ON THE ARITHMETIC AND THE GEOMETRY OF SKEW-RECIPROCAL POLYNOMIALS

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Abstract. We reformulate Lehmer’s question from 1933 and a question due to Schinzel and Zassenhaus from 1965 in terms of a comparison of the Mahler measures and the houses, respectively, of monic integer reciprocal and skew-reciprocal polynomials of the same degree. This entails that understanding the difference between orientation-preserving and orientation-reversing mapping classes is at least as complicated as answering these questions.

1. Introduction

Kronecker’s theorem from 1857 states that if a nonzero algebraic integer is not a root of unity, then it has a Galois conjugate outside the unit circle [5]. Roughly speaking, Lehmer in 1933 and Schinzel and Zassenhaus in 1965 asked whether this statement can be made quantitatively precise using the Mahler measure or the house of polynomials, respectively [7, 10]. The explicit questions they rose remain unanswered to this day.

Our main results (Theorems 1 and 2) provide reformulations of Lehmer’s question and the question of Schinzel and Zassenhaus in terms of a comparison of reciprocal and skew-reciprocal polynomials. A polynomial \( f \in \mathbb{Z}[t] \) of degree \( d \) is reciprocal if \( f(t) = t^d f(t^{-1}) \). Similarly, a polynomial \( f \in \mathbb{Z}[t] \) of even degree \( 2d \) is skew-reciprocal if \( f(t) = (-1)^d t^d f(-t^{-1}) \).

As we discuss in Section 3, reciprocal and skew-reciprocal polynomials arise naturally as the characteristic polynomials of integer symplectic and anti-symplectic matrices, respectively. These are in turn exactly the actions induced on the first homology of closed surfaces by orientation-preserving and orientation-reversing mapping classes, respectively. Via these equivalences, Theorems 1 and 2 are our way of making precise the following statement: Understanding the difference between orientation-preserving and orientation-reversing mapping classes is at least as complicated as answering Lehmer’s question and the question of Schinzel and Zassenhaus.

1.1. The Mahler measure. Let \( f \in \mathbb{Z}[t] \) be a monic polynomial. The Mahler measure \( M(f) \) of \( f \) is the modulus of the product of all zeroes of \( f \) outside the unit circle, counted with multiplicity:

\[
M(f) = \prod_{f(\alpha) = 0} \max(1, |\alpha|).
\]

Question (Lehmer’s question [7]). Does the set of Mahler measures of monic integer polynomials accumulate at 1?

The author was supported by the Swiss National Science Foundation (grant nr. 175260).
The smallest Mahler measure larger than 1 found by Lehmer is the one of the polynomial
\[ L(t) = t^{10} + t^9 - t^7 - t^6 - t^4 - t^3 + t + 1. \]
We call \( \lambda_L = M(L) \approx 1.17628 \) Lehmer’s number. It is still the smallest known Mahler measure larger than 1.

**Question** (Lehmer’s question, strong version \([7]\)). Is Lehmer’s number the smallest Mahler measure larger than 1 among all monic integer polynomials?

Our first result provides a reformulation of Lehmer’s question. Let \( R_i \) be the smallest Mahler measure larger than 1 among monic integer reciprocal polynomials of degree \( 2^i \), and let \( S_i \) be the smallest Mahler measure larger than 1 among monic integer skew-reciprocal polynomials of degree \( 2^i \).

**Theorem 1.** Lehmer’s number \( \lambda_L \) is the smallest Mahler measure larger than 1 among all monic integer polynomials exactly if \( R_i = S_i \) for all \( i \geq 5 \). Furthermore, the set of Mahler measures of monic integer polynomials accumulates at 1 exactly if \( \prod_{i=5}^{N} \frac{R_i}{S_i} \) converges to \( \lambda_L^{-1} \) as \( N \to \infty \).

