DOUBLING PROPERTIES OF CALORIC FUNCTIONS

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"Science is the great antidote to the poison of enthusiasm and superstition" (Adam Smith, Wealth of Nations, 5, III, 3)

Abstract. We obtain quantitative estimates of unique continuation for solutions to parabolic equations: doubling properties and two-sphere one-cylinder inequalities.

1. Introduction

This work is concerned with quantitative estimates of unique continuation for parabolic equations. In the context of elliptic equations the quantitative estimates of unique continuation, which have been derived from the nowadays standard methods to prove qualitative results of unique continuation (Carleman or frequency function methods) are the so called doubling properties and optimal three spheres inequalities. In particular, it is by now well known that when

\[ E = \sum_{i,j=1}^{n} \partial_i(g^{ij}(x)\partial_j) \]

is an elliptic operator in \( \mathbb{R}^n \), there is a constant \( N \) depending on \( n, M \), the parameters of ellipticity and the Lipschitz norm of the matrix of coefficients of the operator \( E \) such that, solutions of the inequality, \( |Eu| \leq M (|u| + |\nabla u|) \) on \( B_4 \), verify the doubling property \[ \int_{B_{2r}(z)} u^2 \, dx \leq (N\Theta)^N \int_{B_r(z)} u^2 \, dx , \]

when \( |z| \leq 1, 0 < r \leq 1/2 \) and

\[ \Theta = \int_{B_4} u^2 \, dx / \int_{B_1} u^2 \, dx \]

and the optimal three sphere inequality \( \[13, 14\] \)

\[ \int_{B_4} u^2 \, dx \leq \left( \int_{B_r} u^2 \, dx \right)^{\frac{1}{1 + N \log (1/r)}} \left( N \int_{B_4} u^2 \, dX \right)^{\frac{N \log (1/r)}{1 + N \log (1/r)}} , \]

when \( 0 < r \leq 1 \).

The three sphere inequality is called optimal for the following reason. If one seeks to find the largest possible function \( \theta : (0,1) \to (0,1) \) such that the three
sphere inequality

\[(1.1) \quad \int_{B_1} u^2 \, dx \leq \left( \int_{B_r} u^2 \, dx \right)^{\theta(r)} \left( N \int_{B_4} u^2 \, dX \right)^{1-\theta(r)} \]

holds for some positive constant \(N\), for all harmonic functions \(u\) in \(B_4\) and \(0 < r < 1\), one finds from the identity

\[\int_{B_r} u^2(x) \, dx = \frac{r^{2k+n}}{2k+n} \int_{\partial B_1} u^2 \, d\sigma ,\]

which holds when \(u\) is a homogeneous harmonic polynomial of degree \(k \geq 1\), and \((1.1)\) that

\[1/4 \leq (r/4)^{\theta} N^{\frac{k-n}{2k+n}} \quad \text{for all} \quad k \geq 1 .\]

Thus, \(\theta(r)\) cannot be larger than \(\log 4/\log (4/r)\). On the other hand, the recent applications to stability issues in Inverse Problems of quantitative estimates of unique continuation \([2, 3]\) show it is useful to know that “harmonic” functions are as small as they can be at intermediate scales when they are known to be small at smaller scales.

The first quantitative result of strong unique continuation for parabolic equations, a two-sphere one-cylinder inequality \([13, \text{Theorem 9}']\) (see the body of the paper for the relevant definitions), was derived in 1974 from the corresponding first qualitative results of strong unique continuation for second order parabolic equations in the literature \([13]\). In \([13]\), E.M. Landis and O.A. Oleinik used a reduction argument to the already established elliptic results of unique continuation, which relays on the representation formula of solutions to parabolic equations with time independent coefficients in terms of the eigenfunctions of the corresponding elliptic operator, and thus their results did not apply to parabolic equations with time dependent coefficients. After 1988 when more results of unique continuation for elliptic equations were available in the literature, the qualitative results in \([13]\) were reproved with less regularity assumptions in \([14]\). The quantitative results in \([13]\) were also reproved with less regularity assumptions in \([5]\). The authors in \([14]\) and \([5]\) used the same reduction argument as in \([13]\) and as a consequence, their results only apply to parabolic operators with time independent coefficients.

In the same way as it has not been a trivial task to derive quantitative estimates of unique continuation for elliptic operators from the corresponding qualitative results (See \([11]\) and \([14]\)), the same happens in the parabolic case.

In \([10]\), \([6]\), \([7]\), \([8]\), \([9]\), \([10]\) and \([11]\) the authors have obtained with standard arguments of unique continuation, frequency functions or Carleman methods, qualitative results of strong unique continuation for parabolic equations with time dependent coefficients. In particular, between \([9]\) and \([10]\) the following qualitative-quantitative result is proved in relation with backward parabolic operators

\[P = \sum_{i,j=1}^{n} \partial_i \left( g^{ij}(X) \partial_j \right) + \partial_t ,\]
which for some positive constants \( \lambda \) and \( M \) satisfy the conditions

\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} g^{ij}(X)\xi_i \xi_j \leq \lambda^{-1}|\xi|^2 , \tag{1.2}
\]

\[
\sum_{i,j=1}^{n} |g^{ij}(X) - g^{ij}(Y)| \leq M \left( |x-y|^2 + |t-s| \right)^{1/2} , \tag{1.3}
\]

when \( X = (x,t) \), \( Y = (y,s) \) are in \( \mathbb{R}^{n+1} \) and \( \xi \) is in \( \mathbb{R}^n \).

**Theorem 1.** Let \( P \) be a backward second order parabolic operator as above. Then, there is a constant \( N = N(\lambda, M, n) \) such that, if \( u \) satisfies

\[
|Pu| \leq M (|u| + |\nabla u|)
\]

in \( Q_2 \) and \( u(x,0) \) has a zero of infinite order at \( x = 0 \), the following holds in \( Q_1 \)

\[
|u(x,t)| \leq Ne^{-1/(N\Theta)}\|u\|_{L^\infty(Q_2)} .
\]

Here and in the sequel \( B_r(z) = \{ x \in \mathbb{R}^n : |x - z| < r \} \), \( B_r = B_r(0) \), \( Q_r(z, \tau) = B_r(z) \times [\tau, \tau + r^2] \) and \( Q_r = Q_r(0,0) \).

