Radiative Bulk Viscosity

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ABSTRACT

Viscous resistance to changes in the volume of a gas arises when different degrees of freedom have different relaxation times. Collisions tend to oppose the resulting departures from equilibrium and, in so doing, generate entropy. Even for a classical gas of hard spheres, when the mean free paths or mean flight times of constituent particles are long, we find a nonvanishing bulk viscosity. Here we apply a method recently used to uncover this result for a classical rarefied gas to radiative transfer theory and derive an expression for the radiative stress tensor for a gray medium with absorption and Thomson scattering. We determine the transport coefficients through the calculation of the comoving entropy generation. When scattering dominates absorption, the bulk viscosity becomes much larger than either the shear viscosity or the thermal conductivity.

1 INTRODUCTION

Both the thermal and mechanical effects of radiation on matter play a significant role in the fluid dynamics of hot objects, from the early universe to early stars. The thermal effects may be reasonably well accounted for by the use of the Eddington approximation (Unno & Spiegel 1966), though special circumstances may call for improved descriptions. In particular, when travel times of photons are not negligible, retardation effects may be significant (Delache & Froeschlé 1972), as indeed they are when we go to treat the dynamical effects of the radiation field.

An early description of the dynamical effects of radiation on matter was provided by L.
H. Thomas (Thomas 1930) who expanded the specific intensity in terms of the mean free path of photons, as Hilbert had previously done for the one-particle distribution in kinetic theory (Kogan 1969). Each of them assumed that the relevant mean free path is much smaller than the characteristic macroscopic length scales of the material medium. Though Thomas’s treatment included only emission and absorption, Masaki later extended it to include the effects of Thomson scattering (Masaki 1971; Hsieh & Spiegel 1976) and Compton scattering (Masaki 1981).

The effectiveness of the dissipation in those studies was characterized by the radiative transport coefficients — shear viscosity and conductivity — about whose evaluation there has been general agreement since Thomas’s derivation of the stress tensor for a matter-radiation mixture with short photon mean free paths (Weinberg 1972; Mihalas & Mihalas 1984). But in recent years, particular attention has been given to the bulk viscous effects, which are of interest for applications to cosmology and, for that case, different expressions have been proposed.

A simple relativistic gas has bulk viscosity (Stewart 1972) because the ultrarelativistic particles and particles with small energies respond to volume changes at different rates and so are driven out of equilibrium with each other under expansion. Analogously, in radiative fluid dynamics, the temperatures of the matter and the radiation may not be equal in nonequilibrium processes. This too can be a cause of a bulk viscosity (Weinberg 1971; Mihalas & Mihalas 1984), even though photons are ultrarelativistic. Weinberg gave an expression for the radiative bulk viscosity in the Thomas case with only emission and absorption (Weinberg 1971). Subsequent work (Anderson & Kox 1977; Stewart 1972), also for media with only emission and absorption but based on other methods, gave different expressions for the forms of the transport coefficients. However, those results have been shown to be equivalent to Weinberg’s in the first order in photon mean free path.

Additional studies have included scattering but, in those more complicated calculations, the bulk viscosity has either not been derived or not extracted explicitly (Masaki 1971; Hsieh & Spiegel 1976; Straumann 1976; Masaki 1981; Thorne 1981). Hence our aim here is to determine the bulk viscosity coefficient from a calculation of the entropy generation in the comoving frame of the matter, once we have obtained suitable approximations for the radiative stress tensor. Since the previous studies have been limited to cases where the mean free paths of photons are much smaller than the prevalent characteristic scales, the problem of treating situations where transparent regions form in the medium has been left
open. This issue must be confronted at the edges stars and disks, in intense turbulence and in photon bubbles. A case of particular interest arises in models of the early universe when the mean flight times of photons are comparable to the age of the universe.

The difficulties posed by long mean free paths are less severe for thermal than for dynamical problems since, as we mentioned, the radiative smoothing of temperature fluctuations is reasonably well described by the Eddington approximation, both in optically thick and thin regions. A reason for this is that the Eddington approximation represents a summation of contributions from terms of all orders in a mean free path expansion (Unno & Spiegel 1966). However, since the Eddington approximation leads to a diagonal stress tensor, it fails completely to describe shear viscosity (Anderson & Spiegel 1972). Moreover, it does not give a good representation of the bulk viscosity that opposes changes in volume.

Methods based on truncation of the moment hierarchy at higher moments than the pressure tensor do lead to shear viscosity (Thorne 1981; Struchtrup 1997), but they give complicated equations and are difficult to use. Another approach is to use a more general resummation procedure than that leading to the Eddington approximation, one that provides a nonlocal viscous shear tensor. Though this has been done for the case of pure absorption and emission (Chen & Spiegel 2000), it has not been carried out with scattering included. Nor is this issue to be considered here. Our purpose in this paper is rather to confront the problem of bulk viscosity for a medium where the photon mean free path is not short and where we also have to deal with scattering processes. For this, we shall adopt a procedure that has recently been introduced in classical kinetic theory (Chen et al. 2000) to generalize the usual Navier-Stokes equations. The generalized equations exhibit resistance to volume changes when the particle mean free paths are long, even for a gas of hard spheres.

