ON VARIETAL CAPABILITY OF INFINITE DIRECT PRODUCTS OF GROUPS

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ABSTRACT. Recently, the authors gave some conditions under which a direct product of finitely many groups is \( V^- \)-capable if and only if each of its factors is \( V^- \)-capable for some varieties \( V \). In this paper, we extend this fact to any infinite direct product of groups. Moreover, we conclude some results for \( V^- \)-capability of direct products of infinitely many groups in varieties of abelian, nilpotent and polynilpotent groups.

1. Introduction

R. Baer [1] initiated an investigation of the question "which conditions a group \( G \) must fulfill in order to be the group of inner automorphisms of a group \( E \)?", that is \( G \cong E/Z(E) \). Following M. Hall and J. K. Senior [5], such a group \( G \) is called capable. Baer [1] determined all capable groups which are direct sums of cyclic groups. As P. Hall [4] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F. R. Beyl, U. Felgner and P. Schmid [2] proved that every group \( G \) possesses a uniquely determined central subgroup \( Z^*(G) \) which is minimal subject to being the image in \( G \) of the center of some central extension of \( G \). This \( Z^*(G) \) is characteristic in \( G \) and is the image of the center of every stem cover of \( G \). Moreover, \( Z^*(G) \) is the smallest central subgroup of \( G \) whose factor group is capable [2]. Hence \( G \) is capable if and only if \( Z^*(G) = 1 \). They showed that the class of all capable groups is closed under the direct products. Also, they presented a condition in which the capability of a direct product of finitely many groups implies the capability of each of the factors. Moreover, they proved that if \( N \)
is a central subgroup of $G$, then $N \subseteq Z^*(G)$ if and only if the mapping $M(G) \rightarrow M(G/N)$ induced by the natural epimorphism, is monomorphism.

Then M. R. R. Moghadam and S. Kayvanfar [10] generalized the concept of capability to $\mathcal{V}$-capability for a group $G$. They introduced the subgroup $(V^*)^*(G)$ which is associated with the variety $\mathcal{V}$ defined by a set of laws $V$ and a group $G$ in order to establish a necessary and sufficient condition under which $G$ can be $\mathcal{V}$-capable. They also showed that the class of all $\mathcal{V}$-capable groups is closed under the direct products. Moreover, they exhibited a close relationship between the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, where $N$ is a normal subgroup contained in the marginal subgroup of $G$ with respect to the variety $\mathcal{V}$. Using this relationship, they gave a necessary and sufficient condition for a group $G$ to be $\mathcal{V}$-capable.

The authors [7] presented some conditions in which the $\mathcal{V}$-capablity of a direct product of finitely many groups implies the $\mathcal{V}$-capablity of each of its factors. In this paper, we extend this fact to direct product of an infinite family of groups. Also, we deduce some new results about the $\mathcal{V}$-capability of direct product of infinitely many groups, where $\mathcal{V}$ is the variety of abelian, nilpotent, or polynilpotent groups.

2. Main Results

Suppose that $\mathcal{V}$ is a variety of groups defined by the set of laws $V$. A group $G$ is said to be $\mathcal{V}$-capable if there exists a group $E$ such that $G \cong E/V^*(E)$, where $V^*(E)$ is the marginal subgroup of $E$, which is defined as follows [6]:

$$\{g \in E \mid v(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, gx_i, x_{i+1}, \ldots, x_n) \text{ for all } x_1, x_2, \ldots, x_n \in E, \forall i \in \{1, 2, \ldots, n\}\}.$$

If $\psi : E \rightarrow G$ is a surjective homomorphism with $\ker\psi \subseteq V^*(E)$, then the intersection of all subgroups of the form $\psi(V^*(E))$ is denoted by $(V^*)^*(G)$. It is obvious that $(V^*)^*(G)$ is a characteristic subgroup of $G$ contained in $V^*(G)$. If $\mathcal{V}$ is the variety of abelian groups, then the subgroup $(V^*)^*(G)$ is the same as $Z^*(G)$ and in this case $\mathcal{V}$-capability is equal to capability [10]. In the following, there are some results which we need them in sequel.

**Theorem 2.1.** [10] (i) A group $G$ is $\mathcal{V}$-capable if and only if $(V^*)^*(G) = 1$.

(ii) If $\{G_i \mid i \in I\}$ is a family of groups, then $(V^*)^*(\prod_{i \in I} G_i) \subseteq \prod_{i \in I}(V^*)^*(G_i)$.

As a consequence, if the $G_i$’s are $\mathcal{V}$-capable groups, then $G = \prod_{i \in I} G_i$ is also $\mathcal{V}$-capable. In the above theorem, the equality does not hold in general (see Example 2.3 of [7]).

