Zero Action on Perfect Crystals for $U_q(G^{(1)}_2)$

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Abstract. The actions of 0-Kashiwara operators on the $U'_q(G^{(1)}_2)$-crystal $B_l$ in [Yamane S., J. Algebra 210 (1998), 440–486] are made explicit by using a similarity technique from that of a $U'_q(D^{(3)}_4)$-crystal. It is shown that $\{B_l\}_{l \geq 1}$ forms a coherent family of perfect crystals.

Key words: combinatorial representation theory; quantum affine algebra; crystal bases

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1 Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Let $I$ be its index set for simple roots, $P$ the weight lattice, $\alpha_i \in P$ a simple root ($i \in I$), and $h_i \in P^\ast (= \text{Hom}(P, \mathbb{Z}))$ a simple coroot ($i \in I$). To each $i \in I$ we associate a positive integer $m_i$ and set $\tilde{\alpha}_i = m_i \alpha_i$, $\tilde{h}_i = h_i/m_i$. Suppose $(\langle \tilde{h}_i, \tilde{\alpha}_j \rangle)_{i,j \in I}$ is a generalized Cartan matrix for another symmetrizable Kac–Moody algebra $\tilde{\mathfrak{g}}$. Then the subset $\tilde{P}$ of $P$ consisting of $\lambda \in P$ such that $\langle \tilde{h}_i, \lambda \rangle$ is an integer for any $i \in I$ can be considered as the weight lattice of $\tilde{\mathfrak{g}}$. For a dominant integral weight $\lambda$ let $B_{\tilde{\mathfrak{g}}}^\theta(\lambda)$ be the highest weight crystal with highest weight $\lambda$ over $U_q(\mathfrak{g})$. Then, in [5] Kashiwara showed the following. (The theorem in [5] is more general.)

**Theorem 1.** Let $\lambda$ be a dominant integral weight in $\tilde{P}$. Then, there exists a unique injective map $S : B_{\tilde{\mathfrak{g}}}^\theta(\lambda) \to B^\theta(\lambda)$ such that

$$ wt S(b) = wt b, \quad S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b). $$

In this paper, we use this theorem to examine the so-called Kirillov–Reshetikhin crystal. Let $\mathfrak{g}$ be the affine algebra of type $D^{(3)}_4$. The generalized Cartan matrix $(\langle h_i, \alpha_j \rangle)_{i,j \in I}$ ($I = \{0, 1, 2\}$) is given by

$$
\begin{pmatrix}
 2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix}.
$$

Set $(m_0, m_1, m_2) = (3, 3, 1)$. Then, $\tilde{\mathfrak{g}}$ defined above turns out to be the affine algebra of type $G^{(1)}_2$. Their Dynkin diagrams are depicted as follows

$D^{(3)}_4 : \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ$

$G^{(1)}_2 : \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ$

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For $G_2^{(1)}$ a family of perfect crystals $\{B_l\}_{l \geq 1}$ was constructed in [7]. However, the crystal elements there were realized in terms of tableaux given in [2], and it was not easy to calculate the action of 0-Kashiwara operators on these tableaux. On the other hand, an explicit action of these operators was given on perfect crystals $\{\hat{B}_l\}_{l \geq 1}$ over $U'_q(D_4^{(3)})$ in [6]. Hence, it is a natural idea to use Theorem 4 to obtain the explicit action of $e_b, f_b$ on $B_l$ from that on $\hat{B}_l$ with suitable $l'$. We remark that Kirillov–Reshetikhin crystals are parametrized by a node of the Dynkin diagram except 0 and a positive integer. Both $B_l$ and $\hat{B}_l$ correspond to the pair $(1, l)$.

Our strategy to do this is as follows. We define $V_l$ as an appropriate subset of $\hat{B}_{3l}$ that is closed under the action of $e_i^{m_i}, f_i^{m_i}$ where $e_i, f_i$ stand for the Kashiwara operators on $\hat{B}_{3l}$. Hence, we can regard $V_l$ as a $U'_q(G_2^{(1)})$-crystal. We next show that as a $U_q(G_2^{(1)})_{\{0,1\}} (= U_q(A_2))$-crystal and as a $U_q(G_2^{(1)})_{\{0,1\}} (= U_q(G_2))$-crystal, $V_l$ has the same decomposition as $B_l$. Then, we can conclude from Theorem 6.1 of [6] that $V_l$ is isomorphic to the $U'_q(G_2^{(1)})$-crystal $B_l$ constructed in [7] (Theorem 2).

The paper is organized as follows. In Section 2 we review the $U'_q(D_4^{(3)})$-crystal $\hat{B}_l$. We then construct a $U'_q(G_2^{(1)})$-crystal $V_l$ in $\hat{B}_{3l}$ with the aid of Theorem 4 and see it coincides with $B_l$ given in [7] in Section 3 Minimal elements of $B_l$ are found and $\{B_l\}_{l \geq 1}$ is shown to form a coherent family of perfect crystals in Section 4. The crystal graphs of $B_1$ and $B_2$ are included in Section 5.