To test the results obtained with the new method, we have calculated the thicknesses of shock waves and the propagation speeds of ultrasonic waves for classical gases. The results found are in good agreement with experiment, whereas those obtained from the Navier-Stokes equations do not fare well. Though the limits of validity of the new approach are still being probed, the results seem good enough to make the application to the problems of radiative fluid dynamics seem worthwhile. Since the methods that have been used heretofore to compute the radiative dissipation terms parallel the Chapman-Enskog procedure (Kogan 1969; Cercignani 1988), we may expect to encounter differences from previous results in this case as well.

In what follows, we describe an expansion in photon mean free path for solutions of the
transfer equation, including the effects of Thomson scattering, and develop a stress tensor for use in media whose photon mean free paths need not be small. From this, we compute the rate of comoving entropy generation and provide a formula for bulk viscosity. In a later paper, we shall use this result to estimate the entropy generation in the standard model of cosmology.

2 TRANSFER THEORY

2.1 The radiation field

We consider radiative transfer in a fluid medium with density $\rho$ and velocity $u^\mu$, each depending on location in space-time, $x^\mu$, where $\mu = 0, 1, 2, 3$. There are two reference frames of basic interest in this work, an inertial frame, or system frame, and the comoving frame of the matter. In the system frame, we have

$$u^\mu = \gamma(1, v), \quad \gamma = (1 - v^2)^{-1/2}$$  \hspace{1cm} (1)

while in the comoving frame $u^\mu = (1, 0)$. Here, as in the following, we assume units in which the speed of light in vacuo and Planck’s constant are each unity. We also adopt the signature in which the Minkowski metric is $\text{diag}(1, -1, -1, -1)$.

The four-momentum, $p^\mu$, of a photon satisfies $p_\mu p^\mu = 0$, so that we may define a basic null vector,

$$n^\mu = p^\mu/\nu = (1, n),$$  \hspace{1cm} (2)

that characterizes the direction of the photon’s motion in spacetime. With $\nu$ as the frequency in the inertial frame, the rest frequency (as seen comoving locally with the medium) is given by the Doppler formula

$$\tilde{\nu} = u_\mu p^\mu = \gamma \nu (1 - v \cdot n).$$  \hspace{1cm} (3)

The coordinates of the phase space of photons are $x^\mu$ and $p^\mu$, which we shall sometimes denote as $x$ and $p$ for brevity. We shall describe the density of photons in phase space by the one-particle distribution function, $f(p, x)$. This quantity is a scalar and its integral over the invariant volume in momentum space gives the photon number density in space. To perform this integration, we introduce the invariant volume in phase space, $dP = \nu dv d\Omega$, where $d\Omega$ is the element of solid angle (Landau & Lifshitz 1984).
A principal quantity of interest in this work is the stress-energy tensor of the radiation field,
\[ T^{\mu\nu} = \int p^\mu p^\nu f \, dP = \int \int n^\mu n^\nu I \, d\Omega d\nu , \]  
(4)
where the last term on the right introduces the specific intensity, defined as
\[ I = \nu^3 f . \]  
(5)
A revealing way of writing the stress tensor in terms of basic moments of the radiation field is obtained by defining the directional vector with respect to the moving matter,
\[ l^\mu = n^\mu - u^\mu , \]  
(6)
so that
\[ l_\mu l^\mu = -1 \quad \text{and} \quad l_\mu u^\mu = 0 . \]  
(7)
Hence \( l^\mu = (0,1) \), where \( l \) is a unit three-vector in the spatial direction of photon motion in the comoving frame. The angular moments of the radiation field are then
\[ E = \int I d\Omega , \quad F^\mu = \int ll^\mu d\Omega , \quad P^{\mu\nu} = \int ll^\mu ll^\nu d\Omega , \]  
(8)
and their frequency-integrated forms are
\[ E = \int_0^\infty E d\nu \quad F^\mu = \int_0^\infty F^\mu d\nu \quad P^{\mu\nu} = \int_0^\infty P^{\mu\nu} d\nu . \]  
(9)
These moments together make up the energy-momentum stress tensor which may now be written as
\[ T^{\mu\nu} = Eu^\mu u^\nu + F^\mu u^\nu + F^\nu u^\mu + P^{\mu\nu} . \]  
(10)

2.2 The Transfer equation

When both absorption and scattering occur, the transfer equation takes the general form (Simon 1963)
\[ p^\mu \partial_\mu f = \rho(\alpha - \beta f) + \rho \int \Re(p, p') f(p')dP' - \rho \int \Re(p', p) f(p)dP' \]  
(11)
where \( \alpha \) and \( \beta \) are scalars characterizing absorption and emission and the kernel \( \Re(p', p) \) is the differential cross-section for scattering a photon from four-momentum \( p' \) into \( p \). For brevity, we have not indicated the dependence of \( f \) on \( x \) in the transfer equation and have omitted the effects of gravity. We here consider only the case of Thomson scattering for which we have (Masaki 1971; Tucker 1973).
\[ \Re(p, p') = \frac{3}{4} [1 + (1 \cdot 1')^2] \delta(\tilde{\nu} - \tilde{\nu}') \frac{\sigma}{4\pi} \] (12)

where \( \sigma \) is the Thomson cross-section.