Theorem 1 gives a reformulation of Lehmer’s question in terms of a comparison of orientation-preserving and orientation-reversing mapping classes. Indeed, from our discussion in Section 3 it follows that \( R_i \) and \( S_i \) are the smallest Mahler measures larger than 1 among all characteristic polynomials of actions induced on the first homology of the closed surface of genus \( 2^{i-1} \) by orientation-preserving and orientation-reversing mapping classes, respectively, compare with Proposition 14.

1.2. The house. Let \( f \in \mathbb{Z}[t] \) be a monic polynomial. The **house** \( \overline{f} \) of \( f \) is the largest modulus among the zeroes of \( f \),
\[
\overline{f} = \max_{f(\alpha) = 0} |\alpha|.
\]

**Question** (Schinzel–Zassenhaus \([10]\)). Does there exists a universal constant \( c > 0 \) so that any house larger than 1 of an irreducible monic integer polynomial is at least \( 1 + \frac{c}{d} \), where \( d \) is the degree of the polynomial?

Our second result is a reformulation of the question of Schinzel and Zassenhaus in terms of a comparison of reciprocal and skew-reciprocal polynomials. Let \( \lambda_i \) and \( \tilde{\lambda}_i \) be the smallest houses larger than 1 among all monic integer reciprocal and skew-reciprocal polynomials of degree \( 2^i \), respectively. Let \( r_i = 2^i \log(\lambda_i) \) and \( s_i = 2^i \log(\tilde{\lambda}_i) \).

**Theorem 2.** There exists a universal constant \( c > 0 \) so that any house larger than 1 of an irreducible monic integer polynomial is at least \( 1 + \frac{c}{d} \), where \( d \) is the degree of the polynomial, exactly if the set \( \left\{ \frac{q_n}{q_m} \in \mathbb{R} : n \in \mathbb{N}, N \in \mathbb{N}, n < N \right\} \subset \mathbb{R} \) is bounded away from zero, where \( q_m = \prod_{i=1}^{m} \frac{n}{n_i} \).

Theorem 2 gives a reformulation of the question of Schinzel and Zassenhaus in terms of a comparison of orientation-preserving and orientation-reversing mapping classes. Again, from our discussion in Section 3 it follows that \( \lambda_i \) and \( \tilde{\lambda}_i \) are equal to \( \delta_{g^{i-1}}^{\text{hom}} \) and \( \tilde{\delta}_{g^{i-1}}^{\text{hom}} \), respectively, compare with Proposition 14. Here, \( \delta_g^{\text{hom}} \) and \( \tilde{\delta}_g^{\text{hom}} \) denote the minimal spectral radii larger than 1 among actions induced on the first homology of the closed surface of genus \( g \) by orientation-preserving and orientation-reversing mapping classes, respectively.
The easiest way to fulfill the second statement in Theorem 2 is if for all but finitely many $i$, we have $r_i \geq s_i$ and hence $\delta_{\text{hom}}^{2i} \geq \delta_{\text{hom}}^{2i-1}$. Combining the results and the conjectures of Hironaka [4], Lanneau and Thiffeault [6], and Strenner and the author [8], this statement seems to have a chance of being true, at least when restricting to the actions induced by pseudo-Anosov mapping classes with an orientable invariant foliation.

Finally, we reformulate the question of Schinzel and Zassenhaus as a comparison of $\delta_{q}^{\text{hom}}$ and the minimal dilatation $\delta_q$ among pseudo-Anosov mapping classes on the closed surface of genus $g$ (defined in Section 3.3).

**Theorem 3.** There exists a universal constant $c > 0$ so that any house larger than $1$ of an irreducible monic integer polynomial is at least $1 + \frac{c}{d}$, where $d$ is the degree of the polynomial, exactly if there exists a universal constant $C > 0$ so that for all $g$, $(\delta_{q}^{\text{hom}})^C \geq \delta_q$.

1.3. **Proof strategy.** Lehmer’s question and hence the question of Schinzel and Zassenhaus is solved in the case of irreducible nonreciprocal polynomials, due to a result of Breusch [2].

