Ultrastrong adhesion in the contact with thin elastic layers on a rigid foundation

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Abstract

In the present short note, we generalize simple approximate Johnson-Jaffar-Barber solutions for the indentation by a rigid punch of a thin elastic layer on a rigid foundation to the case of adhesion. This could be an interesting geometry for an adhesive system, a limit case of the more general class of layered systems, or FGMs (Functionally Graded Materials). We show that ultrastrong adhesion (up to theoretical strength) can be reached both in line contact or in axisymmetric contact for thin layers (typically of nanoscale size), which suggests a new possible strategy for ”optimal adhesion”. In particular, in line contact adhesion enhancement occurs as an increase of the actual pull-off force, while in axisymmetric case the latter is apparently very close to the classical JKR case. However, it appears in closer examination that also for axisymmetric case, the enhancement occurs by reducing the size of contact needed to sustain the pull-off force. These effects are further enhanced by Poisson’s ratio effects in the case of nearly incompressible layer.

Keywords: Adhesion, layer, JKR model, Adhesion enhancement

1. Introduction

Inspired by nature, we are studying and possibly imitating many ways to optimize adhesion devices, one way being by reducing length scales involved in the geometry (Gao & Yao, 2004). A significant amount of study has been devoted to the case of halfspace geometry, for which the optimal shape for maximum pulloff force is found to be concave, although it is not ”robust”
to surface geometry errors (Yao and Gao, 2006). Enhancement of adhesion due to surface geometries is also known in mushroom-shaped fibrils (Peng and Cheng, 2012), rodlike particles (Sundaram and Chandrasekar, 2011), or moving to functionally graded materials (FGMs) which are increasingly used in engineering, and have been also used in nature as a result of evolution (Suresh, 2001, Sherge & Gorb, 2001). Indeed, few authors have explored the behaviour of attachments using FGMs (Chen et al., 2009a, 2009b, Jin et al., 2013), finding interesting results and possible avenues to design “optimal” adhesive systems.

However, curiously a much simpler geometry (which is in a sense a limit case of FGM) is that of adhesion with a layer on a rigid foundation.

In his well known book, Johnson (1985) suggested an elementary formulation to obtain asymptotic results for the contact pressure between a frictionless rigid indenter and a thin elastic layer supported by a rigid foundation. Jaffar (1989) later on used the same technique for the axisymmetric case, and finally Barber (1990) generalized it to the arbitrary, three-dimensional problem for the thin elastic layer.

A typical assumption made is that of the JKR model (Johnson et al., 1971) which corresponds to very short range adhesion where adhesive forces are all within the contact area. Solving the JKR problem is simple generalizing the original JKR energetic derivation assuming calculation of the strain energy in adhesiveless contact, and unloading at constant contact area (see (Ciavarella, 2017)). The underlying assumption of (Ciavarella, 2017) is that the contact area distributions are the same as under adhesiveless conditions (for an appropriately increased normal load). There are no approximations involved if the geometry is that of a single line or axisymmetric contact, as the solution is exact within the JKR assumption of infinitely short adhesion range, and states that the indentation under adhesive conditions for a given surface energy $w$ is

$$\delta = \delta_1 - \sqrt{2wA'/P_1''}$$

where $\delta_1$ is the adhesiveless indentation, $A'$ is the first derivative of contact area and $P_1''$ the second derivative of the adhesiveless load with respect to $\delta_1$. Then, the adhesive load is

$$P = P_1 - P_1'\sqrt{2wA'/P_1''}$$

Hence, the asymptotic solutions for the adhesive thin layer problems are found quite simply from the solutions of Johnson (1985), Jaffar (1989), Bar-
ber (1990). We shall then discuss implications, particularly as some results will be quite surprising, and suggest potential strategies for "optimal" adhesion.

