A LINESEARCH EXTRAGRADIENT ALGORITHM FOR NONMONOTONE EQUILIBRIUM PROBLEMS INVOLVING QUASICONVEX BIFUNCTIONS

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Abstract. We propose an extragradient algorithm for approximating a solution of equilibrium problems involving quasiconvex bifunction. The proposed algorithm is an iterative procedure, where the search direction at each iteration is a normal-subgradient, while the step-size is updated avoiding Lipschitz-type conditions. The sequence generated by the proposed algorithm converges to a $\rho$-quasi-solution with any positive $\rho$ if the bifunction $f$ is semistrictly quasiconvex in its second variable, while it converges to the solution when $f$ is strongly quasiconvex. Neither monotonicity nor Lipschitz property is required.

Keywords. Equilibria, Quasiconvexity, Normal subgradient, Line search

1. INTRODUCTION

Let $C$ be a nonempty closed convex set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a given bifunction such that $f(x, x) = 0$ for every $x \in C$. We consider the problem

$$\text{Find } x^* \in C : f(x^*, y) \geq 0 \quad \forall y \in C. \tag{EP}$$

In what follows we call Problem (EP) a convex (resp. quasiconvex) equilibrium problem if the function $f(x, .)$ is convex (resp. quasiconvex) on $C$ for any $x \in C$.

In recent years this problem attracted much attention of many authors as it contains a lot numbers of important problems such as optimization, variational inequality, Kakutani fixed point, Nash equilibrium problems and others as special cases, see e.g. the interesting monographs [4, 15], the papers [3, 10, 12, 13, 17, 21, 23, 25, 26] and the references cited therein.

Many algorithms have been developed for solving (EP) under the assumption that the bifunction is convex and subdifferentiable with respect to the second variable while the first one being fixed. Almost all of these algorithms are based upon the auxiliary problem principle, which states that when $f(x, .)$ is convex, subdifferentiable on $C$, then the solution-set of (EP) coincides with that of the regularized problem

$$\text{find } x^* \in C : f_{\rho}(x^*, y) := f(x^*, y) + \frac{1}{2\rho} \|y - x^*\|^2 \geq 0 \quad \forall y \in C, \tag{REP}$$

with any $\rho > 0$. The main advantage of the latter problem is that the regularized bifunction $f_{\rho}$ is strongly convex in the second variable when the first one being fixed.

A basic method for solving Problem (REP) is the extragradient one, where at each iteration $k$, having $x^k \in C$, a main operation is of solving the mathematical subprogram

$$\min\{f_{\rho}(x^k, y) := f(x^k, y) + \frac{1}{2\rho} \|y - x^k\|^2 : y \in C\}. \tag{MP}$$
Thanks to convexity of the function $f(x^k, \cdot)$ this is a strongly convex program, and therefore it is uniquely solvable. However, when $f(x, \cdot)$ is quasiconvex rather convex, Problem (MP), in general, is not strongly convex, even not quasiconvex.

In the seminal paper \[6\] in 1972, K. Fan called Problem (EP) a minimax inequality and established solution existence results for it, when $C$ is convex, compact and $f$ is quasiconvex on $C$. To our best knowledge, up to now there does not exist an algorithm for finding a solution of the problem considered in \[6\] by K. Fan.

It worth mentioning that when $f(x, \cdot)$ is convex and subdifferentiable on $C$, the equilibrium problem (EP) can be reformulated as the following multivalued variational inequality

$$\text{find } x^* \in C, v^* \in F(x^*) : \langle v^*, x - x^* \rangle \geq 0 \quad \forall x \in C,$$

(EVI)

where $F(x^*) = \partial_2 f(x^*, x^*)$ with $\partial_2 f(x^*, x^*)$ being the diagonal subdifferential of $f$ at $x^*$, that is the subdifferential of the convex function $f(x^*, \cdot)$ at $x^*$. In the case $f(x, \cdot)$ is semi-strictly quasiconvex rather than convex, Problem (EP) can take the form of (MVI) with $F(x) := Na_{f(x, x)} \setminus \{0\}$, where $Na_{f(x, x)}$ is the normal cone of the adjusted sublevel set of the function $f(x, \cdot)$ at the level $f(x, x)$, see \[1\]. More details about the links between equilibrium problems and variational inequalities can be found in \[2\].

Based upon the auxiliary principle, different methods such as the fixed point, projection, extragradient, regularization, gap function ones have been developed for solving equilibrium problem (EP) by using mathematical programming techniques, where the bifunction involved possesses certain monotonicity properties. Almost all of them require that the bifunction is convex with respect to its second variable, see e.g. the comprehensive monograph \[4\] and the references therein.

