Wave-particle duality in single-photon entanglement

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1. Introduction

It is generally believed that the number of particles in the system where quantum entanglement occurs is greater than or equal to 2 [1, 2]. These particles entangled at some specified degrees of freedom, and there is a strong non-local quantum correlation between them which can not be reproduced by any classical correlation, like the violation different kinds of Bell’s inequalities [2–5]. However, there are some similarities between the nonlocality of quantum entanglement and that of wave function, such as the equivalence between the measurement-induced wave function collapse and the non-signaling theory of an entangled two-part system. Multi-particle entanglement systems sometimes exhibit single-particle behavior, such as the uncertainty relationship of quantum correlations [6]. Conversely, a single particle can also exhibit entanglement characteristics in some situations [7].

It is found that when a quantum particle is incident on a beam splitter, the delocalized quantum state after the beam splitter can be formulated as the entangled state between the vacuum state and the single-particle state [7–9]. It is an entanglement of particle nature in Fock space. This novel quantum state has attracted anormous attention, not only because of its unique physical properties, but also due to its potential role in the field of quantum information [10–15, 14, 16–18]. The most widely used ways to test single-photon entanglement is based on the coincidence measurement of homodyne detection, through which the density matrix and quantum correlated Wigner distributions have been constructed to establish different types of Bell’s inequality [19–24]. It should be noted that the conventional Bell’s inequality, a convincing way to test quantum correlation in one specific basis, is formed by measuring the joint probability in its conjugate space [5, 25]. Thus the single-particle entanglement provides a good platform for studying wave-particle duality of a quantum particle.

In this paper, we present a theoretical study on the wave-particle duality in single-photon entanglement based on the Bell-like inequality via its wave-behavior measurement.
where the bases in Fock space are vacuum state and single-photon state, respectively. In the conjugate space, the wave state is defined as the coherent superposition of the vacuum state and the single-photon state. The wave state detection is accomplished by an interference with another reference wave state, and the wave behavior can be observed by continuously changing their relative phase difference. The joint probability is obtained by the wave-state measurement by two far separated parts, called Alice and Bob, where the reference wave state is replaced by a weak coherent state. Finally, the Bell-like inequality is tested based on the joint wave-state measurement to verify the existence of single-photon entanglement.

2. Main Text

The generation of single-photon entanglement and the wave state measurement is illustrated in figure 1. A photon is incident on the first beam splitter, the two output modes A and B are entangled in Fock space \[7, 8, 19\]

\[
|\Psi_{AB}\rangle = \frac{\sqrt{2}}{2} e^{i\phi}[|\alpha\rangle_A |1\rangle_B + |0\rangle_A |1\rangle_B],
\]

where \(|0\rangle\) and \(|1\rangle\) represent vacuum state and single-photon state, \(e^{i\phi}\) is the accumulated phase difference between the two arms which may be caused by half-wave loss due to reflection or different propagation path lengths, \(e^{i\phi}\) is the global phase which is always omitted. As we know that the degree of entanglement is preserved through a Fourier (unitary) transformation, and after a Fourier transformation the entanglement has an inverse correlation form in the conjugate space.

Through a two-dimensional Fourier transform, we have the single-photon entanglement in wave space

\[
|\Psi_{AB}\rangle = \frac{\sqrt{2}}{2} e^{i(\phi - \alpha)}[|\alpha\rangle_w (|\alpha - \phi\rangle_w - |\alpha + \pi\rangle_w) - |\alpha + \pi\rangle_w (|\alpha - \phi\rangle_w)],
\]

where the subscript \(w\) refers to the wave state, \(\alpha\) is an arbitrary real number in the range of 0 to \(\pi\), the basis vectors for the wave states in the photon number representation are

\[
|\alpha\rangle_w = \frac{\sqrt{2}}{2} [|0\rangle + e^{i\alpha}|1\rangle],
\]

\[
|\alpha + \pi\rangle_w = \frac{\sqrt{2}}{2} [|0\rangle - e^{i\alpha}|1\rangle].
\]

In wave space, we use the phase value \(\alpha\) to characterize a wave state. Now it is can be seen that equation (2) has a Bell state form for the single-photon entanglement in the wave space, the quantum correlation between the modes of Alice and Bob has an inverse correlation compared with that in the photon number space in equation (1). This is a general feature of quantum correlation in conjugate spaces [6, 25]. The above transformation also implies that the wave state may have a behavior that is conjugated to the particle behavior of the photon number state.

