Combinations of Qualitative Winning for Stochastic Parity Games

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Abstract
We study Markov decision processes and turn-based stochastic games with parity conditions. There are three qualitative winning criteria, namely, sure winning, which requires all paths must satisfy the condition, almost-sure winning, which requires the condition is satisfied with probability 1, and limit-sure winning, which requires the condition is satisfied with probability arbitrarily close to 1. We study the combination of these criteria for parity conditions, e.g., there are two parity conditions one of which must be won surely, and the other almost-surely. The problem has been studied recently by Berthon et. al for MDPs with combination of sure and almost-sure winning, under infinite-memory strategies, and the problem has been established to be in NP \( \cap \) coNP. Even in MDPs there is a difference between finite-memory and infinite-memory strategies. Our main results for combination of sure and almost-sure winning are as follows: (a) we show that for MDPs with infinite-memory strategies the problem lie in NP \( \cap \) coNP; (b) we show that for turn-based stochastic games the problem is coNP-complete, both for finite-memory and infinite-memory strategies; and (c) we present algorithmic results for the finite-memory case, both for MDPs and turn-based stochastic games, by reduction to non-stochastic parity games. In addition we show that all the above results also carry over to combination of sure and limit-sure winning, and results for all other combinations can be derived from existing results in the literature. Thus we present a complete picture for the study of combinations of qualitative winning criteria for parity conditions in MDPs and turn-based stochastic games.

Keywords Two-player games, parity winning conditions, Stochastic games

1 Introduction

Stochastic games and parity conditions. Two-player games on graphs are an important model to reason about reactive systems, such as, reactive synthesis [18, 32], open reactive systems [1]. To reason about probabilistic behaviors of reactive systems, such games are enriched with stochastic transitions, and this gives rise to models such as Markov decision processes (MDPs) [23, 33] and turn-based stochastic games [19]. While these games provide the model for stochastic reactive systems, the specifications for such systems that describe the desired non-terminating behaviors are typically \( \omega \)-regular conditions [35]. The class of parity winning conditions can express all \( \omega \)-regular conditions, and has emerged as a convenient and canonical form of specification for algorithmic studies in the analysis of stochastic reactive systems.

Qualitative winning criteria. In the study of stochastic games with parity conditions, there are three basic qualitative winning criteria, namely, (a) sure winning, which requires all possible paths to satisfy the parity condition; (b) almost-sure winning, which requires the parity condition to be satisfied with probability 1; and (c) limit-sure winning, which requires the parity condition to be satisfied with probability arbitrarily close to 1. For MDPs and turn-based stochastic games with parity conditions, almost-sure winning coincide with limit-sure winning, however, almost-sure winning is different from sure winning [8]. Moreover, for all the winning criteria above, if a player can ensure winning, she can do so with memoryless strategies, that do not require to remember the past history of the game. All the above decision problems belong to NP \( \cap \) coNP, and the existence of polynomial-time algorithm is a major open problem.

Combination of multiple conditions. While traditionally MDPs and stochastic games have been studied with a single condition with respect to different winning criteria, in recent studies combinations of winning criteria has emerged as an interesting problem. An example is the beyond worst-case synthesis problem that combines the worst-case adversarial requirement with probabilistic guarantee [6]. Consider the scenario that there are two desired conditions, one of which is critical and cannot be compromised at any cost, and hence sure winning must be ensured, whereas for the other condition the probabilistic behavior can be considered. Since almost-sure and limit-sure provide the strongest probabilistic guarantee, this gives rise to stochastic games where one condition must be satisfied surely, and the other almost-surely (or limit-surely). This motivates the study of different conditions with different qualitative winning criteria, which has been already studied for MDPs with parity conditions in [4].
Finite-memory Limit-sure Almost-sure Limit-sure Almost-sure coNP-complete coNP-complete Reduction to non-stochastic games

Table 1. Summary of Results for Sure–Almost-sure as well as Sure–Limit-sure Winning for Parity Conditions. New results are boldfaced. The reductions give algorithmic results from algorithms for non-stochastic games.

| Model         | Infinite-memory | Finite-memory |
|---------------|-----------------|---------------|
| MDPs          | NP ∩ coNP       | NP ∩ coNP [4] |
| Turn-based stochastic game | coNP-complete | Reduction to non-stochastic parity games |
|               | Reduction to non-stochastic games with conjunction of parity conditions | coNP-complete |

Table 2. Conjunctions of various qualitative winning criteria.

| Criteria 1 | Criteria 2 | Solution Method |
|------------|------------|-----------------|
| Sure $\psi_1$ | Sure $\psi_2$ | Sure ($\psi_1 \land \psi_2$) |
| Sure $\psi_1$ | Almost-sure $\psi_2$ | This work |
| Sure $\psi_1$ | Limit-sure $\psi_2$ | This work |
| Almost-sure $\psi_1$ | Almost-sure $\psi_2$ | Almost-sure ($\psi_1 \land \psi_2$) |
| Almost-sure $\psi_1$ | Limit-sure $\psi_2$ | Almost-sure ($\psi_1 \land \psi_2$) |
| Limit-sure $\psi_1$ | Limit-sure $\psi_2$ | Almost-sure ($\psi_1 \land \psi_2$) |

Previous results and open questions. While MDPs and turn-based stochastic games with parity conditions have been widely studied in the literature (e.g., [20, 22, 2, 13, 14, 8]), the study of combination of different qualitative winning criteria is recent. The problem has been studied only for MDPs with sure winning criteria for one parity condition, and almost-sure winning criteria (also probabilistic threshold guarantee) for another parity condition, and it has been established that even in MDPs infinite-memory strategies are required, and the decision problem lies in NP ∩ coNP [4]. While the existence of infinite-memory strategies represent the general theoretical problem, an equally important problem is the existence of finite-memory strategies, as the class of finite-memory strategies correspond to realizable finite-state transducers (such as Mealy or Moore machines). The study of combination of qualitative winning criteria for parity conditions is open for MDPs with finite-memory strategies, and turn-based stochastic games with both infinite-memory and finite-memory strategies. In this work we present answers to these open questions.

Our results. In this work our main results are as follows:

1. For MDPs with finite-memory strategies, we show that the combination of sure winning and almost-sure winning for parity conditions also belong to NP ∩ coNP, and we present a linear reduction to parity games. Our reduction implies a quasi-polynomial time algorithm, and also polynomial time algorithm as long as the number of indices for the sure winning parity condition is logarithmic. Note that no such algorithmic result is known for the infinite-memory case for MDPs.

2. For turn-based stochastic games, we show that the combination of sure and almost-sure winning for parity conditions is a coNP-complete problem, both for finite-memory as well as infinite-memory strategies. For the finite-memory strategy case we present a reduction to non-stochastic games with conjunction of parity conditions, which implies a fixed-parameter tractable algorithm, as well as a polynomial-time algorithm as long as the number of indices of the parity conditions are logarithmic.

3. Finally, while for turn-based stochastic parity games almost-sure and limit-sure winning coincide, we show that in contrast, while ensuring one parity condition surely, limit-sure winning does not coincide with almost-sure winning even for MDPs. However, we show that all the above results established for combination of sure and almost-sure winning also carries over to sure and limit-sure winning.

Our main results are summarized in Table 1. In addition to our main results, we also argue that our results complete the picture of all possible conjunctions of qualitative winning criteria as follows: (a) conjunctions of sure (or almost-sure) winning with conditions $\psi_1$ and $\psi_2$ is equivalent to sure (resp., almost-sure) winning with the condition $\psi_1 \land \psi_2$ (the conjunction of the conditions); (b) by determinacy and since almost-sure and limit-sure winning coincide for $\omega$-regular conditions, if the conjunction of $\psi_1 \land \psi_2$ cannot be ensured almost-surely, then the opponent can ensure that at least one of them is falsified with probability bounded away from zero; and thus conjunction of almost-sure winning with limit-sure winning, or conjunctions of limit-sure winning coincide with conjunction of almost-sure winning. This is illustrated in Table 2 and shows that we present a complete picture of conjunctions of qualitative winning criteria in MDPs and turn-based stochastic games.

Related works. We have already mentioned the most important related works above. We discuss other related works here. MDPs with multiple Boolean as well as quantitative objectives have been widely studied in the literature [16, 22, 24, 5, 15]. For non-stochastic games combination of various Boolean objectives is conjunction of the objectives, and such games with multiple quantitative objectives have been studied in the literature [37, 10]. For turn-based stochastic games, the general analysis of multiple quantitative objectives is intricate, and they have been only studied for special cases, such as, reachability objectives [17] and almost-sure...
We extend the literature of the study of beyond worst-case synthesis problem for parity objectives by considering turn-based stochastic games and the distinction between finite-memory and infinite-memory strategies.

2 Background

For a countable set $S$ let $\mathcal{D}(S) = \{d : S \to [0, 1] \mid \exists T \subseteq S$ such that $|T| \in \mathbb{N}, \forall s \notin T \cdot d(s) = 0 \text{ and } \Sigma_{s \in T} d(s) = 1\}$ be the set of discrete probability distributions with finite support over $S$. A distribution $d$ is pure if there is some $s \in S$ such that $d(s) = 1$.

Stochastic Turn Based Games We give a short introduction to stochastic turn-based games.

A stochastic turn-based game is $G = (V, (V_0, V_1, V_p), E, \kappa)$, where $V$ is a finite set of configurations, $V_0$, $V_1$, and $V_p$ form a partition of $V$ to Player 0, Player 1, and stochastic configurations, respectively, $E \subseteq V \times V$ is the set of edges, and $\kappa : V_p \to \mathcal{D}(V)$ is a probabilistic transition for configurations in $V_p$ such that $\kappa(v, v') > 0$ implies $(v, v') \in E$. If either $V_0 = \emptyset$ or $V_1 = \emptyset$ then $G$ is a Markov Decision Process (MDP). If both $V_0 = \emptyset$ and $V_1 = \emptyset$ then $G$ is a Markov Chain (MC). If $V_p = \emptyset$ then $G$ is a turn-based game (non-stochastic). Given an MC $M$, an initial configuration $v$, and a measurable set $W \subseteq V^\omega$, we denote by $\Prob_{M, v}(W)$ the measure of the set of paths $W$.

