Free Boson Representation of $q$-Vertex Operators and their Correlation Functions

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ABSTRACT

A bosonization scheme of the $q$-vertex operators of $U_q(\hat{sl}_2)$ for arbitrary level is obtained. They act as intertwiners among the highest weight modules constructed in a bosonic Fock space. An integral formula is proposed for $N$-point functions and explicit calculation for two-point function is presented.

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1 Introduction

One of the central subjects of mathematical physics has been the studies on exactly solvable models in two dimensions for many years. Infinite dimensional symmetries such as conformal and current algebra gives powerful tools to investigate systems just on the critical point [1]. It is now a very important problem how to extend the method developed in the critical theories to massive field theories and lattice models.

A breakthrough was brought by Frenkel and Reshetikhin [2] who studied the $q$-deformation of the vertex operator as an intertwiner between certain modules of quantum affine algebra $U_q(\widehat{sl}_2)$. They showed that the correlation functions satisfy a $q$-difference equation, $q$-deformed Knizhnik-Zamolodchikov equation, and that the resulting connection matrices give rise to the elliptic solution to Yang-Baxter equation of RSOS models [3] [4]. Using the $q$-vertex operators people in Kyoto school [5] succeeded in diagonalization of the XXZ spin chain and showed that the spectra of the XXZ model is completely determined in terms of the representation theory of $U_q(\widehat{sl}_2)$. Furthermore, they found an integral formula for correlation functions of the local operators of the XXZ model [6] by utilizing bosonization of $U_q(\widehat{sl}_2)$ of level one [7] and the bosonized $q$-vertex operators.

In a previous paper [8], one of the authors construct the bosonization of $U'_q(\widehat{sl}_2)$ currents for arbitrary level à la Wakimoto [9]. In this paper we shall introduce a bosonization of the “elementary” $q$-vertex operators, which have exactly the same commutation relations with the generators of $U'_q(\widehat{sl}_2)$ as the bona-fide $q$-vertex operators have. They are well-defined operators acting on a bosonic Fock space, in which all the integrable highest weight modules of a given level can be embedded. Finally $q$-vertex operators as intertwiners among these modules are obtained in terms of the elementary $q$-vertex operators dressed with the screening charges. This technique provides a natural framework to write down an integral formula for correlation functions of the $q$-vertex operators. Our formula will be useful to examine higher spin chain [10].

The present article is organized as follows. In section 2 we construct the currents which give Drinfeld realization of $U'_q(\widehat{sl}_2)$ [11] in terms of free bosons [8]. In section 3 we construct the “elementary” $q$-vertex operator. In section 4 we define the Fock space on which the currents and the elementary $q$-vertex operators act. We also introduce the screening charge, which is necessary to calculate correlation functions. Furthermore we give the expression of the $N$-point function in terms of the bosonized operators. In section 5 we calculate the two-point function in a simple case and show the relevance of our formulation. In section 6 we summarize our results and give some remark.

Three appendices are devoted to the detail of calculation in section 5. In Appendix A OPE formulae among the bosonized operators are listed. In Appendix B we give the normalization
of the elementary vertex operators. In Appendix C we discuss the response of Jackson integrals to $p$-shift of valuables.

2 Free Boson Realization of $U_q'(\hat{\mathfrak{sl}_2})$

In this section we briefly recall the bosonization of $U_q'(\hat{\mathfrak{sl}_2})$. In Appendix C we discuss the response of Jackson integrals of level $k$. Throughout this paper let $q$ be transcendental over $\mathbb{Q}$ with $|q| < 1$. We use the following standard notation:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}},$$

for $m \in \mathbb{Z}$.

The quantum affine algebra $U_q(\hat{\mathfrak{sl}_2})$ is an associative algebra over $\mathbb{Q}(q)$ with 1, generated by $e_0, e_1, f_0, f_1$ and $q^h (h \in P^*)$. The defining relations are as follows

$$q^h q^{h'} = q^{h+h'}, \quad q^0 = 1,$$

$$q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i,$$

$$q^h f_i q^{-h} = q^{-(h,\alpha_i)} e_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_j^{-1}}{q - q^{-1}} \quad (t_i = q^{n_i}),$$

$$e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j),$$

$$f_i^3 f_j - [3] f_i^2 f_j + [3] f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j).$$

(2.1)

The algebra $U_q'(\hat{\mathfrak{sl}_2})$ is the subalgebra of $U_q(\hat{\mathfrak{sl}_2})$ generated by $\{e_i, f_i, t_i \ (i = 0, 1)\}$.

The algebra $U_q(\hat{\mathfrak{sl}_2})$ has a Hopf algebra structure with the following coproduct $\Delta: U_q(\hat{\mathfrak{sl}_2}) \to U_q(\hat{\mathfrak{sl}_2}) \otimes U_q(\hat{\mathfrak{sl}_2})$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad (i = 0, 1)$$

$$\Delta(q^h) = q^h \otimes q^h \quad h \in P^*. $$
2.2 Drinfeld Realization of $U'_q(\hat{\mathfrak{sl}}_2)$ The Chevalley generators $e_i, f_i, t_i$ are not convenient for considering the bosonization. We recall here the Drinfeld realization of $U'_q(\hat{\mathfrak{sl}}_2)$ \[1\] which we will bosonize. The Drinfeld realization of $U_q(\hat{\mathfrak{sl}}_2)$ is an associative algebra generated by the letters $\{J_n^\pm|n \in \mathbb{Z}\}, \{J_n^3|n \in \mathbb{Z}_{\neq 0}\}$, $\gamma^{\pm 1/2}$ and $K$, satisfying the following relations.

\[
\gamma^{\pm 1/2} \in \text{the center of the algebra,}
\]

\[
[J_n^3, J_m^3] = \delta_{n+m,0} \frac{1}{n} [2n] \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},
\]

\[
[J_n^3, K] = 0,
\]

\[
K J_n^\pm K^{-1} = q^{\pm 2} J_n^\pm,
\]

\[
[J_n^3, J_m^\pm] = \pm \frac{1}{n} [2n] \gamma^{\mp |n|/2} J_n^\pm,
\]

\[
J_{n+1}^\pm J_m^\pm - q^\pm 2 J_m^\pm J_{n+1}^\pm = q^\pm 2 J_n^\pm J_{m+1}^\pm - J_{m+1}^\pm J_n^\pm,
\]

\[
[H_n^+, J_m^-] = \frac{1}{q - q^{-1}} (\gamma^{(n-m)/2} \psi_{n+m} - \gamma^{(m-n)/2} \varphi_{n+m}),
\]

where $\{\psi_r, \varphi_s|r, s \in \mathbb{Z}\}$ are related to $\{J_i^3|l \in \mathbb{Z}_{\neq 0}\}$ by

\[
\sum_{n \in \mathbb{Z}} \psi_n z^{-n} = K \exp \left\{ (q - q^{-1}) \sum_{k=1}^{\infty} J_k^3 z^{-k} \right\},
\]

\[
\sum_{n \in \mathbb{Z}} \varphi_n z^{-n} = K^{-1} \exp \left\{ -(q - q^{-1}) \sum_{k=1}^{\infty} J_k^3 z^{k} \right\}.
\]

(2.3)

The standard Chevalley generators $\{e_i, f_i, t_i\}$ are given by the identification

\[
t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = J_0^+, \quad f_1 = J_0^-, \quad e_0 t_1 = J_1^-, \quad t_1^{-1} f_0 = J_1^+.
\]

(2.4)

2.3 Bosonization of $U'_q(\hat{\mathfrak{sl}}_2)$ Let $k$ be a non-negative integer. Let $\{a_n, b_n, c_n, Q_a, Q_b, Q_c|n \in \mathbb{Z}\}$ be a set of operators satisfying the following commutation relations:

\[
[a_n, a_m] = \delta_{n+m,0} \frac{[2n] [(k + 2)n]}{n}, \quad [\tilde{a}_0, Q_a] = 2(k + 2),
\]

\[
[b_n, b_m] = -\delta_{n+m,0} \frac{[2n] [2n]}{n}, \quad [\tilde{b}_0, Q_b] = -4,
\]

\[
[c_n, c_m] = \delta_{n+m,0} \frac{[2n] [2n]}{n}, \quad [\tilde{c}_0, Q_c] = 4,
\]

(2.5)

where

\[
\tilde{a}_0 = \frac{q - q^{-1}}{2 \log q} a_0, \quad \tilde{b}_0 = \frac{q - q^{-1}}{2 \log q} b_0, \quad \tilde{c}_0 = \frac{q - q^{-1}}{2 \log q} c_0,
\]
and others commute.

