Measurement-device-independent measure of steerability and witnesses for all steerable resources

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The fact that nonlocality implies steering enables one to certify steerability by using a Bell inequality violation. Such a certification is device-independent (DI), i.e., one makes no assumption neither on the underlying state nor on the measurements. However, not all steerable states can violate a Bell inequality. Here, we systematically construct a collection of witnesses for steerable resources, defined by assemblages, in a measurement-device-independent (MDI) scenario. The inputs driving the measurement are replaced by a set of tomographically complete quantum states, and neither the detectors nor the underlying state is characterized. We show that all steerable assemblages can be detected by properly chosen witnesses. Furthermore, we introduce the first measure of steerability in an MDI scenario and show that such a measure is a standard one, i.e., a steering monotone, by proving that it is equivalent to the steering robustness.

Introduction.—Entanglement [1], steerability [2], and Bell nonlocality [3] are three types of quantum correlations which play essential roles in quantum cryptography, quantum communication, teleportation, and quantum information processing [4, 5]. The fact that steering is treated as an intermediate quantum correlation between entanglement and nonlocality leads to a hierarchical relation among them. That is, all nonlocal states are steerable, and all steerable states are entangled, but not vice versa [8–10]. During the past decade, there have been many significant experimental works [11–15] and various theoretical results on steering [16–23], including the corresponding characterization of measurement incompatibility [24–28], one-way steering [17, 29–31], measures of EPR steering [6, 32–34], continuous-variable EPR steering [35, 36], as well as temporal steering [40–42].

Bell nonlocality enables one to perform the so-called device-independent (DI) quantum information processing [3, 43, 50], i.e., one makes no assumption neither on the underlying quantum state nor on the quantum measurements performed. From the hierarchical relation [3], it naturally leads to the fact that a Bell inequality can be treated as a DI entanglement witness. Nevertheless, not all entangled states can be detected by using a Bell inequality violation. Recently, based on Buscemi’s semi-quantum nonlocal games [51], Branciard et al. [52] proposed a collection of entanglement witnesses in the so-called measurement-device-independent (MDI) scenario. Compared with the standard DI scenario, there is one more assumption in an MDI scenario: the input of each detector has to be a set of tomographically complete quantum states instead of real numbers. Such a simple relaxation leads to the result that all entangled states can be certified by the proposed MDI entanglement witnesses [51, 52]. This characterization gives rise to recent works providing frameworks for MDI measure of entanglement [54–56], non-classical teleportation [57], and entanglement breaking channel verification [58].

Recently, Cavalcanti et al. [58] introduced another type of nonlocal game, dubbed as quantum refereed steering games (QRSGs). In each of such games, one player is questioned and answers in real numbers, while the other player is questioned in (isolated) quantum states but still answers in real numbers. They showed that there always exists a QRSG with a higher winning probability when the players are correlated by a steerable state. Later, Kocsis et al. [59] experimentally proposed a QRSG to verify the steerability for the Werner states in such a scenario, which is also referred to as an MDI scenario. The characterization of all steerable states by the QRSGs motivates us to construct a collection of MDI steering witnesses (MDI-SWs) through a set of QRSGs. As will be shown later, for any given steerable resource defined by an assemblage [6, 20], there always exists an MDI-SW to certify the steerability. Moreover, we propose the first measure of steerability in an MDI scenario and show that it is a standard measure, i.e., a steering monotone [6], by proving that it is equivalent to the previous proposed measures of steerability — the steering robustness [31] and the steering fraction [32]. Note that, in the MDI scenario, neither the underlying measurements nor the shared state is characterized. The only characterized quantities are: (1) A tomographically complete set of input quantum states for one of the measurement devices, and (2) An observed data table composed of a set of probability distributions. Finally, we provide a proof-of-concept experimental implementation to obtain an MDI steering measure of the maximally entangled state using

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EPR steering and its witnesses.—In this work, we assume all quantum states act on a finite dimensional Hilbert space, denoted by $\mathcal{H}$. The sets of density matrices and operators acting on $\mathcal{H}$ are denoted by $D(\mathcal{H})$ and $L(\mathcal{H})$, respectively. We denote the set of the number of elements, i.e., the index sets, by $A$, $B$, $\mathcal{X}$, and $\mathcal{Y}$. The probability of a specific index, say $a \in A$, is denoted by $p(a)$.

Consider two spatially separated parties, Alice and Bob, sharing an unknown state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. Alice performs a set of uncharacterized measurements, where each of them is described by positive-operator valued measures (POVMs) $\{E_{a|x} \in L(\mathcal{H}_A)\}_x$, satisfying $E_{a|x}^2 = E_{a|x}$, with $x \in \mathcal{X}$ and $a \in A$ being the choice (input) and the outcome of the measurement, respectively. On the other hand, Bob’s measurements are assumed to be fully characterized. Therefore, he is able to perform quantum state tomography on his part of $\rho_{AB}$, and obtains the corresponding subnormalized quantum state $\sigma_{a|x} = \text{tr}_A(\rho_{AB} E_{a|x} \otimes \mathbb{1})$. A set of POVMs $\{E_{a|x}\}_{a,x}$ (i.e., $\{E_{a|x}\}$ in short) gives rise to a collection of subnormalized states $\{\sigma_{a|x}\}_{a,x}$ (i.e., $\{\sigma_{a|x}\}$ in short), called an assemblage [see Fig. 1(a)]. Note that an assemblage includes both the information of Alice’s marginal statistics $p(a|x) = \text{tr}(\sigma_{a|x})$ and the normalized states $\sigma_{a|x} = \sigma_{a|x}/p(a|x) \in D(\mathcal{H}_B)$. An assemblage is said to be unsteerable if it admits a local-hidden-state (LHS) model, i.e.,

$$\sigma_{a|x} = \sigma_{a|x}^{\text{LHS}} = \sum_{\lambda} p(\lambda)p(a|x,\lambda)\sigma_{\lambda} \forall a,x.$$  \hspace{1cm} (1)

