Solving the Brachistochrone Problem by an Influence Diagram

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Abstract

Influence diagrams are a decision-theoretic extension of probabilistic graphical models. In this paper we show how they can be used to solve the Brachistochrone problem. We present results of numerical experiments on this problem, compare the solution provided by the influence diagram with the optimal solution. The R code used for the experiments is presented in the Appendix.

1 Introduction

Formulated by Johan Bernoulli in 1696, the brachistochrone problem is: given two points find a curve connecting them such that a mass point moving along the curve under the gravity reaches the second point in minimum time. See [Bertsekas 2000, Example 3.4.2] for a formulation of this problem as an optimal control problem.

2 The ODE model

The state variable is the vertical coordinate $y$. It is assumed to be a function of the horizontal coordinate $x$. The control variable $u$ controls the derivative of $y$:

$$\frac{dy(x)}{dx} = u(x)$$

The task is to find the control function $u(x)$ so that we get from a point $(0, 0)$ to $(a, b)$, where $a > 0$ and $b < 0$. This means that the boundary conditions are

$$y(0) = 0$$
$$y(a) = b$$

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It is also assumed that the initial speed at the origin is zero.

Speed \( v \) is defined by the law of energy conservation – kinetic energy equals to the change of gravitational potential energy:

\[
\frac{1}{2} \cdot m \cdot (v)^2 = -m \cdot g \cdot y \quad (1)
\]

\[
v = \sqrt{-2 \cdot g \cdot y} . \quad (2)
\]

For an infinitesimal segment of length \( dx \) with an infinitesimal change \( dy \) of the vertical position \( y \) we can write

\[
v = \frac{ds}{dt} = \sqrt{\left(\frac{dy}{dx}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \left(\sqrt{1 + \frac{(dy)^2}{dx^2}}\right) \frac{dx}{dt} \quad (3)
\]

By substituting \((2)\) to \((3)\) we get

\[
dt = \frac{ds}{v} = \left(\frac{1}{\sqrt{-2 \cdot g \cdot y}} \sqrt{1 + \frac{(dy)^2}{dx^2}}\right) dx . \quad (4)
\]

The solution of the brachistochrone problem is a function \( y = f(x) \) that minimizes the total time \( T \) necessary to get from the point \((0,0)\) to the point \((a,b)\)

\[
T = \int_0^a \left(\frac{1}{\sqrt{-2 \cdot g \cdot f(x)}} \sqrt{1 + \frac{(df)^2}{dx^2}}\right) dx . \quad (5)
\]

The solution of the brachistochrone problem is known – it is a part of a cycloid, which can be specified by parametric formulas:

\[
x = \frac{K}{2} (\varphi + \sin \varphi) + L
\]

\[
y = -\frac{K}{2} (1 - \cos \varphi) .
\]

The constants \( K, L \) are specified so that the cycloid goes through points \((0,0)\) and \((a,b)\).

### 3 Discretized version of the problem

We discretize the problem:

- \( n \) ... the number of discrete intervals,
- \( \Delta x = \frac{a}{n} \) ... discretization step of the x-coordinate,
- \( i \) ... the index of the discrete interval,
- \( x_i \) ... x-coordinate \( i \cdot \Delta x, i = 0, 1, \ldots, n \)
- \( y_i \) ... y-coordinate at \( x_i \),
• \( v_i \) ... speed at \( x_i \),
• \( u_i \) ... control at coordinate \( x_i \),
• \( t_i \) ... time to get from \( x_{i-1} \) to \( x_i \).

The state variable \( y_i \) is transformed by the control variable \( u_i \) as

\[
y_{i+1} = y_i + u_i.
\]

In each segment we will assume that the path is a line segment, i.e. for \( x \in [x_i, x_{i+1}] \) and for \( y \in [y_i, y_{i+1}] \) it holds that

\[
y = \frac{u_i}{\Delta x} \cdot x + y_i.
\]

By substituting (6) to (5) and by solving the integral we get the formulas for the time spent at the segment \([x_i, x_{i+1}]\).

\[
t_{i+1} = \begin{cases} 
\frac{\Delta x}{\sqrt{2 \cdot g \cdot y_i}} & \text{if } u_i = 0 \\
-\sqrt{\frac{g}{y_i}} \cdot \left( \frac{(\Delta x)^2 + u_i^2}{u_i} \right) \cdot (\sqrt{-y_i} - \sqrt{-u_i - y_i}) & \text{otherwise.} 
\end{cases}
\]

