ON A TWO-SPECIES CROSS ATTRACTION SYSTEM IN HIGHER DIMENSIONS

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Abstract. We consider a degenerate chemotaxis model with two-species and two-stimuli in dimension \( d \geq 3 \). Under the hypothesis of integrable initial data with finite second moment and energy, we show local-in-time existence for any mass of free-energy solutions, namely weak solutions with some free energy estimates. We exhibit that the qualitative behavior of solutions is decided by a set of critical values: there is a critical value of a parameter pair in the system of equations for which there is a global-in-time energy solution and there exist blowing-up free-energy solutions under a criticality condition is violated for the parameter pair.

1. Introduction

Inspired by [7] (see also [9, 12]), for modeling the interaction and motion of two cell populations in breast cancer cell invasion models in \( \mathbb{R}^d, d \geq 3 \) we propose the following chemotaxis kind of system with two chemicals and a nonlinear diffusion term:

\[
\begin{align*}
\partial_t u &= \alpha_1 \left( \frac{d-2}{d} \right) \frac{1}{d+2} \Delta \left( u^{\frac{2d}{d+2}} \right) - \nabla \cdot (u \nabla v), \quad x \in \mathbb{R}^d, t > 0, \\
-\Delta v &= w, \quad x \in \mathbb{R}^d, t > 0, \\
\partial_t w &= \alpha_2 \left( \frac{d-2}{d} \right) \frac{1}{d+2} \Delta \left( w^{\frac{2d}{d+2}} \right) - \nabla \cdot (w \nabla z), \quad x \in \mathbb{R}^d, t > 0, \\
-\Delta z &= u, \quad x \in \mathbb{R}^d, t > 0,
\end{align*}
\]

with initial data satisfying

\[
\begin{align*}
u(x, 0) &= u_0(x); \quad w(x, 0) = w_0(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

In (1.1) \( u(x, t) \) and \( w(x, t) \) denote the macrophages and the tumor cells; \( v(x, t) \) and \( z(x, t) \) denote the concentration of the chemicals produced by \( w(x, t) \) and \( u(x, t) \), respectively and \( \alpha_1, \alpha_2 \in (0, 1]\). For simplicity the initial data are assumed to satisfy

\[
\begin{align*}
u_0 &\in L^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d), \quad \nabla u_0^{\frac{2d}{d+2}} \in L^2(\mathbb{R}^d), \quad u_0 \geq 0, \\
w_0 &\in L^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d), \quad \nabla w_0^{\frac{2d}{d+2}} \in L^2(\mathbb{R}^d), \quad w_0 \geq 0.
\end{align*}
\]

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We note that \( v(x, t) \) and \( z(x, t) \) are given by

\[
(1.5) \quad v(x, t) = K \ast w = c_d \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{w(y, t)}{|x - y|^{d-2}} \, dy,
\]

\[
(1.6) \quad z(x, t) = K \ast u = c_d \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(y, t)}{|x - y|^{d-2}} \, dy,
\]

with \( K(x) = \frac{c_d}{|x|^{d-2}} \) and \( c_d \) is the surface area of the sphere \( S^{d-1} \) in \( \mathbb{R}^d \). We can rewrite the system (1.1) as follows:

\[
\begin{align*}
\partial_t u &= \alpha_1 \left( \frac{d - 2}{d} \right) \|u\|_{\frac{2d}{d+2}}^2 \Delta \left( \frac{u}{|x|^{d-2}} \right) - \nabla \cdot (u \nabla K \ast w), \\
\partial_t w &= \alpha_2 \left( \frac{d - 2}{d} \right) \|w\|_{\frac{2d}{d+2}}^2 \Delta \left( \frac{w}{|x|^{d-2}} \right) - \nabla \cdot (w \nabla K \ast u),
\end{align*}
\]

with initial data (1.2). Note that mass is conserved for (1.1)(or (1.7)):

\[
M_1 := \int_{\mathbb{R}^d} u(x, t) \, dx = \int_{\mathbb{R}^d} u_0(x) \, dx; \quad M_2 := \int_{\mathbb{R}^d} w(x, t) \, dx = \int_{\mathbb{R}^d} w_0(x) \, dx.
\]

There is a free energy associated with equation (1.1):

\[
(1.9) \quad E_{\alpha}[u(t), w(t)] := \alpha_1 \|u\|_{\frac{2d}{d+2}}^2 + \alpha_2 \|w\|_{\frac{2d}{d+2}}^2 - c_d H[u, w],
\]

where

\[
(1.10) \quad H[u, w] := \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x, t)w(y, t)}{|x - y|^{d-2}} \, dx \, dy.
\]

The energy functional can be rewritten as

\[
(1.11) \quad E_{\alpha}[u(t), w(t)] := \left( \sqrt{\alpha_1} \|u\|_{\frac{2d}{d+2}} - \sqrt{\alpha_2} \|w\|_{\frac{2d}{d+2}} \right)^2 + \left( 2 \sqrt{\alpha_1 \alpha_2} \|u\|_{\frac{2d}{d+2}} \|w\|_{\frac{2d}{d+2}} - c_d H[u, w] \right).
\]

By the HLS inequality, (see (2.3) below), the second term is positive for all \( u, w \) in case

\[
(1.12) \quad \frac{2 \sqrt{\alpha_1 \alpha_2}}{c_d} \geq C_{HLS},
\]

the precise value of which was determined by Lieb for the exponents considered here. Suppose however, that (1.12) is violated. Let

\[
\delta := C_{HLS} - \frac{2 \sqrt{\alpha_1 \alpha_2}}{c_d}.
\]