A set of plays $W \subseteq V^\omega$ is a parity condition if there is a parity priority function $\alpha : V \to \{0, \ldots, d\}$, with $d$ as its index, such that a play $\pi = v_0, \ldots$ is in $W$ iff $\min \{c \in \{0, \ldots, d\} \mid \exists \omega \in V^\omega \cdot \alpha(v_i) = c\}$ is even. A parity condition $W$ where $d = 1$ is a Büchi condition. In such a case, we may identify the condition with the set $B = \alpha^{-1}(0)$. A parity condition $W$ where $d = 2$ and $\alpha^{-1}(0) = 0$ is a co-Büchi condition. In such a case, we may identify the condition with the set $C = \alpha^{-1}(1)$.

A strategy $\sigma$ for Player 0 is $\sigma : V^* \cdot V_0 \to \mathcal{D}(V)$, such that $\sigma(v \cdot v') > 0$ implies $(v, v') \in E$. A strategy $\sigma$ for Player 1 is defined similarly. A strategy is pure if it uses only pure distributions. A strategy uses memory if there is a domain $M$ of size $m$ and two functions $\sigma_0 : M \times V_0 \to \mathcal{D}(V)$ and $\sigma_1 : M \times V_0 \to \mathcal{D}(V)$ such that for every $v \cdot v'$ we have $\sigma(w \cdot v') = \sigma_0(\sigma_1(w, v'), v')$, where $\sigma_0(\epsilon) = \sigma_1(\epsilon, v)$, and $\sigma_1(v \cdot v') = \sigma_0(\sigma_0(v', v), v)$. A strategy is memoryless if it uses memory of size 1. Two strategies $\sigma$ and $\pi$ for both players and an initial configuration $v \in V$ induce a Markov chain $\nu(\sigma, \pi) = (S(v), P, L, v)$, where $S(v) = \{i \cdot v' \mid v' \in V\}$ and if $v \in V_0$ we have $P(wu) = \pi(wu)$, if $v \in V_1$ we have $P(wu) = \pi(wu)$ and if $v \in V_p$ then for every $w \in V^*$ and $v' \in V$ we have $P(wu, vu') = \kappa(v, v')$.

For a game $G$, an $\omega$-regular set of plays $W$, and a configuration $v$, the value $W$ from $v$ for Player 0, denoted $\text{val}_0(W, v)$, and for Player 1, denoted $\text{val}_1(W, v)$, are

$$\text{val}_0(W, v) = \sup_{\pi \in \mathcal{P}} \inf_{r \in \mathbb{R}} \Prob_{\nu(\sigma, \pi)}(W)$$

and

$$\text{val}_1(W, v) = \sup_{\pi \in \mathcal{P}} \inf_{r \in \mathbb{R}} (1 - \Prob_{\nu(\sigma, \pi)}(W)) .$$

We say that Player 0 wins $W$ surely from $v$ if $\exists \sigma \in \Sigma . \forall \pi \in \Pi . \nu(\sigma, \pi) \subseteq W$, where by $\nu(\sigma, \pi) \subseteq W$ we mean that all paths in $\nu(\sigma, \pi)$ are in $W$. We say that Player 0 wins $W$ almost surely from $v$ if $\exists \sigma \in \Sigma . \forall \pi \in \Pi . \Prob_{\nu(\sigma, \pi)}(W) = 1$. We say that Player 0 wins $W$ limit surely from $v$ if $\forall r < 1 . \exists \sigma \in \Sigma . \forall \pi \in \Pi . \Prob_{\nu(\sigma, \pi)}(W) \geq r$. A strategy $\sigma$ for Player 0 is optimal if $\text{val}_0(W, v) = \inf_{\pi \in \mathcal{P}} \Prob_{\nu(\sigma, \pi)}(W)$.

A game with condition $W$ is determined if for every configuration $v$ we have $\text{val}_0(W, v) + \text{val}_1(W, v) = 1$.

Theorem 2.1. Consider a stochastic two-player game $G$, an $\omega$-regular set of plays $W$, and a rational $r$.

- For every configuration $v \in V$ we have $\text{val}_0(W, v) + \text{val}_1(W, v) = 1$.
- If $G$ is finite and $W$ a parity condition, one can decide whether $\text{val}_1(W, v) \geq r$ in $\text{NP} \cap \text{co-NP}$ and $\text{val}_0(W, v)$ can be computed in exponential time [11].
- If $W$ is a parity condition, there is an optimal pure memoryless strategy $\sigma$ such that $\inf_{\pi \in \mathcal{P}} \Prob_{\nu(\sigma, \pi)}(W) = \text{val}_0(W, v)$ [11].
- If $G$ is a non-stochastic game then $\text{val}_1(W, v) \in \{0, 1\}$ for every $v \in V$ and there are optimal pure strategies for both players [35].
- If $G$ is not a non-stochastic game and $W$ is a parity condition, then whether $\text{val}_0(W, v) = 1$ can be decided in $\text{UP} \cap \text{co-UP}$ [27]. Furthermore, the optimal strategies are memoryless [36].
- If $G$ is a non-stochastic game with $n$ configurations and $m$ transitions and $W$ is a parity condition of index $d$ then whether $\text{val}_0(W, v) = 1$ can be decided in time $O(n^{\log d+5} + \log n)$ and if $d \leq \log n$ then in time $O(m^{d} 2^{38})$ [7, 29].
- If $G$ is a non-stochastic game and $W$ is the intersection of two parity conditions, then whether $\text{val}_1(W, v) = 1$ can be decided in co-$\text{NP}$, whether $\text{val}_1(W, v) = 0$ can be decided in $\text{NP}$, and there is a finite-state optimal strategy for Player 0 and a memoryless optimal strategy for Player 1 [12].

3 Sure-Almost-Sure MDPs

Berthom et al. considered the case of MDPs with two parity conditions and finding a strategy that has to satisfy one of the conditions surely and satisfy a given probability threshold with respect to the other [4]. Here we consider the case that the second condition has to hold with probability 1.
Theorem 3.2. In order to decide whether it is possible to win an SAS MDP with \( n \) locations and indices \( d_s \) and \( d_{as} \), with finite memory it is sufficient to solve a parity game with \( O(n \cdot d_s \cdot d_{as}) \) configurations and index \( d_s \). Furthermore, \( d_s \) is a bound on the size of the required memory in case of a win.

The theorem follows from the following Lemmas.

Lemma 3.4. Given an SAS MDP \( M \) with \( n \) configurations and almost-sure index of \( d_{as} \), checking whether Player 0 has a finite-memory winning strategy can be reduced to checking whether Player 0 has a finite-memory winning strategy in an SAS MDP \( M' \) with \( O(n \cdot d_{as}) \) configurations, where the almost-sure winning condition is a Büchi condition and the sure winning condition has the same index.

In Appendix we show that we can think about bottom SCCs where the almost-sure parity condition holds as a Büchi condition involving a commitment not to visit lower parities and to visit a minimal even parity infinitely often.

Lemma 3.5. The finite-memory winning in an SAS MDP where the almost sure winning condition is a Büchi condition can be checked by a reduction to a (non-stochastic) game where the winning condition is the intersection of parity and Büchi conditions.

This Lemma is an instantiation of the result in Theorem 4.4 for the case of almost-sure Büchi. In both cases, the proof is a version of the classical reduction from stochastic parity games to parity games [14]. We include the proof of this lemma in Appendix for completeness of presentation.

The following Lemma is probably folklore. However, we could not find a source for it and include it here with a proof.

Lemma 3.6. A game with a winning condition that is the conjunction of Büchi and parity can be converted to an “equivalent” parity game.

In Appendix we show that we can add to the game a monitor that allows to refresh the best parity priority from one condition that is seen every time that the Büchi set from the other condition is visited.

Based on Lemmas 3.4, 3.5, and 3.6, we can prove Theorem 3.3.

Proof. Starting from an SAS MDP with \( n \) configurations, \( d_s \) priorities in the sure winning condition and \( d_{as} \) priorities in the almost-sure winning condition the reductions produce the following games.

The reduction in Lemma 3.4 produces an MDP with \( O(n \cdot d_{as}) \) configurations with a parity condition with \( d_s \) priorities and a Büchi condition.

The reduction in Lemma 3.5 multiplies the number of configurations by at most four. The resulting game has a winning condition that is the intersection of a parity condition of index \( d_s \) and a Büchi condition. The number of configurations remains \( O(n \cdot d_{as}) \).
Finally, the reduction in Lemma 3.6 produces a parity game with \(O(n \cdot d_{as} \cdot d_s)\) configurations and \(d_s\) priorities.

A winning strategy in the parity game is memoryless. This translated to a strategy with memory \(d_s\) in the SAS MDP where the almost-sure condition is a Büchi condition. In order to use this strategy in the original SAS MDP no additional memory is required.

It follows that in order to decide whether it is possible to win an SAS MDP with indices \(d_s\) and \(d_{as}\) with finite memory we can solve a parity game with \(O(n \cdot d_{as} \cdot d_s)\) configurations and a memory of size \(d_s\) is sufficient. \(\Box\)

Corollary 3.7. Consider an SAS MDP with \(n\) configuration, sure winning condition of index \(d_s\), and almost-sure winning condition of index \(d_{as}\). Checking whether Player 0 can win with finite-memory can be computed in quasi-polynomial time. In case that \(d_s \leq \log n\) it can be decided in polynomial time.

