The blow-up phenomena and exponential decay of solutions for a three-component Camassa–Holm equations

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Abstract

The present paper is mainly concerned with the blow-up phenomena and exponential decay of solution for a three-component Camassa–Holm equation. Comparing with the result of Hu, ect. in the paper [18], a new wave-breaking solution is obtained. The results of exponential decay of solution in our paper cover and extent the corresponding results in [9, 17, 22].

Keywords: A three-component Camassa–Holm equations, blow-up phenomena, the exponential decay, wave-breaking, traveling wave solutions.

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1 Introduction

In this paper, we devote to the study of the Cauchy problem for a three-component Camassa–Holm equation

\[
\begin{aligned}
  m_t - m_x u + 2mu_x + (mv + mw)_x + nv_x + lw_x &= 0, \\
  n_t - n_x v + 2nv_x + (nu + nw)_x + mu_x + lw_x &= 0, \\
  l_t - l_x w + 2lw_x + (lu + lv)_x + mu_x + nw_x &= 0,
\end{aligned}
\]

which was introduced by Qu and Fu in [21] to study multipeakons, where the potential \( m = u - u_{xx}, n = v - v_{xx} \) and \( l = w - w_{xx} \), \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\), and the subscripts denote the partial derivatives. The two peakon solutions of system (1.1) have the following form

\[
\begin{aligned}
  u(x,t) &= p_1(t) \exp(-|x - q_1(t)|) + p_2(t) \exp(-|x - q_2(t)|), \\
  v(x,t) &= r_1(t) \exp(-|x - q_1(t)|) + r_2(t) \exp(-|x - q_2(t)|),
\end{aligned}
\]

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where \( p_i, q_i(t), r_i(t) \) and \( s_i(t), i = 1, 2 \) are functions of \( t \), and the corresponding dynamical system defined in [21].

Let the potential \( v = w = 0 \), system (1.1) becomes the classical Camassa-Holm (CH) equation in form

\[
m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx}.
\]

which comes from an asymptotic approximation to the Hamiltonian for the Green–Naghdi equations in shallow water theory. The CH equation models the unidirectional propagation of shallow water waves over a flat bottom [4], and also is a model for the propagation of axially symmetric waves in hyperelastic rods [12]. It has a bi-Hamiltonian structure [15] and is completely integrable [6], and with a Lax pair based on a linear spectral problem of second order. Also, there exists smooth soliton solutions on a non-zero constant background [5]. Compared with KdV equation, the CH equation not only approximates unidirectional fluid flow in Euler’s equations [16] at the next order beyond the KdV equation [19, 20], but also there exists blow-up phenomena of the strong solution and global existence of strong solution [6, 8, 9, 10]. It is remarkable that the CH equation has peaked solitons of the form

\[
u(t, x) = ce^{-|x-ct|}, \quad c \in \mathbb{R}
\]

which are orbital stable [11], and \( n \)-peakon solutions [1]

\[
\sum_{j=1}^{n} p_j(t) \exp(-|x - q_j(t)|),
\]

where the positions \( q_j \) and amplitudes \( p_j \) satisfy the system of ODEs

\[
\begin{aligned}
q_j &= \sum_{k=1}^{n} p_k \exp(-|q_j - q_k|), \\
p_j &= p_j \sum_{k=1}^{n} p_k \text{sgn}(q_j - q_k) \exp(-|q_j - q_k|),
\end{aligned}
\]

where \( j = 1, \ldots, n \). The CH equation has attracted a lot of interest in the past twenty years for various reasons [2, 3, 7, 8, 11, 14, 23].

If we neglect \( w \) in system (1.1) to obtain 2-component CH equation, which is studied in [13]. They establish the local well-posedness in \( H^s \times H^s, s > \frac{3}{2} \). Also, it has blow-up phenomena, if the initial data satisfy some condition.

Recently, Hu, Lin and Jin investigate the Cauchy problem of the three Camassa-Holm equation (1.1) in [18] on the line. The authors establish the local well-posedness, derive precise blow-up scenario and the conservation law. Moreover, by the conservation law, if the derivative of initial data is negative, they obtain the existence of strong solutions which blows up in
finite time and derive the blow-up rate. In this paper, we give a new blow-up phenomena to system (1.1), as the initial data satisfy
\[ \int_{\mathbb{R}} (u_{0x} + v_{0x} + w_{0x})^3 dx < -9E_0 \sqrt{2E_0}. \]

Next, we study the exponential decay of the solution provided the initial data \( z_0(x) = (u_0, v_0, w_0) \sim \mathcal{O}(e^{-\alpha|x|}), \alpha \in (0, 1) \) as \( x \to \pm \infty \) or the initial potential \( (m_0, n_0, l_0) \sim \mathcal{O}(e^{-(1+\lambda)|x|}), \lambda > 0 \) as \( x \to \pm \infty \). Moreover, we get a class of traveling wave solutions of system (1.1).

The remainder of the paper is organized as follows. In Section 2, we give a new wave-breaking solution of system (1.1). In Section 3, the exponential decay of solution is established, if the initial data satisfy some decay condition. In Section 4, we prove that system (1.1) has a class of traveling wave solution.

Notation: For simplicity, we identify all spaces of functions with function spaces over \( \mathbb{R} \), we drop \( \mathbb{R} \) from our notation. For \( 1 \leq p \leq \infty \), the norm in the Banach space \( L^p(\mathbb{R}) \) will be written by \( \| \cdot \|_{L^p} \), while \( \| \cdot \|_{H^s} \), \( s \in \mathbb{R} \) will stand for the norm in the classical Sobolev spaces \( H^s(\mathbb{R}) \). We shall say for some \( K > 0 \) that
\[ f(x) \sim \mathcal{O}(e^{\alpha x}) \quad \text{as} \ x \uparrow \infty, \quad \text{if} \quad \lim_{x \to \infty} \frac{|f(x)|}{e^{\alpha x}} \leq K, \]
and
\[ f(x) \sim o(e^{\alpha x}) \quad \text{as} \ x \uparrow \infty, \quad \text{if} \quad \lim_{x \to \infty} \frac{|f(x)|}{e^{\alpha x}} = 0. \]

2 The blow-up phenomena of solution

With the potential \( m = u - u_{xx}, \ n = v - v_{xx} \) and \( l = w - w_{xx} \). It is convenient to rewrite the system (1.1) in its formally equivalent differential form
\[
\begin{cases}
  u_t + (u + v + w)u_x + (1 - \partial_x^2)^{-1}(uv_x + uw_x) + \partial_x(1 - \partial_x^2)^{-1}f = 0, \\
  v_t + (u + v + w)v_x + (1 - \partial_x^2)^{-1}(uv_x + vw_x) + \partial_x(1 - \partial_x^2)^{-1}g = 0, \\
  w_t + (u + v + w)w_x + (1 - \partial_x^2)^{-1}(uw_x + vw_x) + \partial_x(1 - \partial_x^2)^{-1}h = 0, \\
  (u, v, w)|_{t=0} = (u_0(x), v_0(x), w_0(x)),
\end{cases}
\]
(2.1)

where the functions \( f, g \) and \( h \) satisfy
\[
\begin{align*}
  f(x, t) &= u^2 + \frac{1}{2}u_x^2 + u_xv_x + u_xw_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 + \frac{1}{2}w^2 - \frac{1}{2}w_x^2, \\
  g(x, t) &= v^2 + \frac{1}{2}v_x^2 + u_xv_x + v_xv_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}w^2 - \frac{1}{2}w_x^2, \\
  h(x, t) &= w^2 + \frac{1}{2}w_x^2 + u_xw_x + v_xw_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}v^2 - \frac{1}{2}v_x^2.
\end{align*}
\]
(2.2)
Lemma 2.1 Assume the initial data \( z_0 = (u_0, v_0, w_0) \in H^s \times H^s \times H^s, s > \frac{3}{2} \). Then there exists a unique strong solution \( z = (u, v, w) \) to Eq. (2.3) and a time \( T = T(z_0) > 0 \), such that
\[
z(t, x) = (u(t, x), v(t, x), w(t, x)) \in C([0, T); H^s \times H^s \times H^s) \cap C^1([0, T); H^{s-1} \times H^{s-1} \times H^{s-1}).
\]
Moreover, the solution \( z(t, x) \) depends continuously on the initial data \( z_0 \), i.e. the mapping \( z_0 \mapsto z(\cdot, z_0) : H^s \times H^s \times H^s \to C([0, T); H^s \times H^s \times H^s) \cap C^1([0, T); H^{s-1} \times H^{s-1} \times H^{s-1}) \) is continuous. Furthermore, the lifespan \( T \) of solution \( z(t, x) \) can be chosen independent of \( s \).