Then if we choose \( u \) and \( w \) to be HLS optimizers such that

\[
\sqrt{\alpha_1} \|u\|_{\frac{2d}{d+2}} = \sqrt{\alpha_2} \|w\|_{\frac{2d}{d+2}},
\]

\[
E_{\alpha}[u(t), w(t)] = -\delta \|u\|_{\frac{2d}{d+2}} \|w\|_{\frac{2d}{d+2}}.
\]

Thus, (1.12) is the necessary and sufficient condition for non-negativity of \( E_{\alpha}[u(t), w(t)] \).
The connection between the free energy (1.9) and the equation (1.1) is that the latter can be written as a gradient flow with respect to Wasserstein metric, and then formally one has the dissipation of this free energy

\[
\frac{d}{dt} E_\alpha[u(t), w(t)] = -\int_{\mathbb{R}^d} u \left( \alpha_1 2 ||u|| \frac{d}{dx^2} u + \frac{d}{dx^2} \right) - \nabla v \right)^2 dx 
- \int_{\mathbb{R}^d} w \left( \alpha_2 2 ||w|| \frac{d}{dx^2} w + \frac{d}{dx^2} \right) - \nabla z \right)^2 dx.
\]

We now define the weak and free energy solutions for (1.1)-(1.2).

**Definition 1.1.** Let \( d \geq 3 \) and \( T > 0 \). Suppose \((u_0, w_0)\) satisfies (1.3)-(1.4). Then \((u, w)\) of nonnegative functions defined in \( \mathbb{R}^d \times (0, T) \) is called a weak solution if

(i) \((u, w)\) is \(C(0, T); L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0, T))\) and \(u, w \in L^2(0, T; H^1(\mathbb{R}^d))\).

(ii) \((u, w)\) satisfies

\[
\int_0^T \int_{\mathbb{R}^d} u \phi_1 \, dx \, dt + \int_{\mathbb{R}^d} u_0(x) \phi_1(x, 0) \, dx = \int_0^T \int_{\mathbb{R}^d} \left( \alpha_1 2 ||u|| \frac{d}{dx^2} u - u \nabla v \right)^2 \nabla \phi_1 \, dx \, dt.
\]

\[
\int_0^T \int_{\mathbb{R}^d} w \phi_2 \, dx \, dt + \int_{\mathbb{R}^d} w_0(x) \phi_2(x, 0) \, dx = \int_0^T \int_{\mathbb{R}^d} \left( \alpha_2 2 ||w|| \frac{d}{dx^2} w - w \nabla z \right)^2 \nabla \phi_2 \, dx \, dt,
\]

for any test functions \(\phi_i \in \mathcal{D}(\mathbb{R}^d) \times [0, T); i = 1, 2\) with \(v = K \ast w\) and \(z = K \ast u\).

**Definition 1.2.** Let \( T > 0 \). Then \((u, w)\) is called a free energy solution with initial data \((u_0, w_0)\) on \((0, T)\) if \((u, w)\) is a weak solution and moreover satisfies

\[
\left(u, w \frac{d}{dx^2} \right) \in \left(L^2(0, T; H^1(\mathbb{R}^d))\right)^2,
\]

and

\[
E_\alpha[u(t), w(t)] + \int_0^t \int_{\mathbb{R}^d} u \left( \alpha_1 2 ||u|| \frac{d}{dx^2} u + \frac{d}{dx^2} \right) - \nabla v \right)^2 \, dx \, ds 
+ \int_0^t \int_{\mathbb{R}^d} w \left( \alpha_2 2 ||w|| \frac{d}{dx^2} w + \frac{d}{dx^2} \right) - \nabla z \right)^2 \, dx \, ds \leq E_\alpha[u_0, w_0],
\]

for all \( t \in (0, T) \) with \(v = K \ast w\) and \(z = K \ast u\).

**Theorem 1.1.** Suppose that the initial data \((u_0, w_0)\) with \(||u_0||_1 = M_1, ||w_0||_1 = M_2\) satisfies (1.3) and (1.4).

(i) If \(\alpha_1, \alpha_2\) satisfy (1.12), then there exists a global free energy solutions.

(ii) If \(\alpha_1, \alpha_2\) do not satisfy (1.12), then it is possible to construct large initial data ensuring blow up in finite time.
2. Approximate System

For the existence of solutions as usual we first consider a regularized system

\begin{equation}
\partial_t u_\varepsilon = \alpha_1 \left( \frac{d-2}{d} \right) \|u_\varepsilon\|^4 \Delta((u_\varepsilon + \varepsilon)^{\frac{2d}{d+2}}) - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon), \quad x \in \mathbb{R}^d, \quad t > 0,
\end{equation}

\[ v_\varepsilon = \mathcal{K} \ast w_\varepsilon, \quad x \in \mathbb{R}^d, \quad t > 0, \]

\begin{equation}
\partial_t w_\varepsilon = \alpha_2 \left( \frac{d-2}{d} \right) \|w_\varepsilon\|^4 \Delta((w_\varepsilon + \varepsilon)^{\frac{2d}{d+2}}) - \nabla \cdot (w_\varepsilon \nabla z_\varepsilon), \quad x \in \mathbb{R}^d, \quad t > 0,
\end{equation}

with initial data satisfying

\begin{equation}
\begin{aligned}
\hspace{5pt} u_\varepsilon(x, 0) &= u_0^\varepsilon(x) \geq 0; \\
\hspace{5pt} w_\varepsilon(x, 0) &= w_0^\varepsilon(x) \geq 0, \quad x \in \mathbb{R}^d,
\end{aligned}
\end{equation}

with $u_0^\varepsilon$ and $w_0^\varepsilon$ being the convolution of $u_0$ and $w_0$ with a sequence of mollifiers and $\|u_0\|_1 = \|u\|_1 = M_1$ and $\|w_0\|_1 = \|w\|_1 = M_2$. As usual we will obtain a priori estimates for the regularized system (2.1)-(2.2) to ensure the existence of weak or free energy solution as $\varepsilon$ tends to 0.

