GRAPH HÖRMANDER SYSTEMS

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Abstract. This paper extends the Bakry-Émery theorem relating the Ricci curvature and log-Sobolev inequalities to the matrix-valued setting. Using tools from noncommutative geometry, it is shown that for a right invariant second order differential operator on a compact Lie group, a lower bound for a matrix-valued modified log-Sobolev inequality is equivalent to a uniform lower bound for all finite dimensional representations. Using combinatorial tools, we obtain computable lower bounds for matrix-valued log-Sobolev inequalities of graph-Hörmander systems using combinatorial methods.

1. Introduction

Estimates for the spectral gap of a Laplace type operator are relevant in several areas of mathematics such as analysis, geometry, combinatorics, and even theoretical computer science. The aim of this paper is to study matrix-valued versions of log-Sobolev inequalities related to representations of compact Lie groups, and the connection to noncommutative geometry.

Log-Sobolev inequalities have been an area of active research for several decades, see [Gro06] for an overview. Let us highlight the local estimates on Sobolev inequalities by Rothchild and Stein [RS76], the new paradigm relating the Ricci curvature and the log-Sobolev inequality [BE86], the so-called Bakry-Émery theory, and the discovery by Meyer and Gross of log-Sobolev inequalities in infinite dimension, see [Yau96, Gro75, Mey83a, Mey83b]. We refer to Ledoux’s article [Led11] for the fundamental connection between log-Sobolev inequalities and concentration inequalities. More recently, Otto and Villani, and Sturm connected geometric insights with the theory of optimal transport and displacement convexity.

Transferring these estimates to the quantum setting is an area of recent ongoing research, see [CM17, CM14, CM20, BBLW12, DR20, BCR20]. In the theory of open quantum systems, it is natural to assume that a diffusion process interacts with an environment system. Mathematically, this implies that a certain diffusion process is no longer ergodic, acting trivially on one part of the space. Bakry-Émery theory in the non-ergodic setting is less developed. This remains particularly true for the connection between Fisher information and log-Sobolev inequalities, see however [GJL18]. We should point out that in the matrix-valued setting, a crucial argument based on uniform convexity of $L_p$ spaces, the famous Rothaus Lemma, is no longer valid and hence the usual approach via hypercontractivity has to be replaced by new methods. Developing these methods is the main technical contribution in this paper.

Log-Sobolev inequalities have also been studied for discrete objects in particular graphs, see the seminal work of Saloff-Coste and Diaconis [DSC96], and also later work by Yau and his collaborators [Yau96, LY93]. Nowadays, it is clear that different variations of log-Sobolev inequalities are relevant. Let us recall that an ergodic system $T_t = e^{-t\Delta}$ on a probability space $(\Omega, \mu)$ satisfies a log-Sobolev
inequality (\(\lambda\)-LSI) if
\[
2\lambda \left( \int f^2 \ln f^2 d\mu - \ln(\int f^2 d\mu) \int f^2 d\mu \right) \leq \int \Delta(f) f d\mu = \mathcal{E}_\Delta(f, f).
\]
The right hand side defines the energy form. A proper formulation of log-Sobolev inequalities for non-ergodic system appears only in hidden form in the existing literature. Indeed, let \(T_t = e^{-tL}\) be a semigroup of self-adjoint positive unital maps. Let \(N_{\text{fix}} = \{f | T_t(f) = f\}\) (some or all \(t > 0\)) be the subalgebra of invariant functions which admits a conditional expectation \(E_{\text{fix}}\). Then \(L\) is said to satisfy a log-Sobolev inequality (\(\lambda\)-LSI) if
\[
\lambda D(f^2 \| E_{\text{fix}}(f^2)) \leq \mathcal{E}_L(f, f).
\]
Here and in the following we will use \(D(f \| g) = \tau(f \ln f) - \tau(f \ln g)\) for the relative entropy, and \(\tau(f) = \int f d\mu\) for the canonical trace induced by the probability measure \(\mu\). We recall that \(L\) satisfies a modified log-Sobolev inequality (\(\lambda\)-MLSI) if
\[
\lambda D(f \| E_{\text{fix}}(f)) \leq \mathcal{E}_L(f, \ln f)
\]
for any positive function \(f\). The right hand side \(I_L(f) = \mathcal{E}_L(f, \ln f)\) is the Fisher information. Let us denote by LSI\((L)\) and MLSI\((L)\) the largest possible such constants, and \(\lambda_2\) the spectral gap. Then
\[
2 \text{LSI}(L) \leq \text{MLSI}(L) \leq 2\lambda_2(L)
\]
holds in general. For smooth manifolds we have 2LSI\((L) = \text{MLSI}(L)\), which however fails for discrete graphs. For Riemanian manifolds there are two approaches to prove log-Sobolov estimates. One can use local Sobolev inequalities from the Euclidean setting, and then use the aforementioned Rothaus Lemma. Alternatively, one can use Bakry-Émery theory for an equivalent measure with satisfies a lower Ricci-curvature bound, i.e. it is enough to satisfy Ricci-curvature at \(\infty\), see [Led11]. The second approach may fail for manifolds not admitting convex functions ([GW79]).

The real power of log-Sobolev inequalities stems from their stability under perturbation, crucial in proving results for spin systems [Yau96] [LY93]. The tensor stability is also crucial in Talagrand’s inequality, as a special case of an LSI-estimates, which has applications in computer science, see [Bou02] [Led19].

Our original motivation for this projects comes from the study of Lindbladians in quantum information theory, or more generally dynamical systems on matrix algebras \(M_m\). Generators of a self-adjoint quantum dynamical system \(T_t = e^{-tL}\) are completely classified and of the form
\[
L(f) = \sum_k a_k^2 f + f a_k^2 - 2a_k f a_k = \sum_k [a_k, [a_k, f]].
\]
Since \(\delta_k(f) = i[a_k, f]\) is a derivation, such Lindblad generators may be considered as analogues of a second order differential operators. We may define the modified log-Sobolev (MLSI\((L) = \sup \{\lambda\}\)) using the normalized matrix trace \(\tau(f) = \frac{\text{tr}(f)}{m}\) such that
\[
\lambda D(f \| E_{\text{fix}}(f)) \leq I_L(f) = \tau(L(f) \ln f)
\]
for any positive \(f\). The complete modified log-Sobolev constant \(\text{CLSI}(L) = \sup \{\lambda\}\), the main notion of this paper, is the best constant such that
\[
\lambda D(f \| E_{\text{fix}}(f)) \leq \tau((id \otimes L)(f) \ln f)
\]
for any positive \(f \in M_m \otimes M\), where \(M\) is a finite von Neumann algebra. In contrast to commutative systems, tensorization with an auxiliary system is not ‘for free’. For the complete version of the modified log-Sobolev inequality, however, we have tensor stability (see [GJL18])
\[
\text{CLSI}(L_1 \otimes 1 + 1 \otimes L_2) \geq \min \{\text{CLSI}(L_1), \text{CLSI}(L_2)\}.
\]
The best way to understand the CLSI constant is via the entropy decay rate, i.e. the best constant such that
\[ D(e^{-tL}(f)\|E_{fix}(f)) \leq e^{-t\text{CLSI}(L)}D(f\|E_{fix}(f)) \]
holds for all positive matrix valued \( f \). We will show that even for commutative systems the complete, i.e. matrix version, provides useful insight. It should be noted, however, that the failure of the Rothaus Lemma is probably responsible for a lack of large classes of examples in the quantum information literature. Before [GJL18] and this paper, all the known results could be deduced from a result by Bardet’s result for \( L = I - E \), or Gaussian systems, see [BR18]. Note, however, that tensor stability will be crucial in analyzing many body systems through the interaction of local systems.

The main focus of this paper is Lindbladians which are “transferred” from a group representation \( u : G \to U(H) \), \( H \) a (finite dimensional) Hilbert space of a finite dimensional Lie group \( G \) with Lie algebra \( \mathfrak{g} \). A vector field \( \mathcal{X} = \{X_1, \ldots, X_m\} \subset \mathfrak{g} \cong T_1M \) is called a Hörmander system if the iterated commutators from \( \mathcal{X} \) generate the Lie algebra. Such a Hörmander system generates an ergodic semigroup of right-invariant maps \( T_t = e^{-t\Delta_{\mathcal{X}}} \) with the sub-Laplacian generator
\[ \Delta_{\mathcal{X}} = \sum_j -X_j^2 = \sum_j X_j^+X_j, \quad X_j(f)(g) = \frac{d}{dt}f(\exp(tX_j)g)|_{t=0}. \]
Here the adjoint \( X_j^+ \) refers to the Haar measure. The Laplace-Beltrami operator on \( G \) is obtained by taking an orthonormal basis of the Lie algebra, and hence is automatically a Hörmander system. Locally the geometry given by the induced Carnot-Caratheodory metric is extremely well-understood thanks to the famous Box-Ball theorem, [Gro96]. It is, however, not easy to obtain good dimension-free estimates for the spectral gap from the Ball-Box theorem, because of the implicit dependence of the dimension of the underlying space and the number of iterations required to generate the Lie algebra, see [Cho39, OL93, RS76]. Thanks to the transference theorem from [GJL18] such estimates also imply estimates for self-adjoint Lindbladians. Indeed, let \( \hat{u} : \mathfrak{g} \to B(H) \) be the induced Lie algebra representation such that
\[ u(\exp(tX)) = e^{it\hat{u}(X)} \]
describes the one-parameter group of unitaries. Let \( a_1, \ldots, a_m \in B(H) \) be images of \( \mathcal{X} \) under \( \hat{u} \). The induced Linbladian
\[ L_H^\mathcal{X}(f) = L_{\mathcal{X}^H}(f) = \sum_j [a_j, [a_j, f]] \]
can be controlled by the original semigroup thanks to the following diagram
\[ \begin{array}{ccc}
L_\infty(G, B(H)) & \xrightarrow{e^{-t\Delta_{\mathcal{X}} \otimes \text{id}}} & L_\infty(G, B(H)) \\
\uparrow & & \uparrow \\
B(H) & \xrightarrow{e^{-t\hat{u}_{\mathcal{X}}^H}} & B(H)
\end{array} \]
given by the trace preserving *- homomorphism \( \pi(T)(g) = u(g)^*Tu(g) \). Therefore \( \text{CLSI}(L_H^\mathcal{X}) \geq \text{CLSI}(\Delta_{\mathcal{X}}) \). Our main result is what might be called an anti-transference result.

