Transformation of the Extended Gamma Function $\Gamma_{0,2}^{2,0}([b, x])$
with Applications to Astrophysical Thermonuclear Functions

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Abstract

Two representations of the extended gamma functions $\Gamma_{0,2}^{2,0}([b, x])$ are proved. These representations are exploited to find a transformation relation between two Fox’s $H$-functions. These results are used to solve Fox’s $H$-function in terms of Meijer’s $G$-function for certain values of the parameters. A closed form representation of the kernel of the Bessel type integral transform is also proved.

1. Introduction

According to Anderson et al. (1994), the stars are gravitationally stabilized fusion reactors changing their chemical composition while transforming light atomic nuclei into heavy ones. The atomic nuclei are supposed to be in thermal equilibrium with the ambient plasma. The majority of reactions among nuclei leading to a nuclear transformation are inhibited by the necessity for the charged particles to tunnel through their mutual Coulomb barrier. The theoretical and experimental verification of nuclear cross-sections leads to the derivation of the closed-form representation of thermonuclear reaction rates (Critchfield, 1972; Haubold and John, 1978; Haubold and Mathai, 1986). These rates are expressed in terms of the four astrophysical thermonuclear functions (Anderson et al., 1994)

\[
I_1(z, \nu) := \int_0^{\infty} y^\nu \exp(-y - z/\sqrt{y})dy,
\]

\[
I_2(z, d, \nu) := \int_0^{d} y^\nu \exp(-y - z/\sqrt{y})dy,
\]

\[
I_3(z, t, \nu) := \int_0^{\infty} y^\nu \exp(-y - z/\sqrt{y + t})dy,
\]
\[ I_4(z, \delta, b, \nu) := \int_0^\infty y^\nu \exp \left( -y - by^\delta - z/\sqrt{y} \right) dy. \] (1.4)

The closed-form representations of these integral functions, asymptotic values and numerical results are discussed in Anderson et al. (1994).

The closed-form representation of these functions in terms of the Meijer’s \( G \)-function is essentially based on the following theorem (Saxena, 1960; Mathai and Haubold, 1988). For \( z > 0, p > 0, \rho \leq 0, \) and integers \( m, n \geq 1, \)

\[
p \int_0^\infty t^{-n\rho} \exp \left( -pt - zt^{-n/m} \right) dt = p^{-\rho}(2\pi)^{(2-n-m)/2}m^{1/2}n^{\left(\frac{1}{2}\right)-n\rho} \times \]

\[ G_{m+n,0}^{m+n,0} \left( \frac{z^m p^n}{m^n n^n} \right| 0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}; 1 - \frac{n\rho}{n}, n - \frac{n\rho}{n} \right). \] (1.5)

The restriction \( \rho \leq 0 \) is essential in the proof of (1.5) whereas one would like to have a closed-form representation free of this restriction. This has been achieved in this paper.

The study of the astrophysical thermonuclear functions led to the development of a new class of special functions (Chaudhry and Zubair, 1998) that extends Meijer’s \( G \) and Fox’s \( H \)-function. In this paper we prove a transformation relation for the extended gamma functions (Chaudhry and Zubair, 1998)

\[ \Gamma(\alpha, x; b, \beta) := \int_x^\infty t^{\alpha-1}e^{-t-b/t^\beta} dt, \] (1.6)

and

\[ \gamma(\alpha, x; b, \beta) := \int_0^x t^{\alpha-1}e^{-t-b/t^\beta} dt, \] (1.7)

and derive the closed-form of the astrophysical thermonuclear functions in terms of Meijer’s \( G \)-function. It is to be noted that the functions (1.6) and (1.7) are special cases of the general class of extended gamma functions introduced in Chaudhry and Zubair (1998).
In fact we have
\[
\Gamma(\alpha, x; b, \beta) = \Gamma_{0,2}^{2,0} \left[ (b, x) \left| \begin{array}{cc}
\begin{array}{c}
\alpha, \beta
\end{array}
\end{array} \right| \left| \begin{array}{cc}
\begin{array}{c}
(0, 1), \ (\alpha, \beta)
\end{array}
\end{array} \right) \right] := \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(s)\Gamma(\alpha + \beta s, x)b^{-s}ds,
\]
(1.8)

