Median bias of M-estimators

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Abstract

In this note, we derive bounds on the median bias of univariate M-estimators under mild regularity conditions. These requirements are not sufficient to imply convergence in distribution of the M-estimators. We also discuss median bias of some multivariate M-estimators.

1 Introduction

Commonly used estimators, known as M-estimators, are obtained as solution to optimization problems. Under certain regularity conditions, properly normalized M-estimators are shown to convergence in distribution to a mean zero Gaussian. This implies that the median of the M-estimator converges to the population parameter, i.e., the M-estimator is asymptotically median unbiased. The aim of this note is to show that in some cases asymptotic median unbiasedness can be proved without proving convergence in distribution. The more interesting aspect is that median unbiasedness can be proved under far less regularity conditions than those required for convergence in distribution.

For any estimator \( \hat{\theta} \) estimating \( \theta_0 \), we define the median bias as

\[
\text{Med-bias}_{\theta_0}(\hat{\theta}) := \left( \frac{1}{2} - \max \left\{ \mathbb{P}(\hat{\theta} \leq \theta_0), \mathbb{P}(\hat{\theta} \geq \theta_0) \right\} \right)_+,
\]

where \((x)_+ := \max\{x, 0\}\). If \(\mathbb{P}(\hat{\theta} \geq \theta_0) \geq 1/2\) and \(\mathbb{P}(\hat{\theta} \leq \theta_0) \geq 1/2\), then \(\text{Med-bias}_{\theta_0}(\hat{\theta}) = 0\) and \(\hat{\theta}\) is median unbiased for \(\theta_0\).

Notation. For any function \(f\), we use \(\dot{f}\) and \(\ddot{f}\) to denote the first and second derivatives of \(f\).

2 Univariate M-estimators

In this section, we consider median bias of univariate M-estimators and Z-estimators.

2.1 Convex M-estimators

Suppose \(\Theta \subseteq \mathbb{R}\) and \(M_n : \Theta \to \mathbb{R}\) is a convex function and define

\[
\hat{\theta}_n := \arg\min_{\theta \in \Theta} M_n(\theta).
\]

If \(M_n(\cdot)\) converges in probability pointwise to \(M(\cdot)\), then we can define the target \(\theta_0\) of \(\hat{\theta}_n\) as the minimizer of \(M(\theta)\). Formally, \(\theta_0 = \arg\min_{\theta \in \Theta} M(\theta)\). With \(\dot{M}_n(\cdot)\) representing the derivative of \(M_n(\cdot)\), convexity of
\( M_n(\cdot) \) and an assumption that \( \theta_0 \) lies in the interior of \( \Theta \) implies that

\[
\dot{M}_n(\theta_0) < 0 \quad \Rightarrow \quad \hat{\theta}_n \geq \theta_0 \quad \Rightarrow \quad \dot{M}_n(\theta_0) \leq 0, \tag{1}
\]

and

\[
\dot{M}_n(\theta_0) > 0 \quad \Rightarrow \quad \hat{\theta}_n \leq \theta_0 \quad \Rightarrow \quad \dot{M}_n(\theta_0) \geq 0. \tag{2}
\]

See, for example, (3.4) of Bentkus et al. (1997) for the proof. Hence, we get that

\[
\mathbb{P}(\hat{\theta}_n \geq \theta_0) \geq \mathbb{P}(\dot{M}_n(\theta_0) < 0) \quad \text{and} \quad \mathbb{P}(\hat{\theta}_n \leq \theta_0) \geq \mathbb{P}(\dot{M}_n(\theta_0) > 0).
\]

These inequalities imply the following result.

**Theorem 1.** If \( \theta_0 \) lies in the interior of \( \Theta \) and \( \theta \mapsto M_n(\theta) \) is convex, then

\[
\text{Med-bias}_{\theta_0}(\hat{\theta}_n) \leq \left( \frac{1}{2} - \max\left\{ \mathbb{P}(\dot{M}_n(\theta_0) < 0), \mathbb{P}(\dot{M}_n(\theta_0) > 0) \right\} \right) _+. \tag{3}
\]

Theorem 1 implies that the median bias of \( \hat{\theta}_n \) can be controlled by studying \( \dot{M}_n(\theta_0) \). The study of \( \dot{M}_n(\theta_0) \) is not enough to prove consistency or convergence in distribution of \( \hat{\theta}_n \). Proving consistency requires assumptions on the curvature of \( M_n(\cdot) \) around \( \theta_0 \) and similarly proving asymptotic normality requires assumptions on the first derivative of \( M_n(\cdot) \).