A physical interpretation of a LHS model can be explained as follows: During each round of the experiment, Bob receives a fixed state $\sigma_{\lambda}$ according to a classical variable $\lambda$ no matter what Alice’s measurement is. What Bob obtains finally is just a classical post-processing of the set of fixed states $\{\sigma_{\lambda}\}$, operated by the probability distribution $p(a|x,\lambda)$ and $p(\lambda)$. An assemblage is called steerable, if it does not obey a LHS model described by Eq. 1.

The set of all unsteerable assemblages LHS forms a convex set $\mathcal{C}$. Therefore, for a given steerable assemblage $\{\sigma_{a|x}^S\}$, there always exists a set of positive semidefinite operators $\{F_{a|x} \geq 0\}_{a,x}$, called a steering witness (SW) or a steering functional, such that $\text{tr}\sum_{a,x} F_{a|x} \sigma_{a|x}^S > \alpha := \max_{\{\sigma_{a|x}\} \in \mathcal{C}} \text{tr}\sum_{a,x} F_{a|x} \sigma_{a|x}$, while $\text{tr}\sum_{a,x} F_{a|x} \sigma_{a|x}^{\text{LHS}} \leq \alpha \forall \{\sigma_{a|x}^{\text{LHS}}\} \in \mathcal{LHS}$. The two conditions can be reformulated as the following, which will be useful in the next sections:

$$\text{tr} \sum_{a,x} \left( F_{a|x} - \frac{\alpha}{|\mathcal{X}|} \mathbb{1} \right) \sigma_{a|x}^S > 0,$$  \hspace{1cm} (2)

while

$$\text{tr} \sum_{a,x} \left( F_{a|x} - \frac{\alpha}{|\mathcal{X}|} \mathbb{1} \right) \sigma_{a|x}^{\text{LHS}} \leq 0 \forall \{\sigma_{a|x}^{\text{LHS}}\} \in \mathcal{LHS},$$  \hspace{1cm} (3)

where $|\mathcal{X}|$ denotes the number of elements in $\mathcal{X}$, i.e., the number of the measurement settings.

Quantum refereed steering games.—Now we give a brief introduction to QRSGs. In a QRSG, consider there are two players who cannot communicate with each other, say Alice and Bob, sharing a quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. During each round of the game, Alice receives a classical number $x \in \mathcal{X}$ with probability $p(x)$ as her question from the referee, while Bob receives a quantum state $\omega_{y}^{B_0} \in D(\mathcal{H}_{B_0})$ ($\omega_y$ in short) with probability $p(y)$ (where $y \in \mathcal{Y}$) as his question. To respond to the referee, Alice performs a quantum measurement, described by a POVM $\{E_{a|x}\}_{a,x}$, on her part of the system $\text{tr}_B(\rho_{AB})$ and sends the measurement outcome $a$ as her answer to the referee, while Bob performs a joint quantum measurement, described by a POVM $\{E_{a|x}\}_{a,x}$, on his part of the system $\text{tr}_A(\rho_{AB})$ together with his received quantum question $\omega_y$ and sends the output $b \in \mathcal{B}$ as his answer to the referee [see Fig. 1(b)]. Finally, according to the questions and answers, the referee gives Alice and Bob a payoff

$$\mu = \mu(a,b,x,y).$$

After many rounds (within the same game), the average payoff they can obtain is

$$\bar{I}(\rho_{AB},\{\mu\}) = \sum_{a,b,x,y} p(x)p(y)\mu(a,b,x,y)p(a,b|x,y),$$  \hspace{1cm} (4)
where
\[ p(a, b|x, y) = \text{tr} \left[ (E_{a|x} \otimes E_{b|y})(\rho_{AB} \otimes \omega_{y}) \right] \quad \forall a, b, x, y. \] (5)

It has been shown that all steerable states always provide more advantage than any unsteerable state, by optimizing Alice’s and Bob’s measurements in Eq. (1) [58].

**MDI witnesses for all steerable assemblages.**—Motivated by the result in the previous section, here we show how to systematically construct a collection of SWs in an MDI scheme, dubbed MDI-SWs. It is MDI since we certify steerability based only on the statistics \( \{ p(a, b|x, y) \} \) and on the fact that \( \{ \omega_{y} \} \) is a tomographically complete set. In what follows, we would like to address the problem under the framework of the resource theory of steering [9], i.e., we will certify steerability of an assemblage \( \{ \sigma_{a|x} \} \) instead of quantum states \( \rho_{AB} \).

