Nonlinear Barabási-Albert Network

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In recent years there has been considerable interest in the structure and dynamics of complex networks. One of the most studied networks is the linear Barabási-Albert model. Here we investigate the nonlinear Barabási-Albert growing network. In this model, a new node connects to a vertex of degree \( k \) with a probability proportional to \( k^\alpha \) (\( \alpha \) real). Each vertex adds \( m \) new edges to the network. We derive an analytic expression for the degree distribution \( P(k) \) which is valid for all values of \( m \) and \( \alpha \leq 1 \). In the limit \( \alpha \to -\infty \) the network is homogeneous. If \( \alpha > 1 \) there is a gel phase with \( m \) super-connected nodes. It is proposed a formula for the clustering coefficient which is in good agreement with numerical simulations. The assortativity coefficient \( r \) is determined and it is shown that the nonlinear Barabási-Albert network is assortative (disassortative) if \( \alpha < 1 \) (\( \alpha > 1 \)) and no assortative only when \( \alpha = 1 \). In the limit \( \alpha \to -\infty \) the assortativity coefficient can be exactly calculated. We find \( r = \frac{7}{2} \) when \( m = 2 \). Finally, the minimum average shortest path length \( l_{\text{min}} \) is numerically evaluated. Increasing the network size, \( l_{\text{min}} \) diverges for \( \alpha \leq 1 \) and it is equal to 1 when \( \alpha > 1 \).

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I. INTRODUCTION

In the late 1990s, the researches on complex networks knew an explosive development \(^1\) \(^2\) \(^3\). The impulse came from the convergence of interests between different sectors like physics, biology, technology and sociology turning the study of complex networks into a new interdisciplinary field.

A network is a set of vertices or nodes provided with some rule to connect them by edges. With such a simple definition, it is not a surprise that it has found so much utility in so many areas. Following the classification introduced by Newman \(^1\), one can divide real networks into four types: social, biological, technological and informational.

Social networks are composed by interacting people with some pattern of contacts like friendship, business or sexual partners. One of the most popular works in social webs was carried out by Milgram \(^4\) who first arrived to the concept of “six degrees of separation” and small-world. Social networks also include problems like the collaboration network of film actors \(^5\) \(^6\) or co-authorship among academics \(^7\).

A number of problems can be mapped into biological networks. Some very good examples are: the 282-neuron neural network of the nematode \( C. Elegans \) \(^8\), blood vessels and vascular networks \(^9\) \(^10\), food webs in an ecosystem with species living in a prey-predator scheme \(^11\) \(^12\), network of metabolic pathways \(^13\) and the genetic regulatory network for the expression of a gene \(^14\).

Technological networks are those built by man in his arduous struggle for progress and welfare. Perhaps, the best known example is the electric power grid (high-voltage three-phase transmission lines network) \(^15\) \(^16\). This category also includes the networks of airline routes \(^15\), railways \(^16\) and internet structures \(^15\).

The fourth type is the information network. One of the oldest example of this kind is the network of citations between academic papers \(^18\). This network is certainly influenced by social relationships and it may be somehow contaminated since putting a citation in a paper does not mean that the author had actually read it \(^19\). Another important example is the World Wide Web, which is the network of informations between Web pages \(^20\). It is a directed network with different power laws for the in and out degree distributions.

Motivated by such a number of applications, a myriad of theoretical models were proposed aiming to reproduce or to describe real-world networks. In many of these models, the degree distribution is power law and the corresponding network is said scale-free \(^2\) \(^21\). A good number of real-world networks are of this kind.

One widely studied scale-free network is the linear Barabási-Albert model (BA) \(^21\) \(^22\). In its general form, that we call the nonlinear Barabási-Albert model (NBA), the network is constructed as follows. A new vertex connects to another (already existing) vertex \( i \) with probability \( \Pi(k_i) = k_i^\alpha / \sum_j k_j^\alpha \), where \( k_i \) is the degree, i. e., the number of edges connected to the vertex \( i \) and \( \alpha \) is a real number. Each new vertex adds \( m \) new edges to the network. In the linear case (\( \alpha = 1 \)), the model has many desired characteristics like to be a scale-free network or to have the small-world effect, but it also has some other features which are not so coveted: it is no assortative and the clustering goes to zero with the increasing size of the network. A network is said assortative if it connects preferentially nodes with almost the same degree. It is something like a social stratification by the incomes. If we think of social networks, assortative mixing and clus-

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tering are both yearned qualities \[23\].

