Derived categories of torsors for abelian schemes

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Abstract

In the first part of our paper, we show that there exist non-isomorphic derived equivalent genus 1 curves, and correspondingly there exist non-isomorphic moduli spaces of stable vector bundles on genus 1 curves in general. Neither occurs over an algebraically closed field. We give necessary and sufficient conditions for two genus 1 curves to be derived equivalent, and we go on to study when two principal homogeneous spaces for an abelian variety have equivalent derived categories. We apply our results to study twisted derived equivalences of the form $D^b(J, \alpha) \simeq D^b(J, \beta)$, when $J$ is an elliptic fibration, giving a partial answer to a question of Căldăraru.

Key Words. Derived equivalence, genus 1 curves, elliptic 3-folds, and Brauer groups.

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1 Introduction

Two smooth projective varieties $X$ and $Y$ over a field $k$ are derived equivalent if there is a $k$-linear triangulated equivalence $D^b(X) \simeq D^b(Y)$. The first example of this phenomenon was discovered by Mukai [17], who showed that $D^b(A) \simeq D^b(\hat{A})$ for any abelian variety $A$, where $\hat{A}$ denotes the dual abelian variety. Finding non-isomorphic derived equivalent varieties has since become a central problem in algebraic geometry, closely bound up with homological mirror symmetry and the study of moduli spaces of vector bundles. For an overview, see the book [14] of Huybrechts.

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Letting $P$ denote the Poincaré line bundle on $A \times_k \tilde{A}$, and denoting by $p$ and $q$ the projections from the product to $A$ and $\tilde{A}$, respectively, Mukai’s equivalence is the functor $\Phi_P : D^b(A) \to D^b(\tilde{A})$, where, for a complex $F \in D^b(A)$,

$$\Phi_P(F) = Rp_*(q^*F \otimes L^P).$$

Orlov [18] showed that any equivalence $D^b(X) \simeq D^b(Y)$ is of the form $\Phi_P$ for some complex $P \in D^b(X \times_k Y)$ when $X$ and $Y$ are smooth projective $k$-varieties. We will refer to these as Fourier-Mukai equivalences and to $P$ as the kernel.

As a consequence of the existence of kernels, derived equivalent varieties have the same dimension, the same $K$-theory spectrum, the same Hochschild homology spectrum, and, by Popa and Schnell [21], closely related Jacobian varieties. Moreover, their canonical classes are simultaneously torsion (of the same order), ample, or anti-ample. And yet, derived equivalence is coarse enough to allow many interesting examples.

If $D^b(X) \simeq D^b(Y)$ and the canonical line bundle $\omega_X$ is either ample or anti-ample, then $X \cong Y$ by a theorem of Bondal and Orlov [5]. If $X$ and $Y$ are genus 1 curves over an algebraically closed field (elliptic curves after choosing basepoints), it is known that $D^b(X) \simeq D^b(Y)$ implies $X \cong Y$. See for instance [14, Corollary 5.46]. Thus, over a separably closed field, derived equivalent curves are isomorphic.

We are interested in two questions in this paper: first when are two genus 1 curves derived equivalent? This is the only remaining unknown situation for curves. Second, when are two homogeneous spaces for a fixed abelian variety derived equivalent? It turns out that the first problem is a special case of the second.

A closely related problem in the curve case is to understand the moduli spaces of stable vector bundles on genus 1 curves. The underlying reason that derived equivalent elliptic curves are isomorphic is the result of Atiyah [2], which shows that the moduli spaces $J_E(r,d)$ of rank $r$ and degree $d$ stable vector bundles on an elliptic curve $E$ are themselves isomorphic to $E$. Indeed, given an equivalence $D^b(F) \simeq D^b(E)$ of two genus 1 curves over an algebraically closed field, it is possible to show that the equivalence sends the structure sheaves of points of $F$ to semi-stable vector bundles on $E$. This realizes $F$ as a moduli space $J_E(r,d)$ for some $r$ and $d$. Pumplün [22] made some extensions of Atiyah’s result in the case of a genus 1 curve without a point, but did not obtain a full classification. In particular, it has remained open whether or not every moduli space $J_X(r,d)$ of vector bundles on a genus 1 curve is isomorphic to $X$.

We show in this paper that there are non-isomorphic derived equivalent genus 1 curves, and that they lead to examples of non-isomorphic moduli spaces of stable vector bundles.

**Theorem 1.1.** Let $k$ be a field, and let $X$ and $Y$ be two smooth projective genus 1 curves over $k$. Suppose that $D^b(X) \simeq D^b(Y)$ as $k$-linear triangulated categories. Then, $X$ and $Y$ have the same Jacobian. If $X$ and $Y$ are principal homogeneous spaces for an elliptic curve $E$, then $D^b(X) \simeq D^b(Y)$ if and only if there exists an automorphism $\phi$ of $E$ such that $X = \phi_*dY$ in $H^1(k,E)$ for some $d$ prime to the order of $Y$. 


Here and in the rest of the paper, $H^i(k, -)$ denotes the étale cohomology group (or pointed set) $H^i_{\text{ét}}(\text{Spec} k, -)$.

**Corollary 1.2.** There exist non-isomorphic derived equivalent genus 1 curves.

In contrast to work of Atiyah [2] when $X$ has a $k$-point, we find the following corollary in the course of our proof.

**Corollary 1.3.** There exists a genus 1 curve $X$ and coprime integers $r \neq 0$ and $d \neq 0$ such that $J_X(r, d)$ and $X$ are not isomorphic.

Our results on derived equivalences of principal homogeneous spaces in higher dimensions rely on the twisted Brauer space, a tool for studying étale twisted forms of derived categories introduced by the first named author in [1]. We give less complete results in higher dimensions, establishing a necessary condition for two principal homogeneous spaces for an abelian variety to be derived equivalent. As the conclusion in the general case is somewhat complicated, we extract one particular case to highlight here.

**Theorem 1.4.** Suppose that $X$ and $Y$ are principal homogeneous spaces for an abelian variety $A$ over a field $k$ such that $\text{End}(A_{k_0}) \cong \mathbb{Z}$ and $\text{NS}(A_{k_0})$ is the constant Galois module $\mathbb{Z}$. Then, if $\mathbb{D}^b(X) \cong \mathbb{D}^b(Y)$, $X$ and $Y$ generate the same subgroup of $H^1(k, A)$. Conversely, if $X$ and $Y$ generate the same subgroup, then $\mathbb{D}^b(X) \cong \mathbb{D}^b(Y, \beta)$ for some $\beta \in \text{Br}(k)$.

The theorem allows one to study derived equivalences of abelian fibrations, and we do so in the elliptic case, to give a partial answer to a question of Căldăraru.

**Theorem 1.5.** Suppose that $p : J \to S$ is a smooth Jacobian elliptic fibration, where $S$ is a connected regular noetherian scheme with characteristic 0 field of fractions, and suppose that the geometric generic fiber $J_\eta$ is geometrically not CM. If there is an equivalence $F : \mathbb{D}^b(J, \alpha) \cong \mathbb{D}^b(J, \beta)$ compatible with $p$ for some $\alpha, \beta \in \text{Br}(J)$, then $\alpha$ and $\beta$ generate the same cyclic subgroup of $\text{Br}(J)/\text{Br}(S)$.

Our work here is in addition closely related to the work of Bridgeland [7] and Bridgeland-Maciocia [8, 9] on derived equivalences of surfaces. In the case where $S$ in the theorem is a smooth curve over the complex numbers, our theorem easily gives the converse of [7, Theorem 1.2], since $\text{Br}(S) = 0$ in this case. This converse also occurs in [8, Section 4]. In particular, they show that derived equivalences $\mathbb{D}^b(X) \cong \mathbb{D}^b(Y)$ of smooth projective elliptic surfaces of non-zero Kodaira dimension are all compatible with appropriate elliptic structures $p : X \to S$ and $q : Y \to S$.

Here is a brief description of the contents of our paper. Our results on genus 1 curves are proved in Section 2. In Section 3, we give background on dg categories, needed for the twisted Brauer space. The twisted Brauer space is introduced in Section 4, and it is used in Section 5 to give restrictions on when two principal homogeneous spaces can be derived equivalent. An example of an application to elliptic fibrations is in Section 6.
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2 Derived equivalences of genus 1 curves

We describe when two genus 1 curves are derived equivalent.

2.1 Elliptic Curves

Fix a field \( k \). An elliptic curve \( E \) is defined to be a smooth geometrically connected genus 1 curve over \( k \) with a distinguished rational point \( e : \text{Spec} \ k \to E \). The derived category \( D^b(E) \) is the bounded derived category of coherent sheaves on \( E \).

By the theorem of Bondal and Orlov, [14, Theorem 5.14], any derived equivalence \( D^b(X) \to D^b(Y) \) (always triangulated and \( k \)-linear) between two smooth projective varieties is isomorphic to the Fourier-Mukai transform \( \Phi = \Phi_P \) associated to a complex \( P \) in \( D^b(X \times_k Y) \). Moreover, the complex \( P \) is unique up to quasi-isomorphism. Unless it is explicitly needed, we suppress the name of the kernel. Given an object \( F \) of \( D^b(X) \) we write \( \Phi_i(F) \) for the \( i \)th cohomology sheaf \( H^i(\Phi(F)) \). We use the notation \( \mathcal{O}_x \) for the skyscraper sheaf at \( x \) with value the residue field \( k(x) \).

**Lemma 2.1.** Suppose that \( p : X \to Y \) is a morphism of noetherian schemes and that \( P \) is a perfect complex on \( X \). If \( y \in Y \) is a closed point such that \( H^i(P_y) = 0 \) for \( i \neq 0 \), then there is a neighborhood \( U \subseteq Y \) containing \( y \) such that \( P_U \) is (quasi-isomorphic to) a coherent \( \mathcal{O}_{U \times_Y X} \)-module flat over \( U \).