We may factor \( f \) out of the second integral of (11) and note that with \( i = 1, 2, 3 \) we have

\[ \int l_i l_j d\Omega = -\frac{4\pi}{3} \delta^{ij} \] (13)

where \( \delta^{ij} \) is the Kronecker symbol. Then the last term in (11) reduces to \( -\rho \sigma f \). Since \( f \) is a scalar, we may use (8) to find the transformation law for the specific intensity. Also, we may compare the covariant form of the transfer equation with the usual one for the comoving frame, as Thomas (Thomas 1930) did, to find \( \beta = \tilde{\nu} \kappa(\tilde{\nu}) \), where \( \kappa \) is the absorption coefficient, and \( \alpha = \tilde{\nu}^{-2} j(\tilde{\nu}) \), where \( j \) is the emissivity. Since \( \alpha, \beta \) and \( \tilde{\nu} \) are scalars, we then know the transformation rules for \( j \) and \( \kappa \) once we introduce the (relativistic) Doppler formula (3).

The equation of transfer may now be written as (Hsieh & Spiegel 1976)

\[ \varepsilon \rho I = \dot{\kappa} S - I + \frac{3\dot{\sigma}}{16\pi} (E + l_\rho l_\sigma P^{\rho\sigma}) \] , (14)

where

\[ S = \nu^3 \alpha / \beta \] , (15)

\[ \varepsilon = [\rho(\kappa + \sigma)]^{-1} \] , (16)

\[ \dot{\kappa} = \frac{\kappa}{\kappa + \sigma} , \quad \dot{\sigma} = \frac{\sigma}{\kappa + \sigma} \] (17)

and a comma with subscript \( \mu \) indicates differentiation with respect to \( x^\mu \). We have replaced \( \delta_{ij} P^{ij} \) by \( \delta_{\rho\sigma} P^{\rho\sigma} \), to which it is equal, for cosmetic reasons.

Expressions for the moments that appear in the transfer equation may be computed from the equation itself. In doing this, we shall assume that the medium is grey, although we could save appearances by introducing suitable mean absorption coefficients. We shall not follow that practice here since it leads to serious complications in moving media, such as tensorial absorption coefficients, and has not repaid the effort involved.

Next we derive expressions for the monochromatic moments defined in (8) by taking suitable moments of (14). For this purpose, we define

\[ h^{\rho\sigma} = -\frac{3}{4\pi} \int l^\rho l^\sigma d\Omega \] , (18)

which, in terms of the basic tensors of the problem, may be expressed as

\[ h^{\rho\sigma} = \eta^{\rho\sigma} - u^\rho u^\sigma \] . (19)

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Similarly, we have
\[\int l^\mu l^\nu l^\rho l^\sigma d\Omega = \frac{4\pi}{15} (h^{\mu\rho} h_{\nu\sigma} + h^{\nu\rho} h_{\mu\sigma} + h^{\mu\sigma} h_{\nu\rho}).\] (20)

Then, when we integrate (14) over all solid angle and note that \(h_{\mu\nu} P^{\mu\nu} = -\mathcal{E}\), we find that
\[\mathcal{E} = 4\pi S - \varepsilon \int n^\rho I_{\rho \sigma} d\Omega.\] (21)

Further, on applying the second definition in (8) to (14), we get
\[\mathcal{F}^\mu = -\varepsilon \int l^\mu n^\rho I_{\rho \nu} d\Omega.\] (22)

Finally, on multiplying (14) by \(l^\mu l^\nu\) and integrating over angle we find that
\[P^{\mu\nu} = -\frac{40\pi \hat{\kappa}}{3(10 - \hat{\sigma})} S h^{\mu\nu} - \frac{3\hat{\sigma}}{10 - \hat{\sigma}} \mathcal{E} h^{\mu\nu} - \frac{10\varepsilon}{10 - \hat{\sigma}} \int l^\mu l^\nu n^\rho I_{\rho \sigma} d\Omega.\] (23)

With the moments expressed in this way, we obtain the expression
\[\mathcal{E} + l^\rho I_{\rho \sigma} P^{\rho\sigma} = \frac{16\pi}{3} S - \frac{2\varepsilon(5 + \hat{\sigma})}{\hat{\kappa}(10 - \hat{\sigma})} \int n^\rho I_{\rho \sigma} d\Omega - \frac{10\varepsilon}{10 - \hat{\sigma}} l^\mu l^\nu \int l^\mu l^\nu n^\rho I_{\rho \sigma} d\Omega.\] (24)

Hence we may write the transfer equation in the compact form
\[\mathcal{I} = S - \varepsilon \mathcal{L}\mathcal{I}\] (25)
where we have introduced the linear operator
\[\mathcal{L} = n^\rho \partial_\rho + c_1 \int d\Omega n^\rho \partial_\rho + c_2 l^\mu l^\nu \int d\Omega l^\mu l^\nu n^\rho \partial_\rho\] (26)
with \(\partial_\rho = \partial / \partial x^\rho\),
\[c_1 = \frac{3\hat{\sigma}(5 + \hat{\sigma})}{8\pi \hat{\kappa}(10 - \hat{\sigma})}\] (27)
and
\[c_2 = \frac{15\hat{\sigma}}{8\pi(10 - \hat{\sigma})}.\] (28)

When there is no scattering, \(\hat{\sigma} = 0\) and the foregoing equations reduce to those for the case of pure absorption.

