STABILITY OF FROBENIUS DIRECT IMAGES OVER SURFACES

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Abstract. Let $X$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $p > 0$ with $\Omega^1_X$ semistable and $\mu(\Omega^1_X) > 0$. For any semistable (resp. stable) bundle $W$ of rank $r$, we prove that $F_*W$ is semistable (resp. stable) when $p \geq r(r-1)^2 + 1$.

1. Introduction

Let $X$ be a smooth projective variety of dimension $n$ over an algebraically closed field $k$ with $\text{char}(k) = p > 0$. The absolute Frobenius morphism $F_{\text{abs}} : X \to X$ is induced by $O_X \to O_X, f \mapsto f^p$. Let $F : X \to X := X \times_k k$ denote the relative Frobenius morphism over $k$. This simple endomorphism of $X$ is of fundamental importance in algebraic geometry over characteristic $p > 0$. One of the themes is to study its action on the geometric objects on $X$.

Recall that a torsion free sheaf $E$ is called semistable (resp. stable) if $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$) for any nontrivial proper subsheaf, where $\mu(E)$ is the slope of $E$ (see Definition 1 in Section 2). Semistable sheaves are basic constituents of torsion free sheaves in the sense that for any torsion free sheaf $E$ admits a unique filtration

$$HN_*(E) : 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_k(E) = E,$$

which is the so called Harder-Narasimhan filtration, such that

1. $\text{gr}_1^{HN}(E) := HN_i(E)/HN_{i-1}(E)$ (1 $\leq$ $i$ $\leq$ $k$) are semistable;
2. $\mu(\text{gr}_1^{HN}(E)) > \mu(\text{gr}_2^{HN}(E)) > \cdots > \mu(\text{gr}_k^{HN}(E))$.

The rational number $I(E) := \mu(\text{gr}_1^{HN}(E)) - \mu(\text{gr}_k^{HN}(E))$, which measures how far a torsion free sheaf from being semistable, is called the instability of $E$. It is clear that $E$ is semistable if and only if $I(E) = 0$.

It is well known that $F_*$ preserves the stability of vector bundles on curves of genus $g \geq 1$ (see [5], [6], [7]). For the high dimension case, it is proved by X. Sun that instability of $F_*W$ is bounded by instability of $W \otimes \Omega^\ell_X(\Omega^1_X)$ ($0 \leq \ell \leq n(p-1)$) for any vector bundle $W$ (see [6], [7]), and an upper bound of the instability $I(W \otimes \Omega^\ell_X(\Omega^1_X))$ is given in [4]. Especially for a surface $X$ with $\Omega^1_X$ semistable and $\mu(\Omega^1_X) > 0$, the
stability of $F_\ast L$ for a line bundle $L$ is proved by X. Sun (see [6]). But it is unknown whether $F_\ast$ preserves the stability of a high rank vector bundle over a smooth projective surface. In this note, we prove that $F_\ast W$ is semistable (resp. stable) when $W$ is semistable (resp. stable) with some restriction on the characteristic $p$ as following:

**Theorem 1.** Let $X$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $p$ with $\Omega^1_X$ semistable and $\mu(\Omega^1_X) > 0$. Let $W$ be a semistable (resp. stable) vector bundle of rank $r$, then $F_\ast W$ is also semistable (resp. stable) if $p \geq r(r-1)^2 + 1$.

Here, we sketch the proof. By [6], there exists a canonical filtration of $F_\ast(F_\ast W)$:

$$0 = V_{2(p-1)+1} \subset V_{2(p-1)} \subset \cdots \subset V_1 \subset V_0 = F_\ast(F_\ast W)$$

with $V_\ell/V_{\ell+1} \cong W \otimes T^\ell(\Omega^1_X)$ for $0 \leq \ell \leq 2(p-1)$. Let $E \subset F_\ast W$ be a nontrivial subsheaf such that $F_\ast W/E$ is torsion free, then the above filtration induces the following filtration (we assume $V_m \cap F_\ast E \neq 0$ and $V_{m+1} \cap F_\ast E = 0$)

$$0 \subset V_m \cap F_\ast E \subset \cdots \subset V_1 \cap F_\ast E \subset V_0 \cap F_\ast E = F_\ast E.$$

