Abstract

We study the Poincaré series of the mixed and pure trace rings of generic matrices. These series are known to be rational functions. We obtain an explicit formula in lowest terms in the case of $2 \times 2$ matrices; a denominator, which we presume but have not been able to prove to be in lowest terms, in the case of $3 \times 3$ matrices; and a conjectured denominator in the general case.
1 Introduction

This paper is concerned with the Poincaré series of two families of algebras, $\bar{C} = \bar{C}(n, k)$ and $\bar{R} = \bar{R}(n, k)$, which in p.i. theory are called the pure and mixed trace rings of generic matrices, and in invariant theory are called the algebras of matrix invariants and concomitants.

Let $X_\alpha$ be the $n \times n$ matrix with each $(i, j)$ entry equal to the commuting indeterminate $x_{ij}^{(\alpha)}$, and let $F$ be a field of characteristic zero. The algebra $F[X_1, \ldots, X_k]$, which is contained in the algebra of $n \times n$ matrices over the polynomial ring $F[x_{ij}^{(\alpha)}]_{i,j,\alpha}$, is called the algebra of generic matrices and it satisfies various universal properties which we do not discuss here. There are three additional algebras that may be constructed from $R$. One is $C$, the center of $R$. The other two are constructed using the trace map $tr : R \to F[x_{ij}^{(\alpha)}]_{i,j,\alpha}$. The algebra $\bar{C} = \bar{C}(n, k)$ is called the pure trace ring and is defined to be the commutative algebra generated by the image of $tr$; and $\bar{R} = \bar{R}(n, k)$ is called the mixed trace ring, and is defined to be the algebra generated by $R$ and $\bar{C}$.

The two algebras $\bar{C}$ and $\bar{R}$, which will be the main focus of this work, can also be defined using invariant theory. A function $f : M_n(F)^k \to F$ is said to be invariant if

$$f(x_1, \ldots, x_k) = f(gx_1g^{-1}, \ldots, gx_kg^{-1})$$

for all $x_1, \ldots, x_k \in M_n(F)$ and all $g \in GL_n(F)$; and it is said to be polynomial in the entries if each entry of $f(x_1, \ldots, x_k)$ is a polynomial function of the entries of $x_1, \ldots, x_k$. An important example would be
\( tr(u) \) where \( u \) is any polynomial in \( x_1, \ldots, x_k \). The First Fundamental Theorem of invariant theory says that the algebra of invariant polynomial functions on \( M_n(F) \) is generated by these trace functions, \( tr(u) \). Using this it can be proven that this algebra is isomorphic to \( \bar{C} \), defined above. A similar construction can be used to show that if one considers functions from \( M_n(F)^k \) to \( M_n(F) \), instead of to \( F \), invariant and polynomial in the entries, that the algebra of such functions is isomorphic to \( \bar{R} \). Obviously, this is a bare-bones description of a deep subject. If the reader wishes to learn more there are many resources including [1] and [13].

Each of the four rings has an \( \mathbb{N} \) grading by total degree and a finer \( \mathbb{N}^k \) grading by degree in the individual matrices. Correspondingly, there are eight Poincaré series one might study: \( P(R), P(C), P(\bar{R}), P(C), P(\bar{R}), P(C), \) and \( P(\bar{C}) \), where \( P \) is used for the series in one variable and \( \tilde{P} \) is used for the \( k \) variable series. Then,

\[
\tilde{P}(X)(t, t, \ldots, t) = P(X)
\]

for \( X \) any one of \( C, R, \bar{C} \) or \( \bar{R} \).

These series have been the focus of a lot of research. One important theorem is that each is a rational function whose denominator can be taken to be a product of terms of the form \((1-t^n)\), where \( t^n \) is a monomial of degree at most \( n \). In [13] Formanek described how these series could be computed using complex integration, but it was Teranishi who first made use of this suggestion in [18], [19], and [20]. The Poincaré series for \( \bar{C}(n, k) \) equals

\[
(2\pi i)^{-n}(n!)^{-1} \int_T \frac{\prod_{i,j=1}^n(1 - \frac{z_i}{z_j})}{\prod_{\alpha=1}^k \prod_{i,j=1}^n(1 - \frac{z_i^{t_{\alpha}}}{z_j^{t_{\alpha}}})} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}
\]

(1)

where \( T \) represents the torus \(|z_1| = \cdots = |z_n| = 1\). The Poincaré series for \( \bar{R}(n, k) \) is the same with an extra factor of \( \sum_{\alpha=1}^k \frac{t_\alpha}{z_1} \sum_{i,j} z_i z_j^{-1} \) in the numerator:

\[
(2\pi i)^{-n}(n!)^{-1} \int_T \frac{\sum_{i,j=1}^n z_i z_j^{-1} \prod_{i,j=1}^n(1 - \frac{z_i}{z_j})}{\prod_{\alpha=1}^k \prod_{i,j=1}^n(1 - \frac{z_i^{t_{\alpha}}}{z_j^{t_{\alpha}}})} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}
\]

(2)

These integrals were used to study the Poincaré series of \( \bar{C}(n, k) \) and \( \bar{R}(n, k) \) in a number of papers including [3], [10], [11], [18], [19], [20] and [21].

By setting all of the \( t_\alpha \) to a single variable \( t \) in (1) and (2) we get the the one variable Poincaré series for \( \bar{C}(n, k) \) and \( \bar{R}(n, k) \). The former would be

\[
(2\pi i)^{-n}(n!)^{-1} \int_T \frac{\prod_{i,j=1}^n(1 - \frac{z_i}{z_j})}{\prod_{i,j=1}^n(1 - \frac{z_i^{t}}{z_j^{t}})} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}
\]

(3)
and the latter

\[(2\pi i)^{-n}(n!)^{-1} \int \frac{\sum_{i,j=1}^{n} z_{i}^{-1} \prod_{i \neq j=1}^{n} (1 - \frac{z_{j}}{z_{i}}) \, dz_{1} \cdots dz_{n}}{\prod_{i,j=1}^{n} (1 - \frac{z_{i}}{z_{j}})^{k} \, z_{1} \cdots z_{n}} \, dz \tag{4}\]

In the case of \( n = 2 \), namely \( 2 \times 2 \) matrices, the series have been computed in various ways, see [13] and [16]. An explicit formula for the one variable Poincaré series \( P(\tilde{C}(2, k)) \) was proven by Teranishi in [18]:

\[P(\tilde{C}(2, k)) = (-1)^{k-1} \frac{1}{2(k-1)!} \frac{1}{(1-t)^{2k}} \left( \frac{d}{dz} \right)^{k-1} \left( z^{2} \right)^{-2} \frac{1}{(tz-1)^{k}} \bigg|_{z=t} \]

In the next two sections of this paper we will prove a different formula for \( P(\tilde{C}(2, k)) \) and also one for \( P(\tilde{R}(2, k)) \) expressing each as a rational function in lowest terms. Theorem 3.1 states that \( C(2, k) \) has Poincaré series

\[(1-t)^{2-2k}(1-t^2)^{1-2k} \left( \sum \binom{k-2}{i} t^{2i} - \sum \binom{k-2}{i+1} t^{2i+1} \right)\]

and Theorem 3.2 states that the Poincaré series of \( \tilde{R}(2, k) \) equals

\[(1-t)^{-2k}(1-t^2)^{3-2k} \sum_{k-2} \binom{k-2}{i} \binom{k-2}{i+1} x^{2i}\]

The coefficients in the numerator of \( P(\tilde{R}(2, k)) \) turn out to be the Narayama numbers. For more on these numbers, see [17].