3 THE RADIATIVE PRESSURE TENSOR

3.1 The Expansion Procedure

We expand the intensity \(\mathcal{I}\) as
\[\mathcal{I} = \sum_{m=0}^{\infty} \mathcal{I}_{(m)} \varepsilon^m.\] (29)
From $\mathcal{I}_{(m)}$, we can calculate the coefficients of the expansions of the monochromatic moments and their frequency-integrated counterparts from the definitions

\[
\mathcal{E}_{(m)} = \int \mathcal{I}_{(m)} d\Omega, \quad \mathcal{F}^\mu_{(m)} = \int \mathcal{I}_{(m)} l^\mu d\Omega, \quad \mathcal{P}^{\mu\nu}_{(m)} = \int \mathcal{I}_{(m)} l^\mu l^\nu d\Omega
\]  

(30)

and

\[
E_{(m)} = \int_0^\infty \mathcal{E}_{(m)} d\nu, \quad F^\mu_{(m)} = \int_0^\infty \mathcal{F}^\mu_{(m)} d\nu, \quad P^{\mu\nu}_{(m)} = \int_0^\infty \mathcal{P}^{\mu\nu}_{(m)} d\nu.
\]  

(31)

When we introduce the expansion (29) into the transfer equation (25) and demand that the expanded equation is satisfied term by term, we get

\[
\mathcal{I}_{(0)} = S,
\]  

(32)

as in comoving local thermodynamic equilibrium. For the higher orders with $m \geq 1$, we obtain

\[
\mathcal{I}_{(m)} = -\mathcal{L}\mathcal{I}_{(m-1)}
\]  

(33)

where $\mathcal{L}$ is defined in (26).

The problem studied is really one of a mixture of radiation and matter and a more complete description would begin with coupled transport equations. In the present version, we simply proceed as if the material properties are given, as did Thomas (Thomas 1930). In doing this, we consider that the properties of the medium are expressed, not as functions of $x^\mu$, but in terms of the basic fields, $T$, $\tilde{\nu}$, $n^\rho$. Therefore, in the simplest case, where we operate on a function of $T$, $\tilde{\nu}$ and $n^\rho$ only, we may write

\[
\partial_\mu = T_\mu \partial_T + \tilde{\nu}_\mu \partial_{\tilde{\nu}} + n^\rho_{\mu} \partial_{n^\rho}.
\]  

(34)

On recalling that $\tilde{\nu} = u_\mu p^\mu$ and that $p^\mu$ does not depend on coordinate, we obtain

\[
\tilde{\nu}_\mu = p_\nu u^\nu_{\mu}.
\]

(35)

Similarly, on differentiating $p^\mu = n^\mu \tilde{\nu}$ and making use of (35) and the identity $u^\mu u_{\mu,\rho} = 0$, we get

\[
n^\mu_{\nu} = -n^\mu n^\rho u_{\rho,\nu}.
\]

(36)

So (34) takes the form

\[
\partial_\mu = T_\mu \partial_T + \tilde{\nu} n^\rho u_{\rho,\mu} \partial_{\tilde{\nu}} - n^\rho n^\rho u_{\rho,\mu} \partial_{n^\rho}.
\]  

(37)

If we further assume that, like the Planck function, $S$ depends on $\tilde{\nu}$ and $T$ only through the combination $\tilde{\nu}/T$, we deduce from (37) that
\[ S_{,\mu} = (T_{,\mu} - T n^\rho u_{,\rho,\mu}) \partial_T S . \] (38)

We note that in the higher orders some of the derived quantities will depend explicitly on the derivatives of \( T \) and \( u^\rho \) as well as on the fields themselves. Thus, more generally, when we go to higher order developments, as we did in computing the shear viscosity (Chen & Spiegel 2000), we must allow for a dependence on those derivatives. However, here we go only to first order in photon mean free path and that complication does not arise; \((37)\) is therefore sufficient for our purposes.

Returning to the calculation of \( E \), we see from \((21)\) that

\[ E^{(0)} = 4 \pi S \] and that, for \( m \geq 1 \),

\[ E^{(m)} = -\frac{1}{\kappa} \int n^\rho \partial_\rho \mathcal{I}_{m-1} \, d\Omega . \] (39)

Also, from \((22)\) we have \( \mathcal{F}_\mu^{(0)} = 0 \) and, for \( m \geq 1 \),

\[ \mathcal{F}_\mu^{(m)} = -\int l^\mu n^\rho \partial_\rho \mathcal{I}_{m-1} \, d\Omega . \] (40)

Then from \((23)\) we find \( \mathcal{P}_{\mu\nu}^{(0)} = -\frac{4\pi}{3} S h^{\mu\nu} \) and, for \( m \geq 1 \),

\[ \mathcal{P}_{\mu\nu}^{(m)} = -\frac{3\tilde{\sigma}}{10 - \tilde{\sigma}} E^{(m)} h^{\mu\nu} - \frac{10}{10 - \tilde{\sigma}} \int l^\mu l^\nu n^\rho \partial_\rho \mathcal{I}_{m-1} \, d\Omega . \] (41)

### 3.2 The First-Order Stress Tensor

From the recursion relation \((33)\) we have

\[ \mathcal{I}_{(1)} = LL^{(0)} = L S . \] (42)

Then from \((38)\) and \((39)\) we get

\[ \mathcal{E}_{(1)} = -\frac{4\pi}{\kappa} (\frac{1}{3} T \theta + \dot{T}) S_T \] (43)

where

\[ \theta = u^{\mu}_{,\mu} \] (44)

and \( \dot{T} = u^{\mu} T_{,\mu} \). On integrating over frequency, we find

\[ E_{(1)} = -\frac{4\pi}{\kappa} (\frac{4}{3} S \theta + \dot{S}) , \] (45)

where

\[ S = \int_0^\infty S d\nu = a T^4 \] (46)

and \( \dot{S} = u^{\mu} S_{,\mu} \).

