**L∞-ALGEBRAS AND DEFORMATIONS OF HOLOMORPHIC MAPS**

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**Abstract.** We construct the deformation functor associated with a pair of morphisms of differential graded Lie algebras, and use it to study infinitesimal deformations of holomorphic maps of compact complex manifolds. In particular, using $L_\infty$ structures, we give an explicit description of the differential graded Lie algebra that controls this problem.

1. Introduction

The aim of this paper is to develop some algebraic tools to study infinitesimal deformations of holomorphic maps.

The modern approach to deformation theory is via differential graded Lie algebras (DGLA for short) or, in general, via $L_\infty$-algebras. A DGLA is a differential graded vector space with a structure of graded Lie algebra, plus some compatibility conditions between the differential and the bracket (of the Lie structure).

Moreover, using the solutions of the Maurer-Cartan equation and the gauge equivalence, we can associate with a DGLA $L$ a deformation functor of Artin rings $\text{Def}_L$, i.e., a functor from the category $\text{Art}$ of local Artinian $\mathbb{C}$-algebras (with residue field $\mathbb{C}$) to the category $\text{Set}$ of sets, that satisfies Schlessinger’s conditions $(H_1)$ and $(H_2)$ of [1, Theorem 2.1].

The guiding principle is the idea due to P. Deligne, V. Drinfeld, D. Quillen and M. Kontsevich (see [13]) that “in characteristic zero every deformation problem is controlled by a differential graded Lie algebra”.

In other words, we can define a DGLA $L$ (up to quasi-isomorphism) from the geometrical data of the problem, such that the deformation functor $\text{Def}_L$ is isomorphic to the deformation functor of Artin rings that describes the formal deformations of the geometric object [1], [7] and [19]. We point out that it is easier to study a deformation functor associated with a DGLA but, in general, it is not an easy task to find the right DGLA (up to quasi-isomorphism) associated with the problem [14].

A first example, in which the associated DGLA is well understood, is the case of deformations of complex manifolds. If $X$ is a complex compact manifold, then its Kodaira-Spencer algebra controls the infinitesimal deformations of $X$ (Theorem 3.4).

The next natural problem is to investigate the embedded deformations of a submanifold in a fixed manifold. Very recently, M. Manetti in [16] studies this problem using the approach via DGLA. More precisely, given a morphism of DGLAs $h : L \longrightarrow M$, he describes a general construction to define...
a new deformation functor $\text{Def}_h$ associated with $h$ (Remark 4.4). Then, by suitably choosing $L, M$ and $h$ he proves the existence of an isomorphism between the functor $\text{Def}_h$ and the functor associated with the infinitesimal deformations of a submanifold in a fixed manifold.

In this paper, we extend these techniques to study not only the deformations of an inclusion but, in general, the deformations of holomorphic maps. These deformations were first studied from the classical point of view (no DGLA) by E. Horikawa [8] and [9], then by M. Namba [17], Z. Ran [18] and, more recently, by E. Sernesi [20].

Roughly speaking, we have a holomorphic map $f : X \to Y$ of compact complex manifolds and we deform both the domain, the codomain and the map itself. Equivalently, we deform the graph of $f$ in the product $X \times Y$, such that the deformation of $X \times Y$ is a product of deformations of $X$ and $Y$. Let $\text{Def}(f)$ be the functor associated with the infinitesimal deformations of the holomorphic map $f$ (Definition 5.3).

To study these deformations, the key point is the definition of the deformation functor $\text{Def}_{(h,g)}$, associated with a pair of morphisms of differential graded Lie algebras $h : L \to M$ and $g : N \to M$. In particular, the tangent and obstruction spaces of $\text{Def}_{(h,g)}$ are the first and second cohomology group of the suspension of the mapping cone $C_{(h,g)}$, associated with the morphism $h - g : L \oplus N \to M$, such that $(h - g)(l, n) = h(l) - g(n)$ (Section 4.2).

By a suitable choice of the morphisms $h : L \to M$ and $g : N \to M$, the functor $\text{Def}_{(h,g)}$ encodes all the geometric data of the problem of infinitesimal deformations of holomorphic maps (Theorem 5.11).

**Theorem (A).** Let $f : X \to Y$ be a holomorphic map of compact complex manifolds. There exist morphisms of DGLAs $h : L \to M$ and $g : N \to M$ such that

$$\text{Def}_{(h,g)} \cong \text{Def}(f).$$

Next, we look for a DGLA that controls the deformations of holomorphic maps and, for this purpose, we use $L_\infty$ structures.

First, using path objects, we define a differential graded Lie algebra $H_{(h,g)}$, for each choice of morphisms $h : L \to M$ and $g : N \to M$. Then, by transferring $L_\infty$ structures, we explicitly describe an $L_\infty$ structure on the cone $C_{(h,g)}$ (Section 6). In particular, the functor $\text{Def}_H_{(h,g)}$ is isomorphic to the deformation functor $\text{Def}_{C_{(h,g)}}^\infty$ associated with this $L_\infty$ structure on $C_{(h,g)}$ (Corollary 6.13).

Finally, we prove that the deformation functor $\text{Def}_{C_{(h,g)}}^\infty$ coincides with the deformation functor $\text{Def}_{(h,g)}$ associated with the pair $(h, g)$ (Theorem 6.17) and so $\text{Def}_H_{(h,g)} \cong \text{Def}_{(h,g)}$ (Corollary 6.18).

Therefore, in particular, we give an explicit description (more than the existence) of a DGLA that controls the deformations of holomorphic maps (Theorem 6.19).

**Theorem (B).** Let $f : X \to Y$ be a holomorphic map of compact complex manifold. Then, there exists an explicit description of a DGLA $H_{(h,g)}$ such
\[ \text{Def}_{H(h,g)} \cong \text{Def}(f). \]

When we developed the techniques of this paper, we had also in mind some applications to the study of obstruction theory. However, since the number of pages grew, we decided to split the material, collecting here the general theory and leaving for the sequel [11] (in preparation) the study of obstructions and of semi-regularity maps that annihilates obstructions.

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2. **Notation**

We will work over the field \( \mathbb{C} \) of complex numbers, although most of the algebraic results are valid over an arbitrary field of characteristic zero. All vector spaces, linear maps, tensor products etc. are intended over \( \mathbb{C} \).

Unless otherwise specified, any (complex) manifold is assumed compact and connected.

Given a manifold \( X \), we denote by \( \Theta_X \) the holomorphic tangent bundle, by \( A^{p,q}_X \) the sheaf of differentiable \((p,q)\)-forms on \( X \) and by \( A^{p,q}_X(\Theta_X) \) the vector space of global sections of \( A^{p,q}_X \). More generally, \( A^{p,q}_X(\Theta_X) = \Gamma(X, A^{p,q}_X(\Theta_X)) \) is the vector space of its global sections.

Let \( f : X \to Y \) be a holomorphic map of manifolds. We denote by \( f^* \) and \( f_* \) the map induced by \( f \), i.e.,

\[
    f^* : A^{p,q}_Y(\Theta_Y) \to A^{p,q}_X(f^*\Theta_Y) \quad \text{and} \quad f_* : A^{p,q}_X(\Theta_X) \to A^{p,q}_X(f^*\Theta_Y).
\]

3. **Background**

Let \( L = (\oplus_i L_i, d, [, ]) \) be a DGLA and \((A, m_A) \in \text{Art}\), where \( m_A \) denotes the maximal ideal of \( A \). The set of Maurer-Cartan elements with coefficients in \( A \) is defined as follows

\[ \text{MC}_L(A) = \{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0 \}, \]

where the DGLA structure on \( L \otimes m_A \) is the natural extension of the DGLA structure on \( L \). For each \( a \in L^0 \otimes m_A \), we define the gauge action \( * : \exp(L^0 \otimes m_A) \times \text{MC}_L(A) \to \text{MC}_L(A) \) by the formula

\[
e^a * x := x + \sum_{n \geq 0} \frac{[a, -]^n}{(n + 1)!} ([a, x] - da).
\]

Given \( x \in \text{MC}_L(A) \), the irrelevant stabilizer \( \text{Stab}_A(x) \) of \( x \) is by definition

\[ \text{Stab}_A(x) = \{ e^{dh + [x, h]} \mid h \in L^{-1} \otimes m_A \}. \]
The set $\text{Stab}_A(x)$ is a subgroup of $\exp(L^0 \otimes A)$, that is contained in the stabilizer of $x$ and it satisfies the following property:

$$\forall \ a \in L^0 \otimes A \quad e^{a} \text{Stab}_A(x)e^{-a} = \text{Stab}_A(y), \quad \text{with} \quad y = e^{a} \ast x.$$  

The deformation functor $\text{Def}_L : \text{Art} \longrightarrow \text{Set}$ associated with a DGLA $L$ is:

$$\text{Def}_L(A) = \left\{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0 \right\} \exp(L^0 \otimes A).$$

**Definition 3.1.** A functor of Artin rings $F : \text{Art} \longrightarrow \text{Set}$ is controlled by a DGLA $L$ if $F$ is isomorphic to $\text{Def}_L$.

**Example 3.2.** Let $X$ be a manifold. The Kodaira-Spencer (differential graded Lie) algebra of $X$ is

$$K_{SX} = \bigoplus_i \Gamma(X, A_X^{0,i}(\Theta_X)) = \bigoplus_i A_X^{0,i}(\Theta_X).$$

The differential $\tilde{d}$ is the opposite of the Dolbeault differential, whereas the bracket $[ , ]$ is defined in local coordinates as the $\Omega^*$-bilinear extension of the standard bracket on $A_X^{0,0}(\Theta_X)$ ($\Omega^* = \ker(\partial : A_X^{0,*} \longrightarrow A_X^{1,*})$ is the sheaf of anti-holomorphic differential forms). Explicitly, if $z_1, \ldots, z_n$ are local holomorphic coordinates on $X$, we have

$$\tilde{d}(f dz_l) = -\partial(f) \wedge dz_l,$$

$$[f \frac{\partial}{\partial z_l}, g \frac{\partial}{\partial z_j}] = (f \frac{\partial g}{\partial z_l} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_l}) \wedge dz_l \wedge dz_j, \quad \forall \ f, g \in A_X^{0,0}.$$  

Then, $A_X^{0,*}(\Theta_X)$ is a sheaf of DGLAs.

Define the **holomorphic Lie derivative**

$$l : A_X^{0,*}(\Theta_X) \longrightarrow \text{Der}^*(A_X^{*,*}),$$

$$l_a(\omega) = \partial(a \lrcorner \omega) + (-1)^{\deg(a)}a \lrcorner \partial \omega,$$

for each $a \in A_X^{0,*}(\Theta_X)$ and $\omega \in A_X^{*,*}$.