Lemma 2.2 Let \( z_0 = (u_0, v_0, w_0) \in H^s \times H^s \times H^s, s > \frac{3}{2} \), and \( T \) be the lifespan of solution \( z = (u, v, w) \) to Eq. (2.3). Then it follows for all \( t \in [0, T) \) that
\[
E(t) := \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2
= \|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2 + \|w_0\|_{H^1}^2 := E_0,
\]
by the conservation law, we have
\[
\|u(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \|w(t)\|_{L^\infty}^2 \leq \frac{1}{2} E_0.
\]
Moreover, the solution \( z = (u, v, w) \) blows up in finite time \( T \) if and only if
\[
\liminf_{t \uparrow T} \inf_{x \in \mathbb{R}} \{u_x(x, t)\} = -\infty,
\]
or
\[
\liminf_{t \uparrow T} \inf_{x \in \mathbb{R}} \{v_x(x, t)\} = -\infty, \text{ or } \liminf_{t \uparrow T} \inf_{x \in \mathbb{R}} \{w_x(x, t)\} = -\infty.
\]
Next, we prove that there exists solutions to system (1.1) which do not exist globally in time. Comparing with the two results of blow-up phenomena which are obtained in [18], we give another new wave-breaking solution.
Thanks to (2.7), it follows that
\[ x \text{ where the functions } \]
By virtue of
\[ T \]
Assume
\[ \text{Theorem 2.1} \]
Let the initial data
\[ \text{Differentiating Eq.}(2.2) \text{ with respective to } \]
Proof. Differentiating Eq. (2.2) with respective to \( x \) variable, we have
\[ u_{tx} = -(u + v + w)_x u_x - (u + v + w)_x u_{xx} - \partial_x G * (uv_x + uw_x) - \partial_x^2 G * f, \]
\[ v_{tx} = -(u + v + w)_x v_x - (u + v + w)_x v_{xx} - \partial_x G * (uv_x + vw_x) - \partial_x^2 G * g, \]
\[ w_{tx} = -(u + v + w)_x w_x - (u + v + w)_x w_{xx} - \partial_x G * (uw_x + vw_x) - \partial_x^2 G * h, \]
where the functions \( f, g \) and \( h \) satisfy equality (2.2).
Thanks to (2.7), it follows that
\[
\frac{\partial}{\partial t} (u_x + v_x + w_x)^3 = 3(u_x + v_x + w_x)^2(u_{tx} + v_{tx} + w_{tx}) \\
= -3(u_x + v_x + w_x)^4 - 3(u_x + v_x + w_x)^2[(u + v + w)(u + v + w)_x \\
+ \partial_x G * (uv + uw + vw)_x + \partial_x^2 G * (f + g + h)].
\]
Integrating Eq. (2.8) with respect to \( x \) variable on \( \mathbb{R} \) yields that
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}} (u_x + v_x + w_x)^3 dx = -2 \int_{\mathbb{R}} (u_x + v_x + w_x)^4 dx \\
- 3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2[\partial_x^2 G * (uv + uw + vw + f + g + h)] dx.
\]
By virtue of \( \partial_x^2 G * f = G * f - f \) we have
\[
-3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2[\partial_x^2 G * (uv + uw + vw + f + g + h)] dx \\
= -3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2[G * (uv + uw + vw + f + g + h)] dx \\
+ 3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2[uv + uw + vw + f + g + h] dx \] (2.10)
\[
= I + II.
\]
Note that
\[ f + g + h = 2(u^2 + v^2 + w^2) - \frac{1}{2}(u_x^2 + v_x^2 + w_x^2) + 2(u_x v_x + u_x w_x + v_x w_x), \]
by Lemma 2.2 and \( \|G\|_{L^\infty} \leq \frac{1}{2} \), the term \( I \) can be bounded by
\[
I \leq 3\| (u_x + v_x + w_x)^2 \|_1 \|G*H\|_{L^\infty} \\
\leq \frac{9}{2} E_0 \|H\|_{L^1} \\
\leq \frac{27}{2} E_0^2, \tag{2.11}
\]
where \( H = uv + uw + vw + f + g + h. \)

We can deal with the term \( II \) as follows
\[
II = 3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2 [uv + uw + vw + 2(u^2 + v^2 + w^2)] dx \\
+ 3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2 [2(u_x v_x + u_x w_x + v_x w_x) - \frac{1}{2}(u_x^2 + v_x^2 + w_x^2)] dx \\
\leq 9(\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \|w\|_{L^\infty}^2) \int_{\mathbb{R}} (u_x + v_x + w_x)^2 dx \\
+ \frac{9}{2} \int_{\mathbb{R}} (u_x + v_x + w_x)^2 (u_x v_x + u_x w_x + v_x w_x) dx \\
\leq \frac{27}{2} E_0^2 + 3 \int_{\mathbb{R}} (u_x + v_x + w_x)^2 [(u_x v_x + u_x w_x + v_x w_x) + \frac{1}{2}(u_x^2 + v_x^2 + w_x^2)] dx \\
= \frac{27}{2} E_0^2 + \frac{3}{2} \int_{\mathbb{R}} (u_x + v_x + w_x)^4 dx. \tag{2.12}
\]

Inserting (2.11) and (2.12) into (2.10). Then combining (2.9) with (2.10) to yield
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}} (u_x + v_x + w_x)^3 dx \leq -\frac{1}{2} \int_{\mathbb{R}} (u_x + v_x + w_x)^4 dx + 27 E_0^2. \tag{2.13}
\]

By the following Hölder inequality
\[
\left| \int_{\mathbb{R}} (u_x + v_x + w_x)^3 dx \right|^2 \leq \int_{\mathbb{R}} (u_x + v_x + w_x)^4 dx \int_{\mathbb{R}} (u_x + v_x + w_x)^2 dx \\
\leq 3 E_0 \int_{\mathbb{R}} (u_x + v_x + w_x)^4 dx, \tag{2.14}
\]
and define
\[
Q(t) = \int_{\mathbb{R}} (u_x + v_x + w_x)^3 dx.
\]

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The inequality (2.13) is changed into
\[
\frac{\partial}{\partial t} Q(t) \leq -\frac{1}{6E_0} Q^2(t) + 27E_0^2 \\
\leq -\frac{1}{6E_0} (Q(t) - 9E_0\sqrt{2E_0}) (Q(t) + 9E_0\sqrt{2E_0}).
\] (2.15)

In view of the assumption \(Q(0) < -9E_0\sqrt{2E_0}\) and (2.14), then \(\partial_t Q(t) < 0\) and \(Q(t)\) is a decreasing function, hence
\[
Q(t) < -9E_0\sqrt{2E_0}.
\]

By solving the inequality (2.15), one can easily check that
\[
\frac{Q(0) + 9E_0\sqrt{2E_0}}{Q(0) - 9E_0\sqrt{2E_0}} e^{\frac{3\sqrt{2} E_0 t}{2}} - 1 \leq \frac{18E_0\sqrt{2E_0}}{Q(t) - 9E_0\sqrt{2E_0}} \leq 0.
\] (2.16)

Observing that
\[
0 < \frac{Q(0) + 9E_0\sqrt{2E_0}}{Q(0) - 9E_0\sqrt{2E_0}} < 1.
\]

In view of (2.16), we can deduce that the lifespan \(T\) of solution \(z\) satisfies
\[
0 < T \leq \frac{\sqrt{2}}{3\sqrt{E_0}} \log \left( \frac{Q(0) - 9E_0\sqrt{2E_0}}{Q(0) + 9E_0\sqrt{2E_0}} \right),
\] (2.17)
such that \(\lim_{t\uparrow T} Q(t) = -\infty\). On the other hand
\[
\left| \int_R (u_x + v_x + w_x)^3 dx \right| \leq \| (u_x + v_x + w_x) \|_{L^\infty} \| (u_x + v_x + w_x) \|_{L^2}^2 \leq 3E_0 \| (u_x + v_x + w_x) \|_{L^\infty}.
\]

which completes the proof Theorem 2.1.

\[\square\]

3 The exponential decay of solution

In this section, our aim is to establish the exponential decay of solution to system (1.1), before stating precisely our main results, we first give two important lemmas, which will be continuously used in the paper.