Let us introduce the free bosonic fields \( a, b, c \) carrying parameters \( L, M, N \in \mathbb{Z}_{>0}, \alpha \in \mathbb{R} \). Define \( a \left( L; M, N \mid z; \alpha \right) \) by

\[
a(L; M, N \mid z; \alpha) = -\sum_{n \neq 0} \frac{[L n] a_n}{[M n][N n]} z^{-n} q^{n|\alpha} + \frac{L a_0}{M N} \log z + \frac{L Q a}{M N},
\]

(2.6)

\( b(L; M, N \mid z; \alpha) \), \( c(L; M, N \mid z; \alpha) \) are defined in the same way. In the case \( L = M \) we also write

\[
a(N \mid z; \alpha) = a(L; L, N \mid z; \alpha)
\]

\[
= -\sum_{n \neq 0} \frac{a_n}{[N n]} z^{-n} q^{n|\alpha} + \frac{a_0}{N} \log z + \frac{Q a}{N},
\]

(2.7)

and likewise for \( b(N \mid z; \alpha) \), \( c(N \mid z; \alpha) \).

Let \( \{a_n, b_n, c_n \mid n \in \mathbb{Z}_{\geq 0}\} \) be annihilation operators, and \( \{a_n, b_n, c_n, Q_a, Q_b, Q_c \mid n \in \mathbb{Z}_{< 0}\} \) creation operators. We denote by : \( O(z) : \) the normal ordering of \( O(z) \). For example,

\[
: \exp \left\{ b(2 \mid z; \alpha) \right\} := \exp \left\{ -\sum_{n < 0} \frac{b_n}{2n} z^{-n} q^{n|\alpha} \right\} \exp \left\{ -\sum_{n > 0} \frac{b_n}{2n} z^{-n} q^{n|\alpha} \right\} e^{Q_b/2} z^{\tilde{b}_0/2}.
\]

Now we define the currents \( J^3(z), J^+(z) \) as follows:

\[
J^3(z) = k_\pm \partial_z a \left( k + 2 \left| q^{-2}z; -1 \right) + 2 \partial_z b \left( 2 \left| q^{-k-2}z; -\frac{k + 2}{2} \right) + 2 \partial_z c \left( 2 \left| q^{-k-2}z; -1 \right) + 2 \partial_z \left( 2 \left| q^{-k-2}z; 1 \right) \right) \right).
\]

(2.8)

Here the \( q \)-difference operator with parameter \( n \in \mathbb{Z}_{>0} \) is defined by

\[
n \partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n} z)}{(q - q^{-1}) z}.
\]

Define further the auxiliary fields \( \psi(z), \varphi(z) \) as

\[
\psi(z) = : \exp \left\{ (q - q^{-1}) \sum_{n > 0} (q^n a_n + q^{(k+2)n/2} b_n) z^{-n} + (\tilde{a}_0 + \tilde{b}_0) \log q \right\} :.
\]

(2.9)

\[
\varphi(z) = : \exp \left\{ -(q - q^{-1}) \sum_{n < 0} (q^{3n} a_n + q^{3(k+2)n/2} b_n) z^{-n} - (\tilde{a}_0 + \tilde{b}_0) \log q \right\} :.
\]
We give the mode expansions of these fields as
\[
\sum_{n \in \mathbb{Z}} J^3_n z^{-n-1} = J^3(z), \quad \sum_{n \in \mathbb{Z}} J^\pm_n z^{-n-1} = J^\pm(z),
\]
(2.10)
and let
\[
K = q^{\delta_0 + \delta_0}, \quad \gamma = q^k.
\]
(2.11)

Then we get the following \([8]\):

**Proposition 2.1** \(\{J^3_n | n \in \mathbb{Z}_{\not=0}\}, \{J^\pm_n | n \in \mathbb{Z}\} \{\varphi_n, \psi_n | n \in \mathbb{Z}\}, K, \text{ and } \gamma \text{ defined by } (2.8), (2.9), (2.10) \text{ and } (2.11) \text{ satisfy the relations } (2.2).\)

**2.4 Finite Dimensional \(U_q(\hat{sl}_2)\) Module** For \(l \in \mathbb{Z}_{\geq 0}\) let \(V^{(l)}\) denote the \((l + 1)\)-dimensional \(U'_q(\hat{sl}_2)\)-module (spin \(l/2\) representation) with basis \(\{v^{(l)}_m | 0 \leq m \leq l\}\) given by
\[
e_1 v^{(l)}_m = [m]v^{(l)}_{m-1}, \quad f_1 v^{(l)}_m = [l-m]v^{(l)}_{m+1}, \quad t_1 v^{(l)}_m = q^{l-2m}v^{(l)}_m,
\]
e_0 = f_1, \quad f_0 = e_1, \quad t_0 = t^{-1}_1 \text{ on } V^{(l)}.

Here \(v^{(l)}_m\) with \(m < 0\) or \(m > l\) is understood to be 0. In the case \(l = 1\) we also write \(v^{(l)}_0 = v_+ \text{ and } v^{(l)}_1 = v_-\).

We equip \(V_z^{(l)} = V^{(l)} \otimes \mathbb{Q}(q)[z, z^{-1}]\) with a \(U_q(\hat{sl}_2)\)-module structure via
\[
e_i (v^{(l)}_m \otimes z^n) = e_i v^{(l)}_m \otimes z^{n+\delta_i 0}, \quad f_i (v^{(l)}_m \otimes z^n) = f_i v^{(l)}_m \otimes z^{n-\delta_i 0},
\]
wt\((v^{(l)}_m \otimes z^n) = n\delta + (l-2m)(\Lambda_1 - \Lambda_0)\)

Namely \(V_z^{(l)}\) is the affinization of \(V^{(l)}\).

We also need the representation of Drinfeld generators on level 0 modules.

**Proposition 2.2** Spin \(l/2\) representation of \(U_q(\hat{sl}_2)\) is given in terms of the Drinfeld generators by
\[
\gamma^{\pm 1/2} v^{(l)}_m = v^{(l)}_m, \quad K v^{(l)}_m = q^{l-2m} v^{(l)}_m, \\
J^+_n v^{(l)}_m = z^n q^{n(l-2m+2)} [l-m+1] v^{(l)}_{m-1}, \\
J^-_n v^{(l)}_m = z^n q^{n(l-2m)} [m+1] v^{(l)}_{m+1}, \\
J^3_n v^{(l)}_m = \frac{z^n}{n} \left\{ [nl] - q^{n(l+1-m)} (q^n + q^{-n}) [nm] \right\},
\]
(2.12)
where \(v^{(l)}_m = 0\) if \(m > l\) or \(m < 0\).
3 Elementary \( q \)-Vertex Operators

In this section we construct the operators which have exactly the same commutation relations with the generators of \( U_q(\widehat{sl}_2) \) as the bona-fide \( q \)-vertex operators have.