The DGLA sheaf morphism $l$ is injective; moreover, using $l$, we define, for any object $(A, m_A) \in \text{Art}$ and $a \in A_X^{0,0}(\Theta_X) \otimes m_A$, the automorphism $e^a$ of $A_X^{0,*} \otimes A$:

$$e^a : A_X^{0,*} \otimes A \longrightarrow A_X^{0,*} \otimes A, \quad f \longmapsto e^a(f) = \sum_{n=0}^{\infty} \frac{l^n}{n!}(f).$$

**Lemma 3.3.** For every local Artinian $\mathbb{C}$-algebra $(A, m_A)$, $a \in A_X^{0,0}(\Theta_X) \otimes m_A$ and $x \in \text{MC}_{KS_X}(A)$ we have

$$e^a \circ (\overline{\partial} + l_x) \circ e^{-a} = \overline{\partial} + e^a \ast l_x : A_X^{0,0} \otimes A \longrightarrow A_X^{0,1} \otimes A,$$

where $\ast$ is the gauge action. In particular, $\ker(\overline{\partial} + e^a \ast l_x) : A_X^{0,0} \otimes A \longrightarrow A_X^{0,1} \otimes A) = e^a(\ker(\overline{\partial} + l_x : A_X^{0,0} \otimes A \longrightarrow A_X^{0,1} \otimes A))$.

**Proof.** See [16, Lemma 5.1] or [10, Lemma II.5.5].
Let \( \text{Def}_X : \text{Art} \to \text{Set} \) be the functor of infinitesimal deformations of \( X \), i.e.,
\[
\text{Def}_X(A) = \{ \text{deformations } X_A \text{ of } X \text{ over } \text{Spec}(A) \}. 
\]

Recall that a deformation \( X_A \) of \( X \) over \( \text{Spec}(A) \) is nothing else that a morphism \( O_{X_A} \to O_X \) of sheaves of \( A \)-algebras such that \( O_{X_A} \) is flat over \( A \) and the induced map \( O_{X_A} \otimes_A C \to O_X \) is an isomorphism. Moreover, \( \text{Def}_X \) has \( H^1(X, \Theta_X) \) and \( H^2(X, \Theta_X) \) as tangent and obstruction space, respectively.

The following theorem is well known and a proof based on the theorem of Newlander-Nirenberg can be found in [2], [6] or more recently in [15]. For a proof that avoid this theorem see [10, Theorem II.7.3].

**Theorem 3.4.** Let \( X \) be a manifold and \( KS_X \) its Kodaira-Spencer algebra. Then there exists an isomorphism of functors
\[
\gamma' : \text{Def}_{KS_X} \to \text{Def}_X,
\]
defined in the following way: given a local Artinian \( \mathbb{C} \)-algebra \( (A,m_A) \) and a solution of the Maurer-Cartan equation \( x \in A^{0,0}_X(\Theta_X) \otimes m_A \), we set
\[
O_{X_A}(x) = \ker(A^{0,0}_X \otimes A \xrightarrow{\nabla + \delta} A^{0,1}_X \otimes A),
\]
and the map \( O_{X_A}(x) \to O_X \) is induced by the projection \( A^{0,0}_X \otimes A \to A^{0,0}_X \otimes \mathbb{C} = A^{0,0}_X \).

**4. Deformation functor of a pair of morphisms of DGLAs**

Let \( h : L \to M \) be a morphism of DGLAs. The suspension of the mapping cone of \( h \) is the complex \( (C_h, \delta) \), where \( C_h^i = L^i \oplus M^{i-1} \) and \( \delta(l, m) = (dl, h(l) - dm) \).

Let \( h : (L, d) \to (M, d) \) and \( g : (N, d) \to (M, d) \) be morphisms of DGLAs:

\[
\begin{array}{c}
N \xrightarrow{g} M. \\
\end{array}
\]

The suspension of the mapping cone of the pair \( (h, g) \) is the differential graded vector space \( (C^i_{(h, g)}, D) \), where
\[
C^i_{(h, g)} = L^i \oplus N^i \oplus M^{i-1}
\]
and the differential \( D \) is defined as follows
\[
L^i \oplus N^i \oplus M^{i-1} \ni (l, n, m) \xrightarrow{D} (dl, dn, -dm - g(n) + h(l)) \in L^{i+1} \oplus N^{i+1} \oplus M^i.
\]

The projection \( C^i_{(h, g)} \to L^i \oplus N^i \) is a morphism of complexes and so there exists the following exact sequence
\[
0 \to (M^{-1}, -d) \to (C^i_{(h, g)}, D) \to (L^i \oplus N^i, d) \to 0
\]
that induces
\[ \cdots \rightarrow H^i(C_{(h,g)}) \rightarrow H^i(L \oplus N') \rightarrow H^i(M') \rightarrow H^{i+1}(C_{(h,g)}) \rightarrow \cdots. \]

Note that, in general, we can not define any bracket on the cone \( C_{(h,g)} \), such that \( C_{(h,g)} = 0 \) is a DGLA and the projection \( C_{(h,g)} \rightarrow L \oplus N \) is a morphism of DGLAs. In Section 6, we will define an \( L_\infty \) structure on \( C_{(h,g)} \).

**Lemma 4.1.** Let \( g : N \rightarrow M \) and \( h : L \rightarrow M \) be morphisms of complexes with \( h \) injective, i.e., there exists the exact sequence of complexes
\[
0 \rightarrow L \xrightarrow{h} M \xrightarrow{\pi} \text{coker}(h) \rightarrow 0 \xrightarrow{g} N.
\]

Then, \( (C_{(h,g)}, D) \) is quasi-isomorphic to \( (C^{\pi \circ g}, \delta) \).

**Proof.** Let \( \gamma : C_{(h,g)} \rightarrow C^{\pi \circ g} \) be defined as
\[ C^i_{(h,g)} \ni (l, n, m) \xrightarrow{\gamma} (-n, \pi(m)) \in C^i_{\pi \circ g}. \]
Then, a straightforward computation shows that \( \gamma \) is a quasi-isomorphism. \( \square \)

4.1. **The functors** \( \text{MC}_{(h,g)} \) **and** \( \text{Def}_{(h,g)} \). The **Maurer-Cartan functor associated with the pair** \((h, g)\) **is defined as follows**
\[
\text{MC}_{(h,g)} : \text{Art} \rightarrow \text{Set},
\]
\[
\text{MC}_{(h,g)}(A) = \{(x, y, e^p) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) | dx + \frac{1}{2}[x, x] = 0, \ dy + \frac{1}{2}[y, y] = 0, \ g(y) = e^p \ast h(x)\}.\]

We note that \( \text{MC}_{(h,g)} \) is a **homogeneous functor**, i.e., for each \( B \rightarrow A \) and \( C \rightarrow A \) in \( \text{Art} \), \( \text{MC}_{(h,g)}(B \times_A C) \cong \text{MC}_{(h,g)}(B) \times_{\text{MC}_{(h,g)}(A)} \text{MC}_{(h,g)}(C) \).

**Remark 4.2.** In [16, Section 2], M. Manetti introduced the functor \( \text{MC}_h : \text{Art} \rightarrow \text{Set} \), associated with a morphism \( h : L \rightarrow M \) of DGLAs, by defining, for each \((A, m_A) \in \text{Art}, \)
\[
\text{MC}_h(A) = \{(x, e^p) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A) | dx + \frac{1}{2}[x, x] = 0, \ e^p \ast h(x) = 0\}.\]

Therefore, if we take \( N = 0 \) and \( g = 0 \), the new functor \( \text{MC}_{(h,g)} \) reduces to the old one \( \text{MC}_h \). By choosing \( M = N = 0 \) and \( h = g = 0 \), \( \text{MC}_{(h,g)} \) reduces to the Maurer-Cartan functor \( \text{MC}_L \) associated with the DGLA \( L \).

Next, we consider on \( \text{MC}_{(h,g)}(A) \) the following relation \( \sim \):
\[
(x_1, y_1, e^{p_1}) \sim (x_2, y_2, e^{p_2})
\]
if and only if there exist \( a \in L^0 \otimes m_A, b \in N^0 \otimes m_A \) and \( c \in M^{-1} \otimes m_A \) such that
\[
x_2 = e^a \ast x_1, \quad y_2 = e^b \ast y_1
\]
and
\[
e^{p_2} = e^{g(b)} e^T e^{p_1} e^{-h(a)}, \quad \text{with} \quad T = dc + [g(y_1), c].
\]
By definition of the irrelevant stabilizer, we note that $e^T \in Stab_A(g(y_1))$. An easy computation shows that $\sim$ is a well defined equivalence relation [10, Lemma III.2.23]. Then, it makes sense to consider the following functor.

**Definition 4.3.** The deformation functor associated with a pair $(h, g)$ of morphisms of differential graded Lie algebras is:

$$Def_{(h, g)} : \mathsf{Art} \rightarrow \mathsf{Set},$$

$$Def_{(h, g)}(A) = \frac{MC_{(h, g)}(A)}{\sim}.$$  

**Remark 4.4.** In [16, Section 2], M. Manetti defined the functor $Def_h$ associated with a morphism $h : L \rightarrow M$ of DGLAs:

$$Def_h : \mathsf{Art} \rightarrow \mathsf{Set},$$

$$Def_h(A) = \frac{MC_h(A)}{\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)},$$

where the gauge action of $\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)$ is given by the formula

$$(e^a, e^{dm}) \cdot (x, e^p) = (e^a \cdot x, e^{dm} e^p e^{-h(a)}), \quad \forall a \in L^0 \otimes m_A, m \in M^{-1} \otimes m_A.$$  

Therefore, if we take $N = 0$ and $g = 0$, the new functor $Def_{(h, g)}$ reduces to the old one $Def_h$.

By choosing $N = M = 0$ and $h = g = 0$, $Def_{(h, g)}$ reduces to the Maurer-Cartan functor $Def_L$ associated with the DGLA $L$.

**Remark 4.5.** Consider the functor $Def_{(h, g)}$. Then the projection $\varrho$ on the second factor:

$$\varrho : Def_{(h, g)} \rightarrow Def_N,$$

$$Def_{(h, g)}(A) \ni (x, y, e^p) \overset{\varrho}{\longrightarrow} y \in Def_N(A)$$

is a morphism of deformation functors.

