COMBINATORICS OF THE BEREZIN-KARPELEVICH INTEGRAL

JONATHAN NOVAK

Abstract. The Berezin-Karpelevich integral is a double integral over unitary matrices which plays the role of the Itzykson-Zuber integral in rectangular matrix models. We obtain a topological expansion of the Berezin-Karpelevich integral in terms of monotone Hurwitz numbers, and obtain from this certain combinatorial identities.

1. Introduction

1.1. Itzykson-Zuber integral. The Itzykson-Zuber integral,

\[ I_N = \int_{U_N} e^{z \text{Tr} A U B U^*} dU, \]

is a unitary matrix integral which first appeared in mathematical physics in the context of multimatrix models [17]. It depends on a coupling parameter \( z \in \mathbb{C} \) and a pair of matrices \( A, B \in \mathbb{C}^{N \times N} \), with the dependence on these matrices only through their eigenvalues \( a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{C} \). Thus, \( I_N \) is an entire function of \( 1 + 2N \) complex variables.

Since \( I_N \) is invariant under independent permutations of \( a_1, \ldots, a_N \) and \( b_1, \ldots, b_N \), and is stable under swapping these two sets of variables, its Maclaurin series can be presented in the form

\[ I_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \in \mathcal{Y}^d} \frac{p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N)}{N^{\ell(\alpha)} N^{\ell(\beta)}} I_N(\alpha, \beta), \]

where the internal sum is over pairs of Young diagrams \( \alpha, \beta \) each consisting of \( d \) cells and \( p_\alpha, p_\beta \) are the corresponding Newton symmetric polynomials, each of which we have normalized by its maximum modulus on the unit polydisc in \( \mathbb{C}^N \). We call this the string expansion of the Itzykson-Zuber integral, and the numbers \( I_N(\alpha, \beta) = I_N(\beta, \alpha) \) its string coefficients. We also have a connected string expansion

\[ \log I_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \in \mathcal{Y}^d} \frac{p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N)}{N^{\ell(\alpha)} N^{\ell(\beta)}} L_N(\alpha, \beta), \]

in which \( L_N(\alpha, \beta) = L_N(\beta, \alpha) \) are the connected string coefficients of the Itzykson-Zuber integral and the series converges absolutely in a neighborhood of the origin in \( \mathbb{C}^{1+2N} \).

The quest for a topological expansion of the string coefficients of \( I_N \) began in [17], where it was shown that the limits

\[ L(\alpha, \beta) = \lim_{N \to \infty} N^{d-2} L_N(\alpha, \beta) \]
exist and are integers for all Young diagrams $\alpha, \beta$ with $d \leq 8$. By analogy with the planar limit of the Hermitian one-matrix model [4], this integrality leads to the hypothesis that $L(\alpha, \beta)$ counts some class of “genus zero” combinatorial objects. More ambitiously, one hopes for the existence of subleading corrections which enumerate objects of higher genus, as in with Hermitian matrix integrals [2]. However, due to the non-Gaussian nature of Haar measure, it is a mistake to insist that the combinatorics of the Itzykson-Zuber integral must be expressed in terms of maps on surfaces [30].

The combinatorial structure underlying the Itzykson-Zuber integral was exposed in [10], where map counting was replaced by its older sibling, the enumeration of branched covers of Riemann surfaces [20]. Let us identify the symmetric group $S^d = \text{Aut}\{1, \ldots, d\}$ with its Cayley graph as generated by the conjugacy class of transpositions. Write $W^r(\alpha, \beta)$ for the number of $r$-step walks on $S^d$ which begin at a permutation of cycle type $\alpha$ and end at a permutation of cycle type $\beta$. The counting function $W^r(\alpha, \beta)$ also enumerates degree $d$ branched covers of the Riemann sphere with $r$ simple branch points and two additional points over which the covering map has ramification profiles $\alpha$ and $\beta$. By the Riemann-Hurwitz formula, $W^r(\alpha, \beta) = 0$ unless $r = 2g - 2 + \ell(\alpha) + \ell(\beta)$ where $g \in \mathbb{Z}$ is the genus of the possibly disconnected covering surface. We write $W^r(\alpha, \beta) = \tilde{H}_g^\bullet(\alpha, \beta)$ when the Riemann-Hurwitz constraint is satisfied and call $\tilde{H}_g^\bullet(\alpha, \beta)$ a disconnected double Hurwitz number of genus $g$. The corresponding connected double Hurwitz number $H_g(\alpha, \beta)$ counts walks whose steps and endpoints generate a transitive subgroup of the symmetric group, or equivalently connected covers. The double Hurwitz numbers $\tilde{H}_g^\bullet(\alpha, \beta)$ and $H_g(\alpha, \beta)$ were first studied by Okounkov [7]. Hurwitz himself had considered the numbers $H_g^\bullet(\alpha)$ and $H_g(\alpha)$ enumerating branched covers with just one point of prescribed ramification.

Now let us enrich the Cayley graph by marking each edge of $S^d$ corresponding to the transposition $(i \ j)$ with the larger symbol $j$. This is the Jucys-Murphy labeling of the symmetric group [19, 23], which proves to be a useful device [5, 27]. The monotone double Hurwitz numbers $\tilde{H}_g^\bullet(\alpha, \beta)$ and $\tilde{H}_g(\alpha, \beta)$, introduced in [8, 9], count those walks counted by the double Hurwitz numbers which have the additional property that the labels of the edges they traverse form a weakly increasing sequence of numbers. This is analogous to the relationship between simple and self-interacting random walks.

Theorem 1.1 ([10]). For any $1 \leq d \leq N$ and any $\alpha, \beta \in \mathcal{Y}^d$ we have

$$I_N(\alpha, \beta) = (-1)^{\ell(\alpha)+\ell(\beta)} N^{-d} \sum_{g=-\infty}^{\infty} N^{2-2g} \tilde{H}_g^\bullet(\alpha, \beta)$$

and

$$L_N(\alpha, \beta) = (-1)^{\ell(\alpha)+\ell(\beta)} N^{-d} \sum_{g=0}^{\infty} N^{2-2g} \tilde{H}_g(\alpha, \beta),$$

where both series converge.

