A general lower bound for collaborative tree exploration

Yann Disser∗,† Frank Mousset‡,§ Andreas Noever‡,¶ Nemanja Škorić‡
Angelika Steger‡

October 7, 2016

Abstract

We consider collaborative graph exploration with a set of $k$ agents. All agents start at a common vertex of an initially unknown graph and need to collectively visit all other vertices. We assume agents are deterministic, vertices are distinguishable, moves are simultaneous, and we allow agents to communicate globally. For this setting, we give the first non-trivial lower bounds that bridge the gap between small ($k \leq \sqrt{n}$) and large ($k \geq n$) teams of agents. Remarkably, our bounds tightly connect to existing results in both domains.

First, we significantly extend a lower bound of $\Omega(\log k / \log \log k)$ by Dynia et al. on the competitive ratio of a collaborative tree exploration strategy to the range $k \leq n \log^c n$ for any $c \in \mathbb{N}$. Second, we provide a tight lower bound on the number of agents needed for any competitive exploration algorithm. In particular, we show that any collaborative tree exploration algorithm with $k = Dn^{1+o(1)}$ agents has a competitive ratio of $\omega(1)$, while Dereniowski et al. gave an algorithm with $k = Dn^{1+\varepsilon}$ agents and competitive ratio $O(1)$, for any $\varepsilon > 0$ and with $D$ denoting the diameter of the graph. Lastly, we show that, for any exploration algorithm using $k = n$ agents, there exist trees of arbitrarily large height $D$ that require $\Omega(D^2)$ rounds, and we provide a simple algorithm that matches this bound for all trees.

1 Introduction

Graph exploration captures the problem of navigating an unknown terrain with a single or multiple autonomous robots. In the abstract setting, we take the perspective of an agent that is located at some vertex of an initially unknown graph, can locally distinguish edges at its current location, and can choose an edge to traverse in its next move. Various scenarios for graph exploration have been studied in the past, for different graph classes and different capabilities of the agent(s). A fundamental goal of exploration is to systematically visit all vertices/edges of the underlying graph. For settings where exploration is possible, we typically ask for efficient exploration algorithms, e.g., in terms of the number of edge traversals.

In this paper, we consider collaborative exploration, where a set of $k$ agents are initially located at some vertex of an unknown undirected graph. We assume agents to move
deterministically, allow them to freely communicate at all times, and to have unlimited computational power and memory at their disposal. In every round each agent may traverse any edge incident to its current location, where the edges incident to a vertex are revealed when that vertex is visited for the first time. The goal is to visit all vertices while minimizing the number of rounds. More precisely, we are interested in the competitive ratio of an exploration strategy, i.e., the worst case ratio between the total number of rounds it needs and the minimum total number of rounds needed to visit all vertices of the same graph, assuming it is known beforehand. We prove new lower bounds for the best-possible competitive ratio of any collaborative exploration algorithm. Our bounds hold even for the much simpler setting of tree exploration. Note that since our results concern trees, it makes no difference whether nodes can be distinguished, and whether the agents need to visit all edges or not.

Let $T_{n,D}$ denote set of all rooted trees with $n$ vertices and height $D$. Each such tree corresponds to an instance of the tree exploration problem in which all $k$ agents start at the root of the tree. Clearly, any offline exploration algorithm needs $\Omega(n/k + D)$ rounds to explore a tree in $T_{n,D}$ using $k$ agents. This is shown to be tight by the following offline exploration algorithm that explores the tree in $\Theta(n/k + D)$ rounds: start with the tree $T$, double its edges, find an Eulerian tour $C$ (of length $2n - 2$), distribute the agents evenly on $C$ (this takes at most $D$ rounds), and explore $T$ by letting each agent walk along $C$ for $O(n/k)$ rounds.

In the online setting, we can explore a tree in $T_{n,D}$ with a single agent using a depth-first traversal in time $O(n)$ and thus we trivially have a competitive ratio of $O(1)$ when $k$ is constant. On the other hand, with $k \geq \Delta^D$ agents, where $\Delta$ is the maximum degree of the tree, we can simply perform a breadth-first traversal, which takes $O(D)$ steps and thus also has competitive ratio $O(1)$. Observe that in the first case $n/k$ dominates the lower bound on the offline optimum, while in the second case $D$ is dominating. We are interested in the best-possible competitive ratios between these two extreme cases.

Surprisingly, Dereniowski et al. [12] showed that already a polynomial number $k = Dn^{1+\varepsilon}$ of agents allows for a BFS-like algorithm that achieves a constant competitive ratio. For smaller teams of agents, Fraigniaud et al. [18, 20] gave a collaborative algorithm with competitive ratio $O(k/\log k)$. This is only slightly better than the trivial upper bound of $O(k)$ that we get by performing a depth first traversal with a single agent. Ortolf and Schindelhauer [23] improved this competitive ratio to $O(k^{\omega(1)})$ for $k = 2^{\omega(\sqrt{\log \log n})}$ and $n = 2^{O(\sqrt{\log n})}$. The only non-trivial lower bound for collaborative tree exploration was given by Dynia et al. [16]. They showed that any deterministic exploration algorithm for $k < \sqrt{n}$ agents has competitive ratio $\Omega(\log k/\log \log k)$.

