Real Forms of the Complex Neumann System: 
Real Roots of Polynomial \( U_S(\lambda) \)

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Abstract

The topology of Liouville sets of the real forms of the complex generic Neumann system depends indirectly on the roots of the special polynomial \( U_S(\lambda) \). For certain polynomials, the existence and positions of the real roots, according to the suitable parameters of the system, is not obvious. In the paper, a novel method for checking the existence and positions of the real roots of the polynomials \( U_S(\lambda) \) is given. The method and algorithm are based on searching of a positive solution of a system of linear equations. We provide a complete solution to the problem of existence of real roots for all special polynomials in case \( n = 2 \). This is a step closer to determining the topology of the Liouville sets.

*Keywords*: roots of polynomial; complex Neumann system; positive solution; system of linear equations.

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1 Introduction

Motivation. In the Hamiltonian mechanics, the geometry of the Liouville sets provides very important information about a Hamiltonian system. It gives the topology of the set of all possible point’s positions of a system in the ambient space.

The Neumann system describes the motion of a particle on the sphere \( S^n \) under the influence of a quadratic potential. For \( n = 2 \), the system is introduced in [5]. Let \( (q, p) = (q_1, \ldots, q_{n+1}, p_1, \ldots, p_{n+1}) \in T^*\mathbb{R}^{n+1} \) be ambient coordinates and \( a_1, \ldots, a_{n+1} \) real constants. Throughout the paper, we assume \( a_1 < a_2 < \ldots < a_{n+1} \).

This corresponds to the so-called generic case of the Neumann system. In the Hamiltonian formalism, the Neumann system can be written as a triple \( (T^*S^n, \omega, H) \) where \( T^*S^n \) is the cotangent bundle of the sphere \( S^n \), \( \omega \) is the canonical 2-form in ambient coordinates, \( \omega = \sum_{j=1}^{n+1} dq_j \wedge dp_j \), and

\[
H = H(q, p) = \frac{1}{2} \left( \sum_{j=1}^{n+1} q_j^2 \sum_{j=1}^{n+1} p_j^2 - \sum_{j=1}^{n+1} q_j p_j \right) + \frac{1}{2} \left( 2 - \sum_{j=1}^{n+1} q_j^2 \right) \sum_{j=1}^{n+1} a_j q_j^2
\]

\(^1\)Liouville set is the regular level set of the system’s moment map
is the Hamiltonian function. The wedge product $\wedge$ is the exterior product of forms and it is clearly explained in [11] (Chapter 7 and Chapter 8). The natural complexification converts real variables $(q, p)$ into complex $(Q, P) \in T^*\mathbb{C}^{n+1}$, and the Neumann system into the complex Neumann system $(T^*(S^n)^C, \omega^C, H^C)$ (see [4][6], see [2] for the meaning of such complex systems). Hence, the classical Neumann system as described above is the real form of the complex Neumann system regard to the standard antiholomorphic involutive automorphism

$$
\tau : T^*\mathbb{C}^{n+1} \rightarrow T^*\mathbb{C}^{n+1}
\quad (Q_1, \ldots, Q_{n+1}, P_1, \ldots, P_{n+1}) \mapsto (\bar{Q}_1, \ldots, \bar{Q}_{n+1}, \bar{P}_1, \ldots, \bar{P}_{n+1})
$$

By $\bar{Q}_j$ and $\bar{P}_j$ we denote the complex conjugate of $Q_j$ and $P_j$, respectively. In [6], the real forms of the complex Neumann system are introduced. Let $S \subseteq \{1, 2, \ldots, n+1\}$ and

$$
\epsilon_{S,j} = \begin{cases} 
1 & ; j \in S \\
-1 & ; j \in S^c.
\end{cases}
$$

Denote by $J_S$ the diagonal matrix with 1 on positions $S$ and $-1$ elsewhere. Other real forms are obtained as real forms of the antiholomorphic involutive automorphisms

$$
\tau_S : T^*\mathbb{C}^{n+1} \rightarrow T^*\mathbb{C}^{n+1}
\quad (Q, P) \mapsto (J_S \bar{Q}, J_S \bar{P}).
$$

According to $\tau_S$, we denote the corresponding real system as the triple $(T^*\mathbb{H}^n_S, \omega_S, H_S)$, where $T^*\mathbb{H}^n_S = \{(q, p) \in T^*\mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1} \epsilon_{S,j} q_j^2 = 1, \sum_{j=1}^{n+1} \epsilon_{S,j} q_j p_j = 0\}$, i.e. the cotangent bundle of the hyperboloid $\mathbb{H}^n_S = \{q \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1} \epsilon_{S,j} q_j^2 = 1\}$, $\omega_S = \sum_{j=1}^{n+1} \epsilon_{S,j} dq_j \wedge dp_j$, and

$$
H_S(q, p) = \frac{1}{2} \left( \sum_{j=1}^{n+1} \epsilon_{S,j} q_j^2 + \sum_{j=1}^{n+1} \epsilon_{S,j} p_j^2 - \sum_{j=1}^{n+1} \epsilon_{S,j} q_j p_j \right) + \frac{1}{2} \left( 2 - \sum_{j=1}^{n+1} \epsilon_{S,j} q_j^2 \right) \sum_{j=1}^{n+1} \epsilon_{S,j} a_j q_j^2.
$$

The topology of the Liouville sets is known only for two families of the real forms, i.e. for all subsets $S = \{1, 2, \ldots, k\}$ for $k \in \{1, 2, \ldots, n+1\}$ and all subsets $S = \{k\}$ for $k \in \{1, 2, \ldots, n+1\}$ (see [6]).

The complex Neumann system is a special example of the Mumford system (see [4][6][8]). It is characterized by the Lax pair $(L^C(\lambda), M^C(\lambda))$ of $2 \times 2$ matrices where

$$
L^C = \begin{pmatrix} 
V^C(\lambda) & W^C(\lambda) \\
U^C(\lambda) & -V^C(\lambda)
\end{pmatrix}
$$

for suitable polynomials $U^C(\lambda), V^C(\lambda)$ and $W^C(\lambda)$. It is known that the topology of a Liouville set of a real form directly depends on the positions of roots of the suitable real form of $U^C(\lambda)$ according to the roots of suitable real form of $f^C(\lambda) = U^C(\lambda) W^C(\lambda) + (V^C(\lambda))^2$, and indirectly depends on the existence and positions of real roots of suitable real form of $U^C(\lambda)$ according to the given constants $a_1, a_2, \ldots, a_n$.

Some generalizations of the Neumann system were studied in [7]. In [9], the Liouville sets of the confluent (non-generic) Neumann system were investigated.

In [6], it is proved that the polynomial

$$
U_S(\lambda) = A(\lambda) \sum_{j=1}^{n+1} \epsilon_{S,j} \frac{q_j^2}{\lambda - a_j}
$$

has $n$ real roots if $S = \{m, m+1, \ldots, k\}$ for some $m \in \{1, 2, \ldots, n+1\}$ and some $k \in \{m, m+1, \ldots, n+1\}$ taking into account the constraint $\sum_{j=1}^{n+1} \epsilon_{S,j} q_j^2 = 1$. If the set $S$ does not contain consecutive integers using the method applied in [6], it is not possible to make any conclusions regarding the real roots positions (and existence) of $U_S(\lambda)$, as the following shows.
Example 1 For $n = 2$ and $S = \{1, 3\}$, we have the polynomial

$$U_{\{1,3\}}(\lambda) = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)\left(\frac{q_1^2}{\lambda - a_1} - \frac{q_2^2}{\lambda - a_2} + \frac{q_3^2}{\lambda - a_3}\right) = q_1^2(\lambda - a_2)(\lambda - a_3) - q_2^2(\lambda - a_1)(\lambda - a_3) + q_3^2(\lambda - a_1)(\lambda - a_2).$$

Since $q_1^2 - q_2^2 + q_3^2 = 1$, the leading coefficient of $U_{\{1,3\}}$ is 1. Furthermore, $\text{sgn}(U_{\{1,3\}}(a_1)) = \text{sgn}(U_{\{1,3\}}(a_2)) = \text{sgn}(U_{\{1,3\}}(a_3)) = 1$, and we can not conclude easily about the existence or positions of real roots of polynomial.