In \[5\], the authors studied an infeasible interior proximal algorithm for solving quasiconvex equilibrium problems with polyhedral constraints. At each iteration $k$ of this algorithm, having $x^k$ it requires globally solving a nonconvex mathematical programming problem, where the objective function is the sum of $f(x^k, \cdot)$ and a strongly convex function defined by a distance function. The convergence of this algorithm is proved under an assumption depending on the iterates $x^k$ and $x^{k+1}$. Very recently, Iusem and Lara \[14\] propose an algorithm for solving quasiconvex equilibrium problem (EP). Their algorithm can be considered as a standard proximal point method for optimization problem applied to the quasiconvex function $f(x, \cdot)$. The convergence has been proved when $f$ is pseudomonotone, Lipschitz-type and strongly quasi-convex.

In our recent papers \[28, 29\], by using the normal subdifferential of quasiconvex functions, we have proposed projection algorithms for Problem (EP) when the bifunction is pseudo and paramonotone.

In this paper, we continue our work by modifying the linesearch extragradient algorithm commonly used for convex equilibrium problem (EP) to solve quasiconvex equilibrium problems without requiring any monotonicity and Lipschitz-type properties of the bifunction involved. More precisely, after the next section that contains preliminaries on normal subdifferentials of a quasiconvex function, in the third section, we describe an extragradient linesearch algorithm for this quasiconvex equilibrium problem. Then by observing that the solution set of the problem coincides with that of the Minty (dual) one for semi-strictly quasiconvex bifunction, we prove that the algorithm converges to a quasi-solution when the bifunction involved is semi-strictly quasiconvex in its second variable, which is the unique solution when the bifunction is strongly quasiconvex in its second variable.

2. Preliminaries on Quasiconvexity, Normal Subdifferentials and Monotonicity

Definition 2.1. ((2, 7, 18)) Let $C$ be a convex set in $\mathbb{R}^n$. Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $C \subseteq \text{dom} \varphi$. The function $\varphi$ is said to be

(i) \textbf{quasiconvex} on $C$ if and only if for every $x, y \in C$ and $\lambda \in [0, 1]$, one has

$$\varphi[(1 - \lambda)x + \lambda y] \leq \max[\varphi(x), \varphi(y)].$$

(2.1)
(ii) **semi-strictly quasi-convex** on $C$ if it is quasiconvex and for any every $x,y \in C$ and $\lambda \in (0,1)$, one has
\[
\varphi(x) < \varphi(y) \Rightarrow \varphi((1-\lambda)x + \lambda y) < \varphi(y).
\] (2.2)

(iii) **strongly quasiconvex** on $C$ with modulus $0 < \gamma < \infty$ if for every $0 \leq \lambda \leq 1$
\[
\varphi(\lambda x + (1-\lambda)y) \leq \max\{\varphi(x), \varphi(y)\} - \lambda (1-\lambda)\frac{\gamma}{2}\|x-y\|^2 \ \forall x,y \in C.
\]

(iv) **essentially quasiconvex** on $C$ if it is quasiconvex and every its local minimum is a global one.

(v) **pseudoconvex** on $C$ if it is quasiconvex and every its local minimum is a global one.

(vi) **proximal convex** on $C$ with modulus $\alpha > 0$ (shortly $\alpha$-prox-convex) if $\text{prox}_{\varphi}(C,z) \neq \emptyset$ and there exists $\alpha > 0$ such that
\[
p \in \text{prox}_{\varphi}(C,z) \Rightarrow \alpha(x-p,z-p) \leq \varphi(x) - \varphi(p) \ \forall x \in C,
\]
where $\text{prox}_{\varphi}(C,z)$ is the proximal mapping of $\varphi$ at $z$ on $C$, that is
\[
\text{prox}_{\varphi}(C,z) := \text{argmin}\{\varphi(x) := \varphi(y) + \frac{1}{2}\|y-x\|^2 : y \in C\}.
\]

It is well known that strongly quasiconvex $\Rightarrow$ semi-strictly quasiconvex $\Rightarrow$ essentially quasiconvex $\Rightarrow$ quasiconvex.

Clearly, $\varphi$ is quasiconvex if and only if, for every $\alpha \in \mathbb{R}$, the strict level set at the level $\alpha$, that is
\[
L_{\alpha} := \{y : \varphi(y) < \alpha\}
\]
is convex .

Recall that the (Hadamard) directional derivative of a function $\varphi$ at $x$ with direction $d$ is defined as
\[
\varphi'(x,d) := \lim_{t \searrow 0, u \to d} \inf \frac{\varphi(x + tu) - f(x)}{t}.
\]

A point $x \in C$ is said to be a stationary point of $\varphi$ on $C$ if $\varphi'(x,d) \geq 0$ for every $d$. A point is a minimizer of $\varphi$ on $C$ then it is a stationary point. The converse direction is true when $\varphi$ is convex or pseudo convex on $C$.

