ARBITRARY POSITIVE POWERS OF
SEMICIRCULANT AND \( r \)-CIRCULANT MATRICES

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Abstract. We provide a novel recursive method, which does not require any assumption, to compute the entries of the kth power of a semicirculant matrix. As an application, a method for computing the entries of the kth power of \( r \)-circulant matrices is also presented.

1. Introduction

\( r \)-Circulant matrices comprise an essential topic in numerous areas of mathematics and natural sciences because of their theoretical and applied aspects. Accordingly, they have gained significant importance owing to their frequent use in various applications. They appear in cryptography, number theory, information processing, coding theory, digital image processing, isotropic Markov chain models, spline approximation, among other domains. The positive integer powers for arbitrary circulant and \( r \)-circulant matrices have been studied [1, 2]. This topic was widely studied in literature [1–11].

An \( n \times n \) \( r \)-circulant matrix \( C_{n,r} \) over a unitary commutative ring \( R \) has the following form:

\[
C_{n,r} = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
r c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
r c_{n-2} & r c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
r c_1 & r c_2 & r c_3 & \cdots & r c_{n-1} & c_0
\end{pmatrix}.
\]

The \( r \)-circulant matrix \( C_{n,r} \) is determined by its first-row elements \( c_0, \ldots, c_{n-1} \) and parameter \( r \in R \). Thus, we denote \( C_{n,r} = \text{circ}_{n,r}(c_0, \ldots, c_{n-1}) \).

When \( r = 1 \), we obtain the circulant matrix \( \text{circ}_n(c_0, \ldots, c_{n-1}) \).

2010 Mathematics Subject Classification. 15B05, 11B83.

Key words and phrases. Semicirculant matrices, \( r \)-Circulant matrices, Powers, Polynomial sequence, Multinomial coefficients.
In [1], Feng provided a method for computing the kth power of an arbitrary circulant matrix by using the multinomial expansion theorem and considering the fact that a circulant matrix can be expressed as a linear combination of the powers of the basic circulant permutation matrix. Their work was later extended by Jiang [2] to arbitrary \( r \)-circulant matrices. The method used by Feng and Jiang is based on the straightforward application of the multinomial expansion theorem and, therefore, requires to solve the difficult problem of determining the set of all \( n \)-tuples \((k_0, \ldots, k_{n-1})\) of nonnegative integers that satisfy the following constraints:

\[
\begin{aligned}
&k_0 + \cdots + k_{n-1} = q \\
&k_1 + 2k_2 + \cdots + (n-1)k_{n-1} \equiv i \pmod{n},
\end{aligned}
\]

where \( i, q \) and \( n \) denote nonnegative integers such that \( i \leq n-1 \).

However, an \( n \times n \) matrix \( A \) is called semicirculant if it has the following form (see e.g., Henrici [12] or Davis [3]):

\[
A = [a_0, a_1, \ldots, a_{n-1}] = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
0 & a_0 & a_1 & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & a_1 \\
& & & 0 & a_0
\end{pmatrix}.
\]

A particular semicirculant matrix is the Jordan block \( J = [0, 1, 0, \ldots, 0] \) of size \( n \) with the eigenvalue of 0. A semicirculant matrix \( A = [a_0, a_1, \ldots, a_{n-1}] \) can be represented as a polynomial in \( J \) as follows:

\[
A = a_0 I_n + a_1 J + \cdots + a_{n-1} J^{n-1},
\]

where \( I_n \) denotes the unit matrix of index \( n \).

Let \( k \) be any nonnegative integer and \( \{a_n\} \) be a sequence of the elements of a unitary commutative ring \( R \). Accordingly, we have the following [14]:

\[
[a_0, a_1, a_2, \ldots]^k = [a_0(k), a_1(k), a_2(k), \ldots],
\]

where

\[
\begin{aligned}
a_0(k) &= a_0^k, \\
a_m(k) &= \frac{1}{ma_0} \sum_{i=1}^{m} (ik - m + i) a_i a_{m-i}(k), \quad m \geq 1.
\end{aligned}
\]

However, this method is valid only if \( m \) and \( a_0 \) are units in ring \( R \).

Here, we propose a novel method, which does not require any assumption, to compute the entries of the kth power of a semicirculant matrix.

As an application, we present a method for directly calculating the entries of the kth power of an \( r \)-circulant matrix \( C_{n,r} = \text{circ}_{n,r}(c_0, \ldots, c_{n-1}) \),
provided that the entries of the kth power of the associated infinite semicirculant matrix \([c_0, \ldots, c_{n-1}, 0, 0, \ldots]\) are known.

Precisely, let \(r\) be an element of \(R\), \(n\) be a positive integer, and \(T_{n,r}\) be a linear map given as follows:

\[
T_{n,r}(a_{ij}) = \left(\frac{r}{\text{lcm}}\right) \odot (a_{ij}),
\]

where \(\odot\) denotes the Hadamard product, \((a_{ij})\) a finite or an infinite matrix with coefficients in \(R\), and \(\lfloor x \rfloor\) the greatest integer less than or equal to \(x\).

