The Generalization of Schottky Inequality and Its Applications

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September 8, 2015

Abstract
This article used Bloch function to derive Schottky inequality, obtained its generalization by using elliptic integral deviation function and demonstrated its applications.

1 Introduction and results

Lemma 1: Let $\zeta = \zeta (z) : C \setminus [1, +\infty) \mapsto D (0, 1)$ is a conformal mappings, the origin remain fixed and $\zeta$ is symmetry about the real axis. $ds = \rho_{0,1} (z)|dz|$ is the Poincare metric in $D (0, 1) \setminus \{0\}$. Then when $|z| < 1, |z| < |z - 1|$, we have $\rho_{0,1} (z)$ in $D (0, 1) \setminus \{0, 1\}$:

$$\rho_{0,1} (z) \geq \frac{1}{|z|(C - \ln |z|)}$$

with $C = [\min \{\rho_{0,1} (z) | |z| = 1\}]^{-1}$.

Proof: Let $\Omega_1 = \{z | |z| < 1, |z| < |z - 1|\}$ consider metric

$$\sigma (z) = \left| \frac{\zeta' (z)}{\zeta (z)} \right| \left[ 4 - \ln |\zeta (z)| \right]^{-1}, \forall z \in C \setminus [1, +\infty)$$

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obviously, $\sigma(z) = \sigma_r(\xi(z))\zeta'(z)$, with $\sigma_r(\zeta(z))$ is the Poincare metric in $D(0, r) \setminus \{0\}$ and $r = e^4$.

As we can see, the curvature of $\sigma(z) \vert dz \vert$ is -1. Take $\rho(z) = \sigma(z)$ in $\Omega_1$, and $\rho(z) = \sigma(1 - \bar{z})$ in $\Omega_2 = \{ z \mid |z - 1| < 1, |z - 1| < |z| \}$.

In addition, choose the analytical continuation of $\rho(z)$ as $\rho(z) = \rho(\frac{1}{z}) \vert z \vert^{-2}$ in $\Omega_3 = \mathbb{C} \setminus (\Omega_1 \cap \Omega_2)$. It is elementary that $\rho(z)$ is continuous in $\mathbb{C} \setminus \{0, 1\}$, and $\rho(z) \in C^2$ in $\Omega_1$, $\Omega_2$, $\Omega_3$ with the curvature equals -1. In order to explain that $\rho(z)$ is ultra-hyperbolic metric, we have to verify that $\rho(z)$ own support metrics in $\partial \Omega_1$ and $\partial \Omega_2$ everywhere. In fact, the support metrics in $\partial \Omega_1$ can be taken as $\sigma(z) \vert dz \vert$, because in the neighborhood of each point in $\partial \Omega_2$ we have $\rho(z) \geq \sigma(z)$.

The function $\zeta(z)$ own its the explicit expression

$$\zeta(z) = \frac{\sqrt{1 - \bar{z} - 1}}{\sqrt{1 - z + 1}}$$

with $\text{Re}(\sqrt{z - 1}) > 0$. Then the inequality we got just now can be written as

$$\rho_{0,1}(z) \geq \frac{1}{|z| |\sqrt{z - 1}|} \left[4 - \ln |\zeta(z)|\right]^{-1}$$

(3)

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(5)

Take $ds = \sigma^*(z) \vert dz \vert$ as the Poincare metric in domain $\{ z \mid 0 < |z| < \exp(C) \}$, in another words,

$$\sigma^*(z) = \frac{1}{|z| |C - \ln |z||$$

According to the definition,

$$\frac{\rho_{0,1}(z)}{\sigma^*(z)} \geq 1$$

when $|z| = 1$ is obviously. In another hand, due to (5) and the definition of $\sigma^*(z)$, we have

$$\lim_{z \to 0} \inf \frac{\rho_{0,1}(z)}{\sigma^*(z)} \geq 1$$

Then, $\rho_{0,1}(z) \geq \sigma^*(z), \forall z \in D(0, 1) \setminus \{0\}$. Of course namely:

$$\rho_{0,1}(z) \geq \frac{1}{|z| |C - \ln |z||}$$
Theorem 1 (Classical Schottky Inequality): If a function $f(z)$ is holomorphic in $|z| < 1$ and the solution is neither 0 nor 1, then

$$\ln |f(z)| \leq [C + \max \{\ln |f(0)|, 0\}] \frac{1 + |z|}{1 - |z|} - C$$

(6)

with $C = [\min \{\rho_{0,1}(z) \mid |z| = 1\}]^{-1}$ named Landau constant. $\rho_{0,1}(z)$ |dz| is the Poincare metric in $D(0,1) \setminus \{0\}$. The best numerical value of Landau constant was confirmed in later 1970s by J.A. Hempel[2]:

$$C = \frac{1}{4\pi^2} \Gamma^4 \left(\frac{1}{4}\right) = 4.3768796 \cdots$$

(7)

