HILBERT SERIES OF ALGEBRAS ASSOCIATED TO DIRECTED GRAPHS

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Abstract. We compute the Hilbert series of some algebras associated to directed graphs and related to factorizations of noncommutative polynomials.

1. Introduction

In [3] we introduced a new class of algebras $A(\Gamma)$ associated to layered directed graphs $\Gamma$. These algebras arose as generalizations of the algebras $Q_n$ (which are related to factorizations of noncommutative polynomials, see [2, 5, 9]), but the new class of algebras seems to be interesting by itself.

Various results have been proven for algebras $A(\Gamma)$. In [3] we constructed a linear basis in $A(\Gamma)$. In [7] we showed that algebras $A(\Gamma)$ are defined by quadratic relations for a large class of directed graphs and proved that in this case they are Koszul algebras. It follows immediately that the dual algebras to $A(\Gamma)$ are also Koszul and that their Hilbert series are related.

In this paper we continue to study algebras $A(\Gamma)$. In Section 2 we recall the definition of the algebra $A(\Gamma)$ and the construction of a basis for $A(\Gamma)$ given in [3]. In Section 3 we prove the main result of the paper, an expression for the Hilbert series, $H(A(\Gamma), t)$ of the algebra $A(\Gamma)$ corresponding to a layered graph $\Gamma$ with a unique element $\ast$ of level 0. In stating this we denote the level of $v$ by $|v|$ and write $v > w$ to indicate that $v$ and $w$ are vertices of the directed graph $\Gamma$ and that there is a directed path from $v$ to $w$. Then we have:

$$H(A(\Gamma), t) = \frac{1 - t}{1 + \sum_{v_1, v_2, \ldots, v_\ell \geq \ast} (-1)^{\ell} t^{\ell |v_1| - |v_\ell| + 1}}.$$

The proof uses matrices $\zeta(t)$ and $\zeta(t)^{-1}$ which generalize the zeta function and the Möbius function for partially ordered sets.

In Section 4 we specialize our results to the case of the Hasse graph of the lattice of subsets of a finite set, giving a derivation of the Hilbert series for the algebras $Q_n$ that is shorter and more conceptual than that in [2]. In Section 5 we treat the case of the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field. Finally, in Section 6, we define the complete layered graph $C[m_n, m_{n-1}, \ldots, m_1, 1]$ and compute the Hilbert series of $A(C[m_n, m_{n-1}, \ldots, m_1, 1])$.

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2. THE ALGEBRA $A(\Gamma)$

We begin by recalling the definition of the algebra $A(\Gamma)$. Let $\Gamma = (V, E)$ be a directed graph. That is, $V$ is a set (of vertices), $E$ is a set (of edges), and $t : E \to V$ and $h : E \to V$ are functions. ($t(e)$ is the tail of $e$ and $h(e)$ is the head of $e$.)

We say that $\Gamma$ is layered if $V = \cup_{i=0}^n V_i$, $E = \cup_{i=1}^n E_i$, $t : E_i \to V_i$, $h : E_i \to V_{i-1}$. If $v \in V_i$, we write $|v| = i$.

We will assume throughout the remainder of the paper that $\Gamma = (V, E)$ is a layered graph with $V = \cup_{i=0}^n V_i$, that $V_0 = \{\star\}$, and that, for every $v \in V_+ = \cup_{i=1}^n V_i$, \{e \in E \mid t(e) = v\} \neq \emptyset. For each $v \in V_+$, fix, arbitrarily, some $e_v \in E$ with $t(e_v) = v$.

If $v, w \in V$, a path from $v$ to $w$ is a sequence of edges $\pi = \{e_1, e_2, \ldots, e_m\}$ with $t(e_1) = v$, $h(e_1) = w$ and $t(e_{i+1}) = h(e_i)$ for $1 \leq i < m$. We write $v = \tau(e)$, $w = h(\tau)$. We also write $v > w$ if there is a path from $v$ to $w$. Define $P_\pi(\tau) = (\tau - e_1)(\tau - e_2)\ldots(\tau - e_m) \in T(E)[[\tau]]$ and write

$$P_\pi(\tau) = \sum_{j=0}^n e(\pi, j)\tau^{m-j}.$$ 

Let $\pi_v$ denote the path $\{e_1, \ldots, e_{|v|}\}$ from $v$ to $\star$ with $e_1 = e_v$, $e_{i+1} = e_{h(e_i)}$ for $1 \leq i < |v|$, and $h(e_{|v|}) = \star$.

