Tent-transformed lattice rules for integration and approximation of multivariate non-periodic functions

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Abstract

We develop algorithms for multivariate integration and approximation in the weighted half-period cosine space of smooth non-periodic functions. We use specially constructed tent-transformed rank-1 lattice points as cubature nodes for integration and as sampling points for approximation. For both integration and approximation, we study the connection between the worst-case errors of our algorithms in the cosine space and the worst-case errors of some related algorithms in the well-known weighted Korobov space of smooth periodic functions. By exploiting this connection, we are able to obtain constructive worst-case error bounds with good convergence rates for the cosine space.

Keywords: Quasi-Monte Carlo methods, Cosine series, Function approximation, Hyperbolic crosses, Rank-1 lattice rules, Spectral methods, Component-by-component construction.

Subject Classification: 65D30, 65D32, 65C05, 65M70 65T40

1 Introduction

In this paper we consider multivariate integration and approximation in the weighted half-period cosine space. We use tent-transformed rank-1 lattice points as cubature nodes for integration and as sampling points for approximation. Lattice rules have been widely studied in the context of multivariate integration, see [5, 24, 28]. Rank-1 lattice point sets are completely described by the number of points $n$ and an integer generating vector $z$, which can be constructed by an algorithm that searches for its elements component by component, see e.g., [6, 15, 25, 26, 29, 30, 31].

We will focus on the non-periodic setting and, as in [7], we will use the half-period cosine space spanned by the cosine series. Cosine series are used for the expansion of non-periodic functions in the $d$-dimensional unit cube. They are the eigenfunctions of the Laplace differential operator with homogeneous Neumann boundary conditions. The half-period cosine functions form a set of orthonormal basis functions of $L_2([0,1])$ and are given by

$$
\phi_0(x) = 1, \quad \text{and} \quad \phi_k(x) = \sqrt{2} \cos(\pi k x) \quad \text{for } k \in \mathbb{N}.
$$

In $d$ dimensions we will use the tensor products of these functions

\begin{equation}
\phi_k(x) := \prod_{j=1}^{d} \phi_{k_j}(x_j) = \sqrt{2^{|k|}} \prod_{j=1}^{d} \cos(\pi k_j x_j),
\end{equation}

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where we denote by $|k|_0$ the number of non-zero elements of $k \in \mathbb{Z}_d^+$, with

$$\mathbb{Z}_+ := \{0, 1, 2, \ldots\}.$$ 

The cosine series expansion of a $d$-variate function $f \in L_2([0,1]^d)$ converges to $f$ in the $L_2$ norm. Additionally, if $f$ is continuously differentiable, we have uniform convergence and $f$ can be expressed as a cosine series expansion as follows, see [1, 10]:

$$f(x) = \sum_{k \in \mathbb{Z}_d^+} \hat{f}(k) \phi_k(x),$$

where $\hat{f}(k)$ are the cosine coefficients of $f$ and are obtained as follows

$$\hat{f}(k) = \int_{[0,1]^d} f(x) \phi_k(x) \, dx.$$ 

Cosine series overcome the well-known Gibbs phenomenon, which traditional Fourier series face in the expansion of non-periodic functions. Cosine series and the spectral methods using them have been studied in depth in [1, 10] and their successors.

The precise definition of the weighted half-period cosine space will be presented in Section 2. For now we mention only that there is a parameter $\alpha > 1/2$ which characterizes the smoothness of the space by controlling the decay of the cosine coefficients, and there is a sequence of weights $1 \geq \gamma_1 \geq \gamma_2 > \cdots > 0$ which models the relative importance between successive coordinate directions.

We will first look at the problem of multivariate integration, where we will use tent-transformed lattice points as cubature nodes. Lattice rules have traditionally been used for the integration of smooth periodic functions. In the Korobov space of smooth periodic functions, it is known that lattice rules with well-chosen generating vectors can achieve the (almost optimal) rate of convergence of $O(n^{-\alpha+\delta})$, for any $\delta > 0$, see, e.g., [6, 15]. Moreover, the result for the case $\alpha = 1$ can be used to prove that randomly-shifted lattice rules can achieve the (almost optimal) rate of convergence of $O(n^{-1+\delta})$ for $\delta > 0$ in the Sobolev spaces of non-periodic functions of dominating mixed smoothness 1. Tent-transformed lattice rules were first used to integrate non-periodic functions in [8], in the setting of unanchored Sobolev spaces of dominating mixed smoothness 1 and 2. It was shown there that when the lattice points are first randomly shifted and then tent-transformed (called bakers’ transform in [8]), they can achieve the convergence rates of $O(n^{-1+\delta})$ and $O(n^{-2+\delta})$, $\delta > 0$, in the Sobolev spaces of smoothness 1 and 2, respectively.

In [7], tent-transformed lattice points were studied for integration in the weighted half-period cosine space without random shifting. It was claimed there that the worst-case error in the cosine space for a tent-transformed lattice rule is the same as the worst-case error in the weighted Korobov space of smooth periodic functions using lattice rules, given the same set of weights $\gamma_j$ and the smoothness parameter $\alpha$. The argument was based on achieving equality in a Cauchy–Schwarz type error bound, however the authors did not realise that this equality is not always possible in this setting. In this paper we correct this by showing that the worst-case error in the Korobov space is in fact an upper bound to the worst-case error in the cosine space and we provide an expression for the scaling factor involved. We also conclude that, with an appropriate rescaling of the weights $\gamma_j$, all the results for integration in Korobov spaces using lattice rules, e.g., [6, 15, 25], also apply to integration in the cosine space using tent-transformed randomly-shifted lattice rules (first randomly shifted and then tent-transformed). Note additionally that the cosine space of smoothness 1 coincides with the unanchored Sobolev space of smoothness 1, see [7]. Thus our results apply to the unanchored Sobolev space of smoothness 1 as well.
The second part of our paper deals with the approximation of non-periodic functions $f: [0,1]^d \to \mathbb{C}$ where the number of variables $d$ is large. Lattice rules have already been used for approximation in weighted Korobov spaces, e.g., in [16] and [18] in the $L_2$ and $L_\infty$ settings, respectively. The use of lattice points for the approximation of periodic functions was also suggested much earlier in [14] and in papers cited there, see also [35, 33, 34]. Lattice points were also used in [19] for a spectral collocation method with Fourier basis where samples of the input function at lattice points were used to approximate the solution of PDEs such as the Poisson equation in $d$ dimensions. The paper [11] also presented an approach for stably reconstructing multivariate trigonometric polynomials (periodic) that have support on a hyperbolic cross, by sampling them on rank-1 lattices. More advances on the topic of reconstruction of trigonometric polynomials using rank-1 lattice sampling can be found, e.g., in [12, 13, 27].

