The mixed joint universality for a class of zeta-functions

Roma Kačinskaite\textsuperscript{*1} and Kohji Matsumoto\textsuperscript{**2}

\textsuperscript{1} Department of Mathematics, Institute of Informatics, Mathematics and E-Studies, Šiauliai University, Višinskio 19, LT-77156 Šiauliai, Lithuania
\textsuperscript{2} Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan

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A mixed joint limit theorem in the space of holomorphic functions, and a mixed joint universality theorem are proved for a rather general class of Euler products and periodic Hurwitz zeta-functions.

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1 Introduction

Let $N, N_0, P, Q, R$ and $\mathbb{C}$ be the set of positive integers, non-negative integers, prime numbers, rational numbers, real numbers, and complex numbers, respectively.

After Voronin’s discovery \cite{26} of the universality theorem for the Riemann zeta-function $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$, where $s = \sigma + it \in \mathbb{C}$, the universality property has been studied for various zeta and $L$-functions (see the survey article \cite{18} of the second author). We say that a certain zeta-function $\varphi(s)$ has the universality property if, roughly speaking, any holomorphic function defined on a given compact set can be approximated by some suitable shift of $\varphi(s)$ (for the rigorous definition, see, e.g., \cite{25}).

The simultaneous approximation by several zeta-functions is called the joint universality. Voronin himself already proved \cite{27} a joint universality theorem for a collection of Dirichlet $L$-functions.

The Riemann zeta-function and Dirichlet $L$-functions are examples of zeta or $L$-functions which have Euler products. On the other hand, there are other zeta-functions which do not have Euler products. A typical example is the Hurwitz zeta-function $\zeta(s, \alpha) = \sum_{m=0}^{\infty} (m+\alpha)^{-s}$ with parameter $\alpha$, where $0 < \alpha \leq 1$. Except for the cases $\alpha = 1/2, 1$, the function $\zeta(s, \alpha)$ does not have an Euler product. More generally, we may consider the periodic Hurwitz zeta-function

$$\zeta(s, \alpha; b) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s},$$

where $0 < \alpha \leq 1$, and $b = \{b_m \mid m \in N_0\}$ is a periodic sequence of complex numbers. It is possible to prove universality, or joint universality, for these zeta-functions under certain conditions (see, e.g., \cite{25}).

It was Mishou \cite{21}, and independently Sander and Steuding \cite{23}, who first obtained a joint universality theorem among a zeta-function with an Euler product and another zeta-function which does not have an Euler product. This type of universality is called the mixed joint universality. To state Mishou’s result, we need some notations.

For any compact set $K \subset \mathbb{C}$, denote by $H^c(K)$ the set of all $\mathbb{C}$-valued functions defined on $K$, continuous on $K$ and holomorphic in the interior of $K$. By $H^c_0(K)$ we mean the subset of $H^c(K)$, consisting of all elements of $H^c(K)$ which are non-vanishing on $K$. Let

$$D(a, b) = \{s \mid a < \sigma < b\}$$

for any $a < b$. The Lebesgue measure on $\mathbb{R}$ is denoted by $\mu$.

\textsuperscript{*} e-mail: r.kacinskaite@fm.su.lt, Phone: +370 652 48 371, Fax: +370 41 590 409
\textsuperscript{**} Corresponding author: e-mail: kohjimat@math.nagoya-u.ac.jp, Phone: +81 52 789 2414, Fax: +81 52 789 5397
Theorem 1.1 (Mishou [21]) Suppose that $\alpha$ is a transcendental number. Let $K_1$, $K_2$ be compact subsets of $D(1/2, 1)$ with connected complements. Then, for any $f_1 \in H^p_0(K_1)$, $f_2 \in H^p(K_2)$, and any $\varepsilon > 0$, it holds that

$$\liminf_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] \mid \max_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \max_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0. \quad (1.2)$$

Roughly speaking, this theorem implies that any two holomorphic functions can be approximated simultaneously by a suitable vertical shift of $\zeta(s)$ and $\zeta(s, \alpha)$. Sander and Steuding [23] proved the same type of result for rational $\alpha$.

Mishou’s theorem was generalized by the first author and Laurinčikas [5] as follows. Let $a = \{a_m \mid m \in \mathbb{N}\}$ be a periodic sequence of complex numbers, with the property that the $a_m$’s are multiplicative. Define the periodic zeta-function $\zeta(s; a)$ by

$$\zeta(s; a) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}. \quad (1.3)$$

Theorem 1.2 (Kačinskaite and Laurinčikas [5]) Let $\alpha, D(1/2, 1), K_1$ and $K_2$ be the same as in Theorem 1.1. Then it holds that

$$\liminf_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] \mid \max_{s \in K_1} |\zeta(s + i\tau; a) - f_1(s)| < \varepsilon, \max_{s \in K_2} |\zeta(s + i\tau, \alpha; b) - f_2(s)| < \varepsilon \right\} > 0. \quad (1.4)$$

Since the $a_m$’s are multiplicative, the function $\zeta(s; a)$ has an Euler product. Therefore Theorem 1.2 is also a mixed joint universality theorem.

