Latent Quaternionic Geometry

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Abstract

In this article we discuss the interaction between the geometry of a quaternion-Kähler manifold $M$ and that of the Grassmannian $G_3(\mathfrak{g})$ of oriented 3-dimensional subspaces of a compact Lie algebra $\mathfrak{g}$. This interplay is described mainly through the moment mapping induced by the action of a group $G$ of quaternionic isometries on $M$. We give an alternative expression for the endomorphisms $I_1, I_2, I_3$, both in terms of the holonomy representation of $M$ and the structure of the Grassmannian’s tangent space. A correspondence between the solutions of respective twistor-type equations on $M$ and $G_3(\mathfrak{g})$ is provided.

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1 Introduction

Let $G$ be a compact Lie group acting by quaternionic isometries on a quaternion-Kähler (QK) manifold $M$. In this case a Killing vector field $X$ satisfies the condition $L_X \Omega = 0$, where $\Omega$ is the parallel 4-form of the QK structure. Recall that the fibre of the standard rank 3 vector bundle over $M$ (whose complexification is often written $S^2H$) is isomorphic to $\mathfrak{sp}(1)$, and is spanned by a basis of endomorphisms $I_1, I_2, I_3$ satisfying the quaternionic relations

$$I_i^2 = -\text{Id} \quad \text{and} \quad I_i I_j = \epsilon^{ijk} I_k$$

with $\epsilon^{ijk}$ the sign of the permutation.

We denote by $\mu$ the moment map for the $G$ action, and by $\mu_A$ the section of $S^2H$ obtained by the contraction of $\mu$ with $A \in \mathfrak{g}$ through the metric induced by the Killing form. It satisfies the equation

$$d\mu_A = i(\tilde{A}) \Omega, \quad (1)$$

where $\tilde{A}$ is the Killing vector field generated by $A$ (see [10], [11]). Another way of describing the sections coming from the moment map is expressed by the formula

$$\mu_A = \pi_{S^2H}(\nabla \tilde{A})$$
up to a constant. The moment map $\mu$ is $G$-equivariant with respect to the
given action of $G$ on $M$ and of the adjoint representation of $G$ on $\mathfrak{g}$: it can be
used to construct the $G$-equivariant morphism

$$
\Psi : M_0 \longrightarrow \mathbb{G}_3(\mathfrak{g}),
$$

where $M_0$ is an appropriate subset of $M$. The morphism $\Psi$ was introduced by
Swann ([22], [23]), who studied the unstable manifolds for the gradient flow of
an appropriate functional $\psi$ on this type of Grassmannians, proving that they
admit a QK structure; we will use the map $\Psi$ in order to relate in various ways
the geometry of QK manifolds to that of Grassmannians of type $\mathbb{G}_3(\mathfrak{g})$.

In Section 2, we introduce the natural first-order differential operator $D$
on the tautological rank $k$ vector bundle over a Grassmannian $\mathbb{G}_k(\mathbb{R}^n)$, which
annihilates projections of constant sections. Indeed, we show that all solutions of $D$
arise in this way (Theorem 2.2). This illustrates a well-known technique, whereby solutions of an overdetermined differential operator may
be interpreted as parallel sections of some connection on a larger bundle ([6]).
Although quaternionic geometry and Lie algebras are not yet involved, we
aim to show that $D$ is completely analogous to the more complicated
twistor operator $\mathcal{D}$ on a QK manifold.

In Section 3, we recall the definition of $D$ on sections of the vector bundle
$S^2H$, and explain that it is satisfied by the moment sections $\mu_A$ defined above.
We then prove that under suitable hypotheses the map $\Psi$ can be used to relate
elements in $\ker \mathcal{D}$ with those in $\ker D$ where $D$ now acts on the tautological
rank 3 vector bundle $V$ over $\mathbb{G}_3(\mathfrak{g})$.

Whilst the tangent space to $\mathbb{G}_3(\mathfrak{g})$ at $V$ is given by

$$
T_V \mathbb{G}_3(\mathfrak{g}) \cong V \otimes V^\perp,
$$

the complexified tangent space to $M$ has the form $H \otimes \mathbb{C} E$, reflecting the
representation of the holonomy group $Sp(1)Sp(n)$. Part of our problem is to
reconcile the roles of the “auxiliary” vector bundles $V$ and $H$ with respective
fibres $\mathbb{R}^3$ and $\mathbb{C}^2$. In Section 4 we give an alternative description of the
imaginary quaternion endomorphisms $I_i$ over a point $x \in M$ in terms of $Sp(1)$
representations of a subgroup $Sp(1)$ diagonally embedded in $Sp(1)Sp(n)$.

In Section 5 we state our main results: we show that it is possible to
push forward the endomorphisms $I_1, I_2, I_3$ so that they can be described as
endomorphisms of the subspace $\Psi_* T_x M$ of (2), where $V = \Psi(x)$. In other
words, if $Z = \sum_{i=1}^3 v_i \otimes p_i$ belongs to $\Psi_* T_x M$, then we can write

$$
I_k Z = \sum_{i=1}^3 v_i \otimes q_i
$$

and we shall explicitly determine the $q_i$s in terms of $v_i$, $p_i$ and $I_k$. This is
accomplished in Proposition 5.4, itself a geometric counterpart to the represen-
tation-theoretic Proposition 4.1.
Finally, in Section 6 we apply the theory to the case of an $Sp(1) \times Sp(1)$ action on $\mathbb{H}P^1$ and to other compatible examples. We describe some natural real 4-dimensional subspaces of (2) which correspond to quaternionic lines in $T_x M$, and are tangent to quaternion projective lines in Wolf spaces.

2 Operators on Grassmannians

Consider an $n$-dimensional real vector space $\mathbb{R}^n$ equipped with an inner product $\langle , \rangle$; we can construct the Grassmannian of oriented $k$-planes $G_k(\mathbb{R}^n)$, whose tangent space at a $k$-plane $V$ can be identified with the linear space

$$\text{Hom}(V, V^\perp) \cong V^* \otimes V^\perp;$$

in fact if $v_1, \cdots, v_k$ is an orthonormal (ON) basis for $V$ and $w_1, \cdots, w_{n-k}$ for $V^\perp$, then each homomorphism $T_{ij}$ defined as $T_{ij}(v_k) = \delta^i_k w_j$, corresponds to an independent tangent direction; more explicitly, the curve

$$\alpha_{ij}(r) := \text{span}\{v_1, \cdots, (\cos r)v_i + (\sin r)w_j, \cdots, v_k\}$$

satisfies $\alpha_{ij}(0) = V$ and $\alpha'_{ij}(0) = T_{ij}$. The presence of a metric on $V$, induced from the ambient space $\mathbb{R}^n$, will allow us to write $V \otimes V^\perp$, using contraction via the metric for the isomorphism $V \cong V^*$.

We will be interested in studying differential operators and sections of vector bundles on $G_k(\mathbb{R}^n)$, so we start by describing some induced objects. Given the metric, we have the splitting of the trivial bundle $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ in two subbundles: the tautological one $V$ and its orthogonal complement:

$$V \oplus V^\perp \overset{\cong}{\to} G_k(\mathbb{R}^n) \times \mathbb{R}^n.$$

The presence of this metric also allows us to define connections on these two subbundles merely by composing $d$ with the two projections $\pi$ and $\pi^\perp$. For instance

$$\nabla^V s = \pi ds,$$

where $s \in \Gamma(V)$ and $d$ is the derivation in $\mathbb{R}^n$. To prove that this is a connection let $a$ be a function, and note that

$$\nabla^V(as) = \pi d(as) = \pi((da)s + a(ds)) = (da)s + a\nabla^V s.$$
as required. Moreover this connection is compatible with the metric induced on the fibres of \( V \) by their ambient space \( \mathbb{R}^n \): in fact if \( s,t \in \Gamma(V) \) and \( X \in T_{V^*G_k}^*(\mathbb{R}^n) \) we have
\[
X\langle s , t \rangle = \langle Xs , t \rangle + \langle s , Xt \rangle = \langle \nabla_X^V s , t \rangle + \langle s , \nabla_X^V t \rangle .
\]

On the other hand we obtain the corresponding second fundamental form by projecting in the opposite way:
\[
\Gamma(V) \longrightarrow \Gamma(T^*G_k(\mathbb{R}^n) \otimes V^\perp)
\]
which sends \( s \) to \( \pi^\perp ds \); analogously \( II^\perp \) sends \( s \in \Gamma(V^\perp) \) to \( \pi ds \). Both \( II \) and \( II^\perp \) are tensors. In fact, if for example \( s \in \Gamma(V^\perp) \) and \( a \) is a function, we get
\[
\pi d(as) = \pi(d(a)s + ad(s)) = \pi ad(s) = a\pi ds
\]
so that we can think to \( II^\perp \) as a section of the bundle
\[
\text{Hom } \left( V^\perp , T^*G_k(\mathbb{R}^n) \otimes V \right) \cong V^\perp \otimes (T^*G_k(\mathbb{R}^n) \otimes V)
\]
(identifying \( V^\perp \cong (V^\perp)^* \) as usual). It turns out that this section determines an immersion of \( V^\perp \) as a subbundle of \( T^*G_k(\mathbb{R}^n) \otimes V \); we shall return to this question later in the Section.