2. The solution

We follow Barber (1990) in denoting the contact surface of the layer by $z = 0$ and choose a two-dimensional Cartesian coordinate system $x_1, x_2$ in the plane of the layer. The original Johnson’s approximation is to assume that plane sections within the layer remain plane, so that the in-plane displacements of the layer with components $u_1, u_2$ is independent of $z$. Barber (1990) shows that in the case the layer is on a frictionless foundation we recover the contact pressure

$$p(x_1, x_2) = \frac{2\mu}{(1-\nu)}w(x_1, x_2)$$

where the indentation $w$ is the local interpenetration between the indenter and the layer if it did not deform, $\mu$ is shear modulus and $\nu$ Poisson’s ratio.

We could transform the adhesionless solution into an adhesive one with an approximation following (Ciavarella, 2017) and the approximation will not be the same as that used for the halfspace by Johnson & Greenwood (2005) which shows the ellipticity of the contact area with adhesion differs from that without.

However, there are no approximations involved in applying (Ciavarella, 2017) and hence (1, 2) in the case of a single line contact, or axisymmetric contact, which we shall consider next. We are already solving the problem with the "asymptotic" assumption for the layer thickness, and prefer in fact not to make further approximations.

2.1. Line contact

2.1.1. Frictionless foundation

As in the original Johnson (1985) derivation, we consider (see Fig.1) a layer indented by a frictionless rigid cylinder of radius $R$, and assume the thickness of the layer $b$ is small compared with the half-width of the contact size $a$, i.e. $b << a$, which is required for the assumptions that plane sections remain plane after compression.
The adhesiveless solution gives for indentation $\delta_1$ and load $P_1$

$$\delta_1 = \frac{a^2}{2R}$$  \hspace{1cm} (4)

$$P_1 = \frac{2}{3} \frac{E^* L}{R b} a^3 = \frac{2^{5/2}}{3} \frac{E^* LR^{1/2}}{b} \delta_1^{3/2}$$  \hspace{1cm} (5)

being $E^*$ the plane strain elastic modulus, $L$ the contact length. Now the adhesive solution is obtained considering $A = 2aL$ and with obvious algebra using \[2\]

$$P = \frac{4}{3} \frac{E^* L \sqrt{2R}}{b} \left( \delta_1 - 3 \sqrt{\frac{b}{2E^* w}} \right)$$  \hspace{1cm} (6)

in terms of the adhesionless indentation $\delta_1$. To find the minimum load (pull-off), the condition $P' = 0$ gives

$$\delta_{1,PO} = \sqrt{\frac{b}{2E^* w}} \quad ; \quad a_{PO} = \sqrt{2R} \left( \frac{b}{2E^* w} \right)^{1/4}$$  \hspace{1cm} (7)

(where notice that we have to assume $a_{PO} >> b$ to be consistent with the
thin layer assumption), and hence substituting

\[ P_{PO} = -\frac{8}{3} \frac{E^* LR^{1/2}}{(2b)^{1/4}} \left( \frac{w}{E^*} \right)^{3/4} \]  

where \( E^* \) is the effective modulus. Whereas the average stress in the contact at pull-off is

\[ \sigma_{PO} = \frac{P_{PO}}{2A_{PO}} = \frac{2}{3} \sqrt{2} \left( \frac{E^* w}{b} \right)^{1/2} = \frac{2}{3} \sqrt{2} \frac{K_{Ic}}{\sqrt{b}} \]  

where \( K_{Ic} \) is toughness of the contact. Hence, notice that the JKR solution simply gives the Griffith condition imposed by a Stress Intensity Factor which scales only with the size the layer \( b \) and not any other length scale (like the radius of the punch). The interesting result that as \( b \to 0 \) the limit of the force also goes to \( \infty \). Since this will be bounded by theoretical strength, the situation is analogous to the well known case of a fibrillar structure in contact with a rigid halfspace, like that discussed for Gecko and many insects who have adopted nanoscale fibrillar structures on their feet as adhesion devices (Gao & Yao, 2004). In our case, to have a design insensitive to small variations in the tip shape, we would simply need to go down in the scale of the layer thickness. In fact, by equating \( \sigma_{PO} \) to theoretical strength, we obtain the order of magnitude of the "critical" thickness below which we expect theoretical strength

\[ b_{cr} = \frac{8}{9} \frac{E^* w}{\sigma_{th}} \]  

