FOUR EXPLICIT FORMULAS FOR THE PROLONGATIONS
OF AN INFINITESIMAL LIE SYMMETRY
AND MULTIVARIATE FA`À DI BRUNO FORMULAS

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ABSTRACT. In 1979, building on S. Lie’s theory of symmetries of (partial) differential equations, P.J. Olver formulated inductive formulas which are appropriate for the computation of the prolongations of an infinitesimal Lie symmetry to jet spaces, for an arbitrary number \( n \geq 1 \) of independent variables \((x^1, \ldots, x^n)\) and for an arbitrary number \( m \geq 1 \) of dependent variables \((y^1, \ldots, y^m)\). This paper is devoted to elaborate a formalism based on multiple Kronecker symbols which enables one to handle these “unmanageable” prolongations and to discover the underlying complicated combinatorics. Proceeding progressively, we write down closed explicit formulas in four cases: \( n = m = 1 \); \( n \geq 1, m = 1 \); \( n = 1, m \geq 1 \); general case \( n \geq 1, m \geq 1 \). As a subpart of the obtained formulas, we recover four possible versions of the (multivariate) Fa`À di Bruno formula. We do not employ the classical formalism based on the symmetric algebra (cf. e.g. H. Federer’s book, p. 222), because it hides several explicit sums in symbolic compactifications, and because the presence of supplementary complexities (e.g. splitting of indices, combinatorics of partial derivatives) impedes us to apply such compactifications coherently. Our method of exposition is inductive: we conduct our reasonings by analyzing several thoroughly organized formulas, by comparing them together and by “drifting” towards generality, in homology with the classical style of L. Euler.

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§1. JET SPACES AND PROLONGATIONS

1.1. Choice of notations for the jet space variables. Let \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Let \( n \geq 1 \) and \( m \geq 1 \) be two positive integers and consider two sets of variables \( x = (x^1, \ldots, x^n) \in \mathbb{K}^n \) and \( y = (y^1, \ldots, y^m) \). In the classical theory of Lie symmetries of partial differential equations, one considers certain differential systems whose (local) solutions should be mappings of the form \( y = y(x) \). We refer to [OL1986] and to [BK1989] for an exposition of the fundamentals of the theory. Accordingly, the variables \( x \) are usually called independent, whereas the variables \( y \) are called dependent. Not to enter in subtle regularity considerations (as in [M2005]), we shall assume \( C^\infty \)-smoothness of all functions throughout this paper.

Let \( \kappa \geq 1 \) be a positive integer. For us, in a very concrete way (without fiber bundles), the \( \kappa \)-th jet space \( \mathcal{J}_n^\kappa \) consists of the space \( \mathbb{K}^{n+m+\kappa m} \) equipped with the affine coordinates

\[
(x^i, y^j, y_{i_1}^j, y_{i_2}^j, \ldots, y_{i_{\kappa}}^j)
\]

having the symmetries

\[
y_{i_1}^j, y_{i_2}^j, \ldots, y_{i_{\kappa}}^j = y_{i_{\sigma(1)}}^j, y_{i_{\sigma(2)}}^j, \ldots, y_{i_{\sigma(\kappa)}}^j,
\]

\( \sigma \) being any permutation of \( \{1, 2, \ldots, \kappa \} \).
for every \( \lambda \) with \( 1 \leq \lambda \leq \kappa \) and for every permutation \( \sigma \) of the set \( \{1, \ldots, \lambda\} \). The variable \( y^j_{i_1, i_2, \ldots, i_\lambda} \) is an independent coordinate corresponding to the \( \lambda \)-th partial derivative \( \frac{\partial^\lambda y^j}{\partial x^{i_1} \cdots \partial x^{i_\lambda}} \), which explains the symmetries (1.3).

In the classical Lie theory (OL1979, OL1986, BK1989), all the geometric objects: point transformations, vector fields, \( etc. \), are local, defined in a neighborhood of some point lying in some affine space \( \mathbb{K}^N \). However, in this paper, the original geometric motivations are rapidly forgotten in order to focus on combinatorial considerations. Thus, to simplify the presentation, we shall not introduce any special notation to speak of certain local open subsets of \( \mathbb{K}^{n+m} \), of the jet space \( J_{n,m}^{\kappa} = \mathbb{K}^{n+m+m(\kappa+n)} \), \( etc. \): we will always work in global affine spaces \( \mathbb{K}^N \).

### 1.4. Prolongation \( \varphi^{(\kappa)} \) of a local diffeomorphism \( \varphi \) to the \( \kappa \)-th jet space.

In this paragraph, we recall how the prolongation of a diffeomorphism to the \( \kappa \)-th jet space is defined (OL1979, OL1986, BK1989).

Let \( x_* \in \mathbb{K}^n \) be a central fixed point and let \( \varphi : \mathbb{K}^{n+m} \to \mathbb{K}^{n+m} \) be a diffeomorphism whose Jacobian matrix is close to the identity matrix at least in a small neighborhood of \( x_* \). Let

\[
J^s_{x_*} := (x^*, y_{i_1}^1, y_{i_2}^2, \ldots, y_{i_{\kappa}}^\kappa) \in J_{n,m}^{\kappa} \bigg|_{x_*}
\]

be an arbitrary \( \kappa \)-jet based at \( x_* \). The goal is to define its transformation \( \varphi^{(\kappa)}(J^s_{x_*}) \) by \( \varphi \).

To this aim, choose an arbitrary mapping \( \mathbb{K}^{n} \ni x \mapsto g(x) \in \mathbb{K}^{m} \) defined at least in a neighborhood of \( x_* \) and representing this \( \kappa \)-jet, i.e. satisfying

\[
y^j_{i_1, i_2, \ldots, i_{\kappa}} = \frac{\partial^\lambda g^j}{\partial x^{i_1} \cdots \partial x^{i_{\kappa}}}(x_*),
\]

for every \( \lambda \in \mathbb{N} \) with \( 0 \leq \lambda \leq \kappa \), for all indices \( i_1, \ldots, i_{\kappa} \) with \( 1 \leq i_1 \leq \ldots \leq i_{\kappa} \leq n \) and for every \( j \in \mathbb{N} \) with \( 1 \leq j \leq m \). In accordance with the splitting \( (x, y) \in \mathbb{K}^{n} \times \mathbb{K}^{m} \) of coordinates, split the components of the diffeomorphism \( \varphi \) as \( \varphi = (\varphi, \psi) \in \mathbb{K}^{n} \times \mathbb{K}^{m} \). Write \( (\overline{x}, \overline{y}) \) the coordinates in the target space, so that the diffeomorphism \( \varphi \) is:

\[
\mathbb{K}^{n+m} \ni (x, y) \mapsto (\overline{x}, \overline{y}) = (\varphi(x, y), \psi(x, y)) \in \mathbb{K}^{n+m}.
\]

Restrict the variables \( (x, y) \) to belong to the graph of \( g \), namely put \( y := g(x) \) above, which yields

\[
\begin{cases}
\overline{x} = \varphi(x, g(x)), \\
\overline{y} = \psi(x, g(x)).
\end{cases}
\]

As the differential of \( \varphi \) at \( x_* \) is close to the identity, the first family of \( n \) scalar equations may be solved with respect to \( x \), by means of the implicit function theorem. Denote \( x = \overline{x}(\overline{x}) \) the resulting mapping, satisfying by definition

\[
\overline{x} \equiv \varphi(\overline{x}(\overline{x}), g(\overline{x}(\overline{x})))
\]

Replace \( x \) by \( \overline{x}(\overline{x}) \) in the second family of \( m \) scalar equations (1.8) above, which yields:

\[
\overline{y} = \psi(\overline{x}(\overline{x}), g(\overline{x}(\overline{x}))).
\]

Denote simply by \( \overline{y} = \overline{y}(\overline{x}) \) this last relation, where \( \overline{y}(\cdot) := \psi(\overline{x}(\cdot), g(\overline{x}(\cdot))). \)

In summary, the graph \( y = g(x) \) has been transformed to the graph \( \overline{y} = \overline{y}(\overline{x}) \) by the diffeomorphism \( \varphi \) whose first order approximation is close to the identity.

Define then the transformed jet \( \varphi^{(\kappa)}(J^s_{x_*}) \) to be the \( \kappa \)-th jet of \( \overline{y} \) at the point \( \overline{x}_* := \varphi(x_*) \), namely:

\[
\varphi^{(\kappa)}(J^s_{x_*}) := \left( \frac{\partial^\lambda \overline{y}^j}{\partial x^{i_1} \cdots \partial x^{i_{\kappa}}} \right)_{1 \leq j \leq m}^{1 \leq i_1, \ldots, i_{\kappa} \leq n, 0 \leq \lambda \leq \kappa} \in J_{n,m}^{\kappa} \bigg|_{\overline{x}_*}\]
It may be shown that this jet does not depend on the choice of a local graph \( y = g(x) \) representing the \( \kappa \)-jet \( J^\kappa_x \) at \( x_* \). Furthermore, if \( \pi_\kappa := J^\kappa \to \mathbb{K}^m \) denotes the canonical projection onto the first factor, the following diagram commutes:

\[
\begin{array}{ccc}
J^\kappa_{n,m} & \xrightarrow{\varphi(\kappa)} & J^\kappa_{n,m} \\
\pi_\kappa \downarrow & & \downarrow \pi_\kappa \\
\mathbb{K}^{n+m} & \xrightarrow{\varphi} & \mathbb{K}^{n+m}
\end{array}
\]

### 1.12. Inductive formulas for the \( \kappa \)-th prolongation \( \varphi(\kappa) \)

To present them, we change our notations. Instead of \((\mathcal{X}, \mathcal{Y})\), as coordinates in the target space \( \mathbb{K}^n \times \mathbb{K}^m \), we shall use capital letters:

\[
(1.13) \quad (X^1, \ldots, X^n, Y^1, \ldots, Y^m).
\]

In the source space \( \mathbb{K}^{n+m} \) equipped with the coordinates \((x, y)\), we use the jet coordinates (1.2) on the associated \( \kappa \)-th jet space. In the target space \( \mathbb{K}^{n+m} \) equipped with the coordinates \((X, Y)\), we use the coordinates

\[
(1.14) \quad (X^i, Y^j, Y^j_{X^i}, Y^j_{X^i, X^j}, \ldots, Y^j_{X^i,\ldots, X^j})
\]

on the associated \( \kappa \)-th jet space. In these notations, the diffeomorphism \( \varphi \) whose first order approximation is close to the identity mapping in a neighborhood of \( x_* \), may be written under the form:

\[
(1.15) \quad \varphi : (x', y') \mapsto (X^i, Y^j) = \left(X^i(x', y'), Y^j(x', y')\right),
\]

for some \( C^\infty \)-smooth functions \( X^i(x', y') \), \( i = 1, \ldots, n \), and \( Y^j(x', y') \), \( j = 1, \ldots, m \). The first prolongation \( \varphi(1) \) of \( \varphi \) may be written under the form:

\[
(1.16) \quad \varphi(1) : (x', y', y'_{i_1}) \mapsto (X^i(x', y'), Y^j(x', y'), Y^j_{X^i}, (x', y', y'_{i_1})),
\]

for some functions \( Y^j_{X^{i_1}} \) which depend on the pure first jet variables \( y_{i_1}' \). The way how these functions depend on the first order partial derivatives functions \( X^i_x \), \( X^i_{x^j} \), \( Y^j_x \), \( Y^j_{x^j} \) and on the pure first jet variables \( y_{i_1}' \) is provided (in principle) by the following compact formulas (BK1989):

\[
(1.17) \quad \begin{pmatrix}
Y^j_{X^{i_1}} \\
\vdots \\
Y^j_{X^n}
\end{pmatrix} = \begin{pmatrix}
D^1_{i_1}X^1 & \cdots & D^1_{i_1}X^n \\
\vdots & \ddots & \vdots \\
D^1_{i_1}X^1 & \cdots & D^1_{i_1}X^n
\end{pmatrix}^{-1} \begin{pmatrix}
D^1_{i_1}Y^j \\
\vdots \\
D^1_{i_1}Y^j
\end{pmatrix},
\]

where, for \( i' = 1, \ldots, n \), the \( D^1_{i'} \) denote the \( i' \)-th first order total differentiation operators:

\[
(1.18) \quad D^1_{i'} := \frac{\partial}{\partial x^{i'}} + \sum_{j'=1}^m y_{i_1}' \frac{\partial}{\partial y^{j'}}.
\]

Strictly speaking, these formulas (1.17) are not explicit, because an inverse matrix is involved and because the terms \( D^1_{i_1}X^i \), \( D^1_{i_1}Y^j \) are not developed. However, it would be elementary to write down the corresponding totally explicit complete formulas for the functions \( Y^j_{X^{i_1}} = Y^j_{X^{i_1}}(x', y', y_{i_1}') \).

Next, the second prolongation \( \varphi(2) \) is of the form

\[
(1.19) \quad \varphi(2) : (x', y', y_{i_1}', y_{i_2}') \mapsto \varphi(1) \left( x', y', y_{i_1}' \right), \quad Y^j_{X^{i_1}, X^{i_2}} = Y^j_{X^{i_1}, X^{i_2}}(x', y', y_{i_1}', y_{i_2}').
\]
for some functions $Y_{X^i_1\ldots X^i_n}^j(x', y', y'_1, y'_{i_{1}}, \ldots, y'_{i_{m}}, Y_{X^i_1\ldots X^i_n}^j)$ which depend on the pure first and second jet variables. For $i = 1, \ldots, n$, the expressions of $Y_{X^i_1\ldots X^i_n}^j$ are given by the following compact formulas (again [BK1989]):

\[
\begin{pmatrix}
Y_{X^i_1\ldots X^i_n}^j \\
Y_{X^i_1\ldots X^i_n}^j \\
\vdots \\
Y_{X^i_1\ldots X^i_n}^j \\
Y_{X^i_1\ldots X^i_n}^j
\end{pmatrix} = 
\begin{pmatrix}
D_1^1 X^1 & \cdots & D_1^1 X^n \\
\vdots & \ddots & \vdots \\
D_n^1 X^1 & \cdots & D_n^1 X^n
\end{pmatrix}^{-1}
\begin{pmatrix}
D_1^n Y_{X^i_1\ldots X^i_n}^j \\
\vdots \\
D_n^n Y_{X^i_1\ldots X^i_n}^j
\end{pmatrix},
\]

where, for $i' = 1, \ldots, n$, the $D_{i'}^2$ denote the $i'$-th second order total differentiation operators:

\[
D_{i'}^2 := \frac{\partial}{\partial x'^{i'}} + \sum_{j'=1}^{m} y'_{i'} \frac{\partial}{\partial y'^{j'}} + \sum_{j'=1}^{m} \sum_{i'_1=1}^{n} y'_{i'_1,i'} \frac{\partial}{\partial y_{i'_1,i'}^{j'}} + \sum_{j'=1}^{m} \sum_{i'_1=1}^{n} \sum_{i'_2=1}^{n} y'_{i'_1,i'_2,i'} \frac{\partial}{\partial y_{i'_1,i'_2,i'}^{j'}} + \cdots + \sum_{j'=1}^{m} \sum_{i'_1=1}^{n} \cdots \sum_{i'_{-1}=1}^{n} y'_{i'_1,i'_2, \ldots, i'_{-1}} \frac{\partial}{\partial y_{i'_1,i'_2, \ldots, i'_{-1}}^{j'}}.
\]

Then, for $i = 1, \ldots, n$, the expressions of $Y_{X^i_1\ldots X^i_{\lambda-1}X^i}^j$ are given by the following compact formulas (again [BK1989]):

\[
\begin{pmatrix}
Y_{X^i_1\ldots X^i_{\lambda-1}X^i}^j \\
\vdots \\
Y_{X^i_1\ldots X^i_{\lambda-1}X^i}^j
\end{pmatrix} = 
\begin{pmatrix}
D_1^1 X^1 & \cdots & D_1^1 X^n \\
\vdots & \ddots & \vdots \\
D_n^1 X^1 & \cdots & D_n^1 X^n
\end{pmatrix}^{-1}
\begin{pmatrix}
D_1^n Y_{X^i_1\ldots X^i_{\lambda-1}X^i}^j \\
\vdots \\
D_n^n Y_{X^i_1\ldots X^i_{\lambda-1}X^i}^j
\end{pmatrix}. 
\]

Again, these inductive formulas are incomplete and unsatisfactory.

**Problem 1.24.** Find totally explicit complete formulas for the $\kappa$-th prolongation $\varphi^{(\kappa)}$.

Except in the cases $\kappa = 1, 2$, we have not been able to solve this problem. The case $\kappa = 1$ is elementary. Complete formulas in the particular cases $\kappa = 2$, $n = 1$, $m \geq 1$ and $n \geq 1$, $m = 1$ are implicitly provided in [M2004a] and in [M2004b], where one observes the appearance of some modifications of the Jacobian determinant of the diffeomorphism $\varphi$, inserted in a clearly understandable combinatorics. In fact, there is a nice dictionary between the formulas for $\varphi^{(2)}$ and the formulas for the second prolongation $\mathcal{L}^{(2)}$ of a vector field $\mathcal{L}$ which were written in equation (43) of [GM2003] (see also equations (2.6), (3.20), (4.6) and (5.3) in the next paragraphs).

In the passage from $\varphi^{(2)}$ to $\mathcal{L}^{(2)}$, a sort of formal first order linearization may be observed and the reverse passage may be easily guessed. However, for $\kappa \geq 3$, the formulas for $\varphi^{(\kappa)}$ explode faster than the formulas for the $\kappa$-th prolongation $\mathcal{L}^{(\kappa)}$ of a vector field $\mathcal{L}$. Also, the dictionary between $\varphi^{(\kappa)}$ and $\mathcal{L}^{(\kappa)}$ disappears. In fact, to elaborate an appropriate dictionary, we believe that one should introduce before a sort of formal $(\kappa - 1)$ order linearizations of $\varphi^{(\kappa)}$, finer than the first order linearization $\mathcal{L}^{(\kappa)}$. To be optimistic, we believe that the final answer to Problem 1.24 is accessible.
The present article is devoted to present totally explicit complete formulas for the $\kappa$-th prolongation $\mathcal{L}^{(\kappa)}$ of a vector field $\mathcal{L}$ to $\mathcal{J}_{n,m}^\kappa$, for $n \geq 1$ arbitrary, for $m \geq 1$ arbitrary and for $\kappa \geq 1$ arbitrary.

