Hyperbolic structures for Artin-Tits groups of spherical type

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The goal of this mostly expository paper is to present several candidates for hyperbolic structures on irreducible Artin-Tits groups of spherical type and to elucidate some relations between them. Most constructions are algebraic analogues of previously known hyperbolic structures on Artin braid groups coming from natural actions of these groups on curve graphs and (modified) arc graphs of punctured disks.

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1 Introduction

Given a group \( G \) and a generating set \( X \) of \( G \), the word metric \( d_X \) turns \( G \) into a metric space; this space is \((1, 1)\)-quasi-isometric to \( \Gamma(G, X) \), the Cayley graph of \( G \) with respect to \( X \), endowed with the usual graph metric where each edge is identified to an interval of length 1. Hyperbolic structures on groups were recently introduced and studied in [1]. A hyperbolic structure on a group \( G \) is a generating set \( X \) of \( G \) such that \((G, d_X)\) is Gromov-hyperbolic; note that \( X \) must be infinite whenever \( G \) is not itself hyperbolic.

In this paper, we are interested in hyperbolic structures on Artin-Tits groups of spherical type. One motivation is trying to prove that irreducible Artin-Tits groups of spherical type are hierarchically hyperbolic [3, 4], where the hierarchical structure should come from the hierarchy of parabolic subgroups of these groups.

After [6, 7], we know that irreducible Artin-Tits groups of spherical type admit non-trivial hyperbolic structures; i.e. each irreducible Artin-Tits group of spherical type \( A \) contains an (infinite) generating set \( X^A_{abs} \) such that the corresponding Cayley graph \( \Gamma(A, X^A_{abs}) \) is a Gromov-hyperbolic metric space with infinite diameter. This hyperbolic structure was defined in a purely algebraic manner, using only the Garside structure on \( A \), and it consists of the set of the so-called absorbable elements (to which we must add the cyclic subgroup generated by the square of the so-called Garside element if \( A \) is of dihedral type). We shall briefly review this construction in Section 4.
Unfortunately, these absorbable elements are poorly understood – for instance we do
not know any polynomial-time algorithm which recognizes whether any given element
belongs to $X_{\text{abs}}^A$ – and this makes it quite difficult to work with the graph $\Gamma(A, X_{\text{abs}}^A)$.

In this paper we generalize to any irreducible Artin-Tits group of spherical type some
well-known hyperbolic structures on Artin’s braid groups with $n + 1$ strands $B_{n+1}$
(a.k.a. Artin-Tits groups of type $A_n$), $n \geq 3$. (The Artin-Tits group of type $A_1$ is
infinite cyclic hence hyperbolic; the case of the Artin-Tits group of type $A_2$ will be
treated separately in Section 2.1 and Proposition 4.8).

Because it can be identified with the mapping class group of a $n + 1$ times punctured
disk $D_{n+1}$, Artin’s braid group on $n + 1$ strands admits nice actions on the curve graph
of $D_{n+1}$ (denoted by $\mathcal{C}(D_{n+1})$), the arc graph of $D_{n+1}$ (denoted by $\mathcal{A}(D_{n+1})$) and the
graph of arcs in $D_{n+1}$ both of whose extremities lie in $\partial D_{n+1}$ (denoted by $\mathcal{A}_0(D_{n+1})$).
All these graphs can be shown to be connected and Gromov-hyperbolic; this was first
shown in [19] but the circle of ideas around [15, 23] provides simpler arguments.

All these actions are cobounded (actually cocompact); according to a standard argu-
ment (Lemma 2.5, close in spirit to Svarc-Milnor’s lemma [1, Section 3.2]), we extract
from each of these actions a hyperbolic structure on $B_{n+1}$, which consists of the union
of the stabilizers of a (finite) family of representatives of the orbits of vertices.

Each of these generating sets can be algebraically described in terms of the parabolic
subgroups of the Artin-Tits group of type $A_n$, allowing to extend the definitions to any
irreducible Artin-Tits group of spherical type. Given an irreducible Artin-Tits group
of spherical type $A$, we define:

- $X_p^A$ is the union of all proper irreducible standard parabolic subgroups of $A$ and
the cyclic subgroup generated by the square of the Garside element;
- $X_{np}^A$ is the union of the normalizers of all proper irreducible standard parabolic
subgroups of $A$;
- $X_{\text{abs}}^A$ is the set of absorbable elements (together with the cyclic subgroup gener-
ated by the square of the Garside element, if $A$ is of dihedral type: 2 generators).

Note that $X_{np}^A$ contains the cyclic subgroup generated by the square of the Garside element (which is central). Similarly for $X_{\text{abs}}^A$: any power of the Garside element can be written as a product of at most 3 absorbable elements, provided $A$ is not of dihedral
type [6, Example 3].

We then study the relationships between $X_p^A$, $X_{np}^A$ and $X_{\text{abs}}^A$. Following [1], given two
generating sets $X, Y$ of a group $G$, we write $X \preceq Y$ if the identity map from $(G, d_Y)$
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...to \((G, d_X)\) is Lipschitz (or equivalently, if \(\sup_{y \in Y} d_X(1_G, y) < \infty\)). The sets \(X\) and \(Y\) are equivalent if both \(X \preceq Y\) and \(Y \preceq X\) hold (or equivalently, if the identity map is a bilipschitz equivalence between \((G, d_X)\) and \((G, d_Y)\)).

