Alternative Size and Lifetime Measurements for High-Energy Heavy-Ion Collisions

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Two-Particle correlations based on the interference of identical particles has provided the chief means for determining the shape and lifetime of sources in relativistic heavy ion collisions. Here, Strong and Coulomb induced correlations are shown to provide equivalent information.

In the collision of highly relativistic heavy ions, a highly excited region of matter is created which occupies thousands of fm$^2$ and explodes and dissolves on a time scale of 10 fm/c. Only the momenta of the collision debris are experimentally accessible. However, space-time information is crucial for understanding the reaction. Most importantly, a latent heat associated with a phase transition would be accompanied by a significant enhancement to the duration of the emitting phase[2, 3, 4, 5]. The correlation function can be analyzed to “correct” their data as to best eliminate the effects of Coulomb from the correlation function so that comparison with simple forms for $g(r)$ is easily accommodated. The aim of this paper is to demonstrate that Coulomb interactions are included in correlation analyses, but that they provide tremendous insight into both the size and shape of $g(r)$.

By performing an inverse Fourier transform of $C(q)$ in Eq. 1, one can obtain $g(r)$. Even though the pions are charged, experimental analyses have tried to ignore the Coulomb interaction as much as possible, and in fact try to “correct” their data as to best eliminate the effects of Coulomb from the correlation function so that comparison with simple forms for $g(r)$ is easily accommodated. The aim of this paper is to demonstrate that Coulomb and strong interactions between the pair should not only be included in correlation analyses, but that they provide tremendous insight into both the size and shape of $g(r)$.

First, we consider two particles which interact only via the Coulomb interaction. Correlations for $pK^+$ from a
Monte Carlo methods. The Coulomb plane waves are the solution to Schrödinger’s equation for a wave where the outgoing wave has momentum $p_{\text{out}}$. The integration described in Eq. (1) was performed with source sizes are chosen $R_{\text{out}} = 8 \text{ fm}$ and $R_{\perp} = 4 \text{ fm}$. The integration described in Eq. (1) was performed with Monte Carlo methods. The Coulomb plane waves are solution to Schrödinger’s equation for a wave where the outgoing wave has momentum $q$.

\begin{equation}
-\nabla^2 \psi(q, r) + \frac{2\eta}{r} \psi(q, r) = q^2 \psi(q, r),
\end{equation}

where $\eta$ is the Sommerfeld parameter,

\begin{equation}
\eta = \frac{Z_a Z_b \alpha e^2}{q}, \quad \mu = \frac{E'_a E'_b}{(E'_a + E'_b)}.
\end{equation}

The usual definition of the reduced mass $\mu$, involving $m_a$ and $m_b$, has been altered to achieve consistency with relativistic treatments for small $\epsilon^2$.

In order to understand the form of the squared wave function, it is insightful to compare to the classical approximation where the squared wave function in Eq. (1) is replaced by the ratio of the initial and final phase space,

\begin{equation}
|\psi(q, r)|^2 \to \frac{d^3q_0}{d^3q} \approx \frac{1 + \cos \theta_{qr} - \eta/(qr)}{\sqrt{(1 + \cos \theta_{qr})^2 - 2(1 + \cos \theta_{qr})\eta/(qr)}} \cdot \Theta(1 + \cos \theta_{qr} - 2\eta/(qr)),
\end{equation}

where $\theta_{qr}$ is the angle between $q$ and $r$. Integrating over $\cos \theta_{qr}$, one finds the angle-averaged correlation weight,

\begin{equation}
\frac{q^2 dq_0}{q^2 dq} \approx \frac{1 - \frac{2\eta}{qr}}{1 - \frac{\eta}{qr}}.
\end{equation}

\begin{align*}
\text{FIG. 1: } pK^+ \text{ correlations are shown for a Gaussian source } (R_{\text{long}} = R_{\text{side}} = 4 \text{ fm}, R_{\text{out}} = 8 \text{ fm}). \text{ The classical approximation well explains Coulomb correlations at large relative momentum. The strong interaction only moderately affects the correlation function.}
\end{align*}

\begin{align*}
\text{FIG. 2: } pK^+ \text{ correlations are shown as a function of the angle of the relative momentum relative to the outwards direction for the Gaussian source } (R_{\text{long}} = R_{\text{side}} = 4 \text{ fm}, R_{\text{out}} = 8 \text{ fm}). \text{ The classical approximation becomes reasonable for large $Q_{\text{inv}}$ where the ratio of the suppression at $\cos \theta = 0$ to the suppression at $\cos \theta = 0$ approaches $(R_{\text{out}}/R_{\text{side}})^2$.}
\end{align*}

In the non-relativistic limit this approaches unity as $1/q^2$. Convoluting this expression with the Gaussian source function gives a classical result for the correlation function which is remarkably close to the quantum correlation function for large $q$ as can be seen in Fig. 1. Thus, the tail of the correlation function provides a measure of the expectation, $\langle 1/r \rangle$.