Remark 2 The discriminant of the polynomial $U_{\{1,3\}}(\lambda)$ above is

$$D = \left(-(a_2 + a_3)q_1^2 + (a_1 + a_3)q_2^2 - (a_1 + a_2)q_3^2\right)^2 - 4(a_2a_3q_1^2 - a_1a_3q_2^2 + a_1a_2q_3^2).$$

Even if we find a set of points $(q_1, q_2, q_3) \in H^0_{\{1,3\}}$ for which the discriminant $D$ is positive (hence real roots exist), we can not determine the positions or real roots according to the given parameters $a_1, a_2$ and $a_3$.

Remark 3 In general, for a polynomial of degree $n$, the necessary condition for the existence of real roots is the positivity of the discriminant. If $n \geq 4$, the positivity of the discriminant is not a sufficient condition.

At this situation it makes sense to ask the following questions.

1. Can we find, in general or in some special case, the set of $(q_1, q_2, \ldots, q_{n+1}) \in H^n_S$ for which the polynomial $U_S(\lambda)$ has only real roots?
2. Can we determine, in general or in some special case, positions of $n$ real roots of polynomial $U_S(\lambda)$ according to the given parameters $a_1, a_2, \ldots, a_{n+1}$?
3. Can we describe, in general or in some special case regarding $n$ or $S$, the topology of a Liouville set?

In our approach the key idea is to replace the roles of constants and variables. Instead of assuming that $q_1, q_2, \ldots, q_{n+1}$ are (given) coordinates of a point on the hyperboloid $H^n_S$ and searching for roots of polynomial $U_S(\lambda)$, we fix values (potential roots) $\lambda_1, \lambda_2, \ldots, \lambda_n$ up to the intervals $(-\infty, a_1), (a_1, a_{i+1})$ for $i = 1, 2, \ldots, n$ and $(a_{n+1}, \infty)$, and ask whether a positive solution $(q_1^2, q_2^2, \ldots, q_{n+1}^2)$ of the system of linear equations

$$\sum_{j=1}^{n+1} \epsilon_{S, j} q_j^2 = 1$$
$$\sum_{j=1}^{n+1} \epsilon_{S, j} \frac{A(\lambda_1)}{\lambda_1 - a_j} q_j^2 = 0$$
$$\sum_{j=1}^{n+1} \epsilon_{S, j} \frac{A(\lambda_2)}{\lambda_2 - a_j} q_j^2 = 0$$
$$\vdots$$
$$\sum_{j=1}^{n+1} \epsilon_{S, j} \frac{A(\lambda_n)}{\lambda_n - a_j} q_j^2 = 0$$

exists. By the setting positions of real parameters (roots) $\lambda_1, \lambda_2, \ldots, \lambda_n$, the signs of all coefficients of system (4) are determined. Recall that from the definition of roots of a polynomial and the additional condition $\sum_{j=1}^{n+1} \epsilon_{S, j} q_j^2 = 1$ for coordinates $q_1, q_2, \ldots, q_{n+1}$, the following statement is obvious. Let $\sum_{j=1}^{n+1} \epsilon_{S, j} q_j^2 = 1$. 3
Real values $\lambda_1, \ldots, \lambda_n$ are the roots of the polynomial $U_S(\lambda)$ (see (3)) for some $q_1, q_2, \ldots, q_{n+1}$ if and only if the system of $n+1$ non-homogeneous linear equations (4) admits a positive solution $(q_1^2, q_2^2, \ldots, q_{n+1}^2)$.

By this simple method we obtain a new result towards completely determining the Liouville sets of the complex Neumann system. In the paper, the method for checking the existence of a positive solution of a linear homogeneous system with real coefficients is recalled (from (3)). As is seen in one of low dimension cases, elaborated in subsection 3.4, the procedure is time-consuming. Hence the problem yell after the computer computation. For the algorithm for checking the existence of a positive solution of the system (4) in MATLAB, two examples of scripts and two main functions are given. For the case $n = 2$, results of the algorithm for all nonempty subsets $S \subseteq \{1, 2, 3\}$ are in Table 1, Table 2 and Table 3. For the subsets $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{2, 3\}$ and $\{1, 2, 3\}$, the results are trivial and expected. For the subset $\{1, 3\}$, it is newly discovered that the polynomial $U_{\{1, 3\}}(\lambda)$ has real roots for some $(q_1, q_2, q_3) \in \mathcal{H}^2_{\{1, 3\}}$. Thus, for two cases, the Liouville set of the Hamiltonian system $(T^*\mathcal{H}^2_{\{1, 3\}}, \omega_{\{1, 3\}}, H_{\{1, 3\}})$ is exactly determined.

After the introduction, the method of finding a positive solution of a system of linear equations is recalled in section 2. In section 3, the scheme of the algorithm is presented. For the example $S = \{1, 3\}$, the procedure of the algorithm is given. Because of the monotonic conversions of equations of the system, the matrix forms of a given system and a reduced system are introduced. At the end of section 3, there is the discussion about the running the algorithm for the case $n = 2$. In section 4, we comment the results of the algorithm and mention open issues. In appendices, concrete operation of the algorithm, results of algorithm and additional analysis are given.

2 Positive solution of a system of linear equation

In (3), an algorithm for determining whether a given system of linear equations with real coefficients admits a positive solution is given. In this section, the idea of the algorithm is summarized with emphasis on a solution of a system of nonhomogeneous equations.

2.1 A single non-homogeneous equation

Assume that a non-homogeneous linear equation is given

$$b_1 x_1 + b_2 x_2 + \ldots + b_n x_n = b.$$

For $b_{n+1} = -b$ and introducing the condition $x_{n+1} = 1$, the equivalent homogeneous equation has a form

$$b_1 x_1 + b_2 x_2 + \ldots + b_n x_n + b_{n+1} x_{n+1} = 0.$$  \hspace{1cm} (5)

Now, temporarily forget both conditions above. A solution of homogeneous equation (5) can be found by the procedure described in (3) that is briefly recalled below. Assume that some of coefficients in (5) are positive and some of coefficients are negative (which is a necessary condition for the existence of a positive solution for (5)). The coefficients can be divided into two sets

- **positive coefficients**: \( \{b_i\} \) for \( i \in I = \{i_1, i_2, \ldots, i_P\} = \{i_p \mid p = 1, 2, \ldots, P\} \)
- **negative coefficients**: \( \{b_j\} \) for \( j \in J = \{j_1, j_2, \ldots, j_Q\} = \{j_q \mid q = 1, 2, \ldots, Q\} \).

