Revisiting Offspring Maxima in Branching Processes

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Abstract
We present a progress report for studies on maxima related to offspring in branching processes. We summarize and discuss the findings on the subject that appeared in the last ten years. Some of the results are refined and illustrated with new examples.

1 Introduction
There is a significant amount of research in the theory of branching processes devoted to extreme value problems concerning different population characteristics. The history of such studies goes back to the works in 50-ies by Zolotarev [26] and Urbanik [23] (see also [6]) who considered the maximum generation size. Our goal here is to summarize and discuss results on maxima related to the offspring. Papers directly addressing this area of study have begun to appear in the last ten years (though see "hero mothers" example in [7].)

Let $M_n$ denote the maximum offspring size of all individuals living in the $(n-1)$-st generation of a branching process. This is a maximum of random number of independent and identically distributed (i.i.d.) integer-valued random variables, where the random index is the population size of the process. $M_n$ has two characteristic features: (i) the i.i.d. random variables are integer-valued and (ii) the distribution of the random index is connected to the distribution of the terms involved through the branching mechanism. These two characteristics distinguish the subject matter maxima among those studied in the general extreme value theory.

The study of the sequence $\{M_n\}$ might be motivated in different ways. It provides a fertility measure characterizing the most prolific individual in one generation. It also measures the maximum litter (or family) size. In the branching tree context, it is the maximum degree of a vertex. The asymptotic behavior of $M_n$ gives us some information about the influence of the largest families on the size and survival of the entire population.
The paper is organized as follows. Next section deals with maxima in simple branching processes with or without immigration. In Section 3 we derive results about maxima of a triangular array of zero-inflated geometric variables. Later we apply these to branching processes with varying geometric environments. Section 4 begins with limit theorems for the max-domain of attraction of bivariate geometric variables. Then we discuss one application to branching processes with promiscuous matting. The final section considers a different construction in which a random score (a continuous random variable) is associated with each individual in a simple branching process. We present briefly limiting results for the score’s order statistics. In the end of the section, we give an extension to two-type processes.

2 Maximum family size in simple branching processes

Define a Bienaymé–Galton–Watson (BGW) branching process and its \( n \)-th generation maximum family size by
\[
Z_0 = 1; \quad Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n) \quad \text{and} \quad M_n = \max_{1 \leq i \leq Z_{n-1}} X_i(n) \quad (n = 1, 2, \ldots),
\]
respectively, where the offspring variables \( X_i(n) \) are i.i.d. nonnegative and integer-valued.

Along with the BGW process \( \{Z_n\} \), we consider the process with immigration \( \{Z_{n}^{im}\} \) and its offspring maximum
\[
Z_{n}^{im} = \sum_{i=1}^{Z_{n-1}^{im}} X_i(n) + Y_n \quad \text{and} \quad M_{n}^{im} = \max_{1 \leq i \leq Z_{n-1}^{im}} X_i(n) \quad (n = 1, 2, \ldots),
\]
respectively, where \( \{Y_n, n = 1, 2, \ldots\} \) are independent of the offspring variables, i.i.d. and integer-valued non-negative random variables.

Finally, let us modify the immigration component such that immigrants may enter the \( n \)-th generation only if the \((n-1)\)-st generation size is zero. Thus, we have the Foster-Pakes process and its offspring maximum
\[
Z_n^0 = \sum_{i=1}^{Z_{n-1}^0} X_i(n) + I_{Z_{n-1}^0 = 0} Y_n \quad \text{and} \quad M_n^0 = \max_{1 \leq i \leq Z_{n-1}^0} X_i(n) \quad (n = 1, 2, \ldots),
\]
where \( I_A \) stands for the indicator of \( A \).

Denote by \( F(x) = P(X_i(n) \leq x) \) the common distribution function of the offspring variables with mean \( 0 < m < \infty \) and variance \( 0 < \sigma^2 \leq \infty \). In this section, we deal with the subcritical \((m < 1)\), critical \((m = 1)\), and supercritical \((m > 1)\) processes separately.

2.1 Subcritical processes

Let \( M_n \) denote the maximum family size in all three processes defined above: \( \{Z_n\} \), \( \{Z_{n}^{im}\} \), and \( \{Z_n^0\} \). Let \( g(s) \) be the immigration p.g.f.. Also, let \( A_n = \{Z_{n-1} > 0\} \) for
processes without immigration, and \( A_n \) be the certain event - otherwise. The following result is true.

**Theorem 1** If \( 0 < m < 1 \), then for \( x \geq 0 \)

\[
\lim_{n \to \infty} P(M_n \leq x | A_n) = \gamma(F(x))
\]  

(1)

and

\[
\lim_{n \to \infty} E(M_n | A_n) = \sum_{k=0}^{\infty} (1 - \gamma(F(k))]
\]  

(2)

where

(i) in case of \( \{Z_n\} \), \( \gamma \) is the unique p.g.f. solution of \( \gamma(f(s)) = m\gamma(s) + 1 - m \) and \( \gamma \) holds if, in addition, \( EX_i(n)\log(1 + X_i(n)) < \infty \).

(ii) in case of process \( \{Z_n^m\} \), \( \gamma \) holds provided \( E\log(1 + Y_n) < \infty \) and \( \gamma \) is the unique p.g.f. solution of \( \gamma(g(s)) = g(s)\gamma(f(s)) \). \( \gamma \) is true if, in addition, \( EY_n < \infty \).

(iii) in case of process \( \{Z_n^m\} \) we assume that \( E\log(1 + Y_n) < \infty \). Then \( \gamma(s) = 1 - \sum_{n=0}^{\infty} [1 - g(f_n(s))] \) \((0 < s \leq 1)\) and \( \gamma(0) = \{1 + \sum_{n=0}^{\infty} [1 - g(f_n(0))]\}^{-1} \).

Also, \( \gamma \) holds if, in addition, \( EY_n < \infty \).

**Example 1** Consider \( \{Z_n\} \) with geometric offspring p.g.f. \( f(s) = p/(1 - qs) \), where \( 1/2 < p = 1 - q < 1 \). Then \( m = q/p < 1 \) and it is not difficult to see that \( \gamma(s) = (1 - m)s/(1 - ms) \). Hence

\[
\lim_{n \to \infty} P(M_n \leq k \mid Z_{n-1} > 0) = \frac{(p - q)(1 - q^{k+1})}{p - q(1 - q^{k+1})}.
\]

It can also be seen \([20]\) that

\[
\frac{m}{1 - pm} \leq \lim_{n \to \infty} E(M_n|Z_n > 0) \leq \frac{m}{1 - m}.
\]

**Example 2** Consider \( \{Z_n^m\} \) \([15]\) with

\[
f(s) = (1 + m - ms)^{-1} \quad (0 < m < 1) \quad \text{and} \quad g(s) = f^\nu(s) \quad (\nu > 0).
\]

Then \( \gamma(s) = ((1 - m)/(1 - ms))^\nu \), a negative binomial p.g.f., and the above theorem yields

\[
\lim_{n \to \infty} P(M_n^m \leq x) = \left( \frac{1 - m}{1 - mF(x)} \right)^\nu \quad \text{and} \quad \lim_{n \to \infty} E(M_n^m) = \sum_{j=0}^{\infty} \left[ 1 - \frac{1 - F(j)}{1 - mF(j)} \right]^\nu \leq \frac{\nu m^2}{1 - m}.
\]

**Example 3** Let \( \mu = EY_n \). Consider \( \{Z_n^0\} \) with

\[
f(s) = (1 + m - ms)^{-1} \quad \text{and} \quad g(s) = 1 - (\mu/m)\log(1 + m - ms) \quad (0 < m < 1).
\]

