An Affine Linear Solution for the 2-Face Colorable Gauss Code Problem in the Klein Bottle and a Quadratic System for Arbitrary Closed Surfaces

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Abstract

Let $\mathcal{P}$ be a sequence of length $2n$ in which each element of $\{1, 2, \ldots, n\}$ occurs twice. Let $P'$ be a closed curve in a closed surface $S$ having $n$ points of simple auto-intersections, inducing a 4-regular graph embedded in $S$ which is 2-face colorable. If the sequence of auto-intersections along $P'$ is given by $\mathcal{P}$, we say that is a $P'$ 2-face colorable solution for the Gauss Code $\mathcal{P}$ on surface $S$ or a lacet for $\mathcal{P}$ on $S$. In this paper we present a necessary and sufficient condition yielding these solutions when $S$ is Klein bottle. The condition take the form of a system of $m$ linear equations in $2n$ variables over $\mathbb{Z}_2$, where $m \leq n(n - 1)/2$. Our solution generalize solutions for the projective plane and on the sphere. In a strong way, the Klein bottle is an extremal case admitting an affine linear solution: we show that the similar problem on the torus and on surfaces of higher connectivity are modelled by a quadratic system of equations.

Keywords: Gauss code problem, lacets, closed surfaces, 4-regular graphs, medial maps (of graphs on surfaces), face colorability

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1 Introduction

A Gauss code $\mathcal{P}$ is a cyclic sequence in the set of labels $E = \{1, 2, \ldots, n\}$ in which each $x \in E$ occurs twice. Let $P'$ be a closed curve in a closed surface $S$ having $n$ points of simple auto-intersections, inducing a 4-regular graph embedded into $S$ such that the cyclic sequence of
auto-intersections reproduces $\mathcal{P}$. We say that $P'$ realizes $\mathcal{P}$ in $S$ and it is called a *lacet* for $\mathcal{P}$. If the embedding of $P'$ produces a 2-face colorable map, then $P'$ is called a *2-colorable lacet* for $\mathcal{P}$. Without the 2-face colorability condition the algebra derived from maps with a single zigzag [6] is not available and an entirely different problem arises. Here we consider only 2-colorable lacets. In this case, $P'$ is the *medial map* ([5]) of a map $M$ formed by a graph $G_M$ embedded into $S$. $M$ has a single zigzag [7]. The dual of $M$ is denoted $D$ and its phial ([7]) is denoted $P$. $G_P$, the graph of $P$, has a single vertex (corresponding to the single zigzag).

Previous work on the Gauss code problem can be found in [13], [11], [12], [9], [6] and [8]. The last two works solve the 2-face colorable problem for the case of the projective plane. The previous works deal with the planar case in which the 2-face colorability is granted. In the present work we algorithmically solve the problem for the Klein bottle.

This problem has been recently tackled by in [2]. In this paper they introduce the terminology *lacet* meaning 2-colorable lacets and develop a theory which has its origin in the basic algebraic fact appearing in [6] and in [8] connecting the surface $S$ and the intersection of the cycle spaces of $G_M$ and $G_D$. They find conditions for their realization in the torus (Theorem 19) and in the Klein bottle (Theorem 20) in terms of the the existence of a pair of $0$ $-$ $1$ vectors with certain properties. However, the verification of the existence of these vectors is left undiscussed. In fact, to verify their existence leads to an exponential number of trials. So the theorems do not provide polynomial algorithms of even good characterizations (in the sense of Edmonds) for realizability. In this technical sense the problems are not solved in [2].

For the the case of the Klein bottle it is solved here. It remains open for the case of $\mathbb{S}^1 \times \mathbb{S}^1$. In a strong way, the Klein bottle is an extremal case admitting an affine linear solution: we show that the similar problem on the torus and on surfaces of higher connectivity are modelled by a quadratic system of equations. We also provide a way for finding the smallest connectivity of a surface realizing a Gauss code as a lacet in terms of deciding whether or not a quadratic system of equations over $\mathbb{Z}_2$ is consistent. We also provide a way for finding the smallest connectivity of a surface realizing a Gauss code as a lacet in terms of deciding whether or not a quadratic system of equations over $\mathbb{Z}_2$ is consistent.

The paper is organized as follows. In Section 2 we show an example to help the reader in understanding our definitions and motivation. This example is used throughout the paper. In Section 3 we give the statement of the main results. Section 4 briefly reviews the theory of Combinatorial Maps as given in [7] to prove the *Parity Theorem* [6], which is needed in this work as a Lemma. In Section 5 we prove some basic lemmas and the main results. Section 6 deals with 2-face colorable lacets in general surfaces. We prove that this general problem is modelled as a quadratic system of equations over $\mathbb{Z}_2$. Finally, short Sections 7 and 8 are concluding remarks and acknowledgements.
2 An example and the algebraic tools

Consider the example of a kleinian map \( M \) with a single zigzag given on the left of Fig. 1. The cyclic sequence of edges visited in the zigzag is

\[
P = (1, 4, 5, 6, 5, 4, 3, 8, 7, 3, 2, -1, -2, 8, 7, -6).
\]

This can be easily followed in the 2-colorable lacet \( P' \). We think of \( P' \) as the medial map of \( M \). The direction of the first occurrence of an edge of \( G_M \) defines the orientation. Edges 3,4,5,7,8 are traversed twice in the positive direction (they correspond to black circles in the medial map) and edges 1,2,6 are traversed once in the positive direction and once in the negative direction (they correspond to white circles). The reason for the notation \( P \) is that the signed cyclic sequence defines the phial map \( P \) (whence also \( M, D \) and \( P' \), as well as the surface of \( M \)) and vice-versa, the phial defines the sequence. For the algebraic concepts we refer to [4]. For the graph terminology to [1] and to [5]. For more background on graphs embedded into surfaces we refer to [3].

Let \( c_p = \kappa_p + i_p \) and \( \mathcal{V}^\perp, \mathcal{F}^\perp \) the cycle spaces of \( G_M \) and \( G_D \), respectively. It is easy to verify that \( c_p(x) \) is the set of edges occurring once in a closed path in \( G_M \). Therefore, \( c_p(x) \in \mathcal{V}^\perp \). In the above figure we see that \( c_p(1) = \emptyset \cup \{2, 8, 7, 6\}, c_p(3) = \{3\} \cup \{8, 7\} \) and, indeed, \( \{2, 8, 7, 6\} \) and \( \{3, 8, 7\} \) are in \( \mathcal{V}^\perp \). From the definitions, it follows that if \( P \) has a single vertex, for any \( x, \kappa_P(x) + \kappa_{P'}(x) = \{x\} \) and that \( c_{P'}(x) + c_p(x) = \{x\} \) and so, \( c_{P'} + c_p \) is the identity linear transformation.

