Numerical solution of Benjamin-Bona-Mahony-Burger (BBMB) and regularized long-wave (RLW) equations using time-space pseudo-spectral method

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Abstract. In this research article, an improved numerical scheme is developed for solving the non-linear Benjamin-Bona-Mahony-Burger (BBMB) and regularized long-wave (RLW) equations. The time-space pseudo-spectral method is developed using Chebyshev polynomials at the CGL points. The algebraic mapping is used to transform the initial-boundary value non-homogeneous problem into the homogeneous problem and further, the BBMB and RLW equations are reduced to a system of nonlinear equations. To demonstrate the performance, the method applied to two model problems and approximate results have been compared with the exact solution and other existing methods. The proposed method succeeds to obtain high order accuracy.

1. Introduction

Existences of nonlinear terms in applied mathematical physics problems are obvious. Therefore, accurate and stable numerical solutions of the nonlinear equations are important. In this paper, a high order pseudo-spectral method in time and space will be employed on BBMB and RLW equations.

Let us consider the nonlinear generalized BBMB equation in the finite space and time intervals:

\[ V_x - V_{xx} - \alpha V_{xx} + \beta V_x + V V_x = F(x,t), \quad x \in [a,b], \quad 0 \leq t \leq T, \]

with initial condition:

\[ V(x,0) = V_0(x), \quad x \in [a,b], \]

and boundary conditions:

\[ V(a,t) = g(t), \quad V(b,t) = h(t), \quad 0 \leq t \leq T, \]

where \( V(x,t) \) represents the fluid velocity. \( V_x \), nonlinear part of Eq (1), represents convection term, \( \alpha V_{xx} \) is a dissipation term and \( V_{xx} \) is a dispersive term. For \( \alpha = 0 \) and \( \beta = 1 \), the Eq (1) is called regularized long wave (RLW) equation. The RLW equation was first time introduced by Peregrine[1] to understand the behaviour of undular bore. For \( \alpha = \beta = 1 \), the Eq (1) is known as the BBMB equation and investigated by Benjamin et al.[2] in 1972. There are various extensive numerical methods to solve RLW and generalized BBMB equations, such as exp-function method[3], finite difference solution[1, 4, 5] and B-splines method[6, 7]. Kamel Al-Khaled et al.[8] obtained solitary-wave solutions of BBM-Burger equation by decomposition method. Salas et al.[9] have studied BBMB equations using the generalization of the tanh-coth method and obtained traveling wave solutions. Qinghua and Zhengzheng[10] have studied the convergence rate of the solutions of the generalized BBMB equations based upon time-space weighted energy method.

The rest of the paper is organized as follows. In section 2, a brief description of pseudo-spectral method in time and space will be discussed. In section 3, numerical results and discussion are described. Finally, the conclusion of the paper is given in section 4.
2. Time-space pseudo-spectral method and Discretization

We know the working domain of Chebyshev polynomials are [-1, 1], therefore, transform the problem into interval [-1, 1], using

\[ x = \frac{b-a}{2} x + \frac{b+a}{2}, \quad t = \frac{T}{T} t + \frac{T}{2} \]

Accordingly, equation (1) is transformed as follows:

\[ V_t - CV_{xx} + \alpha BV_{xx} + A\beta V_x + AVV_x = DF(x,t), \quad x \in [-1,1], \quad t \in [-1,1], \quad (2) \]

where \( A = \frac{T}{b-a}, \quad B = \frac{2T}{(b-a)^2}, \quad C = \frac{2}{(b-a)^2} \) and \( D = \frac{T}{2} \),

with initial condition:

\[ V(x,-1) = \varphi(x), \quad x \in [-1,1], \quad (3) \]

and boundary conditions:

\[ V(-1,t)=\sigma_1(t), \quad V(1,t)=\sigma_2(t), \quad t \in [-1,1]. \quad (4) \]

The problem is non-homogeneous therefore, we consider a mapping for converting the problem from non-homogeneous to homogeneous.

\[ \tilde{\xi}(x,t) = \frac{1-t}{2} \varphi(x) + \frac{1-x}{2} \sigma_1(t) + \frac{1+x}{2} \sigma_2(t) - \left( \frac{1-t}{4} \right) \left( \frac{1-x}{4} \right) \sigma_1(-1) - \left( \frac{1-t}{4} \right) \left( \frac{1+x}{4} \right) \sigma_2(-1). \quad (5) \]

At corner points initial and boundary conditions satisfy, \( \varphi(-1) = \sigma_1(-1) \) and \( \varphi(1) = \sigma_2(-1) \).