2 Review on $U'_q(D_4^{(3)})$-crystal $\hat{B}_l$

In this section we recall the perfect crystal for $U'_q(D_4^{(3)})$ constructed in [6]. Since we also consider $U'_q(G_2^{(1)})$-crystals later, we denote it by $\hat{B}_l$. Kashiwara operators $e_i, f_i$ and $\varepsilon_i, \varphi_i$ on $\hat{B}_l$ are denoted by $\hat{e}_i, \hat{f}_i$ and $\hat{\varepsilon}_i, \hat{\varphi}_i$. Readers are warned that the coordinates $x_i, \bar{x}_i$ and steps by Kashiwara operators in [6] are divided by 3 here, since it is more convenient for our purpose. As a set

$$\hat{B}_l = \left\{ b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in (\mathbb{Z}_{\geq 0}/3)^6 \left| \begin{array}{l}
3x_3 \equiv 3\bar{x}_3 \pmod{2}, \\
\sum_{i=1,2} (x_i + \bar{x}_i) + (x_3 + \bar{x}_3)/2 \leq l/3
\end{array} \right. \right\}. $$

In order to define the actions of Kashiwara operators $\hat{e}_i$ and $\hat{f}_i$ for $i = 0, 1, 2$, we introduce some notations and conditions. Set $(x)_+ = \max(x, 0)$. For $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \hat{B}_l$ we set

$$s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1, \quad \text{(2.1)}$$

and

$$z_1 = \bar{x}_1 - x_1, \quad z_2 = \bar{x}_2 - \bar{x}_3, \quad z_3 = x_3 - x_2, \quad z_4 = (\bar{x}_3 - x_3)/2. \quad \text{(2.2)}$$

Now we define conditions (E1)–(E6) and (F1)–(F6) as follows

$$\begin{align*}
(F_1) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, \quad z_1 + z_2 + 3z_4 \leq 0, \quad z_1 + z_2 \leq 0, \quad z_1 \leq 0, \\
(F_2) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, \quad z_2 + 3z_4 \leq 0, \quad z_2 \leq 0, \quad z_1 > 0, \\
(F_3) \quad & z_1 + z_3 + 3z_4 \leq 0, \quad z_3 + 3z_4 \leq 0, \quad z_4 \leq 0, \quad z_2 > 0, \quad z_1 + z_2 > 0, \\
(F_4) \quad & z_1 + z_2 + 3z_4 > 0, \quad z_2 + 3z_4 > 0, \quad z_4 > 0, \quad z_3 \leq 0, \quad z_1 + z_3 \leq 0, \\
(F_5) \quad & z_1 + z_2 + z_3 + 3z_4 > 0, \quad z_3 + 3z_4 > 0, \quad z_3 > 0, \quad z_1 \leq 0, \\
(F_6) \quad & z_1 + z_2 + z_3 + 3z_4 > 0, \quad z_1 + z_3 + 3z_4 > 0, \quad z_1 + z_3 > 0, \quad z_1 > 0.
\end{align*} \quad \text{(2.3)}$$
The conditions \((F_1)-(F_6)\) are disjoint and they exhaust all cases. \((E_i)\) \((1 \leq i \leq 6)\) is defined from \((F_i)\) by replacing \(>\) (resp. \(\leq\)) with \(\geq\) (resp. \(<\)). We also define

\[
A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4).
\]

Then, for \(b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \hat{B}_l\), \(\hat{e}_i b, \hat{f}_i b, \hat{e}_i(b), \hat{\varphi}_i(b)\) are given as follows

\[
\hat{e}_0 b = \begin{cases}
(x_1 - 1/3, \ldots) & \text{if } (E_1), \\
(x_1 - 1/3, \bar{x}_3 - 1/3, \ldots, \bar{x}_1 + 1/3) & \text{if } (E_2), \\
(x_1 - 2/3, \ldots, \bar{x}_2 + 1/3, \ldots) & \text{if } (E_3), \\
(x_1 - 1/3, \ldots, \bar{x}_3 + 2/3, \ldots) & \text{if } (E_4), \\
(x_1 - 1/3, \ldots, x_3 + 1/3, \bar{x}_3 + 1/3, \ldots) & \text{if } (E_5), \\
(\ldots, \bar{x}_1 + 1/3) & \text{if } (E_6),
\end{cases}
\]

\[
\hat{f}_0 b = \begin{cases}
(x_1 + 1/3, \ldots) & \text{if } (F_1), \\
(x_1 + 1/3, x_3 + 1/3, \ldots, \bar{x}_3 + 1/3, \bar{x}_1 - 1/3) & \text{if } (F_2), \\
(x_1 + 2/3, \ldots, \bar{x}_2 - 1/3, \ldots) & \text{if } (F_3), \\
(x_1 + 1/3, x_2 + x_3 - 2/3, \ldots) & \text{if } (F_4), \\
(x_1 + 1/3, \ldots, x_3 - 1/3, \bar{x}_3 - 1/3, \ldots) & \text{if } (F_5), \\
(\ldots, \bar{x}_1 - 1/3) & \text{if } (F_6).
\end{cases}
\]

\[
\hat{e}_1 b = \begin{cases}
(\ldots, \bar{x}_2 + 1/3, \bar{x}_1 - 1/3) & \text{if } z_2 \geq (-z_3)_{+}, \\
(\ldots, x_3 + 1/3, \bar{x}_3 - 1/3, \ldots) & \text{if } z_2 < 0 \leq z_3, \\
(x_1 + 1/3, x_2 - 1/3, \ldots) & \text{if } (z_2)_+ < (-z_3),
\end{cases}
\]

