Quantum phase liquids: Fermionic superfluid without phase coherence

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Abstract – We investigate the two-dimensional generalized attractive Hubbard model in a bipartite lattice, and find a “quantum phase liquid” phase, in which the fermions are paired but do not have phase coherence at zero temperature, in analogy to quantum spin liquid phase. In addition, two types of topological quantum phase liquids with a small external magnetic field — $Z_2$ quantum phase liquids and chiral quantum phase liquids — are discussed.

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Introduction. – In condensed-matter physics, Landau’s symmetry breaking paradigm had been considered as the foundation to learn all types of orders. Different (quantum) states are characterized by different symmetries and the associated local order parameters. However, in the last 20 years, it has become more and more clear that Landau’s paradigm cannot describe all quantum states of the matter. The first example beyond Landau’s paradigm is the fractional quantum Hall (FQH) effect [1]. The FQH states possess exotic topological orders and cannot be described by symmetry breaking theory [2–4]. Another example is the quantum spin liquid, which is an intriguing possibility for a strongly interacting magnetic system where the magnetically ordered ground state is avoided owing to strong quantum fluctuations. The subtle structures that distinguish different quantum spin liquid (QSL) states are called quantum order [2,5,6]. Physicists have tried to find such exotic quantum states in frustrated spin systems for more than twenty years [7].

In general, such type of QSL can be described in the (repulsive) Hubbard model formalism of the strongly coupling limit. On the other hand, in terms of the canonical particle-hole transformation, a repulsive Hubbard model can be mapped on an attractive one. And the antiferromagnetic magnetic order in repulsive interaction corresponds to the superfluid/charge-density-wave (SF/CDW) order in the attractive one. Are there any novel quantum states corresponding to QSL? We provide some general discussions to the question in this paper.

In this letter we find that the fermionic superfluid (SF) may lose long-range phase coherence due to quite strong quantum fluctuations. The strongly fluctuating fermionic SF possesses exotic quantum orders and also cannot be described by Landau’s symmetry breaking theory. We call it quantum phase liquid (QPL). In the QPL, the fermions are paired and single quasi-particle’s excitation has a finite energy gap. However, the strong quantum fluctuations destroy the long-range phase coherence and the fermionic SF order parameter is still zero. Namely, the QPL state is a fermionic SF without phase coherence, and the excitations may be doublons or holons that correspond to the quantum states with the particle number 2 or 0 on each site, respectively. The single quasi-particle excitations corresponding to the quantum states with the particle number 1 on each site are forbidden. It is found that the physical properties of topological quantum phase liquids are quite different from those of topological quantum spin liquids — the external magnetic field induces a gas of topological excitations which will condense at zero temperature in a topological QPL.

The generalized attractive Hubbard model at large-$U$ limit. – Our starting point is the two-dimensional generalized attractive Hubbard model in a bipartite lattice (for example, the honeycomb lattice and the square lattice), of which the Hamiltonian is given by

$$
\hat{H} = - \sum_{i \in A, j \in B} \left( t_{ij} \hat{c}_{i,\uparrow}^{\dagger} \hat{c}_{j,\uparrow} + t_{ij}^{\ast} \hat{c}_{i,\downarrow}^{\dagger} \hat{c}_{j,\downarrow} + \text{H.c.} \right) \\
- \sum_{i,j \in A/B} \left( t_{ij}' \hat{c}_{i,\uparrow}^{\dagger} \hat{c}_{j,\uparrow} - t_{ij}'^{\ast} \hat{c}_{i,\downarrow}^{\dagger} \hat{c}_{j,\downarrow} + \text{H.c.} \right) - U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} \\
- \mu \sum_{i,\sigma} \hat{c}_{i,\sigma}^{\dagger} \hat{c}_{i,\sigma} - h \sum_i \left( \hat{c}_{i,\uparrow}^{\dagger} \hat{c}_{i,\uparrow} - \hat{c}_{i,\downarrow}^{\dagger} \hat{c}_{i,\downarrow} \right). \tag{1}
$$

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Here, \( i = (i_x, i_y) \) labels the lattice sites, \( A \) and \( B \) denote \( A \)-sublattices and \( B \)-sublattices, \( \sigma = \uparrow, \downarrow \) are spin indices, \( t_{ij} \) are hopping parameters between the sites on different sublattices, \( t'_{ij} \) are hopping parameters between the site on same sublattices that has different signs for different spins, \( U \) is the strength of the attractive interaction, \( \mu \) is the chemical potential, and \( h \) is the strength of the Zeeman field. In the following, we consider the case with \( \mu = -U/2 \), and set the lattice constant to be unity.

