Abstract

A graph is said to be total-colored if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a total monochromatic path if all the edges and internal vertices on the path have the same color. A total-coloring of a graph is a total monochromatically-connecting coloring (TMC-coloring, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph $G$, the total monochromatic connection number, denoted by $tmc(G)$, is defined as the maximum number of colors used in a TMC-coloring of $G$. These concepts are inspired by the concepts of monochromatic connection number $mc(G)$, monochromatic vertex connection number $mvc(G)$ and total rainbow connection number $trc(G)$ of a connected graph $G$. Let $l(T)$ denote the number of leaves of a tree $T$, and let $l(G) = \max\{l(T) \mid T \text{ is a spanning tree of } G \}$ for a connected graph $G$. In this paper, we show that there are many graphs $G$ such that $tmc(G) = m - n + 2 + l(G)$, and moreover, we prove that for almost all graphs $G$, $tmc(G) = m - n + 2 + l(G)$ holds. Furthermore, we compare $tmc(G)$ with $mvc(G)$ and $mc(G)$, respectively, and obtain that there exist graphs $G$ such that $tmc(G)$ is not less than $mvc(G)$ and vice versa, and that $tmc(G) = mc(G) + l(G)$ holds for almost all graphs. Finally, we prove that $tmc(G) \leq mc(G) + mvc(G)$, and the equality holds if and only if $G$ is a complete graph.

Keywords: total-colored graph, total monochromatic connection, spanning tree with maximum number of leaves

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1 Introduction

In this paper, all graphs are simple, finite and undirected. We refer to the book [3] for undefined notation and terminology in graph theory. Throughout this paper, let $n$ and $m$ denote the order (number of vertices) and size (number of edges) of a graph, respectively. Moreover, a vertex of a connected graph is called a leaf if its degree is one; otherwise, it is an internal vertex. Let $l(T)$ and $q(T)$ denote the number of leaves and the number of internal vertices of a tree $T$, respectively, and let $l(G) = \max\{l(T)\mid T \text{ is a spanning tree of } G\}$ and $q(G) = \min\{q(T)\mid T \text{ is a spanning tree of } G\}$ for a connected graph $G$. Note that the sum of $l(G)$ and $q(G)$ is $n$ for any connected graph $G$ of order $n$. A path in an edge-colored graph is a monochromatic path if all the edges on the path have the same color. An edge-coloring of a connected graph is a monochromatically-connecting coloring (MC-coloring, for short) if any two vertices of the graph are connected by a monochromatic path of the graph. For a connected graph $G$, the monochromatic connection number of $G$, denoted by $mc(G)$, is defined as the maximum number of colors used in an MC-coloring of $G$. An extremal MC-coloring is an MC-coloring that uses $mc(G)$ colors. Note that $mc(G) = m$ if and only if $G$ is a complete graph. The concept of $mc(G)$ was first introduced by Caro and Yuster [6] and has been well-studied recently. We refer the reader to [4, 8] for more details.

As a natural counterpart of the concept of monochromatic connection, Cai et al. [5] introduced the concept of monochromatic vertex connection. A path in a vertex-colored graph is a vertex-monochromatic path if its internal vertices have the same color. A vertex-coloring of a graph is a monochromatically-vertex-connecting coloring (MVC-coloring, for short) if any two vertices of the graph are connected by a vertex-monochromatic path of the graph. For a connected graph $G$, the monochromatic vertex connection number, denoted by $mvc(G)$, is defined as the maximum number of colors used in an MVC-coloring of $G$. An extremal MVC-coloring is an MVC-coloring that uses $mvc(G)$ colors. Note that $mvc(G) = n$ if and only if $\text{diam}(G) \leq 2$.

Actually, the concepts of monochromatic connection number $mc(G)$ and monochromatic vertex connection number $mvc(G)$ are natural opposite concepts of rainbow connection number $rc(G)$ and rainbow vertex connection number $rvc(G)$. For details about them we refer to a book [10] and a survey paper [9]. Moreover, the concept of total rainbow connection number $trc(G)$ in [12] was motivated by the rainbow connection number $rc(G)$ and rainbow vertex connection number $rvc(G)$. Thus, here we introduce the concept of total monochromatic connection of graphs. A graph is said to be total-colored if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a total monochromatic path if all the edges and internal vertices on the path have the
same color. A total-coloring of a graph is a total monochromatically-connecting coloring (TMC-coloring, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph $G$, the total monochromatic connection number, denoted by $\text{tmc}(G)$, is defined as the maximum number of colors used in a TMC-coloring of $G$. An extremal TMC-coloring is a TMC-coloring that uses $\text{tmc}(G)$ colors. It is easy to check that $\text{tmc}(G) = m + n$ if and only if $G$ is a complete graph.

