Turbulent Pair Diffusion

F. Nicolleau
The University of Sheffield, Department of Mechanical Engineering, Mappin Street, Sheffield, S1 3JD, UK

J. C. Vassilicos
Imperial College of Science, Technology and Medicine, Department of Aeronautics, Prince Consort Road, South Kensington, London, SW7 2BY, UK

(March 30, 2002)

Kinematic Simulations of turbulent pair diffusion in planar turbulence with a $k^{-5/3}$ energy spectrum reproduce the results of the laboratory measurements of Jullien et al. Phys. Rev. Lett. 82, 2872 (1999), in particular the stretched exponential form of the PDF of pair separations and their correlation functions. The root mean square separation is found to be strongly dependent on initial conditions for very long stretches of times. This dependence is consistent with the topological picture of turbulent pair diffusion where pairs initially close enough travel together for long stretches of time and separate violently when they meet straining regions around hyperbolic points. A new argument based on the divergence of accelerations is given to support this picture.

PACS numbers: 47.27.Eq 47.27.Gs 92.10.Lq 47.27.Qb

The rate with which pairs of points separate in phase space or in physical space is of central importance to the study of dynamical systems. Pairs of points in the phase space of a low-dimensional chaotic dynamical system separate exponentially. This is the celebrated butterfly effect: dynamics are extremely sensitive to initial conditions. Pairs of fluid elements in fully chaotic flows also separate exponentially. However, in fully developed homogeneous and isotropic turbulence, Richardson’s law stipulates that fluid element pairs separate on average algebraically and in such a way that their separation statistics in a certain range of times are the same irrespective of initial conditions. Richardson’s law is therefore a remarkable claim of universality. Specifically, it stipulates that in a range of times where the root mean square separation $\Delta^2/\sigma^2$ is larger than the Kolmogorov length-scale $\eta$ and smaller than the integral length-scale $L$, $\Delta^2$ is increasingly well approximated by

$$\Delta^2 = G_\Delta t^\beta$$

for increasing values of $L/\eta$, where $t$ is time, $\epsilon$ is the kinetic energy rate of dissipation per unit mass and $G_\Delta$ is a universal dimensionless constant.

Richardson accompanied his empirical law with a prediction for the probability density function (PDF) of pair separations $\Delta$. The effective diffusivity approach leading to this prediction was criticised by Batchelor who developed a different approach leading to but also to a different form of the PDF. Kraichnan derived yet another expression for the PDF based on his Lagrangian history direct interaction approximation and so did Shlesinger et al. on the assumption that turbulent pair diffusion is well described by Lévy walks.

Setting $\sigma(t) \equiv \Delta^{21/2}$ and $r \equiv \Delta/\sigma$, the PDFs of $\Delta$ predicted by Richardson, Batchelor and Kraichnan are all of the form

$$P(\Delta, t) \sim \sigma^{-1} \exp(-\alpha r^\beta)$$

with different values of the dimensionless parameters $\alpha$ and $\beta$. Richardson’s prediction for the exponent $\beta$ is $\beta = 2/3$, Batchelor’s is $\beta = 2$ and Kraichnan’s is $\beta = 4/3$. The Lagrangian modelling approach of Shlesinger et al. leads to a totally different, in fact algebraic, PDF form. More recently Jullien et al. reported laboratory measurements of $P(\Delta, t)$ which are well fitted by with $\alpha \approx 2.6$ and $\beta \approx 0.5 \pm 0.1$. These laboratory measurements invalidate the PDF predictions of Batchelor, Kraichnan and Shlesinger et al. and might raise a question mark over the PDF prediction of Richardson even though they can be considered consistent with it if we account for experimental uncertainties. Jullien et al. also observed that fluid element pairs stay close to each other for a long time until they separate quite suddenly, a behaviour which seems qualitatively at odds with the effective diffusivity approach adopted by Richardson to derive with $\beta = 2/3$. In particular, they measured the Lagrangian autocorrelation function of pair separations $R(t, \tau) \equiv <\Delta(t)\Delta(t + \tau)>$ for $-t \leq \tau \leq 0$ and found a Lagrangian pair correlation time $\tau_c \approx 0.6t$ which is surprisingly long. In this paper we report that Kinematic Simulation (KS) reproduces the experimental results of Jullien et al. KS is a Lagrangian model of turbulent diffusion which is distinct from Lévy walks and makes no use of Markovianity assumptions so that it cannot be reduced to an effective diffusivity approach such as
Richardson’s\textsuperscript{[2]}. Furthermore, the observation that fluid element pairs travel close together for long stretches of time until they separate quite suddenly has in fact already been made using KS\textsuperscript{[8].}

