NILPOTENT COVERS OF SYMMETRIC GROUPS

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Abstract. We prove that the symmetric group $S_n$ has a unique minimal cover $\mathcal{M}$ by maximal nilpotent subgroups, and we obtain an explicit and easily computed formula for the order of $\mathcal{M}$. In addition, we prove that the order of $\mathcal{M}$ is equal to the order of a maximal non-nilpotent subset of $S_n$. This cover $\mathcal{M}$ has attractive properties; for instance, it is a normal cover, and the number of conjugacy classes of subgroups in the cover is equal to the number of partitions of $n$ into distinct positive integers.

These results contrast starkly with those for abelian covers, suggesting that nilpotent covers are a richer class of covers to study.

1. Introduction

The principal objective of this paper is to determine the least number of nilpotent subgroups of the symmetric group on $n$ letters $S_n$ that are necessary to cover $S_n$. We establish that there is a unique minimal collection of maximal nilpotent subgroups that cover $S_n$, and the order of this collection can be computed easily from a list of the partitions of $n$ into distinct positive integers. Furthermore, we prove that the order of the collection is equal to the order of a maximal non-nilpotent subset of $S_n$. These results are stronger than similar results for abelian covers, suggesting that nilpotent covers are a richer class of covers to study.

To explain our results in more detail, consider a finite group $G$. A nilpotent cover of $G$ is a finite family $\mathcal{M}$ of nilpotent subgroups of $G$ for which

$$G = \bigcup_{H \in \mathcal{M}} H.$$ 

A nilpotent cover $\mathcal{M}$ of $G$ is said to be minimal if no other nilpotent cover of $G$ has fewer members. Let $\Sigma_\infty(G)$ denote the size of a minimal nilpotent cover of $G$, provided such a cover exists.

Of particular interest to us are nilpotent covers that are invariant under conjugation. A nilpotent cover $\mathcal{M}$ of $G$ is normal if whenever $H \in \mathcal{M}$ and $g \in G$ we have $g^{-1}Hg \in \mathcal{M}$. We seek to ascertain whether a minimal nilpotent cover of $G$ can be found that is normal.

Each normal nilpotent cover of $G$ can be partitioned into conjugacy classes of subgroups. Let $\Gamma_\infty(G)$ denote the least number of such conjugacy classes, among all the normal nilpotent covers of $G$.

There is a parallel notion to that of nilpotent cover: a non-nilpotent subset of $G$ is a subset $X$ of $G$ such that for any two distinct elements $x$ and $y$ of $X$, the subgroup $\langle x, y \rangle$ generated by $x$ and $y$ is not nilpotent. A non-nilpotent subset of $G$ is said to be maximal if no other non-nilpotent subset of $G$ contains more elements. Let $\sigma_\infty(G)$ denote the size of a maximal non-nilpotent subset of $G$. A straightforward consequence of the pigeon-hole principle is that $\sigma_\infty(G) \leq \Sigma_\infty(G)$, provided $G$ has a nilpotent cover.

In this paper we calculate the quantities $\Gamma_\infty(S_n)$, $\Sigma_\infty(S_n)$ and $\sigma_\infty(S_n)$, and prove that the latter two quantities coincide. We use the concept of a distinct partition of a positive integer $n$, which is a set $T = \{t_1, t_2, \ldots, t_k\}$ where $t_1, t_2, \ldots, t_k$ are distinct positive integers and $n = t_1 + t_2 + \cdots + t_k$.

In Section 2 we will prove that if the cycle type of an element $g$ of $S_n$ is a distinct partition of $n$, then $g$ lies within a unique maximal nilpotent subgroup of $S_n$ (Proposition 2.4). We denote by $\mathcal{M}$ the collection of all maximal nilpotent subgroups that arise in this way.

Theorem 1.1. The cover $\mathcal{M}$ is the unique minimal nilpotent cover of $S_n$ by maximal nilpotent subgroups.

The cover $\mathcal{M}$ is by definition a normal cover, so we obtain the following corollary of Proposition 2.4 and Theorem 1.1.

Corollary 1.2. The cover $\mathcal{M}$ is a normal nilpotent cover of $S_n$ and $\Gamma_\infty(S_n)$ is equal to the number of partitions of $n$ into distinct positive integers.

Key words and phrases. alternating group; nilpotent cover; non-nilpotent subset; normal nilpotent cover; symmetric group.
Moreover, using Proposition 2.4 we can see that $\mathcal{M}$ is the unique normal nilpotent cover of $S_n$ containing $\Gamma_\infty(S_n)$ conjugacy classes of maximal nilpotent subgroups of $S_n$.

