GENERALIZED LUZIN SETS

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Abstract. In this paper we investigate the notion of generalized
\((\mathcal{I}, \mathcal{J})\) - Luzin set. This notion generalize the standard notion
of Luzin set and Sierpiński set. We find set theoretical condi-
tions which imply the existence of generalized \((\mathcal{I}, \mathcal{J})\) - Luzin set.
We show how to construct large family of pairwise non-equivalent
\((\mathcal{I}, \mathcal{J})\) - Luzin sets. We find a class of forcings which preserves
the property of being \((\mathcal{I}, \mathcal{J})\) - Luzin set.

1. Notation and Terminology

We will use standard set-theoretic notation following [8]. In partic-
ular for any set \(X\) and any cardinal \(\kappa\), \([X]<^\kappa\) denotes the set of all
subsets of \(X\) with size less than \(\kappa\). Similarly, \([X]^\kappa\) denotes the family
of subsets of \(X\) of size \(\kappa\). By \(\mathcal{P}(X)\) we denote the power set of \(X\).

If \(A \subseteq X \times Y\) then for \(x \in X\) and \(y \in Y\) we put
\[A_x = \{y \in Y : (x, y) \in A\}\]
\[A^y = \{x \in X : (x, y) \in A\}\]

By \(A \triangle B\) we denote the symmetric difference of sets \(A\) and \(B\), i.e.
\[A \triangle B = (A \setminus B) \cup (B \setminus A)\]

In this paper \(\mathcal{X}\) denotes uncountable Polish space. By \(\text{Open}(\mathcal{X})\) we
denote the topology of \(\mathcal{X}\). By \(\text{Borel}(\mathcal{X})\) we denote the \(\sigma\)-field of all
Borel sets. Let us recall that each Borel set can be coded by a function
from \(\omega^\omega\). Precise definition of such coding can be found in [7]. If \(x \in \omega^\omega\)
is a Borel code then by \(#x\) we denote the Borel set coded by \(x\).

\(\mathcal{I}, \mathcal{J}\) are \(\sigma\)-ideals on \(\mathcal{X}\), i.e. \(\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\mathcal{X})\) are closed under
countable unions and subsets. Additionally we assume that \([\mathcal{X}]^\omega \subseteq
\mathcal{I}, \mathcal{J}\). Moreover \(\mathcal{I}, \mathcal{J}\) have Borel base i.e each set from the ideal can
be covered by a Borel set from the ideal. Standard examples of such
ideals are the ideal \(\mathcal{L}\) of Lebesgue measure zero sets and the ideal \(\mathcal{K}\) of
meager sets of Polish space.

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function, meager set, null set.
Definition 1.1. Let $M \subseteq N$ be standard transitive models of ZF. Coding Borel sets from the ideal $I$ is absolute iff

\[(\forall x \in M \cap \omega^n)(M \models \# x \in I \leftrightarrow N \models \# x \in I).\]

We say that $\mathcal{I}$ satisfies $\kappa$ chain condition ($\kappa$-c.c.) if every family $\mathcal{A}$ of Borel subsets of $\mathcal{X}$ satisfying the following conditions:

1. $(\forall A \in \mathcal{A})(A \notin I)$
2. $(\forall A, B \in \mathcal{A})(A \neq B \rightarrow A \cap B \in I)$

has size smaller than $\kappa$. If $I$ is $\omega_1$-c.c. then we say that $I$ is c.c.c.

Let us recall that a function $f : \mathcal{X} \rightarrow \mathcal{X}$ is $I$-measurable if the preimage of every open subset of $\mathcal{X}$ is $I$-measurable i.e. belongs to the $\sigma$-field generated by Borel sets and the ideal $I$.

