PHOTON DISTRIBUTION FUNCTION FOR STOCKS WAVE FOR STIMULATTED RAMAN SCATTERING

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Abstract

New time-dependent integrals of motion are found for stimulated Raman scattering. Explicit formula for the photon-number probability distribution as a function of the laser-field intensity and the medium parameters is obtained in terms of Hermite polynomials of two variables.

1 Introduction

Since the discovery of stimulated Raman scattering in 1962 [1] this phenomenon has been intensively investigated both theoretically and experimentally [2–7]. Quantum-mechanical description of stimulated Raman scattering can be done in the framework of different equations, namely, Heisenberg–Langeven equation [1], Maxwell–Bloch equation [8], and Fokker–Planck equation [9]. Quantum-statistical properties of stimulated Raman scattering were treated using a quadratic Hamiltonian [10]. For studying properties of stimulated Raman scattering with taking into account intermolecular interaction, a cubic Hamiltonian was used [11].

Nonclassical properties of stimulated Raman scattering such as squeezing and sub-Poissonian statistics were considered in a number of papers [12–16]. The purpose of our work is to study the photon distribution function for the Stocks wave in the framework of the method of linear integrals of motion for systems with quadratic Hamiltonians [17–19] using the results of [20–26]. General formulas for matrix elements of the Gaussian density operator for the multimode oscillator in the Fock basis were calculated explicitly in [20]. For one-mode light described by the Wigner function of a generic Gaussian form with five real parameters describing the quadrature means, variances, and covariance, the photon distribution function was obtained explicitly in terms of Hermite polynomials of two variables [21]. In [22], the temperature dependence of oscillations of the photon distribution function for squeezed states was investigated. It was shown that oscillations of the photon distribution function for squeezed and correlated light were decreasing if the temperature increased.

The photon distribution function for $N$-mode mixed state of light described by the Wigner function of the generic Gaussian form was calculated explicitly in terms of Hermite polynomials...
of $2N$ variables in [23, 24], and parameters of the photon distribution function were determined through the dispersion matrix and mean values of quadrature components of the light. The photon distribution for two-mode squeezed vacuum was investigated in [25] where its dependence on four parameters (two squeezing parameters, the relative phase between the two oscillators, and their spatial orientations) was shown. In [26], the case of generic two-mode squeezed coherent states was considered, and the photon distribution function for the states was expressed both through four-variable and two-variable Hermite polynomials dependent on two squeezing parameters, the relative phase between the two oscillators, their spatial orientation, and four-dimensional shift in the phase space of the electromagnetic-field oscillator.

As an application of the theoretical consideration, the linear optical transformer of photon statistics for multimode light was suggested [27]. The transformation coefficient was obtained explicitly in terms of multivariable Hermite polynomials.

In this paper, we find new integrals of motion for the process of stimulated Raman scattering. We describe stimulated Raman scattering with the help of a simple model of a two-dimensional oscillator, the photons of the Stocks mode being described by the one mode of the oscillator and the phonons of the medium being described by the another mode of the oscillator. The interaction of the photons and phonons is taken to be quadratic in creation and annihilation operators of the photons and phonons. We obtain the photon–phonon probability distribution function for the Stocks and phonon modes after the interaction of the laser field with the medium in terms of Hermite polynomials of four variables. The photon distribution function for the Stocks mode is found explicitly and expressed both in terms of Hermite polynomials of two variables with zero arguments and in terms of Legendre polynomials. The mean photon number and its dispersion in the Stocks mode are expressed as functions of the medium parameter (temperature), the laser frequency, and a parameter of interaction (coupling constant).

2 Integrals of Motion

The simplest phenomenological Hamiltonian, which can be used for the description of one-mode Stocks-wave excitation, can be written [4, 28, 29]

$$\hat{H} = \hbar \omega_S \hat{a}^\dagger \hat{a} + \hbar \omega_{31} \hat{b}^\dagger \hat{b} + \hbar \kappa \left[ e^{-i \omega_L t} \hat{a}^\dagger \hat{b}^\dagger + e^{i \omega_L t} \hat{b} \hat{a} \right],$$

(1)

where $\hat{a}$ and $\omega_S$ are the annihilation operator and the frequency of the Stocks photon, $\hat{b}$ and $\omega_{31}$ are the annihilation operator and the frequency of the phonon, $\omega_L$ is the laser frequency, and $\kappa$ is the coupling constant. The laser field is considered as classical one and its frequency is determined by the condition

$$\omega_L = \omega_{31} + \omega_S.$$  

(2)

The damping and depletion of the laser light wave are neglected. Antistocks-mode excitation is also neglected and excitation of only one phonon mode is taken into consideration.

We will show that there exist time-dependent integrals of motion for the model of Stocks-wave excitation. Let us construct four nonhermitian operators

$$\hat{a}(t) = \hat{a} e^{i \omega_{31} t} \cosh \kappa t + i \hat{b}^\dagger e^{-i \omega_{31} t} \sinh \kappa t;$$
\[ \hat{a}^\dagger(t) = \hat{a}^\dagger e^{-i\omega_S t} \cosh \kappa t - i\hat{b}e^{i\omega_S t} \sinh \kappa t; \]
\[ \hat{b}(t) = \hat{b}e^{i\omega_S t} \cosh \kappa t + i\hat{a}^\dagger e^{-i\omega_S t} \sinh \kappa t; \]
\[ \hat{b}^\dagger(t) = \hat{b}^\dagger e^{-i\omega_S t} \cosh \kappa t - i\hat{a}e^{i\omega_S t} \sinh \kappa t. \]

If one introduces two 4-vectors \( \mathbf{A} \) and \( \mathbf{A}(t) \) and uses for the 4-vectors the notation
\[ \mathbf{A} = (\hat{a}, \hat{b}, \hat{a}^\dagger, \hat{b}^\dagger); \]
\[ \mathbf{A}(t) = (\hat{a}(t), \hat{b}(t), \hat{a}^\dagger(t), \hat{b}^\dagger(t)); \]
the above relations (3) may be represented in the matrix form
\[ \mathbf{A}(t) = M(t)\mathbf{A}. \]

