ON THE ASYMMETRY OF STARS AT INFINITY

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Abstract. Given a bordified space, Karlsson defines an incidence geometry of stars at infinity. These stars and their incidence are closely related to well-understood objects when the space is hyperbolic, CAT(0), or a bounded convex domain with the Hilbert metric. A question stemming from Karlsson’s original paper was whether or not the relation of one boundary point being included in a star of another boundary point is symmetric. This paper provides an example demonstrating that this relation in the star boundary of the three-tree Diestel-Leader graph $DL_3(q)$ is not symmetric. In doing so, some interesting bounds on distance in Diestel-Leader graphs are utilized.

1. Introduction

In [5], Karlsson presents a theory on the dynamics of isometries and semicontractions of metric spaces in which he develops and utilizes the idea of “stars at infinity” around boundary points of bordified metric spaces, which essentially extend the notion of half-space to the boundary of the space. For example in a CAT(0) space, the star of a boundary point is the closed ball of radius $\pi/2$ in the angular metric. In a hyperbolic space, stars are singleton boundary points.

For a metric space $X$ with boundary $\partial X$, the star of a boundary point $\eta$ is denoted $S(\eta)$. Karlsson notes in Section 2.1 of [5] that, for $\eta, \xi \in \partial X$, it is unclear whether, or under what conditions, $\xi \in S(\eta)$ implies $\eta \in S(\xi)$; i.e., whether the relation of being included in the star is symmetric. In this paper, we exhibit an example in which this relation is not symmetric by studying the horofunction boundary of the Diestel-Leader graph $DL_3(q)$ (to be introduced in §3), which is a Cayley graph of a kind of generalization of the lamplighter group $L_2$. It should be noted that this example lives outside the context of non-positively curved spaces.

2. Stars at Infinity

2.1. Background. Karlsson introduces the following ideas in [5]. Let $(X, x_0)$ be a based metric space.

Definition 2.1. The halfspace for any $W \subset X$ with constant $C \geq 0$ is given by:

$$H(W, C) = \{ z \mid d(z, W) \leq d(z, x_0) + C \}.$$
For any bordification $\partial X$ of $X$, and any $\xi \in \partial X$ with neighborhood basis $\mathcal{U}$, the \textit{star} of $\xi$ is given by:

$$S(\xi) = \bigcup_{c \geq 0} \bigcap_{U \in \mathcal{U}} H(U, c).$$

This is independent of choice of basepoint for $X$ and of neighborhood basis for $\xi$. One can also consider the star of $\xi$ based at $x_0$, defined by:

$$S^{x_0}(\xi) = \bigcap_{U \in \mathcal{U}} H(U, 0),$$

and note that $S^{x_0}(\xi) \subseteq S(\xi)$.

Bridson \& Haefliger provide an introduction to the horofunction boundary of a metric space in [2], 8.12. We provide a brief overview here. Any based metric space $(X, x_0)$ has a natural embedding into the space $C_0(X)$ of continuous functions $X \to \mathbb{R}$ with $f(x_0) = 0$, via the mapping $x \in X \mapsto f_x(z) = d(x, z) - d(x, x_0)$. We give $C_0(X)$ the compact-open topology and consider the closure $\overline{X}$ of $X$ in this space. This closure is compact when $X$ is proper, as is the space $\partial X = \overline{X} \setminus X$, which we call the \textit{horofunction boundary} of $X$.

**Definition 2.2.** The \textit{horofunction} $f : X \to \mathbb{R}$ defined by a sequence $(x_n)$ is given by:

$$f(z) = \lim_{n \to \infty} d(x_n, z) - d(x_n, x_0).$$

If $(x_n)$ lies along a geodesic ray with $d(x_n, x_0) = n$, we call the induced horofunction a \textit{Busemann function}.

**2.2. A lemma about star-inclusion.** We make the following observation about stars.

**Lemma 2.3.** Let $X$ be a bordified metric space with basepoint $x_0$ and boundary $\partial X$. Let $(x_n)$ and $(y_n)$ be sequences approaching points $\bar{x}$ and $\bar{y}$, respectively, in $\partial X$. If for each $n$, $d(x_n, y_n) \leq d(x_n, x_0)$, then $\bar{x} \in S^{x_0}(\bar{y}) \subseteq S(\bar{y})$.

