THE MISSING PROOF OF PALEY’S THEOREM ABOUT LACUNARY COEFFICIENTS

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To the memory of Frank Forelli, who set me on this path.

Abstract. We modify the classical proof of Paley’s theorem about lacunary coefficients of functions in $H^1$ to work without analytic factorization. This leads to the first direct proof of the extension of Paley’s theorem that we applied to the former Littlewood conjecture about $L^1$ norms of exponential sums.

1. Introduction

Given an integrable function $f$ on the interval $(-\pi, \pi]$, form its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.$$ 

Use the same measure $(1/2\pi) \, dt$ in computing $L^p$ norms. Call a set of nonnegative integers strongly lacunary if it is the range of a sequence, $(k_j)$ say, with the property that

\begin{equation}
\tag{1.1}
k_{j+1} > 2k_j \quad \text{for all } j.
\end{equation}

In Section 2 we give a new proof of the following statement.

Theorem 1.1. There is a constant $C$ so that if $K$ is strongly lacunary, and if $\hat{f}(n) = 0$ when $n < 0$, then

\begin{equation}
\tag{1.2}
\left[ \sum_{k \in K} |\hat{f}(k)|^2 \right]^{1/2} \leq C \|f\|_1.
\end{equation}

Paley’s proof [16] of this used “analytic” factorization of such functions $f$ as products of two measurable functions with the same absolute

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value and with Fourier coefficients that also vanish at all negative integers. We use factors with the same absolute value, but we do not require that their coefficients vanish anywhere. Paley’s proof used orthogonal projections of $L^2$ onto subspaces determined by the set $K$. We use subspaces that may also depend on the choice of factors.

This allows us to give the first direct proofs of some refinements, stated here as Theorems 5.1 and 5.2 of Paley’s theorem. They were proved in a dual way in [10], and used there to give a new proof of “half” of the Littlewood conjecture about $L^1$ norms of exponential sums.

We prove Paley’s theorem in the next section. In Section 3 we extend this to compact abelian groups with partially-ordered duals. We use Riesz products in Section 4 to deduce some of these extensions from previously-known results for totally-ordered dual groups. In Section 5, we show that our new method works with weaker hypotheses. We weaken those further in Section 6, using Riesz products again. Finally, in an appendix, we examine the relation between the method in this paper and the one that was applied to Paley’s theorem in [12].

Remark 1.2. The functions in this paper are scalar-valued. Our methods are applied to some operator-valued functions in [11], and yield a new proof of the main result in [13].

2. Pairs of nested projections

Proof of Paley’s theorem. When $f$ satisfies the hypotheses of Theorem 1.1 factor $f$ as $gh$, where $g$ and $h$ are measurable, and $|g| = |h|$. Let $z$ denote the exponential function mapping each number $t$ in the interval $(-\pi, \pi]$ to $e^{it}$. Then $g$ and the products $z^n h$ belong to $L^2(\pi, \pi]$. Consider the inner products

$$ (g, z^n h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)h(t)e^{-int} dt = \hat{f}(n). $$

It suffices to prove inequality (1.2) when the set $K$ is the range of a finite increasing sequence $(k_j)_{j=1}^J$. Let $A_j$ be the operator on $L^2(-\pi, \pi]$ that multiplies each function by $z^{k_j}$. Then

$$ (g, A_j h) = \hat{f}(k_j). $$

This reduces matters to showing that there is a constant $C$ so that

$$ \left[ \sum_{j=1}^J |(g, A_j h)|^2 \right]^{1/2} \leq C\|g\|_2\|h\|_2 \quad \text{for all } J. $$
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Let $L_j$ be the closure in $L^2$ of the subspace spanned by the products $z^n h$ in which $n < -k_j$. Let $P_j$ project $L^2$ orthogonally onto $L_j$. These projections form a decreasing nest as $j$ increases.

Also consider the subspaces $A_j L_j$ and $A_{j+1} L_j$, where $j < J$ for the latter. Every image $A_j L_j$ is the closure in $L^2$ of the span of the products $z^n h$ for which $n < 0$. By formula (2.1) and the hypothesis that $\hat{f}(n) = 0$ for all $n < 0$, the function $g$ is orthogonal to $A_j L_j$ for all $j$.

The image $A_{j+1} L_j$ is the closure of the span of the products $z^n h$ for which $n < k_{j+1} - k_j$. Strong lacunarity is equivalent to having

$$k_j < k_{j+1} - k_j.$$  

It follows that

$$A_j h \in A_{j+1} L_j \quad \text{when} \quad j < J.$$  

Since $k_j - k_{j-1} \leq k_j < k_{j+1} - k_j$, the subspaces $A_{j+1} L_j$ increase as $j$ increases. Let $Q_j$ project orthogonally onto $A_{j+1} L_j$ when $1 \leq j < J$; let $Q_0 = 0$, and $Q_J = I$. The projections $Q_j$ form an increasing nest.

For each $j$, these choices and the membership condition (2.5) make $A_j h = Q_j A_j h$ and $(g, A_j h) = (g, Q_j A_j h) = (Q_j g, A_j h)$.