Let

\[
\begin{align*}
&\sum_{m=0}^{\infty} \text{circ}_{n,r}(c_{nm}(k), \ldots, c_{n(m+1)-1}(k))
\end{align*}
\]

Then, one has

\[
C_{n,r}^k = \sum_{m=0}^{\infty} \text{circ}_{n,r}(c_{nm}(k), \ldots, c_{n(m+1)-1}(k))
\]

In addition to the notation introduced above, we use the notation \(\Delta(m, q, p)\) for the solution set of the following system of equations:

\[
\begin{align*}
(k_1, \ldots, k_m) &\in \mathbb{N}^m \\
k_1 + \cdots + k_m &= p \\
k_1 + 2k_2 + \cdots + mk_m &= q.
\end{align*}
\]

Here, \(p, q\) and \(m\) denote integers such that \(p \leq q \leq m\). If \(q = m\), \(\Delta(m, q, p)\) will be denoted by \(\Delta(m, p)\).

We also use notation \([C_{n,r}]\) for the infinite semicirculant matrix \([c_0, c_1, \ldots, c_{n-1}, 0, 0, \ldots]\) associated with the \(r\)-circulant matrix \(C_{n,r} = \text{circ}_{n,r}(c_0, \ldots, c_{n-1})\).

Throughout this paper, the symbol \(R\) will be used to denote an arbitrary commutative ring with identity.

2. Explicit expression of the general terms of a recursive polynomial sequence

Let \(R(X_m)_{m \geq 0}\) be an \(R\)-module spanned by the family \(\{X_0, X_1, X_2, \ldots\}\) of indeterminates over \(R\).

Let \(\nabla : R(X_m)_{m \geq 0} \to R(X_m)_{m \geq 0}\) be the shift operator defined as

\[
\nabla(X_m) = X_{m+1}.
\]

For a given sequence \(\{a_m\}_{m \geq 0}\) of the elements of a ring \(R\), we recursively define the sequence

\[
\{a_0(X_0), a_1(X_1), \ldots, a_m(X_1, \ldots, X_m), \ldots\}
\]
of the elements of \( R(X_m)_{m \geq 0} \) as follows:

\[
\begin{align*}
(2.1) & \quad \begin{cases} 
a_0 &= X_0, \\
_{m+1} &= a_{m+1}a_0^m \Delta X_0 + \cdots + a_{i+1}a_0^i \Delta X_{m-i} + \cdots + a_1 \Delta X_m.
\end{cases}
\end{align*}
\]

It is a routine to verify that the sequence

\[
\{a_0(X_0), a_1(X_1), \ldots, a_m(X_1, \ldots, X_m), \ldots\}
\]

is uniquely determined by the recurrence equation (2.1).

The following result provides an explicit expression for \( a_m(X_1, \ldots, X_m) \).

**Theorem 2.1.** The terms of the polynomial sequence defined by the recurrence equation (2.1) assume the following explicit expression:

\[
(2.2) \quad a_m = \sum_{p=1}^m X_p a_0^{m-p} \sum_{\Delta(m,p)} \binom{p}{k_1, \ldots, k_m} a_1^{k_1} \cdots a_m^{k_m} \quad \text{for } m \geq 1.
\]

**Proof.** For simplicity, let us denote

\[
L(i, p) = \sum_{\Delta(i,p)} \binom{p}{k_1, \ldots, k_i} a_1^{k_1} \cdots a_i^{k_i}.
\]

Substitute

\[
Q_m = \begin{cases} 
X_0 & \text{if } m = 0 \\
\sum_{p=1}^m X_p a_0^{m-p} \sum_{\Delta(m,p)} \binom{p}{k_1, \ldots, k_m} a_1^{k_1} \cdots a_m^{k_m} & \text{otherwise.}
\end{cases}
\]

We must prove that sequences \( \{a_m\}_m \) and \( \{Q_m\}_m \) are identical. Because sequence \( \{a_m\}_m \) is uniquely determined by (2.1) and \( \{Q_m\}_m \) begins with \( X_0 \), it suffices to show that

\[
Q_{m+1} = a_{m+1}a_0^m X_1 + \sum_{p=1}^m a_0^{m-p} a_{m-p+1} \Delta X_p.
\]

Accordingly, we argue as follows. For any positive integer \( p \leq m \), we have the following:

\[
(2.3) \quad Q_p = \sum_{i=1}^p X_i a_0^{p-i} L(p, i),
\]

and

\[
a_0^{m-p} a_{m-p+1} \Delta X_p = \sum_{i=1}^p X_{i+1} a_0^{m-i} L(p, i) a_{m-p+1}.
\]

Thus, for all nonnegative integers \( m \), we have

\[
a_{m+1}a_0^m X_1 + \sum_{p=1}^m a_0^{m-p} a_{m-p+1} \Delta X_p = a_{m+1}a_0^m X_1 + \sum_{p=1}^m \sum_{i=1}^p X_{i+1} a_0^{m-i} L(p, i) a_{m-p+1}.
\]
However, it is clear that
\[
\sum_{p=1}^{m} \sum_{i=1}^{p} X_{i+1} a_0^{m-i} L(p, i) a_{m-p+1} = \\
X_2 a_0^{m-1} L(1,1) a_m + X_2 a_0^{m-1} L(2,1) a_{m-1} + \cdots + X_2 a_0^{m-1} L(m,1) a_1.
\]
By rearranging the terms in the right-hand side of this equation, one obtains
\[
(2.4) \quad \sum_{p=1}^{m} \sum_{i=1}^{p} X_{i+1} a_0^{m-i} L(p, i) a_{m-p+1} = \sum_{p=1}^{m} X_{p+1} a_0^{m-p} \sum_{i=p}^{m} L(i, p) a_{m-i+1}.
\]
Because \(L(m+1,1) = a_{m+1}\), we have from (2.3) the following:
\[
Q_{m+1} = a_{m+1} a_0^m X_1 + \sum_{i=2}^{m+1} X_i a_0^{m-i} L(m+1,i)
\]
(2.5) \quad \quad = a_{m+1} a_0^m X_1 + \sum_{p=1}^{m} X_{p+1} a_0^{m-p} L(m+1,p+1).
Comparing (2.4) with (2.5), it is evident that we only need to prove the following:
\[
L(m+1,p+1) = \sum_{i=p}^{m} L(i,p) a_{m-i+1}.
\]
Thus, the proof will be complete with the lemma below. \(\blacksquare\)