**Proof.** Let $\{N_k\}$ be any neighborhood basis about $\bar{y}$. Fix $k$. Then, since $y_n \to \bar{y}$, there exists a subsequence $(s_n)$ of $(y_n)$ contained entirely in $N_k$, and $s_n \to \bar{y}$. Let $(t_n)$ be the corresponding subsequence of $(x_n)$ (i.e., matching indices with $s_n$), so that for each $n$, $d(t_n, s_n) \leq d(t_n, x_0)$ and $t_n \to \bar{x}$. Then

$$(s_n) \subseteq N_k \implies (t_n) \subseteq H(N_k, 0) \implies \bar{x} \in \overline{H(N_k, 0)}.$$

Since $k$ was arbitrary, $\bar{x} \in S^{x_0}(\bar{y})$. Recall, $S^{x_0}(\bar{y}) \subseteq S(\bar{y})$. \hfill $\square$

**3. The Diestel-Leader Graph**

**Definition 3.1 (The graph $DL_d(q)$).** Let $T$ be a regular $q+1$ valent tree, such that each vertex $v$ has a single predecessor and $q$ successors. We think of successors as lying above predecessors. Let each edge have length 1, and label the edges of $T$ so that for each vertex $v$, the $q$ successors of $v$ have labels in one-to-one correspondence with the set $\{0, 1, \ldots, q-1\}$. Choose a basepoint $o$ in $T$. For $v, w \in T$, let $v \triangleright w$ denote the greatest common ancestor of $v$ and $w$ in $T$. Define the following functions $T \to \mathbb{Z}$:

$$l(v) = d(v, o \triangleright v), \quad m(v) = d(o, o \triangleright v), \quad \text{and} \quad h(v) = l(v) - m(v).$$

The function $h$ gives the height in $T$, but we will make heavy use of $m$ and $l$ as well, as they appear in the distance formula provided by Stein and Taback in [6].

For a positive integer $d$, let $\{T_i \mid 1 \leq i \leq d\}$ be a set of copies of $T$ with basepoints $o_i$ and functions $m_i$, $l_i$, and $h_i$. Let $DL_d(q)$ be the graph whose vertices are the $d$-tuples $v = (v_1, v_2, ..., v_d)$, $v_i \in T_i$, satisfying $\sum_{i=1}^{d} h_i(v_i) = 0$. Two vertices $v$ and $w$ in $DL_d(q)$ are joined by an edge if there are $i \neq j$ such that: (i) $v_i$ and $w_i$ are adjacent in $T_i$, (ii) $v_j$ and $w_j$ are adjacent in $T_j$, and (iii) for all $k \notin \{i, j\}$, $v_k = w_k$. That is, two vertices in $DL_d(q)$ are adjacent if you can get from one to the other by simultaneously moving up in one tree and down in another. The graph $DL_d(q)$ has basepoint $o = (o_1, o_2, ..., o_d)$; since we are interested in cases where $DL_d(q)$ is the Cayley graph of a group, we refer to $o$ as $id$. There are natural projections $p_i : DL_d(q) \rightarrow T_i$ sending $v$ to $v_i$. From here out, we will use $p_i(v)$ in lieu of $v_i$.

We will reserve the notation $d(v, w)$ for distance between two vertices in $DL_d(q)$, and we will use $d_i(p_i(v), p_i(w))$ to refer to the distance from the projection $p_i(v)$ to $p_i(w)$ in $T_i$.

Notice that for $v \in DL_d(q)$, since $\sum h_i(v) = 0$, we have $\sum l_i(v) = \sum m_i(v)$; each point $v$ is determined doing the following for each tree $T_i$: first select the value $m_i(v)$, which represents moving downward in $T_i$ to the height $-m_i(v)$, and then select a path upwards from that point that does not backtrack having length $l_i(v)$. This upward path corresponds to an ordered tuple in $\{0, 1, ..., q-1\}^{l_i(v)}$. Figure 1 illustrates an example element of $DL_3(2)$.