First, we map a subset of QRSGs to a set of real numbers \( \{ \beta_{a|x}^{y} \} \) by choosing a set of coefficients \( \beta := \{ \beta_{a|x}^{y} \} \), such that \( I(\{ \rho_{AB} \}, \beta) = \sum_{a,x,y} \beta_{a|x}^{y} p(a, 1|x, y) \). Under the framework of the correlation theory of steering [6], the steering of Bob’s joint measurement on the assemblage, i.e., \( p(a, 1|x, y) = \text{tr}(E_{1a|x}^{B0} \sigma_{a|x} \otimes \omega_{y}) \). The average payoff of an assemblage can then be defined as
\[ I(\{ \sigma_{a|x} \}, \beta) = \sum_{a,x,y} \beta_{a|x}^{y} p(a, 1|x, y). \] (6)

Following the techniques used in Ref. [52], we show that for any given steerable assemblage, one can properly choose a set of coefficients \( \beta := \{ \beta_{a|x}^{y} \} \), such that
\[ I(\{ \sigma_{a|x} \}, \beta) > 0, \] (7)
\[ I(\{ \sigma_{a|x}^{US} \}, \beta) \leq 0 \quad \forall \{ \sigma_{a|x}^{US} \} \in \text{LHS}. \]

**Proof.** Since the set of Bob’s input quantum states \( \{ \omega_{y} \} \) is a tomographically complete set, it can be used to span all Hermitian matrices with the same dimension. Hence, given \( \{ \sigma_{a|x} \} \notin \text{LHS}, \exists \beta := \{ \beta_{a|x}^{y} \} \) such that
\[ I(\{ \sigma_{a|x} \}, \beta) > 0, \] (7)
\[ I(\{ \sigma_{a|x}^{US} \}, \beta) \leq 0 \quad \forall \{ \sigma_{a|x}^{US} \} \in \text{LHS}. \]

First, we prove the second requirement of Eq. (1). From Eq. (1), each component in the correlation \( \{ p(a, 1|x, y) \} \) admitting a LHS model can be expressed as
\[ p(a, 1|x, y) = \text{tr} \left[ (E_{1a|x}^{B0}(\sigma_{a|x} \otimes \omega_{y})) \right] \]
\[ = \sum_{\lambda} p(\lambda) p(a|x, \lambda) \text{tr} \left( \tilde{E}_{1a|x, \lambda}^{B0} \omega_{y} \right), \] (9)

where \( \tilde{E}_{1a|x, \lambda}^{B0} := \text{tr}_{B}[E_{1a|x, \lambda}^{B0} (\sigma_{a|x} \otimes 1)] \) is an effective POVM element. The payoff of the assemblage is then written as
\[ I(\{ \sigma_{a|x} \}, \beta) := \sum_{a,x,y} \beta_{a|x}^{y} p(\lambda, 1|x, y, \lambda), \]
\[ = \sum_{a,x,y} p(\lambda)p(a|x, \lambda) \text{tr} \left[ \tilde{E}_{1a|x, \lambda}^{B0} \right] \sum_{\lambda} \beta_{a|x}^{y} p(\lambda, 1|x, \lambda, \lambda) \text{tr} \left( \tilde{E}_{1a|x, \lambda}^{B0} \omega_{y} \right) \]
\[ = \text{tr} \left[ \left( \sum_{a,x} (F_{a|x} - \frac{\alpha}{|x|} \mathbb{1}) \right) \sum_{\lambda} p(\lambda)p(a|x, \lambda) \frac{(\tilde{E}_{1a|x, \lambda}^{B0})^{\dagger}}{\text{tr} \tilde{E}_{1a|x, \lambda}^{B0}} \right] \text{tr} \tilde{E}_{1a|x, \lambda}^{B0} \omega_{y} \]
\[ \leq 0, \] (10)

where the inequality holds according to Eq. (9).

(ii) Now we prove the first requirement of Eq. (1). We consider the joint measurement performed by Bob to be the projection onto the maximally entangled state \( |\Phi_{+}^{B0}\rangle = 1/\sqrt{d_{B}} \sum_{i=1}^{d_{B}} |i\rangle \otimes |i\rangle \). Therefore, each component of the correlation can be expressed as
\[ p(a, 1|x, y, \lambda) = \text{tr} \left[ (E_{1a|x}^{B0}(\sigma_{a|x} \otimes \omega_{y})) \right] \]
\[ = \text{tr} \left[ \left( |\Phi_{+}^{B0}\rangle \langle \Phi_{+}^{B0}| \right) (\sigma_{a|x} \otimes \omega_{y}) \right] \]
\[ = \text{tr} \left[ \omega_{y}^{\dagger} F_{a|x} / d_{B} \right]. \] (11)

The average payoff is reformulated as
\[ I(\{ \sigma_{a|x} \}, \beta) := \sum_{a,x,y} \beta_{a|x}^{y} p(\lambda, 1|x, y, \lambda) \]
\[ = \sum_{a,x} \text{tr} \left[ \left( \sum_{y} \beta_{a|x}^{y} \omega_{y}^{\dagger} \right) \sigma_{a|x}^{S} \right] / d_{B} \]
\[ = \sum_{a,x} \text{tr} \left[ \left( F_{a|x} - \frac{\alpha}{|x|} \mathbb{1} \right) \sigma_{a|x}^{S} \right] / d_{B} > 0, \] (12)

where the inequality holds according to Eq. (12).

In fact, the average payoff of the assemblage, i.e., Eq. (1), can be seen as a generalization of the standard Bell inequalities and a similar formulation has been considered in the entanglement scenario [52]. In Section A of the Supplementary Material [61], we provide an algorithmic method to numerically construct an MDI-SW for a given steerable assemblage.

**MDI measure of steerability.**—Since we have shown that all steerable assemblage can be certified in an MDI scenario, it is natural to ask if there exists a measure of steerability in such a scheme. In the following, we introduce an MDI steering measure (MDI-SM) and show that it is a standard measure of steerability, i.e., a steering monotone [8]. To this end, we prove that the proposed MDI-SM is equivalent to the steering robustness [31] and the steering fraction [32].