The particle behavior of a photon number state, the single-photon state or vacuum state, is revealed by whether there is a firing of the single-photon detector, while the wave behavior of the wave state is manifested as the existence of an interference fringe when varying the phase difference between the two interference arms. Here, we put forward a method to detect the wave state by interfering it with a reference wave state, which is very
similar to the quarture measurement in homodyne detection. As shown in figure 2, interfering the wave state $|\alpha\rangle_w$ with a reference wave state $|\beta\rangle_w$ on a beam splitter is

$$
|\alpha\rangle_w |\beta\rangle_w = \frac{1}{2} [ |0\rangle + e^{i\alpha} |1\rangle ] [ |0\rangle + e^{i\beta} |1\rangle ]
$$

$$
= \frac{1}{2} |0\rangle + \frac{\sqrt{2}}{4} ( e^{i\alpha} + e^{i\beta} |1\rangle ) |0\rangle + \frac{\sqrt{2}}{4} ( e^{i\alpha} + e^{i\beta} |1\rangle ) |1\rangle + \frac{1}{4} e^{i(\alpha + \beta)} ( |1\rangle |0\rangle + |0\rangle |1\rangle ).
$$

From equation (4), we can see that the photon state behind the beam splitter consists of four parts. The first part is the vacuum state with a probability of $p(0) = \frac{1}{2}$; the second and the third parts are responsible for the wave behavior in which only one photon is detected in the c and d ports, and the probability for these two parts are $p(1, 0) = \frac{1}{4} (1 - \sin(\alpha - \beta))$ and $p(0, 1) = \frac{1}{4} (1 + \sin(\alpha - \beta))$, respectively; the last part is attributed to two-photon interference which is insensitive to the phase difference. By varying the relative phase difference between the wave state and the reference state, an interference fringe can be detected, so the state has wave behavior. The fourth part of detection comes from two-photon interference, which is a kind of particle behavior interference and can be regarded as the background noise.

The detection probability for a wave state $|\beta\rangle_w$ depends on the phase difference between the reference wave state $|\alpha\rangle_w$ and the detected wave state $|\beta\rangle_w$. As for the c-port, by fixing the value of $\alpha$, the detection probability of the wave state $|\beta\rangle_w$ is $\frac{1}{2}$ for $\beta = \alpha + \frac{\pi}{2}$, and the detection probability of the wave state $|\beta\rangle_w$ is 0 for $\beta = \alpha - \frac{\pi}{2}$. The detection probability on the d port is completely opposite. Here, one may wonder why the maximum probability of the successful detection of a wave state is only $\frac{1}{2}$. This is because a successful detection of a quantum state depends on the response of the single-photon detector, which is actually a projection measurement in Fock space. According to the definition of the wave state in equation (3), its detection probability on a single-photon detector is $\frac{1}{2}$, which explains that the detection probability of wave state is in the range of $(0, 0.5)$.

However, the above defined wave state does not exist in nature, nor is there any report about the generation of such wave states. Actually, it is not easy to generate a coherent superposition of vacuum state and single-particle state directly. The state that mostly resembles a wave state is the weak coherent (WC) state which is generated by attenuating the intensity of a laser beam to a value well below that of a single photon. The WC state in the Fock space representation is

$$
|\gamma e^{i\phi}\rangle_c = \exp\left(-\frac{1}{2} |\gamma|^2 \right) \sum_{n=0}^{\infty} \frac{\gamma^n e^{i n \phi}}{\sqrt{n!}} |n\rangle,
$$

where the subscript $c$ represents coherent state, $\gamma$ is a real positive number, $|\gamma|^2$ is the average photon number in the coherent state, $\phi$ is the phase carried by the state. If the condition $|\gamma|^2 \ll 1$ is satisfied, the WC state can be approximated as

$$
|\gamma e^{i\phi}\rangle_c \approx |0\rangle + \gamma e^{i \phi} |1\rangle + O(\gamma).
$$

