On the Bach and Einstein equations in presence of a field

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Abstract

The aim of this paper is to introduce and justify a possible generalization of the classic Bach field equations on a four dimensional smooth manifold $M$ in presence of field $\varphi$, that in this context is given by a smooth map with source $M$ and target another Riemannian manifold. Those equations are characterized by the vanishing of a two times covariant, symmetric, traceless and conformally invariant tensor field, called $\varphi$-Bach tensor, that in absence of the field $\varphi$ reduces to the classic Bach tensor. We provide a variational characterization for $\varphi$-Bach flat manifolds and we do the same also for harmonic-Einstein manifolds, i.e., solutions of the Einstein field equations with source the conservative field $\varphi$. We take the opportunity to discuss a generalization of some related topics: the Yamabe problem, the image of the scalar curvature map, warped product solutions and static manifolds.

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1 Introduction

The Einstein field equations on a four dimensional Lorentzian manifolds $(M, g)$ are given by

$$G + \Lambda g = \alpha T,$$

where

- $G$ is the Einstein tensor of $(M, g)$, that is,

$$G := \text{Ric} - \frac{S}{2} g,$$

where $\text{Ric}$ and $S$ denote, respectively, the Ricci and the scalar curvature of $(M, g)$;
• $T$ is the stress-energy tensor, a symmetric two times covariant tensor, supposed to represent the density of all the energies, momenta and stresses of the sources, that has to be divergence free in order to satisfy the conservation laws, also called equations of motion (geometrically, the fact that $T$ is divergence free follows from the fact that the Einstein tensor is divergence free, since Schur’s identity holds).

• $\Lambda \in \mathbb{R}$ is the cosmological constant;

• $\alpha$ is the positive constant given by

$$\alpha = \frac{8\pi G}{c^4}, \quad (1.1)$$

where $G$ is Newton’s gravitational constant and $c$ is the speed of light in vacuum.

The study of solutions of the Einstein field equations in vacuum, i.e.,

$$G + \Lambda g = 0, \quad (1.2)$$

led to the definition of Einstein manifolds and, when the cosmological constant is not included, of Ricci flat manifolds. Those definitions have been easily extended to smooth manifolds of any dimension $m \geq 2$ endowed with any semi-Riemannian (mostly, Riemannian) metric and, at least for mathematicians, they are geometrical objects of great interest and we refer to [B] for an overview on the topic. We just mention that Einstein manifolds of dimension $m \geq 3$ are characterized by the vanishing of the traceless part of the Ricci tensor and, as a consequence of Schur’s identity (the Einstein tensor $G$ is divergence free), their scalar curvature $S$ is (locally) constant on $M$. Going back to solutions of (1.2), the cosmological constant $\Lambda$ is given by a constant positive multiple of the scalar curvature.

Almost all the equations of mathematical physics admit a variational characterization and the Einstein equations are not an exception. For simplicity, consider a closed and orientable smooth manifold $M$ of dimension $m \geq 3$, and denote by $\mathcal{M}$ the set of all the Riemannian metrics on $M$. Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ of metrics of total volume equal to 1. The functional of total scalar curvature $S : \mathcal{M} \rightarrow \mathbb{R}$, introduced by D. Hilbert, is given by, for every $g \in \mathcal{M}$,

$$S(g) := \int_M S_g \mu_g,$$

where $S_g$ and $\mu_g$ are, respectively, the scalar curvature and the volume form of $(M, g)$. Critical metrics of $S$ on $\mathcal{M}$ coincide with Ricci flat metrics on $M$ while critical metrics of $S$ on $\mathcal{M}_1$ (or equivalently, critical metrics of the normalized total scalar curvature on $\mathcal{M}$) coincide with Einstein metrics on $M$. For the details see [B] or [S].

Since Einstein metrics have constant scalar curvature one may ask weather or not also the largest class of metrics with constant scalar curvature admit a variational characterization. The answer is yes, a metric $g \in \mathcal{M}$ has constant scalar curvature if and only if is a critical point of $S$ in $[g] \cap \mathcal{M}_1$ (see Proposition 4.25 of [B]). This led naturally to the study of the famous Yamabe problem, that consists in finding pointwise conformal metrics with constant scalar curvature on a Riemannian manifold. We refer to [LP] for a detailed proof of its solution.

Another interesting feature of the scalar curvature is that we are able to characterize its image when we look at it as a map $g \in \mathcal{M} \mapsto S_g$, see [FM] or Section 4.E of [B] for an overview. The key step is to prove the surjectivity of its linearization at $g \in \mathcal{M}$, or equivalently, the injectivity of its adjoint. We have that if $u$ belongs to kernel of the adjoint map then either $(M, g)$ is Ricci flat (and in this case $u$ is constant) or the product $M \times \mathbb{R}$, endowed with the metric $g - u^2 dt \otimes dt$, where $t$ is the coordinate on $\mathbb{R}$, is an Einstein manifold outside the zero locus of $u$, where it degenerates. In the latter case, if $(M, g)$ is a three-dimensional Riemannian manifold, then the Lorentzian warped product $M \times_u \mathbb{R}$ is a static spacetime (i.e., it admits a timelike and irrotational Killing vector field) that solves the vacuum Einstein field equations. In [C], J. Corvino studied the same problem for complete non-compact Riemannian manifolds.

In some circumstances one may be interested in field equations for the metric $g$ on $M$ that are conformal invariant, meaning that if $g$ is a solution then every metric that is pointwise conformal equivalent to $g$, i.e., every metric in the conformal equivalence class $[g]$, is a solution too. A disadvantage of the
Einstein field equations and, consequently of Einstein manifolds, is that they are not, in general, conformally invariant. Although it is possible to study conformally Einstein manifolds, that are given by semi-Riemannian manifolds that after a pointwise conformal change of metric are Einstein manifolds, in this work we are more interested to a different approach. A century ago, in [Ba], R. Bach introduced a two times covariant symmetric tensor $B$ on a semi-Riemannian manifold $(M, g)$, nowadays called Bach tensor. The Bach tensor has some particular properties: it is traceless, quadratic in the Riemann tensor and, furthermore, for four dimensional manifolds it is conformally invariant and divergence free. The Bach field equations in vacuum for a four dimensional Lorentzian manifold $(M, g)$ are given by

$$B = 0,$$

they are the conformally invariant counterpart of (1.2) (since $B$ is traceless we cannot have a cosmological constant). Einstein and conformally-Einstein manifolds solves the Bach equations, hence they admits more solutions than Einstein equations, even with non-constant scalar curvature. Bach flat Riemannian metrics on a four dimensional closed orientable smooth manifold $M$ admit the following variational characterization: they are critical points of the functional

$$B : M \to \mathbb{R}, \quad B(g) := \int_M |W_g|^2 \mu_g,$$

where $W_g$ is the Weyl tensor of $(M, g)$, the “conformal invariant part of the Riemann tensor”. Since the (1, 3) version of the Weyl tensor is a conformal invariant tensor it is easy to realize that the functional $B$ is conformal invariant, i.e., $B(\tilde{g}) = B(g)$ for every $\tilde{g} \in [g]$. This provide an easy way to see that the Bach tensor, the gradient of $B$, is a conformal invariant tensor for four dimensional manifolds, at least when they are closed, oriented and Riemannian.

We now leave the vacuum case, coming to the description of our contribution. One of the easiest way to allow the presence of energy and matter is to consider the presence of a field $\varphi$. More precisely: let $(M, g)$ be a four dimensional Lorentzian manifold and $\varphi : M \to N$ a smooth map, where the target $(N, \eta)$ is a Riemannian manifold. To the field $\varphi$ we can associate an energy-stress tensor $T = T_\varphi$, as P. Baird and J. Eells did in [BaE] (see (2.21) for its definition). In order to be an admissible energy-stress tensor it must satisfy the conservation laws. A computation shows that in case $\varphi$ is a wave map (i.e., a harmonic map with source a Lorentzian manifold) then $T_\varphi$ satisfies the conservation laws, compare with [Ba]. The converse implication holds, that is, a smooth map that satisfies the conservation laws is harmonic, providing that $\varphi$ is a submersion a.e., although is not true in general.

Now the Einstein fields equations for the four dimensional Lorentzian manifold $(M, g)$ when the presence of field and matter is described by the wave map $\varphi$ takes the form

$$G + \Lambda g = \alpha T_\varphi,$$

where $G$ is the Einstein tensor of $(M, g)$, $\alpha$ is given by (1.1) and $\Lambda \in \mathbb{R}$ is the cosmological constant.

Now, as did for the vacuum case, we can forget that $(M, g)$ is a four dimensional manifold and that $\alpha$ is given by (1.1) and study semi-Riemannian manifolds $(M, g)$ of dimension $m \geq 3$ satisfying (1.5) for some $\Lambda, \alpha \in \mathbb{R}$ and a wave map $\varphi : M \to N$, where the target $(N, \eta)$ is a Riemannian manifold. We call them harmonic-Einstein manifolds (with respect to $\varphi$ and $\alpha$), see Definition 2.46. Those class of semi-Riemannian manifolds includes the Einstein manifolds and, in dimension $m \geq 3$, also its elements are characterized by the vanishing of the traceless part of a symmetric two times covariant tensor (see Proposition 2.51). This tensor, that plays the role of Ricci’s tensor for harmonic-Einstein manifolds, is the $\varphi$-Ricci tensor, given by

$$\text{Ric}^\varphi := \text{Ric} - \alpha \varphi^* \eta$$

and first used by R. Müller in [M] when dealing with the Ricci-harmonic flow, a combination of the Ricci flow with the heat flow of a smooth map. Notice that in order to define the $\varphi$-Ricci tensor we need a semi-Riemannian metric $g$, a smooth map $\varphi$ and a constant $\alpha$. The trace of the $\varphi$-Ricci tensor is called $\varphi$-scalar curvature and it is given by $S^\varphi = S - \alpha |d\varphi|^2$, where $|d\varphi|^2 = \text{tr}(\varphi^* \eta)$ is the Hilbert-Schmidt norm of the differential of $\varphi$. The generalized Schur’s identity (2.22) gives that harmonic-Einstein manifolds of dimension $m \geq 3$ have constant $\varphi$-scalar curvature and the one with vanishing $\varphi$-scalar curvature are
called ϕ-Ricci flat manifolds. Going back to (1.3), the cosmological constant Λ is a constant positive multiple of the ϕ-scalar curvature.

The similarities between the theory of harmonic-Einstein manifolds and the classical theory of Einstein manifolds are not over. Harmonic-Einstein manifolds admits a variational characterization too. Let M be a closed and oriented smooth manifold of dimension m ≥ 3 and (N, η) be a fixed target Riemannian manifold. We denote by F the set of all the smooth maps ϕ : M → N. Moreover, fix α ∈ R \ {0}. Then harmonic-Einstein structures on M, with respect to α ∈ R, are critical points of the functional of total ϕ-scalar curvature

\[ S : M × F → \mathbb{R}, \quad S(g, ϕ) := \int_M S^ϕ_\alpha g, \]

Considering critical points of S on M × R we obtain ϕ-Ricci flat manifolds. Those results are stated and proved in Section 5.2, where we also discuss the what happens for surfaces, see Remark 5.46.

Considering, for a fixed ϕ ∈ F, the restriction of the total ϕ-scalar curvature to \( M_1 \cap \{ g \} \) its critical points are given by Riemannian metrics of constant ϕ-scalar curvature, see Proposition 5.59. This led us to formulate the ϕ-Yamabe problem: on a compact Riemannian manifold (M, g) of dimension m ≥ 3 there exists \( \bar{g} \in \{ g \} \) such that \( S^ϕ_\alpha \) is constant? In Section 5.3 we just give the definition of the ϕ-Yamabe constant, the first step in the solution of this problem. We expect the solution of the ϕ-Yamabe problem to be very similar and very close to the one of the classic Yamabe problem but we postpone it to some future work.

Proceeding with the similarities, in Section 5.1 we study the adjoint of the linearization of the ϕ-scalar curvature map \((g, ϕ) ∈ M × F ↦→ S^ϕ_\alpha \) on a smooth manifold M. We show that a function u belongs to its kernel if either \((M, g) \) is ϕ-Ricci flat (and in this case u is constant) or the product \( M × R \), endowed with the metric \( g − u^2 dt \otimes dt \) is a harmonic-Einstein manifold with respect to the extension \( \bar{ϕ} \) of ϕ to \( M × R \), outside the zero locus of u, where it degenerates (see Proposition 5.33). This is the first step in the study of the image of the ϕ-scalar curvature map, that we also postpone to some future work. Section 4 is devoted to the study harmonic-Einstein manifolds that arise as semi-Riemannian warped product metric with respect to the lifting \( \bar{ϕ} \) of a smooth map ϕ with base the total of the warped product, see Theorem 4.27. An interesting example of the above situation is given ϕ-static harmonic-Einstein manifold, a concept related to the one of static spacetime: see Definition 4.36 and the remarks below.

Finally, we now come to conformal invariant theories in presence of the field ϕ. At this point the definition of conformally harmonic-Einstein is straightforward: we refer to [A] for it and for results on regarding those manifolds. In this article we do not want to focus on them, instead we are interested in a theory similar to the one of R. Bach mentioned above. To obtain this goal we need to define on a semi-Riemannian manifold \((M, g)\) endowed with a smooth map ϕ a two times covariant symmetric tensor that, at least for four dimensional Lorentzian manifolds, is conformally invariant and that when ϕ is constant reduces to the Bach tensor \( B \). This tensor, that appears for the first time in [A], is the ϕ-Bach tensor \( B^ϕ \), see Section 2.2 for its definition. The proof of its conformal invariance is contained in Corollary 3.44 and it is the main result of Section 3. Furthermore, the ϕ-Bach tensor is traceless for four dimensional manifolds (for higher dimensions see (2.23)) and, analogously to the ϕ-Ricci tensor, it depends on a scale factor \( α ∈ R \). The equation

\[ B^ϕ = 0 \]

seems a natural generalization of (1.3), indeed four dimensional harmonic-Einstein and conformally harmonic-Einstein manifolds are solutions of the above.

To justify that the above equation is indeed the generalization of the Bach field equation in the presence of the field ϕ the only thing that remains is to provide a variational characterization. In Section 2.1, where we briefly recall the definitions and the properties of the ϕ-curvatures (whose detailed proof is contained in [A]), one may find the definition of the ϕ-Weyl tensor \( W^ϕ \), i.e., a generalization of the Weyl tensor in presence of the field ϕ. Assuming that the ϕ-Weyl tensor is the right generalization of the Weyl tensor, one may think that ϕ-Bach flat Riemannian metrics on a four dimensional closed orientable smooth manifold M are critical points of the functional

\[ M × F → \mathbb{R}, \quad (g, ϕ) ↦→ \int_M |W^ϕ|^2 g, \]
This is not true: the appropriate functional is given by
\[ B : \mathcal{M} \times \mathcal{F} \to \mathbb{R}, \quad B(g, \varphi) = \frac{1}{2} \int_M \left( S_2(A_g^\varphi) - \frac{\alpha}{2} |\tau^g(\varphi)|^2 \right) \mu_g, \quad (1.6) \]
where \( \tau^g(\varphi) \) denotes the tension field of \( \varphi \) and \( S_2(A_g^\varphi) \) denotes the second elementary symmetric polynomial in the eigenvalues of \( A_g^\varphi \), the \( \varphi \)-Schouten tensor of \((M, g)\) (see Section 2.1 for the definition of the \( \varphi \)-Schouten tensor and Remark 5.75 the one of \( S^2(A_g^\varphi) \)). By choosing as \( \varphi \) any constant map, for every \( g \in \mathcal{M} \) the above yields
\[ B(g, \varphi) = \frac{1}{2} \int_M S_2(A_g) \mu_g, \]
and, in view of the Chern-Gauss-Bonnet formula for four dimensional manifolds, its critical points are the ones of the standard functional (1.4).

The aim of Section 5.4 is to prove Theorem 5.78: on a four dimensional closed and orientable manifold \( M \) the pair \((g, \varphi)\) is critical for \( B \) if and only if the \( \varphi \)-Bach tensor together with another tensor, denoted by \( J \), vanish. The tensor \( J \), defined in (2.39), is strictly connected to the bi-tension of \( \varphi \) (we can say, in a certain sense, that is a conformal invariant bi-tension for four dimensional manifold). The fact that the equation \( B^\varphi = 0 \) is coupled with another one, involving the map \( \varphi \), is not a surprise: the same happens for harmonic-Einstein manifolds where the equation for \( G + \Lambda g = \alpha T \varphi \) is coupled with \( \tau(\varphi) = 0 \). Instead, the surprising fact is that \( J \) is strictly related to the divergence of \( \varphi \)-Bach and when \( \varphi \) is a submersion a.e. the vanishing of \( J \) is equivalent to the vanishing of \( \text{div}(B^\varphi) \), see Remark 2.38. Since in the definition (1.6) the total bi-energy of the map \( \varphi \), that is,
\[ E_2^\varphi(\varphi) = \frac{1}{2} \int_M |\tau^g(\varphi)| \mu_g, \]
is involved, the tensor \( J \) is related to the bi-tension of \( \varphi \).

2 Conventions, notations and preliminaries

Let \( M \) be a smooth, connected manifold without boundary and of dimension \( m \geq 2 \). Let \( g \) be a pseudo-Riemannian metric on \( M \) and \( \nabla \) the Levi-Civita connection of \((M, g)\). For the Riemann tensor \( \text{Riem} \) of \((M, g)\) we use the sign conventions
\[ R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \quad \text{for every } X,Y,Z \in \mathfrak{X}(M), \]
where \( \mathfrak{X}(M) \) is the \( C^\infty(M) \)-module of vector fields on \( M \) and \([,] \) denotes the Lie bracket, and
\[ \text{Riem}(W,Z,X,Y) = g(\text{Riem}(X,Y)Z,W) \quad \text{for every } X,Y,Z,W \in \mathfrak{X}(M). \]

Let \((N, \eta)\) be a Riemannian manifold of dimension \( n \) and \( \varphi : M \to N \) a smooth map. In order to carry on computations we will mostly use (with some exception in Section 5) the moving frame formalism introduced by E. Cartan. For a crash course in the Riemannian case see Chapter 1 of [AMR] while for the semi-Riemannian case we refer to Chapter 5 of [CCL].

We fix the indexes ranges
\[ 1 \leq i, j, k, t, \ldots \leq m, \quad 1 \leq a, b, c, d \ldots \leq n \]
and from now on we adopt the Einstein summation convention over repeated indexes. In a neighborhood of each point of \( M \) we can write
\[ g = \eta_{ij} \theta^i \otimes \theta^j \]
where \( \{\theta^i\} \) is a local \( g \)-orthonormal coframe and
\[ \eta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \leq k \\ -1 & \text{if } i = j > k, \end{cases} \quad (2.1) \]
The signature of the metric is \((k, m - k)\). When \(g\) is a Riemannian metric then \(\eta_{ij} = \delta_{ij}\), the Kronecker delta. Denoting by \(\{e_i\}\) the local frame dual to \(\{\theta^i\}\) it is easy to see that \(\{e_i\}\) is a local \(g\)-orthonormal frame, i.e., \(g(e_i, e_j) = \eta_{ij}\).

The Levi-Civita connection forms \(\{\theta^i_j\}\) are characterized by the validity of the first structure equations
\[
d\theta^i + \theta^i_j \wedge \theta^j = 0
\] (2.2)
and by the relation
\[
\eta_{ik}\theta^k_j + \eta_{kj}\theta^k_i = 0.
\] (2.3)
In the following we will use the metric to raise and lower the indexes, for instance
\[
\theta_i := \eta_{ij} \theta^j, \quad \theta^i := \eta^{ik} \theta_k.
\] Then the relation (2.3) is equivalent to the skew-symmetry
\[
\theta^i_j + \theta^j_i = 0.
\] (2.4)
The Levi-Civita connection forms \(\{\Theta^i_j\}\) are given by
\[
\Theta^i_j = \frac{1}{2} R^i_{kjt} \theta^k \wedge \theta^t,
\] (2.5)
where \(R^i_{kjt}\) are the components of the \((0, 4)\)-version of the Riemann tensor Riem of \((M, g)\),
\[
\text{Riem} = R^i_{kjt} \theta^k \otimes \theta^t \otimes \theta^i \otimes \theta^j,
\]
and they satisfy the second structure equations
\[
\Theta^i_j = d\theta^i + \theta^i_k \wedge \theta^k_j,
\]
where we denoted
\[
\theta^i_j := \eta^{ik} \theta^j_k, \quad \theta^i := \eta^{ik} \theta^k.
\]
The symmetries of the Riemann tensor are given by
\[
R^i_{kjt} + R^i_{jkt} = 0, \quad R^i_{kjt} + R^i_{jkt} = 0, \quad R^i_{kjt} = R^i_{ktj},
\]
while the first Bianchi identity is reads
\[
R^i_{jkt} + R^i_{ktj} + R^i_{tjk} = 0
\]
and finally the second Bianchi identity is expressed as
\[
R^i_{jkt,l} + R^i_{ijlt,k} + R^i_{ikl,t} = 0,
\]
where, for an arbitrary tensor field \(T\) of type \((r, s)\)
\[
T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \theta^{j_1} \otimes \ldots \otimes \theta^{j_s} \otimes e_{i_1} \otimes \ldots \otimes e_{i_r},
\]
its covariant derivative is defined as the tensor field of type \((r, s + 1)\)
\[
\nabla T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s,k} \theta^{k} \otimes \theta^{j_1} \otimes \ldots \otimes \theta^{j_s} \otimes e_{i_1} \otimes \ldots \otimes e_{i_r},
\]
where the following relation holds
\[
T^{i_1 \ldots i_r}_{j_1 \ldots j_s,k} \theta^k = dT^{i_1 \ldots i_r}_{j_1 \ldots j_s} - \sum_{t=1}^{s} T^{i_1 \ldots i_r}_{j_1 \ldots j_{t-1} h_{j_{t+1}} \ldots j_s} \theta^h_{jt} + \sum_{t=1}^{r} T^{i_1 \ldots i_{t-1} h_{i_{t+1}} \ldots i_r}_{j_1 \ldots j_s} \theta^h_{kt}.
\]
It is possible to prove that the following commutation relation holds
\[
T^{i_1 \ldots i_r}_{j_1 \ldots j_s,kt} = T^{i_1 \ldots i_r}_{j_1 \ldots j_s,tk} + \sum_{t=1}^{s} R^{i_1 \ldots i_r}_{j_1 \ldots j_{t-1} h_{j_{t+1}} \ldots j_s} T^{h_{j_{t+1}} \ldots j_s}_{j_1 \ldots j_{t-1} k} + \sum_{t=1}^{r} R^{i_1 \ldots i_r}_{i_{t+1} \ldots i_r j_1 \ldots j_s} T^{i_1 \ldots i_{t-1} h_{i_{t+1}} \ldots i_r}_{j_1 \ldots j_s,k}.
\] (2.6)
The energy density $V$ is given by

$$V = R_{ijkl} \theta^i \otimes \theta^j; \quad R_{ijkl} = \eta^{ij} R_{iklj}. \tag{2.7}$$

The scalar curvature $S$ of $(M, g)$, denoted also by $S_g$, is given by the trace of the Ricci tensor, locally

$$S = \eta^{ij} R_{ij}. \tag{2.8}$$

The Riemannian volume element of $(M, g)$ is locally given by

$$\mu = \theta^1 \wedge \ldots \wedge \theta^m. \tag{2.8}$$

Let $\{E_a\}, \{\omega^a\}, \{\omega^0_a\}, \{\Omega^p_b\}$ be an orthonormal frame, coframe, the respectively Levi-Civita connection forms and curvature forms on an open subset $V$ on $N$ such that $\varphi^{-1}(V) \subseteq U$. We set

$$\varphi^* \omega^a = \varphi^a_i \theta^i$$

so that the differential $d\varphi$ of $\varphi$, a 1-form on $M$ with values in the pullback bundle $\varphi^{-1}TN$, can be written as

$$d\varphi = \varphi^a_i \theta^i \otimes E_a.$$ 

The energy density $e(\varphi)$, or $e^a(\varphi)$, of the map $\varphi$ is given by

$$e(\varphi) = \frac{1}{2} |d\varphi|^2,$$ 

where $|d\varphi|^2 = \text{tr}(\varphi^* \eta)$ is the square of the Hilbert-Schmidt norm of $d\varphi$. The generalized second fundamental tensor of the map $\varphi$ is given by $\nabla d\varphi$, locally

$$\nabla d\varphi = \varphi^a_i \theta^i \otimes \eta_a,$$

where its coefficient are defined according to the rule

$$\varphi^a_i \theta^j = d\varphi^a_i - \varphi^a_k \theta^k + \varphi^b_i \omega^a_k.$$ 

The tension field $\tau(\varphi)$, or $\tau^a(\varphi)$, of the map $\varphi$ is the section of $\varphi^{-1}TN$ defined by

$$\tau(\varphi) = \text{tr}(\nabla d\varphi), \tag{2.10}$$

locally

$$\tau(\varphi)^a = \eta^{ij} \varphi^a_{ij}.$$ 

The bi-energy density $e_2(\varphi)$, or $e_2^a(\varphi)$, of the map $\varphi$ is defined as

$$e_2(\varphi) = \frac{1}{2} |\tau(\varphi)|^2,$$ 

and the bi-tension field $\tau_2(\varphi)$, or $\tau_2^a(\varphi)$, of the map $\varphi$ is the section of $\varphi^{-1}TN$ locally given by

$$\tau_2(\varphi)^a = \varphi^a_{ijkl} - N R^a_{bcd} \varphi^b_{ij} \varphi^c_{kl}.$$ 

Once again, it is possible to prove

$$\varphi^a_{ijkl} = \varphi^a_{kl} + \sum_{p=1}^r R^a_{ijkl} \varphi^a_{p1} \ldots \varphi^a_{pk} - N R^a_{bcd} \varphi^b_{ij} \varphi^c_{kl}. \tag{2.13}$$

In particular

$$\varphi^a_{ijkl} = \varphi^a_{ik} + R^a_{ijkl} \varphi^a_{l}, - N R^a_{bcd} \varphi^b_{ij} \varphi^c_{kl}$$

and

$$\varphi^a_{ijkl} = \varphi^a_{ijkl} + R^a_{ijkl} \varphi^a_{ij} + R^a_{ijkl} \varphi^a_{is} - N R^a_{bcd} \varphi^b_{ij} \varphi^c_{kl} \varphi^d_{lt} \tag{2.14}$$

hold.

