modules of $\mathcal{F}$. Unlike complexity, $z_\mathcal{F}(\cdot)$ has the advantage of being invariant under category equivalences.

If $P_\bullet \to M$ is a minimal projective resolution of $M$, define $s(P_\bullet)$ to be the rate of growth of the number of indecomposable summands at each step in the resolution. We can easily show that $z_\mathcal{F}(M) = s(P_\bullet)$. 

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3. Computing the complexity for $\mathfrak{osp}(2|2n)$

3.1. The Lie superalgebra $\mathfrak{osp}(2|2n)$. Consider the matrix realization of $\mathfrak{osp}(2|2n)$ given in [11]:

$$\mathfrak{g} = \begin{bmatrix} \alpha & 0 & x & x_1 \\ 0 & -\alpha & y & y_1 \\ y_1^t & x_1^t & a & b \\ -y^t & -x^t & c & -a^t \end{bmatrix},$$

where $x, x_1, y, y_1$ are $1 \times n$ matrices; $a, b, c$ are $n \times n$ matrices with $b$ and $c$ being symmetric; $\alpha$ is a scalar in $\mathbb{C}$. The diagonal blocks form the even part $\mathfrak{g}_0 \cong \mathbb{C} \oplus \mathfrak{osp}(2n)$ while the anti-diagonal blocks form the odd part $\mathfrak{g}_1$. The Lie super-bracket is defined by

$$[A, B] = AB - (-1)^{\bar{A}\bar{B}}BA,$$

for homogeneous elements $A, B \in \mathfrak{g}_0$ or $\mathfrak{g}_1$. We then extend the definition of the bracket to all of $\mathfrak{g}$ by bilinearity.

The Cartan subalgebra $\mathfrak{h}$ is chosen to be the set of diagonal matrices in $\mathfrak{g}$. Let $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ be the linear map that takes an element of $\mathfrak{h}$ to its first diagonal entry. For $1 \leq i \leq n$, let $\delta_i : \mathfrak{h} \to \mathbb{C}$ be the linear map that takes an element of $\mathfrak{h}$ to the $i$th diagonal entry of the matrix in the second diagonal block. The set \{($\varepsilon_1, \delta_1, \delta_2, \ldots, \delta_n$)} forms a basis of $\mathfrak{h}^*$ which is endowed with a nondegenerate symmetric bilinear form $(, )$ given by

$$(\varepsilon_i, \varepsilon_j) = 1, \quad (\delta_i, \delta_j) = -\delta_{ij}, \quad (\varepsilon_i, \delta_i) = 0,$$

for all $1 \leq i, j \leq n$. The set of simple roots is

$$\Delta = \{\delta_i - \delta_{i+1} \mid 1 \leq i \leq n\} \cup \{2\delta_n, \varepsilon_1 - \delta_1\}.$$ 

The even roots of $\mathfrak{g}$ are:

$$\Phi_0 = \{\pm \delta_i \pm \delta_j \mid 1 \leq i \neq j \leq n\} \cup \{\pm 2\delta_i \mid 1 \leq i \leq n\},$$

while the odd roots are

$$\Phi_1 = \{\pm \varepsilon_1 \pm \delta_i \mid 1 \leq i \leq n\}.$$ 

The positive roots of $\mathfrak{g}$ are

$$\Phi^+ = \{\delta_i + \delta_j \mid 1 \leq i \leq j \leq n\} \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_1 \pm \delta_i \mid 1 \leq i \leq n\}.$$ 

Note that $\mathfrak{g}$ has a $\mathbb{Z}$-grading given by $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the subalgebra spanned by the even root vectors and $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$) is the subalgebra spanned by the odd positive (resp. negative) root vectors. Thus $\mathfrak{g}$ is a Type I Lie superalgebra. Note that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are abelian, $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, and $\mathfrak{g}_0 = \mathfrak{g}_0$. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ spanned by the positive root vectors, then we have $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{b}_0$ is spanned by the even positive roots.

The simple modules over $\mathfrak{g} = \mathfrak{osp}(2|2n)$ can be constructed as follows. Let $X_0^+ \subseteq \mathfrak{h}^*$ be the parameterizing set of highest weights for the simple finite dimensional $\mathfrak{g}_0$-modules with respect to the pair $\langle (\mathfrak{h}, \mathfrak{b}_0) \rangle$. An explicit description of $X_0^+$ is

$$X_0^+ = \{\lambda = \lambda_{-1}\varepsilon_1 + \sum_{i=1}^n \lambda_i\delta_i \mid \lambda_{-1} \in \mathbb{C}, \lambda_i \in \mathbb{Z}, \forall i \geq 1; \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0\}.$$
For $\lambda \in X_0^+$, let $L_0(\lambda)$ be the simple finite dimensional $\mathfrak{g}_0$-module of highest weight $\lambda$. Since $\mathfrak{osp}(2|2n)_0 \cong \mathbb{C} \oplus \mathfrak{sp}(2n)$, the simple $\mathfrak{g}_0$-modules are of the form
\[
L_0(\lambda) = \mathbb{C}_{\lambda-1} \otimes L_0(\lambda),
\]
where $L_0(\lambda)$ is the simple $\mathfrak{sp}(2n)$-module of weight $\sum_{i=1}^{n} \lambda_i \delta_i$. Note that there is a slight abuse of notation where we used $\lambda$ for the simple $\mathfrak{sp}(2n)$-module even though we removed the $\lambda_{-1}$-part.

To define the Kac modules, set $\mathfrak{p}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Since $\mathfrak{g}$ is a Type I Lie superalgebra, $\mathfrak{g}_1$ is an abelian ideal of $\mathfrak{p}^+$. We can then view $L_0(\lambda)$ as a simple finite dimensional $\mathfrak{p}^+$-module via inflation through the canonical quotient $\mathfrak{p}^+ \to \mathfrak{g}_0$. For $\lambda \in X_0^+$, the Kac module $K(\lambda)$ is defined by:
\[
K(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} L_0(\lambda).
\]

The Kac module $K(\lambda)$ has a unique maximal submodule. The head of $K(\lambda)$ is the simple finite dimensional $\mathfrak{g}$-module $\lambda(\lambda)$. The set $\{\lambda(\lambda) \mid \lambda \in X_0^+\}$ is a complete set of non-isomorphic simple modules in $\mathcal{F} = \mathcal{F}(\mathfrak{g}, \mathfrak{g}_0)$.

**Remark 3.1.1.** In $\mathfrak{osp}(2|2n)$, we have
\[
\rho = -n\varepsilon_1 + \sum_{i=1}^{n} (n - i + 1)\delta_i.
\]
Using $\rho$ we can show that for any $\lambda \in X_0^+$, atyp$(\lambda)$ is either zero or one. Note that if $\lambda = \lambda_{-1}\varepsilon_1 + \sum_{i=1}^{n} \lambda_i \delta_i = (\lambda_{-1}|\lambda_1, \ldots, \lambda_n)$ is atypical, then $\lambda_{-1} \in \mathbb{Z}$.

### 3.2. Complexity of simple modules for $\mathfrak{g} = \mathfrak{osp}(2|2n)$

For $\lambda \in X_0^+$, we want to find the complexity of $\lambda(\lambda)$. For typical $\lambda$, the simple module $\lambda(\lambda)$ is projective and hence has zero complexity. We only need to consider the case when $\lambda$ is atypical. In this section, we will refer to [6] Sections 1.2, 3.1] to get a description of the projective covers.

Let $W$ be the Weyl group of $\mathfrak{g}$ which, by definition, is generated by the reflections corresponding to the even simple roots of $\mathfrak{g}$. If $\lambda \in X_0^+$ is atypical with respect to the odd positive root $\gamma$, the authors in [6] defined an “$L$-operator” given by:
\[
\lambda^L := \omega(\lambda + \rho - k\gamma) - \rho,
\]
where $k$ is the smallest positive integer such that $(\lambda + \rho - k\gamma, \alpha_i) \neq 0$ for all the even simple roots $\alpha_i$ and $\omega$ is the unique element in the Weyl group of $\mathfrak{sp}(2n)$ rendering $\lambda^L$ dominant. Given an atypical $\lambda \in X_0^+$, we shall write
\[
\lambda^{(0)} = \lambda, \quad \lambda^{(l+1)} = (\lambda^{(l)})^L, \quad l \geq 0. \tag{3.2.1}
\]

Let $\omega_0$ be the longest element in the Weyl group of $\mathfrak{sp}(2n)$ and let $\beta = 2n\varepsilon_1$ be the sum of all positive roots. We can use [6] equation (4)] to get:
\[
\lambda = \omega_0^{-1}(\beta - (\beta - \omega_0\lambda^L)^L),
\]
which proves that the $L$-operator is one-to-one. We will prove that $L$ is onto on the set of atypical weights in $X_0^+$ using a representation theoretical approach (Proposition 1.2.2).
Thus there is no indecomposable module $L$. This shows that the Kac module has two layers: the head of $K$ and the socle of $P$.

We have:

There is a bijection $X^n_0 \rightarrow Y_+^{1|n}$ given by $\lambda \mapsto f_\lambda$, where

$(f_\lambda)_1 = (\lambda + \rho, \varepsilon_1), \quad (f_\lambda)_i = (\lambda + \rho, \delta_i), \ i \geq 1.$

Since $\lambda$ is atypical, then $(f_\lambda)_-1 \in \mathbb{Z}$ and $|f_-| = -f_i$ for some $i \geq 1$. Set $f^L := f_{\lambda^L}$. We can compute $\lambda^L$ for each atypical $\lambda$ by passing over to the set $Y_+^{1|n}$. Using the description of $f^L$ given in [6, Section 1.2], the following computations can be performed.