**Theorem 4** (Breusch [2]). The Mahler measure of any integer nonreciprocal irreducible polynomial other than $(t - 1)$ and $t$ is greater than $1.179$.

The constant of the bound in Theorem 4 is not optimal, but it suffices for our purpose. For the optimal constant and more results on the Mahler measure and the house of integer polynomials, see Smyth’s survey [11].

We use Theorem 4 in order to reduce Lehmer’s question and the question of Schinzel and Zassenhaus to the case of irreducible reciprocal polynomials. From there, the main insight for the proofs of Theorems 1 and 2 consists of the fact that one can in a controlled way compare skew-reciprocal polynomials of degree $2i+1$ with reciprocal polynomials of degree $2i$.

Theorem 3 follows rather directly from the fact that the minimal dilatations $\delta_q$ among pseudo-Anosov mapping classes satisfy an inequality as in the question by Schinzel and Zassenhaus. This is a result due to Penner [9].

**Theorem 5** (Penner [9]). There exist universal constants $R, R' > 1$ so that

$$R \leq (\delta_q)^g \leq R'.$$

1.4. **Organisation.** We prove Theorems 1 and 2 in Section 2. In Section 3, we relate reciprocal and skew-reciprocal polynomials with the characteristic polynomials of the actions induced on the first homology of closed surfaces by mapping classes. We finally prove Theorem 3.

**Acknowledgements:** I would like to thank S. Baader, E. Hironaka, C. McMullen, B. Strenner and an anonymous referee for helpful discussions and comments.

## 2. Reciprocal vs. skew-reciprocal polynomials

Recall that a polynomial $f \in \mathbb{Z}[t]$ of even degree $2d$ is called reciprocal if we have $f(t) = t^{2d}f(t^{-1})$, and skew-reciprocal if $f(t) = (-1)^{d}t^{2d}f(-t^{-1})$.

**Lemma 6.** Let $f \in \mathbb{Z}[t]$ be a monic skew-reciprocal polynomial of degree $2i+1$ with $\prod > 1$. Then either $f(t) = g(t^2)$, where $g(t)$ is a reciprocal polynomial of degree $2i$, or $f$ has a nonreciprocal irreducible factor other than $(t - 1)$.
Proof. Let \( f \in \mathbb{Z}[t] \) be a monic skew-reciprocal polynomial of degree \( 2^{i+1} \) such that \( \prod |a| > 1 \).

Case 1: \( f \) is reciprocal. If a polynomial \( f \in \mathbb{Z}[t] \) of degree \( 2^{i+1} \) is both reciprocal and skew-reciprocal, we have \( f(t) = g(t^2) \), where \( g(t) \) is a reciprocal polynomial of degree \( 2^i \).

Case 2: \( f \) is not reciprocal. If \( f \) is not reciprocal, it must have at least one nonreciprocal irreducible factor. Moreover, \( (t - 1) \) cannot be the only nonreciprocal irreducible factor. Indeed, if \( (t - 1) \) was the only nonreciprocal irreducible factor, then it would have to appear to an even power, since the constant coefficient of \( f \) is \(+1\). This follows directly from the definition of skew-reciprocity and the degree of \( f \) being divisible by four. However, an even power of \( (t - 1) \) is reciprocal and hence so would be the polynomial \( f \), a contradiction. We have shown that the polynomial \( f \) must contain a nonreciprocal irreducible factor other than \( (t - 1) \). \( \square \)

2.1. Mahler measures. Recall that the Mahler measure \( M(f) \) of a monic polynomial \( f \in \mathbb{Z}[t] \) is the modulus of the product of all zeroes of \( f \) outside the unit circle, counted with multiplicity:

\[
M(f) = \prod_{f(\alpha)=0} \max(1, |\alpha|).
\]

We remark that \( M(f(t)) = M(f(t^2)) \) for any polynomial \( f \in \mathbb{Z}[t] \). Let \( R_i \) be the smallest Mahler measure larger than 1 among monic integer reciprocal polynomials of degree \( 2^i \), and let \( S_i \) be the smallest Mahler measure larger than 1 among monic integer skew-reciprocal polynomials of degree \( 2^i \).

