The short-time limit of the Dirichlet partition function and the image method

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Abstract. In this paper we rigorously derive the $t \to 0^+$ asymptotics of the free partition function $Z_\Omega(t)$ for a diffusion process on tessellations of the $d$-dimensional Euclidean space $\mathbb{E}^d$, $d = 1, 2, 3$ with an absorbing boundary. Utilising the path integral approach and the method of images for domains which are compatible with finite reflection subgroups of the orthogonal group $O_d$, we solve this problem following a group theoretic method which was lacking from the literature.

1. Introduction

We study the short-time asymptotics of the partition function $Z_\Omega(t)$ defined by

$$Z_\Omega(t) := \text{Tr}_\Omega(e^{t\Delta_D}) = \int_{P.B.C.} \mathcal{D}x \, e^{-S[x]} = \int_{\Omega} p_t(x, x) \, dx,$$  \hspace{1cm} (1)

where $\Delta_D$ is the Dirichlet Laplacian, the path integral is evaluated with periodic boundary conditions (P.B.C.) and the Euclidean worldline action, for a quantum particle of mass $m$, is given by

$$S[x] = \frac{m}{2\hbar} \int_0^\beta \|\dot{x}\|^2 \, dt.$$ \hspace{1cm} (2)

Utilizing the path integral approach to the heat kernel we apply the image method in $d = 1, 2, 3$ dimensions. The image method was pointed out by Kac in his seminal paper [11] but no explicit calculation has been carried out since then along a group theory direction. The power of this method is based on the connection between the heat kernel in $\Omega$ and the heat kernel in $\mathbb{E}^d$ through the action of the reflection group. The knowledge of the spectrum of the Dirichlet Laplacian (particular cases are studied by [9,10,14]), which conveys all the geometrical information of the domain, is redundant.

As far as the applications is concerned, the most interesting one from the Quantum Field Theory (Q.F.T.) point of view, is the computation of the one-loop effective action for a free, real scalar field living on a tessellation of the Euclidean space. This is represented on a flat manifold, after imposing suitable boundary conditions (Dirichlet or Neumann), by

$$\Gamma[\phi] = -\log \text{Det}^{-\frac{1}{2}}(-\Delta) = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \text{Tr}_\Omega(e^{T\Delta}) = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int_{P.B.C.} \mathcal{D}X \, e^{-S[X]}.$$ \hspace{1cm} (3)
The Q.F.T. action is given by

\[ S_{Q.F.T.}[\phi] = \frac{1}{2} \int_{\Omega} \|\nabla \phi\|^2 \, d\mathbf{X} \quad \text{where} \quad d\mathbf{X} = \prod_{i=1}^{d} dx_i \]  

and the path integral contains the worldline action \( (2) \) with \( \beta \) replaced by \( T \).

The outline of the paper is as follows. In Section 2, we review basic definitions and fundamental theorems which allow us to give a concise introduction to the subject and prepare the mathematical background needed for proving the identification of the function \( p_{\Omega}^{11} \) to the heat kernel on \( \Omega \) (Proposition (4.5)).

In Section 3, we establish the worldline approach to compute the heat kernel for a semi-bounded domain \( \Omega \subset \mathbb{E}^d \). We state that the contribution from the direct paths to the heat kernel is identical to that of the bouncing off the boundary paths, provided that the quantum fluctuations have a linear part in time (Propositions (3.1) and (3.2)). Extending the path space into the complement of \( \Omega \), using the image method, we derive the desired heat kernel on \( \Omega \). The result of this section, with slight modifications, would be applied to our problem.

Section 4 presents the new group theoretic method for attacking the problem. We compute \( Z_{\Omega}(t) \) in the \( t \to 0^+ \) limit for \( d = 1 \), using the image method under the action of the infinite dihedral group \( Dih_{\infty} \). In higher than one dimension, one faces the problem of classification of all the bounded domains dictated by the image method. This issue was partly solved in [13] and a complete solution was recorded in Coxeter’s works [3,4] on discrete groups (see also [12] for an updated treatment of the subject) which are related to the classification of semi-simple Lie algebras according to Cartan and Weyl. The constraint imposed by the absence of a virtual mirror between the two given ones, after successive reflections, limits the angle of the mirrors to be of the form \( \pi/m \), \( m \in \mathbb{N}/\{1\} \). Bearing this in mind, for the infinite two-dimensional wedge, we prove that only elements of the cyclic subgroup of the finite dihedral group of order \( 2m \cdot Dih_{2m} \) contribute to the topological term (Proposition (4.1)). The rest of the elements of \( Dih_{2m} \) contribute to the area and boundary length terms, thus Kac’s result for simply connected domains in \( d = 2 \) is recovered. We also prove, for Dirichlet boundary conditions, that expression (9) is the actual heat kernel on \( \Omega \) in the equilateral case. The nontrivial proof can be extended to more general tessellations of the Euclidean space following similar steps. Repeating the same procedure for the infinite trihedral \( (2,2,r) \) case, as was done in two dimensions, we find a closed formula for the corresponding topological term (Proposition (4.6)). Next, for three-dimensional tessellations which are compatible with the image method, their \( Z_{\Omega}(t) \) in the short-time limit is determined. It is worth noting that, for tessellations, the asymptotic behaviour of the hyperrectangle partition function in \( d \)-dimensions is possibly the only concrete result we have at present [7] (setting \( s = 1 \)).