We use the standard connections and tensors previously introduced in order to construct new differential operators on the tautological bundle \( V \) and on its orthogonal complement \( V^\perp \). First of all, given an element \( A \in \mathbb{R}^n \) we can associate to it two sections of the bundles \( V \) and \( V^\perp \) just using the projections: \( s_A = \pi A \) and \( s_A^\perp = \pi^\perp A \) with \( A = s_A + s_A^\perp \); as \( A \) is constant,
\[
0 = dA = ds_A + ds_A^\perp
\]
so that
\[
ds_A = -ds_A^\perp ;
\]
in the language already deployed
\[
\nabla^V s_A = \pi ds_A = -\pi ds_A^\perp = -II^\perp s_A^\perp .
\]
These equations imply that
\[
ds_A = -II^\perp s_A^\perp + II s_A .
\] (4)

For convenience we will combine the homomorphisms \( II \) and \( II^\perp \) to act upon any \( \mathbb{R}^n \)-valued function on \( G_3(\mathbb{R}^n) \), giving a mapping
\[
i : C^\infty(G_3(\mathbb{R}^n), \mathbb{R}^n) \longrightarrow \Gamma(T^* \otimes \mathbb{R}^n)
\]
defined by
\[ i(S) = II(\pi S) - II^\perp(\pi^\perp S). \] (5)

in a way which is consistent with equation (4). Thus we have
\[ ds_A = i(A) \] (6)
and
\[ ds_A^\perp = -i(A). \] (7)

The image of \( II^\perp \) corresponds to elements of the type
\[ \sum_{i=1}^{k} \lambda y \otimes u_i \otimes v_i \] (8)
with \( y \in V^\perp \) and \( \lambda \in \mathbb{R} \); this can be shown with the following argument: let us consider the decomposition as \( SO(k) \times SO(n-k) \) modules of the involved bundles
\[ V^\perp \otimes V \otimes V \cong V^\perp \otimes \mathbb{R} + V^\perp \otimes (V \otimes V)_0 \] (9)
where \( (V \otimes V)_0 \) is the tracefree part of the tensor product; Schur’s Lemma guarantees that the second summand cannot contain any submodule isomorphic to \( V^\perp \), so the first summand consists of the unique submodule of this type in the right side term of (9). Therefore, as expression (8) provides an \( SO(k) \times SO(n-k) \)-equivariant copy of \( V^\perp \) inside this bundle, it must coincide with \( II^\perp(V^\perp) \). The same argument shows that
\[ II(u) = \sum_{i=1}^{n-k} \lambda u \otimes w_i \otimes w_i \]
with \( u \in V, \lambda \in \mathbb{R} \). We want now to be more precise about these statements, and calculate explicitly the value of \( \lambda \). This is done in the next proposition (in which tensor product symbols are omitted).

**Proposition 2.1.** Let \( A \in \mathbb{R}^n \) so that \( A = u + y \) with \( u \in V \) and \( y \in V^\perp \) at the point \( V \); let \( v_j \) and \( w_i \) denote the elements of ON bases of \( V \) and \( V^\perp \) at \( V \); then
\[ II(u) = \sum_{j} u w_j w_j \] (10)
and
\[ II^\perp(y) = -\sum_{i} y v_i v_i. \] (11)
Proof. We differentiate the section \( s_A \) along the curve \( \alpha_{ij}(t) \) passing through \( V \) and with tangent vector \( v_i w_j \) as in (3); let \( u = \sum_{i=1}^{k} a_i v_i \) and \( y = \sum_{j=1}^{n-k} b_j w_j \); then

\[
s_A(\alpha_{ij})(t) = a_1 v_1 + \cdots + \langle A, \cos r v_i + \sin r w_j \rangle (\cos r v_i + \sin r w_j) + \cdots + v_k = a_1 v_1 + \cdots + (a_i \cos r + b_j \sin r)(\cos r v_i + \sin r w_j) + \cdots + v_k
\]

so that

\[
\frac{d}{dr} s_A(\alpha_{ij})(r)|_{r=0} = d s_A \cdot v_i w_j = b_j v_i + a_i w_j;
\]

therefore, as an \( \mathbb{R}^n \)-valued 1-form,

\[
ds_A = \sum_{ij} b_j v_i v_j + a_i w_j v_i w_j
\]

\[
= \sum_i y v_i v_i + \sum_j u w_j w_j,
\]

where the second summand belongs to \( \mathbf{V} \otimes \mathbf{V}^\perp \otimes \mathbf{V}^\perp \) and coincides with \( II(u) \) as claimed. An analogous calculation for \( s_A^\perp \) gives

\[
ds_A^\perp = -\sum_i y v_i v_i - \sum_j u w_j w_j
\]

as expected from equation (7).

Observation. The opposite signs in (10) and (11) are consistent with the equation

\[
0 = d\langle s_A^\perp, s_A^\perp \rangle|_V = \langle II(u), y \rangle + \langle u, II^\perp(y) \rangle
\]

which expresses the fact that \( II \) and \( II^\perp \) are adjoint linear operators.

Proposition 2.1 shows that \( \nabla^\mathbf{V} s_A \) is of the form seen in (8), or alternatively that if we call \( \pi_2 \) the projection on the second summand in the decomposition (9) and define \( D \equiv \pi_2 \circ \nabla^\mathbf{V} \), the section \( s_A \) satisfies the twistor-type equation

\[
Ds_A = 0.
\]

(12)

Symmetrically we can define another operator \( D^\perp \) such that

\[
D^\perp s_A^\perp = 0.
\]

(13)

Let us choose an orthonormal basis \( e_1, ..., e_n \) of \( \mathbb{R}^n \), every section \( S \) of the flat bundle \( G_k(\mathbb{R}^n) \times \mathbb{R}^n \) is nothing else than an \( n \)-tuple of functions

\[
f_j : G_k(\mathbb{R}^n) \longrightarrow \mathbb{R}^n
\]

so that

\[
S = \sum f_j e_j;
\]
applying the exterior derivative on \( \mathbb{R}^n \) (which is a connection on the flat bundle) we obtain
\[
dS = \sum df_j \otimes e_j
\]
and if \( 1 \wedge i \) denotes an element in Hom \( \left( T^* \otimes \mathbb{R}^n, (\otimes^2 T^*) \otimes \mathbb{R}^n \right) \) (where \( T^* = T^*G_k(\mathbb{R}^n) \) to lighten the notation) acting in the obvious way, we obtain
\[
1 \wedge i (dS) = \sum df_j \wedge i(e_j);
\]
on the other hand
\[
d \sum f_j i(e_j) = \sum df_j \wedge i(e_j) + f_j di(e_j),
\]
so if we can show that
\[
di(e_j) = 0 \quad \forall j
\]
we obtain the commutativity of the following diagram:
\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{d} & T^* \otimes \mathbb{R}^n \\
\downarrow i & & \downarrow 1 \wedge i \\
\mathbb{R}^n & \xrightarrow{d} & \Lambda^2 T^* \otimes \mathbb{R}^n
\end{array}
\]
but equation (6) implies:
\[
di(e_j) = dds_{e_j} = 0,
\]
because the \( e_j \) are constant. A consequence of Proposition 2.1 is that \( i \) is an injective map (because \( II \) and \( II^\perp \) are); if we can show that also \( 1 \wedge i \) is injective (and it happens to be in most part of cases, as we will see) looking at diagram (14) we can deduce the following facts: if \( s \in \Gamma(V) \) satisfies \( Ds = 0 \), then \( ds = i(s + s') \) for some \( s' \in \Gamma(V^\perp) \); this follows by comparing
\[
ds = \nabla s + II(s)
\]
with (5) and noting that \( \pi s = s \) in this case: then \( s' = -(II^\perp)^{-1}(\nabla s) \). Obviously \( dds = 0 \), so \( d(s + s') = 0 \) too, hence it is a constant element \( A \in \mathbb{R}^n \). This implies the main result of this Section:

**Theorem 2.2.** A section \( s \in \Gamma(V) \) satisfies the twistor equation \( Ds = 0 \) if and only if exists another section \( s' \in \Gamma(V^\perp) \) such that \( s + s' = A \) is a constant section of \( \mathbb{R}^n \), provided \( k > 1 \) and \( n - k > 1 \).

In other words sections of type \( s_A \) are the only solutions of equation (12), under these hypotheses.
The missing piece to prove Theorem 2.2 is injectivity of $1 \wedge i$. To prove that we start defining another map:

$$c : \Gamma(T^* \otimes \mathbb{R}^n) \longrightarrow \Gamma(\mathbb{R}^n)$$

acting as a contraction in the following way:

$$c \left( \sum_{ijk} a_{ijk} v_i w_j v_k + \sum_{lmo} b_{lmo} w_l v_m w_o \right) = \sum_{ij} a_{iji} w_j + \sum_{lm} b_{lml} v_m .$$

The same map acts also on $\tau \in (\otimes^q T^*) \otimes \mathbb{R}^n$ in the following way: if $\tau = \tau' \otimes \theta$ with $\tau' \in \otimes^{q-1} T^*$ and $\theta \in T^* \otimes \mathbb{R}^n$ then

$$c(\tau) = \tau' \otimes c(\theta)$$

and then extending linearly.

