On the Influence of the Ionization-Recombination Processes on Hydrogen Plasma Polytropic Index

Todor M. Mishonov,1 Iglika M. Dimitrova,2,1 and Albert M. Varonov1

1Georgi Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences, 72 Tzarigradsko Chaussee Blvd., BG-1784 Sofia, Bulgaria
2Faculty of Chemical Technologies, University of Chemical Technology and Metallurgy, 8 Kliment Ohridski Blvd., BG-1756 Sofia, Bulgaria

(Dated: 8 December 2020, 18:48)

ABSTRACT

The polytropic (adiabatic) index for pure hydrogen plasma is analytically calculated as function of reciprocal temperature and degree of ionization. Additionally, the polytropic index is graphically represented as a function of temperature and density. It is concluded that the partially ionized hydrogen plasma cannot be exactly polytropic. The calculated deviations from the mono-atomic value 5/3 are measurable. The analytical result for pure hydrogen plasma is a test example how this approach can be extended for arbitrary gas cocktail.

1. INTRODUCTION

The polytropes find many applications in astrophysics and related fields (Horedt 2004) and there are a lot of hints (Totten et al. 1995; Kartalev et al. 2006) for deviation of polytropic (or adiabatic) index \( \gamma_{\text{eff}} \) from the mono-atomic value \( \gamma_a = 5/3 \). However, the first measurement of the adiabatic index in the solar corona using time-dependent spectroscopy of HINODE/EIS observations by Doorsselaere et al. (2011) triggered systematic study of this deviation and put in the agenda of physics of plasmas the problem of theoretical understanding. Similar results were obtained in Jacobs & Poedts (2011) and in the recent papers of Prasad et al. (2018); Zavershinskii et al. (2019). The following study was inspired by the paper (Doorsselaere et al. 2011) where the effective adiabatic index in the solar corona is measured for the first time by time-dependent spectroscopy of HINODE/EIS observations.

Let us recall (Goosens 2003) some basic definitions

\[
\gamma_{\text{eff}} \equiv \frac{C_p}{C_v}, \quad C_p \equiv \left( \frac{\partial w}{\partial T} \right)_p, \quad C_v \equiv \left( \frac{\partial \varepsilon}{\partial T} \right)_\rho
\]

which describes relations between small fluctuations of the mass density \( \rho' \), pressure \( p' \) and temperature \( T' \)

\[
\frac{p'}{\rho} = \frac{1}{\gamma_{\text{eff}}} \frac{p'}{p} = \frac{1}{\gamma_{\text{eff}} - 1} \frac{T'}{T},
\]

where \( w \) and \( \varepsilon \) are the enthalpy and free energy per unit mass and \( C_p \) and \( C_v \) are the heat capacities per unit mass at constant pressure \( p \) or volume and mass density \( \rho \). Authors emphasize this first measurement of \( \gamma_{\text{eff}} \) and the clear deviation from the mono-atomic value \( \gamma_a = 5/3 \) has important implications for the solar coronal physics and its modeling (Parker 1963; Roussev et al. 2003; Cohen et al. 2006; Petrie et al. 2007; Chatterjee & Fan 2013; Airapetian & Usmanov 2016). This clear deviation gives a hint that ionization-recombination processes of minority elements as helium, carbon, oxygen and even iron can slightly influence the thermodynamic of the coronal plasma and such a hint has already been found (Basu & Mandel 2004), where it was found that the adiabatic index changes near the second helium ionization. More hints can be found in the measurements of the adiabatic index in solar flaring loops (Wang et al. 2015), whose value is close to 5/3 and investigations of space and laboratory plasmas suggest that although the solar wind electrons have a polytropic index of less than 5/3, their actual transport might be adiabatic (Zhang et al. 2016). At Mega-Kelvin temperatures the solar corona hydrogen is completely ionized. Even in the low-frequency static approximation taking into account the Saha equation requires significant amount of data and numerical calculation. In order to check whether such thermodynamic effects deserve to be studied in detail, in the present comment we represent the textbook like behavior of pure hydrogen plasma where the same effect of deviation of adiabatic index from atomic value can be observed at significantly smaller temperatures, say 30 kK which correspond to the transition region. Even from the beginning the theory should have qualitatively agreement with the experiment.
In order to avoid terminological misunderstandings we will recall some basic thermodynamic relations. Often in hydrodynamics is used notion of liquid particle which means small marked volume $V$ of the fluid with local temperature $T$ and pressure $P \equiv p$ which contains however big enough number of particles (Landau & Lifshitz 1988, Sec. 1). Following (Landau & Lifshitz 1980, Sec. 16) we write