In this paper, we investigate the NBA (the model, not the basketball association) by determining some important quantities like the degree distribution, the clustering coefficient \(C\), the assortativity coefficient \(r\) and the average shortest path length \(l\). This is done in the whole space of the parameters \(m\) and \(\alpha\). We obtain some interesting analytical as well as numerical results. Writing the master equation for the number of vertices with degree \(k\), we derive an expression for the degree distribution \(P(k)\) as

\[
P(k) = \frac{m(k-1)^{\alpha}}{\mu + mk^{\alpha}} P(k-1), \quad k \geq m + 1
\]

which has the solution

\[
P(k) = \frac{\mu}{mk^\alpha} \prod_{j=m}^{k} (1 + \frac{\mu}{mj^\alpha})^{-1}.
\]

We can now establish the dependence of the amplitude \(\mu\) on \(\alpha\). Using the relation \(\mu = \sum_{k \geq m} k^{\alpha} P(k)\) together with the equation above we obtain the intrinsic relation

\[
\mu = m^\alpha [1 - m + \sum_{k=m+1}^{\infty} \prod_{j=m+1}^{k} (1 + \frac{\mu}{mj^\alpha})^{-1}].
\]

If \(\alpha < 0\), the derived expressions are still valid but the NBA model loses its usual preferential attachment or capitalistic interpretation. With \(\alpha\) negative, the new vertex connects to those poorest vertices in a say socialistic way. If \(\alpha = -\|\alpha\|\) with \(\|\alpha\| > 1\) and assuming the connectivity \(k\) to be a continuous variable, the degree distribution exhibits a maximum at \(k_{\text{max}} = \frac{m^\|\alpha\|}{\mu} \frac{1}{\|\alpha\|}\). The degree distribution expanded around this maximum has the gaussian form

\[
P(k) \sim\|\alpha\| \exp (-\|\alpha\|^2 \frac{1}{2} (k - k_{\text{max}})^2).
\]

In the limit \(\|\alpha\| \to \infty\) it approaches the Dirac Delta function \(\delta(k - k_{\text{max}})\). The condition \(\sum_{k} k P(k) = 2m\) fixes \(k_{\text{max}} = 2m\). The network is now homogeneous with all vertices having connectivity \(2m\).

If \(\alpha > 1\) and \(m = 1\), the asymptotic behavior of \(M_{\alpha}\) is \(t^\alpha\) and there arise one super-connected node to which almost every other vertex is connected. When \(m > 1\), we found \(m\) super-connected nodes. This result was confirmed exhaustively by our numerical simulations. For large enough networks, the degree histogram always shows (with no exceptions) \(m\) super-connected nodes. The probability \(B\) that the \(m\) initial vertices are connected to all other remaining sites of the network can be calculated. In the discrete time version process, let \(s = t - 1\) \((t = N - m)\). When the \((m+2)\)-th vertex is being aggregated, the probability that it will be connected to all \(m\) initial sites is \(\theta(s) = \prod_{j=1}^{m} \frac{1}{\frac{1}{j} + m^\alpha}\). In general, we have

\[
\theta(s) = \prod_{j=1}^{m} \frac{1}{1 + \frac{s - m^\alpha}{j}}.
\]

and

\[
B = \prod_{s=1}^{\infty} \theta(s).
\]
Clearly, \( B = 0 \) when \( \alpha \leq 2 \) and \( B > 0 \) otherwise, reproducing the results obtained for \( m = 1 \) \cite{24}. Thus the threshold \( \alpha = 2 \) does not depend on \( m \).

III. DEGREE DISTRIBUTION IN THE MEAN FIELD APPROXIMATION

In the mean field context, the degree distribution was obtained for the linear BA network \cite{22} and for the NBA model with link fluctuations \cite{26}. Here we rederive the results using a very simple and straightforward change of stochastic variable.