In many cases, \( M_n(\cdot) \) is an average of random variables and hence, \( M_n(\theta_0) \) satisfies a central limit theorem under Lindeberg type conditions. This implies that the right hand side of (3) converges to zero and proves that \( \hat{\theta}_n \) is asymptotically median unbiased. Note that it is relatively straightforward to obtain a finite sample bound on the median bias using (3) and the Berry–Esseen bounds for sum of independent or weakly dependent random variables. For examples of such Berry–Esseen bounds, see Petrov (2012, Chap. V) and Hörmann (2009). Below, we provide some simple applications of Theorem 1.

**Median Estimation.** The sample median of \( X_1, \ldots, X_n \) is defined as the minimizer of

\[
M_n(\theta) := \sum_{i=1}^n |X_i - \theta| \quad \Rightarrow \quad \dot{M}_n(\theta) = \sum_{i=1}^n \{2 \mathbb{I}\{X_i \leq \theta\} - 1\}.
\]

If \( X_1, \ldots, X_n \) are independent but possibly non-identically distributed observations and \( \theta_0 \) is a solution of \( \mathbb{E}[M_n(\theta)] = 0 \), then \( n^{-1/2}\dot{M}_n(\theta_0) \) converges in distribution to a mean zero normal distribution. Proving convergence in distribution of the sample median requires an assumption on the Hölder continuity of the distribution functions. See, for example, Knight (1998) and Knight (1999). Note that the calculations above also apply to quantile estimation and proves asymptotic median unbiasedness.

**L_p-median.** Generalizing the sample median, consider \( \hat{\theta}_n \) as a minimizer of \( M_n(\theta) \) over \( \theta \in \mathbb{R} \) where for \( p \geq 1 \),

\[
M_n(\theta) := \sum_{i=1}^n |X_i - \theta|^p \quad \Rightarrow \quad \dot{M}_n(\theta) = p \sum_{i=1}^n |X_i - \theta|^{p-1}\text{sign}(\theta - X_i).
\]

Define \( \theta_0 \) as a solution to the equation \( \mathbb{E}[\dot{M}_n(\theta)] = 0 \). Inequality (3) along with the Berry–Esseen bounds shows that \( \hat{\theta}_n \) is asymptotically median unbiased for \( \theta_0 \). Once again more conditions on the distribution of \( X_i \)'s is needed (for \( p \leq 3 \)) to ensure convergence in distribution; see Bentkus et al. (1997).
Maximum Likelihood Estimator. Suppose \( \{p_\theta : \theta \in \Theta\} \) is a family of parametric densities parametrized by \( \theta \in \Theta \subseteq \mathbb{R} \). Consider the maximum likelihood estimator (MLE) \( \hat{\theta}_n \) as

\[
\hat{\theta}_n := \arg\min_{\theta \in \Theta} - \sum_{i=1}^n \log p_\theta(X_i).
\]

If \( \theta \mapsto -\log p_\theta(x) \) is convex, then \( M_n(\theta) = -\sum_{i=1}^n \log p_\theta(X_i) \) is a convex function of \( \theta \). Assuming differentiability in quadratic mean (DQM) of the parametric family, let \( u_\theta(x) \) be the likelihood score function; see Eq. (7.1) of Van der Vaart (2000) for DQM. Define \( \theta_0 \) to be a solution to the equation \( \sum_{i=1}^n \mathbb{E}[u_\theta(X_i)] = 0 \). Assuming \( X_1, \ldots, X_n \) are independent and the Linderberg condition on \( u_{\theta_0}(X_i), 1 \leq i \leq n \), we get that the MLE is asymptotically median unbiased. Once again a properly normalized MLE need not converge in distribution without further assumptions that guarantee asymptotic equicontinuity of the likelihood score. Further under possible misspecification, the Jacobian also needs to be non-zero at \( \theta_0 \) for convergence in distribution.

