CERTAIN SINGULAR DISTRIBUTIONS AND FRACTALS

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Abstract. In the present article, the main attention is given to fractal sets whose elements have certain restrictions on using digits or combinations of digits in own nega-P-representation. Topological, metric, and fractal properties of images of certain self-similar fractals under the action of some singular distributions, are investigated.

1. Introduction

Nowadays, “pathological” mathematical objects (the notion of pathology in mathematics is described in [18]) such as fractals, functions with complicated (complex) local structure (i.e., singular, non-differentiable, or nowhere monotonic functions), and other mathematical objects, have the applied importance and the interdisciplinary character. A number of researches are devoted to this topic (for example, see [8, 9, 15], [47] - [50], etc.).

Fractals are widely applied in computer design, quantum mechanics, solid-state physics, algorithms of the compression to information, analysis and categorizations of signals of various forms appearing in different areas (e.g. the analysis of exchange rate fluctuations in economics), etc. In addition, such sets are useful for checking preserving the Hausdorff dimension by certain functions [41, 42].

In the present article, the main attention is given to fractals having the Moran structure. Fractal sets considered in this paper, are images of certain fractals under the map which is a some generalization of the Salem function. Thus, in the present research, metric, topological, and fractal properties of certain sets (images of certain fractals under the some singular distribution) are investigated. Also, some fractal properties of the considered singular distribution are studied more detail. In other words, the differences between fractal properties of the considered images and their corresponding preimages are described.

Let us describe the notion of the Moran structure. We will consider two definitions of the notion of Moran sets. The first definition was given by Moran in the paper [21], and the second definition is of Hua et al. ([11]).

Definition 1. (Definition of Moran). Let us consider space $\mathbb{R}^n$. In [21], P. A. P. Moran introduced the following construction of sets and calculated the Hausdorff dimension of the limit set

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1, i_2, \ldots, i_n \in A_0, p} \Delta_{i_1, i_2, \ldots, i_n}.$$ (1)

Here $p$ is a fixed positive integer, $A_0, p = \{1, 2, \ldots, p\}$, and sets $\Delta_{i_1, i_2, \ldots, i_n}$ are basic sets having the following properties:

- any set $\Delta_{i_1, i_2, \ldots, i_n}$ is closed and disjoint;
- for any $i \in A_0, p$, the condition $\Delta_{i_1, i_2, \ldots, i_n} \subset \Delta_{i_1, i_2, \ldots, i_n}$ holds;
- $\lim_{n \to \infty} d(\Delta_{i_1, i_2, \ldots, i_n}) = 0$, where $d(\cdot)$ is the diameter of a set;
- each basic set is the closure of its interior;
- at each level the basic sets do not overlap (their interiors are disjoint);

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any basic set $\Delta_{i_1i_2...i_n}$ is geometrically similar to $\Delta_{i_1i_2...i_n}$; 

\[ \frac{d(\Delta_{i_1i_2...i_n})}{d(\Delta_{i_1i_2...i_n})} = \sigma_i, \]

where $\sigma_i \in (0, 1)$ for $i = \overline{1,p}$.

The Hausdorff dimension $\alpha_0$ of the set $E$ is the unique root of the following equation

\[
\sum_{i=1}^{p} \sigma_i^{\alpha_0} = 1. \tag{2}
\]

It is easy to see that set (1) is a Cantor-like set and is a self-similar fractal. The set $E$ is called the Moran set.

**Definition 2.** (Definition of Hua et al.) Let $(n_k)$ be a sequence of positive integers, $J \in \mathbb{R}^n$ be a compact set with nonempty interior, and $(\Phi_k)$ be a sequence of positive real vectors with $\Phi_k = (\sigma_{k,1}, \sigma_{k,2}, \ldots, \sigma_{k,n_k})$, where $k \in \mathbb{N}$ and

\[
\sum_{j=1}^{n_k} \sigma_{k,j} < 1.
\]

A set of the form

\[
E = \bigcap_{k=1}^{\infty} \bigcup_{i_1,\ldots,i_k \in A_{0,n_k}} \Delta_{i_1i_2...i_n},
\]

where $A_{0,n_k} = \{1,2,\ldots,n_k\}$, is called the Moran set associated with the collection $F$. Here

\[
F = \bigcup_{k=0}^{\infty} F_k = \bigcup_{k=0}^{\infty} \{J_\sigma := \Delta_{i_1i_2...i_{k+1}} : k \in \mathbb{N}, i_k \in \{1,2,\ldots,n_k\}\}
\]

The collection $F$ fulfills the Moran structure provided it satisfies the following Moran Structure Conditions (MSC):

1. $J_\emptyset = J$.
2. An arbitrary $J_\sigma$ is geometrically similar to $J$.
3. For any $i,j \in \{1,2,\ldots,n_{k+1}\}$ such that $i \neq j$, the conditions

\[ \Delta_{i_1i_2...i_{k+1}} \subset \Delta_{i_1i_2...i_k}, \quad \Delta_{i_1i_2...i_k} \cap \Delta_{i_1i_2...i_{k+1}} = \emptyset \]

hold.
4. For any $j \in \{1,2,\ldots,n_{k+1}\}$,

\[ \frac{d(\Delta_{i_1i_2...i_{k+1}})}{d(\Delta_{i_1i_2...i_k})} = \sigma_{k+1,j}. \]

The elements of $F_k$ are called the basic elements of order $k$ of the Moran set $E$, and the elements of $F$ are called the basic elements of the Moran set $E$.

**Remark 1.** Let us note that the main difference between definitions of Moran and Hua is Property 4 in MSC.

Let $M = M(J, (n_k), (\Phi_k))$ be a class of Moran sets satisfying MSC. It is known that one can define a sequence $(\alpha_k)$, where $\alpha_k$ satisfies the equation

\[
\prod_{i=1}^{k} \sum_{j=1}^{n_i} \sigma_{i,j}^{\alpha_k} = 1.
\]

Also, suppose that

\[
\alpha_* = \liminf_{k \to \infty} \alpha_k, \quad \alpha^* = \limsup_{k \to \infty} \alpha_k;
\]

\[
c_* = \inf_{i,j} \sigma_{i,j}, \quad c^* = \sup_{i,j} \sigma_{i,j}.
\]

The following statements are useful for studying the Hausdorff dimension of sets investigated in the present research. Suppose that $\dim_H E$ is the Hausdorff dimension of the set $E$. 

Theorem 1. (Theorem 1.1 in [11]). Let $M = M(J, (n_k), (\Phi_k))$ be a Moran class satisfying $c_\ast = \inf \sigma_{i,j} > 0$, then for any $E \in M$,

$$\dim_H E = \alpha_\ast, \text{ and } E \text{ is a } s\text{-set if and only if } 0 < \liminf_{k \to \infty} \frac{1}{\log n_k} \sum_{i_1, i_2, \ldots, i_k} d(\Delta_{i_1 i_2 \ldots i_k})^{\sigma_\ast} < \infty.$$ 

Theorem 2. (Theorem 1.2 in [11]). Let $M = M(J, (n_k), (\Phi_k))$ be a Moran class. Suppose that the sequences $(n_k), (\Phi_k)$ satisfy the following conditions:

- $\sup \lambda n_k = \lambda < \infty$;
- $0 < \inf \max \{\sigma_{i,j}\} \leq \sup \max \{\sigma_{i,j}\} < 1$.

Then for all $E \in M$, $\dim_H E = \alpha_\ast$.

Theorem 3. (Theorem 1.3 in [11]). Let $M = M(J, (n_k), (\Phi_k))$ be a Moran class. Suppose

$$\lim_{k \to \infty} \frac{\log d_k}{\log M_k} = 0,$$

where $d_k := \min_{1 \leq j \leq n_k} \sigma_{j,k}$, $M_k := \max_{1 \leq i_1, i_2, \ldots, i_k} d(\Delta_{i_1 i_2 \ldots i_k})$. Then for all $E \in M$, $\dim_H E = \alpha_\ast$.

Theorem 4. (Theorem 3.1 in [11]). Let $c_\ast = \inf \sigma_{i,j} > 0$. Then for any $E \in M(J, (n_k), (\Phi_k))$, we have

$$\dim_H E = \alpha_\ast.$$

Let us return to the description of investigations of the present paper.

Let $s > 1$ be a fixed positive integer. Then the $s\text{-adic representation of numbers from } [0, 1]$ is a representation of the following form:

$$x = \Delta_{a_1 a_2 \ldots a_n}^s = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n},$$

where $\alpha_n \in A = \{0, 1, \ldots, s-1\}$.

In addition, the following representation

$$x = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_n}^{-s} = \sum_{n=1}^{\infty} \frac{\alpha_n}{(-s)^n},$$

is the nega-$s\text{-adic representation of numbers from } \left[-\frac{s}{s+1}, \frac{1}{s+1}\right]$. Here $\alpha_n \in A$ as well.

It is easy to see that

$$x = \Delta_{a_1 a_2 \ldots a_n}^{-s} \equiv \frac{1}{s+1} - \Delta_{a_1 \{s-1-a_2\} \ldots \{s-1-a_{2k-1}\} \{s-1-a_{2k}\}}^s,$$

or

$$x = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_n}^{-s} \equiv \Delta_{\alpha_1 \{s-1-a_2\} \ldots \{s-1-a_{2k-1}\} \{s-1-a_{2k}\}}^s - \frac{s}{s+1}.$$

Let us consider the sets

$$S_{(s,u)} = \left\{ x : x = \Delta_{\alpha_1 \ldots \alpha_{u-1} \alpha_{u+1} \ldots \alpha_{n-1}}^s, \alpha_n \in A_0 = \{1, 2, \ldots, s-1\} \setminus \{u\} \right\}$$

and

$$S_{(-s,u)} = \left\{ x : x = \Delta_{\alpha_1 \ldots \alpha_{u-1} \alpha_{u+1} \ldots \alpha_{n-1}}^{-s}, \alpha_n \in A_0 = \{1, 2, \ldots, s-1\} \setminus \{u\} \right\}$$

where $u = 0, s-1$, the parameters $u$ and $2 < s \in \mathbb{N}$ are fixed for the fixed sets $S_{(s,u)}, S_{(-s,u)}$.

