On resolvent approximations of elliptic differential operators with periodic coefficients

S. E. Pastukhova

Department of Mathematics, MIREA – Russian Technological University, Moscow, Russia

ABSTRACT
We consider resolvents \((A_\varepsilon + 1)^{-1}\) of elliptic second-order differential operators \(A_\varepsilon = -\text{div} a(x/\varepsilon) \nabla\) in \(\mathbb{R}^d\) with \(\varepsilon\)-periodic measurable matrix \(a(x/\varepsilon)\) and study the asymptotic behaviour of \((A_\varepsilon + 1)^{-1}\), as the period \(\varepsilon\) goes to zero. We provide a construction for the leading terms of the ‘operator asymptotics’ of \((A_\varepsilon + 1)^{-1}\) in the sense of \(L^2\)-operator-norm convergence and prove order \(\varepsilon^2\) remainder estimates. We apply the modified method of the first approximation with the usage of Steklov’s smoothing. The class of operators covered by our analysis includes uniformly elliptic families with bounded coefficients and also with unbounded coefficients from the John–Nirenberg space \(BMO\) (bounded mean oscillation).

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1. Introduction

1.1. About the topic

This paper relates to homogenization theory which studies heterogeneous media such as small-period composites or porous media in the limit of small period (for introduction to this theory see, for example, books [1–4]). More precisely, the paper relates to the rather new branch of homogenization theory connected with operator-type estimates for the error of homogenization. Among the pioneer publications on operator-type estimates in homogenization of elliptic equations, we mention [5–8], where a number of results have been established concerning the difference, in the operator \(L^2\)-norm, between the resolvent of the elliptic differential operator representing the original heterogeneous medium depending on the small parameter \(\varepsilon\), that is

\[A_\varepsilon = -\text{div} a(x/\varepsilon) \nabla,\]

and the resolvent of the operator

\[A_0 = -\text{div} \ a^0 \nabla\]

representing the limiting (or ‘effective’) medium, as \(\varepsilon \to 0\). Here the matrix function \(a\) is \([-1/2, 1/2]^d\)-periodic, symmetric, measurable, bounded and uniformly positive definite; the constant matrix \(a^0\) is of the same class, and it is found according to a well-known procedure. To study the difference between the resolvents \((A_\varepsilon + 1)^{-1}\) and \((A_0 + 1)^{-1}\) for the above-given operators acting in the space \(L^2(\mathbb{R}^d)\) means, in other words, to study the \(L^2\)-norm of the difference between the solutions to the elliptic problems

\[u^\varepsilon \in H^1(\mathbb{R}^d), \quad A_\varepsilon u^\varepsilon + u^\varepsilon = f, \quad f \in L^2(\mathbb{R}^d),\]
\[ u \in H^1(\mathbb{R}^d), \quad A_0 u + u = f, \quad f \in L^2(\mathbb{R}^d). \]  

In \([5,6]\), the uniform resolvent convergence of \( A_\varepsilon \) to \( A_0 \) in \( L^2(\mathbb{R}^d) \) was maintained together with the rate of this convergence of order \( \varepsilon \). Thus, the resolvent \((A_0+1)^{-1}\) of the homogenized operator turns to be a good approximation for the resolvent \((A_\varepsilon+1)^{-1}\) of the original operator in \( L^2 \)-operator norm with remainder term of order \( \varepsilon \). Naturally, the question arises about similar approximations of \((A_\varepsilon+1)^{-1}\) with remainder term of the next order, i.e. \( \varepsilon^2 \). More exactly, the question is what a correcting term of the form \( \varepsilon C_\varepsilon \) should be added to \((A_0+1)^{-1}\) in order to attain the sharpness of order \( \varepsilon^2 \) for the approximation \((A_0+1)^{-1} + \varepsilon C_\varepsilon \) of \((A_\varepsilon+1)^{-1}\). The answer on this question is also known, thanks to \([7]\) and \([8]\). The authors of both papers have found such type approximations in the framework of more general setups (including the case of systems of elliptic equations \([8]\) or the case of equations in \( L^2 \)-spaces with general Borel measures \([7]\)), acting by spectral method based on the Floquet–Bloch decomposition of the selfadjoint operator \( A_\varepsilon \). Note that this approach is rather restrictive, for it is closely linked with periodic problems since the Floquet–Bloch transformation works well exclusively in the case of operators with periodic coefficients. But homogenization theory is not limited to periodic setups.

As in \([7,8]\), we analyse here the asymptotic behaviour of the resolvent \((A_\varepsilon+1)^{-1}\) with the sharpness of order \( \varepsilon^2 \) in \( L^2 \)-operator norm, but under more general conditions and by another method. First, we allow the operator \( A_\varepsilon \) to be nonselfadjoint with the matrix \( a \) not necessarily symmetric which entails more complicated structure of the correcting term \( \varepsilon C_\varepsilon \) as compared with \([7]\) and \([8]\). Second, we relax the boundedness requirement in ellipticity condition on the matrix \( a \) so that the approximation result remains the same though additional arguments are needed for justification. More precisely, the skew-symmetric part of the diffusion matrix \( a \) is allowed to be unbounded from the John–Nirenberg space \( BMO \) (bounded mean oscillation).

### 1.2. About the method

The present paper can be viewed as following in the footsteps of \([6]\) in that it relies upon the so-called modified method of the first approximation with the usage of the shift parameter (that is why it is called often shortly as the shift method).

This method was proposed by Zhikov \([6]\) as an alternative along with the spectral approach, used in \([5,7,8]\), to prove operator-type homogenization estimates; it turned to be universal in different setups: periodic, locally periodic, quasiperiodic or multiscale. The method has been developed since 2005 in applications to various problems (we refer, e.g. to \([9–25]\) and, in particular, to the overview \([23]\) where other references are given). There have appeared two versions of the method. In the original version, the pure shift is done in the coefficients of the operator \( A_\varepsilon \), this creates a family of perturbed operators with a shift parameter \( \omega \). The subsequent averaging in \( \omega \) gives an additional variable of integration that allows to circumvent the lack of the regularity for the data in Equation \((1)\). In another version, the Steklov smoothing operator (containing the shift implicitly as any other smoothing operator defined by means of convolution) is embedded from the very beginning in the approximation sought. We use here the second version of the shift method.

Since 2005, when \([6–8]\) came up, it has been the challenge to obtain operator-norm resolvent-type homogenization estimates of order \( \varepsilon^2 \) from the point of view close to the classical homogenization theory. We recall that the error of homogenization for Equation \((1)\) is traditionally evaluated by means of direct constructing approximations to the solution \( u^\varepsilon \) via two-scale expansions

\[ u^\varepsilon(x) \approx u^0(x,y) + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + \ldots, \quad y = x/\varepsilon, \]  

with functions \( u^0(x,y), u^1(x,y), \ldots \) periodic in \( y \). A regular way of finding such functions is known. For example, one should take the sum of three terms of the above two-scale expansion and try to
enable
\[(A_\varepsilon + 1)(u^0 + \varepsilon u^1 + \varepsilon^2 u^2) - f = O(\varepsilon).\] (4)

It is quickly seen that \(u^0(x, y) = u(x)\) is independent of \(y\) and turns to be a solution to (2). As for the next terms in the two-scale expansion, they are
\[u^1(x, y) = N^\varepsilon(y)Dju(x), \quad u^2(x, y) = N^{\varepsilon^2}(y)D_jD_lu(x),\]
(summation over repeated indices is assumed from 1 to \(d\)). Here \(D_j = \frac{\partial}{\partial x_j}, j = 1, \ldots, d;\)

\(N^\varepsilon\) is the solution of the periodic problem on the cell \([-1/2, 1/2]^d\) (see (16)); \(N^{\varepsilon^2}\) is the solution of another periodic problem (see it, e.g. in Chapter IV of [3]).

The sum of the first two terms in the above expansion \(u^\varepsilon_1(x) = u(x) + \varepsilon N^\varepsilon(x/\varepsilon)Dj\mu(x)\) is called the first approximation, \(u(x)\) is the zero approximation, and \(\varepsilon N^\varepsilon(x/\varepsilon)Dj\mu(x)\) is a corrector.

All the conclusions derived here about the two-scale expansion (3) are valid assuming that the matrix \(a\) and the right-hand side (RHS) function \(f\) are sufficiently regular. In our case with minimal regularity conditions, even the existence of \(u^\varepsilon_1\) as an element of the space \(H^1(\mathbb{R}^d)\) is under the question and so is the possibility of inserting it into the original equation, as in (4). Estimates
\[
\begin{align*}
\|u^\varepsilon - u\|_{L^2} &\leq C\varepsilon, \\
\|u^\varepsilon - u^\varepsilon_1\|_{H^1} &\leq C\varepsilon
\end{align*}
\]
for the difference of the solution \(u^\varepsilon\) and its zero and first approximations were obtained long ago. However, the constant \(C\) in such estimates appeared to depend on the zero approximation \(u\) being necessarily sufficiently smooth, which was enabled under relevant conditions on the RHS function \(f\). In this case, the estimates have no operator interpretation in terms of resolvents.

Traditionally (see, e.g. Chapter IV in [3]), the \(H^1\)-estimate (5) was derived at the first step from (4) by using the energy estimate \(\|v\|_{H^1} \leq c\|v\|_{H^1}\) with \(c = \text{const}(\lambda)\); then the \(L^2\)-estimate (5) was deduced from the \(H^1\)-estimate as a corollary.

Thus, if the aim is to prove the estimates (5)

(i) under minimal regularity assumptions,
(ii) with majorants that admit operator formulation,
(iii) following more or less standard way of two-scale expansions,

one should sufficiently modify the method. This was done in [6] and [9] where two versions of the modified method of the first approximation were exposed for the first time. The results obtained to date by means of this method are discussed in Section 2. In the present paper, we are aimed to obtain \(\varepsilon^2\) order \(L^2\)-approximations for \(u^\varepsilon\), complying with the demands (i)-(iii), and do this by using the approach of [9].

Now, briefly, about the structure of the paper. The main results are formulated in Theorems 3.1, 3.2 and 6.3. Their proofs are given in Sections 5 and 6. Sections 1–3 are introductory. Sections 4 and 7 are devoted to the Steklov smoothing operator which plays the key role in our approach.

