Posterior consistency for \( n \) in the binomial \((n, p)\) problem with both parameters unknown - with applications to quantitative nanoscopy

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Abstract

The estimation of the population size \( n \) from \( k \) i.i.d. binomial observations with unknown success probability \( p \) is relevant to a multitude of applications and has a long history. Without additional prior information this is a notoriously difficult task when \( p \) becomes small, and the Bayesian approach becomes particularly useful.

In this paper we show posterior contraction as \( k \to \infty \) in a setting where \( p \to 0 \) and \( n \to \infty \). The result holds for a large class of priors on \( n \) which do not decay too fast. This covers several known Bayes estimators as well as a new class of estimators, which is governed by a scale parameter. We provide a comprehensive comparison of these estimators in a simulation study and extent their scope of applicability to a novel application from super-resolution cell microscopy.

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1 Introduction and motivation

Presumably, the binomial distribution $\text{Bin}(n,p)$ is the most fundamental and simple model for the repetition of independent success/failure events. When both parameters $p$ and $n$ are unknown, which is the topic of this paper, it serves as a basic model for many applications. For example, $n$ corresponds to the population size of a certain species (Otis et al., 1978; Royle, 2004; Raftery, 1988), the number of defective appliances (Draper and Guttman, 1971) or the number of faults in software reliability (Basu and Ebrahimi, 2001). In Section 4 we elaborate on a novel application where $n$ is the number of unknown fluorescent markers in quantitative super-resolution microscopy (Hell, 2009; Aspelmeier et al., 2015).

Accordingly, joint estimation of the population size $n$ and the success probability $p$ of a binomial distribution from $k$ independent observations has a long history dating back to Fisher (1941). In contrast to the problem of estimating $p$ or $n$ when one of the parameters is known (Lehmann and Casella, 1996), this is a much more difficult issue. Fisher suggested the use of the sample maximum (which is a consistent estimator for $n$ as $k \to \infty$) and argued that the estimator is always "good", as long as the sample size is large enough. In fact, if $X_1,\ldots,X_k \overset{i.i.d.}{\sim} \text{Bin}(n,p)$ for fixed $n$ and $p$, the sample maximum converges exponentially fast to $n$ as $k \to \infty$ since

$$\mathbb{P}(\max_{i=1}^k X_i = n) = 1 - \mathbb{P}(\max_{i=1}^k X_i < n) = 1 - (1 - p^n)^k. \quad (1.1)$$

While true asymptotically, the maximum very strongly underestimates the true $n$ even for relatively large sample size $k$ if the probability of success is small. This is explicitly quantified in DasGupta and Rubin (2005): if $p = 0.1$ and $n = 10$, then the sample size $k$ needs to be larger than 3635 to ensure that $\mathbb{P}(\max_{i=1}^k X_i \geq n/2) \geq 1/2$. If $p = 0.1$ and $n = 20$, one would need a sample size of more than $k = 900,000$ to guarantee the same probability statement as above.

This fallacy of the sample maximum can be explicitly seen in a refined asymptotic analysis for $n$ and $p$ as well. By Bernoulli inequality and since $1 - x \leq e^{-x}$, it follows from (1.1) that

$$1 - e^{-kp^n} \leq \mathbb{P}(\max_{i=1}^k X_i = n) \leq kp^n,$$

which means that if $kp^n \to 0$, the sample maximum is no longer a consistent estimator of $n$. This occurs, for example, in the domain of attraction of the Poisson distribution, i.e.,
when $n \to \infty$, $p \to 0$ and $np \to \mu \in (0, \infty)$ as $k \to \infty$ and $\log(k) \leq n$, since
\[
kp^n = \exp\{\log(k) + n \log(p)\} \sim \exp\{\log(k) - n \log(n)\} \\
\leq \exp\{\log(k) - \log(k) \log(\log(k))\} \to 0, \quad \text{as } k \to \infty.
\]
In fact, when $np \to \mu$ both parameters become indistinguishable and this asymptotic scenario serves as a limiting benchmark for the $\text{Bin}(n,p)$ problem to become solvable. However, in many applications the small $p$ regime (rare events) is the relevant one (see the references below and Section 4), and this will be the topic of this paper.

A variety of methods addressing this issue and improving over the sample maximum have been provided over the last decades but a final answer remains elusive until today. Broadly speaking, a major lesson from these attempts to obtain better estimators (see Section 2.1 for a detailed discussion) seems that in this difficult regime further information on $n$ and $p$ is required to obtain estimators performing reasonably well. This asks for a Bayesian approach. An early Bayesian estimator of the binomial parameters $(N, P)$, now considered as random, dates back to Draper and Guttman (1971), who suggested the mode of the posterior distribution for a uniform prior on $\{1, \ldots, N_0\}$ for $N$ and a Beta$(a, b)$ prior for $P$. Here, $N_0 \in \mathbb{N}$ is fixed and the parameters $a, b > 0$ are usually chosen as $a = b = 1$, which yields the standard uniform distribution. Later Raftery (1988), Günel and Chilko (1989), Hamedani and Walter (1988) and Berger et al. (2012) provided further estimators, which mainly differ in their choices of loss functions and prior distributions for $N$ and $P$.

A hierarchical Bayes approach is introduced in Raftery (1988) with a Poisson prior on $N$ with mean $\mu$, which implies a Poisson distribution with parameter $\lambda = \mu p$ as the marginal distribution of each observation. The prior for the pair $(\lambda, P)$ is chosen proportional to $1/\lambda$, which is equivalent to a product prior for the pair $(N, P)$ with the prior for $N$ proportional to $1/n$ and the standard uniform prior for $P$. Raftery (1988) suggested to minimize the Bayes risk with respect to the relative quadratic loss, which seems particularly suitable for estimating $n$ and will be adopted in this paper as well. From extensive simulation studies (see the afore mentioned references and Section 3), it is known that such Bayesian estimators of $n$ deliver numerically good results, in general. However, to the best of our knowledge, there is no rigorous theoretical underpinning of these findings. In particular, nothing is known about the posterior concentration of such estimators, and no systematic understanding of the role of the prior has been established.

Our contribution to this topic is threefold: (i) we propose a new class of Bayesian estimators for $n$, generalizing the approach in Raftery (1988), and (ii) we prove the posterior contraction for $n$. The posterior contraction result holds for a wide class of priors for $n$ and does not depend on the choice of the loss function. It implies consistency in a general asymptotic setting of the introduced class of estimators as well as of many (and with small
changes even all) Bayesian estimators mentioned above. Finally (iii), we extend the i.i.d. Bin\((n, p)\) model to a regression setting and apply our Bayes approach to count the number of fluorophores from super-resolution images, which is considered a difficult task.

Ad (i). For the new class of estimators, which we call the *scale estimators*, we consider \(k\) independent random variables \(X_1, \ldots, X_k\) from a Bin\((N, P)\) distribution. Denote \(X_k := (X_1, \ldots, X_k)\) and \(M_k := \max_{i=1, \ldots, k} X_i\). We assume a product prior for the pair \((N, P)\), where the prior for \(P\) is \(\Pi_P \sim \text{Beta}(a, b)\) for some \(a, b > 0\), and \(\Pi_N\), the prior for \(N\), satisfies \(\Pi_N(n) \propto n^{-\gamma}\) for \(\gamma > 1\). Independence of \(N\) and \(P\) is a common assumption and also justified in our example (Section 4) based on physical considerations. The scale estimator is then defined as the minimizer of the Bayes risk with respect to the relative quadratic loss, \(l(x, y) = (x/y - 1)^2\). Following Raftery (1988), it is given by

\[
\hat{n} := \frac{\mathbb{E}\left[\frac{1}{N} | \mathbf{X}_k\right]}{\mathbb{E}\left[\frac{1}{N^2} | \mathbf{X}_k\right]} = \frac{\sum_{n=M_k}^{\infty} \frac{1}{n} L_{a,b}(n) \Pi_N(n)}{\sum_{n=M_k}^{\infty} \frac{1}{n^2} L_{a,b}(n) \Pi_N(n)}, \tag{1.2}
\]

where \(L_{a,b}(n)\) is the beta-binomial likelihood, see, e.g., Carroll and Lombard (1985). In existing literature (Berger et al., 2012; Link, 2013), the Bayesian estimator of \(n\) with the prior \(\Pi_N(n) \propto 1/n\) is often called the scale estimator. Even though we do not allow \(\gamma = 1\) in the above definition since it leads to an improper prior (see, however, Theorem 2 for a proper modification of these estimators for \(0 \leq \gamma \leq 1\), which makes them accessible to our theory), we adopt this name for the new class of estimators.

Ad (ii). We show posterior consistency in a quite general setting, where the prior distribution \(\Pi_N\) can be chosen freely as long as it is a well-defined probability distribution satisfying

\[
\Pi_N(n) \geq \beta e^{-\alpha n^2}, \quad n \in \mathbb{N} \tag{1.3}
\]

for some positive constants \(\alpha\) and \(\beta\). In our asymptotic setting we consider sequences of parameters \((n_k, p_k)_k\) that may depend on the sample size \(k\) and are described by the class

\[
\mathcal{M}_\lambda := \left\{(n_k, p_k)_k : 1/\lambda \leq n_k p_k \leq \lambda, \quad n_k \leq \lambda^{6/\sqrt{k}/\log(k)}\right\}. \tag{1.4}
\]

for \(\lambda > 1\). We show that

\[
\sup_{(n_k^0, p_k^0) \in \mathcal{M}_\lambda} \mathbb{E}_{n_k^0, p_k^0}^{\mathbf{X}_k} \left[ \Pi(N \neq n_k^0 | \mathbf{X}_k) \right] \to 0, \quad \text{as } k \to \infty,
\]

where \(X_1, \ldots, X_k \overset{i.i.d.}{\sim} \text{Bin}(n_k^0, p_k^0)\). This is the main result of the paper and it will be formally stated as Theorem 1 in Section 2.
The recent advances on posterior contraction focus mainly on nonparametric or semiparametric models (Ghosal et al., 2000; Ghosal and van der Vaart, 2017) and posterior contraction for model selection in high-dimensional setups (Castillo and van der Vaart, 2012; Castillo et al., 2015; Gao et al., 2015). Discrete models with complex structure have not yet been studied and it appears difficult to approach them by a general treatment. Our proof uses earlier work on maximum likelihood estimation by Hall (1994) and opens another route to establish posterior consistency beyond the standard approach via testing, see Schwartz (1965).