**Theorem 1.1.** Let \( G \) be a finite dimensional compact Lie group, \( \mathcal{X} \subset \mathfrak{g} \), a vector field. Then
\[ \text{CLSI}^+(\Delta_{\mathcal{X}}) = \inf_{u:G\to U(H)} \text{CLSI}^+(L_H^\mathcal{X}). \]
We conjecture that indeed
\[ \text{CLSI}(\Delta_{\mathcal{X}}) \geq \inf_{H} \text{CLSI}(L_H^\mathcal{X}). \]
In this paper we have to work with a technical variant \( \text{CLSI}^+(L) = \inf_{p>1} C_p\text{SI}(L) \) to justify the use of Connes’ trace formula in our proof of Theorem [1.1]. Our \( p \)-Rényi version of a complete Sobolev
inequality is completely new for quantum dynamical semigroups, although anticipated in [BT06], and defined as the best constant $C_{pSI} = \sup\{\lambda\}$ such that
\[
\lambda \left( \|f\|^p_p - \|E_{\text{fix}}(f)\|^p_p \right) \leq pE_L(f, f^{p-1})
\]
for all positive matrix-valued $f$. In the scalar ergodic case this inequality has been studied by [BT06], and it was proved that the inequality is equivalent to the decay estimate
\[
\|T_t(f)\|^p_p - \|E_{\text{fix}}(f)\|^p_p \leq e^{-tC_{pSI}(L)} \left( \|f\|^p_p - \|E_{\text{fix}}(f)\|^p_p \right).
\]
We refer to [Li20] for a systematic study of the relative entropy $d_p$ associated with $C_{pSI}$. In the existing literature examples of $CLSI$ for quantum and classical systems are very rare, because the usual hypercontractivity argument fails. However, we are able to extend the famous result of Bakry-Émery in this new setting.

**Theorem 1.2** (Complete Bakry-Émery theorem). Let $(M, g, \mu)$ be a smooth Riemannian manifold with a probability measure $\mu$ defined by $d\mu = Z_v e^{-U} \text{dvol}$ with $Z_v = \int_M e^{-U} \text{dvol}$ and $U \in C^\infty(M)$ such that the Bakry-Émery Laplacian
\[
\int \Delta_v(f_1)f_2 d\mu = \int (\nabla f_1, \nabla f_2) d\mu
\]
satisfies $\text{Ric}(\Delta_v) \geq \kappa > 0$. Then
\[
\text{CLSI}(\Delta_v) \geq 2\kappa.
\]

Our key ingredient, motivated from [CM17], is to use Lieb’s concavity theorem, refined by Hiai and Petz [HP12]. This allows us to define ‘Ricci curvature bounded below’ for noncommutative dynamical systems. This result competes with the recent results in [CM20, Wit74], where a notion of transportation Ricci curvature has been introduced. For finite dimensional QMS our notion of geometric Ricci curvature implies complete transportation Ricci curvature. As it turns out to be a source of a large class of examples.

Indeed, the remainder of this paper is to find concrete estimates for the CLSI constant for graphs and related Lindbladian or differential operators. This is motivated form quantum information theory, but certainly interesting in view finite Markov process in the sense of [SC94].

**Theorem 1.3.** Let $G = (\mathcal{V}, \mathcal{E})$ be a connected undirected graph with a uniform distribution on the vertex set. Let
\[
A_\mathcal{E}(f)(x) = 2 \sum_{(x,y) \in \mathcal{E}} (f(x) - f(y))
\]
be the graph Laplacian and $T_s$ be the minimum spanning tree with the number of edges $l(T_s)$ and the maximum degree $d(T_s)$. Then
\[
\text{CLSI}(A_\mathcal{E}) \geq \text{CLSI}^+(A_\mathcal{E}) \geq \frac{2}{45d(T_s)l(T_s)^2}.
\]

This result is optimal for the cyclic graph $\mathbb{Z}_n$ with nearest neighboring interactions. The lower bound is efficiently computable. We refer to [Yau97] (and references therein) for other estimates of the LSI constant that are not directly comparable to our result. Our estimate is not expected to be the best possible because it is modelled after a long a one-dimensional structure. More edges should improve the estimates of the CLSI constant, and this is true for graphs with tensor product structure. Our results support the following conjecture.

**Conjecture 1.4.** For every self-adjoint Lindbladian $L$, we have $\text{CLSI}(L) > 0$. 
We can verify this conjecture for, what we call the graph-Hörmander systems. Indeed, let \( G = (\mathcal{V}, \mathcal{E}) \) be a connected undirected graph with a uniform distribution on \( \mathcal{V} = \{1, \ldots, n\} \). For every edge \( e = (r, s) \) with \( r < s \) we may define the tangent vector \( X_e = \langle r \rangle \langle s \rangle - \langle s \rangle \langle r \rangle \) and
\[
L_\mathcal{E}(x) = \sum_{e = (r, s) \in \mathcal{E}, r < s} [X_e, [X_e, x]]
\]
the corresponding Lindblad transferred from the sub-Laplacian \( \Delta_\mathcal{E} = -\sum_e X_e^2 \) over \( C^\infty(SO_n) \).

**Theorem 1.5.** Let \( G = (\mathcal{V}, \mathcal{E}) \) be a connected undirected graph with a uniform distribution over \( \mathcal{V} \). Then \( \Delta_\mathcal{E} \) is ergodic and
\[
\text{CLSI}^+(A_\mathcal{E}) \geq \text{CLSI}^+(L_\mathcal{E}) \geq \frac{\text{CLSI}^+(A_\mathcal{E})}{1 + 5\pi^2 \text{CLSI}^+(A_\mathcal{E})}.
\]

The paper is organized as follows. In section 2, we introduce important tools, such as derivations and double operator integrals, which we use in later chapters. We give the definitions of CLSI and CLSI\(^+\) and their properties, in particular, tensor stability and stability under perturbation. In section 3, we define the geometric Ricci curvature of a derivation triple and establish the abstract Bakry-Émery theorem. We compare our geometric Ricci curvature with the transportation Ricci curvature previously defined by Carlen and Maas. In section 4, we recapture the Bakry-Émery criterion for matrix-valued functions defined over a smooth manifold. We also include some geometric examples to illustrate derivation triples. In section 5, we briefly review the transference principle and develop the anti-transference principle with the help of representation theory and noncommutative geometry. In section 6, we give computable estimates of CLSI constants of connected graphs via the preorder traversal algorithm and existence of spanning trees. In section 7, we define the graph Hörmander systems and present the relation between CLSI constants of a connected graph and the induced Lindblad operator.

