A REMARK ON REVERSE LITTLEWOOD–PALEY, RESTRICTION AND KAKEYA

ANTHONY CARBERY

Abstract. We show that a certain conjectured optimal reverse Littlewood–Paley inequality would, if true, imply sharp results for the Kakeya maximal function, the Bochner–Riesz means and the Fourier restriction operator.

1. Introduction

Let $\delta > 0$ be a small parameter and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth function of compact support in $[-1,1]$ satisfying $|\Phi^{(k)}(t)| \leq C_k$ for all $k \in \mathbb{N}$. Define the Fourier multiplier operator $S^\delta$ on $\mathbb{R}^n$ by

$$\hat{(S^\delta f)}(\xi) = \Phi \left( \frac{|\xi| - 1}{\delta} \right) \hat{f}(\xi).$$

We decompose the $\delta$-neighbourhood $\{\xi : |\xi| - 1 \leq \delta\}$ of the unit sphere $S^{n-1}$ into coin-shaped pieces $E_\alpha$ of tangential dimensions $\delta^{1/2} \times \cdots \times \delta^{1/2}$ and radial dimension $\delta$, and correspondingly the operator $S^\delta = \sum_\alpha S_\alpha$ where $S_\alpha$ is a Fourier multiplier operator with smooth multiplier $\phi_\alpha$ supported in and adapted to $E_\alpha$.

We consider the reverse Littlewood–Paley inequality

$$\|S^\delta f\|_{L^2(\mathbb{R}^n)} \leq C_n \left( \sum_\alpha |S_\alpha f|^2 \right)^{1/2}$$

where $C_n$ is supposed to be independent of $\delta$ and $f$. When $n = 2$ (and thus $2n/(n-1) = 4$), Fefferman in [11] proved that this inequality is true since we can multiply out the $L^4$ norm and observe that the algebraic differences $E_\alpha - E_{\alpha'}$ are essentially disjoint as $\alpha \neq \alpha'$ vary. In higher dimensions it remains open.

More generally one might ask whether for $q \geq 2$ we have

$$\|S^\delta f\|_{L^q(\mathbb{R}^n)} \leq C_n \left( \sum_\alpha |S_\alpha f|^r \right)^{1/r},$$

where for $2 \leq q \leq 2n/(n-1)$ we take $r = 2$ and when $q \geq 2n/(n-1)$ we take $r' = q(n-1)/n$ (so that when $q = \infty$ we have $r = 1$). The issue of such inequalities with the order the mixed norms reversed, first proposed by Bonami and Garrigós, has recently been studied in [3] where they are termed $L^r$-decoupling inequalities.

There is a maximal function relevant to the study of $S^\delta$, the so-called Nikodym maximal function, for which Fefferman also proved in [11] optimal $L^2$ bounds in two dimensions (in slightly disguised form). By what are very familiar arguments
(see for example [5], [9]) these two ingredients can be combined to prove the optimal two-dimensional result
\[ \| S^\delta f \|_4 \leq C \left( \log \frac{1}{\delta} \right)^{1/4} \| f \|_4 \]
for the operators \( S^\delta \).

In this note we show that (a variant of) (1) actually implies the correct \( L^n \) bound for the maximal function, and thus (1) together with its variant gives the optimal Bochner–Riesz multiplier result

\[ \| S^\delta f \|_{L^{2n/T}(\mathbb{R}^n)} \leq C_n \left( \log \frac{1}{\delta} \right)^{n-1 \over 2n} \| f \|_{L^{2n/T}(\mathbb{R}^n)} \]

in all dimensions upon combining (1) with the maximal function estimate in the familiar way.

For ease of exposition we choose to work in the alternative and essentially equivalent setting of the extension problem for the Fourier transform. Since the kernels of \( S^\delta \) and \( S_\alpha \) in (1) are essentially localised at scale \( 1/\delta \), (1) is equivalent to the corresponding local inequality

\[ \| S^\delta f \|_{L^{2n/T}(B(0,\delta^{-1})))} \leq C_n \left\| \left( \sum_\alpha |S_\alpha f|^2 \right)^{1/2} \right\|_{L^{2n/T}(B(0,\delta^{-1})))} \]

and thus (1) is equivalent to

\[ \| S^\delta f \|_{L^{2n/T}(B(0,\delta^{-1})))} \leq C_n \left\| \left( \sum_\alpha |g_\alpha \sigma|^2 \right)^{1/2} \right\|_{L^{2n/T}(B(0,\delta^{-1})))} \]

