Aspherical Manifolds with Relatively Hyperbolic Fundamental Groups

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Abstract. We show that the aspherical manifolds produced via the relative strict hyperbolization of polyhedra enjoy many group-theoretic and topological properties of open finite volume negatively pinched manifolds, including relative hyperbolicity, nonvanishing of simplicial volume, co-Hopf property, finiteness of outer automorphism group, absence of splitting over elementary subgroups, and acylindricity. In fact, some of these properties hold for any compact aspherical manifold with incompressible aspherical boundary components, provided the fundamental group is hyperbolic relative to fundamental groups of boundary components. We also show that no manifold obtained via the relative strict hyperbolization can be embedded into a compact Kähler manifold of the same dimension, except when the dimension is two.

1. Introduction

The subject of negatively curved manifolds sits at the crossroads of Riemannian geometry, geometric group theory, coarse topology, and dynamics, and there is an extensive literature on (complete Riemannian) negatively curved manifolds of finite volume. In dimensions $>3$ the main source of examples is arithmetic lattices; other known examples are typically obtained from arithmetic ones by various “surgery” constructions along totally geodesic submanifolds [GPS88, MS80, GT87, Der05, FJ89, FJO98]. These constructions are very special, and one could instead try to work in a broader context of aspherical manifolds whose fundamental groups are hyperbolic or relatively hyperbolic hoping, in particular, that this could eventually lead to other sources of examples.

The strict hyperbolization of Charney-Davis [CD95] gives a rich supply of closed aspherical manifolds with hyperbolic fundamental groups. This is a functorial procedure that turns any finite simplicial complex $K$ into a locally-$CAT(-1)$ piecewise hyperbolic finite cell complex $\mathcal{H}(K)$. A key property of the procedure

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is that it preserves links of all the faces, and hence if $K$ is a triangulated manifold, then so is $\mathcal{H}(K)$ (and this also works in smooth category). The closed locally-$C\text{AT}(-1)$ piecewise hyperbolic manifolds obtained via the strict hyperbolization behave in many ways like closed Riemannian manifolds of negative sectional curvature; in fact, if $K$ is a smoothable triangulation of a closed manifold, then for all we know, $\mathcal{H}(K)$ could admit a negatively curved Riemannian metric, and this is what happens e.g. in dimensions $\leq 3$ (by Thurston’s hyperbolization theorem).

The strict hyperbolization has a relative version [DJW01, CD95] (cf. [DJ91, Gro87]), and the main theme of this paper is establishing an analogy between the aspherical manifolds produced via relative strict hyperbolization, and the open complete finite volume negatively pinched manifolds. Roughly, the relative strict hyperbolization, which is explained in detail in Section 4 is a procedure that takes as the input a pair of finite simplicial complexes $(K, L)$, where $L$ is a nonempty subcomplex of $K$, and produces a pair of simplicial complexes $(R_K, R_L)$ with the following properties:

- $R_L$ is isomorphic to a subdivision of $L$,
- $R_K$ is aspherical if and only if each path-component of $L$ is aspherical;
- $R_L$ is incompressible in $R_K$ (i.e. no homotopically nontrivial loop in $R_L$ is null-homotopic in $R_K$);
- if $K$ is a (triangulated, smooth or PL) manifold with boundary $L$, then $R_K$ is a (respectively, triangulated, smooth or PL) manifold with boundary $R_L$;
- the space $Z$ obtained from $R_K$ by attaching a cone to each path-component of $R_L$ admits a piecewise-hyperbolic locally-$C\text{AT}(-1)$ metric.

An appealing feature of relative strict hyperbolization is that it allows to construct compact aspherical manifolds with prescribed boundary: namely, if $L$ bounds and each path-component of $L$ is aspherical, then $L$ bounds an aspherical manifold. By contrast, the problem of prescribing topology of cusps of negatively pinched finite volume complete Riemannian manifolds is still wide open; for recent progress in locally-symmetric case see [LR00, LR02, McR04].

We emphasize that $R_K$ itself carries no obvious locally-$C\text{AT}(-1)$ metric, in fact, geometrically, $R_K$ looks more like a finite volume negatively pinched manifold with cusps chopped off, with $R_L$ corresponding to the union of cusp cross-sections on the boundary. However, negative curvature persists on the group-theoretic level as follows.

**Theorem 1.1.** If $(R_K, R_L)$ is a pair of finite (nonempty) simplicial complexes obtained via relative strict hyperbolization, and $L_1, \ldots, L_m$ are all the path-components of $R_L$ that have infinite fundamental groups, then $\pi_1(R_K)$ is a
non-elementary relatively hyperbolic group in the sense of Bowditch, specifically, \( \pi_1(R_K) \) is hyperbolic relative to the subgroups \( \pi_1(L_1), \ldots, \pi_1(L_m) \). In particular, if each component of \( R_L \) has finite fundamental group, then \( \pi_1(R_K) \) is a non-elementary hyperbolic group.

Relatively hyperbolic groups (see Section 5 for a definition) were introduced by Gromov [Gro87] and since then various characterizations of relatively hyperbolic groups have been obtained by Farb [Far98], Bowditch [Bow], Yaman [Yam04], Osin [Osi06], Drutçu-Osin-Sapir [DS05]. Recently there has been a surge of activity in this subject, and many results about hyperbolic groups have been adapted to the relatively hyperbolic setting. Three basic classes of examples of relatively hyperbolic groups are free products with finitely many factors (which are hyperbolic relative to the factors), hyperbolic groups (which are hyperbolic relative to the empty set of subgroups), and geometrically finite isometry groups of negatively pinched Hadamard manifolds (which are hyperbolic relative to the maximal parabolic subgroups). Finally, any group is hyperbolic relative to itself. A subgroup of a relatively hyperbolic group is called *elementary* if it is finite, or virtually-\( \mathbb{Z} \), or hyperbolic relative to itself; otherwise, the subgroup is called *non-elementary*. Another major source of examples of relatively hyperbolic groups is the small cancellation theory [Osi]. There are however very few (primary) constructions of higher-dimensional relatively hyperbolic groups, e.g. the fundamental groups of compact aspherical manifolds that are hyperbolic relative to the fundamental groups of the boundary components, and this is exactly what Theorem 1.1 does.

Theorem 1.1 has a complicated history. It was predicted by Gromov, and first stated (without proof) in [Gro87, p. 257] for a different hyperbolization procedure. However, results of Charney-Davis [CD95] imply that the claim in [Gro87, p. 257] was incorrect because the Gromov’s hyperbolization procedure does not yield negative curvature in dimensions \( \geq 4 \). This motivated Charney-Davis to introduce the strict hyperbolization whose relative version we study in this paper. Theorem 1.1 answers a question of Szczepański [Sze02] who proved a weaker result that \( \pi_1(R_K) \) of Theorem 1.1 is relatively hyperbolic in the sense of Farb, relative to the same collection of subgroups. Goldfarb in [Gol99, Example 5.5] claimed a result that is stronger than Theorem 1.1 (in the special case when \( R_K \) is an aspherical manifold with boundary \( R_L \)), yet he recently acknowledged that his proof is incorrect; I do not know how to prove what Goldfarb claimed.

We next turn to a general study of aspherical manifolds with relatively hyperbolic fundamental groups; more precisely we make the following assumption.

**Assumption 1.2.** For \( n > 2 \), \( M \) denotes a compact aspherical \( n \)-manifold with non-empty boundary \( \partial M \) such that each component of \( \partial M \) is aspherical
and incompressible in $M$, and $\pi_1(M)$ is non-elementary relatively hyperbolic, relative to the fundamental groups of the components of $\partial M$.

Here are examples of manifolds that do or do not satisfy Assumption 1.2.

- If $R_K$ is a compact $n$-manifold with boundary $R_L$, and if each component of $R_L$ is aspherical, then by Theorem 1.1, $R_K$ satisfies Assumption 1.2.
- Any finite volume complete negatively pinched (e.g. of constant negative curvature) Riemannian manifold with totally geodesic boundary satisfies Assumption 1.2; as we explain in Section 13 this essentially follows from [Bow, Section 7].
- If $M$ is the quotient of a nonpositively curved symmetric space of rank $\geq 2$ by a non-cocompact torsion-free lattice, then $M$ does not satisfy Assumption 1.2 [BDM].
- If $M$ is a 3-manifold that satisfies Assumption 1.2, then we show in Section 12 that $M$ is acylindrical. Note that there are non-acylindrical 3-manifold with incompressible boundary whose interior admits a complete hyperbolic metric (see e.g. [CM04, Example 1.4.5]).

If $G$ is the fundamental group of a closed aspherical manifold, and if $G$ is hyperbolic, then $G$ admits no nontrivial splitting over finite, or virtually-$\mathbb{Z}$ subgroups (as immediately follows from the Mayer-Vietoris sequence in group cohomology), so by Rips theory [BF95] $G$ admit no nontrivial isometric actions on $\mathbb{R}$-trees with finite, or virtually-$\mathbb{Z}$ arc stabilizers, and in particular, $G$ is co-Hopfian [RS94] and $\text{Out}(G)$ is finite [Pau91].

To state a relatively hyperbolic version of these results we need the following definition. We say that a group $G$ has property (m) if $G$ is hyperbolic relative to a family of subgroups none of which is isomorphic to a non-elementary relatively hyperbolic group. For example, if $G$ is not isomorphic to a non-elementary relatively hyperbolic group, then $G$ has property (m), where we consider $G$ hyperbolic relative to itself. It was proved in [BDM] that Dunwoody’s inaccessible group does not have property (m). It is unknown whether there exists a torsion-free or finitely presented group that does not have property (m), and it is conceivable that the fundamental group of a closed aspherical manifold always has property (m).

**Theorem 1.3.** If $M$ satisfies Assumption 1.2, then
(i) $\pi_1(M)$ admits no nontrivial splitting over an elementary subgroup.
(ii) $\pi_1(M)$ is co-Hopfian.
(iii) If the fundamental group of every component of $\partial M$ has property (m), then $\text{Out}(\pi_1(M))$ is finite.
Part (i) of Theorem 1.3 is proved as in [Bel02] where we obtained a similar result for finite volume negatively pinched manifolds, while parts (ii)-(iii) easily follow from (i) and the recent remarkable work of Drutu-Sapir [DS] that extends Rips theory to the relatively hyperbolic setting.

Here is an open question on automorphisms of $\pi_1(M)$.

**Question 1.4.** Let $M$ satisfy Assumption 1.2. Does every automorphism of $\pi_1(M)$ permutes the maximal parabolic subgroups?

Another approach to proving part (ii) of Theorem 1.3 is based on recent work of Mineyev-Yaman [MY], who showed among other things that if $M$ satisfies Assumption 1.2, then $|\partial M| > 0$ where $|\partial M|$ is the relative simplicial volume of $(M, \partial M)$. We use this result of Mineyev-Yaman to prove the following.

**Theorem 1.5.** If $M_1, M_2$ are $n$-manifolds that satisfy Assumption 1.2, and $\phi: \pi_1(M_1) \to \pi_1(M_2)$ is an injective homomorphism that maps each maximal parabolic subgroup to a parabolic subgroup, then $\phi(\pi_1(R_1))$ has finite index in $\pi_1(R_2)$. If in addition $M_1, M_2$ are homeomorphic, then $\phi$ is an isomorphism.

We call a group $G$ *intrinsically elementary* if the image of any injective homomorphism of $G$ into a relatively hyperbolic group is elementary (i.e. finite, virtually-$\mathbb{Z}$, or parabolic). Thus if in Theorem 1.5 all maximal parabolic subgroups of $\pi_1(M_1)$ are intrinsically elementary, and $n > 2$, then the assumption "$\phi$ maps each maximal parabolic subgroup to a parabolic subgroup" holds true. (To see that in this case "elementary" implies "parabolic", note that each boundary component of $M_1$ is a closed aspherical manifold of dimension $\geq 2$ so its fundamental group cannot be finite or virtually-$\mathbb{Z}$).

Below we list several classes of closed aspherical manifolds with intrinsically elementary fundamental groups, and these manifolds can be realized (via the relative strict hyperbolization) as boundary components of manifolds satisfying Assumption 1.2. Examples of intrinsically elementary groups include

- groups with no nonabelian free subgroups [Tuk94] (e.g. fundamental groups of infrasolvmanifolds),
- fundamental groups of closed 3-dimensional graph manifolds [BDM],
- groups with finite dimensional second bounded cohomology [Fuj98] (e.g. cocompact irreducible lattices in higher rank Lie groups [BM99a, BM99b]).
- Since any infinite elementary group has elementary normalizer, any group that contains an intrinsically elementary infinite normal subgroup is intrinsically elementary. For example, the fundamental group of any fiber bundle with aspherical base and fiber is intrinsically elementary, provided the fiber has intrinsically elementary fundamental groups, and
this construction can be iterated since the total space of any such bundle is aspherical.

We next focus on (not necessarily aspherical) compact $n$-manifolds obtained via relative strict hyperbolization. Thus $K$ is an $n$-manifold with boundary $L$, so that $R_K$ is an $n$-manifold with boundary $R_L$. To emphasize that we are dealing with manifolds, we denote $(R_K, R_L)$ by $(R, \partial R)$ and say that $R$ is an $n$-manifold obtained via relative strict hyperbolization. We give $\pi_1(R)$ the structure of a relatively hyperbolic group given by Theorem 1.1, so that the conjugacy classes of maximal parabolic subgroups bijectively correspond to the components of $\partial R$ that have infinite fundamental group.