The rest of this paper is organized as follows: In Section 2, we prove that $\text{tmc}(G) \geq m - n + 2 + l(G)$ for any connected graph and determine the value of $\text{tmc}(G)$ for some special graphs. In Section 3, we prove that there are many graphs with $\text{tmc}(G) = m - n + 2 + l(G)$ which are restricted by other graph parameters such as the maximum degree, the diameter and so on, and moreover, we show that for almost all graphs $G$, $\text{tmc}(G) = m - n + 2 + l(G)$ holds. In Section 4, we compare $\text{tmc}(G)$ with $\text{mvc}(G)$ and $\text{mc}(G)$, respectively, and obtain that there exist graphs $G$ such that $\text{tmc}(G)$ is not less than $\text{mvc}(G)$ and vice versa, and that $\text{tmc}(G) = \text{mc}(G) + l(G)$ for almost all graphs. Moreover, we prove that $\text{tmc}(G) \leq \text{mc}(G) + \text{mvc}(G)$, and the equality holds if and only if $G$ is a complete graph.

2 Preliminary results

In this section, we show that $\text{tmc}(G) \geq m - n + 2 + l(G)$ and present some preliminary results on the total monochromatic connection number. Moreover, we determine the value of $\text{tmc}(G)$ when $G$ is a tree, a wheel, and a complete multipartite graph. The following fact is easily seen.

**Proposition 1.** If $G$ is a connected graph and $H$ is a connected spanning subgraph of $G$, then $\text{tmc}(G) \geq e(G) - e(H) + \text{tmc}(H)$.

Since for any two vertices of a tree, there exists only one path connecting them, we have the following result.

**Proposition 2.** If $T$ is a tree, then $\text{tmc}(T) = l(T) + 1$.

The consequence below is immediate from Propositions 1 and 2.

**Theorem 1.** For a connected graph $G$, $\text{tmc}(G) \geq m - n + 2 + l(G)$.

Let $G$ be a connected graph and $f$ be an extremal TMC-coloring of $G$ that uses a given color $c$. Note that the subgraph $H$ formed by the edges and vertices colored $c$ is connected, or we will give a fresh color to all the edges and vertices colored $c$ in some of these components while still maintaining a TMC-coloring. Moreover, the color of each
internal vertex of $H$ is $c$. Otherwise, let $u_1, \ldots, u_t$ be the internal vertices of $H$ such that each of them is not colored $c$. We obtain the subgraph $H_0$ by deleting the vertices $\{u_1, \ldots, u_t\}$. If $H_0$ is connected, it is possible to choose an edge incident with $u_1$ and assign it with a fresh color while still maintaining a TMC-coloring. If not, we can give a fresh color to all the edges and vertices colored $c$ in some of these components while still maintaining a TMC-coloring. Furthermore, $H$ does not contain any cycle; otherwise, a fresh color can be assigned to any edge of the cycle while still maintaining a TMC-coloring. Thus, $H$ is a tree where the color of each internal vertex is $c$. Now we define the color tree as the tree formed by the edges and vertices colored $c$, denoted by $T_c$. If $T_c$ has at least two edges, the color $c$ is called nontrivial. Otherwise, $c$ is trivial. We call an extremal TMC-coloring simple if for any two nontrivial colors $c$ and $d$, the corresponding trees $T_c$ and $T_d$ intersect in at most one vertex. If $f$ is simple, then the leaves of $T_c$ must have distinct colors different from color $c$. Otherwise, we can give a fresh color to such a leaf while still maintaining a TMC-coloring. Moreover, a nontrivial color tree of $f$ with $m'$ edges and $q'$ internal vertices is said to waste $m' - 1 + q'$ colors. For the rest of this paper we will use these facts without further mentioning them.

The lemma below shows that one can always find a simple extremal TMC-coloring for a connected graph.