**Lemma 3.2.1.** We have:

1. For $d \geq 0$, $(-d|d, 0, \ldots, 0)^L = (-d-1)d+1, 0, \ldots, 0)$.
2. For $d \geq 1$, $(2n + d|d, 0, \ldots, 0)^L = (2n + (d-1)d-1, 0, \ldots, 0)$.
3. $(2n|0, \ldots, 0)^L = (0|0, \ldots, 0)$.
4. For $\lambda = (0|0, \ldots, 0)$ we have $\lambda^{(d)} = (-d|d, 0, \ldots, 0)$ and $\lambda^{(d-1)} = (2n + d|d, 0, \ldots, 0)$.

Let $P(\lambda)$ be the projective cover of $L(\lambda)$. From [6, Theorem 7], we have the following 2-step Kac flag:

$$0 \rightarrow K(\lambda) \rightarrow P(\lambda^L) \rightarrow K(\lambda^L) \rightarrow 0,$$

which implies that $\dim P(\lambda^L) = \dim K(\lambda) + \dim K(\lambda^L)$. Moreover, using [6, Corollary 8], there is a short exact sequence:

$$0 \rightarrow L(\lambda^L) \rightarrow K(\lambda) \rightarrow L(\lambda) \rightarrow 0.$$

This shows that the Kac module has two layers: the head of $K(\lambda)$ is $L(\lambda)$ and the socle is $L(\lambda^L)$. Note that by the Kac filtration of $P(\lambda^L)$ and the composition factors of $K(\lambda)$ we know that the socle of $P(\lambda^L)$ has only one submodule, namely $L(\lambda^L)$. Similarly the head of $P(\lambda^L)$ is the head of $K(\lambda^L)$ which is $L(\lambda^L)$. Using the description of Kazhdan-Lusztig polynomials in [6, Theorem 5, Remark 1], we can show that

$$\dim \text{Ext}^1(L(\lambda), L(\mu)) = 1 \iff \lambda = \mu^L \text{ or } \lambda^L = \mu.$$

Thus there is no indecomposable module $M$ such that the following sequence is exact:

$$0 \rightarrow L(\lambda) \rightarrow M \rightarrow L(\lambda^LL) \rightarrow 0.$$

Therefore, the projective module $P(\lambda^L)$ has the following layer structure:

```
\[ L(\lambda^L) \]
\[ \downarrow \]
\[ L(\lambda) \]
\[ \downarrow \]
\[ L(\lambda LL) \]
\[ \downarrow \]
\[ L(\lambda L) \]
\[ \downarrow \]
\[ L(\lambda) \]
\[ \downarrow \]
\[ \text{GL}_n \]
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To compute the complexity, we need the following bounds on the dimension of the simple \( \mathfrak{sp}(2n) \)-module \( L_0(r, 0, \ldots, 0) \):

**Lemma 3.2.2.** There are positive constants \( C \) and \( C' \) that depend only on \( n \) such that

\[
C r^{2n-1} \leq \dim L_0(r, 0, \ldots, 0) \leq C' r^{2n-1}.
\]

**Proof.** Let \( \delta \) be half the sum of the positive roots in \( \mathfrak{sp}(2n) \), then \( \delta = \sum_{i=1}^{n} (n - i + 1) \delta_i \). By the Weyl-dimension formula ([10, Section 24.3]) we have

\[
\dim L_0(r, 0, \ldots, 0) = \frac{(2n + r) \prod_{j=2}^{n} (r + j - 1)(2n + r - j + 1)}{(2n) \prod_{j=2}^{n} (j - 1)(2n - j + 2)}.
\]

Let

\[
C = \frac{1}{(2n) \prod_{j=2}^{n} (j - 1)(2n - j + 2)}.
\]

Then \( C \) is a positive constant depending only on \( n \) and \( \dim L_0(r, 0, \ldots, 0) \) is a polynomial in \( r \) of degree \( 2n - 1 \) with a positive leading coefficient. Moreover,

\[
\dim L_0(r, 0, \ldots, 0) = C(2n + 2r) \prod_{j=2}^{n} (r + j - 1)(2n + r - j + 1)
\]

\[
\geq C(2n + 2r)(r + 1)^{n-1}(n + r + 1)^{n-1}
\]

\[
\geq C r^{2n-1}.
\]

On the other hand,

\[
\dim L_0(r, 0, \ldots, 0) = C(2n + 2r) \prod_{j=2}^{n} (r + j - 1)(2n + r - j + 1)
\]

\[
\leq C(2n + 2r)(r + n - 1)^{n-1}(2n + r - 1)^{n-1}.
\]

Let us pick positive constants \( C_2, C_3, C_4 \) depending only on \( n \) such that

\[
2n + 2r \leq C_2 r, \quad r + n - 1 \leq C_3 r, \quad 2n + r - 1 \leq C_4 r,
\]

then

\[
\dim L_0(r, 0, \ldots, 0) \leq C C_2 C_3^{n-1} C_4^{n-1} r^{2n-1} = C' r^{2n-1},
\]

where \( C' \) is a positive constant that depends only on \( n \). \( \Box \)

**Theorem 3.2.1.** For atypical \( \lambda \in X_0^+ \), \( c_F(L(\lambda)) = 2n + 1 \).

**Proof.** First, we find the complexity of the trivial module \( \mathbb{C} = L(0|0, \ldots, 0) \). For \( \lambda \in X_0^+ \), the definition of \( \lambda^{(i)} \) was given in equation [3.2.1]. For \( 0 = (0|0, \ldots, 0) \) and \( i \in \mathbb{Z} \) we have by Lemma 3.2.1

\[
0^{(i)} = (-i|i, 0, \ldots, 0) \quad \text{and} \quad 0^{(-i-1)} = (2n + i|i, 0, \ldots, 0).
\]

For \( i \in \mathbb{Z} \), set

\[
[i] = L(0^{(i)}), \quad P^{(i)} = P(0^{(i)}), \quad \text{and} \quad K^{(i)} = K(0^{(i)}).
\]
Using these notations, the projective cover $P^{(i)}$ has the following radical layer structure:

