AUTOEQUIVALENCES OF THE TENSOR CATEGORY OF $U_q\mathfrak{g}$-MODULES

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ABSTRACT. We prove that for $q \in \mathbb{C}^*$ not a nontrivial root of unity the cohomology group defined by invariant 2-cocycles in a completion of $U_q\mathfrak{g}$ is isomorphic to $H^2(P/Q; \mathbb{T})$, where $P$ and $Q$ are the weight and root lattices of $\mathfrak{g}$. This implies that the group of autoequivalences of the tensor category of $U_q\mathfrak{g}$-modules is the semidirect product of $H^2(P/Q; \mathbb{T})$ and the automorphism group of the based root datum of $\mathfrak{g}$. For $q = 1$ we also obtain similar results for all compact connected separable groups.

In a previous paper [6] we showed that if $G$ is a compact connected group then the cohomology group defined by invariant unitary 2-cocycles on $\hat{G}$ is isomorphic to $H^2(\mathbb{Z}(G); \mathbb{T})$ and we conjectured that for semisimple Lie groups a similar result holds for the $q$-deformation of $G$. In the present note we will prove that this is indeed the case using the technique from our earlier paper [5], where we considered symmetric cocycles and were inspired by the proof of Kazhdan and Lusztig of equivalence of the Drinfeld category and the category of $U_q\mathfrak{g}$-modules [2]. For $q = 1$ this gives an alternative proof of the main results in [6, Section 2] and allows us, at least in the separable case, to extend those results to non-unitary cocycles relying neither on ergodic actions nor on reconstruction theorems. At the same time this proof is less transparent than that in [6] and, as opposed to [6], relies heavily on the structure and representation theory of compact Lie groups.

We will follow the notation and conventions of [5]. Let $G$ be a simply connected semisimple compact Lie group, $\mathfrak{g}$ its complexified Lie algebra. Fix a Cartan subalgebra and a system $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots. The weight and root lattices are denoted by $P$ and $Q$, respectively. For $q \in \mathbb{C}^*$ not a nontrivial root of unity consider the quantized universal enveloping algebra $U_q\mathfrak{g}$. Denote by $C_q(\mathfrak{g})$ the tensor category of admissible finite dimensional $U_q\mathfrak{g}$-modules, and by $\mathcal{U}(G_q)$ the endomorphism ring of the forgetful functor $C_q(\mathfrak{g}) \to \text{Vec}$.

An invertible element $\mathcal{E} \in \mathcal{U}(G_q \times G_q)$ is called a 2-cocycle on $\hat{G}_q$ if

$$(\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E}).$$

A cocycle is called invariant if it commutes with elements in the image of $\hat{\Delta}_q$. The set of invariant 2-cocycles forms a group under multiplication, which we denote by $Z^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$. Cocycles of the form $(a \otimes a)^{-1}(\hat{\Delta}_q(a))$, where $a$ is an invertible element in the center of $\mathcal{U}(G_q)$, form a subgroup of the center of $Z^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$. The quotient of $Z^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$ by this subgroup is denoted by $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$.

The center of $\mathcal{U}(G_q)$ is identified with functions on the set $P_+$ of dominant integral weights. By [5, Proposition 4.5] a function on $P_+$ is a group-like element of $\mathcal{U}(G_q)$ if and only if it is defined by a character of $P/Q$. Therefore the Hopf algebra of functions on $P/Q$ embeds into the center of $\mathcal{U}(G_q)$. Hence every 2-cocycle $\mathcal{E}$ on $P/Q$ can be considered as an invariant 2-cocycle $\mathcal{E}_c$ on $\hat{G}_q$. Explicitly, if $U$ and $V$ are irreducible $U_q\mathfrak{g}$-modules with highest weights $\mu$ and $\eta$, then $\mathcal{E}_c$ acts on $U \otimes V$ as multiplication by $c(\mu, \eta)$. We can now formulate our main result.

**Theorem 1.** The homomorphism $c \mapsto \mathcal{E}_c$ induces an isomorphism

$$H^2(P/Q; \mathbb{T}) \cong H^2_{G_q}(\hat{G}_q; \mathbb{C}^*).$$

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In particular, if \( g \) is simple and \( g \not\cong so_{4n}(\mathbb{C}) \) then \( H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \) is trivial, and if \( g \cong so_{4n}(\mathbb{C}) \) then \( H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z} \).