2.2 Non-differentiable M-estimators

We have assumed the existence of a version of the derivative \( \bar{M}_n(\cdot) \) in the previous subsections. It is possible to avoid such assumption. From the convexity of \( M_n(\cdot) \), it follows that for any \( \varepsilon > 0 \),

\[
\{ \hat{\theta}_n > \theta_0 + \varepsilon \} \Rightarrow \{ M_n(\theta_0) \geq M_n(\theta_0 + \varepsilon) \},
\]

and

\[
\{ \hat{\theta}_n < \theta_0 - \varepsilon \} \Rightarrow \{ M_n(\theta_0) \geq M_n(\theta_0 - \varepsilon) \}.
\]

Hence,

\[
\mathbb{P}(\hat{\theta}_n \leq \theta_0 + \varepsilon) \geq \mathbb{P}(M_n(\theta_0) < M_n(\theta_0 + \varepsilon)), \quad \mathbb{P}(\hat{\theta}_n \geq \theta_0 - \varepsilon) \geq \mathbb{P}(M_n(\theta_0) < M_n(\theta_0 - \varepsilon)).
\]

Now observe that

\[
\mathbb{P}(\hat{\theta}_n \leq \theta_0) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(\hat{\theta}_n \leq \theta_0 + \varepsilon) \quad \text{and} \quad \mathbb{P}(\hat{\theta}_n \geq \theta_0) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(\hat{\theta}_n \geq \theta_0 - \varepsilon).
\]

Therefore,

\[
\text{Med-bias}_{\theta_0}(\hat{\theta}) \leq \lim_{\varepsilon \downarrow 0} \left( \frac{1}{2} - \max \{ \mathbb{P}(M_n(\theta_0) < M_n(\theta_0 + \varepsilon)), \mathbb{P}(M_n(\theta_0) < M_n(\theta_0 - \varepsilon)) \} \right) . \quad (4)
\]

In many cases, \( M_n(\theta) \) is an average of random variables and \( M_n(\theta_0) - M_n(\theta_0 + \varepsilon) \) converges to a negative quantity for any fixed \( \varepsilon > 0 \).

Maximum Likelihood Estimation. Consider again the problem of median bias of the maximum likelihood estimator. In this case, \( M_n(\theta) = -\sum_{i=1}^n \log p_\theta(X_i) \) and the target of the MLE \( \theta_0 \) is defined as the minimizer of \( \theta \mapsto M_n(\theta) \) over \( \theta \in \Theta \). If \( \theta_0 \) lies in the interior of \( \Theta \) and \( \theta_0 \pm \varepsilon \in \Theta \), then

\[
\mathbb{P}(M_n(\theta_0) < M_n(\theta_0 + \varepsilon)) = \mathbb{P}\left( \sum_{i=1}^n \log \frac{p_{\theta_0+\varepsilon}(X_i)}{p_{\theta_0}(X_i)} < 0 \right) = \mathbb{P}\left( \sum_{i=1}^n \left( \log \frac{p_{\theta_0+\varepsilon}(X_i)}{p_{\theta_0}(X_i)} - \mathbb{E} \left[ \log \frac{p_{\theta_0+\varepsilon}(X_i)}{p_{\theta_0}(X_i)} \right] \right) < - \sum_{i=1}^n \mathbb{E} \left[ \log \frac{p_{\theta_0+\varepsilon}(X_i)}{p_{\theta_0}(X_i)} \right] \right).
\]

3
Note that, by definition, \( \sum_{i=1}^{n} \mathbb{E} \log(p_{\theta_0 + \varepsilon}(X_i)/p_{\theta_0}(X_i)) \leq 0 \) and is strictly negative if \( \varepsilon > 0 \) by identifiability. Hence, it follows that
\[
\mathbb{P}(\hat{M}_n(\theta_0) < \hat{M}_n(\theta_0 + \varepsilon)) \geq \mathbb{P}\left( \sum_{i=1}^{n} \left\{ \log \frac{p_{\theta_0 + \varepsilon}(X_i)}{p_{\theta_0}(X_i)} - \mathbb{E} \left[ \log \frac{p_{\theta_0 + \varepsilon}(X_i)}{p_{\theta_0}(X_i)} \right] \right\} \leq 0 \right).
\]
Similarly,
\[
\mathbb{P}(\hat{M}_n(\theta_0) < \hat{M}_n(\theta_0 - \varepsilon)) \geq \mathbb{P}\left( \sum_{i=1}^{n} \left\{ \log \frac{p_{\theta_0 - \varepsilon}(X_i)}{p_{\theta_0}(X_i)} - \mathbb{E} \left[ \log \frac{p_{\theta_0 - \varepsilon}(X_i)}{p_{\theta_0}(X_i)} \right] \right\} \leq 0 \right).
\]
If the log-likelihood ratio satisfies the Linderberg condition, then (4) implies that the MLE is again asymptotically median unbiased.