2. Preliminaries

In this section, closely following [6], we provide the mathematical background needed to introduce the reader to the problem and address the way one has to incorporate the method of images in order to determine the trace of the Dirichlet heat kernel.

Let us denote by \( \mathcal{M} \) the Euclidean space equipped with the Euclidean metric \( g_{\mu\nu} = \delta_{\mu\nu}, \mu, \nu = 1, \ldots, d \). The next theorem establishes the existence of an integral kernel for the operator \( \hat{P}_t \).

**Theorem 2.1** For any \( x \in \mathcal{M} \) and for any \( t > 0 \), there exists a unique \( p_{t,x} \in \mathcal{L}^2(\mathcal{M}) \), such that for all \( f \in \mathcal{L}^2(\mathcal{M}) \),

\[ \hat{P}_t f(x) \equiv e^{t\Delta} f(x) = \langle p_{t,x}, f \rangle_{\mathcal{L}^2(\mathcal{M})} = \int_{\mathcal{M}} p_{t,x}(y) f(y) dy. \]  

(5)
More over for any relatively compact set $K \subset \mathcal{M}$ and for any $t > 0$, we have

$$\sup_{x \in \mathcal{M}} \|p_{t,x}\|_{L^2(\mathcal{M})} \leq C(K, g, d)(1 + t^{-\sigma})$$

(6)

where $\sigma$ is the smallest integer larger than $d/4$.

If the function $p_{t,x}$ is defined for every $y$ then it is called the heat kernel of the manifold and satisfies a number of properties which can be found in [6] and [8]. The heat kernel apart from being the integral kernel of the heat semigroup it can also be characterized as the minimal positive fundamental solution of the heat equation. The fundamental solution is given by the well-known Gauss-Weierstrass function

$$p_{0,t}(x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}, \quad t > 0, \ x, y \in \mathbb{R}^d.$$  

(7)

The next theorem helps us to identify whether the fundamental solution is the heat kernel, by using the “boundary condition”.

**Theorem 2.2** Let $u(t, x)$ be a non-negative fundamental solution to the heat equation at the point $y \in \mathcal{M}$. If $u(t, x) \equiv 0$ as $x \to \infty$ where the convergence is uniform in $t \in (0, T)$ for any $T > 0$, then $u(t, x) \equiv p_{t}(x, y)$.

If $\mathcal{M} = \Omega$, where $\Omega$ is a bounded and open subset of $\mathbb{R}^d$, then every compact subset of $\mathcal{M}$ is contained in $\Omega_\delta = \{x \in \Omega : d(x, \partial \Omega) \geq \delta\}$, for some $\delta > 0$. In Theorem (2.2) the convergence $x \to \infty$ in $\mathcal{M}$ means $d(x, \partial \Omega) \to 0$, namely $x \to \partial \Omega$. Also the condition that $u(t, x)$ is non-negative can be replaced by the weaker one

$$\limsup_{k \to \infty} u(t_k, x_k) \geq 0$$

for every sequence $(t_k, x_k)$ such that $t_k \to 0$ and $x_k \to x \in \mathcal{M}$. This theorem will be used in subsection (4.2) for proving that the fundamental solution, in the equilateral triangle case, is actually the heat kernel on $\Omega \subset \mathcal{M}$.

We now speculate on the structure of the heat kernel on $\Omega$. A rigorous proof will be given by Proposition (4.5). Let $l = 0, 1, 2 \ldots$ be an enumeration of the virtual domains generated by successive reflections of the fundamental region in the bounding hyperplanes of $\Omega$. Let also $R_l$ be a composition of reflections that maps $\Omega$ to the region $l$. We denote by $s(l)$ the length of each element $R_l$, namely the minimum number of reflections needed to construct $R_l$. Then, for Dirichlet boundary conditions, one can write the following expression for the heat kernel on $\Omega$

$$p_l^\Omega(x, y) = \sum_{l=0}^{\infty} (-1)^{s(l)} p_{0,l}(x, R_l y) \quad \forall x, y \in \Omega$$

(9)

where $R_0 := id$ is the identity matrix. $R_l y \in \bar{\Omega}$ and $\bar{\Omega}$ is the closure of $\Omega$. The asymptotic behaviour of the Dirichlet partition function in the $t \to 0^+$ limit is dominated by the diagonal elements of (9) which receive contributions from the heat kernels of the unbounded space for the virtual source points clustering around each vertex of the polytope (see Figure (4.2)). This observation reduces expression (9) down to

$$p_l^\Omega(x, x) \xrightarrow{t \to 0^+} \sum_{n=0}^{|\mathcal{G}_v|-1} (-1)^{s(n)} p_{0,n}(x, R_n x)$$

(10)

where $|\mathcal{G}_v|$ is the order of the reflection group at vertex $v$. Thus integrating $p_l^\Omega(x, y)$, as $x \to y$, over a suitable subdomain of $\Omega$ and after taking the $t \to 0^+$ limit we produce the corresponding expressions.
3. The path integral approach to the heat kernel for semi-bounded domains