$$\left( \frac{\partial V}{\partial P} \right)_S = \frac{\partial (V,S)}{\partial (P,S)} = \frac{\partial (V,S)}{\partial (P,T)} \frac{\partial (V,T)}{\partial (P,T)}$$

$$= \frac{T \left( \frac{\partial S}{\partial T} \right)_V (\partial V)T}{T \left( \frac{\partial S}{\partial T} \right)_P (\partial V)_T} = \frac{C_v}{C_p} \left( \frac{\partial V}{\partial P} \right)_T,$$

where $S$ is the entropy and $C_p$ and $C_v$ are the heat capacities of the liquid particle for constant pressure and volume. Substituting then volume via mass $M$ of the liquid particle as $V = M/\rho$ we obtain the relation between ratio of heat capacities and compressibilities

$$\gamma \equiv \frac{C_p}{C_v} = \frac{\left( \frac{\partial V}{\partial P} \right)_T}{\left( \frac{\partial V}{\partial P} \right)_S} = \frac{\left( \frac{\partial p}{\partial \rho} \right)_T}{\left( \frac{\partial p}{\partial \rho} \right)_S} = \frac{\rho}{\partial \rho} = \frac{v_s^2}{v_T^2}. \quad (4)$$

We emphasize that this is only ratio between derivatives and it is not supposed that plasma have polytropic adiabatic equation $PV^\gamma = \text{const}$ which is property of ideal gas with constant heat capacity, see Landau & Lifshitz (1980, Sec. 43). Often is introduced notation

$$\left( \frac{\partial p}{\partial \rho} \right)_{\text{equilibrium}} = \left( \frac{\partial p}{\partial \rho} \right)_S = v_s^2 \quad (5),$$

emphasizing that for slow hydrodynamic and MHD processes plasma follows in every moment Saha equilibrium conditions and this slow process is reversible with negligible entropy production. Another often used notion is the fast adiabatic comprehensibility at constant ionization degree

$$v_\infty^2 \equiv \left( \frac{\partial p}{\partial \rho} \right)_\alpha. \quad (6)$$

As we will see in the next section the heat capacity of partially ionized plasma is temperature and density dependent and can be much bigger than one and definitely chromospheric plasma is not polytropic. In some articles index $\gamma$ is called adiabatic (Basu & Mandel 2004; Doорselsaere et al. 2011; Jacobs & Poedts 2011; Chatterjee & Fan 2013; Zavershinskii et al. 2019) in some polytropic (Roussev et al. 2003; Petrie et al. 2007; Cohen et al. 2006; Jacobs & Poedts 2011; Wang et al. 2015; Zhang et al. 2016; Prasad et al. 2018; Takahashi et al. 2018) but they are synonyms.

2. CALCULATION FOR PURE HYDROGEN PLASMA

The purpose of the present paper is to represent analytical result for the effective adiabatic index $\gamma_{\text{eff}}$ for hydrogen plasma which consists of electrons, protons and hydrogen atoms with volume densities $n_e$, $n_p$ and $n_0$ respectively.

The correlation energy is negligible for atmospheric plasma and with acceptable approximation the pressure and mass density are described by the total density of the particles of an ideal gas

$$p = n_{\text{tot}} T, \quad n_{\text{tot}} = n_e + n_p + n_0, \quad (7)$$

$$\rho = M n_p, \quad n_p = n_0 + n_p, \quad \alpha \equiv n_p/n_p, \quad (8)$$

where $M$ is the proton mass, and $\alpha$ is the degree of ionization.

