Interplay between network structure and self-organized criticality

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We investigate, by numerical simulations, how the avalanche dynamics of the Bak-Tang-Wiesenfeld (BTW) sandpile model can induce emergence of scale-free (SF) networks and how this emerging structure affects dynamics of the system. We also discuss how the observed phenomenon can be used to explain evolution of scientific collaboration.

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Since the discovery of self-organized criticality (SOC) by Bak, Tang and Wiesenfeld (BTW) [1], the phenomenon has received enormous attention among the researchers. During these almost twenty years, dozens of variants of the original sandpile model [1,2] were studied [3,4,5,6,7,8,9,10] and a number of examples of SOC in real world were discovered. One of the most remarkable features that characterize self-organized criticality is the power law distribution of the characteristic events. The feature combined with the abundance of real-world networks with scale-free (SF) degree distribution [7,8,9,10] may give rise to suspicion that there exists mutual relation between the two issues. Although the idea has been already mentioned in several papers (see for example [11,12]), according to our knowledge, so far no-one has established a link between the two phenomena.

At the beginning let us recap the rules of sandpile model which is a simple intuitive example of self-organized criticality. It is a cellular automaton whose configuration is determined by the integer variable \( c_i \) (the height of the "sand column") at every node \( i \) of the network. Depending on network structure minor differences in definition can occur. Here we follow the definition of BTW model for random networks with a given degree distribution \( p(k) \). The dynamics is defined by the following simple rules: A grain of sand is added at a randomly selected node \( i \): \( c_i \rightarrow c_i + 1 \). A sand column with a height \( c_i \geq k_i \), where \( k_i \) is equal to degree of the node \( i \), becomes unstable and collapses by distributing one grain of sand to each of it’s \( k_i \) neighbors. This may cause some of them to become unstable and collapse at the next time step. This in turn can lead to an avalanche of next instabilities. During the evolution a small fraction \( f \) of grains is lost, which prevents system to become overloaded. When avalanche dies another grain of sand is added.

In the sandpile model distributions of avalanche sizes (measured as total number of topplings in the avalanche), avalanche areas (the number of distinct nodes participating in a given avalanche), avalanche durations as well as many other statistics follow power law distributions.

Studies of sandpile dynamics carried out so far show that the characteristic exponents of measured distributions depend on the network topology. For example the avalanche size exponent is \( \tau = 1 \) for 2D square lattice [1] and \( \tau = 1.33 \) for 3D cubic lattice [1]. In Erdős-Rényi (ER) random networks \( \tau = 1.5 \) [13]. Recently, Goh et al. have studied sandpile dynamics on scale free networks \( p(k) \sim k^{-\gamma} \) [14,15] and they have shown that the avalanche area exponent \( \tau \) is independent on the average network connectivity \( \langle k \rangle \) and changes with the exponent \( \gamma \) of the degree distribution. They have obtained

\[ \tau = \frac{\gamma}{\gamma - 1} \]

in the range \( 2 < \gamma < 3 \) and \( \tau = 1.5 \) for \( \gamma > 3 \).

The question we ask in the paper is the following: how the avalanche area exponent behaves when the network topology depends on sandpile dynamics, i.e. when there are mutual interactions between the network structure and network dynamics?

In [11] Bianconi and Marsili proposed a simple model in which the network reorganizes its structure as a consequence of avalanches of rewiring processes. The only parameter of the model which influences the rewiring and in consequence the network structure is a type of probability that a chosen node becomes unstable and has to be rewired. Choosing this probability as a power law one can set the system in a critical state and force the network to take a power law degree distribution.

In the present paper, instead of forcing the network to stay in a critical state, we allow the system to naturally evolve towards the critical region. In our model:

1. the degree distribution of the considered networks changes due to the distribution of sandpile avalanches on this network and
2. the avalanche size distribution changes because the network structure evolves.

These two mechanisms influence each other and lead to the equilibrium point in which the shapes of avalanche distribution and degree distribution become identical. In the second part of the paper we propose how the presented phenomenon can be applied to modelling of evolution of scientific collaboration.

In order to complete the rules of our model, apart from the rules of sandpile model recapitulated above, we define the rewiring process in the following way: each end of a link has been assigned a value specifying the time when
it was rewired for the last time. After an avalanche of area \( A \), the number of 'oldest' ends of links are rewired to the node which has triggered the avalanche off.

In our studies all networks have: \( \langle k \rangle = 4 \), number of nodes \( N = 10^5 \) and \( f = 10^{-4} \). We start our simulation with Erdős-Rényi random network (which corresponds to \( \gamma = \infty \) in static SF networks). Time unit that has been used was simply one avalanche and we have carried out our simulations for \( t = 10000 \) steps. Fig. 1 presents three snapshots of the node degree distribution in three different moments: \( t_{\text{start}} = 0 \), \( t_{\text{mid}} = 2500 \) and \( t_{\text{end}} = 10000 \). Comparing the snapshots, one can see that the network reorganizes itself from Poissonian and scale-free and finally settles on a pure scale-free degree distribution with the well established characteristic exponent \( \gamma \approx 2 \). The same time the characteristic exponent of the avalanche area distribution increases from \( \tau = 1.5 \), which is the known result for ER random networks \([11]\), to \( \tau = 2 \). As one can see at Fig. 2 in the course of simulation the two exponents converge to the common equilibrium value \( \tau = \gamma = 2 \).