**Our Results**

We give the first non-trivial lower bounds on the competitive ratio for collaborative tree exploration in the domain $k \geq \sqrt{n}$ (cf. Figure 1). More precisely, we show that for every constant $c \in \mathbb{N}$, any given deterministic exploration strategy with $k \leq n^{\log c}$ agents has competitive ratio $\Omega(\log k/\log \log k)$ on the set of all trees on $n$ vertices. Note that this extends the range of the bound by Dynia et al. [16] for $k < \sqrt{n}$ significantly.

Secondly, we show that for every constant $\varepsilon > 0$, there is a constant $D = D(\varepsilon)$ such that for any exploration algorithm with $k \leq Dn^{1+\varepsilon}$ agents, there exists a tree in $T_{n,D}$ on which the algorithm needs at least $D/(5\varepsilon)$ rounds. This (almost) tightly matches the algorithm of Dereniowski et al. [12], which can explore any tree in at most $(1 + o(1))D/\varepsilon$ rounds using $k = Dn^{1+\varepsilon}$ agents. Our result implies that any exploration algorithm with $k = D^{1+o(1)}$ agents has competitive ratio $\omega(1)$. More precisely, we get that for any function
$0 \leq f(n) \leq o(1)$, there is a function $D = D(n)$ such that every exploration algorithm with $k = Dn^{1+f(n)}$ agents has competitive ratio $\omega(1)$ on the trees in $T_{n,D}$. In contrast, the algorithm of Dereniowski et al. shows that $k = Dn^{1+\varepsilon}$ agents are sufficient to get a competitive ratio $O(1)$ on such trees.

Finally, for every exploration algorithm with $k = n$, we construct a tree of height $D = \omega(1)$ where the algorithm needs $O(D^2)$ rounds. We give a simple algorithm that achieves this bound in general.

**Further Related Work**

Many variants of graph exploration with a single agent have been studied in the past. Any (strongly) connected graph with distinguishable vertices can easily be explored in polynomial time by systematically building a map of the graph. Regarding the exploration of undirected graphs with indistinguishable vertices, Aleliunas et al. [2] showed that a random walk explores any graph in $O(n^3\Delta^2\log n)$ steps, with high probability. In order to turn this into a terminating exploration algorithm the agent needs $\Omega(\log n)$ bits of memory. Fragniaud et al. [19] showed that every deterministic algorithm needs $\Omega(\log n)$ bits of memory, and Reingold [23] gave a matching upper bound. Disser et al. [14] showed that alternatively $\Theta(\log \log n)$ pebbles and bits of memory are necessary and sufficient for exploration, where a pebble is a device that can be dropped to make a vertex distinguishable and that can be picked up and reused later. Diks et al. [13] showed that trees can be explored with $O(\log \Delta)$ memory, and that $\Omega(\log n)$ memory is required if the agent needs to eventually terminate at the start vertex. Ambühl [4] gave a matching upper bound for the latter result.

For the case of directed graphs with distinguishable vertices, Albers and Henzinger [11] gave an exploration algorithm with subexponential running time $d^{O(\log d)}m$ that learns a map of the graph. Here $m$ denotes the number of edges and $d$ is the deficiency of the graph, i.e., the number of edges missing to make the graph Eulerian. This results narrows the gap between a quadratic lower bound and an exponential upper bound introduced by Deng and Papadimitriou [11].
An even more challenging setting (for the agent) is the exploration of directed, strongly connected graphs with indistinguishable vertices. In general the agent needs exponential time to explore a graph in this setting. On the other hand, Bender and Slonim \cite{BenderSlonim} showed that two agents can explore any directed graph in polynomial time, using a randomized strategy. Bender et al. \cite{BenderSlonim} showed that to accomplish this with a single agent we need $\Theta(\log \log n)$ pebbles, i.e., “a friend is worth $O(\log \log n)$ pebbles”. Remarkably, Bender et al. \cite{BenderSlonim} also showed that if the number of vertices is known beforehand, a deterministic agent with a single pebble can explore any directed graph in polynomial time $O(n^8 \Delta^2)$.

The lower bounds for collaborative tree exploration discussed above carry over to the collaborative exploration of general undirected graphs with distinguishable vertices. Also, the algorithm of Dereniowski et al. \cite{Dereniowski12} for $k = Dn^{1+\varepsilon}$ works on general graphs. Additionally, Ortolf and Schindelhauer \cite{OrtolfSchindelhauer} gave a lower bound on the best-possible competitive ratio for randomized algorithms of $\Omega(\sqrt{\log k / \log \log k})$ for $k = \sqrt{n}$. Collaborative exploration by multiple random walks without communication has been considered by Alon et al. \cite{Alon}, Elsässer and Sauerwald \cite{ElsaesserSauerwald}, and Ortolf and Schindelhauer \cite{OrtolfSchindelhauer}.

Graph exploration has been studied in many other settings. Examples include tethered exploration or exploration with limited fuel \cite{Dereniowski11,Dereniowski13}, exploration of mazes \cite{ElsaesserSauerwald,Dereniowski14}, and exploration of polygonal environments \cite{Dereniowski15,Dereniowski16}.