In the binomial model, posterior consistency for fixed parameters \( n \) and \( p \) with the priors above follows already by Doob’s consistency theorem, see, e.g., van der Vaart (1998). To the best of our knowledge, no refined asymptotic result for a Bayesian approach to estimate \( n \) when \( p \) is unknown exists. Our result shows consistency of the marginal posterior distribution of \( N \) even in the challenging and relevant case of \( n_k \to \infty \) and \( p_k \to 0 \) as \( k \to \infty \) as long as \((n_k, p_k)_k \in \mathcal{M}_\lambda\). The difficulty of this setup comes from the convergence of the binomial distribution to the Poisson distribution with parameter \( \mu = \lim_{k \to \infty} np \) as \( n = n_k \to \infty \) and \( p = p_k \to 0 \). We have seen that the sample maximum is consistent as long as \( kp^n \to \infty \), for which \( e^n = o(k) \) is necessary (but not sufficient, see Lemma 10 for more details). In contrast, the definition of the class \( \mathcal{M}_\lambda \) implies that \( n^{6+\epsilon} = O(k) \) for \( \epsilon > 0 \) is already sufficient for the posterior consistency of the suggested Bayes approach. We stress that a simulation study in Schneider et al. (2018) suggests that the rate in Theorem 1 cannot be relaxed significantly, as numerically posterior consistency is only observed up to \( n^4 = O(k) \).

The posterior contraction result holds for the introduced scale estimators with \( \Pi_{N(n)} \propto n^{-\gamma} \), \( \gamma > 1 \). The improper priors with \( 0 \leq \gamma \leq 1 \) satisfy the assumptions under slight modifications, which are described in Theorem 2 in Section 2. With these modifications (restricting the support of \( N \)) the estimators of Draper and Guttman (1971) and Raftery (1988) are also covered by our theory. Our Theorems are applicable to many other Bayes estimators, as well. For example, Theorem 1 holds for the estimator in Günel and Chilko (1989), where a Gamma prior for \( N \) is suggested, and for the estimator in Hamedani and Walter (1988), which suggests either a poisson prior on \( N \) or an improper prior that can be considered via Theorem 2.

\textbf{Ad (iii).} Modern cell microscopy allows visualizing proteins and their modes of interaction during activity. It has become an indispensable tool for understanding biological function, transport and communication in the cell and its compartments, especially since the development of super-resolution nanoscopy (highlighted by the 2014 Nobel Prize in...
Chemistry). These techniques enable imaging of individual proteins through photon counts obtained from fluorescent markers (fluorophores), which are tagged to the specific protein of interest and excited by a laser beam (see Hell (2015) for a recent survey). In this paper, we are concerned with single marker switching (SMS) microscopy (Betzig et al., 2006; Rust et al. 2006; Hess et al. 2006; Fölling et al. 2008) where the emission of photons, which are then recorded, is inherently random: after laser excitation a fluorophore undergoes a complicated cycling through (typically unknown) quantum mechanical states on different time scales. This severely hinders a precise determination of the number of molecules at a certain spot in the specimen, see, e.g., Lee et al. (2012), Rollins et al. (2015), Aspelmeier et al. (2015). In Section 4 we show how the number of fluorophores can be obtained from a modified \((n, p)\)-Binomial model when they occur in clusters of similar size in the biological sample. Such a modification becomes necessary as the number of active markers decreases over the measurement process due to bleaching effects, and in our application the initial number \(n_0\) is estimated from observations \(X_t \sim \text{Bin}(n_t, p)\) at later times \(t\). We can link \(n_0\) to \(X_t\) by an exponential decay \(n_t = n_0(1 - B)^t\), which is known to be valid on physical grounds, and where the bleaching probability \(B\) of a fluorophore can be estimated within our model. This allows us to determine the number of fluorophores on DNA origami test beds with high accuracy.

This paper is organized as follows. The posterior contraction result and the discussion on the asymptotics of other estimators for \(n\) can be found in Section 2. Section 3 contains an extensive simulation study comparing the finite sample properties of those estimators and investigating robustness against model deviations from the \(\text{Bin}(n, p)\) model relevant to our data example. In Section 4 the data example is presented. The proof of the posterior contraction and the auxiliary results are deferred to Section 5.

## 2 Posterior contraction for \(n\)

Throughout the following \(X_1, \ldots, X_k\) are independent random variables with a \(\text{Bin}(N, P)\) distribution. We assume a product prior \(\Pi_{(N, P)} = \Pi_N \Pi_P\) for the pair \((N, P)\). For \(P\) we choose a \(\text{Beta}(a, b)\) prior with parameters \(a, b > 0\). It is the conjugate prior suggested in Draper and Guttman (1971) and widely used. The prior \(\Pi_N\) for \(N\) can be chosen as any proper probability distribution on the positive integers such that (1.3) holds for some \(\alpha, \beta > 0\). Write \(X^k = (X_1, \ldots, X_k)\), \(M_k = \max_{i=1,\ldots,k} X_i\) and \(S_k := \sum_{i=1}^k X_i\). For \(A \subset [0, 1]\) and \(n \in \mathbb{N}\), the joint posterior distribution for \(P\) and \(N\) is then given by

\[
\Pi( P \in A, N = n \mid X^k) = \frac{\int_A t^{S_k + a - 1} (1 - t)^{kn - S_k + b - 1} \, dt \cdot \prod_{i=1}^k \left( \frac{n}{X_i} \right) \cdot \Pi_N(n)}{\sum_{m=1}^\infty \int_0^1 t^{S_k + a - 1} (1 - t)^{km - S_k + b - 1} \, dt \cdot \prod_{i=1}^k \left( \frac{m}{X_i} \right) \cdot \Pi_N(m)}
\]
if \( n \geq M_k \) and \( \Pi(P \in A, N = n | X^k) = 0 \) otherwise. The marginal posterior likelihood function for \( N \) is thus

\[
\Pi(N = n | X^k) \propto \prod_{i=1}^{k} \left( \frac{n}{X_i} \right) \frac{\Gamma(kn - S_k + b)\Gamma(S_k + a)}{\Gamma(kn + a + b)} 1(n \geq M_k)\Pi_N(n) =: L_{a,b}(n)\Pi_N(n),
\]

where \( 1(\cdot) \) denotes the indicator function and \( L_{a,b}(\cdot) \) is the beta-binomial likelihood, see, e.g., Carroll and Lombard (1985).

The main result is stated in the following theorem and shows posterior contraction for \( n \) in the asymptotic setting described by sequences of parameters \((n_k, p_k)_{k \in M_\lambda} \) as defined in equation (1.4).

**Theorem 1.** Conditionally on \( N = n_0 \) and \( P = p_0 \) let \( X_1, \ldots, X_k \) i.i.d. \( \text{Bin}(n_0, p_0) \). For any prior distribution \( \Pi_{(N, P)} = \Pi_N\Pi_P \) with \( \Pi_P = \text{Beta}(a, b) \), \( a, b > 0 \), and where \( \Pi_N \) is a probability distribution such that (1.3) holds, we have uniform posterior contraction over \( M_\lambda \) in (1.4) for \( \lambda > 1 \), i.e.,

\[
\sup_{(n_0^k, p_0^k)_{k \in M_\lambda}} \mathbb{E}_{n_0^k, p_0^k} \left[ \Pi(N \neq n_0^k | X^k) \right] \rightarrow 0, \quad \text{as } k \rightarrow \infty.
\]

As mentioned in the introduction, Theorem 1 applies to the Bayesian estimators in equation (1.2) and the ones in Hamedani and Walter (1988) and Günel and Chilko (1989). The estimator in Draper and Guttman (1971) is based on a beta prior for \( P \) and a uniform prior on \( \{1, \ldots, N_0\} \) for some \( N_0 \in \mathbb{N} \) for \( N \). Since \( n_k > N_0 \) cannot be excluded for \( k \) large enough, assumption (1.3) is not fulfilled in this case. The estimators in Raftery (1988), Berger et al. (2012) and Link (2013) violate the conditions of Theorem 1 as well, since they are based on an improper prior on \( N \) proportional to \( 1/n \). However, we can still extend our result to modifications of these estimators, where the support of \( N \) is bounded but increases with \( k \).

**Theorem 2.** Theorem 1 holds if we exchange \( \Pi_N \) by \( \Pi_{N,k}(n) \propto \frac{1}{n} 1_{[1, T_k]}(n) \) with \( \gamma \in [0, 1] \), where \( T_k \) satisfies

\[
\lambda \sqrt{k/\log(k)} \leq T_k < \begin{cases} 
(\exp\{\alpha k^{1/3}\} / \beta)^{1/\gamma}, & \gamma < 1, \\
\exp\{\exp\{\alpha k^{1/3}\} / \beta\}, & \gamma = 1,
\end{cases}
\]

for all \( k \) and some positive constants \( \alpha \) and \( \beta \).

**Remark 1.** Theorem 1 and Theorem 2 still hold true if we allow \( \lambda \) in \( M_\lambda \) to increase with \( k \), as long as \( \lambda_k = o\left(\log(k)^{1/4}\right) \). This statement follows by verifying the conditions on the constants in the proof of Theorem 1 and their dependence on \( \lambda \). The strongest restriction results from equation (5.8) and depends on Lemma 8.
2.1 Asymptotic results for frequentist methods: contrasted and compared

In the following we present various existing asymptotic results for frequentist estimators and put them into perspective to Theorem 1, highlighting the differences of their respective asymptotic settings to the one described by the set $\mathcal{M}_\lambda$.

Early estimators based on the method of moments and the maximum likelihood approach can be found in Haldane (1941) and Blumenthal and Dahiya (1981). Their properties are further studied in Olkin et al. (1980), where it is shown that the estimators for $n$, when both $n$ and $p$ are unknown and $p$ is small, are highly irregular and stabilized versions of the two estimators are proposed. Two estimators were introduced more recently in DasGupta and Rubin (2005): the first one is another modification of the method of moments estimator, and the second one is a bias correction of the sample maximum. The asymptotic behavior of these two estimators is also known. For the new moments estimator, $\hat{n}_{NME}$, it holds that, as $k \to \infty$,

$$\sqrt{k}(\hat{n}_{NME} - n) \xrightarrow{D} \mathcal{N}(0, 2\gamma^2 n(n - 1)),$$

where $n$ is fixed and $\gamma > 0$ is a tuning parameter to be chosen by the practitioner. For the bias corrected sample maximum, $\hat{n}_{bias}$ say, it holds for $n$ fixed, as $k \to \infty$:

$$(nk)^{1/(n-1)}(\hat{n}_{bias} - n) \xrightarrow{D} \delta_1,$$

where $\delta_1$ denotes the Dirac measure at 1.