In [10] Berele and Stembridge found denominators for \( P(\tilde{C}(n, k)) \) and \( P(\tilde{R}(n, k)) \) for \( n \leq 4 \) and showed that they were least denominators for \( n \leq 3 \). In the current paper we will be studying the denominators for the one variable Poincaré series. Of course, one could find denominators for the one variable Poincaré series by simply specializing that of the \( k \) variable series, however such a denominator would be far from least. For example, \( P(\tilde{C}(2, k)) \) has denominator

\[\prod_{i} (1-t_{i}) \prod_{i \leq j} (1-t_{i}t_{j}) \]

which specializes to \((1-t)^{k}(1-t^2)^{k+1}\), but the least denominator for \( P(\tilde{C}(2, k)) \) is \((1-t)^{2k-2}(1-t^2)^{2k-1}\). Based on the \( k \) variable case we know that the denominators for the one variable Poincaré series of \( \tilde{C}(n, k) \) and \( \tilde{R}(n, k) \) are products of terms \((1-t^i)\) with \( i \leq n \). For \( n = 3 \) we find denominators which we believe to be least, although we do not prove it. In the case of \( P(\tilde{C}(3, k)) \) the denominator is \((1-t)^{2k-2}(1-t^2)^{4k-4}(1-t^3)^{3k-2}\), and in the case of \( P(\tilde{R}(3, k)) \) the denominator is \((1-t)^{2k}(1-t^2)^{4k-4}(1-t^3)^{3k-4}\). Zeilberger has computed \( P(\tilde{C}(3, k)) \) for \( k \leq 100 \), see [12], and in each of these cases our denominators are least. At any rate, the degrees are again much smaller than those gotten by specializing the \( k \) variable.
series. Finally, in Section 5 we present a conjecture for the denominators of all \( P(\bar{C}(n, k)) \) and \( P(\bar{R}(n, k)) \) which agrees with our results for \( n = 2, 3 \) and with the denominators computed by Doković in [11] for \( n = 4, 5, 6 \) and \( k = 2 \). Since these are all the known denominators at this point we feel confident that our conjecture is correct.

Procesi proved that \( \bar{C}(n, k) \) and \( \bar{R}(n, k) \) each have Gelfand-Kirillov dimension equal to \( (k - 1)n^2 + 1 \) implying this lemma.

**Lemma 1.1.** The Poincaré series for each \( \bar{C}(n, k) \) and \( \bar{R}(n, k) \) has a pole at \( t = 1 \) of order \( (k - 1)n^2 + 1 \).

This lemma will be useful in our computation of the denominators partly because if we can compute the multiplicity of each \( (1 - t^i) \) in the denominator for \( i \geq 2 \), then we can deduce the multiplicity of \( (1 - t) \).

Here is another useful theorem due to Formanek in [14], originally stated for the multiple variable Poincaré series.

**Lemma 1.2.** If \( F(t) \) equals either \( P(\bar{C}(n, k)) \) or \( P(\bar{R}(n, k)) \), then \( F(t) \) satisfies the functional equation \( F(\frac{1}{t}) = \pm t^{kn^2} F(t) \). In particular, if the denominator has degree \( d \) then the numerator will have degree \( d - kn^2 \).

The current work was inspired by a much larger question. Given a p.i. algebra \( A \), perhaps with 1, there are various integer sequences \( c_m(A) \) related to the cocharacter of \( A \) which are known to be asymptotic to some \( \alpha m^d \). In this paper we are focusing on the case in which \( c_m(A) \) equals the growth function for a generic algebra for \( A \). Other sequences known to be asymptotic to a polynomial include the colength sequence and the maximal multiplicity sequence, see [4] and [6]. In the case of verbally prime algebras and prime product algebras the exponent \( d \) has been computed for the growth functions of the generic algebras and for the colength and maximal multiplicity sequences, see [2], [3], [5], [7] and [8], but the coefficients \( \alpha \) can only be computed in those rare cases in which the codimension sequence is completely known. We feel that these coefficients should be of interest.

Generally, if \( f(t) \) and \( g(t) \) are polynomials not divisible by \( (1 - t) \) and if all of the zeros of \( g(t) \) are on the unit circle, each with multiplicity less than \( d \), then if

\[
\frac{f(t)}{(1 - t)^d g(t)} = \sum c_m t^m
\]

the coefficient \( c_m \) will be asymptotic to \( \frac{f(1)}{g(1)} \binom{m+d-1}{d-1} \) or \( \frac{f(1)}{g(1)} d^{-1} m^{d-1} \). It follows that if \( P(\bar{C}(k, n)) = \sum \tilde{c}_m t^m \) and \( P(\bar{R}(n, k)) = \sum \tilde{r}_m t^m \) then each of \( \tilde{c}_m \) and \( \tilde{r}_m \) is asymptotic to some \( \alpha m^{(k-1)n^2} \) where \( \alpha \) is a rational number with denominator of the form \( \alpha' ([k-1]n^2)! \), where \( \alpha' \) has all prime factors less than or equal to \( n \). If our conjecture is true,
one could identify the denominator more precisely. Likewise, if \( \gamma_m = \sum_{i=0}^m \bar{c}_i \) and \( \rho_m = \sum_{i=0}^m \bar{r}_i \), then \( \sum \gamma_m t^m = (1 - t)^{-1} \sum \bar{c}_m t^m \) and \( \sum \rho_m t^m = (1 - t)^{-1} \sum \bar{r}_m t^m \), and so each of \( \gamma_m \) and \( \rho_m \) is asymptotic to a constant times \( m^{(k-1)n^2+1} \) where the constant factor is the same rational number as before, divided by \( (k-1)n^2 + 1 \).

