Bernoulli type polynomials on Umbral Algebra

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Abstract

The aim of this paper is to investigate generating functions for modification of the Milne-Thomson’s polynomials, which are related to the Bernoulli polynomials and the Hermite polynomials. By applying the Umbral algebra to these generating functions, we provide to deriving identities for these polynomials.

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1. Introduction

Throughout of this paper, we use the following notations:

\[ \mathbb{N} := \{1, 2, 3, \ldots \} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \]

\[ \delta_{n,k} = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k, \end{cases} \]

and

\[ (x)_b = x(x-1)\ldots(x-b+1), \]

where \( b \in \mathbb{N} \).

Here, we use the notations and definitions which are related to the umbral algebra and calculus cf. [6].

Let \( P \) be the algebra of polynomials in the single variable \( x \) over the field complex numbers. Let \( P^* \) be the vector space of all linear functionals on \( P \). Let \( \langle L \mid p(x) \rangle \) be the action of a linear functional \( L \) on a polynomial \( p(x) \). Let \( F \) denotes the algebra of formal power series

\[ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k, \]

cf. [6].

This kind of algebra is called an umbral algebra. Each \( f \in F \) defines a linear functional on \( P \) and for all \( k \geq 0 \), \( a_k = \langle f(t) \mid x^k \rangle \). The order \( o(f(t)) \) of a power series \( f(t) \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. A series \( f(t) \) for which
Let \( f(t) \) and \( g(t) \) be in \( F \), we have
\[
\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle ,
\]
(1.1)
cf. [6]. For all \( p(x) \) in \( P \), we have
\[
\langle e^{yt} \mid p(x) \rangle = p(y)
\]
(1.2)
and
\[
e^{yt}p(x) = p(x + y)
\]
(1.3)
cf. [6].

**Theorem 1.** ([6, p. 20, Theorem 2.3.6]) Let \( f(t) \) be a delta series and let \( g(t) \) be an invertible series. Then there exist a unique sequence \( s_n(x) \) of polynomials satisfying the orthogonality conditions
\[
\langle g(t)f(t)^k \mid s_n(x) \rangle = n!\delta_{n,k}
\]
(1.4)
for all \( n, k \geq 0 \).

The sequence \( s_n(x) \) in (1.4) is the Sheffer polynomials for pair \((g(t), f(t))\), where \( g(t) \) must be invertible and \( f(t) \) must be delta series. The Sheffer polynomials for pair \((g(t), t)\) is the Appell polynomials or Appell sequences for \( g(t) \).

The Appell polynomials are defined by means of the following generating function:
\[
\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)}e^{xt},
\]
(1.5)
cf. [6].

The Appell polynomials satisfy the following relations:
\[
s_n(x) = (g(t))^{-1}x^n,
\]
(1.6)
derivative formula
\[
t s_n(x) = s'_n(x) = ns_{n-1}(x)
\]
(1.7)
and
\[
\frac{1}{t} s_n(x) = \frac{1}{n+1} s_{n+1}(x),
\]
(1.8)
recurrence formula
\[
s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) s_n(x),
\]
(1.9)
and multiplication formula, for \( \alpha \neq 0 \)
\[
s_n(\alpha x) = \alpha^n \frac{g(t)}{g(\alpha)} s_n(x).
\]
(see, for details, [6]; and see also [1], [3], [4]).

The remainder of this paper is organized as follows: We modify generating functions for the Milne-Thomson’s polynomials \( \Phi_n^{(a)}(x) \). We give some properties of this functions. By applying the Umbral algebra and Umbral calculus, we derive some identities related to Hermite polynomials, Bernoulli polynomials and Stirling numbers of second kind.
2. NEW TYPE POLYNOMIALS

We modify the Milne-Thomson’s polynomials $\Phi_n^{(a)}(x)$ (see for detail [5]) as $\Phi_n^{(a)}(x, v)$ of degree $n$ and order $a$ by the means of the following generating function:

$$g_1(t, x; a, v) = f(t, a) e^{xt + h(t, v)} = \sum_{n=0}^{\infty} \Phi_n^{(a)}(x, v) \frac{t^n}{n!}$$

(2.1)

where $f(t, a)$ is a function of $t$ and the integer $a$.

