On the gravitational angular momentum of rotating sources

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Abstract

The gravitational energy-momentum and angular momentum satisfy the algebra of the Poincaré group in the full phase space of the teleparallel equivalent of general relativity. The expression for the gravitational energy-momentum may be written as a surface integral in the three-dimensional spacelike hypersurface, whereas the definition for the angular momentum is given by a volume integral. It turns out that in practical calculations of the angular momentum of the gravitational field generated by localized sources like rotating neutron stars, the volume integral reduces to a surface integral, and the calculations can be easily carried out. Similar to previous investigations in the literature, we show that the total angular momentum is finite provided a certain asymptotic behaviour is verified. We discuss the dependence of the gravitational angular momentum on the frame, and argue that it is a measure of the dragging of inertial frames.

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1 Introduction

The teleparallel equivalent of general relativity (TEGR) is a consistent geometrical framework for the dynamics of the gravitational field. The theory is formulated in terms of tetrad fields. It is known that these fields are necessary for establishing the interaction of Dirac spinor fields with the gravitational field, as well as for the description of reference frames in spacetime. In the general case, i.e., if no boundary conditions are imposed on the field quantities, the TEGR is invariant under global SO(3,1) frame transformations. Therefore six degrees of freedom of the tetrad field (with respect to the metric tensor) cannot be removed by gauge transformations as in the Einstein-Cartan theory, for instance, which exhibits local SO(3,1) symmetry. On the other hand, the reference frame is characterized by the six components of the acceleration tensor [1], which characterizes the inertial properties of the frame. It follows that in the TEGR the tetrad field determines both the reference frame and the gravitational field.

The field equations of the theory (Euler-Lagrange and first class constraint equations) are interpreted as equations that define the energy, momentum and angular momentum of the gravitational field. The interpretation of a constraint equation as an energy equation for a physical system is not a specific feature of the TEGR. It takes place, for instance, in the consideration of Jacobi’s action [2, 3] for a parametrized nonrelativistic particle. In order to understand this feature, let us consider a particle of mass $m$ described in the configuration space with generalized coordinates $q^i, i = 1, 2, 3$. The particle is subject to the potential $V(q)$ and has constant energy $E$. Denoting $\dot{q}^i = dq^i/dt$, where $t$ is a monotonically increasing parameter between the (fixed) initial and end points of the path, the Jacobi action integral for this particle can be written as [4]

$$I = \int_{t_1}^{t_2} dt \sqrt{mg_{ij}(q)\dot{q}^i\dot{q}^j} \sqrt{2[E - V(q)]}.$$  

The action is extremized by varying the configuration space path and requiring $\delta q(t_1) = \delta q(t_2) = 0$. We may simplify the integrand by writing $dt \sqrt{mg_{ij}(q)\dot{q}^i\dot{q}^j} = \sqrt{mg_{ij}dq^idq^j}$, which shows that the action is invariant under reparametrizations of the time parameter $t$. Thus in Jacobi’s formulation of the action principle it is the energy $E$ of the particle that is fixed, not its initial and final instants of time. In view of the time reparametrization
of the action integral, the Hamiltonian constructed out of the Lagrangian above vanishes identically, which is a known feature of reparametrization invariant theories. If we denote $p_i$ as the momenta conjugate to $q^i$, we find

$$p_i = g_{ij} \dot{q}^j \sqrt{2(E - V)/\dot{q}^2} \quad \text{(where } \dot{q}^2 = g_{kl}\dot{q}^k \dot{q}^l \text{)},$$

which leads to the constraint

$$C(q, p) \equiv \frac{1}{2} g^{ij}p_ip_j + V(q) - E \approx 0$$

The equation of motion obtained from the action integral has to be supplemented by the constraint equation $C = 0$, in order to be equivalent with Newton’s equation of motion with fixed energy $E$ [4]. Therefore we see that the constraint equation defines the energy of the particle. This is the feature that takes place in the TEGR: the definitions of the energy-momentum and angular momentum of the gravitational field emerge from the constraint equations of the theory [5, 6]. These definitions are viable as long as they yield consistent values in the consideration of well understood gravitational field configurations.

The evaluation of the expression for the gravitational energy-momentum in the TEGR is not difficult because it reduces to a surface integral in the three-dimensional spacelike hypersurface. However, the gravitational angular momentum is given by a volume integral, and we did not succeed in reducing the latter to a surface integral in the general case. Surprisingly, we find that in the framework of static observers the expression for the gravitational angular momentum due to the most general rotating source reduces to a simple surface integral. This feature not only allows the evaluation of the expression in an easy way, but it also points out which is the necessary fall-off of the components of the metric tensor that yields a finite value for the total angular momentum. We find that if the metric tensor obeys asymptotically flat boundary conditions, the total gravitational angular momentum will be finite provided the metric components fall off as $1/r$ or faster in the limit $r \to \infty$. This result agrees with the detailed analysis of Beig and Ó Murchadha [8] and of Szabados [9], who investigated the Poincaré structure of asymptotically flat spacetimes.

