Gauge theories on the noncommutative sphere

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Dedicated to my son Cedrik at the occasion of his birth.

Abstract

Gauge theories are formulated on the noncommutative two-sphere. These theories have only finite number of degrees of freedom, nevertheless they exhibit both the gauge symmetry and the SU(2) "Poincaré" symmetry of the sphere. In particular, the coupling of gauge fields to chiral fermions is naturally achieved.
1 Introduction

There was a reappearing belief through this century that a possible understanding of a short distance behaviour of physical theories should stem from a theory that would incorporate a minimal length. At early times of quantum field theory, such attitude was almost dictated by the pertinent problem of ultraviolet (UV) divergences. With the work of Wilson, we have learned better to get along with the divergences, nevertheless it is still expected that a fundamental theory should be in a sense finite. The divergences should appear in a controlled way in effective approaches to various aspects of the theory.

String theory is often presented as a candidate for a fundamental description of our world. Its perturbation expansion is believed to be UV finite and the string tension sets a scale which is interpreted as the minimal length\(^1\). Insights of last two years even led to remarkable proposals of various matrix models [2, 3] which should constitute the nonperturbative formulation of string theory. We should like to stress, however, that the idea that theories containing large matrices naturally incorporate the concept of minimal length had appeared earlier in the field theoretical context [5, 6, 7, 8, 9, 10, 11]. These works stem from the basic ideas of Connes’ noncommutative geometry [4] and argue that a nonperturbatively well defined path integral of various field theories can be formulated if the fields are replaced by big matrices in a way which correspond to the kinematical quantization of the spacetime on which the fields live. Because this is the crucial point of the whole matrix approach let me be more precise here (a reasoning quite similar in spirit now underlies the matrix approach to superstring theory [3]):

Consider a Riemann sphere as a spacetime of an Euclidean field theory. This is a sufficiently general setting, since we can always effectively decompactify the spacetime by scaling the round metric (or radius) of the sphere and the Minkowski dynamics can be achieved by a suitable modification of the standard Osterwalder-Schrader procedure. An invariance of field theories with respect to the isometry group $SO(3)$ of the sphere then in the limit of infinite scaling play the role of the Euclidean (or Poincaré in the Minkowski case) invariance. Now the crucial observation is that the spacetime $S^2$ is naturally a symplectic manifold; the symplectic form $\omega$ is up to a normalization\(^1\):

\[^1\]However, see a recent progress in understanding substringy scales via D-branes [1].
tion just the round volume form on the sphere. Using the standard complex coordinate $z$ on the Riemann sphere, we have

$$\omega = -\frac{N}{2\pi} \frac{d\bar{z} \wedge dz}{(1 + \bar{z}z)^2},$$

with $N$ a real parameter. If we consider a scalar field theory, then the scalar field $\phi$ is a function on the symplectic manifold or, in other words, a classical observable. The action of the massless (real) scalar field theory on $S^2$ is given by

$$S = -i \int \omega R_i \phi R_i \phi,$$

where $R_i$ are the vector fields which generate the $SO(3)$ rotations of $S^2$ and the Einstein summation convention is understood. The vector fields $R_i$ are Hamiltonian; this means that there exists three concrete observables $r_i$ such that

$$\{r_i, \phi\} = R_i \phi.$$  

Here $\{., .\}$ is the Poisson bracket which corresponds to the symplectic structure $\omega$. The observables $r_i \in \mathbb{R}^3$ are just the coordinates of the embedding of $S^2$ in $\mathbb{R}^3$. Thus we can rewrite the action (ii) as

$$S = -i \int \omega \{r_i, \phi\} \{r_i, \phi\}.$$

Suppose we quantize the symplectic structure on $S^2$ (probably the first who has done it was Berezin [12]). Then the algebra of observables becomes the noncommutative algebra of all square matrices with entries in $\mathbb{C}$; the quantization of $S^2$ can be only performed if $N$ is an integer, the size of the scalar field matrices $\phi$ is then $(N + 1) \times (N + 1)$. This algebra of matrices defines the noncommutative (or fuzzy [6]) sphere. The integration over the phase space volume form $i\omega$ is replaced by taking a properly normalized trace $\text{Tr}$ over the matrices and the Poisson brackets are replaced by commutators (the Hamiltonians $r_i$ are also quantized, of course).

Putting together, we can consider along with (iv) a noncommutative action

$$S = -\frac{1}{N + 1} \text{Tr}([r_i, \phi], [r_i, \phi]).$$

\footnote{Note, that we have chosen a normalization which makes the form $\omega$ purely imaginary. Under quantization, hence, the Poisson bracket is replaced by a commutator without any imaginary unit factor.}
The action (v) has a few nonstandard properties. First of all, the space of all "fields" (=matrices) is finite dimensional and the product of fields is noncommutative. The latter property may seem awkward but in all stages of analysis we shall never encounter a problem which this noncommutativity might create. The former property, however, is highly desirable, since all divergences of the usual field theories are automatically eliminated. We may interpret (v) as the regularized version of (iv); the fact that (v) goes to (iv) in the limit $N \to \infty$ is just the statement that classical mechanics is the limit of the quantum one for the value of the Planck constant $1/N$ approaching zero. Remarkably, unlike lattice regularizations, (v) preserves the "Poincaré" symmetry $SO(3)$ of the spacetime. Indeed, under the variation $\delta \phi = [r_i, \phi]$ the action (v) remains invariant.

Few more words are useful for understanding the nature of the commutative limit $N \to \infty$ (see [8] for more details). Both classical commutative fields and their noncommutative counterparts (matrices) can be decomposed into spherical harmonics. This means that the algebra of observables is a representation of the group $SO(3)$, which infinitesimally acts on $\phi$ via Poisson bracket or commutators (respectively) with the Hamiltonians $r_i$. All spherical harmonics with the quantum number $l$ form the irreducible representation of $SO(3)$ (this is their definition in the noncommutative case). Recall that the Laplace operator on the sphere (the Casimir $R_i R_i$ of $SO(3)$) has the spectrum $l(l + 1)$. Now if a maximal $l$ of $\phi$ in (v) is much smaller than $N$, then the action (v) differs from (iv) by a factor which goes like $1/N$ for $N \to \infty$. This means that, being at large scales (=small momenta $l$), the actions (iv) and (v) are equivalent. They differ only in the short distance limit. Hence we stress that there is no point in saying which one is better, or which action is an approximation of the other. Just from the historical reasons people were using (iv) earlier than (v) (they did not know about the noncommutative geometry). Both (iv) and (v) are extrapolations of the same long distance quantity also to the ultraviolet region. The symmetry principle cannot select one of them. Therefore we tend to believe that (v) might be a better choice, because it is from the outset manifestly regular. Such a statement may call for a criticisms; the most obvious would be: what about the other fields (spinor, vector etc.); can one construct classical field theories on noncommutative manifolds involving those fields? This paper constitutes a step of the program [8, 13, 14] which aims to show that the answer to this question is positive.
Scalar fields, being observables, are naturally defined in the noncommutative context just as elements of the algebra which defines the noncommutative manifold. However, no canonical procedure appears to exist of how to describe, say, spinor fields on a noncommutative manifold. The early approaches by Grosse and Madore [7] and later by Grosse and Prešnajder [15] considered spinors on $S^2$ as two component objects where both components are elements of the noncommutative algebra of matrices $(N + 1) \times (N + 1)$. Although this approach did permit to construct field theories (in the case of [7] also the gauge fields with an inevitable addition of one propagating scalar), the noncommutative actions obtained in this way have somewhat lost their competitiveness with respect to the classical commutative actions. The reason was that the Dirac operator did not anticommute with the chiral grading for finite $N$. A shift of a point of view was presented by Grosse, Klimčík and Prešnajder in [8], where it was argued that spinors should be understood as odd parts of scalar superfields. Then the chirality of the Dirac operator is automatically preserved. Thus the guiding principle for understanding the spinors consists in quantizing the supersymplectic manifolds. If they are compact (like a supersphere in two dimension) then the quantized algebra of observables is just an algebra of supermatrices with a finite size. The odd (off-diagonal) part of the supermatrix then describes a spinor on the bosonic submanifold of the whole supermanifold. In this way, the scalar superfields on the supersphere were shown to describe both scalars and spinors on the sphere [8]. The moral of the story is that, by using supersymmetry, the construction of noncommutative spinors is as canonical as that of noncommutative scalars.

The story of noncommutative gauge theories on $S^2$, compatible with the chirality of the Dirac operator and with the already known description of the scalars and spinors, is so far missing. We are going to fill this gap in this article. Our strategy will consist in converting the standard derivatives into covariant ones in the noncommutative actions like (v). We shall do it in the supersymmetric framework which treats the scalars and spinors on the same footing and which uses only Hamiltonian vector fields as the derivatives appearing in the Lagrangian. This means that we shall covariantize these (odd) derivatives by adding suitable gauge fields. Then we identify a gauge invariant kinetic term for these gauge fields which, remarkably, can be also written solely in terms of the same odd Hamiltonian vector fields. At the commutative level, this means that we will be able to write actions for
scalars and spinors interacting with gauge fields just (like in (iv)) in terms of 1) suitable Poisson brackets 2) integration over a suitable volume form on the spacetime. In this way, we may convert such actions into their noncommutative analogues (like (iv) to (v)). These noncommutative actions will be the same as their commutative counterparts at large distances, they will describe theories possessing the same set of symmetries as in the commutative case but containing only finite number of degrees of freedom!

In chapter 2, we describe a noncommutative complex of differential forms which underlies the notion of the gauge field in the noncommutative case. Then in chapter 3 we construct subsequently scalar and spinor electrodynamics.