1.25. Prolongation of a vector field to the $\kappa$-th jet space. Consider a vector field

\begin{equation}
\mathcal{L} = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \gamma^j \frac{\partial}{\partial y^j},
\end{equation}

defined in $\mathbb{K}^{n+m}$. Its flow:

\begin{equation}
\varphi(t, x, y) := \exp(t\mathcal{L})(x, y)
\end{equation}

constitutes a one-parameter of diffeomorphisms of $\mathbb{K}^{n+m}$ close to the identity. The lift $(\varphi(t))^{(\kappa)}$ to the $\kappa$-th jet space constitutes a one-parameter family of diffeomorphisms of $\mathcal{J}_{n,m}^\kappa$. By definition, the $\kappa$-th prolongation $\mathcal{L}^{(\kappa)}$ of $\mathcal{L}$ to the jet space $\mathcal{J}_{n,m}^\kappa$ is the infinitesimal generator of $(\varphi(t))^{(\kappa)}$, namely:

\begin{equation}
\mathcal{L}^{(\kappa)} := \left. \frac{d}{dt} \right|_{t=0} [(\varphi(t))^{(\kappa)}].
\end{equation}

1.29. Inductive formulas for the $\kappa$-th prolongation $\mathcal{L}^{(\kappa)}$. As a vector field defined in $\mathbb{K}^{n+m}$, the $\kappa$-th prolongation $\mathcal{L}^{(\kappa)}$ may be written under the general form:

\begin{equation}
\mathcal{L}^{(\kappa)} = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \gamma^j \frac{\partial}{\partial y^j} + \sum_{j=1}^{m} \sum_{i_1=1}^{n} Y_{i_1}^{j} \frac{\partial}{\partial y_{i_1}^j} + \sum_{j=1}^{m} \sum_{i_1,i_2=1}^{n} Y_{i_1,i_2}^{j} \frac{\partial}{\partial y_{i_1,i_2}^j} + \cdots + \sum_{j=1}^{m} \sum_{i_1,...i_n=1}^{n} Y_{i_1,...i_n}^{j} \frac{\partial}{\partial y_{i_1,...i_n}^j}.
\end{equation}

Here, the coefficients $Y_{i_1}^{j}$, $Y_{i_1,i_2}^{j}$, ..., $Y_{i_1,...i_n}^{j}$ are uniquely determined in terms of partial derivatives of the coefficients $x^i$ and $\gamma^j$ of the original vector field $\mathcal{L}$, together with the pure jet variables $(y_{i_1}^j, ..., y_{i_n}^j)$, by means of the following fundamental inductive formulas (OL1979, OL1986, BK1989):

\begin{equation}
\begin{cases}
Y_{i_1}^{j} := D_{i_1}^1 (\gamma^j) - \sum_{k=1}^{n} D_{i_1}^1 (x^k) y_{i_1}^k, \\
Y_{i_1,i_2}^{j} := D_{i_2}^2 (Y_{i_1}^{j}) - \sum_{k=1}^{n} D_{i_2}^2 (x^k) y_{i_1,i_2}^{j,k}, \\
\cdots \\
Y_{i_1,...i_n}^{j} := D_{i_n}^n (Y_{i_1,...i_{n-1}}^{j}) - \sum_{k=1}^{n} D_{i_n}^n (x^k) y_{i_1,...i_n}^{j,k},
\end{cases}
\end{equation}

where, for every $\lambda \in \mathbb{N}$ with $0 \leq \lambda \leq n$, and for every $i \in \mathbb{N}$ with $1 \leq i \leq n$, the $i$-th $\lambda$-th order total differentiation operator $D_{i}^\lambda$ was defined in (1.22) above.

**Problem 1.32.** Applying these inductive formulas, find totally explicit complete formulas for the $\kappa$-th prolongation $\mathcal{L}^{(\kappa)}$.

The present article is devoted to provide all the desired formulas.
1.33. Inductive methodology. We have the intention of presenting our results in a purely inductive style, based on several thorough visual comparisons between massive formulae which will be written and commented in four different cases:

(i) \( n = 1 \) and \( m = 1; \kappa \geq 1 \) arbitrary;
(ii) \( n \geq 1 \) and \( m = 1; \kappa \geq 1 \) arbitrary;
(iii) \( n = 1 \) and \( m \geq 1; \kappa \geq 1 \) arbitrary;
(iv) general case: \( n \geq 1 \) and \( m \geq 1; \kappa \geq 1 \) arbitrary.

Accordingly, we shall particularize and slightly lighten our notations in each of the three (preliminary) cases (i) [Section 2], (ii) [Section 3] and (iii) [Section 4].

§2. ONE INDEPENDENT VARIABLE AND ONE DEPENDENT VARIABLE

2.1. Simplified adapted notations. Assume \( n = 1 \) and \( m = 1 \), let \( \kappa \in \mathbb{N} \) with \( \kappa \geq 1 \) and simply denote the jet variables by:

\[
(x, y, y_1, y_2, \ldots, y_\kappa) \in \mathcal{J}_{1,1}^\kappa.
\]

The \( \kappa \)-th prolongation of a vector field will be denoted by:

\[
\mathcal{L} = \mathcal{X} \frac{\partial}{\partial x} + \mathcal{Y} \frac{\partial}{\partial y} + Y_1 \frac{\partial}{\partial y_1} + Y_2 \frac{\partial}{\partial y_2} + \cdots + Y_\kappa \frac{\partial}{\partial y_\kappa}.
\]

The coefficients \( Y_1, Y_2, \ldots, Y_\kappa \) are computed by means of the inductive formulas:

\[
\begin{align*}
Y_1 & := D^1(\mathcal{Y}) - D^1(\mathcal{X}) y_1, \\
Y_2 & := D^2(Y_1) - D^1(\mathcal{X}) y_2, \\
& \quad \quad \vdots \\
Y_\kappa & := D^\kappa(Y_{\kappa-1}) - D^1(\mathcal{X}) y_\kappa,
\end{align*}
\]

where, for \( 1 \leq \lambda \leq \kappa \):

\[
D^\lambda := \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \cdots + y_\lambda \frac{\partial}{\partial y_{\lambda-1}}.
\]

By direct elementary computations, for \( \kappa = 1 \) and for \( \kappa = 2 \), we obtain the following two very classical formulas:

\[
\begin{align*}
Y_1 &= \mathcal{Y}_x + [\mathcal{Y}_y - \mathcal{X}_x] y_1 + [-\mathcal{Y}_y] (y_1)^2, \\
Y_2 &= \mathcal{Y}_{x^2} + [2 \mathcal{Y}_{xy} - \mathcal{X}_{x^2}] y_1 + [\mathcal{Y}_{y^2} - 2 \mathcal{X}_{xy}] (y_1)^2 + [-\mathcal{X}_y] (y_1)^3 + \\
& \quad \quad [\mathcal{Y}_y - 2 \mathcal{X}_x] y_2 + [-3 \mathcal{X}_y] y_1 y_2.
\end{align*}
\]

Our main objective is to devise the general combinatorics. Thus, to attain this aim, we have to achieve patiently formal computations of the next coefficients \( Y_3, Y_4 \) and \( Y_5 \). We systematically use parentheses \([\cdot]\) to single out every coefficient of the polynomials \( Y_3, Y_4 \) and \( Y_5 \) in the pure jet variables \( y_1, y_2, y_3, y_4 \) and \( y_5 \), putting every sign inside these parentheses. We always put the monomials in the pure jet variables \( y_1, y_2, y_3, y_4 \) and \( y_5 \) after the parentheses. For completeness, let us provide the intermediate computation of the third coefficient \( Y_3 \). In detail:

\[
Y_3 = D^3(\mathcal{Y}_2) - D^1(\mathcal{X}) y_3
\]

\[
= \left( \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} \right) \left( \mathcal{Y}_{x^2} + [2 \mathcal{Y}_{xy} - \mathcal{X}_{x^2}] y_1 + \\
+ [\mathcal{Y}_y - 2 \mathcal{X}_x] (y_1)^2 + [-\mathcal{X}_y] (y_1)^3 + [\mathcal{Y}_y - 2 \mathcal{X}_x] y_2 + [-3 \mathcal{X}_y] y_1 y_2 \right)
\]
\[\begin{align*}
&= Y_{x^3} + \left[2 Y_{x^2 y} - X_{x^3}\right] y_1 + \left[Y_{xy^2} - 2 X_{x^2 y}\right] (y_1)^2 + \left[-X_{xy^2}\right] (y_1)^3 + \\
&+ \left[Y_{xy} - 2 X_{xy}\right] y_2 + \left[-3 X_{xy}\right] y_1 y_2 + \left[Y_{y^2}\right] y_1 + \\
&+ \left[2 Y_{xy^2} - X_{x^2 y}\right] (y_1)^2 + \left[Y_{y^3} - 2 X_{xy^2}\right] (y_1)^3 + \left[-X_{y^3}\right] (y_1)^4 + \\
&+ \left[Y_{y^2} - 2 X_{xy}\right] y_1 y_2 + \left[-3 X_{y^2}\right] (y_1)^2 y_2 + \left[2 Y_{xy} - X_{x^2 y}\right] y_2 + \\
&+ \left[Y_{y^2} - 2 X_{xy}\right] y_1 y_2 + \left[-3 X_{y^2}\right] 3(y_1)^2 y_2 + \left[-3 X_{y^2}\right] (y_2)^2 + \\
&+ \left[Y_{y} - 2 X_{xy}\right] y_3 + \left[-3 X_{y^2}\right] y_1 y_3 + \\
&- \left[X_{xy}\right] y_1 - \left[X_{y^2}\right] y_1 y_3 + \left[X_{y}\right] y_3 y_4 \right].
\end{align*}\]

We have underlined all the terms with a number appended. Each number refers to the order of appearance of the terms in the final simplified expression of \(Y_3\), also written in [BK1985] with different notations:

\[\begin{align*}
Y_3 = Y_{x^3} + \left[3 Y_{x^2 y} - X_{x^3}\right] y_1 + \left[3 Y_{xy^2} - 3 X_{x^2 y}\right] (y_1)^2 + \\
+ \left[Y_{y^3} - 3 X_{xy^2}\right] (y_1)^3 + \left[-X_{y^3}\right] (y_1)^4 + \left[3 Y_{xy} - 3 X_{xy}\right] y_2 + \\
+ \left[3 Y_{y^2} - 9 X_{xy^2}\right] y_1 y_2 + \left[-6 X_{y^2}\right] (y_1)^2 y_2 + \left[-3 X_{y^2}\right] (y_2)^2 + \\
+ \left[Y_{y} - 3 X_{xy}\right] y_3 + \left[-4 X_{y}\right] y_1 y_3.
\end{align*}\]

After similar manual computations, the intermediate details of which we will not copy in this LaTeX file, we get the desired expressions of \(Y_4\) and of \(Y_5\). Firstly:

\[\begin{align*}
Y_4 = Y_{x^4} + \left[4 Y_{x^3 y} - X_{x^4}\right] y_1 + \left[6 Y_{x^2 y^2} - 4 X_{x^2 y}\right] (y_1)^2 + \\
+ \left[4 Y_{xy^3} - 6 X_{x^2 y^2}\right] (y_1)^3 + \left[Y_{y^4} - 4 X_{xy^3}\right] (y_1)^4 + \left[-X_{y^4}\right] (y_1)^5 + \\
+ \left[6 Y_{y^2} - 4 X_{xy}\right] y_2 + \left[12 Y_{x^2 y^2} - 18 X_{xy}\right] y_1 y_2 + \\
+ \left[6 Y_{y^3} - 24 X_{xy^2}\right] (y_1)^2 y_2 + \left[-10 X_{y^3}\right] (y_1)^3 y_2 + \\
+ \left[3 Y_{y^2} - 12 X_{xy}\right] (y_2)^2 + \left[-15 X_{y^2}\right] y_4 (y_2)^2 + \\
+ \left[4 Y_{xy} - 6 X_{xy}\right] y_3 + \left[4 Y_{y^2} - 16 X_{xy}\right] y_1 y_3 + \left[-10 X_{y^2}\right] (y_1)^2 y_3 + \\
+ \left[-10 X_{y}\right] y_2 y_3 + \left[Y_{y} - 4 X_{y}\right] y_4 + \left[-5 X_{y}\right] y_1 y_4.
\end{align*}\]

Secondly:

\[\begin{align*}
Y_5 = Y_{x^5} + \left[5 Y_{x^4 y} - X_{x^5}\right] y_1 + \left[10 Y_{x^3 y^2} - 5 X_{x^4 y}\right] (y_1)^2 + \\
+ \left[10 Y_{x^2 y^3} - 10 X_{x^3 y^2}\right] (y_1)^3 + \left[5 Y_{xy^4} - 10 X_{x^2 y^3}\right] (y_1)^4 + \\
+ \left[Y_{y^5} - 5 X_{xy^4}\right] (y_1)^5 + \left[-X_{y^5}\right] (y_1)^6 + \left[10 Y_{y^2} - 5 X_{xy}\right] y_2 + \\
+ \left[30 Y_{x^2 y^2} - 30 X_{x^3 y}\right] y_1 y_2 + \left[30 Y_{xy^3} - 60 X_{x^2 y^2}\right] (y_1)^2 y_2 + \\
+ \left[10 Y_{y^4} - 50 X_{xy^2}\right] (y_1)^3 y_2 + \left[-15 X_{y^4}\right] (y_1)^3 y_2 + \\
+ \left[15 Y_{xy^2} - 30 X_{x^2 y}\right] (y_2)^2 + \left[15 Y_{y^5} - 75 X_{xy^2}\right] y_1 (y_2)^2 + \\
+ \left[-45 X_{y^2}\right] (y_1)^2 (y_2)^2 + \left[-15 X_{y^2}\right] (y_2)^3 + \\
+ \left[10 Y_{xy^2} - 10 X_{y^3}\right] y_3 + \left[20 Y_{x^2 y^3} - 40 X_{xy^2}\right] y_1 y_3 + \\
+ \left[10 Y_{y^2} - 50 X_{xy^2}\right] (y_1)^2 y_3 + \left[-20 X_{y^2}\right] (y_1)^3 y_3 + \\
+ \left[10 Y_{y} - 50 X_{xy}\right] y_2 y_3 + \left[-60 X_{y^2}\right] y_1 y_2 y_3 + \left[-10 X_{y}\right] (y_1)^2 + \\
+ \left[5 Y_{y^2} - 10 X_{xy}\right] y_4 + \left[5 Y_{xy} - 25 X_{x^2 y}\right] y_1 y_4 + \left[-15 X_{y^2}\right] (y_1)^2 y_4 + \\
+ \left[-15 X_{y}\right] y_2 y_4 + \left[Y_{y} - 5 X_{y}\right] y_5 + \left[-6 X_{y}\right] y_1 y_5.
\end{align*}\]
2.11. Formal inspection, formal intuition and formal induction. Now, we have to comment these formulas. We have written in length the five polynomials $Y_1$, $Y_2$, $Y_3$, $Y_4$ and $Y_5$ in the pure jet variables $y_1, y_2, y_3, y_4$ and $y_5$. Except the first “constant” term $Y_{x^\kappa}$, all the monomials in the expression of $Y_\kappa$ are of the general form

\[(y_{\lambda_1})^{\mu_1} (y_{\lambda_2})^{\mu_2} \cdots (y_{\lambda_d})^{\mu_d},\]

for some positive integer $d \geq 1$, for some collection of strictly increasing jets indices:

\[1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq \kappa,
\]

and for some positive integers $\mu_1, \ldots, \mu_d \geq 1$. This and the next combinatorial facts may be confirmed by reading the formulas giving $Y_1$, $Y_2$, $Y_3$, $Y_4$ and $Y_5$. It follows that the integer $d$ satisfies the inequality $d \leq \kappa + 1$. To include the first “constant” term $Y_{x^\kappa}$, we shall make the convention that putting $d = 0$ in the monomial (2.12) yields the constant term $1$. Furthermore, by inspecting the formulas giving $Y_1$, $Y_2$, $Y_3$, $Y_4$ and $Y_5$, we see that the following inequality should be satisfied:

\[\mu_1 \lambda_1 + \mu_2 \lambda_2 + \cdots + \mu_d \lambda_d \leq \kappa + 1.\]

For instance, in the expression of $Y_4$, the two monomials $(y_1)^3 y_2$ and $y_1 (y_2)^2$ do appear, but the two monomials $(y_1)^2 y_2$ and $(y_2)^3$ cannot appear. All coefficients of the pure jet monomials are of the general form:

\[A Y_{x^{\alpha} y^{\beta + 1}} - B X_{x^{\alpha+1} y^{\beta}},\]

for some nonnegative integers $A, B, \alpha, \beta \in \mathbb{N}$. Sometimes $A$ is zero, but $B$ is zero only for the (constant, with respect to pure jet variables) term $Y_{x^\kappa}$. Importantly, $X$ is differentiated once more with respect to $x$ and $Y$ is differentiated once more with respect to $y$. Again, this may be confirmed by reading all the terms in the formulas for $Y_1$, $Y_2$, $Y_3$, $Y_4$ and $Y_5$.

In addition, we claim that there is a link between the couple $(\alpha, \beta)$ and the collection $\{\mu_1, \lambda_1, \ldots, \mu_d, \lambda_d\}$. To discover it, let us write some of the monomials appearing in the expressions of $Y_4$ (first column) and of $Y_5$ (second column), for instance:

\[
\begin{align*}
6 Y_{x^2 y^2} - 4 X_{x^3 y^1} & \quad (y_1)^2, \\
12 Y_{x y^2} - 18 X_{x^2 y^1} & \quad y_1 y_2, \\
-10 X_{y^3} & \quad (y_1)^3 y_2, \\
4 Y_{y^2} - 16 X_{x y^1} & \quad y_1 y_3, \\
-10 Y_{y^2} & \quad (y_1)^2 y_3,
\end{align*}
\]

After some reflection, we discover the hidden intuitive rule: the partial derivatives of $Y$ and of $X$ associated with the monomial $(y_{\lambda_1})^{\mu_1} \cdots (y_{\lambda_d})^{\mu_d}$ are, respectively:

\[
\begin{align*}
\{ Y_{x^{\kappa-\mu_1} \lambda_1 - \cdots - \mu_d \lambda_d y^1 + \cdots + \mu_d} , \\
X_{x^{\kappa-\mu_1} \lambda_1 - \cdots - \mu_d \lambda_d + 1 y^{1} + \cdots + \mu_d - 1} ,
\end{align*}
\]

This may be checked on each of the 10 examples (2.16) above.

Now that we have explored and discovered the combinatorics of the pure jet monomials, of the partial derivatives and of the complete sum giving $Y_\kappa$, we may express that it is of the following
Lemma 2.19. For $\kappa \geq 1$,

$$
Y_\kappa = Y_{2^\kappa} + \sum_{d=1}^{\kappa+1} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d \leq \kappa+1} \left[ A_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)} \cdot Y_{2^{\kappa-\mu_1-\cdots-\mu_d}} y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+1} - B_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)} \cdot X_{2^{\kappa-\mu_1-\cdots-\mu_d}} y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+1} \right] y_{\lambda_1}^{\mu_1} \cdots y_{\lambda_d}^{\mu_d}.
$$

(2.18)

Here, we separate the first term $Y_{2^\kappa}$ from the general sum; it is the constant term in $Y_\kappa$, which is a polynomial with respect to the jet variables $y_{\lambda}$. In this general formula, the only remaining unknowns are the nonnegative integer coefficients $A_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)}$ and $B_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)} \in \mathbb{N}$. In Section 3 below, we shall explain how we have discovered their exact value.

At present, even if we are unable to devise their explicit expression, we may observe that the value of the special integer coefficients $A_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)}$ and $B_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)}$ which are attached to the monomials $ct., y_1, (y_1)^2, (y_1)^3, (y_1)^4$ and $(y_1)^5$ are simple. Indeed, by inspecting the first terms in the expressions of $Y_1, Y_2, Y_3, Y_4$ and $Y_5$, we of course recognize the binomial coefficients. In general:

**Lemma 2.19.** For $\kappa \geq 1$,

$$
Y_\kappa = Y_{2^\kappa} + \sum_{\lambda=1}^{\kappa} \left[ \binom{\kappa}{\lambda} y_{2^{\kappa-\lambda}} \lambda^{\kappa-\lambda} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+1} (y_1)^{\mu_1} \cdots (y_\lambda)^{\mu_d} + \text{remainder},
$$

(2.20)

where the term remainder collects all remaining monomials in the pure jet variables.

In addition, let us remind what we have observed and used in a previous co-signed work.