Elements of these sets have certain restrictions on using combinations of digits in own representations. For example, let $s > 2$ and $u \in \{0, 1, \ldots, s-1\}$ be fixed positive integers. Then the
set $S_{(s,u)}$ is the set whose elements represented in terms of the $s$-adic representation and contain combinations of $s$-adic digits only from the set
$$\left\{1, u2, uu3, \ldots u\ldots u[u-1], u\ldots u[u+1], \ldots u\ldots u[s-2], u\ldots u[s-1]\right\}.$$In the general case, we have a class $\Upsilon_s$ of the sets $S_{(s,u)}$ represented in the form
$$S_{(s,u)} = \left\{x : x = \frac{u}{s-1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{s^{\alpha_1 + \ldots + \alpha_n}}, \alpha_n \neq u, \alpha_n \neq 0\right\},$$where $u = \frac{0}{s-1}$ and parameters $u, s$ are fixed for the set $S_{(s,u)}$. That is, $\Upsilon_s$ contains the sets $S_{(s,0)}, S_{(s,1)}, \ldots, S_{(s,u-1)}, S_{(s,u+1)}, \ldots, S_{(s,s-1)}$. We say that a class $\Upsilon$ of sets contains $\Upsilon_3, \Upsilon_4, \ldots, \Upsilon_n, \ldots$.

Some articles (for example, see [3] [7] [35] [26] [27] [28] [29] [30] [31] [34]) are devoted to sets whose elements have certain restrictions on using combinations of digits in own $s$-adic representation.

Let us discuss properties of the last-mentioned sets.

Theorem 5 ([31]). For an arbitrary $u \in A$, the sets $S_{(s,u)}$ and $S_{(-s,u)}$ are uncountable, perfect, nowhere dense sets of zero Lebesgue measure, and self-similar fractals whose Hausdorff dimension $\alpha_0$ satisfies the following equation
$$\sum_{p \neq u, \ p \in A_0} \left(\frac{1}{s}\right)^{p\alpha_0} = 1.$$

Remark 2. We note that the statement of the last-mentioned theorem is true for all sets $S_{(\pm s,0)}, S_{(\pm s,1)}, \ldots, S_{(\pm s,s-1)}$ (for fixed parameters $u = \frac{0}{s-1}$ and any fixed $2 < s \in \mathbb{N}$) without the sets $S_{(\pm 3,1)}$ and $S_{(\pm 3,2)}$.

Remark 3. Some properties of $S_{(s,u)}$ and $S_{(-s,u)}$ are identical (it follows from the last theorem). However ([31]), certain local properties, e.g., a disposition of cylinders, etc., are different. Also, for the case of $S_{(-s,u)}$, proofs are more difficult.

Remark 4. The sets $S_{(3,0)}, S_{(-3,0)},$ and also sets $S_{(s,u)}$ and $S_{(-s,u)}$ are Cantor-like sets, Moran sets, and self-similar fractals for any positive integer $s > 3$ and an arbitrary integer $0 \leq u < s$. Really, these sets have structure (1), their Hausdorff dimensions are solutions of corresponding equations (2), and their cylinder sets are sets having properties of basic sets $\Delta_{i_1i_2\ldots i_n}$.

Cantor-like sets are important and appear in a number of researches in various areas of mathematics. We present some of them. For example, such sets are important in the study of Diophantine approximation. In [4], it is proven that a large class of Cantor-like sets of $\mathbb{R}^d, d \geq 1$, contains uncountably many badly approximable numbers, respectively badly approximable vectors, when $d \geq 2$. In 1984, Kurt Mahler posed the following fundamental question: How well can irrationals in the Cantor set be approximated by rationals in the Cantor set? In [3], towards such a theory, a Dirichlet-type theorem for this intrinsic Diophantine approximation on Cantor-like sets was proven.

"The resulting approximation function is analogous to that for $\mathbb{R}^d$, but with $d$ being the Hausdorff dimension of the set, and logarithmic dependence on the denominator instead". Let us note that the Cantor set is a set whose elements have a restriction on using ternary digits. This set is the best known example of a fractal in the real line.

"Nowadays, the ternary Cantor set is the paradigmatic model of the fractal geometry [20] [44] and in many branches of physics (see [4]). A large class of Cantor-type sets frequently appear as invariant sets and attractors of many dynamical systems of the real world problems, see [17] - [43]."

1. See Lemmas 1 and 2, Theorems 1–3 which were published with proofs in English in the preprint [51]. These results were published in the papers [29] [30] in Ukrainian.
2. This preprint contains results translated into English without proofs from [29] [30] [31]. See Theorems 4, 6, and 8 in [51].
The papers [2, 15], etc., are devoted to the study of Cantor-type sets in hyperbolic numbers. The study of arithmetical sum of Cantor-type sets plays a special role in dynamical systems (explanations are given in [22]).

To indicate preserving the Hausdorff dimension by singular distribution, Cantor-like sets are useful. For example, in [22], certain self-similar fractals defined in terms of the nega-binary representation and whose elements have some restriction on using binary digits were used for proving a fact that the Minkowski function does not preserve the Hausdorff dimension.

Finally, let us remark that restrictions on using elements of sets $S_{i(±,s,u)}$ are new (they occur for the first time).

Let $s > 1$ be a fixed positive integer and $α_n ∈ A = \{0, 1, . . . , s − 1\}$. Let $P = \{p_0, p_1, . . . , p_{s−1}\}$ be a fixed set whose elements satisfy the following properties: $p_0 + p_1 + \cdots + p_{s−1} = 1$ and $p_i > 0$ for all $i = 0, s − 1$. Then let us consider the following distribution functions.

Let $ξ$ be a random variable defined by the $s$-adic representation

$$ξ = \frac{t_1}{s} + \frac{t_2}{s^2} + \frac{t_3}{s^3} + \cdots + \frac{t_k}{s^k} + \cdots = ∆^s_{t_1t_2...tk...},$$

where digits $t_k$ ($k = 1, 2, 3, . . .$) are random and taking the values $0, 1, . . . , s − 1$ with positive probabilities $p_0, p_1, . . . , p_{s−1}$. That is, $t_k$ are independent and $P\{t_k = α_k\} = p_{α_k}, α_k ∈ A$.

Let $ξ$ be a random variable defined by the $s$-adic representation

$$ξ = ∆^s_{π_1π_2...π_k...} = \sum_{k=1}^{∞} \frac{π_k}{s^k},$$

where

$$π_k = \begin{cases} α_k & \text{if } k \text{ is odd} \\ s - 1 - α_k & \text{if } k \text{ is even} \end{cases}$$

and digits $π_k$ ($k = 1, 2, 3, . . .$) are random and taking the values $0, 1, . . . , s − 1$ with positive probabilities $p_0, p_1, . . . , p_{s−1}$. That is, $π_k$ are independent and $P\{π_k = α_k\} = p_{α_k}, P\{π_k = s - 1 - α_k\} = p_{s−1-α_k}$, where $α_k ∈ A$.

Let us consider the distribution function $f_ξ$ of the random variable $ξ$ and the distribution function $F_ξ$ of the random variable $ξ$ (their particular form will be noted in this paper later).

Let $x ∈ S_{(s,u)}$. Let us consider the image

$$y = F_ξ ∘ f_ξ ∘ f_+(x),$$

where

$$f_+ : x = ∆^s_{α_1α_2...α_n...} ⇒ ∆^s_{α_1α_2...α_n...} = y$$

is not monotonic on the domain and is a nowhere differentiable function (35), $f_1(y) = \frac{1}{1+y} - y$, and $F_ξ$ is the last-mentioned distribution function. That is, in this paper, the main attention is given to properties and a local structure of a set of the form:

$$S_{(−p,u)} = \{y : y = F_ξ ∘ f_1 ∘ f_+(x), x ∈ S_{(s,u)}\}$$

It is easy to see that

$$\{z : z = F_ξ ∘ f_1(x), x ∈ S_{(−s,u)}\} \equiv S_{(−p,u)}.$$

Finally, one can note that

$$S_{(p,u)} = \{y : y = f_1(x), x ∈ S_{(s,u)}\}.$$
great research interest in modeling such sets (in terms of various numeral systems) and studying properties of their images under certain maps.

Let us consider the main properties of the set $S_{(P,u)}$.

**Theorem 6.** [36] The set $S_{(P,u)}$ is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure.

**Theorem 7.** [39] The set $S_{(P,u)}$ is a self-similar fractal whose Hausdorff dimension $\alpha_0(S_{(P,u)})$ satisfies the following equation

$$\sum_{i \in A_0\setminus\{u\}} (p_ip_u^{i-1})^{\alpha} = 1.$$ 

In the present article, the main attention is given to properties of sets $S_{(-P,u)}$. In this paper, topological and metric, local and fractal properties of the set $S_{(-P,u)}$ are investigated.

**Remark 5.** One can note that the present investigations are similar with investigations ([36]) for the set $S_{(P,u)}$, but are more complicated and some techniques for proving the main statements are different. In addition, one can note that if for $S_{(-s,u)}$ and $S_{(s,u)}$, topological, metric, and fractal properties (without some properties of cylinders) are similar, then fractal and some local properties of $S_{(-P,u)}$ and $S_{(P,u)}$ are different. For example, $S_{(P,u)}$ is a self-similar fractal (i.e., this is a Moran set by Moran’s definition, [21]) but $S_{(-P,u)}$ is a non-self-similar set having the Moran structure (i.e., this is a Moran set by the definition of Hua et al. (see the definition in [11])).

Let us formulate the main results of the present paper.

Suppose $d(\cdot)$ is the diameter of a set and a cylinder $\Delta_{c_1c_2\ldots c_n}$ is a set whose elements are elements of $S_{(-P,u)}$ and for these elements the condition $\alpha_i = c_i$ holds for all $i = 1, \ldots, n$ (here $c_1, c_2, \ldots, c_n$ is a fixed tuple). The main results of the present paper are formulated in the following theorems.