**Proof.** Taking \( Z = \text{Spec} \mathcal{O}_{Y,y} \), we can apply [14, Lemma 3.31] over this local ring (since there is only one closed point) to find that \( P_Z \) is quasi-isomorphic to a coherent sheaf flat over \( Z \). Denote by \( j \) the map \( Z \times_Y X \to X \). We have found an equivalence \( j^*P \to F \), where \( F \) is a coherent sheaf on \( Z \times_Y X \) that is flat over \( Z \). As \( j \) is flat, the higher derived functors of \( j^* \) vanish, from which it follows that \( j^*H^i(P) \cong H^i(j^*P) \). Because \( P \) is perfect, there are only finitely many non-zero cohomology sheaves, and \( j^*H^i(P) = 0 \) if \( i \neq 0 \). Hence, there is a neighborhood \( U \) of \( y \) on which \( H^i(P) \) vanishes for each \( i \neq 0 \). The lemma now follows from a second application of [14, Lemma 3.31]. ■

The following theorem is well-known to the experts and appears first (in the algebraically closed case) as far as we can tell in Bridgeland’s thesis [6]. A proof in the case that the base field is \( \mathbb{C} \) appears as [14, Corollary 5.46]. We present a slightly different proof.

**Theorem 2.2.** If \( E \) and \( F \) are elliptic curves over a field \( k \) such that \( \Phi : D^b(E) \simeq D^b(F) \) as \( k \)-linear triangulated categories, then there is an isomorphism of \( k \)-schemes \( E \cong F \).
Proof. Let \( x \) be a closed point of \( E \). We observe that there is a unique value of \( i \) such that \( \Phi^i(\mathcal{O}_x) \neq 0 \). Indeed, since \( F \) is an elliptic curve, the abelian category of quasi-coherent sheaves on \( F \) is hereditary. In particular,

\[
\Phi(\mathcal{O}_x) \cong \bigoplus_i \Phi^i(\mathcal{O}_x)
\]

in \( \mathcal{D}^b(F) \). Since \( \mathcal{O}_x \) is a simple sheaf, it follows that \( \Phi(\mathcal{O}_x) \) is simple and hence that \( \Phi^i(\mathcal{O}_x) \) is non-zero for a unique value of \( i \).

By the lemma, it follows that this \( i \) does not depend on \( x \), and moreover, \( \Phi \cong \Phi_{p_{ji}} \) where \( P \) is a coherent sheaf on \( E \times_k F \) flat over \( E \). If the \( \Phi^i(\mathcal{O}_x) \) are torsion-free, then they are stable vector bundles on \( F \) since they are simple vector bundles on a genus 1 curve. Define \( r = \text{rk}(\Phi^i(\mathcal{O}_x)) \) and \( d = \text{deg}(\Phi^i(\mathcal{O}_x)) \).

In this notation, the kernel realizes \( E \) as the moduli space \( J_F(r,d) \) of semi-stable vector bundles of rank \( r \) and degree \( d \) on \( F \). The geometric points of \( E \) correspond to simple semi-stable vector bundles on \( F_{\bar{k}} \). These are therefore stable bundles, so that \( r \) and \( d \) are relatively prime (see Bridgeland [6, Lemma 6.3.6]). The determinant map \( J_F(r,d) \to J_F(1,d) \) is hence an isomorphism by \( \Phi \cong \Phi_{p_{ji}} \).

But, tensoring with \( \mathcal{O}(d \cdot e) \) where \( e : \text{Spec} k \to F \) defines an isomorphism \( \text{Jac}(F) = J_F(1,0) \to J_F(1,d) \), so that \( E \cong F \). When the sheaves \( \Phi^i(\mathcal{O}_x) \) have torsion, they are supported at points, and since they are simple they must correspond to skyscraper sheaves of single points. Thus, the kernel \( P \) of \( \Phi \) is a translation of a line bundle on the graph of an isomorphism \( E \cong F \) by \([14, \text{Corollary 5.23}]\). \( \blacksquare \)

### 2.2 Genus 1 Curves

The main theorem of this section is an analogue of the main result of the paper in the case of curves. It fully settles when two curves are derived equivalent over any base field. Before stating the theorem we can simplify the problem using the following lemmas.

**Lemma 2.3.** If \( X \) and \( Y \) are genus 1 curves over a field \( k \) such that \( \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \), then \( X \cong J_Y(1,d) \), the fine moduli space of degree \( d \) line bundles on \( Y \).

**Proof.** Fix an equivalence \( \Phi : \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \). As in the proof of Theorem 2.2, \( X \cong J_Y(r,d) \) for some \( r \) and \( d \). We may assume that \( r \geq 1 \) or else they are skyscraper sheaves and hence \( X \cong Y \) by \([14, \text{Corollary 5.23}]\). We may also assume that \( r \) and \( d \) are coprime by \([6, \text{Theorem 6.4.3}]\).

Indeed, by base-change, we obtain an equivalence \( \mathcal{D}^b(X_{\bar{k}}) \cong \mathcal{D}^b(Y_{\bar{k}}) \) from Propositions 3.1 and 3.4. Now, since \( r \) and \( d \) are coprime, the determinant map \( J_Y(r,d) \to J_Y(1,d) \) is geometrically an isomorphism by \([2, \text{Theorem 7}]\). Hence, it is an isomorphism over \( k \). To see that \( J_Y(1,d) \) is fine, note that \( J_Y(r,d) \) is fine because of the existence of the kernel of the equivalence \( \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \). Taking the determinant of the kernel one obtains the universal object on \( J_Y(1,d) \times_k Y \). \( \blacksquare \)
Lemma 2.4. If $X$ and $Y$ are genus 1 curves over a field $k$, and if $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, then $X$ and $Y$ are homogeneous spaces under the same elliptic curve. That is, $\text{Jac}(X) \cong \text{Jac}(Y)$.

Proof. Realizing $X$ as $J_Y(1, d)$ for some $d$ by the previous lemma, we can define an action of $\text{Jac}(Y)$ by $(L, V) \mapsto L \otimes V$. This action is defined over $k$, but over $k^a$ we apply the corollary to [2, Theorem 7] to see that the kernel of the action is trivial. Moreover, the action is geometrically transitive by the same corollary. It follows that $\text{Jac}(Y)$ acts simply transitively on $X$ over $k$. But, this implies that $\text{Jac}(X) \cong \text{Jac}(Y)$ by [24, Theorem X.3.8].

By the lemma, we can focus our attention on derived equivalences among principal homogeneous spaces for a fixed elliptic curve $E$.

Theorem 2.5. If $X$ and $Y$ are principal homogeneous spaces for an elliptic curve $E$ over a field $k$, then $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ if and only if there exists an automorphism $\phi \in \text{Aut}_k(E)$ and an integer $d$ of order coprime to $\text{ord}(d)$ in $H^1(k, E)$ such that $X = \phi_d Y$.

We will prove the theorem after two lemmas.

Lemma 2.6. If $X, Y \in H^1(k, E)$ are genus 1 curves with principal homogeneous space structures under $E$, then $X$ and $Y$ are isomorphic as $k$-curves if and only if they are in the same orbit under the natural action of $\text{Aut}_k(E)$ on $H^1(k, E)$.

Note that for any elliptic curve over any field the group $\text{Aut}_k(E)$ is finite and of order dividing 24.

Proof. Suppose that $\mu_0 : E \times_k X \to X$ and $\mu_1 : E \times_k Y \to Y$ are the homogeneous space structures for $X$ and $Y$. The action of $\phi$ on $H^1(k, E)$ sends $(Y, \mu_1)$ to $(Y, \mu_1 \circ (\phi \times \text{id}_Y))$. The homogeneous spaces $(X, \mu_0)$ and $(Y, \mu_1 \circ (\phi \times \text{id}_X))$ are equal in the Weil-Châtelet group $H^1(k, E)$ if and only if there is a $k$-isomorphism $f : X \to Y$ such that $f \circ \mu_0 = \mu_1 \circ (\phi \times \text{id}_X) \circ (\text{id}_E \times f) = \mu_1 \circ (\phi \times f)$. In particular, this says that any two principal homogeneous spaces in the same $\text{Aut}_k(E)$-orbit have isomorphic underlying curves. Now, given a $k$-isomorphism $f : X \to Y$, define $\phi : E \to E$ by $\mu_0(p, x) = \mu_1(\phi(p), f(x))$. Following the reasoning in [24, Section X.3], one shows that $\phi$ is a $k$-automorphism of $E$. Hence, $X$ and $Y$ are in the same $\text{Aut}_k(E)$-orbit.

Lemma 2.7. If $Y \in H^1(k, E)$ is a principal homogeneous space for $E$, then $dY$ is the homogeneous space $J_Y(1, d)$ of degree $d$ line bundles on $Y$.

Proof. Suppose $E$ is an elliptic curve over $k$, and let $G_k$ be the Galois group of the separable closure $k^a$ over $k$. Let $\alpha \in Z^1(G_k, E(k^a))$ be a cocycle representing $Y$. We assume that $\alpha$ is constructed as follows. Pick a point $p \in E(k^a)$. Then,
\(\alpha(g) = g \cdot p \oplus p\), where subtraction is via the \(k\)-linear map \(Y \times_k Y \to E\). We adopt here a convention by which we use \(\oplus\) and \(\ominus\) for the group operations on \(E\) or the homogeneous space structure on \(Y\), reserving \(+\) and \(−\) for divisors. Hence, if \(q\) and \(r\) are points of \(Y\), then \(q \ominus r\) is the unique point \(x\) of \(E\) such that \(x \oplus r = q\). The point \(p\) defines an isomorphism \(v_p : Y_{k^e} \to E_{k^e}\) via \(q \mapsto q \ominus p\).

