DUALITY IN STRING COSMOLOGY

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Scale factor duality, a truncated form of time dependent T-duality, is a symmetry of string effective action in cosmological backgrounds interchanging small and large scale factors. The symmetry suggests a cosmological scenario ("pre-big-bang") in which all duality related branches, an inflationary branch and a decelerated branch are smoothly joined into one non-singular cosmology. The use of scale factor duality in the analysis of the higher derivative corrections to the effective action, and consequences for the nature of exit transition, between the inflationary and decelerated branches, are outlined. A new duality symmetry is obeyed by the lowest order equations for inhomogeneity perturbations which always exist on top of the homogeneous and isotropic background. In some cases it corresponds to a time dependent version of S-duality, interchanging weak and strong coupling and electric and magnetic degrees of freedom, and in most cases it corresponds to a time dependent mixture of both S- and T-duality. The energy spectra obtained by using the new symmetry reproduce known results of produced particle spectra, and can provide a useful lower bound on particle production when our knowledge of the detailed dynamical history of the background is approximate or incomplete.

1 Introduction

Our starting point is the effective action (in the so-called "string-frame")

\[
S_{eff} = \int d^4x \left\{ \sqrt{-g} e^{-\phi} \left[ \frac{1}{16\pi\alpha'} (R + \partial_\mu \phi \partial^\mu \phi) \right] + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Psi \partial_i \Psi + \ldots \right\}
\]

(1)

The dilaton \(\phi\) is a "Brans-Dicke-like" scalar with \(\omega_{BD} = -1\).

The basic length scales of the theory are the string scale \(\alpha' \equiv \ell_s^2\) and the Planck scale \(e^{\alpha'} \equiv G_N \equiv \ell_P^2\), related by \(e^{\alpha'} \equiv g_{\text{string}}^2 \approx \frac{1}{4\pi} \delta_{\text{GUT}}(1/\ell_s) = \left(\frac{v_s}{\ell_s}\right)^2\). These relations are modified for strongly coupled string theory. We will assume that the theory is weakly coupled throughout the evolution.

At early times fields may have been displaced from their present state, so we look for general FRW type solutions \(g_{\mu\nu} = \text{diag}(-1, a^2(t)dx_i dx_i), \phi = \phi(t)\), and if other fields are present we assume that they have only time dependence. We obtain equations for the Hubble parameter \(H(t) = \dot{a}(t)/a(t)\), \(\dot{\phi}\), and additional fields, in particular moduli, and matter (if present), and solve them, requiring that at late times the evolution has to be that of standard cosmology.

In a general effective action we may represent any contributions in addition to lowest order action by a "matter" Lagrangian

\[
S_{eff} = \int d^4x \left\{ \sqrt{-g} \left[ \frac{e^{-\phi}}{16\pi\alpha'} (R + \partial_\mu \phi \partial^\mu \phi) \right] + \frac{1}{2} \mathcal{L}_m(\phi, g_{\mu\nu}, ...) \right\}.
\]

(2)

The equations of motion are the following

\[
\begin{align*}
\dot{\phi} &= 3H_S \pm \sqrt{3H_S^2 + e^\phi \rho_S} \\
\dot{H}_S &= \pm H_S \sqrt{3H_S^2 + e^\phi \rho_S} \mp \frac{1}{2} e^\phi (\rho_S + \Delta_\phi \mathcal{L}_m) \\
\dot{\rho}_S &= 3H_S(\rho_S + p_S) = -\Delta_\phi \mathcal{L}_m \dot{\phi},
\end{align*}
\]

where

\[
\begin{align*}
T_{\mu\nu} &= \frac{1}{\sqrt{-g}} \delta \mathcal{L}_m \\
\Delta_\phi \mathcal{L}_m &= \frac{1}{2} \frac{1}{\sqrt{-g}} \delta \mathcal{L}_m \\
T_{\mu\nu} &= \text{diag}(\rho, -p, p, -p).
\end{align*}
\]

