MAKING SPANNING GRAPHS

PETER ALLEN*, JULIA BÖTTCHER*, YOSHIHARU KOHAYAKAWA†, HUMBERTO SILVA NAVES‡,
AND YURY PERSON.§

Abstract. We prove that for each $D \geq 2$ there exists $c > 0$ such that whenever $b \leq c (\frac{\log n}{n})^{1/D}$, in the $(1:b)$ Maker-Breaker game played on $E(K_n)$, Maker has a strategy to guarantee claiming a graph $G$ containing copies of all graphs $H$ with $v(H) \leq n$ and $\Delta(H) \leq D$. We show further that the graph $G$ guaranteed by this strategy also contains copies of any graph $H$ with bounded maximum degree and degeneracy at most $\frac{\Delta(H)}{2}$. This lower bound on the threshold bias is sharp up to the log-factor when $H$ consists of $\frac{n}{3}$ vertex-disjoint triangles or $\frac{n}{2}$ vertex-disjoint $K_4$-copies.

1. Introduction

We study Maker-Breaker games on graphs, where the winning sets consist of spanning subgraphs. So far this has been studied only in very special cases such as spanning trees and Hamilton cycles. Here we consider general bounded-degree subgraphs.

Given integers $m$ and $b$, the $(m:b)$ Maker-Breaker game played on a finite ground set $X$ with winning sets $S \subseteq \mathcal{P}(X)$ is played as follows. In each round, first Maker claims up to $m$ unclaimed elements of $X$. Then Breaker claims up to $b$ unclaimed elements of $X$. If at some time Maker claims all elements of some $S \subseteq S$, then Maker wins the game; otherwise Breaker wins. Since this is a finite draw-free game, either Maker or Breaker has a strategy which is guaranteed to win (see for example [9]). We remark that sometimes the game is defined with Breaker playing first rather than Maker, but the results of this paper are insensitive to such a change.

We concentrate on the case $m = 1$, and refer to $b$ as the bias of the game. Since the ability to claim more elements per round can only help Breaker, for any game there is trivially a threshold bias, defined to be the largest $b$ such that Maker has a winning strategy. Note that for some games we have $b = 0$, for example any game in which the winning sets consist of pairwise disjoint sets of size two. We are interested in games played on the ground set $E(K_n)$ in which Maker’s aim is to create copies of (a family of) given graphs, where by a copy of a graph $H$ we mean a not necessarily induced subgraph isomorphic to $H$.

Games of this type received considerable attention. In particular the connectivity game, in which Maker’s aim is to create any spanning tree in $K_n$, was studied in [2, 6, 10]. More recently, the Hamiltonicity game, in which Maker’s aim is to create a Hamiltonian graph, was investigated in [2, 5, 6, 10, 13]. In both cases the threshold bias is known quite accurately, and in fact is roughly the same as if Maker simply chose $\binom{n}{b+1}^2$ random edges. Since Maker and Breaker do not in general play randomly, this is rather surprising, and becomes more so as one observes that a similar random graph intuition, first formulated by Chvátal and Erdős [6], holds for many other graph games.

The first result really harnessing this intuition is due to Bednarska and Łuczak in [4]. Given a graph $H$ on at least three vertices we define

$$m(H) = \max \left\{ \frac{|E(F)|}{|V(F)| - 2} : F \subseteq H \text{ and } |V(F)| \geq 3 \right\}.$$
Bednarska and Luczak proved that the critical bias for the game whose target sets are copies of \( H \) is of order \( \Theta \left( n^{1/m(H)} \right) \).

**Theorem 1.1** (Theorem 2 in [4]). For every graph \( H \) which contains a cycle there exists a constant \( c_0 \) such that for every sufficiently large integer \( n \) and \( b \leq c_0 n^{1/m(H)} \) the following holds. Consider the \((1 : b)\) Maker-Breaker game played on the edges of \( K_n \) with the copies of \( H \) in \( K_n \) as winning sets. Maker has a random strategy for this game that succeeds with probability \( 1 - o(1) \) against any strategy of Breaker.

Maker uses the following simple random strategy. She picks a random ordering of \( E(K_n) \), and in her \( r \)-th round tries to claim the \( r \)-th edge in this ordering; if Breaker already claimed it she does not claim any edge in that round. Notice, that the first \( m \) edges in Maker’s ordering correspond to the random graph \( G(n, m) \), the random graph chosen uniformly at random among all graphs with \( m \) edges. It is well known that \( G(n, m) \) behaves in many respects as \( G(n, p) \) with \( p = m/(\binom{n}{2}) \). It is then easy to check that Maker with high probability successfully claims most of this random graph \( G(n, p) \), where \( p \) is chosen to be a very small constant multiple of \( b^{-1} \). Bednarska and Luczak proved a global resilience result which states that, if \( p \gg n^{-1/m(H)} \), then the random graph \( \Gamma = G(n, p) \) typically has the property that any \( G \subseteq \Gamma \) which contains most of the edges of \( \Gamma \) also contains a copy of \( H \), justifying that Maker wins the \( H \)-game with high probability.

Note that since a Maker-Breaker game is a deterministic game, it follows that if Maker has a random strategy that works with non-zero probability against any given strategy of Breaker, then the game is Maker’s win (otherwise Maker’s strategy should work with probability zero against Breaker’s winning strategy). Thus there exists a deterministic strategy which wins the \( H \)-game for Maker at the same bias, though the proof does not give any hint of what this strategy might be.

It is easy to see that for some different games on \( E(K_n) \) the Bednarska-Luczak strategy will not work close to the threshold bias. For example, it is not a good strategy in the connectivity game: Breaker can choose all his edges at a fixed vertex, in which case Maker must claim an edge at that vertex in the first \( \frac{n-1}{b} \) turns or it will be isolated and she will lose. This is an unlikely event whenever \( b = \omega(1) \), yet the threshold bias for the connectivity game is \( \Theta \left( \frac{n}{\log n} \right) \) (for the best bound see Gebauer and Szabó [10]).

Overcoming this particular obstacle, Ferber, Krivelevich and Naves [8] gave a different randomised strategy which, provided the bias is \( O \left( \frac{n}{\log n} \right) \), ensures that Maker claims many edges at each vertex. A little more formally, for any fixed Breaker strategy, Maker draws a random graph \( \Gamma = G(n, p) \) with \( \frac{\log n}{n} \ll p \ll b^{-1} \), which with high probability is such that Maker can claim a subgraph \( G \) of \( \Gamma \) with minimum degree nearly \( pn \). The advantage of this strategy is that now a local resilience result on \( G(n, p) \) gives a Maker win strategy in the \((1 : b)\) game, and several such results were known at the time [8] was written. This reproves, for example, that the threshold bias in the Hamiltonicity game is \( \Theta \left( \frac{n}{\log n} \right) \), for which the asymptotically optimal bias is shown by Krivelevich [10].

The randomised strategy of [8] is, roughly, to reveal edges of \( G(n, p) \) not arbitrarily but in an order responsive to Breaker’s actions. However, this strategy does not allow Maker to win, for example, the \( K_3 \)-factor game with any serious bias because Breaker can easily focus on a vertex and claim all edges that appear in the vertex neighbourhood; then this vertex will not be in any triangle and hence there is no \( K_3 \)-factor. Our main result overcomes this obstacle.