The aim of the present paper is to prove an analogue of Theorem 1.2, with replacing $\zeta(s; a)$ by a rather general class of zeta-functions which have Euler products. The functions we consider are the so-called Matsumoto zeta-functions, which the second author introduced in [17]. This class includes various important examples of zeta- and $L$-functions, such as automorphic $L$-functions.

2 Statement of results

For $m \in \mathbb{N}$, let $g(m) \in \mathbb{N}$. For $j \in \mathbb{N}$ with $1 \leq j \leq g(m)$, let $f(j, m) \in \mathbb{N}$ and $a_m^{(j)} \in \mathbb{C}$. Denote by $p_m$ the $m$th prime number. We assume

$$g(m) \leq C_1 p_m^{2m}, \quad |a_m^{(j)}| \leq p_m^\alpha \quad (2.1)$$

with a positive constant $C_1$ and non-negative constants $\alpha, \beta$. The zeta-function $\tilde{\zeta}(s)$ introduced by the second author [17] is defined by the Euler product

$$\tilde{\zeta}(s) = \prod_{m=1}^{g(m)} A_m(p_m^{-s})^{-1}, \quad (2.2)$$

where the $A_m$’s are polynomials given by

$$A_m(X) = \prod_{j=1}^{g(m)} \left( 1 - a_m^{(j)} X^{f(j, m)} \right). \quad (2.3)$$

Under the assumption (2.1), we find that $\tilde{\zeta}(s)$ is absolutely convergent in the region $\sigma > \alpha + \beta + 1$. We write the Dirichlet series expansion of $\tilde{\zeta}(s)$ in this region as

$$\tilde{\zeta}(s) = \sum_{k=1}^{\infty} \frac{\tilde{c}_k}{k^s}.$$
For our later convenience, we define the shifted version
\[
\varphi(s) = \tilde{\varphi}(s + \alpha + \beta) = \sum_{k=1}^{\infty} \frac{c_k}{k^{\sigma + \beta}} = \sum_{k=1}^{\infty} \frac{c_k}{k^t},
\]
(2.4)
where \(c_k = \tilde{c}_k k^{-\sigma - \beta}\). Then \(\varphi(s)\) is absolutely convergent in the region \(\sigma > 1\).

In [17], the second author proved two limit theorems on \(\log \tilde{\varphi}(s)\) in the complex plane under the following assumptions (which we write in terms of \(\varphi\) here):

(i) \(\varphi(s)\) can be continued meromorphically to \(\sigma \geq \sigma_0\), where \(1/2 \leq \sigma_0 < 1\), and all poles in this region are included in a compact set which has no intersection with the line \(\sigma = \sigma_0\),

(ii) \(\varphi(\sigma + it) = O(|t|^{C_2})\) for \(\sigma \geq \sigma_0\), where \(C_2\) is a positive constant,

(iii) the mean-value estimate
\[
\int_0^T |\varphi(\sigma_0 + it)|^2 dt = O(T)
\]
(2.5)
holds.

We denote the set of all such \(\varphi(s)\) (defined by (2.4) and satisfies (i), (ii) and (iii)) as \(M\). Laurinčikas [7]–[9] called \(\varphi(s)\) a Matsumoto zeta-function, and proved its limit theorems in various function spaces. We now prepare several notations which are necessary to formulate a limit theorem in the style of Laurinčikas.

Let \(B(S)\) stand for the set of all Borel subsets of a set \(S\). For any open region \(G\) in the complex plane, let \(H(G)\) be the space of holomorphic functions on \(G\) with the uniform convergence topology. It is known that there exists a sequence of compact subsets \(\{K_l\} \in N\) of \(G\) such that \(K_1 \subset K_2 \subset \cdots\), \(G = \bigcup_{l=1}^{\infty} K_l\), and for any compact subset \(K \subset G\), there is some \(l\) for which \(K \subset K_l\) holds. Using this sequence, we can define the metric \(\rho = \rho(G)\) on \(H(G)\) by
\[
\rho(f, g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\rho_l(f, g)}{1 + \rho_l(f, g)}
\]
for \(f, g \in H(G)\) with
\[
\rho_l(f, g) = \sup_{s \in K_l} |f(s) - g(s)|.
\]