We denote by $\Delta$ the Laplace-Beltrami operator acting on functions $u : M \to \mathbb{R}$, defined as $\Delta u = \text{tr} (\text{Hess}(u))$, where $\text{Hess}(u) = u_{ij} \theta^i \otimes \theta^j$, that is, $\Delta u = \eta^{ij} u_{ij}$. Furthermore, if $A$ is a symmetric two times covariant tensor on $(M, g)$ we set

$$\Delta A := - \frac{\text{tr}(A)}{m} g,$$

and the symmetric two times covariant tensors $A^2$ and $\Delta A$ are locally given by

$$A^2_{ij} = \eta^{kt} A_{ik} A_{kt}, \quad \Delta A_{ij} = \eta^{kt} A_{ij,kt}.$$
2.1 \( \varphi \)-Curvatures

Let \((M, g)\) be a semi-Riemannian manifold of dimension \(m \geq 2\), \((N, \eta)\) a Riemannian manifold and \(\alpha \in \mathbb{R} \setminus \{0\}\). In this section we recall the definition of \(\varphi\)-curvatures and we state their properties that shall be useful later on. Their proofs in the Riemannian setting are contained in Section 1.2 of [A]. Those proofs can be easily extended to the semi-Riemannian setting.

The \(\varphi\)-Ricci tensor is defined as
\[
\text{Ric}^\varphi_g \equiv \text{Ric}^\varphi := \text{Ric} - \alpha \varphi^* \eta \tag{2.15}
\]
and the \(\varphi\)-scalar curvature is given by its trace:
\[
S^\varphi_g \equiv S^\varphi := \text{tr}(\text{Ric}^\varphi). \tag{2.16}
\]
Starting from them we define \(\varphi\)-Schouten tensor
\[
A^\varphi_g \equiv A^\varphi := \text{Ric}^\varphi - \frac{S^\varphi}{2(m-1)} g. \tag{2.17}
\]
The \(\varphi\)-Cotton tensor measures the failure of the commutation of the covariant derivatives of the \(\varphi\)-Schouten tensor and in global notation is given by
\[
C^\varphi(X, Y, Z) \equiv C^\varphi \equiv \nabla_Z A^\varphi(X, Y) - \nabla_Y A^\varphi(X, Z) \quad \text{for every } X, Y, Z \in \mathfrak{X}(M). \tag{2.18}
\]
In moving frame notation
\[
C^\varphi_{ijk} = A^\varphi_{ij,k} - A^\varphi_{ik,j}.
\]
Clearly the \(\varphi\)-Cotton tensor is skew-symmetric in the last two entries and satisfies the following Bianchi identity
\[
C^\varphi(X, Y, Z) + C^\varphi(Y, Z, X) + C^\varphi(Z, X, Y) = 0 \quad \text{for every } X, Y, Z \in \mathfrak{X}(M). \tag{2.19}
\]
When \(m \geq 3\) we define the \(\varphi\)-Weyl tensor as
\[
W^\varphi_g = W^\varphi := \text{Riem} - \frac{1}{m-2} A^\varphi \otimes g, \tag{2.20}
\]
where \(\otimes\) denotes the Kulkarni-Nomizu product of two times covariant symmetric tensors and give rise to a \((0,4)\) tensor that has the same symmetries of the Riemann tensor. Recall that, for \(V\) and \(T\) two times covariant symmetric tensors,
\[
(T \otimes V)(X, Y, Z, W) = T(X, Z)V(Y, W) - T(X, W)V(Y, Z) + T(Y, W)V(X, Z) - T(Y, Z)V(X, W),
\]
in moving frame notation
\[
(T \otimes V)_{ijk} = T_{ik}V_{jt} - T_{it}V_{jk} + T_{jt}V_{ik} - T_{jk}V_{it}.
\]
Following P. Baird and J. Eells, see [BE], we define the stress-energy tensor of \(\varphi\) (with a different sign convention) by
\[
T^\eta \equiv T := \varphi^* \eta - \frac{|d\varphi|^2}{2} g. \tag{2.21}
\]
The generalized Schur’s identity is given by
\[
\text{div}(\text{Ric}^\varphi) = \frac{1}{2} dS^\varphi - \alpha \text{div}(T). \tag{2.22}
\]
The divergence of \(\varphi\)-Weyl is related to the \(\varphi\)-Cotton tensor as follows, in a local \(g\)-orthonormal coframe,
\[
\eta^{ks} W^\varphi_{tjs} = \frac{m-3}{m-2} C^\varphi_{tks} + \alpha (\varphi_s \varphi^s - \varphi^s \varphi_s) + \frac{\alpha}{m-2} \tau(\varphi)^s (\varphi^s \eta_{ik} - \varphi^s \varphi^s \eta_{ij}), \tag{2.23}
\]
The traces of \(\varphi\)-Cotton (with respect to the first and the second entries) and of \(\varphi\)-Weyl (with respect to the first and the third entries) are given by, respectively,
\[
\text{tr}(C^\varphi) = \alpha \text{div}(T), \quad \eta^{jk} C^\varphi_{jki} = \alpha \text{div}(T), \tag{2.24}
\]
and
\[
\text{tr}(W^\varphi) = \alpha \varphi^* \eta, \quad \eta^{kt} W^\varphi_{kitj} = \alpha \varphi^s \varphi^s. \tag{2.25}
\]
2.2 The $\varphi$-Bach tensor

It remain another $\varphi$-curvature to introduce: the $\varphi$-Bach tensor $B^\varphi_g \equiv B^\varphi$, whose components are given by, in a local $g$-orthonormal coframe,

\[
(m - 2)B^\varphi_{ij} = \eta^{kt} C^\varphi_{ijk,t} + (R^\varphi)^{jk}(W^\varphi_{ikj} - \alpha \varphi_i^a \varphi_j^a \eta_{jk})
+ \alpha \left( \varphi_i^a \tau(\varphi)^a - \tau(\varphi)^a \varphi_i^a - \frac{1}{m - 2} \tau(\varphi)^2 \eta_{ij} \right).
\]  

(2.26)

It is not immediate to see but the $\varphi$-Bach tensor is symmetric and this is due to the validity of

\[
\eta^{kt} C^\varphi_{kij,t} = \alpha [\varphi_i^a ((R^\varphi)^{jk}_{kt} \varphi_j^a - (R^\varphi)^{jk}_{ij} \varphi_i^a) + \tau(\varphi)^a \varphi_j^a - \tau(\varphi)^a \varphi_i^a].
\]

(2.27)

Furthermore, its trace is given by

\[
(m - 2) \text{tr}(B^\varphi) = \alpha \frac{m - 4}{m - 2} |\tau(\varphi)|^2.
\]

(2.28)

In the following Proposition we provide an alternative, but equivalent, definition for $\varphi$-Bach that shall be useful in the proof of Lemma 5.85.

**Proposition 2.29.** In a local $g$-orthonormal coframe the components of the $\varphi$-Bach tensor can be written as

\[
(m - 2)B^\varphi_{ij} = \Delta R^\varphi_{ij} - \frac{m - 2}{2(m - 1)} S^\varphi_{ij} - \frac{m - 4}{m - 2} (R^\varphi)^2_{ij} - \frac{m}{(m - 1)(m - 2)} S^\varphi R^\varphi_{ij} + 2 R_{kitj}(R^\varphi)^{kt}
+ \left( \frac{(S^\varphi)^2}{(m - 1)(m - 2)} - \frac{1}{2} [\text{Ric}^\varphi]^2 \right) \eta_{ij}
+ \alpha \left[ 2 \varphi_i^a \tau(\varphi)^a - \frac{1}{m - 2} \tau(\varphi)^2 \eta_{ij} - ((R^\varphi)^{jk}_{ki} \varphi_j^a + (R^\varphi)^{jk}_{ij} \varphi_k^a) \varphi_i^a \right].
\]

In particular, if $m = 4$,

\[
B^\varphi_{ij} = \frac{1}{2} \Delta R^\varphi_{ij} - \frac{1}{6} S^\varphi_{ij} - \frac{1}{3} S^\varphi R^\varphi_{ij} + R_{kitj}(R^\varphi)^{kt}
+ \left( \frac{(S^\varphi)^2}{12} - \frac{\Delta S^\varphi}{12} - \frac{1}{4} [\text{Ric}^\varphi]^2 \right) \eta_{ij}
+ \alpha \left[ \varphi_i^a \tau(\varphi)^a - \frac{1}{4} \tau(\varphi)^2 \eta_{ij} - \frac{1}{2} ((R^\varphi)^{jk}_{ki} \varphi_j^a + (R^\varphi)^{jk}_{ij} \varphi_k^a) \varphi_i^a \right].
\]

(2.31)

**Proof.** Using the definitions (2.18) and (2.14) we get

\[
C^\varphi_{ijk,t} = R^\varphi_{ijk,t} - R^\varphi_{ikjt} - \frac{S^\varphi_{kt}}{2(m - 1)} \eta_{ij} + \frac{S^\varphi_{jt}}{2(m - 1)} \eta_{ik}.
\]

Then

\[
\eta^{kt} C^\varphi_{ijk,t} = \Delta R^\varphi_{ij} - \eta^{kt} R^\varphi_{ikjt} - \frac{\Delta S^\varphi}{2(m - 1)} \eta_{ij} + \frac{S^\varphi_{jt}}{2(m - 1)}.
\]

(2.32)

The following relation holds

\[
\eta^{kt} R^\varphi_{ikjt} = \frac{1}{2} S^\varphi_{ij} - R_{kitj}(R^\varphi)^{kt} + (R^\varphi)^2_{ij} + \alpha \left( (R^\varphi)^{jk}_{ij} \varphi_j^a - \tau(\varphi)^a \varphi_j^a - \tau(\varphi)^a \varphi_i^a \right).
\]

(2.33)

To prove (2.33) first notice that, commutating the indexes,

\[
R^\varphi_{ikjt} = R^\varphi_{ikjt} + R^\varphi_{ijkt} - R^\varphi_{kjti} + R^\varphi_{kjit}.
\]

hence contracting the above

\[
\eta^{kt} R^\varphi_{ikjt} = \text{div}(\text{Ric}^\varphi)_{ij} + R^\varphi_{ijkt} + R^\varphi_{ijkt}.
\]
Using (2.22) and (2.16) from the above we infer
\[ \eta^{k} R^\varphi_{ijk,t} = \left( \frac{1}{2} S^\varphi - \alpha \tau(\varphi)^a \varphi^a_i \right)_j + R^\varphi_{ij,t}(R^\varphi)_{jt}^i + (R^\varphi)^2_{ij} + \alpha(R^\varphi)_{ij}^a \varphi^a_j, \]
that is (2.33). By plugging (2.33) into (2.32) we get
\[
\eta^{k} C^\varphi_{ijk,t} = \Delta R^\varphi_{ij,t} - \frac{m - 2}{2(m - 1)} S^\varphi R^\varphi_{ij} + R_{kk,i}(R^\varphi)^{kt} - (R^\varphi)^2_{ij} - \frac{\Delta S^\varphi}{2(m - 1)} \eta_{ij}.
\]
(2.34)

Now, using (2.20) and (2.17),
\[
W^\varphi_{ikj,t}(R^\varphi)^{kt} = R_{ikj,t}(R^\varphi)^{kt} - \frac{1}{m - 2} \left[ \frac{m}{m - 1} S^\varphi R^\varphi_{ij} - 2(R^\varphi)^2_{ij} \right. + \left. \left( |\text{Ric}|^2 - \frac{(S^\varphi)^2}{m - 1} \right) \eta_{ij} \right].
\]
(2.35)

By plugging (2.34) and (2.35) into (2.20), after some computations, we get (2.30).

Motivated by the fact that the Bach tensor is divergence free for four dimensional manifolds in the following Proposition we evaluate the divergence of \( \varphi \)-Bach. The following Proposition will be useful also in the proof of Lemma 5.79.

**Proposition 2.36.** In a local \( g \)-orthonormal coframe the components of the divergence of the \( \varphi \)-Bach tensor are given by
\[
\text{div}(B^\varphi)_i = \frac{m - 4}{m - 2} (R^\varphi_{jk,ki} + \alpha(\tau(\varphi)^a + R^\varphi_{ij} \varphi^a_i) \tau(\varphi)^a)
+ \alpha \varphi^a_i \left[ \frac{m S^\varphi}{(m - 1)(m - 2)} \tau(\varphi)^a - \frac{m - 2}{2(m - 1)} S^\varphi_{ij} \varphi^a_j - 2 R^\varphi_{jk} \varphi^a_{jk} + 2 \tau(\varphi)^b \varphi^b \varphi^a_j - \tau_2(\varphi)^a \right].
\]
(2.37)

By setting
\[
(J_m)^a = J^a := \frac{m S^\varphi}{(m - 1)(m - 2)} \tau(\varphi)^a - \frac{m - 2}{2(m - 1)} S^\varphi_{ij} \varphi^a_j - 2 R^\varphi_{jk} \varphi^a_{jk} + 2 \tau(\varphi)^b \varphi^b \varphi^a_j - \tau_2(\varphi)^a
\]
the above can be written as
\[
\text{div}(B^\varphi)_i = \frac{m - 4}{m - 2} (R^\varphi_{jk,ki} + \alpha(\tau(\varphi)^a + R^\varphi_{ij} \varphi^a_i) \tau(\varphi)^a) + \alpha J_m \varphi^a_i.
\]

**Remark 2.38.** In particular, for \( m = 4 \),
\[
\text{div}(B^\varphi)_i = \alpha J^a \varphi^a_i,
\]
with
\[
J^a = (J_4)^a = \frac{2}{3} S^\varphi \tau(\varphi)^a - \frac{1}{3} S^\varphi \varphi^a_j - 2 R^\varphi_{jk} \varphi^a_{jk} + 2 \tau(\varphi)^b \varphi^b \varphi^a_j - \tau_2(\varphi)^a.
\]
(2.39)

Then, if \( \varphi \)-Bach is divergence free and \( \varphi \) is a submersion a.e.,
\[
J = 0.
\]

**Proof.** We decompose the \( \varphi \)-Bach tensor as follows
\[
(m - 2) B^\varphi_{ij} = \alpha L_{ij} + M_{ij} + N_{ij},
\]
where
\[
L_{ij} := \varphi^a_{kk} \varphi^a_{ij} - \frac{1}{m - 2} |\tau(\varphi)|^2 \delta_{ij},
M_{ij} := R^\varphi_{ik} W^\varphi_{nikj}
\]
and
\[
N_{ij} := C^\varphi_{ijk,k} - \alpha(R^\varphi_{kj} \varphi^a_k + \varphi^a_{kk}) \varphi^a_i.
\]
We proceed by evaluating separately the divergences of \( L \), \( M \) and \( N \) and then we combine them all to obtain (2.30).
• Using the commutation rule (2.13) and the definition of \( \varphi \)-Ricci (2.15) we easily get
\[
\operatorname{div}(L)_j = \frac{m - 4}{m - 2} R^\varphi_{ikj} \varphi^a_{ij} + \varphi^a_{kki} \varphi^a_{ij} + N R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + \alpha \varphi^a_{kki} \varphi^a_{ij}.
\] (2.40)

• Clearly
\[
\operatorname{div}(M)_j = R^\varphi_{ikj} \varphi^a_{ij} + \alpha \frac{S^\varphi}{m - 1} \varphi^a_{ij}.
\]

Using the definition of \( \varphi \)-Schouten and \( \varphi \)-Cotton and the symmetries of \( \varphi \)-Weyl and \( \varphi \)-Schouten we infer
\[
R^\varphi_{ikj} \varphi^a_{ij} = C^\varphi_{ikj} \varphi^a_{ij} + \alpha \frac{S^\varphi}{m - 1} \varphi^a_{ij}.
\]

Using (2.23) we easily get
\[
R^\varphi_{ikj} \varphi^a_{ij} = m - 3 \frac{R^\varphi_{ikj} C^\varphi_{ikj} + \alpha (R^\varphi_{ikj} \varphi^a_{ij} - R^\varphi_{ikj} \varphi^a_{ij} + \alpha (R^\varphi_{ikj} \varphi^a_{ij} + \varphi^a_{ikj}) \varphi^a_{ij} - \alpha (R^\varphi_{ikj} \varphi^a_{ij} + \varphi^a_{ikj}) \varphi^a_{ij}.
\] (2.41)

Combining the equations above we have
\[
\operatorname{div}(M)_j = C^\varphi_{ikj} \varphi^a_{ij} - \alpha \frac{S^\varphi}{m - 1} \varphi^a_{ij} + \frac{m - 3}{m - 2} R^\varphi_{ikj} C^\varphi_{ikj} + \frac{m - 3}{m - 2} R^\varphi_{ikj} C^\varphi_{ikj} + \alpha \frac{S^\varphi}{m - 1} \varphi^a_{ij}.
\] (2.42)

Exchanging the covariant derivatives we obtain
\[
C^\varphi_{ikj} = (C^\varphi_{ikj})_j - R^\varphi_{jik} C^\varphi_{ikj}.
\] (2.43)

Using (2.41) we infer
\[
(C^\varphi_{ikj})_j = \alpha \left[ R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} - R^\varphi_{ikj} \varphi^a_{ij} + (R^\varphi_{ikj} \varphi^a_{ij} - R^\varphi_{ikj} \varphi^a_{ij}) \varphi^a_{ij}.
\] (2.44)

Plugging (2.44) into (2.43) and then plugging it all into (2.42), with the aid of (2.22), we get
\[
\operatorname{div}(N)_j = R^\varphi_{ikj} C^\varphi_{ikj} - \alpha \left[ \frac{1}{2} \left( R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + + R^\varphi_{ikj} \varphi^a_{ij} + \varphi^a_{ikj} \varphi^a_{ij} \right).
\]

Using the decomposition (2.24), the definition of \( \varphi \)-Schouten, the symmetries of \( \varphi \)-Cotton and (2.24) we get
\[
R^\varphi_{ikj} C^\varphi_{ikj} = W^\varphi_{ikj} C^\varphi_{ikj} - \frac{1}{m - 2} \left( R^\varphi_{ikj} C^\varphi_{ikj} + \alpha R^\varphi_{ikj} \varphi^a_{ij} - \frac{\alpha S^\varphi}{m - 1} \varphi^a_{ij} \right),
\]

hence, by plugging into the above we finally get
\[
\operatorname{div}(N)_j = W^\varphi_{ikj} C^\varphi_{ikj} - \frac{1}{m - 2} R^\varphi_{ikj} C^\varphi_{ikj} + \alpha \left[ \frac{1}{2} \left( R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + R^\varphi_{ikj} \varphi^a_{ij} + + R^\varphi_{ikj} \varphi^a_{ij} + \varphi^a_{ikj} \varphi^a_{ij} \right).
\] (2.45)

Combining (2.23), (2.41) and (2.45), after a computation and rearranging the terms, recalling the definition (2.12) of the bi-tension of \( \varphi \), we conclude the validity of (2.37).
2.3 Einstein fields equations in presence of a field

**Definition 2.46.** Let \((M,g)\) be a semi-Riemannian manifold of dimension \(m \geq 3\), \(\varphi : M \to N\) a smooth map with target a Riemannian manifold \((N,\eta)\) and \(\alpha \in \mathbb{R} \setminus \{0\}\). We say that \((M,g)\) is harmonic-Einstein (with respect to \(\varphi\) and \(\alpha\)) if

\[
\begin{align*}
\text{Ric}^\varphi &= 0 \\
\tau(\varphi) &= 0.
\end{align*}
\] (2.47)

We say that \((M,g)\) is \(\varphi\)-Ricci flat (with respect to \(\alpha\)) if

\[
\begin{align*}
\text{Ric}^\varphi &= 0 \\
\tau(\varphi) &= 0.
\end{align*}
\] (2.48)

**Remark 2.49.** Harmonic-Einstein manifolds have constant \(\varphi\)-scalar curvature, it follows from (2.22). Then the \(\varphi\)-Schouten tensor is parallel and, as a consequence, the \(\varphi\)-Cotton tensor vanishes. Furthermore, harmonic-Einstein manifolds are \(\varphi\)-Bach flat.