**Lemma 2.2.** Let \(\{a_n\}_{n \geq 1}\) be a sequence of the elements of \(R\), and \(\{L(i,p)\}_{1 \leq p \leq i}\) be a double sequence defined as
\[
L(i,p) = \sum_{\Delta(i,p) \in \{k_1, \ldots, k_i\}} p \prod_{k=1}^{i} a_k^{k_i}.
\]
Accordingly, for all \(m \geq p \geq 1\), one has
\[
L(m+1,p+1) = \sum_{i=1}^{m-p+1} L(m-i+1,p) a_i.
\]

**Proof.** Consider the following system:
\[
\begin{aligned}
(k_1, \ldots, k_{m+1}) & \in \mathbb{N}^{m+1} \\
k_1 + \cdots + k_{m+1} &= p \\
k_1 + 2k_2 + \cdots + (m+1)k_{m+1} &= q.
\end{aligned}
\]
By subtracting the two equations of the above system, we obtain
\[
k_2 + \cdots + (q-p+1)k_{q-p+1} + \cdots + mk_{m+1} = q-p.
\]
It follows that $k_j = 0$ for all $j \geq q - p + 1$, and hence $k_j = 0$ for all $j \geq q + 1$.

Thus, the following equivalence is valid:

\[
(2.6) (k_1, \ldots, k_{m+1}) \in \Delta(m + 1, q, p) \iff \begin{cases} (k_1, \ldots, k_q) \in \Delta(q, p) \\ k_{q+1} = \cdots = k_{m+1} = 0. \end{cases}
\]

Now consider polynomial $Q(X) = a_1 X + \cdots + a_n X^n$.

Using the multinomial theorem we obtain

\[
Q(X)^p = (a_1 X + \cdots + a_n X^n)^p = \sum_{k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} a_1^{k_1} \cdots a_n^{k_n} X^{k_1 + 2k_2 + \cdots + nk_n}
\]

and considering (2.6), we obtain

\[
Q(X)^p = \sum_{i=p}^{np} \sum_{k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} a_1^{k_1} \cdots a_n^{k_n} X^i,
\]

and considering (2.6), we obtain

\[
Q(X)^p = \sum_{i=p}^{np} \sum_{k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} a_1^{k_1} \cdots a_n^{k_n} X^i
\]

and considering (2.6), we obtain

\[
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\]

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\]

and considering (2.6), we obtain

\[
Q(X)^p = \sum_{i=p}^{np} \sum_{k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} a_1^{k_1} \cdots a_n^{k_n} X^i.
\]

Therefore,

\[
Q(X)^{p+1} = \sum_{i=p+1}^{n(p+1)} L(i, p + 1) X^i.
\]

We can then write $Q(X)^{p+1}$ as

\[
Q(X)^{p+1} = \sum_{i=p+1}^{n(p+1)} L(i, p + 1) X^i.
\]

Now, if we rewrite $Q(X)^{p+1}$ as

\[
Q(X)^{p+1} = Q(X)^p (a_1 X + \cdots + a_n X^n),
\]

the term of degree $m + 1$ of $Q(X)^{p+1}$ is the same as that of the polynomial

\[
( \sum_{i=1}^{m-i+1} L(m - i + 1, p) X^{m-i+1} ) (a_1 X + \cdots + a_n X^n),
\]

that is

\[
L(m + 1, p + 1) X^{m+1} = \sum_{i=1}^{m-p+1} L(m - i + 1, p) X^{m-i+1} a_i X^i
\]

Therefore,

\[
L(m + 1, p + 1) = \sum_{i=1}^{m-p+1} L(m - i + 1, p) a_i.
\]
and the lemma follows. □

3. Simple recursive formula for computing the kth power of semicirculant matrices

From Formula 2.2, we have

\[ a_m(X_1, \ldots, X_m) = \sum_{p=1}^{m} a_0^{m-p} L(m, p) X_p, \]

where

\[ L(m, p) = \sum_{\Delta(m,p)} \binom{p}{k_1, \ldots, k_m} a_1^{k_1} \cdots a_m^{k_m}. \]

Next, we denote by \( \{a_m^{(k)}\}_{m,k \geq 0} \) the double sequence over \( R \) defined as follows:

\[ a_m^{(k)} = \begin{cases} a_0^{k} & \text{if } m = 0 \\ \sum_{p=1}^{m} L(m, p) a_0^{k-p} \binom{k}{p} & \text{otherwise} \end{cases}, \]

Clearly, if \( a_0 \) is a unit in \( R \), then

\[ a_m^{(k)} = a_0^{k-m} a_m \left( \binom{k}{p}, \ldots, \binom{k}{m} \right), \]

for all nonnegative integers \( k \).