2. \(L^2\)- and \(H^1\)-estimates of order \(\varepsilon\)

2.1. \(L^2\)-estimates for the error of homogenization

In the whole space \(\mathbb{R}^d, d \geq 2\), consider a divergent-type second-order elliptic equation
\[
\begin{align*}
u^\varepsilon \in H^1(\mathbb{R}^d), \quad A_\varepsilon u^\varepsilon + u^\varepsilon = f, & \quad f \in L^2(\mathbb{R}^d), \\
A_\varepsilon = -\text{div} \ a_\varepsilon(x) \nabla, & \quad a_\varepsilon(x) = a(\varepsilon^{-1}x),
\end{align*}
\]
with a small parameter \(\varepsilon \in (0, 1)\). Coefficients of the equation are \(\varepsilon\)-periodic and, thus, are rapidly oscillating as \(\varepsilon \to 0\). Here \(a(x) = \{a_{jk}(x)\}_{j,k=1}^d\) is a measurable 1-periodic matrix with real entries. The
periodicity cell is the unit cube \( \Box = [-\frac{1}{2}, \frac{1}{2}]^d \). We suppose that
\[
\lambda |\xi|^2 \leq a \xi \cdot \xi, \quad a \xi \cdot \eta \leq \lambda^{-1} |\xi| |\eta| \quad \forall \xi, \eta \in \mathbb{R}^d
\]
for some \( \lambda > 0 \). The matrix \( a \) is not necessarily symmetric.

We associate with (6) the homogenized equation
\[
\begin{align*}
u & \in H^1(\mathbb{R}^d), \quad A_0 u + u = f, \\
A_0 & = -\text{div} a^0 \nabla,
\end{align*}
\]
where \( a^0 \) is a constant matrix of the same class (7); \( a^0 \) is calculated according to the well known procedure in terms of solutions to auxiliary periodic problems (see below (15), (16)).

Solutions to (6) and (8) are understood in the sense of distributions in \( \mathbb{R}^d \). For example, the following integral identity holds
\[
\int_{\mathbb{R}^d} (A(x) \nabla u^\varepsilon \cdot \nabla \varphi + u^\varepsilon \varphi) \, dx = \int_{\mathbb{R}^d} f \varphi \, dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).
\]

By the closure, the test functions here can be taken from the space \( H^1(\mathbb{R}^d) \). In particular, inserting \( \varphi = u^\varepsilon \) in the integral identity yields the energy inequality
\[
\lambda \| \nabla u^\varepsilon \|^2 + \| u^\varepsilon \|^2 \leq (a \nabla u^\varepsilon, \nabla u^\varepsilon) + (u^\varepsilon, u^\varepsilon) = (f, u^\varepsilon) \leq \| f \| \| u^\varepsilon \|, \quad \lambda \| \nabla u^\varepsilon \|^2 \leq \| f \|^2.
\]

Here and in what follows, we use the notation for the inner product and the norm in \( L^2(\mathbb{R}^d) \)
\[
(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\mathbb{R}^d)}, \quad \| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^d)},
\]
making no difference between the spaces of scalar or vector functions.

The unique solvability of the equations is established by the Lax–Milgram lemma. This fact is true for more general RHS functions, namely, for \( f \in H^{-1}(\mathbb{R}^d) \) (where \( H^{-1}(\mathbb{R}^d) \) is the dual of \( H^1(\mathbb{R}^d) \)), so that the resolvent \( (A_0 + 1)^{-1} : H^{-1}(\mathbb{R}^d) \to H^1(\mathbb{R}^d) \) is a bounded operator. The same is valid for the resolvent \( (A_0 + 1)^{-1} \); but if its action is restricted on the space \( L^2(\mathbb{R}^d) \), the property \( (A_0 + 1)^{-1} : L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d) \) is gained. In other words, the elliptic estimate holds for the solution to the homogenized equation:
\[
\| u \|_{H^2(\mathbb{R}^d)} \leq c \| f \|, \quad c = \text{const}(\lambda),
\]
which can be easily established by means of the Fourier transform because the matrix \( a^0 \) is constant and positive definite.

The homogenization result for (6) is known from long ago and can be formulated, for example, as \( G \)-convergence of operators \( A_\varepsilon \) to \( A_0 \) (see [26] and references therein) which means that
\[
\lim_{\varepsilon \to 0} \langle h, (A_\varepsilon + 1)^{-1} f \rangle = \langle h, (A_0 + 1)^{-1} f \rangle
\]
for any \( f, h \in H^{-1}(\mathbb{R}^d) \). Here the value of a functional \( h \in H^{-1}(\mathbb{R}^d) \) at a \( \nu \in H^1(\mathbb{R}^d) \) is denoted by \( \langle h, \nu \rangle \). In other words, (12) means that, for an arbitrary function \( f \in H^{-1}(\mathbb{R}^d) \), the solutions of Equations (6) and (8) are connected with the the weak convergence in \( H^1(\mathbb{R}^d) \), consequently, in \( L^2(\mathbb{R}^d) \) either. From here, by the energy method and lower semicontinuity arguments, one can derive the
strong convergence $u_\varepsilon \to u$ in $L^2(\mathbb{R}^d)$ which means in operator terms the strong resolvent convergence $(A_\varepsilon + 1)^{-1} \to (A_0 + 1)^{-1}$ in $L^2(\mathbb{R}^d)$. This operator convergence can be further strengthened up to the uniform resolvent convergence with following rate of convergence

$$\| (A_\varepsilon + 1)^{-1} - (A_0 + 1)^{-1} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq c\varepsilon, \quad c = \text{const}(d, \lambda).$$

One can rewrite (13) in terms of the solutions to (6) and (8) as follows

$$\| u_\varepsilon - u \| \leq c\varepsilon \| f \|$$

with the same RHS constant $c$ depending only on the dimension $d$ and the ellipticity constant $\lambda$ from (7). To prove the estimate (13) in the self-adjoint case the authors of [5] used the spectral approach based on operator-theoretic arguments tightly linked to the self-adjoint situation. Quite different method to prove (13) was proposed, first, in [6] and then developed in [9]. This is the modified method of the first approximation with the usage of shift or smoothing operators. From the very beginning of the appearance, this method turned out to be universal for studying various homogenization problems which admit nonselfadjointness, nonlinearity, divergence-form and nondivergence-form equations, different types of degeneracy, high order or vector equations, and others (see, e.g., [6–25] and also references in the overview [23]).

### 2.2. Homogenization attributes

Consider the periodic problem on the unit cube $\square = [-\frac{1}{2}, \frac{1}{2}]^d$

$$N^j \in H^1_{\text{per}}(\square), \quad \text{div}_y a(y)(e^j + \nabla \phi N^j) = 0, \quad (N^j) = 0, \quad j = 1, \ldots, d,$$

where $e^1, \ldots, e^d$ is a canonical basis in $\mathbb{R}^d$, $H^1_{\text{per}}(\square)$ is the Sobolev space of 1-periodic functions,

$$\langle \cdot \rangle = \int_\square \cdot \, dy.$$

Then the homogenized matrix $a^0$ is defined by equalities

$$a^0 e^j = \langle a(e^j + \nabla \phi N^j) \rangle, \quad j = 1, \ldots, d.$$  \hspace{1cm} (16)

A solution to the problem (15) is understood in the sense of the integral identity for smooth periodic functions

$$\langle a(e^j + \nabla \phi N^j) \cdot \nabla \varphi \rangle = 0, \quad \varphi \in C_0^\infty_{\text{per}}(\square),$$

which can be extended by closure to test functions in $H^1_{\text{per}}(\square)$. On the other hand, (15) can be regarded in the sense of distributions in $\mathbb{R}^d$, that is a well-known fact in homogenization theory. Thus, together with (17) the integral identity is satisfied with test functions in $C_0^\infty(\mathbb{R}^d)$.

Introduce the 1-periodic vector

$$g^j(y) := a(y) \left( \nabla N^j(y) + e^j \right) - a^0 e^j, \quad j = 1, \ldots, d,$$

which is solenoidal and has zero mean value, i.e.

$$\text{div} g^j(y) = 0, \quad \langle g^j \rangle = 0,$$

by (17) and (16), respectively. The first property (19) may be understood in both ways: in the sense of the integral identity of the type (17) or in the sense of distributions in $\mathbb{R}^d$. 


Let \( A^*_e \) be the adjoint of \( A_e \) and consider the problem

\[
\begin{align*}
\nu^e & \in H^1(\mathbb{R}^d), \quad A^*_e \nu^e + \nu^e = h, \quad h \in L^2(\mathbb{R}^d), \\
A^*_e & = -\text{div} \, a^*_e(x) \nabla, \quad a^*_e(x) = a^*(\varepsilon^{-1} x),
\end{align*}
\]

where \( a^* \) is the transpose of \( a \). It is known that the homogenized equation for (20) is of the form

\[
v \in H^1(\mathbb{R}^d), \quad A^*_0 v + v = -\text{div} \, (a^0)^* \nabla v + v = h,
\]

where \( A^*_0 \) is the adjoint of \( A_0 \) and has the matrix \((a^0)^*\) transposed to \( a^0 \). Thus,

\[
(a^*)_0 = (a^0)^*.
\]

However, introduce the counterpart of the cell problem (15)

\[
\tilde{N}^j \in H^1_{\text{per}}(\square), \quad \text{div}_y a^*(y)(\epsilon^j + \nabla \tilde{N}^j) = 0, \\
\langle \tilde{N}^j \rangle = 0, \quad j = 1, \ldots, d.
\]

Formally, the solutions to (23) generate the homogenized matrix for Equation (20) similarly, as in (16); thus, in view of (22),

\[
(a^0)^* \epsilon^j = (a^*(\epsilon^j + \nabla \tilde{N}^j)), \quad j = 1, \ldots, d.
\]

Set

\[
\tilde{g}^j(y) := a^*(y) \left( \nabla \tilde{N}^j(y) + \epsilon^j \right) - (a^0)^* \epsilon^j, \quad j = 1, \ldots, d.
\]

Then, thanks to (23) and (24), there hold the relations

\[
\text{div} \tilde{g}^j(y) = 0, \quad \langle \tilde{g}^j \rangle = 0.
\]