2. Notation and background

2.1. Tracial von Neumann algebras and modules. Let \( (\mathcal{N}, \tau) \) be a finite von Neumann algebra equipped with a normal faithful tracial state \( \tau \), and \( \mathcal{N}_+ \) be the set of positive elements in \( \mathcal{N} \). We denote the noncommutative \( L_p \)-space by \( L_p(\mathcal{N}, \tau) \), or \( L_p(\mathcal{N}) \) if the trace \( \tau \) is clear from the context. The Hilbert-Schmidt inner product over \( \mathcal{N} \) is defined by \( \langle x, y \rangle_\tau = \tau(x^* y) \) (also denoted by \( (x, y) \)). Let \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) be two von Neumann algebras. The Hilbert \( \mathcal{N}_1 \)-\( \mathcal{N}_2 \) bimodule \( \mathcal{N}_1 \mathcal{H}_{\mathcal{N}_2} \mathcal{N}_2 \) is a Hilbert space \( \mathcal{H} \) equipped with representations \( \pi_1 : \mathcal{N}_1 \to \mathcal{B}(\mathcal{H}) \) and \( \pi_2^{op} : \mathcal{N}_2^{op} \to \mathcal{B}(\mathcal{H}) \) satisfying \( [\pi_1(\mathcal{N}_1), \pi_2^{op}(\mathcal{N}_2^{op})] = 0 \). Let us recall that the opposite algebra \( \mathcal{N}^{op} \) is obtained by exchanging the left and right multiplications in \( \mathcal{N} \), i.e., \( (ab)^{op} \) in \( \mathcal{N}^{op} \) is given by \( ba \) in \( \mathcal{N} \) for \( a, b \in \mathcal{N} \). We use the notation \( xhy \) to denote the left \( \mathcal{N}_1 \) action and the right \( \mathcal{N}_2 \) actions for \( x \in \mathcal{N}_1 \), \( y \in \mathcal{N}_2 \) and \( h \in \mathcal{H} \). The Hilbert bimodule \( \mathcal{N}_1 \mathcal{H}_{\mathcal{N}_2} \mathcal{N}_2 \) is said to be self-adjoint if \( \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N} \) and there exists an antilinear involution \( J : \mathcal{H} \to \mathcal{H} \) such that \( J(xhy) = y^* J(h)x^* \) for any \( x, y \in \mathcal{N} \) and \( h \in \mathcal{H} \). In many situations we work with the slightly stronger notion of a \( W^* \)-right module \( \mathcal{X} \) which admits an \( \mathcal{N} \)-valued inner product \( \langle x, y \rangle \) such that \( \langle x, ya \rangle = \langle x, y \rangle a \) for any \( a \in \mathcal{N} \) and \( x, y \in \mathcal{X} \). Let us denote by \( \mathcal{L}(\mathcal{X}) \) the von Neumann algebra of left adjointable operators on \( \mathcal{X} \), see [Pasc73, JS05, Lan95]. If in addition there is a weak* continuous *-representation \( \pi : \mathcal{N} \to \mathcal{L}(\mathcal{X}) \), \( X \) becomes an \( \mathcal{N} \)-\( \mathcal{N} \)-\( W^* \)-bimodule. If furthermore there is an antilinear isometry \( J : \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X}) \) such that \( J(\pi(a)x) = xa^* \) for any \( x \in \mathcal{X} \) and \( a \in \mathcal{N} \), we recover all the data from above. Our typical example is given by a trace preserving inclusion \( \mathcal{N} \subset \mathcal{M} \) equipped with a conditional expectation \( E^{\mathcal{N}} : \mathcal{M} \to \mathcal{N} \). Then \( \langle x, y \rangle = E^{\mathcal{N}}(x^* y) \) makes \( \mathcal{M} \) an \( \mathcal{N} \)-valued right module which extends to a complete \( W^* \)-module \( \mathcal{X} = \overline{\mathcal{M}E_{\mathcal{N}}} \subset \mathcal{B}(L_2(\mathcal{M})) \). The left representation is, of course, given by \( \pi(a)\xi = a\xi \) which extends to the closure. The underlying Hilbert space is given by \( \mathcal{H} = L_2(\mathcal{M}, \tau) \). We see that here \( J(x) = x^* \) is an isometry on the Hilbert space, but only densely defined on \( \mathcal{M} \subset \mathcal{X} \).
2.2. Derivations. Let \( N \) be a self-adjoint Hilbert space. A closable derivation of a von Neumann algebra \( N \) is a densely defined closable linear operator \( \delta : L_2(N, \tau) \to H \) such that

1. \( \text{dom}(\delta) \) is a weakly dense \(*\)-subalgebra in \( N \);
2. the identity element \( 1 \in \text{dom}(\delta) \);
3. \( \delta(xy) = x\delta(y) + \delta(x)y \), for any \( x, y \in \text{dom}(\delta). \)

Let \( \tilde{\delta} \) denote the closure of \( \delta \). A derivation \( \delta \) is said to be \( * \)-preserving if \( J(\delta(x)) = \delta(x^*) \). Every closable \( * \)-preserving derivation \( \delta \) determines a positive operator \( \delta^* \delta \) on \( L_2(N, \tau) \). It was shown in [J-L90] that \( T_t = e^{-t\delta^*\delta} : N \to N \) is a strongly continuous semigroup of completely positive, unital and self-adjoint maps. Thus \( T_t \) is also trace preserving since \( \tau(x^* T_t(y)) = \tau(T_t(x^*)y) = \tau(T_t(x)y) \) for any \( x, y \in N \). See [PS10], [dPS04], [dPS07], [BS03], and [BR76] for more details.

Now let \( T_t = e^{-tA} : N \to N \) be a strongly continuous semigroup of completely positive unital self-adjoint maps on \( L_2(N, \tau) \). The generator \( A \) is a positive operator on \( L_2(N, \tau) \) given by

\[
A(x) = \lim_{t \to 0^+} \frac{1}{t}(T_t(x) - x), \forall x \in \text{dom}(A).
\]

It was pointed out in [J-L90] that \( \text{dom}(\delta) = \{ x \in N | \|A^{1/2}x\|_2 < \infty \} \) is indeed a \(*\)-algebra and invariant under the semigroup. The weak gradient form of \( A \) is defined by

\[
\Gamma_A(x, y)(z) = \frac{1}{2}(\tau(A(x)^*yz) + \tau(x^*A(y)z) - \tau(x^*yA(z))).
\]

If the weak gradient form \( \Gamma_A(x, y) \in L_1(N) \) for all \( x, y \in \text{dom}(A^{1/2}) \), we say the generator \( A \) (or \( T_t \)) satisfies \( \Gamma \)-regularity. It was shown in [JRS14] that we may associate the generator \( A \) satisfying the \( \Gamma \)-regularity with a closable \(*\)-preserving derivation \( \delta_A \).

**Theorem 2.1.** If \( A \) satisfies \( \Gamma \)-regularity, then there exists a finite von Neumann algebra \( (M, \tau) \) containing \( N \) and a \(*\)-preserving derivation \( \delta_A : \text{dom}(A^{1/2}) \to L_2(M) \) such that

\[
\tau(\Gamma_A(x, y)(z)) = \tau(\delta_A(x)^*\delta_A(y)z).
\]

Equivalently, \( \Gamma_A(x, y) = E_N(\delta_A(x)^*\delta_A(y)) \), where \( E_N : M \to N \) is the conditional expectation.

Throughout the paper, we always work with a closable \(*\)-preserving derivation \( \delta \) and a strongly continuous semigroup \( T_t = e^{-tA} \) of completely positive unital self-adjoint maps on \( L_2(N, \tau) \) satisfying \( \Gamma \)-regularity.

2.3. Double operator integrals. Let \( \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be bounded and \( \rho, \sigma \in N \) be self-adjoint. The double operator integral is defined by

\[
Q_{\phi}^{\rho, \sigma}(T) := \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(s, t) dE_\rho(s)T dE_\sigma(t),
\]

where \( E_\rho((s, t]) = 1_{(s, t]}(\rho) \) is the spectral projection of \( \rho \). We write \( Q_\phi^\rho \) if \( \rho = \sigma \). The notion of double operator integrals was first introduced by Daleckii and Krein (see [DK51], [kr56]) used for the analytical theory of perturbations. Further construction of double operator integrals was created in a series of papers, (see [dPS04], [dPS07], [BS03]) by Birman and Solomyak. Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be a continuously differentiable function and the difference quotient be \( f_{[1]}(x, y) = \frac{f(x) - f(y)}{x - y} \), then

\[
Q_{f_{[1]}}^{\rho, \sigma}(T) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{f(s) - f(t)}{s - t} dE_\rho(s)T dE_\sigma(t).
\]

See [PS10] for the convergence of the above formula. We abbreviate \( Q_{\ln[1]}^{\rho, \sigma}, Q_{\ln[1]}^\rho \) as \( Q_{f_{[1]}}^{\rho, \sigma}, Q_{f_{[1]}}^{\rho} \), respectively. It was also shown in [PS10] that

\[
\lim_{t \to 0} \frac{f(\rho + t\sigma) - f(\rho)}{t} = Q_{f_{[1]}}^\rho(\sigma).
\]
Thus \( \tau \left( Q_{f[1]}^{\rho}(\sigma) \right) = \tau(f'(\rho)\sigma) \). For \( \rho \in \mathcal{N}_+ \), recall that the functional calculus of derivations is given by
\[
\delta(f(\rho)) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{f(s) - f(t)}{s - t} dE_\rho(s)\delta(\rho)E_\rho(t).
\]
Hence \( \delta(f(\rho)) = Q_{f[1]}^{\rho}((\delta(\rho))) \).

**Example 2.2.** Let \( f(x) = \ln(x) \), recall that \( \frac{\ln(x) - \ln(y)}{x - y} = \int_{\mathbb{R}_+} \frac{1}{(x+r)(y+r)} dr \), then
\[
Q^{\rho,\sigma}(y) = \int_{\mathbb{R}_+} (\rho + r)^{-1} y(\sigma + r)^{-1} dr.
\]
In particular ([CM17]) \( \delta(\ln(\rho)) = \int_{\mathbb{R}_+} (\rho + r)^{-1} \delta(\rho)(\rho + r)^{-1} dr. \)

Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be operator monotone and \( f_{[0]}(x,y) = f\left(\frac{x}{y}\right)y \), and we consider
\[
Q_{f[0]}^{\rho,\sigma}(T) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f_{[0]}(s,t) dE_\rho(s)TdE_\sigma(t).
\]

**Example 2.3.** Let \( f(x) = \frac{x-1}{\ln(x)} \), then \( f \) is operator monotone. Indeed \( f(x) = \int_0^1 x^r dr \) is a convex combination of operator monotone functions \( x^r \). By the integral identity \( \frac{x-y}{\ln(x) - \ln(y)} = \int_{\mathbb{R}_+} x^r y^{1-r} dr \), we obtain that
\[
Q_{f[0]}^{\rho,\sigma}(T) = \int_0^1 \rho^r T\sigma^{1-r} dr.
\]
An important observation is that \( Q^\rho(T) = Q_{f[0]}^{\rho}(T) \).