Taking \( w = 10 mJ/m^2, \sigma_{th} = 20 MPa \) and \( E^* = 1 GPa \), like done in (Gao & Yao, 2004), we estimate \( b_{cr} = \frac{8 \times 10^{10} \times 10^{-2}}{9 \times (20 \times 10^9)^{1/2}} = 22 nm \), which is quite similar to the estimate (of a different geometry) of 64nm robust design diameter of the fiber of the fibrillar structure. Hence, with this size of layer of nanoscopic scale, we would be able to devise a quite strong attachment for any indenter.

In the halfplane limit case, from Barquins (1988), Chaudhury et al. (1996) we have for the cylinder

\[ P_{PO,HP} = -3L \left( \frac{\pi E^* w^2 R}{16} \right)^{1/3} \]  
\[ a_{PO,HP} = \left( \frac{2wR^2}{\pi E^*} \right)^{1/3} \]  
\[ \sigma_{PO,HP} = \frac{P_{PO,HP}}{2A_{PO,HP}} = \frac{3}{2} \left( \frac{\pi^2 E^* w^2}{32R} \right)^{1/3} \]
which does include some dependence on elastic modulus which is not present in the axisymmetric halfspace problem of JKR model (Johnson et al., 1971), but it seems to be quite different in terms of power law dependence from the "layered" case. Indeed, take the ratio

\[
\frac{P_{PO}}{P_{PO,HP}} = \frac{8}{9} \frac{16^{1/3}}{\pi^{1/3}} \frac{R^{1/6}}{b^{1/4}} \left( \frac{w}{E^*} \right)^{1/12}
\]  

which shows how there are really different power law dependences in the layer limit.

The full curve $P - \delta$ is then obtained using (11)

\[
\delta = \frac{a^2}{2R} - \sqrt{2w} \frac{b}{E^*}
\]  

so extracting the equation for the contact area, using $\delta_1 = \frac{a^2}{2R}$, and then substituting back in the solution (6), we get

\[
\tilde{P} = \frac{4}{3} \sqrt{2} \left( \tilde{\delta} + \sqrt{2} \right)^{1/2} \left( \tilde{\delta} + \sqrt{2} - \frac{3}{\sqrt{2}} \right)
\]

where we have defined dimensionless quantities

\[
\tilde{\delta} = \frac{\delta}{\sqrt{w} \frac{b}{E^*}} ; \quad \tilde{P} = \frac{P}{\frac{E^* LR^{1/2}}{b^{1/4}} \left( \frac{w}{E^*} \right)^{3/4}}
\]

so that $\tilde{P}_{PO} = -\frac{8}{3 \times 2^{1/4}} = 2.2424$ and $\tilde{\delta}_{PO} = -\frac{\sqrt{2}}{2} = 0.707$.

Following Fig.2, the solution is plotted in dimensionless terms, and typical loading paths are indicated by arrows: starting from remote locations, one finds contact only when there is contact of the undeformed surfaces (JKR makes it not possible to model long range adhesion) and hence until point O (the origin of the coordinate system) is reached. Then under force control, one would obtain a jump to point B where force remains zero but one finds an effective indentation $\tilde{\delta}_B$. From this point on, one could load in compression and go up in the figure, or start downloading. Unloading then ends at pull-off in point $\tilde{P}_{PO}$. Alternatively, if we were under displacement control, at the point of first contact we would build up adhesive force and jump to point A, with an effective tensile load but then unloading would proceed until the adhesive force is reduced back to zero in point C. Hence, there is no pull-off under displacement control, contrary to the classical JKR case.
Fig 2. Load vs indentation curve for a rigid cylinder indenting an layer on a frictionless rigid foundation. Possible loading path under Force Control (LC) and Displacement Control (DC)