In what follows, we often refer to the energy and elliptic estimates relating to (20) and (21)

\[
\|\nu^e\|_{H^1(\mathbb{R}^d)} \leq c\|f\|, \quad c = \text{const}(\lambda),
\]

\[
\|v\|_{H^2(\mathbb{R}^d)} \leq c\|f\|, \quad c = \text{const}(\lambda).
\]

### 2.3. \(H^1\)-approximations in homogenization

According to (13), the resolvent \((A_0 + 1)^{-1}\) approximates \((A_e + 1)^{-1}\) in \(L^2\)-operator norm with the error of order \(\varepsilon\). If the resolvent \((A_e + 1)^{-1}\) is regarded as an operator from \(L^2(\mathbb{R}^d)\) to \(H^1(\mathbb{R}^d)\), then for its approximation we need the sum \((A_0 + 1)^{-1} + \varepsilon \mathcal{K}_e\), where \(\mathcal{K}_e\) is a correcting operator, and so

\[
\|(A_e + 1)^{-1} - (A_0 + 1)^{-1} - \varepsilon \mathcal{K}_e\|_{L^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq c\varepsilon, \quad c = \text{const}(d, \lambda).
\]

The correcting operator \(\mathcal{K}_e : L^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)\) is defined by

\[
\mathcal{K}_e f = N_e \cdot \nabla S^e u, \quad \text{where} \quad u = (A_0 + 1)^{-1} f,
\]

where \(N_e(x) = N(\varepsilon^{-1} x), N(y) = \{N^j(y)\}_{j=1}^d\) is the periodic vector composed of the solutions to (15) and \(S^e\) is the Steklov smoothing operator defined in (37). Note that

\[
\|\varepsilon \mathcal{K}_e\|_{L^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq c, \quad c = \text{const}(d, \lambda),
\]

by the properties of the smoothing operator (see Lemma 4.1) and the elliptic estimate (11).
Once the smoothing operator $S^\varepsilon$ is included in the corrector, the $H^1$-approximation is well defined and all technical difficulties to estimate its residual in the equation are easily overcome. To this end, we again essentially use the properties of the smoothing operator $S^\varepsilon$ in its interaction with $\varepsilon$-periodic factors (see Lemma 4.3). These properties were first noticed in [9,10].

In the scalar case, the correcting operator can be taken actually without smoothing. Letting
\[
K_\varepsilon f = N_\varepsilon \cdot \nabla u, \quad \text{where} \quad u = (A_0 + 1)^{-1} f,
\]
we have the operator $K_\varepsilon : L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ such that $\|\varepsilon K_\varepsilon f\|_{H^1(\mathbb{R}^d)} \leq c\|f\|$ with $c = \text{cost}(d, \lambda)$, and from (28) the following estimate is derived:
\[
\|K_\varepsilon\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C, \quad c = \text{cost}(d, \lambda).
\]

Estimates (28) and (31) were firstly proved in [11] and [9] by slightly different methods.

### 3. $L^2$-estimate of order $\varepsilon^2$

The operator $K_\varepsilon$ defined in (29) is a bounded operator in $L^2(\mathbb{R}^d)$ with the estimate for the norm $\|K_\varepsilon\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq c$, and the adjoint operator $(K_\varepsilon)^* : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is such that
\[
(K_\varepsilon)^* f := (A_0 + 1)^{-1} S^\varepsilon \text{div}(N_\varepsilon f).
\]

Suppose that the matrix $a$ is symmetric. Then the sum $\varepsilon K_\varepsilon + \varepsilon (K_\varepsilon)^*$ turns to be the true correcting operator of $(A_0 + 1)^{-1}$ in approximations with remainder of order $\varepsilon^2$ for the resolvent $(A_\varepsilon + 1)^{-1}$ in $L^2$-operator norm. The following estimate holds:
\[
\|K_\varepsilon\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\varepsilon^2, \quad c = \text{cost}(d, \lambda).
\]

In the scalar case under the assumption (7), the solution $N^j$ to the problem (15) belongs to $L^{\infty}(\square)$ in view of the generalized maximum principle; therefore, the correcting operator $K_\varepsilon$ in the estimate (32) can be replaced with the simpler one $K_\varepsilon$ defined in (30). Thus,
\[
\|K_\varepsilon\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\varepsilon^2,
\]
with the constant $C$ of the same type as in (32).

The estimate (32) was proved in [27] by using the modified method of the first approximation, and the estimate (33) was derived from (32) as a simple corollary by properties of smoothing.

We make some remarks apropos of the latest estimates.

1. Pay attention on the selfadjointness of the approximations both in (32) and (33) which is contrary to the situation in (28) and (31), where the correcting operators are not selfadjoint.

2. The estimate quite similar to (33) was proved within the framework of more general results in [7] and [8] by using the spectral approach based on the Floquet–Bloch decomposition of selfadjoint differential operators with periodic coefficients.

3. The estimate resembling (32) in structure, but with the different smoothing operator $\Pi^\varepsilon$, was obtained both in [7] and [8]. The pseudodifferential operator $\Pi^\varepsilon$ acting as
\[
\Pi^\varepsilon \varphi(x) = F^{-1} \left(1_{\{\|\xi\| \leq 1/\varepsilon\}}(F \varphi)(\xi)\right)
\]

naturally arises within the scope of the spectral method. Here $F$ denotes the Fourier transform and $F^{-1}$ is its inverse, $1_{\{\|\xi\| \leq 1/\varepsilon\}}$ is a characteristic function of the cube $\{\xi : \|\xi\| \leq 1/\varepsilon\}$. Evidently, $\Pi^\varepsilon$ has smoothing properties, though it emerges as a result of some projection.
Suppose now that the matrix $a(y)$ in (6) is not symmetric. Then the correcting operator in approximations of the resolvent $(A_\varepsilon + 1)^{-1}$ with remainder of order $\varepsilon^2$ is more complicated than in (32) and it is composed of three terms: one of them does not contain oscillating factors, and the remaining two terms are similar to those in (32).

**Theorem 3.1:** Let $N(y) = \{N^j(y)\}_{j=1}^d$, $\tilde{N}(y) = \{\tilde{N}^j(y)\}_{j=1}^d$ be the vectors composed of solutions to (15) and (23), and $S^\varepsilon$ be the Steklov smoothing operator (see (37)). Then the following estimate holds for the resolvents

$$
(A_\varepsilon + 1)^{-1} \text{ and } (A_0 + 1)^{-1} \text{ of the problems } (6) \text{ and } (8):
$$

$$
\| (A_\varepsilon + 1)^{-1} - (A_0 + 1)^{-1} - \varepsilon K_\varepsilon - \varepsilon (\tilde{K}_\varepsilon)^* - \varepsilon L \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq C\varepsilon^2,
$$

(35)

where

$$
K_\varepsilon = N \left( \frac{1}{\varepsilon} \right) \cdot S^\varepsilon \nabla(A_0 + 1)^{-1}, \quad \tilde{K}_\varepsilon = \tilde{N} \left( \frac{1}{\varepsilon} \right) \cdot S^\varepsilon \nabla(A_0^* + 1)^{-1},
$$

$$
L = (A_0 + 1)^{-1} \left( \varepsilon c_{ij}^{jk} - c_{ij}^{jk} \right) D_j D_l D_k (A_0 + 1)^{-1},
$$

(36)

the constant coefficients $c_{ij}^{jk}$, $c_{ij}^{jk}$ are defined in (66) in terms of the functions $N^j$, $\tilde{N}^j$ and its gradients; $D_j = \frac{\partial}{\partial x_j}$, the constant $C$ in (35) depends only on the dimension $d$ and the ellipticity constant $\lambda$.

If the matrix $a(y)$ is symmetric, then the approximation for $(A_\varepsilon + 1)^{-1}$ defined in (35) and (36) reduces into that of (32).

In the scalar case, the solutions $N^j$ and $\tilde{N}^j$ to the cell problems belong to $L^\infty(\Box)$ in view of the generalized maximum principle, and so the smoothing operator in (36) can be dropped.

**Theorem 3.2:** The estimate (35) remains valid if the smoothing operator $S^\varepsilon$ is omitted in correcting operators (36).

Theorems 3.1 and 3.2 are proved in Section 5. In Section 4 we introduce the Steklov smoothing operator and list its properties that are applied in our considerations. Some of these properties have not been noticed before, and so they are proved in Section 7.

**Remark 3.3:** The results similar to Theorems 3.1 and 3.2 are proved in [28] with the difference that, instead of the Steklov smoothing operator $S^\varepsilon$, the smoothing operator (34) is embedded in the correcting terms (36). The operator (34) appears in [28] just like in [7] and [8] as a byproduct of applying the Floquet–Bloch transformation with the purpose to reduce the problem in the whole space $\mathbb{R}^d$ to the problem on the cell of periodicity $\Box = [-1/2, 1/2]^d$.

**Remark 4.4:** It is worth noting that, once the estimate (28) in the operator $(L^2 \rightarrow H^1)$-norm with order $\varepsilon$ remainder is verified, the estimate of the type (35) (or, in selfadjoint setup, its variant (32) with the simpler corrector) in the operator $(L^2 \rightarrow L^2)$-norm with order $\varepsilon^2$ remainder is surely guaranteed by the method we demonstrate here. This idea is implemented in [29–31] for different situations described in the titles of the papers [29–31] in the case of second-order differential operators. As for higher-order operators considered in [21–23], the similar idea is true (see, e.g. [32,33]).

**Remark 5.5:** In the present paper, we restrict ourselves to the scalar case only for the sake of simplicity. We deal with the classical diffusion equation of the type (1) or its appropriate perturbations. Although, in this case, the maximum principle is valid, the latter is not used in our constructions and in the main proof; therefore, the result of Theorem 3.1 carries over to vector models, including, e.g. the elasticity theory system or other systems considered in [24].
4. Smoothing operator

In our method, the Steklov smoothing operator

\[ S^\varepsilon \varphi(x) = \int_{\Box} \varphi(x - \varepsilon \omega) \, d\omega, \]  

(37)

plays an important role, as it was already explained. Its simplest properties are well-known:

\[ \|S^\varepsilon \varphi\| \leq \|\varphi\|, \]  

(38)

\[ \|S^\varepsilon \varphi - \varphi\| \leq (\sqrt{d}/2)\varepsilon \|\nabla \varphi\|, \]  

(39)

\[ \|S^\varepsilon \varphi - \varphi\|_{H^{-1}(\mathbb{R}^d)} \leq (\sqrt{d}/2)\varepsilon \|\varphi\|, \]  

(40)

where the notation (10) is used.