**Remark 2.50.** \(\varphi\)-Ricci flat manifolds (with respect to \(\alpha\)) are no more than harmonic-Einstein manifolds (with respect to \(\varphi\) and \(\alpha\)) with vanishing \(\varphi\)-scalar curvature.

**Proposition 2.51.** Let \((M,g)\) be a semi-Riemannian manifold of dimension \(m \geq 3\), \(\varphi : M \to N\) a harmonic map with target a Riemannian manifold \((N,\eta)\) and \(\alpha \in \mathbb{R} \setminus \{0\}\). Then \((M,g)\) is harmonic-Einstein (with respect to \(\varphi\) and \(\alpha\)) if and only if

\[
G + \Lambda g = \alpha T,
\] (2.52)

where \(G\) is the Einstein tensor of \((M,g)\), that is,

\[
G := \text{Ric} - \frac{S}{2} g,
\]

\(T\) is the energy-stress tensor of the map \(\varphi\) that is given by (2.21). If this the case, the cosmological constant is given by

\[
\Lambda := \frac{m - 2}{2m} S^\varphi \in \mathbb{R}.
\] (2.53)

**Proof.** Notice that, since \(\varphi\) is harmonic, \((M,g)\) is harmonic-Einstein with respect to \(\varphi\) and \(\alpha\) if and only if the first equation of (2.47) holds, that is,

\[
\text{Ric}^\varphi = \frac{S^\varphi}{m} g.
\] (2.54)

Assume (2.51) holds. Using the definition of \(\varphi\)-Ricci tensor, the relation \(S^\varphi = S - \alpha |d\varphi|^2\) and (2.51) we obtain

\[
G = \text{Ric} - \frac{S}{2} g = \text{Ric}^\varphi + \alpha \varphi^\ast \eta - \frac{S^\varphi}{2} g + \alpha \frac{|d\varphi|^2}{2} g = \frac{S^\varphi}{m} g - \frac{S^\varphi}{2} g + \alpha \left( \varphi^\ast \eta - \frac{|d\varphi|^2}{2} g \right).
\]

that gives (2.52), by setting \(\Lambda\) as in (2.53) (that is constant since it is a constant multiple of the \(\varphi\)-scalar curvature) and recalling the definition (2.21) of \(T\). The converse implication follows analogously.

**Remark 2.55.** Four dimensional harmonic-Einstein Lorentzian manifolds \((M,g)\) with respect to a smooth map \(\varphi : M \to N\), with target a Riemannian manifold \((N,\eta)\), and

\[
\alpha = \frac{8\pi G}{c^2},
\] (2.56)

where \(G\) is Newton’s gravitational constant and \(c\) is the speed of light in vacuum, are solutions of the Einstein field equations with cosmological constant \(\Lambda\) given by (2.56) and as field source the wave map (i.e., harmonic map with source a Lorentzian manifold) \(\varphi\), see Section 6.5 of [C-B]. In particular, \(\varphi\)-Ricci flat (with respect to \(\alpha\) given by (2.56)) solves the Einstein field equations with the absence of cosmological constant, i.e., with \(\Lambda = 0\).
of good dual frame, the Levi-Civita connection forms and the curvature forms for the pseudo-Riemannian metric

Using (3.3) the above gives

\[ g := e^{-2h}g. \]  

Since on \( \mathcal{M} \), the Levi-Civita connection forms \( \tilde{\Gamma} = \{ \tilde{\Gamma}_{\alpha \beta}^\gamma \} \) is given by

\[ \tilde{\Gamma}_{\alpha \beta}^\gamma = \Gamma_{\alpha \beta}^\gamma - h_{\alpha \beta} \theta^\gamma + h^\gamma \theta_{\alpha \beta}, \]

The Levi-Civita connection forms \( \tilde{\Gamma}_j^\alpha \) for \( \tilde{g} \) on \( \mathcal{U} \) are given by

\[ \tilde{\Gamma}_j^\alpha = \theta_j^\alpha - h_j \theta^\alpha + h^\alpha \theta_j. \]

The components of the Riemann tensor of \( (\mathcal{M}, \tilde{g}) \) in the local coframe \( \{ \tilde{\theta}^i \} \) are given by

\[ e^{-2h} \tilde{R}_{ijkl} = R_{ijkl} + (h_j k \eta_{it} - h_j t \eta_{ik} + h_i t \eta_{jk} - h_i k \eta_{jt}) \\
+ (h_j k \eta_{it} - h_j t \eta_{ik} + h_i t \eta_{jk} - h_i k \eta_{jt}) - |\nabla h|^2 (\eta_{jk} \eta_{it} - \eta_{jt} \eta_{ik}), \]

where \( R_{ijkl} \) are the components of the Riemann tensor of \( (\mathcal{M}, g) \) in the local coframe \( \{ \theta^i \} \). In global notation

\[ e^{2h} \tilde{\text{Riem}} = \text{Riem} + \left( \text{Hess}(h) + dh \otimes dh - \frac{|\nabla h|^2}{2} \right) \otimes g. \]

The Riemannian volume element of \( \tilde{g} \) is given by

\[ \tilde{\mu} = e^{-mh} \mu. \]

Proof. Since on \( \mathcal{U} \) we have \( g = \eta_{ij} \theta^i \otimes \theta^j, \) \((3.1)\) gives \( \tilde{g} = \eta_{ij} \tilde{\theta}^i \otimes \tilde{\theta}^j \), where \( \{ \tilde{\theta}^i \} \) are given by \((3.3)\). This shows that \( \{ \tilde{\theta}^i \} \) is a local \( \tilde{g} \)-orthonormal coframe. By setting \( \{ \tilde{e}_i \} \) as in \((3.4)\) it is immediate to check that \( \{ \tilde{e}_i \} \) is the dual frame of \( \{ \tilde{\theta}^i \} \).

The first structure equations read

\[ d\tilde{\theta}^i + \tilde{\theta}_j \wedge \tilde{\theta}^i = 0. \]

Using \((3.3)\) the above gives

\[ e^{-h}(d\tilde{\theta}^i - dh \wedge \tilde{\theta}^i) + e^{-h}\tilde{\theta}_j \wedge \theta^i = 0, \]

that is,

\[ d\tilde{\theta}^i = dh \wedge \tilde{\theta}^i + \tilde{\theta}_j \wedge \theta^i = 0. \]
With the aid of the first structure equations for the coframe \{\theta^i\} (see (2.2)) we get

\[-\theta^i_j \wedge \theta^j - h_i \theta^j \wedge \theta^j + \tilde{\theta}^i_j \wedge \theta^j = 0,\]

that is,

\[(\tilde{\theta}^i_j - \theta^i_j + h_j \theta^i) \wedge \theta^j = 0.\]

Assuming that the connection forms \{\tilde{\theta}^i_j\} are given by (3.5) we obtain that the above equation is satisfied.

Indeed, by plugging (3.5) into the above we get \(h^i \theta_j \wedge \theta^j = 0\), that is satisfied. Moreover, using once again (3.5) we deduce

\[\eta_k \theta^k_{ij} + \eta_j \theta^k_{ki} = \eta_k \theta^k + \eta_j \theta^k_i,\]

hence the skew-symmetry \(\tilde{\theta}_{ij} + \tilde{\theta}_{ji} = 0\) follows immediately from the skew-symmetry \(\theta_{ij} + \theta_{ji} = 0\).

Recalling that the connection forms are characterized by the skew symmetry and the validity of the first structure equations we obtain that \{\tilde{\theta}^i_j\} given by (3.5) are the Levi-Civita connection forms for \(\tilde{g}\) on \(\mathcal{M}\).

The second structure equations for the metric are given by

\[
\tilde{\Theta}^i_j = d\tilde{\theta}^i_j + \tilde{\theta}^i_{k} \wedge \theta^k.
\]

Using (3.5) we get

\[
\tilde{\Theta}^i_j = d\theta^i_j - h_i \theta^j + \theta^j \theta_i + (\theta_{ki} - h_i \theta_k + h_k \theta_i) \wedge (\theta^{kj} - h^j \theta^k + h^k \theta^j),
\]

that gives

\[
\tilde{\Theta}^i_j = d\theta^i_j + \theta_{ki} \wedge \theta^k + (dh^i - h_k \theta^{kj}) \wedge \theta^j - (dh_i - h^k \theta_{ki}) \wedge \theta^j
\]

\[- h_i (d\theta^j - \theta^j \wedge \theta_i) - h^j (dh_i - \theta_{ki} \wedge \theta^k) + h_k h_i \theta^k \wedge \theta^k
\]

\[- h_i h^k \theta_k \wedge \theta^j - h_i h^j \theta_i \wedge \theta^k + h_k h^j \theta_i \wedge \theta^j.
\]

From the definition of covariant derivatives we get \(h_{ij} \theta^j = dh_i - h_j \theta^j\) and by plugging it together with the first and the second structure equations for \(g\) and \(\theta_k \wedge \theta^k = 0\) into the above we deduce

\[
\tilde{\Theta}^i_j = \Theta^i_j + h^j_i \theta^k \wedge \theta^i - h_i h^k \theta_{ki} \wedge \theta^j
\]

\[+ h_i (\theta^j_k + \theta^j_i) \wedge \theta^k - h^j (\theta_{ki} + \theta^k_i) \wedge \theta^k
\]

\[- h_i h^k \theta_k \wedge \theta^j - h_i h^j \theta_i \wedge \theta^k + h^j_i h^k \theta_i \wedge \theta^j.
\]

Using the skew symmetry \(\theta_{ij} + \theta_{ji} = 0\) from the above we conclude

\[
\eta_{js} \tilde{\Theta}^i_j = \eta_{js} \Theta^i_j + (h_{sk} \eta_{st} - h_{sk} \eta_{st} - h_i h_k \eta_{st} + h_k h_s \eta_{st} + |\nabla h|^2 \eta_{sk} \eta_{st}) \theta^k \wedge \theta^t,
\]

that gives, recalling (2.5),

\[
\frac{1}{2} \tilde{R}_{jik} \tilde{\theta}^i \wedge \tilde{\theta}^j = \left( \frac{1}{2} R_{jik} + h_{jk} \eta_{jt} - h_{ki} \eta_{jt} - h_i h_k \eta_{jt} + h_k h_j \eta_{jt} + |\nabla h|^2 \eta_{sk} \eta_{st} \right) \theta^k \wedge \theta^t.
\]

Using (3.6), skew-symmetrizing the above we finally get

\[
e^{-2h} \tilde{R}_{jik} = R_{jik} + h_{jk} \eta_{jt} - h_{ki} \eta_{jt} - h_i h_k \eta_{jt} + h_k h_j \eta_{jt} + |\nabla h|^2 (\eta_{sk} \eta_{st} - \eta_{st} \eta_{sk})
\]

that is, (3.6) holds. Since

\[
\tilde{R}_{jik} = \tilde{R}_{jik} \tilde{\theta}^i \tilde{\theta}^j \tilde{\theta}^k \tilde{\theta}^l = e^{-4h} \tilde{R}_{jik} \theta^k \theta^t \theta^i \theta^j,
\]

the validity of (3.7) implies (3.6).

The validity of (3.8) follows easily from (2.8) and (3.6).
Let \((N, \eta)\) be a Riemannian manifold, we denote by \(\{E_a\}, \{\omega^a\}\) and \(\{\omega^a_\xi\}\) the local orthonormal frame, coframe and the corresponding Levi-Civita connection forms on an open set \(\mathcal{V}\) such that \(\varphi^{-1}(\mathcal{V}) \subseteq \mathcal{U}\). Clearly \(d\varphi\) is independent on the choice of the metric on \(M\), it means that

\[
\tilde{\varphi}^a_i = e^h \varphi^a_i, \tag{3.9}
\]

where we used (3.3) and

\[
\varphi^a_i \theta^i \otimes E_a = d\varphi = \tilde{\varphi}^a_i \tilde{\theta}^i \otimes E_a.
\]

As an immediate consequence we get

\[
|d\varphi|^2 = e^{2h}|d\varphi|^2. \tag{10.0}
\]

By definition

\[
\nabla d\varphi = \varphi^a_i \theta^i \otimes \theta^i \otimes E_a, \quad \varphi^a_i \theta^i = d\varphi^a_i - \varphi^a_i \theta^i + \varphi^a_i \omega^b_k
\]

and

\[
\tilde{\nabla} d\varphi = \tilde{\varphi}^a_i \tilde{\theta}^i \otimes \tilde{\theta}^i \otimes E_a, \quad \tilde{\varphi}^a_i \tilde{\theta}^i = d\tilde{\varphi}^a_i - \tilde{\varphi}^a_i \tilde{\theta}^i + \tilde{\varphi}^a_i \tilde{\omega}^b_k.
\]

We denote by \(\tilde{\tau}(\varphi)\) the tension of the map

\[
\varphi : (M, \tilde{g}) \to (N, \eta),
\]

in components

\[
\tilde{\tau}(\varphi)^a = \eta^{ij} \tilde{\varphi}^a_{ij}.
\]

In the next Proposition we determine the transformation laws for the quantities of our interest related to the smooth map \(\varphi\), under the conformal change of the metric (3.1).

**Proposition 3.11.** *In the notations above, in a local orthonormal coframe,*

\[
\tilde{\varphi}^a_{ij} = e^{2h} (\varphi^a_{ij} + \varphi^a_i h_j + \varphi^a_j h_i - \varphi^a_k h^k \eta_{ij}), \tag{3.12}
\]

*in particular*

\[
\tilde{\tau}(\varphi) = e^{2h}[\tau(\varphi) - (m - 2)d\varphi(\nabla h)]. \tag{3.13}
\]

Moreover, *in a local orthonormal coframe,*

\[
\tilde{\tau}(\varphi)^a_i = e^{2h} [\tau(\varphi)^a_i - (m - 2)\varphi^a_i h^i - (m - 2)\varphi^a_i h^i_k + 2\tau^b(\varphi)^a h^b_k - 2(m - 2)\varphi^a_i h^i_k h^k]. \tag{3.14}
\]

**Proof.** The validity of (3.12) follows easily using (3.3), the definition of \(\tilde{\varphi}^a_{ij}\), (3.9), (3.11) and the definition of \(\varphi^a_{ij}\) as follows:

\[
\tilde{\varphi}^a_{ij} e^{-h} \theta^j = \tilde{\varphi}^a_{ij} \tilde{\theta}^j = d\tilde{\varphi}^a_i - \tilde{\varphi}^a_i \tilde{\theta}^j + \tilde{\varphi}^a_i \tilde{\omega}^b_k = d(e^h \varphi^a_i) - e^h \varphi^a_i \left( \theta^j - h^j \theta^j + h^i \theta^i \right) + e^h \varphi^a_i \omega^b_k = e^h (d\varphi^a_i - \varphi^a_i \theta^i + \varphi^a_i \theta^i + \varphi^a_i \omega^b_k) + e^h \varphi^a_i dh + e^h \varphi^a_i (h^i \theta^j - h^j \theta^i). \]

Applying \(\eta^{ij}\) to (3.12) we immediately get (3.13). For convenience we denote

\[
T^a_{ij} = \varphi^a_i + \varphi^a_i h_j + \varphi^a_j h_i - \varphi^a_k h^k \eta_{ij}.
\]

Using the definition of covariant derivative, with the aid of (3.5) we get

\[
\tilde{\varphi}^a_{ij} \tilde{\theta}^k = d\tilde{\varphi}^a_{ij} - \tilde{\varphi}^a_{kj} \tilde{\theta}^i - \tilde{\varphi}^a_{ik} \tilde{\theta}^j + \tilde{\varphi}^a_{ij} \tilde{\omega}^b_k = d(e^h T^a_{ij}) - e^h T^a_{kj} (\theta^i + h^i \theta^i - h^j \theta^j) - e^h T^a_{ik} (\theta^j + h^k \theta^i - h^j \theta^j) + e^h T^a_{ij} \tilde{\omega}^b_k.
\]

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Thus, by plugging (3.3) into the above and doing some calculations,
\[ e^{-3h} \phi^a_{ijk} \theta^k = 2T^a_{ij,k} \theta^k + T^a_{ij,k} \theta^k - T^a_{ij,(h^k \theta_i - h_i \theta^k)} - T^a_{ik}(h^k \theta_j - h_j \theta^k) = (T^a_{ij,k} + 2T^a_{ij,k} + T^a_{ik}h_i - T^a_{ij}h_i \eta_k + T^a_{ik}h_j - T^a_{ik}h_i \eta_j) \theta^k, \]
that gives,
\[ e^{-3h} \phi^a_{ijk} = T^a_{ij,k} + 2T^a_{ij,k} + T^a_{ik}h_i - T^a_{ij}h_i \eta_k + T^a_{ik}h_j - T^a_{ik}h_i \eta_k. \]
Applying \( \eta_{ij} \) and using the relations
\[ \eta^{ij} T^a_{ij,k} = \tau^a_\eta (\phi^a - (m - 2)\phi_i h^i), \]
\[ \eta^{ij} T^a_{ij,k} = \tau_\eta (\phi^a) - (m - 2)\phi_i h^i - (m - 2)\phi_i^a h^i \]
(the first follows immediately from the definition of \( T^a_{ij} \) while the second is obtained taking covariant derivative of the first), we get from the above
\[ e^{-3h} \phi^a_{ijk} = T^a_{ij,k} + 2T^a_{ij,k} + T^a_{ik}h_i - T^a_{ij}h_i \eta_k + T^a_{ik}h_j - T^a_{ik}h_i \eta_k, \]
that is (3.14).

**Proposition 3.15.** The components of the \( \varphi \)-Ricci tensor of \((M, \tilde{g})\) in the coframe \( \{\tilde{\theta}^i\} \) are given by
\[ e^{-2h} \tilde{R}^\varphi_{ij} = R^\varphi_{jk} + (m - 2)h_{ij} + (m - 2)h_i h_j + [\Delta h - (m - 2)|\nabla h|^2] \eta_{ij}, \]
where \( R^\varphi_{jk} \) are the components of the \( \varphi \)-Ricci tensor of \((M, g)\) in the coframe \( \{\theta^i\} \). In global notation
\[ \tilde{\text{Ric}}^\varphi = \text{Ric}^\varphi + (m - 2)\text{Hess}(h) + (m - 2)dh \otimes dh + [\Delta h - (m - 2)|\nabla h|^2] g. \]
Moreover the \( \varphi \)-scalar curvature of \((M, \tilde{g})\) is given by
\[ e^{-2h} \tilde{S}^\varphi = S^\varphi + (m - 1)[2\Delta h - (m - 2)|\nabla h|^2], \]
where \( S^\varphi \) is the \( \varphi \)-scalar curvature of \((M, g)\).

**Proof.** First of all we prove
\[ e^{-2h} \tilde{R}_{jk} = R_{jk} + (m - 2)h_{jk} + (m - 2)h_j h_k + [\Delta h - (m - 2)|\nabla h|^2] \eta_{jk}. \]
From (2.7), using (3.6) we get
\[ e^{-2h} \tilde{R}_{jk} = R_{jk} + (m h_{jk} - h_{k j} + \Delta h_{jk} - h_{jk}) + (m h_j h_k - h_j h_k + |\nabla h|^2 \eta_{jk} - h_j h_k) - |\nabla h|^2 (m \eta_{jk} - \eta_{jk}), \]
or equivalently, (3.19).

The validity of (3.16) follows immediately from (3.19) and (3.9). Using (3.17) and (3.3) we deduce that (3.16) holds.

Applying \( \eta^\varphi \) to (3.16) we immediately get (3.18).

From now on we assume \( m \geq 3 \) and we set
\[ h := \frac{1}{m - 2} f, \]
so that (3.1) reads
\[ \tilde{g} = e^{-\frac{2m - 2}{m - 2}} g. \]
Then (3.17), (3.18), (3.19) and (3.14) reads, respectively,
\[ \tilde{\text{Ric}}^\varphi = \text{Ric}^\varphi + \text{Hess}(f) + \frac{1}{m - 2} (df \otimes df + \Delta f g), \]
\[ e^{-\frac{2}{m-2}} \tilde{S}^\varphi = S^\varphi + \frac{m-1}{m-2} [2\Delta f - |\nabla f|^2], \tag{3.22} \]

\[ \tilde{\tau}(\varphi) = e^{\frac{2}{m-2}} [\tau(\varphi) - d\varphi(\nabla f)] \tag{3.23} \]

and

\[ \eta^{ij} \tilde{\varphi}'_{ijk} = e^{-\frac{2}{m-2} f} \left[ \eta^{ij} \varphi'^{a}_{ijk} - \varphi'^{a}_{ik} f^j - \varphi'^{a}_{ij} f^k \right] + \frac{2}{m-2} (\varphi^a - \varphi^a f^i) f_k]. \tag{3.24} \]

Then it is easy to obtain, using the definition of \( \varphi \)-Schouten and (3.21) and (3.22),

\[ \tilde{\varphi}^a = A^a + \text{Hess}(f) + \frac{1}{m-2} \left( df \otimes df - \frac{|\nabla f|^2}{2} g \right), \tag{3.25} \]

that locally reads

\[ e^{-\frac{2}{m-2} f} \tilde{\varphi}^a_{ij} = A^a_{ij} + f_{ij} + \frac{1}{m-2} \left( f_i f_j - \frac{|\nabla f|^2}{2} \eta_{ij} \right). \tag{3.26} \]

Moreover, from the definition (2.20) of \( \varphi \)-Weyl, using (3.7) and (3.25), we immediately get

\[ e^{\frac{2}{m-2} f} \tilde{W}^\varphi = W^\varphi, \tag{3.27} \]

that is, the \((1,3)\) version of the \( \varphi \)-Weyl is conformal invariant.

In the next Proposition we deal with the transformation laws for the \( \varphi \)-Cotton tensor.

**Proposition 3.28.** In the notations above, the components of the \( \varphi \)-Cotton tensor of \((M, \tilde{g})\) with respect to the coframe \( \{\tilde{\theta}^i\} \) are given by

\[ e^{-\frac{2}{m-2} f} \tilde{C}^\varphi_{ijk} = C^\varphi_{ijk} + W^\varphi_{tijk} f^t, \tag{3.29} \]

where \( C^\varphi_{ijk} \), \( W^\varphi_{tijk} \) and \( f^t \) are, respectively, the components of \( C^\varphi \), \( W^\varphi \) and \( df \) in the coframe \( \{\theta^i\} \).