Hereinafter, we adopt the convention that for any element \( a \in R \) and any nonnegative integers \( k \leq p \),

\[ a^{k-p} \binom{k}{p} = \delta_{k,p}, \]

where

\[ \delta_{k,p} = \begin{cases} 1 & \text{if } k = p \\ 0 & \text{otherwise} \end{cases}. \]

Using the above-mentioned notation, we state the following result.

**Theorem 3.1.** Let \( \{a_n\}_{n \geq 0} \) be a sequence of the elements of a ring \( R \). For all nonnegative integers \( k \), the entries of the kth power of the infinite semicirculant matrix \( A = [a_0, a_1, a_2, \ldots] \) are given as follows:

\[ A^k = [a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \ldots]. \]

**Proof.** From formula (1.3-3) of [12], we may assume, without loss of generality, that \( A = [a_0, a_1, \ldots, a_n] \) is a finite semicirculant matrix. Let
$A^k = [a_0(k), a_1(k), \ldots, a_n(k)]$. We wish to prove that for any nonnegative integer $m \leq n$, $a_m(k) = a_m^{(k)}$.

Because the claim is true for $m = 0$, assume that $m \geq 1$. Because

$$A^k = (a_0 J_{n+1} + a_1 J + \ldots + a_n J^n)^k$$

$$= \sum_{k_0, \ldots, k_n \geq 0} \left( \begin{array}{c} k \\ k_0, \ldots, k_n \end{array} \right) a_0^{k_0} a_1^{k_1} \ldots a_n^{k_n} J^{k_1+2k_2+\ldots+nk_n},$$

we have

$$a_m(k) = \sum_{(k_1, \ldots, k_n) \in \Delta(n, m, k-0)} \left( \begin{array}{c} k \\ k_0, \ldots, k_n \end{array} \right) a_0^{k_0} a_1^{k_1} \ldots a_n^{k_n},$$

$$= \sum_{(k_1, \ldots, k_m) \in \Delta(m, k-0)} \left( \begin{array}{c} k \\ k_0, \ldots, k_m \end{array} \right) a_0^{k_0} a_1^{k_1} \ldots a_m^{k_m}.$$

Therefore,

$$a_m(k) = \sum_{p=1}^{m} a_0^{k-p} \binom{k}{p} \sum_{\Delta(m, p)} \left( \begin{array}{c} p \\ k_1, \ldots, k_m \end{array} \right) a_1^{k_1} \ldots a_m^{k_m}$$

$$= a_m^{(k)}.$$  

Thus, the proof is completed. 

Next, we provide a recursive method to directly compute the coefficients $a_i(k)$ of $[a_0, a_1, a_2, \ldots]^k$ without using polynomials $a_i$; however, only the terms $\binom{k}{p}$ are considered indeterminates.

To do this, let us transform the double sequence $\{a_m^{(k)}\}_{m,k}$, which is defined in (3.1), into another double sequence $\{	ilde{a}_m^{(k)}\}_{m,k}$, which we define as follows:

$$\tilde{a}_m^{(k)} = \begin{cases} 
\binom{k-1}{1} & \text{if } m = 0 \\
\sum_{p=1}^{m} L(m, p) a_0^{k-p-1} \binom{k}{p+1} & \text{otherwise.}
\end{cases}$$

The term $\tilde{a}_m^{(k)}$ can be obtained from $a_m^{(k)}$ by replacing the integer $p$ in each sequence $a_0^{k-p} \binom{k}{p}$ by $p + 1$.

Next, we introduce the following functions of variable $x$:

$$g_{m,k}(x) = \begin{cases} 
x^{k-1} \binom{k}{1} & \text{if } m = 0 \\
\sum_{p=1}^{m} L(m, p) x^{k-p-1} \binom{k}{p+1} & \text{otherwise,}
\end{cases}$$

for $x \in R$. 

Note: The term $\sum_{p=1}^{m} L(m, p)$ is used to adjust the coefficients to ensure the correct form for the function $g_{m,k}(x)$. The Lagrange multiplier $L(m, p)$ is used to balance the contributions of each term in the sum. This is necessary to maintain the consistency of the function across different values of $m$ and $p$.
To adopt above (3.2), functions $f_{m,k}$ are defined for all $x \in R$. Therefore, we write $a_m^{(k)}$ in terms of $a_m((k_1), \ldots, (k_m)) = \sum_{p=1}^{m} L(m,p) a_{0}^{m-p}(k_p)$ even when $a_0$ is not a unit in ring $R$, as follows: $a_m^{(k)} = f_{m,k}(a_0)$.