We are now in position to prove the previously stated assertion, which concludes the proof of Theorem 2.2:

**Lemma 2.3.** The map $1 \wedge i$ is injective, provided $k > 1$ and $n - k > 1$.

**Proof.** Given two bases $v^i$ of $V$ and $w^j$ of $V^\perp$ an element in $T^* \otimes \mathbb{R}^n$ is described by

$$\tau = \sum_{ijh} a_{ijh} v_i w_j v_h + \sum_{lmo} b_{lmo} v_l w_m w_o ;$$

now we will prove that $c \circ 1 \wedge i$ is injective, so that $1 \wedge i$ must be.

So we get

$$1 \wedge i (\tau) = \sum_{ijh} a_{ijh} (v_i w_j \wedge v_h w_\mu) w_\mu + \sum_{lmo} b_{lmo} (v_l v_m \wedge w_o w_\nu) v_\nu$$

$$= \sum_{ijh} a_{ijh} (v_i w_j \otimes v_h w_\mu - v_h w_\mu \otimes v_i w_j) w_\mu$$

$$+ \sum_{lmo} b_{lmo} (v_l v_m \otimes w_o w_\nu - w_o w_\nu \otimes v_l v_m) v_\nu$$

and applying the contraction

$$c(1 \wedge i (\tau)) = \sum_{ijh} a_{ijh} (v_i w_j \otimes v_h - v_h w_\mu \otimes v_i \delta_\mu^j)$$

$$+ \sum_{lmo} b_{lmo} (v_l v_m \otimes w_o - w_o w_\nu \otimes v_l \delta_\nu^m) .$$

Now imposing that it’s zero, we get the following couples of equations:

$$\begin{cases} (n - k) a_{ijh} - a_{hji} = 0 \\ (n - k) a_{hji} - a_{ijh} = 0 \end{cases}$$
and
\[
\begin{align*}
    k l^{l_{mo}} - l^{omp} &= 0 \\
    k l^{omp} - l^{l_{mo}} &= 0
\end{align*}
\]
which imply
\[(n - k)^2 a^{ijh} = a^{ijh}\]
and
\[k^2 b^{l_{mo}} = b^{l_{mo}}\]
which are absurd if \(k > 1\) and \(n - k > 1\). ■

3 The two twistor equations

Let us consider a compact Lie group \(G\) acting by isometries on a QK manifold \(M\); then its moment map \(\mu\) can be described locally as

\[\mu = \sum_{i=1}^{3} \omega_i \otimes B_i \quad (15)\]

with \(\omega_i\) a local orthonormal basis for \(S^2H\) and \(B_i\) belonging to \(\mathfrak{g}\). Suppose that \(V := \text{span}\{B_1, B_2, B_3\}\) is a 3-dimensional subspace of \(\mathfrak{g}\); then \(V\) is independent of the trivialization, as the structure group of \(S^2H\) is \(SO(3)\). Therefore we obtain a well defined map

\[\Psi : M_0 \rightarrow G_3(\mathfrak{g})\]

where \(M_0 \subset M\) is defined as the subset where \(V(x)\) is 3-dimensional; this turns out to be an open dense subset of the union \(\bigcup S\) of \(G\)-orbits \(S\) on \(M\) such that \(\dim S \geq 3\) ([23, Proposition 3.5]). Therefore if the dimension of the maximal \(G\) orbits in \(M\) is big enough, then \(M_0\) is an open dense subset of \(M\).

**Assumption.** From now on we will assume that

\[B_i = \lambda(x)v_i \quad (16)\]

for \(v_i\) an orthonormal basis of \(V\). This hypothesis is not excessively restrictive, in the sense that it is compatible with the existence of open \(G_C\) orbits on the twistor space \(Z = \mathbb{P}(U)\): in fact the projectivization of the complex-contact moment map \(f\) induced on \(Z\) satisfies

\[(\mathbb{P}f)(\omega_1) = \text{span}_C\{B_2 + iB_3\},\]

and in this case this turns out to be a ray of nilpotent elements in \(\mathfrak{g}_C\) (see ([23, §3])). Nilpotent elements belong to the zero set of any invariant symmetric tensor over \(\mathfrak{g}_C\), in particular with respect to the Killing form: for by Engel’s
theorem their adjoint representation can be given in terms of strictly upper triangular matrices, with respect to a suitable basis, and the product of such matrices is still strictly upper triangular and hence traceless; in other words

$$0 = \text{Tr} (ad_{B_2 + iB_3} \circ ad_{B_2 + iB_3}) = \langle B_2 + iB_3, B_2 + iB_3 \rangle$$

$$= \|B_1\|^2 - \|B_2\|^2 + 2i \langle B_2, B_3 \rangle,$$

which implies $B_2 \perp B_3$ and $\|B_2\| = \|B_3\|$, conditions that are equivalent to the assumption, permuting cyclically the indices. Therefore condition (16) holds for all unstable manifolds described in [23], as in that case the twistor bundle $Z$ is $G_C$-homogeneous. We assume throughout the Section that this condition holds for the moment map $\mu$.

Using the map $\Psi$, we can construct on $M_0$ the pullback bundle $\Psi^*(V)$; the latter is unique up to isomorphism of bundles (see [24, Chap. I, Prop. 2.15]). More precisely, any vector bundle $W \rightarrow M_0$ for which there exists a map of bundles $\hat{\Phi} : W \rightarrow V$ which is injective on the fibres, and a commutative diagram

\begin{equation}
\begin{array}{c}
W \\
\downarrow \hat{\Phi} \\
M_0 \cong G_3(\mathfrak{g}),
\end{array}
\end{equation}

is necessarily isomorphic to $\Psi^*(V)$.

**Lemma 3.1.** We have the following isomorphism of bundles on $M_0$:

$$S^2 H \cong \Psi^*(V).$$

**Proof.** To complete the commutative diagram (17), define the morphism of bundles

$$\hat{\Phi} : S^2 H \rightarrow V$$

by

$$(x, \omega_{i}(x)) \rightarrow (\text{span}\{B_1(x), B_2(x), B_3(x)\}, B_i(x))$$

(see (15)), extending linearly on the fibres. This corresponds to the contraction of a vector $v \in S^2 H_x$ with the $S^2 H$ component of $\mu(x)$ using the metric, so it does not depend on the trivialization (the structure group preserves the metric) and is injective on the fibres by definition of $M_0$. ■

We should point out that $\hat{\Phi}$ is not an isometry of Riemannian bundles in general; nevertheless under the hypotheses discussed above, we can assume that $\hat{\Phi}$ is a conformal map of Riemannian bundles, considering $S^2 H$ and $V$ to be equipped with the natural metrics coming respectively from $M$ and from $G_3(\mathfrak{g})$. 10
Let us now recall some well-known differential operators (the symbol $\Gamma$ denoting space of sections is omitted): the Dirac operator

$$\delta : S^2H \xrightarrow{\nabla} E \otimes H \otimes S^2H \hookrightarrow (E \otimes H) \otimes (H \otimes H^*) \twoheadrightarrow T^*$$

where the underlined terms are contracted and $T^* = E \otimes H$; the QK twistor operator is defined as follows:

$$\mathcal{D} : S^2H \xrightarrow{\nabla} E \otimes H \otimes S^2H \xrightarrow{\text{sym}} E \otimes S^3H,$$

where we symmetrize after covariant differentiation. In [19, Lemma 6.5], under the assumption of nonzero scalar curvature, Salamon proved that sections of $S^2H$ belonging to $\ker \mathcal{D}$ are in bijection with the elements in the space $\mathcal{K}$ of Killing vector fields preserving the QK structure; this means that if $\nu$ is in $\ker \mathcal{D}$ then $\delta(\nu)$ is dual to a Killing vector field $\tilde{A} \in \mathcal{K}$, and on the other hand $\nu = \mu_A$, or in other words

$$\mathcal{D} \mu_A = 0 \quad (18)$$

and all elements in $\ker \mathcal{D}$ are of this form.

Recall now what was discussed for Grassmannians in Section 2: there we introduced another differential operator $D$ on the tautological bundle $V$ over $G_3(\mathfrak{g})$; the elements in its kernel were proved to be precisely the sections $s_A$ obtained by projection from the trivial bundle with fibre $\mathfrak{g}$ (see Theorem 2.2). We want to relate the kernels of $\mathcal{D}$ and $D$ through the map $\Psi$ induced by $\mu$; recall that the bundle homomorphism $\hat{\Phi}$ is defined up to a bundle automorphism of $S^2H$; we can for instance introduce a dilation

$$\xi(x, w) = (x, \frac{w}{\|B_i\|}), \quad (19)$$

which is independent of the trivialization; in this way

$$\hat{\Xi}(\omega_i) := \hat{\Phi} \circ \xi(\omega_i) = \frac{B_i}{\|B_i\|},$$

and so an orthonormal basis is sent to another orthonormal basis: this is therefore an isometry of the two bundles compatible with the map $\Psi$ induced by $\mu$.