Theorem 1.1 gives a complete answer to the problem posed in [17], giving a complete genus expansion of the disconnected and connected string coefficients of the Itzykson-Zuber integral. Concerning the analytic functions
\[ I_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} I_N^d \quad \text{and} \quad L_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} L_N^d \]

themselves, Theorem 1.1 says that for all \( 1 \leq d \leq N \) we have

\[ I_N^d = N^{-d} \sum_{g=-\infty}^{\infty} N^{2-2g} \sum_{\alpha, \beta \in \mathcal{Y}} p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N) \frac{(-N)^{\ell(\alpha)}}{(-N)^{\ell(\beta)}} \tilde{H}_g(\alpha, \beta) \]

and

\[ L_N^d = N^{-d} \sum_{g=0}^{\infty} N^{2-2g} \sum_{\alpha, \beta \in \mathcal{Y}} p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N) \frac{(-N)^{\ell(\alpha)}}{(-N)^{\ell(\beta)}} \tilde{H}_g(\alpha, \beta), \]

where both series converge uniformly absolutely on compact subsets of \( \mathbb{C}^{2N} \). Theorem 1.1 thus indicates that as \( N \to \infty \) we have

\[ \log I_N \sim \sum_{g=0}^{\infty} N^{2-2g} F^\text{IZ}_g, \]

with

\[ F^\text{IZ}_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} N^{-d} \sum_{\alpha, \beta \in \mathcal{Y}} p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N) \frac{(-N)^{\ell(\alpha)}}{(-N)^{\ell(\beta)}} \tilde{H}_g(\alpha, \beta) \]

a generating function for connected monotone Hurwitz numbers of genus \( g \). The formal topological expansion (3) of the Itzykson-Zuber free energy is analogous to the formal topological expansion of the Hermitian one-matrix model obtained in the classic papers [4] and [2], but with Hurwitz theory replacing embedded graphs.

1.2. Berezin-Karpelevich integral. The purpose of this paper is to establish the counterpart of Theorem 1.1 for the Berezin-Karpelevich integral

\[ I_{MN} = \int_{U_M} dU \int_{U_N} dV e^{z \text{Tr}(A^*UBV^*+VD^*U^*C)}, \]

where again \( z \in \mathbb{C} \) is a coupling parameter but now \( A, B, C, D \in \mathbb{C}^{M \times N} \) are complex rectangular matrices. By Fubini, we may assume \( M \geq N \). The double integral \( I_{MN} \) plays the role of \( I_N \) in the context of rectangular random matrices [7, 16], and we will see below that it depends on \( A, B, C, D \) only up to the eigenvalues \( x_1, \ldots, x_N \) and \( y_1, \ldots, y_N \) of \( A^*C, D^*B \in \mathbb{C}^{N \times N} \), so is again an entire function of \( 1 + 2N \) complex variables. Furthermore, the MacLaurin series of \( I_{MN} \) can be presented as

\[ I_{MN} = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha, \beta \in \mathcal{Y}} \frac{p_\alpha(x_1, \ldots, x_N) p_\beta(y_1, \ldots, y_N)}{N^{\ell(\alpha)} N^{\ell(\beta)}} I_{MN}(\alpha, \beta), \]
which defines the string coefficients $I_{MN}(\alpha, \beta)$ of the Berezin-Karpelevich integral. The corresponding connected string expansion is

\begin{equation}
\log I_{MN} = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha, \beta \in \mathbb{Y}^d} \frac{p_{\alpha}(x_1, \ldots, x_N) p_{\beta}(y_1, \ldots, y_N)}{N^{\ell(\alpha)}} \frac{1}{N^{\ell(\beta)}} L_{MN}(\alpha, \beta),
\end{equation}

where $L_{MN}(\alpha, \beta)$ are the connected string coefficients of the Berezin-Karpelevich integral.

Our main result is an analogue of Theorem 1.1 giving a topological expansion of the string coefficients of the Berezin-Karpelevich integral and its logarithm in terms of a combinatorial refinement of monotone Hurwitz numbers. Define the disconnected two-legged monotone Hurwitz number $\tilde{H}_g^*(\alpha, \beta; s)$ to be the number of walks from a permutation of cycle type $\alpha$ to a permutation of cycle type $\beta$ in $r = 2g - 2 + \ell(\alpha) + \ell(\beta)$ steps such that the labels of the edges traversed in the first $s$ steps are weakly increasing, as are the labels of the edges traversed in the remaining $r - s$ steps. Thus, we count walks with specified boundary conditions made up of two monotone legs of specified length — virtual histories of a self-interacting random walk on the symmetric group whose memory resets after a specified number of steps. As above, $\tilde{H}_g(\alpha, \beta; s)$ denotes the corresponding connected Hurwitz number.

**Theorem 1.2.** For any $1 \leq d \leq N$, and any $\alpha, \beta \in \mathbb{Y}^d$, we have

\begin{equation}
I_{MN}(\alpha, \beta) = (-1)^{\ell(\alpha) + \ell(\beta)} (MN)^{-d} \sum_{g=-\infty}^{\infty} N^{2g - 2 + \ell(\alpha) + \ell(\beta)} \sum_{s=0}^{2g - 2 + \ell(\alpha) + \ell(\beta)} \left( \frac{N}{M} \right)^s \tilde{H}_g^*(\alpha, \beta; s),
\end{equation}

and

\begin{equation}
L_{MN}(\alpha, \beta) = (-1)^{\ell(\alpha) + \ell(\beta)} (MN)^{-d} \sum_{g=0}^{\infty} N^{2g - 2 + \ell(\alpha) + \ell(\beta)} \sum_{s=0}^{2g - 2 + \ell(\alpha) + \ell(\beta)} \left( \frac{N}{M} \right)^s \tilde{H}_g(\alpha, \beta; s),
\end{equation}

where both series converge.