In the case of \( P(\bar{C}(2, k)) = N_k(t)(1 - t)^{2 - 2k} \), Teranishi computed \( N_k(1) = \frac{1}{24k^2} \). These are the Catalan numbers \( C_k \), and so \( \bar{c}_m \simeq C_k \) for \( n \). It follows from our work that if \( P(R(2, k)) = N_k(t)(1 - t)^{2k}(1 - t^2)^{3 - 2k} \) then \( N_k(1) \) also equals the Catalan number \( C_k \) and so \( \bar{c}_m \) is asymptotic to \( \frac{1}{24} C_k \). These are the Catalan numbers \( C_k \).

In the case of \( P(\bar{C}(3, k)) \), \( N_k(t)(1 - t)^{2 - 2k} \) is asymptotic to \( N_k(1) 2^{k-4} (m+4k-2) \), and so \( \bar{m} \) is asymptotic to \( N_k(1) 2^{k-4} (m+4k-2) \). It is possible to compute \( N_k(1) \) for low values of \( k \) using Zeilberger’s results. Sadly, no pattern is apparent. The first few are \( N_2(1) = 1 \), \( N_3(1) = 21 \) and \( N_4(1) = 1636 \).

We are happy to acknowledge B. Tenner’s help in using oeis; useful conversations with S. Catoiu and K. Liechty; and D. Zeilberger for patiently explaining how to use his algorithm in Maple and, as already mentioned, computing the Poincaré series \( P(\bar{C}(2, k)) \) and \( P(\bar{C}(3, k)) \) for \( k \leq 100 \) in [12]. These computations involved new recurrence relations he found for these functions. The relation for \( n = 2 \) is fairly simple. If \( B(k) = (1 - t)^{2k} P(\bar{C}(2, k)) \) then \( B(k) \) equals \( (1 - t^2)^{2k} (k - 1)^{-1} \) times

\[
2(k^2 - 2t^2 + k - t - 2)B(k - 1) - (k - 3)B(k - 2).
\]

On the other hand, the relation for \( n = 3 \) is anything but simple and takes up hundreds of lines of output!

## 2 Two-By-Two Matrices - Denominators

In [18], a work which served as inspiration for much of this paper, Teranishi computed the asymptotics of the coefficients in Poincaré series in the case of \( 2 \times 2 \) matrices. We now adapt his methods to give a more explicit result.

The Poincaré series for the trace ring \( \bar{C}(2, k) \) of \( 2 \times 2 \) generic matrices for \( k \geq 2 \) is given by the integral

\[
\frac{1}{2!} (2\pi i)^{-2} \oint_{|z_1| = 1} \oint_{|z_2| = 1} \frac{(1 - \frac{z_1}{z_2})(1 - \frac{z_2}{z_1})}{(1 - t)^{2k}(1 - t^{z_1^2})^k(1 - t^{z_2^2})^k} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \tag{5}
\]

Pulling out the factor of \((1 - t)^{2k}\) and clearing fractions we get \(-\frac{1}{2} (2\pi i)^{-2} (1 - t)^{-2k}\) times

\[
\oint_{|z_1| = 1} \oint_{|z_2| = 1} \frac{-z_1^{-k} z_2^{-k} (z_1 - z_2)^2}{(z_1 - z_2)^k (z_2 - z_1)^k} dz_1 \wedge dz_2.
\]

6
Since \( t \) is taken to be less than 1 the only pole in the unit circle is at \( z_1 = tz_2 \) and it is of order \( k \). By Cauchy’s integration formula we can now evaluate \((2\pi i)^{-1}\) times the inner integral in three steps: (1) multiply by \((z_1 - tz_2)^k/(k-1)!\) This step will have the effect of cancelling the \((z_1 - tz_2)^k\) term from the denominator. (2) Take the \((k-1)^{st}\) derivative with respect to \( z_1 \) and (3) take the limit as \( z_1 \to tz_2 \). Letting \( D \) denote the derivative with respect to \( z_1 \), the multiplication by \((z_1 - tz_2)^{k-1}\) derivative \(D^{k-1}\) transforms the integrand to

\[
(k-1)!^{-1} \sum_{a+b+c=k-1} \binom{k-1}{a,b,c} -D^a(z_1^{k-2})z_2^{k-2}D^b(z_1-z_2)^2D^c(z_2-z_1t)^{-k}. \tag{6}
\]

At this point it is useful to use the notation of rising and falling factorials. \((n)_a\) is defined to be \(n(n-1)\cdots(n-a+1)\) or \(n!/(n-a)!\); and \(n^{(a)}\) is defined to be \(n(n+1)\cdots(n+a-1)\) or \((n+a-1)!/(n-1)!\). Taking into account that the \((k-1)!\) coefficient will cancel the \((k-1)!\) from the binomial coefficient, the summand in (6) corresponding to a given \( a, b, c \) equals

\[-\frac{1}{a!b!c!} (k-2)_a z_1^{2-k-a} z_2^{k-2-2} (2)_b (z_1-z_2)^{2-b} k^{(c)} t^c (z_2-tz_1)^{-k-c}.\]

Substituting \( z_1 = tz_2 \) then gives

\[-\frac{1}{a!b!c!} (k-2)_a (2)_b k^{(c)} t^{k-2-a} z_2^{k-2-2} z_2^{k-2} (t-1)^{2-b} t^c z_2^{-k-c} (1-t^2)^{-k-c}.\]

Collecting the powers of \( z_2 \) gives \( z_2 \) to the power of

\[k-2-a+k-2+2-b-k-c = k-2-(a+b+c) = k-2-(k-1) = -1.\]

The integral with respect to \( z_2 \) will merely cancel that term and the remaining \((2\pi i)^{-1}\) in front, giving this theorem:

**Theorem 2.1.** The Poincaré series of \( \bar{C} \) for \( k \) \( 2 \times 2 \) matrices equals the sum over all \( a+b+c = k-1 \), \( b \leq 2 \) of \((1-t)^{-2k}\) times

\[-\frac{1}{2 \cdot a!b!c!} (k-2)_a (2)_b k^{(c)} t^{k-2-a+c} (t-1)^{2-b} (1-t^2)^{-k-c}.\]

In order to identify the least denominator, note that the order of the pole at \( t = 1 \) in each term is \( 2k - 2 + b + k + c \). This is maximal when \( b + c = k-1 \) and so is \( 4k - 3 \), in agreement with Lemma [1.1]. And the order of the pole at \( t = -1 \) in each term is \( k + c \) with unique maximum at \( 2k - 1 \).