As a result of scale factor duality (SFD) which will be discussed in more detail below, the solutions come in pairs or branches, the (+) branch vacuum (without any sources) satisfies

\[
\dot{H}_S = +H_S \sqrt{3H_S^2},
\]

\[
H_S = \frac{1}{\sqrt{\frac{1}{3} - t_0}}, \quad t < t_0 \text{ and is characterized by a future singularity. If the universe starts expanding according to the (+) branch vacuum solution, } H \text{ is positive, and therefore its time derivative is also positive. This branch cannot connect smoothly to radiation dominated (RD) FRW with constant dilaton, } \dot{\phi} = 3H_S + \sqrt{3H_S^2 + e^\phi \rho_S} \Rightarrow \dot{\phi} > 0. \text{ The (-) branch vacuum satisfies}
\]

\[
\dot{H}_S = -H_S \sqrt{3H_S^2},
\]

\[
H_S = \frac{1}{\sqrt{\frac{1}{3} t_0 - t}}, \quad \frac{1}{\sqrt{\frac{1}{3} t_0}} > t_0 \text{ and is characterized by a past singularity. This branch can connect smoothly to RD}
\]
where $\tilde{\phi} \equiv \phi - 3 \ln a$, and we have set the lapse function $n(t)$ to unity.

The action (6) is invariant under the symmetry transformation

\[
a(t) \to 1/a(t),
\]
\[
\phi(t) \to \phi - 6 \ln a(t),
\]
\[
\tilde{\phi} \text{ and } H^2 \text{ are invariant },
\]
\[
\tilde{\phi}(t) \to \tilde{\phi}(t),
\]
\[
H(t) \to -H(t),
\]
and the equations of motion are covariant. The two branches describing an expanding universe are related to each other SFD× Time reversal. In general, for more complicated cosmological backgrounds the symmetry of the action is more complicated.

### 2.2 Leading corrections

The transition between the two duality related branches is called the graceful exit transition. It is known that to lowest order the two branches are separated by a singularity. The emerging scenario for the exit transition requires classical $\alpha'$ corrections which can bound the curvature below the string scale, as well as quantum corrections. The leading classical corrections determine whether the solution can reach a “good” region in $\dot{\phi}, H$ phase space. A model for the exit transition has been presented, and therefore we know that a transition is possible. The question is whether string theory actually determines the coefficients such that a transition occurs.

We have investigated effective classical corrections, and demanded that the action will really be an effective dilaton-gravity action, without additional new degrees of freedom. This means that we have to use actions that produce equations without higher derivatives. Field redefinitions can change that but, it is better to use a “frame” with no higher derivatives ensuring numerical stability and control. We also require an action that reproduces whatever string theoretic information available such as scattering amplitudes, perturbative beta-function calculations etc. As we will see this is not enough to obtain the full corrected action. If scale factor duality could be imposed in a practical way it would help, however, the situation is more complicated.

The action including leading classical corrections is given by

\[
S_{LCC} = \int d^4x \sqrt{-\tilde{g}} e^{-\tilde{\phi}} (R + (\nabla \phi)^2 + \frac{1}{2} A (\nabla \phi)^4 + B R^2_{GB} + \ldots)
\]
\[ C \left( R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \right) \nabla_\mu \phi \nabla_\nu \phi + D \nabla^2 (\phi) \left( \nabla \phi \right)^2 \} \].

In covariant variables, it takes the following form

\[ S_{LCC} = e^{-\phi} \left\{ \frac{3}{n(t)} \left( \frac{\tilde{H}(t)}{n(t)} \right)^2 - \frac{\tilde{\phi}(t)}{n(t)} \right\} + \frac{3 (27 A + 8 B + 9 C + 27 D) \ H(t)^4}{2 n(t)^3} + \frac{(54 A + 4 B + 9 C + 45 D) \ H(t)^3 \ \frac{\tilde{\phi}(t)}{n(t)}}{n(t)} + \frac{3 (18 A + C + 12 D) \ H(t)^2 \ \frac{\tilde{\phi}(t)}{n(t)}}{2 n(t)^3} + \frac{3 (2 A + D) \ H(t) \ \frac{\tilde{\phi}(t)}{n(t)}}{n(t)^3} + \frac{(3 A + D) \ \frac{\phi(t)}{6 n(t)^4}}{\phi(t)} \right\}, \]

where we have performed integration by parts to get rid of the \( h'(t), \phi''(t) \) and \( n'(t) \). This is possible in general due to the 'no higher derivatives' condition.