Proof: Due to the Generalized Schwarz Lemma, we have:

$$\rho_{0,1}(f(z)) \left| \frac{f'(z)}{f(z)} \right| \leq \frac{2}{1 - |z|^2}$$

(8)

Supposed that $z = re^{i\theta} \in D(0,1)$ and $|f(z)| < 1$. We consider the integral path:

$$\gamma : [0, r] \to C, t \mapsto w = f(te^{i\theta})$$

(9)

if $\gamma \subseteq D(0,1)$, then according to lemma 0, we got:

$$\frac{f'(te^{i\theta})}{f(te^{i\theta})} \left[ C - \ln |f(te^{i\theta})| \right]^{-1} \leq \frac{2}{1 - t^2}, t \in [0, r]$$

(10)

Then integral the inequality in both sides, and noticed that $|df| \geq d|f|$, we have:

$$\frac{C - \ln |f(z)|}{C' - \ln |f(0)|} \leq \frac{1 + |z|}{1 - |z|}$$

(11)

if $\gamma \not\subseteq D(0,1)$, we can choose a $r* \in (0, r)$, such that $|f(r*e^{i\theta})| = 1$ , and point $w = f(r*e^{i\theta})$ is the last point of $\gamma$ fall onto the unit circle. Integral the inequality form $r*$ to $r$, and got:

$$\frac{[C - \ln |f(z)|]}{C} \leq \frac{\frac{1+|z|}{1-|z|}}{\frac{1+|r'|}{1-|r'|}} \leq \frac{1 + |z|}{1 - |z|}$$

(12)
According to the both situation that we have listed, when \(|f(z)| < 1\), then
\[
C - \ln |f(z)| \leq \left[ C + \max \left\{ \ln \frac{1}{|f(0)|}, 0 \right\} \right] \frac{1 + |z|}{1 - |z|}
\]
Obviously the inequality is correct when \(|f(z)| \geq 1\), due to we can exchange \(f\) to \(1/f\) while \(|f(z)| \geq 1\), now we explain that the constant \(C\) cannot be modified. In fact, choose \(f_0 : D(0,1) \to \mathbb{C}\setminus \{0,1\}\) as a holomorphic covering mapping, such that \(f_0(0) = -1\), and \(f_0\) map the interval \((-1,1)\) into \((-\infty,0)\). Apply theorem 0 for \(f_0\), then divided both sides by \(|z|\), and take the limit \(|z| \to 0\) we finally got \(f_0'(0) \leq 2C\). However, we had already known that the equality holds in this situation (Landau theorem)[3]. Hence the \(C\) we choose is the best and cannot be modified.

Because classical Schottky inequality is a strong theorem, Picard little theorem and Montel theorem can be proved by it.

Schottky inequality holds great importance in quasi-conformal theory, quasi-regular theory[4], geometric function and composite dynamic analysis systems.

**Theorem 2** : If a function \(f(z)\) is holomorphic in \(|z| < R\) and the solution is neither 0 nor 1, then\([?]\) for \(|z| \leq \theta R\),
\[
|f(z)| \leq S_f = \exp \left( \pi e^{2|F(z)|} \right)
\]
In this equation, \(\theta \in (0,1)\),
\[
F(z) = \frac{1}{2} \ln \left[ 1 + 2 \sqrt{\ln f(z) / 2\pi i} \left( 1 - \frac{\ln f(z)}{2\pi i} \right) \right]
\]