The problem of calculating the number of distinct partitions of a positive integer $n$ is an old one. Values for small $n$ can be found at the OEIS [OEI19], along with a wealth of information about this problem. One well-known fact about distinct partitions (due to Euler) is that the number of distinct partitions of $n$ is equal to the number of partitions of $n$ into odd positive integers.

A further corollary of Theorem 1.2 gives an explicit formula for $\Sigma_\infty(S_n)$ and $\sigma_\infty(S_n)$.

**Corollary 1.3.** We have

$$\Sigma_\infty(S_n) = \sigma_\infty(S_n) = \sum_{T \in \text{DP}(n)} \left( \frac{n!}{\prod_{i=1}^{\ell} (p_i - 1)^{e_i} p_i^{c_i}} \right),$$

where, in the final product, $t = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ is the prime factorisation of $t$ and $e_i = (p_i^{e_i} - 1)/(p_i - 1)$, for $i = 1, 2, \ldots, \ell$.

The product $\prod_{i=1}^{\ell} (p_i - 1)^{e_i} p_i^{c_i}$ in Corollary 1.3 is considered to take the value 1 if $t = 1$.

Let $\text{DP}(n)$ denote the set of all distinct partitions of $n$. Table 1.1 displays the first few values of $\text{DP}(n)$ and $\Sigma_\infty(S_n)$.

| $n$  | $\text{DP}(n)$ | $\Sigma_\infty(S_n)$ |
|------|----------------|----------------------|
| 2    | $\{2\}$        | 1                    |
| 3    | $\{1, 2\}, \{3\}$ | 4                    |
| 4    | $\{1, 3\}, \{4\}$ | 7                    |
| 5    | $\{1, 4\}, \{2, 3\}, \{5\}$ | 31                   |
| 6    | $\{1, 2, 3\}, \{1, 5\}, \{2, 4\}, \{6\}$ | 201                  |
| 7    | $\{1, 2, 4\}, \{1, 6\}, \{2, 5\}, \{3, 4\}, \{7\}$ | 1086                 |
| 8    | $\{1, 2, 5\}, \{1, 3, 4\}, \{1, 7\}, \{2, 6\}, \{3, 5\}, \{8\}$ | 5139                 |
| 9    | $\{1, 2, 6\}, \{1, 3, 5\}, \{2, 3, 4\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}, \{9\}$ | 37507                |

**Table 1.1.** Values of $\text{DP}(n)$ and $\Sigma_\infty(S_n)$, for $n = 2, 3, \ldots, 9$.

When $n = 3$ the minimal nilpotent cover $\mathcal{M}$ comprises the nilpotent subgroups

$$\langle (1, 2), \ (2, 3), \ (3, 1), \ (1, 2, 3) \rangle,$$

where, for example, $\langle (1, 2, 3) \rangle$ is the cyclic group generated by the permutation $(1, 2, 3)$ (written in disjoint cycle notation). When $n = 4$ the minimal nilpotent cover $\mathcal{M}$ comprises the four cyclic subgroups

$$\langle (1, 2, 3), \ (1, 2, 4), \ (1, 3, 4), \ (2, 3, 4) \rangle$$

and the three dihedral subgroups of order 8

$$\langle (1, 2, 3, 4), (1, 3), (1, 3, 4, 2), (1, 4), (1, 4, 2, 3), (1, 2) \rangle.$$

We remark that $n = 6$ is the least value of $n$ for which the corresponding minimal nilpotent cover $\mathcal{M}$ is not the full collection of all maximal nilpotent subgroups of $S_n$. For $\mathcal{M}$ does not contain the maximal nilpotent subgroup

$$\langle (1, 2, 3), (4, 5, 6), (14)(25)(36) \rangle$$

of $S_6$, isomorphic to $C_3 \times C_3 \times C_2$, nor any of the other nine conjugates of this subgroup.

Let us contrast our results to those on abelian covers. An *abelian cover* of a group $G$ is a finite family of abelian subgroups of $G$ whose union is equal to $G$. We write $\Sigma_1(G)$ for the smallest possible size of an abelian cover of $G$. A *non-commuting subset* of a group $G$ is a subset $X$ of $G$ such that any two distinct elements of $X$ generate a non-abelian group (or, equivalently, any two distinct elements of $X$ do not commute). We write $\sigma_1(G)$ for the size of the largest non-commuting subset of $G$. Just as for nilpotent covers, it is clear that $\sigma_1(G) \leq \Sigma_1(G)$. However, in contrast to Corollary 1.3 Brown [Bro88, Bro91] proved that $\Sigma_1(S_n) \neq \sigma_1(S_n)$ for $n \geq 15$. Furthermore, Barrantes et al. showed that, likewise, $\Sigma_1(A_n) \neq \sigma_1(A_n)$ for $n \geq 20$ [BGR16].