Define new variable \( U(x,t) \),

\[ V(x,t) = U(x,t) + \tilde{\xi}(x,t), \quad (6) \]

then above equation (2), is reduced as follows:

\[ (U + \tilde{\xi})_t - C(U + \tilde{\xi})_{xx} - \alpha B(U + \tilde{\xi})_{xx} + A\beta (U + \tilde{\xi})_x + A(U + \tilde{\xi})(U + \tilde{\xi})_x = DF(x,t). \quad (7) \]

Now define the residual \( R(x,t) \),

\[ R(x,t) = (U + \tilde{\xi})_t - C(U + \tilde{\xi})_{xx} - \alpha B(U + \tilde{\xi})_{xx} + A\beta (U + \tilde{\xi})_x + A(U + \tilde{\xi})(U + \tilde{\xi})_x - DF(x,t). \quad (8) \]

We seek spectral approximation \( U^u(x,t) \) of the form:

\[ U^u(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} \Phi_i(x) \Psi_j(t) U_{ij}^u, \]

where the functions \( \Phi_i(x) \) and \( \Psi_j(t) \) are the \( N^{th} \) degree Lagrange polynomials with space and time variables, respectively.

Approximation of partial derivatives at collocation points \( (x_p, t_q) \) are defined as

\[ \frac{\partial}{\partial x} \left[ U^u(x_p, t_q) \right] = \sum_{i=0}^{N} \sum_{j=0}^{N} \Phi_i^{(r)}(x_p) \Psi_j(t_q) U_{ij}^u \]

\[ = \left[ \Delta_{[0,N,r]}^{(r)} \right] \otimes \Psi(t_q) U^u, \quad (10) \]

and

\[ \frac{\partial}{\partial t} \left[ U^u(x_p, t_q) \right] = \sum_{i=0}^{N} \sum_{j=0}^{N} \Phi_i(x_p) \Psi_j^{(s)}(t_q) U_{ij}^u = \sum_{i=0}^{N} \sum_{j=0}^{N} \Phi_i(x_p) \Delta_{[0,N,s]}^{(s)} U_{ij}^u \]

\[ = \left( \Phi(x_p) \otimes \left[ \Delta_{[0,N,s]}^{(s)} \right] \right)^T U^u, \quad (11) \]

where
\[ \Phi(x_p) = \left[ \Phi_0(x_p), \Phi_1(x_p), \ldots, \Phi_N(x_p) \right]^T, \quad \Psi(t_q) = \left[ \Psi_0(t_q), \Psi_1(t_q), \ldots, \Psi_N(t_q) \right]^T, \]

and \( \otimes \) represent the kronecker product of two vectors or matrices. \( U^a \) is a vector defined by:

\[ U^a = \left[ U_{00}^a, \ldots, U_{0N}^a, U_{10}^a, \ldots, U_{1N}^a, \ldots, U_{N0}^a, \ldots, U_{NN}^a \right]^T, \]

and \( \Delta^{(j)} \) is defined \( \tau^j \) order Chebyshev pseudo-spectral differentiation matrix. From the initial boundary conditions, the values of \( \Psi_q(t), \Phi_q(x) \) and \( \Phi_N(x) \) are zero at these CGL points. Therefore using equations (10) and (11), redefine the residual at CGL points in space and time.

\[
R(x_p, t_q) = \theta_1 \left[ \left( I_N \otimes \Delta^i \right) - C \left( \Delta^2 \otimes \Delta^j \right) - \alpha B \left( \Delta^2 \otimes I_N \right) + A \left( \Delta^2 \otimes I_N \right) \left( \beta + \frac{\xi^j}{\tau^i} \right) + A I_{[N^2,(N-1)]} \right] \xi_x^j + \frac{\theta_1 A}{2} \left( \Delta^2 \otimes I_N \right) U^2 + D F^j \xi_x^j - C \xi_x^j - A B \xi_x^j + A \frac{1}{2} \xi_x^j,
\]

where \( \Delta^i \) = \( \Delta^i_{[N-1,N]} \) and \( \Delta^i \) = \( \Delta^i_{[N-1,N]} \), \( \Delta^j \) = \( \Delta^j_{[N-1,N]} \).

\[
\theta_1 = \left( e^T \right)_{\left( (i-1) \times N \right)} \quad \text{and} \quad e^T \quad \text{is a} \quad N(N-1) \quad \text{vectors, i.e.} \quad N^j \quad \text{element is 1 and rest elements are 0.}
\]

