Global weak solution for a coupled compressible Navier-Stokes and Q-tensor system

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Abstract

In this paper, we study a coupled compressible Navier-Stokes/Q-tensor system modeling the nematic liquid crystal flow in a three-dimensional bounded spatial domain. The existence and long time dynamics of globally defined weak solutions for the coupled system are established, using weak convergence methods, compactness and interpolation arguments. The symmetry and traceless properties of the Q-tensor play key roles in this process.

Keywords. Navier-Stokes, Q-tensor, liquid crystals, global weak solution, symmetric, traceless.

Subject Classifications. 35A05, 76A10, 76D03.

1 Introduction

In this paper we consider the following hydrodynamic system modeling the compressible nematic liquid crystal flow in a bounded domain, which is composed of a coupled Navier-Stokes and Q-tensor equations (see [4, 39]):

\begin{align}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla (P(\rho)) &= \mathcal{L} u - \nabla \cdot (L \nabla Q \otimes \nabla Q - \mathcal{F}(Q) I_3) \\
&\quad + L \nabla \cdot (Q \mathcal{H}(Q) - \mathcal{H}(Q) Q), \\
Q_t + u \cdot \nabla Q - \Omega Q + Q \Omega &= \Gamma \mathcal{H}(Q).
\end{align}

The system (1.1)-(1.3) is subject to the following initial conditions:

\[ (\rho, \rho u, Q)|_{t=0} = (\rho_0(x), q_0(x), Q_0(x)), \quad x \in U, \]

with

\[ Q_0 \in H^1(U), \quad Q_0 \in S_0^{(3)} \quad \text{a.e. in } U, \]

and the following boundary conditions

\[ u(x, t) = 0, \quad Q(x, t) = Q_0(x), \quad \text{for } (x, t) \in \partial U \times (0, \infty). \]

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The following compatibility condition is also imposed
\[ \rho_0 \in L^2(U), \quad \rho_0 \geq 0; \quad q_0 \in L^1(U), \quad q_0 = 0 \text{ if } \rho_0 = 0; \quad \frac{|q_0|^2}{\rho_0} \in L^1(U). \]  
(1.7)

Here \( U \subset \mathbb{R}^3 \) is a smooth bounded domain, \( \rho : U \times [0, +\infty) \to \mathbb{R}^1 \) is the density function of the fluid, \( u : U \times [0, +\infty) \to \mathbb{R}^3 \) represents the velocity field of the fluid, \( P = \rho^\gamma \) stands for the pressure function with the adiabatic constant \( \gamma > 1 \), and \( Q : U \times (0, +\infty) \to S_0^{(3)} \) is the order parameter, with \( S_0^{(3)} \subset \mathbb{M}^{3 \times 3} \) representing the space of \( Q \)-tensors in dimension 3, i.e.
\[ S_0^{(3)} = \{ Q \in \mathbb{M}^{3 \times 3}; \ Q_{ij} = Q_{ji}, \ tr(Q) = 0, \ i, j = 1, \cdots, 3 \}. \]

Throughout our paper, div stands for the divergence operator in \( \mathbb{R}^3 \) and \( \mathcal{L} \) stands for the Lamé operator:
\[ \mathcal{L}u = \nu \Delta u + (\nu + \lambda) \nabla \text{div} u, \]
where \( \nu \) and \( \lambda \) are shear viscosity and bulk viscosity coefficients of the fluid, respectively, which satisfy the following physical assumptions:
\[ \nu > 0, \quad 2\nu + 3\lambda \geq 0. \]  
(1.8)

The \((i, j)\)-th entry of the tensor \( \nabla Q \odot \nabla Q \) is \( \sum_{k,l=1}^3 \nabla_i Q_{kl} \nabla_j Q_{kl} \), and \( I_3 \subset \mathbb{M}^{3 \times 3} \) stands for the 3 \( \times \) 3 identity matrix. Furthermore, \( \mathcal{F}(Q) \) represents the free energy density of the director field
\[ \mathcal{F}(Q) = \frac{L}{2} |\nabla Q|^2 + \frac{a}{2} tr(Q^2) - \frac{b}{3} tr(Q^3) + \frac{c}{4} tr^2(Q^2), \]  
(1.9)

and we denote
\[ \mathcal{H}(Q) = L \Delta Q - aQ + b \left[ Q^2 - \frac{I_3}{3} tr(Q^2) \right] - cQtr(Q^2). \]  
(1.10)

Here \( \Omega = \frac{\nabla u - \nabla^T u}{2} \) is the skew-symmetric part of the rate of strain tensor. \( L > 0, \quad \Gamma > 0, \quad a \in \mathbb{R}, \quad b > 0 \) and \( c > 0 \) are material-dependent elastic constants (c.f. [36]).

The celebrated hydrodynamic theory for nematic liquid crystals, namely the Ericksen-Leslie theory, was developed between 1958 and 1968. Afterwards Lin [21] and Lin-Liu [25,26] added a penalization term to the Oseen-Frank energy functional to relax the nonlinear constraint of unit vector length, and made a serious of important analytic work, such as existence of global weak solutions, partial regularity, etc. The corresponding compressible liquid crystal flow was studied in Wang-Yu [40], and also see [31]. On the other hand, quite recently, for a simplified Ericksen-Leslie system with the nonlinear constraint of unit vector length, Lin-Lin-Wang [27] proved the existence of global weak solutions that are smooth away from at most finitely many singular times in any bounded smooth domain of \( \mathbb{R}^2 \), and results on uniqueness of weak solutions were given in [28,41]. Moreover, for the corresponding compressible flow in one-dimensional case, the existence of global regular and weak solutions to the compressible flow of liquid crystals was obtained in [30,7]. The strong solutions in three-dimensional case was also discussed in [18,21].

Besides the Ericksen-Leslie theory, there are alternative theories that attempt to describe the nematic liquid crystal, among which the most comprehensive description is the \( Q \)-tensor theory proposed by P. G. De Gennes in [22]. Roughly speaking, a \( Q \)-tensor is a symmetric and traceless matrix which can be interpreted from the physical point of view as a suitably normalized second-order moment of the probability distribution function describing the orientation of rod-like liquid
crystal molecules (see [1,2] for details). The static theory of $Q$ tensor has been extensively studied in [1,2,32,36]. On the other hand, the mathematical analysis of the corresponding hydrodynamic system was studied in Paiciu-Zarnescu [37,38]. More precisely, they establish the existence of global weak solutions to the coupled system of incompressible Navier-Stokes equations and $Q$-tensors in both two and three dimensional cases, as well as the existence of global regular solutions in two-dimensions.

In this paper, we are interested in the compressible version of the model studied in [38]. In the current case, the fluid flow is governed by the compressible Navier-Stokes equations, and the motion of the order-parameter $Q$ is described by a parabolic type equation. It combines a usual equation describing the flow of compressible fluid with extra nonlinear coupling terms. These extra terms are induced elastic stresses from the elastic energy through the transport, which is represented by the equation of motion for the tensor order parameter $Q$:

$$(\partial_t + u \cdot \nabla)Q - S(\nabla u, Q) = \Gamma \mathcal{H},$$

where $\Gamma > 0$ is a collective rotational diffusion constant. The first term on the left hand side of the above equation is the material derivative of $Q$, which is generalized by a second term

$$S(\nabla u, Q) = (\xi A + \Omega)(Q + \frac{I_3}{3}) + (Q + \frac{I_3}{3})(\xi A - \Omega) - 2\xi (Q + \frac{I_3}{3})tr(Q\nabla u).$$

Here $A = \nabla u + \nabla^T u$ is the rate of strain tensor. The term $S(\nabla u, Q)$ appears in the equation because the order parameter distribution can be both rotated and stretched by the flow gradients. $\xi$ is a constant which depends on the molecular details of a given liquid crystal, which also measures the ratio between the tumbling and aligning effect that a shear flow would exert over the liquid crystal directors. The right hand side of the equation (1) describes the internal relaxation of the order parameter towards the minimum of the free energy. Furthermore, it is noted that in the uniaxial nematic phase, when the magnitude of the order parameter $Q$ remains constant, the coupled hydrodynamic system is reduced to the Ericksen-Leslie system with the validity of Parodi’s relation (see [4]). For the sake of simplicity in mathematical analysis, we take $\xi = 0$ in our system. And we want to point out that the case for $\xi \neq 0$ is mathematically much more challenging. There are no existing results for the coupled system by compressible Navier-Stokes and $Q$-tensors, and the goal of this paper is to establish the existence of global weak solutions for the compressible coupled system. We note that due to higher nonlinearities in the coupled system (1.1)-(1.3), compared to earlier works in [31,40], it is more difficult to study the current system mathematically.

Note that when $Q$ is absent in (1.1)-(1.3), the system is reduced to the compressible Navier-Stokes equations. For the multidimensional compressible Navier-Stokes equations, early work by Matsumura and Nishida [33,35] established the global existence with the small initial data, and later by Hoff [14,16] for discontinuous initial data. To remove the difficulties of large oscillations, Lions in [29] introduced the concept of renormalized solutions and proved the global existence of finite energy weak solutions for $\gamma > 9/5$, where the vacuum is allowed initially, and then Feireisl, et al, in [10] extended the existence results to $\gamma > 3/2$. Since the compressible Navier-Stokes equations is a sub-system to (1.1)-(1.3), one cannot expect better result than those in [10,12]. To this end, in this paper we shall study the initial-boundary value problem for large initial data in certain functional spaces with $\gamma > 3/2$. To achieve our goal, we will use a three-level approximation scheme similar to that in [10,12], which consists of Faedo-Galerkin approximation, artificial viscosity, and artificial pressure (see also [8,9,31,40]). Then, following the idea in [10],
we show that the uniform estimate of the density $\rho^{1+\alpha}$ in $L^1$ for some $\alpha > 0$ ensures the vanishing of artificial pressure and the strong compactness of the density. We will establish the weak continuity of the effective viscous flux for our systems similar to that for compressible Navier-Stokes equations as in Lions and Feireisl in [10,12,29] to remove the difficulty of possible large oscillation of the density. To obtain the related lemma on effective viscous flux, we have to make delicate analysis to deal with the coupling and interaction between $Q-$tensor and the fluid velocity, especially certain higher order terms arising from equation (1.2). It is noted that we have to exploit the structure of the system (1.1)-(1.3), and make use of certain special properties of $Q$-tensor, namely symmetry and trace-free, to obtain the necessary a priori bounds for $Q$ and the weak continuity for the effective viscous flux.

The remaining part of this paper is organized as follows. In Section 2, after the introduction of some preliminaries, we state the main existence result of this paper, namely Theorem 2.1. In Sections 3-5, we study the three-level approximations, namely Faedo-Galerkin, vanishing viscosity, and artificial pressure, respectively. Finally, in section 6, we discuss briefly the long time dynamics of the global weak solution.

2 Preliminaries

Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ the scalar product between two vectors, and

$$A : B = tr(A^T B) = tr(AB^T)$$

represents the inner product between two $3 \times 3$ matrices $A$ and $B$, $\| \cdot \|_{L^2(U)}$ will be shorthanded by $\| \cdot \|$ if necessary. We use the Frobenius norm of a matrix $|Q| = \sqrt{tr(Q^2)} = \sqrt{Q_{ij}Q_{ij}}$ and Sobolev spaces for $Q$-tensors are defined in terms of this norm. For instance,

$$L^2(U,S_0^3) = \{ Q : U \rightarrow S_0^3, \int_U |Q(x)|^2 dx < \infty \}.$$  

Meanwhile, we denote $\mathcal{D}$ as $C_0^\infty$, and $\mathcal{D}'$ in the sense of distributions. We denote by $C$ and $C_i, i = 0, 1, \cdots$ genetic constants which may depend only on $U$, the coefficients of the system (1.1)-(1.3), and the initial data $(\rho_0,u_0,Q_0)$. Special dependence will be pointed out explicitly in the text if necessary. Here and after, the Einstein summation convention will be used. We also denote the total energy by

$$E(t) = \int_U \left( \frac{1}{2} \rho|u|^2(t) + \frac{\rho^\gamma(t)}{\gamma-1} \right) dx + G(Q(t)),  \quad (2.1)$$

where

$$G(Q(t)) = \int_U \left( \frac{L}{2} |\nabla Q|^2 + \frac{a}{2} tr(Q^2) - \frac{b}{3} tr(Q^3) + \frac{c}{4} tr^2(Q^2) \right) dx. \quad (2.2)$$

An important property of the coupling system (1.1)-(1.6) is that it has a basic energy law, which indicates the dissipative nature of the system. It states that the total sum of the kinetic and internal energy are dissipated due to viscosity and internal elastic relaxation.