The internal energy per unit mass $\varepsilon$ and the enthalpy per unit mass $w$ are given by

$$\varepsilon = (c_{v,a} T n_{\text{tot}} + I n_e)/\rho, \quad w = \varepsilon + p/\rho, \quad n_p = \rho/M, \quad n_e = n_p = \alpha n_p, \quad n_0 = (1 - \alpha) n_p, \quad n_{\text{tot}} = (1 + \alpha) n_p, \quad c_{v,a} = \frac{3}{2}, \quad c_{p,a} \equiv c_{v,a} + 1 = \frac{5}{2}, \quad \gamma_a \equiv \frac{c_{p,a}}{c_{v,a}} \equiv \frac{5}{3}.$$ 

Here $c_{p,a}$ and $c_{v,a}$ are just mathematical constants taken from the theory of mono-atomic ideal gasses. Simultaneously the degree of ionization is given by the Saha (Saha 1921) equation

$$\alpha \equiv \frac{n_p}{n_0} = \frac{1}{\sqrt{1 + p/p_s}}, \quad \frac{p}{p_s} = \frac{1}{\alpha^2} - 1, \quad \frac{n_p}{n_s} = \frac{1 - \alpha}{\alpha^2},$$

$$p_s \equiv n_s T, \quad n_s \equiv n_s e^{-I}, \quad I \equiv \frac{T}{T}, \quad n_s = \left( \frac{mT}{2\pi \hbar^2} \right)^{3/2},$$

where $I$ is the hydrogen ionization potential and $m$ is the electron mass. The dependence $\alpha(T/k_B, n_p)$ is given in Fig. 1. The degree of ionization $\alpha$ depends on the density and pressure and that is why the internal energy and enthalpy obtain pressure and mass density dependence

$$\varepsilon = \frac{1}{M} \left[ c_{v,a} (1 + \alpha) T + I \alpha \right], \quad (9)$$

$$w = \frac{1}{M} \left[ c_{p,a} (1 + \alpha) T + I \alpha \right]. \quad (10)$$

We have to emphasize that magnetic pressure and magnetic field in general exactly zero influence on the thermodynamic properties of classical plasma. This result is known as Bohr–Van Leeuwen theorem, see the
Figure 1. Degree of ionization $\alpha$ in vertical direction as a function of temperature and density in logarithmic scale, i.e. as function of $T$ and $\lg n_p$.

well-known monograph by Mattis (1965, Chap. 1) and the cited therein monograph by Vleck (1932, Chap. 4, Sec. 24, p. 94) and PhD theses by Bohr (1972) and Van Leeuwen, H.-J. (1921). Taking the differential from the expression for $\alpha$ the calculation gives

$$T \left( \frac{\partial \alpha}{\partial T} \right)_p = (1 - \alpha^2) \alpha (c_{p,a} + \iota),$$

(11)

$$T \left( \frac{\partial \alpha}{\partial T} \right)_\rho = (1 - \alpha) \alpha^2 - \alpha (c_{v,a} + \iota).$$

(12)

Further differentiation of the thermodynamic potentials with respect to the temperature according to Eq. (1) gives

$$\tilde{\gamma}_p \equiv \left( \frac{\rho C_p}{n_{tot}} \right) = c_{p,a} + (c_{p,a} + \iota)^2 \varphi,$$

(13)

$$\tilde{\gamma}_v \equiv \left( \frac{\rho C_v}{n_{tot}} \right) = c_{v,a} + (c_{v,a} + \iota)^2 \varphi / (1 + \varphi),$$

(14)

$$\tilde{\gamma} = \frac{\tilde{\gamma}_p}{\tilde{\gamma}_v} = \frac{c_{p,a} + (c_{p,a} + \iota)^2 \varphi}{c_{v,a} + (c_{v,a} + \iota)^2 \varphi / (1 + \varphi)},$$

(15)

$$\varphi \equiv \frac{1}{2} (1 - \alpha) \alpha, \quad \frac{\rho}{n_{tot}} = \langle M \rangle \equiv \frac{M}{1 + \alpha},$$

(16)

where $\langle M \rangle$ is the averaged mass of the cocktail, and $\tilde{\gamma}_v(t, \alpha)$ and $\tilde{\gamma}_p(t, \alpha)$ are temperature and ionization dependent heat capacities per particle; the temperature is in energy units.