We can now state the main result of this Section. Let us denote by $\mathcal{K}_g \subset \mathcal{K}$ the subspace of Killing vector fields induced by $\mathfrak{g}$ and by $(\ker \mathcal{D})_g$ the space of the corresponding twistor sections; then

**Proposition 3.2.** There exists a lift $\hat{\Psi}$ of the map $\Psi$ such that

$$\hat{\Psi}(\mu_A) = s_A,$$

inducing a bijective linear map

$$(\ker \mathcal{D})_g \longrightarrow \ker D.$$
Proof. We are looking for a lift \( \hat{\Psi} \) such that the diagram

\[
\begin{array}{ccc}
S^2H & \xrightarrow{\Psi} & V \\
\mu_A & \downarrow & \circlearrowright \\
M_0 & \xrightarrow{\Psi} & \mathcal{G}_\beta(g)
\end{array}
\]

commutes; recall the usual local description (15) of \( \mu \), and let us define \( \hat{\Psi} \) so that

\[
\hat{\Psi}(\omega_i) = \frac{B_i}{\|B_i\|^2},
\]

obtained by composing \( \Phi \) with the dilation \( \xi^2 \) (see (19)); this is again a lift of \( \Psi \); consider as usual \( \mu_A \in \Gamma(S^2H) \) satisfying the twistor equation; then

\[
\hat{\Psi}(\mu_A) = \hat{\Psi} \left( \sum_i \omega_i \langle B_i, A \rangle \right)
\]

\[
= \sum_i \frac{B_i}{\|B_i\|^2} \langle B_i, A \rangle 
\]

\[
= \pi V A = s_A,
\]

as required. As the lift \( \hat{\Psi} \) is injective on the fibres, and as

\[
\dim(\ker D) = \dim \mathcal{H}_g = \dim g = \dim \ker D
\]

the last assertion follows. \( \blacksquare \)

The situation can be summarized in diagram (20):

\[
A \in \mathfrak{g}
\]

\[
s_A \in \ker D \quad \mu_A \in (\ker D)_{\mathfrak{g}}
\]

Observation. We can interpret \( \mu \) as a collection of \( n = \dim \mathfrak{g} \) sections of \( S^2H \); if \( A_i \) are an orthonormal basis for \( \mathfrak{g} \) the moment map \( \mu \) is completely determined by the \( \mu_A \). Locally we get

\[
B_i = \sum_j a^j_i A_j
\]
so that
\[ \mu_{A_i} = \sum_j a^j_i \omega_j. \]

For instance, if a section \( \nu \in \Gamma(S^2 H) \) is given locally by
\[ \nu = \sum_i c^i \omega_i \]
then
\[ \hat{\Phi}(\nu) = \sum_i c^i B_i; \]
with respect to the basis \( A_i \) of \( g \) the local description of the morphism \( \hat{\Phi} \) is encoded in the \((3 \times (n - 3))\) matrix of the coefficients \( a^j_i \) seen in (21).

### 4 The \( Sp(1)Sp(n) \) Structure

We are going now to introduce an alternative description of the endomorphisms \( I_1, I_2, I_3 \) in a purely algebraic setting, using the holonomy representation at a fixed point \( x \in M \).

Let \( h, \hat{h} \) denote a unitary basis of \( H \), in such a way that \( \omega_H(h, \hat{h}) = 1 \); with respect to this basis we have
\[ \omega_H = h \wedge \hat{h} = \frac{1}{2}(h\hat{h} - \hat{h}h). \] (22)

We can in terms of \( h, \hat{h} \) determine a basis of \( S^2 H \):
\[ \begin{align*}
I_1 &= h(h \vee \hat{h}) \\
I_2 &= h^2 + \hat{h}^2 \\
I_3 &= i(h^2 - \hat{h}^2)
\end{align*} \] (23)
are orthogonal of norm \( \sqrt{2} \) with respect to the metric \( \omega_H \otimes \omega_H \) induced on \( S^2 H \); they satisfy the same relations of quaternions:
\[ \begin{align*}
I_k^2 &= -1, \quad I_i I_j = sgn(ijk) I_k \\
\end{align*} \]
with \( sgn(ijk) \) the sign of the permutation; the composition is obtained by contracting again with \( \omega_H \).

Consider now the case where the \( Sp(1) \) representation inside \( Sp(1)Sp(n) \) is such that the projection on the \( Sp(n) \) factor is nonzero: this means that the \( E \) representation is nontrivial under this \( Sp(1) \) action.

In this case it is significant to analyze the quaternionic action from the point of view of these new \( Sp(1) \) representations. First we adopt the following notation: we have the symmetrization map \( S \) acting on tensors as
\[ S(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\pi \in \Pi^n} x_{\pi^1(1)} \otimes \cdots \otimes x_{\pi^1(n)} \]
where \( \pi^n \) varies in the group of permutations on \( n \) elements; the map extends linearly. We give then the following definition: we denote as

\[
\{ \cdot, \cdot \} : \Sigma^k \otimes \Sigma^h \longrightarrow \Sigma^{h+k}
\]

the symmetrization of the two factors, more explicitly if

\[
\alpha = \sum_{\pi^k} \alpha_{\pi^k(1)} \otimes \cdots \otimes \alpha_{\pi^k(k)} \in \Sigma^k, \quad \beta = \sum_{\pi^h} \beta_{\pi^h(1)} \otimes \cdots \otimes \beta_{\pi^h(h)} \in \Sigma^h
\]

then

\[
\{ \alpha \otimes \beta \} = \sum_{\pi^k, \pi^h} S(\alpha_{\pi^k(1)} \otimes \cdots \otimes \alpha_{\pi^k(k)} \otimes \beta_{\pi^h(1)} \otimes \cdots \otimes \beta_{\pi^h(h)}).
\]

In particular we denote by \( \sigma \) the map \( \{ \cdot, \cdot \} \) when the first index is 1:

\[
\sigma := \{ \cdot, \cdot \} : \Sigma^1 \otimes \Sigma^i \longrightarrow \Sigma^{i+1}.
\] (24)

Consider now for simplicity the case that \( E \) corresponds to an irreducible \( Sp(1) \) representation; then

\[
T_x M_C \cong \Sigma^1 \otimes \Sigma^{i-1}
\]

and using Clebsch-Gordan relation, we obtain

\[
T_x M_C \cong \Sigma^i + \Sigma^{i-2} \cong \Sigma^{i+2} + \Sigma^i + \Sigma^{i-2} \cong \Sigma^2 \otimes \Sigma^i; \quad (25)
\]

more precisely \( T_x M_C \) coincides with the kernel of the symmetrization

\[
\{ \cdot, \cdot \} : \Sigma^2 \otimes \Sigma^i \longrightarrow \Sigma^{i+2}.
\]

**Example.** There are (up to conjugation) three non-trivial homomorphisms \( Sp(1) \to Sp(2) \): two correspond to the roots, but in these cases the decomposition of the standard \( Sp(2) \) representation \( C^4 \) is not irreducible; in fact

\[
E = C^4 = \Sigma^0 + \Sigma^1 + \Sigma^0
\]

for the long root, and comparing with the known decomposition of the adjoint representation one has

\[
\mathfrak{sp}(2) = S^2(C^4) = S^2(2\Sigma^0 + \Sigma^1) = \Sigma^2 + 2\Sigma^1 + 3\Sigma^0;
\]

for the short root we have instead

\[
E = C^4 = \Sigma^1 + \Sigma^1
\]

as in fact

\[
\mathfrak{sp}(2) = S^2(C^4) = S^2(2\Sigma^1) = 3\Sigma^2 + \Sigma^0.
\]
There is a third embedding, corresponding to the \( \mathfrak{sl}(2, \mathbb{C}) \) triple

\[
X = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

obtained using the recipe in [7], for which

\[
E = \mathbb{C}^4 = \Sigma^3. \tag{26}
\]

**Observation.** This last can be interpreted in the following way: recall that the decomposition of the Lie algebra \( \mathfrak{g}_2 \) with respect to \( \mathfrak{so}(4) \subset \mathfrak{g}_2 \) is given by

\[
\Sigma_+^2 + \Sigma_-^2 + \Sigma_+^1 \otimes \Sigma_+^3,
\]

where \( \Sigma_{\pm}^k \) denote the representations of the \( \mathfrak{sp}(1) \) corresponding to the long (+) or to the short (−) root; so considering the diagonal embedding

\[
\mathfrak{sp}(1) \Delta \mathfrak{so}(4) = \mathfrak{sp}(1)_+ + \mathfrak{sp}(1)_- \mathfrak{sp}(1)_+ + \mathfrak{sp}(2),
\]

consistently with the \( Sp(1)Sp(2) \) structure of the Wolf space

\[
\frac{G_2}{SO(4)},
\]

we have a description of its tangent space in the \( EH \) formalism as \( H \otimes E \cong \Sigma^1 \otimes \Sigma^3 \), corresponding to the representation in (26).