**Proposition 2.1.** If $(\rho,u,Q)$ is a smooth solution of the problem (1.1)-(1.6), then for any $t > 0$, the following energy dissipative law holds

$$\frac{d}{dt} E(t) + \int_U (\nu|\nabla u|^2 + (\nu + \lambda)|\text{div} u|^2) \, dx + \Gamma \int_U tr^2(\mathcal{H}) \, dx = 0. \quad (2.3)$$
Proof. Multiplying equation (1.2) with \( u \) then integrating over \( U \), using the density equation (1.1) and boundary condition (1.6) for \( u \), we get after integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \int_U \rho |u|^2 \, dx = - (\nu + \lambda) \int_U |\text{div} \, u|^2 \, dx - \nu \int_U |\nabla \, u|^2 \, dx + \int_U \rho^\gamma \text{div} \, u \, dx
\]

\[
- L \int_U (u \cdot \nabla Q) : \Delta Q \, dx + \int_U \left< u, \nabla \left[ \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \right] \right> \, dx
\]

\[
- L \int_U \nabla u : Q \Delta Q \, dx + L \int_U \nabla u : \Delta QQ \, dx. \tag{2.4}
\]

Next, we multiply equation (1.3) with \(- \mathcal{H}\), then take the trace and integrate over \( U \). Since \( \Omega + \Omega^T = 0, Q^T = Q, \text{tr}(Q) = 0 \), after integration by parts we have

\[
\frac{d}{dt} \mathcal{G}(Q(t))
\]

\[
= - \Gamma \int_U tr^2(\mathcal{H}) \, dx + L \int_U (u \cdot \nabla Q) : \Delta Q \, dx - \int_U \left< u, \nabla \left[ \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \right] \right> \, dx
\]

\[
- \frac{L}{2} \int_U (\nabla uQ + Q \nabla u^T) : \Delta Q \, dx + \frac{L}{2} \int_U (\nabla uQ + Q \nabla u) : \Delta Q \, dx
\]

\[
= - \Gamma \int_U tr^2(\mathcal{H}) \, dx + L \int_U (u \cdot \nabla Q) : \Delta Q \, dx - \int_U \left< u, \nabla \left[ \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \right] \right> \, dx
\]

\[
- L \int_U \nabla u : \Delta QQ \, dx + L \int_U \nabla u : Q \Delta Q \, dx. \tag{2.5}
\]

Adding (2.4) and (2.5) together, it yields

\[
\frac{1}{2} \frac{d}{dt} \int_U \rho |u|^2 \, dx + \frac{d}{dt} \mathcal{G}(Q(t))
\]

\[
= - (\nu + \lambda) \int_U |\text{div} \, u|^2 \, dx - \nu \int_U |\nabla \, u|^2 \, dx - \Gamma \int_U tr^2(\mathcal{H}) \, dx + \int_U \rho^\gamma \text{div} \, u \, dx. \tag{2.6}
\]

Using the density equation again, it follows after integration by parts several times that

\[
\int_U \rho^\gamma \text{div} \, u \, dx = - \int_U \left< \gamma \rho^{-1} \nabla \rho, \rho u \right> \, dx = - \frac{\gamma}{\gamma - 1} \int_U \left< \nabla \rho^\gamma, \rho u \right> \, dx
\]

\[
= \frac{\gamma}{\gamma - 1} \int_U \rho^{-1} \text{div} (\rho u) \, dx = - \frac{1}{\gamma - 1} \frac{d}{dt} \int_U \rho^\gamma \, dx. \tag{2.7}
\]

Consequently, we finish the proof after combining (2.6) and (2.7).

\[\square\]

It is worth pointing that the assumption \( c > 0 \) is necessary from a modeling point of view (see [32, 36]) so that the total energy \( \mathcal{E} \) is bounded from below.

Lemma 2.1. For any smooth solution \((\rho, u, Q)\) to the problem (1.1)-(1.6), it holds

\[
\mathcal{E}(t) \geq \int_U \left( \frac{\rho |u|^2}{2} + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + \frac{L}{2} \|\nabla Q(t)\|^2 + \frac{c}{3} \int_U \left[ tr(Q^2) + \frac{2a}{c} - \frac{2b^2}{c^2} \right] \gamma dx - \frac{1}{2c^3} (b^2 - ca)^2 |U|, \tag{2.8}
\]

where \(|U|\) represents the Lebesgue measure of the domain \( U \).
Proof. Since \( Q \in S_0^3 \), \( Q \) has three real eigenvalues at each point: \( \lambda_1, \lambda_2 \) and \(- (\lambda_1 + \lambda_2)\). Hence \( tr(Q^2) = 2(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2), \) \( tr(Q^3) = - 3\lambda_1 \lambda_2 (\lambda_1 + \lambda_2) \). Notice that

\[
tr(Q^3) = - 3\lambda_1 \lambda_2 (\lambda_1 + \lambda_2) \leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2) \left[ \frac{\varepsilon (\lambda_1 + \lambda_2)^2}{4} + \frac{1}{\varepsilon} \right] \\
\leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2) \left[ \frac{\varepsilon (\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2)}{2} + \frac{1}{\varepsilon} \right] \leq \frac{3\varepsilon}{8} tr^2(Q^2) + \frac{3}{2\varepsilon} tr(Q^2). \quad (2.9)
\]

Taking \( \varepsilon = \frac{\Gamma}{3} \) in (2.9), then we infer that

\[
G(Q) \geq \frac{L}{2} \| \nabla Q \|^2 + \int_U \frac{c}{8} tr^2(Q^2) - \frac{b^2}{2\varepsilon} \cdot \frac{a}{2} \cdot \varepsilon \cdot tr(Q^2) \ d\tau \\
= \frac{L}{2} \| \nabla Q \|^2 + \int_U \left[ tr(Q^2) + \frac{2a}{c} - \frac{2b^2}{c^2} \right]^2 \ d\tau - \frac{1}{2\varepsilon} |U| |b - ca|^2 \| \nabla Q \|^2 + E(0), \quad (2.10)
\]

Consequently, using Proposition 2.1 and Lemma 2.1, it is straightforward to deduce the following a priori bounds for \( Q \).

Corollary 2.1. For any smooth solution \((\rho, u, Q)\) to the problem (1.1)-(1.6), it holds

\[
Q \in L^{10}(0, T; U) \cap L^\infty([0, T]; H^1(U)) \cap L^2([0, T]; H^2(U)), \quad \nabla Q \in L^{20}(0, T; U). \quad (2.11)
\]

Proof. First, using Proposition 2.1 and Lemma 2.1, we have

\[
\frac{L}{2} \| \nabla Q(t) \|^2 + \frac{c}{8} \int_U \left[ tr(Q^2) + \frac{2a}{c} - \frac{2b^2}{c^2} \right]^2 \ d\tau \\
\leq \frac{1}{2\varepsilon} |U| |b - ca|^2 \| \nabla Q \|^2 + E(0) \leq \frac{1}{2\varepsilon} |U| |b - ca|^2 \| \nabla Q \|^2 + E(0),
\]

hence \( \nabla Q \in L^\infty(0, T; L^2(U)) \). Meanwhile, using Hodge inequality, it is easy to get from the above inequality that

\[
\| Q(t) \|^4_{L^2(U)} \leq |U| \int_U tr^2(Q^2) \ d\tau \leq 2|U| \int_U \left[ tr(Q^2) + \frac{2a}{c} - \frac{2b^2}{c^2} \right]^2 + \left( \frac{2a}{c} - \frac{2b^2}{c^2} \right)^2 \ d\tau \\
\leq \frac{16|U|^2}{c} \left[ \frac{1}{2\varepsilon} |b - ca|^2 \right] \| \nabla Q \|^2 + E(0) + \frac{8|U|^2}{c^4} (ac - b^2)^2,
\]

which indicates \( Q \in L^\infty(0, T; L^2(U)) \). Next, we observe that

\[
\frac{\Gamma}{2} \int_0^T \int_U L^2 \| \Delta Q(x, t) \|^2 \ d\tau \ d\tau \\
\leq \Gamma \int_0^T \int_U tr^2(\mathcal{H}) \ d\tau + \Gamma \int_0^T \int_U |aQ - b[Q^2 - \left( \frac{I_3}{3} tr(Q^2) \right)] + cQ (tr(Q^2))|^2 \ d\tau \ d\tau \\
\leq E(0) - E(t) + CT \int_0^T \| Q \|^2_{H^1(U)} \ d\tau \\
\leq \frac{1}{2\varepsilon} |b - ca|^2 \| \nabla Q \|^2 + E(0) + CT.
\]

Here \( C > 0 \) depends on \( a, b, c, \Gamma, U \) and \( E(0) \). Consequently, we know \( \Delta Q \in L^2(0, T; L^2(U)) \). Finally, we infer from Gagliardo-Nirenberg inequality that

\[
\| Q \|_{L^{10}(U)} \leq C \| Q \|_{L^6(U)} \| \Delta Q \|_{L^2(U)} + C \| Q \|_{L^6(U)} \leq C \| Q \|_{H^1(U)} \| \Delta Q \|_{L^2(U)} + C \| Q \|_{H^1(U)},
\]

\[\]
Theorem 2.1. Suppose with modified initial conditions:

For any function \( g \)

\[ \text{the proof is complete by noting that } Q \in L^\infty(0, T; H^1(U)) \text{ and } \Delta Q \in L^2(0, T; L^2(U)). \]

Next, we introduce the definition of finite energy weak solutions.

**Definition 2.1.** For any \( T > 0, (\rho, u, Q) \) is called a finite energy weak solution to the problem (1.1) - (1.6), if the following conditions are satisfied.

- \( \rho \geq 0, \rho \in L^\infty([0, T]; L^1(U)), \ u \in L^2([0, T]; H^1(U)), \)
- \( Q \in L^\infty([0, T]; H^1(U)) \cap L^2([0, T]; H^2(U)) \)
- and \( Q \in S^3 \) a.e. \( U \times [0, T] \).

- Equations (1.1) - (1.3) are valid in \( \mathcal{D}'((0, T), U) \). Moreover, (1.1) is valid in \( \mathcal{D}'((0, T), \mathbb{R}^3) \) if \( \rho, u \) are extended to be zero on \( \mathbb{R}^3 \setminus U \);
- The energy \( E \) is locally integrable on \( (0, T) \) and the energy inequality
  \[ \frac{d}{dt} E(t) + \int_U (\nu |\nabla u|^2 + (\nu + \lambda) |\text{div } u|^2 + \Gamma tr^2(\mathcal{H})) \, dx \leq 0, \text{ holds in } \mathcal{D}'(0, T). \]
- For any function \( g \in C^1(\mathbb{R}^+) \) with the property
  
  there exists a positive constant \( M = M(g) \), such that \( g'(z) = 0 \), for all \( z \geq M \),

   \[ \text{the following renormalized form of the density equation holds in } \mathcal{D}'((0, T), U) \]

   \[ g(\rho)_t + \text{div}(g(\rho)u) + (g'(\rho)\rho - g(\rho))\text{div } u = 0. \]  

Now we can state the main result of this paper on the existence of global weak solutions.

**Theorem 2.1.** Suppose \( \gamma > \frac{3}{2} \) and the compatibility condition (1.7) is satisfied. Then for any \( T > 0 \), the problem (1.1) - (1.6) admits a finite energy weak solution \( (\rho, u, Q) \) on \( (0, T) \times U \).

We shall prove Theorem 2.1 via a three-level approximation scheme which consists of Faedo-Galerkin approximation, artificial viscosity, and artificial pressure, as well as the weak convergence method.

### 3 The Faedo-Galerkin Approximation

#### 3.1 Approximate solutions

In this section, our goal is to solve the following problem

\[ \rho_t + \text{div}(\rho u) = \varepsilon \Delta \rho, \]

\[ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) + \delta \nabla \rho^3 + \varepsilon \nabla \rho \cdot \nabla u = L u - \nabla \cdot (L \nabla Q \otimes \nabla Q - \mathcal{F}(Q) I_3) + L \nabla \cdot (Q \mathcal{H}(Q) - \mathcal{H}(Q) Q), \]

\[ Q_t + u \cdot \nabla Q - \Omega Q + Q \Omega = \Gamma \mathcal{H}(Q), \]

with modified initial conditions:

\[ \rho|_{t=0} = \rho_0 \in C^3(\bar{U}), \ 0 < \underline{\rho} \leq \rho_0(x) \leq \overline{\rho}, \frac{\partial \rho_0}{\partial n} |_{\partial U} = 0, \]
\[ \rho u_{|t=0} = q(x) \in C^2(\bar{U}, \mathbb{R}^3), \quad Q_{|t=0} = Q_0(x), \quad Q_0 \in H^1(U), \quad Q_0 \in S^3_0 \text{ a.e. in } U. \]  

(3.5)

Here \( \rho \) and \( \bar{\rho} \) are two positive constants. And it is subject to the following boundary conditions

\[ \frac{\partial \rho}{\partial n} = 0, \quad u_{|\partial U} = 0, \quad Q_{|\partial U} = Q_0(x). \]  

(3.6) \hspace{1cm} (3.7)

**Remark 3.1.** It is noted that (c.f. [12]) the extra term \( \varepsilon \Delta \rho \) appearing on the right-hand side of equation (3.1) represents a “vanishing viscosity” without any physical meaning. On the other hand, such mathematical operations converts the original hyperbolic equation (1.1) to a parabolic one such that one can expect better regularity results for \( \rho \) at this point. Meanwhile, the extra quantity \( \varepsilon \nabla \rho \cdot \nabla u \) in equation (3.2) is added to cancel extra terms to establish necessary energy laws (see [3.22] below). The term \( \delta \rho^3 \) is added to achieve higher integrability for \( \rho \), which is shown in the next section.

To begin with, using a standard argument shown in [10], we have the following existence result.

**Lemma 3.1.** For the initial-boundary value problem (3.1), (3.4) and (3.6), there exists a mapping \( S = S(u) : C([0, T]; C^2(\bar{U}, \mathbb{R}^3)) \to C([0, T]; C^3(\bar{U})) \) with the following properties:

(i) \( \rho = S(u) \) is the unique classical solution of (3.1), (3.4) and (3.6);

(ii) \( \rho \exp \left( -\int_0^t \| \nabla u(s) \|_{L^\infty(U)} \, ds \right) \leq \rho(t, x) \leq \bar{\rho} \exp \left( \int_0^t \| \nabla u(s) \|_{L^\infty(U)} \, ds \right); \)

(iii) For any \( u_1, u_2 \) in the set

\[ M_k = \{ u \in C([0, T]; H^1_0(U)), \quad \text{s.t.} \quad \|u(t)\|_{L^\infty(U)} + \|\nabla u(t)\|_{L^\infty(U)} \leq k, \forall t \}, \]

it holds

\[ \|S(u_1) - S(u_2)\|_{C([0, T]; H^1(U))} \leq Tc(k, T)\|u_1 - u_2\|_{C([0, T]; H^1_0(U))} \]  

(3.8)

Next, we shall provide the following lemma which is useful for subsequent arguments in the Faedo-Galerkin approximate scheme.

**Lemma 3.2.** For each \( u \in C([0, T]; C^3_0(\bar{U}, \mathbb{R}^3)) \), there exists a unique solution \( Q \in L^\infty([0, T]; H^1(U)) \cap L^2([0, T]; H^2(U)) \) to the initial boundary value problem

\[ Q_1 + u \cdot \nabla Q - \Omega Q + Q\Omega = \Gamma H(Q), \]  

(3.9)

\[ Q_{|t=0} = Q_0(x), \quad Q_{|\partial U} = Q_0, \]  

(3.10)

with \( Q_0 \) satisfies (1.3). Moreover, the above mapping \( u \mapsto Q[u] \) is continuous from each bounded set of \( C([0, T]; C^3_0(\bar{U}, \mathbb{R}^3)) \) to \( L^\infty([0, T]; H^1(U)) \cap L^2([0, T]; H^2(U)) \). Furthermore, \( Q[u] \in S^3_0 \) a.e. in \( U \times [0, T] \).