Lemma 3.1 (The Gronwall Lemma) Let \(f(t), g(t), h(t)\) be continuous functions on \(\mathbb{R}^+\) such that
\[
\partial_t f \leq g + h f.
\]
Then the following inequality holds
\[
f(t) \leq e^{\int_0^t h(s) ds} f(0) + \int_0^t g(\tau) e^{\int_\tau^t h(s) ds} d\tau.
\] (3.1)
Moreover, if \( f(t), g(t), h(t) \) are positive, and satisfy
\[
f \leq g + \int_0^t h(s) f(s) ds,
\]
then, we have
\[
f(t) \leq g(t) + \int_0^t h(s) g(s) e^{\int_s^t h(\tau) d\tau} ds. \tag{3.2}
\]

Proof. Applying \( e^{-\int_0^t h(s) ds} \) to the inequality \( \partial_t f \leq g + hf \) to immediately derive (3.1).

Let
\[
\mathcal{R}(t) = \int_0^t h(s) f(s) ds.
\]

Then the derivative \( \mathcal{R}' \) satisfies
\[
\mathcal{R}'(s) - h(s) \mathcal{R}(s) = h(s)(f(s) - \mathcal{R}(s)) \leq h(s) g(s).
\]

Consequently
\[
\frac{d}{ds} \left\{ \mathcal{R}(s) e^{\int_s^t h(\tau) d\tau} \right\} \leq h(s) g(s) e^{\int_s^t h(\tau) d\tau}.
\]

Integrating on \([0, t]\) with respect to \( s \) variable gives
\[
\mathcal{R}(t) \leq \int_0^t h(s) g(s) e^{\int_s^t h(\tau) d\tau} ds.
\]

Adding \( g(t) \) on both sides of the above inequality to obtain (3.2). \( \square \)

Lemma 3.2 Assume the function \( G(x) = \frac{1}{2} e^{-|x|} \). Let the weighted function \( J_N(x) \) be
\[
J_N(x) = \begin{cases} 
  e^{\alpha N}, & x \in (-\infty, -N), \\
  e^{-\alpha x}, & x \in [-N, 0] \\
  1, & x \in (0, \infty), 
\end{cases} \tag{3.3}
\]

where \( N \in \mathbb{Z}^+ \). If the constant \( \alpha \in (0, 1) \), then there exists some constant \( C_0 \), such that
\[
J_N(x)(G * (J_N)^{-1})(x) \leq C_0 \quad \text{and} \quad J_N(x)(\partial_x G * (J_N)^{-1})(x) \leq C_0. \tag{3.4}
\]

Proof. At first, we prove the first inequality. Note that
\[
f(x) := 2J_N(x)(G * (J_N)^{-1})(x) = J_N(x) \int_{\mathbb{R}} \frac{e^{-|x-y|}}{J_N(y)} dy
\]
\[
= J_N(x) \int_x^\infty \frac{e^{x-y}}{J_N(y)} dy + J_N(x) \int_{-\infty}^x \frac{e^{y-x}}{J_N(y)} dy. \tag{3.5}
\]
Case 1: As \( x < -N \), then we have

\[
f(x) = e^{\alpha N} \int_{(x,-N) \cup [-N,0] \cup (0,\infty)} \frac{e^{x-y}}{J_N(y)} dy + e^{\alpha N} \int_{-\infty}^{x} e^{y-x} \frac{dy}{J_N(y)}
\]

\[
= e^{\alpha N} \left( \int_{x}^{-N} e^{x-y-\alpha N} dy + \int_{-N}^{0} e^{x-y+\alpha x} dy + \int_{0}^{\infty} e^{y-x} dy \right)
\]

\[
+ e^{\alpha N} \int_{-\infty}^{x} e^{y-x-\alpha N} dy
\]

\[
= 1 - e^{x+N} + e^{(\alpha+1)(x+N)} - e^{\alpha(x+N)+x} + e^{\alpha N+x} \leq 3.
\] (3.6)

Case 2: If \( x \in [-N,0] \), we can derive

\[
f(x) = e^{-\alpha x} \int_{[x,0] \cup (0,\infty)} \frac{e^{x-y}}{J_N(y)} dy + e^{-\alpha x} \int_{(-\infty,-N) \cup (-N,x)} \frac{e^{y-x}}{J_N(y)} dy
\]

\[
= e^{-\alpha x} \left( \int_{x}^{0} e^{x+y-(\alpha-1)y} dy + \int_{0}^{\infty} e^{y-x} dy \right)
\]

\[
+ e^{-\alpha x} \left( \int_{-\infty}^{-N} e^{y-(x+\alpha N)} dy + \int_{-N}^{x} e^{(\alpha+1)y-x} dy \right)
\]

\[
= \frac{1}{1-\alpha} + e^{(1-\alpha)x} + e^{-(\alpha+1)(x+N)} + \frac{1}{\alpha+1} \left( 1 - e^{-(\alpha+1)(x+N)} \right)
\]

\[
\leq 3 - \frac{\alpha^2}{1-\alpha^2}.
\] (3.7)

Case 3: As \( x \in (0,\infty) \), we can deal with it as follows

\[
f(x) = \int_{x}^{\infty} e^{x-y} dy + \int_{(-\infty,-N) \cup (-N,0) \cup (0,x)} e^{x-y} \frac{dy}{J_N(y)}
\]

\[
= 1 + e^{-(x+\alpha N+N)} + e^{(\alpha-1)x} - e^{(\alpha-1)x-N} + 1 - e^{-x}
\]

\[
\leq 3 + e^{-(\alpha+1)N} - e^{-N}.
\] (3.8)

Let \( 2C_0 \geq \frac{3-\alpha^2}{1-\alpha^2} \). Combining (3.5), (3.6), (3.7) with (3.8) to yield the first inequality. Similarly, the second inequality can be proved. \( \square \)

**Remark 3.1** If we define weighted function for \( \alpha \in (0,1) \),

\[
\varphi_N(x) = \begin{cases} 
1, & x \in (-\infty,0), \\
\alpha x, & x \in [0,N], \\
\alpha N, & x \in (N,\infty),
\end{cases}
\] (3.9)

where \( N \in \mathbb{Z}^+ \). Then there exists some constant \( C_0 \), for all \( N \), it follows that

\[
\begin{cases} 
\varphi_N(x)(G * (\varphi_N)^{-1})(x) \leq C_0, \\
\varphi_N(x)(\partial_x G * (\varphi_N)^{-1})(x) \leq C_0,
\end{cases}
\] (3.10)
Next, as [17, 22], we shall establish the exponential decay of the strong solutions to Eq.(2.3), if the initial data $z_0(x)$ decay at infinity.

**Theorem 3.1** Let the initial data $z_0 = (u_0, v_0, w_0) \in H^s \times H^s \times H^s, s > \frac{3}{2}$ and $T > 0$. Suppose $z = (u, v, w) \in C([0, T]; H^s \times H^s \times H^s)$ is the corresponding solution to Eq.(2.3) with the initial data $z_0$. If there exists some $\alpha \in (0, 1)$ such that

$$
\begin{align*}
|u_0(x)|, |v_0(x)|, |w_0(x)| &\sim O(e^{\alpha x}) \quad \text{as} \quad x \downarrow -\infty, \\
|u_0x(x)|, |v_0x(x)|, |w_0x(x)| &\sim O(e^{\alpha x}) \quad \text{as} \quad x \downarrow -\infty,
\end{align*}
$$

then, it follows that the solutions $z(t, x)$ satisfy

$$
\begin{align*}
|u(t, x)|, |v(t, x)|, |w(t, x)| &\sim O(e^{\alpha x}) \quad \text{as} \quad x \downarrow -\infty, \\
|u_x(t, x)|, |v_x(t, x)|, |w_x(t, x)| &\sim O(e^{\alpha x}) \quad \text{as} \quad x \downarrow -\infty,
\end{align*}
$$

uniformly in the interval $[0, T]$.

**Proof.** For simplicity, let $M = \sup_{t \in [0, T]} \{\|u(t)\|_{H^s}\}$, using the Sobolev embedding theorem, $\|u(t)\|_{L^\infty}, \|u_x(t)\|_{L^\infty} \leq M$.