Let us discuss the global symmetry and the spontaneous symmetry breaking of the original Hamiltonian in eq. (1). The Hamiltonian in eq. (1) has an \( SU(2) \) particle-hole (pseudo-spin) symmetry group when \( \mu = -U/2 \), in which the \( SU(2) \) group elements act on the space of the SF/CDW order parameters. To make the \( SU(2) \) pseudo-spin symmetry more clear, we note that in terms of the canonical particle-hole transformation \[ \hat{c}_{i,\uparrow} \rightarrow \hat{c}_{i,\downarrow}, \quad \hat{c}_{i,\downarrow} \rightarrow (-1)^{i_x+i_y} \hat{c}_{i,\uparrow}^\dagger, \] the original model is mapped onto a repulsive Hubbard model with the effective chemical potential \( \tilde{\mu} = h + U/2 \) and effective Zeeman field \( \tilde{h} = \mu + U/2 \) as \( H \rightarrow \tilde{H} \), where

\[
\tilde{H} = -\sum_{i,j \in B} t_{ij} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} - \sum_{i,j \in A/B} t'_{ij} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} + \text{H.c.} - \tilde{h} \sum_{i,\alpha,\beta} \hat{c}_{i,\alpha}^\dagger \hat{c}_{i,\beta} - \tilde{\mu} \sum_{i,\sigma} \hat{n}_{i,\sigma} + U \sum_{i} \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow}.
\]

One can see that to guarantee the \( SU(2) \) pseudo-spin symmetry \((\tilde{h} = 0)\), the hopping term on the same sublattices must have different signs for different spins.

Now we have omitted the constant term \(-\tilde{(\mu - h)}N\) with \( N \) the total lattice sites number in the Hamiltonian \( \tilde{H} \). Then, for the case of \( \tilde{h} = \mu + U/2 = 0 \), i.e., \( \mu = -U/2 \), we can define an \( SU(2) \) pseudo-spin symmetry of the Hamiltonian in eq. (1), i.e., \( \tilde{H} \rightarrow \tilde{H}' = U \tilde{H} U^{-1} = \tilde{H} \) by doing a pseudo-spin rotation \( \Psi \rightarrow \Psi' \equiv U \Psi \), with \( \Psi = (\hat{c}_{i,\uparrow}^\dagger, \hat{c}_{i,\downarrow}^\dagger)^T \). The \( SU(2) \) pseudo-spin operators of the attractive Hubbard model thus become

\[
\begin{align*}
\hat{\eta}^- & \equiv (-1)^{i_x+i_y} \hat{\Delta}^- = (-1)^{i_x+i_y} \hat{c}_{i,\downarrow}^\dagger \hat{c}_{i,\uparrow}, \\
\hat{\eta}^+ & \equiv (-1)^{i_x+i_y} \hat{\Delta}^+ = (-1)^{i_x+i_y} \hat{c}_{i,\uparrow} \hat{c}_{i,\downarrow}, \\
\hat{\eta}^z & \equiv (\hat{\rho}_i - 1)/2,
\end{align*}
\]

where \( \hat{\rho}_i = \hat{\eta}^z \pm i\hat{\eta}^y \), \( \hat{\rho}_i \) is the particle density operator, and the \( SU(2) \) algebraic relation between the \( SU(2) \) pseudo-spin operators is \[ [\hat{\eta}^\alpha, \hat{\eta}^\beta] = i\epsilon_{\alpha\beta\gamma} \hat{\eta}^\gamma. \] The ground state with \( \langle \hat{\eta}^\alpha \rangle \neq 0 \) or \( \langle \hat{\eta}^- \rangle \neq 0 \) is an SF state and the ground state with \( \langle \hat{\eta}^z \rangle \neq 0 \) is a CDW state.

For the generalized attractive Hubbard model described by eq. (1), with increasing the interaction strength, the ground state turns into a paired state. In large-\( U \) limit, such paired state has a large energy gap about \( U \). For above generalized attractive Hubbard model with \( SU(2) \) particle-hole (pseudo-spin) symmetry, by integrating out the hopping terms, we derive an effective pseudo-spin model with the super-exchange terms

\[
H_s = \sum_{i \in A \cup B} J_{ij} \hat{\eta}_i \cdot \hat{\eta}_j + \sum_{i, j \in A \cup B} J'_{ij} \hat{\eta}_i \cdot \hat{\eta}_j, \tag{4}
\]

where the super-exchange coupling constants between different and same sublattices are \( J_{ij} = 4|t_{ij}|^2/U \) and \( J'_{ij} = 4|t'_{ij}|^2/U \), respectively. One can see that \( H_s \) is really a frustrated pseudo-spin model.