A vector \( | \lambda \rangle \) is called a highest weight vector of weight \( \lambda \) if it satisfies the highest weight condition

\[
e_i | \lambda \rangle = 0, \quad t_i | \lambda \rangle = q^{(h_i, \lambda)} | \lambda \rangle, \quad i = 0, 1.
\]

The left highest weight module \( V(\lambda) \) with the highest weight vector \( | \lambda \rangle \) is defined by

\[
V(\lambda) := U_q(\widehat{sl}_2) | \lambda \rangle.
\]

The right highest weight module is defined in a similar manner.

The left (resp. right) highest weight module with highest weight \( \lambda \in P_k \) will be denoted by \( V(\lambda) \) (resp. \( V^r(\lambda) \)). We fix a highest weight vector \( | \lambda \rangle \in V(\lambda) \) (resp. \( \langle \lambda | \in V^r(\lambda) \)) once for all. There is a unique symmetric bilinear pairing \( V^r(\lambda) \times V(\lambda) \to F \) such that

\[
\langle \lambda | \lambda \rangle = 1, \quad \langle ux | u' \rangle = \langle u | xu' \rangle \quad \forall x \in U_q(\widehat{sl}_2), \quad \forall \langle u | \in V^r(\lambda), \quad \forall | u' \rangle \in V(\lambda).
\]

3.1 Definition of \( q \)-Vertex Operators

We recall below the properties of the \( q \)-vertex operators (\( q \)-VOs) relevant to the subsequent discussions. For more detail, see [5][8].

Fix positive integers \( k, l \) and let \( \lambda, \mu \in P_k \). We set \( \Delta_\lambda = (\lambda, \lambda + 2\rho)/2(k + 2) \).

We shall use the following type of VO

\[
\Phi^V(\lambda)_{\mu}(z) = z^{\Delta_-} \Phi^{V}(\lambda)_{\mu}(z) \quad \Phi^{V}(\lambda)_{\mu}(z) : V(\lambda) \to V(\mu) \otimes V(z)
\]

The map (3.1) means a formal series of the form

\[
\Phi^{V}(\lambda)_{\mu}(z) = \sum_{n} \sum_{m=0}^{l} \Phi_{m,n} \otimes v^{(l)}_{m} z^{-n}
\]

where \( \text{wt}(v^{(l)}_{m}) = (l - 2m)(\Lambda_1 - \Lambda_0) \), \( \delta = \alpha_0 + \alpha_1 \).

By definition, the \( q \)-VO satisfies the intertwining relations

\[
\Phi^{V}(\lambda)_{\mu}(z) \circ x = \Delta(x) \circ \Phi^{V}(\lambda)_{\mu}(z), \quad \forall x \in U_q(\widehat{sl}_2).
\]

\footnote{We do not impose the irreducibility conditions \( f_i^{(h_i, \lambda) + 1} | \lambda \rangle = 0, \quad i = 0, 1. \)}

\footnote{This VO is called “type I” in ref. [3].}
From the general arguments on $q$-VOs \cite{13}, in our case there exists at most one VO up to proportionality. We normalize $\tilde{\Phi}_{\lambda}^{V^{(l)}}(z)$ such that the leading term is $|\mu\rangle \otimes v_{m}^{(l)}$:

$$\tilde{\Phi}_{\lambda}^{V^{(l)}}(z) |\lambda\rangle = |\mu\rangle \otimes v_{m}^{(l)} + \cdots,$$

where $\cdots$ means terms of the form $u \otimes v$, $\text{wt } u \neq \mu$.

**Proposition 3.1** If we write

$$\tilde{\Phi}_{\lambda}^{V^{(l)}}(z) = \sum_{m=0}^{l} \tilde{\Phi}_{\lambda m}^{V^{(l)}}(z) \otimes v_{m}^{(l)},$$

then

$$\tilde{\Phi}_{\lambda m-1}^{V^{(l)}}(z) = \frac{1}{[l-m]} \left\{ \tilde{\Phi}_{\lambda m}^{V^{(l)}}(z) f_{1} - q^{2m-l} f_{1} \tilde{\Phi}_{\lambda m}^{V^{(l)}}(z) \right\}, \quad m = 1, 2, \ldots, l. \quad (3.4)$$

This is easily checked by evaluating the both sides of (3.2) for $x = f_{1}$.

For two vertex operators

$$\tilde{\Phi}_{\mu_{01}}^{V^{(l_{1})}}(z_{1}) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{l} \tilde{\Phi}_{m,n}^{V^{(l_{1})}} \otimes v_{m}^{(l_{1})} z_{1}^{-n},$$

$$\tilde{\Phi}_{\mu_{02}}^{V^{(l_{2})}}(z_{2}) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{l} \tilde{\Phi}_{m,n}^{V^{(l_{2})}} \otimes v_{m}^{(l_{2})} z_{2}^{-n},$$

the composition of these two is defined as a formal series in $z_{1}, z_{2}$:

$$\tilde{\Phi}_{\mu_{1}}^{V^{(l_{2})}}(z_{2}) \circ \tilde{\Phi}_{\mu_{0}}^{V^{(l_{1})}}(z_{1}) = \sum \tilde{\Phi}_{j m} \circ \tilde{\Phi}_{k n} \otimes v_{j}^{(l_{2})} z_{2}^{-m} \otimes v_{k}^{(l_{1})} z_{1}^{-n}.$$

The composition of $N$ $q$-vertex operators are defined in a similar fashion.

**3.2 Elementary $q$-Vertex Operators** In \cite{6} an integral formula for correlation functions of the local operators of the XXZ model is obtained by utilizing bosonization of the $U_{q}(\hat{\mathfrak{sl}_{2}})$ of level one \cite{6} and the bosonized $q$-vertex operators. In the same spirit we derive the formulae for the $q$-vertex operators for arbitrary level $k$ in terms of bosonic fields $a, b$ and $c$.

Since the Drinfeld generators are successfully bosonized, we want to know how the intertwining properties are expressed in those terms \cite{16}.

**Proposition 3.2** For $k \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}_{> 0}$ we have

$$\Delta(J_{k}^{+}) = J_{k}^{+} \otimes \gamma^{k} + \gamma^{2k} K \otimes J_{k}^{+} + \sum_{i=0}^{k-1} \gamma^{(k+3i)/2} \tilde{\Phi}_{i} \otimes \gamma^{k-i} J_{i}^{+} \mod N_{-} \otimes N_{+}^{2},$$

$$\Delta(J_{-l}^{+}) = J_{-l}^{+} \otimes \gamma^{-l} + K^{-1} \otimes J_{-l}^{+} + \sum_{i=1}^{l-1} \gamma^{(l-i)/2} \varphi_{-l+i} \otimes \gamma^{-l+i} J_{i}^{+} \mod N_{-} \otimes N_{+}^{2},$$
\[
\begin{align*}
\Delta(J_{i}^-) &= J_i^- \otimes K + \gamma^i \otimes J_i^- + \sum_{i=1}^{l-1} \gamma^{i-i} J_i^- \otimes \gamma^{(i-1)/2} \psi_{l-i} \\
\Delta(J_{-k}) &= J_{-k} \otimes \gamma^{-2k} K^{-1} + \gamma^{-k} K \otimes J_{-k} + \sum_{i=0}^{k-1} \gamma^{i-k} J_{-i} \otimes \gamma^{-(k+3)/2} \varphi_{i-k} \\
\Delta(J_i^3) &= J_i^3 \otimes \gamma^{i/2} + \gamma^{3l/2} \otimes J_i^3 \\
\Delta(J_{-l}^3) &= J_{-l}^3 \otimes \gamma^{-3l/2} + \gamma^{-l/2} \otimes J_{-l}^3
\end{align*}
\] mod \(N_+^2 \otimes N_+\),

Here \(N_+\) and \(N_+^2\) are left \(Q(q)[\gamma^\pm, \psi, \varphi, r, -s \in \mathbb{Z}_{\geq 0}]\)-modules generated by \(\{J_m^+ \mid m \in \mathbb{Z}\}\) and \(\{J_{m\pm}^+ \mid m, n \in \mathbb{Z}\}\) respectively.