The graph $DL_d(q)$ is a special case of a more general graph, $DL(q_1, q_2, ..., q_d)$ built from $d$ trees having possibly different valences; all of these are called Diestel-Leader graphs after the construction in [3] of an example of a vertex-symmetric graph that they conjectured (in response to the question by Woess) is not quasi-isometric to the Cayley graph of any group. Eskin, Fisher, and Whyte later proved in [4] that when $m \neq n$, this is indeed the case for $DL(m, n)$. In this paper, we will only discuss $DL_3(q)$, which in Corollary 3.15 of [1] is shown to be a Cayley graph of a certain affine matrix group over $\mathbb{Z}/3\mathbb{Z}$ with respect to a certain finite generating set.

Note: Throughout this paper, we are really interested in the vertex set of $DL_d(q)$, representing the corresponding group with the metric structure provided by the edges of the graph. Thus we abuse notation and use $DL_d(q)$ to denote the discrete group.

**Definition 3.2.** Let $x$ and $y$ be vertices of $DL_d(q)$. For $1 \leq i \leq d$, we extend the $m, l$ notation to define $m_i(x, y) = d_i(p_i(x), p_i(y))$, $l_i(x, y) = d_i(p_i(y), p_i(x))$.
Proof. The schematics for each case are illustrated in Figure 2.

Lemma 3.3. The formulas for \( m_i(x,y) \) and \( l_i(x,y) \) are determined by whether \( m_i(x) \) is less than, equal to, or greater than \( m_i(y) \), as follows:

\[
\begin{align*}
    &m_i(x) < m_i(y) : m_i(x,y) = l_i(x) + (m_i(y) - m_i(x)) = m_i(y) + l_i(x) \\
    &l_i(x,y) = l_i(y) \\
    &m_i(x) = m_i(y) : \text{set } D_i \text{ to the length of the common upward path} \\
    &m_i(x,y) = l_i(x) - D_i \\
    &l_i(x,y) = l_i(y) - D_i \\
    &m_i(x) > m_i(y) : m_i(x,y) = l_i(x) \\
    &l_i(x,y) = l_i(y) + (m_i(x) - m_i(y)) = m_i(x) + l_i(y)
\end{align*}
\]

Proof. The schematics for each case are illustrated in Figure 2. □

Definition 3.4. Let \( x \) and \( y \) be vertices in \( DL_d(q) \) and let \( \sigma \in \Sigma_d \), the symmetric group on \( d \) letters. For \( 2 \leq i \leq d - 1 \), define

\[
f_{\sigma,i}(x,y) = m_{\sigma(1)}(x,y) + \cdots + m_{\sigma(i)}(x,y) + l_{\sigma(i)}(x,y) + \cdots + l_{\sigma(d)}(x,y),
\]

and

\[
f_{\sigma,d}(x,y) = 2m_{\sigma(1)}(x,y) + m_{\sigma(2)}(x,y) + \cdots + m_{\sigma(d)}(x,y) + l_{\sigma(d)}(x,y).
\]

Stein and Taback derive the following distance formula in [6] (see Lemma 1 and following discussion, as well as the proof of Corollary 10) in the case that \( DL_d(q) \) is the Cayley graph of a group.

Theorem 3.5 (Stein-Taback). Let \( x \) and \( y \) be vertices in \( DL_d(q) \). For \( \sigma \in \Sigma_d \), let \( f_\sigma(x,y) = \max_{2 \leq i \leq d} \{ f_{\sigma,i}(x,y) \} \). Then \( d(x,y) = \min_{\sigma \in \Sigma_d} \{ f_\sigma(x,y) \} \).

3.2. Distance bounds in Diestel-Leader graphs. One can find a variety of lower bounds on the distance between two points in \( DL_d(q) \).

Observation 3.6. Let \( R > 0 \). Let \( x, y \in DL_d(q) \), and that suppose for each \( \sigma \in \Sigma_d \), there is \( i \in \{ 2, \ldots, d \} \) such that \( f_{\sigma,i}(x,y) \geq R \). Then \( d(x,y) \geq R \).

Observation 3.7. Let \( x, y \in DL_d(q) \), and let \( R = \max \{ d_i(p_i(x), p_i(y)) \mid 1 \leq i \leq d \} \). Then \( d(x,y) \geq R \).
Lemma 3.8. Let $x, y, z \in DL_d(q)$ and $k \geq 0$. Suppose that for all $\sigma \in \Sigma_d$ and all $i \in \{2, \ldots, d\}$, $f_{\sigma,i}(x, z) \geq f_{\sigma,i}(x, y) + k$. Then $d(x, z) \geq d(x, y) + k$. The same is true when the inequalities are not strict.