The last statement follows from the fact that for simple Lie algebras the group \( P/Q \) is cyclic unless \( g \cong so_{4n}(\mathbb{C}) \), in which case \( P/Q \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), see e.g. Table IV on page 516 in [1].

Note that for \( q > 0 \) the same result holds for unitary cocycles. This easily follows by polar decomposition, see [5] Lemma 1.1.

In the proof of the theorem we will assume that \( q \neq 1 \), the case \( q = 1 \) is similar and for unitary cocycles is also proved by a different method in [6].

Our first goal will be to construct a homomorphism \( H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \to H^2(P/Q; \mathbb{T}) \). For every \( \mu \in P_+ \) fix an irreducible \( U_qg \)-module \( V_\mu \) with highest weight \( \mu \) and a highest weight vector \( \xi_\mu \).

Recall [5] Section 2) that for \( \mu, \eta \in P_+ \) there exists a unique morphism

\[
T_{\mu, \eta} : V_{\mu + \eta} \to V_{\mu} \otimes V_{\eta} \quad \text{such that} \quad \xi_{\mu + \eta} \mapsto \xi_\mu \otimes \xi_\eta.
\]

The image of \( T_{\mu, \eta} \) is the isotropic component of \( V_\mu \otimes V_\eta \) with highest weight \( \mu + \eta \). Hence if \( \mathcal{E} \) is an invariant 2-cocycle then it acts on this image as multiplication by a nonzero scalar \( c_{\mathcal{E}}(\mu, \eta) \). As in the proof of [5] Lemma 2.2, identity \( (T_{\mu, \eta} \otimes \iota)_{T_{\mu, \eta} + \nu} = (\iota \otimes T_{\nu, \eta})_{T_{\mu, \eta} + \nu} \) implies that \( c_{\mathcal{E}} \) is a two-cocycle on \( P_+ \).

Furthermore, the cohomology class \( [c_{\mathcal{E}}] \) of \( c_{\mathcal{E}} \) in \( H^2(P_+; \mathbb{C}^*) \) depends only on the class of \( \mathcal{E} \) in \( H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \), since if \( a \in \mathcal{U}(G_q) \) is a central element acting on \( V_\mu \) as multiplication by a scalar \( a(\mu) \) then the action of \( (a \otimes \iota)_{T_{\mu, \eta}} \) on the image of \( T_{\mu, \eta} \) is multiplication by \( a(\mu)a(\eta)a(\mu + \eta)^{-1} \). Thus the map \( \mathcal{E} \mapsto c_{\mathcal{E}} \) defines a homomorphism \( H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \to H^2(P_+; \mathbb{C}^*) \).

Given a cocycle on \( P/Q \), we can consider it as a cocycle on \( P \) and then get a cocycle on \( P_+ \) by restriction. Thus we have a homomorphism \( H^2(P/Q; \mathbb{T}) \to H^2(P_+; \mathbb{C}^*) \). It is injective since the quotient map \( P_+ \to P/Q \) is surjective and a cocycle on \( P/Q \) is a coboundary if it is symmetric.

**Lemma 2.** For every invariant 2-cocycle \( \mathcal{E} \) on \( \hat{G}_q \) the class of \( c_{\mathcal{E}} \) in \( H^2(P_+; \mathbb{C}^*) \) is contained in the image of \( H^2(P/Q; \mathbb{T}) \).

**Proof.** Consider the skew-symmetric bi-quasicharacter \( b : P_+ \times P_+ \to \mathbb{C}^* \) defined by

\[
b(\mu, \eta) = c_{\mathcal{E}}(\mu, \eta)c_{\mathcal{E}}(\eta, \mu)^{-1}.
\]

It extends uniquely to a skew-symmetric bi-quasicharacter on \( P \). To prove the lemma it suffices to show that the root lattice \( Q \) is contained in the kernel of this extension. Indeed, since \( H^2(P/Q; \mathbb{T}) \) is isomorphic to the group of skew-symmetric bi-characters on \( P/Q \), it then follows that there exists a cocycle \( c \) on \( P/Q \) such that the cocycle \( c_{\mathcal{E}}c^{-1} \) on \( P_+ \) is symmetric. Then by [4] Lemma 4.2 the cocycle \( c_{\mathcal{E}}c^{-1} \) is a coboundary, so \( c_{\mathcal{E}} \) and the restriction of \( c \) to \( P_+ \) are cohomologous.