To supplement (38) note that, \( S^\varepsilon \) is a selfadjoint operator in \( L^2(\mathbb{R}^d) \). We also mention the obvious property \( S^\varepsilon (\nabla \varphi) = \nabla (S^\varepsilon \varphi) \). Thereby, \( S^\varepsilon \) and any differential operator with constant coefficients commute with one another. As a corollary, \( S^\varepsilon \) commutes with the resolvent \((A_0 + 1)^{-1}\) either.

The operator \( S^\varepsilon \) displays also the following properties in interaction with \( \varepsilon \)-periodic factors.

**Lemma 4.1:** If \( \varphi \in L^2(\mathbb{R}^d) \), \( b \in L^2_{\text{per}}(\Box) \), \( b_\varepsilon(x) = b(\varepsilon^{-1}x) \), then \( b_\varepsilon S^\varepsilon \varphi \in L^2(\mathbb{R}^d) \) and

\[ \|b_\varepsilon S^\varepsilon \varphi\|^2 \leq \langle b^2 \rangle \|\varphi\|^2. \]  

(41)

**Lemma 4.2:** If \( b \in L^2_{\text{per}}(\Box) \), \( \langle b \rangle = 0 \), \( b_\varepsilon(x) = b(\varepsilon^{-1}x) \), \( \varphi \in L^2(\mathbb{R}^d) \), \( \psi \in H^1(\mathbb{R}^d) \), then

\[ (b_\varepsilon S^\varepsilon \varphi, \psi) \leq C \varepsilon \langle b^2 \rangle^{1/2} \|\psi\| \|\nabla \psi\|, \quad C = \text{const}(d). \]  

(42)

The properties (41), (42) were highlighted and proved in \([9,10]\) (see also \([23]\)).

We formulate the assertions of Lemmas 4.1 and 4.2 in the operator form.

**Lemma 4.3:** Under the conditions of Lemma 4.1, the norms of the operators \( b_\varepsilon S^\varepsilon : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) are uniformly bounded:

\[ \|b_\varepsilon S^\varepsilon\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \langle b^2 \rangle^{1/2}. \]

Furthermore, if \( \langle b \rangle = 0 \), then \( b_\varepsilon S^\varepsilon : L^2(\mathbb{R}^d) \to H^{-1}(\mathbb{R}^d) \) and

\[ \|b_\varepsilon S^\varepsilon\|_{L^2(\mathbb{R}^d) \to H^{-1}(\mathbb{R}^d)} \leq C \varepsilon \langle b^2 \rangle^{1/2}, \quad C = \text{const}(d). \]

The estimates (39) and (42) can be specified under assumptions of higher regularity. For example, if \( \varphi \in H^2(\mathbb{R}^d) \), then

\[ \|S^\varepsilon \varphi - \varphi\| \leq C \varepsilon^2 \|\nabla^2 \varphi\|, \quad C = \text{const}(d). \]  

(43)

Indeed, we write the equality

\[ \varphi(x + h) - \varphi(x) - \nabla \varphi(x) \cdot h = \int_0^1 (1 - t) \nabla (\varphi(x + th) - \varphi(x)) \cdot h \, dt \]

and, setting \( h = -\varepsilon \omega \), integrate it over \( \omega \in \Box = [-\frac{1}{2}, \frac{1}{2}]^d \). As a result, we arrive at the integral representation for the difference \( S^\varepsilon \varphi - \varphi \) in terms of the second-order gradient \( \nabla^2 \varphi \). Consequently,

\[ |S^\varepsilon \varphi(x) - \varphi(x)| \leq \varepsilon^2 \int_{\Box} \int_0^1 |\nabla (\varphi(x - t\varepsilon \omega) - \varphi(x)) \cdot \omega| \, dt \, d\omega, \]

which implies (43) by the Hölder inequality.
As for Lemma 4.2, the following assertions are its extensions.

**Lemma 4.4:** Let \( b \in L^2_{\text{per}}(\mathbb{R}^d), \langle b \rangle = 0, b_\varepsilon(x) = b(x/\varepsilon) \) and \( \varphi, \psi \in H^1(\mathbb{R}^d) \). Then

\[
(b, S^\varepsilon \varphi, S^\varepsilon \psi) \leq C \varepsilon^2 (b^2)^{1/2} \| \nabla \varphi \| \| \nabla \psi \|, \quad C = \text{const}(d).
\]  

(44)

**Lemma 4.5:** Let \( \alpha, \beta \in L^2_{\text{per}}(\mathbb{R}^d), \langle \alpha \beta \rangle = 0, \alpha_\varepsilon(x) = \alpha(x/\varepsilon), \beta_\varepsilon(x) = \beta(x/\varepsilon) \) and \( \varphi, \psi \in H^1(\mathbb{R}^d) \). Then

\[
(\alpha_\varepsilon S^\varepsilon \varphi, \beta_\varepsilon S^\varepsilon \psi) \leq C \varepsilon^2 (\alpha^2)^{1/2} (\beta^2)^{1/2} \| \nabla \varphi \| \| \nabla \psi \|, \quad C = \text{const}(d).
\]  

(45)

Note that the form \((\alpha_\varepsilon S^\varepsilon \varphi, \beta_\varepsilon S^\varepsilon \psi)\) in (45) is well defined since both functions \(\alpha_\varepsilon S^\varepsilon \varphi\) and \(\beta_\varepsilon S^\varepsilon \psi\) belong to \(L^2(\mathbb{R}^d)\), by Lemma 4.1.

**Lemma 4.6:** Let \( \alpha, \beta \in L^2_{\text{per}}(\mathbb{R}^d), \alpha_\varepsilon(x) = \alpha(x/\varepsilon), \beta_\varepsilon(x) = \beta(x/\varepsilon), \varphi \in L^2(\mathbb{R}^d), \psi \in H^1(\mathbb{R}^d) \). Then

\[
|\langle \alpha_\varepsilon S^\varepsilon \varphi, \beta_\varepsilon S^\varepsilon \psi \rangle - \langle \alpha \beta \rangle (\varphi, \psi) | \leq C \varepsilon (\alpha^2)^{1/2} (\beta^2)^{1/2} \| \nabla \varphi \| \| \nabla \psi \|, \quad C = \text{const}(d).
\]  

(46)

The proof of the last three lemmas is given in Section 7.

**5. Proof of the main results**

In this section we prove Theorems 3.1 and 3.2.

**5.1. \( H^1 \)-estimates**

In what follows, we use the notation

\[
u^\varepsilon(x) := S^\varepsilon u(x), \quad N^\varepsilon(x) := N \left( \frac{x}{\varepsilon} \right), \quad U^\varepsilon(x) := N^\varepsilon(x) \cdot \nabla u^\varepsilon(x).
\]

(47)

Then the following estimates hold:

\[
\| u^\varepsilon - u^\varepsilon - \varepsilon U^\varepsilon \|_{H^1(\mathbb{R}^d)} \leq c \varepsilon \| f \|_{L^2(\mathbb{R}^d)}, \quad c = \text{cost}(d, \lambda),
\]

(48)

\[
\| u^\varepsilon - u - \varepsilon U^\varepsilon \|_{H^1(\mathbb{R}^d)} \leq c \varepsilon \| f \|_{L^2(\mathbb{R}^d)}, \quad c = \text{cost}(d, \lambda).
\]

(49)

The latter one is, clearly, equivalent to (28).

We give here the proof of (48) (the estimate (49) is its immediate corollary due to the property (40) of the operator \( S^\varepsilon \) and the elliptic estimate (11)). Further, we systematically use the estimate (48) itself and different elements in its proof either.

We begin with necessary calculations:

\[
\nabla (u^\varepsilon + \varepsilon U^\varepsilon) = \nabla (u^\varepsilon + \varepsilon N^\varepsilon \cdot \nabla u^\varepsilon) = \left( \nabla N^\varepsilon_j + \partial_j \right) \frac{\partial u^\varepsilon}{\partial x_j} + \varepsilon N^\varepsilon_j \nabla \frac{\partial u^\varepsilon}{\partial x_j} + \varepsilon a^\varepsilon \nabla \frac{\partial u^\varepsilon}{\partial x_j},
\]

(50)

\[
a^\varepsilon \nabla (u^\varepsilon + \varepsilon U^\varepsilon) - d^\varepsilon \nabla u^\varepsilon = \frac{g^\varepsilon}{\varepsilon} \frac{\partial u^\varepsilon}{\partial x_j} + \varepsilon a^\varepsilon \nabla \frac{\partial u^\varepsilon}{\partial x_j}
\]

(recall that summation over repeated indices is assumed from 1 to \(d\)), where

\[
\nabla N^\varepsilon_j(x) := (\nabla_j N^\varepsilon) \left( \frac{x}{\varepsilon} \right), \quad \frac{g^\varepsilon}{\varepsilon}(x) := \frac{g^\varepsilon}{\varepsilon} \left( \frac{x}{\varepsilon} \right),
\]

(50)
and the 1-periodic vector $g^j(y)$ is defined in (18). According to (19), $g^j$ is solenoidal and has zero mean value. From (50), we derive

$$A_0u^\varepsilon - A_\varepsilon(u^\varepsilon + \varepsilon U^\varepsilon) = \text{div} \left( a_\varepsilon \nabla(u^\varepsilon + \varepsilon U^\varepsilon) - a^0 \nabla u^\varepsilon \right) = r^\varepsilon + \text{div} R^\varepsilon,$$

$$r^\varepsilon = g^j \cdot \nabla \frac{\partial u^\varepsilon}{\partial x_j}, \quad R^\varepsilon = \varepsilon a_\varepsilon N^j \nabla \frac{\partial u^\varepsilon}{\partial x_j},$$

which enables us to estimate the discrepancy of the approximation $u^\varepsilon + \varepsilon U^\varepsilon$ to Equation (6). Namely,

$$(A_\varepsilon + 1)(u^\varepsilon - u^\varepsilon - \varepsilon U^\varepsilon) = (A_\varepsilon + 1)u^\varepsilon - (A_\varepsilon + 1)(u^\varepsilon + \varepsilon U^\varepsilon)$$

$$= (A_0 + 1)u - (A_\varepsilon + 1)(u^\varepsilon + \varepsilon U^\varepsilon)$$

$$= (A_0 + 1)u^\varepsilon - (A_\varepsilon + 1)(u^\varepsilon + \varepsilon U^\varepsilon) + f - f^\varepsilon$$

$$= A_0u^\varepsilon - A_\varepsilon(u^\varepsilon + \varepsilon U^\varepsilon) - \varepsilon U^\varepsilon + (f - f^\varepsilon)$$

$$\equiv r^\varepsilon + \text{div} R^\varepsilon - \varepsilon U^\varepsilon + (f - f^\varepsilon) =: F^\varepsilon.$$  \hfill (51)

It is easy to show that

$$\|F^\varepsilon\|_{H^{-1}(\mathbb{R}^d)} \leq C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(d, \lambda),$$

using Lemma 4.3 and estimates (40), (11), if the structure of the functions $r^\varepsilon, R^\varepsilon, U^\varepsilon$ (see (47) and (51)) is taken into account.