Just as Theorem 1.1 gives a formal topological expansion of the Itzykson-Zuber integral, Theorem 1.2 gives a formal large topological expansion of the Berezin-Karpelevich integral,

\begin{equation}
\log I_{MN} \sim \sum_{g=0}^{\infty} N^{2g - 2g} F_{g}^{BK}, \quad N \to \infty,
\end{equation}

in which the genus $g$ contribution

\begin{equation}
F_{g}^{BK} = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} (MN)^{-d} \sum_{\alpha, \beta \in \mathbb{Y}^d} \frac{p_{\alpha}(x_1, \ldots, x_N) p_{\beta}(y_1, \ldots, y_N)}{(-N)^{\ell(\alpha)}} \frac{1}{(-N)^{\ell(\beta)}} \sum_{s=0}^{2g - 2 + \ell(\alpha) + \ell(\beta)} \left( \frac{N}{M} \right)^s \tilde{H}_g(\alpha, \beta; s)
\end{equation}
is a generating function for two-legged monotone double Hurwitz numbers of genus $g$. 
2. Character Expansion

In this section we derive the character expansion of the Berezin-Karpelevich integral, which is a known result [6, 7]. The conventional point of view is that character expansions yield determinantal formulas, while in our program they are antecedents of string expansions. We assume familiarity with the representation theory of the general linear and symmetric groups and globally cite [21] as a reference for this material.

2.1. Basic formulas. Isomorphism classes of irreducible polynomial representations of $\text{GL}_N = \text{Aut} \mathbb{C}^N$ are parameterized by the set $\mathcal{Y}_N$ of Young diagrams with at most $N$ rows.

The character

$$s_\lambda(A) = \text{Tr } S^\lambda(A), \quad A \in \text{GL}_N,$$

of any representative $(W^\lambda_N, S^\lambda)$ of the class corresponding to $\lambda \in \mathcal{Y}_N$ is a symmetric homogeneous polynomial function in the eigenvalues of $A$, the Schur polynomial. We will write $s_\lambda(A)$ for the evaluation $s_\lambda(a_1, \ldots, a_N)$ of the Schur polynomial on the eigenvalues of any matrix $A \in \mathbb{C}^{N \times N}$. The pair $(W^\lambda_N, S^\lambda)$ is an irreducible representation of $\text{U}_N \subset \text{GL}_N$, and we have the following basic integration formulas.

**Lemma 2.1.** For any Young diagrams $\lambda, \mu \in \mathcal{Y}_N$ and any matrices $X, Y \in \mathbb{C}^{N \times N}$, we have

$$\int_{\text{U}_N} dV s_\lambda(XVYV^*) = \frac{s_\lambda(X)s_\lambda(Y)}{\dim W^\lambda_N},$$

and

$$\int_{\text{U}_N} dV s_\lambda(XV)s_\mu(YV^*) = \delta_{\lambda\mu} \frac{s_\lambda(XY)}{\dim W^\lambda_N}.$$

**Proof.** Suppose first that $X, Y \in \text{GL}_N$. Then, $XUYU^* \in \text{GL}_N$ and we have

$$s_\lambda(XUYU^*) = \text{Tr } S^\lambda(X)S^\lambda(V)S^\lambda(Y)S^\lambda(V^*) = \sum_{i,j,k,l=1}^N S^\lambda(X)_{ij}S^\lambda(Y)_{jk}S^\lambda(Y)_{kl}S^\lambda(V^*)_{li}.$$ 

We thus have

$$\int_{\text{U}_N} dV s_\lambda(XUYU^*) = \sum_{i,j,k,l=1}^N S^\lambda(X)_{ij}S^\lambda(Y)_{kl} \int_{\text{U}_N} dV S^\lambda(V)_{jk}S^\lambda(V^*)_{li}.$$ 

By orthogonality of matrix elements in an irreducible representation of a compact group, we have

$$\int_{\text{U}_N} dV S^\lambda(V)_{jk}S^\lambda(V^*)_{li} = \frac{\delta_{ij}\delta_{kl}}{\dim W^\lambda_N},$$

and thus
\[
\int_{U_N} dV s_\lambda(XVYV^*) = \frac{1}{\dim W_\lambda} \left( \sum_{i=1}^N S^\lambda(X)_{ii} \right) \left( \sum_{k=1}^N S^\lambda(Y)_{kk} \right) = \frac{1}{\dim W_\lambda} \text{Tr} S^\lambda(X) \text{Tr} S^\lambda(Y).
\]

We now extend to the case where \(X, Y \in \mathbb{C}^{N \times N}\) are arbitrary matrices. Write \(f(V) = s_\lambda(XVYV^*)\). Since \(GL_N\) is dense in \(\mathbb{C}^{N \times N}\), there are two sequences \((X_n)_{n=1}^{\infty}\) and \((Y_n)_{n=1}^{\infty}\) in \(GL_N\) such that

\[
\lim_{n \to \infty} f_n(V) = f(V), \quad V \in U_N,
\]

where \(f_n(V) = s_\lambda(X_n V Y_n V^*)\). Since the Schur polynomials are monomial positive, we have

\[
|f_n(V)| \leq s_\lambda(\|X_n V Y_n V^*\|, \ldots, \|X_n V Y_n V^*\|) \leq \|X_n\|^d \|Y_n\|^d \dim W_\lambda,
\]

where \(\| \cdot \|\) is operator norm and \(d = |\lambda|\) is the number of cells in \(\lambda\). Since \((\|X_n\|_{n=1}^{\infty})\) and \((\|Y_n\|_{n=1}^{\infty})\) are convergent sequences, they are bounded, and we may apply the dominated convergence theorem to conclude

\[
\int_{U_N} dV s_\lambda(XVYV^*) = \lim_{n \to \infty} \frac{s_\lambda(X_n) s_\lambda(Y_n)}{\dim W_\lambda} = \frac{s_\lambda(X) s_\lambda(Y)}{\dim W_\lambda}.
\]

The argument for the other integral is essentially the same, except that one uses orthogonality of matrix elements in non-isomorphic irreducible representations.