For a conventional SF order with spontaneous \( U(1) \) phase symmetry breaking, there exists one Goldstone mode with respect to the quantum phase fluctuation. In two dimensions, there exists a Kosterlitz-Thouless (KT) transition, below which the (quasi-) long-range phase coherence establishes. Now we have an SF/CDW order with spontaneously \( SU(2) \) pseudo-spin rotation symmetry breaking. Hence, the quantum fluctuations around the mean-field ground state are much stronger. In the CDW phase, one has a nonzero particle density modulation at different sublattices \( \langle \hat{\eta}^\alpha \rangle \neq 0 \). According to the commutation relation between the phase \( \hat{\phi}_i \) of \( \Delta^\alpha \) and the particle density operator \( \hat{\rho}_i \), i.e., \( [\hat{\phi}_i, \hat{\rho}_i] = 0 \), the nonzero particle density modulation leads to an uncertainty for the SF phase coherence and even destroys the long-range SF phase coherence. Thus, the nonzero value of \( \Delta_0 = |\Delta^\alpha| \) only means the existence of Cooper pairing. It does not necessarily imply that the ground state is a long-range SF order. As a result, one needs to examine the stability of the SF order against quantum fluctuations based on a formulation by keeping \( SU(2) \) pseudo-spin rotation symmetry.

For the frustrated pseudo-spin model \( H_s \), the SF may possess exotic quantum orders and also cannot be described by Landau’s symmetry breaking theory. We call it QPL, of which the fermions are paired and single quasi-particle excitation has a finite energy gap. However, we do not have long-range SF phase coherence and the SF order parameter is zero, i.e., \( \langle \hat{\eta}^z \rangle = 0 \). Thus in this region, the SF correlation decays exponentially \( \langle \Delta^*(x,y)\Delta(0) \rangle \rightarrow 0 \). The particle number must be 2 or 0 on each site that corresponds to doublon (Copper pair).
or holon (empty state), respectively. The QPL for the attractive Hubbard model in the strongly correlated limit corresponds to the QSL for the repulsive Hubbard model in the strongly correlated limit. See the illustration in fig. 1. In particular, for the system with $SU(2)$ pseudo-spin symmetry, the QPL is a direct duplicate of the idea of the QSL based on the connection between the attractive and the repulsive Hubbard models. However, for the system without $SU(2)$ pseudo-spin symmetry, the situation of QPL differs from that of QSL.

**Projective space construction of QPLs.** – In this section, we are going to use the projective symmetry groups (PSG) to construct quantum phase liquids by the $SU(2)$ slave-boson approach [2,6]. The gauge structure here is just that of the quantum spin liquid. The pseudo-spin operator $\hat{\eta}$ is now represented as $\hat{\eta}_i = \frac{1}{2} \bar{f}_i^\alpha \sigma_\alpha \bar{f}_j^\beta$, where $\bar{f}$ denotes the pseudo-spinor. In terms of the fermion operators, the Hamiltonian $H_s$ can be rewritten as

$$H_s = -\frac{1}{2} \sum_{i \in A,j} J_{ij} \bar{f}_i^\alpha \bar{f}_j^\beta + \frac{1}{2} \sum_{i,j \in A/B} J_{ij} \bar{f}_i^\alpha \bar{f}_j^\beta$$

with the constraints $\bar{f}_i^\alpha \bar{f}_i^\alpha = 1$ and $\bar{f}_i^\alpha \bar{f}_j^\beta \epsilon_{\alpha \beta} = 0$. Such constraints can be enforced by including the site-dependent and the time-independent Lagrangian multipliers: $\eta_0^\alpha(i) \bar{f}_i^\alpha \bar{f}_i^\alpha - 1$, $(\eta_0^\alpha + i \eta_0^\beta) \bar{f}_i^\alpha \bar{f}_j^\beta \epsilon_{\alpha \beta}$ in the Hamiltonian.