Proof. This is a direct result of Theorem 3.3 and the quasi-polynomial algorithm for solving parity games in \([7, 29]\). \(\Box\)

4 Sure-Almost-Sure Parity Games

We now turn our attention to sure-almost-sure parity games, which have not been considered in \([4]\). We have already established the difference between finite and infinite memory for Player 0 even for MDPs. We first show that sure-almost-sure games are determined. Then, considering the decision of winning, we show for Player 1 memoryless strategies are sufficient. It follows that her winning is NP-complete. We show that deciding whether Player 0 has a (general) winning strategy is co-NP-complete. Finally, we show that the case of finite memory (for Player 0) can be solved by a reduction to (non-stochastic) two-player games.

A sure-almost-sure (SAS) parity game is \(G = (V, (V_0, V_1, V_p), E, \kappa, \{W\})\), where all components are as before and \(W\) consists of two parity conditions \(W_s \subseteq V^o\) and \(W_{as} \subseteq V^a\). Strategies and the resulting Markov chains are as before. We say that Player 0 wins \(G\) from configuration \(v\) if she has a strategy \(\sigma\) such that for every strategy \(\pi\) of Player 1 we have \(v(\sigma) \subseteq W_s\) and \(\text{Prob}_{v(\sigma), \pi}(W_{as}) = 1\). That is, Player 0 has to win for sure (on all paths) with respect to \(W_s\) and with probability 1 with respect to \(W_{as}\).

4.1 Determinacy

We start by showing that SAS parity games are determined.

\textbf{Theorem 4.1.} SAS parity games are determined.

In Appendix we prove that a reduction similar to Martin’s proof that Blackwell games are determined by reducing them to turn-based two-player [31] games shows determinacy also for SAS games.

4.2 General Winning

We show that determining whether Player 0 has a (general) winning strategy in an SAS parity game is co-NP-complete and that for Player 1 finite-memory strategies are sufficient and that deciding her winning is NP-complete.

\textbf{Theorem 4.2.} In an SAS parity game memoryless strategies are sufficient for Player 1.

In Appendix we show by an inductive argument over the number of configurations of Player 1 (similar to that done in \([26, 25, 9]\)) that Player 1 has a memoryless optimal strategy.

\textbf{Corollary 4.3.} Consider an SAS parity game. Deciding whether Player 1 wins is \(\text{NP}\)-complete and whether Player 0 wins is co-\(\text{NP}\)-complete.

\textbf{Proof.} Consider the case of Player 1. Membership in \(\text{NP}\) follows from guessing the (memoryless) winning strategy of Player 1 in \(G\). The result is an MDP whose winning can be reduced to a parity game of polynomial size as we have shown in Section 3. It follows that winning for Player 1 in such an MDP can be determined in \(\text{NP}\). An algorithm would guess the two strategies simultaneously and check both in polynomial time over the resulting Markov chain.

Hardness follows from considering SAS games with no stochastic configurations \([12]\).

Consider the case of Player 0. Membership in co-\(\text{NP}\) follows from dualizing the previous argument about membership in \(\text{NP}\) and determinacy.

Hardness follows from considering SAS games with no stochastic configurations \([12]\). \(\Box\)

4.3 Winning with Finite Memory

We show that in order to check whether Player 0 can win with finite memory it is enough to use the standard reduction from almost-sure winning in two-player stochastic parity games to sure winning in two-player parity games \([14]\).

\textbf{Theorem 4.4.} In a finite SAS parity game with \(n\) locations and \(d_{as}\) almost-sure index deciding whether a node \(v\) is winning for Player 0 with finite memory can be decided by a reduction to a two-player game with \(O(n \cdot d_{as})\) locations, where the winning condition is an intersection of two parity conditions of the same indices.

\textbf{Proof.} Let \(G = (V, (V_0, V_1, V_p), E, \kappa, \{W\})\). Let \(p_{as} : V \rightarrow [0..d_{as}]\) be the parity ranking function that induces \(W_{as}\) and \(p_s : V \rightarrow [0..d_s]\) be the parity ranking function that induces \(W_s\). Without loss of generality assume that both \(d_s\) and \(d_{as}\) are even.

Given \(G\) we construct the game \(G'\) where every configuration \(v \in V_p\) is replaced by the gadget in Figure 2. That is, \(G' = (V', (V'_0, V'_1), E', \kappa', \{W'\})\), where the components of \(G'\) are as follows:
The game $G'$ is a linear game whose winning condition (for Player 0) is an intersection of two parity conditions. It is known that such games are determined and that the winning sets can be computed in NP$\cap$co-NP [12]. Indeed, the winning condition for Player 0 can be expressed as a Streett condition, and hence her winning can be decided in co-NP. The winning condition for Player 1 can be expressed as a Rabin condition, and hence her winning can be decided in NP. It follows that $V'$ can be partitioned into $W'_0$ and $W'_1$, the winning regions of Player 0 and Player 1, respectively. Furthermore, Player 0 has a finite-memory winning strategy for her from every configuration in $W'_0$ and Player 1 has a memoryless winning strategy for her from every configuration in $W'_1$. Let $\sigma_0'$ denote the winning strategy for Player 0 on $W'_0$ and $\pi_1'$ denote the winning strategy for Player 1 on $W'_1$. Let $M$ be the memory domain used by $\sigma_0'$. For simplicity, we use the notation $\sigma_0' \subseteq V' \times M \rightarrow V' \times M$, where for every $m \in M$ and $v \in V'_0$ there is a unique $w \in V$ and $m' \in M$ such that $((v, m), (w, m')) \in \sigma'$ and for every $m \in M$ and $v \in V'_1$ and $w$ such that $(v, w) \in E'$ there is a unique $m'$ such that $((v, m), (w, m')) \in \sigma_0'$. By abuse of notation we refer to the successor of a configuration $v$ in $G'$ and mean either $w$ or $\overline{w}$ according to the context.

\[ \equiv \]

We show that every configuration $v \in W'_0$ that is winning for Player 0 in $G'$ is in the winning region $W_0$ of Player 0 in $G$. Let $G' \times M$ denote the product of $G'$ with the memory domain $M$, where moves from configurations in $V'_0 \times M$ are restricted according to $\sigma'_0$. We construct a matching game $G \times M$ restricted to the strategy $\sigma_0$ that is induced by $\sigma'_0$ as follows:

- From a configuration $(v, m) \in V_0 \times M$ the strategy $\sigma'_0$ chooses a successor $(w, m')$.
- From a configuration $(v, m) \in V_1 \times M$ and for every successor $w$ of $v$ there is a memory value $m'$ such that $(w, m')$ is a reachable pair of configuration and memory value in $G' \times M$.

Consider a configuration $(v, m) \in V_0 \times M$. As $\overline{\mathcal{T}}$ is a Player 1 configuration in $G'$, every configuration $(\hat{v}, 2i)$ is a possible successor of $\overline{\mathcal{T}}$ according to $\sigma'_0$.

* If for some $i$ we have that the choice from $(\hat{v}, 2i)$ according to $\sigma'_0$ is $(\hat{v}, 2i-1)$. Then, let $i_0$ be the minimal such $i$ and let $w_0$ be the successor of $v$ such that the choice of $\sigma'_0$ from $(\hat{v}, 2i_0 - 1)$ is $w_0$. We update in $G \times M$ the edge from $(v, m)$ to $(w_0, m')$, where $m'$ is the memory resulting from taking the path $\overline{\mathcal{T}}$, $(\hat{v}, 2i_0)$, $(\hat{v}, 2i_0 - 1)$, $w_0$ in $G' \times M$. We update in $G \times M$ the edge from $(v, m)$ to $(w', m_{w'})$ for $w' \neq w_0$, where $m_{w'}$ is the memory resulting from taking the path $\overline{\mathcal{T}}$, $(\hat{v}, 2i_0 - 2)$, $(\hat{v}, 2i_0 - 2)$, $w'$. Notice that as $i_0$ is chosen to be the minimal the choice from $(\hat{v}, 2i_0 - 2)$ to $(\hat{v}, 2i_0 - 2)$ is compatible with $\sigma'_0$, where $2i_0 - 2$ could be 0.

* If for all $i$ we have that the choice from $(\hat{v}, 2i)$ according to $\sigma'_0$ is $(\hat{v}, 2i)$. Then, for every $w$ successor of $v$ we update in $G \times M$ the edge from $(v, m)$
to \((w, m')\), where \(m'\) is the memory resulting from taking the path \(\mathcal{T}, (\tilde{v}, p_{as}(v)), (\tilde{v}, p_{as}), w\).

Notice that if \(p_{as}(v)\) is odd then the first case always holds as the only successor of \((\tilde{v}, p_{as}(v) + 1)\)

\[(\tilde{v}, p_{as}(v)).\]

The resulting game \(G \times M\) (enforcing the strategy \(\sigma_0\) described above) includes no decisions for Player 0. Consider the winning condition \(W_s\). Every path in \(G \times M\) that is consistent with \(\sigma_0\) corresponds to a path in \(G' \times M\) that is consistent with \(\sigma'_0\) and agrees on the parities of all configurations according to \(p_s\). Indeed, every configuration of the form \((\tilde{v}, 2i)\) or \((\tilde{v}, j)\) in \(G'\) has the same priority according to \(p_s\) as \(T(v)\) (and \(v \in G\)).

As every path in \(G' \times M\) is winning according to \(W_s\), then every path in \(G \times M\) is winning according to \(W_s\).

We turn our attention to consider only the parity condition \(p_{as}\) in both \(G' \times M\) and \(G \times M\). We think about \(G' \times M\) as a parity game with the winning condition \(W'_{s}\) and about \(G \times M\) as a stochastic parity game with the winning condition \(W_{s}\). As \(\sigma'_0\) is winning, all paths in \(G' \times M\) are winning for Player 0 according to \(W'_{s}\).