By using Propositions 2.2, 3.2 and noting that \(N_+ v_0^l = N_- v_0^l = 0, N_+ v_m^l \subset F[z, z^{-1}] v_m^{l+1}\), we get the exact relations

\[
\left[ J_n^+, \tilde{\Phi}^\mu_V^{(l)}(z) \right] = q^{2n} z^n \frac{[nl]}{n} q^{k(n|n)/2} \tilde{\Phi}^\mu_V^{(l)}(z) \quad n \neq 0,
\]

\[\left[ \tilde{\Phi}^\mu_V^{(l)}(z), J^+(w) \right] = 0,
\]

\[K \tilde{\Phi}^\mu_V^{(l)}(z) K^{-1} = q^l \tilde{\Phi}^\mu_V^{(l)}(z),\]

which follows from \(V(\mu) \otimes v_i^l\) components of intertwining relation.

These conditions put stringent constraints on the possible bosonized form \(\phi_i^{(l)}(z)\) of vertex operators \(\tilde{\Phi}^\mu_V^{(l)}(z)\). By the explicit calculation, we can check that if

\[
\phi_i^{(l)}(z) = : \exp \left\{ a \left( l; 2, k + 2 \left| q^{k} z; \frac{k + 2}{2} \right. \right) \right\}:
\]

is substituted for \(\tilde{\Phi}^\mu_V^{(l)}(z)\), then all the commutation relations (3.5) hold. Proposition 3.1 suggest that the other components of the vertex operator should be defined by the following multiple contour integral

\[
\phi_i^{(l)}(z) = \frac{1}{[1][2] \cdots [l-m]} \oint dw_1 \oint dw_2 \cdots \oint dw_{l-m} \times \left[ \left[ \phi_i^{(l)}(z), J_-(w_1)_q, J_-(w_2)_q^2 \cdots J_-(w_{l-m})_q^{l-m} \right] \right].
\]

We will call these operators (3.6), (3.7) as “elementary vertex operators”. A salient feature of these operators is that they are determined solely from the commutation relation with bosonized \(U_q(sl_2)\) currents; this is completely independent of which infinite dimensional modules they intertwine.

Before discussing the relation between “elementary \(q\)-vertex operators” and bona-fide vertex operators, we need to clarify on which space these bosonized operators are acting.

\[\text{3 These elementary \(q\)-vertex operators are determined from a part of the intertwining properties, but it is very likely that they enjoy all of these properties.}\]
4 Fock Module, Screening Charge and Correlation Function

In this section, we define the Fock module of bosons on which the \( U_q(\widehat{sl}_2) \) currents \( J^3(z), J^\pm(z) \), and the elementary \( q \)-vertex operators \( \phi_m^{(l)}(z) \) act. All the integrable highest weight modules are constructed in this Fock module. Further the \( q \)-vertex operators as the intertwiner among these modules are obtained.

4.1 Fock module and Highest Weight Module

From the observation that
\[
\left[ J^3(z), \tilde{b}_0 + \tilde{c}_0 \right] = 0, \quad \left[ J^\pm(z), \tilde{b}_0 + \tilde{c}_0 \right] = 0, \quad \left[ \phi_m^{(l)}(z), \tilde{b}_0 + \tilde{c}_0 \right] = 0, \quad (4.1)
\]
we can restrict the full Fock module of the boson \( a, b, c \) to the sector such that the eigenvalue of the operator \( \tilde{b}_0 + \tilde{c}_0 \) is equal to 0. This requirement does not conflict with any other conditions we shall impose.

Let us introduce a vacuum vector \( |0\rangle \) which has the following properties:
\[
a_n |0\rangle = 0, \quad b_n |0\rangle = 0, \quad c_n |0\rangle = 0, \quad n \geq 0.
\]

Define the vectors \( |r, s\rangle \) by
\[
|r, s\rangle := \exp \left\{ r \frac{Q_a}{k + 2} + s \frac{Q_b + Q_c}{2} \right\} |0\rangle,
\]
where \( r \in \frac{1}{2} \mathbb{Z}, s \in \mathbb{Z} \).

Let \( F \) be a free \( \mathbb{Q}(q) \) module generated by \( \{ a_{-1}, a_{-2}, \ldots, b_{-1}, b_{-2}, \ldots, c_{-1}, c_{-2}, \ldots \} \). Now we define the Fock modules \( F_{r,s} \) as
\[
F_{r,s} := F |r, s\rangle.
\]

We can regard the currents \( J^3(z), J^\pm(z), J^S(z) \), and \( q \)-vertex operators \( \phi_m^{(l)}(z) \) as the following maps:
\[
J^3(z) : F_{r,s} \to F_{r,s}, \quad J^\pm(z) : F_{r,s} \to F_{r,s \pm 1}, \quad (4.3)
\]
\[
\phi_m^{(l)}(z) : F_{r,s} \to F_{r+l/2, s+l-m}.
\]

We can check that \( |i/2, 0\rangle \) satisfies the highest weight condition
\[
t_1 |i/2, 0\rangle = q^i |i/2, 0\rangle, \quad t_0 |i/2, 0\rangle = q^{k-i} |i/2, 0\rangle, \quad e_0 |i/2, 0\rangle = 0, \quad e_1 |i/2, 0\rangle = 0.
\]

\[4\] This kind of decoupling is well known in CFT when we bosonize fermionic ghosts [17].
Thus we can identify 

$$|\lambda_i\rangle = |i/2, 0\rangle.$$ 

We construct the left highest weight representations $V(\lambda_i)$ of $U_q(\hat{sl}_2)$ as follows:

$$V(\lambda_i) := U_q(\hat{sl}_2) |\lambda_i\rangle.$$ 

**Proposition 4.1** Using this highest weight vector, we can embed the left highest weight module $V(\lambda_i)$ in the Fock modules as follows:

$$V(\lambda_i) \hookrightarrow \bigoplus_{s \in \mathbb{Z}} F_{i/2,s}. \quad (4.4)$$

We can not simply use the vector $|r, s\rangle, s \neq 0$, as the highest weight vector since $e_1 |r, s\rangle$ does not vanish.

**4.2 Screening Charge** We see that due to nontrivial charge assignment, naive composition of elementary vertex operators does not define a map between highest weight modules defined in the previous section. This conundrum is solved by introducing screening charge.

Let us define the screening operator $J^S(z)$ as follows $[8]$: \[ J^S(z) = - : 1 \partial_z \exp \left\{ -c \left( 2 \left| q^{-k-2}z; 0 \right) \right\} \exp \left\{ -b \left( 2 \left| q^{-k-2}z; -1 \right) - a \left( k + 2 \left| q^{-2}z; -\frac{k+2}{2} \right) \right\} :. \] (4.5)

Then we get the following:

\[
\begin{align*}
[J^S_n, J^S(z)] &= 0, \\
[J^S_n, J^S(z)] &= 0, \\
[J^S_n, J^S(z)] &= k+2 \partial_z \left[ z^n \exp \left\{ -a \left( k + 2 \left| q^{-2}z; \frac{k+2}{2} \right) \right\} \right],
\end{align*}
\] (4.6)

for all $n \in \mathbb{Z}$.

For $p \in \mathbb{C}$, $|p| < 1$, and $s \in \mathbb{C}^\times$, the Jackson integral is defined as

$$\int_0^{\infty} d_y f(t) = s(1-p) \sum_{m=-\infty}^{\infty} f(sp^m)p^m,$$

whenever the RHS converges $[18]$.