We consider the flat manifold $\Omega = \mathbb{R}^+ \times \partial \Omega$ with local coordinates $X = (x_1, x_2, \ldots, x_d)$ where the boundary $\partial \Omega = \mathbb{R}^{d-1}$ is located at $x_1 = 0, x_1 \in [0, \infty)$ and $x_j \in \partial \Omega, j = 2, \ldots, d - 1$. The trajectories of a free, nonrelativistic particle of mass $m$, are described by orientable and piecewise continuously differentiable mappings: $X : [0, \beta] \to \Omega$. The heat kernel is the solution of the Schrödinger equation in Euclidean time satisfying a Dirac type boundary condition at $\beta = 0$

$$-\hbar \frac{\partial}{\partial \beta} p(Y, \beta; X, 0) = -\frac{\hbar^2}{2m} \Delta_X p(Y, \beta; X, 0),$$

$$p(Y, 0; X, 0) = \delta^{(d)}(X - Y).$$

One method to compute the heat kernel from the Schrödinger equation is to sum over all paths $X(t)$ linking the initial point $X_{in.} := X(0)$ to the final one $X_{fin.} \equiv Y := X(\beta)$, in time $\beta$

$$p(X_{fin.}, \beta; X_{in.}, 0) = \int_{X_{in.}}^{X_{fin.}} D X e^{-S[X]}.$$  

The reader may consult [1] for the inclusion of an interaction term. The Euclidean action $S[X]$ in the exponent, performing a time rescaling of (2) by using the transformation $\tau = t/\beta, \tau \in [0, 1]$, becomes

$$S[X] = \frac{m}{2\beta \hbar} \int_0^1 \| \dot{X} \|^2 d\tau.$$  

There are two distinctive classes of paths denoted by $C_i, i = 1, 2$. The first class $C_1$ has as representative the open, simple and orientable curve $X_{PR}(\tau)$ which starts at the point $X_P = X(0)$ and ends at $X_R = X(1)$ (see Figure (3)). The contribution to the path integral from the paths which bounce off the boundary is the same as that of $X_{PR}(\tau)$ (see Proposition (3.1)).

The second class $C_2$ has as representative the open, simple and orientable curve $X_{PR}'(\tau)$ with endpoints $X_P$ and $X_{PR}' = \tilde{X}(1)$, respectively. The reflected paths, $X_{BR}$, which live in $\Omega$ can be mapped onto their images (the dashed red line in the Figure (3)) and again the paths $X_{PB}(\tau) + \tilde{X}_{BR}(\tau)$ give the same contribution as that of $\tilde{X}_{PR}$.  

![Figure 1. Solid and dashed red line segments represent the classical paths in $\Omega = \mathbb{R}^+ \times \mathbb{R}^2$ and $\mathbb{R}^3$ respectively. The wiggly curve stands for the quantum trajectory in $\Omega$.](image)

Each path can be decomposed into a fixed classical background part, satisfying the geodesic equation and subjected to appropriate homogeneous (or non-homogeneous) Dirichlet boundary
conditions, plus quantum fluctuations [2]. Solving this problem (the interested reader may consult [8] for the proof) we find that the paths are given by the following convex combinations of the initial and final points

\[ X(\tau) = (1 - \tau)X_P + \tau X_R + Q(\tau), \]

\[ \tilde{X}_{PB}(\tau) = \left( 1 - \frac{\tau}{\tau_1} \right) X_P + \frac{\tau}{\tau_1} X_B + Q_{PB}(\tau), \]

\[ \tilde{X}_{BR}(\tau) = \left( 1 - \frac{\tau}{\tau_1} \right) X_B + \frac{\tau}{\tau_1} X_R + \tilde{Q}_{BR}(\tau). \]

The heat kernel for the class \( \mathcal{C}_1 \) of paths is written as

\[ p_{\mathcal{C}_1, \Omega}(X_R, 1; X_P, 0) = e^{-\frac{m}{\Delta_0} ||X_R - X_P||^2} \int_{Q(0)=0}^{Q(1)=0} \mathcal{D}Q e^{-S[Q]}, \]

where

\[ \int_{Q(0)=0}^{Q(1)=0} \mathcal{D}Q e^{-S[Q]} = \left( \frac{m}{2\pi\Delta_0} \right)^{d/2}. \]

**Proposition 3.1 ([8])** The following identity for the heat kernels holds

\[ p_{\mathcal{C}_1, \Omega}(X_R, 1; X_P, 0) = p_{\bar{\mathcal{C}}_1, \Omega}(X_R, 1; X_P, 0) \]

where \( \mathcal{C}_1 \) consists of direct paths with endpoints \( \{X_P, X_R\} \) while \( \bar{\mathcal{C}}_1 \) is composed by paths which bounce off the boundary but have the same endpoints as previously.

**Proposition 3.2 ([8])** The asymptotic behaviour of the paths \( Q^j_{PB}(\tau) \), in the \( Q^j_B \to 0 \) limit, is given by

\[ Q^j_{PB}(\tau) \sim \sqrt{\frac{2}{\tau_1}} \sin(\lambda_{m_j} \tau) + (-1)^{m_j} \frac{\tau}{\tau_1} Q^j_B \cos(\lambda_{m_j} \tau), \lambda_{m_j} = \frac{\pi m_j}{\tau_1}. \]

The heat kernel for the semi-bounded space is then given by

\[ p_t(x, y) = p_{0,1}(\|X - Y\|) - p_{0,1}(\|X + Y\|) \]

where \( p_{0,1} \) is the fundamental solution (7).