KS Lagrangian modelling consists in integrating fluid element trajectories by solving $\frac{dx(t)}{dt} = u(x(t), t)$ in synthesised velocity fields $u(x, t)$. Statistically homogeneous, isotropic and stationary KS velocity fields are superpositions of random Fourier modes\textsuperscript{[6].} KS velocity fields are gaussian but not delta-correlated in time\textsuperscript{[2]}, and this non-Markovianity is an essential ingredient in KS. The Lagrangian measurements of\textsuperscript{[3]} were made in an inverse cascade two-dimensional turbulent flow. Our KS velocity field is therefore prescribed to be planar and given by

$$u = \sum_{m=1}^{m=M} \left[ A_m \times \hat{k}_m \cos (\mathbf{k}_m \cdot \mathbf{x} + \omega_m t) + B_m \times \hat{k}_m \sin (\mathbf{k}_m \cdot \mathbf{x} + \omega_m t) \right]$$

where $M = 500$ is the number of modes, $\mathbf{k}_m$ is a random unit vector ($\mathbf{k}_m = k_m \hat{k}_m$) normal to the plane of the flow whilst the vectors $A_m$ and $B_m$ are in that plane. The random choice of directions for the $m$\textsuperscript{th} wavemode is independent of the choices of the other wavemodes. Note that the velocity field $u$ is incompressible by construction. The amplitudes $A_m$ and $B_m$ of the vectors $A_m$ and $B_m$ are determined by the Kolmogorov energy spectrum $E(k)$ via the relations $A_m^2 = B_m^2 = E(k_m) \Delta k_m$ where $\Delta k_m = (k_{m+1} - k_{m-1})/2$. Finally the unsteadiness frequencies $\omega_m$ are determined by the eddy turnover time of wavemode $m$, that is $\omega_m = 0.5 \sqrt{\frac{m^3}{k_m}} E(k_m)$.

The Lagrangian measurements in\textsuperscript{[3]} were made when the two-dimensional flow had developed an inverse cascade with a well-defined $k^{-5/3}$ energy spectrum. The energy spectrum we have therefore chosen for this study is $E(k) \approx \frac{2u'^2}{3L^2} k^{-5/3}$ in the range $\frac{2\pi}{L} \leq k \leq \frac{2\pi}{\eta}$ and equal to 0 outside this range ($u'^2$ is the total kinetic energy of the turbulence). The $M = 500$ wavenumbers are algebraically distributed between $\frac{2\pi}{L}$ and $\frac{2\pi}{\eta}$. The eddy turnover time at the largest wavenumber $\frac{2\pi}{\eta}$ can be considered to correspond to a Kolmogorov time scale $\tau_\eta$.

![FIG. 1. Semi-log plot of $\sigma_p(r)$ as a function of $r = \frac{x}{\eta}$ in the case $\frac{L}{\eta} = 1691$. $\frac{L}{\eta} = 0.1$ and $\frac{L}{\eta} = + 0.10, \times 0.20, \ast 0.30$, empty box 0.40, black box 0.5. $\Delta x = 1$ and $\frac{L}{\eta} = \bigcirc 0.10$, $\bigcirc 0.20, \bigcirc 0.30$, black triangle 0.4 and $\bigtriangledown 0.5$. The solid line is $\sigma_p(r) \sim e^{2.9s^0.5}$.](image)