Observe that $\Phi_n^{(a)}(x, 0) = \Phi_n^{(a)}(x)$ cf. [5].

Remark 1. Setting $f(t, a) = \left(\frac{t}{e^t - 1}\right)^a$ in (2.1), we obtain the following polynomials by

$$g_2(t, x; a, v) = \left(\frac{t}{e^t - 1}\right)^a e^{xt + h(t, v)} = \sum_{n=0}^{\infty} \beta_n^{(a)}(x; v) \frac{t^n}{n!}.$$  

(2.2)

Observe that the polynomials $\beta_n^{(a)}(x; v)$ are related to not only Bernoulli polynomials but also the Hermite polynomials. For example, if $h(t, 0) = 0$ in (2.2), we have

$$\beta_n^{(a)}(x, 0) = B_n^{(a)}(x),$$

where $B_n^{(a)}(x)$ denotes the Bernoulli polynomials of higher order which is, defined by means of the following generating function

$$f_B(t, x; a) = \left(\frac{t}{e^t - 1}\right)^a e^{xt} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!}.$$  

One can easily see that $B_n^{(a)}(0) = B_n^{(a)}$, that is

$$f_B(t; a) = \left(\frac{t}{e^t - 1}\right)^a = \sum_{n=0}^{\infty} B_n^{(a)} \frac{t^n}{n!}.$$  

If we take $h(t) = -\frac{vt^2}{2}$ in (2.2), we have

$$\left(\frac{t}{e^t - 1}\right)^a e^{xt - \frac{vt^2}{2}} = \sum_{n=0}^{\infty} \left( H_n^{(a)}(x, v) \right) \frac{t^n}{n!}.$$  

Hence, we get

$$H_n^{(0)}(x, v) = H_n^{(v)}(x)$$

where $H_n^{(v)}(x)$ denotes the Hermite polynomials of higher order, which is defined by means of the following generating function:

$$f_H(x, t; v) = e^{xt - \frac{vt^2}{2}} = \sum_{n=0}^{\infty} H_n^{(v)}(x) \frac{t^n}{n!}.$$  

We define the following functional equation:

$$g_2(t, x; a, v) = f_B(t, x; a) e^{h(t, v)}.$$  

(2.3)
By the above functional equation, we get
\[ g_2(t, x; a, v) = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{h(t, v)^n}{n!}. \] (2.4)

If we set \( h(t, v) = -vt \) in (2.4), we have
\[ \beta_n^{(a)}(x, v) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_j^{(a)}(x) v^{n-j}. \]

We define the following functional equation:
\[ g_2(t, x; a, v) = f_B(t; a) e^{xt + h(t, v)}. \] (2.5)

If we set \( h(t, v) = -\frac{v^2 t^2}{2} \) in (2.3), we obtain the following theorem:

**Theorem 2.**
\[ \beta_n^{(a)}(x, v) = \sum_{j=0}^{n} \binom{n}{j} B_j^{(a)}(x) H_{n-j}^{(v)}. \]

From (2.5), we get
\[ \frac{\partial}{\partial x} g_2(t, x; a, v) = tg_2(t, x; a, v). \]

By using the above partial derivative equation, we obtain the following theorem:

**Theorem 3.**
\[ \frac{\partial}{\partial x} \beta_n^{(a)}(x, v) = n\beta_{n-1}^{(a)}(x, v). \]

By using (1.7) and the above theorem, it is easily to see that \( \beta_n^{(a)}(x, v) \) is an Appell-type sequence.

3. Some identities for the polynomials \( H\beta_n^{(a)}(x, v) \)

In this section, by applying the Umbral algebra and Umbral calculus, we derive some identities related to the polynomials \( H\beta_n^{(a)}(x, v) \).