Our study is for non-periodic functions belonging to weighted cosine spaces. In [32], collocation and reconstruction problems were extended to non-periodic functions in the cosine space using tent-transformed lattice point sets; however, that paper did not include error analysis. Multivariate cosine expansions have also been studied alongside hyperbolic cross approximations in [1] and [2]. In [1], however, it was assumed that the error in approximating the cosine coefficients is negligible. We fill this gap by giving a detailed analysis of the error components. We first find the expression of the error for the algorithm using $n$ function values at tent-transformed lattice points for an arbitrary generating vector $z$. We then show that an upper bound for the worst-case error of our algorithm in the cosine space using tent-transformed lattice points is the same as an upper bound presented in [16] for a related algorithm using lattice points in the Korobov space. We can hence inherit all the error bounds as well as the construction algorithms. In [4], it is shown that the convergence rate of rank-1 lattice points for function approximation in the periodic Sobolev space of hybrid mixed smoothness $\alpha$ is $\alpha/2$. This is only half of the optimal rate, achieved for instance by sparse grid sampling. However, as mentioned in [4], rank-1 lattice point sets are still a convenient choice for a number of reasons. The computations in higher dimensions can be reduced to one-dimensional FFT and IFFT. Also, after applying the tent transformation, which is computationally very inexpensive, these point sets become suitable for the non-periodic setting immediately.

We now summarize the content of this paper. In Section 2 we define the weighted cosine space and related function spaces, as well as rank-1 lattice and tent-transformed rank-1 lattice point sets. In Sections 3 and 4 we focus on the problems of integration and approximation, respectively. In both sections, we derive the worst-case errors for our algorithms based on tent-transformed lattice point sets, and relate these errors to those of the Korobov space to obtain results on the construction algorithms and convergence results. Finally, Section 5 provides some concluding remarks.

2 Problem setting

We want to integrate and approximate functions belonging to some weighted $\alpha$-smooth half-period cosine space (henceforth we refer to it as the “cosine space” to be concise) of complex-valued functions, given by

$$C_{d,\alpha,\gamma} := \left\{ f \in L_2([0,1]^d) : \|f\|_{C_{d,\alpha,\gamma}}^2 := \sum_{k \in \mathbb{Z}_d^d} |\hat{f}(k)|^2 r_{\alpha,\gamma}(k) < \infty \right\},$$
where $\alpha > 1/2$ is a smoothness parameter and $\gamma = (\gamma_1, \gamma_2, \ldots)$ is a sequence of weights satisfying $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0$, and where we define

$$r_{\alpha,\gamma}(k) := \prod_{j=1}^{d} r_{\alpha,\gamma_j}(k_j), \quad \text{with} \quad r_{\alpha,\gamma_j}(k) := \begin{cases} 1 & \text{if } k = 0, \\ \frac{|k_j|^{2\alpha}}{\gamma_j} & \text{if } k \neq 0. \end{cases}$$

Here we assume that successive variables have diminishing importance, with each weight $\gamma_j$ moderating the behavior of the $j$th variable. If all $\gamma_j = 1$, we have the unweighted space where all variables are equally important. If, however, $\gamma_j$ is small then the dependence on the $j$th variable is weak. The smoothness parameter $\alpha$ controls the decay of spectral coefficients, measured in the $L_2$ sense. For $\alpha > 1/2$, the cosine space is a reproducing kernel Hilbert space, with the reproducing kernel

$$K_{d,\alpha,\gamma}(x, y) := \sum_{k \in \mathbb{Z}^d_{+}} \frac{\phi_k(x) \phi_k(y)}{r_{\alpha,\gamma}(k)} = \sum_{k \in \mathbb{Z}^d_{+}} \frac{2^{|k|_0}}{r_{\alpha,\gamma}(k)} \prod_{j=1}^{d} \cos(\pi k_j x_j) \cos(\pi k_j y_j)$$

$$= \prod_{j=1}^{d} \left( 1 + 2\gamma_j \sum_{k=1}^{\infty} \frac{\cos(\pi k x_j) \cos(\pi k y_j)}{k^{2\alpha}} \right), \quad x, y \in [0, 1]^d.$$  

Recall that the reproducing kernel satisfies $K_{d,\alpha,\gamma}(x, y) \in C_d$ for all $y \in [0, 1]^d$ as well as the reproducing property $\langle f, K_{d,\alpha,\gamma}(x, y) \rangle_{C_d} = f(y)$ for all $y \in [0, 1]^d$ and all $f \in C_d$, where the inner product is defined by $\langle f, g \rangle_{C_d} := \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \hat{g}(k) r_{\alpha,\gamma}(k)$. (For brevity we have omitted some parameters $\alpha$ and $\gamma$ from our notation in the discussion above.)

When $\alpha = 1$, it is proved in [7] that the cosine space coincides with the unanchored Sobolev space of dominated mixed smoothness 1. For this space the norm is given by

$$\|f\|_{C_{d,1,\gamma}}^2 = \sum_{u \subseteq \{1, \ldots, d\}} \prod_{j \in u} \gamma_j^{-1} \int_{[0,1]^u} \left| \frac{\partial^{[u]} f}{\partial x_u} \right| f d\mathbf{x}_{(1,\ldots,d)\setminus u} d\mathbf{x}_u,$$

where $x_u = (x_j)_{j \in u}$ and $\partial^{[u]} f / \partial x_u$ denotes the mixed first derivatives of $f$ with respect to the variables $x_j$ with $j \in u$, and the reproducing kernel is

$$K_{d,1,\gamma}(x, y) = \prod_{j=1}^{d} \left( 1 + \gamma_j B_1(x_j) B_1(y_j) + \gamma_j \frac{B_2(|x_j - y_j|)}{2} \right), \quad x, y \in [0, 1]^d,$$

where $B_1(x) = x - 1/2$ and $B_2(x) = x^2 - x + 1/6$ are the Bernoulli polynomials of degrees 1 and 2 respectively.

Another function space closely related to the cosine space is the weighted Korobov space of periodic functions defined by

$$E_{d,\alpha,\gamma} := \left\{ f \in L_2([0,1]^d) : \|f\|_{L_2,\alpha,\gamma}^2 := \sum_{h \in \mathbb{Z}^d} |\hat{f}(h)|^2 r_{\alpha,\gamma}(h) < \infty \right\},$$

which, instead of the cosine coefficients, makes use of the Fourier coefficients of $f$ given by

$$\hat{f}(h) := \int_{[0,1]^d} f(x) \exp(-2\pi i h \cdot x) d\mathbf{x} \quad \text{for} \quad h \in \mathbb{Z}^d.$$
(Note that the cosine coefficients are marked with a hat and the Fourier coefficients are marked with a tilde.) Here the smoothness parameter $\alpha > 1/2$ and the weights $\gamma = (\gamma_1, \gamma_2, \ldots)$ have analogous interpretations as in the cosine space. The reproducing kernel is

$$K_{d,\alpha,\gamma}^{\text{per}}(x, y) := \sum_{h \in \mathbb{Z}^d} \frac{\exp(2\pi i h \cdot (x - y))}{r_{\alpha,\gamma}(h)} = \prod_{j=1}^d \left( 1 + 2\gamma_j \sum_{h=1}^{\infty} \frac{\cos(2\pi h(x_j - y_j))}{h^{2\alpha}} \right) x, y \in [0,1]^d. \quad (3)$$

We remark that in many earlier papers the definition of the Korobov space has $2\alpha$ instead of $\alpha$ as the smoothness parameter, and therefore care must be taken when quoting results from these papers.