**Remark 4.6.** If the morphism $h$ is injective, then for each $(A, m_A) \in \mathsf{Art}$ the functor $MC_{(h, g)}$ has the following form:

$$MC_{(h, g)}(A) = \{(x, e^p) \in (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) |$$

$$dx + \frac{1}{2}[x, x] = 0, \quad e^{-p} \cdot g(x) \in L^1 \otimes m_A\}.$$  

If $M$ is also concentrated in non negative degrees, then the gauge equivalence is given by

$$(x, e^p) \sim (e^b \cdot x, e^{g(b)} e^p e^a), \quad \text{with } a \in L^0 \otimes m_A \text{ and } b \in N^0 \otimes m_A.$$  

4.2. Tangent and obstruction spaces of MC\(_{(h,g)}\) and Def\(_{(h,g)}\). By definition, the tangent space of a functor of Artin rings \(F\) is \(F(\mathbb{C}[\varepsilon])\), where \(\varepsilon^2 = 0\). Therefore,

\[
MC_{(h,g)}(\mathbb{C}[\varepsilon]) = \left\{ (x, y, e^p) \in (L^1 \otimes \mathbb{C}[\varepsilon]) \times (N^1 \otimes \mathbb{C}[\varepsilon]) \times \exp(M^0 \otimes \mathbb{C}[\varepsilon]) \mid dx = dy = 0, h(x) - g(y) - dp = 0 \right\}
\]

\[
\cong \left\{ (x, y, p) \in L^1 \times N^1 \times M^0 \mid dx = dy = 0, -dp - g(y) + h(x) = 0 \right\} = \ker(D : C_{(h,g)}^1 \longrightarrow C_{(h,g)}^2),
\]

and

\[
Def_{(h,g)}(\mathbb{C}[\varepsilon]) \cong \left\{ (x, y, p) \in L^1 \times N^1 \times M^0 \mid dx = dy = 0, g(y) = h(x) - dp \right\}
\]

\[
\cong \{ (-da, -db, dc + g(b) - h(a)) \mid a \in L^0, b \in N^0, c \in M^{-1} \} = H^1(C_{(h,g)}).
\]

The obstruction space of Def\(_{(h,g)}\), is naturally contained in \(H^2(C_{(h,g)})\). Indeed, let

\[
0 \longrightarrow J \longrightarrow \hat{A} \xrightarrow{\alpha} A \longrightarrow 0
\]

be a small extension and \((x, y, e^p) \in MC_{(h,g)}(A)\).

Since \(\alpha\) is surjective, there exist \(\tilde{x} \in L^1 \otimes m_{\hat{A}}\) that lifts \(x\), \(\tilde{y} \in N^1 \otimes m_{\hat{A}}\) that lifts \(y\), and \(q \in M^0 \otimes m_{\hat{A}}\) that lifts \(p\). Let

\[
l = d\tilde{x} + \frac{1}{2} [\tilde{x}, \tilde{x}] \in L^2 \otimes m_{\hat{A}}
\]

and

\[
k = d\tilde{y} + \frac{1}{2} [\tilde{y}, \tilde{y}] \in N^2 \otimes m_{\hat{A}}.
\]

It is easy to see that \(\alpha(l) = \alpha(k) = dl = dk = 0\); then \(l \in H^2(L) \otimes J\) and \(k \in H^2(N) \otimes J\).

Let \(r = -g(\tilde{y}) + e^q \circ h(\tilde{x}) \in M^1 \otimes m_{\hat{A}}\); thus, \(\alpha(r) = 0\) or, equivalently, \(r \in M^1 \otimes J\). It can be proved that \(-dr - g(k) + h(l) = 0\) and so \((l, k, r) \in Z^2(C_{(h,g)}) \otimes J\). Let \([l, k, r]\) be the class in \(H^2(C_{(h,g)}) \otimes J\). This class does not depend on the choice of the liftings and it vanishes if and only if there exists a lifting of \((x, y, e^p) \in MC_{(h,g)}(A)\) in \(MC_{(h,g)}(\hat{A})\) ([10, Lemma III.1.19]).

5. Deformations of holomorphic maps

**Definition 5.1.** Let \(f : X \longrightarrow Y\) be a holomorphic map of manifolds and \(A \in \text{Art}\). An **infinitesimal deformation of \(f\) over \(\text{Spec}(A)\)** is a commutative diagram of complex spaces

\[
\begin{array}{ccc}
X_A & \xrightarrow{\mathcal{F}} & Y_A \\
\pi \downarrow & & \downarrow \mu \\
\text{Spec}(A) & &
\end{array}
\]

where \((X_A, \pi, \text{Spec}(A))\) and \((Y_A, \mu, \text{Spec}(A))\) are infinitesimal deformations of \(X\) and \(Y\), respectively, and \(\mathcal{F}\) is a holomorphic map that restricted to the fibers over the closed point of \(\text{Spec}(A)\) coincides with \(f\). If \(A = \mathbb{C}[\varepsilon]\) we have a **first order deformation of \(f\)**.
Definition 5.2. Let

\[ X_A \xrightarrow{\mathcal{F}} Y_A \quad \text{and} \quad X'_A \xrightarrow{\mathcal{F}'} Y'_A \]

be two infinitesimal deformations of \( f \) over \( \text{Spec}(A) \). They are isomorphic if there exist bi-holomorphic maps \( \phi: X_A \rightarrow X'_A \) and \( \psi: Y_A \rightarrow Y'_A \) (that are equivalences of infinitesimal deformations of \( X \) and \( Y \), respectively) such that the following diagram is commutative:

\[ \begin{array}{ccc}
X_A & \xrightarrow{\mathcal{F}} & Y_A \\
\downarrow{\phi} & & \downarrow{\psi} \\
X'_A & \xrightarrow{\mathcal{F}'} & Y'_A
\end{array} \]

Definition 5.3. The functor of infinitesimal deformations of a holomorphic map \( f: X \rightarrow Y \) is

\[ \text{Def}(f): \text{Art} \rightarrow \text{Set}, \]

\[ A \mapsto \text{Def}(f)(A) = \left\{ \text{isomorphism classes of infinitesimal deformations of } f \text{ over } \text{Spec}(A) \right\}. \]

Remark 5.4. Let \( \Gamma \) be the graph of \( f \) in the product \( X \times Y \). The infinitesimal deformations of \( f \) can be interpreted as infinitesimal deformations \( \Gamma_A \) of \( \Gamma \) in the product \( X \times Y \), such that the induced deformations \( (X \times Y)_A \) of \( X \times Y \) are products of infinitesimal deformations of \( X \) and \( Y \). Since not all the deformations of a product are products of deformations ([12, pag. 436]), we are not just considering the deformations of the graph in the product. Moreover, with this interpretation, two infinitesimal deformations \( \Gamma_A \subset (X \times Y)_A \) and \( \Gamma'_A \subset (X \times Y)'_A \) are equivalent if there exists an isomorphism \( \phi: (X \times Y)_A \rightarrow (X \times Y)'_A \) of infinitesimal deformations of \( X \times Y \) such that \( \phi(\Gamma_A) = \Gamma'_A \).

Let \((B, D)\) be the complex with

\[ B^p = A^{(0,p)}_X(\Theta_X) \oplus A^{(0,p)}_Y(\Theta_Y) \oplus A^{(0,p-1)}_X(f^*\Theta_Y) \]

and

\[ D: B^p \rightarrow B^{p+1}, \quad (x, y, z) \mapsto (\partial x, \partial y, \partial z + (-1)^p(f_x + f^*y)). \]

Theorem 5.5 (E. Horikawa). \( H^1(B) \) is in one-to-one correspondence with the first order deformations of \( f: X \rightarrow Y \).

The obstruction space of the functor \( \text{Def}(f) \) is naturally contained in \( H^2(B) \).

Proof. See [17, Section 3.6].

Remark 5.6. Consider a first order deformation \( f_\varepsilon \) of \( f \): in particular, we are considering first order deformations \( X_\varepsilon \) and \( Y_\varepsilon \), of \( X \) and \( Y \), respectively.
Then, we associate with \( X_\varepsilon \) a class \( x \in H^1(X, \Theta_X) \) and with \( Y_\varepsilon \) a class \( y \in H^1(Y, \Theta_Y) \). Therefore, the class in \( H^1(B') \) associated with \( f_\varepsilon \) is \([(x, y, z)]\), with \( z \in A_X^{0,0}(f^*\Theta_Y) \) such that \( \overline{\theta}z = f_\varepsilon x - f^*y \).

Analogously, let \( 0 \to J \to A' \to A \to 0 \) be a small extension and \( \mathcal{F}_A \) an infinitesimal deformation of \( f \) over \( \text{Spec}(A) \). If \( h \in H^2(X, \Theta_X) \) and \( k \in H^2(Y, \Theta_Y) \) are the obstruction classes associated with \( X_A \) and \( Y_A \), respectively, then the obstruction class in \( H^2(B') \) associated with \( \mathcal{F}_A \) is \([(h, k, r)]\), with \( r \in A_X^{0,1}(f^*\Theta_Y) \) such that \( \overline{\theta}r = -(f_\varepsilon h - f^*k) \).

Let \( Z := X \times Y \) be the product of \( X \) and \( Y \) and \( p : Z \to X \) and \( q : Z \to Y \) the natural projections. Defining the morphism

\[
F : X \to \Gamma \subseteq Z, \\
x \mapsto (x, f(x)),
\]

we get the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{F} & Z \\
\downarrow{id} & & \downarrow{f} \\
X & & Y \\
& \downarrow{p} & \downarrow{q} \\
& X & \subseteq Y,
\end{array}
\]

In particular, \( F^* \circ p^* = id \) and \( F^* \circ q^* = f^* \). Since \( \Theta_Z = p^*\Theta_X \oplus q^*\Theta_Y \), it follows that \( F^*(\Theta_Z) = \Theta_X \oplus f^*\Theta_Y \). Define the morphism \( \gamma : \Theta_Z \to f^*\Theta_Y \) as the product

\[
\gamma : \Theta_Z \xrightarrow{F^*} \Theta_X \oplus f^*\Theta_Y, \quad (f^* - id) \quad f^*\Theta_Y;
\]

moreover, let \( \pi \) be the following surjective morphism:

\[
A_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} A_X^{0,*}(f^*\Theta_Y) \to 0,
\]

\[
\pi(\omega u) = F^*(\omega)\gamma(u), \quad \forall \omega \in A_Z^{0,*}, \ u \in \Theta_Z.
\]

Since each \( u \in \Theta_Z \) can be written as \( u = p^*v_1 + q^*v_2 \), for some \( v_1 \in \Theta_X \) and \( v_2 \in \Theta_Y \), we also have

\[
\pi(\omega u) = F^*(\omega)(f_\varepsilon(v_1) - f^*(v_2)).
\]

Since \( F^*\overline{\theta} = \overline{\theta}F^* \), \( \pi \) is a morphism of complexes.

Let \( \mathcal{L} \) be the kernel of \( \pi \):

\[
(4) \quad 0 \to \mathcal{L} \xrightarrow{h} A_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} A_X^{0,*}(f^*\Theta_Y) \to 0
\]

and \( h : \mathcal{L} \to A_Z^{0,*}(\Theta_Z) \) the inclusion.

Since there is a canonical isomorphism between the normal bundle \( N_{\Gamma|Z} \) of \( \Gamma \) in \( Z \) and the pull-back \( f^*T_Y \), (4) reduces to

\[
0 \to \mathcal{L} \xrightarrow{h} A_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} A_{\Gamma|Z}^{0,*}(N_{\Gamma|Z}) \to 0.
\]

Let \( i : \Gamma \to Z \) be the inclusion and \( i^* : A_Z^{0,*} \to A_{\Gamma}^{0,*} \) the induced map. Suppose that \( z_1, \ldots, z_n \) are holomorphic coordinates on \( Z \) such that
Given morphisms of DGLAs $\phi$ and $\beta$, Lemma 5.9.

**Proof.** See [16, Section 5]. It is an easy calculation in local holomorphic coordinates.

Let $L$ be the differential graded Lie algebra of global sections of $\mathcal{L}$.

Let $M$ be the Kodaira-Spencer algebra of the product $Z$, i.e. $M = KS_Z$, and $h : L \to M$ be the inclusion.