We recall some definitions and results from [14]. For \(k \leq d_{as}\), let \(\mathcal{k}\) denote \(k\) if \(k\) is odd and \(k - 1\) if \(k\) is even. A parity ranking for Player 0 is \(\mathcal{r} : V' \times M \rightarrow [n]^{d_{as}/2} \cup \{\infty\}\) for some \(n \in \mathbb{N}\). For a configuration \(v\), we denote by \(\mathcal{r}^k(v)\) the prefix \((r_1, r_3, \ldots, r_{k})\) of \(r(v)\).

Let \(\mathcal{r}(v) = (r_1, \ldots, r_d)\) and \(\mathcal{r}(v') = (r'_1, \ldots, r'_d)\). We write \(\mathcal{r}(v) \preceq_k \mathcal{r}(v')\) if the prefix \((r_1, \ldots, r_{k})\) is at most \((r'_1, \ldots, r'_k)\) according to the lexicographic ordering.

Similarly, we write \(\mathcal{r}(v) <_k \mathcal{r}(v')\) if \((r_1, \ldots, r_{k})\) is less than \((r'_1, \ldots, r'_k)\) according to the lexicographic ordering.

It is well known that in a parity game (here \(G' \times M\)) there is a good parity ranking such that for every \(v \in W'_s\) we have \(\mathcal{r}(v) \neq \infty\) [28]. A parity ranking is good if (i) for every vertex \(v \in W'_0\) there is a vertex \(w \in \text{succ}(v)\) such that \(\mathcal{r}(w) \preceq \mathcal{r}(v)\) and if \(p(v)\) is odd then \(\mathcal{r}(w) \prec \mathcal{r}(v)\) and (ii) for every vertex \(v \in W'_1\) and every vertex \(w \in \text{succ}(v)\) it holds that \(\mathcal{r}(w) \preceq \mathcal{r}(v)\) and if \(p(v)\) is odd then \(\mathcal{r}(w) \prec \mathcal{r}(v)\). Let \(\mathcal{T}\) be the good parity ranking for \(G' \times M\). Consider the same ranking for \(G \times M\). For a configuration \(v \in V'_p \times M\), we write \(\text{Prob}_s(\mathcal{T}_{\geq k}, v)\) for the probability of successors \(w\) of \(v\) such that \(\mathcal{r}(w) \geq_k \mathcal{r}(v)\) and \(\text{Prob}_s(\mathcal{T}_{< k}, v)\) for the probability of successors \(w\) of \(v\) such that \(\mathcal{r}(w) <_k \mathcal{r}(v)\).

**Definition 4.5** (Almost-sure ranking [14]). A ranking function \(r : V \rightarrow [n]^{d_{as}/2} \cup \{\infty\}\) for Player 0 is an almost-sure ranking if there is an \(\epsilon \geq 0\) such that for every vertex \(v\) with \(r(v) \neq \infty\), the following conditions hold:

- If \(v \in V'_0\) there exists a successor \(w\) such that \(\mathcal{r}(w) \leq_{p(v)} \mathcal{r}(v)\) and if \(p(v)\) is odd then \(\mathcal{r}(w) <_{p(v)} \mathcal{r}(v)\).

- If \(v \in V'_1\) then for every successor \(w\) of \(v\) we have \(\mathcal{r}(w) \leq_{p(v)} \mathcal{r}(v)\) and if \(p(v)\) is odd then \(\mathcal{r}(w) <_{p(v)} \mathcal{r}(v)\).

- If \(v \in V'_p\) and \(p(v)\) is even then either \(\text{Prob}_s(\mathcal{T}_{\geq k}, v) = 1\) or

\[\bigvee_{j=2i+1} \text{Prob}_s(\mathcal{T}_{\leq k}, v) = 1 \land \text{Prob}_s(\mathcal{T}_{< k}, v) \geq \epsilon\]

- If \(v \in V'_p\) and \(p(v)\) is odd then

\[\bigvee_{j=2i+1} \text{Prob}_s(\mathcal{T}_{< k}, v) = 1 \land \text{Prob}_s(\mathcal{T}_{\geq k}, v) \geq \epsilon\]

**Lemma 4.6.** [14] A stochastic parity game has an almost-sure ranking iff Player 0 can win for the parity objective with probability 1 from every configuration \(v\) such that \(\mathcal{r}(v) \neq \infty\).

The following lemma is a specialization of a similar lemma in [14] for our needs.

**Lemma 4.7.** The good ranking of \(G' \times M\) induces an almost-sure ranking of \(G \times M\).

We include the proof of this Lemma in Appendix. As Player 0 has no choices in \(G \times M\), it follows that the strategy \(\sigma_0\) defined above is winning in \(G\). That is, once the play reaches \(W'_s\), Player 0 can always win for the parity objective using the strategy \(\sigma_0\) defined above in \(G' \times M\).

Consider now the imposition of the strategy \(\sigma_0\) to \(G'\) using the same memory. We have to show how to update the memory from configurations that result from inclusions of the gadgets that replace probabilistic configurations in \(G'\). Consider a configuration \(\mathcal{T}\) reached with some memory value \(m \in M\). In \(G\) the configuration \((v, m)\) is a probabilistic configuration. It follows that for every successor \(w\) of \(v\) there is a memory value \(m_w\) such that on transition to \(w\) the memory is updated to \(m_w\). We do the same in \(G'\). That is, the memory stays \(m\) while exploring the gadget with configurations \(\mathcal{T}, (\tilde{v}, 2i)\), and \((\tilde{v}, j)\). Once the play reaches a configuration \(w\) the memory is updated to \(m_w\) as in \(G\).
The result is a parity game, which we denote by $G' \times M$. As all paths in $G \times M$ are winning for Player 0 according to the winning condition $W_1$, and the gadgets do not affect the winning condition $W'_1$, it follows that all plays in $G' \times M$ are winning according to $W'_1$ regardless of the choices of Player 0 and Player 1 in the gadgets. We have to show that we can augment the strategy within the gadgets so that all plays are winning according to $W'_{as}$.

According to Lemma 4.6 there exists an almost-sure ranking $r : G \times M$. We show that $r$ induces a good parity ranking over $G' \times M$.

**Lemma 4.8.** The almost-sure ranking of $G \times M$ induces a good ranking of $G' \times M$.

The proof is included in Appendix.

**Corollary 4.9.** Consider an SAS game with $n$ configurations, sure winning condition of index $d_{as}$, and almost-sure condition of index $d_{as}$. Deciding whether Player 0 can win with finite-memory is co-NP-complete. Deciding whether Player 1 can win against finite-memory is NP-complete.

**Proof:** Upper bounds follow from the reductions to Streett and Rabin winning conditions. Completeness follows from the case where the game has no stochastic configurations. □

**Remark 4.10.** The complexity established above in the case of finite-memory is the same as that established for the general case in Corollary 4.3. However, this reduction gives us a clear algorithmic approach to solve the case of finite-memory strategies. Indeed, in the general case, the proof of NP upper bound requires enumeration of all memoryless strategies, and does not present an algorithmic approach, regardless of the indices of the different winning conditions. In contrast our reduction for the finite-memory case to non-stochastic games with conjunction of parity conditions and recent algorithmic results on non-stochastic games with $\omega$-regular conditions of [7] imply the following:

- For the finite-memory case, we have a fixed parameter tractable algorithm that is polynomial in the number of the game configurations and exponential only in the indices to compute the SAS winning region.
- For the finite-memory case, if both indices are constant or logarithmic in the number of configurations, we have a polynomial time algorithm to compute the SAS winning region.

5 **Sure-Limit-Sure Parity Games**

In this section we extend our results to the case where the unsure goal is required to be met with limit-sure certainty, rather than almost-sure certainty.

Sure-limit-sure parity games. A sure-limit-sure (SLS) parity game is, as before, $G = (V, (V_0, V_1, V_p), E, \kappa, W)$. We denote the second winning condition with the subscript $ls$, i.e., $W_{ls}$. We say that Player 0 wins $G$ from configuration $v$ if she has a sequence of strategies $\sigma_i \in \Sigma$ such that for every $i$ for every strategy $\pi$ of Player 1 we have $v(\sigma_i) \subseteq W_1$ and $\Pr_{v(\sigma_i), \pi}(W_{ls}) \geq 1 - \frac{1}{2^i}$. That is, Player 0 has a sequence of strategies that are sure winning (on all paths) with respect to $W_1$ and ensure satisfaction probabilities approaching 1 with respect to $W_{ls}$.

5.1 **Limit-Sure vs Almost-Sure**

In MDPs and stochastic turn-based games with parity conditions almost-sure and limit-sure winning coincide [8]. In contrast to the above result we present an example MDP where in addition to surely satisfying one parity condition limit-sure winning with another parity condition can be ensured, but almost-sure winning cannot be ensured. In other words, in conjunction with sure winning, limit-sure winning does not coincide with almost-sure winning even for MDPs.

**Theorem 5.1.** While satisfying one parity condition surely, winning limit-surely with respect to another parity condition is strictly stronger than winning almost-surely. This holds true already in the context of MDPs.

**Proof:** Consider the MDP in Figure 3. Clearly, Player 0 wins surely with respect to both parity conditions in configuration $r$. In order to win $W_1$ the cycle between $p$ and $c$ has to be taken finitely often. Then, the edge from $c$ to $l$ must be taken eventually. However, $l$ is a sink that is losing with respect to $W_{ls}$. It follows, that Player 0 cannot win almost-surely with respect to $W_{ls}$ while winning surely with respect to $W_1$.