By following the general procedure in the single population chemotaxis systems [14, 17, 6] one can obtain the following lemma for which no proof is provided here.

**Lemma 2.1.** Let $d \geq 3$. There exists $T_{max}^\varepsilon \in (0, \infty]$ denoting the maximal existence time such that the regularized system (2.1)-(2.2) has a unique nonnegative solution $(u_\varepsilon, w_\varepsilon) \in \left( W^{2,1}_p(Q_T) \right)^2$ with some $p > 1$, where $Q_T = \mathbb{R}^d \times (0, T)$ with $T \in (0, T_{max}^\varepsilon)$ and

\[ W^{2,1}_p(Q_T) := \left\{ u \in L^p(0, T; W^{2,p}(\mathbb{R}^d) \cap W^{1,p}(0, T; L^p(\mathbb{R}^d))) \right\}. \]

Moreover, if $T_{max}^\varepsilon < \infty$ then

\[ \lim_{t \to T_{max}^\varepsilon} \left( \|u_\varepsilon(\cdot, t)\|_\infty + \|w_\varepsilon(\cdot, t)\|_\infty \right) = \infty. \]

We also recall here a version of the Hardy-Littlewood-Sobolev inequality (HLS) that if

\[ \frac{1}{p} + \frac{1}{q} = 1 + \frac{\lambda}{d} \]

and $h_1 \in L^p(\mathbb{R}^d)$, $h_2 \in L^q(\mathbb{R}^d)$ with $p, q > 1$ then there exists a constant $C_{HLS} = C_{HLS}(d, \lambda, p) > 0$ such that

\begin{equation}
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{h_1(x)h_2(y)}{|x-y|^{d-\lambda}} \, dy \, dx \right| \leq C_{HLS} \|h_1\|_p \|h_2\|_q.
\end{equation}

**Lemma 2.2.** Let $T \in (0, T_{max}^\varepsilon]$. Suppose that there exists a constant $C > 0$ such that solution $(u_\varepsilon, w_\varepsilon)$ of the system (2.1)-(2.2) with initial data $(u_0^\varepsilon, w_0^\varepsilon)$ being the convolution of $(u_0, w_0)$ satisfies $\|u_\varepsilon(t)\|_{\frac{2d}{d+2}} \leq C$ and $\|w_\varepsilon(t)\|_{\frac{2d}{d+2}} \leq C$ for $t \in (0, T)$. Then there exists a constant $\overline{C} = C(d, u_0^\varepsilon, w_0^\varepsilon) > 0$ such that

\begin{equation}
\| (u_\varepsilon(t), w_\varepsilon(t)) \|_r \leq \overline{C},
\end{equation}

\[ \hspace{5pt} r = \min\{1, \frac{d}{d-\lambda}, \frac{2d}{d+2}\} \]
and
\[ (2.5) \quad \| (v_\varepsilon(t), z_\varepsilon(t)) \|_r + \| (\nabla v_\varepsilon(t), \nabla z_\varepsilon(t)) \|_r \leq C, \]
for \( r \in [1, \infty) \) and \( t \in (0, T) \).

**Proof.** For \( p > 1 \), we test the first equation in (2.1) by \( u_\varepsilon^{p-1} \) and integrate to find that
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^d} u_\varepsilon^p \, dx = -\alpha_1 \frac{d - 2}{d} ||u_\varepsilon||_{\frac{d}{d+2}}^2 \int_{\mathbb{R}^d} \nabla (u_\varepsilon^{p-1}) \cdot \nabla (u_\varepsilon + \varepsilon)^{\frac{2d}{d+2}} \, dx
\]
\[
+ \int_{\mathbb{R}^d} u_\varepsilon \nabla (u_\varepsilon^{p-1}) \cdot \nabla v_\varepsilon \, dx.
\]
(2.6)

It is not difficult to see that
\[
I = -2\alpha_1 \frac{d - 2}{d + 2} (p - 1) ||u_\varepsilon||_{\frac{d}{d+2}}^2 \int_{\mathbb{R}^d} u_\varepsilon^{p-2}(u_\varepsilon + \varepsilon)^{\frac{2d}{d+2}-1} |\nabla u_\varepsilon|^2 \, dx
\]
\[
\leq -2\alpha_1 \frac{d - 2}{d + 2} (p - 1) ||u_\varepsilon||_{\frac{d}{d+2}}^2 \int_{\mathbb{R}^d} u_\varepsilon^{p+\frac{d-2}{d+2}-3} |\nabla u_\varepsilon|^2 \, dx
\]
\[
= -4\alpha_1 \frac{d - 2}{d + 2} (p - 1) \frac{1}{p + \frac{d-2}{d+2}} \int_{\mathbb{R}^d} \left| \nabla \left( u_\varepsilon^{\frac{p+\frac{d-2}{2}}{2}} \right) \right|^2 \, dx.
\]
(2.7)

Similarly, by multiplying the third equation in (2.1) by \( u_\varepsilon^{p-1} \) and using \(-\Delta z_\varepsilon = u_\varepsilon\)
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^d} u_\varepsilon^p \, dx \leq -4\alpha_2 \frac{d - 2}{d + 2} (p - 1) \frac{1}{p + \frac{d-2}{d+2}} \int_{\mathbb{R}^d} \left| \nabla \left( u_\varepsilon^{\frac{p+\frac{d-2}{2}}{2}} \right) \right|^2 \, dx
\]
\[
+ \frac{p - 1}{p} \int_{\mathbb{R}^d} u_\varepsilon u_\varepsilon^p \, dx.
\]
(2.9)