**Proof.** For each \( u \in C([0, T]; C^3_0(\bar{U}, \mathbb{R}^3)) \), the existence of such \( Q \) is guaranteed by standard parabolic theory (c.f. [30]). To prove \( Q \) lies in \( L^\infty([0, T]; H^1(U)) \cap L^2([0, T]; H^2(U)) \), suppose \( \|u\|_{C([0, T]; C^3_0(U))} \leq M \) for some positive constant \( M \). We multiply equation (3.9) with \(-\Delta Q\), then take the trace and integrate over \( U \), using Young’s inequality, we get

\[ \frac{1}{2} \frac{d}{dt} \|\nabla Q\|^2 + \Gamma L \|\Delta Q\|^2 = \int_U \langle u \cdot \nabla Q \rangle : \Delta Q \, dx + \int_U (Q\Omega) : \Delta Q \, dx - \int_U (\Omega Q) : \Delta Q \, dx \]

\[ + \int_U \left( aQ - bQ^2 + \frac{b}{3} \text{tr}(Q^2)I_3 + cQ \text{tr}(Q^2) \right) : \Delta Q \, dx \]

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Next, multiplying equation (3.9) with $Q$, in a similar way we have

\[
\frac{1}{2} \frac{d}{dt} \|Q\|^2 = \Gamma L \int_U \Delta Q : \nabla Q \, dx - \int_U (u \cdot \nabla Q) : Q \, dx + \int_U \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \, dx \\
\leq \frac{\Gamma L}{4} \|\Delta Q\|^2 + C_2 \|Q\|^2_{H^1}.
\]

Here $C_1 > 0$ and $C_2 > 0$ are two constants which may depend on $M$, $a$, $b$, $c$, $\Gamma$ and $L$. Summing up the above two equations, we obtain

\[
\frac{d}{dt} \|Q\|^2_{H^1} + \Gamma L \|\Delta Q\|^2 \leq C \|Q\|^2_{H^1}.
\]

Using Gronwall’s inequality again, we infer that

\[
\|Q\|_{L^\infty(0,T;H^1(U))} + \|Q\|_{L^2(0,T;H^2(U))} \leq C^*, \tag{3.11}
\]

where $C^* > 0$ is a constant which may depend on $M$, $\|Q_0\|_{H^1(U)}$, $a$, $b$, $c$, $\Gamma$, $L$ and $T$.

To prove uniqueness, suppose $Q_1$ and $Q_2$ are two different solutions, then $\bar{Q} = Q_1 - Q_2$ satisfies

\[
\bar{Q}_t + u \cdot \nabla \bar{Q} - \Omega \bar{Q} + \bar{Q} \Omega = \Gamma \left( L\Delta \bar{Q} - a\bar{Q} + b[Q_1^2 - Q_2^2 - \frac{I_0}{3} \text{tr}(Q_1^2 - Q_2^2)] \right) \\
- cQ_1 \text{tr}(Q_1^2) + cQ_2 \text{tr}(Q_2^2), \tag{3.12}
\]

\[
\|\bar{Q}\|_{\partial U} = 0, \quad \|Q\|_{\partial U} = 0. \tag{3.13}
\]

Multiplying both sides of equation (3.12) with $\bar{Q}$, then taking its trace and integrating over $U$, due to the assumption $u \in C([0,T];C_0^2(\bar{U}, \mathbb{R}^3))$ and the fact that $\|Q\|_{L^\infty(0,T;H^1(U))} \leq C^*$, for $Q = Q_1, Q_2$, we get

\[
\frac{1}{2} \frac{d}{dt} \|\bar{Q}\|^2 + \Gamma L \|\nabla \bar{Q}\|^2 \\
= - \int_U (u \cdot \nabla \bar{Q}) : \bar{Q} \, dx - \Gamma a \|\bar{Q}\|^2 + \Gamma b \int_U [\bar{Q}(Q_1 + Q_2) : : \bar{Q} \, dx - \frac{\Gamma b}{3} \int_U [\bar{Q}(Q_1 + Q_2)] \text{tr}(\bar{Q}) \, dx \\
- \Gamma c \int_U \text{tr}(\bar{Q})^2 \, dx + (Q_2 - \bar{Q})(Q_2 - (Q_1 + Q_2)) \, dx \\
\leq M \|\nabla \bar{Q}\| \|\bar{Q}\| + \Gamma |a| \|\bar{Q}\|^2 + \frac{4\Gamma b}{3} \|\bar{Q}\|_{L^6(U)} \|\bar{Q}\| \|Q_1 + Q_2\|_{L^3(U)} \\
+ 2\Gamma c \|Q_2\|_{L^6(U)} \|\bar{Q}\| \left( \|Q_1\|^2_{L^6(U)} + \|Q_2\|^2_{L^6(U)} \right) \\
\leq \frac{\Gamma L}{2} \|\nabla \bar{Q}\|^2 + C \|\bar{Q}\|^2, \tag{3.14}
\]

where we used Sobolev embedding inequality, Poincaré inequality and Young’s inequality to obtain the last inequality. Here $C$ is a positive constant which $U$, $M$, $a$, $b$, $c$, $\Gamma$ and $L$. Hence we arrive at the uniqueness result by applying Gronwall’s inequality.

Then we let \( \{u_n\} \) be a bounded sequence in $C_0^2(\bar{U}, \mathbb{R}^3)$, with $\|u_n\|_{C(0,T;C_0^2(\bar{U}))} \leq M$, $\forall n \in \mathbb{N}$, and

\[
\lim_{n \to \infty} \|u_n - u\|_{C(0,T;C_0^2(\bar{U}))} = 0, \tag{3.15}
\]
for some $u \in C(0,T;C_0^0(\bar{U}))$. For the mappings $u_n \rightarrow Q_n$, $u \rightarrow Q$, we denote by $\bar{Q}_n = Q_n - Q$ and we are going to show that
\[ \lim_{n \rightarrow \infty} \| \bar{Q}_n \|_{L^\infty(0,T;H^1(U))} + \| \bar{Q}_n \|_{L^2(0,T;H^2(U))} = 0. \] (3.16)

Taking the difference of the equations given by $Q_n$ and $Q$, then taking the inner product with $-\Delta \bar{Q}_n$, we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \bar{Q}_n \|^2 + \Delta \| \bar{Q}_n \|^2 = \int_U (u_n \cdot \nabla Q_n - u \cdot \nabla Q) : \Delta \bar{Q}_n \, dx - \int_U (Q_n \Omega_n - Q \Omega) : \Delta \bar{Q}_n \, dx + \int_U (\nabla Q_n - \nabla Q) : \Delta \bar{Q}_n \, dx + \int_U \Delta \bar{Q}_n \, dx
\]
\[+ \int_U (\Omega_n \bar{Q}_n - \Omega Q) : \Delta \bar{Q}_n \, dx + \Gamma a \int_U |\bar{Q}_n| \Delta \bar{Q}_n \, dx - \Gamma b \int_U |\bar{Q}_n(Q_n + Q)| : \Delta \bar{Q}_n \, dx
\]
\[+ \epsilon \int_U (Q_n tr(Q_n) - Q tr(Q^2)) : \Delta \bar{Q}_n \, dx \]
\[= I_1 + \cdots + I_6, \] (3.17)
with $\Omega_n = \frac{\nabla u_n - \nabla u}{2}$, $n = 1, 2, \ldots$. Notice that $\|Q_n\|_{L^\infty(0,T;H^1(U))} + \|\bar{Q}_n\|_{L^2(0,T;H^2(U))} \leq C^*$ uniformly for $n \in \mathbb{N}$, we can estimate $I_1$ to $I_6$ as follows:

\[I_1 \leq \int_U (|u_n \cdot \nabla \bar{Q}_n| |\Delta \bar{Q}_n| + |u_n - u||\nabla Q||\Delta \bar{Q}_n|) \, dx \]
\[\leq \int_U (|u_n||\nabla \bar{Q}_n||\Delta \bar{Q}_n| + |u_n - u||L^\infty(U)||\nabla Q||\Delta \bar{Q}_n|) \, dx \]
\[\leq M\|\nabla \bar{Q}_n\|\|\Delta \bar{Q}_n\| + \|u_n - u\|_{L^\infty(U)} \int_U \left( \frac{12M}{1\ell L} |\nabla \bar{Q}_n|^2 + \frac{M L}{48M} |\Delta \bar{Q}_n|^2 \right) \, dx \]
\[\leq \frac{M L}{12} |\Delta \bar{Q}_n|^2 + \frac{12M C^*}{1\ell L} |u_n - u|_{L^\infty(U)} + C |\nabla \bar{Q}_n|^2. \]

\[I_2 \leq \int_U (|Q_n||\Omega_n||\Delta \bar{Q}_n| + |Q||\Omega_n - \Omega||\Delta \bar{Q}_n|) \, dx \]
\[\leq M\|Q_n\|\|\Delta \bar{Q}_n\| + \|\nabla u_n - \nabla u||L^\infty(U)\| \int_U \left( \frac{12M}{1\ell L} |Q|^2 + \frac{M L}{48M} |\Delta \bar{Q}_n|^2 \right) \, dx \]
\[\leq \frac{M L}{12} |\Delta \bar{Q}_n|^2 + C |\nabla u_n - \nabla u|_{L^\infty(U)} + C |\nabla \bar{Q}_n|^2, \]
where we used Poincaré inequality in the last step since $\bar{Q}_n|_{\partial U} = 0$. In the same way as $I_2$, we get
\[I_3 \leq \frac{M L}{12} |\Delta \bar{Q}_n|^2 + C |\nabla u_n - \nabla u|_{L^\infty(U)} + C |\nabla \bar{Q}_n|^2. \]

For $I_4$ and $I_5$, using Poincaré inequality again, it yields
\[I_4 \leq \frac{M L}{12} |\Delta \bar{Q}_n|^2 + C |\nabla \bar{Q}_n|^2, \]
\[I_5 \leq \Gamma b |Q_n + Q||\bar{Q}_n||L^3(U)||\Delta \bar{Q}_n|| \leq \frac{M L}{12} |\Delta \bar{Q}_n|^2 + C |\nabla \bar{Q}_n|^2. \]

And
\[I_6 \leq \Gamma c |Q_n||L^6(U)||\bar{Q}_n||L^6(U)||\Delta \bar{Q}_n|| + \Gamma c |Q||L^6(U)||Q_n + Q||L^6(U)||\bar{Q}_n||L^6(U)||\Delta \bar{Q}_n|| \leq \frac{M L}{12} |\Delta \bar{Q}_n|^2 + C |\nabla \bar{Q}_n|^2. \]
Here the existence and uniqueness of the solution \( Q \).

Next, following the idea in [10], we introduce a family of operators \( U \).

\[ \forall \] \( X \)

Hence we can prove (3.16) by passing \( n \to \infty \).

To finish the proof of this lemma, we finally show that

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\[ \{\psi_n\}_{n=1}^{\infty} \subset C^\infty(U, \mathbb{R}^3) \] be the eigenfunctions of the Laplacian operator that vanish on the boundary:

\[ -\Delta \psi_n = \lambda_n \psi_n \quad \text{in} \quad U, \quad \psi_n|_{\partial U} = 0. \]

Here \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) are eigenvalues and \( \{\psi_n\}_{n=1}^{\infty} \) forms an orthogonal basis of \( H^1_0(U) \). Let \( X_n := \text{span}\{\psi_1, \ldots, \psi_n\}, n = 1, 2, \ldots \) be a sequence of finite dimensional spaces.

Then we consider the following variational approximate problem for \( u_n \in C([0,T], X_n) \):

\[ \forall t \in [0,T], \forall \psi \in X_n, \]

\[ \int_U \langle \rho u_n(t), \psi \rangle \, dx - \int_U \langle q, \psi \rangle \, dx \]

\[ = \int_0^t \int_U \langle L u_n - \text{div}(\rho u_n \otimes u_n) - (\rho^\gamma + \delta \rho^\beta) - \varepsilon \nabla \rho \cdot \nabla u_n, \psi \rangle \, dxds \]

\[ - \int_0^t \int_U \langle \nabla \cdot (L \nabla Q_n \otimes \nabla Q_n - F(Q_n) I_3), \psi \rangle \, dxds \]

\[ - L \int_0^t \int_U \langle \nabla \cdot (Q_n \mathcal{H}(Q_n) - \mathcal{H}(Q_n) Q_n), \psi \rangle \, dxds. \]

Next, following the idea in [10], we introduce a family of operators

\[ \mathcal{M}[\rho] : X_n \to X_n^*, \quad \mathcal{M}[\rho] v(w) = \int_U \langle \rho v, w \rangle \, dx, \quad \forall v, w \in X_n. \]

Here the existence and uniqueness of the solution \( Q_n \) to (3.3) is guaranteed by Lemma 3.2, while \( \rho = S(u_n) \) is the unique classical solution to (3.1) given by Lemma 3.1.
Therefore, in view of (3.8) and (3.20), using standard fixed point theorem on \( \rho \) obtain a local solution \((3.3), (3.19), \) with initial and boundary conditions \((3.4)-(3.7)\).

Consequently, combined with Lemma 2.1, we have

\[
\rho_n = \mathcal{S}(u_n), \quad Q_n = Q_n[S_n], \quad q^* \in X_n^*, \quad \text{and } q^*(\psi) = \int_U (q, \psi) \, dx.
\]

Therefore, using the energy inequality (3.22) again, we know from \( N_\eta = \{ \rho \in L^1(U) \mid \inf_{x \in U} \rho \geq \eta > 0 \} \) is well defined and satisfies

\[
\| \mathcal{M}^{-1}[\rho^1] - \mathcal{M}^{-1}[\rho^2] \|_{L(X_\eta, X_n)} \leq C(n, \eta) \| \rho^1 - \rho^2 \|_{L^1(U)}. \tag{3.20}
\]

Meanwhile, due to Lemma 3.1, we may rewrite the variational problem (3.19) as: \( \forall t \in [0, T], \forall \psi \in X_n \),

\[
u u + \delta \rho \nabla \psi \rangle = \int_U \langle \mathcal{L}u_n - \text{div}(\rho_n u_n \otimes u_n) - (\rho_n^\gamma + \delta \rho_n^\beta) - \epsilon \nabla \rho_n \cdot \nabla u_n, \psi \rangle \, dx
- \int_U \langle \nabla \cdot (L \nabla Q_n \otimes \nabla Q_n - \mathcal{F}(Q_n)I_3), \psi \rangle \, dx
- L \int_U \langle \nabla \cdot (Q_n \mathcal{H}(Q_n) - \mathcal{H}(Q_n)Q_n), \psi \rangle \, dx,
\]

\[
\rho_n = \mathcal{S}(u_n), \quad Q_n = Q_n[S_n], \quad q^* \in X_n^*, \quad \text{and } q^*(\psi) = \int_U (q, \psi) \, dx.
\]

Therefore, in view of (3.8) and (3.20), using standard fixed point theorem on \( C([0, T], X_n) \), we obtain a local solution \((\rho_n, u_n, Q_n)\) on a short time interval \([0, T_n], T_n \leq T\) to the problem (3.11), (3.3), (3.19), with initial and boundary conditions (3.4)–(3.7).