Note that the RHS of (4.6) is a total $p = q^{2(k+2)}$ difference. Therefore, the following Jackson integral of the screening operator (screening charge)

$$\int_0^{\infty} d_y f^S(t) \quad (4.7)$$

\footnote{As is well known in CFT, $f_1^{2i+1} |i/2, 0\rangle = 0$ but $f_0^{k-2i+1} |i/2, 0\rangle \neq 0$. So our module is reducible.}
commutes with all the generators of $U_q(\mathfrak{sl}_2)$ exactly.

The screening operator enjoys the same relations

$$[J^S(z), \tilde{b}_0 + \tilde{c}_0] = 0,$$  

(4.8)

as \([4,7]\), and is a map among Fock modules as follows

$$J^S(z) : F_{r,s} \rightarrow F_{r-1,s-1}.$$  

(4.9)

We want to construct a $U_q(\mathfrak{sl}_2)$-homomorphism $V(\lambda) \rightarrow V(\mu) \otimes V_z^{(l)}$. Let us consider the following combination of operators

$$J^S(t_1)J^S(t_2) \cdots J^S(t_{l-m})\phi_m^{(l)}(z) : F_{r,s} \rightarrow F_{r-l/2+m,s}.$$  

By performing Jackson integral of this operator we obtain a $U_q(\mathfrak{sl}_2)$-linear map

$$\sum_m \int_0^{s_1} d_p t_1 J^S(t_1) \cdots \int_0^{s_1} d_p t_{l-m} J^S(t_{l-m})\phi_m^{(l)}(z) \otimes v_m^{(l)} : V(\lambda) \rightarrow V(\mu) \otimes V_z^{(l)},$$

for arbitrary $\lambda, \mu \in P_k$. Note that this operator depends on $k$ and $l$ but is independent of the choice of $\lambda, \mu$. Since we fixed the normalization of $q$-vertex operator in \([3,3]\), we have to choose an appropriate normalization factor for each $\lambda, \mu \in P_k$ and $V^{(l)}$.

Now we are in a position to state our main proposition:

**Proposition 4.2** The $q$-vertex operator is bosonized as

$$\tilde{\Phi}^{\mu V^{(l)}}_\lambda(z) = \sum_{m=0}^l \tilde{\Phi}^{\mu V^{(l)}}_{\lambda m}(z) \otimes \upsilon_m^{(l)},$$

(4.10)

where

$$\tilde{\Phi}^{\mu V^{(l)}}_{\lambda m}(z) = g^{\mu V^{(l)}}(z) \int_0^{s_1} d_p t_1 J^S(t_1) \cdots \int_0^{s_1} d_p t_{l-m} J^S(t_{l-m})\phi_m^{(l)}(z),$$

(4.11)

and $g^{\mu V^{(l)}}_\lambda(z)$ is the normalization factor mentioned above.

$N$-point function of the $q$-vertex operators is by definition the expectation value of the composition

$$\Phi^{\mu_N V_N}_{\mu_{N-1}}(z_N) \circ \cdots \circ \Phi^{\mu_1 V_1}_{\mu_0}(z_1) : V(\mu_N) \otimes V_z^{(l_N)} \otimes \cdots \otimes V_z^{(l_1)}.$$  

(4.12)

As a corollary of Proposition 4.2 we have

**Proposition 4.3** If we expand $N$-point function of $q$-vertex operators as

$$\langle \mu_N | \Phi^{\mu_N V_N}_{\mu_{N-1}}(z_N) \circ \cdots \circ \Phi^{\mu_1 V_1}_{\mu_0}(z_1) | \mu_0 \rangle$$

$$= \sum_{m_1,\ldots,m_N} f_{m_1,\ldots,m_N}^{(l_N)}(z_1,\ldots,z_N) \upsilon_m^{(l_N)} \otimes \cdots \otimes \upsilon_m^{(l_1)} \in V^{(l_N)} \otimes \cdots \otimes V^{(l_N)},$$
where \( \mu_0, \ldots, \mu_N \in \mathbb{P}_k \), then each component has the following integral form

\[
\begin{align*}
    f_{m_1, \ldots, m_N}(z_1, \ldots, z_N) &= \prod_{i=1}^{N} z_i^{\Delta_{m_i} - \Delta_{m_i - 1}} g_{\mu_i} v_i(z_i) \times \\
    &\times \langle \mu_N | \int_{0}^{s_1^{(N)}} d_p t_{1}^{(N)} J^S(t_{1}^{(N)}) \cdots \int_{0}^{s_{N-m_N}^{(N)}} d_p t_{N-m_N}^{(N)} J^S(t_{N-m_N}^{(N)}) \phi_{m_N}^{(N)}(z_N) \\
    &\times \cdots \cdots \\
    &\times \int_{0}^{s_{1-m_1}^{(1)}} d_p t_{1-m_1}^{(1)} J^S(t_{1-m_1}^{(1)}) \cdots \int_{0}^{s_{1}^{(1)}} d_p t_{1}^{(1)} J^S(t_{1}^{(1)}) \phi_{m_1}^{(1)}(z_1) | \mu_0 \rangle. \tag{4.13}
\end{align*}
\]

5 Calculation of Two-Point Function

In what follows we denote \( V := V^{(1)} = C_{v_+} \oplus C_{v_-} \), and \( z := \frac{z_1}{z_2} \), for short. Let \( \Psi(z_1, z_2) \in V \otimes V \otimes z^{3/4(k+2)} Q(q)[[z]] \) be the following two-point function

\[
\Psi(z_1, z_2) := \langle \lambda_0 | \Phi_{\lambda_1}^{\lambda_0 V}(z_2) \circ \Phi_{\lambda_0}^{\lambda_1 V}(z_1) | \lambda_0 \rangle, \tag{5.1}
\]

where,

\[
\Phi_{\lambda_0}^{\lambda_1 V}(z_1) = z^{3/4(k+2)} \Phi_{\lambda_0}^{\lambda_1 V}(z_1), \quad \Phi_{\lambda_1}^{\lambda_0 V}(z_2) = z^{-3/4(k+2)} \Phi_{\lambda_1}^{\lambda_0 V}(z_2).
\]

In this section by evaluating this correlation function, we prove Proposition 4.3 for \( N = 2, l_1 = l_2 = 1 \), and \( k \in \mathbb{Z}_{>0} \).

5.1 Jackson Integral Formula for Two-Point Function

From Proposition 4.2 q-VOs have the following bosonization

\[
\begin{align*}
    \Phi_{\lambda_0}^{\lambda_1 V}(z_1) &= g_{\lambda_0}^{\lambda_1 V}(z_1) (\phi_+(z_1) \otimes v_+ + \phi_-(z_1) \otimes v_-), \tag{5.2} \\
    \Phi_{\lambda_1}^{\lambda_0 V}(z_2) &= g_{\lambda_1}^{\lambda_0 V}(z_2) \int_{0}^{s_{\infty}} d_p t \ J^S(t) (\phi_+(z_2) \otimes v_+ + \phi_-(z_2) \otimes v_-), \tag{5.3}
\end{align*}
\]

where, \( \phi_+(z_i) = \phi_0^{(1)}(z_i) \), and \( \phi_-(z_i) = \phi_1^{(1)}(z_i) \) \((i = 1, 2)\).