It seems likely that results similar to those presented here should hold for nilpotent covers of the alternating groups and perhaps for other finite simple groups also. There has been progress on this already; for instance, Azad et al. [ABG15] (following earlier work of Azad [Aza11]) proved that for a finite simple group $G$ the size
of a maximal non-nilpotent subset is approximately equal to the number of maximal tori in $G$. Moreover, they proved that $\Sigma_\infty(G) = \sigma_\infty(G)$ when $G$ is a finite group of Lie type of rank 1.

This discussion motivates the following question.

**Question 1.4.** Which almost simple groups $G$ satisfy some or all of the following properties:

(i) $G$ has a unique minimal cover by maximal nilpotent subgroups,
(ii) a minimal cover of $G$ by maximal nilpotent subgroups is necessarily normal,
(iii) $\Sigma_\infty(G) = \sigma_\infty(G)$?

At present we are unaware of any almost simple groups for which these properties do not hold.

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### 2. Proof of Theorem 1.2

In the course of the proof we use a variety of well known properties of permutation groups and nilpotent groups. In particular, we use the fact that a finite nilpotent group is a direct product of its Sylow subgroups. A consequence of this observation is that elements of coprime order in such a group commute. Another fact we use is that if $P$ is a Sylow $p$-subgroup of $S_n$, for some prime $p$, and $n = a_0 + a_1p + \cdots + a_dp^k$ is the base-$p$ expansion of $n$, then $P$ has $a_i$ orbits on $\{1, 2, \ldots, n\}$ of size $p^i$, for $i = 0, 1, \ldots, k$ (and this accounts for all orbits of $P$). For example, if $p = 3$ and $n = 16$, then $P$ has orbits of sizes 1, 3, 3, 9.

We make use of the following notation and terminology. Given a subset $X$ of $\{1, 2, \ldots, n\}$, we denote by $\text{Sym}(X)$ the full group of symmetries of $X$ within $S_n$. That is, $\text{Sym}(X)$ is the pointwise stabilizer of the complement of $X$. Also, we refer to the ‘orbits of a permutation $g$’ as a shorthand for the orbits of the cyclic group generated by $g$.

For an integer $k$ and a prime $p$, we define $|k|_p$ to be the largest power of $p$ that is a factor of $k$, and we define $|k|_{p'} = |k|/|k|_p$.

**Lemma 2.1.** Let $O_1, O_2, \ldots, O_k$ be the orbits in $\{1, 2, \ldots, n\}$ of an element $g$ of $S_n$, and suppose that the orders $t_1, t_2, \ldots, t_k$ of these orbits form a distinct partition of $n$. Any nilpotent subgroup $N$ of $S_n$ containing $g$ satisfies

$$N \leq \text{Sym}(O_1) \times \text{Sym}(O_2) \times \cdots \times \text{Sym}(O_k).$$

**Proof.** Let $M = \text{Sym}(O_1) \times \text{Sym}(O_2) \times \cdots \times \text{Sym}(O_k)$, a subgroup of $S_n$. Since $N$ is a direct product of its Sylow subgroups, it is enough to prove that any Sylow $p$-subgroup $P$ of $N$ is contained in $M$. Let $h \in P$. We will prove that $O_i^h = O_i$, for $i = 1, 2, \ldots, k$, from which it follows that $h \in M$.

Let $m$ denote the order of $g$ and let $d = |m|_p$ and $e = |m|_{p'}$ (so $m = de$). We define $g_1 = g^e$, which has order $d$. The orbits of $g_1$ are a refinement of those of $g$. More specifically, for each index $i$, there are $|t_i|_{p'}$ orbits of $g_1$ within $O_i$, each of size $|t_i|_{p'}$.

Since $g_1$ and $h$ have coprime orders, they commute. Consequently, for any $x \in \{1, 2, \ldots, n\}$, the orbit of $x$ under $g_1$ has the same size as the orbit of $x^h$ under $g_1$. It follows that if $x \in O_i$ and $x^h \in O_j$, then $|t_i|_{p'} = |t_j|_{p'}$.

Consider a complete set of orbits $O_i$ for which the values $|t_i|_{p'}$ are all equal; after relabelling we can assume that $O_1, O_2, \ldots, O_t$ is such a set. The preceding argument shows that $h$ preserves the union $O_1 \cup O_2 \cup \cdots \cup O_t$. For the remainder of the argument, we focus on the restriction of $g$ and $h$ to this union. To simplify notation, we assume that this union is in fact the full set $\{1, 2, \ldots, n\}$.