**Lemma 1.** Every connected graph $G$ has a simple extremal TMC-coloring.

**Proof.** We are given an extremal TMC-coloring $f$ of $G$ with the most number of trivial colors, and then we prove that this coloring must be simple. Suppose that there exist two nontrivial colors $c$ and $d$ such that $T_c$ and $T_d$ contain $k$ common vertices denoted by $u_1, u_2, \ldots, u_k$, where $k \geq 2$. Now we divide our discussion into two cases.

**Case 1.** For $1 \leq i \leq k$, $u_i$ is an internal vertex of $T_c$ or $T_d$.

For $1 \leq i \leq k$, if $u_i$ is an internal vertex of $T_c$, $u_i$ must be a leaf of $T_d$ and then set $e_i = u_iw_i$ where $w_i$ is the neighbor of $u_i$ in $T_d$; otherwise, $u_i$ must be a leaf of $T_c$ and then put $e_i = u_iv_i$ where $v_i$ is the neighbor of $u_i$ in $T_c$. Let $H$ denote the subgraph consisting of the edges and vertices of $T_c \cup T_d$. Clearly, $H$ is connected. We obtain a spanning tree $H_0$ of $H$ by deleting the edges $\{e_2, e_3, \ldots, e_k\}$. Now we change the total-coloring of $H$ while still maintaining the colors of the leaves in $H_0$ unchanged. Assign the edges and internal vertices of $H_0$ with color $c$ and the remaining edges $\{e_2, e_3, \ldots, e_k\}$ with distinct new colors. Obviously, the new total-coloring is also a TMC-coloring and uses $k - 2$ more colors than our original one. So, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on $f$.

**Case 2.** There exists a vertex among $u_1, \ldots, u_k$, say $u_1$, which is a leaf of both $T_c$ and $T_d$. 

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Let \( v_1 \) and \( w_1 \) be the neighbors of \( u_1 \) in \( T_e \) and \( T_d \), respectively. There must be another color tree \( T_e \) (including a single edge) connecting \( v_1 \) and \( w_1 \). For \( 1 \leq i \leq k \), if \( u_i \) is a leaf of \( T_e \), then set \( e_i = u_iv_i \) where \( v_i \) is the neighbor of \( u_i \) in \( T_e \); otherwise, \( u_i \) must be a leaf of \( T_d \) and then put \( e_i = u_iw_i \) where \( w_i \) is the neighbor of \( u_i \) in \( T_d \). Let \( H_1 \) denote the subgraph consisting of the edges and vertices of \( T_e \cup T_d \). We obtain a spanning subgraph \( H_2 \) of \( H_1 \) by deleting the edges \( \{e_1, e_2, \ldots, e_k\} \). If \( T_e \) and \( H_2 \) do not have common leaves, let \( E_0 = \{e_1, e_2, \ldots, e_k\} \). Otherwise, let \( u'_1, \ldots, u'_t \) denote the common leaves of \( T_e \) and \( H_2 \). Set \( e'_i = u'_iv'_i \) where \( v'_i \) is the neighbor of \( u'_i \) in \( T_e \) for \( 1 \leq i \leq t \). And then let \( E_0 = \{e_1, \ldots, e_k, e'_1, \ldots, e'_t\} \). Let \( H \) denote the subgraph consisting of the edges and vertices of \( T_e \cup T_d \cup T_e \). Clearly, \( H \) is connected. We obtain a spanning connected subgraph \( H_0 \) of \( H \) by deleting the edges of \( E_0 \). Now we change the total-coloring of \( H \) while still maintaining the colors of the leaves in \( H_0 \) unchanged. Assign the edges and internal vertices of \( H_0 \) with color \( c \) and the remaining edges of \( H \) (i.e., the edges of \( E_0 \)) with distinct new colors. Note that if \( v \) is a common leaf of either \( T_e \) and \( T_d \) or \( T_e \) and \( H_2 \), it is also a leaf of \( H_0 \). Obviously, the new total-coloring is also a TMC-coloring and uses at least \( k + t - 2 \) more colors than our original one. So, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on \( f \).

Now we use the above results to compute the total monochromatic connection numbers of wheel graphs and complete multipartite graphs.