Combining together
\[
\frac{1}{p} \left( \frac{d}{dt} \left( \int_{\mathbb{R}^d} u_\varepsilon^p \, dx + \int_{\mathbb{R}^d} u_\varepsilon^p \, dx \right) \right)
\]
\[
+ 4\alpha_1 \frac{d - 2}{d + 2} (p - 1) \frac{1}{p + \frac{d-2}{d+2}} \int_{\mathbb{R}^d} \left| \nabla \left( u_\varepsilon^{\frac{p+\frac{d-2}{2}}{2}} \right) \right|^2 \, dx
\]
\[
+ 4\alpha_2 \frac{d - 2}{d + 2} (p - 1) \frac{1}{p + \frac{d-2}{d+2}} \int_{\mathbb{R}^d} \left| \nabla \left( u_\varepsilon^{\frac{p+\frac{d-2}{2}}{2}} \right) \right|^2 \, dx
\]
\[
\leq \frac{p - 1}{p} \left( \int_{\mathbb{R}^d} u_\varepsilon^p \, dx + \int_{\mathbb{R}^d} u_\varepsilon^p \, dx \right).
\]
(2.10)

Now,
\[
\int_{\mathbb{R}^d} u_\varepsilon^p \, dx \leq \left( \int_{\mathbb{R}^d} u_\varepsilon^{p-1} \, dx \right)^{\frac{1}{p-1}} \left( \int_{\mathbb{R}^d} u_\varepsilon^{r_1} \, dx \right)^{\frac{1}{r_1}}
\]
(2.11)
and

\[
\int_{\mathbb{R}^d} u_\varepsilon w_\varepsilon^p \, dx \leq \left( \int_{\mathbb{R}^d} u_\varepsilon^r \, dx \right)^{\frac{1}{r_2}} \left( \int_{\mathbb{R}^d} w_\varepsilon^{pr_2} \, dx \right)^{\frac{1}{r_2}}
\]

by Hölder inequality with \( r_1, r_2 > 1, r'_1 = \frac{r_1}{r_1-1} \) and \( r'_2 = \frac{r_2}{r_2-1} \). We have

\[
\frac{2d}{d+2} < pr_1 < \frac{p + \frac{d-2}{d+2}}{d-2}
\]

and

\[
\frac{1}{r_1} > \max \left\{ \frac{d-2}{2d}, \frac{d-2}{d}, \frac{p}{p + \frac{d-2}{d+2}} \right\}.
\]

We now recall a variant of Gagliardo-Nirenberg inequality (see Lemma 6 in [16])

\[
||\Phi||_{k_2} \leq C^{2/r+m-1} ||\Phi||_{k_1}^{1-\sigma} ||\nabla\Phi||_{r_k-m-1}^{\sigma} \leq C^{\frac{2\sigma}{d+m-1}}
\]

with \( m \geq 1, k_1 \in [1, r + m - 1] \) and \( 1 \leq k_1 \leq k_2 \leq \frac{(r+m-1)d}{d-2} \) with \( d \geq 3 \) and

\[
\sigma = \frac{r+m-1}{r_k} \left( \frac{1}{k_1} - \frac{1}{k_2} \right) \left( \frac{d}{2} - \frac{1}{2} + \frac{r+m-1}{2k_1} \right)^{-1}.
\]

We pick \( r = p, m = \frac{2d}{d+2}, k_1 = \frac{2d}{d+2}, k_2 = pr_1 \) and use the above inequalities to deduce

\[
||u_\varepsilon||_{pr_1} \leq C ||u_\varepsilon||_{\frac{p}{d+2}}^{\frac{p}{d+2} - \sigma} ||\nabla u_\varepsilon||_{\frac{p}{d+2}}^{\frac{p}{d+2} \sigma},
\]

with

\[
\sigma = \frac{p + \frac{d-2}{d+2}}{2} \cdot \frac{d+2 - pr_1}{1} \cdot \frac{1}{\frac{d}{2} - \frac{1}{2} + \frac{p + \frac{d-2}{d+2}}{\frac{d+2}{2}}}
\]

Using the bound \( ||u_\varepsilon(t)||_{\frac{2d}{d+2}} \leq C \) for \( t \in (0, T) \) we deduce that

\[
\left( \int_{\mathbb{R}^d} u_\varepsilon^{pr_1} \, dx \right)^{\frac{1}{pr_1}} \leq C ||\nabla u_\varepsilon||_{\frac{p}{d+2}}^{\frac{p}{d+2} \sigma} \cdot \frac{\frac{d+2}{d} \cdot \frac{1}{r_1}}{\frac{1}{2} + \frac{p + \frac{d-2}{d+2}}{\frac{d+2}{2}}}
\]

Likewise, using

\[
\frac{2d}{d+2} < r'_1 < \frac{(p + \frac{d-2}{d+2})d}{d-2}
\]

and Gagliardo-Nirenberg inequality with \( ||w_\varepsilon(t)||_{\frac{2d}{d+2}} \leq C \)

\[
||w_\varepsilon||_{r'_1} \leq C ||\nabla w_\varepsilon||_{\frac{d+2}{d+2}}^{\frac{d+2}{d+2} \cdot \frac{1}{r_1}} \cdot \frac{\frac{d+2}{d} \cdot \frac{1}{r_1}}{\frac{1}{2} + \frac{p + \frac{d-2}{d+2}}{\frac{d+2}{2}}}
\]

Then
\[
\left( \int_{\mathbb{R}^d} u_\varepsilon^{pr_1} \, dx \right)^{1/r_1} \leq C \left\| \nabla u_\varepsilon \right\|_{L^2}^{\frac{p^2}{d+2}} \left( \int_{\mathbb{R}^d} w_\varepsilon^{r_1'} \, dx \right)^{1/r_1'}
\]