**Corollary 2.2.** The Poincaré series of \( \bar{C} \) for \( k \) \( 2 \times 2 \) matrices is a rational function with denominator \((1-t)^{2k-2}(1-t^2)^{2k-1}\)
We record the first few numerators, which we denote $N_k$:

\[
\begin{align*}
N_2 &= 1 \\
N_3 &= t^2 - t + 1 \\
N_4 &= t^4 - 2t^3 + 4t^2 - 2t + 1 \\
N_5 &= t^6 - 3t^5 + 9t^4 - 9t^3 + 9t^2 - 3t + 1 \\
N_6 &= t^8 - 4t^6 + 16t^4 - 24t^3 + 36t^2 - 24t + 1 \\
N_7 &= t^{10} - 5t^9 + 25t^8 - 50t^7 + 100t^6 - 100t^5 + 100t^4 - 50t^3 + 25t^2 - 5t + 1 \\
N_8 &= t^{12} - 6t^{11} + 36t^{10} - 90t^9 + 225t^8 - 300t^7 + 400t^6 - 300t^5 + 225t^4 - 90t^3 \\
&\quad + 36t^2 - 6t + 1 \\
\end{align*}
\]

With some help from oeis we found that the coefficients in $N_k$ alternate between \((n-2)^2\) and \(-(n-2)(n-2)\). To see what this means consider, for example, the fourth row of Pascal’s triangle:

\[
1, \quad 4, \quad 6, \quad 4, \quad 1
\]

Then the coefficients in $N_6$, up to sign, equal

\[
1 \cdot 1, \quad 1 \cdot 4, \quad 4 \cdot 4, \quad 4 \cdot 6, \quad 6 \cdot 6, \quad 6 \cdot 4, \quad 4 \cdot 4, \quad 4 \cdot 1, \quad 1 \cdot 1
\]

Before proving this in the next section, we turn to the case of $\bar{R}$.

In order to compute the Poincaré series of $\bar{R}$ the computation is almost identical, except that the integrand has an extra factor of \((2 + \frac{z}{z_2} + \frac{z_2}{z})\) in the numerator. Since the denominator will follow from the computations in the next section we leave the proof of the following to the interested reader.

**Theorem 2.3.** The Poincaré series of $\bar{R}$ for $k$ $2 \times 2$ matrices is a rational function with denominator $(1 - t)^{2k}(1 - t^2)^{2k-3}$.

Again denoting the numerator as $N_k$ we record:

\[
\begin{align*}
N_2 &= 1 \\
N_3 &= 1 \\
N_4 &= t^2 + 1 \\
N_5 &= t^4 + 3t^2 + 1 \\
N_6 &= t^6 + 6t^4 + 6t^2 + 1 \\
N_7 &= t^8 + 10t^6 + 20t^4 + 10t^2 + 1 \\
N_8 &= t^{10} + 15t^8 + 50t^6 + 50t^4 + 15t^2 + 1 \\
N_9 &= t^{12} + 21t^{10} + 105t^8 + 175t^6 + 105t^4 + 21t^2 + 1 \\
N_{10} &= t^{14} + 28t^{12} + 196t^{10} + 490t^8 + 490t^6 + 196t^4 + 28t^2 + 1
\end{align*}
\]
With some help from oeis we found that these are Narayana numbers. The coefficient of $t^{2k}$ in $N_n$ is equal to
\[ \frac{1}{n-2} \binom{n-2}{k} \binom{n-2}{k+1}. \]

3 Two-By-Two Matrices - Numerator

Following Teranishi, we can write the Poincaré series for $\bar{C}(2, k)$ and $\bar{R}(2, k)$ as integrals over one complex variable instead of two. $\bar{C}(2, k)$ has one variable Poincaré series equal to the integral
\[ \frac{1}{2} \left( 1 - t \right)^{-2k} (2\pi i)^{-1} \oint_{|z|=1} \frac{(1-z)(1-z^{-1})}{(1-zt)^k(1-t/z)^k} \frac{dz}{z} \]
and $\bar{R}(2, k)$ has Poincaré series
\[ \frac{1}{2} \left( 1 - t \right)^{-2k} (2\pi i)^{-1} \oint_{|z|=1} \frac{(1-z^2)(1-z^{-2})}{(1-zt)^k(1-t/z)^k} \frac{dz}{z}, \]
where $|t| < 1$. By Cauchy’s theorem, in order to evaluate an integral of the form $(2\pi i)^{-1} \oint_{|z|=1} f(z) \frac{dz}{z}$ we can expand $f(z)$ as a Taylor series in $z$ and take the constant coefficient.

Expanding equation (7) as power series in $t$ we get $\frac{1}{2} \left( 1 - t \right)^{-2k}$ times the integral of
\[ (1-z)(1-z^{-1}) \sum_{a=0}^{\infty} \binom{a+k-1}{k-1} t^a z^a \sum_{b=0}^{\infty} \binom{b+k-1}{k-1} t^b z^{-b} \]
times $\frac{dz}{z}$ which picks out the constant terms in $z$. The product of the first two terms is $2 - z - z^{-1}$, so in the product of the two summations we need only consider the terms with $a = b$ or with $a = b \pm 1$. For the former the $\frac{1}{2}$ and 2 cancel and we get
\[ \sum_{a=0}^{\infty} \binom{a+k-1}{k-1}^2 t^{2k}. \]

For the latter we get twice (also cancelling the $\frac{1}{2}$) the sum
\[ \sum_{a=1}^{\infty} \binom{a+k-1}{k-1} \binom{a+k-2}{k-2} t^{2k-1} \]
At this point we wish to prove two combinatorial identities:
\[ \sum_a \binom{a+k-1}{k-1}^2 t^{2a} = (1 - t^2)^{1-2k} \sum_a \binom{k-1}{a} t^{2a}, \]
(9)
\[
\sum_a \left( \frac{a+k}{k-1} \right) \left( \frac{a+k-1}{k-1} \right) t^{2a+1} = (1-t^2)^{1-2k} \sum_a \left( \frac{k-2}{a} \right) \left( \frac{k}{a+1} \right) t^{2a}. \tag{10}
\]

These can now be proven with software instead of erudition and intelligence. To prove (9) we run the following Maple commands (kindly supplied to us by Doron Zeilberger):

```maple
Z := SumTools[Hypergeometric][Zeilberger];
ope1 := Z(binomial(a+k-1,k-1)^2*t^a,a,n,K)[1];
```

The program returns the output

\[
\text{ope1} := (kt^2 - 2kt + t^2 + k - 2t + 1)K^2 + (-2kt - 2k - t - 1)K + k
\]

Which is Maple’s way of saying that if \( F(k) = \sum_n (\frac{a+k-1}{a})^2 t^n \) then

\[
(kt^2 - 2kt + t^2 + k - 2t + 1)F(k+2) + (-2kt - 2k - t - 1)F(k+1) + kF(k) = 0
\]

or

\[
F(k+2) = \frac{2kt + 2k + t + 1}{kt^2 - 2kt + t^2 + k - 2t + 1} F(k+1) - \frac{k}{kt^2 - 2kt + t^2 + k - 2t + 1} F(k)
\]

for all \( k \geq 0 \). If we then run the same program on

```maple
ope2 := Z((1-t)^{(1-2*k)}*binomial(k-1,a)^2*t^a,k,a,k)[1];
```

we get that \( G(k) = \sum_n (\frac{k-2}{n})^2 t^n \) satisfies the exact same recurrence. Since \( F(0) = G(0) = 0 \) and \( F(1) = G(1) = (1-t)^{-1} \) this proves the \( G(k) = F(k) \) for all \( k \geq 0 \) and so proves (9).