According to [Gro82], there exist positive constants $C_1 = C_1(n)$, $C_2 = C_2(n, a)$ such that if $N$ is a compact $n$-manifold and $V = \text{Int}(N)$ admits a complete finite volume Riemannian metric of sectional curvature within $[-a^2, -1]$, then $C_1 \text{vol}(V) \geq \|N, \partial N\| \geq C_2 \text{vol}(V)$. We prove a similar statement for manifolds obtained via the relative strict hyperbolization.

**Theorem 1.6.** There are constants $C_1 \geq C_2 > 0$ depending only on $n$ such that if $R$ is a compact $n$-manifold obtained by the relative strict hyperbolization, then $C_1 \mathcal{H}^n(R) \geq \|R, \partial R\| \geq C_2 \mathcal{H}^n(R)$.

Here $\mathcal{H}^n(R)$ denotes the $n$-dimensional Hausdorff measure of $R$, and the metric on $R$ is induced by the inclusion $R \to Z$. As we explain in Section 7, $\mathcal{H}^n(R)$ is roughly the same as the number of $n$-simplices in some natural triangulation of $R$. Since there are only finitely many ways to glue finitely many simplices, we get the following finiteness theorem (which also holds in PL and smooth categories).

**Theorem 1.7.** For any $C > 0$, there are only finitely many homeomorphism types of triangulated $n$-manifolds $R$ obtained by the relative strict hyperbolization and satisfying $\|R, \partial R\| < C$.

One immediate implication of Theorem 1.6 is that $\|R, \partial R\| > 0$. (If $R$ is aspherical, this also follows from Mineyev-Yaman [MY], but here we give a direct proof). Note that any compact orientable $n$-manifold $M$ satisfies $\|M, \partial M\| \geq \|\partial M\|/n$ (see e.g. [Kue]), hence $\|R, \partial R\| > 0$ holds trivially when $\|\partial R\| > 0$.

It is easy to see that the fundamental groups of the manifolds obtained via relative or non-relative strict hyperbolization split as nontrivial amalgamated products. In particular, they do not have Kazhdan property (T), and hence these manifolds are not homotopy equivalent to a quaternionic or Cayley hyperbolic manifold. Furthermore, using by now standard harmonic map technology, we show the following.
**Theorem 1.8.** If $K$ is a finite simplicial complex, then $\mathcal{H}(K)$ is not homotopy equivalent to a compact Kähler manifold of real dimension $\geq 4$.

**Theorem 1.9.** If $R$ is a manifold obtained by relative strict hyperbolization, then $\text{Int}(R)$ is not homeomorphic to an open subset of a compact Kähler manifold of real dimension $\geq 4$.

By contrast, it is much harder to decide when $R$ admits a real hyperbolic metric (see Section 11 for some examples).

An important (and still poorly understood) invariant of a relatively hyperbolic group is the Bowditch boundary, which generalizes both the ideal boundary of a hyperbolic group, and the limit set of a geometrically finite isometry group of a negatively pinched Hadamard manifold. Any relatively hyperbolic group $G$ acts on its Bowditch boundary as a geometrically finite convergence group whose maximal geometrically parabolic subgroups are exactly the maximal parabolic subgroups of $G$. In fact, Yaman [Yam04] characterized relatively hyperbolic groups as geometrically finite convergence groups, generalizing a similar characterization of hyperbolic groups due to Bowditch. For the fundamental groups of complete finite volume negatively pinched manifolds the Bowditch boundary is a sphere, so we ask the following.

**Question 1.10.** Let $R$ be an aspherical manifold of dimension $\geq 4$ obtained by the relative strict hyperbolization. What is the Bowditch boundary of $\pi_1(R)$? Is it ever a sphere? Is it a sphere when the maximal parabolic subgroups are virtually nilpotent?

The structure of the paper is as follows. In Section 2-3 we review the strict hyperbolization, and then in Section 4 we discuss the relative strict hyperbolization. Theorem 1.1 is proved in Section 5. Section 6 contains background on relative simplicial volume. Section 7 discusses relations between the relative strict hyperbolization and the simplicial volume; Theorems 1.6–1.7 are proved in this section. Section 8 is devoted to applications to the co-Hopf property. Theorem 1.3 on non-existence of elementary splittings is proved in Section 9. Sections 10-11 discuss when manifolds obtained by relative or non-relative strict hyperbolization admit negatively curved Riemannian metrics, in particular Kähler ones. In Section 12 we prove general acylindricity results for CW-pairs with relatively hyperbolic fundamental groups. Section 13 contains a proof that compact hyperbolic manifolds with totally geodesic boundary satisfy Assumption 1.2.
The (non-relative) strict hyperbolization is defined in \[\text{CD95, Theorem 7.6}\] as the combination of two independent constructions due to Gromov and Charney-Davis. The Gromov’s construction is described in \[\text{DJ91, 4c}, \text{CD95, pp 348–349}\]. It takes as the input an arbitrary \( n \)-dimensional simplicial complex \( K \) and turns it into a locally \( \text{CAT}(0) \) cubical complex denoted by \( G(K) \). By \[\text{CD95, Lemma 7.5}\] there is a folding map \( p: G(K) \rightarrow \square^n \), where by a folding map we mean a cellular map whose restriction on every cell is a combinatorial isomorphism.

The strict hyperbolization of Charney-Davis \[\text{CD95, Proposition 7.1}\] associates to an \( n \)-dimensional cubical complex \( C \) a piecewise-hyperbolic \( n \)-dimensional cell complex \( S(C) \) such that the links of the corresponding cells in \( C \) and \( S(C) \) are isometric piecewise-spherical polyhedra. Gromov showed \[\text{BH99, Proposition II.4.14}\] that a piecewise-Euclidean (or piecewise-hyperbolic) cell complex is \( \text{CAT}(0) \) (or \( \text{CAT}(-1) \), respectively) if and only if the link of every vertex is \( \text{CAT}(1) \). It follows that if \( C \) is \( \text{CAT}(0) \), then \( S(C) \) is \( \text{CAT}(-1) \).

A key idea in constructing \( S(C) \) is that for each \( k \) there exists a compact orientable hyperbolic \( k \)-manifold with corners \( X^k \) such that the boundary of \( X^k \) is subdivided into totally geodesic \((k-1)\)-dimensional faces that intersect orthogonally so that the poset of faces of \( X^k \) is the isomorphic to the poset of faces of \( \square^k \). To obtain \( S(C) \) one glues together the \( X^k \)'s in the same combinatorial pattern as \( k \)-cubes of \( C \). Furthermore, by \[\text{CD95, Lemma 5.9}\] for each \( k \) there exists a smooth face-preserving degree one maps \( f_k: X^k \rightarrow \square^k \), and if \( C \) admits a folding map \( p: C \rightarrow \square^n \), one can describe \( S(C) \) as the subset of \( C \times X^n \) consisting of all \((c, x)\) that satisfy \( p(c) = f_n(x) \).

We let \( \mathcal{H}(K) := S(G(K)) \), and refer to \( \mathcal{H}(K) \) as the strict hyperbolization of \( K \).

The strict hyperbolization has the following properties which can be deduced from \[\text{DJ91, CD95, Dav02}\].

- (Negative curvature) \( \mathcal{H}(K) \) is locally \( \text{CAT}(-1) \) piecewise-hyperbolic cell complex. If \( \text{dim}(K) \leq 1 \), then \( \mathcal{H}(K) \) is the barycentric subdivision of \( K \).
- (Hyperbolized cell) If \( \sigma \) is a simplex of dimension \( n > 0 \), then \( \mathcal{H}(\sigma) \) is a connected, compact, orientable \( n \)-manifold with boundary.
- (Functoriality) If \( i: J \rightarrow K \) is an embedding onto a subcomplex, then there is an isometric embedding \( \mathcal{H}(i): \mathcal{H}(J) \rightarrow \mathcal{H}(K) \) onto a locally convex subspace. (Charney-Davis \[\text{CD95}\] called locally convex subspaces “totally geodesic”).
- (Preserving local structure) The link of \( \mathcal{H}(\sigma) \) in \( \mathcal{H}(K) \) is PL-homeomorphic to the link of \( \sigma \) in \( K \). Furthermore, if \( v \) is vertex of \( K \), then the link of the vertex \( h^{-1}(v) \) is isomorphic to a subdivision of the link of \( v \).
• (Small balls centered at vertices are cones) If $v$ is a vertex of $\mathcal{H}(K)$, then the link $L_v$ of $v$ is a $CAT(1)$ piecewise-spherical complex, and by a result of Berestovskii, a sufficiently small $\epsilon$-ball centered at $v$ is isometric to the $\epsilon$-ball centered at the cone point in the $-1$-cone over $L_v$ [BH99, page 59 and Theorem 3.14]. Alternatively, the ball can be described as the warped product $[0, \epsilon) \times_{\sinh(r)} L_v$ with base $[0, \epsilon)$, warping function $\sinh(r)$, and fiber $L_v$. (For information on warped products see e.g. [AB05] where the above warped product is called an elliptic cone over $L_v$).

• (Hyperbolization map) There is a PL-map $h: \mathcal{H}(K) \rightarrow K$ that restricts to a degree one map $(\mathcal{H}(\sigma), \partial \mathcal{H}(\sigma)) \rightarrow (\sigma, \partial \sigma)$ for each simplex $\sigma$. One can choose $h$ so that for each vertex $v$ of $K$ it induces a PL-isomorphism between the small conical neighborhoods of $h^{-1}(v)$ and $v$.

• (Surjectivity) $h$ induces surjections on homology and fundamental groups.

• (Preserving invariants of manifolds) If $K$ is a triangulated manifold, or PL manifold, or a smoothly triangulated smooth manifold, then so is $\mathcal{H}(K)$, and $h$ pulls back the rational Pontrjagin classes, and the first Stiefel-Whitney class. (This boils down to the fact that $X^n$ is orientable and has trivial rational Pontrjagin classes). In particular, if $K$ is oriented, then so is $\mathcal{H}(K)$.

3. ON THE BUILDING BLOCK IN THE STRICT HYPERBOLIZATION

The manifold $X^k$ is obtained from a carefully chosen closed orientable hyperbolic $k$-manifold by cutting it open along a family of $k$ connected totally geodesic $(k-1)$-dimensional submanifolds that intersect orthogonally. There is a considerably flexibility in choosing $X^k$ and its structure is not yet well understood. Of course, $X^2$ is just a compact orientable hyperbolic surface whose boundary is a piecewise geodesic “square”. In higher dimensions visualizing $X^k$ seems much harder, and it is of some interest to get a better grip on the topology of $X^k$. Here are some open questions:

• Can one arrange $X^k$ to be stably parallelizable? This question was asked in [CD95, p. 348], and if the answer is "yes", then the strict hyperbolization construction with this building block preserves stable tangent bundle.

• The poset of faces of $X^k$ is the poset of faces of the $k$-cube, yet for $i > 0$ the $i$-dimensional faces of $X^k$ are generally not connected. Can one build $X^k$ with connected faces? If yes, visualizing $X^k$ would be considerably easier.

• Can one arrange $X^k$ to be topologically “complicated”, e.g. to have all Betti numbers nonzero? This becomes relevant in Section 10.
• As we note below $X^k$ is never simply-connected. What is the “simplest possible” $X^k$?

Lemma 3.1. If $k \geq 2$, then $\pi_1(X^k)$ contains an nonabelian free subgroup.

Proof. By functoriality of the strict hyperbolization [CD95, p. 347], any $i$-dimensional face $\Box^i \subset \Box^k$ gives rise to an embedding $X^i \subset X^k$ onto a locally convex submanifold, where $X^i$ is contained in (but not necessarily equal to) the $i$-dimensional face of $X^k$ that corresponds to $\Box^i$. Since $X^i$ a locally convex subspace, the embedding $X^i \subset X^k$ is $\pi_1$-injective [BH99, Proposition II.4.14], and in particular, $\pi_1(X^k)$ contains a nonabelian free subgroup isomorphic to $\pi_1(X^2)$.

Lemma 3.2. There exists a cubical complex structure on the $n$-torus $T^n$ such that $S(T^n)$ is a closed hyperbolic manifold, i.e. a Riemannian manifold of constant negative curvature. In particular, there is a $\pi_1$-injective embedding of $X^n$ into closed hyperbolic $n$-manifold.

Proof. Let $T^n$ be the $n$-torus built by identifying the opposite sides of the cube $[-1,1]^n$ in $\mathbb{R}^n$. Every quadrant of $\mathbb{R}^n$ intersects $[-1,1]^n$ in a cube of sidelength 1, and these $2^n$ cubes give $T^n$ the structure of a cubical complex. (The definition of a cubical complex that we use follows [CD95, p.330], and is slightly more general than the definition in [BH99]: namely, we require that each cube in a cubical complex is embedded, yet we allow distinct cubes to intersect in several faces, rather than in just one face). Note that the corresponding piecewise-Euclidean metric on $T^n$ is flat, or equivalently the link at each face is a round sphere. Then by [CD95, p.347] the link at each face of $S(T^n)$ is also a round sphere, therefore, the piecewise-hyperbolic metric on $S(T^n)$ is in fact hyperbolic. The inclusion $X^n$ onto a building block of $S(T^n)$ is locally convex, and hence $\pi_1$-injective [BH99, Proposition II.4.14].

Lemma 3.3. If $K$ is connected and $\dim(K) = n$, then $\mathcal{H}(K)$ retracts onto $X^n$. If $n \geq 2$, then $\pi_1(\mathcal{H}(K))$ does not have Kazhdan property $(T)$.