Recall that $R$ is the ideal of $T(E)$ generated by

$\{e(\pi_1, k) - e(\pi_2, k) \mid t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2), 1 \leq k \leq |l(\pi_1)|\}$.

The algebra $A(\Gamma)$ is the quotient $T(E)/R$.

For $v \in V_+$ and $1 \leq k \leq |v|$ we define $\hat{e}(v, k)$ to be the image in $A(\Gamma)$ of the product $e_1 \ldots e_k$ in $T(E)$ where $\pi_v = \{e_1, \ldots, e_{|v|}\}$.

If $(v, k), (u, l) \in V \times N$ we say $(v, k)$ covers $(u, l)$ if $v > u$ and $k = |v| - |u|$. In this case we write $(v, k) \triangleright (u, l)$. (In [3] we used different terminology and notation: if $(v, l) \triangleright (u, l)$ we said $(v, l)$ can be composed with $(u, l)$ and wrote $(v, l) \models (u, l)$.)

The following theorem is proved in [3] Corollary 4.5.

**Theorem 1.** Let $\Gamma = (V, E)$ be a layered graph, $V = \cup_{i=0}^n V_i$, and $V_0 = \{\star\}$ where $\star$ is the unique minimal vertex of $\Gamma$. Then

$$\{\hat{e}(v_1, k_1) \ldots \hat{e}(v_t, k_t) \mid 0 \geq v_1, \ldots, v_t \in V_+, 1 \leq k_i \leq |v_i|, (v_i, k_i) \triangleright (v_{i+1}, k_{i+1})\}$$

is a basis for $A(\Gamma)$.

3. THE HILBERT SERIES OF $A(\Gamma)$

Let $h(t)$ denote the Hilbert series $H(A(\Gamma), t)$, where $\Gamma$ is a layered graph with unique minimal element $\star$ of level 0. If $X \subseteq A(\Gamma)$ is a set of homogeneous elements (so $X = \cup_{i=0}^\infty X_i$ where $X_i = X \cap A(\Gamma)_i$), denote the ”graded cardinality” $\sum_{i=0}^\infty |X_i|t^i$ of $X$ by $||X||$. Let $B$ denote the basis for $A(\Gamma)$ described in Theorem 1 and, for $v \in V_+$, let $B_v = \{\hat{e}(v_1, k_1) \ldots \hat{e}(v_t, k_t) \in B \mid |v_1| = v\}$. Then $B = \{1\} \cup \bigcup_{v \in V_+} B_v$. Let $h_v(t)$ denote the graded dimension of the subspace of $A(\Gamma)$ spanned by $B_v$. Since $B$ is linearly independent, we have $||B|| = h(t)$ and $||B_v|| = h_v(t)$. Then

$$||B|| = h(t) = 1 + \sum_{v \in V_+} h_v(t).$$
Let $C_v = \bigcup_{k=1}^{[v]} \delta(v, k)B$. Then

$$\|C_v\| = (t + \cdots + t^{[v]})h(t) = t \frac{t^{[v]} - 1}{t - 1} h(t).$$

Now $C_v \supseteq B_v$. Let $D_v$ denote the compliment of $B_v$ in $C_v$. Then

$$D_v = \{ \delta(v, k)\delta(v_1, k_1)\cdots \delta(v_k, k_{e_{kl}})|1 \leq k \leq [v],$$

$$(v, k) \succ (v_1, k_1), \delta(v_1, k_1)\cdots \delta(v_k, k_{e_{kl}}) \in B \}$$

and so

$$D_v = \bigcup_{v > v_1 > *} \delta(v, |v| - |v_1|)B_{v_1}.$$ 

Then $\|D_v\| = \sum_{v > v_1 > *} t^{[v] - |v_1|}h_{v_1}(t)$ and so

$$h_v(t) = ||B_v|| = ||C_v|| - ||D_v|| = t \frac{t^{[v]} - 1}{t - 1} h(t) - \sum_{v > w > *} t^{[v] - |w|}h_w(t).$$