4. The \( Z_\Omega(t) \) in the \( t \to 0^+ \) limit

4.1. The \( d = 1 \) case

In \( d = 1 \), the reflection group, \( \mathcal{G} \), coincides to the infinite dihedral group \( \text{Dih}_\infty \) with defining relations: \( R_{a_1}^2 = R_{a_2}^2 = I \) where \( R_{a_i} \)'s are involuntary transformations and represent reflections with respect to the boundary points of \( \Omega \subset \mathbb{R} \). In this presentation \( \text{Dih}_\infty \) is the free product of two \( \mathbb{Z}/2\mathbb{Z} \). Alternatively \( \text{Dih}_\infty \) is isomorphic to the semi-direct product \( \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = \langle g, h| h^2 = I, hgh = g^{-1} \rangle \). Assuming that \( \Omega = (0, L) \), for every \( y \in \Omega \), we generate the following two infinite sequences of virtual source points depending on whether we start reflection from the left or right fixed point of the isometry \( R_{a_i} \) (see Figure (2)).
Table 1. The locations of the virtual image points derived from the action of the elements of the Dih$_{\infty}$ group.

| Group Element | Location of v.i.p. | Group Element | Location of v.i.p. |
|---------------|------------------|---------------|------------------|
| $R_{a_1}$     | $-y$             | $R_{a_3}$     | $-y + 2L$        |
| $R_{a_1} \cdot R_{a_2}$ | $y + 2L$       | $R_{a_2} \cdot R_{a_1}$ | $y - 2L$        |
| $R_{a_2} \cdot R_{a_2}$ | $-y - 2L$       | $R_{a_1} \cdot R_{a_2} \cdot R_{a_1}$ | $-y + 4L$        |
| $(R_{a_1} \cdot R_{a_2})^2$ | $y + 4L$       | $(R_{a_2} \cdot R_{a_1})^2$ | $y - 4L$        |
| ...           | ...              | ...           | ...              |

Figure 2. The fundamental region $\Omega = (0, L)$ and the virtual domains generated by the elements $R_i := R_{a_i}$ of the infinite dihedral group.

The heat kernel in this case is written as

$$p_t(x, y) = p_{0,t}(|x - y|) - \sum_{n \in \mathbb{Z}} p_{0,t}(|x + y + 2nL|) + \sum_{n \in \mathbb{Z}\backslash\{0\}} p_{0,t}(|x - y + 2nL|)$$  \hspace{1cm} (23)

where in the first sum we have grouped the contributions of odd number of group elements while in the second sum that of even number of group elements. Relation (9) then becomes

$$p_t(y, y) = \frac{1}{\sqrt{4\pi t}} \left( 1 - \sum_{n \in \mathbb{Z}} e^{-\frac{|y + nL|^2}{t}} + 2 \sum_{n \in \mathbb{N}\backslash\{0\}} e^{-\frac{|nL|^2}{t}} \right).$$  \hspace{1cm} (24)

Integrating (24) over the fundamental domain, taking the $t \to 0^+$ limit and maintaining $t^{-1/2}$ terms we obtain

$$Z_{\Omega}(t) \xrightarrow{t \to 0^+} \frac{L}{\sqrt{4\pi t}} - \frac{1}{2}. \hspace{1cm} (25)$$

4.2. The $d = 2$ case

The case $d = 2$ is more involved. Let $\theta$ be the angle between two intersecting mirror rays in $\mathbb{R}^2$. If we require the absence of a virtual mirror ray between the two given ones, after successive reflections of the fundamental domain, then $\theta = \pi/q$ where $q \in \mathbb{N}\backslash\{1\}$. This remark facilitates the enumeration of bounded tessellations of the plane through reflections. For triangular domains with angles $\pi/p$, $\pi/q$, $\pi/r$ and $p, q, r \in \mathbb{N}\backslash\{1\}$, we have $\pi/p + \pi/q + \pi/r = \pi$. The previous relation is satisfied for the congruent equilateral triangles $(3, 3, 3)$, the isosceles
right triangles \((2, 4, 4)\) and the bisected equilateral triangles \((2, 3, 6)\). Moreover, the only other admissible polygon is the rectangle.

We now study the case of an infinite plane wedge (see Figure (4.2)) since this is the guiding principle for investigating the topological term. If the angle \(\theta = \pi/m\), \(m \in \mathbb{N} \setminus \{1\}\) then there is a unique, up to isomorphism, group generated by two involutions \(R_{a_1}, R_{a_2}\) such that their product \(R_{a_1} \cdot R_{a_2}\) has order \(m\). The group is denoted by \(\text{Dih}_{2m}\), called the dihedral group of order \(2m\) and has the presentation

\[
\text{Dih}_{2m} = \langle R_{a_1}, R_{a_2} | R_{a_1}^2 = R_{a_2}^2 = (R_{a_1} \cdot R_{a_2})^m = I \rangle. \tag{26}
\]

An alternative way to define \(\text{Dih}_{2m}\) would be the semidirect product

\[
\text{Dih}_{2m} = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \tag{27}
\]

where if \(h\) generates \(\mathbb{Z}/2\mathbb{Z}\) then \(hgh = g^{-1}, \forall g \in \mathbb{Z}/m\mathbb{Z}\). In this set up the presentation reads

\[
\text{Dih}_{2m} = \langle g, h | h^2 = g^m = (hg)^2 = I \rangle \tag{28}
\]

and it can be proved that (28) is equivalent to (26) under the substitutions \(R_{a_1} = h\) and \(R_{a_2} = hg\).