**Example 1.** Let \( G \) be a wheel \( W_{n-1} \) of order \( n \geq 5 \). Then \( tmc(G) = m - n + 2 + l(G) \).

**Proof.** We are given a simple extremal TMC-coloring \( f \) of \( G \). Note that \( m - n + 2 + l(G) = m + 1 \) and \( tmc(G) \geq m + 1 \) by Theorem \( \numref{thm:trivial} \). Suppose that \( f \) consists of \( k \) nontrivial color trees, denoted by \( T_1, \ldots, T_k \). In fact, we can always find two vertices with degree at least 4 if \( k \geq 3 \), a contradiction. Likewise, if \( k = 2 \), \( G \) must be \( W_4 \) and \( tmc(W_4) = m + 1 \). Thus, assume that \( k = 1 \) and \( T_1 \) is not spanning \( \) (Otherwise, \( tmc(G) = n - n + 2 + l(G) \)). Note that for every vertex \( v \notin T_1 \), there exist the total monochromatic paths connecting \( v \) and the \(|T_1| \) vertices of \( T_1 \). As \( f \) is simple, these paths are internally vertex-disjoint. Hence, \( deg(v) \geq |T_1| \). If \( |T_1| \geq 4 \), the \( n - 1 \) vertices with degree 3 of \( G \) must be in \( T_1 \) and then \( T_1 \) is a path. Thus, \( tmc(G) = m + n - (n - 3) - (n - 3) = m + 6 - n \leq m + 1 \). If \( |T_1| = 3 \), then \( G \) must be \( W_5 \) while \( n \geq 5 \). Therefore, the proof is complete.

**Example 2.** Let \( G = K_{n_1, \ldots, n_r} \) be a complete multipartite graph with \( n_1 \geq \ldots \geq n_t \geq 2 \) and \( n_{t+1} = \ldots = n_r = 1 \). Then \( tmc(G) = m + r - t \).

**Proof.** The case that \( r = 2 \) is a special case of Theorem \( \numref{thm:multipartite} \) whose proof is given in Section 3, so assume that \( r \geq 3 \). Let \( f \) be a simple extremal TMC-coloring of \( G \) with maximum
trivial colors. Suppose that \( f \) consists of \( k \) nontrivial color trees, denoted by \( T_1, \ldots, T_k \), where \( t_i = |V(T_i)| \) and \( q_i = q(T_i) \) for \( 1 \leq i \leq k \). Now we divide our discussion into two cases.

**Case 1.** \( t = r \).

In this case, every vertex appears in at least one of the nontrivial color trees. Note that \( m - n + 2 + l(G) = m \) and \( tmc(G) \geq m \) by Theorem 1. If \( \sum_{i=1}^{k} (t_i - 1) \geq n \), then we have that \( tmc(G) \leq m + n - n - \sum_{i=1}^{k} q_i + k = m - \sum_{i=1}^{k} q_i + k \leq m \). Thus, \( tmc(G) = m \).

Suppose that \( \sum_{i=1}^{k} (t_i - 1) \leq n - 1 \). Now consider the subgraph \( G' \) consisting of the union of the \( T_i \) and let \( C_1, \ldots, C_s \) denote its components.

Now we may assume that there exists a component, say \( C_1 \), such that each nontrivial color tree in \( C_1 \) is a star. Let \( S \) be a star of \( C_1 \) with center \( u \) and leaves \( u_1, \ldots, u_p \), where \( u_1, \ldots, u_p \) are in the same vertex class, say \( V_1 \). Suppose that \( p' \geq 2 \). Indeed, if \( p' = 1 \), we can give a new color to the edge \( uu_1 \) while still maintaining a TMC-coloring. We claim that \( C_1 \) contains a cycle. If \( p' < |V_1| \), there exists a vertex \( u_{p+1} \) of \( V_1 \) not adjacent to \( u \) in \( S \). Then \( u_1 \) and \( u_{p+1} \) must be in a same nontrivial color tree and the same happens for \( u_p \) and \( u_{p+1} \). These nontrivial color trees containing \( u_1, u_p \) and \( u_{p+1} \) must form a cycle. If \( p' = |V_1| \), we have that the vertices of the vertex class containing \( u \) must be in a same nontrivial color tree, or we will get a cycle in a similar way. By that analogy, we obtain a cycle formed by some centers of the nontrivial color trees in \( C_1 \). Now we change the total-coloring of \( C_1 \). We obtain a spanning tree \( T' \) of \( C_1 \) by connecting \( u_1 \) to the vertices in the same class with \( u \) and \( u \) to the other vertices of \( C_1 \). We color the edges and internal vertices of \( T' \) with the same color and all other edges and vertices with distinct new colors. Clearly, this new total-coloring is also a TMC-coloring. However, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on \( f \).