In this paper we analyze in detail the gravitational angular momentum of a pulsar model that describes a rotating neutron star. If $J$ represents the angular momentum of the source, we find that the gravitational angular momentum is given by $2J/3$. According to our analysis this result seems to be general: if $J$ is the angular momentum of an arbitrary rotating source, the
Gravitational angular momentum is given by $2J/3$. The constraint equation that defines the gravitational angular momentum is not related to the energy-momentum tensor. Thus we interpret the angular momentum discussed here as the angular momentum of the field, not of the source.

In the Newtonian description of classical mechanics the angular momentum of the source is frame dependent. This feature also holds in relativistic mechanics. If the angular momentum of the source in general is frame dependent, in principle one cannot exclude the possibility that the angular momentum of the field is frame dependent as well. Differently from other definitions of gravitational angular momentum that are formulated in terms of surface integrals at spacelike infinity and depend only on the asymptotic behaviour of the metric tensor, the definition considered here naturally depends on the frame, as it is invariant under global (rather than local) SO(3,1) transformations of the tetrad field. In the present framework, observers that are in rotational motion around the rotating source measure the gravitational angular momentum differently from static observers. These observers also measure the angular momentum of the source differently from stationary observers (in Newtonian mechanics the angular momentum of the source in the frame of observers that rotate at the same angular frequency vanishes). We will show that the concept of the present definition of gravitational angular momentum is closely related to the dragging of inertial frames.

In section 2 we present a summary of the formulation of the TEGR. In section 3 we address the gravitational angular momentum of a rotating neutron star, and then of an arbitrary gravitational field configuration. In section 4 we argue that the dependence of the gravitational angular momentum on the frame is a natural feature of the present formalism. Observers whose angular velocity around the rotating source is the same as the dragging velocity do not measure dragging effects (the dragging velocity, for instance) as do static observers. We show that for such observers the gravitational angular momentum vanishes. Finally, in section 5 we present the final remarks.

Notation: space-time indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Latin indices from the middle of the alphabet $(i, j, k)$ run from 1 to 3. Time and space indices are indicated according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field is denoted by $e^a_\mu$, and the torsion tensor reads $T_{a\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu$. ($T_{a\mu\nu}$ is related via $T_{a\mu\nu} = e^a_\lambda T^\lambda_\mu\nu$ to the torsion tensor of Cartan’s connection $\Gamma^\lambda_{\mu\nu} = e^a_\lambda \partial_\mu e^a_\nu$). The flat, Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-+++)$.
The determinant of the tetrad field is represented by $e = \det(e^a_\mu)$.

## 2 The Hamiltonian form of the TEGR

The equivalence of the TEGR with Einstein’s general relativity may be understood by means of an identity between the scalar curvature $R(e)$ constructed out of the tetrad field and a combination of quadratic terms of the torsion tensor (see, for example, ref. [10]),

$$e R(e) \equiv -e\left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a\right) + 2 \partial_\mu(e T^\mu).$$  \hspace{1cm} (1)

The Lagrangian density of the TEGR in empty spacetime is given by the combination of the quadratic terms on the right hand side of eq. (1),

$$L = -k e\left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a\right) \equiv -k e \Sigma^{abc} T_{abc},$$  \hspace{1cm} (2)

where $k = c^3/16\pi G$, $T_a = T^{b}_{\ b a}$, $T_{abc} = e^b_\mu e^c_\nu T_{a\mu\nu}$ and

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac} T^b - \eta^{ab} T^c).$$  \hspace{1cm} (3)

The field equations derived from eq. (2) are equivalent to Einstein’s equations.

The Hamiltonian formulation of the theory has been investigated in detail in ref. [11]. The Hamiltonian density $H$ may be obtained from $L$ by rewriting it as $L = \dot{p} q - H$, and expressing $H$ in terms of the canonical variables. There is no time derivative of $e_{a0}$, and therefore this field quantity arises as a Lagrange multiplier. The momenta canonically conjugated to $e_{ak}$ is denoted by $\Pi^{ak}$. In the configuration space we have

$$\Pi^{ai} = -4k e \Sigma^{ai}.$$  \hspace{1cm} (4)

We find that $H$ may indeed be expressed in terms of $e_{ak}$, $\Pi^{ak}$ and Lagrange multipliers. In ref. [6] we have redefined some Lagrange multipliers and
constraints in order to arrive at simpler form of \( H \). Except for a surface term it reads

\[
H = e_{a0} C^a + \frac{1}{2} \lambda_{ab} \Gamma^{ab},
\]

(5)

where \( e_{a0} \) and \( \lambda_{ab} = -\lambda_{ba} \) are Lagrange multipliers, and \( C^a \) and \( \Gamma^{ab} \) are first class constraints. After solving one set of Hamilton’s field equations we may identify \( \lambda_{ik} = 1/2(T_{i0k} - T_{k0i}) \) and \( \lambda_{0k} = T_{00k} \). The quantities \( \lambda_{ik} \) and \( \lambda_{0k} \) are components of \( \lambda_{\mu\nu} = e^a_{\mu} e^b_{\nu} \lambda_{ab} \). For further details see refs. [6, 11].