Rewrite the latter in the form

$$\| (a_j) \|_2^2 \leq \left( \sum_{j=1}^{J} \| \{Q_j - Q_{j-1}\} g \|_2^2 \right) \| h \|_2^2 \leq \| g \|_2^2 \| h \|_2^2.$$  

By the Cauchy-Schwarz inequality, the fact that the operators $A_j$ are contractions, and the nesting of the projections $Q_j$,

$$\| (b_j) \|_2^2 \leq \left( \sum_{j=1}^{J} \| \{Q_j - Q_{j-1}\} g \|_2^2 \right) \| h \|_2^2 \leq \| g \|_2^2 \| h \|_2^2.$$  

Now $b_1 = 0$, because $Q_0 = 0$. Since $A_j$ is unitary, $A_j P_{j-1} = Q_{j-1} A_j$ when $j > 1$, and then

$$b_j = (g, Q_{j-1} A_j h) = (g, A_j P_{j-1} h).$$  

The fact that $g \perp A_j L_j$ then makes $(g, A_j P_j h) = 0$, and it follows that $b_j = (g, A_j \{ P_{j-1} - P_j \} h)$. So $\| (b_j) \|_2^2 \leq \| g \|_2^2 \| h \|_2^2$ too, and inequality (2.3) holds with $C = 2$. \qed

The refinements in Theorems 5.1 and 5.2 below were proved in [10] using dual methods. Here, Theorem 5.1 will follow from an analysis of the direct proof above. The notions in the next two sections will then allow us to deduce Theorem 5.2.
Remark 2.1. To organize Paley’s proof in the same way, require that the factors $g$ and $h$ both be analytic. Replace the subspaces $L_j$ above with the closures in $L^2$ of the spans of the functions $z^n$ for which $n < -k_j$; in many cases, these subspaces are larger than the ones used above, but they nest as before, as do their images $A_{j+1}L_j$, which are generated by the functions $z^n$ for which $n < k_{j+1} - k_j$. Assuming that $\hat{g}(n) = 0$ for all $n < 0$ guarantees that $g \perp A_jL_j$ for all $j$, because $A_jL_j$ is generated by the function $z^n$ for which $n < 0$. Assuming that $\hat{h}$ is analytic makes $\hat{h}(n) = 0$ for all $n > 0$. It follows that $A_jh \in A_{j+1}L_j$ when $j < J$, and the rest of the proof above applies.

Remark 2.2. The subspaces $L_j$ that we used in the proof of Theorem 1.1 are invariant under multiplication by $z$, and their conjugates are invariant under multiplication by $\overline{z}$. In [6], it was observed that those conjugate subspaces must be simply invariant when $f$ is analytic, and the characterization of simply invariant subspaces of $L^2(\mathbb{T})$ was then used to show that both factors $\overline{h}$ and $g$ of $f$ can be chosen to be analytic too.

3. Partially ordered dual groups

Our proof of Theorem 1.1 resembles the one given in [7, Section 2] for the following statement, which differs only in the set where $\hat{f}$ is required to vanish.

**Theorem 3.1.** There is a constant $C$ so that if $K$ is strongly lacunary, and if $\hat{f}(n) = 0$ for all positive integers $n$ lying outside $K$, then

$$
(3.1) \quad \left[ \sum_{k \in K} |\hat{f}(k)|^2 \right]^{1/2} \leq C\|f\|_1.
$$

Various other methods in [14, p. 533–4], [20] and [7, Theorem 10] derive this conclusion from weaker conditions on $K$ or $f$. In Remark 3.3 below, we outline our direct proof of Theorem 3.1. That proof extended to compact abelian groups with partially ordered duals.

As in [18, Section 8.1], where the dual group $\Gamma$ is written additively, total orders arise when there is an additive semigroup $P$ with the two properties

$$
(3.2) \quad P \cap (-P) = \{0\}, \quad P \cup (-P) = \Gamma.
$$

We then write that $\gamma \leq \gamma'$ when $\gamma' - \gamma \in P$. Partial orders arise in the same way when the nonnegative cone $P$ need only satisfy the first condition above. We now confirm that our new proof of Paley’s theorem extends to that setting.
Call a subset $K$ of $P$ strongly lacunary if for each pair $\gamma$ and $\gamma'$ of distinct members of $K$, one of the differences $\gamma - 2\gamma'$ or $\gamma' - 2\gamma$ belongs to the strictly positive cone $P' = P \setminus \{0\}$. The following extension of Theorem 1.1 is known [18, Section 8.6], with a different proof, in the cases where the partial order on $\Gamma$ is a total order.

**Theorem 3.2.** There is a constant $C$ with the following property. Let $G$ be a compact abelian group with a partially ordered dual $\Gamma$. Let $K$ be strongly lacunary relative to that order. If $f \in L^1(G)$, and $\hat{f}(\gamma) = 0$ for all characters $\gamma$ in the strictly negative cone $-P'$, then

$$
(3.3) \quad \left[ \sum_{\gamma \in K} |\hat{f}(\gamma)|^2 \right]^{1/2} \leq C \|f\|_1.
$$

**Proof.** Without loss of generality, $K$ is finite. Enumerate it in increasing order as $(\gamma_j)_{j=1}^\ell$. Factor $f$ measurably as $gh$ with $|g| = |h|$. Make the following choices for each $j$. Let $A_j$ be the operator that multiplies each function in $L^2(G)$ by $\gamma_j$. Let $L_j$ be the closure in $L^2(G)$ of the subspace spanned the products $\gamma_j h$ in which $\gamma < -\gamma_j$. Define the nested projections $P_j$ and $Q_j$ as before, and split the inner product $(g, A_j h)$ in the same way to get inequality (3.3) with $C = 2$. \hfill \Box