As before, we let $m$ denote the order of $g$ and let $d = |m|_{p'}$, $e = |m|_p$ and $g_1 = g^e$, of order $d$. Since each of the orbits of $g_1$ has the same size, we see that $g_1$ is a product of $n/d$ disjoint $d$-cycles. Thus $g_1$ has $n/d$ orbits, each of which comprises the $d$ entries of some $d$-cycle of $g$.

Any element of $S_n$ that commutes with $g_1$ must permute these $n/d$ orbits. In this way we obtain a homomorphism $\theta$ from the centralizer $C_{S_n}(g_1)$ of $g_1$ in $S_n$ to $S_{n/d}$. The Sylow $p$-subgroup $P$ is contained in $C_{S_n}(g_1)$, so we can define $P' = \theta(P)$, a Sylow $p$-subgroup of $S_{n/d}$.

Now define $g_2 = g^d$. This has order $e$, a power of $p$, so $g_2 \in P$. Let us consider how the cyclic group $\langle g_2 \rangle$ acts on the orbits of $g_1$. For any index $i \in \{1, 2, \ldots, t\}$, we know that $g_2$ fixes $O_i$. Choose $x, y \in O_i$; then $y = g^r(x)$ for some integer $r$. Since $d$ and $e$ are coprime we can find integers $a$ and $b$ such that $r = ad + be$, in which case $y = g^{ad+be}(x)$, so $g_2^{b}(x) = g_2^{-b}(y)$. It follows that $g_2^b(x)$ and $y$ are in the same orbit of $g_1$. Therefore any orbit of $g_1$ in $O_i$ can be mapped to any other under the action of $\langle g_2 \rangle$. Consequently, $\theta(g_2)$ is a product of
disjoint cycles of orders $t_1/d, t_2/d, \ldots, t_ℓ/d$. Each of these integers is a power of $p$, and they are distinct, since the integers $t_1, t_2, \ldots, t_ℓ$ are distinct. Thus $n/d = t_1/d + t_2/d + \cdots + t_ℓ/d$ is the $p$-ary decomposition of $n/d$.

Consider next the action of the Sylow $p$-subgroup $P'$ on $S_{n/d}$. Using the observation about orbits of Sylow $p$-subgroups stated before the lemma we see that the orbits of $P'$ must be exactly those of $θ(g)$. In particular, since $θ(h) ∈ P'$ we see that, for any $i ∈ \{1, 2, \ldots, ℓ\}$, the permutation $h$ maps any orbit of $g_1$ within $O_i$ to another orbit of $g_1$ in $O_i$. Hence $h$ fixes $O_i$, as required.

Let $p$ be a prime, $a$ a positive integer, and $q = p^a$. The next lemma uses the known result that all $q$-cycles in a Sylow $p$-subgroup $P$ of $S_q$ are conjugate in the normalizer $N_{S_q}(P)$ of $P$ in $S_q$.

**Lemma 2.2.** Let $g$ be a $q$-cycle in $S_q$, where $q = p^a$, a prime power. There is a unique Sylow $p$-subgroup of $S_q$ that contains $g$.

*Proof.* Let $P$ be a Sylow $p$-subgroup of $S_q$ that contains $g$, and suppose that $g$ also belongs to another Sylow $p$-subgroup $h^{-1}Ph$, for some $h ∈ S_q$. Then $hgh^{-1}$ is a $q$-cycle in $P$, so there exists $k ∈ N_{S_q}(P)$ with $hgh^{-1} = k^{-1}gk$. Consequently, $(kh)g = g(kh)$, so $kh$ belongs to the centralizer of $g$ in $S_q$. Now, the centralizer of $g$ is the cyclic group generated by $g$, so $h ∈ N_{S_q}(P)$. Hence $h^{-1}Ph = P$. Thus $g$ is contained in a unique Sylow $p$-subgroup of $S_q$. □

Next we introduce some concepts about permutation groups and Sylow subgroups.

Suppose that $G$ is a subgroup of $S_m$ and $H$ is a subgroup of $S_n$. Then $G × H$ acts faithfully on $\{1, 2, \ldots, m\} × \{1, 2, \ldots, n\}$ by the formula $(g, h): (x, y) ↦ (x^g, y^h)$. By choosing some identification of $\{1, 2, \ldots, m\} × \{1, 2, \ldots, n\}$ with $\{1, 2, \ldots, mn\}$ we obtain a subgroup of $S_{mn}$, which we denote by $G × H$ (the freedom to choose an identification means that this group is defined only up to conjugation in $S_{mn}$).