In a similar way, we find for the frequency-integrated first-order radiative flux,

\[ F_\mu^{(1)} = \frac{4\pi}{3} [S_{,\rho} - 4 S u_{,\rho}] h^{\mu\rho} , \] (47)
while for the frequency-integrated pressure tensor in first order, we obtain
\[
P_{\mu\nu}^{(1)} = \frac{40\pi}{10 - \sigma} \left[ -\frac{3\sigma}{40\pi} h^{\mu\nu} E_{(1)} + \frac{h^{\mu\nu}}{3} \dot{S} + \frac{4S}{15} (\tau^{\mu\nu\rho\sigma} u_{\rho\sigma} + \frac{5}{3} h^{\mu\nu} \theta) \right] \tag{48}
\]
where
\[
\tau^{\mu\nu\rho\sigma} = h^{\mu\rho} h^{\nu\sigma} + h^{\mu\sigma} h^{\nu\rho} - \frac{2}{3} h^{\mu\nu} h^{\rho\sigma} . \tag{49}
\]
We may then introduce (45) and so write the pressure tensor to this order as
\[
P_{\mu\nu} = -\frac{4\pi}{3} h^{\mu\nu} \left[ S - \frac{\varepsilon}{\kappa} \left( \frac{4}{3} S \theta + \dot{S} \right) \right] + \Xi_{\mu\nu} \tag{50}
\]
where
\[
\Xi_{\mu\nu} = \mu \tau^{\mu\nu\rho\sigma} u_{\rho\sigma} \tag{51}
\]
is the shear tensor and the shear viscosity coefficient is that found by Thomas and Masaki,
\[
\mu = \frac{32\pi S}{3\rho(10\kappa + 9\sigma)} . \tag{52}
\]
As mentioned, we have elsewhere computed a nonlocal expression for the shear tensor for the case of pure absorption (Chen & Spiegel 2000) and expect that a similar generalization may apply in the case with scattering, though we have not yet carried out the necessary calculations to confirm this.

From (47), we may identify the thermal conductivity as the coefficient of \( T_{,\rho} \):
\[
\chi = \frac{16\pi S}{3T\rho(\kappa + \sigma)} . \tag{53}
\]
The second term in the flux, with coefficient \( \xi = T\chi \), has been considered to be a purely relativistic effect (Mihalas 1983), but there is a classical counterpart when the particles have long mean free paths (Chen et al. 2000). We shall see in the next section how these transport coefficients, as well as the bulk viscosity, can be deduced from the entropy generation rate.

### 4 ENTROPY GENERATION AND BULK VISCOSITY

To clarify the roles of the various terms in the radiative stress tensor, we compute their contributions to the entropy generation. In leading order, this is related to the deviation of the stress tensor from its equilibrium value, which we approximate to first order in \( \varepsilon \) as
\[
\Delta T^{\mu\nu} = T^{\mu\nu} - T_{(0)}^{\mu\nu} = \varepsilon T_{(1)}^{\mu\nu} , \tag{54}
\]
where
\[
T_{(1)}^{\mu\nu} = E_{(1)} u^\mu u^\nu + F_{(1)}^{\mu\nu} + F_{(1)}^{\nu\mu} + P_{(1)}^{\mu\nu} . \tag{55}
\]
The rate of entropy production is the four-divergence of the entropy flux, \( \Sigma^\mu \), whose expression as given by Weinberg (Weinberg 1972) and recalled in Appendix A is
\[
T \Sigma^\mu_{,\mu} = \Delta T^{\mu\nu} [u_{\mu,\nu} - u_\mu (\log T)_\nu].
\] (56)

For the components of \( T^{\mu\nu}_{(1)} \) we introduce (45), (47) and (48) and obtain, after some simple rearrangements,
\[
\rho (\kappa + \sigma) T \Sigma^\mu_{,\mu} = -E^{(1)} \left[ u^\nu (\log T)_{,\nu} + \frac{1}{3} \theta \right] + \Xi^{\mu\nu} u_{\mu,\nu} + F^{(1)}_\mu [u^\nu u_{\mu,\nu} - (\log T)_{,\mu}].
\] (57)

Since \( S = a T^4 \), we can rewrite (45) as
\[
E^{(1)} = -\frac{16\pi S}{\kappa} \left[ u^\rho (\log T)_{,\rho} + \frac{1}{3} \theta \right],
\] (58)

which we use in the first term of (57). To simplify the second term in (57), we note that, on account of the form of \( \Xi^{\mu\nu} \), we encounter the expression \( u_{\mu,\nu} \tau_{\mu\nu\rho\sigma} u_{\rho,\sigma} \). We may use the identity \( h_{\mu\rho} \tau_{\mu\nu\rho\sigma} = 0 \) to show that
\[
u_{\mu,\nu} \tau_{\mu\nu\rho\sigma} u_{\rho,\sigma} = \frac{1}{2} (u_{\mu,\nu} + u_{\nu,\mu} - \frac{2}{3} h_{\mu\nu} \theta) \tau_{\mu\nu\rho\sigma} u_{\rho,\sigma}.
\] (59)

