Nonlinear dynamics of wave packets in $\mathcal{PT}$-symmetric optical lattices near the phase transition point

Sean Nixon, Yi Zhu, Jianke Yang

1Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05401, USA
2Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing, China

Nonlinear dynamics of wave packets in $\mathcal{PT}$-symmetric optical lattices near the phase-transition point are analytically studied. A nonlinear Klein-Gordon equation is derived for the envelope of these wave packets. A variety of novel phenomena known to exist in this envelope equation are shown to also exist in the full equation including wave blowup, periodic bound states and solitary wave solutions.

Linear Schrödinger operators with complex but parity-time ($\mathcal{PT}$)-symmetric potentials have the unintuitive property that their spectra can be completely real. \[1]\] This phenomenon was first studied in quantum mechanics where a real spectrum is required to guarantee real energy levels. The same phenomenon was later investigated in paraxial optics, where $\mathcal{PT}$-symmetric potentials could be realized by employing symmetric index guiding and an antisymmetric gain/loss profile. \[2,3\] In this optical setting, $\mathcal{PT}$ potentials have been experimentally realized. \[4,5\] In temporal optics, $\mathcal{PT}$-symmetric lattices have been experimentally obtained as well. \[6\] So far, a number of novel physical phenomena in optical $\mathcal{PT}$ systems have been reported, including phase transition ($\mathcal{PT}$-symmetry breaking), nonreciprocal Bloch oscillation, unidirectional propagation, distinct pattern of diffraction, formation of soliton families, and so on. \[7,8\] The search of additional new behaviors in optical $\mathcal{PT}$ systems is still ongoing.

In this Letter, we analytically study nonlinear dynamics of wave packets in $\mathcal{PT}$-symmetric optical lattices near the phase-transition point (where bandgaps close and Bloch bands intersect transversely like the letter 'x'). A nonlinear Klein-Gordon equation is derived for the envelope of these wave packets near the band intersection. Based on this envelope equation, we predict a variety of novel phenomena such as wave splitting, wave blowup, periodic bound states and solitary wave states. We further show these predicted phenomena occur in the full model as well.

The paraxial model for nonlinear propagation of light beams in $\mathcal{PT}$-symmetric optical lattices is

\[i\Psi_z + \Psi_{xx} + V(x)\Psi + \sigma|\Psi|^2\Psi = 0,\]  
\[(1)\]

where $z$ is the propagation axis, $x$ is the transverse axis,

\[V(x) = V_0^2 [\cos(2x) + iW_0 \sin(2x)]\]  
\[(2)\]

is a $\mathcal{PT}$-symmetric potential, $V_0^2$ is the potential depth, $W_0$ is the relative gain/loss strength, and $\sigma = \pm 1$ is the sign of nonlinearity. All variables are non-dimensionalized.

First we discuss the linear diffraction relation of Eq. \[(1)\] at the phase-transition point $W_0 = 1$. \[1,2,11\] In this case, the linear equation of \[(1)\] can be solved exactly. \[11\]

Its Bloch solutions are

\[\Psi_\pm(x,z;\mu) = I_{\pm(k+2m)}(V_0e^{iz})e^{-i\mu z},\]  
\[(3)\]

where $I_k$ is the modified Bessel function, $\mu = (k + 2m)^2$ is the diffraction relation, $k$ is in the first Brillouin zone $k \in [-1,1]$, and $m$ is any nonnegative integer. This diffraction relation is depicted in Fig. \[(1)A\], where different colors indicate different Bloch bands. At $k = 0$ and $\pm 1$, adjacent Bloch bands intersect each other transversely like the letter 'x'. Due to this intersection (degeneracy), wave packets near these points will exhibit novel dynamics which we will reveal next.