2.4. **Lieb’s concavity Theorem.** Lieb ([LR02]) proved that the map \( (A,B) \mapsto \tau(K^*A^{1-t}KB^t) \) with \( t \in [0,1] \) is jointly concave in the positive definite matrix pair \( (A,B) \), usually referred to as **Lieb’s concavity Theorem.** Petz ([Pet85]) discovered the following generalized Lieb’s concavity theorem by using the Jensen inequality of operator concave functions.

**Theorem 2.4.** Let \( \beta : \mathcal{N} \to \mathcal{N} \) be a completely positive trace preserving map and \( f : (0,\infty) \to (0,\infty) \) be an operator monotone function. Then for any \( \rho, \sigma \in \mathcal{N}_+ \), we have
\[
\beta^* Q_{f[1]}^{\beta(\rho),\beta(\sigma)} \beta \leq Q_{f[1]}^{\rho,\sigma}.
\]
Furthermore, \( (\rho, \sigma, x) \mapsto \langle x, Q_{f[0]}^{\rho,\sigma}(x) \rangle \) is jointly convex for \( \rho, \sigma \in \mathcal{N}_+ \) and \( x \in \mathcal{N} \).

**Lemma 2.5.** Let \( f : (0,\infty) \to (0,\infty) \) and \( \beta : \mathcal{N} \to \mathcal{N} \) be a completely positive trace preserving map. The conditions
\[
\beta^* Q_{f[1]}^{\beta(\rho),\beta(\sigma)} \beta \leq Q_{f[1]}^{\rho,\sigma}
\]
and
\[
\beta Q_{f[0]}^{\rho,\sigma} \beta^* \leq Q_{f[0]}^{\beta(\rho),\beta(\sigma)}
\]
are equivalent for any \( \rho, \sigma \in \mathcal{N}_+ \).

See the proof of Lemma 1 and Theorem 5 in [HP12], and they assumed that \( \rho, \sigma, \beta(\rho), \beta(\sigma) \) are invertible additionally. It is enough to assume the positivity by perturbation argument \( \rho + \varepsilon I \) for \( \varepsilon \to 0^+ \). Theorem 2.4 remains true for a larger family of functions. (For details, see [Li20].)

**Theorem 2.6.** Let \( \beta : \mathcal{N} \to \mathcal{N} \) be a completely positive trace preserving mapping and \( f(x) = x^p \), where \( p \in (0,1) \). Assume that \( \rho, \sigma \in \mathcal{N}_+ \). Then
\[
\beta^* Q_{f[1]}^{\beta(\rho),\beta(\sigma)} \beta \leq Q_{f[1]}^{\rho,\sigma}.
\]
2.5. Formulation of CLSI.

2.5.1. Quantum relative entropy. Recall that the quantum relative entropy of $\rho, \sigma \in \mathcal{N}_+$ is

$$D^\tau(\rho\|\sigma) = \begin{cases} \tau(\rho \ln(\rho)) - \tau(\rho \ln(\sigma)), & \text{if } \text{supp}(\rho) \supset \text{supp}(\sigma); \\ +\infty, & \text{otherwise}, \end{cases}$$

where supp$(\rho)$ is the support projection of $\rho$. We denote the relative entropy by $D(\rho\|\sigma)$ if the trace $\tau$ is clear from the context. Equivalently $D(\rho\|\sigma) = \lim_{\epsilon \to 0^+} D(\rho\|\sigma + \epsilon 1)$. See e.g. [Wil13] and [NC10] for more entropy properties. Relative entropy is monotone decreasing under the application of quantum channels (also known as data processing inequality)

$$D(\beta(\rho)\|\beta(\sigma)) \leq D(\rho\|\sigma),$$

where $\beta : \mathcal{N} \to \mathcal{N}$ is a completely positive trace preserving linear map. Let $\mathcal{K}$ be a von Neumann subalgebra of $\mathcal{N}$ and $E_\mathcal{K} : \mathcal{N} \to \mathcal{K}$ be the conditional expectation onto $\mathcal{K}$. The relative entropy $D_\mathcal{K}$ with respect to $\mathcal{K}$ is given by

$$D_\mathcal{K}(\rho) = D(\rho\|E_\mathcal{K}(\rho)) = \inf_{\tau(\rho) = \tau(\sigma), \sigma \in \mathcal{K}} D(\rho\|\sigma). \quad (2.4)$$

Lemma 2.7. Let $E_1, \ldots, E_m$ be pairwise commuting conditional expectations on $\mathcal{N}$. Then

$$D\left(\rho\|\left(\prod_{j=1}^m E_j\right)(\rho)\right) \leq \sum_{j=1}^m D(\rho\|E_j(\rho)).$$

Proof. Let us define the subalgebras $\mathcal{M}_{m-k} = (\prod_{j=1}^k E_j)(\mathcal{N})$, so that $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \mathcal{M}_m = \mathcal{N}$ is a filtration. Then we deduce from the data processing inequality

$$D(\rho\|E_{\mathcal{M}_0}(\rho)) = D(\rho\|E_{\mathcal{M}_1}(\rho)) + D(E_{\mathcal{M}_1}(\rho)\|E_{\mathcal{M}_0}(\rho))$$

$$= D(\rho\|E_{\mathcal{M}_1}(\rho)) + D\left(\prod_{j=1}^{m-1} E_j(\rho)\|\left(\prod_{j=1}^m E_j\right)(\rho)\right)$$

$$\leq D(\rho\|E_{\mathcal{M}_1}(\rho)) + D(\rho\|E_m(\rho)).$$

Repeating the argument for the first term $m - 1$ times yields the assertion. $\blacksquare$

Lindblad extended the relative entropy to positive functionals

$$D_{\text{Lin}}(\rho\|\sigma) = \tau(\rho \ln(\rho) - \rho \ln(\sigma) - \rho + \sigma), \forall \rho, \sigma \in \mathcal{N}_+.$$  

It follows from the definition that $D_{\text{Lin}}(\rho\|\sigma) \geq 0$, with equality if and only if $\rho = \sigma$. ( [Lin73] Thus $D(\rho\|\sigma)$ is non-negative when $\tau(\rho) \geq \tau(\sigma)$. By the nonnegativity of Lindblad relative entropy, we rewrite the relative entropy with respect to $\mathcal{K}$ as

$$D_K(\rho) = \inf_{\sigma \in \mathcal{K}} D_{\text{Lin}}(\rho\|\sigma). \quad (2.5)$$

Recall that for any finite von Neumann algebra $\mathcal{N}$, there exists a $\sigma$-finite measure space $(X, \mu)$ such that $\mathcal{Z}(\mathcal{N}) \cong L_\infty(X, \mu)$ and $\mathcal{N} = \int_X \mathcal{N}_x d\mu(x)$, where $\mathcal{Z}(\mathcal{N})$ is the center of $\mathcal{N}$ and $\mathcal{N}_x$ is a factor for any $x \in X$. Now we rewrite the relative entropy by using the direct integral

$$D_{\text{Lin}}(\rho\|\sigma) = \int_X D_{\text{Lin}}(\rho_x\|\sigma_x) d\mu(x).$$

Lemma 2.8. Let $\tau_1$ and $\tau_2$ be two normal faithful traces over a finite von Neumann algebra $\mathcal{N}$ such that $\frac{\tau_1}{\tau_2} \leq c$ for $c > 0$. For any $\rho, \sigma \in \mathcal{N}_+$,

$$D_{\text{Lin}}^{\tau_1}(\rho\|\sigma) \leq c D_{\text{Lin}}^{\tau_2}(\rho\|\sigma).$$

In particular, we have $D_{\mathcal{K}}^{\tau_1}(\rho) \leq c D_{\mathcal{K}}^{\tau_2}(\rho)$. 

[Wil13]: William F. Young, “Entropic Inequalities and Their Applications,” Communications in Mathematical Physics, 2013.

[NC10]: Narutaka Ozawa, “Entropy Inequalities and Their Applications,” Journal of Mathematical Physics, 2010.
Proof. Note that two traces only differ by two measures $\mu_1$ and $\mu_2$ over the center $L_\infty(X,\mu_1) \cong L_\infty(X,\mu_2) \cong Z(\mathcal{N})$. Also note that $\frac{d\mu_1}{d\mu_2} \leq c$ if and only if $\frac{d\mu_1}{d\mu_2} \leq c$. Again by the non-negativity of the Lindblad relative entropy, we have

$$D_{\text{Lin}}^2(\rho||\sigma) = \int_X D_{\text{Lin}}^2(\rho_x||\sigma_x)d\mu_1(x) \leq c \int_X D_{\text{Lin}}^2(\rho_x||\sigma_x)d\mu_2(x) = cD_{\text{Lin}}^2(\rho||\sigma).$$

The second assertion follows from (2.5).