### 2.3 Non-convex M-estimators

Convexity of the objective function \( \hat{M}_n(\cdot) \) is not very crucial for (3). All that is required is that \( \theta \mapsto \hat{M}_n(\theta) \) is convex in the neighborhood of \( \theta_0 \) and with some positive probability the estimator \( \hat{\theta}_n \) belongs to that neighborhood. These conditions are same as convexity and consistency assumptions (1.4), (1.5) in Bentkus et al. (1997). Formally, for set
\[
\eta_{1,n}(\delta) := 1 - \mathbb{P}(\theta \mapsto \hat{M}_n(\theta) \text{ is convex on } [\theta_0 - \delta, \theta_0 + \delta]),
\]
\[
\eta_{2,n}(\delta) := \mathbb{P}(\hat{\theta}_n - \theta_0 > \delta).
\]
On the event \( \hat{\theta}_n \in [\theta_0 - \delta, \theta_0 + \delta] \subseteq \Theta \), inequalities (1) and (2) hold true. Therefore, we get
\[
\text{Med-bias}_{\theta_0}(\hat{\theta}_n) \leq \left( \frac{1}{2} - \max \left\{ \mathbb{P}(\hat{M}_n(\theta_0) < 0), \mathbb{P}(\hat{M}_n(\theta_0) > 0) \right\} \right) + \min_{\delta \geq 0} [\eta_{1,n}(\delta) + \eta_{2,n}(\delta)]. \tag{5}
\]
Note that if \( \theta \mapsto \hat{M}_n(\theta) \) is convex on \( \Theta \), then \( \eta_{1,n}(\infty) = 0 \) and \( \eta_{2,n}(\infty) = 0 \). Inequality (5) follows from (3.3) of Bentkus et al. (1997).

Alternatively, one can consider the usual Taylor series expansion way and prove a better result. For this, we additionally require absolute continuity of the first derivative of \( \hat{M}_n(\cdot) \). If \( \hat{\theta}_n \) solves the equation \( \hat{M}_n(\theta) = 0 \) and using absolute continuity of \( \theta \mapsto \hat{M}_n(\theta) \), we get that
\[
0 = \hat{M}_n(\theta_0) + \int_{0}^{1} \hat{M}_n(\theta_0 + t(\hat{\theta}_n - \theta_0))dt(\hat{\theta}_n - \theta_0).
\]
Now assuming that \( \hat{\theta}_n \) is a locally unique solution with probability 1, we conclude that \( \int_{0}^{1} \hat{M}_n(\theta_0 + t(\hat{\theta}_n - \theta_0))dt \neq 0 \) and hence,
\[
\hat{\theta}_n - \theta_0 = -\frac{\hat{M}_n(\theta_0)}{\int_{0}^{1} \hat{M}_n(\theta_0 + t(\hat{\theta}_n - \theta_0))dt} \tag{6}.
\]
This implies that
\[
\text{Med-bias}_{\theta_0}(\hat{\theta}_n) = \left( \frac{1}{2} - \max \left\{ \mathbb{P}(\hat{M}_n(\theta_0) \geq 0), \mathbb{P}(\hat{M}_n(\theta_0) \leq 0) \right\} \right) \+ . \tag{7}
\]
Equation (6) provides a more intuitive reason for why median bias of univariate M-estimators can be controlled without requiring conditions to imply convergence in distribution. To prove convergence in distribution, we would require \( \hat{\theta}_n \) to be consistent for \( \theta_0 \) along with conditions to ensure that the denominator on the right hand side of (6) can be replaced with \( \hat{M}_n(\theta_0) \).