Here the \( 4 \times 4 \)-matrix \( M(t) \) is
\[ M(t) = \begin{pmatrix} e^{i\omega_S t} \cosh \kappa t & 0 & 0 & ie^{-i\omega_S t} \sinh \kappa t \\ 0 & e^{i\omega_S t} \cosh \kappa t & ie^{-i\omega_S t} \sinh \kappa t & 0 \\ 0 & -ie^{i\omega_S t} \sinh \kappa t & e^{-i\omega_S t} \cosh \kappa t & 0 \\ -ie^{i\omega_S t} \sinh \kappa t & 0 & 0 & e^{-i\omega_S t} \cosh \kappa t \end{pmatrix}. \]

One can see that two components of the vector \( \mathbf{A} \), which are \( \hat{a}, \hat{b}^\dagger \) operators, are transformed independently of the other two components.

In view of the commutation relations between the photon creation and annihilation operators \( \hat{a}, \hat{a}^\dagger \) and the phonon creation and annihilation operators \( \hat{b}, \hat{b}^\dagger \), one can check that the operators constructed above satisfy boson commutation relations
\[ [\hat{a}(t), \hat{a}^\dagger(t)] = 1; \quad [\hat{b}(t), \hat{b}^\dagger(t)] = 1, \]
and the operators \( \hat{a}(t), \hat{b}(t) \), and their hermitian conjugates commute as
\[ [\hat{a}(t), \hat{b}(t)] = 0; \quad [\hat{a}(t), \hat{b}^\dagger(t)] = 0; \]
\[ [\hat{a}^\dagger(t), \hat{b}(t)] = 0; \quad [\hat{a}^\dagger(t), \hat{b}^\dagger(t)] = 0. \]

It can be shown, that the total time derivatives
\[ \frac{d\hat{a}(t)}{dt} = \frac{\partial \hat{a}(t)}{\partial t} + \frac{i}{\hbar} [\hat{H}, \hat{a}(t)]; \]
\[ \frac{d\hat{b}(t)}{dt} = \frac{\partial \hat{b}(t)}{\partial t} + \frac{i}{\hbar} [\hat{H}, \hat{b}(t)] \]
of the operators (3) are equal to zero, i.e.,
\[ \frac{d\hat{a}(t)}{dt} = 0; \quad \frac{d\hat{a}^\dagger(t)}{dt} = 0; \]
\[ \frac{d\hat{b}(t)}{dt} = 0; \quad \frac{d\hat{b}^\dagger(t)}{dt} = 0. \]
Consequently, the time-dependent operators \( \hat{a}(t) \), \( \hat{a}^\dagger(t) \), \( \hat{b}(t) \), and \( \hat{b}^\dagger(t) \) considered in the Schrödinger representation are the integrals of motion (linear with respect to the photon and phonon creation and annihilation operators) for Stocks-mode excitation in the framework of the model with the Hamiltonian \((\text{I})\). The operators \((\text{III})\) are equal to standard photon and phonon creation and annihilation operators at the initial time moment and their commutators are time-independent at all time moments.

Let us introduce quadrature components of photon and phonon creation and annihilation operators

\[
\hat{p}_a = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}; \quad \hat{q}_a = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}; \\
\hat{p}_b = \frac{\hat{b} - \hat{b}^\dagger}{i\sqrt{2}}; \quad \hat{q}_b = \frac{\hat{b} + \hat{b}^\dagger}{\sqrt{2}}.
\]

For Stocks-mode excitation, one can write four additional integrals of motion using the properties of the integrals of motion \((\text{I})\), \((\text{III})\)

\[
\hat{p}_a(t) = \hat{p}_a \cosh \kappa t \cos \omega_S t + \hat{q}_a \cosh \kappa t \sin \omega_S t \\
- \hat{p}_a \sinh \kappa t \sin \omega_{31} t + \hat{q}_a \sinh \kappa t \cos \omega_{31} t; \\
\hat{q}_a(t) = -\hat{p}_a \cosh \kappa t \cos \omega_S t + \hat{q}_a \cosh \kappa t \cos \omega_S t \\
- \hat{p}_a \sinh \kappa t \cos \omega_{31} t + \hat{q}_a \sinh \kappa t \sin \omega_{31} t; \\
\hat{p}_b(t) = -\hat{p}_b \sinh \kappa t \sin \omega_S t + \hat{q}_b \sinh \kappa t \cos \omega_S t \\
+ \hat{p}_b \cosh \kappa t \cos \omega_{31} t + \hat{q}_b \cosh \kappa t \sin \omega_{31} t; \\
\hat{q}_b(t) = -\hat{p}_b \sinh \kappa t \cos \omega_S t + \hat{q}_b \sinh \kappa t \sin \omega_S t \\
- \hat{p}_b \cosh \kappa t \cos \omega_{31} t + \hat{q}_b \cosh \kappa t \cos \omega_{31} t.
\]

The physical meaning of the invariants \((\text{III})\) is that their eigenvalues determine the initial values of classical quadrature components in the phase space of mean values \(\langle \hat{p}_a \rangle\), \(\langle \hat{p}_b \rangle\), \(\langle \hat{q}_a \rangle\), and \(\langle \hat{q}_b \rangle\).