In the power series expansion of the WC state in the Fock space, only the vacuum state and the single-photon state is retained, all the high order infinitesimal terms are represented by $O(\gamma)$. Furthermore, the interference...
between two wave states approximated by two WC states is the most fundamental physics in phase-matching quantum key distribution [26, 27].

In fact, similar devices as in figure 1 have been reported in previous works for testing the nonlocality of the single-photon wave function [19, 20, 22, 23, 24]. In this paper, we would like to show the wave-particle duality is inherent in single-photon entanglement. Equipped with the technique of wave state detection given above, we derive the joint probability distribution through the wave state measurement for single-photon entanglement, which may provide the clues for the conjugated behaviors for the wave state detection and photon number detection. The reference WC states possessed by Alice and Bob are \( |\gamma_A e^{i\alpha}\rangle \) and \( |\gamma_B e^{i\beta}\rangle \), respectively. To maximize the resolution of the interference pattern, we first fix \( \gamma_A = \gamma_B = \gamma \). According to the definition of wave state measurement in figure 2 and the detection device for single-photon entanglement in figure 1, for the reference wave states of \( |\gamma_A e^{i\alpha}\rangle \) and \( |\gamma_B e^{i\beta}\rangle \), used by Alice and Bob, the wave states detected in single-photon entanglement are \( \alpha + \frac{\pi}{2} \) and \( \beta + \frac{\pi}{2} \) at ports \( A_1 \) and \( B_1 \), and \( \alpha - \frac{\pi}{2} \) and \( \beta - \frac{\pi}{2} \) at ports \( A_2 \) and \( B_2 \) respectively. By overlapping two reference WC states with the single-photon entangled state \( |\Psi_{A,B}\rangle \) in the beam splitters by two spatially separated parts, Alice and Bob, the joint state \( |\Psi\rangle \) is

\[
|\Psi\rangle = |\gamma_A e^{i\alpha}\rangle \gamma_B e^{i\beta} |\Psi_{A,B}\rangle
\]

\[
\approx \frac{1}{2} [ie^{i\beta} |1_A\rangle + i|1_A\rangle] + [|1_B\rangle + i|1_B\rangle] \\
+ \frac{\sqrt{2}}{4} \gamma e^{i(\alpha+\beta)} [|1_A\rangle + i|1_A\rangle][i|1_B\rangle + |1_B\rangle] \\
+ \frac{\sqrt{2}}{4} \gamma e^{i\alpha} [|1_A\rangle + |1_A\rangle][i|1_B\rangle + i|1_B\rangle] \\
+ \frac{\sqrt{2}}{4} \gamma e^{i\beta} [|1_B\rangle + i|1_B\rangle][i|1_A\rangle + |1_A\rangle] \\
+ O(\gamma).
\]

(7)

Here we mainly focus on the quantum correlation between A and B, so only the coincidence counts between them are considered. According to equation (4), a part of the counting in the wave state detection comes from two-photon interference, which is insensitive to phase difference. For the WC state as the reference wave state, the contribution of this part to coincidence counting is in the order of \( \frac{\gamma^2}{4} \). In consideration of \( |\gamma|^2 \ll 1 \), this part of coincidence count can be ignored. By setting \( \alpha' = \alpha + \frac{\pi}{2} \) and \( \beta' = \beta + \frac{\pi}{2} \), the joint probabilities between wave states detected at the four single-photon detectors can be approximated to

\[
p(A_1, B_1) = \gamma^2 \left[ 1 + \cos (\alpha' - \beta' - \phi) \right]
\]

\[
p(A_1, B_2) = \gamma^2 \left[ 1 - \cos (\alpha' - \beta' - \phi) \right]
\]

\[
p(A_2, B_1) = \gamma^2 \left[ 1 - \cos (\alpha' - \beta' - \phi) \right]
\]

\[
p(A_2, B_2) = \gamma^2 \left[ 1 + \cos (\alpha' - \beta' - \phi) \right].
\]