**Lemma 2.21.** (GM2003, p. 536) For $\kappa \geq 4$, nine among the monomials of $Y_\kappa$ are of the following general form:

$$
Y_\kappa = Y_{2^\kappa} + \left[ C_{1_1}^1 Y_{2^{\kappa-1}} - C_{1_2}^1 X_{2^\kappa} \right] y_1 + \left[ C_{1_1}^2 Y_{2^{\kappa-2}} - C_{1_2}^1 X_{2^{\kappa-1}} \right] y_2 + \left[ C_{1_1}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-1}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+1} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+2} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+3} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+4} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+5} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+6} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+7} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+8} + \left[ C_{1_2}^2 Y_{2^{\kappa-2}} - C_{1_2}^2 X_{2^{\kappa-2}} \right] y_{\lambda_1+y_{\mu_1}+\cdots+y_{\mu_d}+9}
$$

(2.22)

where the term remainder denotes all the remaining monomials, and where $C_{\kappa}^\lambda := \frac{\kappa!}{(\kappa-\lambda)! \lambda!}$ is a notation for the binomial coefficient which occupies less space in Latex “equation mode” than the classical notation

$$
\binom{\kappa}{\lambda}
$$

(2.23)

Now, we state directly the final theorem, without further inductive or intuitive information.
Theorem 2.24. For \( \kappa \geq 1 \), we have:

\[
Y_\kappa = Y_\kappa + \sum_{d=1}^{\kappa+1} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \cdots, \mu_d \geq 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d \leq \kappa+1} \frac{\kappa \cdots (\kappa - \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{(\lambda_1!)^{\mu_1} \cdots (\lambda_d!)^{\mu_d}} Y_{\kappa-\mu_1 \lambda_1 - \cdots - \mu_d \lambda_d} y^{\mu_1 + \cdots + \mu_d} \]

Once the correct theorem is formulated, its proof follows by accessible induction arguments which will not be developed here. It is better to continue through and to examine thoroughly the case of several variables, since it will help us considerably to explain how we discovered the exact values of the integer coefficients \( A_{\kappa}(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d) \) and \( B_{\kappa}(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d) \).

2.26. Verification and application. Before proceeding further, let us rapidly verify that the above general formula (2.25) is correct by inspecting two instances extracted from \( Y_5 \).

Firstly, the coefficient of \( y_1^3 y_3 \) in \( Y_5 \) is obtained by putting \( \kappa = 5, d = 2, \lambda_1 = 1, \mu_1 = 3, \lambda_2 = 3 \) and \( \mu_2 = 1 \) in the general formula (2.25), which yields:

\[
(2.27) \quad \left[ 0 - \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 6}{(1!)^3 3! (3!)^1} \right] X_{y^3} = \left[ -20 X_{y^3} \right].
\]

This value is the same as in the original formula (2.10); confirmation.

Secondly, the coefficient of \( y_1(y_2)^2 \) in \( Y_5 \) is obtained by putting \( \kappa = 5, d = 2, \lambda_1 = 1, \mu_1 = 1, \lambda_2 = 2 \) and \( \mu_2 = 2 \) in the general formula (2.25), which yields:

\[
(2.28) \quad \left[ \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(1!)^1 1! (2!)^2 2!} \right] Y_{y^3} - \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5}{(1!)^2 1! (2!)^2 2!} X_{xy^2} = \left[ 15 Y_{y^3} - 75 X_{xy^2} \right].
\]

This value is the same as in the original formula (2.10); again: confirmation.

Finally, applying our general formula (2.25), we deduce the value of \( Y_6 \) without having to use \( Y_5 \) and the induction formulas (2.4), which shortens substantially the computations. For the
pleasure, we obtain:

\[
\begin{align*}
Y_0 &= \mathcal{Y}_0 + \left[6 \mathcal{Y}_{x^2 y} - \mathcal{X}_{x^2 y}\right] y_1 + \left[15 \mathcal{Y}_{x^4 y^2} - 6 \mathcal{X}_{x^2 y^3}\right] (y_1)^2 + \\
&
+ \left[20 \mathcal{Y}_{x^2 y^3} - 15 \mathcal{X}_{x^2 y^4}\right] (y_1)^3 + \left[15 \mathcal{Y}_{x^4 y^4} - 20 \mathcal{X}_{x^2 y^5}\right] (y_1)^4 + \\
&
+ \left[6 \mathcal{Y}_{x^2 y^5} - 15 \mathcal{X}_{x^2 y^6}\right] (y_1)^5 + \left[\mathcal{Y}_{x^6} - 6 \mathcal{X}_{x^2 y^3}\right] (y_1)^6 + \left[-\mathcal{X}_{y^6}\right] (y_1)^7 + \\
&
+ \left[15 \mathcal{Y}_{x^4 y} - 6 \mathcal{X}_{x^2 y^2}\right] y_2 + \left[60 \mathcal{Y}_{x^2 y^2} - 45 \mathcal{X}_{x^2 y^3}\right] y_1 y_2 + \\
&
+ \left[90 \mathcal{Y}_{x^2 y^3} - 120 \mathcal{X}_{x^2 y^4}\right] (y_1)^2 y_2 + \left[60 \mathcal{Y}_{x^2 y^4} - 150 \mathcal{X}_{x^2 y^5}\right] (y_1)^3 y_2 + \\
&
+ \left[15 \mathcal{Y}_{x^6} - 90 \mathcal{X}_{x^2 y^3}\right] (y_1)^4 y_2 + \left[-21 \mathcal{X}_{y^6}\right] (y_1)^5 y_2 + \\
&
+ \left[45 \mathcal{Y}_{x^4 y^2} - 60 \mathcal{X}_{x^2 y^3}\right] (y_2)^2 + \left[90 \mathcal{Y}_{x^2 y^3} - 225 \mathcal{X}_{x^2 y^4}\right] y_1 (y_2)^2 + \\
&
+ \left[45 \mathcal{Y}_{x^4 y^3} - 270 \mathcal{X}_{x^2 y^4}\right] (y_1)^2 (y_2)^2 + \left[-210 \mathcal{X}_{y^6}\right] (y_1)^3 (y_2)^2 + \\
&
+ \left[15 \mathcal{Y}_{x^6} - 90 \mathcal{X}_{x^2 y^3}\right] (y_2)^3 + \left[-105 \mathcal{X}_{y^6}\right] y_1 (y_2)^3 + \\
&
+ \left[20 \mathcal{Y}_{x^2 y^3} - 15 \mathcal{X}_{x^2 y^4}\right] y_3 + \left[60 \mathcal{Y}_{x^2 y^4} - 80 \mathcal{X}_{x^2 y^5}\right] y_1 y_3 + \\
&
+ \left[60 \mathcal{Y}_{x^2 y^5} - 150 \mathcal{X}_{x^2 y^6}\right] (y_1)^2 y_3 + \left[20 \mathcal{Y}_{x^2 y^4} - 120 \mathcal{X}_{x^2 y^5}\right] (y_1)^3 y_3 + \\
&
+ \left[-35 \mathcal{X}_{x^6}\right] (y_1)^4 y_3 + \left[60 \mathcal{Y}_{x^2 y^4} - 150 \mathcal{X}_{x^2 y^5}\right] y_2 y_3 + \\
&
+ \left[60 \mathcal{Y}_{x^2 y^5} - 360 \mathcal{X}_{x^2 y^6}\right] y_1 y_2 y_3 + \left[-210 \mathcal{X}_{y^6}\right] (y_1)^3 y_2 y_3 + \\
&
+ \left[-105 \mathcal{X}_{x^6}\right] (y_2)^2 y_3 + \left[10 \mathcal{Y}_{x^2 y^3} - 60 \mathcal{X}_{x^2 y^4}\right] (y_3)^2 + \\
&
+ \left[-70 \mathcal{X}_{x^6}\right] y_1 (y_3)^2 + \left[15 \mathcal{Y}_{x^2 y^3} - 20 \mathcal{X}_{x^2 y^4}\right] y_4 + \\
&
+ \left[30 \mathcal{Y}_{x^2 y^4} - 75 \mathcal{X}_{x^2 y^5}\right] y_1 y_4 + \left[15 \mathcal{Y}_{x^2 y^5} - 90 \mathcal{X}_{x^2 y^6}\right] (y_1)^2 y_4 + \\
&
+ \left[-35 \mathcal{X}_{x^6}\right] (y_1)^4 y_4 + \left[15 \mathcal{Y}_{x^2 y^4} - 90 \mathcal{X}_{x^2 y^5}\right] y_2 y_4 + \\
&
+ \left[-105 \mathcal{X}_{x^6}\right] y_1 y_3 y_4 + \left[-35 \mathcal{X}_{x^6}\right] y_3 y_4 + \left[6 \mathcal{Y}_{x^2 y^3} - 15 \mathcal{X}_{x^2 y^4}\right] y_5 + \\
&
+ \left[6 \mathcal{Y}_{x^2 y^4} - 36 \mathcal{X}_{x^2 y^5}\right] y_1 y_5 + \left[-21 \mathcal{X}_{x^6}\right] (y_1)^2 y_5 + \left[-21 \mathcal{X}_{x^6}\right] y_2 y_5 + \\
&
+ \left[\mathcal{Y}_{y} - 6 \mathcal{X}_{y}\right] y_6 + \left[-7 \mathcal{X}_{y}\right] y_1 y_6. \\
\end{align*}
\]

\[2.29\]

2.30. Deduction of the classical Faà di Bruno formula. Let \(x, y \in \mathbb{R}\) and let \(g = g(x)\), \(f = f(y)\) be two \(C^{\infty}\)-smooth functions \(\mathbb{R} \to \mathbb{R}\). Consider the composition \(h := f \circ g\), namely \(h(x) = f(g(x))\). For \(\lambda \in \mathbb{N}\) with \(\lambda \geq 1\), simply denote by \(g_\lambda\) the \(\lambda\)-th derivative \(\frac{d^\lambda}{dy^\lambda}\) and similarly for \(h_\lambda\). Also, abbreviate \(f_\lambda := \frac{d^\lambda}{dy^\lambda}\).

By the classical formula for the derivative of a composite function, we have \(h_1 = f_1 g_1\). Further computations provide the following list of subsequent derivatives of \(h\):

\[
\begin{align*}
h_1 &= f_1 g_1, \\
h_2 &= f_2 (g_1)^2 + f_1 g_2, \\
h_3 &= f_3 (g_1)^3 + 3 f_2 g_1 g_2 + f_1 g_3, \\
h_4 &= f_4 (g_1)^4 + 6 f_3 (g_1)^2 g_2 + 3 f_2 (g_2)^2 + 4 f_2 g_1 g_3 + f_1 g_4, \\
h_5 &= f_5 (g_1)^5 + 10 f_4 (g_1)^3 g_2 + 15 f_3 (g_1)^2 g_3 + 10 f_3 g_1 (g_2)^2 + \\
&+ 10 f_2 g_2 g_3 + 5 f_2 g_1 g_4 + f_1 g_5, \\
h_6 &= f_6 (g_1)^6 + 15 f_5 (g_1)^4 g_2 + 45 f_4 (g_1)^2 (g_2)^2 + 15 f_3 (g_2)^3 + \\
&+ 20 f_4 (g_1)^3 g_3 + 60 f_3 g_1 g_2 g_3 + 10 f_2 (g_3)^2 + 15 f_3 (g_1)^2 g_4 + \\
&+ 15 f_2 g_2 g_4 + 6 f_2 g_1 g_5 + f_1 g_6.
\end{align*}
\]

\[2.31\]

**Theorem 2.32.** For every integer \(\kappa \geq 1\), the \(\kappa\)-th derivative of the composite function \(h = f \circ g\) may be expressed as an explicit polynomial in the partial derivatives of \(f\) and of \(g\) having integer
coefficients:

\[
\frac{d^kh}{dx^k} = \sum_{d=1}^{\kappa} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \frac{k!}{(\lambda_1!)^{\mu_1} \mu_1! \cdots (\lambda_d!)^{\mu_d} \mu_d!} \frac{d^{\mu_1 + \cdots + \mu_d} f}{d y_{\mu_1 + \cdots + \mu_d}} \left( \frac{d^{\lambda_1} g}{dx^{\lambda_1}} \right)^{\mu_1} \cdots \left( \frac{d^{\lambda_d} g}{dx^{\lambda_d}} \right)^{\mu_d}.
\]

This is the classical Faà di Bruno formula. Interestingly, we observe that this formula is included as subpart of the general formula for \( Y_\kappa \), after a suitable translation. Indeed, in the formulas for \( Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \) and in the general sum for \( Y_\kappa \), pick only the terms for which \( \mu_1 \lambda_1 + \cdots + \mu_d \lambda_d = \kappa \) and drop \( x' \), which yields:

\[
\sum_{d=1}^{\kappa} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \frac{k!}{(\lambda_1!)^{\mu_1} \mu_1! \cdots (\lambda_d!)^{\mu_d} \mu_d!} \frac{d^{\mu_1 + \cdots + \mu_d} f}{d y_{\mu_1 + \cdots + \mu_d}} \left( y_{\lambda_1} \right)^{\mu_1} \cdots \left( y_{\lambda_d} \right)^{\mu_d}.
\]

The similarity between the two formulas (2.33) and (2.34) is now clearly visible.

The Faà di Bruno formula may be established by means of substitutions of power series ([H1969], p. 222), by means of the umbral calculus ([CS1996]), or by means of some induction formulas, which we write for completeness. Define the differential operators

\[
F_2 := g_2 \frac{\partial}{\partial g_1} + g_1 \left( f_2 \frac{\partial}{\partial f_1} \right),
\]

\[
F_3 := g_2 \frac{\partial}{\partial g_1} + g_3 \frac{\partial}{\partial g_2} + g_1 \left( f_2 \frac{\partial}{\partial f_1} + f_3 \frac{\partial}{\partial f_2} \right),
\]

\[
F_\lambda := g_2 \frac{\partial}{\partial g_1} + g_3 \frac{\partial}{\partial g_2} + \cdots + g_\lambda \frac{\partial}{\partial g_{\lambda-1}} + g_1 \left( f_2 \frac{\partial}{\partial f_1} + f_3 \frac{\partial}{\partial f_2} + \cdots + f_\lambda \frac{\partial}{\partial f_{\lambda-1}} \right).
\]

Then we have

\[
h_2 = F^2(h_1),
\]

\[
h_3 = F^3(h_2),
\]

\[
\ldots
\]

\[
h_\lambda = F^\lambda(h_{\lambda-1}).
\]

§3. SEVERAL INDEPENDENT VARIABLES AND ONE DEPENDENT VARIABLE

3.1. Simplified adapted notations. As announced after the statement of Theorem 2.24, it is only after we have treated the case of several independent variables that we will understand perfectly the general formula (2.25), valid in the case of one independent variable and one dependent variable. We will discover massive formal computations, exciting our computational intuition.

Thus, assume \( n \geq 1 \) and \( m = 1 \), let \( \kappa \in \mathbb{N} \) with \( \kappa \geq 1 \) and simply denote (instead of (1.2)) the jet variables by:

\[
(x^i, y_1, y_{i_1, i_2}, \ldots, y_{i_1, i_2, \ldots, i_n}).
\]
Also, instead of (1.30), denote the $\kappa$-th prolongation of a vector field by:

$$
L^{(\kappa)} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} Y^i \frac{\partial}{\partial y^i} + \sum_{i_1, i_2=1}^{n} Y_{i_1,i_2} \frac{\partial}{\partial y_{i_1,i_2}} + \sum_{i_1,i_2,\ldots,i_n=1}^{n} Y_{i_1,i_2,\ldots,i_n} \frac{\partial}{\partial y_{i_1,i_2,\ldots,i_n}}.
$$

(3.3)

The induction formulas are

$$
\begin{align*}
Y_{i_1} & := D^1_{i_1} (Y) - \sum_{k=1}^{n} D^1_{i_1} (X^k) y_k, \\
Y_{i_1,i_2} & := D^2_{i_2} (Y_{i_1}) - \sum_{k=1}^{n} D^1_{i_2} (X^k) y_{i_1,k}, \\
& \ldots \\
Y_{i_1,i_2,\ldots,i_n} & := D^n_{i_n} (Y_{i_1,i_2,\ldots,i_{n-1}}) - \sum_{k=1}^{n} D^1_{i_n} (X^k) y_{i_1,i_2,\ldots,i_{n-1},k},
\end{align*}
$$

(3.4)

where the total differentiation operators $D^j_i$ are defined as in (1.22), dropping the sums $\sum_{j'=1}^{m}$ and the indices $j'$.

### 3.5. Two instructing explicit computations.

To begin with, let us compute $Y_{i_1}$. With $D^1_{i_1} = \frac{\partial}{\partial x^{i_1}} + y_{i_1} \frac{\partial}{\partial y}$, we have:

$$
Y_{i_1} = D^1_{i_1} (Y) - \sum_{k=1}^{n} D^1_{i_1} (X^k) y_k \\
= Y_{x^{i_1}} + Y_y y_{i_1} - \sum_{k=1}^{n} X^k_{x^{i_1}} y_k - \sum_{k=1}^{n} X^k_{y} y_{i_1} y_k.
$$

(3.6)

Searching for formal harmony and for coherence with the formula (2.6)$_1$, we must include the term $Y_y y_{i_1}$ inside the sum $\sum_{k=1}^{n} [.] y_k$. Using the Kronecker symbol, we may write:

$$
Y_y y_{i_1} = \sum_{k=1}^{n} \delta_{i_1}^{k_1} Y_y y_{k_1}.
$$

(3.7)

Also, we may rewrite the last term of (3.6) with a double sum:

$$
- \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \delta_{i_1}^{k_1} Y_y y_{k_1} y_{k_2} = \sum_{k_1,k_2=1}^{n} \left[ -\delta_{i_1}^{k_1} \lambda_{y}^{k_2} \right] y_{k_1} y_{k_2}.
$$

(3.8)

From now on and up to equation (3.39), we shall abbreviate any sum $\sum_{k=1}^{n}$ from 1 to $n$ as $\sum_{k}$. Putting everything together, we get the final desired perfect expression of $Y_{i_1}$:

$$
Y_{i_1} = Y_{x^{i_1}} + \sum_{k_1=1}^{n} \left[ \delta_{i_1}^{k_1} Y_y y_{k_1} - \lambda_{x^{i_1}}^{k_1} \right] y_{k_1} + \sum_{k_1,k_2=1}^{n} \left[ -\delta_{i_1}^{k_1} \lambda_{y}^{k_2} \right] y_{k_1} y_{k_2}.
$$

(3.9)

This completes the first explicit computation.