**Our results.** Our main result states that if \( p \gg n^{-1/2} \) and \( b \ll p^{-1} \), in the \((1 : b)\) Maker-Breaker game played on \( K_n \), Maker has a randomised strategy which with high probability generates a subgraph of \( G(n, p) \) with minimum degree close to \( pn \), in which every vertex neighbourhood contains nearly \( p^3 n^2/2 \) edges.

**Theorem 1.2.** For every \( n, \epsilon \in (0, 1) \), \( p \geq 10^8 \epsilon^{-2} n^{-1/2} / \log n \), and \( b \leq 10^{-24} \epsilon^6 p^{-1} \) the following holds. In the \((1 : b)\) Maker-Breaker game played on \( K_n \), for any fixed strategy of Breaker, if Maker draws a random graph \( \Gamma = G(n, p) \), then with high probability \( \Gamma \) is such that Maker can claim a subgraph \( G \) of \( \Gamma \) with \( \delta(G) \geq (1 - \epsilon)pn \) and \( e(G[N_T(v)]) \geq (1 - \epsilon)p^3 n^2/2 \).

It is easy to see that, up to constant factors, this result is sharp in the following strong sense. Whatever strategy Maker plays, if \( b \gg p^{-1} \) then Breaker can simply choose edges at a fixed vertex until none remain, so Maker claims at most \( \frac{n^{1/2}}{b^{1/2}} \ll pm \) edges at that vertex. Similarly, if \( b \gg n^{-1/2} \), Breaker has a strategy which stops Maker creating any triangles (whatever her strategy) and in particular Maker’s vertex neighbourhoods do not contain any edges claimed by Maker.

Using recent results from [4], we show that if \( \Gamma \) is a typical sample from \( G(n, p) \) then any subgraph \( G \) of \( \Gamma \) satisfying these conditions contains all low-maximum-degree and low-degeneracy
subgraphs. Recall that a graph is $D$-degenerate if all of its subgraphs contain a vertex of degree at most $D$. Since a randomised strategy for Maker which wins with positive probability in particular implies Breaker has no winning strategy, we get the following.

**Corollary 1.3.** For every $D$ and $\Delta$, there exists $c > 0$ such that the following holds for any $b \leq c(\frac{n}{\log n})^{1/2}$. In the $(1 : b)$ Maker-Breaker game played on $E(K_n)$, Maker has a deterministic strategy to claim a graph $G$ which contains copies of all graphs $H$ on at most $n$ vertices such that

1. $\Delta(H) \leq D$, or
2. $\Delta(H) \leq \Delta$ and $H$ is $\frac{D-1}{2}$-degenerate.

In particular, there is a $c > 0$ such that Maker can obtain a $K_3$-factor provided Breaker’s bias is at most $c(\frac{n}{\log n})^{1/2}$, and a $K_4$-factor if the bias is at most $c(\frac{n}{\log n})^{1/3}$. We show that both these results are optimal up to the log-factor, by giving Breaker strategies that win for biases $4n^{1/2}$ and $8n^{1/3}$ respectively.

**Theorem 1.4.**

(a) If $b \geq 4n^{1/2}$ and $3|n$, then Breaker wins the $(1 : b)$ $K_3$-factor game on $E(K_n)$.

(b) If $b \geq 8n^{1/3}$ and $4|n$, then Breaker wins the $(1 : b)$ $K_4$-factor game on $E(K_n)$.

We believe that the log-factor in Corollary 1.3 can be removed by proving a stronger resilience result in $G(n, p)$, at least for $H$ being a $K_3$-factor or $K_4$-factor. This would then give the threshold bias for the $(1 : b)$ Maker-Breaker game for these two graphs. We intend to return to this in future work.

In general we do not believe Corollary 1.3 is very close to optimal, because we expect that the results of [1] can be substantially improved in general. We suspect that it is possible (though perhaps technically hard) to remove the log-factors from those results, but we also believe that the polynomial term can in general be much better. In the spirit of inciting further research, however, we conjecture that the strategy of Theorem 1.2 is optimal up to a constant for these problems.

**Conjecture 1.5.** For each $\Delta$, there is a constant $C$ such that, given an $n$-vertex graph $H$ with $\Delta(H) \leq \Delta$, provided Maker has a winning strategy to claim a copy of $H$ in the $(1 : \bar{c})$ game on $E(K_n)$, also with positive probability when Maker plays the strategy of Theorem 1.2 in the $(1 : b)$ game on $E(K_n)$ she claims $H$ with positive probability.

We remark that if Maker plays the strategy of Theorem 1.2 using the random graph $G(n, p)$ when $\frac{\log n}{n} \ll p \ll n^{-1/2}$, while she will not necessarily claim edges in the neighbourhood of all vertices, she will at least succeed in making a subgraph of $G(n, p)$ with minimum degree close to $p n$. Similarly, if $\omega(n^{-2}) = p \ll \frac{\log n}{n}$, she will not achieve this minimum degree, but she will at least claim most of the edges of $G(n, p)$.

Finally, we observe that while we prove the existence of a deterministic winning strategy for Maker in the game of Corollary 1.3 our proof gives no hint of what that strategy might be. It would be interesting to find such deterministic winning strategies, even for a simple case such as the $K_3$-factor game.

**Organisation.** In Section 2 we collect some tools that we need in our proofs. In Section 3 we outline the proof of Theorem 1.2, for which we need an auxiliary game that is defined and analysed in Section 4. The details of the proof of Theorem 1.2 are then provided in Section 5. In Section 6 we prove a resilience result, which together with Theorem 1.2 implies Corollary 1.3. The proof of Theorem 1.4 finally, is given in Section 7.

2. Preliminaries

In this section we present some auxiliary results that will be used throughout the paper. We use extensively the following well-known bound on the tails of the binomial distribution due to Chernoff (see, for instance, [12]).

**Lemma 2.1.** If $X \sim \text{Bin}(n, p)$, then for every $0 < a < \frac{3}{4}$ we have

$$P[X \neq (1 \pm a)np] \leq 2 \exp \left( -\frac{a^2 np}{3} \right).$$

We also use the following graph partitioning lemma, derived in [1] Lemma 2.4] from the Hajnal-Szemerédi Theorem [11].

**Lemma 2.2.** Given any graph $F$ and a subset $Y$ of $V(F)$, there is a partition $V(F) = V_1 \cup \cdots \cup V_{\Delta(F)}$ such that each part is independent, the parts differ in size by at most one, and the sets $V_i \cap Y$ differ in size by at most one.
Finally, we need to know that dense subgraphs of typical random graphs contain all sparse graphs. This is one of the main results of [1]. The statement we give is a weakening and combination of Lemmas 1.21 and 1.23 from that paper. To state it, we need the following definition. We say that a pair $(X, Y)$ of disjoint vertex sets in a graph $G$ is $(\varepsilon, d, p)$-regular if for any $X' \subseteq X$ and $Y' \subseteq Y$, we have $\varepsilon |X' \cap Y'| \leq |X'\cup Y'| \leq (1+\varepsilon) |X' \cap Y'|$. For a graph $\Gamma$, a vertex $v \in V(\Gamma)$, and a vertex set $A \subset V(\Gamma)$ we also write $N_\Gamma(v; A)$ for the neighbours of $v$ in $A$, and we simply write $N_\Gamma(v)$ for $N_\Gamma(v; V(\Gamma))$.