To obtain (48) it remains to apply the following energy inequality $\|\varepsilon\|_{H^1(\mathbb{R}^d)} \leq c\|F^\varepsilon\|_{H^{-1}(\mathbb{R}^d)}$ with $c = \text{const}(\lambda)$ to the solution of the equation $(A_\varepsilon + 1)z^\varepsilon = F^\varepsilon$, where $z^\varepsilon = u^\varepsilon - u^\varepsilon - \varepsilon U^\varepsilon$.

### 5.2. $L^2$-estimates

From (49) we have, in particular,

$$\|u^\varepsilon - u - \varepsilon U^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq c\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \quad c = \text{cost}(d, \lambda),$$

wherefrom the $L^2$-estimate (14) follows, since $\|U^\varepsilon\| \leq c\|f\|$ by properties of smoothing.

On the next step we would like to estimate the difference $u^\varepsilon - u - \varepsilon U^\varepsilon$ more accurately, investigating the $L^2$-form

$$(u^\varepsilon - u - \varepsilon U^\varepsilon, h), \quad h \in L^2(\mathbb{R}^d).$$

To this end, insert $u^\varepsilon - u - \varepsilon U^\varepsilon$ as a test function into the integral identity for the solution of the adjoint equation (20) with an arbitrary RHS function $h \in L^2(\mathbb{R}^d)$.

We recall some facts about the adjoint equation (20) and its solution $v^\varepsilon$. First, the homogenized equation associated with (20) is of the form

$$v \in H^1(\mathbb{R}^d), \quad (A_0^* + 1)v = h;$$

second, the approximation in $H^1$-norm to $v^\varepsilon$ can be chosen as

$$v^\varepsilon(x) + \varepsilon V^\varepsilon(x), \quad \text{where } V^\varepsilon(x) = \tilde{N}_e(x) \cdot \nabla v^\varepsilon(x), \quad v^\varepsilon(x) = S^\varepsilon v(x),$$

$\tilde{N}$ is the vector composed of the solutions to the adjoint cell problem (23). What is more, the following estimate (that is a counterpart of (49)) is valid

$$\|v^\varepsilon - v^\varepsilon - \varepsilon V^\varepsilon\|_{H^1(\mathbb{R}^d)} \leq c\varepsilon\|h\|_{L^2(\mathbb{R}^d)}, \quad c = \text{const}(d, \lambda).$$

Thus, writing the integral identity for $v^\varepsilon$ with the test function $u^\varepsilon - u - \varepsilon U^\varepsilon$, we make restructuring in it:

$$(u^\varepsilon - u - \varepsilon U^\varepsilon, h) \overset{(20)}{=} (u^\varepsilon - u - \varepsilon U^\varepsilon, (A_\varepsilon^* + 1)v^\varepsilon)$$
\[
\begin{align*}
  &= ((A \varepsilon + 1) u^\varepsilon - (A \varepsilon + 1)(u + \varepsilon U^\varepsilon), v^\varepsilon) = ((A_0 + 1) u - (A \varepsilon + 1)(u + \varepsilon U^\varepsilon), v^\varepsilon) \\
  &= (A_0 u^\varepsilon - A \varepsilon (u^\varepsilon + \varepsilon U^\varepsilon), v^\varepsilon) + (A_0 (u - u^\varepsilon), v^\varepsilon) - (A \varepsilon (u - u^\varepsilon), v^\varepsilon) - \varepsilon (U^\varepsilon, v^\varepsilon) \\
  &=: T_1 + T_2 - T_3 - T_4. \quad (56)
\end{align*}
\]

Our goal is to estimate the terms \( T_i \). We begin with the simplest term

\[
T_4 := \varepsilon (U^\varepsilon, v^\varepsilon) \quad (47) \leq \varepsilon^2 C (|N|) \|\nabla u\| \|\nabla v^\varepsilon\|,
\]

where the final inequality is due to Lemma 4.2. To apply it note that \( \langle N \rangle = 0 \) (see the cell problem (15)), \( \nabla u \in L^2(\mathbb{R}^d) \), and \( v^\varepsilon \in H^1(\mathbb{R}^d) \). Hence, in view of (11) and (26), we obtain

\[
T_4 \cong 0. \quad (57)
\]

Here and hereafter we use the sign \( \cong \) to denote any equality modulo terms \( T \) estimated as follows:

\[
|T| \leq c \varepsilon^2 \|f\| \|h\|, \quad c = \text{const}(d, \lambda),
\]

and let us call such terms \( T \) inessential.

Next, the term \( T_3 \) in (56) admits the following presentation:

\[
T_3 = (u - u^\varepsilon, A^*_0 v^\varepsilon) \quad (20) \cong (u - u^\varepsilon, h - v^\varepsilon) \cong 0,
\]

by (43), (11) and (26). The similar arguments are applicable to the term \( T_2 \):

\[
T_2 := (A_0 (u - u^\varepsilon), v^\varepsilon) \quad (8) \cong (f - f^\varepsilon, v^\varepsilon) - (u - u^\varepsilon, v^\varepsilon) \quad (43) \cong (f - f^\varepsilon, v^\varepsilon).
\]

We engage now the \( H^1 \)-approximation (54) and continue our changes:

\[
T_2 \cong (f - f^\varepsilon, v^\varepsilon - v^\varepsilon - \varepsilon V^\varepsilon) + (f - f^\varepsilon, v^\varepsilon + \varepsilon V^\varepsilon) \cong (f - f^\varepsilon, v^\varepsilon + \varepsilon V^\varepsilon),
\]

where one term has been dropped, because it is inessential in view of the estimates

\[
\|f - f^\varepsilon\|_{H^{-1}} \quad (40) \leq C \varepsilon \|f\|, \quad \|v^\varepsilon - v^\varepsilon - \varepsilon V^\varepsilon\|_{H^1} \quad (55) \leq c \varepsilon \|h\|.
\]

Therefore,

\[
T_2 \cong ((A_0 + 1)(u - u^\varepsilon), v^\varepsilon + \varepsilon V^\varepsilon) \\
= (u - u^\varepsilon, (A^*_0 + 1)v^\varepsilon) - \varepsilon (f^\varepsilon, V^\varepsilon) + \varepsilon (f, V^\varepsilon) \\
= (u - u^\varepsilon, h^\varepsilon) - \varepsilon (f^\varepsilon, N^\varepsilon \cdot \nabla v^\varepsilon) + \varepsilon (f, V^\varepsilon),
\]
where \((u - u^\varepsilon, h^\varepsilon) \cong 0\) by (43), and the next term is also inessential, by Lemma 4.2 (note that \(\langle \tilde{N} \rangle = 0, f \in L^2(\mathbb{R}^d), \nabla v^\varepsilon \in H^1(\mathbb{R}^d)\)). Consequently,

\[
T_2 \cong \varepsilon(f, V^\varepsilon).
\]

We proceed now to the most difficult term \(T_1\) in (56). Using the presentation (51), we write

\[
T_1 := (A_0 u^\varepsilon - A_\varepsilon(u^\varepsilon + \varepsilon U^\varepsilon), v^\varepsilon) = \left( g^j_e \cdot \nabla \frac{\partial u^\varepsilon}{\partial x_j}, v^\varepsilon \right) - \left( \varepsilon a_\varepsilon N^j_e \frac{\partial u^\varepsilon}{\partial x_j}, \nabla v^\varepsilon \right) := I + II. \tag{59}
\]

Engaging the approximation (54), we have the sum

\[
I = \left( g^j_e \cdot \nabla \frac{\partial u^\varepsilon}{\partial x_j}, v^\varepsilon + \varepsilon V^\varepsilon \right) + \left( g^j_e \cdot \nabla \frac{\partial u^\varepsilon}{\partial x_j}, (v^\varepsilon + \varepsilon V^\varepsilon) \right),
\]

where the first summand is inessential due to Lemma 4.2 and relations (19)\(_2\), (55) and (11). Hence, using the fact that the vector \(g^j_e\) is solenoidal, we obtain

\[
I \cong \left( g^j_e \cdot \nabla \frac{\partial u^\varepsilon}{\partial x_j}, v^\varepsilon + \varepsilon V^\varepsilon \right) = - \left( g^j_e \cdot \nabla \frac{\partial u^\varepsilon}{\partial x_j}, (v^\varepsilon + \varepsilon V^\varepsilon) \right)
\]

\[
= - \left( g^j_e \frac{\partial u^\varepsilon}{\partial x_j}, (\nabla \tilde{N}^k_e + \varepsilon^k) \frac{\partial v^\varepsilon}{\partial x_k} + \varepsilon \tilde{N}^k_e \nabla \frac{\partial v^\varepsilon}{\partial x_k} \right)
\]

\[
= - \left( \nabla \tilde{N}^k_e + \varepsilon^k \right) \cdot g^j_e \frac{\partial u^\varepsilon}{\partial x_j} - \varepsilon \left( \tilde{N}^k_e \frac{\partial u^\varepsilon}{\partial x_j}, \nabla \frac{\partial v^\varepsilon}{\partial x_k} \right),
\]

where the gradient \(\nabla (v^\varepsilon + \varepsilon V^\varepsilon)\) has been calculated in the same way as in (50).