In this paper we study multivariate integration and approximation in the cosine space using “tent-transformed lattice rules”. For a given $n \in \mathbb{N}$ and $z \in \mathbb{Z}_n^d$ where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, a rank-1 lattice point set $\Lambda(z, n)$ is given by

$$\Lambda(z, n) := \left\{ \frac{iz}{n} \mod 1 : i = 1, 2, \ldots, n \right\}, \quad (4)$$

and $z$ is called the generating vector. The tent transformation $\psi : [0,1] \to [0,1]$, is given by

$$\psi(x) := 1 - |2x - 1|, \quad x \in [0,1],$$

and we write $\psi(x) := (\psi(x_1), \psi(x_2), \ldots, \psi(x_d))$ to denote a tent-transformed point $x \in [0,1]^d$, where the transformation $\psi$ is applied component-wise to all coordinates in $x$. We obtain a tent-transformed point multiset $\Lambda_{\psi}(z, n)$ by applying the tent transformation component-wise to all the points of the rank-1 point set $\Lambda(z, n)$, that is,

$$\Lambda_{\psi}(z, n) := \left\{ \psi\left( \frac{iz}{n} \mod 1 \right) : i = 1, 2, \ldots, n \right\}. \quad (6)$$

We may also consider a shifted point set, and a tent-transformed shifted point multiset (the points are first shifted and then tent-transformed), that is, given a shift $\Delta \in [0,1]^d$ we define

$$\Lambda(z, \Delta, n) := \left\{ \left( \frac{iz}{n} + \Delta \right) \mod 1 : i = 1, 2, \ldots, n \right\},$$

$$\Lambda_{\psi}(z, \Delta, n) := \left\{ \psi\left( \frac{iz}{n} + \Delta \right) \mod 1 : i = 1, 2, \ldots, n \right\}. \quad (6)$$

In the forthcoming sections, we will provide more details specific to the problems of integration and approximation.

3 Integration

We first study multivariate integration

$$\text{INT}_d(f) := \int_{[0,1]^d} f(x) \, dx \quad (7)$$

for functions $f$ from the cosine space $C_{d,\alpha,\gamma}$. We will approximate the integral (7) by some weighted cubature rule

$$Q_n(f) := \sum_{i=1}^n w_i f(t_i),$$

and...
where \(d, n \in \mathbb{N}, t_1, \ldots, t_n \in [0, 1]^d\) are the sampling points, and \(w_1, \ldots, w_n \in \mathbb{R}\) are the cubature weights.

A lattice rule is a cubature rule which uses points from a lattice \(\Lambda(z, n)\), see (4), with equal weights \(w_i = 1/n\), and we will denote its application to a function \(f\) by \(Q_n(f; z)\). Likewise, a tent-transformed lattice rule uses points from the tent-transformed point multiset \(\Lambda_{\psi}(z, n)\), see (6), again with equal weights \(1/n\), and we will denote it by \(Q_n(f \circ \psi; z)\). Note that transforming the input argument to a function is equivalent to transforming the function itself, i.e., \(f(\psi(x)) = (f \circ \psi)(x)\), hence our notation \(Q_n(f \circ \psi; z)\).

Analogously, we denote a shifted lattice rule by \(Q_n(f; z, \Delta)\), and a tent-transformed shifted lattice rule by \(Q_n(f \circ \psi; z, \Delta)\). If the shift \(\Delta\) is generated randomly from the uniform distribution on \([0, 1]^d\), then we denote the corresponding randomized methods by \(Q_n^{\text{ran}}(f; z)\) and \(Q_n^{\text{ran}}(f \circ \psi; z)\), respectively.

The set of indices for those Fourier frequencies that are not integrated exactly by the lattice rule, together with the index 0, is called the dual of the lattice and is given by
\[
\Lambda(z, n)\perp := \{h \in \mathbb{Z}^d : h \cdot z \equiv 0 \pmod{n}\}.
\]

More precisely, from [20, Lemma 5.21], we have
\[
\frac{1}{n} \sum_{t \in \Lambda(z, n)} \exp(2\pi i h \cdot t) = \begin{cases} 1 & \text{if } h \in \Lambda(z, n)\perp, \\ 0 & \text{otherwise}. \end{cases} \tag{8}
\]

We will make use of this property in our analysis below.

In general, if \(K\) is the reproducing kernel of some reproducing kernel Hilbert space \(H_d\) of functions on \([0, 1]^d\), then the squared worst-case error of \(Q_n\) is given by (see, e.g., [9])
\[
e_{\text{wor}}^2(Q_n; H_d) := \left( \sup_{f \in H_d, \|f\|_{H_d} \leq 1} \|\text{INT}_d(f) - Q_n(f)\| \right)^2
= \int_{[0,1]^d} K(x, y) \, dx \, dy - 2 \sum_{i=1}^n w_i \int_{[0,1]^d} K(x, t_i) \, dx + \sum_{i, i' = 1}^n w_i w_i' K(t_i, t_i'). \tag{9}
\]

In Subsections 3.2 and 3.3 below, we will make use of this formula to derive and analyse the worst-case error for a tent-transformed lattice rule and the root-mean-squared worst-case error for a tent-transformed randomly-shifted lattice rule. We further need the following lemma and an identity following the lemma.

**Lemma 1.** Let \(\psi(x)\) be the tent transform function as in (5). For any \(k \in \mathbb{Z}_+^d\) and the corresponding basis function \(\phi_k\) as in (1), we have
\[
\phi_k(\psi(x)) = (\sqrt{2} |k|_0^d \sum_{j=1}^d \cos(2\pi k_j x_j) = \frac{\sqrt{2} |k|_0^d }{2^d} \sum_{\sigma \in \{\pm 1\}^d} \exp(2\pi i \sigma(k) \cdot x), \tag{10}
\]
where \(\sigma \in \{\pm 1\}^d\) are sign combinations and \(\sigma(k)\) denotes the application of these signs on the indices of \(k\) element-wise, i.e., \(\sigma(k) = (\sigma_1 k_1, \ldots, \sigma_d k_d)\), and as before \(|k|_0\) denotes the number of non-zero elements in \(k\).

