PROBABILITY MEASURES AND EFFECTIVE RANDOMNESS

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Abstract. We study the question, “For which reals \( x \) does there exist a measure \( \mu \) such that \( x \) is random relative to \( \mu \)?” We show that for every nonrecursive \( x \), there is a measure which makes \( x \) random without concentrating on \( x \). We give several conditions on \( x \) equivalent to there being continuous measure which makes \( x \) random. We show that for all but countably many reals \( x \) these conditions apply, so there is a continuous measure which makes \( x \) random. There is a meta-mathematical aspect of this investigation. As one requires higher arithmetic levels in the degree of randomness, one must make use of more iterates of the power set of the continuum to show that for all but countably many \( x \)’s there is a continuous \( \mu \) which makes \( x \) random to that degree.

1. Introduction

Most studies on algorithmic randomness focus on reals random with respect to the uniform distribution, i.e. the \((1/2, 1/2)\)-Bernoulli measure, which is measure theoretically isomorphic to Lebesgue measure on the unit interval. The theory of uniform randomness, with all its ramifications (e.g. computable or Schnorr randomness) has been well studied over the past decades and has led to an impressive theory.

Recently, a lot of attention focused on the interaction of algorithmic randomness with recursion theory: What are the computational properties of random reals? In other words, which computational properties hold effectively for almost every real? This has led to a number of interesting results, many of which will be covered in a forthcoming book by Downey and Hirschfeldt [4].

While the understanding of “holds effectively” varied in these results (depending on the underlying notion of randomness, such as computable, Schnorr, or weak randomness, or various arithmetic levels of Martin-Löf randomness, to name only a few), the meaning of “for almost every” was usually understood with respect to Lebesgue measure. One reason for this can surely be seen in the fundamental relation between uniform Martin-Löf tests and descriptive complexity in terms of (prefix-free) Kolmogorov complexity: A real is not covered by any Martin-Löf test (with respect to the uniform distribution) if and only if all of its initial segments are incompressible (up to a constant additive factor).

However, one may ask what happens if one changes the underlying measure. This question is virtually as old as the theory of randomness. Martin-Löf [15] defined randomness not only for Lebesgue measure but also for arbitrary Bernoulli distributions. Levin’s contributions in the 1970’s [24, 11, 12, 13] extended this to arbitrary probability measures. He obtained a number of remarkable results and principles such as the existence of uniform tests, conservation of randomness, and the existence of neutral measures.
In this paper we will survey a recent line of research by the authors which dealt with the question for which reals \( x \) does there exist a probability measure which makes \( x \) random without concentrating on \( x \). We consider two kinds of measures – arbitrary probability measures, which may have atoms (reals other than \( x \) on which the measure concentrates), and continuous measures, i.e. non-atomic measures. The investigations exhibit an interesting, and quite unexpected, connection between the randomness properties of a real and its logical complexity, in the sense of recursion or set theoretic hierarchies. In the following we will try to describe this connection in some detail. We will sketch proofs to provide some intuition, but for a full account we have to refer the reader to the forthcoming research papers [19, 20].

2. Measures and Randomness

In this section we introduce the basic notions of measure on the Cantor space \( 2^\omega \) and define randomness for arbitrary probability measures.

The Cantor space \( 2^\omega \) is the set of all infinite binary sequences, also called reals. The topology generated by the cylinder sets

\[
N_\sigma = \{ x : x[n] = \sigma \},
\]

where \( \sigma \) is a finite binary sequence, turns \( 2^\omega \) into a compact Polish space. We will occasionally use the notation \( N(\sigma) \) in place of \( N_\sigma \) to avoid multiple subscripts. \( 2^{<\omega} \) denotes the set of all finite binary sequences. If \( \sigma, \tau \in 2^{<\omega} \), we use \( \subseteq \) to denote the usual prefix partial ordering. This extends in a natural way to \( 2^{<\omega} \cup 2^\omega \). Thus, \( x \in N_\sigma \) if and only if \( \sigma \subset x \). Finally, given \( U \subseteq 2^{<\omega} \), we write \( N_U \) to denote the open set induced by \( U \), i.e. \( N_U = \bigcup_{\sigma \in U} N_\sigma \).

