Loop inequalities and confinement

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We consider correlation inequalities that follow from the well-known loop equations of LGT, and their analogues in spin systems. They provide a way of bounding long range by short or intermediate range correlations. In several cases the method easily reproduces results that otherwise require considerable effort to obtain. In particular, in the case of the 2-dimensional O(N) spin model, where large N analytical results are available, the absence of a phase transition and the exponential decay of correlations for all $\beta$ is easily demonstrated. We report on the possible application of this technique to the analogous 4-dimensional problem of area law for the Wilson loop in LGT at large $\beta$.

1. Loop equations in LGT

The loop equations of gauge theory, i.e. the SD equations for Wilson loops, are well-known both on the lattice and, at least formally, also in the continuum [1]. On the lattice:

$$= \beta \sum_\mu \biggl( - \beta \sum_\mu \biggr) \tag{1}$$

Here, and in the following, circles denote general but non-self-intersecting loops. (For self-intersecting loops, the l.h.s. contains additional terms involving multiple-loop expectations generating the infinite sequence of coupled SD loop equations. For a simple loop, however, one has the closed equation [1].) The deformations on the r.h.s. involve one plaquette protruding in direction $\mu$. Also: $\beta = 1/N g^2$ for $U(N)$, $\beta = N/[(N^2 - 1) g^2]$ for $SU(N)$.

2. Basic idea

In strong coupling expansion for $U(N)$ one finds for the ‘curly’ deformation term

$$\sum_\mu \biggl( - \beta \sum_\mu \biggr) \tag{2}$$

so the loop equation (1) gives

$$= \frac{\beta}{1 + \beta^2 (2d - 3) + \cdots} \sum_\mu \biggr) \tag{3}$$

Now one may iterate [3] any number of times up to the number of plaquettes in the minimal loop area $|A|$, and obtain [3]:

$$\leq \lambda^{|A|} \max_{\text{all loops}} \biggl( \biggr) \tag{4}$$

i.e. area law

$$\leq \text{Const. exp} \left[-(\ln \frac{1}{\lambda}) |A| \right] \tag{5}$$

provided $\lambda \equiv \frac{2(d - 1) \beta}{1 + \beta^2 (2d - 3)} < 1$, i.e. for $\beta < \frac{1}{(2d - 3)}$. Note that this estimate is comparable to estimates of the convergence radius of the strong coupling expansion obtained after rather more involved arguments.

Abstracting from the strong coupling case suggests the following general approach [3]. Assume that the ‘curly’ term satisfies

$$\sum_\mu \biggr( \biggr) \geq \alpha(\beta) \tag{6}$$
for some function $\alpha(\beta)$ (for sufficiently large loops). Then the loop equation (1) gives

$$\leq \frac{\beta}{1 + \beta\alpha(\beta)} \sum_{\mu} \alpha(\beta) \quad (7)$$

and area law follows by the above iteration argument, previously applied to (3), provided

$$\frac{\beta(2d - 1)}{1 + \beta\alpha(\beta)} < 1 \quad (8)$$

Attempts to extract area-law from the continuum loop equations have not been particularly successful. Very little seems to have been done with the lattice version. Before proceeding to our main case of interest, i.e. 4-dimensional LGT at large $\beta$, we illustrate this general approach in some simpler examples.

3. U(N) LGT in 2 dimensions

In this case we have the equality:

$$\sum_{\mu} \alpha(\beta) = \left[ \square + \square \right] \quad \equiv \alpha(\beta) \quad (9)$$

Substituting in the loop equation gives (1) as an equality, and, from (8), area law follows if $\alpha(\beta) > 2 - \frac{1}{\beta}$. Now, explicit computation in the large $N$ limit gives:

$$\alpha(\beta) = \beta, \quad \beta < \frac{1}{2}$$

$$= 2 - \frac{1}{\beta} + \frac{1}{4\beta(4\beta - 1)}, \quad \beta > \frac{1}{2}.$$  

So the above bound on $\alpha(\beta)$ is satisfied for all $0 \leq \beta < \infty$.

4. O(N) spin model in 2 dimensions

In the case of the spin model the analog of loop equation (1) is

$$\langle S_n \cdot S_m \rangle = \beta \sum_{\mu = \pm 1}^{\pm 2} \langle S_{n+\mu} \cdot S_m \rangle \quad (10)$$

where $S_n$ denotes an $N$-component unit length spin at site $n$. Note that the second term on the r.h.s corresponds to the ‘curly’ term and comes with a minus sign. We work in the large $N$ limit. To leading order in $1/N$:

$$\langle S_{n+\mu} \cdot S_n \rangle = \langle S_{n+\mu} \cdot S_n \rangle \langle S_n \cdot S_m \rangle \quad (11)$$

So (10) becomes

$$\langle S_n \cdot S_m \rangle = \frac{\beta}{1 + \beta\alpha(\beta)} \sum_{\mu = \pm 1}^{\pm 2} \langle S_{n+\mu} \cdot S_m \rangle \quad (12)$$

This is the analog of (1), and the same reasoning implies that $\langle S_n \cdot S_m \rangle$ decays exponentially as long as:

$$\alpha(\beta) \equiv \sum_{\mu = \pm 1}^{\pm 2} \langle S_{n+\mu} \cdot S_n \rangle > 4 - \frac{1}{\beta} \quad (13)$$

Now it is a well-known result [3] that to leading order

$$\alpha(\beta) = 4 - \frac{1}{\beta} + 2e^{-2\pi\beta} + \cdots, \quad \beta \to \infty \quad (14)$$

Hence condition (13) is fulfilled all the way to $\beta \to \infty$ proving the exponential decay of correlations and absence of a phase transition in the model. Note that this happens by virtue of the nonperturbative (exponential) term in (14) signaling the presence of a spin condensate [4]. The next to leading $1/N$ order is rather more involved [4] with technical complications of the sort one encounters in the 4-dim gauge theory case.