2 Hamiltonian de Rham complex of $S^2$

2.1 The commutative case

Consider an algebra $\mathcal{A}$ of functions in two conjugated bosonic variables $z, \bar{z}$ and two conjugated fermionic ones $b, \bar{b}$ with the standard graded commutative multiplication but linearly generated (over $\mathbb{C}$) only by the functions of the form

$$\frac{\bar{z}^k z^l \bar{b}^l b^l}{(1 + \bar{z}z + \bar{b}b)^m}, \quad \text{max}(k + \bar{l}, k + l) \leq m, \quad k, \bar{k}, l, \bar{l}, m \geq 0. \quad (1)$$

The algebra is equipped with the graded involution

$$z^\dagger = \bar{z}, \quad \bar{z}^\dagger = z, \quad b^\dagger = \bar{b}, \quad \bar{b}^\dagger = -b; \quad (2)$$

$$(AB)^\dagger = (-1)^{AB} B^\dagger A^\dagger, \quad (A^\dagger)^\dagger = (-1)^A A \quad (3)$$

and it is known as the algebra of functions on the supersphere $\mathbb{CP}(1, 1)$. We can define an integral over an element $f$ in the algebra as follows

$$I[f] \equiv -\frac{i}{2\pi} \int \frac{d\bar{z} \wedge dz \wedge d\bar{b} \wedge db}{1 + \bar{z}z + \bar{b}b} f. \quad (4)$$

(Note $I[1] = 1$.) Now an inner product on $\mathcal{A}$ is defined simply as

$$(f, g) = I(f^\dagger g). \quad (5)$$
The remaining structure to be defined is the graded symplectic structure, given by a non-degenerate super-Poisson bracket \( \{ ., . \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \)

\[
\{ f, g \} = (1 + \bar{z}z)(1 + \bar{z}z + \bar{b}b)(\partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_z g) \\
+(-1)^l(1 + \bar{z}z)\bar{b}z(\partial_z f \partial_{\bar{b}} g - \partial_{\bar{b}} f \partial_z g) + (1 + \bar{z}z)b\bar{z}(\partial_{\bar{b}} f \partial_z g - (1)^l \partial_z f \partial_{\bar{b}} g) \\
+(-1)^{(l+1)}(1 + \bar{z}z - \bar{b}b\bar{z})z(\partial_{\bar{b}} f \partial_{\bar{b}} g + \partial_{\bar{b}} f \partial_{\bar{b}} g). \quad (6)
\]

Now let us introduce four odd vector fields on \( \mathcal{A} \):

\[
T_1 = \bar{z}\partial_b - b\partial_z, \quad T_2 = \partial_b + zb\partial_z, \quad \bar{T}_1 = \bar{b}\partial_{\bar{z}} - \bar{z}\partial_b, \quad \bar{T}_2 = \partial_{\bar{b}} - \bar{z}\bar{b}\partial_{\bar{z}}. \quad (7)
\]

It turns out that \( T_i, \bar{T}_i \) are the Hamiltonian vector fields with respect to the super-Poisson bracket (6). They are generated by the Hamiltonians \( t_i, \bar{t}_i \):

\[
T_i f = \{ t_i, f \}, \quad \bar{T}_i f = \{ \bar{t}_i, f \}; \quad (8)
\]

\[
t_1 \equiv \frac{\bar{z}b}{1 + \bar{z}z + \bar{b}b}, \quad t_2 \equiv \frac{b}{1 + \bar{z}z + \bar{b}b}, \quad \bar{t}_1 \equiv \frac{\bar{b}z}{1 + \bar{z}z + \bar{b}b}, \quad \bar{t}_2 \equiv \frac{\bar{b}}{1 + \bar{z}z + \bar{b}b}. \quad (9)
\]

Note also the properties of \( T_i, \bar{T}_i \) with respect to the graded involution:

\[
(T_i f)^t = -\bar{T}_i f^t, \quad (\bar{T}_i f)^t = T_i f^t. \quad (10)
\]

Define a complexified Hamiltonian de Rham complex \( \Omega \) over the standard sphere \( S^2 \) as the graded associative algebra with unit

\[
\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2, \quad (11)
\]

where

\[
\Omega_0 = \Omega_2 = \mathcal{A}_e \quad (12)
\]

and

\[
\Omega_1 = \mathcal{A}_b \oplus \mathcal{A}_{\bar{b}} \oplus \mathcal{A}_b \oplus \mathcal{A}_{\bar{b}}. \quad (13)
\]

Here \( \mathcal{A}_e \) is the even subalgebra of \( \mathcal{A} \) linearly generated over \( \mathbb{C} \) only by elements with \( l = \bar{l} \) (cf. Eq. (1)) and \( \mathcal{A}_b (\mathcal{A}_{\bar{b}}) \) are (odd) bimodules over \( \mathcal{A}_e \) linearly generated by the elements of the form (1) with \( \bar{l} = 0, \ l = 1 \) (\( \bar{l} = 1, \ l = 0 \)). The multiplication in \( \Omega \) is entailed by one in \( \mathcal{A} \), the only non-obvious thing is to define the product of 1-forms. Here it is

\[
(A_1, A_2, \bar{A}_1, \bar{A}_2)(B_1, B_2, \bar{B}_1, \bar{B}_2) \equiv A_1\bar{B}_1 + A_2\bar{B}_2 + \bar{A}_1B_1 + \bar{A}_2B_2. \quad (14)
\]
Of course, the r.h.s. is viewed as an element of $\Omega_2$. The product of a 1-form and a 2-form is set to zero by definition. Now the coboundary operator $d$ is given by

$$df \equiv (T_1 f, T_2 f, \bar{T}_1 f, \bar{T}_2 f), \ f \in \Omega_0; \quad (15)$$

$$d(A_1, A_2, \bar{A}_1, \bar{A}_2) \equiv T_1 A_1 + T_2 A_2 + T_1 A_1 + T_2 A_2, \quad (A_1, A_2, \bar{A}_1, \bar{A}_2) \in \Omega_1; \quad (16)$$

$$dh = 0, \quad h \in \Omega_2. \quad (17)$$

It maps $\Omega_i$ to $\Omega_{i+1}$ and it satisfies

$$d^2 = 0, \quad d(AB) = (dA)B + (-1)^A A(dB). \quad (18)$$

Now we show that the complex $\Omega$ resembles very much the standard (complexified) de Rham complex $\Omega_{dR}$ of the commutative sphere $S^2$ (not of the supersphere!). The latter can be defined again as the graded associative algebra with unit given by

$$\Omega_{dR} = \omega_0 \oplus \omega_1 \oplus \omega_2, \quad (19)$$

where

$$\omega_{0,2} = \mathcal{B}, \quad \omega_1 = \mathcal{B}_2 \oplus \bar{\mathcal{B}}_2. \quad (20)$$

Here $\mathcal{B}$ is unital algebra linearly generated (over $\mathbb{C}$) by elements of the form

$$\frac{z^k \bar{z}^k}{(1 + z\bar{z})^m}, \quad \max(\bar{k}, k) \leq m, \quad k, \bar{k}, m \geq 0, \quad (21)$$

that is, $\mathcal{B}$ is the algebra of complex functions on $S^2$. $\mathcal{B}_2$ and $\bar{\mathcal{B}}_2$ are $\mathcal{B}$-bimodules linearly generated by the following elements of $\mathcal{B}$

$$\frac{z^k \bar{z}^k}{(1 + z\bar{z})^m}, \quad \max(\bar{k}, k + 2) \leq m, \quad k, \bar{k}, m \geq 0 \quad (22)$$

and

$$\frac{\bar{z}^k z^k}{(1 + \bar{z}z)^m}, \quad \max(\bar{k} + 2, k) \leq m, \quad k, \bar{k}, m \geq 0, \quad (23)$$

respectively. One often writes the elements $(V, \bar{V})$ of $\omega_1$ as

$$Vdz + \bar{V}d\bar{z} \quad (24)$$
and \( h \) of \( \omega_2 \) as
\[
\frac{2}{i(\bar{z}z + 1)^2}d\bar{z} \wedge dz, \quad h \in \omega_2.
\] (25)

The multiplication in \( \Omega_{dR} \) is now given by the standard wedge product in the representation (24,25) and the de Rham coboundary operator \( d_{dR} \) is given in the standard way.

One can easily verify the following

**Fact:** The complex \( \Omega_{dR} \) can be injected into the complex \( \Omega \). The injection \( \nu \) is a homomorphism which preserves all involved structures (i.e. linear, multiplicative and differential ones). It is given explicitly as follows
\[
f(z, \bar{z}) \in \omega_0 \rightarrow f(z, \bar{z}) \in \Omega_0; \quad (26)
\]
\[
(V(z, \bar{z}), \bar{V}(z, \bar{z})) \in \omega_1 \rightarrow (-V(z, \bar{z})b, V(z, \bar{z})\bar{b} + \bar{V}(z, \bar{z})\bar{b}, -\bar{V}(z, \bar{z})\bar{b}) \in \Omega_1; \quad (27)
\]
\[
h(z, \bar{z}) \in \omega_2 \rightarrow \frac{2i}{(1 + \bar{z}z)}h(z, \bar{z})\bar{b}b \in \Omega_2. \quad (28)
\]

The injection (26) of 0-forms requires, perhaps, some explanation. The elements of \( \Omega_0 \) were said to be linearly generated by the expressions of the form (1) with \( l = \bar{l} \). However, by noting the identity
\[
\frac{1}{\bar{z}z + 1} \equiv \frac{1}{\bar{z}z + bb + 1} + \frac{\bar{b}b}{(\bar{z}z + bb + 1)^2}
\] (29)
one can easily see that any element of \( \mathcal{B} \equiv \omega_0 \) (cf. (21)) can be written as a linear combination of the quantities (1).

Thus we have injected the standard de Rham complex into the bigger Hamiltonian de Rham complex \( \Omega \) which has the virtue that all vector fields (i.e. \( T, \bar{T} \)) needed for the definition of the exterior derivative \( d \) are now Hamiltonian. This means that we have good chance to quantize the structure while maintaining all its properties (except graded commutativity of the multiplication in \( \Omega \)).

