DYNKIN OPERATORS, RENORMALIZATION AND THE GEOMETRIC $\beta$
FUNCTION

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Abstract. In this paper, I show a close connection between renormalization and a generalization of the Dynkin operator in terms of logarithmic derivations. The geometric $\beta$ function, which describes the dependence of a Quantum Field Theory on an energy scale defines is defined by a complete vector field on a Lie group $G$ defined by a QFT. It also defines a generalized Dynkin operator.

1. Introduction

The Dynkin operator has recently become an important object in the study of dynamical systems. The classical Dynkin operator defines a bijection from a Lie group to it’s Lie algebra, the inverse of the exponential map. It is key in the closed form expansion of the Baker-Campbell-Hausdorff formula. In [11], the authors generalize Dynkin operators in terms of logarithmic derivatives on a Lie algebra, and connect it to Magnus-type formulas. The classical Magnus formula provides a solution to the system of differential equations of the form

$$X'(t) = A(t)X(t).$$

Systems of this form appear in the study of renormalization of quantum field theories (QFTs). In [5], the authors define a $\beta$ function, a Lie algebra element representing how a dimensionally regularized QFT depends on the energy scale. The $\beta$ function for dimensional regularization and momentum cutoff regularization satisfies an equation of the form (1). In [6, 2], the authors show that this $\beta$ function defines a connection that also satisfies (1). In this note, I show that there is a much deeper connection between the Dynkin operator and renormalization.

As in the literature on the Hopf algebraic approach to renormalization, initiated by [4], consider a regularized perturbative Quantum Field Theory (QFT), $\phi$, as a map from Feynman diagrams to an algebra $\mathcal{A}$. The divergence structure of the Feynman diagrams is encoded in a Hopf algebra $\mathcal{H}$, as initially introduced by Connes and Kreimer in [4]. I wish to keep the discussion in the paper general, but for specific examples, one can consider the Hopf algebra structure on scalar field theories, developed in [4], on QED developed in [12], on gauge theories developed in [10]. The algebras in all these cases have been the algebra of formal Laurent series, $\mathcal{A} = \mathbb{C}[z^{-1}][z]$ [3]. However, if one is interested in momentum cut-off renormalization, $\mathcal{A} = \mathbb{C}[\log z, z^{-1}][z]$ is appropriate [3]. For a scalar field theory over a curved, compact Euclidean background, use $\mathcal{A} = \mathbb{D}'(\mathcal{M})[z^{-1}, z]$.

In section 2 I generalize the $\beta$ function defined in [5] [1] [3]. I generalize regularized Feynman rules as elements of an affine Lie group associated to a Hopf algebra, $\mathcal{H}$. The action of the renormalization scale generalizes to a flow on this group. Specifically, it defines an one parameter family of diffeomorphisms. The $\beta$ function defining the action of the renormalization scale action is the vector field of the flow pulled back to the Lie algebra. In section 3 I recall the Dynkin operator, $D : T(V) \to V$, a map from a tensor algebra to the underlying Lie algebra that defines a map from the Lie group $G = \exp(V)$ to $V$. In [9], the authors showed that the $\beta$ function of [5] can be written as a variation of this map on $G$. I generalize this map for the class of geometric $\beta$ function defined in section 2. In [11], the authors define a generalization of the classical Dynkin operator using logarithmic derivatives with regards to a Lie derivative. I show that the geometric $\beta$ function, as defined in [7], is compatible with the Dynkin variant defined in [9].

2. The perturbative $\beta$ function

The literature on renormalization theory is often confusing because of different nomenclature referring to slightly different things in different parts of the community. To avoid this confusion, I use this section to
set up a dictionary of what I mean when I use different terms commonly found in the physics literature, and what mathematical generalizations they correspond to. In this way, I motivate why the definition of a geometric $\beta$ function is the appropriate object of study.