The Greenberg-Pierskalla subgradient of a quasiconvex function [9] is defined as
\[
\partial^{GP} \varphi(x) := \{g \in \mathbb{R}^n : (g,y-x) > 0 \Rightarrow \varphi(y) \geq \varphi(x)\}.
\]

A variation of this subdifferential is the star-subdifferential that is defined as
\[
\partial^* \varphi(x) := \{g \in \mathbb{R}^n : (g,y-x) < 0 \ \forall y \in L_{\varphi}(x)\},
\]
where $L_{\varphi}(x)$ stands for the strict level set of $\varphi$ with level $\varphi(x)$. It is well known [9, 22] that if $\varphi$ is continuous on $\mathbb{R}^n$, then $\partial^* \varphi(x)$ contains nonzero vector and
\[
\partial^* \varphi(x) \cup \{0\} = \text{cl}(\partial^* \varphi(x)) = \partial^{GP} \varphi(x),
\]
where $\text{cl}(A)$ stands for the closure of the set $A$. Thus the star-subdifferential is also called the normal-subdifferential. Various calculus rules for normal subdifferential can be found in [22].

The following concepts are commonly used in the field of equilibrium problem [4].

**Definition 2.2.** Let $f : C \times C \to \mathbb{R}$ and $S \subseteq C$

(i) $f$ is said to be strongly monotone on $S$ with modulus $\eta \geq 0$ (shortly $\eta$-strongly monotone) if
\[
f(x,y) + f(y,x) \leq -\eta \|x-y\|^2 \ \forall x,y \in S.
\]
If $\eta = 0$ it is also called monotone on $S$. 

(ii) $f$ is said to be paramonotone on $C$ if $x$ is a solution of (EP) and $y \in C$, $f(x,y) = f(y,x) = 0$ then $y$ is also a solution of (EP).

(iii) $f$ is said to be pseudomonotone on $C$ if $f(x,y) \geq 0$ then $f(y,x) \leq 0$ for every $x, y \in C$.

(iv) $f$ is said to be Lipschitz-type on $C$ if
\[ f(x,y) + f(y,z) \geq f(x,z) - L_1\|x-y\|^2 - L_2\|y-z\|^2 \forall x,y,z \in C. \]

Clearly, in the case of optimization when $f(x,y) := \varphi(y) - \varphi(x)$ it possesses both the paramonotonicity and Lipschitz-type property.

3. ALGORITHM AND ITS CONVERGENCE

A problem closely related to Problem (EP) is the Minty (or dual) equilibrium one that is defined as
\[
\text{Find } z^* \in C \text{ such that } f(y,z^*) \leq 0. \quad (\text{DEP})
\]

Let us denote by $S$ and $S_d$ the solution set of (EP) and (DEP) respectively. It is clear that if $f$ is pseudomonotone on $C$ then $S \subseteq S_d$. Conversely, $S_d \subseteq S$ if $f$ is upper semi-continuous with respect to the first variable and convex with respect to the second variable (see [20]).

In what follows we always suppose that $f(\cdot, y)$ is upper semi-continuous for any $y \in C$.

In the following lemma, we prove that the inclusion $S_d \subseteq S$ still holds true when $f$ is semi-strictly quasiconvex with respect to the second variable.

**Lemma 3.1.** Assume that $f(x,\cdot)$ is semi-strictly quasiconvex on $C$ for any $x \in C$. Then $S_d \subseteq S$.

**Proof.** Let $z^* \in S_d$. If $z^* \notin S$, then there would exist $y \in C$ such that $f(z^*,y) < 0$.

For $\lambda \in (0,1)$, set $y_{\lambda} = \lambda z^* + (1-\lambda)y$. Since $f(\cdot, y)$ is upper semi-continuous, there exists $0 < \lambda < 1$ such that $f(y_{\lambda}, y) < 0$. Since $z^* \in S_d$, $f(y_{\lambda}, z^*) \leq 0$.

We consider two cases

- **Case 1:** $f(y_{\lambda}, z^*) < 0$. By the quasiconvexity of $f(y_{\lambda}, \cdot)$,
\[
0 = f(y_{\lambda}, y_{\lambda}) \leq \max\{f(y_{\lambda}, z^*), f(y_{\lambda}, y)\} < 0.
\]

This is a contradiction.

- **Case 2:** $f(y_{\lambda}, z^*) = 0$. By the semi-strictly quasiconvexity of $f(y_{\lambda}, \cdot)$ and the fact that $f(y_{\lambda}, y) < 0 = f(y_{\lambda}, z^*)$, which would imply
\[
0 = f(y_{\lambda}, y_{\lambda}) < \max\{f(y_{\lambda}, z^*), f(y_{\lambda}, y)\} = 0.
\]

This is also a contradiction. \qed

The following algorithm can be considered as a modification of the one in [23] for solving Problem (EP) when the bifunction is quasiconvex with respect to its second variable.