The proof of (10) is similar. If we now let \( F(k) \) equal the right hand side of (10) and \( G(k) \) equal the left then each satisfies

\[
(k^2 + k)X(k+2) + (-2k^2 t^2 - k^2 - 2k^2 - k)X(k+1) + (k^2 t^4 - 2k^2 t^2 - t^4 + k^2 + 2t^2 - 1)X(k) = 0
\]

It now follows that \( \tilde{C}(2,k) \) has Poincaré series

\[
(1-t)^{-2k}(1-t^2)^{1-2k} \left( \sum \left( \frac{k-1}{i} \right)^2 t^{2i} - \sum \left( \frac{k-2}{i+1} \right) \left( \frac{k}{i} \right) t^{2i+1} \right)
\]

It turns out that this is not in lowest terms and one more simplification is possible.

**Theorem 3.1.** \( \tilde{C}(2,k) \) has Poincaré series

\[
(1-t)^{2-2k}(1-t^2)^{1-2k} \left( \sum \left( \frac{k-2}{i} \right)^2 t^{2i} - \sum \left( \frac{k-2}{i} \right) \left( \frac{k-2}{i+1} \right) t^{2i+1} \right)
\]
Proof. We need to show that
\[
\sum_{i} \left( k_i - 1 \right)^2 t^{2i} - \sum_{i+1} \left( k_i - 2 \right) t^{2i+1} =
\]
\[
(1-t)^2 \left[ \sum_{i} \left( k_i - 2 \right)^2 t^{2i} - \sum_{i+1} \left( k_i - 2 \right) \left( k_i - 1 \right) t^{2i+1} \right]
\]
Expanding \((1-t)^2 = 1 - 2t + t^2\) and first comparing coefficients of \(t^{2i}\) on both sides, we see that we need
\[
\left( k_i - 1 \right)^2 = \left( k_i - 2 \right)^2 + 2 \left( k_i - 2 \right) \left( k_i - 1 \right) + \left( k_i - 1 \right)^2.
\]
This is simply the square of \(\binom{k-1}{i} = \binom{k-2}{i} + \binom{k-2}{i-1}\). Finally, comparing the coefficients of \(t^{2i+1}\) we need to prove
\[
\binom{k}{i+1} \left( k_i - 2 \right) = \binom{k-2}{i} \left( k_i - 1 \right) + \binom{k-2}{i} \left( k_i - 1 \right) + 2 \left( k_i - 2 \right)^2.
\]
Dividing both sides by \(\binom{k-1}{i+1}\) the left hand side becomes \(\binom{k}{i+1}\) and the right side becomes
\[
\binom{k-2}{i+1} + \binom{k-1}{i-1} + 2 \binom{k-1}{i} =
\]
\[
\left( k_i - 2 \right) + \binom{k-1}{i-1} + \binom{k-1}{i} =
\]
\[
\left( k_i - 1 \right) + \binom{k-1}{i-1} + \binom{k-1}{i} =
\]
\[
\binom{k}{i+1}.
\]
Remarks. We remark that this fraction is in lowest terms. By Lemma 1, the order of the pole at \(t = 1\) is \(4k - 3\), which equals \(2k - 2 + 2k - 1\), so no factor of \((1-t)\) can cancel. Moreover, the numerator cannot have a factor of \((1+t)\) else it would have a root at \(t = -1\), which it clearly does not.

The case of \(\bar{R}(2, k)\) is similar and slightly easier. The integral \(\widehat{R}(2, k)\) equals
\[
\frac{1}{2} (1-t)^{-2k} (2\pi i)^{-1} \oint_{|z|=1} \frac{2 - z^2 - z^{-2}}{(1-zt)^k (1-t/z)^k} \frac{dz}{z}.
\]
The integrand then equals
\[(2 - z^2 - z^{-2}) \sum_{a=0}^{\infty} \binom{a + k - 1}{k - 1} t^a z^a \sum_{b=0}^{\infty} \binom{b + k - 1}{k - 1} t^b z^{-b} \] (12)
times \frac{dz}{z},
and so we need to pick out the constant term in (12). These terms will occur when \(a = b\) and \(a = b \pm 2\) and so the integral equals:
\[
\sum \left[ \left( \binom{a + k - 1}{k - 1} - \binom{a + k - 2}{k - 1} \right) t^a \right].
\] (13)
The term in brackets simplifies to \(\frac{1}{a+k-1} \binom{a+k-1}{k-2} \binom{a+k-1}{k-1}\). For fixed \(k\) this is a polynomial in \(a\) of degree \(2k - 4\), so the series equals \((1 - t^2)^{3-2k} N_k(t^2)\) where \(N_k\) is a polynomial. With some help from Maple we now prove:
\[
\sum \frac{1}{a+k-1} \binom{a+k-1}{k-2} \binom{a+k-1}{k-1} t^a =
(1-t)^{3-2k} \sum \frac{1}{k-2} \binom{k-2}{i} \binom{k-2}{i+1} t^i
\] (14)
for \(k \geq 3\). Using Zeilberger’s algorithm on Maple we see that each side of the equation satisfies
\[(kt^2 - 2kt + t^2 + k - 2t + 1)X(k+2) + (-2kt - 2k + t + 1)X(k+1) + (k - 2)X(k) = 0.
\]
That the two sides agree for the initial values \(k = 3, 4\) follows from the computations following Corollary 2.3.

**Theorem 3.2.** For \(k \geq 3\) the Poincaré series of \(\bar{R}(2, k)\) equals
\[(1-t)^{-2k} (1 - t^2)^{3-2k} \sum \frac{1}{k-2} \binom{k-2}{i} \binom{k-2}{i+1} t^i
\]

**Remarks.** This fraction is in lowest terms, since the numerator is positive if \(t = \pm 1\). If \(k\) is even the numerator has a factor of \(1 + t^2\).