**Definition 1.** The Hopf algebra of Feynman diagrams, $\mathcal{H}$ is a commutative Hopf algebra over a field $k$ of characteristic 0 associated to the Feynman diagrams for some QFT, as originally constructed in [4]. The Hopf algebra is constructed to encode the subdivergence structure of the Feynman integrals in a manner that is compatible with BPHZ renormalization.

Recall a few useful properties of a Hopf algebra of Feynman diagrams. The Hopf algebra $\mathcal{H}$ is generated by all 1PI graphs of the QFT. For more details on this Hopf algebra, see [4] [7] [2]. The coproduct of a graph $\Gamma \in \mathcal{H}$ is given by

$$\Delta(\Gamma) = \sum_{\gamma \subseteq \Gamma} \gamma / \gamma \otimes \Gamma / \gamma ,$$

where $\Gamma / \gamma$ is the obtained from $\Gamma$ by the contraction of the connected components of $\gamma$ to a point. This coproduct encodes the divergence structure found in BPHZ renormalization. Multiplication of graphs is given by disjoint union. The counit is written

$$\varepsilon(h) = \begin{cases} h & h \in \mathcal{H}_0 \\ 0 & \text{else.} \end{cases}$$

The Hopf algebra is graded by loop number, with the grading operator $Y(\Gamma) = n\Gamma$ if $\Gamma$ has $n$ loops. The antipode is defined recursively as

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subseteq \Gamma, \gamma / \gamma \in \mathcal{H}} S(\gamma) \Gamma / \gamma .$$

The coproduct structure on $\mathcal{H}$ induces a convolution product on the associated affine group scheme $G = \text{Spec} \ \mathcal{H}$. For a given $k$-algebra $A$, the Lie group $G(A) = \text{Hom}_{k \text{alg}}(\mathcal{H}, A)$. That is, for $g, g' \in G(A)$ and $\Gamma \in \mathcal{H}$,

$$g \ast g'(\Gamma) = (g \otimes g')(\Delta \Gamma) .$$

Note that $\mathcal{H} \simeq k[G]$, the ring of regular functions on $G$.

**Definition 2.** In this paper, the renormalization group is $G$. It is the group of evaluations of the Hopf algebra of the QFT $\mathcal{H}$.

Given a QFT, there are well established Feynman rules that assign a divergent integral to each Feynman diagram. Given a regularization scheme, the regularized Feynman rules assign to each diagram a integral that evaluates into some algebra $A$. This is a linear map. If $A$ is a $k$-algebra, the regularized Feynman rules define an algebra homomorphisms from $\mathcal{H}$ to $A$.

**Definition 3.** The elements of $G(A)$, with $G = \text{Spec} \ \mathcal{H}$ are the generalized regularized Feynman rules for a QFT.

Regularized Feynman rules can be written as elements of $G(A)$ for some appropriately defined $A$. These are the physical regularization theories. The general elements of the renormalization group $\phi \in G(A)$ need not have any physical interpretation at all.

**Definition 4.** The renormalization mass scale of a physical theory is represented by $\mathbb{R}_+$. It is the energy scale at which a physical theory is evaluated. In this paper, I follow the convention of [4] and complexify the energy scale, and write it $e^s \in \mathbb{C}^s$ for $s \in \mathbb{C}$.

The regularized Feynman integrals are functions of the renormalization mass scale.

**Definition 5.** The renormalization scale action describes the dependence of the generalized regularized theory on the renormalization mass scale.
Definition 7.