Lemma 2.3. For all $r, \Delta, D \geq 2$ and $d > 0$ there exists $\varepsilon \in (0, 1)$ and $C$ such that if $p \geq C(\log n/n)^{1/D}$, then the random graph $\Gamma = G_{n, p}$ asymptotically almost surely has the following property.

Let $G \subseteq \Gamma$ and $H$ be graphs on $n$ vertices. Suppose that $V(G) = V_1 \cup \cdots \cup V_r$ is a partition with parts differing in size by at most one, and that $H$ has an $r$-colouring with colour classes $X_1, \ldots, X_r$ where $|V_i| = |X_i|$ for each $i$. Suppose that

(i) $\Delta(H) \leq D$, or

(ii) $\Delta(H) \leq \Delta$, and $H$ has degeneracy at most $\frac{D-1}{2}$, and at least $\frac{1}{n^2} |X_i|$ vertices of each $X_i$ have degree at most $2D$.

Suppose furthermore that the following conditions hold.

(a) For each $i < j$, the pair $(V_i, V_j)$ is $(\varepsilon, d, p)$-regular in $G$.

(b) For each $i \neq j$ and each $v \in V_i$, we have $|N_G(v; V_j)| \geq (d - \varepsilon) p |V_j|$.

(c) For each $i \neq j \neq k$ and each $v \in V_i$, the pair $(N_\Gamma(v; V_j), V_k)$ is $(\varepsilon, d, p)$-regular in $G$.

(d) For all distinct indices $i, j, k$ and all $v \in V_i$, the pair $(N_\Gamma(v; V_j), N_\Gamma(v; V_k))$ is $(\varepsilon, d, p)$-regular in $G$.

Then $H$ is a subgraph of $G$.

3. OUTLINE OF THE STRATEGY

We now briefly describe Maker’s strategy, which will prove Theorem 1.2. Maker will not initially reveal the graph $\Gamma$. Rather in each round she will pick an edge of $K_n$ not yet chosen by Breaker, and query whether this edge is in $\Gamma$. If so, she will claim it; if not, she will pick another edge, and so on, until she finds an edge of $\Gamma$. For convenience, we suppose that Maker reveals to Breaker not only which edges she claimed, but also which edges she queried. Note that under this strategy Breaker never has an incentive to claim any edge which Maker has queried and found not to be in $\Gamma$. We therefore can and will assume that Breaker never does claim such edges. In particular, this means that the edges Breaker claims are not revealed in the game, and we stop when all edges of $K_n$ are either queried by Maker or claimed by Breaker.

The strategic part of Maker’s play is thus to determine the order in which she queries edges of $\Gamma$. The idea is as follows. In alternating turns, Maker will try to avoid Breaker claiming too many edges at a given vertex (the degree game), or in a given vertex neighbourhood (the neighbourhood game), respectively. In both cases, she does this by (privately) playing an auxiliary game called box game. A box game is played on the vertex set (for us, corresponding to $E(K_n)$) of a hypergraph $H$, whose edges are the boxes. The idea of using box games goes back to Chvátal and Erdős [3]. In their box game, Maker’s aim was simply to claim at least one vertex in each box; in this paper, following Gebauer and Szabó [10] and Ferber, Krivelevich and Naves [5], Maker aims for more, namely to claim almost a ‘fair share’ of each box. Since Breaker might choose to concentrate on only one of the degree and neighbourhood games, in effect Maker plays both these games with a $1 : 2b$ bias, and thus a ‘fair share’ is a $\frac{1}{1+2b}$ fraction of each box.

As is well known, the potential function strategy of Beck [3] is a good strategy for Maker to win box games. This strategy assigns to each vertex of $H$ a score, and Maker claims the highest-scoring unclaimed vertex in each round. However there are two complicating factors in the games we consider. First, Maker typically cannot claim the highest-scoring unclaimed edge, because she can claim only edges of $\Gamma$. Second, in the neighbourhood game the boxes change over time. This is the case because during the game the $\Gamma$-neighbourhoods of vertices are not known yet, but only their portion which Maker revealed so far by claiming edges.

In the following Section 4 we define a more general box game which takes into account these complicating factors, which we model as the actions of a third player called Ghost. We prove (in Lemma 4.1) that a modification of Beck’s strategy allows Maker to win this game. In this proof it turns out that we do not need to know that the Ghost player results from a combination of the randomness of $\Gamma$ and Maker’s own actions. Instead, it suffices to require boxes to change over time only by adding unclaimed vertices of $H$ (which correspond to edges of $K_n$), and that this change has to take place immediately before Maker’s next turn.
Analysing these two box games, we will show that typically Breaker claims only a tiny fraction of the edges of $K_n$ at each vertex or in each vertex neighbourhood. Since by our assumption these edges claimed by Breaker were not revealed in $\Gamma$ during the game, with high probability $\Gamma$ also only contains a tiny fraction of these edges at each vertex or in each vertex neighbourhood, proving Theorem 1.2.

4. A GAME WITH SPIRIT

*In which a Ghost appears in the game, but in the end nothing really changes.*

In this section we define and analyse the auxiliary box game used in the proof of Theorem 1.2. This is a perfect information game played by three players, Maker, Breaker and Ghost, and we call it the $(m, b, V, e, \ell, M)$-*SpookyBox game*. It is played on a vertex set $V$. Initially a multihypergraph $H_0$ on $V$ is given, which we call the underlying multihypergraph, with $e$ edges, some of which may be the empty set and all of which have at most $M$ vertices. We refer to the vertices in $V$ taken by Maker or Breaker as claimed, whereas the vertices taken by the Ghost are haunted. At the beginning all vertices in $V$ are unclaimed and unhaunted. Now, in each round $r \geq 1$, play goes as follows. First, the Ghost creates $H_r$ from $H_{r-1}$ by, for each edge of $H_{r-1}$, adding arbitrarily some (possibly empty) set of vertices of $V$ which are neither claimed nor haunted, subject to the constraint that no edge of $H_r$ has more than $M$ vertices. Next, Maker chooses an unclaimed and unhaunted vertex. The Ghost either allows her to claim the vertex she chooses, or itself haunts the vertex. The Ghost may choose in addition to haunt (arbitrarily) other vertices as well. In each round, Maker continues to choose vertices until she has claimed $m$ vertices or all vertices are either claimed or haunted. Finally, Breaker chooses $b$ vertices which are neither claimed nor haunted, and claims all of them. The game stops when all vertices of $V$ are either claimed or haunted.

The game is won by Maker if at every round $r$ and for each edge $S \in E(H_r)$, the following holds. Let $c$ be the number of vertices of $S$ that are claimed. Then at least $\frac{mc}{m+b} - \ell$ vertices of $S$ are claimed by Maker.