One way to view the cocycle is that it describes the difference between the \(G_k\)-actions on \(E(k_s)\) using the natural action and using the isomorphism \(v_p\).

Indeed, given \(x \in E(k^e)\), we can compute

\[
v_p(g \cdot v_p^{-1}(x)) = v_p(g \cdot (x \oplus p)) = v_p(g \cdot x \oplus g \cdot p) = g \cdot x \oplus g \cdot p \ominus p = g \cdot x \ominus \alpha(g).
\]

Now, perform the same calculation for \(J_Y(1, d)\). Namely, \(v_p\) defines an isomorphism \(J_{Y_{k^e}}(1, d) \to J_{E_{k^e}}(1, d)\), and we can compose with an isomorphism \(J_{E_{k^e}}(1, d) \to E_{k^e}\) to find a cocycle representation of \(J_Y(1, d)\). Let \(e\) denote the identity element of \(E(k^e)\). Then, we view \(J_{E}(1, d)\) as being isomorphic to \(J_{E}(1, 0) \cong E\) via subtraction of the divisor \(de\). By abuse of notation, we write \(v_p\) for the composite isomorphism \(J_{Y_{k^e}}(1, d) \to J_{E_{k^e}}(1, 0)\). We compute for an element \(x \in E(k^e)\) that

\[
v_p(g \cdot v_p^{-1}(x - e)) = v_p(g \cdot ((x \oplus p) + (d - 1)p))
\]

\[
= v_p((g \cdot x \oplus g \cdot p) + (d - 1)(g \cdot p))
\]

\[
= (g \cdot x \oplus g \cdot p \ominus p) + (d - 1)(g \cdot p \ominus p) - de
\]

\[
= g \cdot x + (g \cdot p \ominus p) - e + (g \cdot p \ominus p) \oplus (d - 1) + (d - 2)e - de
\]

\[
= g \cdot x + (g \cdot p \ominus p) \oplus d - 2e
\]

\[
= g \cdot x \oplus (g \cdot p \ominus p) \oplus d - e,
\]

using the definition of the group law on \(E(k^e)\) and the fact that \(e\) is \(G_k\)-fixed.

The lemma follows. \(\blacksquare\)

Now, we prove the main result of the section.

**Proof of Theorem 2.5.** Assume that \(D^\delta(X) \simeq D^\delta(Y)\). Then, by Lemma 2.3, as a variety, \(X\) is isomorphic to \(J_Y(1, d)\) for some \(d\), but this isomorphism might not take into account the homogeneous space structures on each side. It follows from Lemma 2.6 that \(X = \phi_* J_Y(1, d)\) for some \(\phi \in Aut_k(E)\), and now from Lemma 2.7 that \(X = \phi_* dY\). As this argument is entirely symmetric, \(Y = \psi_* eX\) for some \(\psi \in Aut_k(E)\). Hence, the orders of \(Y\) and \(\psi_* \phi_* edY\) in \(H^1(k, E)\) agree. As the order of \(\psi_* \phi_* edY\) is that of \(edY\), we must have that \(ed\) is coprime to \(\text{ord}(|Y|)\), as desired.

Now, suppose that \(X = \phi_* dY\) for some \(\phi \in Aut_k(E)\) and some \(d\) prime to \(\text{ord}(|Y|)\). In order to prove that \(D^\delta(X) \simeq D^\delta(Y)\), using Lemma 2.6, we see that it is enough to consider the case when \(\phi\) is the identity, which is to say, by Lemma 2.7, when \(X = J_Y(1, d)\). We conclude by showing that there
is a universal sheaf on $J_Y(1, d) \times_k Y$, which induces an equivalence $D^b(X) \simeq D^b(Y)$.

As $J_Y(1, d)$ is the moduli space of a $G_m$-gerbe, the obstruction class is an element $\alpha \in Br(J_Y(1, d)) \subseteq Br(K)$, where $K$ is the function field of $J_Y(1, d)$. In particular, $\alpha = 0$ if and only if $\alpha|_\eta = 0$, where $\eta : \text{Spec } K \to J_Y(1, d)$ is the generic point. If $d$ is relatively prime to $\text{ord}(|Y|)$, then $Y$ and $J_Y(1, d)$ generate the same subgroup of $H^1(k, E)$. As $J_Y(1, d)$ is in the kernel of $H^1(k, E) \to H^1(K, E_K)$, it follows that $Y$ has a $K$-rational point as well. But, as soon as there is a $K$-rational point, $J_{Y_K}(1, d)$ is a fine moduli space. Using the rational point, there is a factorization of $Br(J_Y(1, d)) \to Br(K)$ through $Br(J_Y(1, d)) \to Br(J_{Y_K}(1, d))$.

Hence, $\alpha = 0$.

### 2.3 Examples and Applications

In this section we show that in some typical situations it is still the case that genus 1 curves are derived equivalent if and only if they are isomorphic. We also show that when this is not the case, it provides counterexamples to an open problem about moduli of stable vector bundles on genus 1 curves.

**Example 2.8.** If $X$ and $Y$ are derived equivalent genus 1 curves over a finite field $k = \mathbb{F}_q$, then $X \cong Y$.

**Proof.** Lang’s theorem tells us in general that if $X/k$ is geometrically an abelian variety, then $X(k) \neq \emptyset$. Thus $X$ and $Y$ are elliptic curves over $k$. Now by Theorem 2.2 we have that $X \cong Y$. ■

**Example 2.9.** If $X$ and $Y$ are derived equivalent genus 1 curves over $\mathbb{R}$, then $X \cong Y$.

**Proof.** Let $E = \text{Jac}(X)$. If $X$ and $Y$ are not isomorphic, then they must represent two distinct $\text{Aut}_k(E)$-orbits in $H^1(\mathbb{R}, E)$. Our goal is to show that $H^1(\mathbb{R}, E)$ is too small to allow non-isomorphic torsor classes. There are two cases corresponding to whether or not $E$ has full 2-torsion defined over $\mathbb{R}$. We merely check that $|H^1(\mathbb{R}, E)| \leq 2$. This suffices because the two classes cannot be derived equivalent by Theorem 2.2.

Since $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$, we know that $H^1(\mathbb{R}, E)$ is killed by 2. This tells us computing the whole group is equivalent to computing the 2-torsion part. In the case that $E$ has full 2-torsion defined over $\mathbb{R}$, consider the Kummer sequence

$$0 \to E[2] \to E \xrightarrow{2} E \to 0.$$ 

This induces an exact sequence

$$0 \to E(\mathbb{R})/2E(\mathbb{R}) \to H^1(\mathbb{R}, E[2]) \to H^1(\mathbb{R}, E) \to 0.$$ 

The middle group consists of non-twisted homomorphisms $\text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2 \times \mathbb{Z}/2)$ because the 2-torsion is fully defined over $\mathbb{R}$, so the Galois action is trivial. Since the Weil pairing is non-degenerate and Galois invariant, we know
that the full 4-torsion is not defined over $\mathbb{R}$ otherwise $\mathbb{R}^\times$ would contain four distinct roots of unity. This means that $[2] : E(\mathbb{R}) \to E(\mathbb{R})$ is not surjective and hence $E(\mathbb{R})/2E(\mathbb{R})$ is non-trivial. This proves the inequality.

The other case is that $E[2](\mathbb{R}) \cong \mathbb{Z}/2$. In this case we can explicitly write down in terms of elements $E[2] = \{1, a, b, c\}$. Without loss of generality, we assume $a^\sigma = a$, $b^\sigma = c$ and $c^\sigma = b$ where $\sigma$ is the non-trivial element of $G_{\mathbb{R}}$. The condition on $\rho : G_{\mathbb{R}} \to E[2]$ being a twisted homomorphism forces $\rho(1) = 1$ and $\rho(\sigma) = 1$ or $a$. Thus there are only two possible cocycles. The non-trivial one is actually also a coboundary, since $\rho(\sigma) = b^\sigma - b$. Thus $H^1(\mathbb{R}, E[2]) = 0$, which forces $H^1(\mathbb{R}, E) = 0$. ■

**Example 2.10.** There exist non-isomorphic derived equivalent genus 1 curves.

*Proof.* Fix $E/\mathbb{Q}$ a non-CM elliptic curve with $j(E) \neq 0, 1728$. Consider a genus 1 curve $X \in H^1(\mathbb{Q}, E)$ with period 5. The cyclic subgroup generated by $X$ has order 5 and hence all four non-split classes are generators. Only one other of these generators can be isomorphic as a $\mathbb{Q}$-curve by Lemma 2.6. But by Theorem 2.5 any non-isomorphic generator is a non-isomorphic derived equivalent curve. ■

**Example 2.11.** For any $N > 0$, there exists a genus 1 curve $Y$ that admits at least $N$ distinct moduli spaces of stable vector bundles $\mathcal{J}_Y(r, d)$.

*Proof.* We can again fix $E/\mathbb{Q}$ a non-CM elliptic curve with $j(E) \neq 0, 1728$ and $N > 0$. Choose a prime $p > 3N$. There exists a cyclic subgroup of $H^1(\mathbb{Q}, E)$ of order $p$. Since $\text{Aut}_\mathbb{Q}(E) \cong \mathbb{Z}/2$, there are more than $p/2 > N$ non-isomorphic generators. By Theorem 2.5, any two of these generators, $X$ and $Y$, are derived equivalent. Thus $X \cong \mathcal{J}_Y(r, d)$ for some $r$ and $d$ coprime. ■

This result is a rather surprising contrast to the results of Atiyah which says that fine moduli spaces of vector bundles on an elliptic curve are always isomorphic to the elliptic curve. More recently, Pumplün [22] even extended some of these results to work in a more general genus 1 setting suggesting that these types of examples might not exist.