```
  [i]
 /   /
[i-1] [i+1]
 /   /
[i]
```

The minimal projective resolution of $L(0)$ is

$$\ldots P_d \xrightarrow{f_d} \ldots \xrightarrow{f_1} P(0) \xrightarrow{f_0} L(0) \to 0,$$

(3.2.2)

where the $d^{th}$ term in this resolution is given as follows. If $d$ is even,

$$P_d = P(-d) \oplus P(-d+2) \oplus \ldots \oplus P(-2) \oplus P(0) \oplus P(d) \oplus P(d-2) \oplus \ldots \oplus P(2),$$

and if $d$ is odd,

$$P_d = P(-d) \oplus P(-d+2) \oplus \ldots \oplus P(-1) \oplus P(d) \oplus P(d-2) \oplus \ldots \oplus P(1).$$

The kernel of $f_d$ is

$$\text{Ker}(f_d) = [-d-1] \oplus [-d+1] \oplus \ldots \oplus [-1] \oplus [1] \oplus [d-1] \oplus [d+1] \oplus [d].$$

An inductive argument proves the result using the diagrammatic method for modular representations given in [1]. In particular, we use the description given in [1, Section 10.3] for the kernel of the surjective map $P(M) \to M$ where $P(M)$ is the projective cover of $M \in F$.

We have $\dim P(\lambda^Z) = \dim K(\lambda^Z) + \dim K(\lambda)$. Then, for $i \in \mathbb{Z}$,

$$\dim P^{(i)} = \dim K^{(i)} + \dim K^{(i-1)}.$$  

(3.2.3)

By the PBW basis of $U(\mathfrak{g})$, we have

$$\dim K(\lambda) = 2^{\dim \mathfrak{g} - 1}. \dim L_0(\lambda),$$

(3.2.4)

where $L_0(\lambda)$ is the simple $\mathfrak{sp}(2n)$-module of highest weight $\lambda$ (where we omit the $\lambda_{-1}$ from $\lambda$). By equations (3.2.2), (3.2.3), (3.2.4), and by Lemma [3.2.2] we can show that

$$kd^{2n} \leq \dim P_d \leq k'd^{2n} \quad \text{for all } d,$$

(3.2.5)

where $k$, $k'$ are positive constants that depend only on $n$. Therefore,

$$c_F(L(0|0, \ldots, 0)) = 2n + 1.$$

By [14, Theorem 4.1.1], all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple $\mathfrak{osp}(2|2n)$-modules is $2n + 1$. \qed
3.3. Complexity of \( K(0|0,\ldots, 0) \) for \( g = \mathfrak{osp}(2|2n) \) using projective resolutions.

Using the computations done in the above theorem, we compute the complexity of the Kac module \( K(0|0,\ldots, 0) \).

**Proposition 3.3.1.** \( c_F(K(0|0,\ldots, 0)) = 2n \).

**Proof.** Using the same notation as in Theorem 3.2.1 the minimal projective resolution of \( K(0) \) is given by:

\[
\ldots \to P^{(-1)} \to P^{(0)} \to K(0) \to 0. \tag{3.3.1}
\]

By equations (3.2.3), (3.2.4), and by Lemma 3.2.2 we can show that

\[
k d^{2n-1} \leq \dim P_d \leq k' d^{2n-1}
\]

for all \( d \), where \( k, k' \) are positive constants that depend only on \( n \). Therefore,

\[
c_F(K(0|0,\ldots, 0)) = 2n.
\]

\( \square \)

3.4. Complexity of Kac modules. Recall that if \( \lambda \) is typical, then \( K(\lambda) \) is projective and hence has zero complexity. We will use the complexity of \( K(0|0,\ldots, 0) \) to compute the complexity of any atypical Kac module. First we show that that the \( L \)-operator on the weights does not change the complexity of the Kac modules. Then we give an explicit description of the principal block \( F^{\chi_0} \), where \( \chi_0 \) is the central character corresponding to the weight \( \lambda = (0|0,\ldots, 0) \). This description will be obtained using the characterization of blocks and the notion of weight diagrams given in [9, Sections 5, 6]. We then use the fact that translation functors preserve the complexity to show that \( c_F(K(\lambda)) = 2n \).

**Lemma 3.4.1.** For \( \lambda \in X_0^+ \), \( c_F(K(\lambda)) = c_F(K(\lambda^l)), l \in \mathbb{Z} \).

**Proof.** It is sufficient to prove that

\[
c_F(K(\lambda)) = c_F(K(\lambda^L)).
\]

The complexity of any module \( M \in \mathcal{F} \) is given by:

\[
c_F(M) = \text{r}(\text{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{g}_0)}(M, \bigoplus L(\mu)^{\dim P(\mu)})), \tag{3.4.1}
\]

where the sum is over all simple modules in \( \mathcal{F} \). Let \( S = \bigoplus L(\mu)^{\dim P(\mu)} \). By applying the functor Hom( , \( S \)) to the following 2-step Kac flag

\[
0 \to K(\lambda) \to P(\lambda^L) \to K(\lambda^L) \to 0,
\]

we get a long exact sequence in cohomology. Since \( P(\lambda^L) \) is projective, then \( \text{Ext}^d(P(\lambda^L), S) \) vanishes for all \( d \geq 1 \), which gives

\[
\text{Ext}^d(K(\lambda^L), S) = \text{Ext}^{d+1}(K(\lambda), S).
\]

This implies that \( \text{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{g}_0)}(K(\lambda), S) \) and \( \text{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{g}_0)}(K(\lambda^L), S) \) will have the same rate of growth, hence the theorem follows. \( \square \)
3.5. **Weight diagrams and translation functors.** Assume \( \lambda \) is atypical and let

\[
\lambda + \rho = a_1 \varepsilon_1 + b_1 \delta_1 + \ldots + b_n \delta_n,
\]

The weight diagram of \( \lambda \) is the function \( \tilde{f}_\lambda : \mathbb{Z}_{\geq 0} \rightarrow \{ >, <, \times, 0 \} \) represented by a diagram according to the following algorithm:

1. Put the symbol \( > \) in position \( t \) if \( t = |a_1| \).
2. Put the symbol \( < \) in position \( t \) for all \( i \) such that \( t = b_i \).
3. If there are both \( > \) and \( < \) in the same position replace them by the symbol \( \times \), repeat if possible.
4. Put 0 otherwise.

For example, the weight diagram of \( 0 = (0 | 0, \ldots, 0) \) is given by:

\[
0, <, <, \ldots, <, \times, \ldots,
\]

where the first 0 is at position 0, the first < is at position 1, the last < is at position \( n - 1 \), \( \times \) is at position \( n \), the dots after that stand for empty positions (or zeros).

It is important to note the relation with the notation developed in Subsection 3.2 to compute \( \lambda^L \). In fact, \( f_i = (\lambda + \rho, \delta_i) = -b_i \). Define the core of \( \lambda \) to have the same diagram as \( \tilde{f}_\lambda \) but with replacing all \( \times \)'s by zeros. Thus the weight diagrams of \( \lambda \) and its core have \( <, > \) at the same positions. For example, \( \tilde{f}_{\text{core}}(0) \) is given by:

\[
0, <, <, \ldots, <, 0, \ldots.
\]

Using the characterization of the blocks in \( \mathcal{F} \) given in [9, Section 5], we have

**Corollary 3.5.1.** Let \( \chi_\lambda \) be the central character corresponding to the weight \( \lambda \) and \( \mathcal{F}^{\chi_\lambda} \) be the corresponding block. Then \( \lambda \) and \( \lambda^L \), \( l \in \mathbb{Z} \), belong to the same block \( \mathcal{F}^{\chi_\lambda} \).

**Proof.** It is sufficient to prove that \( \lambda \) and \( \lambda^L \) are in the same block. by [9, Section 5], two dominant weights \( \lambda \) and \( \mu \) belong to the same block if and only if they have the same atypicality and the same core. By the definition of the \( L \)-operator, \( \lambda^L \) and \( \lambda \) have the same atypicality. We have to show that \( \lambda \) and \( \lambda^L \) have the same core. We consider the three cases that defined \( f^L \) in [6, Section 1.2] and suppose \( f_{-1} = \pm f_i \) for some \( 1 \leq i \leq n \). In the weight diagram of \( \text{core}(\lambda) \), we will have \( < \) at the positions \( -f_j \), \( j \neq i \). However, by the definition of \( f^L \), we will have \( < \) at the same \( -f_j \), \( j \neq i \) in the weight diagram of \( \text{core}(\lambda^L) \). The corollary follows.

In the following we have an explicit description of the block \( \mathcal{F}^{\chi_0} \):

**Lemma 3.5.1.** \( \mathcal{F}^{\chi_0} = \{ 0(l) \mid l \in \mathbb{Z} \} \).

**Proof.** The proof follows from Corollary 3.5.1 and Lemma 3.2.1.

**Lemma 3.5.2.** Let \( \lambda \in \mathcal{F}^{\chi_\lambda} \) with \( \text{atyp}(\lambda) = 1 \). Then

\[
c_{\mathcal{F}}(K(\lambda)) = c_{\mathcal{F}}(K(\mu)),
\]

for some \( \mu \in \mathcal{F}^{\chi_0} \).