To prove that \( Q \) is contained in the kernel of \( b \), recall [5] Section 2 that for every simple root \( \alpha_i \) and weights \( \mu, \eta \in P_+ \) with \( \mu(i), \eta(i) \geq 1 \) we can define a morphism

\[
\tau_{\iota, \mu, \eta} : V_{\mu + \eta - \alpha_i} \to V_{\mu} \otimes V_{\eta} \quad \text{such that} \quad \xi_{\mu + \eta - \alpha_i} \mapsto [\mu(i)]_q \xi_\mu \otimes F_i \xi_\eta - q_\eta^{\mu(i)}[\eta(i)]_q F_i \xi_\mu \otimes \xi_\eta.
\]

The image of \( \tau_{\iota, \mu, \eta} \) is the isotypic component of \( V_\mu \otimes V_\eta \) with highest weight \( \mu + \eta - \alpha_i \). Since the element \( \mathcal{E} \) is invariant, it acts on this space as multiplication by a nonzero scalar \( c_{\mathcal{E}}(\mu, \eta) \). As in the proof of [5] Lemma 2.3, consider now another weight \( \nu \) with \( \nu(i) \geq 1 \). The isotypic component of \( V_\mu \otimes V_\eta \) with highest weight \( \mu + \eta + \nu - \alpha_i \) has multiplicity two, and is spanned by the images of \( (\iota \otimes T_{\nu, \eta})_{\tau_{\iota, \mu, \eta, \nu}} \) and \( (\iota \otimes \tau_{\iota, \mu, \eta, \nu})_{\tau_{\iota, \mu, \eta, \nu}} \), as well as by the images of \( (T_{\nu, \eta} \otimes \iota)_{\tau_{\iota, \mu, \eta, \nu}} \) and \( (\tau_{\iota, \mu, \eta} \otimes \iota)_{\tau_{\iota, \mu, \eta, \nu}} \). We have

\[
[q(\nu(i))_q(T_{\nu, \eta} \otimes \iota)_{\tau_{\iota, \mu, \eta, \nu}} - [\nu(i)]_q(\iota \otimes T_{\nu, \eta})_{\tau_{\iota, \mu, \eta, \nu}}]T_{\mu + \eta - \alpha_i, \nu} = [\mu(i) + \eta(i)]_q(\iota \otimes \tau_{\iota, \mu, \eta, \nu})_{\tau_{\iota, \mu, \eta, \nu}}.
\]

Apply the element

\[
\Omega := (\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E})
\]
to this identity. The morphisms \((T_{\mu,\eta} \otimes \iota)\tau_{\mu+\eta,\lambda} \), \((\iota \otimes \tau_{\mu,\eta})T_{\mu,\eta+\lambda-\alpha_i}\), and \((\iota \otimes \tau_{\mu,\eta})T_{\mu,\eta+\lambda-\alpha_i}\) are eigenvectors of the operator of multiplication by \(\Omega\) on the left with eigenvalues \(c_\xi(\mu,\eta)c_\xi(\mu+\eta,\lambda)\), \(c_\xi(\mu,\eta)\), and \(c_\xi(\mu,\eta)\), respectively. Since the morphisms \((T_{\mu,\eta} \otimes \iota)\tau_{\mu+\eta,\lambda} \) and \((\iota \otimes \tau_{\mu,\eta})T_{\mu,\eta+\lambda-\alpha_i}\) are linearly independent, by applying \(\Omega\) to (1) we conclude that these three eigenvalues coincide. In particular, 
\[c_\xi(\mu,\eta)\xi(\mu+\eta-\alpha_i,\lambda) = c_\xi(\mu,\eta)\xi(\mu+\eta-\alpha_i,\lambda).\]

Applying this to \(\eta = \nu = \mu\) we get 
\[b(2\mu - \alpha_i, \mu) = 1.\]

Since \(b\) is skew-symmetric, this gives \(b(\alpha_i, \mu) = 1\). The latter identity holds for all \(\mu \in P_+\) with \(\mu(i) \geq 1\). Since every element in \(P\) can be written as a difference of two such elements \(\mu\), it follows that \(\alpha_i\) is contained in the kernel of \(b\).