Let

$$\mathcal{F}_\ell := \frac{V_\ell \cap F_\ast E}{V_{\ell+1} \cap F_\ast E} \subset \frac{V_\ell}{V_{\ell+1}}, \quad r_\ell = \text{rk}(\mathcal{F}_\ell).$$

Then, taking $n = 2$ in the formula (4.10) of [7], we have

$$\mu(E) - \mu(F_\ast W) = \sum_{\ell=0}^m r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \text{rk}(E)} - \frac{\mu(\Omega^1_X)}{p \cdot \text{rk}(E)} \sum_{\ell=0}^m (p-1-\ell)r_\ell.$$

If $r_{2(p-1)} = r_0$, there exists a subsheaf $W' \subset W$ of rank $r_{2(p-1)}$ such that $\mathcal{F}_\ell \supseteq W' \otimes T^\ell(\Omega^1_X)$ for $0 \leq \ell \leq 2(p-1)$ by [7]. The local computations in the proof of Theoren 4.7 of [7] imply $r_\ell = \text{rk}(W' \otimes T^\ell(\Omega^1_X))$ for $0 \leq \ell \leq 2(p-1)$. Then, by (4.22) of [7], we have

$$\mu(E) - \mu(F_\ast W) \leq \frac{r_{2(p-1)}(\text{rk}(F_\ast W) - \text{rk}(E))}{p \cdot \text{rk}(E) \cdot \text{rk}(W)}(\mu(W') - \mu(W/W')).$$

Otherwise, we have $r_0 > r_{2(p-1)}$ and

$$\mu(E) - \mu(F_\ast W) \leq \sum_{\ell=0}^m r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \text{rk}(E)} - \frac{(p-1)\mu(\Omega^1_X)}{p \cdot \text{rk}(E)}$$

by (4.10), (4.11) and (4.12) of [7].
The main part of this note is to give an upper bound of
\[ \sum_{\ell=0}^{m} r_{\ell}(\mu(F_{\ell}) - \mu(V_{\ell + 1})), \]
which depends only on \( r \) and \( \mu(\Omega_{X}^{1}) \).

2. Preliminaries

Let \( X \) be a smooth projective surface. Fixed an ample divisor \( H \), for a torsion free sheaf \( E \) on \( X \), we define the slope of \( E \) by:
\[ \mu(E) = \frac{c_{1}(E) \cdot H}{\text{rk}(E)}, \]
where \( c_{1}(E) \) is the first Chern class of \( E \) and \( \text{rk}(E) \) is the rank of \( E \).

**Definition 1.** A torsion free sheaf \( E \) on \( X \) is called semistable (resp. stable) if for any subsheaf \( 0 \neq E' \subset E \) with \( E/E' \) torsion free, we have
\[ \mu(E') \leq \mu(E) \quad (\text{resp. } \mu(E') < \mu(E)). \]

Let \( F : X \to X_{1} \) be the relative \( k \)-linear Frobenius morphism, where \( X_{1} := X \times_{k} k \) is the base change of \( X/k \) under the Frobenius \( \text{Spec}(k) \to \text{Spec}(k) \). Let \( W \) be a vector bundle on \( X \) and \( V = F^{*}(F_{*}W) \).

**Definition 2.** Let \( V_{0} := V = F^{*}(F_{*}W), V_{1} = \ker(F_{*}F_{*}W) \to W) \)
\[ V_{\ell + 1} := \ker(V_{\ell} \sum V \otimes_{S} \Omega_{X}^{1} \to (V/V_{\ell}) \otimes_{S} \Omega_{X}^{1}) \]
where \( \nabla : V \to V \otimes_{S} \Omega_{X}^{1} \) is the canonical connection (see [1, Theorem 5.1]).

The above filtration has been fully studied in [6, Section 3], and the following theorem is a special case of [6, Theorem 3.7, Corollary 3.8] for surfaces.