\[
\hat{f}_1 b = \begin{cases}
(x_1 - 1/3, x_2 + 1/3, \ldots) & \text{if } (z_2)_+ \leq (-z_3), \\
(\ldots, x_3 - 1/3, \bar{x}_3 + 1/3, \ldots) & \text{if } z_2 \leq 0 < z_3, \\
(\ldots, \bar{x}_2 - 1/3, \bar{x}_1 + 1/3) & \text{if } z_2 > (-z_3),
\end{cases}
\]

\[
\hat{e}_2 b = \begin{cases}
(\ldots, \bar{x}_3 + 2/3, \bar{x}_2 - 1/3, \ldots) & \text{if } z_4 \geq 0, \\
(\ldots, x_2 + 1/3, x_3 - 2/3, \ldots) & \text{if } z_4 < 0,
\end{cases}
\]

\[
\hat{f}_2 b = \begin{cases}
(\ldots, x_2 - 1/3, x_3 + 2/3, \ldots) & \text{if } z_4 \leq 0, \\
(\ldots, \bar{x}_3 - 2/3, \bar{x}_2 + 1/3, \ldots) & \text{if } z_4 > 0,
\end{cases}
\]

\[
\hat{\varphi}_0(b) = l - 3s(b) + 3 \max A - 3(2z_1 + z_2 + z_3 + 3z_4), \\
\hat{\varphi}_1(b) = l - 3s(b) + 3 \max A, \\
\hat{\varphi}_1(b) = 3\bar{x}_1 + 3(\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_{+})_{+}, \\
\hat{\varphi}_1(b) = 3x_1 + 3(x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_{+})_{+}, \\
\hat{\varphi}_2(b) = 3\bar{x}_2 + 3\bar{x}_3 \geq \frac{3}{2}(\bar{x}_3 - x_3)_{+}, \\
\hat{\varphi}_2(b) = 3x_2 + \frac{3}{2}(\bar{x}_3 - x_3)_{+}.
\]

If \(\hat{e}_i b\) or \(\hat{f}_i b\) does not belong to \(\hat{B}_l\), namely, if \(x_j\) or \(\bar{x}_j\) for some \(j\) becomes negative or \(s(b)\) exceeds \(l/3\), we should understand it to be 0. Forgetting the 0-arrows,

\[
\hat{B}_l \simeq \bigoplus_{j=0}^{l} B^{G_2^l}(j\Lambda_1),
\]

where \(B^{G_2^l}(\lambda)\) is the highest weight \(U_q(G_2^l)\)-crystal of highest weight \(\lambda\) and \(G_2^l\) stands for the simple Lie algebra \(G_2\) with the reverse labeling of the indices of the simple roots \((\alpha_1\) is the short
root). Forgetting 2-arrows,

$$\hat{B}_l \simeq \bigoplus_{i=0}^{\lfloor \frac{l}{3} \rfloor} \bigoplus_{i \leq j_0, j_1 \leq l-i \pmod{3}} B^{A_2}(j_0\lambda_0 + j_1\lambda_1),$$

where $B^{A_2}(\lambda)$ is the highest weight $U_q(A_2)$-crystal (with indices $\{0,1\}$) of highest weight $\lambda$.

3 \hspace{1em} \textbf{$U'_q(G_2^{(1)})$-crystal}

In this section we define a subset $V_l$ of $\hat{B}_{3l}$ and see it is isomorphic to the $U'_q(G_2^{(1)})$-crystal $B_l$.

The set $V_l$ is defined as a subset of $\hat{B}_{3l}$ satisfying the following conditions:

$$x_1, \bar{x}_1, x_2 - x_3, \bar{x}_3 - \bar{x}_2 \in \mathbb{Z}. \quad (3.1)$$

For an element $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_1)$ of $V_l$ we define $s(b)$ as in (2.11). From (3.1) we see that $s(b) \in \{0, 1, \ldots, l\}$.

**Lemma 1.** For $0 \leq k \leq l$

$$\sharp \{ b \in V_l \mid s(b) = k \} = \frac{1}{120} (k+1)(k+2)(2k+3)(3k+4)(3k+5).$$

**Proof.** We first count the number of elements $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ satisfying the conditions of coordinates as an element of $V_l$ and $x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 = m$ $(m = 0, 1, \ldots, k)$. According to $(a, b, c, d) (a, d \in \{0, 1/3, 2/3\}, b, c \in \{0, 1/3, 2/3, 1, 4/3, 5/3\})$ such that $x_2 \in \mathbb{Z} + a, x_3 \in 2\mathbb{Z} + b, \bar{x}_3 \in 2\mathbb{Z} + c, \bar{x}_2 \in \mathbb{Z} + d$, we divide the cases into the following 18:

- (i) $(0, 0, 0, 0)$,
- (ii) $(0, 0, 2/3, 2/3)$,
- (iii) $(0, 0, 4/3, 1/3)$,
- (iv) $(0, 1, 1/3, 1/3)$,
- (v) $(0, 1, 1, 0)$,
- (vi) $(0, 1, 5/3, 2/3)$,
- (vii) $(1/3, 1/3, 1/3, 1/3)$,
- (viii) $(1/3, 1/3, 1, 0)$,
- (ix) $(1/3, 1/3, 5/3, 2/3)$,
- (x) $(1/3, 4/3, 0, 0)$,
- (xi) $(1/3, 4/3, 2, 2/3)$,
- (xii) $(1/3, 4/3, 4/3, 1/3)$,
- (xiii) $(2/3, 2/3, 0, 0)$,
- (xiv) $(2/3, 2/3, 2, 2/3)$,
- (xv) $(2/3, 2/3, 4/3, 1/3)$,
- (xvi) $(2/3, 5/3, 1, 1/3)$,
- (xvii) $(2/3, 5/3, 1, 0)$,
- (xviii) $(2/3, 5/3, 5/3, 2/3)$.

The number of elements $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ in a case among the above such that $a + (b + c)/2 + d = e$ ($e = 0, 1, 2, 3$) is given by $f(e) = \binom{m-e+3}{3}$. Since there is one case with $e = 0$ (i) and $e = 3$ (xviii) and 8 cases with $e = 1$ and $e = 2$, the number of $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ such that $x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 = m$ is given by

$$f(0) + 8f(1) + 8f(2) + f(3) = \frac{1}{2} (2m + 1)(3m^2 + 3m + 2).$$

For each $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ such that $x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 = m$ $(m = 0, 1, \ldots, k)$ there are $(k - m + 1)$ cases for $(x_1, \bar{x}_1)$, so the number of $b \in V_l$ such that $s(b) = k$ is given by

$$\sum_{m=0}^{k} \frac{1}{2} (2m + 1)(3m^2 + 3m + 2)(k - m + 1).$$

A direct calculation leads to the desired result.
We define the action of operators $e_i, f_i$ ($i = 0, 1, 2$) on $V_l$ as follows.

$$e_{0b} = \begin{cases} 
(x_1 - 1, \ldots) & \text{if } (E_1), \\
(x_1 - 1, \ldots) & \text{if } (E_2), \\
(x_2 - 2, \ldots, \bar{x}_2 + 1, \ldots) & \text{if } (E_3) \text{ and } z_4 = -\frac{1}{3}, \\
(x_2 - 1, \ldots, \bar{x}_3 + 2, \ldots) & \text{if } (E_4), \\
(x_1 - 1, \ldots, x_3 + 1, \bar{x}_3 + 1, \ldots) & \text{if } (E_5), \\
(\ldots, \bar{x}_1 + 1) & \text{if } (E_6),
\end{cases}$$

$$f_{0b} = \begin{cases} 
(x_1 + 1, \ldots) & \text{if } (F_1), \\
(x_3 + 1, \bar{x}_3 + 1, \ldots) & \text{if } (F_2), \\
(x_3 + 2, \ldots, \bar{x}_2 - 1, \ldots) & \text{if } (F_3), \\
(x_2 + \frac{1}{3}, \bar{x}_3 + \frac{4}{3}, \bar{x}_3 - \frac{2}{3}, \bar{x}_2 - \frac{2}{3}, \ldots) & \text{if } (F_4) \text{ and } z_4 = \frac{1}{3}, \\
(x_2 + \frac{2}{3}, \bar{x}_3 + \frac{5}{3}, \bar{x}_3 - \frac{4}{3}, \bar{x}_2 - \frac{1}{3}, \ldots) & \text{if } (F_4) \text{ and } z_4 = \frac{2}{3}, \\
(x_2 + 1, \ldots, \bar{x}_3 - 2, \ldots) & \text{if } (F_4) \text{ and } z_4 \neq \frac{1}{3}, \frac{2}{3}, \\
(x_1 + 1, \ldots, x_3 - 1, \bar{x}_3 - 1, \ldots) & \text{if } (F_5), \\
(\ldots, \bar{x}_1 - 1) & \text{if } (F_6),
\end{cases}$$

$$e_{1b} = \begin{cases} 
(\ldots, \bar{x}_2 + 1, \bar{x}_1 - 1) & \text{if } \bar{x}_2 - \bar{x}_3 \geq (x_2 - x_3)_+, \\
(x_1 + 1, \bar{x}_3 - 1, \ldots) & \text{if } \bar{x}_2 - \bar{x}_3 < 0 \leq x_3 - x_2, \\
(x_1 + 1, x_2 - 1, \ldots) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ < x_2 - x_3,
\end{cases}$$

$$f_{1b} = \begin{cases} 
(x_1 - 1, x_2 + 1, \ldots) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ \leq x_2 - x_3, \\
(x_3 - 1, \bar{x}_3 + 1, \ldots) & \text{if } \bar{x}_2 - \bar{x}_3 \leq 0 < x_3 - x_2, \\
(\ldots, \bar{x}_2 - 1, \bar{x}_1 + 1) & \text{if } \bar{x}_2 - \bar{x}_3 > (x_2 - x_3)_+,
\end{cases}$$

$$e_{2b} = \begin{cases} 
(\ldots, \bar{x}_3 + \frac{2}{3}, \bar{x}_2 - \frac{4}{3}, \ldots) & \text{if } \bar{x}_3 \geq x_3, \\
(\ldots, \bar{x}_2 + \frac{1}{3}, \bar{x}_3 - \frac{2}{3}, \ldots) & \text{if } \bar{x}_3 < x_3,
\end{cases}$$

$$f_{2b} = \begin{cases} 
(\ldots, x_2 - \frac{1}{3}, x_3 + \frac{2}{3}, \ldots) & \text{if } \bar{x}_3 \leq x_3, \\
(\ldots, \bar{x}_3 - \frac{2}{3}, \bar{x}_2 + \frac{4}{3}, \ldots) & \text{if } \bar{x}_3 > x_3.
\end{cases}$$

We now set $(m_0, m_1, m_2) = (3, 3, 1)$.