Table 1 summarizes the main contents of this paper, for any irreducible Artin-Tits group of spherical type \(A\) with at least 3 generators. Vertical arrows indicate the identity of \(A\). The graph of irreducible parabolic subgroups and the additional length graph (denoted by \(C_{parab}(A)\) and \(C_{AL}(A)\), respectively) were defined in [10] and [6], respectively. For Artin-Tits groups of type \(A\), all the mentioned generating sets are hyperbolic structures. In any case, all spaces under consideration have infinite diameter (Corollary 4.10).

| Gen. set | Cayley graph | Hyperbolicity |
|----------|--------------|---------------|
| \(X^A_P\) | \(\Gamma(A, X^A_P)\) | generalized \(\mathcal{A}_\delta(D_{n+1})\) | Conjectured |
| \(\preceq\) | Lipschitz. Conj. 4.11(ii): not equivalent |
| \(X^A_{NP}\) | \(\Gamma(A, X^A_{NP})\) | generalized \(\mathcal{C}(D_{n+1})\), q.isom. to \(C_{parab}\) [10] | Conjectured |
| \(\preceq\) | Lipschitz [6, 2]. Conj. 4.11(i): equivalent |
| \(X^A_{abs}\) | \(\Gamma(A, X^A_{abs})\) | q.isom. to \(C_{AL}(A)\) [6] | Proved [6, 7] |

Table 1: Summary of the main results presented in the paper when \(A\) has at least 3 generators.

A space similar to \(\Gamma(A, X^A_P)\) was defined in [8]; that paper gives necessary and sufficient conditions on an Artin-Tits group (not necessarily of spherical type) for the union of its standard parabolic subgroups of spherical type to be a hyperbolic structure. However, when the group is itself of spherical type, this results in a Cayley graph of finite diameter.

When \(A\) is of spherical dihedral type, all three sets \(X^A_P\), \(X^A_{NP}\) and \(X^A_{abs}\) are hyperbolic structures but their orders differ slightly, namely \(X^A_P\) and \(X^A_{NP}\) are equivalent to each other and smaller than \(X^A_{abs}\) – see Proposition 4.8; also, all corresponding Cayley graphs have infinite diameter.
The plan of the paper is as follows. In Section 2 we review necessary definitions and facts about Artin-Tits groups of spherical type; we also prove Lemma 2.5. The latter is used in Section 3 to describe hyperbolic structures on $B_{n+1}$; at the same time, these structures are reinterpreted algebraically in terms of parabolic subgroups of $B_{n+1}$. In Section 4, we justify the assertions and present the conjectures from Table 1.

2 Preliminaries

2.1 Preliminaries on Artin-Tits groups of spherical type

Let $S$ be a finite set. A Coxeter matrix over $S$ is a symmetric matrix $M = (m_{s,t})_{s,t \in S}$ with $m_{s,s} = 1$ for $s \in S$ and $m_{s,t} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t \in S$. The Coxeter graph $\Gamma = \Gamma_S = \Gamma(M)$ associated to $M$ is a labeled graph whose vertices are the elements of $S$ and in which two distinct vertices are connected by an edge labeled by $m_{s,t}$ whenever $m_{s,t} \geq 3$.

For two symbols $a, b$ and an integer $m$, define

$$\Pi(a, b; m) = \begin{cases} (ab)^k & \text{if } m = 2k, \\ (ab)^ka & \text{if } m = 2k + 1. \end{cases}$$

The Artin-Tits system associated to $M$ (or to $\Gamma$) is the pair $(A, S)$ where $A = A_S = A_\Gamma$ is the group presented by

$$\langle S \mid \Pi(s, t; m_{s,t}) = \Pi(t, s; m_{s,t}), \forall s \neq t \in S, \text{with } m_{s,t} < \infty \rangle.$$ 

As relations are homogeneous, there is a well-defined homomorphism $\ell : A_S \rightarrow \mathbb{Z}$ which sends each $s \in S$ to 1. The rank of $A_S$ is the number of elements of $S$.

Given a subset $T$ of $S$, denote by $\Gamma_T$ the subgraph of $\Gamma_S$ induced by the vertices in $T$ and by $A_T$ the subgroup of $A_S$ generated by $T$. It is a theorem of Van der Lek [25] that $(A_T, T)$ is the Artin-Tits system associated to the graph $\Gamma_T$. Such a subgroup $A_T$ ($T \subset S$) is called a standard parabolic subgroup of $A_S$. A parabolic subgroup of $A_S$ is a subgroup which is conjugate to some standard parabolic subgroup.

The group $A_S$ is said to be irreducible if the graph $\Gamma_S$ is connected, equivalently if $A_S$ cannot be expressed as the direct product of two standard parabolic subgroups. The group $A_S$ is of spherical type if the Coxeter group obtained as the quotient of $A_S$ by the normal closure of the squares of elements in $S$ is finite. The group $A_S$ is said to
Irreducible Artin-Tits groups of spherical type have been classified [9] (see also [16] for a modern exposition) into type $A_n (n \geq 1)$, $B_n (n \geq 2)$, $D_n (n \geq 4)$, $E_6$, $E_7$, $E_8$, $F_4$, $H_3$, $H_4$ and $I_{2m} (m \geq 5)$. Figure 1 summarizes this classification. Note that for $n \geq 1$, the Artin-Tits group of type $A_n$ is precisely Artin’s braid group $B_{n+1}$. In this paper, we will be concerned only with irreducible Artin-Tits groups of spherical type.

For an integer $m = 3, 4$, we write also $I_{2m}$ for $A_2$ and $B_2$ respectively.

We recall that each irreducible Artin-Tits groups of spherical type $A = A_S$ can be equipped with a finite Garside structure – this was proved in [5, 11]. The reader is referred to [14, Section 2] for a gentle introduction to Garside structures. We just recall the existence of a Garside element $\Omega_S$; either $\Delta_S$ or $\Delta_S^2$ generates the center of $A_S$ and we denote by $\Omega_S$ this minimal central element. (It is known that $\Omega_S = \Delta_S$ for groups of type $A_1$, $B_n (n \geq 2)$, $D_n (n$ even$)$, $E_7$, $E_8$, $F_4$, $H_3$, $H_4$ and $I_{2m} (m$ even$)$, and $\Omega_S = \Delta_S^2$ for groups of type $A_n (n \geq 2)$, $D_n (n$ odd$)$, $E_6$, and $I_{2m} (m$ odd$)$).