In other words $f$ is $I$-measurable iff

\[(\forall U \in \text{Open}(\mathcal{X}))(\exists B \in \text{Borel}(\mathcal{X}))(\exists I \in I)(f^{-1}[U] = B \triangle I).\]

Let us recall the following cardinal coefficients:

Definition 1.2 (Cardinal coefficients).

\[
\non(I) = \min\{|A| : A \subseteq \mathcal{X} \land A \notin I\}
\]
\[
\add(I) = \min\{|A| : A \subseteq \mathcal{I} \land \bigcup A \notin \mathcal{I}\}
\]
\[
\cov(I) = \min\{|A| : A \subseteq \mathcal{I} \land \bigcup A = \mathcal{X}\}
\]
\[
\covh(I) = \min\{|A| : A \subseteq \mathcal{I} \land (\exists B \in \text{Borel}(\mathcal{X}) \setminus I)(B \subseteq \bigcup A)\}
\]
\[
\cof(I) = \min\{|A| : A \subseteq \mathcal{I} \land A \text{ is a base of } \mathcal{I}\}
\]

where $\mathcal{A}$ is a base of $\mathcal{I}$ iff $\mathcal{A} \subseteq \mathcal{I} \land (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)$.

Let us remark that above coefficients can be defined for larger class of families (not only ideals).

Definition 1.3. We say that $L \subseteq \mathcal{X}$ is a $(\mathcal{I}, \mathcal{J})$- Luzin set if

- $L \notin \mathcal{I}$,
- $(\forall B \in \mathcal{I})(B \cap L \in \mathcal{J})$.

Assume that $\kappa$ is a cardinal number. We say that $L \subseteq \mathcal{X}$ is a $(\kappa, \mathcal{I}, \mathcal{J})$- Luzin set iff $L$ is a $(\mathcal{I}, \mathcal{J})$- Luzin set and $|L| = \kappa$.

The above definition generalizes the standard notion of Luzin and Sierpiński sets. Namely, $L$ is Luzin set iff $L$ is generalized $(\mathbb{L}, \mathbb{R}^{\leq \omega})$ - Luzin set and $S$ is Sierpiński set iff $S$ is generalized $(\mathbb{K}, \mathbb{R}^{\leq \omega})$ - Luzin set. The above notion generalizes also notions from [2].

Definition 1.4. We say that ideals $\mathcal{I}$ and $\mathcal{J}$ are orthogonal if

\[
(\exists A \in \mathcal{P}(\mathcal{X})) A \in I \land A^c \in J.
\]

In such case we write $\mathcal{I} \perp \mathcal{J}$.

Definition 1.5. Let $\mathcal{F} \subseteq \mathcal{P}(\mathcal{X})$ be a family of functions. We say that $A, B \subseteq \mathcal{X}$ are equivalent with respect to $\mathcal{F}$ if

\[
(\exists f \in \mathcal{F}) (B = f[A] \lor A = f[B])
\]
Definition 1.6. We say that $A, B \subseteq X$ are Borel equivalent if $A, B$ are equivalent with respect to the family of all Borel functions.

Definition 1.7. We say that $I$ has Fubini property iff for every Borel set $A \subseteq X \times X$

\[
\{x \in X : A_x \notin I\} \in I \implies \{y \in X : A^y \notin I\} \in I
\]

Natural examples of ideals fulfilling Fubini property are the ideal of null sets $L$ (by Fubini theorem) and the ideal of meager sets $K$ (by Kuratowski-Ulam theorem).

By definition we can obtain the following properties:

Fact 1.1. Assume that $I \perp J$.

1. There exist a $(I, J)$ - Luzin set.
2. If $L$ is a $(I, J)$ - Luzin set then $L$ is not $(J, I)$ - Luzin set.

Proof. (Part 1) By the definition of $I \perp J$ we can find two sets $I \in I$ and $J \in J$ such that $I \cup J = X$. We will show that $J$ is $(I, J)$ - Luzin set. $J$ is not in $I$. Let us fix any set $A \in I$. We have that $A \cap J \subseteq J \notin J$. By the property of being $(J, I)$ - Luzin set

$L \cap J \subseteq J \notin J$ and $L \cap I \subseteq I \in I$.

By the property of being $(J, I)$ - Luzin set

$L \cap J \in I$.

So $L = (L \cap J) \cup (L \cap I) \in I$. what is a contradiction with being $(J, I)$ - Luzin set.

We will try to find a wide class of forcings which preserves the property of being $(I, J)$ - Luzin set. We are mainly interested in so called definable forcings (see [11]). Let us recall that $P$ is definable forcing if $P$ is of the form $Borel(X) \setminus I$, where $X$ and $I$ have absolute definition for standard transitive models of ZF of the same hight.