Thus, suppose that there exists a nontrivial color tree of \( C_i \), say \( T_{i1} \), having two adjacent internal vertices \( u_i \) and \( v_i \) for \( 1 \leq i \leq s \). We obtain a spanning tree \( T \) by connecting \( v_1 \) to each vertex in the same class with \( u_1 \) of \( G \) and \( u_1 \) to the other vertices of \( G \). Now we give a new total-coloring \( f' \) of \( G \). Color the edges and internal vertices of \( T \) with the same color and all other edges and vertices of \( G \) with distinct new colors. Obviously, \( f' \) is still a TMC-coloring. If \( s \geq 2 \), then it either uses more colors or uses the same number of colors but more trivial colors than \( f \), a contradiction. Thus, \( s = 1 \). Moreover, we can check that \( f' \) is a simple extremal TMC-coloring with maximum trivial colors. Therefore, \( tmc(G) = m \).

**Case 2.** \( t < r \).

We obtain a star \( S \) by connecting a vertex of \( \bigcup_{i=t+1}^{r} V_i \) to each vertex of \( \bigcup_{i=1}^{r} V_i \). Color
the edges and the center vertex of $S$ with the same color and all other edges and vertices of $G$ with distinct new colors. Clearly, this new total-coloring is still a TMC-coloring, denoted by $f'$. Thus, $tmc(G) \geq m + r - t$. If $\sum_{i=1}^{k}(t_{i} - 1) \geq n - r + t$, then we have that $tmc(G) \leq m + n - (n - r + t) - \sum_{i=1}^{k}q_{i} + k = m + r - t - \sum_{i=1}^{k}q_{i} + k \leq m + r - t$. Hence, $tmc(G) = m + r - t$. Suppose that $\sum_{i=1}^{k}(t_{i} - 1) \leq n - r + t - 1$. Next consider the subgraph $G'$ consisting of the union of the $T_{i}$'s and suppose that it has $s$ components, say $C_{1}, \ldots, C_{s}$. Note that $|V(G')| \geq n - r + t$ since any two vertices of the same class must be covered in a nontrivial color tree. The case that $|V(G')| = n - r + t$ can be verified by a similar discussion to Case 1. Thus, suppose that $|V(G')| > n - r + t$. It is obvious that $s \geq 2$. Moreover, there must exist a vertex $x$ of $\cup_{i=t+1}^{r}V_{i}$, which is contained in a component of $G'$, say $C_{1}$. For $2 \leq j \leq s$, there does not exist a vertex of $\cup_{i=t+1}^{r}V_{i}$ in $C_{j}$. Otherwise, let $x$ be the center of $S$ and then $f'$ either uses more colors or uses the same number of colors but more trivial colors than $f$, a contradiction. By a similar discussion to Case 1, we can obtain that there exists a nontrivial color tree of $C_{j}$ having two adjacent internal vertices for $2 \leq j \leq s$. We obtain a star $S_{1}$ by joining the vertices of $\cup_{i=2}^{r}C_{i}$ to one internal vertex of $C_{1}$. We give a new total-coloring of $G$ while still maintaining the total-coloring of $C_{1}$ unchanged. Assign the edges and the center vertex of $S_{1}$ with one color and the other edges and vertices of $G \setminus C_{1}$ with distinct new colors. This new total-coloring is still a TMC-coloring and it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on $f$. Therefore, we have finished the proof.

\[\Box\]

3 Graphs with $tmc(G) = m - n + 2 + l(G)$

In this section, we prove that there are many graphs $G$ for which $tmc(G) = m - n + 2 + l(G)$, even for almost all graphs.

Lemma 2. \[\Box\] Let $G$ be a connected graph of order $n > 3$. If $G$ satisfies any of the following properties, then $mc(G) = m - n + 2$.