Similarly, for each irreducible standard parabolic subgroup $A_T (T \subset S)$, there is a Garside element $\Delta_T$ and $\Omega_T = \Delta_T$ or $\Delta_T^2$ generates the center of $A_T$. Moreover, if $P$ is any irreducible parabolic subgroup of $A_S$, then $P = a^{-1}A_T a$ for some element $a \in A$ and some $T \subset S$ – note that this expression is by no means unique. However there is a well-defined element $\Omega_P = a^{-1}\Omega_T a$ which does not depend on $a$ nor on $T$ in the above expression of $P$ and generates the center of $P$ (see [10]).

The two following results on conjugacy of parabolic subgroups will be useful later.

**Proposition 2.1** [22, Theorem 6.1] Let $A = A_S$ be an irreducible Artin-Tits group of spherical type and let $T \subset S$ such that $A_T$ is an irreducible Artin-Tits group. Then

$$N_{A_S}(A_T) = Z_{A_S}(\Omega_T).$$
Proposition 2.2 [10, Section 11] Let \( A = A_S \) be an irreducible Artin-Tits group of spherical type. Let \( P, Q \) be two irreducible parabolic subgroups of \( A_S \) and suppose that \( \Omega_P \) and \( \Omega_Q \) commute. Then there exists \( g \in A \) and \( T_1, T_2 \subset S \) such that \( g^{-1}Pg = A_{T_1} \) and \( g^{-1}Qg = A_{T_2} \).

We conclude this section with a few words on the case of dihedral type Artin-Tits groups.

Proposition 2.3 [21, Theorem 6.2 (2)] Let \( m \geq 3 \) be an integer. Let \( A \) be an Artin-Tits group of type \( I_{2m} \). Then the quotient \( A/\langle \Delta^2 \rangle \) has a finite index subgroup isomorphic to a free group of rank \( m - 1 \).

Corollary 2.4 Let \( m \) and \( A \) be as in Proposition 2.3. Let \( X \) be any finite generating set of \( A \). Then the generating set \( X \cup \langle \Delta^2 \rangle \) is a hyperbolic structure on \( A \). All hyperbolic structures obtained in this way are equivalent.

Proof Let \( \overline{X} \) be the set of cosets of elements of \( X \) mod \( \langle \Delta^2 \rangle \); this generates \( A/\langle \Delta^2 \rangle \). Observe that \( \Gamma(A, X \cup \langle \Delta^2 \rangle) \) and \( \Gamma(A/\langle \Delta^2 \rangle, \overline{X}) \) are quasi-isometric spaces. From Proposition 2.3 it follows applying Svarc-Milnor lemma that \( \Gamma(A/\langle \Delta^2 \rangle, \overline{X}) \) is quasi-isometric to a \( m - 1 \)-regular tree, hence hyperbolic. Therefore, for any \( X \), \( \Gamma(A, X \cup \langle \Delta^2 \rangle) \) is quasi-isometric to a regular tree, whence the claim. \( \square \)

2.2 Cocompact actions on connected graphs

The unique result in this subsection will allow us to exhibit a number of hyperbolic structures on Artin’s braid group \( \mathcal{B}_{n+1} \). Although this is a standard fact, we include a stand-alone proof for the convenience of the reader.

Lemma 2.5 [19, Lemma 3.2][17, Lemma 10] Let \( X \) be a connected graph and denote by \( d_X \) the graph-metric on \( X \) obtained by identifying each edge to an interval of length 1. Let \( G \) be a group acting by isometries on \( X \) such that

(i) there exists a finite set \( A \) of vertices of \( X \) such that every vertex orbit has a (non-necessarily unique) representative in \( A \),

(ii) there exists a finite set \( B \) of edges of \( X \) such that every edge orbit has a (non-necessarily unique) representative in \( B \),

(iii) the set \( T = \bigcup_{a \in A} \text{Stab}_G(a) \) generates \( G \).

Then \( (G, d_T) \) is quasi-isometric to \( (X, d_X) \).
Let $\alpha_t \cdot u_i \in A$ and fix $\beta_t \cdot v_i \in A$. Let
$$M_3 = \max_{1 \leq i \leq m} \{ \| \alpha_i \|_T, \| \beta_i \|_T \}.$$ 

Fix $a_0 \in A$ and define a map $\psi : G \rightarrow \mathcal{V}_X$ by $g \mapsto g \cdot a_0$. We first prove that the map $\psi : (G, d_T) \rightarrow (\mathcal{V}_X, d_X)$ is Lipschitz. Let $g, h \in G$, let $n = \| g^{-1}h \|_T$ and write $g^{-1}h = t_1 \ldots t_n$, with $t_i \in T$. We have
$$d_X(\psi(g), \psi(h)) = d_X(a_0, g^{-1}h \cdot a_0) = d_X(a_0, t_1 \ldots t_n \cdot a_0)$$
and
$$d_X(a_0, t_1 \ldots t_n \cdot a_0) \leq d_X(a_0, t_1 \cdot a_0) + d_X(t_1 \cdot a_0, t_1t_2 \cdot a_0) + \ldots + d_X(t_1 \ldots t_{n-1} \cdot a_0, t_1 \ldots t_n \cdot a_0)
= \sum_{i=1}^n d_X(a_0, t_i \cdot a_0).$$

But for any $t \in T$ there is at least some $a_t \in A$ so that $t \cdot a_t = a_t$; so we have
$$d_X(a_0, t \cdot a_0) \leq d_X(a_0, a_t) + d_X(a_t, t \cdot a_0) = d_X(a_0, a_t) + d_X(t \cdot a_t, t \cdot a_0) = 2d_X(a_0, a_t) \leq 2M_1.$$