**Algorithm 3.1.** Take $\alpha, \theta \in (0,1)$ and two sequences $\{\rho_k\}_{k \geq 0}, \{\sigma_k\}_{k \geq 0}$ of positive numbers such that $\rho_k$ nonincreasingly converges to some $\overline{\rho} > 0$,
\[
\sum_{k=0}^{\infty} \sigma_k = \infty, \quad \sum_{k=0}^{\infty} \sigma_k^2 < \infty.
\]

**Initialization:** Pick $x^0 \in C$.

**Iteration** $k = 0, 1, \ldots$

- **Find** $y^k$ such that
\[
y^k \in \arg\min_{y \in C} \left\{ f(x^k, y) + \frac{1}{2\rho_k} \|x^k - y\|^2 \right\}, \quad (3.1)
\]
Proof. (i) For Proposition 3.1, while in the rest part it is only a stationary point. But, part (i) of the following proposition shows that it is a solution restricted on a part of $\Omega$.

\[ f(z_k^m, x^k) - f(z_k^m, y_k) \geq \frac{\alpha}{2\rho_k} \|y_k - x^k\|^2, \]  

(3.2)

and set $z_k = z_k^m$.

(ii) If $f$ is pseudoconvex on $C$ or strongly quasiconvex on $C$, then $x^k$ is a solution. Take

\[ g_k \in \partial f(z_k^m, x^k) := \left\{ g \in \mathbb{R}^n : \langle g, y - x^k \rangle < 0 \text{ if } f(z_k^m, y) < f(z_k^m, x^k) \right\}, \]  

(3.3)

and normalize it to obtain $\|g_k\| = 1$ ($g_k \neq 0$, see Proposition 3.2 below). Compute

\[ x^{k+1} = P_C(x^k - \sigma g_k^k). \]  

(3.4)

If $x^{k+1} = x^k$ then stop: $x^k$ is a solution, else set $k := k + 1$.

Remark 3.1. (i) The existence of solution for (3.1) can be guaranteed under the assumption that the function $f(x^k, \cdot)$ is lower semicontinuous and $2$–weakly coercive (see [7]), i.e.,

\[ \liminf_{\|y\| \to +\infty} \frac{f(x^k, y)}{\|y\|^2} \geq 0. \]

If $C$ is bounded, the $2$–weakly coercivity assumption can be dropped. If $f(x, \cdot)$ is strongly quasi-convex on $C$, then it is $2$–weakly coercive (see [14] Lemma 2).

Another example is the $\alpha$-proximal convex function introduced in [7], where it has been proved that if $h$ is $\alpha$-proximal convex on $C$ then $\text{Pro}_{x_h}(C, z)$ is a singleton. It can be seen from Lemma 2.2 in [14] that if $f(x, \cdot)$ is strongly quasi-convex on $C$ for any $x \in C$, then it is proximal convex on $C$. Note that if $h$ is convex, then $\alpha = 1$.

(ii) When $f$ is quasiconvex rather convex on $C$, problem (3.1), in general is not convex, even not quasi-convex. However, in some special cases (see examples below) one can choose regularization parameter $\rho_k$ such that problem (3.1) is strongly convex, and therefore it is uniquely solvable.

In contrast to the convex case, in the algorithm, $y^k = x^k$ does not necessarily implies that $x^k$ is a solution. But, part (i) of the following proposition shows that it is a solution restricted on a part of $C$, while in the rest part it is only a stationary point.

Proposition 3.1. Suppose that $y^k = x^k$.

(i) If $x^k$ is not a solution of (EP), that means

\[ \Omega(x^k) := \left\{ y \in C : f(x^k, y) < 0 \right\}, \]

is nonempty, then for any $y \in \Omega(x^k)$

\[ f_{x^k}(x^k, y - x^k) = 0, \]

where $f_{x^k} := f(x^k, \cdot)$.

(ii) If $f(x^k, \cdot)$ is pseudoconvex on $C$ or strongly quasiconvex on $C$, then $x^k$ is a solution of (EP).

Proof. (i) For $y \in \Omega(x^k)$, set $d = y - x^k$. For $\lambda \in (0, 1)$, set $y_\lambda = x^k + \lambda d = \lambda y + (1 - \lambda)x^k$. Since $f(x^k, x^k) = 0$, by the semi-strictly quasiconvexity of $f(x^k, \cdot)$, we have

\[ f(x^k, y_\lambda) < 0. \]

So,

\[ \frac{f(x^k, x^k + \lambda d) - f(x^k, x^k)}{\lambda} < 0. \]  

(3.5)
From 
\[ x^k = y^k \in \arg\min \{ f(x^k, y) + \frac{1}{2\rho_k} \| y - x^k \|^2 : y \in C \}, \]
it implies that for any \( y \in C \),
\[ f(x^k, y) + \frac{1}{2\rho_k} \| y - x^k \|^2 \geq 0. \]

Let \( y = y^\lambda \),
\[ f(x^k, y^\lambda) + \frac{1}{2\rho_k} \lambda^2 \| y - x^k \|^2 \geq 0. \]

Therefore,
\[ \frac{f(x^k, x^k + \lambda d) - f(x^k, x^k)}{\lambda} \geq -\frac{1}{2\rho_k} \| d \|^2. \] (3.6)

By combining (3.5) and (3.6) and let \( \lambda \to 0^+ \), we obtain \( f'_x(x^k, d) = 0 \).