Lemma 7. \( S_{i+1} = R_i \) for \( i \geq 4 \).

Proof. Let \( \lambda_L \approx 1.17628 \) be Lehmer’s number, an algebraic integer of degree 10. We have \( R_i \leq \lambda_L \) for \( i \geq 4 \): the minimal polynomial of \( \lambda_L \) is reciprocal, so we can multiply it with a power of \( (t + 1) \) to obtain a reciprocal polynomial of arbitrary degree and Mahler measure equal to \( \lambda_L \). Furthermore, we have \( S_{i+1} \leq R_i \leq \lambda_L \) for \( i \geq 4 \). Indeed, if \( g \) is a reciprocal polynomial of degree \( 2^i \), then \( f(t) = g(t^2) \) is a skew-reciprocal polynomial of degree \( 2^{i+1} \) and \( M(f) = M(g) \).

In order to prove \( S_{i+1} \geq R_i \) for \( i \geq 4 \), let \( f(t) \) be a monic skew-reciprocal polynomial of degree \( 2^{i+1} \) and of Mahler measure \( > 1 \). By Lemma 6, \( f(t) \) either has an irreducible non-reciprocal factor other than \( (t - 1) \), or equals \( g(t^2) \) for some reciprocal polynomial \( g(t) \). In the former case, Theorem 4 implies \( M(f) \geq \lambda_L \geq R_i \). In the latter case, we have \( M(f) = M(g) \geq R_i \).

Proof of Theorem 1. By Theorem 4, Lehmer’s question can be reduced to monic irreducible reciprocal polynomials. By multiplication with factors \( (t + 1) \), one sees that Lehmer’s question is in turn equivalent to the same question for (not necessarily irreducible) monic reciprocal polynomials of some degree \( 2^i \), that is, for \( R_i \). Now, some number \( R_N \) is smaller than \( \lambda_L \) exactly if \( R_i < S_i \) for some \( i \geq 5 \). This follows directly from

\[
R_N = \lambda_L \prod_{i=5}^{N} \frac{R_i}{R_{i-1}} = \lambda_L \prod_{i=5}^{N} \frac{R_i}{S_i},
\]
where we use Lemma 7 to prove the second equality. Furthermore, since we have \( S_i = R_{i-1} \geq R_i \), it holds that \( R_i \leq S_i \) for all \( i \geq 5 \), and the set of all \( R_N \) accumulates at 1 if and only if \( \prod_{i=5}^{N} R_i^{-1} \) converges to \( \lambda_1^{-1} \) as \( N \to \infty \). \( \square \)

2.2. Houses. Recall that the house of a polynomial is the largest modulus among its roots. Let \( \lambda_i \) and \( \tilde{\lambda}_i \) be the smallest houses larger than 1 among all monic integer reciprocal and skew-reciprocal polynomials of degree \( 2^i \), respectively. Furthermore, let \( r_i = 2^i \log(\lambda_i) \) and \( s_i = 2^i \log(\tilde{\lambda}_i) \).

**Lemma 8.** For \( i \geq 1 \), we have \( s_{i+1} \geq \min \{ r_i, \log(1.179) \} \).

**Proof.** Let \( f \in \mathbb{Z}[t] \) be a monic skew-reciprocal polynomial of degree \( 2^{i+1} \) such that \( \prod r_i > 1 \). We use Lemma 6 to distinguish two cases. Assume for the first case that \( f(t) = g(t^2) \), where \( g(t) \) is a reciprocal polynomial of degree \( 2^i \). In this case, we have \( \prod r_i^2 = \prod q_i \). It follows that \( 2^{i+1} \log \prod r_i = 2^i \log \prod q_i \geq r_i \). On the other hand, if \( f \) has a nonreciprocal irreducible factor that is not \((t - 1)\), then Theorem 4 implies \( 2^{i+1} \log \prod r_i \geq \log(1.179) \). \( \square \)

**Lemma 9.** The answer to the question of Schinzel and Zassenhaus is positive exactly if the sequence \( \{ r_i \} \) is bounded strictly away from zero.