Next, we consider the sequence $\{x^m\}_{m \geq 1}$ defined by $x_0 = x$ and $x_m = a_m$ for all $m \geq 1$. From Theorem 2.1, one has

$$f_{m,k}(x) = \begin{cases} x^k \binom{k}{0} & \text{if } m = 0 \\ x^{k-m} \sum_{p=1}^{m} L(m,p)x^{m-p} \binom{k}{p} & \text{otherwise} \end{cases}$$

and

$$g_{m,k}(x) = \begin{cases} x^{k-1} \nabla(x_0) \binom{k}{1} & \text{if } m = 0 \\ x^{k-m-1} \nabla(x_m) \binom{k}{1}, \ldots, \binom{k}{m+1} & \text{otherwise} \end{cases}$$

Thus,

$$f_{m+1,k}(x) = x^{k-m-1}x_0^{m+1}(\binom{k}{1}, \ldots, \binom{k}{m+1})$$

$$= x^{k-m-1}[a_{m+1}x_0^m \nabla(x_0) \binom{k}{1} + \cdots + a_{i+1}x_0^i \nabla(x_{m-i}) \binom{k}{1}, \ldots, \binom{k}{m-i+1}] + \cdots + a_1 \nabla(x_m) \binom{k}{1}, \ldots, \binom{k}{m+1}]$$

$$= a_{m+1}x_0^{k-1} \nabla(x_0) \binom{k}{1} + \cdots + a_{i+1}x_0^{k-m-i-1} \nabla(x_{m-i}) \binom{k}{1}, \ldots, \binom{k}{m-i+1} + \cdots + a_1 x^{k-m-1} \nabla(x_m) \binom{k}{1}, \ldots, \binom{k}{m+1})$$

Therefore,

$$f_{m+1,k}(x) = a_{m+1}g_{m,k}(x) + \cdots + a_{1}g_{0,k}(x).$$

Considering $x = a_0$ in (3.4), we obtain the following recursive formula:

$$a_m^{(k)} = a_{m+1}a_0^{(k)} + \cdots + a_{i+1}a_{m-i}^{(k)} + \cdots + a_1 a_0^{(k)}.$$

Conclusively, we have proved the following theorem.

**Theorem 3.2.** The double sequence $\{a_{n,k}^{(k)}\}_{n,k \geq 0}$ defined in (3.1) can be recursively determined as follows:

$$a_{m+1}^{(k)} = a_{m+1}a_0^{(k)} + \cdots + a_{i+1}a_{m-i}^{(k)} + \cdots + a_1 a_0^{(k)}.$$
where \( \widehat{a}_m^{(k)} \) denotes the transform of the term \( a_n^{(k)} \) obtained by replacing the integer \( p \) in each sequence \( a_0^{k-p}\binom{k}{p} \) by \( p+1 \).

**Remarks 3.3.**

1. Notably, \([a_1^{(k)}, a_2^{(k)}, \ldots] = [a_1, a_2, \ldots][\widehat{a}_0^{(k)}, \widehat{a}_1^{(k)}, \ldots]\). This formula can help us remember the manner to compute the double sequence \( a_k^{(m)} \).

2. Notably, in formula (3.6), the elements \( \binom{k}{m} \), \( m \geq 0 \), must be considered independent indeterminates over ring \( R \) during all the formal operations, and only restored as binomial coefficients when operations are completed.

3. Notably, for some nonnegative integer \( m \), \( (a_m^{(k)})_{k \geq 0} \) is an identically zero sequence, then polynomial \( a_m \) is identically equal to 0 and, therefore, so is its transform \( \nabla(a_m) \). Particularly, \( (\widehat{a}_m^{(k)})_{k \geq 0} \) is the identically zero sequence.

4. If matrix \( C \) has the form \([0, \ldots, 0, a_p, a_{p+1}, \ldots]\), it becomes significantly easy to begin by computing the powers of the left-shift matrix of \( C \) by \( p \) positions and then deduce those of matrix \( C \).

5. Clearly, if \( a_n = 0 \) for all \( n \geq p + 1 \), then \( a_m^{(k)} = \sum_{i=[m/p]}^{m} L(m, i)a_0^{k-i}\binom{k}{i} \), where \( [x] \) denotes the smallest integer greater than or equal to \( x \). Particularly, \( a_m^{(k)} = 0 \) for all \( m \geq kp + 1 \).

The following two simple examples illustrate the application of Theorem 3.2.

**Example 3.4.**

Consider a semicirculant matrix \( A = [2, 4, 2, 3] \) over the ring \( \mathbb{Z}/8\mathbb{Z} \) of integers modulo 8. Let \( k \) be any nonnegative integer. Accordingly,
Consider another semicirculant matrix. Example 3.5. ARBITRARY POSITIVE POWERS OF SEMICIRCULANT AND r-CIRCULANT MATRICES

\[ A^k = [a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, a_3^{(k)}], \] where

\[
\begin{align*}
    a_0^{(k)} &= 2^k \binom{k}{0} \\
    a_1^{(k)} &= 4 \times 2^{k-1} \binom{k}{1} = 2^{k+1} \binom{k}{1} \\
    a_2^{(k)} &= 4 \times 2^k \binom{k}{2} + 2 \times 2^{k-1} \binom{k}{1} = 2^{k+2} \binom{k}{2} + 2^k \binom{k}{1} \\
    a_3^{(k)} &= 4 \times (2^{k+1} \binom{k}{3} + 2^{k-1} \binom{k}{2}) + 2 \times 2^k \binom{k}{2} + 3 \times 2^{k-1} \binom{k}{1} \\
        &= 2^{k+3} \binom{k}{3} + 2^{k+2} \binom{k}{2} + 3 \times 2^{k-1} \binom{k}{1}
\end{align*}
\]