In the same way we can take subgroups $G_1, G_2, \ldots, G_k$ of the symmetric groups $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$, in order, and define the product $G_1 × G_2 × \cdots × G_k$ a subgroup of $S_n$, where $n = n_1n_2 \cdots n_k$. Observe that the operation $×$ is both associative and commutative. Observe also that if $G_1, G_2, \ldots, G_k$ are transitive subgroups of $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$, then $G_1 × G_2 × \cdots × G_k$ is a transitive subgroup of $S_n$.

For a positive integer $t$, we write $t = \prod p_1^{α_1}p_2^{α_2} \cdots p_{α_ℓ}^{α_ℓ}$, where $p_1 < p_2 < \cdots < p_ℓ$ are primes and $α_1, α_2, \ldots, α_ℓ$ are positive integers. Let $q_i = p_1^{α_i}$, for $i = 1, 2, \ldots, ℓ$. We define $P(t) = \prod p_1^{α_i}p_2^{α_i} \cdots p_{α_ℓ}^{α_ℓ}$ and $PP(t) = Q_i^{q_1, q_2, \ldots, q_ℓ}$, the lists of primes and the corresponding prime powers in the prime factorisation of $t$, respectively, each written in increasing order.

Let $P_{p_i, q_i}$ be a Sylow $p_i$-subgroup of $S_{q_i}$, for $i = 1, 2, \ldots, ℓ$. Since $t = q_1q_2 \cdots q_ℓ$, we see that

$$P_{p_i, q_i} × P_{p_{i+1}, q_{i+1}} × \cdots × P_{p_ℓ, q_ℓ}$$

is a subgroup of $S_t$. Indeed, it is a transitive subgroup of $S_t$, because $P_{p_i, q_i}$ is a transitive subgroup of $S_{q_i}$.

The following lemma generalises Lemma 2.2.

**Lemma 2.3.** Let $g$ be a $t$-cycle in $S_t$. Let $P(t) = \prod p_1^{α_1}p_2^{α_2} \cdots p_{α_ℓ}^{α_ℓ}$ and $PP(t) = Q_i^{q_1, q_2, \ldots, q_ℓ}$. Then $g$ lies in a unique maximal nilpotent subgroup $N$ of $S_t$ and

$$N = P_{p_1, q_1} × P_{p_2, q_2} × \cdots × P_{p_ℓ, q_ℓ},$$

for some Sylow $p_i$-subgroups $P_{p_i, q_i}$ of $S_{q_i}$, $i = 1, 2, \ldots, ℓ$.

*Proof.* By conjugating, we can assume that $g = (1, 2, \ldots, t)$. Let $N$ be a nilpotent subgroup of $S_t$ that contains $g$. Then $N$ is the direct product of its Sylow $p_i$-subgroups $P_i$, for $i = 1, 2, \ldots, ℓ$.

We identify the permutation set $\{1, 2, \ldots, t\}$ with $Z/tZ$ by sending $x$ to $[x]_t$, the congruence class of integers congruent to $x$ modulo $t$. We identify $Z/tZ$ with $Z/q_1Z × Z/q_2Z × \cdots × Z/q_ℓZ$ by sending $[x]_t$ to $([x]_{q_1}, [x]_{q_2}, \ldots, [x]_{q_ℓ})$. On $Z/tZ$ the action of $g$ is given by $[x]_t ↦ [x + 1]_t$.

Let $N_1 = P_2P_3 \cdots P_ℓ$ and choose $h ∈ N_1$. Consider any element $x = (x_1[1], x_2[2], \ldots, x_t[ℓ])$ of $Z/q_1Z × Z/q_2Z × \cdots × Z/q_ℓZ$ and define

$$([y_1]_{q_1}, [y_2]_{q_2}, \ldots, [y_t]_{q_t}) = h([x_1]_{q_1}, [x_2]_{q_2}, \ldots, [x_t]_{q_t}).$$

We will prove that $[y_1]_{q_1} = [x_1]_{q_1}$.

Define $g_1 = g^{1/s_1}$, which has order $q_1$, so it commutes with $h$. Hence $h(g_1^k(x)) = g_1^k(h(x))$, for any integer $k$. Evaluating each side of this equation we obtain

$$h(\{x_1 + kt/q_1[1], x_2[2], \ldots, x_t[ℓ]\}) = \{y_1 + kt/q_1[1], y_2[2], \ldots, y_t[ℓ]\}.$$


Now define $g_2 = g^9$, which has order $t/q_1$, so $g_2 \in N_1$. We can choose an integer $m$ such that $[y_i + m q_1]_{q_1} = [x_i]_{q_1}$, for $i = 2, 3, \ldots, r$. It follows that
\[ g_2^m h([x_1 + kt/q_1], [x_2], \ldots, [x_i]) = ([y_1 + kt/q_1], [x_2], \ldots, [x_i]), \]
for any integer $k$. We obtain an action of $g_2^m h$ on $\mathbb{Z}/q_1 \mathbb{Z}$. However, the order of $g_2^m h$ is coprime to $q_1$, so $g_2^m h$ is the identity permutation. Hence $[x_1]_{q_1} = [y_1]_{q_1}$, as required.