Define a weighted function

$$J_N(x) = \begin{cases} 
E^{-N}, & x \in (-\infty, -N), \\
E^{-\alpha x}, & x \in [-N, 0], \\
1, & x \in (0, \infty),
\end{cases} \quad (3.11)$$

where $N \in Z^+$. One can easily check that for all $N$

$$0 \leq -J_N(x) \leq J_N(x) \quad \text{a.e.} \quad x \in \mathbb{R}. \quad (3.12)$$

Applying Eq.(2.3) by $J_N$ to deduce

$$
\begin{align*}
(uJ_N)_t + (u + v + w)J_N u_x + J_N [G * (uw_x + uw_x)] + J_N [\partial_x G * f] &= 0, \quad (3.13) \\
(vJ_N)_t + (u + v + w)J_N v_x + J_N [G * (uv_x + uv_x)] + J_N [\partial_x G * g] &= 0, \quad (3.14)
\end{align*}
$$

and

$$
\begin{align*}
(wJ_N)_t + (u + v + w)J_N w_x + J_N [G * (uw_x + uw_x)] + J_N [\partial_x G * h] &= 0. \quad (3.15)
\end{align*}
$$

Taking the scalar product of $(uJ_N)^{2p-1}$ and Eq.(3.13), integration by parts is given by the following equality

$$
\int_{\mathbb{R}} (uJ_N)_t (uJ_N)^{2p-1} dx = - \int_{\mathbb{R}} J_N (u + v + w)u_x (uJ_N)^{2p-1} dx
$$

$$
- \int_{\mathbb{R}} (J_N [G * (uw_x + uw_x)] + J_N [\partial_x G * f]) (uJ_N)^{2p-1} dx.
$$

(3.16)
In view of (3.16) and the above relations, we have

\[ \int_{\mathbb{R}} (uJ_N)(uJ_N)^{2p-1} dx = \| uJ_N \|_{L^{2p}}^2 \frac{d}{dt} \| uJ_N \|_{L^{2p}}, \]

\[ \int_{\mathbb{R}} J_N(u + v + w)u_x(uJ_N)^{2p-1} dx \leq \| u_x \|_{L^\infty} \| J_N(u + v + w) \|_{L^{2p}} \| uJ_N \|_{L^{2p}}^{2p-1} \]

\[ \leq M \| (uJ_N, vJ_N, wJ_N) \|_{L^{2p}} \| uJ_N \|_{L^{2p}}^{2p-1}, \]

and

\[ \int_{\mathbb{R}} (J_N[G * (uv_x + uw_x)] + J_N[\partial_x G * f])(uJ_N)^{2p-1} dx \]

\[ \leq \| uJ_N \|_{L^{2p}}^{2p-1}(\| J_N[G * (uv_x + uw_x)] \|_{L^{2p}} + \| J_N[\partial_x G * f] \|_{L^{2p}}). \]

In view of (3.16) and the above relations, we have

\[ \frac{d}{dt} \| uJ_N \|_{L^{2p}} \leq M \| (uJ_N, vJ_N, wJ_N) \|_{L^{2p}} + \| J_N[G*(uv_x+uw_x)] \|_{L^{2p}} + \| J_N[\partial_x G * f] \|_{L^{2p}}, \]

(3.17)

where \( \| (uJ_N, vJ_N, wJ_N) \|_{L^{2p}} = \| uJ_N \|_{L^{2p}} + \| vJ_N \|_{L^{2p}} + \| wJ_N \|_{L^{2p}}. \) Since the function \( \varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) implies

\[ \lim_{n \uparrow \infty} \| \varphi \|_{L^n} = \| f \|_{L^\infty}. \]

Let \( p \uparrow \infty \) in (3.17), it follows that

\[ \frac{d}{dt} \| uJ_N \|_{L^\infty} \leq M \| (uJ_N, vJ_N, wJ_N) \|_{L^\infty} + \| J_N[G*(uv_x+uw_x)] \|_{L^\infty} + \| J_N[\partial_x G * f] \|_{L^\infty}. \]

(3.18)

Similar to the process of (3.18), multiplying Eq.(3.14), Eq.(3.15) by \((vJ_N)^{2p-1}\) and \((wJ_N)^{2p-1}\), respectively, integrating the result on \( \mathbb{R} \) with respect to \( x \)-variable, we end up with

\[ \frac{d}{dt} \| vJ_N \|_{L^\infty} \leq M \| (uJ_N, vJ_N, wJ_N) \|_{L^\infty} + \| J_N[G*(vu_x+vw_x)] \|_{L^\infty} + \| J_N[\partial_x G * g] \|_{L^\infty}, \]

(3.19)

and

\[ \frac{d}{dt} \| wJ_N \|_{L^\infty} \leq M \| (uJ_N, vJ_N, wJ_N) \|_{L^\infty} + \| J_N[G*(wu_x+vw_x)] \|_{L^\infty} + \| J_N[\partial_x G * h] \|_{L^\infty}, \]

(3.20)

where the functions \( f, \ g \) and \( h \) satisfy equality (2.2).

By virtue of Lemma 3.2, there exists a constant \( C_0 \) such that

\[ |J_N[G * (uv_x + uw_x)]| = \left| J_N(x) \int_{\mathbb{R}} \frac{e^{-|x-y|}}{2J_N(y)} J_N(y)(uv_x + uw_x)(y)dy \right| \]

\[ \leq C_0 |uv_x + uw_x|_{L^\infty} \| uJ_N \|_{L^\infty} \]

\[ \leq 2C_0 M \| uJ_N \|_{L^\infty}. \]

(3.21)
and

\[ |J_N[\partial_x G * f]| = \left| J_N(x) \int_{\mathbb{R}} \frac{e^{-|x-y|}}{2J_N(y)} J_N(y)f(y)dy \right| \leq C\|(uJ_N, vJ_N, wJ_N, u_x J_N, v_x J_N, w_x J_N)\|_{L^\infty}. \]  

(3.22)

Similarly, we have

\[ \|J_N[G * (vx_x + vw_x)]\|_{L^\infty} + \|J_N[\partial_x G * g]\|_{L^\infty} \leq C(\|vJ_N\|_{L^\infty} + \|(uJ_N, vJ_N, wJ_N, u_x J_N, v_x J_N, w_x J_N)\|_{L^\infty}), \]  

(3.23)

\[ \|J_N[G * (wu_x + vw_x)]\|_{L^\infty} + \|J_N[\partial_x G * h]\|_{L^\infty} \leq C(\|wJ_N\|_{L^\infty} + \|(uJ_N, vJ_N, wJ_N, u_x J_N, v_x J_N, w_x J_N)\|_{L^\infty}). \]  

(3.24)

Add up (3.18), (3.19) with (3.20), plugging (3.21), (3.22), (3.23) and (3.24) into the inequality, by the Gronwall lemma to yield

\[ \|(uJ_N, vJ_N, wJ_N)\|_{L^\infty} \leq e^{Ct}\|(u_0J_N, v_0J_N, w_0J_N)\|_{L^\infty} + \int_0^t \|(uJ_N, vJ_N, wJ_N, u_x J_N, v_x J_N, w_x J_N)\|_{L^\infty}(\tau)d\tau. \]  

(3.25)

Differentiating Eq.(2.3) with respect to x variable, after multiplying the result by \( J_N \) it follows that

\[ (u_x J_N)_t + [(u+v+w)x_x + (u+v+w)u_{xx}]J_N + [\partial_x G*(uv_x + uw_x) + \partial_x^2 G*f]J_N = 0. \]  

(3.26)

\[ (v_x J_N)_t + [(u+v+w)v_x + (u+v+w)v_{xx}]J_N + [\partial_x G*(vu_x + vw_x) + \partial_x^2 G*h]J_N = 0. \]  

(3.27)

\[ (w_x J_N)_t + [(u+v+w)w_x + (u+v+w)w_{xx}]J_N + [\partial_x G*(wu_x + wv_x) + \partial_x^2 G*h]J_N = 0. \]  

(3.28)