Let $\phi_{dr} \in G(A)$, the dimensionally regularized Feynman rules for an (integer) $d$-dimensional scalar QFT. Let $z$ be a complex parameter. For a given diagram $\Gamma$ with $I(\Gamma)$ internal edges and $L(\Gamma)$ loops,

$$\phi_{dr}(z)(\Gamma) = A(d + z)^{l_0} \prod_{k=1}^{I(\Gamma)} \frac{1}{p_k^2 + m^2} \prod_{i=1}^{L(\Gamma)} p_i^{d+z-1} dp_i .$$

Momentum cutoff regularization in the same theory gives

$$\varphi_{mc}(z)(\Gamma) = A \int_0^\infty \prod_{k=1}^{I(\Gamma)} \frac{1}{f_k(p_k, e_j)^2 + m^2} \prod_{i=1}^{L(\Gamma)} d^d p_i .$$

The action of the renormalization scale maps the momenta $p_i \to e^s p_i$ and thus

$$\phi_{dr}(z) \mapsto e^{sy} \phi_{dr}(z)$$

$$\phi_{mc}(z) \mapsto \phi_{dr}(e^s z) .$$

Dimensionally regularized Feynman rules are elements of the group $\phi_{dr}(z) \in G(\mathbb{C}[z^{-1}][[z]])$. Momentum cutoff Feynman rules are in $\phi_{mc} \in G(\mathbb{C}[z^{-1}, \log(z)][[z]])$. For more details on this renormalization scale action, see [7].

Definition 6.
The action of the renormalization scale on a physical regularized QFT, $\phi \in G(A)$ defines a one parameter path in $G(A)$. This is called the renormalization flow of $\phi$.

The action of the renormalization scale on a particular physical $\phi$ can be extended to an action of the renormalization scale on $G(A)$.

Definition 7. Let $\sigma$ be an action of $\mathbb{C}$ on $G(A)$

$$\sigma : \mathbb{C} \times G(A) \to G(A)$$

$$(s, \phi) \to \sigma(s)(\phi) .$$

In the examples above, I extend the dependence of dimensional regularization and momentum cutoff regularization to an generalized regularized theories as $\sigma_{dr}(s)(\phi) = e^{sy} \phi(z)$ and $\sigma_{mc}(s)(\phi) = \phi(e^s z)$.

For physical reasons, one expects the paths defined by the renormalization scale to be integral; they are related to the solutions of the renormalization group equations, which describe the dependence of the observables of the theory on the energy scale. To mimick this mathematically, I am interested in extensions of the renormalization flows of physical theories to an action on $G(A)$ such that for each $\phi \in G(A)$, the renormalization flow, $\sigma(s)\phi$ is an integral path in $G(A)$. In other words the renormalization group action on $G(A)$ defines a one parameter family of diffeomorphisms on $G(A)$.

Definition 8. An action $\sigma$ on $G(A)$ defines a renormalization group flow if it generates a one parameter family of diffeomorphisms on $G(A)$.

For the next theorem, let $A = \mathbb{C}[z^{-1}, \log(z)][[z]]$. Both $\phi_{dr}$ and $\phi_{mc}$ can be written as elements of $G(A)$.

Proposition 2.1. The actions $\sigma_{dr}$ and $\sigma_{mc}$ both define one parameter families of diffeomorphism on $G(A)$.

Proof. Let $* \in \{mc, dr\}$. Since $\sigma_*$ is an action on $G(A)$,

$$\sigma_*(s) \circ \sigma_*(u)(\phi(z)) = \sigma_*(s + u)\phi(z) .$$

The action $\sigma_{dr}$ induces an automorphism on $G(A)$ [8]

$$e^{sy}(\phi \ast \psi) = e^{sy} \phi \ast e^{sy} \psi .$$

Since the action is smooth, the result follows.

It is easy to check that $\sigma_{mc}$ is a smooth map. It remains to check that it is bijective. To see surjectivity, notice that for any fixed $s \in \mathbb{C}$ and any $\phi(z) \in G(A)$, one can define $\phi'(s)(z) = \phi(e^{-s} z)$, and

$$\phi(z) = \sigma_{mc}(s)\phi'(z) .$$

For injectivity, if there exists and $s \in \mathbb{C}$, and $\phi, \psi \in G(A)$, such that $\phi(sz)(\Gamma) = \psi(sz)(\Gamma)$ for every $\Gamma \in \mathcal{H}$, then $\phi(z)(\Gamma) = \psi(z)(\Gamma)$ for every $\Gamma \in \mathcal{H}$. This implies that $\phi(z) = \psi(z)$.