The boundary conditions are

\[
(x_0, y_0) = (0, 0) \\
(x_n, y_n) = (a, b).
\]

The goal is to find the control strategy \( u = (u_0, \ldots, u_{n-1}) \), \( u_i \in \mathbb{R}, i = 0, 1 \ldots, n - 1 \) so that we get from the initial point \((x_0, y_0)\) to the terminal point \((x_n, y_n)\) in the shortest possible time

\[
J(u) = \sum_{i=1}^{n} t_i
\]

and satisfy the state conditions (the gravitational potential energy corresponding to the value of \( y \) cannot be more than it was at the initial point):

\[
y_i \leq y_0 \text{ for } i = 1, \ldots, n.
\]

4 The influence diagram

We will illustrate how an influence diagram can be used to find an arbitrary precise solution of the problem. An influence diagram (Howard and Matheson, 1981) is a Bayesian network augmented with decision variables and utility functions. For details see, e.g., Jensen (2001).

The structure of a segment of the influence diagram for the discrete version of the Brachistochrone Problem is presented in Figure 1. The utility function for node \( t_{i+1} \) is defined by formula (7). The conditional probability \( P(Y_{i+1}|U_i, Y_i) \) is deterministic and defined as:

\[
P(Y_{i+1} = y_{i+1}|U_i = u_i, Y_i = y_i) = \begin{cases} 
1 & \text{if } y_{i+1} = y_i + u_i \\
0 & \text{otherwise.}
\end{cases}
\]
In Figure 2 we compare the optimal trajectory (full red line) with the solution found by the influence diagram (circles connected by lines) for $\Delta x = 0.25$, $\Delta y = 0.1$ and $(a, b) = (10, -5)$. The difference between the optimal trajectory and the influence diagram solution can be reduced by reducing the discretization steps $\Delta x$ and $\Delta y$. The experiments were performed using R (R Core Team, 2014) – we present the code in Appendix A.

Figure 2: Comparison of the optimal solution with the influence diagram solution.
5 Conclusions

We have shown how influence diagrams can be used to solve the Brachistochrone problem. The numerical experiments reveal that the solution found by influence diagrams approximates well the optimal solution. In future we plan to apply influence diagrams to other trajectory optimization problems where the optimal solution is not known. These problems are traditionally solved by methods of optimal control theory but influence diagrams offer an alternative that can bring several benefits over the traditional approaches.

References

Bertsekas, D. P. (2000). *Dynamic Programming and Optimal Control*. Athena Scientific, 2nd edition.

Howard, R. A. and Matheson, J. E. (1981). Influence diagrams. In Howard, R. A. and Matheson, J. E., editors, *Readings on The Principles and Applications of Decision Analysis*, volume II, pages 721–762. Strategic Decisions Group.

Jensen, F. (2001). *Bayesian Networks and Decision Graphs*. Springer-Verlag.

R Core Team (2014). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

A The R code

```
n.x <- 41 # number of x values
n <- 101 # number of y values
a <- 10 # the x-coordinate of the goal state
b <- -5 # the y-coordinate of the goal state
delta.x <- a/(n.x-1) # the discretization step of x
delta.y <- 2*(-b)/(n-1) # the discretization step of y
g <- 9.81 # the gravitation constant
eps <- 10^-12

# time spent at one segment of length delta.x assuming linear path
time.step <- function(u,y){
  if (((y>0) || (y+u>0) || ((y==0)&(u==0))){
    return(Inf)
  }else{
    if (u==0){
      return(delta.x/sqrt(-2*g*y))
    }else{
      s <- sqrt(delta.x^2 + u^2)
      return(sqrt(2/g)*(s/u)*(sqrt(-y) - sqrt(-(u + y))))
    }
  }
}
```
address.y <- function(y){
  stopifnot(y <= 0)
  stopifnot(y >= 2*b)
  return(round(1+(y-2*b)/delta.y))
}

value.y <- function(address){
  return(2*b+(address-1)*delta.y)
}

address.u <- function(u){
  stopifnot(u <= -b)
  stopifnot(u >= b)
  return(round(1+(u-b)/delta.y))
}

value.u <- function(address){
  return(b+(address-1)*delta.y)
}

address.x <- function(x){
  stopifnot(x <= a+eps)
  stopifnot(x >= 0)
  return(round(1+(x/delta.x)))
}