The number of photons does not conserve in the process of stimulated Raman scattering. But since any function of integrals of motion is the integral of motion \((\text{I})\), \((\text{III})\), one can find some time-dependent combinations of the photon and phonon numbers, which are integrals of motion. Thus, the observable

\[
N_a(t) = \hat{a}^\dagger(t) \hat{a}(t) \\
= \hat{a}^\dagger \hat{a} \cosh^2 \kappa t + \left( \hat{b}^\dagger \hat{b} + 1 \right) \sinh^2 \kappa t \\
+ \frac{i}{2} \left\{ \hat{a}^\dagger \hat{b} \exp \left[ -it \left( \omega_S + \omega_{31} \right) \right] - \hat{a} \hat{b} \exp \left[ it \left( \omega_S + \omega_{31} \right) \right] \right\} \sinh 2\kappa t
\]

is the integral of motion, which has the physical meaning of the initial number of photons in the system state. The observable

\[
N_b(t) = \hat{b}^\dagger(t) \hat{b}(t) \\
= \hat{b}^\dagger \hat{b} \cosh^2 \kappa t + \left( \hat{a}^\dagger \hat{a} + 1 \right) \sinh^2 \kappa t \\
+ \frac{i}{2} \left\{ \hat{a}^\dagger \hat{b} \exp \left[ -it \left( \omega_S + \omega_{31} \right) \right] - \hat{a} \hat{b} \exp \left[ it \left( \omega_S + \omega_{31} \right) \right] \right\} \sinh 2\kappa t
\]
is the integral of motion, which has the physical meaning of the initial number of phonons in the system state. The difference of the two integrals of motion

\[ N_a(t) - N_b(t) = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \]

is the time-independent integral of motion, which has the physical meaning of the difference of photon and phonon numbers in the system, which is constant of the motion for the phenomenological Hamiltonian \( \hat{H} \). Thus, for stimulated Raman scattering we have found new integrals of motion.

3 Photon–Phonon Probability Distribution Function

In this section, we obtain explicit expression for photon distribution function of the Stocks mode. Let us introduce the vector column constructed from quadrature components of photon and phonon creation and annihilation operators in the medium at the initial time moment

\[ \hat{Q} = (\hat{p}_a, \hat{p}_b, \hat{q}_a, \hat{q}_b) \]

and the vector column, constructed from the integrals of motion

\[ \hat{I}(t) = (\hat{p}_a(t), \hat{p}_b(t), \hat{q}_a(t), \hat{q}_b(t)) . \]

Then the relation between the integrals of motion (5) and the initial quadrature components is of the form

\[ \hat{I}(t) = \Lambda(t) \hat{Q}, \]

where the real symplectic matrix \( \Lambda(t) \) is determined by the equation

\[ \Lambda(t) = \begin{pmatrix} \cosh kt \cos \omega St & - \sinh kt \sin \omega_{13} t & \cosh kt \sin \omega St & \sinh kt \cos \omega_{13} t \\ - \sinh kt \sin \omega St & \cosh kt \cos \omega_{13} t & \sinh kt \cos \omega St & \cosh kt \sin \omega_{13} t \\ - \cosh kt \cos \omega St & - \sinh kt \cos \omega_{13} t & \cosh kt \cos \omega St & \sinh kt \sin \omega_{13} t \\ - \sinh kt \cos \omega St & - \cosh kt \cos \omega_{13} t & - \sinh kt \sin \omega St & \cosh kt \cos \omega_{13} t \end{pmatrix} . \]

Let us introduce the dispersion matrix of quadrature components

\[ \sigma(0) = \begin{pmatrix} \sigma_{p_a p_b} & \sigma_{p_a q_a} & \sigma_{p_a q_b} \\ \sigma_{p_b p_a} & \sigma_{p_b q_a} & \sigma_{p_b q_b} \\ \sigma_{q_a p_a} & \sigma_{q_a p_b} & \sigma_{q_a q_b} \\ \sigma_{q_b p_a} & \sigma_{q_b p_b} & \sigma_{q_b q_b} \end{pmatrix} , \]

where the matrix elements are determined through the density matrix as follows

\[ \sigma_{p_i p_j} = \operatorname{Tr} \hat{\rho} \hat{p}_i \hat{p}_j - \langle \hat{p}_i \rangle \langle \hat{p}_j \rangle ; \]
\[ \sigma_{q_i q_j} = \operatorname{Tr} \hat{\rho} \hat{q}_i \hat{q}_j - \langle \hat{q}_i \rangle \langle \hat{q}_j \rangle ; \]
\[ \sigma_{p_i q_j} = \frac{1}{2} \operatorname{Tr} \hat{\rho} (\hat{q}_j \hat{p}_i + \hat{p}_i \hat{q}_j) - \langle \hat{p}_i \rangle \langle \hat{q}_j \rangle . \]
(indices \(i\) and \(j\) can be equal to \(a\) and \(b\)). The dispersion matrix at the initial time moment \(t\) can be expressed through the initial dispersion matrix of quadrature components of medium photons and phonons in the form of matrix equation

\[
\sigma(t) = \Lambda^{-1}(0)\Sigma\Lambda^T\Sigma,
\]

where the \(4\times4\)-block matrix \(\Sigma\) consists of \(2\times2\)-zero matrices and unity matrices \(I_2\),

\[
\Sigma = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.
\]

If the medium photons are in the ground state and phonons are in the state of thermodynamical equilibrium with temperature \(T\) at the initial time moment, then

\[
\sigma_{p_2}(0) = \frac{1}{2}; \quad \sigma_{q_2}(0) = \frac{1}{2}; \quad \sigma_{p_b}(0) = \frac{1}{2}\coth\frac{\beta}{2}; \quad \sigma_{q_b}(0) = \frac{1}{2}\coth\frac{\beta}{2}; \quad \beta = \frac{\hbar \omega_3}{T},
\]

and the matrix elements of the matrix \(\sigma(t)\) can be found in the explicit form