(ii) Now, assume that \( f(x^k, \cdot) \) is pseudoconvex on \( C \), then \( f(x^k, \cdot) \) is differentiable on an open set containing \( C \) and for any \( y, y' \in C \), we have
\[ \nabla f(x^k, y)(y' - y) \geq 0 \Rightarrow f(x^k, y') \geq f(x^k, y). \]

From \( x^k = y^k \), it implies that \( \langle \nabla f(x^k, y^k), y - y^k \rangle \leq 0 \) for every \( y \in C \). Therefore, \( f(x^k, y) \geq 0 \) for \( y \in C \).

If \( f(x^k, \cdot) \) is strongly quasiconvex, then subproblem (3.1) is uniquely solvable. Since \( x^k = y^k \) with \( y^k \) being the solution of subproblem (3.1), we have
\[ 0 \leq f(x^k, y) + \frac{1}{2\rho_k} \| y - x^k \|^2 \ \forall y \in C. \] (3.7)

Let \( y := \lambda x + (1 - \lambda)x^k \) with any \( x \in C \) and \( \lambda \in [0, 1] \). Then applying (3.7), by the strong quasiconvexity of \( f(x^k, \cdot) \) we obtain
\[
0 \leq f(x^k, \lambda x + (1 - \lambda)x^k) + \frac{1}{2\rho_k} \| \lambda x + (1 - \lambda)x^k - x^k \|^2 \\
\leq \max \{ f(x^k, x^k), f(x^k, x) \} - \lambda (1 - \lambda) \frac{\rho_k}{2} \| x - x^k \|^2 + \frac{1}{2\rho_k} \| \lambda x + (1 - \lambda)x^k - x^k \|^2.
\]

Thus for any \( \lambda \in [0, 1] \) we have
\[ 0 \leq \max \{ f(x^k, x), 0 \} + \left[ \frac{\lambda^2}{2\rho_k} - \frac{\lambda(1 - \lambda)}{2} \right] \| x - x^k \|^2 \ \forall x \in C. \]

Since \( \rho_k \downarrow \rho > 0 \), one can choose \( \lambda > 0 \) small enough so that \( \frac{\lambda^2}{2\rho_k} - \frac{\lambda(1 - \lambda)}{2} < 0 \). Hence \( f(x^k, x) \geq 0 \) for every \( x \in C \).

\( \square \)

**Proposition 3.2.** Assume that \( f(\cdot, y) \) is continuous on \( C \) for any \( y \in C \). If \( y^k \neq x^k \) then the following statements hold:

(i) There exists a positive integer \( m \) satisfying (3.2).
(ii) If \( f(x, \cdot) \) is semi-strictly quasiconvex on \( C \) for any \( x \in C \), then \( f(z^k, x^k) > 0 \).
(iii) \( 0 \notin \partial_x^2 f(z^k, x^k) \).
Proof. (i) If there does not exist $m$ satisfying (3.2), then for every positive integer $m$, we have

$$f(z_k, y_k) - f(z_k^m, y_k) < \frac{\alpha}{2p_k} \|y_k - x^k\|^2. \quad (3.8)$$

Let $m \to +\infty$, we have $z_k^m \to x^k$ and (3.8) becomes

$$- f(x^k, y_k) \leq \frac{\alpha}{2p_k} \|y_k - x^k\|^2. \quad (3.9)$$

On the other hand, (3.1) means that for all $y \in C$,

$$f(x^k, y_k) + \frac{1}{2p_k} \|y_k - x^k\|^2 \leq f(x^k, y) + \frac{1}{2p_k} \|y - x^k\|^2.$$

By choosing $y = x^k$, we obtain

$$f(x^k, y_k) + \frac{1}{2p_k} \|y_k - x^k\|^2 \leq 0. \quad (3.10)$$

Combining (3.9) with (3.10), it follows that $\alpha \geq 1$. This is a contradiction because $\alpha \in (0, 1)$.

(ii) From (3.2), $f(z_k^m, x^k) > f(z_k, y_k)$. By the semi-strictly quasiconvexity of $f(z_k, \cdot)$ on $C$, it follows

$$0 = f(z_k, z_k) < f(z_k, x^k).$$

(iii) It follows from part (ii) that

$$0 = f(z_k, z_k) < f(z_k, x^k).$$

By the definition of $\partial^2 f(z_k, x^k)$, it is clear that $0 \notin \partial^2 f(z_k, x^k)$.