Now we shall extend the local existence time \( T_n \) to \( T \). First we can derive an energy law in a similar manner as Proposition 2.1

\[
\frac{d}{dt} \int_U \left[ \frac{\rho_n u_n^2}{2} + \frac{\rho_n^\gamma}{\gamma - 1} + \frac{\delta \rho_n^\beta}{\beta - 1} + \mathcal{G}(Q_n) \right] \, dx + \int_U \left( \nu |\nabla u|^2 + (\nu + \lambda) |\text{div } u|^2 + \Gamma |\text{tr} \mathcal{H}(n)| \right) \, dx
+ \epsilon \int_U \left( \gamma \rho_n^\gamma - 2 + \delta \beta \rho_n^2 \right) |\nabla \rho_n|^2 \, dx \leq 0, \quad \forall t \in (0, T_n).
\]

Consequently, combined with Lemma 2.1, we have

\[
\int_0^{T_n} \| \nabla u_n \|^2 \, dt \leq \frac{2}{\nu} E_3[\rho_0, q_0, Q_0],
\]

with

\[
E_3[\rho_0, q_0, Q_0] = \int_U \left[ \frac{|q_0|^2}{2 \rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} + \frac{\delta \rho_0^\beta}{\beta - 1} + \mathcal{G}(Q_0) \right] \, dx + \frac{(b^2 - ca)^2}{2c^3} |U|.
\]

Meanwhile, since the \( L^2 \) norm and \( H^2 \) norm are equivalent on each finite dimensional space \( X_n \), we can deduce from Lemma 3.1 that there exists \( C_2 = C_2(n, \rho_0, q_0, Q_0, a, b, c, U) \), such that

\[
0 < C_2 \leq \rho_n(t, x) \leq \frac{1}{C_2}, \quad \forall t \in (0, T_n), \quad x \in U.
\]

Therefore, using the energy inequality (3.22) again, we know

\[
\| u_n(t) \|_{L^\infty(U)} + \| \nabla u_n(t) \|_{L^\infty(U)} \leq C_3 = C_3(n, \rho_0, q_0, Q_0, a, b, c, U), \quad \forall t \in [0, T_n],
\]

And it follows from the arguments in [10] that the map

\[
\rho \mapsto \mathcal{M}^{-1}[\rho]
\]
which allows us to extend the existence interval \((0, T_n)\) of \(u_n\) to \([0, T]\). Further, we know from Lemma 3.1 and Lemma 3.2 that the local solution \(Q_n\) and \(\rho_n\) can also be extended up to \(T\).

To finish this subsection, we summarize all the results in the following lemma, part of which is based on (3.22), arguments in Lemma 2.1 and Corollary 2.1, while (3.28) and (3.29) are due to interpolation inequalities (see [10] for details).

**Lemma 3.3.** Suppose \(\beta \geq 4\), there exists solution \((\rho_n, u_n, Q_n)\) to (3.1), (3.19), (3.9) in \((0, T) \times U\), and

\[
\sup_{t \in [0, T]} \|\rho_n(t)\|_{L^\infty(U)}^\beta \leq C[E_\delta[\rho_0, q_0, Q_0], \gamma],
\]

\[
\sup_{t \in [0, T]} \|\rho_n(t)\|_{L^\infty(U)} \leq C(E_\delta[\rho_0, q_0, Q_0], \beta),
\]

\[
\sup_{t \in [0, T]} \|\sqrt{\rho_n(t)} u_n(t)\|_{L^2(U)} \leq 2E_\delta[\rho_0, q_0, Q_0],
\]

\[
\|u_n\|_{L^2((0, T); H^1_0(U))} \leq C(E_\delta[\rho_0, q_0, Q_0], \lambda, \nu),
\]

\[
\|\rho_n\|_{L^{\beta+1}(0, T) \times U} \leq C(E_\delta[\rho_0, q_0, Q_0], \varepsilon, \delta, U),
\]

\[
\beta \|\nabla \rho_n\|_{L^2((0, T); L^2(U))} \leq C(E_\delta[\rho_0, q_0, Q_0], \beta, \delta, U, T),
\]

\[
\|Q_n\|_{L^{\infty}(0, T; H^2(U))} \leq C(E_\delta[\rho_0, q_0, Q_0], a, b, c, L, \Gamma, U, T),
\]

\[
\|Q_n\|_{L^{\infty}(0, T; H^1(U))} \leq \frac{2}{L}E_\delta[\rho_0, q_0, Q_0],
\]

\[
\|\nabla Q_n\|_{L^2((0, T) \times U)} \leq C(E_\delta[\rho_0, q_0, Q_0], a, b, c, L, \Gamma, U, T),
\]

\[
\|Q_n\|_{L^2(0, T; H^2(U))} \leq C(E_\delta[\rho_0, q_0, Q_0], a, b, c, L, \Gamma, U, T),
\]

\[
\|Q_n\|_{L^2(0, T; H^2(U))} \leq C(E_\delta[\rho_0, q_0, Q_0], a, b, c, L, \Gamma, U, T).
\]

### 3.2 Passing to limit

Now we shall employ the estimate in Lemma 3.3 to pass to the limit as \(n \to \infty\) of the solution sequence \((\rho_n, u_n, Q_n)\) to obtain a solution to the problem (3.1)-(3.7). To this end, we have to ensure that all these a priori estimates are independent of \(n\). Here and after, for the sake of convenience, we do not distinguish sequence convergence and subsequence convergence.

To begin with, it follows from [10] that if \(\beta > 4\), \(\gamma > \frac{\beta}{2}\),

\[
u_n \to \nu \quad \text{weakly in} \quad L^2(0, T; H^1_0(U, \mathbb{R}^3)),
\]

\[
\rho_n \to \rho \quad \text{in} \quad L^4((0, T) \times U),
\]

\[
\rho_n^\gamma \to \rho^\gamma, \quad \rho_n^\beta \to \rho^\beta \quad \text{in} \quad L^1((0, T) \times U);
\]

Meanwhile, using Sobolev inequality, we deduce from (3.30)-(3.33) that

\[
\left\| \frac{\partial Q_n}{\partial t} \right\|_{L^\infty(U)} \leq \|u_n\|_{L^6(U)} \|\nabla Q_n\|_{L^2(U)} + C\|\nabla u_n\|_{L^2(U)} \|Q_n\|_{L^6(U)} + \Gamma\|H_n\|_{L^\frac{4}{3}(U)}
\]

\[
\leq C\|\nabla u_n\| + C\|H_n\|,
\]

which infers that

\[
\left\| \frac{\partial Q_n}{\partial t} \right\|_{L^2((0, T); L^\frac{4}{3}(U))} \leq C(E_\delta[\rho_0, q_0, Q_0], a, b, c, L, \Gamma, \lambda, \nu, U, T).
\]

Combined with (3.33), we know from the well-known Aubin-Lions compactness theorem that

\[
\{Q_n\} \text{ is precompact in } L^2(0, T; H^1(U)).
\]
Therefore, we conclude that
\[ Q_n \to Q \text{ weakly in } L^2(0,T;H^2(U)), \text{ strongly in } L^2(0,T;H^1(U)). \]

Hence it is easy to show that \( Q \) is a weak solution to \( (3.3) \). Furthermore, we get from \( (3.24) \), \( (3.25) \) and \( (3.27) \) that \( \{\rho_n u_n\} \) is uniformly bounded in \( L^\infty(0,T;L^{2r}(U)) \). Consequently, using \( (3.34) \) and \( (3.35) \), we have
\[ \rho_n u_n \to \rho u \text{ weakly star in } L^\infty(0,T;L^{2r}(U)), \]
then we can pass to limit in the continuity equation \( (3.1) \).

Lemma 3.4. There exist \( r > 1, s > 2 \) such that \( \partial_t \rho_n, \Delta \rho_n \) are uniformly bounded in \( L^r((0,T) \times U), \nabla \rho_n \) is uniformly bounded in \( L^s((0,T) \times U) \). And the limit function \( \rho \) satisfies equation \( (3.1) \) almost everywhere on \((0,T) \times U\) and the boundary condition \( (3.6) \) in the trace sense.

We now show that for any fixed test function \( \psi \) in \( (3.19) \), \( \int_U \langle \rho_n u_n(t), \psi \rangle \, dx \) is equi-continuous in \( t \). By Lemma \( 3.3 \) and Lemma \( 3.4 \) we get for any \( 0 < \zeta < 1 \), it holds
\[
\left| \int_t^{t+\zeta} \int_U (L \nabla Q_n \odot \nabla Q_n - F(Q_n)I_3 - LQ_n \mathcal{H}(Q_n) + L \mathcal{H}(Q_n)Q_n, \nabla \psi) \, dx \, ds \right|
\leq C\zeta \|\nabla \psi\|_{L^\infty(U)} \sup_{0 \leq t \leq T} \|\nabla Q_n(t)\|^2 + 1 + C \int_t^{t+\zeta} \|Q_n\| \|\Delta Q_n\| \|\nabla \psi\|_{L^\infty(U)} \, ds
\leq C\zeta + C \|\nabla \psi\|_{L^\infty(U)} \left( \int_t^{t+\zeta} 2 \|\Delta Q_n\|^2 \, ds \right)^{\frac{1}{2}} \left( \int_t^{t+\zeta} 2 \|Q_n\|^2 \, ds \right)^{\frac{1}{2}}
\leq C\zeta^{\frac{1}{2}},
\]
\[
\left| \int_t^{t+\zeta} \int_U \langle \rho_n^2 + \delta \rho_n^2, \psi \rangle \, dx \, dt \right| \leq C\zeta \|\nabla \psi\|_{L^\infty(U)} \sup_{0 \leq t \leq T} \int_U \left( \rho_n^2 + \delta \rho_n^2 \right) \, dx \leq C\zeta,
\]
\[
\left| \int_t^{t+\zeta} \int_U \langle \mathcal{L} u_n, \psi \rangle \, dx \, ds \right| \leq C\zeta \|\nabla \psi\|_{L^2(U)} \left( \int_0^T \|\nabla u_n(t)\|^2 \, dt \right)^{\frac{1}{2}} \leq C\zeta.
\]
\[
\left| \int_t^{t+\zeta} \int_U \langle \nabla \cdot (\rho_n u_n \otimes u_n), \psi \rangle \, dx \, ds \right| \leq C\zeta \|\nabla \psi\|_{L^\infty} \sup_{0 \leq t \leq T} \int_U \rho_n \|u_n\|^2 \, dx \leq C\zeta.
\]
\[
\left| \int_t^{t+\zeta} \int_U \langle \varepsilon \nabla \rho_n \cdot \nabla u_n, \psi \rangle \, dx \, ds \right|
\leq \varepsilon \|\nabla \psi\|_{L^\infty(U)} \left( \int_t^{t+\zeta} \int_U |\nabla u_n|^2 \, dt \right)^{\frac{1}{2}} \left( \int_t^{t+\zeta} \left( \int_U |\nabla \rho_n|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \zeta \frac{1}{2} \left( \int_t^{t+\zeta} \left( \int_U |\nabla \rho_n|^2 \right)^{\frac{1}{2}} \right)^{-\frac{1}{2}}
\leq C\zeta^{\frac{1}{2} - \frac{1}{2}},
\]
where we used Lemma \( 3.3 \) for the last estimate. Hence we know (c.f. Corollary 2.1 in \( [12] \))
\[
\rho_n u_n \to \rho u \text{ in } C([0,T];L^{\frac{2r}{r+1}}_{weak}).
\]
(3.39)

Due to the compact embedding \( L^{\frac{2r}{r+1}}(U) \hookrightarrow H^{-1}(U) \) if \( \gamma > \frac{3}{2} \), we infer from \( (3.39) \) that
\[
\rho_n u_n \to \rho u \text{ in } C([0,T];H^{-1}(U)),
\]
which together with (3.34) indicates
\[ \rho_n u_n \otimes u_n \to \rho u \otimes u \quad \text{in} \quad D'((0,T) \times U). \]

Finally, the convergence of the remaining term \( \nabla \rho_n \cdot \nabla u_n \to \nabla \rho \cdot \nabla u \) in \( D'((0,T) \times U) \) follows [10].

In all, we summarize the above results as follows.

**Proposition 3.1.** The problem (3.1)–(3.7) admits a weak solution \((\rho, u, Q)\) which satisfies all estimates in Lemma 3.3. Moreover, the energy inequality (3.22) holds in \( D'((0,T) \times U) \) and there exists \( r > 1 \), such that \( \rho_t, \Delta \rho \in L^r((0,T) \times U) \) and the equation (3.1) is satisfied pointwisely in \((0,T) \times U\). In addition, \( Q \in S_0^3 \) a.e. in \([0,T] \times U\).

4 Vanishing artificial viscosity

Our next aim is to let \( \varepsilon \to 0 \) in the modified continuity equation (3.1) and velocity equation (3.2) for passing to the limit. We denote by \((\rho_\varepsilon, u_\varepsilon, Q_\varepsilon)\) the corresponding solution of the problem (3.1)–(3.7). At this point, we are lack in the bound of \( \varepsilon \nabla \rho_\varepsilon \) (see (3.29)) and consequently, it is essential for the study of strong compactness of \( \{\rho_\varepsilon\}_\varepsilon > 0 \) in \( L^1((0,T) \times U) \).

4.1 Density estimates independent of viscosity

To begin with, we deduce from (3.27) and (3.29) that
\[ \varepsilon \nabla \rho_\varepsilon \cdot \nabla u_\varepsilon \to 0 \quad \text{in} \quad L^1((0,T) \times U), \quad \varepsilon \Delta \rho_\varepsilon \to 0 \quad \text{in} \quad L^2(0,T; H^{-1}(U)). \] (4.1, 4.2)

And in the same way as last section, we get
\[ Q_\varepsilon \to Q \quad \text{weakly in} \quad L^2(0,T; H^2(U)) \] and strongly in \( L^2(0,T; H^1(U)) \). (4.3)

**Remark 4.1.** Since \( Q_\varepsilon \in S_0^3 \) a.e. in \([0,T] \times U\), it is also true that its limit \( Q \in S_0^3 \) a.e. in \([0,T] \times U\) because of the above convergence result (4.3).

