Equivalence of Geometric Engineering and Hanany-Witten

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Abstract

We show the equivalence of three different realisations of gauge theory in string theory. These are the Hanany-Witten brane constructions, the use of branes as probes and geometric engineering. We illustrate the equivalence via T- and S-dualities with the simplest non-trivial examples in four dimensions: $\mathcal{N} = 2$ SYM with gauge groups $\prod SU(N_i)$. 
1 Introduction

This talk is based on work in collaboration with A. Karch and D. Lüst. More details can be found in [1]. Recently there have been three main methods used to describe non-perturbative supersymmetric gauge theories using string theory. Each method has its advantages and disadvantages and appears to be a very different way of describing the same low energy physics. We will show that in fact all three methods are equivalent under string dualities. We begin by describing the Hanany-Witten brane configurations [2] and branes at an orbifold fixed point [3] which we then relate by T-duality. We then provide some evidence for this relation by comparing moduli spaces and deformations at the classical level. Next we describe the correspondence between quantum effects in these two descriptions. Finally we relate the branes at an orbifold to geometric engineering [4, 5, 6] via a series of dualities. One of the crucial ingredients in the duality relations is the use of fractional branes [7, 8, 9].

2 Hanany-Witten configurations

In this section we will review the features of the Hanany-Witten [2] brane construction for four dimensional $\mathcal{N} = 2$ gauge theories with unitary gauge groups [10]. We will consider the so-called elliptic models which means that the 6 direction is compact. Let us first consider the Higgs branch which is described by $k$ NS5-branes in type IIA string theory with worldvolume 012345, separated in the 6 direction and at the origin of the 789 space. We also have $N$ D4-branes wrapped around the 6 direction with worldvolume 01236. When these D4-branes are together but not at the origin of the 789 space, the low energy theory on their worldvolume will be $\mathcal{N} = 4$ SYM with gauge group $U(N)$. We can move to the Coulomb branch by taking the D4-branes to the origin of the 789 space where they can split on the NS5-branes. This gives the by now familiar $\mathcal{N} = 2$ Hanany-Witten setup with gauge group $U(N)^k$ and hypermultiplets in bifundamental representations of each neighbouring gauge group. The D4-brane segments which end on consecutive NS5-branes can be moved independently and so in general we will have gauge group $U(1)^{Nk}$.

3 T-Duality and Branes on Orbifold

We can perform at T-duality in the 6 direction on the Higgs branch of the Hanany-Witten configuration. The D4-branes will become D3-branes with worldvolume 0123 and the $k$ NS5-branes will become $k$ Kaluza-Klein 5-branes which we will interpret as a multi-Taub-NUT space [11]. So at low energies we have $N$ D3-branes moving in the background of 6789 space with an $A_{k-1}$ singularity $^{1}$

$k - 1$ U(1) factors will be frozen out in the quantum theory but for simplicity we will mainly discuss the classical theory in this talk.

1
at the origin. This can be described by choosing complex coordinates:

\[ z_1 = x^6 + ix^7 \quad z_2 = x^8 + ix^9 \]

and modding out by the orbifold action:

\[ z_1 \rightarrow e^{2\pi i k} z_1 \quad z_2 \rightarrow e^{-2\pi i k} z_2 \]

The low energy theory on the \( N \) D3-branes is \( \mathcal{N} = 4 \) SYM with gauge group \( U(N) \), exactly as for the T-dual configuration. The D3-branes can be moved to the orbifold fixed point at the origin of the Higgs branch and then the low energy theory will be \( \mathcal{N} = 2 \) SYM with gauge group \( U(N)^k \) which is the T-dual description of the origin of the Higgs branch in the Hanany-Witten configuration. Now we want to understand how to T-dualise the Hanany-Witten configuration in the Coulomb branch, where the D4-branes end on NS5-branes. Our proposal is that the T-dual description will be the Coulomb branch described by the D3-branes at the orbifold fixed point. The description of this branch involves the novel concept of fractional branes \[7, 8, 9\].

Fractional branes are branes with some fraction of the mass and charge of a full brane. In this example a fractional D3-brane can exist at the orbifold fixed point with a mass and charge \( 1/k \) of a full D3-brane. These fractional branes cannot move away from the fixed point but are free to move in the 45 directions. \( k \) of these fractional branes can combine to form a complete D3-brane which is then free to move away from the fixed point. The fractional branes are interpreted as D5-branes wrapping the non-trivial intersecting 2-cycles of the resolved orbifold fixed point. These 2-cycles intersect according to the extended Dynkin diagram for \( A_{k-1} \). The sum of these \( k \) 2-cycles is a trivial cycle which explains why \( k \) fractional branes can combine to form a full D3-brane.

It is clear that these fractional branes have the same properties as the segments of D4-branes ending on NS5-branes. By moving the fractional D3-branes in the 45 directions we will generically have gauge group \( U(1)^{N_k} \). So it is natural to postulate that this is the T-dual description of the Coulomb branch of the Hanany-Witten configuration. This can be summarised by the rule that the T-dual of a \( D_p \)-brane along a direction where it ends on two consecutive NS5-branes is a fractional \( D(p - 1) \)-brane. Using this rule we can now T-dualise a general Hanany-Witten configuration. It should be noted that this does not require any new assumptions since we can produce a general elliptic configuration from our special case (with the same number \( N \) of D4-branes between each consecutive NS5-brane) by simply moving some of these D4-brane segments to infinity in the 45 directions. In this way we can leave any number \( N_i \) of D4-branes between the \( i \)-th and \( (i+1) \)-th NS5-branes. This is exactly reproduced in the T-dual picture by moving the corresponding fractional D3-branes to infinity in the 45 directions.
4 Coupling Constants and Fayet-Iliopoulos Terms

We have already described how the Higgs and Coulomb branches of the Hanany-Witten setup are mapped to those of the branes in the orbifold space. Now we will consider the mapping of gauge coupling constants and Fayet-Iliopoulos terms.