2.1.2. Bonded layer

For bonded layer, a similar procedure finds

\[ P_{PO} = -\frac{8}{3} E^* LR^{1/2} \frac{(1 - \nu)^{1/2}}{2^{1/4} (1 - 2\nu)^{1/4}} \left( \frac{w}{E^*} \right)^{3/4} \]  

\[ a_{PO} = \sqrt{2R} \left( \frac{(1 - 2\nu) b}{(1 - \nu)^2 E^* w} \right)^{1/4} \]

and therefore for the bonded layer the Poisson’s effect appears, which only changes a prefactor in the result for the frictionless foundation — but notice this prefactor makes the load diverge towards the incompressible limit \( \nu = 0.5 \). Hence, in this case the average stress in the contact at pull-off is

\[ \sigma_{PO} = \frac{P_{PO}}{2A_{PO}} = -\frac{4}{3} \frac{(E^* w)^{1/2}}{b^{1/2}} \frac{1 - \nu}{2 (1 - 2\nu)^{1/2}} \]
and hence here by equating $\sigma_{PO}$ to theoretical strength, we obtain

$$b_{cr} = \left[ \frac{1}{2 \left( 1 - 2\nu \right)} \right] \frac{8 E^*w}{9 \sigma_{th}^2} \left[ \frac{1}{2 \left( 1 - 2\nu \right)} \right] b_{cr,frictionless}$$

(18)

and therefore this time the critical layer thickness becomes dependent on Poisson’s ratio, rendering the layer adhesive much more effective.

### 2.1.3. Incompressible bonded layer

The results of the previous paragraph hold until the layer is nearly incompressible, in which case a similar procedure yields

$$P_{PO} = -\frac{8}{5}L \frac{(3Rw)^{2/3}}{(2b)^{1/2}} w^{1/6}$$

(19)

and $\delta_{1,PO} = b \left( \frac{w}{3E^*R} \right)^{1/3}$ while $a_{PO} = \sqrt{6Rb \left( \frac{w}{3E^*R} \right)^{1/3}}$, which is therefore rather different from the frictionless counterpart. Hence, in this case the average stress in the contact at pull-off is

$$\sigma_{PO} = \frac{P_{PO}}{2A_{PO}} = -\frac{4}{5} \frac{3^{5/6}}{(12)^{1/2}} \frac{w^{2/3}E^{*1/6}}{b} R^{1/3}$$

and we return to see effects of the radius of the indenter (i.e. qualitative effects on the geometry) like in the halfplane problem.

### 3. Axysimmetric problems

Jaffar (1989) has studied axisymmetric contact problems involving bonded and unbounded layers indented by a frictionless rigid body. For the case of unbounded frictionless foundation, the total load is given by

$$P_1 = \frac{\pi}{4} E^* \frac{a^4}{Rb}$$

(20)

For the value of the penetration depth $\delta_1$ when $a/b >> 1$, no asymptotic formula exists as for line contacts; but it can be obtained on the interpretation that ”it represents zero change in volume of the material under the indenter”

$$\delta_1 = \frac{a^2}{4R}$$

(21)
Hence we have for the repulsive solution \( P_1 = 4\pi E^* \frac{R}{b} \delta_1^2 \) and therefore repeating the standard procedure, the adhesive solution is obtained as

\[
P = P_1 - P'_1 \sqrt{2wA'/P_1''} = 4\pi E^* \frac{R}{b} \delta_1 \left( \delta_1 - 2 \sqrt{\frac{w}{E^* b}} \right)
\] (22)

Finding the minimum,

\[
\delta_{1,PO} = \sqrt{\frac{w}{E^* b}}; \quad a_{PO} = 4R \left( \frac{w}{E^* b} \right)^{1/4}
\] (23)

