Automorphisms of one-relator groups

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Abstract

It is a well known fact that every group $G$ has a presentation of the form $G = F/R$, where $F$ is a free group and $R$ the kernel of the natural epimorphism from $F$ onto $G$. Driven by the desire to obtain a similar presentation of the group of automorphisms $\text{Aut}(G)$, we consider the subgroup $\text{Stab}(R) \subseteq \text{Aut}(F)$ of those automorphisms of $F$ that stabilize $R$ and ask whether the natural homomorphism $\text{Stab}(R) \rightarrow \text{Aut}(G)$ is onto; if it is, we can try to determine its kernel.

Both parts of this task are usually quite hard. The former part received considerable attention in the past, whereas the more difficult part (determining the kernel) seemed unapproachable. Here we approach this problem for a class of one-relator groups with a special kind of small cancellation condition. Then, we address a somewhat easier case of 2-generator (not necessarily one-relator) groups and determine the kernel of the above-mentioned homomorphism for a rather general class of those groups.

1. Introduction

Let $F = F_n$ be the free group of a finite rank $n \geq 2$ with a set $X = \{x_i\}, 1 \leq i \leq n$, of free generators. Let $R$ be a normal subgroup of $F$ and $\text{Stab}(R) \subseteq \text{Aut}(F)$ the group of those automorphisms $\varphi$ of $F$ that stabilize $R$, i.e. $\varphi(R) = R$ (this does not necessarily mean that $\varphi$ fixes every element of $R$).

Then there is the natural homomorphism $\rho: \text{Stab}(R) \rightarrow \text{Aut}(G)$, where $G = F/R$ (when we say the natural homomorphism, it means that every element $s \in F$ maps onto the coset $sR$ and this extends to the mapping between groups of automorphisms). In some important cases, this homomorphism $\rho$ is known to be onto. This was established by Nielsen for surface groups and by Zieschang for somewhat more general one-relator and Fuchsian groups (see [17] and [18]); more recently, a number of results in this direction have been published, of which we mention only a couple, which are in line with the subject of the present paper. Lustig, Moriah and Rosenberger [9] completely characterized Fuchsian groups with the property above. Bachmuth, Formanek and Mochizuki [1] established this property for a class of 2-generator groups. On the other hand, it is known that for many groups $G$, the homomorphism $\rho$ is not onto (see [8] or [15] for a survey).

When $\rho: \text{Stab}(R) \rightarrow \text{Aut}(G)$ is onto, it is natural to try to determine its kernel, since that would give a presentation of the group $\text{Aut}(G)$ as $\text{Stab}(R)/\text{Ker}(\rho)$, and
this, in turn, could give a presentation of \( \text{Aut}(G) \) by generators and defining relations (at least in the case where \( G \) is finitely presented) if one uses results of McCool [12] on presentation of \( \text{Stab}(R) \).

It is clear that the group \( \text{Inn}_R \) of inner automorphisms of \( F \) induced by elements of \( R \) is always contained in \( \ker(\rho) \). In many cases, this is the whole of \( \ker(\rho) \) (see Theorem 1.1 below). However, in several important situations \( \ker(\rho) \) happens to be bigger than \( \text{Inn}_R \). This is the case, in particular, with surface groups:

Example 1. (a) Let \( r = x_1^2x_2^2\ldots x_n^2 \) and \( \varphi : x_i \to x_1^{-2}r^{-x_i^2} = x_1^{-1}x_n^{-2} \ldots x_2^{-2}; x_i \to x_i, i \not\equiv 1 \) (our notation is \( x^y = yxy^{-1} \)). Then \( \varphi \) is obviously a non-inner automorphism of the free group \( F_n \), but it induces the identical automorphism of the group \( F_n/R \).