Let \(\gamma\) be the unit circle in the complex plane, and define
\[
\Omega_1 = \prod_{p \in P} \gamma_p,
\]
where \(\gamma_p = \gamma\) for all \(p\). With the product topology and the pointwise multiplication, \(\Omega_1\) becomes a compact topological group, and hence there exists a Haar measure \(m_{1H}\) on \(\Omega_1\) with the normalization \(m_{1H}(\Omega_1) = 1\). Thus we obtain a probability space \((\Omega_1, B(\Omega_1), m_{1H})\). Let \(\omega_1(p)\) stand for the projection of \(\omega_1 \in \Omega_1\) to the coordinate space \(\gamma_p\). Define \(\omega_1(k)\) for any \(k \in N\) by
\[
\omega_1(k) = \prod_{j=1}^{r} \omega_1(p_j)^{c_j},
\]
when the decomposition of \(k\) into prime divisors is \(k = p_1^{c_1} \cdots p_r^{c_r}\).

Associated to the expression (2.4) of \(\varphi(s)\), we define
\[
\varphi(s, \omega_1) = \sum_{k=1}^{\infty} \frac{c_k \omega_1(k)}{k^t}.
\]
(2.6)
Then \(\varphi(s, \omega_1)\) is convergent almost surely, uniformly in any compact subset of
\[
D = \{s \mid \sigma > 1/2\}
\]
(see [8], [9]). Therefore $\varphi(s, \omega_1)$ is an $H(D)$-valued random element defined on $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. This random element was introduced by Laurinčikas [8], [9], and is the main tool in those papers.

Next, let

$$\Omega_2 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m$, and $m_{2H}$ be the probability Haar measure on the probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$.

Define

$$\zeta(s, \alpha, \omega_2; \mathbf{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m + \alpha)^s},$$

where $\omega_2(m)$ is the projection of $\omega_2 \in \Omega_2$ to the coordinate space $\gamma_m$. This is convergent almost surely, uniformly in any compact subset of $D$ (see [4]; the proof is similar to that of [11, Lemma 5.2.1]), so it is an $H(D)$-valued random element defined on $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$.

Further, let $\Omega = \Omega_1 \times \Omega_2$, and $m_H$ its probability Haar measure defined as a product of measures $m_{1H}$ and $m_{2H}$. Let $H^2(D) = H(D) \times H(D)$. Define

$$Z(\mathbf{s}, \omega) = (\varphi(s_1, \omega_1), \zeta(s_2, \alpha, \omega_2; \mathbf{b})),
$$

where $\mathbf{s} = (s_1, s_2) \in D \times D$ and $\omega = (\omega_1, \omega_2) \in \Omega$. This is an $H^2(D)$-valued random element defined on $(\Omega, \mathcal{B}(\Omega), m_H)$.

Let $D_1, D_2 \subset D$ be two open subsets of $D$. Then $\varphi(s_1, \omega_1)$ may be regarded as an $H(D_1)$-valued random element, by just restricting the region of definition to $D_1$. Similarly $\zeta(s_2, \alpha, \omega_2; \mathbf{b})$ may be regarded as an $H(D_2)$-valued random element. Therefore $Z(\mathbf{s}, \omega)$ may be regarded as an $H(D_1) \times H(D_2)$-valued random element.

Denote by $P_Z = P_{Z(\omega)}$ the distribution of $Z(s, \omega)$ as an $H(D_1) \times H(D_2)$-valued random element, that is

$$P_Z(A) = m_H(\omega \in \Omega \mid Z(s, \omega) \in A) \quad \text{for} \quad A \in \mathcal{B}(H(D_1) \times H(D_2)).$$

The function $\varphi(s)$ has only finitely many poles by condition (i). Denote those poles by $s_1(\varphi), \ldots, s_l(\varphi)$, and define

$$D_\varphi = \{ s \mid \sigma > \sigma_0, \sigma \neq \Re s_j(\varphi) \ (1 \leq j \leq l) \}.$$

Then $\varphi(s)$ and its vertical shift $\varphi(s + i \tau)$ are holomorphic in $D_\varphi$. The function $\zeta(s, \alpha; \mathbf{b})$ can be written as a linear combination of Hurwitz zeta-functions, therefore it is entire, or has a simple pole only at $s = 1$. Therefore $\zeta(s, \alpha; \mathbf{b})$ and its vertical shift $\zeta(s + i \tau, \alpha; \mathbf{b})$ are holomorphic in

$$D' = \begin{cases} D & \text{if } \zeta(s, \alpha; \mathbf{b}) \text{ is entire,} \\ \{ s \mid \sigma > 1/2, \sigma \neq 1 \} & \text{if } s = 1 \text{ is a pole of } \zeta(s, \alpha; \mathbf{b}). \end{cases}$$

Let $T > 0$, $D_1 \subset D_\varphi$, $D_2 \subset D'$, and define the probability measure $P_T = P_{T(\omega)}$ on $H(D_1) \times H(D_2)$ by

$$P_T(A) = \frac{1}{T} \mu_{\{ t \in [0, T] \mid Z(s + i \tau) \in A \}} \quad \text{for} \quad A \in \mathcal{B}(H(D_1) \times H(D_2)),$$

where $s + i \tau = (s_1 + i \tau, s_2 + i \tau)$ with $s_1 \in D_1, s_2 \in D_2$, and

$$Z(s) = (\varphi(s_1), \zeta(s_2, \alpha; \mathbf{b})).$$

Our first main result is the following functional limit theorem.