(where notice that we have to assume \( a_{PO} \gg b \) to be consistent with the thin layer assumption). Hence, substituting \( \delta_{1,PO} \)

\[
P_{PO} = -4\pi Rw
\] (24)

and simply this is much larger than the JKR classical solution, but shows the same independence on elastic modulus, and, surprisingly, also on thickness of the layer (contrary to the line contact case). Notice however that

\[
\sigma_{PO} = \frac{P_{PO}}{\pi a_{PO}^2} = \frac{\sqrt{wE^*}}{\sqrt{b}} = \frac{K_{IC}}{\sqrt{b}}
\] (25)

and equalling \( \sigma_{PO} = \sigma_{th} \) we get

\[
b_{cr} = \frac{E^* w}{\sigma_{th}^2}
\] (26)

which is of the same order of \( b_{cr,frictionless} \) in the line contact case. Although the pull-off load seems close to the JKR value and does not depend on layer thickness, this can be made still ”optimal” as it is clear from a more complete comparison with the JKR classical solution

\[
P_{JKR} = \sqrt{8\pi E^* wa^2} - \frac{4}{3} E^* a^3
\] (27)

whose minimum (force control) is at \( P_{JKR,PO} = -\frac{3}{2}\pi Rw \) and \( a_{JKR,PO} = \frac{9\pi R^2 w}{8\pi E^*} \). Hence, the ratio of contact areas at pull-off

\[
\frac{a_{PO}}{a_{JKR,PO}} = \sqrt{\frac{4}{3}} \frac{b^{1/4}}{\sqrt{\frac{9\pi}{8} R^{1/6} \left( \frac{w}{E^*} \right)^{1/12}}}
\] (28)
and hence this shows for thin layers, the same pull-off load of the JKR solution is reached with a much smaller contact area (tending to zero — but should not be smaller than thickness for not violating the thin-film assumption): this therefore could serve to optimize the adhesive system. In particular, for the "optimal" layer thickness, the ratio of contact areas at pull-off

\[
\left( \frac{a_{PO}}{a_{JKR,PO}} \right)_{b=b_{cr}} = \frac{\sqrt{4} \left( \frac{E^*}{\sigma_{th}} \right)^{1/2} \left( \frac{w}{E^* R} \right)^{1/6}}{\sqrt{\frac{3}{2}} \left( \frac{9\pi}{8} \right)^{1/2} \left( w/E^* \right)^{1/12}} \tag{29}
\]

Therefore taking \( w = 10mJ/m^2, \sigma_{th} = 20MPa \) and \( E^* = 1GPa \), again as in the example of (Gao & Yao, 2004), we estimate \( b_{cr} = \frac{10^9 10^{-2}}{20 \cdot 10^6} = 2.5 \times 10^{-8}m \). However, we also have to check that this is feasible within the thin layer assumption. However, the two conditions, \( a_{PO} >> b \) (thin layer) and \( \frac{a_{PO}}{a_{JKR,PO}} << 1 \) (optimal adhesion) lead to the same dependences

\[
b << 2^{4/3} R^{2/3} \left( \frac{w}{E^*} \right)^{1/3} \text{ (thin layer)}
\]

\[
b << \frac{1}{24} \left( \frac{9\pi}{8} \right)^{4/3} R^{2/3} \left( \frac{w}{E^*} \right)^{1/3} \text{ (optimal adhesion)}
\]

of which the second dominates. Hence, indeed, having \( b < b_{cr} \) leads to a feasible regime of optimal adhesion.