In summary, we have
$$d_X(\psi(g), \psi(h)) \leq 2M_1 d_T(g, h).$$

On the other hand, given any vertex $x$ of $X$, there exist $a \in A$ and $g \in G$ so that $x = g \cdot a$. If $a, a' \in A$ and $g, g' \in G$ are such that $g \cdot a = g' \cdot a' = x$, we have $g_a \cdot a = g_a \cdot a' \in A_0$ whence $g_a^{-1}gg_a^{-1}g_a$ fixes $a'$ and so lies in $T$. It follows that the set of elements $g \in G$ such that $g \cdot a = x$ for some $a \in A$ has diameter at most $1 + 2M_2$ in $(G, d_T)$. We define $\varphi : \mathcal{V}_X \rightarrow G$ in the following way: to each $x \in \mathcal{V}_X$ we associate some $g$ in $G$ such that $g^{-1} \cdot x \in A$. It is now clear that $\varphi \circ \psi$ is at distance at most $2M_2 + 1$ from $Id_G$ and $\psi \circ \varphi$ is at distance at most $M_1$ from $Id_{\mathcal{V}_X}$.

We now show that $\varphi : (\mathcal{V}_X, d_X) \rightarrow (G, d_T)$ is also Lipschitz. Let $x_1, x_2$ be adjacent vertices of $X$. There is some $g$ in $G$ and $1 \leq i \leq m$ such that $\{g \cdot x_1, g \cdot x_2\} = \{u_i, v_i\}$ is
an edge belonging to $B$. By construction, $\alpha_i^{-1}g \cdot x_1 = \alpha_i^{-1}u_i \in A$ and $\beta_i^{-1}g \cdot x_2 = \beta_i^{-1}v_i \in A$; it follows that $d_T(\varphi(x_1), g^{-1}\alpha_i) \leq 2M_2 + 1$ and $d_T(\varphi(x_2), g^{-1}\beta_i) \leq 2M_2 + 1$.

We then have
\[
d_T(\varphi(x_1), \varphi(x_2)) \leq d_T(\varphi(x_1), g^{-1}\alpha_i) + d_T(g^{-1}\alpha_i, g^{-1}\beta_i) + d_T(g^{-1}\beta_i, \varphi(x_2)) \\
\leq 4M_2 + 2 + d_T(\alpha_i, \beta_i) \\
\leq 4M_2 + 2 + 2M_3.
\]

In summary, if $x_1, x_2$ are arbitrary vertices of $X$ then
\[
d_T(\varphi(x_1), \varphi(x_2)) \leq 2(2M_2 + M_3 + 1) \cdot d_X(x_1, x_2).
\]

3 Hyperbolic structures for $B_{n+1}$

In this section, we review some Gromov-hyperbolic graphs on which Artin’s braid group $B_{n+1}$ acts. Lemma 2.5 is put to work in order to derive hyperbolic structures on $B_{n+1}$ from these actions. These hyperbolic structures are interpreted in an algebraic context.

3.1 The curve graph

Fix an integer $n \geq 3$. Artin’s braid group on $n + 1$ strands is identified with the mapping class group of a closed, $n + 1$ times punctured disk $D_{n+1}$ (with boundary fixed pointwise). We consider the following model for $D_{n+1}$: the disk of diameter $[0, n + 2]$ in $\mathbb{C}$ with the standard orientation, and $n + 1$ punctures at $1, \ldots, n + 1$ numbered accordingly $p_1, \ldots, p_{n+1}$. For $1 \leq i \leq n$, write $s_i$ for the half-Dehn twist along an horizontal arc connecting $p_i$ and $p_{i+1}$. Then $(B_{n+1}, \{s_1, \ldots, s_n\})$ is exactly the Artin-Tits system associated to the Coxeter graph $A_n$.

The curve graph $C(D_{n+1})$ of the $n + 1$ times punctured disk $D_{n+1}$ is the (locally-infinite) graph defined as follows:

- its vertices are isotopy classes of essential simple closed curves in $D_{n+1}$, that is isotopy classes of simple closed curves in $D_{n+1}$ surrounding at least 2 and at most $n$ punctures;
- two distinct vertices are connected by an edge if and only if the corresponding isotopy classes can be realized disjointly.
According to [19], this graph is connected and Gromov-hyperbolic; a simpler proof can be found in [15]. The group $B_{n+1}$ acts naturally by isometries (simplicial automorphisms) on $C(D_{n+1})$.

An essential simple closed curve in $D_{n+1}$ (a vertex of $C(D_{n+1})$) is said to be round or standard if it is isotopic to a geometric circle; note that there are exactly $\frac{n(n+1)}{2} - 1$ such vertices. Each of them can be described by two integers $1 \leq i < j \leq n+1$: $p_i$ ($p_j$ respectively) is the first (the last respectively) puncture surrounded; we denote this curve by $c_{ij}$. For $1 \leq i < j \leq n+1$, we denote by $B_{ij}$ the set of braids whose support is enclosed by $c_{ij}$; this is the standard irreducible parabolic subgroup generated by $\{s_t, i \leq t \leq j-1\}$, which is isomorphic to a braid group on $j - i + 1$ strands. The square of the Garside element of this group $\Delta_{ij}$ is the Dehn twist around the curve $c_{ij}$.

**Lemma 3.1** Let $X_C$ be the union of stabilizers of all standard curves. Then $B_{n+1}$ equipped with the word metric $d_{X_C}$ is quasi-isometric to the curve graph of the $n+1$ times punctured disk.