**Proof.** By Theorem 4, the question of Schinzel and Zassenhaus is equivalent to the same question restricted to reciprocal polynomials. Furthermore, any reciprocal polynomial \( f(t) \) can be multiplied by \( (t + 1)^k \), where \( k \) is at most the degree of \( f(t) \), so that it becomes reciprocal of degree \( 2^i \), for some \( i \geq 1 \), keeping its house. This means that the question of Schinzel and Zassenhaus is equivalent to the same question for (not necessarily irreducible) reciprocal polynomials of degree \( 2^i \). The statement of the lemma now follows from the fact that \( \{ r_i \} = \{ 2^i \log(\lambda_i) \} \) is strictly bounded away from zero exactly if there exists a constant \( c \) such that \( \lambda_i > 1 + \frac{c}{2^i} \) for all \( i \). \( \square \)

**Proof of Theorem 2.** For one direction, we assume there exists a sequence \( \{ \frac{q_n}{q_{n_j}} \} \), where \( 0 < n_j < N_j \), that converges to zero. If \( f(t) \) is a reciprocal polynomial of even degree, then \( f(t^2) \) is a skew-reciprocal polynomial. This implies \( s_i \leq r_{i-1} \).

In particular, we have:

\[
\frac{r_{N_j}}{r_{n_j}} = \prod_{i=n_j+1}^{N_j} \frac{r_i}{r_{i-1}} \leq \prod_{i=n_j+1}^{N_j} \frac{r_i}{s_i} \leq 4 \log(\varphi) \frac{q_{N_j}}{q_{n_j}},
\]

where \( \varphi \) is the golden ratio. For the last inequality, we use \( r_{n_j} \leq r_1 = 4 \log(\varphi) \). The numbers \( r_{N_j} \) converge to 0 as \( j \to \infty \), giving a negative answer to the question of Schinzel and Zassenhaus by Lemma 9.

For the other direction, we assume the set \( \left\{ \frac{q_n}{q_N} \in \mathbb{R} : n, N \in \mathbb{N}, n < N \right\} \subset \mathbb{R} \) is bounded away from zero.

**Claim.** \( \frac{\log(1.179)q_N}{\max \{ q_1, \ldots, q_{N-1} \}} \geq r_N \).

We admit the claim for a moment. By our assumption, there is a constant bounding all fractions \( \frac{q_n}{q_N} \) with \( 0 < n < N \) away from zero. In particular, by the claim, there exists a constant bounding \( r_N \) strictly away from zero for all \( N \). This
is equivalent to a positive answer to the question of Schinzel and Zassenhaus by Lemma 9.

We now prove the claim by induction on N.

**Base case:** For \( N = 2 \), we verify

\[
q_2 \geq \frac{q_2}{q_1} \min \{ r_1, \log(1.179) \} = \frac{\log(1.179)q_2}{\max \{ q_1 \}},
\]

where the inequality is due to Lemma 8, and the equality on the right follows from \( r_1 = 4 \log(\phi) \), which is larger than \( \log(1.179) \).