In this case \( \gamma = (m - \mu \log(1 - ms))/(m - \mu \log(1 - m)) \) and by the theorem

\[
\lim_{n \to \infty} P(M_n^0 \leq x) = \frac{m - \mu \log(1 - mF(x))}{m - \mu \log(1 - m)},
\]
and
\[
\lim_{n \to \infty} E\mathcal{M}_n^0 = \mu \frac{m + \sum_{k=0}^{\infty} \log \frac{1 - m[(1 + m)^{k+1} - m^{k+1}]}{1 - m}}{m - \mu \log(1 - m)} \leq \frac{\mu m}{m - \mu \log(1 - m)} \frac{m}{1 - m}.
\]

### 2.2 Critical processes

In the rest of this section we need some asymptotic results for the maxima of i.i.d. random variables. Recall that a distribution function \( F(x) \) belongs to the max-domain of attraction of a distribution function \( H(x, \theta) \) (i.e., \( F \in D(H) \)) if and only if there exist sequences \( a(n) > 0 \) and \( b(n) \) such that
\[
\lim_{n \to \infty} F_n(a(n)x + b(n)) = H(x, \theta),
\]
weakly. According to the classical Gnedenko’s result, \( H(x; \theta) \) has the following (von Mises) form
\[
H(x; \theta) = \exp\{-h(x; \theta)\} = \exp\{-1 + x\theta^{-1} - \theta^2\}, \quad 1 + x\theta^{-1} > 0; \quad -\infty < \theta < \infty.
\]

Necessary and sufficient conditions for \( F \in D(H) \) are well-known. In particular, \( F \in D(\exp\{-x^{-a}\}), \ a > 0 \) if and only if for \( x > 0 \) the following regularity condition on the tail probability holds
\[
1 - F(x) = x^{-a}L(x),
\]
where \( L(x) \) is a slowly varying at infinity function (s.v.f.).

**A. Processes without immigration.** In case of a simple BGW process, the following result holds.

**Theorem 2** Let \( m = 1 \) and \( \sigma^2 < \infty \). (i) If (3) holds, then
\[
\lim_{n \to \infty} P\left(\frac{\mathcal{M}_n - b(n)}{a(n)} \leq x | Z_{n-1} > 0\right) = \frac{1}{1 + \sigma^2 h(x, \theta)/2}.
\]

(ii) If (3) holds, then
\[
\lim_{n \to \infty} \frac{E(\mathcal{M}_n | Z_{n-1} > 0)}{n^{1/a}L_1(n)} = \frac{\pi/a}{\sin(\pi/a)} \quad (a \geq 2),
\]
where \( L_1(x) \) is certain s.v.f. with known asymptotics.

The theorem implies that if \( F \in D(\exp\{-e^{-x}\}) \) then the limiting distribution is logistic with c.d.f. \( (1 + e^{-x})^{-1} \); and if \( F \in D(\exp\{-x^{-a}\}) \) then the limiting distribution is log-logistic with c.d.f. \( (1 + x^{-a})^{-1} \).
**Theorem 3** Let $m = 1$, $\sigma^2 = \infty$, and (4) holds. Then for $x \geq 0$ and $1 < a \leq 2$

$$
\lim_{n \to \infty} P \left( \frac{M_n}{n^{1/|a(a-1)|}} L_2(n) \leq x \mid Z_{n-1} > 0 \right) = 1 - \frac{1}{(1 + x^a(a-1))^{1/(a-1)}},
$$

(8)

which is a Burr Type XII distribution (e.g. [22]) and

$$
\lim_{n \to \infty} \frac{E(M_n | Z_{n-1} > 0)}{n^{1/|a(a-1)|} L_2(n)} = \frac{1}{a - 1} B \left( \frac{1}{a - 1} - \frac{1}{a(a-1)}, 1 + \frac{1}{a(a-1)} \right) \quad (1 < a \leq 2),
$$

(9)

where $B(u, v)$ is the Beta function and $L_2(x)$ is certain s.v.f. with known asymptotics.

Note that for $a = 2$ the right-hand sides in (6) (under assumption (5)) and (7) coincide with those in (8) and (9), respectively. The right-hand side in (9) is the expected value of the limit in (8) (see [22]).

**Example 4** Let $1 - F(x) \sim x^{-2} \log x$. In this case one can check (see [20]) that Theorem 3 with $a = 2$ implies

$$
\lim_{n \to \infty} P \left( \frac{M_n}{n^{1/2} (\log n)^{3/2}} \leq x \mid Z_{n-1} > 0 \right) = \frac{4x^2}{1 + 4x^2}.
$$

for $x \geq 0$ and

$$
\lim_{n \to \infty} \frac{E(M_n | Z_{n-1} > 0)}{n^{1/2} (\log n)^{3/2}} = \frac{\pi}{2}.
$$

**B. Processes with immigration** $\{Z_{n}^{im}\}$. Let $\mu = EY_n$. We have the following theorem.

**Theorem 4** Assume that $m = 1$, $0 < \sigma^2 < \infty$, and $0 < \mu < \infty$. (i) If (3) holds, then

$$
\lim_{n \to \infty} P \left( \frac{M_{n}^{im} - b(n)}{a(n)} \leq x \right) = \frac{1}{(1 + \sigma^2 h(x, \theta)/2)^{2\mu/\sigma^2}}.
$$

(10)

(ii) If (5) is true, then

$$
\lim_{n \to \infty} \frac{E(M_{n}^{im})}{n^{1/2} L_2(n)} = \frac{2\mu}{\sigma^2} B \left( \frac{2\mu}{\sigma^2} + 1 - \frac{1}{a} \right) \quad (a \geq 2),
$$

(11)

where $B(u, v)$ is the Beta function and $L_2(x)$ is certain s.v.f. with known asymptotics.

The theorem implies that if $F \in D(\exp\{-e^{-x}\})$ then the limiting distribution is generalized logistic with c.d.f. $(1 + \sigma^2 e^{-x}/2)^{-2\mu/\sigma^2}$; if $F \in D(\exp\{-x^{-a}\})$ then the limiting distribution is a Burr Type III (e.g. [22]) with c.d.f. $(1 + \sigma^2 x^{-a}/2)^{-2\mu/\sigma^2}$. The right-hand side in (11) is the expected value of the limit in (10) (see [22]).
Theorem 5  Let \( m = 1, \sigma^2 = \infty, \) and (4) holds. In addition, suppose

\[
\Theta(x) := -\int_0^x \log[1 - P(Z_t^{im} > 0)]dt = c \log x + d + \varepsilon(x),
\]  

where \( \lim_{x \to \infty} \varepsilon(x) = 0, \) \( c > 0, \) and \( d \) are constants. Then for \( x \geq 0, \)

\[
\lim_{n \to \infty} P\left( \frac{M_{im}^n}{n^{1/(a(a-1))}L_2(n)} \leq x \right) = \frac{1}{(1 + x^{-a(a-1)})^c} (1 < a \leq 2),
\]

which is a Burr Type III distribution (e.g. [22]) and

\[
\lim_{n \to \infty} \frac{EM_{im}^n}{n^{1/(a(a-1))}L_2(n)} = cB \left( c + \frac{1}{a(a-1)}, 1 - \frac{1}{a(a-1)} \right) (1 < a \leq 2),
\]

where \( B(u,v) \) is the Beta function and \( L_2(x) \) is certain s.v.f. with known asymptotics. The right-hand side in (14) is the expected value of the limit in (13).

Note that for \( c = 1 \) and \( a = 2 \) the right-hand sides in (13) and (14) coincide with those in (8) and (9), respectively. The condition (12) holds even when the immigration mean is not finite. Next example illustrates this point.