Let \( b_p = c_{P'} \circ c_p = c_p \circ c_{P'} \). The image of \( b_p \) is \( \mathcal{V}^\perp \cap \mathcal{F}^\perp \). Moreover the dimension of this subspace is the connectivity of \( S \). These facts were first proved in [6]. They also appear in
For the example of Fig. 1 the parts are Shank[1987]. For the Klein bottle the proof is given in Section 5.

If \( S \) is the Klein bottle, then \( \dim(\text{Im}(b_P)) = 2 = \xi(S) \). Since the underlying field is \( \mathbb{Z}_2 \), there are at most 4 distinct values in the image of \( b_P \). Indeed we can be more specific. Let \( \mathcal{O} = \{ x \in E \mid |i_T(x)| \text{ is odd} \} \) and \( \mathcal{E} \) be the complementary subset of edges.

**Proposition 1 (Main Lemma)** If \( P' \) is the 2-colored lacet for \( T \) which is in the 2-sphere, \( S^2 \), in the real projective plane, \( \mathbb{R}P^2 \), or in the Klein bottle, \( S^1 \times S^1 \), then there are partitions \((\mathcal{O}_0,\mathcal{O}_1)\) of \( \mathcal{O} \) and \((\mathcal{E}_0,\mathcal{E}_1)\) of \( \mathcal{E} \) satisfying

\[
b_P(x) = \begin{cases} \mathcal{O}_0 \cup \mathcal{E}_1, & \text{if } x \in \mathcal{O}_0 \\ \mathcal{O}_1 \cup \mathcal{E}_1, & \text{if } x \in \mathcal{O}_1 \\ \emptyset, & \text{if } x \in \mathcal{E}_0 \\ \mathcal{O}, & \text{if } x \in \mathcal{E}_1. \end{cases}
\]

Moreover, if \( P' \) is in the Klein bottle, then \( \mathcal{O}_0 \neq \emptyset \) and at most one of \( \mathcal{O}_1 \) and \( \mathcal{E}_1 \) is empty.

This proposition is already proved for the cases of \( S^2 \) and \( \mathbb{R}P^2 \): Lins[1980], Lins, Richter and Shank[1987]. For the Klein bottle the proof is given in Section 5.

For the example of Fig. 1 the parts are \( \mathcal{E}_0 = \{3\}, \mathcal{E}_1 = \{1\}, \mathcal{O}_0 = \{2, 6, 7, 8\}, \mathcal{O}_1 = \{4, 5\} \).

**Proposition 2 (Main Theorem)** Let \( T \) be a given Gauss code with \( E = \{1, 2, \ldots, n\} \). For some \( m \leq n(n-1)/2 \), there exists an \((m \times 2n)\)-matrix \( L = L(T) \in \mathbb{Z}_2^{m \times 2n} \) (computable from \( T \)) and a column \( 2n \)-vector \( r = r(T) \in \mathbb{Z}_2^{2n} \) (also computable from \( T \)) such that the following characterization holds: each 2-face colorable realization of \( T \) in \( S^2 \), in \( \mathbb{R}P^2 \) or in
\( S^1 \times S^1 \) corresponds in an \( 1-1 \) way to a solution \( \xi \in \mathbb{Z}_2^n \) of the linear system \( L\xi = r \). If the code is not realizable in these surfaces, then there exists an \( m \)-column vector \( \nu \in \mathbb{Z}_2^m \) such that \( \nu^T L = 0 \) and \( \nu^T r = 1 \).

The proof of the Main Theorem is also postponed to Section 5.

4 Combinatorial maps and a Parity Theorem

A topological map \( M^t = (G, S) \) is an embedding of a graph \( G \) into a closed surface \( S \) such that \( S \setminus G \) is a collection of disjoint open disks, called faces. By going around the boundary of a face and recalling the edges traversed we define a facial path of \( M^t \), which is a closed path in \( G \). Note that a facial path is obtained starting in an edge and by choosing at each vertex always the rightmost or always the leftmost possibility for the next edge. If we alternate the choice, then the result is a zigzag path, or simply a zigzag. Even if the surface is non-orientable these left-right choices are well defined, because they are local. For more background on graphs embedded into surfaces see [3]. To make our objects less dependent on topology we use a combinatorial counterpart for topological maps introduced in [7]. A combinatorial map or simply a map \( M \) is an ordered triple \((C_M, v_M, f_M)\) where: (i) \( C_M \) is a connected finite cubic graph; (ii) \( v_M \) and \( f_M \) are disjoint perfect matchings in \( C_M \), such that each component of the subgraph of \( C_M \) induced by \( v_M \cup f_M \) is a polygon (i.e. a non-empty connected subgraph with all the vertices having two incident edges) with 4 edges and it is called an \( M \)-square.

From the above definition, it follows that \( C_M \) may contain double edges but not loops. A third perfect matching in \( C_M \) is \( E(C_M) - (v_M \cup f_M) \) and is denoted by \( a_M \). The set of diagonals of the \( M \)-squares, denoted by \( z_M \), is a perfect matching in the complement of \( C_M \). The edges in \( v_M, f_M, z_M, a_M \) are called respectively \( v_M \)-edges, \( f_M \)-edges, \( z_M \)-edges, \( a_M \)-edges. The graph \( C_M \cup z_M \) is denoted by \( Q_M \), and is a regular graph of valence 4. A component induced by \( a_M \cup v_M \) is a polygon with an even number of vertices and it is called a \( v \)-gon. Similarly, we define an \( f \)-gon, and a \( z \)-gon, by replacing \( v \) for \( f \) and \( v \) for \( z \). Clearly, the \( f \)-gons and \( z \)-gons of \( C_M \) correspond to the facial paths and the zigzags of \( M^t \). To avoid the use of colors the \( M \)-squares are presented in the pictures as rectangles in which the short sides \((s)\) are \( v_M \)-edges, the long sides \((\ell)\) are \( f_M \)-edges and the diagonals \((d)\) are \( z_M \) edges. An \( M \)-rectangle with diagonals or simply an \( M \)-rectangle (being understood that the diagonals are present) is a component induced by \( v_M, f_M, z_M \). The set of \( M \)-rectangles is denoted by \( R \). If \( \pi \) is a permutation of the symbols \( sld \), and \( R' \subseteq R \) subset of rectangles of \( M \), then \( M(R' : \pi) \) denotes the map obtained from \( M \) by permuting the short sides, the long sides and the diagonals according to \( \pi \) in all \( r \subset R \). Let \( M_{r} \) denote \( M(\{r\} : \pi) \). The dual map of \( M \) is the map \( D = M(R : \ell sd) \); \( D \) and \( M \) have the same \( z \)-gons and the \( v \)-gons and \( f \)-gons interchanged. The phial map of \( M \) is the map \( P = M(R : dls) \); \( P \) and \( M \) have the same \( f \)-gons and the \( v \)-gons and \( z \)-gons interchanged. The anti-map of \( M \) is the map \( M^\sim = M(R : sdl) \); \( M \) and \( M^\sim \) have the same \( v \)-gons and the \( f \)-gons and \( z \)-gons interchanged. The pairs \((M, D), (M, P), (M, M^\sim)\) constitute the map dualities introduced in [Lins, 1982].
The dual of $P \circ \Omega$ from the set of $R$ from the set of $M$ from the set of maps onto the set of $E \circ \Omega$. Given a map $M$, disjoint closed disks. Each edge of $C_M$ occurs twice in the boundary of this collection of disks. Identify the collection of disks along the two occurrences of each edge. The result is a closed surface and $C_M$ is faithfully embedded on it, meaning that the boundaries of the faces are bicolored polygons or bigons. Similarly, there are surfaces $\text{Surf}(D^\sim, P)$ and $\text{Surf}(P^\sim, M^\sim)$.