2.1. Probability measures. A probability measure on \( 2^\omega \) is a countably additive, monotone function \( \mu : \mathcal{F} \to [0, 1] \), where \( \mathcal{F} \subseteq \mathcal{P}(2^\omega) \) is \( \sigma \)-algebra and \( \mu(2^\omega) = 1 \). \( \mu \) is called a Borel probability measure if \( \mathcal{F} \) is the Borel \( \sigma \)-algebra of \( 2^\omega \). It is a basic result of measure theory that a probability measure is uniquely determined by the values it takes on an algebra \( \mathcal{A} \subseteq \mathcal{F} \) that generates \( \mathcal{F} \). It is not hard to see that the Borel sets are generated by the algebra of clopen sets, i.e. finite unions of basic open cylinders. Normalized, monotone, countably additive set functions on the algebra of clopen sets are induced by any function \( \rho : 2^{<\omega} \to [0, 1] \) satisfying

\[
\rho(\epsilon) = 1 \quad \text{and} \quad \rho(\sigma) = \rho(\sigma^{-0}) + \rho(\sigma^{-1}) \tag{2.1}
\]

for all finite sequences \( \sigma \). Then \( \mu(N_\sigma) = \rho(\sigma) \) yields an monotone, additive function on the clopen sets, which in turn uniquely extends to a Borel probability measure on \( 2^\omega \). In the following, we will deal exclusively with Borel probability measures, and hence we will identify such measures with the underlying function on cylinders satisfying (2.1), and write, in slight abuse of notation, \( \mu(\sigma) \) instead of \( \mu(N_\sigma) \). Besides, we will mostly speak of measures, understanding Borel probability measures.

The Lebesgue measure \( \mathcal{L} \) on \( 2^\omega \) is obtained by distributing a unit mass uniformly along the paths of \( 2^\omega \), i.e. by setting \( \mathcal{L}(\sigma) = 2^{-|\sigma|} \). A Dirac measure, on the other hand, is defined by putting a unit mass on a single real, i.e. for \( x \in 2^\omega \), let

\[
\delta_x(\sigma) = \begin{cases} 
1 & \text{if } \sigma \subset x, \\
0 & \text{otherwise.}
\end{cases}
\]
If, for a measure \( \mu \) and \( x \in 2^\omega \), \( \mu(\{x\}) > 0 \), then \( x \) is called an atom of \( \mu \). Obviously, \( x \) is an atom of \( \delta_x \). A measure that does not have any atoms is called continuous.

3. MARTIN-LÖF RANDOMNESS

It was Martin-Löf’s fundamental idea to define randomness by choosing a countable family of nullsets. For any non-trivial measure, the complement of the union of these sets will have positive measure, and any point in this set will be considered random. There are of course many possible ways to pick a countable family of nullsets. In this regard, it is very benefiting to use the framework of recursion theory and effective descriptive set theory.

3.1. Nullsets. Before we go on to define Martin-Löf randomness formally, we note that every nullset is contained in a \( G_\delta \)-nullset.

**Proposition 3.1.** Suppose \( \mu \) is a measure. Then a set \( A \subseteq 2^\omega \) is \( \mu \)-null if and only if there exists a set \( U \subseteq \mathbb{N} \times 2^{<\omega} \) such that for all \( n \),

\[
A \subseteq N(U_n) \quad \text{and} \quad \sum_{\sigma \in U_n} \mu(N_\sigma) \leq 2^{-n},
\]

where \( U_n = \{ \sigma : (n, \sigma) \in U \} \).

Of course, the \( G_\delta \)-cover of \( A \) is given by \( \bigcap_n U_n \).

3.2. Martin-Löf tests and randomness. Essentially, a Martin-Löf test is an effectively presented \( G_\delta \)-nullset (relative to some parameter \( z \)).

**Definition 3.2.** Suppose \( z \in 2^\omega \) is a real. A test relative to \( z \), or simply a \( z \)-test, is a set \( W \subseteq \mathbb{N} \times 2^{<\omega} \) which is recursively enumerable in \( z \). Given a natural number \( n \geq 1 \), an \( n \)-test is a test which r.e. in \( \emptyset^{(n-1)} \), the \( (n-1) \)st Turing jump of the empty set. A real \( x \) passes a test \( W \) if \( x \not\in \bigcap_n N(W_n) \).

Passing a test \( W \) means not being contained in the \( G_\delta \) set given by \( W \). The condition ‘r.e. in \( z \)’ implies that the open sets given by the sets \( W_n \) form a uniform sequence of \( \Sigma^0_1(z) \) sets, and the set \( \bigcap_n N(W_n) \) is a \( \Pi^0_2(z) \) subset of \( 2^\omega \).