Theorem \([1]\) gives a qualitative-quantitative result of strong unique continuation but is missing the proper and natural quantitative estimates of strong unique continuation for second order parabolic equations: doubling properties within characteristic hyperplanes and two-sphere one-cylinder inequalities. Here, we prove these quantitative estimates with new arguments, which are based on the frequency functions or Carleman inequalities appearing in \([10], [9] \) and \([10] \) and extend their applicability to solutions of parabolic equations with time dependent coefficients. In particular, the following theorem is proved:

**Theorem 2.** Let \( P \) be a backward second order parabolic operator verifying (1.2) and (1.3) and \( u \) satisfy \( |Pu| \leq M (|u| + |\nabla u|) \) in \( Q_1 \). Then, there is a constant \( N = N(\lambda, M, n) \) such that the following properties hold when \( 0 < r \leq 1/2 \)

1. \[
\int_{B_r} u^2(x,0) \, dx \leq (N\Theta)^N \int_{B_1} u^2(x,0) \, dx .
\]
2. \[
\int_{B_1} u^2(x,0) \, dx \leq \left( \int_{B_2} u^2(x,0) \, dx \right)^{\frac{N \log (1/r)}{N \log (1/\Theta)}} \left( N \int_{Q_1} u^2 \, dX \right)^{\frac{1}{N \log (1/\Theta)}} .
\]
3. If \( u_{|z=0} \) is not identically zero, then \( |u|_{L^1} \) is a Muckenhoupt weight in \( B_{1/2} \).

Here, \( \Theta = \int_{Q_1} u^2 \, dX/ \int_{B_1} u^2(x,0) \, dx \) and \( dX = dxdt \).

Recall that a nonzero and locally integrable function \( w \) in \( B_1 \) is called a Muckenhoupt weight when there are numbers \( p \in (1, +\infty) \) and \( C > 0 \) such that

\[
\left( \int_{B_r(z)} w^p \, dx \right)^{\frac{1}{p}} \leq C \int_{B_r(z)} w \, dx \quad \text{and} \quad \int_{B_{2r}(z)} w \, dx \leq C \int_{B_r(z)} w \, dx
\]

for all balls \( B_r(z) \) such that \( B_{3r}(z) \subset B_1 \). Property 2 generalizes the analogue result for elliptic operators in \([11] \).

In regard to space-time like doubling properties we have the following result.

**Theorem 3.** Assume that \( P \) and \( u \) satisfy the conditions in Theorem 2. Then, there is \( N = N(\lambda, M, n) \) such that the following holds when \( 0 < r \leq 1/\sqrt{N \log (N\Theta)} \)

1. \[
\int_{Q_r} u^2 \, dX \leq e^{N \log (N\Theta) \log (N \log (N\Theta))} r^2 \int_{B_r} u^2(x,0) \, dx .
\]
2. \[
\int_{Q_{2r}} u^2 \, dX \leq e^{N \log (N\Theta) \log (N \log (N\Theta))} \int_{Q_r} u^2 \, dX .
\]
We do not know if Theorem 2 is optimal. In fact, if \( u \) is a backward caloric polynomial of degree \( k \geq 1 \), (i.e. \( u(\lambda x, \lambda^2 t) = \lambda^k u(x, t) \) when \( (x, t) \) is in \( \mathbb{R}^{n+1} \), \( \lambda > 0 \)) the following happens when \( r > 0 \)
\[
\frac{\int_{Q_r} u^2 \, dX}{\int_{B_r} u^2(x, 0) \, dx} = \frac{\int_{Q_1} u^2 \, dX}{\int_{B_1} u^2(x, 0) \, dx}.
\]

Theorems 2 and 3 also hold when \( u \) verifies \( |Pu| \leq M (|u| + |\nabla u|) \) in \( Q_4 \cap D \times [0, +\infty) \), \( D = \{ (x', x_n) : x_n > \varphi(x') \} \), where \( \varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) is a \( C^{1,1} \) function verifying \( \varphi(0) = 0 \) and when either the Dirichlet or Neumann (conormal derivative) data of \( u \) vanishes identically on \( \partial D \times [0, +\infty) \cap Q_4 \). In fact, when the matrix of coefficients of \( P \) is time independent and \( u \) has zero Dirichlet condition on the lateral boundary the results hold when \( \varphi = \varphi_1 + \varphi_2 \), where \( \varphi_1 \) is a convex function and \( \varphi_2 \) is \( C^{1,\alpha} \) for some \( \alpha \) in (0,1). Of course, one must replace \( B_r \) by \( B_r \cap D \) and so on in the corresponding statement of the boundary version of the Theorems. We do not prove these results here but they follow combining the arguments in this work with others in [11].

In section 3 we prove Theorems 2 and 3 when \( P = \Delta + \partial_t \) with the frequency method. With some extra work it is possible to prove Theorems 2 and 3 with the same method and when the main coefficients of \( P \) are time independent. In section 4, the more general case is proved using a more detailed version of the Carleman inequalities appearing in [9] and [10].

2. Constant coefficients

The proof of Theorems 2 and 3 when \( P = \Delta + \partial_t \) is a consequence of the following four Lemmas.

**Lemma 1.** Assume that \( u \) satisfies \( |\Delta u + \partial_t u| \leq M (|u| + |\nabla u|) \) in \( Q_4 \). Then, there is \( N = N(n, M) \) such that the inequalities
\[
N \log(N \Theta) \geq 1 \quad \text{and} \quad N \int_{B_2} u^2(x, t) \, dx \geq \int_{B_1} u^2(x, 0) \, dx
\]
hold when \( t \leq 1/(N \log(N \Theta)) \) and \( \Theta = \int_{Q_2} u^2 \, dX / \int_{B_1} u^2(x, 0) \, dx \).