Proof. For each $\sigma \in \Sigma_d$,

$$f_{\sigma}(x, z) = \max_{2 \leq i \leq d} \{f_{\sigma,i}(x, z)\} \geq \max_{2 \leq i \leq d} \{f_{\sigma,i}(x, y)\} + k = f_{\sigma}(x, y) + k.$$ 

So,

$$d(x, z) = \min_{\sigma \in \Sigma_d} \{f_{\sigma}(x, z)\} \geq \min_{\sigma \in \Sigma_d} \{f_{\sigma}(x, y)\} + k = d(x, y) + k.$$

\[\square\]

Lemma 3.9. Let $x, y, z \in DL_d(q)$ and suppose there are nonnegative numbers $c_j$, $1 \leq j \leq d$, such that $m_j(x, z) \geq m_j(x, y) + c_j$ and $l_j(x, z) \geq l_j(x, y) + c_j$. Then $d(x, z) \geq d(x, y) + \sum_{j=1}^{d} c_j$.

Proof. Investigating Definition 3.4, we see that for each $\sigma \in \Sigma_d$ and $2 \leq i \leq d$, $f_{\sigma,i}(x, z)$ is a sum that can be decomposed into $d$ terms, one for each tree, such that for $1 \leq j \leq d$, tree $T_j$ contributes exactly one of: $m_j(x, z)$, $2m_j(x, z)$, $l_j(x, z)$, or $m_j(x, z) + l_j(x, z)$. Note that the term contributed depends only on $\sigma$ and $i$ and does not depend on $x$ or $z$.

The assumption that $m_j(x, z) \geq m_j(x, y) + c_j$ and $l_j(x, z) \geq l_j(x, y) + c_j$ for each $j$ ensures that $f_{\sigma,i}(x, z) \geq f_{\sigma,i}(x, y) + \sum_{j=1}^{d} c_j$. By Lemma 3.8, $d(x, z) \geq d(x, y) + \sum_{j=1}^{d} c_j$. \[\square\]

The next results deal with points $z$ that have $h_i(z) = 0$ for each $i$. Such points are special because $h_i(z) = 0$ implies that $m_i(z) = l_i(z)$, which can be useful in understanding distance. Also, since $id$ is such a point, it is easier to compare distances to these points with distances to $id$.

Lemma 3.10 (Cases where $d(x, z) \leq d(x, id)$). Let $x, z \in DL_d(q)$ and suppose $h_i(z) = 0$ for $1 \leq i \leq d$.

(i) If $m_i(z) < m_i(x)$ for each $i$ such that $m_i(x) \neq 0$, and $m_i(z) = 0$ for each $i$ such that $m_i(x) = 0$, then $d(x, z) = d(x, id)$.

(ii) If $m_i(z) \leq m_i(x)$ for each $i$, then $d(x, z) \leq d(x, id)$.

Proof. We apply Lemma 3.3, where we establish the parameter $D_i$ for path overlap in cases where $m_i(z) = m_i(x)$.

For part (i), if $m_i(x) \neq 0$, $m_i(z) < m_i(x)$ implies $m_i(z, x) = m_i(x) + h_i(z) = m_i(x)$ and $l_i(z, x) = l_i(x)$. If $m_i(x) = 0$, the path overlap $D_i = 0$ since $l_i(z) = 0$, so that $m_i(z, x) = l_i(z) = m_i(x)$ and $l_i(z, x) = l_i(x)$. Thus for each $\sigma$ and $j$, $f_{\sigma,j}(z, x) = f_{\sigma,j}(id, x)$, so that $d(x, z) = d(x, id)$.

For part (ii), for $i$ such that $m_i(x) = m_i(z) > 0$, we replace the equations from the previous paragraph with inequalities $m_i(z, x) = l_i(z) - D_i = -D_i \leq m_i(x)$, and $l_i(z, x) = l_i(x) - D_i \leq l_i(x)$. By Lemma 3.9, $d(x, z) \leq d(x, id)$. \[\square\]

Lemma 3.11 (Cases where $d(x, z) \geq d(x, id) + k$). Let $x, z \in DL_d(q)$ and suppose $h_i(z) = 0$ for each $i$. Assume $m_i(z) \neq m_i(x)$ except in cases where $m_i(x) = 0$. Let $c_i = \max\{0, m_i(z) - m_i(x)\}$. Then $d(x, z) \geq d(x, id) + \sum_{i=1}^{d} c_i$. 