In the Hanany-Witten configuration we can turn on Fayet-Iliopoulos terms by moving the $i$-th NS5-brane a distance $\zeta_i$ in the 789 space. Since any D4-branes connecting to this NS5-brane will no longer be parallel to the other D4-branes, this will break supersymmetry. Indeed it is easy to see that this will increase the energy of the configuration by an amount of order $\zeta_i^2$ (for small $\zeta_i$). This means that part of the Coulomb branch, corresponding to D4-branes ending on the $i$-th NS5-brane, will be removed. The T-dual description of this is the (partial) resolution of the orbifold singularity. This is realised by blowing up the $i$-th 2-cycle and so reducing the singularity type to $A_{k-2}$. This 2-cycle has a radius of order $\zeta_i$ and so a D5-brane wrapping it will be interpreted as a fractional D3-brane with mass of order $\zeta_i^2$, the area of the 2-cycle. Clearly this will break supersymmetry and so we again see that part of the Coulomb branch is removed.

Now we will discuss the T-dual descriptions of the gauge coupling constants. Consider first the Hanany Witten setup with the $i$-th and $(i+1)$-th NS5-branes separated in the 6 direction by some distance $\Delta_i$. Then the gauge coupling of the associated gauge group factor is given by:

$$\frac{1}{g_i^2} = \frac{\Delta_i}{g_s^A L_s}$$ (1)

where $g_s^A$ and $L_s$ are the type IIA string coupling and string length respectively. T-dualising this we get the relation between the gauge coupling and type IIB string coupling:

$$\frac{1}{g_i^2} = \frac{\Delta_i}{g_s^B L_6}$$ (2)

where $L_6$ is the period around the 6 direction in type IIA. But we know that for branes in an orbifold background that the gauge coupling is given by [12]:

$$\frac{1}{g_i^2} \sim \frac{1}{g_s^B} \int_{\sigma_i} F$$ (3)

where $\sigma_i$ is the $i$-th 2-cycle and $F = F - B^{NS}$ is the gauge invariant 2-form field strength on the D5-brane (as appears in the Born-Infeld action.) So we see the expected relations from T-duality: the 6 position of an NS5-branes is translated into a Wilson line. So the seemingly different determinations of the gauge coupling are actually just T-dual.


5 Comments on Quantum Effects

Although what we have described so far has been at the classical level, the T-duality should also relate quantum properties of the two descriptions of the gauge theory. For $\mathcal{N} = 2$ theories in four dimensions there are two types of quantum effects: 1-loop effects and non-perturbative instanton contributions.

In the Hanany-Witten description the gauge theory instantons are Euclidean D0-branes whose worldline is in the 6 direction \[13\]. They end on the NS5-branes and so they should T-dualise into fractional D($-1$)-branes at the orbifold fixed point. This is indeed what is expected since D-instantons on D3-branes are simply Yang-Mills instantons.

The 1-loop effects have a simple geometric interpretation in the Hanany-Witten setup. Here the NS5-branes bend logarithmically since the D4-branes ending on them have a tension. The logarithmic behaviour is simply because the ends of the D4-branes have codimension 2 on the NS5-branes. Since the distance between the NS5-branes determines the gauge coupling the variation of the distance is interpreted as the running of the gauge coupling. In the T-dual picture the distance between the NS5-branes becomes a Wilson line for $\mathcal{F}$ which again determines the gauge coupling. The logarithmic running can be seen directly since the fractional D3-branes are charged objects in the 2-dimensional space transverse to them and the orbifold. This means that the 4-form potential has a logarithmic dependence. Since this couples to $\mathcal{F}$ in the D5-brane action, when we compactify the D5-brane on a 2-cycle to get the fractional D3-brane we will find that the effective gauge coupling has a logarithmic dependence.

6 From Branes on Orbifold to Geometric Engineering

It is now quite straightforward to complete the duality relation between the Hanany-Witten description and geometric engineering. If we consider the situation with an $A_{k-1}$ singularity and $N_i$ fractional D3-branes, i.e. $N_i$ D5-branes wrapped on each 2-cycle $\sigma_i$, then we can make an S-duality transformation. This doesn’t affect the singularity but will transform the D5-branes into NS5-branes wrapping the same cycles. We can now perform a T-duality in one of the two directions transverse to the NS5-branes and the orbifold. This will transform the NS5-branes into $A_{N_i-1}$ singularities fibred over the 2-cycles of the original $A_{k-1}$ singularity. This is precisely the description of a non-compact Calabi-Yau threefold used in geometric engineering to describe the same gauge theory we have been considering. More details of this duality can be found in \[1\].

7 Conclusions

We have shown the dualities connecting Hanany-Witten setups, via branes at an orbifold, to geometric engineering in this simplest non-trivial example. This
can easily be extended to more general examples of gauge theories in various dimensions and with orthogonal or symplectic groups by including orientifold planes. Theories with reduced supersymmetry can also be analysed in this way. For example the brane box models [14] have been studied by dualising to branes at orbifolds using this method [15]. There are of course some limitations to each method. For example exceptional groups cannot be constructed using the Hanany-Witten setup but can by using geometric engineering. This is translated here into the fact that we cannot T-dualise an E-type singularity.

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