There remains to clarify the issues of reality, Hodge star and cohomology. As already the name suggests, the complexified Hamiltonian complex has a real subcomplex \( \Omega_R \) given by all elements of \( \Omega \) real under an involution \( \dagger \) defined as follows
\[
f^\dagger = f^\dagger, \quad f \in \Omega_0, \quad h^\dagger = -h^\dagger, \quad h \in \Omega_2; \quad (30)
\]
\[(A_i, \bar{A}_i)^\dagger = (\bar{A}_i^\dagger, -A_i^\dagger), \ (A_i, \bar{A}_i) \in \Omega_1. \tag{31}\]

The involution \(\dagger (\dagger^2 = 1)\) preserves the linear combinations with real coefficients and the multiplication, and commutes with the coboundary operator \(d\):

\[(af + bg)^\dagger = af^\dagger + bg^\dagger, \ a, b \in \mathbb{R}, \ f, g \in \Omega; \tag{32}\]
\[(fg)^\dagger = f^\dagger g^\dagger, \ f, g \in \Omega; \tag{33}\]
\[(df)^\dagger = df^\dagger, \ f \in \Omega. \tag{34}\]

We see that \(\Omega_R\) is indeed the real subcomplex which may be called the Hamiltonian de Rham complex of \(S^2\).

We recall that the standard involution \(\dagger\) on \(\Omega_{dR}\) is given by

\[f^\dagger = f^*, \ f \in \omega_0, \ g^\dagger = g^*, \ g \in \omega_2; \tag{35}\]
\[(V, \bar{V})^\dagger = (\bar{V}^*, V^*), \ (V, \bar{V}) \in \omega_1, \tag{36}\]

where * is the standard complex conjugation. It has also the property of preserving the real linear combinations and the multiplication in \(\Omega_{dR}\) and it commutes with the standard de Rham coboundary operator. Thus the real elements of \(\Omega_{dR}\) form the real de Rham complex of \(S^2\).

It is now easy thing to check that the homomorphism \(\nu : \Omega_{dR} \to \Omega\) preserves the involution, i.e.

\[(\nu(f))^\dagger = \nu(f^\dagger). \tag{37}\]

This is also the reason, why we have chosen the same symbol for both involutions.

It is instructive to compute the cohomology of the both real complexes:

1) The standard de Rham case.

The only non-trivial class occurs in the second cohomology and it is given by an element 1 in \(\omega_2\).

2) The Hamiltonian de Rham case:

Using the homomorphism \(\nu\), the de Rham class above can be injected into \(H^*(\Omega_R)\). It is not very difficult to verify, that this is the only non-trivial class there.
The Hodge star and the inner product:

The Hodge star $*_{H}$ on the standard de Rham complex $\Omega_{dR}$ is given by

$$*_{H} : f(z, \bar{z}) \in \omega_{0} \to f(z, \bar{z}) \in \omega_{2}, \quad h(z\bar{z}) \in \omega_{2} \to h(z, \bar{z}) \in \omega_{0};$$  \hspace{1cm} (38)

$$*_{H} : (V, \bar{V}) \in \omega_{1} \to (iV, -i\bar{V}) \in \omega_{1}.$$  \hspace{1cm} (39)

Note that $*_{H}$ send real forms into real ones. The Hodge star $*$ on the Hamiltonian de Rham complex is given by

$$* : f(\bar{z}, z, \bar{b}, b) \in \Omega_{0} \to \frac{2if(\bar{z}, z, \bar{b}, b)b\bar{b}}{\bar{z}z + 1} \in \Omega_{2};$$  \hspace{1cm} (40)

$$* : (A_{1}, A_{2}, \bar{A}_{1}, \bar{A}_{2}) \in \Omega_{1} \to (iA_{1}, iA_{2}, -i\bar{A}_{1}, -i\bar{A}_{2}) \in \Omega_{1};$$  \hspace{1cm} (41)

$$* : h(\bar{z}, z, \bar{b}, b) \in \Omega_{2} \to \frac{1}{4i}(\bar{T}_{1}T_{1} - T_{i}\bar{T}_{i} - 2)h(\bar{z}, z, \bar{b}, b) \in \Omega_{0}.$$  \hspace{1cm} (42)

This Hodge star is also compatible with the involution $\dagger$ and has the property

$$\nu(f) = \nu(*_{H}f).$$  \hspace{1cm} (43)

This in turn means, that the natural inner product in $\Omega_{dR}$

$$(X, Y)_{dR} \equiv \frac{1}{4\pi} \int (\nu X^{\dagger})Y, \quad X, Y \in \omega_{0}, \omega_{1}, \omega_{2}$$  \hspace{1cm} (44)

does respect the natural inner product in $\Omega$

$$(X', Y') \equiv \frac{i}{2}I[(\nu X'^{\dagger})Y'], \quad X', Y' \in \Omega_{0}, \Omega_{1}, \Omega_{2}.$$  \hspace{1cm} (45)

In other words:

$$\nu(X), \nu(Y)) = (X, Y)_{dR}.$$  \hspace{1cm} (46)

It should be perhaps noted, for clarity, that in (45) and in all the rest of the paper the integral $I$ is applied always on element of $\mathcal{A}_{e}$. Though in (45) the argument of $I$ is always a 2-form (to be understood as an element of $\mathcal{A}_{e}$) in subsequent applications we shall encounter also situations in which the argument will be a 0-form.
The action of \( SU(2) \)

The standard action of the group \( SU(2) \) on \( S^2 \) induces the action of the same group on the Hamiltonian de Rham complex. The latter respects the grading of the complex; on the 0-forms (from \( \Omega_0 \equiv A_e \)) and 2-forms (from \( \Omega_2 \equiv A_e \)) it is given by even Hamiltonian vector field \( R_\pm, R_3 \) obtained by taking suitable anticommutators of the odd vector fields \( T_j, \bar{T}_j \):

\[
R_+ = [T_1, \bar{T}_2]_+ = -\partial_z - \bar{z}^2 \partial_{\bar{z}} - \bar{z} b \partial_b; \quad (47)
\]

\[
R_- = [T_2, \bar{T}_1]_+ = \partial_z + z^2 \partial_z + zb \partial_b; \quad (48)
\]

\[
R_3 = \frac{1}{2}([T_1, \bar{T}_1]_+ - [T_2, \bar{T}_2]_+) = \bar{z} \partial_z - z \partial_z + \frac{1}{2} \bar{b} \partial_b - \frac{1}{2} b \partial b. \quad (49)
\]

The \( SU(2) \) Lie algebra commutation relations

\[
[R_3, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = 2R_3 \quad (50)
\]

then directly follows. The Hamiltonians \( r_j \) of the vector fields \( R_j \) are obtained by taking the corresponding Poisson brackets (6) of the Hamiltonians \( t_i, \bar{t}_i \) given in (9).

The vector fields \( R_j \) acting on the algebra \( A \) realize a (highly reducible) representation of \( SU(2) \). This representation is unitary with respect to the inner product (5) and the representation space \( A \) has several invariant subspaces which are of interest for us. They are \( A_e, A_b \) and \( A_{\bar{b}} \); all of them give rise to smaller unitary representations of \( SU(2) \) than \( A \). In particular, since both \( \Omega_0 \) and \( \Omega_2 \) can be identified with \( A_e \), we have an action of \( SU(2) \) on the 0-forms and 2-forms of the Hamiltonian de Rham complex \( \Omega \).

Now we realize that the space \( \Omega_1 \) of the Hamiltonian 1-forms can be written as

\[
\Omega_1 = A_b \otimes \mathbb{C}^2 \oplus A_{\bar{b}} \otimes \mathbb{C}^2. \quad (51)
\]

The group \( SU(2) \) can be represented on the second copy of \( \mathbb{C}^2 \) in (51) by the standard spin 1/2 representation generated by the Pauli matrices

\[
\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (52)
\]

and on the first copy of \( \mathbb{C}^2 \) by its (equivalent) complex conjugated representation \( (\sigma^+ \rightarrow -\sigma^-, \quad \sigma^3 \rightarrow -\sigma^3) \). Now we can define the action of \( SU(2) \) on
the 1-forms from $\Omega_1$ again by the formula (51) where $\oplus$ and $\otimes$ are understood to be the direct product and the direct sum of the $SU(2)$ representations.

Now it is easy to check that
1) the standard de Rham complex $\Omega_{dR}$ injected in $\Omega$ by the homomorphism $\nu$ is also an invariant subspace of the just defined $SU(2)$ action on $\Omega$;
2) the $SU(2)$ action on $\Omega$ preserves the inner product (45) and restricted to the image of $\nu$ gives the standard (unitary) action of $SU(2)$ on the de Rahm complex $\Omega_{dR}$;
3) the coboundary operator $d$ of $\Omega$ and the Hodge star $*$ are both $SU(2)$ invariant.
4) the generators $R_i$ of $SU(2)$ act on $\Omega$ as derivations with respect to the product on $\Omega$, i.e. $R_i(\Phi \Psi) = (R_i \Phi) \Psi + \Phi (R_i \Psi)$.