**Remark 3.3.** In [7], we proved Theorem 3.1 using the same factorization and the same operators $A_j$ as in our proof of Theorem 1.1 but using the subspaces $M_j$ spanned by the products $z^m h$ in which $-k_j \leq n < 0$. Those subspaces form an increasing nest, as do their images $A_j M_j$. We used orthogonal projections, $P_j'$ and $Q_j'$ say, with ranges $M_j$ and $A_j M_j$ respectively, also letting $P_0' = 0$, and $Q_{J+1}' = I$. When $j < J$, the subspace $A_{j+1} M_j$ is spanned by the products $z^m h$ for which $m$ lies in the half-open interval $[k_{j+1} - k_j, k_{j+1})$. By strong lacunarity, these integers $m$ all fall in the gap between $k_j$ and $k_{j+1}$. Since $\hat{f}$ vanishes in these gaps, $g \perp A_{j+1} M_j$ for all $j < J$. Also, $A_j h \in A_{j+1} M_{j+1}$ for these values of $j$. So $(g, A_j h)$ splits here as $a_j' + b_j'$, where

$$
(3.4) \quad a_j' = (\{Q_{j+1} - Q_j\} g, A_j h), \quad \text{and} \quad b_j' = (Q_j g, A_j h).
$$

Then $b_j' = (g, A_j P_j h)$. This can be rewritten as $(g, A_j \{P_j' - P_{j-1}'\} h)$, because $P_{j-1}' = 0$ when $j = 1$, and $g$ is orthogonal to $A_j M_{j-1}$ in the remaining cases. Estimate $\ell^2$ norms as above.

**Remark 3.4.** The proof just above was derived from Paley’s proof of his Theorem 1.1 but it no longer worked for that theorem. Our new proof of the latter resulted from a study of the argument in Remark 3.3.
and the proof, using analytic factorization and projections onto finite-dimensional subspaces, of the version of Theorem 1.1 in [12]. See Appendix A for more about the latter proof.

**Remark 3.5.** The dual method in [10] shows that the best constant in Theorems 1.1 and 3.2 is \(\sqrt{2}\). The dual method in [3] and [9] shows that the best constant in Theorem 3.1 is at most \(\sqrt{e}\). Theorem 3.2 also follows, with constant 2, by the dual method in [17] and [19].

**Remark 3.6.** In Theorem 3.2, the set where the coefficients are required to vanish is no larger than a half space. Other methods [15] work when that set is significantly larger than a half space, and yield inequality (3.3) for more sets \(K\).

## 4. Finite Riesz Products

We consider Fourier coefficients of certain measures in the proof of Theorem 5.2. We confirm here that Theorem 3.2 extends to regular Borel measures, with the usual convention that

\[\hat{u}(\gamma) = \int_G \overline{\gamma(x)} \, d\mu(x),\]

for such a measure \(\mu\). We also show how Theorem 3.2 follows in most cases of interest, with a larger constant \(C\), from its special case where the order is total.

Denote the total variation of \(\mu\) by \(\|\mu\|\). Continue to work with a partial order on \(\Gamma\). Suppose throughout this section that \(\hat{u}\) vanishes on the strictly negative cone \(-P'\).

Given a finite subset \(K\) of \(\Gamma\), let \(K' = K\setminus\{0\}\). Recall some properties of the product

\[R_K := \prod_{\gamma' \in K'} \left(1 + \frac{\gamma' + \bar{\gamma'}}{2}\right),\]

of nonnegative factors. It expands as as

\[\sum_{\gamma} c(\gamma) \gamma\]

in which \(c(\gamma) \neq 0\) only when \(\gamma = \prod_{\gamma' \in K'} (\gamma')^{\varepsilon_{\gamma'}}\), where \(\varepsilon_{\gamma'} \in \{-1, 0, 1\}\) in all cases. In the additive notation for \(\Gamma\),

\[\gamma = \sum_{\gamma' \in K'} \varepsilon_{\gamma'} \gamma'.\]

Denote the set of such characters \(\gamma\) by \(\text{Rsz}(K)\); this includes the identity element 0 of \(\Gamma\), written as the empty sum. Then
• Each member $\gamma$ of $K'$ has a representation (4.1) with $\varepsilon_{\gamma} = 1$ and with $\varepsilon_{\gamma'} = 0$ otherwise.
• $c(\gamma) = 1/2$ if there are no other representations of $\gamma$.
• $c(\gamma) > 1/2$ if there are other representations of $\gamma$.

Similarly, $c(0) \geq 1$.

Now assume that $K$ is strongly lacunary. Then

• $\text{Rsz}(K) \subset P \cup (-P')$.
• The only representation (4.1) of 0 is the empty sum.

Hence $c(0) = 1$. Since $\hat{R}_K = c$, it vanishes off $P \cup (-P)$, while

$$\hat{R}_K(0) = 1, \quad \text{and} \quad \hat{R}_K(\gamma) \geq \frac{1}{2} \quad \text{when} \quad \gamma \in K'.$$

Since $R_K \geq 0$,

$$\|R_K\|_1 = \hat{R}_K(0) = 1.$$  \hspace{1cm} (4.3)

Let $f_K = \mu \ast R_K$. Then $\hat{f}_K = \hat{\mu} \hat{R}_K$, which vanishes on $-P'$ because $\hat{\mu}$ does. Also,

$$|\hat{f}_K(\gamma)| = |\hat{\mu}(\gamma) \hat{R}_K(\gamma)| \geq \frac{1}{2} |\hat{\mu}(\gamma)| \quad \text{for all} \ \gamma \ \text{in} \ K.$$

Applying Theorem 3.2 to $f_K$ yields that

$$\|\hat{\mu}|K\|_2 \leq 2 \|\hat{f}_K|K\|_2 \leq 4 \|f_K\|_1 \leq 4 \|R_K\|_1 \|\mu\| = 4\|\mu\|. \hspace{1cm} (4.4)$$

In many cases, the partial order on $\Gamma$ extends to a total order. That is, the cone $P$ imbeds in a cone $\tilde{P}$ which satisfies both conditions in line (3.2). Then the set $K$ is strongly lacunary relative to $\tilde{P}$.