The second one is about $Y_{i_1,i_2}$. It becomes more delicate, because several algebraic transformations must be achieved until the final satisfying formula is obtained. Our goal is to present each step very carefully, explaining every tiny detail. Without such a care, it would be impossible to claim that some of our subsequent computations, for which we will not provide the intermediate steps, may be redone and verified. Consequently, we will expose our rules of formal computation thoroughly.
Replacing the value of $Y_1$ just obtained in the induction formula (3.4)_2 and developing, we may conduct the very first steps of the computation:

$$Y_{i_1,i_2} = D^2_{x^2} (Y_{i_1}) - \sum_{k_1} D^2_{x^2} (\chi^{k_1}) y_{i_1,k_1}$$

$$= \left( \frac{\partial}{\partial x^2} + y_{i_2} \frac{\partial}{\partial y} + \sum_{i_1,k_1} y_{i_2,k_1} \frac{\partial}{\partial y_{k_1}} \right) \left( Y_{i_1} + \sum_{k_1} \left[ \sum_{k_1} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1} + \sum_{k_1,k_2} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1,k_2} \right] \right)$$

$$+ \sum_{k_1} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1} + \sum_{k_1,k_2} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1,k_2}$$

$$= \left( \frac{\partial}{\partial x^2} + y_{i_2} \frac{\partial}{\partial y} + \sum_{i_1,k_1} y_{i_2,k_1} \frac{\partial}{\partial y_{k_1}} \right) \left( Y_{i_1} + \sum_{k_1} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1} + \sum_{k_1,k_2} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1,k_2} \right)$$

Some explanations are needed about the computation of the last two terms of line 11, i.e. about the passage from line 7 of (3.10) just above to line 11. We have to compute:

$$\left( \sum_{k_1} y_{i_2,k_1} \frac{\partial}{\partial y_{k_1}} \right) \left( \sum_{k_1,k_2} \left[ \delta^{k_1}_{x_1} A_{x_1,y} - \chi^{k_1}_{x_1} \right] y_{k_1,k_2} \right)$$

This term is of the form

$$\left( \sum_{k_1} A_{k_1} \frac{\partial}{\partial y_{k_1}} \right) \left( \sum_{k_1,k_2} B_{k_1,k_2} y_{k_1,k_2} \right),$$

where the terms $B_{k_1,k_2}$ are independent of the pure first jet variables $y_{x^2}$. By the rule of Leibniz for the differentiation of a product, we may write

$$\left( \sum_{k_1} A_{k_1} \frac{\partial}{\partial y_{k_1}} \right) \left( \sum_{k_1,k_2} B_{k_1,k_2} y_{k_1,k_2} \right) =$$

$$= \sum_{k_1} \sum_{k_2} y_{k_2} B_{k_1,k_2} A_{k_1} + \sum_{k_1,k_2} \left( \sum_{k_1} A_{k_1} \frac{\partial}{\partial y_{k_1}} \right) y_{k_1,k_2}$$

$$= \sum_{k_1,k_2} [B_{k_1,k_2} y_{k_2} A_{k_1} + \sum_{k_1,k_2} [B_{k_1,k_2} y_{k_1} A_{k_2}].$$
This is how we have written line 11 of (3.10).

Next, the first term $\mathcal{Y}_{x'_{1}y}y_{l_{2}}$ in line 10 of (3.10) is not in a suitable shape. For reasons of harmony and coherence, we must insert it inside a sum of the form $\sum_{k_{1}} \left[ \cdot \right] y_{k_{1}}$. Hence, using the Kronecker symbol, we transform:

\begin{equation}
\mathcal{Y}_{x'_{1}y}y_{l_{2}} \equiv \sum_{k_{1}} \left[ \delta^{k_{1}}_{l_{2}} \mathcal{Y}_{x'_{1}y} \right] y_{k_{1}}.
\end{equation}

Also, we must “summify” the seven other terms, remaining in lines 10, 11 and 12 of (3.10). Sometimes, we use the symmetry $y_{l_{2},k_{1}} \equiv y_{k_{1},l_{2}}$ without mention. Similarly, we get:

\begin{align*}
\sum_{k_{1}} \left[ \delta^{k_{1}}_{l_{1}} \mathcal{Y}_{yy} - \mathcal{A}^{k_{1}}_{x'_{1}y} \right] y_{k_{1}}y_{l_{2}} & \equiv \sum_{k_{1},k_{2}} \left[ \delta^{k_{1}}_{l_{1}} \delta^{k_{2}}_{l_{2}} \mathcal{Y}_{yy} - \delta^{k_{2}}_{l_{1}} \mathcal{A}^{k_{1}}_{x'_{1}y} \right] y_{k_{1}}y_{k_{2}}, \\
\sum_{k_{1},k_{2}} \left[ -\delta^{k_{1}}_{l_{1}} \mathcal{A}^{k_{2}}_{yy} \right] y_{k_{1}}y_{k_{2}}y_{l_{2}} & \equiv \sum_{k_{1},k_{2},k_{3}} \left[ -\delta^{k_{1}}_{l_{1}} \delta^{k_{3}}_{l_{2}} \mathcal{A}^{k_{2}}_{yy} \right] y_{k_{1}}y_{k_{2}}y_{k_{3}}, \\
\sum_{k_{1}} \left[ \delta^{k_{1}}_{l_{1}} \mathcal{Y}_{y} - \mathcal{A}^{k_{1}}_{x'_{1}1} \right] y_{k_{1}}y_{l_{2}} & \equiv \sum_{k_{1},k_{2}} \left[ \delta^{k_{1}}_{l_{1}} \delta^{k_{2}}_{l_{2}} \mathcal{Y}_{y} - \delta^{k_{2}}_{l_{1}} \mathcal{A}^{k_{1}}_{x'_{1}1} \right] y_{k_{1}}y_{k_{2}}, \\
\sum_{k_{1},k_{2}} \left[ -\delta^{k_{1}}_{l_{1}} \mathcal{A}^{k_{2}}_{y} \right] y_{k_{1}}y_{k_{2}}y_{l_{2}} & = \sum_{k_{1},k_{2}} \left[ -\delta^{k_{2}}_{l_{1}} \mathcal{A}^{k_{1}}_{y} \right] y_{k_{1}}y_{k_{2}}, \\
\sum_{k_{1},k_{2}} \left[ -\delta^{k_{1}}_{l_{1}} \mathcal{A}^{k_{2}}_{y} \right] y_{k_{1}}y_{k_{2}}y_{l_{2}} & \equiv \sum_{k_{1},k_{2},k_{3}} \left[ -\delta^{k_{2}}_{l_{1}} \delta^{k_{3}}_{l_{2}} \mathcal{A}^{k_{1}}_{y} \right] y_{k_{1}}y_{k_{2}}y_{k_{3}}, \\
\sum_{k_{1}} \left[ -\mathcal{A}^{k_{1}}_{x'_{1}2} \right] y_{k_{1}}y_{l_{2}}i_{1} & \equiv \sum_{k_{1},k_{2}} \left[ -\mathcal{A}^{k_{2}}_{x'_{1}2} \mathcal{A}^{k_{1}}_{y} \right] y_{k_{1}}y_{k_{2}}, \\
\sum_{k_{1}} \left[ -\mathcal{A}^{k_{1}}_{y} \right] y_{l_{2}}y_{k_{1}}i_{1} & \equiv \sum_{k_{2}} \left[ -\mathcal{A}^{k_{2}}_{y} \right] y_{l_{2}}y_{k_{2}}, \\
& \equiv \sum_{k_{1},k_{2},k_{3}} \left[ -\delta^{k_{1}}_{l_{1}} \delta^{k_{2}}_{l_{2}} \mathcal{A}^{k_{3}}_{y} \right] y_{k_{1}}y_{k_{2}}y_{k_{3}}.
\end{align*}

In the sequel, for products of Kronecker symbols, it will be convenient to adopt the following self-evident contracted notation:

\begin{equation}
\delta^{k_{1}}_{l_{1}} \delta^{k_{2}}_{l_{2}} \equiv \delta^{k_{1},k_{2}}_{l_{1},l_{2}}, \quad \text{generally} \quad \delta^{k_{1}}_{l_{1}} \delta^{k_{2}}_{l_{2}} \cdots \delta^{k_{\lambda}}_{l_{1}} \equiv \delta^{k_{1},k_{2},\ldots,k_{\lambda}}_{l_{1},l_{2},\ldots,l_{\lambda}}.
\end{equation}
Re-inserting plainly these eight summified terms (3.14), (3.15) in the last expression (3.10) of \( Y_{i_1,i_2} \) (lines 10, 11 and 12), we get:

\[
Y_{i_1,i_2} = Y_{x_1 x_2} + \sum_{k_1} \left[ \delta_{k_1} Y_{y_{x_1 y}} - \delta_{k_1} X_{x_1 x_2} \right] y_{k_1} + \sum_{k_1,k_2} \left[ -\delta_{k_1} \delta_{k_2} Y_{x_{x_1 y}} \right] y_{k_1} y_{k_2} +
\]

\[
+ \sum_{k_1} \left[ \delta_{k_1} \delta_{k_2} Y_{x_{x_1 y}} \right] y_{k_1} + \sum_{k_1,k_2} \left[ \delta_{k_1} \delta_{k_2} Y_{y_{x_1 y}} \right] y_{k_1} y_{k_2} +
\]

\[
+ \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{y_y} \right] y_{k_1} y_{k_2} y_{k_3} + \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{x_{x_1 y}} \right] y_{k_1} y_{k_2} y_{k_3} +
\]

\[
+ \sum_{k_1,k_2} \left[ -\delta_{k_1} \delta_{k_2} X_{y_y} \right] y_{k_1} y_{k_2} + \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{y_y} \right] y_{k_1} y_{k_2} y_{k_3} .
\]

Next, we gather the underlined terms, ordering them according to their number. This yields 6 collections of sums of monomials in the pure jet variables:

\[
Y_{i_1,i_2} = Y_{x_1 x_2} + \sum_{k_1} \left[ \delta_{k_1} Y_{y_{x_1 y}} + \delta_{k_1} X_{x_1 x_2} \right] y_{k_1} +
\]

\[
+ \sum_{k_1,k_2} \left[ \delta_{k_1} \delta_{k_2} Y_{y_{x_1 y}} \right] y_{k_1} y_{k_2} + \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{y_y} \right] y_{k_1} y_{k_2} y_{k_3} +
\]

\[
+ \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{x_{x_1 y}} \right] y_{k_1} y_{k_2} y_{k_3} .
\]

To attain the real perfect harmony, this last expression has still to be worked out a little bit.

Lemma 3.19. The final expression of \( Y_{i_1,i_2} \) is as follows:

\[
Y_{i_1,i_2} = Y_{x_1 x_2} + \sum_{k_1} \left[ \delta_{k_1} Y_{y_{x_1 y}} + \delta_{k_1} X_{x_1 x_2} \right] y_{k_1} +
\]

\[
+ \sum_{k_1,k_2} \left[ \delta_{k_1} \delta_{k_2} Y_{y_{x_1 y}} \right] y_{k_1} y_{k_2} + \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{y_y} \right] y_{k_1} y_{k_2} y_{k_3} +
\]

\[
+ \sum_{k_1,k_2,k_3} \left[ -\delta_{k_1} \delta_{k_2} \delta_{k_3} X_{x_{x_1 y}} \right] y_{k_1} y_{k_2} y_{k_3} .
\]

Proof. As promised, we explain every tiny detail.

The first lines of (3.18) and of (3.20) are exactly the same. For the transformations of terms in the second, in the third and in the fourth lines, we use the following device. Let \( \sum_{k_1,k_2} \) be an indexed quantity which is symmetric: \( \sum_{k_1,k_2} = \sum_{k_2,k_1} \). Let \( A_{k_1,k_2} \) be an arbitrary indexed
quantity. Then obviously:

\[ (3.21) \quad \sum_{k_1, k_2} A_{k_1, k_2} Y_{k_1, k_2} = \sum_{k_1, k_2} A_{k_2, k_1} Y_{k_1, k_2}. \]

Similar relations hold with a quantity \( Y_{i_1, i_2, \ldots, i_3} \) which is symmetric with respect to its \( \lambda \) indices. Consequently, in the second, in the third and in the fourth lines of (3.18), we may permute freely certain indices in some of the terms inside the brackets. This yields the passage from lines 2, 3 and 4 of (3.18) to lines 2, 3 and 4 of (3.20).

It remains to explain how we pass from the fifth (last) line of (3.18) to the fifth (last) line of (3.20). The bracket in the fifth line of (3.18) contains three terms: \([-T_1 - T_2 - T_3]\). The term \( T_3 \) involves the product \( \delta^{k_2, k_3}_{i_2, i_3} \), which we rewrite as \( \delta^{k_3, k_2}_{i_3, i_2} \), in order that \( i_1 \) appears before \( i_2 \). Then, we rewrite the three terms in the new order \([-T_2 - T_3 - T_1]\), which yields:

\[ (3.22) \quad \sum_{k_1, k_2, k_3} \left[ -\delta^{k_1, k_3}_{i_1, i_2} \delta^k_y - \delta^{k_3, k_1}_{i_2, i_3} \delta^k_y - \delta^{k_2, k_3}_{i_3, i_2} \delta^k_y \right] y_{k_1, k_2, k_3}. \]

It remains to observe that we can permute \( k_2 \) and \( k_3 \) in the first term \(-T_2\), which yields the last line of (3.20). The detailed proof is complete. \( \square \)

3.23. Final perfect expression of \( Y_{i_1, i_2, i_3} \). Thanks to similar (longer) computations, we have obtained an expression of \( Y_{i_1, i_2, i_3} \), which we consider to be in final harmonious shape. Without copying the intermediate steps, let us write down the result. The comments which are necessary to read it and to interpret it start just below.

\[ Y_{i_1, i_2, i_3} = Y_{x_1 x_2 x_3} + \sum_{k_1} \left[ \delta^{k_1}_{i_1} Y_{x_1 x_2 x_3} + \delta^{k_1}_{i_2} Y_{x_1 x_2 x_3} + \delta^{k_1}_{i_3} Y_{x_1 x_2 x_3} - X^{k_1}_{x_1 x_2 x_3} \right] y_{k_1} + \]

\[ + \sum_{k_1, k_2} \left[ \delta^{k_1, k_2}_{i_1, i_2} y_{x_1 x_2 x_3} + \delta^{k_1, k_2}_{i_1, i_3} y_{x_1 x_2 x_3} + \delta^{k_1, k_2}_{i_2, i_3} y_{x_1 x_2 x_3} - \delta^{k_2, k_1}_{i_1, i_2} X^{k_2}_{x_1 x_2 x_3} \right] y_{k_1, k_2} + \]

\[ + \sum_{k_1, k_2, k_3} \left[ \delta^{k_1, k_2, k_3}_{i_1, i_2, i_3} y_{x_1 x_2 x_3} + \delta^{k_2, k_1, k_3}_{i_1, i_2, i_3} y_{x_1 x_2 x_3} + \delta^{k_3, k_1, k_2}_{i_1, i_2, i_3} y_{x_1 x_2 x_3} - X^{k_3}_{x_1 x_2 x_3} \right] y_{k_1, k_2, k_3} + \]

\[ + \sum_{k_1, k_2} \left[ \delta^{k_1, k_2}_{i_1, i_2} y_{x_1 x_2 x_3} + \delta^{k_1, k_2}_{i_1, i_3} y_{x_1 x_2 x_3} + \delta^{k_1, k_2}_{i_2, i_3} y_{x_1 x_2 x_3} - \delta^{k_2, k_1}_{i_1, i_2} X^{k_2}_{x_1 x_2 x_3} \right] y_{k_1, k_2} + \]

\[ + \sum_{k_1, k_2, k_3} \left[ \delta^{k_1, k_2, k_3}_{i_1, i_2, i_3} y_{x_1 x_2 x_3} + \delta^{k_2, k_1, k_3}_{i_1, i_2, i_3} y_{x_1 x_2 x_3} + \delta^{k_3, k_1, k_2}_{i_1, i_2, i_3} y_{x_1 x_2 x_3} - X^{k_3}_{x_1 x_2 x_3} \right] y_{k_1, k_2, k_3} + \]

\[ + \sum_{k_1, k_2, k_3, k_4} \left[ \delta^{k_1, k_2, k_3, k_4}_{i_1, i_2, i_3, i_4} y_{x_1 x_2 x_3 x_4} + \delta^{k_2, k_1, k_3, k_4}_{i_1, i_2, i_3, i_4} y_{x_1 x_2 x_3 x_4} + \delta^{k_3, k_1, k_2, k_4}_{i_1, i_2, i_3, i_4} y_{x_1 x_2 x_3 x_4} - X^{k_3, k_1, k_2, k_4}_{x_1 x_2 x_3 x_4} \right] y_{k_1, k_2, k_3, k_4} + \]

\[ + \sum_{k_1, k_2, k_3, k_4} \left[ \delta^{k_1, k_2, k_3, k_4}_{i_1, i_2, i_3, i_4} y_{x_1 x_2 x_3 x_4} + \delta^{k_1, k_2, k_3, k_4}_{i_1, i_2, i_3, i_4} y_{x_1 x_2 x_3 x_4} + \delta^{k_1, k_2, k_3, k_4}_{i_1, i_2, i_3, i_4} y_{x_1 x_2 x_3 x_4} - X^{k_3, k_1, k_2, k_4}_{x_1 x_2 x_3 x_4} \right] y_{k_1, k_2, k_3, k_4} + \]
3.25. Comments, analysis and induction. First of all, by comparing this expression of $Y_{i_1,i_2,i_3}$ with the expression (2.8) of $Y_3$, we easily guess a part of the (inductional) dictionary between the cases $n = 1$ and the case $n \geq 1$. For instance, the three monomials $[y_1] y_2$ and $[y_1] (y_1)^2 y_2$ in $Y_3$ are replaced in $Y_{i_1,i_2,i_3}$ by the following three sums:

$$(3.26) \sum_{k_1,k_2,k_3} [y_1] y_{k_1} y_{k_2} y_{k_3}, \quad \sum_{k_1,k_2,k_3} [y_1] y_{k_1} y_{k_2} y_{k_3}, \quad \text{and} \quad \sum_{k_1,k_2,k_3,k_4} [y_1] y_{k_1} y_{k_2} y_{k_3} y_{k_4}.$$ 

Similar formal correspondences may be observed for all the monomials of $Y_1$, $Y_{i_1}$, of $Y_2$, $Y_{i_1,i_2}$ and of $Y_3$, $Y_{i_1,i_2,i_3}$. Generally, and inductively speaking, the monomial

$$(3.27) \left[ y_{\lambda} \right]^{\mu_1} \cdots \left( y_{\lambda} \right)^{\mu_d}$$

appearing in the expression (2.25) of $Y_{\iota}$ should be replaced by a certain multiple sum generalizing (3.26). However, it is necessary to think, to pause and to search for an appropriate formalism before writing down the desired multiple sum.