In addition to the above integration formulas, we will use Frobenius’s formula for Schur polynomials in terms of Newton polynomials. Isomorphism classes of irreducible representations of \(S^d = \text{Aut}\{1, \ldots, d\}\), or equivalently of the group algebra \(\mathbb{C}S^d\), are indexed by the set \(Y^d\) of Young diagrams with exactly \(d\) cells. For each \(\lambda \in Y^d\), we choose a representative \((V^\lambda, R^\lambda)\) of the irreducible representations corresponding to \(\lambda \in Y^d\). Moreover, for each \(\alpha \in Y^d\) we identify the conjugacy class \(C_\alpha \subseteq S^d\) of permutations of cycle type \(\alpha\) with the formal sum of its elements in \(\mathbb{C}S^d\). By Schur’s Lemma, \(R^\lambda(C_\alpha)\) is a scalar operator in \(\text{End} V^\lambda\), and we write \(\omega_\alpha(\lambda)\) for its eigenvalue. The expansion of Schur polynomials on Newton polynomials is then

\[
s_\lambda = \frac{\dim V^\lambda}{d!} \sum_{\alpha \in Y^d} p_{\alpha} \omega_\alpha(\lambda), \quad \lambda \in Y^d_N.
\]

2.2. **BGW character expansion.** We now derive the character expansion of the Bars-Green/Brézin-Gross-Witten/Wadia integral,

\[
J_N = \int_{U_N} dV e^{z \text{Tr}(X^*V + V^*Y)},
\]

the basic special function of \(U_N\) gauge theory on a lattice of any dimension \([1, 3, 15, 28, 29]\). This series expansion is due to Bars and Green \([1]\), who considered the case where \(X = Y\) and the action \(V \mapsto \text{Tr}(X^*V + V^*X)\) is real-valued.
**Theorem 2.2.** For any $z \in \mathbb{C}$ and $X, Y \in \mathbb{C}^{N \times N}$, we have

$$J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!d!} \sum_{\lambda \in \mathbb{Y}_N^d} s_\lambda(t_1, \ldots, t_N) \frac{(\dim V^\lambda)^2}{\dim W_N^\lambda},$$

where $t_1, \ldots, t_N$ are the eigenvalues of $X^*Y$ and the series converges absolutely.

**Proof.** We view $J_N$ as an entire function of the complex variable $z$, with the matrices $X, Y \in \mathbb{C}^{N \times N}$ arbitrary but fixed. The Maclaurin series of $J_N$ is

$$J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!d!} \int_{\mathbb{U}_N} (\text{Tr} X^*V)^d (\text{Tr} YV^*)^d dV,$$

and

$$(\text{Tr} X^*V)^d (\text{Tr} YV^*)^d = \left( \sum_{\lambda \in \mathbb{Y}_N^d} s_\lambda(X^*V) \dim V^\lambda \right) \left( \sum_{\mu \in \mathbb{Y}_N^d} s_\mu(YV^*) \dim V^\mu \right).$$

The result now follows from Lemma 2.1. \hfill \Box

2.3. **BK character expansion.** We now derive the character expansion of the Berezin-Karpelevich integral.

**Theorem 2.3.** For any $z \in \mathbb{C}$ and $A, B, C, D \in \mathbb{C}^{M \times N}$, we have

$$I_{MN} = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!d!} \sum_{\lambda \in \mathbb{Y}_N^d} s_\lambda(x_1, \ldots, x_N)s_\lambda(y_1, \ldots, y_N) \frac{\dim V^\lambda}{\dim W_M^\lambda} \frac{\dim V^\lambda}{\dim W_N^\lambda},$$

where $x_1, \ldots, x_N$ are the eigenvalues of $A^*C$ and $y_1, \ldots, y_N$ are the eigenvalues of $D^*B$ and the series is absolutely convergent.

**Proof.** In the Berezin-Karpelevich integral (10), the inner integral over the lower-rank unitary group,

$$\int_{\mathbb{U}_N} dVe^{z \text{Tr}(A^*UBV^*+VD^*U^*C)},$$

is the BGW integral (18) with

$$X = C^*UD \quad \text{and} \quad Y = A^*UB.$$ 

Thus, by Theorem 2.2 the Maclaurin series of the Berezin-Karpelevich integral as a holomorphic function of $z$ is

$$I_{MN} = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!d!} \sum_{\lambda \in \mathbb{Y}_N^d} \frac{(\dim V^\lambda)^2}{\dim W_N^\lambda} \int_{\mathbb{U}_M} dUS_\lambda(D^*U^*CA^*UB).$$

It remains to compute the integral...
\[
\int_{U_M} \mathrm{d} U s_\lambda(D^* U^* C A^* U B),
\]
where \( D^* U^* C A^* U B \in \mathbb{C}^{N \times N} \). Recall that \( M \geq N \). From the characteristic polynomial identity
\[
x^{M-N} \det(x I_N - Z_1 Z_2) = \det(x I_M - Z_2 Z_1),
\]
which holds for arbitrary \( Z_1 \in \mathbb{C}^{N \times M} \) and \( Z_2 \in \mathbb{C}^{M \times N} \), the spectrum of \( D^* U^* C A^* U B \) coincides with that of \( C A^* U B D^* U^* \) up to \( M - N \) additional zero eigenvalues. Because the Schur polynomials are stable,

\[
s_\lambda(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N, 0, 0, \ldots, 0),
\]
we have

\[
s_\lambda(C^* U^* D A^* U B) = s_\lambda(D A^* U B C^* U^*),
\]
and therefore

\[
\int_{U_M} \mathrm{d} U s_\lambda(D^* U^* C A^* U B) = \int_{U_M} \mathrm{d} U s_\lambda(C A^* U B D^* U^*).
\]
Note also that \( \lambda \in \mathcal{Y}^d_N \) implies \( \lambda \in \mathcal{Y}^d_M \) because \( M \geq N \), so that \( \lambda \) indexes an irreducible representation \( W^\lambda_M \) of \( U_M \), and

\[
\int_{U_M} \mathrm{d} U s_\lambda(C A^* U B D^* U^*) = \frac{s_\lambda(C A^*) s_\lambda(B D^*)}{\dim W^\lambda_M} = \frac{s_\lambda(A^* C) s_\lambda(D^* B)}{\dim W^\lambda_M}.
\]

\[\square\]

### 3. Topological Expansion

In this Section we prove our main result, Theorem 1.2 which gives a topological expansion for the string coefficients and connected string coefficients of the Berezin-Karpelevich integral. We pair this with a topological expansion for the string coefficients of the BGW integral.