Let $N = KS_X \times KS_Y$ be the product of the Kodaira-Spencer algebras of $X$ and of $Y$ and $g = p^* + q^* : KS_X \times KS_Y \to KS_Z$, i.e., $g(n_1, n_2) = p^* n_1 + q^* n_2$ (for $n = (n_1, n_2)$ we also use the notation $g(n)$).

Therefore, we get the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{g=(p^*, q^*)} & M = KS_Z \\
\downarrow{h} & & \\
N = KS_X \times KS_Y & \to & M = KS_Z.
\end{array}
$$

**Remark 5.8.** Given morphisms of DGLAs $h : L \to KS_Z$ and $g : KS_X \times KS_Y \to KS_Z$, we can consider the complex $(C_{(h,g)}, D)$, with $C_{(h,g)}^i = L^i \oplus KS_X^i \oplus KS_Y^i \oplus KS_Z^{i-1}$ and differential is given by $D(l, n_1, n_2, m) = (-\partial_i l, -\partial_i n_1, -\partial_i n_2, -\partial_m n_1 - q^* n_2 + h(l))$.

Using the morphism $\pi : KS_{X \times Y} \to A_X^{0,*}(f^* \Theta_Y)$, we can define a morphism

$$
\beta : (C_{(h,g)}, D) \to (B^i, D_\eta),
$$

$$
\beta(l, n_1, n_2, m) = ((-1)^i n_1, (-1)^i n_2, -\pi(m)) \quad \forall (l, n_1, n_2, m) \in C_{(h,g)}^i.
$$

**Lemma 5.9.** $\beta : (C_{(h,g)}, D) \to (B^i, D_\eta)$ is a morphism of complexes which is a quasi-isomorphism.

**Proof.** It follows from an easy computation.

Let us consider the functor $\text{Def}_{(h,g)}$ associated with diagram (5). Since $h$ is injective and $M$ is concentrated in non negative degrees, by Remark 4.6, for each $(A, m_A) \in \text{Art}$, we have

$$
\text{Def}_{(h,g)}(A) = \{(n, e^m) \in (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) | dn + \frac{1}{2} [n, n] = 0, e^{-m} * g(n) \in L^1 \otimes m_A) \}/ \sim,
$$

where $(x, e^p) \sim (e^b * x, e^{g(b)} e^p e^a)$, with $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$. 

$Z \supset \Gamma = \{z_{t+1} = \cdots = z_n = 0\}$. Then, $\omega = \sum_{j=1}^n \omega_j \frac{\partial}{\partial z_j} \in A_Z^{0,*}(\Theta_Z)$ lies in $\mathcal{L}$ if and only if $\omega_j \in \ker i^*$ for $j \geq t + 1$. In particular, $\mathcal{L}^0$ is the sheaf of differentiable vector fields on $Z$ that are tangent to $\Gamma$.

**Lemma 5.7.** $\mathcal{L}$ is a sheaf of differential graded Lie subalgebras of $A_Z^{0,*}(\Theta_Z)$ such that $\mathcal{L}(\ker i^*) \subset \ker i^*$ if and only if $a \in \mathcal{L} \subset A_Z^{0,*}(\Theta_Z)$. Moreover, consider the automorphism $e^a$ of $A_Z^{0,*} \otimes A$ defined in (1): if $a \in L^0 \otimes m_A$ then $e^a(\ker(i^*) \otimes A) = \ker(i^*) \otimes A$.
Remark 5.10. Let \((n, e^m) \in \text{Def}_{(h, g)}\). In particular, \(n = (n_1, n_2)\) satisfies the Maurer-Cartan equation and so \(n_1 \in \text{MC}_{K/S_X}\) and \(n_2 \in \text{MC}_{K/S_Y}\). Therefore, there are associated with \(n\) infinitesimal deformations \(X_A\) of \(X\) (induced by \(n_1\)) and \(Y_A\) of \(Y\) (induced by \(n_2\)). Moreover, since \(g(n)\) satisfies the Maurer-Cartan equation in \(M = KS_Z\), it defines an infinitesimal deformation \(Z_A\) of \(Z\). By construction, the deformation \(Z_A\) is the product of the deformations \(X_A\) and \(Y_A\).

Consider an infinitesimal deformation of the holomorphic map \(f\) over \(\text{Spec}(A)\) as an infinitesimal deformation \(\Gamma_A\) of \(\Gamma\) over \(\text{Spec}(A)\) and \(Z_A\) of \(Z\) over \(\text{Spec}(A)\), with \(Z_A\) product of deformations of \(X\) and \(Y\) over \(\text{Spec}(A)\).

By applying Remark 5.10 and Theorem 3.4, the condition on the deformation \(Z_A\) is equivalent to requiring \(\mathcal{O}_{Z_A} = \mathcal{O}_{Z_A}(g(n))\), for some Maurer-Cartan element \(n \in KS_X \times KS_Y\). Let \(i^*: \mathcal{A}_Z^{0,n} \rightarrow \mathcal{A}_Z^{0,1}\) be the restriction morphism and let \(I = \ker i^* \cap \mathcal{O}_Z\) be the holomorphic ideal sheaf of the graph \(\Gamma\) of \(f\) in \(Z\). The deformations \(\Gamma_A\) of the graph \(\Gamma\) correspond to infinitesimal deformations \(I_A \subset \mathcal{O}_{Z_A}\) of \(\Gamma\) over \(\text{Spec}(A)\), with \(\mathcal{I}_A\) ideal sheaves of \(\mathcal{O}_{Z_A}\), flat over \(A\) and such that \(\mathcal{I}_A \otimes_A \mathbb{C} \cong I\).

In conclusion, to give an infinitesimal deformation of \(f\) over \(\text{Spec}(A)\) (an element in \(\text{Def}(f)(A)\)), it is sufficient to give an ideal sheaf \(\mathcal{I}_A \subset \mathcal{O}_{Z_A}(g(n))\) (for some \(n \in \text{MC}_{K/S_X \times K/S_Y}\)) with \(\mathcal{I}_A\) \(A\)-flat and \(\mathcal{I}_A \otimes_A \mathbb{C} \cong I\).

Theorem 5.11. Let \(h, g\) and \(i^*\) be as above. Then, there exists an isomorphism of functors
\[
\gamma: \text{Def}_{(h, g)} \rightarrow \text{Def}(f).
\]
Given a local Artinian \(\mathbb{C}\)-algebra \(A\), and an element \((n, e^m) \in \text{MC}_{(h, g)}(A)\), we define a deformation of \(f\) over \(\text{Spec}(A)\) as a deformation \(\mathcal{I}_A(n, e^m)\) of the holomorphic ideal sheaf of the graph of \(f\) in the following way
\[
\gamma(n, e^m) = \mathcal{I}_A(n, e^m) := (\ker(\mathcal{A}_Z^{0,0} \otimes A \xrightarrow{i^*} \mathcal{A}_Z^{0,1} \otimes A)) \cap e^m(\ker i^* \otimes A)
\]
\[
= \mathcal{O}_{Z_A}(g(n)) \cap e^m(\ker i^* \otimes A),
\]
where \(\mathcal{O}_{Z_A}(g(n))\) is the infinitesimal deformation of \(Z\), given by Theorem 3.4, that corresponds to \(g(n) \in \text{MC}_{K/S_X \times K/S_Y}\).

Proof. For each \((n, e^m) \in \text{MC}_{(h, g)}(A)\) we have defined
\[
\mathcal{I}_A(n, e^m) = \mathcal{O}_{Z_A}(g(n)) \cap e^m(\ker i^* \otimes A).
\]
First of all, we verify that this sheaf \(\mathcal{I}_A(n, e^m) \subset \mathcal{O}_{Z_A}(g(n))\) defines an infinitesimal deformation of \(f\); therefore, we need to prove that it is flat over \(A\) and \(\mathcal{I}_A(n, e^m) \otimes_A \mathbb{C} \cong I\). It is equivalent to verify these properties for \(e^{-m}\mathcal{I}_A(n, e^m)\). Applying Lemma 3.3, yields
\[
e^{-m}(\mathcal{O}_{Z_A}(g(n))) = \ker(\overline{D} + e^{-m} \ast g(n) : \mathcal{A}_Z^{0,0} \otimes A \longrightarrow \mathcal{A}_Z^{0,1} \otimes A)
\]
and also
\[
e^{-m}\mathcal{I}_A(n, e^m) = e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A) = \ker(\overline{D} + e^{-m} \ast g(n)) \cap (\ker i^* \otimes A).
\]
This implies that the deformations for each \( a \). Moreover, we recall that \( e \).

Thus, \( b \) in particular, the deformations induced on \( n,e \) \( L \).

Therefore, 
\[
e^{-m}I_A(n,e^m) = \ker(\mathcal{D} + e^\nu \ast 0) \cap (\ker i^* \otimes A) = \mathcal{O}_Z(e^\nu \ast 0) \cap (\ker i^* \otimes A)
\]
\[
= e^\nu(\mathcal{O}_Z(0)) \cap e^\nu(\ker i^* \otimes A) = e^\nu(I \otimes A).
\]

Thus, \( I_A(n,e^m) \) defines a deformation of \( f \) and the morphism
\[
\gamma : MC_{(h,g)} \longrightarrow \text{Def}(f)
\]
is well defined, such that
\[
\gamma(A) : MC_{(h,g)}(A) \longrightarrow \text{Def}(f)(A)
\]
\[
(n,e^m) \longmapsto \gamma(n,e^m) = I_A(n,e^m).
\]

Moreover, \( \gamma \) is well defined on \( \text{Def}_{(h,g)}(A) = MC_{(h,g)}(A)/\text{gauge} \). Actually, for each \( a \in L^0 \otimes m_A \) and \( b \in N^0 \otimes m_A \), we have
\[
\gamma(e^b \ast n, e^g(b) e^m) = O_Z(e^g(b) \ast g(n)) \cap e^g(b) e^m \ast (\ker i^* \otimes A) = e^g(b) O_Z(g(n)) \cap e^g(b) e^m \ast (\ker i^* \otimes A) = e^g(b) \gamma(n,e^m).
\]

This implies that the deformations \( \gamma(n,e^m) \) and \( \gamma(e^b \ast n, e^g(b) e^m) \) are isomorphic (Remark 5.4).

In conclusion, \( \gamma : \text{Def}_{(h,g)} \longrightarrow \text{Def}(f) \) is a well defined natural transformation of functors.

In order to prove that \( \gamma \) is an isomorphism it is sufficient to prove that
i) \( \gamma \) is injective;
ii) \( \gamma \) induces a bijective map on the tangent spaces;
iii) \( \gamma \) induces an injective map on the obstruction spaces.

i) \( \gamma \) is injective. Suppose that \( \gamma(n,e^m) = \gamma(r,e^s) \), then we want to prove that \( (n,e^m) \) is gauge equivalent to \( (r,e^s) \), i.e., there exist \( a \in L^0 \otimes m_A \) and \( b \in N^0 \otimes m_A \) such that \( e^b \ast r = n \) and \( e^g(b) e^s e^a = e^m \).