First, we introduce an MDI-SM, denoted by
\[ S_{\text{MDI}}(\{ \sigma_{a|x} \}), \]
\[ S_{\text{MDI}}(\{ \sigma_{a|x} \}) := \max \left\{ S_{0, \text{MDI}}^{\text{MDI}}(\{ \sigma_{a|x} \}) - 1, 0 \right\}, \] (13)
where

\[ S_{0}^{\text{MDI}}(\{\sigma_{a|x}\}) := \sup_{\beta,\omega} \frac{\sum_{a,x,y} \beta_{a,1}^{x,y} p(a,1|x,\omega_y)}{\sup_{P \in \text{LHS}'} \sum_{a,x,y} \beta_{a,1}^{x,y} p(a,1|x,\omega_y)} \]

with \( P := \{p(a,1|x,\omega_y)\} \) and \( \bar{P} := \{\bar{p}(a,1|x,\omega_y)\} \). Here, by \( P \in \text{LHS}' \) we mean that the correlation is obtained from an assemblage admitting a LHS model, i.e., \( \bar{p}(a,1|x,\omega_y) = \text{tr}[E_{1}^{BB} (\tau_{a|x} \otimes \omega_y)] \) with \( \{\tau_{a|x}\} \in \text{LHS} \). One should note that \( \beta_{a,1}^{x,y} \) are not arbitrary real numbers, but instead, they become once satisfied the satisfied relation \( F_{a|x} = \sum y \beta_{a,1}^{x,y} \omega_{y}^{T} \) for all positive semidefinite operators \( \{F_{a|x}\} \geq 0 \) [32]. We can reformulate Eq. (14) as

\[ S_{0}^{\text{MDI}}(\{\sigma_{a|x}\}) = \sup_{\beta,\omega,E} \frac{\sum_{a,x,y} \beta_{a,1}^{x,y} \text{tr}[E_{1}^{BB} (\sigma_{a|x} \otimes \omega_y)]}{\sup_{\tau \in \text{LHS}} \sum_{a,x,y} \beta_{a,1}^{x,y} \text{tr}[E_{1}^{BB} (\tau_{a|x} \otimes \omega_y)]} = \sup_{F \geq 0,E} \frac{\sum_{a,x} \text{tr}[E_{1}^{BB} (\sigma_{a|x} \otimes F_{a|x}^{\tau})]}{\sum_{a,x} \text{tr}[E_{1}^{BB} (\tau_{a|x} \otimes F_{a|x}^{\tau})]}, \]

where \( \tau \) denotes \( \{\tau_{a|x}\} \) for brevity. We can derive the following lemma.

**Lemma 1.** The supremum in Eq. (14) is achieved if \( E_{1}^{BB} = \text{proj} \) onto the maximally entangled state, i.e., \( E_{1}^{BB} = |\Phi_{+}^{BB}\rangle \langle \Phi_{+}^{BB}| \), with \( |\Phi_{+}^{BB}\rangle = 1/\sqrt{d_B} \sum_{i=1}^{d_B} |i\rangle \otimes |i\rangle \). Moreover, it is independent of the chosen tomographically complete set \( \{\omega_{y}\} \).

The proof is given in Section C of the Supplementary Material [61]. By using Lemma 1 one can simplify Eq. (15) and further show the following result:

**Theorem 1.** The proposed MDI-SM \( S^{\text{MDI}}(\{\sigma_{a|x}\}) \) is equivalent to the steering fraction [32], and hence the steering robustness [61]. Therefore, it is a steering monotone [6].

The detailed proof is provided in Sections B and C of the Supplementary Material.

**Examples.**—In the following, we give two explicit examples to demonstrate the proposed MDI-SM. First, we consider the two-qubit Werner state\footnote{See Section D of the supplementary material [61] for the derivation.}, where \( |\Phi^{-}\rangle = (|10\rangle - |01\rangle)/\sqrt{2} \) is the singlet state. Alice’s measurements are considered to be Pauli X and Z. Obtaining an assemblage through such a setting, one can get a steering witness \( \{F_{a|x} = \frac{1}{2} (X^{a} \otimes Z^{a}) \} \) (see Section D of the supplementary material [61] for the derivation), where \( \sigma_{x=1} = X \) and \( \sigma_{x=2} = Z \). We take the tomographically complete set \( \{\omega_{y}\} \) to be the eigenstates of the three Pauli matrices. Then, the set \( \beta \) can be obtained from the spanned relation \( F_{a|x} = \sum y \beta_{a,1}^{x,y} \omega_{y}^{T} \).

\[ \sum y \beta_{a,1}^{x,y} \omega_{y}^{T} \]
steering monotone by proving the equivalence with the steering fraction as well as the steering robustness. To our knowledge, this is the first work providing a measure of steerability based only on the observed statistics.

In the standard steering scenario, one has to perform quantum state tomography on Bob’s side to obtain an assemblage and the degree of steerability. In the MDI steering scenario, on the other hand, the requirement of the tomography is replaced by the optimization of Bob’s all possible joint measurements. It looks like the proposed strategy is even more complicated at first glance. However, we show that the projection onto the maximally entangled state is always an optimal measurement for any steerable resource. Such a result greatly simplifies the complexity of the realization of experiments. We also perform a proof-of-concept experiment on the IBM Quantum Experience.

Some open questions follow. Is there any other optimal measurement for Bob to achieve the MDI-SM? To calculate the value of the MDI-SM or obtain an MDI-SW for a steerable assemblage, can one directly obtain an optimal set of coefficients $\beta$ instead of obtaining it through the standard steering witness? Can one also construct a collection of MDI-SWs for all steerable states, instead of all steerable assemblages?

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[61] See Supplemental Material at [URL will be inserted by publisher], which contains also references to xxx, for further details on yyy.