**Inductive step:** We again use Lemma 8. We have

\[
r_{N+1} = \frac{r_{N+1}}{s_{N+1}} \geq \frac{q_{N+1}}{q_N} \min \{ r_N, \log(1.179) \}.
\]

Using the induction hypothesis on \( r_N \), this yields

\[
r_{N+1} \geq \frac{q_{N+1}}{q_N} \min \left\{ \frac{\log(1.179)q_N}{\max \{ q_1, \ldots, q_{N-1} \}}, \log(1.179) \right\} = \log(1.179)q_{N+1} \min \left\{ \frac{1}{\max \{ q_1, \ldots, q_{N-1} \}}, \frac{1}{q_N} \right\} = \frac{\log(1.179)q_{N+1}}{\max \{ q_1, \ldots, q_N \}},
\]

which completes the inductive step. \( \square \)

**3. Symplectic matrices and mapping classes**

The goal of this section is to illustrate that monic integer reciprocal and skew-reciprocal polynomials arise naturally in geometry: as characteristic polynomials of symplectic and anti-symplectic matrices, respectively. These in turn arise as the actions induced on the first homology of closed surfaces by orientation-preserving and orientation-reversing mapping classes, respectively.

**3.1. Symplectic matrices.** An integer matrix \( A \) of size \( 2g \times 2g \) is symplectic if it preserves the standard symplectic form

\[
\Omega = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},
\]

that is, if \( A^T \Omega A = \Omega \). An integer matrix \( A \) of size \( 2g \times 2g \) is anti-symplectic if it reverses the standard symplectic form \( \Omega \), that is, if \( A^T \Omega A = -\Omega \).

The next two lemmas characterise monic integer reciprocal and skew-reciprocal polynomials of even degree as the characteristic polynomials of integer symplectic and anti-symplectic matrices, respectively.

**Lemma 10.** The characteristic polynomial of an integer symplectic matrix is reciprocal, and the characteristic polynomial of an integer anti-symplectic matrix is skew-reciprocal.

**Proof.** The statement for symplectic matrices is a standard fact. An adaptation of the proof to anti-symplectic matrices is given by Strenner and the author [8]. \( \square \)
Lemma 11. Any monic reciprocal polynomial \( f \in \mathbb{Z}[t] \) of even degree is the characteristic polynomial of an integer symplectic matrix, and any monic skew-reciprocal polynomial \( f \in \mathbb{Z}[t] \) of even degree is the characteristic polynomial of an integer anti-symplectic matrix.

Proof. On page 686, Ackermann [1] defines a symplectic companion matrix

\[
B = \begin{pmatrix}
0 & \cdots & 0 & -1 \\
1 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 1 -a_1 \\
1 & -a_1 & -a_2 & \cdots & -a_g \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

for the monic reciprocal polynomial

\[
h(t) = a_g t^g + \sum_{i=0}^{g-1} a_i (t^i + t^{2g-i}),
\]

where \( a_0 = 1 \). Multiplying \( B \) from the right with \( R = \begin{pmatrix} -I_g & 0 \\ 0 & I_g \end{pmatrix} \) yields an anti-symplectic matrix, since

\[
(RB)^\top \Omega RB = B^\top R\Omega RB = B^\top (-\Omega)B = -\Omega.
\]

The characteristic polynomial of \( B \) can be calculated by successively developing the first row, until the matrix is of size \( g \times g \), then successively developing the last column. Multiplication with \( R \) changes only the first \( g \) rows of \( B \), and one can see that it only possibly changes the first \( g \) coefficients of the characteristic polynomial. More precisely, one can explicitly check that multiplication with \( R \) changes the coefficient of \( t^i \) by a sign \((-1)^i (-1)^g\), for \( i < g \). Hence, the characteristic polynomial of \( RB \) is the monic skew-reciprocal polynomial

\[
f(t) = a_g t^g + \sum_{i=0}^{g-1} a_i ((-1)^g(-1)^i + t^{2g-i}),
\]

where \( a_0 = 1 \). Altogether, we get any monic skew-reciprocal polynomial \( f \in \mathbb{Z}[t] \) of even degree as the characteristic polynomial of an anti-symplectic matrix of the type \( RB \). \( \square \)

3.2. Mapping classes. Let \( \Sigma_g \) be the orientable closed surface of genus \( g \). A mapping class of \( \Sigma_g \) is a homeomorphism \( \phi : \Sigma_g \to \Sigma_g \), up to isotopy. Mapping classes of a fixed surface form a group under composition, the extended mapping class group.