2.2 The non-commutative case.

In the previous section, we have described the Hamiltonian de Rham complex, by using substantially the structure of the algebra of functions on the supersphere (1). This algebra can be described in an alternative way, which makes the explicit form of the Poisson structure (6) much less cumbersome, though it makes more involved the relation between the standard de Rham complex and its Hamiltonian counterpart. Here are the details:

Consider the algebra of functions on the complex $\mathbb{C}^{2,1}$ superplane, i.e. algebra generated by bosonic variables $\bar{\chi}^i, \chi^i, i = 1, 2$ and by fermionic ones $\bar{a}, a$. The algebra is equipped with the graded involution

$$(\chi^i)^\dagger = \bar{\chi}^i, \quad (\bar{\chi}^i)^\dagger = \chi^i, \quad a^\dagger = \bar{a}, \quad \bar{a}^\dagger = -a$$

and with the super-Poisson bracket

$$\{f, g\} = \partial_{\chi^i} f \partial_{\bar{\chi}^i} g - \partial_{\bar{\chi}^i} f \partial_{\chi^i} g + (-1)^{f+1}[\partial_{a} f \partial_{\bar{a}} g + \partial_{\bar{a}} f \partial_{a} g].$$

Here and in what follows, the Einstein summation convention applies. We can now apply the (super)symplectic reduction with respect to a moment map $\bar{\chi}^i \chi^i + \bar{a} a$. The result is a smaller algebra $\mathcal{A}$, that by definition consists of all functions $f$ with the property

$$\{f, \bar{\chi}^i \chi^i + \bar{a} a\} = 0.$$
Moreover, two functions obeying (55) are considered to be equivalent if they differ just by a product of \((\bar{\chi}^i \chi^i + \bar{a}a - 1)\) with some other such function. The algebra \(\mathcal{A}\) is just the same algebra (1) that we have considered in the previous section. The Poisson bracket (6) becomes the bracket (54) for the functions in \(\mathcal{A}\). The relation between the generators is as follows

\[
z = \frac{\chi^1}{\chi^2}, \quad \bar{z} = \frac{\bar{\chi}^1}{\bar{\chi}^2}, \quad b = \frac{a}{\chi^2}, \quad \bar{b} = \frac{\bar{a}}{\bar{\chi}^2}. \tag{56}
\]

The integral (4) can be written as

\[
I[f] = -\frac{1}{4\pi^2} \int d\bar{\chi}^1 \wedge d\chi^1 \wedge d\chi^2 \wedge d\bar{a} \wedge da \delta(\bar{\chi}^i \chi^i + \bar{a}a - 1)f. \tag{57}
\]

The vector fields \(T_i, \bar{T}_i\) turn out to be

\[
T_i = \bar{\chi}^i \partial_{\bar{a}} - a \partial_{\chi^i}, \quad \bar{T}_i = \bar{a} \partial_{\bar{\chi}^i} + \chi^i \partial_a. \tag{58}
\]

Of course, they annihilate the moment map \((\bar{\chi}^i \chi^i + \bar{a}a)\), otherwise they would not be well defined differential operators acting on \(\mathcal{A}\). Their Hamiltonians are

\[
t_i = \bar{\chi}^i a, \quad \bar{t}_i = \chi^i \bar{a}. \tag{59}
\]

Now we are ready to quantize the infinitely dimensional algebra \(\mathcal{A}\) with the goal of obtaining its (noncommutative) finite dimensional deformation. The quantization was actually performed in [8] using the representation theory of \(osp(2, 2)\) superalgebra. Here we adopt a different procedure, namely the quantum symplectic reduction (or, in other words, quantization with constraints). This method should be more transparent for anybody who knows the elements of quantum mechanics. We start with the well-known quantization of the complex plane \(\mathbb{C}^{2,1}\). The generators \(\bar{\chi}^i, \chi^i, \bar{a}, a\) become creation and annihilation operators on the Fock space whose commutation relations are given by the standard replacement

\[
\{, , \} \rightarrow \frac{1}{\hbar} [, , ]. \tag{60}
\]

Here \(\hbar\) is a real parameter (we have absorbed the imaginary unit into the definition of the Poisson bracket) referred to as the "Planck constant". Explicitly

\[
[\chi^i, \bar{\chi}^j]_- = \hbar \delta^{ij}, \quad [a, \bar{a}]_+ = \hbar \tag{61}
\]
and all remaining graded commutators vanish. The Fock space is built up as usual, applying the creation operators $\bar{\chi}^i, \bar{a}$ on the vacuum $|0\rangle$, which is in turn annihilated by the annihilation operators $\chi, a$. The scalar product on the Fock space is fixed by the requirement that the barred generators are adjoint of the unbarred ones. We hope that using the same symbols for the classical and quantum generators will not confuse the reader; it should be fairly obvious from the context which usage we have in mind.

Now we perform the quantum symplectic reduction with the self-adjoint moment map $(\bar{\chi}^i \chi^i + \bar{a} a)$. First we restrict the Hilbert space only to the vectors $\psi$ satisfying the constraint

$$(\bar{\chi}^i \chi^i + \bar{a} a - 1) \psi = 0.$$  \hspace{1cm} (62)

Hence operators $\hat{f}$ acting on this restricted space have to fulfil

$$[\hat{f}, \bar{\chi}^i \chi^i + \bar{a} a] = 0$$  \hspace{1cm} (63)

and they are to form our deformed version $\mathcal{A}_N$ of $\mathcal{A}$.

The spectrum of the operator $(\bar{\chi}^i \chi^i + \bar{a} a - 1)$ in the Fock space is given by a sequence $m \hbar - 1$, where $m$’s are integers. In order to fulfil (62) for a non-vanishing $\psi$, we observe that the inverse Planck constant $1/\hbar$ must be an integer $N$. The constraint (62) then selects only $\psi$’s living in the eigenspace $H_N$ of the operator $(\bar{\chi}^i \chi^i + \bar{a} a - 1)$ with the eigenvalue 0. This subspace of the Fock space has the dimension $2N + 1$ and the algebra $\mathcal{A}_N$ of operators $\hat{f}$ acting on it is $(2N + 1)^2$-dimensional.

When $N \to \infty$ (the dimension $(2N + 1)^2$ then also diverges) we have the Planck constant approaching 0 and, hence, the algebras $\mathcal{A}_N$ tend to the classical limit $\mathcal{A}$. The fact that the resulting finite-dimensional noncommutative algebras $\mathcal{A}_N$ are deformations of $\mathcal{A}$ is thus clear since the latter is just the classical limit of the former. The interested reader may find a rigorous proof of this fact in [8].

The Hilbert space $H_N$ is naturally graded. The even subspace $H_{eN}$ is created from the Fock vacuum by applying only the bosonic creation operators:

$$(\bar{\chi}^1)^{n_1} (\bar{\chi}^2)^{n_2}|0\rangle, \quad n_1 + n_2 = N,$$  \hspace{1cm} (64)

\footnote{Note that (63) is just a quantum version of (55) and it says that elements of the deformed algebra have to commute with the particle number operator.}
while the odd one $H_{oN}$ by applying both bosonic and fermionic creation operators:

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}\bar{a}|0\rangle, \ n_1 + n_2 = N - 1. \quad (65)$$

Correspondingly, the algebra of operators $A_N$ on $H_N$ consists of an even part $A_{eN}$ (operators respecting the grading) and an odd part (operators reversing the grading). The odd part can be itself written as a direct sum $A_{aN} \oplus A_{\bar{a}N}$. The two components in the sum are distinguished by their images:

$$A_{aN}H_N = H_{eN} \quad \text{while} \quad A_{\bar{a}N}H_N = H_{oN}. \quad (66)$$

and $A_{eN}$ by

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}(\chi^1)^{m_1}(\chi^2)^{m_2}a, \ n_1 + n_2 = m_1 + m_2 + 1 = N, \quad (67)$$

From this and (56), it is obvious that $A_e$ from the previous section got deformed to $A_{eN}$, while $A_b$ and $A_{\bar{b}}$ to $A_{aN}$ and $A_{\bar{a}N}$, respectively.

The inner product (5) on $A$ is given by the integral $I$ (4) or (57). Its representation (57) is more convenient for finding its noncommutative deformation. At the level of supercomplex plane $C^{2,1}$ it is the textbook fact from quantum mechanics that the integral $\int d\bar{\chi}i d\chi i d\bar{a}da$ (this is the Liouville integral over the superphase space) is replaced under the quantization procedure by the supertrace in the Fock space. (The supertrace is the trace over the indices of the zero-fermion states minus the trace over the one-fermion states). The $\delta$ function of the operator $(\bar{\chi}^i \chi^i + \bar{a}a - 1)$ just restrict the supertrace to the trace over the indices of $H_{eN}$ minus the trace over the indices of $H_{oN}$. Hence

$$(\hat{f}, \hat{g})_N \equiv \text{STr}[\hat{f}^\dagger \hat{g}], \quad \hat{f}, \hat{g} \in A_N. \quad (69)$$

Here the graded involution $\dagger$ in the noncommutative algebra $A_N$ is defined exactly as in (53). It is now obvious that this inner product approaches in the limit $N \to \infty$ the commutative one. This detailed proof of this fact was furnished in [8].
Define a non-commutative Hamiltonian de Rham complex $\Omega_N$ of the fuzzy sphere $S^2$ as the graded associative algebra with unit

$$\Omega_N = \Omega_{0N} \oplus \Omega_{1N} \oplus \Omega_{2N},$$

where

$$\Omega_{0N} = \Omega_{2N} = \mathcal{A}_{eN}$$

and

$$\Omega_{1N} = \mathcal{A}_{aN} \oplus \mathcal{A}_{aN} \oplus \mathcal{A}_{aN} \oplus \mathcal{A}_{aN}.$$  

The multiplication in $\Omega_N$ with the standard properties with respect to the grading is entailed by one in $\mathcal{A}_N$. The product of 1-forms is given by the same formula as in the graded commutative case (14)

$$\left(A_1, A_2, \bar{A}_1, \bar{A}_2\right) \left(B_1, B_2, \bar{B}_1, \bar{B}_2\right) = A_1 B_1 + A_2 B_2 + \bar{A}_1 \bar{B}_1 + \bar{A}_2 \bar{B}_2.$$  

Of course, by definition, the r.h.s. is viewed as an element of $\Omega_{2N}$. Here we note an important difference with the graded commutative case: the product $AA$ of a 1-form $A$ with itself automatically vanishes in the commutative case but may be a non-vanishing element of $\Omega_{2N}$ in the deformed picture. The product of a 1-form and a 2-form is again set to zero by definition. Now the coboundary operator $d$ is given by

$$df \equiv (T_1 f, T_2 f, \bar{T}_1 f, \bar{T}_2 f), \quad f \in \Omega_{0N};$$

$$d(A_1, A_2, \bar{A}_1, \bar{A}_2) = T_1 \bar{A}_1 + T_2 \bar{A}_2 + \bar{T}_1 A_1 + \bar{T}_2 A_2, \quad (A_1, A_2, \bar{A}_1, \bar{A}_2) \in \Omega_{1N};$$

$$dh = 0, \quad h \in \Omega_{2N},$$

where the action of $T_i, \bar{T}_i$ is given by the noncommutative version of (8):

$$T_i X \equiv N(t_i X - (-1)^X t_i X), \quad \bar{T}_i X \equiv N(\bar{t}_i X - (-1)^X \bar{t}_i X), \quad X \in \mathcal{A}_N,$$

where

$$t_i = \chi^i a, \quad \bar{t}_i = \chi^i \bar{a}.$$  

$d$ maps $\Omega_{iN}$ to $\Omega_{i+1,N}$ and it satisfies

$$d^2 = 0, \quad d(AB) = (dA)B + (-1)^A A(dB).$$

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Using the graded involution $\dagger$, we define the standard involution $\check{\dagger}(\check{\dagger}^2 = 1)$ on the noncommutative complex $\Omega_N$:

\[ f^{\dagger} = f^{\dagger}, \quad f \in \Omega_{0N}, \quad g^{\dagger} = -g^{\dagger}, \quad g \in \Omega_{2N}; \]

\[ (A_i, \bar{A}_i)^{\dagger} = (\bar{A}_i^\dagger, -A_i^\dagger), \quad (A_i, \bar{A}_i) \in \Omega_{1N}. \]

The coboundary map $d$ is compatible with the involution, however, due to noncommutativity, it is no longer true that the product of two real elements of $\Omega_N$ gives a real element. Thus we cannot define the real noncommutative Hamiltonian de Rham complex. For field theoretical applications this is not a drawback, nevertheless, because for the formulation of the field theories we shall not need the structure of the real subcomplex, but only the involution on the complex Hamiltonian de Rham complex.