**Theorem 3** : If a function \(f(z) : D(0,1) \to D(0,1)\) is holomorphic in \(|z| < R\) and the solution is not 0 or 1, then
\[
|f(z)| \leq S_f^{(k)} = \exp \left[ 2Ku \left( r' \right) - \frac{2}{K} u(r) \right] \prod_{n=0}^{\infty} \left[ 1 + \frac{\varphi(K)}{1 + \varphi(K)} \left( r_n \right) \right]^{2^{1-n}}
\]
In this equation, \(r = r_0 = |z|, r' = r'_0 = \sqrt{1 - r^2}, r_n = \frac{2\sqrt{r_{n-1}}}{1+r_{n-1}}, r'_n = \frac{2\sqrt{r_{n-1}}}{1+r_{n-1}}, K = \frac{1+r}{1-r}\),
\[
\kappa(r) = \int_0^\frac{\pi}{2} \frac{dt}{\sqrt{1 - t^2 \sin^2 t}}
\]
is the first type of complete elliptical integral while
\[ \varepsilon (r) = \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2 t} \, dt \tag{17} \]
is the second type;
\[ u (r) = \frac{\pi \kappa' (r)}{2\kappa (r)} \tag{18} \]
is the conformal modulus of the plane Grotzsch[1] ring \( D^2 (0, 1) \setminus [0, r] \); and
\[ \varphi_K (r) = u^{-1} \left( \frac{u (r)}{K} \right) \tag{19} \]
is the Hersch-Pfluger \( \varphi \)-discrepancy function.

**Theorem 4**: If a function \( f (z) : D (0, 1) \rightarrow D (0, 1) \) is a K-quasi-conformal mapping, \( 0 < \alpha < f (0) \leq \beta \) and \( 0 < \alpha < 1 < \beta \), then
\[ |z|^K \prod_{n=0}^\infty (1 + r_n)^{(1-K)2^{1-n}} \leq |f (z) - f (0)| \leq |z|^\frac{1}{K} \prod_{n=0}^\infty (1 + r_n)^{(1-\frac{1}{K})2^{1-n}} \tag{20} \]

## 2 Proof of theorem 2

Because the function \( f (z) \) does not take 0 and 1, it can be denoted as
\[ E = \{ z \mid |z| < R, F (z) = \pm \ln (\sqrt{n} - \sqrt{n-1}) + 2m\pi i, n \in N^+, m \in Z \} \]
\[ \forall \zeta \in \{ z \mid |z| < R \} \tag{21} \]
Suppose \( F' (z) \neq 0 \) and let \( F' (z) \neq 0 \), then it’s obvious that \( G (z) \) is holomorphic in \( |z - \zeta| \leq (1 - \varepsilon) (R - |\zeta|) \); hence
\[ G (z) = (z - \zeta) + \sum_{n=2}^\infty a_n (z - \zeta)^n \tag{22} \]
According to Bloch theorem, the value of \( G \) in \( |z - \zeta| \leq (1 - \varepsilon) (R - |\zeta|) \) covers a circle with a radius of \( B (1 - \varepsilon) (R - |\zeta|) \) on a plane. Then the results of \( F (z) \) in \( |z| < R \) will cover a circle with a radius of \( B (1 - \varepsilon) (R - |\zeta|) \left| F' (z) \right| \).
\[ \therefore F (z) \notin E \]
hence
\[ : B(1 - \varepsilon)(R - |\zeta|)|F'(z)| \leq d \]
which means
\[
\left| F'(z) \right| \leq \frac{d}{B_1(R - |\zeta|)} \tag{23}
\]
When \( F'(z) = 0 \), the inequality also holds true, and thus
\[
\forall \zeta \in \{ z \mid |z| < R \}
\]
\[
|F(\zeta) - F(0)| = \left| \int_0^\zeta F'(t) \, dt \right| \leq \frac{d}{B_1} \int_0^\zeta \frac{dr}{(R - r)} \tag{24}
\]
When \( |z| \leq \theta R \), \( |F(z)| \leq |F(0)| + \frac{d}{B_1} \ln \frac{1}{1 - \theta} \),
\[
|f(z)| = \left| \exp \left( \frac{\pi i}{2} e^{2|F(z)|} + e^{-2|F(z)|} \right) \right| \leq \exp \left( \pi e^{2|F(z)|} \right) = S_f \tag{25}
\]
When \( 0 < \alpha < f(0) \leq \beta \) and \( 0 < \alpha < 1 < \beta \),
\[
|F(0)| \leq \ln \left[ \sqrt{\frac{\ln |f(0)|}{2\pi}} + \frac{1}{2} - \sqrt{\frac{\ln |f(0)|}{2\pi}} + \frac{3}{2} \right] \tag{26}
\]
and
\[
|\ln |f(0)|| \leq \ln \beta - \ln \alpha \tag{27}
\]
then \( |f(z)|'s \) exact upper bound \( S_f(\alpha, \beta) \) can be obtained. The \( 0 < \alpha < 1 < \beta \) presupposed here can be generalized to other numbers. For if \( \alpha \geq 1 \), then \( \alpha \) can be replaced by \( \frac{1}{\alpha + 1} \); if \( \beta \leq 1 \), then \( \beta \) can be replaced by \( \beta + 1 \). Because \( \left| \frac{1}{f(z)} \right| \) also meets the requirement of Theorem 1, \( \left| \frac{1}{f(z)} \right| \leq S_1 \) can be obtained. Hence when \( |z| \leq \theta R \), the exact upper bound of \( |f(z)| \) is \( \frac{1}{S_1} \leq |f(z)| \leq S_f \).