## 3 Background on dg categories

In this section we give a brief introduction to dg categories. For details and further references, consult Keller [15].

### 3.1 Definitions

A *dg category* $\mathcal{C}$ over a commutative ring $R$, also called an $R$-linear dg category, consists of

1. a class of objects $\text{ob}(\mathcal{C})$;
2. for each pair of objects \( x \) and \( y \) a chain complex \( \text{Map}_\mathcal{C}(x, y) \) of \( R \)-modules;  
3. for each triple of objects \((x, y, z)\) a morphism of degree 0  
\[ \text{Map}_\mathcal{C}(y, z) \otimes_R \text{Map}_\mathcal{C}(x, y) \rightarrow \text{Map}_\mathcal{C}(x, z) \]

of chain complexes of \( R \)-modules

satisfying obvious analogues of the unit and associativity axioms of a category. Note that the tensor product is the underived tensor product. For the sake of concreteness, we fix the sign conventions appearing in Keller’s ICM talk [15].

When \( \text{ob}(\mathcal{C}) \) is a set (and not a proper class), we say that \( \mathcal{C} \) is small. An example is the dg category \( \mathcal{C} \) consisting of a single point \(*\) where \( \text{Map}_\mathcal{C}(*, *) \) is any fixed dg algebra. Let \( \text{dgcat}_R \) denote the category of small dg categories over \( R \). A morphism in \( \text{dgcat}_R \) is a dg functor, i.e., the data of a function \( F : \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D}) \) together with functorial morphisms of chain complexes \( \text{Map}_\mathcal{C}(x, y) \rightarrow \text{Map}_\mathcal{D}(F(x), F(y)) \) for all objects \( x, y \).

The homotopy category \( \text{Ho}(\mathcal{C}) \) of a dg category \( \mathcal{C} \) is the additive \( R \)-linear category obtained by taking the same objects as \( \mathcal{C} \), but where \( \text{Hom}_{\text{Ho}(\mathcal{C})}(x, y) = H^0 \text{Map}_\mathcal{C}(x, y) \). A quasi-equivalence of dg categories is a dg functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) such that \( \text{Map}_\mathcal{C}(x, y) \rightarrow \text{Map}_\mathcal{D}(F(x), F(y)) \) is a quasi-equivalence for all \( x, y \in \mathcal{C} \) and such that \( \text{Ho}(F) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D}) \) is essentially surjective. Note that these conditions imply that \( \text{Ho}(F) \) is an equivalence of categories.

Another construction is \( \text{Z}^0(\mathcal{C}) \), a category with the same class of objects as \( \mathcal{C} \), but in which \( \text{Hom}_{\text{Z}^0(\mathcal{C})}(x, y) = \text{Z}^0 \text{Map}_\mathcal{C}(x, y) \).

### 3.2 Big and small dg categories

The key notion in the study of dg categories is the idea of a right module over a small dg category \( \mathcal{C} \). This is simply a dg functor

\[ M : \mathcal{C}^{\text{op}} \rightarrow \text{Ch}_\text{dg}(R), \]

where \( \text{Ch}_\text{dg}(R) \) is the dg category of complexes of \( R \)-modules. The right modules over \( \mathcal{C} \) form a dg category we will call \( \text{Mod}_\text{dg}(\mathcal{C}) \). We refer to [15] for the definition of the mapping complexes in \( \text{Mod}_\text{dg}(\mathcal{C}) \), and for the important projective model category structure on \( \text{Z}^0 \text{Mod}_\text{dg}(\mathcal{C}) \).

In the special case of a dg algebra \( A \) viewed as a dg category with one object, giving a right module \( M : A^{\text{op}} \rightarrow \text{Ch}_\text{dg}(R) \) is the same as giving a chain complex of \( R \)-modules \( M \) together with a map of dg algebras \( A^{\text{op}} \rightarrow \text{End}_R(M) \). That is, we recover the usual sense of right \( A \)-module.

There is a model category structure on \( \text{dgcat}_R \) in which the weak equivalences are the quasi-equivalences of dg categories [25]. For the purposes of this paper, the derived category of a small \( R \)-linear dg category \( \mathcal{C} \) is the dg category \( \text{D}_\text{dg}(\mathcal{C}) = \text{Mod}_\text{dg}(\mathcal{C}^{\text{op}}) \), the full dg subcategory of cofibrant objects (with respect to the projective model structure) in \( \text{Mod}_\text{dg}(\mathcal{C}^{\text{op}}) \), where \( \mathcal{C}^{\text{op}} \) is a cofibrant replacement for \( \mathcal{C} \) in \( \text{dgcat}_R \). This is a large dg category.
The dg categories \(D_{\text{dg}}(\mathcal{C})\) that arise in this way are very special. Their homotopy categories \(\text{Ho}(D_{\text{dg}}(\mathcal{C}))\) are triangulated categories that are closed under arbitrary coproducts, compactly generated, and locally small. We call any such dg category a **compactly generated stable presentable** \(R\)-linear dg category. This terminology differs slightly from that used in most literature on dg categories, and is derived instead from the language of stable \(\infty\)-categories, as developed in \([16]\).

Here is another way to describe \(D_{\mathcal{C}}\). A map \(M \to N\) of right modules over \(\mathcal{C}\) is a quasi-isomorphism if the induced map \(M(X) \to N(X)\) is a quasi-isomorphism in \(\text{Ch}_{\text{dg}}(R)\) for each object \(X\) of \(\mathcal{C}\). The derived category of \(\mathcal{C}\) is the localization of \(\text{Ho}(\text{Mod}_{\text{dg}}(\mathcal{C}))\) at the quasi-isomorphisms. It is in fact equivalent to \(D_{\mathcal{C}}\). Besides providing a dg model for \(D_{\mathcal{C}}\), the construction of \(D_{\text{dg}}(\mathcal{C})\) above serves to show that this localization actually exists. Details can be found in \([15]\).

Note the correspondence between the small dg category \(\mathcal{C}\) and its category of right modules \(\text{Mod}_{\text{dg}}(\mathcal{C})\). This correspondence becomes tighter if we use pretriangulated small dg categories. Call a small dg category \(\mathcal{C}\) pretriangulated if the image of the Yoneda embedding \(\text{Ho}(\mathcal{C}) \to D(\mathcal{C})\) is stable under shifts and extensions. A pretriangulated small dg category \(\mathcal{C}\) is idempotent-complete if this image is also stable under summands.

### 3.3 Sheaves and dg categories

If \(X\) is an \(R\)-scheme, there is an \(R\)-linear dg category \(\text{QC}_{\text{dq}}(X)\) of complexes of \(\mathcal{O}_X\)-modules with quasi-coherent cohomology sheaves. This dg category has a full dg subcategory \(\text{Perf}_{\text{dq}}(X) \subseteq \text{QC}_{\text{dq}}(X)\) consisting of the perfect complexes. These are the complexes of \(\mathcal{O}_X\)-modules that are Zariski-locally quasi-isomorphic to bounded complexes of vector bundles. By a theorem of Bondal and van den Bergh \([4]\), \(\text{QC}_{\text{dq}}(X)\) is quasi-equivalent to \(D_{\text{dg}}(\text{Perf}_{\text{dq}}(X))\).

There is an equivalence \(D(\text{QC}_{\text{dq}}(X)) \simeq D_{\text{qc}}(X)\), where \(D_{\text{qc}}(X)\) is the usual triangulated category of complexes of \(\mathcal{O}_X\)-modules with quasi-coherent cohomology sheaves. The full subcategory of \(D(\text{QC}_{\text{dq}}(X))\) consisting of objects quasi-isomorphic to objects in \(\text{Perf}_{\text{dq}}(X)\) is \(\text{Perf}(X)\), the triangulated category of perfect complexes. If \(X\) is quasi-compact and quasi-separated, the perfect complexes \(\text{Perf}(X) \subseteq D_{\text{qc}}(X)\) have a purely categorical description. Namely, by Bondal and van den Bergh \([4]\, \text{Theorem 3.1.1}\) they are the complexes \(x \in D_{\text{qc}}(X)\) such that the functor \(\text{Hom}_{D_{\text{qc}}(X)}(x, -) : D_{\text{qc}}(X) \to \text{Mod}_R\) commutes with coproducts.

When \(X\) is regular and noetherian, the natural inclusion of triangulated categories \(\text{Perf}(X) \to D^b(X)\) is an equivalence. This is because any bounded complex of coherent \(\mathcal{O}_X\)-modules has a finite-length resolution by vector bundles Zariski-locally.

**Proposition 3.1.** If \(X\) and \(Y\) are quasi-compact and quasi-separated and if \(D_{\text{qc}}(X) \simeq D_{\text{qc}}(Y)\), then \(\text{Perf}(X) \simeq \text{Perf}(Y)\). If \(X\) and \(Y\) are smooth projective schemes over a field \(k\), then the following are equivalent:
1. $\mathcal{D}_{qc}(X) \simeq \mathcal{D}_{qc}(Y)$ as $k$-linear triangulated categories;

2. $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ as $k$-linear triangulated categories;

3. $\text{Perf}_{dg}(X) \simeq \text{Perf}_{dg}(Y)$ as $k$-linear dg categories;

4. $\text{QC}_{dg}(X) \simeq \text{QC}_{dg}(Y)$ as $k$-linear dg categories.