The jet variable $y_{\lambda_1}$ should be replaced by a jet variable corresponding to a $\lambda_1$-th partial derivative, say $y_{k_1,\ldots,k_{\lambda_1}}$, where $k_1,\ldots,k_{\lambda_1} = 1,\ldots,n$. For the moment, to simplify the discussion, we leave out the presence of a sum of a form $\sum_{k_1,\ldots,k_{\lambda_1}}$. The $\mu_1$-th power $\left( y_{\lambda_1} \right)^{\mu_1}$ should be replaced not by $\left( y_{k_1,\ldots,k_{\lambda_1}} \right)^{\mu_1}$, but by a product of $\mu_1$ different jet variables $y_{k_1,\ldots,k_{\lambda_1}}$ of length $\lambda_1$, with all indices $k_\alpha = 1,\ldots,n$ being distinct. This rule may be confirmed by inspecting the expressions of $Y_1$, of $Y_{i_1,i_2}$ and of $Y_{i_1,i_2,i_3}$. So $y_{k_1,\ldots,k_{\lambda_1}}$ should be developed as a product of the form

$$(3.28) y_{k_1,\ldots,k_{\lambda_1}} y_{k_1+1,\ldots,k_{\lambda_1}} \cdots y_{k_{(\mu_1-1)\lambda_1+1,\ldots,k_{\mu_1\lambda_1}},}$$

where

$$(3.29) k_1,\ldots,k_{\lambda_1},\ldots,k_{\mu_1\lambda_1} = 1,\ldots,n.$$ 

Consider now the product $\left( y_{\lambda_1} \right)^{\mu_1} \left( y_{\lambda_2} \right)^{\mu_2}$. How should it develope in the case of several independent variables? For instance, in the expression of $Y_{i_1,i_2,i_3}$, we have developed the product $(y_1)^2 y_2$ as $y_{k_1} y_{k_2} y_{k_3} y_{k_4}$. Thus, a reasonable proposal of formalism would be that the product $\left( y_{\lambda_1} \right)^{\mu_1} \left( y_{\lambda_2} \right)^{\mu_2}$ should be developed as a product of the form

$$(3.30) y_{k_1,\ldots,k_{\lambda_1}} y_{k_{\lambda_1+1,\ldots,k_{2\lambda_1}}} \cdots y_{k_{(\mu_1-1)\lambda_1+1,\ldots,k_{\mu_1\lambda_1}}} y_{k_{\mu_1\lambda_1+1,\ldots,k_{\mu_1\lambda_1+1+\mu_1\lambda_1}}} \cdots y_{k_{\mu_1\lambda_1+\mu_2\lambda_2,\ldots,k_{\mu_1\lambda_1+\mu_2\lambda_2}},}$$

where

$$(3.31) k_1,\ldots,k_{\lambda_1},\ldots,k_{\mu_1\lambda_1},\ldots,k_{\mu_1\lambda_1+\mu_2\lambda_2} = 1,\ldots,n.$$ 

However, when trying to write down the development of the general monomial $\left( y_{\lambda_1} \right)^{\mu_1} \left( y_{\lambda_2} \right)^{\mu_2} \cdots \left( y_{\lambda_d} \right)^{\mu_d}$, we would obtain the complicated product

$$(3.32) y_{k_1,\ldots,k_{\lambda_1}} y_{k_{\lambda_1+1,\ldots,k_{2\lambda_1}}} \cdots y_{k_{(\mu_1-1)\lambda_1+1,\ldots,k_{\mu_1\lambda_1}}} y_{k_{\mu_1\lambda_1+1,\ldots,k_{\mu_1\lambda_1+1+\mu_1\lambda_1}}} \cdots y_{k_{\mu_1\lambda_1+\mu_2\lambda_2,\ldots,k_{\mu_1\lambda_1+\mu_2\lambda_2}},} \cdots y_{k_{\mu_1\lambda_1+\mu_2\lambda_2,\ldots,k_{\mu_1\lambda_1+\mu_2\lambda_2}}} \cdots y_{k_{\mu_1\lambda_1+\mu_2\lambda_2,\ldots,k_{\mu_1\lambda_1+\mu_2\lambda_2}},}$$

Essentially, this product is still readable. However, in it, some of the integers $k_\alpha$ have a too long index $\alpha$, often involving a sum. Such a length of $\alpha$ would be very inconvenient in writing down and in reading the general Kronecker symbol $\delta_{k_1,\ldots,k_{\lambda_1}}$ which should appear in the final
expression of $Y_{i_1,\ldots,i_n}$. One should read in advance Theorem 3.73 below to observe the presence of such multiple Kronecker symbols. Consequently, for $\alpha = 1,\ldots,\mu_1\lambda_1,\ldots,\mu_1\lambda_1 + \cdots + \mu_d\lambda_d$, we have to denote the indices $k_\alpha$ differently.

**Notational convention 3.33.** We denote $d$ collection of $\mu_d$ groups of $\lambda_d$ (a priori distinct) integers $k_\alpha = 1,\ldots,n$ by

\[
\begin{array}{c}
\underbrace{k_1;1:1,\ldots,k_1;1:1}_{\lambda_1}
\underbrace{k_1;\mu_1;1,\ldots,k_1;\mu_1;1}_{\lambda_1}
\underbrace{k_1;\mu_1;1,\ldots,k_1;\mu_1;1}_{\lambda_1}
\vdots
\underbrace{k_d;1:1,\ldots,k_d;1:1}_{\lambda_d}
\end{array}
\]

(3.34)

Correspondingly, we identify the set

\[
\{1,\ldots,\lambda_1,\lambda_1,\ldots,\mu_1\lambda_1,\ldots,\mu_1\lambda_1 + \mu_2\lambda_2,\ldots,\mu_1\lambda_1 + \mu_2\lambda_2 + \cdots + \mu_d\lambda_d\}
\]

of all integers $\alpha$ from 1 to $\mu_1\lambda_1 + \mu_2\lambda_2 + \cdots + \mu_d\lambda_d$ with the following specific set

\[
\{1:1;1,\ldots,1:1;1,\ldots,1:1;\lambda_1,\ldots,1:1;1,\ldots,1:1;\mu_1\lambda_1,\ldots,1:1;1,\ldots,1:1;\mu_1\lambda_1 + \mu_2\lambda_2,\ldots,1:1;1,\ldots,1:1;\mu_1\lambda_1 + \mu_2\lambda_2 + \cdots + \mu_d\lambda_d\}
\]

written in a lexicographic way which emphasizes clearly the subdivision in $d$ collections of $\mu_d$ groups of $\lambda_d$ integers.

With this notation at hand, we see that the development, in several independent variables, of the general monomial $(y_{\lambda_1})^{\mu_1} \cdots (y_{\lambda_d})^{\mu_d}$, may be written as follows:

\[
y_{k_1;1:1,\ldots,k_1;1:1,\ldots,k_1;\mu_1;1,\ldots,k_1;\mu_1;1,\ldots,k_d;1:1,\ldots,k_d;1:1,\ldots,k_d;\mu_d\lambda_d}\]

Formally speaking, this expression is better than (3.32). Using product symbols, we may even write it under the slightly more compact form

\[
\prod_{1\leq \nu_1 \leq \mu_1} \cdots \prod_{1\leq \nu_d \leq \mu_d} y_{k_{d.;\nu_d};1,\ldots,k_{d.;\nu_d};\lambda_d}
\]

Now that we have translated the monomial, we may add all the summation symbols: the general expression of $Y_\alpha$ (which generalizes our three previous examples (3.26)) will be of the form:

\[
Y_\alpha = \mathcal{Y}_{x^1,\ldots,x^\alpha} + \sum_{d=1}^{n+1} \sum_{1\leq \lambda_1 < \cdots < \lambda_d \leq \alpha} \sum_{\mu_1 \geq 1,\ldots,\mu_d \geq 1} \sum_{\mu_1\lambda_1 + \cdots + \mu_d\lambda_d \leq \alpha+1} \sum_{k_1;1:1,\ldots,k_1;1:1} \cdots \sum_{k_1;\mu_1;1,\ldots,k_1;\mu_1;1} \cdots \sum_{k_d;1:1,\ldots,k_d;1:1} \cdots \sum_{k_d;\mu_d\lambda_d;1,\ldots,k_d;\mu_d\lambda_d;1} \prod_{1\leq \nu_1 \leq \mu_1} y_{k_{1.;\nu_1};1,\ldots,k_{1.;\nu_1};\lambda_1} \cdots \prod_{1\leq \nu_d \leq \mu_d} y_{k_{d.;\nu_d};1,\ldots,k_{d.;\nu_d};\lambda_d}
\]

(3.39)
From now on, up to the end of the article, to be very precise, we will restitute the bounds $\sum_{k=1}^{n}$ of all the previously abbreviated sums $\sum_{k}$. This is justified by the fact that, since we shall deal in Section 5 below simultaneously with several independent variables $(x^1, \ldots, x^n)$ and with several dependent variables $(y^1, \ldots, y^m)$, we shall encounter sums $\sum_{l=1}$, not to be confused with sums $\sum_{k=1}$.

3.40. Combinatorics of the Kronecker symbols. Our next task is to determine what appears inside the brackets [?] of the above equation. We will treat this rather delicate question very progressively. Inductively, we have to guess how we may pass from the bracketed term of (2.25), namely from

$$
\left[ \frac{\kappa \cdots (\kappa - \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\lambda_1!^{\mu_1} \cdots (\lambda_d!)^{\mu_d} \mu_d!} \cdot \mathcal{Y}_{x^\kappa - \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d y^{\mu_1 \cdots + \mu_d}} - \frac{\kappa \cdots (\kappa - \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d)}{\lambda_1!^{\mu_1} \cdots (\lambda_d!)^{\mu_d} \mu_d!} \cdot \mathcal{X}_{x^\kappa - \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1 y^{\mu_1 \cdots + \mu_d + 1}} \right],
$$

(3.41)

to the corresponding (still unknown) bracketed term [?].

First of all, we examine the following term, extracted from the complete expression of $\mathbf{Y}_{i_1,i_2,i_3}$ (first line of (3.24)):

$$
\sum_{k_1=1}^{n} \left[ \delta_{i_1}^{k_1} \mathcal{Y}_{x^{i_2} x^{i_3} y} + \delta_{i_2}^{k_1} \mathcal{Y}_{x^{i_1} x^{i_3} y} + \delta_{i_3}^{k_1} \mathcal{Y}_{x^{i_1} x^{i_2} y} - \mathcal{X}_{x^{i_1} x^{i_2} x^{i_3}} \right] y_{k_1}.
$$

(3.42)

Here, the coefficient $[3 \mathcal{Y}_{x^{i_2} y} - \mathcal{X}_{x^{i_3}}]$ of the monomial $y_1$ in $\mathbf{Y}_3$ is replaced by the above bracketed terms.

Let us precisely analyze the combinatorics. Here, $\mathcal{X}_{x^{i_3}}$ is replaced by $\mathcal{X}_{x^{i_1} x^{i_2} x^{i_3}}$, where the lower indices $i_1, i_2, i_3$ come from $\mathbf{Y}_{i_1,i_2,i_3}$ and where the upper index $k_1$ is the summation index. Also, the integer 3 in $3 \mathcal{Y}_{x^{i_2} y}$ is replaced by a sum of exactly three terms, each involving a single Kronecker symbol $\delta_{i_1}^{k_1}$, in which the lower index is always an index $i = i_1, i_2, i_3$ and in which the upper index is always equal to the summation index $k_1$. By the way, more generally, we immediately observe that all the successive positive integers

$$1, 3, 1, 3, 3, 1, 3, 1, 3, 3, 3, 9, 3, 1, 3, 4$$

appearing in the formula (2.8) for $\mathbf{Y}_3$ are replaced, in the formula (3.24) for $\mathbf{Y}_{i_1,i_2,i_3}$, by sums of exactly the same number of terms involving Kronecker symbols. This observation will be a precious guide. Finally, in the symbol $\delta_{i_1}^{k_1}$, if $i$ is chosen among the set $\{i_1, i_2, i_3\}$, for instance if $i = i_1$, it follows that the development of $\mathcal{Y}_{x^{i_2} y}$ necessarily involves the remaining indices, for instance $\mathcal{Y}_{x^{i_2} x^{i_3} y}$. Since there are three choices for $i = i_1, i_2, i_3$, we recover the number 3.

Next, comparing $[\mathcal{Y}_{y y} - 2 \mathcal{X}_{x y}] (y_1)^2$ with the term

$$
\sum_{k_1,k_2=1}^{n} \left[ \delta_{i_1,i_2}^{k_1,k_2} \mathcal{Y}_{y y} - \delta_{i_1}^{k_1} \mathcal{X}_{x^{i_2} y} - \delta_{i_2}^{k_1} \mathcal{X}_{x^{i_1} y} \right] y_{k_1} y_{k_2},
$$

(3.44)

extracted from the complete expression of $\mathbf{Y}_{i_1,i_2}$ (second line of (3.18)), we learn and we guess that the number of Kronecker symbols before $\mathcal{Y}_{x^{i_2} y}$ must be equal to the number of indices $k_{\alpha}$ minus $\gamma$. This rule is confirmed by examining the term (second and third line of (3.24))

$$
\sum_{k_1,k_2} \left[ \delta_{i_1,i_2}^{k_1,k_2} \mathcal{Y}_{x^{i_3} y^2} + \delta_{i_1,i_3}^{k_1} \mathcal{Y}_{x^{i_2} y^2} + \delta_{i_2,i_3}^{k_1,k_2} \mathcal{Y}_{x^{i_1} y^2} - \delta_{i_1}^{k_1} \mathcal{X}_{x^{i_2} x^{i_3} y} - \delta_{i_2}^{k_1} \mathcal{X}_{x^{i_1} x^{i_3} y} - \delta_{i_3}^{k_1} \mathcal{X}_{x^{i_1} x^{i_2} y} \right] y_{k_1} y_{k_2},
$$

(3.45)

developing $[3 \mathcal{Y}_{y^2} - 3 \mathcal{X}_{x y}] (y_1)^2$. 
Also, we may examine the following term

$$
\sum_{k_1, k_2=1}^n \left[ \delta_{k_1, k_2}^{k_1, k_2} Y_{x^1 x^2 x^4 y^2} + \delta_{k_1, k_2}^{k_1, k_2} Y_{x^2 x^4 y^2} + \delta_{k_1, k_2}^{k_1, k_2} Y_{x^2 x^3 x^4 y^2} + \right.
$$

$$
\left. + \delta_{k_2, k_3}^{k_1, k_2} Y_{x^1 x^2 x^4 y^2} + \delta_{k_2, k_4}^{k_1, k_2} Y_{x^1 x^1 x^3 y^2} + \right.
$$

$$
\left. + \delta_{k_1, k_4}^{k_1, k_2} Y_{x^1 x^1 x^1 x^3 y^2} - \delta_{k_1, k_4}^{k_1, k_2} Y_{x^1 x^2 x^2 x^3 y^2} - \right.
$$

$$
\left. - \delta_{k_1, k_4}^{k_1, k_2} Y_{x^1 x^2 x^3 x^4 y^2} \right] y_{k_1} y_{k_2},
$$

extracted from $Y_{i_1, i_2, i_3, i_4}$ and developing $[6 Y_{x^2 y^2} - 4 X_{x^3 y}] (y_1)^2$. We would like to mention that we have not written the complete expression of $Y_{i_1, i_2, i_3, i_4}$, because it would cover two and a half printed pages.

By inspecting the way how the indices are permuted in the multiple Kronecker symbols of the first two lines of this expression (3.46), we observe that the six terms correspond exactly to the six possible choices of two complementary ordered couples of integers in the set $\{1, 2, 3, 4\}$, namely

$$
\{1, 2\} \cup \{3, 4\}, \quad \{1, 3\} \cup \{2, 4\}, \quad \{1, 4\} \cup \{2, 3\},
$$

$$
\{2, 3\} \cup \{1, 4\}, \quad \{2, 4\} \cup \{1, 3\}, \quad \{3, 4\} \cup \{1, 2\}.
$$

At this point, we start to devise the general combinatorics. Before proceeding further, we need some notation.

### 3.48. Permutation groups

For every $p \in \mathbb{N}$ with $p \geq 1$, we denote by $\mathfrak{S}_p$ the full permutation group of the set $\{1, 2, \ldots, p-1, p\}$. Its cardinal equals $p!$. The letters $s$ and $t$ will be used to denote an element of $\mathfrak{S}_p$. If $p \geq 2$, and if $q \in \mathbb{N}$ satisfies $1 \leq q \leq p-1$, we denote by $\mathfrak{S}_p^q$ the subset of permutations $\sigma \in \mathfrak{S}_p$ satisfying the two collections of inequalities

$$
\sigma(1) < \sigma(2) < \cdots < \sigma(q) \quad \text{and} \quad \sigma(q+1) < \sigma(q+2) < \cdots < \sigma(p).
$$

The cardinal of $\mathfrak{S}_p^q$ equals $C_p^q = \frac{p!}{q!(p-q)!}$.

**Lemma 3.50.** For $\kappa \geq 1$, the development of (2.20) to several independent variables $(x^1, \ldots, x^n)$ is:

$$
Y_{i_1, i_2, \ldots, i_n} = \sum_{k_1=1}^n \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1}^{k_1} Y_{x^{\tau(1)} x^{\tau(2)} \cdots x^{\tau(n)} y} - \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1}^{k_1} X_{x^{\tau(1)} x^{\tau(2)} \cdots x^{\tau(n)} y} + \sum_{k_1, k_2=1}^n \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1, k_2}^{k_1, k_2} Y_{x^{\tau(2)} \cdots x^{\tau(n)} y^2} + \delta_{k_1, k_2}^{k_1, k_2} X_{x^{\tau(2)} \cdots x^{\tau(n)} y^2} + \sum_{k_1, k_2, k_3=1}^n \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1, k_2}^{k_1, k_2} X_{x^{\tau(2)} \cdots x^{\tau(n)} y^2} + 
$$

$$
\sum_{k_1, \ldots, k_n=1}^n \left[ \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1, \ldots, k_n}^{k_1, \ldots, k_n} Y_{x^{\tau(1)} x^{\tau(2)} \cdots x^{\tau(n)} y} - \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1, \ldots, k_n}^{k_1, \ldots, k_n} X_{x^{\tau(1)} x^{\tau(2)} \cdots x^{\tau(n)} y} \right] \prod_{k=1}^n y_{k_1} y_{k_2} y_{k_3} + \sum_{k_1, \ldots, k_n=1}^n \left[ \sum_{\tau \in \mathfrak{S}_n} \delta_{k_1, \ldots, k_n}^{k_1, \ldots, k_n} X_{x^{\tau(1)} x^{\tau(2)} \cdots x^{\tau(n)} y} \right] \prod_{k=1}^n y_{k_1} \cdots y_{k_n} + \text{remainder}.
$$
Here, the term remainder collects all remaining monomials in the pure jet variables $y_{k_1,\ldots,k_\lambda}$.

**3.5.2. Continuation.** Thus, we have devised how the part of $Y_{i_1,\ldots,i_\lambda}$ which involves only the jet variables $y_{k_0}$ must be written. To proceed further, we shall examine the following term, extracted from $Y_{i_1,i_2,i_3}$ (lines 12 and 13 of (3.24))

$$
\sum_{k_1,k_2,k_3,k_4} \left[ -\delta_{i_1,i_2,i_3} k_1 k_3 Y_{y_{k_1} y_{k_3}} - \delta_{i_1,i_2,i_3} k_1 k_4 Y_{y_{k_1} y_{k_4}} - \delta_{i_1,i_2,i_3} k_2 k_3 Y_{y_{k_2} y_{k_3}} - \delta_{i_1,i_2,i_3} k_2 k_4 Y_{y_{k_2} y_{k_4}} \right] y_{k_1} y_{k_2} y_{k_3} y_{k_4},
$$

(3.53)

which develops the term $-6X_{y_{k_3} y_{k_2} y_{k_1}} (y_{k_1} y_{k_2})$ of $Y_3$ (third line of (2.8)). During the computation which led us to the final expression (3.24), we organized the formula in order that, in the six Kronecker symbols, the lower indices $i_1,i_2,i_3$ are all written in the same order. But then, what is the rule for the appearance of the four upper indices $k_1,k_2,k_3,k_4$?

In April 2001, we discovered the rule by inspecting both (3.53) and the following complicated term, extracted from the complete expression of $Y_{i_1,i_2,i_3,i_4}$ written in one of our manuscripts:

$$
\sum_{k_1,k_2,k_3} \left[ \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \right. \\
+ \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \right. \\
+ \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \\
- \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \\
- \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \\
- \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \\
- \delta_{i_1,i_2,i_3} k_1 k_3 X_{x_{k_1} x_{k_3}} + \delta_{i_1,i_2,i_3} k_1 k_4 X_{x_{k_1} x_{k_4}} + \delta_{i_1,i_2,i_3} k_2 k_3 X_{x_{k_2} x_{k_3}} + \delta_{i_1,i_2,i_3} k_2 k_4 X_{x_{k_2} x_{k_4}} \\
\left. \right] y_{k_1} y_{k_2} y_{k_3} y_{k_4},
$$

(3.54)

This sum develops the term $12X_{x_{k_3} x_{k_2} y_{k_1}} (y_{k_1} y_{k_2})$ of $Y_4$ (third line of (2.9)). Let us explain what are the formal rules.