2. Existence of Luzin sets

Let us start with a theorem which under suitable assumptions guarantees existence of uncountably many pairwise different $(I, J)$ - Luzin sets.

Theorem 2.1. Assume that $\kappa = cov(I) = cof(I) \leq non(J)$. Let $F$ be a family of functions from $X$ to $X$. Assume that $|F| \leq \kappa$. Then we can find a sequence $(L_\alpha)_{\alpha < \kappa}$ such that

1. $L_\alpha$ is $(\kappa, I, J)$ - Luzin set,
2. for $\alpha \neq \beta$, $L_\alpha$ is not equivalent to $L_\beta$ with respect to the family $F$.
Proof. Let us enumerate the family $F$:
\[ F = \{ f_\alpha : \alpha < \kappa \}. \]

Now, let us enumerate Borel base of ideal $\mathcal{I}$:
\[ B_\mathcal{I} = \{ B_\alpha : \alpha < \kappa \}. \]

Now without loss of generality we can assume that
\[ (\forall f \in F)(\forall \lambda < \kappa)(\kappa \leq |f[\bigcup_{\xi<\lambda} B_\xi]^c|) \]
Indeed, since $\text{cov}(\mathcal{I}) = \kappa$ a set $\bigcup_{\xi<\lambda} B_\xi$ is not in the ideal $\mathcal{I}$. If the function $f$ does not have the above property and $L$ is a $(\mathcal{I}, \mathcal{J})$-Luzin set then
\[ f[L] = f[L \cap \bigcup_{\xi<\lambda} B_\xi] \cup f[L \cap (\bigcup_{\xi<\lambda} B_\xi)^c] \]
and both sets has cardinality less than $\kappa$. So $f[L]$ is not $(\mathcal{I}, \mathcal{J})$-Luzin set.

By induction we will construct the family $\{ x_{\eta, \zeta, \alpha} : \eta, \zeta, \alpha < \kappa \}$ and $\{ d_{\eta, \zeta, \alpha} : \eta, \zeta, \alpha < \kappa \}$ such that
\[ d_{\eta, \zeta, \alpha} = f_\zeta(x_{\eta, \zeta, \alpha}) \]
and for any different $\eta, \eta' < \kappa$
\[ \{ x_{\eta, \zeta, \alpha} : \zeta, \alpha < \kappa \} \cap \{ d_{\eta', \zeta, \alpha} : \zeta, \alpha < \kappa \} = \emptyset \]
and
\[ x_{\eta, \zeta, \alpha} \in \mathcal{F} \setminus \left( \{ d_{\eta, \zeta, \alpha} : \eta, \xi, \zeta < \alpha \} \cup \{ x_{\eta, \zeta, \alpha} : \eta, \xi, \zeta < \alpha \} \cup \bigcup_{\xi<\alpha} B_\xi \right) \]
for every $\eta, \zeta < \alpha$.

Assume that we are in $\alpha$-th step of construction. Fix $\eta, \zeta < \alpha$. It means that we have constructed the following set
\[ \text{Old} = \{ x_{\beta, \xi, \delta} : \beta, \xi, \delta < \alpha \} \cup \{ x_{\lambda, \xi, \delta} : \lambda < \eta \land (\lambda = \eta \land \xi < \zeta) \}. \]

Since $|f_\zeta[\bigcup_{\xi<\alpha} B_\xi]^c| \geq \kappa$ and $|\text{Old}| < \kappa$ we get that
\[ |f_\zeta[\bigcup_{\xi<\alpha} B_\xi \cup \text{Old}]^c| \geq \kappa. \]

That’s why we can find
\[ d_{\eta, \zeta, \alpha} \in f_\zeta[\bigcup_{\xi<\alpha} B_\xi \cup \text{Old}]^c \setminus \text{Old}. \]

Let $x_{\eta, \zeta, \alpha}$ be such that $d_{\eta, \zeta, \alpha} = f_\zeta(x_{\eta, \zeta, \alpha})$. In this way we can finish the $\alpha$-th step of construction.

Now, let us define $L_\alpha = \{ x_{\eta, \zeta, \alpha} : \xi, \zeta < \kappa \}$.