Replacing the operators $\bar{f}_i^\alpha \bar{f}_j^\beta$ and $\bar{f}_i^\alpha \bar{f}_j^\beta$ by their mean-field values $\Delta_{ij}$, $\chi_{ij} = 2(\delta_{\alpha \beta} \bar{f}_i^\alpha \bar{f}_j^\beta)$, we arrive at the mean-field Hamiltonian

$$H_{\text{mean}} = \frac{3}{8} \sum_{i \in A,j \in B} J_{ij} (\chi_{ij} \bar{f}_i^\alpha \bar{f}_j^\beta + \Delta_{ij} \bar{f}_i^\alpha \bar{f}_j^\alpha \epsilon_{\alpha \beta} + \text{H.c.})$$

$$- \frac{3}{8} \sum_{i,j \in A/B} J_{ij} (\chi_{ij} \bar{f}_i^\alpha \bar{f}_j^\beta + \Delta_{ij} \bar{f}_i^\alpha \bar{f}_j^\beta \epsilon_{\alpha \beta} + \text{H.c.})$$

$$+ \frac{3}{8} \sum_{i \in A,j \in B} J_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2)$$

$$+ \frac{3}{8} \sum_{i,j \in A/B} J_{ij} (|\chi_{ij}|^2 + |\Delta_{ij}|^2)$$

$$+ \sum_i \left( \eta_0^\alpha (\bar{f}_i^\alpha \bar{f}_i^\alpha - 1) + [\eta_0^\alpha + i \eta_0^\beta] \bar{f}_i^\alpha \bar{f}_j^\beta \epsilon_{\alpha \beta} + \text{H.c.} \right),$$

(5)

where $\chi_{ij}, \Delta_{ij}$ and $\eta_0^\alpha$ can be derived by the mean-field approach for a given QPL. Let $|\Psi_{\text{mean}}\rangle$ be the ground state of $H_{\text{mean}}$. Then a many-body state can be obtained from the mean-field state $|\Psi_{\text{mean}}\rangle$ by projecting $|\Psi\rangle = P|\Psi_{\text{mean}}\rangle$ into the subspace with a single occupation.

In addition, we introduce the SU(2) doublet $\psi = (f_1^\alpha, f_2^\alpha)$ and the matrix $U_{ij} = \left( \begin{array}{cc} \chi_{ij} & \Delta_{ij} \\ \Delta_{ij}^* & \chi_{ij}^* \end{array} \right) = U_{ji}^\dagger$. Using them, we can rewrite the mean-field Hamiltonian of the QPL and the constraints into a more compact formula as

$$H_{\text{mean}} = \frac{3}{8} \sum_{i \in A,j \in B} J_{ij} \left[ \frac{1}{2} \text{Tr}(U_{ij}^\dagger U_{ij}) - (\psi_i^\dagger U_{ij} \psi_j + \text{H.c.}) \right]$$

$$+ \frac{3}{8} \sum_{i,j \in A/B} J_{ij} \left[ \frac{1}{2} \text{Tr}(U_{ij}^\dagger U_{ij}) - (\psi_i^\dagger U_{ij} \psi_j + \text{H.c.}) \right]$$

$$+ \sum_i \delta_{ij} \psi_i^\dagger \tau^l \psi_i,$$

(6)

and $\psi_i^\dagger \tau^l \psi_i = 0$, where $\tau^l$ are the Pauli matrices with $l = 1, 2, 3$.

Similarly to the quantum spin liquids, the mean-field Hamiltonian of QPL is invariant under a local $SU(2)$ transformation $W(i)$ [2,6]: $\psi_i \rightarrow W(i) \psi_i, U_{ij} \rightarrow W(i) U_{ij} W^\dagger(j)$. In particular, two mean-field ansatzs that have different ($U_{ij}, a_0^\alpha$) and ($U'_{ij}, a_0'^\alpha$) may be the same QPL by an $SU(2)$ gauge transformation. To understand the gauge fluctuations around the above mean-field state of a QPL, we may observe the nontrivial $SU(2)$ flux through plaquettes. These fluxes may break $SU(2)$ gauge structure down to $U(1)$ or $Z_2$ gauge structures and play the role of the Higgs fields. Thus, we may characterize different quantum phase liquids by different PSGs and different gauge structures as one understands the quantum spin liquids.