Proof: For each $i$, we apply Lemma 3.3 to consider $m_i(z, x)$ and $l_i(x, z)$ in the following cases.

If $m_i(z) < m_i(x)$, then
\[ m_i(z, x) = m_i(x) + l_i(z) = m_i(x), \quad \text{and} \quad l_i(z, x) = l_i(x). \]

If $m_i(z) = m_i(x) = 0$, then the path overlap $D_i = 0$ since $l_i(z) = 0$, so that
\[ m_i(z, x) = l_i(z) - D_i = m_i(x) \quad \text{and} \quad l_i(z, x) = l_i(x) - D_i = l_i(x). \]

Finally, if $m_i(z) > m_i(x)$, then $m_i(z, x) = l_i(z) = m_i(z) = m_i(x) + c_i$ and
\[ l_i(z, x) = m_i(z) + h_i(x) = (m_i(x) + c_i) + (l_i(x) - m_i(x)) = l_i(x) + c_i. \]

By Lemma 3.9, $d(x, z) \geq d(x, id) + \sum_{i=1}^{d} c_i$. \hfill \Box

4. Horofunctions and Stars in $DL_d(q)$

4.1. $m$-invariance of horofunctions.

Definition 4.1 (The point $\zeta^i_k$). For $1 \leq i \leq d$ and $k > 0$, let $\zeta^i_k \in DL_d(q)$ be the point given by $m_i(\zeta^i_k) = l_i(\zeta^i_k) = k$, following upward edges labeled “1”; and $m_j(\zeta^i_k) = l_j(\zeta^i_k) = 0$ for $j \neq i$.

Lemma 4.2. Let $(x_n)$ be a sequence defining a horofunction $h_x$. Then for each $1 \leq i \leq d$, $\lim m_i(x_n)$ exists in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Proof. Suppose for some $i$ that $\lim m_i(x_n)$ does not exist in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then since $m_i$ takes nonnegative integer values, there exist two subsequences $(y_n)$ and $(z_n)$ of $(x_n)$, both approaching $h_x$, such that $m_i(y_n)$ is a constant $m_y$ and $m_i(z_n)$ and is bounded below by $m_y + 1$.

Let $k = m_y + 1$. Lemma 3.11 gives $d(y_n, \zeta^i_k) > d(y_n, id)$ for each $n$, implying $h_x(\zeta^i_k) > 0$. But by Lemma 3.10, $d(z_n, \zeta^i_k) \leq d(z_n, id)$ for each $n$, implying $h_x(\zeta^i_k) \leq 0$, which is a contradiction. \hfill \Box

The proof of Lemma 4.2 also proves:

Theorem 4.3 ($m_i$-Invariance). If for sequences $(y_n)$ and $(z_n)$ defining horofunctions $h_y$ and $h_z$, there is an index $i$ such that $\lim m_i(y_n) \neq \lim m_i(z_n)$, then $h_y \neq h_z$.

In light of Theorem 4.3, for a horofunction $h$, we extend our notation to let $m_i(h) = \lim m_i(x_n)$ for any $(x_n)$ approaching $h$.

Corollary 4.4. If a sequence $(h_n)$ of horofunctions converges to a horofunction $h$, then $m_i(h_n)$ converges to $m_i(h)$ for each $i$.

Proof. For each $z \in DL_d(q)$, the sequence $(h_n(z))$ must become constant on $z$ for large enough $n$, since the codomain, $\mathbb{Z}$, is discrete. By $m$-invariance, as in the proof of Lemma 4.2, if there was any $i$ such that $m_i(h_n)$ did not also eventually become constant, there would exist $z$ for which that did not occur. \hfill \Box
Lemma 4.5. The sequence \( \alpha_n \) where each \( \alpha \) is a Busemann function, and therefore a horofunction (\([2]\), 8.17-8.18). While it is not difficult to calculate \( \alpha \) in general, we will not need to.

Let \( (\beta_n) \) be the sequence \( m_3(\beta_n) = l_3(\beta_n) = n, m_i(\beta) = l_i(\beta) = 0 \) for \( i = 1, 2 \), and the path upward in \( T_3 \) always selects the “1” edge.