This equation may be written in matrix form. Arrange the elements of $V$ in nonincreasing order and index the elements of vectors and matrices by this ordered set. Let $h(t)$ denote the column vector with entry $h_v(t)$ in the $v$-position (where we set $h_v(t) = 1$), let $u$ denote the vector with $t^{[v]}$ in the $v$-position, $e_v$ denote the vector with $\delta_{vv}$ in the $v$-position, let $1$ denote the column vector all of whose entries are $1$, and let $\zeta(t)$ denote the matrix with entries $\zeta_{v,w}(t)$ for $v, w \in V$ where $\zeta_{v,w}(t) = t^{[v] - |w|}$ if $v \geq w$ and $0$ otherwise. Note that

$$\zeta(t)e_v = u.$$

Then we have

$$\zeta(t)(h(t) - e_v) = \frac{t}{t - 1}(u - 1)h(t)$$

and so

$$h(t) - e_v = \frac{t}{t - 1}(u - \zeta(t)^{-1}1)h(t).$$

Then

$$1^T(h(t) - e_v) = \frac{t}{t - 1}(1^Tu - 1^T\zeta(t)^{-1}1)h(t)$$

or

$$h(t) - 1 = \frac{t}{t - 1}(1 - 1^T\zeta(t)^{-1}1)h(t).$$

Consequently, we have

**Lemma 1.**

$$\frac{1 - t}{h(t)} = 1 - t1^T\zeta(t)^{-1}1.$$ 

Now $N(t) = \zeta(t) - I$ is a strictly upper triangular matrix and so $\zeta(t)$ is invertible. In fact, $\zeta(t)^{-1} = I - N(t) + N(t)^2 - \cdots$ and so the $(v, w)$-entry of $\zeta(t)^{-1}$ is

$$\sum_{v = v_1 \succ \cdots \succ v_l = w \succ *} (-1)^{l+1}t^{[v] - |w|}.$$ 

Combining this remark with Lemma 1 we obtain the following result.
Theorem 2. Let $\Gamma$ be a layered graph with unique minimal element $*$ of level 0 and $h(t)$ denote the Hilbert series of $A(\Gamma)$. Then

$$\frac{1-t}{h(t)} = 1 + \sum_{v_1 \supset v_2 \supset \cdots \supset v_{\ell} \supset *} (-1)^{\ell} t^{|v_1|-|v_{\ell}|+1}.$$ 

We remark that the matrices $\zeta(1)$ and $\zeta(1)^{-1}$ are well-known as the zeta-matrix and the Möbius-matrix of $V$ (cf. [8]).

In the remaining sections of this paper we will use Theorem 2 to compute the Hilbert series of the algebras $A(\Gamma)$ associated with certain layered graphs.

4. The Hilbert series of the algebra associated with the Hasse graph of the lattice of subsets of $\{1, \ldots, n\}$

Let $\Gamma_n$ denote the Hasse graph of the lattice of all subsets of $\{1, \ldots, n\}$. Thus the vertices of $\Gamma_n$ are subsets of $\{1, \ldots, n\}$, the order relation $>$ is set inclusion $\supset$, the level $|v|$ of a set $v$ is its cardinality, and the unique minimal vertex $*$ is the empty set $\emptyset$. Then the algebra $A(\Gamma_n)$ is the algebra $Q_n$ defined in [5]. In this section we will prove the following theorem (from [2]). The present proof is much shorter and more conceptual than that in [2].

Theorem 3.

$$H(Q_n, t) = \frac{1-t}{1-t(2-t)^n}.$$ 

Our computations depend on the following lemma and corollary.

Lemma 2. Let $w$ be a finite set. Then

$$\sum_{w \supset w_2 \supset \cdots \supset w_{\ell} \supset \emptyset} (-1)^{\ell} = (-1)^{|w|+1}.$$ 

Proof. If $|w| = 1$, both sides are $+1$. Assume the result holds for all sets of cardinality $< |w|$. Then

$$\sum_{w \supset w_2 \supset \cdots \supset w_{\ell} \supset \emptyset} (-1)^{\ell} = \sum_{w \supset w_2 \supset \emptyset} \sum_{w_2 \supset \cdots \supset w_{\ell} \supset \emptyset} (-1)^{\ell}$$

and, by the induction assumption, this is equal to

$$\sum_{w \supset w_2 \supset \emptyset} (-1)^{|w_2|}.$$ 

Since

$$\sum_{w \supset w_2 \supset \emptyset} (-1)^{|w_2|} = \sum_{w \supset w_2 \supset \emptyset} (-1)^{|w_2|} - (-1)^{|w|} = 0 + (-1)^{|w|+1} = (-1)^{|w|+1}$$

the proof is complete. 