The inertial range ratio $\frac{L}{\eta}$ is $O(10)$ in the laboratory experiment of\textsuperscript{[3]} but here we have also run simulations with $\frac{L}{\eta} = 0, 100, 1691, 11180, 38748, 250000$. For initial pair separations $\Delta_0$ smaller or equal to $\eta$ our KS integrations lead to $\sigma P(\Delta, t) \sim \exp(-\alpha \Delta^2)$ where $r = \Delta/\sigma(t)$ with $2.6 \leq \alpha \leq 3$ and $0.46 \leq \beta \leq 0.5$ in very good agreement with the laboratory results of\textsuperscript{[3]} and for all the $\frac{L}{\eta}$ values that we tried (see example in Figure 1). (We record, however, that this PDF does seem to depend on the initial separation $\Delta_0$ when $\Delta_0 > \eta$.) It has already been noted in\textsuperscript{[3]} that KS gives non-gaussian-stretched exponential PDFs of pair separations without, however, estimating $\alpha$ and $\beta$. The synthetic velocity fields of\textsuperscript{[3]} lead to the Richardson stretched exponential form with $\beta = 2/3$. An approach based on asymmetric Levy walks\textsuperscript{[11]} gives rise to stretched exponential forms of $\sigma P(\Delta, t)$ where $\beta$ can be tuned as a function of a persistence parameter.
FIG. 2. Lagrangian separation correlation factor \( R(t, \tau) / \sigma^2(t) \) as a function of \( \frac{\tau}{t} \) in the case \( \frac{L}{\eta} = 1691, \Delta_0 = 1 \) and \( \frac{tu}{L} = +2.01, \times 1.51, * 1.01, \) empty box 0.76, black box 0.11, \( \circ \) 0.06. \( \Delta_0 / \eta = 0 \). 1 and \( \frac{tu}{L} = \bullet 1.25, \triangle 0.62, \) black triangle 0.16, \( \nabla 0.08, \) black triangle down 0.04.

Following \[6\] we also calculate correlation functions of pair separations, i.e. \( R(t, \tau) \equiv \langle \Delta(t) \Delta(t+\tau) \rangle \) for \(- t \leq \tau \leq 0\) and with \( \Delta_0 \) equal to \( \eta \) in one set of runs and 0.1\( \eta \) in another (the choice of \( \Delta_0 \) in \[6\] is within this range). The laboratory results \[6\] show that \( R(t, \tau) / \sigma^2(t) \) is a function of \( \tau/t \) and exactly the same collapse is found here with KS (Figure 2). We calculate a Lagrangian correlation time from \( R(t, \tau) \) in the way done in \[6\] and we obtain \( \tau_c \approx 0.45t \) from Figure 2. This value 0.45 is indeed the constant asymptotic value that we obtain for all large enough scale ratios \( \frac{L}{\eta} \), i.e. \( \frac{L}{\eta} \geq O(10) \), and it compares sufficiently well with \( \tau_c \approx 0.6t \) in the laboratory experiment \[6\]. The agreement is therefore good and KS leads to the same conclusion, effectively as the laboratory experiment, that pair separations remember about half their history.

FIG. 3. Non dimensional diagonal Lagrangian velocity correlation \( D_{11}(t, \tau)/D_{11}(t, 0) \) as a function of \( \tau/t \). Same case as Figure 2. \( \frac{tu}{L} = +0.03, \times 0.06, * 0.08, \) empty box 0.11, black box 0.25, \( \circ \) 0.50, \( \bullet \) 0.76, \( \triangle 1 \).