2.5.2. Quantum Fisher information. The Fisher information $I_A$ of $A$ is defined by

$$I_A^\tau(\rho) = \tau(A(\rho)\ln(\rho)), \quad \forall \rho \in \text{dom}(A^{1/2}) \cap L_2(\mathcal{N}) \text{ and } \ln(\rho) \in L_\infty(\mathcal{N}).$$

Equivalently $I_A(\rho) = \lim_{\epsilon \to 0^+} \tau(A(\rho)(\ln(\rho + \epsilon1)))$. The Fisher information $I_A$ is also called the entropy production ([Spo08]). For a derivation $\delta$, the Fisher information is defined by

$$I_\delta(\rho) = \tau(\delta(\rho)Q^\rho(\delta(\rho))), \quad \forall \rho \in \text{dom}(\delta) \subset \mathcal{N}.$$ 

Then $I_A(\rho) = I_{A^\delta}(\rho)$. We use $I_A$ or $I_\delta$ if the trace is clear from the context. By Theorem 2.1 for any $A$ satisfying $\Gamma$-regularity, there exists a closable $\delta$-preserving derivation $\delta_A : \text{dom}(A^{1/2}) \to L_2(\mathcal{M})$ such that $\Gamma_A(x, y) = E_N(\delta_A(x)^*\delta_A(y))$ where $E_N : \mathcal{M} \to \mathcal{N}$. Thus

$$I_A(\rho) = I_{\delta_A}(\rho).$$

The choice of $\delta_A$ is not necessarily unique, but $I_A$ is uniquely determined by (2.1).

**Proposition 2.9.** The Fisher information $I_A$ ($I_\delta$) is non-negative and convex.

**Proof.** Recall Example 2.2 that

$$I_A(\rho) = \int_{\mathbb{R}^+} \tau(\delta_A(\rho)(\rho + r)^{-1}\delta_A(\rho)(\rho + r)^{-1}) dr.$$ 

Let us define the differential form

$$w_r = (\rho + r)^{-1/2}\delta_A(\rho)(\rho + r)^{-1/2},$$

then

$$I_A(\rho) = \int_{\mathbb{R}^+} \tau(E_N(w_r,w_r)) dr \geq 0.$$ 

The convexity was proved in [HP12] by using Theorem 2.1. Similar argument applies for $I_\delta$.

**Lemma 2.10.** Let $E : \mathcal{N} \to \mathcal{N}_{\text{fix}}$ be the conditional expectation onto the fixed-point algebra $\mathcal{N}_{\text{fix}} \subset \mathcal{N}$ of the semigroup $T_t = e^{-tA}$, then $E \circ T_t = T_t \circ E = E$. We also have

$$I_A(T_t(\rho)) = -\frac{d}{dt}D_{\text{Nfix}}(T_t(\rho)).$$

See [GJL18] for the proof. This result, especially the classical case, goes back [KL51].

**Lemma 2.11.** Let $\tau_1$ and $\tau_2$ be two normal faithful traces over a finite von Neumann algebra $\mathcal{N}$ and $\frac{d\tau_1}{d\tau_2} \geq c$ for $c > 0$. Then for any $\rho \in \mathcal{N}_+$,

$$cI_A^{\tau_2}(\rho) \leq I_A^{\tau_1}(\rho).$$

It remains true for $I_\delta$. 

Proof. We use the direct integral argument and notations in Lemma 2.8. Again \( \frac{d\mu_1}{d\mu_2} \geq c \) if and only if \( \frac{d\mu_1}{d\mu_2} \geq c \). Let us consider the pointwise differential form
\[
w_{x,r} = (\rho_x + r)^{-1/2} \delta_A(\rho_x + r)^{-1/2}.
\]
By the non-negativity of the Fisher information at any point \( x \in X \) (Proposition 2.3), we have
\[
cI_A^2(\rho) = c \int_X \int_0^\infty \tau_{2x}(E_N(w_{x,r}w_{x,r})) \, d\mu_2(x) \, dr
\]
\[
\leq \int_X \int_0^\infty \tau_{1x}(E_N(w_{x,r}w_{x,r})) \, d\mu_1(x) \, dr = I_A^2(\rho).
\]
Similar argument applies for \( I_\delta \).

2.5.3. Log-Sobolev type inequalities.

Definition 2.12. The semigroup \( T_t = e^{-tA} \) or the generator \( A \) with the fixed-point algebra \( N_{\text{fix}} \) is said to satisfy:

1. the modified log-Sobolev inequality \( \lambda \)-MLS, with respect to the trace \( \tau \): if there exists a constant \( \lambda > 0 \) such that
   \[
   \lambda D_{N_{\text{fix}}} \leq I_A, \quad \forall \rho \in \text{dom}(\delta) \cap N_+;
   \]
   (or equivalently \( D_{N_{\text{fix}}}(T_t) \leq e^{-\lambda t} D_{N_{\text{fix}}}(\rho), \forall \rho \in N_+.)

2. the complete log-Sobolev inequality \( \lambda \)-CLSI, with respect to the trace \( \tau \): if \( A \otimes \text{id}_\mathcal{F} \) satisfies \( \lambda \)-MSLI for any finite von Neumann algebra \( \mathcal{F} \).

Let CLSI\( (A, \tau) \) be the supremum of \( \lambda \) such that \( A \) satisfies \( \lambda \)-CLSI, or denoted by CLSI\( (A) \) if there is no ambiguity. We also use CLSI\( (T_t) \) for convenience. The derivation \( \delta \) is said to satisfy \( \lambda \)-MSLI \( \lambda \)-CLSI if \( \delta^* \delta \) satisfies \( \lambda \)-MSLI \( \lambda \)-CLSI. Similarly, we define CLSI\( (\delta, \tau) \) and CLSI\( (\delta) \) for the derivation \( \delta \).

An edge of CLSI over the log-Sobolev inequality for quantum systems is tensorization stability (see [GJL18]).

Proposition 2.13. Let \( T_t^j : N_j \rightarrow N_j \) be a family of semigroups with fixed-point algebras \( N_{\text{fix},j} \subset N_j \) for \( 1 \leq j \leq k \). Then the tensor semigroup \( T_t = \otimes_{j=1}^k T_t^j \) has the fixed-point algebra \( N_{\text{fix}} = \otimes_{j=1}^k N_{\text{fix},j} \). Moreover, we have
\[
\text{CLSI}(T_t) \geq \inf_{1 \leq j \leq k} \text{CLSI}(T_t^j).
\]

Combining Lemma 2.8 and Lemma 2.11, we obtain the following change of measure principle. The following observation is also referred to as Holley Stroock argument in the literature.

Theorem 2.14 (change of measure principle). Let \( \tau_1 \) and \( \tau_2 \) be normal faithful traces over \( \mathcal{N} \) and \( c_2 \leq \frac{d\tau_1}{d\tau_2} \leq c_1 \) for some \( c_1, c_2 > 0 \). Then CLSI\( (A, \tau_1) \geq \frac{c_2}{c_1} \text{CLSI}(A, \tau_2) \).

Lemma 2.15. Let \( \lambda > 0 \). If the Fisher information decays exponentially
\[
I_A(T_t(\rho)) \leq e^{-t\lambda} I_A(\rho), \quad \forall \rho \in N_+;
\]
then CLSI\( (A) \geq \lambda \).

Proof. Let \( f(t) = D_{N_{\text{fix}}}(T_t(\rho)), \) then \( f'(t) = -I_A(T_t(\rho)) \) by Lemma 2.10. Integrating both sides over \([0, \infty)\) yields the assertion.

\[\blacksquare\]
2.6. \(C^*_p\text{SI} \text{ and } \text{CLSI}^+\). Now we give a brief introduction of complete \(p\)-Sobolev type inequalities and refer to \[Li20\] for details and generalized results. In the sequel, we assume that \(p \in (1, 2)\). For any \(\rho, \sigma \in \mathcal{N}_+ \cap L_p(\mathcal{N})\), the quantum \(p\)-relative entropy is defined as
\[
d^p(\rho\|\sigma) = \tau(\rho^p - \sigma^p) - p \tau((\rho - \sigma)\sigma^{p-1}).
\]
It follows from the definition that \(d^p(\rho\|\sigma) \geq 0\), with the equality if and only if \(\rho = \sigma\). The \(p\)-relative entropy with respect to the von Neumann subalgebra \(\mathcal{K} \subset \mathcal{N}\) is defined by
\[
d^p_\mathcal{K}(\rho) = d^p(\rho\|E_\mathcal{K}(\rho)) = \inf_{\tau(\rho) = \tau(\sigma), \sigma \in \mathcal{K}} d^p(\rho\|\sigma).
\]
Lemma 2.17 remains true for the \(p\)-relative entropy, see \[Li20\] for the proof.