**Proof.** For simplicity of notation we set

\[ T_{ij} := A^a_{ij} + f_{ij} + \frac{1}{m-2} \left( f_i f_j - \frac{|\nabla f|^2}{2} \eta_{ij} \right), \tag{3.30} \]

so that (3.20) reads

\[ \tilde{A}^a_{ij} = e^{\frac{2}{m-2} f} T_{ij}. \tag{3.31} \]

We claim the validity of

\[ e^{-\frac{2}{m-2} f} \tilde{A}^a_{ijk} = \frac{2}{m-2} T_{ijk} f_k + T_{ij,k} + \frac{1}{m-2} (T_{kji} f_i - T_{ij} f^t \eta_{ki} + T_{ik} f_j - T_{it} f^t \eta_{jk}). \tag{3.32} \]

To prove the claim we use the definition of covariant derivative, (3.31) and (3.5) (with \( h \) given by (3.20)), obtaining

\[ \tilde{A}^a_{ijk} \tilde{\theta}^k = d\tilde{A}^a_{ijk} - \tilde{A}^a_{ij} \tilde{\theta}^k - \tilde{A}^a_{ik} \tilde{\theta}^j \]

\[ = d(e^{\frac{2}{m-2} f} T_{ij}) - e^{\frac{2}{m-2} f} T_{kij} \left( \theta^k - \frac{f_i}{m-2} \theta^k + \frac{f^k}{m-2} \theta_i \right) \]

\[ - e^{\frac{2}{m-2} f} T_{ik} \left( \theta^j - \frac{f_j}{m-2} \theta^k + \frac{f^j}{m-2} \theta_k \right), \]

that is,

\[ e^{-\frac{2}{m-2} f} \tilde{A}^a_{ijk} \tilde{\theta}^k = \frac{2}{m-2} T_{ij} df + (dT_{ij} - T_{kji} \theta^t - T_{ik} \theta^j) + \frac{1}{m-2} [T_{kij} (f_i \theta^k - f^k \theta_i) + T_{ikj} (f_j \theta^k - f^k \theta_j)]. \]

Using (3.33) and the definition of \( T_{ij,k} \) the above yields

\[ e^{-\frac{2}{m-2} f} \tilde{A}^a_{ijk} \tilde{\theta}^k = \left[ \frac{2}{m-2} T_{ij} f_k + T_{ij,k} + \frac{1}{m-2} (T_{kji} f_i - T_{ij} f^t \eta_{ki} + T_{ik} f_j - T_{it} f^t \eta_{jk}) \right] \theta^k, \]

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hence (3.32) holds.

Now, using the definition of the \( \varphi \)-Cotton tensor, (3.32) twice and the symmetry of \( T \) we get
\[
e^{-\frac{m-\tau}{m}} \tilde{C}^\varphi_{ijk} = e^{-\frac{m-\tau}{m}} (\tilde{A}^\varphi_{ij,k} - \tilde{A}^\varphi_{ik,j})
\]
\[
= T_{ij,k} - T_{ik,j} + \frac{2}{m-2} (T_{ij} f_k - T_{ik} f_j) + \frac{1}{m-2} [T_{ik} f_j - T_{ij} f_k + (T_{ik} \eta_{ji} - T_{ij} \eta_{ki}) f^l],
\]
that is,
\[
e^{-\frac{m-\tau}{m}} \tilde{C}^\varphi_{ijk} = T_{ij,k} - T_{ik,j} + \frac{1}{m-2} (T_{ij} \eta_{kt} - T_{ik} \eta_{jt} + T_{ik} \eta_{ji} - T_{ij} \eta_{ki}) f^l. \quad (3.33)
\]
To express the right hand side of the above in terms of \( C^\varphi \) we first observe that, from the definition
(3.30) of \( T \),
\[
T_{ij,k} = A^\varphi_{ij,k} + f_{ijk} + \frac{1}{m-2} (f_{ik} f_j + f_{ij} f_k - f^l f_{ik} \delta_{ij}),
\]
so that, using the commutation rule (see (2.6))
\[
\text{hence (3.32) holds.}
\]

Moreover an easy computation using (3.30) shows that
\[
(V_{ij} := \eta^{kt} C^\varphi_{ij,k,t} - \alpha [(R^\varphi)^{ij}_k \varphi^o_k + \tau (\varphi)^o_j] \varphi^o_i, \quad (3.34)
\]
that is the content of

Lemma 3.35. In the above notations,
\[
e^{-\frac{m-\tau}{m}} \tilde{V}_{ij} = V_{ij} + f^{lk} W_{ij,k}^\varphi - \frac{m-4}{m-2} f^l f^k W_{ij}^\varphi + \frac{m-4}{m-2} (C^\varphi_{jk,i} + C^\varphi_{ij,k}) f^k + \alpha \varphi^o_l f^o_k f^k
\]
\[
+ \frac{\alpha}{m-2} [(\varphi^o_k f^k - \tau (\varphi)^o_k) (\varphi^o_i f_j + \varphi^o_j f_i) - \tau (\varphi)^o_k \varphi^o_k f^k \eta_{ij} - \Delta f \varphi^o_i \varphi^o_j]. \quad (3.36)
\]
Applying \( \eta^k \) to the above an easy calculation shows that
\[
e^{-\frac{4\pi}{7}f} \eta^k \tilde{C}_{ijk,s} = \eta^k T_{ijk,s} + f^k T_{ijks} + \frac{1}{m-2} f^k (T_{ijk} T_{jsk} + f_j T_{iks} + f_k T_{ij}) - \frac{f^k}{m-2} T_{ijk} (\eta_{ls} + T_{iks} + T_{ij} \eta_{jk}).
\]

Using the definition of \( T_{ijk} \) and by plugging (2.24) into the above we infer
\[
\eta^k T_{ijk,s} = \eta^k C_{ijk,s} + f^k W_{ijks} + \frac{1}{m-2} C_{ikj} f^k + \alpha (\varphi_{ij}^0 \varphi_{jk}^0 f^k - \varphi_{ij}^k \varphi_{jk}^0 f^0) + \frac{\alpha}{m-2} \tau (\varphi)^0 (\varphi_i^0 f_j - \varphi_j^0 f_i). \tag{3.38}
\]

Clearly
\[
T_{ijk} f^k = C_{ijk} f^k + f^k f^k W_{ijk}. \tag{3.39}
\]

The traces of \( T \) are given by, using (2.24), (2.25) and the symmetries of tensors involved,
\[
\eta^k T_{kjs} = \eta^k C_{kjs} + f^k \eta^k W_{tkjs} = -\alpha \tau (\varphi)^0 \varphi_i^0 + \alpha \varphi_i^0 \varphi_j^0 f^0 = \alpha(\varphi_i^0 f^k - \tau (\varphi)^0) \varphi_i^0 \tag{3.40}
\]
and
\[
\eta^k T_{kis} = 0,
\]
then we easily get
\[
\eta^k T_{kjs} f_i + \eta^k T_{kis} f_j = \alpha(\varphi_i^0 f^k - \tau (\varphi)^0) \varphi_j^0 f_i. \tag{3.41}
\]

Using once again the definition of \( T \), the skew symmetry in the first two indices of \( W^0 \) and the identity (2.19) for \( C^0 \) we evaluate
\[
f^k T_{kji} + f^k T_{kij} = f^k (C_{kji} + W_{tkji} f^t) + f^k (C_{tkj} + W_{tikj} f^t)
= f^k (C_{kji} + C_{tkj}) + f^k f^k W_{tikj}
= - f^k C_{tkj} + f^k f^k W_{tikj}
= f^k C_{tkj} + f^k f^k W_{tikj}.
\]

Plugging the above together with (3.38), (3.39) and (3.40) in (3.37) we finally get
\[
e^{-\frac{4\pi}{7}f} \eta^k \tilde{C}_{ijk,s} = \eta^k C_{ijk,s} + f^k W_{ijks} + \frac{1}{m-2} f^k W_{ijks}
\]
\[
+ \frac{m-4}{m-2} (C_{ikj} + C_{tkj}) f^k + \alpha (\varphi_{ij}^0 \varphi_{jk}^0 f^k - \varphi_{ij}^k \varphi_{jk}^0 f^0) + \frac{\alpha}{m-2} \tau (\varphi)^0 (\varphi_i^0 f_j - \varphi_j^0 f_i) + \alpha \varphi_i^0 \varphi_j^0 f_i - \tau (\varphi)^0 \varphi_i^0 f_i].
\]

To conclude the proof notice that, with the aid of (3.21) and (3.36), we get
\[
e^{-\frac{4\pi}{7}f} ((R^0)^k_{ij} \varphi_i^0 \varphi_j^0 + \tau (\varphi)^0 \varphi_i^0) = (R^0)^k_{ij} \varphi_i^0 \varphi_j^0 + \tau (\varphi)^0 \varphi_i^0 - \alpha \varphi_i^0 \varphi_j^0 f^k \varphi_i^0 + \frac{1}{m-2} (\Delta f \varphi_i^0 \varphi_j^0 f_j + \tau (\varphi)^0 \varphi_i^0 f_j).
\]

Inserting the relations obtained so far into the definition (3.34) we obtain the validity of (3.36).
Now we are finally ready to prove

**Theorem 3.41.** In the above notations, the components of the $\varphi$-Bach tensor of $(M, \tilde{g})$ in the local coframe $\{\tilde{e}^a\}$ are given by

$$ e^{-\frac{1}{2}m-2f}(m-2)	ilde{B}^{\varphi}_{ij} = (m-2) B^{\varphi}_{ij} - \frac{m-4}{m-2} f^k (C^{\varphi}_{ijk} + f^l W^{\varphi}_{ljk} - C^{\varphi}_{jki}). \tag{3.42} $$

**Proof.** From the definition of $\varphi$-Bach (2.26) and (3.34)

$$(m-2)B^{\varphi}_{ij} = V_{ij} + W^{\varphi}_{ijk} (R^{\varphi})^{tk} + \alpha \tau (\varphi)^a \left( \tau^{\varphi}_{ij} - \frac{1}{m-2} \tau (\varphi)^a \eta_{ij} \right) \tag{3.43}$$

Using (3.16), the conformal invariance of $\varphi$-Weyl (3.27) and (3.28) we infer

$$ e^{-\frac{1}{2}m-2f}\tilde{W}^{\varphi}_{tikj} (\tilde{R}^{\varphi})^{tk} = W^{\varphi}_{tikj} \left( (R^{\varphi})^{tk} + f^l \frac{f^k}{m-2} + \Delta f f \frac{\tau^{\varphi} a}{m-2} \right) $$

$$ = W^{\varphi}_{tikj} (R^{\varphi})^{tk} + W^{\varphi}_{tikj} f^k + \frac{1}{m-2} W^{\varphi}_{tikj} f^l f^k + \alpha \frac{\Delta f f}{m-2} \varphi_j^a \varphi_j^a $$

$$ = W^{\varphi}_{tikj} (R^{\varphi})^{tk} - W^{\varphi}_{tikj} f^k - \frac{1}{m-2} W^{\varphi}_{tikj} f^l f^k + \alpha \frac{\Delta f f}{m-2} \varphi_j^a \varphi_j^a. $$

Using (3.12) three times a computation yields

$$ e^{-\frac{1}{2}m-2f}\tilde{\tau} (\varphi)^a \left( \tilde{\varphi}^a_{ij} - \frac{1}{m-2} \tilde{\tau} (\varphi)^a \eta_{ij} \right) = \tau (\varphi)^a \left( \tau^{\varphi}_{ij} - \frac{1}{m-2} \tau (\varphi)^a \eta_{ij} \right) $$

$$ + \frac{1}{m-2} \tau (\varphi)^a \varphi_k^f f^k \eta_{ij} - \varphi_j^a f^k \varphi_j^a $$

$$ + \frac{1}{m-2} (\tau (\varphi)^a - \varphi_j^a f^k) (\varphi_j^a f_j + \varphi_j^a f_j). $$

Combining these two relations with (3.36), from (3.43) we deduce the validity of (3.42). \qed

As an immediate consequence of the transformation law for $\varphi$-Bach we generalize the conformal invariance in the four dimensional case of the Bach tensor in the following.

**Corollary 3.44.** Let $(M, g)$ be a four dimensional pseudo-Riemannian manifold, $\varphi : M \to N$ a smooth map, where $(N, \eta)$ is a target Riemannian manifold, and $\alpha \in \mathbb{R} \setminus \{0\}$. Then the $\varphi$-Bach tensor is a conformal invariant tensor.

**Definition 3.45.** Let $(M, g)$ be a semi-Riemannian manifold of dimension $m \geq 3$, $\varphi : M \to N$ a smooth map, where $(N, \eta)$ is a target Riemannian manifold, and $\alpha \in \mathbb{R} \setminus \{0\}$. We denote

$$ [g] := \{ v^2 g : v \in C^\infty (M), v > 0 \text{ on } M \} $$

and we say that $(M, g)$ is conformally harmonic-Einstein (with respect to $\varphi$ and $\alpha$) if there exists $\tilde{g} \in [g]$ such that $(M, \tilde{g})$ is harmonic-Einstein (with respect to $\varphi$ and $\alpha$). We say that $(M, g)$ is conformally $\varphi$-Ricci flat (with respect to $\alpha$) if there exists $\tilde{g} \in [g]$ such that $(M, \tilde{g})$ is $\varphi$-Ricci flat (with respect to $\alpha$).

**Remark 3.46.** Four dimensional conformally harmonic-Einstein manifolds are $\varphi$-Bach flat, this is due to the conformal invariance of $\varphi$-Bach for four dimensional semi-Riemannian manifolds and the trivial fact that harmonic-Einstein manifolds are $\varphi$-Bach flat, see Remark 2.49.

**Remark 3.47.** Let $(M, g)$ be a semi-Riemannian manifold of dimension $m \geq 3$, $\varphi : M \to N$ a smooth map, where $(N, \eta)$ is a target Riemannian manifold, and $\alpha \in \mathbb{R} \setminus \{0\}$. Using (3.21), (3.22) and (3.23) it is immediate to realize that $(M, g)$ is conformally harmonic-Einstein (with respect to $\varphi$ and $\alpha$) if and only if

$$ \begin{cases} 
(\operatorname{Ric}^\varphi + \operatorname{Hess} (f) + \frac{1}{m-2} df \otimes df) = 0 \\
\tau (\varphi) = d (\varphi (\nabla f)),
\end{cases} $$

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for some \( f \in C^\infty(M) \). Indeed the second equation of the above is clearly equivalent to \( \tilde{\tau}(\varphi) = 0 \), using (3.23), while the first equation of can be rewritten as

\[
\text{Ric}^\varphi + \text{Hess}(f) + \frac{1}{m-2} df \otimes df = \frac{1}{m} \left( S^\varphi + \Delta f + \frac{1}{m-2} |\nabla f|^2 \right) g,
\]

or also as

\[
\text{Ric}^\varphi + \text{Hess}(f) + \frac{1}{m-2} (df \otimes df + \Delta f) g = \frac{1}{m} \left( S^\varphi + \frac{2(m-1)}{m-2} \Delta f - \frac{m-1}{m-2} |\nabla f|^2 \right) g.
\]

Now the equivalence between the above and \( \tilde{\text{Ric}}^\varphi \) is evident, using (3.21), (3.22).

4 Harmonic-Einstein warped products

Let \((M, g)\) and \((F, g_F)\) be two semi-Riemannian manifolds of dimension \( m \) and \( d \) respectively. Let \( u \in C^\infty(M) \), \( u > 0 \) on \( M \).

Definition 4.1. We denote by \( \bar{M} = M \times F \) the product manifold, by

\[
\bar{g} := \pi_M^* g + (u \circ \pi_M)^2 \pi_F^* g_F,
\]

where \( \pi_M : \bar{M} \to M \) and \( \pi_F : \bar{M} \to F \) are the canonical projections, and by

\[
\bar{M} \times_u F := (M, \bar{g})
\]

the semi-Riemannian warped product with base \((M, g)\), fibre \((F, g_F)\) and warping function \( u \).

We are going to identify \( T\bar{M} \) with \( TM \oplus TF \). Via the identification

\[
\bar{g} \equiv g + u^2 g_F.
\]

We use the following indexes conventions

\[
1 \leq i, j, \ldots \leq m, \quad 1 \leq \alpha, \beta, \ldots \leq d, \quad 1 \leq A, B, \ldots \leq m + d.
\]

Let \( \{ e_i \} \), \( \{ \theta^i \} \), \( \{ \Theta^i \} \) be, respectively, a local \( g \)-orthonormal frame, the dual coframe, the relative connection and curvature forms on an open subset \( U \) of \( M \) and let \( \{ e_\alpha \} \), \( \{ \psi_\beta \} \), \( \{ \Psi_\beta \} \) be the same quantities on an open subset \( W \) of \( F \) with respect to \( g_F \).

In the next well known Proposition we determine the local \( \bar{g} \)-orthonormal frame, the dual coframe, the relative connection and curvature forms on \( \bar{U} := U \times W \) induced by the choices above, that we denote by \( \{ \bar{\tau}_A \} \), \( \{ \bar{\theta}^A \} \), \( \{ \bar{\Theta}^A \} \), respectively. We have

\[
\bar{\tau}_A = \eta_{ij} \theta^i \otimes \theta^j, \quad \bar{\theta}^A = F_{\eta_{\alpha\beta}} \psi^\alpha \otimes \psi^\beta,
\]

where \( \eta_{ij} \) and \( F_{\eta_{\alpha\beta}} \) and defined as in (2.1), according to the signature of \( g \) and \( g_F \), respectively.

Proposition 4.2. In the notations above

- The local \( g \)-orthonormal frame \( \{ \bar{\tau}_A \} \) is given by

\[
\bar{e}_i = \pi_M^* (e_i) \equiv e_i, \quad \bar{e}_{m+i} = \frac{1}{u \circ \pi_M} \pi_F^* (e_\alpha) \equiv \frac{1}{u} \bar{e}_\alpha.
\]

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• The corresponding dual coframe \( \{ \psi^\alpha \} \) is given by
\[
\bar{\theta}^j = \pi_M^*(\theta^j) \equiv \theta^j, \quad \bar{\theta}^{m+\alpha} = u \circ \pi_M \circ \pi_F^* \psi^\alpha \equiv u \psi^\alpha.
\] (4.4)

Then
\[
\bar{g} = \bar{\eta}_{j\alpha} \bar{\theta}^j \otimes \bar{\theta}^\alpha,
\]
where
\[
\bar{\eta}_{ij} = \eta_{ij}, \quad \bar{\eta}_{m+\alpha} = 0 = \bar{\eta}_{m+\alpha} = 0 = \bar{\eta}_{m+\alpha} = \bar{\eta}_{m+\alpha} = \bar{\eta}_{m+\alpha}.
\]

• The Levi-Civita connection forms \( \{ \bar{\theta}^A \} \) are given by
\[
\bar{\theta}^j = \theta^j, \quad \bar{\theta}^{m+\alpha} = \psi^\alpha, \quad \bar{\theta}^{m+\alpha} = u \psi^\alpha, \quad \bar{\theta}^{m+\alpha} = -u^j \theta^j \wedge \psi^\alpha.
\] (4.5)

• The curvature forms \( \{ \bar{\Omega}^A \} \) are given by
\[
\bar{\Omega}^j_i = \Theta^j_i, \quad \bar{\Omega}^{m+\alpha} = \Psi^\alpha + |\nabla u|^2 \psi^\alpha \wedge \psi^\beta, \quad \bar{\Omega}^{m+\alpha} = u \iota \theta^j \wedge \psi^\alpha, \quad \bar{\Omega}^{m+\alpha} = -u^j \theta^j \wedge \psi^\alpha.
\] (4.6)

Then the non-vanishing components of \( \bar{\text{Riem}} \) are determined by
\[
\bar{R}_{ijk\ell} = \bar{R}_{ijk\ell}, \quad \bar{R}_{ikm+\alpha j m+\beta} = -\frac{u^j F}{u} \psi^\beta, \quad \bar{R}_{ikm+\alpha m+\beta} = \frac{1}{u^2} \bar{\psi}^\ell \bar{\psi}^\beta \bar{\psi}^\gamma - \frac{u^j F}{u^2} \bar{\psi}^\beta \bar{\psi}^\gamma - F \psi^\beta \psi^\gamma,
\] (4.7)
where \( R_{ijkl} \) and \( F \bar{R}_{\alpha\beta\gamma} \) are the components of the Riemann tensors of \( (M, g) \) and \( (F, g_F) \), respectively.

**Proof.** It is clear that \( \{ \bar{\tau}^A \} \) defined as in 4.3 is a local orthonormal frame, indeed
\[
\bar{g}(\bar{\tau}^i, \bar{\tau}^j) = g(\tau^i, \tau^j) = \eta_{ij}, \quad \bar{g}(\bar{\tau}^i, \bar{\tau}^m) = 0
\]
and
\[
\bar{g}(\bar{\tau}^m, \bar{\tau}^m) = u^2 g_F \left( \frac{\varepsilon^\alpha}{u}, \frac{\varepsilon^\beta}{u} \right) = g_F (\varepsilon^\alpha, \varepsilon^\beta) = F \eta_{\alpha\beta}.
\]
The relations (4.5) follows immediately from (4.3).

To show the validity of (4.5) recall that the first structure equation on \( M \times \bar{F} \) are given by
\[
d \bar{\theta}^A = -\bar{\theta}^A \wedge \bar{\theta}^B.
\]
For \( A = i \) we obtain, using (4.3),
\[
d \bar{\theta}^i = -\bar{\theta}^i \wedge \bar{\theta}^B \equiv -\bar{\theta}^i \wedge \bar{\theta}^j - \bar{\theta}^{m+\alpha} \wedge \bar{\theta}^{m+\alpha} \equiv -\bar{\theta}^i \wedge \bar{\theta}^j - \bar{\theta}^{m+\alpha} \wedge \bar{\theta}^{m+\alpha}
\]
and since, from the first structure equation on \( M \),
\[
d \bar{\theta}^j = d \theta^j \equiv -\theta^j \wedge \theta^j,
\]
we conclude from the above
\[
(\bar{\theta}^j - \theta^j) \wedge \theta^j + u \bar{\theta}^{m+\alpha} \wedge \psi^\alpha = 0.
\] (4.8)

For \( A = m + \alpha \) we obtain, using (4.3),
\[
d \bar{\theta}^{m+\alpha} = -\bar{\theta}^{m+\alpha} \wedge \bar{\theta}^B \equiv -\bar{\theta}^{m+\alpha} \wedge \bar{\theta}^j - \bar{\theta}^{m+\alpha} \wedge \bar{\theta}^{m+\beta} \equiv -\bar{\theta}^{m+\alpha} \wedge \theta^j - u \bar{\theta}^{m+\alpha} \wedge \psi^\beta
\]
and since, from the first structure equation for \( F \),
\[
d\theta^{m+\alpha} = d(u \psi^\alpha) = du \wedge \psi^\alpha + u d\psi^\alpha = u_\alpha \theta^\alpha \wedge \psi^\beta - u_\alpha \psi^\beta \wedge \psi^\beta = -u_\alpha \psi^\beta \wedge \theta^\alpha = 0
\]
we conclude from the above
\[
u(\theta^{m+\alpha} - \psi^\beta) \wedge \psi^\beta = (\theta^{m+\alpha} - u_\alpha \psi^\alpha) \wedge \theta^\alpha = 0 \quad (4.9)
\]
It is immediate to verify that \( \{ \theta_B \} \) given by \( (4.5) \) satisfies \( (4.8), (4.9) \) and
\[
\eta_{CB} \theta_A + \eta_{AC} \theta_B = 0,
\]
hence they are the connection forms associated to the coframe \( \{ \theta_A \} \).