Example 3.5.
Consider another semicirculant matrix \( B = [0, 2, 1, 1, 0] \) over the ring \( \mathbb{Z}/8\mathbb{Z} \) of integers modulo 8. Let \( k \) be any nonnegative integer. Accordingly, \( B^k = [b_0^{(k)}, b_1^{(k)}, b_2^{(k)}, b_3^{(k)}] \), where

\[
\begin{align*}
    b_0^{(k)} &= 0^k \binom{k}{0} \\
    b_1^{(k)} &= 2 \times 0^{k-1} \binom{k}{1} \\
    b_2^{(k)} &= 4 \times 0^{k-2} \binom{k}{2} + 0^{k-1} \binom{k}{1} \\
    b_3^{(k)} &= 2 \times (4 \times 0^{k-3} \binom{k}{3} + 0^{k-2} \binom{k}{2}) + 2 \times 0^{k-2} \binom{k}{2} + 0^{k-1} \binom{k}{1}
\end{align*}
\]
\[ \mathcal{B}_3^{(k)} = 8 \times 0^{k-4} \binom{k}{4} + 4 \times 0^{k-3} \binom{k}{3} + 0^{k-2} \binom{k}{2}; \]
\[ b_4^{(k)} = 2 \times (8 \times 0^{k-4} \binom{k}{4} + 4 \times 0^{k-3} \binom{k}{3} + 0^{k-2} \binom{k}{2}) + 8 \times 0^{k-3} \binom{k}{3} + 5 \times 0^{k-2} \binom{k}{2}. \]

4. **KTH POWER OF MATRIX \( C_{n,r} \)**

Let \( E_{n,r} \) be the basic \( r \)-circulant permutation matrix of order \( n \) over ring \( R \) defined as \( E_{n,r} = \text{circ}_{n,r}(0, 1, 0, \ldots, 0) \). Because \( E_{n,r} \) is the companion matrix of polynomial \( X^n - r \), it follows that \( X^n - r \) is the characteristic polynomial of matrix \( E_{n,r} \). Accordingly, we have \( E_{n,r}^k = r \) (see, e.g., Brown [13]). Using this result, we may deduce that \( E_{n,r}^{k+n} = r E_{n,r}^k \) for all \( k \geq 0 \). Hence,

\[ E_{n,r}^k = \begin{cases} \frac{1}{r^{[\frac{k}{r}]}} E_{s,r}^s, & \text{if } k \equiv s \pmod{n} \\ \frac{1}{r^{[\frac{k}{r}]}} I_n, & \text{if } k \equiv 0 \pmod{n} \end{cases} \]

where \( 1 \leq r \leq n - 1 \).

The next theorem shows that the entries of the kth power of any \( r \)-circulant matrix \( C_{n,r} \) can be deduced from those of the kth power of the infinite semicirculant matrix \([C_{n,r}]\).

**Theorem 4.1.** Let \( C_{n,r} = \text{circ}_{n,r}(c_0, \ldots, c_{n-1}) \) be an \( r \)-circulant matrix and let

\[ [C_{n,r}] = [c_0, \ldots, c_{n-1}, 0, 0, \ldots] \]

be the associated infinite semicirculant matrix. Put

\[ T_{n,r}( [C_{n,r}]^k ) = [b_0(k), b_1(k), b_2(k), \ldots], \]

where \( k \) denotes any nonnegative integer and \( T_{n,r} \) a linear map defined as

\[ T_{n,r}(a_{ij}) = (r^{[\frac{|i-j|}{n}]}) \otimes (a_{ij}). \]

Accordingly, we have

\[ C_{n,r}^k = \sum_{m \geq 0} \text{circ}_{n,r}(c_{nm}(k), \ldots, c_{n(m+1)-1}(k)). \]
Proof. It is well-known that
\[ C_{n,r} = c_0 I_n + c_1 E_{n,r} + \cdots + c_{n-1} E_{n,r}^{n-1}. \]
Therefore, for any nonnegative integer \( k \),
\[ C_{n,r}^k = (c_0 I_n + c_1 E_{n,r} + \cdots + c_{n-1} E_{n,r}^{n-1})^k. \]
Using the multinomial theorem, we have
\[ C_{n,r}^k = \sum_{k_0, \ldots, k_{n-1} \in \mathbb{N}} \left( \binom{k}{k_0, \ldots, k_{n-1}} \right) c_0^{k_0} c_1^{k_1} \cdots c_{n-1}^{k_{n-1}} E_{n,r}^{k_0 + 2k_2 + \cdots + (n-1)k_{n-1}}. \]
Because the family
\[ \{(k_0, \ldots, k_{n-1}) \in \mathbb{N}^n / k_0 + \cdots + k_{n-1} = k \text{ and } k_1 + \cdots + (n-1)k_{n-1} = i + mn\}, m = 0, 1, \ldots \]
forms a partition of the set
\[ \{(k_0, \ldots, k_{n-1}) \in \mathbb{N}^n / k_0 + \cdots + k_{n-1} = k \text{ and } k_1 + \cdots + (n-1)k_{n-1} = i \pmod{n}\}, \]
it follows from (4.1) that
\[ C_{n,r}^k = \sum_{p=0}^{n-1} \sum_{m=k_1 + \cdots + (n-1)k_{n-1} = p \pmod{n}} \left( \binom{k}{k_0, \ldots, k_{n-1}} \right) c_0^{k_0} c_1^{k_1} \cdots c_{n-1}^{k_{n-1}} E_{n,r}^p. \]
Consequently, for all \( p = 0, \ldots, n-1 \), we have
\[ (C_{n,r}^k)_p = \sum_{m=k_1 + \cdots + (n-1)k_{n-1} = p \pmod{n}} \left( \binom{k}{k_0, \ldots, k_{n-1}} \right) c_0^{k_0} c_1^{k_1} \cdots c_{n-1}^{k_{n-1}} E_{n,r}^p, \]
where \((C_{n,r}^k)_p\) denotes the \( p \)th strip of matrix \( C_{n,r}^k \). The conclusion immediately follows from the fact that
\[ \sum_{k_0 + \cdots + k_{n-1} = k} \left( \binom{k}{k_0, \ldots, k_{n-1}} \right) c_0^{k_0} c_1^{k_1} \cdots c_{n-1}^{k_{n-1}} \]
is the \((1, m+1)\) entry of matrix \([C_{n,r}]^k\). \( \blacksquare \)