where \( g \) is a smooth function defined on \( S^{n-1} \), \( \sigma \) is the Lebesgue measure on \( S^{n-1} \), \( g_\alpha = g\chi_{E_\alpha} \), the \( E_\alpha \) are spherical caps of radius \( \delta^{1/2} \) decomposing \( S^{n-1} \) and where

\[ \hat{h}\sigma(x) = \int_{S^{n-1}} h(\xi)e^{2\pi ix\cdot\xi}d\sigma(\xi) \]

is the Fourier transform of the density \( h\sigma \). Note that the inverse Fourier transform of \( g\hat{\sigma} \Phi(\cdot) \) is essentially supported in a \( \delta \) neighbourhood of \( S^{n-1} \), and that we may assume \( g \) is constant at scale \( \delta \).

The appropriate maximal function in this setting is the Kakeya maximal function which is defined as follows: let \( N \gg 1 \) be a large parameter, and for \( f \) defined on \( \mathbb{R}^n \) and \( \omega \in S^{n-1} \) let

\[ M_N f(\omega) = \sup_{T \ni \omega} \frac{1}{|T|} \int_T |f| \]

where the sup is taken over all tubes \( T \) of dimensions \( 1 \times \cdots \times 1 \times N \) whose axis is parallel to \( \omega \). It is conjectured that

\[ \| M_N f \|_{L^n(S^{n-1})} \leq C_n (\log N)^{n-1 \over n} \| f \|_{L^n(\mathbb{R}^n)} \]

\[ \frac{1}{\delta} ] \) on both sides, ensuring that we may assume that the integrand on the right hand side is indeed essentially supported in \( B(0, \delta^{-1}) \). Without this interpretation it is not clear what meaning \( \delta \) may have in general. However, as we shall see below, we shall actually use \( \delta \) only in the case that \( g \) is constant at scale \( \delta^{1/2} \), in which case there is no ambiguity.
and this is also known to be true in two dimensions by work of Fefferman [11] but is open in higher dimensions. Once again, (4) and (5) fit into a standard machine which can be used to establish, *inter alia*, restriction estimates such as

\begin{equation}
\| \hat{g} \sigma \|_{L^{2n/(n-1)}(B(0,R))} \leq C_n \left( \log R \right)^{\frac{n-1}{2n}} \| g \|_{L^{2n/(n-1)}(\mathbb{R}^n)}.
\end{equation}

**Proposition 1.** Suppose that (4) holds\(^2\), that is

\[ \| \hat{g} \sigma \|_{L^{2n/(n-1)}(B(0,\delta^{-1}))} \leq C_n \left( \sum_{\alpha} |c_{\alpha}|^2 \right)^{1/2} \]

Then (5) holds, that is

\[ \| M_N f \|_{L^n(\mathbb{R}^n)} \leq C_n \left( \log N \right)^{\frac{n-1}{2n}} \| f \|_{L^n(\mathbb{R}^n)}, \]

and hence

\[ \| \hat{g} \sigma \|_{L^{2n/(n-1)}(B(0,R))} \leq C_n \left( \log R \right)^{\frac{n-1}{2n}} \| g \|_{L^{2n/(n-1)}(\mathbb{R}^n)} \]

also holds.

The proof uses a two-scale technique and is similar to Fefferman’s argument for the disc multiplier [10] and to the argument in [1] in which it was first noted that restriction estimates imply estimates for maximal functions, and which was later treated more formally by Bourgain [2]. Also see the remark at the end for another argument by-passing the maximal function.

Since we know that the Kakeya maximal conjecture implies the Nikodym maximal conjecture (see Theorem 4.10 of [14], which is in turn based upon [5]) we conclude that (4) and (1) (together) also imply the Bochner–Riesz conjecture [2].

**Remark.** This work was done in the early 1990’s but was never presented for publication. The author has communicated its essence to a number of harmonic analysts over the years and its existence has apparently become known, resulting in occasional requests for it. It is hoped that this informal presentation will satisfy such demand.