Perturbative string calculation can provide two of the coefficients, one additional coefficient sets the overall scale at which the leading corrections kick in, so one coefficient remains unknown. If some symmetry principle, such as SFD could be used, the leading corrections action would be determined completely. For example, to impose naive SFD we would have to eliminate the odd parts of the action. We can set, for example,

\[ D = -2 A \]
\[ C = 4 A - \frac{4}{9} B. \]

However, this is only possible in a homogeneous background. In an inhomogeneous background we get more equations and the only consistent solution is \( A = B = C = D = 0 \). The conclusion is that, at the very least, the SFD transformation has to be modified at this order, making it less useful for determination of the one remaining coefficient. If we insist on naive SFD in the homogeneous case, then it is possible to show that if a stable algebraic fixed point exists, another non-stable fixed point will also exist, and that the generic solution will encounter the unstable fixed point first and run into a singularity.

### 2.3 All orders

As we have seen, additional input is required to determine the behaviour of solutions when classical stringy corrections are included. The best would be to establish the existence of an exact conformal field theory solution corresponding to the algebraic fixed point. In general this requires working with 2-d conformal field theories rather than with effective actions. I outline here some of the possibilities to achieve progress in that direction.

For highly symmetric backgrounds, such as the linear-dilaton deSitter background, it is possible to use the isometries of the background to impose additional symmetries on world-sheet operators, and constrain the beta-function coefficients. Another possibility is to use conformal perturbation theory to add \((1,1)\) operators to an established conformal field theory, a linear dilaton flat-space background. Yet another possibility is to start with known exact solutions in higher dimensions and compactify down to 4-d.

### 3 “S”-Duality

The quadratic action for perturbations of any tensor field expanded around a cosmological dilaton-gravity background is given by

\[ S_{\text{pert}} = \frac{1}{2} \int d^3 x \eta S(\eta) \left[ \psi'^2 + (\nabla \psi)^2 \right]. \] (7)

The prefactor \( S(\eta) \) is given by \( a^2 e^{C_\ell} \), where \( m \) and \( \ell \) depend on the type of field. For example, gravitons, dilatons and moduli have \( m = 1, \ell = -1 \), model independent axions have \( m = 1, \ell = 1 \), while Ramond-Ramond axions have \( m = 1, \ell = 0 \), and so on. We would like to compute the evolution of perturbation and eventually compute an important physical observable quantity: the spectrum of produced particles at late times.

#### 3.1 “S”-Duality symmetry

To discuss duality symmetry of the action (6) it is more convenient to to use the Hamiltonian formalism. The Hamiltonian density corresponding to (6) is given by

\[ H = \frac{1}{2} \int d\eta \left\{ S^{-1} \Pi'^2 + S(\nabla \Psi)^2 \right\}, \] (8)

where the momentum conjugate to \( \Psi \) is given by

\[ \Pi = S \Psi'. \] (9)

The Hamilton equations of motion are first order

\[ \Pi' = - \frac{\delta H}{\delta \Psi} = S \nabla^2 \Psi \]
\[ \Psi' = \frac{\delta H}{\delta \Pi} = S^{-1} \Pi, \] (10)
and lead to second order equations
\[ \Pi'' - \frac{S'}{S} \Pi' - \nabla^2 \Pi = 0 \]  
\[ \Psi'' + \frac{S'}{S} \Psi' - \nabla^2 \Psi = 0. \]  
The second equation (12) is commonly used in analysis of perturbation spectra.

In Fourier space the Hamiltonian density is given by
\[ H = \frac{1}{2} \int d\eta \left\{ S^{-1} \Pi_k \Pi_{-k} + Sk^2 \Psi_k \Psi_{-k} \right\}, \]  
and the equations of motion are given by
\[ \Pi_k' = -Sk^2 \Psi_{-k} \]  
\[ \Psi_k' = S^{-1} \Pi_{-k}. \]  

“S”-duality exchanges the variables and momenta and at the same time sends \( S \) to its inverse,
\[ \Pi_k \to \Pi_{-k} = k \Psi_k \]  
\[ k \Psi_k \to k \Psi_{-k} = -\Pi_k \]  
\[ S \to \bar{S} = S^{-1}, \]  
leaving the Hamiltonian, equations of motion and Poisson brackets invariant.

We are interested in a situation in which the initial conditions correspond to zero-point vacuum fluctuations of the field \( \Psi \), and therefore
\[ \langle S^{-1} \Pi^2 \rangle = \langle S (\nabla \Psi)^2 \rangle, \]  
where \( \langle \cdots \rangle \) denotes ensemble average.