Multiplying Eq.(3.26) by \( (uJ_N)^{2p-1} \) with \( p \in \mathbb{Z}^+ \) and integrating the result on \( \mathbb{R} \) with respect to \( x \)-variable, applying Holder’s inequality, we have

\[ \|u_x J_N\|_{L^{2p}}^{2p-1} \frac{d}{dt}\|u_x J_N\|_{L^{2p}} \leq M\|(u_x J_N, v_x J_N, w_x J_N)\|_{L^{2p}}\|u_x J_N\|_{L^{2p}}^{2p-1} \]  

\[ + \|\partial_x G*(uv_x + uw_x) + \partial_x^2 G*f\|_{L^{2p}}\|u_x J_N\|_{L^{2p}}^{2p-1} \]  

\[ - \int_{\mathbb{R}} J_N(u+v+w)u_{xx}(u_x J_N)^{2p-1}dx. \]  

(3.29)

Observing that

\[ \int_{\mathbb{R}} J_N(u+v+w)u_{xx}(u_x J_N)^{2p-1}dx = \int_{\mathbb{R}} (u+v+w)(u_x J_N)^{2p-1}[(u_x J_N)_x - u_x J_N]dx \]  

\[ = -\frac{1}{2p} \int_{\mathbb{R}} (u+v+w)(u_x J_N)^{2p}dx - \int_{\mathbb{R}} (u+v+w)u_x J_N(u_x J_N)^{2p-1}dx \]  

\[ \leq C\|u_x J_N\|_{L^{2p}}^{2p}, \]  

(3.30)
where we have applied \( |J'_N| \leq J_N \). Substituting (3.30) into (3.29), letting \( p \uparrow \infty \) to obtain

\[
\frac{d}{dt} \| u_x J_N \|_{L^\infty} \leq M \| (u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty} + \| [\partial_x G * (v w_x + u w_x) + \partial^2_x G * f] J_N \|_{L^\infty}.
\]  

(3.31)

Multiplying Eq.(3.27) and Eq.(3.28) by \((v_x J_N)^{2p-1}\) and \((w_x J_N)^{2p-1}\), respectively, integrating the result on \( \mathbb{R} \) with respect to \( x \)-variable, it follows that

\[
\frac{d}{dt} \| v_x J_N \|_{L^\infty} \leq M \| (u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty} + \| [\partial_x G * (v u_x + v w_x) + \partial^2_x G * g] J_N \|_{L^\infty},
\]

and

\[
\frac{d}{dt} \| w_x J_N \|_{L^\infty} \leq M \| (u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty} + \| [\partial_x G * (w u_x + w v_x) + \partial^2_x G * h] J_N \|_{L^\infty}.
\]

(3.32)

(3.33)

In view of Lemma 3.2, we can derive

\[
\| [\partial_x G * (v u_x + v w_x)] J_N \|_{L^\infty} \leq C \| u J_N \|_{L^\infty}.
\]

(3.34)

Thanks to \( \partial^2_x G * f = G * f - f \), by Lemma 3.2 again to yield

\[
\| [\partial^2_x G * f] J_N \|_{L^\infty} \leq \| f J_N \|_{L^\infty} + \| [G * f] J_N \|_{L^\infty} + C \| (u J_N, v J_N, w J_N, u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty}.
\]

(3.35)

Consequently,

\[
\| [\partial_x G * (v u_x + v w_x) + \partial^2_x G * g] J_N \|_{L^\infty} \leq C \| (u J_N, v J_N, w J_N, u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty},
\]

(3.36)

\[
\| [\partial_x G * (w u_x + w v_x) + \partial^2_x G * h] J_N \|_{L^\infty} \leq C \| (u J_N, v J_N, w J_N, u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty}.
\]

(3.37)

Add up (3.31), (3.32) with (3.33), plugging (3.34), (3.35), (3.36) and (3.37) into the inequality. Then by virtue of Gronwall’s inequality implies

\[
\| (u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty} \leq e^{C t} \| (u_{0x} J_N, v_{0x} J_N, w_{0x} J_N) \|_{L^\infty} + C \int_0^t \| (u J_N, v J_N, w J_N, u_x J_N, v_x J_N, w_x J_N) \|_{L^\infty} (\tau) d\tau.
\]

(3.38)

where \( C \) is constant depending only on \( C_0, M \).

Let

\[
Z(t) = (\| (J_N u(t), J_N v(t), J_N w(t)) \|_{L^\infty} + \| (J_N u_x(t), J_N v_x(t), J_N w_x(t)) \|_{L^\infty}).
\]
Applying Lemma 3.1 to (3.38), for all \( t \in [0, T] \), there exists a constant \( \tilde{C} = \tilde{C}(C_0, M, T) \) such that
\[
Z(t) \leq \tilde{C}Z(0) \leq \tilde{C}(\| (u_0, v_0, w_0) \max(1, e^{-\alpha x}) \|_{L^\infty} + \| (u_{0x}, v_{0x}, w_{0x}) \max(1, e^{-\alpha x}) \|_{L^\infty}).
\] (3.39)

Letting \( N \uparrow \infty \), from (3.19), for all \( t \in [0, T] \), it follows for \( x \leq 0 \) that
\[
(\| (u, v, w)e^{-\alpha x} \|_{L^\infty} + \| (u_x, v_x, w_x)e^{-\alpha x} \|_{L^\infty}) \leq \tilde{C}(\| (u_0, v_0, w_0)e^{-\alpha x} \|_{L^\infty} + \| (u_{0x}, v_{0x}, w_{0x})e^{-\alpha x} \|_{L^\infty}),
\]
which obtains the desired result of Theorem 3.1.

If we choose the weighted function \( \varphi_N(x) \) for \( \alpha \in (0, 1) \) as
\[
\varphi_N(x) = \begin{cases} 1, & x \in (-\infty, 0), \\ e^{\alpha x}, & x \in [0, N], \\ e^{\alpha N}, & x \in (N, \infty), \end{cases}
\] (3.40)
where \( N \in \mathbb{Z}^+ \), then by virtue of Remark 3.1, by the method of proof of Theorem 3.1, we have the following result.

**Corollary 3.1** Assume \( z_0 = (u_0, v_0, w_0) \in H^s \times H^s \times H^s, s > \frac{3}{2} \) and \( T > 0 \). Suppose \( z(t, x) = (u, v, w) \in C([0, T]; H^s \times H^s \times H^s) \) is the corresponding solution to Eq.(2.3) with the initial data \( z_0 \). If there exists some \( \alpha \in (0, 1) \) such that
\[
\begin{align*}
|u_0(x)|, |v_0(x)|, |w_0(x)| & \sim \mathcal{O}(e^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty, \\
|u_{0x}(x)|, |v_{0x}(x)|, |w_{0x}(x)| & \sim \mathcal{O}(e^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty,
\end{align*}
\]
then the solutions \( z \) satisfy
\[
\begin{align*}
|u(t, x)|, |v(t, x)|, |w(t, x)| & \sim \mathcal{O}(e^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty, \\
|u_x(t, x)|, |v_x(t, x)|, |w_x(t, x)| & \sim \mathcal{O}(e^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty,
\end{align*}
\]
uniformly in the interval \([0, T]\).

**Remark 3.2** In fact, let \( \alpha \in (0, 1) \) and \( j = 0, 1, 2, \cdots \), if the initial data \( z_0 \) satisfy
\[
(\partial_x^j u_0, \partial_x^j v_0, \partial_x^j w_0) \sim \mathcal{O}(e^{-\alpha |x|}) \quad \text{as} \quad |x| \rightarrow \infty,
\]
then the solutions \( z \) to Eq.(2.3) satisfy
\[
(\partial_x^j u, \partial_x^j v, \partial_x^j w) \sim \mathcal{O}(e^{-\alpha |x|}) \quad \text{as} \quad |x| \rightarrow \infty.
\]

Theorem 3.1 and Corollary 3.1 tell us that the solution \( z \) can only decay as \( e^{\alpha x} \) as \( x \rightarrow -\infty \) and \( e^{-\alpha x} \) as \( x \rightarrow \infty \) for \( \alpha \in (0, 1) \). Whether the decay is optimal? the next result tell us some information.
Theorem 3.2 Given \( z_0 = (u_0, v_0, w_0) \in H^s \times H^s \times H^s, s \geq 3 \). Let \( T = T(z_0) \) be the maximal existence time of the solutions \( z(t, x) = (u, v, w) \) to system (1.1) with the initial data \( z_0 \). If for some \( \lambda \geq 0 \) and \( p \geq 1 \),

\[
\| (m_0, n_0, l_0) e^{(1+\lambda)|x|} \|_{L^{2p}} \leq C,
\]  