One can see in Fig. 2 that the relative adiabatic index $\tilde{\gamma}/\gamma_a$ can differ significantly from 1 even when the temperature is high enough and the degree of ionization is almost 1. The dependency $\tilde{\gamma}/\gamma_a$ in Fig. 2 is shown only up to 30 kK temperature, which roughly corresponds to the beginning of the solar transition region (Eddy 1979; Avrett & Loeser 2008) in order the deviation from 1 to be seen in detail. For higher temperatures its value is clearly 1, which is well-known and of course anticipated since we have included only pure hydrogen in our treatment. Our analytical results for the heat capacities $\tilde{c}_p$, $\tilde{c}_v$ and their ratio $\gamma_{eff} = \tilde{c}_p/\tilde{c}_v$ are depicted in Figs. 3 and 4. This correction will not change qualitatively the uncountable MHD simulations but let be quantitatively correct. MHD is science not a model and the nature of the effective polytropic index was discussed in the excellent monograph by Goosens (2003). In great detail hydrodynamics and MHD of fluid with chemical reactions was discussed also in the monographs Groot & Masur (1974, Chap. 12), Rudenko & Soluyan (1977, Sec. 4) and references therein. Both heat capacities have almost identical behavior, the only visible difference being the vertical scales. The symmetry of the heat capacities and the relative polytropic index about the maximal value $\alpha = 0.5$ is governed by $\varphi$. Despite the large values of the heat capacities, their quite similar behavior limits the values of the relative polytropic index to within around 10% of $\gamma_a$.

3. CONCLUSIONS

Let us summarize the novelty of our results. We have derived for the first time explicit expressions for the adiabatic index and heat capacities of pure hydrogen plasma. Our results are directly applicable for the solar chromosphere where ionization-recombination processes of heavy elements have negligible contribution. Our approach is also applicable for pure argon (Takahashi et al. 2018), the solar corona and arbitrary plasma cocktail. It is necessary to solve the corresponding Saha equations.
One can see significant increase of both heat capacities at small temperatures $T$ related to energy of ionization $I$ of the plasma and both heat capacities have almost identical behavior with the only clearly visible difference being the scales of the vertical direction.

And the polytropic index again will be different from the single-atomic value $\gamma_a = 5/3$ from the computer simulations.

The solar corona and stellar atmospheres in general contain heavy elements and even ionization of helium can create significant changes of the polytropic index (Basu & Mandel 2004). It is a routine task for every plasma cocktail to include the Saha ionization equation in its thermodynamics. For pure argon used in the laboratory experiments (Takahashi et al. 2018) the task is even simplified.

In conclusion, we consider that the experimental data processing of the astrophysical observations has to start with the equilibrium thermodynamics of realistic chemical compound for which it is possible to make state of the art theoretical evaluation of $\gamma_{\text{eff}}$. Our analytical result for pure hydrogen plasma Eq. (15) is just the illustration of the first step. And this first step is a necessary ingredient for the explanation of the physical processes in the solar chromosphere, for instance what causes the hydrogen ionization there.

The next problem of the physics of solar corona is to recalculate the dispersion relations of magneto-hydrodynamic waves taking into account the influence of ionization-recombination processes on the kinetic coefficients. Wave propagation and kinetic effects related to frequency dependent misbalance requires even more sophisticated treatment. For example, even the second viscosity of the hydrogen plasma and its dispersion is still an open problem in astrophysics.

Acknowledgments. The authors thanks to Valery Nakariakov and Kris Murawski for the interest to the paper, correspondence and valuable remarks.

Apropos: The experimental set-up presented in Fig. 1 of the commented article Takahashi et al. (2018) remains a propulsion engine of a magneto-plasma rocket. We use the opportunity to mention a new idea that not only helicon waves but antennas exciting Alfvén waves (AW) can be even the better solution for heating of hot dense plasma by viscosity friction. The area of AW damping will be similar to the combustion chamber of chemical jet engines. And creation of propulsion will be analogous to the launching of solar wind by absorption of AW as Hannes Alfvén suggested many years ago (Alfvén 1942; Alfvén & Lindblad 1947).