The duality (13), contains strong-weak coupling duality as a special case. For perturbative heterotic 4-d gauge bosons the function \( S \) is given simply by \( S(\eta) = e^{\phi(\eta)} \). Recall that \( e^{\phi(\eta)} = g_{\text{string}} \), so the transformation \( S \to \bar{S} = S^{-1} \) is, at each time \( \eta \), simply the celebrated strong-weak coupling duality \( g_{\text{string}} \to g_{\text{string}}^{-1} \), which appears as a part of the \( SL(2,\mathbb{Z}) \) group, usually called \( S \)-duality. The transformation (13) exchanges in this case electric and magnetic degrees of freedom.

3.2 Approximate solutions

To construct approximate solutions define \( \hat{\Psi}, \hat{\Pi} \), whose Fourier modes are given by
\[ \hat{\Psi}_k = S^{1/2} \Psi_k \]  
\[ \hat{\Pi}_k = S^{-1/2} \Pi_k. \]  
The new variables have simple transformation law under “S”-duality
\[ k \hat{\Psi}_k \to \frac{\hat{\Psi}_k}{k} = -\Pi_k \]  
\[ k \hat{\Pi}_k \to \frac{\hat{\Pi}_k}{k} = k \Psi_k \]  
\[ S \to \bar{S} = S^{-1}. \]  
The variables \( \hat{\Psi}, \hat{\Pi} \) satisfy the following Schrödinger-like equations
\[ \hat{\Psi}_k'' + \left( k^2 - (S^{1/2})^n S^{-1/2} \right) \hat{\Psi}_k = 0 \]  
\[ \hat{\Pi}_k'' + \left( k^2 - (S^{-1/2})^n S^{1/2} \right) \hat{\Pi}_k = 0. \]  
Since \( S(\eta) \sim \eta^n \), the potentials \( V_\Phi = (S^{1/2})^n S^{-1/2}, \) \( V_\Pi = (S^{-1/2})^n S^{1/2} \), if non-vanishing, are proportional to \( 1/\eta^n \). For \( k^2 > V_\Phi, V_\Pi \), or equivalently \( (k\eta)^2 > 1 \) (inside the horizon), we look for WKB-like approximate solutions
\[ \hat{\Psi}_k(\eta) = (k^2 - V_\Phi)^{-1/4} e^{-i \int d\eta' (k^2 - V_\Phi)^{1/2}} \]  
\[ \hat{\Pi}_k(\eta) = (k^2 - V_\Pi)^{-1/4} e^{-i \int d\eta' (k^2 - V_\Pi)^{1/2}}. \]  
The advantage of looking at solutions (20) is that they manifestly preserve the “S”-duality symmetry of the equations, because the potentials \( V_\Phi, V_\Pi \) get interchanged under \( S \to \bar{S} \).

For very large \( k^2, k^2 \gg V_\Phi, V_\Pi \) solutions (20) reduce to correctly normalized vacuum fluctuations
\[ \hat{\Psi}_k(\eta) = k^{-1/2} e^{-ik\eta + i\varphi_0} \]  
\[ \hat{\Pi}_k(\eta) = k^{+1/2} e^{-ik\eta + i\varphi_0}, \]  where \( \varphi_0, \varphi_0' \) are random phases, originating from the random initial conditions. Note that because of the random phases, “S”-duality holds only on the average in the sense of eq.(16).

For \( k^2 < V_\Phi, V_\Pi \), or equivalently \( (k\eta)^2 < 1 \) (outside the horizon), it is possible to write “exact” solutions (21). It is convenient to define the functions \( T \cos(S^{-1}, S), T \sin(S^{-1}, S) \)
\[ T \cos(S^{-1}, S) = 1 - k \int_{\eta_e}^{\eta} d\eta' S^{-1}(\eta') k \int_{\eta_1}^{\eta} d\eta S(\eta_2) + \cdots \]  
\[ + (-1)^{n+1} k^{2(n-1)} \prod_{n=1}^{n-1} \int S^{-1} \int S \cdots \int S \cdots \int S + \cdots \]  
\[ T \sin(S^{-1}, S) = k \int_{\eta_e}^{\eta} d\eta' S^{-1}(\eta_1) - \cdots \]  
\[ + (-1)^{n+1} k^{2n-3} \int S^{-1} \int S \cdots \int S \cdots \int S + \cdots, \]
in terms of which the “exact” solutions take the following form
\[
\hat{\Psi}_k(\eta) = \sqrt{S}\left\{ A_k T\cos(S^{-1}, S) + B_k T\sin(S^{-1}, S) \right\}
\]
\[
\hat{\Pi}_k(\eta) = \frac{k}{\sqrt{S}}\left\{ B_k T\cos(S, S^{-1}) - A_k T\sin(S, S^{-1}) \right\}.
\]
(23)