**Theorem 8.** An arbitrary set $S_{(-P,u)}$ is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure.

One can note that for proving topological and metric properties of $S_{(-P,u)}$, cylinders of this set and their properties are used. Such investigations includes local properties of the set $S_{(-P,u)}$.

**Theorem 9.** In the general case, the set $S_{(-P,u)}$ is not a self-similar fractal, the Hausdorff dimension $\alpha_0(S_{(-P,u)})$ of which can be calculated by the formula:

$$\alpha_0 = \liminf_{k \to \infty} \alpha_k,$$

where $(\alpha_k)$ is a sequence of numbers satisfying the equation

$$\left(\sum_{c_1 \text{ is odd } \atop c_1 \in A} (\omega_{2,c_1})^{\alpha_1} + \sum_{c_1 \text{ is even } \atop c_1 \in A} (\omega_{4,c_1})^{\alpha_1}\right) \times \prod_{j=2}^{k} \left(\sum_{c_1 \text{ is odd } \atop c_1 \in A} N_{1,c_1} (\omega_{1,c_1})^{\alpha_1} + \sum_{c_1 \text{ is odd } \atop c_1 \in A} N_{2,c_1} (\omega_{2,c_1})^{\alpha_1} + \sum_{c_1 \text{ is even } \atop c_1 \in A} N_{3,c_1} (\omega_{3,c_1})^{\alpha_1} + \sum_{c_1 \text{ is even } \atop c_1 \in A} N_{4,c_1} (\omega_{4,c_1})^{\alpha_1}\right) = 1.$$

Here $N_{j,c_1}$ $(j = \overline{1,4}, 1 < i \in \mathbb{N})$ is the number of cylinders $\Delta_{c_1c_2\ldots c_n}$ for which

$$\frac{d(\Delta_{c_1c_2\ldots c_n})}{d(\Delta_{c_1c_2\ldots c_{n-1}})} = \omega_{j,c_1}.$$ 

Also,

$$\omega_{1,c_1} = \frac{p_{s-1-u}p_{u} \cdots p_{s-1-u}p_{0}p_{s-1-c_i}}{c_i-1} \frac{d(S_{(P,u)})}{d(S_{(P,u)})}$$ for an odd number $c_i$.\[\]
where following form

\[ \omega_{2, c_i} = \frac{p_u p_{u-1} \cdots p_u p_{u-1} c_i}{c_i-1} \] for an odd number \( c_i \),

\[ \omega_{3, c_i} = \frac{p_{u-1} u p_u \cdots p_{u-1} u p_{u-1} c_i}{c_i-1} \] for an even number \( c_i \),

\[ \omega_{4, c_i} = \frac{p_u p_{u-1} u p_u \cdots p_{u-1} u p_{u-1} c_i}{c_i-1} \] for an even number \( c_i \).

In addition, \( N_{1, c_i} + N_{2, c_i} = (m+l)^{i-1} \) and \( N_{3, c_i} + N_{4, c_i} = m(m+l)^{i-1} \), where \( l \) is the number of odd numbers in the set \( \mathbb{A} = \mathbb{A} \setminus \{0, u\} \) and \( m \) is the number of even numbers in \( \mathbb{A} \).

2. The notion of the Salem type functions

A function of the form

\[ f(x) = f \left( \Delta^2_{\alpha_1 \alpha_2 \cdots \alpha_n} \right) = \beta_0 + \sum_{n=2}^{\infty} \left( \beta_{n-1} \prod_{i=1}^{n-1} p_i \right) = y = \Delta^2 p_0 p_1 \cdots p_{n-1}, \]

where \( p_0 > 0 \), \( p_1 > 0 \), and \( p_0 + p_1 = 1 \), is called the Salem function. Here \( \beta_0 \), \( \beta_1 \), and \( \beta_0 + \beta_1 = p_0 \). This is one of the simplest examples of singular functions but its generalizations can be non-differentiable functions or do not have the derivative at points from a certain set. In [25], Salem modeled this function for the case when the argument represented in terms of the \( s \)-adic representation. Let us note that many researches are devoted to the Salem function and its generalizations (for example, see \([1, 16, 32, 39, 43]\) and references in these papers).

Let us consider a technique for modeling the Salem type function. Such functions are the main functions of the present investigation.

Let \( \zeta \) be a random variable defined by the \( s \)-adic representation

\[ \zeta = \frac{t_1}{s} + \frac{t_2}{s^2} + \frac{t_3}{s^3} + \cdots + \frac{t_k}{s^k} + \cdots = \Delta_{t_1 t_2 \cdots t_k}, \]

where digits \( t_k \) \((k = 1, 2, 3, \ldots)\) are random and taking the values \( 0, 1, \ldots, s-1 \) with positive probabilities \( p_0, p_1, \ldots, p_{s-1} \). That is, \( t_k \) are independent and \( P\{t_k = \alpha_k\} = p_{\alpha_k}, \alpha_k \in \mathbb{A} \).

From the definition of a distribution function and the following expressions

\[ \{\zeta < x\} = \{\xi_1 < \alpha_1(x)\} \cup \{\xi_1 = \alpha_1(x), \xi_2 < \alpha_2(x)\} \cup \cdots \]

\[ \cup \{\xi_1 = \alpha_1(x), \xi_2 = \alpha_2(x), \ldots, \xi_{k-1} = \alpha_{k-1}(x), \xi_k < \alpha_k(x)\} \cup \cdots, \]

\[ P\{\xi_1 = \alpha_1(x), \xi_2 = \alpha_2(x), \ldots, \xi_{k-1} = \alpha_{k-1}(x), \xi_k < \alpha_k(x)\} = \beta_{\alpha_k(x)} \prod_{j=1}^{k-1} p_{\alpha_j(x)}, \]

where

\[ \beta_{\alpha_k} = \begin{cases} \sum_{i=0}^{\alpha_k(x)-1} p_i(x) & \text{whenever } \alpha_k(x) > 0 \\ 0 & \text{whenever } \alpha_k(x) = 0, \end{cases} \]

it is easy to see that the following statement is true.

Statement 1. The distribution function \( f_\zeta \) of the random variable \( \zeta \) can be represented in the following form

\[ f_\zeta(x) = \begin{cases} 0 & \text{whenever } x < 0 \\ \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left( \beta_{\alpha_k(x)} \prod_{j=1}^{k-1} p_{\alpha_j(x)} \right) & \text{whenever } 0 \leq x < 1 \\ 1 & \text{whenever } x \geq 1, \end{cases} \]

where \( p_{\alpha_i(x)} > 0 \).
It is easy to see that
\[ x = \Delta^{s}_{\alpha_1, \alpha_2, \ldots, \alpha_k} = \frac{1}{s+1} - \Delta^{-s}_{\alpha_1, \alpha_2, \ldots, \alpha_k} = \frac{1}{s+1} - \sum_{k=1}^{\infty} \frac{(-1)^{\alpha_k}}{s^k} = \sum_{k=1}^{\infty} \frac{\alpha_{2k-1}}{s^{2k-1}} + \sum_{k=1}^{\infty} \frac{s-1-\alpha_{2k}}{s^{2k}}. \]

Let us consider the following distribution function. By analogy, let \( \varsigma \) be a random variable defined by the s-adic representation
\[ \varsigma = \Delta^{s}_{\pi_2, \ldots, \pi_k} = \sum_{k=1}^{\infty} \frac{\pi_k}{s^k}, \]
where
\[ \pi_k = \begin{cases} \alpha_k & \text{if } k \text{ is odd} \\ s-1-\alpha_k & \text{if } k \text{ is even} \end{cases} \]
and digits \( \pi_k \) \((k = 1, 2, 3, \ldots)\) are random and taking the values \(0, 1, \ldots, s-1\) with positive probabilities \(p_0, p_1, \ldots, p_{s-1}\). That is, \( \pi_k \) are independent and \( P\{\pi_k = \alpha_k\} = p_{\alpha_k}, P\{\pi_k = s-1-\alpha_k\} = p_{s-1-\alpha_k}, \) where \( \alpha_k \in A \). From the definition of a distribution function and the following expressions
\[ \{\varsigma < x\} = \{\pi_1 < \alpha_1(x)\} \cup \{\pi_1 = \alpha_1(x), \pi_2 < s-1-\alpha_2(x)\} \cup \ldots \]
\[ \ldots \cup \{\pi_1 = \alpha_1(x), \pi_2 = s-1-\alpha_2(x), \ldots, \pi_{2k-1} < \alpha_{2k-1}(x)\} \cup \ldots \]
\[ \cup\{\pi_1 = \alpha_1(x), \pi_2 = s-1-\alpha_2(x), \ldots, \pi_{2k-1} = \alpha_{2k-1}(x), \pi_{2k} < s-1-\alpha_{2k}(x)\} \cup \ldots, \]
\[ P\{\pi_1 = \alpha_1(x), \pi_2 = s-1-\alpha_2(x), \ldots, \pi_{2k-1} < \alpha_{2k-1}(x)\} = \beta_{\alpha_{2k-1}(x)} \prod_{j=1}^{2k-2} \tilde{p}_{\alpha_j(x)} \]
and
\[ P\{\pi_1 = \alpha_1(x), \pi_2 = s-1-\alpha_2(x), \ldots, \pi_{2k} < s-1-\alpha_{2k}(x)\} = \beta_{s-1-\alpha_{2k}(x)} \prod_{j=1}^{2k-1} \tilde{p}_{\alpha_j(x)}, \]
we have the following statement.