Proof. We think of $\mathcal{H}(K)$ as the subset of $G(K) \times X^n$ consisting of all $(c,x)$ that satisfy $p(c) = f_n(x)$, where $p: G(K) \to \Box^n$ is a folding map, and $f_n: X^n \to \Box^n$ is the map of [CD95, Lemma 5.9]. Fix an $n$-cube $i: \Box^n \hookrightarrow G(K)$ in $G(K)$, which exists as $\dim(K) = n$, and let $\sigma = i(\Box^n)$. Then $\mathcal{H}(K) \cap (\sigma \times X^n)$ is a copy of $X^n$ that can be thought of as the graph of $i \circ f: X^n \to \sigma$. Restrict to $\mathcal{H}(K)$ the projection of $G(K) \times X^n$ onto the second factor, and then compose it with the inclusion $X^n \to \sigma \times X^n$ given by $x \to (x,i(f(x)))$. This defines a retraction of $\mathcal{H}(K)$ onto $\mathcal{H}(K) \cap (\sigma \times X^n)$, which is a copy of $X^n$. Since retractions are $\pi_1$-surjective, Lemmas 3.1-3.2 show that $\pi_1(\mathcal{H}(K))$ surjects onto
the nontrivial group \( \pi_1(X^n) \) that acts freely on the hyperbolic \( n \)-space. Thus \( \pi_1(\mathcal{H}(K)) \) cannot have property (T) \([dHV89]\).

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\square
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4. Relative strict hyperbolization

In this section we adapt the discussion \([DJW01, \text{Section 2}]\) of the relative (non-strict) hyperbolization to the strict case.

Let \( K \) be a connected finite-dimensional simplicial complex, and let \( L \) be a (not necessarily connected, nonempty) subcomplex of \( K \). Let \( P \) be the simplicial complex obtained by attaching a cone \( C L_i \) to each path-component \( L_i \) of \( L \). Denote by \( o_i \) the cone point of \( C L_i \). Let \( h: \mathcal{H}(P) \to P \) be the strict hyperbolization map as in Section 2. By functoriality, \( \mathcal{H}(K) \) is a subcomplex of \( \mathcal{H}(P) \), and none of the vertices \( h^{-1}(o_i) \) lies in \( \mathcal{H}(K) \), because again by functoriality \( h(\mathcal{H}(K)) = K \). Recall that we have chosen \( h \) so that each \( h^{-1}(o_i) \) has some small open \( \epsilon \)-neighborhood \( O_i \) such that \( h: O_i \to h(O_i) \) is a homeomorphism. Let

\[
R_K := \mathcal{H}(P) \setminus \left( \bigcup_i O_i \right)
\]

and let \( R_L \) be the (topological) boundary of \( R_K \) in \( \mathcal{H}(P) \), which is also the union of the boundaries of \( O_i \)'s. Since the boundary of \( O_i \) is a subdivision of \( L_i \), we conclude that \( R_L \) is a subdivision of \( L \).

We refer to the pair \((R_K, R_L)\) as the relative strict hyperbolization of \((K, L)\). For future use we note basic properties of the pair \((R_K, R_L)\).

\( \textbf{(A)} \) Since \( h: O_i \to h(O_i) \) is a homeomorphism, contracting along cone directions defines a deformation retraction and a homeomorphism of \((h(R_K), h(R_L))\) onto \((K, L)\).

\( \textbf{(B)} \) The restriction of \( h \) to the map \( R_K \to h(R_K) \) is surjective on homology and fundamental groups, because \( h(R_K) \) deformation retracts onto \( K \) and \( h: \mathcal{H}(K) \to K \) is surjective on homology and fundamental groups.

\( \textbf{(C)} \) The facts \( C1, C2, C3 \) below holds for triangulated, PL and smooth manifolds (where in the smooth case one works with smooth triangulations), but we just state all results in PL category; other cases are similar. Let \( K \) be a PL manifold with boundary \( L \), and let \( P \) be the simplicial complex obtained by attaching a cone \( C L_i \) to each path-component \( L_i \) of \( L \). By \([CD95, \text{p 348}] \) and \([DJ91, \text{pp 356–357}] \), \( \mathcal{H}(P) \) can be identified with \( \phi^{-1}(0) \), where \( \phi: P \times X^n \to \mathbb{R}^n \) is the difference of the projections \( P \to \Box^n, X^n \to \Box^n \), where \( \phi \) is transverse to 0, and where the restriction of the projection \( P \times X^n \to P \) to \( \phi^{-1}(0) \) corresponds to the map \( h: \mathcal{H}(P) \to P \). Restricting \( \phi \) to \( K \times X^n \),
we get the map $\phi: K \times X^n \to \mathbb{R}^n$ where $R_K$ is identified with $\phi^{-1}(0)$. Now just like in [CD95, p 348] and [DJ91, pp 356–357] we deduce the following.

(C1) $R_K$ is a PL manifold with boundary $R_L$, because $\phi$ is transverse to 0.

(C2) The normal (block) bundle of $R_K$ in $K \times X^n \to \mathbb{R}^n$ is trivial, so that the tangent bundle $T R_K$ to $R_K$ is stably isomorphic to the restriction of $T K \times T X^n$ to $R_K$. Since $X^n$ is orientable, and has trivial rational Pontrjagin classes, the map $h: R_K \to h(R_K)$ of compact manifolds pulls back the first Stiefel-Whitney class and the rational Pontrjagin classes.

(C3) If the manifold $(K, L)$ is orientable, then so is $(R_K, R_L)$, because $h$ pulls back the first Stiefel-Whitney class.

(D) The facts D1, D2, D3 below are proved in [DJW01, Section 2] for non-strict relative hyperbolization, and the same proof works verbatim if throughout the proof piecewise-Euclidean complexes are replaced with piecewise hyperbolic complexes.

(D1) The inclusion of each connected component of $R_L$ into $R_K$ is $\pi_1$-injective.

(D2) $R_K$ is aspherical if and only if each component of $R_L$ is aspherical.

(D3) For a small positive $\delta$, the $\delta$-neighborhood of $R_L$ in $R_K$ is the warped product $[\epsilon, \epsilon + \delta] \times_{\sinh(r)} R_L$, where $R_L$ is given the $CAT(1)$ piecewise-spherical metric induced by the inclusion $R_L \subset H(K)$. If $q: \tilde{R}_K \to R_K$ is the universal cover with pullback metric, then each component $C_L$ of $q^{-1}(R_L)$ is a $CAT(1)$ piecewise-spherical complex, whose $\delta$-neighborhood is the warped product $[\epsilon, \epsilon + \delta] \times_{\sinh(r)} C_L$. Therefore, the space $\tilde{R}_K$ obtained by attaching the elliptic cone $[0, \epsilon + \delta] \times_{\sinh(r)} C_L$ along $(\epsilon, \epsilon + \delta) \times_{\sinh(r)} C_L$ is locally $CAT(-1)$; hence $\tilde{R}_K$ is $CAT(-1)$ because $\tilde{R}_K$ is simply-connected.

Remark 4.1. Since $h_*$ is surjective on homology, $h^*$ is injective on rational cohomology, and hence if $K$ is a manifold that has a nontrivial rational Pontrjagin class, $R_K$ has the same property. Note that by replacing $K$ with the connected sum of $K$ and a manifold with a nonzero rational Pontrjagin class, one can always arrange that the rational Pontrjagin class of $K$ is nonzero. Also if $n \geq 4$, then by attaching handles one can prescribe the fundamental group of $K$ to be any given finitely presented group, so one can arrange that $\pi_1(R_K)$ surjects on any given finitely presented group. Thus, by varying $K$ while keeping $L$ fixed, one can produce $R_K$ with boundary $L$ and fairly complicated topology.

Remark 4.2. There is another natural way to define the relative strict hyperbolization of $(K, L)$, where instead of $H(K \cup_i C L_i)$ we look at $H(K \cup C L)$, and then remove a small conical neighborhood of the vertex $H(o)$, which is the preimage of the cone vertex $o$ of $C L$ under the hyperbolization map
$\mathcal{H}(K \cup CL) \to K \cup CL$. As we now explain this new procedure yields the same pair $(R_K, R_L)$ as above. Indeed, by functoriality $\mathcal{H}(K \cup CL)$ is $\mathcal{H}(K)$ with $\mathcal{H}(CL)$ attached along $\mathcal{H}(L)$, and in turn, $\mathcal{H}(CL)$ is the union of $\mathcal{H}(CL_i)$’s that are all identified at the vertex $\mathcal{H}(o)$. Thus $\mathcal{H}(K \cup CL)$ is obtained from $\mathcal{H}(K \cup L)$ by attaching a cone over $R_L$, in particular, $\mathcal{H}(K \cup CL)$ is homeomorphic to $R_K/R_L$.

Even more generally, the same proof shows the following. Suppose we partition the set $\{L_i\}$ of path-components of $L$ into the disjoint subsets $S_1, \ldots, S_r$, and then attach $r$ cones to $K$ with cone points $o_k$, $k = 1, \ldots, r$, so that the cone with vertex $o_k$ is attached over the union of $L_i$’s that belong to $S_k$. Then after applying $\mathcal{H}$, we get a locally $\text{CAT}(-1)$ complex with cone points $h^{-1}(o_1), \ldots, h^{-1}(o_r)$, and after removing the cone points we get $R_K$.

**Lemma 4.3.** If $\dim(L) < \dim(K) = n$, then $R_K$ retracts onto $X^n$. If $n \geq 2$, then $\pi_1(R_K)$ does not have Kazhdan property (T).

**Proof.** Note that $\mathcal{H}(K \cup CL)$ has dimension $\leq n$ as $\dim(L) < \dim(K) = n$. Since there is an $n$-simplex that does not lie in $L$, there exists an inclusion $i: X^n \to \mathcal{H}(K \cup CL)$ such that $i(X^n)$ does not contain the cone vertex $\mathcal{H}(o)$. Hence the retraction of $\mathcal{H}(K \cup CL)$ onto $i(X^n)$ given by Lemma 3.3 restricts to a retraction of $R_K$ onto $i(X^n)$. Finally, as in Lemma 3.3 we conclude that $\pi_1(R_K)$ does not have property (T). $\square$

5. Proving relative hyperbolicity

One of the equivalent definitions of a relatively hyperbolic group is due to Bowditch [Bow, Definition 2]. Let $P_1, \ldots, P_k$ be finitely generated infinite subgroups of a group $G$. Let $\mathcal{P}$ be the set of subgroups of $G$ that are conjugate to $P_i$, for some $i = 1, \ldots, k$. Then $G$ is called hyperbolic relative to $P_1, \ldots, P_k$ if $G$ admits an action on a connected fine hyperbolic graph $\mathcal{G}$ such that each edge has finite stabilizer, there are only finitely many orbits of edges, and elements of $\mathcal{P}$ are precisely the stabilizers of vertices of infinite valency. Here a graph is called fine if for each $L > 0$ every edge of $\mathcal{G}$ is contained in only finitely many circuits of length $\leq L$. If in the above situation $P_1, \ldots, P_k$ are not specified, we shall say, with a slight abuse of language, that $G$ hyperbolic relative to $\mathcal{P}$. Elements of $\mathcal{P}$ are called maximal parabolic subgroups; any subgroup of a maximal parabolic subgroup is called parabolic.

**Example 5.1.** If all vertices of $\mathcal{G}$ have finite valency (i.e. $\mathcal{P}$ is empty), then $G$ is quasi-isometric to $\mathcal{G}$ and hence $G$ is a hyperbolic group.
Example 5.2. If \( G \) is the fundamental group of a finite graph of groups whose edge groups are finite, then the Bass-Serre theory gives a \( G \)-tree \( G \) with finite edge stabilizers, and \( G/G \) is the original graph. Since any tree is hyperbolic and fine, \( G \) is hyperbolic relative to the collection of infinite vertex stabilizers.

Proof of Theorem 1.1. Since \( R_L \) is a subdivision of a finite complex \( L \), we know that \( R_L \) has finitely many (connected) components \( L_i, i = 1, \ldots, r \), where we assume that \( \pi_1(L_i) \) is infinite if and only if \( i = 1, \ldots, m \) where \( m \leq r \). Each inclusion \( L_i \to R_K \) is \( \pi_1 \)-injective hence it defines a conjugacy class of subgroups of \( G := \pi_1(R_K) \) which are isomorphic to \( \pi_1(L_i) \), and let \( H_i \) be a subgroup in the conjugacy class. Let \( \mathcal{P} \) be the union of conjugacy classes that contain infinite \( H_i \)’s.

We need to show that \( G \) is hyperbolic relatively to \( \mathcal{P} \), and that \( G \) is non-elementary. Look at \( G \)-action on the universal cover \( q: \tilde{R}_K \to R_K \). Conjugates of \( H_i \) are in one-to-one correspondence with components of \( q^{-1}(L_i) \), namely, each subgroup conjugate to \( H_i \) is the stabilizer of some components of \( q^{-1}(L_i) \). These components are simply-connected since each inclusion \( L_i \to R_K \) is \( \pi_1 \)-injective. As in Section 4 for every \( i \) we attach an elliptic cone along each component of \( q^{-1}(L_i) \), denote the result by \( \tilde{R}_K \), and let \( G \) acts isometrically and simplicially on \( \tilde{R}_K \) in the obvious way so that every conjugate of \( H_i \) fixes the vertex of the corresponding cone. Since \( \tilde{R}_K \) is \( CAT(-1) \), \( \tilde{R}_K \) is a hyperbolic metric space.