and
\[
\gamma(\alpha, x; b, \beta) = \gamma_{0,2}^{2,0} \left[ (b, x) \left| \begin{array}{cc}
\begin{array}{c}
\alpha, \beta
\end{array}
\end{array} \right| \left| \begin{array}{cc}
\begin{array}{c}
(0, 1), \ (\alpha, \beta)
\end{array}
\end{array} \right) \right] := \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(s)\gamma(\alpha + \beta s, x)b^{-s}ds.
\]
(1.9)

It is important to note that the astrophysical thermonuclear functions (1.1) and (1.4) are special cases of the extended gamma functions (1.8) and (1.9). As a matter of fact we have
\[
I_1(z, \nu) = \Gamma \left( \nu + 1, 0; z; \frac{1}{2} \right), \quad (1.10)
\]
\[
I_2(z, d, \nu) = \gamma \left( \nu + 1, d; z; \frac{1}{2} \right), \quad (1.11)
\]
\[
I_3(z, t, \nu) = e^t \sum_{r=0}^{\nu} \left( \frac{\nu}{r} \right) (-t)^{\nu-r} \Gamma \left( \nu + 1, t; z; \frac{1}{2} \right), \quad (1.12)
\]
\[
I_4(z, \delta, b, \nu) = \sum_{r=0}^{\infty} \frac{(-b)^r}{r!} \Gamma \left( \nu + r\delta + 1, 0; z; \frac{1}{2} \right). \quad (1.13)
\]

2. The Transformation Theorem

**Theorem (2.1).** For \( x \geq 0, b \geq 0, \beta \geq 0 \),
\[
\Gamma(\alpha, x; b, \beta) = \frac{1}{\beta} \Gamma_{0,2}^{2,0} \left[ (b^{1/\beta}, x) \left| \begin{array}{cc}
\begin{array}{c}
\alpha, \beta
\end{array}
\end{array} \right| \left| \begin{array}{cc}
\begin{array}{c}
(0, \frac{1}{\beta}), \ (\alpha, 1)
\end{array}
\end{array} \right) \right] \quad (2.1)
\]

**Proof.** Replacing \( b \) by \( b^\beta \) in (1.6) yields
\[
\Gamma(\alpha, x; b^\beta; \beta) = \int_0^\infty f_1(b/t)f_2(t)\frac{dt}{t^\beta}, \quad (2.2)
\]

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where

\[ f_1(t) := e^{-t^\beta}, \quad (2.3) \]

\[ f_2(t) := t^\alpha e^{-t} H(t - x), \quad (2.4) \]

and

\[ H(t - x) := \begin{cases} 
1 & \text{if } t > x, \\
0 & \text{if } t < x.
\end{cases} \quad (2.5) \]

The Mellin transforms of the functions (2.3) and (2.4) are readily found to be

\[ \overline{f}_1(s) := \frac{1}{\beta} \Gamma(s/\beta) \quad (2.6) \]

\[ \overline{f}_2(s) := \Gamma(\alpha + s, x). \quad (2.7) \]

However, according to Erdélyi et al. (1954)

\[ M \left( \int_0^\infty f_1(b/t) f_2(t) \frac{dt}{t}; s \right) = \overline{f}_1(s) \overline{f}_2(s). \quad (2.8) \]

