The Elephant Quantum Walk

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We explore the impact of long-range memory on the properties of a family of quantum walks in a one-dimensional lattice and discrete time, which can be understood as the quantum version of the classical ‘Elephant Random Walk’ non-Markovian process. This Elephant Quantum Walk is robustly superballistic with the standard deviation showing a constant exponent, $\sigma \propto t^2$, whatever the quantum coin operator, on which the diffusion coefficient is dependent. On the one hand, this result indicates that contrarily to the classical case, the degree of superdiffusivity in quantum non-Markovian processes of this kind is mainly ruled by the extension of memory rather than other microscopic parameters that explicitly define the process. On the other hand, these parameters reflect on the diffusion coefficient.

Introduction. The Markovian property, according to which a memoryless random process, $X = \{X_t\}$, is defined as a orderly succession of events where the conditional probability distribution of the future state $X_t$ (discrete time $t > t_0$) does only depend on its present state, $P(X_t | X_{t-1}, \ldots, X_{t_0}) = P(X_t | X_{t-1})$, is one of the cornerstones for the statistical analysis of a enormous state of natural processes within a physical context [1]. For instance, the Stosszahlansatz in Boltzmann’s kinetic theory – which paved the way from the mechanical traits of a system to its thermodynamical properties – or Einstein’s theory of Brownian motion heavily rely on the Markovian property [2]. Moreover, several computational techniques either considering discrete or continuous space(-time) – e.g., Hamiltonian Monte Carlo, belief propagation, genetic and search algorithms [3–7] – are built upon Markovian/Brownian processes that can be viewed as a random walk in phase-space of the system.

Although systematically ignored due to the matching of the theoretical predictions with experimental results, the validity of the Markovian approach explicitly depends on the time scale of relaxation of the interaction between the particles and the observational time; in other words, when the latter is less than the former it is possible to find ballistic regimes [3]. Still, there are processes for which the Markovian property does not hold, e.g., if we replace Einstein’s scenario by a situation where the particles of the bath have a similar density as the particle we are tracking, the respective Langevin Equation becomes a function of the past velocities corresponding to a non-Markovian process [8]. History-dependent processes are often related to anomalous diffusion where the variance of the stochastic process, $\sigma_t^2 \equiv \langle (t - \langle x_t \rangle)^2 \rangle$, grows as $t^\alpha$ with the diffusion exponent, $\alpha$, different from unity. Instances of physical and biological systems exhibiting subdiffusion ($0 < \alpha < 1$) or superdiffusion ($1 < \alpha < 2$) are galore [10–13]. Moreover, in several cases such as random search strategies – namely foraging – non-Markovian processes have proven to outperform Markovian proposals [14].

Moving into the quantum realm, we do not have a proper generalization of conditional probability to define the Markovian property since we need to measure the state of the system at different times and the measurement process itself affects the state. In this sense, the definition of conditional probability would depend not only on the dynamics but on the measurement scheme at different times as well. To circumvent such problem, the definition of Markovianity must be changed to one that is either similar to the classical definition or to one that presents qualitatively the same features.

Among the definitions found in literature [15–17], the Rivas-Huelga-Plenio (RHP) [18] and the Breuer-Laine-Piilo (BLP) [19] are two most notorious. The former is mathematically closer to the classical definition of Markovianity since it preserves the positivity of the intermediate dynamics and obeys the Chapman-Kolmogorov equation. The latter states that in a quantum Markovian process any information quantifier is a monotonically decreasing func-
tion in time, i.e., it keeps no memory of the previous states. A good measure of such property is the trace distance of two different initial states. In this scenario, the trace distance shows that the Markovian evolution turns these states indistinguishable. Thus, any increase of its value indicates that some information is flowing back to the system, breaking the memoryless property. It is important to notice that a RHP Markovian process is also a BLP Markovian process \[15\] \[17\].