The last set of statistics measured by \[6\] are Lagrangian correlations of pair velocity differences, i.e. \( D_{ij} \equiv \langle V_i^L(t)V_j^L(t+\tau) \rangle \) with \(- t \leq \tau \leq 0\), where \( V_i^L(t) \) denotes the \( i \)th component of the Lagrangian relative velocity between a pair of fluid elements. We calculate these same statistics using our KS model and find that \( D_{ij} \) remains close to 0 for \( i \neq j \), that \( D_{11}(t, \tau)/D_{11}(t, 0) \) and \( D_{22}(t, \tau)/D_{22}(t, 0) \) are functions of \( \tau/t \) (see Figure 3) and that this collapse is the same for \( D_{11} \) and \( D_{22} \) again in agreement with the laboratory results of \[6\].
L/η = 0.1, 0.01 and 0.001. The deviations from Richardson’s law (1) observed when 

\[ \frac{L}{\eta} \] as a function of \( t \), from top to bottom \( \frac{\Delta L}{\eta} \) = 1000, 10, 1, 0.1, 0.01 and 0.001.

\[ \frac{(\Delta - \Delta_0)^2}{f} \] as a function of \( t \) for \( \frac{\Delta L}{\eta} = 1, 0.1, 0.01 \) and 0.001.

Having validated our KS Lagrangian model of pair diffusion in planar turbulence against the laboratory experiment of [1], we now turn our attention to Richardson’s law (1) and the claim of universality that it is based on. We do indeed observe this law over the entire inertial range of times \( \tau_3 < t < L/u' \), but only for initial separations \( \Delta_0 \) between \( \eta \) and 0.1\( \eta \), and this for all the ratios \( \frac{L}{\eta} \) that we tried (see Figure 4a). Of course this ratio should be large enough, otherwise Richardson’s law is not observed for any \( \Delta_0 \), but it is surprising that Richardson’s law is so \( \Delta_0 \)-specific even at enormous values of \( \frac{L}{\eta} \) reaching \( O(10^5) \).

When \( \Delta_0 \leq \eta \), Richardson’s law (1) is observed over the limited large scale range \( 0.2 \frac{L}{\eta} \) to \( \frac{L}{\eta} \) (Figure 4), and the coefficient 0.2 seems to have no dependence on \( \frac{L}{\eta} \) in our simulations as long as \( L/\eta \) is order \( 10^3 \) or larger, so if it has one it must be weak. In the remainder of the inertial range between \( \tau_2 \) and \( \tau_3 \), the time dependencys of \( \frac{(\Delta - \Delta_0)^2}{f} \) and \( \Delta^2 \) are different from Richardson’s (1) and different for different values of \( \Delta_0 \leq \eta \) (Figure 4), even at extremely high \( \frac{L}{\eta} \). We tried to replace \( t \) by \( t - t_0(\Delta_0) \), where \( t_0(\Delta_0) \) is a virtual origin significantly smaller than \( 0.2 \frac{L}{\eta} \) but did not recover Richardson’s law (1), particularly since the discrepancies we observe are over such wide time ranges. When \( \Delta_0 \) is significantly larger than \( \eta \) there is no clear indication of a Richardson law at all (Figure 4).

We have carefully studied the time dependence of \( \frac{(\Delta - \Delta_0)^2}{t} \) in the range between \( \tau_2 \) to \( \frac{L}{\eta} \) for different values of \( \Delta_0 \leq \eta \) and have found the following formula to collapse the data in that range (see Figure 4b): \[ (\Delta - \Delta_0)^2 = G_{\Delta} \frac{u'^3}{L} f(t, \Delta_0) \] (4) where the dimensionless function \( f \) is given by (using \( T \equiv 0.2 \frac{L}{\eta} \))

\[ f(t, \Delta_0) = \exp \left[ \frac{\ln \left( \frac{t}{T} \right)}{2 \ln(\tau_3/T)} \left( \ln(t/T) - \sqrt{\ln^2(t/T) + 2} \right) \right] . \] (5)

Note that \( f(t, \Delta_0) \) tends to 1 when \( t \) is between \( T \) and \( \frac{L}{\eta} \) and \( L/\eta \rightarrow \infty \) (i.e. \( T/\tau_3 \rightarrow \infty \)). The Richardson constant \( G_{\Delta} \) is determined from the value of \( \frac{(\Delta - \Delta_0)^2}{t} \) in the range \( 0.2 \frac{L}{\eta} \) to \( \frac{L}{\eta} \) and we find \( G_{\Delta} \approx 0.03 \) for large enough scale ratio \( \frac{L}{\eta} \) (of order \( 10^3 \) and larger). We should stress that in KS, \( G_{\Delta} \) effectively contains both the original Richardson constant as in \( G_{\Delta} t^3 \) but also the constant of proportionality relating the kinetic energy dissipation rate to \( u'^3 \). We therefore retain the orders of magnitude of \( G_{\Delta} \) obtained by KS but not the actual values.