### 4 Three-By-Three Matrices

We now turn to the three-by-three case. It will simplify the computations if instead of taking each \(|z_i|\) to be 1, we instead take
\[t \ll R_1 = |z_1| < R_2 = |z_2| < R_3 = |z_3|.
\]
The justification for doing this is based on Weyl’s integration formula. If \(s_{\lambda}(x_1, \ldots, x_n)\) and \(s_{\mu}(x_1, \ldots, x_n)\) are Schur functions, then their
inner product, which equals \( \delta_{\lambda, \mu} \) can be computed as the coefficient of 1 in
\[
(n!)^{-1} s_{\lambda}(x_1, \ldots, x_n) s_{\mu}(x_1^{-1}, \ldots, x_n^{-1}) \prod_{i \neq j} (1 - \frac{z_i}{z_j}).
\]
Traditionally, this coefficient is computed by integrating over the torus \( |z_i| = R_i \) and our choice of torus will simplify the computation. Then the Poincaré series of \( \bar{C} \) for \( k \) \( 3 \times 3 \) matrices equals \( \frac{1}{6} (2\pi i)^{-3} (1 - t)^{-3k} \) times the integral over \( |z_i| = R_1, |z_2| = R_2, |z_3| = R_3 \) of
\[
\frac{(1 - \frac{z_1}{z_2})(1 - \frac{z_1}{z_3})(1 - \frac{z_2}{z_3})(1 - \frac{z_2}{z_1})(1 - \frac{z_3}{z_1})}{(1 - t \frac{z_1}{z_2})^k(1 - t \frac{z_2}{z_3})^k(1 - t \frac{z_3}{z_1})^k(1 - t \frac{z_1}{z_2})^k(1 - t \frac{z_2}{z_3})^k(1 - t \frac{z_3}{z_1})^k}
\]
times \( \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3} \). Clearing fractions gives \( (z_1 z_2 z_3)^{2k - 3} \) times the integral of
\[
-(z_1 - z_2)^2(z_2 - z_3)^2(z_1 - z_3)^2 dz_1 \wedge dz_2 \wedge dz_3
\frac{(z_1 - tz_2)^k(z_2 - tz_3)^k(z_2 - z_3)^k(z_3 - tz_1)^k(z_3 - z_1)^k}{(z_1 - tz_2)^k(z_2 - tz_3)^k(z_2 - z_3)^k(z_3 - tz_1)^k(z_3 - z_1)^k}
\]
(15)
The poles for \( z_3 \) are at \( tz_2, tz_3, z_2/t \) and \( z_3/t \). However, only the first two have absolute value less than or equal to \( R_1 \), so only these contribute to the integral. The residue at \( z_1 = tz_2 \) is gotten by cancelling one factor of \( (2\pi i)^{-1} \) and the \( (z_1 - tz_2)^k \), taking the \( (k - 1) \)-st derivative with respect to \( z_1 \), dividing by \( (k - 1)! \), and then substituting \( z_1 = tz_2 \). Let \( D_1 \) be the partial derivative with respect to \( z_1 \). Then the derivative in question will \( (z_2 z_3)^{2k - 3}(z_2 - z_3)^2(z_2 - tz_3)^{-k}(z_3 - tz_2)^{-k} \) times the sum over \( a + \cdots + f = k - 1 \) of \((-a, b, c, d, e, f)\) times
\[
D_1^a(z_1 z_2 z_3)^b D_1^c(z_1 - z_3)^2 D_1^d(z_1 - z_2)^{-k} D_1^e(z_2 - tz_3)^{-k} D_1^f(z_3 - tz_1)^{-k}
\]
This summand equals a constant times a power of \( z_1 \) and a power of \( t \) times
\[
(z_1 - z_2)^{2 - b + c}(z_1 - z_3)^{2 - c}(z_1 - tz_3)^{-k - d}(z_2 - tz_1)^{-k - e}(z_3 - tz_1)^{-k - f}.
\]
If we are only concerned with the denominator we can investigate only terms with \( a = 0 \), because the other derivatives increase the degrees of terms in the denominator and the derivatives of \( z_1^{2k - 3} \) do not. Ignoring this term and specializing \( z_1 = tz_2 \) we get the sum of terms
\[
(z_2 t - z_2)^{2 - b}(z_2 t - z_3)^{2 - c}(z_2 t - tz_3)^{-k - d}(z_2 - t^2 z_2)^{-k - e}(z_3 - t^2 z_2)^{-k - f}
\]
or, \( \pm (z_2)^{2 - k - e} t^{-k} \) times
\[
(1 - t)^{2 - b}(1 - t^2)^{-k - e}(z_3 - z_2 t)^{2 - c}(z_2 - z_3)^{-k - d}(z_3 - t^2 z_2)^{-k - f}
\]
(16)
all times \((z_2 - z_3)^2(z_2 - tz_3)^{-k}(z_3 - z_2t)^{-k}\). The only pole for \(z_2\) with absolute value less than \(R_2\) is \(z_2 = tz_3\), and the order of the pole is \(k\). For a given \(a, \ldots, f\) the residue will be a constant times a power of \(z_3\) times a power of \(t\) times \((1 - t^2)^{2a-b}(1 - t^2)^{-k-c}\) times

\[
(1-t^2)^2-cg(1-t)^{-k-d-b}(1-t^3)^{-k-f-i}(1-t)^2-j(1-t^2)^{-k-l}
\]

where \(g + h + i + j + l = k - 1\). The pole at \(t = -1\) has order \(k + e - 2 + c + g + k + l\). Since \(c + e \leq k - 1\) and \(g + l \leq k - 1\) this is at most \(2k - 2 + 2(k - 1) = 4k - 4\). The pole at \(t = \sqrt[3]{3}\) has order \(k + f + i\) which is maximized by taking \(f = i = k - 1\) in which case it equals \(3k - 2\).

**Lemma 4.1.** Taking the residue of \((15)\) with respect to \(z_1 = tz_2\) gives a fraction with denominator a power of \((1 - t)^3\) times \((1 - t^2)^{4k-4}(1 - t^3)^{3k-2}\).

We next turn to the residue of \((15)\) at \(z_1 = tz_3\). The first steps in the computation are identical to those of the residue at \(z_1 = tz_2\), with \(z_2\) and \(z_3\) switched. Analogous to \((16)\) we get that the denominator is the same as that of the sum of terms \((z_2 - z_3)^2(z_2 - tz_3)^{-k}(z_3 - z_2t)^{-k}\) times

\[
(z_3t - z_3)^2-b(z_3t - z_2)^2-c(z_3t - tz_3)^{-k-d}(z_3 - t^2z_3)^{-k-e}(z_2 - t^2z_3)^{-k-f}
\]

where again \(b + \cdots + f \leq k - 1\), or, \(\pm(z_3)^{2-b-k-e}t^{-k-c}\) times

\[
(1-t)^2-b(1-t^2)^{-k-c}(z_3 - z_2)^2-c(z_3 - z_2)^{-k-d}(z_2 - t^2z_3)^{-k-f}
\]

all times \((z_3 - z_2)^2(z_3 - tz_2)^{-k}(z_2 - z_3t)^{-k}\). Now there are two poles for \(z_2\) inside of \(|z_2| = R_2\): namely \(z_2 = tz_3\) and \(z_2 = t^2z_3\). In the former case the order of the pole is \(k - 2 + c\) and there will be no factors of \((1 - t^3)\). The factors of \((1 - t^2)\) will come from the terms

\[
(1-t^2)^{-k-c}(z_3 - z_2)^{-k}
\]

and that differentiation will increase the order of the factor by as much as \(k - 3 + c\). Keeping in mind that \(b + e \leq k - 1\) the order will be

\[
k + e + k + k - 3 + a \leq 3k - 3 + a + e
\]

Since \(a + e \leq k - 1\) this implies the following:

**Lemma 4.2.** The residue of \((15)\) at \(z_1 = tz_3\) and then at \(z_2 = tz_3\) is a rational function with denominator a power of \((1 - t)^3\) times \((1 - t^2)^{4k-4}\).