Therefore the map \(E \mapsto c_\xi\) induces a homomorphism \(H^2_{\text{G}_q}(\hat{G}_q; \mathbb{C}^*) \rightarrow H^2(P/Q; \mathbb{T})\). Clearly, it is a left inverse of the homomorphism \(H^2(P/Q; \mathbb{T}) \rightarrow H^2_{\text{G}_q}(\hat{G}_q; \mathbb{C}^*)\) constructed earlier. Thus it remains to prove that the homomorphism \(H^2_{\text{G}_q}(\hat{G}_q; \mathbb{C}^*) \rightarrow H^2(P/Q; \mathbb{T})\) is injective.

Assume \(E\) is an invariant 2-cocycle such that the cocycle \(c_\xi\) on \(P_+\) is a coboundary. Then the considerations in [5, Section 2] following Lemma 2.2 apply and show that replacing \(E\) by a cohomologous cocycle we may assume that 
\[\mathcal{E}T_{\mu,\eta} = T_{\mu,\eta}\] and \(\mathcal{E} \tau_{\iota,\eta,\omega} = \tau_{\iota,\eta,\omega}\) for all \(\mu, \eta \in P_+, 1 \leq i \leq r\) and \(\nu, \omega \in P_+\) such that \(\nu(i), \omega(i) \geq 1\). Therefore to prove injectivity it suffices to show the following result.

**Proposition 3.** If \(E\) is an invariant 2-cocycle on \(\hat{G}_q\) with property (2) then \(E = 1\).

By [5] Corollary 4.4 the result is true under the additional assumption that \(E\) is symmetric, that is, \(R_hE = E_2R_h\) for an \(R\)-matrix \(R_h \in \mathcal{U}(G_q \times G_q)\), which depends on the choice of a number \(h \in \mathbb{C}\) such that \(q = e^{\pi i h}\). We will show that this assumption is automatically satisfied for any \(h\).

The results of [5, Section 4] up to (but not including) Lemma 4.3 apply to any invariant cocycle satisfying (2). To formulate these results recall some notation.

For every weight \(\mu \in P_+\) fix an irreducible \(U_q\mathfrak{g}\)-module \(V_\mu\) with lowest weight \(-\mu\) and a lowest weight vector \(\xi_\mu\). For \(\lambda \in P\) and \(\mu, \eta \in P_+\) such that \(\lambda + \mu \in P_+\) there exists a unique morphism 
\[\text{tr}^\eta_{\mu,\lambda+\mu} : \bar{V}_\mu \otimes V_{\lambda+\mu} \rightarrow \bar{V}_\mu \otimes V_{\lambda+\mu} \] such that \(\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta} \mapsto \bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta}\).

Using these morphisms define an inverse limit \(U_q\mathfrak{g}\)-module 
\[M_\lambda = \lim_{\mu} V_\mu \otimes V_{\lambda+\mu}.\]

Denote by \(\text{tr}_{\mu,\lambda+\mu}\) the canonical map \(M_\lambda \rightarrow V_\mu \otimes V_{\lambda+\mu}\). The module \(M_\lambda\) is considered as a topological \(U_q\mathfrak{g}\)-module with a base of neighborhoods of zero formed by the kernels of the maps \(\text{tr}_{\mu,\lambda+\mu}\), while all modules in our category \(\mathcal{C}_q(\mathfrak{g})\) are considered with discrete topology. Then \(\text{Hom}_{U_q\mathfrak{g}}(M_\lambda, V)\) is the inductive limit of the spaces \(\text{Hom}_{U_q\mathfrak{g}}(V_\mu \otimes V_{\lambda+\mu}, V)\). The vectors \(\bar{\xi}_{\mu} \otimes \xi_{\lambda+\mu}\) define a topologically cyclic vector \(\Omega_\lambda \in M_\lambda\). For any finite dimensional admissible \(U_q\mathfrak{g}\)-module \(V\) the map 
\[\eta_\nu : \text{Hom}_{U_q\mathfrak{g}}(\oplus \lambda M_\lambda, V) \rightarrow V, \quad \eta_\nu(f) = \sum_\lambda f(\Omega_\lambda),\]

is an isomorphism.