We define a function $\psi$ which turns out to be a bijection from the set of maps onto the set of $t$-maps. We denote $\psi(M)$ by $M^t$. Given a map $M$, to obtain $M^t$ we proceed as follows. Consider the $t$-map $(C_M, S)$, where $S = \text{Surf}(M, D)$, given by the faithful embedding of $M$. The $v$-gons, the $f$-gons and the $M$-squares are boundaries of (closed, in this case) disks embedded (and forming) the surface $S(M)$. Shrink to a point the disjoint closed disks bounded by $v$-gons. The $M$-squares, then, become bounding digons. Shrink each such bounding digon to a line, maintaining unaffected its vertices. With these contractions, effected in $S$, $t$-map $(C_M, S)$ becomes, by definition, $M' = (G_M, S)$. Graph $G_M$ is called the graph induced by $M$. A combinatorial description of $G_M$ can be given as follows: the vertices of $G_M$ are the $v$-gons of $M$; its edges are the squares of $M$; the two ends of an edge of $G_M$ are the two $v$-gons (which may coincide and the edge is a loop) that contain the $v_M$-edges of the corresponding $M$-square. It is evident that $\psi$ is inversible: given a $t$-map we replace each edge by a bounding digon in its surface, and then expand each vertex to a disc in order to obtain a cellular embedding of a cubic graph. Therefore, $\psi^{-1}$ is well-defined; in fact, it is the dual of a useful construction in topology, namely, barycentric division. Thus, $\psi$ is a bijection from the set of maps onto the set of $t$-maps. It can be observed that $\psi$ induces a bijection from the set of $M$-rectangles onto the set of edges of $G_M$. We use this bijection to identify the sets $R$ and $E(G_M)$. Via $R$, which is invariant for the members of $\Omega^*(M)$, we identify $E(G_M)$ and $E(G_M')$ for $M' \in \Omega^*(M)$. Denote these identified sets of edges by $E$.

Consider the function $\psi^M_v$, from the cycle space of $C_M$, $CS(C_M)$, onto the cycle space of $G_M$, $CS(G_M) = V^\perp$. It is defined as follows: for $S \in CS(C_M)$, an edge $s \in E$ is in $\psi^M_v(S)$ if each square $s \in SQ(M)$ meets $S$ in exactly one $f_M$-edge. With this definition, it is evident that $\psi^M_v(S)$ is a cycle in $G_M$ and that $\psi^M_v$ is surjective.

**Proposition 3 (Lins 1980)** $\psi^M_v$ is a homomorphism. Its kernel is the subspace of $CS(C_M)$ generated by the $v$-gons and the squares of $M$. 
Proof: Let $S_1$ and $S_2$ be cycles in $C_M$. We must show that $\psi_c^M(S_1 + S_2) = \psi_c^M(S_1) + \psi_c^M(S_2)$. An edge $e$ of $G_M$ is in $\psi_c^M(S_1 + S_2)$ if, and only if, exactly one $f_M$-edge of square $e \in SQ(M)$ is in $S_1 + S_2$. Therefore, $e \in \psi_c^M(S_1 + S_2)$ if, and only if, an even number of $f_M$-edges of square $e$ belongs to a fixed member of the set $\{S_1, S_2\}$, and one $f_M$-edge of square $e$ belongs to the other. The latter statement is equivalent to $e \in \psi_c^M(S_1) + \psi_c^M(S_2)$. This proves that $\psi_c^M$ is homomorphism. The image under $\psi_c^M$ of any square or any $v$-gon of $M$ is the null cycle in $G_M$. Thus, the space generated by the squares and $v$-gons is contained in $\text{Ker}(\psi_c^M)$. Conversely, suppose that $S \in \text{Ker}(\psi_c^M)$. The intersection of $S$ with the edges of an arbitrary square has zero or two $f_M$-edges. Denote by $T$ the cycle formed by the union of the squares $Q \in SQ(M)$ such that $eQ \cap S$ contains the two $f_M$-edges of $Q$. It follows that $S + T$ has no $f_M$-edges. Since $S + T$ is the edge set of a collection of polygons, $(C_M$ is cubic), it follows that $S + T$ is formed by a collection of $v$-gons whose edge set we denote by $U$. Hence, $S = T + U$, with $T$ induced by squares, $U$ induced by $v$-gons. Therefore, we conclude that $\text{Ker}(\psi_c^M)$ is contained in the space generated by the edge sets of squares and $v$-gons of $M$. The proof is complete.

Since an element of the kernel of $\psi_c^M$ has an even number of edges of $C_M$, it follows that if $\psi_c^M(S_1) = \psi_c^M(S_2)$, then $|S_1| \equiv |S_2| \mod 2$. This observation makes the following definition meaningful. A cycle $S$ in $C_M$ is called an $r$-cycle in $M^t$ if $\psi_c^M(S') = S$ and $|S'|$ is odd, for some cycle $S'$ in $C_M$. If $|S'|$ is even and $\psi_c^M(S') = S$, then we say that $S$ is an $s$-cycle in $M^t$. $r$-Circuits are minimal $r$-cycles. We observe that the $r$-circuits in $M^t$ are precisely the orientation-reversing polygons in $M^t$. This topological notion is not used; we work with our parity definition of $r$-cycle. Observe that a subset $T \subseteq E$ is a boundary in $M^t$ if, and only if, there exists a cycle $T'$ of $C_M$, such that $\psi_c^M(T') = T$, and $T'$ can be written as the mod 2 sum of some subsets of $v$-gons, $f$-gons, and squares. For a map $M$, two cycles in $C_M$ are homologous mod 2 if their symmetric difference is a boundary in $M^t$. Homology mod 2 is, thus, an equivalence relation. It follows from Lemma 3.5.b of [6] applied to $c_{p-}$- shows that $b_P(x) = b_P(y)$ if, and only if, the cycles $c_P(x)$ and $c_P(y)$ are homologous mod 2.

The following proposition shows that the type of $c_P(x)$ depends only on the parity of $\iota(M(x))$. It is an important tool on lacet theory.