Example 5  Following [16], we consider offspring and immigrants generated by

\[
f(s) = 1 - (1 - s)(1 + (a - 1)(1 - s))^{-1/(a-1)} \text{ and } g(s) = \exp\{-\lambda(1 - s)^{a-1}\},
\]

respectively. Then (5) holds and (12) yields

\[
\Theta(t) = \left( \frac{\lambda}{a-1} \right) \left[ \log t + \log(a-1) + \log(1 + (a-1)t)^{-1} \right].
\]

Therefore,

\[
\lim_{n \to \infty} P\left( \frac{M_{im}^n}{n^{1/(a(a-1))}L_3(n)} \leq x \right) = \frac{1}{(1 + x^{-a(a-1)})^{\lambda/(a-1)}} (1 < a \leq 2),
\]

and

\[
\lim_{n \to \infty} \frac{EM_{im}^n}{n^{1/(a(a-1))}L_3(n)} = \frac{\lambda}{a-1} B \left( \frac{\lambda}{a-1} - \frac{1}{a(a-1)}, 1 - \frac{1}{a(a-1)} \right) (1 < a \leq 2),
\]

where \( B(u,v) \) is the Beta function and \( L_3(x) \) is certain s.v.f. with known asymptotics.

C. Foster-Pakes processes \( \{Z_n^0\} \). The following limit theorem for \( M_0^n \) under a non-linear normalization holds.
Theorem 6 Assume that \( m = 1, \ 0 < \sigma^2 < \infty, \) and \( 0 < \mu < \infty. \) If
\[
\lim_{n \to \infty} \frac{P(X_1(1) > n)}{P(X_1(1) > n + 1)} = 1
\] (15)
then for \( 0 < x < 1, \)
\[
\lim_{n \to \infty} P \left( \frac{\log U(M_0^n)}{\log n} \leq x \right) = x,
\] (16)
where \( U(y) = \frac{1}{1 - F(y)}. \)

Note that (15) is a necessary condition for \( X_1(n) \) to be in a max-domain of attraction.

2.3 Supercritical processes
Denote by \( \hat{M}_n \) (as in the subcritical case above) the maximum family size in all three processes: \( \{Z_n\}, \{Z_{im}^n\}, \) and \( \{Z_0^n\}. \) The following result is true.

Theorem 7 Assume that \( m > 1 \) and \( EX_i(n) \log(1 + X_i(n)) < \infty. \) If (14) holds, then
\[
\lim_{n \to \infty} P \left( \frac{\hat{M}_n - b(m^n)}{a(m^n)} \leq x \right) = \psi(h(x, \theta))
\]
If (14) is true, then
\[
\lim_{n \to \infty} \frac{E \hat{M}_n}{m^{-n/a}L_1(m^{-n/a})} = \int_0^\infty 1 - \psi(x^{-a})dx,
\]
where \( L_1(x) \) is certain s.v.f. with known asymptotics.

(i) in case of \( \{Z_n\}, \) \( \psi \) is the unique, among the Laplace transforms, solution of
\[
\psi(u) = f(\psi(um^{-1})), \quad (u > 0).
\] (17)

(ii) in case of \( \{Z_{im}^n\}, \) we assume in addition that \( E \log(1 + Y_n) < \infty \) and
\[
\psi(u) = \prod_{k=1}^{\infty} g(\varphi(um^{-1})) \quad (u > 0),
\]
where \( \varphi(u) \) is the unique, among the Laplace transforms, solution of (14).

(iii) in case of \( \{Z_0^n\}, \) we assume in addition that \( EY_n < \infty \) and
\[
\psi(u) = g(\varphi(u)) - \sum_{n=0}^{\infty} [1 - f(\varphi(um^{-n}))]P(Z_0^n = 0) \quad (u > 0)
\]
and \( \varphi(u) \) is the unique, among the Laplace transforms, solution of (17).
It is interesting to compare the limiting behavior of the maximum family size in the processes allowing immigration with that when the processes evolve in "isolation", i.e., without immigration. In the supercritical case, as might be expected, the immigration has little effect on the asymptotics of the maximum family size. The limits differ only in the form of the Laplace transform $\psi(u)$. In the subcritical and critical cases the mechanism of immigration eliminates the conditioning on non–extinction. Theorem 6 for the Foster-Pakes process differs from the rest of the results by the non-linear norming of $M_n$. The study of the limiting behavior of the expectation in this case needs additional efforts.

It is known that some of the most popular discrete distributions, like geometric and Poisson, do not belong to any max-domain of attraction. This restricts the applicability of the results in the critical and supercritical cases above. A general construction of discrete distributions attracted in a max-domain is given in Wilms (1994). As it is proved there, if $X$ is attracted by a Gumbel or Fréchet distributions, then the same holds for the integer part $\lfloor X \rfloor$. Next we follow a different approach considering triangular arrays of geometric variables which leads to branching processes with varying environments.

The results in this section are published in [9], [12], and [13]-[20]. In [25] an extension for order statistics is considered.

3 Maximum family size in processes with varying environments

It is well-known that the geometric law is not attracted to any max-stable law. Therefore, the limit theorems for maxima in the critical and supercritical cases above do not apply to geometric offspring. In this section we utilize a triangular array of zero-modified geometric (ZMG) offspring distributions, instead.

3.1 Maxima of arrays of zero-modified geometric variables

In this subsection we prove limit theorems for maximum of ZMG with p.m.f.

$$P(X_1(n) = j) = \begin{cases} a_n p_n (1 - p_n)^{j-1} & \text{if } j \geq 1, \\ 1 - a_n & \text{if } j = 0, \end{cases} \quad (n = 1, 2, \ldots)$$

For a positive integer $\nu_n$ consider the triangular array of variables

$$X_1(1), X_2(1), \ldots, X_{\nu_1}(1)$$

$$X_1(2), X_2(2), \ldots, X_{\nu_2}(2)$$

\[ \vdots \]

$$X_1(n), X_2(n), \ldots, X_{\nu_n}(n)$$

We prove limit theorems as $\nu_n \to \infty$ for the row maxima

$$M_n = \max_{1 \leq i \leq \nu_n} X_i(n).$$
Let $\Lambda$ has the standard Gumbel law with c.d.f. $\exp(-e^{-x})$ for $-\infty < x < \infty$.

**Theorem 8** Assume that for some real $c$

$$\lim_{n \to \infty} p_n = 0 \quad \text{and} \quad \lim_{n \to \infty} p_n \log(\nu_n a_n) = 2c.$$

A. If $\lim_{n \to \infty} \log(\nu_n a_n) = \infty$, then $c \geq 0$ and

$$p_n M_n - \log(\nu_n a_n) \xrightarrow{d} \Lambda - c.$$

B. If $\lim_{n \to \infty} \log(\nu_n a_n) = \alpha$, $(-\infty < \alpha < \infty)$, then

$$p_n M_n \xrightarrow{d} (\Lambda + \alpha)^+. $$

The idea of the proof is to exploit: (i) the exponential approximation to the zero-modified geometric law when its mean $a_n/p_n$ is large; (ii) the fact that exponential law is attracted by Gumbel distribution.

### 3.2 Processes with varying geometric environments

Consider a branching process with ZMG offspring law defined over the triangular array above. Thus, we have a simple branching process with geometric varying environments. For this process we prove limit theorems for the offspring maxima in all three classes: subcritical, critical, and supercritical. Define $\mu_0 = 1$,

$$\mu_n = E(Z_n|Z_0 = 1) = \prod_{j=1}^{n} m_j \quad (n \geq 1).$$

If the environments are weakly varying, i.e., $\mu = \lim_{n \to \infty} \mu_n$ exists, then the processes can be classify (see [11]) as follows.

$\{Z_n\}$ is

\[
\begin{cases} 
\text{supercritical} & \text{if } \mu = \infty \quad \text{i.e. } \sum_n (m_n - 1) \to \infty \\
\text{critical} & \text{if } \mu \in (0, \infty) \quad \text{i.e. } \sum_n (m_n - 1) < \infty \\
\text{subcritical} & \text{if } \mu = 0 \quad \text{i.e. } \sum_n (m_n - 1) \to -\infty
\end{cases}
\]

Define the maximum family size for the process with varying geometric environments as

$$M_n^{ge} = \max_{1 \leq i \leq Z_n} X_i(n), \quad (n = 1, 2, \ldots)$$

In the result below the role played by $\nu_n$ before is played by $B_{n-1}$ where

$$B_n = \mu_n \sum_{j=1}^{n} \frac{p_j^{-1} - 1}{\mu_j}.$$ 

Let $V$ be a standard logistic random variable with c.d.f. $(1 + e^{-x})^{-1}$ for $-\infty < x < \infty$. 