**Statement 2.** The distribution function \( \tilde{F}_\varsigma \) of the random variable \( \varsigma \) can be represented in the following form
\[ \tilde{F}_\varsigma(x) = \begin{cases} 0 & \text{whenever } x < 0 \\ \tilde{\beta}_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left( \tilde{\beta}_{\alpha_k(x)} \prod_{j=1}^{k-1} \tilde{p}_{\alpha_j(x)} \right) & \text{whenever } 0 \leq x < 1 \\ 1 & \text{whenever } x \geq 1, \end{cases} \]
where \( p_{\alpha_j(x)} > 0, \)
\[ \tilde{p}_{\alpha_k} = \begin{cases} p_{\alpha_k} & \text{if } k \text{ is odd} \\ p_{s-1-\alpha_k} & \text{if } k \text{ is even}, \end{cases} \]
and
\[ \tilde{\beta}_{\alpha_k} = \begin{cases} \beta_{\alpha_k} & \text{if } k \text{ is odd} \\ \beta_{s-1-\alpha_k} & \text{if } k \text{ is even}. \end{cases} \]
The function
\[ \tilde{F}(x) = \beta_{\alpha_1(x)} + \sum_{n=2}^{\infty} \left( \tilde{\beta}_{\alpha_n(x)} \prod_{j=1}^{n-1} \tilde{p}_{\alpha_j(x)} \right), \]
is a partial case of the function investigated in [43].
Definition 3. Any function is said to be a function of the Salem type if it is modeled by the mentioned technique. That is, the Salem type function is a distribution function of the random variable \( \eta = \Delta_{\xi_1 \xi_2 ... \xi_k} \) defined by a certain representation \( \Delta_{i_1 i_2 ... i_k} \) of real numbers, where digits \( \xi_k \) \((k = 1, 2, 3, \ldots)\) are random and taking the values \(0, 1, \ldots, m_k\) with positive probabilities \(p_0, p_1, \ldots, p_{m_k}\) \((\xi_k\) are independent and \(P\{\xi_k = i_k\} = p_{i_k}, i_k \in A_k = \{0, 1, \ldots, m_k\}, m_k \in \mathbb{N} \cup \{\infty\}\)). In addition, some generalizations of the Salem type functions (for example, the last functions for which \(p_i \in (-1, 1)\)) also are called functions of the Salem type.

By analogy, let \( \eta \) be a random variable defined by the \( s \)-adic representation

\[
\eta = \Delta_{\xi_1 \xi_2 ... \xi_k} = \sum_{k=1}^{\infty} \frac{\xi_k}{s^k},
\]

where

\[
\xi_k = \begin{cases} 
\alpha_k & \text{if } k \text{ is even} \\
 s - 1 - \alpha_k & \text{if } k \text{ is odd}
\end{cases}
\]

and digits \( \xi_k \) \((k = 1, 2, 3, \ldots)\) are random and taking the values \(0, 1, \ldots, s - 1\) with positive probabilities \(p_0, p_1, \ldots, p_{s-1}\). Then the distribution function \( \tilde{F}_\eta \) of the random variable \( \eta \) can be represented in the following form

\[
\tilde{F}_\eta(x) = \begin{cases} 
0 & \text{whenever } x < 0 \\
\beta_{s-1-\alpha_1(x)} + \sum_{k=2}^{\infty} \left( \tilde{\beta}_{\alpha_k(x)} \prod_{j=1}^{k-1} \tilde{p}_{\alpha_j(x)} \right) & \text{whenever } 0 \leq x < 1 \\
1 & \text{whenever } x \geq 1,
\end{cases}
\]

where \( p_{\alpha_i(x)} > 0 \),

\[
\tilde{p}_{\alpha_k} = \begin{cases} 
p_{\alpha_k} & \text{if } k \text{ is even} \\
p_{s-1-\alpha_k} & \text{if } k \text{ is odd},
\end{cases}
\]

and

\[
\tilde{\beta}_{\alpha_k} = \begin{cases} 
\beta_{\alpha_k} & \text{if } k \text{ is even} \\
\beta_{s-1-\alpha_k} & \text{if } k \text{ is odd}.
\end{cases}
\]

3. \( P \) - and nega-\( P \)-representations

Let us note that in the last decades, there exists the tendency to modeling and studying numeral systems defined in terms of alternating expansions of real numbers under the condition that numeral systems defined in terms of the corresponding positive expansions of real numbers are known. For example, a number of researchers investigate positive and alternating \( \beta \)-expansions, Lüroth series, Engel series, etc. Let us remark that positive and alternating \( \beta \)-expansions are the \( s \)-adic and nega-\( s \)-adic representations with \( s = \beta > 1, \beta \in \mathbb{R} \). The notion of \( \beta \)-expansions was introduced by A. Rényi in 1957 in the paper [21], but \((-\beta)\)-expansions were introduced in the paper [12] in 2009. In addition, in 1883, the German mathematician J. Lüroth introduced an expansion of a real number in the form of a special series (now these series are called Lüroth series). However, in 1990, S. Kalpazidou, A. Knopfmacher, and J. Knopfmacher introduced alternating Lüroth series in the paper [13].

Peculiarities of such numeral systems are the following: the similarity of some properties for mathematical objects defined in terms of corresponding positive and alternating expansions; the complexity of proofs of corresponding statements for the case of such alternating representations. Let us note that discovering differences in properties of fractals which are defined in terms of a certain type of positive and alternating expansions of real numbers (\( P \)- and nega-\( P \)-representations), is one of the aims of this paper.

The function

\[
f(x) = \beta_{\alpha_1(x)} + \sum_{n=2}^{\infty} \left( \beta_{\alpha_n(x)} \prod_{j=1}^{n-1} p_{\alpha_j(x)} \right),
\]
can be used as a representation (the P-representation) of numbers from \([0, 1]\). That is,

\[
y = \Delta^P_{x_{1}(x)}(x) = f(x) = \beta x_1(x) + \sum_{n=2}^{\infty} \left( \beta x_n(x) \prod_{j=1}^{n-1} \rho_j(x) \right),
\]

where \(P = \{p_0, p_1, \ldots, p_{s-1}\}, p_0 + p_1 + \cdots + p_{s-1} = 1\), and \(p_i > 0\) for all \(i = 0, s - 1\). In other words, this function “preserves” digits of representations of numbers:

\[
f : x = \Delta^s_{x_1(x)}(x) \rightarrow \Delta^P_{x_1(x)}(x) = \Delta^P_{x_1(y)}(y) = y.
\]

We begin with the nega-P- and P-representations of numbers from \([0, 1]\):

\[
x = \sum_{i=0}^{s-1} p_i + \sum_{n=2}^{\infty} \left( (-1)^{n-1} \tilde{\delta}_n \prod_{j=1}^{n-1} \tilde{\rho}_j \right) + \sum_{n=1}^{\infty} \left( 2^{n-1} \prod_{j=1}^{n-1} \tilde{\rho}_j \right) = \Delta^-\!
\]

and

\[
x = \Delta^P_{x_1(x)}(x) = \beta x_1(x) + \sum_{n=2}^{\infty} \left( \beta x_n(x) \prod_{j=1}^{n-1} \rho_j(x) \right)
\]

for which

\[
\Delta^-\!
\]

where

\[
\tilde{\delta}_n = \begin{cases} 
\sum_{i=s-1}^{s-1} p_i & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd and } n = 0 \\
\sum_{i=0}^{n-1} p_i & \text{if } n \text{ is odd and } n \neq 0,
\end{cases}
\]

and

\[
\tilde{\rho}_n = \begin{cases} 
\rho_n & \text{if } n \text{ is odd} \\
\rho_{s-1-n} & \text{if } n \text{ is even}.
\end{cases}
\]

Here \(s > 1\) is a fixed positive integer and \(\alpha_n \in A = \{0, 1, \ldots, s - 1\}\). Also, \(P = \{p_0, p_1, \ldots, p_{s-1}\}\) is a fixed set whose elements satisfying the following properties: \(p_0 + p_1 + \cdots + p_{s-1} = 1\) and \(p_i > 0\) for all \(i = 0, s - 1\).

**Definition 4.** A representation of form \([3]\) is called the nega-P-representation of numbers from \([0, 1]\) (this representation is a partial case of the nega-\(Q\)-representation considered in the papers \([37, 43]\) and of the quasi-nega-\(Q\)-representation \([33]\)).

A representation of form \([4]\) is called the P-representation of numbers from \([0, 1]\). This representation is a partial case of some representations noted in \([33, 43]\).

In the present article, the main attention is given to properties of sets of the following form:

\[
S_{(-P,u)} = \left\{ x : x = \Delta^P_{x_1(x)}(x) = \beta x_1(x) + \sum_{n=2}^{\infty} \left( \beta x_n(x) \prod_{j=1}^{n-1} \rho_j(x) \right), \alpha_n \in A_0 = \{1, 2, \ldots, s - 1\} \setminus \{u\} \right\},
\]

where \(s > 3\) is a fixed positive integer, \(u = 0, s - 1\), and the parameters \(u\) and \(s\) are fixed for the fixed set \(S_{(-P,u)}\).

Let us consider the P-representation and the nega-P-representation more detail. Let \(s\) be a fixed positive integer, \(s > 1\), and \(c_1, c_2, \ldots, c_m\) be an ordered tuple of integers such that \(c_i \in \{0, 1, \ldots, s - 1\}\) for \(i = 1, m\).
Definition 5. A \(P\)-cylinder (or nega-\(P\)-cylinder) of rank \(m\) with base \(c_1c_2\ldots c_m\) is a set \(\Delta_{c_1c_2\ldots c_m}^P\) (or \(\Delta_{c_1c_2\ldots c_m}^{-P}\)) formed by all numbers of the segment \([0,1]\) with \(P\)-representations (or nega-\(P\)-representations) in which the first \(m\) digits coincide with \(c_1, c_2, \ldots, c_m\), respectively, i.e.,

\[
\Delta_{c_1c_2\ldots c_m}^P = \{ x : x = \Delta_{\alpha_1\alpha_2\ldots \alpha_m}, \alpha_j = c_j, j = 1, m \}
\]

or

\[
\Delta_{c_1c_2\ldots c_m}^{-P} = \{ x : x = \Delta_{-\alpha_1\alpha_2\ldots \alpha_m}, \alpha_j = c_j, j = 1, m \}
\]

Lemma 1. Cylinders \(\Delta_{c_1c_2\ldots c_m}^P, \Delta_{c_1c_2\ldots c_m}^{-P}\) have the following properties:

1. an arbitrary cylinder \(\Delta_{c_1c_2\ldots c_m}^P\) or \(\Delta_{c_1c_2\ldots c_m}^{-P}\) is a closed interval;
2. the following relationships hold:

\[
\inf \Delta_{c_1c_2\ldots c_m}^P = \Delta_{c_1c_2\ldots c_m}^P[0,000\ldots], \sup \Delta_{c_1c_2\ldots c_m}^P = \Delta_{c_1c_2\ldots c_m}^P[1[s-1][s-1][s-1]\ldots];
\]

\[
\inf \Delta_{c_1c_2\ldots c_m}^{-P} = \Delta_{c_1c_2\ldots c_m}^{-P}[s-1][s-1][s-1]\ldots, \sup \Delta_{c_1c_2\ldots c_m}^{-P} = \Delta_{c_1c_2\ldots c_m}^{-P}[0[s-1][s-1][s-1]\ldots];
\]

\(\text{if } n \text{ is odd, } \sup \Delta_{c_1c_2\ldots c_m}^{-P} = \Delta_{c_1c_2\ldots c_m}^{-P}[0[s-1][s-1][s-1]\ldots];\)

\(\text{if } n \text{ is even, } \sup \Delta_{c_1c_2\ldots c_m}^{-P} = \Delta_{c_1c_2\ldots c_m}^{-P}[0[s-1][s-1][s-1]\ldots];\)

3. \(|\Delta_{c_1c_2\ldots c_m}^P| = p_1p_2\ldots p_{c_m}, |\Delta_{c_1c_2\ldots c_m}^{-P}| = \tilde{p}_1\tilde{p}_2\ldots \tilde{p}_{c_m};\)

4. \(\Delta_{c_1c_2\ldots c_m}^P \subset \Delta_{c_1c_2\ldots c_m}^{-P}, \Delta_{c_1c_2\ldots c_m}^{-P} \subset \Delta_{c_1c_2\ldots c_m}^P;\)

5. \(\Delta_{c_1c_2\ldots c_m}^P = \bigcup_{c=0}^{s-1} \Delta_{c_1c_2\ldots c_m}^P c, \Delta_{c_1c_2\ldots c_m}^{-P} = \bigcup_{c=0}^{s-1} \Delta_{c_1c_2\ldots c_m}^{-P} c;\)

6. \(\lim_{m \to \infty} |\Delta_{c_1c_2\ldots c_m}^P| = \lim_{m \to \infty} |\Delta_{c_1c_2\ldots c_m}^{-P}| = 0;\)

7. \(|\Delta_{c_1c_2\ldots c_m c m+1}^P| = p_{c m+1}, |\Delta_{c_1c_2\ldots c_m c m+1}^{-P}| = \tilde{p}_{c m+1};\)

8. for any \(m \in \mathbb{N}\)

\(\sup \Delta_{c_1c_2\ldots c_m c}^{-P} = \inf \Delta_{c_1c_2\ldots c_m c}^P, \text{ where } c \neq s - 1;\)

\(\sup \Delta_{c_1c_2\ldots c_m c-1}^{-P} = \inf \Delta_{c_1c_2\ldots c_m c-1}^P \text{ if } m \text{ is odd, }\)

\(\sup \Delta_{c_1c_2\ldots c_m c-1}^{-P} = \inf \Delta_{c_1c_2\ldots c_m c-1}^P \text{ if } m \text{ is even; }\)

9. for an arbitrary \(x \in [0,1]\)

\[\bigcap_{m=1}^{\infty} \Delta_{c_1c_2\ldots c_m}^P = x = \Delta_{c_1c_2\ldots c_m}^{-P} \text{ and } \bigcap_{m=1}^{\infty} \Delta_{c_1c_2\ldots c_m}^{-P} = x = \Delta_{c_1c_2\ldots c_m}^P;\]

10. for any \(x_1, x_2 \in [0,1]\), the following equalities are true:

\(x_1 = \Delta_{\alpha_1\alpha_2\ldots \alpha_m}, x_2 = \Delta_{\alpha_1\alpha_2\ldots \alpha_m}.\)

Definition 6. A number \(x \in [0,1]\) is called \(P\)-rational if

\[x = \Delta_{\alpha_1\alpha_2\ldots \alpha_m}^P 000\ldots\]

or

\[x = \Delta_{\alpha_1\alpha_2\ldots \alpha_m-1}^P \alpha_{m-1}[s-1][s-1][s-1]\ldots;\]

The other numbers in \([0,1]\) are called \(P\)-irrational.
Definition 7. Numbers from some countable subset of \([0, 1]\) have two different nega-P-representations, i.e.,
\[
\Delta^{-P}_{\alpha_1, \alpha_2, \ldots, \alpha_n} u = \Delta^{-P}_{\alpha_1, \alpha_2, \ldots, \alpha_n} [\alpha_n - 1]_{0, s-1} [u]_{0, s-1}, \quad \alpha_n \neq 0.
\]
These numbers are called nega-P-rational, and other numbers from \([0, 1]\) are called nega-P-irrational.

4. Sets \(S_{(-P,u)}\) as images of certain fractals under the action of the Salem type function

Let us consider the sets
\[
S_{(s,u)} = \left\{ x : x = \Delta^u_{\alpha_1, \alpha_2, \ldots, \alpha_n} u, \quad \alpha_n \in A_0 = \{1, 2, \ldots, s - 1\} \setminus \{u\} \right\}
\]
and
\[
S_{(-s,u)} = \left\{ x : x = \Delta^{-s}_{\alpha_1, \alpha_2, \ldots, \alpha_n} u, \quad \alpha_n \in A_0 = \{1, 2, \ldots, s - 1\} \setminus \{u\} \right\}
\]
where \(u = 0, s - 1\), the parameters \(u\) and \(2 < s \in \mathbb{N}\) are fixed for the fixed sets \(S_{(s,u)}, S_{(-s,u)}\).

Let \(x \in S_{(s,u)}\). Let us consider the image
\[
\tilde{y} = \tilde{F} \circ f_1 \circ f_+(x),
\]
where
\[
f_+ : x = \Delta^u_{\alpha_1, \alpha_2, \ldots, \alpha_n} \rightarrow \Delta^{-s}_{\alpha_1, \alpha_2, \ldots, \alpha_n} = y
\]
is not monotonic on the domain and is a nowhere differentiable function \((35)\), \(f_1(y) = \frac{1}{s+1} - y\), and \(\tilde{F}\) is the distribution function described earlier. That is, in this paper, the main attention is given to properties and a local structure of a set of the form:
\[
S_{(-P,u)} = \left\{ \tilde{y} : \tilde{y} = \tilde{F} \circ f_1 \circ f_+(x), x \in S_{(s,u)} \right\}
\]
\[
= \left\{ x : x = \Delta^u_{\alpha_1, \alpha_2, \ldots, \alpha_n} u, \quad \alpha_n \neq u, \alpha_n \neq 0 \right\}
\]
In other words,
\[
\Delta^{-P}_{\tilde{u}, \alpha_1, \alpha_2, \ldots, \alpha_n} u = \Delta^{-P}_{\tilde{u}, \alpha_1, \alpha_2, \ldots, \alpha_n} u, \quad \tilde{u} = \begin{cases} u & \text{whenever } n \text{ is odd} \\ s - 1 - u & \text{whenever } n \text{ is even} \end{cases}
\]
and
\[
\tilde{u} = \begin{cases} u & \text{whenever } u \text{ is situated at an odd position in the representation} \\ s - 1 - u & \text{whenever } u \text{ is situated at in an even position in the representation} \end{cases}
\]
Remark 6. Since properties of the set
\[
S_{(-s,u)} = \left\{ x : x = \Delta^u_{\alpha_1, \alpha_2, \ldots, \alpha_n} u, \quad \alpha_n \neq u, \alpha_n \neq 0 \right\}
\]
(here also \(u = 0, s - 1\), the parameters \(u\) and \(s\) are fixed for the set \(S_{(-s,u)}\)) were investigated (see \(31, 34, 35\)), one can consider the set of images
\[
\{ z : z = \tilde{F} \circ f_1(x), x \in S_{(-s,u)} \} \equiv S_{(-P,u)}.
\]
Let us remark that
\[
\inf S_{(-s,u)} = \begin{cases} 
\Delta^*_{(02)} & \text{if } u = 0 \\
\Delta^*_{13(12)} & \text{if } u = 1 \\
\Delta^*_{(u2)} & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]
and
\[
\sup S_{(-s,u)} = \begin{cases} 
\Delta^*_{(u2)} & \text{if } u \in \{0, 1\} \\
\Delta^*_{(u2)} & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]

Here (α) is period.

5. Some auxiliary notes

In this section, the main attention is given to notes and calculations which are useful for proving the main results.

Suppose \( x \in S_{(-s,u)} \). Then consider the set
\[
\overline{S}_{(s,u)} \equiv \left\{ \tilde{y} : \tilde{y} = \Delta^*_{(s-1)(0)} + x, \ x \in S_{(-s,u)} \right\}.
\]
We have
\[
\inf \overline{S}_{(s,u)} = \begin{cases} 
\Delta^*_{(s-2)(0(s-3))} & \text{if } u = 0 \\
\Delta^*_{(s-2)(1(s-4)(1(s-3))} & \text{if } u = 1 \\
\Delta^*_{(s-1-u)(2)} & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]
and
\[
\sup \overline{S}_{(s,u)} = \begin{cases} 
\Delta^*_{(s-1-u)(2)} & \text{if } u \in \{0, 1\} \\
\Delta^*_{(s-2)(u(s-3))} & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]

Suppose \( x \in S_{(-s,u)} \). Then consider the set
\[
\overline{\bar{S}}_{(s,u)} \equiv \left\{ \bar{y} : \bar{y} = \Delta^*_{(0(s-1))} - x, \ x \in S_{(-s,u)} \right\}.
\]
We obtain
\[
\inf \overline{\bar{S}}_{(s,u)} = \begin{cases} 
\Delta^*_{(s-3)} & \text{if } u \in \{0, 1\} \\
\Delta^*_{(s-1-u)(2)} & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]
and
\[
\sup \overline{\bar{S}}_{(s,u)} = \begin{cases} 
\Delta^*_{(s-1)(2)} & \text{if } u = 0 \\
\Delta^*_{(s-3)(s-2)(2)} & \text{if } u = 1 \\
\Delta^*_{(s-2)(s-3)} & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]