**Proof.** It is trivial to verify that for \(k \in \mathbb{Z}_+\) we have \(\cos(\pi k \psi(x)) = \cos(2\pi k x)\). This yields the first equality in (10). Next we write
\[
\prod_{j=1}^d \cos(2\pi k_j x_j) = \frac{1}{2^d} \prod_{j=1}^d (\exp(2\pi i k_j x_j) + \exp(-2\pi i k_j x_j)).
\]
Expanding the product then yields the second equality in (10). \hfill ■

We will repeatedly use the following identity: for any two functions \( G_1, G_2 : \mathbb{Z}^d \to \mathbb{C} \),

\[
\sum_{k \in \mathbb{Z}_d^+} \left( G_1(k) \sum_{\sigma \in \{\pm 1\}^d} G_2(\sigma(k)) \right) = \sum_{k \in \mathbb{Z}^d} G_1(|k|) G_2(k) 2^{d-|k|},
\]

where \(|k|\) indicates that the absolute value function is applied component-wise to the vector.

### 3.1 Lower bound

A lower bound for the worst-case error for integration in the cosine space is known from [7], and is given in the following theorem.

**Theorem 1.** For arbitrary points \( t_1, \ldots, t_n \in [0,1]^d \) and weights \( w_1, \ldots, w_n \in \mathbb{R} \), we have

\[
e_{\text{wor}}(Q_n; C_{d,\alpha,\gamma}) \geq c(d, \alpha, \gamma) \left( \frac{\log n}{n^a} \right),
\]

where \( c(d, \alpha, \gamma) > 0 \) depends on \( d, \alpha \), and \( \gamma \), but not on \( n \), the points \( t_1, \ldots, t_n \), or the weights \( w_1, \ldots, w_n \).

### 3.2 Upper bound for tent-transformed lattice rules

The following theorem gives the formula for the worst-case integration error for a tent-transformed lattice rule in the cosine space.

**Theorem 2.** The squared worst-case error for a tent-transformed lattice rule in the cosine space is given by

\[
e_{\text{wor}}(Q_n(\cdot \circ \psi; z); C_{d,\alpha,\gamma})^2 = \sum_{0 \neq k \in \Lambda(z,n)^+} \frac{1}{r_{\alpha,\gamma}(k)} \left( \frac{1}{2^d} \sum_{\sigma \in \{\pm 1\}^d} \mathbb{1}_{\sigma(k) \in \Lambda(z,n)^+} \right).
\]

**Proof.** Using (9) and (2), and then applying (10), (8), (11) in turn, we obtain

\[
e_{\text{wor}}(Q_n(\cdot \circ \psi; z); C_{d,\alpha,\gamma})^2 = -1 + \frac{1}{n^2} \sum_{t, t' \in \Lambda(z,n)} \sum_{k \in \mathbb{Z}_d^+} \frac{\phi_k(\psi(t)) \phi_k(\psi(t'))}{r_{\alpha,\gamma}(k)}
\]

\[
\times \left( \frac{1}{2^d} \sum_{\sigma \in \{\pm 1\}^d} \exp(2\pi i \sigma(k) \cdot t) \right) \left( \frac{1}{2^d} \sum_{\sigma' \in \{\pm 1\}^d} \exp(2\pi i \sigma'(k) \cdot t') \right)
\]

\[
= -1 + \sum_{k \in \mathbb{Z}_d^+} \frac{2^{|k|}}{r_{\alpha,\gamma}(k)} \left( \frac{1}{2^d} \sum_{\sigma \in \{\pm 1\}^d} \mathbb{1}_{\sigma(k) \in \Lambda(z,n)^+} \right) \left( \frac{1}{2^d} \sum_{\sigma' \in \{\pm 1\}^d} \mathbb{1}_{\sigma'(k) \in \Lambda(z,n)^+} \right)
\]

\[
= -1 + \sum_{k \in \mathbb{Z}_d^+} \frac{1}{r_{\alpha,\gamma}(|k|)} \mathbb{1}_{k \in \Lambda(z,n)^+} \left( \frac{1}{2^d} \sum_{\sigma \in \{\pm 1\}^d} \mathbb{1}_{\sigma(|k|) \in \Lambda(z,n)^+} \right),
\]

which yields (12). \hfill ■
In comparison, the squared worst-case error for a lattice rule in the Korobov space is (see, e.g., [6, 15])

\[ e_{\text{wor}}(Q_n(\cdot; z); E_{d,\alpha,\gamma})^2 = \sum_{0 \neq k \in \Lambda(z,n)} \frac{1}{r_{\alpha,\gamma}(k)} \]  

(13)

Clearly (13) is an upper bound for (12), since the formula (12) involves an additional factor which is always \( \leq 1 \). This was not recognized in [7]. Nevertheless, it is true that one may borrow the result from the Korobov space for the cosine space. We formalize this conclusion in the corollary below. For simplicity we state the result only for a prime \( n \), but a similar result for general \( n \) is also known, see [6, 15, 25, 26].

**Corollary 1.** A fast component-by-component algorithm can be used to obtain a generating vector \( z \in \mathbb{Z}_n^d \) in \( O(dn \log n) \) operations, using the squared worst-case error for a lattice rule in the Korobov space \( E_{d,\alpha,\gamma} \) as the search criterion, such that the worst-case error for the resulting tent-transformed lattice rule in the cosine space \( C_{d,\alpha,\gamma} \) satisfies

\[ e_{\text{wor}}(Q_n(\cdot; z); C_{d,\alpha,\gamma}) \leq e_{\text{wor}}(Q_n(\cdot; z); E_{d,\alpha,\gamma}) \leq \left( \frac{1}{n-1} \left( \prod_{j=1}^{d} \left( 1 + 2\zeta(2\alpha)\gamma_j \right) \right) - 1 \right)^{1/(2\lambda)} \]

for all \( 1/(2\alpha) < \lambda \leq 1 \), where \( \zeta(x) = \sum_{k=1}^{\infty} k^{-x} \) is the Riemann zeta function. Hence, the convergence rate is \( O(n^{-1/(2\lambda)}) \), with the implied constant independent of \( d \) if \( \sum_{j=1}^{\infty} \gamma_j < \infty \). As \( \lambda \to 1/(2\alpha) \), the method achieves the optimal rate of convergence close to \( O(n^{-\frac{1}{2}}) \).

Ideally we would like to be able to perform the fast component-by-component algorithm using the formula (12) as the search criterion directly, rather than using its upper bound (13). However, to do this we must identify a strategy to handle the evaluation of the sum over all sign changes which is of order \( 2^d \). This is left for future research.