(2.19)

\[
\leq C \left\| \nabla w_\varepsilon \right\|_{L^2}^{\frac{p(d+2)}{2d}} \left( \int_{\mathbb{R}^d} u_\varepsilon^{p_2} \, dx \right)^{\frac{p_2}{2d}} \left( \int_{\mathbb{R}^d} w_\varepsilon^{p_2} \, dx \right)^{\frac{p_2}{2d}}
\]

We have

\[
\frac{2d}{d+2} < r_2 < \frac{(p+\frac{d-2}{d+2})d}{d-2}.
\]

Then by Gagliardo-Nirenberg inequality

\[
\left( \int_{\mathbb{R}^d} u_\varepsilon^{r_2} \, dx \right)^{1/r_2} \leq C \left\| \nabla u_\varepsilon \right\|_{L^2}^{\frac{p^2}{d+2}} \left( \int_{\mathbb{R}^d} w_\varepsilon^{r_2'} \, dx \right)^{1/r_2'}
\]

(2.20)

as \( \|u_\varepsilon(t)\|_{\frac{2d}{d+2}} \leq C \) and \( \|w_\varepsilon(t)\|_{\frac{2d}{d+2}} \leq C \). Now,

\[
\frac{2d}{d+2} < p r_2' < \frac{(p+\frac{d-2}{d+2})d}{d-2}.
\]

One has

\[
\|w_\varepsilon\|_{pr_2}^p \leq C \|w_\varepsilon\|_{\frac{2d}{d+2}}^{p(1-\sigma)} \left\| \nabla w_\varepsilon \right\|_{L^2}^{\frac{p^2}{d+2}} \left( \int_{\mathbb{R}^d} w_\varepsilon^{p_2} \, dx \right)^{\frac{p_2}{2d}}
\]

(2.21)

\[
\leq C \left\| \nabla w_\varepsilon \right\|_{L^2}^{\frac{p^2}{2d}} \left( \frac{p^2}{d+2} + \frac{p_2}{2d} \right)^{-\frac{p_2}{2d}},
\]

with

\[
\sigma = \frac{(d+2)\frac{d-2}{4d} - \frac{1}{pr_2'}}{\frac{1}{d} - \frac{1}{2} + (d+2)\frac{d-2}{4d}}.
\]

Then,

\[
\left( \int_{\mathbb{R}^d} u_\varepsilon^{r_2} \, dx \right)^{1/r_2} \left( \int_{\mathbb{R}^d} w_\varepsilon^{pr_2} \, dx \right)^{1/r_2'} \leq C \left\| \nabla u_\varepsilon \right\|_{L^2}^{\frac{d+2}{2d}} \left( \frac{1}{2} + (d+2)\frac{d^2}{4d^2} \right)^{-\frac{d+2}{2d}}
\]

(2.22)

\[
\times \left\| \nabla w_\varepsilon \right\|_{L^2}^{\frac{d+2}{2d}} \left( \frac{1}{2} + (d+2)\frac{d^2}{4d^2} \right)^{-\frac{d+2}{2d} - \frac{d+2}{2d}}.
\]
Combining, one gets
\[
\frac{1}{p} \frac{d}{dt} \left( \int_{\mathbb{R}^d} u^p_w + w^p dx + \delta_1(d, \alpha_1) \int_{\mathbb{R}^d} \left| \nabla u^p_w \right|^2 \right) + \delta_2(d, \alpha_2) \int_{\mathbb{R}^d} \left| \nabla w^p \right|^2 dx 
\leq \frac{p-1}{p} \left( \int_{\mathbb{R}^d} u^p_w \right)^{1/r_1} \left( \int_{\mathbb{R}^d} w^p \right)^{1/r_1} + \frac{p-1}{p} \left( \int_{\mathbb{R}^d} w^p \right)^{1/r_2} \left( \int_{\mathbb{R}^d} u^p_w \right)^{1/r_2} 
(2.23)
\]

where \( p > 0 \), \( \gamma > 0 \), and can be calculated explicitly.

We now deal with the boundedness of \( u_w \) and \( w^p \) in \( L^p \)-space: Let \( \gamma_1 > 0, \gamma_2 > 0 \) be such that \( \gamma_1 + \gamma_2 < 2 \). For \( \varepsilon > 0 \) by Young’s inequality we have
\[
\alpha^{\gamma_1} \beta^{\gamma_2} \leq \varepsilon (\alpha^2 + \beta^2) + C.
\]

From the above calculations, there exists some \( p > \bar{p} \) with some \( \bar{p} > 1 \) such that
\[
\frac{\mu(d+2)}{2d} - \frac{1}{r_1} + \frac{1}{d - \frac{1}{2} + \frac{1}{4d/(d+2)}} < 2.
\]

The above two inequalities can be simplified further (left to the reader). Now,
\[
\frac{1}{p} \left( \int_{\mathbb{R}^d} u^p_w dx + \int_{\mathbb{R}^d} w^p dx + \delta_1(d, \alpha_1) \int_{\mathbb{R}^d} \left| \nabla u^p_w \right|^2 \right) + \delta_2(d, \alpha_2) \int_{\mathbb{R}^d} \left| \nabla w^p \right|^2 dx \leq C,
(2.24)
\]

with \( \delta_1, \delta_2 > 0 \). Using Gagliardo-Nirenberg inequality with \( \|u\|_1 = M_1 \) and \( \|w\|_1 = M_2 \) and Young’s inequality
\[
\frac{1}{p} \left\| u^p_w \right\|_p^p \leq C \left\| \nabla u^p_w \right\|_{\frac{p+1}{p-1}} \left\| \frac{p+1}{p-1} \right\|_{\frac{p+1}{p-1}} \leq \frac{p+1}{p-1} \left( \int_{\mathbb{R}^d} \left| \nabla u^p_w \right|^2 \right)^{1/2} dx + C
(2.25)
\]

\[
\leq \frac{4d}{d+2} \left( \frac{p-1}{p+1} \right)^2 \int_{\mathbb{R}^d} \left| \nabla u^p_w \right|^2 dx + C.
\]
and
\begin{equation}
\frac{1}{p} \int_{\mathbb{R}^d} w_\varepsilon^p \, dx \leq \frac{4d (p-1)}{(p + \frac{d-2}{2})^2} \int_{\mathbb{R}^d} |\nabla w_\varepsilon|^{\frac{p+\frac{d}{2}}{2}} \, dx + C,
\end{equation}
by
\begin{equation}
\frac{p-1}{\frac{d}{2} - \frac{1}{2} + \frac{d+2}{2}} < 2.
\end{equation}