To test for randomness with respect to a measure, we have to ensure two things: First that a test \( W \) actually describes a nullset. Second, that the information present in a measure is available to the test. The first criterion we call correctness.

**Definition 3.3.** Suppose \( \mu \) is a measure on \( 2^\omega \). A test \( W \) is correct for \( \mu \) if

\[
\sum_{\sigma \in W_n} \mu(N_\sigma) \leq 2^{-n}.
\]

To incorporate measures into an effective test for randomness we have to represent it in a form that makes it accessible for recursion theoretic methods. Essentially, this means to code a measure via an infinite binary sequence or a function \( f : \mathbb{N} \to \mathbb{N} \). Unfortunately, there are many possible such representations. Hence, strictly speaking, we will deal with randomness with respect to a representation of a measure, not the measure itself. However, we will see that for one of our main topics, randomness for continuous measures, representational issues can be resolved quite elegantly.

The most straightforward representation of a measure is the following.
Definition 3.4. Given a measure $\mu$, define its rational representation $r_\rho$ by letting, for all $\sigma \in 2^{<\omega}$, $q_1, q_2 \in \mathbb{Q}$,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \iff q_1 < \rho(\sigma) < q_2.$$  

(3.3)

The rational representation does not reflect the topological properties of the space of probability measures on $2^\omega$. The space of probability measures $P$ on $2^\omega$ is a compact polish space (see Parthasarathy [17]). The topology is the weak topology, which can be metrized by the Prokhorov metric, for instance. There is an effective dense subset, given as follows: Let $Q$ be the set of all reals of the form $\sigma \uparrow 0^{\omega}$. Given $\bar{q} = (q_1, \ldots, q_n) \in Q^{<\omega}$ and non-negative rational numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum \alpha_i = 1$, let

$$\delta_{\bar{q}} = \sum_{k=1}^{n} \alpha_k \delta_{q_k},$$

where $\delta_x$ denotes the Dirac point measure for $x$. Then the set of measures of the form $\delta_{\bar{q}}$ is dense in $P$.

The recursive dense subset $\{\delta_{\bar{q}}\}$ and the effectiveness of the metric $d$ between measures of the form $\delta_{\bar{q}}$ suggests that the representation reflects the topology effectively, i.e. the set of representations should be $\Pi_0^1$. However, this is not true for the set of rational representations of probability measures. Instead, we have to resort to other representations in metric spaces, such as Cauchy sequences. Using the framework of effective descriptive set theory, as for example presented in Moschovakis [16], one can obtain the following.

Theorem 3.5. There is a recursive surjection

$$\pi: 2^\omega \to P$$

and a $\Pi_0^1$ subset $P$ of $2^\omega$ such that $\pi[P]$ is one-one and $\pi(P) = P$.

In the following sections, we will always assume that a measure is either represented by its rational representation or via the the set $P \subseteq 2^\omega$ of the previous theorem. The definition of randomness, however, works for any representation.

Definition 3.6. Suppose $\mu$ is a probability measure on $2^\omega$, $\rho_\mu \in 2^\omega$ is a representation of $\mu$, and $z \in 2^\omega$ is a real. A real is Martin-Löf $n$-random for $\mu$ relative to $\rho_\mu$ and $z$, or simply $(n, z)$-random for $\rho_\mu$ if it passes all $(\rho_\mu \oplus z)^{(n-1)}$-tests which are correct for $\mu$.

If the representation is clear from the context, we speak of $(n, z)$-randomness for $\mu$. If $\mu$ is Lebesgue measure $\mathcal{L}$, we drop reference to the measure and simply say “$x$ is $(n, z)$-random”. We also drop the index 1 in case of $(1, z)$-randomness and simply speak of “randomness relative to $z$” or z-randomness.

Since there are only countably many Martin-Löf $n$-tests, it follows from countable additivity that the set of Martin-Löf $n$-random reals for $\mu$ has $\mu$-measure 1. Hence there always exist $(n, z)$-random reals for any measure $\mu$.

3.3. Image measures and conservation of randomness. One can obtain new measures from given measures by transforming them with respect to a sufficiently regular function. Let $f: 2^\omega \to 2^\omega$ be a Borel (measurable) function, i.e. for every Borel set $A$, $f^{-1}(A)$ is Borel, too. If $\mu$ is a measure on $2^\omega$ and $f$ is Borel, then the image measure $\mu_f$ is defined by

$$\mu_f(A) = \mu(f^{-1}(A)).$$
It can be shown that every probability measure can be obtained from Lebesgue measure $\mathcal{L}$ by means of a measurable transformation.