Denote by \( G \) the 1-skeleton of \( \tilde{R}_K \) and check that the \( G \)-action on \( G \) satisfies the Bowditch’s definition of relative hyperbolicity. Since \( G \) is quasi-isometric to \( \tilde{R}_K \), the graph \( G \) is hyperbolic. Since \( K \) and \( L \) are finite complexes, each \( H_i \) is finitely generated, and \( G \) has only finitely many orbits of edges. By construction, elements of conjugacy classes of \( H_i \)'s correspond bijectively to cone points. The cone points are the only vertices that can have nontrivial stabilizers, so since no edge joins two cone points, each edge has trivial stabilizer.

Now we check that \( G \) is fine, which is the only part of the proof that is really new. By [Bow, Proposition 2.1(F5)] it suffices to show that for each vertex \( v \), any infinite set of vertices that are adjacent to \( v \) is unbounded in the induced metric on \( G \setminus \{v\} \). The only vertices that can have infinitely many adjacent ones are the cone points stabilized by some subgroup in \( \mathcal{P} \), so we can assume that \( v \) is one of such cone points, and denote the corresponding elliptic cone in \( \tilde{R}_K \) by \( C_v \). Note that \( C_v \) is convex because \( C_v \) equals to a metric ball \( \tilde{R}_K \) centered at \( v \) and \( \tilde{R}_K \) is \( CAT(-1) \) [BH99, Proposition II.1.4]. Let \( S \) be an infinite set of vertices adjacent to \( v \), and arguing by contradiction, assume that \( S \) is bounded in \( G \setminus \{v\} \), or equivalently that \( S \) is bounded in \( G \setminus \text{Int}(C_v) \). Because \( \tilde{R}_K \) is \( CAT(-1) \) (in fact \( CAT(0) \) is enough), the nearest point retraction \( \tilde{R}_K \to C_v \) is distance-nonincreasing [BH99, Proposition II.2.4], in particular,
it does not increase the lengths of curves, and hence any bounded subset of $\mathcal{G} \setminus \text{Int}(C_v)$ projects to a subset of $\partial C_v$ that is bounded in the induced metric on $\partial C_v$. Since the retraction restricts to the identity on $S$, we conclude that $S$ is bounded in the induced metric on $\partial C_v$, and since $S$ is discrete, it must be finite, which is a contradiction.

Finally, we show that $\pi_1(R_K)$ is non-elementary. If $\pi_1(R_K)$ equals to a parabolic subgroup, then $\mathcal{H}(P)$ is simply-connected, and if $\pi_1(R_K)$ is finite or virtually-$\mathbb{Z}$, then so is $\pi_1(\mathcal{H}(P))$. (Recall that $\mathcal{H}(P)$ is $R_K$ with cone attached over $R_L$). Take a 2-simplex $\sigma^2$ in $P$. By functoriality $\mathcal{H}(\sigma^2)$ is a locally convex subset of $\mathcal{H}(P)$. Therefore, by [BH99, Proposition II.4.14] the inclusion $\mathcal{H}(\sigma^2) \to \mathcal{H}(P)$ is $\pi_1$-injective, so we conclude that $\pi_1(\mathcal{H}(\sigma^2))$ is finite or virtually-$\mathbb{Z}$. On the other hand, $\mathcal{H}(\sigma^2)$ is a compact surface with boundary of negative Euler characteristic, so its fundamental group is free nonabelian. □

6. Simplicial volume basics

Here we review the properties of simplicial volume that are relevant to what we are going to do next. We refer to [Gro82, BP92] for details. The complex $C_*(X)$ of (singular real-valued) chains on a topological space $X$ has a natural $l^1$-norm given by $||\sum_{i=1}^s r_i \sigma_i|| = \sum_{i=1}^s |r_i|$. This norm gives rise to a pseudonorm on the homology of $X$, namely $||h|| = \inf \{ ||z|| : h = [z] \}$, i.e. the pseudonorm of a homology class $h$ is the infimum of the norms of the cycles representing $h$. More generally, for a pair of spaces $(X, Y)$ the relative chain complex $C_*(X, Y) = C_*(X)/C_*(Y)$ has the quotient $l^1$-norm, and the infimum of the norms of the cycles representing a relative homology class is the a pseudonorm on $H_*(X, Y)$.

If $M$ is a compact oriented manifold with (possibly empty) boundary, the relative simplicial volume $||M, \partial M||$ of $(M, \partial M)$ is the pseudonorm of the fundamental class $[M, \partial M]$. For non-orientable compact manifold $M$ with the orientation cover $\tilde{M}$, one defines $||M, \partial M|| = ||\tilde{M}, \partial \tilde{M}||/2$. For example, if the interior of $M$ carries a complete finite volume hyperbolic metric $g$, then $||M, \partial M|| = \text{Vol}(M, g)/V_n$, where $V_n$ is the maximum of the volumes of ideal regular simplices in the hyperbolic $n$-space [Thu, Fra04].

Remark 6.1. It is proved in [MY] that if $M$ satisfies Assumption 1.2, then any class in $H^k(M, \partial M)$ is bounded for $k \geq 2$. In particular, the $n$-dimensional cohomology class dual to the fundamental class $[M, \partial M]$ is bounded, which by a standard argument [BP92, p. 278] implies that $||M, \partial M|| > 0$.

7. Hyperbolization and simplicial volume

Let $K$ be a triangulated compact connected oriented $n$-manifold with boundary $L$. To avoid trivialities we assume $n \geq 2$. Denote by $s = s(K)$ the number of
n-simplices in the triangulation. Let $R$ be the the compact oriented $n$-manifold with boundary $\partial R$ produced via relative strict hyperbolization of $(K, L)$. The goal of this section is to show that

$$C_1(n) \geq \frac{||R, \partial R||}{s} \geq C_2(n) > 0,$$

for some positive constants $C_1(n), C_2(n)$ depending only on $n$.

It will be convenient to replace $K$ with its second barycentric subdivision; this increases the number of $n$-simplices by a factor that only depends on $n$. The number of $n$-simplices of $K \cup CL$ is the sum of the number of $n$-simplices of $K$ and the number of $(n-1)$-simplices of $L$. Since $K$ is a manifold with boundary $L$, any $(n-1)$-simplex of $L$ is a face of a unique $n$-simplex of $K$. Thus the number of $n$-simplices of $K \cup CL$ is $\geq s$ and $\leq$ than a factor of $s$ that only depends on $n$.

Let $Z = R/\partial R$, let $q: R \to Z$ be the quotient map, and let $S = q(\partial R)$. Alternatively, $Z$ can be thought of as the result of attaching a cone to $R$ along $\partial R$. By Remark 4.2, $Z = \mathcal{H}(K \cup CL)$. Let $i: S \to Z$ and $j: Z \to (Z, S)$ be the inclusions. Then since $S$ is a point and $n \geq 2$, the long exact sequence of the pair $(Z, S)$ gives the isomorphism

$$i_*: H_n(Z) \to H(Z, S).$$

Denote $[Z, S] = q_*[R, \partial R] \in H_n(Z, S)$ and $[Z] = i_*^{-1}([Z, S])$. Note that $Z$ is an oriented pseudomanifold with the fundamental class $[Z]$. Denote the simplicial norms of $[Z, S], [Z]$ by $||Z, S||, ||Z||$, respectively.

Denote by $\mathcal{H}^n(Z)$ the $n$-dimensional Hausdorff measure of $Z$. As before $\mathcal{X}^n$ denotes the compact hyperbolic manifold with corners constructed in [CD95] that is used as a building block in the strict hyperbolization. If $Z$ is built from $N$ copies of $\mathcal{X}^n$, then clearly $\mathcal{H}^n(Z) = N \cdot \text{Vol}(\mathcal{X}^n)$. It follows from the definition of strict hyperbolization that $s \leq N \leq C(n)s$ where $C(n)$ is a constant that only depends on $n$. This estimate combined with Proposition 7.1 below yields Theorem 1.6.

**Proposition 7.1.** There exist constants $C_1(n) \geq C_2(n) > 0$ depending only on $n$ such that $C_1(n)s \geq ||R, \partial R|| \geq ||Z, S|| \geq \frac{||Z||}{n+2} \geq \frac{(n-1)^{n-1}}{\pi(n+2)^2} \mathcal{H}^n(Z) \geq C_2(n)s$.

**Proof.** Triangulate $\mathcal{X}^n$, look at the induced triangulation of $Z$, and furthermore pass to the second barycentric subdivision of $Z$, so that $R$ is homeomorphic to $Z$ with the open star of each cone vertex removed. Note that the number of $n$-simplices of the triangulation of $R$ is at most the number of $n$-simplices in the triangulation of $Z$ that we just constructed, which is $\leq s \cdot C_1(n)$ where $C_1(n)$ is some constant depending only on $n$. 
The fundamental class $[R, \partial R]$ is represented by a cycle in which each $n$-simplex of the above triangulation enters only once. This gives the first inequality $||R, \partial R|| \leq s \cdot C_1(n)$. Since the simplicial norm does not increase under continuous maps, we get the second inequality: $||R, \partial R|| \geq ||q_*[R, \partial R]|| = ||Z, S||$.

Let $i_\#, j_\#$ be the chain maps induced by $i, j$ in the short exact sequence of complexes

$$0 \to C_*(S) \xrightarrow{i_\#} C_*(Z) \xrightarrow{j_\#} C_*(Z, S) \to 0.$$ 

Since $C_*(Z, S)$ carries the quotient norm, $||Z, S||$ is the infimum of $||c||$ where $c$ is an $n$-chain in $Z$ and $j_\#(c)$ is a cycle that represents $[Z, S]$. Note that $c$ need not be a cycle in $C_n(Z)$. The strategy of the proof is to modify $c$ into a cycle $c'$ such that $j_\#(c')$ still represents $[Z, S]$, and $||c'|| \leq (n + 2)||c||$; then we would get $||Z|| \leq ||c'|| \leq (n + 2)||c||$ which gives the third inequality after taking infimum over all $c$'s.

The cycle $c'$ produced after an elementary diagram chase which we present below. Since $j_\#$ is a chain map, $j_\# \partial(c) = \partial j_\#(c) = 0$, hence $\partial e \in \ker(j_\#) = \text{Im}(i_\#)$, and so $\partial c = i_\# e$ for some $e \in C_{n-1}(S)$. Write $e = se^{n-1}$ where $s \in \mathbb{R}$, and $e^k: \Delta^k \to S$ is the unique singular $k$-simplex with image $S$. Now $i_\# \partial(e) = \partial i_\#(e) = \partial \partial c = 0$, so by injectivity of $i_\#$ we get $\partial e = 0$. Since $H_{n-1}(S) = 0$, the cycle $e$ is the boundary of some chain $f = te^n \in C_n(S)$, so that $se^{n-1} = t\partial e^n$. Since $\partial i_\#(f) = i_\# \partial(f) = i_\#(e) = \partial c$, we deduce that $c - i_\# f$ is a cycle, which represents $[Z] = j_*^{-1}([Z, S])$ because $j_\#(c - i_\# f) = j_\#(c)$ represents $[Z, S]$.

To compute $f$ notice that $\partial e^n = 0$ if $n$ is odd, and $\partial e^n = e^{n-1}$ if $n$ is even. Thus if $n$ is odd, then $\partial: C_n(S) \to C_{n-1}(S)$ is the zero map, so that $0 = \partial c = i_\# e$ which implies $e = 0$, and therefore we may take $f = 0$. If $n$ is even, then $se^{n-1} = e = \partial f = t\partial e^n = te^{n-1}$, i.e. $s = t$, and therefore in either case $||e|| = ||f||$. Also since $i_\#$ is norm-preserving, $||i_\# f|| = ||f||$ and $||\partial c|| = ||i_\# e|| = ||e||$, thus all these chains have equal norms. Therefore,

$$||Z|| \leq ||c - i_\# f|| \leq ||c|| + ||i_\# f|| = ||c|| + ||f|| = ||c|| + ||\partial c|| \leq (n + 2)||c||,$$

where the last inequality follows from the estimate $||\partial c|| \leq (n + 1)||c||$ coming from the definition of the boundary homomorphism. Taking infimum over all $c$'s we get $||Z|| \leq (n + 2) \cdot ||Z, S||$.

The fourth inequality follows by a result of Yamaguchi [Yam97] who, using Thurston’s straightening as in [Gro82], proved the bound when $Z$ is a compact locally-CAT$(-1)$ geodesically complete orientable pseudomanifold. That $Z$ is geodesically complete follows as in [BH99, Proof of Proposition II.5.12] because the local $n$-homology group is nontrivial at each point of $Z$.

The fifth inequality follows because $\mathcal{H}^n(Z) = N \cdot \text{Vol}(\mathcal{X}^n) \geq s \cdot \text{Vol}(\mathcal{X}^n)$ by letting $C_2(n) = \text{Vol}(\mathcal{X}^n)$. □
Remark 7.2. If \((R, \partial R)\) is a nonorientable compact manifold obtained via relative strict hyperbolization, we still get the estimate
\[
C_1(n)s \geq \|R, \partial R\| \geq C_2(n)s,
\]
for some \(C_1(n) \geq C_2(n) > 0\). Indeed, let \(\hat{R}\) be the orientable 2-fold cover of \(R\); this is a manifold with boundary with a natural triangulation that is pulled backed from \(R\). If \(\hat{Z}\) is the space obtained by attaching a cone to each component of \(\partial R\), then \(\hat{Z}\) is locally-\(CAT(-1)\) (see the proof of [DJW01, Lemma 2.6]). Then the proof of Proposition 7.1 applies to \(\hat{R}\) and \(\hat{Z}\), so we get the above estimates for \(\|\hat{R}, \partial \hat{R}\|\), and hence for \(\|R, \partial R\|\), perhaps for some other \(C_1(n), C_2(n)\).