The second structure equation reads as
\[
\Theta^B_A = d\theta^B_A + \eta_{CA} \theta^C_B.
\]
For \( A = i \) and \( B = j \), using \( (4.5) \), \( \psi^\alpha \) and \( \psi^\beta = 0 \) and the second structure equations of \( (M, g) \) we obtain
\[
\Theta^{m+\beta}_i = d\theta_i^{m+\beta} + \theta_{ki} \wedge \theta^{k+\beta} + \theta_{m+\alpha i} \wedge \theta^{m+\alpha j} = d\theta_i^{m+\beta} + \theta_{ki} \wedge \theta^{k+\beta} + u_\alpha \psi^\alpha \wedge \psi^\beta = \psi^\beta + \nabla u^2 \psi^\beta
\]
For \( A = m + \alpha \) and \( B = m + \beta \), using \( (4.5) \) and the second structure equations of \( (F, g_F) \) we get
\[
\Theta^{m+\beta}_{m+\alpha} = d\theta^{m+\beta}_{m+\alpha} + \theta_{km+\alpha} \wedge \theta^{m+\beta} + \theta^{m+\gamma m+\alpha} \wedge \theta^{m+\gamma m+\beta} = d\psi^\beta + u^k \psi^\beta \wedge \psi^\gamma \wedge \psi^\gamma = \psi^\beta + \nabla u^2 \psi^\beta
\]
For \( A = i \) and \( B = m + \alpha \), using \( (4.5) \), the definition of \( u_{ik} \) and the first structure equations of \( (F, g_F) \) we have
\[
\Theta^{m+\alpha}_i = d\theta_i^{m+\alpha} + \theta_{ki} \wedge \theta^{m+\alpha} + \theta_{m+\gamma i} \wedge \theta^{m+\gamma m+\alpha} = (du_i - u^k \theta_{ki}) \wedge \psi^\alpha + u_i (d\psi^\alpha - \psi^\gamma \wedge \psi^\gamma) = u_{ik} \theta^k \wedge \psi^\alpha.
\]
Since
\[
\Pi_{CB} \theta_A + \Pi_{CA} \theta_B = 0
\]
we deduce that
\[
\Theta^{m+\alpha}_{m+\alpha} = -\eta^{ij} F_{\alpha \beta} \Theta^{m+\beta}_j = -\eta^{ij} F_{\alpha \beta} u_{jk} \theta^k \wedge \psi^\beta = -u^{jk} \theta_k \wedge \psi_\alpha.
\]
By definition of \( R \), the Riemann tensor of \( M \times_u F \), we have
\[
\Theta^4_B = \frac{1}{2} R_{B C D} \theta^C \wedge \theta^D.
\]
For \( A = m + \alpha \) and \( B = i \), with the aid of \( (4.4) \),
\[
\Theta^{m+\alpha}_i = \frac{1}{2} R^{m+\alpha C D} \theta^C \wedge \theta^D = \frac{1}{2} \left( R^{m+\alpha C D} \theta^C \wedge \theta^D + u R^{m+\alpha}_{\alpha \beta \theta^k} \wedge \psi^\beta + u R^{m+\alpha}_{\alpha \beta \psi^\gamma} \wedge \theta^k + u^2 R^{m+\alpha}_{\alpha \beta \gamma} \wedge \psi^\beta \right).
\]
Using (4.6) and the symmetries of $\overline{\text{Riem}}$ the above gives
\[
 u_{ij} \theta^k \wedge \psi^\alpha = \overline{\Theta}_{i}^{\alpha} = \frac{1}{2} R_{i k l}^{m+\alpha} \theta^k \wedge \theta^l + u R_{i k m+\alpha} \theta^k \wedge \psi^\delta + \frac{1}{2} u^2 \bar{R}_{i m+\gamma} \theta^{\gamma} \wedge \psi^\delta,
\]
hence we deduce
\[
 R_{m+\alpha i j k} = 0, \quad R_{m+\alpha i m+\beta m+\gamma} = 0, \quad R_{m+\alpha i j m+\beta} = \frac{u_{ij}}{u} F \eta_{\alpha \beta}. \tag{4.10}
\]
For $A = i$ and $B = j$, using (4.4),
\[
 \overline{\Theta}_{j}^{i} = \frac{1}{2} \overline{R}_{i j k}^{m+\alpha} \theta^k \wedge \theta^l + u \overline{R}_{i j k m+\alpha} \theta^k \wedge \psi^\alpha + u \overline{R}_{i j m+\alpha k} \psi^\alpha \wedge \theta^k + u^2 \overline{R}_{i j m+\alpha m+\beta} \psi^\alpha \wedge \psi^\beta).
\]
Using the (4.6), the symmetries of $\text{Riem}$ and (4.10) the above gives
\[
 \frac{1}{2} \overline{R}_{i j k}^{m+\alpha} \theta^k \wedge \theta^l = \overline{\Theta}_{j}^{i} = \frac{1}{2} \overline{R}_{i j k}^{m+\alpha} \theta^k \wedge \theta^l + \frac{1}{2} u^2 \overline{R}_{i j m+\alpha m+\beta} \psi^\alpha \wedge \psi^\beta,
\]
so that
\[
 \overline{R}_{i j k} = R_{i j k}, \quad \overline{R}_{i j m+\alpha m+\beta} = 0. \tag{4.11}
\]
For $A = m + \alpha$ and $B = m + \beta$, \(\overline{\Theta}_{m+\alpha}^{m+\beta} = \frac{1}{2} \left( R_{m+\beta m+\alpha}^{m+\gamma} \theta^l \wedge \theta^l + u R_{m+\beta m+\gamma}^{m+\alpha} \theta^l \wedge \psi^\gamma + u R_{m+\alpha m+\gamma}^{m+\alpha} \psi^\gamma \wedge \theta^l + u^2 \overline{R}_{m+\alpha m+\beta m+\gamma} \psi^\gamma \wedge \psi^\gamma \right).
\]
Using the (4.6), the symmetries of $\text{Riem}$, (4.10) and (4.11) from the above we infer
\[
 \frac{1}{2} F R_{3}^{\alpha \beta} \psi^\gamma \wedge \psi^\delta + |\nabla u|^2 \psi^\alpha \wedge \psi^\beta = \psi_{\beta}^3 + |\nabla u|^2 \psi^\alpha \wedge \psi^\gamma = \overline{\Theta}_{m+\beta}^{m+\alpha} = \frac{1}{2} u^2 \overline{R}_{m+\beta m+\gamma m+\delta} \psi^\gamma \wedge \psi^\delta.
\]
Skew-symmetrizing the above we obtain
\[
 u^2 \overline{R}_{m+\alpha m+\beta m+\gamma m+\delta} = F R_{m+\beta m+\gamma}^\alpha \delta + |\nabla u|^2 \left( F \eta_{\alpha \delta} F \eta_{\beta \gamma} - F \eta_{\alpha \gamma} F \eta_{\beta \delta} \right)
\]
that is,
\[
 \overline{R}_{m+\alpha m+\beta m+\gamma m+\delta} = \frac{1}{u^2} F R_{m+\beta m+\gamma}^\alpha \delta - \frac{|\nabla u|^2}{u^2} \left( F \eta_{\alpha \gamma} F \eta_{\beta \delta} - F \eta_{\alpha \delta} F \eta_{\beta \gamma} \right).
\]
We are finally able to conclude the validity of (4.7). \(\square\)

Since $u > 0$ on $M$ there exists $f \in C^\infty(M)$ such that
\[
 u = e^{-\frac{t}{2}}, \tag{4.12}
\]
As an immediate consequence of the above Proposition we have

**Corollary 4.13.** In the notations above, the non-vanishing components of $\overline{\text{Ric}}$, the Ricci tensor of $(M, \overline{g})$, are given by
\[
 \overline{R}_{i j} = R_{i j} - d \frac{u_{ij}}{u}, \quad \overline{R}_{m+\alpha m+\beta} = - \left( \frac{\Delta u}{u} + (d-1) \frac{|\nabla u|^2}{u^2} \right) F \eta_{\alpha \beta} + \frac{1}{u^2} F R_{\alpha \beta}, \tag{4.14}
\]
where $R_{i j}$ and $F R_{\alpha \beta}$ are the components of the Ricci tensors of $(M, g)$ and $(F, g_F)$, respectively. Moreover
\[
 S = S + \frac{F S}{u^2} - d \left[ \frac{2 \Delta u}{u} + (d-1) \frac{|\nabla u|^2}{u^2} \right].
\]
Equivalently, in terms of $f$, where $f$ is defined by (4.11),
\[
 \overline{R}_{i j} = R_{i j} + f_{i j} - \frac{1}{d} f f_{i j}, \quad \overline{R}_{m+\alpha m+\beta} = \frac{\Delta f}{d} F \eta_{\alpha \beta} + e^{\frac{2 f}{d}} F R_{\alpha \beta} \tag{4.15}
\]
and
\[
 S = S + \frac{2 f}{d} S + 2 \Delta f - \frac{d+1}{d} |\nabla f|^2.
\]
Let \( \varphi : M \to N \) be a smooth map with source the base manifold and target a Riemannian manifold \((N, \eta)\) and denote
\[
\tilde{\varphi} := \varphi \circ \pi_M : \tilde{M} \to N.
\] (4.16)
We use the indexes convention
\[
1 \leq a, b, \ldots \leq n,
\]
where \( n \) is the dimension of \( N \). Let \( \{E_a\}, \{\omega^a\}, \{\omega_\alpha^a\}, \{\Omega^a_\alpha\} \) be, respectively, a \( \eta \)-orthonormal frame, \( \eta \)-orthonormal coframe, connection forms and curvatures form on a open subset \( \mathcal{V} \) of \( N \) such that \( \varphi^{-1}(\mathcal{V}) \subseteq \tilde{U} \). We denote
\[
d\tilde{\varphi} = \tilde{\varphi}^a_i \theta^i \otimes E_a,
\]
so that
\[
\tilde{\varphi}^*_\eta = \tilde{\varphi}^a_i \theta^i \otimes \theta^B.
\]

**Proposition 4.17.** In the assumptions and the notations above

- The components of \( d\tilde{\varphi} \) are given by
  \[
  \tilde{\varphi}^a_i = \varphi^a_i, \quad \tilde{\varphi}^a_{m+\alpha} = 0,
  \] (4.18)
  as a consequence
  \[
  |d\tilde{\varphi}|^2 = |d\varphi|^2.
  \] (4.19)
- The tension \( \tilde{\tau}(\tilde{\varphi}) \) of \( \tilde{\varphi} : M \times_u F \to (N, \eta) \) is given by
  \[
  \tilde{\tau}(\tilde{\varphi}) = \tau(\varphi) + \frac{d\varphi(u)}{u}(\nabla u),
  \] (4.20)
in terms of \( f \) given by \((4.12)\),
  \[
  \tilde{\tau}(\tilde{\varphi}) = \tau(\varphi) - d\varphi(\nabla f).
  \] (4.21)

**Proof.** Using \((4.1)\),
\[
d\tilde{\varphi} = \tilde{\varphi}^a_i \theta^i \otimes E_a = \varphi^a_i \theta^i \otimes E_a + u\tilde{\varphi}^a_{m+\alpha} \psi^\alpha \otimes E_a.
\]
From \((4.16)\) we deduce \( d\tilde{\varphi} = \pi^*_M d\varphi \equiv d\varphi \), that is,
\[
d\tilde{\varphi} = \pi^*_M d\varphi \equiv d\varphi = \varphi^a_i \theta^i \otimes E_a,
\]
and thus \((4.18)\) follows by comparison with the above. In particular
\[
\tilde{\psi}^A_{AB} \tilde{\psi}^B_{\alpha} = \eta_{ij} \varphi^a_i \varphi^a_j,
\]
hence \((4.19)\) holds. Moreover
\[
\varphi^a_i \theta^i \otimes E_a = \varphi^a_i \theta^i \otimes \theta^B = \varphi^a_i \theta^i \otimes \theta^B + \varphi^a_i \omega_\alpha^a,
\]
that is, using \((4.5)\),
\[
\varphi^a_{A} \theta^i + u\varphi^a_{m+\alpha} \psi^\alpha = d\varphi^a_A - \varphi^a_j \theta^j_A - \varphi^a_{m+\alpha} \theta^A_{m+\alpha} + \varphi^a_\alpha \omega_\alpha^a,
\]
For \( A = i \) we obtain, using \((4.18)\) and \((4.5)\),
\[
\varphi^a_i \theta^i + u\varphi^a_{m+\alpha} \psi^\alpha = d\varphi^a_i - \varphi^a_j \theta^j_i + \varphi^a_\alpha \omega_\alpha^a = \varphi^a_i \theta^i,
\]
hence
\[
\varphi^a_i = \varphi^a_i, \quad \varphi^a_{m+\alpha} = 0.
\] (4.22)
For \( A = m + \beta \) we obtain using \((4.18)\) and \((4.5)\),
\[
\varphi^a_{m+\beta} \theta^i + u\varphi^a_{m+\beta m+\alpha} \psi^\alpha = u^j \varphi^a_j \psi_\beta,
\]
hence,
\[
\varphi^a_{m+\beta} = 0, \quad \varphi^a_{m+\alpha m+\beta} = \varphi^a_i \frac{u^j}{u} \eta_{\alpha \beta}.
\] (4.23)
Then, using (4.22) and (4.23), we infer

$$\tau(\bar{\varphi})^a = \eta^{AB} \varphi^a_B = \eta^{ij} \varphi^a_{ij} + F \eta^{\alpha\beta} \varphi_{m+\alpha m+\beta} = \tau(\varphi)^a - d\varphi^a_f u^j,$$

that is, (4.20). Finally (4.21), where $f \in C^\infty(M)$ given by (4.12), follows from (4.20).

Combining Corollary 4.13 with Proposition 4.17 we immediately get

**Corollary 4.24.** In the notations above, the non-vanishing components of $\text{Ric}^\varphi$, the $\varphi$-Ricci tensor of $M \times_u F$, are given by

$$R^\varphi_{ij} = R^\varphi_{ij} - d\omega_{ij} / u, \quad R^\varphi_{m+\alpha m+\beta} = -\left(\frac{\Delta u}{u} + (d-1) \frac{\left|\nabla u\right|^2}{u^2}\right) F\eta_{\alpha\beta} + \frac{1}{u^2} F R_{\alpha\beta},$$

(4.25)

where $R^\varphi_{ij}$ and $F R_{\alpha\beta}$ are the components of the $\varphi$-Ricci tensors of $(M, g)$ and of the Ricci tensor of $(F, g_F)$, respectively. Moreover

$$S^\varphi = S^\varphi + \frac{FS}{u^2} - d \left[\frac{2 \Delta u}{u} + (d-1) \frac{\left|\nabla u\right|^2}{u^2}\right].$$

Equivalently, in terms of $f$, where $f$ is defined by (4.12),

$$R^\varphi_{ij} = R^\varphi_{ij} + f_{ij} - \frac{1}{d} f_i f_j, \quad R^\varphi_{m+\alpha m+\beta} = \frac{\Delta f}{d} F \eta_{\alpha\beta} + (d+2) F R_{\alpha\beta},$$

(4.26)

and

$$S^\varphi = S^\varphi + \frac{2}{F} F S + 2\Delta f - \frac{d+1}{d} \left|\nabla f\right|^2.$$

**Theorem 4.27.** Let $(M, g)$ and $(F, g_F)$ be pseudo-Riemannian manifolds of dimension $m$ and $d$, respectively, with $m + d \geq 3$, and $(N, \eta)$ a Riemannian manifold. Let $f \in C^\infty(M)$ and $\varphi : M \to N$ smooth. Set $u$ as in (4.12) and $\bar{\varphi} : M \to \bar{N}$ as in (4.16). Then the following are equivalent

- $M \times_u F$ is harmonic-Einstein with respect to $\alpha \in \mathbb{R} \setminus \{0\}$ and $\bar{\varphi}$ and with $\bar{\varphi}$-scalar curvature $\lambda \in \mathbb{R}$.
- $(M, g)$ satisfies, for some $\alpha \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$,

$$\begin{cases} \text{Ric}^\varphi + \text{Hess}(f) - \frac{1}{d} df \otimes df = -\frac{\lambda}{m+d} g \\ \tau(\varphi) = df(\nabla f), \end{cases}$$

(4.28)

$(F, g_F)$ is Einstein with scalar curvature $\Lambda \in \mathbb{R}$ and the following equation holds

$$\Delta f - \frac{d}{m+d} \lambda + \Lambda e^{2\varphi} = 0,$$

(4.29)

**Proof.** The warped product $M \times_u F$ is harmonic-Einstein with $\bar{\varphi}$-scalar curvature $\lambda$ if and only if

$$\begin{cases} \text{Ric}^\varphi = \frac{\lambda}{m+d} g \\ \tau(\bar{\varphi}) = 0. \end{cases}$$

(4.30)

The first equation above gives, via (4.26),

$$R^\varphi_{ij} + f_{ij} - \frac{1}{d} f_i f_j = \frac{\lambda}{m+d} \delta_{ij},$$

(4.31)

and

$$F R_{\alpha\beta} = \left(\frac{\lambda}{m+d} - \frac{\Delta f}{d}\right) e^{2\varphi} F \eta_{\alpha\beta},$$

(4.32)
while the second, using (1.21), implies \( \tau(\varphi) = d\varphi(\nabla f) \). Hence (1.28) holds. It is possible to prove that, since \( M \) is connected and even thought \( g \) is a pseudo-Riemannian metric, the Hamilton-type identity holds for the system (1.28) holds, that is, (1.29) holds. From (1.29) we infer

\[
FS = \left( \frac{d}{m+d} \lambda - \Delta f \right) e^{-\frac{2f}{m+d}} = \Lambda.
\]

The converse implication is trivial.

The following is immediate from the above.

**Corollary 4.33.** Let \((M, g)\) be a semi-Riemannian manifold of dimension \( m \geq 2 \) and \((N, \eta)\) a Riemannian manifold. Let \( f \in C^\infty(M) \) and \( \varphi : M \to N \) smooth. Set \( u \) as

\[
u = e^{-f}
\]

and \( \bar{\varphi} := \varphi \circ \pi_M : \bar{M} \to N \). Let \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( \lambda \in \mathbb{R} \). Then the following are equivalent

- \( \bar{M} := M \times \mathbb{R} \) is harmonic-Einstein with respect to \( \alpha \) and \( \bar{\varphi} \), with \( \bar{\varphi} \)-scalar curvature \( \lambda \) and with respect to the metric

\[
\bar{g} = g \pm u^2 dt \otimes dt,
\]

where \( t \) is the coordinate of \( \mathbb{R} \).

- \((M, g)\) satisfies

\[
\begin{cases}
\text{Ric}^\varphi + \text{Hess}(f) - df \otimes df = \frac{\lambda}{m+1} g \\
\tau(\varphi) = d\varphi(\nabla f)
\end{cases}
\]

and the following equation holds

\[
\Delta f = \frac{\lambda}{m+1},
\]

or equivalently, in terms of \( u \),

\[
-\Delta u = \frac{\lambda}{m+1} u,
\]

that is, \( u \) is an eigenfunction of \(-\Delta\) for the eigenvalue \( \frac{\lambda}{m+1} \).

**Definition 4.36.** We say that a semi-Riemannian manifold \((M, g)\) of dimension \( m \geq 2 \) is \( \varphi \)-static harmonic-Einstein with respect to a smooth map \( \varphi : M \to N \), where \((N, \eta)\) is a target Riemannian manifold, and \( \alpha \in \mathbb{R} \setminus \{0\} \) if it satisfies one the equivalent conditions of the Proposition above for some \( f \in C^\infty(M) \) and \( \lambda \in \mathbb{R} \).

**Remark 4.37.** If \((M, g)\) is \( \varphi \)-static harmonic-Einstein then it has constant \( \varphi \)-scalar curvature given by

\[
S^\varphi = \frac{m-1}{m+1} \lambda.
\]

The validity of the above follows easily taking the trace of the first equation in (4.34) and plugging by (4.35) into it.

Recall that a four dimensional Lorentzian manifold \((\bar{M}, \bar{g})\) of dimension is called static spacetime if it admits a timelike and irrotational Killing vector field \( K \). Locally any static spacetime is isometric to \( M \times \mathbb{R} \) endowed with the metric \( g - e^{-2f} dt \otimes dt \), where \((M, g)\) is a three-dimensional Riemannian manifold, \( f \in C^\infty(M) \) and \( t \) is the coordinate in \( \mathbb{R} \) where \( K \) is given by \( \partial/\partial t \) (see [W] for details).

Furthermore, let \((N, \eta)\) be a Riemannian manifold and \( \bar{\varphi} : (M, \bar{g}) \to (N, \eta) \) a smooth map.

**Definition 4.38.** We say that \((\bar{M}, \bar{g})\) is a \( \bar{\varphi} \)-static spacetime if it admits a timelike and irrotational Killing vector field \( K \) such that \( d\bar{\varphi}(K) = 0 \).

**Remark 4.39.** It is clear that in a (connected) neighborhood of each point any static spacetime is isometric to \( M \times \mathbb{R} \) endowed with the metric \( g - e^{-2f} dt \otimes dt \), where \((M, g)\) is a three-dimensional Riemannian manifold, \( f \in C^\infty(M) \) and \( t \) is the coordinate in \( \mathbb{R} \), but it can also be proved that in that neighborhood \( \bar{\varphi} \) is given by the lifting of a time independent map \( \varphi : M \to N \), i.e., \( \bar{\varphi} = \varphi \circ \pi_M \) where \( \pi_M : M \times \mathbb{R} \to M \) is the canonical projection.
Remark 4.40. The name $\varphi$-static harmonic-Einstein in [Definition 4.36] is justified by the fact that, if $(M, g)$ is a three dimensional $\varphi$-static Riemannian manifold with respect to $\alpha$ given by (2.56), then the four dimensional manifold $\tilde{M} := M \times \mathbb{R}$ endowed with the Lorentzian metric $\tilde{g} := g - e^{-2f} dt \otimes dt$, where $t \in \mathbb{R}$ represent the time, is a $\varphi$-static spacetime, where $\varphi = \varphi \circ \pi_M$ and $\pi_M : \tilde{M} \to M$ is the canonical projection, that solves the Einstein field equations with source the wave map $\varphi$. This fact follows easily from [Remark 2.55].

5 Variational characterizations

Let $M$ be a closed, i.e. a compact smooth manifold without boundary, and orientable manifold of dimension $m \geq 2$. We denote by $\mathcal{M}$ the set of all the Riemannian metrics of $M$.

Remark 5.1. All the results of this Section can be extended to non-compact manifolds by considering compactly supported variations, but for simplicity we deal with closed and orientable manifolds in this work.

Let $g \in \mathcal{M}$, the volume form associated to $g$ is denoted by $\mu_g$ and is locally given by

$$\mu_g = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^m,$$

where $(x^1, \ldots, x^m)$ are local coordinates on an open subset $U$ of $M$ and $g$ is, in turn, locally given by

$$g = g_{ij} dx^i \otimes dx^j.$$

We set

$$\text{vol}_g(M) := \int_M \mu_g.$$

Recall that for every $g \in \mathcal{M}$ the tangent space $T_g \mathcal{M}$ of $\mathcal{M}$ at $g$ can be identified with $S^2(M)$, the set of two-times covariant tensor fields on $M$. The identification is the following: for $g \in \mathcal{M}$ and $h \in S^2(M)$ we define

$$g_t := g + th \quad \text{for} \quad t \in (-\varepsilon, \varepsilon),$$

(5.2)

where $\varepsilon > 0$ is sufficient small so that $g_t$ is positive definite for every $t \in (-\varepsilon, \varepsilon)$. Then $g_0 = g$ and

$$\frac{d}{dt} \bigg|_{t=0} g_t = h,$$

In a local chart the components of $g_t$ are given by

$$g_{ij}(t) = g_{ij} + th_{ij}, \quad g_t = g_{ij}(t) dx^i \otimes dx^j,$$

where

$$h = h_{ij} dx^i \otimes dx^j.$$

Clearly

$$\dot{g} = \frac{d}{dt} \bigg|_{t=0} g_t = h,$$

(5.3)

locally,

$$\dot{g} = \dot{g}_{ij} dx^i \otimes dx^j, \quad \dot{g}_{ij} = h_{ij}.$$

We denote by $\mathcal{F}$ the set of all the smooth maps $\varphi : M \to (N, \eta)$, where the target $(N, \eta)$ is a fixed Riemannian manifold and we fix $\alpha \in \mathbb{R} \setminus \{0\}$. The results on the following Proposition are well known or follows easily from well known results. We sketch their proof for the sake of completeness.