The results of [5, Section 4] up to Lemma 4.3 can be summarized by saying that for every invariant cocycle \(E\) satisfying (2) there exist a character \(\chi\) of \(P/Q\), an invertible morphism \(\mathcal{E}_0\) of \(\oplus \lambda M_\lambda\) onto itself preserving the direct sum decomposition, and an invertible element \(c\) in the center of \(\mathcal{U}(G_q)\) such that 
\[\text{tr}_{\mu,\lambda+\mu} \mathcal{E}_0 = \chi(\mu)^{-1} \mathcal{E} \text{tr}_{\mu,\lambda+\mu} \quad \text{and} \quad \eta_\nu(f \mathcal{E}_0) = c \eta_\nu(f) \] (3)
for all $\mu \in P_+$, $\lambda \in P$ such that $\lambda + \mu \in P_+$, all finite dimensional admissible $U_q\mathfrak{g}$-modules $V$ and $f \in \text{Hom}_{U_q\mathfrak{g}}(M_\lambda, V)$.

**Proof of Proposition** 4. As we have already remarked, by [5] Corollary 4.4 it suffices to show that $\mathcal{R}_h\mathcal{E} = \mathcal{E}_2\mathcal{R}_h$, for some $h$ such that $q = e^{\pi i h}$. We will prove a stronger statement: $\sigma\mathcal{E} = \mathcal{E}\sigma$ for any braiding $\sigma$ on $C_q(\mathfrak{g})$.

By (3), since $\text{tr}_{\mu,\lambda+\mu}(\Omega_\lambda) = \xi_\mu \otimes \xi_{\lambda+\mu}$, for any $\mu, \eta, \nu \in P_+$ and $f \in \text{Hom}_{U_q\mathfrak{g}}(V_\mu \otimes V_\eta, V_\nu)$ we have

$$\chi(\mu)^{-1} f(\mathcal{E}(\xi_\mu \otimes \xi_\eta)) = c(\nu) f(\xi_\mu \otimes \xi_\eta).$$

As the vector $\xi_\mu \otimes \xi_\eta$ is cyclic, this means that $f(\mathcal{E}) = \chi(\mu)c(\nu)f$. Since this is true for all $f$, we conclude that $\mathcal{E}$ acts on the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\nu$ as multiplication by $\chi(\mu)c(\nu)$. In other words, $\mathcal{E}$ acts on the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\nu$ as multiplication by $\chi(\mu)c(\nu)$. It follows that

$$\sigma\mathcal{E} = \chi(\bar{\mu} - \bar{\eta})\mathcal{E}\sigma \text{ on } V_\mu \otimes V_\eta.$$

But by assumption (2) the element $\mathcal{E}$ is the identity on the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu + \eta$; so by considering the above identity on this isotypic component we conclude that $\chi(\bar{\mu} - \bar{\eta}) = 1$. Thus $\chi$ is the trivial character and $\sigma\mathcal{E} = \mathcal{E}\sigma$. This finishes the proof of Proposition 4 and hence of Theorem 4.

By a result of McMullen [9] any automorphism of the fusion ring of $C_q(\mathfrak{g})$, mapping irreducibles into irreducibles, is implemented by an automorphism of the based root datum of $\mathfrak{g}$, hence by an automorphism of the Hopf algebra $U_q\mathfrak{g}$. Hence, similarly to [6] Theorem 2.5, we get the following consequence of Theorem 4.

**Theorem 4.** The group of $\mathbb{C}$-linear monoidal autoequivalences of the tensor category $C_q(\mathfrak{g})$ is canonically isomorphic to $H^2(P/Q; T) \rtimes \text{Aut}(\Psi)$, where $\Psi$ is the based root datum of $\mathfrak{g}$.

The group $P/Q$ is canonically identified with the dual of the center $Z(G)$ of the group $G$, so for $q = 1$ Theorem 4 can be formulated as $H^2_0(G; \mathbb{C}^*) \cong H^2(\widehat{Z(G)}; \mathbb{C}^*)$. In this form it can be extended to a larger class of groups.