Put the residual equal to zero which leads to the following system of nonlinear algebraic equation:

\[
\Gamma_1 U + \Gamma_2 U^2 = \Gamma_3, \tag{12}
\]

where,

\[
\Gamma_1 = \left[ \left( I_N \otimes \Delta^i \right) - C \left( \Delta^2 \otimes \Delta^j \right) - \alpha B \left( \Delta^2 \otimes I_N \right) + A \left( \Delta^2 \otimes I_N \right) \left( \beta + \frac{\xi^j}{\tau^i} \right) + A I_{[N^2,(N-1)]} \right],
\]

\[
\Gamma_2 = \frac{\alpha}{2} \left( \Delta^2 \otimes I_N \right), \quad \text{and} \quad \Gamma_3 = DF - G_1 + CG_2 + \alpha BG_3 - \beta AG_4 - \frac{A}{2} G_5.
\]

\[
U = \left[ \left[ U_{11}, \ldots, U_{1N} \right] \ldots \left[ U_{(N-1)1}, \ldots, U_{(N-1)N} \right] \right]^T, \quad F = \left[ \left[ F_{11}, \ldots, F_{1N} \right] \ldots \left[ F_{(N-1)1}, \ldots, F_{(N-1)N} \right] \right]^T,
\]

\[
G_1 = \left[ \xi_x^1, \ldots, \xi_x^N \right] \ldots \left[ \xi_x^{(N-1)1}, \ldots, \xi_x^{(N-1)N} \right]^T, \quad G_2 = \left[ \xi_x^{11}, \ldots, \xi_x^{1N} \right] \ldots \left[ \xi_x^{(N-1)(N-1)} \right]^T,
\]

\[
G_3 = \left[ \xi_x^{21}, \ldots, \xi_x^{2N} \right] \ldots \left[ \xi_x^{(N-1)(N-1)} \right]^T, \quad G_4 = \left[ \xi_x^{31}, \ldots, \xi_x^{3N} \right] \ldots \left[ \xi_x^{(N-1)(N-1)} \right]^T,
\]

\[
G_5 = \left[ \xi_x^{21}, \ldots, \xi_x^{2N} \right] \ldots \left[ \xi_x^{2(N-1)(N-1)} \right]^T.
\]

The system of nonlinear equation (12) can be solved by using Newton Raphson method.

3. Numerical results and discussion

In this section, we consider numerical examples to obtain approximate solutions using pseudo-spectral method in time and space and demonstrate the sup norm and relative errors norm, which are defined by:

\[
L_2 = \| U - U \|_2 = \text{Sup}_{(\rho, \phi)} \left[ U^a(x_p, t_q) - U(x_p, t_q) \right]
\]

and

\[
L_2 = \left( \frac{\sum_{\rho=0}^{N} \sum_{\phi=0}^{N} \left[ U^a(x_p, t_q) - U(x_p, t_q) \right]^2}{\sum_{\rho=0}^{N} \sum_{\phi=0}^{N} U(x_p, t_q)^2} \right)^{1/2},
\]

where \( U^a \) and \( U \) are the spectral approximations and analytical solutions, respectively.

3.1. Example 1

In this example, consider RLW equation, i.e. \( \alpha = 0 \) and \( \beta = 1 \) defined by Eq (1) with \( F = 0 \).

Initial condition: \( V(x, 0) = 3 \sec h^2 \left( k(x - x_0) \right) \), \( x \in [a, b] \), \( \alpha = 0 \).

and boundary conditions:
\[ V(a,t) = 3c \sec h^{2}(k(x - va - x_0)), \quad V(b,t) = 3c \sec h^{2}(k(x - vb - x_0)), \quad 0 \leq t \leq T, \quad (14) \]

the exact solution of this problem is [11]:
\[ V(x,t) = 3c \sec h^{2}(k(x - \nu x - x_0)). \quad (15) \]

Here wave velocity \( \nu = 1 + c \) and \( k = \frac{c}{\sqrt{4\nu}} \), where \( c \) denotes solitary wave amplitude. For computation purpose, we consider the space interval \( [a,b] = [-40,60], \; x_0 = 0 \). The results for \( c = 0.1 \) and \( c = 0.03 \) at a different time \( T = 1,5,10 \) and 20 have shown in Figure 1 and 2. The exact solutions have shown by solid line and numerical solutions by blocks, which have a very good agreement with exact solutions. Figure 3 contains 3D surface plots for the numerical solution with different wave amplitude at \( T = 20 \). The tabulated results have presented for the proposed method and other existing methods[5, 11, 12, 13] in Table 1 and 2 with different \( c \).