From (2.2), (2.6) and (2.7) we have

\[ \Gamma(\alpha, x; b^\beta, \beta) = \frac{1}{\beta} M^{-1} \left\{ \Gamma \left( \frac{s}{\beta} \right) \Gamma(\alpha + s, x); b \right\} \quad (2.9) \]

\[ = \frac{1}{\beta} \frac{1}{2\pi i} \int_{C-\infty}^{C+\infty} \Gamma \left( \frac{s}{\beta} \right) \Gamma(\alpha + s, x) b^{-s} ds \quad (2.10) \]

\[ = \frac{1}{\beta} \Gamma_{0.2}^{2.0} \left[ b, x \right] \left| \begin{array}{cc} - & - \\ (0, 1), (\alpha, \beta) \end{array} \right| \left| \begin{array}{cc} - & - \\ (\alpha, 1), (0, 1/\beta) \end{array} \right|. \quad (2.11) \]

Replacing \( b \) by \( b^{1/\beta} \) in (2.11) yields (2.1).

3. Applications of the Transformation to Astrophysical Thermonuclear Functions

Theorem (3.1).

\[ \Gamma(\alpha, 0; b; \beta) = H_{0.2}^{2.0} \left[ b \right| \begin{array}{c} - \\ (0, 1), (\alpha, \beta) \end{array} \] = \frac{1}{\beta} H_{0.2}^{2.0} \left[ b^{1/\beta} \right| \begin{array}{c} - \\ (\alpha, 1), (0, 1/\beta) \end{array}. \quad (3.1) \]
Proof. This is a direct consequence of (1.8) and (2.1).

Corollary (3.1).

\[ \Gamma(\alpha, 0; b; 1_n) = H_{0.2}^{2.0} \left[ b \left| \begin{array}{c} -1, -1 \\ 0, (\alpha, n) \end{array} \right. \right] = (2\pi)^{(1-n)/2} \sqrt{n} \times G_{0,n+1}^{n+1,0} \left[ \left( \frac{b}{n} \right)^n \left| \begin{array}{c} -1, -1, \ldots, -1 \\ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \alpha \end{array} \right. \right] \] (3.2)

Proof. The substitution \( \beta = \frac{1}{n} \) in (3.1) leads to

\[ \Gamma(\alpha, 0; b; 1_n) = nH_{0,2}^{2.0} \left[ b \left| \begin{array}{c} -1, -1 \\ (\alpha, 1), (0, n) \end{array} \right. \right] = n \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(\alpha + s)\Gamma(ns)b^{-ns}ds. \] (3.3)

However, the use of the multiplication formula (Anderson et al., 1994; Mathai and Haubold, 1988)

\[ \Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left( z + \frac{k}{m} \right) \] (3.4)

for the gamma function yields

\[ \Gamma(\alpha, 0; b; \frac{1}{n}) = (2\pi)^{(1-n)/2} \sqrt{n} \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(\alpha + s) \prod_{k=0}^{n-1} \Gamma\left( s + \frac{k}{n} \right) (b/n)^{-ns}ds, \] (3.5)

which is exactly (3.2).

Corollary (3.2).

\[ \Gamma(\alpha, 0; b; \frac{1}{2}) = \pi^{-1/2} G_{0.3}^{3,0} \left[ \frac{b^2}{4} \left| 0, \frac{1}{2}, \alpha \right. \right]. \] (3.6)

Proof. This is a special case of (3.2) when we take \( n = 2 \).

Remark. The closed-form representation (3.6) is important in view of the relation (1.10) that yields (Anderson et al., 1994)

\[ I_1(z, \nu) = \Gamma(\nu + 1, 0; z; \frac{1}{2}) = \pi^{-1/2} G_{0.3}^{3,0} \left[ \frac{z^2}{4} \left| 0, \frac{1}{2}, 1+\nu \right. \right]. \] (3.7)
Moreover,

\[ I_2(z, d, \nu) = \Gamma \left( \nu + 1, 0; z; \frac{1}{2} \right) - \Gamma \left( \nu + 1, d; z; \frac{1}{2} \right), \quad (3.8) \]

and the functions \( I_3(z, t; \nu) \) and \( I_4(z, \delta, b, \nu) \) are expressible in terms of \( I_1 \) and \( I_2 \) functions. Therefore, it seems important to search for the closed-form of the extended function \( \Gamma \left( \alpha, x; b; \frac{1}{2} \right) \) in terms of classical special functions. In view of the results proved in Anderson et al. (1994), it seems impossible to have such type of representations. Thus, the extended gamma functions (Chaudhry and Zubair, 1998) provide the unique closed form-representation of the astrophysical thermonuclear functions given by (1.10) – (1.13).