**Proposition 4 (Parity Theorem - [6])** If $M$ is a map with a single $z$-gon, then $c_P(x)$ is an $s$-cycle in $M^t$ if and only if $|\iota(M(x))|$ is even.

Proof: To prove the result, we define a function $c'_P$ on the vertices of $C_M$ whose image is $CS(C_M)$. For a vertex $x$ of $C_M$, the cycle $c'_P(x)$ is defined by the edges of $C_M$ occurring once in the reentrant path which starts at $z_M(X)$ and proceeds by using $a_M$, $v_M$, and $f_M$-edges (in this order) until it reaches another vertex of $X$, which denotes the square to which $x$ belongs. This vertex can be $v_M(X)$, in which case we close the path by using the $f_M$-edge linking $v_M(X)$ to $z_M(X)$; it also can be $f_M(X)$, in which case we close the path by using the $v_M$-edge which links $f_M(X)$ to $z_M(X)$. The fact that $x$ is a loop in $G_P$ and a parity argument show that the first vertex of $x$ reached by the path is not $X$, and is not $z_M(X)$. This completes the definition of $c'_P$. Observe that $\Psi_c^M(c'_P(X')) = c_P(x)$ for any vertex.
X′ of the square of M containing X, corresponding to edge x of G_M. Since C_M is cubic, the cycle c_p(X) induces a subgraph of C_M which consists of a certain number of disjoint polygons, whose set is denoted by Ω. We count the vertices of these polygons. In the square corresponding to x there are two vertices which are vertices of a polygon in Ω. If a square is met twice by the reentrant path which defines c_p(X), then all its four vertices are vertices of polygons in Ω. If a square is met once by the reentrant path, then three of its vertices are vertices of a polygon in Ω. Evidently, if a square is not met by the reentrant path, none of its vertices is in a polygon in Ω. Hence, the parity of the number of vertices of the polygons in Ω, which is the same as the parity of |c_p(X)|, is equal to the parity of |i_Ψ(x)|. This establishes the theorem.

5 Basic results for the proof of the Main Lemma

Let A ∈ E and e ∈ E. Define s_e(A) = {f ∈ A | κ(f) = κ(e)}.

Proposition 5 Let M be any map with a single zigzag with interlace function i_Ψ. Then, for x ∈ E, b_p(x) = i_Ψ(x) + s_x(i_Ψ(x)).

Proof: b_p(x) = c_p ∩ c_p(x) = [c_p + c_p](x) = [(i_Ψ + κ_p) + (i_Ψ + κ)](x). By expanding we get [i_Ψ^2 + i_Ψκ_p + κ_p + i_Ψ^2 + i_Ψ + κ](x) = [i_Ψ^2 + i_Ψκ_p + κ_p + i_Ψ + i_Ψ](x), since κ^2 = κ. If x is black, then i_Ψκ_p(x) = i_Ψ(x) and [i_Ψ^2 + i_Ψκ_p + κ_p + i_Ψ + i_Ψ](x) = [i_Ψ^2 + κ_p + i_Ψ](x) = i_Ψ(x) + s_x(i_Ψ(x)). If x is white, then [i_Ψ^2 + i_Ψκ_p + κ_p + i_Ψ](x) = [i_Ψ^2 + κ_p + i_Ψ](x) = i_Ψ(x) + s_x(i_Ψ(x)).

Proposition 6 (a) For x, y ∈ E, y ∈ i_Ψ(x) ⇐ i_Ψ(x) ∩ i_Ψ(y) is odd. Therefore x ∈ i_Ψ(y) ⇐ y ∈ i_Ψ(x). (b) For x ∈ E, x ∈ b_p(x) ⇐ x ∈ i_Ψ(x) ⇐ x ∈ O.

Proof: Part (a) is straightforward. For part (b) we have b_p(x) = i_Ψ^2(x) + s_x(i_Ψ(x)). Since x ∈ s_x(i_Ψ(x)) it follows that x ∈ b_p(x) ⇐ x ∈ i_Ψ(x). By part (a), x ∈ i_Ψ(x) ⇐ i_Ψ(x) ∩ i_Ψ(x) = |i_Ψ(x)| is odd ⇐ x ∈ O.

Proposition 7 x ∈ E, y ∈ O ⇒ b_p(x) ≠ b_p(y).

Proof: Assume that b_p(x) = b_p(y). This implies that c_p(x) + c_p(y) = 0. Therefore c_p(x) + c_p(y) ∈ F, by Lemma 3.5.b of [Lins 1980] applied to c_p-. It follows that c_p(x) + c_p(y) is an s-cycle. This is a contradiction since, by Theorem 2, c_p(x) is an s-cycle, c_p(y) is an r-cycle and their sum must be an r-cycle.

Proposition 8 For x, y ∈ E, y ∈ b_p(x) ⇐ x ∈ b_p(y).
Proof: Consider the equivalence $x \in b_P(y) \iff [(x \in i_P^2(y) \text{ and } x \notin s_y(i_P(y))] \text{ or } (x \notin i_P^2(y) \text{ and } x \in s_y(i_P(y)))$. From Proposition 6, $x \in i_P^2(y) \iff y \in i_P^2(x)$. As $x \in i_P(y) \iff y \in i_P(x)$, $x \in s_y(i_P(y)) \iff y \in s_x(i_P(x))$.

Proposition 9 If $P'$ is kleinian and $x, y \in \mathcal{E}, b_P(x) \neq \emptyset \neq b_P(y)$, then $b_P(x) = b_P(y)$.

Proof: If $c_P(z)$ is an $s$-cycle for all $z \in E$, then $C_M$ would be bipartite and $P'$ would not be kleinian. Thus, there exists $z \in \mathcal{O} \neq \emptyset$. Consider the non-null vectors of $\text{Im}(b_P)$: $b_P(x), b_P(y)$ and $b_P(z)$. As $\dim(\text{Im}(b_P)) = 2$ and by Proposition 7, either $b_P(x) = b_P(y)$ or $b_P(z) = b_P(x) + b_P(y)$. We show that the second possibility leads to a contradiction. This possibility implies that $b_P(x), b_P(y)$ and $b_P(z)$ are the 3 distinct non-empty elements of $\text{Im}(b_P)$.

As $z \in b_P(z)$ we may adjust notation and suppose that $z \in b_P(x) \setminus b_P(y)$. Define $X = \{x' \in E \mid b_P(x') = b_P(x)\}, Y = \{y' \in E \mid b_P(y') = b_P(y)\}$ and $Z = \{z' \in E \mid b_P(z') = b_P(z)\}$. The three sets $X, Y$ and $Z$ are non-empty, since $x \in X$, $y \in Y$ and $z \in Z$. Moreover, $X \cup Y \subseteq \mathcal{E}$, because of Proposition 7 and $x, y \in \mathcal{E}$. Also, $Z \subseteq \mathcal{O}$, because $z \in \mathcal{O}$.