The bound on median bias (7) readily applies to Z-estimators which are obtained as solutions of estimating equations rather than minimizers of objective functions.
3 Multivariate M-estimation

The calculations from previous sections can be used trivially when estimating a parameter in presence of nuisance parameters. Suppose \( M_n : \Theta \times \Lambda \to \mathbb{R} \) be an objective function and define

\[
(\hat{\theta}_n, \hat{\lambda}_n) := \arg\min_{\theta, \lambda} M_n(\theta, \lambda).
\]

Setting \( M(\theta, \lambda) \) as the pointwise limit in probability of \( M_n(\theta, \lambda) \), define

\[
(\theta_0, \lambda_0) := \arg\min_{\theta, \lambda} M(\theta, \lambda).
\]

To apply the results from previous section to study the median bias of \( \hat{\theta}_n \), observe that

\[
\hat{\theta}_n := \arg\min_{\theta \in \Theta} \lambda \min M_n(\theta, \lambda).
\]

If \( (\theta, \eta) \mapsto M_n(\theta, \eta) \) is a convex function, then \( \theta \mapsto \min_\lambda M_n(\theta, \lambda) \) is also a convex function. This is called the inf-projection of \( M_n(\theta, \eta) \). However, \( \mathbb{W}_n(\theta) = \min_\lambda M_n(\theta, \lambda) \) and its derivative (sub-gradient) \( \mathbb{W}_n(\theta) \) are complicated functions to study, in general. In some special cases, \( \mathbb{W}_n(\theta) \) is available in closed form and might be easily analysed.

3.1 Application 1: Least Squares Linear Regression

Suppose \( (Y_i, T_i, X_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \), \( 1 \leq i \leq n \) represent the set of observations on the treatment variable \( (T) \), covariates \( (X) \), and the response \( (Y) \). Consider

\[
(\tilde{\theta}_n, \tilde{\lambda}_n) := \arg\min_{\theta, \lambda} \sum_{i=1}^{n} (Y_i - \theta T_i - \lambda^\top X_i)^2.
\]

The targets of \( \tilde{\theta}_n, \tilde{\lambda}_n \) are defined as

\[
(\theta_0, \lambda_0) := \arg\min_{\theta, \lambda} \sum_{i=1}^{n} \mathbb{E}[(Y_i - \theta T_i - \lambda^\top X_i)^2].
\]

This can be written as

\[
\tilde{\theta}_n := \arg\min_{\theta} \sum_{i=1}^{n} (Y_i - \beta_{Y,n}^\top X_i - \theta (T_i - \beta_{T,n}^\top X_i))^2,
\]

where

\[
\beta_{Y,n} := \arg\min_{\beta \in \mathbb{R}^d} \sum_{i=1}^{n} (Y_i - \beta^\top X_i)^2, \quad \text{and} \quad \beta_{T,n} := \arg\min_{\beta \in \mathbb{R}^d} \sum_{i=1}^{n} (T_i - \beta^\top X_i)^2.
\]

Hence \( \tilde{\theta}_n \) is the minimizer of a quadratic convex objective function and (3) (or (5)) yields a bound on the median bias of \( \tilde{\theta}_n \). For notational convenience, set \( \tilde{R}_{Y,i} = Y_i - \beta_{Y,n}^\top X_i \) and \( \tilde{R}_{T,i} = T_i - \beta_{T,n}^\top X_i \). Also, let \( \beta_Y \) and \( \beta_T \) denote the targets of \( \beta_{Y,n} \) and \( \beta_{T,n} \), and set \( R_{Y,i} = Y_i - \beta_Y^\top X_i \) and \( R_{T,i} = R_i - \beta_T^\top X_i \). Then (5) yields

\[
\text{Med-bias}_{\theta_0}(\tilde{\theta}_n) = \left( \frac{1}{2} - \max \left\{ \mathbb{P}\left( \sum_{i=1}^{n} \tilde{R}_{T,i}(\tilde{R}_{Y,i} - \theta_0 \tilde{R}_{T,i}) \leq 0 \right), \mathbb{P}\left( \sum_{i=1}^{n} \tilde{R}_{T,i}(\tilde{R}_{Y,i} - \theta_0 \tilde{R}_{T,i}) \geq 0 \right) \right\} \right) .
\]