Integrations of \( \Delta^2 \) in Direct Numerical Simulations (DNS) of two-dimensional turbulence in the inverse energy cascade regime also show a strong dependence on \( \Delta_0 \), even when the simulations are very highly resolved [1]. Nowadays, such DNS cannot reach well-defined \( -5/3 \) ranges over more than two decades, i.e. \( L/\eta = O(100) \), and this at the very highest resolutions currently available. The deviations from Richardson’s law (1) observed when \( L/\eta = O(100) \) might perhaps be due to edge effects (\( L/\eta \) too small to reach the asymptotic Richardson’s law expected to be valid for \( L/\eta \gg 1 \)). But can this also be the case in our KS where \( L/\eta \) reaches \( O(10^5) \)? Clearly one cannot
answer this question with numerical simulations except if in the future KS and/or DNS runs with even higher \( L/\eta \) eventually converge to Richardson’s law without \( \Delta_0 \) - dependencies.

Nevertheless, the success of our KS to reproduce the laboratory observations of [2] and its failure to retrieve Richardson’s law without \( \Delta_0 \) - dependencies even at extremely high \( L/\eta \) does raise the question of the validity of Richardson’s universality and of the locality assumption that it is based on [2], even asymptotically for arbitrarily high \( L/\eta \). In general, \( \nabla^2 \) is a function of \( t, L, \eta, \Delta_0 \) and \( \omega' \) in KS, and the Richardson locality assumption adapted for KS states that, for large enough \( L/\eta \), \( \nabla^2 \) should only depend on \( \nabla^2 \) and \( E(k) \) at \( k = 2\pi/\sqrt{\nabla^2} \) when \( \max(\eta, \Delta_0) \ll \sqrt{\nabla^2} \ll L \). Fung & Vassilicos (1998) [3] found this assumption to be valid in planar KS for different spectral exponents \( p \) between 1 and 2 (\( E(k) \sim k^{-p} \)) but specifically for \( \Delta_0 = \eta/2 \) and unsteadiness parameter \( \lambda = O(1) \) and smaller than 1 in \( \omega_m = \lambda \sqrt{K_m E_n(k_m)} \). The direct consequence of this assumption is that \( \nabla^2 \sim t^\gamma \) with \( \gamma = \frac{1}{3-p} \) which is indeed observed in KS for different values of \( p \) but only for \( \Delta_0 \) close to and below \( \eta \) [3]. What could invalidate locality and Richardson’s law for \( \Delta_0 \) very different from \( \eta \)?

The low values of \( G_\Delta \) and the very large Lagrangian flatness factors of \( V^L \) also observed in KS [3] are consistent with the observation that fluid element pairs travel close to each other for long stretches of time and separate in sudden bursts [3]. Fluid element accelerations \( \mathbf{a} \equiv \frac{D}{Dt} \mathbf{u} \) (where \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \)) are such that \( \nabla \cdot \mathbf{a} = \mathbf{s}^2 - \omega^2 \) where \( \mathbf{s} \) is the strain rate matrix and \( \omega \) the vorticity vector. Hence, \( \nabla \cdot \mathbf{a} \) is large and positive most often in straining regions around hyperbolic points of the flow where \( \mathbf{s}^2 \) is large and \( \omega^2 \) close to 0. Close fluid element pairs can separate violently where \( \nabla \cdot \mathbf{a} \) is large and positive, and the separation is effective if the streamline structure of the turbulence is persistent enough in time. Hence, such violent separation events will most often occur where close fluid element pairs meet hyperbolic points that are persistent enough.