Using the relations
\[
[T\cos(S^{-1}, S)]' = -\frac{k}{S} T\sin(S, S^{-1}),
\]
\[
[T\sin(S^{-1}, S)]' = \frac{k}{S} T\cos(S, S^{-1})
\]
and similar relations for \([T\cos(S, S^{-1})]'\) and \([T\sin(S, S^{-1})]'\), it is possible to verify explicitly that \(\hat{\Psi}_k, \hat{\Pi}_k\) in eq. (23) are indeed solutions of eqs. (19).

Formally, these solutions are valid also inside the horizon, but the functions \(T\cos, T\sin\) are not well defined there.

We need to match the solutions inside and outside the horizon and do it such that “S”-duality is respected. One way of doing so is to use solutions (20) inside the horizon, and (23) outside the horizon and match them at some time near horizon exit time \(\eta_{ex}\), for which \(k\eta_{ex} \sim 1\).

Taking advantage of the phenomenon of “freezing of perturbations” outside the horizon we obtain the following result,
\[
\hat{\Psi}_k(\eta) = \frac{1}{\sqrt{k}} \left( \frac{S_{ex}}{S_{re}} \right)^{-1/2} \cos(k\eta) + \left( \frac{S_{ex}}{S_{re}} \right)^{1/2} \sin(k\eta)
\]
\[
\hat{\Pi}_k(\eta) = \sqrt{k} \left( \frac{S_{ex}}{S_{re}} \right)^{1/2} \cos(k\eta) - \left( \frac{S_{ex}}{S_{re}} \right)^{-1/2} \sin(k\eta)
\]
(25)

where \(S_{re} = S(\eta_{re})\). The reentry time \(\eta_{re}\) is the second time at which \(k\eta_{re} \sim 1\).

### 3.3 Energy Spectrum

We compute an important physical observable, the Hamiltonian density,
\[
\langle H_k \rangle = \frac{1}{2} \left( \langle |\hat{\Pi}_k|^2 \rangle + k^2 \langle |\hat{\Psi}_k|^2 \rangle \right)
\]
(26)

Using the approximate solutions (25) we obtain
\[
\langle H_k \rangle = k \left( \frac{S_{ex}}{S_{re}} + \frac{S_{re}}{S_{ex}} \right).
\]
(27)

It is invariant under \(S_{ex} \rightarrow S_{ex}^{-1}, S_{re} \rightarrow S_{re}^{-1}\), and overall rescaling of \(S\). Note that, for a given \(k\), \(\langle H_k \rangle\) depends only on \(S_{ex}\) and \(S_{re}\), and not on the whole evolution.

The spectral energy distribution, \(d\rho_k/d\ln k = (k^3/a^4)(\langle H_k \rangle)\), is given by
\[
\frac{d\rho(\omega)}{d\ln \omega} = \omega^4 \left( \frac{S_{ex}}{S_{re}} \right) \left( \frac{S_{re}}{S_{ex}} \right)
\]
\[
\cong \omega^4 \max \left\{ \frac{S_{ex}}{S_{re}}, \frac{S_{re}}{S_{ex}} \right\}.
\]
(28)

It has the same invariance properties as the Hamiltonian density.

From eq. (28) we obtain model independent lower bound on energy density of cosmologically produced particles. The spectrum (28) is a sum of two terms, one being the inverse of the other. Therefore it is not possible to decrease the contribution of one term without increasing the contribution of the other. The physical origin of this lower bound is indeed the uncertainty principle. Recall that one term originates from the contribution of the perturbation conjugate momentum and the other from the contribution of the perturbation itself. The uncertainty principle says that it is not possible to decrease both without limits. For specific cases, the lower bound may be improved using some particular properties of the background.

The result (28) provides an easy and a very general way of computing \(d\rho(\omega)/d\ln \omega\). The prescription is simple. Once the function \(S(\eta)\) is known for all times, substitute for \(\eta_{ex} \rightarrow k^{-1}\), and for \(\eta_{re}\) substitute the properly redshifted \(k^{-1}\). The results obtained using this simple method reproduce known results obtained by explicit complicated calculations.

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