(b) A similar automorphism works in the orientable case, where

\[ r = [x_1, x_2] \cdot [x_3, x_4] \cdot \ldots \cdot [x_{2m-1}, x_{2m}] \]

we assume \( n = 2m \geq 4 \) and our commutator notation is \( [x, y] = x^{-1}y^{-1}xy \):

\[ \varphi \circ : x_i \to x_1^{-2}r^{-x_i^2}x_1x_2 \cdot [x_3, x_4] \cdot \ldots \cdot [x_{2m-1}, x_{2m}] ; x_i \to x_i, i \not\equiv 1. \]

Example 2. (a) A slightly more sophisticated example is provided (in the non-orientable case) by the following automorphism (here \( r = x_1^2x_2^2\ldots x_n^2 \)):

\[ \psi : x_i \to r^{-x_i^{-2}}x_1r^{-x_i^{-2}}r^{x_i^{-2}}x_1 \to x_i, i \not\equiv 1, n; x_n \to r^{-x_i^{-2}}x_n r^{x_i^{-2}}. \]

(b) Based on a similar idea, we get the following automorphism in the orientable case:

\[ \psi_0 : x_1 \to r^{[x_2, x_1]}x_1; x_2 \to x_2^{r^{x_1^{-1}}}; x_i \to x_i, i \not\equiv 1, 2. \]

It is not quite obvious that the automorphisms from Examples 1 and 2 belong to \( \text{Stab}(R) \); it is however obvious that we have \( \varphi(R) \subseteq R \) for any of them; now \( \varphi(R) = R \) follows from the fact (established by Hopf himself) that surface groups are hopfian.

A description of the group \( \ker(\rho) \) for orientable surface groups was obtained by Maclachlan [11] based on earlier results of Birman [4] (who used topology) and Bers [3] (who used analysis) on presentations of mapping class groups. Recently, Dicks and Formanek [7, corollary 5.3] obtained a generating set for \( \ker(\rho) \) (which happens to be free of rank \( n \)) using a purely algebraic approach. We also note that the group \( \ker(\rho)/\text{Inn}_R \) for an orientable surface group \( F/R \) of rank \( > 2 \), is isomorphic to \( F/R \) (see e.g. [7, proposition 5.4]).

In the non-orientable case, there is a description of the group \( \ker(\rho) \) due to Zieschang [19] who used topological methods.

One more remark about automorphisms of surface groups is in order. The group of outer automorphisms of a surface group is known to be isomorphic to the \textit{mapping class group} of the corresponding surface and this latter group was studied extensively by a number of people. In particular, Lickorish, Birman and others obtained several different presentations of mapping class groups based on various geometric ideas (see [5]). Recently, a very 'economical' presentation was found by Wajnryb [16].

Here we suggest a possible approach to the problem of describing \( \ker(\rho) \) which works for many one-relator groups that satisfy a strong type of small cancellation condition, in particular for groups that are in some sense close to surface groups.

Before we give the statement of our first result, we need to say a couple of words
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about the Whitehead graph \( Wh(u) \) of a free group element \( u \). The vertices of this graph correspond to the elements of the generating set \( X \) and their inverses. If the word \( u \) has a subword \( x_i x_j \), then there is an edge in \( Wh(u) \) that connects the vertex \( x_i \) to the vertex \( x_j^{-1} \); if \( u \) has a subword \( x_i x_j^{-1} \), then there is an edge that connects \( x_i \) to \( x_j \), etc. We note that, usually, there is one more edge (the external edge) included in the definition of the Whitehead graph: this is the edge that connects the vertex corresponding to the last letter of \( u \), to the vertex corresponding to the inverse of the first letter. We shall not include the external edge in \( Wh(u) \); instead, we shall consider the Whitehead graph \( Wh(\pi) \) of a cyclic word \( \pi \) when necessary, in which case, of course, the external edge is included.

**Theorem 1.1.** Let \( G = F_n/R \), \( n \geq 3 \), be a one-relator group with a relator \( r \) and such that: \( G \) satisfies a small cancellation condition \( C'(\lambda), \lambda \leq \frac{1}{6} \), and the Whitehead graph of any subword of length \( \geq (1-3\lambda)|r| \) of the word \( r \) or any of its cyclic permutations is 2-connected (i.e. is connected and does not have a cut vertex). Let \( \rho : Stab(R) \to Aut(G) \) be the natural homomorphism. Then \( Ker(\rho) = Inn_R \).

If \( n = 2 \), there might be some additional automorphisms in \( Ker(\rho) \), namely, inner automorphisms induced by elements \( u \in F \) such that \( uR \) is central in \( G = F/R \). We treat 2-generator groups separately, in Theorem 1.3.

We note that it is easy and straightforward to check if a given one-relator group satisfies the conditions of Theorem 1.1.