At these intersection points, $\Psi_+ = \Psi_-$, thus Bloch solutions \[(3)\] are degenerate and $2\pi$-periodic in $x$. Posed as an eigenvalue problem for $\Psi = \phi(x)e^{-i\mu z}$ in the linear equation of \[(1)\], we get $L\phi = -\mu\phi$, where $L = \partial_{xx} + V_0(x)$, and $V_0(x)$ is the $\mathcal{PT}$ lattice at the phase-transition point $W_0 = 1$. Then at these band-intersection points, the eigenvalues are $\mu = n^2$, where $n$ is any positive integer. The corresponding eigenfunctions are

\[\phi(x) = I_n(V_0e^{iz}) = \sum_{j=0}^{\infty} \frac{(V_0e^{iz}/2)^{2j+n}}{j!(j+n)!}.\]  
\[(4)\]

These eigenvalues $\mu = n^2$ all have geometric multiplicity 1 and algebraic multiplicity 2, thus there exist a generalized eigenfunction $\phi^g$ satisfying

\[(L + \mu)\phi^g = \phi.\]  
\[(5)\]

For $V_0 = \sqrt{6}$ and $n = 1$, this eigenfunction and generalized eigenfunction are plotted in Fig. \[(1)B\].

Now we consider nonlinear dynamics of wave packets near these band-intersection points. For convenience, we first take the $\mathcal{PT}$ lattice to be exactly at the phase-transition point (i.e., $W_0 = 1$). Generalization to lattices near the phase-transition point will be made afterwards.

Nonlinear wave-packet solutions near a band-intersection point can be expanded into a perturbation series

\[\Psi = e^{-i\mu z}[eA(X,Z)\phi(x) + e^2\psi_1 + e^3\psi_2 + \ldots],\]  
\[(6)\]

where $\mu = n^2$ is the propagation constant at the band intersection, $\phi(x)$ is the degenerate Bloch mode given...
in Eq. (4), \(A(X, Z)\) is the envelope of this Bloch mode, \(X = \epsilon x, Z = \epsilon z\) are slow variables, and \(0 < \epsilon < 1\) is a small positive parameter. Substituting the above perturbation series into the original equation (1), this equation at \(O(\epsilon)\) is automatically satisfied. At order \(\epsilon^2\) we have

\[
(L + \mu)\psi_1 = -iA_Z \phi - 2A_X \phi_x. 
\] (7)

The solution to this equation is

\[
\psi_1 = -iA_Z \phi^g - 2A_X \phi^d, 
\] (8)

where \(\phi^g\) is the generalized Bloch mode defined in (5), and

\[
\phi^d = (L + \mu)^{-1}\phi_x. 
\] (9)

This \(\phi^d\) solution exists and can be determined by Fourier series [11].

At \(O(\epsilon^3)\) we get

\[
(L + \mu)\psi_2 = -A_{ZZ} \phi^g + A_{XX} \left(4\phi^d_x - \phi\right) + i2A_{ZX} \left(\phi^d_x + \phi^g_x\right) - \sigma|A|^2A|\phi|^2 \phi. 
\] (10)

The solvability condition of this equation is that its right hand side be orthogonal to the adjoint homogeneous solution \(\phi^g\). Using Fourier expansions of solutions \(\phi^g\) and \(\phi^d\) (see [11]), this solvability condition then yields the following nonlinear Klein-Gordon equation for the envelope function \(A(X, Z)\):

\[
A_{ZZ} - 4n^2A_{XX} + \gamma|A|^2A = 0, 
\] (11)

where

\[
\gamma = (-1)^{n+1}2\sigma n^2 \int_{-\pi}^{\pi} |\phi|^2|\phi|^2 dx. 
\] (12)

In view of the formula (4) for \(\phi\), it is easy to see that \(\text{sgn}(\gamma) = (-1)^{n+1}\gamma\). For the values of \(V_0 = \sqrt{6}\) and \(n = 1\) which we will use in later numerical simulations, \(\gamma \approx 11.0430\sigma\).