The action of \( S^2 H \cong \Sigma^2 \) on \( T_x M_C \) can be therefore expressed suitably exploiting this new formulation, involving the \( \Sigma^2 \) factor instead of the \( \Sigma^1 = H \); to understand more deeply this \( \Sigma^2 \)-approach we need to define more explicitly the invariant immersion (25). Let us define the map as

\[
Q : \Sigma^1 \otimes \Sigma^{i-1} \overset{\omega_H \otimes}{\longrightarrow} \Sigma^1 \otimes \Sigma^1 \otimes \Sigma^{i-1} \overset{\{\cdot, \cdot\}}{\longrightarrow} \Sigma^{2} \otimes \Sigma^{i} \tag{27}
\]

acting in the following way: if

\[
Y = h \otimes \beta + \tilde{h} \otimes \tilde{\beta} \in \Sigma^1 \otimes \Sigma^{i-1}, \quad \beta, \tilde{\beta} \in \Sigma^{i-1}
\]

then

\[
Q(Y) = \frac{1}{2}\{hh\}\{\hat{h}\beta\} + \frac{1}{4}(\hat{h}h + \hat{h}h)(\{h\tilde{\beta}\} - \{h\beta\}) - \frac{1}{2}\{\hat{h}h\}\{h\tilde{\beta}\} \tag{28}
\]
is obtained, after tensorizing with the invariant element $\omega_H$, by symmetrization of the tensorial factors in accordance with the simple or double underlining marks in (27).

Our next aim is to express the quaternionic action in terms of this description: a first guess in this sense is that for $Q(Y) = \sum v_i \otimes p_i$ then

$$Q(I_1 Y) = v_2 \otimes p_3 + v_3 \otimes p_2,$$

mimicking the adjoint representation of $su(2)$ on itself; but this is not correct, as at the second step

$$Q(I_2 Y) = -v_2 \otimes p_2 - v_3 \otimes p_3,$$

which is not $-Id$. Something more is needed to “reconstruct” the missing term $-v_1 \otimes p_1$.

The next Proposition gives the correct answer in order to express the quaternionic action from the $\Sigma^2$ viewpoint:

**Proposition 4.1.** Let $Y \in T_x M = \Sigma^1 \otimes \Sigma^i$; if $Q(Y) = \sum v_i \otimes p_i$ then

$$Q(I_1 Y) = v_1 \otimes \frac{1}{4} \sigma(Y) + v_2 \otimes p_3 - v_3 \otimes p_2. \quad (29)$$

**Proof.** We have the definition of $Q(Y)$ as in (28); then if we identify $v_i$ with the basis $I_i$ defined in (23), grouping the terms properly we obtain

$$p_1 = -\frac{i}{4} (\{\hat{h}\beta\} - \{h\beta\})$$

$$p_2 = \frac{1}{4} (\{\hat{h}\beta\} - \{h\beta\})$$

$$p_3 = -\frac{i}{4} (\{\hat{h}\beta\} + \{h\beta\});$$

the quaternionic action of $I_1$ on $Y$ is given, in the $\Sigma^1$ context, by

$$I_1 Y = -ih \otimes \beta + ih \otimes \hat{\beta};$$

so we obtain

$$Q(I_1 Y) = -\frac{i}{2} \{hh\} \{\hat{h}\beta\} + \frac{i}{4} (\hat{h}h + hh)(\{h\beta\} + \{\hat{h}\hat{\beta}\}) - \frac{i}{2} \{h\hat{h}\} \{\hat{h}\beta\}$$

and in the form $Q(I_1 Y) = \sum_{i=1}^3 v_i \otimes q_i^1$ we have

$$q_1^1 = \frac{i}{4} (\{h\beta\} + \{\hat{h}\beta\})$$

$$q_2^1 = -\frac{i}{4} (\{h\beta\} + \{\hat{h}\beta\})$$

$$q_3^1 = -\frac{1}{4} (\{\hat{h}\beta\} - \{h\beta\});$$
the conclusion follows by the definition of $\sigma$ (24) and comparing the two sets of equalities.

In the same way we obtain for the other quaternionic elements

$$I_2Y = -\hat{h} \otimes \beta + h \otimes \hat{\beta}$$

$$I_3Y = i\hat{h} \otimes \beta + i\h \otimes \hat{\beta}$$

so that

$$Q(I_2Y) = \frac{1}{2}\{hh\}\{\hat{h}\} - \frac{1}{2}\{h\hat{h}\}\{h\beta\} + \frac{1}{2}\{\hat{h}\hat{h}\}\{h\beta\}$$

$$Q(I_3Y) = \frac{1}{2}\{hh\}\{\hat{h}\} + \frac{1}{2}\{h\hat{h}\}\{h\beta\} - \frac{1}{2}\{\hat{h}\hat{h}\}\{h\beta\}$$

which imply the equalities

$$q^i_j = \eta_{ijk}p_k - \delta^i_j \frac{1}{4}\sigma(Y),$$

where $\eta_{ijk} = \text{sgn}i$ if $i \neq j$, otherwise $\eta_{iik} = 0$; moreover

$$p_i = -\frac{1}{4}\sigma(I_i Y).$$

We can therefore state the quaternionic relations in terms of this description: for example

$$Q(I_2^2Y) = Q(I_1I_1Y) = -v_1 \otimes \frac{1}{4}\sigma(I_1Y) - v_2 \otimes p_2 - v_3 \otimes p_3$$

$$= -v_1 \otimes p_1 - v_2 \otimes p_2 - v_3 \otimes p_3$$

$$= -Q(Y)$$

and also

$$Q(I_1I_2Y) = -v_1 \otimes \frac{1}{4}\sigma(I_2Y) - v_2 \otimes q_3^2 - v_3 \otimes q_2^2$$

$$= -v_1 \otimes p_2 + v_2 \otimes p_1 - v_3 \otimes \frac{1}{4}\sigma(Y)$$

$$= Q(I_3Y)$$

as expected.

5 The Coincidence Theorem

Another way of expressing the twistor equation (1) is given by

$$\nabla^{S^2H}_{\mu A} = k \sum_{i=1}^{3} I_i \bar{A}^\nu \otimes I_i,$$  \hspace{1cm} (30)
where \( \tilde{A} \) is the Killing vector field generated by \( A \) in \( \mathfrak{g} \), the symbol \( \flat \) means passing to the corresponding 1-form via the metric and \( k \) is the scalar curvature, which is constant as the metric is Einstein (for simplicity we can put \( k = 1 \)). On the other hand on \( \mathbf{V} \) we have defined the sections \( s_A \) and the natural connection \( \nabla^\mathbf{V} \) so that (see (8) and Proposition 2.1)

\[
\nabla^\mathbf{V} s_A = \sum_{i=1}^{3} s_A^\perp \otimes v_i \otimes v_i .
\]

In general, given a differentiable map \( \Psi : M \to N \) of manifolds, and an isomorphism \( \hat{\Phi} \) between vector bundles \( E \to F \) on the manifold \( M \) and \( N \) respectively, the second one equipped with a connection \( \nabla^F \), we can define the pullback connection \( \hat{\Psi}^* \nabla^F \) acting in the following way on elements \( s \) of \( \Gamma(E) \):

\[
(\hat{\Psi}^* \nabla^F)_Y(s) := \hat{\Psi}^* (\nabla^F_{\Psi_* Y}(\hat{\Psi} s))
\]

where \( Y \in T_x M \) and \( \hat{\Psi}^* \) means taking the pullback section.

We want to apply this construction in our case, with the map \( \Psi : M \to \mathbb{G}_3(\mathfrak{g}) \) induced by \( \mu \), \( N = \mathbb{G}_3(\mathfrak{g}) \), \( E = S^2 H \), \( F = \mathbf{V} \); our aim is to relate, at a fixed point \( x \in M \), the action of the quaternionic structure on 1-forms induced by \( G \) (the duals of the Killing vector fields) with special cotangent vectors on the Grassmannian \( \mathbb{G}_3(\mathfrak{g}) \):

**Lemma 5.1.** Let \( M, \mathfrak{g}, \mathbb{G}_3(\mathfrak{g}), \mu \) be defined as usual, with

\[
\mu = \sum_{i=1}^{3} I_i \otimes B_i
\]

where \( B_i = \lambda v_i \), \( \lambda \) a differentiable \( G \)-invariant function on \( M \) and \( v_i \) an orthonormal basis of a point \( V \in \mathbb{G}_3(\mathfrak{g}) \); let us choose \( A \in V^\perp \subset \mathfrak{g} \); then at the point \( x \) such that \( \Psi(x) = V \), for \( \Psi \) induced by \( \mu \) as usual, we have

\[
\frac{1}{\lambda} I_i \tilde{A}^\flat = \Psi^*(A \otimes v_i)^\flat ,
\]

where \( A \otimes v_i \in T_x \mathbb{G}_3(\mathfrak{g}) \). Moreover we have \( \|\mu\|^2 = 3\lambda^2 \).