Let us check that $L_\alpha$ is $(\mathcal{I}, \mathcal{J})$ - Luzin set. Indeed, if $A \in \mathcal{I}$ then there exists $\beta < \kappa$ s.t. $A \subset B_\beta$. Then we have
\[ A \cap L_\alpha \subset B_\beta \cap L_\alpha = B_\beta \cap \{ x_{\xi, \zeta, \alpha} : \xi, \zeta < \beta \} \subseteq \{ x_{\xi, \zeta, \alpha} : \xi, \zeta < \beta \} \in \mathcal{J} \]
because $|\{x^\alpha_{\xi,\zeta} : \xi, \zeta < \beta\}| \leq |\beta| < \kappa \leq \text{non}(\mathcal{J})$.

What is more, for every function $f = f_\alpha \in \mathcal{F}$ and every $\beta \neq \gamma$ we have that

$$\kappa \leq |f[L_\gamma] \setminus L_\beta|$$

because $\{d^\alpha_{\xi,\alpha} : \alpha < \xi < \kappa\} \subseteq f[L_\gamma] \setminus L_\beta$. So $L_\beta \neq f[L_\gamma]$. □

In fact we have proved a little stronger result.

Remark 2.1. Assume that $\kappa = \text{cov}(\mathcal{I}) = \text{cof}(\mathcal{I}) \leq \text{non}(\mathcal{J})$.

Let $\mathcal{F}$ be a family of functions from $X$ to $X$.

Assume that $|\mathcal{F}| \leq \kappa$.

Then we can find a sequence $(L_\alpha)_{\alpha < \kappa}$ such that

1. $L_\alpha$ is $(\kappa, \mathcal{I}, \mathcal{J})$ - Luzin set,
2. for $\alpha \neq \beta$ and $f \in \mathcal{F}$ we have that $\kappa \leq |f[L_\alpha] \triangle L_\beta|$.

Let us notice that for every ideal $\mathcal{I}$ we have the inequality $\text{cov}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$.

This gives the following corollary.

Corollary 2.1. If $2^\omega = \text{cov}(\mathcal{I}) = \text{non}(\mathcal{J})$ then there exists continuum many different $(\mathcal{I}, \mathcal{J})$ - Luzin sets which aren’t Borel equivalent.

In particular, if CH holds then there exists continuum many different $(\omega_1, \mathcal{I}, \mathcal{J})$ - Luzin sets which aren’t Borel equivalent.

We can extend above corollary to a wilder class of functions - namely, $\mathcal{I}$-measurable functions.

Corollary 2.2. If $2^\omega = \text{cov}(\mathcal{I}) = \text{non}(\mathcal{J})$ then there exists continuum many different $(\mathcal{I}, \mathcal{J})$ - Luzin sets which aren’t equivalent with respect to all $\mathcal{I}$-measurable functions.

In particular, if CH holds then there exists continuum many different $(\omega_1, \mathcal{I}, \mathcal{J})$ - Luzin sets which aren’t equivalent with respect to all $\mathcal{I}$-measurable functions.

Proof. First, let us notice that if a function $f$ is $\mathcal{I}$-measurable then there exists a set $I \in \mathcal{I} \cap \text{Borel}(\mathcal{X})$ such that $f \upharpoonright (\mathcal{X} \setminus I)$ is Borel. Indeed, it is enough to consider a countable base $\{U_n\}_{n \in \omega}$ of topology of $\mathcal{X}$. Then $f^{-1}[U_n] = B_n \triangle I_n$, where $B_n$ is Borel and $I_n$ is from the ideal $\mathcal{I}$. Now, put $I = \bigcup_{n \in \omega} I_n$.

So we can consider a family of partial Borel functions which domain is Borel set with complement in the ideal $\mathcal{I}$. This family is naturally of size continuum. So we can use Corollary 2.1 and Remark 2.1 to finish the proof. □

Now, let us concentrate on ideal of null and meager sets.

Corollary 2.3. (1) Assume that $\text{cov}(\mathcal{L}) = 2^\omega$. There exists continuum many different $(2^\omega, \mathcal{L}, \mathcal{K})$ - Luzin sets which aren’t equivalent with respect to the family of Lebesgue - measurable functions.
(2) Assume that \( \text{cov}(K) = 2^\omega \). There exists continuum many different \( (2^\omega, K, L) \)-Luzin sets which aren’t equivalent with respect to the family of Baire-measurable functions.