(a) The complement $\overline{G}$ of $G$ is 4-connected.
(b) $G$ is $K_{3}$-free.
(c) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. In particular, this holds if $\Delta(G) \leq (n + 1)/2$, and this also holds if $\Delta(G) \leq n - 2m/n$.
(d) $diam(G) \geq 3$.
(e) $G$ has a cut vertex.

We can obtain that $tmc(G) \leq mc(G) + l(G)$ for a noncomplete graph, whose proof is
contained in the proof of Theorem 3 in Section 4. In addition with Theorem 1 and Lemma 2, we have the following results.

**Theorem 2.** Let \( G \) be a connected graph of order \( n > 3 \). If \( G \) satisfies any of the following properties, then \( \text{tmc}(G) = m - n + 2 + l(G) \).

(a) The complement \( \overline{G} \) of \( G \) is 4-connected.
(b) \( G \) is \( K_3 \)-free.
(c) \( \Delta(G) < n - \frac{2m - 3(n-1)}{n-3} \).
(d) \( \text{diam}(G) \geq 3 \).
(c) \( G \) has a cut vertex.

One cannot hope to strengthen Theorem 2(c) by improving the upper bound of \( \Delta(G) \). In fact, let \( G = K_{n-2,1,1} \). Then we have that \( \text{tmc}(G) = m - n + 3 + l(G) \) while the maximum degree is \( n - 1 = n - \frac{2m - 3(n-1)}{n-3} \).

From Theorem 2(a), we can get a stronger result. For a property \( P \) of graphs and a positive integer \( n \), define \( \text{Prob}(P,n) \) to be the ratio of the number of graphs with \( n \) labeled vertices having \( P \) over the total number of graphs with these vertices. If \( \text{Prob}(P,n) \) approaches 1 as \( n \) tends to infinity, then we say that almost all graphs have the property \( P \). See [1] for example.

**Theorem 3.** For almost all graphs \( G \), we have that \( \text{tmc}(G) = m - n + 2 + l(G) \).

In order to prove Theorem 3, we need the following lemma.

**Lemma 3.** [1] For every nonnegative integer \( k \), almost all graphs are \( k \)-connected.

**Proof of Theorem 3:** For any given nonnegative integer \( n \), let \( \mathcal{G}_n \) denote the set of all graphs of order \( n \), and let \( \mathcal{G}_4^n \) denote the set of all 4-connected graphs of order \( n \). Moreover, let \( \mathcal{B}_n \) denote the set of all graphs \( G \) of order \( n \) such that the complement \( \overline{G} \) of \( G \) is 4-connected. Note that for any two graphs \( G \) and \( H \), \( G \cong H \) if and only if \( \overline{G} \cong \overline{H} \). Then, it is easy to check that the map: \( G \to \overline{G} \) is a bijection from \( \mathcal{B}_n \) to \( \mathcal{G}_4^n \). Therefore, we have

\[
\frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} = \frac{|\mathcal{G}_4^n|}{|\mathcal{G}_n|}.
\]

By Lemma 3 it follows that almost all graphs are 4-connected. Then, we get that almost all graphs have 4-connected complements. Furthermore, since almost all graphs are connected, we have that \( \text{tmc}(G) = m - n + 2 + l(G) \) by Theorem 2(a).

**Remark 1.** For the monochromatic connection number \( \text{mc}(G) \), from Lemma 2(a) and Lemma 3, one can deduce, in a similar way, that for almost all graphs \( G \), \( \text{mc}(G) = m-n+2 \) holds.
Remark 2. To use the parameter \( l(G) \) in the above formulas looks good. However, from [7, p.206] we know that it is NP-hard to find a spanning tree that has maximum number of leaves in a connected graph \( G \).

4 Compare \( tmc(G) \) with \( mvc(G) \) and \( mc(G) \)

Let \( G \) be a nontrivial connected graph. Firstly, we compare \( tmc(G) \) with \( mvc(G) \). The question we may ask is, can we bound one of \( tmc(G) \) and \( mvc(G) \) in terms of the other? The following two theorems give sufficient conditions for \( tmc(G) > mvc(G) \).

**Theorem 4.** Let \( G \) be a connected graph with diameter \( d \). If \( m \geq 2n - d - 2 \), then \( tmc(G) > mvc(G) \).