Fig. 3 presents the convergence process in a more detailed way. We define there a new parameter \( \tilde{\gamma}(t) \) that in some sense may be understood as the characteristic exponent of fat-tailed degree distributions and may be compared to \( \gamma \). \( \tilde{\gamma}(t) \) is simply obtained from the second moment of the degree distribution that is know from simulations. Given \( \langle k^2 \rangle \), we numerically solve the below equation for \( \tilde{\gamma}(t) \)

\[
\langle k^2 \rangle = \sum_{k=1}^{N} k^2 p_{\text{an}}(k) \tag{2}
\]

where

\[
p_{\text{an}}(k) = \frac{\langle k \rangle^{\tilde{\gamma} - 1} \Gamma(k - \tilde{\gamma} + 1, \langle k \rangle^{\tilde{\gamma} - 2})}{\Gamma(k - 1)^{\tilde{\gamma} - 2} \Gamma(k + 1)} \tag{3}
\]

is the known analytic solution of the static model \([17]\). The new parameter has been introduced because in intermediate times of the simulation the degree distribution does not follow a pure power law. At the mentioned figure one can see that the value of the exponent \( \tilde{\gamma} \) (open triangles) decreases from \( \infty \) to 2.1. Simultaneously, the parameter \( \tau \) characterizing the sandpile dynamics (solid squares) increases from 1.5 and finally settles at the equilibrium value of 2.1. The values of \( \tau \) fairly good agree with the values of \( \tau_{\text{theor}} \) (solid line) calculated from the relation \([14]\) derived for static SF networks by Goh et al. \([14]\). The last observation let us suspect that during simulation the system moves along the trajectory given by the formula (see Fig. 3)

\[
\tau(t) = \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t) - 1}, \tag{4}
\]

and respectively the equilibrium point may be calculated from the above equation when one assumes \( \tilde{\gamma} = \tau \).

The exponents \( \tilde{\gamma} = \tau = 2 \) characterizing the final critical state seem to be robust against different threshold assignment strategies in sandpile dynamics. In order to support the last statement let us mention two papers \([12, 15]\) in which we have found probable symptoms of such a universality. In \([12]\), a class of sandpile models is studied. In this class the threshold height of a node is set as \( k^{1-\eta} \), where \( 0 \leq \eta < 1 \) is a parameter of the class. The avalanche size exponent is received as \( \tau = (\gamma - 2\eta)/(\gamma - 1 - \eta) \). If one assumes that due to self-organization and rewiring process in our model \( \tau = \gamma \), then one obtains \( \tau = \gamma = 2 \) independently on \( \eta \). The second example is a model of rapid rearrangements in the network of the magnetic field flows in the Sun corona \([12]\). The authors show that the avalanches of link reconnections and scale free structure of the considered network co-generate each other. They also show that for the equilibrium the degree distribution exponent \( \gamma = 2 \). Unfortunately, they do not present reconnection distribution exponent which corresponds to \( \tau \).

In fact, the precise value of the fixed point is a bit larger than 2, about 2.1. It can result from the finite size effects or from the fact, that in the vicinity of \( \gamma = 2 \) the considered networks are highly correlated but the way we perform rewiring includes a small random contribution (it is known that in this range of parameters the network should be correlated disasssortatively, so instead of rewiring the oldest end of the link we should perhaps rewrite the less disassorative link).

In the last part of the paper we would like to show how the observed phenomenon can be used to construct a simple model of evolution of scientific connections.

In the model each node represents a scientist. A link between two scientists represents the fact that they can exchange new ideas and draw inspiration from each other. A scientist has its own potential (hidden variable) which describes his/her ability to produce an interesting paper. The potential is a non decreasing function of time (we rather collect ideas, do not lose them). If the potential...
FIG. 2: Distributions of avalanche area and node degree in time \( t_{\text{end}} \). Data are logarithmically binned. Lines are linearly fitted with the values indicated at the figure.

FIG. 3: Process of equilibration of exponents \( \tilde{\gamma} \) and \( \tau \). Solid line presents theoretical \( \tau_{\text{theor}} \) obtained from \( \tilde{\gamma} \) and eq. (1). Inset: second moment of the degree distribution in time.

FIG. 4: Dependence of avalanche area exponent \( \tau \) on parameter \( \tilde{\gamma} \) of generated scale-free network. The arrow shows direction of a movement in space of parameters during the process of equilibration. Measurements are done in equal time steps \( \Delta t = 500 \) and marked as open circles. The black dot depicts the fixed point of the process. Dashed line presents theoretical \( \tau_{\text{theor}} \) obtained from \( \tilde{\gamma} \) and eq. (1).

Observation of other less interesting one.

In the model that has been presented above one can find mechanisms which can be modeled by the phenomenon described in the first part of the paper. The potential may be considered as a column of grains and every grain represents a quantum of idea. To make the potential equal for all scientists one can normalize it by a scientist’s degree. By an avalanche we understand an occurrence of some scientific sub-domain (like physics of complex networks or self-organized criticality) so the timescale of such an avalanche will be expressed in years.

The above model just gives an idea where the presented phenomenon can be found. However, since the term ‘collaboration’ does not only mean common papers but every manifestation of exchange of ideas (appeared as co-authorship, citing and even common discussions), the applicability of the model to real data may suffer a number of difficulties.

To conclude, in this paper we have presented by numerical simulations how the avalanche dynamics of the Bak-Tang-Wiesenfeld sandpile model and the network structure may influence each other. Such an interplay between dynamics and structure leads to self-organization in which the shapes of avalanche distribution and degree distribution become identical. We suspect that the value of both exponents \( \gamma = \tau = 2 \) may be universal for a large class of SOC phenomena in which the critical behavior occurs not ‘on’ the network structure but ‘in’ the structure. We also show how the observed phenomenon can be used to construct a simple model of evolution of scientific collaborations.
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