On the other hand, for every $\epsilon > 0$ there is a finite-memory strategy that is sure winning with respect to $W_1$ and wins with probability at least $1 - \epsilon$ with respect to $W_{ls}$. Indeed, Player 0 has to choose the edge from $c$ to $p$ at least $N$ times, where $N$ is large enough such that $\frac{1}{N} < \epsilon$, and then choose the edge from $c$ to $l$. Then, Player 0 wins surely with respect to $W_1$ (every play eventually reaches either $l$ or $r$) and wins with probability more than $1 - \epsilon$ with respect to $W_{ls}$. □

5.2 **Solving SLS Games**

We now present the solution to winning in SLS games, which is achieved in two parts: (a) first, solve the SAS component, and (b) then compute limit-sure reachability to the SAS-winning region, while respecting the sure parity condition. The memory used in the SLS part has to match the memory used for winning in the SAS part. That is, if Player 0 is restricted to finite-memory in the SAS part of the game she has to consider finite-memory strategies in the SLS part.
SAS winning region A. Consider a game \( G = (V, (V_0, V_1, V_p), E, \xi, W) \), where \( W = (W_s, W_z) \). Consider \( G \) as an SAS game and compute the set of configurations from which Player 0 can win \( G \). Let \( A \subseteq V \) denote this winning region and \( B = V \setminus A \) be the complement region. Clearly, \( A \) is closed under Player 1 moves and under probabilistic moves. That is, if \( v \in (V_1 \cup V_p) \cap A \) then for every \( v' \) such that \( (v, v') \in E \) we have \( v' \in A \). Furthermore, under Player 0’s winning strategy, Player 0 does not use edges going back from \( A \) to \( B \). It follows that we can consider \( A \) as a sink in the game \( G \).

Reduction to limit-sure reachability. Consider a memoryless optimal strategy \( \pi \in \Pi \) (we have already established that Player 1 has an optimal memoryless strategy on all her winning region) for the region \( B \). We present the argument for finite-memory strategies for Player 0, and the argument for infinite-memory strategies is similar. Consider an arbitrary finite-memory strategy \( \sigma \in \Sigma \), and consider the Markov chain that is the result of restricting Player 0 and Player 1 moves according to \( \sigma \) and \( \pi \), respectively.

- **Bottom SCC property.** Let \( S \) be a bottom SCC that intersects with \( B \) in the Markov chain. As explained above, it cannot be the case that this SCC intersects \( A \) (since we consider \( A \) as sink due to the closed property). Thus the SCC \( S \) must be contained in \( B \). Thus, either \( S \) must be losing according to \( W_s \) or the minimal parity in \( S \) according to \( W_{is} \) is odd, as otherwise in the region \( S \) Player 0 ensures sure winning wrt \( W_s \) and almost-sure winning wrt \( W_{is} \) against an optimal strategy \( \pi \), which means that \( S \) belongs to the SAS winning region \( A \). This contradicts that \( S \) is contained in \( B \).

- **Reachability to A.** In a Markov chain bottom SCCs are reached with probability 1, and from above item it follows that the probability to satisfy the \( W_{is} \) goal along with ensuring \( W_s \) while reaching bottom SCCs in \( B \) is zero. Hence, the probability to satisfy \( W_{is} \) along with ensuring \( W_s \) is at most the probability to reach \( A \). On the other hand, after reaching \( A \), the SAS goal can be ensured by switching to an appropriate SAS strategy in the winning region \( A \), which implies that the SLS goal is ensured. Hence it follows that the SLS problem reduces to limit-sure reachability to \( A \), while ensuring the sure parity condition \( W_s \).

**Remark 5.2.** Note that for finite-memory strategies the argument above is based on bottom SCCs. Given the optimal strategy \( \pi \in \Pi \), we have an MDP, and the SAS region for MDPs wrt to infinite-memory strategies is achieved by characterizing certain strongly connected components (called Ultradend-components [4, Definition 5]), and hence a similar argument as above also works for infinite-memory strategies to show that SLS for infinite-memory strategies for two parity conditions reduces to limit-sure reachability to the SAS region while ensuring the sure parity condition (however, in this case the SAS region has to be computed for infinite-memory strategies).

Limit-sure reachability and sure parity. Consider an SLS game, with a set \( T \) of sink target states, such that the limit-sure goal is to reach \( T \), and every state in \( T \) has even priority with respect to \( W_s \). As argued above, the general SLS problem can be reduced to the limit-sure reachability and sure parity problem after computing the SAS region \( A \) (which can be treated as the absorbing sink \( T \)). We now present solution to this limit-sure reachability with sure parity problem. The computational steps are as follows:

- First, compute the sure winning region \( X \) in the game. Note that \( T \subseteq X \) as \( T \) represent sink states with even priority for \( W_s \).
- Second, restrict the game to \( X \) and compute limit-sure reachability region to \( T \), and let the region be \( Y \). Note that the game restricted to \( X \) is a turn-based stochastic game where almost-sure and limit-sure reachability coincide.

Let us denote by \( Z \) the desired winning region (i.e., from where sure parity can be ensured along with limit-sure reachability to \( T \)). We argue the correctness proof that \( Y \) computes the desired winning region \( Z \) as follows:

- First, note that since the sure parity condition \( W_s \) must be ensured, the sure winning region \( X \) must never be left. Thus without loss of generality, we can restrict the game to \( X \). Since \( Y \) is the region in \( X \) to ensure limit-sure reachability to \( T \), clearly \( Z \) that must ensure both limit-sure reachability as well as sure parity is a subset of \( Y \). Hence \( Z \subseteq Y \).
- Second, for any \( \varepsilon > 0 \), there is a strategy in \( Y \) to ensure that \( T \) is reached with probability at least \( 1 - \varepsilon \) within \( N_{s} \) steps staying in \( X \) (since in the subgame restricted to \( X \), almost-sure reachability to \( T \) can be ensured). Consider a strategy that plays the above strategy for \( N_{s} \) steps, and if \( T \) is not reached, then switches to a sure winning strategy for \( W_s \) (such a strategy exist since \( X \) is never left, and parity conditions are independent of finite prefixes). It follows that from \( Y \) both
limit-sure reachability to $T$ as well as sure parity condition $W_s$ can be ensured. Hence $Y \subseteq Z$.

It follows from above that to solve SLS games, the following computation steps are sufficient: (i) solve SAS game, (ii) compute sure winning region for parity conditions, and (iii) compute almost-sure (limit-sure) reachability in turn-based stochastic games. The second step is a special case of the first step, and the third step can be achieved in polynomial time [21]. Hence it follows that all the complexity and algorithmic upper bounds we established for the SAS games carry over to SLS games. Moreover, as the complexity lower bound results for SAS parity games follow from games with no stochastic transitions, they apply to SLS parity games as well. Hence we have the following corollary.

**Corollary 5.3.** All complexity and algorithmic results established in Section 4 (resp., Section 3) for SAS turn-based stochastic parity games (resp., MDPs) also hold for SLS turn-based stochastic parity games (MDPs).

### 6 Conclusions and Future Work

In this work we considered MDPs and turn-based stochastic games with two parity winning conditions, with combinations of qualitative winning criteria. In particular, we study the case where one winning condition must be satisfied surely, and the other almost-surely (or limit-surely). We present results for MDPs with finite-memory strategies, and turn-based stochastic games with finite-memory and infinite-memory strategies. Our results establish complexity results, as well as algorithmic results for finite-memory strategies by reduction to non-stochastic games. Some interesting directions of future work are as follows. First, while our results establish algorithmic results for finite-memory strategies, whether similar results can be establishes for infinite-memory strategies is an interesting open question. Second, the study of beyond worst-case synthesis problem for turn-based stochastic games with quantitative objectives is another interesting direction of future work.

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A Appendix

A.1 Proofs from Section 3

We include the proof of Theorem 3.2:

Proof. Consider the MDP in Figure 1. Here \( W_s \) is induced by the Büchi winning condition \( \{ l, r \} \) and \( W_{as} \) is induced by the co-Büchi winning condition \( \{ r \} \). That is, in order to win surely, either \( l \) or \( r \) has to be visited infinitely often in every path and, in order to win almost surely, \( r \) has to be visited finitely often with probability 1.

We show that every finite-memory strategy that wins almost-surely with respect to \( W_{as} \) is losing according to \( W_s \). Consider a finite-memory strategy \( \sigma \). Consider the Markov chain \( \tau_0(\sigma) \). As \( \sigma \) is finite memory, \( \tau_0(\sigma) \) is a finite-state Markov chain.

It follows that bottom strongly connected components are reached with probability 1 and once a bottom SCC is reached all configurations in that SCC are visited with probability 1. Suppose that \( r \) appears in some reachable bottom SCC. Then, clearly, the almost-sure winning condition cannot have the probability 1. It follows that \( r \) does not appear in bottom SCCs of \( \tau_0(\sigma) \) and that in bottom SCCs the choice of Player 0 from configuration \( c \) is to go to configuration \( p \). This implies that the infinite suffix \((c \cdot p)\omega \) appears in \( \tau_0(\sigma) \). However, this path violates the sure winning condition.

We show that Player 0 has an optimal strategy that uses infinite memory. Consider the following strategy. Player 0 plays in rounds. In the \( j \)-th round Player 0 plays as follows. She maintains a counter of how many times the play visited \( r \) without visiting \( l \). As long as the counter is smaller than \( j \) she continues moving left from \( c \) to \( p \). Once she visits \( l \) she moves to round \( j + 1 \). Once the counter reaches \( j \) she chooses to visit \( r \) by going right from \( c \) and moves to round \( j + 1 \). It follows that there are infinitely many rounds and in every round parity 0 for \( W_s \) is visited. Hence, Player 0 wins surely according to \( W_s \). Furthermore, the probability to reach \( r \) in the \( j \)-th round is bounded by \( \frac{1}{2^j} \). Dually, the probability to not visit \( r \) in the \( j \)-th round is at least \( 1 - \frac{1}{2^j} \). It follows that the probability to visit \( r \) infinitely often is 0.