3 Proof of Theorem 3

According to Theorem 2, the geometric significance of quasi-conformal mapping focus on whether the derivative \( f' \) of every point projects the circle to an oval with an eccentricity of \( K \).

The value of \( F(z) \) is dependent on the elliptical function of \( r \):
∀t ∈ (0, ∞), f (0) = t, suppose
\[ \frac{2KK' (r)}{K (r)} = 1 + 2 \sqrt{\frac{\ln f (z)}{2\pi i} \left( 1 - \frac{\ln f (z)}{2\pi i} \right)} \]  
(28)

and let \( r = \sqrt{\frac{t}{1 + t}} \), then
\[ |f (z)| \leq S_f^{(K)} = S (r) \exp \left[ 2Ku (r') - \frac{2}{K} u (r) \right] \]  
(29)

In this equation, \( S (r) \) is a positive number that is related to \( r \). Then this hypothesis can be tested: Suppose \( a \in (0, \frac{1}{2}] \); according to the Ramanujan constant theory,
\[ R (a) = - \left[ 2 \frac{\Gamma' (1)}{\Gamma (1)} + \frac{\Gamma' (a)}{\Gamma (a)} + \frac{\Gamma' (1 - a)}{\Gamma (1 - a)} \right] \]  
(30)

In this equation, \( -\frac{\Gamma' (1)}{\Gamma (1)} \) is an Euler constant
\[ \gamma = - \frac{\Gamma' (1)}{\Gamma (1)} = \lim_{n \to \infty} \left[ \sum_{k=0}^{\infty} \frac{1}{k} - \ln n \right] \]  
(31)

However, whether this number is an irrational number or not is still under argument. The present hypothesis argues that it is a number of transcendental number.

Let \( C = \frac{1}{4} e^\frac{R(a)}{2} \) and \( C' = \exp \left( a - \frac{1}{2} \right)^2 \), then the plane Grotzsch ring model can be derived is
\[ u_a (r) = \frac{\pi}{2 \sin (\pi a)} \frac{F (a, 1 - a, 1, r'^2)}{F (a, 1 - a, 1, r^2)} \]  
(32)

In this equation, \( F \) is a hypergeometric function.

**Lemma 2**: If \( C_1 = \frac{R(a) - \log 16}{2} \), \( C_2 = \frac{C_3}{\log 4} \), \( C_3 = \frac{(1 - 2a)^2}{(1 - a)x} \), \( C_4 = \exp C_1 \), \( C_5 = \exp C_2 \), \( C_6 = \frac{C_4}{C_1} \),
\[ A (r) = \frac{r'^2 \arctan r}{r} \]
\[ B (r) = r'^2 \log \frac{4}{r} \]

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\[ P(r) = \prod_{n=0}^{\infty} (1 + r_n)^2 \]

then:

1. \( C_2 B(r) < u_a(r) - u(r) < C_1 \)
2. \( C_1 A(r) < u_a(r) - u(r) < C_2 [1 - C_6 (1 - A(r))] \)
3. \( P(r) \max \left\{ C_4^A(r), C_5^B(r) \right\} \leq \exp (u_a(r) - \log r) \leq C_4 P(r) e^{-C_6 [1 - A(r)]} \)

Proof:

1. To check the monotonicity of \( G_1(r) = \frac{u_a(r) - u(r)}{B(r)} \) (33)

   to disassemble \( G_1(r) = G_3(r) G_4(r) \). In this equation,

   \[ G_3(r) = \frac{u_a(r) - u(r)}{\varepsilon(r) - 1} \] (34)

   and

   \[ G_4(r) = \frac{\varepsilon(r) - 1}{B(r)} \] (35)