Remark 7.3. Proposition 7.1 immediately implies Theorem 1.7 because there are only finitely many ways to glue finitely many simplices. The same is true in smooth category by using smooth triangulations, and smooth relative hyperbolization.

Remark 7.4. Since the relative simplicial volume is invariant under homotopy equivalences of pairs, Theorem 1.7 gives a finiteness result for manifolds pairs obtained by relative strict hyperbolization that are in the same homotopy type of pairs. Furthermore, if \(R, R'\) are homotopy equivalent manifolds obtained by relative strict hyperbolization, and each component of \(\partial R\) is aspherical and has intrinsically elementary fundamental group, then there is homotopy equivalence of pairs \((R, \partial R) \rightarrow (R', \partial R')\) (see the proof of Theorem 8.1). Hence we deduce the following corollary which also holds in the smooth category.

Corollary 7.5. Every homotopy type contains at most finitely many pairwise PL-non-homeomorphic aspherical \(n\)-manifolds \(R\) obtained by the relative strict hyperbolization and such that each component of \(\partial R\) has intrinsically elementary fundamental group.

Remark 7.6. An analog of Proposition 7.1 is also true for manifolds obtained via (non-relative) strict hyperbolization. To see this let \(K\) be a closed \(n\)-manifold triangulated with \(q\) simplices of dimension \(n\). Replace \(K\) with its second barycentric subdivision; this changes \(q\) by a factor that only depends on \(n\). Let \(St_v\) be an open star of a vertex \(v\) of \(K\), and let \(Lk_v\) be the link of \(v\), which we think of the topological boundary of \(s\). Then the relative hyperbolization \((R, \partial R)\) of \((K\setminus St_v, Lk_v)\) is obtained by removing from \(\mathcal{H}(K)\) a small conical neighborhood of the vertex corresponding to \(v\), and in particular, \(Z = R/\partial R\) equals to \(\mathcal{H}(K)\). By Proposition 7.1 we know that \(\|Z\|/s\) is bounded between two positive constants that only depend on \(n\), where \(s\) is the number of \(n\)-simplices in the triangulation of \(K\setminus St_v\). It is straightforward to check that \(s/q\) is bounded between two positive constants depending only on \(n\), so the desired assertion follows.
The following theorem implies Theorem 1.5 because homeomorphic manifolds have equal relative simplicial volumes.

**Theorem 8.1.** If $M_1$, $M_2$ are compact aspherical $n$-manifolds satisfying Assumption 1.2, and $\phi: \pi_1(M_1) \to \pi_1(M_2)$ is an injective homomorphism that maps each maximal parabolic subgroup to a parabolic subgroup, then $\phi(\pi_1(M_1))$ has finite index in $\pi_1(M_2)$. If in addition $||M_1, \partial M_1|| \leq ||M_2, \partial M_2||$, then $\phi$ is induced by a homotopy equivalence of pairs $(M_1, \partial M_1) \to (M_2, \partial M_2)$, so that $\phi$ is onto.

**Proof.** Denote the image of $\phi$ by $\tilde{G}$, and look at the $\tilde{G}$-action on the universal covering $\tilde{M}_2$ of $M_2$. To prove that $\phi(\pi_1(M_1))$ has finite index in $\pi_1(M_2)$, we need to show that $\tilde{M}_2 := \tilde{M}_2/\tilde{G}$ is compact.

Let $H_1, \ldots, H_k$ are maximal parabolic subgroups of $\pi_1(M_1)$ which are in one-to-one correspondence with path-components $Q_1, \ldots, Q_k$ of $\partial M_1$. Since the group $\phi(H_i)$ is parabolic, it stabilizes a path-component $\tilde{Y}_i$ of $\partial M_2$. In fact $\phi(H_i)$ acts cocompactly on $\tilde{Y}_i$, else the cohomological dimension of $H_i$ would be $< n - 1$. Each closed aspherical manifold $\tilde{Y}_i := \tilde{Y}_i/\phi(H_i)$ is an incompressible boundary component of $\tilde{M}_2$, for if $\tilde{Y}_i$ were compressible, so would be its projection to $M_2$. A priori $\partial M_2$ could have components other than $\tilde{Y}_i$'s. To see that this does not happen, we double $\tilde{M}_2$ along the union of $\tilde{Y}_i$'s. The result is an aspherical manifold $\hat{D}_2$, which has empty boundary if and only if $\partial \hat{M}_2$ is the union of $\tilde{Y}_i$'s. The fundamental group of $\hat{D}_2$ is isomorphic to the fundamental group of the double $DM_1$ of $M_1$ along $\partial M_1$, hence $\hat{D}_2$ and $DM_1$ are homotopy equivalent. So by looking at top-dimensional $\mathbb{Z}_2$-homology, we get $H_n(\hat{D}_2; \mathbb{Z}_2) \cong H_n(DM_1; \mathbb{Z}_2) \cong \mathbb{Z}_2$, which implies that $\hat{D}_2$ is a closed manifold, and in particular $\hat{M}_2$ is a compact manifold and $\partial \hat{M}_2$ is the union of $\tilde{Y}_i$'s. This proves that $\phi(\pi_1(M_1))$ has finite index in $\pi_1(M_2)$.

Next we show that $\phi$ induces a homotopy equivalence of pairs $(M_1, \partial M_1) \to (\hat{M}_2, \partial \hat{M}_2)$. The proof relies on the following standard result which can be found e.g. in [Spa81, Theorem 8.1.9]: given connected CW-complexes $(X, x)$, $(Y, y)$ where $Y$ is aspherical, there is a natural bijective correspondence between homotopy classes of maps $X \to Y$ and the conjugacy classes of the induced homomorphisms $\pi_1(X, x) \to \pi_1(Y, y)$.

Since $\hat{M}_2$ is aspherical, $\phi$ induces a homotopy equivalence $f: M_1 \to \hat{M}_2$, as well as homotopy equivalences $f_i: Q_i \to \tilde{Y}_i$, where $f_i$ and $f_{iQ_i}$ are homotopic for each $i = 1, \ldots, k$. One can use the homotopy of $f_i$ and $f_{iQ_i}$ to modify $f$ on a collar neighborhood of $\partial M_1$ so that we get a map of pairs $f: (M_1, \partial M_1) \to (\hat{M}_2, \partial \hat{M}_2)$ that induces $\phi$, and equals to $f_i$ when restricted.
to \( Q_i \). The same argument applied to a homotopy inverse of \( f \) yields a map of pairs \( g \colon (\hat{M}_2, \partial \hat{M}_2) \to (M_1, \partial M_1) \) inducing \( \phi^{-1} \). The composition \( g \circ f \) induces the identity on \( \pi_1(M_1) \) and on each \( H_i \). In particular, \( g \circ f \) is homotopic to the identity as a selfmap of \( M_1 \), and each \( Q_i \). Yet we need a stronger conclusion that \( g \circ f \) is homotopic to the identity as a selfmap of \((M_1, \partial M_1)\). As before, we modify \( g \circ f \) on a collar neighborhood of \( \partial M_1 \) so that \( g \circ f \) becomes a selfmap of \((M_1, \partial M_1)\) that restricts to the identity of \( \partial M_1 \). Now by a standard fact [Hat02, Proposition 0.19], \( g \circ f \) is homotopic to the identity \( \text{rel } \partial M_1 \). The same argument applies to \( f \circ g \). Thus \( f \) is a homotopy equivalence of pairs.

Now we are ready to show that \( \phi \) is onto provided \( ||M_2, \partial M_2|| \geq ||M_1, \partial M_1|| \). There are three cases to consider as follows.

If \( M_2 \) is orientable, then so is \( \hat{M}_2 \). Since any homotopy equivalence of pairs preserves the Stiefel-Whitney classes [Spa81, Theorem 6.10.7], \( M_1 \) is orientable. Furthermore, homotopy equivalent pairs have equal relative simplicial volumes: \( ||M_1, \partial M_1|| = ||M_2, \partial \hat{M}_2|| \). Finally, since \( \hat{M}_2 \to M_2 \) is a finite cover of degree \( |G: \hat{G}| \), we get \( ||M_2, \partial M_2|| \geq |G: \hat{G}| \cdot ||M_1, \partial M_1|| \). Combining this information, we get \( ||M_1, \partial M_1|| \geq |G: \hat{G}| \cdot ||M_1, \partial M_1|| \) which forces \( |G: \hat{G}| = 1 \), because \( ||M_1, \partial M_1|| \geq 0 \) by Theorem 1.6.

If \( M_2 \) and \( \hat{M}_2 \) are not orientable, then the covering \( q \colon \hat{M}_2 \to M_2 \) does not factor through the orientation two-fold cover \( M_2' \to M_2 \), so the \( q \)-pullback of \( M_2' \to M_2 \) is a certain two-fold cover over \( \hat{M}_2 \), which is orientable, because it is a cover of \( M_2' \). Since the orientation two-fold cover of a manifold is unique, the two-fold cover over \( \hat{M}_2 \) defined in the previous sentence has to coincide with the orientation cover \( \hat{M}_2' \to \hat{M}_2 \). Thus \( q \) lifts to a map of the orientation two-fold covers \( q' \colon \hat{M}_2' \to M_2' \). Again by [Spa81, Theorem 6.10.7], the non-orientability of \( \hat{M}_2 \) implies that \( M_1 \) is not orientable and that \( f \) lifts to the homotopy equivalence \( f' \colon (M_1', \partial M_1') \to (\hat{M}_2', \partial M_2') \) of the two-fold orientation covers of \((M_1, \partial M_1)\) and \((\hat{M}_2, \partial \hat{M}_2)\). Now the argument of the previous paragraph applied to \( M_1', \hat{M}_2', M_2' \) implies that \( q' \) is one-to-one, and hence so is \( q \).

Finally, if \( M_2 \) is not orientable, while \( \hat{M}_2 \) is orientable, then the covering \( q \colon \hat{M}_2 \to M_2 \) also factors through the orientation two-fold cover \( M_2' \to M_2 \). Since the covering \( \hat{M}_2 \to M_2' \) has degree \( |G: \hat{G}|/2 \), we conclude that

\[
||\hat{M}_2, \partial \hat{M}_2|| \geq ||M_2', \partial M_2'|| \cdot |G: \hat{G}|/2 = ||M_2, \partial M_2|| \cdot |G: \hat{G}|,
\]

where the last equality holds since \( ||M_2', \partial M_2'|| = 2||M_2, \partial M_2|| \). Now the argument is concluded as in the case when \( \hat{M}_2 \) is orientable. \( \square \)

9. No splitting over elementary subgroups

Proof of Theorem 1.3. The assertion of (i) was proved in [Bel02, Section 3] in case \( M \) is negatively pinched, and the same proof works verbatim if one replaces
“virtually nilpotent” by “elementary” throughout the proof. For completeness we now review the argument in [Bel02, Section 3].

The proof starts off by making the splitting “relative”, and this is done in [Bel02, Lemma 3.1], that is essentially due to Bowditch, which applies provided every maximal parabolic subgroup of $\pi_1(M)$ is one-ended. One-endedness holds in the setting of Theorem 1.3 because any maximal parabolic subgroup is the fundamental group of a component of $\partial M$, which is a closed aspherical manifold of dimension $> 1$, and any such group is one ended, as can be seen via a straightforward Mayer-Vietoris argument in group cohomology. After relativizing the splitting, we double $M$ along appropriate part of its boundary, and then again look at the Mayer-Vietoris sequence of the splitting. Then a fairly tedious and lengthy elementary cohomological arguments lead to a contradiction, which implies (i).

To prove (ii) note that according to [DS, Theorem 1.14] if a non-elementary relatively hyperbolic group $G$ is not co-Hopfian, then either $G$ is isomorphic to a parabolic subgroup, or $G$ splits over an elementary subgroup. The latter possibility is ruled out by (i). Arguing by contradiction, consider a component $B$ of $\partial M$ and a monomorphism $\phi: \pi_1(M) \to \pi_1(B)$. Since $B$ is a closed aspherical manifold, $\phi(\pi_1(B))$ has finite index in $\pi_1(B)$. Since $\phi(\pi_1(M))$ sits between $\phi(\pi_1(B))$ and $\pi_1(B)$, the index of $\phi(\pi_1(B))$ in $\phi(\pi_1(M))$ is finite, so that $\pi_1(B)$ has finite index in $\pi_1(M)$. Then the intersection of all conjugates of $\pi_1(B)$ is a finite index subgroup of $\pi_1(B)$ that is normal in $\pi_1(M)$. But a maximal parabolic subgroup always contains the normalizers of its infinite subgroups, so $\pi_1(M) = \pi_1(B)$, contradicting the assumption that $\pi_1(M)$ is non-elementary.

Finally, we prove (iii). According to [DS05, Corollary 1.14] if $G$ is hyperbolic relative to subgroups $P_i$ and each $P_i$ is hyperbolic relative to subgroups $P_{ij}$, then $G$ is hyperbolic relative to $P_{ij}$. Since the fundamental group of each component of $\partial M$ has property (m), we can assume that $\pi_1(M)$ is hyperbolic relative to a family of subgroups $Q_l$ where each $Q_l$ is contained in the fundamental group of a component of $\partial M$, and furthermore, no $Q_l$ is isomorphic to a non-elementary relatively hyperbolic group. By [DS, Theorem 1.12] if a non-elementary relatively hyperbolic group $G$ has infinite outer automorphism group and if no maximal parabolic subgroup of $G$ is isomorphic to a non-elementary relatively hyperbolic group, then $G$ splits over an elementary subgroup. Thus (iii) follows from (i).