As noted above, $\hat{f}_K$ vanishes on $-P'$. Because of its factor $\hat{R}_K$, it also vanishes off $\text{Rsz}(K)$, and hence off $P \cup (-P')$. So $\hat{f}_K$ vanishes off $P$, and hence off the larger set $\tilde{P}$.

Theorem 3.2 is already known for the total order given by $\tilde{P}$, and yields that $\|\hat{f}_K|K\|_2 \leq C\|f_K\|_1$. It follows as above that

$$\|\hat{\mu}|K\|_2 \leq 2C\|\mu\|. \hspace{1cm} (4.5)$$

Remark 4.1. In the same cases, the version of Theorem 3.1 for partial orders follows as above from the instance of it for total orders, which has other proofs.

Remark 4.2. We do not know how to use the method above to prove Theorem 5.1 below, but it will allow us to then deduce Theorem 5.2.
4.3. Replacing the Riesz product $R_K$ above with a suitable sequence of the trigonometric polynomials discussed in [1] gives the part $\|\hat{\mu}|K\|_2 \leq 4\|\mu\|$ of inequality (4.4) with the constant 4 replaced by 2. The use of finite Riesz products to pass from more general objects to trigonometric polynomials goes back at least as far as [2, pp.133–134], and also occurs in [5].

5. Analysing our method

Theorems 1.1 and 3.1 both state that if $\hat{f}$ vanishes on a suitable part of the complement of a strongly lacunary set $K$, then

\begin{equation}
\|\hat{f}|K\|_2 \leq C\|f\|_1.
\end{equation}

In [8, Remark 3], an examination of the proof in Remark 3.3 of Theorem 3.1 revealed that inequality (5.1) follows, with $C = 2$, if $\hat{f}(n) = 0$ whenever $n$ is equal to an alternating sum

$$k_{j_1} - k_{j_2} + \cdots + k_{j_{2i-1}} - k_{j_{2i}} + k_{j_{2i+1}},$$

with at least 3 terms and with a strictly increasing index sequence $(j_i)$. There is no requirement here that $K$ be strongly lacunary, or that it be enumerated monotonically.

We examine our new proof of Theorem 1.1 with a similar goal. Given a subset $D$ of the integer group $\mathbb{Z}$, let $V(D)$ denote the closed subspace of $L^2(\mathbb{T})$ spanned by the products $z^n h$ for which $n \in D$. The subspaces $L_j$ used to prove Paley’s theorem had the form $V(D_j)$ where $D_j = \{n : n < -k_j\}$.

For any choice of sets $D_j$, let $L_j = V(D_j)$. Then $A_j L_j = V(D_j + k_j)$, and $A_{j+1} L_j = V(D_j + k_{j+1})$, where $j < J$ in the latter case. We required that $g$ be orthogonal to the subspace $A_j L_j$ for all $j > 1$. By formula (2.1), this happens if only if

\begin{equation}
\hat{f}(n) = 0 \quad \text{for all integers } n \text{ in the set } \bigcup_{j=2}^J (D_j + k_j).
\end{equation}

Our proof uses three properties of the subspaces $L_j$ and their images.

(1) $A_j h \in A_{j+1} L_j$ when $j < J$.
(2) $L_j \supset L_{j+1}$ when $j < J$.
(3) $A_j L_{j-1} \subset A_{j+1} L_j$ when $1 < j < J$.

The membership condition (11) holds if

\begin{equation}
k_j \in D_j + k_{j+1}
\end{equation}
when \( j < J \). The subspaces \( L_j \) and their images \( A_{j+1}L_j \) nest suitably if

\[
D_j \supset D_{j+1},
\]

\[
D_{j-1} + k_j \subset D_j + k_{j+1},
\]

where \( j < J \) in both cases, and \( j > 1 \) in the second case.

Extend the finite sequence \((k_j)_{j=1}^J\) to a doubly-infinite sequence, in the integers or some larger abelian group, with no monotonicity or disjointness requirement, and seek sets \( D_j \) satisfying the three conditions above for all values of \( j \). The lack of special conditions at endpoints for \( j \) makes it easier to find a pattern that works.