Here \( g_{\lambda_0}^{\lambda_1 V}(z_1) = 1 \) and \( g_{\lambda_1}^{\lambda_0 V}(z_2) =: g(z_2) \) are the normalization factors of \( \Phi_{\lambda_0}^{\lambda_1 V}(z_1) \) and \( \Phi_{\lambda_1}^{\lambda_0 V}(z_2) \), respectively. Explicitly, (as for detail calculation, see Appendix B.)

\[
g(z_2) = -q^{-2-(k+8)/2(k+2)} z_2^{-1/2(k+2)} \int_{0}^{s_{\infty}} d_p t \ t^{-1-2/(k+2)} \left( \frac{p^2 q_2/t}{(pq^{-1} z_2/t; p)_{\infty}} \right)^{-1}, \tag{5.4}
\]

where, \( p = q^{2(k+2)} \), and

\[
(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - a q^n).
\]

Since the vertex operators preserve the weight modulo \( \delta \) we have

\[
\Psi(z_1, z_2) = f_1(z_1, z_2) v_+ \otimes v_- + f_2(z_1, z_2) v_- \otimes v_+.
\]
Using the free boson representation of the $q$-vertex operators we can rewrite $f_1(z_1, z_2)$ as

$$f_1(z_1, z_2) = z^{3/4(k+2)} g(z_2) \int_0^{\infty} dt \langle \lambda_0 | J^S(t) \phi_+(z_2) \phi_-(z_1) | \lambda_0 \rangle$$

$$= z^{3/4(k+2)} g(z_2) \int_0^{\infty} dt \int \frac{dw}{2\pi \sqrt{-1}} \langle \lambda_0 | J^S(t) [\phi_-(z_2), J^-(w)] \phi_-(z_1) | \lambda_0 \rangle. \tag{5.5}$$

Thanks to the formulae of the OPE given in Appendix A we obtain

$$f_1(z_1, z_2) = z^{3/4(k+2)} g(z_2) \int_0^{\infty} dt \left\{ \int_{q^{k+3}z_1} \int \frac{dw}{2\pi \sqrt{-1}} \langle \lambda_0 | J^S(t) \phi_-(z_2) J^-(w) \phi_-(z_1) | \lambda_0 \rangle \right. \right.

$$- q \int_{q^{k+3}z_1, i=1,2} \int \frac{dw}{2\pi \sqrt{-1}} \langle \lambda_0 | J^S(t) J^-(w) \phi_-(z_2) \phi_-(z_1) | \lambda_0 \rangle \right) \right)

$$= -q^{1+4/(k+2)} z^{3/4(k+2)} g(z_2) G(z_1, z_2) \int_0^{\infty} dt \varphi_1(z_1, z_2, t), \tag{5.6}$$

where $G(z_1, z_2)$ comes from OPE of the $q$-vertex operators

$$\phi_-(z_2) \phi_-(z_1) = G(z_1, z_2) : \phi_-(z_2) \phi_-(z_1) :,$$

$$G(z_1, z_2) = (q f z_2)\left(2^{1/2(k+2)} \prod_{m=1}^{\infty} \frac{(p^m z; q^4)(p^m q^4 z; q^4)_{\infty}}{(p^m q^2 z; q^4)_{\infty}}, \tag{5.7} \right.$$

while, the integrand of the Jackson integral is given as follows:

$$\varphi_1(z_1, z_2, t) = q t^{-1-2/(k+2)} \frac{(qp^2 z_1/t; p)_{\infty} (qp^2 z_2/t; p)_{\infty}}{(q^{-1} p^2 z_1/t; p)_{\infty} (q^{-1} p^2 z_2/t; p)_{\infty}}. \tag{5.8}$$

Let us check that $f_1(z_1, z_2)$ depends upon $z = z_1/z_2$ only and hence we may denote $f_1(z_1, z_2) = f_1(z)$. Using the freedom of redefinition $t \mapsto z_2 t$ in the Jackson integral, we can rewrite

$$g(z_2) = -q^{-2-(k+4)/2(k+2)} z^{3/2(k+2)} \left[ \int_0^{\infty} dt \left( t^{-1-2/(k+2)} \frac{(p^2 q/t; p)_{\infty}}{(p^2 q^{-1} z/t; p)_{\infty}} \right) \right]^{-1} \tag{5.9}$$

$$\int_0^{\infty} dt \varphi_1(z_1, z_2, t) = q z_2^{-2/(k+2)} \int_0^{\infty} dt t^{-1-2/(k+2)} \frac{(p^2 q z/t; p)_{\infty} (p^2 q/t; p)_{\infty}}{(p^2 q^{-1} z/t; p)_{\infty} (p^2 q^{-1} t/p)_{\infty}}. \tag{5.10}$$

Therefore we can regard $f_1(z_1, z_2)$ as a function of $z$:

$$f_1(z) = z^{3/4(k+2)} \prod_{m=1}^{\infty} \frac{(p^m z; q^4)(p^m q^4 z; q^4)_{\infty}}{(p^m q^2 z; q^4)_{\infty}} \times$$

$$\times \int_0^{\infty} dt t^{-1-2/(k+2)} \frac{(p^2 q z/t; p)_{\infty} (p^2 q/t; p)_{\infty}}{(p^2 q^{-1} z/t; p)_{\infty} (p^2 q^{-1} t/p)_{\infty}}. \tag{5.11}$$

Let us repeat the same argument with respect to $f_2(z_1, z_2)$.

$$f_2(z_1, z_2) = z^{3/4(k+2)} g(z_2) \int_0^{\infty} dt \langle \lambda_0 | J^S(t) \phi_-(z_2) \phi_+(z_1) | \lambda_0 \rangle \tag{5.12}$$

$$= z^{3/4(k+2)} g(z_2) \int_0^{\infty} dt \int \frac{dw}{2\pi \sqrt{-1}} \langle \lambda_0 | J^S(t) \phi_-(z_2) [J^-(w), \phi_-(z_1)]_q | \lambda_0 \rangle.$$
Similarly we have

\[ f_2(z_1, z_2) = -q^{1+4/(k+2)}z^{3/4(k+2)}g(z_2)G(z_1, z_2) \int_0^{s_{\infty}} dp t \varphi_2(z_1, z_2, t), \]  

(5.13)

where,

\[ \varphi_2(z_1, z_2, t) = t^{-1-2/(k+2)} \frac{(q^p z_1/t; p)_{\infty}(qp z_2/t; p)_{\infty}}{(q^{-1} p z_1/t; p)_{\infty}(q^{-1} p z_2/t; p)_{\infty}}. \]  

(5.14)

Thus we obtain

\[
\begin{align*}
  f_2(z) &= q^{-1}z^{3/4(k+2)} \prod_{m=1}^{\infty} \frac{(p^m z; q^4)_{\infty}(p^m q^4 z; q^4)_{\infty}}{(p^m q^2 z; q^4)_{\infty}} \\
  &\quad \times \int_0^{s_{\infty}} dp t^{-1-2/(k+2)} \frac{(p^2 q z/t; p)_{\infty}(pq/t; p)_{\infty}}{(pq^{-1} z/t; p)_{\infty}(pq^{-1}-t/p)_{\infty}} \\
  &\quad \times \int_0^{s_{\infty}} dp t^{-1-2/(k+2)} \frac{(p^2 q/t; p)_{\infty}}{(pq^{-1}-t/p)_{\infty}}.
\end{align*}
\]  

(5.15)

Note that \( z^{-3/4(k+2)}f_i(z) \) is analytic around \( z = 0 \) (\( i = 1, 2 \)).

### 5.2 q-KZ Equation

Now we show that the \( q \)-difference system for \( f_1(z_1, z_2) \), \( f_2(z_1, z_2) \) gives the \( q \)-KZ equation for two-point function. Let us study the effect of \( p \)-shift \( z_1 \mapsto z_1, z_2 \mapsto p z_2 \). The change of \( f_i(z_1, z_2) \) results from \( z^{3/4(k+2)}, G(z_1, z_2) \) and \( \phi_i(z_1, z_2, t) \). (\( i = 1, 2 \)) First \( G(z_1, z_2) \) transforms as follows:

\[ G(z_1, p z_2) = q^{1/2} \rho(pz)G(z_1, z_2), \]  

(5.16)

where,

\[ \rho(z) = q^{-1/2} \frac{(q^2 z; q^4)_{\infty}}{(q^2 z; q^4)_{\infty}(q^2 z; q^4)_{\infty}} \]  

(5.17)

is precisely the same factor appeared in the image of the universal \( R \) matrix of \( U_q(\mathfrak{sl}_2) \) \([2]\).