Based on their KS results which were limited to \( \Delta_0 = \eta/2 \), Fung & Vassilicos (1998) [3] rephrased Richardson’s locality assumption as follows: “in the inertial range, the dominant contribution to the turbulent diffusivity \( \frac{\nabla^2}{\eta^2} \) comes from straining regions of size \( \sqrt{\nabla^2} \); these straining regions are embedded in a fractal-eddy structure of cat’s eyes within cat’s eyes and therefore straining regions exist with a variety of length-scales over the entire inertial range.” Davila & Vassilicos [12] have related \( \gamma \) to the fractal dimension \( D \) of this fractal-eddy streamline structure of straining regions when \( \Delta_0 \) is close to and below \( \eta \) (\( \gamma = 4/D \)). These results suggest that when \( \Delta_0 \) is between \( \eta \) and 0.1\( \eta \), the evolution of fluid element pairs by bursts when they meet straining regions somehow tunes into the straining fractal streamline structure of the flow and gives rise to Richardson’s law. This requires some persistence of the streamline structure, and indeed Richardson’s law is lost when the unsteadiness parameter \( \lambda \) is made significantly larger than 1 [3].

This topological picture of turbulent pair diffusion suggested by results in previous papers and our argument concerning \( \nabla \cdot \mathbf{a} \) could also explain the strong \( \Delta_0 \) - dependence of \( \nabla^2 \). As \( \Delta_0 \) decreases well below \( \eta \), the probability for fluid element pairs to encounter a hyperbolic point and be separated by it also decreases and can become so small for \( \Delta_0 \ll \eta \) that pairs may travel close to each other for very long times. Eventually, at times nearing \( L/\omega' \), the eddy turnover time of the turbulence, the two fluid elements will be separated by the unsteadiness of the flow rather than by its streamline structure as they will have to become independent at times \( t \gg L/\omega' \). They therefore largely bypass the relatively persistent straining fractal streamline structure of the turbulence and also Richardson’s law as a result.

For initial conditions \( \Delta_0 \gg \eta \), the argument based on \( \nabla \cdot \mathbf{a} \) does not apply and the separation of fluid element pairs cannot be considered to be dominated by straining events in the vicinity of hyperbolic regions. In the framework of the topological turbulent pair diffusion picture, this is consistent with the absence of a Richardson law for \( \Delta_0 \gg \eta \).

F.N. and J.C.V. are grateful for financial support from the Royal Society and EPSRC.

[1] J.M. Ottino, The kinematics of mixing: stretching, chaos and transport (Cambridge University Press, Cambridge, 1989).
[2] L.F. Richardson, Proc. R. Soc. London Sect. A 110, 709 (1926).
[3] G.K. Batchelor, Proc. Cambridge Philos. Soc. 48, 345 (1952).
[4] R.H. Kraichnan, Phys. Fluids 9, 1937 (1966).
[5] M.F. Shlesinger, B.J. West and J. Klafter, Phys. Rev. Lett. 58, 1100 (1987).
[6] M.-C. Jullien, J. Paret and P. Tabeling, Phys. Rev. Lett. 82, 2872 (1999).
[7] J.C.H. Fung et al., J. Fluid Mech. 236 281 (1992); J.C.H. Fung and J.C. Vassilicos, Phys. Rev. E 57 1677 (1998); N.A. Malik and J.C. Vassilicos, Phys. Fluids 6 1572 (1999).
[8] P. Flohr and J.C. Vassilicos, J. Fluid Mech. 407 315 (2000).
[9] G. Boffetta, A. Celani, A. Crisanti and A. Vulpiani, Phys. Rev. E 60 6734 (1999).
[10] I.M. Sokolov, J. Klafter and A. Blumen, Phys. Rev. E 61 2717 (2000).
[11] G. Boffetta, In “Advances In Turbulence VIII” (ed. C. Dopazo et al.), CIMNE, Barcelona (2000).
[12] J. Davila & J.C. Vassilicos, preprint to be submitted, 2002.