**Proof.** Let \( f = w\varphi \), where \( \varphi \in C_0^\infty(B_2), \ 0 \leq \varphi \leq 1, \ \varphi = 1 \) in \( B_{3/2} \). The standard estimate [12]
\[
\|u\|_{L^\infty(Q_3)} + \|\nabla u\|_{L^\infty(Q_3)} \leq N\|u\|_{L^2(Q_4)}
\]
for solutions of the inequality
\[
|\Delta u + \partial_t u| \leq M (|u| + |\nabla u|) \text{ in } Q_4
\]
gives
\[
|\Delta f + \partial_t f| \leq N \left[ \|f\| + |\nabla f| + \|u\|_{L^2(Q_4)} \chi_{B_2 \setminus B_{3/2}} \right]
\]
and shows that the first claim holds. Setting \( H(t) = \int f^2(x, t)G(x - y, t) \, dx \), where \( G(x, t) = t^{-n/2}e^{-|x|^2/4t} \) and \( y \in B_1 \) we have
\[
\dot{H}(t) = 2 \int f(\Delta f + \partial_t f)G(x - y, t) \, dx + 2 \int |\nabla f|^2 G(x - y, t) \, dx
\]
and from (2.3), (2.4) and the Cauchy-Schwarz’s inequality
\[ \dot{H}(t) \geq -N \left( H(t) + e^{-1/Nt} \|u\|_{L^2(\Omega_t)}^2 \right). \]
Integration of this inequality in \((0, t)\) for \(t \in (0, 16)\) gives
\[ N \int f^2(x, t) G(x - y, t) \, dx \geq u^2(y, 0) - Ne^{-1/Nt} \|u\|_{L^2(\Omega_t)}^2. \]
Integrating this inequality over \(B_1\) and recalling that \(\int G(x - y, t) \, dy = 1\) we get
\[ N \int_{B_1} u^2(x, t) \, dx \geq \int_{B_1} u^2(x, 0) \, dx - Ne^{-1/Nt} \|u\|_{L^2(\Omega_t)}^2, \]
and the Lemma follows from this inequality and (2.1). \(\square\)

**Lemma 2.** Given \(a > 0\) and \(f \in W^{2,\infty}(\mathbb{R}^{n+1}_+)\) set \(G_a = (t + a)^{-n/2} e^{-|x|^2/(4(t+a))}\),
\[ H_a(t) = \int_{\mathbb{R}^n} f^2 G_a \, dx, \quad D_a(t) = \int_{\mathbb{R}^n} |\nabla f|^2 G_a \, dx \text{ and } N_a(t) = \frac{2(t + a) D_a(t)}{H_a(t)}. \]
Then,
\[ \dot{N}_a(t) \geq -\frac{(t + a)}{H_a(t)} \int (\Delta f + \partial_t f)^2 G_a \, dx. \]

**Proof.** The identities \(\partial_a G_a - \Delta G_a = 0, \nabla G_a = -\frac{x}{2(t+a)} G_a, \Delta = \text{div}(\nabla)\) and integration by parts imply the following identities
(2.5) \[ \dot{H}_a(t) = 2 \int f(\Delta f + \partial_t f) G_a \, dx + 2 D_a(t), \]
\[ \dot{H}_a(t) = 2 \int f \left( \partial_t f + \frac{x}{2(t+a)} \cdot \nabla f - \frac{1}{2} (\Delta f + \partial_t f) \right) G_a \, dx + \int f (\Delta f + \partial_t f) G_a \, dx, \]
\[ D_a(t) = \int f \left( \partial_t f + \frac{x}{2(t+a)} \cdot \nabla f - \frac{1}{2} (\Delta f + \partial_t f) \right) G_a \, dx + \frac{1}{2} \int f (\Delta f + \partial_t f) G_a \, dx \]
and
(2.6) \[ \dot{H}_a(t) D_a(t) = 2 \left( \int f \left( \partial_t f + \frac{x}{2(t+a)} \cdot \nabla f - \frac{1}{2} (\Delta f + \partial_t f) \right) G_a \, dx \right)^2 \]
\[ - \frac{1}{2} \left( \int f (\Delta f + \partial_t f) G_a \, dx \right)^2. \]
The Rellich-Nécas identity with vector field \(\nabla G_a\)
\[ \text{div}(\nabla G_a | \nabla f|^2) - 2 \text{div}(\nabla f \cdot \nabla G_a) \nabla f \]
\[ = \Delta G_a |\nabla f|^2 - 2 D^2 G_a \nabla f \cdot \nabla f - 2 \nabla f \cdot \nabla G_a \Delta f \]
and integration by parts give
(2.7) \[ \int \Delta G_a |\nabla f|^2 \, dx = 2 \int D^2 G_a \nabla f \cdot \nabla f \, dx + 2 \int \nabla f \cdot \nabla G_a \Delta f \, dx \]
\[ = 2 \int \left( \frac{x}{2(t+a)} \cdot \nabla f \right)^2 G_a \, dx - 2 \int \frac{x}{2(t+a)} \cdot \nabla f \Delta f G_a \, dx - D_a(t)/(t+a). \]
Again, the fact that \(G_a\) is a caloric function, integration by parts, (2.7) and the completion of the square of \(\partial_t f + \frac{x}{2(t+a)} \cdot \nabla f - \frac{1}{2} (\Delta f + \partial_t f)\) yields the formula
\[
\dot{D}_a(t) = 2 \int \left( \partial_t f + \frac{x}{2(t+a)} \cdot \nabla f - \frac{1}{2} \left( \Delta f + \frac{1}{t} \right) \right)^2 G_a \, dx \\
- \frac{1}{2} \int \left( \Delta f + \frac{1}{t} \right)^2 G_a \, dx - D_a(t)/(t+a).
\]

Then, from (2.6), (2.8) and the quotient rule

\[
\dot{N}_a(t) = 4 \left( t + a \right)^2 H_a \left( t \right)^2 \left\{ \int \left( \partial_t f + \frac{x}{2(t+a)} \cdot \nabla f - \frac{1}{2} \left( \Delta f + \frac{1}{t} \right) \right) G_a \, dx \right\}
\]

and Lemma 2 follows from (2.9), the Cauchy-Schwarz inequality and the positivity of the third term on the right hand side of (2.9).

Lemma 3. The inequality

\[
\int \frac{|x|^2}{8a} h^2 e^{-|x|^2/4a} \, dx \leq 2a \int |\nabla h|^2 e^{-|x|^2/4a} \, dx + \frac{n}{2} \int h^2 e^{-|x|^2/4a} \, dx
\]

holds for all \( h \in C^\infty_0(\mathbb{R}^n) \) and \( a > 0 \).