Note that $P + Q = n + 1$. As soon as the equation (5) is written in the form

$$\sum_{i \in I} b_i x_i = - \sum_{j \in J} b_j x_j$$
one among the positive solutions is obvious

\[ x_i = - \sum_{j \in J} b_j \quad \text{for } i \in I \]

\[ x_j = \sum_{i \in I} b_i \quad \text{for } j \in J. \]

### 2.2 A system of non-homogeneous linear equations

Consider first the system of \( m \) homogeneous equations given by

\[ \sum_{s=1}^{n+1} b_{r,s} x_s = 0 \quad ; \quad r = 1, 2, \ldots, m. \]  \hspace{1cm} (6)

As in the previous subsection, the coefficients of the first equation can be separated into two subsets

positive coefficients : \( \{ b_{1,i} \} \) for \( i \in I = \{ i_1, i_2, \ldots, i_P \} = \{ i_p \mid p = 1, 2, \ldots, P \} \)

negative coefficients : \( \{ b_{1,j} \} \) for \( j \in J = \{ j_1, j_2, \ldots, j_Q \} = \{ j_q \mid q = 1, 2, \ldots, Q \} \),

where \( P + Q = n + 1 \), and the first equation can be written in the form

\[ \sum_{i \in I} b_{1,i} x_i = - \sum_{j \in J} b_{1,j} x_j. \]  \hspace{1cm} (7)

Then, the remaining equations have to be written in the form

\[ \sum_{j \in J} b_{r,j} x_j = - \sum_{i \in I} b_{r,i} x_i \quad ; \quad r = 2, 3, \ldots, m. \]  \hspace{1cm} (8)

We multiply both sides of each equation of (8) by the corresponding sides of (7) and obtain \( m - 1 \) equations

\[ \sum_{i \in I, j \in J} b_{r,i,j} x_{i,j} = 0 \quad \text{for } r = 2, 3, \ldots, m, \]  \hspace{1cm} (9)

where

\[ b_{r,i,j} = b_{1,i} b_{r,j} - b_{1,j} b_{r,i} \quad \text{and} \quad x_{i,j} = x_i x_j. \]  \hspace{1cm} (10)

Recall the next theorem that is proven in [3].

**Theorem 4 (Dines [3])** To every positive solution of the system (6) there corresponds a positive solution of the system (9), and conversely.

If a positive solution \( (x_1, \ldots, x_{n+1}) \) of the system (6) is given, the solution of the system (9) is obvious. Conversely, if the system (9) admits a positive solution \( (x_{i,j}) \), then

\[ x_i = - \sum_{j \in J} b_{1,j} x_{i,j}, \quad \text{and} \quad x_j = \sum_{i \in I} b_{1,i} x_{i,j}. \]

Next corollary is related to the existence of a positive solution of a system of non-homogeneous linear equations.

**Corollary 5** To every positive solution of the system (6) where \( x_{n+1} = 1 \) there corresponds a positive solution of the system (9), and conversely.
Proof: If \((\mathbf{x}_1, \ldots, \mathbf{x}_n, 1)\) is a positive solution of the system \((6)\), then \((\mathbf{x}_1, \ldots, \mathbf{x}_n, 1)\) also satisfies \((7)\) and \((8)\). Therefore, \(x_{i,j} = \mathbf{x}_i \mathbf{x}_j\) for \(i, j \in \{1, \ldots, n\}\) and \(x_{i,n+1} = x_{n+1,i} = \mathbf{x}_i\) satisfy \((9)\).

Conversely, let \((\mathbf{x}_{i,j})\) be a positive solution of the system \((9)\). Without the condition on the value of \(x_{n+1}\), a positive solution of \((6)\) is known from the above theorem, i.e.

\[
x_i = -\sum_{j \in I} b_{1,j} x_{i,j}, \quad \text{and} \quad x_j = \sum_{i \in I} b_{1,i} x_{i,j}.
\]

Without loss of generality, set \(n+1 \in J\). Since \(P \neq 0\), \(Q \neq 0\) and \(x_{i,j} > 0\), we have

\[
\sum_{i \in I} b_{1,i} x_{i,n+1} > 0. \tag{11}
\]

Hence, we can "normalize" the given solution of the system \((6)\) by dividing the solution above by the sum \((11)\) as follows

\[
x_{n+1} = \frac{\sum_{i \in I} b_{1,i} x_{i,n+1}}{\sum_{i \in I} b_{1,i} x_{i,n+1}} = 1, \quad x_j = \frac{\sum_{i \in I} b_{1,i} x_{i,j}}{\sum_{i \in I} b_{1,i} x_{i,j}} \quad \text{and} \quad x_i = -\frac{\sum_{j \in J} b_{1,j} x_{i,j}}{\sum_{i \in I} b_{1,i} x_{i,n+1}}.
\]

3 Algorithm

First, we choose an arbitrary nonempty subset \(S \subseteq \{1, 2, \ldots, n+1\}\) and write the system \((4)\) in the homogeneous form

\[
\sum_{s=1}^{n+2} b_{r,s} q_s^2 = 0; \quad r = 0, 1, \ldots, n, \tag{12}
\]

where \(b_{0,s} = \epsilon_s s\) for \(s = 1, 2, \ldots, n+1\) (see \((1)\)), \(b_{0,n+2} = -1, b_{r,s} = \epsilon_s s A(\lambda_r)\) for \(s = 1, 2, \ldots, n+1\) and \(r = 1, 2, \ldots, n\), and \(b_{r,n+2} = 0\) for \(r = 1, 2, \ldots, n\).

3.1 The basic concept

Set the positions of the parameters \(\lambda_1 < \lambda_2 < \ldots < \lambda_n\) with respect to the values \(a_1 < a_2 < \ldots < a_{n+1}\).

**Step 1.** Verify the existence of the solution (every equation contains both positive and negative coefficients). If there exists one equation with all positive or all negative coefficients, go to **Step 3**. Else, if there is only one equation, go to **Step 4**, else, go to **Step 2**.

**Step 2.** Set \(k\) as the number of equations. In the first equation, separate the values; the positive values on the left side, the negative values on the right side. In all other equations, separate the values by the indices as: on the left side stay indices which go on the right side in the first equation, on the right side go indices which stay on the left side in the first equation. Form a new linear homogeneous system as:

a) the left side of the \(j\)-th equation is obtained by multiplying the left side of the first equation (in the original system) and the left side of the \((j + 1)\)-th equation (in the original system), the right side of the \(j\)-th equation is obtained by multiplying the right side of the first equation (in the original system) and the right side of the \((j + 1)\)-th equation (in the original system)

b) arranging new equations into the homogeneous linear system, new variables are the products of the previous ones.

**Step 3.** The system does not have any positive solution.

**Step 4.** The system has a positive solution.
3.2 Example \( n = 2 \) and \( S = \{1, 3\} \)

Let \( a_1 < a_2 < a_3 \) and \((q_1, q_2, q_3)\) be coordinates in \( \mathbb{R}^3 \). Take \( S = \{1, 3\} \) (see Example 1 and Example 2.1. in [6]). We would like to characterize the roots of the polynomial

\[
U_{\{1,3\}}(\lambda) = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3) \left( \frac{q_1^2}{\lambda - a_1} - \frac{q_2^2}{\lambda - a_2} + \frac{q_3^2}{\lambda - a_3} \right)
\]

according to the constraint \( q_1^2 - q_2^2 + q_3^2 = 1 \). There are 10 possible positions of real roots \( \lambda_1, \lambda_2 \) (according to the ordered constants \( a_1, a_2, a_3 \)) of the polynomial \( U_{\{1,3\}} \):

(i) \( \lambda_1 < \lambda_2 < a_1 < a_2 < a_3 \),

(ii) \( \lambda_1 < a_1 < \lambda_2 < a_2 < a_3 \),

(iii) \( \lambda_1 < a_1 < a_2 < \lambda_2 < a_3 \),

(iv) \( \lambda_1 < a_1 < a_2 < a_3 < \lambda_2 \),

(v) \( a_1 < \lambda_1 < \lambda_2 < a_2 < a_3 \),

(vi) \( a_1 < \lambda_1 < a_2 < \lambda_2 < a_3 \),

(vii) \( a_1 < \lambda_1 < a_2 < a_3 < \lambda_2 \),

(viii) \( a_1 < a_2 < \lambda_1 < \lambda_2 < a_3 \),

(ix) \( a_1 < a_2 < \lambda_1 < a_3 < \lambda_2 \),

(x) \( a_1 < a_2 < a_3 < \lambda_1 < \lambda_2 \).