**Proposition 1.**

1. For any $b \in V_l$ we have $e_{ib}, f_{ib} \in V_l \sqcup \{0\}$ for $i = 0, 1, 2$.
2. The equalities $e_i = e_i^{m_i}$ and $f_i = f_i^{m_i}$ hold on $V_l$ for $i = 0, 1, 2$.

**Proof.** (1) can be checked easily.

For (2) we only treat $f_i$. To prove the $i = 0$ case consider the following table

| $(F_1)$ | $(F_2)$ | $(F_3)$ | $(F_4)$ | $(F_5)$ | $(F_6)$ |
|--------|--------|--------|--------|--------|--------|
| $z_1$  | $-1/3$ | $-1/3$ | $0$    | $0$    | $-1/3$ | $-1/3$ |
| $z_2$  | $0$    | $-1/3$ | $-1/3$ | $2/3$  | $1/3$  | $0$    |
| $z_3$  | $0$    | $1/3$  | $2/3$  | $-1/3$ | $-1/3$ | $0$    |
| $z_4$  | $0$    | $0$    | $-1/3$ | $-1/3$ | $0$    | $0$    |
This table signifies the difference \((z_j \text{ for } \hat{f}_0 b) - (z_j \text{ for } b)\) when \(b\) belongs to the case \((F_i)\). Note that the left hand sides of the inequalities of each \((F_i)\) \((2,3)\) always decrease by \(1/3\). Since \(z_1, z_2, z_3 \in \mathbb{Z}, z_4 \in \mathbb{Z}/3\) for \(b \in V_l\), we see that if \(b\) belongs to \((F_i)\), \(\hat{f}_0 b\) and \(\hat{f}_0^2 b\) also belong to \((F_i)\) except two cases: (a) \(b \in (F_4)\) and \(z_4 = 1/3\), and (b) \(b \in (F_4)\) and \(z_4 = 2/3\). If (a) occurs, we have \(\hat{f}_0 b, \hat{f}_0^2 b \in (F_3)\). Hence, we obtain \(f_0 = f_0^3\) in this case. If (b) occurs, we have \(f_0 b \in (F_4)\), \(f_0^2 b \in (F_3)\). Therefore, we obtain \(f_0 = f_0^3\) in this case as well.

In the \(i = 1\) case, if \(b\) belongs to one of the 3 cases, \(f_1 b\) and \(f_1^2 b\) also belong to the same case. Hence, we obtain \(f_1 = f_1^3\). For \(i = 2\) there is nothing to do. ■

Proposition \(\text{[I]}\) together with Theorem \(\text{[I]}\) shows that \(V_l\) can be regarded as a \(U_q(G_2^{(1)})\)-crystal with operators \(e_i, f_i\) \((i = 0, 1, 2)\).

**Proposition 2.** As a \(U_q(G_2^{(1)})\)-crystal

\[
V_l \simeq \bigoplus_{k=0}^l B^{G_2}(k\Lambda_1),
\]

where \(B^{G_2}(\lambda)\) is the highest weight \(U_q(G_2)\)-crystal of highest weight \(\lambda\).

**Proof.** For a subset \(J\) of \(\{0, 1, 2\}\) we say \(b\) is \(J\)-highest if \(e_j b = 0\) for any \(j \in J\). Note from \((2,5)\) that \(b_k = (k, 0, 0, 0, 0)\) \((0 \leq k \leq l)\) is \(\{1, 2\}\)-highest of weight \(3k\Lambda_1\) in \(B_{3l}\). By setting \(g = G_2^\dagger\) (= \(G_2\) with the reverse labeling of indices), \((m_1, m_2) = (3, 1), \ g = G_2\) in Theorem \(\text{[I]}\) we know that the connected component generated from \(b_k\) by \(f_1 = f_1^3\) and \(f_2 = f_2^3\) is isomorphic to \(B^{G_2}(k\Lambda_1)\). Hence by Proposition \(\text{[I]}(1)\) we have

\[
\bigoplus_{k=0}^l B^{G_2}(k\Lambda_1) \subset V_l.
\]

Now recall Weyl’s formula to calculate the dimension of the highest weight representation. In our case we obtain

\[
\sharp B^{G_2}(k\Lambda_1) = \frac{1}{120} (k+1)(k+2)(2k+3)(3k+4)(3k+5).
\]

However, this is equal to \(\sharp \{b \in V_l \mid s(b) = k\}\) by Lemma \(\text{[I]}\) Therefore, \(\subset\) in \((3.2)\) should be =, and the proof is completed. ■

**Proposition 3.** As a \(U_q(G_2^{(1)})\)-crystal

\[
V_l \simeq \bigoplus_{i=0}^{[l/2]} \bigoplus_{i \leq j_0, j_1 \leq l-i} B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1),
\]

where \(B^{A_2}(\lambda)\) is the highest weight \(U_q(A_2)\)-crystal (with indices \(\{0, 1\}\)) of highest weight \(\lambda\).