In the bracketed terms of (3.53), there are no permutation of the indices $i_1,i_2,i_3$, but there is a certain known subset of all the permutations of the four indices $k_1,k_2,k_3,k_4$. In the bracketed terms of (3.54), two combinatorics are present:

- there are some permutations of the indices $i_1,i_2,i_3,i_4$ and
- there are some permutations of the indices $k_1,k_2,k_3$.

Here, the permutations of the indices $i_1,i_2,i_3,i_4$ are easily guessed, since they are the same as the permutations which were introduced in §3.48 above. Indeed, in the first four lines of (3.54), we see the four decompositions

$$
\{i_1,i_2,i_3\} \cup \{i_4\}, \quad \{i_1,i_2,i_4\} \cup \{i_3\}, \quad \{i_1,i_3,i_4\} \cup \{i_2\}, \quad \{i_2,i_3,i_4\} \cup \{i_1\},
$$

(3.55) of the set $\{i_1,i_2,i_3,i_4\}$, and in the last six lines of (3.54), we see the six decompositions

$$
\{i_1,i_2\} \cup \{i_3,i_4\}, \quad \{i_1,i_3\} \cup \{i_2,i_4\}, \quad \{i_1,i_4\} \cup \{i_2,i_3\}, \\
\{i_2,i_3\} \cup \{i_1,i_4\}, \quad \{i_2,i_4\} \cup \{i_1,i_3\}, \quad \{i_3,i_4\} \cup \{i_1,i_2\},
$$

(3.56)
so that (3.54) may be written under the form
\[
\sum_{k_1, k_2, k_3} \left[ \sum_{r \in \mathcal{P}_3} \sum_{\sigma \in \mathcal{S}_3} \delta_{r_1, r_2, r_3} \varepsilon_{r_1, r_2, r_3} \sum_{s \in \mathcal{P}_4} \sum_{\tau \in \mathcal{S}_4} \delta_{s_1, s_2, s_3} \chi_{s_1, s_2, s_3, s_4} \right] y_{k_1} y_{k_2} y_{k_3},
\]
where in the two above sums \( \sum_{\sigma \in \mathcal{S}_3} \), the letter \( \sigma \) denotes a permutation of the set \( \{1, 2, 3\} \) and where the sign \( ? \) refers to two (still unknown) subset of the full permutation group \( \mathcal{S}_3 \). The only remaining question is to determine how the indices \( k_a \) are permuted in (3.53) and in (3.54).

The answer may be guessed by looking at the permutations of the set \( \{k_1, k_2, k_3, k_4\} \) which stabilize the monomial \( y_{k_1} y_{k_2} y_{k_3} k_4 \) in (3.53): we clearly have the following four symmetry relations between monomials:
\[
y_{k_1} y_{k_2} y_{k_3} k_4 \equiv y_{k_2} y_{k_1} y_{k_3} k_4 \equiv y_{k_2} y_{k_1} y_{k_4} k_3 \equiv y_{k_2} y_{k_1} y_{k_4} k_3,
\]
and nothing more. Then the number 6 of bracketed terms in (3.53) is exactly equal to the cardinal \( 24 = 4! \) of the full permutation group of the set \( \{k_1, k_2, k_3, k_4\} \) divided by the number 4 of these symmetry relations. The set of permutations \( \sigma \) of \( \{1, 2, 3, 4\} \) satisfying these symmetry relations
\[
y_{k_{\sigma(1)}} y_{k_{\sigma(2)}} y_{k_{\sigma(3)}} y_{k_{\sigma(4)}} \equiv y_{k_1} y_{k_2} y_{k_3} k_4
\]
constitutes a subgroup of \( \mathcal{S}_4 \) which we will denote by \( \mathcal{S}_4^{(2,1),(1,2)} \). Furthermore, the coset
\[
\mathcal{S}_4^{(2,1),(1,2)} := \mathcal{S}_4 / \mathcal{S}_4^{(2,1),(1,2)}
\]
possesses the six representatives
\[
\begin{array}{ccc}
(1 & 2 & 3 & 4) & , & (1 & 2 & 3 & 4) & , & (1 & 2 & 3 & 4) \\
(1 & 2 & 3 & 4) & , & (1 & 2 & 3 & 4) & , & (1 & 2 & 3 & 4) \\
3 & 4 & 1 & 2 & , & 3 & 4 & 1 & 2 & , & 3 & 4 & 1 & 2
\end{array}
\]
which exactly appear as the permutations of the upper indices of our example (3.53). Of course, the question arises whether the choice of such six representatives in the quotient \( \mathcal{S}_4 / \mathcal{S}_4^{(2,1),(1,2)} \) is legitimate.

Fortunately, we observe that after conjugation by any permutation \( \sigma \in \mathcal{S}_4^{(2,1),(1,2)} \), we do not perturb any of the six terms of (3.53), for instance the third term of (3.53) is not perturbed, as shown by the following computation
\[
\begin{aligned}
\sum_{k_1, k_2, k_3, k_4} & \left[ - \delta_{k_{1(1)}, k_{2(1)}, k_{3(1)}; k_{4(1)}, k_{4(2)}, k_{4(3)}} \chi_{k_{4(2)}, k_{4(3)}; k_{4(2)}, k_{4(3)}} \right] y_{k_1} y_{k_2} y_{k_3} k_4 = \\
= & \sum_{k_1, k_2, k_3, k_4} \left[ - \delta_{k_{2(1)}, k_{2(2)}, k_{2(3)}; k_{4(1)}, k_{4(2)}, k_{4(3)}} \chi_{k_{4(2)}, k_{4(3)}; k_{4(2)}, k_{4(3)}} \right] y_{k_1} y_{k_2} y_{k_3} k_4
\end{aligned}
\]
thanks to the symmetry (3.59). Thus, as expected, the choice of 6 arbitrary representatives \( \sigma \in \mathcal{S}_4^{(2,1),(1,2)} \) in the bracketed terms of (3.53) is free. In conclusion, we have shown that (3.53) may be written under the form:
\[
\begin{aligned}
\sum_{k_1, k_2, k_3, k_4} & \left[ - \delta_{k_{1(1)}, k_{2(1)}, k_{3(1)}; k_{4(1)}, k_{4(2)}, k_{4(3)}} \chi_{k_{4(2)}, k_{4(3)}; k_{4(2)}, k_{4(3)}} \right] y_{k_1} y_{k_2} y_{k_3} k_4 = \\
= & \sum_{\sigma \in \mathcal{S}_4^{(2,1),(1,2)}} \left[ - \delta_{\sigma_{1(1)}, \sigma_{2(1)}, \sigma_{3(1)}; \sigma_{4(1)}, \sigma_{4(2)}, \sigma_{4(3)}} \chi_{\sigma_{4(2)}, \sigma_{4(3)}; \sigma_{4(2)}, \sigma_{4(3)}} \right] y_{k_1} y_{k_2} y_{k_3} k_4
\end{aligned}
\]
This rule is confirmed by inspecting (3.54) (as well as all the other terms of \( \mathcal{Y}_{1, 2, i_3, i_4} \) and of \( \mathcal{Y}_{1, i_2, i_3, i_4} \)). Indeed, the permutations \( \sigma \) of the set \( \{k_1, k_2, k_3\} \) which stabilize the monomial
uniquely decomposes as the composition of

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix},
\]

which appear in the upper index position of each of the ten lines of (3.54). It follows that (3.54) may be written under the form

\[
\sum_{k_1, k_2, k_3} \left[ \sum_{\tau \in \mathcal{S}_3} \sum_{\sigma \in \mathcal{S}_3^{(1.1), (1.2)}} \delta_{\sigma(1), \tau(2), \tau(3)}^{k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}} \mathcal{Y}_x^{\tau(4)} y^2 - \sum_{\sigma \in \mathcal{S}_3} \sum_{\tau \in \mathcal{S}_3^{(1.1), (1.2)}} \delta_{\sigma(1), \tau(2)}^{k_{\sigma(1)}, k_{\sigma(2)}} \mathcal{X}_{\sigma(3)}^{k_{\sigma(3)}} x^{\tau(3)} x^{\tau(4)} y \right] y_{k_1} y_{k_2} y_{k_3}.
\]

**3.66. General complete expression of \( Y_{\lambda_1, \ldots, \lambda_n} \).** As in the incomplete expression (3.39) of \( Y_{\lambda_1, \ldots, \lambda_n} \), consider integers \( 1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa \) and \( \mu_1 \geq 1, \ldots, \mu_d \geq 1 \) satisfying \( \mu_1 \lambda_1 + \cdots + \mu_d \lambda_d \leq \kappa + 1 \). By \( \mathcal{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} \), we denote the subgroup of permutations \( \tau \in \mathcal{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} \) that leave unchanged the general monomial (3.38), namely that satisfy

\[
\prod_{1 \leq \mu_1 \leq \mu_2} y_{k_{\sigma(1), \mu_1 \lambda_1}, \ldots, k_{\sigma(1), \mu_1 \lambda_1}} \cdots \prod_{1 \leq \mu_3 \leq \mu_d} y_{k_{\sigma(d, \mu_d \lambda_d), \ldots, k_{\sigma(d, \mu_d \lambda_d)}}} = \prod_{1 \leq \mu_1 \leq \mu_2} y_{k_{\sigma(1, \mu_1 \lambda_1), \ldots, k_{\sigma(1, \mu_1 \lambda_1)}}} \cdots \prod_{1 \leq \mu_3 \leq \mu_d} y_{k_{\sigma(d, \mu_d \lambda_d), \ldots, k_{\sigma(d, \mu_d \lambda_d)}}}.
\]

The structure of this group may be described as follows. For every \( e = 1, \ldots, d \), an arbitrary permutation \( \sigma \) of the set

\[
\{e : 1; 1, \ldots, e : e; 1, \ldots, e : 2; 1, \ldots, e : 2; \lambda_1, \cdots, e : \mu_e : 1, \ldots, e : \mu_e : \lambda_e\}
\]

which leaves unchanged the monomial

\[
\prod_{1 \leq \nu_1 \leq \mu_2} y_{k_{\sigma(1, \nu_1 \lambda_1), \ldots, k_{\sigma(1, \nu_1 \lambda_1)}}} = \prod_{1 \leq \nu_1 \leq \mu_2} y_{k_{\sigma(1, \nu_1 \lambda_1), \ldots, k_{\sigma(1, \nu_1 \lambda_1)}}}
\]

uniquely decomposes as the composition of

- \( \mu_e \) arbitrary permutations of the \( \mu_e \) groups of \( \lambda_e \) integers \( \{e : \nu_e : 1, \ldots, e : \nu_e : \lambda_e\} \), of total cardinal \( (\lambda_e)_e^{\nu_e} \);
- an arbitrary permutation between these \( \mu_e \) groups, of total cardinal \( \mu_e! \).

Consequently

\[
\text{Card} \left( \mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} \right) = \mu_1!(\lambda_1)! \cdots \mu_d!(\lambda_d)! \mu_e!
\]

Finally, define the coset

\[
\mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} := \mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} / \mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d}
\]

with

\[
\text{Card} \left( \mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} \right) = \frac{\text{Card} \left( \mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} \right)}{\text{Card} \left( \mathfrak{S}_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d} \right)} = \frac{(\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d)!}{\mu_1!(\lambda_1)! \cdots \mu_d!(\lambda_d)! \mu_e!}.
\]
In conclusion, by means of this formalism, we may write down the complete generalization of Theorem 2.24 to several independent variables.

**Theorem 3.73.** For every $\kappa \geq 1$ and for every choice of $\kappa$ indices $i_1, \ldots, i_\kappa$ in the set $\{1, 2, \ldots, n\}$, the general expression of $Y_{i_1, \ldots, i_\kappa}$ is as follows:

\[
Y_{i_1, \ldots, i_\kappa} = \mathcal{Y}_i x_i \mathcal{Y}_k x_k + \sum_{d=1}^{n+1} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d \leq \kappa + 1} 
\]

\[
\left[ \prod_{1 \leq \nu_1 \leq \mu_1} y_{k_1, i_1} \cdots \prod_{1 \leq \nu_d \leq \mu_d} y_{k_d, i_d} \right] 
\]

3.75. **Deduction of a multivariate Faà di Bruno formula.** Let $n \in \mathbb{N}$ with $n \geq 1$, let $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $g = g(x^1, \ldots, x^n)$ be a $C^\infty$-smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, let $y \in \mathbb{R}$ and let $f = f(y)$ be a $C^\infty$ function from $\mathbb{R}$ to $\mathbb{R}$. The goal is to obtain an explicit formula for the partial derivatives of the composition $h := f \circ g$, namely $h(x^1, \ldots, x^n) := f(g(x^1, \ldots, x^n))$. For $\lambda \in \mathbb{N}$ with $\lambda \geq 1$ and for arbitrary indices $i_1, \ldots, i_\lambda = 1, \ldots, n$, we shall abbreviate the partial derivative $\frac{\partial^\lambda g}{\partial x^1 \cdots \partial x^\lambda}$ by $g_{i_1, \ldots, i_\lambda}$ and similarly for $h_{i_1, \ldots, i_\lambda}$. The derivative $\frac{\partial^\lambda f}{\partial y^\lambda}$ will be abbreviated by $f_\lambda$.

Applying the chain rule, we may compute:

\[
h_{i_1} = f_1 [g_{i_1}], \\
h_{i_1, i_2} = f_2 \left[ g_{i_1, i_2} \right] + f_1 \left[ g_{i_1, i_2} \right], \\
h_{i_1, i_2, i_3} = f_3 \left[ g_{i_1, i_2, i_3} \right] + f_2 \left[ g_{i_1, i_2, i_3} + g_{i_1, i_3, i_2} + g_{i_2, i_1, i_3} + g_{i_3, i_1, i_2} + f_1 \left[ g_{i_1, i_2, i_3} \right] \right], \\
h_{i_1, i_2, i_3, i_4} = f_4 \left[ g_{i_1, i_2, i_3, i_4} \right] + f_3 \left[ g_{i_1, i_2, i_3, i_4} + g_{i_1, i_2, i_4, i_3} + g_{i_2, i_1, i_4, i_3} + g_{i_3, i_1, i_2, i_4} + g_{i_4, i_1, i_2, i_3} \right] + f_2 \left[ g_{i_1, i_2, i_3, i_4} + g_{i_1, i_3, i_2, i_4} + g_{i_2, i_1, i_4, i_3} + g_{i_3, i_1, i_2, i_4} + g_{i_4, i_1, i_2, i_3} + g_{i_1, i_2, i_3, i_4} + g_{i_1, i_3, i_2, i_4} + g_{i_2, i_1, i_4, i_3} + g_{i_3, i_1, i_2, i_4} + g_{i_4, i_1, i_2, i_3} + g_{i_1, i_2, i_3, i_4} \right] + f_1 \left[ g_{i_1, i_2, i_3, i_4} \right].
\]

Introducing the derivations

\[
F_3 := \sum_{k_1, k_2 = 1}^{n} g_{k_1, k_2} \frac{\partial}{\partial y_{k_1}} + \sum_{k_1, k_2, k_3 = 1}^{n} g_{k_1, k_2, k_3} \frac{\partial}{\partial y_{k_1, k_2}} + \cdots + g_1 \left( f_2 \frac{\partial}{\partial f_1} + f_3 \frac{\partial}{\partial f_2} \right),
\]

\[
F_4 := \sum_{k_1, k_2, k_3, k_4 = 1}^{n} g_{k_1, k_2, k_3, k_4} \frac{\partial}{\partial y_{k_1, k_2, k_3}} + \cdots + \cdots,
\]

(3.77)
we observe that the following induction relations hold:

\[
\begin{align*}
  h_{i_1, i_2} &= F^2_{i_1, i_2}(h_i), \\
  h_{i_1, i_2, i_3} &= F^3_{i_1, i_2, i_3}(h_{i_1, i_2}), \\
  &\quad \ldots \quad \ldots \quad \ldots \\
  h_{i_1, i_2, \ldots, i_\lambda} &= F^\lambda_{i_1, i_2, \ldots, i_\lambda}(h_{i_1, i_2, \ldots, i_\lambda}).
\end{align*}
\]

(3.78)

To obtain the explicit version of the Faà di Bruno in the case of several variables \((x^1, \ldots, x^n)\) and one variable \(y\), it suffices to extract from the expression of \(Y_{i_1, \ldots, i_n}\) provided by Theorem 3.73 only the terms corresponding to \(\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d = \kappa\), dropping all the \(X\) terms. After some simplifications and after a translation by means of an elementary dictionary, we obtain a statement.

**Theorem 3.79.** For every integer \(\kappa \geq 1\) and for every choice of indices \(i_1, \ldots, i_\kappa\) in the set \(\{1, 2, \ldots, n\}\), the \(\kappa\)-th partial derivative of the composite function \(h = h(x^1, \ldots, x^n) = f(g(x^1, \ldots, x^n))\) with respect to the variables \(x^1, \ldots, x^n\) may be expressed as an explicit polynomial depending on the derivatives of \(f\), on the partial derivatives of \(g\) and having integer coefficients:

\[
\frac{\partial^\kappa h}{\partial x^{i_1} \cdots \partial x^{i_\kappa}} = \sum_{d=1}^\kappa \sum_{\lambda_1, \ldots, \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d = \kappa} \frac{\partial^\mu_1 \cdots + \partial^\mu_d f}{\partial y^{\mu_1} \cdots + \partial y^{\mu_d}} \prod_{\sigma \in \mathcal{S}_{\kappa}} \frac{\partial^{\lambda_\sigma} g}{\partial x^{\sigma(1:1)} \cdots \partial x^{\sigma(\lambda_\sigma-1)}} \ldots
\]

(3.80)

In this formula, the coset \(\mathcal{S}_{\kappa}^{(\mu_1, \lambda_1), \ldots, (\mu_d, \lambda_d)}\) was defined in equation (3.71); we have made the identification:

\[
\{1, \ldots, \kappa\} \equiv \{1:1, 1:1, \ldots, 1:1, \ldots, d:1, 1:1, \ldots, d:1, \ldots, d:d\}
\]

and also, for the sake of clarity, we have restituted the complete (not abbreviated) notation for the (partial) derivatives of \(f\) and of \(g\).