**Proof** First observe that the action of each generator $s_i$ fixes some standard curve; therefore, $X_C$ generates $B_{n+1}$. According to [12, Section 1.3.1], the set $A$ of all standard curves contains a representative of each vertex orbit under the action of $B_{n+1}$. According to [18, Proposition 4.4], the finite set of edges in $C(D_{n+1})$ both of whose extremities are standard curves contains a representative of each edge orbit under the action of $B_{n+1}$. Hence we see that the isometric action of $B_{n+1}$ on the curve graph $C(D_{n+1})$ satisfies the hypotheses of Lemma 2.5. □

**Proposition 3.2** Let $X_{NP}^{B_{n+1}}$ be the union of all normalizers of proper irreducible standard parabolic subgroups of $B_{n+1}$. Then $B_{n+1}$ equipped with the word metric $d_{X_{NP}^{B_{n+1}}}$ is quasi-isometric to the curve graph of the $n+1$ times punctured disk. In particular, $X_{NP}^{B_{n+1}}$ is a hyperbolic structure on $B_{n+1}$.

**Proof** According to [12, Fact 3.8], the stabilizer of any curve $c$ is exactly the centralizer of the Dehn twist around $c$; if $c = c_{ij}$ is standard, this is the centralizer of the square of the Garside element $\Delta_{ij}$ of $B_{ij}$ (this is the same as the centralizer of $s_i$ if $j = i + 1$). In other words, in view of Proposition 2.1, $\text{Stab}_{B_{n+1}}(c_{ij})$ is the same as the normalizer of the standard parabolic subgroup $B_{ij}$. □

### 3.2 The boundary connecting arc graph

Again, we fix an integer $n \geq 3$; we keep the same model for the punctured disk $D_{n+1}$ and $s_i$, $1 \leq i \leq n$, are the usual Artin generators of $B_{n+1}$.
The boundary connecting arc graph $\mathcal{A}_\partial(D_{n+1})$ of the $n+1$ times punctured disk $D_{n+1}$ is the (locally-infinite) graph defined as follows:

- its vertices are isotopy classes of essential (i.e. not homotopic into a subset of $\partial D_{n+1}$) simple arcs in $D_{n+1}$ both of whose endpoints lie in the boundary $\partial D_{n+1}$ and where in addition the endpoints are allowed to slide freely along the boundary during the isotopy.
- two distinct vertices are connected by an edge if and only if the corresponding isotopy classes of arcs can be realized disjointly.

The argument given in [15] can be easily adapted to show that $\mathcal{A}_\partial(D_{n+1})$ is a Gromov-hyperbolic connected graph. The group $B_{n+1}$ acts by isometries on $\mathcal{A}_\partial(D_{n+1})$.

Given $1 \leq i \leq n$, let $a_i$ be the arc consisting simply of the vertical line with real part $i + \frac{1}{2}$; let $A = \{a_i, 1 \leq i \leq n\}$. Then $A$ exhausts all the orbits of vertices of $\mathcal{A}_\partial(D_{n+1})$ under the action of $B_{n+1}$ (the orbit of a boundary connecting arc $a$ is uniquely characterized by the number of punctures in any of the two connected components of $D_{n+1} \setminus a$). Two different disjoint boundary connecting arcs (forming an edge of $\mathcal{A}_\partial(D_{n+1})$) can be transformed simultaneously into two arcs $a_i, a_j$ for some $1 \leq i \neq j \leq n$ by the action of a braid: this says that there is a finite system $B$ of representatives of the orbits of edges. Finally, each half-twist $s_i$ stabilizes some arc in $A$. (Indeed, $s_i$ fixes $a_{i-1}$ if $i \geq 2$, and $s_1$ fixes $a_2$.) Therefore, all three hypotheses of Lemma 2.5 are satisfied.

It follows from Lemma 2.5 that the union $X_{\mathcal{A}_\partial}$ of stabilizers of arcs $a_i$, $1 \leq i \leq n$, is a hyperbolic structure on $B_{n+1}$ and that $(B_{n+1}, d_{X_{\mathcal{A}_\partial}})$ is quasi-isometric to $\mathcal{A}_\partial(D_{n+1})$.

**Claim.** If $n$ is odd and $i = \frac{n+1}{2}$, the stabilizer of $a_i$ is generated by $\Delta$ and the direct product of standard parabolic subgroups $\langle s_1, \ldots, s_{i-1} \rangle \times \langle s_{i+1} \ldots s_n \rangle$. In any other case, the stabilizer of $a_i$ is generated by $\Delta^2$ and the direct product of standard parabolic subgroups $\langle s_1, \ldots, s_{i-1} \rangle \times \langle s_{i+1} \ldots s_n \rangle$.

**Proof** The arc $a_i$ cuts $D_{n+1}$ into two connected components which are punctured disks with $i$ and $n+1-i$ punctures respectively. Except if $2i = n+1$, an element $g$ in the stabilizer of $a_i$ must preserve the boundary of each of these two disks. Following the conventions in [18], this means that $g$ can be written as a product of an interior braid (which is an element of $\langle s_1, \ldots, s_{i-1} \rangle \times \langle s_{i+1} \ldots s_n \rangle$) and a tubular braid of the form

$$
(s_1 \ldots s_{i}) (s_{i+1} \ldots s_{2}) \ldots (s_n \ldots s_{n-i+1}) (s_{n-i+1} \ldots s_{1}) \ldots (s_n \ldots s_{i+1})^k
$$
for some \( k \in \mathbb{Z} \) (a pure braid on 2 fat strands). We observe however that the above braid can be written as

\[
\Delta^{2k} \Delta^{2k}_{\langle s_1, \ldots, s_{i-1} \rangle} \Delta^{-2k}_{\langle s_{i+1} \ldots s_n \rangle}.
\]

If \( 2i = n + 1 \), an element of the stabilizer of \( a_i \) can be written as the product of an interior braid as above and a tubular braid (on 2 fat strands) which needs not be pure, that is a power of

\[
(s_i \ldots s_1)(s_{i+1} \ldots s_2) \ldots (s_n \ldots s_{n-i+1}),
\]

which we can rewrite as

\[
\Delta \Delta^{-1}_{\langle s_1, \ldots, s_{i-1} \rangle} \Delta^{-1}_{\langle s_{i+1} \ldots s_n \rangle}.
\]

**Proposition 3.3** Let \( \mathcal{B}^{E_{n+1}} \) be the subset of \( \mathcal{B}_{n+1} \) consisting of the union of all proper irreducible standard parabolic subgroups together with the center \( \langle \Delta^2 \rangle \). Then \( \mathcal{B}_{n+1} \) equipped with the word metric \( d_{\mathcal{B}^{E_{n+1}}} \) is quasi-isometric to \( A_\partial(D_{n+1}) \). In particular, \( \mathcal{B}^{E_{n+1}} \) is a hyperbolic structure on \( \mathcal{B}_{n+1} \).