**Proof.** The translation functors defined in [9, Section 5] move a simple module \( L(\lambda) \in \mathcal{F}^{\chi} \) to \( L(\mu) \in \mathcal{F}^{7} \). Let \( \chi_\lambda \) be the central character corresponding to the weight \( \lambda \) and \( \mathcal{F}^{\chi_\lambda} \) be the corresponding block. Let \( \mathcal{F}^{\chi_0} \) be the block containing the trivial module. The translation
functors define an equivalence of blocks between $F^\chi$ and $F^{\chi_0}$. To see this, we note that $core(\lambda)$ has < at $n-1$ positions, the same as $core(0)$. Assume the $\times$ in $f_\lambda$ is at position $i$ for some $i \in \mathbb{N}$, then the translation functors allow us to move the $<$'s to the positions $1, 2, \ldots, n-1$ and move $\times$ to some position $k \geq n$. The diagram we get is $f_\mu$ where $\mu \in F^{\chi_0}$. The same discussion as in [2, Section 6.3] shows that the translation functors preserve the complexity of any $g$-module. This completes the proof. □

4. Support, rank and associated varieties

The dimensions of the support varieties of the simple and the Kac modules in $F$ were computed in [14, Corollary 4.4.2] and [5, Corollary 3.2]. Next we introduce rank varieties and compute their dimensions for the Kac modules. This will be used to give a geometric interpretation of the complexity.

4.1. Rank variety. Let $F(g_{\pm 1})$ be the category of finite dimensional $g_{\pm 1}$-modules. Define the $g_{\pm 1}$ support variety of $M \in F(g_{\pm 1})$ by:

$$V_{g_{\pm 1}}(M) := V_{g_{\pm 1}, 0}(M).$$

From [3 Subsection 3.2], we have $V_{g_{\pm 1}}(M)$ is canonically isomorphic to the following rank variety:

$$V_{g_{\pm 1}}^{rank}(M) := \{x \in g_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)-\text{module} \} \cup \{0\},$$

where $U(\langle x \rangle)$ denotes the enveloping algebra of the Lie superalgebra generated by $x \in g_{\pm 1}$. By [3 Proposition 5.4], $M$ will be projective as $U(\langle x \rangle)$-module if and only if it does not contain a direct summand which is isomorphic to the trivial module.

4.2. Rank variety of Kac modules. We start by computing the rank variety of an atypical Kac module in $g = \mathfrak{osp}(2|2)$:

**Proposition 4.2.1.** For $g = \mathfrak{osp}(2|2)$ and for an atypical $\lambda \in X_0^+$, $V_{g_1}^{rank}(K(\lambda)) = g_1$.

**Proof.** By [3 Subsection 3.8.4], the action of $G_0 = C^* \times \text{Sp}(2n)$ on $g_1$ has two orbits $\{0\}$ and $g_1 \setminus \{0\}$. Since $V_{g_1}^{rank}(M)$ is a closed $G_0$-stable subvariety of $V_{g_1}^{rank}(C) = g_1$, it is enough to find $0 \neq x \in g_1$ such that $K(\lambda)$ contains a direct summand isomorphic to the trivial module when viewed as a $U(\langle x \rangle)$-module. Note that $g_1$ is spanned by the root vectors $x_1$ and $x_2$ with weights $wt(x_1) = \varepsilon_1 - \delta_1$ and $wt(x_2) = \varepsilon_1 + \delta_1$. On the other hand, $g_{-1}$ is spanned by
$y_1$ and $y_2$ with weights $wt(y_1) = -wt(x_1) = -\varepsilon_1 + \delta_1$ and $wt(y_2) = -wt(x_2) = -\varepsilon_1 - \delta_1$. We will need the following elements of $\mathfrak{g}_0$ as we proceed:

$$
z_{11} = 1 \oplus h, \quad z_{22} = -1 \oplus h, \quad z_{12} = 0 \oplus -2f, \quad \text{and} \quad z_{21} = 0 \oplus 2e,
$$

where $\{e, f, h\}$ is the standard basis of $\mathfrak{sl}_2$. Since $\mathfrak{g}_0 \cong \mathbb{C} \oplus \mathfrak{sl}_2$, the simple $\mathfrak{g}_0$-modules are of the form $L_0(\lambda) = \mathbb{C}_{\lambda_1} \boxtimes L(d)$ where $L(d)$ is the simple $\mathfrak{sl}_2$-module of dimension $d + 1$.

Let $\{v_k \mid 0 \leq k \leq d\}$ be the basis for $L(d)$ given in [10, Subsection 7.2] with $v_0$ being the highest weight vector such that:

$$
h.v_k = (d - 2k)v_k, \quad e.v_k = (d - k + 1)v_{k-1}, \quad f.v_k = (k + 1)v_{k+1},
$$

with $v_{-1} = v_{d+1} = 0$. Let $w_k = 1 \otimes v_k$ then $\{w_k \mid 0 \leq k \leq d\}$ is a basis of $L_0(\lambda)$ with $w_0$ being the highest weight vector. We can show that for $0 \leq k \leq d$,

$$
z_{11}.w_k = -2kw_k, \quad z_{12}.w_k = -2(k + 1)w_{k+1}, \quad z_{21}.w_k = 2(d - k + 1)w_{k-1},
$$

with $w_{-1} = w_{d+1} = 0$.

By the PBW-basis theorem for $U(\mathfrak{g})$, $\dim K(\lambda) = 4(d + 1)$ with basis $\{y^a_b \otimes w_k \mid a, b \in \{0, 1\}, 0 \leq k \leq d\}$. We have $\rho = (-1|1)$ then $\lambda = (\lambda_{-1}|d)$ is atypical if $(\rho + \lambda, \varepsilon_1 \pm \delta_1) = 0$. Thus we have two cases:

**Case 1:** Using the odd root $wt(x_1) = \varepsilon_1 - \delta_1$, we have $\lambda = -(d|d)$. Then, $x_1(1 \otimes w_0) = 1 \otimes x_1w_0 = 0$ since $w_0$ is annihilated by $\mathfrak{g}_1$. By degrees, $y_i \otimes w_k$ could be in the $U(\langle x_1 \rangle)$-submodule generated by $1 \otimes w_0$, but using weights, we can check that only $y_1 \otimes w_0$ could land in that submodule. However,

$$
x_1(y_1 \otimes w_0) = -y_1 \otimes x_1w_0 + 1 \otimes z_{11}w_0 = 0 + (d - d)(1 \otimes w_0) = 0.
$$

Therefore, the $U(\langle x_1 \rangle)$-submodule generated by $1 \otimes w_0$ forms a trivial direct summand of $K(\lambda)$ when viewed as a $U(\langle x_1 \rangle)$-module. Thus $0 \neq x_1 \in \mathcal{V}^{rank}_{\mathfrak{g}_1}(K(\lambda))$, the theorem follows in this case.

**Case 2:** Using the odd root $wt(x_2) = \varepsilon_1 + \delta_1$, $\lambda = (d + 2|d)$. Then we can show $x_2(y_1y_2 \otimes w_0) = -(y_1x_1 + z_{21})y_2 \otimes w_0 = -y_1(-y_2x_2 + z_{22}) \otimes w_0 + (2y_1 + y_1z_{21}) \otimes w_0 = 0$.

In the above we used the fact the $x_2$ and $z_{21}$ both annihilate $w_0$. Since $y_1y_2 \otimes w_0$ has degree $-2$, the $U(\langle x_2 \rangle)$-submodule generated by $y_1y_2 \otimes w_0$ forms a trivial direct summand of $K(\lambda)$ when viewed as a $U(\langle x_2 \rangle)$-module. Thus $0 \neq x_2 \in \mathcal{V}^{rank}_{\mathfrak{g}_1}(K(\lambda))$, the theorem follows in this case.

To generalize the above theorem, we need to use the equivalence of blocks stated in [9, Theorem 2]. Indeed, we have:

**Theorem 4.2.1.** For $\mathfrak{osp}(2|2n)$ and when $\lambda$ is atypical, $\mathcal{V}^{rank}_{\mathfrak{g}_1}(K(\lambda)) = \mathfrak{g}_1$.

**Proof.** Assume $\text{atyp}(\lambda) = 1$. Let $\chi_\lambda$ be the central character corresponding to the weight $\lambda$ and let $F^{\chi_\lambda}$ be the corresponding block. [9, Theorem 2] implies that $F^{\chi_\lambda}$ is equivalent to the maximal block of $\mathfrak{g}' = \mathfrak{osp}(2|2)$ containing the trivial module.

As discussed in [7], the equivalence of blocks is a composition of translation functors between the blocks of $F$, followed by a restriction functor from $\mathfrak{g}$ to $\mathfrak{g}'$. Upon restricting to $\mathfrak{g}'$, we have:

$$
K(\lambda) = K(\lambda') \oplus M,
$$

where $\lambda' = \chi_\lambda - \chi_\lambda'$.
where $K(\lambda')$ is an atypical Kac $g'$-module (the above equivalence of blocks takes Kac modules to Kac modules) and $M$ is a $g'$-module. By Proposition 4.2.1, there exists $0 \neq x' \in \mathcal{V}_{g'_1}^{\text{rank}}(K(\lambda')) = g'_1$. By [4, Theorem 6.6] we have

$$
\mathcal{V}_{g'_1}^{\text{rank}}(K(\lambda)) = \mathcal{V}_{g'_1}^{\text{rank}}(K(\lambda')) \cup \mathcal{V}_{g'_1}^{\text{rank}}(M).
$$

This shows that there exists $0 \neq x' \in g' \subseteq \mathcal{V}_{g'_1}^{\text{rank}}(K(\lambda))$. However, we can embed $g' \hookrightarrow g$ such that the odd elementary matrices $E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4} \in g'$ are sent respectively to the odd elementary matrices $E_{1,3}, E_{1,n+1}, E_{2,3}, E_{2,n+1} \in g$. Therefore, we can find $0 \neq x \in \mathcal{V}_{g'_1}^{\text{rank}}(K(\lambda))$ but the $G_0$-orbits of $g_1$ are $\{0\}$ and $g_1 \setminus \{0\}$, hence $\mathcal{V}_{g'_1}^{\text{rank}}(K(\lambda)) = g_1$. □

Thus we can interpret the complexity of Kac modules geometrically as follows:

**Corollary 4.2.1.** For $\lambda \in X_0^+$, $c_F(K(\lambda)) = \dim \mathcal{V}_{g_1}^{\text{rank}}(K(\lambda))$.

**Proof.** If $\lambda$ is typical, the Kac module $K(\lambda)$ is projective, then $\dim \mathcal{V}_{g_1}^{\text{rank}}(K(\lambda)) = 0$ by [3, Theorem 3.5.1], and hence

$$
c_F(K(\lambda)) = \dim \mathcal{V}_{g_1}^{\text{rank}}(K(\lambda)) = 0.
$$

If $\lambda$ is atypical, Theorem 3.5.1 and Theorem 4.2.1 imply that

$$
c_F(K(\lambda)) = \dim \mathcal{V}_{g_1}^{\text{rank}}(K(\lambda)) = \dim g_1 = 2n.
$$

□