**Theorem 2.** [6, Theorem 3.7, Corollary 3.8] Let \( X \) be a smooth projective surface over \( k \), then the filtration defined above is
\[ 0 = V_{2(p-1)+1} \subset V_{2(p-1)} \subset \cdots \subset V_{1} \subset V_{0} = V = F^{*}(F_{*}W) \] (1)
which has the following properties

(i) \( \nabla(V_{\ell}) \subset V_{\ell-1} \otimes \Omega_{X}^{1} \) for \( \ell \geq 1 \), and \( V_{0}/V_{1} \cong W \).

(ii) \( V_{\ell}/V_{\ell+1} \sum (V_{\ell-1}/V_{\ell}) \otimes \Omega_{X}^{1} \) are injective morphisms of vector bundles for \( 1 \leq \ell \leq 2(p-1) \), which induced isomorphisms
\[ V_{\ell}/V_{\ell+1} = W \otimes T^{\ell}(\Omega_{X}^{1}) \]
where
\[ T^{\ell}(\Omega_{X}^{1}) = \begin{cases} \text{Sym}^{\ell}(\Omega_{X}^{1}) & \text{when } \ell < p \\ \text{Sym}^{2(p-1)-\ell}(\Omega_{X}^{1}) \otimes \omega_{X}^{p-1} & \text{when } \ell \geq p. \end{cases} \]
Let \( \mathcal{E} \subset F_*W \) be a nontrivial subsheaf such that \( F_*W/\mathcal{E} \) is torsion free, then the canonical filtration (1) induces the filtration (we assume \( V_m \cap F^*\mathcal{E} \neq 0 \) and \( V_{m+1} \cap F^*\mathcal{E} = 0 \))

\[
0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}. \tag{2}
\]

Let

\[
\mathcal{F}_\ell := \frac{V_\ell \cap F^*\mathcal{E}}{V_{\ell+1} \cap F^*\mathcal{E}} \subset \frac{V_\ell}{V_{\ell+1}}, \quad r_\ell = \text{rk}(\mathcal{F}_\ell).
\]

Then \( \mu(F^*\mathcal{E}) = \frac{1}{\text{rk}(\mathcal{E})} \sum_{\ell=0}^{m} r_\ell \cdot \mu(\mathcal{F}_\ell) \) and

\[
\mu(\mathcal{E}) - \mu(F_*W) = \frac{1}{p \cdot \text{rk}(\mathcal{E})} \sum_{\ell=0}^{m} r_\ell (\mu(\mathcal{F}_\ell) - \mu(F^*F_*W)). \tag{3}
\]

**Lemma 1.** ([7, Lemma 4.5]) With the same notations in Theorem 2, we have

\[
\mu(F^*F_*W) = p \cdot \mu(F_*W) = \frac{p-1}{2} K_X \cdot H + \mu(W),
\]

\[
\mu(V_\ell/V_{\ell+1}) = \mu(W \otimes T^\ell(\Omega^1_X)) = \frac{\ell}{2} K_X \cdot H + \mu(W).
\]

By using the above lemma, we have

**Lemma 2.** ([6, Lemma 4.4]) Keep the above notations. Then we have

\[
\mu(\mathcal{E}) - \mu(F_*W) = \sum_{\ell=0}^{m} r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(V_\ell)}{p \cdot \text{rk}(\mathcal{E})} - \frac{\mu(\Omega^1_X)}{p \cdot \text{rk}(\mathcal{E})} \sum_{\ell=0}^{m} (p - 1 - \ell) r_\ell. \tag{4}
\]

The numbers \( r_\ell (0 \leq \ell \leq m) \) are related by the following fact that \( V_\ell/V_{\ell+1} \to (V_{\ell-1}/V_\ell) \otimes \Omega^1_X \) induces injective morphisms

\[
\mathcal{F}_\ell \to \mathcal{F}_{\ell-1} \otimes \Omega^1_X \quad (1 \leq \ell \leq m).
\]

Using this fact, it is proved in [6] that

\[
r_{2(p-1)-\ell} r_\ell \geq 0 \quad (\ell \geq p - 1).
\]

Especially for \( \ell = 2(p - 1) \), we have \( r_0 \geq r_{2(p-1)} \). The following lemma is implicit in [7, Lemma 4.6].