From (27) observe that the dihedral group has the cyclic group \(\mathbb{C}_m = \mathbb{Z}/m\mathbb{Z}\) as a subgroup. Also \(\text{Dih}_{2m}\) is the group of automorphisms of the regular planar \(m\)-gon.

Now we adopt the traditional approach to finite reflection groups using root systems. Let us choose the normalised root vectors to be

\[
a_1 = (-\sin \theta, \cos \theta)^\top, \quad a_2 = (0, -1)^\top \tag{29}
\]

where “\(^\top\)” stands for the transpose. The reflection matrices in \(d = 2\) are

\[
R_{a_1}(\theta) = \begin{pmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{pmatrix}
\quad \text{and} \quad
R_{a_2} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\tag{30}
\]

A simple computation reveals that the element \(R_{a_2} \cdot R_{a_1}\), which represents a rotation through twice the angle between the two rays, generates the cyclic group \(\mathbb{C}_m\) with elements

\[
(R_{a_2} \cdot R_{a_1})^k(\theta) = \begin{pmatrix}
\cos(2k\theta) & \sin(2k\theta) \\
-\sin(2k\theta) & \cos(2k\theta)
\end{pmatrix}, \quad k = 0, 1, 2, \ldots, m - 1. \tag{31}
\]
**Proposition 4.1** In two dimensions, only the elements of the cyclic group $C'_m \subset \text{Dih}_{2m}$ contribute to the topological term which is present in the short-time limit of the partition function.

**Proof** The contribution of the elements (31), excluding the identity element (therefore the notation $C'_m$), to the heat kernel at coincident points is

$$p_t(r, r) = \frac{1}{4\pi t} \sum_{k=1}^{m-1} e^{-\frac{r^2}{4t} \sin^2\left(\frac{k\pi}{m}\right)}$$

which after integration over the infinite angular region in polar coordinates gives

$$\int_0^{\infty} p_t(r, r) r dr \int_0^{\pi/m} d\theta = \frac{1}{8m} \sum_{k=1}^{m-1} \sin^2\left(\frac{k\pi}{m}\right) = \frac{1}{24m} (m^2 - 1).$$

We now study in detail the equilateral triangle $\Omega := \Delta$ of side $L$ and height $h = \frac{L \sqrt{3}}{2}$. The symbol $\Delta$ should not be confused to the one used to denote the Dirichlet Laplacian. In proving that $p^\Delta_t$ is actually the heat kernel of the equilateral triangle we will need the following three lemmas.

**Lemma 4.2** Let $u(t, x)$ be a non-negative smooth function on $\mathbb{R}^+ \times M$ satisfying $\int_M u(t, \cdot) dx \leq 1$ and such that $u(t, \cdot) \xrightarrow{D^\alpha} \delta_y$ as $t \to 0$. Then for every open set $\Omega \subseteq M$ and every continuous and bounded function $f \in C_b(\Omega)$ holds

$$\int_{\Omega} u(t, x) f(x) dx \to \begin{cases} f(y), & \text{if } y \in \Omega \\ 0, & \text{if } y \in M/\Omega \end{cases}$$

as $t \to 0$. In (34) the symbol $A \Subset B$ means “compact inclusion”, namely the closure $\overline{A}$ of the set $A$ is compact and $\overline{A} \subset B$.

**Lemma 4.3** Let $M$ be a Riemannian manifold, $a, b \in \mathbb{R}$. Suppose $\{u_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of solutions of the heat equation on $(a, b) \times M$, such that

$$\int_M |u_k(t, x)| dx \leq C,$$

where the constant $C$ is independent of $k$ and $t \in (a, b)$. Then $u = \lim_{k \to \infty} u_k$ is a smooth solution of the heat equation on $M \times (a, b)$ and $u_k \to u$ uniformly on compacta together with derivatives of all orders.

The proof can be found in [5].

**Lemma 4.4** ([8]) For every $\varepsilon > 0$ the function $\nabla p_t(x)$ is bounded on $|x| \geq \varepsilon, t > 0$ and

$$\sup_{t>0 \atop |x| \geq \varepsilon} |\nabla p_t(x)| \leq \frac{\text{const.}}{\varepsilon^{d+1}}$$

where $p_t(x)$ is the Gauss-Weierstrass function (7).

**Proposition 4.5** The function

$$p^\Delta_t(x, y) = \sum_{l=0}^{\infty} (-1)^s(l) p_{0, l}(x, R_ly) \quad x, y \in \Delta$$

is the heat kernel of the equilateral triangle.
Proof Proceeding stepwise, we shall prove:

(i) The function \( p_t^{\Delta}(x, y) \) is a smooth solution of the heat equation on \((0, \infty) \times \Delta\). We first split the sum into two groups of sums carrying the same sign. Let \( p_t^{\Delta}(x, y) = \sum_{l=0}^{\infty} p_{0,t}(x, R_l y) \) be the sum of positive terms where \( R_0 = id \). The partial sums \( \sum_{l=0}^{n} p_{0,t}(x, R_l y) \) constitute an increasing sequence of solutions to the heat equation on \( \Delta \times (0, \infty) \). If we consider an increasing sequence of triangles on the plane which is similar and concentric to the initial triangle (see Figure (i)), then the plane is divided into sectors \( T_n, n = 0, 1, 2 \ldots \) where \( T_0 = \Delta \). Each sector consists of \( 18n - 3 \) virtual domains from which \( 9n - 1 \), at most, correspond to terms with positive sign.