9
Theorem 9 Suppose that \( \lim_{n \to \infty} B_n = \infty \) and for \( c \) real

\[
\lim_{n \to \infty} p_n = 0 \quad \text{and} \quad \lim_{n \to \infty} p_n \log(B_n - 1/a_n) = 2c.
\]

A. If \( \lim_{n \to \infty} \log(B_n - 1/a_n) = \infty \), then

\[
(p_n \mathcal{M}_{n}^{ge} - \log(B_n - 1/a_n) | Z_{n-1} > 0) \overset{d}{\to} \mathcal{V} - c.
\]

B. If \( \lim_{n \to \infty} \log(B_n - 1/a_n) = \alpha, \ (-\infty < \alpha < \infty) \), then

\[
(p_n \mathcal{M}_{n}^{ge} | Z_{n-1} > 0) \overset{d}{\to} (\mathcal{V} + \alpha)^{+}.
\]

Referring to the above theorem, we can say that the branching mechanism transforms Gumbel to logistic distribution. It is interesting to notice that this is in parallel with results for maximum of i.i.d. random variables with random geometrically distributed index discussed in [5].

Example 6 Let us sample a linear birth and death process \((B_t)\) at irregular times. Let \( Z_n = B_{t_n} \) where \( 0 < t_n < t_{n+1} \to t_\infty \leq \infty \). If \( \lambda \) and \( \mu \) are the birth and death rates, respectively, and \( a_n = t_n - t_{n-1} \), then \( a_n = m_n p_n \).

\[
p_n = \begin{cases} \frac{l - \mu}{\lambda m_n - \mu} & \text{if } \lambda \neq \mu, \\ \frac{l}{1 + \lambda d_n} & \text{if } l = \mu, \end{cases} \quad m_n = e^{(\lambda - \mu) t_n}.
\]

and

\[
B_n = \begin{cases} \frac{\lambda(\mu_n - 1)}{\lambda - \mu} & \text{if } \lambda \neq \mu, \\ \frac{\lambda}{l_n} & \text{if } \lambda = \mu, \end{cases} \quad \mu_n = e^{(\lambda - \mu) t_n}.
\]

A. If \( \lambda > \mu \) and

\[
\lim_{n \to \infty} \frac{t_n}{m_n} = \frac{2c}{\lambda - \mu} \in [0, \infty),
\]

then

\[
\left( \frac{\mathcal{M}_{n}^{ge}}{m_n} - (\lambda - \mu) t_n \mid Z_{n-1} > 0 \right) \overset{d}{\to} \mathcal{V} - c.
\]

B. If \( \lambda = \mu \) and \( t_n = n^{\delta} l(n) \) \((\delta \geq 1)\), then

\[
\left( \frac{\mathcal{M}_{n}^{ge}}{\lambda d_n^{\delta} l(n)} - \log n \mid Z_{n-1} > 0 \right) \overset{d}{\to} \mathcal{V}.
\]

The results in this section can be found in [13].

4 Maxima in bisexual processes

In this section we consider maxima of triangular arrays of bivariate geometric random vectors. The obtained results are applied to a class of bisexual branching processes.
Theorem 10

Let $\{U,V\}$ having Bernoulli marginals, i.e., it takes on four possible values $(0,0), (0,1), (1,0),$ and $(1,1)$ with probabilities $p_{00}, p_{01}, p_{10},$ and $p_{11},$ respectively. Thus the marginal probabilities for $U$ and $V$ are

$$P(U = 0) = p_{0+} = p_{00} + p_{01}, \quad P(U = 1) = p_{1+} = p_{10} + p_{11}$$
$$P(V = 0) = p_{+0} = p_{00} + p_{10}, \quad P(V = 1) = p{+1} = p_{01} + p_{11}. $$

Consider a sequence $\{(U_n, V_n)\}_{n=1}^\infty$ of independent and identically distributed with $\{(U, V)\}$ random vectors. Let $\xi$ and $\eta$ be the number of zeros preceding the first 1 in the sequences $(U_n)_{n=1}^\infty$ and $(V_n)_{n=1}^\infty$, respectively. Both $\xi$ and $\eta$ follow a geometric distribution and, in general, they are dependent variables. The vector $(\xi, \eta)$ has a bivariate geometric distribution with probability mass function for integer $l$ and $k$

$$P(\xi = l, \eta = k) = \begin{cases} p_{00}p_{10}p_{k-1}^{l-1}p_{+1} & \text{if } 0 \leq l < k, \\ p_{00}p_{11} & \text{if } l = k, \\ p_{00}p_{01}p_{k-1}^{l-1}p_{1+} & \text{if } 0 \leq k < l. \end{cases} \quad (18)$$

and

$$P(\xi > l, \eta > k) = \begin{cases} p_{00}p_{10}p_{l+1}^{k} & \text{if } 0 \leq l \leq k, \\ p_{00}p_{11} & \text{if } 0 \leq k < l. \end{cases} \quad (19)$$

The marginals of $\xi$ and $\eta$ for integer $l$ and $k$ are $P(\xi = l) = p_{l+1}p_{0+}^{l}$ ($l \geq 0$) and $P(\eta = k) = p_{1+}p_{k+0}^{k}$ ($k \geq 0$), respectively and

$$\bar{F}_{\xi}(l) = P(\xi > l) = p_{l+1}^{l+1} \quad (l \geq 0), \quad \bar{F}_{\eta}(k) = P(\eta > k) = p_{k+1}^{k+1} \quad (k \geq 0). \quad (20)$$

For $n = 1, 2, \ldots$, let $\nu_n$ be a positive integer and $\{ (\xi_i(n), \eta_i(n)) : i = 1, 2, \ldots, \nu_n \}$ be a triangular array of independent random vectors with the same bivariate geometric distribution $(18)$ where $p_{ij}$ are replaced by $p_{ij}(n)$ ($i, j = 0, 1$) for $n = 1, 2, \ldots$. That is,

$$\begin{align*}
(\xi_1(1), \eta_1(1)), (\xi_2(1), \eta_2(1)), \ldots, (\xi_{\nu_1}(1), \eta_{\nu_1}(1)) \\
(\xi_1(2), \eta_1(2)), (\xi_2(2), \eta_2(2)), \ldots, (\xi_{\nu_2}(2), \eta_{\nu_2}(2)) \\
\vdots \\
(\xi_1(n), \eta_1(n)), (\xi_2(n), \eta_2(n)), \ldots, (\xi_{\nu_n}(n), \eta_{\nu_n}(n))
\end{align*}$$

Below we prove a limit theorem as $\nu_n \to \infty$ for the bivariate row maximum

$$(\mathcal{M}_n^\xi, \mathcal{M}_n^\eta) = \left( \max_{1 \leq i \leq \nu_n} \xi_i(n), \max_{1 \leq i \leq \nu_n} \eta_i(n) \right).$$

**Theorem 10** Let $\lim_{n \to \infty} \nu_n = \infty$. If there are constants $0 \leq a, b, c < \infty$, such that

$$\lim_{n \to \infty} p_{11}(n) \log \nu_n = 2c \quad \lim_{n \to \infty} \frac{p_{00}(n)}{p_{11}(n)} \log \nu_n = a \quad \text{and} \quad \lim_{n \to \infty} \frac{p_{01}(n)}{p_{11}(n)} \log \nu_n = b, \quad (21)$$
then for \( x, y \geq 0 \)
\[
\lim_{n \to \infty} P \left( p_{11}(n)M_n^0 - \log \nu_n \leq x, \ p_{11}(n)M_n^n - \log \nu_n \leq y \right)
= \exp \left\{ -e^{-x-a-c} - e^{-y-b-c} + e^{-\max(x,y) - a-b-c} \right\}.
\]