For the present investigation, the following sets are auxiliary sets:
\[
\overline{S}_{(P,u)} \equiv \left\{ \bar{z} : \bar{z} = \tilde{F}(\bar{y}), \ \bar{y} \in \overline{\bar{S}}_{(s,u)} \right\} \equiv \left\{ \bar{z} : \bar{z} = \Delta^*_P \frac{\bar{u}_{\alpha_1} \bar{u}_{\alpha_2} \ldots \bar{u}_{\alpha_n}}{\alpha_n} \right\},
\]
\[
\overline{S}_{(P,u)} \equiv \left\{ \bar{z} : \bar{z} = \tilde{F}(\bar{y}), \ \bar{y} \in \overline{\bar{S}}_{(s,u)} \right\} \equiv \left\{ \bar{z} : \bar{z} = \Delta^*_P \frac{\bar{u}_{\alpha_1} \bar{u}_{\alpha_2} \ldots \bar{u}_{\alpha_n}}{\alpha_n} \right\},
\]
where
\[
\bar{\alpha}_n = \begin{cases} 
\alpha_n & \text{whenever } n \text{ is even} \\
\frac{s - 1 - \alpha_n}{\alpha_n} & \text{whenever } n \text{ is odd}
\end{cases}
\]
and
\[
\bar{u} = \begin{cases} 
u & \text{whenever } u \text{ is situated at an even position in the representation} \\
\frac{s - 1 - u}{\alpha_n} & \text{whenever } u \text{ is situated at in an odd position in the representation}
\end{cases}
\]

So, one can note the following statement.
Lemma 2. For the sets $S(P, u)$ and $\overline{S}(P, u)$, the following equalities hold:

\[
\inf S(P, u) = \begin{cases} 
\Delta_{[s-2](0|s-3)}^P & \text{if } u = 0 \\
\Delta_{[s-2](1|s-4)(1|s-3)}^P & \text{if } u = 1 \\
\Delta_{(s-1-u)(2)]}^P & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]

and

\[
\sup S(P, u) = \begin{cases} 
\Delta_{([s-1-u)(2)]}^P & \text{if } u \in \{0, 1\} \\
\Delta_{([s-2]u|s-3))}^P & \text{if } u \in \{2, 3, \ldots, s - 1\},
\end{cases}
\]

\[
\inf S(P, u) \equiv \inf \overline{S}(P, u) = \begin{cases} 
\Delta_{[u|s-3)}^P & \text{if } u \in \{0, 1\} \\
\Delta_{([s-1-u)2]}^P & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]

and

\[
\sup S(-P, u) \equiv \sup \overline{S}(P, u) = \begin{cases} 
\Delta_{1(|s-1)2]}^P & \text{if } u = 0 \\
\Delta_{([s-2]|3|(s-2))}^P & \text{if } u = 1 \\
\Delta_{([u|s-3)]}^P & \text{if } u \in \{2, 3, \ldots, s - 1\}
\end{cases}
\]

Proof. Since $\tilde{F}$ and $\tilde{F}$ are continuous and strictly increasing when the inequality $p_i > 0$ holds for all $i = 0, s - 1$, our statement is true. \qed

6. Proof of Theorem 8

One can begin with the following statement.

Lemma 3. An arbitrary set $S_{(-P, u)}$ is an uncountable set.

Proof. It is known ([31]) that the set $S_{(-s, u)}$ is uncountable. Really, it follows from using the mapping $h$:

\[x = \Delta_{\mu \mu_1 \mu_2 \mu_3 \ldots \mu_{n-1}}^P \rightarrow \Delta_{\alpha_1 \alpha_2 \ldots \alpha_n}^h(x) = y.\]

It is easy to see that the set \{ $y : y = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_n}^h, \alpha_n \in A \setminus \{0, u\}$\} is an uncountable set.

In our case, we have

\[S_{(-P, u)} \equiv \overline{S}(P, u) \ni \tilde{y} = \tilde{F} \circ f_1(x), \ x \in S_{(-s, u)}.\]

Since the functions $\tilde{F}, f_1$ are continuous and monotonic functions determined on $[0, 1)$, we obtain that $S_{(-P, u)}$ is uncountable. \qed

Let $P = \{p_0, p_1, \ldots, p_{s-1}\}$ be a fixed set of positive numbers such that $p_0 + p_1 + \cdots + p_{s-1} = 1$. Let us consider the class $\Phi$ containing classes $\Phi_{-P}$ of sets $S_{(-P, u)}$ represented in the form

\[S_{(-P, u)} = \left\{ x : x = \Delta_{\mu \mu_1 \mu_2 \mu_3 \ldots \mu_{n-1}}^P, \mu_1 \neq u, \mu_n \neq 0 \right\}, \quad (5)\]

where $u = \overline{0, s-1}$, the parameters $u$ and $s$ are fixed for the set $S_{(-P, u)}$. That is, for a fixed positive integer $s > 3$, the class $\Phi_{-P}$ contains the sets $S_{(-P, 0)}, S_{(-P, 1)}, \ldots, S_{(-P, s-1)}$.

To investigate topological and metric properties of $S_{(-P, u)}$, we study properties of cylinders.

By $(a_1 a_2 \ldots a_k)$ denote the period $a_1 a_2 \ldots a_k$ in the representation of a periodic number.

Let $c_1, c_2, \ldots, c_n$ be a fixed ordered tuple of integers such that $c_i \in A = A \setminus \{0, u\}$ for $i = 1, n$.

Definition 8. A cylinder of rank $n$ with base $c_1 c_2 \ldots c_n$ is a set $\Delta_{c_1 c_2 \ldots c_n}^{-P}$ of the form:

\[\Delta_{c_1 c_2 \ldots c_n}^{-P} = \left\{ x : x = \Delta_{c_1 c_2 \ldots c_n}^{-P}, \alpha_1, \alpha_2 \ldots, \alpha_n = c_j, \ j = 1, n \right\}.\]
By analogy, we have

\[
\Delta_{c_1 \ldots c_n}^{(P,u)} = \left\{ x : x = \Delta_{\overbrace{\ldots \overbrace{u \ldots u}}^{c_1} \overbrace{\ldots \overbrace{u \ldots u}}^{c_2} \ldots \overbrace{\ldots \overbrace{u \ldots u}}^{c_n} \ \text{if} \ \alpha_{n+1} \ldots \alpha_{n+k}, C_j \in A, j = 1, \ldots, k \in \mathbb{N} \right\} .
\]

Remark 7. It is easy to see that

\[
\Delta_{c_1 \ldots c_n}^{(-P,u)} = \Delta_{\overbrace{\ldots \overbrace{u \ldots u}}^{c_1} \overbrace{\ldots \overbrace{u \ldots u}}^{c_2} \ldots \overbrace{\ldots \overbrace{u \ldots u}}^{c_n}}^{(-P,u)} \cap S(\mathcal{U})
\]

and

\[
\Delta_{c_1 \ldots c_n}^{(-P,u)} = \Delta_{\overbrace{\ldots \overbrace{u \ldots u}}^{c_1} \overbrace{\ldots \overbrace{u \ldots u}}^{c_2} \ldots \overbrace{\ldots \overbrace{u \ldots u}}^{c_n}}^{(-P,u)} \cup S(\mathcal{U})
\]

By definition, put

\[
\tilde{p}_{u,i} = \begin{cases} p_u & \text{whenever } i \text{ is odd} \\
p_{u-1} & \text{whenever } i \text{ is even} \end{cases},
\]

\[
\tilde{\beta}_{u,i} = \begin{cases} \beta_u & \text{whenever } i \text{ is odd} \\
\beta_{u-1} & \text{whenever } i \text{ is even} \end{cases},
\]

and

\[C_n = \{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \ldots + c_n\} .\]

Lemma 4. Cylinders \(\Delta_{c_1 \ldots c_n}^{(-P,u)}\) have the following properties:

1. \(\inf\Delta_{c_1 \ldots c_n}^{(-P,u)} = \left\{ \tau_n + \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \left( \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \inf S(p_u) \right) \right\} \) if \(c_1 + \ldots + c_n\) is even,

\[
\inf\Delta_{c_1 \ldots c_n}^{(-P,u)} = \left\{ \tau_n + \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \left( \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \inf S(p_u) \right) \right\} \) if \(c_1 + \ldots + c_n\) is odd,
\]

2. \(\sup\Delta_{c_1 \ldots c_n}^{(-P,u)} = \left\{ \tau_n + \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \left( \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \sup S(p_u) \right) \right\} \) if \(c_1 + \ldots + c_n\) is even,

\[
\sup\Delta_{c_1 \ldots c_n}^{(-P,u)} = \left\{ \tau_n + \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \left( \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \sup S(p_u) \right) \right\} \) if \(c_1 + \ldots + c_n\) is odd,
\]

where

\[
\tau_n = \Delta_{\overbrace{\ldots \overbrace{u \ldots u}}^{c_1} \overbrace{\ldots \overbrace{u \ldots u}}^{c_2} \ldots \overbrace{\ldots \overbrace{u \ldots u}}^{c_n}}^{(-P,u)}(0).
\]

3. \(d(\Delta_{c_1 \ldots c_n}^{(-P,u)}) = \left\{ \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \left( \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} d(S(p_u)) \right) \right\} \) if \(c_1 + \ldots + c_n\) is even,

\[
d(\Delta_{c_1 \ldots c_n}^{(-P,u)}) = \left\{ \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} \left( \prod_{i=1}^{n} \tilde{p}_{c_i, c_i + \ldots + c_i} d(S(p_u)) \right) \right\} \) if \(c_1 + \ldots + c_n\) is odd,
\]

\[
\frac{d(\Delta_{c_1 \ldots c_n}^{(-P,u)})}{d(\Delta_{c_1 \ldots c_n})} = \left\{ \begin{array}{ll}
p_{u-1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ are even} \\
p_{u+1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ is odd} \\
p_{u-1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ are odd} \\
p_{u+1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ is even} \end{array} \right.
\]

\[
\frac{d(\Delta_{c_1 \ldots c_n}^{(-P,u)})}{d(\Delta_{c_1 \ldots c_n})} = \left\{ \begin{array}{ll}
p_{u-1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ are even} \\
p_{u+1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ is odd} \\
p_{u-1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ are odd} \\
p_{u+1} & \text{if } c_1 + \ldots + c_n, c_{n+1} \text{ is even} \end{array} \right.
\]
(4) \[
\Delta_{c_1 \ldots c_n} = \bigcup_{c \in \mathcal{A}} \Delta_{c_1 \ldots c_n} \quad \forall c_n \in \mathcal{A}, \ n \in \mathbb{N}.
\]