We end this subsection by providing another insight into why the error (12) in the cosine space is smaller than the error (13) in the Korobov space. From (9), we can derive that the worst-case error of a tent-transformed lattice rule in the cosine space is the same as the worst-case error of the lattice rule in the *tent-transformed cosine space*, which is a reproducing kernel Hilbert space with kernel

\[ K_{d,\alpha,\gamma}^{\text{tr}}(x, y) := K_{d,\alpha,\gamma}(\psi(x), \psi(y)) = \sum_{k \in \mathbb{Z}_n^d} \frac{2|k|_0}{r_{\alpha,\gamma}(k)} \prod_{j=1}^{d} \cos(2\pi k_j x_j) \cos(2\pi k_j y_j). \]  

(14)

Indeed, we have

\[ e_{\text{wor}}(Q_n(\cdot; \psi; z); C_{d,\alpha,\gamma})^2 = -1 + \frac{1}{n^2} \sum_{t, t' \in \Lambda(z,n)} K_{d,\alpha,\gamma}(t, t') \]

\[ = -1 + \frac{1}{n^2} \sum_{t, t' \in \Lambda(z,n)} K_{d,\alpha,\gamma}(\psi(t), \psi(t')). \]

It can be shown that the kernel of the tent-transformed cosine space is smaller than the kernel of the Korobov space, i.e., \( K_{d,\alpha,\gamma}^{\text{tr}}(x, y) - K_{d,\alpha,\gamma}^\psi(x, y) \) is positive definite. From the theory of reproducing kernels [3], we then know that the tent-transformed cosine space is a subspace of the Korobov space, and hence the worst-case error of a lattice rule in the tent-transformed cosine space is at most its worst-case error in the Korobov space.
3.3 Upper bound for tent-transformed randomly-shifted lattice rules

We now consider the randomized method \( Q_n^\text{ran}(f \circ \psi; z) \). Recall that in a tent-transformed shifted lattice rule \( Q_n(f \circ \psi; z, \Delta) \) we first shift the lattice point set and then apply the tent transformation. In the randomized method the shift \( \Delta \) is generated randomly from the uniform distribution on \([0, 1]^d\). To show the existence of good shifts \( \Delta \), we analyze the root-mean-squared worst-case error defined by

\[
 e_{\text{rms}}^{\text{wor}}(Q_n^\text{ran}(\cdot \circ \psi; z); C_{d,\alpha,\gamma}) := \left( \int_{[0,1]^d} e_{\text{rms}}^{\text{wor}}(Q_n(\cdot \circ \psi; z, \Delta); C_{d,\alpha,\gamma}) \, d\Delta \right)^{1/2}.
\]

From [8] we know that

\[
e_{\text{rms}}^{\text{wor}}(Q_n^\text{ran}(\cdot \circ \psi; z); C_{d,\alpha,\gamma})^2 = -1 + \frac{1}{n^d} \sum_{t, t' \in \Lambda(z, n)} K_{d,\alpha,\gamma}^{sh}(t, t'),
\]

where \( K_{d,\alpha,\gamma}^{sh} \) is the shift-invariant tent-transformed kernel associated with \( K_{d,\alpha,\gamma} \), given by

\[
 K_{d,\alpha,\gamma}^{sh}(x, y) := \int_{[0,1]^d} K_{d,\alpha,\gamma}(\psi(x + \Delta), \psi(y + \Delta)) \, d\Delta, \quad x, y \in [0,1]^d.
\]

**Theorem 3.** The shift-invariant tent-transformed kernel defined in (16) can be written as

\[
 K_{d,\alpha,\gamma}^{sh}(x, y) = \sum_{k \in \mathbb{Z}^d} \frac{2^{-|k_0|}}{r_{\alpha,\gamma}(k)} \exp(2\pi i k \cdot (x - y)) = K_{d,\alpha,\gamma/2}^\text{per}(x, y), \quad x, y \in [0,1]^d.
\]

That is, it is precisely the kernel for the Korobov space with weights \( \gamma \) replaced by \( \gamma/2 \).

**Proof.** Starting from (16) and (14), we have

\[
 K_{d,\alpha,\gamma}^{sh}(x, y) = \int_{[0,1]^d} \sum_{k \in \mathbb{Z}^d} \frac{2^{d-|k_0|}}{r_{\alpha,\gamma}(k)} \prod_{j=1}^d \cos(2\pi k_j (x_j + \Delta_j)) \cos(2\pi k_j (y_j + \Delta_j)) \, d\Delta
\]

\[
 = \sum_{k \in \mathbb{Z}^d} \frac{2^{d-|k_0|}}{r_{\alpha,\gamma}(k)} \prod_{j=1}^d \left( \int_0^1 \cos(2\pi k_j (x_j + \Delta_j)) \cos(2\pi k_j (y_j + \Delta_j)) \, d\Delta_j \right)
\]

\[
 = \sum_{k \in \mathbb{Z}^d} \frac{1}{r_{\alpha,\gamma}(k)} \prod_{j=1}^d \cos(2\pi k_j (x_j - y_j))
\]

\[
 = \sum_{k \in \mathbb{Z}^d} \frac{1}{r_{\alpha,\gamma}(k)} \left( \frac{1}{2^d} \sum_{\sigma \in \{\pm\}^d} \exp(2\pi i \sigma(k) \cdot (x - y)) \right).
\]

Applying the identity (11) then yields the first equality in (17). The second equality in (17) follows immediately by a comparison with the formula (3), noting that \( 2^{-|k_0|} r_{\alpha,\gamma}(k) = r_{\alpha,\gamma/2}(k) \) for all \( k \in \mathbb{Z}^d \).

**Theorem 4.** The root-mean-squared worst-case error for a tent-transformed randomly-shifted lattice rule in the cosine space is given by

\[
e_{\text{rms}}^{\text{wor}}(Q_n^\text{ran}(\cdot \circ \psi; z); C_{d,\alpha,\gamma})^2 = \sum_{0 \neq k \in \Lambda(z, n)^d} \frac{2^{-|k_0|}}{r_{\alpha,\gamma}(k)} = e_{\text{rms}}^{\text{wor}}(Q_n(\cdot; z); E_{d,\alpha,\gamma/2})^2.
\]
That is, it is precisely the squared worst-case error of the lattice rule in the Korobov space with weights $\gamma$ replaced by $\gamma/2$.

**Proof.** The first equality in (18) follows by combining (15) with (17) and using (8). The second equality in (18) then follows immediately by comparison with the formula (13), noting again that $2^{k_0} r_{\alpha,\gamma}(k) = r_{\alpha,\gamma/2}(k)$ for all $k \in \mathbb{Z}^d$.

Due to the precise connection with the Korobov space, we can again borrow all results from the Korobov space for the cosine space as mentioned in [7], but this time with all weights scaled by a factor of 2. We summarize this conclusion in the corollary below.