**Proof** Set \( x_n = (x + \log \nu_n)/p_{11}(n) \) and \( y_n = (y + \log \nu_n)/p_{11}(n) \).
\[
P \left( M_n^0 \leq x_n, M_n^n \leq y_n \right) = \left( F(x_n, y_n) \right)^{\nu_n}
= (1 - \bar{F}_\xi(x_n) - \bar{F}_\eta(y_n) + P(\xi_i(n) > x_n, \eta_i(n) > y_n))^{\nu_n}.
\]

Let \( x < y \) and thus, \( x_n < y_n \). Taking logarithm, expanding in Taylor series, and using (19) and (20), we obtain
\[
\log P \left( M_n^0 \leq x_n, M_n^n \leq y_n \right) = \nu_n \log \left( 1 - \bar{F}_\xi(x_n) - \bar{F}_\eta(y_n) + P(\xi_i(n) > x_n, \eta_i(n) > y_n) \right)
\]
\[
= \nu_n \left\{ \frac{\bar{F}_\xi(x_n)}{1} + \frac{\bar{F}_\eta(y_n)}{1} - P(\xi_i(n) > x_n, \eta_i(n) > y_n) \right\} (1 + o(1))
\]
\[
= -\left( \nu_n p_{00} + n \right) [x_n]^+ + \nu_n p_{10} + n [y_n]^+ - \nu_n p_{00} [x_n] + \nu_n p_{10} [y_n] - [x_n] \right\} (1 + o(1)).
\]

Write \( [x_n] = x_n - \{x_n\} \), where \( 0 \leq \{x_n\} < 1 \) is the fractional part of \( x_n \). It is easily seen that \( \lim_{n \to \infty} (p_{00} + n)^{[x_n]} = \lim_{n \to \infty} (p_{00} + n) = \lim_{n \to \infty} (p_{00} + n)^{x_n} \) as \( n \to \infty \). Furthermore, taking into account (21), we have
\[
\log \left( \nu_n p_{00}^x(n) \right) = \log \nu_n + \frac{x + \log \nu_n}{p_{11}(n)} \log (1 - p_{11})(n)
\]
\[
= \log \nu_n - \frac{x + \log \nu_n}{p_{11}(n)} \left( p_{11}(n) + p_{10}(n) + \frac{1}{2} (p_{11}(n) + p_{10}(n))^2 + O(p_{11}(n)) \right)
\]
\[
= -x(1 + o(1)) - \frac{p_{10}(n)}{p_{11}(n)} \log \nu_n - \frac{(p_{11}(n) + p_{10}(n))^2}{2p_{11}(n)} \log \nu_n + O(p_{11}(n))
\]
\[
= -x(1 + o(1)) - \frac{p_{10}(n)}{p_{11}(n)} + \frac{1}{2} p_{11}(n) \log \nu_n (1 + o(1)) + O(p_{11}(n))
\]
\[
\to -x - a - c.
\]

Therefore
\[
\lim_{n \to \infty} \nu_n p_{00}^x(n) = e^{-x - a - c}.
\]

Similarly we arrive at
\[
\lim_{n \to \infty} \nu_n p_{10}^y(n) = e^{-y - b - c} \quad \text{and} \quad \lim_{n \to \infty} \nu_n p_{00}^y(n) = e^{-x - a - b - c}.
\]

Finally,
\[
\log p_{10}^y(n) = \frac{(y - \log \nu_n) - (x - \log \nu_n)}{p_{11}(n)} \log (1 - p_{11}(n) - p_{01}(n))
\]

12
\[= - \frac{y - x}{p_{11}(n)} \left( p_{11}(n) + p_{01}(n) + \frac{1}{2} (p_{11}(n) + p_{01}(n))^2 + O(p_{+1}^2(n)) \right) \]

\[= x - y - (y - x) \left( \frac{p_{01}(n)}{p_{11}(n)} (1 + o(1)) + \frac{1}{2} p_{11}(n)(1 + o(1)) + O(p_{+1}^2(n)) \right) \]

\[\to x - y \]

Thus,

\[\lim_{n \to \infty} p_{+0}(n) \left[ y_n \right] - \left[ x_n \right] = e^x - y. \]  

The assertion of the theorem for \( x < y \) follows from (22)-(25). The case \( y < x \) is treated similarly. This completes the proof.

In particular, if \( a = b = 0 \) then

\[\lim_{n \to \infty} P \left( p_{11}(n) \mathcal{M}_n^x - \log \nu_n \geq x, p_{11}(n) \mathcal{M}_n^y - \log \nu_n \leq y \right) = \exp \left\{ -e^{-\min\{x,y\}-c} \right\}. \]

Note that in this case the limit is proportional to the upper bound for the possible asymptotic distribution of a multivariate maximum given in [4], Theorem 5.4.1.

For the componentwise maxima, applying Theorem 10, one can obtain the following limiting results. If \( p_{1+}(n) \log \nu_n \to 2c_1 < \infty \), then

\[\lim_{n \to \infty} P \left( p_{11}(n) \mathcal{M}_n^x - \log \nu_n \leq x \right) = \exp \left\{ -e^{-c} \right\}. \]

If \( p_{1+}(n) \log \nu_n \to 2c_2 < \infty \), then

\[\lim_{n \to \infty} P \left( p_{11}(n) \mathcal{M}_n^y - \log \nu_n \leq y \right) = \exp \left\{ -e^{-c} \right\}. \]

4.2 Bisexual processes with varying geometric environments

Consider the array of bivariate random vectors \( \{(\xi_i(n),\eta_i(n)) : i = 1, 2, \ldots ; n = 0, 1, \ldots \} \), which are independent with respect to both indexes. Let \( L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) be a mating function. A bisexual process with varying environments is defined (see [14]) by the recurrence: \( Z_0 = N > 0, \)

\[\left( Z_{n+1}^F, Z_{n+1}^M \right) = \sum_{i=1}^{Z_n} (\xi_i(n),\eta_i(n)) \]

and

\[Z_{n+1} = L(Z_n^F, Z_n^M) \quad (n = 0, 1, \ldots). \]

Define the mean growth rate per mating unit

\[r_{nj} = j^{-1} E(Z_{n+1} | Z_n = j) \quad (j = 1, 2, \ldots) \quad \text{and} \quad \mu_n = \prod_{i=0}^{n-1} r_{i1}, \mu_0 = 1 \quad (n = 1, 2, \ldots)\]
Lemma \((14)\) If
\[\sum_{n=0}^{\infty} \left(1 - \frac{r_{n+1}}{r_n}\right)\]
then
\[\lim_{n \to \infty} \frac{Z_n}{\mu_n} = W \text{ a.s.,}\]
where \(W\) is a nonnegative random variable with \(E(W) < \infty\).

If, in addition, there exist constants \(A > 0\) and \(c > 1\) such that
\[\prod_{i=j}^{n+j-1} r_{ii} \geq Ac^n \quad j = 1, 2, \ldots; \quad n = 0, 1, \ldots\]
and there exists a random variable \(X\) with \(E(X) < \infty\) such that for any \(u\)
\[P(X \leq u) \leq P\left(\frac{L(\xi_i(n), \eta_i(n))}{r_{n1}} \leq u\right) \quad (n = 0, 1, \ldots),\]
then \(P(W > 0) > 0\).

Further on we assume that \((\xi_i(n), \eta_i(n))\) are i.i.d. copies of the bivariate geometric vector \((\xi, \eta)\) introduced above and that the mating is promiscuous, i.e.,
\[L(\xi(n), \eta(n)) = \xi(n) \min\{1, \eta(n)\}.\]

Theorem 11 Let \(\{Z_n\}\) be a bisexual branching process with varying geometric environments and mating function \((29)\). If
\[\prod_{j=1}^{\infty} p_{+0}(j)p_{0+}(j) \neq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} p_{+1}(n) < \infty,\]
then
\[\lim_{n \to \infty} \frac{Z_n}{\mu_n} = W \quad \text{a.s.,}\]
where \(W\) is a nonnegative random variable with \(E(W) < \infty\) and \(P(W > 0) > 0\).