We claim that $b_P(y) \cap Z = \emptyset$. If $z' \in b_P(y) \cap Z$, then $z' \notin b_P(x)$. Equivalently by Proposition 8, $x \notin b_P(z') = b_P(z)$, a contradiction because $x \in b_P(z)$. We also claim that $b_P(y) \cap Y = \emptyset$. If $y' \in b_P(y) \cap Y$, then $y \in b_P(y') = b_P(y)$, contradicting Proposition 6. From the 2 claims we may conclude that $b_P(y) \subseteq X$. Let $x' \in b_P(y) \subseteq X$ and $x''$ be an arbitrary element of $X$. We have that $y \in b_P(x') = b_P(x'')$ and so by Proposition 8, $x'' \in b_P(y)$. It follows that $X \subseteq b_P(y)$, so that $b_P(y) = X$.

Next we show that $b_P(x) = Y \cup Z$. We already know that $y, z \in b_P(x)$. Let $y' \in Y$ and $z' \in Z$. We have $y \in b_P(x) \iff x \in b_P(y) = b_P(y') \iff y' \in b_P(x)$ and $z \in b_P(x) \iff x \in b_P(z) = b_P(z') \iff z' \in b_P(x)$. Therefore, $b_P(x) \supseteq Y \cup Z$. Let $W = \{w \in E \mid b(w) = \emptyset\}$, so that $E = W \cup X \cup Y \cup Z$. If $x' \in X \cap b_P(x)$, we get a contradiction with Proposition 6, because $x \in b_P(x') = b_P(x)$. It follows that $b_P(x) \cap (X \cup W) = \emptyset$. Thus, $b_P(x) \subseteq Y \cup Z$ and $b_P(x) = Y \cup Z$.

We show that $b_P(z) = X \cup Z$ as follows. We already know that $x, z \in b_P(z)$. Let $x' \in X$ and $z' \in Z$. We have $x \in b_P(z) \iff z \in b_P(x) = b_P(x') \iff x' \in b_P(z)$ and $z \in b_P(z) = b_P(z') \iff z' \in b_P(z)$. Therefore, $b_P(z) \supseteq X \cup Z$. As $b_P(z) \cap (Y \cup W) = \emptyset$, we get $b_P(z) = X \cup Z = X + Z$. Also, $b_P(z) = b_P(y) + b_P(x) = X + (Y \cup Z) = X + Y + Z$.

These two expressions for $b_P(z)$ imply that $X + Z = X + Y + Z$, or $Y = \emptyset$, which is a contradiction. Thus, the only possibility is $b_P(x) = b_P(y)$.

Proposition 10 If $P'$ is kleinian, $x \in \mathcal{E}$ and $b_P(x) \neq \emptyset$, then $b_P(x) = \mathcal{O}$.

Proof: Let $z \in b_P(x)$. Suppose $z \in \mathcal{E}$. Since $x \in b_P(z)$, $b_P(x)$ and $b_P(z)$ are non-empty. By Proposition 9, $b_P(x) = b_P(z)$. Thus $x \in b_P(z) = b_P(x)$, contradicting Proposition 6.
Therefore $b_P(x) \subseteq \mathcal{O}$. Let $z \in b_P(x) \subseteq \mathcal{O}$ and $z' \in \mathcal{O}\setminus b_P(x)$. Note that $b_P(z) \neq b_P(z')$, since $x \in b_P(z)\setminus b_P(z')$. By Proposition 7, $b_P(x) \neq b_P(z)$ and $b_P(x) \neq b_P(z')$. It follows that $b_P(x), b_P(z)$ and $b_P(z')$ are the 3 non-null vectors in the image of $b_P$. Thus, they satisfy $b_P(x) = b_P(z) + b_P(z')$. From this equality, since $x \in b_P(z)$ and $x \notin b_P(z')$, it follows that $x \in b_P(x)$, a contradiction to Proposition 6, because $x \in \mathcal{E}$. So, $\mathcal{O}\setminus b_P(x)$ must be empty, that is, $b_P(x) = \mathcal{O}$.

\[ \Box \]

Proof of the Main Lemma (Proposition 1): Let $z_0 \in \mathcal{O} \neq \emptyset$. Suppose that $b_P(z_0) = \mathcal{O}_0 \cup \mathcal{E}_1$, with $\mathcal{O}_0 \subseteq \mathcal{O}$ and $\mathcal{E}_1 \subseteq \mathcal{E}$. Note that $z_0 \in \mathcal{O}_0$. Let $\mathcal{O}_1 = \mathcal{O}\setminus \mathcal{O}_0$. If both $\mathcal{O}_1$ and $\mathcal{E}_1$ are empty, then $\text{Im}(b_P)$ would have dimension 1, and we know it is 2. Assume that $\mathcal{E}_1 \neq \emptyset$. By the Proposition 10, for $x \in \mathcal{E}_1$, then $b_P(x) = \mathcal{O}$ and for $x \in \mathcal{E}_0 = \mathcal{E}\setminus \mathcal{E}_1$, $b_P(x) = \emptyset$. The fourth vector in the image of $b_P$ is $b_P(z_0 + x_1) = \mathcal{O}_1 \cup \mathcal{E}_1$, for $x_1 \in \mathcal{E}_1$. Clearly, $b_P(z_1) = \mathcal{O}_1 \cup \mathcal{E}_1, \forall z_1 \in \mathcal{O}_1$.

If $\mathcal{E}_1 = \emptyset$, then $\exists z_1 \in \mathcal{O}_1$. Note that $\{b_P(z_0), b_P(z_1)\}$ is a basis of $\text{Im}(b_P)$. Let $Z_h = \{z \in \mathcal{O} \mid b_P(z) = b_P(z_h), h = 1, 2\}$. The pair $(Z_0, Z_1)$ is a partition for $\mathcal{O}$. We claim that $Z_h \subseteq b_P(z_h)$, $h = 0, 1$: indeed, as $Z_h \subseteq \mathcal{O}$, $z \in Z_h \Rightarrow z \in b_P(z) = b_P(z_h)$. Since $b_P(z_0) \neq b_P(z_1)$, $\exists z'_0 \in b_P(z_0)\setminus b_P(z_1)$ or $\exists z'_1 \in b_P(z_1)\setminus b_P(z_0)$. Note that $z'_0 \in Z_0$, if it exists, because $z'_0 \in Z_1$ leads to a contradiction: $z_1 \in b_P(z_1)$ and $z_1 \notin b_P(z'_0)$. Analogously, $z'_1 \in Z_1$, if it exists. By replacing $z_0$ by $z'_0$ in the first case, or replacing $z_1$ by $z'_1$ in the second and restoring the original notation we may assume that $z_1 \notin b_P(z_0)$. For arbitrary $z'_0 \in Z_0$ and $z'_1 \in Z_1$ we claim that $z'_0 \notin b_P(z'_1)$. If $z'_0 \in b_P(z'_1)$, then $z'_0 \in b_P(z'_1) = b_P(z_1)$ and $z_1 \in b_P(z'_0) = b_P(z_0)$ contradicting what we have established. In consequence, $b_P(z_0) \cap Z_1$ and $b_P(z_1) \cap Z_0$ are both empty. We can conclude that $b_P(z_h) = Z_h = \mathcal{O}_h$, $h = 0, 1$.

\[ \Box \]

Let a pair of variables $\gamma_k, \delta_k \in \mathbb{Z}_2$ be associated with each $k \in E$. The value of $\gamma_k$ is 1 if $k$ is black and is 0 if it is white. The value of $\delta_k$ is $h \in \{0, 1\}$ if $k \in \mathcal{O}_h \cup \mathcal{E}_h$.