**Proof** Let \( X_{A_\partial} \) be the union of stabilizers of all arcs \( a_i \), \( 1 \leq i \leq n \), as above. Then \( X_{A_\partial} \) contains the generating set \( \mathcal{B}^{E_{n+1}} \) (every element in a proper irreducible standard parabolic subgroup fixes some arc \( a_i \)). On the other hand, each element in \( X_{A_\partial} \) can be written as a product of at most 6 elements of \( \mathcal{B}^{E_{n+1}} \), according to the above claim (observe that \( \Delta \) can be decomposed as a product of three elements of proper irreducible standard parabolic subgroups). This shows that both generating sets are equivalent. The remaining part of the statement follows as \( X_{A_\partial} \) is a hyperbolic structure and \( (\mathcal{B}_{n+1}, d_{X_{A_\partial}}) \) is quasi-isometric to \( A_\partial(D_{n+1}) \). \( \square \)

### 4 Generalizing to Artin-Tits groups of spherical type

Throughout this section, \( A = A_S \) is some fixed non-cyclic irreducible Artin-Tits group of spherical type.

#### 4.1 Definitions

##### 4.1.1 Generalized boundary connecting arc graph

Let \( \mathcal{B}^A \) be the union of all proper irreducible standard parabolic subgroups of \( A \) and the cyclic subgroup generated by the square of \( \Delta_S \). It is clear that \( \mathcal{B}^A \) generates \( A \).
because each generator in $S$ belongs to some proper irreducible standard parabolic subgroup.

**Proposition 4.1** Suppose that one of the following holds.

(i) $A$ is of dihedral type, or  
(ii) $A$ is of type $A_n$ ($n \geq 3$).

Then $X^A_P$ is a hyperbolic structure on $A$.

**Proof** We refer to Proposition 4.8 for the proof of (i). For (ii), this is Proposition 3.3: in this case, the Cayley graph $\Gamma(A, X^A_P)$ is quasi-isometric to the graph $A_{\partial}(D_{n+1})$. 

**Conjecture 4.2** Let $A$ be any non-cyclic irreducible Artin-Tits group of spherical type. The set $X^A_P$ is a hyperbolic structure on $A$.

4.1.2 Generalized curve graph

Let $X^A_{NP}$ be the union of the normalizers of all proper irreducible standard parabolic subgroups of $A$. Of course, $X^A_{NP}$ generates $A$, as each $s \in S$ normalizes any parabolic subgroup containing it and hence belongs to $X^A_{NP}$. Note also that any power of $\Delta^2_S$ normalizes every parabolic subgroup.

Recall from [10] the definition of the graph of parabolic subgroups of $A$, denoted $C_{parab}(A)$:

- its vertices are proper irreducible parabolic subgroups of $A$,
- two distinct vertices $P, Q$ are connected by an edge if the minimal central elements $\Omega_P$ and $\Omega_Q$ commute.

The group $A$ acts naturally (by conjugation on vertices) by isometries on $C_{parab}(A)$. Observing that the set of standard parabolic subgroups is finite and using Proposition 2.2, Lemma 2.5 implies along the same lines as Lemma 3.1:

**Proposition 4.3** The graphs $\Gamma(A, X^A_{NP})$ and $C_{parab}$ are quasi-isometric.

In the special case where $A_S$ is of type $A_n$ ($n \geq 3$), it is shown in [10] that $C_{parab}(A_S)$ is exactly the same graph (isometric) as the curve graph of a $n + 1$ times punctured disk. We have:

**Proposition 4.4** Suppose that one of the following holds.
(i) $A$ is of dihedral type, or
(ii) $A$ is of type $A_n$ ($n \geq 3$).

Then $X^A_{NP}$ is a hyperbolic structure on $A$.

Proof We refer to Proposition 4.8 for the proof of (i). For (ii), this is Proposition 3.2: in this case, the Cayley graph $\Gamma(A, X^A_{NP})$ is quasi-isometric to the graph $C(D_{n+1})$. \qed

Conjecture 4.5 Let $A$ be any non-cyclic irreducible Artin-Tits group of spherical type. The set $X^A_{NP}$ is a hyperbolic structure on $A$.

4.1.3 Absorbable elements and additional length graph

Let $X^A_{abs}$ be the set of absorbable elements (together with the cyclic subgroup generated by $\Delta^2$ if $A$ is of dihedral type). The definition of absorbable element of a Garside group was proposed in [6]. It is not difficult to see that each element in $S$ is absorbable, hence $X_{abs}$ generates $A$.

Recall from [6] the definition of the additional length graph $C_{AL}(A)$: its vertices are right cosets of $A$ modulo the cyclic subgroup generated by $\Delta$; there is an edge between the vertices $a\langle \Delta \rangle$ and $b\langle \Delta \rangle$ if there is some $u$ which is either simple (i.e. a positive prefix of $\Delta$) or absorbable such that $au$ belongs to the coset $b\langle \Delta \rangle$.

Lemma 4.6 $C_{AL}(A)$ and $\Gamma(A, X^A_{abs})$ are quasi-isometric.