The constraint \( C^a \) may be written in the form \( C^a = -\partial_i \Pi^{ai} + h^a \), where \( h^a \) is an intricate expression of the field quantities. We note that \( -\partial_i \Pi^{ai} \) is the only total divergence of the momenta \( \Pi^{ai} \) that arises in the expression of \( C^a \). The integral form of the equation \( C^a = 0 \) motivates the definition of the gravitational energy-momentum \( P^a \) [5],

\[
P^a = - \int_V d^3 x \partial_i \Pi^{ai},
\]

(6)

where \( V \) is an arbitrary volume of the three-dimensional space. The emergence of a nontrivial total divergence is a feature of theories with torsion. Metric theories of gravity do not share this feature. The integration of this total divergence yields a surface integral. If we consider the \( a = (0) \) component of eq. (6) and adopt asymptotic boundary conditions for the tetrad field we find [5] that the resulting expression is precisely the surface integral at infinity that defines the ADM energy [12]. This fact is a strong indication (but no proof) that eq. (6) does indeed represent the gravitational energy-momentum.

The constraint \( \Gamma^{ab} \) is defined by [6, 11]

\[
\Gamma^{ab} = M^{ab} + 4ke(\Sigma^{a0b} - \Sigma^{b0a}),
\]

(7)

where \( M^{ab} = e^a_{\mu} e^b_{\nu} M^{\mu\nu} = -M^{ba} \), and \( M^{\mu\nu} \) is given by

\[
M^{ik} = 2\Pi^{ik} = e_a^i \Pi^{ak} - e_k^a \Pi^{ai},
\]

(8)

\[
M^{0k} = \Pi^{0k} = e_a^0 \Pi^{ak}.
\]

(9)

Similar to the definition of \( P^a \), the integral form of the constraint equation \( \Gamma^{ab} = 0 \) motivates the definition of the space-time angular momentum. The equation \( \Gamma^{ab} = 0 \) implies
\[ M^{ab} = -4ke(\Sigma^{a0b} - \Sigma^{b0a}) . \]  

Therefore we define [6]

\[ L^{ab} = - \int_V d^3x \, e^a_\mu e^b_\nu M^{\mu\nu} , \]

as the 4-angular momentum of the gravitational field. This definition differs by a sign from the definition presented in ref. [6].

The evaluation of definitions (6) and (11) are carried out in the configuration space, by means of eqs. (4) and (10), respectively. However, if we consider (6) and (11) in the phase space of the theory, with the right hand side of the latter equations given in terms of the momenta \( \Pi^{ai} \) (with the help of eqs. (8) and (9)), we find that \( P^a \) and \( L^{ab} \) constitute a representation of the Poincaré group. By working out the Poisson brackets between \( P^a \) and \( L^{ab} \) in the full phase space of the theory we find [6]

\[
\begin{align*}
\{P^a, P^b\} &= 0, \\
\{P^a, L^{bc}\} &= \eta^{ab} P^c - \eta^{ac} P^b, \\
\{L^{ab}, L^{cd}\} &= \eta^{ac} L^{bd} + \eta^{bd} L^{ac} - \eta^{ad} L^{bc} - \eta^{bc} L^{ad} .
\end{align*}
\]

The Poincaré algebra of \( P^a \) and \( L^{ab} \) confirms the consistency of the definitions. We recall that the evaluation of \( P^a \) and \( L^{ab} \) is carried out in an arbitrary volume \( V \) of the three-dimensional space.

Two important points must be noted. First, the Poincaré algebra above is rigorously verified irrespective of the specification of any surface term (surface integral) on the boundary of \( V \). Thus it is not necessary to impose boundary conditions to obtain (12). Second, the procedure for obtaining (12) is exactly the same procedure for obtaining the algebra of constraints of a theory in Hamiltonian form (as in ref. [11], for instance). Therefore the validity of the constraint algebra and of eq. (12) are on equal footing.

Definitions (6) and (11) are invariant under (i) general coordinate transformations of the three-dimensional space, (ii) time reparametrizations, and (iii) global SO(3,1) transformations. The non-invariance of these definitions under the local SO(3,1) group reflects the frame dependence of the definitions. We have argued [1, 6, 7] that this dependence is a natural feature of the gravitational energy-momentum defined by eq. (6). In the TEGR each set
of tetrad fields is interpreted as a reference frame in spacetime. The tetrad frame is adapted to observers whose four-velocity $u^\mu$ in spacetime is identified with $e_\mu^{(0)}$, namely, $e_\mu^{(0)} = u^\mu$, and is characterized by the acceleration tensor $\phi_{ab}$ defined by [1]

$$\phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}].$$

(13)

The tensor $\phi_{ab}$ characterizes the inertial properties of the frame. It is clearly not invariant under the local SO(3,1) group. It provides the values of the inertial accelerations that are necessary to maintain the frame in a given orientation and inertial state, in the gravitational field configuration defined by $g_{\mu\nu}$.