Proposition 11 (Linear vectorial equations up to the Klein bottle) Let a Gauss code $\overline{P}$ be given and $E = \{1, 2, \ldots, n\}$. Each 2-face colorable realization of $\overline{P}$ in $\mathbb{S}^2$, in $\mathbb{R}P^2$ or in $\mathbb{S}^1 \times \mathbb{S}^1$ corresponds to a solution $(\delta_1, \ldots, \delta_n, \gamma_1, \ldots, \gamma_n)$ of the following system of $n$ linear vectorial equations over $\mathbb{Z}_2$:

\[
\begin{align*}
k \in \mathcal{O} & \Rightarrow i_\mathcal{O}^2(k) + \sum_{\ell \in i_\mathcal{O}(k)}(1 + \gamma_k + \gamma_\ell)\ell = \sum_{\ell \in \mathcal{E}} \delta_\ell \ell + \sum_{\ell \in \mathcal{O}} (1 + \delta_k + \delta_\ell)\ell \\
k \in \mathcal{E} & \Rightarrow i_\mathcal{E}^2(k) + \sum_{\ell \in i_\mathcal{E}(k)}(1 + \gamma_k + \gamma_\ell)\ell = \delta_k \sum_{\ell \in \mathcal{O}} \ell.
\end{align*}
\]

\[ \Box \]

Proof: Observe that the left hand side of both equations of Proposition 11 yields the value of $b_P(k)$, according to Proposition 5. The value of $\gamma_k$ is 1 if $k$ is black and is 0 if it is white. The value of $\delta_k$ is $h \in \{0, 1\}$ if $k \in \mathcal{O}_h \cup \mathcal{E}_h$. Given partitions $(\mathcal{O}_0, \mathcal{O}_1)$ and $(\mathcal{E}_0, \mathcal{E}_1)$ and this interpretation of the variables $\gamma_k, \delta_k$ in $\mathbb{Z}_2$, Proposition 11 says the same as Proposition 1.
Proof of Proposition 2 (Main Theorem): Let $\xi = (\gamma^T, \delta^T)^T$, where $\delta$ and $\gamma$ are $n$-column vectors in $\mathbb{Z}_2^n$. The rows of $L$ are indexed by the pairs $k\ell = (k, \ell) \in E^2$. The first $n$ columns of $L$ correspond to the variables $(\gamma_1, \ldots, \gamma_n)$. The last $n$ columns of $L$ correspond to the variables $(\delta_1, \ldots, \delta_n)$. A solution $(\gamma, \delta)$ for the Gauss code in those surfaces satisfies, by expanding the vectorial equations given in Proposition 11 to their components, to the following twelve classes of implications:

1. $k \in O, \quad \ell \in O, \quad \ell \in i^2_{\gamma}(k) \cup i^2_{\delta}(k) \Rightarrow 0\gamma_k + 0\gamma_\ell + 1\delta_k + 1\delta_\ell = 0$
2. $k \in O, \quad \ell \in O, \quad \ell \in i^2_{\gamma}(k) \cap i_{\gamma}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 1\delta_k + 1\delta_\ell = 1$
3. $k \in O, \quad \ell \in O, \quad \ell \in i_{\gamma}(k) \cup i_{\delta}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 1\delta_k + 1\delta_\ell = 0$
4. $k \in O, \quad \ell \in E, \quad \ell \in i^2_{\gamma}(k) \cap i_{\gamma}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 0\delta_k + 1\delta_\ell = 1$
5. $k \in O, \quad \ell \in E, \quad \ell \in i^2_{\gamma}(k) \cup i_{\gamma}(k) \Rightarrow 0\gamma_k + 0\gamma_\ell + 0\delta_k + 1\delta_\ell = 0$
6. $k \in E, \quad \ell \in E, \quad \ell \in i^2_{\gamma}(k) \cap i_{\gamma}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 0\delta_k + 1\delta_\ell = 1$
7. $k \in E, \quad \ell \in O, \quad \ell \in i^2_{\gamma}(k) \cup i_{\gamma}(k) \Rightarrow 0\gamma_k + 0\gamma_\ell + 1\delta_k + 0\delta_\ell = 0$
8. $k \in E, \quad \ell \in O, \quad \ell \in i^2_{\gamma}(k) \cap i_{\gamma}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 1\delta_k + 0\delta_\ell = 0$
9. $k \in E, \quad \ell \in O, \quad \ell \in i_{\gamma}(k) \cup i_{\delta}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 1\delta_k + 0\delta_\ell = 1$
10. $k \in E, \quad \ell \in E, \quad \ell \in i^2_{\gamma}(k) \cap i_{\gamma}(k) \Rightarrow 0\gamma_k + 0\gamma_\ell + 0\delta_k + 0\delta_\ell = 0$
11. $k \in E, \quad \ell \in E, \quad \ell \in i^2_{\gamma}(k) \cup i_{\gamma}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 0\delta_k + 0\delta_\ell = 1$
12. $k \in E, \quad \ell \in E, \quad \ell \in i_{\gamma}(k) \cup i_{\delta}(k) \Rightarrow 1\gamma_k + 1\gamma_\ell + 1\delta_k + 0\delta_\ell = 1$

Note that the pairs $(k, \ell)$ which do not appear in the left hand side of the above implications are precisely the ones in which $\ell \notin [i_{\gamma}(k) \cup i^2_{\gamma}(k)]$. These pairs imply no restriction and so it is safe to write the thirteenth class of implications

$I_{13} \quad \ell \notin [i_{\gamma}(k) \cup i^2_{\gamma}(k)] \Rightarrow 0\gamma_k + 0\gamma_\ell + 0\delta_k + 0\delta_\ell = 0,$

producing a partition of $E^2$ into 13 classes. It is now an easy matter to display the unique matrix $L \in \mathbb{Z}_2^{n^2 \times 2n}$ and the unique column vector $r \in \mathbb{Z}_2^{2n}$, such that the $n^2$ implications above (classified in 13 types) are equivalent to the system $L\xi = r$. The only possibility for $(L, r)$ is to have them implying that the $k\ell$-th equation of the system $L\xi = r$ is

$$[(L)_{k\ell,k}] \cdot \gamma_k + [(L)_{k\ell,\ell}] \cdot \gamma_\ell + [(L)_{k\ell,k+n}] \cdot \delta_k + [(L)_{k\ell,\ell+n}] \cdot \delta_\ell = (r)_{k\ell}. $$

Thus, $(L)_{k\ell,p} = 0$, if $p \notin \{k, \ell, k + n, \ell + n\}$. Moreover, if $k\ell$ induces an implication of type $I_q, 1 \leq q \leq 13$, then $(L)_{k\ell,k}, (L)_{k\ell,\ell}, (L)_{k\ell,k+n}, (L)_{k\ell,\ell+n}$ and $(r)_{k\ell}$ coincide with the coefficients in the right hand side of implication $I_q$.