By hypothesis, \( \gamma(n,e^m) \) and \( \gamma(r,e^s) \) are isomorphic deformations; then, in particular, the deformations induced on \( Z \) are isomorphic. This implies that there exists \( b \in N^0 \otimes m_A \) such that \( e^b \ast r = n \) and \( e^g(b)(O_Z(g(r))) = O_Z(g(n)) \). Up to substituting \( r,e^s \) with its equivalent \( (e^b \ast r, e^g(b) e^s) \), we can assume to be in the following situation
\[
O_Z(g(n)) \cap e^m(\ker i^* \otimes A) = O_Z(g(n)) \cap e^m'(\ker i^* \otimes A).
\]
Let \( e^a = e^{-m'} e^m \), then
\[
e^a(e^{-m}(O_Z(g(n)))) \cap (\ker i^* \otimes A)) = e^{-m'}(O_Z(g(n))) \cap (\ker i^* \otimes A).
\]
In particular, \( e^a(e^{-m}(O_Z(g(n)))) \cap (\ker i^* \otimes A)) \subseteq \ker i^* \otimes A.\]
Next, we prove, by induction, that \( a \in L^0 \otimes m_A \) (thus \( e^m = e^{m'}e^a = e^{g(b)}e^a \)).

Let \( z_1, \ldots, z_n \) be holomorphic coordinates on \( Z \) such that \( Z \supset \Gamma = \{ z_{t+1} = \cdots = z_n = 0 \} \). Consider the projection on the residue field \( e^{-m}(\mathcal{O}_Z(g(n))) \cap (\ker i^* \otimes A) \rightarrow \mathcal{O}_Z \cap \ker i^* \).

Then, \( z_i \in \ker i^* \cap \mathcal{O}_Z \), for \( i > t \). Since \( e^{-m}(\mathcal{O}_Z(g(n))) \cap (\ker i^* \otimes A) \) is flat over \( A \), we can lift \( z_i \) to \( \tilde{z}_i = z_i + \phi_i \in e^{-m}(\mathcal{O}_Z(g(n))) \cap (\ker i^* \otimes A) \), with \( \phi_i \in \ker i^* \otimes m_A \). By hypothesis,

\[
(6) \quad e^a(\tilde{z}_i) = e^a(z_i) + e^a(\phi_i) \in \ker i^* \otimes A.
\]

By Lemma 5.7, in order to prove that \( a \in L^0 \otimes m_A \) it is sufficient to verify that \( e^a(z_i) \in \ker i^* \otimes A \) and so, by (6), that \( e^a(\phi_i) \in \ker i^* \otimes A \).

If \( A = \mathbb{C}[\varepsilon] \), then \( \phi_i \in \ker i^* \otimes \mathbb{C}[\varepsilon] \) and \( a \in A^0_0 \otimes \mathbb{C}[\varepsilon] \); this implies \( e^a(\phi_i) = \phi_i \in \ker i^* \otimes \mathbb{C}[\varepsilon] \).

Next, let \( 0 \rightarrow J \rightarrow \tilde{A} \xrightarrow{\alpha} A \rightarrow 0 \) be a small extension. By hypothesis, \( \alpha(a) \in L^0 \otimes m_A \), that is, \( \alpha(a) = \sum_{j=1}^{n} \pi_j \partial \frac{\partial}{\partial z_j} \) with \( \pi_j \in \ker i^* \otimes m_A \) for \( j > t \).

Let \( a_j' \) be liftings of \( \pi_j \). Then \( a_j' \in \ker i^* \otimes m_{\tilde{A}} \), for \( j > t \), \( a' = \sum_{j=1}^{n} a_j' \partial \frac{\partial}{\partial z_j} \in L^0 \otimes m_{\tilde{A}} \) and \( e^{a'}(\phi_i) \in \ker i^* \otimes m_{\tilde{A}} \). Since \( \alpha(a) = \alpha(a') \), then \( a = a' + j \) with \( j \in M^0 \otimes J \). This implies that \( e^{a}(\phi_i) = e^{a'+j}(\phi_i) = e^{a'}(\phi_i) \in \ker i^* \otimes m_{\tilde{A}} \).

As to \( ii) \) and \( iii) \), a straightforward computation shows that the maps induced by \( \gamma \) on tangent and obstruction spaces are the isomorphisms induced by \( \beta \) of Lemma 5.9.

\[\square\]

**Remark 5.12.** Consider the diagram

\[
\begin{array}{ccc}
K S_X \times K S_Y & \xrightarrow{g} & K S_{X \times Y} \\
\downarrow h & & \downarrow \pi \\
A^0_X(f^*T_Y)
\end{array}
\]

Since \( h \) is injective, Lemma 4.1 implies the existence of a quasi-isomorphism of complexes \( (C_{(h,q)}, D) \) and \( (C_{\pi \circ q}, \delta) \).

Then, we get the following exact sequence

\[
(7) \quad \cdots \rightarrow H^1(C_{\pi \circ q}) \xrightarrow{\partial^1} H^1(X, \Theta_X) \oplus H^1(Y, \Theta_Y) \rightarrow H^1(X, f^*\Theta_Y) \rightarrow \cdots,
\]

where \( \partial^1 \) and \( \partial^2 \) are the projections on the second factor and they are induced by the projection morphism \( \gamma : \text{Def}(f) \rightarrow \text{Def}_{K S_X \times K S_Y} \) (see Remark 4.5).

In particular, \( \partial : \text{Def}(f) \rightarrow \text{Def}_{K S_X \times K S_Y} \) associates with an infinitesimal deformation of \( f \) the induced infinitesimal deformations of \( X \) and \( Y \).
6. $L_\infty$ STRUCTURE ON THE CONE $C_{(h,g)}^\infty$

In this section we explicitly describe an $L_\infty$ structure on the cone $C_{(h,g)}^\infty$, associated with the pair of morphisms of DGLAs $h : L \to M$ and $g : N \to M$. In particular, we prove that the deformation functor $\text{Def}_{\text{f}}$ coincides with the functor $\text{Def}_{\text{f}}^\infty$, associated with this $L_\infty$ structure on $C_{(h,g)}^\infty$ (Theorem 6.17). Finally, we show the existence of a DGLA $H_{(h,g)}$ that controls the deformations of holomorphic maps (Corollary 6.18).

First of all, we briefly recall the definition of an $L_\infty$ structure on a graded vector space $V$. For a complete description of such structures, see, for example, [4], [5], [13] or [15, Chapter IX].

We denote by $V[1]$ the complex $V$ with degrees shifted by 1. More precisely, for $\mathbb{C}[1]$ we have

$$\mathbb{C}[1]^i = \begin{cases} \mathbb{C} & \text{if } i + 1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $V[1] = \mathbb{C}[1] \otimes V$, which implies $V[1]^i = V^{i+1}$.

Let $\epsilon(\sigma; v_1, \ldots, v_n)$ be the Koszul sign. We denote by $\bigodot^n V$ the space of co-invariant elements for the action of $\Sigma_n$ on $\bigotimes^n V$ given by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = \epsilon(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

When $(v_1, \ldots, v_n)$ are clear by the context we simply write $\epsilon(\sigma)$ instead of $\epsilon(\sigma; v_1, \ldots, v_n)$.

**Definition 6.1.** The set of unshuffles of type $(p, q)$ is the subset $S(p, q)$ of $\Sigma_n$ of permutations $\sigma$, such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(p + q)$.

**Definition 6.2.** An $L_\infty$ structure on a graded vector space $V$ is a system $\{q_k\}_{k \geq 1}$ of linear maps $q_k \in \text{Hom}^1(\bigodot^k(V[1]), V[1])$ such that the map

$$Q : \bigoplus_{n \geq 1} \bigodot^n V[1] \to \bigoplus_{n \geq 1} \bigodot^n V[1],$$

defined as

$$Q(v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k, n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) \otimes v_{\sigma(k+1)} \otimes \cdots \otimes v_{\sigma(n)},$$

is a co-derivation on the graded co-algebra $\bigoplus_{n \geq 1} \bigodot^n V[1]$, i.e., $Q \circ Q = 0$.

**Remark 6.3.** Let $(L, d, [ , ])$ be a differential graded Lie algebra. Let $q_1$ be the suspension of the differential $d$, i.e.,

$$q_1 := d[1] = Id_{\mathbb{C}[1]} \otimes d : V[1] \to V[1]$$

$$q_1(v[1]) = -(dv)[1].$$

Therefore, $g^1$ associates with a first order deformation of $f$ the induced first order deformations of $X$ and $Y$ and $g^2$ is a morphism of obstruction theories: the obstruction to deform $f$ is mapped to the induced obstructions to deform $X$ and $Y$ (see also Remark 5.10).
Then, define \(q_2 \in \text{Hom}^1(\otimes^2(V[1]), V[1])\) in the following way
\[
q_2(v_1 \otimes w_1) = (-1)^{\deg v[v, w]}[1].
\]
Finally, defining \(q_k = 0\), for each \(k \geq 3\), we endow the DGLA \(L\) with an \(L_\infty\) structure, i.e., every DGLA is an \(L_\infty\)-algebra with zero higher multiplications.

**Example 6.4.** Consider the DGLA \(M[t, dt] = M \otimes \mathbb{C}[t, dt]\), where \(\mathbb{C}[t, dt]\) is the differential graded algebra of polynomial differential forms over the affine line. For every \(a \in \mathbb{C}\), define the evaluation morphism in the following way
\[
e_a : M[t, dt] \rightarrow M,
\]
\[
e_a(\sum m_i t^i + n_i t^i dt) = \sum m_i a^i.
\]
The evaluation morphism is a morphism of DGLAs which is a left inverse of the inclusion and it is a surjective quasi-isomorphism, for each \(a\).

Next, define \(K \subset L \times N \times M[t, dt] \times M[s, ds]\) as follows
\[
K = \{(l, n, m_1(t, dt), m_2(s, ds)) | h(l) = e_1(m_2(s, ds)), g(n) = e_0(m_1(t, dt))\}.
\]
\(K\) is a DGLA with bracket and differential defined as the natural ones on each component.

Consider the following morphisms of DGLAs:
\[
e_1 : K \rightarrow M, \quad (l, n, m_1(t, dt), m_2(s, ds)) \mapsto e_1(m_1(t, dt))
\]
and
\[
e_0 : K \rightarrow M, \quad (l, n, m_1(t, dt), m_2(s, ds)) \mapsto e_0(m_2(s, ds)).
\]

Let \(H \subset K\) be defined as follow
\[
H = \{k \in K | e_1(k) = e_0(k)\},
\]
or written in detail
\[
H = \{(l, n, m_1(t, dt), m_2(s, ds)) \in L \times N \times M[t, dt] \times M[s, ds] | h(l) = e_1(m_2(s, ds)), g(n) = e_0(m_1(t, dt)), e_1(m_1(t, dt)) = e_0(m_2(s, ds))\}.
\]
For each \(k = (l, n, m_1(t, dt), m_2(s, ds)) \in K\), the pair \(m_1(t, dt)\) and \(m_2(s, ds)\) has fixed values at one of the extremes of the unit interval. More precisely, the value of \(m_1(t, dt)\) is fixed at the origin and \(m_2(s, ds)\) is fixed at 1. If \(k\) also lies in \(H\), then there are conditions on the other extremes: the value of \(m_1(t, dt)\) at 1 has to coincide with the value of \(m_2(s, ds)\) at 0.