In Carroll and Lombard (1985) a further modification of the maximum likelihood estimator is introduced. The estimator is the maximizer of the beta-binomial likelihood for $n$, where a beta density is assumed for $p$ and $p$ is integrated out. The Carroll-Lombard estimator is nearly equivalent to the Bayesian Draper-Guttman estimator (e.g., for $N_0$ large they produce the same estimates), since the Carroll-Lombard estimator can be understood as a maximum a posteriori (MAP) Bayesian estimator of $n$ with an improper uniform prior on $\mathbb{N}$. That means, if we bound the set of values where a maximum can be attained to $\{1, \ldots, T_k\}$, then Theorem 2 with $\gamma = 0$ applies to the Caroll-Lombard estimator as well. This extends the classical asymptotic normality result of the Carroll-Lombard estimator $\hat{n}_{CL}$, which holds for $p$ constant, $n \to \infty$ and $\sqrt{k}/n \to 0$ as $k \to \infty$:

$$\sqrt{k} \left( \frac{\hat{n}_{CL} - n}{n} \right) \xrightarrow{D} \mathcal{N}(0, 2(1 - p)^2/p^2).$$

All of the results above hold for either $n$ or $p$ fixed and hence provide only limited insight into the situation when $p$ is small. A notable extension is discussed in Hall (1994). There,
it is shown for \( n = n_k \to \infty \) and \( p = p_k \to 0 \) as \( k \to \infty \) that
\[
\frac{p\sqrt{k}}{2} \left( \frac{\hat{n}_{\text{CL}} - n}{n} \right) \xrightarrow{D} \mathcal{N}(0, 1),
\]
if \( np \to \mu \in (0, \infty) \) and \( kp^2 \to \infty \) as \( k \to \infty \). Note that this result, like the previous, studies the limiting distribution of the relative difference, where \( \hat{n} - n \) is scaled by \( n \). In contrast, we show posterior contraction to the exact value \( \hat{n} = n \). This explains that Hall \((1994)\) can allow a faster rate \( (n = o(k^2)) \) than in our setting where \( n = O\left( \sqrt[3]{k}/\log(k) \right) \).

Also note that the above result is one specific scenario in a broader context and relies on further technical conditions, like \( n \) to be lower bounded by some positive power of \( k \).

### 3 Simulation study

In this section we investigate the finite sample performance of Bayesian estimators numerically for different choices of priors \( \Pi_P \) and \( \Pi_N \). We compare the following estimators that we introduced in the previous sections.

(SE) The scale estimator \( \text{SE}(\gamma) \) with \( \Pi_P = \text{Beta}(a, b) \) and \( \Pi_N(n) \propto n^{-\gamma} \). We consider both proper prior distributions \( (\gamma > 1) \), and improper ones \( (0 \leq \gamma \leq 1) \). The beta prior is chosen such that \( P \) has expectation \( \hat{p} \), where \( \hat{p} \in (0, 1] \) is a first guess for the probability of success, which might roughly be known beforehand. We select \( a \) to be 1 or 2 and set \( b = b(\hat{p}) := a/\hat{p} - a \). The scale factor \( \gamma \) needs to be chosen. Note that the Raftery estimator is equivalent to the scale estimator with \( \gamma = 1 \) and \( a = b = 1 \).

(DGE) The Draper-Guttman estimator \( \text{DGE}(N_0) \). The parameters \( a \) and \( b \) of the beta distribution are selected in the same way as for the scale estimator. The upper bound \( N_0 \) should be selected sufficiently large to avoid underestimation.

We look at the SE with \( \gamma \in \{0, 0.5, 1, 2, 3\} \) and the DGE with \( N_0 = 500 \). In case of an improper prior \( (\gamma \leq 1) \), Theorem 2 applies, and the posterior distribution is well defined as long as \( a + \gamma > 1 \) (see Kalm \((1987)\) for a cautionary note on this problem). We also employ the estimator SE(0) with \( a = 1 \), for which the posterior does not exist (so it is no Bayes estimator), but which still produces finite estimates.

**General performance.** Our first simulation study is based on 1000 samples of size \( k \in \{30, 100, 300\} \) from a binomial distribution \( \text{Bin}(n_0, p_0) \) for \( n_0 \in \{20, 50\} \) and \( p_0 \in \{0.05, 0.1, 0.3\} \). For all pairs \( (n_0, p_0) \) and each estimator \( \hat{n} \) we simulate:
• the relative mean squared error (RMSE), given by $\mathbb{E}\left(\left(\frac{\hat{n}}{n_0} - 1\right)^2\right)$,
• the bias $\mathbb{E}[\hat{n}] - n_0$ of the estimator.

We set $\hat{p} = p_0$ in the beta prior for this simulation and study the influence of the parameter $\gamma$. In Table 1 we present the estimators that have the lowest RMSE and the lowest bias for the different choices of $k$. The outcome advises to select smaller values of $\gamma$, the smaller $p_0$ is expected to be. Note that the DG estimator with large values $N_0$ is similar to the MAP estimator with the improper prior $\gamma = 0$. Thus, it is not surprising that there is only little difference between the performance of DG(500) and SE(0) in the simulations. Both of them perform superior in the regime of very small $p_0$. Still, one should be aware that a small $\gamma$ increases the variance of the posterior and therefore of the estimates. For this reason, higher choices of $\gamma$ become preferable for low RMSEs as $k$ increases. The similarity of Table 1 (A) and (B) for $n_0 = 20$ and $n_0 = 50$ suggests that the influence of $n_0$ is much weaker than the one of $p_0$ for the optimal estimator choice.

| $n_0 = 20$ | $n_0 = 50$ |
|---|---|
| $p_0$ | $k$ | RMSE | bias | $p_0$ | $k$ | RMSE | bias |
| 0.05 | 30 | DGE(500) | SE(0) | 0.05 | 30 | DGE(500) | SE(0) |
| 0.05 | 100 | DGE(500) | SE(0) | 0.05 | 100 | DGE(500) | SE(0) |
| 0.05 | 300 | SE(0.5) | DGE(500) | 0.05 | 300 | SE(0.5) | DGE(500) |
| 0.1 | 30 | DGE(500) | SE(0) | 0.1 | 30 | DGE(500) | SE(0) |
| 0.1 | 100 | SE(0.5) | DGE(500) | 0.1 | 100 | SE(0.5) | DGE(500) |
| 0.1 | 300 | SE(1) | DGE(500) | 0.1 | 300 | SE(1) | SE(0.5) |
| 0.3 | 30 | SE(2) | SE(1) | 0.3 | 30 | SE(1) | SE(0.5) |
| 0.3 | 100 | SE(3) | DGE(500) | 0.3 | 100 | SE(3) | DGE(500) |
| 0.3 | 300 | SE(3) | DGE(500) | 0.3 | 300 | SE(3) | DGE(500) |

Table 1: Overview of the estimators with the smallest RMSE and the smallest absolute bias for $a = 2$ and $b = 2/p_0 - 2$.

Our next numerical study covers a setting that is motivated by the data example in Section 4 where $p_0 \approx 0.0339$ and $k = 94$. We therefore set $p_0 = 0.0339$ and $k = 94$, and we select $n_0 = 15$. Our focus lies on the effect of the parameters $a$ and $b$, and particularly on the stability of the results with respect to misspecification of the guess $\hat{p}$. To this end, we consider four different scenarios: no information about $p_0$ (setting $\hat{p} = 0.5$), perfect information ($\hat{p} = p_0$), underestimation ($\hat{p} = 0.5p_0$), and overestimation ($\hat{p} = 1.5p_0$).

The results in Table 2 show that it is advantageous to choose a small $\gamma$ and a unimodal
beta prior (i.e., $a = 2$) if $p_0$ is known. If we have no information or are overestimating, it is again advisable to select $\gamma = 0$, while choosing a less confident prior for $P$ with $a = 1$. In contrast, underestimation of $p_0$ leads to high instabilities and substantial overestimation of $n_0$ if $\gamma$ is small. Here, estimators with proper priors for $\gamma = 1$ and 2 perform very well: the tendency for overestimation caused by the choice $\hat{p} = 0.5 p_0$ is compensated by the tendency for underestimation in case of higher values of $\gamma$.

| $\hat{p}$ | $a$ | estimator | RMSE | bias |
|-----------|-----|-----------|------|------|
| 0.5       | 1   | SE(0.5)   | 0.478| -10.17|
| 1         | SE(0) | 0.395   | -9   |
| $p_0$     | 2   | DGE(500)  | 0.034| -0.266|
| 2         | SE(0) | 0.036   | -0.043|

| $\hat{p}$ | $a$ | estimator | RMSE | bias |
|-----------|-----|-----------|------|------|
| 1.5 $p_0$ | 1   | SE(0)     | 0.12 | -3.73|
| 2         | SE(0) | 0.121 | -4.69|
| 0.5 $p_0$ | 1   | SE(1)     | 0.036| -0.032|
| 2         | SE(2) | 0.025   | -0.55|

Table 2: The two estimators that perform best under different choices of $\hat{p}$ for $n_0 = 15$, $p_0 = 0.0339$, and $k = 94$. The respective values of $b$ are given by $b(\hat{p}) = a/\hat{p} - a$.

The general lesson seems to be that the smaller $p_0$, the more difficult it becomes to estimate $n_0$ and the smaller we want to choose $\gamma$. A smaller $\gamma$, however, increases the variance of the posterior distribution and leads to estimators that are more sensitive against misspecification of $\hat{p}$ in the beta prior. This is investigated in Table 3, where we compare the sensitivity of estimators corresponding to $\gamma = 0$ and $\gamma = 1$. We see that misspecifying $\hat{p} = 0.5 p_0$ leads to severe overestimates $\mathbb{E}[\hat{n}] \approx 2 n_0$ for DGE(500), while SE(1) is less sensitive. Selecting $\gamma = 0$ can therefore help to estimate $n_0$ in very difficult scenarios, but it can also lead to heavily biased results if $\hat{p}$ is chosen too small.

| estimator | $\hat{p}$ | RMSE | bias |
|-----------|-----------|------|------|
| SE(1)     | $p_0$     | 0.122| -4.85|
|           | 0.5 $p_0$ | 0.129| 4.43 |
|           | 1.5 $p_0$ | 0.279| -7.73|
| DGE(500)  | $p_0$     | 0.034| -0.27|
|           | 0.5 $p_0$ | 1.002| 14.32|
|           | 1.5 $p_0$ | 0.139| -5.09|

Table 3: Sensitivity of SE(1) and DGE(500) against misspecification of $\hat{p}$. The value $a$ is set to 2. All other parameters are selected like in Table 2. Note that the behavior of DGE(500) and SE(0) is comparable in this setting.
Robustness. Motivated by our data example in Section 4, we also investigate the situation where \( n \) may slightly vary within the sample. This appears to be relevant in many other situations as well, e.g., the (unknown) population size of a species may vary from experiment to experiment in the capture-recapture method. Whereas varying probabilities \( p \) have been investigated in Basu and Ebrahimi (2001), models with a varying population size \( n \) have not received any attention in the previous research.