Corollary 1. Let $v \supset w$ be finite sets. Then

$$\sum_{v=v_1 \supset v_2 \supset \cdots \supset v_{\ell} = w} (-1)^{\ell} = (-1)^{|v|-|w|+1}.$$
Proof. Let \( w' \) denote the complement of \( w \) in \( v \). Sets \( u \) satisfying \( v \subseteq u \subseteq w \) are in one-to-one correspondence with subsets of \( w' \) via the map \( u \mapsto u \cap w' \). Thus
\[
(-1)^{\ell} = \sum_{v'_1 \supset v'_2 \supset \cdots \supset v'_{k-1}} (-1)^{\ell},
\]
By the lemma, this is \((-1)^{|w'|+1}\), giving the result.

To prove the theorem we observe that
\[
\sum_{v_1 \supset v_2 \supset \cdots \supset v_k \supset \emptyset} (-1)^{|v_1|} (-1)^{|v_k|} = \sum_{\{1, \ldots, n\} \supset v_1 \supset \emptyset} t^{|v_1|} \sum_{v_2 \supset \cdots \supset v_k \supset \emptyset} (-1)^{\ell},
\]
By Corollary 1, this is
\[
\sum_{\{1, \ldots, n\} \supset v_1 \supset \emptyset} t^{|v_1|} (-1)^{|v_1|} = \sum_{\{1, \ldots, n\} \supset v_1 \supset \emptyset} t^{|v_1|} (-1)^{|v_1| + 1}.
\]
Let \( u \) denote the complement of \( v_k \) in \( v_1 \) and \( u' \) denote the complement of \( u \) in \( \{1, \ldots, n\} \). Then the coefficient of \( t^{k+1} \) in the above expression is the number of ways of choosing a \( k \)-element subset \( u \subseteq \{1, \ldots, n\} \) times the number of ways of choosing a subset \( v \subseteq u' \). This is \( \binom{n}{k} 2^{n-k} \). Thus
\[
\sum_{v_1 \supset v_2 \supset \cdots \supset v_k \supset \emptyset} (-1)^{\ell} t^{|v_1|} (-1)^{|v_1| + 1} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^{k+1} = -t(2 - t)^k.
\]
In view of Theorem 2, this completes the proof of the theorem.

5. The Hilbert Series of Algebras Associated with the Hasse Graph of the Lattice of Subspaces of a Finite-Dimensional Vector Space over a Finite Field

We will denote by \( L(n, q) \) the Hasse graph of the lattice of subspaces of an \( n \)-dimensional space over the field \( F_q \) of \( q \) elements. Thus the vertices of \( L(n, q) \) are subspaces of \( F_q^n \), the order relation \( \supset \) is inclusion of subspaces \( \supset \), the level \( |U| \) of a subspace \( U \) is its dimension, and the unique minimal vertex \( * \) is the zero subspace \( (0) \).

Theorem 4.

\[
\frac{1 - t}{H(A(L(n, q)), t)} = 1 - t \sum_{m=0}^{n} \binom{n}{m} q^{-m} (1 - t)(1 - tq) \cdots (1 - tq^{n-m-1}).
\]

Our proof depends on the following lemma and corollary.

Lemma 3. Let \( U \) be a finite-dimensional vector space over \( F_q \). Then
\[
\sum_{U = U_1 \supset U_2 \supset \cdots \supset U_k = (0)} (-1)^{\ell} = (-1)^{|U|+1} q^{\binom{|U|}{2}}.
\]

Proof. If \( |U| = 0 \), the sum occurring in the lemma has a single term corresponding to \( l = 1, U = U_1 = (0) \). Then both sides of the expression in the lemma are equal to \(-1\). Now let \( U \) be a finite-dimensional vector space and assume the result holds for all spaces of dimension less than \( |U| \). Then
By the induction assumption, this is equal to

$$\sum_{U=U_1 \supset U_2 \supset \cdots \supset U_\ell = (0)} (-1)\ell q^{U_\ell / 2}.$$  

It is well-known that the number of $m$-dimensional subspaces of the space $U$ is given by the $q$-binomial coefficient $\binom{|U|}{m}_q$.