More importantly, we can prove the following estimate of density independent of \( \varepsilon \).

**Lemma 4.1.** Suppose \((\rho_\varepsilon, u_\varepsilon, Q_\varepsilon)\) is a sequence of solutions to the problem (3.1)–(3.7) constructed in Proposition 3.1. Then
\[ \|\rho_\varepsilon\|_{L^{\gamma+1}((0,T) \times U)} + \|\rho_\varepsilon\|_{L^{\beta+1}((0,T) \times U)} \leq C(E_{\delta}(\rho_0, q_0, Q_0), a, b, c, \delta, \lambda, \nu, L, U, T). \] (4.4)

**Proof.** The proof is similar to [10] (c.f. Lemma 3.1). We introduce an operator \( (3.13) \)
\[ B : \{ f \in L^p(U) : \int_U f \, dx = 0 \} \to [H_{00}^{1,p}(U)]^3, \]
such that \( v = B(f) \) solves the following problem
\[ \text{div} \, v = f \quad \text{in} \, U, \quad v|_{\partial U} = 0. \]

Then we take the test function for (3.2) as
\[ \psi(t)B(\rho_\varepsilon - m_0), \psi \in D(0,T), \, 0 \leq \psi \leq 1, \, m_0 = \frac{1}{|U|} \int_U \rho(t) \, dx. \]
We note that the total mass $m_0$ is a constant such that the test function is well defined. Then direct calculations lead to

$$\int_0^T \int_U \psi(\rho_e^{\gamma+1} + \delta \rho_e^{\beta+1}) dx dt = m_0 \int_0^T \int_U \psi \left( \int_U \rho_e^\gamma + \delta \rho_e^\beta \right) dx dt + (\lambda + \nu) \int_0^T \int_U \psi u_e \nabla \cdot \mathbf{u}_e \ dx dt$$

and by (3.27), (3.28), the property of operator $B$ yields

$$\int_0^T \int_U \psi \left( \int_U \rho_e u_e \mathbf{B} \right) dx dt = \int_0^T \int_U \psi \left( \int_U \rho_e u_e \mathbf{B} \right) dx dt - \varepsilon \int_0^T \int_U \psi \left( \int_U \rho_e u_e \mathbf{B} \nabla \rho_e \right) dx dt$$

Now we estimate $I_1, \cdots, I_{10}$. By (3.23), (3.25), (3.27) and (3.28), we get

$$|I_1| \leq |m_0| T \left( \sup_{t \in [0,T]} \| \rho_e(t) \|_{L^\gamma(U)}^\gamma + \delta \sup_{t \in [0,T]} \| \rho_e(t) \|_{L^\beta(U)}^\beta \right) \leq C(E_\delta(\rho_0, q_0, Q_0), \gamma, \beta, T).$$

$$|I_2| \leq (\lambda + \nu) \| \rho_e \|_{L^2(0,T;L^2(U))} \| \nabla u_e \|_{L^2(0,T;L^2(U))} \leq C(E_\delta(\rho_0, q_0, Q_0), \delta, \gamma, \beta, \lambda, \nu, U).$$

By the property of the operator $B$, we know

$$\| B(\rho_e - m_0) \|_{H^1_0(U)} \leq C(\beta, U) \| \rho_e - m_0 \|_{L^2(U)}.$$
with
\[ g \cdot \mathbf{n}|_{\partial U} = 0, \]
we infer from (3.27) and (3.28) that
\[
|I_7| \leq \int_0^T \|\rho_\varepsilon\|_{L^3(U)} \|u_\varepsilon\|_{L^6(U)} \|\rho_\varepsilon u_\varepsilon\|_{L^2(U)} \, dt \leq \int_0^T \|\rho_\varepsilon\|_{L^3(U)}^2 \|\nabla u_\varepsilon\|_{L^2(U)}^2 \, dt
\leq C(E_\delta(\rho_0, q_0, Q_0), \delta, \beta, \lambda, \nu, U).
\]
Further, by (3.27) and (3.29), we obtain
\[
|I_8| \leq \varepsilon \|\nabla \rho_\varepsilon\|_{L^2(0,T;L^2(U))} \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(U))} + \|\nabla B(\rho_\varepsilon - m_0)\|_{L^\infty(0,T;L^\infty(U))}
\leq C(\beta, U) \varepsilon \|\nabla \rho_\varepsilon\|_{L^2(0,T;L^2(U))} \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(U))} \|\rho_\varepsilon\|_{L^\infty(0,T;L^\beta(U))}
\leq C(E_\delta(\rho_0, q_0, Q_0), \delta, \beta, \lambda, \nu, U, T), \quad \text{for } \varepsilon \leq 1.
\] (4.5)

Then by (3.25), (3.30), and (3.32), we know
\[
|I_9| \leq \int_0^T \|\nabla Q_\varepsilon\|_{L^6(U)}^2 \|\nabla B(\rho_\varepsilon - m_0)\|_{L^\frac{5}{2}(U)} + \|\nabla F(Q_\varepsilon)\|_{L^\frac{5}{2}(U)} \|\nabla B(\rho_\varepsilon - m_0)\|_{L^\frac{5}{2}(U)} \, dt
\leq C(L, U) \int_0^T (\|\nabla Q_\varepsilon\|_{L^6(U)}^2 + 1) \|\nabla B(\rho_\varepsilon - m_0)\|_{L^\frac{5}{2}(U)} \, dt
\quad + C(a, b, c) \int_0^T (\|\nabla^2(Q_\varepsilon)\|_{L^\frac{5}{2}(U)} + 1) \|\nabla B(\rho_\varepsilon - m_0)\|_{L^\frac{5}{2}(U)} \, dt
\leq C(E_\delta(\rho_0, q_0, Q_0), a, b, c, \delta, \beta, \lambda, \nu, L, U, T).
\]

Finally, we deduce from (3.25), (3.30) and (3.33)
\[
|I_{10}| \leq 2L \int_0^T \|\nabla Q_\varepsilon\|_{L^1(U)} \|\Delta Q_\varepsilon\|_{L^2(U)} \|\nabla B(\rho_\varepsilon - m_0)\|_{L^\frac{5}{2}(U)} \, dt
\leq C(E_\delta(\rho_0, q_0, Q_0), a, b, c, \delta, \beta, \lambda, \nu, L, U, T).
\]

Hence we finish the proof by summing up all previous results for $I_1, \cdots, I_{10}$.

Lemma 4.1 together with (3.25) imply that
\[ \rho_\varepsilon \to \rho \text{ in } C(0, T; L^\beta_{weak}(U)) \text{ and weakly in } L^{\beta+1}((0,T) \times U). \] (4.6)

Moreover,
\[ u_\varepsilon \to u \text{ weakly in } L^2(0, T; H^1_0(U)), \] (4.7)

which together with (3.24) and (4.6) yield
\[ \rho_\varepsilon u_\varepsilon \to \rho u \text{ in } C([0, T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(U)). \] (4.8)

Applying the same arguments as in the last section, and noting that $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$, it then follows from (4.7) and (4.8) that
\[ \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \rho u \otimes u \text{ in } D'((0, T) \times U). \] (4.9)

Meanwhile, (4.3) implies that
\[ -\nabla \cdot (\nabla Q_\varepsilon \otimes \nabla Q_\varepsilon - F(Q_\varepsilon)I_3) + L \nabla \cdot (Q_\varepsilon \mathcal{H}(Q_\varepsilon) - \mathcal{H}(Q_\varepsilon)Q_\varepsilon). \]
\[ -\nabla \cdot (\nabla Q \odot \nabla Q - \mathcal{F}(Q)I_3) + L \nabla \cdot (Q \mathcal{H}(Q) - \mathcal{H}(Q)Q) \in \mathcal{D}'((0, T) \times U). \quad (4.10) \]

In conclusion, we prove the limit \((\rho, u, Q)\) satisfies the following equations in \(\mathcal{D}'((0, T) \times U)\):

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \quad (4.11) \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p &= \mathcal{L}u - \nabla \cdot (L \nabla Q \odot \nabla Q - \mathcal{F}(Q)I_3) \\
&\quad + L \nabla \cdot (Q \mathcal{H}(Q) - \mathcal{H}(Q)Q), \quad (4.12) \\
Q_t + u \cdot \nabla Q - \Omega Q + Q\Omega &= \Gamma \mathcal{H}(Q), \quad (4.13)
\end{align*}
\]

with the initial data

\[
\rho(0) = \rho_0, \quad (\rho u)(0) = q_0, \quad Q(0) = Q_0.
\]

**Remark 4.2.** Using Lemma 4.1 and the assumption \(\beta > \gamma\) we know the pressure \(p\) in the above system (4.11)-(4.13) has the property

\[
\rho^\gamma + \delta\rho^\beta \to p \text{ weakly in } L^{\frac{\beta+1}{\beta}}((0, T) \times U). \quad (4.14)
\]

The remaining part of this section is to improve the convergence in (4.11) to be strong in \(L^1((0, T) \times U)\), such that

\[
p = \rho^\gamma + \delta\rho^\beta.
\]

### 4.2 The effective viscous flux

The quantity \(\rho^\gamma + \delta\rho^\beta - (\lambda + 2\nu)\text{div} u\) is usually referred to as the effective viscous flux. We shall find that it plays an essential role on our coupled system (see also [15, 29]).

**Lemma 4.2.** Let \((\rho_\varepsilon, u_\varepsilon, Q_\varepsilon)\) be a sequence of solutions constructed in Proposition 3.1 and \((\rho, u, Q)\) be its limit satisfying (4.11)-(4.13), respectively. Then for any \(\psi \in \mathcal{D}(0, T), \phi \in \mathcal{D}(U)\), it holds

\[
\lim_{\varepsilon \to 0^+} \int_0^T \int_U \phi(\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta - (\lambda + 2\nu)\text{div} u_\varepsilon) \rho_\varepsilon \, dx \, dt = \int_0^T \int_U \phi(p - (\lambda + 2\nu)\text{div} u) \rho \, dx \, dt \quad (4.15)
\]

**Remark 4.3.** It is worth pointing out that from the fluid mechanics point of view, the quantity \(P - (\lambda + 2\nu)\text{div} u\) appearing in (4.15) is the amplitude of the normal viscous stress augmented by the hydrostatic pressure.

**Proof.** We consider the singular integral operator

\[ A_i = \partial_x^i \Delta^{-1}, \]

or equivalently in terms of its Fourier symbol

\[ A_j(\xi) = \frac{-\sqrt{-1} \xi_j}{|\xi|^2}. \]

By Proposition 3.1, \(\rho_\varepsilon, u_\varepsilon\) satisfy (3.1) a.e. on \((0, T) \times U\) with the boundary condition (3.6). In particular, we extend \(\rho_\varepsilon, u_\varepsilon\) to be zero outside \(U\). Then it yields

\[
\partial_t \rho_\varepsilon + \text{div}(\rho_\varepsilon u_\varepsilon) = \varepsilon \text{div}(1_U \nabla \rho_\varepsilon) \in \mathcal{D}'((0, T) \times \mathbb{R}^3) \quad (4.16)
\]
with $1_U$ the characteristic function on $U$. Next, we consider the vector-valued test function

$$\varphi(t, x) = \psi(t)(\phi(x)A(\rho_z) = \psi(t)(\phi(x)(A_1(\rho_z), A_2(\rho_z), A_3(\rho_z))), \text{ where } \psi \in \mathcal{D}(0, T), \phi \in \mathcal{D}(U).$$

Analogously, after direct calculations we derive

$$\int_0^T \psi \int_U \phi(\rho_z^+ + \delta \rho_z^+ - (\lambda + 2\nu)\text{div } u) \rho_z \, dx \, dt = (\lambda + \nu) \int_0^T \psi \int_U \text{div } u(\nabla \phi, A(\rho_z)) \, dx \, dt - \int_0^T \psi \int_U (\rho_z^+ + \delta \rho_z^+)(\nabla \phi, A(\rho_z)) \, dx \, dt$$

$$- \int_0^T \psi \int_U \rho_z u \otimes u : \nabla \phi \otimes A(\rho_z) \, dx \, dt - \int_0^T \psi \int_U \phi(\rho_z u, u_A(\rho_z)) \, dx \, dt$$

$$+ \nu \int_0^T \psi \int_U (\rho_z u_A(\rho_z) \otimes \nabla \phi : \nabla u dx - \nu \int_0^T \psi \int_U \nabla A(\rho_z) : u \otimes \nabla \phi dx dt$$

$$+ \nu \int_0^T \psi \int_U \rho u \cdot \nabla \phi dx + \int_0^T \psi \int_U \rho [\rho \nabla_j A_i(\rho u)] dx$$

$$- \int_0^T \psi \int_U (L \nabla Q \otimes \nabla Q - F(Q)I_3 : \nabla (\phi A(\rho_z)) \, dx \, dt$$

$$\equiv I_1 + \cdots + I_{12}. \quad (4.17)$$

In the meantime, we can repeat the above procedures to the limit equations (4.11) and (4.12), since we have the following result from [10].

**Lemma 4.3.** Suppose $\rho \in L^2((0, T) \times U), u \in L^2(0, T; H_0^1(U))$ is a solution of (4.11) in $\mathcal{D}'((0, T) \times U)$. Then the equation (4.11) still holds in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$, provided $(\rho, u)$ are extended to be 0 in $\mathbb{R}^3 \setminus U$.