Remarks 4.2.
1. The canonical mapping
\[ c_0 + c_1 x + c_2 x^2 + \cdots \rightarrow [c_0, c_1, c_2, \ldots] \]
is an isomorphism from the ring of formal power series onto the ring of infinite semicirculant matrices. Thus, another way of establishing Theorem 4.1 is to observe that \( C_{n,r}^k = P^k(E_{n,r}) \), where \( P(x) \) denotes the representer polynomial of the \( r \)-circulant matrix \( C_{n,r} \).
2. The sequence \( r^\left\lfloor \frac{m}{n} \right\rfloor \), which appears in the pth strip

\[
(C_{n,r}^k)_p = \sum_{m=p} c_m(k) r^\left\lfloor \frac{m}{n} \right\rfloor
\]

of the \( r \)-circulant matrix \( C_{n,r}^k \), is a geometric sequence with the common ratio of \( r \).

Let us illustrate our method using the following examples. The first example is similar to the one used in [2].

**Example 4.3.** Let \( C = circ_{5,-1}(5,4,3,2,1) \) and let \([C] = [5, 4, 3, 2, 1, 0, 0, \ldots]\) be the associated infinite semicirculant matrix. Put

\[
[C]^k = [c_0^{(k)}, c_1^{(k)}, c_2^{(k)}, \ldots].
\]

Using the method provided in Theorem 3.2, we obtain

\[
c_0^{(k)} = 5^k \binom{k}{0}; \quad c_0^{(3)} = 125
\]

\[
c_1^{(k)} = 4 \times 5^{k-1} \binom{k}{1}; \quad c_1^{(3)} = 300
\]

\[
c_2^{(k)} = 4^2 \times 5^{k-2} \binom{k}{2} + 3 \times 5^{k-1} \binom{k}{1}; \quad c_2^{(3)} = 465
\]

\[
c_3^{(k)} = 4(4^2 \times 5^{k-3} \binom{k}{3} + 3 \times 5^{k-2} \binom{k}{2}) + 3 \times 4 \times 5^{k-2} \binom{k}{2} + 2 \times 5^{k-1} \binom{k}{1}; \quad c_3^{(3)} = 574
\]

\[
c_4^{(k)} = 4c_3^{(k)} + 3c_2^{(k)} + 2c_1^{(k)} + c_0^{(k)}; \quad c_4^{(3)} = 594
\]

\[
c_5^{(k)} = 4c_4^{(k)} + 3c_3^{(k)} + 2c_2^{(k)} + c_1^{(k)}; \quad c_5^{(3)} = 504
\]

\[
c_6^{(k)} = 4c_5^{(k)} + 3c_4^{(k)} + 2c_3^{(k)} + c_2^{(k)}; \quad c_6^{(3)} = 369
\]

\[
c_7^{(k)} = 4c_6^{(k)} + 3c_5^{(k)} + 2c_4^{(k)} + c_3^{(k)}; \quad c_7^{(3)} = 234
\]

\[
c_8^{(k)} = 4c_7^{(k)} + 3c_6^{(k)} + 2c_5^{(k)} + c_4^{(k)}; \quad c_8^{(3)} = 126
\]

\[
c_9^{(k)} = 4c_8^{(k)} + 3c_7^{(k)} + 2c_6^{(k)} + c_5^{(k)}; \quad c_9^{(3)} = 56
\]

\[
c_{10}^{(k)} = 4c_9^{(k)} + 3c_8^{(k)} + 2c_7^{(k)} + c_6^{(k)}; \quad c_{10}^{(3)} = 21
\]

\[
c_{11}^{(k)} = 4c_{10}^{(k)} + 3c_9^{(k)} + 2c_8^{(k)} + c_7^{(k)}; \quad c_{11}^{(3)} = 6
\]

\[
c_{12}^{(k)} = 4c_{11}^{(k)} + 3c_{10}^{(k)} + 2c_9^{(k)} + c_8^{(k)}; \quad c_{12}^{(3)} = 1
\]

\[
c_s^{(3)} = 0 \quad \text{for all } s \geq (3 \times 4) + 1 = 13.
\]

Hence,

\[
[C]^3 = [125, 300, 465, 574, 594, 504, 369, 234, 126, 56, 21, 6, 1, 0, 0, \ldots].
\]
Therefore,
\[
C_0^3 = 125(-1)^0 + 504(-1)^1 + 21(-1)^2 = -358 \\
C_1^3 = 300(-1)^0 + 369(-1)^1 + 6(-1)^2 = -63 \\
C_2^3 = 465(-1)^0 + 234(-1)^1 + (-1)^2 = 232 \\
C_3^3 = 574(-1)^0 + 126(-1)^1 = 448 \\
C_4^3 = 594(-1)^0 + 56(-1)^1 = 538.
\]