Each solution of the system $L\xi = r$ satisfies all the $n^2$ implications. Conversely, given $(\gamma, \delta)$ satisfying all the $n^2$ implications, $\xi = (\gamma^T, \delta^T)^T$ is a solution of $L\xi = r$. Therefore, if there is
no solution for the set of implications, the system \( L \xi = r \) is inconsistent. In this case, row
operations produce a \( \nu \in \mathbb{Z}_2^n \) such that \( \nu^T L = 0 \) and \( \nu^T r = 1 \), giving a short proof of
the inconsistence: \( L \xi = r \Rightarrow \nu^T L \xi = \nu^T r \Rightarrow 0 = 0 = (\nu^T L) \xi = \nu^T (L \xi) = \nu^T r = 1. \)

In fact we have used with all the \( n^2 \) pairs of elements of \( E^2 \) just for conciseness of the
argument. It is easy to show that the pairs of equal elements \( (k, k) \) induce trivial restrictions
and that \( (k, \ell) \) and \( (\ell, k) \) induce the same restriction. Thus we have at most \( (n^2 - n)/2 \)
restrictions. Of these, each one coming from \( k \ell \) inducing an implication in the class \( I_{13} \) clearly
does not need to be considered. So we need to use only \( m \) of the potential \( n^2 \) equations, where
\( m \leq n(n - 1)/2. \)

As an example, the system of \( n^2 = 8^2 = 64 \) equations for the example in Fig. 1 simplifies,
when the trivial and duplicated equations are discarded, to the following system, consisting
of only 16 equations in the 16 variables \( \gamma_1, \ldots, \gamma_8, \delta_1, \ldots, \delta_8: \)

\[
\begin{array}{cccc}
\delta_1 &=& 1 + \gamma_1 + \gamma_2 & \delta_1 = 1 \\
\delta_1 &=& \gamma_1 + \gamma_8 & \delta_2 + \delta_6 = 0 \\
\delta_3 &=& \gamma_3 + \gamma_7 & \delta_4 + \delta_5 = 0 \\
\delta_5 + \delta_6 &=& \gamma_5 + \gamma_6 & \delta_7 + \delta_8 = 0
\end{array}
\]

It is interesting to observe that the solution is by no means unique. In this case, the dimension
of the solution space is 4. The specific solution corresponding to Fig. 1 is the second on the
left table below. All the sixteen solutions are displayed below. As expected, the solution set
is closed if we interchange black and white vertices: \( \gamma'_k = 1 - \gamma_k \), for all \( k \in E. \)

A basic question about a Gauss code \( \mathcal{P} \), apparently not considered before, is to determine
its \textit{connectivity}, \( \text{conn}(\mathcal{P}) \), defined as the minimum connectivity among the connectivities of
the surfaces of \( \mathcal{P}' \), which realize \( \mathcal{P} \) as a lacet (2-face colorable or not).

Given a Gauss code \( \mathcal{P} \), let \( \gamma \) be any \( 0-1 \) vector indexed by \( E. \) Vector \( \gamma \) induces a map \( P_\gamma \) with

\section{2-Face colorable lacets on general surfaces}

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the surfaces of \( \mathcal{P}' \), which realize \( \mathcal{P} \) as a lacet (2-face colorable or not).

Given a Gauss code \( \mathcal{P} \), let \( \gamma \) be any \( 0-1 \) vector indexed by \( E. \) Vector \( \gamma \) induces a map \( P_\gamma \) with
a single vertex as follows: the edges around the single vertex are as in $\mathcal{P}$ and the orientation reversing loops are the edges $x$ of $\gamma_x = 1$; the others, with $\gamma_x = 0$ are orientation preserving. The phial map, $M_\gamma$, of $P_\gamma$ is a map with a single zigzag and its medial map $P'_\gamma$ is a lacet for $\mathcal{P}$ on a surface $S_\gamma$ which has $B_\gamma = \{ x \in E \mid \gamma_x = 1 \}$ and $W_\gamma = \{ x \in E \mid \gamma_x = 0 \}$ as its sets of black and white vertices. The dimension of the image of $b_{P_\gamma}$ is the connectivity of $S_\gamma$. The important fact is that both $b_{P_\gamma}$ and $S_\gamma$ are defined from $\mathcal{P}$ and $\gamma$.

**Proposition 12** Given a Gauss code $\mathcal{P}$ over $E$ and an arbitrary $0 - 1$ vector $\gamma$ indexed by $E$, let $p = \dim(\text{Im}(b_{P_\gamma})) = \text{conn}(S_\gamma)$. Then the following quadratic system of $n$ vectorial equations on the $2np$ variables $\delta_{11}, \delta_{12}, \ldots, \delta_{1p}, \ldots, \delta_{n1}, \delta_{n2}, \ldots, \delta_{np}, \ldots, \epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{1p}, \ldots, \epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{np}$ is solvable: for each $x \in E$,

$$
\sum_{z \in E}(\sum_{j=1}^{p} \delta_{xz}\epsilon_{xj})z = \tilde{i}_{P_{\gamma}}^2(x) + \sum_{y \in \gamma(x)}(1 + \gamma_x + \gamma_y)y = b_{P_\gamma}(x).
$$

**Proof:** Since the right hand side of each of the above equations is the value of $b_{P_\gamma}(x)$ it is enough to prove that the coefficient of $z$ on the right hand side is 1 or 0 according to $z \in b_{P_\gamma}(x)$ or not. Henceforth, to simplify the notation, we drop the subscripts $\gamma$. Observe that $z \in b_{P}(x) = c_{P}(c_{P^{-}}(x))$ if and only if $|c_{P}(z) \cap c_{P^{-}}(x)|$ is odd. Moreover, $c_{P}(z) \cap c_{P^{-}}(x)$ and $c_{P}(x) \cap c_{P^{-}}(z)$ have the same parity. The crucial observation is that the parity of $c_{P}(z) \cap c_{P^{-}}(x)$ is the intersection number of the mod 2 homology classes (Giblin[1977]) of the cycle $c_{P}(z)$ on the map $M$ and of the cycle $c_{P^{-}}(x)$ on the dual map $D$. As we only use homology mod 2, henceforth we drop the mod 2.