Next the contribution from the Jackson integral is given as follows: (See Appendix C, as for detail.)

\[ \left( \int_0^{s_{\infty}} dp t \varphi_1(z_1, p z_2, t), \int_0^{s_{\infty}} dp t \varphi_2(z_1, p z_2, t) \right) \]

\[ = \left( \int_0^{s_{\infty}} dp t \varphi_1(z_1, z_2, t), \int_0^{s_{\infty}} dp t \varphi_2(z_1, z_2, t) \right) \mathcal{R}(pz), \]  

(5.18)

where,

\[ \mathcal{R}(z) = \frac{1}{1-q^2 z} \begin{pmatrix} 1 - z & q^{-1} - q \\ (q^{-3} - q^{-1})z & q^{-2}(1-z) \end{pmatrix}. \]  

(5.19)

is just the zero-weight part of the \( R \) matrix of the six vertex model up to a similarity transformation.
By combining eqs. (5.16), (5.18), (5.19), and the factor from $z^{3/4(k+2)}$ we obtain
\[
\begin{pmatrix}
  f_1(pz) \\
  f_2(pz)
\end{pmatrix}
= \frac{\rho(pz)}{1 - pq^2 z}
\begin{pmatrix}
  q^2 (1 - pz) & pq^{-1} (1 - q^2) z \\
  q (1 - q^2) & (1 - zp)
\end{pmatrix}
\begin{pmatrix}
  f_1(z) \\
  f_2(z)
\end{pmatrix}.
\] (5.20)

It coincides with the $q$-KZ equation [2] for the two-point function.

This recursion formula implies
\[
q f_1(pz) + f_2(pz) = \rho(pz)(q f_1(z) + f_2(z)).
\]

Compare the coefficients of $z^{3/4(k+2)}$ of both sides of (5), then we obtain
\[
q f_1(z) + f_2(z) = 0.
\]

Therefore we have $q$-difference equation of the first order
\[
\begin{align*}
\frac{f_1(pz)}{f_1(z)} &= q^{3/2} \frac{(pq^{-2} z; q^4)^\infty (pq^6 z; q^4)^\infty}{(pq z; q^4)^\infty (pq^4 z; q^4)^\infty}, \\
\frac{f_2(z)}{f_2(z)} &= -q f_1(z).
\end{align*}
\] (5.21) (5.22)

In particular if we put $k = 1$, by solving the above $q$-difference equation we have
\[
\Psi(z_1, z_2) = z^{1/4} \frac{(q^6 z; q^4)^\infty}{(q^4 z; q^4)^\infty} (v_+ \otimes v_+ - q v_- \otimes v_+),
\] (5.23)

which reproduces the known results [6].

6 Conclusion

In this paper we discuss a bosonization of $q$-vertex operator on the basis of the Fock representation of $U_q(\widehat{sl}_2)$. We propose an integral formula for $N$-point functions of the $q$-vertex operators with the help of the screening charges. Matsuo [19] and Reshetikhin [20] have obtained integral formulae from the viewpoint of the $q$-KZ equation. The relations among these three integral formulae should be clarified.

After performing all the residue calculus of two-point function, we have Jackson integral of Jordan-Pochhammer type [21] [18]. It is intriguing that the scalar factor which arises in the image of the universal $\mathcal{R}$ matrix naturally appears in the OPE of elementary vertex operators.

We would like to check all the intertwining properties of the elementary $q$-vertex operators for general case. The analogy with CFT is quite remarkable; we can deform $\widehat{sl}_2$ currents, screening current, and vertex operators à la Tsuchiya-Kanie [22]. However, we have no

\[\text{We use the opposite ordering of two } V \text{s to that of ref. [3].}\]
counterpart of Virasoro algebra, and the meaning of the spectral parameters of the \( q \)-vertex operators is not yet obvious.

Recently Matsuo \[23\] constructed another bosonization of \( U_q'(\widehat{sl}_2) \). It is interesting to investigate the connection between his bosonization and ours. After completing this work we received a preprint by Abada et al. \[24\].

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**A  Operator Product Expansion Formulae**

In this appendix we list the operator product expansion formulae among the \( U_q'(\widehat{sl}_2) \)-current \( J^-(z) \), the screening current \( J^S(z) \) and the elementary \( q \)-vertex operator \( \phi^{(l)}(z) \).

We split \( J^-(z) \) into two parts

\[
J^-(z) = \left\{ J^+_I(z) - J^+_II(z) \right\},
\]

\[
J^+_I(z) = \exp \left\{ a (k + 2 | q^{k + 2} z; -\frac{k + 2}{2} ) - a (k + 2 | q^{-k} z; \frac{k + 2}{2} ) + b (2 | z; -1) + c (2 | q^{-1} z; 0) \right\} ;,
\]

\[
J^+_II(z) = \exp \left\{ a (k + 2 | q^{-k} z; -\frac{k + 2}{2} ) - a (k + 2 | q^{-k} z; \frac{k + 2}{2} ) + b (2 | q^{-k} z; -1) + c (2 | q^{-k} z; 0) \right\} ;.
\]

Similarly, we put

\[
J^S(z) = - \left\{ J^+_I(z) - J^+_II(z) \right\},
\]

\[
J^+_I(z) = \exp \left\{ -a (k + 2 | q^{-k} z; -\frac{k + 2}{2} ) - b (2 | q^{k-k} - 2 z; -1) - c (2 | q^{-k-1} z; 0) \right\} ;,
\]

\[
J^+_II(z) = \exp \left\{ -a (k + 2 | q^{-k} z; -\frac{k + 2}{2} ) - b (2 | q^{k-k} - 2 z; -1) - c (2 | q^{-k-3} z; 0) \right\} .
\]
\begin{align*}
J^I_I(z) \phi^{(l)}_i(w) &= \frac{q^l z - q^{k+2} w}{z - q^{k+2} w} : J^I_I(z) \phi^{(l)}_i(w) : \quad |z| > q^{k+2-l}|w| \\
J^H_H(z) \phi^{(l)}_i(w) &= q^{-l} : J^H_H(z) \phi^{(l)}_i(w) :
\end{align*}

(A.1)

\begin{align*}
\phi^{(l)}_i(w) J^I_I(z) &= : \phi^{(l)}_i(w) J^I_I(z) : \\
\phi^{(l)}_i(w) J^H_H(z) &= \frac{w - q^{-l-k-2} z}{w - q^{-l-k-2} z} : \phi^{(l)}_i(w) J^H_H(z) : \quad |w| > q^{-l-k-2}|z| \\
J^S_I(w) J^I_I(z) &= q^{-1} : J^I_I(z) J^S_I(w) :
\end{align*}

(A.2)

\begin{align*}
J^S_I(w) J^H_H(z) &= \frac{q^{-1} w - q^{-k-1} z}{w - q^{-k-2} z} : J^H_H(z) J^S_I(w) : \quad |w| > q^{-k-2}|z| \\
J^H_H(w) J^I_I(z) &= \frac{q w - q^{k+1} z}{w - q^{k+2} z} : J^I_I(z) J^H_H(w) : \quad |w| > q^k|z| \\
J^H_H(w) J^H_H(z) &= q : J^H_H(z) J^H_H(w) :
\end{align*}