Hence

$$\sum_{U=U_1 \supset U_2 \supset \cdots \supset U_\ell = (0)} (-1)\ell q^{U_\ell / 2} = \left(\frac{|U|}{m}\right)_{q, m} (-1)^{|U_2|} q^{U_2 / 2}.$$  

Recall the $q$-binomial theorem

$$\prod_{i=0}^{m-1} (1 + xq^i) = \sum_{j=0}^m \left(\begin{array}{c} m \\ j \end{array}\right)_q x^j q^{j^2/j}.$$  

Set $x = -1$. Then the $i = 0$ factor in the product is 0 and so we have

$$\sum_{j=0}^m \left(\begin{array}{c} m \\ j \end{array}\right)_q (-1)^j q^{j^2/j} = (-1)^m + 1 q^{m^2/m}.$$

Thus

$$\sum_{U=U_1 \supset U_2 \supset \cdots \supset U_\ell = (0)} (-1)\ell = (-1)^{|U|+1} q^{U / 2}$$

as required. \qed

**Corollary 2.** Let $V \supseteq W$ be subspaces of $F_q$. Then

$$\sum_{V=V_1 \supset V_2 \supset \cdots \supset V_\ell = W} (-1)^\ell q^{V_\ell / 2} = q^{|V/W| + 1}.$$  

**Proof.** Since subspaces $Y, V \supseteq Y \supseteq W$, are in one-to-one correspondence with subspaces of $V/W$ via the map $Y \mapsto Y/W$, this is immediate from the lemma. \qed

To prove the theorem, we observe that

$$\sum_{V_1 \supset V_2 \supset \cdots \supset V_\ell = (0)} (-1)^\ell t |V_1/V_\ell| + 1 = \sum_{V_1 \supset V_2 \supset \cdots \supset V_\ell = (0)} t |V_1/V_\ell| + 1 \sum_{V_1 \supset V_2 \supset \cdots \supset V_\ell = (0)} (-1)^\ell.$$  

By Corollary 2, this is equal to

$$\sum_{V_1 \supset V_2 \supset \cdots \supset V_\ell = (0)} t |V_1/V_\ell| + 1 (-1)^{V_1/V_\ell} q^{V_1/V_\ell}.$$  

Set $|v_\ell| = m$ and $|V_1/V_\ell| = k$. Then the number of possible $V_\ell$ is $\binom{n}{m}$, and, for fixed $V_\ell$, the number of possible $V_1$ is the number of $k$-dimensional subspaces of $F_q^n / V_\ell$ which is $\binom{n-m}{k}$. Thus
In view of Theorem 2, the theorem is proved.

Therefore

\[
\sum_{V_1 \cup V_2 \cup \ldots \cup V_{n} \geq V_{0}} (-1)^{t|V/V_i|+1} = \sum_{0 \leq k, m \leq n} \binom{n}{m} \binom{n-m}{k} (-t)^{k+1} q^{\ell(2)}.
\]

Setting \( x = -t \) in the \( q \)-binomial theorem shows that

\[
\sum_{k=0}^{n-m} \binom{n-m}{k} (-t)^k q^{\ell(2)} = \prod_{i=0}^{n-m-1} (1 - tq^i).
\]

Therefore

\[
\sum_{V_1 \cup V_2 \cup \ldots \cup V_{n} \geq V_{0}} (-1)^{t|V/V_i|+1} = (-t) \sum_{m=0}^{n-m} \binom{n}{m} \prod_{i=0}^{n-m-1} (1 - tq^i).
\]

In view of Theorem 2, the theorem is proved.

Note that setting \( q = 1 \) in the expression in Theorem 4 gives \( 1 - t(2 - t)^n \). By Theorem 3, this is \( H(A, t) = \frac{1}{1+t} \).