\[
\sigma_{p_2}(t) = \frac{1}{2} \left( \cosh^2 \kappa t + \coth \frac{\beta}{2} \sinh^2 \kappa t \right);
\]
\[
\sigma_{q_2}(t) = \frac{1}{2} \left( \sinh^2 \kappa t + \coth \frac{\beta}{2} \cosh^2 \kappa t \right);
\]
\[
\sigma_{p_2p_b}(t) = \frac{1}{4} \left( 1 + \coth \frac{\beta}{2} \right) \sinh 2\kappa t \sin \omega_S t;
\]
\[
\sigma_{q_2}(t) = \cosh^2 \kappa t \cos \omega_S t + \frac{1}{2} \sinh^2 \kappa t \coth \frac{\beta}{2};
\]
\[
\sigma_{q_b}(t) = \frac{1}{2} \sinh^2 \kappa t + \cosh^2 \kappa t \cos^2 \omega_{13} t \coth \frac{\beta}{2};
\]
\[
\sigma_{q_2q_b}(t) = \frac{1}{4} \sinh 2\kappa t \left[ \cos \omega_S t \left( \cos \omega_{13} t - \sin \omega_{13} t \right) + \coth \frac{\beta}{2} \cos \omega_{13} t \left( \cos \omega_S t - \sin \omega_S t \right) \right];
\]
\[
\sigma_{p_2q_b}(t) = \frac{\sinh^2 \kappa t}{2} \sin 2\omega_S t \coth \frac{\beta}{2} + \frac{\cosh^2 \kappa t}{2} \cos \omega_S t \left( \cos \omega_S t - \sin \omega_S t \right);
\]
\[
\sigma_{p_2q_b}(t) = \frac{\sinh^2 \kappa t}{2} \sin 2\omega_{13} t + \frac{\cosh^2 \kappa t}{2} \coth \frac{\beta}{2} \cos \omega_{13} t \left( \cos \omega_{13} t - \sin \omega_{13} t \right);
\]
\[
\sigma_{p_2q_b}(t) = \frac{1}{4} \sinh 2\kappa t \left( \cos (\omega_S - \omega_{13}) t + \cos \omega_{13} t \coth \frac{\beta}{2} \left( \sin \omega_S t - \cos \omega_S t \right) \right);
\]
\[
\sigma_{p_2q_b}(t) = \frac{1}{4} \sinh 2\kappa t \left( \cos (\omega_S - \omega_{13}) t \coth \frac{\beta}{2} + \cos \omega_S t \left( \sin \omega_{13} t - \cos \omega_{13} t \right) \right).
\]

One can see that dispersions of the photon and phonon quadratures become larger in the process of stimulated Raman scattering. Being initially noncorrelated the quadratures became statistically-dependent observables, since the Hamiltonian (11) is quadratic.
After interacting with the laser field, the state of the system can be described by the Wigner function of the Gaussian type

$$W(Q) = \frac{1}{\sqrt{\det \sigma(t)}} \exp \left( -\frac{1}{2} Q \sigma^{-1}(t) Q \right), \quad (9)$$

where matrix $\sigma(t)$ is determined by formulas (8). We can express the inverse matrix $\sigma^{-1}(t)$ at the time moment $t$ through the initial inverse matrix $\sigma^{-1}(0)$ using (7) and the known property of symplectic matrices, namely,

$$\sigma^{-1}(t) = \Lambda^T \sigma^{-1}(0) \Lambda. \quad (10)$$

The matrix elements of the inverse dispersion matrix $\sigma^{-1}(t)$ (11) have the explicit form

\[
\begin{align*}
\sigma_{p_a p_a}^{-1}(t) &= 4 \cosh^2 \kappa t \cos^2 \omega_S t + 2 \tanh \beta \sinh^2 \kappa t; \\
\sigma_{p_b p_b}^{-1}(t) &= 4 \tanh \frac{\beta}{2} \cosh^2 \kappa t \cos^2 \omega_{13} t + 2 \sinh^2 \kappa t; \\
\sigma_{p_a p_b}^{-1}(t) &= \sinh 2 \kappa t \left[ \cos \omega_S t \left( \cos \omega_{13} t - \sin \omega_{13} t \right) + \cos \omega_{13} t \tanh \frac{\beta}{2} \left( \cos \omega_S t - \sin \omega_S t \right) \right]; \\
\sigma_{p_a q_a}^{-1}(t) &= 2 \cosh^2 \kappa t \cos \omega_S t \left( \sin \omega_S t - \cos \omega_S t \right) - 2 \tanh \frac{\beta}{2} \sinh^2 \kappa t \sin 2 \omega_S t; \\
\sigma_{p_a q_b}^{-1}(t) &= \sinh 2 \kappa t \left[ \cos \omega_S t \cos \omega_{13} t \left( 1 - \tanh \frac{\beta}{2} \right) - \sin \omega_{13} \left( \tanh \frac{\beta}{2} \sin \omega_S t + \cos \omega_S t \right) \right]; \\
\sigma_{p_b q_a}^{-1}(t) &= \sinh 2 \kappa t \left[ \cos \omega_{13} t \cos \omega_S t \left( \tanh \frac{\beta}{2} - 1 \right) - \sin \omega_S t \left( \sin \omega_{13} t + \tanh \frac{\beta}{2} \cos \omega_{13} t \right) \right]; \\
\sigma_{p_b q_b}^{-1}(t) &= -2 \sinh^2 \kappa t \sin 2 \omega_{13} t + \tanh \frac{\beta}{2} \cosh^2 \kappa t \left( \sin 2 \omega_{13} t - 2 \cos^2 \omega_{13} t \right); \\
\sigma_{q_a q_a}^{-1}(t) &= 2 \left( \cosh^2 \kappa t + \tanh \frac{\beta}{2} \sinh^2 \kappa t \right); \\
\sigma_{q_a q_b}^{-1}(t) &= 2 \left( \sinh^2 \kappa t + \tanh \frac{\beta}{2} \cosh^2 \kappa t \right); \\
\sigma_{q_b q_b}^{-1}(t) &= \sinh 2 \kappa t \sin \left( \omega_S + \omega_{13} \right) \left( 1 + \tanh \frac{\beta}{2} \right).
\end{align*}
\]