They dictate 10 verifications of the solvability of the system

\[
\begin{align*}
q_1^2 - q_2^2 + q_3^2 &= 1 \\
(\lambda_1 - a_2)(\lambda_1 - a_3)q_1^2 - (\lambda_1 - a_1)(\lambda_1 - a_3)q_2^2 + (\lambda_1 - a_1)(\lambda_1 - a_2)q_3^2 &= 0 \\
(\lambda_2 - a_2)(\lambda_2 - a_3)q_1^2 - (\lambda_2 - a_1)(\lambda_2 - a_3)q_2^2 + (\lambda_2 - a_1)(\lambda_2 - a_2)q_3^2 &= 0.
\end{align*}
\]  \( (13) \)

The system \( (13) \) is equivalent to the homogeneous system of four variables

\[
\begin{align*}
q_1^2 - q_2^2 + q_3^2 - q_4^2 &= 0 \\
(\lambda_1 - a_2)(\lambda_1 - a_3)q_1^2 - (\lambda_1 - a_1)(\lambda_1 - a_3)q_2^2 + (\lambda_1 - a_1)(\lambda_1 - a_2)q_3^2 + 0 \cdot q_4^2 &= 0 \\
(\lambda_2 - a_2)(\lambda_2 - a_3)q_1^2 - (\lambda_2 - a_1)(\lambda_2 - a_3)q_2^2 + (\lambda_2 - a_1)(\lambda_2 - a_2)q_3^2 + 0 \cdot q_4^2 &= 0.
\end{align*}
\]  \( (14) \)

with the requirement \( q_4 = 1 \). In the notation of the homogeneous system \( \sum_{s=1}^{4} b_{r,s}q_s^2 = 0 \), \( r = 0, 1, 2 \) (see (12)) we have

\[
\begin{align*}
b_{0,1} &= 1 \quad &b_{0,2} &= -1 \quad &b_{0,3} &= 1 \quad &b_{0,4} &= -1 \\
b_{1,1} &= (\lambda_1 - a_2)(\lambda_1 - a_3) \quad &b_{1,2} &= -(\lambda_1 - a_1)(\lambda_1 - a_3) \quad &b_{1,3} &= (\lambda_1 - a_1)(\lambda_1 - a_2) \quad &b_{1,4} &= 0 \\
b_{2,1} &= (\lambda_2 - a_2)(\lambda_2 - a_3) \quad &b_{2,2} &= -(\lambda_2 - a_1)(\lambda_2 - a_3) \quad &b_{2,3} &= (\lambda_2 - a_1)(\lambda_2 - a_2) \quad &b_{2,4} &= 0.
\end{align*}
\]

To illustrate the algorithm procedure take one of ten possibilities of the roots positions, for example case (i), i.e. \( \lambda_1 < \lambda_2 < a_1 < a_2 < a_3 \). Hence \( b_{1,1} > 0, b_{1,2} < 0, b_{1,3} > 0 \) and \( b_{2,1} > 0, b_{2,2} < 0, b_{2,3} > 0 \) and this (together with the obvious positive and negative coefficients in the first line) allows a positive solution of the system. We reorganize the first equation as proposed in \( (17) \)

\[
b_{0,1}q_1^2 + b_{0,3}q_3^2 = -b_{0,2}q_2^2 - b_{0,4}q_4^2,
\]

7
and with respect to the first equation the remaining two (see (8)) are

\[ b_{1,2}q_2^2 + b_{1,4}q_4^2 = -b_{1,1}q_1^2 - b_{1,3}q_3^2 \]
\[ b_{2,2}q_2^2 + b_{2,4}q_4^2 = -b_{2,1}q_1^2 - b_{2,3}q_3^2 . \]

According to the notation in Section 2, indices of positive and negative coefficients are \( I = \{i_1, i_2\} = \{1, 3\} \), \( J = \{j_1, j_2\} = \{2, 4\} \) and \( P = Q = 2 \). By multiplying we get the new system of two linear equations (see (9)). Orderly, we can write

\[
\begin{align*}
(b_{0,1}b_{1,2} - b_{0,2}b_{1,1})q_1^2q_2^2 + (b_{0,1}b_{1,4} - b_{0,4}b_{1,1})q_1^2q_4^2 + (b_{0,3}b_{1,2} - b_{0,2}b_{1,3})q_2^2q_3^2 + \\
+ (b_{0,3}b_{1,4} - b_{0,4}b_{1,3})q_3^2q_4^2 &= 0 \\
(b_{0,1}b_{2,2} - b_{0,2}b_{2,1})q_1^2q_2^2 + (b_{0,1}b_{2,4} - b_{0,4}b_{2,1})q_1^2q_4^2 + (b_{0,3}b_{2,2} - b_{0,2}b_{2,3})q_2^2q_3^2 + \\
+ (b_{0,3}b_{2,4} - b_{0,4}b_{2,3})q_3^2q_4^2 &= 0 \tag{15}
\end{align*}
\]

We check the signs of coefficients of the first equation (taking into account the order of \( \lambda_1, \lambda_2, a_1, a_2, a_3 \)):

\[ b_{0,1}b_{1,2} - b_{0,2}b_{1,1} = 1 \cdot (- (\lambda_1 - a_1)(\lambda_1 - a_3)) + 1 \cdot (\lambda_1 - a_2)(\lambda_1 - a_3) = \]
\[ = (\lambda_1 - a_3)(a_1 - a_2) \]

therefore \( \text{sgn}(b_{0,1}b_{1,2} - b_{0,2}b_{1,1}) = 1 \),

\[ b_{0,1}b_{1,4} - b_{0,4}b_{1,1} = 0 - (\lambda_1 - a_2)(\lambda - a_3) = (\lambda_1 - a_2)(\lambda - a_3) \]

and therefore \( \text{sgn}(b_{0,1}b_{1,4} - b_{0,4}b_{1,1}) = 1 \),

\[ b_{0,3}b_{1,2} - b_{0,2}b_{1,3} = -(\lambda_1 - a_1)(\lambda_1 - a_3) + (\lambda_1 - a_1)(\lambda_1 - a_2) = \]
\[ = (\lambda_1 - a_1)(-\lambda_1 + a_3 + \lambda_1 - a_2) = (\lambda_1 - a_1)(a_3 - a_2) \]

thus \( \text{sgn}(b_{0,3}b_{1,2} - b_{0,2}b_{1,3}) = -1 \), and

\[ b_{0,3}b_{1,4} - b_{0,4}b_{1,3} = 0 + (\lambda_1 - a_1)(\lambda_1 - a_2) = (\lambda_1 - a_1)(\lambda_1 - a_2) \]

and it follows \( \text{sgn}(b_{0,3}b_{1,4} - b_{0,4}b_{1,3}) = 1 \).