## 2 Results

Our first result extends the lower bound for $k < \sqrt{n}$ agents of Dynia et al. \cite{Dynia16} to the much larger range $k \leq n \log^{O(1)} n$. We prove the following theorem:

**Theorem 2.1.** Let $c$ be any positive integer constant. Then for every $n$ and every $1 \leq k \leq n \log^c n$ there is some $D = D(n, k, c)$ such that the following holds: for any given deterministic exploration strategy with $k$ agents, there exists a tree $T$ on $n$ vertices and with height $D$ on which the strategy needs

$$\Omega\left(\frac{\log k}{\log \log k} \cdot \left(\frac{n}{k} + D\right)\right)$$

rounds.

As mentioned above, there is an offline algorithm that explores any graph with $n$ vertices and height $D$ in time $\Theta(n/k + D)$. From this, we obtain the following corollary to Theorem 2.1.

**Corollary 2.2.** Let $c$ be any positive integer constant. Then any deterministic exploration strategy using $k \leq n \log^c n$ has a competitive ratio of

$$\Omega\left(\frac{\log k}{\log \log k}\right).$$

Our second main result shows that the algorithm of Dereniowski et al. \cite{Dereniowski12} that explores a graph with $k = Dn^{1+\varepsilon}$ agents in time $(1 + o(1))D/\varepsilon$ is almost optimal: using $k \leq Dn^{1+\varepsilon}$ agents it is generally impossible to explore the graph in fewer than $D/(5\varepsilon)$ rounds.

**Theorem 2.3.** Given any constant $\varepsilon > 0$ there is an integer $D = D(\varepsilon)$ such that for sufficiently large $n$ and for every deterministic exploration strategy using $k \leq D \cdot n^{1+\varepsilon}$ agents, there exists a tree on $n$ vertices and with height $D$ on which the strategy needs at least $D/(5\varepsilon)$ rounds.
In the range where \( k \geq n \), the offline optimum is determined by the height \( D \) of the tree. Therefore, the result of Dereniowski et al. mentioned above implies that the competitive ratio is constant when \( k = D \cdot n^{1+\Omega(1)} \). Theorem 2.3 shows in particular that in some sense this is tight:

**Corollary 2.4.** For any function \( 0 \leq f(n) \leq o(1) \), there is a function \( D = D(n) \) such that the competitive ratio of any deterministic exploration strategy using \( k = D \cdot n^{1+f(n)} \) agents is \( \omega(1) \) on the the set \( T_{n,D} \) of all rooted trees with \( n \) vertices and height \( D \).

Finally, it is possible for \( k = n \) agents to explore any tree on \( n \) vertices and of height \( D \) in \( D^2 \) rounds using a breadth-first exploration strategy. More precisely, we can split the \( D^2 \) rounds in \( D \) phases of length \( D \), and in each phase \( 1 \leq i \leq D \) do the following. Let \( A_i \) be the set of unvisited leaves of the tree that is revealed at the start of phase \( i \). Then we send one agent to each vertex in \( A_i \) along a shortest path. This is clearly doable in \( D \) rounds, and after \( D \) such phases, the tree is completely explored. We show that the running time of \( D^2 \) is optimal up to a constant factor:

**Theorem 2.5.** For every \( n \) and every deterministic exploration strategy using \( k = n \) agents, there exists a tree \( T \) on \( n \) vertices and with height \( D = \omega(1) \) such that the strategy needs at least \( D^2/3 \) rounds to explore \( T \).

In all the results above, we have considered the worst-case performance of an exploration strategy on any tree. However, by looking at the proofs of Theorems 2.1 and 2.3 one can see that the heights of our lower bound constructions are typically quite small. We believe it is also natural to ask about the competitive ratio on the set of trees of height at least \( D \), for a given \( D \). We show that at least for subpolynomial heights, the competitive ratio with \( k = \Theta(n) \) agents is unbounded:

**Theorem 2.6.** For any function \( D \leq n^{o(1)} \) and any exploration strategy using \( k = \Theta(n) \) agents, the competitive ratio on the set of all trees of size \( n \) and height at least \( D \) is \( \omega(1) \).

We stress that Theorem 2.6 differs from the other results in that it applies to any height \( D \leq n^{o(1)} \), while in the other results merely state that there exists some height with the desired property.