[62] Here, \( \{ \omega_y \} \) is used to span \( F_{a|x} \) instead of \( F_{a|x} - \alpha/|x| \), therefore the set of coefficients \( \{ \beta_{a,y} \} \) is different from the one in Eq. (3).
Supplemental material

Appendix A: Construction of MDI-SWs and MDI-SMs from the standard steering witnesses

In this section, we provide an algorithmic method for constructing a set of coefficients $$\beta$$ of the MDI-SW and MDI-SM from the standard steering witness $$\{F_{a|x}\}$$. For a target steerable assemblage $$\{\sigma_{a|x}\}$$, one can construct a MDI-SW through the following steps:

1. Choose a tomographically complete set $$\{\omega_y\}$$ to be Bob’s quantum inputs.

2. Consider the optimal standard steering witness $$\{F_{a|x}\}$$ of the target assemblage $$\{\sigma_{a|x}\}$$, which can be obtained either from the dual SDP program of the steerable weight $$S_W$$ [30],

$$S_W + 1 = \min \sum a,x F_{a|x} \sigma_{a|x}$$

such that

$$\sum a,x D(a|x, \lambda) F_{a|x} \geq I \quad \forall \lambda$$

or from the dual SDP program of the steering robustness $$S_R$$ [31],

$$S_R + 1 = \max \sum a,x F_{a|x} \sigma_{a|x}$$

such that

$$\sum a,x D(a|x, \lambda) F_{a|x} \leq I \quad \forall \lambda$$

3. Choose a set of coefficients $$\beta := \{\beta_{a,y}^{x}\}$$ satisfying the spanned relation:

$$F_{a|x} - \frac{1}{|X|} = \sum_{y} \beta_{a,y}^{x} \omega_{y}^{x} \quad \forall a, x.$$ (A3)

4. Finally, $$I(\{\sigma_{a|x}\}, \{\beta\}) := \sum a,x,y \beta_{a,y}^{x} \omega_{y}^{x}$$ is a MDI-SW. The negative value certifies the steerability if we consider the program Eq. (A1) of the steerable weight in the second step, while the positive value certifies the steerability if we consider the steering robustness described by Eq. (A2).

To construct the MDI-SM of the given assemblage, we must follow these steps:

1. Choose a tomographically complete set $$\{\omega_y\}$$ to be Bob’s quantum inputs.

2. Choose Bob’s measurement to be in the basis $$\{E_{B1}^{1|B0}, 1 - E_{B}^{1|B0}\}$$, with $$E_{B1}^{1|B0}$$ being the projection onto the maximally entangled state $$(1/\sqrt{d}) \sum_{i} |ii\rangle$$.

3. From the above two steps, one obtains the optimal correlation $$\{\rho(a, 1|x, \omega_y) = tr(E_{B1}^{1|B} \sigma_{a|x} \otimes \omega_y\}$$.

4. Consider the optimal standard steering witness $$\{F_{a|x}\}$$ of the assemblage $$\{\sigma_{a|x}\}$$, which is obtained from the dual SDP program Eq. (A2) of the steering robustness.

5. Choose a set of coefficients $$\beta := \{\beta_{a,y}^{x}\}$$ satisfying the spanned relation:

$$F_{a|x} = \sum_{y} \beta_{a,y}^{x} \omega_{y}^{x} \quad \forall a, x.$$ (A4)

6. Finally,

$$\max \left\{ \sum_{a,x,y} \beta_{a,y}^{x} \rho(a, 1|x, \omega_y) \right\} \sum_{a,x,y} \beta_{a,y}^{x} \rho(a, 1|x, \omega_y) - 1, 0 \right\}$$ (A5)

is the MDI-SM, where the denominator [see Eq. (C6)]

$$\sup_{\rho \in LHS} \sum_{a,x,y} \beta_{a,y}^{x} \rho(a, 1|x, \omega_y) = \frac{1}{d}$$ (A6)

One may find out that the algorithmic method for constructing the MDI-SM is not genuine MDI since the assemblage has to be known. We have mentioned this in the last section of the main text, i.e., obtaining an optimal set $$\beta$$ in an MDI scenario is one of the open problems. We would also like to stress that the definition of the MDI-SM itself and the proof of the equivalence with the steering robustness are still in the MDI scheme.

Appendix B: Proof of the equivalence between the steering fraction and the steering robustness

In this section, we explicitly prove the equivalence between the steering fraction and the steering robustness, although their equivalence is implicitly mentioned in some references (see, e.g., Ref. [33]). The steering robustness of a given assemblage can be obtained by the dual program described in Eq. (A2). On the other hand, the steering fraction $$S_F$$ of the given assemblage is defined as [32]

$$S_F + 1 = \max_{F \geq 0} \frac{\sum a,x F_{a|x} \sigma_{a|x}}{\max_{\tau \in LHS} \sum a,x F_{a|x} \tau_{a|x}}.$$ (B1)

We can rewrite it as

$$S_F + 1 = \max_{F \geq 0} \frac{\sum a,x F_{a|x} \sigma_{a|x}}{\sum a,x F_{a|x}^{\tau_{a|x}}} \geq 0.$$ (B3)

Therefore, to prove the equivalence between Eqs. (A2) and (B1), it is equivalent to prove

$$\sum a,x D(a|x, \lambda) F_{a|x} \leq 1 \quad \forall \lambda.$$ (B4)
Proof. For each $\lambda$, the quantity $\mathbb{1} - \sum_{a,x} D(a|x, \lambda) F_{a|x}$ is multiplied by a subnormalized quantum state $\rho_\lambda \geq 0$.