Then we find that
\[
u_{\mu,\nu} \tau_{\mu\nu\rho\sigma} u_{\rho,\sigma} = \frac{1}{4} (u_{\mu,\nu} + u_{\nu,\mu} - \frac{2}{3} h_{\mu\nu} \theta) \tau_{\mu\nu\rho\sigma} (u_{\rho,\sigma} + u_{\sigma,\rho} - \frac{2}{3} h_{\rho\sigma} \theta).
\] (60)

All this adds up to
\[
\Sigma^\mu_{,\mu} = -\eta h^{\mu\nu} \left( \frac{T_{\mu}}{T} - u^\rho u_{,\rho} \right) \left( \frac{T_{\nu}}{T} - u^\rho u_{,\rho} \right) + \zeta \theta^2
+ \frac{2\mu}{T} h^{\mu\rho} h^{\nu\sigma} (u_{\mu,\nu} + u_{\nu,\mu} - \frac{2}{3} h_{\mu\nu} \theta) (u_{\rho,\sigma} + u_{\sigma,\rho} - \frac{2}{3} h_{\rho\sigma} \theta)
\] (61)

where
\[
\chi = \frac{16\pi S}{3T \rho (\kappa + \sigma)},
\] (62)
\[
\mu = \frac{32\pi S}{3(10\kappa + 9\sigma)}
\] (63)
and
\[
\zeta = \frac{16\pi S}{\rho T} \left[ \frac{1}{3} + \frac{u^\alpha T_{,\alpha}}{T \theta} \right]^{2}.
\] (64)

To see the meaning of these results, we note that for any four-vector \( v^\mu \), whose components in the comoving frame are \( (v^0, v) \), we have
\[
h^{\mu\nu} v^\mu v^\nu = \delta^{ij} v_i v_j = - \| \vec{v} \|^2 \leq 0.
\] (65)

Similarly, for any rank-2 tensor \( w^{\mu\nu} \), we can show that \( h^{\mu\rho} h^{\nu\sigma} w_{\mu\nu} w_{\rho\sigma} \geq 0 \). So we find that the rate of entropy generation is never negative as long as all these coefficients are positive.
The first term in (61) is the entropy generation produced by the drag of the radiative flux, which is an effect of radiative thermal conductivity, with due allowance for the acceleration of the matter. The second term represents the entropy generation produced by volume changes and the coefficient in this term, $\zeta$, is the radiative bulk viscosity. Finally, the third term is a rate of entropy increase caused by shear, with the coefficient of radiative shear viscosity, $\mu$.

We see from (64) that the bulk viscosity depends on $u \rho T, \rho = \dot{T}$. In depending on a rate, our bulk viscosity differs qualitatively from that derived by Weinberg (Weinberg 1972) who found a bulk viscosity of the form

$$\zeta_W = \frac{16\pi S}{\rho\kappa T} \left[ \frac{1}{3} - \left( \frac{\partial P}{\partial e} \right)_n \right]^2.$$  

For a radiation-dominated situation, in which $\left( \frac{\partial P}{\partial e} \right)_n \approx \frac{1}{3}$, we get $\zeta_W \approx 0$.

The principal reason for the difference between the results (64) and (66) is that Weinberg followed the standard practice in transport theory of using a lower order condition (here $T_{(0),\nu}^{\mu\nu} = 0$) to eliminate the time derivative from the pressure tensor (Weinberg 1972; Bernstein 1988). Thus he obtained $\dot{S} = -4S \left( \frac{\partial e}{\partial e} \right)_n \theta$ (see Appendix B) and was led to formula (66) for the bulk viscosity. While this does simplify the derivation of (66), it also shrinks its domain of validity. In avoiding the use of this condition in our derivation, we obtain a result that is capable of describing relatively rapid processes and narrow structures. Moreover, we find that the bulk viscous pressure can be large even for the radiation-dominated case.

It is interesting that with both approaches, as the absorption coefficient gets smaller, the bulk viscosity grows larger. This is not so surprising since, for large absorption coefficient, the temperature difference between radiation and matter is small. Still, the tendency toward large bulk viscosity at small absorption coefficient is noteworthy and is reminiscent of the sharply different behaviors of fluids with zero viscosity and those with viscosity tending to zero. Of course, since the effect of Compton scattering will be like that of absorption, the limit of zero absorption strictly does not apply to the general case. Nevertheless, we may expect to find effective heating of expanding media in some situations.

5 CONCLUSION

From its very beginnings, the subject of radiative viscosity has been vexatious, as several of the founders of modern radiative transfer theory — Milne, Eddington, Jeans — found in their attempts to compute the shear viscosity (Mihalas 1983). It was not until Thomas
(Thomas 1930) did the problem relativistically that agreement was reached on the value of the shear viscosity, to leading order. What is interesting is that the agreement seems to have been reached on purely theoretical grounds since there do not appear to be direct measurements confirming Thomas’s results.