Extracting the shape of the source requires measuring the correlation function as a function of the direction of $q$. Since the correlation approaches unity as $1/q^2$, we recommended plotting $q^2[C(q) - 1]$ rather than $C(q)$ so that the main $q$ dependence can be ignored, allowing the use of larger bins in $q$.

As a function of the direction of $q$, correlations are shown in Fig. 2 for $Q_{\text{inv}} = 30$ and 150 MeV/c alongside the analogous results calculated with the classical form described in Eq. (3). The negative correlation at $\cos \theta = \pm 1$ derives from the fact that the relative momentum of the two positive particles are deflected away from the direction defined by their relative position by the repulsive Coulomb force. Careful analysis of the classical expression, Eq. (3), shows that for small $\eta$, the negative correlation is confined to $q$ and $r$ being anti-parallel. Since the chance of the particles being separated along the outwards axis is larger by a factor of $R_{\text{out}}^2/R_{\perp}^2$, and since the strength of the correlation in the two directions scales as $R_{\perp}/R_{\text{out}}$, the strength of the Coulomb induced correlation will be stronger along the $R_{\text{out}}$ direction by a
factor of $R_{\text{out}}^2/R_{\text{core}}^2$ at large $q$.

It should not be particularly difficult to obtain the required statistics even though the correlation is of the order of 1%. Although the correlation falls as $1/Q^2_{\text{inv}}$, phase space rises as $Q^2_{\text{inv}}$. Unlike identical-particle correlations at small relative momentum, there are no issues with two-track resolution. However, one has to consider large-scale correlations, e.g., collective flow, jets and charge conservation. Correlation from charge conservation can be neglected by considering only same-sign pairs. Collective flow can be eliminated by carefully constructing the correlation denominator with pairs from events with the same reaction plane. Such competing correlations ultimately limit the range in $Q_{\text{inv}}$ that is useful for the analysis. For instance, if the uncertainty of the competing correlations for $pK^+$ is of the order of 1%, the analysis should be restricted to $Q_{\text{inv}} < 150$ MeV/c so that Coulomb remains the dominant factor.

The angular sensitivity of Coulomb-induced correlations have been investigated in intermediate-energy heavy-ion collisions. These studies involved light fragments, e.g. Carbon-Carbon, which can be treated with classical trajectories. The Coulomb mean field from the remainder of the source was found to distort the shape information. However, the residual Coulomb interaction is expected to play a much smaller role at RHIC where the hadrons move much faster and spend less time interacting with the mean field. Furthermore, analyses can be performed with both positive and negative pairs, e.g., both $pK^+$ and $\bar{p}K^-$. Averaging the two correlation functions should largely cancel the effects of the residual interaction. At high energy, the angular dependence of Coulomb correlations of non-identical particles have been used to determine the degree to which one species is displaced relative to another. These studies involved comparing correlations for $\theta < \pi/2$ with correlations with $\theta > \pi/2$.

Incorporating the strong interaction into the wave function in Eq. (10) is straightforward if the strong interaction can be expressed as a non-relativistic potential. One must simply solve the Schrödinger equation. However, many interactions cannot be expressed in terms of such potentials. For instance, the interaction of a $\pi^+$ and proton through the delta resonance involves a quantum rearrangement of the participating quarks. Thus, a wave function is not a meaningful quantity for separations below $\epsilon \sim 1$ fm, but it is certainly a well defined object for $r > \epsilon$, and can be expressed in terms of a Coulomb wave with the incoming partial waves are modified by phase shifts.

$$\psi(q,r) = \psi_0(q,r) + \sum_{\ell} \sqrt{4\pi(2\ell+1)} \frac{i^{\ell+1}}{2q^2} e^{-i\sigma_{\ell}} \cdot (F_\ell(q,r) - iG_\ell(q,r)) \left( e^{-2i\delta_\ell} - 1 \right) Y_{\ell,m=0},$$

where $F_\ell$ and $G_\ell$ are the regular and irregular partial Coulomb waves and the second term describes the distortion of the incoming partial wave, $F_\ell - iG_\ell$.

For the $p\pi^+$ and $pK^+$ examples discussed in this study, the plane wave also has a spin index, $m_s = \pm 1/2$, which is not conserved for partial waves with $\ell > 0$. The states of a given $m_s$ must be decomposed in terms of eigenstates of total angular momentum which are phase shifted by eigen- phases, $\delta_{\ell,J}$. After applying some angular momentum algebra, Eq. (10) can be modified to include flipping the spin. For $\ell = 1$,

$$\frac{e^{-2i\delta_\ell} - 1}{2} Y_{\ell,m=0} \rightarrow \left( \frac{2}{3} e^{-2i\delta_\ell} + \frac{1}{3} e^{-2i\delta_\ell} - 1 \right) Y_{\ell,m=0} \uparrow$$

$$+ \sqrt{2} \left( e^{-2i\delta_\ell} - e^{-2i\delta_\ell} \right) Y_{\ell,m=1} \downarrow$$

Thus, the wave function for $r > \epsilon$ can be evaluated if given the phase shifts.