3.1. **String expansions.** String expansions for the BGW integral \( J_N \) and the Berezin-Karpelevich integral \( I_{M,N} \) follow immediately from their character expansions, Theorems 2.2 and 2.3 together with Frobenius's formula (17) which expresses Schur polynomials in terms of Newton polynomials. For the BGW integral, we have

\[
J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha \in \mathcal{Y}^d} \frac{p_\alpha(t_1, \ldots, t_N)}{N^{l(\alpha)}} J_N(\alpha)
\]
with
For the Berezin-Karpelevich integral, we obtain

\begin{equation}
I_{MN} = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha, \beta \in \Lambda} p_\alpha(x_1, \ldots, x_N) p_\beta(y_1, \ldots, y_N) I_{MN}(\alpha, \beta)
\end{equation}

with

\begin{equation}
I_{MN}(\alpha, \beta) = \frac{1}{d!d!} \sum_{\lambda \in \Lambda_N} \frac{(\dim V_\lambda)^2}{d!} \omega_\alpha(\lambda) \dim V_\lambda \dim W_\lambda \omega_\beta(\lambda).
\end{equation}

Using the standard dimension formulas \cite{21}, for any \( \lambda \in \Lambda_N \) we have

\begin{equation}
\frac{\dim V_\lambda}{\dim W_\lambda} = \frac{d!}{N^d} \Omega_1^{-1}(\lambda),
\end{equation}

with

\begin{equation}
\Omega_h(\lambda) = \prod_{\square \in \lambda} (1 + hc(\square))
\end{equation}

the content polynomial of \( \lambda \), in which \( c(\square) \) is the column index minus the row index of a given cell \( \square \in \lambda \). Note that \( \Omega_N(\lambda) > 0 \) for any \( \lambda \in \Lambda_N \). Using \cite{21} the BGW and Berezin-Karpelevich string coefficients become

\begin{equation}
J_N(\alpha) = N^{\ell(\alpha) - d} \sum_{\lambda \in \Lambda_N} \frac{(\dim V_\lambda)^2}{d!} \omega_\alpha(\lambda) \Omega^{-1}_1(\lambda)
\end{equation}

and

\begin{equation}
I_{MN}(\alpha, \beta) = M^{-d}N^{\ell(\alpha) + \ell(\beta) - d} \sum_{\lambda \in \Lambda_N} \frac{(\dim V_\lambda)^2}{d!} \omega_\alpha(\lambda) \Omega^{-1}_1(\lambda) \Omega^{-1}_1(\lambda) \omega_\beta(\lambda).
\end{equation}

These formulas are already enough to obtain \( 1/N \) expansions for the string coefficients of the BGW and Berezin-Karpelevich integrals.

**Proposition 3.1.** For any Young diagram \( \alpha \) with \( d \leq N \) cells, we have

\begin{equation}
J_N(\alpha) = N^{\ell(\alpha) - d} \sum_{r=0}^{\infty} \frac{(-1)^r}{N^r} \sum_{\lambda \in \Lambda_N} \frac{(\dim V_\lambda)^2}{d!} \omega_\alpha(\lambda) h_r(\lambda),
\end{equation}

where the series converges absolutely and \( h_r(\lambda) \) denotes the evaluation of the complete symmetric polynomial of degree \( r \) on the multiset of contents of \( \lambda \).
Proof. The contents of any Young diagram \( \lambda \) contained in the \( N \times N \) square diagram are all strictly less than \( N \) in absolute value. Thus, for any such diagram we have

\[
\Omega^{-1}_{\frac{N}{M}}(\lambda) = \prod_{\Box \in \lambda} \left( 1 + \frac{c(\Box)}{N} \right)^{-1} = \sum_{r=0}^{\infty} \frac{(-1)^r}{N^r} h_r(\lambda),
\]

where the series is absolutely convergent, and plugging this absolutely convergent expansion into (36) yields the result.

Proposition 3.2. For any Young diagrams \( \alpha, \beta \) with \( d \leq N \) cells, we have

\[
I_{MN}(\alpha, \beta) = M^{-d} N^{\ell(\alpha) + \ell(\beta) - d} \sum_{r=0}^{\infty} \frac{(-1)^r}{N^r} \sum_{s=0}^{r} \sum_{\lambda \in \mathcal{Y}_N^d} \frac{1}{d!} \omega_{\alpha}(\lambda) h_s(\lambda) h_{r-s}(\lambda) \omega_{\beta}(\lambda),
\]

where the series converges absolutely.

Proof. The proof is the same as the proof of the preceding proposition, except that we have the double product

\[
\Omega^{-1}_{\frac{1}{M}}(\lambda) \Omega^{-1}_{\frac{1}{N}}(\lambda) = \prod_{\Box \in \lambda} \left( 1 + \frac{c(\Box)}{M} \right)^{-1} \left( 1 + \frac{c(\Box)}{N} \right)^{-1},
\]

which we write as

\[
\Omega^{-1}_{\frac{1}{M}}(\lambda) \Omega^{-1}_{\frac{1}{N}}(\lambda) = \prod_{\Box \in \lambda} \left( 1 + \frac{vc(\Box)}{N} \right)^{-1} \left( 1 + \frac{c(\Box)}{N} \right)^{-1}
\]

with \( v = \frac{N}{M} \leq 1 \). Then, for any Young diagram \( \lambda \) contained in the \( N \times N \) square we have

\[
\Omega^{-1}_{\frac{1}{M}}(\lambda) \Omega^{-1}_{\frac{1}{N}}(\lambda) = \sum_{r_1, r_2=0}^{\infty} \frac{(-v)^{r_1} (-1)^{r_2}}{N^{r_1} N^{r_2}} h_{r_1}(\lambda) h_{r_2}(\lambda) = \sum_{r=0}^{\infty} \frac{(-1)^r}{N^r} \sum_{s=0}^{r} v^s h_s(\lambda) h_{r-s}(\lambda),
\]

where the series converges absolutely and can be substituted into (37). 