**Lemma 2.16.** Let \(E_1, \ldots, E_m\) be pairwise commuting conditional expectations on \(\mathcal{N}\), then
\[
d^p \left( \rho \| (\prod_{j=1}^m E_j)(\rho) \right) \leq \sum_{j=1}^m d^p(\rho\|E_j(\rho)).
\]

In \[Li20\], we defined the \(p\)-Fisher information \(I^p_A\) of the generator \(A\) of a semigroup \(e^{-tA}\) by
\[
I^p_A(\rho) = p \tau(A(\rho)\rho^{p-1}), \quad \forall \rho \in \text{dom}(A^{1/2}) \cap L_{p-1}(\mathcal{N}, \tau)
\]
and the \(p\)-information \(I^p_\delta\) of \(\delta\) by
\[
I^p_\delta(\rho) = p \tau\left( \delta(\rho)Q^\rho_{(xp-1)[1]}(\delta(\rho)) \right), \quad \forall \rho \in \text{dom}(\delta) \subset \mathcal{N}.
\]
Note that
\[
I_A(\rho) = \lim_{p \to 1^+} \frac{I^p_A(\rho)}{p - 1} \quad \text{and} \quad D_{\text{lin}}(\rho\|\sigma) = \lim_{p \to 1^+} \frac{d^p(\rho\|\sigma)}{p - 1}.
\]
See \[Li20\] for the nonnegativity and convexity of the quantum \(p\)-Fisher information.

**Definition 2.17.** The semigroup \(T_t = e^{-tA}\) or the generator \(A\) with the fixed-point algebra \(\mathcal{N}_{\text{fix}}\) is said to satisfy:

1. the modified \(p\)-Sobolev inequality \(\lambda \cdot Mp\text{SI}\) (with respect to the trace \(\tau\)) if there exists a constant \(\lambda > 0\) such that
   \[
   \lambda d^p_{\mathcal{N}_{\text{fix}}}(\rho) \leq I^p_A(\rho), \quad \forall \rho \in \text{dom}(\delta) \cap \mathcal{N}_+;
   \]
   (or equivalently \(d^p_{\mathcal{N}_{\text{fix}}}((T_t(\rho)) \leq e^{-\lambda t}d^p_{\mathcal{N}_{\text{fix}}}(\rho), \forall \rho \in \mathcal{N}_+\))
2. the complete \(p\)-Sobolev inequality \(\lambda \cdot C^*_p\text{SI}\) (with respect to the trace \(\tau\)) if \(A \otimes id_F\) satisfies \(\lambda \cdot Mp\text{SI}\) for any finite von Neumann algebra \(F\).
3. the enhanced complete log-Sobolev inequality \(\lambda \cdot \text{CLSI}^+\) if \(A\) satisfies \(\lambda \cdot C^*_p\text{SI}\) for \(p \in (1, 2)\).

Let \(C^*_p\text{SI}(A, \tau)\) (\(\text{CLSI}^+(A, \tau)\)) be the supremum of \(\lambda\) such that \(A\) satisfies \(\lambda \cdot C^*_p\text{SI}(\text{CLSI}^+)\), or denoted by \(C_p\text{SI}(A)\) (\(\text{CLSI}^+(A)\)) if there is no ambiguity. The derivation \(\delta\) is said to satisfy \(\lambda \cdot Mp\text{SI}\) (\(\lambda \cdot C^*_p\text{SI}\)) if \(\delta^* \cdot \delta\) satisfies \(\lambda \cdot Mp\text{SI}\) (\(\lambda \cdot C^*_p\text{SI}\)). Similarly we define \(C_p\text{SI}(\delta, \tau)\), \(C_p\text{SI}(\delta)\), \(\text{CLSI}^+(\delta, \tau)\), and \(\text{CLSI}^+(\delta)\) for the derivation \(\delta\).

It is enough to define \(\text{CLSI}^+\) for \(p \in (1, 1 + \varepsilon)\) for some \(\varepsilon > 0\) to retrieve the log-Sobolev inequality. The limiting case \(p \to 2^+\) reduces to the Poincaré inequality.

**Theorem 2.18.** Let \(E_\mathcal{K}\) be the conditional expectation onto the von Neumann subalgebra \(\mathcal{K} \subset \mathcal{N}\), then we have
\[
C_p\text{SI}(I - E_\mathcal{K}) \geq p.
\]
Proof. It is obvious that $\mathcal{N}_{fix} = \mathcal{K}$. By the operator concavity of $x^{p-1}$, we have

$$(E_K(\rho))^{p-1} \leq E_K(\rho^{p-1}).$$

Thus

$$d^p_{\mathcal{N}_{fix}}(\rho) = \tau(\rho^p - E_K(\rho)(E_K(\rho))^{p-1}) \leq \tau(\rho^p - E_K(\rho)E_K(\rho^{p-1})) = \frac{1}{p} I^p_{\mathcal{K}} - E_K(\rho).$$

Here is a list of important properties of $C_p\mathcal{SI}$ and refer to [Li20] for proofs.

**Theorem 2.19.** Let $T_t = e^{-tA} : \mathcal{N} \to \mathcal{N}$, then

1. $I^p_A(e^{-tA}(\rho)) = -\frac{d}{dt} d^p_{\mathcal{N}_{fix}}(e^{-tA}(\rho))$;
2. the exponential decay of Fisher information $I^p_A(T_t(\rho)) \leq e^{-t\lambda} I^p_A(\rho)$ implies $C_p\mathcal{SI}(A) \geq \lambda$;
3. CLSI$(A) \geq$ CLSI$^+(A)$;
4. $C_p\mathcal{SI}$ and CLSI$^+$ are stable under tensorization;
5. $C_p\mathcal{SI}$ and CLSI$^+$ are stable under change of measure.

Similar results remain true for $\delta$.

## 3. Derivation triple

Let $\mathcal{N}$ be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau$, and $\delta$ be a closable *-preserving derivation on $\mathcal{N}$. Suppose there exists a larger finite von Neumann algebra $(\mathcal{M}, \tau)$ containing $\mathcal{N}$ and a weakly dense *-subalgebra $\mathcal{A} \subset \mathcal{N}$ such that

1. $\mathcal{A} \subset \text{dom}(\delta)$;
2. $\delta : \mathcal{A} \to L_2(\mathcal{M}, \tau)$.

We call such $(\mathcal{N} \subset \mathcal{M}, \tau, \delta)$ a *derivation triple*. This notion is closely related to, and inspired by Connes’ notion of a spectral triple $(\mathcal{A}, H, D)$ given by a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ and a (usually unbounded) self-adjoint operator $D$ such that

$$\delta(a) = [D, \pi(a)].$$

is bounded. In classical geometry $D$ is the Dirac operator and $[D, \pi(a)] \in C_0(\mathcal{C}^{\ell}(\mathcal{M}))$, where $C_0(\mathcal{C}^{\ell}(\mathcal{M}))$ is $C^*$-bundle of Clifford algebras of the dimension of $\mathcal{M}$ over $\mathcal{M}$. This Clifford bundle admits a natural faithful trace hence is contained in the von Neumann algebra $\mathcal{C}^{\ell}(\mathcal{M})$ given by the GNS construction, see section 4.1 for more details.

Thus our notion of derivation triple requires additional conditions on the algebra of differential forms to admit a tracial state. On the other hand we allow for slightly more general derivations, because in many situations it is difficult to identify a good choice of $D$. In order to understand the role of differential forms, we recall Connes’ abstract definition

$$\Omega^1(\mathcal{A}) = \{ \sum_j (a_j \otimes b_j - 1 \otimes a_j b_j) | a_j, b_j \otimes \mathcal{A} \} \subset \mathcal{A} \otimes \mathcal{A}.$$

The induced differential representation $\pi_\delta : \Omega^1(\mathcal{A}) \to \mathcal{M}$ is defined by

$$\pi_\delta(a \otimes b - 1 \otimes ab) = \delta(ab),$$

and we denote the range $\pi_\delta(\Omega^1(\mathcal{A}))$ by $\Omega_\delta(\mathcal{A})$. Thus $\Omega_\delta(\mathcal{A})$ is Hilbert $\mathcal{A}$-bimodule with inner product

$$(\delta(a_1)b_1, \delta(a_2)b_2)_\mathcal{A} = b_1^* E_\mathcal{N}(\delta(a_1^*)\delta(a_2))b_2,$$

where $E_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ is the conditional expectation and $(\cdot, \cdot)_\mathcal{A}$ is the $\mathcal{N}$-valued inner product. Indeed, $\Omega_\delta(\mathcal{A})$ is also left $\mathcal{A}$-module since $a\delta(b) = \delta(ab)b - \delta(a)b.$
Definition 3.1. Let \((N \subset \mathcal{M}, \tau, \delta)\) and \((\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta})\) be two derivation triples with \(\tilde{E}_N : \tilde{N} \to N\). Let \(N_{\text{fix}} \subset N\) and \(\tilde{N}_{\text{fix}} \subset \tilde{N}\) be the fixed-point algebras of \(e^{-t\delta^2}\) with the corresponding conditional expectations \(E\) and \(\tilde{E}\). We say \((N \subset \mathcal{M}, \tau, \delta)\) is a sub-triple of \((\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta})\), denoted by \((N \subset \mathcal{M}, \tau, \delta) \subset (\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta})\), if the first two diagrams are commuting and the last diagram is a commuting square.