Since in quantum theory the concept of probability is intrinsic \[20\], Quantum Walks (QW) emerge as the formal quantum analogous of Random Walks \[21\] \[22\]. Physically, QWs describe situations where a quantum particle is moving on a discrete grid, which allows simulating a wide range of transport phenomena \[23\]. Explicitly, the particle dynamically explores a large Hilbert space, \( \mathcal{H}_P \), spanned by the positions of the particle on a lattice corresponding to basis states \( \{|l\rangle, l \in \mathbb{Z}\} \), that is augmented by a Hilbert space, \( \mathcal{H}_C \), spanned by the particle internal states – e.g., a two-dimensional basis \( \{|\uparrow\rangle, |\downarrow\rangle\} \). The evolution of a QW on the full Hilbert space, \( \mathcal{H} \equiv \mathcal{H}_C \otimes \mathcal{H}_P \), is ruled by the combined application of two unitary operators \( \hat{U} \otimes \hat{P} \), where \( \hat{P} \) is the identity operator on the \( \mathcal{H}_P \) subspace.

\[
\hat{U} = \hat{S} \left[ \hat{C} \otimes \hat{I} \right],
\]

where \( \hat{I} \) is the identity operator on the \( \mathcal{H}_P \) subspace. Bearing in mind the analogy of QWs with the classical random walk, the operator \( \hat{C} \) acts on subspace \( \mathcal{H}_C \) and plays the same role as the coin. For that reason, it is named quantum coin and the internal states related to the subspace \( \mathcal{H}_C \) the coin states. On the other hand, the shift operator, \( \hat{S} \), is state-dependent and following Ref. \[21\] reads

\[
\hat{S} = \sum_l (|l+1\rangle \langle l| \otimes |\uparrow\rangle \langle \uparrow|) + \sum_l (|l-1\rangle \langle l| \otimes |\downarrow\rangle \langle \downarrow|).
\]

Assuming the quantum coin as

\[
\hat{C} = \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix},
\]

the successive application of the time-evolution operator, \( \hat{U} \), \( t \) times to the initial state

\[
|\Psi\rangle_0 = |l_0\rangle \otimes |s\rangle \equiv \left( \psi_0^T(l) \right) \psi_0(l),
\]

where the internal state is defined as

\[
|s\rangle = \cos \left( \frac{\gamma}{2} \right) |\uparrow\rangle + \exp^{-i\phi} \sin \left( \frac{\gamma}{2} \right) |\downarrow\rangle = \begin{pmatrix} \cos \frac{\gamma}{2} \\ e^{-i\phi} \sin \frac{\gamma}{2} \end{pmatrix},
\]

allows us to obtain the normalised probability at time \( t \), \( P(t) = \text{Tr}_s \langle \Psi | \Psi \rangle \). A straightforward computation (see e.g. \[21\]) shows that \( \sigma_t \propto t \); in other words, the standard QW diffuses ballistically in opposition to the random walk. Similarly to the classical case, QWs are short of providing a full account of transport phenomena \[24\] \[25\] and yields an important approach in quantum computing processes \[20\] \[26\] \[28\].

Even so, quantum models tend to follow in the same approach as most of the classical models: to make non-Markovianity emerge from analytically treatable system-environment interactions rather than properties of microscopic nature related to the dynamics \[19\]. One of the few (simple) classical cases where microscopic dynamical rules are translated into non-Markovian statistical properties of the walker is the so called Elephant Random Walk (ERW) \[29\]. In the present Letter, we introduce a Quantum version of that stochastic process and assess to what extent both non-Markovian models differ one another. Particularly, we are interested in understanding what kind of behaviour such non-Markovian QW yields bearing in mind that the standard QW already provides ballistic motion and in addition to related such properties to physical systems on which these findings can be made.

**The classical Elephant.** According to Shütz and Trimper \[29\], the ERW describes displacements on the infinite discrete lattice \( X_t \in \mathbb{Z} \) assuming discrete time as well. The ‘elephant’ starts its walk at some specific point \( X_0 \) at time \( t = 0 \) and for \( t > t_0 \) the stochastic evolution, \( X_t = X_{t-1} + \Delta_t \), occurs as follows:

(A) For \( t = 1 \), the ‘elephant’ moves to the right (\( \Delta_1 = 1 \)) with probability \( q \) and to the left (\( \Delta_1 = -1 \)) with probability \( 1 - q \);

(B) for \( t > 1 \), an instant in the past \( t' < t \) is first randomly and independently chosen abiding by a uniform probability and then \( \Delta_j \) is determined by the rule: \( \Delta_j = \Delta_{j'} \) with probability \( p \) and \( \Delta_j = -\Delta_{j'} \) with probability \( 1 - p \).