Proposition 5.4. Let $g \in \mathcal{M}$, $\varphi \in \mathcal{F}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Let $h \in S^2(M)$ and $g_t$ as in (5.2).

• Let $(g^{ij}(t))_{ij}$ be the inverse matrix of $(g_{ij}(t))_{ij}$. Then

$$\dot{g}^{ij} = \frac{d}{dt} \bigg|_{t=0} g^{ij}(t) = -h^{ij},$$

(5.5)

where the indexes of $h$ are raised with the aid of the metric $g$. 

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• The variation of the Riemannian volume element in the direction $h$ is given by

$$\dot{\mu}_g := \left. \frac{d}{dt} \right|_{t=0} \mu_{g_t} = \frac{1}{2} tr_g(h) \mu_g,$$

(5.6)

• The variation of the volume in the direction $h$ is given by

$$vol_g(M) := \left. \frac{d}{dt} \right|_{t=0} vol_{g_t}(M) = \frac{1}{2} \int_M tr_g(h) \mu_g.$$

(5.7)

• The variation of the $\varphi$-Ricci tensor is given by

$$\dot{\mathcal{R}}^\varphi_{ij} := \left. \frac{d}{dt} \right|_{t=0} R^\varphi_{ij}(t) = \frac{1}{2} g^{pq}(\nabla_q \nabla_j h_{ip} - \nabla_i \nabla_j h_{pq} + \nabla_q \nabla_j h_{jp} - \nabla_q \nabla_p h_{ij}).$$

(5.8)

where

$$\text{Ric}^\varphi_{ij} = R^\varphi_{ij}(t) dx^i \otimes dx^j, \quad \nabla^g(\nabla^g h) = \nabla_k h_{ij} dx^k \otimes dx^i \otimes dx^j.$$

• The variation of the density of energy of $\varphi$ in the direction $h$ is given by

$$e^g(\varphi) := \left. \frac{d}{dt} \right|_{t=0} e^g(\varphi) = -\frac{1}{2} \left(h, \varphi^* \eta \right)_g.$$

(5.9)

• The variation of the $\varphi$-scalar curvature in the direction $h$ is given by

$$\dot{S}^\varphi_g := \left. \frac{d}{dt} \right|_{t=0} S^\varphi_{g_t} = -\Delta_g(tr_g(h)) + \text{div}_g(\text{div}_g(h)) - \left(h, \text{Ric}^\varphi_g \right)_g.$$

(5.10)

**Proof.** The validity of (5.5) is trivial. For the proof of (5.6) and (5.7) see Proposition 1.186 of [B]. Recall 1.1774 (d) of [B]:

$$\dot{R}_{ij} = \left. \frac{d}{dt} \right|_{t=0} R_{ij}(t) = \frac{1}{2} g^{pq}(\nabla_q \nabla_j h_{ip} - \nabla_i \nabla_j h_{pq} + \nabla_q \nabla_j h_{jp} - \nabla_q \nabla_p h_{ij}),$$

where locally

$$\text{Ric}_{g_t} = R_{ij}(t) dx^i \otimes dx^j.$$

Since $\alpha \varphi^* \eta$ does not depend on the metric $g$ on $M$, the above gives the validity of (5.8).

The above equation gives

$$\dot{S}_g := \left. \frac{d}{dt} \right|_{t=0} S_{g_t} = -\Delta_g(tr_g(h)) + \text{div}_g(\text{div}_g(h)) - \left(h, \text{Ric}_g \right)_g,$$

(5.11)

where we denoted by $\left< \cdot, \cdot \right>_g$ the extension of $g$ to the bundle of the two-times covariant symmetric tensors on $M$, that is locally given by

$$\left< h, \text{Ric}_g \right>_g = g^{ij} h_{ij}.$$

Notice that, if we choose local coordinates $y^1, \ldots, y^n$ on a open subset $\mathcal{V}$ of $N$ such that $\varphi(\mathcal{U}) \subset \mathcal{V}$, then $|d\varphi|_{g_t}^2$ is locally given by

$$|d\varphi|^2_{g_t} = g^{ij}(t) \eta_{ab} \varphi^a_t \varphi^b_j,$$

(5.12)

where $\eta$ is locally given by

$$\eta_{ab} dy^a \otimes dy^b.$$

Using (5.12) and (5.5) we easily get (5.9). Combining (5.11) and (5.9) and recalling the definition of the $\varphi$-Ricci tensor we obtain the variation formula (5.10) for the $\varphi$-scalar curvature. □
On the other hand we may vary the map $\varphi \in \mathcal{F}$. The tangent space $T_{\varphi} \mathcal{F}$ of $\mathcal{F}$ at $\varphi$ can be identified with $\Gamma(\varphi^{-1}TN)$, the set of smooth sections of $\varphi^{-1}TN$. The identification is the following: let $v$ be a smooth section of $\varphi^{-1}TN$. We define

$$\Phi : M \times (-\varepsilon, \varepsilon) \to N, \quad \Phi(x, t) = \exp_{\varphi(x)}^N(tv_x),$$

where $\exp^N_y : T_y N \to N$ denotes the exponential map of $(N, \eta)$ at $y \in N$ and $\varepsilon > 0$ is sufficiently small. Then, by setting

$$\varphi_t := \Phi(\cdot, t)$$

for every $t \in (-\varepsilon, \varepsilon)$, we have $\varphi_0 = \varphi$ and

$$\frac{d}{dt} \bigg|_{t=0} \varphi_t = v.$$

Recall that the total energy of $\varphi$ is given by

$$E^g(\varphi) := \int_M e^g(\varphi) \mu_g = \frac{1}{2} \int_M |d\varphi|^2 g_{ij} \mu_g,$$

where $e^g(\varphi)$ is the density of energy of $\varphi$, defined in (2.9) while the total bi-energy of $\varphi$ is given by

$$E^2_\varphi(\varphi) := \int_M e^2_\varphi(\varphi) \mu_g = \frac{1}{2} \int_M |\tau^g(\varphi)|^2 \mu_g,$$

where $e^2_\varphi(\varphi)$ is the density of bi-energy of $\varphi$, defined in (2.11).

The results on the following Proposition are well known. We sketch their proof for completeness and to show how the method of the moving frame makes computation easier.

**Proposition 5.16.** Let $g \in \mathcal{M}$, $\varphi \in \mathcal{F}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Let $h \in S^2(M)$ and $g_t$ as in (5.2).

- The variation of the energy of $\varphi$ in the direction $h$ is given by
  $$E^g(\varphi) = \frac{d}{dt} \bigg|_{t=0} E^g(\varphi) = -\frac{1}{2} \int_M \langle T^g, h \rangle_g \mu_g,$$
  where $T^g$ is the energy stress tensor (2.21) of the map $\varphi : (M, g) \to (N, \eta)$.

- The variation of the bi-energy of $\varphi$ in the direction $h$ is given by
  $$E^2(\varphi) = \frac{d}{dt} \bigg|_{t=0} E^2_\varphi(\varphi) = \frac{1}{2} \int_M \langle T^2_\varphi, h \rangle_g \mu_g,$$
  where, in a local orthonormal coframe for $g$,
  $$(T^2_\varphi)_{ij} = \tau(\varphi)^a_i \varphi^a_j + \tau(\varphi)^a_j \varphi^a_i - (e^2_\varphi(\varphi) + \tau(\varphi)^a_i \varphi^a_j) \delta_{ij}.$$

Let $v \in \Gamma(\varphi^{-1}TN)$ and $\varphi_t$ such that (5.13) holds.

- The variation of total energy of $\varphi$ in the direction $v$ is given by
  $$\frac{d}{dt} \bigg|_{t=0} E^g(\varphi_t) = -\int_M \langle \tau^g(\varphi), v \rangle_g \mu_g,$$
  where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\varphi^{-1}TN$ and $\tau^g(\varphi)$ is the tension field (2.10) of the map $\varphi : (M, g) \to (N, \eta)$.

- The variation of total bi-energy of $\varphi$ in the direction $v$ is given by
  $$\frac{d}{dt} \bigg|_{t=0} E^2_\varphi(\varphi_t) = \int_M \langle \tau^2_\varphi(\varphi), v \rangle_g \mu_g,$$
  where $\tau^2_\varphi(\varphi)$ is the bi-tension field of the map $\varphi : (M, g) \to (N, \eta)$, given by (2.12).
Proof. The validity of (5.17) follows from the definition of energy of \( \varphi \) and the formulas (5.20) and (5.6), since
\[
\frac{d}{dt} \bigg|_{t=0} \int_M e^{\varphi} \mu_g = \int \left( \frac{d}{dt} \right)_{t=0} e^{\varphi} \mu_g + \int_M e^{\varphi} \frac{d}{dt} \bigg|_{t=0} \mu_g.
\]
Recall that
\[
\nabla \frac{\partial \varphi}{\partial x^j} = \Gamma^k_{ij} \frac{\partial \varphi}{\partial x^k}
\]
and
\[
\Gamma^k_{ij} = \frac{1}{2} g^{lk} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right),
\]
Notice that \( \Gamma^k_{ij} \) does not define a tensor while \( \hat{\Gamma}^k_{ij} \) does, where
\[
\hat{\Gamma}^k_{ij} = \frac{1}{2} g^{lk} (\nabla_i h_{lj} + \nabla_j h_{li} - \nabla_l h_{ij}).
\] (5.21)
Recall that the components of the generalized second fundamental form of \( \varphi \) are given by
\[
\nabla d\varphi^a_{ij} = \frac{\partial^2 \varphi^a}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \varphi^a}{\partial x^k} + \frac{N}{2} \nabla a \frac{\partial \varphi^b}{\partial x^i} \frac{\partial \varphi^c}{\partial x^j}.
\] (5.22)
Using (5.21) we get
\[
\nabla d\varphi^a_{ij} = -\hat{\Gamma}^k_{ij} \frac{\partial \varphi^a}{\partial x^k} = -\frac{1}{2} g^{lk} (\nabla_i h_{lj} + \nabla_j h_{li} - \nabla_l h_{ij}) \frac{\partial \varphi^a}{\partial x^k}.
\] (5.23)
The components of the tension field of \( \varphi \) are given by
\[
\tau^g(\varphi)^a = g^{ij} \nabla d\varphi^a_{ij}.
\]
Using (5.5) and (5.23)
\[
\tau^g(\varphi)^a = g^{ij} \nabla d\varphi^a_{ij} = -h^{ij} \nabla d\varphi^a_{ij} - \frac{1}{2} g^{lk} g^{ij} (\nabla_i h_{lj} + \nabla_j h_{li} - \nabla_l h_{ij}) \frac{\partial \varphi^a}{\partial x^k}.
\]
In a local orthonormal coframe for \( g \) the above equation reads
\[
\tau^g(\varphi)^a = -h_{ij} \varphi^a_{ij} - \frac{1}{2} (2 h_{ki,i} - h_{ii,k}) \varphi^a_k
\]
Hence
\[
e^2_{ij}(\tau) = \tau^g(\varphi)^a \tau^g(\varphi)^a = -h_{ij} \varphi^a_{ij} \tau(\varphi)^a - h_{ki,i} \tau(\varphi)^a \varphi^a_k + \frac{1}{2} h_{ii,k} \tau(\varphi)^a \varphi^a_k.
\]
Then, using (5.6),
\[
E^2_{ij}(\varphi) = \int_M e^2_{ij}(\tau) \mu_g + \frac{1}{2} \int_M e^2_{ij}(\tau) \mu_g = \int_M \left( e^2_{ij}(\tau) + \frac{1}{2} e^2_{ij}(\tau) \mu_g \right) \mu_g.
\] (5.24)
Notice that
\[
e^2_{ij}(\tau) + \frac{1}{2} e^2_{ij}(\tau) (h, g)_g = -h_{ij} \varphi^a_{ij} \tau^g(\varphi)^a - h_{ki,i} \tau^g(\varphi)^a \varphi^a_k + \frac{1}{2} h_{ii,k} \tau^g(\varphi)^a \varphi^a_k + \frac{1}{2} e^2_{ij}(\varphi) h_{ij} \delta_{ij}
\]
\[
= -h_{ij} \varphi^a_{ij} \tau^g(\varphi)^a + h_{ki,i} \tau^g(\varphi)^a \varphi^a_k - \frac{1}{2} h_{ii} (\tau^g(\varphi)^a \varphi^a_k) + \frac{1}{2} \varphi^a_{ij} + \ldots
\]
\[
= h_{ij} \left( \tau^g(\varphi)^a \varphi^a_{ij} + \frac{1}{2} e^2_{ij}(\varphi) - \tau^g(\varphi)^a \varphi^a_k - |\tau^g(\varphi)|^2 \delta_{ij} \right) + \ldots
\]
where with the lower dots we denote divergences terms. Then, integrating by parts we obtain (5.18).
Now we deal with variations of $\varphi$. Clearly
\begin{equation}
\frac{d}{dt}
\bigg|_{t=0}
\varphi_1^a \frac{d}{dt}
\bigg|_{t=0}
(\varphi_1)^a
= e^g(\varphi_t)
= \varphi_1^a \frac{d}{dt}
(\varphi_1)^a
\tag{5.25}
\end{equation}
hence exchanging the covariant derivatives of $\Phi : (-\varepsilon, \varepsilon) \times M \to N$, where $\Phi(t, x) = \varphi_t(x)$, we have
\begin{equation}
\frac{d}{dt}
\bigg|_{t=0}
(\varphi_1)^a
= e^g(\varphi_t)
= \varphi_1^a v_1^a = (\varphi_t^a v^a)_i - \varphi_i^a v^a.
\tag{5.26}
\end{equation}
Integrating the above, using the divergence theorem, we conclude the validity of (5.19).

Finally
\begin{equation}
\frac{d}{dt}
\bigg|_{t=0}
\varphi_1^a = \tau(\varphi_1)^a = \tau(\varphi_1)^a.
\tag{5.26}
\end{equation}
Using (2.13), for the map $\Phi$, since the components $\bar{R}_{\alpha\beta\gamma\delta}$, for $1 \leq \alpha, \beta, \ldots \leq m + 1$, of the Riemann tensor of $M = (-\varepsilon, \varepsilon) \times M$ satisfies (it can be seen using $\bar{A}$ with $u \equiv 1$)
\begin{align*}
\bar{R}_{k_j m+1 \tau} &= 0, \\
\bar{R}_{m+1 j \tau} &= 0, \\
\bar{R}_{kjst} &= R_{kjst},
\end{align*}
we obtain
\begin{equation}
\frac{d}{dt}
(\varphi_1)^a = \Phi_{\alpha}^a \Phi_{\alpha}^a - \bar{R}_{\alpha \beta \gamma \delta} \Phi_{\alpha}^a \Phi_{\alpha}^a + \bar{N} \bar{R}_{\alpha \beta \gamma \delta} \Phi_{\alpha}^a \Phi_{\alpha}^a = 0,
\tag{5.27}
\end{equation}
hence, evaluating at $t = 0$,
\begin{equation}
\frac{d}{dt}
\bigg|_{t=0}
(\varphi_1)^a = v_1^a.
\tag{5.28}
\end{equation}

Then we infer
\begin{equation}
\frac{d}{dt}
E^g_2(\varphi_1) = \int_M (\varphi_1)^a \tau^a(\varphi_1) \mu_g,
\tag{5.29}
\end{equation}
that integrating by parts twice gives (5.26).

\textbf{Remark 5.27.} It is well known, see for instance Proposition 1.1.17 of [A] for a proof, that
\begin{equation}
\text{div}(T_2^a) = \tau^a(\varphi_1)^a \varphi_1^a.
\tag{5.30}
\end{equation}
In particular harmonic maps are conservative. The analogous happens also for the bi-energy, that is,
\begin{equation}
\text{div}(T_2^a) = \tau^a(\varphi_1)^a \varphi_1^a.
\tag{5.31}
\end{equation}

To prove (5.30) notice that
\begin{equation}
(T_2^a)_{ij, j} = \tau^a(\varphi_1)^a \varphi_1^a + \tau^a(\varphi_1)^a \tau^a(\varphi_1)^a + \tau^a(\varphi_1)^a \tau^a(\varphi_1)^a + \tau^a(\varphi_1)^a \varphi_1^a - (\varphi_1^a)^2 + \tau^a(\varphi_1)^a \varphi_1^a,
\tag{5.32}
\end{equation}
that is,
\begin{equation}
(T_2^a)_{ij, j} = (\tau^a(\varphi_1)^a - \tau^a(\varphi_1)^a) \varphi_1^a + \tau^a(\varphi_1)^a \varphi_1^a,
\tag{5.33}
\end{equation}
and notice that (2.13) implies
\begin{equation}
\tau^a(\varphi_1)^a = \tau^a(\varphi_1)^a + 2R_{\bar{N} \beta \gamma \delta} \varphi_1^a - \bar{N} \bar{R}_{\beta \gamma \delta} \varphi_1^a \varphi_1^a,
\tag{5.34}
\end{equation}
that is, using the symmetries of Riem and of $\nabla d\varphi$,
\begin{equation}
\tau^a(\varphi_1)^a - \tau^a(\varphi_1)^a = -\bar{N} \bar{R}_{\beta \gamma \delta} \varphi_1^a \varphi_1^a.
\tag{5.35}
\end{equation}
Plugging the above into (5.30) we get
\[
\text{div}(T^g_2)_{i} = (\tau^g(\varphi)^a_{ij} - N R^a_{\beta\alpha\gamma\delta} \varphi^\beta_j \varphi^\gamma_i \varphi^\delta_i) \varphi^a_i,
\]
that is (5.29), recalling (2.12).

For a detailed study of the stress-energy tensor $T^g_2$ for biharmonic maps we refer [LMO]. Notice that the stress-energy tensor $T^g$ for harmonic maps play a special role for bidimensional manifolds: indeed it is well known that $T^g = 0$ if and only if $\varphi : (M,g) \to (N,\eta)$ is weakly conformal and $m = 2$, where $m$ is the dimension of $M$. We expect that the the stress-energy tensor for biharmonic maps could play a special role for four dimensional manifolds and we will investigate it in future works.

### 5.1 The linearization of the $\varphi$-scalar curvature map

We can consider the $\varphi$-scalar curvature as a map
\[
\mathcal{S} : \mathcal{M} \times \mathcal{F} \to C^\infty(M), \quad (g,\varphi) \mapsto \mathcal{S}(g,\varphi) \equiv \mathcal{S}^g(\varphi) \equiv \mathcal{S}^g(\varphi) = S^g.
\]
Then, for every $(g,\varphi) \in \mathcal{M} \times \mathcal{F}$, the linearization of the $\varphi$-scalar curvature map $\mathcal{S}$ at $(g,\varphi)$
\[
d_{(g,\varphi)}\mathcal{S} : T_{(g,\varphi)}(\mathcal{M} \times \mathcal{F}) \to C^\infty(M)
\]
is given by, using the identification $T_{g,\varphi}(\mathcal{M} \times \mathcal{F}) \equiv S^2(M) \oplus \Gamma(\varphi^{-1}TN)$,
\[
(d_{(g,\varphi)}\mathcal{S})(h,v) = (d_g\mathcal{S}^g)(h) + (d_\varphi\mathcal{S}^g)(v) \quad \text{for every } (h,v) \in S^2(M) \oplus \Gamma(\varphi^{-1}TN).
\]
Using (5.10) and (5.26) (that will be proved below),
\[
(d_{(g,\varphi)}\mathcal{S})(h,v) = -\Delta_g(\text{tr}_g(h)) + \text{div}_g(\text{div}_g(h)) - \langle h,\text{Ric}_g^\varphi \rangle_g - 2\alpha \varphi^a_i v^a_i. \tag{5.31}
\]
It is easy to see that the adjoint $(d_{(g,\varphi)}\mathcal{S})^* : C^\infty(M) \to S^2(M) \times \Gamma(\varphi^{-1}TN)$ of $d_{(g,\varphi)}\mathcal{S}$ is given by
\[
(d_{(g,\varphi)}\mathcal{S})^*(u) = [(d_{(g,\varphi)}\mathcal{S})^*(u)_1, (d_{(g,\varphi)}\mathcal{S})^*(u)_2], \tag{5.32}
\]
where
\[
\begin{cases}
(d_{(g,\varphi)}\mathcal{S})^*(u)_1 := \text{Hess}_g(u) - u\text{Ric}_g^\varphi - \Delta_g u g 
\in S^2(M) \\
(d_{(g,\varphi)}\mathcal{S})^*(u)_2 := 2\alpha[u\tau^g(\varphi) + d_\varphi(\nabla_g u)] \in \Gamma(\varphi^{-1}TN).
\end{cases}
\]
To prove (5.32) it is sufficient to prove that for every $u \in C^\infty(M)$
\[
\int_M u \left(-\Delta_g(\text{tr}_g(h)) + \text{div}_g(\text{div}_g(h)) - \langle h,\text{Ric}_g^\varphi \rangle_g - 2\alpha \varphi^a_i v^a_i \right) \mu_g
= \int_M \left(\langle h,\text{Hess}_g(u) - u\text{Ric}_g^\varphi - \Delta_g u g \rangle_g + 2\alpha(u, u\tau^g(\varphi) + d_\varphi(\nabla_g u)) \right) \mu_g,
\]
that follows easily from the divergence theorem.

In the following Proposition we show that $d_{(g,\varphi)}\mathcal{S}$ is surjective (or equivalently, $(d_{(g,\varphi)}\mathcal{S})^*$ is injective), unless some particular conditions on $(g,\varphi) \in \mathcal{M} \times \mathcal{F}$ are satisfied.

**Proposition 5.33.** Let $M$ be a compact manifold and let $(g,\varphi) \in \mathcal{M} \times \mathcal{F}$. If $(d_{(g,\varphi)}\mathcal{S})^*$ is not injective, then one of the following hold.

- $(M,g)$ is $\varphi$-Ricci flat with respect to $\alpha$ and
  \[\ker(d_{(g,\varphi)}\mathcal{S})^* = \mathbb{R}.\] \tag{5.34}

- There exists a non-constant function $u \in C^\infty(M)$ such that $\Sigma := u^{-1}\{0\}$ is a total umbilical hypersurface of $(M,g)$ and if $\mathcal{U}$ is a connected component of $M \setminus \Sigma$, then $\mathcal{U} := \mathcal{U} \times \mathbb{R}$ endowed with the metric $\hat{g} = g \pm u^2 dt \otimes dt$, where $t$ is the coordinate on $\mathbb{R}$, is harmonic-Einstein with respect to $\alpha$ and $\hat{\varphi} := \varphi \circ \pi_M$, where $\pi_M : M \to M$ is the canonical projection.