□

Proposition 3.3. If $x^{k+1} = x^k$ then $z^k$ is a solution of (EP) provided $f(x, \cdot)$ is semi-strictly quasiconvex on $C$ for any $x \in C$.

Proof. By the algorithm, $x^{k+1} = x^k$ means that $x^k = P_C(x^k - \sigma_k g^k)$, which is equivalent to

$$\langle g^k, y - x^k \rangle \geq 0 \quad \forall y \in C. \quad (3.11)$$

Remember that, by (3.3),

$$g^k \in \partial^2 f(z_k, x^k) := \left\{ g \in \mathbb{R}^n : \langle g, y - x^k \rangle < 0 \text{ if } f(z_k, y) < f(z_k, x^k) \right\}.$$

Thus, by (3.11), $f(z_k, y) \geq f(z_k, x^k)$ for $y \in C$.

Note that, in part (ii), Proposition 3.2, we have proved that if $x^k \neq y^k$, then $f(z_k, x^k) > 0$. So, we can conclude that $f(z_k, y) \geq f(z_k, x^k) \geq 0$ for every $y \in C$, which means that $z^k$ is a solution of (EP).

□

Proposition 3.4. Suppose that the solution-set $S_d$ of the Minty problem is nonempty. Let $z^* \in S_d$, then

$$\|x^{k+1} - z^*\|^2 \leq \|x^k - z^*\|^2 + \sigma_k^2, \quad (3.12)$$

and

$$\liminf_{k \to +\infty} \langle g^k, x^k - z^* \rangle = 0. \quad (3.13)$$

Proof. For $y \in C$, we have

$$\|x^{k+1} - y\|^2 = \|P_C(x^k - \sigma_k g^k) - y\|^2 \leq \|x^k - \sigma_k g^k - y\|^2 \leq \|x^k - y\|^2 + \sigma_k^2 + 2\sigma_k \langle g^k, y - x^k \rangle.$$
With $y = z^* \in S_d$, we have
\[ \|x^{k+1} - z^*\|^2 \leq \|x^k - z^*\|^2 + \sigma_k^2 + 2\sigma_k (g^k, z^* - x^k). \]  
(3.14)

Since $f(z^k, z^*) \leq 0 < f(z^k, x^k)$ and $g^k \in \partial^*_S f(z^k, x^k)$, it follows that
\[ (g^k, z^* - x^k) < 0. \]

Therefore,
\[ \|x^{k+1} - z^*\|^2 < \|x^k - z^*\|^2 + \sigma_k^2. \]

Following [8] we say that a point $x \in C$ is $\rho$- quasi-solution (prox-solution) to Problem (EP) if $f(x, y) + \frac{1}{2\rho} \|y - x\|^2 \geq 0$ for every $y \in C$.

For the convergence of the proposed algorithm we need the following assumptions.

(A0) $f$ is continuous jointly in both variables on an open set containing $C \times C$;

(A1) $f(x, .)$ is semistrictly quasiconvex on $C$ for every $x \in C$;

(A2) the solution-set $S_d$ of the Minty problem is nonempty;

(A3) The sequence $\{x^k\}$ is bounded.

**Theorem 3.1.** Suppose that the algorithm does not terminate. Let $\{x^k\}$ be the infinite sequence generated by the algorithm. Under the assumptions (A0),(A1),(A2),(A3), there exists a subsequence of $\{x^k\}$ converging to a $\overline{\rho}$- quasi solution $\overline{x}$. If in addition, $f(x, .)$ is strongly quasi-convex for every $x \in C$, then $\{x^k\}$ converges to the unique solution of (EP).

**Proof.** Let $z^* \in S_d$. By part (i) Proposition 3.4 and $\sum_{k=1}^{+\infty} \sigma_k^2 < +\infty$, the sequence $\{\|x^k - z^*\|^2\}$ is convergent. Hence, $\{x^k\}$ is bounded.

Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ such that $x^{k_j}$ converges to some point $\overline{x}$ and
\[ \lim_{j \to +\infty} \langle g^{k_j}, x^{k_j} - z^* \rangle = \lim_{j \to +\infty} \langle g^k, x^k - z^* \rangle = 0. \]  
(3.15)

Since the sequence $\{x^{k_j}\}$ is bounded, $\{x^{k_j}\}$ is bounded too. By taking subsequences if necessary, without loss of generality, we can assume that $x^{k_j}$ converges to $\overline{x}$ and $z^{k_j}$ converges to $z^*$.