We leave the case of \(z_2 = t^2z_3\) to the reader. The result is

**Lemma 4.3.** The residue of \((15)\) at \(z_1 = tz_3\) and then at \(z_2 = t^2z_3\) is a rational function with denominator a power of \((1 - t)\) times \((1 - t^2)^{4k-4}(1 - t^3)^{3k-2}\).
Combining the three lemmas we know have

**Theorem 4.4.** The Poincaré series of $\bar{C}(3, k)$ is a rational function with denominator $(1 - t)^{2k-2}(1 - t^2)^{4k-4}(1 - t^3)^{3k-2}$.

*Proof.* The powers of $(1 - t^2)$ and $(1 - t^3)$ follow from the lemmas and the power of $(1 - t)$ follows from the fact that the order of the pole at $t = 1$ is $9k - 8$ by Lemma 1.1.

Taking into account Lemma 1.2 we can say this about the numerator:

**Corollary 4.5.** If $P(\bar{C}(3, k))$ is written as a fraction with denominator as above, then the numerator is a symmetric polynomial of degree $10k - 16$.

Doron Zeilberger [12] computed $P(\bar{C}(3, k))$ for $k \leq 100$. In each case our denominator appears to be least and we conjecture that it always is. Here are the first few numerators:

$$N_2(t) = 1 - t^2 + t^4$$
$$N_3(t) = t^{14} - t^{13} - 2t^{12} + 6t^{11} + 6t^{10} - 9t^9 + t^8 + 17t^7$$
$$+ t^6 - 9t^5 + 6t^4 + 6t^3 - 2t^2 - t + 1$$
$$N_4(t) = t^{24} - 2t^{23} - t^{22} + 18t^{21} + 6t^{20} - 30t^{19}$$
$$+ 75t^{18} + 150t^{17} - 30t^{16} + 30t^{15} + 401t^{14} + 238t^{13}$$
$$- 76t^{12} + 238t^{11} + 401t^{10} + 30t^9 - 30t^8 + 150t^7$$
$$+ 75t^6 - 30t^5 + 6t^4 + 18t^3 - t^2 - 2t + 1$$

The computation of the denominator for the Poincaré series of $\bar{R}(3, k)$ can be carried out similarly. In this case the numerator of the integrand has an additional factor $\sum \frac{1}{z_i}$ which can also be written $\sum z_i^{-1}$. Again, there are three pairs of residues to consider. If we first take $z_1 = tz_2$ and then $z_2 = tz_3$ then in the terms we consider, the extra term in the numerator will become

$$(1 + t + t^2)(1 + t^{-1} + t^{-2}) = t^{-2}(1 + t + t^2)^2.$$

Hence, in the denominator the power of $(1 - t^3)$ will be $3k - 4$ instead of $3k - 2$.

For the next case we take the residue at $z_1 = tz_3$ and then at $z_2 = tz_3$. The extra term in the numerator contributes $(z_3 + 2tz_3)^2$, which has no effect on the denominator. So there are no factors of $(1 - t^3)$ and $4k - 4$ factors of $(1 - t^2)$, as before.

Finally, if we first take $z_1 = tz_3$ and then $z_2 = t^2z_3$ we again pick up two factors of $(1 + t + t^2)$ in the numerator, cancelling two in the denominator. Altogether we now have
Theorem 4.6. The Poincaré series of $\bar{R}(3, k)$ is a rational function with denominator $(1 - t)^{2k}(1 - t^2)^{4k-4}(1 - t^3)^{3k-4}$ and numerator a symmetric polynomial of degree $10k - 20$.

5 A Conjecture

In this section we present a conjecture for the denominator for $\bar{C}(n, k)$ and $\bar{R}(n, k)$. The computations are based on the following function:

$$F(t) = \prod \{1 - t^{i-j}|1 \leq i, j \leq n, i \neq j\}$$

$$= \prod \{(1 - t^{i-j+1})^k|1 \leq i, j \leq n, i \neq j - 1\}$$

We leave the proof of the following to the interested reader.

Lemma 5.1. $F(t)$ in the above equation equals $\prod_{i=1}^{n} (1-t^i)^{-\alpha(i)}$, where

$$\alpha(i) = \begin{cases} 2(k-1)(n-i), & 1 \leq i \leq n-1 \\ k, & i = n \end{cases}$$

Here then is our conjecture for the denominator of $\bar{C}(k, n)$:

Conjecture 1. $\bar{C}(n, k)$ is a rational function with least denominator the same as the denominator of the $(n-1)(k-1)$-st derivative of $F(t)$.

This denominator can be expressed more explicitly in two different ways. One is that it is the least common multiple of $\{(1 - t^i)^{\alpha(i)+(n-1)(k-1)}\}$. For the other, let $\phi_i(t)$ be the $i$-th cyclotomic polynomial, i.e., the minimal polynomial of a primitive $i$-th root of 1. It is known that $1 - t^i$ is the product of the $\phi_j(t)$ for $j|i$, and so the denominator of $F(t)$ is a product of the $\phi_i(t)$, $i \leq n$, to various powers. Then it follows from the product rule that the conjectured denominator for $\bar{C}(n, k)$ will be the denominator of $F(t)$ times $(\phi_1(t) \cdots \phi_n(t))^{(n-1)(k-1)}$.

We stumbled across this denominator when attempting the substitutions $z_1 = t z_2, z_2 = t z_3, \ldots, z_{n-1} = t z_n$ in (3), but the real evidence in its favor is that it agrees with the known denominators for $n = 2$ and $n = 3$ computed here, and for $n = 4, 5, 6$ and $k = 2$ computed in [11].

Let us check the $n = 3$ case. The rational function $F(t)$ would be the reciprocal of

$$(1 - t)^4(1 - t^2)^2(1 - t^3)^k.$$
orders of $9k - 8, 4k - 4,$ and $3k - 2,$ respectively. This is in agreement with Theorem 4.4.