**Theorem 3.7** (folklore, see e.g. Billingsley [1]). If $\mu$ is a Borel probability measure on $2^\omega$, then there exists a measurable $f : 2^\omega \to 2^\omega$ such that $\mu = \mathcal{L} f$.

If the transformation of $\mathcal{L}$ is effective, then $f$ maps an $\mathcal{L}$-random real to a $\mathcal{L} f$-random real. This principle is called *conservation of randomness*, first introduced by Levin. We can use it to construct measures for which a given real is random, as we will see in the next sections.

4. Randomness of Non-Recursive Reals

If $x$ is an atom of some probability measure $\mu$, it is trivially $\mu$-random. Interestingly, the recursive reals are exactly those for which this is the only way to become random.

**Theorem 4.1** (Reimann and Slaman [20]). For any real $x$, the following are equivalent.

(i) There exists a (representation of a) probability measure $\mu$ such that $\mu(\{x\}) = 0$ and $x$ is $\mu$-random.

(ii) $x$ is not recursive.

**Proof sketch.** If $x$ is recursive and $\mu$ is a measure with $\mu(\{x\}) = 0$, then we can obviously construct a $\mu$-test that covers $x$, by computing (recursively in $\mu$) the measure of initial segments of $x$, which tends to 0.

Now assume $x$ is not recursive. A fundamental result by Kučera [10] ensures that every Turing degree above $\emptyset'$ contains a $\mathcal{L}$-random real. This result relativizes. Hence one can combine it with the *Posner-Robinson Theorem* [18], which says that for every non-recursive real $x$ there exists a $z$ such that $x \oplus z =_T z'$. This way we obtain a real $R$ which is

(1) $\mathcal{L}$-random relative to some $z \in 2^\omega$, and

(2) $T(z)$-equivalent to $x$.

There are Turing functionals $\Phi$ and $\Psi$ recursive in $z$ such that

$$\Phi(R) = x \quad \text{and} \quad \Psi(x) = R.$$ 

We can use the functionals to define a class of measures that are possible candidates to render $x$ random. Given $\sigma \in 2^\omega$, define the set $\text{Pre}(\sigma)$ to be the set of minimal elements of

$$\{ \tau \in 2^{<\omega} : \Phi(\tau) \supseteq \sigma \text{ and } \Psi(\sigma) \subseteq \tau \}.$$ 

We define a set of measures $M$ by requiring that $\mu \in M$ if and only if

$$\forall \sigma [\mathcal{L} (\text{Pre}(\sigma)) \leq \mu(\sigma) \leq \mathcal{L} (\Psi(\sigma))]. \quad (4.1)$$

The first inequality ensures that $\mu$ dominates an image measure induced by $\Phi$. This will ensure that any Martin-Löf random real is mapped by $\Phi$ to a $\mu$-random real. The second inequality guarantees that $\mu$ is non-atomic on the domain of $\Psi$.

One can show the topological representations of the measures in $M$ (Theorem 3.5) form a non-empty $\Pi^0_1$ class $M$ in $2^\omega$ relative to $z$.

In order to apply conservation of randomness, we have to know that one of the measures in $M$, when given as an additional information to a $\mathcal{L}$-test, will not destroy the randomness of $R$. This is ensured by the following basis result for $\Pi^0_1$ sets regarding relative randomness (essentially a consequence of *compactness*). □
Theorem 4.2 (Reimann and Slaman [20], Downey, Hirschfeldt, Miller, and Nies [3]). Let $S$ be $\Pi^0_1(z)$. If $R$ is $\mathcal{L}$-random relative to $z$, then there exists $y \in S$ such that $R$ is $\mathcal{L}$-random relative to $y \oplus z$.

5. Randomness for continuous measures

A natural question arising in the context Theorem 4.1 is whether the measure making a real random can be ensured to have certain regularity properties; in particular, can it be chosen continuous?

Reimann and Slaman [20] gave an explicit construction of a non-recursive real not random with respect to any continuous measure. Call such reals 1-ncr. In general, let $NCR_n$ be the set of reals which are not $n$-random with respect to any continuous measure.

Kjos-Hanssen and Montalban [8] observed that any member of a countable $\Pi^0_1$ class is an element of $NCR_1$.

Proposition 5.1. If $A \subseteq 2^\omega$ is $\Pi^0_1$ and countable, then no member of $A$ can be in $NCR_1$.