Let
\[
H_{(h, g)} = \{(l, n, m(t, dt)) \in L \times N \times M[t, dt] | h(l) = e_1(m(t, dt)), g(n) = e_0(m(t, dt))\}.
\]
Since \(e_i\) are morphisms of DGLAs, it is clear that \(H_{(h, g)}\) is a DGLA.

Moreover, considering the barycentric subdivision, we get an injective quasi-isomorphism
\[
H_{(h, g)} \hookrightarrow H,
\]
\[
(l, n, m(t, dt)) \mapsto (l, n, m(\frac{1}{2} t, dt), m(\frac{s + 1}{2}, ds)).
\]
\(H_{(h, g)}\) is the differential graded Lie algebra associated with the pair \((h, g)\).
Then, by Remark 6.3, an $L_\infty$ structure on $H_{(h,g)}$ is defined by the following system of linear maps $q_k \in Hom^1(\odot^k(H_{(h,g)}[1], H_{(h,g)}[1]))$:

- $q_1(l, n, m(t, dt)) = (-dl, -dn, -dm(t, dt))$;
- $q_2((l_1, n_1, m_1(t, dt)) \circ (l_2, n_2, m_2(t, dt))) = (-1)^{deg_{h,g}(l_1, n_1, m_1(t, dt))} ((l_1, l_2), [n_1, n_2], [m_1(t, dt), m_2(t, dt))$;
- $q_k = 0$, for every $k \geq 3$.

A $L_\infty$-morphism $f_\infty : (V, q_1, q_2, \ldots) \to (W, p_1, p_2, p_3, \ldots)$ of $L_\infty$-algebras is a sequence of degree zero linear maps

$$f_n : \bigodot^n V[1] \to W[1], \quad n \geq 1,$$

such that the morphism of coalgebra

$$F : \bigoplus_{n \geq 1} \bigodot^n V[1] \to \bigoplus_{n \geq 1} \bigodot^n W[1],$$

induced by $\sum_n f_n : \bigoplus_{n \geq 1} \bigodot^n V[1] \to W[1]$, commutes with the codifferentials.

Sometimes this is the definition of a weak $L_\infty$-morphism; the strong (or linear) $L_\infty$-morphisms are the ones with $f_n = 0$, for each $n \geq 2$.

In particular, the linear part $f_1 : V[1] \to W[1]$ of an $L_\infty$-morphism $f_\infty : (V, q_1, q_2, q_3, \ldots) \to (W, p_1, p_2, p_3, \ldots)$ satisfies the condition $f_1 \circ q_1 = p_1 \circ f_1$, i.e., $f_1$ is a map of differential complexes $(V[1], q_1) \to (W[1], p_1)$.

A quasi-isomorphism of $L_\infty$-algebra is an $L_\infty$-morphism, whose linear part is a quasi-isomorphism of differential complexes.

The key result in this setting is the homotopical transfer of $L_\infty$ structures.

**Theorem 6.5.** Let $(V, q_1, q_2, q_3, \ldots)$ be an $L_\infty$-algebra and $(C, \delta)$ a differential graded vector space. If there exist two morphisms

$$\pi : (V[1], q_1) \to (C[1], \delta_{[1]}), \quad \iota : (C[1], \delta_{[1]}) \to (V[1], q_1)$$

which are homotopy inverses, then there exists an $L_\infty$-algebra structure $(C, \langle \_ \rangle_1, \langle \_ \rangle_2, \ldots)$ on $C$ extending its differential complex structure, and making $(V, q_1, q_2, q_3, \ldots)$ and $(C, \langle \_ \rangle_1, \langle \_ \rangle_2, \ldots)$ be quasi-isomorphic $L_\infty$-algebra, via an $L_\infty$-quasi-isomorphism $\iota_\infty$ extending $\iota$.

**Proof.** See [4, Theorem 4.1], [5] or [13]. \hfill \Box

Next, we use this theorem to transfer the $L_\infty$-structures of $H_{(h,g)}$, given by Example 6.4, to the cone $C_{(h,g)}$. We recall that $C^i_{(h,g)} = L^i \oplus N_i \oplus M^{i-1}$ and $D(l, n, m) = (dl, dn, -dm - g(n) + h(l))$.

Denote by $\langle \_ \rangle_1 \in Hom^1(C_{(h,g)}[1], C_{(h,g)}[1])$ the suspended differential, i.e.,

$$\langle(l, n, m) \rangle_1 = (-dl, -dn, dm + g(n) - h(l))$$

First of all, we note that we can define an integral operator $\int_a^b$ on $M[t, dt]$, that is a linear map of degree -1 and extends the usual integration $\int_a^b : \mathbb{C}[t, dt] \to \mathbb{C}$, i.e.,

$$\int_a^b : M[t, dt] \to M \quad \int_a^b (\sum_i t^i m_i + t^i dt \cdot n_i) = \sum_i \left( \int_a^b t^i dt \right) n_i.$$
Next, define the following linear maps of degree 0:
\[ \iota : C_{(h,g)}[1] \longrightarrow H_{(h,g)}[1] \]
\[ \iota(l, n, m) = (l, n, (1 - t)g(n) + th(l) + dtm) \]

and
\[ \pi : H_{(h,g)}[1] \longrightarrow C_{(h,g)}[1] \]
\[ \pi(l, n, m(t, dt)) = (l, n, \int_0^1 m(t, dt)) \].

Finally, define the homotopy \( K \in \text{Hom}^{-1}(H_{(h,g)}[1], H_{(h,g)}[1]) \) in the following way
\[ K : H_{(h,g)}[1] \longrightarrow H_{(h,g)}[1] \]
\[ K(l, n, m(t, dt)) = (0, 0, t \int_0^1 m(t, dt) - \int_0^t m(t, dt)) \].

**Lemma 6.6.** \( \iota \) and \( \pi \) are quasi-isomorphisms of complexes such that
\[ \pi \circ \iota = \text{id}_{C_{(h,g)}[1]} \quad \text{and} \quad \text{id}_{H_{(h,g)}[1]} - \iota \circ \pi = K \circ q_1 + q_1 \circ K \]

**Proof.** See [4, Lemma 3.2]. It is a straightforward computation. \( \square \)

Thus, applying Theorem 6.5, we get an \( L_\infty \)-algebra structure \( C_{(h,g)} \).

**Corollary 6.7.** There exists an \( L_\infty \)-algebra structure \( (C_{(h,g)}, \langle \rangle_1, \langle \rangle_2, \ldots) \) on the complex \( C_{(h,g)} \) that extends its differential structure, and it makes \( (H_{(h,g)}, q_1, q_2, 0, \ldots) \) and \( (C_{(h,g)}, \langle \rangle_1, \langle \rangle_2, \ldots) \) quasi-isomorphic \( L_\infty \)-algebras, via an \( L_\infty \)-quasi-isomorphism \( \iota_\infty \) extending \( \iota \).

**Remark 6.8.** As explained in [3], [4], [5] and [13], we have an explicit description of the higher multiplication \( \langle \rangle_n \) on \( C_{(h,g)} \) in terms of rooted trees. Since
\[ q_2(\text{Im } K \otimes \text{Im } K) \subseteq \ker \pi \cap \ker K \quad \text{and} \quad q_k = 0, \ \forall k \geq 3, \]
it can be proved that we have to consider just the following rooted trees

[Diagram of rooted trees decorated with operators]

It is well defined, since \( e_0(l, n, (1 - t)g(n) + th(l) + dtm) = g(n) \) and \( e_1(l, n, (1 - t)g(n) + th(l) + dtm) = h(l) \).
Explicitly, for each $n \geq 2$, these diagrams give us the following formula for the higher multiplications $(\gamma)_n$:

$$\langle \gamma_1 \circ \cdots \circ \gamma_n \rangle_n = \frac{(-1)^{n-2}}{2} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) \pi q_2(\varepsilon(\gamma_1(1)) \circ K q_2(\varepsilon(\gamma_2(2)) \circ \cdots \circ K q_2(\varepsilon(\gamma_{n-1}(n)) \circ \varepsilon(\gamma_n(n+1))))).$$

The factor $1/2$ in the above formula accounts for the cardinality of the automorphisms group of the graph involved.

In particular, for each $(l_1, n_1, m_1)$ and $(l_2, n_2, m_2)$ in $C_{(h,g)}$ we have

$$\langle (l_1, n_1, m_1) \circ (l_2, n_2, m_2) \rangle_2 = \pi q_2(\varepsilon(l_1, n_1, m_1) \circ \varepsilon(l_2, n_2, m_2)) = (-1)^{\text{deg} C_{(h,g)}(l_1, n_1, m_1)} ([l_1, l_2], [n_1, n_2],$$

$$(g(n), m_2) + [m_1, g(n_2)]) \int_0^1 (1-t)dt + ([h(l_1), m_2] + [m_1, h(l_2)]) \int_0^1 tdt = (-1)^{\text{deg} C_{(h,g)}(l_1, n_1, m_1)}$$

$$([l_1, l_2], [n_1, n_2], \frac{1}{2} ([g(n_1), m_2] + [m_1, g(n_2)] + [h(l_1), m_2] + [m_1, h(l_2)]))c.$$  

If $(l_1, n_1, m_1) = (l_2, n_2, m_2) = (l, n, m) \in C_{(h,g)}^0[1]$, then

$$\langle (l, n, m) \circ^2 \rangle_2 = (\gamma_1 - [n, n], -[m, g(n)] - [m, h(l)]).$$

**Remark 6.9.** All higher multiplications (for $n \geq 3$) vanish except the following ones:

$$(10) \quad \langle m_1 \circ \cdots \circ m_j \circ l \rangle_{j+1} \quad \text{and} \quad \langle m_1 \circ \cdots \circ m_j \circ n \rangle_{j+1}$$

or their linear combinations (for each $j \geq 2$). Here we use the notation $\gamma_i = m_i$ instead of $\gamma_i = (0, 0, m_i)$ and analogously for $\gamma_i = l$ or $\gamma_i = n$.