Assuming the connectivity \( k_i \) to be a continuous variable defined on a site \( i \) of a network with \( N \) vertices, one can write

\[
\frac{\partial k_i}{\partial t} = m \frac{k_i^\alpha}{\sum_{j=1}^{N} k_j^\alpha},
\]

where \( t_i \) is the time when the site \( i \) was incorporated to the network and the initial condition \( k_i(t_i) = m \) was used. Now, if one vertex is to be picked randomly then the probability that the chosen site is \( i \) is \( P(t_i) = \frac{1}{N} \), so the equation above can be seen as a change of stochastic variables \( t_i \rightarrow k_i \) and the respective distributions are related by \( P(k) = \int \delta(k - k_i(t_i))P(t_i)dt_i \). The result is

\[
P(k) = \frac{\mu}{mk^\alpha} \exp\left(-\frac{\mu}{m^\alpha(1-\alpha)}\left[\frac{k}{m}\right]^{1-\alpha} - 1\right),
\]

which is the asymptotic behavior of the exact \( P(k) \) given by Eq.(4) (see also \cite{26}). We will later use this mean field result to derive the clustering coefficient in the next section.

If the same procedure is applied to the BA with the modified preferential attachment

\[
\frac{\partial k_i}{\partial t} = m \frac{(k_i + \lambda)}{N \sum_{j=1}^{N} (k_j + \lambda)}, \quad \lambda > -m,
\]

then the degree distribution is given by

\[
P(k) = \left(2 + \frac{\lambda}{m}\right) \left(\frac{k + \lambda}{m + \lambda}\right)^{-3-\frac{\lambda}{m}},
\]

which, in the asymptotic limit \( k \gg 1 \), is a power law \( k^{-3-\frac{\lambda}{m}} \) with the exponent tuned by the parameter \( \lambda \).

IV. CLUSTERING COEFFICIENT

The clustering coefficient is a transitivity property of the network. If a vertex 1 is connected with vertex 2 and the vertex 2 with the vertex 3 then there is a high probability that the vertex 1 is also connected to 3. In social networks, this can be easily interpreted as the fact that the friend of your friend is likely to be your friend. The clustering coefficient of order \( x \) of a site \( i \), \( C_x(i) \), is defined as the probability that there is a distance of length \( x \) (measured without passing through \( i \)) between two nearest neighbors of the site \( i \) \cite{27}. It is given by

\[
C_x(i) = \frac{2y_i}{k_i(k_i - 1)} ,
\]

where \( y_i \) is the number of such \( x \) distances and \( k_i \) is the degree of the vertex \( i \). For all vertices of the network, the average clustering coefficient is \( C(x) = \frac{\sum_{i} C_x(i)}{N} \).

In this paper, we will only treat the order \( x = 1 \), so we drop the index \( x \) and use simply \( C \) as being the average clustering coefficient. Of course, when \( m = 1 \), \( C \) is always equal to zero (independently of the value of \( \alpha \)) since the graph is a tree like.

We can derive an analytic expression for \( C \) in the mean field approximation. Let \( P(j \rightarrow i) \) be the probability that at the time \( t = j \) the vertex \( j \) is connected to the vertex \( i \). This means that \( P(j \rightarrow i) = \frac{k_j(i)}{\sum_{l=1}^{N} k_j(l)} \), where \( k_j(i) \) is the degree of the vertex \( i \) at the time \( j \). For \( \alpha \leq 1 \), the denominator of this probability is equal to \( \mu_j \) and using Eq.(10) we get

\[
P(j \rightarrow i) = \frac{m^{1+\alpha}[1 + \frac{\mu\alpha\ln(\frac{k_j(i)}{\mu_j})}{\mu}]^{\frac{1-\alpha}{\alpha}}}{\mu_j} j > i.
\]

When \( \alpha \) is equal to one, the expression above reduces to \( \frac{2}{\lambda}(j)\lambda^{-1/2} \) which was first obtained by Klemm and Eguíluz \cite{28}.