We record the denominators for $\overline{C}(n, 2)$ for $n = 4, 5, 6$ computed by Dokovic in [11], and which he denoted $D(C_n, 2; t),$ should the reader wish to check them.

$$D(C_4, 2; t) = (1 - t)^3(1 - t^2)^4(1 - t^3)^5(1 - t^4)^5$$
$$D(C_5, 2; t) = (1 - t)^2(1 - t^2)^6(1 - t^3)^8(1 - t^4)^6(1 - t^5)^6$$
$$D(C_6, 2; t) = (1 - t)^5(1 - t^2)^3(1 - t^3)^6(1 - t^4)^9(1 - t^5)^7(1 - t^6)^7$$

The case of $\overline{R}(n, k)$ is similar. The definition of $F(t)$ from (17) gains a factor of $\sum z_i = \sum z_i \sum z_j$ in the numerator, which, up to a power of $z,$ is $(\sum z_i)^2$ suggesting

$$G(t) = \frac{\left(\sum_{i,j=1}^{n} t^{i-j}\right)^2 \prod \{1 - \ell t^{i-j} | 1 \leq i, j \leq n, \ i \neq j\}}{\prod \{1 - \ell t^{i-j+1} | 1 \leq i, j \leq n, \ i \neq j - 1\}}$$

(18)

The extra factor in the numerator equals $(1 + t + \cdots + t^{n-1})^2.$ Hence, if $G(t) = \prod (1 - t^i)^{-\beta(i)}$, then $G(t)$ has two fewer factors of $(1 - t^n)$ and two more factors of $(1 - t)$ than $F(t)$, namely

$$\beta(i) = \begin{cases} (2(k - 1)(n - 1) + 2), & i = 1 \\ 2(k - 1)(n - i), & 2 \leq i \leq n - 1 \\ k - 2, & i = n \end{cases}$$

Here is our conjecture for the denominator of $\overline{R}(n, k)$.

**Conjecture 2.** $\overline{R}(n, k)$ is a rational function with least denominator the same as the denominator of the $(n-1)/(k-1)$-st derivative of $G(t)$.

Just as in the previous case the conjecture holds for all known cases. And, just as in the previous case, this denominator can be computed either as the least common multiple of $\{(1 - t)^{\beta(i) + (n-1)/(k-1)}\}$ or as the denominator of $G(t)$ times $(\phi(1) \cdots \phi(n))^{(n-1)/(k-1)}.$ Here are the denominators computed by Dokovic in [11] for $n = 4, 5, 6$.

$$D(T_4, 2; t) = (1 - t)^5(1 - t^2)^4(1 - t^3)^5(1 - t^4)^3$$
$$D(T_5, 2; t) = (1 - t)^2(1 - t^2)^6(1 - t^3)^8(1 - t^4)^6(1 - t^5)^6$$
$$D(T_6, 2; t) = (1 - t)^7(1 - t^2)^3(1 - t^3)^6(1 - t^4)^9(1 - t^5)^7(1 - t^6)^5$$

If these conjectures are true they have an application to the growth functions of $C(n, k)$ and $\overline{R}(n, k).$ In order to apply them need this lemma.

**Lemma 5.2.** Let $f(x)$ and $g(x)$ be monic polynomials with no factors of $(1 - x),$ let

$$\frac{f(x)}{(1 - x)^a g(x)} = \sum a_n x^n,$$
and assume that all of the poles of \( g(x) \) are on the unit circle and are of order less than \( d \). Then \( a_n \) is asymptotic to \( \frac{f(1)}{g(1)} (\frac{n+1}{n-1})^d \).

**Proof.** By partial fractions

\[
\frac{f(x)}{(1-x)^d g(x)} = \frac{A}{(1-x)^d} + \sum B_{\omega,i} \frac{1}{(1-\omega x)^i}
\]

where each \( \omega \) is a root of 1 and where each \( i < d \). Multiplying both sides by \( (1-x)^d \) yields

\[
\frac{f(x)}{g(x)} = A + \text{terms with factors of } (1-x)
\]

and setting \( x = 1 \) gives \( A = f(1)/g(1) \). The Taylor series of \((1-x)^{-d}\) is \( \sum (\frac{n+d-1}{d-1}) x^d \) and the Taylor series of each \((1-\omega x)^{-i}\) has coefficients polynomial of degree \( i - 1 \) which is less than \( d-1 \).

The algebra \( \tilde{C}(n,k) \) is graded by degree. If we let \( \tilde{c}_i \) be the dimension of the degree \( i \) part then \( P(\tilde{C}(n,k)) = \sum \tilde{c}_i t^i \), by definition of Poincaré series. It follows from Lemma 5.2 that \( \tilde{c}_i \) is asymptotic to a rational number times \( i \) to the power of \( (k-1)n^2 \). Moreover, since the generating function for \( \sum j \tilde{c}_j \) is \( P(\tilde{C}(n,k))(1-t)^{-1} \), the sums \( \sum \tilde{c}_j \) are asymptotic to a rational number times \( i \) to the power of \((k-1)n^2 + 1 \). If Conjecture 1 is true, then we can identify a denominator for those rational numbers.

**Theorem 5.3.** If Conjecture 1 is true then \( \tilde{c}_i \) is asymptotic to a rational number times \( i \) to the power of \((k-1)n^2 \), and the rational number can be written with denominator

\[
[n^2(k-1)]! \prod_{j=1}^n j^{a(j)} (\phi_1 \cdots \phi_n)^{(k-1)(n-1)},
\]

moreover \( \sum_{m=1}^i \tilde{c}_m \) is asymptotic to a rational number times \( i \) to the power of \((k-1)n^2 + 1 \), and the rational number can be written with denominator

\[
[n^2(k-1) + 1]! \prod_{j=1}^n j^{a(j)} (\phi_1 \cdots \phi_n)^{(k-1)(n-1)}
\]

Likewise, if we let \( \bar{r}_i \) be the dimension of the degree \( i \) part of \( \bar{R}(n,k) \) then we have this theorem.
Theorem 5.4. If the Conjecture 2 is true then $\bar{r}_i$ is asymptotic to a rational number times $i$ to the power of $(k-1)n^2$, and the rational number can be written with denominator

$$[n^2(k-1)]! \prod_{j=1}^{n} j^{\beta(j)}(\phi_1 \cdots \phi_n)^{(k-1)(n-1)},$$

moreover $\sum_{m=1}^{i} \bar{r}_m$ is asymptotic to a rational number times $i$ to the power of $(k-1)n^2 + 1$, and the rational number can be written with denominator

$$[n^2(k-1) + 1]! \prod_{j=1}^{n} j^{\beta(j)}(\phi_1 \cdots \phi_n)^{(k-1)(n-1)}.$$

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