(5) The following relationships are satisfied:

(a) If \( u \in \{0,1\} \), then

\[
\begin{align*}
&\inf \Delta_{c_1 \ldots c_n+1} > \sup \Delta_{c_1 \ldots c_n} \quad \text{whenever } c_1 + \cdots + c_n + c \text{ is even} \\
&\inf \Delta_{c_1 \ldots c_n} > \sup \Delta_{c_1 \ldots c_n+1} \quad \text{whenever } c_1 + \cdots + c_n + c \text{ is odd} \\
&c \neq s - 1;
\end{align*}
\]

(b) if \( u \in \{2,3,\ldots,s-3\} \), then for an odd \( c_1 + \cdots + c_n + c \)

\[
\begin{align*}
&\sup \Delta_{c_1 \ldots c_n} < \inf \Delta_{c_1 \ldots c_n} + 1 \quad \text{for all } c + 1 \leq u \\
&\inf \Delta_{c_1 \ldots c_n+1} > \sup \Delta_{c_1 \ldots c_n+1} \quad \text{for all } u < c;
\end{align*}
\]

(c) if \( u \in \{s-2, s-1\} \), then

\[
\begin{align*}
&\inf \Delta_{c_1 \ldots c_n+1} > \sup \Delta_{c_1 \ldots c_n} \quad \text{whenever } c_1 + \cdots + c_n + c \text{ is odd} \\
&\inf \Delta_{c_1 \ldots c_n} > \sup \Delta_{c_1 \ldots c_n+1} \quad \text{whenever } c_1 + \cdots + c_n + c \text{ is even}.
\end{align*}
\]

Proof. The first property follows from equality (3) and the definition of the set \( S_{(-P,u)} \). The second property follows from the first property, and the third property is a corollary of the first and second properties. Property 4 follows from the definition of the set.

Let us prove Property 5. By definition, put

\[
P_n = \prod_{j=1}^{n} \tilde{p}_{c_j,1+c_j+\cdots+c_j}, \quad P^{(u)}_{c_1+\cdots+c_n+1} = \prod_{i=1, c_1+\cdots+c_n+1} \tilde{p}_{u,i}.
\]

Let \( c_1 + c_2 + \cdots + c_n + c \) be an even number and \( u \in \{0,1\} \). Then

\[
\inf \Delta_{c_1 \ldots c_n+1} - \sup \Delta_{c_1 \ldots c_n} = P_n \cdot P^{(u)}_{c_1+\cdots+c_n+1} \cdot (\tilde{p}_{c_1+\cdots+c_n+1} + \tilde{p}_{c_1+\cdots+c_n+1}) \sup \Delta_{c_1 \ldots c_n}
\]

\[
= P_n \cdot P^{(u)}_{c_1+\cdots+c_n+1} \left( \beta_c + p_u \inf \Delta_{c_1 \ldots c_n+1} - \beta_u - \beta_{c-2} - p_u - p_{c-2} \right) \sup \Delta_{c_1 \ldots c_n}
\]

Since \( u \in \{0,1\}, \ c > u \), and \( \beta_c = 1 - p_{c-1} - p_{c-2} - \cdots - p_k \).

Let \( c_1 + c_2 + \cdots + c_n + c \) be an odd number and \( u \in \{0,1\} \). Then

\[
\inf \Delta_{c_1 \ldots c_n+1} - \sup \Delta_{c_1 \ldots c_n} = P_n \cdot P^{(u)}_{c_1+\cdots+c_n+1} \cdot (\tilde{p}_{c_1+\cdots+c_n+1} + \tilde{p}_{c_1+\cdots+c_n+1}) \sup \Delta_{c_1 \ldots c_n}
\]

\[
= P_n \cdot P^{(u)}_{c_1+\cdots+c_n+1} \left( \beta_c + p_u \inf \Delta_{c_1 \ldots c_n+1} - \beta_u - \beta_{c-2} - p_u - p_{c-2} \right) \sup \Delta_{c_1 \ldots c_n}
\]

Since \( \beta_u + p_u = \beta_{u+1} \) and \( c > u \), i.e., \( c \geq u + 1 \).
Let us prove the second system of inequalities. Let \( c_1 + c_2 + \cdots + c_n + c \) be an odd number and \( u \in \{2, 3, \ldots, s - 3\} \). Then for all \( c + 1 \leq u \) let us consider the difference

\[
\sup_{\Delta_c} (\cdot P_n) - \inf_{\Delta_c} (\cdot P_n) = P_n \cdot (\cdot P_n) (u) (c_1 + c_2 + \cdots + c_n + c) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

\[
= P_n \cdot (\cdot P_n)(u)(c_1 + c_2 + \cdots + c_n + c) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

since \( c + 1 \leq u \) and \( \beta_1 = 1 - p_{s-1} - p_{s-2} - \cdots - p_{c+1} - p_c = \beta_1 - 1 + p_{c+1} \).

If \( c_1 + c_2 + \cdots + c_n + c \) is an odd number and \( u \in \{2, 3, \ldots, s - 3\} \), and \( u + 1 \leq c \), then

\[
\sup_{\Delta_c} (\cdot P_n) - \inf_{\Delta_c} (\cdot P_n) = P_n \cdot (\cdot P_n)(u)(c_1 + c_2 + \cdots + c_n + c) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

since \( c \geq u + 1 \) and \( \beta_1 - c = 1 - p_{s-1} - \cdots - p_{s-1} - p_{s-2} - p_{s-3} - p_{s-4} - \cdots - p_{s-5} - p_{s-6}

Let us prove the third system of inequalities. Suppose \( c_1 + c_2 + \cdots + c_n + c \) is even and \( u \in \{2, 3, \ldots, s - 3\} \).

Then

\[
\sup_{\Delta_c} (\cdot P_n) - \inf_{\Delta_c} (\cdot P_n) = P_n \cdot (\cdot P_n)(u)(c_1 + c_2 + \cdots + c_n + c) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

since \( c + 1 \leq u \), i.e., \( s - c - 1 \geq s - c \).

Let us consider the difference

\[
\sup_{\Delta_c} (\cdot P_n) - \inf_{\Delta_c} (\cdot P_n) = P_n \cdot (\cdot P_n)(u)(c_1 + c_2 + \cdots + c_n + c + 1) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

since \( u < c \), i.e., \( s - 1 - u \geq s - c \).

Let us prove the 4th system of inequalities. Let \( c_1 + c_2 + \cdots + c_n + c \) be an odd number and \( u \in \{2, 3, \ldots, s - 1\} \). Then

\[
\sup_{\Delta_c} (\cdot P_n) - \inf_{\Delta_c} (\cdot P_n) = P_n \cdot (\cdot P_n)(u)(c_1 + c_2 + \cdots + c_n + c + 1) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

since \( u + 1 \leq c + 1 \).

Suppose \( c_1 + c_2 + \cdots + c_n + c \) is even. Then

\[
\sup_{\Delta_c} (\cdot P_n) - \inf_{\Delta_c} (\cdot P_n) = P_n \cdot (\cdot P_n)(u)(c_1 + c_2 + \cdots + c_n + c + 1) \sup_{\Delta_c} S_{(P_n)(u)} - \inf_{\Delta_c} S_{(P_n)(u)}
\]

since \( u + 1 \leq c + 1 \).
that there exist cylinders \( \Delta_{c_1 \cdots c_n} \) of rank \( n \) in an arbitrary subinterval of the segment \( I = [\inf S_{(-P,u)}, \sup S_{(-P,u)}] \). Since Property 5 from Lemma 5 is true for these cylinders, we have that for any subinterval of \( I \) there exists a subinterval such that does not contain points from \( S_{(-P,u)} \). So \( S_{(-P,u)} \) is a nowhere dense set.

Let us prove that the set \( S_{(-P,u)} \) is a set of zero Lebesgue measure. Suppose that \( I_{c_1c_2 \cdots c_n} \) is a closed interval whose endpoints coincide with endpoints of the cylinder \( \Delta_{c_1c_2 \cdots c_n} \). It is easy to see that

\[
|I_{c_1c_2 \cdots c_n}| = d(\Delta_{c_1c_2 \cdots c_n}).
\]

Also,

\[
S_{(-P,u)} = \bigcap_{k=1}^{\infty} S_k,
\]

where

\[
S_1 = \bigcup_{c_1 \in A} J_{c_1},
\]

\[
S_2 = \bigcup_{c_1, c_2 \in A} J_{c_1c_2},
\]

\[
\cdots
\]

\[
S_k = \bigcup_{c_1, c_2, \ldots, c_k \in A} I_{c_1c_2 \cdots c_k},
\]

\[
\cdots
\]

In addition, since \( S_{k+1} \subset S_k \), we have

\[
S_k = S_{k+1} \cup S_{k+1}.
\]

Suppose

\[
I_P = [\inf S_{(P,u)}, \sup S_{(P,u)}]
\]

and

\[
I_u = [\inf S_{(P,u)}, \sup S_{(P,u)}]
\]

are initial closed intervals, \( \lambda(\cdot) \) is the Lebesgue measure of a set. Then

\[
0 < \lambda(S_1) = \sum_{c_1 \text{ is odd}} \left( \prod_{i=1}^{c_1-1} \tilde{p}_{u,i} \right) \lambda(I_0) + \sum_{c_1 \text{ is even}} \left( \prod_{i=1}^{c_1-1} \tilde{p}_{u,i} \right) \lambda(I_{\overline{\eta}}).
\]