**Corollary 2.** Let $z \in \mathbb{Z}^d_1$ be the generating vector obtained by a fast component-by-component algorithm in $O(dn \log n)$ operations, using the squared worst-case error for a lattice rule in the Korobov space $E_{d,\alpha,\gamma/2}$ as the search criterion. Then there exists a shift $\Delta \in [0,1]^d$ such that the worst-case error for the resulting tent-transformed shifted lattice rule with the generating vector $z$ in the cosine space $C_{d,\alpha,\gamma}$ satisfies

$$
eq_{\text{wor}}(Q_n(\cdot \circ \psi; z, \Delta); C_{d,\alpha,\gamma}) \leq \neq_{\text{wor}}(Q_n^{\text{ran}}(\cdot \circ \psi; z); C_{d,\alpha,\gamma}) = \neq_{\text{wor}}(Q_n(\cdot; z); E_{d,\alpha,\gamma/2})$$

$$\leq \left( \frac{1}{n-1} \prod_{j=1}^d \left( 1 + 2^{1-\lambda} \zeta(2\alpha \lambda) \gamma_j^2 - 1 \right) \right)^{1/(2\lambda)}$$

for all $1/(2\alpha) < \lambda \leq 1$, where $\zeta(\cdot)$ is again the Riemann zeta function. Hence, the convergence rate is $O(n^{-1/(2\lambda)})$, with the implied constant independent of $d$ if $\sum_{j=1}^{\infty} \gamma_j^2 < \infty$. As $\lambda \rightarrow 1/(2\alpha)$, the method achieves the optimal rate of convergence close to $O(n^{-\alpha})$.

## 4 Function Approximation

We define approximation in terms of the operator which is the embedding from the cosine space to the $L_2$ space, i.e., $\text{APP}_d : C_{d,\alpha,\gamma} \rightarrow L_2([0,1]^d)$, and

$$\text{APP}_d(f) := f.$$  

To approximate $\text{APP}_d$, we study linear algorithms of the form

$$A_{n,d}(f)(x) = \sum_{i=1}^n f(t_i) a_i(x), \quad (19)$$

for some functions $a_i \in L_2([0,1]^d)$ and deterministically chosen sample points $t_i \in [0,1]^d$. In particular, we are interested in tent-transformed rank-$1$ lattice points for sampling.

For approximating the function $f$ from its samples, we consider a hyperbolic cross index set for truncating the cosine series expansion. As cosine series have spectral support only on the positive hyperoctant, we define the *weighted hyperbolic cross* $H_M$ on the positive hyperoctant by

$$H_M := H_{M,\alpha,\gamma}^d := \left\{ k \in \mathbb{Z}_+^d : r_{\alpha,\gamma}(k) \leq M \right\}, \quad (20)$$

with $M \in \mathbb{R}$ and $M \geq 1$. We approximate $f$ by first truncating its cosine series expansion to $H_M$ and then approximating the cosine coefficients for $k \in H_M$ by an $n$-point tent-transformed rank-$1$ lattice rule. So we have

$$A_{n,d,M}(f)(x) := \sum_{k \in H_M} \left( \frac{1}{n} \sum_{t \in \Lambda_0(z,n)} f(t) \phi_k(t) \right) \phi_k(x). \quad (21)$$
That is, $f$ is approximated by a linear algorithm of the form (19) with $t_i$ from $\Lambda_\psi(z, n)$ and
\[
a_i(x) = \frac{1}{n} \sum_{k \in H_M} \phi_k(t_i) \phi_k(x).
\]

We are then interested in the worst-case error of the algorithm $A_{n,d,M}$, which is defined as follows
\[
e_{\text{wor}}(A_{n,d,M}; C_{d,\alpha,\gamma}) := \sup_{f \in C_{d,\alpha,\gamma}, \|f\|_{C_{d,\alpha,\gamma}} \leq 1} \|f - A_{n,d,M}(f)\|_{L^2([0,1]^d)}.
\]

4.1 Upper bound on the worst-case error

The following theorem gives the expression for the $L^2$ error of the algorithm.

**Theorem 5.** The $L^2$ error of approximating $f$ by first truncating the spectral expansion to a hyperbolic cross $H_M$ and then using a tent-transformed rank-1 lattice rule with points $\Lambda_\psi(z, n)$ to approximate the cosine coefficients is given by
\[
\|f - A_{n,d,M}(f)\|_{L^2([0,1]^d)}^2 = \sum_{k \not\in H_M} |\hat{f}(k)|^2 + \sum_{k \in H_M} \left| \sum_{\sigma \in \{\pm 1\}^d} \hat{f}(\sigma(h) + k) \left( \sqrt{2} - |\sigma(h)| + |k|_1 \right) \right|^2,
\]

where $\Lambda(z, n)^\perp$ is the dual of $\Lambda(z, n)$.

**Proof.** Clearly the approximation error of our algorithm (21) is
\[
(f - A_{n,d,M}(f))(x) = \sum_{k \not\in H_M} \hat{f}(k) \phi_k(x) + \sum_{k \in H_M} \left( \hat{f}(k) - \hat{f}_a(k) \right) \phi_k(x),
\]

where we denote by $\hat{f}_a(k)$ the approximation of $\hat{f}(k)$, i.e.,
\[
\hat{f}_a(k) := \frac{1}{n} \sum_{t \in \Lambda_\psi(z, n)} f(t) \phi_k(t).
\]

Since $\phi_k$ is a set of orthonormal basis functions, we conclude that
\[
\|f - A_{n,d,M}(f)\|_{L^2([0,1]^d)}^2 = \sum_{k \not\in H_M} |\hat{f}(k)|^2 + \sum_{k \in H_M} |\hat{f}(k) - \hat{f}_a(k)|^2.
\]

To complete the proof we need to derive an explicit expression for $\hat{f}_a(k)$.

We can write
\[
\hat{f}_a(k) = \frac{1}{n} \sum_{t \in \Lambda(z, n)} f(\psi(t)) \phi_k(\psi(t)) = \frac{1}{n} \sum_{t \in \Lambda(z, n)} \left( \sum_{\ell \in \mathbb{Z}_+^d} \hat{f}(\ell) \phi_\ell(\psi(t)) \right) \phi_k(\psi(t))
\]
\[
= \sum_{\ell \in \mathbb{Z}_+^d} \hat{f}(\ell) \frac{1}{n} \sum_{t \in \Lambda(z, n)} \phi_\ell(\psi(t)) \phi_k(\psi(t)).
\]
Using Lemma 1 and (8), we obtain

\[
\frac{1}{n} \sum_{t \in \Lambda(z, n)} \phi_d(\psi(t)) \phi_k(\psi(t)) = \frac{1}{n} \sum_{t \in \Lambda(z, n)} \left( \frac{\sqrt{2} |\ell| + |k|}{2^d} \right) \sum_{\sigma, \sigma' \in \{\pm 1\}^d} \exp(2\pi i (\sigma(\ell) - \sigma'(k)) \cdot t) \\
= \left( \frac{\sqrt{2} |\ell| + |k|}{2^d} \right) \sum_{\sigma, \sigma' \in \{\pm 1\}^d} \mathbb{I}_{\sigma(\ell) = \sigma'(k) \in \Lambda(z, n)^+}.
\]

Note that for any function \(G : \mathbb{Z}^d \to \mathbb{C}\) and any \(\ell, k \in \mathbb{Z}^d_+\),

\[
\sum_{\sigma, \sigma' \in \{\pm 1\}^d} G(\sigma(\ell) - \sigma'(k)) = \sum_{\sigma' \in \{\pm 1\}^d} \sum_{\sigma \in \{\pm 1\}^d} G(\sigma'(-1)(\sigma(\ell)) - k))
\]