We include the proof of Lemma 3.4:

Proof. Consider an MDP \( G = (V, (V_0, V_p), E, \kappa, W) \), where \( W = (W_s, W_{as}) \). Suppose that there is an winning finite-memory strategy \( \sigma \) for \( G \). Let \( \sigma \) be composition of \( \sigma_s \) and \( \sigma_u \), where \( M = \{ 1, \ldots, m \} \) is the memory domain used by \( \sigma \). Then, the Markov chain \( G^M = (M \times V, V_p^M, E^M, \kappa^M, W^M) \), where \( E^M = \{ ((m, v), (m', v')) | (v, v') \in E \text{ and } m' = \sigma_u(m, v') \} \), for \( (m, v) \) such that \( v \in V_p \) we have \( \kappa^M \) induced by \( \kappa \) in the obvious way and for \( (m, v) \) such that \( v \in V_0 \) we have \( \kappa^M \) induced by \( \sigma_s(m, v) \). The condition \( W^M \) is obtained by considering the projection of a play on \( V^\omega \). Clearly, \( G^M \) is a finite-state Markov chain. It follows that in \( G^M \) bottom SCCs are reached with probability 1 and in every bottom SCC every configuration is visited infinitely often with probability 1. Consider a bottom SCC in \( G^M \). There is some minimal parity appearing in this bottom SCC, which will be visited infinitely often. It follows that we can consider this minimal parity as a Büchi condition. We reduce the optimal winning with finite-memory in \( G \) to the optimal winning with finite-memory in an MDP \( G' \), where the almost-sure winning condition is a Büchi condition and the sure winning condition has the same index.

Formally, we have the following. Let \( \omega_{as} : V \to \{ 0, \ldots, 2d_{as} \} \) be the parity function inducing \( W_{as} \) and let \( \omega_s : V \to \{ 0, \ldots, 2d_s \} \) be the parity function inducing \( W_s \). Let \( G_b = (V', (V_0', V_p'), E', \kappa', W') \), where \( G_b \) has the following components. For a set \( V \) let \( \hat{V} \) denote the set \( \{ \hat{v} | v \in V \} \).

- \( V' = V \cup \hat{V} \cup V \times \{ 0, 2, \ldots, 2d_{as} \} \cup \{ \perp \} \)
- \( V_0' = V_0 \cup \hat{V} \cup V_0 \times \{ 0, 2, \ldots, 2d_{as} \} \)
- \( V_p' = V'' \setminus V_0' \)
- \( E' = \{ ((v, \hat{w}) | (v, w) \in E \} \cup \{ (\perp, \perp) \} \cup \{ ((\hat{v}, (v, i)), (\hat{v}, v), (\hat{v}, \perp) | v \in V) \} \cup \{ ((v, i), (w, i)) | (v, w) \in E \text{ and } \omega_{as}(v) \geq i \} \cup \{ ((v, i), \perp) | \omega_{as}(v) < i \} \)
- For \( v \in V_p \) we have \( \kappa'(v, \hat{w}) = \kappa'((v, i), (w, i)) = \kappa(v, w) \) and \( \kappa'((v, i), \perp) = 1 \).
- \( W' = (W_0', W'_{as}) \), where \( W_0' \) is induced by \( \alpha'_{0}(q) = \alpha'_{0}(q) = \alpha'_{0}(q, i) = \omega_s(q) \) and \( \omega_s(\perp) = 1 \) and \( W'_{as} \) is induced by \( \alpha'_{as} \) where \( \alpha'_{as}(q, i) = 0 \) if \( \omega_{as}(q) = i \) and \( \alpha'_{as} \) is 1 for all other configurations.

The MDP \( G_b \) has an original (extended) copy of \( G \) and additional \( d_{as} + 1 \) restricted copies of \( G \). In the original copy \( (V \cup \hat{V}) \), every move in \( G \) is mimicked by two moves in \( G_b \). Before getting to a location in \( V \) we pass through a location in \( \hat{V} \), where Player 0 is given a choice between progressing to \( V \) (and effectively staying in the original copy) or moving on to one of the restricted copies (or giving up by going to \( \perp \)). Each of the other \( d_{as} + 1 \) copies corresponds to a bottom strongly connected
We include the proof of Lemma 3.5:

**Proof.** Consider an MDP $G = (V, (V_0, V_p), E, \kappa, W)$, where $W = (W_s, W_{as})$ is such that $W_{as}$ is a Büchi condition. Let $B \subseteq V$ denote the set of Büchi configurations of the almost-sure winning condition. Let $p_s : V \rightarrow \{1..d_s\}$ be the parity ranking function that induces $W_s$. We create the (non-stochastic) game $G'$ by replacing every configuration $\nu \in V_p$ by one of the gadgets in Figure 4. Formally, $G' = (V', (V'_0, V'_p), E, 'W')$, where the components of $G'$ are as follows:

- $V'_0 = V_0 \cup \{(\hat{\nu}, 0), (\hat{\nu}, 2), (\hat{\nu}, 1) | \nu \in V_p\}$
- $V'_1 = \{(\pi, (\hat{\nu}, 0)) | \nu \in V_p\}$
- $E' = \{(\nu, w) | (\nu, w) \in E \cap (V_0 \times V_0)\} \cup \{(\nu, w) | (\nu, w) \in E \cap (V_p \times V_0)\} \cup \{(\nu, w) | (\nu, w) \in E \cap (V_0 \times V_p)\} \cup \{(\nu, w) | (\nu, w) \in E \cap (V_p \times V_p)\}$
- $\nu \in V \rightarrow [1..d_s]$, where $\nu' = \nu_s(\nu)$ for $\nu \in V_p$ and $\nu' = \nu_{as}(\nu)$. We show that if Player 0 has a finite-memory winning strategy in $M$ she wins in $G$. Consider the combination of the winning strategy for Player 0 in $M$ with $M$. It follows that every bottom SCC is winning according to both $W_s$ and $W_{as}$. We define the following strategy in $G'$ that uses the same memory. We have to extend the strategy by a decision from configurations of the form $(\hat{\nu}, 1)$, which belong to Player 0 in $G'$. If the configuration $(\hat{\nu}, 1)$ is in a bottom SCC, Player 0 chooses the successor of $\nu$ that minimizes the distance to $B$. By assumption every bottom SCC has some configuration in $B$ appearing in it. If the configuration $(\hat{\nu}, 1)$ is in a non-bottom SCC, Player 0 chooses the successor of $\nu$ with minimal distance to exit the SCC. Consider a play consistent with this strategy. Clearly, this play is possible also in the MDP $G$. By assumption the play satisfies $\alpha_s$. If a play visits infinitely often configurations of the type $(\hat{\nu}, 0)$, then this play satisfies the Büchi winning condition. Consider a play that visits finitely often configurations of the type $(\hat{\nu}, 0)$. It follows that whenever it visits a configuration of the type $\pi$ the distance to leave non-bottom SCC decreases and hence the play eventually reaches a bottom SCC. Similarly, after reaching a bottom SCC whenever a configuration of the type $\pi$ is visited the distance to $B$ decreases. 

We show that if Player 0 has a finite-memory winning strategy in $M$ she wins in $G$. Consider the combination of the winning strategy for Player 0 in $M$ with $M$. It follows that every bottom SCC is winning according to both $W_s$ and $W_{as}$. We define the following strategy in $G'$ that uses the same memory. We have to extend the strategy by a decision from configurations of the form $(\hat{\nu}, 1)$, which belong to Player 0 in $G'$. If the configuration $(\hat{\nu}, 1)$ is in a bottom SCC, Player 0 chooses the successor of $\nu$ that minimizes the distance to $B$. By assumption every bottom SCC has some configuration in $B$ appearing in it. If the configuration $(\hat{\nu}, 1)$ is in a non-bottom SCC, Player 0 chooses the successor of $\nu$ with minimal distance to exit the SCC. Consider a play consistent with this strategy. Clearly, this play is possible also in the MDP $G$. By assumption the play satisfies $\alpha_s$. If a play visits infinitely often configurations of the type $(\hat{\nu}, 0)$, then this play satisfies the Büchi winning condition. Consider a play that visits finitely often configurations of the type $(\hat{\nu}, 0)$. It follows that whenever it visits a configuration of the type $\pi$ the distance to leave non-bottom SCC decreases and hence the play eventually reaches a bottom SCC. Similarly, after reaching a bottom SCC whenever a configuration of the type $\pi$ is visited the distance to $B$ decreases.

![Figure 4. Gadget to replace probabilistic configurations in the reduction.](image-url)
We show that if Player 0 wins the game $G'$ she has a finite-memory winning strategy in $M$. As $G'$ does not have stochastic elements and as the winning condition there is $\omega$-regular, Player 0 has a finite-memory pure winning strategy. In case that from $\square$ Player 1 chooses the successor $(d, 0)$ then the strategy instructs Player 0 what to do from every successor of $d$. In case that from $\Box$ Player 1 chooses the successor $(d, 2)$ then the strategy instructs Player 0 to choose a unique successor of $d$.

The strategy of Player 0 in $M$ follows the strategy of Player 0 in $G$. Whenever after a configuration $v \in V_p$ we discover that the successor is the one that is consistent with the choice from $(d, 1)$ we update the memory accordingly. Whenever after a configuration $v \in V_p$ we discover that the successor is not consistent with the choice from $(d, 1)$ we update the memory according to the choice from $(d, 0)$.