In the remainder of this section we shall show that given $m, b, V, e$ and $M$, if $\ell$ is large enough then Maker has a winning strategy for the $(m, b, V, e, \ell, M)$-*SpookyBox game*. This strategy is a potential function strategy with parameters $\lambda, \tau > 0$, which we now describe. If Maker has claimed $X \subseteq V$ and Breaker has claimed $Y \subseteq V$, we define the potential of a set $A \subseteq V$ to be

$$\Phi(A, X, Y) := (1 - \lambda)^{|A \cap X|} (1 + \tau)^{|A \cap Y|}.$$ 

We define the potential of a multihypergraph $H$ on $V$ to be

$$\Phi(H, X, Y) := \sum_{S \in E(H)} \Phi(S, X, Y),$$

and the effect of $v \in V$ (with respect to $H$) to be

$$\Phi(v, H, X, Y) := \Phi(H, X, Y) - \Phi(H, X \cup \{v\}, Y).$$

Maker’s strategy is simple: when Maker has a choice, if $H$ is the current multihypergraph, and $X$ and $Y$ are the sets currently claimed by Maker and Breaker respectively, then she chooses a vertex $v$ which maximises $\Phi(v, H, X, Y)$ among all unclaimed and unhaunted vertices. We call this strategy the $($λ, τ$)$-potential strategy.

We remark that, if the Ghost never takes any action (i.e. allows Maker to claim all the vertices she chooses, and does not add vertices to edges of $H_0$), this game is a Balancer-Unbalancer game (see e.g. [3]), and the potential function strategy is well known to be good for Maker (playing Balancer). The following lemma states that the Ghost’s actions do not have any material effect on the game—as one would expect of a Ghost—and the proof is essentially the observation that the standard proof for the Balancer-Unbalancer game goes through unaffected by the actions of the Ghost.

**Lemma 4.1.** Given $m, b, e, M \in \mathbb{N}$, a vertex set $V$ and an initial multihypergraph $H_0$ on $V$ with $e$ edges, suppose we have $M \geq 9(m + b) \log e$ and

$$\ell \geq \frac{5mb}{m+b} \sqrt{\frac{M \cdot \log e}{m+b}}.$$ 

Then there exist $\lambda, \tau > 0$ such that the $($λ, τ$)$-potential strategy is a winning strategy for Maker in the $(m, b, V, e, \ell, M)$-*SpookyBox game*.

**Proof.** Given $m, b, e$ and $M$ we set

$$\lambda = \frac{1}{m} \sqrt{\frac{(m+b) \log e}{M}},$$

and the effect of $v \in V$ (with respect to $H$) to be

$$\Phi(v, H, X, Y) := \Phi(H, X, Y) - \Phi(H, X \cup \{v\}, Y).$$

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and let \( \tau \) be the unique positive solution of the equation \((1 + \tau)^b = 1 + m\lambda\).

Fix arbitrary strategies of Breaker and Ghost, and let Maker play the \((\lambda, \tau)\)-potential strategy. Let \( \mathcal{H}_0 \) be given and let \( X_0 = Y_0 = \emptyset \). For each \( r \geq 1 \), let \( X_r \) be the set of vertices claimed by Maker after round \( r \), let \( Y_r \) be the set of vertices claimed by Breaker after round \( r \), and let \( Z_r \) be the haunted vertices after round \( r \), and recall that \( \mathcal{H}_r \) is the multihypergraph created by the Ghost in round \( r \). Observe that \( \Phi(\mathcal{H}_0, X_0, Y_0) = \epsilon \).

**Claim 4.2.** \( \Phi(\mathcal{H}_r, X_r, Y_r) \) is monotone decreasing in \( r \).

**Proof.** Fix \( r \geq 1 \). We aim to show \( \Phi(\mathcal{H}_{r-1}, X_{r-1}, Y_{r-1}) \geq \Phi(\mathcal{H}_r, X_r, Y_r) \).

First observe that \( \Phi(\mathcal{H}_{r-1}, X_{r-1}, Y_{r-1}) = \Phi(\mathcal{H}_r, X_{r-1}, Y_{r-1}) \) since the Ghost only adds unclaimed vertices to edges. Suppose that in round \( r \), Maker claims vertices \( u_1, \ldots, u_m \) and Breaker then claims vertices \( v_1, \ldots, v_b \). Since the effect of Maker claiming a vertex \( u_i \) on the potential of any given edge \( S \in E(\mathcal{H}_r) \) is either nothing (if \( u_i \not\in S \)) or to multiply it by a factor \((1 - \lambda)\) (if \( u_i \in S \)), it follows that the effect of unclaimed vertices decreases as Maker claims the vertices \( u_1, \ldots, u_m \) in turn.

Let
\[
\overline{\mathcal{T}}_r = \max_{v \in V \setminus (X_r \cup Y_r \cup Z_r)} \overline{\mathcal{T}}(v, H_r, X_r, Y_{r-1}) .
\]
Since Maker chooses an unclaimed and unaunted vertex maximising the effect, we have
\[
\Phi(\mathcal{H}_r, X_r, Y_{r-1}) - \Phi(\mathcal{H}_r, X_r, Y_{r-1}) \leq -m \cdot \overline{\mathcal{T}}_r.
\]
for each \( v \in V \setminus (X_r \cup Y_{r-1} \cup Z_r) \), so in particular
\[
\Phi(\mathcal{H}_r, X_r, Y_{r-1}) - \Phi(\mathcal{H}_r, X_{r-1}, Y_{r-1}) \leq -m \cdot \overline{\mathcal{T}}_r.
\]
Now by definition, for any \( H \) and \( X, Y \), and any \( v \not\in X \cup Y \), we have
\[
\Phi(H, X, Y \cup \{v\}) - \Phi(H, X, Y) = \frac{1}{\lambda} \overline{\mathcal{T}}(v, H, X, Y).
\]
Furthermore, for any \( Y' \supseteq Y \) and any \( A \subseteq V \), we have \( \Phi(A, X, Y') \leq (1 + \tau)^{|Y'| - |Y|} \Phi(A, X, Y) \).

We conclude that for each \( 1 \leq i \leq b \) we have
\[
\Phi(\mathcal{H}_r, X_r, Y_{r-1} \cup \{v_1, \ldots, v_i\}) - \Phi(\mathcal{H}_r, X_r, Y_{r-1} \cup \{v_1, \ldots, v_{i-1}\}) \\
= \frac{1}{\lambda} \overline{\mathcal{T}}(v_i, H_r, X_r, Y_{r-1} \cup \{v_1, \ldots, v_{i-1}\}) \\
\leq \frac{1}{\lambda} (1 + \tau)^{i-1} \overline{\mathcal{T}}(v_i, H_r, X_r, Y_{r-1}) \leq \frac{1}{\lambda^i} (1 + \tau)^{i-1} \overline{\mathcal{T}}_r.
\]
Summing over \( 1 \leq i \leq b \), we have a geometric series, so
\[
\Phi(\mathcal{H}_r, X_r, Y_r) - \Phi(\mathcal{H}_r, X_r, Y_{r-1}) \leq \frac{\tau}{\lambda} \cdot \frac{(1 + \tau)^b - 1}{(1 + \tau) - 1} \overline{\mathcal{T}}_r = \frac{(1 + \tau^b - 1)}{\lambda} \overline{\mathcal{T}}_r .
\]
At last, we have
\[
\Phi(\mathcal{H}_r, X_r, Y_r) - \Phi(\mathcal{H}_r, X_{r-1}, Y_{r-1}) \leq \frac{(1 + \tau^b - 1)}{\lambda} \overline{\mathcal{T}}_r - m \cdot \overline{\mathcal{T}}_r = 0 ,
\]
where the final equality uses our choice of \( \tau \) such that \((1 + \tau)^b = 1 + m\lambda \).