We determine the photon–phonon probability distribution function $P_{nm}$ as the probability to obtain $n$ photons in the Stocks mode and $m$ phonons in the phonon mode after the interaction of the laser field with the medium. The function $P_{nm}$ is the matrix element of the density matrix of the system in the Fock basis

$$P_{nm} = \langle n, m | \hat{\rho} | n, m \rangle,$$

where $| n, m \rangle$ is eigenstate of the set of the photon and phonon number operators $\hat{a}^\dagger \hat{a}$ and $\hat{b}^\dagger \hat{b}$

\[
\begin{align*}
\hat{a}^\dagger \hat{a} | n, m \rangle &= n | n, m \rangle; \\
\hat{b}^\dagger \hat{b} | n, m \rangle &= m | n, m \rangle.
\end{align*}
\]
Using the scheme of calculations developed in [20–26] for multimode coupled oscillators we arrived at the expression for the photon–phonon probability distribution function $P_{nm}$ in terms of Hermite polynomials of four variables with zero arguments

$$P_{nm} = \frac{H_{n\min}^{R}(0, 0, 0)}{[\det (\sigma(t) + I_4/2)]^{1/2} n! m!},$$  \hspace{1cm} (12)

where the matrix $R$ is expressed through the dispersion matrix at the time moment $t$

$$R = U^\dagger (I_4 - 2\sigma(t)) (I_4 + 2\sigma(t))^{-1} U^*,$$

the matrix $U$ is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & i & 0 \\ 0 & -i & 0 & i \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

and the matrix $I_4$ is four-dimensional unity matrix.

4 Photon Probability Distribution Function

In experiments, the photon number in the Stocks mode is usually measured. So, it is interesting to average the photon–phonon probability distribution function over the phonon mode and to obtain the probability to have $n$ photons in the Stocks mode.

The photon–phonon probability distribution function can be described by the Wigner function \([9]\). One has to integrate the Wigner function over the variables $q_b$, $p_b$ in order to obtain the Wigner function (averaged over the phonon mode) describing the photon state

$$W_{\text{ph}}(q_a, p_a) = \frac{1}{2\pi} \int \int_{-\infty}^{\infty} W(Q) dq_b dp_b$$

$$= \frac{1}{2\pi \sqrt{\det \sigma(t)}} \int \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} Q\sigma^{-1}(t)Q\right) dq_b dp_b, \hspace{1cm} (13)$$

where $\sigma^{-1}(t)$ is determined by \([10]\). The Wigner function $W_{\text{ph}}(q_a, p_a)$ \([13]\) describes the photon mode.

It is convinient to change the places of quadrature components. For this purpose, we introduce a vector

$$X = PQ,$$

where the matrix $P$ is of the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, in view of \([10]\), for the argument of exponential function in \([13]\) one has the equality

$$Q\sigma^{-1}(t)Q = XPA^T\sigma^{-1}(0)\Lambda PX.$$
By introducing the block matrix

$$A = \frac{1}{2} PA^T \sigma^{-1}(0) \Lambda P = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the integral in (13) can be rewritten in the form

$$\int \int_{-\infty}^{\infty} \exp (-XAX) \, dy, \quad y = (p_b, q_b).$$

One can easily see that the integral in (13) has the form of the Gaussian integral calculated in Appendix A. We obtain, that the Wigner function of the photon state of the Stocks mode is described in explicit form by the formula

$$W_{ph}(p_a, q_a) = \frac{1}{2 \sqrt{\det \sigma(t)} \det d} \exp \left[ -\frac{1}{2} (p_a, q_a) \sigma_{ph}^{-1} (p_a, q_a) \right], \quad (14)$$

where

$$\sigma_{ph}^{-1} = 2a - \frac{1}{2} (c^T + b) d^{-1} (b^T + c) \quad (15)$$

with the matrix elements

$$\sigma_{ph}^{-1} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

expressed through the matrix elements of the inverse photon–phonon matrix $\sigma^{-1}(t)$ in the following form

$$s_{11} = \sigma_{p_a}^{-1} - \sigma_{q_a}^{-2} \left( \sigma_{p_a p_b}^{-1} \right)^2 + \sigma_{p_b}^{-1} \left( \sigma_{p_a q_b}^{-1} \right)^2 - 2 \sigma_{p_a p_b}^{-1} \sigma_{p_a q_b}^{-1} \sigma_{p_b q_b}^{-1} \quad (16)$$

$$s_{22} = \sigma_{q_a}^{-1} - \sigma_{p_a}^{-2} \left( \sigma_{p_b p_a}^{-1} \right)^2 + \sigma_{q_b}^{-1} \left( \sigma_{q_a q_b}^{-1} \right)^2 - 2 \sigma_{p_a q_b}^{-1} \sigma_{p_b q_a}^{-1} \sigma_{p_b q_b}^{-1} \quad (16)$$

$$s_{12} = s_{21} = -\sigma_{p_a q_a}^{-1} - \sigma_{q_a}^{-1} \left( \sigma_{p_a p_b}^{-1} \sigma_{p_a q_b}^{-1} \sigma_{p_b q_b}^{-1} \right)^2 + \sigma_{q_b}^{-1} \left( \sigma_{p_a q_b}^{-1} \sigma_{p_b q_a}^{-1} \sigma_{p_b q_b}^{-1} \right)^2 - 2 \sigma_{p_a q_b}^{-1} \sigma_{p_b q_a}^{-1} \sigma_{q_a q_b}^{-1} \sigma_{p_b q_b}^{-1}.$$

The averaged probability distribution function of photons in the Stocks mode can be expressed through the Hermite polynomials of two variables with zero arguments using the scheme developed in [20–27] (see general formulas in Appendix B)

$$P_n = \left[ \det \left( \sigma_{ph} + \frac{I_2}{2} \right) \right]^{-1/2} \frac{H_n(\hat{R})}{n!}, \quad (17)$$

where the matrix $\hat{R}$ is given by the formula

$$\hat{R} = U^+(I_2 - 2\sigma_{ph})(I_2 + 2\sigma_{ph})^{-1}U^*$$
with

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}. \]

One can write the photon probability distribution function as a function of Legendre polynomials using the relations between Hermite and Legendre polynomials [19].

The symmetric matrix \( \tilde{R} \) has the form

\[ \tilde{R} = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}. \]

Using the relation [19]

\[ H_{nn}^{\{\tilde{R}\}}(0,0) = (r_{11}r_{22})^{n/2} H_{nn}^{\{\beta\}}(0,0), \]

where the matrix

\[ \beta = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \]

has the matrix element

\[ r = (r_{11}r_{22})^{-1/2} r_{12}, \]

we arrive at

\[ P_n = \left[ \text{det} \left( \sigma_{ph} + \frac{I_2}{2} \right) \right]^{-1/2} (-1)^n \left( r_{12}^2 - r_{11}r_{22} \right)^{n/2} L_n \left( \frac{r}{\sqrt{r^2 - 1}} \right), \quad (18) \]

where \( L_n \) are Legendre polynomials.