□
Definition 9. The physical $\beta$ function for a renormalized QFT calculates the dependence of the coupling constant on the renormalization scale

$$\beta(g) = \frac{1}{\mu} \frac{d\mu}{dg}.$$ 

The physical $\beta$ function is calculated perturbatively by loop number. In this Hopf algebraic picture of renormalization, a related object exists if the action $\sigma$ defines a renormalization group flow on $G(A)$.

**Theorem 2.2.** If $\sigma$ defines a renormalization group flow on $G(A)$, it defines a complete vector field $X_\sigma \in \mathfrak{x}(G(A))$.

**Proof.** By hypothesis, $\sigma$ defines a one parameter family of diffeomorphisms on $G(A)$. Then $\sigma(s)\phi$ is an integral curve in $G(A)$ defined for all $s \in \mathbb{C}$, with $\sigma(0)\phi = \phi$. Define a the vector field

$$X_\sigma(\sigma(s)\phi) = \frac{d}{ds}\sigma(s)\phi.$$ 

This is is complete. \(\Box\)

**Definition 10.** The geometric $\beta$ function for a renormalization group flow, $\sigma$, is defined

$$\beta_\sigma : G(A) \rightarrow \mathfrak{g}(A)$$

$$\phi \mapsto \phi^{-1} \star \frac{d}{ds}(\sigma(s)\phi)|_{s=0} =: \phi^{-1} \star X_\sigma(\phi).$$

To see that $\beta_\sigma(\phi) \in \mathfrak{g}(A)$ for all $\phi \in G(A)$, note that $\beta_\sigma(\phi)$ is formed by left translating the vector $X_\sigma(\phi) \in T_\phi G(A)$ to $T_\phi G(A) = \mathfrak{g}$.

**Remark 1.** In \cite{4}, the authors show that $z\beta_{\sigma_{dr}}(\phi) \in \mathfrak{g}(\mathbb{C})$, and is the generator of the one parameter subgroup of $G(A)$ defined $F_s(\phi) = \lim_{z \rightarrow 0} \phi^{-1} \star \sigma_{dr}(s)\phi$. This is a happy accidental property of dimensional regularization. It does not generalize to all regularization schemes or regularization group actions.

For more details on the geometric $\beta$ function, especially in the case of $\sigma_{mc}$ and $\sigma_{dr}$, see \cite{3}. In the next section, we related the geometric $\beta$ function to the Dynkin operator that appears in the study of dynamical systems.

### 3. Generalized Dynkin Operators and Geometric $\beta$ Functions

Let $S$ be a set and $k$ a field of characteristic 0. Let $V = k[\{S\}]$ be the vector space generated by this set. One can write $(V, [\cdot, \cdot])$ as a Lie algebra generated by $S$. The $T(V)$, the tensor algebra on $V$, is the universal enveloping algebra of $V$, $T(V) = \mathcal{U}(V)$. The classical (left) Dynkin operator $D$ is a map

$$D : T(V) \rightarrow (V, [\cdot, \cdot])$$

$$x_1 \otimes \cdots \otimes x_n \rightarrow [x_1, [\cdots, [x_{n-1}, x_n] \cdots]].$$