**Step 1:** We will prove that
\[ f(\overline{z}, \overline{x}) = 0. \]  
(3.16)

Indeed, from part (ii) Proposition 3.2, $f(\overline{z}, x^k) > 0$. In addition, by Assumption (A0), $f(., .)$ is continuous on $C \times C$, we have
\[ f(\overline{z}, \overline{x}) = \lim_{j \to +\infty} f(\overline{z}^{k_j}, x^{k_j}) \geq 0. \]

Now, assume that $f(\overline{z}, \overline{x}) = \varepsilon > 0$. Then there exists $j_0$ such that $f(\overline{z}^{k_j}, x^{k_j}) > \frac{\varepsilon}{2}$ for all $j \geq j_0$.

Since $z^* \in S_d$, we have $f(\overline{z}, z^*) \leq 0$. Again by (A0), there exists $\varepsilon_1, \varepsilon_2 > 0$ such that for all $z \in B(\overline{z}, \varepsilon_1)$, $z' \in B(z^*, \varepsilon_2)$:
\[ f(z, z') < \frac{\varepsilon}{2}. \]

Since $\{z^{k_j}\}$ converges to $\overline{z}$, there exists $j_1$ such that for any $j \geq j_1$ we have $z^{k_j} \in B(\overline{z}, \varepsilon_1)$, from which it follows that for $j \geq \max(j_0, j_1)$, and $z' = B(z^*, \varepsilon_2)$, we have
\[ f(z^{k_j}, z') < f(z^{k_j}, x^{k_j}). \]

By taking $z' = z^* + \varepsilon_2 g^{k_j}$ and thanks to (3.3), we have for $j \geq \max(j_0, j_1)$,
\[ \langle g^{k_j}, x^{k_j} - z^* \rangle > \varepsilon_2, \]
which contracts to (3.15). Thus \( f(\overline{x}, \overline{x}) = 0 \).

**Step 2:** We prove that

\[
\lim_{j \to +\infty} \| x_j^k - y_j^k \| = 0.
\]

From (3.2), we know that for any \( k \)

\[
f(x_j^k, x_j^k) > f(x_j^k, y_j^k).
\]

So, \( f(\overline{x}, \overline{y}) \leq f(\overline{x}, \overline{x}) = 0 \) (by Step 1).

Let \( \theta_j = \theta^{m_j} \) such that \( z_j^{m_j} = (1 - \theta_j)x_j^{m_j} + \theta_jy_j^{m_j} \). Clearly, \( 0 < \theta \leq \theta_j < 1 \). Therefore, \( \overline{x} \) is a convex combination of \( \overline{x} \) and \( \overline{y} \) and \( \overline{x} \neq \overline{y} \).

Now if \( f(\overline{x}, \overline{y}) < 0 \), then \( \overline{y} \neq \overline{y} \). By the semi-strictly quasiconvexity of \( f(\overline{z}, .) \), we have

\[
0 = f(\overline{x}, \overline{y}) < f(\overline{x}, \overline{x}),
\]

which is impossible. It implies that

\[
f(\overline{x}, \overline{y}) = 0.
\]

From (3.2),

\[
f(x_j^k, x_j^k^k) - f(x_j^k, y_j^k) \geq \alpha \rho f \| y_j^k - x_j^k \|^2.
\]

Let \( j \to +\infty \), and note that \( \lim_{k \to +\infty} \rho_k = \overline{\rho} > 0 \), we obtain

\[
\lim_{j \to +\infty} \| x_j^k - y_j^k \| = 0.
\]

This means \( \overline{x} = \overline{y} \). Note that

\[
y_j^k \in \arg\min_{y \in C} \left\{ f(x_j^k, y) + \frac{1}{2\rho_k} \| x_j^k - y \|^2 \right\}.
\]

then for any \( y \in C \), we have

\[
f(x_j^k, y_j^k) + \frac{1}{2\rho_k} \| x_j^k - y_j^k \|^2 \leq f(x_j^k, y) + \frac{1}{2\rho_k} \| x_j^k - y \|^2.
\]

Let \( j \to +\infty \), by the continuity of \( f \) and \( \overline{x} = \overline{y} \), we obtain for any \( y \in C \),

\[
0 \leq f(\overline{x}, y) + \frac{1}{2\overline{\rho}} \| \overline{x} - y \|^2.
\]

Assume, in addition, that \( f(x, .) \) is strongly quasiconvex on \( C \) with modulus \( \gamma > 0 \). For any \( x \in C \) and \( \lambda \in [0, 1] \), take \( y = \lambda x + (1 - \lambda)\overline{x} \). Then and replacing to (3.18) we obtain

\[
0 \leq f(\overline{x}, \lambda x + (1 - \lambda)\overline{x}) + \frac{1}{2\overline{\rho}} \| \overline{x} - (\lambda x + (1 - \lambda)\overline{x}) \|^2.
\]

Then using the definition of strong quasiconvexity, by the same argument as in the proof of part (ii) in Proposition 3.1 we can see that \( f(\overline{x}, x) \geq 0 \) for every \( x \in C \).