Consider a play in $M$ consistent with this strategy. Clearly, a version of this play appears also in $G$. As both the game and the MDP agree on $W_s$ and the strategy in $G$ is winning it follows that $W_s$ is satisfied in $M$ as well. Consider now the winning condition $W_{sa}$. If the play visits only finitely many configurations in $V_p$ then it is identical to the play in $G$ and satisfies also $W_{sa}$. If the play visits infinitely many configurations in $V_p \cap B$ then it is clearly winning according to $W_{as}$. Otherwise, the play visits infinitely many configurations in $V_p$ but only finitely many in $V_p \cap B$. The play cannot stay forever in a non-bottom SCC of $G$ (combined with the strategy of Player 0). Indeed, there is a non-zero probability for the probabilistic choices to agree with the strategy of Player 0 and leave this non-bottom SCC. Similarly, when reaching a bottom SCC, as all configurations of the SCC will be visited with probability 1 and from the strategy being winning in $G'$ we conclude that there is a configuration in $B$ appearing in this bottom SCC and it will be visited infinitely often.

We include the proof of Lemma 3.6:

**Proof.** Let the parity condition be induced by the priority function $\alpha : V \to [0..d]$ and let the Büchi condition be $B \subseteq V$. For a parity $p$ let $[p]$ denote $p$ if $p$ is odd and $p + 1$ otherwise. We take the product of the game with the deterministic monitor $M = (V, \{0, \ldots, [d]\} \times \{\bot, \top\}, \delta, \alpha')$, where $\alpha'(i, \top) = i$ and $\alpha'(i, \bot) = [i]$ and $\delta$ is defined as follows. We note that the monitor "reads" the configuration of the game.

$$\delta((i, \beta), v) = \begin{cases} (\alpha(v), \top) & v \in B \text{ or } \alpha(v) < i \\ (i, \bot) & v \notin B \text{ and } \alpha(v) \geq i \end{cases}$$

That is, the monitor memorizes the lowest parity that has been seen since a visit to the Büchi set. The monitor signals a "real" visit to parity $i$ whenever its second component is $\top$. The second component can become $\top$ whenever a Büchi state is visited or when the visited priority is lower than all priorities visited since the last visit to the Büchi set.

Consider a play $p$ in $G$ that is winning according to both the Büchi condition and the original parity condition. Let $e$ be the least (even) parity to be visited infinitely often in $p$ according to $\alpha$. Let $j$ be the index such that $p_i$ is a visit to $B$ and after $j$ there are no visits to parities smaller than $e$. Then, for every $j' \geq j$ there is a $j'' \geq j$ such that $p_{j''} \in B$. If $p_{j''}$ also has parity $e$ then the state of the monitor after reading $p_{j''}$ is $(e, \top)$. If $p_{j''}$ has parity greater than $e$ then let $j'' > j'''$ be the minimal such that $p_{j'''}$ has parity $e$. Then the state of the monitor after reading $p_{j'''}$ is $(e, \top)$. It follows that $(e, \top)$ is visited infinitely often. Also, from the structure of the monitor, the only way the monitor signals a parity smaller than $e$ is if parities smaller than $e$ are visited in the game. It follows that the monitor signals acceptance.

Consider a play $p$ in $G$ for which the minimal parity indicated by $\alpha'$ to be visited infinitely often is even. The only states of the monitor for which $\alpha'()$ is even are of the form $(i, \top)$. States of the form $(i, \bot)$ can appear at most $d/2$ times without a visit to a configuration in $B$, indeed, without a visit to $B$ $i$ can only decrease. It follows that $p$ visits $B$ infinitely often. Let $e$ be the least (even) parity to be visited infinitely often in $p$ according to $\alpha'$. Let $j$ be the location such that for every $j' > j$ we have $\alpha'() \geq e$. By the structure of the monitor, for every $j' > j$ we have $\alpha'()$ has parity at least $e$. Furthermore, the only way for $\alpha'()$ to signal parity $e$ is after a visit to a state $p_{j''}$ for which $\alpha(p_{j''}) = e$. It follows that $B$ is visited infinitely often and that the minimal parity to be visited infinitely often is even.

**A.2 Proofs from Section 4**

We include the proof of Theorem 4.1:

**Proof.** We define a Martin-like reduction that annotates configurations with the values that Player 0 can get from them according to $W_{sa}$ [31]. Formally, we have the following. Let $G = (V, (V_0, V_1, V_p), E, \kappa, W)$. We define the following two-player game: Let $G' = (V', (V'_0, V'_1), E', W')$, where $G'$ has the following components:

- The set of configurations $V' = V'_0 \cup V'_1$, where $V'_0 = \{ (v, r) \mid v \in V \text{ and } r \in [0, 1) \}$ and $V'_1 = \{ (v, f) \mid v \in V \text{ and } f : \text{succ}(v) \rightarrow ([0, 1) \cup \{\bot\}) \}$
• The transitions of $G'$ are as follows.

\[
E = \left\{ ((v, r), (v, f)) \begin{array}{l} \forall v \in V_G, \exists w \in \text{succ}(v) : f(w) \neq \bot \\
\text{and } f(w) \geq r \\
(v, r, (v, f)) \end{array} \begin{array}{l} \forall v \in V_G, \forall w \in \text{succ}(v) : f(w) \neq \bot \\
\text{and } f(w) \geq r \\
((v, r), (v, f)) \end{array} \right\} \cup
\left\{ ((v, f), (w, f(w))) \mid w \in \text{succ}(v) \text{ and } f(w) \neq \bot \right\}
\]

• The winning condition is:

\[
W' = \left\{ w \mid w \in (V \times (0, 1))^* \cdot (V \times \{0\})^\omega \text{ and } w \left\|_V \in W_s \right\} \cup
\left\{ w \mid w \in (V \times (0, 1))^\omega \text{ and } w \left\|_V \in W_s \cap W_{as} \right\}
\]

That is, every configuration is annotated by the value that Player 0 can win from it according to $W_{as}$. Then, Player 0 chooses a function that shows what the value is for the successors of $v$ followed by a choice of Player 1 of which successor to follow. In configurations in $V_0$, Player 0 is allowed to use $\bot$ to some successors to signify that she has to win from at least one successor. In configurations in $V_1$, Player 0 must choose numerical values for all successors. She must choose a value larger than current value for all of them. Indeed, if the value is lower than current value for one of them Player 1 would choose that successor and win. In configurations in $V_p$, Player 0 must choose numerical values for all successors. The weighted average of the values of successors must be at least the current value. This means that some successors can have the value 0. If the value is 0, Player 0 still needs to win according to $W_s$. That is, the game proceeds with the value 0 forever. If the value is non-zero, she needs to show that she can win according to $W_s$ and obtain that value for the measure of winning paths according to $W_{as}$.

This is a turn-based two-player game (albeit somewhat large) and as such is determined [30]. Let $W'_s$ be the set of configurations winning for Player 0 in $G'$. We show that $v$ is winning for Player 0 in $G$ iff $(v) \times [0, 1) \subseteq W'_s$. That is, Player 0 wins from $(v, r)$ for every possible value of $r$ (note that 1 is not a possible value of $r$).

Suppose that Player 0 wins from $(v, r)$ for every $r \in [0, 1)$. It follows that for every $\epsilon > 0$ there is a strategy of Player 0 winning from $(v, 1 - \epsilon)$. All the plays consistent with this strategy are winning according to $W_s$. According to Martin’s proof [31] the measure of plays winning for Player 0 for $W_{as}$ is at least $1 - \epsilon$. The two together show that Player 0 wins from $v$.

Suppose otherwise, then there is some value $r$ such that Player 1 wins from $(v, r)$. In $G'_s$ if $r = 0$, then the strategy of Player 1 in $G'$ can be used to show that Player 1 wins according to the winning condition $W_s$. If $r > 0$ then Martin’s proof can be used to show that Player 1 has a strategy that forces the measure of plays not in $W_{as}$ to be at least $1 - r > 0$.

We include the proof of Theorem 4.2:

Proof. WLOG we consider that every Player 1-configuration $v$ has two successors $v_a$ and $v_b$: if $v$ has one successor it can be considered as Player 0-configuration; and the case where there are more than two successors can be reduced to the case of two successors by adding more configurations for Player 1. We prove the result by induction on the number of configurations controlled by Player 1. This technique has used in the context of non-stochastic games [26], as well as stochastic games [25, 9], but only for single objectives, whereas we use it for combination of qualitative criteria.

If there are no configurations controlled by Player 1 then clearly a memoryless strategy suffices for her. Assume by induction that in a game with less than $k$ configurations controlled by Player 1 she can win with memoryless strategies. Consider a game with $k$ configurations in $V_1$. Let $v$ be such a configuration that has two successors $v_a$ and $v_b$. Let $G_a$ be the game $G$ where the edge from $v$ to $v_b$ is removed and let $G_b$ be the game $G$ where the edge from $v$ to $v_a$ is removed. Assume that Player 1 wins in $G$ but requires memory to do so. It follows that Player 1 cannot win both $G_a$ and $G_b$, otherwise, by induction, she would have a memoryless winning strategy and the same strategy would be extended to a memoryless winning strategy in $G$. It follows that Player 0 has winning strategies $\sigma_a$ and $\sigma_b$ in $G_a$ and $G_b$, respectively. We compose $\sigma_a$ and $\sigma_b$ to a winning strategy in $G$. Player 0 starts playing according to $\sigma_a$. Considering a history $h = v_0 \cdots v_n$, $h$ can be partitioned to $h_0, h_1, \ldots, h_k$, where for every $i$ the last configuration of $h_i$ is $v$ and $v$ does not appear elsewhere in $h_i$. Let $h_{ab}$ be the restriction of $h$ to those $h_i$ such that $v_b$ is the first configuration in $h_i$ (including $h_0$). Let $h_b$ be the restriction of $h$ to those $h_i$ such that $v_b$ is the first configuration in $h_i$ (excluding $h_0$). Then, the strategy $\sigma$ in $G$ plays according to $\sigma_a$ if the last move from $v$ was to $v_a$ (or $v$ has not been visited) and according to $\sigma_b$ if the last move from $v$ was to $v_b$. When playing according to $\sigma_a$ the strategy $\sigma$ considers the history $h_a$ and whenever playing according to $\sigma_b$ the strategy $\sigma$ considers the history $h_b$. It follows that $\sigma$ is well defined.