The mean number of photons in the Stocks mode is the function of temperature and the coupling constant (of the laser field with the Stocks mode) and it can be calculated in explicit form

\[ \langle n \rangle = \frac{1}{2} \left( \sigma_{pp} + \sigma_{qq} - 1 \right) \]

\[ = \frac{1}{2} \left[ \cosh^2 \kappa t \left( \frac{1}{2} + \cos^2 \omega_{St} \right) + \sinh^2 \kappa t \coth \frac{\beta}{2} - 1 \right]. \quad (19) \]

The dispersion of mean photon number in the Stocks mode is

\[ \sigma_{n^2}(t) = \frac{1}{2} \cosh^4 \kappa t \left[ \frac{1}{4} + \cos^4 \omega_{St} + \cos^2 \omega_{St} \left( \frac{1}{2} - \sin \omega_{St} \right) \right] \]

\[ + \frac{1}{4} \sinh^4 \kappa t \coth \frac{\beta}{2} (1 + \sin^2 2\omega_{St}) + \frac{1}{2} \sinh^2 \kappa t \sin 2\omega_{St} \coth \frac{\beta}{2} \]

\[ + \frac{1}{8} \sinh^2 2\kappa t \coth \frac{\beta}{2} \left( \frac{1}{2} + \cos^2 \omega_{St} \right) \]

\[ - \frac{1}{2} \cosh^2 \kappa t \cos \omega_{St} (\cos \omega_{St} - \sin \omega_{St}) - \frac{1}{4}. \quad (20) \]

Thus we have found evolution of the quadrature dispersion matrix due to photon–phonon interaction. The propagator of the system under consideration in explicit form is presented in Appendix C.
5 Conclusion

We have shown that in the framework of simple quadratic model there exist new time-dependent integrals of motion for the process of stimulated Raman scattering. The linear (in photon and phonon quadratures) integrals of motion describe the initial values of the mean quadratures of the system trajectory in the phase space. The quadratic (in photon and phonon creation and annihilation operators) integrals of motion describe the initial numbers of photons and phonons in the system state.

Another result of our work is the calculated photon distribution function, which may be expressed either in terms of Hermite polynomials of two variables or in terms of Legendre polynomials.

The dependence of the photon number distribution function on the parameters of the laser field and the coupling constant shows the possibility of processing the statistics of stimulated Raman scattering by varying the laser field and medium parameters (e.g., temperature). So, the stimulated Raman scattering can be used for production of nonclassical light.

Analogous method of investigation can be applied for studying the stimulated Brillouin scattering.

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Appendix A

Here we calculate the Gaussian integral used in Section 4. We consider the integral in (13)

\[ \int \exp \left[ -\mathbf{X} \mathbf{A} \mathbf{X} \right] dy_1 \cdots dy_m, \]

where

\[ \mathbf{X} = (x, y) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{pmatrix} \]

and the matrix \( \mathbf{A} \) has four blocks

\[ \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]
We introduce notation
\[
XAX = \sum_{k=1}^{n} \sum_{l=1}^{n} x_k a_{kl} x_l + \sum_{s=1}^{m} \sum_{p=1}^{m} y_s d_{sp} y_p + \sum_{k=1}^{n} \sum_{s=1}^{m} x_k (b + c^T)_{ks} y_s .
\]
The matrix $c^T$ is transposed matrix $c$.

Using the general formula for $N$-dimensional Gaussian integral (see, for example, [19])
\[
\int \cdots \int \exp (-ZMZ + KZ) \, dz_1 \cdots dz_N = \frac{(\sqrt{\pi})^N}{\sqrt{\det M}} \exp \left( \frac{1}{4} KM^{-1}K \right),
\]
we get
\[
\int \exp (-XAX) \, dy = \frac{(\sqrt{\pi})^m}{\sqrt{\det d}} \exp (-xg x),
\tag{21}
\]
where the matrix $g$ is
\[
g = a - \frac{1}{4} \left( c^T + b \right) d^{-1} \left( b^T + c \right)
\]
and
\[
dy = dy_1 \cdots dy_m.
\]

Using the result obtained one can formulate the following theorem.

Given an arbitrary quantum state of a composed system with a Gaussian Wigner function $W(Q_1, Q_2)$ depending on the set of $2N$ phase-space variables $Q_1$ describing the first subsystem and on the set of $2M$ phase-space variables $Q_2$ describing the second subsystem. Then the Wigner function of the first subsystem
\[
W_1(Q_1) = \langle W(Q_1, Q_2) \rangle_2,
\]
which is the given Wigner function of the composed system averaged over variables of the second subsystem $Q_2$, has the Gaussian form.

The statement takes place for both finite and infinite numbers of the degrees of freedom $N$ and $M$ of the first and second subsystems.

**Appendix B**

In Appendix B, we derive general relations of the distribution function for a system in a Gaussian state to the distribution function of its subsystem. The most general mixed squeezed state of the $N$-mode system with a *Gaussian* density operator $\hat{\rho}$ is described by the Wigner function $W(p, q)$ of the generic Gaussian form (see, for example, [19, 23])
\[
W(p, q) = (\det \sigma)^{-1/2} \exp \left[ -\frac{1}{2} (Q - \langle Q \rangle) \sigma^{-1} (Q - \langle Q \rangle) \right],
\tag{22}
\]
where $2N$-dimensional vector $Q = (p, q)$ consists of $N$ components $p_1, \ldots, p_N$ and $N$ components $q_1, \ldots, q_N$, operators $\hat{p}$ and $\hat{q}$ being the quadrature components of the creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators (we use dimensionless variables and assume $\hbar = 1$):
\[
\hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}; \quad \hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}.
\]
The $2N$ parameters $\langle p_i \rangle$ and $\langle q_i \rangle$, $i = 1, 2, \ldots, N$, combined into the vector $\langle Q \rangle$, are the average values of the quadratures,

\[
\langle p \rangle = \text{Tr} \hat{p}; \quad \langle q \rangle = \text{Tr} \hat{q},
\]