Similarly, we can calculate the signs of the coefficients of the second equation. We realize that the solution of the system may occur. We use the triple indexation (see (3) and (10))

\[ b_{r,i,p,j,q} = b_{0,i,p}b_{r,j,q} - b_{0,j,q}b_{r,i,p} \quad \text{ for } r = 1, 2 . \]

Hence, the triple indexation can be switch to the double indexation as follows

\[ b_{r,i,p,j,q} = b_{r-1,(q-1)P+p}^{(1)} \quad \text{and} \quad q_i^2q_j^2 = q_{i,j}^2 = q_{i,p,j,q}^2 . \]

As above (for the coefficients) the double indexation in variables can become single by

\[ q_{i,p,j,q}^2 = q_{(q-1)P+p}^2 . \]

Then, the system (15) can be simply written as

\[ b_{0,1}q_1^2 + b_{0,2}q_2^2 + b_{0,3}q_3^2 + b_{0,4}q_4^2 = 0 \]
\[ b_{1,1}q_1^2 + b_{1,2}q_2^2 + b_{1,3}q_3^2 + b_{1,4}q_4^2 = 0 . \]
Thus, the procedure has to be repeated for the smaller system of equations. With respect to the signs of the coefficient the first equation has to be written as

$$b^{(1)}_{0,1} q_1^2 + b^{(1)}_{0,2} q_2^2 + b^{(1)}_{0,4} q_4^2 = -b^{(1)}_{0,3} q_3^2$$

and accordingly the second equation as

$$b^{(1)}_{1,3} q_3^2 = -b^{(1)}_{1,1} q_1^2 - b^{(1)}_{1,2} q_2^2 - b^{(1)}_{1,4} q_4^2.$$

By multiplying (left side with left side and right side with right side) and reordering, we obtain the last linear equation in the procedure

$$\left(b^{(1)}_{0,1} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,1}\right) q_1^2 q_3^2 + \left(b^{(1)}_{0,2} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,2}\right) q_2^2 q_3^2 + \left(b^{(1)}_{0,4} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,4}\right) q_3^2 q_4^2 = 0.$$ 

By straightforward but tedious calculation the first and the third coefficient can be factorized as

$$b^{(1)}_{0,1} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,1} = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(\lambda_2 - \lambda_1)$$

and

$$b^{(1)}_{0,4} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,4} = -(a_3 - a_2)(a_1 - \lambda_1)(a_1 - \lambda_2)(\lambda_2 - \lambda_1).$$

Thus, the signs of both coefficients are

$$\text{sgn}(b^{(1)}_{0,1} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,1}) = -1$$

and

$$\text{sgn}(b^{(1)}_{0,4} b^{(1)}_{1,3} - b^{(1)}_{0,3} b^{(1)}_{1,4}) = 1,$$

and hence a positive solution of system exists.

Since all these tedious calculations are time-consuming, especially for large systems, the need for use of computer computation is obvious. For writing a code in MATLAB, matrix forms of the systems are needed.

### 3.3 Matrix form of algorithm

For reasons of transparency, it would make sense to write the system of equations (6) (or (12) for an appropriate number of indices) in the matrix form. To a system of $m$ homogeneous linear equations (6) corresponds the matrix of the system

$$B = \left[ b_{r,s} \right]_{r=1,2,\ldots,m}^{s=1,2,\ldots,n+1}.$$

In this discussion, coefficients $b_{r,s}$ can be arbitrary real numbers. According to the separation of equations (see (7) and (8)) four submatrices of dimensions $1 \times P$, $1 \times Q$, $(m-1) \times P$ and $(m-1) \times Q$ are defined respectively as

$$C_{11} := \left[ b_{1,ik} \right]_{k=1,2,\ldots,P}^{i=1,2,\ldots,m}$$

$$C_{12} := \left[ b_{1,jk} \right]_{k=1,2,\ldots,P}^{j=1,2,\ldots,Q}$$

$$C_{21} := \left[ b_{r,1k} \right]_{k=1,2,\ldots,P}^{r=2,3,\ldots,m}$$

$$C_{22} := \left[ b_{r,jk} \right]_{k=1,2,\ldots,P}^{r=2,3,\ldots,m}$$

---

\(^1\)In Mathematica one can use the functions Assume and Factor:

Assume(\(\lambda_1 < \lambda_2 < a_1 < a_2 < a_3\)), Factor((-\((\lambda_1 - a_1)(\lambda_1 - a_3) + (\lambda_1 - a_2)(\lambda_1 - a_3)\))-\((\lambda_2 - a_1)(\lambda_2 - a_3) + (\lambda_2 - a_2)(\lambda_2 - a_3)\))

\(^2\)In MATLAB: assume(\(\lambda_1 < \lambda_2 < a_1 < a_2 < a_3\)); factor((-\((\lambda_1 - a_1)(\lambda_1 - a_3) + (\lambda_1 - a_2)(\lambda_1 - a_3)\))-\((\lambda_2 - a_1)(\lambda_2 - a_3) + (\lambda_2 - a_2)(\lambda_2 - a_3)\))

---

9
Define two operations on the set of \( m \times t \) matrix \( M^{m,t} \). The first one returns the \( i \)-th row of a matrix

\[
R_i : \quad M^{m,t} \quad \rightarrow \quad M^{1,t}
\]

\[
R_i : \quad [a_{r,s}]_{r=1,2,\ldots,m} \quad \mapsto \quad [a_{i,s}]_{s=1,2,\ldots,t}.
\]

The second one converts the matrix into a column vector (vectorization) as

\[
\text{vec} : \quad M^{m,t} \quad \rightarrow \quad M^{mt,1}
\]

\[
\text{vec} : \quad [a_{r,s}]_{r=1,2,\ldots,m} \quad \mapsto \quad [a_{1,1}, a_{2,1}, \ldots, a_{m,1}, a_{1,2}, a_{2,2}, \ldots, a_{m,2}, \ldots, a_{1,t}, a_{2,t}, \ldots, a_{m,t}]^T.
\]

**Proposition 6** The matrix of the reduced system \((9)\) of the main system \((6)\) is the \((m - 1) \times PQ\) matrix \( D \), which rows are

\[
R_i(D) = \text{vec} \left( C_{11}^T \cdot R_i(C_{22}) - (R_i(C_{21}))^T \cdot C_{12} \right)
\]

for \( i = 1, 2, \ldots, m - 1 \).

**Proof:** By direct calculation we obtain

\[
C_{11}^T \cdot R_i(C_{22}) - (R_i(C_{21}))^T \cdot C_{12} =
\]

\[
= \begin{bmatrix}
  b_{1,i_1} & b_{1,i_2} & \cdots & b_{1,i_{Q_1}}
\end{bmatrix} - \begin{bmatrix}
  b_{1,i_1} & b_{1,i_2} & \cdots & b_{1,i_{Q_1}}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  b_{1,i_1} & b_{1,i_2} & \cdots & b_{1,i_{Q_1}}
\end{bmatrix} - \begin{bmatrix}
  b_{1,i_1} & b_{1,i_2} & \cdots & b_{1,i_{Q_1}}
\end{bmatrix}
\]

Therefore, all elements in the \((m - 1) \times PQ\) matrix \( \text{vec} \left( C_{11}^T \cdot R_i(C_{22}) - (R_i(C_{21}))^T \cdot C_{12} \right) \) are in the form

\[
b_{1,i_p}b_{i,j_q} - b_{i,i_p}b_{1,j_q}
\]

for \( p \in \{1, 2, \ldots, P\} \) and \( q \in \{1, 2, \ldots, Q\} \). These elements are exactly the coefficients of the reduced system \((9)\).

### 3.4 Algorithm for \( n = 2 \)

At the beginning of this section, the course of the algorithm for checking the existence of a positive solution of the system \((12)\) is given. Since the signs of coefficients of some reduced systems cannot be easily determined in general (see examples in \((3)\), we restricted our investigation to the low dimension \( n = 2 \) and all polynomials \( U_S(\lambda) \) for nonempty subset \( S \subseteq \{1, 2, 3\} \). For coding the algorithm we use MATLAB. To recording the matrix of the main system for all nonempty subsets \( S \) and all possible positions of roots \( \lambda_1 \) and \( \lambda_2 \), according to the given values \( a_1, a_2 \) and \( a_3 \), 70 scripts are prepared. Every script is denoted by \( S\text{num}_{\text{num1}}\text{num}_{\text{num2}} \), where

- \( \text{num}_{\text{num1}} \) denotes the subset \( S \) as: \( \text{num}_{\text{num1}} = 1 \) determines \( \{1\} \), \( \text{num}_{\text{num1}} = 12 \) determines \( \{1, 2\} \), etc.
- \( \text{num}_{\text{num2}} \) denotes the positions of roots \( \lambda_1 \) and \( \lambda_2 \) respectively: \( 0 \) means that a root lies in \((-\infty, a_1)\), the value \( 1 \) means that a root lies in \((a_1, a_2)\), etc. (in MATLAB, roots \( \lambda_1 \) and \( \lambda_2 \) are denoted by \( l_1 \) and \( l_2 \)).