In the matter of bulk viscosity, the situation is even more troubling in that the calculations are subtler and there are by now many contributors to the growing literature on this issue. It is conceivable that disagreements over bulk viscosity may also be settled by common consent, but we feel this is unlikely. As we have noted, in previous work, the simplification of introducing equilibrium approximations into the lower order theory is used, in the spirit of the Chapman-Enskog procedure (Kogan 1969; Cercignani 1988). For gases of classical particles, this approximation leads to the conclusion that a simple classical gas does not have a bulk viscosity. However, when this restrictive approximation is not made, we have found (Chen et al. 2000), on using the relaxation (or BGKW) model (Kogan 1969; Cercignani 1988), that a classical rarefied gas may have an effective bulk viscosity and that the gas obeys macroscopic equations that differ from the Navier-Stokes equations when the particle mean free paths become long. In the classical case, one has recourse to experiments to test the conclusions and these have lent to support to our conclusion. In the radiative problem, we may hope for a rough empirical check in the photon density of the present universe. This is a calculable quantity, though it is model dependent, and we shall report elsewhere on the results obtained for this case (or see (Chen 2000)).

But this is not the whole story. There has been some previous disagreement on the correct form of transport coefficients (Anderson & Kox 1977), especially, the bulk viscosity. The calculations are usually done in the frame defined by Eckart and the coefficients are identified by comparison with the general form of the stress tensor. But there are usually two temperatures in the expression of the stress tensor (for matter and radiation) and the difference between them is of the same order as the bulk viscosity. Hence the derived bulk viscosity is dependent on which temperature is actually adopted in the stress tensor. But with our present method, we needed to go only to first order to get good results, so the temperature difference does not figure significantly in the outcome. A calculation of the entropy generation rate then leads to an unambiguous determination of the transport coefficients.

Another way to attack this problem is to use the moment method, which was first formally developed in kinetic theory by Grad (Grad 1963); in transfer theory, it may be
traced back to Krook’s (Krook 1955) formulation of Eddington’s methods. In modern times, the moment method has been elaborated for use in radiative fluid dynamics by several authors in attempting to deal with viscous effects, especially nonlocal ones (Anderson & Spiegel 1972; Thorne 1981; Kato et al. 1993; Uday & Israel 1982; Struchtrup 1997). However, in the context of classical kinetic theory, the introduction of higher moments beyond those used by Grad does not lead to rapid improvement of the macroscopic description of rarefied gases (Chen et al. 2000). The reason is that the microscopic theory admits only a few slow variables, corresponding to the lowest moments of the distribution function.

Higher moments are fast variables and, as we know from the experience of dynamical systems theory, their introduction does not lead quickly to improved representation of the dynamics. Rather, it is best to attempt to improve the description in terms of the slow variables. Therefore, we have pursued here an approach in which we compute only approximations for the energy density, flux and pressure tensor that may be derived by an expansion in the style of Thomas. The next step in this approach would be to seek a closure relation for the pressure tensor in terms of the radiative variables rather than in terms of a material quantity like the source function $S$. We shall take this up in a later paper where we shall see that, to improve our description in terms of the slow variables, we should include their derivatives in the closure relation, since the derivatives of slow variables are also slow variables.

**APPENDIX A: DERIVATION OF ENTROPY GENERATION RATE**

Consider a medium made up of $N$ interacting species in a state of near thermal equilibrium. Let us represent the one-particle distribution of the $i$th species as

$$i f = if_0 + if_1$$  \hspace{1cm} (A1)

where

$$if_0 = \exp[-\beta_\mu p^\mu - i\alpha - i\epsilon]$$  \hspace{1cm} (A2)

is the equilibrium distribution, with $i\epsilon = 1, -1$ for a Bose-Einstein or a Fermi-Dirac gas, respectively. Also,

$$\beta_\mu = \frac{u_\mu}{T}$$  \hspace{1cm} (A3)

and $i\alpha$ is the chemical potential for the $i$th species, where $i = 1, 2, \cdots, N$. The corresponding 4-current and stress tensor are defined as
\[ i_{\mu} \equiv ig \int \frac{p^\mu}{p^0} i f d^3p , \quad iT^{\mu\nu} \equiv ig \int \frac{p^\mu p^\nu}{p^0} i f d^3p \] (A4)

while the corrections are, similarly,

\[ i_{\mu}^1 \equiv ig \int \frac{p^\mu}{p^0} if_1 d^3p , \quad iT^{\mu\nu}_1 \equiv ig \int \frac{p^\mu p^\nu}{p^0} if_1 d^3p \] (A5)

where \( g \) is a normalization constant depending on the nature of the statistics of the particles.