**Proof.** Let \( \Psi \) denote the conformal lift of the map \( \mu \) so that

\[
\Psi(I_i) = \frac{1}{\lambda^2} B_i ;
\]

hence as seen in Proposition 3.2

\[
\Psi(\mu A) = s_A ;
\]

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then applying the $\Psi^* \nabla^V$ connection of $S^2H$ to $\mu_A$ we obtain

$$(\Psi^* \nabla^V)\mu_A = \Psi^*(\nabla^V(\Psi(\mu_A)))$$

$$= \Psi^*(\nabla^V s_A)$$

$$= \Psi^*\left(\sum_{i=1}^{3} s^1_A \otimes v_i \otimes v_i\right)$$

$$= \lambda \sum_{i=1}^{3} \Psi^*(s^1_A \otimes v_i) \otimes I_i ; \quad (33)$$

on the other hand the difference of two connections on the same vector bundle is a tensor, so given any section $s \in S^2H$ which vanishes at a point $x \in M$

$$(\nabla^{S^2H} - \Psi^* \nabla^V)s(x) = 0.$$  

This is precisely the case for the section $\mu_A$ at the point $x$ for which $\Psi(S^2H_x) = V$, because $A \in V^\perp$ by hypothesis; in other words

$$\nabla^{S^2H} \mu_A|_x = (\Psi^* \nabla^V)\mu_A|_x .$$

In the light of the calculations in (33) and of the twistor equation (30), we can deduce

$$\sum_{i=1}^{3} I_i \tilde{A}^i \otimes I_i = \lambda \sum_{i=1}^{3} \Psi^*(s^1_A \otimes v_i) \otimes I_i ;$$

the result follows considering that $s^1_A = A$ at $V$. ■

Lemma 5.1 leads to various ways of relating elements in the respective spaces $T_x M$ and $T_V G_3(g)$ and the quaternionic elements $I_i$; nevertheless it is stated merely in terms of 1-forms, whereas we are interested in involving the two metrics in this interplay. To this aim, let us define a linear transformation $\sharp$ of $T_x M$ by

$$(X)^\sharp := (\Psi^*((\Psi_* X)^\flat))^\sharp \quad (34)$$

in $\text{End}(T_x M)$. This corresponds to moving in a counterclockwise sense around the following diagram, starting from bottom left:

$$(35)$$

Thus the linear endomorphism $(\cdot)^\sharp$ measures the noncommutativity of the diagram (35), and the difference between the pullbacked Grassmannian metric from the quaternionic one.

We are in position now to prove the Co"{i}ncidence Theorem:
Theorem 5.2. Let \( Y \in T_x M \) such that
\[
\Psi_*Y = \sum v_i \otimes p_i;
\]
for \( p_i \in V^\perp \) with \( V = \Psi(x) \); then
\[
(Y)^2 = \frac{1}{\lambda} \sum_i I_i \tilde{p}_i.
\]

Proof. Using the definitions and (31) we obtain
\[
(\Psi_*Y)^b (\Psi_*Z) = \langle \sum v_i \otimes p_i, \Psi_*Z \rangle_{G_3}
\]
\[
= \frac{1}{\lambda} \langle \sum I_i \tilde{p}_i, Z \rangle_M
\]
for any \( Z \in T_x M \), hence the conclusion. \( \blacksquare \)

Theorem 5.2 provides a memorable way of “converting” tangent vectors of \( G_3(g) \) to tangent vectors on \( M \) by means of the correspondence
\[
v_i \longrightarrow I_i
\]
\[
p_i \longrightarrow \tilde{p}_i
\]
for \( p_i \in V^\perp \).

The equivariance of the moment map \( \mu \) implies that Killing vector fields on \( M \) map to Killing vector fields on \( G_3(g) \): in other words if \( \tilde{A} \) is induced by \( A \in g \) on \( M \), then
\[
\Psi_* \tilde{A} = \sum_{i=1}^3 v_i \otimes [A, v_i]^\perp.
\]

Let now \( \alpha = (\sum_{i=1}^3 v_i \otimes p_i)^b \in T^*_x G_3(g) \) and let \( A_r \) be an orthonormal basis of \( V^\perp \); then
\[
\sum_{r=1}^{n-3} (\Psi^* \alpha, \tilde{A}_r) A_r = \sum_{r=1}^{n-3} (\alpha, \Psi_* \tilde{A}_r) A_r = \sum_{i,r} (p_i, [v_i, A_r]^\perp) A_r
\]
\[
= \sum_{i,r} (p_i, [v_i, A_r]) A_r = \sum_{i,r} ([p_i, v_i], A_r) A_r
\]
\[
= \sum_i [p_i, v_i]^\perp.
\]

We can therefore define a mapping
\[
\rho: T_x^* M \longrightarrow V^\perp
\]
by \( \rho(\zeta) = \sum_r \langle \zeta, \tilde{A}_r \rangle A_r \); so if \( \alpha \in T^*_x \mathbb{G}_3(\mathfrak{g}) \), then \( \Psi^* \alpha \in T^*_e M \), and the composition \( \tilde{\gamma} = \rho \circ \Psi^* \) is a map

\[
\tilde{\gamma} : T^*_x \mathbb{G}_3(\mathfrak{g}) \rightarrow V^\perp.
\]
defined by \( \tilde{\gamma}(\alpha) = \sum_i [v_i, p_i]^\perp \); this operator can be described as

\[
\tilde{\gamma} = \pi^\perp \circ \gamma
\]
where \( \gamma(\alpha) = \sum_i [v_i, p_i] \) is the obstruction to the orthogonality of \( \alpha \) to the \( G \)-orbit: in fact

**Lemma 5.3.** A tangent vector \( P = \sum_{i=1}^3 v_i \otimes p_i \in T_V \mathbb{G}_3(\mathfrak{g}) \) is orthogonal to the \( G \)-orbit through the point \( V \) if and only if \( \gamma(P) = 0 \).

**Proof.** For any \( A \in \mathfrak{g} \) let us consider the Killing vector field \( \tilde{A} \) on \( \mathbb{G}_3(\mathfrak{g}) \); the condition of orthogonality of \( P \) is expressed by

\[
0 = \langle \tilde{A}, P \rangle = \sum_{i=1}^3 \langle [A, v_i]^\perp, p_i \rangle = \\
= \sum_{i=1}^3 \langle [A, v_i], p_i \rangle = \sum_{i=1}^3 \langle A, [v_i, p_i] \rangle = \\
= \langle A, \gamma(P) \rangle .
\]

We give now a more explicit description of the quaternionic endomorphisms:

**Proposition 5.4.** Let \( Y \in T^*_e M \) so that

\[
\Psi^* Y = v_1 \otimes p_1 + v_2 \otimes p_2 + v_3 \otimes p_3 ;
\]
then we have

\[
\Psi^* I_1 Y = \frac{1}{\lambda} v_1 \otimes \rho(Y^b) - v_2 \otimes p_3 + v_3 \otimes p_2 .
\]

(37)

**Proof.** Consider any \( A \in V^\perp \), then

\[
\langle p_1, A \rangle_K = \langle \Psi^* Y, A \otimes v_1 \rangle_{\mathbb{G}_3} = \frac{1}{\lambda} \langle I_1 \tilde{A}^b, Y \rangle \\
= \frac{1}{\lambda} \langle I_1 \tilde{A}, Y \rangle_M = - \frac{1}{\lambda} \langle \tilde{A}, I_1 Y \rangle_M \\
= - \frac{1}{\lambda} \langle I_1 Y^b, \tilde{A} \rangle,
\]

(38)
where \( \langle , \rangle_{M,G} \) denote the respective Riemannian metrics, \( \langle , \rangle_K \) minus the Killing form on \( g \) and \( \langle , \rangle \) without subscript is merely the contraction of a cotangent and tangent vector; then considering (38) and (36)

\[
p_1 = \sum_r \langle p_1, A_r \rangle_K A_r = -\frac{1}{\lambda} \sum_r \langle I_1 Y^\flat, \tilde{A}_r \rangle A_r
\]

\[
= -\frac{1}{\lambda} \rho(I_1 Y^\flat)
\]

and analogously

\[
p_i = -\frac{1}{\lambda} \rho(I_i Y^\flat), \quad i = 2, 3;
\]

in consequence

\[
\Psi^* I_1 Y = \frac{1}{\lambda} v_1 \otimes \rho(Y^\flat) - \frac{1}{\lambda} v_2 \otimes \rho(I_2 Y^\flat) + \frac{1}{\lambda} v_3 \otimes \rho(I_2 Y^\flat)
\]

\[
= \frac{1}{\lambda} v_1 \otimes \rho(Y^\flat) - v_2 \otimes p_3 + v_3 \otimes p_2.
\]

Clearly analogous assertions are valid for \( I_2 \) and \( I_3 \).

**Remark.** Assuming that \( \Psi^* \) is injective at the point \( x \), we can define the push forward of the endomorphisms \( I_k \) in the obvious way, namely via the equation:

\[
(\Psi^* I_k)Z := \Psi^*(I_k(\Psi^{-1}_x Z))
\]

and Proposition 5.4. A striking feature of (37) is that in the expression obtained the first summand is independent from \( I_1 \). The operators \( \rho, \gamma \) appear as the essential ingredient to reconstruct the quaternionic action; the other summands \(-v_2 \otimes p_3 + v_3 \otimes p_2\) are obtained from the adjoint representation and (as explained in Section 4) are not sufficient. Nevertheless proposition 5.4 predicts that if \( Y \) is perpendicular to the \( G \)-orbit on \( M \), then \( \rho(Y^\flat) = 0 \), thanks to the definition of \( \rho \) (see Lemma 5.3); in that case

\[
\Psi^* I_1 Y = -v_2 \otimes p_3 + v_3 \otimes p_2
\]

which coincides with the irreducible representation of \( \mathfrak{sp}(1) \) on \( V = \mathbb{R}^3 \).