Theorem 3.2. Let \((N \subset \mathcal{M}, \tau, \delta) \subset (\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta})\), then

\[ \text{CLSI}(N \subset \mathcal{M}, \tau, \delta) \geq \text{CLSI}(\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta}) \quad \text{and} \quad \text{CpSI}(N \subset \mathcal{M}, \tau, \delta) \geq \text{CpSI}(\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta}). \]

Proof. Let \(\iota_N : L_1(N, \tau) \to L_1(\tilde{N}, \tilde{\tau})\) be the trace preserving inclusion. Then we compare \(D_{N_{\text{fix}}} (\rho)\) and \(D_{\tilde{N}_{\text{fix}}} (\rho)\).

\[
D_{N_{\text{fix}}} (\rho) = D(\tilde{E}_N \circ \iota_N (\rho) \| \tilde{E}_N \circ \iota_N (\rho)) = D(\tilde{E}_N (\iota_N (\rho)) \| \tilde{E}_N (\tilde{E}(\rho)))
\leq D(\iota_N (\rho) \| \tilde{E}(\rho)) \quad \text{(by data processing inequality)}
= D_{\tilde{N}_{\text{fix}}} (\iota_N (\rho)).
\]

Thanks to the first condition, we see that \(\iota_N (a \delta(f) b) = \iota_N (a) \delta(\iota_N (f)) \iota_N (b)\). This implies \(I_\delta (\rho) = I_{\tilde{\delta}} (\iota_N (\rho))\). The proof for \(\text{CpSI}\) is the same.

A linear operator \(Rc : \Omega_\delta (\mathcal{A}) \to \mathcal{M}\) is called the (geometric) Ricci operator of \((N \subset \mathcal{M}, \tau, \delta)\) provided that

1. \(Rc\) is bimodule over \(\mathcal{A}\)

\[ Rc(ab \rho) = a Rc(\rho)b, \quad \forall a, b \in \mathcal{A}, \rho \in \Omega_\delta (\mathcal{A}); \]

2. there exists a strongly continuous semigroup \(\tilde{T}_t = e^{-tL} : \mathcal{M} \to \mathcal{M}\) of completely positive trace preserving maps such that

\[ \Gamma_L (a, b) = E_{\mathcal{N}} (\delta^* (a) \delta (b)), \quad \forall a, b \in \mathcal{N}. \]

3. \(\delta (\tilde{a}) \in \text{dom}(L)\) if there exists \(a \in \mathcal{A}\) and \(t \geq 0\) such that \(\tilde{a} = T_t (a) \in \mathcal{A}\) and

\[ \delta (\tilde{\delta} (\tilde{a}^*) - L(\delta (\tilde{a})) = Rc (\delta (\tilde{a})). \]

The derivation \(\delta\) is said to admit a Ricci curvature \(Rc \geq \lambda\) bounded below by a constant \(\lambda\), if \((Rc(\rho), \rho)_A \geq \lambda E_{\mathcal{N}} (\rho^* \rho)\) for any \(\rho \in \Omega_\delta (\mathcal{A})\). We say the generator \(A\) of \(T_t = e^{-tA}\) admits \(Rc \geq \lambda\) if there exists a derivation triple \((\tilde{N} \subset \tilde{\mathcal{M}}, \tilde{\tau}, \tilde{\delta})\) such that

\[ \Gamma_A (a, b) = E_{\tilde{\mathcal{N}}}(\delta (a^* \delta (b)), \quad \forall a, b \in \mathcal{A}. \]

and \(\delta\) admits \(Rc \geq \lambda\). It shall be noted that the choice of \(\delta\) is not unique, thus we may find a larger Ricci lower bound of \(A\) by choosing a better \(\delta\).

3.1. **Abstract Bakry-Émery criterion.** We establish an operator-valued Bakry-Émery criterion relating the Ricci curvature and the log-Sobolev inequality.

**Theorem 3.3.** Let \((N \subset \mathcal{M}, \tau, \delta)\) be a derivation triple with \(Rc \geq \lambda > 0\). Then

\[ \text{CLSI}(N \subset \mathcal{M}, \tau, \delta) \geq 2\lambda. \]
As an application of Theorem 2.4, with the Ricci curvature $Rc$ bounded below by $\lambda$, we may define
\[ h(t) = I_\delta(\rho_t) \]
and
\[ k(t) = \left\langle \hat{T}_t(\delta(\rho)), Q^{\mu_t} \left( \hat{T}_t(\delta(\rho)) \right) \right\rangle. \]

We compute the derivatives of $h$ and $k$ by Example 2.2 and obtain:
\[ h'(t) = -2 \int_{\mathbb{R}_+} \tau (\delta(A\rho_t)(r + \rho_t)^{-1}\delta(\rho_t)(r + \rho_t)^{-1}) dr \]
\[ - 2 \int_{\mathbb{R}_+} \tau (\delta(\rho_t)(r + \rho_t)^{-1}\delta(\rho_t)(r + \rho_t)^{-1}) dr, \]
and
\[ k'(t) = -2 \int_{\mathbb{R}_+} \tau \left( L\hat{T}_t(\delta(\rho))(r + \rho_t)^{-1}\hat{T}_t(\delta(\rho))(r + \rho_t)^{-1} \right) dr \]
\[ - 2 \int_{\mathbb{R}_+} \tau \left( \hat{T}_t(\delta(\rho))(r + \rho_t)^{-1}\delta(\rho_t)(r + \rho_t)^{-1}\hat{T}_t(\delta(\rho))(r + \rho_t)^{-1} \right) dr. \]

The key observation is that the second lines of both derivatives coincide at $t = 0$. It remains to compare the first lines:
\[ h'(0) - k'(0) = -2 \int_{\mathbb{R}_+} \tau ((\delta A - L\delta)(\rho)(r + \rho)^{-1}\delta(\rho)(r + \rho)^{-1}) dr. \]

Thanks to the commutator identity (3.2), the Ricci curvature $Rc$ finally shows up
\[ h'(0) - k'(0) = -2 \int_{\mathbb{R}_+} \tau (Rc(\delta(\rho))(r + \rho)^{-1}\delta(\rho)(r + \rho)^{-1}) dr. \]

Let us define $\omega_r = (\rho + r)^{-1/2}\delta(\rho)(\rho + r)^{-1/2}$, then $\omega_r = \omega^*_r \in \Omega_\delta(A)$. By (3.1), we have
\[ Rc(\omega_r) = (\rho + r)^{-1/2}\delta(\rho)(\rho + r)^{-1/2}. \]
We rewrite $h'(0) - k'(0)$ as the trace of $\mathcal{A}$-valued inner product over $\Omega_\delta(A)$,
\[ h'(0) - k'(0) = -2 \int_{\mathbb{R}_+} \tau (\langle Rc(\omega_r), \omega_r \rangle_A) dr. \]

Since Ricci curvature $Rc$ is bounded below by $\lambda$, we deduce that
\[ h'(0) - k'(0) \leq -2\lambda \int_{\mathbb{R}_+} \tau (E_N(\omega^*_r, \omega_r)) dr = -2\lambda h(0). \]

As an application of Theorem 2.4, $k'(0) \leq 0$. Indeed, by Example 2.3
\[ k(t) = \left\langle \hat{T}_t(\delta(\rho)), Q^{\mu_t}_{\frac{\lambda}{10}} \left( \hat{T}_t(\delta(\rho)) \right) \right\rangle \]
with the $f(x) = \frac{x-1}{m(x)}$. Applying Theorem 2.4 with $\beta = \hat{T}_t$ yields
\[ k(t) \leq k(0) \]
for $t \geq 0$. Together with $k'(0) \leq 0$, we deduce that
\[ h'(0) \leq -2\lambda h(0). \]

This inequality remains true by replacing the initial state $\rho$ with $\rho_s$. Let us define
\[ h_s(t) = I_\delta(\rho_{t+s}) \]

Proof. To simplify the notation, we denote $A_\delta, T_t(\rho), \hat{T}_t(\sigma) = e^{-tL}\sigma$ by $A, \rho_t, \hat{\sigma}_t$, respectively in this proof, where $L$ is given in the definition of Ricci operator $Rc$. For any fixed $\rho \in \mathcal{N}_+$, we may consider two functions:
for fixed $s \geq 0$, then
\[ h_s'(0) \leq -2\lambda h_s(0) = -2\lambda h(0). \]
Note that $h_s'(0) = h'(s)$, and consequently for any $s \geq 0$,
\[ h'(s) \leq -2\lambda h(s). \]
By Grönwall’s lemma, this implies the exponential decay of Fisher information
\[ h(t) \leq e^{-2\lambda t} h(0). \]
Using Lemma 2.15, the theorem is established.

In the proof, we actually show that Fisher information of $(\mathcal{N} \subset \mathcal{M}, \tau, \delta)$ decays exponentially with the decay rate $2\lambda$, and this is a stronger condition than CLSI$(\mathcal{N} \subset \mathcal{M}, \tau, \delta) \geq 2\lambda$ (Lemma 2.15). See [Led11]. Theorem 3.3 remains true for CpSI and CLSI+, and we refer to [Li20] for the proof.

**Theorem 3.4.** Let $(\mathcal{N} \subset \mathcal{M}, \tau, \delta)$ be a derivation triple with $Rc \geq \lambda > 0$. For $p \in (1, 2)$, then
\[ \text{CpSI}(\mathcal{N} \subset \mathcal{M}, \tau, \delta) \geq 2\lambda. \]

Thus CLSI$(\mathcal{N} \subset \mathcal{M}, \tau, \delta) \geq 2\lambda$.