**Topological QPLs with an external magnetic field.** – Now we consider the generalized attractive Hubbard model with an external magnetic field, of which the Hamiltonian becomes

$$\hat{H} = -\sum_{i \in A,j \in B} e^{i(\theta_i - \theta_j)} \left( t_{ij} \hat{c}_{i,j}^\dagger \hat{c}_{j,i} + t_{ij}' \hat{c}_{i,j}^\dagger \hat{c}_{j,i}' \right)$$

$$- \sum_{i,j \in A/B} e^{i(\theta_i - \theta_j)} \left( t_{ij}' \hat{c}_{i,j}^\dagger \hat{c}_{j,i} - t_{ij}' \hat{c}_{i,j}' \hat{c}_{j,i} \right) - \mu \sum_{i \in A} \hat{n}_{i\sigma}$$

$$- U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - h \sum_i \left( \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\downarrow} - \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\uparrow} \right) + \text{H.c.},$$

(7)

where the spatial variation of the phase $e^{i(\theta_i - \theta_j)}$ can be interpreted as an effective Aharanov-Bohm phase induced by the external magnetic field. With the choice of the symmetric gauge, we can define the flux ratio $\alpha = \Phi/\Phi_0$ with $\Phi$ the magnetic flux through a plaquette, and $\Phi_0 = h/(2e)$ the half of flux quantum. The sum of the phases along a closed loop surrounding the plaquette is $\alpha$, which is actually a $\pi$-flux per plaquette. The strength of the external magnetic field is $B = \pi \alpha / S_0$, where $S_0$ is the area of a plaquette. In this paper we only consider the limit of a small external field, $B \rightarrow 0$ ($\alpha \ll 1$).

In the large-$U$ limit, the generalized attractive Hubbard model with an external magnetic field turns into an effective pseudo-spin model with external magnetic field,
of which the Hamiltonian is reduced into

\[ H_s = J_{ij} \sum_{i \in A, j \in B} (e^{2(\theta_i - \theta_j)} \eta_i^+ \eta_j^- + e^{-2(\theta_i - \theta_j)} \eta_i^- \eta_j^+) 
+ \sum_{i \in A, j \in B} J_{ij}' \eta_i^\dagger \eta_j^+ + \sum_{i \in A} \left( e^{2(\theta_i - \theta_j)} \eta_i^+ \eta_j^- + e^{-2(\theta_i - \theta_j)} \eta_i^- \eta_j^+ \right) + \sum_{i, j \in A/B} J_{ij}' \eta_i^\dagger \eta_j^- . \] (8)

Note that in eq. (3) we have used the definition of the \( SU(2) \) pseudo-spin operators. Then, we find that the mean-field Hamiltonian \( H'_\text{mean} \) of slave-particles becomes

\[ H'_\text{mean} = \frac{3}{8} \sum_{i, j \in A/B} [J_{ij}(\chi_{ij} e^{2i(\theta_i - \theta_j)} f_i^\dagger f_j) + \Delta_{ij} e^{2i(\theta_i + \theta_j)} f_i^\dagger f_j] + \frac{3}{8} \sum_{i \in A, j \in B} J_{ij} \left( |\chi_{ij}|^2 + |\Delta_{ij}|^2 \right) + \frac{3}{8} \sum_{i, j \in A/B} J_{ij}' \left( |\chi_{ij}|^2 + |\Delta_{ij}|^2 \right) + \sum_{i, j, k} \left( a_0^i (f_j^\dagger f_i - 1) \right) \] (9)

From this effective Hamiltonian of the slave-particles, it can be seen that the slave-particles \( f_i \) see \( 2 \pi \) \( \pi \)-flux per plaquette. While, for the quantum spin liquid, the situation is much different: the external magnetic field never changes the effective spin model and the mean-field Hamiltonian of slave-particles.

Since quantum phase liquids with topological order are protected by the finite energy gaps of excitations, they are stable against arbitrary local perturbations [2-4]. As a result, for the topological phase liquids, we can assume that the small external magnetic field will not change their mean-field ansatz and the corresponding PSG of the topological order. Hence, for the topological phase liquids, since the mean-field Hamiltonian of slave-particles in eq. (6) changes the original PSG, it is not quite right. For this case, the external magnetic field will induce quantized vortices. In the topological phase liquids in a lattice, the induced vortex must be \( \pi \)-flux and the density of the induced quantized vortices \( \rho_v \) is \( BS_0/\pi \). See fig. 2. In the following subsections we will discuss two kinds of topological phase liquids with a finite density of quantized vortices.