After establishing the equivalence of blocks between $g$ and $g'$, we can show that the operator $L$ is onto without using its combinatorial definition.

**Proposition 4.2.2.** The operator $L$ is surjective on the set of atypical weights in $X_0^+$.

**Proof.** First, we will show that $L$ is onto in the case of $g' = \mathfrak{osp}(2|2)$. Let $\mu$ be an atypical weight, then $\mu = (d+2d)$ or $\mu = (-d)d$ for some $d \geq 0$. By Theorem 3.2.1 $(0|0) = (2|0)^L$, $(d + 2|d) = (d + 3|d + 1)^L$ for any $d \geq 0$ and $(-d|d) = (-d + 1|d + 1)^L$ for any $d \geq 1$. Thus $\mu$ is the image under $L$ of some atypical weight.

Now Let $\mu$ be an atypical weight in the case of $g = \mathfrak{osp}(2|2n)$. Let $L(\mu')$ be the image of the simple module $L(\mu)$ under the above equivalence of blocks between $g$ and $g'$. Then $\mu'$ is an atypical weight for $g'$. Thus there exists an atypical weight $\lambda'$ with $\lambda' = \mu'$. The head of $K(\lambda')$ is $L(\lambda')$ and the socle is $L(\lambda'|\lambda') = L(\mu')$. This Kac module corresponds to a Kac module $K(\lambda)$ (where $\lambda$ corresponds to $\lambda'$ under the same equivalence) which has $L(\lambda)$ as its head and $L(\mu)$ as its socle. But the socle of $K(\lambda)$ is $L(\lambda|\lambda)$, thus $L(\lambda|\lambda) \cong L(\mu)$ which shows $\mu = \lambda|\lambda$ for some atypical $\lambda$. □

Next, we find the dimension of the associated variety of the simple and Kac modules:

**Lemma 4.2.1.** For $\lambda \in X_0^+$, let $X(\lambda)$ be $L(\lambda)$ or $K(\lambda)$, then

$$
\dim X(\lambda) = \begin{cases} 
2n, & \text{if atyp}(\lambda) = 1; \\
0, & \text{if atyp}(\lambda) = 0.
\end{cases}
$$
Proof. When \( \lambda \) is typical, \( L(\lambda) \) and \( K(\lambda) \) are projective, hence \( \dim X_{(\lambda)} = 0 \) by [7 Theorem 3.4]. For the rest of the proof, assume \( \lambda \) is atypical. [17 Corollary 2.5] implies that \( X_{(\lambda)} = X \) when \( \lambda \) is atypical. In \( \mathfrak{osp}(2\mid 2n) \), \( X \) has two irreducible components each with dimension equals to \( \frac{\dim g_1}{2} = 2n \). Thus \( \dim X = 2n \), ([7 Corollary 4.8, 4.9]).

Now assume \( X(\lambda) = K(\lambda) \). From the definitions of the rank variety and the associated variety, we have
\[
\mathcal{X}_{(\lambda)}^{\mathfrak{g}_1}(K(\lambda)) = X(\lambda) \cap \mathfrak{g}_1 \subseteq X(\lambda).
\]
Following the same proof of [2 Theorem 6.4.1], we show that the inclusion is in fact an equality. Thus \( \dim X_{(\lambda)} = \dim \mathcal{X}_{(\lambda)}^{\mathfrak{g}_1}(K(\lambda)) = \dim g_1 = 2n \), (using Theorem 4.2.1).

Combining the computations about the complexity, support variety, and the associated variety of the simple and Kac modules, we can conclude that:

**Theorem 4.2.2.** For \( \lambda \in X^+_0 \), let \( X(\lambda) \) be \( L(\lambda) \) or \( K(\lambda) \), then
\[
c_{\mathcal{X}}(X(\lambda)) = \dim X_{(\lambda)} + \dim \mathcal{X}_{(\lambda)}^{\mathfrak{g}_1}(X(\lambda)).
\]

5. **z-complexity over \( \mathfrak{osp}(2\mid 2n) \)**

5.1. **z-complexity of simple modules.** The \( z \)-complexity is defined in Subsection 2.6.

**Lemma 5.1.1.** \( z_{\mathcal{X}}(C) = 2 \).

**Proof.** The proof follows directly by counting the number of summands in each \( P_d \) in the resolution ([3.2.2]) of the trivial module. By doing so, we can see that the number of summands in each \( P_d \) is a polynomial in \( d \) of degree 1. Hence, the rate of growth of this number is \( s(P_\bullet) = 2 \). The result follows. \( \square \)

To find the \( z \)-complexity of all atypical simple modules, we will show that simple modules of the same atypicality have the same \( z \)-complexity.

**Lemma 5.1.2.** For any module \( X \in \mathcal{F} \) and a Kac module \( K(\lambda) \), there exists a constant \( D_X \) depending only on \( X \) such that
\[
K(\lambda) \otimes X \cong \bigoplus_{\gamma \in I} K(\gamma),
\]
where \( |I| \leq D_X \).

**Proof.** By the definition of Kac modules, we have
\[
K(\lambda) = U(\mathfrak{g}) \otimes_{U(p^+)} L_0(\lambda).
\]
As a \( \mathfrak{g}_0 \)-module, \( L_0(\lambda) \otimes X \) decomposes into a direct sum \( \bigoplus_{\mu \in I_X} L_0(\lambda) \otimes L_0(\mu) \), where \( I_X \) is a finite indexing set depending only on \( X \). By the generalized Littlewood-Richardson formula [15 Subsection A.4], each summand \( L_0(\lambda) \otimes L_0(\mu) \) decomposes into a finite direct sum \( \bigoplus_{\gamma \in I_{\lambda,\mu}} L_0(\gamma) \) indexed by the \( \mathfrak{sp}(2n) \)-standard Young tableaux of shape \( p(\mu) \) which are
\( \lambda \)-dominant (see [15], Subsection A.4 for definitions). Let \( l(\mu) \) be the number of boxes in \( p(\mu) \), then

\[ |I_{\lambda, \mu}| \leq (2n)^l(\mu). \]

By using the functor \( U(g) \otimes_{U(p^+)} - \), we have the following isomorphisms as \( U(g) \)-modules:

\[ K(\lambda) \otimes X \cong \bigoplus_{\mu \in I_{\lambda, \mu}} \bigoplus_{\gamma \in I_{\lambda, \mu}} U(g) \otimes_{U(p^+)} L_0(\gamma) \cong \bigoplus_{\mu \in I_{\lambda, \mu}} K(\gamma). \]

The number of summands is at most \( D_X := \sum_{\mu \in I_{\lambda, \mu}} (2n)^l(\mu) \). This completes the proof. \( \square \)

**Corollary 5.1.1.** For any module \( X \in \mathcal{F} \) and a projective cover \( P(\lambda) \), there exists a constant \( E_X \) depending only on \( X \) such that

\[ P(\lambda) \otimes X \cong \bigoplus_{\alpha \in J} P(\alpha), \]

where \( |J| \leq E_X \).

**Proof.** Recall the following 2-step Kac flag:

\[ 0 \rightarrow K(\lambda) \rightarrow P(\lambda^L) \rightarrow K(\lambda^L) \rightarrow 0, \]

which can be rewritten as

\[ 0 \rightarrow K(\lambda') \rightarrow P(\lambda) \rightarrow K(\lambda) \rightarrow 0, \]

with \( \lambda'^L = \lambda \). This Kac filtration of \( P(\lambda) \) together with the exactness of the tensor functor (over \( \mathbb{C} \)) imply the exact sequence:

\[ 0 \rightarrow K(\lambda') \otimes X \rightarrow P(\lambda) \otimes X \rightarrow K(\lambda) \otimes X \rightarrow 0. \]

The factors \( K(\lambda) \otimes X \) and \( K(\lambda') \otimes X \) decompose into Kac modules by the previous lemma. Thus \( P(\lambda) \otimes X \) has a Kac filtration. Moreover, there are two numbers \( D_X \) and \( D'_X \) such that the number of Kac modules in this filtration is at most \( E_X := D_X + D'_X \). On the other hand, the projective module \( P(\lambda) \otimes X \) decomposes into a finite direct sum of projective indecomposables \( \bigoplus_{\alpha \in R} P(\alpha) \) where each summand has a Kac filtration. However Kac filtrations of the same module will have the same number of Kac modules. In fact, by [4, Proposition 3.3], the number of times \( K(\lambda) \) appears in a Kac filtration of a module \( M \) is equal to \( \dim \hom(M, K'(\lambda)) \), where \( K'(\lambda) \) is the dual Kac module (cf. [5, Subsection 3.4]). This dimension is not dependent on the choice of the filtration. Thus the number of the projective indecomposables, \( P(\alpha) \), is bounded by \( E_X \). \( \square \)

**Lemma 5.1.3.** Let \( M, N, X, T \in \mathcal{F} \) such that \( M \otimes X \cong N \oplus T \). Then \( z_\mathcal{F}(M) \geq z_\mathcal{F}(N) \).

**Proof.** Let \( P_\bullet \rightarrow M \otimes X \) and \( Q_\bullet \rightarrow M \) be the minimal projective resolutions of \( M \otimes X \) and \( M \) respectively. Then \( Q_\bullet \otimes X \rightarrow M \otimes X \) is a projective resolution of \( M \otimes X \). Using the above corollary, the \( d \)-th term in this resolution decomposes as follows:

\[ Q_d \otimes X \cong \bigoplus_{\lambda \in R_d} P(\lambda) \otimes X \cong \bigoplus_{\lambda \in R_d} \bigoplus_{\gamma \in I} P(\gamma), \]

where \( |I| \leq D_X \). Thus, for each \( d \), the number of summands in \( Q_d \otimes X \) is at most \( |R_d| \cdot D_X \) which is a constant multiple of the number of summands in each \( Q_d \). Recall that \( s(P_\bullet) \)
denotes the rate of growth of the number of summands in $P_d$, the $d^{th}$ term in the resolution $P_\bullet$, then by minimality of the resolution, we have $s(P_\bullet) \leq s(Q_\bullet \otimes X)$. Thus,