We take the trace, and summation over all $\lambda$:

$$
\tr \sum_\lambda \left( \mathbb{1} - \frac{\sum_{a,x} D(a|x, \lambda) F_{a|x}}{\max_{\tau \in \text{LHS}} \tr \sum_{a|x} F_{a|x} \tau_{a|x}} \right) \rho_\lambda
= 1 - \frac{\sum_{a,x} D(a|x, \lambda) F_{a|x}}{\max_{\tau \in \text{LHS}} \tr \sum_{a|x} F_{a|x} \tau_{a|x}},
$$

which is non-negative for all $\rho_\lambda \geq 0$ and $\lambda$. Since the only constraint between the free parameters $\rho_\lambda$ is $\tr \sum_\lambda \rho_\lambda = 1$, we arrive at the fact that

$$
\mathbb{1} - \frac{\sum_{a,x} D(a|x, \lambda) F_{a|x}}{\max_{\tau \in \text{LHS}} \tr \sum_{a|x} F_{a|x} \tau_{a|x}} \geq 0 \ \forall \lambda.
$$

(\text{B5})

\hfill \blacksquare

Appendix C: The equivalence between the MDI measure of steerability and the steering robustness

Let us now rewrite the definition of the MDI steering measure (MDI-SM), i.e., Eq. (14) in the main text

$$
S_{\text{MDI}}^\text{SM}(\{\sigma_{a|x}\}) := \max \{ S_{\text{MDI}}(\{\sigma_{a|x}\}) - 1, 0 \}, \tag{C1}
$$

where

$$
S_{0,\text{MDI}}^\text{MDI}(\{\sigma_{a|x}\}) := \sup_{\beta, p} \frac{\sum_{a,x,y} \beta_{a|x}^y p(a, 1|x, \omega_y)}{\sup_{\tau \in \text{LHS}} \sum_{a,x,y} \beta_{a|x}^y p(a, 1|x, \omega_y)}. \tag{C2}
$$

By replacing $p(a, b|x, \omega_y)$ with $\tr[ E_{1}^{BB_{0}} \sigma_{a|x} \otimes \omega_y ]$ and using the spanned relation $F_{a|x} = \sum_y \beta_{a|x}^\tau \omega_y$, then $S_{0,\text{MDI}}^\text{MDI}(\{\sigma_{a|x}\})$ can be reformulated as [i.e., Eq. (15)]:

$$
S_{0,\text{MDI}}^\text{MDI}(\{\sigma_{a|x}\})
= \sup_{F \geq 0, E_1} \frac{\sum_{a,x} \tr[ E_{1}^{BB_0}(\sigma_{a|x} \otimes F_{a|x}^T) ]}{\sup_{\tau \in \text{LHS}} \sum_{a,x} \tr[ E_{1}^{BB_0}(\tau_{a|x} \otimes F_{a|x}^T) ]}, \tag{C3}
$$

where $F \geq 0$ denotes $\{ F_{a|x} \geq 0 \}$ for brevity. Since $E_{1}^{BB_{0}}$ is a POVM element, it is diagonalizable and can be taken as a linear combination of rank-1 projectors with coefficients lying between 0 and 1. Since any rank-$k$ projector can be produced by acting a separable operation on the maximally entangled state, $E_{1}^{BB_{0}}$ can be written as

$$
E_{1}^{BB_{0}} = \sum_{k,i} u(k) A^k_i \otimes B^k_i \langle \Phi | A^k_i \otimes B^k_i \rangle,
$$

(C4)

where $u(k)$ denotes the coefficients between 0 and 1, $A^k_i \otimes B^k_i = \sqrt{u(k)} A^k_i \otimes B^k_i$ is the redefined Kraus operators for each $i$, and (for brevity) $| \Phi \rangle$ denotes $| \Phi^{BB_{0}} \rangle$. Then, we can proceed to write $S_{0,\text{MDI}}^\text{MDI}(\{\sigma_{a|x}\})$ as

$$
\begin{align*}
S_{0,\text{MDI}}^\text{MDI}(\{\sigma_{a|x}\}) & = \sup_{F \geq 0, E_1} \frac{\sum_{a,x} \tr[ E_{1}^{BB_0}(\sigma_{a|x} \otimes F_{a|x}^T) ]}{\sup_{\tau \in \text{LHS}} \sum_{a,x} \tr[ E_{1}^{BB_0}(\tau_{a|x} \otimes F_{a|x}^T) ]}, \\
& = \sup_{F \geq 0, E_1} \frac{\sum_{a,x} \tr[ E_{1}^{BB_0}(\sigma_{a|x} \otimes F_{a|x}^T) ]}{\sum_{a,x} \tr[ E_{1}^{BB_0}(\tau_{a|x} \otimes F_{a|x}^T) ]}, \\
& \leq \sup_{F \geq 0} \frac{\sum_{a,x} \tr[ \sigma_{a|x} F_{a|x} ]}{\sum_{a,x} \tr[ \tau_{a|x} F_{a|x} ]}.
\end{align*}
$$

The inequality is due to the fact that the convex set $F$ is a superset of the one after performing the completely
positive map, i.e., \( F' := \{ \sum_{k,i} A_{k,i}^k B_{k,i}^T F_{a,1}^i B_{k,i}^T A_{k,i}^k \}_{a,x} \).