3.2.Disconnected topological expansion. The Fourier transform gives an algebra isomorphism from the center of \( \mathbb{CS}^d \) to the pointwise algebra of complex-valued functions on Young diagrams: if \( A \in \mathbb{CS}^d \) is a central function, its Fourier transform \( \hat{A}(\lambda) \) is the unique eigenvalue of the scalar operator \( R^\lambda(A) \) acting in \( V^\lambda \). Furthermore, the canonical trace \( \langle \cdot \rangle \) on \( \mathbb{CS}^d \), i.e. the normalized character of the regular representation, is for central elements implemented by the Plancherel formula

\[
\langle A \rangle = \frac{\dim V^\lambda}{d!} \hat{A}(\lambda).
\]

By definition, \( \omega_\alpha(\lambda) \) is the Fourier transform of a conjugacy class,

\[
\hat{\omega}_\alpha(\lambda) = \omega_\alpha(\lambda).
\]
The theorem of Jucys [19] and Murphy [23] says that any symmetric polynomial $f(J_1, \ldots, J_d)$ in the Jucys-Murphy elements

$$J_j = \sum_{i=1}^{j} (i \ j), \quad 1 \leq j \leq d,$$

is a central element in $\mathbb{C}S^d$, and that its Fourier transform is the function $f(\lambda)$ obtained by evaluation of $f$ on the multiset of contents of $\lambda$. We thus have that

$$\langle C_\alpha h_r \rangle = \sum_{\lambda \in Y^d} \frac{(\dim V^\lambda)^2}{d!} \omega_\alpha(\lambda) h_r(\lambda)$$

is the coefficient of the identity permutation in the product $C_\alpha h_r(J_1, \ldots, J_d)$, which is precisely the number $\tilde{W}^r(\alpha)$ of monotone $r$-step walks from the identity permutation to a permutation of cycle type $\alpha$ on the Cayley graph of $S^d$. Thus, Proposition 3.1 implies that for any Young diagram $\alpha$ with $d \leq N$ cells, we have

$$J_N(\alpha) = N^{\ell(\alpha) - d} \sum_{r=0}^{\infty} \frac{(-1)^r}{N^r} \tilde{W}^r(\alpha),$$

the series being absolutely convergent. That is, the string coefficients of the BGW integral are generating functions enumerating monotone walks of specified length from the identity to a specified conjugacy class. Similarly,

$$\langle C_\alpha h_s h_{r-s} C_\beta \rangle = \sum_{\lambda \in Y^d} \frac{(\dim V^\lambda)^2}{d!} \omega_\alpha(\lambda) h_s(\lambda) h_{r-s}(\lambda) \omega_\beta(\lambda)$$

is the number $\tilde{W}^r(\alpha, \beta; s)$ of $r$-step walks $C_\alpha \rightarrow C_\beta$ on $S^d$ consisting of two monotone legs, one of length $s$ followed by one of length $r - s$. Therefore Proposition 3.2 is equivalent to the statement that for all $\alpha, \beta$ with $d \leq N$ cells we have

$$I_{MN}(\alpha, \beta) = M^{-d} N^{\ell(\alpha) + \ell(\beta) - d} \sum_{r=0}^{\infty} \frac{(-1)^r}{N^r} \sum_{s=0}^{r} u^s \tilde{W}^r(\alpha, \beta; s),$$

so that the string coefficients of the Berezin-Karpelevich integral are generating functions for two-legged monotone walks of specified length between specified conjugacy classes.

According to the Riemann-Hurwitz formula, $\tilde{W}^r(\alpha)$ vanishes unless $r = 2g - 2 + \ell(\alpha) + d$, where $|\alpha| = d$. Thus, we obtain the following genus expansion for the string coefficients of the BGW integral.

**Theorem 3.3.** For any Young diagram $\alpha$ with $d \leq N$ cells, we have

$$J_N(\alpha) = (-1)^{\ell(\alpha) + d} N^{-2d} \sum_{g=-\infty}^{\infty} N^{2-2g} \tilde{H}_g^*(\alpha),$$
where $\vec{H}_g^0(\alpha) = W^{2g - 2 + \ell(\alpha) + d(\alpha)}$ is the disconnected monotone single Hurwitz number of genus $g$, and the series converges.

Applying the Riemann-Hurwitz formula in (44), we likewise obtain a genus expansion for the string coefficients of the Berezin-Karpelevich integral.

**Theorem 3.4.** For any Young diagrams $\alpha, \beta$ with $d \leq N$ cells, we have

$$I_{MN}(\alpha, \beta) = (-1)^{\ell(\alpha) + \ell(\beta)} (MN)^{-d} \sum_{g=-\infty}^{\infty} N^{2g - 2 + \ell(\alpha) + \ell(\beta)} \sum_{s=0}^{\infty} v^s \vec{H}_g^0(\alpha, \beta; s)$$

where $\vec{H}_g^0(\alpha, \beta; s) = W^{2g - 2 + \ell(\alpha) + \ell(\beta)}(\alpha, \beta; s)$ is the disconnected two-legged monotone double Hurwitz number of genus $g$, and the series converges.