Now we extend the above envelope equation to the case where the \(PT\) lattice is near the phase-transition point (i.e., \(W_0 \sim 1\)). Following similar perturbation analysis, we find that when \(\mu = n = 1\) (the lowest band-intersection point) and \(W_0 = 1 - \epsilon^2\), the envelope is governed by a slightly more general nonlinear Klein-Gordon equation,

\[
A_{ZZ} - 4n^2A_{XX} + \alpha A + \gamma|A|^2A = 0, 
\] (13)

where \(\gamma\) is as given in (12), \(\alpha = \epsilon V_0^2/2\), and the \(\psi_1\) solution in (8) is still given by Eq. (9). When \(\mu = n^2\) with \(n = 1\) and \(W_0 = 1 - \epsilon^2\), the envelope equation (13) will contain an additional term proportional to \(iA_Z\). But this \(iA_Z\) term can be eliminated through a gauge transformation \(A \rightarrow A e^{-i\epsilon \bar{V}_0^2/4(n^2 - 1)}\), and the transformed equation remains the same as (13), except that \(\alpha = \epsilon^2 V_0^2/64\) for \(n = 2\) and \(\alpha = 0\) for \(n > 2\). If \(c = 0\) (i.e., at the phase-transition point), then \(\alpha = 0\), hence Eq. (13) reproduces Eq. (11) as a special case. When \(n > 1\), the \(\psi_1\) solution [8] will also contain an additional term proportional to \(iA_Z\).

The nonlinear Klein-Gordon equation (13) is second-order in \(Z\), which means that two initial conditions, \(A(X, 0)\) and \(A_Z(X, 0)\), are needed. These two initial conditions can be obtained from the initial conditions of the perturbation series [6] at orders \(\epsilon\) and \(\epsilon^2\), i.e., from the initial envelope \(A(X, 0)\) as well as \(A_Z|_{z=0}\). This \(A_Z|_{z=0}\) generally contains many eigenmodes of the operator \(L\), but only the \(\phi^g\) component in it affects the dynamics of envelope \(A\). If we denote \(B(X)\) as the envelope function of the \(\phi^g\) component in \(a|z=0\), then by projecting \(a|z=0\) (such as from (8)) onto \(\phi^g\), we find that

\[
B(X) = -i[A_Z(X, 0) + 2nA_X(X, 0)]. 
\] (14)

This relation holds for all \(n\) values. Thus, initial conditions for the original \(PT\) model (1) and those for the envelope equation (13) can be related as

\[
\Psi(x, 0) = \epsilon A(X, 0)\sigma(x) + \epsilon^2 B(X)\phi^g(x). 
\] (15)

This initial-value connection will be used in our numerical simulations later.

Now we examine the envelope dynamics in the nonlinear Klein-Gordon equation (13), and show that the corresponding wavepacket dynamics occurs in the original \(PT\) model (1) too. In this discussion, we take \(n = 1, V_0 = \sqrt{6}\) and \(\epsilon = 0.1\) for definiteness.

First we consider the linear Klein-Gordon equation (13) at the phase-transition point, i.e., \(A_{ZZ} - 4n^2A_{XX} =\)
0. This is the familiar second-order wave equation. It admits unidirectional wave solutions \( F(X - 2nZ) \) as well as bidirectional wave solutions \( F(X - 2nZ) + G(X + 2nZ) \). In the original linear \( \mathcal{PT} \) model \([1]\), we find that the corresponding wavepacket solutions also exist. Examples are displayed in Fig. 2(C,D). Note that the wave splitting in Fig. 2(D) is manifestation of the ‘x’-shaped diffraction structure at the band-intersection point, while the unidirectional propagation in Fig. 2(C) occurs for \( B(X) = 0 \).

FIG. 2: Nonlinear wavepacket solutions below the phase-transition point under self-defocusing nonlinearity. Top: a periodic bound state. Left: envelope solutions \(|A|\); right: full solutions \(|\Psi|\).

Next we consider envelope solutions in the nonlinear Klein-Gordon equation \([13]\) below the phase-transition point, i.e., \( W_0 < 1 \), or \( \alpha > 0 \) (above the phase-transition point, infinitesimal linear waves are unstable, thus it is not pursued). When the nonlinearity is self-defocusing (\( \sigma = -1 \)), \( \gamma \approx -11.0430 < 0 \). In this case, envelope solutions can blow up to infinity in finite distance \([12]\). One such example is shown in Fig. 2(upper left panel) for \( c = 1 \) (\( \alpha = 18 \)) and initial conditions \( A(X,0) = 1.2 \text{sech}(X), A_Z(X,0) = -2A_X(X,0) (B(X) = 0) \). In the full nonlinear \( \mathcal{PT} \) model \([1]\), we have found similar blowup solutions which are displayed in Fig. 2(upper right panel). This solution-blowup under self-defocusing nonlinearity is very surprising. Note that in the full model \([1]\), our asymptotic envelope approximation breaks down as the singular (blowup) point is approached. In this case, the amplitude of the full-model solution eventually saturates, but its power still grows unbounded \([11]\).