Form the sets \( G_{j+1} = k_{j+1} + D_j \). Making them minimal will do the same for the sets \( D_j \). The three conditions on the latter hold for all \( j \) if and only if

\[
k_j \in G_{j+1},
\]

\[
G_{j+1} - k_{j+1} \subset G_j - k_j,
\]

\[
\text{and } G_j \subset G_{j+1}
\]

for all \( j \). Rewrite the second condition above as

\[
G_{j+1} - \Delta k_j \subset G_j,
\]

where \( \Delta k_j = k_{j+1} - k_j \). Since \( G_j \subset G_{j+1} \), it follows that

\[
G_{j+1} - 2\Delta k_j = (G_{j+1} - \Delta k_{j+1}) - \Delta k_j \\
\subset G_j - \Delta k_j \subset G_{j+1} - \Delta k_j \subset G_j
\]

Let \( i \) and \( i' \) be integers for which \( i < i' \), and let \((m_{j'})_{j'=1}^{i'}\) be a sequence of strictly positive integers. Iterate the reasoning above to get that

\[
G_{i'+1} - \sum_{j'=i}^{i'} m_{j'} \Delta k_{j'} \subset G_i.
\]

Combine this with condition (5.6) to get that

\[
k_{i'} - \sum_{j'=i}^{i'} m_{j'} \Delta k_{j'} \in G_i.
\]

Since \( k_{i'} - \sum_{j'=i}^{i'-1} \Delta k_j = k_i \), the expression on the left above is equal to

\[
k_i - \sum_{j'=i}^{i'-1} (m_{j'} - 1) \Delta k_{j'} - m_{i'} \Delta k_{i'} = k_i - \sum_{j'=i}^{i'} n_{j'} \Delta k_{j'}
\]

say, where \( n_{j'} \geq 0 \) for all \( j' \) and \( n_{i'} > 0 \).
Condition (5.8) forces $G_{j+1}$ to contain combinations of the form (5.10) when $i \leq j + 1$. By conditions (5.6) and (5.8), it must also contain $k_i$ when $i \leq j$. So $G_{j+1}$ must contain all combinations $k_i - s_i$ in which

1. $i \leq j + 1$.
2. $s_i$ is a sum of finitely many copies of $\Delta k_{j'}$ in which $j' \geq i$.
3. Repetitions are allowed in the sum $s_i$.
4. That sum is nonempty if $i = j + 1$.

Let each set $G_{j+1}$ contain nothing else. Then it is obvious that conditions (5.6) and (5.8) hold. For the remaining condition (5.9), suppose that the four statements listed above hold for $k_i - s_i$. In the cases where $i \leq j$,

$$ (k_i - s_i) - \Delta k_j = k_i - (s_i + \Delta k_j), $$

which belongs to $G_i$, and hence to $G_j$. When $i = j + 1$ instead,

$$ (k_i - s_i) - \Delta k_j = (k_{j+1} - s_{j+1}) - \Delta k_j = (k_{j+1} - \Delta k_j) - s_{j+1} = k_j - s_{j+1}, $$

which also belongs to $G_j$, since the sum $s_{j+1}$ is nonempty.

The conclusion that $\left[ \sum_j |\hat{f}(k_j)|^2 \right]^{1/2} \leq 2\|f\|_1$ follows if $\hat{f}$ vanishes on all the sets $D_j + k_j$. They coincide with the difference sets $G_{j+1} - \Delta k_j$ considered above. There, expressions of the form $k_j - s_j$ arose in two ways, as $k_j - (s_j + \Delta k_j)$, and as $(k_{j+1} - \Delta k_j) - s_{j+1}$. In both cases, the sum $s_j$ is nonempty. All nonempty sums $s_j'$ of differences $\Delta k_{j'}$ in which $j' \geq j$ arise in these ways.

Call such a combination $k_j - s_j'$ a top member of the set $G_{j+1} - \Delta k_j$. The other members of that set have the form $k_{j'} - s_{j'}$ where $j' < j$ and $s_{j'}$ contains a copy of $\Delta k_{j'+1}$. Then $k_{j'} - s_{j'}$ is a top member of $G_{j'+1} - \Delta k_{j'}$.

Denote the union of the sets $G_{j+1} - \Delta k_j$, or their subsets of top members, by $\text{Sch}((k_j))$. It comprises all combinations $k_j - s_j'$ as above where the sum $s_j'$ is nonempty. Rewrite $k_j - s_j'$ as

$$ \sum_{j'} \varepsilon_{j'j} k_{j'}, $$

where the coefficients $\varepsilon_{j'}$ are integers, and only finitely-many of them differ from 0. Such sums belong to $\text{Sch}((k_j))$ if and only if these coefficients satisfy the following conditions, which arose in the dual method in [10], and also arise in the one used in [17, 19, 21].

- The full sum $\sum_{j'} \varepsilon_{j'j}$ is equal to 1.
- All partial sums of that full sum are nonnegative.
- All partial sums after the first positive one are positive.
Some partial sum is greater than 1.

Specify $G_{j+1}$ and $\text{Sch}((k_j))$ in the same way for enumerations of the form $(k_j)_{j=I}^\infty$, where $I$ is finite, except for requiring that $j \geq I$. Given an enumeration of form $(k_j)_{j=-\infty}^I$ or $(k_j)_{j=I}^\infty$ where $J$ is finite, specify $G_{j+1}$ as above when $j < J$, and let $\text{Sch}((k_j))$ be the union of the sets $G_{j+1} - \Delta k_j$ for these values of $j$. In all cases, this union is still the set of sums (5.11) with the four properties listed above.

Conditions (5.6), (5.8) and (5.9) hold for the same reasons as before.

Let $D_j$ be $G_{j+1} - k_j + 1$ when this difference set is defined. Conditions (5.3), (5.4) and (5.5) then hold except when $j = J - 1$ and $J$ is the largest index in the enumeration. These cases are not required in putting $L_j = V(D_j)$ and applying the method in our proof of Theorem 1.1. Doing that yields the following.