3 The gravitational angular momentum of rotating sources

3.1 A simple model for rotating neutron stars

The starting point of the present analysis is the analytic pulsar model described in ref. [13], which is taken to represent a rotating neutron star. The model represents a star that is approximately rigidly rotating. In the latter reference the authors argue that although the stellar rotation rate is a function of radius, it can be made arbitrarily close to rigid rotation. For our purposes the model is important because it is simple and singularity free, and allows to carry out integration over the whole three-dimensional space.

It is represented by the metric tensor

$$ds^2 = -\alpha^2 dt^2 + \beta^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta (d\phi - \Omega dt)^2. \quad (14)$$

We denote by $R$ the stellar radius. For $r \leq R$ we have

$$\alpha = \frac{1}{2} [3(1 - 8\pi \rho R^2 / 3)^{1/2} - (1 - 8\pi \rho r^2 / 3)^{1/2}],$$

$$\beta^2 = (1 - 8\pi \rho r^2 / 3)^{-1},$$

$$\Omega = \Omega(0)[1 - b(r/R)^2 - b\tau(r/R)^4],$$

(15)

and for $r \geq R$,
\[ \alpha^2 = \left[1 - 2m(R)/r\right], \]
\[ \beta^2 = \left[1 - 2m(R)/r\right]^{-1}, \]
\[ \Omega = \frac{2J}{r^3} = \frac{2GJ_s}{c^3r^3}, \]  \hspace{1cm} (16)

where \( \Omega \) is the angular velocity of inertial frames along the rotation axis and \( J_s \) represents the angular momentum of the star; \( \rho \) is the uniform density and \( m(r) = 4\pi \rho r^3/3 \). The quantity \( b \) is defined by \( b = 3/(5 + 7\tau) \), where \( \tau \) is a free parameter. The case \( \tau = 0 \) yields completely rigid rotation in the Newtonian limit. For further details, see ref. [13].

The evaluation of definition (11) is conceptually simple. It requires the calculation of the tensors \( T_{\lambda \mu \nu} \) and \( \Sigma^{\lambda \mu \nu} \). We only need to determine the tetrad field adapted to a particular class of observers. We choose to work with the tetrad field adapted to stationary observers. Such observers have the velocity field \( u^\lambda = (u^0, 0, 0, 0) \). Therefore we identify \( e_{(0)}^\mu = u^\mu \) and require
\[ e_{(0)}^i = 0. \]  \hspace{1cm} (17)

Moreover we choose the \( e_{(3)}^\mu \) component to coincide asymptotically \( (r \to \infty) \) with the unit vector \( \hat{z} \) along the z axis, namely,
\[ e_{(3)}^\mu(t, x, y, z) \cong (0, 0, 0, 1). \]  \hspace{1cm} (18)

These two conditions were imposed on the set of tetrad fields for the Kerr spacetime, discussed in ref. [1]. They do not fix the tetrad field completely because of the axial symmetry of the spacetime. The orientations of the \( x \) and \( y \) directions are arbitrary, except that the latter are orthogonal.

The requirement of the conditions above on \( e_{a\mu} \) constructed out of (14) yields

\[ e_{a\mu}(t, r, \theta, \phi) = \begin{pmatrix}
-A & 0 & 0 & -C \\
0 & B \sin \theta \cos \phi & r \cos \theta \cos \phi & -D r \sin \theta \sin \phi \\
0 & B \sin \theta \sin \phi & r \cos \theta \sin \phi & D r \sin \theta \cos \phi \\
0 & B \cos \theta & -r \sin \theta & 0
\end{pmatrix}, \]  \hspace{1cm} (19)
with the following definitions,

\[
A(r, \theta) = (-g_{00})^{1/2}, \\
B(r) = (g_{11})^{1/2}, \\
C(r, \theta) = \frac{g_{03}}{\sqrt{-g_{00}}}, \\
D(r, \theta) = \left(\frac{-\delta}{g_{00}g_{33}}\right)^{1/2}, \\
\delta = g_{03}g_{03} - g_{00}g_{33}.
\]  

(20)

The gravitational angular momentum density along the \(z\) direction is given by \(M^{(1)(2)}\). Taking into account the definition of \(\Sigma^{abc}\) in (10) we find

\[
M^{(1)(2)} = -2ke e^{(1)}_{\mu} e^{(2)}_{\nu} [T^{0\mu\nu} - g^{0\mu} T^{0\nu} + \tilde{g}^{0\nu} T^\mu].
\]  