$$z_F(M \otimes X) = s(P_\bullet) \leq s(Q_\bullet \otimes X) \leq s(Q_\bullet) = z_F(M).$$

By using the definition of the $z$-complexity we can easily show that

$$z_F(N \oplus T) = \max \left( z_F(N), z_F(T) \right).$$

Then

$$z_F(M \otimes X) \geq z_F(N),$$

thus

$$z_F(M) \geq z_F(M \otimes X) \geq z_F(N).$$

\textbf{Theorem 5.1.1.} Let $\lambda \in X_0^+$. 

(1) If $\lambda$ is typical, then $z_F(L(\lambda)) = 0$.

(2) If $\lambda$ is atypical, then $z_F(L(\lambda)) = 2$.

\textbf{Proof.} If $\lambda$ is typical, then $L(\lambda)$ is projective, and hence $z_F(L(\lambda)) = 0$. Let $\lambda$ be atypical, then [14 Corollary 3.2.2] implies that if $\lambda$ and $\mu$ have the same atypicality, there are modules $X_1$ and $X_2$ such that $L(\mu)$ is a direct summand of $L(\lambda) \otimes X_1$ and $L(\lambda)$ is a direct summand of $L(\mu) \otimes X_2$. Then Lemma 5.1.3 shows that $z_F(L(\lambda)) = z_F(L(\mu))$. Thus, for an atypical $\lambda$, $z_F(L(\lambda)) = z_F(C) = 2$. □

5.2. $z$-complexity of Kac modules.

\textbf{Theorem 5.2.1.} Let $\lambda \in X_0^+$. 

(1) If $\lambda$ is typical, then $z_F(K(\lambda)) = 0$.

(2) If $\lambda$ is atypical, then $z_F(K(\lambda)) = 1$.

\textbf{Proof.} If $\lambda$ is typical, then $K(\lambda)$ is projective, and hence $z_F(K(\lambda)) = 0$. Assume $\lambda$ is atypical. For $\lambda = 0$, we can use the projective resolution (3.3.1) to see that $z_F(K(\lambda)) = 1$. Following the same proof as in Lemma 3.4.1, we can show that

$$z_F(K(\lambda)) = z_F(K(\lambda^L)).$$

A similar argument as in Subsection 3.4 is used to show that

$$z_F(K(\lambda)) = z_F(K(\mu)) \quad \text{for some } \mu \in F^{\lambda_0}$$

$$= z_F(K(0^{(l)})) \quad \text{for some } l \in \mathbb{Z}$$

$$= z_F(K(0)) = 1.$$

□
5.3. Detecting subsuperalgebra. As we interpreted the complexity of the simple and the Kac modules geometrically through the dimensions of the associated variety and the support variety, we can also find a geometric interpretation of the $z$-complexity. To do so, a detecting subalgebra is introduced. Let $f_1 \subseteq g_1$ be the span of the root vectors $x_\alpha$, $x_-\alpha$ where $\alpha = \xi_1 - \delta_1$. In the matrix realization, $x_\alpha = E_{1,3} - E_{n+3,2}$ and $x_-\alpha = E_{2,n+3} + E_{3,1}$. Set $f_0 = [f_1, f_1]$. Then $f_0$ is spanned by the diagonal matrix $E_{1,1} - E_{2,2} + E_{3,3} - E_{n+2,n+2}$. We define a three-dimensional subalgebra of $g$ by

$$\tilde{f} := f_0 \oplus f_1.$$  

The Lie superalgebra $\tilde{f}$ is classical and so has a support variety theory. Furthermore, as $[f_0, f_1] = 0$, it follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $f_1$, i.e.,

$$V_{(f_0)}(M) = V_{f_1}^{rank}(M) = \{y \in f_1 \mid M \text{ is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}.$$  

For example, $V_{(f_0)}(C) = f_1$. If $M$ is projective, then $V_{(f_0)}(M) = 0$.

**Proposition 5.3.1.** Let $\lambda \in X_0^+$. If $\lambda \in X_0^+$ is typical, then

$$V_{(f_0)}(K(\lambda)) = V_{(f_0)}(L(\lambda)) = 0.$$  

If $\lambda$ is atypical, then

$$\dim V_{(f_0)}(K(\lambda)) = 1, \quad \dim V_{(f_0)}(L(\lambda)) = 2.$$  

**Proof.** If $\lambda$ is typical, then $L(\lambda)$ and $K(\lambda)$ are projective and the result follows. Let $\lambda$ be an atypical weight. As argued in [2, Theorem 6.4.1,Theorem 9.2.1] we can show

$$V_{(f_0)}(K(\lambda)) = f_1 \cap g_1,$$

which implies $\dim V_{(f_0)}(K(\lambda)) = 1$.

Since $\lambda$ and $(0|0,\ldots,0)$ have the same atypicality, [14, Theorem 4.1.1] implies that

$$V_{(f_0)}(L(\lambda)) = V_{(f_0)}(L(0|0,\ldots,0)) = f_1,$$

thus $\dim V_{(f_0)}(L(\lambda)) = 2$. \qed

Note that Theorem 5.1.1, Theorem 5.2.1, and Proposition 5.3.1 imply the following geometric interpretation of the $z$-complexity:

**Theorem 5.3.1.** If $X(\lambda)$ is a simple or a Kac module over $\mathfrak{osp}(2|2n)$ then

$$z_F(X(\lambda)) = \dim V_{(f_0)}(X(\lambda)).$$

6. Additional examples

In this section we assume that $g$ is either $\mathfrak{osp}(3|2)$, $D(2,1;\alpha)$ for some $\alpha \in \mathbb{C} \setminus \{0,-1\}$, $G(3)$, or $F(4)$. In these cases, $g_0$ is semisimple and hence our category $\mathcal{F}$ is the category of finite-dimensional supermodules. In this section we refer to [8] to describe the atypical blocks over these Lie superalgebras. We will also use the notation developed in [8]. In particular, let $X^+$ be the set of isomorphism classes of simple finite-dimensional $g$-modules. For $\lambda \in X^+$ we choose a simple module $S(\lambda)$ in $\lambda$.  

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Germoni [8] denotes by $\mathfrak{g}$-mod the category of finite-dimensional $\mathfrak{g}$-modules with even morphisms of representations. In [8, Lemma 1.1.1], the category $\mathfrak{g}$-mod is shown to contain enough projective modules where projective and injective modules coincide. However, we will be considering the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-modules with all morphisms. The choice of even morphisms has no effect on the radical layer structure of the projective indecomposable modules.

In this section, we compute the complexity and the $z$-complexity of the atypical simple modules. The typical ones are projective, and hence they have zero complexity and $z$-complexity. We then interpret these complexities geometrically as before.

6.1. Case I: $\mathfrak{osp}(3|2)$. Let $\mathfrak{g} = \mathfrak{osp}(3|2)$ (see [11]). The even part of $\mathfrak{g}$ is $\mathfrak{g}_0 \cong \mathfrak{so}(3) \oplus \mathfrak{sp}(2) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, hence $\dim \mathfrak{h}^* = 2$ with a basis $\{\varepsilon, \delta\}$.

By proposition 2.3 in [12], the set of dominant weights for $\mathfrak{g}$ is $X^+ = \{(a, b) \in \mathbb{N}/2 \times \mathbb{N} \mid b = 0 \Rightarrow a = 0\}$. The atypical dominant weights are $\lambda_0 = (0, 0)$ and $\lambda_l = (l - 1, l)$ (for $l \in \mathbb{N}^*$). In fact, $S(\lambda_0)$ is the trivial module. According to ([8, Theorem 2.1.1]), the principal block $\Gamma_0 = \{\lambda_l \mid l \in \mathbb{N}\}$ is the unique atypical block.

Let $P(\mu)$ be the projective cover of $S(\mu)$. As a $\mathfrak{g}_0$-module, $S(\mu)$ contains a simple $\mathfrak{g}_0$-module $S_0(\mu)$ as a composition factor. Using the discussion in [2, Subsection 5.1], we have the following bounds:

$$\dim S_0(\mu) \leq \dim P(\mu) \leq 2^{\dim \mathfrak{g}_0} \dim S_0(\mu), \quad (6.1.1)$$

Note that $S_0(\lambda_0)$ is the trivial $\mathfrak{g}_0$-module and for $\lambda_l = (l - 1, l) \in X^+$, $S_0(\lambda_l) = V_{l-1} \otimes V_l$ where $V_m$ denotes the simple $\mathfrak{sl}_2$-module of dimension $m + 1$. Thus

$$l(l + 1) \leq \dim P(\lambda_l) \leq 2^6l(l + 1). \quad (6.1.2)$$

6.2. Complexity of Simple $\mathfrak{osp}(3|2)$-modules. We give a minimal projective resolution of the trivial module $S(\lambda_0)$ to compute its complexity, then we use the generalized Kac-Wakimoto conjecture to show that any typical simple module will have the same complexity as the trivial module.

**Theorem 6.2.1.** For $\lambda_l \in X^+$, $c_F(S(\lambda_l)) = 4$.

**Proof.** Using the radical layer structure of the projective indecomposable modules given in [8, Theorem 2.1.1] and following the same diagrammatic techniques used in proving Theorem 3.2.1 the minimal projective resolution of $S(\lambda_0)$ is given by:

$$\ldots \rightarrow P_d \rightarrow \ldots \rightarrow P(\lambda_0) \rightarrow S(\lambda_0) \rightarrow 0, \quad (6.2.1)$$

where the $d^{th}$ term in this resolution is given by:

$$P_d = P(\lambda_{d+1}) \oplus P(\lambda_{d-1}) \oplus \cdots \oplus P(\lambda_0),$$

such that

$$r = \begin{cases} 2 & \text{if } d \text{ is odd}, \\ 0 & \text{if } d \equiv 0 \mod 4, \\ 1 & \text{if } d \equiv 2 \mod 4. \end{cases}$$

Then, by (6.1.2), we have for all $d$:

$$C' d^3 \leq \dim P_d \leq C d^3,$$
for some positive constants $C, C'$. This shows that $c_F(S(\lambda_0)) = 4$. By [14] Theorem 4.1.1, all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple $\mathfrak{osp}(3\mid 2)$-modules is 4. □