Consequently, the counterpart to (4.17) is

$$\int_0^T \psi \int_U \phi(p - (\lambda + 2\nu)\text{div } u) \rho \, dx \, dt = (\lambda + \nu) \int_0^T \psi \int_U \text{div } u(\nabla \phi, A(\rho_z)) \, dx \, dt - \int_0^T \psi \int_U P(\nabla \phi, A(\rho_z)) \, dx \, dt$$

$$- \int_0^T \psi \int_U \rho u \otimes u : \nabla \phi \otimes A(\rho_z) \, dx \, dt - \int_0^T \psi \int_U \phi(\rho u, A(\rho_z)) \, dx \, dt$$

$$+ \nu \int_0^T \psi \int_U A(\rho_z) \otimes \nabla \phi : \nabla u dx - \nu \int_0^T \psi \int_U \nabla A(\rho_z) : u \otimes \nabla \phi dx dt$$

$$+ \nu \int_0^T \psi \int_U \rho u \cdot \nabla \phi dx + \int_0^T \psi \int_U \rho [\rho \nabla_j A_i(\rho u)] dx$$

$$- \int_0^T \psi \int_U (L \nabla Q \otimes \nabla Q - F(Q)I_3 : \nabla (\phi A(\rho_z)) \, dx \, dt$$

$$\equiv J_1 + \cdots + J_{10}. \quad (4.18)$$
Due to the classical $L^p$-theory for elliptic problems, we have
\[ \|A(v)\|_{H^{1,s}(U)} \leq C(s,U)\|v\|_{L^s(\mathbb{R}^3)}, \quad 1 < s < \infty, \tag{4.19} \]
which combined with (4.6) lead to
\[ A(\rho_\varepsilon) \to A(\rho) \quad \text{in} \quad C((0,T) \times U), \tag{4.20} \]
and henceforth
\[ \nabla A(\rho_\varepsilon) \to \nabla A(\rho) \quad \text{in} \quad C([0,T]; L^\beta_{\text{weak}}(U)). \tag{4.21} \]
Therefore, direct derivations from (4.7) and (4.20) show that
\[ I_1 \to J_1, \quad I_5 \to J_5, \quad \text{as} \quad \varepsilon \to 0. \]
Meanwhile, (4.8) and (4.20) indicate that
\[ I_4 \to J_4, \quad \text{as} \quad \varepsilon \to 0. \]
By (4.7) and (4.8), we know $\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \in L^2(0,T; L^{\frac{6\gamma}{2\gamma-3}}(U))$. Then it infers from (4.9) that
\[ \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \rho u \otimes u \quad \text{weakly in} \quad L^2(0,T; L^{\frac{6\gamma}{2\gamma-3}}(U)). \]
Consequently, we infer from (4.20) that
\[ I_3 \to J_3, \quad \text{as} \quad \varepsilon \to 0, \]
provided $\beta \geq \frac{6\gamma}{2\gamma-3}$. Note that (4.21) indicates $\nabla A(\rho_\varepsilon) \to \nabla A(\rho)$ strongly in $C([0,T], H^{-1}(U))$, hence we get from (4.7) that
\[ I_6 \to J_6, \quad \text{as} \quad \varepsilon \to 0. \]
Analogously, since $\beta > 4$, we can apply similar argument as for $I_6$ to conclude
\[ I_7 \to J_7, \quad \text{as} \quad \varepsilon \to 0. \]
For $I_8$, it follows from (4.6), (4.8) and (4.19) that if $\beta > \frac{6\gamma}{2\gamma-3}$, then
\[ \rho_\varepsilon \nabla_j A_i(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j \nabla_j A_i(\rho_\varepsilon) \in L^\infty(0,T; L^\alpha(U)), \quad \text{with} \quad \frac{\gamma + 1}{2\gamma} + \frac{1}{\beta} = \frac{1}{\alpha} < \frac{5}{6}. \]
Hence we infer from the celebrated Div-Curl Lemma and compact embedding $L^\alpha(U) \hookrightarrow H^{-1}(U)$ that
\[ \rho_\varepsilon \nabla_j A_i(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j \nabla_j A_i(\rho_\varepsilon) \to \rho \nabla_j A_i(\rho u) - \rho u \nabla_j A_i(\rho) \quad \text{strongly in} \quad H^{-1}(U). \]
Then applying Lebesgue convergence theorem, we obtain
\[ \rho_\varepsilon \nabla_j A_i(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j \nabla_j A_i(\rho_\varepsilon) \to \rho \nabla_j A_i(\rho u) - \rho u \nabla_j A_i(\rho) \quad \text{strongly in} \quad L^2(0,T; H^{-1}(U)), \]
which combined with (4.7) yields
\[ I_8 \to J_8, \quad \text{as} \quad \varepsilon \to 0. \]
Using (3.26), (3.27), (3.29) and (4.19), we get
\[ I_{11} \to 0, \quad I_{12} \to 0, \quad \text{as} \quad \varepsilon \to 0. \]
It remains to prove the corresponding convergence results for $I_9$ and $I_{10}$, which are related to the order parameter $Q$. Notice that both $I_9$ and $J_9$ can be decomposed in the following manner:

$$
I_9 = - \int_0^T \psi \int_U (L \nabla Q_\varepsilon \otimes \nabla Q_\varepsilon - \mathcal{F}(Q_\varepsilon) I_3) : (A(\rho_\varepsilon) \otimes \nabla \phi) \, dx \, dt \\
- \int_0^T \psi \int_U \phi (L \nabla Q_\varepsilon \otimes \nabla Q_\varepsilon - \mathcal{F}(Q_\varepsilon) I_3) : \nabla A(\rho_\varepsilon) \, dx \, dt \\
= I_{9a} + I_{9b}.
$$

(4.22)

$$
J_9 = - \int_0^T \psi \int_U (L \nabla Q \otimes \nabla Q - \mathcal{F}(Q) I_3) : A(\rho) \otimes \nabla \phi \, dx \, dt \\
- \int_0^T \psi \int_U \phi (L \nabla Q \otimes \nabla Q - \mathcal{F}(Q) I_3) : \nabla A(\rho) \, dx \, dt \\
= J_{9a} + J_{9b}.
$$

(4.23)

Due to (4.3) and (4.20), the convergence of $I_{9a}$ to $J_{9a}$ is straightforward. While for $I_{9b}$ and $J_{9b}$, by the property of the singular integral operator $A$, it holds

$$
I_{9b} - J_{9b} = -L \int_0^T \psi \int_U \phi (\nabla Q_\varepsilon - \nabla Q) \otimes \nabla A(\rho_\varepsilon) \, dx \, dt \\
- L \int_0^T \psi \int_U \phi \nabla Q \otimes (\nabla Q_\varepsilon - \nabla Q) : \nabla A(\rho_\varepsilon) \, dx \, dt \\
- L \int_0^T \psi \int_U \phi \nabla Q \otimes \nabla Q : \nabla (A(\rho_\varepsilon) - A(\rho)) \, dx \, dt \\
+ \int_0^T \psi \int_U \phi (\mathcal{F}(Q_\varepsilon) - \mathcal{F}(Q)) \rho_\varepsilon \, dx \, dt + \int_0^T \psi \int_U \phi \mathcal{F}(Q)(\rho_\varepsilon - \rho) \, dx \, dt \\
= K_{9a} + K_{9b} + K_{9c} + K_{9d} + K_{9e}.
$$

(4.24)

Using (3.25), (4.3) and (4.19), we find $K_{9a} \to 0$, $K_{9b} \to 0$. By (3.30), (3.31), (4.3), (4.6) and Lemma 4.1, we know $K_{9d} \to 0$, $K_{9e} \to 0$. As for $K_{9c}$, we deduce from (4.21) that for a.e. fixed $t \in [0, T]$, it holds

$$
\psi(t) \int_U \phi(x) \nabla Q \otimes \nabla Q : (\nabla A(\rho_\varepsilon) - \nabla A(\rho)) \, dx \to 0, \text{ as } \varepsilon \to 0.
$$

Meanwhile, since $\beta > 4$, using Holder’s inequality, we obtain from (4.19) and Lemma 4.1 that $\forall \varepsilon > 0, \forall t \in [0, T],$

$$
\left| \psi(t) \int_U \phi(x) \nabla Q \otimes \nabla Q : (\nabla A(\rho_\varepsilon) - \nabla A(\rho)) \, dx \right| \\
\leq C \| \nabla Q \|_{L^2(U)}^2 \| \nabla A(\rho_\varepsilon) - \nabla A(\rho) \|_{L^2(U)} \\
\leq C(E_\beta(p_0, q_0, Q_0), a, b, c, \beta, L, U, T) \| \nabla Q \|_{L^2(U)}^2,
$$

with the right hand side term being integrable on $(0, T)$ due to (3.32). Hence we conclude that $K_{9c} \to 0$ after applying Lebesgue’s convergence theorem. In all, we prove

$$
I_9 \to J_9 \text{ as } \varepsilon \to 0.
$$
For $I_{10}$, we have

$$I_{10} = L \int_{0}^{T} \psi \int_{U} (Q_{e} \mathcal{H}(Q_{e}) - \mathcal{H}(Q_{e})Q_{e}) : A(\rho_{e}) \otimes \nabla \phi \, dx \, dt$$

$$+ L \int_{0}^{T} \psi \int_{U} (Q_{e} \mathcal{H}(Q_{e}) - \mathcal{H}(Q_{e})Q_{e}) : \phi \nabla A(\rho_{e}) \, dx \, dt$$

$$\equiv I_{10a} + I_{10b}. \quad (4.25)$$

Notice that $Q_{e} = Q_{e}^{T}$, hence $Q_{e} \mathcal{H}(Q_{e}) - \mathcal{H}(Q_{e})Q_{e}$ is skew-symmetric. And it is observed that $\nabla A$ is symmetric. Therefore, we conclude

$$I_{10b} = 0. \quad (4.26)$$

**Remark 4.4.** We want to point out that the special property of $Q$-tensor is of great importance here, for otherwise we are not able to control the higher order terms in $I_{10b}$.

We proceed to show the convergence of $I_{10}$ to $J_{10}$.

$$I_{10} - J_{10} = I_{10a} - J_{10}$$

$$= L \int_{0}^{T} \psi \int_{U} (Q_{e} \Delta Q_{e} - \Delta Q_{e}Q_{e}) : (A(\rho_{e}) - A(\rho)) \otimes \nabla \phi \, dx \, dt$$

$$+ L \int_{0}^{T} \psi \int_{U} ((Q_{e} - Q) \Delta Q_{e} - \Delta Q_{e}(Q_{e} - Q)) : A(\rho) \otimes \nabla \phi \, dx \, dt$$

$$+ L \int_{0}^{T} \psi \int_{U} (Q_{e}(\Delta Q_{e} - \Delta Q) - (\Delta Q_{e} - \Delta Q)Q_{e}) : A(\rho) \otimes \nabla \phi \, dx \, dt$$

$$\equiv K_{10a} + K_{10ab} + K_{10ac}. \quad (4.28)$$

By (3.30), (3.31), (3.33), (4.3) and (4.20), it is easy to see that

$$K_{10a} \to 0, \quad K_{10ab} \to 0, \quad K_{10ac} \to 0, \quad \text{as} \ \varepsilon \to 0,$$

hence

$$I_{10} \to J_{10}, \quad \text{as} \ \varepsilon \to 0.$$

Summing up all the above convergence results, we finish the proof of Lemma 4.2.

**4.3 Strong convergence of density**

In this subsection we shall show that

$$p = \rho^{\gamma} + \delta \rho^{3},$$

and consequently the strong convergence of $\rho_{e}$ in $L^{1}((0, T) \times U)$. By Lemma 4.3, we can take the standard mollifier $\vartheta_{m} = \vartheta_{m}(x)$ to equation (4.11), such that

$$\partial_{t} S_{m}(\rho) + \text{div} (S_{m}(\rho)u) = r_{m}, \quad \text{on} \ (0, T) \times \mathbb{R}^{3}, \quad (4.27)$$

with $S_{m}(\rho) = \vartheta * \rho$ and $r_{m} \to 0$ in $L^{1}((0, T) \times U)$ (c.f. [?]). Then for any $g$ satisfying (2.12), we can multiply (4.27) with $g(S_{m}(\rho))$ and pass to the limit as $m \to \infty$. Then we may argue that (5) $(\rho, u)$ solve (4.11) in the sense of renormalized solutions, namely, (2.13) holds in $\mathcal{D}'((0, T) \times U)$.  

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Proposition 4.1. Suppose \( \beta > \max\{\frac{\gamma}{2}, \gamma, 4\} \). Then for any given \( T > 0 \) and \( \delta > 0 \), there exists a finite energy weak solution \((\rho, u, Q)\) to the problem

\[
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla (\rho^\gamma + \delta \rho^\beta) = L u - \nabla \cdot (L \nabla Q \otimes \nabla Q - F(Q) I_3) + L \nabla \cdot (Q H(Q) - H(Q) Q), \\
Q_t + u \cdot \nabla Q - Q \Omega + Q \Omega = \Gamma H(Q),
\]

with initial and boundary conditions \((3.4)\) - \((3.7)\). Furthermore, \( \rho \in L^{3+1}((0,T) \times U) \) and the equation \((4.30)\) is satisfied in the sense of renormalized solutions on \(D'((0,T) \times \mathbb{R}^3)\) provided \( \rho, u \) are extended to be zero on \( \mathbb{R}^3 \setminus U \). In addition, the following estimates are valid:

\[
\sup_{t \in [0,T]} \| \rho(t) \|_{L^\gamma(U)} \leq C(E_\delta(\rho_0, q_0, Q_0), \gamma),
\]

Instead of the strong restrictions on \( g \) in \((2.12)\), one can use the Lebesgue convergence theorem to relax the assumptions in Definition \((2.1)\) to any function \( b \in C^1(0, \infty) \cap C[0, \infty) \) with

\[
|g'(z)z| \leq C(z^\theta + z^{\frac{\gamma}{2}}), \quad \forall z > 0 \text{ and some } 0 < \theta < \frac{\gamma}{2}.
\]

Hence we may choose \( g(z) = z \ln(z) \) and integrate \((2.13)\) to obtain

\[
\int_0^T \int_U \rho \text{div} u \, dx dt = \int_U \rho_0 \ln(\rho_0) \, dx - \int_U \rho(T) \ln(\rho(T)) \, dx.
\]

(4.28)

Meanwhile, using Lemma \((3.4)\) and the convexity of \( g(z) = z \ln(z) \), we know

\[
\partial_t g(\rho_e) + \text{div}(g(\rho_e) u_e) + \rho_e \text{div} u_e - \varepsilon \Delta g(\rho_e) \leq 0,
\]

which leads to

\[
\int_0^T \int_U \rho_e \text{div} u_e \, dx dt = \int_U \rho_0 \ln(\rho_0) \, dx - \int_U \rho_e(T) \ln(\rho_e(T)) \, dx.
\]

(4.29)

Taking two nondecreasing sequences \( \phi_n \in D(0,T), \phi_n \in D(U) \) of nonnegative functions with \( \psi_n \to 1, \phi_n \to 1 \) as \( n \to \infty \). By Lemma \((1.2)\), \((4.28)\) and \((4.29)\), one can apply standard arguments to show that

\[
\lim \sup_{\varepsilon \to 0^+} \int_0^T \psi_n \int_U \phi_n \rho_e^\gamma + \delta \rho_e^\beta \rho_e \, dx dt \leq \int_0^T \int_U P \rho \, dx dt, \quad \text{for all } n = 1, 2, \ldots
\]