**Theorem 2.2.** [10] Let $\mathcal{V}$ be a variety of groups with a set of laws $V$. Let $G$ be a group and $N$ be a normal subgroup with the property $N \subseteq V^*(G)$. Then $N \subseteq (V^*)^*(G)$ if and only if the homomorphism induced by the natural map $\mathcal{V}M(G) \rightarrow \mathcal{V}M(G/N)$ is a monomorphism.
We recall that the Baer-invariant of a group $G$, with the free presentation $F/R$, with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} ,$$

where $V(F)$ is the verbal subgroup of $F$ with respect to $\mathcal{V}$ and

\[ [RV^*F] = \langle v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots, f_n)^{-1} | r \in R, \quad f_i \in F, v \in V, 1 \leq i \leq n, n \in \mathbb{N} \rangle. \]

It is known that the Baer-invariant of a group $G$ is always abelian and independent of the choice of the presentation of $G$. Also if $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of $G$ will be $R \cap F'/[R, F] \cong M(G)$, where $M(G)$ is the Schur multiplier of $G$ (see [6]).

**Theorem 2.3.** [7] Let $\mathcal{V}$ be a variety, $A$ and $B$ be two groups with $\mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B)$, then $(V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B)$. Consequently, $A \times B$ is $\mathcal{V}$-capable if and only if $A$ and $B$ are both $\mathcal{V}$-capable.

**Theorem 2.4.** [8] Let $\{G_i; \phi_i^j, I\}$ be a directed system of groups. Then, for a given variety $\mathcal{V}$, the Baer-invariant preserves direct limit, that is $\mathcal{V}M(\varprojlim G_i) = \varprojlim \mathcal{V}M(G_i)$.

**Lemma 2.5.** For any family of groups $\{G_i\}_{i \in I}$, consider the directed system $\{G_I, \phi^\Lambda, \Lambda\}$ consisting of all finite direct products $G_I = \prod_{i \in I} G_i$ ($I_\Lambda$ is a finite subset of $I$), with the natural embedding homomorphisms $\phi^\Lambda : G_I \to G_I^\Lambda$ ($I_\Lambda \subseteq I^{\Lambda'}$). Also, the index set $\Lambda$ is ordered in a directed way so that for any $\Lambda, \Lambda' \in \Lambda$, $\Lambda \leq \Lambda'$ if and only if $I_\Lambda \subseteq I^{\Lambda'}$. Then the direct product $G_I = \prod_{i \in I} G_i$ is a direct limit of this directed system.

**Proof.** Let $G = \varprojlim G_I$, be a direct limit of this directed system, with homomorphisms $\phi_\Lambda : G_I \to G$. Also, for any $\Lambda \in \Lambda$, consider the embedding homomorphism $\tau_\Lambda : G_I \to G_I$. Clearly, for any $\Lambda, \Lambda' \in \Lambda$ with $\Lambda \leq \Lambda'$, $\tau_\Lambda \tau^\Lambda_\Lambda = \tau_\Lambda$. Now, by universal property of $G$, there exists a unique homomorphism $\phi : G \to G_I$ such that for any $\Lambda \in \Lambda$, $\phi \phi_\Lambda = \tau_\Lambda$. To define the inverse homomorphism $\tau : G_I \to G$, recall that for any $x = \{x_i\}_{i \in I} \in G_I$, there exists a finite subset $I_\Lambda$ of $I$ that for any $i \in I \setminus I_\Lambda$, $x_i$ is trivial in $G_i$. Hence we can consider $x$ as an element of $G_I$, and define $\tau(x) = \phi_\Lambda(x)$. It is easy to see that for any $\Lambda \in \Lambda$, $\tau \tau_\Lambda = \phi_\Lambda$. Finally, we see that for any $x \in G_I$, $\phi \tau(x) = \phi \phi_\Lambda(x)$, for some $\Lambda \in \Lambda$; and so $\phi \tau(x) = \tau_\Lambda(x) = x$. Conversely, the equation $\tau \phi = id_G$ holds because of the universal property of the direct limit $G$. \qed

By the above notations, we conclude that $\prod_{i \in I} G_i$, $\prod_{i \in I} V^{**}(G_i)$, and $\prod_{i \in I} G_i/V^{**}(G_i)$ are direct limits of directed systems $\{\prod_{i \in I}^\Lambda G_i, \phi^\Lambda, \Lambda\}$, $\{\prod_{i \in I} V^{**}(G_i), \phi^\Lambda, \Lambda\}$, and $\{\prod_{i \in I} G_i/V^{**}(G_i), \psi^\Lambda, \Lambda\}$ respectively, where $\phi^\Lambda$’s are restrictions of $\phi$’s and $\psi^\Lambda$’s are quotient homomorphisms induced by $\phi$’s.
Now, suppose that \( \{G_i\}_{i \in I} \) is a family of groups in which for any \( G_i \) and \( G_j \) \( (i, j \in I) \), \( \mathcal{V}M( G_i \times G_j ) \cong \mathcal{V}M( G_i ) \times \mathcal{V}M( G_j ) \). By Theorem 2.3, \( \prod_{i \in I_{\lambda}} (V^*)^*( G_i ) \subseteq (V^*)^*( \prod_{i \in I_{\lambda}} G_i ) \), for any finite subset \( I_{\lambda} \) of \( I \). Thus, using Theorem 2.2, we have the following monomorphism

\[
\mathcal{V}M( \prod_{i \in I_{\lambda}} G_i ) \hookrightarrow \mathcal{V}M( \prod_{i \in I_{\lambda}} (V^*)^*( G_i ) ).
\]