**Theorem 2.1** Suppose $\alpha$ is a transcendental number, $0 < \alpha < 1$. Let $T > 0$, $D_1$ be an open subset of $D_\varphi$, and $D_2$ be an open subset of $D'$. Then the measure $P_T = P_{T(\omega)}$ converges weakly to $P_Z = P_{Z(\omega)}$ as $T \to \infty$.

This is a generalization of [21, Theorem 1] and an analogue of [5, Theorem 6], and is fundamental in the present paper.

The standard proof of universality theorems usually consists of two parts; a limit theorem, and a denseness lemma. The limit theorem (Theorem 2.1) can be shown for any $\varphi \in \mathcal{M}$. However, it is difficult to prove the necessary denseness lemma for general $\varphi$. Therefore we need further assumptions.
The first attempt to prove a universality theorem for the class $\mathcal{M}$ was carried out by Laurinčikas [10], in which he assumed a certain lower bound of sums of coefficients $a_m^{(j)}$. We may assume the same type of assumption to show a joint universality theorem.

However, in the present paper we prefer to adopt another type of assumptions, due to Steuding [25].

Steuding first considered a certain subclass of the Selberg class, and proved the universality for zeta-functions belonging to that class in [24]. Later, he extended that class to introduce the class $\tilde{S}$ in [25], which is now sometimes called the Steuding class. We say that $\varphi$ belongs to the class $\tilde{S}$ if it satisfies the following five conditions:

(a) Dirichlet series expansion $\varphi(s) = \sum_{m=1}^{\infty} a(m) m^{-s}$ with the estimate $a(m) = O(m^\varepsilon)$ for any $\varepsilon > 0$;

(b) (from (a) we see that the Dirichlet series is convergent absolutely for $\sigma > 1$, but) there exists $\sigma_\varphi < 1$ such that $\varphi(s)$ can be continued meromorphically to the region $\sigma > \sigma_\varphi$, and holomorphic there except for at most a pole at $s = 1$;

(c) there exists $C \geq 0$ such that $\varphi(\sigma + it) = O(|t|^{C+\varepsilon})$ for any fixed $\sigma > \sigma_\varphi$ and any $\varepsilon > 0$;

(d) there is the Euler product expansion

$$\varphi(s) = \prod_{\mathfrak{p}} \prod_{j=1}^{l} \left( 1 - \frac{a_j(p)}{p^s} \right)^{-1};$$

(e) there exists a constant $\kappa > 0$ for which

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa$$

holds, where $\pi(x)$ denotes the number of prime numbers up to $x$, and the summation runs over prime numbers.

The class $\tilde{S}$ is not a subclass of the Selberg class. On the other hand, we can observe that $\tilde{S}$ is a subclass of the class $\mathcal{M}$. In fact, let $\sigma^*$ be the infimum of all $\sigma_\varphi$ for which

$$\frac{1}{2T} \int_{-T}^{T} |\varphi(\sigma + it)|^2 dt \sim \sum_{m=1}^{\infty} \frac{|a(m)|^2}{m^{2\sigma}}$$

holds for any $\sigma \geq \sigma_\varphi$. Then [25, Theorem 2.4] says that $1/2 \leq \sigma^* < 1$. Therefore we may choose $\sigma_0 = \sigma^* + \varepsilon$ with any small $\varepsilon$ to find that, with this $\sigma_0$, the function $\varphi(s)$ satisfies the assumptions (i), (ii) and (iii) of the class $\mathcal{M}$.

Now we state the second main result in the present paper, the mixed joint universality theorem among the elements of Steuding class and periodic Hurwitz zeta-functions.

**Theorem 2.2** Suppose that $\varphi \in \tilde{S}$, and $\alpha$ is a transcendental number. Let $K_1$ be a compact subset of $D(\sigma^*, 1)$, $K_2$ be a compact subset of $D(1/2, 1)$, both with connected complements. Then, for any $f_1 \in H^c_\mathcal{S}(K_1)$, $f_2 \in H^c(K_2)$, and any $\varepsilon > 0$, it holds that

$$\liminf_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] \left| \max_{x \in K_1} |\varphi(s + i\tau) - f_1(s)| < \varepsilon, \max_{x \in K_2} |\zeta(s + i\tau, \alpha; \mathbf{b}) - f_2(s)| < \varepsilon \right. \right\} > 0. \quad (2.8)$$