Proof. If $\mathcal{D}_{qc}(X) \simeq \mathcal{D}_{qc}(Y)$, then the subcategories of compact objects coincide, as these are preserved by any equivalence of triangulated categories. Since these subcategories are precisely the categories of perfect complexes, we have $\text{Perf}(X) \simeq \text{Perf}(Y)$. If $X$ and $Y$ are regular and noetherian, then $\mathcal{D}_{qc}(X) \simeq \mathcal{D}_{qc}(Y)$ implies $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ by the first part and by our remark in the previous paragraph. Hence, (1) implies (2).

Suppose now that $F : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ is an equivalence. Then, by an important theorem of Orlov [18], there is a complex $P \in \mathcal{D}^b(X \times_k Y)$ such that the Fourier-Mukai functor $\Phi_P : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ agrees with $F$, where $\Phi_P$ is defined by

$$\Phi_P(x) = R\pi_{Y,*}(P \otimes^L \pi^*_X x),$$

and where $\pi_X$ and $\pi_Y$ denote the projections from $X \times_k Y$. Because we have this nice model for $F$, $\Phi_P$ extends to a functor $\text{Perf}_{dg}(X) \rightarrow \text{Perf}_{dg}(Y)$ which is by definition a quasi-equivalence, so that (2) implies (3). That (3) implies (4) is immediate from the description of $\text{QC}_{dg}(X)$ as $\mathcal{D}_{dg}(\text{Perf}_{dg}(X))$. Finally, that (4) implies (1) follows by taking homotopy categories.

Remark 3.2. Because of the second statement of the proposition, we work everywhere with $\mathcal{D}_{qc}(X)$ and its dg enhancement $\text{QC}_{dg}(X)$ in the rest of this paper.

Remark 3.3. The converse to the first statement would hold if we required a possibly stronger condition, namely that the dg enhancements $\text{Perf}_{dg}(X)$ and $\text{Perf}_{dg}(Y)$ are quasi-equivalent. Indeed, $\text{QC}_{dg}(X)$ can be constructed from $\text{Perf}_{dg}(X)$ as we have remarked above.

3.4 Base change

If $\mathcal{C}$ is a small $R$-linear dg category and $S$ is a commutative $R$-algebra, we denote by $\mathcal{C}_S$ the $S$-linear dg category with the same objects as $\mathcal{C}$, but where

$$\text{Map}_{\mathcal{C}_S}(x, y) = \text{Map}_{\mathcal{C}}(x, y) \otimes_R S$$

for $x, y \in \text{ob}(\mathcal{C})$.

If $\mathcal{Q}_{dg} = \mathcal{D}_{dg}(\mathcal{C})$, we define $\mathcal{Q}_{dg,S} = \mathcal{D}_{dg}(\mathcal{C}_S)$. If $\mathcal{C}$ is a small idempotent-complete pretriangulated dg category, we define $\mathcal{C}_S$ as the full subcategory of compact objects in $\mathcal{Q}_{dg,S}$. Note that in this case we have now overloaded the definition of the base change of a small idempotent-complete pretriangulated dg category. We will however always mean the latter when $\mathcal{C}$ is of this form.
We choose this definition so that the result is also pretriangulated and idempotent complete, and so that the following examples hold. If $A$ is an $R$-algebra, then $D_{dg}(A)_S \simeq D_{dg}(A \otimes_R S)$. If either $X$ is a flat $R$-scheme or $S$ is a flat commutative $R$-algebra, then $QC_{dg}(X)_S \simeq QC_{dg}(X_S)$, and $Perf_{dg}(X)_S = Perf_{dg}(X_S)$.

### 3.5 Stacks of dg categories

Later in the paper, we will need to use stacks of dg categories. We take the perspective of [26, Section 3], and explain briefly what we mean here.

A stack of stable presentable dg categories $\mathcal{Q}_{dg}$ on a scheme $X$ in some topology $\tau$ is an assignment of a stable presentable dg category (what Toën calls locally presentable) $\mathcal{Q}_{dg}(\text{Spec } S)$ for each map $\text{Spec } S \rightarrow X$, together with the assignment of pullback maps $f^*: \mathcal{Q}_{dg}(\text{Spec } S) \rightarrow \mathcal{Q}_{dg}(\text{Spec } T)$ that preserve homotopy colimits for each map $f: \text{Spec } T \rightarrow \text{Spec } S$ of affine $X$-schemes. One needs to fix moreover the various data that encode the composition functions and so on. Finally, one requires that whenever $S \rightarrow T^\bullet$ is a hypercover in the $\tau$-topology, the natural map

$$\mathcal{Q}_{dg}(\text{Spec } S) \rightarrow \text{holim}_{\Delta} \mathcal{Q}_{dg}(\text{Spec } T^\bullet)$$

is a quasi-equivalence. This homotopy limit is taken in an appropriate model category or $\infty$-category of big $R$-linear dg categories. See Toën [26, Section 3] for details and references.

There are a few remarks about how we will use these that need to be made. First of all, define a stable presentable $R$-linear dg category with descent to be a stable presentable $R$-linear category $Q_{dg}$ such that if $R \rightarrow S$ is a map of commutative rings, and if $S \rightarrow T^\bullet$ is a $\tau$-hypercover, then the natural map

$$Q_{dg,S} \rightarrow \text{holim}_{\Delta} Q_{dg,T^\bullet}$$

is a quasi-equivalence. If $X = \text{Spec } R$ is itself affine, then giving a stack of stable presentable dg categories $\mathcal{Q}_{dg}$ is equivalent to giving the stable presentable $R$-linear dg category with descent $Q_{dg} = Q_{dg}(\text{Spec } R)$.

Second, any dg category $D_{dg}(\mathcal{C})$, where $\mathcal{C}$ is a small $R$-linear dg category, is a stable presentable $R$-linear dg category with fpqc descent. This important fact follows from [26, Corollary 3.8].

Third, as a consequence, if $X$ is a quasi-compact and quasi-separated $R$-scheme, the dg category $QC_{dg}(X)$ is a stable presentable $R$-linear dg category with descent. Indeed, in this case, $QC_{dg}(X) = D_{dg}(\text{Perf}_{dg}(X))$ by [4]. It follows that if $X \rightarrow B$ is a flat quasi-compact and quasi-separated morphism of schemes, then the functor $\mathcal{Q}_{dg}(\mathcal{C})^X$ on $B$, which assigns to any Spec $S \rightarrow B$ the $S$-linear dg category $\mathcal{Q}_{dg}(\text{Spec } S) = QC_{dg}(X_S)$ is an fpqc stack of stable presentable dg categories on $B$. 


Fourth, by restricting the class of affines we test on, given a stack \( \mathcal{Q} \) of stable presentable \( R \)-linear categories on \( X \) and a map \( f : Y \to X \), we can obtain a stack \( f^* \mathcal{Q} \) on \( X \).

Fifth, there is an obvious notion of an equivalence of stacks \( \mathcal{Q} \simeq \mathcal{P} \), which results by specifying equivalences \( \mathcal{Q}(\text{Spec } S) \to \mathcal{P}(\text{Spec } S) \), specifying compatibilities with the pullback functors in each stack, and specifying various higher homotopy coherences. Of particular importance is that if \( X \) and \( Y \) are \( R \)-schemes at least one of which is flat over \( R \), then any complex \( P \) in \( D_{\text{qc}}(X \times_R Y) \) gives rise to a Fourier-Mukai functor \( \Phi_P : \mathcal{D}_X \to \mathcal{D}_Y \) by the usual formula.

**Proposition 3.4.** Let \( \Phi : \text{QC}_{\text{dg}}(X) \to \text{QC}_{\text{dg}}(Y) \) be a morphism of dg categories over \( k \) such that \( \Phi_{k^s} : \text{QC}_{\text{dg}}(X_{k^s}) \to \text{QC}_{\text{dg}}(Y_{k^s}) \) is an equivalence. Then, \( \Phi \) is an equivalence.

**Proof.** This follows from the fact that the assignment \( I \mapsto \text{QC}_{\text{dg}}(X_I) \) is actually an fpqc stack of stable presentable dg categories on \( \text{Spec } k \) by Toën [26, Proposition 3.7]. In particular, there is a commutative diagram

\[
\begin{array}{ccc}
\text{QC}_{\text{dg}}(X) & \longrightarrow & \text{holim}_\Delta \text{QC}_{\text{dg}}(X(k^s)^{\otimes n}) \\
\Phi & & \Phi_{(k^s)^{\otimes n}} \\
\text{QC}_{\text{dg}}(Y) & \longrightarrow & \text{holim}_\Delta \text{QC}_{\text{dg}}(Y(k^s)^{\otimes n})
\end{array}
\]

in which the horizontal arrows are equivalences, and where the tensor products \( (k^s)^{\otimes n} \) are taken over \( k \). By base change, the right-hand vertical arrow is also an equivalence by our hypothesis. It follows that \( \Phi \) is an equivalence. ■

### 4 The twisted Brauer space

This paper can be seen as a contribution to the arithmetic theory of derived categories. To begin this study, one must understand how to twist a given derived category. This problem was posed in [1], where the twisted Brauer space was introduced as a computational tool for classifying twists\(^1\). As motivation, consider the following fact due to Toën [26]: the Brauer group \( \text{Br}(k) \) of a field classifies the stable presentable \( k \)-linear categories with descent \( \mathcal{Q}_{\text{dg}} \), such that there is a finite separable field extension \( k \to l \) where \( \mathcal{Q}_{\text{dg}}(l) \simeq \mathcal{D}_{\text{dg}}(\text{Spec } l) \).