   To prove \( G_3(r) \) is monotonically increasing, let \( G_5(r) = u_a(r) - u(r) \) and \( G_6(r) = \varepsilon(r) - 1 \); then

   \[ G_5(1) = G_6(1) = 0 \] (36)

   \[ \frac{G_5'(r)}{G_6'(r)} = \frac{\pi^2 [\kappa(r) + \kappa_a(r)] [\kappa(r) - \kappa_a(r)]}{4 [r^2 \kappa^2(r) \kappa_a^2(r)] [\kappa(r) - \varepsilon(r)]} \] (37)

   According to the monotonicity of elliptic integral, \( G_3(r) \) increase monotonically on \((0,1)\). \( G_3(0) = \frac{C_1}{2 - 1}, G_3(1^-) = \infty \); According to the same theorem, \( G_4(r) \) increase monotonically and thus, the monotonicity of \( G_1(r) \) can be acquired.

2. Let \( G_2(r) = \frac{C_1 - G_5(r)}{1 - A(r)} \), \( G_7(r) = C_1 - G_5(r) \), \( G_8(r) = 1 - A(r) \), then

   \[ G_2(r) = \frac{G_7(r)}{G_8(r)} \] (38)
\[G_7(r) = G_8(r) = 0 \quad (39)\]
\[
G_7'(r) = \pi^2 \left[ \kappa(r) + \kappa_a(r) \right] \left[ \kappa(r) - \kappa_a(r) \right] \\
G_8'(r) = \frac{\pi^2}{4} \left[ r^2 \kappa^2(r) \kappa_a^2(r) \right] [A(r) - 1] \quad (40)
\]

Hence the monotonicity of \(G_2(r)\) can also be obtained. Obviously \(G_2(1^-) = C_1\); according to the L'Hospital Theorem,
\[
\lim_{r \to 0} G_2(r) = \lim_{r \to 0} \frac{G_7(r)}{G_8(r)} = \frac{(1 - 2a)^2}{(1 - a) \pi} = C_3 \quad (41)
\]

3. According to [1], [2] and [15] When \(\forall a \in (0, \frac{1}{2})\) and \(r \in (0, 1)\,
\[C^{1-r^2} P(r) \leq \exp \left[ u_a(r) - \log r \right] \leq CC_1^{-r^2} P(r) \quad (42)\]
and this equality holds if and only if \(a = \frac{1}{2}\). From the equations above,
\[
\eta_K(t) = \exp \left\{ 2 \left[ u(s) - u(s') \right] + 2 \left[ u(s) + \log s \right] - 2 \left[ u(s') - \log s \right] \right\} \\
= \left[ \frac{P(s)}{P(s')} \right]^2 \exp \left\{ 2Ku \left( r' \right) - \frac{2u(r)}{K} \right\} \\
= S(r) \exp \left\{ 2Ku \left( r' \right) - \frac{2u(r)}{K} \right\} \quad (43)
\]

From [16], \(|f(z)| \leq \eta_K(t)|. Hence, theorem 3 is validated.

4 Proof of Theorem 4
\[s = \varphi_1^a(a, r) = u^{-1}_a(pu_a(r)) \quad (44)\]
\[f_K(r) = \varphi_K(a, r) r^\frac{1}{K} \quad (45)\]

Lemma 3 : When \(K > 1, f_K(r)\) decreases monotonically on \((0, 1)\)

Proof : From[17],
\[
\frac{\partial \varphi_K(a, r)}{\partial r} = \frac{s}{Kr} \left[ \frac{s' F(a, 1 - a, 1, s^2)}{r' F(a, 1 - a, 1, r^2)} \right]^2 \quad (46)
\]
\[
\frac{\partial \varphi_K (a,r)}{\partial K} = \frac{\pi}{2K \sin (\pi a)} s' \, 2 \, F (a,1-a,1,s^2) \, F \left( a,1-a,1,s'^2 \right)
\]

Let
\[ g_1 (z) = z' \, F (a,1-a,1,z^2) \]
then
\[ K \frac{f_K' (r)}{f_K (r)} = \left[ \frac{g_1 (s)}{g_1 (r)} \right]^2 - 1 \quad (47) \]

From [18], \( g_1 (z) \) decreases monotonically and the extreme points of \( f_K (r) \) can be obtained.