**Theorem 5.1.** Let $K$ be a subset of the group $\mathbb{Z}$, and let $f \in L^1(\mathbb{T})$. If $\hat{f}$ vanishes on $\text{Sch}((k_j))$ for some enumeration $(k_j)$ of $K$, then

$$\|\hat{f}|_K\|_2 \leq 2\|f\|_1.$$  

Again, there is no requirement that $K$ be strongly lacunary, or that it be enumerated in increasing order. In many cases, $\text{Sch}((k_j))$ overlaps with $K$, and the hypothesis in the theorem then forces $\hat{f}$ to vanish on that overlap. When $K$ is strongly lacunary and enumerated in increasing order, however, no such overlap can occur, because $\text{Sch}((k_j))$ is then included in the strictly negative cone. In most cases, that inclusion is strict, and Theorem 5.1 sharpens Theorem 1.1.

As in Remark 4.3, Theorem 5.1 extends, with the same constant 2, to Fourier coefficients of measures. One can also replace $\text{Sch}((k_j))$ by a significantly smaller set, at the cost of using a larger constant in inequality (5.12). Let $S((k_j))$ consist of all integers $m$ with at least one representation (5.11) in which the coefficients $\varepsilon_j$ belong to the set $\{-1, 0, 1\}$ and satisfy the four conditions for membership of $m$ in $\text{Sch}((k_j))$.

Consider the corresponding notion on abelian groups. Recall the definition of the set $\text{Rsz}(K)$ in Section 4. Clearly,

$$S((\gamma_j)) = S((\gamma_j)) \cap \text{Rsz}(K).$$

The following statement is proved in the next section.

**Theorem 5.2.** Let $K$ be a subset of a discrete abelian group with dual $G$, and let $\mu$ be a regular Borel measure on $G$. If $\hat{\mu}$ vanishes on $S((\gamma_j))$ for some enumeration $(\gamma_j)$ of $K$, then

$$\|\hat{\mu}|_K\|_2 \leq 4\|\mu\|.$$
Remark 5.3. The two theorems above were proved in the late 1970’s in [10] via a dual construction using the Schur algorithm. That method yielded inequalities (5.12) and (5.13) with the smaller constants $\sqrt{2}$ and $2\sqrt{2}$. The utility of the methods used in the present paper was understood by the early 1970’s, however, so that the application in [10] to “half” of the Littlewood conjecture for exponential sums could have been obtained somewhat earlier.

Remark 5.4. The dual construction in [17] and [19] can also be used to prove Theorem 5.1, with constant 2.

Remark 5.5. In the case where the sequence $(k_j)$ is doubly infinite, the sets $\text{Sch}((k_j))$, $G_{j+1}$ and $D_j$ can also be described using suitable partial orders or preorders that are compatible with addition. For each index $j$, let $P_j$ be the semigroup generated by the differences $\Delta k_i$ in which $i \geq j$. Write $m <_j n$ when $n - m \in P_j$, with no requirement that $0 \notin P_j$. Then

\begin{enumerate}
  \item $m \in \text{Sch}((k_j))$ if and only if $m <_j k_j$ for some $j$.
  \item $m \in G_{j+1}$ if and only if $m <_{j+1} k_{j+1}$ or $m \leq_i k_i$ for some $i \leq j$.
  \item $m \in D_j$ if and only if $m <_{j+1} 0$ or $m \leq_i k_i - k_{j+1}$ for some $i \leq j$.
\end{enumerate}

Remark 5.6. In the second part of the description of $D_j$ just above, write $k_i - k_{j+1}$ as $-\sum_{j'=1}^i \Delta k_{j'}$. It follows that the members of $D_j$ are the combinations $-\sum_{j'} n_{j'} \Delta k_{j'}$ with integer coefficients $n_{j'}$ having the following properties.

- $n_{j'} \geq 0$ for all $j'$.
- $n_{j'} > 0$ for some $j'$.
- The set of indices $j' < j$ for which $n_{j'} \neq 0$ has no gaps, and contains $j - 1$ unless that set is empty.

The antinesting property (5.4) of the sets $D_j$ is then easy to check.

Remark 5.7. So is the fact that each set $D_j$ is an additive semigroup. Define preorders by saying that $m <^*_{j} n$ when $m - n \in D_j$. Rewrite conditions (5.3) to (5.5) as follows.

- Membership: $k_j <^*_j k_{j+1}$.
- Antinesting: If $m <^*_j n$, then $m <^*_n n$.
- Nesting: If $m <^*_j k_j$, then $m <^*_j k_{j+1}$.

The hypothesis in Theorem 5.1 is that $\hat{f}$ vanishes on the union of the sets $D_j + k_j$, that is $\hat{f}(m) = 0$ whenever there is some index $j$ for which $m <^*_j k_j$.\[\]
6. Direct Proof of Theorem 5.2

We work initially with stronger hypotheses.