(21)

The determinant \(e\) reads \(e = r (g_{11} \delta)^{1/2}\). We recall that \(T^\mu = T^\lambda \_\lambda^\mu\). The following relations are valid for the metric tensor in spherical coordinates for the spacetime of an arbitrary rotating source,

\[
g^{00} = \frac{-g_{33}}{\delta}, \\
g^{03} = \frac{g_{03}}{\delta}, \\
g^{33} = \frac{-g_{00}}{\delta}.
\]  

(22)

In view of (19) we have

\[
e^{(1)}_{1} e^{(2)}_{2} - e^{(1)}_{2} e^{(2)}_{1} = 0, \\
e^{(1)}_{1} e^{(2)}_{3} - e^{(1)}_{3} e^{(2)}_{1} = BD r \sin^2 \theta, \\
e^{(1)}_{2} e^{(2)}_{3} - e^{(1)}_{3} e^{(2)}_{2} = D r^2 \sin \theta \cos \theta.
\]  

(23)

and \(M^{(1)(2)}\) is first simplified as

\[
M^{(1)(2)} = -2ke BD r \sin^2 \theta (T^{013} + g^{03} T^1)
\]
\[-2k\varepsilon D r^2 \sin \theta \cos \theta (T^{023} + g^{03}T^2) = \]
\[-2k\varepsilon BD r \sin^2 \theta [g^{11}(g^{00}g^{33} - g^{03}g^{03})T_{013} + g^{03}g^{11}g^{22}T_{212}] \]
\[-2k\varepsilon D r^2 \sin \theta \cos \theta [g^{22}(g^{00}g^{33} - g^{03}g^{03})T_{023} + g^{03}g^{11}g^{22}T_{112}] . \]

The necessary torsion tensor components are

\[
T_{013} = -A\partial_1 C , \quad T_{023} = -A\partial_2 C , \quad T_{112} = 0 , \quad T_{212} = r(1 - B) ,
\]

where \(\partial_1 = \partial_r\) and \(\partial_2 = \partial_\theta\). Considering \(g^{00}g^{33} - g^{03}g^{03} = -1/\delta\) we find that eq. (24) is written as

\[
M^{(1)(2)} = -2k\varepsilon (BDr \sin^2 \theta) \left[ g^{11}_\delta A\partial_1 C - g^{03}g^{11}g^{22} r(1 - B) \right] \\
-2k\varepsilon (Dr^2 \sin \theta \cos \theta) \left[ g^{22}_\delta A\partial_2 C \right]. \quad (26)
\]

After substitution of eq. (20) into (26) we obtain

\[
M^{(1)(2)} = 2kr \sin \theta \partial_1 \left[ \frac{g_{03}}{-g_{00}} \right]^{1/2} + 2k \sin \theta \left[ \frac{g_{03}}{-g_{00}} \right]^{1/2} \left[ 1 - (g_{11})^{1/2} \right] \\
+2k \cos \theta \left( g_{11} \right)^{1/2} \partial_2 \left[ \frac{g_{03}}{-g_{00}} \right]^{1/2} , \quad (27)
\]

which leads to the final form of \(M^{(1)(2)}\),

\[
M^{(1)(2)} = 2k \frac{\partial}{\partial r} \left[ r \sin \theta g_{03} \right] \left( -g_{00} \right)^{1/2} + 2k \frac{\partial}{\partial \theta} \left[ \cos \theta \left( g_{11} \right)^{1/2} g_{03} \right] \left( -g_{00} \right)^{1/2} . \quad (28)
\]

From eq. (14) we have \(g_{03} = -\Omega(r) r^2 \sin^2 \theta\). Taking into account definitions (15) and (16) we find...
\[
\int_0^\infty dr \frac{\partial}{\partial r} \left[ \frac{r \sin \theta g_{03}}{(-g_{00})^{1/2}} \right] = -2J \sin^3 \theta
\]
\[
\int_0^\pi d\theta \frac{\partial}{\partial \theta} \left[ \frac{\cos \theta (g_{11})^{1/2} g_{03}}{(-g_{00})^{1/2}} \right] = 0. \tag{29}
\]

Therefore we have
\[
L^{(1)(2)} = -\int d^3x M^{(1)(2)} = -2k \int_0^{2\pi} d\phi \int_0^\pi d\theta (-2J \sin^3 \theta) = \frac{32\pi}{3} k J. \tag{30}
\]

Considering that \( k = c^3/(16\pi G) \) and \( J = (G/c^3)J_s \), where \( J_s \) is the angular momentum of the star, we finally obtain

\[
L^{(1)(2)} = \frac{2}{3} J_s. \tag{31}
\]

The evaluation of \( L^{(1)(2)} \) is easy because \( M^{(1)(2)} \) can be expressed as a total divergence, and eventually \( L^{(1)(2)} \) is given by a surface integral. We see that the expression of \( L^{(1)(2)} \) is finite provided \(-g_{00} \approx 1+O(1/r) \) and \( g_{03} \approx O(1/r) \) at spacelike infinity.