**Remark 9.1.** In the case $M$ is negatively pinched, the part (i) of Theorem 1.3 could be also deduced from results of Bowditch that relate the existence of splittings with absence of global cut points on the boundary, and this depends on the facts that $n > 2$, and that the Bowditch boundary of $\pi_1(M)$ is the $(n-1)$-sphere in the negatively pinched case. This argument of Bowditch is
much more conceptual than the above proof. Unfortunately, it does not work in the generality of Theorem 1.3, because in general we know nothing about the Bowditch boundary of $\pi_1(M)$.

10. Strict hyperbolization and Kähler manifolds

The following elementary proposition helps to prove that the manifolds obtained by relative (as well as by non-relative) strict hyperbolization are not Kähler.

**Proposition 10.1.** Let $K$ be a finite connected simplicial complex of dimension $n \geq 2$ that contains at least one $n$-simplex $\sigma$, and such that not every 2-simplex of $K$ is a face of $\sigma$. Then $\pi_1(H(K))$ splits as

$$\pi_1(H(\sigma)) \ast_{\pi_1(H(\partial \sigma))} \pi_1(H(K \setminus \text{Int}(\sigma)))$$

and $\pi_1(H(\partial \sigma))$ has infinite index in both $\pi_1(H(\sigma))$ and $\pi_1(H(K \setminus \text{Int}(\sigma)))$. In particular, $\pi_1(N)$ does not have Serre’s property FA.

**Proof.** The space $H(K)$ is the union of $H(\sigma)$ and $H(K \setminus \text{Int}(\sigma))$ intersecting along $H(\partial \sigma)$, which is locally-convex (as $\partial \sigma$ is a subcomplex of $K$), and hence $H(\partial \sigma)$ is $\pi_1$-injectively embedded into $H(K)$ [BH99, Proposition II.4.14]. Since $H(K)$ and $H(\partial \sigma)$ are path-connected, so is $H(K \setminus \text{Int}(\sigma))$. So $\pi_1(H(K))$ is an amalgamated product of $\pi_1(H(\sigma))$ and $\pi_1(H(K \setminus \text{Int}(\sigma)))$ over $C = \pi_1(H(\partial \sigma))$.

Let up show that the index of $C$ in $A = \pi_1(H(\sigma))$ is infinite. Since $H(\sigma)$ is oriented $n$-manifold with boundary $H(\partial \sigma)$, the homology boundary map is an isomorphism in the dimension $n$, so that by exactness $H(\partial \sigma) \to H(\sigma)$ is zero in the $(n-1)$-homology. If the index $|A : C|$ were finite, then since both $H(\partial \sigma)$ and $H(\sigma)$ are aspherical, $H(\partial \sigma)$ would be homotopy equivalent to a finite cover of $H(\sigma)$, so the map $H(\partial \sigma) \to H(\sigma)$ would induce in the $(n-1)$-homology the multiplication by the degree in the of the cover, while by the previous sentence the map is zero.

Let $B = \pi_1(H(K \setminus \text{Int}(\sigma)))$, and let $\tau$ be a 2-simplex that is not a face of $\sigma$. Then $\pi_1(H(\partial \sigma \cup \tau))$ is a subgroup of $B$ in which $C = \pi_1(H(\partial \sigma))$ has infinite index because each component of $H(\partial \sigma \cap \tau) = \partial \sigma \cap \tau$ is a simplex of dimension $\leq 1$, and $\pi_1(H(\tau))$ is infinite. Thus the index $|B : C|$ is infinite. $\square$

**Remark 10.2.** A similar argument shows that if $K$ is a finite connected simplicial complex of dimension $n \geq 2$, and $L$ is a subcomplex satisfying $\dim(L) < \dim(K)$, then for any $n$-simplex $\sigma$ of $K$, the group $\pi_1(R_K)$ splits as an amalgamated product over $H(\partial \sigma)$, where $H(\partial \sigma)$ has infinite index in each of the factors.

By a *compact Kähler manifold* we mean a closed Riemannian manifold with a Kähler metric.
**Theorem 10.3.** If $K$ is a finite simplicial complex, then $\mathcal{H}(K)$ is not homotopy equivalent to a compact Kähler manifold of real dimension $\geq 4$.

**Proof.** Arguing by contradiction assume that $\mathcal{H}(K)$ is homotopy equivalent to a compact Kähler manifold $M$ of real dimension $n \geq 4$. First note that $\dim(K) = n$ and that $K$ has at least two $n$-simplices. (Indeed, $\dim(K) = \dim(\mathcal{H}(K)) \geq n$ because $\mathcal{H}(K)$ is homotopy equivalent to a closed manifold $M$. If $\dim(K) > n$, then $K$ has a simplex $\sigma$ of dimension $n+1$, so $\pi_1(M) \cong \pi_1(\mathcal{H}(K))$ must contain the fundamental groups of the closed aspherical $n$-manifold $\pi_1(\partial\sigma)$. Since $\mathcal{H}(K)$ is aspherical, so is $M$, therefore $\pi_1(\partial\sigma)$ must have finite index in $\pi_1(M)$ which is impossible as $\pi_1(\partial\sigma)$ has infinite index in $\pi_1(\sigma) \leq \pi_1(\mathcal{H}(K))$ by the proof of Proposition 10.1. If $K$ has only one $n$-simplex $\sigma$, then $\mathcal{H}(K)$ is the union of $\mathcal{H}(\sigma)$ and a subcomplex of dimension $< n$. Then $\mathcal{H}(\sigma)$ and $(\mathcal{H}(K), \mathcal{H}(\sigma))$ have trivial $n$th homology, and hence so does $\mathcal{H}(K)$ by the long exact sequence of the pair which contradicts to the assumption that $\mathcal{H}(K)$ is homotopy equivalent to a closed $n$-manifold).

Hence by Proposition 10.1, $G = \pi_1(\mathcal{H}(K))$ splits over $\pi_1(\mathcal{H}(\partial\sigma))$. The splitting $G = A \ast_C B$ gives rise to a nontrivial $G$-action on the corresponding Bass-Serre tree $T$, which has no fixed end. (If $G$ fixes an end, then $A$ fixes the end and a vertex $v$ of $T$, so that $A$ fixes the edge adjacent to $v$ that lies on a unique ray joining $v$ and the end. Thus the inclusion $C \to A$ is onto contradicting nontriviality of the splitting). Therefore, by [KS97, Corollary 2.3.2] there exists a $G$-equivariant harmonic map $u: \hat{M} \to T$, where $\hat{M}$ is the universal cover of $M$. In fact, the image of $u$ locally lies in a finite subtree of $T$ [GS92, p.243] and [Sun03, Theorem 3.9], so $u$ is pluriharmonic by [GS92, Theorem 7.3]. Then by the factorization theorem in [GS92, Section 9], $u$ factors through the hyperbolic plane $\mathbf{H}^2$ equipped with an isometric discrete (possibly ineffective) $G$-action, and moreover the map $\hat{M} \to \mathbf{H}^2$ is $G'$-equivariant, where $G'$ is a subgroup of $G$ of index $\leq 2$. (By [GS92, Lemma 9.4] the discreteness of the $G$-action on $\mathbf{H}^2$ depends on the fact that $G$ is not virtually solvable, which is true since $G$ contains free nonabelian subgroup).

The kernel $K'$ of the $G'$-action on $\mathbf{H}^2$ lies in the kernel of the $G'$-action on $T$, which in turn lies in $C$. The subgroup $C$ is a quasiconvex subgroup because $\mathcal{H}(\partial\sigma)$ is locally convex in $\mathcal{H}(K)$. Note that $K'$ is infinite, else $G \cong \pi_1(M)$ would have cohomological dimension $\leq 2$. Thus the quasiconvex subgroup $C$ contains $K'$, which is an infinite normal subgroup of the hyperbolic group $G'$, which is impossible by a standard “limit set” argument. (Indeed, $K'$, $G'$, and $G$ have the same limit sets, and the limit set of $K'$ lies in the limit set of $C$. Thus $C$ and $G$ have the same limit set, hence the quasiconvexity of $C$ implies that $C$ has finite index in $G$ [Swe01]). \qed
Remark 10.4. The proof of Theorem 10.3 was inspired by recent work of Delzant-Gromov [DG05] where more general situation is considered and where the proofs are unfortunately sketchy.

Theorem 10.5. If $R$ is an $n$-manifold obtained by relative strict hyperbolization, then $\text{Int}(R)$ is not homeomorphic to an open subset of a compact Kähler manifold of real dimension $\geq 4$.

Proof. We think of $R$ as obtained from the $\mathcal{H}(K \cup CL)$ by removing a small conical neighborhood of the cone point. Since $K$ is an $n$-manifold with boundary $L$, there is at least one $n$-simplex $\sigma$ of $K$ that does not lie in $L$, and $K \cup CL$ contains a 2-simplex that contains the cone point and is not a face of $\sigma$. Thus we get a splitting of $\pi_1(\mathcal{H}(K \cup CL))$ as in Proposition 10.1.

Arguing by contradiction assume that $U$ is an open subset of a compact Kähler manifold $M$, and $f: U \to \text{Int}(R)$ is a homeomorphism. Note that $f$ extends to a continuous map $\bar{f}: M \to R/\partial R = \mathcal{H}(K \cup CL)$ that maps $M \setminus U$ to the cone point $\partial R/\partial R$, because $f$ is proper and continuous, and $R/\partial R$ is the one-point-compactification of $\text{Int}(R)$. The map $\bar{f}: M \to R/\partial R$ has degree one (recall that $R/\partial R$ is a pseudomanifold), and in particular, $\bar{f}$ is $\pi_1$-surjective.

We can write $M$ as the union of $f^{-1}(\mathcal{H}(\sigma))$ and $M \setminus f^{-1}(\text{Int}(\mathcal{H}(\sigma)))$ which intersect along $f^{-1}(\mathcal{H}(\partial \sigma))$. Note that $f^{-1}(\mathcal{H}(\partial \sigma))$ is $\pi_1$-injectively embedded in both $f^{-1}(\mathcal{H}(\sigma))$ and $M \setminus f^{-1}(\text{Int}(\mathcal{H}(\sigma)))$: indeed, if a homotopically nontrivial loop in $f^{-1}(\mathcal{H}(\partial \sigma))$ bounds a disk in $M$, its $f$-image is a homotopically nontrivial loop in $\mathcal{H}(\partial \sigma)$ that bounds a disk in $\mathcal{H}(K)$, contradicting the fact that $\mathcal{H}(\partial \sigma)$ is $\pi_1$-injectively embedded in $\mathcal{H}(K)$. Thus by Van Kampen theorem we get a splitting $A *_C B$ induced by the decomposition of $M$ into $f^{-1}(\mathcal{H}(\sigma))$ and $M \setminus f^{-1}(\text{Int}(\mathcal{H}(\sigma)))$ with $C$ is the fundamental group of $f^{-1}(\mathcal{H}(\partial \sigma))$. The splitting is nontrivial because it is mapped via $\bar{f}$ onto the splitting of $R/\partial R$ along $\mathcal{H}(\partial \sigma)$ that is nontrivial by Proposition 10.1.

As in the proof of Theorem 10.3 we see that $C$ contains an infinite subgroup that is normal in a subgroup $G' \leq \pi_1(M)$ of index $\leq 2$. Because $f_*$ maps $C$ isomorphically onto $\pi_1(\mathcal{H}(\partial \sigma))$, we conclude that $\pi_1(\mathcal{H}(\partial \sigma))$ contains an infinite subgroup that is normal in $f_*(G') \leq \pi_1(R/\partial R)$ that has index $\leq 2$, because $f_*$ is surjective. The proof is now concluded as in Theorem 10.3 by looking at limits sets.

Remark 10.6. In the statement of Theorem 10.5 it should be possible to replace “not homeomorphic” by “not properly homotopy equivalent”. This is done in Theorem 10.9 under the assumption that the building block $\mathcal{X}^n$ of the strict hyperbolization has nonzero $i$th homology group for some $i > 2$.

Example 10.7. By Theorem 10.5, $\text{Int}(R_K)$ is not homeomorphic to a quasiprojective manifold (i.e. the complement $M \setminus S$ where $M$ is compact Kähler and
S is a subvariety with normal crossings, see discussion of quasiprojective manifolds in [JY86, p.292]). It was shown in [Yeu91] that any complete finite volume Kähler manifold with negative Ricci curvature and two sided bounds on sectional curvature is quasiprojective. In particular, any complete finite volume Kähler manifold $V$ of pinched negative curvature is quasiprojective, and hence is not homeomorphic to $\mathrm{Int}(R_K)$.

There is somewhat different way to see that $H_{\mathbb{C}}(K)$ and $R_K$ are not Kähler that is also based on harmonic maps technology, and depends on the assumption that the building block $X^n$ of the strict hyperbolization has nonzero $i$th homology group for some $i \geq 3$. This assumption might be always true if $n \geq 4$ (which is an obvious necessary condition), but I do not know how to prove it. It is almost certain that one should be able to choose $X^n$ with nonzero $i$th homology group for some $i \geq 3$ (e.g. it would suffice to find $X^n$ that contains two closed totally geodesic submanifolds of dimensions $i$, $n-i$ that intersect transversely at one point).