A similar argument holds with the $i$th component instead of the first component. Thus, if $h \in P_j$, then $h$ fixes each component of $\mathbb{Z}/q_i \mathbb{Z} \times \mathbb{Z}/q_2 \mathbb{Z} \times \cdots \times \mathbb{Z}/q_t \mathbb{Z}$ other than the $j$th component. In this way we can identify $P_j$ with a subgroup of $\text{Sym}(\mathbb{Z}/q_i \mathbb{Z})$, and $N = P_1 \times P_2 \times \cdots \times P_t$. Now, $g^{n/q_i} \in P_j$ and it is a $q_i$-cycle in $\text{Sym}(\mathbb{Z}/q_i \mathbb{Z})$. By Lemma 2.3 there is a unique Sylow $p_j$-subgroup $Q_{j}$ of $S_{q_i}$ that contains $g^{n/q_i}$. Taking the product $Q_1 \times Q_2 \times \cdots \times Q_t$ of all such groups we obtain a maximal nilpotent group containing $g$ (and $N$), uniquely specified by the subgroups $Q_j$, as required. \hfill $\Box$

The following proposition is an immediate consequence of Lemmas 2.1 and 2.3

**Proposition 2.4.** Let $O_1, O_2, \ldots, O_k$ be the orbits in $\{1, 2, \ldots, n\}$ of an element $g$ of $S_n$, and suppose that the orders $t_1, t_2, \ldots, t_k$ of these orbits form a distinct partition of $n$. Then there is a unique maximal nilpotent subgroup $N$ of $S_n$ that contains $g$. Furthermore, there are subgroups $N_i$ of $\text{Sym}(O_i)$, for $i = 1, 2, \ldots, k$, with $N = N_1 \times N_2 \times \cdots \times N_k$, where, for $i = 1, 2, \ldots, k$, we write $\text{P}(t_i) = [p_{j_1}, p_{j_2}, \ldots, p_{j_s}]$ and $\text{PP}(t_i) = [q_{j_1}, q_{j_2}, \ldots, q_{j_s}]$, and we have $N_i = P_{p_{j_1}, q_{j_1}} \times P_{p_{j_2}, q_{j_2}} \times \cdots \times P_{p_{j_s}, q_{j_s}}$, for some Sylow $p_j$-subgroups $P_{p_{j_i}, q_{j_i}}$ of $S_{q_i}$, $s = 1, 2, \ldots, \ell_i$.

Let $g$ be an element of a subgroup $G$ of $S_n$ and let $h$ be an element of a subgroup $H$ of $S_n$. Suppose that $g$ can be expressed as a product of disjoint cycles of lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$ (not necessarily distinct) and $h$ can be expressed as a product of disjoint cycles of lengths $\mu_1, \mu_2, \ldots, \mu_k$. Then the permutation $(g, h)$: $(x, y) \mapsto (x^g, y^h)$ acting on $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ can be expressed as a product of disjoint cycles of lengths $\lambda_j \mu_j$, for $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, \ell$. In this manner we can determine the cycle types of all members of $G \times H$ from those of $G$ and $H$. This observation is used in the proof of Proposition 2.5, to follow.

Now, let $T$ be a distinct partition of the positive integer $n$; that is, $T \in \text{DP}(n)$. Given an element $g$ of $S_n$ with cycle type $T$ we let $N_g$ denote the unique maximal nilpotent subgroup of $S_n$ containing $g$. Then we define $N(T) = \{ h^{-1} N_{g} h : h \in S_n \}$.

Clearly, this definition does not depend on the choice of permutation $g$ of cycle type $T$.

We are now able to state our final result.

**Proposition 2.5.** We have $S_n = \bigcup_{T \in \text{DP}(n)} N(T)$.

**Proof.** Observe that if $g$ is a power of a prime $p$, then the group $P_{p,q}$ contains a $q$-cycle. This implies, in the notation of Proposition 2.4, that the group $N_i$ contains $C_{q_{j_1}} \times C_{q_{j_2}} \times \cdots \times C_{q_{j_s}}$, where each group $C_{q_{j_i}}$ is a cyclic permutation group generated by a $q_{j_i}$-cycle. Hence $N_i$ contains a $t_i$-cycle. It follows, in turn, that the group $N$ contains the product of $k$ disjoint cycles of lengths $t_1, t_2, \ldots, t_k$.