**Lemma 4.5.** The sequence \( (\beta_n) \) defines a horofunction \( \beta \), and
\[
\beta(z) = m_1 + m_2 + \min_{j=1,2} \{m_j + h_3, h_j + h_3, l_j\},
\]
where each \( m_i = m_i(z), l_i = l_i(z), h_i = h_i(z) \).

**Proof.** Let \( z \in D_{L_3}(q) \) and denote \( m_i(z), l_i(z), h_i(z) \) by \( m_i, l_i, \) and \( h_i \) for convenience. Assume \( n >> m_i, l_i \), and after this translation all terms have an additional \( m_1 + m_2 \) in common. This yields:
\[
d(\beta_n, z) = 2n + m_1 + m_2 + \min\{m_1 + h_3, m_2 + h_3, l_1, h_2 + h_3, h_1 + h_3, l_2\}.
\]
This yields \( d(\beta_n, id) = 2n \), and \( \beta(z) = \lim_{n \to \infty} d(\beta_n, z) - 2n \). \( \square \)

### 4.2. Two Horofunctions

We will demonstrate \( \alpha, \beta \in \partial D_{L_3}(q) \) with \( \beta \in S(\alpha) \) and \( \alpha \notin S(\beta) \).

Let \( (\alpha_n) \) be the sequence such that \( l_1(\alpha_n) = m_2(\alpha_n) = n, \) all other \( m_i, l_i = 0 \), and \( (\alpha_n) \) moves upward in \( T_1 \) choosing edges labeled “1”. The function \( \alpha(z) = \lim_{n \to \infty} d(\alpha_n, z) - n \) is a Busemann function, and therefore a horofunction (\([2]\), 8.17-8.18). While it is not difficult to calculate \( \alpha \) in general, we will not need to.

Let \( (\beta_n) \) be the sequence \( m_3(\beta_n) = l_3(\beta_n) = n, m_i(\beta) = l_i(\beta) = 0 \) for \( i = 1, 2 \), and the path upward in \( T_3 \) always selects the “1” edge.

**Lemma 4.5.** The sequence \( (\beta_n) \) defines a horofunction \( \beta \), and
\[
\beta(z) = m_1 + m_2 + \min_{j=1,2} \{m_j + h_3, h_j + h_3, l_j\},
\]
where each \( m_i = m_i(z), l_i = l_i(z), h_i = h_i(z) \).

**Proof.** Let \( z \in D_{L_3}(q) \) and denote \( m_i(z), l_i(z), h_i(z) \) by \( m_i, l_i, \) and \( h_i \) for convenience. Assume \( n >> m_i, l_i \) for each \( i \). We have \( m_3(\beta_n, z) = n, l_3(\beta_n, z) = n + h_3, \) and \( m_i(\beta_n, z) = m_i \) and \( l_i(\beta_n, z) = l_i \) for \( i = 1, 2 \). Figure 3 shows the various \( f_\sigma(\beta_n, z) \). The only term all \( f_\sigma \) have in common is \( 2n \), but we can rewrite \( l_i = m_i + h_i \), and after this translation all terms have an additional \( m_1 + m_2 \) in common. This yields:
\[
d(\beta_n, z) = 2n + m_1 + m_2 + \min\{m_1 + h_3, m_2 + h_3, l_1, h_2 + h_3, h_1 + h_3, l_2\}.
\]
This yields \( d(\beta_n, id) = 2n \), and \( \beta(z) = \lim_{n \to \infty} d(\beta_n, z) - 2n \). \( \square \)

### 4.3. Neighborhoods and Stars

**Lemma 4.6.** \( \beta \in S(\alpha) \).

**Proof.** As noted, \( d(\beta_n, id) = 2n \), by moving as necessary in \( T_3 \) and compensating in either \( T_1 \) or \( T_2 \). We cannot apply Lemma 4.5, as the parameters of \( \alpha_n \) grow along with \( \beta_n \). But we can find \( d(\beta_n, \alpha_n) \) directly: we can move from \( \beta_n \) to \( \alpha_n \) in \( 2n \) steps as well, by moving in \( T_3 \) as necessary, compensating in \( T_2 \) for the first \( n \) steps.
and then in $T_1$ for the final $n$ steps. So $d(\beta_n, \alpha_n) = 2n = d(\beta_n, id)$. By Lemma 2.3, $\beta \in S(\alpha)$. 