(A.3)

\begin{align*}
J^S_I(z) \phi^{(l)}_i(w) &= \frac{(q^l pw/z; p)_\infty}{(q^{-l} pw/z; p)_\infty} (q^{2-l} z)^{-1/2(k+2)} : J^S_I(z) \phi^{(l)}_i(w) : \quad |z| > q^{-l-p}|w| \\
J^S_H(z) \phi^{(l)}_i(w) &= \frac{(q^l pw/z; p)_\infty}{(q^{-l} pw/z; p)_\infty} (q^{2-l} z)^{-1/2(k+2)} : J^S_H(z) \phi^{(l)}_i(w) : \quad |z| > q^{-l-p}|w| \\
\phi^{(l)}_i(w) J^S_I(z) &= \frac{(q^l w/z; p)_\infty}{(q^{-l} w/z; p)_\infty} (q^k w)^{-1/2(k+2)} : J^S_I(z) \phi^{(l)}_i(w) : \quad |w| > q^{-l}|z| \\
\phi^{(l)}_i(w) J^S_H(z) &= \frac{(q^l w/z; p)_\infty}{(q^{-l} w/z; p)_\infty} (q^k w)^{-1/2(k+2)} : J^S_H(z) \phi^{(l)}_i(w) : \quad |w| > q^{-l}|z|
\end{align*}

(A.4)

\begin{align*}
\phi^{(l)}_i(z) \phi^{(l)}_i(w) &= (q^k z)^{2/2(k+2)} \prod_{m=1}^{\infty} \frac{(p^m q^{(1-l)} w/z; q^4)_\infty (p^m q^{(1-l)} w/z; q^4)_\infty}{(p^m q^2 w/z; q^4)_\infty} : \phi^{(l)}_i(z) \phi^{(l)}_i(w) : 
\end{align*}

(A.5)
B Normalization of $q$-Vertex Operators

Here we consider the normalization of the following VOs

$$
\tilde{\Phi}_{\lambda_0}^{V^{(1)}}(z) : V(\lambda_0) \rightarrow V(\lambda_1) \otimes V_z^{(1)},
$$

$$
\tilde{\Phi}_{\lambda_1}^{V^{(1)}}(z) : V(\lambda_1) \rightarrow V(\lambda_0) \otimes V_z^{(1)}.
$$

(B.1)

These VOs have the following leading terms

$$
\tilde{\Phi}_{\lambda_0}^{V^{(1)}}(z) | \lambda_0 \rangle = | \lambda_1 \rangle \otimes v_- + \cdots,
$$

$$
\tilde{\Phi}_{\lambda_1}^{V^{(1)}}(z) | \lambda_1 \rangle = | \lambda_0 \rangle \otimes v_+ + \cdots.
$$

(B.2)

These VOs are bosonized as

$$
\tilde{\Phi}_{\lambda_0}^{V^{(1)}}(z) = g_{\lambda_0}^{\lambda_1} V^{(1)}(z) [\phi_+ (z) \otimes v_+ + \phi_- (z) \otimes v_-],
$$

$$
\tilde{\Phi}_{\lambda_1}^{V^{(1)}}(z) = g_{\lambda_1}^{\lambda_0} V^{(1)}(z) \left[ \int_0^{s\infty} d\mu t J^S(t) \phi_+ (z) \otimes v_+ + \int_0^{s\infty} d\mu t J^S(t) \phi_- (z) \otimes v_- \right].
$$

(B.3)

We can get these normalization functions $g_{\lambda_0}^{\lambda_1} V^{(1)}(z), g_{\lambda_1}^{\lambda_0} V^{(1)}(z)$ by calculating the leading term explicitly.

First we have

$$
\phi_-(z) | 0, 0 \rangle
$$

$$
= : \exp \left\{ a \left( 1; 2, k + 2 \mid q^k z; \frac{k + 2}{2} \right) \right\} : | 0, 0 \rangle
$$

$$
= | 1/2, 0 \rangle + \cdots
$$

(B.4)

then we get

$$
g_{\lambda_0}^{\lambda_1} V^{(1)}(z) = 1.
$$

(B.5)

Next we can see

$$
\int_0^{s\infty} d\mu t J^S(t) \phi_+ (z) | 1/2, 0 \rangle
$$

$$
= \int_0^{s\infty} d\mu t J^S(t) \int \frac{dw}{2\pi \sqrt{-1}} \left[ \phi_- (z), J^- (w) \right]_q \exp \left\{ \frac{Q_a}{2(k + 2)} \right\} | 0, 0 \rangle
$$

$$
= -q^{2 + (k + 8)/2(k + 2)} z^{1/2(k + 2)} \int_0^{s\infty} d\mu t^{1-2/(k+2)} \frac{(qp^2 z/t;p)_\infty}{(q^{-1}p z/t;p)_\infty} | 0, 0 \rangle + \cdots,
$$

(B.6)

likewise. Then in this case

$$
g_{\lambda_1}^{\lambda_0} V^{(1)}(z) = -q^{-2 - (k + 8)/2(k + 2)} z^{-1/2(k + 2)} \left[ \int_0^{s\infty} d\mu t^{1-2/(k+2)} \frac{(qp^2 z/t;p)_\infty}{(q^{-1}p z/t;p)_\infty} \right]^{-1}
$$

(B.7)

holds.
In this appendix, we calculate the difference system satisfied by the following functions:

\[
\int_{0}^{\infty} d\rho t \varphi_1(z_1, z_2, t), \quad \int_{0}^{\infty} d\rho t \varphi_2(z_1, z_2, t),
\]

where \( \varphi_1(z_1, z_2, t) \) and \( \varphi_2(z_1, z_2, t) \) are given as

\[
\varphi_1(z_1, z_2, t) = \frac{(qp^2 z_1 / t; p)_{\infty} (qp^2 z_2 / t; p)_{\infty}}{(q^{-1} p z_1 / t; p)_{\infty} (q^{-1} p z_2 / t; p)_{\infty}} t^{-1-2/(k+2)},
\]

\[
\varphi_2(z_1, z_2, t) = \frac{(qp z_1 / t; p)_{\infty} (qp z_2 / t; p)_{\infty}}{(q^{-1} p z_1 / t; p)_{\infty} (q^{-1} p z_2 / t; p)_{\infty}} t^{-1-2/(k+2)}.
\]

We note that these functions are the Jackson integrals of Jordan-Pochhammer type. For the general theory of the difference system for the Jackson integrals of Jordan-Pochhammer type, we refer the reader to [21][18].

To find the difference equation, we use the following identity:

\[
\int_{0}^{\infty} d\rho t \varphi_i(z_1, z_2, t) = \int_{0}^{\infty} d\rho t \ p \varphi_i(z_1, z_2, pt).
\]

Since we have

\[
p \varphi_1(z_1, z_2, pt) = p \varphi_1(z_1, z_2, pt) \frac{1-pz}{1-pq^2z} + \varphi_2(z_1, z_2, t) \frac{(1-q^2) pq^{-2} z}{1-pq^2z},
\]

\[
p \varphi_2(z_1, z_2, pt) = p \varphi_1(z_1, z_2, pt) \frac{(1-q^2) q^{-1}}{1-pq^2z} + \varphi_2(z_1, z_2, t) \frac{(1-zp) q^{-2}}{1-pq^2z},
\]

we get the following difference equation:

\[
\left( \int_{0}^{\infty} d\rho t \varphi_1(z_1, z_2, t), \int_{0}^{\infty} d\rho t \varphi_2(z_1, z_2, t) \right)

= \left( \int_{0}^{\infty} d\rho t \varphi_1(z_1, z_2, t), \int_{0}^{\infty} d\rho t \varphi_2(z_1, z_2, t) \right)

\times \begin{pmatrix}
\frac{1-pz}{1-pq^2z} & \frac{(1-q^2) q^{-1}}{1-pq^2z} \\
\frac{(1-q^2) pq^{-2} z}{1-pq^2z} & \frac{(1-zp) q^{-2}}{1-pq^2z}
\end{pmatrix}.
\]

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