We define $\Omega = \bigcup_{T \in \text{DP}(n)} N(T)$. Choose an an arbitrary element $h$ of $S_n$. We wish to show that $h \in \Omega$, and since $\Omega$ is a union of conjugacy classes of $S_n$, it is enough to show that $\Omega$ contains an element conjugate to $h$. Let $R = \lambda_1^{a_1} \lambda_2^{a_2} \cdots \lambda_k^{a_k}$ be the cycle type of $h$, using the usual partition notation: $h$ is a product of $a_i$ disjoint $\lambda_i$-cycles, for $i = 1, 2, \ldots, k$, and $a_1 \lambda_1 + a_2 \lambda_2 + \cdots + a_k \lambda_k = n$. (This notation will be used only in this paragraph and the next.)

Now we describe a process for amalgamating parts of the partition $R$ “in pairs”: for $i = 1, 2, \ldots, k$, whenever $a_i > 1$ we replace $\lambda_i^{a_i}$ by \[
\begin{cases}
(2\lambda_i)^{a_i/2}, & \text{if } a_i \text{ is even,} \\
(2\lambda_i)^{(a_i-1)/2} \lambda_i, & \text{if } a_i \text{ is odd.}
\end{cases}
\]
We repeat this process until we have a distinct partition $T$. For example, if $R = 2^23^24^16^18^116^1$, a partition of 52, then
\[
2^23^24^16^18^116^1 \rightarrow 4^16^18^14^16^18^1 = 4^26^28^116^1 \rightarrow 8^112^116^116^1 = 8^112^116^2 \rightarrow 8^112^132^1.
\]
We claim that the groups in \( N(T) \) contain elements of cycle type \( R \). To see this, observe that, given a part \( t \) of \( T \), we can work backwards through the algorithm from \( T \) to \( R \) to obtain a list \( R_t \) of parts of \( T \) whose sum is \( t \) (not necessarily unique). Each element of \( R_t \) is a factor of \( t \) with quotient a power of 2. By working through the parts of \( T \) one by one, we can choose the lists \( R_t \), for \( t \in T \), to be a partition of \( R \). For instance, using the example above with \( T = 8^112^132^1 \), we can choose

\[
R_8 = [2, 2, 4], \quad R_{12} = [3, 3, 6] \quad R_{32} = [4, 4, 8, 16].
\]

Let \( t \) be a part of \( T \) and let \( R_t = [r_1, r_2, \ldots, r_l] \) be the corresponding parts of \( R \), listed in ascending order. Then \( \sum t/r_1 = 2^d \), for some positive integer \( d \). Let \( s_i = r_i/r_1 \), for \( i = 1, 2, \ldots, l \), so \( s_1 + s_2 + \cdots + s_l = 2^d \). Each integer \( s_i \) is a power of 2. Hence any element \( g \) of the symmetric group \( S_{2^d} \) with cycle type \([s_1, s_2, \ldots, s_l]\) has order a power of 2. It follows that such an element is contained in a Sylow 2-subgroup \( P_{2^d} \). Consequently, the group \( C_{r_1} \times P_{2^d} \) contains an element of cycle type \( R_t \), where \( C_{r_1} \) is generated by an \( r_1 \)-cycle.

Let \( r_1 = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k} \) be the prime decomposition of \( r_1 \) and, as usual, set \( q_i = p_i^{q_i} \), for \( i = 1, 2, \ldots, k \). We observed at the start of the proof that \( C_{r_1} \) is a subgroup of \( P_{p_1^{a_1}} \times P_{p_2^{a_2}} \times \cdots \times P_{p_k^{a_k}} \), so the group

\[
P_{p_1^{a_1}q_1} \times P_{p_2^{a_2}} \times \cdots \times P_{p_k^{a_k}q_k}
\]

contains an element of cycle type \( R_t \). If \( p_1, p_2, \ldots, p_k \) are all odd, then we call this group \( N_1 \). If this is not the case, then we relabel so that \( p_\ell = 2 \) and we make this group bigger, by defining

\[
N_1 = P_{p_1^{a_1}} \times P_{p_2^{a_2}} \times \cdots \times P_{p_{\ell-1}^{a_{\ell-1}}q_{\ell-1}} \times P_{p_\ell^{a_\ell}q_\ell}
\]

Observe that, again, \( N_1 \) contains an element of cycle type \( R_t \).

If \( T \) has \( e \) parts, then we repeat this process, and obtain groups \( N_1, N_2, \ldots, N_e \). Notice that \( N_1 \times N_2 \times \cdots \times N_e \) embeds in \( S_n \) (naturally and intrinsically), and observe that it lies in \( N(T) \) and contains an element of cycle type \( R \), as required.