## 3 Tree exploration games

In order to prove a lower bound on the competitive ratio, we consider a tree exploration game defined as follows. By a *tree exploration game with \( k \) agents* we mean a game with two players, the explorer (the online algorithm) and the revealer (the adversary), played according to the following rules. The game proceeds in rounds which we index by the variable \( t \) (‘time’), the first round being \( t = 0 \). The state of the game at time \( t \) is described by a triple \( (T_t,A_t,\phi_t) \), where \( T_t \) is a rooted tree (the tree revealed at the beginning of round \( t \)), \( A_t \) is a subset of the vertices of \( T_t \) (the subset of visited vertices by round \( t \)), and \( \phi_t : \{1,\ldots,k\} \rightarrow A_t \) is an assignment of the agents to the vertices (where \( \phi_t(i) \) is the location of the \( i \)-th agent at time \( t \)). In round \( t = 0 \) the revealer decides on the initial tree \( T_0 \). The state at time 0 is then given by \( (T_0,A_0,\phi_0) \) where \( A_0 = \{\text{root}(T_0)\} \) and \( \phi_0(x) = \text{root}(T_0) \) for all \( 1 \leq x \leq k \) — that is to say, all agents are initially at the root of \( T_0 \). In every round \( t > 0 \), each player can make a move. First, the explorer creates a new assignment \( \phi_t \) by moving each agent \( i \) to a neighbor of \( \phi_{t-1}(i) \) in \( T_{t-1} \) or by keeping the location of the agent same, i.e., \( \phi_t(i) = \phi_{t-1}(i) \). Then the revealer decides on the new
tree $T_t$, where $T_t$ must be obtained from $T_{t-1}$ by attaching (possibly empty) trees at some vertices $v \in V(T_{t-1}) \setminus A_{t-1}$, where $V(T_{t-1})$ is the set of vertices of $T_{t-1}$. We then let $A_t = A_{t-1} \cup N_t$ where $N_t = \{ \phi_t(i) : 1 \leq i \leq k \}$ is the set of the new agent locations. The game ends in round $t^*$ if all vertices of $T_{t^*}$ are visited at the beginning of round $t^*$, i.e., if $A_{t^*} = V(T_{t^*})$.

This type of game naturally lends itself to proving lower bounds for the time in which $k$ agents can explore an unknown tree. Specifically, consider any deterministic strategy for exploring an unknown tree $T$ with $k$ agents. Such a strategy can be interpreted as a strategy for the explorer in the tree exploration game with $k$ agents. If the revealer can play so that the game lasts for at least $t^*$ rounds, then this means that the proposed exploration strategy needs $t^*$ rounds to explore the tree $T_{t^*}$. We will use this observation to prove lower bounds for the online graph exploration in the following section.

As a side remark, here it is crucial that the strategy is deterministic: if the strategy were allowed to make random choices, then the tree $T_{t^*}$ would turn out to be a random variable that might be highly correlated with the random choices made by the explorer, and it could not serve as an instance on which the strategy performs badly.

4 Lower bound construction

We now give our lower bound construction that establishes the following technical lemma.

**Lemma 4.1.** Let $n, L, m$ be positive integers such that $n \geq L \cdot 16^m$. Then for any deterministic exploration strategy using $k \leq \frac{n^{1+1/m}}{6L(m+1)^2(2L)^{1/m}}$ agents, there exists a tree $T$ on $n$ vertices and of height $Lm$ such that the strategy needs at least $L \binom{m}{2}$ rounds to explore $T$.

**Proof.** Assume that integers $n$, $L$ and $m$ as above are given. Let $k$ be any integer such that $1 \leq k \leq \frac{n^{1+1/m}}{6L(m+1)^2(2L)^{1/m}}$. To prove the lemma, we will describe a strategy for the revealer in the tree exploration game with $k$ agents such that

- the game does not end before round $t^* := L \cdot \binom{m}{2}$, and
- the tree $T_{t^*}$ has height $Lm$ and at most $n$ vertices,

where the notation is as in Section 3. Note that this is enough to prove the lemma.

Before explaining the strategy, we fix some notation. Let

$$\alpha := (2L/n)^{1/m} \quad \text{and} \quad t_i := L \cdot \binom{i+1}{2}$$

for $0 \leq i < m$. For each $t \geq 0$ we can consider the equivalence relation $\sim_t$ on $V(T_t)$ where $u \sim_t v$ if there exists a path between $u$ and $v$ in $T_t$ that avoids the root of $T_t$ (i.e., if they have a common ancestor that is not the root). Since $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ are trees with the same root, we will just write $u \sim v$ instead of $u \sim_t v$ without causing confusion. Then we define $a_t(v) := |\{ x \mid \phi_t(x) \sim v \}|$. In other words, $a_t(v)$ counts the total number of agents that could reach vertex $v$ without passing through the root (under the assignment $\phi_t$). We give the strategy for the revealer in Algorithm 4.
Algorithm 1: The strategy for the revealer.

\begin{algorithm}
\begin{algorithmic}
\State let \( T_0 \) be a ‘star’ consisting of \( \lceil n/(2L) \rceil \) paths of length \( L \) from the root;
\ForEach {round \( t = 1, 2, 3, \ldots \)}
\State let the explorer choose \( \phi_t \);
\If {\( t = t_i \) for some \( 1 \leq i < m \)}
\State let \( K_i \) be a maximal set of vertices in \( V(T_{t_i-1}) \setminus A_{t_i-1} \) s.t.
\State \hspace{1em} (i) every vertex in \( K_i \) has distance \( L \cdot i \) to the root in \( T_{t_i-1} \)
\State \hspace{1em} (ii) there are no two distinct vertices \( u, v \in K_i \) with \( u \sim v \).
\State let \( S_i \subseteq K_i \) be the \( \lceil \alpha|K_i| \rceil \) vertices \( v \in K_i \) with least \( a_{t_i}(v) \);
\State define \( T_{t_i} \) by attaching at each \( v \in S_i \) a path of length \( L - 1 \) with a star with \( L \cdot (i + 1) \cdot a_{t_i}(v) \) leaves at the end;
\Else
\State let \( T_t = T_{t-1} \);
\EndIf
\EndFor
\end{algorithmic}
\end{algorithm}

Figure 2: A sketch of the tree generated by the revealing strategy, for artificial values \( \alpha = 1/3 \) and \( \lceil n/(2L) \rceil = 9 \) (degrees in the actual construction are much larger). The actual shape depends on the distribution of the agents at times \( t_1, t_2 \). Dashed lines represent paths of the specified length.