Recall, the point $\zeta^t_k$, introduced in Definition 4.1, has $m_i = l_i = k$, and all other parameters trivial.

**Lemma 4.7.** Let $z \in DL_3(q)$ have $m_i(z) > 0$ for $i = 1$ or $2$. Then $d(z, \zeta^1_k) - d(z, id) \neq \beta(\zeta^1_k)$.

**Proof.** By Lemma 3.10, $d(z, \zeta) \leq d(z, id)$, so $d(z, \zeta) - d(z, id) \leq 0$, while Lemma 3.11 ensures $\beta(\zeta) > 0$. 

**Definition 4.8.** For $j \in \{1, 2\}$, $k \in \mathbb{Z}_{\geq 0}$, and $\epsilon \in \{0, 1, ..., q - 1\}$, let $\nu^{j, \epsilon}_k$ be the point in $DL_3(q)$ with $d(\nu^{j, \epsilon}_k, id) = k$ obtained by moving $k$ edges (all labeled $\epsilon$) upward in $T_j$ and $k$ edges downward in $T_3$. So $l_j(\nu^{j, \epsilon}_k) = m_3(\nu^{j, \epsilon}_k) = k$, while all other $m_i, l_i = 0$.

**Lemma 4.9.** Let $z \in DL_3(q)$ have $m_1(z) = m_2(z) = 0$, and assume $l_j(z) > 0$ for some $j \in \{1, 2\}$. Choose label $\epsilon$ not equal to the first edge joining $o_j$ to $p_j(z)$. Then $\nu = \nu^{j, \epsilon}_k$ has $d(z, \nu) - d(z, id) \neq \beta(\nu)$.

**Proof.** By Lemma 4.5, $\beta(\nu) = -1$. Lemma 3.6 will ensure that $d(z, \nu) - d(z, id) \geq 0$ if for each $\sigma \in \Sigma_d$, there is at least one $i$ such that $f_{\sigma,i}(z, \nu) \geq d(z, id)$.

For each $k = 1, 2, 3$, denote $m_k(z)$ and $l_k(z)$ by $m_k$ and $l_k$ respectively. Assume for concreteness that $l_1 > 0$. Note that $m_3 = \sum_{k=1}^3 l_k$. Also, $d(z, id) = d_3(o_3, p_3(z)) = l_1 + l_2 + 2l_3$ since Observation 3.7 ensures this is a lower bound on $d(z, id)$, and it can be realized by moving in $T_3$ and compensating in $T_1$ and $T_2$ as appropriate.

We have the following distances, by Lemma 3.3.

$m_1(z, \nu) = l_1 \quad m_2(z, \nu) = l_2 \quad m_3(z, \nu) = l_3$

$l_1(z, \nu) = 1 \quad l_2(z, \nu) = 0 \quad l_3(z, \nu) = \sum l_k - 1$

The following table provides, for each $\sigma$, an $f_{\sigma,i}(z, \nu) \geq l_1 + l_2 + 2l_3$:

| $f_{(1),3}$ | $f_{(12),3}$ | $f_{(13),3}$ | $f_{(2),3}$ | $f_{(23),3}$ | $f_{(123),2}$ |
|------------|--------------|-------------|------------|-------------|-------------|
| 2l_1 + l_2 + l_3 + (\sum l_k - 1) | 2l_2 + l_1 + l_3 + (\sum l_k - 1) | 2l_3 + l_2 + l_1 + 1 | l_1 + l_3 + (\sum l_k - 1) + 1 | 2l_3 + l_1 + l_2 |

If $l_2 > 0$ instead, the calculation is symmetric. 

**Definition 4.10.** For any finite set $F$, let

$$B(F) = \{ f \in DL_3(q) \mid f|_F = \beta|_F \}.$$ 

and set

$$F_0 = \{ \zeta^1_1, \zeta^2_2, \nu^{1,0}_1, \nu^{1,1}_1, \nu^{2,0}_1, \nu^{2,1}_1 \}$$

Because the functions in $DL_3(q)$ map from the discrete space $DL_3(q)$ to the discrete space $\mathbb{Z}$, when we consider the compact-open topology on $DL_3(q)$, the collection $\{ B(F) \}$, $F$ a finite set, is a neighborhood basis for $\beta$. Thus $B(F_0)$ is an open set about $\beta$.