![Figure 4. The partition of the plane into sectors \( T_n, n = 0, 1, 2 \ldots \) each containing \( 18n - 3 \) virtual domains.](image)

The distance of each sector from the initial triangle is \((n - 1)h\). Let \( 0 < a < b \) then for every \( n \) and \( t \in (a, b) \) we have:

\[
\int_{\Delta} \sum_{l=0}^{k} p_{0,t}(x, R_l y) \, dx \leq \frac{1}{4\pi \alpha} \int_{\Delta} \left( 1 + \sum_{n=1}^{\infty} (9n - 1) \exp \left( - \frac{(n - 1)^2 h^2}{4b} \right) \right) \, dx = C. \tag{38}
\]

Therefore from Lemma (4.3) the function \( p_t^{\Delta}(x, y) \) is a smooth solution of the heat equation on \( M \times (a, b) \). Since \( (a, b) \) is arbitrarily chosen this property can be extended to the whole \((0, \infty) \times \Omega\) space. The second sum \( p_t^{-\Delta}(x, y) \) can be treated similarly and therefore the claim is true.

(ii) \( p_t^{\Delta}(\cdot, y) \xrightarrow{p(\Delta)} \delta_y \) as \( t \rightarrow 0 \). Let \( f \in C_0^\infty(\Omega) \) then we have

\[
\lim_{t \rightarrow 0^+} \int_{\Delta} \left( \sum_{l=1}^{\infty} (-1)^{s(l)} p_{0,t}(x, R_l y) \right) f(x) \, dx = 0, \tag{39}
\]

where the first equality is justified by the uniform convergence of both the sum into the parenthesis of (39) on \( x \in \Delta \) and \( \sum_{l=1}^{k} \int_{\Delta} (-1)^{s(l)} p_{0,t}(x, R_l y) f(x) \, dx \) on \( t \in (0, c) \), for \( c \) small enough. The last equality is due to Lemma (4.2) since for every \( l \) the point \( R_l y \notin \overline{\Delta} \).

The term \( p_{0,t}(x, R_l y) \) from Lemma (4.2) also leads to \( p_{0,t}(\cdot, y) \xrightarrow{p(\Delta)} \delta_y \) as \( t \rightarrow 0 \) given that \( y \in \Delta \). The first two statements assure that \( p_t^{\Delta}(x, y) \) is the fundamental solution of the heat equation.

(iii) \( \limsup_{t \rightarrow \infty} p_t^{\Delta}(x_k, y_k) \geq 0 \) for every sequence \( (t_k, x_k) \) such that \( t_k \rightarrow 0 \) and \( x_k \rightarrow x \in \Delta \). This statement is the weaker condition satisfied by the fundamental solution in Theorem (2.2). For \( x \in \Delta, t \in (0, c) \) and \( c \) small, we have

\[
\sum_{l=1}^{\infty} p_{0,t}(x, R_l y) \leq \sum_{k=1}^{\infty} \frac{(18k - 3)}{4\pi c} \exp \left( - \frac{(k - 1)^2 h^2}{4c} \right) < \infty. \tag{40}
\]
From the previous relation we conclude that the series \( \sum_{l=1}^{\infty} (-1)^{s(l)} p_{0,t}(x, R_l y) \) uniformly converges for \( x \in \triangle, t \in (0, c) \), therefore for every sequence \((t_k, x_k)\) such that \( t_k \to 0 \) and \( x_k \to x \in \Omega \), we have

\[
\lim_{k \to \infty} \sum_{l=1}^{\infty} (-1)^{s(l)} p_{0,t_k}(x_k, R_l y) = \sum_{l=1}^{\infty} (-1)^{s(l)} \lim_{k \to \infty} p_{0,t_k}(x_k, R_l y) = 0. \tag{41}
\]

Due to the inequality \( \limsup_{k \to \infty} p_{0,t_k}(x_k, y) \geq 0 \) we end up with the desired result.

(iv) \( p^\triangle_t(x, y) \equiv 0 \) as \( x \to \partial \triangle \) where the convergence is uniform w.r.t. \( t \in (0, T) \) for every \( T > 0 \).

The function \( p^\triangle_t(x, y) \) vanishes on the boundary \( \partial \triangle \) by construction. Let \( x' \) be a point of the boundary \( \partial \triangle \) such that \( |x - x'| = d(x, \partial \triangle) \). Using the mean value theorem we obtain:

\[
\left| p^\triangle_t(x, y) \right| \leq |\nabla p_{0,t}(z - y)| |x - x'| + \sum_{l=1}^{\infty} |\nabla p_{0,t}(z_l - R_l y)| |x - x'|,
\]

where \( z_l, z \in [x, x'] \). For \( d(x, \partial \triangle) < d(y, \partial \triangle)/2 \) we have \( |z - y| \geq d(y, \partial \triangle)/2 \). Also for those \( l \)'s satisfying \( R_l y \in T_n \), it holds: \( |z_l - R_l y| > (n - 1)h + d(y, \partial \triangle) \). Using the previous relations and Lemma (4.4) we get:

\[
\sup_{t>0} \left| p^\triangle_t(x, y) \right| \leq \left( \frac{\text{const.}}{(d(y, \partial \triangle)/2)^3} + \sum_{n=1}^{\infty} \frac{(18n - 3)\text{const.}}{(n - 1)h + d(y, \partial \triangle))^3} \right) d(x, \partial \triangle) = Cd(x, \partial \triangle). \tag{43}
\]

Therefore by Theorem (2.2), \( u(t, x) \equiv p^\triangle_t(x, y) \).