**Proof.** Let us notice that the equality \( \text{cov}(L) = 2^\omega \) implies that \( 2^\omega = \text{cov}(L) = \text{cov}(L) = \text{non}(K) \). Similarly, the equality \( \text{cov}(K) = 2^\omega \) implies that \( 2^\omega = \text{cov}(K) = \text{cov}(K) = \text{non}(L) \) (see [1]). Corollary 2.2 finishes the proof. \( \square \)

3. **Luzin sets and forcing**

Now, let us focus on the class of forcings which preserves being \((\mathcal{I}, \mathcal{J})\)-Luzin set. Let us start with a technical observation.

**Lemma 3.1.** Assume that \( \mathcal{I} \) has Fubini property. Suppose that \( P_{\mathcal{I}} = \text{Borel}(X) \setminus \mathcal{I} \) is a proper definable forcing. Let \( B \in \mathcal{I} \) be a set in \( V^{P_{\mathcal{I}}}[G] \). Then \( B \cap X \subseteq \mathcal{I} \).

**Proof.** Let \( \dot{B} \) – name for \( B \), \( \dot{r} \) – canonical name for generic real, \( C \subseteq X \times X \) - Borel set from the ideal \( \mathcal{I} \). \( C \) is coded in ground model \( V \) and \( B = C_{\dot{r}} \).

Now by Fubini property:
\[
\{ x : C^x \notin \mathcal{I} \} \in \mathcal{I}.
\]

Let \( x \in B \cap X \subseteq \mathcal{I} \).

Then we have:
\[
B \cap X \subseteq \{ x : C^x \notin \mathcal{I} \} \in \mathcal{I}.
\]

But the last set is coded in ground model because the set \( C \) was coded in \( V \). \( \square \)

**Theorem 3.1.** Assume that \( \omega < \kappa \) and \( \mathcal{I} \), \( \mathcal{J} \) are c.c.c. and have Fubini property. Suppose that \( P_{\mathcal{I}} = \text{Borel}(X) \setminus \mathcal{I} \) and \( P_{\mathcal{J}} = \text{Borel}(X) \setminus \mathcal{J} \) are definable forcings. Then \( P_{\mathcal{J}} \) preserves \((\kappa, \mathcal{I}, \mathcal{J})\)-Luzin set property.

**Proof.** Let \( L \) be a \((\kappa, \mathcal{I}, \mathcal{J})\)-Luzin set in \( V \). In \( V[G] \) take any \( B \in \mathcal{I} \) then \( L \cap B \cap V = L \cap B \) but \( L \cap B \in \mathcal{I} \) in \( V \) so \( L \cap B \in \mathcal{I} \) in \( V \) by definition of \( L \). Finally, by Lemma 3.1
\[
L \cap B = L \cap B \cap V \in \mathcal{J} \text{ in } V[G].
\]

\( \square \)

**Theorem 3.2.** Let \( (P, \leq) \) be a forcing notion such that
\[
\{ B : B \in \mathcal{I} \cap \text{Borel}(X), B \text{ is coded in } V \}
\]
is a base for \( \mathcal{I} \) in \( V^P[G] \). Assume that Borel codes for sets from ideals \( \mathcal{I}, \mathcal{J} \) are absolute. Then \( (P, \leq) \) preserve being \((\mathcal{I}, \mathcal{J})\)-Luzin sets.
Proof. Let $L$ be a $(\mathcal{I}, \mathcal{J})$ - Luzin set in ground model $V$. We will show that $V^P[G] \models L$ is $(\mathcal{I}, \mathcal{J})$ - Luzin set.

Let us work in $V^P[G]$. Fix $I \in \mathcal{I}$. $I$ has Borel base consisting of sets coded in $V$. So, there exists $b \in \omega^\omega \cap V$ such that $I \subseteq \#b \in \mathcal{I}$.

By absoluteness of Borel codes from $I$ we have that $V \models \#b \in \mathcal{I}$.

$L$ is a $(\mathcal{I}, \mathcal{J})$ - Luzin set in the model $V$. So, there is $c \in \omega^\omega \cap V$ which codes Borel set from the ideal $\mathcal{J}$ such that $V \models L \cap \#b \subseteq \#c$.