Proof idea. If $\mu$ is a continuous measure, then obviously $\mu(A) = 0$. One can use a recursive tree $T$ such that $[T] = A$ to obtain a $\mu$-test for $A$. □

It follows from results of Cenzer, Clote, Smith, Soare, and Wainer [2] that members of $NCR_1$ can be found throughout the hyperarithmetical hierarchy of $\Delta^1_1$, whereas Kreisel [9] had shown earlier that each member of a countable $\Pi^0_1$ class is in fact hyperarithmetical.

Quite surprisingly, $\Delta^1_1$ turned out to be the precise upper bound for $NCR_1$. An analysis of the proof of Theorem 4.1 shows that if $x$ is truth-table equivalent to a $\mathcal{L}$-random real, then the “pull-back” procedure used to devise a measure for $x$ yields a continuous measure. More generally, we have the following.

Theorem 5.2 (Reimann and Slaman [20]). Let $x$ be a real. For any $z \in 2^\omega$ and any $n \geq 1$, the following are equivalent.

(i) $x$ is $(n, z)$-random for a continuous measure $\mu$ recursive in $z$.
(ii) $x$ is $(n, z)$-random for a continuous dyadic measure $\nu$ recursive in $z$.
(iii) There exists a functional $\Phi$ recursive in $z$ which is an order-preserving homeomorphism of $2^\omega$ such that $\Phi(x)$ is $(n, z)$-random.
(iv) $x$ is truth-table equivalent relative to $z$ to a $(n, z)$-random real.

Here dyadic measure means that the values of $\mu$ on the open cylinders are of the form $\mu(\sigma) = m/2^n$ with $m, n \in \mathbb{N}$. The theorem can be seen as an effective version of the classical isomorphism theorem for continuous probability measures (see for instance Kechris [7]).

Woodin [23], using a variation on Prikry forcing, was able to prove that if $x \in 2^\omega$ is not hyperarithmetic, then there is a $z \in 2^\omega$ such that $x \oplus z \equiv_{tt(z)} z'$, i.e. outside $\Delta^1_1$ the Posner-Robinson theorem holds with truth-table equivalence. Hence we can infer the following result.

Theorem 5.3 (Reimann and Slaman [20]). If a real $x$ is not $\Delta^1_1$, then there exists a continuous measure $\mu$ such that $x$ is $\mu$-random.

1The theorem suggests that for continuous randomness representational issues do not really arise, since there is always a measure with a computationally minimal representation.
It is on the other hand an open problem whether every real in $\text{NCR}_1$ is a member of a countable $\Pi^0_1$ class.

One may ask how the complexity and size of $\text{NCR}_n$ grows with $n$. It turned out all levels of $\text{NCR}$ are countable.

**Theorem 5.4** (Reimann and Slaman [19]). For all $n$, $\text{NCR}_n$ is countable.

**Proof idea.** The first step is to use Borel determinacy to show that the complement of $\text{NCR}_n$ contains an upper Turing cone. This follows from the fact that the complement of $\text{NCR}_n$ contains a Turing invariant and cofinal (in the Turing degrees) Borel set, which can be seen as follows.

If for two reals $x, y$, $x \equiv_{T(z)} y$, then $x \equiv_{T(z')} y$. Suppose $x \equiv_{T(z)} R$ where $R$ is $(n+1)$-random relative to $z$. Then, since $R$ is $n$-random relative to $z'$, it follows from Theorem 5.2 that $x$ is random with respect to some continuous measure.

So if we let $B \subseteq 2^\omega$ be the set

$$\{x \in 2^\omega : \exists z \exists R (x \equiv_T z \oplus R \& R \text{ is } (n+1)\text{-random relative to } z)\},$$

$B$ is a Turing invariant Borel set cofinal in the Turing degrees. It follows from Borel Determinacy [14] that $B$ contains an upper cone in the Turing degrees.

The next step is to show that the elements of $\text{NCR}_n$ show up at a countable level of the constructible universe $L$. It holds that $\text{NCR}_n \subseteq L_{\beta_n}$, where $\beta_n$ is the least ordinal such that

$$L_{\beta_n} \models \text{ZFC}^-_{\omega},$$

where $\text{ZFC}^-_{\omega}$ is $\text{ZFC}$ with the power set axiom replaced by the existence of $n$ iterates of the power set of $\omega$. Note that $L_{\beta_n}$ is the level of constructiblility capturing Martin’s construction of a winning strategy in a $\Sigma^0_n$-game.