We consider 1000 repetitions of size \( k = 100 \), where each observation \( X_i, i = 1, \ldots, k \), is generated from a Bin\((n_i, p_0)\) distribution. Each \( n_i \) is in turn a realization of a binomial random variable \( N \sim \text{Bin}(\tilde{n}, \tilde{p}) \). For each sample, \( p_0 \) is drawn from a Beta\((2, 38)\) distribution with expectation 0.05. To test the influence of the varying parameter \( n_i \), we compare the performance of the estimators in the described scenario to their performance on binomial samples with a constant \( n_0 \) (chosen as the integer nearest to \( E[N] = \tilde{n} \tilde{p} \)) and the same realizations \( p_0 \). We calculate the RMSE with respect to \( n_0 \) for both scenarios and present the RMSE for \( X_i \sim \text{Bin}(n_i, p) \) divided by the RMSE in the i.i.d. case. The ratios in Table 4 verify a stable performance of the estimators in this setting since all values are close to 1. The parameters in Table 4 are chosen close to the data example in Section 4 with \( \tilde{n} \in \{8, 22\} \) and \( \tilde{p} = 0.7 \), but further simulations (not shown) confirmed the stability for other parameter choices, like \( \tilde{p} = 0.5 \) or \( \tilde{p} = 0.9 \), as well. Hence, in summary, we find that for inhomogeneous (random) \( N \) all estimators perform quite similar to the situation of a homogeneous (constant) \( n_0 \) (\( \approx E[N] \)).

| \( \tilde{n} \) | \( \tilde{n} = 8 \) | \( \tilde{n} = 22 \) |
|----------------|----------------|----------------|
| estimator      | RMSE-R         | RMSE-R         |
| SE(0.5)        | 1.022          | 1.130          |
| SE(1)          | 1.011          | 1.067          |
| SE(2)          | 1.020          | 1.010          |
| DGE(500)       | 1.032          | 1.073          |
| RE             | 0.988          | 0.981          |

Table 4: Ratios of the RMSE for i.i.d. and non-i.i.d. samples (RMSE-R) for the estimators SE(\(\gamma\)), DGE\((N_0)\), and the Raftery estimator RE. The beta prior in SE and DGE is defined by \( a = 2 \) and \( b = 38 \).

4 Data example

In this section we extend the previously described Bayesian estimation methods to quantify the number of fluorescent molecules in a specimen recorded with super-resolution mi-
microscopy. Reliable methods to count such molecules are highly relevant to quantitative cell biology, for example, to determine the number of proteins of interest in a compartment of the cell, see, e.g., Lee et al. (2012), Rollins et al. (2015), Ta et al. (2015), Aspelmeier et al. (2015) or Karathanasis et al. (2017) and references therein.

**Experimental setup.** Data has been recorded at the Laser-Laboratorium Göttingen e.V. During experimental preparation so called DNA origami (Schmied et al., 2014), tagged with the fluorescent marker Alexa647, were dispersed on a cover slip. DNA origami are nucleotide sequences which have been engineered in such a way that the origami folds itself into a desired shape (see Fig. 1A). Fluorescent molecules (fluorophores), which are equipped with an “anchor” that sticks to a specific region of the origami, are attached to the origami molecules. In the experiment, Alexa647 fluorophores with 22 different types of anchors were used, each one matching a different anchor position on the origami (see Fig. 1A). Therefore, at most 22 fluorophores can be attached to a single origami. The pairing itself is random (so not every possible anchor position needs to be occupied) and is expected to occur with a probability between 0.6 and 0.75, according to producer specifications.

Fundamental to super-resolution microscopy is the switching behavior of the fluorophores. A fluorophore can be in two different states (“on” or “off”) but only emits light in the “on” state. When excited with a laser beam, it switches between these “on” and “off” states until it bleaches, i.e., reaches an irreversible “off” state. During the course of the experiment, an image sequence of several origami distributed on a cover slip is recorded over a period of a few minutes (see the movie supplement material). The exposure time for one image (denoted as frame) is 15 ms. Switching of fluorophores between “on” and “off” states is necessary to achieve super-resolution, which denotes the ability to discern markers with distance below the diffraction limit achievable with visible light of about 250 – 500 nm (Hell, 2009). Such fluorophores could not be discerned by conventional microscopy. Super-resolution becomes possible by separating photon emissions of spatially close molecules in time. This is realized by applying a low laser intensity, such that only a small fraction of fluorophores switches in the “on” state for a given frame. Hence, it is very unlikely that nearby fluorophores emit photons at the same time (see, e.g., Betzig et al. (2006), Rust et al. (2006), Hess et al. (2006), Fölling et al. (2008) for different variants of this principle). By this method, an increased resolution of up to 20 – 30 nm can be achieved.

The experiment was prepared in such a way that most fluorophores are guaranteed to be “on” in the first frame, and all origami are thus visible as bright spots in Fig. 1B. Note that individual fluorophores occupying the same origami cannot be discerned in this frame. This becomes possible only when most of the fluorophores are switched “off” at later times,
Figure 1: (A) Schematic drawing of the DNA-origami used in the experiment. The origami is a tube-like structure that consists of 12 suitably folded DNA helices. In each of the two highlighted green regions up to 11 fluorescence markers can anchor. (B) First frame from the sequence of microscopic images. The 94 regions of interest (ROIs) that were chosen for analysis are identified by white boxes. The selection was done algorithmically. No overlap between ROIs was allowed, and it was made sure that no excessive background noise and disturbances affected the ROI during the course of the experiment.

such that markers show up individually (see the supplementary movie for illustration).

Quantitative biology. Quantitative biology addresses the issue of counting the number of fluorophores from measurements like the one described above. The brightness of each spot is proportional to the number of fluorophores in the “on” state within the respective origami. An origami is invisible if all of its fluorophores are “off”, but its location is still known from the first frame, which allows us to register 94 regions of interest (ROIs) in a preparational step (see Figure 1B). Exemplarily, six microscopic frames (out of 14,060) recorded at different times $t \in \{1500, 3000, 4500, 6000, 7500, 9000\}$, which show the influence of switching and bleaching on the observations, are visualized in Figure 2.

We aim to estimate the number of fluorophores attached to each origami, which we expect to be between 13 and 16 according to the producer specification. In order to make our model accessible to the data, we assume for simplicity that each origami carries the same number $n_0$ of fluorophores and we only model the mean number $n_t$ of unbleached fluorophores at time $t$. The physical relation between $n_0$ and $n_t$ is given by

$$n_t = n_0 (1 - B)^t,$$
Figure 2: Six selected frames from the dataset of recorded origami. The (physical) time difference between two consecutive images in this figure is roughly 22.5 seconds. Bleaching causes the number of visible origami to decrease with increasing frame number, and switching causes that unbleached origami are visible only in some frames.

where $B$ denotes the bleaching probability. Now, the brightness observed for a spot in frame $t$ is proportional to the number $X_t$ of “on” fluorophores during the frame’s exposure. This number $X_t$ is binomially distributed $\text{Bin}(n_t, p)$, where $p$ denotes the (time-independent) probability that an unbleached fluorophore is in its “on” state. We can estimate $n_0$ and $B$ by fitting a log-linear model to equation (4.1), where the respective population sizes $n_t$ are in turn estimated from the 94 realizations of $X_t$ observed in frame $t$.

To get a sense for the magnitude of $p$, we use data from a similar experiment: in this case, each origami has been designed in such a way that it carries exactly one fluorophore. We estimate $p$ as the average ratio $\hat{p}$ of the number of frames where the fluorophore is “on”
(a bright spot is seen) and the total number of observed frames before bleaching (no spot is seen for any time in the future), and we find $\hat{p} \approx 0.0339$. This indicates that we are in the difficult “small $p$” regime of the $\text{Bin}(n, p)$ problem, and we will therefore apply the Bayesian estimators introduced in Section 3 (SE, DGE) to estimate $n_t$. The beta prior for SE and DGE uses the parameters $a = 2$ and $b = 2/\hat{p} - 2 \approx 56.99$. We choose the unimodal prior with $a = 2$, as suggested by Table 2, since we assume that our guess $\hat{p}$ is reasonably accurate. Note that a finer degree of modeling would require to view $n_0$, $n_t$ and $p$ as random variables (with small variances) instead of constants. However, as shown in Section 3, the Bayesian estimators we consider are robust against fluctuations in the parameters and are therefore suited to estimate the respective mean values.

Since most fluorophores are deliberately forced to be “on” in the first frame, the relation $X_t \sim \text{Bin}(n_t, p)$ does not hold initially. It only becomes valid after the initial state has relaxed to an equilibrium, which is why we only take into account data after frame 1500 ($\approx 22.5$ seconds). To mitigate the influence of correlations between observations (i.e., $X_t$ and $X_{t+1}$ for a spot cannot be considered independent), we also add a waiting time of 1500 frames between the frames we use for our analysis. In total, we use the six frames with $t \in \{1500, 3000, 4500, 6000, 7500, 9000\}$ depicted in Figure 2. The 94 realizations of $X_t$ are extracted from the image data as follows: at each registered origami position, represented by a $6 \times 6$ pixel ROI, the total brightness is measured and then divided by the brightness of a single fluorophore. We determined the brightness of a single fluorophore from the late frames of the experiment, where at most one fluorophore of each origami is active with high probability.

| estimator | $n_0$ | $B \cdot 10^3$ |
|-----------|------|--------------|
| SE(0)     | 16   | 0.152        |
| SE(0.5)   | 13   | 0.148        |
| SE(1)     | 11   | 0.139        |
| SE(2)     | 9    | 0.163        |
| SE(3)     | 6    | 0.123        |
| SE(5)     | 5    | 0.114        |
| DGE(500)  | 16   | 0.167        |

Table 5: Estimates of the bleaching probability $B$ and the number $n_0$ of fluorophore molecules on single DNA origami.