As in [4, Section 5], we can use Bernoulli's numbers to give an explicit description of the multiplications of (10). First of all, we recall that the Bernoulli's numbers $B_n$ are defined by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160} + \cdots.$$

Next, consider the multiplications

$$q_2(\varepsilon(m \circ \varepsilon(l)) = q_2((0, 0, dtm) \circ (l, 0, tth(l)))$$

$$= (-1)^{\text{deg} q_2(0, 0, dtm)} (0, 0, tdt[m, h(l)]).$$

Define recursively $\phi_j(t) \in \mathbb{Q}[t]$ and $I_j \in \mathbb{Q}$ as

$$\phi_1(t) = t, \quad I_j = \int_0^t \phi_j(t)dt, \quad \phi_{j+1}(t) = \int_0^t \phi_j(s)ds - tI_j.$$  

Then,

$$K(\phi_j(t)dt)m = -\phi_{j+1}(t)m$$

and so

$$K q_2((dtm) \circ \phi_j(t)m) =$$
\(-(-1)^{\deg M_{m_1}} \phi_{j+1}(t)[m_1, m_2] = (-1)^{\deg M_{m_1}} \phi_{j+1}(t)[m_1, m_2].\)

**Lemma 6.10.** The following formula holds

\[
\langle m_1 \circ \cdots \circ m_j \circ l \rangle_{j+1} = (-1)^{\sum_{i=1}^{j} \deg M(m_i)} I_j \sum_{\sigma \in \Sigma_j} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \cdots [m_{\sigma(j)}, h(l)] \cdots]]
\]

\[
= -(-1)^{\sum_{i=1}^{j} \deg M(m_i)} B_j \sum_{\sigma \in \Sigma_j} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \cdots [m_{\sigma(j)}, h(l)] \cdots]].
\]

**Proof.** See [4, Theorem 5.5].

In particular, if \(m_{\sigma(i)} = m \in M^0\), for each \(i\), then

\[
\langle m^\circ j \circ l \rangle_{j+1} = -\frac{B_j}{j!} \sum_{\sigma \in \Sigma_j} [m, [m, \cdots [m, h(l)] \cdots]] = -B_j \text{ad}_{m}^{j}(h(l)).
\]

Next, consider the multiplication

\[
q_2((m) \circ g(n)) = q_2((0, 0, dtm) \circ (0, g(n), (1 - t)g(n)) = (-1)^{\deg M_{h,g}^{(0,0,dtm)}}(0, 0, (1 - t)dt[m, g(n)]).
\]

In this case, define recursively \(\varphi_j(t) \in \mathbb{Q}[t]\) as follows

\[
\varphi_1(t) = 1 - t = 1 - \phi_1(t)
\]

and

\[
\varphi_{j+1}(t) = \int_0^t \varphi_j(s)ds - t \int_0^1 \varphi_j(s)ds.
\]

We note that

\[
\varphi_2(t) = \int_0^t \varphi_1(s)ds - t \int_0^1 \varphi_1(s)ds = \int_0^t (1 - \phi_1(s))ds - t \int_0^1 (1 - \phi_1(s))ds = (\int_0^t ds - t \int_0^1 ds) - (\int_0^t \phi_1(s)ds - t \int_0^1 \phi_1(s)ds) = -\phi_2(t).
\]

Therefore, for \(j \geq 2\) we get

\[
\varphi_j(t) = -\phi_j(t).
\]

Then,

\[
K((\varphi_j(t)dt)m) = -\varphi_{j+1}(t)m
\]

and so

\[
Kq_2((dtm_1) \circ \varphi_j(t)m_2) = -(1)^{\deg M_{m_1}^1} \varphi_{j+1}(t)[m_1, m_2] = (-1)^{\deg M_{m_1}^1} \varphi_{j+1}(t)[m_1, m_2].
\]
Lemma 6.11. The following formula holds
\[
\langle m_1 \odot \cdots \odot m_j \odot n \rangle_{j+1} = \\
-(-1)^{n + \sum_{i=1}^{j} \deg \gamma(m_i)} I_j \sum_{\sigma \in S_j} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \ldots [m_{\sigma(j)}, g(n)] \ldots]]
\]
\[
= (-1)^{j} \sum_{i=1}^{j} \deg \gamma(m_i) B_j \sum_{\sigma \in S_j} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \ldots [m_{\sigma(j)}, g(n)] \ldots]].
\]

Proof. Analogous of Lemma 6.10. \( \square \)

In particular, if \( m_{\sigma(i)} = m \in M_0 \) for each \( i \), then
\[
\langle m^{\odot j} \odot n \rangle_{j+1} = B_j \sum_{\sigma \in S_j} [m, [m, \ldots [m, g(n)] \ldots]] = B_j \text{ad}_m^j(g(n)).
\]

6.1. The Maurer-Cartan functor on the cone. Let \( T \) be an \( L_\infty \)-algebra. Then we can define (see [5],[13]) the Maurer-Cartan functor \( MC^\infty_T \) associated with \( T \) in the following way
\[
MC^\infty_T : \text{Art} \to \text{Set}
\]
\[
MC^\infty_T(A) = \left\{ \gamma \in T^1 \otimes m_A \left| \sum_{j \geq 1} \frac{\langle \gamma^{\odot j} \rangle_j}{j!} = 0 \right. \right\}.
\]

Next, we want to prove that the Maurer-Cartan functor \( MC_{(h,g)} \) associated with the pair of morphisms of DGLAs \( h : L \to M \) and \( g : N \to M \) (introduced in Section 4.1) is exactly the Maurer-Cartan functor associated with the \( L_\infty \) structure on \( C_{(h,g)} \) defined before.

By definition,
\[
MC^\infty_{C_{(h,g)}} : \text{Art} \to \text{Set}
\]
\[
MC^\infty_{C_{(h,g)}}(A) = \left\{ \gamma \in C_{(h,g)}[1]^0 \otimes m_A \left| \sum_{j \geq 1} \frac{\langle \gamma^{\odot j} \rangle_j}{j!} = 0 \right. \right\}.
\]

Let \( \gamma = (l, n, m) \in C_{(h,g)}[1]^0 \otimes m_A \), thus \( l \in L^1 \otimes m_A \), \( n \in N^1 \otimes m_A \) and \( m \in M^0 \otimes m_A \). Then,
\[
\langle (l, n, m) \rangle_1 = (-dl, -dn, dm + g(n) - h(l))
\]
and
\[
\langle (l, n, m) \rangle_{\odot 2}^2 = (-[l, l], -[n, n], -[m, g(n)] - [m, h(l)]).
\]

Therefore, the Maurer-Cartan equation
\[
\sum_{j \geq 1} \frac{\langle (l, n, m) \rangle_{\odot j}^j}{j!} = 0
\]
splits into
\[
dl + \frac{1}{2} [l, l] = dn + \frac{1}{2} [n, n] = 0
\]
and
\[
g(n) - h(l) + dm - \frac{1}{2} [m, g(n)] - \frac{1}{2} [m, h(l)] + \sum_{j \geq 3} \frac{\langle (l, n, m) \rangle_{\odot j}^j}{j!} = 0.
\]
Since
\[ \sum_{j \geq 3} \frac{(l, m) \circ j}{j!} = \sum_{j \geq 3} \frac{(j + 1)}{(j + 1)!} (m \circ j \otimes l)_{j+1} + \sum_{j \geq 3} \frac{(j + 1)}{(j + 1)!} (m \circ j \otimes n)_{j+1}, \]
applying Lemma 6.10 and Lemma 6.11, Equation (11) becomes
\[
g(n) - h(l) + dm - \frac{1}{2}[m, g(n)] - \frac{1}{2}[m, h(l)] + \sum_{j \geq 2} B_j \frac{j!}{j!} \text{ad}_m^j(g(n)) - \sum_{j \geq 2} B_j \frac{j!}{j!} \text{ad}_m^j(h(l)) = 0.
\]
Since \( B_0 = 1 \) and \( B_1 = -\frac{1}{2} \), we can write
\[
0 = g(n) - h(l) + dm - [m, h(l)] + \sum_{j \geq 1} B_j \frac{j!}{j!} \text{ad}_m^j(g(n)) - \sum_{j \geq 1} B_j \frac{j!}{j!} \text{ad}_m^j(h(l))
\]
and so
\[
0 = dm - [m, h(l)] + \sum_{j \geq 0} B_j \frac{j!}{j!} \text{ad}_m^j(g(n)) - \sum_{j \geq 0} B_j \frac{j!}{j!} \text{ad}_m^j(h(l)) =
\]
\[
dm - [m, h(l)] + \sum_{j \geq 0} B_j \frac{j!}{j!} \text{ad}_m^j(g(n) - h(l)).
\]
This implies that
\[
0 = dm - [m, h(l)] + \frac{\text{ad}_m}{\text{ad}_m - id}(g(n) - h(l)).
\]
Applying the operator \( \frac{e^{\text{ad}_m} - id}{\text{ad}_m} \), we get
\[
g(n) = h(l) + \frac{e^{\text{ad}_m} - id}{\text{ad}_m}([m, h(l)] - dm) = e^{m \ast h(l)}.
\]
In conclusion, the Maurer-Cartan equation for the \( L_\infty \) structure on \( C_{(h,g)} \) is
\[
\begin{cases}
  dl + \frac{1}{2}[l, l] = 0 \\
  dn + \frac{1}{2}[n, n] = 0 \\
  e^{m \ast h(l)} = g(n).
\end{cases}
\]
**Corollary 6.12.** \( MC_{(h,g)} \cong MC_{C_{(h,g)}}^\infty \).

Given an \( L_\infty \)-algebra \( T \), two elements \( x \) and \( y \) \( MC_T^\infty(A) \) are *homotopy equivalent* if there exists \( g[s, ds] \in MC_T^\infty(T[s, ds]) \) with \( g(0) = x \) and \( g(1) = y \). Then, the *deformation functor* \( \text{Def}_T^\infty \) associated with \( T \) is \( MC_T^\infty \) / homotopy. Moreover, if two \( L_\infty \)-algebras are quasi-isomorphism, then there exists an isomorphism between the associated deformation functors [5]. In particular, Corollary 6.7 implies the following result.

**Corollary 6.13.** \( \text{Def}_{C_{(h,g)}}^\infty \cong \text{Def}_{H_{(h,g)}}^\infty \).
Next, we prove that there is also an isomorphism between the functor \( \text{Def}_{(h,g)} \) and the deformation functor \( \text{Def}_{C(h,g)}^\infty \). First of all, we state two useful lemmas. We will use the notation \( e_a(p(t,dt)) := p(a) \), for each \( p(t,dt) \in M[t,dt] \) and \( a \in \mathbb{C} \).

**Lemma 6.14.** Let \( M \) be a differential graded Lie algebra and let \( A \in \text{Art} \). Then, for any \( x \) in \( \text{MC}_M(A) \) and any \( g(t) \in M^0[t] \otimes m_A \), with \( g(0) = 0 \), the element \( e^g(t)*x \) is an element of \( \text{MC}_{M[t,dt]}(A) \). Moreover all the elements of \( \text{MC}_{M[t,dt]}(A) \) are obtained in this way.

**Proof.** See [4, Corollary 7.2]. \( \square \)

**Lemma 6.15.** Let \( x(t,dt) \in \text{MC}_{M[t,dt]}(A) \), \( \mu(t,dt) \in M[t,dt]^0 \otimes m_A \), such that \( \mu(0) = 0 \), and

\[
e^{\mu(t,dt)}*x(t,dt) = x(t,dt).
\]

Then, \( e^{\mu_1(t)} \in \text{Stab}_A(x(1)) \), i.e., there exists \( C \in M^{-1} \) such that \( \mu(1) = d_M C + [x(1),C] \), where \( d_M \) is the differential in \( M \).