### 2. Sketch of proof of Proposition 1

We shall establish (5) in a certain dual form, see for example [6] and [4]. For \( N \) fixed there are essentially \( N^{n-1} \) ‘distinct’ directions that a \( 1 \times \cdots \times 1 \times N \) tube can occupy. Suppose we have a collection \( \{ T_\alpha \} \) of such tubes, one in each direction. Then the best constant \( A_r \) in the inequality

\begin{equation}
\| \sum_{\alpha} c_{\alpha} \chi_{T_\alpha} \|_{L^r} \leq A_r N^{1/r} \left( \sum_{\alpha} |c_{\alpha}|^r \right)^{1/r}
\end{equation}

(over all such families of tubes) is equivalent to the \( L^r \to L^{r'} \) operator norm of \( M_N \). The equivalent scaled version of (7) for families of \( \lambda \times \cdots \times \lambda \times \lambda N \) tubes with distinct directions is

\[ \| \sum_{\alpha} c_{\alpha} \chi_{T_\alpha} \|_{L^r} \leq A_r N^{1/r} \lambda^{n/r} \left( \sum_{\alpha} |c_{\alpha}|^r \right)^{1/r} \].
It is easy to verify that if the tubes \{T_\alpha\} form a bush (i.e. all pass through a common centre) then (7) holds for \(r = n/(n-1)\) in the optimal form

\[
\| \sum_\alpha c_\alpha \chi_{T_\alpha} \|_{\frac{n}{n-1}} \leq C_n (\log N)^{\frac{n-1}{n}} N^{\frac{(n+1)(n-1)}{2n}} \left( \sum_\alpha |c_\alpha|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}.
\]

**Proof. Warning:** The treatment is somewhat informal.

Let \(T_\alpha\) be a collection of \(\delta^{-1/2} \times \cdots \times \delta^{-1/2} \times \delta^{-1}\) tubes in \(\mathbb{R}^n\) (so \(\lambda = N = \delta^{-1/2}\) here), one in each of the \(\delta^{-(n-1)/2}\) directions. By the remarks above, it suffices to show the appropriately scaled form of (7) which is in this case

\[
\| \sum_\alpha c_\alpha \chi_{T_\alpha} \|_{\frac{n}{n-1}} \leq C_n (\log N)^{\frac{n-1}{n}} N^{\frac{(n+1)(n-1)}{2n}} \left( \sum_\alpha |c_\alpha|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}.
\]

We may assume that all the \(T_\alpha\) are supported in a ball \(B(0, \delta^{-1})\) in \(\mathbb{R}^n\) and that \(T_\alpha\) is the translate by \(\lambda_\alpha\) of the tube passing through the origin in the \(\alpha\)’th direction. Note that \(|\lambda_\alpha| \lesssim \delta^{-1}\). We may assume that \(c_\alpha \geq 0\).

Let \(\{E_\alpha\}\) be \(\delta^{1/2}\)-cells on \(S^{n-1}\) and let

\[g(\xi) = \sum_\alpha \pm e^{i\lambda_\alpha \cdot \xi} \phi_\alpha(\xi)\]

where \(\phi_\alpha\) is a smooth bump function associated to the cell \(E_\alpha\). Standard stationary phase calculations give

\[
\left| \left( e^{i\lambda_\alpha \cdot \xi} \phi_\alpha(\xi) d\sigma(\xi) \right)^\wedge (x) \right| \gtrsim \delta^{(n-1)/2} \chi_{T_\alpha}(x).
\]

Since \(|\lambda_\alpha| \lesssim \delta^{-1}\) we have that \(g\) is roughly constant at scale \(\delta\) and so we can further decompose \(g\) as

\[g = \sum_\beta a_\beta c_{(\alpha(\beta)} \psi_\beta\]

where \(\psi_\beta\) are smooth bump functions associated to a decomposition of the sphere into \(\delta\)-cells \(F_\beta\), where \(\alpha(\beta)\) is the \(\alpha\) such that \(F_\beta \subseteq E_\alpha\) and where \(|a_\beta| \sim 1\).