More generally, over a quasi-compact and quasi-separated scheme \( X \), stacks of locally presentable dg categories on \( X \) that are étale locally equivalent to the canonical stack \( \text{QC}_{\text{dg}} \) are classified up to equivalence of stacks by the derived Brauer group \( H^1_{\text{ét}}(X, \mathbb{Z}) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \). Note that we use the entire group

\(^1\)In [1], stable ∞-categories were used instead of dg categories. No changes are needed to carry out the same arguments in the dg setting.
$H^2_{ét}(X, G_m)$. If $X$ is normal, then $H^1_{ét}(X, \mathbb{Z}) = 0$. If $X$ is regular and noetherian, then in addition $H^2_{ét}(X, G_m)_{\text{tors}} = H^2_{ét}(X, G_m)$, so that the derived Brauer group of $X$ coincides with the cohomological Brauer group in this case.

Let $Y \to X$ be a flat quasi-compact and quasi-separated morphism of schemes, and let $\mathcal{D}^Y_{dg}$ be the associated stack of stable presentable dg categories on $X$, as described in Section 3.5, which we view as an étale stack on $X$. Another étale stack $\mathcal{D}_Y$ of locally presentable dg categories over $X$ is étale locally equivalent to $\mathcal{D}^Y_{dg}$ if there is an étale cover $p : \text{Spec } S \to X$ such that $p^* \mathcal{D}_Y \simeq p^* \mathcal{D}^Y_{dg}$ as $S$-linear dg categories. Equivalently, we should have $\mathcal{D}_Y(\text{Spec } S) \simeq D_{dg}(Y_S)$.

**Definition 4.1.** Let $f : Y \to X$ be a flat map of schemes. Define $\text{Br}^Y(X)$ to be the set of étale stacks of locally presentable dg categories on $X$ that are étale locally equivalent to $\mathcal{D}^Y_{dg}$, modulo equivalence of stacks over $X$. We view $\text{Br}^Y(X)$ as a pointed set, with the point being the equivalence class defined by $\mathcal{D}^Y_{dg}$.

**Example 4.2** ([1]). Suppose that $C$ is the genus 0 curve defined by the equation $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_R$. Then, $C(R) = \emptyset$, and $C$ is not rational. In particular, $D_{qc}(C)$ is not equivalent to $D_{qc}(\mathbb{P}^1_R)$. However, after extension to the complex numbers, we do have $D_{qc}(\mathbb{C}_C) \simeq D_{qc}(\mathbb{P}^1_C)$. Hence, $\mathcal{D}_{dg}^C$ defines a non-trivial element $\text{Br}^{\mathbb{P}^1}(\mathbb{R})$.

This pointed set was defined and realized as the set of connected components $\pi_0 \text{Br}^Y(X)$ in [1], where $\text{Br}^Y(X)$ is a certain topological space. In fact, this space is itself the space of global sections of an étale hypersheaf of spaces on the big étale site of $X$. This observation is useful for actually computing $\text{Br}^Y(X)$.

As discussed in [1, Section 3.2], there is a fibre sequence of sheaves of spaces on $X$

$$K(\text{HH}^0(Y)^\times, 2) \to \text{Br}^Y \to K(\text{Aut}_{\text{QC}_{dg}}(Y), 1),$$

where $K(A, n)$ denotes the Eilenberg-MacLane sheaf of spaces for the sheaf $A$, $\text{HH}^0(Y)^\times$ denotes the sheaf of units in the degree 0 Hochschild cohomology ring of $Y$, and $\text{Aut}_{\text{QC}_{dg}}(Y)$ is the sheaf of autoequivalences of the stack $\mathcal{D}^Y_{dg}$ on $X$. Since $\text{HH}^0(Y)^\times$ is a sheaf of abelian groups, when the action of $\text{Aut}_{\text{QC}_{dg}}(Y)$ on $\text{HH}^0(Y)^\times$ is trivial the sequence can be delooped, and we can identify $\text{Br}^Y$ with the fiber in

$$\text{Br}^Y \to K(\text{Aut}_{\text{QC}_{dg}}(Y), 1) \to K(\text{HH}^0(Y)^\times, 3).$$

Recall that if $K(A, n)$ is an Eilenberg-MacLane sheaf, then

$$\pi_i(\Gamma(X, K(A, n))) \cong H^{n-i}(X, A)$$

for $0 \leq i \leq n$ and 0 otherwise. Suppose now that $Y$ is smooth, proper, and geometrically connected over $X$. Then, $\text{HH}^0(Y)^\times \cong G_{m,X}$. By taking sections of
the fiber sequence above, and then taking the long exact sequence in homotopy, we obtain the exact sequence
\[
0 \to G_{m,X}(X) \to \pi_2 Br^Y(X) \to 0
\]
\[
\to H^1_{et}(X, G_{m,X}) \to \pi_1 Br^Y(X) \to \text{Aut}_{QC_{dg}(Y)}(X) \to H^2_{et}(X, G_{m,X})
\]
\[
\to \pi_0 Br^Y(X) \to H^1_{et}(X, \text{Aut}_{QC_{dg}(Y)}) \to H^3_{et}(X, G_{m,X}).
\]

Exactness is a slightly touchy matter here, as the last line is an exact sequence of pointed sets. It means that there is an action of \(H^2_{et}(X, G_{m,X})\) on \(\pi_0 Br^Y(X)\) and the fibers of \(\pi_0 Br^Y(X) \to H^1_{et}(X, \text{Aut}_{QC_{dg}(Y)})\) are precisely the orbits. Exactness at \(H^1_{et}(X, \text{Aut}_{QC_{dg}(Y)})\) says only that the fiber of the map to \(H^3_{et}(X, G_{m,X})\) over 0 is the image of \(\pi_0 Br^Y(X)\).

The action of \(H^2_{et}(X, G_{m})\) on \(\pi_0 Br^Y(X)\) can be described as follows. Given \(\alpha \in H^2_{et}(X, G_{m})\) and \(\mathcal{C}_{dg} \in \pi_0 Br^Y(X)\), we simply form the tensor product \(\mathcal{C}_{dg}^a \otimes \mathcal{C}_{dg}^X\), where \(\mathcal{C}_{dg}^a\) is the stack that assigns to \(\text{Spec } S \to X\) the dg category \(QC_{dg}(\text{Spec } S, \alpha)\), the dg category of complexes of \(\alpha\)-twisted \(O_X\)-modules with quasi-coherent \(\alpha\)-twisted cohomology sheaves. The basic example is \(\mathcal{C}_{dg}^a \otimes \mathcal{C}_{dg}^X \simeq \mathcal{C}_{dg}^a(X, \alpha)\), the stack whose sections over \(\text{Spec } S \to X\) is the dg category \(QC_{dg}(X, S, \alpha)\). Since \(\alpha\) and \(\mathcal{C}\) are both \(\text{etale}\) locally trivial, so is their tensor product.

## 5 Derived equivalences of principal homogeneous spaces

Orlov [19] and Polishchuk [20] have demonstrated that the group \(U(A \times_k \hat{A})\) of isometric automorphisms of \(A \times_k \hat{A}\) plays a central role in the study of derived autoequivalences of \(D^b(A)\). Recall that \(U(A \times_k \hat{A})\) is the group of automorphisms
\[
\sigma = \begin{pmatrix} x & y \\ z & w \end{pmatrix}
\]
of the abelian \(k\)-variety \(A \times_k \hat{A}\) such that
\[
\sigma^{-1} = \begin{pmatrix} \hat{w} & -\hat{y} \\ -\hat{z} & \hat{x} \end{pmatrix},
\]
where \(x\) is a homomorphism \(A \to A\), \(y\) is a homomorphism \(\hat{A} \to A\), and so forth.

Orlov [19] showed that there is a representation of \(\text{Aut}_{D^b(A)}\) on \(U(A \times_k \hat{A})\) with kernel precisely the subgroup \(Z \times (A \times_k \hat{A})(k)\). Moreover, when \(k\) is algebraically closed, the map \(\text{Aut}_{D^b(A)} \to U(A \times_k \hat{A})\) is surjective. This follows
from Orlov [19] when \( k \) has characteristic 0, and from Polishchuk [20, Theorem 15.5] in general.

Note that \( \text{Aut}_{D^b(A)} \) is isomorphic to \( \text{Aut}_{\text{QC}_{\text{dg}}(A)} \). Indeed, by Orlov’s representability theorem, every derived automorphism of \( D^b(A) \) has a kernel and hence lifts to an automorphism of \( \text{Perf}_{\text{dg}}(A) \). But, this then extends to an autoequivalence of \( \text{QC}_{\text{dg}}(A) \). The inverse map is given by observing that any autoequivalence of \( \text{QC}_{\text{dg}}(A) \) must preserve compact objects.

Returning to the case where \( k \) is an arbitrary field, from the work of Orlov, we have an exact sequence

\[
0 \to \mathbb{Z} \times A \times \hat{k} \to \text{Aut}_{\text{QC}_{\text{dg}}(A)} \to U^A
\]

of sheaves of groups over \( \text{Spec} k \), where \( U^A \) denotes the sheaf with \( U^A(l) = U((A \times \hat{k} \hat{A}))_l \) for an extension \( l/\mathbb{k} \). Since the right-hand map is surjective on algebraically closed fields, it follows that when \( k \) is perfect, the map is a surjective map of sheaves. In any case, let \( V^A \) denote the image as \( \text{étale} \) sheaf. So, \( V^A \) is a subsheaf of \( U^A \), and its sections over a field \( l \) is some subgroup of the group of unitary automorphisms of \( (A \times \hat{k} \hat{A})_l \). We have an exact sequence

\[
0 \to \mathbb{Z} \times A \times \hat{k} \to \text{Aut}_{\text{QC}_{\text{dg}}(A)} \to V^A \to 1
\]

of \( \text{étale} \) sheaves of groups on \( \text{Spec} k \).