For a better intuition, we refer the reader to Figure 4, which shows what the tree constructed by this strategy might look like. We establish three claims which are used to show that the algorithm indeed runs for at least \( t^* = t_{m-1} \) rounds and that the tree
constructed in this way has the right properties.

**Claim 1.** For every $0 \leq i < m$ the following holds. The height of $T_{t_i}$ is at most $L \cdot (i + 1)$. Moreover, if $S_1, \ldots, S_i$ are all non-empty, then the height of $T_{t_i}$ is exactly $L \cdot (i + 1)$.

*Proof.* The tree $T_{t_i}$ differs from $T_{t_{i-1}}$ if and only if $S_i$ is non-empty, and in this case it is obtained by attaching trees of height $L$ at some vertices with distance $L \cdot i$ to the root in $T_{t_{i-1}}$. Since $T_0 = T_0$ has height $L$, this implies the claim by induction. \[
\]

**Claim 2.** For all $1 \leq i < m$ and every $v \in S_i$, there exists at least one descendant of $v$ at depth $L \cdot (i + 1)$ in $T_{t_{i+1} - 1}$ that does not belong to $A_{t_{i+1} - 1}$. In particular, for all $1 \leq i < m$ we have $|K_{i+1}| = |S_i|$.

*Proof.* Each vertex at depth $L(i + 1)$ is a descendant of some vertex $v \in S_i$. Moreover, we have $u \sim v$ for any two distinct $u, v \in S_i$. Thus, the second claim follows directly from the first.

For the first claim, consider any $1 \leq i < m$ and $v \in S_i$. Note that

1. at time $t_i$ we create $L \cdot (i + 1) \cdot a_{t_i}(v)$ descendants of $v$ at depth $L \cdot (i + 1)$;
2. $t_{i+1} - t_i = L \cdot (i + 1)$.

Because of this, no agent passing through the root can visit any descendant of $v$ at depth $L \cdot (i + 1)$ before round $t_{i+1}$. On the other hand, the $a_{t_i}(v)$ agents that could visit a descendant at this depth without passing through the root cannot visit all descendants before round $t_{i+1}$. Thus at least one descendant at depth $L \cdot (i + 1)$ must be unvisited at the end of round $t_{i+1} - 1$. \[
\]

**Claim 3.** For every $1 \leq i < m$ we have the bounds

$$|S_i| \geq \frac{\alpha^i n}{2L} \geq \frac{1}{\alpha} \quad \text{and} \quad |S_i| \leq \frac{(2\alpha)^i n}{2L}.$$  

*Proof.* By definition we have $\alpha = (2L/n)^{1/m} < 1$ and thus $\alpha^m = 2L/n$, which gives us

$$\frac{\alpha^i n}{2L} \geq \frac{\alpha^{m-1} n}{2L} = 1/\alpha$$

for all $1 \leq i < m$.

For the lower bound, note that since $A_0$ contains only the root, we have $|K_1| = \lceil n/(2L) \rceil$. By the definition of $S_i$, we have $|S_i| \geq \alpha|K_i|$ for all $1 \leq i < m$. Moreover, if $2 \leq i < m$ then by Claim 2 we have $|K_i| = |S_{i-1}|$. The lower bound then follows by induction.

For the upper bound, note that $K_1 \leq n/(2L) + 1 \leq n/L$, where the last inequality uses $n \geq 2L$. Moreover, using $|K_i| \geq 1/\alpha$ we have $|S_i| \leq \alpha|K_i| + 1 \leq 2\alpha|K_i|$ for all $1 \leq i < m$. Finally, if $2 \leq i < m$ then $|K_i| = |S_{i-1}|$ by Claim 2 and the upper bound follows by induction. \[
\]

Since $|S_i| > 0$ implies in particular that $A_{t_{i-1}} \neq V(T_{t_{i-1}})$, we conclude from Claim 3 that the game does not stop before reaching round $t_{m-1} = L \cdot \binom{m}{2} = t^*$. Moreover, from
Claim 4 and Claim 5 we see that $T_{t_{m-1}}$ is a tree with height $L \cdot m$. To complete the proof we need to show that $|V(T_{t_{m-1}})| \leq n$. We have

$$|V(T_{t_{m-1}})| \leq \left\lceil n/(2L) \right\rceil \cdot L + 1 + \sum_{i=1}^{m-1} \sum_{v \in S_i} (L - 1 + L \cdot (i + 1) \cdot a_t(v))$$

$$\leq n/2 + L + 1 + \sum_{i=1}^{m-1} L(i + 1) \sum_{v \in S_i} a_t(v) + \sum_{i=1}^{m-1} |S_i|(L - 1). \quad (1)$$