Accordingly, the conditioned probability distribution of the classical walker displacement at time \( t \) was calculated in Ref. \[29\] and reads

\[
P(\Delta_{t+1} = \ell | \Delta_t, \ldots, \Delta_1) = \sum_{j=1}^t \frac{1 - (1 - 2p) \ell \Delta_j}{2t},
\]

wherefrom the conditioned moments of the displacement can be computed. Thus, it was proven that
the memory parameter $p$ governs the long-term behaviour of such process: for $p < 1/2$, the ‘elephant’ is a Brownian Walker (BW), $\mu_t \equiv \langle X_t \rangle \sim X_0$ and $\sigma_t^2 \propto t$; for $1/2 < p < 3/4$ it becomes a biased BW with $\mu_t \propto t^{2p-1}$ whereas for $p > 3/4$ besides the bias behaviour, the motion becomes superdiffusive with $\sigma_t^2 \propto t^{4p-2}$. Yet, it can be shown that the distribution is Gaussian and the Fokker-Planck Equation is equivalent to that if BW subjected to a time-dependent drift force $f_t (X) = (2p - 1) (X_t - X_0)/t$.

The quantum analogue of the ERW, we extend the definition of the QW to a time dependent drift force $f_t (X) = (2p - 1) (X_t - X_0)/t$.

As the quantum version of rule (A), we consider a set of random variables $\Delta_j$, uniformly distributed at each time step of the QW evolution. The quantum equivalent to rule (B) is to consider that at time $t+1$, the amplitude $\psi_{t+1}^j (t+\Delta_j)$ encodes the probability to move towards the right by a step of size $\Delta_j$ and the amplitude $\psi_{t+1}^j (t-\Delta_j)$ encodes the probability to move towards the left by a step of size $\Delta_j$.

Having said that, the dynamical evolution defined by Eq. (1) for the EWR is replaced by

$$\hat{U}_t = \hat{S}^E_t \left[ \hat{\mathcal{E}} \otimes T \right].$$

Therefore, a significant difference immediately emerges: the evolution operator ceases being the simple iteration of the one-step operator, i.e., $\hat{U}_t \neq \hat{U}_1^t$, a property that hints at the non-Markovian nature of the EQW, which we shall prove shortly. Actually, the equality, $\hat{U}_t = \hat{U}_1^t$, can be understood as the quantum analogue of the classical condition that the transition matrix $T$ of a Markov chain after $t$ steps yields, $T_t = T_1^t$. Moreover, a direct calculation, allows us to introduce the conditioned probability density functions for the jumps in the EQW similarly to Eq. (6). For $t = 2$ that reads

$$P (\Delta_2 | \Delta_1) = \left[ \cos \left( \frac{1-\Delta_2}{4} \pi - \theta \right) \right]^2 \left\{ 1 + \Delta_1 \left[ \cos (\gamma) \cos (2\theta) + \sin (\gamma) \sin (2\theta) \sin (\phi) \right] \right\}$$

and for $t \geq 2$,

$$P (\Delta_{t+1} = \Delta | \Delta_t, \Delta_{t-1}, \ldots, \Delta_1) = \frac{(t-1)!}{t!} \sum_{j=1}^{t} \prod_{k=2}^{t} \left[ \cos \left( \frac{\Delta_{k+1} - \Delta_k}{4} \pi - \theta \right) \right]^2 P (\Delta_2 | \Delta_1) \delta_{\Delta, \Delta_j}$$

where the coefficients $A (t, \{ \Delta_t \})$ and $B (t, \{ \Delta_t \})$ represent the number of sequences yielding an even(odd) number of contrarian steps composing the sequence so that the condition $A (t, \{ \Delta_t \}) + B (t, \{ \Delta_t \}) = (t - 1)!$ is verified. Notice that for $t \geq 2$, and due to the quantum nature of this process, we cannot have the conditioned probability abiding by a simple superposition of contributions involving the gauging value $\Delta_{t+1} = \ell$ and a past chosen step. Each term involves all the values of the chain. This contains a signature of a long-range (non-Markovian) memory effect on the walker dynamics, giving rise to a completely different behavior from the usual QW, as we can see in Fig. 1.