Let $N$ be an arbitrary map with a single vertex and a single face defined on the same surface $S = S_\gamma$. $N$ is formed by loops $v_1, v_2, \ldots, v_p$. Denote by $N^*$ the geometrical dual of $N$ formed by loops $v_1^*, v_2^*, \ldots, v_p^*$. Loop $v_j$ crosses loop $v_j^*$ once and is disjoint from the others. The $v_j$’s form a basis for the homology of $S$ and the $v_j^*$’s the dual basis. We are considering all the maps $M$, $D$, $N$ and $N^*$ simultaneously on the same surface $S$. Each crossing between $N$ and $D$ or between $N$ and $N^*$ is between dual edges. All the other crossings are irrelevant for our argument. Let $\sim$ denote the relation “is homologous to”. Since the $v_j$’s form a basis for the homology of $S$, there are unique scalars $\delta_{xz}$ such that $c_{P}(z) \sim \sum_{j=1}^{p}\delta_{xz}v_j$. Similarly, the $v_j^*$’s form a basis for the homology of $S$. Thus, there are unique scalars $\epsilon_{xj}$ such that $c_{P^{-}}(x) \sim \sum_{j=1}^{p}\epsilon_{xj}v_j^*$. The crossing number between the homology classes of $c_{P}(z)$ and $c_{P^{-}}(x)$ can be computed from the crossing number between $\sum_{j=1}^{p}\delta_{xz}v_j$ and $\sum_{j=1}^{p}\epsilon_{xj}v_j^*$. This number is simply $\sum_{j=1}^{p}\delta_{xz}\epsilon_{xj}$, proving the proposition. 

**Proposition 13 (Vectorial quadratic equations for $P'$ on arbitrary surfaces)** Let a Gauss code $\mathcal{P}$ over $E$ be given. There exists a $0 - 1$ vector $\gamma$ indexed by $E$ inducing a medial $P'$ which is a lacet for $\mathcal{P}$ in a surface of connectivity at most $p$ if and only if the following quadratic system of $n$ vectorial equations on the $(1 + 2p)n$ variables $\gamma_1, \gamma_2, \ldots, \gamma_n, \delta_{11}, \delta_{12}, \ldots, \delta_{1p}, \ldots, \delta_{n1}, \delta_{n2}, \ldots, \delta_{np}, \ldots, \epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{1p}, \ldots, \epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{np}$ is solvable: for each $x \in E$,

$$
\sum_{y \in \gamma(x)}(1 + \gamma_x + \gamma_y)y + \sum_{z \in E}(\sum_{j=1}^{p}\delta_{xz}\epsilon_{xj})z = \tilde{i}_{P_{\gamma}}^2(x).
$$
Proof: If there exists a $\gamma$ with $\dim(\text{Im}(b_{P_{\gamma}})) = \text{conn}(S_{\gamma}) = q \leq p$, then apply the previous proposition with $P$ and this $\gamma$ as the input parameters. We get values for $\delta_{xj}$ and $\epsilon_{xj}$ with $x \in E$ and $j \in \{1, 2, \ldots, q\}$. By defining $\delta_{xj} = 0$ and $\epsilon_{xj} = 0$ for $x \in E$ and $j \in \{q+1, \ldots, p\}$, we produce a solution of the above system.

To prove the opposite implication, assume that the system has a solution $(\gamma, \delta, \epsilon)$: $\gamma$ is an $n$-vector indexed by $E$, $\delta$ and $\epsilon$ are $n \times p$ matrices. It is enough to show that $\text{conn}(S_{\gamma}) = \dim(\text{Im}(b_{P_{\gamma}})) \leq p$. Let $\{x_1, \ldots, x_p, x_{p+1}\}$ be an arbitrary subset with $p+1$ elements of $E$ whose only restriction is that $b_{P_{\gamma}}(x_i) \neq 0$, $1 \leq i \leq p+1$. Consider the $(p+1) \times p$ matrix $K$ whose $(i, j)$ entry is $\epsilon_{x_i,j}$. Since $K$ has more rows than columns, there exists a subset $I \subseteq \{1, \ldots, p, p+1\}$ such that the sum of the $K$-rows indexed by $I$ is the zero row. Obviously $I \neq \emptyset$. Since $(\gamma, \delta, \epsilon)$ is a solution, we get $\sum_{i \in I} b_{P_{\gamma}}(x_i) = \sum_{z \in E}(\sum_{j=1}^p \delta_{zj}(\sum_{i \in I} \epsilon_{x_i,j}))z = \sum_{z \in E}(\sum_{j=1}^p \delta_{zj})0z = \sum_{z \in E}0z = 0$. Thus, any set of $p+1$ $b_{P_{\gamma}}(x_i)$’s is linearly dependent. Therefore, $\dim(\text{Im}(b_{P_{\gamma}}))$ is at most $p$, proving the result.

Let $\text{conn}_2(P)$ denote the minimum connectivity of a surface where $P$ is realizable as a 2-face colorable lacet.

**Proposition 14 (Quadratic equations for $P'$ on arbitrary surfaces)** Given a Gauss code $P$, $\text{conn}_2(P) \leq p$ if and only if the following system of $n^2$ quadratic equations (on the $n(2p+1)$ variables of the previous theorem) over $\mathbb{Z}_2$ is solvable: for $x, y \in E$,

$$\alpha_{xy}(\gamma_x + \gamma_y) + \sum_{j=1}^p \delta_{yj}\epsilon_{xj} = \beta_{xy},$$

where the constants $\alpha_{xy}$ and $\beta_{xy}$ are defined as $\alpha_{xy} = 1$, if $y \in i_{\overline{P}}(x)$, $\alpha_{xy} = 0$, if $y \notin i_{\overline{P}}(x)$, $\beta_{xy} = 1$, if $y \in i_{\overline{P}}(x) + i_{\overline{\overline{P}}}(x) + i_{\overline{\overline{\overline{P}}}}(x)$ and $\beta_{xy} = 0$, if $y \notin i_{\overline{P}}(x) + i_{\overline{\overline{P}}}(x) + i_{\overline{\overline{\overline{P}}}}(x)$.

Proof: The result follows from the previous proposition: just expand the vectorial equations to their components.

7 Concluding Remarks

Whether or not the above quadratic system can be efficiently solved is an open question. Of course, if it can, then we also can efficiently obtain $\text{conn}_2(P)$. In the small examples we have tested, they are easy to solve via a package like MAPLE. We have made some theoretical progress in the case of the torus. This will be reported in a new paper currently under preparation. Another line of research is to solve the lacet problem when the condition of face 2-colorability is not imposed. This has been recently achieved in the case of the projective plane [10].
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