Notice that \( P(z) = z^\gamma + \delta z^\beta \) is monotone, by Minty’s trick, we have

\[
\int_0^T \psi_m(t) \int_U \phi_m(x) (P(\rho_e) - P(v))(\rho_e - v) \, dx dt \geq 0.
\]

Consequently, taking \( n \to \infty \), we obtain after rearrangement that for any \( v = \rho + \kappa \phi, \phi \in D(U) \), it holds

\[
\int_0^T \int_U (p - P(v))(\rho - v) \, dx dt \geq 0.
\]

Let \( \kappa \to 0 \), we come to the conclusion

\[
p = \rho^\gamma + \delta \rho^\beta.
\]

In all, we may summarize the above results in the following proposition.
\[
\delta \sup_{t \in [0,T]} \| \rho(t) \|_{L^2(U)}^2 \leq C(E_\delta(\rho_0, q_0, Q_0), \beta), \quad (4.34)
\]
\[
\sup_{t \in [0,T]} \left\| \sqrt{\rho}(t) u(t) \right\|_{L^2(U)}^2 \leq 2E_\delta(\rho_0, q_0, Q_0), \quad (4.35)
\]
\[
\| u \|_{L^2(0,T;H^1_0(U))} \leq C(E_\delta(\rho_0, q_0, Q_0), \lambda, \nu), \quad (4.36)
\]
\[
\| Q \|_{L^1((0,T) \times U)} \leq C(E_\delta(\rho_0, q_0, Q_0), a, b, c, L, \Gamma, U, T), \quad (4.37)
\]
\[
\| \nabla Q \|_{L^2((0,T) \times U)} \leq C(E_\delta(\rho_0, q_0, Q_0), a, b, c, L, \Gamma, U, T), \quad (4.38)
\]
\[
\| Q \|_{L^2(0,T;H^2(U))} \leq C(E_\delta(\rho_0, q_0, Q_0), a, b, c, L, \Gamma, U, T). \quad (4.39)
\]

**Remark 4.5.** The initial conditions (3.4)–(3.5) are satisfied in the weak sense, since we infer from (4.3) and (4.8) that
\[
\rho_\varepsilon \to \rho \text{ in } C(0,T; L^2_{\text{weak}}(U)), \quad \rho_\varepsilon u_\varepsilon \to \rho u \text{ in } C([0,T]; L^{2+\gamma}_{\text{weak}}(U)).
\]

### 5 Vanishing artificial pressure

In this section, we denote by \((\rho_\delta, u_\delta, Q_\delta)\) the corresponding approximate solutions constructed in Proposition 4.1. We are going to finish the third level approximation, namely, we shall provide the convergence of solutions of \((\rho_\delta, u_\delta, Q_\delta)\) to the solution of the original problem (1.1)–(1.3) as \(\delta\) goes to 0.

To begin with, we relax the conditions on the general initial data \((\rho_0, u_0, Q_0)\). It is easy to find a sequence \(\rho_\delta \in C_0^0(\bar{U})\) with the property
\[
0 \leq \rho_\delta(x) \leq \frac{1}{2} \delta^{-\frac{1}{2}}, \quad \text{and} \quad \| \rho_\delta - \rho_0 \|_{L^2(U)} < \delta.
\]

Taking \(\rho_{0,\delta} = \rho_\delta + \delta\), due to (3.4), then we have
\[
0 < \delta \leq \rho_{0,\delta} \leq \delta^{-\frac{1}{2}}, \quad \frac{\partial \rho_{0,\delta}}{\partial n} = 0, \quad (5.1)
\]
with
\[
\rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma(U) \quad \text{as} \quad \delta \to 0. \quad (5.2)
\]
Set
\[
\tilde{q}_\delta(x) = \begin{cases} q(x) \sqrt{\frac{\rho_{0,\delta}}{\rho_0}}, & \text{if } \rho_0(x) > 0, \\ 0, & \text{if } \rho_0(x) = 0. \end{cases} \quad (5.3)
\]

Then it follows from (1.7) that \(\frac{\left| q_\delta \right|^2}{\rho_{0,\delta}}\) is uniformly bounded in \(L^1(U)\). At the same time, it is easy to find \(h_\delta \in C^2(\bar{U})\) such that
\[
\left\| \frac{\tilde{q}_\delta}{\sqrt{\rho_{0,\delta}}} - h_\delta \right\|_{L^2(U)} < \delta.
\]

Consequently, we choose \(q_\delta = h_\delta \sqrt{\rho_{0,\delta}}\) and one can readily check that
\[
\frac{|q_\delta|^2}{\rho_{0,\delta}} \text{ are uniformly bounded in } L^1(U), \quad (5.4)
\]
\[ q_\delta \to q \text{ in } L^1(U) \text{ as } \delta \to 0. \]  

(5.5)

In what follows, we shall deal with the sequence of approximate solutions \((\rho_\delta, u_\delta, Q_\delta)\) to the problem (1.30)-(1.32) with the initial data \((\rho_\delta, q_\delta, Q_0)\).

**Remark 5.1.** We want to point out that due to the above modifications, the estimates (4.33)-(4.40) are independent of \(\delta\) because the constant \(E_\delta(\rho_0, q_0, Q_0)\) defined in (3.23) is independent of \(\delta\).

Now we shall develop some pressure estimates independent of \(\delta > 0\). Notice that the continuity equation (4.30) is satisfied in the sense of renormalized solutions in \(D'((0, T) \times \mathbb{R}^3)\), hence we may apply the standard mollifying operator to both sides of (2.13) and get

\[ \partial_t S_m[g(\rho)] + \text{div}(S_m[g(\rho)u]) + S_m[(g'(\rho)\rho - g(\rho))\text{div}u] = r_m, \]  

with \(r_m \to 0\) in \(L^2(0, T; L^2(\mathbb{R}^3))\) as \(m \to \infty\). Using the operator \(B\) introduced in the proof of Lemma 4.1, we take the test function to (4.31) to be

\[ \phi_i(t, x) = \psi(t)B_i\left\{ S_m[g(\rho_\delta)] - \frac{1}{|U|} \int_U S_m[g(\rho_\delta)]dx \right\}, \quad i = 1, 2, 3, \psi \in D(0, T). \]

Next, we can approximate the function \(g(z)\) by a sequence of function \(\{z^\theta \chi_n(z)\}\), where each \(\chi_n(z)\) being a cutoff function such that \(\chi_n(z) = 1\) on \([0, n]\) and \(\chi_n(z) = 0\) on \(z > 2n\). Then using all the estimates (4.33)-(4.40), we have

**Lemma 5.1.** For \(\gamma > \frac{3}{2}\), there exists a constant \(\theta\) that only depends on \(\gamma\), such that

\[ \int_0^T \int_U \left( \rho_\delta^{\gamma+\theta} + \delta \rho_\delta^{\beta+\theta} \right) dxdt \leq C(\rho_0, q_0, Q_0, a, b, c, \lambda, \nu, \gamma, \beta, \Gamma, L, U, T), \]

provided \(0 < \theta < \min \{1, \frac{\gamma}{\gamma+1}, \frac{2}{\gamma} - 1\}\)

**Proof.** Since the technique is quite similar to Lemma 4.1, we shall skip the details of proof and leave it to interested readers. It is noted that the right hand side bound is independent of \(\delta\). \(\square\)

### 5.1 The limit passage and the effective viscous flux

We conclude from the uniform estimates (4.33)-(4.40) in Proposition 4.1 and Lemma 5.1 that

\[ \rho_\delta \to \rho \text{ in } C([0, T]; L^\gamma_{\text{weak}}(U)), \]  

(5.7)

\[ \rho_\delta \to \rho^\gamma \text{ weakly in } L^{2+\gamma}_w((0, T) \times U), \]  

(5.8)

\[ u_\delta \to u \text{ weakly in } L^2(0, T; H^1_0(U)), \]  

(5.9)

\[ \rho_\delta u_\delta \to \rho u \text{ in } C([0, T]; L^\gamma_{\text{weak}}(U)), \]  

(5.10)

\[ Q_\delta \to Q \text{ weakly in } L^2(0, T; H^2(U)), \]  

(5.11)

\[ Q_\delta \to Q \text{ strongly in } L^2(0, T; H^1(U)), \]  

(5.12)
which infers
\[ \rho_\delta u_\delta \otimes u_\delta \rightarrow \rho u \otimes u \text{ in } \mathcal{D}'((0,T) \times U), \] (5.13)
and
\[ \nabla Q_\delta \otimes \nabla Q_\delta - \mathcal{F}(Q_\delta) I_3 - L(Q_\delta \mathcal{H}(Q_\delta) - \mathcal{H}(Q_\delta)Q) \rightarrow \nabla Q \otimes \nabla Q - \mathcal{F}(Q) I_3 - L(Q \mathcal{H}(Q) - \mathcal{H}(Q)Q) \text{ in } L^1((0,T) \times U). \] (5.14)

Further, Lemma 5.1 implies that
\[ \delta \rho_\delta^3 \rightarrow 0 \text{ in } L^1((0,T) \times U). \] (5.15)

Therefore, the limit \((\rho, u, Q)\) satisfies
\[ \rho_t + \nabla \cdot (\rho u) = 0, \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \] (5.16)
\[ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \rho \gamma = \mathcal{L} u - \nabla \cdot (L \nabla Q \otimes \nabla Q - \mathcal{F}(Q) I_3) + L \nabla \cdot (Q \mathcal{H}(Q) - \mathcal{H}(Q)Q), \] (5.17)
\[ Q_t + u \cdot \nabla Q - \Omega Q + Q \Omega = \Gamma \mathcal{H}(Q), \] (5.18)
in \( \mathcal{D}'((0,T) \times U) \). And the initial data (1.4) is satisfied due to (5.2) and (5.5).

In what follows, our ultimate goal is to show that \( \Omega = 0 \) in \( L^1 \). Consider a family of cut-off functions by \( T_k(x) = kT(\frac{x}{k}) \) for \( z \in \mathbb{R}, k = 1, 2, 3 \ldots \) and \( T \in C^\infty(R) \) is chosen to be
\[ T(z) = z \text{ for } z \leq 1, \quad T(z) = 2 \text{ for } z \geq 3, \quad T \text{ is concave.} \]

Since \((\rho_\delta, u_\delta)\) is a normalized solution to (5.16), it holds
\[ T_k(\rho_\delta)_t + \nabla \cdot (T_k(\rho_\delta) u_\delta) + (T_k'(\rho_\delta) - T_k(\rho_\delta)) \text{div } u_\delta = 0, \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \] (5.19)
from which we get after passing to limit for \( \delta \rightarrow 0 \) that
\[ \overline{T_k(\rho)}_t + \nabla \cdot (\overline{T_k(\rho)} u) + (\overline{T_k'(\rho)} - \overline{T_k(\rho)}) \text{div } u = 0, \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3). \] (5.20)

Here
\[ (T_k'(\rho_\delta) - T_k(\rho_\delta)) \text{div } u_\delta \rightarrow (T_k'(\rho) - T_k(\rho)) \text{div } u \text{ weakly in } L^2((0,T) \times U), \] (5.21)
and
\[ T_k(\rho_\delta) \rightarrow \overline{T_k(\rho)} \text{ in } C(0,T; L^p_{weak}(U)), \quad \forall 1 \leq p < \infty. \] (5.22)

By similar arguments as in the proof of Lemma 4.2, we have the following auxiliary result:

**Lemma 5.2.** Suppose \((\rho_\delta, u_\delta)\) is a sequence of approximate solutions constructed in Proposition 4.7, then for any \( \psi \in \mathcal{D}(0,T), \phi \in \mathcal{D}(U) \), it holds
\[ \lim_{\delta \rightarrow 0^+} \int_0^T \psi(t) \int_U \phi(x) (\rho_\delta^ - (\lambda + 2\nu) \text{div } u_\delta) T_k(\rho_\delta) \, dx \, dt = \int_0^T \psi(t) \int_U \phi(x) (\overline{\rho}^- - (\lambda + 2\nu) \text{div } u) \overline{T_k(\rho)} \, dx \, dt \] (5.23)
5.2 The renormalized solutions and strong convergence of density

As in \[10\], we introduce a quantity namely oscillations defect measure. To consider the weak convergence of the sequence \(\{\rho_\delta\}_{\delta>0}\) in \(L^1((0,T) \times U)\), we define

\[
\text{osc}_{\gamma+1}[\rho_\delta - \rho] \equiv \sup_{k \geq 1} \left( \limsup_{\delta \to 0} \int_0^T \int_U |T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1} \, dx \, dt \right),
\]

where \(T_k\) are the cut-off functions defined above. First by virtue of Lemma \[5.2\] we claim the following result concerning the oscillation defect measure.

**Lemma 5.3.** There exists a constant \(C\) independent of \(k\), such that

\[
\text{osc}_{\gamma+1}[\rho_\delta - \rho] \leq C.
\]

**Proof.** Notice that \(z^\gamma\) is a convex function for \(\gamma > \frac{1}{2}\), we have (see Theorem 2.11 in \[12\])

\[
\rho^\gamma \leq \rho^\gamma, \quad z^\gamma - y^\gamma \geq (z - y)^\gamma, \quad \text{for} \quad z, y \geq 0.
\]

Meanwhile, since \(T_k(z)\) is concave, we know

\[
|T_k(z) - T_k(y)| \leq |z - y|, \quad T_k(\rho) \geq \overline{T}_k(\rho), \quad \forall k \geq 1,
\]

and henceforth

\[
|T_k(z) - T_k(y)|^{\gamma+1} \leq |z - y|^\gamma |T_k(z) - T_k(y)| \leq (z^\gamma - y^\gamma)(T_k(z) - T_k(y)).
\]

Consequently, it yields

\[
\limsup_{\delta \to 0} \int_0^T \int_U |T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1} \, dx \, dt
\leq \lim_{\delta \to 0} \int_0^T \int_U (\rho_\delta^\gamma - \rho^\gamma)(T_k(\rho_\delta) - T_k(\rho)) \, dx \, dt + \int_0^T \int_U (\overline{\rho}^\gamma - \rho^\gamma)(T_k(\rho) - \overline{T}_k(\rho)) \, dx \, dt
\leq \lim_{\delta \to 0} \int_0^T \int_U \rho_\delta^\gamma T_k(\rho_\delta) - \rho^\gamma \overline{T}_k(\rho) \, dx \, dt
= \nu \lim_{\delta \to 0} \int_0^T \int_U \text{div} u_\delta T_k(\rho_\delta) - \text{div} u T_k(\rho) \, dx \, dt
\leq C \sup_{\delta \geq 0} \|\text{div} u_\delta\|_{L^2((0,T) \times U)} \limsup_{\delta \to 0} \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\gamma+1}((0,T) \times U)},
\]

where we applied Lemma \[5.2\] in the third step.