Proof. The inequality follows setting \( v = he^{-|x|^2/8a} \) and from the identity

\[
2a \int |\nabla h|^2 e^{-|x|^2/4a} \, dx + \frac{n}{2} \int h^2 e^{-|x|^2/4a} \, dx - \int \frac{|x|^2}{8a} h^2 e^{-|x|^2/4a} \, dx = 2a \int |\nabla v|^2 \, dx.
\]

\[\square\]

Lemma 4. Assume that \( N \) and \( \Theta \) verify \( N \log(N\Theta) \geq 1 \), \( h \in C^\infty_0(\mathbb{R}^n) \), and that the inequality

\[
2a \int |\nabla h|^2 e^{-|x|^2/4a} \, dx + \frac{n}{2} \int h^2 e^{-|x|^2/4a} \, dx \leq N \log(N\Theta) \int h^2 e^{-|x|^2/4a} \, dx
\]

holds when \( a \leq \frac{1}{12N \log(N\Theta)} \). Then,

\[
\int_{B_{2r}} h^2 \, dx \leq (N\Theta)^N \int_{B_r} h^2 \, dx \text{ when } 0 < r \leq 1/2.
\]

Proof. The inequality satisfied by \( h \) and Lemma 3 show that

\[
\int \frac{|x|^2}{8a} h^2 e^{-|x|^2/4a} \, dx \leq N \log(N\Theta) \int h^2 e^{-|x|^2/4a} \, dx
\]
when \( a \leq 1/(12N \log (N\Theta)) \). For given \( 0 < r \leq 1/2 \) and \( 0 < a \leq \frac{r^2}{16N \log (N\Theta)} \), the last inequality implies

\[
\int \frac{|x|^2}{8a}h^2e^{-|x|^2/4a} \, dx \leq N \log (N\Theta) \left[ \int_{B_r} h^2 \, dx + \frac{8a}{r^2} \int_{B_r \setminus B_r} \frac{|x|^2}{8a}h^2e^{-|x|^2/4a} \, dx \right] \\
\leq N \log (N\Theta) \int_{B_r} h^2 \, dx + \frac{1}{2} \int \frac{|x|^2}{8a}h^2e^{-|x|^2/4a} \, dx 
\]

Thus,

\[
(2.10) \quad \int \frac{|x|^2}{16a}h^2e^{-|x|^2/4a} \, dx \leq N \log (N\Theta) \int_{B_r} h^2 \, dx \quad \text{when} \quad 0 < a \leq \frac{r^2}{16N \log (N\Theta)} .
\]

Now, \( e^{-|x|^2/4a}|x|^2/(16a) \geq (N\Theta)^{-N}N \log (N\Theta) \) when \( r \leq |x| \leq 2r \) and \( a = \frac{r^2}{16N \log (N\Theta)} \). This and (2.10) imply

\[
\int_{B_{2r}} h^2 \, dx \leq (N\Theta)^N \int_{B_r} h^2 \, dx \quad \text{when} \quad 0 < r \leq 1/2 .
\]

\[\square\]

**Proof of Theorems 2 and 3** Constant coefficients. Let \( u \) satisfy (2.2) and set \( f = u\psi \) in Lemma 2 where \( \psi \in C_0^\infty (B_1) \), \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) in \( B_3 \) and \( \psi = 0 \) outside \( B_2 \). Then,

\[
(2.11) \quad |\Delta f + \partial_t f| \leq N \left[ |f| + |\nabla f| + \|u\|_{L^2(Q_1)}\chi_{B_1 \setminus B_1} \right] 
\]

and from Lemmas 1, 2 and 2.11, we have

\[
(2.12) \quad H_a(t) \geq N^{-1}(t+a)^{-n/2}e^{-1/(t+a)} \int_{B_1} u^2(x,0) \, dx , \\
\hat{N}_a(t) \geq \frac{N}{1 + N_a(t) + e^{-5/(t+a)}\Theta} 
\]

when \( 0 \leq t + a \leq 1/N \log (N\Theta) \) and where \( \Theta \) was defined in Lemma 1. The choice in Lemma 1 of the constant \( N \) gives, \( e^{-5/(t+a)}\Theta \leq 1 \) when \( 0 \leq t + a \leq 1/N \log (N\Theta) \). This fact and the second inequality in (2.12) imply that

\[
(2.13) \quad e^{Nt}N_a(t) + Ne^{Nt} \text{ is non-decreasing when} \quad t + a \leq 1/N \log (N\Theta) .
\]

Multiplying (2.5) by \( \frac{1+a}{H_a(t)} \), we find from (2.11), Lemma 1 and the first inequality in (2.12) that there is some constant \( N = N(n, M) \) such that

\[
(2.14) \quad N_a(t) \leq N \left[ 1 + (t+a) \partial_t \log H_a(t) \right] , \quad \text{when} \quad 0 \leq t + a \leq 1/N \log (N\Theta) 
\]

and

\[
(2.15) \quad \partial_t \log H_a(t) \leq N \left[ 1 + N_a(t) \right]/(t+a) , \quad \text{when} \quad 0 \leq t + a \leq 1/N \log (N\Theta) .
\]

Setting \( \beta = \frac{1}{N \log (N\Theta)} \), we get from (2.13), (2.11) and the first inequality in (2.12) that

\[
(2.16) \quad N_a(t) \leq N_a(\beta/4) + 1 \leq 1 + \int_{\beta/4}^{\beta/2} \frac{N_a(t)}{(t+a)} \, dt \leq 1 + \int_{\beta/4}^{\beta/2} \partial_t \log H_a(t) \, dt \\
= 1 + \log \left( \frac{H_a(\beta/2)}{H_a(\beta/4)} \right) \leq N \log (N\Theta) , \quad \text{when} \quad t + a \leq \beta/12
\]
and with constants depending only on $M$ and $n$. In particular, the inequality

$$2a \int |\nabla f(x,0)|^2 e^{-|x|^2/4a} \, dx + \frac{a}{2} \int f^2(x,0) e^{-|x|^2/4a} \, dx$$

(2.17)

$$\leq N \log (N \Theta) \int f^2(x,0) e^{-|x|^2/4a} \, dx$$

holds for all $0 < a \leq 1/(12N \log (N \Theta))$.

From Lemma 1 with $h = f(\cdot,0)$ and recalling that $f(x,0) = u(x,0)$ in $B_3$ we obtain the doubling property claimed in Theorem 2 when $z = 0$

$$\int_{B_{2r}} u^2(x,0) \, dx \leq (N \Theta)^N \int_{B_r} u^2(x,0) \, dx \quad \text{when } 0 < r \leq 1/2$$

(2.18)

For the same values of $r$ choose $k \geq 2$ such that $2^{-k} < r \leq 2^{-k+1}$ and iterate (2.18) when $r = 2^{-j}$, $j = 0, \ldots, k-1$. It gives

$$\int_{B_1} u^2(x,0) \, dx \leq (N \Theta)^{N \log(1/r)} \int_{B_r} u^2(x,0) \, dx$$

but writing the value of $\Theta$ in the above inequality, which was defined in Lemma 1, one gets

$$\int_{B_1} u^2(x,0) \, dx \leq \left( \int_{B_r} u^2(x,0) \, dx \right)^{1 + N \log(1/r)} \left( N \int_{Q_{4r}} u^2 \, dX \right)^{N \log(1/r)}$$

which proves the two-sphere one-cylinder inequality.