A real symmetric dispersion matrix $\sigma$ consists of $(2N^2 + N)$ variances and covariances

\[
\sigma_{\alpha\beta} = \frac{1}{2} \left[ \langle \hat{Q}_\alpha \hat{Q}_\beta + \hat{Q}_\beta \hat{Q}_\alpha \rangle - \langle \hat{Q}_\alpha \rangle \langle \hat{Q}_\beta \rangle \right], \quad \alpha, \beta = 1, 2, \ldots, 2N.
\]

They obey certain constraints, which are nothing but the generalized uncertainty relations [19]. From the theorem formulated in Appendix A, it follows that the Wigner function of a subsystem of the $N$-mode system with $m$ degrees of freedom $(1 \leq m < N)$ has the same form (22), namely,

\[
W(\hat{p}, \hat{q}) = (\det \bar{\sigma})^{-1/2} \exp \left[ -\frac{1}{2} (\hat{Q} - \langle \hat{Q} \rangle) \bar{\sigma}^{-1} (\hat{Q} - \langle \hat{Q} \rangle) \right],
\]

where the $2M \times 2M$-matrix $\bar{\sigma}$ is obtained from the matrix $\sigma$ by crossing out the rows and columns with indices larger than $m$. The $2m$-vector $\bar{Q}$ is obtained from the vector $Q$ by crossing out components of vectors $p$ and $q$ with indices larger than $m$. It means that

\[
W(\hat{p}, \hat{q}) = (2\pi)^{N-m} \int W(p, q) dq_{m+1} \cdots dq_N dp_{m+1} \cdots dp_N.
\]

The number distribution function is nothing but the probability to have $n_1$ quanta in the first mode, $n_2$ quanta in the second mode, and so on. We shall designate it by the symbol $P_n$, where the vector $n$ consists of $N$ nonnegative integers: $n = (n_1, n_2, \ldots, n_N)$. This probability is given by the formula

\[
P_n = \text{Tr} \hat{\sigma} | n \rangle \langle n |,
\]

where $\hat{\sigma}$ is the density operator of the system under study, and the state $| n \rangle$ is a common eigenstate of the set of number operators $\hat{a}_i \hat{a}_i^\dagger$, $i = 1, 2, \ldots, N$.

\[
\hat{a}_i \hat{a}_i^\dagger | n \rangle = n_i | n \rangle.
\]

We use designation

\[
n! = n_1! n_2! \cdots n_N!
\]

and introduce the $2N$-dimensional unitary matrix

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} -iI_N & iI_N \\ I_N & I_N \end{pmatrix}.
\]

The symmetric $2N$-dimensional matrix $R$ and the $2N$-dimensional vector $y$, given by the relations

\[
R = U^\dagger (I_{2N} - 2\sigma) (I_{2N} + 2\sigma)^{-1} U^*; \quad y = 2U^\dagger (I_{2N} - 2\sigma)^{-1} \langle Q \rangle,
\]

determine the number distribution.

The factor $P_0$ is nothing but the probability to register no quanta. It equals [23]

\[
P_0 = \left[ \det \left( \sigma + \frac{1}{2} I_{2N} \right) \right]^{-1/2} \exp \left[ -\langle Q \rangle (2\sigma + I_{2N})^{-1} \langle Q \rangle \right].
\]
If \( \langle Q \rangle = 0 \), the probability to have no quanta depends on \((2N-1)\) parameters, which coincide up to numerical factors with the coefficients of the characteristic polynomial of the dispersion matrix.

The photon distribution function \( P_n \) can be expressed through the “diagonal” multidimensional Hermite polynomials \[23\] :

\[ P_n = P_0 \frac{H_{mm}(R)(y)}{n!}. \] (27)

One can derive that

\[ \sigma = \frac{1}{2} \left( I_{2N} - U R U^t \right) \left( I_{2N} + U R U^t \right)^{-1} \] (28)

and

\[ \langle Q \rangle = \frac{1}{2} (1 - 2\sigma) \left( U^t \right)^{-1} y. \] (29)

The number distribution function for the subsystem has the same form \[27\] but with replacement

\[ \tilde{P}_n = \tilde{P}_0 \frac{H_{mm}(\tilde{R})(\tilde{y})}{n!}; \quad n = (n_1, n_2, \ldots, n_m). \] (30)

Here

\[ \tilde{P}_0 = \left[ \det \left( \tilde{\sigma} + \frac{1}{2} I_{2m} \right) \right]^{-1/2} \exp \left[ -\langle \tilde{Q} \rangle (2\tilde{\sigma} + I_{2m})^{-1} \langle \tilde{Q} \rangle \right]. \] (31)

The distribution function for a subsystem is completely determined by the matrix \( \sigma \) and the vector \( Q \) since the matrix \( \tilde{\sigma} \) and the vector \( \tilde{Q} \) are determined by these quantities.