Every script determines two matrices, System and SignsSystem. The matrix System is the matrix of system \((14)\) and SignSystem is the matrix of signs of the elements in System, respectively. For the visualization, see Appendix 1, Script S1L00 and Script S13L12.
The algorithm contains two essential functions. The first one is \texttt{eops} (existence of a positive solution). Its inputs are the matrix $B = \text{SignsSystem}$ and the integer $n = ((\text{the number of equations}) - 1)$. Its output is a vector $D$, which components are

- the first component is 1 and the second component is an empty vector if a positive solution exists. In MATLAB, this is equal to value 1.
- the first component is 0 and the second component is the row of the matrix $B = \text{SignsSystem}$, therefore the system does not allow a positive solution.

The second one is \texttt{newsys} function. Its inputs are submatrices of the matrix of the system, i.e. matrices (16), (17), (18) and (19), respectively. By the for loop function \texttt{newsys} constructs the matrix of the reduced system as is presented in Proposition 6. The output of \texttt{newsys} function is the matrix of the reduced system $\text{Sys}$ and the matrix of signs $B$ of elements in $\text{Sys}$, respectively.

For all 70 scripts, the algorithm confirms or denies the existence of a positive solution of the inputted system. All results are presented in Appendix 2 (Table 1 Table 2 and Table 3). There, each subtable contains three columns. In the first column, there are the names of the scripts. In the Y/N column, 1 declares the existence of a positive solution, and 0 declares the non-existence of a positive solution. The third column step tells us in which step of the algorithm the existence of a positive solution is denied. Additional pf along with the number means that after running the algorithm it returns a row of $B = \text{SignsSys}$ which values of elements (1 and $-1$) can not be determined by a factorization. The elements remain expressions in variables $\lambda_1$, $\lambda_2$, $a_1$, $a_2$ and $a_3$. Below the tables in Appendix C all these examples are further analyzed. After checking, by exhaustive calculations, it is confirmed that all examples can be partially factorized and the signs of expressions can be determined.

4 Conclusion

The designed algorithm confirms the known facts about the roots of the polynomials $U_S(\lambda)$ for all nonempty sets $S \subseteq \{1, 2, 3\}$ and $S \neq \{1, 3\}$. For $S = \{1, 3\}$, a new result is obtained. We realize that the polynomial $U_{\{1,3\}}(\lambda)$ has real roots for some $(q_1, q_2, q_3) \in \mathcal{H}_{\{1, 3\}}^2$. Moreover, the new results of this work include possible positions of roots according to the values $a_1$, $a_2$ and $a_3$ (see Table 2 the subs of subset $\{1, 3\}$). More precisely, we show that there are points in the hyperboloid $\mathcal{H}_{\{1, 3\}}^2$ at which the polynomial $U_{\{1,3\}}(\lambda)$ has real roots:

(a) for some $(q_1, q_2, q_3) \in \mathcal{H}_{\{1,3\}}^2$ polynomial $U_{\{1,3\}}(\lambda)$ has real roots $\lambda_1, \lambda_2 \in (-\infty, a_1)$,

(b) for some $(q_1, q_2, q_3) \in \mathcal{H}_{\{1,3\}}^2$ polynomial $U_{\{1,3\}}(\lambda)$ has real roots $\lambda_1, \lambda_2 \in (a_1, a_2)$,

(c) for some $(q_1, q_2, q_3) \in \mathcal{H}_{\{1,3\}}^2$ it has real roots $\lambda_1, \lambda_2 \in (a_2, a_3)$ and

(d) for some $(q_1, q_2, q_3) \in \mathcal{H}_{\{1,3\}}^2$ it has real roots $\lambda_1, \lambda_2 \in (a_3, \infty)$.

In addition, for two cases the structure of Liouville sets is precisely determined below.

The topology of a Liouville set. Following [4, 5, 8], let

\[ f_S(\lambda) = U_S(\lambda)W_S(\lambda) + V^2_S(\lambda) = -(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - b_1)(\lambda - b_2) \]

be a characteristic polynomial; i.e. the real form of $f^C(\lambda) = U^C(\lambda)W^C(\lambda) + (V^C(\lambda))$ according to a subset $S \subseteq \{1, 2, 3\}$ and the automorphism $\tau_S$ (see (2)). The topology of a Liouville set is determined by the position of roots $\lambda_1, \lambda_2$ of polynomial $U_S(\lambda)$ with respect to the positions of the roots $a_1, a_2, a_3, b_1, b_2$ of polynomial $f_S(\lambda)$. As we here consider only the example $S = \{1, 3\}$, let $f(\lambda) = f_{\{1,3\}}(\lambda)$. 

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Case (b), \( \lambda_1, \lambda_2 \in (a_1, a_2) \): since \( f(\lambda_1) > 0, f(\lambda_2) > 0, f(a_1) = f(a_2) = f(a_3) = 0, f(\lambda) < 0 \) for \( \lambda \gg 0 \) and \( f(\lambda) > 0 \) for \( \lambda \ll 0 \), the roots \( b_1 \) and \( b_2 \) of the polynomial \( f(\lambda) \) has to be real and \( b_1 \in (-\infty, \lambda_1) \) and \( (b_2, \infty) \). Thus the roots \( \lambda_1 \) and \( \lambda_2 \) lie in the common closed interval \( (a_1, a_2) \) and the Liouville set is isomorphic to two discs (see [8] and Fig. 1 where \( c_1, c_2, c_3, c_4 \) and \( c_5 \) are ordered roots of polynomial \( f(\lambda) \)).

![Figure 1](image1.png)

**Figure 1:** Real part of the spectral curve \( \mu^2 = f^C(\lambda) \) in regard to the involution \( \tau_{\{1,3\}} \) and case \( \lambda_1, \lambda_2 \in (a_1, a_2) \)

Case (d), \( \lambda_1, \lambda_2 \in (a_3, \infty) \): from the properties of polynomial \( f(\lambda) \) as noted in the previous case, the roots \( b_1 \) and \( b_2 \) of the polynomial \( f(\lambda) \) have to be real. In this case \( b_1 \in (-\infty, u_1) \) and \( b_2 \in (u_2, \infty) \). Thus the roots \( \lambda_1 \) and \( \lambda_2 \) lie in the common closed interval \( (a_3, b_2) \) and the Liouville set is isomorphic to two discs (see [8] and Fig. 2 where \( c_1, c_2, c_3, c_4 \) and \( c_5 \) are ordered roots of polynomial \( f(\lambda) \)).

![Figure 2](image2.png)

**Figure 2:** Real part of the spectral curve \( \mu^2 = f^C(\lambda) \) in regard to the involution \( \tau_{\{1,3\}} \) and case \( \lambda_1, \lambda_2 \in (a_3, \infty) \)

In the other two cases ((a) and (c)), our results do not allow to exactly determine the roots \( b_1 \) and \( b_2 \) of characteristic polynomial. We can not even perceive whether the roots \( b_1 \) and \( b_2 \) are real.