Rewrite the integral on the left hand side of the following inequalities as the sum of the corresponding integrals over $B_r$ and $\mathbb{R}^n \setminus B_r$ and assume that $a \leq r^2/(16N \log (N \Theta))$, then

$$\int f^2(x,0) e^{-|x|^2/4a} \, dx \leq \int_{B_r} f^2(x,0) \, dx + \frac{16a}{r^2} \int_{\mathbb{R}^n \setminus B_r} \frac{|x|^2}{16a} f^2(x,0) e^{-|x|^2/4a} \, dx$$

$$\leq \int_{B_r} f^2(x,0) \, dx + \frac{1}{N \log (N \Theta)} \int_{\mathbb{R}^n} \frac{|x|^2}{16a} f^2(x,0) e^{-|x|^2/4a} \, dx$$

and recalling that (2.10) holds when $h = f(\cdot,0)$ and $a \leq r^2/(16N \log (N \Theta))$, it follows that

$$\int f^2(x,0) e^{-|x|^2/4a} \, dx \leq 2 \int_{B_r} u^2(x,0) \, dx \quad \text{when } 0 < a \leq \frac{r^2}{16N \log (N \Theta)}$$

(2.19)

The fact that $f(x,0) = u(x,0)$ when $|x| \leq 2$ and choosing $a = \frac{r^2}{16N \log (N \Theta)}$ in the last inequality and (2.17) implies

$$\int_{B_r} |\nabla u(x,0)|^2 \, dx \leq (N \Theta)^N r^{-2} \int_{B_r} u^2(x,0) \, dx$$

(2.20)

Then, (2.20) and the Sobolev inequality

$$\left( \int_{B_r} |\varphi - f|^{2n} \, dx \right)^{\frac{n-2}{n}} \leq Nr \left( \int_{B_r} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}}, \ \varphi \in C^\infty(B_r)$$

give

$$\left( \int_{B_r} |u(x,0)|^{2n} \, dx \right)^{\frac{n-2}{n}} \leq (N \Theta)^N \int_{B_r} u^2(x,0) \, dx.$$
which shows that $|u|^2_{t=0}$ satisfies a reverse Hölder inequality centered at 0. When $n = 2,$ replace the previous Sobolev inequality by

$$\left( \int_{B_r} |\varphi|^2 dx \right)^{\frac{1}{2}} \leq N r \left( \int_{B_r} |\nabla \varphi|^2 dx \right)^{\frac{1}{2}}, \quad \varphi \in C^\infty(B_r).$$

These and other standard arguments show that $|u|^2_{t=0}$ is a Muckenhoupt weight in $B_{1/2},$ which satisfies a reverse Hölder inequality from $L^{n-2}$-averages to $L^1$-averages and with constant

$$\left( N \int_{Q_{4r}} u^2 dX \right)^N \left( \int_{B_{1/2}} u^2(x,0) dx \right).$$

To prove Theorem 3, (2.16) and (2.15) imply

$$\partial_t \log H_a(t) \leq N \log(N\Theta) / (t + a), \quad \text{when } 0 < t + a \leq 1/N \log(N\Theta)$$

and integration of this inequality over $[0,t]$ gives

$$\int f^2(x,t) e^{-|x|^2/4(a+t)} dx \leq \left( 1 + \frac{t}{a} \right)^{N \log(N\Theta)} \int f^2(x,0) e^{-|x|^2/4a} dx,$$

when $0 < t + a \leq 1/12N \log(N\Theta).$ From (2.19) and the last inequality

$$\int_{B_{2r}} u^2(x,t) dx \leq (N\Theta)^N \left( 1 + \frac{t}{a} \right)^{N \log(N\Theta)} \int_{B_r} u^2(x,0) dx,$$

when $0 < t < 4r^2 \leq 1/12N \log(N\Theta)$ and $a = \frac{r^2}{16N \log(N\Theta)}.$ The integration of (2.21) over $[0,4r^2]$ gives

$$\int_{Q_{2r}} u^2 dX \leq (N\Theta)^N a \int_{B_r} u^2(x,0) dx \int_0^{4r^2/a} (1 + s)^{N \log(N\Theta)} ds.$$

Thus,

$$\int_{Q_{2r}} u^2 dX \leq e^{N \log(N\Theta) \log(N \log(N\Theta))} \int_{B_r} u^2(x,0) dx.$$

The second part in Theorem 3 is a consequence of (2.22), the doubling property in Theorem 2 and the standard local $L^\infty$-bounds of solutions of (2.2) in terms of their $L^2$-averages [12].

### 3. Variable coefficients

In this section $P$ is a backward parabolic operator verifying (1.2) and (1.3) and $u$ a solution of the inequality

$$|Pu| \leq M (|u| + |
abla u|) \text{ in } Q_4.$$  

As in section 2 we need a few Lemmas to prove Theorems 2 and 3. The first one, Lemma 5, is the variable coefficients version of Lemma 1. The second, Lemma 6, is a more detailed version of the Carleman inequalities established in [9] and [10]. These two imply Lemma 7, a result analogous to (2.17), which as we saw in section 2 implies Theorems 2 and 3.
Lemma 5. There is $N = N(\lambda, M, n)$ such that the inequalities
\[ N \log(N\Omega_\rho) \geq 1 \quad \text{and} \quad N \int_{B_{2\rho}} u^2(x, t) \, dx \geq \int_{B_{\rho}} u^2(x, 0) \, dx \]
hold when $0 < t \leq \rho^2/(N \log(N\Omega_\rho))$ and $0 < \rho \leq 1$. Here,
\[ \Omega_\rho = \frac{\int_{Q_{4\rho}} u^2 \, dX}{\rho^2 \int_{B_{\rho}} u^2(x, 0) \, dx}. \]

Proof. When $\rho = 1$ set as in Lemma 1 $f = u\varphi$, where $\varphi \in C_0^\infty(B_2)$, $0 \leq \varphi \leq 1$ and $\varphi = 1$ in $B_{3/2}$. Define
\[ H(t) = \int f^2(x, t) G(x, t; y, 0) \, dx, \]
where now $G(x, t; y, s)$ is the fundamental solution in $\mathbb{R}^{n+1}$ of the parabolic operator $P^*$, the adjoint of $P$, $y \in B_1$ and $P^*G(x, t; y, s) = -\delta(y, s)$ is the Dirac operator at $(y, s)$.