Since $T(V) \simeq \mathcal{U}(V)$, $T(V)$ is isomorphic to a graded cocommutative Hopf algebra. Let $Y$ be the grading operator. The elements of $S$ are primitive, which defines comultiplication. Multiplication is defined by concatenation. The antipode is defined

$$S(x_1 \otimes \cdots \otimes x_n) = (-1)^n x_n \otimes \cdots \otimes x_1.$$ 

Under this change of notation, the Dynkin operator $D = S \ast Y$ \cite{13}

$$S \ast Y : \mathcal{U}(V) \rightarrow (V, [\cdot, \cdot]).$$

The grading operator $Y$ is a derivation on $\mathcal{U}(V)$. Let $G = \exp(V)$. The Baker-Campbell-Hausdorff (BCH) formula provides an inverse map from $G \rightarrow V$. The Dynkin operator defines a closed form for the BCH formula \cite{9}

$$\log(\exp X \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{r_1+s_1 > 0} \cdots \sum_{r_n+s_n > 0} \frac{(\sum_{i=1}^n (r_i + s_i))^{-1}}{r_1! s_1! \cdots r_n! s_n!} D(X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \cdots X^{r_n} Y^{s_n}).$$

In fact, the Dynkin operator, $D$, defines a bijection from $G$ to $V$. I call this the Dynkin map.
In [9], the authors show that given any derivation $\delta$ on a graded commutative Hopf algebra, $\mathcal{H}$ the map $D_\delta = S \ast \delta$ defines a bijection between $G(A) = \text{Hom}_k \text{alg}(\mathcal{H}, A)$ and $g(A) = \text{Lie}(G(A))$, by defining
\[
D_\delta(\phi)(\mathcal{H}) := \phi(S \ast \delta)(\Delta(h)) .
\]
It is easy to check that the grading operator $Y$ is a derivation on $\mathcal{H}$. Using the notation established in this paper, they show that $z\beta_{\sigma_\delta}$ is a derivation $\text{def}$.

**Theorem 3.1.** The geometric $\beta$ function, $\beta_\sigma$ is a generalized Dynkin map, $D_{X_\sigma}$ from $G(A)$ to $g(A)$
\[
\beta_\sigma(\phi) = \phi^{-1} \ast X_\sigma(\phi) = \phi \circ D_{X_\sigma} .
\]

**Proof.** The map $\beta_\sigma$ is defined by the vector field $X_\sigma$ on $G(A)$. Vector fields on a Lie group define derivations on the algebra of regular functions on that group. Since $G = \text{Spec} \mathcal{H}$, the algebra of regular functions $k[G] \simeq \mathcal{H}$. Therefore $X_\sigma$ defines a derivation on $\mathcal{H}$, call it $\delta_\sigma$. Specifically, for $h \in k[G]$,
\[
X_\sigma(\phi) \leftrightarrow \delta_\sigma(h)(\phi) := \frac{d}{ds} h(\sigma_s(\phi))|_{s=0} .
\]
Recall that the product on $k[G]$ is defined pointwise
\[
hh'(\phi) = h(\phi)h'(\phi) .
\]
It is easy to check that $\delta_\sigma$ is a derivation
\[
\delta_\sigma(hh')(\phi) = \frac{d}{ds} (hh'(\sigma_s(\phi)))|_{s=0} = \frac{d}{ds} (h(\sigma_s(\phi))h'(\sigma_s(\phi))) = \frac{d}{ds} (h(\sigma_s(\phi)))|_{s=0}h'(\phi) + h(\phi)\frac{d}{ds} (h(\sigma_s(\phi)))|_{s=0} = (\delta_\sigma(h)h')(\phi) + (h\delta_\sigma(h'))(\phi) .
\]
The first equality is from the definition of $\delta_\sigma$, the second from the definition of $k[G]$. Under this set of definitions,
\[
\beta_\sigma(\phi)(h) = \phi^{-1} \ast X_\sigma(\phi)(h) = \phi \circ (S \ast \delta_\sigma)(\Delta h) .
\]
In other words, $\beta_\sigma = \phi \circ D_{X_\sigma}$. \hfill $\square$

**Remark 2.** Note that this implies that the geometric $\beta$ function $\beta_\sigma$ defines a set bijection from $G(A)$ to $g(A)$.