§4. SEVERAL INDEPENDENT VARIABLES AND ONE DEPENDENT VARIABLE

**4.1. Simplified adapted notations.** Assume \(n = 1\) and \(m \geq 1\), let \(\kappa \in \mathbb{N}\) with \(\kappa \geq 1\) and simply denote the jet variables by (instead of (1.2)):

\[
(x, y^1, y^2_1, y^2_2, \ldots, y^m_\kappa) \in J^\kappa_{1,m}.
\]

(4.2)
Instead of (1.30), denote the $\kappa$-th prolongation of a vector field by:

\[
\mathcal{L}^{(\kappa)} = X \frac{\partial}{\partial x} + \sum_{j=1}^{m} \mathcal{Y}^j \frac{\partial}{\partial y^j} + \sum_{j=1}^{m} Y^j_1 \frac{\partial}{\partial y^j_1} + \sum_{j=1}^{m} Y^j_2 \frac{\partial}{\partial y^j_2} + \ldots + \sum_{j=1}^{m} Y^j_\kappa \frac{\partial}{\partial y^j_\kappa}.
\]

The induction formulas are:

\[
\begin{align*}
Y^j_1 &:= D^1 (\mathcal{Y}^j) - D^1 (X) y^j_1, \\
Y^j_2 &:= D^2 (Y^j_1) - D^1 (X) y^j_2, \\
& \\
Y^j_\lambda &:= D^\lambda \left( Y^j_{\lambda-1} \right) - D^1 (X) y^j_\lambda,
\end{align*}
\]

where the total differentiation operators $D^\lambda$ are denoted by (instead of (1.22)):

\[
D^\lambda := \frac{\partial}{\partial x} + \sum_{l=1}^{m} y^l_1 \frac{\partial}{\partial y^l_1} + \sum_{l=1}^{m} y^l_2 \frac{\partial}{\partial y^l_2} + \ldots + \sum_{l=1}^{m} y^l_\lambda \frac{\partial}{\partial y^l_\lambda}.
\]

Applying the definitions in the first two lines of (4.4), we compute, we simplify and we organize the results in a harmonious way, using in an essential way the Kronecker symbol. Here, the computations are more elementary than the computations of $Y_{i_1}$ and of $Y_{i_1,i_2}$ achieved thoroughly in the previous Section 3, so that we do not provide a Latex track of the details. Firstly and secondly:

\[
\begin{align*}
Y^j_1 &= \mathcal{Y}^j + \sum_{l_1=1}^{m} \left[ \mathcal{Y}^j_{y^l_1} - \delta^j_{l_1} X_{y^l_1} \right] y^j_1 + \sum_{l_1,l_2=1}^{m} \left[ -\delta^j_{l_1} X_{y^l_2} \right] y^j_1 y^j_2, \\
Y^j_2 &= \mathcal{Y}^j_{y^l_1} + \sum_{l_1=1}^{m} \left[ 2 \mathcal{Y}^j_{x y^l_1} - \delta^j_{l_1} X_{x} \right] y^j_1 + \sum_{l_1,l_2=1}^{m} \left[ \mathcal{Y}^j_{y^l_1 y^l_2} - \delta^j_{l_1} 2 X_{y^l_2} \right] y^j_1 y^j_2 + \\
& \quad + \sum_{l_1,l_2,l_3} \left[ -\delta^j_{l_1} X_{y^l_2 y^l_3} \right] y^j_1 y^j_2 y^j_3 + \sum_{l_1} \left[ \mathcal{Y}^j_{y^l_1} - \delta^j_{l_1} 2 X_{x} \right] y^j_2 + \\
& \quad + \sum_{l_1,l_2=1}^{m} \left[ -\delta^j_{l_1} X_{y^l_2} - \delta^j_{l_2} 2 X_{y^l_1} \right] y^j_1 y^j_2.
\end{align*}
\]

Thirdly:

\[
\begin{align*}
Y^j_3 &= \mathcal{Y}^j_{y^l_3} + \sum_{l_1=1}^{m} \left[ 3 \mathcal{Y}^j_{x y^l_1} - \delta^j_{l_1} X_{x} \right] y^j_1 + \sum_{l_1,l_2=1}^{m} \left[ 3 \mathcal{Y}^j_{x y^l_1 y^l_2} - \delta^j_{l_1} 3 X_{x y^l_2} \right] y^j_1 y^j_2 + \\
& \quad + \sum_{l_1,l_2,l_3} \left[ \mathcal{Y}^j_{y^l_1 y^l_2 y^l_3} - \delta^j_{l_1} X_{x y^l_2 y^l_3} \right] y^j_1 y^j_2 y^j_3 + \\
& \quad + \sum_{l_1,l_2,l_3,l_4} \left[ -\delta^j_{l_1} X_{x y^l_2 y^l_3 y^l_4} \right] y^j_1 y^j_2 y^j_3 y^j_4 + \sum_{l_1=1}^{m} \left[ 3 \mathcal{Y}^j_{x y^l_1} - \delta^j_{l_1} 3 X_{x} \right] y^j_2
\end{align*}
\]
Fourthly:

\[ Y'_4 = Y'_4 + \sum_{l_1, l_2} \left[ 3 Y'_{y' y' 2} - \delta'_{l_1} 3 X_{y' y' 2} - \delta'_{l_2} 6 X_{x' y'} \right] y_{l_1} y_{l_2} + \]

\[ + \sum_{l_1, l_2, l_3} \left[ -\delta'_{l_1} 3 X_{y' y' y'} - \delta'_{l_2} 3 X_{x' y' y'} \right] y_{l_1} y_{l_2} y_{l_3} + \sum_{l_1, l_2} \left[ -\delta'_{l_1} 3 X_{y' y'} - \delta'_{l_2} 3 X_{x' y'} \right] y_{l_1} y_{l_2} + \]

\[ + \sum_{l_1} \left[ Y'_{y' y' 4} - \delta'_{l_1} 3 X_{y'} \right] y_{l_1} + \sum_{l_1, l_2} \left[ -\delta'_{l_1} X_{y' y' 2} - \delta'_{l_2} 3 X_{x' y} \right] y_{l_1} y_{l_2}. \]

(4.8)

4.9. Inductive elaboration of the general formula. Now we compare the formula (2.9) for \( Y_4 \) with the above formula (4.8) for \( Y'_4 \). The goal is to find the rules of transformation and of development by inspecting several instances, in order to devise how to transform and to develop the formula (2.25) to several dependent variables.

First of all, we have to develop the general monomial \((y_{\lambda_1})^{\mu_1} \cdots (y_{\lambda_n})^{\mu_n}\). In every monomial present in the expressions of \( Y'_1 \), of \( Y'_2 \), of \( Y'_3 \) and of \( Y'_4 \) above, we see that the number \( \alpha \) of
indices $l_\beta$ appearing in all the sums $\sum_{i_1,\ldots,i_n=1}^{m}$ is exactly equal to $\mu_1 + \cdots + \mu_d$. To denote these $\mu_1 + \cdots + \mu_d$ indices $l_\beta$, we shall use the notation:

$$l_{1:1}, \ldots, l_{1:\mu_1}, \ldots, l_{d:1}, \ldots, l_{d:\mu_d},$$

inspired by Convention 3.33. With such a choice of notation, we may avoid long subscripts in the indices $l_\beta$, like $l_{\mu_1+\cdots+\mu_d}$. It follows that the development of the general monomial $(y_{\lambda_1})^{\mu_1} \cdots (y_{\lambda_d})^{\mu_d}$ to several dependent variables yields $m^{\mu_1+\cdots+\mu_d}$ possible choices:

$$\prod_{1 \leq i_1 \leq \mu_1} y_{\lambda_{1_{i_1}}} \cdots \prod_{1 \leq i_d \leq \mu_d} y_{\lambda_{d_{i_d}}}$$

where the indices $l_{1:1}, l_{1:\mu_1}, \ldots, l_{d:1}, l_{d:\mu_d}$ take their values in the set $\{1, 2, \ldots, m\}$. Consequently, the general expression of $Y_\kappa^y$ must be of the form:

$$Y_\kappa^y = \sum_{d=1}^{m+1} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d \leq \kappa+1} \sum_{l_{1:1}=1}^{m} \cdots \sum_{l_{1:\mu_1}=1}^{m} \cdots \sum_{l_{d:1}=1}^{m} \cdots \sum_{l_{d:\mu_d}=1}^{m} [?] \prod_{1 \leq i_1 \leq \mu_1} y_{\lambda_{1_{i_1}}} \cdots \prod_{1 \leq i_d \leq \mu_d} y_{\lambda_{d_{i_d}}}$$

where the term in brackets $[?]$ is still unknown. To determine it, let us examine a few instances.

According to (4.8) (fourth line), the term $[6 \mathcal{Y}_{x^2y} - 4 \mathcal{X}_{x^3}] y_2$ of $Y_4$ develops as $\sum_{i=1}^{m} \left[ 6 \mathcal{Y}_{x^2y}^i - \delta_{1i}^4 \mathcal{X}_{x^3} \right] y_2$ in $Y_4^y$. Here, $6 \mathcal{Y}_{x^2y}$ just becomes $6 \mathcal{Y}_{x^2y}^i$. Thus, we suspect that the term $\frac{\kappa(\kappa-1) \cdots (\kappa-\mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{(\lambda_1 !)^{\mu_1} \cdots (\lambda_d !)^{\mu_d} \cdot \mu_1 ! \cdot \cdots \cdot \mu_d !} \frac{\partial^{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d + 1} y_2}{\partial x^{\mu_1 \lambda_1 - \cdots - \mu_d \lambda_d} \partial y_{i_1}^{\mu_1} \cdots \partial y_{i_d}^{\mu_d} \cdots \partial y^{i_1 \cdots i_d}}$ of the second line of (2.25) should simply be developed as

$$\frac{\kappa(\kappa-1) \cdots (\kappa-\mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{(\lambda_1 !)^{\mu_1} \cdots (\lambda_d !)^{\mu_d} \cdot \mu_1 ! \cdot \cdots \cdot \mu_d !} \frac{\partial^{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d + 1} y_2}{\partial x^{\mu_1 \lambda_1 - \cdots - \mu_d \lambda_d} \partial y_{i_1}^{\mu_1} \cdots \partial y_{i_d}^{\mu_d} \cdots \partial y^{i_1 \cdots i_d}}$$

This rule is confirmed by inspecting all the other monomials of $Y_4^y$, of $Y_3^y$ and of $Y_2^y$.

It remains to determine how we must develop the term in $\mathcal{X}$ appearing in the last two lines of (2.25). To begin with, let us rewrite in advance this term in the slightly different shape, emphasizing a factorization:

$$\frac{\kappa \cdots (\kappa-\mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 2)}{(\lambda_1 !)^{\mu_1} \cdots (\lambda_d !)^{\mu_d} \cdot \mu_1 ! \cdot \cdots \cdot \mu_d !} \left[ (\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d) \mathcal{X}_{x^2y}^{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d + 1} y_{i_1} \cdots y_{i_d} \cdot \partial y_{i_1} \cdots \partial y_{i_d} \right]$$
Then we examine four instances extracted from the complete expression of \( \mathbf{Y}_4 \):

\[
\begin{align*}
\sum_{l_1,l_2,\ldots,l_m=1}^m \left[ 4 \mathcal{Y}_{x_1x_2y_1y_2} - \delta^l_1 6 \mathcal{X}_{x_1x_2y_1y_2} \right] y_1^{l_1} y_2^{l_2}, \\
\sum_{l_1,l_2=1}^m \left[ 12 \mathcal{Y}_{x_1y_1y_2} - \delta^l_1 6 \mathcal{X}_{x_1y_1y_2} - \delta^l_2 12 \mathcal{X}_{x_1y_1y_2} \right] y_1^{l_1} y_2^{l_2}, \\
\sum_{l_1,l_2,l_3=1}^m \left[ -\delta^l_1 6 \mathcal{X}_{x_1y_1y_2y_3} - \delta^l_2 4 \mathcal{X}_{y_1y_2y_3} \right] y_1^{l_1} y_2^{l_2} y_3^{l_3}, \\
\sum_{l_1,l_2,l_3=1}^m \left[ -\delta^l_1 6 \mathcal{X}_{y_1y_2y_3} - \delta^l_2 6 \mathcal{X}_{y_1y_2y_3} \right] y_1^{l_1} y_2^{l_2} y_3^{l_3},
\end{align*}
\]

(4.15)

and we compare them to the corresponding terms of \( \mathbf{Y}_4 \):

\[
\begin{align*}
4 \mathcal{Y}_{x_1y_1} - 6 \mathcal{X}_{x_1y_1} (y_1)^3, \\
12 \mathcal{Y}_{x_1y_2} - 18 \mathcal{X}_{x_1y_2} y_1y_2, \\
-10 \mathcal{X}_{y_1y_2} (y_1)^3 y_2, \\
-10 \mathcal{X}_{y_2} (y_1)^3 y_3.
\end{align*}
\]

(4.16)

In the development from (4.16) to (4.15), we see that the four integers just before \( \mathcal{X} \), namely 6, 6, 18, 6 + 12, 10 = 6 + 4 and 10 = 4 + 6, are split in a certain manner. Also, a single Kronecker symbol \( \delta_{\mu}^{l_1} \) is added as a factor. What are the rules?

In the second splitting \( 18 = 6 + 12 \), we see that the relative weight of 6 and of 12 is the same as the relative weight of 1 and 2 in the splitting \( 3 = 1 + 2 \) issued from the lower indices of the corresponding monomial \( y_1^{l_1} y_2^{l_2} \). Similarly, in the third splitting \( 10 = 6 + 4 \), the relative weight of 6 and of 4 is the same as the relative weight of 1 + 1 + 1 and of 2 issued from the lower indices of the corresponding monomial \( y_1^{l_1} y_2^{l_2} y_3^{l_3} \). This rule may be confirmed by inspecting all the other monomials of \( \mathbf{Y}_2 \), \( \mathbf{Y}_3 \), of \( \mathbf{Y}_4 \), \( \mathbf{Y}_5 \), and of \( \mathbf{Y}_4 \). For a general \( \kappa > 1 \), the splitting of integers just amounts to decompose the sum appearing inside the brackets of (4.14) as \( \mu_1 \lambda_1, \mu_2 \lambda_2, \ldots, \mu_d \lambda_d \). In fact, when we wrote (4.14), we emphasized in advance the decomposable factor \( \mu_1 \lambda_1 + \cdots + \mu_d \lambda_d \).

Next, we have to determine what is the subscript \( \alpha \) in the Kronecker symbol \( \delta_{\mu}^{l_1} \). We claim that in the four instances (4.15), the subscript \( \alpha \) is intrinsically related to weight splitting. Indeed, recall that in the second line of (4.15), the number 6 of the splitting \( 18 = 6 + 12 \) is related to the number 1 in the splitting \( 3 = 1 + 2 \) of the lower indices of the monomial \( y_1^{l_1} y_2^{l_2} \). It follows that the index \( l_\alpha \) must be the index \( l_1 \) of the monomial \( y_1^{l_1} \). Similarly, also in the second line of (4.15), the number 12 of the splitting \( 18 = 6 + 12 \) being related to the number 2 in the splitting \( 3 = 1 + 2 \) of the lower indices of the monomial \( y_1^{l_1} y_2^{l_2} \), it follows that the index \( l_\alpha \) attached to the second \( \mathcal{X} \) term must be the index \( l_2 \) of the monomial \( y_2^{l_2} \).

This rule is still ambiguous. Indeed, let us examine the third line of (4.15). We have the splitting \( 10 = 6 + 4 \), homologous to the splitting of relative weights \( 5 = (1 + 1 + 1) + 2 \) in the monomial \( y_1^{l_1} y_1^{l_2} y_1^{l_3} y_2^{l_2} \). Of course, it is clear that we must choose the index \( l_4 \) for the Kronecker symbol associated to the second \( \mathcal{X} \) term \( -4 \mathcal{X}_{y_1y_2y_3} \), thus obtaining \( -\delta_{l_4}^l 4 \mathcal{X}_{y_1y_2y_3} \). However, since the monomial \( y_1^{l_1} y_2^{l_2} y_3^{l_3} \) has three indices \( l_1, l_2, \) and \( l_3 \), there arises a question: what index \( l_\alpha \) must we choose for the Kronecker symbol \( \delta_{\mu}^{l_1} \), attached to the first \( \mathcal{X} \) term \( 6 \mathcal{X}_{y_1y_2y_3} \): the index \( l_1 \), the index \( l_2 \) or the index \( l_3 \)?

The answer is simple: any of the three indices \( l_1, l_2 \) or \( l_3 \) works. Indeed, since the monomial \( y_1^{l_1} y_1^{l_2} y_1^{l_3} \) is symmetric with respect to all permutations of the set of three indices \( \{l_1, l_2, l_3\} \), we
Theorem 4.18. For one independent variable \( x \), for several dependent variables \( (y^1, \ldots, y^m) \) and for \( \kappa \geq 1 \), the general expression of the coefficient \( Y^i_\kappa \) of the prolongation (4.3) of a vector field is:

\[
Y^j_\kappa = Y^j_{\kappa, \alpha} + \sum_{d=1}^{\kappa+1} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \geq 1, \ldots, \mu_d \geq 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d = \kappa + 1} \frac{\kappa (\kappa - 1) \cdots (\kappa - \mu_1 \lambda_1 + \cdots + \mu_d \lambda_d + 2)}{\lambda_1!^{\mu_1} \cdots \lambda_d!^{\mu_d} \mu_1! \cdots \mu_d!} \\
\left[ \delta^\lambda_{1:1} \mu_1 \lambda_1 \prod_{i=1}^{\lambda_1} \frac{\partial (\delta^\lambda_{1:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \frac{\partial (\delta^\lambda_{1:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \cdots \frac{\partial (\delta^\lambda_{1:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \right] \\
- \delta^\lambda_{l:1} \mu_1 \lambda_1 \prod_{j=1}^{\lambda_1} \frac{\partial (\delta^\lambda_{l:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \frac{\partial (\delta^\lambda_{l:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \cdots \frac{\partial (\delta^\lambda_{l:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \\
- \cdots \\
- \delta^\lambda_{l:d} \mu_1 \lambda_1 \prod_{j=1}^{\lambda_1} \frac{\partial (\delta^\lambda_{l:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \frac{\partial (\delta^\lambda_{l:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \cdots \frac{\partial (\delta^\lambda_{l:1} \mu_1 \lambda_1 - \cdots - \mu_d \lambda_d + 1)}{\partial y_i^{1:1}} \\
\prod_{1 \leq i_1 \leq \nu_1} y_{i_1, \nu_1} \cdots \prod_{1 \leq i_d \leq \nu_d} y_{i_d, \nu_d}.
\]

Here, the notation \( \partial y_i^{1:1} \) means that the partial derivative is dropped.