**Lemma 6.1.** Let $K$ be a strongly lacunary set in a partially ordered discrete abelian group $\Gamma$, and let $\mu$ be a regular Borel measure on the dual of $\Gamma$. Enumerate $K$ in increasing order as $(\gamma_j)$. If $\hat{\mu}$ vanishes on $\text{Sch}((\gamma_j)) \cap \text{Rsz}(K)$, then

\begin{equation}
\|\hat{u}K\|_2 \leq 4\|\mu\|.
\end{equation}

**Proof.** Denote the group dual to $\Gamma$ by $G$. The proof of Theorem 5.1 applies to functions in $L^1(G)$ whose coefficients vanish on $\text{Sch}((\gamma_j))$. The methods in Section 4 then yield inequality (6.1) when $\hat{\mu}$ vanishes on $\text{Sch}((\gamma_j)) \cap \text{Rsz}(K)$. \hfill \qed

**Proof of Theorem 5.2.** Drop the requirement that the finite set $K$ be strongly lacunary. Form the product group $G \times \mathbb{T}$ and its dual $\Gamma \times \mathbb{Z}$. Define a partial order on that dual group by declaring that $(\gamma', n') < (\gamma, n)$ when $n' < n$.

The set $\tilde{K}$ of pairs $(\gamma_j, 3^j)$ is strongly lacunary relative to this partial order. Note that if $(\gamma, n) \in S(\tilde{K})$, then $\gamma \in S(K)$.

Identify $\mathbb{T}$ with the interval $(-\pi, \pi]$ with addition modulo $2\pi$. Identify $G$ with the subgroup $G \times \{0\}$ of $G \times \mathbb{T}$. Given a measure $\mu$ on $G$ form a measure $\tilde{\mu}$ on $G \times \mathbb{T}$ by first transferring $\mu$ to $G \times \{0\}$, and then extending it to vanish outside that subgroup of $G \times \mathbb{T}$. Note that $\|\tilde{\mu}\| = \|\mu\|$, and that

$$
\tilde{\mu}(\gamma, n) = \hat{\mu}(\gamma)
$$

in all cases.

Suppose that $\hat{\mu}$ vanishes on $S(K)$. Then $\tilde{\mu}$ vanishes on $S(\tilde{K})$. Since $\tilde{K}$ is strongly lacunary, Lemma 6.1 applies to $\tilde{\mu}$, and yields that

$$
\|\hat{\mu}K\|_2 = \|\hat{\mu}\tilde{K}\|_2 \leq 4\|\tilde{\mu}\| = 4\|\mu\|.
\end{equation}

**Remark 6.2.** The idea of adding one dimension to remove some unwanted frequencies goes back at least as far as [4].

**Appendix A.** Other nestings

For the classical Paley theorem, the authors of [12] used analytic factorization and projections into finite-dimensional subspaces. A version of their argument, without analytic factors, runs as follows.

Factor $f$ as before, and form the subspaces $L_j$. As in Section 5, given any set $S$ of integer, let $V(S)$ be the closure in $L^2$ of the span of the products $z^n h$ for which $n \in S$. Also denote the subspaces $V(Z \cap (\infty, 0))$.
and \( V(\mathbb{Z} \cap (\infty, 0]) \) by \( M \) and \( M'' \) respectively. When \( 1 \leq j \leq J \), let \( M_j'' \) be the part of \( M'' \) that is orthogonal to \( L_j \); let \( M_0'' \) be the trivial subspace. More generally, denote the part of \( V(\mathbb{Z} \cap (-\infty, b]) \) that is orthogonal to \( V(\mathbb{Z} \cap (-\infty, a)) \) by \( W\{a^\perp, b]\} \), and denote the corresponding part of \( V(\mathbb{Z} \cap (-\infty, b]) \) by \( W\{a^\perp, b\} \}. Then \( M''_j = W\{\!\{-k_j\}^\perp, 0]\}. \)

Like the subspaces \( M_j \) in Remark \([3.3]\) the subspaces \( M_j'' \) are finite-dimensional, and form an increasing nest. The shifted subspaces \( A_jM''_j \) and \( A_jM''_j \) are equal to \( V(Z \cap (\infty, k_j]) \) and \( W\{0^\perp, k_j]\} \) respectively. In particular, \( A_jh \in A_jM''_j \).

Denote the orthogonal projection onto \( A_jM''_j \) by \( Q_j'' \). Since \( g \) vanishes on the negative integers, \( g \perp M \). Split \( A_jh \) as \( u + v \), where \( u \in M \) and \( v \in A_jM''_j \). Then

\[
(g, A_jh) = (g, u) + (g, v) = (g, Q_j''A_jh) = (Q_j''g, A_jh).
\]

Much as in Remark \([3.3]\) write this as \( a_j'' + b_j'' \), where

\[
a_j'' = (\{Q_j'' - Q_{j-1}''\}g, A_jh), \quad \text{and} \quad b_j'' = (g, Q_{j-1}''A_jh),
\]

with the convention that \( Q_0'' = 0 \). Estimate \( \|a''\|_2 \) as before.

When \( j > 1 \), the range of \( Q_{j-1}'' \) is \( W\{0^\perp, k_{j-1}\}\}, which is the image under \( A_j \) of \( W\{\!\{-k_j\}^\perp, k_{j-1} - k_j\} \). Denote the orthogonal projection onto the latter subspace by \( R_j'' \). Then \( b_j'' = (g, A_jR_j''h) \).

Now \( W\{\!\{-k_j\}^\perp, k_{j-1} - k_j\} \) is included in \( M_j''_j \). By strong lacunarity, it is also included in \( L_{j-1} \), and hence is orthogonal to \( M_j''_{j-1} \). The orthogonal projections \( P_j'' \) and \( P_{j-1}'' \) onto \( M_j'' \) and \( M_{j-1}'' \) therefore have the properties that \( R_j''P_j'' = R_j'' \) and \( R_j''P_{j-1}'' = 0 \). So

\[
A_jR_j''h = A_jR_j''(P_j'' - P_{j-1}''h),
\]

and \( \|b''\|_2 \leq \|g\|_2\|h\|_2 \).