(8)

The remaining counts are phase-insensitive, regardless of single counts or coincidence counts on the same side. In equation (8), the phase \( \phi \) arises from the optical path difference between the two path modes \( A \) and \( B \) propagating from the light source to Alice and Bob, and its value can be considered as a constant once the whole light path is stable. The coincidence counts between the single-photon detectors owned by Alice and Bob is a cosine function of the phase difference of the wave states between them. Also the visibility of the joint probability curve reaches the maximum value of 1 under the condition that the average photon number of the WC state \( \gamma^2 \) is far below than 1. Thus the derivation of joint probability in wave space gives the wave-particle duality in one-photon entanglement accurately from another aspect.

By removing the single-photon counts that do not contribute to the coincidence counts, from equation (8), it is can be seen that the count rate of each single-photon detector, which is the marginal probability of \( p(A_j, B_j) \) with \( i,j = 1,2 \), is a constant

\[
p(A_1) = p(A_2) = p(B_1) = p(B_2) = \frac{\gamma^2}{4}.
\]

(9)
Thus it is can be seen that the correlation between the wave state measurements of two far separated parts can be regarded as the first order coherence, which is a common feature of quantum correlation with maximum entanglement. The completely different behaviors between the single-detector count rate and the coincidence count rate also demonstrate the quantum nonlocality of a single-photon state.

To give a conclusive conjugation between wave behavior and particle behavior, we need to verify the existence of single photon entanglement of particle behavior by using the joint probability derived from wave space. This can be done by testing the violation of the Bell’s inequality based on the joint probability measurement of wave states in equation (9). The most commonly used CHSH-type Bell’s inequality for two-dimensional entanglement can be written as [4, 28]

\[ S = |E(\alpha_1', \beta_1') + E(\alpha_1', \beta_2') + E(\alpha_2', \beta_1') - E(\alpha_2', \beta_2')| \leq 2, \]

where \(\alpha_{1(2)}'\) and \(\beta_{1(2)}'\) represent the wave states in the single-photon entanglement detected by Alice and Bob, respectively; \(E(\alpha_{1, i}', \beta_{1, j}')\) is the correlation function

\[ E(\alpha_{1, i}', \beta_{1, j}') = P(A_1, B_i) + P(A_2, B_j) - P(A_1, B_j) - P(A_2, B_i). \]

Here \(P(A_{i, j})\) is the normalized joint probability

\[ P(A_{i, j}) = \frac{p(A_1, B_i)}{\sum_{i=1}^{2, 2} \sum_{m=1}^{2, 2} p(A_m, B_n)}, \]

where the subscripts \(i, j, m, n\) represent the indices of the single-photon detectors. If we assume the phase difference between the modes propagating to Alice and Bob \(\phi = 0\), by setting \(\alpha_1' = 0, \alpha_2' = \frac{\pi}{2}\) and \(\beta_1' = \frac{\pi}{4}, \beta_2' = -\frac{\pi}{4}\), a result of \(S = 2\sqrt{2}\) can be almost obtained. The violation of Bell’s inequality in equation (8) demonstrate not only the existence of delocalized single-photon entanglement but also the wave-particle duality in single-photon entanglement.

It should be noted that the post-selection method is indispensible for the test of single-photon entanglement test based on wave states detection, where the single detector clicks and triple detector clicks events are all omitted. When the intensity of the reference WC state satisfies \(|\gamma| \ll 1\), the probability of multi-detector click events approaches to 0. In addition, although single detector clicks account for most of the detection events, it is also abandoned because it does not contribute to coincidence counting. Therefore, we can find that the entanglement of the wave states is different from the standard electron-spin based entanglement [29] and the photon polarization [30] and orbital angular momentum based entanglement [31]. This is because a wave state contains a vacuum state, and the final detection of the wave state is still by means of a single-photon detector, which is of particle nature. In the future, perhaps a detection method that is completely aimed at wave states can solve this difference. Considering that the loophole free Bell’s inequality test experiments eliminate all LHV theories [29, 30, 32, 33], the wave state entanglement can be assured even though the post-selection method is used in our scheme.