3.3. **Connected topological expansion.** The BGW integral $J_N$ is an analytic function on $\mathbb{C}^{1+N}$, and its logarithm $\log J_N$ is analytic on an open neighborhood of the origin. Writing

$$J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} J_N^d \quad \text{and} \quad \log J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} K_N^d$$

each coefficient $K_N^d$ is a polynomial in $J_N^1, \ldots, J_N^d$ given explicitly by the Exponential Formula [13]. This in turn gives an explicit relation between the string coefficients $J_N(\alpha)$ of $J_N$ and the string coefficients $K_N(\alpha)$ of its logarithm, which are defined by the expansion

$$K_N^d = \sum_{\alpha \in Y^d} \frac{p_{\alpha}(t_1, \ldots, t_N)}{N(\alpha)} K_N(\alpha).$$

As a consequence this [11], Theorem 3.3 is equivalent to the following result.

**Theorem 3.5.** For any Young diagram $\alpha$ with $d \leq N$ cells, we have

$$J_N(\alpha) = (-1)^{\ell(\alpha) + d} N^{-2d} \sum_{g=0}^{\infty} N^{2g} \vec{H}_g(\alpha),$$

where $\vec{H}_g(\alpha)$ is the monotone single Hurwitz number of genus $g$, and the series converges.

An explicit formula for $\vec{H}_0(\alpha)$ is given in [8], and it is equivalent to the first-order asymptotics of the BGW integral derived by O’Brien and Zuber [21, 25] in the context of lattice gauge theory; see also [14]. Going further, an explicit formula for $\vec{H}_1(\alpha)$ is given in [9], so that the genus one correction to the formula of O’Brien and Zuber follows from Theorem 3.5. Likewise, applying the moment-cumulant formula to Theorem 3.4 completes the proof of Theorem 1.2, giving a topological expansion for both the string coefficients and connected string coefficients of the Berezin-Karpelevich integral.
4. Combinatorial Identities from Matrix Integrals

4.1. Itzykson-Zuber case. Taking $B$ to be the identity matrix in the Itzykson-Zuber integral, we obtain an exponential function: in this specialization

\[ \log I_N = z(a_1 + \cdots + a_N). \]

Together with Theorem 1.1, this degeneration of $I_N$ implies the following cancellation identity for monotone double Hurwitz numbers.

**Theorem 4.1.** For any $(d, g) \in \mathbb{N} \times \mathbb{N}_0$ except $(1, 0)$, we have

\[ \sum_{\beta \in \mathcal{Y}_d} (-1)^{\ell(\alpha)} \mathcal{H}_g(\alpha, \beta) = 0 \]

for all $\alpha \in \mathcal{Y}_d$.

**Proof.** The case where $d = 1$ and $g > 0$ is combinatorially obvious: the sum consists of the single term $\mathcal{H}_g(1, 1)$, which vanishes as there are no walks of positive length in a graph with a single vertex.

For $d > 1$, the result is a consequence of Theorem 1.1 together with the degeneration (47). More precisely, writing

\[ \log I_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} L_N^d, \]

in the degeneration (47) we have that $L_N^d = 0$ for all $d > 1$. Let $d > 1$ be arbitrary but fixed. Then by Theorem 1.1 for all $N \geq d$ we have that

\[ \sum_{\alpha, \beta \in \mathcal{Y}_d} p_{\alpha}(a_1, \ldots, a_N) \frac{1}{(-N)^{\ell(\alpha)}} (-1)^{\ell(\beta)} \sum_{g=0}^{\infty} N^{2-2g} \mathcal{H}_g(\alpha, \beta) = 0, \]

which forces

\[ \sum_{\beta \in \mathcal{Y}_d} (-1)^{\ell(\beta)} \sum_{g=0}^{\infty} N^{-2g} \mathcal{H}_g(\alpha, \beta) = 0 \]

for each $\alpha \in \mathcal{Y}_d$ and all $N \geq d$, by linear independence of the degree $d$ Newton polynomials in $N \geq d$ variables.

We now proceed by induction in $g$. For $g = 0$, take the $N \to \infty$ limit in (50) to obtain

\[ \sum_{\beta \in \mathcal{Y}_d} (-1)^{\ell(\beta)} \mathcal{H}_0(\alpha, \beta) = 0 \]

for each $\alpha \in \mathcal{Y}_d$. Assuming the result holds up to genus $k$, (50) becomes
(52) \[
\sum_{g=k+1}^{\infty} N^{-2g} \sum_{\beta \in Y^d} (-1)^{\ell(\beta)} \bar{H}_g(\alpha, \beta) = 0,
\]
for each \(\alpha \in Y^d\). Multiply (52) by \(N^{2k}\) and take the \(N \to \infty\) limit to obtain

(53) \[
\sum_{\beta \in Y^d} (-1)^{\ell(\beta)} \bar{H}_{k+1}(\alpha, \beta) = 0
\]
for each \(\alpha \in Y^d\).

\[\square\]

4.2. Berezin-Karpelevich case. We now consider an analogous specialization of the Berezin-Karpelevich integral which, via Theorem 1.2, produces a combinatorial identity for two-legged monotone Hurwitz numbers. We consider the case of equal dimensions, \(M = N\), and take the matrices \(B\) and \(D\) to be the identity. In this specialization we see that, by invariance of Haar measure, the Berezin-Karpelevich integral degenerates to the BGW integral,

(54) \[
I_{NN} = \int_{U_N} \int_{U_N} dV e^{z \text{Tr}(A^*UU^* + VV^*)} = \int_{U_N} dU e^{z \text{Tr}(A^*U + U^*C)} = J_N.
\]

Thus, for each \(d \in \mathbb{N}\) we have the polynomial identity

(55) \[
\sum_{\alpha, \beta \in Y^d} \frac{p_\alpha(x_1, \ldots, x_N)}{N^{\ell(\alpha)}} L_{NN}(\alpha, \beta) = \sum_{\alpha \in Y^d} \frac{p_\alpha(x_1, \ldots, x_N)}{N^{\ell(\alpha)}} K_N(\alpha),
\]

where \(x_1, \ldots, x_N\) are the eigenvalues of \(A^*C \in \mathbb{C}^{N \times N}\), and \(L_{NN}(\alpha, \beta)\) and \(K_N(\alpha)\) are the connected string coefficients of \(I_{NN}\) and \(J_N\), respectively, which implies that for any \(\alpha \in Y^d\) and all \(N \geq d\) we have the numerical identity

(56) \[
\sum_{\beta \in Y^d} L_{NN}(\alpha, \beta) = K_N(\alpha).
\]

By Theorems 1.2 and 3.3 this in turn gives

(57) \[
\sum_{g=0}^{\infty} N^{2-2g} \sum_{\beta \in Y^d} (-1)^{\ell(\beta)} \sum_{s=0}^{2g-2+\ell(\alpha)+\ell(\beta)} \bar{H}_g(\alpha, \beta; s) = (-1)^d \sum_{g=0}^{\infty} N^{2-2g} \bar{H}_g(\alpha),
\]

for each \(\alpha \in Y^d\) and all \(N \geq d\), both series being convergent. We thus obtain the following summation formula, which is useful in the theory of the rectangular \(R\)-transform [22].