Under self-defocusing nonlinearity, the envelope equation \([13]\) also admits \( X \)-localized and \( Z \)-periodic bound states. One example with \( c = 1 \) (below the phase-transition point) is shown in Fig. 2(lower left panel). The initial condition for this solution is \( A(X,0) = \text{sech}(X) \) and \( A_Z(X,0) = 0 \) (\( B(X) = -2iA_X(X,0) \)). In the full nonlinear \( \mathcal{PT} \) model \([1]\), we have found similar periodic bound states as well, see Fig. 2(lower right panel).

Under self-defocusing nonlinearity and below the phase-transition point, the envelope equation \([13]\) also admits stationary solitary waves \( A(X,Z) = F(X)e^{i\omega Z} \) when \( \omega \) lies inside the bandgap \(-\sqrt{\alpha} < \omega < \sqrt{\alpha} \). These solitary envelope solutions correspond to the solitons of the full nonlinear \( \mathcal{PT} \) model \([1]\) reported in \([11]\).

If the nonlinearity is self-focusing (\( \sigma = 1 \)), envelope solutions do not blow up, periodic bound states cannot be found, and stationary solitary waves do not exist in the envelope equation. In this case, nonlinear diffracting solutions similar to the linear diffracting waves reported in \([9]\) exist. In addition, solutions periodic in both \( X \) and \( Z \) can be found.

In summary, we have shown that the nonlinear Klein-Gordon equation governs the envelope dynamics of wavepackets in \( \mathcal{PT} \) lattices near the phase-transition point. We have also shown that a variety of novel phenomena in this envelope equation (such as wave blowup and periodic bound states) occur in the full \( \mathcal{PT} \) model too. These findings open new possibilities for wave engineering in \( \mathcal{PT} \) lattices.

This work is supported in part by AFOSR.

[1] C. Bender and S. Boettcher, “Real spectra in non-Hermitian Hamiltonians having PT symmetry,” Phys. Rev. Lett. 80, 5243-5246 (1998).
[2] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, “Theory of coupled optical PT-symmetric structures,” Opt. Lett. 32, 2632-2634 (2007).
[3] K. G. Makris, R. El-Ganainy, and D.N. Christodoulides, “Beam dynamics in PT-symmetric optical lattices,” Phys. Rev. Lett. 100, 103904 (2008).
[4] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, “Observation of PT-Symmetry Breaking in Complex Optical Potentials,” Phys. Rev. Lett. 103, 093902 (2009).
[5] C. E. Ruefer, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, “Observation of parity-time-symmetry in optics,” Nature Physics 6, 192–195 (2010).
[6] A. Regensburger, C. Bersch, M.A. Miri, G. Onishchukov, D.N. Christodoulides, and U. Peschel, “Paritytime synthetic photonic lattices”, Nature 488, 167-171 (2012).
[7] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, “Optical solitons in PT periodic potentials,” Phys. Rev. Lett. 100, 030402 (2008).
[8] S. Longhi, “Bloch oscillations in complex crystals with PT symmetry”, Phys. Rev. Lett. 103, 123601 (2009).
[9] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, “PT-symmetric optical lattices”, Phys. Rev. A 81, 063807 (2010).

[10] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D.N. Christodoulides, “Unidirectional invisibility induced by PT-symmetric periodic structures”, Phys. Rev. Lett. 106, 213901 (2011).

[11] S. Nixon, L. Ge, and J. Yang, “Stability analysis for solitons in PT-symmetric optical lattices,” Phys. Rev. A 85, 023822 (2012).

[12] F. John, *Nonlinear Wave Equations, Formation of Singularities* (American Mathematical Society, 1990).