The expressions for \( L^{(1)(3)} \) and \( L^{(2)(3)} \) vanish because \( e^{(1)}_i e^{(3)}_j - e^{(1)}_j e^{(3)}_i \) and \( e^{(2)}_i e^{(3)}_j - e^{(2)}_j e^{(3)}_i \) contain the functions \( \sin \phi \) or \( \cos \phi \), and the integration of these functions from 0 to \( 2\pi \) vanishes. We note that the quantities above arise in the expressions of \( M^{(1)(3)} \) and \( M^{(2)(3)} \), respectively, exactly like \( e^{(1)}_i e^{(2)}_j - e^{(1)}_j e^{(2)}_i \) does in eq. (21).

For the same reason above, we conclude that the center-of-mass moments \( L^{(0)(1)} \) and \( L^{(0)(2)} \) vanish. The only component that does not vanish on account of integrals like \( \int_0^{2\pi} d\phi \sin \phi = \int_0^{2\pi} d\phi \cos \phi = 0 \) is \( L^{(0)(3)} \). From definition (10) we obtain

\[
M^{(0)(3)} = -2k e_0^i e^{(0)}_0 e^{(3)}_i (T^{001} - g^{00} T^1) + e_0^i e^{(0)}_0 e^{(3)}_i (T^{002} - g^{00} T^2)
\]
\[-e^{(0)}_3 e^{(3)}_1(T^{013} - g^{03}T^{1})
-e^{(0)}_3 e^{(3)}_2(T^{023} - g^{03}T^{2})\].

(32)

After a long calculation and several simplifications we arrive at

\[M^{(0)}(3) = 2k[\partial_1(D r^2 \sin \theta \cos \theta) - \partial_2(BD r \sin^2 \theta)].\]

(33)

Since \(D\) is a function of \(\sin^2 \theta\) we conclude that

\[\int_0^\pi d\theta \partial_1(D r^2 \sin \theta \cos \theta) = \int_0^\pi d\theta \partial_2(BD r \sin^2 \theta) = 0,\]

(34)

and therefore \(L^{(0)}(3) = 0\). Thus the only nonvanishing component of \(L^{ab}\) is given by eq. (31).

### 3.2 An arbitrary rotating source with axial symmetry

The most general form of the metric tensor that describes the spacetime generated by an arbitrary stationary rotating source with axial symmetry is represented by the line element (see section 6.4 of ref. [14])

\[ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 + 2g_{03}d\phi dt,\]

(35)

where all metric components depend on \(r\) and \(\theta\): \(g_{\mu\nu} = g_{\mu\nu}(r, \theta)\). Denoting by \(g\) the determinant of the metric tensor, we have \(\sqrt{-g} = [g_{11}g_{22}(g_{03}g_{03} - g_{00}g_{33})]^{1/2}\). The inverse metric components \(g^{00}, g^{03}\) and \(g^{33}\) are given by eq. (22). Here we also have the quantity \(\delta\) defined by \(\delta = g_{03}g_{03} - g_{00}g_{33}\).

The tetrad field adapted to stationary observers in spacetime (i.e., that satisfies conditions (17) and (18)) is written as

\[e_{a\mu}(t, r, \theta, \phi) = \begin{pmatrix}
-A & 0 & 0 & -C \\
0 & \sqrt{g_{11}} \sin \theta \cos \phi & \sqrt{g_{22}} \cos \theta \cos \phi & -D r \sin \theta \sin \phi \\
0 & \sqrt{g_{11}} \sin \theta \sin \phi & \sqrt{g_{22}} \cos \theta \sin \phi & -D r \sin \theta \cos \phi \\
0 & \sqrt{g_{11}} \cos \theta & -\sqrt{g_{22}} \sin \theta & 0
\end{pmatrix}.\]

(36)

The functions \(A, C\) and \(D\) are quite similar to the corresponding quantities in eq. (20),

12
\begin{align*}
A(r, \theta) &= (-g_{00})^{1/2}, \\
C(r, \theta) &= -\frac{g_{03}}{(-g_{00})^{1/2}}, \\
D(r, \theta) &= \left[\frac{-\delta}{(r^2 \sin^2 \theta)g_{00}}\right]^{1/2}.
\end{align*}

The same remarks about the fixation of the tetrad field given by (19) hold in the present case.