Now we apply our assumption (41) – with \(\delta^2\) now playing the role of \(\delta\) in (41) – to obtain

\[
\| \hat{g} d\sigma \|_{L^{\frac{2n}{n-1}}(B(0,\delta^{-2}))} \leq C_n \left( \sum_\beta c_{(\alpha(\beta)} |\psi_\beta d\sigma|^2 \right)^{1/2} \| \psi_\beta d\sigma \|_{L^{\frac{2n}{n-1}}(B(0,\delta^{-2}))}.
\]

The main contribution to \(|\psi_\beta d\sigma|\) is given by \(\delta^{(n-1)} \chi_{R_\beta}\) where \(R_\beta\) is a \(\delta^{-1} \times \cdots \delta^{-1} \times \delta^{-2}\) tube passing through the origin. Hence, ignoring lower order contributions,

\[
\left| \sum_\alpha \pm e^{i\lambda_\alpha \cdot \xi} \phi_\alpha(\xi) d\sigma(\xi) \right|^\wedge \leq C_n \delta^{(n-1)} \left( \sum_\beta c_{(\alpha(\beta)} \chi_{R_\beta} \right)^{1/2} \| \chi_{R_\beta} \|_{L^{\frac{2n}{n-1}}(B(0,\delta^{-2}))}.
\]
Next we use Khintchine’s inequality and (10) to deduce that
\[ \delta^{(n-1)/2} \left\| \left( \sum_{\alpha} c_{\alpha} \chi_{T_{\alpha}} \right)^{1/2} \right\|_{L^{2n/(n-1)}(\mathbb{S}^{n-1})} \leq C_n \delta^{(n-1)} \left\| \left( \sum_{\beta} c_{\alpha(\beta)} \chi_{R_{\beta}} \right)^{1/2} \right\|_{L^{2n/(n-1)}(B(0,\delta^{-2}))} \]

or equivalently
\[ \left\| \sum_{\alpha} c_{\alpha} \chi_{T_{\alpha}} \right\|_{n^{-1}} \leq C_n \delta^{(n-1)} \left\| \sum_{\beta} c_{\alpha(\beta)} \chi_{R_{\beta}} \right\|_{n^{-1}}. \]

Now the tubes \( R_{\beta} \) all pass through the origin, so by the scaled version of the remark immediately preceding the proof we have
\[ \left\| \sum_{\beta} d_{\beta} \chi_{R_{\beta}} \right\|_{n^{-1}} \leq C_n \left( \log \frac{1}{\delta} \right) \delta^{(n-1)} \left( \sum_{\beta} |d_{\beta}|_{n^{-1}} \right)^{n-1}. \]

Taking \( d_{\beta} = c_{\alpha(\beta)} \) and noting that
\[ \sum_{\beta} |d_{\beta}|_{n^{-1}} = \delta^{-(n-1)/2} \sum_{\alpha} |c_{\alpha}|_{n^{-1}}, \]
we obtain
\[ \left\| \sum_{\alpha} c_{\alpha} \chi_{T_{\alpha}} \right\|_{n^{-1}} \leq C_n \left( \log \frac{1}{\delta} \right) \delta^{(n-1)} \left( \sum_{\beta} |d_{\beta}|_{n^{-1}} \right)^{n-1} \]
\[ = C_n \left( \log \frac{1}{\delta} \right) \delta^{-(n+1)(n-1)/2n} \left( \sum_{\alpha} |c_{\alpha}|_{n^{-1}} \right)^{n-1} \cdot \]

\[ \square \]

**Remark.** Note that we did not need the full force of hypothesis (4) for general \( g \), only for those \( g \) constant on \( \delta^{1/2} \)-cells.

Indeed, suppose we assume (4) for such \( g \) constant on \( \delta^{1/2} \) cells \( E_{\alpha} \). We can then conclude inequality (11) directly without passing through the maximal function. To see this, we begin by observing that in order to prove inequality (10) we may assume that \( g \) is a step function on \( \mathbb{S}^{n-1} \), which is constant at scale \( R^{-1} \) (this is the effect of restricting attention to \( B(0,R) \)). So, since \( B(0,R) \subseteq B(0,R') \), in order to prove (11) for such \( g \) it suffices to prove the ostensibly stronger inequality
\[ \left\| \bar{g} \sigma \right\|_{L^{2n/(n-1)}(\mathbb{S}^{n-1})} \leq C_n \left( \log R \right)^{n-1} \left\| \bar{g} \right\|_{L^{2n/(n-1)}(\mathbb{S}^{n-1})} \]
for such \( g \). But if we relabel \( R' \) as \( R \), we see it therefore suffices to prove that inequality (10) holds for \( g \) constant at scale \( R^{-1/2} \); that is, we may assume \( g = \)

\[ ^{3} \text{but not really, as } \bar{g} \sigma \text{ is pretty much supported in } B(0,R) \]
\[ \sum_{\alpha} c_{\alpha} \phi_{\alpha}. \]  