The results for the scale estimator with $\gamma = 0.5$ are depicted in Figure 3, which also shows the log-linear fit for model (4.1). This provides us with estimates for $n_0$ and $B$. The point estimates of $n_0$ and $B$ for different estimators are summarized in Table 5. Given that it is
Figure 3: The log-linear fit described by $n_t = n_0 (1 - B)^t$ for the SE with $\gamma = 0.5$.

to be expected that the true $n_0$ in this experiment lies between 13 and 16, we can see in Table 5 that the SEs with an improper prior ($\gamma \leq 1$) produce reasonable results, and the DGE also performs well. This confirms that we are indeed in the critical case of Bin($n, p$) with small $p$, so that the prior putting a lot of weight on large values on $n$ gives best results by correcting for the usual tendency to underestimate, see also the results of the simulation study performed under comparable conditions in Table 2. To illustrate the difficulty of this problem, Figure 4 shows exemplary counting results we obtained for $t \in \{1500, 7500\}$. Note that the final estimates for $n_0$ are exclusively based on observations $X_t \leq 3$, where a great majority of these observations is already 0.

Figure 4: Bar charts of the observed numbers of fluorophore molecules for time frames 1500 and 7500.
Proof of Theorem 1. First observe that
\[
  t > \text{falling factorial for } j \text{ for }
\]
follows that \( \lambda, a \) and arguments do not depend on the specific choice of \((\lambda_n, p_n)\) for notational simplification. Our arguments do not depend on the specific choice of \((n_k, p_k)\) but only rely on the parameters \(\lambda, a \) and \(b\).

**Proof of Theorem 1.** First observe that
\[
\Pi(N \neq n_k | X^k) = \frac{\sum_{n \neq n_k, n \geq M_k} L_{a,b}(n) \Pi_N(n)}{\sum_{n=M_k}^{\infty} L_{a,b}(n) \Pi_N(n)} \leq \sum_{n \neq n_k, n \geq M_k} \frac{L_{a,b}(n) \Pi_N(n)}{L_{a,b}(n_k) \Pi_N(n_k)}.
\]
Under the assumption that \( S_k \geq 2 \) (which we justify below), we can apply Lemma 4.1 and find
\[
\frac{L_{a,b}(n)}{L_{a,b}(n_k)} \leq c_1 k n_k \frac{L_{|a|,b}(n)}{L_{|a|,b}(n_k)}
\]
for \( c_1 = 1 + \lceil a \rceil + b \), where \( \lceil \cdot \rceil \) and \( \lfloor \cdot \rfloor \) denote the ceil and floor functions, respectively. It follows that
\[
\Pi(N \neq n_k | X^k) \leq c_1 k n_k \sum_{n \neq n_k, n \geq M_k} \exp \left( k \int_{n_k}^{n} f'(m) \, dm \right) \frac{\Pi_N(n)}{\Pi_N(n_k)} \tag{5.1}
\]
with \( f(m) = \frac{1}{k} \log L_{|a|,b}(m) \). If \( n < n_k \), we can write \( \int_{n_k}^{n} f'(m) \, dm = -\int_{a_k}^{a_k} f'(m) \, dm \). For an upper bound on the posterior we thus need a lower bound of \( f'(m) \) if \( m \leq n_k \) and an upper bound if \( m \geq n_k \). Since \( f \) only depends on \( a \) via \( |a| \), we assume that \( a \in \mathbb{N}_0 \) in the following. Then we can apply Lemma 4.1 from [Hall (1994)] and find
\[
f'(m) = \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{X_i} \frac{1}{m-j+1} - \frac{1}{km + a - j} = \sum_{j=1}^{M_k} \frac{T_j - U_j}{j} - \sum_{j=M_k+1}^{S_k+a} U_j \tag{5.2}
\]
with
\[
T_j := \frac{1}{k} \sum_{i=1}^{k} \frac{(X_i)_j}{(m)_j} \quad \text{and} \quad U_j := \frac{(S_k+a)_j}{(km+a+b-1)_j}
\]
for \( j \leq M_k \) and \( j \leq S_k + a \) respectively, where \( (t)_j = t(t-1) \cdots (t-j+1) \) denotes the falling factorial for \( t > 0 \). For convenience we define \( T_j := 0 \) for all \( j > M_k \). Next, we introduce the events
\[
\mathcal{U}_k := \left\{ M_k = n_k \text{ or } M_k \geq l_k \right\}, \quad \mathcal{R}_k := \left\{ M_k \leq 2 \log(k) \right\},
\]
\[ T_{kj} := \left\{ (m) \mid T_j - ET_j \leq \sqrt{(c_2j)^3 l_k \log(k)/k} \right\}, \quad T_k := \bigcap_{j \in \mathbb{N}} T_{kj}, \]

\[ S_k := \left\{ \left| S_k - kn_pk \right| \leq \sqrt{\lambda k \log(k)} \right\}, \]

and denote the intersection \( U_k \cap R_k \cap T_k \cap S_k \) by \( A_k \). The constant \( c_2 = 2 \lambda (\lambda + 2) \) is chosen to satisfy Lemma 4 for each sequence \((n_k, p_k)_k \in \mathcal{M}_\lambda\), and \( l_k \) is a fixed sequence with \( l_k = o(\sqrt{\log(k)}) \). Note that the sets \( T_{kj} \) are in fact independent of \( m \) due to the definition of \( T_j \). On the event \( S_k \), Lemma 5 grants us the additional property

\[ \left| U_j - \tilde{U}_j \right| \leq j \sqrt{\frac{\lambda \log(k)}{k}} \left( \frac{c_3}{m} \right)^j \quad \text{with} \quad \tilde{U}_j := \frac{(kn_pk + a)_j}{(km + a + b - 1)_j}, \]

for \( j \leq S_k + a \) and \( c_3 = 2e^2(3\lambda + a + 1) \). Also note that \( S_k \leq 2k\lambda \) holds and that \( S_k \geq 2 \) is guaranteed for \( k/\lambda - \sqrt{\lambda k \log(k)} \geq 2 \) on \( S_k \). Thus, equations (5.1) and (5.2) apply on \( A_k \) if \( k \) is sufficiently large.

Indeed, we can restrict our attention to \( A_k \), since

\[ \mathbb{E}_{n_k,p_k} \left[ \Pi(N \neq n_k \mid X^k) \right] - \mathbb{E}_{n_k,p_k} \left[ \mathbf{1}_{A_k} \Pi(N \neq n_k \mid X^k) \right] \leq \mathbb{P}_{n_k,\mathbb{P}_k}(A_k^c) \rightarrow 0 \quad (5.3) \]

for \( k \rightarrow \infty \). To see this, one can bound \( \mathbb{P}(A_k^c) := \mathbb{P}_{n_k,p_k}(A_k^c) \) by

\[ \mathbb{P}(A_k^c) \leq \mathbb{P}(S_k^c) + \mathbb{P}(U_k^c) + 2 \mathbb{P}(R_k^c) + \mathbb{P}(T_k^c \cap R_k). \]

The first contribution vanishes due to Chebyshev’s inequality (see e.g., DeGroot and Schervish (2012)), and the second and third terms are controlled by Lemma 8 and Lemma 9 respectively. The last contribution satisfies

\[ \mathbb{P}(T_k^c \cap R_k) = \mathbb{P} \left( \bigcup_{j=1}^{[2 \log(k)]} T_{kj}^c \right) \leq \frac{[2 \log(k)]}{l_k \log(k)} \rightarrow 0 \]

for \( k \rightarrow \infty \) due to Lemma 4. It is important to note that the upper bounds in these considerations only depend on \( \lambda \) if suitable choices for the involved constants are made. In the following, we always assume that \( X^k \in A_k \).

**Auxiliary lower bound.** For \( M_k \leq m < n_k \), we prove a lower bound of \( f'(m) \). We may assume that \( M_k \geq l_k \rightarrow \infty \) for \( k \rightarrow \infty \) in this case, since \( X^k \in U_k \). For \( k \) such that \( l_k \geq 4 \) we can bound equation (5.2) by

\[ f'(m) \geq \sum_{j=1}^{4} \frac{T_j - U_j}{j} - S_{k+a} \sum_{j=5}^{U_j} j, \quad (5.4) \]

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as \( T_j \geq 0 \) for all \( j \). In case of \( j = 1 \) we obtain

\[
T_1 - U_1 = \frac{S_k}{km} - \frac{S_k + a}{km + a + b - 1} \geq - \frac{a + 1}{km - 1} \geq -2 \left( \frac{a + 1}{km} \right) \geq -2 \frac{\lambda(a + 1)}{m^2} \sqrt{\frac{\log(k)}{k}},
\]

where we used the upper bound \( m < n_k \leq \lambda \sqrt{k \log(k)} \) in the last inequality, which is guaranteed in \( \mathcal{M}_\lambda \). To handle the terms with \( j \geq 2 \), we exploit that \( X^k \in T_k \) and apply \( (m)_j \geq (m/e^2)_j \) from Lemma 2 in order to derive

\[
\sum_{j=2}^{4} \frac{|T_j - \mathbb{E}T_j|}{j} \leq \sqrt{\frac{\log(k)}{k}} \sum_{j=2}^{4} \left( \frac{\sqrt{c_2}}{m/e^2} \right)^j \leq 2 \frac{4 c_2 e^4}{m^2} \sqrt{\frac{\log(k)}{k}}
\]

for \( k \) large enough such that \( \sqrt{4 c_2 e^2}/l_k < 1/2 \). Similarly, we find

\[
\sum_{j=2}^{S_k+a} \frac{|U_j - \mathbb{U}_j|}{j} \leq \lambda \frac{\log(k)}{k} \sum_{j=2}^{S_k+a} \left( \frac{c_3}{m} \right)^j \leq 2 \frac{\sqrt{\lambda c_3^2}}{m^2} \sqrt{\frac{\log(k)}{k}}
\]

and

\[
\sum_{j=3}^{S_k+a} \frac{\mathbb{U}_j}{j} \leq \sum_{j=3}^{S_k+a} \frac{1}{j} \left( \frac{c_4}{m} \right)^j \leq 2 \left( \frac{c_4}{m} \right)^5
\]

for \( k \) (and thus \( m \)) sufficiently large. In the latter equation, we applied the first part of Lemma 6 with \( c_4 = 6 e^2(\lambda + a) \), using \( S_k \leq 2 k \lambda \) on \( S_k \). Next, we note that \( \mathbb{E}(X_1)_j = (n_k)_j p^j \) and consult the first result of Lemma 7 to establish

\[
\sum_{j=2}^{4} \frac{\mathbb{E}T_j - \mathbb{U}_j}{j} \geq \frac{1}{2 \lambda^2 \frac{2c_5}{n_k m^3}} \geq \frac{1}{2 \lambda^2} \frac{n_k - m}{n_k m^3} - 2 \frac{\lambda^2 c_5^4}{m^2} \sqrt{\log(k) / k},
\]