The Hodge star $\ast$ in $\Omega_N$ is defined precisely as in the commutative case:

\[ \ast : \ f \in \Omega_{0N} \rightarrow 2i\bar{a}af \in \Omega_{2N}; \]

\[ \ast : \ (A_1, A_2, \bar{A}_1, \bar{A}_2) \in \Omega_{1N} \rightarrow (iA_1, iA_2, -i\bar{A}_1, -i\bar{A}_2) \in \Omega_{1N}; \]

\[ \ast : \ h \in \Omega_{2N} \rightarrow \frac{1}{4i}(\bar{T}_iT_i - T_i\bar{T}_i - 2)h \in \Omega_{0N}. \]

This Hodge star is compatible with the involution $\dagger$. Note that the definition of the Hodge star $\ast$ on the 0-forms does not involve any ordering problem, since the operator $\bar{a}a$ commutes with the elements of $\Omega_{0N}$.

The natural inner product on $\Omega_N$, whose commutative limit is (45), is

\[ (X,Y)_N \equiv \frac{i}{2} \text{STr}[(\ast X)^\dagger Y], \quad X,Y \in \Omega_{N0}, \Omega_{N1}, \Omega_{N2}; \]

where $\text{STr}$ is the standard supertrace (cf. (69)).

**The action of $SU(2)$**

The study of the $SU(2)$ action on the deformed Hamiltonian de Rham complex $\Omega_N$ is important in view of our field theoretical applications. We require that such a $SU(2)$ action gives in the commutative limit the $SU(2)$ action on the undeformed de Rham complex, described in the previous section. This can be easily arranged, however, because the action on the undeformed
The commutative complex is generated entirely in terms of the Hamiltonian vector fields \( R_j \) whose Hamiltonians are \( r_j \):

\[
\begin{align*}
    r_+ &= \chi_1^\dagger \chi_2, \\
    r_- &= \chi_2^\dagger \chi_1, \\
    r_3 &= \frac{1}{2}(\chi_1^\dagger \chi_1 - \chi_2^\dagger \chi_2).
\end{align*}
\]  

(86)

Hence the deformed action on \( \mathcal{A}_N \) will be generated by the same Hamiltonians (86) (now understood as operators on the Fock space) but the Poisson bracket will be replaced by the commutator:

\[
R_j X = N[r_j, X], \quad X \in \mathcal{A}_N.
\]  

(87)

It is trivial to check that the commutation relations (50) are fulfilled for this definition and that the \( SU(2) \) representation so generated is unitary with respect to the deformed inner product (69) on \( \mathcal{A}_N \). Since both \( \Omega_{0N} \) and \( \Omega_{2N} \) can be identified with \( \mathcal{A}_{eN} \), which is (as in the nondeformed case) an invariant subspace of \( \mathcal{A}_N \), we have just obtained the \( SU(2) \) action on the even forms of the deformed Hamiltonian complex \( \Omega_N \). Recall that the space \( \Omega_{1N} \) of the 1-forms can be written as

\[
\Omega_1 = \mathcal{A}_{aN} \otimes \mathbb{C}^2 \oplus \mathcal{A}_{aN} \otimes \mathbb{C}^2.
\]  

(88)

As in the nondeformed case, the spaces \( \mathcal{A}_{aN} \) and \( \mathcal{A}_{bN} \) are invariant subspaces of the \( SU(2) \) action on \( \mathcal{A}_N \), given by (87), thus the formula (88) makes sense at the level of representations of \( SU(2) \). In other words, \( \oplus \) and \( \otimes \) are operations on the \( SU(2) \) representations thus defining \( \Omega_{1N} \) as a \( SU(2) \) representations which we look for. (As in the nondeformed case (51), the \( SU(2) \) acts by (52) on the second copy of \( \mathbb{C}^2 \) in (88) and on the first copy it acts in the complex conjugated way.)

The final three facts, we shall need, read: 1) the inner product (85) on \( \Omega_N \) is invariant with respect to the just defined \( SU(2) \) action; 2) the coboundary operator \( d \) on \( \Omega_N \) and the Hodge star \( * \) are both \( SU(2) \) invariant; 3) the generators \( R_i \) of \( SU(2) \) act on \( \Omega_N \) as derivations with respect to the deformed product on \( \Omega_N \).

We have constructed the non-commutative deformation of the complexified commutative Hamiltonian de Rham complex. We have identified the noncommutative counterparts of all structures of the latter and shown that they have the correct commutative limit. In particular, we observed that
the multiplication in $\Omega_N$ approaches for $N \to \infty$ the standard commutative product in $\Omega$. We have described the involution, the inner product, the Hodge star and the $SU(2)$ action on $\Omega_N$ which have also the correct commutative limits and we have established that the "$d$" in the noncommutative context has all the basic properties (79) in order to deserve to be called the coboundary operator. Thus, whatever commutative construction which we perform by using these structures can be rewritten in the deformed finite-dimensional case. In particular, we shall write the field theoretical actions in this way.

Note that we did not injected any smaller deformed de Rham complex into the deformed Hamiltonian complex. If we could do this we would not have had to bother ourselves with the Hamiltonian case! The point of our construction is that we will be able to formulate the dynamics of the standard gauge theories using the commutative Hamiltonian complex; this means that we also can deform those theories directly at this level. The price to pay is relatively low: few auxiliary fields will appear on the top of the fields present in the more standard constructions. We are here in a similar position as people who introduce auxiliary fields in trying to achieve a closure of algebras of supersymmetry without imposing the equations of motion. It turns out that our auxiliary fields are just the same as those used for the closure of the algebra of supersymmetry! Thus the same auxiliary fields do the double job. We believe that this is not just a coincidence.

Noncommutative Poincaré lemma and cohomology

Although the following few lines will be of no immediate use for our discussion of the field theoretical actions in this article, it is interesting to remark, that the finite deformed complex $\Omega_N$ preserves faithfully the cohomological content of $\Omega$ as the following noncommutative generalization of the Poincaré lemma says:

**Theorem:**

i) Let $f \in \Omega_{0N}$, $df = 0$. Then $f$ is the unit element of $\Omega_{eN}$ (unit matrix acting on $H_N$) multiplied by some complex number.

ii) Let $A \equiv (A_1, A_2, \bar{A}_1, \bar{A}_2) \in \Omega_{1N}$, $dA = 0$, $A = A^\dagger$. Then $A = dg$ for some $g \in \Omega_{0N}$, $g = g^\dagger$. 

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iii) Let $F \in \Omega_{2N}$ (i.e. $dF$ automatically vanishes), $F = F^\dagger$. Then $F$ can be written as $F = p Id + dB$, where $B \in \Omega_{1N}$, $B = B^\dagger$ is some 1-form, $Id$ is the unit element in $\mathcal{A}_{eN}$ and $p$ is an imaginary number. $Id \in \Omega_{2N}$ itself cannot be written as a coboundary of some 1-form.

Thus the theorem implies that $Id$ is the only nontrivial cohomology class in $H^2(\Omega_N)$ and $H^0(\Omega_N)$, and $H^1(\Omega_N)$ vanishes.

**Proof:**

i) One notices that the Hamiltonians $t_i, \bar{t}_i$ of the vector fields $T_i, \bar{T}_i$ generates the whole algebra $\mathcal{A}_N$ and therefore also its subalgebra $\mathcal{A}_{eN} = \Omega_{0N}$. According to (77), the $T_i, \bar{T}_i$ act on an element $f \in \Omega_{0N}$ as commutators $N[t_i, f], N[\bar{t}_i, f]$, respectively. Thus vanishing of the commutators means that $f$ commutes with all matrices in $\mathcal{A}_N$. Hence $f$ is a multiple of the unit matrix $Id$.

ii) It is easy to see that an arbitrary real 1-form $A \in \Omega_{1N}$ has $2N(N + 1)$ complex components. The condition $dA = 0$ sets on them $(N + 1)^2 + N^2 - 1$ real independent constraints (the term $-1$ comes from the fact that $\text{STr}(dA) \equiv 0$ for every $A$). Thus a solution $A = A^\dagger$ of this condition must depend on

$$2N(N + 1) = (N + 1)^2 + N^2 - 1$$

real parameters. The explicit form of the equation $dA = 0$ can be written by using the matrices $t_i, \bar{t}_i$:

$$(t_1)_{jk} = \sqrt{N - j + 1}\delta_{jk}, \quad j = 1, \ldots, N + 1; \quad k = 1, \ldots, N;$$

$$(t_2)_{jk} = \sqrt{j - 1}\delta_{j-1,k}, \quad j = 1, \ldots, N + 1; \quad k = 1, \ldots, N;$$

$$(\bar{t}_1)_{jk} = \sqrt{N - j + 1}\delta_{jk}, \quad j = 1, \ldots, N; \quad k = 1, \ldots, N + 1;$$

$$(\bar{t}_2)_{jk} = \sqrt{j}\delta_{j+1,k}, \quad j = 1, \ldots, N; \quad k = 1, \ldots, N + 1.$$ 

Now the direct inspection reveals that indeed the remaining $(N + 1)^2 + N^2 - 1$ real parameters organizes into one real element $g = g^\dagger$ of $\mathcal{A}_{0N} = \Omega_{0N}$ ($A = dg$). Note a small redundancy, however: If some $g'$ differs from $g$ by a real multiple of $Id$, it gives the same $A$; the term $(-1)$ in (89) precisely corresponds to this (cohomological) ambiguity. This means, for example,
that the solution \( g \) (\( dg = A \)) can be chosen to have a vanishing supertrace, i.e. it really has precisely \((N + 1)^2 + N^2 - 1\) free real parametres.

iii) We can do a similar counting as in ii). Again, an arbitrary real 1-form \( A \in \Omega_{1\mathbb{R}} \) has in total \(2N(N+1)\) complex components. If we consider varying \( A \), then \( dA \) sweeps a submanifold \( S \) in the manifold of real 2-forms. The real dimension of this submanifold \( S \) cannot exceed \((N + 1)^2 + N^2 - 1\). The counting is easy: since \( A = df \) is in the kernel of \( dA \), varying of \((N + 1)^2 + N^2 - 1 = 2N(N + 1)\) real parameters of \( f \) does not show up in \( dA \). There remains \((N + 1)^2 + N^2 - 1\) real parameters to vary in \( S \).