**Proof.** For integers \(i, j_0, j_1\) such that \(0 \leq i \leq l/2, i \leq j_0, j_1 \leq l - i\), define an element \(b_{i,j_0,j_1}\) of \(V_l\) by

\[
b_{i,j_0,j_1} = \begin{cases} (0, y_1, 3y_0 - 2y_1 + i, y_0 + i, y_0 + j_0, 0) & \text{if } j_0 \leq j_1, \\ (0, y_0, y_0 + i, 2y_1 - y_0 + i, 2y_0 - y_1 + j_0, 0) & \text{if } j_0 > j_1. \end{cases}
\]

Here we have set \(y_a = (l - i - j_a)/3\) for \(a = 0, 1\). From \((2,5)\) one notices that \(b_{i,j_0,j_1}\) is \(\{0, 1\}\)-highest of weight \(3j_0\Lambda_0 + 3j_1\Lambda_1\) in \(B_{3l}\). For instance, \(\hat{e}_0(b_{i,j_0,j_1}) = 0\) and \(\hat{e}_0(b_{i,j_0,j_1}) = 3j_0\) since
Then we have
\[ \psi(0) \]
For Lemma 2. \( Z \) in [6], although \( z \) where

\[ U \]
our representation are given by \( V \)
Therefore, the proof is completed.

Moreover, it is already established in [7] that
\[ \sharp V_l = \sum_{k=0}^{l} \sharp B^{G_2}(k\Lambda_1). \]
Moreover, it is already established in [7] that
\[ \sum_{k=0}^{l} \sharp B^{G_2}(k\Lambda_1) = \sum_{i=0}^{[l/2]} \sum_{i \leq j_0, j_1 \leq l-i} \sharp B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1). \]
Therefore, the proof is completed.

Theorem 6.1 in [6] shows that if two \( U'_q(G_2^{(1)}) \)-crystals decompose into \( \bigoplus_{0 \leq k \leq l} B^{G_2}(k\Lambda_1) \) as \( U_q(G_2) \)-crystals, then they are isomorphic to each other. Therefore, we now have

**Theorem 2.** \( V_l \) agrees with the \( U'_q(G_2^{(1)}) \)-crystal \( B_l \) constructed in [7]. The values of \( \varepsilon_i, \varphi_i \) with our representation are given by
\[ \varepsilon_0(b) = l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4), \quad \varphi_0(b) = l - s(b) + \max A, \]
\[ \varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+, \quad \varphi_1(b) = x_1 + (x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+)_+, \]
\[ \varepsilon_2(b) = 3\bar{x}_2 + \frac{3}{2}(x_3 - x_3)_+, \quad \varphi_2(b) = 3x_2 + \frac{3}{2}(x_3 - x_3)_+. \]

### 4 Minimal elements and a coherent family

The notion of perfect crystals was introduced in [3] to construct the path realization of a highest weight crystal of a quantum affine algebra. The crystal \( B_l \) was shown to be perfect of level \( l \) in [4]. In this section we obtain all the minimal elements of \( B_l \) in the coordinate representation and also show \( \{B_l\}_{l \geq 1} \) forms a coherent family of perfect crystals. For the notations such as \( P_d, (P_d^+)_l \), see [3].

#### 4.1 Minimal elements

From (3.3) we have
\[ \langle c, \varphi(b) \rangle = \varphi_0(b) + 2\varphi_1(b) + \varphi_2(b) \]
\[ = l + \max A + 2(z_3 + (z_2)_+)_+ + (3z_4)_+ - (z_1 + z_2 + 2z_3 + 3z_4), \]
where \( z_j \) (1 \( \leq j \leq 4 \)) are given in (2.2) and \( A \) is given in (2.4). The following lemma was proven in [6], although \( \mathbb{Z} \) is replaced with \( \mathbb{Z}/3 \) here.

**Lemma 2.** For \((z_1, z_2, z_3, z_4) \in (\mathbb{Z}/3)^4\) set
\[ \psi(z_1, z_2, z_3, z_4) = \max A + 2(z_3 + (z_2)_+)_+ + (3z_4)_+ - (z_1 + z_2 + 2z_3 + 3z_4). \]

Then we have \( \psi(z_1, z_2, z_3, z_4) \geq 0 \) and \( \psi(z_1, z_2, z_3, z_4) = 0 \) if and only if \((z_1, z_2, z_3, z_4) = (0, 0, 0, 0)\).
From this lemma, we have \( \langle c, \varphi(b) \rangle - l = \psi(z_1, z_2, z_3, z_4) \geq 0 \). Since \( \langle c, \varphi(b) - \epsilon(b) \rangle = 0 \), we also have \( \langle c, \epsilon(b) \rangle \geq l \).