6.3. $z$-complexity over $\mathfrak{osp}(3\mid 2)$. We compute the $z$-complexity of the simple $\mathfrak{osp}(3\mid 2)$-modules. We start by computing this complexity for the trivial module:

**Proposition 6.3.1.** $z_F(S(\lambda_0)) = 2$.

*Proof.* Using the resolution in the proof of Theorem 6.2.1 the number of summands in $P_d$ is either $d/2 + 1$ if $d$ is even or $(d + 1)/2$ if $d$ is odd. The rate of growth of this number is 2 and the result follows. □

We will also show that:

**Proposition 6.3.2.** For $l \geq 1$, we have $z_F(S(\lambda_l)) = 2$.

*Proof.* Since the projective cover of $S(\lambda_1)$ has the same structure as the projective cover of $S(\lambda_0)$, one can write an analogous minimal projective resolution to show that $z_F(S(\lambda_1)) = 2$. The minimal projective resolution of $S(\lambda_2)$ is given by:

$$
\ldots \to P_d \to \ldots \to P(\lambda_2) \to S(\lambda_2) \to 0, \quad (6.3.1)
$$

where the $d^{th}$ term in this resolution is given by:

$$
P_d = \begin{cases} 
P(\lambda_{d+2}) \oplus 2.P(\lambda_d) \oplus \cdots \oplus 2.P(\lambda_2) & \text{if } d \geq 2 \text{ is even}, \\
P(\lambda_{d+2}) \oplus 2.P(\lambda_d) \oplus \cdots \oplus 2.P(\lambda_3) \oplus P(\lambda_1) \oplus P(\lambda_0) & \text{if } d \geq 1 \text{ is odd.}
\end{cases}
$$

In the above, $2.P(\lambda)$ means $P(\lambda) \oplus P(\lambda)$. Thus the number of direct summands in $P_d$ is either $d + 1$ if $d \geq 2$ is even or it is $d + 2$ if $d \geq 1$ is odd. This shows that $z_F(S(\lambda_l)) = 2$.

The $d^{th}$ term in the minimal projective resolution of $S(\lambda_3)$ is given by:

$$
P_d = \begin{cases} 
P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) & \text{if } d = 1, \\
P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) \oplus P(\lambda_1) \oplus P(\lambda_0) & \text{if } d = 2, \\
P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) \oplus 2.P(\lambda_{d-1}) \oplus \cdots \oplus 2.P(\lambda_2) & \text{if } d \geq 3 \text{ is odd,} \\
P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) \oplus 2.P(\lambda_{d-1}) \oplus \cdots \oplus 2.P(\lambda_3) \oplus P(\lambda_1) \oplus P(\lambda_0) & \text{if } d \geq 4 \text{ is even.}
\end{cases}
$$

Thus the number of summands in $P_d$ is either $d + 2$ if $d \geq 2$ is even or it is $d + 1$ if $d \geq 1$ is odd. This shows that $z_F(S(\lambda_l)) = 2$.

For $l \geq 3$, the projective cover of $S(\lambda_l)$ has the same structure as the projective cover of $S(\lambda_3)$. This gives $z_F(S(\lambda_l)) = 2$. □

6.4. **Case II:** $D(2, 1; \alpha)$. Let $\alpha \in \mathbb{C} \setminus \{0, -1\}$, and let $\mathfrak{g}$ be the basic classical Lie superalgebra $D(2, 1; \alpha)$ (see [12]). The even part of $\mathfrak{g}$ is $\mathfrak{g}_0 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. By [12] Proposition 2.2, the set of dominant weights for $\mathfrak{g}$ is: $X^+ = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid a = 0 \Rightarrow b = c = 0 \text{ and } a + 1 \Rightarrow (b + 1) = \pm \alpha(c + 1)\}$.

Atypical dominant weights are described in [8] Subsection 3.1. Notice that if $\alpha \not\in \mathbb{Q}$, the only atypical dominant weights are $\lambda_0 = (0, 0, 0)$ (corresponding to the trivial module) and $\lambda_l = (l + 1, l - 1, l - 1)$ for $l \geq 1$. 

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If $\alpha \in \mathbb{Q}$, we assume $\alpha = p/q$, with $p$ and $q$ relatively prime positive integers. For $k \in \mathbb{N}$, let $\Gamma_k$ be the set of atypical simple modules $\lambda \in X^+$ such that the Casimir element has the eigenvalue $p(p + q)k^2/2$. For $l \in \mathbb{Z}$ we set moreover:

$$
\lambda_{k,l} = \begin{cases} 
(-l + 2, -l - kp, -l + kq) & \text{if } l \leq -kp, \\
(-l + 1, l + kp - 1, -l + kq - 1) & \text{if } -kp + 1 \leq l \leq 0, \\
(l + 1, l + kp - 1, -l + kq - 1) & \text{if } 0 \leq l \leq kq - 1, \\
(l + 2, l + kp, -kq) & \text{if } kq \leq l.
\end{cases}
$$

According to [3] Theorem 3.1.1, the principal block of $D(2,1;\alpha)$ is $: \Gamma_0 = \{\lambda_l \mid l \in \mathbb{N}\}$ and it is equivalent to the principal block of $\mathfrak{osp}(3|2)$. Moreover, the projective covers of the simple modules in the principal block of $D(2,1;\alpha)$ have the same radical layer structure as those over $\mathfrak{osp}(3|2)$.

Let $S_0(\mu)$ be the simple $\mathfrak{g}_0$-module of weight $\mu$. Note that $S_0(\lambda_0)$ is the trivial $\mathfrak{g}_0$-module and for $\lambda_l = (l + 1, l - 1, l) \in X^+$, $S_0(\lambda_l) = V_{l+1} \otimes V_{l-1} \otimes V_l$ where $V_m$ denotes the simple $\mathfrak{sl}_2$-module of dimension $m + 1$. Then by (6.4.1) we have

$$
(l + 2)(l)(l + 1) \leq \dim P(\lambda_l) \leq 2^8(l + 2)(l)(l + 1).
$$

6.5. **Complexity of Simple $D(2,1;\alpha)$-modules.** Since the principal block $\Gamma_0 = \{\lambda_l \mid l \in \mathbb{N}\}$ is equivalent to the principal block of $\mathfrak{osp}(3|2)$ and the projective covers have the same structure in both cases, we will have the same minimal projective resolution (6.2.1) for $S(\lambda_0)$. Then

**Theorem 6.5.1.** If $S$ is an atypical simple $D(2,1;\alpha)$-module, we have $c_F(S) = 5$.

*Proof.* Using the projective resolution (6.2.1) and the bounds in equation (6.4.1), we have

$$
C' \cdot d^3 \leq \dim P_d \leq C \cdot d^4,
$$

for some positive constants $C$, $C'$. This shows that $c_F(S(\lambda_0)) = 5$. On the other hand, by [13] Example 3.4, we see that the Kac-Wakimoto conjecture holds for simple modules over $D(2,1;\alpha)$. Since this conjecture holds, then [13] Theorem 4.1.1 holds over $D(2,1;\alpha)$. Thus all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple $D(2,1;\alpha)$-modules is 5. □

6.6. **$z$-complexity over $D(2,1;\alpha)$**. The $z$-complexity is a categorical invariant, thus using the equivalence between the principal blocks of $\mathfrak{osp}(3|2)$ and $D(2,1;\alpha)$ we have:

**Proposition 6.6.1.** If $S(\lambda)$ is a simple $D(2,1;\alpha)$-module in the principal block, then $z_F(S(\lambda)) = 2$.

For the simple modules in the other atypical blocks $\Gamma_k$, we can compute the $z$-complexity by writing an explicit minimal projective resolution.

**Proposition 6.6.2.** For $\lambda_{k,l} \in \Gamma_k$, we have $z_F(S(\lambda_{k,l})) = 2$.

*Proof.* The $d^{th}$ term in the minimal projective resolution of $S(\lambda_{k,l})$ is given by:

$$
P_d = \begin{cases} 
P(\lambda_{k,l \pm d}) \oplus P(\lambda_{k,l \pm (d-2)}) \oplus \cdots \oplus P(\lambda_{k,l \pm 1}) & \text{if } d \geq 1 \text{ is odd}, \\
P(\lambda_{k,l \pm d}) \oplus P(\lambda_{k,l \pm (d-2)}) \oplus \cdots \oplus P(\lambda_{k,l}) & \text{if } d \geq 2 \text{ is even}.
\end{cases}
$$
In the above, \( P(\lambda_{k,l+i}) \) means \( P(\lambda_{k,l+i}) \oplus P(\lambda_{k,l-i}) \). Thus the number of direct summands in \( P_d \) is \( d + 1 \). This shows that \( z_F(S(\lambda_{k,l})) = 2 \). \( \square \)

6.7. **Case III:** \( G(3) \). Let \( \mathfrak{g} \) be the basic classical Lie superalgebra \( G(3) \) (see [12]). The even part of \( \mathfrak{g} \) is : \( \mathfrak{g}_0 \cong \mathfrak{sl}_2 \oplus \mathfrak{g}_2 \). We identify the set of dominant weights \( \omega \) with \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) by means of the fundamental weights \( (\omega_1 = \varepsilon_1 + \varepsilon_2; \omega_2 = \varepsilon_1 + 2\varepsilon_2; \omega_3 = \delta) \). The dimension of the simple \( G_2 \)-module, \( L(\lambda) = L(m_1\omega_1 + m_2\omega_2) \), can be computed by the Weyl-dimension formula [10, Section 24.3]:

\[
\dim L(\lambda) = \frac{1}{5!} (m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(m_1 + 2m_2 + 3) (m_1 + 3m_2 + 4)(m_1 + 3m_2 + 4)(2m_1 + 3m_2 + 5). \quad (6.7.1)
\]

By [12] Proposition 2.2, the set of dominant weights for \( \mathfrak{g} \) is : \( X^+ = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid a = 0 \Rightarrow b = c = 0, a \neq 1 \text{ and } a = 2 \Rightarrow b = 0\} \).