**Theorem 10.8.** Suppose that $X^n$ has nonzero $i$th homology group for some $i \geq 3$. If $K$ is a finite simplicial complex of dimension $n$, then $H(K)$ is not homotopy equivalent to a compact Kähler manifold.

**Proof.** Arguing by contradiction consider a compact Kähler manifold $M$ and a homotopy equivalence $M \to H(K)$. By Lemmas 3.2-3.3, $H(K)$ retracts onto $X^n$, and $X^n$ embeds into a closed hyperbolic $n$-manifold $Q = S(T^n)$. Composing the homotopy equivalence $M \to H(K)$, the retraction $H(K) \to X^n$, and the inclusion $X^n \to Q$, we get a map $r: M \to Q$. By the factorization theorem of Sampson and Carlson-Toledo [ABC+96, Theorem 6.21], $r$ is homotopic to the map $g: M \to Q$ that factors through a circle or a compact 2-manifold. The inclusion $X^n \to Q$ is a $\pi_1$-injective map of aspherical spaces, hence $X^n \to Q$ lifts to a cover $\bar{Q}$ of $Q$ such that the inclusion $X^n \to \bar{Q}$ is a homotopy equivalence. The image of $r$ is $X^n$, so $r$ lifts to $M \to X^n \subset \bar{Q}$, which is a homotopy retraction, and therefore is surjective in the $i$th homology. By assumption the $i$th homology group of $X^n$ is nontrivial, so $M \to X^n \subset \bar{Q}$ is nontrivial on the $i$th homology. On the other hand, by the covering homotopy theorem the map $M \to X^n \subset \bar{Q}$ is homotopic to a lift $\bar{g}: M \to \bar{Q}$ of $g$ that factors through a circle or a compact 2-manifold, and is therefore trivial on the $i$th homology as $i \geq 3$.

**Theorem 10.9.** Suppose that $X^n$ has nonzero $i$th homology group for some $i \geq 3$. If $L$ is a subcomplex of a finite connected simplicial complex $K$ with $\dim(L) < \dim(K) = n$, then $\mathrm{Int}(R_K)$ is not properly homotopy equivalent to an open subset of a compact Kähler manifold.
Proof. Arguing by contradiction assume that $U$ is an open subset of a compact Kähler manifold $M$, and $f: U \to \text{Int}(R_K)$ is a proper homotopy equivalence. Note that $f$ extends to a continuous map $\bar{f}: M \to R_K/R_L$ with $\bar{f}(M \setminus U) = R_L/R_L$, because $f$ is proper and continuous, and $R_K/R_L$ is the one-point-compactification of $\text{Int}(R_K)$. By Lemma 4.3 there is a retraction $r: R_K/R_L \to X_n$ where $X_n$ is a block that does not contain the cone vertex $R_L/R_L$. The composition $r \circ \bar{f}: M \to X_n$ is surjective on homology, because its restriction to $U$ is $\bar{f}$ which is surjective on homology as the composition of a homotopy equivalence $U \to \text{Int}(R_K)$ and the retraction $\text{Int}(R_K) \to X_n$. The proof is then finished as in the proof of Theorem 10.8. □

11. Strict hyperbolization and negatively curved manifolds

In this section we discuss several examples of aspherical manifolds of the form $\mathcal{H}(K)$ and $\text{Int}(R_K)$ that admit no complete Riemannian metrics of pinched negative curvature and finite volume.

Example 11.1. (Davis–Januszkiewicz) The first example of a closed smoothable manifold of the form $\mathcal{H}(K)$ that carries no Riemannian metric of negative curvature was constructed in [DJ91, p.384]. The construction works in each dimension $n \geq 5$, and $K$ is the double suspension of a smooth homology $(n-2)$-sphere with finite nontrivial fundamental group. By Edward’s theorem, $K$ is a topological manifold, yet its obvious triangulation is not PL. It is proved in [DJ91, p.385] that the universal cover of $\mathcal{H}(K)$ is homeomorphic to $\mathbb{R}^n$ yet its ideal boundary as a $\text{CAT}(-1)$ space (or equivalently the ideal boundary of the hyperbolic group $\pi_1(\mathcal{H}(K))$), is not a sphere, hence $\mathcal{H}(K)$ carries no Riemannian metric of negative curvature. (Strictly speaking, the proof in [DJ91] was written for the Gromov’s hyperbolization procedure that in fact does not yield negative curvature, which was not known at the time when [DJ91] was written, but the same proof holds for the strict hyperbolization of [CD95]).

Remark 11.2. This kind of examples are impossible in dimensions 2 and 3.

• Indeed, in dimension 2, the manifolds $\mathcal{H}(K)$ and $R_K$ retract onto $X^2$ that has free fundamental group, hence $\mathcal{H}(K)$ and $R_K$ are surfaces of negative Euler characteristics, so their interiors admit hyperbolic metrics of finite volume.

• In dimension 3 the manifold $\mathcal{H}(K)$ is hyperbolizable by the Thurston’s hyperbolization theorem: indeed, $\mathcal{H}(K)$ is Haken as it contains the incompressible surface $\mathcal{H}(\partial \sigma)$ where $\sigma$ is a 3-simplex of $K$; also $\mathcal{H}(K)$ contains no fake 3-cells because its universal cover is $\mathbb{R}^3$ [DJ91, Theorem 3b.2], and finally, $\mathcal{H}(K)$ is atoroidal since its fundamental group is hyperbolic.
• Assuming the Poincaré Conjecture, one can show that the aspherical (let us say orientable) 3-manifold \( R^K \) is also hyperbolizable, in fact this holds for any 3-dimensional orientable manifold \( M \) satisfying Assumption 1.2. Indeed, by Theorem 1.3, the group \( \pi_1(M) \) is freely indecomposable, so \( M \) is irreducible. Since \( M \) has nonempty boundary, \( M \) is Haken. Since any \( \mathbb{Z} \times \mathbb{Z} \) subgroup of a relatively hyperbolic group must be parabolic, \( M \) is atoroidal. By Section 12, \( M \) is acylindrical. Since \( \pi_1(M) \) is non-elementary relatively hyperbolic, it is not virtually abelian, which excludes various special cases e.g. \( M \) is not an \( I \)-bundle over the torus or the Klein bottle, or that \( M \) is not contractible. Hence by the Thurston’s hyperbolization theorem [Kap01, Theorem 1.43] \( M \) is homeomorphic to the convex core of a geometrically finite Kleinian group whose maximal parabolic subgroups correspond to the boundary tori of \( M \). Furthermore, if \( \partial M \) consists of tori, then \( M \) has the complete hyperbolic metric of finite volume.

Example 11.3. Long-Reid noted in [LR00] that by the Atiyah-Patodi-Singer formula the \( \eta \)-invariant of the cusp cross-section of any 1-cusped hyperbolic 4-manifold must be an integer, while there exists a closed orientable flat 3-manifold that has a nonintegral \( \eta \)-invariant. However, since any flat manifold bounds [HR82], \( F \) bounds some \( R \) obtained via the relative strict hyperbolization, which by [LR00] admits no hyperbolic metric. For related work in complex and quaternion hyperbolic setting see [McR04].

Example 11.4. (Gromov-Thurston examples with large pinching) Consider an \( \mathbb{Z}_m \)-action on \( S^{k+2} := \mathbb{R}^{k+2} \cup \{ \infty \} \) generated by a rotation by \( 2\pi/m \) about \( \mathbb{R}^k \cup \{ \infty \} \). Fix a \( \mathbb{Z}_m \)-invariant triangulation of \( \mathbb{R}^{k+2} \), (and pass to the first barycentric subdivision to ensure that the complex can be folded onto a \((k+2)\)-simplex). The result, denoted by \( K_m \), is a simplicial complex homeomorphic to the \((k+2)\)-sphere and equipped with a simplicial semifree \( \mathbb{Z}_m \)-action whose fixed point set is homeomorphic to the \( k \)-sphere, which we denote by \( S \). Let \( P \) be the subcomplex of \( K_m \) corresponding to a closed \((k+1)\)-dimensional half-plane in \( \mathbb{R}^{k+2} \cup \{ \infty \} \) whose boundary is \( \mathbb{R}^k \cup \{ \infty \} \). An obvious fundamental domain \( F_m \) for the \( \mathbb{Z}_m \)-action is the “lens” bounded by \( P \) and its image under the generator of \( \mathbb{Z}_m \). Since the strict hyperbolization is functorial, we get the induced \( \mathbb{Z}_m \)-action on \( \mathcal{H}(K_m) \) with the fundamental domain \( \mathcal{H}(F_m) \) and the fixed-point-set \( \mathcal{H}(S) \). Note that \( \mathcal{H}(P) \) is a compact manifold whose boundary is \( \mathcal{H}(S) \), and the \( \mathbb{Z}_m \)-images of \( \mathcal{H}(P) \) all intersect in \( \mathcal{H}(S) \). Recall that the strict hyperbolization takes subcomplexes to subcomplexes that are \( \pi_1 \)-injectively embedded. Thus \( \mathcal{H}(K_m) \) contains the amalgamated product of the fundamental groups of the \( \mathbb{Z}_m \)-images of \( \mathcal{H}(P) \) which are amalgamated over \( \pi_1(\mathcal{H}(S)) \). If \( k \geq 2 \), then \( \pi_1(\mathcal{H}(S)) \) is non-elementary hyperbolic, so the
fundamental class of $\mathcal{H}(S)$ is noncuspidal [GT87, p.10], yet it bounds the relative chain given by $P$, so we can apply [GT87, p.11] to deduce the following: for each $n$ there exists a sequence $\{a_m\}$ with $a_m \to \infty$ as $m \to \infty$ such that $\mathcal{H}(K_m)$ is not homotopy equivalent to a complete Riemannian $n$-manifold with $-a_m^2 \leq \sec \leq -1$. Note that we necessarily have $n \geq k+2 \geq 4$, because $\mathcal{H}(K_m)$ is a closed aspherical manifold and $k \geq 2$.

Example 11.5. (Gromov-Thurston examples with no locally symmetric metric) The results in [GT87, p.8–11], used in Example 11.4, depend on some delicate volume estimates, and it is much easier to see (again following [GT87, p.1–2]) that $\mathcal{H}(K_m)$ admits no locally symmetric negatively curved metric for all large $m$. This is what we do in this example: the argument is essentially taken from [GT87] (with a few details added).

Let $h$ be a generator of the group $\mathbb{Z}_m$ that acts by simplicial homeomorphisms on $K_m$ and $\mathcal{H}(K_m)$. Note that $h$ is not homotopic to the identity. (Indeed, $h$ moves some top-dimensional simplex $\sigma$ of $K_m$ with a vertex $v \in S$ to a different simplex $\sigma'$, so $h(\mathcal{H}(\sigma)) = \mathcal{H}(\sigma')$ and $h(v) = v$. If $\alpha$ is a homotopically non-trivial loop in $\mathcal{H}(\sigma)$ based in $v$, then the loops $\alpha$ and $h(\alpha)$ are not homotopic.) Clearly $h$ is orientation-preserving on $K_m$, and hence also on $\mathcal{H}(K_m)$. Suppose $\mathcal{H}(K_m)$ is homotopy equivalent to a closed locally symmetric negatively curved manifold $X/\Gamma$, where $X$ is the corresponding negatively curved symmetric space and $\Gamma$ is a discrete isometry group isomorphic to $\pi_1(\mathcal{H}(K_m),\star)$. Let $\phi$ be the automorphism of $\Gamma$ induced by $h$. By Mostow Rigidity, there exists an isometry $\tilde{i}$ of $X$ such that $\phi(\gamma) = \tilde{i} \circ \gamma \circ \tilde{i}^{-1}$ for all $\gamma \in \pi_1(\mathcal{H}(K_m),\star)$, and $\tilde{i}$ descends to an isometry $i$ of $X/\Gamma$ that becomes homotopic to $h$ when $\mathcal{H}(K_m)$ is identified with $X/\Gamma$. Since $h_*$ is the identity on $\pi_1(\mathcal{H}(S),\star)$, we conclude that $\tilde{i}$ lies in the centralizer of the subgroup $\Gamma_S$ that corresponds $\pi_1(\mathcal{H}(S),\star)$. If $\dim(S) \geq 2$, then $\Gamma_S$ is non-elementary, hence $\tilde{i}$ is a rotation about the totally geodesic subspace $X_S$ of $X$ that is the convex hull of the limit set of $\mathcal{G}_S$.

(In fact, it is easy to see that $\dim(X_S) = k$ even though we do not need this fact here. Indeed, $X_S/\Gamma_S$ is an aspherical manifold homotopy equivalent to a closed aspherical manifold $\mathcal{H}(S)$, so we get that $\dim(X_S) \geq \dim S = k$. By the same reason $\dim(X) = \dim(X/\Gamma) = \dim(\mathcal{H}(K_M)) = k+2$. If $\dim(X_S) = k+2$, then $\tilde{i}$ is the identity, contradicting the fact $h$ is not homotopic to the identity, and if $\dim(X_S) = k+1$, then $\tilde{i}$ is a reflection in $X_S$, so that $i$ reverses the orientation, contradicting the fact $h$ is orientation-preserving.)