Suppose \( \langle c, \epsilon(b) \rangle = l \). It implies \( \psi = 0 \). Hence from the lemma one can conclude that such element \( b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \) should satisfy \( x_1 = \bar{x}_1, x_2 = x_3 = \bar{x}_3 = \bar{x}_2 \). Therefore the set of minimal elements \( (B_t)_{\min} \) in \( B_t \) is given by

\[
(B_t)_{\min} = \{ (\alpha, \beta, \beta, \beta, \beta, \alpha) | \alpha \in \mathbb{Z}_{\geq 0}, \beta \in (\mathbb{Z}_{\geq 0})/3, 2\alpha + 3\beta \leq l \}.
\]

For \( b = (\alpha, \beta, \beta, \beta, \beta, \alpha) \in (B_t)_{\min} \) one calculates

\[
\epsilon(b) = \varphi(b) = (l - 2\alpha - 3\beta)\Lambda_0 + \alpha\Lambda_1 + 3\beta\Lambda_2.
\]

### 4.2 Coherent family of perfect crystals

The notion of a coherent family of perfect crystals was introduced in [1]. Let \( \{B_t\}_{t \geq 1} \) be a family of perfect crystals \( B_t \) of level \( l \) and \( (B_t)_{\min} \) be the subset of minimal elements of \( B_t \). Set \( J = \{(l, b) | l \in \mathbb{Z}_{\geq 0}, b \in (B_t)_{\min}\} \). Let \( \sigma \) denote the isomorphism of \( (P_{cl}^+)_l \) defined by \( \sigma = \epsilon \circ \varphi^{-1} \).

For \( \lambda \in P_{cl} \), \( T_{cl} \) denotes a crystal with a unique element \( t_{\lambda} \) defined in [1]. For our purpose the following facts are sufficient. For any \( P_{cl}^{-} \)-weighted crystal \( B \) and \( \lambda, \mu \in P_{cl} \) consider the crystal

\[
T_{\lambda} \otimes B \otimes T_{\mu} = \{ t_{\lambda} \otimes b \otimes t_{\mu} | b \in B \}.
\]

The definition of \( T_{\lambda} \) and the tensor product rule of crystals imply

\[
\bar{e}_i(t_{\lambda} \otimes b \otimes t_{\mu}) = t_{\lambda} \otimes \bar{e}_i b \otimes t_{\mu}, \quad \bar{f}_i(t_{\lambda} \otimes b \otimes t_{\mu}) = t_{\lambda} \otimes \bar{f}_i b \otimes t_{\mu},
\]

\[
\epsilon_i(t_{\lambda} \otimes b \otimes t_{\mu}) = \epsilon_i(b) - \langle h_i, \lambda \rangle, \quad \varphi_i(t_{\lambda} \otimes b \otimes t_{\mu}) = \varphi_i(b) + \langle h_i, \mu \rangle,
\]

\[
\text{wt}(t_{\lambda} \otimes b \otimes t_{\mu}) = \lambda + \mu + \text{wt}b.
\]

**Definition 1.** A crystal \( B_{\infty} \) with an element \( b_{\infty} \) is called a limit of \( \{B_t\}_{t \geq 1} \) if it satisfies the following conditions:

- \( \text{wt}b_{\infty} = 0, \epsilon(b_{\infty}) = \varphi(b_{\infty}) = 0 \),
- for any \( (l, b) \in J \), there exists an embedding of crystals

\[
\tilde{f}_{(l,b)} : T_{\epsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \rightarrow B_{\infty}
\]

sending \( t_{\epsilon(b)} \otimes b \otimes t_{-\varphi(b)} \) to \( b_{\infty} \),
- \( B_{\infty} = \bigcup_{(l,b) \in J} \mathrm{Im} \tilde{f}_{(l,b)} \).

If a limit exists for the family \( \{B_t\} \), we say that \( \{B_t\} \) is a coherent family of perfect crystals.

Let us now consider the following set

\[
B_{\infty} = \left\{ b = (\nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1) \in (\mathbb{Z}/3)^6 \left| \begin{array}{c}
\nu_1, \nu_2, \nu_3 \in \mathbb{Z}, \\
3\bar{\nu}_3 \equiv 3\bar{\nu}_3 \quad \text{(mod 2)}
\end{array} \right. \right\},
\]

and set \( b_{\infty} = (0, 0, 0, 0, 0, 0, 0) \). We introduce the crystal structure on \( B_{\infty} \) as follows. The actions of \( e_i, f_i \) (\( i = 0, 1, 2 \)) are defined by the same rule as in Section 3 with \( x_i \) and \( \bar{x}_i \) replaced with \( \nu_i \) and \( \bar{\nu}_i \). The only difference lies in the fact that \( e_i b \) or \( f_i b \) never becomes 0, since we allow a coordinate to be negative and there is no restriction for the sum \( s(b) = \sum_{i=1}^{2}(\nu_i + \bar{\nu}_i) + (\nu_3 + \bar{\nu}_3)/2 \).

For \( \epsilon_i, \varphi_i \) with \( i = 1, 2 \) we adopt the formulas in Section 3. For \( \epsilon_0, \varphi_0 \) we define

\[
\epsilon_0(b) = -s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4), \quad \varphi_0(b) = -s(b) + \max A,
\]
where $A$ is given in (2.3) and $z_1, z_2, z_3, z_4$ are given in (2.2) with $x_i, \bar{x}_i$ replaced by $\nu_i, \bar{\nu}_i$. Note that $\text{wt } b_\infty = 0$ and $\varepsilon_i(b_\infty) = \varphi_i(b_\infty) = 0$ for $i = 0, 1, 2$.