To bound the double sum note that $|K_i| \geq |S_i| \geq 1/\alpha$ (Claim 3) implies that $[\alpha|K_i|] \leq 2\alpha|K_i|$. Note also that the sum $\sum_{v \in K_i} a_t(v)$ in (1) is at most $k$, as no two vertices $u, v$ from $K_i$ are in the same subtree, i.e., $u \not\sim v$. Since $S_i$ contains the $[\alpha|K_i|] \leq 2\alpha|K_i|$ vertices of $K_i$ with least $a_t(v)$, we thus have

$$\sum_{v \in S_i} a_t(v) \leq 2\alpha \sum_{v \in K_i} a_t(v) \leq 2\alpha k,$$

and therefore

$$\sum_{i=1}^{m-1} L(i + 1) \sum_{v \in S_i} a_t(v) \leq L(m + 1)^2 \alpha k. \quad (2)$$

To bound the simple sum in (1), we use the upper bound from Claim 3 and obtain

$$\sum_{i=1}^{m-1} |S_i|(L - 1) \leq (L - 1) \sum_{i=1}^{\infty} \frac{(2\alpha)^i n}{2L} = \frac{L - 1}{2L} \cdot 2\alpha n \sum_{i=0}^{\infty} (2\alpha)^i \leq \frac{2\alpha n}{2 - 4\alpha}. \quad (3)$$

Combining (1) with (2) and (3), we get

$$|V(T_{t_{m-1}})| \leq n/2 + L + 1 + L(m + 1)^2 \alpha k + \frac{2\alpha n}{2 - 4\alpha}. \quad (4)$$

Since $n \geq L \cdot 16^m \geq 12L$ we have $L + 1 \leq 2L \leq n/6$. By the definition $\alpha = (2L/n)^{1/m}$ and the assumption $k \leq n^{1+1/m}/(6L(m + 1)^2(2L)^{1/m})$ we have

$$L(m + 1)^2 \alpha k = L(m+1)^2(2L/n)^{1/m} k \leq n/6.$$

Finally, $n \geq L \cdot 16^m$ implies that $\alpha \leq 1/8$ and so the last term in (4) is also at most $n/6$. Hence $|V(T_{t_{m-1}})| \leq n/2 + 3n/6 = n$. \qed

5 Consequences for competitiveness

We now use Lemma 4.1 to derive consequences for best-possible competitive ratios of collaborative tree exploration algorithms. In the proofs below, log is always to the natural base $e$.

**Theorem 2.1.** Let $c$ be any positive integer constant. Then for every $n$ and every $1 \leq k \leq n \log^c n$ there is some $D = D(n, k, c)$ such that the following holds: for any given deterministic exploration strategy with $k$ agents, there exists a tree $T$ on $n$ vertices and with height $D$ on which the strategy needs

$$\Omega\left(\frac{\log k}{\log \log k} \cdot (n/k + D)\right)$$

dounds.
Proof. By the result of Dynia et al. [10] it suffices to consider the case where \( k \geq \sqrt{n} \). Let \( c > 0 \) be a constant and assume \( k \leq n \log^c n \). We apply Lemma 4.1 with \( m = \lceil (8 + c) \log \log n \rceil \) and \( L = \lceil n/(mk) \rceil \). Using \( k \geq \sqrt{n} \), we have \( L = O(\sqrt{n}) \) and \( m = o(\log n) \) and thus \( n \geq L \cdot 16^m \) holds for sufficiently large \( n \). The lemma states that if

\[
k \leq \frac{n^{1+1/m}}{6L(m+1)^2(2L)^{1/m}} \geq \frac{n^{1+1/m}}{24m^2} \geq \frac{n \log^{8+c} n}{24 \log^2 n} \geq k,
\]

then there is a tree of height \( D := Lm \) on which the strategy needs at least \( L(m) = \Omega((n/k + D) \cdot \log k / \log \log k) \) rounds. To complete the proof, we need to show that (5) holds for all \( 1 \leq k \leq n \log^c n \). We split the analysis to two cases. Let us first assume \( k \geq n/m \) and thus \( L = 1 \). This implies

\[
\frac{n^{1+1/m}}{6L(m+1)^2(2L)^{1/m}} \geq \frac{n^{1+1/m}}{24m^2} \geq \frac{n \log^{8+c} n}{24 \log^2 n} \geq k,
\]

when \( k \leq n \log^c n \) and for sufficiently large \( n \).