The periodic vector \(\nabla \tilde{N}^k_e + \varepsilon^k\) - \(g^j\) \cdot \varepsilon = 0,

thanks to (19). Thereby, Lemma 4.5 and the elliptic estimates for the solutions \(u\) and \(v\) yield

\[
\left( \nabla \tilde{N}^k_e + \varepsilon^k \right) \cdot g^j_e \frac{\partial u^\varepsilon}{\partial x_j} + \frac{\partial v^\varepsilon}{\partial x_k} \cong 0 \quad \Rightarrow \quad I \cong -\varepsilon \left( \tilde{N}^k_e g^j_e \frac{\partial u^\varepsilon}{\partial x_j}, \nabla \frac{\partial v^\varepsilon}{\partial x_k} \right). \tag{60}
\]

To estimate the term \(II\) in (59) we write it as the sum

\[
II = -\varepsilon \left( a_\varepsilon N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, \nabla (v^\varepsilon - v^\varepsilon - \varepsilon V^\varepsilon) \right) - \varepsilon \left( a_\varepsilon N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, \nabla (v^\varepsilon + \varepsilon V^\varepsilon) \right),
\]

where the first summand is inessential, which is shown by the Hölder inequality, Lemma 4.1 and (55). Next, the calculation of the type of (50) for the gradient \(\nabla (v^\varepsilon + \varepsilon V^\varepsilon)\) is made, whence

\[
II \cong -\varepsilon \left( a_\varepsilon N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, (\nabla \tilde{N}^k_e + \varepsilon^k) \frac{\partial v^\varepsilon}{\partial x_k} + \varepsilon \tilde{N}^k_e \nabla \frac{\partial v^\varepsilon}{\partial x_k} \right)
\]

\[
= -\varepsilon \left( a_\varepsilon N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, (\nabla \tilde{N}^k_e + \varepsilon^k) \frac{\partial v^\varepsilon}{\partial x_k} \right) - \varepsilon^2 \left( a_\varepsilon N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, \tilde{N}^k_e \nabla \frac{\partial v^\varepsilon}{\partial x_k} \right),
\]

where the last term is inessential due to the Hölder inequality, Lemma 4.1 and the elliptic estimates for the solutions \(u\) and \(v\). Then

\[
II \cong -\varepsilon \left( a_\varepsilon N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, (\nabla \tilde{N}^k_e + \varepsilon^k) \frac{\partial v^\varepsilon}{\partial x_k} \right) - \varepsilon \left( N^j_e \nabla \frac{\partial u^\varepsilon}{\partial x_j}, a_\varepsilon (\nabla \tilde{N}^k_e + \varepsilon^k) \frac{\partial v^\varepsilon}{\partial x_k} \right).
\]
where we have inserted the vector $\tilde{g}^k$ (see its definition in (25)) using the equality
\[ a^*(\nabla \tilde{N}^k + e^k) = g^k + (a^0)^* e^k. \]

Note that
\[ -\varepsilon \left( N^j_i \frac{\partial u^e}{\partial x_j}, (a^0)^* \nabla v^e \right) \equiv 0, \]
by Lemma 4.2. In conclusion, we obtain
\[ II \equiv -\varepsilon \left( N^j_i \frac{\partial u^e}{\partial x_j}, \tilde{g}^k \frac{\partial v^e}{\partial x_k} \right). \] (61)

From (59)–(61), it follows that
\[ T_1 \equiv -\varepsilon \left( \tilde{N}^k_{\varepsilon} g^j - N^k \tilde{g}^j, \frac{\partial u^e}{\partial x_j}, \nabla \frac{\partial v^e}{\partial x_k} \right) - \varepsilon \left( N^j_i \frac{\partial u^e}{\partial x_j}, \tilde{g}^k \frac{\partial v^e}{\partial x_k} \right) \]
\[ = -\varepsilon \left( \tilde{N}^k_{\varepsilon} g^j - N^k \tilde{g}^j, \frac{\partial u^e}{\partial x_j}, \nabla \frac{\partial v^e}{\partial x_k} \right) - \varepsilon \left( \tilde{N}^k_{\varepsilon} g^j - N^k \tilde{g}^j, \frac{\partial u^e}{\partial x_j}, \nabla \frac{\partial v^e}{\partial x_k} \right). \] (62)

From now on, we distinguish between two cases: selfadjoint and nonselfadjoint.

1° Let the matrix $a$ be symmetric. Hence $\tilde{N}^k_{\varepsilon} = N^k, \tilde{g}^j = g^j$, thereby, in (62) there stand
\[ b^{jk} := N^k g^j = N^k \tilde{g}^j = \tilde{N}^k_{\varepsilon} g^j \] (63)
such that $b^{jk} \in L^2_{\text{per}}(\square)$, since $N^k \in L^\infty_{\text{per}}(\square)$ due to the maximum principle valid in the scalar problem. Subsequent investigation of the term $T_1$ can be based on Lemma 4.2. But we avoid using the maximum principle in order to make our arguments universal and independent of it. We rely on Lemma 4.6. For the latter it is enough to have $b^{jk} \in L^1_{\text{per}}(\square)$ with $N^k, g^j \in L^2_{\text{per}}(\square)$ which surely holds. So by Lemma 4.6,
\[ T_1 \overset{(62)+(63)}{=} -\varepsilon \left( \langle b^{jk} \rangle \frac{\partial u}{\partial x_j}, \nabla \frac{\partial v}{\partial x_k} \right) - \varepsilon \left( \langle b^{jk} \rangle \cdot \nabla \frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_j} \right) = 0 \]
with constant vectors $\langle b^{jk} \rangle$. The final equality to zero holds, due to the relations
\[ \left( \frac{\partial \varphi}{\partial x_j}, \frac{\partial^2 \psi}{\partial x_i \partial x_k} \right) = - \left( \frac{\partial^2 \varphi}{\partial x_i \partial x_k}, \frac{\partial \psi}{\partial x_j} \right) \quad \forall \varphi, \psi \in H^2(\mathbb{R}^d). \]
Thus, all the terms $T_1$ in (56) have been considered. They are shown to be inessential except for $T_2$ (see (58)). As a result, the equality
\[ (u^e - u - \varepsilon U^e, h) \equiv (f, \varepsilon V^e) \] (64)
is proved, where, according to (47) and (54),
\[ U^e(x) = N^e(x) \cdot S^e \nabla u(x), \quad V^e(x) = N^e(x) \cdot S^e \nabla v(x). \]

We give the operator form to (64). Since $u^e = (A_\varepsilon + 1)^{-1} f$, $u = (A_0 + 1)^{-1} f$,
\[ \varepsilon U^e = \varepsilon N^e \cdot S^e \nabla (A_0 + 1)^{-1} f =: \varepsilon K_{\varepsilon} f, \quad \varepsilon V^e = \varepsilon N^e \cdot S^e \nabla (A_0 + 1)^{-1} h =: \varepsilon K_{\varepsilon} h, \]
we get
\[(A_\varepsilon + 1)^{-1} f - (A_0 + 1)^{-1} f - \varepsilon K_\varepsilon f - \varepsilon (K_\varepsilon)^* f, h) \lesssim 0.\]

Recalling the convention about the notation \(\cong\) (it is given after (57)), we deduce that
\[
\| (A_\varepsilon + 1)^{-1} f - (A_0 + 1)^{-1} f - \varepsilon K_\varepsilon f - \varepsilon (K_\varepsilon)^* f \| \leq C \varepsilon^2 \| f \|
\]
with the constant \(C = \text{const}(d, \lambda)\), whence (32) immediately follows.

2° In the case, where the matrix \(a\) is nonsymmetric, the term \(T_1\) in (56) cannot be considered as inessential, thereby, it contributes to the correcting operator. In fact, regarding the last two forms in (62), we see there \(\varepsilon\)-periodic vectors \(N_\varepsilon^{k,j} g^j\) and \(\tilde{N}_\varepsilon^{k,j} g^j\) that are distinct. For the corresponding \(1\)-periodic vectors, we introduce their mean values
\[
\bar{c}^{jk} = \langle N_\varepsilon^{k,j} \rangle, \quad \bar{\tilde{c}}^{jk} = \langle \tilde{N}_\varepsilon^{k,j} g^j \rangle.
\]
By definitions of \(g^j, g^j\) (see (18), (25)), we can rewrite
\[
\bar{c}^{jk} = \langle N_\varepsilon^{k} a^* (\nabla \tilde{N}_\varepsilon^j + e^j) \rangle, \quad \bar{\tilde{c}}^{jk} = \langle \tilde{N}_\varepsilon^{k} a (\nabla N_\varepsilon^j + e^j) \rangle.
\]
Indeed, for instance,
\[
\bar{\tilde{c}}^{jk} = \langle \tilde{N}_\varepsilon^{k,j} g^j \rangle \overset{(18)}{=} \langle \tilde{N}_\varepsilon^{k} (a (\nabla N_\varepsilon^j + e^j) - a^0 e^j) \rangle
\]
\[
= \langle \tilde{N}_\varepsilon^{k} a (\nabla N_\varepsilon^j + e^j) \rangle - \langle \tilde{N}_\varepsilon^{k} a^0 e^j \rangle \overset{(23)}{=} \langle \bar{\tilde{c}}^{jk} \rangle.
\]
The same arguments that were used in the selfadjoint case now show that \(\varepsilon\)-periodic vectors \(N_\varepsilon^{k,j} g^j\) and \(\tilde{N}_\varepsilon^{k,j} g^j\) in (62) can be replaced with the constant vectors \(c^{jk}\) and \(\tilde{c}^{jk}\) defined in (65) or (66); this replacement gives a negligible error. As a result,
\[
T_1 \overset{\cong}{=} -\varepsilon \left( c^{jk}_{\varepsilon} \frac{\partial u}{\partial x_j} \cdot \nabla \frac{\partial v}{\partial x_k} \right) - \varepsilon \left( c^{jk} \cdot \nabla \frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_k} \right)
\]
\[
= \varepsilon \left( u, \bar{c}^{jk} \frac{\partial^3 v}{\partial x_j \partial x_i \partial x_k} \right) + \varepsilon \left( \bar{\tilde{c}}^{jk} \frac{\partial^3 v}{\partial x_j \partial x_i \partial x_k} \right) := \varepsilon (u, \tilde{L} v) + \varepsilon (Lu, v),
\]
where we have introduced the third-order differential operators \(L\) and \(\tilde{L}\) with the constant coefficients and, thus, completed studying the term \(T_1\) in (56).