**Theorem 4.11.** A sequence $(b_n) \subset DL_3(q)$ approaches $\beta$ if and only if it has a tail whose projections are trivial in $T_1$ and $T_2$ and $m_3(b_n) \rightarrow \infty$. 

Proof. By Lemmas 4.7 and 4.9, no \( x \in DL_3(q) \) that is nontrivial in \( T_1 \) or \( T_2 \) lies in \( B(F_0) \). This proves the forward direction.

Now suppose \( (b_n) \) is trivial in \( T_1 \) and \( T_2 \) and has \( m_3(b_n) \to \infty \). Then \( l_3(b_n) = m_3(b_n) \). The calculations in the proof of Lemma 4.5 show that when \( n \) is large relative to the parameters of \( z \in DL_3(q) \), the choice of upward path in \( T_3 \) is irrelevant to the calculation. We can replace \( n \) with \( m_3(b_n) \) in that calculation, and we obtain the same result. \( \square \)

**Corollary 4.12.** The horofunction \( \beta \) is isolated in \( \partial DL_3(q) \).

**Proof.** Let \( (h_n) \subset \partial DL_3(q) \) be a sequence of horofunctions approaching \( \beta \). Then \( (h_n) \) has a tail \( (t_n) \) lying in \( B(F_0) \). For each \( n \), any sequence \( (x_k) \in DL_3(q) \) approaching \( t_n \) also has a tail in \( B(F_0) \). By Theorem 4.11. \( (t_n) \) is the constant sequence \( \beta \). \( \square \)

It is worth noting that while \( \beta \) is isolated in the boundary, \( S(\beta) \neq \{ \beta \} \). For example, let \( Y \subseteq \{ 1, 2, 3 \} \) contain 3 and at least one other index. Let \( (\gamma_n) \) be any sequence with \( m_i(\gamma_n) = l_i(\gamma_n) = n \) for \( i \in Y \), and \( m_i(\gamma_n) = l_i(\gamma_n) = 0 \) otherwise. It can be shown that \( (\gamma_n) \) defines a horofunction \( \gamma \neq \beta \); and one can calculate \( d(\gamma_n, id) \geq d(\gamma_n, \beta_n) \) for each \( n \), so that Lemma 2.3 ensures \( h_\gamma \in S(\beta) \).

**Corollary 4.13.** For \( k \geq 0 \), let \( N_k(\beta) = \{ z \in DL_3(q) \mid m_i(z) = l_i(z) = 0 \text{ for } i = 1, 2 \text{ and } m_3(z) = l_3(z) \geq k \}. \) Then the family \( \{ N_k(\beta) \} \), \( k \geq 0 \), is a neighborhood basis for \( \beta \in DL_3(q) \).

Note that \( \beta_n \) and \( \beta \) are not the only elements of \( N_k(\beta) \), since alternate choices of edge labels may be made.

**Theorem 4.14.** Let \( h \in \partial DL_3(q) \) have \( m_3(h) = 0 \). Then \( h \notin S(\beta) \). As a consequence, \( \alpha \notin S(\beta) \).

**Proof.** By \( m \)-invariance, any sequence of points in \( DL_3(q) \) approaching \( h \) has a tail \( (a_n) \) such that for each \( n \), \( m_3(a_n) = 0 \). For any \( C \geq 0 \), when \( k > C \), the structure of \( N_k(\beta) \) and Lemma 3.11 together imply that

\[
d(a_n, N_k(\beta)) \geq d(a_n, id) + k > d(a_n, id) + C.
\]

So \( a_n \notin H(N_k(\beta), C) \), and so \( h \notin \bigcap_{k>0} H(N_k(\beta), C) \).

By Corollary 4.4, any sequence \( (h_n) \) of horofunctions approaching \( h \) must also eventually have \( m_3(h_n) = 0 \) as well, so \( h \) is also not a limit of any sequence of horofunctions in the union of \( \bigcap_{k>0} H(N_k(\beta), C) \) over all \( C \).

Thus

\[
h \notin \bigcup_{C \geq 0} \left( \bigcap_{k>0} H(N_k(\beta), C) \right) = S(\beta).
\]

\( \square \)

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