Now we consider the case \( k < n/m \). Using that assumption and the definition of \( L \) we obtain \( L(m+1)^2 \leq 4mn/k \) and \( 2L \leq 4n/(mk) \). Putting it all together we have

\[
\frac{n^{1+1/m}}{6L(m+1)^2(2L)^{1/m}} = \frac{n}{6L(m+1)^2} \left( \frac{n}{2L} \right)^{1/m} \geq \frac{k}{24m} \left( \frac{mk}{4} \right)^{1/m} \geq k,
\]

where the last inequality holds for \( k \geq \sqrt{n} \) because, for sufficiently large \( n \),

\[
(mk)^{1/m} \geq k^{1/m} \geq \frac{\log n}{2m} \geq e^{\frac{(8+c) \log \log n}{4}} \geq (\log n)^2 \geq 100m.
\]

\[\square\]

**Theorem 2.3.** Given any constant \( \varepsilon > 0 \) there is an integer \( D = D(\varepsilon) \) such that for sufficiently large \( n \) and for every deterministic exploration strategy using \( k \leq D \cdot n^{1+\varepsilon} \) agents, there exists a tree on \( n \) vertices and with height \( D \) on which the strategy needs at least \( D^2/3 \) rounds.

Proof. We choose \( L = 1 \) and \( m = \lceil 1/2\varepsilon \rceil \) in Lemma 4.1. The claim is trivial unless \( \varepsilon < 1/5 \), so we can eliminate rounding and assume generously that \( 1/m \geq 1.4\varepsilon \). The condition \( n \geq L \cdot 16^m \) is clearly satisfied for sufficiently large \( n \).

By Lemma 4.1 there is a tree \( T \) of height \( m \) that needs time \( \binom{m}{2} \geq m/(5\varepsilon) \) to be explored, provided the team has size at most (for \( n \) sufficiently large)

\[
k \leq n^{1+1.4\varepsilon}/(12(m+1)^2) \leq m \cdot n^{1+\varepsilon} = D \cdot n^{1+\varepsilon}.
\]

\[\square\]

**Theorem 2.5.** For every \( n \) and every deterministic exploration strategy using \( k = n \) agents, there exists a tree \( T \) on \( n \) vertices and with height \( D = \omega(1) \) such that the strategy needs at least \( D^2/3 \) rounds to explore \( T \).

Proof. We choose \( L = 1 \) and \( m = \lfloor \sqrt{\log n} \rfloor \) in Lemma 4.1. Then \( n \geq L \cdot 16^m \) holds for sufficiently large \( n \). Note also that for sufficiently large \( n \),

\[
\frac{n^{1+1/m}}{12(m+1)^2} = \Omega(n \cdot e^{\sqrt{\log n} / \log n}) \geq n.
\]

The lemma now states that there exists a tree \( T \) of height \( m \) such that the given strategy with \( k = n \) agents needs at least \( \binom{m}{2} \) rounds to explore \( T \). Since for large enough \( n \) we have \( \binom{m}{2} \geq m^2/3 \), this implies the theorem.
Theorem 2.6. For any function $D \leq n^{o(1)}$ and any exploration strategy using $k = \Theta(n)$ agents, the competitive ratio on the set of all trees of size $n$ and height at least $D$ is $\omega(1)$.

Proof. Suppose that $D \leq n^{o(1)}$, i.e., $D = n^{1/f(n)}$, where $f(n)$ is a function which tends to infinity with $n$. Let $L = D$ and note that we have

$$\frac{16L^{1/m}}{n^{1/m}} \leq \frac{L^{1+1/m}(m+1)^2}{n^{1/m}} \leq \frac{4m^2n^{2/f(n)}}{n^{1/m}}.$$ \(\text{If we choose } m = m(n) = \omega(1) \text{ as a function growing sufficiently slowly such that we have } m \leq \min\{(f(n))^{1/2}, (\log n)^{1/2}\}, \text{ then the following is true:}

$$\frac{4m^2n^{2/f(n)}}{n^{1/m}} = 4 \cdot e^{2\log m + 2(\log n)/f(n) - \log n/m} \to 0.$$ \(\text{This implies } 16L^{1/m} = o(n^{1/m}) \text{ and } L^{1+1/m}(m+1)^2 = o(n^{1/m}). \text{ In particular, } n \geq L \cdot 16^m \text{ for sufficiently large } n. \text{ Moreover, if } n \text{ is large enough then } k = \Theta(n) \text{ implies}

$$\frac{n^{1+1/m}}{6L(m+1)^2(2L)^{1/m}} = \frac{n^{1+1/m}}{o(n^{1/m})} \geq k.$$ \(\text{By Lemma 4.1 there exists a tree } T \text{ with height } Lm \geq D \text{ on which the strategy needs } L(m) = \omega(Lm) \text{ rounds. Since } k = \Theta(n), \text{ the offline optimum is } O(Lm + n/k) = O(Lm), \text{ so the competitive ratio on the set of trees of height at least } D \text{ is } \omega(1), \text{ as claimed.} \)

6 Conclusions

In this paper we presented new lower bounds for collaborative tree exploration. Including our results, the following bounds are now known. For $k = O(1)$ or $k \geq D \cdot n^{1+\varepsilon}$ agents, a competitive ratio of $\Theta(1)$ can be achieved. For $\omega(1) \leq k \leq n \log^c n$, the best-possible competitive ratio is bounded by $\Omega((\log k)/\log \log k)$, and no constant competitive ratio is possible when $n \log^c n \leq k \leq D \cdot n^{1+o(1)}$. On the other hand, the best exploration algorithms for trees in the domain $k \leq D \cdot n^{1+o(1)}$ stay close to the trivial competitive ratio of $k$ (the best ratios are $k/\log k$ and $k^{o(1)}$, depending on the domain).