Thus,
\[
C^3 = circ_{5,-1}(-358, -63, 232, 448, 538).
\]

**Example 4.4.** Let \( p, q \) and \( n \) denote integers such that \( 0 \leq p < q < n \) and let \( a, b \in \mathbb{R} \). Consider an \( r \)-circulant matrix \( C_{n,r} = circ_{n,r}(0, \ldots, 0, a, 0, \ldots, 0, b, 0, \ldots, 0) \), where \( a \) and \( b \) denote the \( p \)th and \( q \)th strips of the \( r \)-circulant matrix \( C_{n,r} \), respectively. Clearly, the \((1, m + 1)\) entry of the semicirculant matrix \([C_{n,r}]^k\) is

\[
[C_{n,r}]_m^k = a^{k - \frac{m-kp}{q-p}} b^{\frac{m-kp}{q-p}} \left( \frac{k}{m-kp} \right)^{1_{\mod n}},
\]

where we have adopted the convention that if \( s \notin \mathbb{N} \), then \( \binom{s}{n} = 0 \).

From Theorem 4.1 and formula (4.2), the \( i \)th step of \( r \)-circulant matrix \( C_{n,r}^k \) is

\[
(C_{n,r}^k)_i = \sum_{m=i \mod n} a^{k - \frac{m-kp}{q-p}} b^{\frac{m-kp}{q-p}} \left( \frac{k}{m-kp} \right)^{1_{\mod n}}.
\]

Especially, for \( p = 1 \) and \( q = n - 1 \), we find the following formula that gives the \( i \)th step of the \( k \)th power of \( r \)-circulant matrix \( C_{n,r} = circ_n(0, a, 0, \ldots, 0, b) \)

\[
(C_{n,r}^k)_i = \sum_{m=i \mod n} a^{k - \frac{m-k}{n-2}} b^{\frac{m-k}{n-2}} \left( \frac{k}{m-k} \right)^{1_{\mod n}},
\]

which is the same as that found by Jiang [2] with \( h \) replaced by \( \frac{m-k}{n-2} \).

5. Conclusion

We proposed a method that does not require any assumption to compute the arbitrary positive integer powers for semicirculant matrices over an arbitrary unitary commutative ring. Accordingly, we derived an easy method to compute the arbitrary positive integer powers for \( r \)-circulant matrices. The advantage of this method is that it does not require to solve the difficult problem of determining the solution set
of equation 1.2. By comparing this method with [1–11] we see clearly that our method is much simpler than those used in [1–11].

References

[1] J. Feng, A note on computing of positive integer powers for circulant matrices, Appl. Math. Comput. 223 (2013) 472–475, doi:http://dx.doi.org/10.1016/j.amc.2013.08.016.

[2] Z. Jiang, H. Xin, H. Wang, On computing of positive integer powers for r-circulant matrices, Appl. Math. Comput. 265(C) (2015) 409–413, doi:http://dx.doi.org/10.1016/j.amc.2015.05.022.

[3] P. Davis, Circulant Matrices, Ams Chelsea Publishing, Providence, Rhode Island, 2012.

[4] J. Gutiérrez-Gutiérrez, Positive integer powers of complex skew-symmetric circulant matrices, Appl. Math. Comput. 202(2) (2008) 798–802, doi:10.1016/j.amc.2008.03.024.

[5] J. Gutiérrez-Gutiérrez, Positive integer powers of complex symmetric circulant matrices, Appl. Math. Comput. 202(2) (2008) 877–881, doi:10.1016/j.amc.2008.02.010.

[6] F. Köken, D. Bozkurt, Positive integer powers for one type of odd order circulant matrices, Appl. Math. Comput. 217(9) (2011) 4377–4381, doi:10.1016/j.amc.2010.10.030.

[7] J. Rimas, On computing of arbitrary positive integer powers for one type of odd order symmetric circulant matrices-I, Appl. Math. Comput. 165(1) (2005) 137–141, doi:10.1016/j.amc.2004.04.023.

[8] J. Rimas, On computing of arbitrary positive integer powers for one type of odd order symmetric circulant matrices-II, Appl. Math. Comput. 169(2) (2005) 1016–1027, doi:10.1016/j.amc.2004.11.003.

[9] J. Rimas, On computing of arbitrary positive integer powers for one type of even order symmetric circulant matrices-I, Appl. Math. Comput. 172(1) (2006) 86–90, doi:10.1016/j.amc.2005.01.121.

[10] J. Rimas, On computing of arbitrary positive integer powers for one type of even order symmetric circulant matrices-II, Appl. Math. Comput. 174(1) (2006) 511–523, doi:10.1016/j.amc.2005.04.102.

[11] G. Zhao, A cogredient algorithm for the m-th power of r-circulant matrices, volume 2, International Conference on Computer Technology and Development (2009) 557–560. IEEE, doi:10.1109/ICCTD.2009.220.

[12] P. Henrici, Applied and Computational Complex Analysis, volume 1, John Wiley, New York, 1974.

[13] W. Brown, Matrices Over Commutative Rings, Marcel Dekker, Inc., New York, 1993.

[14] Wikipedia, Formal power series, https://en.wikipedia.org/wiki/Formal_power_series, 2020 (accessed May 31, 2020).