The Steuding class includes a lot of arithmetically important zeta and $L$-functions, such as Dirichlet $L$-functions $L(s, \chi)$ and automorphic $L$-functions $L(s, f)$ attached to holomorphic cusp forms $f$ of $SL(2, \mathbb{Z})$ or its congruence subgroups. Therefore the above theorem includes the results when $\varphi(s) = L(s, \chi)$ and $\varphi(s) = L(s, f)$. In those special cases, such results were already studied by several papers written by Laurinčikas and his colleagues [2], [3], [12], [15], [16], [22] in a more generalized form (that is, joint universality theorems for several Dirichlet $L$-functions, and/or several periodic Hurwitz zeta-functions).
3 Proof of Theorem 2.1

The proof of our Theorem 2.1 goes along the same line as the argument developed in Mishou [21] and Kačinskaite and Laurinčikas [5], so we just give a sketch of the proof.

First, let
\[ a_t = ((p^{-i\tau}; p \in P), ((m + \alpha)^{-i\tau}; m \in N_0)) \in \Omega. \]
and define the probability measure \( Q_T \) on \( \Omega \) by
\[
Q_T(A) = \frac{1}{T} \mu \{ \tau \in [0, T] \mid a_\tau \in A \}
\]
for \( A \in \mathcal{B}(\Omega). \)

**Lemma 3.1** Suppose that \( \alpha \) is transcendental. Then the probability measure \( Q_T \) converges weakly to the Haar measure \( m_H \) as \( T \to \infty. \)

**Proof.** This is [21, Lemma 3]. An essential point in the proof of this lemma is the fact that \( \log p (p \in P) \) and \( \log(m + \alpha) (m \in N_0) \) are linearly independent over \( Q \), because \( \alpha \) is transcendental. \( \square \)

Let \( \sigma_1 > 1/2, \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2) \in \Omega \) and
\[
v_1(m, n) = \exp \left\{ -\left( \frac{m}{n} \right)^{\sigma_1} \right\}, \quad v_2(m, n, \alpha) = \exp \left\{ -\left( \frac{m + \alpha}{n + \alpha} \right)^{\sigma_1} \right\}.
\]
For \( n \in N \), define
\[
\phi_n(s) = \sum_{k=1}^{\infty} \frac{c_k v_1(k, n)}{k^s}, \quad \phi_{n, \alpha}(s) = \sum_{k=1}^{\infty} \frac{c_k \tilde{\omega}_1(k) v_1(k, n)}{k^s}.
\]
These series were introduced in Laurinčikas [7], [9], and are absolutely convergent for \( s \in D \), because \( v_1(m, n) = O(m^{-\sigma_1}) \) (see [7, p. 153]). Similarly define
\[
\xi_n(s, \alpha; \mathbf{b}) = \sum_{m=0}^{\infty} b_m v_2(m, n, \alpha), \quad \xi_n(s, \alpha, \tilde{\omega}_2; \mathbf{b}) = \sum_{m=0}^{\infty} \frac{b_m \tilde{\omega}_2(m) v_2(m, n, \alpha)}{(m + \alpha)^s},
\]
which are absolutely convergent for \( s \in D \).

Let \( Z_n(s) = (\phi_{n, s}(s_1), \xi_{n, s}(s_2, \alpha; \mathbf{b})) \)
and
\( Z_n(s, \tilde{\omega}) = (\phi_n(s_1, \tilde{\omega}_1), \xi_n(s_2, \alpha, \tilde{\omega}_2; \mathbf{b})). \)

Define the probability measures \( P_{\tau, n} = P_{\tau, n,(D_1, D_2)} \) and \( \hat{P}_{\tau, n} = \hat{P}_{\tau, n,(D_1, D_2)} \) by
\[
P_{\tau, n}(A) = \frac{1}{T} \mu \{ \tau \in [0, T] \mid Z_n(s + i\tau) \in A \}
\]
and
\[
\hat{P}_{\tau, n}(A) = \hat{P}_{\tau, n}(s, \tilde{\omega}) = \frac{1}{T} \mu \{ \tau \in [0, T] \mid Z_n(s + i\tau, \tilde{\omega}) \in A \}
\]
for \( A \in \mathcal{B}(H(D_1) \times H(D_2)). \)

**Lemma 3.2** There exists a probability measure \( P_n \) on \((H(D_1) \times H(D_2), \mathcal{B}(H(D_1) \times H(D_2))) \) such that the measures \( P_{\tau, n} \) and \( \hat{P}_{\tau, n} \) (for any \( \tilde{\omega} \)) both converge weakly to \( P_n \) as \( T \to \infty. \)