It is easy to see that

\[
0 < \lambda(S_1) \leq \sum_{c_1 \in A} \left( \prod_{i=1}^{c_1-1} \tilde{p}_{u,i} \right) \max \left\{ \lambda(I_P), \lambda(I_u) \right\} < 1
\]

since

\[
\lambda \left( \bigcup_{c_1, \ldots, c_n \in A = \{0, 1, \ldots, s-1\}} \Delta_{c_1c_2 \cdots c_n}^P \right) = 1.
\]

Also, one can denote

\[
v_{c_1} = \sum_{c_1 \in A} \left( \prod_{i=1}^{c_1-1} \tilde{p}_{u,i} \right) < 1 = p_0 + p_1 + \cdots + p_{s-1}.
\]
In the second step, we get

\[ 0 < \lambda(S_2) = \sum_{\substack{c_1, c_2 \text{ is odd} \in A \\ c_1, c_2 \in A}} \left( \prod_{j=1}^{n} \tilde{p}_{c_j, c_1 + c_2 + \ldots + c_j} \right) \left( \prod_{i=1}^{\alpha} \tilde{p}_{u, i} \right) \lambda(I_2) \]

\[ + \sum_{\substack{c_1, c_2 \text{ is even} \in A \\ c_1, c_2 \in A}} \left( \prod_{j=1}^{n} \tilde{p}_{c_j, c_1 + c_2 + \ldots + c_j} \right) \left( \prod_{i=1}^{\alpha} \tilde{p}_{u, i} \right) \lambda(I_2) \]

\[ \leq \sum_{\substack{c_1, c_2 \text{ is even} \in A}} \left( \prod_{j=1}^{n} \tilde{p}_{c_j, c_1 + c_2 + \ldots + c_j} \right) \left( \prod_{i=1}^{\alpha} \tilde{p}_{u, i} \right) \max \{ \lambda(I_2), \lambda(I_2) \} \]

\[ \leq \max \{ \lambda(I_2), \lambda(I_2) \} \left( \max \{ v_{c_1}, v_{c_2} \} \right)^2, \]

where

\[ v_{c_2} = \sum_{c_2 \in A} \left( \prod_{i=c_1+1}^{c_1+c_2-1} \tilde{p}_{u, i} \right) < 1. \]

In the nth step, we have

\[ 0 < \lambda(S_n) = \sum_{\substack{c_1, c_2, \ldots, c_n \text{ is odd} \in A \\ c_1, c_2, \ldots, c_n \in A}} \left( \prod_{j=1}^{n} \tilde{p}_{c_j, c_1 + c_2 + \ldots + c_j} \right) \left( \prod_{i=1}^{\alpha} \tilde{p}_{u, i} \right) \lambda(I_2) \]

\[ + \sum_{\substack{c_1, c_2, \ldots, c_n \text{ is even} \in A \\ c_1, c_2, \ldots, c_n \in A}} \left( \prod_{j=1}^{n} \tilde{p}_{c_j, c_1 + c_2 + \ldots + c_j} \right) \left( \prod_{i=1}^{\alpha} \tilde{p}_{u, i} \right) \lambda(I_2) \]

\[ \leq \sum_{\substack{c_1, c_2, \ldots, c_n \text{ is even} \in A}} \left( \prod_{j=1}^{n} \tilde{p}_{c_j, c_1 + c_2 + \ldots + c_j} \right) \left( \prod_{i=1}^{\alpha} \tilde{p}_{u, i} \right) \max \{ \lambda(I_2), \lambda(I_2) \} \]

\[ \leq \max \{ \lambda(I_2), \lambda(I_2) \} \left( \max_{k=1}^{n} \left( \sum_{c_k \in A} \left( \prod_{i=c_1+c_2+\ldots+c_{k-1}+1}^{c_1+c_2+\ldots+c_k-1} \tilde{p}_{u, i} \right) \right)^n \right) \]

\[ = \max \{ \lambda(I_2), \lambda(I_2) \} \left( \max_{k=1}^{n} \{ v_{c_k} \} \right)^n < 1. \]

Here \( c_{k-1} = 0 \) for \( k = 1 \).

So,

\[ \lim_{n \to \infty} \lambda(S_n) \leq \lim_{n \to \infty} \left( \max \{ \lambda(I_2), \lambda(I_2) \} \left( \max_{k=1}^{n} \{ v_{c_k} \} \right)^n \right) = 0. \]

Hence the set \( S_1 \cup \cdots \cup S_\infty \) is a set of zero Lebesgue measure.

Let us prove that \( S_1 = \ldots = S_\infty \) is a perfect set. Since

\[ S_k = \bigcup_{c_1, c_2, \ldots, c_k \in A} I_{c_1, c_2, \ldots, c_k} \]

is a closed set (\( S_k \) is a union of segments), we see that

\[ S_\infty = \bigcap_{k=1}^{\infty} S_k \]

is a closed set.
Suppose $x \in S(-P,u)$, $R$ is any interval containing $x$, and $J_n$ is a segment of $S_n$ such that contains $x$. Let us choose a number $n$ such that $J_n \subset R$. Suppose that $x_n$ is the endpoint of $J_n$ such that the condition $x_n \neq x$ holds. Hence $x_n \in S(-P,u)$ and $x$ is a limit point of the set.

Since $S(-P,u)$ is a closed set and does not contain isolated points, we obtain that $S(-P,u)$ is a perfect set.

### 7. Proof of Theorem \[7\]

Since $S(-P,u) \subset J$ and $S(-P,u)$ is a perfect set, we obtain that $S(-P,u)$ is a compact set. In addition,

$$
\Delta(\frac{-P,u}{c_1e_2\ldots c_{n-1}c_n}) = \begin{cases} 
\omega_{1,c_n} = \text{pf}_{a-1-n}p_{a-1-n}p_{a-1-n}d(S(-P,u)) & \text{if } c_1 + \cdots + c_{n-1} \text{ is odd, } c_n \text{ is odd} \\
\omega_{2,c_n} = \text{pf}_{a-1-n}p_{a-1-n}p_{a-1-n}d(S(-P,u)) & \text{if } c_1 + \cdots + c_{n-1} \text{ is even, } c_n \text{ is odd} \\
\omega_{3,c_n} = \text{pf}_{a-1-n}p_{a-1-n}p_{a-1-n}p_{a-1-n}d(S(-P,u)) & \text{if } c_1 + \cdots + c_{n-1} \text{ is odd, } c_n \text{ is even} \\
\omega_{4,c_n} = \text{pf}_{a-1-n}p_{a-1-n}p_{a-1-n}p_{a-1-n}d(S(-P,u)) & \text{if } c_1 + \cdots + c_{n-1} \text{ is even, } c_n \text{ is even}
\end{cases}
$$

and

$$
S(-P,u) = \bigcap_{n=1}^{\infty} \bigcup_{c_1,\ldots,c_n \in \mathcal{A}} \Delta(\frac{-P,u}{c_1e_2\ldots c_{n-1}c_n}).
$$

Suppose $l$ is the number of odd numbers in the set $\mathcal{A} = \{0, 1, \ldots, s - 1\}$ \(\setminus\{0, u\}\) and $m$ is the number of even numbers in $\mathcal{A}$. We have $(l + m)^n$ cylinders $\Delta(\frac{-P,u}{c_1e_2\ldots c_{n-1}c_n})$.

Let $\dot{N}_{j,n}$ ($j = \overline{1,4}, 1 < n \in \mathbb{N}$) be the number of cylinders $\Delta(\frac{-P,u}{c_1e_2\ldots c_{n-1}c_n})$ for which

$$
d \left( \Delta(\frac{-P,u}{c_1e_2\ldots c_{n-1}c_n}) \right) = \omega_{j,c_n}.
$$

**In the first step,** we have $\dot{N}_{2,1} = l$ (the unique cylinder for any odd $c_1 \in \mathcal{A}$) and $\dot{N}_{4,1} = m$ (the unique cylinder for any even $c_1 \in \mathcal{A}$) because for $n = 0$ we get $J \cap S(P,u)$.

**In the second step,** we have:

- $\dot{N}_{1,2} = l^2$ (l cylinders for any odd $c_2 \in \mathcal{A}$);
- $\dot{N}_{2,2} = lm$ (m cylinders for any odd $c_2 \in \mathcal{A}$);
- $\dot{N}_{3,2} = lm$ (l cylinders for any even $c_2 \in \mathcal{A}$);
- $\dot{N}_{4,2} = m^2$ (m cylinders for any even $c_2 \in \mathcal{A}$).

**In the third step,** we have:

- $\dot{N}_{1,3} = 2l^2m$ (2m cylinders for any odd $c_3 \in \mathcal{A}$);
- $\dot{N}_{2,3} = l(l^2 + m^2)$ (l^2 cylinders for any odd $c_3 \in \mathcal{A}$);
- $\dot{N}_{3,3} = 2lm^2$ (2lm cylinders for any even $c_3 \in \mathcal{A}$);
- $\dot{N}_{4,3} = m(l^2 + m^2)$ (l^2 cylinders for any even $c_3 \in \mathcal{A}$).

**In the fourth step,** we have:

- $\dot{N}_{1,4} = l^4 + 3l^2m^2$ (l^3 cylinders for any odd $c_4 \in \mathcal{A}$);
- $\dot{N}_{2,4} = lm^3 + 3lm^2$ (m^3 cylinders for any odd $c_4 \in \mathcal{A}$);
- $\dot{N}_{3,4} = l^3m + 3lm^2$ (l^3 cylinders for any even $c_4 \in \mathcal{A}$);
- $\dot{N}_{4,4} = m^4 + 3l^2m^2$ (m^4 cylinders for any even $c_4 \in \mathcal{A}$).

Let us remark that, in the general case, values of $\omega_{j,c_n}$ are different for the unique $j$ but different $c_n \in \mathcal{A}$. Hence the Hausdorff dimension of our set depends on the numbers of $\omega_{j,c_n}$ for all different $c_n \in \mathcal{A}$.

Using arguments described in \[11\] and auxiliary Theorems \[11\], this completes the proof.
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