Writing $g(t) := \frac{1}{p} (\int_{\mathbb{R}^d} (u_\varepsilon^p + w_\varepsilon^p) \, dx)$ we get
\begin{equation}
y'(t) + y(t) \leq C \quad \text{for} \quad t \in (0, T).
\end{equation}
Then,
\begin{equation}
||u_\varepsilon(t)||_p, \ ||w_\varepsilon(t)||_p \leq C \quad \text{for} \quad r \in [1, \infty) \quad \text{and} \quad t \in (0, T).
\end{equation}
From this we deduce that
\begin{equation}
||(u_\varepsilon(t), w_\varepsilon(t))||_r \leq C, \quad \text{for} \quad r \in [1, \infty) \quad \text{and} \quad t \in (0, T).
\end{equation}

We now try to improve the regularities of $v$ and $z$. Since
\begin{equation}
v_\varepsilon(x, t) = K \ast w_\varepsilon = c_d \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{w_\varepsilon(y, t)}{|x - y|^{d-2}} \, dy,
\end{equation}
\begin{equation}
z_\varepsilon(x, t) = c_d \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u_\varepsilon(y, t)}{|x - y|^{d-2}} \, dy,
\end{equation}
by HLS inequality we have
\begin{equation}
||\nabla v_\varepsilon||_r \leq c_d (d-2) ||I_1(w_\varepsilon)||_r \leq C ||w_\varepsilon||_{\frac{dr}{d+r}}
\end{equation}
\begin{equation}
||\nabla z_\varepsilon||_r \leq C ||u_\varepsilon||_{\frac{dr}{d+r}}.
\end{equation}
On the other hand, Calderon-Zygmund inequality implies that
\begin{equation}
||\partial_x \partial_x v_\varepsilon||_r \leq C ||w_\varepsilon||_r,
\end{equation}
\begin{equation}
||\partial_x \partial_x z_\varepsilon||_r \leq C ||u_\varepsilon||_r, \quad 1 \leq i, j \leq d.
\end{equation}
The above estimates and the Morrey’s inequality imply that
\begin{equation}
||(v_\varepsilon(t), z_\varepsilon(t))||_r + ||(\nabla v_\varepsilon(t), \nabla z_\varepsilon(t))||_r \leq C, \quad \text{for} \quad r \in [1, \infty] \quad \text{and} \quad t \in (0, T).
\end{equation}

\begin{lemma}
Under the assumptions of Lemma 2.2, there exists $C > 0$ independent of $\varepsilon$ such that the strong solution of (2.1) satisfies
\begin{equation}
||(u_\varepsilon(t), w_\varepsilon(t))||_\infty \leq C, \quad \forall t \in (0, T).
\end{equation}
Moreover, there exists a global weak solution $(u, w)$ of (1.1)-(1.2) which also satisfies a uniform bound.
\end{lemma}

\begin{proof}
Using Lemma 2.2 one can apply the Moser’s iteration to obtain a priori estimate of solution in $L^\infty$. Then this solution can be extended globally in time from the extensibility criterion in Lemma 2.1 establishing (2.31),
one can refer to Proposition 10 of [16]. From (2.31) there is \((u, v, w, z)\) with regularities given in Definition 1.1 such that, up to a subsequence, \(\varepsilon \to 0\),

\[
2.32 \quad u_{\varepsilon_n} \to u \text{ strongly in } C([0, T); L^p_{\text{loc}}(\mathbb{R}^d)) \text{ and a.e. in } \mathbb{R}^d \times (0, T), \\
\nabla u_{\varepsilon_n}^{2d/(d+2)} \to \nabla u^{2d/(d+2)} \text{ weakly } - \ast \text{ in } L^\infty((0, T); L^2(\mathbb{R}^d)), \\
w_{\varepsilon_n} \to w \text{ strongly in } C([0, T); L^p_{\text{loc}}(\mathbb{R}^d)) \text{ and a.e. in } \mathbb{R}^d \times (0, T), \\
\nabla w_{\varepsilon_n}^{2d/(d+2)} \to \nabla w^{2d/(d+2)} \text{ weakly } - \ast \text{ in } L^\infty((0, T); L^2(\mathbb{R}^d)), \\
v_{\varepsilon_n}(t) \to v(t) \text{ strongly in } L^r_{\text{loc}} \text{ and a.e. in } (0, T), \\
\Delta v_{\varepsilon_n}(t) \to \Delta v(t) \text{ weakly in } L^r_{\text{loc}}(\mathbb{R}^d) \text{ and a.e. in } (0, T), \\
z_{\varepsilon_n}(t) \to z(t) \text{ strongly in } L^r_{\text{loc}} \text{ and a.e. in } (0, T), \\
\Delta z_{\varepsilon_n}(t) \to \Delta z(t) \text{ weakly in } L^r_{\text{loc}}(\mathbb{R}^d) \text{ and a.e. in } (0, T),
\]

where \(p \in (1, \infty), \ r \in (1, \infty) \) and \(T \in (0, \infty)\). Since the above convergence can be established following [15, Section 4], and the details are left to the reader. Therefore, we have a global weak solution \((u, v, w, z)\) over \(\mathbb{R}^d \times (0, T)\) with \(T > 0\).