For $r < \epsilon$, one must use an effective form for $|\psi(q,r)|^2$. Since $\epsilon$ is much smaller than any characteristic dimension of the source, only the integral of $\psi^2$ matters in the region less than $\epsilon$, and one can safely choose,

$$|\psi(q,r < \epsilon)|^2 = |\psi_0(q,r)|^2 + W(\epsilon,q),$$

where $\psi_0$ is the Coulomb wave and $W$ is independent of $r$. The change in the density of states can be expressed both in terms of phase shifts and wave functions.

$$\Delta \frac{dN}{dq} = \frac{4\pi q^2}{(2\pi)^3} \int d^3r \left( |\phi(q,r)|^2 - |\phi_0(q,r)|^2 \right)$$

$$= \sum_{\ell} \frac{(2\ell+1)}{\pi} \int dq \left( |\phi(\eta,qr)|^2 - |F_\ell^2(\eta,qr)|^2 \right),$$

$$\phi_\ell = F_\ell + \frac{1}{2} (e^{-2i\delta_\ell} - 1) (F_\ell - iG_\ell).$$

Thus, $W$ can be expressed in terms of derivatives of the phase shifts and integrals of the type,

$$I_{\ell}(\epsilon,q,\delta_\ell) \equiv \int_\epsilon^\infty dr |\phi_\ell(\eta,qr,\delta_\ell)|^2,$$

where $\phi_\ell$ is a solution to the Schrödinger equation, one can rewrite $I_{\ell}$, assuming that $\phi$ and $\phi^*$ are solutions with eigenvalues $q$ and $q' \sim q$,

$$(q'^2 - q^2) I_{\ell}(\epsilon,q,\delta_\ell) = \Re \int_\epsilon^\infty dr \left( \partial^2_\ell \phi^*_\ell - \partial^2_\ell \phi_{\ell} \right)$$

$$I_{\ell}(\epsilon,q,\delta_\ell) = \frac{1}{2q} \Re \left( \partial_\ell \phi^*_\ell \partial_\ell \phi_{\ell} - \phi^*_\ell \partial_\ell \partial_\ell \phi_{\ell} \right) \epsilon \delta_\ell.$$


Transforming the derivatives to the variables $\eta$ and $x = qr$ facilitates use of the recursion relations for the Coulomb wave functions \[ I_{\ell=0}(\epsilon, q, \delta_t) = \binom{\phi_0^* \phi_0 + \phi_1^* \phi_1}{2q} x (1 + \eta^2) \\
- \Re(\phi_1^* \phi_0) \left(1 + \eta^2\right) 2q \sqrt{1 + \eta^2} x \bigg|_{x=q\epsilon} \\
+ \Re(\phi_1^* \partial_\eta \phi_0 - \phi_0^* \partial_\eta \phi_1) \eta \sqrt{1 + \eta^2} 2q x \bigg|_{x=q\epsilon}. \]
\[ (16) \]
The upper limit, $x = \infty$, need not be evaluated since it would be canceled by an equal and opposite contribution from the non-phaseshifted term in Eq. (16). For the case with no Coulomb interactions, one can derive a simpler form involving spherical Bessel functions rather than Coulomb wave functions. For the no-Coulomb example, one can show that $W = 0$ as $q \to 0$.

Using experimentally tabulated phase shifts, and the arbitrary choice of $\epsilon = 1$ fm, wave functions were numerically generated and convoluted with the source function through Eq. (11) to generate correlation functions. Correlations for $p\pi^+$ were only slightly affected by the strong interaction as can be seen in Figs. 8 and 9. For this example, there are no resonant channels and the correlation is dominated by Coulomb. Figure 3 shows results for $p\pi^+$ correlations which are dominated by the $\Delta^{++}$ for $Q_{inv}$ near the resonant momentum, $Q_{inv} = 450$ MeV.

A strong angular correlation is caused by shadowing in the forward direction. Like Coulomb-induced correlation, there is a dip when the relative momentum is aligned with the long-axis of the source.

Strong and Coulomb induced correlations had been previously studied for their ability to to unfold the angle-averaged source function, $q(r)$, for both high-energy and low-energy collisions [13, 14, 15]. Our findings show that such correlations also have tremendous potential to discern shape characteristics and thus provide an estimate of source lifetimes. Determining shape and lifetime characteristics had been previously confined to analyses of identical-particle correlations. As strong and Coulomb correlations are of a manifestly different character than correlations from identical-particle interference, the analyses described here represent a truly independent strategy for determining space-time characteristics of hadronic sources.

FIG. 3: $p\pi^+$ angular correlations for a Gaussian source ($R_{out} = R_{side} = 4$ fm, $R_{long} = 8$ fm) with $Q_{inv} = 450$ MeV/c. At this $Q_{inv}$ the correlation is dominated by the $\Delta^{++}$ resonance. The suppression for $\cos \theta = \pm 1$ can be explained by shadowing.

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