Through the construction of wave state entanglement, we can find that the wave state is not only a quantum state that shows different behaviors from the particle state, it may also be used as a carrier of quantum information. The most typical example is quantum key distribution (QKD) protocol. The original QKD schemes are all based on the single-photon state, such as BB84-QKD [34], measurement device independent (MDI)-QKD [35] and device independent (DI)-QKD [36], where the information carrier is of particle character. Until the WC state based twin field (TF)-QKD scheme is proposed [26], in which the information carrier is of wave character. In addition, one of our previous work has established the relationship between BB84-QKD and TF-QKD from the perspective of single-photon entanglement [37], which is the relationship between particle property and wave property.

3. Discussion and conclusion

Now, we would like to point out the difference between the single-photon entanglement test based on a Bell-like inequality, like phase correlation [19], Wigner distribution correlation [20, 23], quadrature correlation [22, 24] and wave state correlation in our paper. The similarity of these methods is that single-photon quantum correlation is detected by interfering the path modes with local coherent states. Here we only focus on the different manifestations of quantum correlation, regardless of the specific forms of Bell’s inequality in different methods. Because wave states are obtained by the Fourier transform of the particle states characterized by the phase gradient, thus the quantum correlation between far separated wave states can be well exhibited by the quantum correlation between the phases of two local oscillators [19]. In fact, this phenomenon can not be interpreted as the quantum correlation between the phases of the two far separated local oscillators, as they are independent of each other. It is the quantum correlation between the wave states that leads to this novel phenomenon. The phase measurement of local oscillator can also be represented by quadrature measurement of
The electromagnetic field in homodyne detection [38], thus another representation of phase correlation can also be represented by quadrature correlation [22]. Furthermore, the Wigner distribution of a quantum state can be measured by the quadrature measurement of a homodyne detection [39–44], thus the quadrature correlation will lead to the Wigner distribution correlation as well as correlation of the density of states [20]. The difference between these methods lie in that entanglement is carried by different types of carriers. Through wave state measurement proposed in this paper, we can better understand the relationship between wave entanglement and particle entanglement in a single-photon entanglement.

In summary, we have proposed a method to demonstrate the wave-particle duality in a single-photon entanglement by constructing the CHSH-type Bell’s inequality based on the joint measurement of wave states. By Fourier transform, the entanglement of single-photon in Fock space is transformed into the wave space, which has the opposite diagonalized form. Our results show that the entanglement is not only restricted to multi-particle systems and the carrier of the entanglement is not necessarily a single particle. The single-photon entanglement based on Bell-like equality points out the intrinsic relationship between the nonlocality of wave function and that of quantum entanglement. The introduction of wave state adds a new degree of freedom to the quantum system, which can be used as a new information carrier in the field of quantum information. The definition and detection method of a wave state presented in this paper can be extended to any other quantum single-particle systems.

Acknowledgments

The authors declare no conflicts of interest. This work is supported by China Postdoctoral special funding project (2020T130289), the National Natural Science Foundation of China (No. 61 871 234).