We now follow [14] to establish that a global weak solution is also a global free energy solution.

**Lemma 2.4.** Consider a global weak solution in Lemma 2.3. Then it is also a global free energy solution \((u, w)\) of (1.1) given in Definition 1.2.

**Proof.** The proof can be done following the proof of Proposition 2.1 in [2] and [14]. The details are omitted. □

**Theorem 2.5.** Under the assumption (1.3)-(1.4) on the initial data with \(|u_0| = M_1, \ |w_0| = M_2, \ |v_0| = M_3, \ |z_0| = M_4\), there exists a \(T_{\text{max}} \in (0, \infty)\) and a free energy solution \((u, w)\) over \(\mathbb{R}^d \times [0, T_{\text{max}}]\) of (1.1) such that either \(T_{\text{max}} = \infty\) or \(T_{\text{max}} < \infty\) and

\[
2.33 \quad \lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_\infty + \|w(\cdot, t)\|_\infty) = \infty.
\]

Moreover, let \(\alpha_1, \alpha_2\) satisfy (1.12), Then, if \(T_{\text{max}} < \infty\)

\[
2.34 \quad \lim_{t \to T_{\text{max}}} \|u(\cdot, t)\|_{\frac{2d}{d+2}} = \infty = \lim_{t \to T_{\text{max}}} \|w(\cdot, t)\|_{\frac{2d}{d+2}}.
\]

**Proof.** For \((u_0, w_0)\) satisfying (1.3)-(1.4) local existence and (2.33) can be established as in the proof of Theorem 1.1. of [14] by employing approximation arguments.

\[
2.35 \quad \alpha_1 \|u\|_{\frac{2d}{d+2}}^2 + \alpha_2 \|w\|_{\frac{2d}{d+2}}^2 \leq c_d H[u, w] + E_\alpha[u_0, w_0].
\]

Modifying Lemma 3.1 of [7] we deduce that
\[
\begin{align*}
\alpha_1 & \|u\|_{\frac{2d}{d+2}}^2 + \alpha_2 \|w\|_{\frac{2d}{d+2}}^2 \leq c_d \eta \|u\|_{\frac{4d}{d+2}}^\frac{2d}{d+2} + c_d C \eta^{-\frac{d+2}{d-2}} M_2^{\frac{4d^2}{d+2} + \frac{8d}{(d-2)(d+2)}} \|w\|_{\frac{4d}{d+2}}^\frac{4d-10d}{d+2} \\
& + E_\alpha[u_0, w_0] \leq c_d \eta \|u\|_{\frac{2d}{d+2}}^2 + c_d C \eta^{-\frac{d+2}{d-2}} \|w\|_{\frac{2d}{d+2}}^2 + C,
\end{align*}
\]

where we have used (1.12) in the last inequality to discard \( E_\alpha[u_0, w_0] \). Taking \( \eta \) small enough, one has

\[
\|u(t)\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} \leq C \|w(t)\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} + C, \quad \text{for} \quad t \in (0, T_{\text{max}}),
\]

and if \( \eta \) is large enough one observes that

\[
\|w(t)\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} \leq C' \|u(t)\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} + C', \quad \text{for} \quad t \in (0, T_{\text{max}}).
\]

Hence, (2.34) holds by (2.33), (2.37) and (2.38).

\[ \square \]

**Theorem 2.6.** Let \( d \geq 3 \) and let \((u_0, w_0)\) be initial data with \( \|u_0\|_1 = M_1 \) and \( \|w_0\|_1 = M_2 \) satisfying (1.3)-(1.4). Then, if \( \alpha_1, \alpha_2 \) satisfy (1.12), then (1.1) has a global free energy solution given in Definition 1.2.

**Proof.** We have

\[
|H[u, w]| \leq \frac{(d+2)}{2c_d(d-2)} \|u\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} + C \|w\|_1^{\frac{2d}{d+2}} \|w\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} \leq \frac{d+2}{2c_d(d-2)} \left[ \|u\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} + \|w\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} \right] + C,
\]

using the Young’s inequality. Combining this with

\[
\alpha_1 \|u\|_{\frac{2d}{d+2}}^2 + \alpha_2 \|w\|_{\frac{2d}{d+2}}^2 \leq c_d H[u, w] + E_\alpha[u_0, w_0]
\]

one deduces that

\[
\|u\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} \leq C \quad \text{and} \quad \|w\|_{\frac{2d}{d+2}}^\frac{2d}{d+2} \leq C,
\]

and this implies, by employing Theorem 2.5, the existence of a free energy solution.

\[ \square \]

### 3. Blow-up

In this section we deal with the finite-time blow-up phenomenon. We begin with a simple lemma stating the time-evolution of second moment of solutions.
Lemma 3.1. Let \((u_0, w_0)\) satisfy (1.3) and (1.4) and let \((u, w)\) be a free energy solution of (1.1) on \([0, T_{\text{max}}]\) with \(T_{\text{max}} \in (0, \infty)\). Then,

\[
\frac{d}{dt} I(t) = G(t), \quad \forall t \in (0, T_{\text{max}}),
\]

where

\[
I(t) := \int_{\mathbb{R}^d} |x|^2 (u(x, t) + w(x, t)) \, dx
\]

and

\[
G(t) := C(\alpha_1, u) \int_{\mathbb{R}^d} u(x, t)\frac{2d}{d+2} \, dx + C(\alpha_2, w) \int_{\mathbb{R}^d} w(x, t)\frac{2d}{d+2} \, dx
\]  

\[-2c_d(d-2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x, t)w(y, t)}{|x-y|^{d-2}} \, dy \, dx.
\]

with

\[
C(\alpha_1, u) := 2(d-2)\alpha_1 ||u||^{\frac{4d}{d+2}},
\]

\[
C(\alpha_2, w) := 2(d-2)\alpha_2 ||w||^{\frac{4d}{d+2}}.
\]