(3.41)

then we have for all \( t \in [0, T) \) that

\[
\| (m, n, l) e^{(1+\lambda)|x|} \|_{L^{2p}} \leq C.
\]  

(3.42)

Moreover, if the initial data satisfy

\[
\partial_x^j u_0, \partial_x^j v_0, \partial_x^j w_0 \sim \mathcal{O}(e^{-(1+\lambda)|x|}) \quad \text{as} \quad |x| \to \infty, \quad j = 0, 1, 2,
\]  

(3.43)

then for any \( t \in [0, T) \), it follows that

\[
(m, n, l) \sim \mathcal{O}(e^{-1+\lambda)|x|}) \quad \text{as} \quad |x| \to \infty
\]  

and there exists some \( \alpha \in (0, 1) \) such that

\[
\lim_{x \to +\infty} \| (\partial_x^j u, \partial_x^j v, \partial_x^j w) e^{\alpha x} \| \leq C, \quad \lim_{x \to -\infty} \| (\partial_x^j u, \partial_x^j v, \partial_x^j w) e^{-\alpha x} \| \leq C.
\]

Proof. Multiplying system (1.1) by \( e^{(1+\lambda)|x|} \), after taking inner product with \( (me^{(1+\lambda)|x|})^{2p-1} \) we have

\[
\| me^{(1+\lambda)|x|} \|_{L^{2p}}^{2p-1} \frac{\partial}{\partial t} \| me^{(1+\lambda)|x|} \|_{L^{2p}} + \int_\mathbb{R} (m_x u + 2mu_x)e^{(1+\lambda)|x|} (me^{(1+\lambda)|x|})^{2p-1} dx
\]

\[
+ \int_\mathbb{R} e^{(1+\lambda)|x|} (mv + mw)_x (me^{(1+\lambda)|x|})^{2p-1} dx
\]

\[
\leq M \| (m, n, l)e^{(1+\lambda)|x|} \|_{L^{2p}} \| me^{(1+\lambda)|x|} \|_{L^{2p}}^{2p-1}.
\]

(3.44)

Due to

\[
\int_\mathbb{R} (m_x u + 2mu_x)e^{(1+\lambda)|x|} (me^{(1+\lambda)|x|})^{2p-1} dx = -\frac{2p-1}{2p} \int_\mathbb{R} u \partial_x (me^{(1+\lambda)|x|})^{2p} dx
\]

\[
- \int_\mathbb{R} [-u_x + (1+\lambda) \text{sgn} |x|] (me^{(1+\lambda)|x|})^{2p} dx
\]

\[
\leq \left( \| u_x \|_{L^\infty} + (1+\lambda) \| u \|_{L^\infty} \right) \| me^{(1+\lambda)|x|} \|_{L^{2p}}^{2p}
\]

\[
\leq C \| me^{(1+\lambda)|x|} \|_{L^{2p}},
\]

(3.45)

and

\[
\int_\mathbb{R} e^{(1+\lambda)|x|} (mv + mw)_x (me^{(1+\lambda)|x|})^{2p-1} dx \leq C \| me^{(1+\lambda)|x|} \|_{L^{2p}},
\]

(3.46)
Combining (3.44), (3.45) with (3.46) to imply
\[
\frac{\partial}{\partial t} \| me^{(1+\lambda)|x|} \|_{L^2 p} \leq C \|(m, n, l)e^{(1+\lambda)|x|}\|_{L^2 p}.
\]  (3.47)

As the process of the estimation to (3.47), we deal with system (1.1)\_2 and system (1.1)\_3 is given by
\[
\frac{\partial}{\partial t} \| ne^{(1+\lambda)|x|} \|_{L^2 p} \leq C \|(m, n, l)e^{(1+\lambda)|x|}\|_{L^2 p}.
\]  (3.48)
\[
\frac{\partial}{\partial t} \| le^{(1+\lambda)|x|} \|_{L^2 p} \leq C \|(m, n, l)e^{(1+\lambda)|x|}\|_{L^2 p}.
\]  (3.49)

Add up (3.47), (3.48) with (3.49), then by the Gronwall inequality yields that
\[
\|(m, n, l)e^{(1+\lambda)|x|}\|_{L^2 p} \leq C \|(m_0, n_0, l_0)e^{(1+\lambda)|x|}\|_{L^2 p}.
\]  (3.50)

By virtue of the assumption (3.41), it follows that (3.42). In view of the assumption (3.43) to obtain
\[
(m_0(x), n_0(x), l_0(x)) \sim \mathcal{O}(e^{-(1+\lambda)|x|}) \quad \text{as} \quad |x| \uparrow \infty.
\]  (3.51)

Let \( p \uparrow \infty \) in (3.50). Combining (3.50) with (3.51), let \( |x| \) large enough, we have
\[
(m, n, l)(t, x) \sim \mathcal{O}(e^{-(1+\lambda)|x|}) \quad \text{as} \quad |x| \uparrow \infty.
\]

On the other hand, by virtue of (3.43), Theorem 3.1 and Corollary 3.1, we deduce for any \( \alpha \in (0, 1) \) that
\[
(\partial^j_x u, \partial^j_x v, \partial^j_x w) \sim \mathcal{O}(e^{-\alpha|x|}) \quad \text{as} \quad |x| \uparrow \infty, \quad j = 0, 1, 2.
\]  (3.52)

This means that for all \( t \in [0, T) \) and \( j = 0, 1, 2 \).
\[
\lim_{x \to +\infty} (\partial^j_x u, \partial^j_x v, \partial^j_x w)e^{\alpha x} \leq C, \quad \lim_{x \to -\infty} (\partial^j_x u, \partial^j_x v, \partial^j_x w)e^{-\alpha x} \leq C.
\]

This completes the proof of Theorem 3.2. \( \square \)

**Remark 3.3** As long as the solution \( z(t, x) \) exists, the result of Theorem 3.2 tells us that the solutions \( (z, z_x) \) decay as \( e^{\alpha x} \) as \( x \to -\infty \) and \( e^{-\alpha x} \) as \( x \to \infty \) for \( \alpha \in (0, 1) \). However, the potential \( (m, n, l) \) can decay as \( e^{-(1+\lambda)|x|} \) as \( |x| \to \infty \) for \( \lambda \in (0, \infty) \).
4 Traveling wave solutions

In the subsection, we will establish a family of traveling wave solutions to system (1.1).

At first, we gives two important definitions and an useful lemma.

Definition 4.1 The solution \( z(t, x) = (u, v, w) \) to system (1.1) is \( x \)-symmetric if there exists a function \( b(t) \in C^1(\mathbb{R}^+) \) such that

\[
z(t, x) = z(t, 2b(t) - x), \quad \forall t \in [0, \infty),
\]

for almost every \( x \in \mathbb{R} \), then the function \( b(t) \) is called the symmetric axis of \( z(t, x) \).

Definition 4.2 Let \( \mathcal{N}(\mathbb{R}) = \{ z : z = (u, v, w) \in C(\mathbb{R}^+, H^1 \times H^1 \times H^1) \} \). If for all \( \phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}) \) and \( z(t, x) \in \mathcal{N}(\mathbb{R}) \) satisfy

\[
\begin{align*}
\langle u, (1 - \partial_x^2)\phi_t \rangle + \langle u_x(v + w), \phi_{xx} \rangle - \frac{1}{2} \langle u^2, \phi_{xxx} \rangle + \\
\left\langle \frac{3}{2}u^2 + \frac{1}{2}u_x^2 + u(v + w) + u_x(v + w)_x + \frac{1}{2}(v^2 + w^2 - v_x^2 - w_x^2), \phi_x \right\rangle &= 0, \tag{4.1}
\end{align*}
\]

\[
\begin{align*}
\langle v, (1 - \partial_x^2)\phi_t \rangle + \langle v_x(u + w), \phi_{xx} \rangle - \frac{1}{2} \langle v^2, \phi_{xxx} \rangle + \\
\left\langle \frac{3}{2}v^2 + \frac{1}{2}v_x^2 + v(u + w) + v_x(u + w)_x + \frac{1}{2}(u^2 + w^2 - u_x^2 - w_x^2), \phi_x \right\rangle &= 0, \tag{4.2}
\end{align*}
\]

\[
\begin{align*}
\langle w, (1 - \partial_x^2)\phi_t \rangle + \langle w_x(u + v), \phi_{xx} \rangle - \frac{1}{2} \langle w^2, \phi_{xxx} \rangle + \\
\left\langle \frac{3}{2}w^2 + \frac{1}{2}w_x^2 + w(u + v) + w_x(u + v)_x + \frac{1}{2}(u^2 + v^2 - u_x^2 - v_x^2), \phi_x \right\rangle &= 0, \tag{4.3}
\end{align*}
\]

then \( z(t, x) \) is a weak solution to system (1.1), where \( \langle \cdot, \cdot \rangle \) denotes the distributions on \((t, x)\).