To find the clustering coefficient \( C \), care must be taken with the ordering \( j > i \). They cannot be interchanged. If \( i \) and \( j \) are two nearest neighbors vertices of the node \( l \) and we assume \( i < j \), then the contributions come from 3 regimes: \( l < i < j \), \( i < l < j \) and \( i < j < l \). In this way, the average clustering coefficient can be written

\[
C = \frac{1}{N} \left[ \sum_{j=3}^{N} \sum_{i=2}^{j-1} \sum_{l=1}^{i} \frac{P(i \rightarrow l)P(j \rightarrow l)P(j \rightarrow i)}{n_l(N)} + \sum_{j=3}^{N} \sum_{l=2}^{j-1} \sum_{i=1}^{l-1} \frac{P(l \rightarrow i)P(j \rightarrow l)P(j \rightarrow i)}{n_l(N)} + \sum_{i=3}^{N} \sum_{j=2}^{j-1} \sum_{l=1}^{j-1} \frac{P(l \rightarrow i)P(l \rightarrow j)P(j \rightarrow i)}{n_l(N)} \right],
\]

where \( n_l(N) = \frac{k_l(N)[k_l(N)-1]}{2} \) is the total number of pairs of neighbors that the node \( l \) has at the time (or size) \( N \).
For all values of $\alpha \leq 1$, the clustering coefficient goes to zero with $N$. There are two particular cases in which the asymptotic behavior of $C$ is known exactly: when $\alpha = 0$ (random graph) $C \propto N^{-\frac{1}{2}}$ and $\alpha = 1$, $C(N) \propto N^{-1}(\ln N)^2$ \cite{28}. The latter form is more consistent with the numerical simulations results than an earlier theoretical prediction ($C(N) \propto N^{-0.75}$ \cite{29}).

We simulate the nonlinear NBA model in networks with sizes varying from $N = 100$ up to $N = 102,400$ and $m = 2$. For each size, the average clustering coefficient was determined and then average it over 20 to 24,000 independent runs. In the Fig.1, we plot the dependence of $C$ with $\alpha$ for two network sizes $N = 102,400$ and $N = 25,600$.

In the Figure 2, we compare the results of the analytical expression (eq.(16)) with those of the simulated NBA model.

For $\alpha > 1$, the clustering coefficient rapidly approaches the value 1 as we increase the size $N$. This result comes from the fact that the $m$ super-connected nodes are also inter-connected. Thus, when a new vertex is added to the network, the strong preferential dynamics (alpha > 1) forces it to connect with the $m$ super-connected nodes, leading to a clustering coefficient value close to one.

V. MIXING PATTERNS

One important aspect of the networks is how the vertices are linked. If there is a tendency to connect vertices with almost the same degree, we call it an assortative mixing. When the links favor nodes of very different degrees the system is said disassortative. The neutral situation is named no assortative \cite{30,31}. To measure the assortative mixing level, we slightly change the definitions given by Callaway et al. \cite{32}. Let $e_{jk}$ be the joint probability distribution that a randomly chosen edge has degrees $j$ and $k$ at either end. It obeys the sum rules

$$\sum_{jk} e_{jk} = 1, \quad \sum_{jk} e_{jk} = q_k. \quad (17)$$

Originally, $j$ and $k$ were defined as the remaining degrees - the number of edges leaving the vertex other than the one we arrived along. \cite{32}. We can define the assortativity coefficient $r$ as \cite{30,31}.

$$r = \frac{1}{\sigma^2_q} \sum_{jk} jk(e_{jk} - q_jq_k), \quad (18)$$

where $\sigma^2_q = \sum_k k^2q_k - [\sum_k kq_k]^2$ is the variance of the distribution $q_k$. The quantity $r$ lies in the interval $-1 \leq r \leq 1$. It has the value $+1$ ($-1$) if the system is perfectly assortative (disassortative) and it is zero in the no assortative case.

In our simulations, we determine $r$ and average it in a number of 20 to 24,000 runs depending on the network size $N$. In the Figure 3 we plot the assortative coefficient $r$ against the exponent $\alpha$. In the limit of an infinite lattice, $r = 0$ if $\alpha = 1$, a result already known \cite{31}. Our numerical data show that this is the unique case. The NBA model is assortative ($r > 0$) for $\alpha < 1$ and disassortative ($r < 0$) when $\alpha > 1$.