We can now prove the results stated in the introduction, beginning with Theorem 1.1.

Proposition 2.4 tells us that if the cycle type of an element \( g \) of \( S_n \) is a distinct partition of \( n \), then \( g \) lies within a unique maximal nilpotent subgroup of \( S_n \). Recall from the introduction that we denote the collection of all such maximal nilpotent subgroups by \( \mathcal{M} \). Thus \( \mathcal{M} = \{ X : X \subseteq N(T) \) and \( T \in \text{DP}(n) \} \). By Proposition 2.5 \( \mathcal{M} \) is a cover of \( S_n \). Furthermore, the uniqueness property of Proposition 2.4 implies that \( \mathcal{M} \) is the unique minimal nilpotent cover of \( S_n \) by maximal nilpotent subgroups. This concludes the proof of Theorem 1.1.

Corollary 1.2 follows immediately from Proposition 2.4 and Theorem 1.1.

It remains only to prove Corollary 1.1. First we establish that \( \Sigma_{\infty}(S_n) = \sigma_{\infty}(S_n) \), observing that we already know that \( \sigma_{\infty}(S_n) \leq \Sigma_{\infty}(S_n) \) (which, as noted in the introduction, is true more generally). For the reverse inequality, given \( N \in \mathcal{M} \) we can find an element \( g_N \) (with cycle type a distinct partition of \( n \)) for which \( N \) is the unique maximal nilpotent subgroup of \( S_n \) containing \( g_N \). The set \( \{g_N : N \in \mathcal{M}\} \) is a non-nilpotent subset of \( S_n \), of size \( |\mathcal{M}| \), so \( \Sigma_{\infty}(S_n) \leq \sigma_{\infty}(S_n) \), as required.

Next, observe that

\[
|\mathcal{M}| = \sum_{T \in \text{DP}(n)} |N(T)|,
\]

where \( N(T) \) is equal to the index of the normalizer \( N_{S_n}(N_g) \) of \( N_g \) in \( S_n \), for any permutation \( g \) of cycle type \( T \). Let \( O_1, O_2, \ldots, O_k \) be the orbits in \([1, 2, \ldots, n]\), of \( O_i \), for \( i = 1, 2, \ldots, k \). Using Proposition 2.4 we can write \( N_g \) as a direct product \( N_1 \times N_2 \times \cdots \times N_k \), where \( N_i \) is a subgroup of \( \text{Sym}(O_i) \), for \( i = 1, 2, \ldots, k \). Now, if \( h \in N_{S_n}(N_g) \), then \( h \) must permute the orbits \( O_i \), and since they are of distinct orders we see that \( h \) fixes each orbit. Consequently, the normalizer \( N_{S_n}(N_g) \) is the direct product of the normalizers of the subgroups \( N_i \) in \( \text{Sym}(O_i) \), for \( i = 1, 2, \ldots, k \).

Let \( N \) be any one of the subgroups \( N_i \) and let \( t = |O_i| \); thus \( t \) is one of the parts of \( T \). We write \( P(t) = [p_1, p_2, \ldots, p_t] \) and \( \text{PP}(t) = [q_1, q_2, \ldots, q_j] \); then Proposition 2.4 tells us that

\[
N = P_{p_1^{a_1}q_1} \times P_{p_2^{a_2}} \times \cdots \times P_{p_t^{a_t}q_t},
\]

for Sylow \( p_j \)-subgroups \( P_{p_j^{a_j}q_j} \) of \( S_{q_j} \), \( j = 1, 2, \ldots, \ell \). Now, if \( h \) belongs to the normalizer of \( N \) (in \( \text{Sym}(O_i) \)), then, for each \( j = 1, 2, \ldots, \ell \), the permutation \( h \) must preserve the unique system of imprimitivity of \( N \) comprising \( t/q_j \) sets of size \( q_j \). Consequently, the normalizer of \( N \) is the direct product of the normalizers \( N_{S_{q_j}}(P_{p_j^{a_j}q_j}) \), for \( j = 1, 2, \ldots, \ell \). The normalizer \( N_{S_{q_j}}(P_{p_j^{a_j}q_j}) \) is known to have order \( (p_j - 1)^{a_j}q_j^{e_j} \), where \( q_j = p_j^{q_j} \) and \( e_j = (p_j^{a_j} - 1)/(p_1 - 1) \). Hence

\[
|N(T)| = \frac{|S_n|}{|N_{S_n}(N_g)|} = \frac{n!}{\prod_{t \in T} \prod_{i=1}^{t} (p_i - 1)^{a_i} p_i^{e_i}},
\]

as required.
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