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References

[1] Pan J-W, Chen Z-B, Lu C-Y, Weinfurter H, Zeilinger A and Żukowski M 2012 Rev. Mod. Phys. 84 777
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[3] Brunner N, Cavalcanti D, Pironio S, Scarani V and Wehner S 2014 Rev. Mod. Phys. 86 419
[4] Clauser J F, Horne MA, Shimony A and Holt R A 1969 Phys. Rev. Lett. 23 880
[5] Collins D, Gisin N, Linden N, Massar S and Popescu S 2002 Phys. Rev. Lett. 88 040404
[6] Jin R B, Saito T and Shimizu R 2018 Physical Review Applied 10 034011
[7] Tan S, Walla D and Collett M 1991 Phys. Rev. Lett. 66 232
[8] Enk S Van 2005 Phys. Rev. A 72 064306
[9] Enk S Van 2006 Phys. Rev. A 74 026302
[10] Wildfeuer C F, Lund A P and Dowling J P 2007 Phys. Rev. A 76 052101
[11] Sangouard N, Simon C, Minář J, Zbinden H, De Riedmatten H and Gisin N 2007 Phys. Rev. A 76 050301
[12] Yin J O and van Enk S 2008 Phys. Rev. A 77 062333
[13] Sangouard N, Simon C, Coudreau T and Gisin N 2008 Phys. Rev. A 78 050301
[14] Salart D et al 2010 Phys. Rev. Lett. 104 180504
[15] Wildfeuer C F and Dowling J P 2008 Phys. Rev. A 78 032113
[16] Guerreiro T et al 2016 Phys. Rev. Lett. 117 070404
[17] Di Fidio C and Vogel W 2009 Phys. Rev. A 79 050303
[18] Brask J B, Chaves R and Brunner N 2013 Phys. Rev. A 88 012111
[19] Banaszek K and Wodkiewicz K 1999 Phys. Rev. Lett. 82 2009
[20] Babichev S, Appel J and Ivlevsky A 2004 Phys. Rev. Lett. 92 193601
[21] D’Angelo M, Zavatta A, Parisi V and Bellini M 2006 Phys. Rev. A 74 052114
[22] Morin O, Bancal J-D, Ho M, Sekatski P, D’Auria V, Gisin N, Laurat J and Sangouard N 2013 Phys. Rev. Lett. 110 130401
[23] Fuwa M, Takeda S, Zwierz M, Wiseman H M and Furusawa A 2015 Nat. Commun. 6 6665
[24] Ho M, Morin O, Bancal J-D, Gisin N, Sangouard N and Laurat J 2014 New J. Phys. 16 103035
[25] Li W and Zhao S 2018 Sci. Rep. 8 4812
[26] Lucamarini M, Yuan Z L, Dykes J F and Shields A J 2018 Nature 557 400
[27] Ma X, Zeng P and Zhou H 2018 Phys. Rev. X 8 031043
[28] Yarnall T, Abouraddy A F, Saleh B E and Teich M C 2007 Phys. Rev. Lett. 99 170408
[29] Hensen B et al 2015 Nature 526 682
[30] Giustina M et al 2015 Phys. Rev. Lett. 115 250401
[31] Jack B, Leach J, Romero J, Franke-Arnold S, Kitsch-Marte M, Barnett S and Padgett M 2009 Phys. Rev. Lett. 103 083602
[32] Abellán C, Amaya W, Mitran D, Pruneri V and Mitchell M W 2015 Phys. Rev. Lett. 115 250403
[33] Shalm L K et al 2015 Phys. Rev. Lett. 115 250402
[34] Bennett C H and Brassard G 1984 Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India, IEEE New York 175–9
[35] Lo H-K, Curty M and Qi B 2012 Phys. Rev. Lett. 108 130503
[36] Acín A, Brunner N, Gisin N, Massar S, Pironio S and Scarani V 2007 Phys. Rev. Lett. 98 230501
[37] Li W, Wang L and Zhao S 2019 Sci. Rep. 9 1
[38] Vogel K and Risken H 1989 Phys. Rev. A 40 2847
[39] Cahill K E and Glauber R J 1969a Phys. Rev. 177 1882
[40] Cahill K E and Glauber R J 1969b Phys. Rev. 177 1857
[41] Yuen H and Shapiro J 1980 IEEE Trans. Inf. Theory 26 78
[42] Yuen H P and Chan V W 1983 Opt. Lett. 8 177
[43] Schumaker B L 1984 Opt. Lett. 9 189
[44] Yurke B and Stoler D 1987 Phys. Rev. A 36 1955