Proof To prove the theorem it is sufficient to verify the assumptions \((26)-(28)\) in the above lemma. First, we prove that \((26)\) holds. Indeed, for \(j \geq 1\)
\[j r_{nj} = E(Z_{n+1}^{F} \min\{1, Z_{n+1}^{M}\})\]
\[= EE(Z_{n+1}^{F} \min\{1, Z_{n+1}^{M}\} | Z_{n+1}^{M})\]
\[= (1 - P(Z_{n+1}^{M} = 0))EZ_{n+1}^{F}\]
\[= (1 - p_{1+1}(n)) \frac{jp_{0+}(n)}{p_{1+}(n)} ,\]
where we have used that both $Z_{n+1}^M$ and $Z_{n+1}^F$ are negative binomial with parameters $(j, p_{+1}(n))$ and $(j, p_{+1}(n))$, respectively. Thus,

$$r_n = \lim_{j \to \infty} r_{nj} = \lim_{j \to \infty} \left(1 - p_{+1}^j(n)\right) \frac{p_{0+}(n)}{p_{1+}(n)} = \frac{p_{0+}(n)}{p_{1+}(n)}.$$  \hspace{1cm} (33)

Now, (32) and (33) imply $1 - r_{n1}/r_n = p_{+1}(n)$, which along with (30) leads to (26).

Let us prove (28). Indeed, for $k \geq 1$

$$P(L(\xi(n), \eta(n)) = k) = \sum_{j=1}^{\infty} P(\xi(n) \cap \{1, \eta(n) = j\}) P(\eta(n) = j)$$

$$= P(\xi(n) = k) \sum_{j=1}^{\infty} P(\eta(n) = j)$$

$$= p_{+1}(n) p_{0+}^k(n) \sum_{j=1}^{\infty} p_{+1}(n) p_{10}^j(n)$$

$$= p_{+1}(n) p_{1+}(n) p_{0+}^k(n).$$

Therefore, $P(L(\xi(n), \eta(n))/r_{n1} \geq u) = p_{[ur_{n1}]^{+1}}(n)$ and hence, similarly to (28), taking into account (30), we obtain

$$\log P\left(\frac{L(\xi(n), \eta(n))}{r_{n1}} \geq u\right) \sim ur_{n1} \log p_{0+}(n)$$

$$= -u \frac{p_{+1}(n) p_{0+}(n)}{p_{1+}(n)} (1 + o(1))$$

$$\rightarrow -u$$

Thus, $\lim_{n \to \infty} P(L(\xi(n), \eta(n))/r_{n1} \geq u) = e^{-u}$, which implies (28).

Finally, to prove (27), observe that (32) implies for any $j$ and $n$

$$\prod_{i=j}^{n+j-1} r_{i1} = \prod_{i=j}^{n+j-1} p_{0+}(i) p_{10+}(i) \prod_{i=j}^{n+j-1} p_{11-1}(i)$$

$$\geq \prod_{i=1}^{\infty} p_{0+}(i) p_{10+}(i) \prod_{i=j}^{n+j-1} p_{11-1}(i)$$

$$\geq A e^c,$$

where $A = \prod_{i=1}^{\infty} p_{+1}(i) > 0$ (provided that the product in (30) is finite) and $c = \min_{i \geq j} p_{11}(i) > 1$ ($p_{11}(i) \rightarrow 0$ under (30)). (27) also holds if the product in (30) is infinite. Now, referring to the above lemma we complete the proof of the theorem.

Define offspring maxima in the bisexual process $\{Z_n\}$ by

$$\{M_n^F, M_n^M\} = \left(\max_{1 \leq i \leq Z_n} x_i(n), \max_{1 \leq i \leq Z_n} y_i(n)\right).$$
Theorem 12 Assume that \( \mu_n \to \infty \) and there are constants \( 0 \leq a, b, c < \infty \), such that

\[
\lim_{n \to \infty} p_{11}(n) \log \mu_n = 2c \quad \lim_{n \to \infty} \frac{p_{10}(n)}{p_{11}(n)} \log \mu_n = a \quad \lim_{n \to \infty} \frac{p_{01}(n)}{p_{11}(n)} \log \mu_n = b. \tag{34}
\]

Also assume that

\[
\prod_{j=1}^{\infty} p_{+0}(j)p_{0+}(j) \neq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} p_{+1}(n) < \infty. \tag{35}
\]

Then

\[
\lim_{n \to \infty} P \left( p_{11}(n), M_n^F - \log \mu_n \leq x, p_{11}(n), M_n^M - \log \mu_n \leq y \right) = \int_{0}^{\infty} (G(x,y))^z dP(W \leq z),
\]

where

\[
G(x,y) = \exp \left\{ -e^{-x-a-c} - e^{-y-b-c} + e^{-\max(x,y)-a-b-c} \right\}.
\]

Proof Set \( x_n = (x + \log \mu_n)/p_{11}(n) \) and \( y_n = (y + \log \mu_n)/p_{11}(n) \). Under assumption (34), Theorem 11 implies

\[
P \left( M_n^F \leq x_n, M_n^M \leq y_n \mid Z_n = k \right) = (F(x_n, y_n))^k \to H(x,y). \tag{36}
\]

Under (35), Theorem 12 implies

\[
\lim_{n \to \infty} P \left( \frac{Z_n}{\mu_n} \leq x \right) = P(W \leq x). \tag{37}
\]

Therefore, by (36) and (37),

\[
P \left( M_n^F \leq \frac{x + \log \mu_n}{p_{11}(n)}, M_n^M \leq \frac{y + \log \mu_n}{p_{11}(n)} \right) = \sum_{k=0}^{\infty} P(Z_n = k) (F(x_n, y_n))^k
\]

\[
= \sum_{k=0}^{\infty} P \left( \frac{Z_n}{\mu_n} = \frac{k}{\mu_n} \right) (F(x_n, y_n))^k/\mu_n
\]

\[
= \int_{0}^{\infty} (G(x,y))^z dP(W \leq z).
\]

Next example, adopted from [10], shows that the various conditions in Theorem 13 can be satisfied.

Example 7 Let \( \alpha > 1 \) and \( \beta > 1 \). Set

\[
p_{11}(n) = n^{-\alpha} \quad \text{and} \quad p_{01}(n) = p_{10}(n) = n^{-(\alpha+\beta)} \quad (n \geq 2).
\]

It is not difficult to see that with this choice of \( p_{ij}(n) \) \((i, j = 0, 1)\), we have

\[
\log \mu_n \sim an \log n \quad \text{as} \quad n \to \infty
\]

and both (34) (with \( a = b = c = 0 \)) and (35) are satisfied.

The exposition in this section follows [10], extending some of the results there.
5 Maximum score

In this section we assume that every individual in a Galton-Watson family tree has a continuous random characteristic which maximum is of interest.

5.1 Maximum scores in Galton-Watson processes

Let us go back to the simple BGW process and attach random scores to each individual in the family tree. More specifically, associate with the \( j \)-th individual in the \( n \)-th generation a continuous random variable \( Y_{j}(n) \). Arnold and Villaseñor (1996) published the first paper studying the maxima individual scores ("heights"). Pakes (1998) proves more general results concerning the laws of offspring score order statistics. Quoting [17], "these results provide examples of the behavior of extreme order statistics of observations from samples of random size." Define by \( M_{k,n} \) the \( k \)-th largest score within the \( n \)-th generation and by \( \bar{M}_{k,n} \) the \( k \)-th largest among the random variables \( \{ Y_{i}(n) : 1 \leq i \leq Z_{n}, 0 \leq \nu \leq n \} \), i.e., the \( k \)-th largest score up to and including the \( n \)-th generation.