Now a real dimension of the whole manifold of the real two forms is \((N + 1)^2 + N^2\), i.e. bigger by a factor 1. Using the explicit form (90-93) of \( t_i, \bar{t}_i \), it is straightforward to check that \( S \) consists of all real 2-forms in \( \Omega_{2\mathbb{R}} \) with a vanishing supertrace. Thus the real dimension of \( S \) is, in fact, precisely \((N + 1)^2 + N^2 - 1\). The only nontrivial cohomology class in \( H^2(\Omega_N) \) is therefore supertraceful and can be chosen to be an imaginary multiple of \( \text{Id} \). The theorem is proved.

3 Gauge theories on noncommutative \( S^2 \)

3.1 Theories with a scalar matter.

Consider a complex scalar field \( \phi \) on \( S^2 \) (i.e. \( \phi \in \mathcal{B} \); cf. (21)) and an \( U(1) \) gauge connection described on the complement of the north (south) pole by a real 1-form field \( v^N = V^N(\bar{z}, z)dz + V^{*N}(\bar{z}, z)d\bar{z} \) \( (v^S = V^S(\bar{w}, w)dw + V^{*S}(\bar{w}, w)d\bar{w}) \) where \( z, w \) \( (z = 1/w) \) are the complex coordinates on the corresponding patches. In what follows, we understand \( \phi \) to be always a section of the trivial line bundle on \( S^2 \), i.e. the standard complex function. Thus we can describe the connection by one globally defined 1-form on \( S^2 \). Here we shall work only with one patch -the complement of the north pole. We encode the global character of the 1-form \( v \)

\[
v = Vdz + V^*d\bar{z}
\]

(94)
on the patch by demanding that \( V(V^*) \) is an element of a \( \mathcal{B} \)-bimodule \( \mathcal{B}_2(\mathcal{B}_2) \) (cf. (22,23)). This ensures that \( z^2V \) does not diverge for \( z \to \infty \) and, thus, the form \( v \) is well defined globally over the whole sphere \( S^2 \).
The scalar electrodynamics is defined by an action

\[ S = \frac{1}{4\pi i} \int d\bar{z} \wedge dz \{(\partial\bar{z} + iV^*)(\partial_z - iV)\phi + (\partial\bar{z} - iV^*)(\partial_z + iV)\phi^* - \frac{1}{2g^2}(1 + \bar{z}z)^2(\partial_z V - \partial_2 V^*)^2\}. \] (95)

Here \( g \) is a real coupling constant of the theory. As usual, the kinetic term of the scalar field in two dimension does not "remember" the conformal factor of the round metric \( ds^2 \) on the sphere (i.e. \( ds^2 = 4d\bar{z}dz(1 + \bar{z}z)^{-2} \)) but the kinetic term of the gauge fields must be multiplied by the inverse power of the conformal factor.

The action (95) can be written in the language of forms (elements of the standard de Rham complex) as follows

\[ S = (d\phi - iv\phi, d\phi - iv\phi)_{dR} + \frac{1}{g^2}(dv, dv)_{dR}, \] (96)

where the inner product \((.,.)_{dR}\) was defined in (44).

Consider now a theory whose multiplet of fields is given by a complex Hamiltonian 0-form \( \Phi \in \mathcal{A}_0 \) (cf. (12)) and by a real Hamiltonian 1-form \( A \in \Omega_1 \) \( (A = (A_1, A_2, \bar{A}_1, \bar{A}_2), A^\dagger = A) \) and whose action is given by

\[ S_\infty = (d\Phi - iA\Phi, d\Phi - iA\Phi) + \frac{1}{g^2}(dA, dA). \] (97)

Here the inner product \((.,.)\) on \( \Omega \) was defined in (45). (The index \( \infty \) refers to the fact that this action will be soon recovered as an \( N \to \infty \) limit of an action \( S_N \), defined on the deformed Hamiltonian complex.) In order to understand the content of this theory, let us parametrize the Hamiltonian forms \( \Phi \) and \( A \) as

\[ \Phi = \phi + \frac{Fbb}{\bar{z}z + 1}; \] (98)

\[ A_1 = -(V + \frac{P\bar{z}}{\bar{z}z + 1})b, \quad A_2 = (Vz - \frac{P}{\bar{z}z + 1})b, \]

\[ \bar{A}_1 = (V^* + \frac{P^*\bar{z}}{\bar{z}z + 1})\bar{b}, \quad \bar{A}_2 = -(V^*\bar{z} - \frac{P^*}{\bar{z}z + 1})\bar{b}, \] (99)

where all the fields \( \phi, F, V \) and \( P \) depend only on the variables \( \bar{z}, z \). It is easy to verify that if \( \phi, F \) and \( P \) belong to the algebra \( \mathcal{B} \) (cf. (21)) and \( V \) and \( V^* \) to
the bimodules $B_2$ and $B_2$ respectively, then $A$ given by (99) sweeps the space $\Omega_1$. In other words, the multiplet consisting of the complex Hamiltonian 0-form $\Phi$ and the real Hamiltonian 1-form $A$ contains three standard complex de Rham 0-forms (the scalar fields $\phi, P, F$) and one real de Rham 1-form $v = Vdz + V^*d\bar{z}$. Moreover note, that the 1-form $v$ is injected in $A$ by using the homomorphism $\nu$ (cf.(27)).

By using the ansatz (98) and (99), we first compute

$$dA = (P^* - P) + \frac{\overline{bb}}{1 + \bar{z}z}[(\partial_z V^* - \partial_{\bar{z}} V)(1 + \bar{z}z)^2 + (P^* - P)]$$

(100)

and

$$*dA = \frac{i}{2}(1 + \bar{z}z)^2[(\partial_{\bar{z}} V - \partial_z V^*) + \frac{\overline{bb}}{\bar{z}z + 1} \partial_z \partial_{\bar{z}}(P^* - P)]$$

(101)

and then for the action (97) we have

$$S_\infty = (d\phi - iv\phi, d\phi - iv\phi)_{dR} + \frac{1}{g^2}(dv, dv)_{dR}$$

$$+ \frac{1}{4g^2}(d(P^* - P), d(P^* - P))_{dR} + \frac{1}{4}((P^* - P)\phi, (P^* - P)\phi)_{dR}$$

$$+ \frac{1}{4}(2F + i(P^* + P)\phi, 2F + i(P^* + P)\phi)_{dR}$$

(102)

Here the the combination $2F + i(P^* + P)\phi$ plays the role of an auxiliary filed and therefore the last line in (102) can be obviously dropped out.

We observe that the result (102) is not quite the action (96) of the scalar electrodynamics but, apart from $\phi$ and $v$, there is one more propagating interacting field present, namely the imaginary part of $P$. This new field is neutral and it couples to the field $\phi$ only. It is not difficult to recognize, that the action (102) describes nothing but the bosonic sector of the supersymmetric extension of the Schwinger model [16]. Thus, we have a good chance that upon adding fermions to our framework we shall recover the whole supersymmetric theory!

It is not difficult to construct also the ”pure” scalar electrodynamics (96) in our approach. For doing this, we have to note that, except of applying the Hodge star $*$, there is another way how to convert the 2-form $dA \in \Omega_2$ into a 0-form from $\Omega_0$. Indeed, since both $\Omega_0$ and $\Omega_2$ can be identified with $A_e$, the

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identity map does the job. In what follows, we shall understand by the symbol $(dA)_0$ the corresponding 0-form. The standard scalar electrodynamics is then described by the action

$$S_{\infty} = (d\Phi - iA\Phi, d\Phi - iA\Phi) - \frac{1}{4g^2} I[(dA)^2_0] + \frac{i}{8} I[* (\Phi^\dagger (dA)^2_0 \Phi)].$$

(103)

Indeed, apart from the combination $2F + i(P^* + P)\phi$, also the field $(P^* - P)$ becomes auxiliary and it ought to be integrated away hence giving the action (96).