It is easy to verify that \( S_{0}^{\text{MDI}}(\{\sigma_{a|x}\}) \) can achieve the upper bound of Eq. (C6) when considering \( E_{1}^{BB_0} \) in Eq. (C3) to be the projection onto the maximally entangled state \( |\Phi^{BB_0}\rangle \), i.e.,

\[
S_{0}^{\text{MDI}}(\{\sigma_{a|x}\}) \bigg|_{E_1 = |\Phi\rangle\langle\Phi|} = \sup_{F \geq 0} \frac{\sum_{a,x} \text{tr} [E_{1}^{BB_0}(\sigma_{a|x} \otimes \Phi^T_{a|x})] |E_1 = |\Phi\rangle\langle\Phi|}{\sum_{a,x} \text{tr} [E_{1}^{BB_0}(\tau_{a|x} \otimes \Phi^T_{a|x})]} = \sup_{F \geq 0} \frac{\sum_{a,x} \text{tr} [\Phi \sigma_{a|x} \otimes \Phi^T_{a|x}]}{\sum_{a,x} \text{tr} [\sigma_{a|x} F_{a|x}]}.
\]

The last quantity is exactly the steering fraction of Eq. (B1). From the result of the previous section, we obtain that \( S_{0}^{\text{MDI}}(\{\sigma_{a|x}\}) \) is the same as the steering robustness.

**Appendix D: Analytical construction of the MDI steering measure for Werner states**

In this section, we provide an analytical construction of the MDI steering measure of an assemblage obtained from the two-qubit Werner state. The procedure is the same as the algorithmic method in Section A of this supplementary material. To obtain the optimal standard steering witness \( \{ F_{a,1} \} \), we use a similar technique with the one used in Ref. [35]. The two-qubit Werner state is written as:

\[
\rho_{AB}^w = v|\Phi^-\rangle\langle\Phi^-| + (1 - v)\frac{I}{4}, \quad v \in [0,1],
\]

where \( |\Phi^-\rangle = ([10] - [01])/\sqrt{2} \) is the singlet state. We take the measurements performed by Alice to be in the bases of Pauli X and Z. The corresponding assemblage is then given by [35]

\[
\sigma_{a|x} = \frac{I + (-1)^{a+1}n_x \cdot \sigma}{2} + \frac{1 - v}{4} \frac{I}{4}, \quad \forall a,x.
\]

where \( n_1 = (1,0,0) \) and \( n_2 = (0,0,1) \) are vectors on the Bloch sphere, and \( \sigma = (X,Y,Z) \) is the set of Pauli matrices. Any two-dimensional Hermitian matrix \( F_{a | x} \) can be expressed as \( F_{a | x} = \gamma_{a | x} I + \kappa_{a | x} \cdot \delta \), with \( \gamma_{a | x} \) being a real number and \( \kappa_{a | x} \) being a three dimensional vector. Then, we arrive at

\[
\text{tr} \sum_{a,x} F_{a|x} \sigma_{a|x} = \frac{1}{2} \left( \gamma_{1|1} + \gamma_{2|1} + \gamma_{1|2} + \gamma_{2|2} \right) + \frac{v}{2} \left[ \hat{n}_1 \cdot (\hat{k}_{1|1} - \hat{k}_{2|1}) + \hat{n}_2 \cdot (\hat{k}_{1|2} - \hat{k}_{2|2}) \right].
\]

Finally, from Eq. (D7) and the symmetric rule, we obtain the two inequalities as

\[
1 \geq \kappa \geq 0 \quad \text{and} \quad 1 - \frac{\sqrt{2}k^2}{2} \geq \gamma \geq \kappa.
\]

Since Eq. (D4) is a linear function of \( \gamma \) and \( \kappa \), the local maximal value takes place at the extremal points \( \gamma =

| \begin{array}{cccc}
\beta_{0,1}^1 & \beta_{1,1}^1 & \beta_{1,0}^1 & \beta_{0,0}^1 \\
\beta_{0,1}^2 & \beta_{1,1}^2 & \beta_{1,0}^2 & \beta_{0,0}^2 \\
\beta_{0,1}^3 & \beta_{1,1}^3 & \beta_{1,0}^3 & \beta_{0,0}^3 \\
\beta_{0,1}^4 & \beta_{1,1}^4 & \beta_{1,0}^4 & \beta_{0,0}^4
\end{array} |
| \begin{array}{cccc}
0 & 0 & 0 & 0 \\
2 \kappa & \kappa & \kappa & \kappa \\
0 & 0 & \kappa & \kappa \\
\kappa & \kappa & \kappa & \kappa
\end{array} |

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
\hline
\beta_{0,1}^1 & \beta_{1,1}^1 & \beta_{1,0}^1 & \beta_{0,0}^1 & \beta_{0,1}^2 & \beta_{1,1}^2 & \beta_{1,0}^2 & \beta_{0,0}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 \kappa & \kappa & \kappa & \kappa & \kappa & \kappa & \kappa & \kappa \\
0 & \kappa & 0 & \kappa & \kappa & \kappa & \kappa & \kappa \\
\kappa & \kappa & \kappa & \kappa & \kappa & \kappa & \kappa & \kappa \\
\hline
\end{tabular}
\caption{The set \{ \beta_{a,1} \} of the MDI-SM of the assemblage produced from the two-qubit Werner state.}
\end{table}
\( \kappa = (1/2 + \sqrt{2})^{-1} \) of constraint (DN). Therefore, the optimal \( \{ F_a | \} \) in Eq. (A2) is analytically constructed as

\[
F_a | x = \frac{1}{2 + \sqrt{2}} [I + (-1)^{a+1} \sigma_x] \quad \forall a, x. \tag{D9}
\]

Now we take Bob’s input quantum states \( \{ \omega_y \} \) are eigenstates of Pauli matrices, which form a tomographically complete set. The above steering functional can be spanned by this set:

\[
F_a | x = \sum_y \beta_{a,1}^{x,y} \omega_y^T \quad \forall a, x. \tag{D10}
\]

A choice of the set \( \{ \beta_{a,1}^{x,y} \} \) is listed in Table III. One should note that \( \{ \beta_{a,1}^{x,y} \} \) is not unique. The steerability of the assemblage created by the measurements on the two-qubit Werner state can then be obtained in an MDI scenario, which is shown in Fig. 1 in the main text.