Lemma 4.1 Assume that \( Z(x) = (U, V, W) \in \mathcal{N}(\mathbb{R}) \) and satisfies

\[
\begin{align*}
\langle -cU, (1 - \partial_x^2)\phi_x \rangle + \langle U_x(V + W), \phi_{xx} \rangle - \frac{1}{2} \langle U^2, \phi_{xxx} \rangle + \\
\left\langle \frac{3}{2}U^2 + \frac{1}{2}U_x^2 + U(V + W) + U_x(V + W)_x + \frac{1}{2}(V^2 + W^2 - V_x^2 - W_x^2), \phi_x \right\rangle &= 0, \tag{4.4}
\end{align*}
\]
Proof. Since \((−c, \phi) + \langle V_x(U + W), \phi_{xx} \rangle - \frac{1}{2} \langle V^2, \phi_{xxx} \rangle + \langle 3^2 V^2 + \frac{1}{2} V_x^2 + V(U + W) + V_x(U + W)_x + \frac{1}{2}(U^2 + W^2 - U_x^2 - W_x^2), \phi_x \rangle = 0, \) (4.5)

\(<-c, \phi) + \langle W_x(U + V), \phi_{xx} \rangle - \frac{1}{2} \langle W^2, \phi_{xxx} \rangle + \langle 3^2 W^2 + \frac{1}{2} W_x^2 + W(U + V) + W_x(U + V)_x + \frac{1}{2}(U^2 + V^2 - U_x^2 - V_x^2), \phi_x \rangle = 0, \) (4.6)

for all \(\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R})\). Then the function \(z\) is given by

\[ z(t, x) = Z(x - c(t - t_0)) \] (4.7)

is a weak solution of system (1.1), for any fixed \(t_0 \in \mathbb{R}^+\).

Proof. Since \(C_0^\infty(\mathbb{R}^+ \times \mathbb{R})\) is dense in \(C_0^1(\mathbb{R}^+ \times \mathbb{R})\), by the density argument, we only consider the test functions belongs to \(C_0^1(\mathbb{R}^+ \times \mathbb{R})\). Without loss of generality, let \(t_0 = 0\). Choosing \(\psi \in C_0^1(\mathbb{R}^+ \times \mathbb{R})\), let \(\psi_c = \psi(t, x + ct)\), we can derive

\[ \partial_x(\partial_x) = (\psi_x) c, \quad \text{and} \quad \partial_t(\psi) = (\psi_t) + c(\psi_x) c, \] (4.8)

Assume \(z(t, x) = Z(x - c(t - t_0))\). It is easy to check

\[ \left\{ \begin{array}{l}
\langle u, \psi \rangle = \langle U, \psi \rangle, \quad \langle u^2, \psi \rangle = \langle U^2, \psi \rangle, \\
\langle u_x v, \psi \rangle = \langle U_x V, \psi \rangle, \quad \langle u_x^2, \psi \rangle = \langle U_x^2, \psi \rangle,
\end{array} \right. \] (4.9)

where \(Z = Z(x)\). In view of (4.8) and (4.9), it follows that

\[ \langle u, (1 - \partial_x^2)\psi \rangle + \langle u_x(v + w), \psi_{xx} \rangle - \frac{1}{2} \langle u_x, \psi_{xxx} \rangle \\
= \langle U, (1 - \partial_x^2)(\psi_t) c \rangle + \langle U_x(V + W), (\psi_{xx}) c \rangle - \frac{1}{2} \langle U^2, (\psi_{xxx}) c \rangle \\
= \langle U, (1 - \partial_x^2)(\partial_x \psi - c \partial_x \psi_c) \rangle + \langle U_x(V + W), \partial_x^2 \psi_c \rangle - \frac{1}{2} \langle U^2, \partial_x^2 \psi_c \rangle, \] (4.10)

and

\[ \frac{1}{2} (3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)_x + (v^2 + w^2 - v_x^2 - w_x^2), \psi_x) \\
= \frac{1}{2} \langle 3U^2 + U_x^2 + (V^2 + W^2 - V_x^2 - W_x^2), \partial_x \psi_c \rangle \\
\quad + \langle U(V + W) + U_x(V + W)_x, \partial_x \psi_c \rangle. \] (4.11)
Combining (4.10), (4.11) with (4.12), it follows that
\[ \langle U, (1 - \partial_x^2)\partial_t \psi_c \rangle = \int_\mathbb{R} U(x) \int_{\mathbb{R}^+} \partial_t (1 - \partial_x^2) \psi_c \, dt \, dx \]
\[ = \int_\mathbb{R} U(x) [(1 - \partial_x^2)\psi_c(T, x) - (1 - \partial_x^2)\psi_c(0, x)] \, dx \]
\[ = 0. \quad (4.12) \]

Combining (4.10), (4.11) with (4.12), it follows that
\[ \langle u, (1 - \partial_x^2)\psi_t \rangle + \langle u_x(v + w), \psi_{xx} \rangle - \frac{1}{2} \langle u^2, \psi_{xxx} \rangle + \frac{1}{2} \langle 3u^2 + u_x^2 + 2(u(v + w)) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \psi_x \rangle \]
\[ = \langle -cU, (1 - \partial_x^2)\partial_t \psi_c \rangle + \langle U_x(V + W), \partial_x^2 \psi_c \rangle - \frac{1}{2} \langle U^2, \partial_x^2 \psi_c \rangle + \langle \frac{3}{2}U^2 + \frac{1}{2}U_x^2 + U(V + W) + U_x(V + W)x + \frac{1}{2}(V^2 + W^2 - V_x^2 - W_x^2), \partial_x \psi_c \rangle = 0, \]

where we have applied (4.4) with \( \phi(x) = \psi_c(t, x) \). Therefore \( u(t, x) = U(x - c(t - t_0)) \) is a weak solution of system (1.1). Similarly, thanks to (4.5) and (4.6), we imply that \( v(t, x) = V(x - c(t - t_0)) \), \( w(t, x) = W(x - c(t - t_0)) \) is weak solutions to system (1.1)_2, (1.1)_3 respectively. This completes the proof of Lemma 4.1.

Finally, we state the main result in this subsection.

**Theorem 4.1** Assume \( z(t, x) \) be \( x \)-symmetric. If \( z = (u, v, w) \) is a unique weak solution of system (1.1), then \( z(t, x) \) is a traveling wave.

**Proof.** It is necessary to consider the test function \( \varphi \in C^1_0(\mathbb{R}^+, C^1_0(\mathbb{R})) \). Let
\[ \varphi_b(t, x) = \varphi(t, 2b(t) - x), \quad b(t) \in C^1(\mathbb{R}). \]

One can easily check that \( (\varphi_b)_b = \varphi \) and
\[ \begin{cases} 
\partial_t \varphi_b = (\partial_t \varphi)_b + 2\dot{b}(\partial_x \varphi)_b, \\
\partial_x u_b = - (\partial_x u)_b, \quad \partial_x \varphi_b = -(\partial_x \varphi)_b. 
\end{cases} \quad (4.13) \]