value.x <- function(address){
  return((address-1)*delta.x)
}

is.addmissible <- function(y,k){
  if (k==(n.x-1)){
    return(abs(y-b)<eps)
  }else{
    return((y <= 0) & (y >= 2*b))
  }
}

find.best.policy <- function(y.start=0){
  policy <- array(0,dim=c(n.x-1,n))
  expected.utility <- rep(0,times=n)
  cat("\n")
  for (k in (n.x-1):1){
    x <- value.x(k)
    expected.utility.new <- rep(Inf,times=n)
    for (i in 1:n) {
      y <- value.y(i)
      for(j in 1:n){
        cat("\r k=",k," i=",i," j=",j," ")
        u <- value.u(j)
        y.next <- y+u
        # if y.next is within the admissible region
        if (is.addmissible(y.next,k)){
          # do something
        }
    }
  }
}
exp.util <- time.step(u=u,y=y) + expected.utility[address.y(y.next)]
if (exp.util < expected.utility.new[address.y(y)]){
    expected.utility.new[address.y(y)] <- exp.util
    policy[address.x(x), address.y(y)] <- u
}
}
}
expected.utility <- expected.utility.new
}
return(list(policy=policy,
    expected.utility=expected.utility[address.y(y.start)]))
}

# The construction of the state (vertical position y) profile.
# Note: since u and y have the same discretization step it is assured that
# by the application of u at state y we stay at the grid of y
construct.y.profile <- function(policy, y.start=0){
    x <- 0
    y <- y.start
    profile.y <- array(0,dim=c(n.x))
    profile.y[1] <- y
    for (i in 1:(n.x-1)){
        u <- policy[i,address.y(y)]
        y <- y+u
        profile.y[i+1] <- y
    }
    return(profile.y)
}

# The construction of the control profile.
# Note: since u and y have the same discretization step it is assured that
# by the application of u at state y we stay at the grid of y
construct.u.profile <- function(policy, y.start=0){
    x <- 0
    y <- y.start
    profile.u <- array(0,dim=c(n.x-1))
    for (i in 1:(n.x-1)){
        u <- policy[i,address.y(y)]
        y <- y+u
        profile.u[i] <- u
    }
    return(profile.u)
}

evaluate.u.profile <- function(profile.u, y.start=0){
    val <- 0
    y <- y.start
    for(i in 1:length(profile.u)){
        val <- val + profile.u[i]
    }
    return(val)
}
u <- profile.u[i]
val <- val + time.step(u,y)
y <- y + u
}
return(val)
}

# Brachistochrone (the solution found by the Mathematica FindRoot function)
theta.max <- 3.50837
a.val <- 2.586
theta.val <- (0:100)*(theta.max/100)
brachistochrone.x <- a.val * (theta.val - sin(theta.val))
brachistochrone.y <- - a.val * (1 - cos(theta.val))

# The actual computations
res <- find.best.policy()
profile.y <- construct.y.profile(res$policy)

# Plot results
plot(x=(0:(n.x-1))*delta.x, y=profile.y, type="b", xlab="x", ylab="y")
lines(x=brachistochrone.x, y=brachistochrone.y, col="red")
grid()

profile.u <- construct.u.profile(res$policy)
plot(x=0:(n.x-2), y=profile.u, type="l", xlab="x", ylab="u")
grid()

# reconstruction of the optimal control profile
# from the values found by the Mathematica FindRoot function
brachistochrone.x <- (0:40)*delta.x
# values found by the Mathematica FindRoot function
brachistochrone.y <- c(0, -0.86755, -1.34679, -1.73084, -2.05946,
-2.34941, -2.60981, -2.84634, -3.06283, -3.26204,
-3.44602, -3.61637, -3.77433, -3.92094, -4.05702,
-4.13327, -4.30028, -4.40854, -4.50848, -4.60047,
-4.68481, -4.76179, -4.83164, -4.89456, -4.95074,
-5.00031, -5.04342, -5.08018, -5.11067, -5.13497,
-5.15315, -5.16523, -5.17125, -5.17123, -5.16517,
-5.15304, -5.13483, -5.11049, -5.07995, -5.04316, -5.0)
brachistochrone.u <- -brachistochrone.y[-length(brachistochrone.y)]
+brachistochrone.y[-1]

# compute the total time for the path found by the influence diagram
evaluate.u.profile(profile.u)
# compute the total time for the optimal path at the same discrete scale
evaluate.u.profile(brachistochrone.u)