By substituting
\[ g(t) = \left( e^t - \frac{1}{t} \right)^a e^{\frac{vt^2}{2}} \] (3.1)
into \( (1.6) \), one can easily obtain the following lemma:

**Lemma 1.** Let \( n \in \mathbb{N}_0 \). The following relationship holds true:
\[ H\beta_n^{(a)}(x, v) = \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{vt^2}{2}} x^n. \]

By using (1.7) and (1.8), we arrive at the following lemma:
Lemma 2.
\[ t_H \beta_n^{(a)}(x, v) = n_H \beta_{n-1}^{(a)}(x, v), \tag{3.2} \]
and
\[ \frac{1}{t} \beta_n^{(a)}(x, v) = \frac{1}{n + 1} \beta_{n+1}^{(a)}(x, v). \tag{3.3} \]

The action of a linear operator \( (e^t - 1) \) on the polynomial \( H \beta_n^{(a)}(x, v) \) is given by the following lemma:

Lemma 3.
\[ (e^t - 1) H \beta_n^{(a)}(x, v) = n_H \beta_{n-1}^{(a-1)}(x, v). \]

Proof. By using Lemma 1, we obtain
\[ (e^t - 1) H \beta_n^{(a)}(x, v) = (e^t - 1) \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{t^2}{2}} x^n. \]

After some calculations in the above equation, we get
\[ (e^t - 1) H \beta_n^{(a)}(x, v) = t H \beta_{n-1}^{(a-1)}(x, v). \]
Using (3.2) in the above equation, we arrive at the desired result. \( \square \)

From Lemma 3, we arrive at the following result:

Corollary 1.
\[ e^t H \beta_n^{(a)}(x, v) = n_H \beta_{n-1}^{(a-1)}(x, v) + H \beta_n^{(a)}(x, v). \tag{3.4} \]

Theorem 4.
\[ H \beta_n^{(a)}(x + 1, v) = n_H \beta_{n-1}^{(a-1)}(x, v) + H \beta_n^{(a)}(x, v). \]

Proof. Using (1.3), we get
\[ e^t H \beta_n^{(a)}(x, v) = H \beta_n^{(a)}(x + 1, v). \]
Combining the above equation with (3.4), we complete the proof. \( \square \)

By applying \( \frac{1}{e^t - 1} \) to the polynomial \( H \beta_n^{(a)}(x, v) \), we give the following lemma

Lemma 4.
\[ \frac{1}{e^t - 1} \beta_n^{(a)}(x, v) = \frac{1}{n + 1} \beta_{n+1}^{(a+1)}(x, v). \]

Proof. From Lemma 1, we get
\[ \frac{1}{e^t - 1} \beta_n^{(a)}(x, v) = \frac{1}{e^t - 1} \left( \frac{t}{e^t - 1} \right)^a e^{-\frac{t^2}{2}} x^n. \]

After some calculations, we obtain
\[ \frac{1}{e^t - 1} \beta_n^{(a)}(x, v) = \frac{1}{t} \beta_{n+1}^{(a+1)}(x, v). \]
By using (3.3) in the above equation, we arrive at the desired result. \( \square \)
Theorem 5 (Recurrence formula).

\[ H\beta_{n+1}^{(a)}(x, v) = \frac{1}{n - a + 1} (x - a) (n + 1) H\beta_n^{(a)}(x, v) - a H\beta_{n+1}^{(a+1)}(x, v) - n (n + 1) H\beta_n^{(a)}(x, v) \].

Proof. By using (3.1) into (1.9), we obtain

\[ H\beta_n^{(a)}(x, v) = \left( x - \frac{ae^t}{e^t - 1} + \frac{a}{t} - t \right) \left( H\beta_n^{(a)}(x, v) \right) \].

After elementary manipulations in this equation by using (3.2), (3.3), (3.4) and Lemma 4, we arrive at the last result.