In the limit $\alpha \to -\infty$ the assortativity coefficient can be analytically calculated. For example, if $m = 2$ then the number of vertices $N(k)$ with degree $k$ is $N(2) = 1$, $N(3) = 6$, $N(4) = N - 7$ and zero for $k \geq 5$. The mean degree $< k >$ is equal to $4 - \frac{8}{N}$. The joint probability distribution $e_{ij}$ ($i, j = 2, 3, 4$) can be easily evaluated.
Let us now discuss the dependence of $l_{\text{min}}$ with $N$. We conjecture that both $l$ and $l_{\text{min}}$ scale with $N$ in the same way. It is well known that random networks, such as Erdős-Rényi networks \cite{34,35}, or partially random like the small-world networks \cite{8}, have an average shortest path length scaling as $l \sim \ln(N)$. For scale-free networks having a degree distribution $P(k) \sim k^{-\lambda}$, it was proved that $l \sim \ln(\ln(N)) (l \sim \ln(N))$ if $2 < \lambda < 3$ \cite{3,22}. In the frontier value $\lambda = 3$, the expected dependence is $l \sim \frac{\ln(N)}{\ln(\ln(N))} = \frac{\ln(N)}{\ln(\ln(N))}$.

In the mean field approximation, the linear Barabási-Albert model ($\alpha = 1$) has an exponent $\lambda = 3$ which is independent of $m$. However, in the real linear Barabási-Albert model, simulated in networks with a few millions nodes, the exponent $\lambda$ is actually less than $3$ \cite{22}. Through a careful analysis of our numerical data, we got the exponent $\lambda = 2.91 \pm 0.03$. This gives us an opportunity to check two theoretical predictions: if $m = 1$, the linear BA is a graph tree and the average shortest path length scales as $l \sim \ln(N)$ \cite{3}; if $m \geq 2$, the expected dependence is $l \sim \ln(\ln(N))$ \cite{32}, once actually $\lambda < 3$. Our numerical results corroborate these theoretical previsions as can be seen in the Fig. 5.

\section{Conclusions}

We studied many properties of the nonlinear Barabási-Albert network. Using the master equation, we derive an analytic expression for the degree distribution $P(k)$ which holds for all integer $m \geq 1$ and real $\alpha \leq 1$. In particular, the situation $\alpha < 0$ was focused for the first time. This sector had been neglected in most of the previous

\begin{equation}
    l_{\text{min}} \leq l \leq 2l_{\text{min}}.
\end{equation}
works. We claim that it might be useful in some social networks (since the system is highly assortative there) or in the study of the crossover from the small-world regime, where the shortest path length grows up logarithmically with the network size, to the large-world regime, where it grows up faster with some power of the network size. In the limit $\alpha \to -\infty$, the network is homogeneous with connectivity $2m$. If $\alpha > 1$, there are $m$ super-connected vertices in the gel phase. The probability that the $m$ initial vertices are connected to all the other sites is non null only when $\alpha \geq 2$ and this threshold $\alpha = 2$ does not depend on $m$.

We proposed an analytic formula for the average clustering coefficient $C$. Its validity was verified by numerical simulations. For any fixed network size, $C$ is a monotonically increasing function of $\alpha$. If $N \to \infty$, the clustering coefficient falls to zero for all $\alpha \leq 1$ and rapidly approaches its maximum value $C = 1$ when $\alpha > 1$.

The mixing patterns of the NBA model were determined by the assortativity coefficient $r$. If $\alpha < 1$, the assortativity coefficient $r$ increases with $N$ and converges asymptotically to some real positive value which is smaller than one; if $\alpha = 1$, $r$ decreases with the size $N$ and goes to zero in the limit of an infinite network; if $\alpha > 1$, $r$ diminishes with the size but now converges to the value $-1$. In other words, NBA is assortative (disassortative) if $\alpha < 1$ ($\alpha > 1$) and no assortative only when $\alpha = 1$. In the limit $\alpha \to -\infty$ the assortativity coefficients can be exactly calculated. We find $r = \frac{7}{13}$ when $m = 2$.

Through the minimum average shortest path length $l_{\text{min}}$ we estimate the diameter of the NBA. For a fixed network size, $l_{\text{min}}$ is a monotonically decreasing function of $\alpha$. It goes to infinite if $\alpha \leq 1$ and is equal to 1 when $\alpha > 1$. In the particular (linear) case $\alpha = 1$, $l_{\text{min}} \sim \ln(N)$ if $m = 1$ and $l_{\text{min}} \sim \ln(\ln(N))$ when $m \geq 2$.

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