The Gaussian bounds for the fundamental solution
\[ N^{-1}(t-s)^{-\frac{n}{2}} e^{-\frac{N|x-y|^2}{t-s}} \leq G(x, t; y, s) \leq N(t-s)^{-\frac{n}{2}} e^{-\frac{N|x-y|^2}{t-s}}, \]
and the fact that (2.1) is satisfied in this setting, show that the arguments in the proof of Lemma 1 can be repeated again, which proves Lemma 5 when $\rho = 1$. The case $0 < \rho \leq 1$ follows rescaling to the case $\rho = 1$ and from the observation
\[ \int_{Q_{4\rho}} u^2 \, dX \leq \int_{Q_4} u^2 \, dX. \]

In Lemma 6 we use the following notation: for a given function $\sigma$ defined on some interval and $a > 0$, $\sigma_a(t) = \sigma(t+a)$, $G(x, t) = t^{-n/2} e^{-|x|^2/4t}$ is the Gauss kernel and $G_a(x, t) = G(x, t+a)$.

Lemma 6. Assume $g^{ij}(0,0) = \delta_{ij}$. Then, there is a constant $N = N(\lambda, M, n)$ verifying the following property: for each number $\alpha \geq 2$ there is a nondecreasing function $\sigma : (0, +\infty) \rightarrow \mathbb{R}_+$ satisfying $t/N \leq \sigma(t) \leq t$ when $0 < ct \leq 4$ and such that the inequality
\[ \alpha^2 \int \sigma_a^{-\alpha} v^2 G_a \, dX + \alpha \int \sigma_a^{1-\alpha} |\nabla v|^2 G_a \, dX \]
\[ \leq N \int \sigma_a^{1-\alpha} |Pv|^2 G_a \, dX + N^\alpha \alpha \sup_{t \geq 0} \int v^2 + |\nabla v|^2 \, dx \]
\[ + \sigma(a)^{-\alpha} \left[ -(a/N) \int |\nabla v(x, 0)|^2 G(x, a) \, dx + N^\alpha \left( \int v^2(x, 0) G(x, a) \, dx \right) \right] \]
holds for all $0 < a \leq \frac{1}{\alpha}$ and $v \in C_0^\infty(\mathbb{R}^n \times [0, \frac{1}{\alpha}])$.

The proof of Lemma 6 is sketched at the end of this section. We now prove Lemma 7 where as in (2.4), $f = u\psi$ with $\psi \in C_0^\infty(B_4)$, $0 \leq \psi \leq 1$, $\psi = 1$ in $B_3$ and $\psi = 0$ outside $B_4$. 
Lemma 7. There are constants $N = N(\lambda, M, n) \geq 1$ and $\rho = \rho(\lambda, M, n)$ in $(0,1)$ such that
\[
2a \int \left| \nabla f(x,0) \right|^2 e^{-|x|^2/4a} \, dx + \frac{\rho}{2} \int f^2(x,0) e^{-|x|^2/4a} \, dx \\
\leq N \log(N\Theta_\rho) \int f^2(x,0) e^{-|x|^2/4a} \, dx ,
\]
when $0 < a \leq 1/(12N \log(N\Theta_\rho))$ and $N \log(N\Theta_\rho) \geq 1$. Here,
\[
\Theta_\rho = \int_{Q_\rho} u^2 \, dx / \int_{B_\rho} u^2(x,0) \, dx .
\]

Proof. For fixed $\alpha \geq 2$ and $a \leq \frac{1}{\alpha}$ take as $v$ in the Carleman inequality the function $v = f(x,t) \varphi(t) = u(x,t) \psi(x) \varphi(t)$, where $\varphi = 1$ when $t \leq \frac{1}{\alpha}$ and $\varphi = 0$ when $t \geq \frac{2}{\alpha}$. From (3.1) and (2.1) we have
\[
(3.2)
\]
\[
|Pv| \leq N \left[ |v| + \left| \nabla v \right| + \|u\|_{L^2(Q_\rho)} \chi_{B_4 \times [0,\frac{1}{\alpha}] / B_3 \times [0,\frac{1}{\alpha}]} \right] .
\]
Then, (3.2), the observation that $\sigma_a^{-1} G_a \leq N^{\alpha + \frac{\alpha}{2} a + 1 - \frac{1}{a}}$ in the region $B_4 \times [0,\frac{2}{a}] \setminus B_3 \times [0,\frac{1}{a}]$, standard arguments used with Carleman inequalities and the fact that $t/N \leq \sigma(t) \leq t$ on $(0,\frac{1}{\alpha})$, imply that the inequality
\[
(3.3)
\]
\[
\alpha^2 \int_0^1 \int_{B_2} (t + a)^{-\alpha} u^2 e^{-|x|^2/4(t+a)} \, dX \leq N^{\alpha} \alpha^\alpha \|u\|_{L^2(Q_\rho)}^2
\]
holds when $\alpha \geq \frac{2}{2\alpha}$ and $0 < a \leq \frac{1}{\alpha}$. Then, (3.4)
\[
\alpha^2 \int_0^1 \int_{B_2} (t + a)^{-\alpha} u^2 e^{-|x|^2/4(t+a)} \, dX \geq \alpha^2 \int_0^{\rho^2} \int_{B_{2\rho}} (t + a)^{-\alpha} e^{-\frac{2}{a} t} \, u^2 \, dX \\
\geq \alpha^2 \int_a^{\rho^2} s^{-\alpha} e^{-\frac{2}{a} s} \, ds \int_{B_{\rho}} u^2(x,0) \, dx \geq \frac{\alpha^2}{N} \int_a^{\rho^2} s^{-\alpha} e^{-\frac{2}{a} s} \, ds \, \int_{B_{\rho}} u^2(x,0) \, dx \\
\geq \frac{\alpha^{\alpha+1}}{2N} \left( \frac{1}{\rho^2} \right)^{2\alpha} \left\| u \right\|_{L^2(Q_\rho)}^2 \left( \frac{\Omega_{\rho}}{\Omega} \right) .
\]
The inequalities (3.3) and (3.4) show that to make sure that the left hand side of (3.3) is greater than four times the first term on right hand side of (3.3) when $\alpha \geq N \log(N\Omega_\rho)$ and $0 < a \leq \frac{2}{2\alpha}$, suffices to know that
\[
(3.5)
\]
\[
\left( \frac{1}{\rho^2} \right)^{2\alpha} \geq 8N (Ne^2)^\alpha \Omega_\rho .
\]
Choose then $\rho$ as the solution of the equation $\frac{1}{\rho^2} = 8Ne^2$. Then, (3.5) holds when $8^{\alpha-1} \geq N\Omega_\rho$. Thus, there are fixed constants, $\rho = \rho(\lambda, M, n)$ in $(0,1)$ and
$N = N(\lambda, M, n) \geq 1$, such that