**Lemma 3.** If \( r_0 > r_{2(p-1)} \), then we have

\[
\sum_{\ell=0}^{m} (p - 1 - \ell) r_\ell \geq (p - 1).
\]

**Proof.** When \( m \leq p - 1 \), it is (4.11) of [7]. When \( m > p - 1 \), it follows from (4.12) of [7] and the assumption \( r_0 > r_{2(p-1)} \). \( \square \)
Lemma 4. If \( r_0 = r_{2(p-1)} \), then there exists a subsheaf \( W' \subset W \), such that
\[
\mu(E) - \mu(F_\ell W) \leq \frac{r_{2(p-1)}(\text{rk}(F_\ell W) - \text{rk}(E))}{p \cdot \text{rk}(E) \cdot \text{rk}(W)} (\mu(W') - \mu(W/W'))
\]

Proof. It is proved in [7] that there exists a subsheaf \( W' \subset W \) of rank \( r_{2(p-1)} \) such that \( F_\ell^2 (\Omega^1_X) \cong W' \otimes T^{2(p-1)}(\Omega^1_X) \) and \( W' \otimes T^\ell(\Omega^1_X) \hookrightarrow F_\ell \).

By (4.22) of [7], it is enough to show \( r'_\ell = 0 \), i.e. \( \text{rk}(F_\ell) = \text{rk}(W' \otimes T^\ell(\Omega^1_X)) \), which follows from the local computations in the proof of Theorem 4.7 of [7].

For the convenience of readers, we repeat the arguments here. To show the assertion is a local problem. Let \( K = K(X) \) be the function field of \( X \) and consider the \( K \)-algebra
\[
R = \frac{K[\alpha_1, \alpha_2]}{(\alpha_1^p, \alpha_2^p)} = \bigoplus_{\ell=0}^{2(p-1)} R^\ell,
\]
where \( R^\ell \) is the \( K \)-linear space generated by
\[
\{ \alpha_1^{k_1} \alpha_2^{k_2} | k_1 + k_2 = \ell, 0 \leq k_i \leq p - 1 \}.
\]
The quotients in the filtration (1) can be described locally
\[
V_\ell/V_{\ell+1} = W \otimes_K R^\ell
\]
as \( K \)-vector spaces. Then the homomorphism
\[
\nabla : W \otimes_K R^\ell \rightarrow W \otimes_K R^{\ell-1} \otimes_K \Omega^1_X
\]
in Theorem 2 is locally the \( k \)-linear homomorphism defined by
\[
\nabla(w \otimes \alpha_1^{k_1} \alpha_2^{k_2}) = -w \otimes k_1 \alpha_1^{k_1-1} \alpha_2^{k_2} \otimes_K dx_1 - w \otimes k_2 \alpha_1^{k_1} \alpha_2^{k_2-1} \otimes_K dx_2.
\]
And the fact that \( F_\ell \nabla \rightarrow F_{\ell-1} \otimes \Omega^1_X \) for \( F_\ell \subset W \otimes R^\ell \) is equivalent to
\[
\forall \sum_j w_j \otimes f_j \in F_\ell \Rightarrow \sum_j w_j \otimes \frac{\partial f_j}{\partial \alpha_i} \in F_{\ell-1} (1 \leq i \leq 2).
\]

The polynomial ring \( P = K[\partial_{\alpha_1}, \partial_{\alpha_2}] \) acts on \( R \) through partial derivations, which induces a D-module structure on \( R \), where
\[
D = \frac{K[\partial_{\alpha_1}, \partial_{\alpha_2}]}{(\partial_{\alpha_1}^\ell, \partial_{\alpha_2}^\ell)} = \bigoplus_{\ell=0}^{2(p-1)} D^\ell
\]
and \( D_\ell \) is the linear space of degree \( \ell \) homogeneous elements. In particular, \( W \otimes R \) has the induced D-module structure with \( D \) acts on \( W \) trivially. Using this notation, (5) is equivalent to \( D_1 \cdot F_\ell \subseteq F_{\ell-1} \).
Locally, \( \mathcal{F}_{2(p-1)} \) is equal to \( W' \otimes R^{2(p-1)} \) as \( K \)-vector spaces. Combining with \( D_1 \cdot \mathcal{F}_\ell \subseteq \mathcal{F}_{\ell-1} \), we have