By absoluteness of Borel codes from $\mathcal{J}$ we get that $V^P[G] \models L \cap B \subseteq L \cap \#b \subseteq \#c \in \mathcal{J}$, what proves that $L$ is a $(\mathcal{I}, \mathcal{J})$ - Luzin set in generic extension. □

The above theorem gives us a series of corollaries.

**Corollary 3.1.** Let $(\mathbb{P}, \leq)$ be any forcing notion which does not change the reals i. e. $(\omega^\omega)^V = (\omega^\omega)^{V[G]}$. Assume that Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserve being $(\mathcal{I}, \mathcal{J})$ - Luzin sets.

**Corollary 3.2.** Assume that $(\mathbb{P}, \leq)$ is a $\sigma$-closed forcing and Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserve $(\mathcal{I}, \mathcal{J})$ - Luzin sets.

**Corollary 3.3.** Let $\lambda \in \text{On}$ be an ordinal number. Let $\mathbb{P}_\lambda = \langle (P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda \rangle$ be iterated forcing with countable support. Spouse that

1. for any $\alpha < \lambda P_\alpha \Vdash \dot{Q}_\alpha$ - $\sigma$ closed,
2. Borel codes for sets from ideals $\mathcal{I}, \mathcal{J}$ are absolute,

then $\mathbb{P}_\lambda$ preserve $(\mathcal{I}, \mathcal{J})$ - Luzin sets.

**Proof.** Our forcing $\mathbb{P}_\lambda$ is $\sigma$-closed because it is countable support iteration of $\sigma$ -closed forcings. So, we can apply Corollary 3.2 to finish the proof. □

Now, let us consider some properties of countable support iteration connected with preservation of some relation. We will follow notation given by Goldstern (see [4]).

First, let us consider measure case. Let $\Omega$ is a family of clopen sets of Cantor space $2^\omega$ and

$$C_{\text{random}} = \{f \in \Omega^\omega : (\forall n \in \omega)\mu(f(n)) < 2^{-n}\}$$

with discrete topology. If $f \in C_{\text{random}}$ then let us define the following set $A_f = \bigcap_{n \in \omega} \bigcup_{k \geq n} f(k)$.

Now, we are ready to define the following relation $\subseteq = \bigcup_{n \in \omega} \subseteq_n$ where

$$(\forall f \in C_{\text{random}})(\forall g \in 2^\omega)(f \subseteq_n g \iff (\forall k \geq n) g \notin f(k)).$$

Definition of the notion of preservation of relation $\subseteq_{\text{random}}$ by forcing notion $(\mathbb{P}, \leq)$ can be found in paper [4]. Let us focus on the following consequence of that definition.
Fact 3.1 (Goldstern). If \((\mathbb{P}, \leq)\) preserves \(\sqsubseteq_{\text{random}}\) then \(\mathbb{P} \Vdash \mu^*(2^\omega \cap V) = 1\).

Now, we say that forcing notion \(\mathbb{P}\) preserves outer measure iff \(\mathbb{P}\) preserves \(\sqsubseteq_{\text{random}}\).

It is well known that Laver forcing preserves some stronger property than \(\sqsubseteq_{\text{random}}\) (see [5]). So, Laver forcing preserves outer measure.

In [4] we can find the following theorem:

**Theorem 3.3** (Goldstern). Let \(\mathbb{P}_\lambda = ((\mathbb{P}_\alpha, \mathbb{Q}_\alpha) : \alpha < \gamma)\) be any countable support iteration such that

\[
(\forall \alpha < \gamma) \mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha \text{ preserves } \sqsubseteq_{\text{random}}
\]

then \(\mathbb{P}_\gamma\) preserves the relation \(\sqsubseteq_{\text{random}}\).

**Theorem 3.4.** Assume that \(\mathbb{P}\) is a forcing notion which preserves \(\sqsubseteq_{\text{random}}\). Then \(\mathbb{P}\) preserves being \((\mathbb{L}, \mathbb{K})\)-Luzin set.

**Proof.** Assume that \(V \models L\) is \((\mathbb{L}, \mathbb{K})\)-Luzin set. Let us work in \(V^\mathbb{P}[G]\). Take any null set \(A \in \mathbb{L}\). Then there is a null set \(B \in V\) such that \(A \cap V \subseteq B\).