Gathering the essential terms in (56), we obtain the ‘equality’
\[
(u^\varepsilon - u - \varepsilon U^\varepsilon, h) \overset{\cong}{=} (f, \varepsilon V^\varepsilon) + \varepsilon (Lu, v) + \varepsilon (u, \tilde{L} v),
\]
which should be rewritten in the operator form. To this end, recall that
\[
\begin{align*}
&u^\varepsilon = (A_\varepsilon + 1)^{-1} f, \quad u = (A_0 + 1)^{-1} f, \quad v = (A_0^* + 1)^{-1} h, \\
&U^\varepsilon = N_\varepsilon \cdot S^\varepsilon \nabla (A_0 + 1)^{-1} f =: K_\varepsilon f, \quad V^\varepsilon = \tilde{N}_\varepsilon \cdot S^\varepsilon \nabla (A_0^* + 1)^{-1} h =: \tilde{K}_\varepsilon h
\end{align*}
\]
and coin a new operator

$$(A_0 + 1)^{-1} \left( L + \tilde{L}^* \right) (A_0 + 1)^{-1} f =: \mathcal{L} f,$$

where

$$L + \tilde{L}^* \equiv \left( \xi^j_k - \xi^j_i \right) \frac{\partial^3}{\partial x_j \partial x_i \partial x_k}$$

is the third-order differential operator with the constant coefficients. Then

$$(A_\varepsilon + 1)^{-1} f - (A_0 + 1)^{-1} f - \varepsilon K_\varepsilon f - \varepsilon (\tilde{K}_\varepsilon)^* f - \varepsilon L f, h \right) \cong 0. \quad (68)$$

Finally, recalling the convention about the notation $\cong$, we establish the estimate

$$\| (A_\varepsilon + 1)^{-1} f - (A_0 + 1)^{-1} f - \varepsilon K_\varepsilon f - \varepsilon (\tilde{K}_\varepsilon)^* f - \varepsilon L f \| \leq C \varepsilon^2 \| f \| \quad (69)$$

with the constant $C = \text{const}(d, \lambda)$, whence the estimate (35) follows.

Since the solutions of the cell problems (15) and (23) belong to the space $L^\infty(\Box)$ (recall that we consider the scalar case under the condition (7)), the functions $N_\varepsilon \cdot \nabla u$ and $\tilde{N}_\varepsilon \cdot \nabla v$ are well defined as elements of $L^2(\mathbb{R}^d)$. If we omit smoothing in the definitions (47) and (54), we pass to $N_\varepsilon \cdot \nabla u$ and $\tilde{N}_\varepsilon \cdot \nabla v$ in the place of the correctors $U^\varepsilon$ and $V^\varepsilon$ with an admissible error, due to the property (39) for the operator $S^\varepsilon$ and the elliptic estimates for $u$ and $v$ (see (11) and (27)). Hence we successively find (68) and (69), where smoothing is omitted in $K_\varepsilon$ and $\tilde{K}_\varepsilon$. As a result, we arrive at (33). This completes the proof of Theorem 3.2.

6. Extension to the case of unbounded coefficients

6.1. Problem setup

Let us try to weaken the conditions (7) on the matrix $a(y)$ so that the main results of Section 3 (we have in mind the operator $L^2$-estimates (32) and (33)) are still valid. Assuming that the measurable 1-periodic matrix $a(y)$ is not symmetric, we decompose it into the symmetric and skew-symmetric parts:

$$a(y) = a^s(y) + a^c(y), \quad (70)$$

and suppose that the symmetric part $a^s$ satisfies the elliptic inequality

$$\lambda |\xi|^2 \leq a^s \xi \cdot \xi \leq \lambda^{-1} |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \lambda > 0. \quad (71)$$

A condition on the skew-symmetric part $a^c$ is imposed to ensure, first of all, the unique solvability of the resolvent equation (6). According to the Lax–Milgram lemma, for this purpose it is sufficient to ensure the boundedness of the form $(a \nabla u, \nabla \phi)_{L^2(\mathbb{R}^d)}$ w.r.t. $u, \phi \in H^1(\mathbb{R}^d)$:

$$(a \nabla u, \nabla \phi)_{L^2(\mathbb{R}^d)} \leq c_0 \| \nabla u \|_{L^2(\mathbb{R}^d)} \| \nabla \phi \|_{L^2(\mathbb{R}^d)}. \quad (72)$$

Note that the coercivity of this form, that is, the inequality

$$(a \nabla u, \nabla u)_{L^2(\mathbb{R}^d)} \geq \lambda \| \nabla u \|_{L^2(\mathbb{R}^d)}^2$$

is already ensured by the ellipticity of the matrix $a^s$. Moreover, (71) implies also the boundedness of the $L^2$-form with the matrix $a^s$, and so we need to investigate only the form

$$(a^c \nabla u, \nabla \phi)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} a^c_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} \, dx = \frac{1}{2} \int_{\mathbb{R}^d} a^c_{ij} \left( \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} - \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i} \right) \, dx$$
\[\frac{1}{2} \int_{\mathbb{R}^d} a_{ij}^c I_{ij}(u, \varphi) \, dx. \tag{73}\]

Note that the necessary and sufficient conditions on the matrix \(a\) for the continuity property (72) were investigated in [34]. Dealing with homogenization, we have to reproduce some details of this investigation.

The skew-symmetric difference \(I_{ij}(u, \varphi)\) in parentheses of (73) have 'better than expected' regularity: it belongs surely to the space \(L^1(\mathbb{R}^d)\), but the algebraic structure makes this non-linear expression lie in the narrower Hardy space

\[\mathcal{H}^1(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d) : R_j f \in L^1(\mathbb{R}^d), 1 \leq j \leq d\},\]

where \(R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2}\) are the Riesz operators (see Proposition 4.4 in [23] which proved relying upon the results from [35]).

The space \(BMO\) (bounded mean oscillation) is dual to \(\mathcal{H}^1(\mathbb{R}^d)\) [36]. We recall that a measurable function \(g\) on \(\mathbb{R}^d\) lies in \(BMO\) if

\[\|g\|_{BMO} = \sup_B \int_B |g - g_B| \, dx < \infty \quad \text{with} \quad g_B = \int_B g \, dx = \frac{1}{|B|} \int_B g \, dx, \tag{74}\]

where the supremum is taken over all balls \(B \subset \mathbb{R}^d\), or alternatively, over cubes with faces parallel to coordinate hyperplanes. Obviously, elements of the space BMO are defined up to a constant.

By duality arguments (see more details in [23, Section 4]), we come to

**Proposition 6.1:** If the entries of the matrix \(a^c\) lie in BMO, then the form (73) is bounded:

\[(a^c \nabla u, \nabla \varphi)_{L^2(\mathbb{R}^d)} \leq c_0 \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla \varphi\|_{L^2(\mathbb{R}^d)},\]

where the constant \(c_0\) depends only on the norm \(\|a^c\|_{BMO}\).

Thus, from now on we assume:

\(\text{(C) the symmetric part } a^s \text{ of the matrix } a \text{ satisfies the ellipticity condition (71); its skew-symmetric part } a^c \text{ belongs to the space BMO.}\)

Then the whole form \((a \nabla u, \nabla \varphi)_{L^2(\mathbb{R}^d)}\) is bounded, and the estimate (72) holds with the constant \(c_0\) depending only on \(\lambda\) and \(\|a^c\|_{BMO}\). A homothety does not change the \(BMO\) norm: if \(b_\varepsilon(x) = b(x/\varepsilon)\), then \(\|b_\varepsilon\|_{BMO} = \|b\|_{BMO}\). Hence, the form with an \(\varepsilon\)-periodic matrix \(a_\varepsilon(x)\), namely, \((a_\varepsilon \nabla u, \nabla \varphi)_{L^2(\mathbb{R}^d)}\), is bounded and satisfies an estimate of type (72) with the same constant \(c_0\).

Therefore, Equation (6) is uniquely solvable and the uniform (in \(\varepsilon\)) estimate of the type (9) is valid for its solution. Parallelly, one can show that the cell problem (15) (and also (23)) is well posed, thereby, the homogenized matrix \(a_0\) is well defined in (16) in terms of the solutions \(N_j\) to (15) (see details in [23, Section 4]).

### 6.2. Estimates of order \(\varepsilon\)

Under condition (C), homogenization results stated in Section 2 remain true, including the estimate (31). The maximum principle holds for the cell problems (15), (23), and its solutions \(N_j^\varepsilon\) and \(\tilde{N}_k\) belong to \(L^\infty(\Box)\) (for the proof see arguments in [37]).

To justify the operator estimates (13) and (28) under condition (C), look through the derivation of the main estimate (48) from which (13) and (28) easily follow. One of the key points here is the estimate (53) for the residual \(F_\varepsilon\) defined in (52). To obtain this estimate we are to benefit from
Lemma 4.3. For this purpose, the terms \( r_\varepsilon \) and \( R_\varepsilon \) defined in (51) should have a proper structure, that is, its components \( g^j \) and \( aN^j \) should be sufficiently regular, namely,

\[
g^j, aN^j \in L^2(\Box).
\]

To verify \( g^j \in L^2(\Box) \) (see definition (18)) we are to invoke

**Proposition 6.2:** If \( N^j \) is the solution of the problem (15), then \( a\nabla N^j \) belongs to \( L^2(\Box) \) and satisfies the estimate \( \|a\nabla N^j\|_{L^2(\Box)} \leq C \), where the constant \( C \) depends only on \( \lambda \) and \( \|b\|_{BMO} \).