![Figure 1](image1.png) ![Figure 2](image2.png)

**Figure 1:** Exact and numerical solutions at \( c = 0.1 \). **Figure 2:** Exact and numerical solutions at \( c = 0.03 \)

![Figure 3](image3.png)

**Figure 3:** Numerical solutions for (a) \( c = 0.1 \) and (b) \( c = 0.03 \) at \( T = 20 \).

| T | Errors | Proposed Method | [12] | [11] | [5] | [13] |
|---|---|---|---|---|---|---|
| 4 | \( L_x \) | 3.634e-05 | 3.982e-02 | 1.485e-05 | 1.2e-04 | 6.646e-05 |
| | \( L_y \) | 1.314e-05 | 1.374e-02 | 5.956e-06 | 5.0e-05 | 2.496e-05 |
| 8 | \( L_x \) | 9.234e-04 | 7.946e-02 | 2.706e-05 | 2.3e-04 | 1.282e-04 |
| | \( L_y \) | 4.665e-04 | 2.766e-02 | 1.051e-05 | 9.0e-05 | 5.049e-05 |
| 12 | \( L_x \) | 1.229e-04 | 1.188e-01 | 3.640e-05 | 3.4e-04 | 1.900e-04 |
| | \( L_y \) | 4.141e-05 | 4.135e-02 | 1.318e-05 | 1.4e-04 | 7.470e-05 |

**Table 1:** Comparison of proposed method with existing method at \( c = 0.1 \).
Table 2: Comparison of proposed method with existing method at \( c = 0.03 \).

| T  | Errors     | Proposed Method | [12] | [11] | [13] |
|----|------------|-----------------|------|------|------|
| 4  | \( L_z \)  | 5.986e-05       | 2.928e-03 | 4.120e-04 | 4.097e-04 |
|    | \( L_{e} \) | 1.313e-05       | 7.86e-04  | 2.299e-04 | 2.299e-04 |
| 8  | \( L_z \)  | 7.596e-05       | 5.816e-03 | 5.112e-04 | 5.109e-04 |
|    | \( L_{e} \) | 3.464e-05       | 1.582e-03 | 2.210e-04 | 2.210e-04 |
| 12 | \( L_z \)  | 3.186e-04       | 8.698e-03 | 5.356e-04 | 5.359e-04 |
|    | \( L_{e} \) | 6.641e-04       | 1.961e-01 | 2.125e-04 | 2.125e-04 |
| 16 | \( L_z \)  | 1.456e-04       | 1.158e-02 | 5.429e-04 | 5.446e-04 |
|    | \( L_{e} \) | 1.219e-04       | 3.190e-03 | 2.139e-04 | 2.120e-04 |
| 20 | \( L_z \)  | 3.935e-04       | 3.996e-03 | 5.624e-04 | 5.684e-04 |
|    | \( L_{e} \) | 3.339e-04       | 4.315e-04 | 4.278e-04 |

3.2. Example 2

Let us consider the BBMB equation i.e. \( \alpha = 1 \) and \( \beta = 1 \) defined by Eq (1) with

\[
F(x,t) = \exp(-t) \left[ \cos x - \sin x + \frac{1}{2} \exp(-t) \sin 2x \right].
\]

Initial condition:

\[
V(x,0) = \sin x, \quad x \in [a,b],
\]

and boundary conditions:

\[
V(a,t) = \exp(-t) \sin a, \quad V(b,t) = \exp(-t) \sin b, \quad 0 \leq t \leq T,
\]

the exact solution of this problem is\([10, 14]\).

\[
V(x,t) = \exp(-t) \sin x.
\]

Figure 4 has shown a plot of the numerical solution by dots and exact solution by solid lines at different times intervals, namely \( T = 3.0, 4.0, 5.0, 6.0 \) with space interval \([a,b]=[0,\pi]\). Figure 5 contains 3D surface plots for the numerical solution at the time \( T = 6 \), which also depict the exactness of method. The computation results of the proposed method with existing methods \([14, 15]\) at the time \( T = 10 \) have presented in Table 3, which clearly shows that the proposed method has achieved higher accuracy with minimum numbers of grid points as compared to other existing methods.

Table 3: Comparison of proposed method with existing method at time \( T = 10 \).

| N  | Proposed Method | N  | [14] | [15] |
|----|-----------------|----|------|------|
|    | \( L_z \)       | \( L_{e} \) | \( L_z \) | \( L_{e} \) |
| 08 | 5.3251e-05      | 3.2597e-05 | 10  | 1.7147e-04 | 2.1800e-02 |
| 16 | 4.0646e-05      | 9.0876e-06 | 20  | 5.6341e-05 | 5.3000e-03 |
| 64 | 4.1282e-06      | 1.1177e-06 | 80  | 7.2635e-06 | 3.3291e-04 |
| 128| 5.2201e-07      | 2.7547e-07 | 320 | 8.1631e-07 | 2.0766e-05 |
In this research paper, the numerical results of non-linear Benjamin-Bona-Mahony-Burger (BBMB) and regularized long-wave (RLW) equations have obtained using time-space pseudo-spectral method with Chebyshev polynomial as basis functions and achieved spectral accuracy. The performance of the proposed method has been demonstrated by considering two examples. The computed results have shown very good agreement with the exact solutions and compete with existing results.

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