**Example 1:** \( Z_2 \) topological QPLs with a small external magnetic field. In this section we will discuss \( Z_2 \) topological QPLs with a small external magnetic field. \( Z_2 \) topological QPLs have the simplest topological order — \( Z_2 \) topological order [9,10], of which the ground state has topological degeneracy [10,11]. The low-energy effective theory of \( Z_2 \) topological QPL is a \( Z_2 \) gauge theory. The mutual \( \pi \) statistics between quasi-particles, the pseudo-spinons and the bosonic \( Z_2 \)-vortices (quantized vortices with \( \pi \)-flux on a plaquette) truly reflect the properties of a \( Z_2 \) gauge theory.

We introduce the \( SU(2) \) flux operator, \( P(C_i) = U_{i,j}^0 U_{j,k}^1 \cdots U_{n,l}^0 \), to characterize the local gauge symmetry. The flux operator corresponds to gauge field strength in the continuum limit, and under gauge transformations, \( P(C_i) \) transform as \( P(C_i) \) and \( W_i P(C_i) W_i^\dagger \). In general, the flux operator has the form \( P(C_i) = \chi^0(C_i) + \epsilon^1(C_i) \pi^1 \). Moreover, if we consider the flux operator has different orientation (noncollinear), the noncollinear \( SU(2) \) flux operator breaks the \( SU(2) \) gauge symmetry down to \( Z_2 \). As a result, the ground state becomes a stable \( Z_2 \) topological phase liquid, of which the gauge fluctuations have finite energy gap. An example of the \( Z_2 \) topological QPLs is given by the following ansatzs:

\[ U_{i,i+\epsilon_x} = U_{i,i+\epsilon_y} = -e^{-\chi \tau^3}, \quad U_{i,i+\epsilon_x+\epsilon_y} = \eta \tau^1 + \lambda \tau^2, \]

\[ U_{i,i-\epsilon_x+\epsilon_y} = \eta \tau^1 - \lambda \tau^2, \quad \eta^2 = 0, a_0^1 \neq 0, \]

where \( \chi, \eta, \lambda \) are all real parameters. The fermionic spinons are fully gapped and the \( SU(2) \) gauge structure is obviously broken down to \( Z_2 \) gauge structure. Thus, the gauge fluctuations have energy gap due to Higgs mechanism. As a result, the \( Z_2 \) topological QPLs are stable against arbitrary perturbation.

For the \( Z_2 \) topological QPLs with a small external magnetic field, there exist the induced \( Z_2 \)-vortices with the density \( B/\pi \) (here and in the following, we assume a plaquette area \( S_0 \equiv 1 \)). For the dilute gas of the bosonic \( Z_2 \)-vortices, the ground state becomes a Bose-Einstein condensation (BEC) state. In the BEC state of the \( Z_2 \)-vortices, the pseudo-spinons are confined at zero temperature. In particular, the BEC state of the \( Z_2 \)-vortices is a super-solid state. Let us calculate the “charge” conductance \( \sigma_c \) of the system. The origin of the dissipation in
this BEC state is the flow of the $Z_2$-vortices as response
to an external “electric” field. Using the relation
between vortex conductance $\sigma_v$ and “charge” conductance $\sigma_c$ [12-14]; $\sigma_v\sigma_c = 1/\pi^2$ (see detailed proof in ref. [15]),
we can find that the diverge vortex conductance $\sigma_v = \infty$
leads to a zero “charge” conductance $\sigma_c = 0$. That means
$Z_2$ topological QPLs with a small external magnetic field
is really an insulating super-solid state. In addition, there
exists an anomalous Nernst effect. The Nernst effect refers
to a transverse “electric” field induced by applying a tem-
perature gradient on the system [16]. The mechanism
leading to a significant Nernst signal is the $Z_2$-vortex flow
in the topological QPL phase. In the BEC state of the
$Z_2$-vortices, the Nernst signal will also diverge.

Example 2: Topological chiral QPLs with a small external magnetic field. 
Another topological QPL is the topological chiral QPL breaking time reversal symmetry,
of which the elementary excitations are anyons with fractional
statistics [17].