Recall from [23, Section I.5.5] that in this setting there is an action of \( V^A(l) \subseteq U((A \times \hat{k} \hat{A})_l) \) on \( H^1_{\text{ét}}(l, A \times \hat{k} \hat{A}) \), which we will denote by \( \sigma \circ \left( \begin{array}{c} X \\ Y \end{array} \right) \), when \( \sigma \) is a unitary isomorphism defined over \( l \), \( X \) is a principal homogeneous space for \( A \), and \( Y \) is a principal homogeneous space for \( \hat{A} \). Note that \( U((A \times \hat{k} \hat{A})_l) \) also acts in an obvious way on \( H^1_{\text{ét}}(l, A \times \hat{k} \hat{A}) \), via automorphisms of the coefficients. We will denote this action by \( \sigma \cdot \left( \begin{array}{c} X \\ Y \end{array} \right) \). These two actions coincide if and only if the boundary map

\[
\delta : V^A(l) \to H^1_{\text{ét}}(l, A \times \hat{k} \hat{A})
\]

vanishes by [23, Proposition 40].

The following theorem and its corollaries give the main application of the theory of twisted Brauer spaces in our paper.

**Theorem 5.1.** Let \( A \) be an abelian variety over a field \( k \). Suppose that \( X \) and \( Y \) are principal homogeneous spaces for \( A \). If \( D^b(X) \simeq D^b(Y) \), then there exists an isometric automorphism \( \sigma \) of \( A \times \hat{k} \hat{A} \) such that

\[
\sigma \circ \left( \begin{array}{c} X \\ \hat{A} \end{array} \right) = \left( \begin{array}{c} Y \\ \hat{A} \end{array} \right)
\]

in \( H^1(k, A \times \hat{k} \hat{A}) \). Conversely, if \( k \) is perfect and if such an automorphism exists, then \( D^b(X) \simeq D^b(Y, \beta) \) for some \( \beta \in \text{Br}(k) \).
Proof. Combining the exact sequence computing \( \text{Br}^A(k) \) and the medium-length exact sequence in nonabelian cohomology for the extension (1), we obtain the commutative diagram

\[
\begin{array}{cccc}
\text{H}^0(k, \text{Aut}_{\text{QC} \text{dg}(A)}) & \rightarrow & \text{H}^0(k, V^A) & \rightarrow \\
\downarrow & & \downarrow & \\
\text{H}^1(k, A) & \rightarrow & \text{H}^1(k, A \times_k \hat{A}) & \\
\downarrow & & \downarrow & \\
\text{Br}(k) & \rightarrow & \text{Br}^A(k) & \rightarrow \\
\rightarrow & & \rightarrow & \\
\text{H}^1(k, \text{Aut}_{\text{QC} \text{dg}(A)}) & \rightarrow & \text{H}^3(k, G_m) & \\
\downarrow & & & \\
\text{H}^1(k, V^A) & & &
\end{array}
\]

where the row and column are exact, and where \( j \) sends a homogeneous space \( X \) for \( A \) to the stack \( \mathcal{E}^X_{\text{dg}} \). The fibers of the map \( i \) are precisely the orbits for the \( \diamond \)-action of \( V^A(k) \) on \( H^1_{\text{ét}}(k, A \times_k \hat{A}) \) by \([23, \text{Proposition 39}]\).

Suppose that \( D^b(X) \simeq D^b(Y) \). Then, \( \text{QC} \text{dg}(X) \simeq \text{QC} \text{dg}(Y) \) by Proposition 3.1. Since \( j(X) = j(Y) \), we see that \( X \) and \( Y \) are in the same \( V^A(k) \)-orbit in \( H^1(k, A \times_k \hat{A}) \). This proves the first part of the theorem.

If \( k \) is perfect, then the discussion before the theorem says that \( V^A = U^A \). Hence, if there exists \( \sigma \in U^A(k) = U(A \times_k \hat{A}) \) such that

\[
\sigma \circ \left( \begin{array}{c} X \\ A \end{array} \right) = \left( \begin{array}{c} Y \\ A \end{array} \right),
\]

then the images of \( X \) and \( Y \) in \( H^1_{\text{ét}}(k, \text{Aut}_{\text{QC} \text{dg}(A)}) \) coincide. It follows that \( \mathcal{E}^X \) and \( \mathcal{E}^Y \) are in the same \( \text{Br}(k) \)-orbit in \( \text{Br}^A(k) \), which proves the second part of the theorem. \( \blacksquare \)

The theorem reduces entirely the problem of finding derived equivalences of principal homogeneous spaces for \( A \) to that of understanding the \( \diamond \)-action. In many interesting cases, this can be accomplished by appealing to the special geometry of \( A \). We include two examples.

**Corollary 5.2.** Let \( A \) be an abelian variety over a field \( k \) of characteristic \( 0 \) such that \( \text{End}(A_{k'}) = \mathbb{Z} \) and such that the inclusion \( \text{NS}(A) \subseteq \text{NS}(A_{k'}) = \mathbb{Z} \) is an equality. These conditions are satisfied by generic polarized abelian varieties. Suppose that \( X \) and \( Y \) are principal homogeneous spaces for \( A \). If \( D^b(X) \simeq D^b(Y) \), then \( aX = Y \) in \( H^1(k, A) \) for some integer \( a \) coprime to the order of \( X \).
Proof. Since we work in characteristic 0, $V^A = U^A$. Moreover, as explained in [19, Example 4.16], $U^A(k^e) = \Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ for some integer $N$, where $\Gamma_0(N)$ is the congruence subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{N}$.

Our assumption that the Neron-Severi group of $A_{k^e}$ is defined over $k$ implies that $U^A(k) = \Gamma_0(N)$ as well, and that $\text{Aut}_{QC dg}(A(l)) \to U^A(l)$ is surjective for all finite extensions $l/k$. See for example the discussion in [14, Section 9.5]. Hence, the boundary map $\delta$ vanishes in diagram in the proof of Theorem 5.1.

By our discussion preceding the theorem, the vanishing of $\delta$ implies that the $\diamond$-action of $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ on $H^1_{\text{ét}}(k, A \times \hat{k})$ is the same as the action induced by $\Gamma_0(N)$ acting on the coefficient group.

It follows that there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left( \begin{array}{c} X \\ \hat{A} \end{array} \right) = \left( \begin{array}{c} aX \\ c\phi \ast X \end{array} \right) = \left( \begin{array}{c} Y \\ \hat{A} \end{array} \right),$$

where $\phi$ is the polarization of $A$. Thus, $Y$ is in the subgroup of $H^1(k, A)$ generated by $X$. Since the argument is symmetric, it follows that $X$ and $Y$ generate the same subgroup. The theorem follows.

Corollary 5.3. Let $A = E^n$ be the $n$-fold product of a geometrically non-CM elliptic curve $E$ over a field $k$ of characteristic 0. Suppose that $X$ and $Y$ are principal homogeneous spaces for $A$. If $D^b(X) \simeq D^b(Y)$, then there is an integral symplectic matrix $\phi \in \text{Sp}_{2n}(\mathbb{Z})$ such that

$$\phi \cdot \left( \begin{array}{c} X \\ \hat{A} \end{array} \right) = \left( \begin{array}{c} Y \\ \hat{A} \end{array} \right)$$

in $H^1(k, A \times \hat{A})$.

Proof. The corollary follows as above from the fact that $V^A = U^A$ and that $U^A(k) = U^A(k^e) = \text{Sp}_{2n}(\mathbb{Z})$. See [19, Example 4.15].

6 Derived equivalences of elliptic schemes

By an elliptic fibration, we mean a flat projective morphism $p : X \to S$ onto an integral scheme $S$ such that the generic fiber $X_\eta$ is a smooth genus 1 curve. If $p$ has a section, we say that it is a Jacobian elliptic fibration.

Let $p : J \to S$ be a Jacobian elliptic fibration defined over the complex numbers with smooth base $S$. Then, there is a small resolution in the analytic category, resulting in a smooth complex analytic manifold $\overline{J} \to J$. In [10, 6.4], Căldăraru asks when $D^b(\overline{J}, a) \simeq D^b(\overline{J}, \beta)$ for two Brauer classes $a$ and $\beta$ in $\text{Br}(\overline{J})$. He found that if $J \to S$ is a so-called generic elliptic 3-fold, and if $\beta = aa$, where $a$ is coprime to the order of $a$, then such an equivalence does exist. The definition of a generic elliptic 3-fold is not important for us, as we will work in far greater generality, and we give a partial answer to Căldăraru’s question.
Definition 6.1. Let \( p : J \to S \) be a Jacobian elliptic fibration. We say that a triangulated equivalence \( F : \text{Perf}(J, \alpha) \simeq \text{Perf}(J, \beta) \) is compatible with \( p \) if \( F \) is isomorphic to \( \Phi_p \) for some \( P \in \text{Perf}(J \times_S J, \alpha^{-1} \boxtimes \beta) \).

Work of Canonaco and Stellari [11] shows that if \( J \) is smooth and projective over a field, then every such triangulated equivalence \( F \) is represented by a kernel \( P \in D^b(J \times_k J, \alpha^{-1} \boxtimes \beta) \). The more important criterion is that \( P \) be supported scheme-theoretically on the closed subscheme \( J \times_S J \subseteq J \times_k J \).

Căldăruş’s proof that \( D^b(J, \alpha) \simeq D^b(J, \beta) \) as above shows that in fact the equivalence is compatible with the morphism \( p \). The equivalence is defined by a Fourier-Mukai transform defined by a specific sheaf, in this case a universal sheaf for some moduli problem. Since the moduli problem is relative to \( S \), it is automatic that it is supported not just on \( J \times_S J \) but scheme-theoretically on \( J \times_S J \). The importance of this notion is encoded in the following proposition.