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Proof. From (5.32), \( u \in \ker (d_{(g, \varphi)} S)^* \) if and only if
\[
\begin{align*}
\text{Hess}_g(u) &= u \text{Ric}_g^\varphi - \Delta_g u g = 0 \\
\tau^g(\varphi) + d \varphi(\nabla_g u) &= 0.
\end{align*}
\] (5.35)

Notice that a non-zero constant \( u \) belongs to \( \ker (d_{(g, \varphi)} S)^* \) if and only if \((M, g)\) is \( \varphi \)-Ricci flat and, if this is the case, then (5.34) holds. The equivalence follows immediately from (5.35) and if \((M, g)\) is \( \varphi \)-Ricci flat, (5.35) reduces to
\[
\begin{align*}
\text{Hess}_g(u) &= \Delta_g u g \\
d \varphi(\nabla_g u) &= 0.
\end{align*}
\] (5.36)

Tracing the first equation above we conclude that \( u \) is harmonic and, since \( M \) is compact, is constant.

Assume that \( u \) is non-constant. Taking the trace of the first equation of (5.35) we get
\[
-\Delta u = \frac{\lambda}{m + 1} u,
\] (5.37)
where
\[
\lambda := \frac{m + 1}{m - 1} S^\varphi.
\]

Then \( u \) satisfies a unique continuation property and thus, since it is not identically zero, it cannot vanish on an open subset of \( M \).

Now we show that \( S^\varphi \) is constant. Taking the divergence of the first equation of (5.36) we get
\[
\eta^{ik} u_{ijk} - u^i R^\varphi_{ij} - u \eta^{ik} R^\varphi_{ij,k} - (\Delta u)_j = 0.
\] (5.38)

Using (2.22) and commutating the indexes we obtain
\[
R_{ij} u^t - u^i R^\varphi_{ij} - u^j \frac{1}{2} S^\varphi_j + \alpha u \tau(\varphi) u^a \varphi^a_j = 0.
\] (5.39)

Using the second equation of (5.35) and the definition of \( \varphi \)-Ricci we conclude
\[
udS^\varphi = 0.
\]

Then \( S^\varphi \) is constant on \( M \).

It can be easily proved that \( \Sigma := u^{-1}(\{0\}) \), if it is not empty, is a total umbilical hypersurface of \((M, g)\) (it follows from the fact that \( du \neq 0 \) on \( \Sigma \), see the proof of Proposition 2.3 of \cite{C}). Notice that, since \( u \) is non-constant and \( M \) is compact, from (5.37) we deduce that \( S^\varphi \) must be a positive constant and \( u \) must change sign, hence \( \Sigma \neq \emptyset \).

By setting, on a fixed connected component \( U \) of \( M \setminus \Sigma \),
\[
u := \pm e^{-f},
\]
according to the sign of \( u \) on \( U \), then the validity of (5.33) gives, on \( U \),
\[
\begin{align*}
\text{Ric}_g^\varphi + \text{Hess}(f) - df \otimes df &= (\Delta_g f - |\nabla f|_g^2) g \\
\tau^g(\varphi) &= d \varphi(\nabla_g f).
\end{align*}
\]

Notice that, taking the trace of the first equation of the above,
\[
S^\varphi = (m - 1) \Delta f f,
\]

hence the above can be rewritten as
\[
\begin{align*}
\text{Ric}_g^\varphi + \text{Hess}(f) - df \otimes df &= \frac{\lambda}{m + 1} g \\
\tau^g(\varphi) &= d \varphi(\nabla_g f).
\end{align*}
\]

Then, using Corollary 4.33 we have that \( \bar{U} := U \times \mathbb{R} \) endowed with the metric \( \bar{g} = g \pm u^2 dt \otimes dt \), where \( t \) is the coordinate on \( \mathbb{R} \), is harmonic-Einstein with respect to \( \alpha \) and \( \bar{\varphi} := \varphi \circ \pi_M \), where \( \pi_M : \bar{M} \to M \) is the canonical projection. 

\[\square\]
The following Corollary follows automatically from the above Proposition

**Corollary 5.40.** Let \((M, g)\) be a compact Riemannian manifold, \(\alpha \in \mathbb{R} \setminus \{0\}\) and \(u \in C^\infty(M)\) a non-constant function. Then \(u \in \text{Ker}(d_{\varphi, \varphi}S)^*\) if and only if \(\Sigma := u^{-1}(\{0\})\) is a total umbilical hypersurface of \((M, g)\) and the (possibly disconnected) Riemannian manifold \(M \setminus \Sigma\) is \(\varphi\)-static harmonic-Einstein, in the sense of Definition 4.39 with respect to \(\alpha\) and \(f\), where \(f = -\log|u|\) on \(M \setminus \Sigma\). In other words, \(\tilde{M} := M \times \mathbb{R}\) endowed with the metric \(\tilde{g} = g - u^2dt \otimes dt\), where \(t\) is the coordinate on \(\mathbb{R}\), is harmonic-Einstein with respect to \(\alpha\) and \(\tilde{\varphi} := \varphi \circ \pi_M\), where \(\pi_M: \tilde{M} \to M\) is the canonical projection, outside of \(\Sigma\) (where \(\tilde{g}\) degenerates).

**Remark 5.41.** The Corollary above shows that compact Riemannian manifolds admitting a non-constant smooth function in \(u \in \text{Ker}(d_{\varphi, \varphi}S)^*\) are (possibly disconnected) \(\varphi\)-static harmonic-Einstein manifolds endowed with a “horizon” given by the zero-locus of \(u\). Notice that Corollary 5.40 is an extension, in the compact case, of Proposition 2.7 of [C]. It is possible also to deal with the non-compact case and to study in more detail the image of the \(\varphi\)-scalar curvature in the compact case, as done in [FM]. Those tasks will be addressed, possibly, in some future works.

### 5.2 Variational derivation of the harmonic-Einstein equations

**Definition 5.42.** The functional of total \(\varphi\)-scalar curvature, for every \((g, \varphi) \in M \times F\) and \(\alpha \in \mathbb{R} \setminus \{0\}\), is given by

\[
S(g, \varphi) \equiv S^\varphi(g) \equiv S^\varphi := \int_M S^\varphi_g. \tag{5.43}
\]

**Remark 5.44.** Denoting by \(S(g)\) the total scalar curvature of \((M, g)\), from the relation \(S^\varphi = S - \alpha|d\varphi|^2\) and of total energy of \(\varphi\) we immediately deduce

\[
S(g, \varphi) = S(g) - 2\alpha E^\varphi(\varphi). \tag{5.45}
\]

**Remark 5.46.** For \(m = 2\) the total \(\varphi\)-scalar curvature is given by

\[
S(g, \varphi) = 4\pi\chi(M) - 2\alpha E^\varphi(\varphi),
\]

where \(\chi(M)\) is the Euler characteristic of \(M\). Indeed, from Gauss-Bonnet formula,

\[
\frac{1}{2} \int_M S_g \mu_g = 2\pi\chi(M),
\]

hence the above follows easily from (5.45).

As a consequence, \((g, \varphi) \in S \times F\) is a critical point for \(S\) if and only if it is a critical point for the total energy of \(\varphi\). We have characterized critical point to the total energy of \(\varphi\) in Proposition 5.16 they satisfy

\[
\begin{align*}
\tau^\varphi(\varphi) &= 0 \\
T^\varphi &= 0.
\end{align*}
\]

As mentioned in Remark 5.27 \(T^\varphi = 0\) if and only if \(m = 2\) and \(\varphi\) is weakly conformal. Recall that \(\varphi : (M, g) \to (N, \eta)\), where \(M\) is a surface, is called branched minimal immersion (see [BW], Section 3.5) if it is weakly-conformal and harmonic. In conclusion, critical points of \(S\) for \(m = 2\) are given by \((g, \varphi) \in S \times F\) such that \(\varphi : (M, g) \to (N, \eta)\) is a branched minimal immersion.

Notice that the fact of being a branched minimal immersion does not depend only on the Riemannian metric \(g\) but depends on the conformal class \([g]\) of \(g\). Indeed, using (5.10) and (5.8), we get that

\[
E^\varphi(\varphi) = E^\varphi(\varphi).
\]

This can be seen also directly: from (3.13) we have

\[
\tau^\varphi(\varphi) = e^{2h}\tau^\varphi(\varphi) \tag{5.47}
\]

and clearly \(\varphi\) is weakly conformal with respect to \(\tilde{g}\) if and only if it is with respect to \(g\).

Finally (3.18) and (3.3) give

\[
S^\varphi_{\tilde{g}} \mu_{\tilde{g}} = S^\varphi_g \mu_g + 2\Delta_g h \mu_g, \tag{5.48}
\]

and this in another equivalent way to see that \(S^\varphi(g) = S^\varphi(\tilde{g})\).
From now on assume that \( m \geq 3 \).

**Remark 5.49.** Notice that \( S^\varphi \) is not scale invariant, it is homogeneous of degree \( \frac{m-2}{2} \), that is, for every \( \lambda > 0 \),

\[
S^\varphi(\lambda g) = \lambda^{\frac{m-2}{2}} S^\varphi(g). \tag{5.50}
\]

To prove (5.50) we set \( \tilde{g} := \lambda g \) and we use (3.18) and (3.8) with \( h \in \mathbb{R} \) such that \( \lambda = e^{-2h} \), that are \( \lambda S^\varphi_{\tilde{g}} = S^\varphi_g \) and \( \mu_{\tilde{g}} = \lambda^{\frac{m}{2}} \mu_g \) to get

\[
\int_M S^\varphi_{\tilde{g}} \mu_{\tilde{g}} = \lambda^{\frac{m}{2}-1} \int_M S^\varphi_g \mu_g.
\]

To overcome this issue we will study also another functional.

**Definition 5.51.** We set the rescaled total \( \varphi \)-scalar curvature of \((M, g)\) as

\[
\bar{S}(g, \varphi) = \bar{S}^\varphi(g) := \text{vol}_g(M)^{-\frac{m-2}{2}} S^\varphi(g). \tag{5.52}
\]

**Remark 5.53.** It is easy to see that, proceeding as in **Remark 5.49**, \( \text{vol}_\lambda g(M) = \lambda^{\frac{m}{2}} \text{vol}_g(M) \) for every \( g \in M \) and \( \lambda > 0 \). Combining it with (5.50) we immediately get, for every \( g \in M \) and \( \lambda > 0 \)

\[
\bar{S}^\varphi(\lambda g) = \bar{S}^\varphi(g),
\]

that is, \( \bar{S}^\varphi \) is scale invariant.

**Proposition 5.54.** Let \( M \) be a compact manifold of dimension \( m \geq 3 \) and let \((N, \eta)\) be a Riemannian manifold.

- The pair \((g, \varphi) \in M \times F\) is a critical point of the functional \( S \) on \( M \times F \) if and only if

\[
\begin{align*}
\{ \text{Ric}^\varphi_g &= 0 \\
\tau(\varphi) &= 0,
\end{align*}
\]

that is, if and only if \((M, g)\) is \( \varphi \)-Ricci flat with respect to \( \alpha \).

- The pair \((g, \varphi) \in M \times F\) is a critical point of the functional \( \bar{S} \) on \( M \times F \) if and only if

\[
\begin{align*}
\{ \text{Ric}^\varphi_g &= 0 \\
\tau(\varphi) &= 0,
\end{align*}
\]

that is, if and only if \((M, g)\) is harmonic-Einstein with respect to \( \varphi \) and \( \alpha \).

**Proof.** Clearly

\[
\left. \frac{d}{dt} \right|_{t=0} \text{S}^\varphi(g_t) = \int_M \left( \left. \frac{d}{dt} \right|_{t=0} \text{S}^\varphi_{g_t} \right) \mu_g + \int_M \text{S}^\varphi_{g_t} \left. \frac{d}{dt} \right|_{t=0} \mu_{g_t},
\]

so that, using (5.10) and (5.33),

\[
\left. \frac{d}{dt} \right|_{t=0} \text{S}^\varphi(g_t) = \int_M \left[ -\Delta_g(\text{tr}_g(h)) + \text{div}_g(\text{div}_g(h)) - \langle h, \text{Ric}^\varphi_g \rangle_g \mu_g + \frac{1}{2} \int_M \text{S}^\varphi_{g_t} \text{tr}_g(h) \mu_{g_t} \right].
\]

Using the divergence theorem, from the above we get

\[
\left. \frac{d}{dt} \right|_{t=0} \text{S}^\varphi(g_t) = \int_M \left( \langle h, \frac{\text{S}^\varphi_g}{2} g - \text{Ric}^\varphi_g \rangle_g \right) \mu_g. \tag{5.55}
\]

In particular, \( g \) is a critical point for \( S^\varphi \) if and only if

\[
\text{Ric}^\varphi_g = \frac{\text{S}^\varphi_g}{2} g.
\]

Since \( m \geq 3 \) the above is equivalent to

\[
\text{Ric}^\varphi = 0.
\]
Moreover, using the definition \((5.52)\),
\[
\frac{d}{dt} \bigg|_{t=0} \tilde{S}^\varphi(g_t) = -\frac{m-2}{m} \frac{\text{vol}_\varphi(M) - 2(m-1)}{m} S^\varphi(g) \frac{d}{dt} \bigg|_{t=0} \text{vol}_\varphi(M) + \frac{m-2}{m} \frac{d}{dt} \bigg|_{t=0} S^\varphi_g.
\] (5.56)

Then, by plugging \((5.7)\) and \((5.55)\) into \((5.56)\) we obtain
\[
\frac{d}{dt} \bigg|_{t=0} S^\varphi(g_t) = \int_M \left\langle \text{vol}_\varphi(M) \frac{-m-2}{m} \left( \frac{S^\varphi_g}{2} - \frac{m-2}{2m} \frac{S^\varphi(g)}{\text{vol}_\varphi(M)} \right) g - \text{Ric}_g \right\rangle_d, h \right\rangle_g \mu_g,
\] (5.57)
and thus \(g\) is critical for \(S^\varphi\) if and only if
\[
\text{Ric}_g = \left( \frac{S^\varphi_g}{2} - \frac{m-2}{2m} \frac{S^\varphi(g)}{\text{vol}_\varphi(M)} \right) g.
\]

The above gives
\[
\text{Ric}_g = 0
\]
and
\[
S^\varphi(g) = S^\varphi_g \text{vol}_\varphi(M).
\]

From \((5.43)\) it is easy to see that \(\varphi\) is a critical point of \(S^\varphi\) or \(\tilde{S}^\varphi\) if and only if it is a critical point for \(E^\varphi\), that is, if and only if \(\varphi : (M, g) \rightarrow (N, \eta)\) is harmonic. Combining with the results obtained above we conclude the proof. \(\square\)

**Remark 5.58.** We denote by \(\mathcal{M}_1\) the subset of \(\mathcal{M}\) determined by the Riemannian metrics \(g \in \mathcal{M}\) such that \(\text{vol}_\varphi(M) = 1\). We claim that \(g \in \mathcal{M}_1\) is critical for \(S^\varphi\) in \(\mathcal{M}_1\) if and only if it is critical for \(\tilde{S}^\varphi\) in \(\mathcal{M}\). Indeed, \(T_g \mathcal{M}_1\) can be identified with \(S^2_2(M, g)\), the set of traceless two times covariant tensor fields on \((M, g)\) (see [13] at page 118). Hence, proceeding as in the proof of the Proposition above we get, for \(g \in \mathcal{M}_1\) and \(h \in S^2_2(M, g)\),
\[
\frac{d}{dt} \bigg|_{t=0} S^\varphi(g + th) = \int_M \left( \frac{d}{dt} \bigg|_{t=0} S^\varphi_{g+th} \right) \mu_g = -\int_M \langle h, \text{Ric}_g \rangle_d \mu_g,
\]
where we integrated by parts and we used that
\[
\frac{d}{dt} \bigg|_{t=0} \mu_{g+th} = \frac{1}{2} \text{tr}_g(h) \mu_g = 0.
\]
Then \(g\) is critical in \(\mathcal{M}_1\) if and only if \(\text{Ric}_g = 0\), hence the claim.

### 5.3 The total \(\varphi\)-scalar curvature restricted to a conformal class of metrics

The following Proposition shows that the problem of finding a conformal metric with constant \(\varphi\)-scalar curvature on a compact Riemannian manifold admit a variational characterization.

**Proposition 5.59.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(m \geq 3\), \(\varphi : M \rightarrow N\) a smooth map, where \((N, \eta)\) is a target Riemannian manifold and \(\alpha \in \mathbb{R} \setminus \{0\}\). Then the following are equivalent:

- The \(\varphi\)-scalar curvature is constant;
- The metric \(g\) is a critical point of the rescaled total \(\varphi\)-scalar curvature \(\tilde{S}^\varphi\) restricted to conformal class \([g] \subseteq \mathcal{M}\) of \(g\).

**Proof.** We have to characterize the critical points of \(\tilde{S}^\varphi\) restricted to \([g]\). Let \(\tilde{g} \in [g]\), that is,\[
\tilde{g} = f^2 g
\]
for some positive function \(f\) on \(M\). We set \(\eta \in C^\infty(M)\) and we define, for \(t\) sufficiently small,
\[
f_t^2 := f^2 + t \eta > 0 \quad \text{on} \ M.
\]
By setting $\tilde{g}_t = f_t^2 g$

we have

$$\tilde{g}_t = \tilde{g} + th,$$

(5.60)

where $h := \eta g$. In particular $\tilde{g}_t$ is a variation of $\tilde{g}$ that lies in $[g]$ and all such variations are of this form. Then $\tilde{g}$ is critical for $\mathcal{S}^\varphi$ on $[g]$ if and only if, for every $\eta \in C^\infty(M)$,

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{S}^\varphi(\tilde{g}_t) = 0.$$

Using (5.57), since $h = \eta g$ we immediately get

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{S}^\varphi(\tilde{g}_t) = \hat{\mathcal{M}} \text{vol}(\tilde{g}_t) + m - \frac{2}{m} \eta \text{tr}_{\tilde{g}_t} \left( S^\varphi(\tilde{g}_t) \right) - \frac{1}{2} \eta \text{tr}_{\tilde{g}_t} \left( S^\varphi(\tilde{g}_t) \right) \text{vol}(\tilde{g}_t).$$

Then $\tilde{g}$ is critical for $\mathcal{S}^\varphi$ on $[g]$ if and only if

$$\mathcal{S}^\varphi(\tilde{g}) = S^\varphi \text{vol}(\tilde{g}),$$

(5.61)

that is, if and only if $S^\varphi \tilde{g}$ is constant.

Let $(M, g)$ be a compact Riemannian manifold of dimension $m \geq 3$, $\varphi : M \to N$ a smooth map, where $(N, \eta)$ is a target Riemannian manifold and $\alpha \in \mathbb{R} \setminus \{0\}$.

**Definition 5.62.** The $\varphi$-Yamabe invariant of $(M, g)$ as

$$Y^\varphi(g) := \inf_{\tilde{g} \in [g]} \mathcal{S}^\varphi.$$

(5.63)

We are going to show that the Definition above makes sense.

**Definition 5.64.** For every $u \in C^\infty(M)$, the $\varphi$-conformal Laplacian is given by

$$L^\varphi_g(u) := -\frac{4(m-1)}{m-2} \Delta_g u + S^\varphi_g u.$$

We denote by $\lambda_1(L^\varphi_g)$ the first eigenvalue of $L^\varphi_g$. By the variational characterization of $\lambda_1(L^\varphi_g)$

$$\lambda_1(L^\varphi_g) = \inf_{u \in C^\infty(M), u \neq 0} \frac{\int_M \left( \frac{4(m-1)}{m-2} |\nabla_g u|^2_g + S^\varphi_g u^2 \right) \mu_g}{\int_M u^2 \mu_g},$$

it is immediate to get

$$\lambda_1(L^\varphi_g) \geq \inf_{\tilde{g} \in [g]} \mathcal{S}^\varphi > -\infty.$$

(5.65)

**Proposition 5.66.** Let $(M, g)$ be a compact Riemannian manifold of dimension $m \geq 3$, $\varphi : M \to N$ a smooth map, where $(N, \eta)$ is a target Riemannian manifold and $\alpha \in \mathbb{R} \setminus \{0\}$. For every $\tilde{g} \in [g]$ we have

$$\mathcal{S}^\varphi(\tilde{g}) \geq \min\{0, \lambda_1(L^\varphi_g)\}.$$

(5.67)

In particular, the $\varphi$-Yamabe invariant of $(M, g)$ is well defined.

**Proof.** Recall the validity of (3.22). By setting $u = e^{-\frac{\varphi}{2}}$, so that

$$\tilde{g} := u^{-\frac{2}{m-2}} g$$
the validity of (3.22) gives
\[
\frac{4(m-1)}{m-2} \Delta u - S^g u + \bar{S}^\varphi u = 0.
\] (5.68)

We set
\[
(L^\varphi_g(u), u) := \int_M L^\varphi_g(u) u \mu = -\frac{4(m-1)}{m-2} \int_M u \Delta_g u \mu_g + \int_M \bar{S}^\varphi_g u^2 \mu_g.
\]
Notice that, since $M$ is compact,
\[
(L^\varphi_g(u), u) = \hat{M} L^\varphi_g(u) \|u\|^2_{L^2(M,g)}.
\] (5.69)

Recalling the validity of (3.8), that in terms of $\tilde{\mu} = u^{\frac{2m}{m-2}}\mu$, we immediately get
\[
\text{vol}_M^g(M) = \int_M u^{\frac{2m}{m-2}} \mu.
\]

Combining the above and (5.68) with the definition of $\bar{S}^\varphi$ we have
\[
\mathfrak{S}^\varphi(\bar{g}) = \left( \int_M u^{\frac{2m}{m-2}} \mu \right)^{-\frac{m-2}{m}} (L^\varphi u, u).
\]

Then, using (5.69), from the above we deduce
\[
\mathfrak{S}^\varphi(\bar{g}) \geq \lambda_1(L^\varphi_g) \left( \int_M u^{\frac{2m}{m-2}} \mu \right)^{-\frac{m-2}{m}} \int_M u^2 \mu.
\] (5.70)

From (5.70) we deduce the validity of (5.67). Indeed, if $\lambda_1(L^\varphi_g) \geq 0$, from (5.70) we immediately get
\[
\mathfrak{S}^\varphi(\bar{g}) \geq 0.
\]

Notice that, from Jensen’s inequality applied to the convex function $t^p$, for $p > 1$, we have
\[
\left( \int u^2 \right)^{\frac{p}{2}} \leq \int u^{2p},
\] (5.71)
that is,
\[
\left( \int u^2 \right)^{\frac{p}{2}} \geq \int u^2.
\] (5.72)

Then, if $\lambda_1(L^\varphi_g) < 0$, using (5.70) and (5.72) for
\[
p = \frac{m}{m-2} > 1,
\]
we obtain
\[
\mathfrak{S}^\varphi(\bar{g}) \geq \lambda_1(L^\varphi_g).
\]

In conclusion, (5.67) holds.

The validity of (5.67) shows that
\[
\inf_{u \in C^\infty(M), u > 0} \mathfrak{S}^\varphi(u^{\frac{1}{m-2}}g) > -\infty.
\]

To conclude notice that
\[
Y^\varphi(g) = \inf_{u \in C^\infty(M), u > 0} \mathfrak{S}^\varphi(u^{\frac{1}{m-2}}g).
\]

Remark 5.73. We have just given the definition of the $\varphi$-Yamabe invariant in the compact case. The study of his property (also in the complete non-compact case) will be the subject of some future works.
5.4 Variational characterization of four dimensional \( \varphi \)-Bach flat manifolds

Let \( M \) be a closed smooth manifold of dimension \( m \geq 3 \), \( \varphi : M \to N \) a smooth map with target a Riemannian manifold \( (N, \eta) \) and \( \alpha \in \mathbb{R} \setminus \{0\} \).