Notice that with the definitions,

\[(3.1) \quad G(t) = 2(d-2) E_\alpha [u(t), v(t)] .\]

Proof. Straightforward calculations lead to

\[
\frac{d}{dt} I(t) = \int_{\mathbb{R}^d} |x|^2 (u(x, t) + w(x, t)) \, dx
\]

\[
= \int_{\mathbb{R}^d} |x|^2 \left[ C(\alpha_1, u) \Delta \left( u^{\frac{2d}{d+2}} \right) - \nabla \cdot (u \nabla v) \right. \\
\left. + C(\alpha_2, w) \Delta \left( w^{\frac{2d}{d+2}} \right) - \nabla \cdot (w \nabla z) \right] \, dx
\]

\[
= 2dC(\alpha_1, u) \int_{\mathbb{R}^d} u(x, t)^{\frac{2d}{d+2}} d x + C(\alpha_2, w) \int_{\mathbb{R}^d} w(x, t)^{\frac{2d}{d+2}} d x
\]

\[
+ 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla K(x-y)] u(x, t)w(y, t) \, dy \, dx
\]

\[
+ 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla K(x-y)] u(y, t)w(x, t) \, dy \, dx,
\]
with \( K(x) = \frac{c_d}{|x|^{d-2}} \) and \( c_d \) is the surface area of the sphere \( S^{d-1} \) in \( \mathbb{R}^d \). Now,

\[
2 \int_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla K(x - y)] u(x, t)w(y, t) \, dy \, dx
\]

\[
= -2c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(x - y) \cdot x}{|x - y|^d} u(x, t)w(y, t) \, dy \, dx
\]

\[
= -2c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{|x - y|^d} u(x, t)w(y, t) \, dy \, dx
\]

\[
+ 2c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y \cdot x}{|x - y|^d} u(x, t)w(y, t) \, dy \, dx
\]

Similarly,

\[
2 \int_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla K(x - y)] u(y, t)w(x, t) \, dx \, dy
\]

\[
= -c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{|x - y|^d} u(y, t)w(x, t) \, dx \, dy
\]

\[
- c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|y|^2}{|x - y|^d} u(y, t)w(x, t) \, dx \, dy
\]

\[
+ 2c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y \cdot x}{|x - y|^d} u(y, t)w(x, t) \, dx \, dy
\]

Combining, we deduce that

\[
\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 (u(x, t) + w(y, t)) \, dx = 2dC(\alpha_1, u) \int_{\mathbb{R}^d} u^{\frac{2d}{d+2}} \, dx
\]

\[
+ 2dC(\alpha_2, w) \int_{\mathbb{R}^d} w^{\frac{2d}{d+2}} \, dx
\]

\[
- c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2 + |y|^2}{|x - y|^d} u(x, t)w(y, t) \, dy \, dx
\]

\[
- c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2 + |y|^2}{|x - y|^d} u(y, t)w(x, t) \, dx \, dy
\]

\[
+ 4c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y \cdot x}{|x - y|^d} u(x, t)w(y, t) \, dy \, dx
\]

\[
= 2dC(\alpha_1, u) \int_{\mathbb{R}^d} u^{\frac{2d}{d+2}} \, dx + 2dC(\alpha_2, w) \int_{\mathbb{R}^d} w^{\frac{2d}{d+2}} \, dx
\]

\[
- 2c_d(d - 2) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x, t)w(y, t)}{|x - y|^{d-2}} \, dy \, dx.
\]

□
We now observe that
\[
\frac{d}{dt} I(t) = G(t) = 2(d - 2)E_\alpha[u(t), w(t)] \\
\leq 2(d - 2)E_\alpha[u_0, w_0] = G(0).
\]  
(3.2)

As we have seen above, \(\alpha_1\) and \(\alpha_2\) are such that (1.12) is violated, then we may take the initial data \((u_0, w_0)\) to consist of appropriate multiples of HLS optimizers, and then we shall have \(E_\alpha[u_0, w_0] < 0\), and hence \(G(0) < 0\). By the monotonicity if the energy, we will then have \(G(t) \leq G(0) < 0\) for all \(t\) such that a regular solution exists. But then (3.2) will imply that the second moment will be negative after some time which is impossible by the non-negativity of \(u\) and \(w\). Hence blow-up occurs in a finite time.

**Theorem 3.2.** Let \(\alpha_1, \alpha_2\) be such that (1.12) is violated. Then one can find some initial data \((u_0, w_0)\) satisfying (1.3) and (1.4) such that free energy solution \((u, w)\) of (1.1) with \((u, w)|_{t=0} = (u_0, w_0)\) blows up in finite time.

**Proof.** We consider an initial data \((u_0, w_0)\) with \(G(0) < 0\). By a continuity argument there exists a \(T^* > 0\) such that
\[
G(t) < \frac{G(0)}{2}, \quad \text{for all} \quad t \in [0, T^*].
\]
(3.3)

Integrating (3.3), it follows that
\[
I(T^*) < I(0) + \frac{G(0)}{2}T^*.
\]
(3.4)

We may choose the initial data in such a way that the right hand side of (3.4) is negative. But this leads to a contradiction since it implies that \(I(T^*) < 0\) but \(I(t)\) is always non-negative for all \(t > 0\). Hence the solution \((u(t), w(t))\) blows-up in finite time.

\[\square\]

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