We take the following ansatz as an example to learn the proper-
ties of topological chiral QPLs with a small external magnetic
field, of which the ansatz is given by
\begin{align}
\chi_{i,i+\hat{e}_x} &= -x^3 - x^3, \chi_{i,i+\hat{e}_y} = -x^3 + x^3, \quad (11) \\
\chi_{i,i+\hat{e}_y} &= \eta^2, \chi_{i,i-\hat{e}_y} = -\eta^2, \quad a_0 = 0,
\end{align}
where $\chi, \eta, \lambda$ are all real parameters. The $SU(2)$ gauge
symmetry is unbroken. The low-energy effective theory is
an $SU(2)$ Chern-Simons theory (of level 1). The pseudo-
anyons are gapped and have a semionic statistics.

Firstly, to calculate the topological invariants in mo-
mentum space, the above formulation in eq. (11) is re-
duced into $H_f = \sum_k \psi_k^\dagger[\hat{u}(k) \cdot \tau]\psi_k + H.c.$, where $\psi_k = (\hat{f}_k, \hat{\bar{f}}_k)^T$ and $\hat{u}(k) = u/|u|$ is the unit vector in the
momentum space. By introducing the Chern number
$C = \int d\mathbf{k} \, d\mathbf{k}_0 \frac{1}{2} \epsilon_{\mathbf{k},\mathbf{k}_0} \epsilon_{\mathbf{k},\mathbf{k}_0}$, we may have a
topological chiral QPL with the nonzero Chern number
$C = 1$. Consequently, the topological chiral QPL has a
nonzero chiral order parameter which is defined by [17]
$\chi_{1,2} = \langle \eta_1 \cdot (\eta_2 \times \eta_0) \rangle \neq 0$.
Secondly, we calculate the induced quantum number on the
$Z_2$-vortices. Let us consider a single $Z_2$-vortex excita-
tion in the topological chiral QPLs. The vacuum expec-
tation value of the fermion number $\langle N_v \rangle$ is related
to the spectral asymmetry of the Dirac Hamiltonian $\langle N_v \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} dE \frac{1}{2} \text{Im} \text{Tr} \left( \frac{1}{2} \mathbf{D} - \mathbf{D}^\dagger \right) \sigma_y(E) \right|$ [18]. The Atiyah-
Patodi-Singer index theorem states that the induced quan-
tum number on single $Z_2$-vortex is
$\langle N_v \rangle = \frac{m}{2\pi}$, where $m$ is the mass gap of fermionic spinons. This means that a
$Z_2$-vortex is really a bound state of $\pi$-flux and a fermionic
spinon. Due to nontrivial $AB$ phases upon adiabatic ex-
change of charge and flux, the $\pi$-flux turns into semion,
of which the statistical angle $\theta = \frac{\pi}{2}$. As a result, the
low-energy effective theory for the topological chiral phase
liquid is the Chern-Simons (CS) theory [19,20]
\begin{equation}
\mathcal{L}_{CS} = \frac{iCN}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \quad (12)
\end{equation}
where $N = 2$ is the flavor number of the fermionic pseudo-
anyons. Here $a_\mu$ is the the auxiliary $U(1)$ gauge fields.
Due to the CS term, the gauge fluctuations of topological
chiral QPLs have energy gap. As a result, the topological
chiral QPLs are stable against arbitrary perturbation.

Thirdly, we discuss the topological chiral QPLs with a small external magnetic field. For the topological chiral
QPL with a small external magnetic field, there also exist the induced vortices with the density $B/\pi$. To calcu-
late the many-body system of anyons, we will use a dual
description where we introduce a $U(1)$ gauge field $b_\mu$ to
describe the density $j^0$ and current $j^i$ of the bosonic field
$\varphi$ denoting anyons: $j^i = \frac{1}{\pi} r^\mu b_\mu a_\lambda$. Now the effective continuum Lagrangian of the topological CPL with a small external magnetic field turns into
\begin{equation}
\mathcal{L}_{eff} = \mathcal{L}_v + \mathcal{L}_f + \mathcal{L}_{CS} + \frac{i}{2\pi} \epsilon^{\mu\nu\lambda} b_\mu \partial_\nu a_\lambda, \quad (13)
\end{equation}
where $\mathcal{L}_f$ is the effective Lagrangian of the pseudo-
spinon coupling to $a_\lambda$, and $\mathcal{L}_v$ is the Lagrangian of anyons $\mathcal{L}_v = \varphi^* D_\mu \varphi + \frac{\mu^2}{2m^2} - \mu_\varphi \varphi$. Here, the bosonic field $\varphi$ denotes anyons that couple to both $a_\nu$ and $b_\nu$. $D_\mu = \partial_\mu - i a_\mu/2 - ib_\mu$ is the covariant derivative. $\mu_\varphi$ is the mass of anyons.