### 6 Examples and applications

The apparent distinction between the points of view we have adopted in Section 4 and Section 5 disappears as soon as one compares (37) and (29). This
suggests that an intimate relationship exists between the two descriptions of
the quaternionic structure: we are going to discuss now some examples which
throw light on this link.

Let us consider the Wolf space
\[ \mathbb{H}P^1 \cong \frac{Sp(2)}{Sp(1) \times Sp(1)} \cong \frac{SO(5)}{SO(4)} \cong S^4 \]
and the action of the stabilizer \( Sp(1) \times Sp(1) \) of a point \( N \), with Lie algebra
\( \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- = \mathfrak{so}(4) \); this is a cohomogeneity 1 action, with generic orbits isomorphic to
\[ S^3 \cong \frac{Sp(1) \times Sp(1)}{Sp(1)_\Delta} \]
where \( Sp(1)_\Delta \) is the diagonal representation, and 2 singular orbits corresponding to a couple of antipodal points \( N,S \). Let us choose at the point \( N \) any closed geodesic \( \beta(t) \) connecting \( N \) to \( S \); this will be orthogonal to any \( Sp(1) \times Sp(1) \) orbit, and will intersect all of them (a normal geodesic in the language of [4], which in higher cohomogeneity is generalized by submanifolds called sections, see [12]). For instance, we can choose \( N = eSp(1) \times Sp(1) \), and take the geodesic corresponding to following copy of \( U(1) \subset Sp(2) \):

\[
g(t) = \begin{pmatrix}
\cos t & \sin t & 0 & 0 \\
-\sin t & \cos t & 0 & 0 \\
0 & 0 & \cos t & \sin t \\
0 & 0 & -\sin t & \cos t \\
\end{pmatrix} = \exp \begin{pmatrix} 0 & t & 0 & 0 \\
-t & 0 & 0 & 0 \\
0 & 0 & 0 & t \\
0 & 0 & -t & 0 \end{pmatrix}, \tag{39}
\]

where the matrix on the right is denoted by \( tu \). This subgroup generates a geodesic \( \beta(t) \) connecting \( N \ (t = 0) \) with the south pole \( S \ (t = \pi/2) \) passing through the equator \( t = \pi/4 \), and then backwards to \( N \ (t = \pi) \). The stabilizer of the \( Sp(1) \times Sp(1) \) action is constant along \( \beta(t) \) on points that are different from \( N \) and \( S \), and coincides with \( Sp(1)_\Delta \), both along \( \beta(t) \) in \( \mathbb{H}P^1 \) and along \( u(1) \) for the isotropy representation.

Let now \( e_i \) and \( f_i \) denote orthonormal bases of \( \mathfrak{sp}(1)_+ \) and \( \mathfrak{sp}(1)_- \) respectively; as \( \mathfrak{so}(4) \) is a subalgebra of \( \mathfrak{sp}(2) \) corresponding to the longest root, the elements of the two copies of \( \mathfrak{sp}(1) \) correspond to the following matrices:

\[
e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad f_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \end{pmatrix}, \tag{40}
\]
\[
e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \end{pmatrix}. \tag{41}
\]
and
\[
e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; \tag{42}
\]

so if \(e_i(t)\) and \(f_i(t)\) denote an orthonormal basis of the isotropy subalgebra at \(\beta(t)\) (given by \(Ad_{g(t)}\mathfrak{so}(4)\)), we get via the Killing metric:
\[
\langle e_i, f_j(t) \rangle = \delta_i^j \sin^2 t
\]
\[
\langle e_i, e_j(t) \rangle = \delta_i^j \cos^2 t
\]
\[
\langle f_i, e_j(t) \rangle = \delta_i^j \sin^2 t
\]
\[
\langle f_i, f_j(t) \rangle = \delta_i^j \cos^2 t;
\]
in terms of Killing vector fields this implies
\[
\pi_{S^2 H}(\nabla e_i) = \sin^2 t \ f_i(t) \quad \text{and} \quad \pi_{S^2 H}(\nabla f_i) = \cos^2 t \ f_i(t).
\]

if we identify \(S^2 H \cong Ad_{g(t)}\mathfrak{sp}(1)_-\).

The conclusion is that along \(\beta(t)\) the moment map for the action of \(Sp(1) \times Sp(1)\) on \(\mathbb{H}P^1\) is given by
\[
\mu(\beta(t)) = \sum_i \omega_i \otimes (\cos^2 t \ f_i + \sin^2 t \ e_i), \tag{43}
\]
up to a constant. This is the only information that we need to reconstruct the moment map on the whole \(\mathbb{H}P^1\), as \(\beta(t)\) intersects all the orbits and the moment map is equivariant.

We can now interpret these facts in terms of the induced map
\[
\Psi : \mathbb{H}P^1 \longrightarrow G_3(\mathfrak{so}(4));
\]
first of all we note that in this case \(M_0 = M\), as the three vectors
\[
B_i(t) = \cos^2 t \ f_i + \sin^2 t \ e_i \tag{44}
\]
are linearly independent for all \(t\); moreover we observe that \(\hat{\Phi}\) is a conformal mapping of bundles, as asked in the general hypotheses discussed in Section 3.

Recall from [23] that the critical manifolds for the gradient flow of the functional
\[
\psi = \langle [v_1, v_2], v_3 \rangle
\]
defined on \(G_3(\mathfrak{so}(4))\) are given by the maximal points \(\mathfrak{sp}(1)_+\), \(\mathfrak{sp}(1)_-\) and the submanifold
\[
C_\Delta = \mathbb{R}P^3 \cong \frac{Sp(1) \times Sp(1)}{\mathbb{Z}_2 \times Sp(1)_\Delta}.
\]
corresponding to the 3-dimensional subalgebra \( \mathfrak{sp}(1) \), for \( \psi > 0 \); the unstable manifold \( M_\Delta \) emanating from this last one is 4-dimensional and isomorphic to \( \mathbb{HP}^1 \setminus \{N, S\} / \mathbb{Z}_2 \).

A trajectory for the flow of \( \nabla \psi \) is given by

\[
V(x, y) = \text{span}\{xe_i + yf_i \mid x^2 + y^2 = 1, \ i = 1 \cdots 3\},
\]

therefore, comparing (45) with (44) we obtain that \( \Psi(\mathbb{HP}^1) = M_\Delta \cup \mathfrak{sp}(1)_+ \cup \mathfrak{sp}(1)_- \); in particular:

\[
\Psi(N) = \mathfrak{sp}(1)_-
\]

\[
\Psi(S) = \mathfrak{sp}(1)_+
\]

\[
\Psi(\beta(\pi/4)) = \mathfrak{sp}(1)_\Delta.
\]

\[\text{Observation.} \quad \text{The map } \Psi \text{ is not injective. The points corresponding to } t \text{ and } \pi - t \text{ are sent to the same 3-plane; so the principal orbits of type } S^3 \text{ in } \mathbb{HP}^1 \text{ are sent to the orbits of type } \mathbb{RP}^3 \text{ in } M_\Delta. \text{ The map } \Psi \text{ becomes injective on the orbifold } \mathbb{HP}^1/\mathbb{Z}_2, \text{ nevertheless } \Phi_* \text{ is injective away from } N, S.\]

Therefore the \( Sp(1) \times Sp(1) \) orbit through \( x_\Delta = \beta(\pi/4) \) is sent through \( \Psi \) to the critical orbit \( C_\Delta \); we have

**Proposition 6.1.** The differential

\[
T_{x_\Delta} \mathbb{HP}^1 \xrightarrow{\Psi_*} T_{\mathfrak{sp}(1)_\Delta} \mathbb{G}_3(\mathfrak{so}(4))
\]

is a linear \( Sp(1)_\Delta \)-invariant injective map. It coincides (up to a constant) with the map \( Q \) defined in (27), in terms of \( Sp(1)_\Delta \) modules.

**Proof.** Let \( \alpha(t) \) be any curve through \( x_\Delta \), then

\[
g_* \cdot \Psi_* \alpha'(0) = \frac{d}{dt} g \cdot \Psi(\alpha(t)) = \frac{d}{dt} \Psi(g \cdot \alpha(t)) = \Psi_* g_* \cdot \alpha'(0)
\]

for \( g \in Sp(1)_\Delta \subset Sp(1) \times Sp(1)_{x_\Delta} \) where this last is the isotropy subgroup at \( x_\Delta \); in this case the Lie algebra \( \mathfrak{sp}(1)_\Delta \) of \( Sp(1)_\Delta \), which is the stabilizer at \( \beta(t) \) for any \( t \), turns out to coincide with the image \( \Psi(x_\Delta) \). The decomposition of the holonomy representation in terms of \( Sp(1)_\Delta \)-modules is given in this case by

\[
E \otimes H \cong \Sigma^1 \otimes \Sigma^1 \cong \Sigma^2 + \Sigma^0;
\]

correspondingly, the decomposition of the Grassmannian’s tangent space at \( V = \mathfrak{sp}(1)_\Delta \) is given by

\[
T_V \mathbb{G}_3(\mathfrak{so}(4)) \cong V \otimes V^\perp \cong \mathfrak{sp}(1)_\Delta \otimes \Sigma^2 \cong \Sigma^2 \otimes \Sigma^2
\]

\[
\cong \Sigma^4 + \Sigma^2 + \Sigma^0,
\]

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and \( \Psi \) sends injectively \( \Sigma^2 + \Sigma^0 \) in \( \Sigma^4 + \Sigma^2 + \Sigma^0 \); then, as a consequence of Schur’s lemma, an isomorphism of \( Sp(1) \Delta \)-modules is unique up to a constant for each irreducible submodule, hence

\[
\begin{align*}
\Psi_* &= aQ \quad \text{on } \Sigma^2 \\
\Psi_* &= bQ \quad \text{on } \Sigma^0
\end{align*}
\]

for some constants \( a, b \in \mathbb{R} \).