Proof It is enough to check the quasi-isometry of the respective vertex sets. Any coset $a\langle \Delta \rangle$ has diameter at most 4 in $\Gamma(A, X^A_{abs})$: any two elements of such a coset differ by a power of $\Delta$; and any power of $\Delta$ can be written as a product of at most 4 elements of $X^A_{abs}$ in the dihedral case (because $\Delta^q = \Delta^{2\lfloor \frac{q}{2} \rfloor} \cdot a \cdot b \cdot b^{-1}a^{-1}\Delta$), and of at most 3 absorbable elements in the general case: (the argument of [6, Example 3] for braid groups with at least 4 strands can be easily adapted to any Artin-Tits group of rank $\geq 3$). For any coset $a\langle \Delta \rangle$, choose an arbitrary representative; this defines a map $\varphi$ from the vertices of $C_{AL}$ to $A$. Conversely, given $a \in A$, we associate to $a$ the coset $\psi(a) = a\langle \Delta \rangle$. It is straightforward to check that these maps are quasi-inverses of each other with respect to the graph metrics on $C_{AL}(A)$ and $\Gamma(A, X^A_{abs})$. It is also clear that $\psi$ is 1-Lipschitz. On the other hand, as the set of simple elements (positive divisors of $\Delta$) is finite, we can define $M$ as the maximum of $d^A_{abs}(1, s\Delta^j)$, for $s$ simple and $j \in \mathbb{Z}$. It then follows that $\varphi$ is $M$-Lipschitz. \qed
The graph $C_{sl}(A)$ was shown to be hyperbolic in [6]; therefore Proposition 4.6 shows:

**Proposition 4.7** Let $A$ be any non-cyclic irreducible Artin-Tits group of spherical type. The set $X_{abs}^A$ is a hyperbolic structure on $A$.

### 4.2 Comparisons and further results

In this subsection, we compare the different generating sets for $A$ described above. We will first look at Artin-Tits groups of dihedral type, and then at all other types.

**Proposition 4.8** Assume that $A$ is of type $I_{2m}$ ($m \geq 3$). Then we have

$$X_P^A \sim X_{NP}^A \preceq X_{abs}^A \sim S \cup \langle \Delta^2 \rangle,$$

where the middle inequality is strict. All the sets mentioned are hyperbolic structures on $A$ and the corresponding Cayley graphs all have infinite diameter.

**Proof** Denote by $a, b$ the generators of $A$. In order to prove that $X_P^A \sim X_{NP}^A$, we first observe that $X_P^A \subset X_{NP}^A$ whence $X_{NP}^A \preceq X_P^A$. For the converse inequality, it will be enough to show that the normalizer of $\langle a \rangle$ ($\langle b \rangle$ respectively) is generated by $\Omega$ and $a$ ($\Omega$ and $b$ respectively). In order to do so, we first observe that for any $g \in N_A(\langle a \rangle)$ we have $g^{-1}ag \in \langle a \rangle$, and using the abelianisation we see that $g^{-1}ag = a$, i.e., the normalizer of $\langle a \rangle$ coincides with $Z_A(a)$, the centraliser of $a$. This centraliser can be very easily computed using the algorithm in [13]. For $m$ even, we get that $Z_A(a) = \langle a, \Delta \rangle$. For $m$ odd, we obtain $\{a, \Pi(b, a; m-2)a^2\Pi(b, a; m-2)\}$ as a generating set; up to right and left multiplying the second generator by $a$, this gives the generating set $\{a, \Delta^2\}$. In any case, an element of $N_A(\langle a \rangle)$ can be written as $\Omega^n a^m$, which has length at most $(2 + \|\Delta\|_{X_P^A})$ with respect to $d_{X_P^A}$ and similarly for $N_A(\langle b \rangle)$. It follows that the identity map from $(A, d_{X_{NP}^A})$ to $(A, d_{X_P^A})$ is $(2 + \|\Delta\|_{X_P^A})$-Lipschitz. This shows the first equivalence.

In order to prove the middle inequality, we note that the set of absorbable elements is finite with $4m - 8$ elements (see [6, Example 2.2] for $m = 3$). Letting $M = \sup\{\|x\|_{X_P^A}, x \in X_{abs}^A \setminus \langle \Delta^2 \rangle\}$, we see that the identity is an $M$-Lipschitz map from $(A, d_{X_{NP}^A})$ to $(A, d_{X_P^A})$, whence the middle inequality. This inequality is not an equivalence, because we have $d_{X_P^A}(Id, a^n) = 1$ for all $n$ but $d_{X_{abs}^A}(Id, a^n) \rightarrow_{n \rightarrow \infty} +\infty$.

For the second equivalence, we use again that there are only finitely many absorbable elements. It follows from Corollary 2.4 that $X_{abs}^A$ and $S \cup \langle \Delta^2 \rangle$ are equivalent hyperbolic structures on $A$. 

Next we prove that $X_A^A$ is a hyperbolic structure. The proof uses the “guessing geodesics lemma” [20, Theorem 3.15].

Given $g, h$ vertices of $X_A^A$, consider the subgraph $A_{gh}$ consisting of the projections to $\Gamma(A, X_A^A)$ of the set of geodesics in $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ between $g$ and $h$. Because the identity from $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ to $\Gamma(A, X_A^A)$ is Lipschitz and $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ is hyperbolic, the different subgraphs $A_{g, h}$ form thin triangles in $\Gamma(A, X_A^A)$ so they satisfy the second condition of [20, Theorem 3.15].

What happens if $g, h$ are 1 apart in $\Gamma(A, X_A^A)$? Observe first that if $g, h \in A$ satisfy $g^{-1}h = a^n$ (or $b^n$, with $n \in \mathbb{Z}$), then in $\Gamma(A, S \cup \langle \Delta^2 \rangle)$, there is a unique geodesic between $g$ and $h$, namely $g, ga, ga^2, \ldots, h$. Thus the subgraph $A_{gh}$ has diameter 1 as well; hence the first condition in [20, Theorem 3.15] is also satisfied.