**Proof.** This is analogous to [21, Lemma 5] and [5, Theorem 8]. The proof in [21] is based on the approximation to the above series by their finite truncations. The argument in [5] is more direct. Here we follow the way of [5] and sketch the proof.
Define $u_n : \Omega \to H(D_1) \times H(D_2)$ by

$$u_n(\omega_1, \omega_2) = \left( \sum_{k=1}^{\infty} c_k \omega_1(k) v_1(k, n), \sum_{m=0}^{\infty} b_m \omega_2(m) v_2(m, n, \alpha) \right).$$

Then we have

$$u_n(a_\tau) = Z_n(s + i \tau),$$

and hence $P_{T,n} = Q_T \circ u_n^{-1}$. Therefore, using Lemma 3.1, we see that $P_{T,n}$ converges weakly to $m_H \circ u_n^{-1}$ as $T \to \infty$.

Similarly, we have

$$(u_n \circ v)(a_\tau) = Z_n(s + i \tau, \tilde{\omega}),$$

where $v : \Omega \to \Omega$ is defined by $v(\omega) = \omega \tilde{\omega}$. Therefore $\tilde{P}_{T,n}$ converges weakly to $m_H \circ (u_n \circ v)^{-1}$ as $T \to \infty$. However, $m_H \circ (u_n \circ v)^{-1} = m_H \circ u_n^{-1}$ because $m_H$ is the Haar measure. Therefore the lemma follows with $P_n = m_H \circ u_n^{-1}$.

Let $\rho_1 = \rho(D_1), \rho_2 = \rho(D_2)$. For $f = (f_1, f_2), g = (g_1, g_2) \in H(D_1) \times H(D_2)$, define

$$\rho(f, g) = \max\{\rho_1(f_1, g_1), \rho_2(f_2, g_2)\}.$$

This is a metric on $H(D_1) \times H(D_2)$.

**Lemma 3.3** For $s = (s_1, s_2) \in D_1 \times D_2$, we have

$$\lim_{n \to \infty} \sup_{T \to \infty} \frac{1}{T} \int_0^T \rho(Z(s + i \tau), Z_n(s + i \tau)) d\tau = 0$$

(3.1)

and, for almost all $\omega \in \Omega$,

$$\lim_{n \to \infty} \sup_{T \to \infty} \frac{1}{T} \int_0^T \rho(Z(s + i \tau, \omega), Z_n(s + i \tau, \omega)) d\tau = 0.$$  

(3.2)

**Proof.** Let $K$ be a compact subset of $D_1$. Then

$$\lim_{n \to \infty} \sup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K} |\varphi(s + i \tau) - \varphi_n(s + i \tau)| d\tau = 0.$$  

(3.3)

This is a special case of [7, formula (16)]. In [7], the influence of the poles is not rigorously considered. However, now we restrict our consideration to the region $D_1 \subset D_\omega$, so there is no pole. Therefore (3.3) is valid.

Also, as a special case of [9, Lemma 11], we have

$$\lim_{n \to \infty} \sup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K} |\varphi(s + i \tau, \omega_1) - \varphi_n(s + i \tau, \omega_1)| d\tau = 0.$$  

(3.4)

for almost all $\omega_1$. In [9], the formula is stated not for $\varphi(s)$, but for

$$\Psi(s) = \varphi(s) \prod_{j=1}^{l} (1 - 2^j/j^{s-j}) ,$$

which is holomorphic in the region $\sigma > \sigma_0$. But the same argument can be applied to $\varphi(s)$ in the region $D_1$, so (3.4) is valid.

The formulas, corresponding to (3.3) and (3.4), for $\zeta(s, \alpha; b)$ and $\xi(s, \alpha, \omega_2; b)$ have been shown in Javtokas and Laurinčikas [4]. Gathering these results we obtain the assertion of the lemma.

Define the probability measure $\tilde{P}_T = \tilde{P}_{T, (D_1, D_2)}$ on $H(D_1) \times H(D_2)$ by

$$\tilde{P}_T(A) = \tilde{P}_T(A, \omega) = \frac{1}{T} \mu(\tau \in [0, T] \mid Z(s + i \tau, \omega) \in A)$$

for $A \in B(H(D_1) \times H(D_2))$. 

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Lemma 3.4 There exists a probability measure $P$ on $(H(D_1) \times H(D_2), \mathcal{B}(H(D_1) \times H(D_2)))$ such that the measures $P_T$ and $P_T^{\alpha}$ (for almost all $\alpha$) both converge weakly to $P$ as $T \to \infty$.

Proof. This is an analogue of [21, Lemma 6] and [5, Theorem 10]. Consider the case of $P_T$.

The first task is to show that the family $\{P_n \mid n \in \mathbb{N}\}$ is tight. The argument is standard, based on mean value estimates of $\phi_n(s)$ and $\rho_n(s, \alpha; b)$. We omit the details; see [5, pp. 269–270].