It is important to note that the actions (97) and (103) have a bigger gauge symmetry than the standard actions (102) and (96). Indeed, let $G_e$ be a group of unitary elements of $A_e$, i.e.

$$G_e = \{U \in A_e; U^\dagger U = UU^\dagger = 1\}. \quad (104)$$

An element $U$ of $G_e$ acts on $(\Phi, A)$ as follows

$$\Phi \rightarrow U\Phi, \quad A \rightarrow A - idU U^{-1}. \quad (105)$$

With the transformation law (105), the actions (97) and (103) are gauge invariant with respect to $G_e$. By inspecting (104), it is not difficult to find that the group $G_e$ decomposes in the direct product of the standard $U(1)$ gauge group consisting of all elements of the form $e^{i\lambda}, \lambda \in B$ and a $\mathbb{R}$ gauge group (gauged real line) which acts only on the auxiliary fields. This $\mathbb{R}$-subgroup drops out from the formulation involving only the dynamical fields and we are left with the standard $U(1)$ gauge transformations

$$\phi \rightarrow e^{i\lambda} \phi, \quad V \rightarrow V + \partial z \lambda, \quad V^* \rightarrow V^* + \partial \bar{z} \lambda. \quad (106)$$

We have rewritten the standard action of the scalar electrodynamics in terms of the structures of the Hamiltonian de Rham complex. This in turn means that we can directly write down a finite dimensional deformation of the field theoretical model (103) by replacing all structures occurring in (103) by their noncommutative counterparts:

$$S_N = (d\Phi - iA\Phi, d\Phi - iA\Phi)_N - \frac{1}{4g^2} STr[(dA- iA^2)^2] + \frac{i}{8} STr[* (\Phi^\dagger (dA- iA^2)^2 \Phi)].$$

(107)
Here $\Phi \in \Omega_{0N}$ is a noncommutative Hamiltonian 0-form, $A \in \Omega_{1N}$ a real noncommutative Hamiltonian 1-form and the inner product $(.,.)_N$ is given by (85).

Now we should examine the gauge invariance of this action. Consider a group $G_{eN}$ consisting of the unitary elements of $\mathcal{A}_{eN}$:

$$G_{eN} = \{ U \in \mathcal{A}_{eN}; \; U^\dagger U = UU^\dagger = 1 \}. \quad \text{(108)}$$

An element $U$ of $G_{eN}$ acts on $(\Phi, A) \in \Omega_N$ as follows

$$\Phi \rightarrow U\Phi, \quad A \rightarrow UAU^{-1} - idU \; U^{-1}. \quad \text{(109)}$$

Note that due to noncommutativity the gauge transformation looks like a non-Abelian one (only in the limit $N \rightarrow \infty$ it reduces to the standard Abelian one). In the action (107), the term $A^2$ (which identically vanishes in the commutative limit) is crucial for the gauge invariance for only with it the field strength $dA - iA^2$ transforms homogeneously:

$$dA - iA^2 \rightarrow U(dA - iA^2)U^{-1}. \quad \text{(110)}$$

It is important to keep in mind that the gauge group $G_{eN}$ is a deformation of the commutative gauge group $G_e$. The latter has its local $U(1)$ subgroup acting as in (106). In the noncommutative case, we cannot say which part of the Hamiltonian connection $A$ is auxiliary (i.e. $P$-part) and which dynamical ($v$-part). Thus neither we can identify a noncommutative "local" $U(1)$ subgroup of $G_{eN}$. This means that the full $G_{eN}$ group plays a role in the deformed theory. I believe that this fact will be crucial in getting a nonperturbative insight on the problem of chiral anomaly.

The noncommutative deformation of the model (97) is slightly more involved than the one of the standard scalar electrodynamics (103), for we have to add more terms like $A^2$ which in the commutative limit trivially vanish but they are required for the gauge invariance of the deformed model. The easiest way to proceed consists first in rewriting the undeformed action in the form

$$S_\infty = \langle d\Phi - iA\Phi, d\Phi - iA\Phi \rangle + \frac{1}{4g^2} (d(dA)_0, d(dA)_0) - \frac{1}{4g^2} I[(dA)_0^2]. \quad \text{(111)}$$

In order to derive (111) from (97) we have used the explicit form (41,42) of the Hodge star $\ast$ and the integration "per partes". Now it is easy to write a
deformation of the model (111), which is gauge invariant with respect to the transformation laws (109):

\[ S_N = (d\Phi - iA\Phi, d\Phi - iA\Phi)_N - \frac{1}{4g^2} \text{STr}[(dA - iA^2)_0^2]\]

\[ + \frac{1}{4g^2} (d(dA - iA^2)_0 - i[A, (dA - iA^2)_0], d(dA - iA^2)_0 - i[A, (dA - iA^2)_0])_N. \]

As the term \( A^2 \), also \( i[A, (dA - iA^2)_0] \) vanishes identically in the commutative limit, thus converting (112) into (111) for \( N \to \infty \).

So far we did not mention a very important property of the classical scalar electrodynamics on the sphere. Namely, the non-deformed actions (97) and (103) are invariant with respect to the \( SU(2) \) group which rotates the sphere (this \( SU(2) \) symmetry is a compact euclidean version of the standard Poincaré symmetry of field theories). This statement follows from the invariance of the inner product (45) on \( \Omega \) (recall that the action of the group \( SU(2) \) on \( A_e \) and on \( \Omega \) was defined in section 2.1), the \( SU(2) \) invariance of the coboundary operator \( d \) and of the Hodge star \( * \) and the fact that the \( SU(2) \) generators act as derivations with respect to the product on \( \Omega \) (i.e. \( R_j(\Phi\Psi) = (R_j\Phi)\Psi + \Phi(R_j\Psi) \)). But all these properties hold also in the deformed case, thus we conclude that also our deformed actions (107) and (112) are \( SU(2) \) invariant.

We end up this section by noting that one can easily formulate theories with a nontrivial potential energy of the scalar field (like an Abelian Higgs model) by adding to the action (103) a term of the form \((\ast W(\Phi^\dagger\Phi),1) \equiv \frac{i}{2} \text{STr}[\ast W(\Phi^\dagger\Phi)]\). In the deformed case we have to add to (107) \( \frac{i}{2} \text{STr}[\ast W(\Phi^\dagger\Phi)] \). Here the potential \( W \) is some real entire function. After eliminating the auxiliary fields in the commutative case, a standard potential term \((\ast W(\phi^\dagger\phi),1)_{dR} \) turns out to be added to (96). In particular, the linear function \( W \) corresponds to assigning a mass to the charged field.

### 3.2 Spinor electrodynamics

The standard (chiral or Weyl) spinor bundle on \( S^2 \) can be identified with the complex line bundle with the winding number \( \pm 1 \) (this is to say that the

\[^4\text{It may seem that the commutator gives a Poisson bracket in the commutative limit but, in fact, this is false. The truth is that only the commutator multiplied by } N \text{ gives for } N \to \infty \text{ the Poisson bracket.}\]
transition function on the overlap \( N \cap S \) of the patches is \((\xi)^{\pm \frac{1}{2}} = e^{\pm i\varphi}\) where \(\varphi\) is the asimutal angle on the sphere). The plus (minus) sign corresponds to the right (left) chirality of the spinor. We shall work only with the patch \( N \) (the complement of the north pole) parametrized by the complex coordinate \(z\). A Dirac spinor is then a sum of right and left handed Weyl spinors:

\[
\psi_D = \psi_R + \psi_L. \tag{113}
\]

Here \(\psi_R\) and \(\psi_L\) have each one complex Grassmann valued component and they represent globally well defined sections of the corresponding chiral spinor bundles iff they are elements of a \(\mathcal{B}\)-bimodules \(\mathcal{B}_1\) and \(\mathcal{B}_1\) linearly generated by the elements of the form

\[
\frac{\bar{z}^\bar{k} z^k}{(1 + \bar{z}z)^{m + \frac{1}{2}}}, \quad \max(\bar{k} - 1, k) \leq m, \quad k, \bar{k}, m \geq 0 \tag{114}
\]

and

\[
\frac{z^k \bar{z}^\bar{k}}{(1 + \bar{z}z)^{m + \frac{1}{2}}}, \quad \max(k, \bar{k} - 1) \leq m, \quad k, \bar{k}, m \geq 0, \tag{115}
\]

respectively. The fact that, say, a right handed spinor \(\psi_R\) should be an element of the bimodule (114) follows from the fact that a spinor bilinear composed of spinors of the same chirality must be in a line bundle with winding number 2. The elements of the latter, multiplied by the zweibein component

\[
e^u_z = \frac{2}{1 + \bar{z}z} \tag{116}
\]

are therefore components of holomorphic 1-forms \(Vdz\) and we already know that \(V\) must belong to \(\mathcal{B}_2\) (cf. (22)). The action of the free euclidean massless Dirac field on \(S^2\) is given by

\[
S = \frac{i}{8\pi} \int d\bar{z} \wedge dz \ e \ \bar{\psi}_D \gamma^\mu e^\mu_c (\partial_\mu + \frac{1}{8} \omega_{\mu,ab}[\gamma^a, \gamma^b]) \psi_D, \tag{117}
\]

where \(e\) is the determinant of the zweibein \(e^a_\mu\) (or square root of the determinant of the metric), \(\gamma^c\) are flat euclidean Hermitian \(\gamma\) matrices satisfying

\[
\{\gamma^a, \gamma^b\} = 2\delta^{ab} \tag{118}
\]

and

\[
\bar{\psi}_D = \bar{\psi}_R + \bar{\psi}_L \tag{119}
\]
is the conjugated Dirac spinor ($\bar{\psi}_{R(L)}$ is now left(right) handed).