where \( c_5 = 3 \lambda + 2 a + 2 \). Similar to the case \( j = 1 \), we applied \( m^2 < \lambda^2 \sqrt{k \log(k)} \) in the last inequality. All bounds calculated above can be inserted into inequality (5.4), yielding

\[
f'(m) \geq (T_1 - U_1) + \sum_{j=2}^{4} \frac{T_j - \mathbb{E}T_j}{j} + \sum_{j=2}^{S_k+a} \frac{\mathbb{U}_j - U_j}{j} + \sum_{j=2}^{4} \frac{\mathbb{E}T_j - \mathbb{U}_j}{j} - \sum_{j=5}^{S_k+a} \frac{\mathbb{U}_j}{j}
\]

\[
\geq \frac{1}{4 \lambda^2} \frac{n_k - m}{n_k m^3} - \frac{C_2}{m^2} \sqrt{\frac{l_k \log(k)}{k}} + \left[ \frac{1}{4 \lambda^2} \frac{n_k - m}{n_k m^3} - 2 \left( \frac{c_4}{m} \right)^5 \right]
\]

\[
\geq \frac{C_1}{m^3} \frac{n_k - m}{n_k} - \frac{C_2}{m^2} \sqrt{\frac{l_k \log(k)}{k}} + h(m),
\]

where \( h(m) \) is defined as the expression in square brackets. The constants in this bound are \( C_1 = 1/4 \lambda^2 \) and \( C_2 = 2 (\lambda(a + 1) + 4 e^2 e^4 + \sqrt{\lambda c_3^2} + 2 \lambda^2 c_5) \).
**Auxiliary upper bound.** We next provide an upper bound for \( f'(m) \) if \( m > n_k \geq M_k \). Unlike for the lower bound, we cannot assume that \( m \) becomes larger than any given constant with increasing \( k \). Since \( U_j \) is nonnegative, we can bound

\[
f'(m) \leq \sum_{j=1}^{M_k} \frac{T_j - U_j}{j}.
\]

in equation (5.2). We look at the case \( j = 1 \) first, and see

\[
T_1 - U_1 = \frac{S_k}{km} - \frac{S_k + a}{km + a + b - 1} \leq \frac{S_k (a + b)}{km (km - 1)} \leq \frac{4 \lambda (a + b)}{m^2} \leq \frac{4 \lambda (a + b)}{m^2} \sqrt{\frac{\log^3(k)}{k}},
\]

where we used that \( S_k \leq 2 \lambda k \) on the event \( S_k \). Next we set \( \bar{m} := 4 c_2 e^4 \) and derive

\[
\sum_{j=2}^{M_k} \left| \frac{T_j - \mathbb{E} T_j}{j} \right| \leq \sqrt{\frac{\lambda \log(k)}{k}} \sum_{j=2}^{M_k} \left( \frac{\sqrt{c_2} j}{m/e^2} \right)^j \\
\leq \frac{c_2 M_k e^4}{m^2} \sqrt{\frac{\lambda \log(k)}{k}} \sum_{j=0}^{[m]} \left( \frac{e^2 \sqrt{c_2}}{\sqrt{m}} \right)^j \\
\leq \frac{c_2 M_k e^4}{m^2} \sqrt{\frac{\lambda \log(k)}{k}} \cdot \left\{ \begin{array}{ll} 
2 & \text{if } m > \bar{m} \\
\bar{m} \left( e^2 \sqrt{c_2} + 1 \right) \bar{m} & \text{if } m \leq \bar{m}
\end{array} \right.
\]

for \( c_6 = 2 c_2 e^4 (\bar{m} \left( e^2 \sqrt{c_2} + 1 \right) \bar{m} + 2) \). In the last step, we used that \( M_k \leq 2 \log(k) \) on the event \( R_k \). In a similar fashion, we can establish the bound

\[
\sum_{j=2}^{M_k} \left| \frac{U_j - \bar{U}_j}{j} \right| \leq \sqrt{\frac{\lambda \log(k)}{k}} \sum_{j=2}^{M_k} \left( \frac{c_3}{m} \right)^j \leq \frac{c_7}{m^2} \sqrt{\frac{\log^3(k)}{k}},
\]

where \( c_7 = 4 \sqrt{\lambda c_2} \left( c_3^{2 \gamma + 1} + 1 \right) \). Finally, we apply the second claim of Lemma 7 and obtain

\[
\sum_{j=2}^{M_k} \frac{\mathbb{E} T_j - \bar{T}_j}{j} \leq -C_1' \frac{m - n_k}{n_k m^3}
\]

with \( C_1' = \left( 2 \lambda^2 (a + b + 1)^2 \right)^{-1} \) for sufficiently large \( k \). We conclude

\[
f'(m) \leq (T_1 - U_1) + \sum_{j=2}^{M_k} \frac{T_j - \mathbb{E} T_j}{j} + \sum_{j=2}^{M_k} \frac{U_j - \bar{U}_j}{j} + \sum_{j=2}^{M_k} \frac{\mathbb{E} T_j - \bar{T}_j}{j}
\leq -\frac{C_1' m - n_k}{n_k m^3} + \frac{C_2'}{m^2} \sqrt{\frac{\log^3(k)}{k}}
\]

for \( C_2' = 4 \lambda (a + b) + c_6 + c_7 \).
Posterior bound. By applying the two inequalities \((5.5)\) and \((5.6)\) for \(m < n_k\) and \(m > n_k\), we can now bound the posterior probability \(\Pi(N \neq n_k | X^k)\) on the event \(\mathcal{A}_k\) through equation \((5.1)\). First, we observe that for \(n \in \mathbb{N}\) with \(n \neq n_k\)

\[
\int_{n_k}^{n} \frac{m - n_k}{n_k m^3} \, dm = \frac{1}{2} \frac{(n - n_k)^2}{(nk)^2} \quad \text{and} \quad \int_{n_k}^{n} \frac{1}{m^2} \, dm = \frac{1}{2} \frac{(n - n_k)^2}{(nk)^2} \frac{2kn}{n - n_k}.
\]

It also holds for \(n \neq n_k\) that

\[
\left| \frac{n}{n_k} - 1 \right| \geq \frac{1}{2n}.
\]

Therefore, if \(l_k \leq n < n_k\), the function \(h(m)\) introduced in equation \((5.5)\) satisfies

\[
\int_{n}^{n_k} h(m) \, dm = \frac{C_1}{2} \frac{(n - n_k)^2}{(nk)^2} - \frac{c_4^5 n_k^4 - n^4}{2} \geq \frac{C_1}{2} \frac{(n - n_k)^2}{(nk)^2} \left( 1 - \frac{4c_4^5}{C_1} \frac{1}{1 - n/n_k} \frac{1}{n^2} \right) \geq 0
\]

for \(k\) such that \(l_k \geq 8 c_4^5/C_1\). Employing bound \((5.5)\) thus yields

\[
-k \int_{n}^{n_k} f'(m) \, dm \leq -k \frac{C_1}{2} \frac{(n - n_k)^2}{(nk)^2} \left( 1 - C' \frac{n_k n}{k - n_k} \sqrt{\frac{l_k \log(k)}{k}} \right),
\]

where the constant \(C\) is given by \(2C_2/C_1\). On the other hand, for \(n_k < n\), bound \((5.6)\) similarly leads to

\[
k \int_{n_k}^{n} f'(m) \, dm \leq -k \frac{C_1'}{2} \frac{(n - n_k)^2}{(nk)^2} \left( 1 - C' \frac{n_k n}{n - n_k} \sqrt{\frac{l_k \log^3(k)}{k}} \right)
\]

for \(C' = 2C_2/C_1'\).

Finally, let \(\tilde{C}_1 = \min \{ C_1, C_1' \} \) and \(\tilde{C} = \max \{ C, C' \} \). We can apply inequality \((5.7)\) (with \(n\) and \(n_k\) switched) to find for any \(n \in \mathbb{N}\) with \(n \neq n_k\) and \(n \geq M_k\) that

\[
k \int_{n_k}^{n} f'(m) \, dm \leq -k \tilde{C}_1 \frac{C_1}{2} \frac{(n - n_k)^2}{(nk)^2} \left( 1 - \tilde{C} \frac{n_k}{1 - n_k/n} \sqrt{\frac{l_k \log^3(k)}{k}} \right)
\]

\[
\leq -k \tilde{C}_1 \frac{C_1}{8n_k^2} \left( 1 - 2 \tilde{C} \frac{n_k^2}{k} \sqrt{\frac{l_k \log^3(k)}{k}} \right) \leq -\tilde{C}_1 \frac{k}{16 n_k^4}
\]

for \(k\) large enough such that \(n_k^2 \leq \sqrt{k/(l_k \log^3(k))}/4\tilde{C}\) for each sequence in \(\mathcal{M}_k\). Consulting inequality \((5.1)\) and using the constraint \(\Pi_N(n_k) \geq \beta \exp \left( -\alpha n_k^2 \right)\) of the prior yields

\[
1_{\mathcal{A}_k} \Pi(N \neq n_k | X^k) \leq c_1 k n_k \sum_{n \neq n_k} \exp \left( -\frac{\tilde{C}_1}{16 n_k^4} \right) \frac{\Pi_N(n)}{\Pi_N(n_k)}
\]

(5.9)
\[ \leq \frac{c_1}{\beta} \exp \left( - \tilde{C}_1 \frac{k}{16 n_k^4} + \alpha n_k^2 + \log (k n_k) \right) \]

\[ \leq \frac{c_1}{\beta} \exp \left( - \tilde{C}_1 \frac{k^{1/3} \log(k)^{2/3}}{\alpha \lambda^2 \log(k)^{1/3}} + \alpha \lambda^2 \log(k)^{2/3} \right) \rightarrow 0 \]

for \( k \to \infty \). Due to statement (5.3) this is sufficient to prove Theorem 1. \( \square \)

**Proof of Theorem 2.** The result follows from the proof of Theorem 1, where we only need to handle the inequalities in (5.9) differently. Bounding the sum over the prior in (5.9) by an integral, one sees that the upper bound on \( T_k \) in (2.1) is sufficient to ensure convergence towards 0, if \( \Pi_{N,k} \) is considered instead of \( \Pi_N \). \( \square \)

**Lemma 1.** For \( k, n, s \in \mathbb{N} \) and \( b > 0 \) such that \( 2 \leq s \leq kn \) define the function

\[ f(a) = \frac{\Gamma(kn - s + b) \Gamma(s + a)}{\Gamma(kn + a + b)} \]

for \( a \geq 0 \). Then \( f \) is monotonically decreasing and \( f([a]) / f(\lceil a \rceil) \leq c kn \) for \( c \geq 1 + [a] + b \).