Pakes (1998) studies the limiting behavior of "near maxima", i.e., (upper) extreme order statistics \( M_{k,n} \) and \( \bar{M}_{k,n} \) when \( n \to \infty \) and \( k \) remains fixed. The two general cases that arise are whether the law of \( Z_{n} \) (or the total progeny \( T_{n} = \sum_{\nu=0}^{n} Z_{\nu} \)), conditional on survival, do not require or do require, normalization to converge to non-degenerate limits.

If no normalization is required then no particular restriction need to be placed on the score distribution function \( S \), but the limit laws are rather complex mixtures of the laws of extreme order statistics. The principal result states that

\[
\lim_{n \to \infty} P(M_{k,n} \leq x | A_{n}) = \sum_{j=1}^{\infty} \sum_{i=0}^{k-1} \binom{j}{i} (1 - S(x))^i S^{j-i}(x) g_j,
\]

where it is assumed that the conditional law \( G_{n} \) of \( Z_{n} \) given \( A_{n} \) (\( A_{n} \) includes non-extinction) converges to a discrete and non-defective limit \( G \) and \( g_j \) denote the masses attributed to \( j \) by \( G \).

If normalization is required then one must assume that the score distribution function \( S \) is attracted to an extremal law, and then the limit laws are mixtures of the classical limiting laws of extreme order statistics. let us assume that there are positive constants \( C_{n} \uparrow \infty \) such that for the conditional law \( G_{n} \) we have \( G_{n}(xC_{n}) \Rightarrow N(x) \), where \( N(x) \) is a non-defective but possibly degenerate distribution function. Assume also that the score distribution function \( S \) is in the domain of attraction of an extremal law given by (4).

The general result in [17] is

\[
\lim_{n \to \infty} P \left( \frac{M_{k,n} - b(C_{n})}{a(C_{n})} \leq x | A_{n} \right) = \sum_{i=0}^{k-1} \frac{(h(x,\theta))^{i}}{i!} \int_{0}^{\infty} y^{i} e^{-y h(x,\theta)} dN(y). \quad (38)
\]

Example 8 Consider an immortal (i.e., \( P(X = 0) = 0 \)) supercritical process with shifted geometric offspring law given by its p.g.f. \( f(s) = s/(1+m-ms) \) \((m > 1)\), then (see
Pakes (1998) becomes
\[
\lim_{n \to \infty} P \left( \frac{M(k)_{n} - b(C_n)}{a(C_n)} \leq x | A_n \right) = 1 - \left( \frac{h(x, \theta)}{1 + h(x, \theta)} \right)^k.
\]
Thus the limit has a generalized logistic law when the score law is attracted to Gumbel law, \( h(x, \theta) = e^{-x} \); and a Pareto-type law results when \( S \) is attracted to the Fréchet law.

Phatarford (see [17]) has raised the question (in the context of horse racing), "What is the probability that the founder of a family tree is better than all its descendants?" The answer turns out to be
\[
P(\tau_n = k) = E \left( \frac{Z_k}{T_n} \right), \quad (k = 0, 1, \ldots, n),
\]
as well as limit theorems for \( \tau_n \) as \( n \to \infty \).

This subsection is based on [2] and [17].

5.2 Maximum scores in two-type processes
Let each individual in a two-type branching process be equipped with a non-negative continuous random variable - individual score. We present limit theorems for the maximum individual score. Consider two independent sets of independent random vectors with integer nonnegative components
\[
\{X^1(n)\} = \{(X^1_{1j}(n), X^1_{2j}(n))\} \quad \text{and} \quad \{X^2(n)\} = \{(X^2_{1j}(n), X^2_{2j}(n))\} \quad (j \geq 1; n \geq 0).
\]
A two-type branching process \( \{Z(n)\} = \{(Z_1(n), Z_2(n))\} \) is defined as follows: \( Z(0) \neq 0 \) a.s. and for \( n = 1, 2, \ldots \)
\[
\begin{align*}
Z_1(n) &= \sum_{j=1}^{Z_1(n-1)} X^1_{1j}(n) + \sum_{j=1}^{Z_2(n-1)} X^2_{1j}(n), \\
Z_2(n) &= \sum_{j=1}^{Z_1(n-1)} X^1_{2j}(n) + \sum_{j=1}^{Z_2(n-1)} X^2_{2j}(n).
\end{align*}
\]
Here \( X^i_{kj}(n) \) refers to the number of offspring of type \( k \) produced by the \( j \)-th individual of type \( i \). With the \( j \)-th individual of type \( i \) living in the \( n \)-th generation we associate a non-negative continuous random variable \( \zeta_{ij}(n), (i = 1, 2) \) "score", say. Assume that the offspring of type 1 and type 2 have scores, which are independent and identically distributed within each type. Define the maximum score within the \( n \)-th generation by
\[
M^\zeta_n = \max \{M^\zeta^{1}_n, M^\zeta^{2}_n\}, \quad \text{where} \quad M^\zeta^{i}_n = \max_{1 \leq j \leq Z_i(n-1)} \zeta_{ij}(n) \quad (i = 1, 2).
\]
Note that this is maximum of random number, independent but non-identically distributed random variables. Let \( F_i(x) = P(\zeta_i \leq x) \) \((i = 1, 2)\) be the c.d.f.’s of the scores of type 1 and type 2 individuals, respectively.

**Assumption 1** (tail-equivalence) We assume that \( F_1 \) and \( F_2 \) are tail equivalent, i.e., they have the same right endpoint \( x_0 \) and for some \( A > 0 \)

\[
\lim_{x \uparrow x_0} \frac{1 - F_1(x)}{1 - F_2(x)} = A.
\]

**Assumption 2** (max-stability) Suppose \( F_1 \) is in a max-domain of attraction, i.e., \((3)\) holds.

We consider the critical branching process \( Z(n) \) with mean matrix \( M \), which is positively regular and nonsingular. Let \( M \) has maximum eigenvalue 1 and associated right and left eigenvectors \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \), normalized such that \( u \cdot v = 1 \) and \( u \cdot 1 = 1 \).

**Theorem 13** Let \( \{Z(n)\} \) be the above critical two-type branching process. If the offspring variance \( 2B < \infty \) and both Assumptions 1 and 2 hold, then

\[
\lim_{n \to \infty} P \left( \frac{M^c_n - b(v_1 Bn)}{a(v_1 Bn)} \leq x | Z(n) \neq 0 \right) = \frac{1}{1 + h(x, \theta) + (v_2/v_1)h(cx + d, \theta)},
\]

where if \( -\infty < \theta < \infty \) is fixed, then \( c = A^{1/|\theta|} \) and \( d = 0 \); if \( \theta \to \pm \infty \), then \( c = 1 \) and \( d = \ln A \).

**Proof.** Since \( F_1(x) \) and \( F_2(x) \) are tail-equivalent, we have (see [21], p.67)

\[
\lim_{n \to \infty} (F_2(a(n)x + b(n)))^n \to H(cx + d, \theta),
\]

where the constants \( c \) and \( d \) are as in \((3)\). On the other hand, it is well-known (see [3], p.191) that for \( x > 0 \) and \( y > 0 \)

\[
\lim_{n \to \infty} P \left( \frac{Z_1(n)}{v_1 Bn} \leq x, \frac{Z_2(n)}{v_2 Bn} \leq y | Z(n) \neq 0 \right) = G(x, y),
\]

where the limiting distribution has Laplace transform

\[
\psi(\lambda, \mu) = \frac{1}{1 + \lambda + \mu} \quad (\lambda > 0, \mu > 0).
\]