Indeed, let us assume that there is no such \(B \in V\). Then without loss of generality \((2^\omega \setminus A) \cap V \in \mathbb{L}\). But \(A \in \mathbb{L}\) then we have that \(2^\omega \cap V \subseteq A \cup ((2^\omega \setminus A) \cap V)\) which is a null set. But by Fact 3.1 \(\mu^*(2^\omega \cap V) = 1\). So we have a contradiction.

Then intersection \(A \cap L \subseteq B \cap L \in \mathbb{K}\) is a meager set in ground model. Then by absoluteness of borel codes of meager sets the set \(A \cap L\) is a meager set what finishes the proof. \(\square\)

**Remark 3.1.** In constructible universe \(L\) let us consider the countable forcing iteration \(\mathbb{P}_{\omega_2} = ((\mathbb{P}_\alpha, \mathbb{Q}_\alpha) : \alpha < \omega_2)\) of the length \(\omega_2\) as follows, for any \(\alpha < \omega_2\)

- if \(\alpha\) is even then \(\mathbb{P}_\alpha \Vdash \text{"}Q_\alpha\text{ is random forcing"},\)
- in odd case \(\mathbb{P}_\alpha \Vdash \text{"}Q_\alpha\text{ is Laver forcing"}\).

Previously we noticed that both random and Laver forcing, preserves \(\sqsubseteq_{\text{random}}\) and then by Theorem 3.3 \(\mathbb{P}_{\omega_2}\) preserves relation \(\sqsubseteq_{\text{random}}\). By Theorem 3.4 the \((\mathbb{L}, \mathbb{K})\)-Luzin sets are preserved by our iteration \(\mathbb{P}_{\omega_2}\). Moreover, in generic extension we have \(\text{cov}(\mathbb{L}) = \omega_2\) and \(2^\omega = \omega_2\) (for details see [4]).

Assume that in the ground model \(A\) is \((\mathbb{L}, \mathbb{K})\)-Luzin set with outer measure equal to one. Then in generic extension it has outer measure one and \(|A| = \omega_1\). So, it does not contain any Lebesgue positive Borel set. Thus \(A\) is completely \(\mathbb{L}\)-nonmeasurable set.

The analogous machinery can be used for ideal of meager sets \(\mathbb{K}\). Let us recall the necessary definitions (see [4]).

Let \(C^{\text{Cohen}}\) be set of all functions from \(\omega^{<\omega}\) into itself. Then \(C^{\text{Cohen}} = \bigcup_{n \in \omega} C_n^{\text{Cohen}}\) and for any \(n \in \omega\) let

\[
(\forall f \in C^{\text{Cohen}})(\forall g \in \omega^n)(f \sqsubseteq_n^{\text{Cohen}} g \text{ iff } (\forall k < n)(g \upharpoonright k \subseteq f(g \upharpoonright k) \subseteq g)).
\]
Then finally we have the following theorem:

**Theorem 3.5.** Assume that \( P \) is a forcing notion which preserves \( \subseteq^{\text{Cohen}} \). Then \( P \) preserves being \((K, L)\)-Luzin set.

The another preservation theorem which is due to Shelah (see [9] and also [10]) is as follows

**Theorem 3.6** (Shelah). Let \( P_\lambda = ((P_\alpha, Q_\alpha) : \alpha < \lambda) \) be any countable support iteration such that (\( \forall \alpha < \gamma \)) \( P_\alpha \Vdash Q_\alpha \) is proper and

\[
P_\alpha \Vdash Q_\alpha \Vdash \text{ every new open dense set contains old open dense set}
\]

then \( P_\lambda \Vdash \text{ every new open dense contains old open dense set} \).

We can easily derive

**Corollary 3.4.** Let \( P_\lambda = ((P_\alpha, Q_\alpha) : \alpha < \lambda) \) be any countable support iteration such that (\( \forall \alpha < \lambda \)) \( P_\alpha \Vdash Q_\alpha \) is proper and

\[
P_\alpha \Vdash Q_\alpha \Vdash \text{ every new open dense set contains old open dense set}
\]

Then \( P_\lambda \) preserves being \((K, L)\)-Luzin set.

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