Although surface groups do not satisfy all those conditions, there are 'similar' groups that do; for example, one-relator groups with the relator of the form \( (x_1^2 x_2^2 \ldots x_n^2)^p \), \( p \geq 2 \), or \( ([x_1, x_2], [x_3, x_4], \ldots, [x_{2m-1}, x_{2m}])^p \), \( p \geq 2 \). (In general, one-relator groups with a 'very long' relator tend to satisfy these conditions).

Note also that the stabilizer of a (cyclic) word

\[
(x_1^2 x_2^2 \ldots x_n^2)^p \quad \text{or} \quad ([x_1, x_2], [x_3, x_4], \ldots, [x_{2m-1}, x_{2m}])^p
\]

is the same as that of \( x_1^2 x_2^2 \ldots x_n^2 \) or \([x_1, x_2], [x_3, x_4], \ldots, [x_{2m-1}, x_{2m}]\) respectively, and that the homomorphism \( \rho : Stab(R) \to Aut(G) \) is onto for any of the corresponding one-relator groups \( G = F/R \) (see [14] or [15]). Therefore, we have:

**Corollary 1.2.** Let \( G = F_n/R \), \( n \geq 3 \), be a one-relator group with the relator of the form \( (x_1^2 x_2^2 \ldots x_n^2)^p \), \( p \geq 2 \), or \( ([x_1, x_2], [x_3, x_4], \ldots, [x_{2m-1}, x_{2m}])^p \), \( p \geq 2 \). Then \( Aut(G) = Stab(R)/Inn_R \).

Theorem 1.1 is proved the following way: first we apply small cancellation theory to make sure that we have a sufficiently large fragment of \( r^\pm 1 \) (or some of its conjugates) in every element of \( R \) and then use this large fragment to show that the Whitehead graph of a cyclically reduced element of the form \( x_i s \), \( s \in R \), cannot be the Whitehead graph of a primitive element of a free group, since those large fragments appear to be 'primality-blocking' because the Whitehead graph of such a fragment does not have a cut vertex. This latter observation is essentially due to E. Turner (informal communication).

Finally, we consider the problem of determining \( Ker(\rho) \) for 2-generator (not necessarily one-relator) groups. These groups are easier to handle because of a very convenient criterion of primitivity for an element of the free group of rank 2, which
Theorem 1.3. Let $R \subseteq [F_2, F_2]$ and let $\rho : \text{Stab}(R) \to \text{Aut}(G)$ be the natural homomorphism. If $\varphi \in \text{Ker}(\rho)$ then $\varphi$ is an inner automorphism induced by an element $s \in F_2$ such that $sR$ is central in $F_2/R$.

In particular, if the group $G = F_2/R$ has trivial centre then $\text{Ker}(\rho) = \text{Inn}_R$.

We emphasize again that when the rank of a group $G$ is bigger than 2, the situation becomes much more complicated. In particular, it is not known what $\text{Ker}(\rho)$ is when $G$ is a free metabelian group of rank $> 2$. It is not even known whether or not $\text{Ker}(\rho)$ is finitely generated as a normal subgroup of $\text{Aut}(F) = \text{Stab}(R)$ in that case. We draw special attention to free metabelian groups because the problem of determining $\text{Ker}(\rho)$ for those groups is closely related to a notorious problem of combinatorial group theory and algebraic topology – that of the Gassner representation of a pure braid group being faithful (see [5, section 3-3]).

2. One-relator groups

Proof of Theorem 1.1. We start by recalling a well known property of the Whitehead graph of a free group element. If an element is primitive and cyclically reduced, then its Whitehead graph is either disconnected or has a cut vertex, i.e. a vertex that, having been removed from the graph together with all incident edges, makes the graph disconnected.

Therefore, if we want to prove that some element of a free group is not primitive, it is sufficient to show that this element has a subword whose Whitehead graph is 2-connected.

Now, by way of contradiction, suppose there is an automorphism $\varphi$ of the free group $F_n$ that takes $x_i$ to $x_i \cdot s_i$, $1 \leq i \leq n$, where $s_i \in R$ and at least one of the $s_i$, say, $s_1$, is non-trivial.