This assertion is proved in [23] relying on the higher integrability of the gradient \( \nabla N^j \), that is, \( \nabla N^j \in L^{2+\delta}(\Box) \) for some \( \delta > 0 \) (see Lemma 4.2 in [23]), and the John–Nirenberg inequality [36]

\[
\int_B |g - g_B|^p \, dx \leq c_p \|g\|^p_{BMO} \quad \forall \, p > 1,
\]

valid for any \( g \in BMO \). We apply (76) to the matrix function \( a^\varepsilon \) on the unit cube \( B = \Box \). It is appropriate here to refer to the fact that the form (73) will not change its value on subtracting a constant skew-symmetric from \( a^\varepsilon \). In the case of \( a^\varepsilon \in BMO \), a suitable integral mean (see (74)) is taken for this constant matrix, which allows to invoke the John–Nirenberg inequality.

To show \( aN^j \in L^2(\Box) \) we apply the assertion (76) w.r.t. the matrix \( a^\varepsilon \) and the property \( N^j \in L^t(\Box) \), \( t > 2 \), by the Sobolev embedding theorem (or the deeper property \( N^j \in L^\infty(\Box) \)).

In conclusion of this subsection, we note that the more detailed proof of the estimates (13) and (28) for the operator \( A^\varepsilon \) with the coefficients from \( BMO \) is given in [23].

### 6.3. \( L^2 \)-estimate of order \( \varepsilon^2 \)

Assuming the condition \( (C) \) on the matrix \( a \) stated in Section 6.1, let us show that the operator estimate (35) remains valid. We can repeat without any changes reasoning in Section 5 up to the 'equality' (62), in particular, taking into account (75). Then, following the lines of the nonsymmetric case, we come to (67) by Lemma 4.6 and afterwards duplicate the end of the proof of the estimate (35).

We formulate finally the main result of this section that has been just verified.

**Theorem 6.3:** Let the matrix \( a \) in (6) satisfy (70), (71) with the skew-symmetric part \( a^\varepsilon \in BMO \). Then there holds the estimate (35) with the correcting operators defined in (36) and the RHS constant \( C \) which depends only on the dimension \( d \), the ellipticity constant \( \lambda \) in (71) and the norm \( \|a^\varepsilon\|_{BMO} \).

**Remark 6.4:** The operator \( A^\varepsilon \) considered in this section can be written in the form

\[
A^\varepsilon = -\text{div } a^\varepsilon(x/\varepsilon) \nabla - \varepsilon^{-1} b(x/\varepsilon) \cdot \nabla
\]

with \( b(x) := \text{div } a^\varepsilon(x) \), whence \( \text{div } b = 0 \), \( b \in BMO^{-1}(\mathbb{R}^d) \). Assume \( a^\varepsilon \) is the identity matrix. Then \( A^\varepsilon \) corresponds to the convection-diffusion operator in a stationary incompressible flow, \( b \) is called the drift vector. Elliptic and parabolic equations with the operator \( A^\varepsilon \) having the divergence-free drift with low regularity is of particular interest for applications to incompressible flows (see, e.g. [37–39] and references in the latter). Here we present some homogenization result for the operator (77) of this type. Similar resolvent approximations, or approximations for the operator exponential \( \exp(-tA^\varepsilon) \), of higher sharpness than \( \varepsilon \) were studied in [40,41] by spectral approach assuming the drift is sufficiently regular but not necessarily divergence-free.
7. Auxiliaries

In this section we prove some properties of the Steklov smoothing operator $S^e$ formulated in §4.

Proof of Lemma 4.4: To estimate the form $I := (b, S^e \varphi, S^e \psi)$, where

$$S^e \varphi(x) = \int_{\partial} \varphi(x - \varepsilon \omega) \, d\omega, \quad S^e \psi(x) = \int_{\partial} \psi(x - \varepsilon \sigma) \, d\sigma,$$

we make standard transformations:

$$I = \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} \right) \varphi(x - \varepsilon \omega) \psi(x - \varepsilon \sigma) \, d\omega \, d\sigma \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} + \omega + \sigma \right) \varphi(x + \varepsilon \sigma) \psi(x + \varepsilon \omega) \, d\omega \, d\sigma \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} + \omega + \sigma \right) \varphi(x + \varepsilon \sigma) (\psi(x + \varepsilon \omega) - \psi(x)) \, d\omega \, d\sigma \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} + \omega + \sigma \right) \varphi(x + \varepsilon \sigma) \left( \int_0^1 \nabla \psi(x + t\varepsilon \omega) \cdot \varepsilon \omega \, dt \right) \, d\omega \, d\sigma \, dx,$$

where we have used the condition $\langle b \rangle = 0$ and the integral representation

$$\psi(x + h) - \psi(x) = \int_0^1 \nabla \psi(x + th) \cdot h \, dt. \quad (78)$$

We continue the standard transformations:

$$I = \int_0^1 \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} + \omega + \sigma \right) (\varphi(x + \varepsilon \sigma) - \varphi(x)) \nabla \psi(x + t\varepsilon \omega) \cdot \varepsilon \omega \, d\omega \, d\sigma \, dx \, dt$$

$$= \int_0^1 \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} + \omega + \sigma \right) \left( \int_0^1 \nabla \varphi(x + s\varepsilon \sigma) \cdot \varepsilon \sigma \, ds \right) \nabla \psi(x + t\varepsilon \omega) \cdot \varepsilon \omega \, d\omega \, d\sigma \, dx \, dt$$

$$= \varepsilon^2 \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} b \left( \frac{x}{\varepsilon} + \omega + \sigma \right) (\nabla \varphi(x + s\varepsilon \sigma) \cdot \sigma) \nabla \psi(x + t\varepsilon \omega) \cdot \omega \, d\omega \, d\sigma \, dx \, dt \, ds,$$

where we again use the condition $\langle b \rangle = 0$ and an integral representation for the difference $\varphi(x + \varepsilon \sigma) - \varphi(x)$ similar to (78). Applying Hölder’s inequality to the last multi-dimensional integral, we find

$$I^2 \leq \varepsilon^4 \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} |b \left( \frac{x}{\varepsilon} + \omega + \sigma \right)|^2 |\nabla \varphi(x + s\varepsilon \sigma) \cdot \sigma|^2 \, d\omega \, d\sigma \, dx \, dt \, ds$$

$$\times \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_{\partial} \int_{\partial} |\nabla \psi(x + t\varepsilon \omega) \cdot \omega|^2 \, d\omega \, d\sigma \, dx \, dt \, ds, \quad (79)$$

where both integral factors can be easily estimated. Thereby,

$$I^2 \leq \varepsilon^4 C(|b|^2) \|\nabla \varphi\|_{L^2(\mathbb{R}^d)}^2 \|\nabla \psi\|_{L^2(\mathbb{R}^d)}^2, \quad C = \text{const}(d), \quad (80)$$

and we come to the estimate (44). The lemma is proved.

Proof of Lemma 4.5: Deriving the estimate (45), one can assume that $\varphi, \psi \in C^\infty_0(\mathbb{R}^d)$ and, considering the oscillating factor $b = \alpha \beta$, repeat the standard transformations of the form $I$ made above up to formula (79). Before we use the Hölder inequality at this moment, we recall that $b = \alpha \beta$ and
distribute the functions $\alpha \beta$ among the different integral factors. Thus, instead of (79), we come to the inequality

$$I^2 \leq \epsilon^4 \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_0^1 |\alpha(\frac{x}{\epsilon} + \omega + \sigma)|^2 |\nabla \varphi(x + \varepsilon \omega) \cdot \sigma|^2 \, d\omega \, d\sigma \, dx \, dt \, ds \times \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_0^1 |\beta(\frac{x}{\epsilon} + \omega + \sigma)|^2 |\nabla \psi(x + t\varepsilon \omega) \cdot \omega|^2 \, d\omega \, d\sigma \, dx \, ds.$$

Here, both integral factors can be easily estimated, and instead of (80) we obtain

$$I^2 \leq \epsilon^4 C \langle |\alpha|^2 \rangle \|\nabla \varphi\|^2_{L^2(\mathbb{R}^d)} \langle |\beta|^2 \rangle \|\nabla \psi\|^2_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(d),$$

which is equivalent to (45). The lemma is proved.

**Disclosure statement**

No potential conflict of interest was reported by the author.

**Proof of Lemma 4.6:** The same transformations that are used in the proof of Lemma 4.4 yield

$$I : = (\alpha S^\varepsilon \varphi, \beta S^\varepsilon \psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b \left( \frac{x}{\epsilon} \right) \varphi(x - \varepsilon \omega) \psi(x - \varepsilon \sigma) \, d\omega \, d\sigma \, dx \, dt \, ds,$$

where we set $b = \alpha \beta$. Decomposing $\psi(x + \varepsilon \omega) = \psi(x) + (\psi(x + \varepsilon \omega) - \psi(x))$, we write the representation $I = I_1 + I_2$, where

$$I_1 : = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b \left( \frac{x}{\epsilon} \right) \varphi(x + \varepsilon \sigma) \psi(x) \, d\omega \, d\sigma \, dx = \langle b \rangle \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x + \varepsilon \sigma) \psi(x) \, d\sigma \, dx = \langle b \rangle \langle \varphi, S^\varepsilon \psi \rangle$$

and

$$I_2 : = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b \left( \frac{x}{\epsilon} \right) \varphi(x + \varepsilon \sigma)(\psi(x + \varepsilon \omega) - \psi(x)) \, d\omega \, d\sigma \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b \left( \frac{x}{\epsilon} \right) \varphi(x + \varepsilon \sigma) \nabla \psi(x + \tau \varepsilon \omega) \cdot \varepsilon \omega \, dt \, d\omega \, d\sigma \, dx$$

thanks to the integral formula (78). By arguments used in the proof of Lemma 4.5, we show that

$$I_2 \leq C \varepsilon \langle |\alpha|^2 \rangle^{1/2} \langle |\beta|^2 \rangle^{1/2} \|\nabla \varphi\|_{L^2(\mathbb{R}^d)} \|\nabla \psi\|_{L^2(\mathbb{R}^d)}.$$

Returning to $I_1$, it is clear that $I_1 = \langle b \rangle \langle \varphi, \psi \rangle + \langle b \rangle \langle \varphi, S^\varepsilon \psi - \psi \rangle$, where, by properties of smoothing, the second summand admits the estimate from above with the same majorant as in (46). Eventually, gathering all the relations proved above, we come to (46). The lemma is proved.

**Disclosure statement**

No potential conflict of interest was reported by the author.
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