Lemma 3 is fully verified so far.

Let \( s' = \sqrt{1-s^2} = s_0, s_n = \frac{2\sqrt{s_{n-1}}}{1+s_{n-1}} \) and \( M (r) = A (s) A^{-m} (r) \) Because
\[ sr^{-m} = \exp \left[ (u_a (s) + \log s) - m (u_a (r) + \log r) \right] \]
\[ C^{1-m-s^2} C^{mr^2} M (r) \leq sr^{-m} \leq C^{1-m+mr^2} C_1^{-r^2} M (r) \]
Let \( a = \frac{1}{2} \), then
\[ (1-2a)^2 \max \left\{ \frac{A}{4}, \frac{1}{a} \right\} \leq 2C_1 \leq A \frac{(1-2a)^2}{8a} \quad (48) \]

In this equation, \( A = 14 \zeta (3) \) and \( \zeta (n) \) is a Riemann \( \zeta \)-function. Then
\[ \nu_K (r) = \left[ \frac{r}{P (r)} \right]^\frac{1}{4} \prod_{n=0}^\infty \left( 1 + \nu_{\frac{1}{n}} (r_n) \right)^{-n} \quad (49) \]

According to the quasi-conformal mapping theorem of Schwarz,
\[ |z|^K \, P \left( |z| \right)^{1-K} \leq |f (z) - f (0)| \leq |z|^\frac{1}{K} \, P \left( |z| \right)^{1-\frac{1}{K}} \quad (50) \]

Theorem 3 is fully tested so far.

5 An application of Theorem 3

Definition:
\[ \alpha = \arcsin \frac{|z_2 - z_1|}{|z_2 - z_0| + |z_1 - z_0|} \quad (51) \]
\[ \beta = \arcsin \frac{|f (z_2) - f (z_1)|}{|f (z_2) - f (z_0)| + |f (z_1) - f (z_0)|} \quad (52) \]
In these equations, \( z_0, z_1, z_2 \) are three different points. From the deduction process of Theorem 3 \( \sin \frac{\beta}{2} \leq \varphi_K \left( \frac{\alpha}{2} \right) \) can be obtained. Because

\[
\cos \frac{\beta}{2} = \sqrt{1 - \sin^2 \frac{\beta}{2}} \geq \sqrt{1 - \varphi_K^2 \left( \frac{\alpha}{2} \right)} = \varphi_{\frac{1}{K}} \left( \cos \frac{\alpha}{2} \right)
\]

can be acquired. Because

\[
\sin \beta = \frac{2 \tan \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}
\]
changes monotonically and

\[
\tan \frac{\beta}{2} \leq \frac{\varphi_K \left( \sin \frac{\alpha}{2} \right)}{\varphi_{\frac{1}{K}} \left( \sin \frac{\alpha}{2} \right)}
\]

\[
\sin \beta = \frac{2 \tan \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}} \leq 2 \frac{\varphi_1 \left( \sin \frac{\alpha}{2} \right) \varphi_{\frac{1}{K}} \left( \cos \frac{\alpha}{2} \right)}{\varphi_{\frac{1}{K}} \left( \sin \frac{\alpha}{2} \right) + \varphi_K \left( \cos \frac{\alpha}{2} \right)} = 2 \varphi_K \left( \sin \frac{\alpha}{2} \right) \varphi_{\frac{1}{K}} \left( \sin \frac{\alpha}{2} \right)
\]

at the same time we have

\[
\left( \sin \frac{\beta}{2} + \cos \frac{\beta}{2} \right)^2 = 1 + \sin \beta \leq 1 + 2 \varphi_K \left( \sin \frac{\alpha}{2} \right) \varphi_{\frac{1}{K}} \left( \sin \frac{\alpha}{2} \right) = \left[ \varphi_{\frac{1}{K}} \left( \sin \frac{\alpha}{2} \right) + \varphi_K \left( \cos \frac{\alpha}{2} \right) \right]^2
\]

(53)

Hence we have:

**Theorem 5** (the deviation theorem of Mori): The function \( f \left( z \right) : D \left( 0, 1 \right) \rightarrow D \left( 0, 1 \right) \) is a \( K \)-quasi-conformal mapping and \( \exists K_0 > 1, \alpha \geq \alpha_0 > 0 \). When \( K \geq K_0 \),

\[
\sin \beta \leq 2^{1 - \frac{1}{K}} \sin \frac{1}{\pi} \alpha
\]