For such \( g \) we are assuming that we have inequality \([4]\) with \( \delta = 1/R \) and so

\[
\| \hat{g} \|_{L^{2n/(n-1)}(B(0,R))} \leq C_{\alpha} \left( \sum_{\alpha} |c_{\alpha}|^2 |\hat{\phi}_{\alpha}|^2 \right)^{1/2} \left\| \int_{n/R}^{n/(n-1)} (\hat{A}_{t} |g|^2)^{n/(n-1)} \, d\sigma \frac{dt}{t} \right\|^{(n-1)/2n}
\]

where the main contributions of the terms \( |\hat{\phi}_{\alpha}| \) come from \( R^{1/2} \times \cdots \times R^{1/2} \times R \)-tubes centred at the origin, and which we may therefore calculate and estimate directly. Indeed, for such \( g \) constant at scale \( R^{-1/2} \), we have\(^4\)

\[
\left( \sum_{\alpha} |c_{\alpha}|^2 |\hat{\phi}_{\alpha}|^2 \right)^{1/2} \left\| \int_{n/R}^{n/(n-1)} (\hat{A}_{t} |g|^2)^{n/(n-1)} \, d\sigma \frac{dt}{t} \right\|^{(n-1)/2n}
\]

where \( A_{t} \) denotes a local average on the sphere at scale \( t \). Now by the triangle inequality we have

\[
\left( \int_{S^{n-1}} (\hat{A}_{t} |g|^2)^{n/(n-1)} \, d\sigma \right)^{(n-1)/n} \leq \| |g|^2 | \|_{n/(n-1)}
\]

and so

\[
\left( \int_{R^{-1/2}}^{R^{1/2}} \int_{S^{n-1}} (\hat{A}_{t} |g|^2(\omega))^{n/(n-1)} \, d\sigma(\omega) \frac{dt}{t} \right)^{(n-1)/2n} \leq C_{n} (\log R) \| \hat{g} \|_{L^{2n/(n-1)}}.
\]

See also the articles \([7, 12, 13]\) for related discussions.

References

[1] W. Beckner, A. Carbery, S. Semmes and F. Soria, A note on restriction of the Fourier transform to spheres, *Bull. London Math. Soc.* 21 (1989) 394–398.

[2] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, *Geom. Funct. Anal.* 1 (1991), 147–187.

[3] J. Bourgain and C. Demeter, The proof of the \( l^2 \)-decoupling conjecture, *arXiv:1403.5335 [math.CA]*

[4] A. Carbery, Covering lemmas revisited, *Proc. Edinburgh Math. Soc. (2)* 31 (1988) 145–150.

[5] A. Carbery, Restriction implies Bochner–Riesz for paraboloids, *Math. Proc. Cambridge Philos. Soc.* 111 (1992) 525-529.

[6] A. Córdoba, The Kakeya maximal function and the spherical summation multipliers, *Amer J. Math.* 99 (1977) 1–22.

[7] A. Córdoba, Multipliers of \( \mathcal{F}(L^p) \), (Proc. Sem., Univ. of Maryland, College Park, Maryland, 1979), 162–177, *Lecture Notes in Math.*, 779 Springer, Berlin, 1980.

[8] A. Córdoba, A note on Bochner–Riesz operators, *Duke Math. J.* 46 (1979) 505–511.

[9] A. Córdoba, Translation invariant operators, (Proc. Sem., El Escorial, 1979), 117–176, Asociación Matemática Española, Madrid, 1980.

[10] C. Fefferman, The multiplier problem for the ball, *Ann. of Math.* 94 (1971) 330–336.

[11] C. Fefferman, A note on spherical summation multipliers, *Israel J. Math.* 15 (1973) 44–52.

[12] A. Moyua, A. Vargas and L. Vega, Schrödinger maximal function and restriction properties of the Fourier transform, *Internat. Math. Res. Notices* (1996) 793–815.

[13] A. Moyua, A. Vargas and L. Vega, Restriction theorems and maximal operators related to oscillatory integrals in \( \mathbb{R}^3 \), *Duke Math. J.* 96 (1999) 547–574.

\(^4\)There are similar expressions valid for other values of \( q \) and \( r \).
[14] T. Tao, The Bochner–Riesz conjecture implies the restriction conjecture, *Duke Math. J.* **96** (1999) 363–375.

**Anthony Carbery**, School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, JCMB, King’s Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland.

*E-mail address: A.Carbery@ed.ac.uk*