4. Application to Bessel Type Integral Transforms

Kilbas et al. (1998) have studied the integral transform

\[ K_\rho^\nu(f)(x) = \int_0^\infty z_\rho^\nu(xt)f(t)dt, \quad (x > 0), \quad (4.1) \]

with the kernel

\[ z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} \exp \left( -t^\rho - \frac{x}{t} \right) dt, \quad (\rho > 0, \nu \in \mathbb{C}), \quad (4.2) \]

on the spaces \( F_{p,\mu} \) and \( F'_{p,\mu} \) \( (1 \leq p \leq \infty, u \in \mathbb{C}) \) of tested and generalized functions. When \( \rho = 1 \) and \( x = t^2/4 \)

\[ z_1^\nu(t^2/4) = 2(t/2)^\nu K_\nu(t), \quad (4.3) \]

where \( K_\nu(t) \) is the modified Bessel function of the third kind. For other values of \( \rho \), the authors considered the integral representation (4.2) of the kernel without the closed-form representation, and proved the compositions of the operator \( K_\rho^\nu \) with the left and right-sided Liouville fractional integrals and derivatives.
The left-sided \( I_{0+}^\alpha, D_{0+}^\alpha \) and right-sided \( I_{-}^\alpha, D_{-}^\alpha \) Liouville fractional derivatives are defined for \( x > 0 \) by (Kilbas et al., 1998)

\[
\left( I_{0+}^\alpha \varphi \right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}}, \quad (\alpha > 0),
\]

\[
\left( D_{0+}^\alpha \varphi \right)(x) = \left( \frac{d}{dx} \right)^{[\alpha]+1} \left( I_{0+}^{1-\{\alpha\}} \varphi(x) \right), \quad (\alpha > 0),
\]

\[
\left( I_{-}^\alpha \varphi \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{\varphi(t)dt}{(t-x)^{1-\alpha}}, \quad (\alpha > 0),
\]

\[
\left( D_{-}^\alpha \varphi \right)(x) = \left( -\frac{d}{dx} \right)^{[\alpha]+1} \left( I_{-}^{1-\{\alpha\}} \varphi \right)(x), \quad (\alpha > 0),
\]

respectively, where \([\alpha]\) and \(\{\alpha\}\) are the integral and fractional parts of \(\alpha > 0\).

The following relations are then proved:

\[
\left( K_\rho^\nu I_{0+}^\alpha \right) \varphi = \left( x^{-\alpha} K_\rho^\nu \right) \varphi,
\]

\[
\left( K_\rho^\nu D_{0+}^\alpha \right) \varphi = \left( x^\alpha K_\rho^{-\nu} \right) \varphi,
\]

\[
\left( I_{-}^\alpha D_\nu^\rho \right) \varphi = K_\rho^\nu \left( x^{-\alpha} \varphi \right),
\]

\[
\left( D_{-}^\alpha K_\rho^\nu \right) \varphi = K_\rho^{-\nu} \left( x^\alpha \varphi \right).
\]

It is to be noted that the kernel (4.2) of the integral transform (4.1) can be simplified in terms of the extended gamma function

\[
\Gamma(\alpha, 0; b; \beta) = H_{0,2}^{2,0} \left[ b \left| \begin{array}{c} -, - \\ 0, 1, \alpha, \beta \end{array} \right. \right].
\]