Moreover,
\[ \begin{cases} 
\langle u_b, \varphi \rangle = \langle u, \varphi_b \rangle, \quad \langle u_b^2, \varphi \rangle = \langle u^2, \varphi_b \rangle, \\
\langle v_b \partial_x u_b, \varphi \rangle = - \langle v \partial_x u, \varphi_b \rangle, \quad \langle (\partial_x u_b)^2, \varphi \rangle = \langle (\partial_x u)^2, \varphi_b \rangle, 
\end{cases} \quad (4.14) \]

where \( \dot{b} \) denotes the time derivative of \( b \).
Since $z$ is $x$-symmetric, in view of (4.13) and (4.14), we imply that
\[
\langle u, (1 - \partial_x^2)\varphi \rangle + \langle u_x(v + w), \varphi_{xx} \rangle - \frac{1}{2} \langle u^2, \varphi_{xxx} \rangle = \langle u, ((1 - \partial_x^2)\partial_t\varphi)_b \rangle - \langle u_x(v + w), \partial_x^2\varphi_b \rangle - \frac{1}{2} \langle u^2, \partial_x^2\varphi_b \rangle = \langle u, (1 - \partial_x^2)\varphi_b + 2\partial_x\partial_x\varphi_b \rangle - \langle u_x(v + w), \partial_x^2\varphi_b \rangle + \frac{1}{2} \langle u^2, \partial_x^2\varphi_b \rangle,
\]
and
\[
\frac{1}{2} \langle 3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \varphi_x \rangle
= -\frac{1}{2} \langle 3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \partial_x\varphi_b \rangle.
\]
Add up (4.15) with (4.16), by (4.1) we have
\[
\langle u, (1 - \partial_x^2)\varphi \rangle + \langle u_x(v + w), \varphi_{xx} \rangle - \frac{1}{2} \langle u^2, \varphi_{xxx} \rangle + \frac{1}{2} \langle 3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \varphi_x \rangle
= \langle u, (1 - \partial_x^2)(\partial_t\varphi_b + 2\partial_x\partial_x\varphi_b) \rangle - \langle u_x(v + w), \partial_x^2\varphi_b \rangle + \frac{1}{2} \langle u^2, \partial_x^2\varphi_b \rangle - \frac{1}{2} \langle 3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \partial_x\varphi_b \rangle
= 0.
\]
Thus, taking place $\varphi$ by $\varphi_b$ in (4.17), due to $(\varphi_b)_b = \varphi$, it follows that
\[
-\frac{1}{2} \langle 3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \partial_x\varphi \rangle
+ \langle u, (1 - \partial_x^2)(\partial_t\varphi + 2\partial_x\partial_x\varphi) \rangle - \langle u_x(v + w), \partial_x^2\varphi \rangle + \frac{1}{2} \langle u^2, \partial_x^2\varphi \rangle = 0.
\]
Subtracting (4.18) from (4.17) to derive
\[
\langle u, 2\partial_t(1 - \partial_x^2)\partial_x\varphi \rangle - 2 \langle u_x(v + w), \partial_x^2\varphi \rangle + \langle u^2, \partial_x^2\varphi \rangle - \langle 3u^2 + u_x^2 + 2u(v + w) + 2u_x(v + w)x + (v^2 + w^2 - v_x^2 - w_x^2), \partial_x\varphi \rangle = 0.
\]
If we choose $\varphi_\varepsilon(t, x) = \psi(x)\varrho_\varepsilon(t)$ in (4.19), where $\psi \in C_0^\infty(\mathbb{R})$ and $\varrho_\varepsilon \in C_0^\infty(\mathbb{R}^+)$ is a mollifier with the property that $\varrho_\varepsilon \to \delta(t - t_0)$, the Dirac mass
at \( t_0 \), as \( \varepsilon \to 0 \). This implies that

\[
\int_{\mathbb{R}} (1 - \partial^2_x \partial_x \psi) \int_{\mathbb{R}^+} b \varphi_{\varepsilon}(t) dt dx + \frac{1}{2} \int_{\mathbb{R}} \partial^2_x \psi \int_{\mathbb{R}^+} u^2 \varphi_{\varepsilon}(t) dt dx

- \int_{\mathbb{R}} \partial^2_x \psi \int_{\mathbb{R}^+} u_x (v + w) \varphi_{\varepsilon}(t) dt dx - \frac{1}{2} \int_{\mathbb{R}} \partial_x \psi \int_{\mathbb{R}^+} [2u_x (v + w)] \varphi_{\varepsilon}(t) dt dx

- \frac{1}{2} \int_{\mathbb{R}} \partial_x \psi \int_{\mathbb{R}^+} [3u^2 + u_x^2 + 2u (v + w) + (v^2 + w^2 - v_x^2 - w_x^2)] \varphi_{\varepsilon}(t) dt dx = 0.
\]

(4.20)

Note that

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} b \varphi_{\varepsilon}(t) dt = \hat{b}(t_0) u(t_0, x) \quad \text{in} \; L^2(\mathbb{R}),
\]

Therefore, letting \( \varepsilon \to 0 \), (4.20) is given by

\[
\langle -\hat{b}(t_0) u(t_0, x), (1 - \partial^2_x \partial_x \psi) + \langle u_x (v + w)(t_0, x), \partial^2_x \psi \rangle - \frac{1}{2} \langle u^2(t_0, x), \partial^2_x \psi \rangle + \frac{1}{2} \langle [3u^2 + u_x^2 + 2u (v + w) + 2u_x (v + w) + (v^2 + w^2 - v_x^2 - w_x^2)](t_0, x), \partial_x \psi \rangle

= 0.
\]

(4.21)

Hence set \( c = \hat{b}(t_0) \), we prove that \( u(t_0, x) \) satisfies (4.4). As the process of (4.21), \( v(t_0, x), w(t_0, x) \) is the solution to (4.5), (4.6) respectively. By virtue of Lemma 4.1, \( \tilde{z}(t, x) = z(t_0, x - \hat{b}(t_0)(t - t_0)) \) is a traveling wave solution of system (1.1). In view of \( \tilde{z}(t, x) = z(t_0, x) \) and the uniqueness of the solution of system (1.1), it follows that \( \tilde{z}(t, x) = z(t, x) \), for any \( t > 0 \), which conclude the proof of Theorem 4.1. \( \square \)

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References

[1] R. Beals, D. Sattinger and J. Szmigielski, *Multipeakons and a theorem of Stieltjes*, Inverse Problems, **15** (1999), 1–4.

[2] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa- Holm equation*, Arch. Rat. Mech. Anal., **183** (2007), 215–239.
[3] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl., 5 (2007), 1–27.

[4] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Letters, 71 (1993), 1661–1664.

[5] R. Camassa, D. Holm and J. Hyman, *An integrable shallow water equation*, Adv. Appl. Mech., 31 (1994), 1–33.

[6] A. Constantin, *Global existence and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble), 50 (2000), 321–362.

[7] A. Constantin, *On the scattering problem for the Camassa-Holm equation*, Proc. R. Soc. London A, 457 (2001), 953–970.

[8] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., 181 (1998), 229-243.

[9] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Ann. Scuola Norm. Sup. Pisa, 26 (1998), 303-328.

[10] A. Constantin and J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z., 233 (2000), 75–91.

[11] A. Constantin and W.A. Strauss, *Stability of peakons*, Comm. Pure Appl. Math., 53 (2000), 603–610.

[12] H.H. Dai, *Model equations for nonlinear dispersive waves in a compressible Mooney–Rivlin rod*, Acta Mechanica, 127 (1998), 193–207.

[13] Y. Fu and C. Qu, *Well posedness and blow-up solution for a new coupled Camassa–Holm equations with peakons*, J. Math. Phys. 50 (2009), 012906, 1–25.

[14] B. Fuchssteiner, *Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa–Holm equation*, Physica D, 4 (1996), 229–6243.

[15] A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäcklund transformation and hereditary symmetries*, Physica D, 4 (1981), 47–66.

[16] A.E. Green and P.M. Naghdii, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech., 78 (1976), 237–246.

[17] A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, *Persistence properties and unique continuation of solutions of the Camassa–Holm equation*, Comm. Math. Phys., 271 (2007), 511–522.
[18] Q.Y. Hu, L.Y. Lin and J. Jin, Well-posedness and blowup phenomena for a three-component Camassa–Holm system with peakons, J. Hyperbolic Differential Equations, 9 (2012), 451–467.

[19] T. Kato, On the Korteweg-de Vries equation, Manuscripta Math., 28 (1979), 89–99.

[20] T. Kato, On the Cauchy problem for the generalized Korteweg-de Vries equation, in: Studies in Applied Mathematics, in: Adv. Math. Suppl. Stu., 8, Academic Press, New York, (1983), 93–128.

[21] C. Qu and Y. Fu, On a Three-component Camassa–Holm equation with peakons, Commun. Theor. Phys. 53 (2010), 223–230.

[22] X. Wu and B. Guo, Persistence properties and infinite propagation for the modified 2-component Camassa–Holm equation, Discrete Contin. Dyn. Syst. A, 33 (2013), 3211–3223.

[23] X. Wu and Z. Yin, Well-posedness and global existence for the Novikov equation, Annali Sc. Norm. Sup. Pisa, XI (2012) 707–727.