Now suppose that Maker does not win the \((m, b, V, \epsilon, \ell, M)\)-SpookyBox game. Then there is some stage \( r \) and edge \( S \in E(\mathcal{H}_r) \) such that \(|X_r \cap S| \leq \frac{m}{m^b} |(X_r \cup Y_r) \cap S| - \ell \). In particular, we have
\[
\Phi(S, X_r, Y_r) = (1 - \lambda)^{|X_r \cap S|} (1 + \tau)^{|Y_r \cap S|} \geq \left(1 + \frac{\lambda m}{m^b} \right)^\ell \cdot \left(1 - \lambda \right)^m (1 + \tau^b)^{|X_r \cup Y_r \cap S|/(m^b + b)} .
\]
By choice of \( \tau \), we have \((1 + \tau)^b = 1 + m\lambda \), and we have
\[
(1 - \lambda)^m (1 + m\lambda) = (1 - \lambda)^m - (1 + (m - 1)\lambda - m^2\lambda^2) < (1 - \lambda)^{m-1} (1 + (m - 1)\lambda)
\]
so that for each \( m \geq 1 \) we have \((1 - \lambda)^m (1 + \tau^b) < 1 \). In particular, this gives
\[
\Phi(S, X_r, Y_r) = (1 - \lambda)^{|X_r \cap S|} (1 + \tau)^{|Y_r \cap S|} \geq \left(1 + \frac{\lambda}{m^b} \right)^\ell \cdot \left(1 - \lambda \right)^m (1 + \tau^b)^{|S|/(m^b + b)} .
\]
Finally, because \( S \in E(\mathcal{H}_r) \) we have \(|S| \leq M \). Plugging in our chosen \( \tau \), we have
\[
\log \Phi(S, X_r, Y_r) \geq \ell \left(\frac{1}{m^b} \log(1 + m\lambda) - \log(1 - \lambda)\right) + \frac{M}{m^{b+1}} (m \log(1 - \lambda) + \log(1 + m\lambda)) .
\]
Since \( \log(1 + m\lambda) \geq m\lambda - m^2\lambda^2/2 \) and \( \log(1 - \lambda) \leq -\lambda \) hold, we thus have
\[
\frac{1}{m^b} \log(1 + m\lambda) - \log(1 - \lambda) \geq \frac{m^b \lambda - m^2\lambda^2/2}{m^{b+1}} + \frac{m \lambda - m^2\lambda^2/2}{m^{b+1}} \geq \frac{m^b \lambda}{m^{b+1}} - \frac{m^2\lambda^2}{m^{b+1}} \geq \frac{m^b \lambda}{m^{b+1}} - \frac{3m^2\lambda^2}{m^{b+1}} .
\]
where the final inequality uses \( m\lambda \leq \frac{1}{2} \).

Similarly, we use \( \log(1 - \lambda) \geq -\lambda - \lambda^2/2 - \lambda^3 \) for \( \lambda \in (0, 1/3) \) to show
\[
m \log(1 - \lambda) + \log(1 + m\lambda) \geq -m\lambda - m^2\lambda^2/2 - m\lambda^3 + m\lambda - \frac{m^2\lambda^2}{2} > -\frac{3m^2\lambda^2}{2} .
\]

\[
\frac{1}{m^b} \log(1 + m\lambda) - \log(1 - \lambda) \geq \frac{m^b \lambda - m^2\lambda^2/2}{m^{b+1}} + \frac{m \lambda - m^2\lambda^2/2}{m^{b+1}} \geq \frac{m^b \lambda}{m^{b+1}} - \frac{m^2\lambda^2}{m^{b+1}} \geq \frac{m^b \lambda}{m^{b+1}} - \frac{3m^2\lambda^2}{m^{b+1}} .
\]
where the final inequality uses \( m\lambda \leq \frac{1}{2} \).

Similarly, we use \( \log(1 - \lambda) \geq -\lambda - \lambda^2/2 - \lambda^3 \) for \( \lambda \in (0, 1/3) \) to show
\[
m \log(1 - \lambda) + \log(1 + m\lambda) \geq -m\lambda - m^2\lambda^2/2 - m\lambda^3 + m\lambda - \frac{m^2\lambda^2}{2} > -\frac{3m^2\lambda^2}{2} .
\]
Plugging these approximations into (1), and using the equalities $\ell = \frac{m + b}{m + 6r} \log e$ and $m^2\lambda^2 = \frac{m}{2r + 1} \log e$, we have

$$\log \Phi(S, X_r, Y_r) > \ell \frac{m + b}{2r + 1} - \frac{3m^2\lambda^2}{2r + 1} \cdot \frac{m}{2r + 1} \geq \frac{\lambda}{2} \log e - \frac{3}{2} \log e = \log e.$$ 

By Claim 4.2 we have $e \geq \Phi(H_r, X_r, Y_r) \geq \Phi(S, X_r, Y_r) > e$, which is a contradiction, proving the lemma.

5. Making most of a random graph

In this section we prove Theorem 1.2 by analysing the following strategy of Maker.

**Maker’s strategy: degree and neighbourhood games.** Given $p, b$ and $\varepsilon > 0$, let $\delta = 10^{-6}\varepsilon^2$. Maker fixes a graph $\Gamma$, and considers two auxiliary games. In odd turns Maker plays the degree game, which is a $(1, 2b, E(K_n), n, \ell_d, n)$-SpookyBox game with $\ell_d = \delta pn$. The underlying (multi)hypergraph $H_d$ of the degree game is on vertex set $E(K_n)$, with edges $E_{K_n}(v) := \{uw: u \in [n] \setminus \{v\}\}$ for each $v \in [n]$. The actions of the Ghost in this game are simply to haunt any member of $E(K_n)$ which Maker chooses and which is either not in $\Gamma$ or was claimed by Maker in the neighbourhood game. In particular, in this game the underlying hypergraph $H_d$ does not change over the real. The simulated Breaker in this game simply claims the same subset of $E(K_n)$ in each turn that the real Breaker claims in his two turns between successive odd turns of Maker.