The evaluation of $M^{ab}$ is carried out exactly as in the previous subsection, and for this reason we will not present the details of the calculations. The expression below for $M^{(1)(2)}$ does not immediately arise as a total divergence. A number of simple manipulations has to be made in order to obtain its final form. We find that the two nonvanishing components of $M^{ab}$ are

\begin{align*}
M^{(1)(2)} &= 2k \left[\partial_1 \left(\frac{g_{03}\sqrt{g_{22}} \sin \theta}{\sqrt{-g_{00}}}\right) + \partial_2 \left(\frac{g_{03}\sqrt{g_{11}} \cos \theta}{\sqrt{-g_{00}}}\right)\right], \\
M^{(0)(3)} &= 2k \left[\partial_1 \left(\frac{\delta^{1/2}\sqrt{g_{22}} \cos \theta}{\sqrt{-g_{00}}}\right) - \partial_2 \left(\frac{\delta^{1/2}\sqrt{g_{11}} \sin \theta}{\sqrt{-g_{00}}}\right)\right].
\end{align*}

It is not difficult to verify that if we reduce the metric components to the values of eq. (14) we arrive at eqs. (28) and (33).

In order to calculate the total gravitational angular momentum we transform definition (11) into a surface integral such that the surface of integration $S$, determined by the condition $r = \text{constant}$, is located at spacelike infinity. Therefore we have

\begin{align*}
L^{(1)(2)} &= -\int d^3 x \, M^{(1)(2)} = -2k \oint_{S \rightarrow \infty} d\theta d\phi \left(\frac{g_{03}\sqrt{g_{22}} \sin \theta}{\sqrt{-g_{00}}}\right),
\end{align*}

assuming that the components of the metric tensor are regular at the origin, and that $g_{03} = 0$ for $r = 0$.

At this point we may evaluate whether for a given spacetime metric tensor the total gravitational angular momentum is finite, vanishes or diverges. If
$g_{\mu\nu}$ is given in spherical coordinates and if the following asymptotic behaviour is verified,

\begin{align*}
g_{03} &\cong O(1/r) + \cdots \\
g_{22} &\cong r^2 + O(r) + \cdots \\
-g_{00} &\cong 1 + O(1/r) + \cdots ,
\end{align*}

expression (40) will be finite. This result agrees with the analysis of Beig and Ó Murchadha [8] and of Szabados [9].

The quantity $M^{(0)(3)}$ is interpreted as the gravitational center of mass moment. It vanished for the model described by (14) because the latter is spherically symmetric in the limit $\Omega \to \infty$, and a nonvanishing $\Omega(r)$ given by (15) and (16) does not alter the gravitational center of mass along the $z$ axis. The model determined by (36) is arbitrary in the sense that the metric tensor depends on $\theta$. In view of the axial symmetry of the model, it is natural that the gravitational center of mass vanishes along the $x$ and $y$ directions. Because of the $\theta$ dependence of the metric tensor, (39) does not vanish a priori.

Finally we mention that the angular momentum of the Kerr spacetime will be addressed elsewhere. In the spacetime of a rotating black hole there exists the ergosphere, and inside the ergosphere it is not possible to define stationary reference frames. The tetrad field determined by conditions (17) and (18) and that yields the Kerr spacetime is not (mathematically) defined inside the ergosphere.

4 The frame dependence of the gravitational angular momentum

Definition (11) for the gravitational angular momentum is not invariant under local Lorentz transformations, and therefore it is frame dependent. In this section we will discuss a physical aspect of this feature. Before addressing this issue, let us consider the frame dependence of the gravitational energy-momentum. For this purpose we consider a black hole of mass $m$ and an observer that is very distant from the black hole. The black hole will appear to this observer as a particle of mass $m$, with energy $mc^2$ ($m$ is the rest
mass of the black hole, i.e., the mass of the black hole in the frame where the black hole is at rest). If, however, the black hole is moving at velocity \( v \) with respect to the observer, then its total gravitational energy will be \( \gamma m c^2 \), where \( \gamma = (1 - v^2/c^2)^{-1/2} \). This example is a consequence of the special theory of relativity, and demonstrates the frame dependence of the gravitational energy-momentum. We note that the frame dependence is not restricted to observers at spacelike infinity. It holds for observers located everywhere in the three-dimensional space.

The dependence of the gravitational angular momentum on the frame is an intrinsic feature of definition (11). In order to understand some consequences of this feature, we will investigate a special set of tetrad fields. Let us consider the frame for the spacetime of the rotating neutron star that satisfies Schwinger’s time gauge condition,

\[
e_{(i)}^0 = 0.
\]

It reads

\[
e_{\alpha\mu} = \begin{pmatrix}
-\alpha & 0 & 0 & 0 \\
\Omega r \sin \theta \sin \phi & \beta \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
-\Omega r \sin \theta \cos \phi & \beta \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
0 & \beta \cos \theta & -r \sin \theta & 0
\end{pmatrix},
\]

where \( \alpha, \beta \) and \( \Omega \) are the quantities defined in (14-16). This frame is adapted to the field of observers whose velocity \( e_{(0)}^\mu \) is given by

\[
e_{(0)}^\mu(t, r, \theta, \phi) = \frac{1}{\alpha}(1, 0, 0, \Omega(r)).
\]