Here \(\sigma^{-1}\) is such that for any \(k \in \mathbb{Z}^d\), \((\sigma^{-1} \circ \sigma)(k) = k\). We thus arrive at

\[
\hat{f}_a(k) = \sum_{\ell \in \mathbb{Z}^d_+} \hat{f}(\ell) \left( \frac{\sqrt{2} |\ell| + |k|}{2^d} \right) \sum_{\sigma, \sigma' \in \{\pm 1\}^d} \mathbb{I}_{\sigma(\sigma'(\ell)) \in \Lambda(z, n)^+} \\
= \sum_{\ell \in \mathbb{Z}^d} \hat{f}(\ell) \left( \frac{\sqrt{2} |\ell| + |k|}{2^d} \right) \sum_{\sigma' \in \{\pm 1\}^d} \mathbb{I}_{\sigma(\ell - k) \in \Lambda(z, n)^+},
\]

where we applied (11) to change the index set from \(\mathbb{Z}^d_+\) to \(\mathbb{Z}^d\). Taking \(\ell - k = h\) so that \(\ell = h + k\), we get

\[
\hat{f}_a(k) = \sum_{h \in \mathbb{Z}^d} \hat{f}(|h + k|) \left( \frac{\sqrt{2} |h + k| + |k|}{2^d} \right) \sum_{\sigma' \in \{\pm 1\}^d} \mathbb{I}_{\sigma(\sigma'(h)) \in \Lambda(z, n)^+}.
\]

Since we sum over all \(h \in \mathbb{Z}^d\) as well as over all sign combinations \(\sigma' \in \{\pm 1\}^d\), the above expression can be regrouped as

\[
\hat{f}_a(k) = \sum_{h \in \mathbb{Z}^d} \hat{f}(|h + k|) \left( \frac{\sqrt{2} |h + k| + |k|}{2^d} \right) \sum_{\sigma' \in \{\pm 1\}^d} \mathbb{I}_{\sigma(\sigma'(h)) \in \Lambda(z, n)^+} \\
= \hat{f}(k) + \sum_{0 \neq h \in \Lambda(z, n)^+} \sum_{\sigma \in \{\pm 1\}^d} \hat{f}(\sigma(h) + k) \left( \frac{\sqrt{2} |\sigma(h) + k| + |k|}{2^d} \right).
\]

Substituting this into (23) then completes the proof.
Proof. By the definition of \( H_M \) in (20) we have \( r_{\alpha,\gamma}(k) > M \) for \( k \notin H_M \), and thus the truncation error in (22) satisfies
\[
\sum_{k \in H_M} |\hat{f}(k)|^2 = \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \frac{r_{\alpha,\gamma}(k)}{r_{\alpha,\gamma}(k)} < \frac{1}{M} \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 r_{\alpha,\gamma}(k) = \frac{1}{M} \|f\|_{C_{d,\alpha,\gamma}}^2.
\]

For the aliasing error in (22), we first apply the Cauchy–Schwarz inequality to obtain
\[
\left| \sum_{0 \neq h \in \mathcal{A}(z,n)^\perp} \sum_{\sigma \in \{\pm 1\}^d} \hat{f}(\sigma(h) + k) \frac{(\sqrt{2})^{-|\sigma(h)+k|_0}}{2^d} \right|^2 
\leq \left( \sum_{0 \neq h \in \mathcal{A}(z,n)^\perp} \sum_{\sigma \in \{\pm 1\}^d} 2^{-|\sigma(h)+k|_0} r_{\alpha,\gamma}(\sigma(h) + k) |\hat{f}(\sigma(h) + k)| \right)^2 \times \left( \sum_{0 \neq h \in \mathcal{A}(z,n)^\perp} \sum_{\sigma \in \{\pm 1\}^d} 2^d r_{\alpha,\gamma}(\sigma(h) + k) \right) \tag{25}
\]

The first factor in (25) can be bounded from above by relaxing the condition on \( h \) and instead summing over all \( h \in \mathbb{Z}^d \):
\[
\sum_{\sigma \in \{\pm 1\}^d} \sum_{h \in \mathbb{Z}^d} 2^{-|\sigma(h)+k|_0} r_{\alpha,\gamma}(\sigma(h) + k) |\hat{f}(\sigma(h) + k)| \leq \sum_{h \in \mathbb{Z}^d} 2^{-|h|_0} r_{\alpha,\gamma}(h) |\hat{f}(h)| = \sum_{h \in \mathbb{Z}^d} r_{\alpha,\gamma}(h) |\hat{f}(h)| = \|f\|_{C_{d,\alpha,\gamma}}^2,
\]
where the first equality holds since the vectors \( \sigma(h) + k \) are precisely all of \( \mathbb{Z}^d \) as we sum over all \( h \in \mathbb{Z}^d \). These bounds for the two sources of errors lead to the worst-case error bound in the theorem. \( \square \)

4.2 Connection with the weighted Korobov space

In [16], a similar approximation algorithm was developed for the weighted Korobov space with rank-1 lattice points instead of tent-transformed rank-1 lattice points. It was shown there that the squared worst-case error for the corresponding algorithm, which we denote by \( \tilde{A}_{n,d,M} \) here, is bounded by
\[
e_{\text{wor}}(\tilde{A}_{n,d,M}; E_{d,\alpha,\gamma})^2 \leq \frac{1}{M} + \tilde{E}_{n,d,M}(z), \tag{26}
\]
with
\[
\tilde{E}_{n,d,M}(z) := \sum_{k \in H_M} 0 \neq h \in \mathcal{A}(z,n)^\perp 1 \frac{1}{r_{\alpha,\gamma}(h + k)} \tag{27}
\]
where \( \tilde{H}_M \) denotes the hyperbolic cross used with the Fourier basis functions. The set \( \tilde{H}_M \) differs from \( H_M \) in that it is defined over all the hyperoctants instead of just the positive hyperoctant:
\[
\tilde{H}_M := \tilde{H}_M^{d,\alpha,\gamma} := \{ k \in \mathbb{Z}^d : r_{\alpha,\gamma}(k) \leq M \}.
\]

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Theorem 7. The bounds of the squared worst-case errors in (24) and (26)–(27) are equal, given that the weights \( \gamma \) and the decay parameters \( \alpha \) of the two spaces are the same:

\[
\sum_{k \in H_M} \sum_{0 \neq h \in \Lambda(z,n)} \sum_{\sigma \in \{\pm 1\}^d} 2^{k|\sigma|} r_{\alpha,\gamma}(\sigma(h) + k) = \sum_{k \in H_M} \sum_{0 \neq h \in \Lambda(z,n)} \sum_{\sigma \in \{\pm 1\}^d} \frac{1}{r_{\alpha,\gamma}(h + k)},
\]

(28)