**Proof.** It is sufficient to look at \( h(a) := \Gamma(y + a) / \Gamma(z + a) \), where \( 2 \leq y < z \) are fixed. For \( \epsilon > 0 \), we find that \( \log h(a + \epsilon) \leq \log h(a) \) is equivalent to

\[ \gamma(y + a + \epsilon) - \gamma(y + a) \leq \gamma(z + a + \epsilon) - \gamma(z + a) \]

with \( \gamma(t) = \log \Gamma(t) \) for \( t > 0 \). This inequality is true since \( \gamma \) is convex, see Merkle (1996), which therefore establishes monotonicity. We also find

\[ \frac{h([a])}{h(\lceil a \rceil)} = \frac{z + [a] - 1}{y + [a] - 1} \leq z + [a]. \]

Substituting \( y = s \) and \( z = kn + b \), and using that \( kn + [a] + b \leq (1 + [a] + b) kn \) yields the second result. \( \square \)

**Lemma 2.** Let \( j \in \mathbb{N} \) and \( n, m > 1 \) with \( j \leq \min \{m, n\} \). Let \( (a)_j = a(a - 1) \ldots (a - j + 1) \) denote the falling factorial for \( a \in \mathbb{R} \).

1. For \( 0 < c \leq e^{-2} \) it holds that \( (cm)_j \leq (m)_j \leq m^j \).

2. For \( n \geq m \) and \( j > 1 \) it holds that \( \frac{m^j(n)_j}{n^j(m)_j} \geq 1 + \frac{n - m}{nm} \).

**Proof.** 1. From Theorem 1 in Jameson (2015) follows that

\[ \sqrt{2\pi} m^{m+1/2} e^{-m} \leq \Gamma(m + 1) \leq e \sqrt{2\pi} m^{m+1/2} e^{-m}. \]
We apply this to obtain
\[
(m)_j = \frac{\Gamma(m+1)}{\Gamma(m-j+1)} \geq \frac{1}{e} \left( \frac{m}{m-j} \right)^{m-j+1/2} \left( \frac{m}{e} \right)^j \geq \left( \frac{m}{e} \right)^j.
\]

2. For \( n \geq m \) and \( j > 1 \), we bound
\[
\frac{m^j(n)_j}{n^j(m)_j} = \prod_{i=0}^{j-1} \frac{m(n-i)}{(m-i)n} = \prod_{i=0}^{j-1} \left( 1 + i \frac{n-m}{n(m-i)} \right) \geq 1 + \frac{n-m}{nm}.
\]

**Lemma 3.** Let \( X \) be a binomial random variable, \( X \sim \text{Bin}(n,p) \), for \( n \in \mathbb{N} \) and \( p \in (0,1) \). Then
\[
\mathbb{E}[X^r] \leq B_r \cdot \max\{np,(np)^r\},
\]
where \( B_r \) is the \( r \)-th Bell number.

**Proof.** Let \( q = (1-p) \) and let \( M_{n,p} \) be the moment generating function of the binomial distribution,
\[
M_{n,p}(t) = (pe^t + q)^n = f(g(t)),
\]
where \( f(s) = s^n \) and \( g(t) = pe^t + q \). To obtain the moments of \( X \), we look at the derivatives of \( M_{n,p} \) at \( t = 0 \). The \( r \)-th derivatives of \( f \) and \( g \) are
\[
f^{(r)}(s) = (n)_r s^{n-r} \quad \text{and} \quad g^{(r)}(t) = pe^t
\]
for \( r \in \mathbb{N} \). Since \( g(0) = 1 \), it holds that \( f^{(r)}(g(0)) = (n)_r \). Furthermore, \( g^{(r)}(0) = p \) for all \( r \). We employ the Bell polynomial version of Faà di Bruno’s formula, see Johnson (2002) equation (2.2), which is
\[
(f \circ g)^{(r)}(t) = \sum_{k=1}^{r} f^{(k)}(g(t)) B_{r,k}(g^{(1)}(t), g^{(2)}(t), \ldots, g^{(r-k+1)}(t)).
\]  \( \text{(5.10)} \)

The Bell polynomials \( B_{r,k} \) are homogeneous of degree \( k \). Therefore,
\[
\mathbb{E}[X^r] = (f \circ g)^{(r)}(0) = \sum_{k=1}^{r} f^{(k)}(g(0)) B_{r,k}(g^{(1)}(0), g^{(2)}(0), \ldots, g^{(r-k+1)}(0))
\]
\[
= \sum_{k=1}^{r} B_{r,k}B(1, \ldots, 1) (n)_kp^k
\]
\[
\leq B_r \cdot \max\{np,(np)^r\},
\]
where \( B_r = \sum_{k=1}^{r} B_{r,k} \) is the \( r \)-th Bell number. \( \Box \)
Lemma 4. Let \( n \in \mathbb{N} \) and \( p \in (0, 1) \). Define \( k \) i.i.d. binomial random variables \( X_1, \ldots, X_k \) with distribution \( \text{Bin}(n, p) \). For each \( j \in \mathbb{N} \) with \( j \leq n \), the inequality

\[
P \left( \left| \frac{1}{k} \sum_{i=1}^{k} (X_i)_j - \mathbb{E}[(X_1)_j] \right| > \sqrt{\frac{l(cj)^j}{k}} \right) \leq \frac{1}{l}
\]

holds for any \( l > 0 \) and \( c \geq 2np (np + 2) \).

Proof. We define the random variable \( \tilde{X} \sim \text{Bin}(n - j, p) \) and note that \( \mathbb{E}[(X_i)_j] = (n)_j p^j \). Invoking Lemma 3, we derive the upper bound

\[
\text{Var}[(X_i)_j] \leq (n)_j p^j \mathbb{E}[(\tilde{X} + j)_j]
\]

\[
\leq (np)^2 \mathbb{E}[(\tilde{X} + j)^j]
\]

\[
\leq 2^j (np)^2 (\mathbb{E}[\tilde{X}^j] + j^j)
\]

\[
\leq 2^j (np)^2 (B_j (np + 1)^j + j^j)
\]

\[
\leq (2jn p(np + 2))^j
\]

on the variance of \( (X_i)_j \). The second inequality becomes transparent from expanding the expectation as a sum, and the last inequality is valid due to the relation \( B_j \leq j^j \) that can be found in Berend and Tassa (2010). For \( c \geq 2np (np + 2) \), we obtain by Chebyshev’s inequality that

\[
P \left( \left| \frac{1}{k} \sum_{i=1}^{k} (X_i)_j - \mathbb{E}[(X_1)_j] \right| > \sqrt{\frac{l(cj)^j}{k}} \right) \leq \frac{\text{Var}[(X_1)_j]/k}{l(cj)^j/k} \leq \frac{1}{l}.
\]

Lemma 5. Let \( k, n, s \in \mathbb{N} \), \( m > 0 \) and \( p \in (0, 1) \) such that \( k \geq 2 \) and \( km \geq s \). Let furthermore \( a \geq 0, b > 0 \) and define

\[
u_j = \frac{(s + a)_j}{(km + a + b - 1)_j}, \quad \tilde{u}_j = \frac{(kn p + a)_j}{(km + a + b - 1)_j}
\]

for \( j \in \mathbb{N} \) with \( j \leq s + a \). Then it holds that

\[
|s - knp| \leq \sqrt{\lambda k \log k} \implies |\nu_j - \tilde{u}_j| \leq j \sqrt{\frac{\lambda \log k}{k} \left( \frac{c}{m} \right)^j}
\]

for any \( \lambda \geq np \) and \( c \geq 2e^2(3\lambda + a + 1) \).
Proof. Let \( t = \sqrt{\lambda k \log k} \) and assume that \( |s - knp| = |t'| \leq t \). Then, by applying a telescoping sum, we find

\[
|u_j - \tilde{u}_j| = \frac{|(knp + t' + a)_j - (knp + a)_j|}{(km + a + b - 1)_j} \\
\leq \sum_{l=0}^{j-1} \frac{|knp + t' + a| \ldots |knp + t' + a - l| - (knp + a - l)| \ldots |knp + a - j + 1|}{(km + a + b - 1)_j} \\
\leq \sum_{l=0}^{j-1} \frac{(c_1 k)^{j-l} t}{(c_2 km)^l} \leq \frac{j t}{k} \left( \frac{c_1}{m} \right)^j.
\]

(5.11)

In the second inequality, we bound the numerator from above by noting that \( t \leq k\lambda \) and thus \( j \leq s + a \leq 2k\lambda + a \). Therefore,

\[
knp + t + a + 1 + j \leq (3\lambda + a + 1) k =: c_1 k.
\]

The denominator is bound from below by applying the first statement of Lemma 2, yielding

\[
(km + a + b - 1)_j \geq \left( (km - 1)/e^2 \right)^j \geq (c_2 km)^j
\]

(5.12)

for \( c_2 = 1/2e^2 \). In the final inequality of equation (5.11), \( c \) can be chosen as \( c_1/c_2 \) since \( c_1 > 1 \).

\[\square\]

Lemma 6. Let \( k, n \in \mathbb{N}, m > 0 \) and \( p \in (0, 1) \) such that \( k \geq 2 \), and let \( a \geq 0, b > 0 \). For any \( \lambda \geq np \) it holds that

\[
|\tilde{u}_j| := \left| \frac{(knp + a)_j}{(km + a + b - 1)_j} \right| \leq \left( \frac{c_1}{m} \right)^j
\]

if \( j \in \mathbb{N} \) with \( j \leq 2k\lambda + a \) and \( c_1 \geq 6e^2(\lambda + a) \). Furthermore, if \( j \leq m \), then

\[
\tilde{u}_j \leq \left( \frac{np}{m} \right)^j + \frac{j c_2^j}{k}
\]

for any \( c_2 \geq 3np + 2a + 2 \).