**Definition 5.74.** We define the *Bach operator* \( B : \mathcal{M} \times \mathcal{F} \to \mathbb{R} \) as

\[
B(g, \varphi) = B^\varphi(g) := \int_M (S_2(A^\varphi_g) - \alpha e_2(\varphi)) \mu_g = S_2(g, \varphi) - \alpha E^g_2(\varphi),
\]

where

\[
S_2(g, \varphi) \equiv S^\varphi_2(g) := \int_M S^\varphi_2 \mu_g
\]

and \( S_2(A^\varphi_g) \) is the second elementary symmetric polynomial in the eigenvalues of the \( \varphi \)-Schouten tensor \( A^\varphi_g \) of \( (M, g) \).

**Remark 5.75.** The Bach operator can be equivalently written as

\[
B(g, \varphi) = \int_M \left( \frac{m}{8(m-1)} (S^\varphi_g)^2 - \frac{1}{2} |\text{Ric}^\varphi_g|^2 - \frac{\alpha}{2} |\tau^\varphi_g|^2 \right) \mu_g.
\]

Indeed, recall that if \( A \) is a two times covariant symmetric tensor with eigenvalues \( \lambda_1 \leq \ldots \leq \lambda_m \), then

\[
\text{tr}(A) = \sum_i \lambda_i, \quad S_2(A) = \sum_{i<j} \lambda_i \lambda_j.
\]

Clearly

\[
\left( \sum_i \lambda_i \right)^2 = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j,
\]

hence we have the validity of

\[
\text{tr}(A)^2 = |A|^2 + 2S_2(A).
\]

Applying the above formula to the \( \varphi \)-Schouten tensor \( A^\varphi_g \) of \( (M, g) \) we get

\[
S_2(A^\varphi_g) = \frac{1}{2} [\text{tr}(A^\varphi_g)^2 - |A^\varphi_g|^2].
\]

Using the definition of the \( \varphi \)-Schouten tensor we infer

\[
\text{tr}(A^\varphi_g) = \frac{m - 2}{2(m-1)} S^\varphi_g, \quad |A^\varphi_g|^2 = |\text{Ric}^\varphi_g|^2 - \frac{3m - 4}{4(m-1)^2} (S^\varphi_g)^2,
\]

then, by plugging those relations into the above we finally conclude

\[
S_2(A^\varphi_g) = \frac{m}{8(m-1)} (S^\varphi_g)^2 - \frac{1}{2} |\text{Ric}^\varphi_g|^2,
\]

and thus (5.77) follows immediately, using also the definition (5.15) of total bi-energy of \( \varphi \).

We are ready to state our main Theorem.

**Theorem 5.78.** Let \( M \) be a closed orientable four dimensional smooth manifold. Then \( (g, \varphi) \in \mathcal{M} \times \mathcal{F} \) is a critical point for \( B \) on \( \mathcal{M} \times \mathcal{F} \) if and only if \( B^\varphi = 0 \) and \( J = 0 \). Notice that, if \( \varphi \) is a submersion a.e. then \( (g, \varphi) \) is critical if and only if \( (M, g) \) is \( \varphi \)-Bach flat.

From now on, when is clear from the context, we omit the dependence on \( g \). To prove Theorem 5.78 we need to evaluate variations of \( B \) both with respect to \( \varphi \) and to \( g \). In the next Lemma we compute the variations of the Bach functional with respect to variations of the map.
Lemma 5.79. Let \((g, \varphi) \in S \times F\). Let \(v \in \Gamma(\varphi^{-1}TN)\) and \(\varphi_t\) as in \((5.13)\). Then, in the notations above,
\[
\frac{d}{dt} \bigg|_{t=0} B_{\varphi_t}^v(g) = \alpha \int_M \left( \frac{m}{2(m-1)} S^v \tau^a \phi - \frac{m-2}{2(m-1)} S^v \phi_a^a - 2R_{ij}^v \phi_a^a - \tau_2(\phi) \right) v^a \mu_g + 2\alpha^2 \int_M \tau(\phi) \phi_i^a \phi_i^a v^a \mu_g.
\]
(5.80)

In particular, if \(m = 4\)
\[
\frac{d}{dt} \bigg|_{t=0} B_{\varphi_t}^v(g) = \alpha \int_M (J, v) \mu,
\]
(5.81)
where \(J = J_4\) is defined as in \((2.30)\).

Proof. Using the relation between \(S^\phi = S - \alpha |d\phi|^2\) we get
\[
\frac{d}{dt} \bigg|_{t=0} S^\varphi = -2\alpha \phi_i^a v_i^a,
\]
(5.82)
hence
\[
\frac{d}{dt} \bigg|_{t=0} (S^\varphi)^2 = 2S^\varphi \frac{d}{dt} \bigg|_{t=0} S^\varphi = -4\alpha S^\varphi \phi_i^a v_i^a.
\]
Then, using the divergence theorem
\[
\frac{d}{dt} \bigg|_{t=0} \int_M (S^\varphi)^2 \mu = 4\alpha \int_M (S^\varphi \phi_i^a + S^\varphi \phi_i^a) v^a.
\]
(5.83)

Using the relation between \(\text{Ric}^\phi\) and \(\text{Ric}\) we get
\[
\frac{d}{dt} \bigg|_{t=0} \text{Ric}^\varphi = -\alpha (\phi_i^a v_j^a + \phi_j^a v_i^a),
\]
hence
\[
\frac{d}{dt} \bigg|_{t=0} |\text{Ric}^\varphi|^2 = 2\text{Ric}^\varphi \frac{d}{dt} \bigg|_{t=0} \text{Ric}^\varphi = -4\alpha R_{ij}^\varphi v_i^a v_j^a.
\]
Then, using the divergence theorem
\[
\frac{d}{dt} \bigg|_{t=0} \int_M |\text{Ric}^\varphi|^2 \mu = 4\alpha \int_M (R_{ij}^\varphi \phi_i^a + R_{ij}^\varphi \phi_j^a) v^a.
\]
Using \((2.22)\) the above yields
\[
\frac{d}{dt} \bigg|_{t=0} \int_M |\text{Ric}^\varphi|^2 \mu = 2\alpha \int_M (S^\varphi \phi_i^a - 2\alpha \tau(\phi) \phi_i^a \phi_i^a + 2R_{ij}^\varphi \phi_i^a) v^a.
\]
(5.84)

Using \((5.83), (5.84)\) and the definition of bi-tension we get \((5.80)\).

Now we deal with variations of the metric.

Lemma 5.85. Let \((g, \varphi) \in S \times F\). Let \(h \in S^2(M)\) and \(g_t\) as in \((5.2)\), then
\[
\frac{d}{dt} \bigg|_{t=0} B^\varphi(g_t) = \int_M \left[ R_{ik}^\varphi R_{kj}^\varphi + \frac{1}{2} \Delta R_{ij}^\varphi - \frac{m-2}{4(m-1)} S_{ij}^\varphi - \frac{m}{4(m-1)} S^\varphi R_{ij}^\varphi \right] h_{ij} \mu
\]
\[
+ \alpha \int_M \left( \phi_{ik}^a \phi_{kj}^a - R_{ij}^\varphi \phi_i^a \phi_j^a \right) h_{ij} \mu
\]
\[
+ \int_M \left( \frac{m}{16(m-1)} (S^\varphi)^2 - \frac{1}{4(m-1)} \Delta S^\varphi - \frac{1}{4} |\text{Ric}^\varphi|^2 - \frac{\alpha}{4} |\tau(\varphi)|^2 \right) \delta_{ij} h_{ij} \mu.
\]
(5.86)

In particular, if \(m = 4\),
\[
\frac{d}{dt} \bigg|_{t=0} B^\varphi(g_t) = \int_M \langle B^\varphi, h \rangle \mu.
\]
(5.87)
Proof. By definition

$$|\text{Ric}^\varphi_{g_t}|^2_{g_t} = R^\varphi_{ij}(t)R^\varphi_{jk}(t)g^{it}(t)g^{jk}(t),$$

hence

$$\left.\frac{d}{dt}\right|_{t=0} |\text{Ric}^\varphi_{g_t}|^2_{g_t} = 2(\dot{R}^\varphi_{ij}g^{it} + R^\varphi_{ij}\dot{g}^{it})R^\varphi_{jk}g^{jk}.\tag{5.88}$$

In a local orthonormal coframe, using (5.59) and (5.53), we get

$$\left.\frac{d}{dt}\right|_{t=0} |\text{Ric}^\varphi_{g_t}|^2_{g_t} = 2h_{ik,jk}R^\varphi_{ij} - \text{tr}(h)R^\varphi_{ij} - (\Delta h)_{ij}R^\varphi_{ij} - 2(R^\varphi)^2_{ij}h_{ij}.\tag{5.89}$$

We have

$$\left.\frac{d}{dt}\right|_{t=0} \int_M |\text{Ric}^\varphi_{g_t}|^2_{g_t} \mu_{g_t} = \int_M \left( \left.\frac{d}{dt}\right|_{t=0} |\text{Ric}^\varphi_{g_t}|^2_{g_t} \right) \mu + \int_M |\text{Ric}^\varphi|^2 \left.\frac{d}{dt}\right|_{t=0} \mu_{g_t},$$

so that, using (5.88) and (5.53),

$$\left.\frac{d}{dt}\right|_{t=0} \int_M |\text{Ric}^\varphi_{g_t}|^2_{g_t} \mu_{g_t} = \int_M \left( 2h_{ik,jk}R^\varphi_{ij} - \text{tr}(h)R^\varphi_{ij} - (\Delta h)_{ij}R^\varphi_{ij} - 2(R^\varphi)^2_{ij}h_{ij} \right) \mu + \frac{1}{2} \int_M |\text{Ric}^\varphi|^2 \text{tr}(h) \mu,\tag{5.90}$$

that is, using the divergence theorem,

$$\left.\frac{d}{dt}\right|_{t=0} \int_M |\text{Ric}^\varphi_{g_t}|^2_{g_t} \mu_{g_t} = \int_M \left( 2h_{ik,jk}R^\varphi_{ij} - \delta_{ij}R^\varphi_{ik,tk} - R^\varphi_{ij,kk} - 2(R^\varphi)^2_{ij}h_{ij} + \frac{1}{2} |\text{Ric}^\varphi|^2 \delta_{ij} \right) h_{ij} \mu.\tag{5.91}$$

The following commutation relation holds (see (2.60))

$$R^\varphi_{ik,jk} = R^\varphi_{ik,kj} + R^\varphi_{ij}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{it}.\tag{5.92}$$

Using the generalized Schur’s identity and the definition of $\varphi$-Ricci the above reads

$$R^\varphi_{ik,jk} = \frac{1}{2} S^\varphi_{ij} - \alpha(\varphi^{\alpha}_{kk}\varphi^{\alpha}_{ij}) + R^\varphi_{ij}R^\varphi_{tk} + R^\varphi_{ij}R^\varphi_{tt} + \alpha \varphi^{\alpha}_{ij}\varphi^{\alpha}_{tt} R^\varphi_{tt},$$

that is,

$$R^\varphi_{ik,jk} = \frac{1}{2} S^\varphi_{ij} - R^\varphi_{ik,kj} + R^\varphi_{ij} + \alpha R^\varphi_{ik,kj}\varphi^{\alpha}_{ij} - \alpha \varphi^{\alpha}_{kk}\varphi^{\alpha}_{ij} - \alpha \varphi^{\alpha}_{kk}\varphi^{\alpha}_{ij}.\tag{5.93}$$

Moreover, taking the trace of the above we infer

$$R^\varphi_{ik,tk} = \frac{1}{2} \Delta S^\varphi - \alpha \varphi^{\alpha}_{kk}\varphi^{\alpha}_{tt} - \alpha |\tau(\varphi)|^2.\tag{5.94}$$

By plugging (5.91) and (5.92) into (5.50) we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \int_M |\text{Ric}^\varphi_{g_t}|^2_{g_t} \mu_{g_t} = \int_M \left( S^\varphi_{ij} - 2R^\varphi_{ik,tk}R^\varphi_{ij} + 2\alpha R^\varphi_{ik,kj}\varphi^{\alpha}_{ij} - 2\alpha \varphi^{\alpha}_{kk}\varphi^{\alpha}_{ij} - \alpha \varphi^{\alpha}_{kk}\varphi^{\alpha}_{ij} - \Delta R^\varphi_{ij} \right) h_{ij} \mu$$

$$- \int_M \left( \frac{1}{2} \Delta S^\varphi - \alpha \varphi^{\alpha}_{kk}\varphi^{\alpha}_{ij} - \alpha |\tau(\varphi)|^2 - \frac{1}{2} |\text{Ric}^\varphi|^2 \right) \delta_{ij} h_{ij} \mu.$$

Using (5.10) and (5.60) we have

$$\left.\frac{d}{dt}\right|_{t=0} \int_M (S^\varphi)_{g_t}^2 \mu_{g_t} = \int_M \left( 2\text{Hess}(S^\varphi) - 2S^\varphi\text{Ric}^\varphi + \left( \frac{(S^\varphi)^2}{2} - 2\Delta S^\varphi \right) g, h \right) \mu.\tag{5.95}$$

In conclusion, since

$$\left.\frac{d}{dt}\right|_{t=0} B^\varphi(g_t) = \frac{m}{8(m-1)} \left.\frac{d}{dt}\right|_{t=0} \int_M (S^\varphi)_{g_t}^2 \mu_{g_t} - \frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \int_M |\text{Ric}^\varphi_{g_t}|^2_{g_t} \mu_{g_t} - \frac{\alpha}{2} \left.\frac{d}{dt}\right|_{t=0} \int_M |\tau^\varphi(\varphi)|^2 \mu_{g_t},\tag{5.96}$$
by plugging the two relations above and using (5.18) we obtain the validity of (5.86).

For \( m = 4 \) the above gives
\[
\frac{d}{dt} \bigg|_{t=0} B^\varphi(g_t) = \int_M \left[ R_{ikj}^\varphi R_{ik}^\varphi + \frac{1}{2} \Delta R_{ij}^\varphi - \frac{1}{6} S_{ij}^\varphi - \frac{1}{3} S^\varphi R_{ij}^\varphi + \alpha \left( \varphi_{jk} \varphi_{ij} - R_{ik}^\varphi \varphi_{ij} \right) \right] h_{ij} \mu_g \\
+ \int_M \left( \frac{1}{12} (S^\varphi)^2 - \frac{1}{12} \Delta S^\varphi - \frac{1}{4} |\text{Ric}^\varphi|^2 - \frac{\alpha}{4} |\tau(\varphi)|^2 \right) \delta_{ij} h_{ij} \mu_g
\]
Recalling (2.31), that is,
\[
B^\varphi_{ij} = \frac{1}{2} R_{ij,kk}^\varphi + R_{ikj}^\varphi R_{ik}^\varphi - \frac{1}{6} S_{ij}^\varphi - \frac{1}{3} S^\varphi R_{ij}^\varphi + \left( \frac{(S^\varphi)^2}{12} - \frac{\Delta S^\varphi}{12} - \frac{1}{4} |\text{Ric}^\varphi|^2 \right) \delta_{ij}
\]
we get
\[
(B^\varphi, h) = \left[ \frac{1}{2} \Delta R_{ij}^\varphi + R_{ikj}^\varphi R_{ik}^\varphi - \frac{1}{6} S_{ij}^\varphi - \frac{1}{3} S^\varphi R_{ij}^\varphi + \left( \frac{(S^\varphi)^2}{12} - \frac{\Delta S^\varphi}{12} - \frac{1}{4} |\text{Ric}^\varphi|^2 \right) \delta_{ij} \right] h_{ij}
\]
\[
+ \alpha \left( \varphi_{ij} \varphi_{kk} - \frac{1}{2} R_{ikj}^\varphi \varphi_{kk} \varphi_{ij} - \frac{1}{2} R_{ikj}^\varphi \varphi_{ij} \varphi_{kk} - \frac{1}{4} |\tau(\varphi)|^2 \delta_{ij} \right) h_{ij},
\]
Then we finally obtain the validity of (5.87).

We are finally ready to give the

**Proof (of Theorem 5.78).** The proof follows immediately from (5.81) and (5.87). It remains only to observe that, if \( \varphi \) is a submersion a.e. and \( \varphi \)-Bach vanishes on \( M \), then \( \varphi \)-Bach is divergence free and from Remark 2.38 we automatically get that \( J = 0 \).

Recall that Corollary 3.44 tell us that \( \varphi \)-Bach is a conformal invariant tensor for \( m = 4 \). Now we provide an alternative proof of the conformal invariance of \( \varphi \)-Bach, at least when \((M, g)\) is a closed orientable four dimensional Riemannian manifold.

**Proposition 5.94.** For \( m = 4 \) the functional \( B \) is conformal invariant, that is,
\[
B^\varphi(\tilde{g}) = B^\varphi(g),
\]
where
\[
\tilde{g} = e^{-f} g,
\]
for some smooth function \( f \) on \( M \). As a consequence its gradient, that is \( \varphi \)-Bach, is a conformal invariant tensor.

**Proof.** For \( m = 4 \) the functional \( B \) is given by
\[
B^\varphi(g) = \frac{1}{2} \int_M \left( \frac{1}{3} (S^\varphi)^2 - |\text{Ric}^\varphi|^2 - \alpha |\tau^\varphi|^2 \right) \mu_g.
\]
To prove (5.95) it is sufficient to show the validity of
\[
Q^\varphi_{\tilde{g}} = Q^\varphi_{g} + \text{div}_{\tilde{g}} [P_{\tilde{g}}(f)] \mu_{\tilde{g}},
\]
where the 4-form \( Q^\varphi_{\tilde{g}} \) is given by, for every Riemannian metric \( g \) on \( M \),
\[
Q^\varphi_{\tilde{g}} := \left( \frac{1}{3} (S^\varphi)^2 - |\text{Ric}^\varphi|^2 - \alpha |\tau^\varphi|^2 \right) \mu_g
\]
and the vector field \( P_{\tilde{g}}(f) \) is defined as
\[
P_{\tilde{g}}(f) = \left( S^\varphi + \Delta f - \frac{1}{2} |\nabla f|^2 \right) \nabla f - (2\text{Ric}^\varphi + \text{Hess}(f))(\nabla f, \cdot)^2.
\]
• For $m = 4$ (3.22) reads
\[ e^{-f} S^g = S^\varphi + 3\Delta f - \frac{3}{2} |\nabla f|^2, \]
hence
\[ e^{-2f} \frac{1}{3} (S^g)^2 = \frac{1}{3} (S^\varphi)^2 + S^\varphi (2\Delta f - |\nabla f|^2) + 3(\Delta f)^2 + \frac{3}{4} |\nabla f|^4 - 3|\nabla f|^2 \Delta f. \] (5.98)

• For $m = 4$ (3.21) reads
\[ \text{Ric}^g = \text{Ric}^\varphi + \text{Hess}(f) + \frac{1}{2} df \otimes df + \frac{1}{2} (\Delta f - |\nabla f|^2)g. \]
hence
\[ e^{-2f} |\text{Ric}^g|_g^2 = |\text{Ric}^\varphi|^2 + |\text{Hess}(f)|^2 + \frac{1}{4} |\nabla f|^4 + (\Delta f - |\nabla f|^2)^2 + 2R^g_{ij} f_{ij} + R^g_{ij} f_i f_j \]
\[ + S^\varphi (\Delta f - |\nabla f|^2) + f_{ij} f_i f_j + (\Delta f - |\nabla f|^2) \Delta f + \frac{1}{2} (\Delta f - |\nabla f|^2)|\nabla f|^2, \]
that is, using the definition of $\varphi$-Ricci
\[ e^{-2f} |\text{Ric}^g|_g^2 = |\text{Ric}^\varphi|^2 + S^\varphi (\Delta f - |\nabla f|^2) + 2R^g_{ij} f_{ij} \]
\[ + |\text{Hess}(f)|^2 + R^g_{ij} f_{ij} + \frac{3}{4} |\nabla f|^4 + 2(\Delta f)^2 - \frac{5}{2} |\nabla f|^2 \Delta f + f_{ij} f_i f_j - \alpha |d\varphi(\nabla f)|^2. \]
Using Bochner formula (see, for instance, (1.176) of [AMR])
\[ \frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess}(f)|^2 + R^g_{ij} f_i f_j + f_{ij} f_j, \]
from the above we conclude
\[ e^{-2f} |\text{Ric}^g|_g^2 = |\text{Ric}^\varphi|^2 + S^\varphi (\Delta f - |\nabla f|^2) + 2R^g_{ij} f_{ij} + \frac{1}{2} \Delta |\nabla f|^2 - f_{ij} f_j \]
\[ + \frac{3}{4} |\nabla f|^4 + 2(\Delta f)^2 - \frac{5}{2} |\nabla f|^2 \Delta f + f_{ij} f_i f_j - \alpha |d\varphi(\nabla f)|^2. \] (5.99)

• For $m = 4$ (3.23) reads
\[ e^{-f} \tau^g(\varphi) = \tau^\varphi(\varphi) - d\varphi(\nabla f), \]
hence
\[ e^{-2f} |\tau^g(\varphi)|^2 = |\tau(\varphi)|^2 + |d\varphi(\nabla f)|^2 - 2\tau(\varphi)^a \varphi^a f_i. \] (5.100)

• For $m = 4$ (3.3) with $f = 2h$ gives
\[ e^{2f} \mu^g = \mu. \] (5.101)

Using (5.98), (5.99), (5.100) and (5.101) and also the definition (5.97) we get
\[ Q^g = Q^\varphi + \left( S^\varphi \Delta f + (\Delta f)^2 - \frac{1}{2} |\nabla f|^2 \Delta f - \frac{3}{2} |\nabla f|^2 \Delta f - 2R^g_{ij} f_{ij} + f_{ij} f_i f_j + 2\alpha \tau(\varphi)^a \varphi^a f_i \right) \mu. \]

To conclude notice that, using (2.22),
\[ R^g_{ij} f_{ij} = (R^g_{ij} f_i)_j - R^g_{ij} f_i = (R^g_{ij} f_i)_j - \frac{1}{2} S^\varphi g_{ij} f_j + \alpha \varphi^a_{ij} \varphi^a f_i, \]
\[ S^\varphi \Delta f = (S^\varphi g_{ij}) j - S^\varphi f_j, \]
\[ f_{ij} f_j = (\Delta f) f_j - (\Delta f)^2, \]
and, finally,
\[ f_{ij} f_i f_j = \frac{1}{2} |\nabla f|^2 f_j = \left( \frac{1}{2} |\nabla f|^2 f_j \right)_j - \frac{1}{2} |\nabla f|^2 \Delta f \] (5.102)
hold and thus
\[ Q^g = Q^\varphi + \left( S^\varphi f_j - \frac{1}{2} |\nabla f|^2 - 2R^g_{ij} f_i + \Delta f f_j - \frac{1}{2} |\nabla f|^2 f_j \right) \mu. \]

This concludes the proof.
Remark 5.103. We choose the notation $Q^\varphi_g$ in the proof above because when $\varphi$ is constant we recover the $Q$-curvature introduced by Branson, see [Br].

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