For \( k \in \mathbb{N} \), we denote by \( \Gamma_k \) the set of dominant weights \( \lambda \in X^+ \) such that the Casimir element has the eigenvalue \( 6k(k + 1) \). For \( l \in \mathbb{N} \), we set :

\[
\begin{align*}
\lambda_{0,0} &= (0, 0, 0), \\
\lambda_{0,1} &= (5, 0, 0), \\
\lambda_{k,0} &= (2, 0, k - 1) \quad \text{if } k \geq 1, \\
\lambda_{k,1} &= (3, 0, k - 1) \quad \text{if } k \geq 1, \\
\lambda_{k,l} &= \begin{cases} 
(l + 2, 2l - 2, k - l) & \text{if } 2 \leq l \leq k, \\
(l + 3, 3k - l, l - k - 1) & \text{if } k + 1 \leq l \leq 3k, \\
(l + 4, l - 3k - 1, 2k) & \text{if } 3k + 1 \leq l.
\end{cases}
\end{align*}
\]

By [8] Theorem 4.1.1], every atypical block of \( \mathfrak{g} \) is one of the \( \Gamma_k \) which is equivalent to the principal block of \( \mathfrak{osp}(3|2) \). Moreover, the projective covers of the simple modules in \( \Gamma_k \) have the same radical layer structure as those in the principal block of \( \mathfrak{osp}(3|2) \).

6.8. **Complexity of Simple \( G(3) \)-modules.** Since the principal block \( \Gamma_0 = \{\lambda_{0,l} \mid l \in \mathbb{N}\} \) is equivalent to the principal block of \( \mathfrak{osp}(3|2) \) and the projective covers have the same radical layer structure as in \( \mathfrak{osp}(3|2) \), we will have the same minimal projective resolution (6.2.1) for \( S(\lambda_{0,0}) \). Then,

**Theorem 6.8.1.** If \( S \) is an atypical simple \( G(3) \)-module, we have \( c_F(S) = 8 \).

**Proof.** Using the projective resolution (6.2.1), the bounds in equation (6.1.1), and the dimension formula in (6.7.1), we have

\[
C'.d^7 \leq \dim P_d \leq C.d^7,
\]

for some positive constants \( C, C' \). These bounds are obtained by multiplying the dimension formula given in (6.7.1) by an extra factor from the \( \mathfrak{sl}_2 \)-part. This shows that \( c_F(S(\lambda_0)) = 8 \).

On the other hand, by [13] Example 3.4, we see that the Kac-Wakimoto conjecture holds for simple modules over \( G(3) \). Since this conjecture holds, then [14] Theorem 4.1.1] holds over \( G(3) \). Thus all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple \( G(3) \)-modules is 8. \( \square \)
6.9. \( z \)-complexity over \( G(3) \). The \( z \)-complexity is a categorical invariant, thus using the equivalence between the principal block of \( \mathfrak{osp}(3|2) \) and the atypical blocks in \( G(3) \) we have:

**Proposition 6.9.1.** If \( S \) is an atypical simple \( G(3) \)-module, then \( z_F(S) = 2 \).

6.10. **Case IV:** \( F(4) \). Let \( g \) be the basic classical Lie superalgebra \( F(4) \) (see [12]). The even part of \( g \) is \( g \cong \mathfrak{sl}_2 \oplus \mathfrak{so}_7 \). The set of dominant weights and the atypical blocks are described in [16]. The fundamental weights are \( \omega_1 = \varepsilon_1, \omega_2 = \varepsilon_1 + \varepsilon_2, \omega_3 = (1/2)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \) and \( \omega_4 = (1/2)\delta \). For \( \lambda = m_1\omega_1 + m_2\omega_2 + m_3\omega_3 \), the dimension of the simple \( \mathfrak{so}_7 \)-module, \( L(\lambda) \), can be computed by the Weyl-dimension formula [10, Section 24.3]:

\[
\dim L(\lambda) = \frac{1}{720}(m_1 + 1)(m_2 + 1)(m_3 + 1)(m_1 + m_2 + 2)(m_2 + m_3 + 2)(2m_2 + m_3 + 3)(m_1 + m_2 + m_3 + 3)(m_1 + 2m_2 + m_3 + 4)(2m_1 + 2m_2 + m_3 + 5). \quad (6.10.1)
\]

6.11. **Complexity of Simple \( F(4) \)-modules.** The description of the projective indecomposable modules over \( F(4) \) given in [16, Lemma 11.1] is the same as the one given in [8] over \( G(3) \), except for a small difference in the notation. For example, in [16], \( \lambda_1 \) corresponds to the trivial module, while in [8] \( \lambda_0 \) corresponds to the trivial module. This similarity means that the projective resolution (6.2.1) over \( \mathfrak{osp}(3|2) \) will carry over to \( F(4) \).

**Theorem 6.11.1.** If \( S \) is an atypical simple \( F(4) \)-module, we have \( c_F(S) = 9 \).

**Proof.** Using the projective resolution (6.2.1), the bounds in (6.1.1), and the dimension formula in (6.10.1) we have

\[
C'd^8 \leq \dim P_d \leq C'd^8,
\]

for some positive constants \( C, C' \). These bounds are obtained by multiplying the dimension formula in equation (6.10.1) by an extra factor from the \( \mathfrak{sl}_2 \)-part. This shows that \( c_F(S(\lambda_1)) = 9 \). On the other hand, by [13, Example 3.4], we see that the Kac-Wakimoto conjecture holds for simple modules over \( F(4) \). Since this conjecture holds, then \( \mathfrak{g} \) holds over \( F(4) \). Thus all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple \( F(4) \)-modules is 9. \( \square \)

6.12. \( z \)-complexity over \( F(4) \). The radical layer structure of the projective indecomposable modules over \( F(4) \) is the same as that over \( G(3) \). This will give the same projective resolutions over \( F(4) \). Thus

**Proposition 6.12.1.** If \( S \) is an atypical simple \( F(4) \)-module, then \( z_F(S) = 2 \).

6.13. **Geometric Interpretation of the Complexity.** Let \( g = \mathfrak{osp}(3|2), D(2,1;\alpha), G(3), \) or \( F(4) \). We interpret the complexity of simple modules geometrically:

**Theorem 6.13.1.** If \( S \) is a simple \( g \)-module, then

\[
c_F(S) = \dim \mathcal{X}_S + \dim \mathcal{V}_r(\mathfrak{g};\mathfrak{g}_0)(S).
\]

**Proof.** If \( S \) is typical then \( S \) is projective and both sides are zero. Let \( S \) be an atypical simple module (i.e. \( \text{atyp}(S) = \text{defect}(g) = 1 \)). By [17, Corollary 2.5], we have \( \mathcal{X}_S = \mathcal{X} \) by
the stratification described in [17, Section 2]. Using [7, Theorem 4.5, Corollary 4.8],

\[ \dim \mathcal{X} = \begin{cases} 
  3 & \text{if } g = \mathfrak{osp}(3|2) \\
  4 & \text{if } g = D(2,1;\alpha) \\
  7 & \text{if } g = G(3) \\
  8 & \text{if } g = F(4). 
\end{cases} \]

On the other hand, by [14, Corollary 4.4.2] we have \( \dim V_{(\emptyset,\emptyset_0)}(S) = \text{atyp}(S) = 1 \). The result then follows.

6.14. Geometric Interpretation of the \( z \)-complexity. Let \( g = \mathfrak{osp}(3|2) \), \( D(2,1;\alpha) \), \( G(3) \), or \( F(4) \). In each we give an explicit detecting subsuperalgebra that will be used to interpret the \( z \)-complexity geometrically. Let \( f_1 \subseteq g_1 \) be the span of the root vectors \( x_\alpha, x_{-\alpha} \) where

\[ \alpha = \begin{cases} 
  \varepsilon_1 + \delta & \text{if } g = \mathfrak{osp}(3|2) \\
  \varepsilon_1 + \varepsilon_2 + \varepsilon_3 & \text{if } g = D(2,1;\alpha) \\
  \varepsilon_3 + \delta & \text{if } g = G(3) \\
  \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta) & \text{if } g = F(4). 
\end{cases} \]

Set \( f_0 = [f_1, f_1] \). We define a three-dimensional subalgebra of \( g \) by

\[ f := f_0 \oplus f_1. \]

The Lie superalgebra \( f \) is classical and so has a support variety theory. Furthermore, as \( [f_0, f_1] = 0 \), it follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of \( f_1 \), i.e.,

\[ V_{(f_0)}(M) = V_{f_1}^{\text{rank}}(M) = \{ y \in f_1 \mid M \text{ is not projective as } U(\langle y \rangle)\text{-module} \} \cup \{0\}. \]

Using this detecting subsuperalgebra, we have the following geometric interpretation of the \( z \)-complexity:

**Theorem 6.14.1.** If \( S \) is a simple module over \( \mathfrak{osp}(3|2), D(2,1;\alpha), G(3), \) or \( F(4) \), then

\[ \dim V_{(f_0)}(S) = z_{\mathcal{F}}(S). \]

**Proof.** Note that \( V_{(f_0)}(\mathbb{C}) = f_1 \). Thus

\[ \dim V_{(f_0)}(\mathbb{C}) = z_{\mathcal{F}}(\mathbb{C}) = 2. \]

Moreover, [14, Theorem 4.1.1] implies that for any atypical simple module \( S \), we have

\[ V_{(f_0)}(S) = V_{(f_0)}(\mathbb{C}) \]

since \( S \) and \( \mathbb{C} \) have the same atypicality. The result follows for atypical simple modules. If \( S \) is typical, then it is projective, hence

\[ \dim V_{(f_0)}(S) = z_{\mathcal{F}}(S) = 0. \]

\[ \square \]
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Department of Mathematics and Physics, University of New Haven, 300 Boston Post Road, West Haven, CT 06516

E-mail address: helturkey@newhaven.edu