*Proof.* The case that \( d = 1 \) is trivial, so assume that \( d \geq 2 \). We can check that if \( l(G) = 2 \), then \( tmc(G) > mvc(G) \). Thus, suppose that \( l(G) \geq 3 \). By Theorem 1 it follows that \( tmc(G) \geq m - n + 2 + l(G) \geq 2n - d - 2 - n + 2 + 3 = n - d + 3 \). Moreover, we have that \( mvc(G) \leq n - d + 2 \) by [5, Proposition 2.3]. Therefore, \( tmc(G) > mvc(G) \). \( \Box \)

**Theorem 5.** Let \( G \) be a connected graph of diameter 2 with maximum degree \( \Delta \). If \( \Delta \geq \frac{n+1}{2} \), then \( tmc(G) > mvc(G) \).

Before proving Theorem 5, we need the lemma below.

**Lemma 4.** [2] Let \( G \) be a connected graph of diameter 2 with maximum degree \( \Delta \). Then

\[
m \geq \begin{cases} 
  n + \Delta - 2, & \text{if } \Delta = n - 2 \text{ or } n - 3 \\
  2n - 5, & \text{if } \Delta = n - 4 \\
  2n - 4, & \text{if } \frac{2n-2}{3} \leq \Delta \leq n - 5 \\
  3n - \Delta - 6, & \text{if } \frac{3n-3}{9} \leq \Delta < \frac{2n-2}{3} \\
  5n - 4\Delta - 10, & \text{if } \frac{5n-3}{9} \leq \Delta < \frac{3n-3}{5} \\
  4n - 2\Delta - 11, & \text{if } \frac{n+1}{2} \leq \Delta < \frac{5n-3}{9} 
\end{cases}
\]  

(1)

*Proof of Theorem 5:* The case that \( n \leq 7 \) can be easily verified. Suppose that \( n \geq 8 \).

Since the diameter of \( G \) is 2, we have that \( mvc(G) = n \). By Theorem 4 and Lemma 4, \( tmc(G) \geq m - n + 2 + l(G) > n \). Thus, \( tmc(G) > mvc(G) \). \( \Box \)

Actually, we have that \( tmc(C_5) = 4 < mvc(C_5) = 5 \), where \( m < 2n - d - 2 \) and \( \Delta < \frac{n+1}{2} \). This implies that the conditions of Theorems 4 and 5 cannot be improved. Moreover, if \( G \) is a star, then \( tmc(G) = mvc(G) = n \). Therefore, there exist graphs \( G \) such that \( tmc(G) \) is not less than \( mvc(G) \) and vice versa. However, we cannot show whether there exist other graphs with \( tmc(G) \leq mvc(G) \). Thus, we propose the following problem.
Problem 1. Dose there exist a graph of order \( n \geq 6 \) except a star such that \( \text{tmc}(G) \leq \text{mvc}(G) \)?

Next we compare \( \text{tmc}(G) \) with \( \text{mc}(G) \). If \( G \) satisfies one of the conditions in Theorem 2 then we have \( \text{mc}(G) = m - n + 2 \) and so \( \text{tmc}(G) = \text{mc}(G) + l(G) \). For a complete graph \( G \), \( \text{tmc}(G) > \text{mc}(G) + l(G) \). From [5, Corollary 13], if \( G \) is a wheel \( W_{n-1} \) of order \( n \geq 5 \), we have that \( \text{mc}(G) = m - n + 3 \) and then \( \text{tmc}(G) < \text{mc}(G) + l(G) \). However, by Theorem 3 and Remark 1, it follows that almost all graphs have that \( \text{tmc}(G) > \text{mc}(G) \), which implies that almost all graphs have that \( \text{tmc}(G) > \text{mc}(G) \). Thus, we propose the following conjecture.

Conjecture 1. For a connected graph \( G \), it always holds that \( \text{tmc}(G) > \text{mc}(G) \).

Finally, we compare \( \text{tmc}(G) \) with \( \text{mc}(G) + \text{mvc}(G) \).

Theorem 6. Let \( G \) be a connected graph. Then \( \text{tmc}(G) \leq \text{mc}(G) + \text{mvc}(G) \), and the equality holds if and only if \( G \) is a complete graph.

In order to prove Theorem 6 we need the following lemma.