As in \([12]\), simpler choices work when \( g \) and \( h \) are analytic. Replace the subspaces \( L_j \) with the closures in \( L^2 \) of the spans of the functions \( z^n \) for which \( n < -k_j \). Let \( M''_j \) be the closure in \( L^2 \) of the span of the functions \( z^n \) for which \( n \leq 0 \). Form the orthogonal complements \( M_j'' \) in \( M'' \), and estimate as above.

In general, one can also use the orthogonal complements of each \( L_{j+1} \) in each \( L_j \), that is \( W\{\!\{-k_{j+1}\}^\perp, -k_j\} \). These subspaces are not nested, but their images \( A_{j+1}W\{\!\{-k_{j+1}\}^\perp, -k_j\} \) are, because they coincide with the spaces \( W\{0^\perp, k_{j+1} - k_j\} \).

Denote the projection onto the latter by \( Q_j'' \). Much as above, the facts that \( A_jh \in V(Z \cap (\infty, k_{j+1} - k_j]) \), and \( g \perp V(Z \cap (\infty, 0]) \) make \((g, A_jh)\) equal to \((Q_j''g, A_jh)\). In turn, that splits as \( a_j'' + b_j'' \),
where \( a'''_j = ((Q'''_j - Q'''_{j-1})g, A_jh), \) and \( b'''_j = (g, Q'''_{j-1}A_jh), \) again with the convention that \( Q'''_0 = 0. \)

When \( j > 1, \) denote the projection onto \( W\{−k_j, −k_j−1)\} \) by \( P'''_{j-1}. \)
Then \( b'''_j = (g, A_jP'''_{j-1}h). \) The ranges \( W\{−k_j, −k_j−1)\} \) of the various projections \( P'''_{j-1} \) are orthogonal, because the corresponding intervals \( [−k_j, −k_j−1) \) are disjoint. It follows that \( \|b'''\|_2 \leq \|g\|_2\|h\|_2. \)

References

1. G.F. Bachelis, W.A. Parker and K.A. Ross, Local Units in \( L^1(G), \) Proc. Amer. Math. Soc. 31 (1972), 312–313.
2. S. Bochner, Beiträge zur Theorie der fastperiodischen Funktionen. I: Funktionen einer Variablen, Math. Ann. 96 (1926), 119-147.
3. J.M. Clunie, On the derivative of a bounded function, Proc. London Math. Soc. 14A (1965), 58–68.
4. S.W. Drury, Sur les ensembles de Sidon, C. R. Acad. Sci., Paris, Sér. A 271 (1970), 162-163
5. Alessandro Figà-Talamanca, An example in the theory of lacunary Fourier series, Boll. Unione Mat. Ital. IV. Ser. 3 (1970) 375-378.
6. Frank Forelli, Invariant subspaces in \( L^1, \) Proc. Amer. Math. Soc. 14 (1963), 76–79.
7. John J.F. Fournier, Extensions of a Fourier multiplier theorem of Paley, Pacific J. Math. 30, (1969), 415–431.
8. ______, Fourier coefficients after gaps, J. Math. Anal. Appl. 42, (1973), 255–270.
9. ______, An interpolation problem for coefficients of \( H^\infty \) functions, Proc. Amer. Math. Soc. 42, (1974), 402–408.
10. ______, On a theorem of Paley and the Littlewood conjecture, Ark. Mat. 17, (1979), 199–216.
11. ______, Noncommutative Khintchine and Paley inequalities via generic factorization, arXiv:1407.2578 [math.CA].
12. S. Kwapień and A. Pełczyński, Some linear topological properties of the Hardy spaces \( H^p, \) Compositio Math. 33 (1976), 261–288.
13. Françoise Lust-Piquard and Gilles Pisier, Noncommutative Khintchine and Paley inequalities, Ark. Mat. 29 (1991), 241–260.
14. Yves Meyer, Endomorphismes des idéaux fermés de \( L^1(G), \) classes de Hardy et séries de Fourier lacunaires, Ann. Sci. École Norm. Sup. (4) 1 (1968), 499–580.
15. Daniel M. Oberlin, Two multiplier theorems for \( H^1(U^2). \) Proc. Edinburgh Math. Soc. (2) 22 (1979), 43–47.
16. R.E.A.C. Paley, On the lacunary coefficients of power series, Ann. of Math. (2) 34 (1933), 615–616.
17. Louis Pigno, and Brent Smith, Quantitative behaviour of the norms of an analytic measure, Proc. Amer. Math. Soc. 86 (1982), 581–585.
18. Walter Rudin, Fourier analysis on groups, Interscience, New York, 1962.
19. Brent Smith, Two trigonometric designs: one-sided Riesz products and Littlewood products. General inequalities, 3 (Oberwolfach, 1981), 141–148, Internat. Schriftenreihe Numer. Math., 64, Birkhäuser, Basel, 1983.
20. S. A. Vinogradov, *Interpolation theorems of Banach-Rudin-Carleson and norms of imbedding operators for certain classes of analytic functions*. (Russian) Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 19 (1970), 6–54.

21. V. A. Yudin, *Multidimensional versions of Paley’s inequality*. (Russian. Russian summary) Mat. Zametki 70 (2001), 941–947; translation in Math. Notes 70 (2001).

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