Let

\[
S_{\overline{\rho}} := f(\overline{x}, y) + \frac{1}{2\overline{\rho}} \| \overline{x} - y \|^2.
\]

**Remark 3.2.** (i) In virtue of Lemma 3.1 we have \( S_d \subseteq S \). Thus, if \( S_{\overline{\rho}} = S_d \), then \( \overline{x} \in S_d = S \). Remember that the sequence \( \{ \| x^k - \overline{x} \|^2 \} \) is convergent we can conclude that the whole sequence \( \{ x^k \} \) converges to \( \overline{x} \) which is a solution of (EP).

(ii) Since for any \( \overline{\rho} \), one can choose a sequence \( \{ \rho_k \} \) such that \( \rho_k \to \overline{\rho} \). Thus, from \( f(\overline{x}, y) + \overline{\rho} \| \overline{x} - y \|^2 \geq 0 \forall y \in C \), it can be seen that for any \( \varepsilon > 0 \), there exists \( \overline{\rho} > 0 \) small enough such that \( f(\overline{x}, y) \geq -\varepsilon \) provided \( C \) is bounded. So one can considered \( \overline{x} \) as approximate solution.
In the case \( f \) is pseudomonotone, then by Lemma 3.1 \( S = S_d \). Clearly, \( S \subseteq S_\rho \) for every \( \rho > 0 \). In addition, if \( f(x,.) \) is pseudoconvex and continuously differentiable for any \( x \in C \), then \( S = S_\rho \) for every \( \rho > 0 \). Hence \( \bar{x} \) is a solution.

(iii) Clearly, Assumption (A3) may be dropped if \( C \) is bounded (often in practice), moreover, from the proof one can see that this assumption is not needed if the optimization problem (3.1) admits a unique solution for every \( k \).

The following simple example shows that a \( \rho \)- quasi-solution with any \( \rho > 0 \) may not be a solution.

Let \( C := [-1,0] \), \( f(x,y) := y^3 - x^3 \). Clearly, with \( x^* = 0 \), we have \( f(x^*,y) + \frac{1}{2\rho}(y-x)^2 = y^3 + \frac{1}{2\rho}y^2 \geq 0 \), \( \forall y \in C \) if \( \rho > 0 \) small enough, for example \( \rho < 1/2 \). Thus, 0 is \( \rho \)- prox-solution, but \( f(x^*,y) = y^3 < 0 \) with \( y = -1 \in C \). So the auxiliary problem principle fails to apply to semi-strictly quasiconvex equilibrium problems.

Now we consider an example [16] in which the optimization problem (3.1) can be solved efficiently. Suppose that bifunction \( f(x,y) := \max_{i \in I} g_i(x,y) \), where \( I \subset \mathbb{R} \) is compact, and each \( g_i(x,.) \) is quasiconvex on \( C \) for every fixed \( x \in C \). Then \( f(x,.) \) is quasiconvex. Suppose that each \( g_i(x,.) \) \( (i \in I) \) is differentiable and its derivative is Lipschitz with constant \( L_i(x) > 0 \). Let \( \rho < \frac{1}{L_i(x)} \) with \( L(x) := \max_{i \in I} L_i(x) \) and

\[
  f_\rho(x,y) := f(x,y) + \frac{1}{2\rho} \|y-x\|^2.
\]

Then \( f_\rho(x,.) \) is strongly convex on \( C \). Indeed, for any \( u,v \in C \) and \( i \in I \), consider the function \( g_{i,\rho}(x,y) := g_i(x,y) + \frac{1}{2\rho} \|y-x\|^2 \). Then we have

\[
  \langle \nabla g_{i,\rho}(x,u) - \nabla g_{i,\rho}(x,v), u-v \rangle \\
  = \langle \nabla g_i(x,u) + \frac{1}{\rho} (u-x) - \nabla g_i(x,v) - \frac{1}{\rho} (v-x), u-v \rangle \\
  \geq -L(x)\|u-v\|^2 + \frac{1}{\rho}\|u-v\|^2 = \left( \frac{1}{\rho} - L(x) \right)\|u-v\|^2.
\]

Hence \( f_\rho(x,.) \) is strongly convex whenever \( \rho < \frac{1}{L_i(x)} \). The above example belongs to the class of the lower -\( C^2 \) functions considered by some some authors see e.g. [19, 24, 27].

Conclusion. We have proposed an extragradient linesearch algorithm for approximating a solution of the Minty equilibrium problem involved quasiconvex bifunction. The sequence of the iterates generated by the proposed algorithm converges to a proximal-solution when the bifunction is semi-strictly quasiconvex with respect to its second variable, which is an equilibrium solution provided the bifunction is strongly quasiconvex. Neither monotonicity nor Lischitz properties are required. Thus the algorithm could be considered as an iterative scheme for a solution of the problem considered by K. Fan in [6] with the bifunction being semi-strictly quasiconvex.

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