Lemma 5. For a noncomplete connected graph \( G \), let \( f \) be a simple extremal TMC-coloring of \( G \) and \( T_1, \ldots, T_k \) denote all the nontrivial color trees of \( f \), where \( t_i = |V(T_i)| \) and \( q_i = q(T_i) \) for \( 1 \leq i \leq k \). Then, \( \sum_{i=1}^{k} q_i \geq q(G) \).

Proof. For any \( v \in G \), if \( v \notin \bigcup_{i=1}^{k} T_i \), \( v \) must be adjacent to an internal vertex \( w_0 \) of a nontrivial color tree and then set \( E_v = \{vw \suchthat w \in N(v) \setminus \{w_0\}\} \). If \( v \) is an internal vertex of a nontrivial color tree containing \( v \), set \( E_v = \emptyset \). Otherwise, \( v \) is a leaf of any nontrivial color tree containing \( v \). Let \( T_1, \ldots, T_s \) denote the nontrivial color trees containing \( v \) and \( v_1, \ldots, v_s \) be the neighbors of \( v \) in \( T_1, \ldots, T_s \), respectively. Let \( E_v = \{v_1v_2, \ldots, vv_s\} \). We obtain a spanning subgraph \( G' \) by deleting the edges of \( \bigcup_{v \in G} E_v \). Note that every vertex of \( \{v : E_v = \emptyset\} \) is connected to each other. For any two vertices \( u_1 \) and \( u_2 \) of \( \{v : E_v = \emptyset\} \), there exists a total monochromatic path \( P \) of \( G \) connecting them. For each vertex \( u \) of \( P \), we have \( E_u = \emptyset \). Thus, \( G' \) also contains \( P \) from \( u_1 \) to \( u_2 \). Moreover, every vertex of \( \{v : E_v \neq \emptyset\} \) is connected to a vertex of \( \{v : E_v = \emptyset\} \). Hence, \( G' \) is connected and each vertex of \( \{v : E_v \neq \emptyset\} \) cannot be an internal vertex of \( G' \). Then \( \sum_{i=1}^{k} q_i \geq q(G') \geq q(G) \). \( \Box \)

Now, we are ready to prove Theorem 6.

Proof of Theorem 6: If \( G \) is a complete graph, we have that \( \text{tmc}(G) = \text{mc}(G) + \text{mvc}(G) \). Thus, suppose that \( G \) is not complete. We are given a simple extremal TMC-coloring \( f \) of \( G \). Suppose that \( f \) consists of \( k \) nontrivial color trees denoted by \( T_1, \ldots, T_k \), where
$t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Then $tmc(G) = m + n - \sum_{i=1}^{k} (t_i - 2) - \sum_{i=1}^{k} q_i$.

Now we take a copy $G'$ of $G$. Then $G'$ contains the trees $T_1', \ldots, T_k'$ corresponding to $T_1, \ldots, T_k$, respectively. Define an edge-coloring $f_e$ of $G'$ as follows: color the edges of $T_i$ with color $i$ for $1 \leq i \leq k$ and the other edges of $G'$ with distinct new colors. Then $f_e$ is an MC-coloring of $G'$ with $m - \sum_{i=1}^{k} (t_i - 2)$ colors. Thus, $mc(G) = mc(G') \geq m - \sum_{i=1}^{k} (t_i - 2) = tmc(G) - n + \sum_{i=1}^{k} q_i$. By Lemma 5 we have that $\sum_{i=1}^{k} q_i \geq q(G)$. Then $tmc(G) \leq mc(G) + n - q(G) = mc(G) + l(G)$. Moreover, it is easy to obtain that $mvc(G) \geq l(G) + 1$. Hence, $tmc(G) < mc(G) + mvc(G)$. Therefore, the proof is complete.

**Remark 3.** For the total rainbow connection number $trc(G)$, we cannot bound one of $trc(G)$ and $rc(G) + rvc(G)$ in terms of the other. For a connected graph $G$, $trc(G) = rc(G) + rvc(G)$ if $G$ is a complete graph or a star. Moreover, if $G$ is a complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 6$, then $trc(G) = 7 > rc(G) + rvc(G) = 4 + 1$ [9, 11, 12]. In [12], for every $s \geq 1481$, there exists a graph $G$ with $trc(G) = rvc(G) = s$ which implies that $trc(G) < rc(G) + rvc(G)$. This is one thing that the total monochromatic connection differs from the total rainbow connection.

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