The ground state of the dilute anyon gas is really an anyon superfluids. To obtain the universal features from
the anyon superfluids, we define $a_\mu = a_{+\mu} - a_{-\mu}, b_\mu = (a_{+\mu} + a_{-\mu})/2$, and rewrite the effective Lagrangian as
\begin{equation}
\mathcal{L}_{eff} = \mathcal{L}_f + \varphi^* D_\mu \varphi + \frac{|D\varphi|^2}{2m^2} - \mu_\varphi \varphi + i \frac{C}{4\pi} a_{+\mu} \partial_\mu a_{-\lambda} \epsilon^{\mu\nu\lambda} - i \frac{C}{4\pi} a_{+\mu} \partial_\nu a_{-\lambda} \epsilon^{\mu\nu\lambda}, \quad (14)
\end{equation}
At zero temperature, with $\langle \varphi \rangle = \sqrt{\frac{\mu_\varphi}{\mu^2}} \neq 0$, we get
\begin{equation}
\mathcal{L}_v = i \rho_\varphi (\partial_\mu \varphi - a_{+\mu} - a_{-\mu}) + \frac{\mu_\varphi}{2m^2} (\nabla \phi - a_{+\mu})^2. \quad (15)
\end{equation}
After integrating $a_{+\mu}$, we get
\begin{equation}
\mathcal{L}_v = \frac{i(N - 1)}{4\pi} a_{+\mu} \partial_\nu a_{-\lambda} \epsilon^{\mu\nu\lambda} + \frac{\mu_\varphi}{2m^2} (\nabla \phi)^2 + \frac{m_\varphi CN}{4\pi B^2} |E_r|^2 - \frac{iCN}{2\pi} \epsilon^{0\mu\nu\lambda} a_0 \partial_\nu a_\lambda \varphi - \mu_\varphi \rho_\varphi, \quad (15)
\end{equation}
where $E_r = \partial_\mu a_\mu - \nabla a_{-\mu}$ and $B^2 = 2\pi \rho_\varphi/CN$. In the process of integration of gauge field $a_\mu$, we only consider the term quadratic term $\frac{m_\varphi CN}{4\pi B^2} a_\mu^2$ for $a_\mu$, and neglect the term $\frac{iCN}{4\pi B^2} a_\mu (E_r \times a_\mu)$ due to the fact that this term corresponds to a low-energy part. Due to $CN \neq 1$, the gauge field $a_{-\mu}$ has an energy gap, which implies that the pseudo-spinons are deconfined.

Conclusion. – We have studied the two-dimensional
generalized attractive Hubbard model on a bipartite lat-
tice, and by particle-hole transformation, we find that
the model has $SU(2)$ pseudo-spin rotation symmetry when the
chemical potential $\mu = -U/2$. With a further analysis,
we observe that the strong phase fluctuations destroy the
long-range SF phase coherence, the strong phase fluctuations phase corresponds to the “quantum phase liquid”. Analogously to spin liquid, we make a projective construction of quantum phase liquids in terms of SU(2) slave-boson approach. Then, $Z_2$ QPLs and CPLs with a small external magnetic field are discussed. For $Z_2$ QPLs, the ground state for dilute $Z_2$-vortices gas turns into a BEC state with confined fermionic pseudo-spinons. In table 1, we show the comparison between the quantum spin liquid and quantum phase liquid.

Finally, we address the possible physical realizations of the topological QPLs. For the ansatz in eq. (10), a possible realization is the effective frustrated pseudo-spin model (for example, $J - J'$ model). It is proposed that the attractive Hubbard model on square optical lattice with nearest-neighbor and next-nearest-neighbor hoppings can be realized in the cold atoms. When two-component fermions with attractive interaction are put into such optical lattice, one can get an effective frustrated attractive Hubbard model. It is easy to change the potential barrier by varying the laser intensities to tune the Hamiltonian parameters including the hopping strength and the particle interaction ($U$-term). In the large $U$ limit, we may get the effective frustrated pseudo-spin model, of which the ground state may be $Z_2$ QPL. For the ansatz in eq. (11), a possible realization is from a topological Hubbard model on honeycomb lattice. In ref. [21], it is proposed that the Haldane model on honeycomb optical lattice can be realized in the cold atoms. When two-component fermions with attractive interaction are put into such optical lattice, one can get an effective topological Hubbard model, of which the ground state may be CPLs.

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