\[
D_\ell \cdot \mathcal{F}_{2(p-1)} = W' \otimes D_\ell \cdot R^{2(p-1)} = W' \otimes R^{2(p-1)-\ell} \subseteq \mathcal{F}_{2(p-1)-\ell}
\]

(6)

for \( 0 \leq \ell \leq 2(p-1) \), and the following sequence

\[
W' = D_{2(p-1)} \cdot \mathcal{F}_{2(p-1)} \subseteq D_{2(p-1)-1} \cdot \mathcal{F}_{2(p-1)-1} \subseteq \cdots \subseteq D_1 \cdot \mathcal{F}_1 \subseteq \mathcal{F}_0.
\]

But \( r_0 = r_{2(p-1)} \), so \( \mathcal{F}_0 = W' \) and \( D_\ell \cdot \mathcal{F}_\ell = \mathcal{F}_0 \) for \( 1 \leq \ell \leq 2(p-1) \). For any element \( \alpha \in \mathcal{F}_\ell \subseteq W \otimes R^\ell \), it can be written as

\[
\alpha = \sum w_{i_1 i_2} \otimes (\alpha_1^{i_1} \alpha_2^{i_2}),
\]

where \( w_{i_1 i_2} \in W \) and the sum runs over \( i_1 + i_2 = \ell, 0 \leq i_1, i_2 \leq p - 1 \). Meanwhile, we have

\[
\partial_{\alpha_1}^{i_1} \partial_{\alpha_2}^{i_2} \cdot \sum w_{i_1 i_2} \otimes (\alpha_1^{i_1} \alpha_2^{i_2}) = w_{i_1 i_2} \in \mathcal{F}_0 = W'
\]

from \( D_\ell \cdot \mathcal{F}_\ell = \mathcal{F}_0 \). Consequently, \( \alpha \in W' \otimes R^\ell \), which implies that

\[
\mathcal{F}_\ell \subseteq W' \otimes R^\ell.
\]

Together with the conclusion \( W' \otimes R^\ell \subseteq \mathcal{F}_\ell \) in (6), we have

\[
\mathcal{F}_\ell = W' \otimes R^\ell
\]

for \( 0 \leq \ell \leq 2(p-1) \). Thus \( \text{rk}(\mathcal{F}_\ell) = \text{rk}(W' \otimes T^\ell(\Omega_1^X)) \) for \( 0 \leq \ell \leq 2(p-1) \).

3. Proof of the main theorem

For any torsion free sheaf \( \mathcal{E} \), we denote

\[
s(\mathcal{E}) = \max_{\mathcal{F}} \{ \text{rk}(\mathcal{F})(\mu(\mathcal{F}) - \mu(\mathcal{E})) \mid \mathcal{F} \subseteq \mathcal{E} \}.
\]

Then it is easy to see that \( s(\mathcal{E}) \geq 0 \) and \( \mathcal{E} \) is semistable if and only if \( s(\mathcal{E}) = 0 \).

In this section, we always assume that \( X \) is a surface with \( \Omega_X^1 \) semistable and \( \mu(\Omega_X^1) > 0 \), \( W \) is a semistable bundle on \( X \) with \( \text{rk}(W) = r \). In order to simplify the symbols, we denote \( A_\ell = \text{Sym}^\ell(\Omega_X^1) \otimes W \) and \( s(\ell) = s(A_\ell) \) for all \( \ell \), then we have the following lemmas.