Let $b_0 = (\alpha, \beta, \beta, \beta, \alpha)$ be an element of $(B_l)_{\text{min}}$. Since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\sigma = \text{id}$. Let $\lambda = \varepsilon(b_0)$. For $b = (x_1, x_2, x_3, \bar{x}_2, \bar{x}_1) \in B_l$ we define a map

$$f_{(l,b_0)} \colon T_\lambda \otimes B_l \otimes T_{-\lambda} \longrightarrow B_\infty$$

by

$$f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda}) = b' = (\nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1),$$

where

$$\nu_1 = x_1 - \alpha, \quad \nu_2 = x_1 - \bar{x}_1,$$
$$\nu_3 = x_2 - \beta, \quad \bar{\nu}_2 = \bar{x}_2 - \beta \quad (j = 2, 3).$$

Note that $s(b') = s(b) - (2\alpha + 3\beta)$. Then we have

$$\text{wt } (t_\lambda \otimes b \otimes t_{-\lambda}) = \text{wt } b = \text{wt } b',$$
$$\varphi_0(t_\lambda \otimes b \otimes t_{-\lambda}) = \varphi_0(b) + \langle h_0, \lambda \rangle$$
$$= \varphi_0(b') + (l - s(b)) + s(b') - (l - 2\alpha - 3\beta) = \varphi_0(b'),$$
$$\varphi_1(t_\lambda \otimes b \otimes t_{-\lambda}) = \varphi_1(b) + \langle h_1, \lambda \rangle = \varphi_1(b') + \alpha - \alpha = \varphi_1(b'),$$
$$\varphi_2(t_\lambda \otimes b \otimes t_{-\lambda}) = \varphi_2(b) + \langle h_2, \lambda \rangle = \varphi_2(b') + 3\beta - 3\beta = \varphi_2(b').$$

$\varepsilon_i(t_\lambda \otimes b \otimes t_{-\lambda}) = \varepsilon_i(b')$ ($i = 0, 1, 2$) also follows from the above calculations.

From the fact that $(z_j$ for $b) = (z_j$ for $b')$ it is straightforward to check that if $b, e_i b \in B_l$ (resp. $b, f_i b \in B_l$), then $f_{(l,b_0)}(e_i(t_\lambda \otimes b \otimes t_{-\lambda})) = e_i(f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda})$ (resp. $f_{(l,b_0)}(f_i(t_\lambda \otimes b \otimes t_{-\lambda})) = f_i(f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda})$. Hence $f_{(l,b_0)}$ is a crystal embedding. It is easy to see that $f_{(l,b_0)}(t_\lambda \otimes b_0 \otimes t_{-\lambda}) = b_\infty$. We can also check $B_\infty = \bigcup_{(l,b) \in j \text{ Im } f_{(l,b)}} B_l$. Therefore we have shown that the family of perfect crystals $\{B_l\}_{l \geq 1}$ forms a coherent family.

## 5 Crystal graphs of $B_1$ and $B_2$

In this section we present crystal graphs of the $U'_q(G_2^{(1)})$-crystals $B_1$ and $B_2$ in Figs. 1 and 2.

In the graphs $b \xrightarrow{i} b'$ stands for $b' = f_i b$. Minimal elements are marked as *. Recall that as a $U_q(G_2)$-crystal

$$B_1 \simeq B(0) \oplus B(\Lambda_1), \quad B_2 \simeq B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1).$$

We give the table that relates the numbers in the crystal graphs to our representation of elements according to which $U_q(G_2)$-components they belong to.

$B(0)$: $\phi^0 = (0, 0, 0, 0, 0, 0)$

$B(\Lambda_1)$:

\[
\begin{array}{cccccc}
1 & = & (1, 0, 0, 0, 0, 0) & 2 & = & (0, 1, 0, 0, 0, 0) & 3 & = & (0, \frac{2}{3}, \frac{2}{3}, 0, 0, 0) & 4 & = & (0, \frac{1}{3}, \frac{4}{3}, 0, 0, 0) \\
5 & = & (0, \frac{1}{3}, \frac{1}{3}, 1, 0, 0) & 6^* & = & (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) & 7 & = & (0, 0, 1, 1, 0, 0) & 8 & = & (0, 0, 1, \frac{1}{3}, \frac{1}{3}, 0) \\
9 & = & (0, 0, 0, \frac{4}{3}, \frac{1}{3}, 0) & 10 & = & (0, 0, 0, \frac{2}{3}, \frac{2}{3}, 0) & 11 & = & (0, 0, 0, 0, 1, 0) & 12 & = & (0, 0, 0, 0, 0, 1) \\
13 & = & (0, 0, 2, 0, 0, 0) & 14 & = & (0, 0, 0, 2, 0, 0) & & & & \\
\end{array}
\]
Comparing our crystal graphs with those in [7] we found that some 2-arrows are missing in Fig. 3 of [7].
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Figure 2. Crystal graph of $B_2$. $\vee$ is $f_0$, $\swarrow$ is $f_1$ and others are $f_2$.

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