The transformations

\[
t^\rho = u, \quad \frac{dt}{t} = \frac{1}{\rho} \frac{du}{u}
\]

in (4.2) lead to

\[
z_\rho^\nu(x) = \frac{1}{\rho} \int_0^\infty u^{\nu/\rho-1} \exp \left( -u - \frac{x}{u^{1/\rho}} \right) du,
\]

which is expressible in terms of the extended gamma function to give (Chaudhry and Zubair, 1998)

\[
z_\rho^\nu(x) = \frac{1}{\rho} \Gamma \left( \frac{\nu}{\rho}, 0; x; \frac{1}{\rho} \right),
\]
which can further be simplified in terms of the Fox’s $H$-function to give

$$z^\nu_{\rho}(x) = \frac{1}{\rho} H^{2,0}_{0,2} \left[ x \left| \begin{array}{c} -\frac{1}{\rho}, \frac{1}{\rho} \\ 0, 1 \end{array} \right. \right].$$  \hspace{1cm} (4.16)

Hence the transformation (4.1) can be defined in a closed-form as follows:

$$\left( K^\nu_{\rho} f \right)(x) = \frac{1}{\rho} \int_0^\infty H^{2,0}_{0,2} \left[ xt \left| \begin{array}{c} -\frac{1}{\rho}, \frac{1}{\rho} \\ 0, 1 \end{array} \right. \right] f(t) dt. \hspace{1cm} (4.17)$$

The representation (4.15) – (4.17) and the relations (4.7) – (4.10) can be exploited to find the closed-form representations of the integrals involving the extended gamma function (4.11). In particular, the substitution $\varphi(x) = \delta(x)$ in (4.8) yields

$$\int_0^\infty t^{\alpha-1} H^{2,0}_{0,2} \left[ xt \left| \begin{array}{c} -\frac{1}{\rho}, \frac{1}{\rho} \\ 0, 1 \end{array} \right. \right] dt = \Gamma(\alpha) \Gamma \left( \frac{\nu + \alpha}{\rho} \right) x^{-\alpha}, \hspace{1cm} (\alpha > 0). \hspace{1cm} (4.18)$$

The closed form representation of the kernel (4.2) in terms of Fox’s $H$-function has provided a compact and useful representation (4.17) of the integral transformation (4.1). It will facilitate the study of the integrals involving astrophysical thermonuclear functions.

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References

Anderson, W.J., Haubold, H.J., and Mathai, A.M. (1994). Astrophysical thermonuclear functions. *Astrophys. Space Sci.*, **214**, 49–70 (http://xxx.lanl.gov/abs/astro-ph/9402020).

Chaudhry, M.A. and Zubair, S.M. (1998). Extended incomplete gamma functions with applications, *Journal of London Mathematical Society* (submitted).

Critchfield, C.L. (1972). In: Cosmology, Fusion and Other Matters, George Gamow Memorial Volume. Ed. F. Reines, Colorado, Colorado: Associated University Press 1972.

Erdélyi et al. (1954). *Table of Integral Transforms*, Volume 1, McGraw-Hill, New York.

Haubold, H.J. and Mathai, A.M. (1986). Analytic representations of thermonuclear reaction rates. *Studies in Applied Mathematics*, **LXXV** (2), 123–137.

Haubold, H.J., John, R.W. (1978). On the evaluation of an integral connected with the thermonuclear reaction rate in closed form. *Astron. Nachr*. **299**, 225–232; **300**, 173.

Haubold, H.J. and Mathai, A.M. (1986). Analytic results for screened non-resonant nuclear reaction rates, *Astrophys. Space Sci.*, **127**, 45–53.

Kilbas, A.A., Bonilla, B., Rivero, M., Rodrigues, J. and Trujillo, J. (1998). Composition of Bessel type integral transform with fractional operators on spaces \( F_{p,\mu} \) and \( F'_{p,\mu} \). *Fractional Calculus and Applied Analysis*, **1**, no. 2, 135.

Mathai, A.M. and Haubold, H.J. (1988). *Modern Problems in Nuclear and Neutrino Astrophysics*, Akademie-Verlag, Berlin.

Saxena, R.K. (1960). *Proc. Nat. Acad. Sci. India*, **26**, 400–413.