In even turns, Maker plays the **neighbourhood game**, which we will construct as a SpookyBox game with parameters $(1, 2b, E(K_n), n, \ell_{nbh}, p^n n^{-2})$ with $\ell_{nbh} = \delta pn^2$. In this game, we start with an underlying multihypergraph $H_{nbh}^0$ on $E(K_n)$ consisting of $n$ empty edges, one for each vertex in $[n]$. The actions of the Ghost are as follows. First, in each round $r \geq 1$ of the neighbourhood game (which corresponds to round $2r$ of the real game), the Ghost creates $H_{nbh}^r$. For this, it looks at each vertex $v \in [n]$ in the real game, immediately before Maker’s $2r$-th turn in the real game, and adds to the hyperedge of $H_{nbh}^r$ corresponding to $v$ any edge $xy$ of $K_n$ such that $xv$ and $yv$ have been claimed by Maker, and such that $xy$ is unclaimed. Second, the Ghost will haunt any member of $E(K_n)$ which Maker chooses and which is either not in $\Gamma$ or was claimed by Maker in the degree game. Again, the simulated Breaker in this game simply claims the same subset of $E(K_n)$ in each turn that the real Breaker claims in his two turns between successive even turns of Maker.

In both games, Maker plays according to the potential function strategy which Lemma 4.1 states is a winning strategy for her. In both even and odd turns, Maker claims the edge in the real game that she claims in the corresponding auxiliary game.

We now prove that against any strategy of Breaker, when $\Gamma$ is drawn from the distribution $G(n, p)$, with high probability Maker wins the real game, proving Theorem 1.2.

**Proof of Theorem 1.2.** Given $\varepsilon \in (0, 1)$, let $\delta = 10^{-6}\varepsilon^2$. Given $p \geq 100m^{-1}n^{-1/2}$, suppose $b \leq 10^{-6}\delta p^{-1}$. Fix a strategy of Breaker. We will reveal a graph $\Gamma$ drawn from the distribution $G(n, p)$ as the game progresses: Maker’s actions will reveal edges and non-edges of $\Gamma$. Breaker’s actions will not reveal any edges or non-edges.

Before we analyse the result of the game, we pause to observe two properties $\Gamma$ satisfies with high probability. First, for each $v \in [n]$ we have $\deg_T(v) = (1 + \delta)pn$. Second, for each $v \in [n]$ and $S \subseteq N_{T}(v)$ with $|S| \geq pm/2$, we have $e(G[S]) = (1 + \delta)p|S|^2/2$. The first is a straightforward consequence of Lemma 2.1 while to show the second, we begin by fixing $v$ and revealing $N_{T}(v)$. Assuming $\deg_T(v) \leq (1 + \delta)pn$, we next choose $S \subseteq N_{T}(v)$ with $|S| \geq pm/2$. We reveal the edges of $\Gamma$ in $S$, and by Lemma 2.1 with input $\delta/2$, we find $(1 + \delta)p|S|^2/2$ edges with probability at least $1 - 2 \exp(-\frac{(2\delta^2 B^2 n)}{20}) \geq 1 - 2 \exp(-10p)$, where the final inequality is by choice of $p$. Taking the union bound over the at most $2^p|S|$ choices of $S$, we see that with probability at least $1 - 2 \exp(-10p)$ all subsets $S$ of $N_{T}(v)$ with $|S| \geq pm/2$ satisfy $e(G[S]) = (1 + \delta)p|S|^2/2$. Taking finally the union bound over choices of $v$, and since with high probability indeed $\deg_T(v) \leq (1 + \delta)pn$ for each $v$, we conclude that indeed the second property of $\Gamma$ also holds with high probability.

Now let Maker play the above strategy. By Lemma 4.1 she wins both auxiliary games: the degree game and the neighbourhood game. To see this, we simply need to observe that (respectively) the lower bounds on $M$ and $\ell$ for given $m, b$ and $e$ and by the choice of $p$ are satisfied. This is easy to check for $M$, and we have

$$\delta pn = \ell_d \geq \frac{100}{2b + 1} \sqrt{\frac{\log n}{2b + 1}} \quad \text{and} \quad \delta p^2n^2 = \ell_{nbh} \geq \frac{100}{2b + 1} \sqrt{\frac{p^2n^2 \log n}{2b + 1}},$$

where in the latter we observe that at most $p^2n^2$ edges are in any vertex neighbourhood since no vertex neighbourhood has size more than $(1 + \delta)pn$. Note that these inequalities hold with a great deal of room to spare; we only need $p \gg (\log n)^{1/3}n^{-2/3}$. 


Our aim now is to show that Breaker claims only a tiny fraction of edges of $K_n$ at any given vertex, or in any given vertex neighbourhood. We will see that this implies that Maker claims most of the edges of the random graph $\Gamma$ in each set.

For each $r \geq 1$, let $G^{m}_r$ be the graph of edges claimed by Maker after round $r$, and $G^{b}_r$ the graph of edges claimed by Breaker. It is convenient to view $(G^{m}_r, G^{b}_r)$ as a random process. We introduce one new quantity for the analysis. For $v \in [n]$, we let $S_0(v) = \emptyset$. For each $r$, if in round $r$, Maker claims the edge $uv$ and we have $|N_{G^{m}_{r-1}}(v) \cap N_{G^{b}_{r-1}}(u)| \geq \sqrt{2}pn$, then we set $S_r(v) = S_{r-1}(v) \cup \{u\}$, and otherwise we set $S_r(v) = S_{r-1}(v)$. In other words, we add $u$ to $S_r(v)$ when Maker claiming $uv$ adds exceptionally many Breaker-edges to the Maker-neighbourhood of $v$. We stop the process with failure at the first $r$ when for some $v$ we have either $\deg_{G^{m}_{r-1}}(v) \geq 2pn$ or $|S_r(v)| \geq 8\sqrt{3}pn$. Otherwise we terminate the process when all edges of $K_n$ are claimed by either Maker or Breaker. We now show that neither failure event is likely to occur.

For each $r$ and $v$, let $T_r(v)$ be the set of vertices $u$ such that $|N_{G^{m}_{r-1}}(v) \cap N_{G^{b}_{r-1}}(u)| \geq \sqrt{2}pn$. In other words, $T_r(v)$ is the set of vertices which send exceptionally many Breaker-edges to the Maker-neighbourhood of $v$. Observe that if we have not stopped the process, we have $|N_{G^{m}_{r-1}}(v)| \leq 2pn$ for each $v \in [n]$. Because Maker wins the degree game, it follows that $|N_{G^{b}_{r-1}}(v)| < 4bpn + (2b+1)\delta pn < \delta n$. In particular, at most $2b^2pn^2$ edges in $G_{r-1}^{b}$ are incident to $N_{G^{m}_{r-1}}(v)$, so $|T_r(v)| \leq 4\sqrt{3}n$. Furthermore, observe that $T_{r-1}(v) \subseteq T_r(v)$ for each $r$, $v$. Observe that both $\deg_{G^{m}_{r-1}}(v)$ and $|S_r(v)|$ are given by a sum of independent Bernoulli random variables with probability $p$. In the former case, trivially at most $n - 1$ random variables are summed, while in the latter case at most $|T_r(v)| \leq 4\sqrt{3}n$ random variables are summed. Since these sums are thus stochastically dominated by binomial random variables with expectations respectively $pn$ and $4\sqrt{3}pn$, we conclude that by Lemma 2.1 with high probability the process does not stop with failure. From now on we assume that the process does not stop with failure.