**Corollary 3.2.** The geometric $\beta$ function
\[
\beta_\sigma : G(A) \rightarrow g(A) ,
\]
is defined by the Maurer-Cartan connection on the Lie group $G(A)$ contracted with $X_\sigma$.

**Proof.** The Maurer-Cartan connection is a $g(A)$ valued one form defined
\[
\theta : \phi^{-1} \ast d\phi
\]
for $\phi \in G(A)$. Contracting with a vector field, $X_\sigma$
\[
(X_\sigma(\phi), \theta) = \phi^{-1} \ast X_\sigma(\phi) = \beta_\sigma(\phi) .
\]

In [11], the authors define a generalization of the classical Dynkin operator, $D_\delta = S \ast \delta$ that is defined by a Lie derivation $\delta$ on a free Lie algebra $g$
\[
D_\delta : \mathcal{U}(g) \rightarrow g .
\]
It is a Lie idempotent in the sense that if $x \in g$, then $D_\delta(x) = \delta(x)$.

In the context of renormalization, the Hopf algebra of Feynman diagrams $\mathcal{H}$, is of finite type. The Lie algebra $g = \text{Lie}(G)$ is freely generated, and the graded dual, $\mathcal{H}^\vee \simeq \mathcal{U}(g)$. Vector fields on $G$ exactly define Lie derivatives on $g$. This gives the following theorem.
Theorem 3.3. The renormalization group flow defining action $\sigma$ defines a generalized Dynkin operator in the sense of [11].

Proof. If the action $\sigma$ defines a renormalization group flow on $G(A)$, then it defines a one parameter family of diffeomorphisms on $G(A)$, and thus a complete vector field $X_\sigma \in \mathfrak{X}(G(A))$. The derivative $\delta_\sigma$ on $H$ is exactly the Lie derivative on $g(A)$ defined by $X_\sigma$.

The action $\sigma$ induces a path through $g(A)$ defined by the map $\beta_\sigma$. Since $\beta_\sigma$ is a bijection from $G(A)$ from $g(A)$, for any $\alpha \in g(A)$, one can find a $\phi \in G(A)$ such that $\alpha = \beta_\sigma(\phi)$. The action of $\sigma$ on $G(A)$ lifts to an action on $g(A)$ as

$$\sigma(s)(\alpha) = \sigma(s)(\phi) \frac{d}{ds}(\sigma(s)(\phi)) .$$

The Lie derivative $\delta_\sigma$ gives

$$\delta_\sigma(\alpha)(h) = \alpha(\delta_\sigma(h)) = \frac{d}{ds}\sigma(s)(\alpha)(h) .$$

Let $\gamma \in g(A)$. Writing $\Delta(h) = \sum(h) h_{(1)} \otimes h_{(2)}$,

$$\delta_\sigma(\alpha \star \gamma)(h) = \frac{d}{ds}(\sum(h) \sigma(s)(\alpha)(h_{(1)}) \sigma(s)(\gamma)(h_{(2)}))$$

$$= \frac{d}{ds}(\sigma(s)(\alpha)) \star \sigma(s)(\gamma)(h) + \sigma(s)(\alpha) \frac{d}{ds}(\sigma(s)(\gamma))(h) = \delta_\sigma(\alpha) \star \gamma(h) + \alpha \delta_\sigma(\gamma)(h) .$$

To summarize, I relate the generalizations of the Dynkin map defined in [9] and the generalized Dynkin map defined in [11].

Theorem 3.4. Each action $\sigma$ on $G(A)$ that defines a one parameter family of diffeomorphism on $G(A)$ and thus induces the vector field $X_\sigma$, defines a generalized Dynkin operator

$$D_{X_\sigma} : U(g) \rightarrow g .$$

The associated geometric $\beta$ function $\beta_\sigma$ defines a generalized Dynkin map defined by the Maurer-Cartan connection, $\theta$,

$$\beta_\sigma : G(A) \rightarrow g(A)$$

$$\phi \rightarrow <X_\sigma(\phi), \theta> .$$

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