Given $x \not\in L_{\beta_n}$, construct a set $G$ such that $L_{\beta_n}[G]$ is a model of $\text{ZFC}^-_{\omega}$, and for all $y \in L_{\beta_n}[G] \cap 2^\omega$, $y \leq_T x \oplus G$. $G$ is constructed by Kumabe-Slaman forcing (see [22]). This notion of forcing provides a method to extend the Posner-Robinson Theorem to higher levels of the jump and beyond. The existence of $G$ allows to conclude: If $x$ is not in $L_{\beta_n}$, it will belong to every cone with base in $L_{\beta_n}[G]$. In particular, it will belong to the cone given by the Borel Turing determinacy argument (relativized to $G$, here one has to use absoluteness), i.e. the cone avoiding $\text{NCR}_n$. Hence $x$ is random relative to $G$ for some continuous $\mu$, an thus in particular $\mu$-random.

The proof of the countability of $\text{NCR}_n$ makes essential use of Borel determinacy. It is known from a result by Friedman [5] that the use of $\omega_1$-many iterates of the power set of $\omega$ are necessary to prove Borel determinacy. In the simplest case, Friedman showed that $\text{ZFC}^-$ does not prove the statement “All $\Sigma^0_n$-games on countable trees are determined.” The proof works by showing that there is a model of $\text{ZFC}^-$ for which $\Sigma^0_n$-determinacy does not hold. This model is just $L_{\beta_0}$. The analysis extends to higher levels of the Borel hierarchy, applying to more and more iterates of the power set.

The question suggests itself whether the proof of the countability of $\text{NCR}_n$ requires a similar set theoretic complexity.

**Theorem 5.5** (Reimann and Slaman [19]). For every $k$, the statement

For every $n$, $\text{NCR}_n$ is countable.

cannot be proven in $\text{ZFC}^-_{k}$.
Proof sketch. We show that for every fixed $k$, some $\text{NCR}_n$ is cofinal in the Turing degrees of $L_{\beta_k}$. In fact, Jensen’s master codes [6] for $L$, the universe of constructible sets, are the cofinal set.

$L$ is generated by transfinite recursion in which the recursion steps are closing under first order definability and forming unions. The master codes represent the initial segments of $L$ and are generated by iterating the Turing jump and taking $L$-least representations of direct limits. In short, a master code is either definable relative to an earlier master code or is the code for the well-founded limit of structures each of which is coded by an earlier master code. The number of iterates of the power set present in the initial segment of $L$ which is being coded is linked to the complexity of describing the direct limit used to form its master code. For $\alpha$ less than $\beta_k$, there is a fixed bound on this complexity. We let $M_\alpha$ denote the master code for $L_{\beta_k}$.

Neither of these cases is consistent with randomness, as indicated by the following lemmas.

Lemma 5.6. Suppose that $n \geq 2$, $y \in 2^{\omega}$, and $x$ is $n$-random for $\mu$. If $i < n$, $y$ is recursive in $(x \oplus \mu)$ and recursive in $\mu^{(i)}$, then $y$ is recursive in $\mu$.

Lemma 5.7. Suppose that $x$ is $(n + 5)$-random for $\mu$, $\prec$ is a linear ordering that is $\Delta^0_{n+1}$ relative to $\mu$, and $I$ is the largest initial segment of $\prec$ which is well-founded. If $i < n$ and $I$ is $\Sigma^0_i$ in $(x \oplus \mu)$, then $I$ is recursive in $\mu$.

Suppose $\alpha$ less than $\beta_k$ and a continuous measure $\mu$ are given so that $M_\alpha$ is random relative to $\mu$. Heuristically, we argue as follows. We proceed by induction on $\beta \leq \alpha$ to prove that $M_\beta$ is recursive in $\mu$. If $\beta$ is a successor, then $M_\beta$ is arithmetic in some earlier master code, with a uniform upper bound on the complexity of the definition depending on $k$. Then, $M_\beta$ is uniformly arithmetic in $\mu$ and recursive in $M_\alpha$, Lemma 5.6 applies. Otherwise, $M_\beta$ is the well-founded direct limit of structures recursive in $\mu$ and recursive in in $M_\alpha$, so Lemma 5.7 applies. In either case, $M_\beta$ is recursive in $\mu$. By induction, $M_\alpha$ is itself recursive in $\mu$ and not $\mu$-random, a contradiction.

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