Let $G^m$ and $G^b$ be the graphs of edges claimed by respectively Maker and Breaker when the process terminates, and let $S(v)$ be the final set $S_r(v)$. We will now show that $G^b$ contains only a tiny fraction of edges of $K_n$ at each vertex and in each $G^m$-neighbourhood. Observe that none of the edges of $G^b$ has been revealed when the process terminates. We already showed $\deg_{G^{m}}(v) < \delta n$ for each $v$.

We claim that for each $v$ we have $e\left(G^{b}[N_{G^{m}}(v) \setminus S(v)]\right) \leq \delta p^2n^2 + 2\sqrt{\delta p^2n^2 + \delta p^2n^2}$. First, observe that with high probability we have $e\left(G^{m}[N_{G^{m}}(v) \setminus S(v)]\right) \leq 4p^3n^2$, using both properties of $\Gamma$ we observed at the beginning. Now, the edges of $G^{b}[N_{G^{m}}(v) \setminus S(v)]$ fall into three classes. There are the Breaker edges which appear in the neighbourhood game. Because Maker wins the neighbourhood game, and because Maker claims at most $4p^3n^2$ edges in $N_T(v)$ in this game, the number of edges Breaker claims in the neighbourhood game in $N_T(v)$ is at most $2b \cdot 4p^3n^2 + (2b + 1)\delta p^2n^2 \leq \delta p^2n^2$. There are the old edges, which were claimed by Breaker before being revealed to be in $N_T(v)$. By definition of $S(v)$ and since $\deg_{G^{m}}(v) \leq 2pn$, there are at most $2pn \cdot \sqrt{\delta p}n$ such edges. Finally, there are the grabbed edges (which are potential edges for $\Gamma$), Breaker claimed after they were revealed to be in $N_T(v)$ but before Maker had the chance to play for them in the neighbourhood game. There are at most $2b$ such edges per vertex of $N_T(v)$, so in total at most $2b \cdot n \leq \delta p^2n^2$ such edges.

Finally, we reveal the edges of $\Gamma$ in $G_b$. Using Lemma 2.1 and the union bound, we see that with high probability at most $2b\cdot pn$ edges of $\Gamma$ appear in $G_b$ at any vertex $v \in [n]$, so $\deg_{G^{m}}(v) = (1 \pm 3\delta)pn = (1 \pm \eps)pn$. Again using Lemma 2.1 and the union bound, with high probability Breaker claimed at most $6\sqrt{\delta p^3n^2}$ edges of $\Gamma$ in $N_{G^{m}}(v) \setminus S(v)$. Since the process did not stop with failure, we have $|S(v)| \leq 8\sqrt{\delta p}n$, so $|N_{G^{m}}(v) \setminus S(v)| \geq (1 - 3\delta - 8\sqrt{\delta}p)n$. By the second property of $\Gamma$, the total number of edges of $\Gamma$ in $N_{G^{m}}(v) \setminus S(v)$ is at least $(1 - \eps/2)n^2/2$, of which at most $6\sqrt{\delta p^3n^2}$ are claimed by Breaker. The remaining at least $(1 - \eps)p^3n^2/2$ edges are claimed by Maker, giving the desired edges claimed by Maker in $N_T(v)$. \hfill \square

Let us finish this section with a few remarks about simple modifications to the setup considered in Theorem 1.2. Firstly, it is easy to check that this strategy also works if $K_n$ is replaced by a typical random graph $G(n, q)$ for some $q > 0$. In that case the random graph $G(n, p)$ is obtained from $G(n, q)$ by keeping edges independently with probability $p/q$ (which must be at most one) and Breaker’s bias satisfies $b \ll q/p$. Secondly, it is similarly easy to check that this proof goes through if Breaker, rather than Maker, plays first. To see this, observe that removing any single edge from Maker’s graph (corresponding to Maker losing her first turn) only affects the number of edges Maker claims at a vertex or in a vertex neighbourhood by one; this off-by-one error is absorbed by the slack in the constant choices.
6. Universal graphs

The following resilience result for random graphs together with Theorem 6.1 immediately implies Corollary 1.3.

**Theorem 6.1.** For all $D, \Delta \geq 2$ there exist $\varepsilon > 0$ and $C$ such that if $p \geq C\left(\frac{\log n}{n}\right)^{1/D}$, then the random graph $\Gamma = G_{n,p}$ asymptotically almost surely has the following property.

Let $G$ be any spanning subgraph of $\Gamma$ such that $\text{deg}_{G}(v) \geq (1 - \varepsilon)p n$ and $e_{G}(N_{\Gamma}(v)) \geq (1 - \varepsilon)p n^{2}/2$. Then $G$ contains all graphs $H$ on at most $n$ vertices with $\Delta(H) \leq D$, and all graphs $H$ on at most $n$ vertices with $\Delta(H) \leq \Delta$ and degeneracy at most $\frac{D+1}{2}$.

**Proof.** Given $D$ and $\Delta$, without loss of generality we assume $\Delta \geq D$ and we set $r = 8\Delta$. We set $d = \frac{r}{2}$. Let $\varepsilon' > 0$ and $C$ be returned by Lemma 2.3. We set $\varepsilon = 10^{-6}r^{-3}(\varepsilon')^{3}$. Let $p \geq C\left(\frac{\log n}{n}\right)^{1/D}$.

Partition $[n]$ into $r$ parts $V_{1}, \ldots, V_{r}$, in size order, differing in size by at most one. We now generate $\Gamma = G(n, p)$. Using Lemma 2.3, we can easily check that with high probability, the following all hold. First, for each $v \in V(\Gamma)$ we have $\text{deg}(v) = (1 \pm \varepsilon)p n$ and $N(v)$ contains $(1 \pm \varepsilon)p n^{2}/2$ edges. Second, for each $i \in [r]$ and $v \notin V_{i}$ we have $\text{deg}(v; V_{i}) = (1 \pm \varepsilon)p|V_{i}|$. Third, for each $i \neq j \in [r]$, and each $v \notin V_{i}$, the pairs $(V_{i}, V_{j})$, and $(N(v; V_{i}), V_{j})$ are $(\varepsilon, 1, p)$-regular, and if $v \notin V_{j}$ in addition $(N(v; V_{i}), N(v; V_{j}))$ is $(\varepsilon, 1, p)$-regular. Suppose now that $\Gamma$ satisfies this good event and in addition the good event of Lemma 2.3 with the inputs as above.

Let $G \subseteq \Gamma$ be such that $\text{deg}_{G}(v) \geq (1 - \varepsilon)p n$ and $e_{G}(N_{\Gamma}(v)) \geq (1 - \varepsilon)p n^{2}/2$. Then $G$ and $\Gamma$ differ by at most $2\varepsilon p n$ edges at each vertex, and at most $\varepsilon p n^{2}$ edges in each vertex neighbourhood. In particular, for any $i$ and $S \subseteq V_{i}$, at most $2\varepsilon p m|S|$ edges were removed from the vertices of $S$, so that for any $j \neq i$ and $T \subseteq V_{j}$ with $|T| \geq \varepsilon'|V_{j}|$, the pair $(S, T)$ has density at least $p/2$. It follows that $(V_{i}, V_{j})$ and $(N_{\Gamma}(v; V_{i}), V_{j})$ are $(\varepsilon', 1, p)$-regular in $G$ for each $i \neq j$ and $v \notin V_{i}$. Finally, since $(N_{\Gamma}(v; V_{i}), N_{\Gamma}(v; V_{j}))$ is $(\varepsilon, 1, p)$-regular in $\Gamma$ for any $i \neq j$ and $v \notin V_{i} \cup V_{j}$, and since at most $\varepsilon p n^{2}$ edges were removed from $N_{\Gamma}(v)$ to form $G$, also the pair $(N_{\Gamma}(v; V_{i}), N_{\Gamma}(v; V_{j}))$ is $(\varepsilon, 1, p)$-regular in $G$. We conclude that $G$, with the partition $V_{1}, \ldots, V_{r}$, meets the conditions of Lemma 2.3.