$$\frac{1}{t} \int_{0}^{t} \int_{B_{2}} (t + a)^{-\alpha} u^{2} e^{-|x|^{2}/4(t + a)} dX + \frac{\alpha}{\rho} \int_{Q_{s}} u^{2} dX + \sigma \int_{Q_{s}} |f|^{2} e^{-|x|^{2} / 4a} dX + \frac{\alpha}{\rho} \int_{Q_{s}} f^{2} e^{-|x|^{2} / 4a} dX$$

$$\leq \sigma(a)^{-\alpha} \left[-(a/N) \int |\nabla f (x, 0)|^{2} e^{-|x|^{2} / 4a} dX + N \alpha \int f^{2} (x, 0) e^{-|x|^{2} / 4a} dX \right],$$

when $\alpha \geq N \log(N^{2} \rho^{2})$ and $0 < a \leq \frac{\rho^{2}}{12N}$. In particular, the inequality

$$2a \int |\nabla f(x, 0)|^{2} e^{-|x|^{2} / 4a} dX + \frac{\alpha}{2} \int f^{2}(x, 0) e^{-|x|^{2} / 4a} dX$$

$$\leq N \log(N^{2} \rho^{2}) \int f^{2}(x, 0) e^{-|x|^{2} / 4a} dX$$

(3.6)

is true when $0 < a \leq \rho^{2} / (12N \log(N^{2} \rho^{2}))$ and $N \log(N^{2} \rho^{2}) \geq 1$. Recall that $\Omega_{p} = \Theta / \rho^{2}$ and rename the fraction $N^{2} \rho^{2}$ as $N$ in the above claim. Then, Lemma 7 follows.

Proof of Theorems 2 and 3. Variable coefficients. Lemmas 4 and 7 imply that

$$\int_{B_{2r}} u^{2}(x, 0) dX \leq (N \Theta)^{N} \int_{B_{r}} u^{2}(x, 0) dX \text{ when } 0 < r \leq 1 / 2$$

and (3.7) and the argument used in section 2 to prove the interpolation inequality or two-sphere one-cylinder inequality (2) in Theorem 2 show that the interpolation inequality

$$\int_{B_{p}} u^{2}(x, 0) dX \leq \left( \int_{B_{r}} u^{2}(x, 0) dX \right)^{\frac{1}{1 + N \log(\rho / r)}} \left( N \int_{Q_{1}} u^{2} dX \right)^{\frac{N \log(\rho / r)}{1 + N \log(\rho / r)}}$$

(3.8)

holds when $r \leq \frac{\rho}{2}$. On the other hand, translations of $u$ in the space variable show that the same interpolation inequality is true when we replace in (3.8) the balls $B_{p}$ and $B_{r}$ by $B_{p}(x)$ and $B_{r}(x)$ respectively, and where $x$ is any point in $B_{1}$. Then, standard covering arguments combined with these interpolation inequalities at large scales (3.8) and its translated analogues when $r = \rho / 2$, show that we can find $\beta$ in $(0, 1)$ such that

$$\int_{B_{1}} u^{2}(x, 0) dX \leq \left( \int_{B_{1}} u^{2}(x, 0) dX \right)^{\beta} \left( N \int_{Q_{1}} u^{2} dX \right)^{1 - \beta},$$

inequality which can be rewritten as $(N \Theta)^{\beta} \leq N \Theta$. This fact, (3.7) and the arguments in section 2 finish the prove of Theorem 2.

Following the notation in Lemma 2, the arguments in section 2 give that the claims 3 and 4 in Theorem 3 follow if we know that

$$(9) \quad \partial_{t} \log H_{a}(t) \leq N \log(N^{2} \Theta) / (t + a) \text{ when } 0 < t + a \leq 1 / N \log(N \Theta).$$

On the other hand, translations of $u$ in the time variable, Lemma 5 and Lemma 7 (In particular, applying to $u(x, t + s)$ the arguments leading to the proof of (3.6), which satisfies (3.1) for some backward parabolic operator $P$ when $s \leq \rho^{2} / N \log(N^{2} \rho^{2})$, then recalling that in this case $f(x, 0) = u(x, s) \psi(x)$ and then, rewriting the variables $a$ and $s$ as $a + t$ and $t$ respectively), show that with a possibly a larger $N$ and
a smaller $\rho$

\[
(a + t) \int |\nabla f(x, t)|^2 e^{-|x|^2/(a + t)} \, dx \leq N \log (N \Omega_\rho) \int f^2(x, t) e^{-|x|^2/(a + t)} \, dx
\]

when $0 < a \leq \rho^2 / N \log (N \Omega_\rho)$. The last inequality, Lemma 3 and the inequality $(N \Theta)^{\beta} \leq N \Theta$ imply that

\[
N_a(t) \leq N \log (N \Theta),
\]

(3.10)

\[
\int |x|^2_{a + t} f^2(x, t) G_a \, dx \leq N \log (N \Theta) \int f^2(x, t) G_a \, dx,
\]

when $0 < a \leq 1 / N \log (N \Theta)$.

Only using the Lipschitz regularity in the space-variable of the matrix of coefficients of $P$ to bound the first order term arising in the formal calculation of $P^* G_a$, gives that $|P^* G_a| \leq N(|x|^2 + t + a)/(t + a)^2$. This fact, (3.10) and a calculation similar to the one leading to (2.5), which was done to compute $\dot{\sigma}$, when $0 < t < a$, imply that (3.9) holds and prove Theorem 3.

\[\square\]

**Proof of Lemma 6** Here we do not give a complete proof of Lemma 6 since it is basically already done in [9] and [10]. We only outline the main details.