Then by Prokhorov’s theorem, we can find a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that $P_{n_k}$ converges weakly to a certain measure $P$ on $H(D_1) \times H(D_2)$ as $k \to \infty$. That is, $X_{n_k} \to P$ in distribution, where $X_n$ denotes an $H(D_1) \times H(D_2)$-valued random element with the distribution $P_n$.

Let $\eta$ be a random variable defined on a certain probability space $(\Omega, \mathcal{B}(\Omega), m)$ and uniformly distributed on the interval $[0, 1]$. Define the $H(D_1) \times H(D_2)$-valued random element $X_{T,n}$ by

$$X_{T,n} = X_{T,n}(s) = Z_n(s + iT\eta).$$

Then the distribution of $X_{T,n}$ is $P_{T,n}$, so by Lemma 3.2, $X_{T,n} \to X_T$ in distribution.

Now define

$$X_T = X_T(s) = Z(s + iT\eta).$$

This is an $H(D_1) \times H(D_2)$-valued random element, whose distribution is $P_T$. For any $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \limsup_{T \to \infty} m \left( \rho(X_T, X_{T,n}) \geq \varepsilon \right) = \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \left\{ \tau \in [0, T] \mid \rho(Z(s + iT\eta), Z_n(s + iT\eta)) \geq \varepsilon \right\} \leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon T} \int_0^T \rho(Z(s + iT\eta), Z_n(s + iT\eta)) d\tau = 0$$

by (3.1).

Collecting all the above results and using [1, Theorem 4.2], we obtain $X_T \to P$ in distribution as $T \to \infty$. Therefore $P_T$ converges weakly to $P$ as $T \to \infty$.

The above conclusion also implies that $P$ is independent of the choice of $\{n_k\}$. Therefore we find that $X_T \to P$ in distribution as $n \to \infty$ (by [6, Theorem 1.1.9]). Noting this fact, and repeating the above argument in the case of $P_T$ (using (3.2) instead of (3.1)), we obtain that $P_T$ also converges weakly to $P$ as $T \to \infty$. \hfill \Box

To complete the proof of Theorem 2.1, the only remaining thing is to show that $P = P_Z$. This can be done in a standard way, using ergodic theory.

Define $\Phi : \Omega \to \Omega$ by $\Phi(\omega) = \alpha_\omega \omega (\omega \in \Omega, \tau \in \mathbb{R})$. Then $\{\Phi_\tau \mid \tau \in \mathbb{R}\}$ is a one-parameter group of measurable transformations on $\Omega$. We can show that this group is ergodic ([21, Lemma 7], [5, Lemma 11]). Here, again, the assumption that $\alpha$ is transcendental is essentially used.

Therefore, similarly to the argument in pp. 77–78 of [21], or pp. 272–273 of [5], based on the Birkhoff-Khinchin theorem in ergodic theory, we obtain the assertion $P = P_Z$. The proof of Theorem 2.1 is complete. \hfill \Box

4 Proof of Theorem 2.2

Assume that $\varphi$, $K_1$, $K_2$, $f_1$ and $f_2$ are as in the statement of Theorem 2.2. Then we can find a real number $\sigma_0$ with $\sigma^* < \sigma_0 < 1$ and a positive number $M > 0$, such that $K_1$ is included in the open rectangle

$$D_M = \{s \mid \sigma_0 < \sigma < 1, |t| < M\}.$$

Since now $\varphi \in \tilde{S}$, the pole of $\varphi$ is at most at $s = 1$, so in this case we find that

$$D_\varphi = \{s \mid \sigma > \sigma_0, \sigma \neq 0\}.$$

Therefore $D_M$ is an open subset of $D_\varphi$. Also we can find $N > 0$ such that $K_2$ is included in the open rectangle

$$D_N = \{s \mid 1/2 < \sigma < 1, |t| < N\}.$$
We use Theorem 2.1 with $D_1 = D_M$ and $D_2 = D_N$ to obtain

**Lemma 4.1** Suppose that $\alpha$ is transcendental. Then the measure $P_T = P_{T, (D_M, D_N)}$ converges weakly to $P_\mathbb{Z} = P_{\mathbb{Z}, (D_M, D_N)}$ as $T \to \infty$.

**Remark 4.2** In many previous articles, this type of limit theorems was first shown in some wider regions, and then deduced by “restrictions”. In the present paper, we prove it more directly, without using such “restriction” argument.

Now we determine the support of the measure $P_\mathbb{Z}$. Let $S_\phi$ be the set of all $f \in H(D_M)$ which is non-vanishing on $D_M$, or constantly $0$ on $D_M$.