**Theorem 4.2.** For any degree \(d \in \mathbb{N}\), genus \(g \in \mathbb{N}_0\), and profile \(\alpha \in Y^d\), we have

\[
\sum_{\beta \in Y^d} (-1)^{\ell(\beta)} \sum_{s=0}^{2g-2+\ell(\alpha)+\ell(\beta)} \bar{H}_g(\alpha, \beta; s) = (-1)^d \bar{H}_g(\alpha).
\]
4.2.1. Conflict of interest statement. The author states that there is no conflict of interest.

4.2.2. Data availability statement. The author states that all relevant data is included.

References

[1] I. Bars, F. Green, Complete integration of $U(N)$ lattice gauge theory in a large $N$ limit, Phys. Rev. D 20 (1979), 3311-3330.
[2] D. Bessis, C. Itzykson, J.-B. Zuber, Quantum field theory techniques in graphical enumeration, Adv. in Appl. Math. 1 (1980), 109-157.
[3] E. Brezin, D. Gross, The external field problem in the large $N$ limit of QCD, Phys. Lett. 97 (1980), 120-124.
[4] E. Brézin, C. Itzykson, G. Parisi, J. B. Zuber, Planar diagrams, Commun. Math. Phys. 59 (1978), 35-51.
[5] P. Diaconis, C. Greene, Applications of Murphy’s elements, Technical Report 335 (1989), Department of Statistics, Stanford University.
[6] A. Ghaderipoor, C. Tellumbra, Generalizations of some integrals over unitary matrices by character expansions of groups, J. Math. Phys. 49 (2008), 073519.
[7] A. Ghaderipoor, C. Tellumbra, On the application of character expansions for MIMO capacity analysis, IEEE Transactions on Information Theory 58 (2012), 2950-2962.
[8] I. Goulden, M. Guay-Paquet, J. Novak, Monotone Hurwitz numbers in genus zero, Canad. J. Math. 65 (2013), 1020-1042.
[9] I. Goulden, M. Guay-Paquet, J. Novak, Polynomiality of monotone Hurwitz numbers in higher genera, Adv. Math. 238 (2013), 1-23.
[10] I. Goulden, M. Guay-Paquet, J. Novak, Monotone Hurwitz theory and the IZ integral, Ann. Math. Blaise Pascal 21 (2014), 71-89.
[11] I. Goulden, M. Guay-Paquet, J. Novak, Toda equations and piecewise polynomiality for mixed double Hurwitz numbers, SIGMA 12 (2016), 1-10.
[12] I. Goulden, M. Guay-Paquet, J. Novak, On the convergence of monotone Hurwitz generating functions, Ann. Comb. 21 (2017), 73-81.
[13] I. Goulden, D. Jackson, Combinatorial Enumeration, Dover Publications, 2004.
[14] D. Gross, M. Newman, Unitary and hermitian matrices in an external field, Phys. Lett. B 166 (1991), 291-297.
[15] D. Gross, E. Witten, Possible third-order phase transition in the large-$N$ lattice gauge theory, Phys. Rev. D 21 (1980), 446-453.
[16] A. Guionnet, J. Huang, Asymptotics of rectangular spherical integrals, J. Funct. Anal. 285 (2023), 110144.
[17] C. Itzykson, J.-B. Zuber, The planar approximation. II, J. Math. Phys. 21 (1980), 411-421.
[18] K. Johansson, Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices, Commun. Math. Phys. 215 (2001), 683-705.
[19] A. Jucys, Symmetric polynomials and the center of the symmetric group ring, Rep. Math. Phys. 5 (1974), 107-112.
[20] S. Lando, Hurwitz numbers: on the edge between combinatorics and geometry, ICM Proceedings 2010.
[21] I. G. Macdonald, Symmetric Functions and Hall Polynomials, Second Edition. Oxford Mathematical Monographs, 1995.
[22] C. McSwiggen, J. Novak, High-dimensional approximations of $R$-transforms, in preparation.
[23] G. Murphy, A new construction of Young’s seminormal representation of the symmetric group, J. Algebra 69 (1981), 287-297.
[24] K. O’Brien, J.-B. Zuber, A note on $U_N$ integrals in the large $N$ limit, Phys. Lett. 144B (1984), 407-408.
[25] K. O’Brien, J.-B. Zuber, Strong coupling expansion of large $N$ QCD and surfaces, Nucl. Phys. B 253 (1985), 621-634.
[26] A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Lett. 7 (2000), 447-453.

[27] A. Okounkov, A. Vershik, *A new approach to the representation theory of the symmetric groups*, Selecta Math. 2 (1996), 581-605.

[28] S. Samuel, *$U(N)$ integrals, $1/N$, and the De Wit-'t Hooft anomalies*, J. Math. Phys. 21 (1980), 2695-2703.

[29] S. Wadia, *A study of $U(N)$ lattice gauge theory in two dimensions*, 
https://arxiv.org/abs/1212.2906v1

[30] J.-B. Zuber, P. Zinn-Justin, *On some integrals over the $U(N)$ unitary group and their large $N$ limits*, 
J. Phys. A: Math. Gen. 36 (2003), 3173-3193.