Theorem 6. Let \( k, a \in \mathbb{N} \) and \( k > a \). We have

\[ \left\langle (e^t - 1)^{-k} \mid H\beta_n^{(a)}(x, v) \right\rangle = \sum_{m=0}^{\infty} \frac{(-v)^{2m} \ (k - a)! \ n_{2m+a} \ S(n - 2m - a, k - a)}{(m!)^{2m} \ 2^m x^n} \],

where \( S(n - 2m - a, k - a) \) denotes the Stirling numbers of second kind.

Proof. Using Lemma 1, we get

\[ \left\langle (e^t - 1)^{-k} \mid H\beta_n^{(a)}(x, v) \right\rangle = \left\langle (e^t - 1)^{-k} \mid \left( t \frac{1}{e^t - 1} e^{-\frac{vt}{t}} x^n \right) \right\rangle. \]

By using (1.1), we obtain

\[ \left\langle (e^t - 1)^{-k} \mid H\beta_n^{(a)}(x, v) \right\rangle = \left\langle (e^t - 1)^{-k} \mid t^a e^{-\frac{vt}{t}} x^n \right\rangle. \]

After some calculations, we have

\[ \left\langle (e^t - 1)^{-k} \mid H\beta_n^{(a)}(x, v) \right\rangle = \left\langle (e^t - 1)^{-k-a} \mid \sum_{m=0}^{\infty} \frac{(-v)^{2m} \ n_{2m+a} \ x^n}{(m!)^{2m} \ 2^m} \right\rangle. \]

Thus, using (3.2) in the above equation, we get

\[ \left\langle (e^t - 1)^{-k} \mid H\beta_n^{(a)}(x, v) \right\rangle = \sum_{m=0}^{\infty} \frac{(-v)^{2m} \ (k - a)! \ n_{2m+a} \ S(n - 2m - a, k - a)}{(m!)^{2m} \ 2^m x^n}. \] (3.5)

By substituting

\[ S(n - 2m - a, k - a) = \frac{1}{(k - a)!} \left\langle (e^t - 1)^{-k-a} \mid x^{n-2m-a} \right\rangle \]

cf. [9] pp. 59] the above equation into (3.5), we obtain

\[ \left\langle (e^t - 1)^{-k} \mid H\beta_n^{(a)}(x, v) \right\rangle = \sum_{m=0}^{\infty} \frac{(-v)^{2m} \ (k - a)! \ n_{2m+a} \ S(n - 2m - a, k - a)}{(m!)^{2m} \ 2^m x^n}. \]

A relationship between \( B_n^{(a)}(x) \) and \( H\beta_n^{(a)}(x, v) \) is given by the following theorem:

Theorem 7. The following relationship holds true:

\[ e^{-\frac{vt}{t}} B_n^{(a)}(x) = H\beta_n^{(a)}(x, v). \]
Proof. Setting
\[ B_n^{(a)}(x) = \left( \frac{t}{e^t - 1} \right)^a x^n, \]
we obtain,
\[ e^{-\frac{vt^2}{2}} B_n^{(a)}(x) = e^{-\frac{vt^2}{2}} \left( \frac{t}{e^t - 1} \right)^a x^n. \]
Using Lemma 1 we arrive at the final result.

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REFERENCES

[1] P. Blasiak, G. Dattoli, A. Horzela and K. A. Penson, Representations of monomiality principle with Sheffer-type polynomials and boson normal ordering, Phys. Lett. A 352 (2006) 7-12.
[2] G. Bretti and P. E. Ricci, Multidimensional extensions of the Bernoulli and Appell polynomials, Taiwanese Journal of Mathematics 8 (2004) 415-428.
[3] G. Dattoli, M. Migliorati and H. M. Srivastava, Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials, Math. Comput. Modelling 45 (2007) 1033-1041.
[4] R. Dere and Y. Simsek, Genocchi polynomials associated with the Umbral algebra, In press, accepted manuscript, Appl. Math. Comput. 217 (2011), doi:10.1016/j.amc.2011.01.078.
[5] L. M. Milne-Thomson, Two classes of generalized polynomials, Proc. London Math. Soc. 2-35(1) (1933) 514-522.
[6] S. Roman, The Umbral Calculus, Dover Publ. Inc. New York, 2005.