(9)
This equality holds true if $T_i$ is not perfectly collinear with $X_i$. Note that unlike the examples discussed in previous sections, the sums on the right hand side of (9) are not of independent random variables. Observe that

$$
\sum_{i=1}^n \hat{R}_{T,i}(\hat{R}_{Y,i} - \theta_0 \hat{R}_{T,i}) = \sum_{i=1}^n R_{T,i}(R_{Y,i} - \theta_0 R_{T,i}) + \sum_{i=1}^n (R_{T,i} - \hat{R}_{T,i})(R_{Y,i} - \theta_0 R_{T,i}) - \sum_{i=1}^n \hat{R}_{T,i}(R_{Y,i} - \theta_0(\hat{R}_{T,i} - R_{T,i})).
$$

From the definition of $\hat{\beta}_{T,n}$ it follows that $\sum_{i=1}^n \hat{R}_{T,i}X_i = 0$. Noting that $\hat{R}_{Y,i} - R_{Y,i} = X_i^\top(\hat{\beta}_{Y,n} - \beta_Y)$ and $\hat{R}_{T,i} - R_{T,i} = X_i^\top(\hat{\beta}_{T,n} - \beta_T)$, we conclude that

$$
\sum_{i=1}^n \hat{R}_{T,i}(\hat{R}_{Y,i} - R_{Y,i} - \theta_0(\hat{R}_{T,i} - R_{T,i})) = 0.
$$

We obtain that

$$
\sum_{i=1}^n \hat{R}_{T,i}(\hat{R}_{Y,i} - \theta_0 \hat{R}_{T,i}) = \sum_{i=1}^n R_{T,i}(R_{Y,i} - \theta_0 R_{T,i}) + (\hat{\beta}_{T,n} - \beta_T)^\top \sum_{i=1}^n X_i(R_{Y,i} - \theta_0 R_{T,i}).
$$

From the definition of $\theta_0$, it can be easily verified that $\sum_{i=1}^n \mathbb{E}[R_{T,i}(R_{Y,i} - \theta_0 R_{T,i})] = 0$ and from the definitions of $\beta_{Y,n}, \beta_T$ that $\sum_{i=1}^n \mathbb{E}[X_i(R_{Y,i} - \theta_0 R_{T,i})] = 0$. Consider the event

$$
\mathcal{E}_\eta := \left\{ (\hat{\beta}_{T,n} - \beta_T)^\top \sum_{i=1}^n X_i(R_{Y,i} - \theta_0 R_{T,i}) \leq \eta \right\},
$$

and the sum

$$
S_n := \sum_{i=1}^n R_{T,i}(R_{Y,i} - \theta_0 R_{T,i}).
$$

Using this event and inequality (9), we obtain the following result.

**Proposition 1.** For any sequence of random vectors $(Y_i, T_i, X_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, the estimator $\hat{\theta}_n$ defined by (8) satisfies

$$
\text{Med-bias}_0(\hat{\theta}_n) \leq \inf_{\eta > 0} \left[ \left( \frac{1}{2} - \max \{ \mathbb{P}(S_n \leq -\eta), \mathbb{P}(S_n \geq \eta) \} \right)_+ + \mathbb{P}(\mathcal{E}_\eta) \right].
$$

Under mild moment conditions as well as weak dependence assumptions, it can be proved that $\mathbb{P}(\mathcal{E}_\eta)$ converges to 1 as $n \to \infty$ for $\eta = O(d)$. With $\eta = O(d)$, it suffices for $d = o(\sqrt{n})$ to ensure that the median bias converges to zero. If $d = O(\sqrt{n})$, then the median bias converges to a constant bounded away from 0 and 1/2. The calculations above can also be used with the Neyman orthogonal estimating equation in a partial linear model (Chernozhukov et al., 2018).