**Lemma 4.3** The support of the measure $P_\mathbb{Z}$ is the set $S = S_\phi \times H(D_N)$.

**Proof.** This lemma is an analogue of [5, Lemma 12], and the proof is also analogous. Since $H(D_M)$ and $H(D_N)$ are separable, we have

$$B(H(D_M) \times H(D_N)) = B(H(D_M)) \times B(H(D_N))$$

(4.1)

(see [1, p. 20, p. 225]). Let $g = (g_1, g_2) \in H(D_M) \times H(D_N)$. Then $g \in S$ if and only if $P_\mathbb{Z}(A) > 0$ for any neighborhood $A$ of $g$. In view of (4.1), it is enough to consider the case $A = A_1 \times A_2$, where $A_1 \in H(D_M)$ and $A_2 \in H(D_N)$. Then

$$P_\mathbb{Z}(A) = m_H(\{ \omega \in \Omega | Z(s, \omega) \in A \}) = m_H(\{ \omega \in \Omega | \varphi(s, \omega_1) \in A_1, \zeta(s_2, \alpha, \omega_2; b) \in A_2 \}) = m_{1H}(\{ \omega_1 \in \Omega \mid \varphi(s_1, \omega_1) \in A_1 \}) \times m_{2H}(\{ \omega_2 \in \Omega_2 \mid \zeta(s_2, \alpha, \omega_2; b) \in A_2 \}).$$

Here, $A_1$ is a neighborhood of $g_1$ and $A_2$ is a neighborhood of $g_2$. Therefore, $P_\mathbb{Z}(A)$ is positive for any $A$, if and only if $g_1$ belongs to the support of $\varphi(s_1, \omega_1)$ and $g_2$ belongs to the support of $\zeta(s_2, \alpha, \omega_2; b)$. The support of $\varphi(s_1, \omega_1)$ is $S_\phi$ ([25, Lemma 5.12]), and the support of $\zeta(s_2, \alpha, \omega_2; b)$ is the whole of $H(D_N)$ ([4, Lemma 5]). This implies $S = S_\phi \times H(D_N)$.

**Remark 4.4** In the above proof we use the result on $S_\phi$ due to Steuding ([25, Lemma 5.12]). This follows from the denseness lemma ([25, Theorem 5.10]). To prove the latter denseness lemma, the so-called positive density method, whose origin goes back to [14] (and also [13]), is applied. The “prime-number-theorem type” condition (e) of the Steuding class is essentially used here. This is the reason why we cannot prove the universality theorem for general $\phi \in \mathcal{M}$ by the method of the present paper.

Now we complete the proof of Theorem 2.2. First we assume that $f_1(s)$ and $f_2(s)$ have holomorphic continuations to $D_M$ and $D_N$, respectively, and the continuation of $f_1(s)$ is non-vanishing on $D_M$. Define

$$G = \left\{ g = (g_1, g_2) \in H(D_M) \times H(D_N) \mid \max_{\tau \in \mathbb{K}_i} |g_j(s) - f_j(s)| < \varepsilon \quad (j = 1, 2) \right\}.$$ 

Then $G$ is an open subset of $H(D_M) \times H(D_N)$. Therefore by Lemma 4.1 we have

$$\lim_{T \to \infty} \frac{1}{T} \mu([0, T] \mid Z(s + i \tau) \in G) = \lim_{T \to \infty} \mu_T(G) \geq P_\mathbb{Z}(G).$$

(4.2)

Since $(f_1, f_2) \in S$ by Lemma 4.3 and $G$ is an open neighborhood of $(f_1, f_2)$, we have $P_\mathbb{Z}(G) > 0$. Hence from (4.2) the assertion of Theorem 2.2 follows.

The proof of the general case can be reduced to the above special case. We sketch the argument (for the details, see [5], pp. 274–275). First, by Mergelyan’s approximation theorem ([19], [20], see also [28]), we find polynomials $p_j(s)$ which approximate $f_j(s)$ uniformly on $K_j$ ($j = 1, 2$). Further we may assume that $p_1(s)$ is non-vanishing on $K_1$, or on a region $K_1$ slightly wider than $K_1$. Then we can define log $f_1(s)$ on $K_1$. Again using Mergelyan’s theorem we find a polynomial $\tilde{p}_1(s)$ such that $\exp(\tilde{p}_1(s))$ approximates $p_1(s)$ uniformly on $K_1$.

Then, $\exp(\tilde{p}_1(s))$ and $p_2(s)$ are holomorphic and $\exp(\tilde{p}_1(s))$ does not vanish on $D_M$. Therefore we can apply the result on the above special case to obtain that (2.8), replacing $f_1(s)$ and $f_2(s)$ by $\exp(\tilde{p}_1(s))$ and $p_2(s)$ respectively, is valid.
Combining this result with the facts that \( \exp(\tilde{p}_1(s)) \) approximates \( f_1(s) \) while \( p_2(s) \) approximates \( f_2(s) \), we arrive at the desired conclusion.

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