To finally assess the non-Markovianity of the EQW we have followed [19, 30] where it is shown that if the discrete analogous of the trace distance velocity, $v_1 \equiv D_{t+1} - D_t$, is positive at least once, then the process is non-Markovian.\(^1\)

\(^1\) $D_t = \frac{1}{2} \text{tr} \{ \hat{\mathcal{E}} t \} - \omega^B t$, where $\omega_j^{(t)}$ represents the trace of the density matrix operator at time $t$ given the initial condition $\langle \rangle$, defined as $\rho_t = |\Psi_t\rangle \langle \Psi_t|$, for the pure states defined in Eq. (9).
In Fig. 2 we present the computation of the trace distance and its velocity for two initial pure states representing opposite poles on the Bloch sphere (north and south) with $\gamma_A = 0$ and $\gamma_B = \pi$, $\phi = 0$, for an initial Gaussian packet of width $\delta = 0.001$. The inset in that figure shows that $v_1$ is positive more than once, implying the EQW is clearly non-Markovian.

![FIG. 1. (Color online) Time evolution of the probability distribution for the QW and the EQW for $\theta = \frac{\pi}{4}$. The initial state is localized at the origin, with a coin state $\frac{1}{\sqrt{2}} (1, 1)^T$.](image1)

![FIG. 2. (Color online) Left panel: the squares (blue) represent the trace distance, $D(t)$, as a function of the number of time steps between two initial states with $\gamma_A = 0$ and $\gamma_B = \pi$, $\phi = 0$, for an initial Gaussian packet of width $\delta = 0.001$. In case of noise, the trace distance decreases rapidly after the second time step (circles, red). The inset shows the velocity of the trace distance. Right panel: Diffusion coefficient and the exponent $\eta$, in function of different values of $\theta$, for the same initial condition.](image2)

As occurs in the classical elephant walk, long range memory effects play a crucial role in the dynamics, and analogously to the superdiffusivity observed in the EW, in the QEW we prove analytically and numerically a strong superballistic behaviour. Indeed, following the same techniques introduced by [31, 32], we computed the first and the second moment of the distribution (see the EPAPS for the analytical details):

$$\langle \ell \rangle = \int_{\ell = -\infty}^{\ell = \infty} \ell \langle r(t) \rangle d\ell = 0, \quad (11)$$

and

$$\langle \ell^2 \rangle = \int_{\ell = -\infty}^{\ell = \infty} \ell^2 \langle r(t) \rangle d\ell = 2\sqrt{2\pi}(C_1 + 2C_2)t^3, \quad (12)$$

where $C_1, C_2$ a real coefficients depending on the coin parameter $\theta$. That clearly shows that the EQW is robustly superballistic, with the standard deviation showing a constant exponent, whatever the quantum coin operator, $\sigma \propto t^2$. This result, confirmed numerically in Fig. 2 indicates that, contrarily to the classical case, the degree of superballistic in non-Markovian processes of this kind is mainly ruled by the extension of the memory and form of the memory, rather than the microscopic parameters that define the dynamics of the problem.

**Concluding Remarks.** In this work, we have established a long-range memory Quantum Walk that can be understood as the quantum version of the so-called ‘Elephant Random Walk’. We have found two unexpected features for our model: first, its diffusion takes place superballistically, and second, the bias of the coin does only affects the diffusion constant so that the diffusion exponent (for the standard deviation) is constant and equal to 3, which constitutes a sheer difference to the ERW that exhibits a transition between normal and superdiffusion. Interestingly, the same kind of behaviour was found in Refs. [33, 34] (Figs. 3 and 4) of the wave packet spreading in several disordered and quasiperiodic systems as well as in tight-binding lattices if a sublattice with on-site potential is embedded in a uniform lattice without on-site potential, which is likely to take place because of an energy band mismatch between the sublattice and the rest uniform lattice, and by the structure of the underlying eigenstates. That said, our EQW can be employed as the quantum version of the so-called ‘Elephant Random Walk’. We have found two unexpected features for our model: first, its diffusion takes place superballistically, and second, the bias of the coin does only affects the diffusion constant so that the diffusion exponent (for the standard deviation) is constant and equal to 3, which constitutes a she
dependence passing through exponential and power-law.

From a quantum implementation perspective, apparatus equivalent to those considering cold atoms in optical lattices [37,39], which manage to yield superballistic diffusion are straightforward options to performing EQWs. Alternatively, we can understand the introduction of such memory as the storage process on the state of the system that is cyclically isolated and put in contact with a surrounding environment which acts as the tossing of the coin. In practical terms, this can be carried out considering an apparatus close to that present in Ref. [40] that accommodates that long-time storage.

Last, taking into account that classical superdiffusive search approaches are robustly more efficient than Brownian searches, we consider it is worth looking at the EQW within the context of search algorithms as an (improved) extension of the random walk search algorithm [41,42], which fits in the Grover class.

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