4.20. Deduction of a multivariate Faà di Bruno formula. Let \( m \in \mathbb{N} \) with \( m \geq 1 \), let \( y = (y^1, \ldots, y^m) \in \mathbb{K}^m \), let \( f = f(y^1, \ldots, y^m) \) be a \( C^\infty \)-smooth function from \( \mathbb{K}^m \) to \( \mathbb{K} \), let \( x \in \mathbb{K} \) and let \( g^1 = g^1(x), \ldots, g^m = g^m(x) \) be \( C^\infty \) functions from \( \mathbb{K} \) to \( \mathbb{K} \). The goal is to obtain an explicit formula for the derivatives, with respect to \( x \), of the composition \( h := f \circ g \), namely \( h(x) := f(g^1(x), \ldots, g^m(x)) \). For \( \lambda \in \mathbb{N} \) with \( \lambda \geq 1 \), and for \( j = 1, \ldots, m \), we shall abbreviate the derivative \( \frac{\partial^\lambda f}{\partial y^1 \cdots \partial y^\lambda} \) by \( g^j_\lambda \) and similarly for \( h_\lambda \). The partial derivatives \( \frac{\partial^\lambda f}{\partial y^1 \cdots \partial y^\lambda} \) will be abbreviated by \( f_{11, \ldots, 1, \lambda} \).
Applying the chain rule, we may compute:

\[ h_1 = \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{1}, \]

\[ h_2 = \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{2} + \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{1}, \]

\[ h_3 = \sum_{l_1,l_2,l_3} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} + 3 \sum_{l_1,l_2,l_3} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} + \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{3}, \]

\[ h_4 = \sum_{l_1,l_2,l_3,l_4} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} g_{l_4}^{4} + 6 \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} + \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{3}, \]

\[ h_5 = \sum_{l_1,l_2,l_3,l_4,l_5} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} g_{l_4}^{4} g_{l_5}^{5} + 10 \sum_{l_1,l_2,l_3,l_4,l_5} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} g_{l_4}^{4} + \]

\[ + 15 \sum_{l_1,l_2,l_3,l_4,l_5} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} g_{l_4}^{4} h_{l_5} + 10 \sum_{l_1,l_2,l_3,l_4,l_5} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} + \]

\[ + 10 \sum_{l_1,l_2,l_3,l_4,l_5} f_{l_1} g_{l_1}^{1} g_{l_2}^{2} g_{l_3}^{3} + \sum_{l_1=1}^{m} f_{l_1} g_{l_1}^{5}. \]

Introducing the derivations

\[ F^2 := \sum_{l_1=1}^{m} g_{l_2}^{1} \frac{\partial}{\partial g_{l_1}^{2}} + \sum_{l_1=1}^{m} g_{l_2}^{1} \left( \sum_{l_2=1}^{m} f_{l_1} g_{l_2}^{1} \frac{\partial}{\partial f_{l_1}^{2}} \right), \]

\[ F^3 := \sum_{l_1=1}^{m} g_{l_2}^{1} \frac{\partial}{\partial g_{l_1}^{3}} + \sum_{l_1=1}^{m} g_{l_2}^{1} \frac{\partial}{\partial g_{l_1}^{2}} + \sum_{l_1=1}^{m} g_{l_2}^{1} \left( \sum_{l_2=1}^{m} f_{l_1} \frac{\partial}{\partial f_{l_1}^{3}} + \sum_{l_2=1}^{m} f_{l_1} \frac{\partial}{\partial f_{l_1}^{2}} \right), \]

\[ F^\lambda := \sum_{l_1=1}^{m} g_{l_2}^{1} \frac{\partial}{\partial g_{l_1}^{\lambda}} + \sum_{l_1=1}^{m} g_{l_2}^{1} \frac{\partial}{\partial g_{l_1}^{\lambda-1}} + \cdots + \sum_{l_1=1}^{m} g_{l_2}^{1} \frac{\partial}{\partial g_{l_1}^{1}} + \]

\[ + \sum_{l_1=1}^{m} g_{l_2}^{1} \left( \sum_{l_2=1}^{m} f_{l_1} \frac{\partial}{\partial f_{l_1}^{\lambda}} + \sum_{l_2=1}^{m} f_{l_1} \frac{\partial}{\partial f_{l_1}^{\lambda-1}} + \cdots + \sum_{l_2=1}^{m} f_{l_1} \frac{\partial}{\partial f_{l_1}^{1}} \right), \]

we observe that the following induction relations hold:

\[ h_2 = F^2 (h_1), \]

\[ h_3 = F^3 (h_2), \]

\[ \cdots \cdots \cdots \]

\[ h_\lambda = F^\lambda (h_{\lambda-1}). \]

To obtain the explicit version of the Faa di Bruno in the case of one variable \( x \) and several variables \( (y^1, \ldots, y^m) \), it suffices to extract from the expression of \( Y^\lambda \), provided by Theorem 4.18 only the terms corresponding to \( \mu_1 \lambda_1 + \cdots + \mu_\lambda \lambda_\lambda = \kappa \), dropping all the \( X \) terms. After some simplifications and after a translation by means of an elementary dictionary, we may formulate a statement.

**Theorem 4.24.** For every integer \( \kappa \geq 1 \), the \( \kappa \)-th partial derivative of the composite function \( h = h(x) = f (g^1(x), \ldots, g^m(x)) \) with respect to \( x \) may be expressed as an explicit polynomial
depending on the partial derivatives of $f$, on the derivatives of $g$ and having integer coefficients:

\[
\frac{d^n h}{dx^n} = \sum_{d=1}^{\kappa} \sum_{\lambda_1 \cdots \lambda_d \mu_1 \cdots \mu_d \geq 1} \frac{\kappa!}{(\lambda_1!)^{\mu_1} \cdots (\lambda_d!)^{\mu_d}} \frac{\partial^{\mu_1 + \cdots + \mu_d} f}{\partial y^{\lambda_{11}} \cdots \partial y^{\lambda_{1u_1}} \cdots \partial y^{\lambda_{d1}} \cdots \partial y^{\lambda_{dv_d}}},
\]

\[
\sum_{l_1,1 \leq l_1 \leq \lambda_1} \cdots \sum_{l_d,1 \leq l_d \leq \lambda_d} \prod_{1 \leq u_1 \leq \mu_1} \frac{\partial^{\lambda_{11}}}{\partial x^{\lambda_{11}}} \cdots \prod_{1 \leq v_d \leq \mu_d} \frac{\partial^{\lambda_{d1}}}{\partial x^{\lambda_{d1}}},
\]

### §5. Several Independent Variables and Several Dependent Variables

#### 5.1. Expression of $Y^j_{i_1}$, of $Y^j_{i_1,i_2}$ and of $Y^j_{i_1,i_2,i_3}$

Applying the induction (1.31) and working out the obtained formulas until they take a perfect shape, we obtain firstly:

\( Y^j_{i_1} = Y^j_{i_1} + \sum_{l_1=1}^{m} \sum_{k_1=1}^{n} \left[ \delta^{k_1}_{i_1} Y^j_{x_{i_2}y^{l_1}_{i_1}} - \delta^{k_1}_{i_1} X^{k_1}_{x_{i_2}y^{l_1}_{i_1}} \right] y^{l_1}_{k_1} + \sum_{l_1,l_2=1}^{m} \sum_{k_1,k_2=1}^{n} \left[ \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}y^{l_1}_{i_1}} - \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}y^{l_1}_{i_1}} \right] y^{l_1}_{k_1} y^{l_2}_{k_2} + \sum_{l_1,l_2,i_2=1}^{m} \sum_{k_1,k_2,k_3=1}^{n} \left[ \delta^{k_1}_{i_1,i_2} X^{k_2}_{x_{i_2}y^{l_1}_{i_1}} - \delta^{k_1}_{i_1,i_2} X^{k_2}_{x_{i_2}y^{l_1}_{i_1}} \right] y^{l_1}_{k_1} y^{l_2}_{k_2} y^{l_3}_{k_3} \)

Secondly:

\( Y^j_{i_1,i_2} = Y^j_{i_1,i_2} + \sum_{l_1=1}^{m} \sum_{k_1=1}^{n} \left[ \delta^{k_1}_{i_1} Y^j_{x_{i_2}x_{i_2}y^{l_1}_{i_1}} - \delta^{k_1}_{i_1} X^{k_1}_{x_{i_2}x_{i_2}y^{l_1}_{i_1}} \right] y^{l_1}_{k_1} + \sum_{l_1,l_2=1}^{m} \sum_{k_1,k_2=1}^{n} \left[ \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}y^{l_1}_{i_1}} - \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}y^{l_1}_{i_1}} \right] y^{l_1}_{k_1} y^{l_2}_{k_2} + \sum_{l_1,l_2,i_2=1}^{m} \sum_{k_1,k_2,k_3=1}^{n} \left[ \delta^{k_1}_{i_1,i_2} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}y^{l_1}_{i_1}} - \delta^{k_1}_{i_1,i_2} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}y^{l_1}_{i_1}} \right] y^{l_1}_{k_1} y^{l_2}_{k_2} y^{l_3}_{k_3} \)

Thirdly:

\( Y^j_{i_1,i_2,i_3} = Y^j_{i_1,i_2,i_3} + \sum_{l_1=1}^{m} \sum_{k_1=1}^{n} \left[ \delta^{k_1}_{i_1} Y^j_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_1}} + \delta^{k_1}_{i_2} Y^j_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_2}} + \delta^{k_1}_{i_3} Y^j_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_3}} - \delta^{k_1}_{i_2} X^{k_1}_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_2}} \right] y^{l_1}_{k_1} + \sum_{l_1,l_2=1}^{m} \sum_{k_2=1}^{n} \left[ \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_2}} + \delta^{k_1}_{i_2} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_2}} \right] y^{l_1}_{k_1} y^{l_2}_{k_2} + \sum_{l_1,l_2,i_3=1}^{m} \sum_{k_1,k_2,k_3=1}^{n} \left[ \delta^{k_1}_{i_1,i_2} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_2}} - \delta^{k_1}_{i_1,i_2} \delta^{k_2}_{i_2} X^{k_2}_{x_{i_2}x_{i_2}x_{i_2}y^{l_1}_{i_2}} \right] y^{l_1}_{k_1} y^{l_2}_{k_2} y^{l_3}_{k_3} \)
\[ + \sum_{l_1, l_2, l_3, l_4 = 1}^{m} \sum_{k_1, k_2, k_3, k_4 = 1}^{m} \left[ -\delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} \right] y_{k_1}^{l_1} y_{k_2}^{l_2} y_{k_3}^{l_3} y_{k_4}^{l_4} + \\
+ \sum_{l_1 = 1}^{m} \sum_{k_1, k_2 = 1}^{m} \left[ \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \sum_{l}^{2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} \right] y_{l_1}^{l_2} y_{l_2}^{l_2} y_{k_1}^{l_1} y_{k_2}^{l_2} - \\
- \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \\
- \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \\
- \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \\
- \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \\
- \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2} - \\
- \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} \delta_{l_1}^{l_2} X^{k_1 k_2} y_{l_1}^{l_2} y_{l_2}^{l_2} y_{l_1}^{l_2} y_{l_2}^{l_2}
\]

\[ (5.4) \]

5.5. **Final synthesis.** To obtain the general formula for \( Y_{i_1 \ldots i_n}^{j_1 \ldots j_m} \), we have to achieve the synthesis between the two formulas (3.74) and (4.19). We start with (3.74) and we modify it until we reach the final formula for \( Y_{i_1 \ldots i_n}^{j_1 \ldots j_m} \).

We have to add the \( \mu_1 + \ldots + \mu_d \) sums \( \sum_{l_1 = 1}^{m} \sum_{l_2 = 1}^{m} \sum_{l_3 = 1}^{m} \sum_{l_4 = 1}^{m} \sum_{l_5 = 1}^{m} \sum_{l_6 = 1}^{m} \sum_{l_7 = 1}^{m} \sum_{l_8 = 1}^{m} \sum_{l_9 = 1}^{m} \sum_{l_{10} = 1}^{m} \),
together with various indices \( l_i \). About these indices, the only point which is not obvious may be analyzed as follows.

A permutation \( \sigma \in \mathfrak{S}^{(\mu_1, \lambda_1) \ldots (\mu_d, \lambda_d)} \) yields the list:

\[ \sigma(1:1:1), \ldots, \sigma(1:1:1), \ldots, \sigma(1:1:1), \ldots \]

\[ \ldots, \sigma(d:1:1), \ldots, \sigma(1:1:1), \ldots, \sigma(d:1:1), \ldots, \sigma(d:1:1), \ldots \]

(6.6)

In the end of the sixth line of (3.74), the last term \( \sigma(d: \mu_d: \lambda_d) \) of the above list appears as the subscript of the upper index \( k_{\sigma(d: \mu_d: \lambda_d)} \) of the term \( X^{k_{\sigma(d: \mu_d: \lambda_d)}} \). According to the formal rules of Theorem 4.19, we have to multiply the partial derivative of \( X^{k_{\sigma(d: \mu_d: \lambda_d)}} \) by a certain Kronecker symbol \( \delta_{l_i}^{l_i} \). The question is: what is the subscript \( \alpha \) and how to denote it?

As explained before the statement of Theorem 4.19, the subscript \( \alpha \) is obtained as follows. The term \( \sigma(d: \mu_d: \lambda_d) \) is of the form \( (e: \nu_e: \gamma_e) \), for some \( e \) with \( 1 \leq e \leq d \), for some \( \nu_e \) with \( 1 \leq \nu_e \leq \mu_e \) and for some \( \gamma_e \) with \( 1 \leq \gamma_e \leq \lambda_e \). The single pure jet variable

\[ \left\{ \begin{array}{l} y_{k_{\sigma(d: \mu_d: \lambda_d)}}^{l_i} \\
\end{array} \right. \]

(7.7)
appears inside the total monomial

\[(5.8) \prod_{1 \leq i_1 \leq \mu_1} y_{k_1 \cdot \cdot \cdot k_{1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·,
5.15. Deduction of the most general multivariate Faà di Bruno formula. Let $n \in \mathbb{N}$ with $n \geq 1$, let $x = (x^1, \ldots, x^n) \in \mathbb{K}^n$, let $m \in \mathbb{N}$ with $m \geq 1$, let $g^j = g^j(x^1, \ldots, x^n)$, $j = 1, \ldots, m$, be $C^\infty$-smooth functions from $\mathbb{K}^n$ to $\mathbb{K}^m$, let $y = (y^1, \ldots, y^m) \in \mathbb{K}^m$ and let $f = f(y^1, \ldots, y^m)$ be a $C^\infty$ function from $\mathbb{K}^m$ to $\mathbb{K}$. The goal is to obtain an explicit formula for the partial derivatives of the composition $h := f \circ g$, namely

\[
(5.16) \quad h(x^1, \ldots, x^n) := f \left( g^1(x^1, \ldots, x^n), \ldots, g^m(x^1, \ldots, x^n) \right).
\]

For $j = 1, \ldots, m$, for $\lambda \in \mathbb{N}$ with $\lambda \geq 1$ and for arbitrary indices $i_1, \ldots, i_\lambda = 1, \ldots, n$, we shall abbreviate the partial derivative $\frac{\partial^{\lambda} g^j}{\partial x^{\lambda}}$ by $g^j_{i_1, \ldots, i_\lambda}$ and similarly for $h_{i_1, \ldots, i_\lambda}$. For arbitrary indices $l_1, \ldots, l_\lambda = 1, \ldots, m$, the partial derivative $\frac{\partial^{\lambda} f}{\partial y^{\lambda}}$ will be abbreviated by $f_{l_1, \ldots, l_\lambda}$.

Applying the chain rule, we may compute:

\[
(5.17) \quad h_{i_1} = \sum_{l_1 = 1}^m f_{i_1} \left[ g^1_{i_1, l_1} \right],
\]

\[
h_{i_1, i_2} = \sum_{l_1, l_2 = 1}^m f_{i_1, i_2} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} \right] + \sum_{l_1 = 1}^m f_{i_1} \left[ g^1_{i_1, i_2} \right],
\]

\[
h_{i_1, i_2, i_3} = \sum_{l_1, l_2, l_3 = 1}^m f_{i_1, i_2, i_3} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} g^3_{i_3, l_3} \right] + \sum_{l_1, l_2 = 1}^m f_{i_1, i_2} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} + g^1_{i_1, l_1} g^2_{i_2, l_2} + g^1_{i_1, l_1} g^2_{i_2, l_2} \right] +
\]

\[
+ \sum_{l_1 = 1}^m f_{i_1} \left[ g^1_{i_1, i_2, i_3} \right],
\]

\[
h_{i_1, i_2, i_3, i_4} = \sum_{l_1, l_2, l_3, l_4 = 1}^m f_{i_1, i_2, i_3, i_4} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} g^3_{i_3, l_3} g^4_{i_4, l_4} \right] +
\]

\[
\sum_{l_1, l_2, l_3, l_4 = 1}^m f_{i_1, i_2, i_3} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} g^3_{i_3, l_3} \right] + \sum_{l_1, l_2, l_3, l_4 = 1}^m f_{i_1, i_2, i_3} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} g^3_{i_3, l_3} \right] +
\]

\[
+ \sum_{l_1, l_2, l_3, l_4 = 1}^m f_{i_1, i_2} \left[ g^1_{i_1, l_1} g^2_{i_2, l_2} + g^1_{i_1, l_1} g^2_{i_2, l_2} + g^1_{i_1, l_1} g^2_{i_2, l_2} \right] +
\]

\[
+ \sum_{l_1 = 1}^m f_{i_1} \left[ g^1_{i_1, i_2, i_3, i_4} \right].
\]
Introducing the derivations

\[ F_i^2 := \sum_{k_1=1}^{n} \sum_{i_1=1}^{m} g_{k_1}^{i_1,i} \frac{\partial}{\partial g_{k_1}^{i_1}} + \sum_{i_1=1}^{m} g_{i_1}^{i_1} \left( \sum_{l_2=1}^{m} f_{l_2,i_2} \frac{\partial}{\partial f_{l_2,i_2}} \right), \]

\[ F_i^3 := \sum_{k_1=1}^{n} \sum_{i_1=1}^{m} g_{k_1}^{i_1,i} \frac{\partial}{\partial g_{k_1}^{i_1}} + \sum_{k_1,k_2=1}^{n} \sum_{i_1=1}^{m} g_{k_1,k_2}^{i_1,i} \frac{\partial}{\partial g_{k_1,k_2}^{i_1}} + \quad \]

\[ + \sum_{i_1=1}^{m} g_{i_1}^{i_1} \left( \sum_{l_2=1}^{m} f_{l_2,i_2} \frac{\partial}{\partial f_{l_2,i_2}} + \sum_{l_2,l_3=1}^{m} f_{l_2,i_2,l_3} \frac{\partial}{\partial f_{l_2,i_2,l_3}} \right), \]

(5.18)

\[ \text{we observe that the following induction relations hold:} \]

\[ h_{i_1,i_2} = F_{i_2}^2 (h_{i_1}), \]
\[ h_{i_1,i_2,i_3} = F_{i_3}^3 (h_{i_1,i_2}), \]
\[ \quad \]
\[ \text{.........} \]
\[ h_{i_1,i_2,...,i_{\lambda}} = F_{i_{\lambda}}^\lambda (h_{i_1,i_2,...,i_{\lambda-1}}). \]

(5.19)

To obtain the explicit version of the Faà di Bruno in the case of several variables \((x^1, \ldots, x^n)\) and several variables \((y^1, \ldots, y^m)\), it suffices to extract from the expression of \(Y_j^{i_1,...,i_{\lambda}}\) provided by Theorem 5.12 only the terms corresponding to \(\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d = \kappa\), dropping all the \(X\) terms. After some simplifications and after a translation by means of an elementary dictionary, we obtain the fourth and the most general multivariate Faà di Bruno formula.

**Theorem 5.20.** For every integer \(\kappa \geq 1\) and for every choice of indices \(i_1, \ldots, i_\kappa\) in the set \(\{1, 2, \ldots, n\}\), the \(\kappa\)-th partial derivative of the composite function

\[ h = h(x^1, \ldots, x^n) = f (g^1(x^1, \ldots, x^n), \ldots, g^m(x^1, \ldots, x^n)) \]

(5.21)

with respect to the variables \(x^{i_1}, \ldots, x^{i_\kappa}\) may be expressed as an explicit polynomial depending on the partial derivatives of \(f\), on the partial derivatives of the \(g^j\) and having integer coefficients:
\[ \frac{\partial^\kappa h}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} = \sum_{d=1}^{\kappa} \sum_{1 \leq \lambda_1 < \cdots < \lambda_d \leq \kappa} \sum_{\mu_1 \lambda_1 + \cdots + \mu_d \lambda_d = \kappa} \sum_{l_1, \ldots, l_{\mu_1} = 1}^{\mu_1} \cdots \sum_{l_d, \ldots, l_{\mu_d} = 1}^{\mu_d} \frac{\partial^{\mu_1 + \cdots + \mu_d} f}{\partial y_1^{\lambda_1} \cdots \partial y_d^{\lambda_d} \cdots \partial y_d^{\lambda_d}} \]

\[ \left[ \sum_{\sigma \in \sigma_\kappa(\lambda_1, \ldots, \lambda_d)} \prod_{1 \leq \nu_1 \leq \mu_1} \frac{\partial^{\lambda_1} g^{1: \nu_1}}{\partial x^{\sigma(1: \nu_1)}} \cdots \prod_{1 \leq \nu_d \leq \mu_d} \frac{\partial^{\lambda_d} g^{d: \nu_d}}{\partial x^{\sigma(d: \nu_d)}} \right]. \]

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