(55)

and

\[
|f \left( z_2 \right) - f \left( z_1 \right)| \leq 16^{1 - \frac{1}{K}} \left| z_2 - z_1 \right|^\frac{1}{\pi}
\]

(56)

**Proof:** We know \( \sin \beta \leq 2 \varphi_K \left( \sin \frac{\alpha}{2} \right) \varphi_{\frac{1}{K}} \left( \sin \frac{\alpha}{2} \right) H \left( K, \alpha \right) \sin^\frac{1}{\pi} \alpha \) and in this equation,

\[
H \left( K, \alpha \right) = \frac{1}{\sin \frac{1}{\pi} \alpha} = \frac{1}{2 \pi \sin^\frac{1}{\pi} \left( \frac{\alpha}{2} \right) \cos^\frac{1}{\pi} \left( \frac{\alpha}{2} \right)}
\]

(57)
For $\sin \beta \leq 2^{1-\frac{1}{\pi}} \sin^{\frac{1}{\pi}} \alpha$ to be true, we first need to specify that $\exists K_0 > 1$ when $K \geq K_0$

\[
\frac{\varphi_K \left( \frac{\sin \alpha}{2} \right) \varphi^{\frac{1}{\pi}} \left( \frac{\sin \frac{\alpha}{2}}{2} \right)}{\sin^{\frac{1}{\pi}} \left( \frac{\alpha}{2} \right) \cos^{\frac{1}{\pi}} \left( \frac{\alpha}{2} \right)} \leq 1
\]  

(58)

From [22], [23],

\[
\varphi_K \left( \sin \frac{\alpha}{2} \right) \leq 4^{1-\frac{1}{\pi}} \sin^{\frac{1}{\pi}} \left( \frac{\alpha}{2} \right)
\]  

(59)

and

\[
\varphi^{\frac{1}{\pi}} \left( \sin \frac{\alpha}{2} \right) \leq \cos^{\frac{1}{\pi}} \left( \frac{\alpha}{2} \right)
\]  

(60)

both have limits and

\[
\frac{\varphi^{\frac{1}{\pi}} \left( \sin \frac{\alpha}{2} \right)}{\cos^{\frac{1}{\pi}} \left( \frac{\alpha}{2} \right)} \leq \cos^{\frac{1}{\pi}} \left( \frac{\alpha}{2} \right)
\]  

(61)

is an infinitesimal ($K \to \infty$). Hence $\exists K_0 > 1$ and when $K \geq K_0$, $\sin \beta \leq 2^{1-\frac{1}{\pi}} \sin^{\frac{1}{\pi}} \alpha$.

Then

\[
\frac{|f (z_2) - f (z_1)|}{|z_2 - z_1|^{\frac{1}{\pi}}} \leq 2^{1-\frac{1}{\pi}} \frac{|f (z_2) - f (z_1)|}{(|z_2| + |z_1|)^{\frac{1}{\pi}}}
\]  

(62)

According to Theorem 3, $|f (z)| \leq \varphi_K (|z|)$ Then through the method we used to derive extreme values we can know that,

\[
2^{1-\frac{1}{\pi}} \frac{|f (z_2) - f (z_1)|}{(|z_2| + |z_1|)^{\frac{1}{\pi}}}
\]

is maximized when $|z_2| = |z_1| = r < 1$:

\[
2^{1-\frac{1}{\pi}} \varphi_K (|z|) \leq 8^{1-\frac{1}{\pi}}
\]  

(63)

Hence $\exists K_0 > 1$, $\alpha \geq \alpha_0 > 0$ and when $K \geq K_0$,

\[
\frac{|f (z_2) - f (z_1)|}{|z_2 - z_1|^{\frac{1}{\pi}}} \leq 16^{1-\frac{1}{\pi}}
\]  

(64)

Theorem 5 is fully validated so far. Besides, from

\[
\varphi_K (r) \leq 4^{1-\frac{1}{\pi}} r^{\frac{1}{\pi}}
\]

and

\[
\sin \frac{\beta}{2} \leq \varphi_K \left( \sin \frac{\alpha}{2} \right) \leq \cos \frac{\beta}{2}
\]

we can know that

\[
\frac{|f (z_2) - f (z_1)|}{|z_2 - z_1|^{\frac{1}{\pi}}} \leq 64^{1-\frac{1}{\pi}}
\]  

(65)
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