Consider a play $r$ consistent with $\sigma$. Clearly, $r$ can be partitioned to $r_a, \ldots$ according to the visits to $v$ similar to the way a history is partitioned and to the projections $r_a$ and $r_b$. Then, $r_a$ and $r_b$ are plays in $G_a$ and $G_b$ consistent with $\sigma_a$ and $\sigma_b$. It
follows that $r_a$ and $r_b$ are winning for Player 0 according to $W_s$. As $W_s$ is a parity condition the minimum parity occurring infinitely often in $r_a$ is even and the minimum parity occurring in $r_b$ is even. It follows that the minimal parity occurring infinitely often in $r$ is even and it is winning according to $W_s$.

Given a strategy for Player 1 and the resulting Markov chain $w(\sigma, \pi)$ we can consider the same partition to $w_a(\sigma, \pi)$ and $w_b(\sigma, \pi)$ of the Markov chains. The Markov chain $w_a(\sigma, \pi)$ is consistent with $\sigma_a$ and $w_b(\sigma, \pi)$ is consistent with $\sigma_b$. It follows that the probability of winning according to $W_{as}$ is 1 in both $w_a(\sigma, \pi)$ and $w_b(\sigma, \pi)$.

The shuffle of two plays $r_1$ and $r_2$ is a play $r = u_0 u_1 \ldots$, where $u_i \in V^+$ and $r_1 = u_0 \cdot u_2 \cdots$ and $r_2 = u_1 \cdot u_3 \cdots$. A winning condition is shuffle closed if for every two plays $r_1, r_2 \in W$ we have that every shuffle $r$ is in $W$ as well. Clearly, parity condition is shuffle closed. For shuffle-closed winning conditions we know that

$$ \text{Prob}_{w(\sigma, \pi)}(W_{as}) \geq \text{min}(\text{Prob}_{w_a(\sigma, \pi)}(W_{as}), \text{Prob}_{w_b(\sigma, \pi)}(W_{as})). $$

As $\sigma_a$ and $\sigma_b$ are winning it follows that $\text{Prob}_{w_a(\sigma, \pi)}(W_{as}) = 1$, $\text{Prob}_{w_b(\sigma, \pi)}(W_{as}) = 1$ and $\text{Prob}_{w(\sigma, \pi)}(W_{as}) \geq 1$. $\square$

We include the proof of Lemma 4.7:

**Proof.** Let $\epsilon$ be the minimal probability of a transition in $G$. As $G$ is finite $\epsilon$ exists. For configurations in $V_0 \cup V_1$ the definitions of good parity ranking and almost-sure ranking coincide.

Consider a configuration $v \in V_2$ and the matching configuration $\mathcal{V}$. Let $p = p_{as}(v)$. Consider the appearance of $(v, m)$ in $G \times M$ and $(\mathcal{V}, m)$ in $G' \times M$ for some memory value $m$. We consider the cases where $p$ is even and when $p$ is odd.

- Suppose that $p$ is even. If there is some minimal $i$ such that the choice of $\sigma'_0$ from $((\hat{v}, 2i), m')$ in $G' \times M$ is $((\hat{v}, 2i - 1), m'')$. Then, there is some $w \in \text{succ}(v)$ and some $m'''$ such that

  $$ \tilde{r}(w, m''') < 2i - 1 \tilde{r}((\hat{v}, 2i - 1), m'') \leq_p \tilde{r}((\hat{v}, 2i), m') \leq_p \overline{r}(\mathcal{V}, m). $$

  It follows that $\text{Prob}_{\tilde{r}}(\overline{r} <_{2i-1} \tilde{r}) \geq \epsilon$. Furthermore, as $i$ is minimal it follows that $i \neq 0$ and that the choice of $\sigma'_0$ from $((\hat{v}, 2i - 2), n)$ is $((\hat{v}, 2i - 2), n')$, which belongs to Player 1 in $G' \times M$. Then, for every successor $(w, n''')$ of $((\hat{v}, 2i - 2), n')$ we have

  $$ \tilde{r}(w, n''') \leq_{2i-2} \tilde{r}((\hat{v}, 2i - 2), n'') \leq_p ((\hat{v}, 2i - 2), n') \leq_p (\mathcal{V}, m). $$

  If there is no such $i$, then the choice of $\sigma'_0$ from $((\hat{v}, p), m')$ in $G' \times M$ is $((\hat{v}, p), m'')$ and for every $w \in \text{succ}(v)$ there is some $m'''$ such that

  $$ \tilde{r}(w, m''') \leq_p \tilde{r}((\hat{v}, p), m') \leq_p (\mathcal{V}, m). $$

- Suppose that $p$ is odd. In this case there must be some minimal $i$ such that the choice of $\sigma'_0$ from $((\hat{v}, 2i), m')$ is $((\hat{v}, 2i - 1), m'')$. We can proceed as above.

$\square$

We include the proof of Lemma 4.8:

**Proof.** The conditions for parity ranking and almost-sure ranking are the same for Player 0 or Player 1 configurations. Consider a configuration $(v, m)$ in $G \times M$ and the matching $(\mathcal{V}, m)$ in $G' \times M$.

- Suppose that $v$ is such that $p(v)$ is even. We set the ranking $r$ for all configurations in the gadget replacing $v$ to $r(v)$. By definition 4.5 for $v$ we know that either $\text{Prob}_{v}(\overline{r} <_{p(v)-1} \tilde{r}) = 1$ or there is some odd $j$ such that $\text{Prob}_{v}(\overline{r} <_{j} \tilde{r}) \geq \epsilon$.

  - In the first case, from $\mathcal{V}$ and for every $i$ we know that $r(\mathcal{V}) \geq_{p(v)-1} r(\hat{v}, 2i)$. Similarly, for $(\hat{v}, 2i)$ we know that $r(\hat{v}, 2i) \geq_{p(v)-1} r(\hat{v}, 2i)$. Finally, as $p(\hat{v}, 2i) = 2i$ and for every successor $w$ we are ensured that $r(\hat{v}, 2i) \geq_{p(v)-1} r(w)$, which implies that $r(\hat{v}, 2i) \geq_{2i-1} r(w)$. It follows, that for every Player 1 configuration in the gadget we have that the ranking does not increase for its parity for all successors and for Player 0 configurations there is a successor such that the ranking does not increase.

  - In the second case, let $j$ be the minimal odd number such that $\text{Prob}_{v}(\overline{r} <_{2i} \tilde{r}) = 1$ and $\text{Prob}_{v}(\overline{r} <_{j} \tilde{r}) \geq \epsilon$. It follows that there is some successor $w$ for which $\tilde{r}(w) <_j \overline{r}(w)$. As before, from $\mathcal{V}$ and for every $i$ we know that $r(\mathcal{V}) \geq_{p(v)-1} r(\hat{v}, 2i)$. For $(\hat{v}, 2i)$ such that $2i < j$ we know that $r(\hat{v}, 2i) \geq_{p(v)-1} r(\hat{v}, 2i)$. For $(\hat{v}, 2i)$ such that $2i > j$ we know that $r(\hat{v}, 2i) \geq_{p(v)-1} r(\hat{v}, 2i - 1)$. Finally, for $i < j$ we know that for all successors $w$ of $v$ we have $r(w) \geq_{2i-1} r(w)$. It follows that $r(\hat{v}, 2i) \geq_{2i} r(w)$. For $i \geq j$ we know that for some successor $w$ of $v$ we have $r(w) >_j r(w)$. It follows that $r(\hat{v}, 2i - 1) >_{2i-1} r(w)$. 

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• Suppose that \( v \) is such that \( p(v) \) is odd. We set the ranking \( r \) for all configurations in the gadget replacing \( v \) to \( r(v) \). By definition 4.5 for \( v \) we know that there is some odd \( j \) such that \( \Pr_{v}(\hat{r}_{\preceq j-2}) = 1 \) and \( \Pr_{v}(\hat{r}_{< j}) \geq \epsilon \).

Let \( j \) be the minimal odd number such that \( \Pr_{v}(\hat{r}_{\preceq j-2}) = 1 \) and \( \Pr_{v}(\hat{r}_{< j}) \geq \epsilon \). It follows that there is some successor \( w \) for which \( r(w) < j \).

As before, from \( \overline{v} \) and for every \( i \) we know that \( r(\overline{v}) \geq p(\overline{v})-1 \) \( r(\hat{v}, 2i) \). For \((\hat{v}, 2i)\) such that \( 2i < j \) we know that \( r(\hat{v}, 2i) \geq p(\overline{v})-1 \) \( r(\hat{v}, 2i) \). For \((\hat{v}, 2i)\) such that \( 2i > j \) we know that \( r(\hat{v}, 2i) \geq p(\overline{v})-1 \) \( r(\hat{v}, 2i-1) \). Finally, for \( i < j \) we know that for all successors \( w \) of \( v \) we have \( r(v) \geq j \) \( r(w) \). It follows that \( r(\hat{v}, 2i) \geq 2i \) \( r(w) \). For \( i \geq j \) we know that for some successor \( w \) of \( v \) we have \( r(v) > j \) \( r(w) \). It follows that \( r(\hat{v}, 2i-1) > 2i-1 \) \( r(w) \).

It follows that \( r \) induces a good parity ranking on \( G' \times M \). Then, Player 0 wins in \( G' \times M \) according to \( W_{a_{4}} \). \( \square \)