### 3.2 Application 2: Partial Linear Regression with Sample Splitting

Suppose $(Y_i, T_i, X_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, 1 \leq i \leq n$ satisfy the partial linear model

$$
Y_i = \theta_0 T_i + g_0(X_i) + U_i, \quad \text{and} \quad T_i = m_0(X_i) + V_i, \quad \text{where} \quad \mathbb{E}[U_i|T_i, X_i] = 0, \mathbb{E}[V_i|X_i] = 0.
$$

The parameter of interest is still $\theta_0$ (the treatment effect). Split the data into two parts $\mathcal{D}_1$ and $\mathcal{D}_2$. Let $\hat{m}(\cdot)$ and $\hat{g}(\cdot)$ be estimators of $m_0(\cdot)$ and $g_0(\cdot)$, respectively. We will assume that these estimators are computed
from $D_1$. For $i \in D_2$, set $\hat{R}_{T,i} = T_i - \hat{m}(X_i)$ and $\hat{R}_{Y,i} = Y_i - \hat{g}(X_i)$. Then consider the estimator $\hat{\theta}_n$ as a solution to the equation

$$Z_n(\theta) := \sum_{i \in D_2} \hat{R}_{T,i}(\hat{R}_{Y,i} - D_i \theta) = 0,$$

where $D_2$ is the second part of the data. From (7), it follows that

$$\text{Med-bias}_{\theta_0}(\hat{\theta}_n) = \left( \frac{1}{2} - \max \{ \mathbb{P}(Z_n(\theta_0) \leq 0), \mathbb{P}(Z_n(\theta_0) \geq 0) \} \right)_+ \prod_{i \in D_2}.$$ \hspace{1cm} (10)

Now set $Z_n^c(\theta_0) = Z_n(\theta_0) - \mathbb{E}[Z_n(\theta_0)|D_1]$. Then, (10) implies

$$\text{Med-bias}_{\theta_0}(\hat{\theta}_n) \leq \left( \frac{1}{2} - \max \left\{ \mathbb{P}(Z_n^c(\theta_0) \leq -\mathbb{E}[Z_n(\theta_0)|D_1]), \mathbb{P}(Z_n^c(\theta_0) \geq \mathbb{E}[Z_n(\theta_0)|D_1]) \right\} \prod_{i \in D_2} \right)_+. \hspace{1cm} (11)$$

Note that conditional $D_1$ (the first split of the data), $Z_n^c(\theta_0)$ is a sum of centered (mean zero) random variables. We can write

$$Z_n(\theta_0) = \sum_{i \in D_2} R_{T,i}(R_{Y,i} - D_i \theta_0) + \sum_{i \in D_2} R_{T,i}(\hat{R}_{Y,i} - R_{Y,i})$$

$$+ \sum_{i \in D_2} (\hat{R}_{T,i} - R_{T,i})(R_{Y,i} - D_i \theta_0) + \sum_{i \in D_2} (\hat{R}_{T,i} - R_{T,i})(\hat{R}_{Y,i} - R_{Y,i}).$$

Conditional on $D_1$, the first three terms above are mean zero and hence,

$$\mathbb{E}[Z_n(\theta_0)|D_1] = \sum_{i \in D_2} \mathbb{E}[(\hat{g}(X_i) - g_0(X_i))(\hat{m}(X_i) - m_0(X_i))|D_1].$$

This can be bounded as

$$|\mathbb{E}[Z_n(\theta_0)|D_1]| \leq |D_2| \mathbb{E}[(g - g_0)_2,n] \mathbb{E}[(\hat{m} - m_0)_2,n],$$

where for $P_{X_i}(\cdot)$ representing the probability measure of $X_i$,

$$\|g - g_0\|_2^2 := |D_2|^{-1} \sum_{i \in D_2} \int (g(x) - g_0(x))^2 dP_{X_i}(x),$$

$$\|\hat{m} - m_0\|_2^2 := |D_2|^{-1} \sum_{i \in D_2} \int (\hat{m}(x) - m_0(x))^2 dP_{X_i}(x).$$

Hence if $|D_2|^{1/2}\|g - g_0\|_2,n \|\hat{m} - m_0\|_2,n = o_p(1)$, then inequality (11) implies that the median bias of $\hat{\theta}_n$ converges to zero. Further, if $|D_2|^{1/2}\|g - g_0\|_2,n \|\hat{m} - m_0\|_2,n = O_p(1)$, then the median bias of $\hat{\theta}_n$ is bounded away from 0 to 1/2.

4 Conclusion

In this note, we proved that median bias of several M/Z-estimators can be controlled under conditions weaker than those required for convergence in distribution of these estimators. The control of the median bias implies that the recently proposed confidence interval methodology HulC (Kuchibhotla et al., 2021) can be applied to these estimators. Note that without convergence in distribution none of the usual methods of inference, including bootstrap and subsampling, apply.
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