5. 4-dim. SU(N) gauge theory at large $\beta$

We begin by trivially rewriting the curly loop in the loop equation (1) as

$$A = (1 - \kappa) A + \kappa$$  

with $\kappa$ an arbitrary parameter to be adjusted later for optimizing bounds.
Inserting (11) in (10), and by a series of manipulations involving adding and subtracting appropriately chosen loop terms on the r.h.s. and use of reflection positivity, one may derive from (10) the inequality:

\[ \beta 2(d-1) \left( 1 - \frac{r(\kappa)}{\beta 2(d-1)} + \kappa \right) \]

where \( c = 1 - \kappa \), and \( r \) is a somewhat complicated expression involving differences of ratios of loops whose explicit form need not be given here. \( r = O(1) \) but is \( \kappa \)-dependent. In fact alternative versions of this type of inequality with somewhat different forms of \( r \) may be obtained. Writing for the two- and one-plaquette expectations in (16):

\[ \square = 1 - \bar{\Sigma}(g), \quad \square = 1 - \Sigma(g) \]

one has:

\[ [1 + c^2 \square - 2c \square^{1/2} ] \leq \beta \sum_{\mu} \left( 1 + \frac{1 - r(\kappa)}{\beta 2(d-1)} + \kappa \right) \]

(16)

Now expanding for small lattice spacing \( a \) and using OPE:

\[ (2\Sigma - \bar{\Sigma}) = -a^4 \left\langle g^2 F_{\mu \nu}(x)F^{\mu \nu}(x + a) \right\rangle + \ldots \]

\[ \sim C_1(a, \mu) \mathbf{1} + C_{F^2}(a, \mu) \left( g^2 \mathbf{F}^2(\mu) \right) + \ldots \]

with leading singular behavior:

\[ C_1(a, \mu) \sim a^{-4}, \quad C_{F^2} \sim a^0. \]

We now write the r.h.s. of (17) as:

\[ [1 + c^2 \square - 2c \square^{1/2} ] \equiv \left[ \mathcal{O}(\kappa) - a^4 \mathcal{O}_1 \right] \]

(18)

The \( \kappa \) independent terms proportional to \( a^4 \) on the r.h.s. of (18) are uniquely picked out, and arise from the gluon condensate:

\[ \mathcal{O}_1 \sim \text{const } \Lambda^4 \sim \text{const } \mu^4 e^{-2b_0/g^2(\mu)} \]

(19)

After some manipulation the expression multiplying the loop on the l.h.s. in (16) can be written as

\[ \beta 2(d-1) \left[ 1 + \frac{1}{2} a^2 \mathcal{O}_1^{1/2} + f(g^2, \mu, a, \kappa) \right], \]

(20)

where \( f \) is given in terms of \( \Sigma, \bar{\Sigma} \) and the other quantities entering in \( r(\kappa) \). It involves a series in \( g^2 \) ‘perturbative’ part. Note that all \( \mu \) dependence, other than in RG invariant combination (as in \( \Lambda \)), arising through the use of OPE must, in principle, cancel among the various terms in \( f \). (10) is optimized by choosing \( \kappa \) to maximize \( f \). Now since the inequality is rigorously valid, \( f \) must be negative for all \( \kappa \) not equal to the maximizing \( \kappa_{\text{max}} \), thus rendering the iteration argument inapplicable. Otherwise, as it is easily seen, unphysical loop behavior would result. For \( \kappa_{\text{max}} \), the optimum that could be achieved is that \( f \) vanishes (to within powers of \( a \Lambda \)). In such a case (10) would give:

\[ \leq \frac{1}{2(d-1)} \left[ 1 + \frac{1}{2} a^2 \mathcal{O}_1^{1/2} \right] \sum_{\mu} \left( 1 - \frac{r(\kappa)}{\beta 2(d-1)} + \kappa \right) \]

which may be iterated as above to give area law with string tension \( \sim \text{const } \Lambda^2 \).

Now \( f \) at \( \kappa_{\text{max}} \) depends rather delicately on the structure of \( r \) in (10), and is, unfortunately, not easily estimated accurately enough to ascertain whether it vanishes. This is of course only to be expected. The inequality (10) must clearly be very sharp in order to obtain the right behavior. As noted there are different versions of it with somewhat different forms of \( r \) that may be explored, as well as possible ways of successive refinement of a given form. These are currently under investigation.

REFERENCES

1. D. Foerster, Phys. Lett. 87B (1979) 87; T. Eguchi, Phys. Lett. 87B (1979) 91; Y.M. Maakeenko and A.A. Migdal, Phys. Lett. 88B (1979) 135; A.M. Polyakov, Phys. Lett. 82B (1979) 247.
2. E.T. Tomboulis, A. Ukawa, and P. Windey, Nucl. Phys. B180 [FS2] (1981) 294.
3. T.H. Berlin and M. Kac, Phys. Rev. 86 (1952) 821.
4. V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B249 (1985) 445.