An analogous situation holds for appropriate orbits in the following cases, which are all cohomogeneity 1 actions on classical Wolf spaces:

- \( Sp(n)Sp(1) \) acting on \( \mathbb{H}^n \);
- \( Sp(n) \) acting on \( G_2(\mathbb{C}^{2n}) \);
- \( SO(n - 1) \) acting on \( G_4(\mathbb{R}^n) \).

In the first case the orbit sent through \( \Psi \) to a critical submanifold of type \( C_\Delta \) in the corresponding Grassmannian is one of the principal orbits \( S^{4n-1} \), in the second and third case is one of the singular orbits, more precisely

\[
\frac{Sp(n)}{Sp(n - 2) \times U(2)} \quad \text{and} \quad G_3(\mathbb{R}^{n-1}) \cong \frac{SO(n - 1)}{SO(n - 4) \times SO(3)}
\]

respectively.

This situation can be generalized in the following sense: let \( G \) be a compact group acting by quaternionic isometries on a QK manifold \( M \); let \( G_x \) denote the stabilizer at the point \( x \in M \); then \( G_x \subset SO(T_xM) \) with respect to the quaternionic metric. Since QK manifolds are characterized by the condition \( Hol(M)_x \subset Sp(n)Sp(1) \subset SO(T_xM) \), we have by hypothesis that \( G_x \subset Sp(n)Sp(1) \). Now suppose that \( G_x \) contains some copy of \( Sp(1) \) with nontrivial projection on the \( Sp(n) \) factor. In the case that

\[ \Psi(x) = sp(1) \]

and that a tubular neighborhood of \( G_x \) is sent to the unstable manifold (for \( \psi > 0 \) emanating from the critical manifold \( C \subset G_3(g) \) corresponding to \( sp(1) \), then we have a corresponding decomposition of \( T_xM \) and \( T_{sp(1)}G_3(g) \) in \( Sp(1) \)-modules, and the differential \( \Psi_* \) coincides with \( Q \) up to determining 2 constants, 2 for each \( Sp(1) \)-irreducible summand of the standard \( Sp(n) \) module \( E \).

Let us now decompose the holonomy representation in the case that \( Sp(1) \) is the standard quaternionic subgroup, hence with trivial projection on the \( Sp(n) \) factor. In this case \( E \) turns out to be a direct sum of trivial representations:

\[ E \otimes H \cong (2n \Sigma^0) \otimes \Sigma^1 \cong 2n \Sigma^1 \]
where $2\Sigma^1$ can be identified with the complexified algebra of Quaternions $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. Therefore going back to the real tangent bundle, we obtain the $Sp(1)$ invariant decomposition

$$T_x M \cong n \mathbb{H}.$$ (49)

The presence of the $G$-action allows to single out a quaternionic line of $T_x M$: this determines a quaternionic 1-dimensional distribution $\mathcal{N}_H$ on $M$, or a section $\tau : M \longrightarrow \mathbb{HP}(TM)$ of the associated $\mathbb{HP}^{n-1}$-bundle.

The distribution $\mathcal{N}_H$ arises in the following way: recall that at a point $V \in G_3(\mathfrak{g})$ with $v_1, v_2, v_3$ ON basis, we have

$$\text{grad } \psi = v_1 \otimes [v_2, v_3]^\perp + v_2 \otimes [v_3, v_1]^\perp + v_3 \otimes [v_1, v_2]^\perp.$$ Maintaining the general hypotheses considered in Sections 3 and 5, and assuming that $\Psi_\ast$ is injective, let us define $X := \Psi_\ast^{-1}(\text{grad } \psi)$; then we have:

**Lemma 6.2.** Suppose that $\Psi(x) = V$. Then the subspaces

$$\text{span}\{\text{grad } \psi, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \subset T_V G_3(\mathfrak{g})$$

$$\text{span}\{X, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \subset T_x M$$

are $Sp(1)$ invariant, hence quaternionic.

**Proof.** We need to prove that the endomorphisms of $S^2 H$ over $x$ (or equivalently those of $V$ over $V$) preserve the respective subspaces; let us recall the description of $I_1, I_2, I_3$ given in Proposition 5.4, then

$$I_1(\text{grad } \psi) = \frac{1}{\lambda} v_1 \otimes \rho((\text{grad } \psi)^{\flat}) - v_2 \otimes [v_1, v_2]^\perp + v_3 \otimes [v_3, v_1]^\perp$$

$$= -v_2 \otimes [v_1, v_2]^\perp + v_3 \otimes [v_3, v_1]^\perp$$

$$= -\tilde{v}_1,$$ (50)

where the first summand vanishes thanks to the $G$-invariance of $\psi$, which implies that $\text{grad } \psi$ is orthogonal to the $G$ orbits. Analogously, $I_2(\text{grad } \psi) = -\tilde{v}_2$ and $I_3(\text{grad } \psi) = -\tilde{v}_3$, and the quaternionic identities imply that the whole span$\{\text{grad } \psi, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is preserved; the second inclusion follows from the injectivity and equivariance of $\Psi$. ■

In all the examples discussed above the distribution $\mathcal{N}_H$ turns out to be integrable, with integral manifolds isomorphic to $\mathbb{HP}^1$ embedded quaternionically in $\mathbb{HP}^n$, $G_2(\mathbb{C}^{2n})$ or $G_4(\mathbb{R}^n)$ respectively.

For $Sp(1) \times Sp(1)$ acting on $\mathbb{HP}^1$ the distribution $\mathcal{N}_H$ clearly coincides with the tangent bundle; in this case it is possible to describe the relationship between the two metrics and the $(\cdot)^2$ endomorphism:
Proposition 6.3. Let $M = \mathbb{H}P^1 \setminus \{N, S\}$; consider the decomposition
\[ T_x M \cong \text{span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \oplus \text{span}\{X\} =: C_1 \oplus C_2 \]
induced by the $\text{Sp}(1) \times \text{Sp}(1)$ action; then the map $\Psi : M \rightarrow \mathbb{G}_3(\mathfrak{so}(4))$ satisfies the condition
\[ \Psi^* (\langle \cdot, \cdot \rangle_{\mathbb{G}_3})|_{C_i} = \eta_i(x)(\cdot, \cdot)_M \quad i = 1, 2 \]
where $\eta_i(x)$ two real-valued $\text{Sp}(1) \times \text{Sp}(1)$ invariant function defined on $M$.

The endomorphism (34) is just the multiplication by $\eta_i(x)$ on $C_i$.

Proof. The tangent space $T_Y \mathbb{G}_3(\mathfrak{so}(4))$ along the unstable manifold can be seen as an irreducible $\text{Sp}(1)_\Delta$-module, and $\Psi_*$ as a morphism of $\text{Sp}(1)$-modules. Schur’s Lemma guarantees the uniqueness of an invariant bilinear form (up to a constant), for every irreducible submodule. Recall that
\[ T_x M \cong \Sigma^2 \oplus \Sigma^0 \]
as $\text{Sp}(1)_\Delta$ representations, corresponding to the splitting (51): therefore equation (52) holds, as both metrics are $\text{Sp}(1)_\Delta$ invariant. For the second assertion, let $Y \in C_i$:
\[
(Y)^\sharp = (\Psi^*((\langle \Psi_* Y, \cdot \rangle_{\mathbb{G}_3})))^\sharp \\
= (\Psi^*((\langle \Psi_* Y, \cdot \rangle_{\mathbb{G}_3})))^\sharp \\
= \eta_i(x)((Y, \cdot)_M)^\sharp \\
= \eta_i(x) Y
\]
as required. ■

Observation. Equation (50) together with the equality $\|\text{grad } \psi\| = 3/2 \|\tilde{v}_1\|$ confirms that the endomorphisms $I_i$ are not isometries for Grassmannian metric; hence $\Psi^* (\cdot, \cdot)_{\mathbb{G}_3}$ and $\langle \cdot, \cdot \rangle_M$ can not coincide. Indeed,
\[
\|\text{grad } \psi\|_{\mathbb{G}_3}^2 = \frac{3}{2} \|\tilde{v}_1\|^2_{\mathbb{G}_3} = \frac{3}{2} \eta_2 \|\tilde{v}_1\|^2_M ;
\]
moreover
\[
\|\text{grad } \psi\|_{\mathbb{G}_3}^2 = \eta_1 \|X\|^2_M
\]
and $\|X\|_M = \|I_1 X\|_M = \|\tilde{v}_1\|_M$. Thus $\frac{\eta_1}{\eta_2} = \frac{3}{2}$. An analogous result is expected to hold in general.

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