Proof. The first result follows from bounding the numerator of \( \tilde{u}_j \) from above by \( (3k(\lambda + a))^j \) and the denominator from below by equation (5.12) of the previous lemma. In case of \( j \leq m \) it holds that

\[
\frac{knp + a - i + 1}{km + a + b - i} - \frac{np}{m} = \frac{m(a - i + 1) - np(a + b - i)}{m(km + a + b - i)} \\
\leq \frac{inp + (a + 1)m}{m(km - i)} \\
\leq \frac{m(np + a + 1)}{m^2(k - 1)} \leq 2 \frac{np + a + 1}{k} =: \tilde{c}_2
\]

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for each $i = 1, \ldots, j$. This inequality yields the upper bound

$$\tilde{u}_j \leq \left( \frac{np}{m} + \frac{\tilde{c}_2}{k} \right)^j .$$

We then apply the relation

$$(x + y)^j \leq x^j + j y (x + y)^{j-1}$$

for $x, y > 0$, which can be obtained from expanding $(x + y)^j$ as binomial sum, and conclude

$$\tilde{u}_j \leq \left( \frac{np}{m} \right)^j + j \frac{\tilde{c}_2}{k} \left( \frac{np}{m} + \frac{\tilde{c}_2}{k} \right)^{j-1} \leq \left( \frac{np}{m} \right)^j + j c^j \frac{j}{k}$$

for $c_2 \geq \tilde{c}_2 + np$.  \hfill \square

**Lemma 7.** Let $n, k \in \mathbb{N}$, $m > 0$ and $p \in (0, 1)$ such that $k \geq 2$, and let $a \geq 0, b > 0$. Define

$$t_j = \frac{(n)_j p^j}{(m)_j} \quad \text{and} \quad \tilde{u}_j = \frac{(knp + a)_j}{(km + a + b - 1)_j} .$$

for $j \in \mathbb{N}$ with $1 < j \leq m$. If $n > m$, it holds that

$$t_j - \tilde{u}_j \geq \frac{(np)^j}{n} \frac{n - m}{nm + 1} - j c_j \frac{j}{k},$$

where $c_2 = 3np + 2a + 2$ from Lemma 6. If $n < m$, $j \leq knp + a$, and $n \leq \lambda \sqrt{k}$ for some $\lambda > 0$, we also have

$$t_j - \tilde{u}_j \leq -c \frac{m - n}{nm^3} \quad \text{and} \quad t_j - \tilde{u}_j \leq 0$$

for $k \geq (1 + 1/np)^2 (2 \lambda (a + b + 1))^4$ and $c \leq (np/(a + b + 1))^2/2$.

**Proof.** Applying the respective second statements of Lemma 6 and Lemma 2, we establish

$$t_j - \tilde{u}_j \geq \left[ \frac{(n)_j p^j}{(m)_j} - \left( \frac{np}{m} \right)^j \right] - j c_j \frac{j}{k} \geq \left( \frac{np}{m} \right)^j \frac{n - m}{nm} - j c_j \frac{j}{k}$$

for $n > m$, which shows the first result. For the second result, assume $m > n$. We look at the case $j = 2$ first. Direct calculation shows

$$t_2 - \tilde{u}_2 = \frac{n(n - 1)p^2}{m(m - 1)} - \frac{(knp + a)(knp + a - 1)}{(km + a + b - 1)(km + a + b - 2)}$$

$$\leq \frac{np}{m} \left( \frac{n - 1}{m - 1} p - \frac{1 - 1/knp}{(1 + (a + b)/km)^2} \right)$$

$$\leq -\frac{np}{nm} \frac{nm - 1}{m (m - 1) (1 + (a + b)/km)^2}$$

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with $\tilde{c} = (1 + np)(1 + a + b)^2$. Under the assumed conditions, the numerator of the last expression can be bounded by

$$np(m - n) - \frac{\tilde{c} nm}{k} = np(m - n) \left(1 - \frac{\tilde{c} nm}{np(m - n)^k}\right) \geq \frac{np}{2} (m - n)$$

for $k \geq k_0 := (4\tilde{c}\lambda^2/np)^2$. This follows from

$$\frac{\tilde{c} nm}{np(m - n)^k} \leq \frac{\tilde{c}}{np} \cdot \frac{2n}{2n^2} \text{ if } m > 2n$$

$$\frac{\tilde{c}}{np} \cdot \frac{2n^2}{m} \text{ if } m \leq 2n$$

(5.13)

if $n \leq \lambda\sqrt{k}$, and it implies that

$$t_2 - \tilde{u}_2 \leq -\frac{(np)^2}{2(a + b + 1)^2} \frac{m - n}{nm^3} \leq 0.$$ 

Finally, for $2 \leq j \leq knp + a$ and $k \geq k_0$ we can derive

$$\frac{t_j}{\tilde{u}_j} = \frac{t_2}{\tilde{u}_2} \prod_{i=2}^{j-1} \frac{p(n - i)}{m - i} \frac{km + a + b - 1 - i}{knp + a - i} = \frac{t_2}{\tilde{u}_2} \prod_{i=2}^{j-1} \frac{r_i}{v_i} \leq 1.$$ 

This statement is true due to $t_2/\tilde{u}_2 \leq 1$, and because $r_i \leq v_i$ is equivalent to

$$p(n - i)(a + b - 1 - i) - (m - i)(a - i) \leq ikp(m - n),$$

which follows from

$$np(a + b) + i(a + m) \leq i2\tilde{c}m \leq ikp(m - n).$$

The two inequalities hold because of the choice of $\tilde{c}$ and equation (5.13).

\textbf{Lemma 8.} Let $n \in \mathbb{N}$ and $p \in (0,1)$, and let $M_k$ denote the maximum of $k$ independent binomial variables $X_1, \ldots, X_k \sim \text{Bin}(n,p)$. Let $(l_k)_{k \in \mathbb{N}}$ be such that $l_k \to \infty$ and $l_k = o(\sqrt{\log(k)})$. Then, for each $k$ with $l_k > \max\{1, 4np\}$,

$$\mathbb{P}(M_k < \min\{l_k, n\}) \leq e^{-d_k},$$

where

$$d_k = \min \left\{ \frac{k}{e^{l_k \log(l_k/np)}}, \frac{knp}{8\pi l_k^2 e^{l_k/np}} \right\} \to +\infty \text{ as } k \to \infty.$$ 

\textbf{Proof.} We have that

$$\mathbb{P}(M_k < \min\{l_k, n\}) = \begin{cases} \mathbb{P}(M_k < n) & \text{if } n \leq l_k, \\ \mathbb{P}(M_k < l_k) & \text{if } n > l_k. \end{cases}$$

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In case of $l_k \geq n$, we derive the upper bound
\[
\log \mathbb{P}(M_k < n) \leq -kp^n \leq -k e^{-l_k \log(l_k/np)} \to -\infty, \text{ as } k \to \infty,
\]
by applying Bernoulli’s inequality and for $l_k = o(\sqrt{\log(k)})$. If $n > l_k$, we find that $p \leq 1/4$, and thus Slud’s bound from Telgarsky (2010) can be applied to yield
\[
\mathbb{P}(M_k < l_k) = (1 - \mathbb{P}(X_1 \geq l_k))^k
\leq \Phi \left(\frac{\sqrt{2l_k}}{\sqrt{np}}\right)^k \Phi \left(\frac{\sqrt{l_k}}{\sqrt{np}}\right),
\]
where $\Phi$ is the cumulative standard normal distribution function. By the lower tail bound in Gordon (1941), which states that $1 - \Phi(t) \geq \frac{1}{2\pi} t e^{-t^2/2}$ for $t > 0$, we obtain
\[
\Phi \left(\frac{\sqrt{2l_k}}{\sqrt{np}}\right)^k \leq \left(1 - \exp\left(-\frac{k \frac{np}{8\pi l_k^2} e^{-\frac{l_k^2}{np}}}{l_k^2 e^{l_k^2/np}}\right)\right) \to 0, \text{ as } k \to \infty,
\]
where we set $t = \sqrt{2l_k}/\sqrt{np} > 1$ (by assumption) and used $t/(t^2 + 1) \geq 1/2t^2$ for $t \geq 1$ and $l_k = o(\sqrt{\log(k)})$.

**Lemma 9.** Let $n \in \mathbb{N}$ and $p \in (0, 1)$. Define $k$ i.i.d. binomial random variables $X_1, \ldots, X_k$ with distribution $\text{Bin}(n, p)$, and let $M_k := \max_{i=1,\ldots,k} X_i$. Then
\[
\mathbb{P}(M_k \leq 2 \log(k)) \geq \left(1 - \frac{1}{k^2}\right)^k
\]
if $k \geq e^{3np}$. Consequently, $\mathbb{P}(M_k > 2 \log(k)) \to 0$, as $k \to \infty$.

**Proof.** We can write $X_1$ as a sum of $n$ i.i.d. Bernoulli random variables bounded by 1. By Bernstein’s inequality (see e.g., van der Vaart and Wellner (1996))
\[
\mathbb{P}(M_k \leq 2 \log(k)) = (1 - \mathbb{P}(X_1 - np > 2 \log(k) - np))^k
\geq \left(1 - \exp\left(-\frac{(2 \log(k) - np)^2}{2(np(1 - p) + \log(k)/3)}\right)\right)^k
\geq \left(1 - e^{-2 \log(k)}\right)^k,
\]
where the last inequality holds for $\log(k) \geq 3np$. \qed
Lemma 10. Let \((n_k, p_k)_{k \in \mathbb{N}}\) be a sequences with \(n_k \in \mathbb{N}\), \(p_k \in (0,1)\) and \(n_k p_k \to \mu > 0\). Define the independent random variables \(X_1, \ldots, X_k \sim \text{Bin}(n_k, p_k)\) and let \(M_k := \max_{i=1,\ldots,k} X_i\).

(i) If \(n_k \log(n_k) < c \log(k)\) for \(c < 1\), then \(\mathbb{P}(M_k = n_k) \to 1\), as \(k \to \infty\).

(ii) If \(n_k \log(n_k) > c \log(k)\) for \(c > 1\), then \(\mathbb{P}(M_k = n_k) \to 0\), as \(k \to \infty\).

Proof. (i): We have convergence of the sample maximum towards the parameter \(n_k\) if

\[
\mathbb{P}(M_k = n_k) = 1 - (1 - p_k^{n_k})^k \geq 1 - e^{-kp_k^{n_k}} \to 1, \text{ as } k \to \infty,
\]

where we applied Bernoulli’s inequality. This holds if \(\log(k) - n_k \log(n_k/n_k p_k) \to \infty\), which follows from

\[
\frac{n_k \log(n_k)}{\log(k)} < c < 1 \quad \text{and} \quad \frac{n_k |\log(n_k p_k)|}{\log(c \log(k))} \leq \frac{c |\log(n_k p_k)|}{\log(c \log(k))}.
\]

(ii): It holds that \(P(M_k = n_k) \leq kp_k^{n_k} \leq \exp\left(\log(k) - n_k \log(n_k/n_k p_k)\right)\). Similar to the argument above, the right hand side in this inequality converges to 0 since

\[
\frac{n_k \log(n_k)}{\log(k)} > c > 1 \quad \text{and} \quad \frac{n_k |\log(n_k p_k)|}{\log(k)} \leq \frac{c |\log(n_k p_k)|}{\log(c \log(k))}.
\]

\(\square\)

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