Based on the uniform bound for oscillation defect measure shown in Lemma \[5.3\], we can apply the same argument in \[10\] to show that the limit functions \((\rho, u)\) satisfy \((5.16)\) in the sense of renormalized solutions.

**Lemma 5.4.** The limit functions \((\rho, u)\) satisfy equation \((5.16)\) in the sense of renormalized solutions, namely,

\[
g(\rho)_{t} + \text{div}(g(\rho)u) + (g'(\rho)\rho - g(\rho))\text{div} u = 0,
\]

holds in \(D\left((0,T) \times \mathbb{R}^3\right)\) for any \(g\) satisfying \((5.12)\).
Finally, we shall discuss the propagation of oscillations, whose amplitude in the sequence \( \{\rho_\delta\}_{\delta > 0} \) is measured by the following quantity

\[
\text{df}[\rho_\delta \to \rho](t) \equiv \int_U \left( \rho \ln(\rho) - \rho \ln(\rho) \right)(t, x) dx, \quad t \in [0, T].
\]

To this end, we introduce the auxiliary functions

\[
L_k(\rho) = \rho \int_1^\rho \frac{T_k(z)}{z^2} dz,
\]

where \( T_k \) are cutoff functions defined above. Now the equation

\[
\partial_t L_k(\rho_\delta) + \text{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta)\text{div} u_\delta = 0
\]

holds in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \). Letting \( \delta \to 0 \) we obtain

\[
\partial_t L_k(\rho) + \text{div}(L_k(\rho)u) + T_k(\rho)\text{div} u = 0. \tag{5.27}
\]

Here \( L_k(\rho) \in C([0, T]; L^1(U)) \) and

\[
L_k(\rho_\delta) \to L_k(\rho) \quad \text{in} \quad C(0, T; L^1_{\text{weak}}(U)), \quad T_k(\rho_\delta)\text{div} u_\delta \to T_k(\rho)\text{div} u \quad \text{weakly in} \quad L^2((0, T) \times U).
\]

By Lemma 5.3, the limits \((\rho, u)\) satisfy

\[
\partial_t L_k(\rho) + \text{div}(L_k(\rho)u) + T_k(\rho)\text{div} u = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^3). \tag{5.28}
\]

Taking the difference between (5.27) and (5.28), then taking the inner product of the resultant with a test function \( \psi(t)\phi(x) \), with \( \psi \in \mathcal{D}(0, T) \) and \( \phi \in \mathcal{D}(\mathbb{R}^3) \) with \( \phi \equiv 1 \) on an open neighborhood of \( \bar{U} \), we get after integrating from 0 to \( t \) that

\[
\int_U \left( \frac{L_k(\rho) - L_k(\rho_\delta)}{T_k(\rho_\delta) - T_k(\rho)} \right)(t) dx
= \int_0^t \int_U (T_k(\rho) - T_k(\rho_\delta)) \text{div} u dxdt + \int_0^t \int_U (T_k(\rho) - T_k(\rho_\delta)) \text{div} u dt. \tag{5.29}
\]

Notice that \( T_k(z) \) is a convex function of \( z \geq 0 \), by Lemma 5.2 again, we deduce from (5.29) that for all \( t \in [0, T] \),

\[
0 \leq \int_U \left( \frac{L_k(\rho) - L_k(\rho_\delta)}{T_k(\rho_\delta) - T_k(\rho)} \right)(t) dx
\]

\[
= \lim_{\delta \to 0^+} \frac{1}{\nu} \int_0^t \int_U \rho_\delta^2 T_k(\rho_\delta) - \rho^2 T_k(\rho) dx dt + \int_0^t \int_U (T_k(\rho) - T_k(\rho_\delta)) \text{div} u dx dt
\]

\[
\leq \int_0^t \int_U (T_k(\rho) - T_k(\rho_\delta)) \text{div} u dx dt
\]

\[
\leq \|\text{div} u\|_{L^2((0, T) \times U)} \|T_k(\rho) - T_k(\rho_\delta)\|_{L^{\frac{3}{2}}_{\text{weak}}((0, T) \times U)} \|T_k(\rho) - T_k(\rho_\delta)\|_{L^{\frac{3}{2}}_{\text{weak}}((0, T) \times U)}
\]

\[
\leq I. \tag{5.30}
\]

By (4.33), (4.36), Lemma 5.3 letting \( k \to \infty \) in (5.30), we get

\[
0 \leq \text{df}[\rho_\delta \to \rho](t) \leq I
\]

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Consequently, we know from Corollary 2.1 that coercivity of $G$. To begin with, we obtain from Theorem 2.1 that $G$ theory. The infimum energy of $G$.

Finally, in this section we discuss briefly the long time behavior of any finite energy global weak solution $(\rho, u, Q)$. The main result is as follows.

**Theorem 6.1.** Suppose $\gamma > \frac{3}{2}$, for any finite weak energy solution to the problem (1.1) - (1.4), there exists a steady state solution $(\rho_s, 0, Q_s)$, with

$$\rho_s = \frac{m_0}{|U|}, \quad \mathcal{H}(Q_s) = 0 \text{ for } x \in U, \quad Q_s|_{\partial U} = Q_0,$$

where $m_0 = \int_U \rho_0 \, dx$, such that

$$\rho(t) \to \rho_s \text{ weakly in } L^1(U) \text{ as } t \to \infty,$$

and

$$\lim_{t \to \infty} \mathcal{E}(t) = \mathcal{E}_s,$$

where $\mathcal{E}_s$ is defined in (6.23). Furthermore, there exists an increasing sequence $\{t_n\}$ tending to infinity, for $t \in [0, 1]$, it holds as $n \to \infty$

$$u(t + t_n) \to 0 \text{ weakly in } L^2(0, 1; H^1(U)), \quad Q(t_n) \to Q_s \text{ strongly in } L^2(0, 1; H^1(U)) \text{ and weakly in } L^2(0, 1; H^2(U)).$$

**Remark 6.1.** The existence of a classical solution $Q_s$ in (6.1) is guaranteed from elliptic PDE theory. The infimum energy of $G(Q)$ can be achieved, due to the weak lower semi-continuity and coercivity of $G(Q)$.

**Proof.** To begin with, we obtain from Theorem 2.1 that

$$\esssup_{t>0} \mathcal{E}(t) + \int_0^\infty \int_U \left( \nu |\nabla u|^2 + (\lambda + \nu) |\text{div} u|^2 + \Gamma \text{tr}^2(\mathcal{H}) \right) \, dx \, dt \leq \mathcal{E}(0). \quad (6.6)$$

Consequently, we know from Corollary 2.1 that

$$\esssup_{t>0} \left( \|\rho\|_{L^1(U)} + \|\sqrt{\rho} u\|_{L^2(U)} + \|Q\|_{H^1(U)} \right) + \int_0^\infty \int_U \|\nabla u\|_{L^2(U)}^2 + \text{tr}^2(\mathcal{H}) \, dx \, dt \leq C(\mathcal{E}_0, a, b, c, U). \quad (6.7)$$
For the sake of convenience, we introduce the following sequences
\[
\rho_n(x,t) \doteq \rho(x,t + n), \quad u_n(x,t) \doteq u(x,t + n), \quad Q_n(x,t) \doteq Q(x,t + n),
\]
\[
\mathcal{H}_n(x,t) = L\Delta Q_n - aQ_n - cQ_n\text{tr}(Q_n^2),
\]
for all integer \(n\) and \(t \in (0,1), x \in U\). Then it follows immediately from (6.7) that for any \(n\), we have
\[
\rho_n \in L^\infty(0,1; L^\gamma(U)), \quad \sqrt{\rho_n}u_n \in L^\infty(0,1; L^2(U)), \quad Q_n \in L^\infty(0,1; H^1(U)),
\]
\[
\lim_{n \to \infty} \int_0^1 \left( \|\nabla u_n\|^2_{L^2(U)} + \|\nabla^2\|_{L^1(U)} \right) dt = 0.
\]

Therefore, choosing a subsequence if necessary, we know as \(n \to \infty\) that
\[
\rho_n(x,t) \to \rho_s \text{ weakly in } L^\gamma((0,1) \times U),
\]
\[
u_n(x,t) \to 0 \text{ weakly in } L^2(0,1; H^1_0(U)),
\]
\[
Q_n(x,t) \to Q_s \text{ weakly in } L^2(0,1; H^2(U)),
\]
\[
H_n(x,t) \to 0 \text{ weakly in } L^2(0,1; L^2(U)).
\]

On the other hand, it is easy to deduce from (6.7) and (6.9) that
\[
\lim_{n \to \infty} \int_0^1 \left( \|\rho_n|u_n|^2\|^2_{L^{2\gamma}(U)} + \|\rho_n u_n|^2\|^2_{L^{2\gamma}(U)} \right) dt = 0.
\]

Since \(\rho, u\) are solutions to (1.1) in the sense of renormalized solutions, we take the test function sequence \(\eta(x,t) = \psi(t)\phi(x)\) in (1.1), with \(\phi(x) \in \mathcal{D}(U), \psi(t) \in \mathcal{D}(0,1)\), to have
\[
\int_0^1 \left( \int_U \rho_n(x,t)\phi(x)dx \right) \psi'(t) dt + \int_0^1 \int_U \rho_n(x)u_n(x)\nabla \phi(x)\psi(t) dxdt = 0.
\]
Taking \(n \to \infty\) and using (6.14), we get
\[
\int_0^1 \left( \int_U \rho_s\phi(x)dx \right) \psi'(t) dt = 0,
\]
which indicates \(\rho_s\) is a function independent of \(t\), and henceforth \(m(\rho) = \int_U \rho(x,t)dx\) is a constant. On the other hand, by (6.9), (6.12) and (6.13), we have
\[
\mathcal{H}(Q_s) = 0.
\]

Hence if we apply the test function \(\eta(x,t)\) again to equation (1.3), we know that \(Q_s\) is also a function independent of \(t\). Moreover, we infer from equation (1.3) and (6.7) that
\[
\partial_t Q_n \in L^2((0,1); L^2(U)),
\]
combined with (6.12), we deduce by Aubin-Lions compactness theorem that
\[
Q_n \to Q_s \text{ strongly in } L^2(0,1; H^1(U)),
\]
with \(Q_s\) satisfying
\[
\mathcal{H}(Q_s) = 0, \quad Q_s \in S_0^1, \text{ a.e. in } U, \quad Q_s|_{\partial U} = Q_0.
\]
Lemma 6.1. For $\gamma > 1$, there exists $\theta > 0$, such that for all $n$, it holds
\[ \int_0^1 \int_U \rho_n^{\gamma+\theta}(x,t) \, dx \, dt \leq C. \]

By Lemma 6.1, we may assume
\[ \rho_n^{\gamma} \to \rho^{\gamma} \text{ weakly in } L^{\frac{\gamma}{\gamma-1}}((0,1) \times U). \]  (6.18)

Thus, passing to the limit in equation (1.2), using (6.8), (6.9), and (6.14), we obtain
\[ \nabla \rho^{\gamma} = -\nabla \cdot \left( L \nabla Q_s \circ \nabla Q_s - \mathcal{F}(Q_s) I_3 \right) \]
\[ = -\nabla Q_s : \left[ L \Delta Q_s - a Q_s + b Q_s^2 - c Q_s tr(Q_s^2) \right] \]
\[ = -\nabla Q_s : \left[ H(Q_s) + \frac{b}{3} tr(Q_s^2) I_3 \right] \]
\[ = -\nabla Q_s : H(Q_s) - \frac{b}{3} tr(Q_s^2) \nabla tr(Q_s) = 0 \text{ in } D'(U). \]  (6.19)

Next, following the same argument as in [11], that is, using the $L^p$-version of the celebrated div-curl lemma argument as in [11], we can actually show that the convergence in (6.18) is strong, and henceforth
\[ \rho_n \to \rho_s \text{ strongly in } L^{\gamma}((0,1) \times U). \]  (6.20)

Note that we already claim that $\rho_s$ is a function independent of $t$, thus (6.19)-(6.20) indicate that
\[ \rho_s = \frac{m_0}{|U|}, \]  (6.21)
where we used a fact that $m(\rho) = \int_U \rho \, dx$ is a constant, and $m_0 = \int_U \rho_0 \, dx$.

On the other hand, by the basic energy law (2.3) and Lemma 2.1, we may assume
\[ \mathcal{E}_\infty = \lim_{t \to \infty} \mathcal{E}(t) = \lim_{t \to \infty} \left( \int_U \left[ \frac{1}{2} |u(t)|^2 + \frac{\rho^{\gamma}(t)}{\gamma - 1} \right] \, dx + \mathcal{G}(Q(t)) \right). \]  (6.22)

And we define the energy for the limit functions $(\rho_s, 0, Q_s)$ by
\[ \mathcal{E}_s = \int_U \frac{\rho_s^\gamma}{\gamma - 1} \, dx + \mathcal{G}(Q_s). \]  (6.23)

Using (6.14), (6.16) and (6.20), we get
\[ \mathcal{E}_\infty = \lim_{n \to \infty} \int_0^1 \mathcal{E}(\tau + n) \, d\tau = \lim_{n \to \infty} \int_0^1 \left\{ \int_U \left[ \frac{1}{2} \rho_n |u_n|^2 + \frac{\rho_n^{\gamma}}{\gamma - 1} \right] \, dx + \mathcal{G}(Q_n) \right\} \, d\tau = \mathcal{E}_s. \]  (6.24)

Finally, it is easy to derive from equation (1.1) that
\[ \rho(t) \to \rho_s \text{ weakly in } L^\gamma(U), \quad \text{as } t \to \infty. \]

Acknowledgments: The authors would like to thank Professors Arghir Zarnescu and Colin Denniston for their valuable discussions. D. Wang’s research was supported in part by the National Science Foundation under Grant DMS-0906160 and by the Office of Naval Research under Grant N00014-07-1-0668. C. Yu’s research was supported in part by the National Science Foundation under Grant DMS-0906160. Xu was partially supported by NSF grant DMS-0806703.
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