Turning back to the complete solution, the indentation is obtained using

\[
\delta = \frac{a^2}{4R} - \sqrt{w \frac{b}{E^*}} \tag{30}
\]

and so extracting the equation for the contact area, and \( \frac{a^2}{4R} = \delta + \sqrt{w \frac{b}{E^*}} \),

using \( \delta_1 = \frac{a^2}{4R} = \delta + \sqrt{w \frac{b}{E^*}} \) and substituting back in the solution (22), we obtain the quite simple relationship between adhesive load and indentation

\[
\hat{P}_{axi} = \hat{\delta}^2 - 1 \tag{31}
\]

using the same dimensionless notation of line contact for \( \hat{\delta} \), but obviously not that for \( \hat{P} \)

\[
\hat{\delta} = \frac{\delta}{\sqrt{w \frac{b}{E^*}}} ; \quad \hat{P}_{axi} = \frac{P}{4\pi R w} \tag{32}
\]
so that $\hat{P}_{PO} = -1$ and $\hat{\delta}_{PO} = 0$. This is plotted in Fig.3 with the remark that $\delta_{PO} = 0$ and that there seems to be no instability under displacement control on separation.

![Fig.3. Load vs indentation curve for a rigid sphere indenting a layer on a frictionless rigid foundation. Only force control path is indicated.](image)

3.1. Bonded layer

For bonded layer the Poisson’s effect appears. In fact, Jaffar (1989) shows

$$P_1 = \frac{\pi}{4} E^* \frac{(1 - \nu)^2 a^4}{(1 - 2\nu) R b}$$

(33)

whereas we continue to use $\delta_1 = \frac{a^2}{4R}$. Therefore, $P_1 = 4\pi E^* \frac{(1 - \nu)^2 R b}{(1 - 2\nu) \delta_1^2}$ and the adhesive solution is obtained as

$$P = P_1 - P_1' \sqrt{2w/\alpha'} P_1'' = 4\pi E^* \frac{(1 - \nu)^2 R}{(1 - 2\nu) b} \left( \delta_1^2 - 2\delta_1 \sqrt{\frac{w}{(1 - \nu)^2 E^*}} \right)$$

(34)
Finding the minimum,
\[ \delta_{1,PO} = \sqrt{\frac{(1 - 2\nu)w}{(1 - \nu)^2 E^* b}} ; \quad a_{PO} = \sqrt{4R \left( \frac{(1 - 2\nu)w}{E^* (1 - \nu)^2 b} \right)^{1/4}} \] (35)
and hence we obtain the same pull-off of the frictionless foundation,
\[ P_{PO} = -4\pi Rw \] (36)
It is only the area of contact at pull-off which reduces and therefore the contact is indeed more "efficient" than the frictionless case.

3.1.1. Bonded incompressible layer
The last case that needs to be considered is the bonded case with incompressible layer, \( \nu = 0.5 \).

From Jaffar (1989) we have for the repulsive solution
\[ P_1 = \frac{\pi}{3 \times 2^5 E^*} \frac{a^6}{Rb^3} \]
therefore
\[ P = \frac{2}{3} \pi E^* R^2 \delta_1^{3/2} \left( \delta_1^{3/2} - 3 \sqrt{\frac{2w}{RE^* b^3}} \right) \] (37)
and pull-off is found at
\[ \delta_{1,PO} = \left( \frac{9b^3 w}{2E^* R} \right)^{1/3} \] (38)
and hence
\[ P_{PO} = -3\pi Rw \] (39)
Notice that for the case of incompressible layer the pull-off is lower and the indentation at pull-off depends on the radius of the sphere.

4. Conclusions
In this short communication, we show that ultrastrong adhesion can be reached both in line contact or in axisymmetric contact for contact of a Hertzian indenter with ultrathin layers, suggesting a new possible strategy for "optimal adhesion". There are some details which differ in plane contact vs axisymmetric contact: indeed, in line contact adhesion enhancement occurs as an increase of the actual pull-off force, while in axisymmetric case...
the latter is apparently very close to the classical JKR case. However, in both cases the enhancement occurs because the dominant length scale for the stress intensity factor at the contact edge is the layer thickness, and this induces a reduction of the size of contact needed to sustain the pull-off force. These effects are remarkably further enhanced by Poisson’s ratio effects in the case of nearly incompressible layer.

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