In the sequel we calculate the asymptotic behaviour of the partition function for the equilateral triangle and show how the reflection elements of Dih\(_6\) group conspire to give the surface area and the length of its perimeter. Each vertex, its two adjacent median points and the centroid form three quadrilateral subdomains \( D_i, i = 1, 2, 3 \). The Cartesian coordinates of the triangle vertices are: \( A = (0, 0), B = (L, 0), C = (L/2, \sqrt{3}L/2) \) and the reflection matrices are given by

\[ R_{a_1} = \begin{pmatrix} \frac{2\pi}{3} \\ \frac{-\pi}{3} \end{pmatrix} \quad \text{and} \quad R_{a_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{44} \]

The contribution of the identity element \( I \equiv \text{id} \) to \( Z_\Omega \) is \( |\Omega|/4\pi t \), where \( |\Omega| \) is the area of the equilateral triangle. The contribution of the reflection elements \( R_{a_1}, R_{a_2} \) and \( R_{a_1} \cdot R_{a_2} \cdot R_{a_1} \) from the neighbouring to a vertex virtual domains (see Figure (4.2)) is

![Figure 5](image-url)
\[
\sum_{i=1}^{3} \int \int_{D_1} p_t^{ref}_1(x,y) dx \, dy \underset{t \to 0^+}{\sim} -\frac{[\partial \Omega]}{8\sqrt{\pi t}}, \quad \text{where} \quad p_t^{ref}_1(x,y) = -\frac{1}{4\pi t} \left( e^{-\frac{(\sqrt{r}x+y)^2}{4t}} + e^{-\frac{x^2}{t}} + e^{-\frac{(\sqrt{r}y-x)^2}{4t}} \right). (45)
\]

The rotations \( R_{a_2} \cdot R_{a_1} \) and \( (R_{a_2} \cdot R_{a_1})^2 \) give the constant 1/3.

4.3. The \( d = 3 \) case

In three dimensions the bounded regions compatible with the image method are: triangular prisms with \( \theta = 3.4.3. \), the admissible ones through the requirement of three-dimensional tessellations but the method of images puts severe constraints on the previous two conditions are: (2

Proposition 4.6 ([8]) If the infinite trihedral angle is \( \theta_{12}, \theta_{23}, \theta_{31} = (\pi/3, \pi/2, \pi/2) \), right triangular prisms (both with five faces) and rectangular parallelepiped (six faces). (69x466) Again we study the infinite trihedral angle by applying the image method to determine the time independent term in \( \Omega \). Setting \( \theta_i \) denote the angle between the \( i \)- and \( j \)-plane. Setting \( \theta_{23} = \theta_{31} = \pi/2 \), the reflection matrices associated with the roots \( \alpha_i \), \( i = 1, 2, 3 \) orthogonal to the \( i \)-planes are found to be

\[
R_{a_1}(\theta_{12}) = \begin{pmatrix} \cos(2\theta_{12}) & \sin(2\theta_{12}) & 0 \\ -\sin(2\theta_{12}) & \cos(2\theta_{12}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{a_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{a_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

(66)

where \( \theta_{12} = \pi/p, p \in \mathbb{N}/\{1\} \). The interior of the trihedral domain does not contain any virtual mirror generated by multiple reflections in the three bounding hyperplanes only if \( (p,q,r) \) (or permutation of them) satisfy: \( \pi/p + \pi/q + \pi/r > \pi \) and \( r,q,p \in \mathbb{N}/\{1\} \). The solutions to the previous two conditions are: \( (2,2,r), (2,3,3), (2,3,4) \) and \( (2,3,5) \). The defining relations of the corresponding group become: \( (R_{a_1})^2 = (R_{a_2})^2 = (R_{a_3})^2 = (R_{a_1} \cdot R_{a_2})^p = (R_{a_2} \cdot R_{a_3})^q = (R_{a_3} \cdot R_{a_1})^r = I. \)

Proposition 4.6 ([8]) If the infinite trihedral angle is \( \theta_{12}, \theta_{23}, \theta_{31} = (\pi/r, \pi/2, \pi/2), r \in \mathbb{N} \setminus \{1\} \) then only the elements of \( C_r' \times \{R_{a_3}\} \subset \text{Dih}_{2}\mathbb{R}^\infty \times \mathbb{Z}/2\mathbb{Z} \) contribute to the constant term of \( \Omega \). In this case the contribution reads

\[
I_{\text{const.}}(r) = -\frac{1}{32r} \sum_{k=1}^{r-1} \frac{1}{\sin^2 \left( \frac{k\pi}{2r} \right)} = -\frac{1}{96r}(r^2 - 1). (47)
\]

The rest of the group elements give the following transition densities:

### Table 2. The heat kernels of group elements which do not belong to \( C_r' \times \{R_{a_3}\} \).

| Reflections | Rotations |
|-------------|-----------|
| \( R_{a_1} \) | \( (4\pi)^{\frac{k}{2}} p_t(x,y,z) \) | \( (4\pi)^{\frac{k}{2}} p_t(x,y,z) \) |
| \( R_{a_2} \) | \( -e^{-\frac{1}{4}(\sin(\frac{\pi}{r})x-\cos(\frac{\pi}{r})y)^2} \) | \( (R_{a_1} \cdot R_{a_2})^k, 1 \leq k \leq r - 1 \) |
| \( R_{a_3} \) | \( -e^{-\frac{r^2}{2}} \) | \( R_{a_3} \cdot R_{a_1} \) |