Proof. Starting with the left-hand side of (28), we can write

\[
\text{LHS of (28)} = \sum_{k \in H_M} \sum_{0 \neq h \in \Lambda(z,n)} \sum_{\sigma \in \{\pm 1\}^d} 2^{k|\sigma|} r_{\alpha,\gamma}(\sigma(h) + \sigma^{-1}(k))
\]

\[
= \sum_{k \in H_M} \sum_{0 \neq h \in \Lambda(z,n)} \sum_{\sigma^{-1} \in \{\pm 1\}^d} 2^{k|\sigma|} r_{\alpha,\gamma}(\sigma(h) + \sigma^{-1}(k)) = \text{RHS of (28),}
\]

where in the second equality we have used the sign invariance of \( r_{\alpha,\gamma} \) and the fact that for any function \( G : \mathbb{Z}^d \to \mathbb{C} \) and any \( k \in \mathbb{Z}^d \),

\[
\sum_{\sigma \in \{\pm 1\}^d} G(\sigma^{-1}(k)) = \sum_{\sigma^{-1} \in \{\pm 1\}^d} G(\sigma^{-1}(k)),
\]

as it is the same sum in a different order. For the final equality we have used the identity in (11) but with the sets \( \mathbb{Z}^d_1 \) and \( \mathbb{Z}^d \) replaced by \( H_M \) and \( \tilde{H}_M \). The result is hence proved.

The quantity \( \tilde{E}_{n,d,M}(z) \) in (26)–(27) was used in [16] as the search criterion in a component-by-component search algorithm to construct the generating vector for a lattice rule that satisfies a proven worst-case error bound for approximation in the Korobov space. We see from the above theorem that this quantity coincides with the second term in (24). Thus the generating vector constructed by the algorithm for the Korobov space can also be used in a tent-transformed lattice rule for approximation in the cosine space. A fast implementation of this construction in the spirit of [26] is discussed in [17]. We summarize this conclusion in the corollary below.

Corollary 3. Let \( \kappa > 1 \) be some fixed number and suppose \( n \) is a prime number satisfying \( n \geq \kappa M^{1/(2\alpha)} \). A fast component-by-component search algorithm can be used to obtain a generating vector \( z \in \mathbb{Z}^d_1 \) in \( O(\tilde{H}_M \cdot n \log n) \) operations, using the expression \( \tilde{E}_{n,d,M}(z) \) in (27) as the search criterion, such that the worst-case error in the cosine space \( C_{d,\alpha,\gamma} \) for the algorithm \( A_{n,d,M} \) defined by (21) with the resulting tent-transformed lattice points satisfies

\[
e^{\text{wor}}(A_{n,d,M}; C_{d,\alpha,\gamma})^2 \leq \frac{1}{M} + \frac{M^{7/\lambda}}{n - 1} \frac{1}{\mu} \prod_{j=1}^{d} \left( 1 + 2\zeta(2\alpha \tau) \gamma_j^5 \right) \left( 1 + 2(1 + \mu^4)\zeta(2\alpha \lambda) \gamma_j^5 \right)^{1/\lambda}
\]

for all \( \tau > 1/(2\alpha) \), \( 1/(2\alpha) < \lambda \leq 1 \), and \( 0 < \mu \leq (1 - 1/\kappa)^{2\alpha} \), where \( \zeta(\cdot) \) again denotes the Riemann zeta function. Hence, upon setting \( \tau = \lambda \) and choosing \( M = \mathcal{O}(n^{1/(2\lambda)}) \) to balance the order of the two error contributions, we conclude that the convergence rate is \( \mathcal{O}(n^{-1/(4\lambda)}) \), with the implied constant independent of \( d \) if \( \sum_{j=1}^{\infty} \gamma_j^{5} < \infty \). As \( \lambda \to 1/(2\alpha) \), the method achieves the convergence rate close to \( \mathcal{O}(n^{-a/2}) \), while the optimal rate is believed to be close to \( \mathcal{O}(n^{-a}) \).

It was proved in [16] that \( |\tilde{H}_M| \leq M^q \prod_{j=1}^{d} \zeta(2\alpha q) \gamma_j^q \) for all \( q > 1/(2\alpha) \), and this quantity can be bounded independently of \( d \) if \( \sum_{j=1}^{\infty} \gamma_j^{q} < \infty \).

Tractability analysis for approximation in the cosine space can be carried out following exactly the same argument as in [16] for the Korobov space. Roughly speaking, tractability means
that the minimal number of function evaluations required to achieve an error $\varepsilon$ in $d$ dimensions is bounded polynomially in $\varepsilon^{-1}$ and $d$, while strong tractability means that this bound is independent of $d$. Tractability depends on the problem setting and on the type of information used by algorithms, see the books [21, 22, 23]. In the corollary below we provide just an outline of tractability results for approximation in the cosine space.

**Corollary 4.** Consider the approximation problem for weighted cosine spaces in the worst-case setting.

(a) Let $p^* = 2 \max\left(\frac{1}{2\alpha}, s_\gamma\right)$, where $s_\gamma = \inf\{s > 0 : \sum_{j=1}^{\infty} \gamma_j^s < \infty\}$ and suppose that

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$ 

Given $\varepsilon > 0$, the approximation algorithm $A_{n,d,M}$ defined by (21), with appropriately chosen values of $n$ and $M$ and specially constructed generating vector $z$, achieves the error bound $e_{1\text{wor}}(A_{n,d,M}; C_{d,\alpha,\gamma}) \leq \varepsilon$ using $n = O(\varepsilon^{-p})$ function values. The implied factor in the big $O$ notation is independent of $d$ and the exponent $p$ is arbitrarily close to $2p^*$.

(b) Suppose that

$$a := \limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j}{\log(d+1)} < \infty.$$ 

Given $\varepsilon > 0$, the approximation algorithm $A_{n,d,M}$ defined by (21), with appropriately chosen values of $n$ and $M$ and specially constructed generating vector $z$, achieves the error bound $e_{1\text{wor}}(A_{n,d,M}; C_{d,\alpha,\gamma}) \leq \varepsilon$ using $n = O(\varepsilon^{-d^q})$ function values. The implied factor in the big $O$ notation is independent of $\varepsilon$ and $d$, and the exponent $q$ can be arbitrarily close to $4\zeta(2\alpha)a$.

## 5 Conclusions

We have studied the problems of integration and approximation in the weighted cosine space of smooth non-periodic functions using tent-transformed lattice points. For the integration problem, we provided a precise formula for the squared worst-case error of a tent-transformed lattice rule, amending the result in [7]. We also derived the root-mean-squared worst-case error of a tent-transformed randomly-shifted lattice rule. By exploiting the connection with the weighted Korobov space of smooth periodic functions, we show that these methods can be constructed to achieve the optimal rate of convergence in the cosine space. For the approximation problem, we showed that the worst-case error for our algorithm in the cosine space has an upper bound which is identical to a previously analyzed upper bound on the worst-case error for a related algorithm in the Korobov space, and this allowed us to apply known constructive results for approximation in the Korobov space to the cosine space.

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