\( \Omega(r) \) is the dragging velocity of inertial frames that rotate under the action of the neutron star. The expression above for \( e_{(0)}^\mu \) describes the velocity field of observers in circular motion around the star. It turns out that the angular momentum of the gravitational field calculated out of (43) vanishes: \( L^{(1)(2)} = 0 \). It is not difficult to arrive at this result. Out of eq. (43) we find

\[
T_{013} = -(1 - \beta)\Omega r \sin^2 \theta, \\
T_{212} = (1 - \beta)r, \\
T_{313} = (1 - \beta)r \sin^2 \theta.
\]
Working out and simplifying the expression for $M^{(1)(2)}$ we obtain

$$M^{(1)(2)} = -2ke\left[(e^{(1)}_0e^{(2)}_1 - e^{(1)}_1e^{(2)}_0)g^{11}[g^{00}g^{33} - g^{03}g^{03}]T_{313}
+ g^{00}g^{22}T_{212}]
+ (e^{(1)}_1e^{(2)}_3 - e^{(1)}_3e^{(2)}_1)g^{11}[g^{00}g^{33} - g^{03}g^{03}]T_{013}
- g^{03}g^{22}T_{212}\right]. \tag{46}$$

Taking into account

$$\begin{align*}
(g^{00}g^{33} - g^{03}g^{03})T_{313} + g^{00}g^{22}T_{212} &= -\frac{2}{\delta} T_{212} \sin^2 \theta, \\
(g^{00}g^{33} - g^{03}g^{03})T_{013} - g^{03}g^{22}T_{212} &= \frac{2}{\delta} \Omega T_{212} \sin^2 \theta, \tag{47}
\end{align*}$$

and also that

$$\begin{align*}
e^{(1)}_0e^{(2)}_1 - e^{(1)}_1e^{(2)}_0 &= \Omega \beta r \sin^2 \theta, \\
e^{(1)}_1e^{(2)}_3 - e^{(1)}_3e^{(2)}_1 &= \beta r \sin^2 \theta, \tag{48}
\end{align*}$$

we find that $M^{(1)(2)} = 0$. The other components also vanish. We note that observers with four-velocity given by (44) are known in the literature as ZAMOs (zero angular momentum observers) (see Ref. [15], section 11.3). These observers follow trajectories with constant radial coordinate $r$ and with angular velocity given by the dragging velocity of inertial frames.

The result above shows that observers that are in rotational motion around the rotating source measure the gravitational angular momentum differently from static observers. An explanation for this result must take into account the angular momentum of the source, which is different for observers at rest and for those that rotate around the source. In the Newtonian limit of the theory the angular momentum of the source in the frame of observers that rotate at the same angular frequency vanishes. We know that this feature holds for a rigid body in Newtonian mechanics, where the angular momentum depends not only on the frame, but also on the origin of the frame.
We conclude that definition (11) for the angular momentum of the field is a measure of the dragging of inertial frames. Observers whose angular velocity around the rotating source is the same as the dragging velocity $\Omega(r)$ do not measure the dragging velocity itself (and possibly other dragging effects), and therefore for these observers the gravitational angular momentum vanishes.

Because of the frame dependence, the interpretation of definition (11) is different from the standard interpretation of the gravitational angular momentum [8, 9] based on the idea of conserved field quantities. The latter are obtained by means of surface integrals at spacelike infinity. In our opinion these are just two different interpretations that do not seem to be in conflict.

## 5 Conclusions

The major reason for considering the present definition of gravitational angular momentum is that it satisfies the Poincaré algebra. In the framework of the TEGR the energy-momentum and angular momentum of the gravitational field are defined by suitably interpreting the constraint equations of the theory, not by means of boundary integrals of the Hamiltonian. Definitions (6) and (11) are evaluated in the configuration space of the theory.

The definition for the gravitational angular momentum investigated here yields consistent results in the consideration of a rotating neutron star. In the context of isolated rotating sources, we observe that the angular momentum of the gravitational field is related to the dragging velocity of inertial frames $\Omega(r)$. By inspecting eqs. (28) and (38) we see that the angular momentum density depends on $g_{03}$, and typically we have $g_{03} \cong -\Omega(r)r^2\sin^2\theta$, at least at great distances from the source. Therefore the angular momentum of the spacetime is a closely related to the dragging of inertial frames.

One main result of the paper is eq. (40), which gives the gravitational angular momentum generated by an isolated rotating source. Equation (40) allows to verify precisely which is the asymptotic behaviour of the metric tensor that leads to a finite value for the angular momentum in the frame of stationary observers. In the consideration of a rotating neutron star, we found that if $J$ is the angular momentum of the star, the angular momentum of the gravitational field is $2J/3$, in the limit of rigid rotation. So far we do not have an explanation as to why the angular momentum of the field is
$2J/3$ (and not $J$). This issue will be addressed in the future. However, to our knowledge there is no physical requirement for the two angular momenta, of the field and of the source, to be the same.

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