If $H$ satisfies $v(H) \leq n$ and $\Delta(H) \leq D$, we can add isolated vertices if necessary to obtain a graph with exactly $n$ vertices. By Lemma 2.2 with the input $Y = \emptyset$ we can let $X_{1}, \ldots, X_{r}$ be a partition of $V(H)$ into independent sets, in size order, differing in size by at most one, so $|V_{i}| = |X_{i}|$ for each $i$. Now Lemma 2.3 states that $H \subseteq G$, as desired.

Finally, if $H$ satisfies $v(H) \leq n$, and $\Delta(H) \leq \Delta$, and $H$ has degeneracy at most $\frac{D+1}{2}$, then again we can add isolated vertices to $H$ if necessary to obtain an $n$-vertex graph. Let $Y$ be the set of vertices in $H$ with degree at most $2D$. Since $H$ has less than $Dn$ edges, we have $|Y| \leq \frac{n}{2(2D+1)}$. Now let $X_{1}, \ldots, X_{r}$ be a partition of $V(H)$ into independent sets, in size order, obtained by applying Lemma 2.2 with input $Y$. We have $|V_{i}| = |X_{i}|$ for each $i$, and $|X_{i} \cap Y| \leq \frac{1}{2(2D+1)}|X_{i}|$ for each $i$. Thus again Lemma 2.3 states that $H \subseteq G$, as desired.

7. Breaker strategies

In this section we give strategies for Breaker in the $K_{3}$-factor and $K_{4}$-factor games, proving Theorem 1.4.

**Proof of Theorem 1.4.** The Breaker strategy for the $K_{3}$-factor game is well known. In fact, Breaker can even guarantee that Maker creates no triangles at all: In each round, when Maker claims $uv$, Breaker claims $n^{1/2}$ edges at each of $u$ and $v$. Breaker then claims any dangerous edge which would otherwise enable Maker to complete a triangle in the next round. It is easy to see that Maker will never claim more than $n^{1/2}$ edges at any given vertex. In a given round, since Maker claims $uv$ rather than creating a triangle, all dangerous edges must form a triangle with two of Maker’s edges, one of which is $uv$. Since Maker claimed at most $n^{1/2}$ edges at each of $u$ and $v$, there are at most $2n^{1/2}$ such dangerous edges.

The $K_{4}$-factor game is a little harder. Breaker cannot avoid Maker creating any $K_{4}$, but he can ensure that there is no copy of $K_{4}$ that contains a vertex $v_{0}$ that he may choose. Roughly speaking, Breaker’s strategy is to bound the maximum degree of $v_{0}$ in Maker’s graph by $n^{2/3}$ while preventing a copy of $K_{4}$ in the growing neighbourhood of $v_{0}$ in Maker’s graph. We will be denoting Maker’s graph by $G_{M}$ at any stage of the game.

Thus, Breaker uses the following strategy.

- Breaker plays $n^{1/3}$ edges in each round at $v_{0}$. In this way the degree of $v_{0}$ in $G_{M}$ stays bounded by $n^{2/3}$.
- Breaker aims to prevent Maker from creating a triangle in Maker’s (growing) neighbourhood $N_{G_{M}}(v_{0})$ of $v_{0}$ as follows:
(i) If Maker chooses some edge $xy$ outside of her neighbourhood of $v_0$ then Breaker claims edges $v_0x$ and $v_0y$ and thus prevents $xy$ to lie in a copy of $K_4$ with $v_0$ in $G_M$. So this move of Maker is wasted.

(ii) If Maker chooses some edge $v_0y$, then Breaker connects $y$ to $2n^{1/3}$ highest degree vertices in Maker’s current neighbourhood of $v_0$. Observe that $y$ is not connected to any vertex in Maker’s neighbourhood at this point.

(iii) If Maker chooses some edge $xy$ from her neighbourhood of $v_0$, then Breaker connects $x$ and $y$ to $2n^{1/3}$ available highest degree vertices in $N_{G_M}(v_0)$. Then Breaker claims all available edges which lie in $N_{G_M}(v_0) \cap N_{G_M}(x)$ and $N_{G_M}(v_0) \cap N_{G_M}(y)$. There are at most $3n^{1/3}$ such edges, since every vertex $z$ satisfies $|N_{G_M}(v_0) \cap N_{G_M}(z)| \leq \frac{3}{2}n^{1/3}$ throughout the game, as we will argue below.

In total Breaker plays with bias at most $n^{1/3} + 4n^{1/3} + 3n^{1/3} = 8n^{1/3}$ and wins the $K_4$-factor game.

It thus remains to show that any vertex $z$ satisfies $|N_{G_M}(v_0) \cap N_{G_M}(z)| \leq \frac{3}{2}n^{1/3}$ during the game. Recall that when a vertex is added to $N_{G_M}(v_0)$ its degree within $N_{G_M}(v_0)$ is zero. After $n^{2/3}$ rounds, all edges are claimed at $v_0$ and thus, Maker’s graph has at most $n^{2/3}$ edges within $N_{G_M}(v_0)$, so that at most $2n^{1/3}$ vertices have degree at least $n^{1/3}$. Certainly any vertex $z$ whose degree within $N_{G_M}(v_0)$ eventually exceeds $\frac{3}{2}n^{1/3}$ must at some point have more than $n^{1/3}$ neighbours in $N_{G_M}(v_0)$. From the time at which the degree of a vertex $z \in N_{G_M}(v_0)$ within $N_{G_M}(v_0)$ is at least $n^{1/3}$, the edge from $z$ to each new vertex entering $N_{G_M}(v_0)$ is claimed by Breaker. Thus, from the time when $z$ has $n^{1/3}$ neighbours in $N_{G_M}(v_0)$, in order to increase the degree of $z$ in $N_{G_M}(v_0)$, Maker can only claim edges to a fixed set of size at most $n^{2/3}$. Each time Maker claims such an edge at $z$, Breaker claims up to $2n^{1/3}$ other such edges and thus Maker cannot claim more than $\frac{1}{2}n^{1/3}$ additional edges at $z$, as desired. \qed

Note that this strategy does not generalise to show that Breaker wins the $K_5$-factor game if $b \gg n^{1/4}$. Indeed, it follows from Theorem 1.2 and the positive solution to the KLR conjecture \cite{lor} that when $b \ll n^{2/7}$, Maker can guarantee to make a graph in which every vertex is contained in a copy of $K_5$.



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