By the fact that direct limit of a directed system preserves exactness of a sequence [8], we obtain the following monomorphism

\[
\lim_{\rightarrow} \mathcal{V}M( \prod_{i \in I_{\lambda}} G_i ) \hookrightarrow \mathcal{V}M( \lim_{\rightarrow} ( \prod_{i \in I_{\lambda}} (V^*)^*( G_i ) ) ).
\]

Using Theorem 2.4, we conclude the monomorphism

\[
\mathcal{V}M( \lim_{\rightarrow} \prod_{i \in I_{\lambda}} G_i ) \hookrightarrow \mathcal{V}M( \lim_{\rightarrow} \prod_{i \in I_{\lambda}} (V^*)^*( G_i ) ),
\]

and so we have the monomorphism

\[
\mathcal{V}M( \prod_{i \in I} G_i ) \hookrightarrow \mathcal{V}M( \prod_{i \in I} (V^*)^*( G_i ) ).
\]

Finally, by Theorem 2.2, we conclude that

\[
\prod_{i \in I} (V^*)^*( G_i ) \subseteq (V^*)^*( \prod_{i \in I} G_i ).
\]

Using these notes, we deduce the following theorem.

**Theorem 2.6.** Let \( \mathcal{V} \) be a variety, \( \{G_i\}_{i \in I} \) be a family of groups such that for any \( i, j \in I \), \( \mathcal{V}M( G_i \times G_j ) \cong \mathcal{V}M( G_i ) \times \mathcal{V}M( G_j ) \). Then \( (V^*)^*( \prod_{i \in I} G_i ) = \prod_{i \in I} (V^*)^*( G_i ) \). Consequently, \( \prod_{i \in I} G_i \) is \( \mathcal{V} \)-capable if and only if each \( G_i \) is \( \mathcal{V} \)-capable.

**Remark 2.7.** (i) In the above theorem, the sufficient condition

\[
\mathcal{V}M( A \times B ) \cong \mathcal{V}M( A ) \times \mathcal{V}M( B )
\]

is not necessary (see Example 2.3(iii) of [7]). Also, this condition is essential and can not be omitted (see Example 2.3(i), (ii) of [7]).

(ii) It is known that for varieties of abelian and nilpotent groups, and for any groups \( A \) and \( B \), \( \mathcal{V}M( A \times B ) \cong \mathcal{V}M( A ) \times \mathcal{V}M( B ) \times T \), where \( T \) is an abelian group whose elements are tensor products of the elements of \( A^{ab} \) and \( B^{ab} \) [3, 9]. Hence in these known varieties, the isomorphism \( \mathcal{V}M( A \times B ) \cong \mathcal{V}M( A ) \times \mathcal{V}M( B ) \) holds, where both \( A^{ab} \) and \( B^{ab} \) have finite exponent with \( (\exp(A^{ab}), \exp(B^{ab})) = 1 \).

In the following, using the main theorem and the above remark, we deduce some corollaries which are generalizations of some results of [7] (Remark 2.4(ii), Corollary 2.5 and Example 2.2).

**Corollary 2.8.** Let \( \{G_i\}_{i \in I} \) be a family of groups whose abelianizations have mutually coprime exponents. Then \( \prod_{i \in I} G_i \) is capable (\( N_c \)-capable) if and only if each \( G_i \) is capable (\( N_c \)-capable).
Corollary 2.9. Suppose that \( \{G_i\}_{i \in I} \) is a family of groups whose abelianizations have mutually co-prime exponents. If \( \prod_{i \in I} G_i \) is nilpotent of class at most \( c_1 \), then it is \( N_{c_1,\ldots,c_s} \)-capable if and only if every \( G_i \) is \( N_{c_1,\ldots,c_s} \)-capable.

Corollary 2.10. If \( \{G_i\}_{i \in I} \) is a family of perfect groups, then \( \prod_{i \in I} G_i \) is \( V \)-capable if and only if each \( G_i \) is \( V \)-capable, where \( V \) may be each of these three varieties:

1. variety of abelian groups,
2. variety of nilpotent groups,
3. variety of polynilpotent groups.

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