Lemma 12. The action induced on the first homology \( H_1(\Sigma_g) \) by an orientation-preserving or orientation-reversing mapping class is given by an integer symplectic or anti-symplectic matrix, respectively.

Proof. The action of an orientation-preserving or orientation-reversing mapping class preserves or reverses, respectively, the intersection form on \( H_1(\Sigma_g) \). \( \square \)
Lemma 13. Any integer symplectic or anti-symplectic matrix of size $2g \times 2g$ is obtained as the action induced on the first homology $H_1(\Sigma_g)$ by some orientation-preserving or orientation-reversing mapping class, respectively.

Proof. The statement for orientation-preserving mapping classes and symplectic matrices is a standard fact about the symplectic representation of mapping class groups, see, for example, [3]. The statement for anti-symplectic matrices follows from the fact that multiplication by an anti-symplectic matrix induces an automorphism of the group of symplectic and anti-symplectic matrices. This automorphism sends a symplectic matrix to an anti-symplectic one and vice-versa. In particular, by composing mapping classes which represent all symplectic matrices by an orientation-reversing mapping class, we obtain all anti-symplectic matrices as actions on the first homology.

The following proposition summarises the results of Section 3 so far.

Proposition 14. The monic integer reciprocal and skew-reciprocal polynomials of degree $2g$ are exactly the characteristic polynomials of the actions induced on the first homology of the closed surface of genus $g$ by orientation-preserving and orientation-reversing mapping classes, respectively.

3.3. Pseudo-Anosov mapping classes. A mapping class $f$ of a surface $\Sigma_g$ is pseudo-Anosov if there exists a pair of transverse, singular measured $f$-invariant foliations of $\Sigma_g$ such that $f$ stretches one of them by a factor $\lambda > 1$ and the other one by a factor $\lambda^{-1}$. The number $\lambda$ is called the dilatation of $f$ and is an algebraic integer [12]. Let $\delta_g$ be the smallest dilatation among all pseudo-Anosov mapping classes on $\Sigma_g$. Recall that $\delta_{\text{hom}}^g$ is the minimal spectral radius larger than 1 among actions induced on the first homology of the closed surface of genus $g$ by orientation-preserving mapping classes.

We finish this section by proving Theorem 3.

Proof of Theorem 3. By Theorem 4, the question of Schinzel and Zassenhaus is equivalent to the same question restricted to reciprocal polynomials. This is in turn equivalent to the statement for the characteristic polynomials of actions on the first homology induced by orientation-preserving mapping classes, by Proposition 14. Thus, a positive answer to the question of Schinzel and Zassenhaus is equivalent to the statement: there exists a universal constant $c' > 0$ such that $\delta_{\text{hom}}^g$ satisfies $\delta_{\text{hom}}^g \geq 1 + \frac{c'}{g}$. This is in turn equivalent to the existence of a constant $c > 1$ so that for all $g$,

$$(\delta_{g}^{\text{hom}})^g \geq c.$$  

For one direction, we assume that this inequality holds. Setting $C = \log_e(R')$ yields $(\delta_{g}^{\text{hom}})^g \geq (\log_e(R'))^g = R' \geq (\delta_g)^g$, where $R'$ is the constant from Theorem 5. This implies $(\delta_{g}^{\text{hom}})^g \geq \delta_g$.

For the other direction, assume there exists a universal constant $C$ such that

$$(\delta_g^{\text{hom}})^g \geq \delta_g.$$  

Then, we have $(\delta_g^{\text{hom}})^g \geq (\delta_g)^g \geq R$, where $R > 1$ is the constant in Theorem 5. In particular, it follows that $(\delta_g^{\text{hom}})^g \geq R \frac{1}{g} > 1$, which finishes the proof. 

□
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