Set \( x_n = a(v_1 Bn)x + b(v_1 Bn) \), \( s_n = k/v_1 Bn \), and \( t_n = l/v_2 Bn \). Referring to the definition of both \( M^c_n \) and process \( \{Z(n)\} \) we obtain

\[
P \left( M^c_n \leq x_n | Z(n) \neq 0 \right) = \sum_{(k,l) = 0}^{\infty} P \left( Z(n) = (k, l) | Z(n) \neq 0 \right) P \left( \max \left\{ M^c_n^1, M^c_n^2 \right\} \leq x_n \right)
\]

\[
= \sum_{(k,l) = 0}^{\infty} P \left( \frac{Z_1(n)}{v_1 Bn} = \frac{k}{v_1 Bn}, \frac{Z_2(n)}{v_2 Bn} = \frac{l}{v_2 Bn} | Z(n) \neq 0 \right) [F_1(x_n)]^k [F_2(x_n)]^l
\]
\[
\sum_{(k,l)=0}^{\infty} P\left( \frac{Z_1(n)}{v_1 B_n} = s_n, \frac{Z_2(n)}{v_2 B_n} = t_n | Z(n) \neq 0 \right) \left[ F_1(x_n) \right]^{(v_1 B_n)s_n} \left[ F_2(x_n) \right]^{(v_2 B_n)t_n(v_2/v_1)} \\
\rightarrow \int_0^{\infty} \int_0^{\infty} H(x,\theta)^s H(c x + d, \theta)^{(v_2/v_1) t} dG(s,t) \\
= \int_0^{\infty} \int_0^{\infty} \exp \left\{ -s h(x, \theta) - t \frac{v_2}{v_1} h(c x + d, \theta) \right\} dG(s,t) \\
= \left[ 1 + h(x, \theta) + \frac{v_2}{v_1} h(c x + d, \theta) \right]^{-1},
\]

where in the last formula we used the Laplace transform of \( G(u,v) \) given in (40). The proof is complete.

The two examples below illustrate the kind of limit laws that can be encountered.

**Example 9** Let \( F_1 \) and \( F_2 \) be Pareto c.d.f.’s given for \( x > \theta > 0 \) and \( c > 0 \) by

\[
F_i(x_i) = 1 - \left( \frac{\theta_i}{x_i} \right)^c \quad (i = 1, 2).
\]

Note that the two distributions share the same value of the parameter \( c \). It is not difficult to check that the limit is log-logistic given by

\[
\lim_{n \to \infty} P \left\{ \frac{\mathcal{M}_n^c}{\theta_1 (v_1 B_n)^{1/c}} \leq x | Z(n) \neq 0 \right\} = \left[ 1 + \left( 1 + \frac{v_2}{v_1} \left( \frac{\theta_1}{\theta_2} \right)^{-c} \right) x^{-c} \right]^{-1}.
\]

**Example 10** Let \( F_1 \) and \( F_2 \) be logistic and exponential c.d.f.'s given by

\[
F_1(x_1) = 1 - e^{-x_1} \quad (0 < x_1 < \infty) \quad \text{and} \quad F_2(x_2) = \frac{1}{1 + e^{-x_2}} \quad (-\infty < x_2 < \infty),
\]

respectively. It is known that both are in the max-domain of attraction of \( H(x) = \exp \{-\exp(-x)\} \) and share (see [1], p.91) the same normalizing constants \( a(n) = 1 \) and \( b(n) = \ln n \). This fact, after inspecting the proof of the theorem, allows us to bypass the tail-equivalence assumption and obtain a logistic limiting distribution, i.e., for \(-\infty < x < \infty\)

\[
\lim_{n \to \infty} P \left\{ \mathcal{M}_n^c - \log(v_1 B_n) \leq x | Z(n) \neq 0 \right\} = \left[ 1 + \left( 1 + \frac{v_2}{v_1} \right) e^{-x} \right]^{-1}.
\]

The results in this subsection are modifications of those in [13].

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References

[1] Ahsanullah, M. and Nevzorov, V. B., *Ordered Random Variables*, Nova Science Publishers, Inc., Huntington, NY, 2001.

[2] Arnold, B. C. and Villaseñor, J. A., The tallest man in the world., In: *Statist. Theory and Appl.: papers in honor of Herbert A. David*, Eds. Nagaraja, H. N., Sen, P. K. and Morrison, D. F, pp. 81–88, Springer, Berlin, 1996.

[3] Athreya, K. B. and Ney, P. E., *Branching Processes*, Springer, New York, 1972.

[4] Galambos, J. *The Asymptotic Theory of Extreme Order Statistics*, 2nd Edn., Krieger, Melbourne, Florida, 1987.

[5] Gnedenko, B.V. and Gnedenko, D.B. Laplace distributions and the logistic distribution as limit distributions in probability theory. *Serdica*, 8(1982), 2:229–234 (In Russian).

[6] Harris, T.E. *The Theory of Branching Processes*, Springer, Berlin, 1963.

[7] Jagers, P. and Nerman, O. Limit theorems for sums determined by branching and exponentially growing processes. *Stoch. Proc. Appl.* 17(1984), 47-71.

[8] Marshall, A.W. and Olkin, I. A family of bivariate distributions generated by the Bernoulli distribution. *J. Amer. Statist. Assoc.*, 80(1985), 332-338.

[9] Mitov, K.V. The maximal number of offspring of one particle in a branching process with state-dependent immigration. *Proc. of 27th Spring Conference of the Union of Bulgarian Mathematicians, Math. and Math. Education*, 1998, 92-97.

[10] Mitov, K.V. Extremes of bivariate geometric variables with application to bisezual branching processes. *Pliska Studia Math. Bulgarica*, 17(2005), 349-362.

[11] Mitov, K.V., Pakes, A.G., and Yanev, G.P. Extremes of geometric variables with applications to branching processes. *Statist and Probab Letters*, 65(2003), 379-388.

[12] Mitov, K.V. and Yanev G.P. Maximum family size in branching processes with state–dependent immigration. *Proc. of 28th Spring Conference of the Union of Bulgarian Mathematicians, Montana, Math. and Math. Education*, 1999, 142-144.

[13] Mitov, K.V. and Yanev, G.P. Maximum individual score in critical two-type branching processes. *C. R. Acad. Bulg. Sci.*, 55(2002), 11:17-22.

[14] Molina, M., Mota, M., and Ramos, A. Limiting behaviour for superadditive bisezual Galton-Watson processes in varying environments. *Test*, 13(2004), 2:481-499.

[15] Pakes, A.G. Branching processes with immigration. *J. Appl. Prob.*, 8(1971), 32-42.

[16] Pakes, A.G. Some new limit theorems for the critical branching process allowing immigration. *Stoch. Proc. Appl.* 3(1975), 175-185.
[17] Pakes, A.G. Extreme order statistics on Galton–Watson trees. *Metrika*, 47(1998), 95-117.

[18] Rahimov, I. and Yanev G.P. On a maximal sequence associated with simple branching processes. *Institute of Mathematics and Informatics*, Sofia, Preprint no.6, 1996, pp.14.

[19] Rahimov, I. and Yanev, G.P. Maximal number of direct offspring in simple branching processes. *Nonlinear Analysis, Theory, Methods and Applications*, 30(1997), 2015-2023.

[20] Rahimov, I. and Yanev, G.P. On maximum family size in branching processes. *J. Appl. Probab.*, 36(1999), 632–643.

[21] Resnick, S., *Extreme Value Distributions, Regular Variations, and Point Processes*, Springer, Berlin, 1987.

[22] Tadikamalla, P.R. A look at the Burr and related distributions. *International Statistical Review*, 48(1980) 337-344.

[23] Urbanik, K. On a problem concerning birth and death processes. *Acta Math. Acad. Sci. Hungar.*, 7(1956), 99-106 (In Russian.)

[24] Wilms, R. Fractional parts of random variables. Limit theorems and infinite divisibility. Ph.D. Thesis, Technical University of Eindhoven, Eindhooven, Holland, 1994.

[25] Yanev, G.P. and Tsokos, C.P. Family size order statistics in branching processes with immigration. *Stoch. Anal. Appl*. 18(2000), 4:655-670.

[26] Zolotarev, V.M. On a problem in the theory of branching rprocesses. *Uspehi Matemat. Nauk* 9(1954), 147-156 (In Russian.)