**Theorem 5.** For any compact connected separable group $G$ we have a canonical isomorphism

$$H^2_0(G; \mathbb{C}^*) \cong H^2(\widehat{Z(G)}; \mathbb{C}^*).$$

**Proof.** For Lie groups the proof is essentially the same as above, with $P$ replaced by the weight lattice of a maximal torus of $G$. In the general case we have a homomorphism $H^2(\widehat{Z(G)}; \mathbb{C}^*) \to H^2_0(G; \mathbb{C}^*)$ obtained by considering $\mathcal{U}(Z(G))$ as a subring of $\mathcal{U}(G)$. To construct the inverse homomorphism, for every quotient $H$ of $G$ which is a Lie group consider the composition

$$H^2_0(G; \mathbb{C}^*) \to H^2(H; \mathbb{C}^*) \to H^2(\widehat{Z(H)}; \mathbb{C}^*),$$

where the first homomorphism is defined using the quotient map $\mathcal{U}(G) \to \mathcal{U}(H)$. The map $Z(G) \to Z(H)$ is surjective (since this is true for Lie groups), so $Z(G)$ is the inverse limit of the groups $Z(H)$. Then $H^2(\widehat{Z(G)}; \mathbb{C}^*)$ is the inverse limit of the groups $H^2(\widehat{Z(H)}; \mathbb{C}^*)$. Therefore the above maps $H^2_0(G; \mathbb{C}^*) \to H^2(\widehat{Z(H)}; \mathbb{C}^*)$ define a homomorphism $H^2_0(G; \mathbb{C}^*) \to H^2(\widehat{Z(G)}; \mathbb{C}^*)$. It is clearly a left inverse of the map $H^2(\widehat{Z(G)}; \mathbb{C}^*) \to H^2_0(G; \mathbb{C}^*)$, so it remains to show that it is injective.

In other words, we have to check that if $\mathcal{E}$ is an invariant cocycle on $\hat{G}$ such that its image in $\mathcal{U}(H \times H)$ is a coboundary for every Lie group quotient $H$ of $G$, then $\mathcal{E}$ itself is a coboundary. If $\mathcal{E}$ were unitary, this could be easily shown by taking a weak operator limit point of cochains, see the proof of [6] Theorem 2.2], and would not require separability of $G$. In the non-unitary case we can argue as follows.

Since $G$ is separable, there exists a decreasing sequence of closed normal subgroups $N_n$ of $G$ such that $\cap_{n=1}^\infty N_n = \{e\}$ and the quotients $H_n = G/N_n$ are Lie groups. Let $\mathcal{E}_n$ be the image
of $\mathcal{E}$ in $\mathcal{U}(H_n \times H_n)$. By assumption there exist invertible central elements $c_n \in \mathcal{U}(H_n)$ such that $\mathcal{E}_n = (c_n \otimes c_n) \hat{\Delta}(c_n)^{-1}$. For a fixed $n$ consider the image $a$ of $c_{n+1}$ in $\mathcal{U}(H_n)$. Then $c_n a^{-1}$ is a central group-like element in $\mathcal{U}(H_n)$. By [5, Theorem A.1] it is therefore defined by an element of the center of the complexification $(H_n)_C$ of $H_n$. Since the homomorphism $(H_{n+1})_C \to (H_n)_C$ is surjective, we conclude that there exists a central group-like element $b$ in $\mathcal{U}(H_{n+1})$ such that its image in $\mathcal{U}(H_n)$ is $c_n a^{-1}$. Replacing $c_{n+1}$ by $c_{n+1} b$ we get an element such that $\mathcal{E}_{n+1} = (c_{n+1} \otimes c_{n+1}) \hat{\Delta}(c_{n+1})^{-1}$ and the image of $c_{n+1}$ in $\mathcal{U}(H_n)$ is $c_n$. Applying this procedure inductively we can therefore assume that the image of $c_{n+1}$ in $\mathcal{U}(H_n)$ is $c_n$ for all $n \geq 1$. Then the elements $c_n$ define a central element $c \in \mathcal{U}(G)$ such that $\mathcal{E} = (c \otimes c) \hat{\Delta}(c)^{-1}$. □

In [6, Theorem 2.5] we computed the group of autoequivalences of the $C^*$-tensor category of finite dimensional unitary representations of $G$. The above theorem allows us to get a similar result ignoring the $C^*$-structure.

**Theorem 6.** For any compact connected separable group $G$, the group of $C^*$-linear monoidal autoequivalences of the category of finite dimensional representations of $G$ is canonically isomorphic to $H^2(\mathbb{Z}(G); C^*) \rtimes \text{Out}(G)$.

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