As in [9], if $P$ is a backward parabolic operator satisfying (1.2), (1.3) and $g^{ij}(0, 0) = \delta_{ij}$, we consider the time independent backward parabolic operator $Q$

\[
Q = \sum_{i,j=1}^n \partial_i (g^{ij}(x) \partial_j \, + \partial_t \quad \text{where} \quad g^{ij}(x) = g^{ij}(x, 0), \ x \in \mathbb{R}^n.
\]

In [9] Theorem 4] it is shown with an integration by parts argument that there are $N = N(\lambda, M, n)$ and $\delta_0 = \delta_0(\lambda, M, n)$ such that the inequality

\[
\delta^2 \int \sigma^{-2} |D^2 v|^2 G \, dX + \alpha \int \sigma^{-\alpha} \theta(t) v^2 G \, dX + \int \sigma^{1-\alpha} \frac{\theta(t)}{t} |\nabla v|^2 G \, dX
\]

\begin{equation}
(3.11)
\end{equation}

\[
\leq N \int \sigma^{1-\alpha} |Q v|^2 G \, dX + N^\alpha \gamma \alpha - N \int (v^2 + t|\nabla v|^2) \, dX
\]

is satisfied when $v \in C^\infty_0(\mathbb{R}^n \times (0, 1/4])$, $\gamma = \alpha / \beta^2$, $\alpha \geq 2$, $0 < \delta \leq \delta_0$ and where $\sigma$ is the solution of the ordinary differential equation

\[
\frac{d}{dt} \left[ \log \left( \frac{\sigma}{\sigma(0)} \right) \right] = \frac{\theta(t)}{t}, \ \sigma(0) = 0, \ \dot{\sigma}(0) = 1, \ \text{where} \ \theta(t) = t^{\frac{1}{2}} \left( \log \frac{2}{t} \right)^{\frac{3}{2}}.
\]

In fact, $\sigma(t) = \beta(t) / \gamma$ with

\[
\beta(t) = t \exp \left[ - \int_0^t \left( 1 - \exp \left( - \int_0^s \frac{\theta(u)}{u} \, du \right) \right) \frac{ds}{s} \right].
\]

The main point here is that for some $N$, $t/N \leq \beta(t) \leq t$, when $0 < t \leq 4$ and

\begin{equation}
(3.12)
\end{equation}

\[
t/N \leq \sigma(t) \leq t, \ \text{when} \ 0 < t \leq 4 / \gamma.
\]

If for fixed $0 < a \leq 1 / \gamma$ one repeats the calculations which led to (3.11) in [9], but now working with $v \in C^\infty_0(\mathbb{R}^n \times (0, 1/4))$ and carrying out the integration over $\mathbb{R}^n \times [a, +\infty)$, one must take into account the boundary terms which occur when integrating by parts with respect to the time-variable and add up on the on the right hand side of (3.11) certain new terms. In fact, an analysis of the proof of
Theorem 4 in [9] shows that there is $N = N(\lambda, M, n)$, such that the sum of those boundary terms is bounded from above by

$$\sigma(a)^{-\alpha} \left[ -(a/N) \int |\nabla v(x, a)|^2 G(x, a) \, dx + N\alpha \int v^2(x, a) G(x, a) \, dx \right]$$

$$+ N \int \frac{|x|^2}{a} v^2(x, a) G(x, a) \, dx \right].$$

(3.12) implies, $\frac{\alpha^2}{\alpha} G(x, a)\sigma(a)^{-\alpha} \leq N^\alpha \gamma^{\alpha+N}$, when $|x| \geq \delta$, $0 < \alpha < 1$. This fact, Lemma 3 and the writing of the third integral in (3.13) as the sum of the same integral over $B_\delta$ and $\mathbb{R}^n \setminus B_\delta$ show that (3.13) can be bounded by

$$\sigma(a)^{-\alpha} \left[ -(a/N) \int |\nabla v(x, a)|^2 G(x, a) \, dx + N\alpha \int v^2(x, a) G(x, a) \, dx \right]$$

$$+ N^\alpha \gamma^{\alpha+N} \int v^2(x, a) \, dx ,$$

when $\delta$ is sufficiently small. Thus, there is $N = N(\lambda, M, n)$ and $\delta_0 = \delta(M, \lambda, n)$ such that if $v \in C_0^\infty(\mathbb{R}^n \times [0, \frac{1}{\gamma}])$, $\alpha \geq 2$, $0 < \alpha \leq \frac{1}{\gamma}$ and $\delta \leq \delta_0$, then

$$\delta^{-2} \int \sigma^{-\alpha} |D^2 v|^2 G \, dx + \alpha \int \sigma^{-\alpha} \frac{\alpha(t^2)}{2} v^2 G \, dx + \int \sigma^{1-\alpha} \frac{\alpha(t^2)}{2} |\nabla v|^2 G \, dx$$

$$\leq N \int \sigma^{1-\alpha} |Qv|^2 G \, dx + N^\alpha \gamma^{\alpha+N} \sup_{t \geq a} \int v^2 + t|\nabla v|^2 \, dx$$

$$+ \sigma(a)^{-\alpha} \left[ -(a/N) \int |\nabla v(x, a)|^2 G(x, a) \, dx + N\alpha \int v^2(x, a) G(x, a) \, dx \right] ,$$

and where the integration in the integrals with respect to Lebesgue measure $dx$ is done over $\mathbb{R}^n \times [a, +\infty]$. If we plug in the last inequality the function $v(x, t - a)$, consider the change of variables $t = s + a$ and observe that $\frac{\delta(t^2)}{2} \geq \gamma$ when $0 < \gamma t < 4$, we get upon renaming the new variable $s$ as $t$ that the inequality

$$\delta^{-2} \int \sigma^{2-\alpha} |D^2 v|^2 G_a \, dx + \alpha^2 \int \sigma^{-\alpha} v^2 G_a \, dx + \alpha \int \sigma^{1-\alpha} |\nabla v|^2 G_a \, dx$$

$$\leq N \int \sigma^{1-\alpha} |Q(v)|^2 G_a \, dx + N^\alpha \gamma^{\alpha+N} \sup_{t \geq a} \int v^2 + (t + a)|\nabla v|^2 \, dx$$

$$+ \sigma(a)^{-\alpha} \left[ -(a/N) \int |\nabla v(x, 0)|^2 G(x, a) \, dx + N\alpha \int v^2(x, 0) G(x, a) \, dx \right]$$

is satisfied when $\alpha \geq 2$, $v \in C_0^\infty(\mathbb{R}^n \times [0, \frac{1}{\gamma}])$, $0 < \alpha \leq \frac{1}{\gamma}$, $\delta \leq \delta_0$, $\delta_0$ is sufficiently small, and where the integration with respect to Lebesgue measure $dx$ is done over $\mathbb{R}^n \times [0, +\infty]$. On the other hand, $|g^{ij}(x, t) - g^{ij}(x)| \leq M \sqrt{t}$. Then, choosing $\delta$ sufficiently small it is possible to replace $Q$ by $P$ on the right hand side of the above inequality, which finishes the proof of Lemma 8.
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