But there are still some outstanding issues. How does the position of roots determine a subset in \( H_2^2 \) and vice versa? Can we answer the questions in Introduction for some other case or in general? For a further study, the authors propose the consider of the case \( n = 3 \) first, to find out if there is any possibility to determined the signs of coefficients in a systems matrices. And perhaps to perceive whether the determination of the signs can be generalized to systems of arbitrary dimensions.

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**Appendix A Scripts and Functions in MATLAB**

Bellow, two scripts are presented. Scripts differ only in the fifth and sixth lines. In fifth line, a nonempty subset \( S \subseteq \{1, 2, 3\} \) is determined, and in sixth line, the positions of roots \( \lambda_1 \) and \( \lambda_2 \) (in MATLAB \( l_1 \) and \( l_2 \)) are fixed. After calling a script the matrix System is the matrix of coefficients of the system \( (12) \) for \( n = 2 \) and
the matrix SignsSystem is the matrix of signs of the coefficients in System, respectively. Recall that the notation $S_{num_1}L_{num_2}$ is described in subsection 3.4.

**Script S1L00**

```matlab
syms a1 a2 a3;
syms l1 l2;
a = [a1 a2 a3];
l = [l1 l2];
S = [1];
assume(l1<l2<a1<a2<a3);
n=2;
A1=(l1-a1).*(l1-a2).*(l1-a3);
A2=(l2-a1).*(l2-a2).*(l2-a3);
A=[A1,A2];
for r=1:2
    for s=1:3
        T(r,s) = (2.*ismember(s,S)-1).*A(r)/(l(r)-a(s));
    end;
end;
T0=[T, zeros(2,1)];
System=[[2.*ismember([1 2 3],S)-1 -1];T0];
SignsSystem=sign(System);
```

**Script S13L12**

```matlab
syms a1 a2 a3;
syms l1 l2;
a = [a1 a2 a3];
l = [l1 l2];
S = [1 3];
assume(a1<l1<a2<l2<a3);
n=2;
A1=(l1-a1).*(l1-a2).*(l1-a3);
A2=(l2-a1).*(l2-a2).*(l2-a3);
A=[A1,A2];
for r=1:2
    for s=1:3
        T(r,s) = (2.*ismember(s,S)-1).*A(r)/(l(r)-a(s));
    end;
end;
T0=[T, zeros(2,1)];
System=[[2.*ismember([1 2 3],S)-1 -1];T0];
SignsSystem=sign(System);
```

Two main functions of the algorithm are eops and newsys. Their functionality is explained in subsection 3.4.

```matlab
function [D] = eops(B,n)
i=0;
while (i < n+1) && (ismember(1,B(i+1,:))==1) && (ismember(-1,B(i+1,:))==1)
    i=i+1;
end;
```

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function [Sys,B] = newsys(B11,B12,B21,B22)
    for j=1:2
        A=((B11.')*B22(j,:)-((B21(j,:)).')*B12)';
        Sys(j,:)=B(:)';
    end
    B=sign(factor(Sys));
end

Appendix B  Results of Algorithm

Table 1: Yes/No (1/0) table for the existence of real roots of the systems for the subsets \{1\}, \{2\} and \{3\}.

| system roots | Y/N | step | system roots | Y/N | step | system roots | Y/N | step |
|--------------|-----|------|--------------|-----|------|--------------|-----|------|
| S1L00        | 0   | 2    | S2L00        | 0   | 3    | S3L00        | 0   | 3    |
| S1L01        | 0   | 1    | S2L01        | 0   | 2    | S3L01        | 1   |      |
| S1L02        | 0   | 2    | S2L02        | 0   | 2    | S3L02        | 0   | 1    |
| S1L03        | 0   | 2    | S2L03        | 1   |      | S3L03        | 0   | 2    |
| S1L04        | 0   | 1    | S2L04        | 0   | 2    | S3L04        | 0   | 3    |
| S1L05        | 0   | 1    | S2L05        | 0   | 2    | S3L05        | 0   | 1    |
| S1L06        | 0   | 3    | S2L06        | 0   | 2    | S3L06        | 0   | 1    |
| S1L07        | 0   | 3    | S2L07        | 0   | 3    | S3L07        | 0   | 2    |
Table 2: Yes/No (1/0) table for the existence of real roots of the systems for the subsets \{1, 2\}, \{1, 3\} and \{2, 3\}.

| system roots | Y/N | step | system roots | Y/N | step | system roots | R/NR | step |
|--------------|-----|------|--------------|-----|------|--------------|------|------|
| S12L00       | 0   | 2    | S13L00       | 1   |      | S23L00       | 0    | 3pf  |
| S12L01       | 0   | 2    | S13L01       | 0   | 3pf  | S23L01       | 0    | 1    |
| S12L02       | 0   | 1    | S13L02       | 0   | 3pf  | S23L02       | 1    |      |
| S12L03       | 0   | 2    | S13L03       | 0   | 3pf  | S23L03       | 0    | 2    |
| S12L11       | 0   | 3pf  | S13L11       | 1   |      | S23L11       | 0    | 1    |
| S12L12       | 0   | 1    | S13L12       | 0   | 3pf  | S23L12       | 0    | 1    |
| S12L13       | 1   |      | S13L13       | 0   | 3pf  | S23L13       | 0    | 1    |
| S12L22       | 0   | 1    | S13L22       | 1   |      | S23L22       | 0    | 3pf  |
| S12L23       | 0   | 1    | S13L23       | 0   | 3pf  | S23L23       | 0    | 2    |
| S12L33       | 0   | 3pf  | S13L33       | 1   |      | S23L33       | 0    | 2    |

Table 3: Yes/No (1/0) table for the existence of real roots of the system for the subset \{1, 2, 3\}.

| system roots | Y/N | step |
|--------------|-----|------|
| S123L00      | 0   | 1    |
| S123L01      | 0   | 1    |
| S123L02      | 0   | 1    |
| S123L03      | 0   | 1    |
| S123L11      | 0   | 3    |
| S123L12      | 1   |      |
| S123L13      | 0   | 1    |
| S123L22      | 0   | 3    |
| S123L23      | 0   | 1    |
| S123L33      | 0   | 1    |

Appendix C  Additional analysis of the results in Table 2

Example S12L11: In the third step of the algorithm, the matrix of signs (i.e. the matrix $B$) is

$$
\begin{bmatrix}
-1 & -\text{sgn}(a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2) & -1 \\
\text{sgn}(a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2) & -1
\end{bmatrix},
$$

The expressions $a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2$ and $a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2$ can be partially factorized as

$$
a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2 = (\lambda_1 - a_1)(a_3 - a_2) + (a_3 - \lambda_2)(a_2 - \lambda_1)
$$

and

$$
a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2 = (a_2 - \lambda_1)(a_1 - a_3) + (a_1 - \lambda_1)(a_3 - \lambda_2).
$$

Since $a_1 < \lambda_1 < \lambda_2 < a_2 < a_3$, the first expression is the sum of two positive summands and the second expression is the sum of two negative summands. Thus, the matrix of signs is $[-1, -1, -1, -1]$ and a positive solution does not exist.
Example S12L33: In the third step of the algorithm, the matrix of signs is

\[
\begin{bmatrix}
1 \\
\text{sgn}(a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2) \\
\text{sgn}(a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2) \\
1
\end{bmatrix}^T.
\]

As in Example S12L11, the expressions can be partially factorized as

\[a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2 = (a_2 - \lambda_1)(a_1 - a_3) + (a_1 - \lambda_1)(a_3 - \lambda_2)\]

and

\[a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2 = (\lambda_1 - a_1)(a_3 - a_2) + (a_3 - \lambda_2)(a_2 - \lambda_1)\]

Since \(a_1 < a_2 < a_3 < \lambda_1 < \lambda_2\), the first expression is the sum of two positive summands (both are the product of two negative factors) and the second expression is also the sum of two positive summands. Therefore, the matrix of signs is \([1, 1, 1, 1]\) and a positive solution does not exist.