The entropy four-vector is

\[ i\Sigma^\mu = -ig \int \frac{d^3p}{p^0} p^\mu [if \ln if - i\epsilon(1 + i\epsilon) if \ln(1 + i\epsilon if)] . \] (A6)

If we put equation (A1) into this definition and calculate the entropy generation rate, accurate to second order in \( if_1 \), we have

\[ i\Sigma_{\mu,\nu} = -i\alpha i_{\mu,\nu} - i\beta_{\mu} iT^{\mu\nu}_1 + i\epsilon \alpha_{\mu} i_{\nu,\mu}^1 + i\epsilon \beta_{\nu,\mu} iT^{\nu\mu}_1 . \] (A7)

So the total entropy generation rate of the mixture is

\[ \Sigma^\mu_{\mu} = -\sum_i i\alpha i_{\mu,\mu}^i - \sum_j \beta_{\mu} iT^{\mu\nu}_j + \sum_i i\epsilon \alpha_{\mu} i_{\nu,\mu}^i + \epsilon \beta_{\nu,\mu} iT^{\nu\mu}_1 . \] (A8)

Chemical equilibrium leads to \( \sum_i i\alpha i_{\mu,\mu}^i = 0 \) and total energy conservation results in \( \sum_j \beta_{\mu} iT^{\mu\nu}_j = 0 \). With these two constraints, equation (A8) reduces to:

\[ \Sigma^\mu_{\mu} = \sum_i i\epsilon \alpha_{\mu} i_{\nu,\mu}^i + \epsilon \beta_{\nu,\mu} iT^{\nu\mu}_1 . \] (A9)

With formula (A9) we can evaluate the entropy generation rate for radiating media by considering only two species in our mixture. The material medium, is one of these and is assumed to be in thermal equilibrium in the absence of photons. For this component, \( f_1 = 0 \). The other constituent is the radiation field, whose chemical potential, \( \alpha \), is zero. So we obtain for the entropy generation formula of the mixture

\[ \Sigma^\mu_{\mu} = \beta_{\nu,\mu} T^{\nu\mu}_1 , \] (A10)

where \( T^{\nu\mu}_1 \) is the correction to the radiative stress tensor. With the definition for \( \beta_{\nu} \), equation (A10) readily reduces to the formula reported in the text except that, in this appendix, we have not introduced a fiducial \( \epsilon \).

**APPENDIX B: THE RELATION BETWEEN \( \dot{S} \) AND \( \theta \)**

In order to evaluate the bulk viscosity, we need to express \( \dot{S} \) in terms of \( \theta \). This can be done by analyzing the zeroth order entropy of the entire system — medium plus radiation — and using the conservation laws. We have the thermodynamic relation

\[ \dot{S} \text{ } \frac{c}{2000 \text{ RAS, MNRAS 000, 1–17}} \]
\[ T d\sigma = d(ev) + pdv , \]  
(B1)

where \( p \) is the total pressure, \( \sigma \) is the entropy per particle, \( v \) is the volume per particle and \( e \) is the total energy density.

If \( T \) and \( v \) are chosen to be the independent variables and \( F \) is the Helmholtz energy per particle, we can write
\[ dF = -\sigma dT - pdv . \]  
(B2)

Since \( dF \) is a perfect differential, we have the Maxwell relation
\[ \frac{\partial \sigma}{\partial v} \bigg|_T = -\frac{\partial p}{\partial T} \bigg|_v . \]  
(B3)

On writing \( F = ev - T\sigma \), we find \( \sigma = \frac{1}{T}(ev - F) \). If we put this into the left of the foregoing equation and notice that from (B2), \( \frac{\partial F}{\partial v} \bigg|_T = -p \), we get
\[ v \left( \frac{\partial e}{\partial v} \bigg|_T \right) + e + p = T \left( \frac{\partial p}{\partial T} \bigg|_v \right) . \]  
(B4)

We also have the continuity equation
\[ \dot{n} = u^\mu n_{,\mu} = -nu^\mu_{,\mu} \]  
(B5)

where \( \dot{n} = u^\mu n_{,\mu} \). In terms of \( v(= 1/n) \), the continuity equation can be written as
\[ \dot{v} = u^\mu v_{,\mu} = vu^\mu_{,\mu} . \]  
(B6)

If we assume that the bare matter stress tensor satisfies \( \Theta^{\mu\nu}_{(0),\nu} = 0 \) with \( \Theta^{\mu\nu}_{(0)} = eu^\mu u^\nu - ph^{\mu\nu} \), we find
\[ \dot{\theta} = -(e + p)\dot{v}/v , \]  
(B7)

where we have used (B6). Now we are in a position to calculate \( \dot{T} \) in terms of \( \theta(= u^\mu_{,\mu}) \), starting with the identity,
\[ e(v, T)_{,\mu} = \left( \frac{\partial e}{\partial v} \bigg|_T \right) v_{,\mu} + \left( \frac{\partial e}{\partial T} \bigg|_v \right) T_{,\mu} . \]  
(B8)

We have
\[ \dot{e} = \left( \frac{\partial e}{\partial v} \bigg|_T \right) \dot{v} + \left( \frac{\partial e}{\partial T} \bigg|_v \right) \dot{T} . \]  
(B9)

Hence
\[ \dot{T} = \left( \frac{\partial T}{\partial e} \bigg|_v \right) \left[ \frac{\partial e}{\partial v} \bigg|_T \right] \dot{v} . \]  
(B10)

On replacing \( \dot{\theta} \) from (B7), \( \left( \frac{\partial e}{\partial v} \bigg|_T \right) \) from (B4), and \( \dot{\theta} \) from (B6) in the above relation, we get
\[ \dot{T} = u^\mu T_{,\mu} = -T \left( \frac{\partial p}{\partial e} \bigg|_v \right) u^\mu_{,\mu} . \]  
(B11)
With $S = aT^4$, we find that, in the comoving frame, equation (B11) reduces to
\[ \dot{S} = -4S \left( \frac{\partial p}{\partial e} \right)_v \theta . \]  

(B12)

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