**Lemma 5.** As the above notations, we have

\[
s(\ell) - s(\ell - 1) \leq s(\ell + 1) - s(\ell).
\]

**Proof.** Consider the exact sequence

\[
0 \to A_{\ell-1} \otimes \omega_X \to A_\ell \otimes \Omega_X^1 \to A_{\ell+1} \to 0
\]
where all of the bundles have the same slope \((\ell + 1) \cdot \mu(\Omega^1_X) + \mu(W)\).

Assume \(E_\ell\) is the subsheaf of \(A_\ell\) such that
\[
\text{rk}(E_\ell) \cdot (\mu(E_\ell) - \mu(A_\ell)) = s(\ell).
\]
Then the above exact sequence induces an exact sequence
\[
0 \to E'_\ell \to E_\ell \otimes \Omega^1_X \to E''_\ell \to 0,
\]
where
\[
E'_\ell \subset A_{\ell-1} \otimes \omega_X, \quad E''_\ell \subset A_{\ell+1}.
\]
A direct computation implies
\[
\text{rk}(E_\ell \otimes \Omega^1_X)\mu(E_\ell \otimes \Omega^1_X) - \mu(A_\ell \otimes \Omega^1_X)) = \text{rk}(E'_\ell)(\mu(E'_\ell) - \mu(A_{\ell-1} \otimes \omega_X)) + \text{rk}(E''_\ell)(\mu(E''_\ell) - \mu(A_{\ell+1})))
\]
Consequently, we have
\[
2s(\ell) \leq s(\ell - 1) + s(\ell + 1)
\]
by the definition of \(s(\ell)\). Thus
\[
s(\ell) - s(\ell - 1) \leq s(\ell + 1) - s(\ell).
\]

Taking \(\ell = p\) in (ii) of Proposition 3.5 of \cite{6}, we have the following exact sequence
\[
0 \to W \otimes F^*\Omega^1_X \to A_p \to A_{p-2} \otimes \omega_X \to 0,
\]
we obtain a upper bound for \(s(\ell)\) by using the above exact sequence. For simplicity, we define \(t = s(W \otimes F^*\Omega^1_X)\).

**Lemma 6.** Assume \(p \geq r\). Then we have
\[
s(\ell) \leq \frac{t}{2} \cdot (\ell - (p - r))
\]
for \(p - r \leq \ell \leq p - 1\).

**Proof.** Consider the exact sequence
\[
0 \to W \otimes F^*\Omega^1_X \to A_p \to A_{p-2} \otimes \omega_X \to 0
\]
where all the bundles have the same slope \(p \cdot \mu(\Omega^1_X) + \mu(W)\). As the same argument in Lemma 5, we have \(s(p) \leq t + s(p - 2)\). Combining with \(2s(p - 1) \leq s(p) + s(p - 2)\), we have
\[
s(p - 1) - s(p - 2) \leq \frac{s(p) + s(p - 2)}{2} - s(p - 2) \leq \frac{t}{2}.
\]
Then Lemma 5 implies that
\[
s(\ell) - s(\ell - 1) \leq s(\ell + 1) - s(\ell) \leq \cdots \leq s(p - 1) - s(p - 2) \leq \frac{t}{2}.
\]
But $\text{Sym}^\ell(\Omega^1_X)$ is semistable for $\ell \leq p - 1$ and
$$\text{rk}(\text{Sym}^\ell(\Omega^1_X)) + \text{rk}(W) = \ell + 1 + r \leq p + 1$$
for $\ell \leq p - r$, thus we have $A_\ell$ is semistable for $\ell \leq p - r$ by a theorem of Ilangovan-Mehta-Parameswaran (see Section 6 of [3] for the precise statement): If $E_1$, $E_2$ are semistable with $\text{rk}(E_1) + \text{rk}(E_2) \leq p + 1$, then $E_1 \otimes E_2$ is semistable. Consequently, we have $s(\ell) = 0$ for $\ell \leq p - r$. Then the result is a direct computation.