Example S13L01: In the third step of the algorithm, the matrix of signs is

\[
[1, -\text{sgn}(a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2), 1].
\]

If we use the partial factorization in Example S12L11

\[a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2 = (a_2 - \lambda_1)(a_1 - a_3) + (a_1 - \lambda_1)(a_3 - \lambda_2),\]

from the ordering \(\lambda_1 < a_1 < \lambda_2 < a_2 < a_3\) we can not deduce on the sign of the expression. But the expression can be also partially factorized as

\[a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2 = (a_2 - \lambda_1)(a_1 - \lambda_2) + (a_3 - \lambda_2)(a_1 - a_2),\]

and the negativity of the expression is obvious. The matrix of signs is \([1, 1, 1]\) and a positive solution does not exist.

Example S13L02: In the third step of the algorithm, the matrix of signs is

\[
[1, -\text{sgn}(a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2), 1].
\]

As in the previous example, the expression \(a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2\) can be partially factorized as

\[a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2 = (a_2 - \lambda_1)(a_1 - \lambda_2) + (a_3 - \lambda_2)(a_1 - a_2).\]

Since \(\lambda_1 < a_1 < a_2 < \lambda_2 < a_3\), the expression is the sum of two negative summands. The matrix of signs is \([1, 1, 1]\) and a positive solution does not exist.

Example S13L03: In the third step of the algorithm, the matrix of signs is

\[
[1, -\text{sgn}(a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2), 1].
\]

Since

\[a_1a_2 + a_1a_3 - a_2a_3 - a_1\lambda_1 - a_1\lambda_2 + \lambda_1\lambda_2 = (a_2 - \lambda_1)(a_1 - \lambda_2) + (a_3 - \lambda_2)(a_1 - a_2)\]

(see previous two examples) and \(\lambda_1 < a_1 < a_2 < a_3 < \lambda_2\), the matrix of signs is \([1, 1, 1]\) and a positive solution does not exist.
Example S13L12: In the third step of the algorithm, the matrix of signs is
\[
\begin{bmatrix}
\text{sgn}(a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} + a_{3}\lambda_{1} + a_{3}\lambda_{2} - \lambda_{1}\lambda_{2}), & 1, & 1
\end{bmatrix}.
\]
The expression \(a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} + a_{3}\lambda_{1} + a_{3}\lambda_{2} - \lambda_{1}\lambda_{2}\) can be partially factorized as
\[
a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} + a_{3}\lambda_{1} + a_{3}\lambda_{2} - \lambda_{1}\lambda_{2} = (\lambda_{1} - a_{1})(a_{3} - a_{2}) + (\lambda_{1} - a_{3})(a_{2} - \lambda_{2}).
\]
Since \(a_{1} < \lambda_{1} < a_{2} < \lambda_{2} < a_{3}\), the considered expression is the sum of two positive summands. The matrix of signs is \([1, 1, 1]\) and a positive solution does not exist.

Example S13L13: In the third step of the algorithm, the matrix of signs is
\[
\begin{bmatrix}
\text{sgn}(a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} + a_{3}\lambda_{1} + a_{3}\lambda_{2} - \lambda_{1}\lambda_{2}), & 1, & 1
\end{bmatrix}.
\]
Since
\[
a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} + a_{3}\lambda_{1} + a_{3}\lambda_{2} - \lambda_{1}\lambda_{2} = (\lambda_{1} - a_{1})(a_{3} - a_{2}) + (\lambda_{1} - a_{3})(a_{2} - \lambda_{2})
\]
(see the previous example) and \(a_{1} < \lambda_{1} < a_{2} < a_{3} < \lambda_{2}\), the considered expression is the sum of two positive summands. The matrix of signs is \([1, 1, 1]\) and a positive solution does not exist.

Example S13L23: In the third step of the algorithm, the matrix of signs is
\[
\begin{bmatrix}
1, & \text{sgn}(a_{1}a_{2} + a_{1}a_{3} - a_{2}a_{3} - a_{1}\lambda_{1} - a_{1}\lambda_{2} + \lambda_{1}\lambda_{2}), & 1
\end{bmatrix}.
\]
Since
\[
a_{1}a_{2} + a_{1}a_{3} - a_{2}a_{3} - a_{1}\lambda_{1} - a_{1}\lambda_{2} + \lambda_{1}\lambda_{2} = (a_{2} - \lambda_{1})(a_{1} - a_{3}) + (a_{1} - \lambda_{1})(a_{3} - \lambda_{2})
\]
(see Example S12L11) and \(a_{1} < a_{2} < \lambda_{1} < a_{3} < \lambda_{2}\), the considered expression is the sum of two positive summands. The matrix of signs is \([1, 1, 1]\) and a positive solution does not exist.

Example S23L00: In the third step of the algorithm, the matrix of signs is
\[
\begin{bmatrix}
-1, & -\text{sgn}(a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} - a_{1}\lambda_{1} - a_{2}\lambda_{2} + \lambda_{1}\lambda_{2}), & 1
\end{bmatrix}^T.
\]
If we use the partial factorization in Example S12L11
\[
a_{1}a_{2} - a_{1}a_{3} + a_{2}a_{3} - a_{2}\lambda_{1} - a_{2}\lambda_{2} + \lambda_{1}\lambda_{2} = (\lambda_{1} - a_{1})(a_{3} - a_{2}) + (a_{3} - \lambda_{2})(a_{2} - \lambda_{1}),
\]
from the ordering \(\lambda_{1} < \lambda_{2} < a_{1} < a_{2} < a_{3}\) we can not deduce on the sign of the expression. After rearranging
\[
a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} - a_{2}\lambda_{1} - a_{2}\lambda_{2} + \lambda_{1}\lambda_{2} = (\lambda_{1} - a_{1})(\lambda_{2} - a_{2}) + (a_{3} - \lambda_{2})(a_{2} - a_{1}),
\]
the positivity of the expression obvious. Since
\[
a_{1}a_{2} - a_{1}a_{3} - a_{2}a_{3} + a_{3}\lambda_{1} + a_{3}\lambda_{2} - \lambda_{1}\lambda_{2} = (\lambda_{1} - a_{1})(a_{3} - a_{2}) + (\lambda_{1} - a_{3})(a_{2} - \lambda_{2})
\]
(see Example S13L12), the expression is the sum of two negative summands. Therefore, the matrix of signs is \([-1, -1, -1, -1]\) and a positive solution does not exist.
Example S23L22: In the third step of the algorithm, the matrix of signs is
\[
\begin{bmatrix}
1 & sgn(a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2) \\
sgn(a_1a_2 - a_1a_3 - a_2a_3 + a_3\lambda_1 + a_3\lambda_2 - \lambda_1\lambda_2) & 1
\end{bmatrix}^T
\]
Since
\[a_1a_2 - a_1a_3 + a_2a_3 - a_2\lambda_1 - a_2\lambda_2 + \lambda_1\lambda_2 = (\lambda_1 - a_1)(\lambda_2 - a_2) + (a_3 - \lambda_2)(a_2 - a_1)\]
and
\[a_1a_2 - a_1a_3 - a_2a_3 + a_3\lambda_1 + a_3\lambda_2 - \lambda_1\lambda_2 = (\lambda_1 - a_1)(a_3 - a_2) + (\lambda_1 - a_3)(a_2 - \lambda_2),\]
(see the previous example) and \(a_1 < a_2 < \lambda_1 < \lambda_2 < a_3\), both expressions are the sums of two positive summands. The matrix of signs is \([1, 1, 1, 1]\) and a positive solution does not exist.

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