Recall (cf. [10]) that if \( A \) is a quadratic algebra it has a dual quadratic algebra, denoted \( A' \) and that if \( A \) is a Koszul algebra the Hilbert series of \( A \) and \( A' \) are related by

\[
H(A, t)H(A', -t) = 1
\]

Since by \([7]\) \( A(\mathbf{L}(n, q)) \) is a Koszul algebra, we have the following

**Corollary 3.**

\[
H(A(\mathbf{L}(n, q))^t) = 1 + \sum_{m=0}^{n-1} \binom{n}{m} (1 + tq) \ldots (1 + tq^{n-m-1}).
\]

6. **The Hilbert series of algebras associated with complete layered graphs**

We say that a layered graph \( \Gamma = (V, E) \) with \( V = \bigcup_{i=0}^{n} V_i \) is **complete** if for every \( i, 1 \leq i \leq n \), and every \( v \in V_i, w \in V_{i-1} \), there is a unique edge \( e \) with \( t(e) = v, h(e) = w \). A complete layered graph is determined (up to isomorphism) by the cardinalities of the \( V_i \). We denote the complete layered graph with \( V = \bigcup_{i=0}^{n} V_i, |V_i| = m_i \) for \( 0 \leq i \leq n \) by \( C[m_n, m_{n-1}, \ldots, m_1, m_0] \). Note that the graph \( C[m_n, m_{n-1}, \ldots, m_1, 1] \) has a unique minimal vertex of level 0 and so Theorem 2 applies to \( A(C[m_n, m_{n-1}, \ldots, m_1, 1]) \). We will show:

**Theorem 5.**

\[
\frac{1 - t}{H(A(C[m_n, m_{n-1}, \ldots, m_1, 1]), t)} = 1 - \sum_{k=0}^{n} \sum_{a=k}^{n} (-1)^k m_a(m_{a-1} - 1)(m_{a-2} - 1) \ldots (m_{a-k+1} - 1)m_{a-k}t^{k+1}.
\]
Proof. We first compute

$$\sum_{v_1 > v_2 > \cdots > v_\ell \geq \emptyset, \ell \geq 1} (-1)^\ell t^{|v_1| - |v_\ell| + 1}.$$ 

The coefficient of $t^{k+1}$ in the sum is

$$\sum_{v_1 > v_2 > \cdots > v_\ell \geq \emptyset, \ell \geq 1, |v_1| = k} (-1)^\ell.$$

Note that the number of chains $v_1 > \cdots > v_\ell$ with $|v_i| = a_i$ for $1 \leq i \leq \ell$ is $m_1 m_2 \cdots m_\ell$. Then, writing $k = |v_1| - |v_\ell|$ and $a_1 = a$ we have

$$\sum_{v_1 > v_2 > \cdots > v_\ell \geq \emptyset, \ell \geq 1} (-1)^\ell t^{|v_1| - |v_\ell| + 1} = \sum_{k=0}^{n} \left( \sum_{v_1 > v_2 > \cdots > v_\ell \geq \emptyset, \ell \geq 1, |v_1| = k} (-1)^\ell \right) t^{k+1}$$

$$= \sum_{k=0}^{n} \left( \sum_{a_1 > \cdots > a_{\ell-1} > a_{\ell-1} - k \geq 0} (-1)^\ell m_1 m_2 \cdots m_{\ell-1} m_{a_{\ell-1} - k} \right) t^{k+1}$$

$$= \sum_{k=0}^{n} \left( \sum_{a=0}^{m} m_a (1 - m_a) \cdots (1 - m_a - k + 1) m_a - k \right) t^{k+1}.$$ 

The theorem now follows from Theorem 2. \(\square\)

This result applies, in particular, to the case $m_0 = m_1 = \cdots = m_n = 1$. The resulting algebra $A(C[1, \ldots, 1])$ has $n$ generators and no relations. Theorem 5 shows that

$$\frac{1 - t}{H(A(C[1, \ldots, 1]), t)} = 1 - \sum_{a=0}^{n} t + \sum_{a=1}^{n} t^2 = (1 - t)(1 - nt).$$

Thus $H(A(C[1, \ldots, 1]), t) = \frac{1}{1 - nt}$ and we have recovered the well-known expression for the Hilbert series of the free associative algebra on $n$ generators.

Since by [7] the algebras associated to complete directed graphs are Koszul algebras, we have the following

Corollary 4.

$$H(A(C[m_n, m_{n-1}, \ldots, m_1, 1]^t), t) =$$

$$1 + \sum_{k=1}^{n} \sum_{a=k}^{n} m_a (m_a - 1)(m_a - 2) \cdots (m_a - k + 1) - 1) t^k.$$
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