We conclude that $\Gamma(A, X_A^A)$ is hyperbolic.

Finally, we have to prove that $\Gamma(A, X_A^A)$ has infinite diameter. Let us assume, for a contradiction, that there exists some number $L$ such that every element $x$ of $A$ can be written in the form

$$x = \Delta^{k_0}a^{k_1}b^{k_2}a^{k_3}b^{k_4}\ldots c^{k_L} \quad \text{or} \quad x = \Delta^{k_0}b^{k_1}a^{k_2}b^{k_3}a^{k_4}\ldots c^{k_L}$$

with $k_0, k_1, \ldots, k_L \in \mathbb{Z}$, and $c = a$ or $c = b$, according to the parity of $L$. Now, we recall from Proposition 2.3 that $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ is quasi-isometric to an infinite regular $2(m - 1)$-valent tree – in particular, as a metric space it has uncountably many ends. Also, powers of $a$ and powers of $b$ represent quasi-geodesics in this quasi-tree. However, the set of points in the tree represented by words as above has only countably many ends, leading to a contradiction.

For the rest of the section, we now assume that $A$ has rank at least 3.

**Proposition 4.9** Suppose that $A$ is an irreducible Artin-Tits group of spherical type with rank at least 3. Then $X_A^{A_{abs}} \preccurlyeq X_A^{A_{NP}} \preccurlyeq X_A^A$.

**Proof** For the second inequality, it is enough to observe that $X_A^A \subset X_A^{A_{NP}}$ whence the identity map $(A, d_{X_A^A}) \to (A, d_{X_A^{A_{NP}}})$ is 1-Lipschitz.

The first inequality was shown for braids in [6]: any braid in the stabilizer of a standard curve can be written as a product of 9 absorbable braids [6, Lemma 11]. This result has been generalized to the case of a general Artin-Tits group $A$ of spherical type in the forthcoming paper [2], where it is shown that any element which normalizes a proper irreducible standard parabolic subgroup of $A$ is a product of at most 9 absorbable elements. This is to say that the identity map from $(A, d_{X_A^{A_{NP}}})$ to $(A, d_{X_A^{A_{abs}}})$ is 9-Lipschitz.
Corollary 4.10  The metric spaces \((A, d_{X_{NP}})\) and \((A, d_{X_A})\) have infinite diameter.

Proof  This is just a combination of Propositions 4.9 and 4.6 together with [7, Theorem 1.1], which asserts that \(C_{AL}(A)\) has infinite diameter.

Conjecture 4.11  
(i) The inequality \(X_{abs}^A \preceq X_{NP}^A\) is not strict, i.e. the converse inequality \(X_{NP}^A \preceq X_{abs}^A\) also holds, and the identity map \(\Gamma(A, X_{NP}^A) \to \Gamma(A, X_{abs}^A)\) is a quasi-isometry. (In the case of Artin braid group, this is claiming that \(C_{AL}(B_n)\) is quasi-isometric to the curve graph of the punctured disk).

(ii) The inequality \(X_{NP}^A \preceq X_P^A\) is strict, i.e., the identity map \(\Gamma(A, X_P^A) \to \Gamma(A, X_{NP}^A)\) is Lipschitz but not a quasi-isometry.

Conjecture 4.11(i) is not even known to hold in the specific case of Artin’s braid groups – see [6, Conjecture 1]. This conjecture would of course imply that \(X_{NP}^A\) is a hyperbolic structure (and hence hyperbolicity of the graph of irreducible parabolic subgroups \(C_{parab}(A)\)). This would also imply that Garside normal forms are unparameterized quasi-geodesics in \(C_{parab}(A)\) (since they are in \(C_{AL}(A)\)); this would contrast with the recently announced example in [24] of a family of geodesics in a mapping class group whose shadows in the corresponding curve graph are not unparameterized quasi-geodesics.

By contrast, Conjecture 4.11(ii) does hold when \(A\) is a braid group with at least 4 strands: for \(n \geq 3\), the natural map \(A_\beta(D_{n+1}) \to C(D_{n+1})\) is not a quasi-isometry. To see this, we refer to [20, Section 5]. Consider the open disk \(D\) of radius 1 centered at \(\frac{3}{2}\) containing the first two punctures of \(D_{n+1}\) (see the beginning of Section 3.1); let \(X = D_{n+1} \setminus D\); this subsurface is a hole for \(A_\beta(D_{n+1})\) according to [20, Definition 5.2]. Note also that \(X\) is homeomorphic to \(D\). Now, let \(\beta\) be a pseudo-Anosov braid on \(n\) strands and consider the \(n+1\)-strand braid \(\hat{\beta}\) obtained from \(\beta\) by doubling the first strand. Fix \(c\) greater than the constant \(C_0\) relative to the complex \(A_\beta(D_{n+1})\) in [20, Theorem 5.14]. It is known [19, Proposition 4.6] that \(\beta\) acts in a loxodromic way on the curve graph of \(D\), hence it acts loxodromically on \(C(X)\), the curve graph of \(X\). In other words, we can find \(\alpha > 0\) so that

\[d_{C(X)}(1, \beta^k) \geq \alpha k,\]

for every integer \(k\), and in particular \(d_{C(X)}(1, \beta^k) \geq c\), for \(k\) big enough. Now, [20, Theorem 5.14] implies that there exists a constant \(A = A(c) \geq 1\) such that for \(k\) big enough, \(d_{A_\beta(D_{n+1})}(1, \beta^k) \geq \frac{\alpha k - A}{A}\), whence \(\hat{\beta}\) acts loxodromically on \(A_\beta(D_{n+1})\). However, as it preserves the round curve bounding \(D\), \(\hat{\beta}\) acts elliptically on \(C(D_{n+1})\).
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