**Lemma 7.** Assume $p \geq r + 1$. Then we have
$$t \leq (2r - 1) \cdot \mu(\Omega^1_X).$$

**Proof.** By the proposition 3.9 of [7], We have $I(F^*\Omega^1_X) \leq \mu(\Omega^1_X)$, If $p \geq r + 1$, then it is easy to check that
$$I(W \otimes F^*\Omega^1_X) = I(F^*\Omega^1_X).$$
Thus we have
$$t \leq (2r - 1) \cdot I(W \otimes F^*\Omega^1_X) \leq (2r - 1) \cdot \mu(\Omega^1_X).$$

Now, we finish the proof of Theorem 1.

**Proof of Theorem 1:** Let us assume that $W$ is semistable firstly.
If $r_0 = r_{2(p-1)}$, then Lemma 4 implies that there exists a subsheaf $W' \subset W$ such that
$$\mu(\mathcal{E}) - \mu(F_\ell W) \leq \frac{r_{2(p-1)}(\text{rk}(F_\ell W) - \text{rk}(\mathcal{E}))}{p \cdot \text{rk}(\mathcal{E}) \cdot \text{rk}(W)}(\mu(W') - \mu(W/W')) \leq 0$$
If $r_0 > r_{2(p-1)}$, then we have
$$\sum_{\ell=0}^m (p - 1 - \ell)r_\ell \geq (p - 1)$$
by Lemma 3. Consider formula (4), it is enough to prove that
$$\sum_{\ell=0}^m r_\ell(\mu(F_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) \leq (p - 1) \cdot \mu(\Omega^1_X).$$
Recall that $V_\ell/V_{\ell+1} = W \otimes T^\ell(\Omega^1_X)$, where
$$T^\ell(\Omega^1_X) = \begin{cases} 
\text{Sym}^\ell(\Omega^1_X) & \text{when } \ell < p \\
\text{Sym}^{2(p-1)-\ell}(\Omega^1_X) \otimes \omega_X^{\ell-(p-1)} & \text{when } \ell \geq p.
\end{cases}$$
Consequently, we have $V_\ell/V_{\ell+1}$ is semistable for $\ell \leq p - r$ and $\ell \geq p + r - 2$, and we only need to prove
\[
\sum_{\ell=p-r+1}^{p+r-3} r_\ell (\mu(F_\ell) - \mu(V_\ell/V_{\ell+1})) \leq (p-1) \cdot \mu(\Omega^1_X).
\]

But
\[
r_\ell (\mu(F_\ell) - \mu(V_\ell/V_{\ell+1})) \leq s(2(p-1) - \ell)
\]
for $p \leq \ell \leq p + r - 3$. Combining with Lemma 6 and Lemma 7, we obtain that
\[
\sum_{\ell=p-r+1}^{p+r-3} r_\ell (\mu(F_\ell) - \mu(V_\ell/V_{\ell+1})) = \sum_{\ell=p-r+1}^{p-1} r_\ell (\mu(F_\ell) - \mu(V_\ell/V_{\ell+1})) + \sum_{\ell=p}^{p+r-3} r_\ell (\mu(F_\ell) - \mu(V_\ell/V_{\ell+1}))
\]
\[
\leq \sum_{\ell=p-r+1}^{p-1} s(\ell) + \sum_{\ell=p}^{p+r-3} s(2(p-1) - \ell)
\]
\[
\leq \frac{t}{2} \cdot (1 + \cdots + r - 1) + \frac{t}{2} \cdot (r - 2 + \cdots + 1)
\]
\[
\leq \frac{1}{2}(2r - 1)(r - 1)^2 \cdot \mu(\Omega^1_X)
\]
\[
\leq (p-1) \cdot \mu(\Omega^1_X)
\]

If $W$ is stable, we can prove that $F_*W$ is stable similarly. The proof is completed.

\[\square\]

**Remark 1.** *Keep the assumption of Theorem 1. For $r = 1$, the stability of $F_*W$ is proved by X. Sun in [7]. As a slightly generalized version of [2, Theorem 3.1], it is proved by X. Sun that $F_*(L \otimes \Omega^1_X)$ is semistable when $L$ is a line bundle; moreover, if $\Omega^1_X$ is stable, then $F_*(L \otimes \Omega^1_X)$ is stable (see [7, Theorem 4.9]). There is no restriction on the characteristic $p$ for these results.*

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