In summary, we now fully understand the domain where constant competitive ratios are possible, but, outside this domain, a wide gap persists.

Acknowledgments

We would like to thank Rajko Nenadov for useful discussions.
Bibliography

[1] S. Albers and M. R. Henzinger. Exploring unknown environments. *SIAM Journal on Computing*, 29(4):1164–1188, 2000.

[2] R. Aleliunas, R. M. Karp, R. J. Lipton, L. Lovász, and C. Rackoff. Random walks, universal traversal sequences, and the complexity of maze problems. In *Proceedings of the 20th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 218–223, 1979.

[3] N. Alon, C. Avin, M. Koucký, G. Kozma, Z. Lotker, and M. R. Tuttle. Many random walks are faster than one. *Combinatorics, Probability and Computing*, 20(4):481–502, 2011.

[4] C. Ambühl, L. Gąsieniec, A. Pelc, T. Radzik, and X. Zhang. Tree exploration with logarithmic memory. *ACM Transactions on Algorithms*, 7(2):1–21, 2011.

[5] B. Awerbuch, M. Betke, R. L. Rivest, and M. Singh. Piecemeal graph exploration by a mobile robot. *Information and Computation*, 152(2):155–172, 1999.

[6] M. A. Bender, A. Fernández, D. Ron, A. Sahai, and S. Vadhan. The power of a pebble: Exploring and mapping directed graphs. *Information and Computation*, 176(1):1–21, 2002.

[7] M. A. Bender and D. K. Slonim. The power of team exploration: Two robots can learn unlabeled directed graphs. In *Proceedings 35th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 75–85, 1994.

[8] M. Blum and D. Kozen. On the power of the compass (or, why mazes are easier to search than graphs). In *Proceedings of the 19th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 132–142, 1978.

[9] J. Chalopin, S. Das, Y. Disser, M. Mihalák, and P. Widmayer. Mapping simple polygons: How robots benefit from looking back. *Algorithmica*, 65(1):43–59, 2011.

[10] J. Chalopin, S. Das, Y. Disser, M. Mihalák, and P. Widmayer. Mapping simple polygons. *ACM Transactions on Algorithms*, 11(4):1–16, 2015.

[11] X. Deng and C. H. Papadimitriou. Exploring an unknown graph. *Journal of Graph Theory*, 32(3):265–297, 1999.

[12] D. Dereniowski, Y. Disser, A. Kosowski, D. Pająk, and P. Uznański. Fast collaborative graph exploration. *Information and Computation*, 243:37–49, 2015.

[13] K. Diks, P. Fraigniaud, E. Kranakis, and A. Pelc. Tree exploration with little memory. *Journal of Algorithms*, 51(1):38–63, 2004.

[14] Y. Disser, J. Hackfeld, and M. Klimm. Undirected graph exploration with $\Theta(\log \log n)$ pebbles. In *Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 25–39, 2016.

[15] C. A. Duncan, S. G. Kobourov, and V. S. A. Kumar. Optimal constrained graph exploration. *ACM Transactions on Algorithms*, pages 380–402, 2006.
[16] M. Dynia, J. Łopuszański, and C. Schindelhauer. Why robots need maps. In Proceedings of the 14th International Colloquium on Structural Information and Communication Complexity (SIROCCO), pages 41–50, 2007.

[17] R. Elsässer and T. Sauerwald. Tight bounds for the cover time of multiple random walks. Theoretical Computer Science, 412(24):2623–2641, 2011.

[18] P. Fraigniaud, L. Gąsieniec, D. R. Kowalski, and A. Pelc. Collective tree exploration. Networks, 48(3):166–177, 2006.

[19] P. Fraigniaud, D. Ilcinkas, G. Peer, A. Pelc, and D. Peleg. Graph exploration by a finite automaton. Theoretical Computer Science, 345(2-3):331–344, 2005.

[20] Y. Higashikawa, N. Katoh, S. Langerman, and S.-i. Tanigawa. Online graph exploration algorithms for cycles and trees by multiple searchers. Journal of Combinatorial Optimization, 28(2):480–495, 2012.

[21] F. Hoffmann. One pebble does not suffice to search plane labyrinths. In Proceedings of the 3rd International Symposium on Fundamentals of Computation Theory (FCT), pages 433–444, 1981.

[22] C. Ortolf and C. Schindelhauer. Online multi-robot exploration of grid graphs with rectangular obstacles. In Proceedings of the 24th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA), pages 27–36, 2012.

[23] C. Ortolf and C. Schindelhauer. A recursive approach to multi-robot exploration of trees. In Proceedings of the 21st International Colloquium on Structural Information and Communication Complexity (SIROCCO), pages 343–354, 2014.

[24] C. Ortolf and C. Schindelhauer. Strategies for parallel unaware cleaners. Theoretical Computer Science, 608:178–189, 2015.

[25] O. Reingold. Undirected connectivity in log-space. Journal of the ACM, 55(4):1–24, 2008.