To apply the property of the Whitehead graph discussed above, we need to have some of the elements $x_i \cdot s_i$ cyclically reduced. Suppose, for example, that $u_1 = x_1 \cdot s_1$ is cyclically reduced (in particular, that there is no cancellation between $x_1$ and $s_1$). Then, $u_1$ being primitive, the Whitehead graph $\text{Wh}(u_1)$ must have a cut vertex. However, there is a subword of $u_1$ whose Whitehead graph has no cut vertex. Indeed, a small cancellation condition $C'(\lambda)$, $\lambda \leq \frac{1}{6}$, implies that every element of $R$ has a subword whose length is more than $(1 - 3\lambda)|r|$ and which is a subword of some cyclic permutation of $r^{\pm 1}$ (see [10, theorem V-4-4]). Now the conditions of Theorem 1-1 imply that the Whitehead graph of such a subword has no cut vertex, and therefore neither does $\text{Wh}(u_1)$, a contradiction.

If there is a cancellation between $x_1$ and $s_1$, i.e. if $s_1$ is of the form $x_1^{-1}s'_1$ (but not of the form $s'_1x_1^{-1}$), then we might lose one letter in our long subword described in the previous paragraph; that is why we require the condition on the Whitehead graph to be satisfied by subwords of length $\geq (1 - 3\lambda)|r|$, not just $> (1 - 3\lambda)|r|$ as it appears in Theorem V-4-4 of [10].

Suppose now that all of the $x_i \cdot s_i$ are cyclically reducible; in particular, that there might be cancellations between $x_i$ and $s_i$. Then start composing our automorphism
φ with automorphisms ψ_{i,k}, where k runs through integers, 2 ≤ i ≤ n, and ψ_{i,k} takes $x_i$ to $x_i x_i^{-k}$ and fixes other generators. Thus, $ψ_{i,k} ∘ φ$ takes $x_i$ to $v_i = x_i ∘ s_i ∘ (x_i ∘ s_i)^k$. If for some i, k the element $v_i$ is cyclically reduced (of course, it is still primitive), then we are done since the previous argument applies.

If for all pairs (i, k) the element $v_i$ is not cyclically reduced, this can only mean that every $φ(x_i)$ has a form $w_i = (x_i ∘ y_i)^g$ for some $g ∈ F$, $y_i ∈ R$. In that case, we see that the element $x_i ∘ y_i$ is itself primitive (as a conjugate of a primitive element), but we have seen that this is only possible if $y_i = 1$, i.e. if $φ$ is just the conjugation by $g$.

Thus, we have shown that only inner automorphisms of a free group can belong to $\text{Ker}(ρ)$. Since every n-generator one-relator group has trivial centre provided $n ≥ 3$ (see [10, proposition II-5-22]), this implies $\text{Ker}(ρ) = \text{Inn}_F$ and completes the proof of Theorem 1-1.

3. Two-generator groups

Proof of Theorem 1-3. We start by recalling a convenient necessary condition of primitivity in $F_2$ (see [6]):

if $w$ is a primitive element of $F_2$, then some conjugate of $w$ can be written in the form $x_1^{m_1} x_2^{m_2} \cdots x_i^{m_i} x_2^{l_m}$, so that some $x_i$ occurs either solely with exponent 1 or solely with exponent $−1$.

To prove Theorem 1-3, it is sufficient to prove that if a cyclically reduced primitive element of $F_2$ has the form $x_1 ∘ c \in [F_2, F_2]$, then $c = 1$. For it will follow that the only situation where an automorphism of $F_2$ may induce the identical automorphism of $G$, is where both generators $x_1$ and $x_2$ are taken to their conjugates. In that case, by a well-known result of Nielsen (see e.g. [10, proposition I-4-5]), this automorphism must be inner and the result follows.

Now we prove the statement. If $c \in [F_2, F_2]$, then both $x_1$ and $x_2$ occur in $c$ both with positive and negative exponents. Therefore, the only way for an element of the form $x_1 ∘ c$ to be primitive is to have a cancellation of $x_1$ with the first letter of $c$, which has to be $x_1^{-1}$. Thus, let $c = x_1^{-1} ∘ c_1$, where $c_1$ has only positive occurrences of $x_1$. But this is possible only if $c_1 = x_2^{l_2} x_1 x_2^{-k}$; in that case, we have $x_1 ∘ c = x_2^{l_2} x_1 x_2^{-k}$, a cyclically reducible element, contrary to our assumption. This completes the proof.

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