Since $h^m$ is the identity, $i^m$ is homotopic to the identity, and hence by Mostow Rigidity, $i^m$ is the identity. We conclude that $\tilde{i}$ is an order $m$ rotation about $X_S$. Therefore, the orbifolds obtained as $\mathbb{Z}_m$-quotients of $X/\Gamma$ are pairwise non-isometric. The volume of any such orbifold is $\text{vol}(X/\Gamma)/m$, and
\[ \text{vol}(X/\Gamma) = C(k)||X/\Gamma|| \]
where \( C(k) \) is a constant depending only on \( k \) \cite{BP92}.

On the other hand, simplicial volume is invariant under homotopy equivalences so \( ||X/\Gamma|| = ||\mathcal{H}(K_m)|| \) which is bounded above by (a constant multiple of) the number of simplices of \( K_m \), which in turn is bounded above by a constant multiple of \( m \). Thus there is an upper bound depending only on the dimension on the volume of the orbifolds obtained as \( \mathbb{Z}_m \)-quotients of \( X/G \). By Wang’s finiteness theorem there can be only finitely many isometry types of such orbifolds, which shows that for all large \( m \), the manifold \( \mathcal{H}(K_m) \) is not homotopy equivalent to a closed locally symmetric manifold of negative curvature.

**Remark 11.6.** Similarly to Example 11.5 one can use relative strict hyperbolization to build examples of manifolds \( R \) such that each component of \( \partial R \) is a flat manifold and \( R \) is not homotopy equivalent to a complete locally symmetric negatively curved manifold of finite volume.

### 12. Spaces of maps and acylindricity

It is well-known that two maps into an aspherical space are homotopic if and only if the induced \( \pi_1 \)-homomorphisms are conjugate. In this section we take a closer look at the case when the aspherical space has relatively hyperbolic fundamental group, and the \( \pi_1 \)-homomorphisms have parabolic images.

A sample application is that if \( M \) be a compact aspherical 3-manifold satisfying Assumption 1.2, then \( M \) is **acylindrical**.

In fact, the argument used to prove acylindricity has nothing to do with dimension three, and more generally, instead of \((M, \partial M)\) we consider an arbitrary CW-pair \((X, Y)\) such that
- \( X \) is aspherical and locally compact,
- each path-component of \( Y \) is aspherical and incompressible in \( X \),
- the group \( \pi_1(X) \) is non-elementary relatively hyperbolic, relative to fundamental groups of path-components of \( Y \).

Given a connected CW-complex \( Z \), and a continuous map \( f: Z \to Y \subset X \), we consider the inclusion \( i: C(Z, Y; f) \to C(Z, X; f) \), where \( C(Z, Y; f) \) denotes the path-component of \( f \) in the space of continuous maps from \( Z \) to \( Y \) with compact-open topology, and similarly for \( C(Z, X; f) \).

**Proposition 12.1.** If \( f: Z \to Y \) is not null-homotopic, then the inclusion \( i: C(Z, Y; f) \to C(Z, X; f) \) is a weak homotopy equivalence.

Recall that \( i \) is a weak homotopy equivalence if and only if for any finite-dimensional CW-pair \((K, J)\) every map \((K, J) \to (C(Z, X; f), C(Z, Y; f))\) is homotopic rel \( J \) to a map with image in \( C(Z, Y; f) \). For example, applying Proposition 12.1 for \( K = [0,1], J = \{0,1\} \), we conclude that two maps \( Z \to Y \)
are homotopic if and only if they are homotopic in \( X \). In particular if \( X \) is a compact 3-manifold with boundary \( Y \), and \( Z = S^1 \), then \( X \) is acylindrical.

**Remark 12.2.** If \( B \) is the component of \( Y \) that contains \( f(Z) \), then of course the inclusion \( C(Z; B; f) \to C(Z; Y; f) \) is a bijection, so that \( C(Z; X; f) \) is actually weakly homotopy equivalent to \( C(Z; B; f) \), unless \( f \) is null-homotopic. The weak homotopy type of \( C(Z; B; f) \) was computed by Gottlieb (see [Han81]), namely \( C(Z; B, h) \) is aspherical (i.e. its universal cover is weakly homotopy equivalent to a point), and its fundamental group is isomorphic to the centralizer of \( f_* \pi_1(Z) \) in \( \pi_1(B) \). Since \( f_* \pi_1(Z) \) is parabolic, the centralizer of \( f_* \pi_1(Z) \) is a parabolic subgroup of \( \pi_1(B) \) that contains the center of \( \pi_1(B) \).

**Proof of Proposition 12.1.** Denote by \( \tilde{Y} \) the preimage of \( Y \) under the universal covering \( p: \tilde{X} \to X \). Since components of \( Y \) are aspherical and incompressible, the components of \( \tilde{Y} \) are contractible. The identification of \( \pi_1(X) \) and the automorphism group of the covering \( p \) can be chosen to induce a one-to-one correspondence between the stabilizers in \( \pi_1(X) \) of components of \( \tilde{Y} \) (or equivalently, the maximal parabolic subgroups of \( \pi_1(X) \)), and the subgroups of \( \pi_1(X) \) conjugate to the fundamental groups of the components of \( Y \).

Fix an arbitrary path-component \( \tilde{B} \) of \( p^{-1}(B) \), and denote by \( P \) the stabilizer of \( \tilde{B} \) in \( \pi_1(X) \). Since \( \pi_1(X) \) is torsion-free and relatively hyperbolic, any two maximal distinct parabolic subgroups have trivial intersection. In particular, no element of \( P \) can stabilize two distinct components of \( \tilde{Y} \). Hence the only non-contractible component of \( \tilde{Y} := \tilde{Y}/P \) is \( \tilde{B} := \tilde{B}/P \), which is projected homeomorphically to \( B \) by the covering projection \( \tilde{p}: \tilde{X} \to X \), where \( \tilde{X} := \tilde{X}/P \). Since \( X \) and \( \tilde{B} \) are aspherical CW-complexes, and the inclusion \( \tilde{B} \to \tilde{X} \) is a \( \pi_1 \)-isomorphism, this inclusion is a homotopy equivalence. Since \( (\tilde{X}, \tilde{B}) \) is a CW-pair, it has the homotopy extension property, so there is a deformation retraction \( F_t: \tilde{X} \to \tilde{B} \). For the rest of the proof by a “map” we always mean a “continuous map”. To see that \( \nu: C(Z, Y; f) \to C(Z, X; f) \) is a weak homotopy equivalence, it suffices to show that every map

\[
H: (D^{k+1}, S^k) \to (C(Z; X; f), C(Z; Y; f))
\]

can be homotoped rel \( S^k \) into \( C(Z; Y; f) \) [Whi78, Lemma II.3.1]. Since \( C(Z; Y; f) \), \( C(Z; X; f) \) are path-connected, we may assume \( k + 1 > 0 \). Fix a basepoint \( * \in S^k \). Since \( X \) is locally-compact, \( H \) gives rise to a map \( h: D^{k+1} \times Z \to X \) that takes \( S^k \times Z \) to \( Y \) and whose restriction to \( * \times Z \) is a map in \( C(Z; Y; f) = C(Z, B; f) \). Since \( D^{k+1} \) is simply-connected, \( h \) lifts to \( \tilde{h}: D^{k+1} \times Z \to \tilde{X} \), whose restriction to \( * \times Z \) is a map in \( C(Z, \tilde{B}; \tilde{f}) \), where \( \tilde{f}: \tilde{Z} \to \tilde{B} \subset \tilde{X} \) be the unique lift of \( f \) to \( \tilde{B} \). If \( k > 0 \), then \( S^k \times Z \) is path-connected, and so \( \tilde{h}(S^k \times Z) \) lies in \( \tilde{B} \). Then \( \tilde{p} \circ F_t \circ \tilde{h} \) is a homotopy rel \( S^k \times Z \) from \( h \) to a map with image in \( B \). If \( k = 0 \), and \( * = 0 \in \{0, 1\} \), then \( \tilde{h}: [0, 1] \times Z \to \tilde{X} \)
is a homotopy between the maps which we denote \( \tilde{h}_0, \tilde{h}_1 \) where \( \tilde{h}_0 \) lies in \( C(Z, B; f) \). The image of \( \tilde{h}_1 \) lies in \( \bar{p}^{-1}(B) \), because it projects to \( B \). If the image of \( \tilde{h}_1 \) were in a contractible component of \( p^{-1}(B) \), then \( \tilde{h}_1 \) would be null-homotopic in \( X \), and so \( f \) would be null-homotopic in \( \bar{B} \) which contradicts the assumption. Thus the image of \( \tilde{h}_1 \) lies in \( \bar{B} \), and again \( p \circ F_i \circ \tilde{h} \) is a homotopy rel \( S^k \times Z \) from \( h \) to a map with image in \( B \). Thus \( H \) is always homotopic rel \( S^k \) to a map with image in \( C(Z, Y; f) \), as promised. \( \square \)

13. Hyperbolic manifolds with geodesic boundary

The following result can be deduced from [Bow, Section 7]. We find it worthwhile to spell out the details.

**Proposition 13.1.** If \( M \) is a compact real hyperbolic manifold with (smooth) totally geodesic boundary, then \( M \) satisfies Assumption 1.2.

*Proof.* The manifold \( M \) is clearly locally convex, so the universal cover \( \tilde{M} \) of \( M \) can be identified with the convex complete codimension zero submanifold of the real hyperbolic \( n \)-space \( \mathbb{H}^n \) (see e.g. Corollary 1.3.7 and Proposition 1.4.2 in [CEG87]). The boundary components of \( M \) are locally convex, hence they are \( \pi_1 \)-injectively embedded and aspherical, because they are totally geodesic and \( \sec(M) \leq 0 \). Then the boundary components of \( \tilde{M} \) are simply-connected totally geodesic codimension one submanifolds of \( H^n \), thus each component of \( \partial \tilde{M} \) is a copy of \( H^{n-1} \). It follows that \( H^n \setminus \tilde{M} \) is the union of disjoint open \( n \)-dimensional halfspaces in \( H^n \). We denote the halfspaces by \( Q(p), p \in \Pi \). Clearly, \( G \) acts on the set \( Q \) of \( Q(p) \)'s, and the stabilizer of each \( Q(p) \) is isomorphic to the fundamental group of the corresponding boundary component of \( M \). Bowditch proved in Proposition 7.12 and Lemma 7.13 of [Bow] that given a quasidense locally finite collection of uniformly convex subsets with bounded penetration in a hyperbolic metric space, the nerve of the collection is a fine hyperbolic graph. (The terms “quasidense”, “locally finite”, and “uniformly convex” are self-explanatory, while “bounded penetration” will be defined below.) Thus it is enough to show that \( Q \) satisfy the the above conditions. Each \( Q(p) \) is convex, hence uniformly convex. Since \( M \) is compact, \( Q \) is quasidense. Local finiteness of \( Q \) easily follows from the fact that \( G \)-action on \( H^n \) is properly discontinuous. By definition \( Q \) has bounded penetration if for any \( r \geq 0 \) there exists \( D(r) \) such that for any distinct \( Q(p), Q(q) \in Q \) the intersection of the \( r \)-neighborhoods of \( Q(p), Q(q) \) has diameter \( \leq D(r) \). Arguing by contradiction assume that \( Q \) does not have bounded penetration. This means that for some \( r \) there exist a sequence \( x_i, y_i \in H^n \) and \( Q(p_i), Q(q_i) \in Q \) such that \( x_i, y_i \) lie in the intersection of the \( r \)-neighborhoods of \( Q(p_i), Q(q_i) \), while the distance \( d(x_i, y_i) \) between \( x_i \) and \( y_i \) is \( > i \). Given \( z \in H^n \), denote by \( z(p) \) the orthogonal projection of \( z \) to \( Q(p) \). The geodesic segments \( [x_i(p_i), y_i(p_i)] \)
and $[x_i(q_i), y_i(q_i)]$ have endpoints within $2r$, while their lengths are $\geq i - 2r$. Because we are in a hyperbolic space the segments become arbitrary close near the middle for sufficiently large $i$, more precisely, their midpoints $m_{p_i}$, $m_{q_i}$ satisfy $d(m_{p_i}, m_{q_i}) \to 0$ as $i \to \infty$, and the same is true for any pair of corresponding points on the segments that are within say $i/4$ of $m_{p_i}$, $m_{q_i}$. Since $M$ is compact, we can assume after passing to a subsequence that $m_{p_i}$ is nearly a constant sequence in $H^n$. Then using that $Q$ is locally finite, we can pass to a subsequence to arrange that $Q(p_i), Q(q_i)$ are constant sequences. Thus $Q(p_i), Q(q_i)$ have to touch along the geodesic that is the pointed Hausdorff limit of (either of) the segments. If two halfspaces in $H^n$ touch along a geodesic, they intersect. So $Q(p_i), Q(q_i)$ intersect, which contradicts the fact that distinct elements of $Q$ are disjoint. □

**Remark 13.2.** An analog of Proposition 13.1 remains true (with a similar proof) for complete finite volume Riemannian manifolds with pinched negative curvature and compact totally geodesic boundary, namely, $\pi_1(M)$ is hyperbolic relative to the fundamental groups of boundary components and cusps of $M$. More generally, it should be possible to prove the following relatively hyperbolic analog of Theorem 7.11 in [Bow]: if $G$ is hyperbolic relative to $\mathcal{P}$, and $\mathcal{G}$ is a conjugacy invariant family of infinite quasiconvex subgroups such that any element of $\mathcal{G}$ equals to its normalizer, and the intersection of any two elements of $\mathcal{G}$ is finite, then $G$ is hyperbolic relative to $\mathcal{P} \cup \mathcal{G}$. However, the focus of this paper is on manifolds, so I will not attempt to proof this more general result here.

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