According to Coxeter’s classification scheme there are polyhedra which fall into the class of three-dimensional tessellations but the method of images puts severe constraints on the admissible ones through the requirement \( \theta_{ij} = \pi/p_{ij}, p_{ij} \in \mathbb{N}/\{1\}, \forall i,j \).
The first candidate in our investigation is the prism \((\theta_{12}, \theta_{23}, \theta_{31}) = (\pi/3, \pi/2, \pi/2)\) which in the light of Proposition (4.6) gives the topological term \(-1/6\). The asymptotic behaviour of the partition function is found to be

\[
Z_{\Omega, (2,2,3)}(t) \overset{t \to 0^+}{\sim} \frac{3a^2c}{16\pi t} - \frac{a(a\sqrt{3}/2 + 3c)}{16\pi t} - \frac{3(2a + c)}{32\sqrt{\pi t}} - \frac{1}{6}
\]

where \(a\) is the side length of the equilateral triangle and \(c\) the height of the prism.

The next candidates are the right triangular prisms with \((\theta_{12} = \theta_{23} = \theta_{31} = \pi/2)\) at two vertices. We distinguish the following two possibilities

\[
Z_{\Omega, (2,4,4)}(t) \overset{t \to 0^+}{\sim} \frac{a^2c}{2(4\pi t)^{3/2}} - \frac{a(a + c(2 + \sqrt{2}))}{16\pi t} + \frac{2a(2 + \sqrt{2}) + 3c}{32\sqrt{\pi t}} - \frac{3}{16}
\]

\[
Z_{\Omega, (2,3,6)}(t) \overset{t \to 0^+}{\sim} \frac{a^2c\sqrt{3}}{2(4\pi t)^{3/2}} - \frac{a(\sqrt{3} + c(3 + \sqrt{3}))}{16\pi t} + \frac{2a(3 + \sqrt{3}) + 3c}{32\sqrt{\pi t}} - \frac{5}{24}
\]

where \(a\) in (50) is the length of the side opposite to the \(\pi/6\) angle and \(c\) as in the previous case.

Finally, we study the rectangular parallelepiped with edge lengths \((a, b, c)\). Dividing it into eight sub-rectangular parallelepipeds we arbitrarily choose one of them and perform our computation. Their contributions are listed in the following table:

**Table 3.** The heat kernels and the associated reflection and rotation group elements.

| Reflections | Contributions |
|-------------|---------------|
| \(R_{\alpha_1}\) | \(-\frac{bc}{64\pi t} \text{erf} \left( \frac{a}{2\sqrt{t}} \right)\) |
| \(R_{\alpha_2}\) | \(-\frac{ac}{64\pi t} \text{erf} \left( \frac{b}{2\sqrt{t}} \right)\) |
| \(R_{\alpha_3}\) | \(-\frac{ab}{64\pi t} \text{erf} \left( \frac{c}{2\sqrt{t}} \right)\) |

| Rotations | Contributions |
|-----------|---------------|
| \(R_{\alpha_1} \cdot R_{\alpha_2}\) | \(-\frac{a}{64\sqrt{\pi t}} \text{erf} \left( \frac{c}{2\sqrt{t}} \right) \text{erf} \left( \frac{b}{2\sqrt{t}} \right)\) |
| \(R_{\alpha_2} \cdot R_{\alpha_3}\) | \(-\frac{b}{64\sqrt{\pi t}} \text{erf} \left( \frac{a}{2\sqrt{t}} \right) \text{erf} \left( \frac{c}{2\sqrt{t}} \right)\) |
| \(R_{\alpha_3} \cdot R_{\alpha_2}\) | \(-\frac{c}{64\sqrt{\pi t}} \text{erf} \left( \frac{a}{2\sqrt{t}} \right) \text{erf} \left( \frac{b}{2\sqrt{t}} \right)\) |

The identity element again gives the volume of the domain, the reflection elements \(R_{\alpha_i}\), \(i = 1, 2, 3\) provide the area of the bounding surfaces, the rotations contribute to the lengths of the edges and the reflection \(R_{\alpha_1} \cdot R_{\alpha_2} \cdot R_{\alpha_3}\) supports the topological term. In the short-time limit the error functions converge to unity and if we sum up the contributions from all the vertices we recover the correct result

\[
Z_{\Omega}(t) \overset{t \to 0^+}{\sim} \frac{abc}{(4\pi t)^{3/2}} - \frac{2(ab + bc + ac)}{16\pi t} + \frac{4(a + b + c)}{32\sqrt{\pi t}} - \frac{1}{8}
\]

5. Conclusions

In this paper, using a group theoretic approach, we studied analytically the short-time asymptotics of the free partition function corresponding to the Dirichlet Laplacian on tessellations which possess mirror symmetry through the hyperplanes bounding the fundamental domain. Implementing the method of images along with the path integral representation of the heat kernel up to three-dimensions, we established the connection of the geometrical quantities \(|\Omega|, |\partial\Omega|\) and the topological term, with certain elements of the orthogonal group. We proved rigorously that (9) is the actual heat kernel on \(\Omega\). This method can be generalised to higher dimensions and may help to solve the same problem with Dirichlet fractional Laplacians without requiring the knowledge of its spectra.
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