**Remark 6.16.** The proof of the case \( x(t,dt) = 0 \) is contained in [4, Theorem 7.4].

**Proof.** First, suppose that \( x(t,dt) = x \in M \). Thus, we have \( e^{\mu(t,dt)}*x = x \) and so

\[
d(\mu(t,dt)) + [x,\mu(t,dt)] = 0 \in (M[t,dt])^1 \otimes m_A.
\]

If we write \( \mu(t,dt) = \mu_0(t) + \mu_1(t)dt \), with \( \mu_0(t) \in M[t]^0 \) and \( \mu_1(t) \in M[t]^{-1} \), then Equation (12) becomes

\[
\dot{\mu}_0(t)dt + d_M \mu_0(t) + d_M \mu_1(t)dt + [x,\mu_0(t)] + [x,\mu_1(t)]dt = 0,
\]

or, equivalently,

\[
\begin{cases}
\dot{\mu}_0 + d_M \mu_1 + [x,\mu_1] = 0 \\
d_M \mu_0(t) + [x,\mu_0(t)] = 0.
\end{cases}
\]

Thus, for any fixed \( \mu_1(t) \), we get

\[
\mu_0(t) = -\int_0^t d_M \mu_1(s)ds - \int_0^t [x,\mu_1(s)]ds = -d_M \int_0^t \mu_1(s)ds - [x,\int_0^t \mu_1(s)ds].
\]

Let \( C = -\int_0^t \mu_1(s)ds \in M^{-1} \). Therefore, \( \mu(1) = \mu_0(1) = d_M C + [x,C] \) or, analogously, \( e^{\mu_1(t)} \in \text{Stab}_A(x) \). This concludes the proof in the case \( x(t,dt) = x \).

Next, consider the general case of a Maurer-Cartan element \( x(t,dt) \in \text{MC}_{M[t,dt]}(A) \). Lemma 6.14 implies the existence of \( g(t) \in M^0[t] \) such that \( g(0) = 0 \) and

\[
x(t,dt) = e^g(t) * x(0).
\]

Therefore, the hypothesis \( e^{\mu(t,dt)}*x(t,dt) = x(t,dt) \), implies

\[
e^{-g(t)}e^{\mu(t,dt)}e^g(t)*x(0) = x(0).
\]

Let \( q(t,dt) = -g(t) \bullet \mu(t,dt) \bullet g(t) \). If we write \( \mu(t,dt) = \mu_0(t) + \mu_1(t)dt \) and \( q(t,dt) = q_0(t) + q_1(t)dt \), then \( q_0(t) = -g(t) \bullet \mu_0(t) \bullet g(t) \).
By the previous consideration applied to $e^{q(t, dt)} \ast x_0 = x_0$, we conclude that $e^{q(1)} \in Stab_A(x(0))$.

The main property of irrelevant stabilizer asserts that

$$\forall a \in M^0 \otimes A \quad e^aStab_A(x)e^{-a} = Stab_A(y), \quad \text{with} \quad y = e^a \ast x.$$  

Therefore, $e^{\mu(1)} = e^{q(1)} e^{-q(1)} \in Stab_A(y)$, with $y = e^{q(1)} \ast x(0) = x(1)$.

Equivalently, there exists $C \in M^{-1}$ such that $\mu(1) = d_M C + [x(1), C]$. 

\[ \square \]

**Theorem 6.17.**

$$\text{Def}^\infty_{C,(h,g)} = \frac{\text{MC}^\infty_{C,(h,g)}}{\text{homotopy}} \simeq \frac{\text{MC}(h,g)}{\text{gauge}} = \text{Def}_{(h,g)}.$$  

**Proof.** This theorem is a generalization of [4, Theorem 7.4].

First, we show that gauge equivalence implies homotopy equivalence. Let $(l_0, n_0, m_0)$ and $(l_1, n_1, m_1)$ in $\text{MC}(h,g)(A)$, for some $A \in \text{Art}$; in particular, $e^{m_0} \ast h(l_0) = g(n_0)$ and $e^{m_1} \ast h(l_1) = g(n_1)$.

Suppose that they are gauge equivalent elements, i.e., there exist $a \in L^0 \otimes m_A$, $b \in N^0 \otimes m_A$ and $c \in M^{-1} \otimes m_A$ such that

$$l_1 = e^a \ast l_0, \quad n_1 = e^b \ast n_0, \quad m_1 = g(b) \ast T \ast m_0 \ast (-h(a)),$$

with $T = dc + [g(n_0), c]$ (and so $e^T \in Stab_A(g(n_0)))$.

Let $\tilde{l}(s, ds) = e^{sa} \ast l_0 \in L[s, ds] \otimes m_A$, $\tilde{n}(s, ds) = e^{sb} \ast n_0 \in N[s, ds] \otimes m_A$ and $T(s) = d(sc) + [g(n_0), sc]$. By Lemma 6.14, $\tilde{l}$ and $\tilde{n}$ satisfy the Maurer-Cartan equation and $h(\tilde{l}) = e^{g(sa)} \ast h(l_0)$ and $g(\tilde{n}) = e^{g(sb)} \ast g(n_0)$.

Define $\tilde{m} = g(sb) \ast T(s) \ast m_0 \ast (-h(sa))$; then,

$$e^{\tilde{m}} \ast h(\tilde{l}) = e^{g(sb) \ast T(s) \ast m_0 \ast (-h(sa))} \ast h(l_0) = e^{g(sb) \ast T(s) \ast m_0} \ast h(l_0) = e^{g(sb) \ast T(s)} \ast g(n_0) = g(\tilde{n}).$$

Therefore, $(\tilde{l}, \tilde{n}, \tilde{m}) \in \text{MC}^\infty_{C,(h,g)}[t,dt](A)$. Moreover, $\tilde{l}(0) = l_0$, $\tilde{l}(1) = l_1$, $\tilde{n}(0) = n_0$, $\tilde{n}(1) = n_1$, $\tilde{m}(0) = m_0$ and $\tilde{m}(1) = g(b) \ast T \ast m_0 \ast (-h(a)) = m_1$, i.e., $(l_0, n_0, m_0)$ and $(l_1, n_1, m_1)$ are homotopy equivalent.

Next, we show that homotopy equivalence implies gauge equivalence. Let $(l_0, n_0, m_0)$ and $(l_1, m_1, m_1)$ be homotopy equivalent elements in $\text{MC}^\infty_{C,(h,g)}(A)$.

Thus, there exists $(\tilde{l}, \tilde{n}, \tilde{m}) \in \text{MC}^\infty_{C,(h,g)}[t,dt](A)$ such that

$$d\tilde{l} + \frac{1}{2} [\tilde{l}, \tilde{l}] = 0, \quad d\tilde{n} + \frac{1}{2} [\tilde{n}, \tilde{n}] = 0, \quad g(\tilde{n}) = e^{\tilde{m}} \ast h(\tilde{l})$$

and

$$\begin{cases} (\tilde{l}(0), \tilde{n}(0), \tilde{m}(0)) = (l_0, n_0, m_0) \\ (\tilde{l}(1), \tilde{n}(1), \tilde{m}(1)) = (l_1, n_1, m_1). \end{cases}$$

In particular, $\tilde{l}$ and $\tilde{n}$ satisfy the Maurer-Cartan equation in $L[t, dt]$ and $N[t, dt]$, respectively. Applying Lemma 6.14, there exist degree zero elements $\lambda(t) \in L[t] \otimes m_A$ and $\nu(t) \in N[t] \otimes m_A$, such that $\lambda(0) = 0$, $\tilde{l} = e^{\lambda} \ast l_0$, $\nu(0) = 0$ and $\tilde{n} = e^{\nu} \ast n_0$.

This implies that $h(\tilde{l}) = e^{h(\lambda)} \ast h(l_0)$, $g(\tilde{n}) = e^{g(\nu)} \ast g(n_0)$ and, for $s = 1$, that

$$l_1 = e^{\lambda(1)} \ast l_0 \quad \text{and} \quad n_1 = e^{\nu(1)} \ast n_0.$$
Moreover, we note that
\[ e^{g(\nu) \cdot m_0} \cdot (-h(\lambda)) \cdot h(\tilde{f}) = e^{g(\nu) \cdot m_0} \cdot h(0) = e^{g(\nu)} \cdot g(0) = g(\tilde{n}). \]

Let \( \mu = \tilde{m} \cdot h(\lambda) \cdot (-m_0) \cdot (-g(\nu)) \in M^0[t, dt] \otimes m_A \) so that \( \mu = \mu \cdot g(\nu) \cdot m_0 \cdot (-h(\lambda)) \). Then \( \mu(0) = 0 \) and \( e^\mu \cdot g(\tilde{n}) = g(\tilde{n}). \)

Therefore, by Lemma 6.15, there exists \( C \in M^{-1} \), such that \( \mu(1) = d_M(C) + [g(n_1), C] \) and so \( e^{\mu(1)} \in Stab_A(g(n_1)) \). Then, \( m_1 = \tilde{m}(1) = \mu(1) \cdot g(\nu(1)) \cdot m_0 \cdot (-h(\lambda_1)). \) Applying the main property of the irrelevant stabilizers, there exists \( C' \in M^{-1} \) such that
\[
\mu(1) \cdot g(\nu(1)) = g(\nu(1)) \cdot T',
\]
with \( T' = dC' + [g(n_0), C'] \) and \( e^{T'} \in Stab_A(g(n_0)). \) Thus, \( m_1 = g(\nu(1)) \cdot T' \cdot m_0 \cdot (-h(\lambda_1)). \)

In conclusion, if \( (l_0, n_0, m_0) \) and \( (l_1, n_1, m_1) \) are homotopy equivalent, then there exists \( (\lambda(1), \nu(1)) \in (L^0 \otimes m_A) \times (N^0 \otimes m_A) \) and \( T' = dC' + [g(n_0), C'] \), for some \( C' \in M^{-1} \), such that
\[
\begin{align*}
l_1 &= e^{\lambda(1)} \cdot l_0 \\
n_1 &= e^{\nu(1)} \cdot n_0 \\
m_1 &= g(\nu(1)) \cdot T' \cdot m_0 \cdot (-h(\lambda_1)),
\end{align*}
\]
i.e., \( (l_0, n_0, m_0) \) and \( (l_1, n_1, m_1) \) are gauge equivalent.

\[ \square \]

**Corollary 6.18.** \( \text{Def}_{(h,g)} \cong \text{Def}_{H(h,g)} \).

Therefore, by suitably choosing \( L, M, \) and \( h, g \), we have an explicit description of the DGLA that controls the infinitesimal deformations of holomorphic maps.

**Theorem 6.19.** Let \( f : X \rightarrow Y \) be a holomorphic map. Then, the DGLA \( H_{(h,g)} \) associated with the morphisms \( h : L \leftarrow KS_{X \times Y} \) and \( g = (p^*, q^*) : KS_X \times KS_Y \rightarrow KS_{X \times Y} \) (introduced in Section 4) controls infinitesimal deformations of \( f \), i.e.,
\[ \text{Def}_{H(h,g)} \cong \text{Def}(f). \]

**Proof.** It is sufficient to apply Theorem 5.11 and Corollary 6.18. \( \square \)

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