Proposition 6.2. Suppose that \( F : \text{Perf}(J, \alpha) \simeq \text{Perf}(J, \beta) \) is compatible with \( p \), say equal to \( \Phi_p \) for some \( P \in \text{Perf}(J \times_S J, \alpha^{-1} \boxtimes \beta) \). Then, for any map of schemes \( T \to S \), \( P \) restricts to a complex \( P_T \in \text{Perf}(J_T \times_T J_T, \alpha^{-1} \boxtimes \beta) \) such that the induced map

\[
\Phi_{P_T} : \text{Perf}(J_T, \alpha) \to \text{Perf}(J_T, \beta).
\]

is an equivalence.

Proof. By [3, Theorem 1.2(2)], the hypothesis guarantees in fact that \( F \) is the global section over \( S \) of an equivalence of stacks \( \mathcal{D}_\text{dg}^e(J, \alpha) \simeq \mathcal{D}_\text{dg}^e(J, \beta) \). The claim follows from this.

Theorem 6.3. Suppose that \( p : J \to S \) is a smooth Jacobian elliptic fibration, where \( S \) is a connected regular noetherian scheme with characteristic 0 field of fractions, and suppose that the geometric generic fiber \( J_\eta \) is geometrically not CM. If there is an equivalence \( F : D^b(J, \alpha) \simeq D^b(J, \beta) \) compatible with \( p \) for some \( \alpha, \beta \in \text{Br}(J) \), then \( \alpha \) and \( \beta \) generate the same cyclic subgroup of \( \text{Br}(J) / \text{Br}(S) \).

Proof. Under these conditions, there is an inclusion \( \text{Br}(J) \to \text{Br}(J_\eta) \). By our hypotheses and the proposition, \( F \) restricts to an equivalence \( F_\eta : D^b(J_\eta, \alpha) \simeq D^b(J_\eta, \beta) \). The scheme \( J_\eta \) is an elliptic curve over \( \eta = \text{Spec} k \), where \( k \) is the field of fractions of \( S \). There is an exact sequence

\[
0 \to \text{Br}(k) \to \text{Br}(J_\eta) \to H^1(k, J_\eta) \to 0
\]

since \( J_\eta \) has a rational point. Moreover, this sequence is split for the same reason. So, we can write every class of \( \text{Br}(J_\eta) \) as \( (X, \gamma) \) for \( X \) in \( H^1(k, J_\eta) \) and \( \gamma \in \text{Br}(k) \). Let \( \alpha = (X, \gamma) \), and \( \beta = (Y, \epsilon) \). Now, consider the commutative diagram

\[
\begin{array}{ccc}
\text{Br}(k) & \longrightarrow & \text{Br}(J_\eta) \\
\downarrow & & \downarrow \\
\text{Br}(k) & \longrightarrow & \text{Br}^{J_\eta}(k) \longrightarrow H^1(k, \text{Aut}_{\mathcal{Q}_\text{dg}}(J_\eta)),
\end{array}
\]
where the vertical map $\text{Br}(J_f) \to \text{Br}^h(k)$ sends a class $(X, \gamma)$ to $D^h(X, \gamma)$. The action of $\sigma \in \text{Br}(k)$ on $(X, \gamma)$ is simply $\sigma \cdot (X, \gamma) = (X, \gamma + \sigma)$. It follows that the image of $(X, \gamma)$ in $H^1(k, \text{Aut}_{\text{QCoh}}(J_f))$ is the same as $(X, 0)$. In particular, $(X, 0)$ and $(Y, 0)$ have the same image. Now, the exact same argument as in the proof of Theorem 5.1 shows that $aX = Y$ in $H^1(k, J_f)$ for some integer $a$ prime to the order of $X$. Now, $aa = (ax, ay) = (y, e + (a\gamma - \epsilon)) = \beta + (0, a\gamma - \epsilon)$.

As $\alpha$ and $\beta$ are Brauer classes that are unramified over $J$, it follows that $a\gamma - \epsilon$ is too. But, this means that $a\gamma - \epsilon \in \text{Br}(S)$. Since the argument is symmetric, this completes the proof.

**Remark 6.4.** The situation in [10] is more special in that the assumption is that the classes $\alpha$ and $\beta$ are of the form $(X, 0)$ and $(Y, 0)$ in the notation of the proof. For these classes, the same proof yields the stronger statement that $aa = \beta$ for some $a$ coprime to the order of $\alpha$.

**Remark 6.5.** If the fibration is not smooth, but the rest of the hypotheses remain the same, then the same argument, using the compatibility of $F$ with $p$, shows that $a$ and $\beta$ generate the same subgroup of $\text{Br}(J - D)/\text{Br}(S - D)$, where $D \subseteq S$ is the (reduced) subscheme of points where $p$ is not smooth and $D = \Delta \times_S J$. In the situation where $\alpha = (X, 0)$ and $\beta = (Y, 0)$, we get that $aa = \beta$ in $\text{Br}(J - D)$ for $a$ coprime to the order of $\alpha$. But, since $\alpha, \beta \in \text{Br}(J) \subseteq \text{Br}(J - D)$, this is already true in $\text{Br}(J)$.

**Remark 6.6.** One can also consider the special case of $\alpha = (X, \gamma)$ and $\beta = (X, 0)$. In this case, we are asking about derived equivalences $D^h(X, \gamma) \simeq D^h(X)$ for $\gamma \in \text{Br}(S)$. When $S = \text{Spec} k$, Han [13] and Ciperiani-Krashen [12] have shown that the map $\text{Br}(S) \to \text{Br}(X)$ need not be injective. The elements $\gamma$ in the kernel of this map, the relative Brauer group $\text{Br}(X/S)$, obviously give examples. In [1, Conjecture 2.13], it is suggested that these are the only examples.

The admittedly strong hypotheses of the theorem are satisfied in the examples in [10] and also for example in Bridgeland and Maciocia’s treatment [8, Section 4] of derived equivalences of elliptic surfaces of non-zero Kodaira dimension.

**References**

[1] B. Antieau, Étale twists in noncommutative algebraic geometry and the twisted Brauer space, ArXiv e-prints (2012), available at [http://arxiv.org/abs/1211.6161](http://arxiv.org/abs/1211.6161). ↑1, 4, 1, 4.2, 4, 6.6

[2] M. F. Atiyah, Vector Bundles over an Elliptic Curve, Proc. London Math. Soc. 7 (1957), 414-452. ↑1, 1, 2.1, 2.2, 2.2

[3] D. Ben-Zvi, J. Francis, and D. Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry, J. Amer. Math. Soc. 23 (2010), no. 4, 909–966. ↑6.2

[4] A. Bondal and M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. 3 (2003), no. 1, 1–36, 258. ↑3.3, 3.5
REFERENCES

[5] A. Bondal and D. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, Compositio Math. 125 (2001), no. 3, 327–344.

[6] T. Bridgeland, Fourier-Mukai transforms for surfaces and moduli spaces of stable sheaves, Ph.D. thesis, University of Edinburgh (1998).

[7] ———, Fourier-Mukai transforms for elliptic surfaces, J. Reine Angew. Math. 498 (1998), 115–133.

[8] T. Bridgeland and A. Maciocia, Complex surfaces with equivalent derived categories, Math. Z. 236 (2001), no. 4, 677–697.

[9] ———, Fourier-Mukai transforms for $K_3$ and elliptic fibrations, J. Algebraic Geom. 11 (2002), no. 4, 629–657.

[10] A. Căldăraru, Derived categories of twisted sheaves on elliptic threefolds, J. Reine Angew. Math. 544 (2002), 161–179.

[11] A. Canonaco and P. Stellari, Twisted Fourier-Mukai functors, Adv. Math. 212 (2007), no. 2, 484–503.

[12] M. Ciperiani and D. Krashen, Relative Brauer groups of genus 1 curves, Israel J. Math. 192 (2012), 921–949.

[13] I. Han, Relative Brauer groups of function fields of curves of genus one, Comm. Algebra 31 (2003), no. 9, 4301–4328.

[14] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2006.

[15] B. Keller, On differential graded categories, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[16] J. Lurie, Higher algebra (2012), available at http://www.math.harvard.edu/~lurie/.

[17] S. Mukai, Semi-homogeneous vector bundles on an Abelian variety, J. Math. Kyoto Univ. 18 (1978), no. 2, 239–272.

[18] D. O. Orlov, Equivalences of derived categories and $K3$ surfaces, J. Math. Sci. (New York) 84 (1997), no. 5, 1361–1381.

[19] ———, Derived categories of coherent sheaves on abelian varieties and equivalences between them, Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), no. 3, 131–158; English transl., Izv. Math. 66 (2002), no. 3, 569–594.

[20] A. Polishchuk, Abelian varieties, theta functions and the Fourier transform, Cambridge Tracts in Mathematics, vol. 153, Cambridge University Press, Cambridge, 2003.

[21] M. Popa and C. Schnell, Derived invariance of the number of holomorphic 1-forms and vector fields, Ann. Sci. Éc. Norm. Supér. (4) 44 (2011), no. 3, 527–536 (English, with English and French summaries).

[22] S. Pumplün, Vector bundles and symmetric bilinear forms over curves of genus one and arbitrary index, Math. Z. 246 (2004), no. 3, 563–602.

[23] J.-P. Serre, Cohomologie galoisienne, 5th ed., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994.

[24] J. H. Silverman, The arithmetic of elliptic curves, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986.

[25] G. Tabuada, Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories, C. R. Math. Acad. Sci. Paris 340 (2005), no. 1, 15–19.

[26] B. Toën, Derived Azumaya algebras and generators for twisted derived categories, Invent. Math. 189 (2012), no. 3, 581-652.