It is convenient to introduce the flat holomorphic index $u$ by

$$\gamma^u \equiv \gamma^0 + i\gamma^1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \quad (120)$$

Then $\gamma^\dot{u}$ is defined as the Hermitian conjugate of $\gamma^u$. The elements of the zweibeins are

$$e^u_z = \frac{2}{1 + \bar{z}z} = e^\dot{u}_\bar{z}. \quad (121)$$

The last thing to be explained in (117) is the notion of the spin connection $\omega_{\mu, ab}$. It is defined by the requirement

$$\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega^a_{\mu b} e^b_\nu = 0, \quad (122)$$

where $\Gamma^\lambda_{\mu\nu}$ are the standard Christoffel symbols. For the round metric on the sphere one computes

$$\left(\omega_z\right)_{\dot{u}u} = \frac{1}{2} \frac{\bar{z}}{1 + \bar{z}z}, \quad \left(\omega_{\bar{z}}\right)_{\dot{u}u} = \frac{1}{2} \frac{z}{1 + \bar{z}z}. \quad (123)$$

It is important to note that the 1-form $\omega_z dz + \omega_{\bar{z}} d\bar{z}$ (with values in the Lie algebra $\text{so}(2) \equiv u(1)$) given by (123) is not globally defined in the sense of (22,23) since it is singular at the north pole $N$ (it is easy to see the singularity by looking at $\omega$ in the coordinate patch $w$). This should be the case, however, since $\omega$ is the connection on the nontrivial spinor bundle. An arbitrary $U(1)$ connection on the spinor bundle can be achieved by adding a globally well-defined form $v = V dz + V^* d\bar{z}$ (cf. (22,23)) to the spin connection $\omega$. Thus, the interaction of the Dirac field on $S^2$ with an external $U(1)$ field $v$ is described by the action

$$S_v = \frac{i}{2\pi} \int d\bar{z} \wedge dz \left\{ \bar{\psi}_R (\partial_z - \frac{1}{2} \frac{z}{1 + \bar{z}z} - iV) \psi_L + \bar{\psi}_L (\partial_{\bar{z}} - \frac{1}{2} \frac{\bar{z}}{1 + \bar{z}z} + iV^*) \psi_R \right\}. \quad (124)$$

In the two-dimensional context, one often “gets rid” of the spin connection by renormalizing the spinors $\bar{\psi}_D, \psi_D$:

$$\psi_D = \sqrt{1 + \bar{z}z} \xi_D, \quad \bar{\psi}_D = \sqrt{1 + \bar{z}z} \xi_D. \quad (125)$$

The action (124) then becomes

$$S_v = \frac{i}{2\pi} \int d\bar{z} \wedge dz \left\{ \bar{\xi}_R (\partial_z - iV) \xi_L + \bar{\xi}_L (\partial_{\bar{z}} + iV^*) \xi_R \right\}. \quad (126)$$
One should remember, however, that the spinors \( \xi_R, \bar{\xi}_L \) now belong to a \( \mathcal{B} \)-bimodule \( \mathcal{B}_1 \) linearly generated (over Grassmann numbers) by the elements of \( \mathcal{B} \) of the form
\[
\frac{z^k \bar{z}^k}{(1+\bar{z}z)^m}, \quad \text{max}(\bar{k}, k + 1) \leq m, \quad k, \bar{k}, m \geq 0
\] (127)
and \( \xi_L, \bar{\xi}_R \) to a bimodule \( \mathcal{B}_{\bar{1}} \) generated by elements
\[
\frac{\bar{z}^k z^k}{(1+\bar{z}z)^m}, \quad \text{max}(\bar{k} + 1, k) \leq m, \quad k, \bar{k}, m \geq 0.
\] (128)

The action of the massless spinor electrodynamics (the Schwinger model) on \( S^2 \) can be then written as
\[
S = \frac{i}{2\pi} \int d\bar{z} \wedge dz \{ \xi_R (\partial_z - iV) \xi_L + \bar{\xi}_L (\partial_{\bar{z}} + iV^*) \xi_R + \frac{1}{4g^2}(1+\bar{z}z)^2(\partial_z V - \partial_{\bar{z}} V^*)^2 \}.
\] (129)

Our next task will be to rewrite the spinor electrodynamics (129) in the form which would use the Hamiltonian vector fields \( T_j, \bar{T}_\bar{j} \) instead of \( \partial_z, \partial_{\bar{z}} \) and the (real) Hamiltonian 1-form \( A \in \Omega_1 \) instead of \( v \). To do this we have first to encode the spinors \( \xi_D, \bar{\xi}_D \) as elements of the algebra \( \mathcal{A} \) (cf.(1)). This is easy: define
\[
\Phi \equiv b \bar{\xi}_L + \bar{b} \xi_L, \quad \bar{\Phi} \equiv b \xi_R + \bar{b} \bar{\xi}_R;
\] (130)
the fact that \( \xi_R, \bar{\xi}_L \in \mathcal{B}_1 \) and \( \xi_L, \bar{\xi}_R \in \mathcal{B}_{\bar{1}} \) implies that \( \Phi \) and \( \bar{\Phi} \) belong to \( \mathcal{A} \). We also have to identify how the 1-form \( v \) enters into the real Hamiltonian 1-form \( A \). This is given as before in (99). With the ansatz (99) and (130), it turns out that the standard Schwinger model on \( S^2 \) can be rewritten as
\[
S = -\frac{1}{4g^2} I[(dA)_{\bar{0}}^2] + \frac{1}{2} I[\bar{T}_2 \bar{\Phi} \bar{T}_1 \Phi + T_2 \Phi \bar{T}_1 \bar{\Phi} + \bar{T}_2 \bar{\Phi} T_1 \Phi + T_2 \Phi \bar{T}_1 \bar{\Phi}].
\] (131)

The quantity \( P \) (coming from (99)) plays the role of a nondynamical auxiliary field in (131); it can be eliminated by its equation of motion to yield (129). The meaning of the symbols in (131) is the following: \( T_j, \bar{T}_j \) are covariant derivatives acting on \( \Phi \) and \( \bar{\Phi} \) as follows
\[
T_j \Phi = T_j \Phi - iA_j \Phi, \quad \bar{T}_j \Phi = \bar{T}_j \Phi - i\bar{A}_j \Phi;
\] (132)
\[
T_j \bar{\Phi} = T_j \bar{\Phi} + \bar{\Phi} iA_j, \quad \bar{T}_j \bar{\Phi} = \bar{T}_j \bar{\Phi} + \bar{\Phi} i\bar{A}_j;
\] (133)
the integral $I$ was defined in (5).

It is easy to check that the action (131) has the gauge symmetry with the gauge group $G_e$ (cf. (104)). An element $U$ of $G_e$ acts on $\Phi, \bar{\Phi}$ and $A$ as follows

$$\Phi \to U\Phi, \quad \bar{\Phi} \to \bar{\Phi}U^{-1}, \quad A \to A - idUU^{-1}. \quad (134)$$

Now it is straightforward to write the action of the deformed Schwinger model:

$$S_N = -\frac{1}{4g^2} \text{Str}[(dA - iA^2)_{0}^2] + \frac{1}{2} \text{Str}[\mathcal{T}_2 \Phi \bar{T}_1 \Phi + \mathcal{T}_2 \Phi \mathcal{T}_1 \Phi + \mathcal{T}_2 \Phi \bar{T}_1 \Phi]. \quad (135)$$

As in the case of the scalar electrodynamics, here $A \in \Omega_{1N}$ is a real noncommutative Hamiltonian 1-form and the deformed spinor fields $\Phi$ and $\bar{\Phi}$ are elements of $A_{aN} \oplus A_{a\bar{N}}$ (cf. (66) and (67)) with Grassmann coefficients. The operators $\mathcal{T}_j, \bar{T}_j$ (by a little abuse of notation we denote them in the same way as the undeformed quantities appearing in (131)) act as

$$\mathcal{T}_j \Phi = N[t_j, \Phi]_+ - iA_j \Phi, \quad \bar{T}_j \Phi = N[\bar{t}_j, \Phi]_+ - i\bar{A}_j \Phi; \quad (136)$$

$$\mathcal{T}_j \bar{\Phi} = N[t_j, \bar{\Phi}]_+ + \bar{\Phi}iA_j, \quad \bar{T}_j \bar{\Phi} = N[\bar{t}_j, \bar{\Phi}]_+ + \bar{\Phi}i\bar{A}_j. \quad (137)$$

The quantities $t_j, \bar{t}_j$ were defined in (9).

It is obvious that for $N \to \infty$ the deformed action (135) gives the undeformed one (131). The gauge symmetry group in the noncommutative case is $G_{eN}$ (cf. (108)) and the deformed fields transforms as

$$\Phi \to U \Phi, \quad \bar{\Phi} \to \bar{\Phi}U^{-1}, \quad A \to UAU^{-1} - idUU^{-1}. \quad (138)$$

A proof of the ”Poincaré” symmetry $SU(2)$ of the undeformed action (131) and of the deformed one (135) is easy: 1) the invariance of the terms not containing the fermions was already proved in the case of the scalar electrodynamics; 2) the invariance of the fermionic terms follows from the $SU(2)$ invariance of the inner products (5) and (69) and from the following commutation relations:

$$[R_3, T_1] = \frac{1}{2} T_1, \quad [R_3, T_2] = -\frac{1}{2} T_2, \quad [R_3, \bar{T}_1] = -\frac{1}{2} \bar{T}_1, \quad [R_3, \bar{T}_2] = \frac{1}{2} \bar{T}_2; \quad (139)$$

$$[R_+, T_1] = 0, \quad [R_+, T_2] = T_1, \quad [R_+, \bar{T}_1] = \bar{T}_2, \quad [R_+, \bar{T}_2] = 0; \quad (140)$$
\[ [R_-, T_1] = T_2, \quad [R_-, T_2] = 0, \quad [R_-, T_1] = 0, \quad [R_-, T_2] = -T_1. \quad (141) \]

It is interesting to remark that we can add to the Hamiltonian vector fields \( R_i, T_i, \bar{T}_1 \) one more even vector field \( Z \), generated by the Hamiltonian \( \bar{a}a = \left( \frac{\bar{b}b}{\bar{z}z+1} \right) \) and obeying

\[ [Z, R_i] = 0, \quad [Z, T_i] = -T_i, \quad [Z, \bar{T}_i] = \bar{T}_i. \quad (142) \]

Then the generators \( R_i, T_i, \bar{T}_1 \) and \( Z \) fulfil the \( osp(2, 2) \) superalgebra commutation relations (47)-(50) and (139)-(142).

Note that we can construct also the chiral electrodynamics by setting the fields \( \xi_L \) and \( \xi_R \) to zero. In the noncommutative situation the latter case corresponds to saying that both matrices \( \Phi \) and \( \bar{\Phi} \) are in \( \mathcal{A}_{aN} \). The action will continue to be (131) in the commutative case and (135) in the noncommutative one and the gauge transformations will be (134) and (138), respectively. Thus we have achieved quite an interesting result: we have naturally coupled the gauge field to a chiral fermion while having only a finite number of degrees of freedom and no fermion doubling. Perhaps it would be somewhat premature to draw too optimistic conclusions from this two-dimensional story, nevertheless, there is a clear promise that the method might work also in higher dimensions.

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