**Appendix E: Experimental MDI-SM for maximally entangled state**

In this section, we explain in detail the experimental setup on the back-end of the IBM Quantum Experience to experimentally measure the steerability of the maximally entangled two-qubit state \( \rho_{AB} = |\Phi^+\rangle\langle\Phi^+| \), where \( |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \). This state can be created by the Hadamard and CNOT gates. More precisely, one can perform the Hadamard gate on a control qubit before utilizing a CNOT gate on both qubits to create the maximally entangled state. The input quantum states \( \{ \omega_y \} \), which Bob receives, are the six eigenstates of the three Pauli matrices. One can create these input states by using the Pauli \( X \) gate, phase gate (\( S \) and \( S^\dagger \)), and Hadamard gates. We note that the initial state of each qubit on the IBM Quantum Experience is prepared in \(|0\rangle\). The state \(|1\rangle\) can be directly obtained by performing the \( X \) gate on \(|0\rangle\). The states \(|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) and the state \(|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \) can be created, respectively, by performing the Hadamard gate and \( X \) gate followed by the Hadamard gate on the initial state. The states \(|R\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle) \) and \(|L\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle) \) are attained by performing the Hadamard gate before the \( S \) and \( S^\dagger \) gates, respectively.

We consider that Alice measures in the bases of Pauli \( X \) and \( Z \) on her quantum state \( \rho_{AB} \). Although the measurements on the IBM Quantum Experience can only be performed in the basis of Pauli \( Z \), we can still produce the same measurement by properly choosing suitable bases. The measurement in the basis of Pauli \( X \) can be achieved by performing the Hadamard gate before \( Z \) measurement. Bob’s measurement in the Bell basis can be realized by performing the CNOT gate followed by the Hadamard gate (on control qubit) before \( Z \) measurement. Note that we only take into account the outcome \( 00 \) corresponding to the projection \( |\Phi^\perp\rangle \) and assign it as \( \alpha = 1 \).

| \( p(1,1|1,1) \) | \( p(2,1|1,1) \) | \( p(1,1|1,2) \) | \( p(2,1|1,2) \) | \( p(1,1|1,3) \) | \( p(2,1|1,3) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.05            | 0.28            | 0.18            | 0.03            | 0.19            | 0.14            |
| 0.08            | 0.17            | 0.13            | 0.13            | 0.04            | 0.25            |
| 0.18            | 0.15            | 0.09            | 0.13            | 0.13            | 0.14            |
| 0.12            | 0.12            | 0.04            | 0.25            | 0.27            | 0.06            |

**TABLE II.** Table of probability distributions \( p(a, b|x, y) \) of the simplest MDI scenario for the maximally entangled state. Here, the index sets \( x = 1, 2, \) and \( y = 1, ..., 6 \) are Alice’s measurement settings and Bob’s input quantum states with the corresponding measurement outcomes \( a = 1, 2 \) and \( b = 1, 2, \) respectively.

**TABLE III.** The set of coefficients \( \{ \beta_{a,1}^{x,y} \} \) for the maximally entangled state \( |\Psi^+\rangle \). Due to the fact that there is only one unitary transformation between \( |\Phi^+\rangle \) and \( |\Phi^\perp\rangle \), the coefficients here are basically the same as those in Table III up to some permutations of \( a, x, \) and \( y \).

For illustrative purposes, in Fig 3 we show a schematic diagram of the circuit used in our experiment, where Alice measures in the Pauli \( X \) basis and Bob receives \( |L\rangle \). Other measurement settings can be similarly obtained. With this setup, in each pair of the setting \((x, y)\), we run 8192 times to obtain the probability distributions \( \{ p(a, |x, \omega_y\rangle \}_a\), and Bob’s measurement probabilities \( \{ p(a, |x, \omega_y\rangle \}_b \) are obtained, which is shown in Table III. Then, by using the coefficients \( \beta \) in Table III, we can obtain the MDI-SM.
FIG. 3. The quantum circuit for one pair of settings \((x, y)\) in our experiments. Same as the notation with the IBM Quantum Experience, each node (in blue color), except the rightmost one, represents a quantum gate, while the rightmost one represents the measurement, which can perform only in the basis of \(Z\). The qubits are all initially prepared in the state \(|0\rangle\), sent from left to right, undergoing some quantum gates, and finally being measured in the basis of \(Z\). In our experimental setup, the top and the middle qubits undergoing the first two gates (from left to right) are the state \(\rho_{AB} = |\Phi^+\rangle\langle \Phi^+|\) shared between Alice and Bob, where 

\[
|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\]

The bottom qubit, undergoing the first two gates (from left to right) is \(|L\rangle\), is one of the input quantum states for Bob. Alice’s measurement in the basis of Pauli \(X\) can be performed effectively by the Hadamard gate followed by the measurement, i.e., the rightmost two operations on the top qubit. Bob’s projective measurement in the Bell basis are effectively performed by the CNOT gate followed by the Hadamard gate together with the measurements on the middle and bottom qubits, i.e., the last three operations on the middle qubit and the last two operations on the bottom qubit.