In view of the physical meaning of the joint distribution functions, one has the relation

\[ \tilde{P}_n = \sum_{n_{m+1}=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} P_n. \] (32)

Due to this, a sum rule for the multivariable Hermite polynomials appears, namely,

\[ \tilde{P}_0 \frac{H_{mm}(\tilde{R})}{n!} \left( \tilde{n}_1, \ldots, \tilde{n}_m \right) = \tilde{P}_0 \sum_{n_{m+1}=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{H_{mm}(R)}{n_{m+1}! \cdots n_N!}. \] (33)

Thus, given symmetric \( 2N \times 2N \)-matrix \( R \) and the \( 2N \)-vector \( y \). It means, in view of \[28\] and \[29\], that we have the \( 2N \times 2N \)-matrix \( \sigma \) and the \( 2N \)-vector \( \langle Q \rangle \). Using the rule formulated for obtaining the \( 2m \times 2m \)-matrix \( \tilde{\sigma} \) and the \( 2m \)-vector \( \langle \tilde{Q} \rangle \) from the matrix \( \sigma \) and the vector \( \langle Q \rangle \), we construct the \( 2m \times 2m \)-matrix \( \tilde{R} \) and the \( 2m \)-vector \( \tilde{y} \):

\[ \tilde{R} = U^\dagger (I_{2m} - 2\tilde{\sigma}) (I_{2m} + 2\tilde{\sigma})^{-1} U^*; \] (34)

\[ \tilde{y} = 2U^\dagger (I_{2m} - 2\tilde{\sigma})^{-1} \langle \tilde{Q} \rangle. \] (35)

Here \( U \) is the \( 2m \times 2m \)-matrix.
In the left-hand side of the sum rule (33), the constructed arguments and parameters are used. By multiplying the both sides of the sum rule (33) by the factor
\[
\frac{\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_m^{n_m}}{n_1! n_2! \cdots n_m!}
\]
and doing summation over indices \(n_i\), we arrive at the relation for generating functions for quanta distributions of the system and the subsystem:

\[
G (\lambda_1, \ldots, \lambda_N) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_N^{n_N}}{n_1! n_2! \cdots n_N!} H_{n_1,n_2,\ldots,n_N,n_1,n_2,\ldots,n_N} (R^{-1} y)
\]

\[
= \frac{1}{2} y (\Lambda \Sigma_x R + I_{2N})^{-1} \Sigma_x \Lambda y
\]

\[ (36) \]

\[
\tilde{G} (\lambda_1, \ldots, \lambda_m) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_m^{n_m}}{n_1! n_2! \cdots n_m!} H_{n_1,n_2,\ldots,n_m,n_1,n_2,\ldots,n_m} (\tilde{R}^{-1} \tilde{y})
\]

\[
= \frac{1}{2} \tilde{y} (\tilde{\Lambda} \tilde{\Sigma}_x \tilde{R} + I_{2m})^{-1} \tilde{\Sigma}_x \tilde{\Lambda} \tilde{y}
\]

\[ (37) \]

Here \( y = (y_1, y_2, \ldots, y_{2N}) \), the \( 2N \times 2N \)-matrix \( \Sigma_x \) is the \( 2N \)-dimensional analog of the Pauli matrix \( \sigma_x \),

\[
\Sigma_x = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix} \equiv U^\dagger U^* = U^t U,
\]

and the diagonal \( 2N \times 2N \)-matrix \( \Lambda \) reads

\[
\Lambda = \sum_{j=1}^{N} \lambda_j \Lambda_j,
\]

where each matrix \( \Lambda_j \) has only two nonzero elements:

\[
(\Lambda_j)_{jj} = (\Lambda_j)_{j+N,j+N} = 1.
\]

The matrices \( \tilde{\Sigma}_x \) and \( \tilde{\Lambda}_x \) are defined by analogous formulas but in \( 2m \)-dimensional space.

The relation for the generating functions is

\[
\tilde{P}_0 \tilde{G} (\lambda_1, \ldots, \lambda_m) = P_0 G (\lambda_1, \ldots, \lambda_m, 1, 1, \ldots, 1).
\]

We have shown that for any Gaussian state (a state with a Gaussian Wigner function) of \( N \)-mode system, the number distribution function of a \( m \)-mode subsystem is described by the diagonal Hermite polynomials with \( 2m \) indices. The parameters of these polynomials are determined explicitly by quadrature dispersion matrix and quadrature means of the \( N \)-mode system. The discussed model of the stimulated Raman scattering is the two-mode example of the suggested construction for its partial case corresponding to zero values of the quadrature means.
Appendix C

In this Appendix, we present the propagator of the two-mode system containing photons and phonons in coordinate representation. We introduce four $2 \times 2$-matrices

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix}
\cosh \kappa t \cos \omega_S t & -\sinh \kappa t \sin \omega_{13} t \\
-\sinh \kappa t \sin \omega_S t & \cosh \kappa t \cos \omega_{13} t
\end{pmatrix}; \\
\lambda_2 &= \begin{pmatrix}
\cosh \kappa t \sin \omega_S t & \sinh \kappa t \cos \omega_{13} t \\
\sinh \kappa t \cos \omega_S t & \cosh \kappa t \sin \omega_{13} t
\end{pmatrix}; \\
\lambda_3 &= \begin{pmatrix}
-\cosh \kappa t \cos \omega_S t & -\sinh \kappa t \sin \omega_{13} t \\
-\sinh \kappa t \sin \omega_S t & -\cosh \kappa t \cos \omega_{13} t
\end{pmatrix}; \\
\lambda_4 &= \begin{pmatrix}
\cosh \kappa t \cos \omega_S t & \sinh \kappa t \sin \omega_{13} t \\
\sinh \kappa t \sin \omega_S t & \cosh \kappa t \cos \omega_{13} t
\end{pmatrix}.
\end{align*}
\]

The matrix (6) determining the linear integrals of motion $\mathbf{\hat{I}}(t)$ is expressed in terms of these $2 \times 2$-matrices

\[\Lambda(t) = \begin{pmatrix}
\lambda_1 & \lambda_2 \\
\lambda_3 & \lambda_4
\end{pmatrix}.\]

Then, in view of general formula for the propagator of quadratic systems [19], $G(x_1, x_2, t)$ reads in coordinate representation

\[G(x_1, x_2, t) = \left[\det (-2\pi i \lambda_3)\right]^{-1/2} \exp \left\{ -\frac{i}{2} \left[ x_2 \lambda_3^{-1} \lambda_4 x_2 - 2x_2 \lambda_3^{-1} x_1 + x_1 \lambda_1 \lambda_3^{-1} x_1 \right] \right\},\]

where $x_1 = (x_{11}, x_{12})$ and $x_2 = (x_{21}, x_{22})$ are the initial and final quadrature components of the photon and phonon modes.

Thus the Green function of the system under consideration is constructed.

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