### 3.2. Connection to $\lambda$-convexity.
Otto and Villani ([OV00]) show that the Ricci curvature on a Riemannian manifold $M$ is bounded below by $\lambda \in \mathbb{R}$ if and only if the entropy is geodesically $\lambda$-convex in the space of probability measures $P(M)$ endowed with the Kantorovich metric $W_2$. Carlen and Maas use this characterization as a starting point and define the lower bound of the Ricci curvature in the noncommutative setting through a transportation condition, see [CM20] for details.

Indeed, the key ingredient of the characterization may be considered as the noncommutative adaptation of Bakry-Émery’s $\Gamma_2$ condition. We will indicate here that in finite dimension our geometric definition of Ricci curvature bounded below implies the complete lower bound of transportation definition.

**Theorem 3.5** (Carlen and Maas). A differential structure $(\mathcal{A}, \nabla, \sigma)$ has (transportation) Ricci curvature bounded from below by $\lambda > 0$ if and only if the following gradient estimate holds for $\rho \in \mathfrak{B}$, $a \in \mathcal{A}_0$ and $t \geq 0$:
\[
\| \nabla (\mathcal{P}_t a) \|_{\rho}^2 \leq e^{-2\lambda t} \| \nabla a \|_{\mathcal{P}_t}^2.
\]

Let us point out that we have chosen $\nabla = \delta$, $\mathcal{P}_t = T_t = e^{-tA}$ and $\mathcal{P}_t^\dagger = \hat{T}_t = e^{-tL}$ in our setting. However, the results remain true for every other choice of $\delta$ as well, as long as $\Gamma_A(x, y) = E(\delta(x)\sigma\delta(y))$ is still satisfied. Carlen and Maas did not consider a semigroup acting on the space of differential forms. The $\rho$-inner $\| \cdot \|_\rho$ product can be interpreted as
\[
\| \sigma \|^2_\rho = \tau \left( \sigma Q_{f[0]}^\rho(\sigma) \right),
\]
where $f(x) = \frac{1}{\ln(x)}$.

**Theorem 3.6.** Let $A = \delta^*\delta$ be a (finite dimensional) generator over $(\mathcal{N} \subset \mathcal{M}, \tau, \delta)$ with geometry Ricci curvature $Rc$ bounded below by $\lambda$. Then $A$ also satisfies (3.3), equivalently a lower bound on the transportation Ricci curvature.

**Proof.** Again, we denote $T_t(\rho)$, $\hat{T}_t(\sigma) = e^{-tL}(\sigma)$ by $\rho_t$, $\hat{\sigma}_t$, respectively, with $L$ given in the definition of Ricci operator $Rc$. We follow the spirit proof of Theorem 3.5 and define $F : [0, t] \to \mathbb{R}$ for any fixed $t > 0$,
\[
F(s) = e^{-2\lambda s} \tau \left( \delta(T_{t-s}(\sigma))Q_{f[0]}^\rho(\delta(T_{t-s}(\sigma))) \right).
\]
If $F'(s) \geq 0$ for any $s \geq 0$, then $F(0) \leq F(t)$ yields the assertion. In the rest of the proof, we want to show that $F'(s) \geq 0$. Differentiating $F(s)$, we obtain that

$$
F'(s) = -2\lambda F(s) + 2e^{-2\lambda s} \tau \left( \delta(A(\sigma_{t-s}))Q_{f[0]}^{\rho_s}(\delta(\sigma_{t-s})) \right) + e^{-2\lambda s} \left( \delta(\sigma_{t-s})D(Q_{f[0]}^{\rho_s})(\delta(\sigma_{t-s})) \right),
$$

where $D(Q_{f[0]}^{\rho_s})$ is given by differentiating $Q_{f[0]}^{\rho_s}$ in terms of $s$. It is convenient to define two more functions $h$ and $k$ for fixed $\rho$ and $\sigma$:

$$
h(x) = \tau \left( \hat{T}_x(\sigma_{t-s})Q_{f[0]}^{\rho_s} \hat{T}_x(\delta(\sigma_{t-s})) \right)
$$

and

$$
k(x) = \tau \left( \delta(\sigma_{t-s})Q_{f[0]}^{\rho_s+\epsilon}(\delta(\sigma_{t-s})) \right).
$$

Plugging $f(x) = \frac{t-x}{|t-x|}$ into Theorem 2.4 and Lemma 2.5 implies that $h(x) \leq k(x)$. Noting that $h(0) = k(0)$, we infer that $h'(0) \leq k'(0)$, i.e.,

$$(3.4) \quad -2\tau \left( L(\delta(\sigma_{t-s}))Q_{f[0]}^{\rho_s}(\delta(\sigma_{t-s})) \right) \leq \tau \left( \delta(\sigma_{t-s})D(Q_{f[0]}^{\rho_s})(\delta(\sigma_{t-s})) \right).$$

Applying (3.2) implies that

$$(3.5) \quad 2\tau \left( (\delta(A(\sigma_{t-s})) - \text{Rc}(\delta(\sigma_{t-s})))Q_{f[0]}^{\rho_s}(\delta(\sigma_{t-s})) \right) + \tau \left( \delta(\sigma_{t-s})D(Q_{f[0]}^{\rho_s})(\delta(\sigma_{t-s})) \right) \geq 0$$

Since Ricci curvature is bounded below by $\lambda$, we deduce that

$$(3.6) \quad \tau \left( \text{Rc}(\delta(\sigma_{t-s}))Q_{f[0]}^{\rho_s}(\delta(\sigma_{t-s})) \right) \geq \tau \left( \delta(\sigma_{t-s})Q_{f[0]}^{\rho_s}(\delta(\sigma_{t-s})) \right).$$

Putting pieces (3.4), (3.5) and (3.6) together, we obtain $F'(s) \geq 0$. 

**Remark 3.7.** We refer to [Wir18] for a discussion of lower bounds on the transportation Ricci curvature in infinite dimension.

### 4. Geometric Applications

#### 4.1. Clifford bundle.

Let us recall that the Clifford algebra $\mathbb{C}l_n$ is generated by $n$ self-adjoint unitaries $\{e_k\}_{k=1}^n$ satisfying

$$
e_k^* = -e_k, \quad e_k^2 = -1, \quad \text{and} \quad e_k e_l = -e_l e_k \quad \text{for} \quad k \neq l.$$

Equivalently, we may use the Clifford function from real Hilbert spaces to von Neumann algebras $c : H \to \mathbb{C}l(H)$ such that $c(h)$ is self-adjoint and

$$c(h)c(k) + c(k)c(h) = -2(h,k),$$

where $(,)$ is the inner product over $H$. Let $(M, g)$ be an $n$-dimensional smooth Riemannian manifold without boundary. Let $\mu$ be the probability measure defined by $d\mu = \frac{1}{Z_g} e^{-U} dvol$ over the manifold $M$ with $Z_g = \int_M e^{-U(x)} dvol(x)$. We may consider $\mathbb{C}l_x \cong \mathbb{C}l_n$ the Clifford algebra generated $\{c(e_k)\}$, where $\{e_k\}_{k=1}^n$ is an orthogonal basis of the cotangent space $T_x^*M$ at a point $x \in M$. Recall that $\mathbb{C}l_x$ and $T_x^*M$ are also identified as vector spaces, see [Cho39], [LM16]. We denote by $C_0(\mathbb{C}l(M))$ ($C_0^{\infty}(\mathbb{C}l(M))$) the space of continuous (respectively smooth) sections vanishing at infinity. Let $CL(M)$ be the von Neumann algebra generated by the GNS construction with respect to the trace $\tau(a) = \int \tau_x(a(x)) d\mu(x)$, where $\tau_x$ is the unique trace satisfying $\tau(c_{k_1} \cdots c_{k_m}) = 0$ for $m \leq n$ and mutually different indices $1 \leq k_1, \ldots, k_m \leq n$.

We now explain the **derivation triple for the Riemannian manifold**. Let $\nabla$ be the Levi-Civita connection (covariant derivative) and $\{X_1, \ldots, X_n\}$ be an orthonormal basis of the tangent space
TM, then $\nabla_{X_k} : C^\infty(M) \to C^\infty(M)$ defines a family of differential operators. We may combine $\{\nabla_{X_k}\}$ and $\{e_k\}$ and define

$$\delta(f) = \sum_{k=1}^n e_k \nabla_{X_k}(f), \quad \forall f \in C^\infty(M).$$

(4.1)

It is obvious that $\delta : C^\infty_0(M) \to L_2(C\ell(M), \mu)$ is a $*$-preserving closable derivation. Thus we obtain the derivation triple

$$\langle 4.2 \rangle$$

The Bakry-Émery Ricci $R_{c} \tau$ the Bochner Weitzenböck formula

$$L \mathcal{W} \text{we can also identify the restriction of } \mathcal{R}_{c} \mathcal{U} \text{ preserves semigroup (see e.g. } \mathcal{L}\mathcal{M}_{16} \text{)}, \text{i.e.}$$

remains a derivation with respect to Clifford multiplication (see e.g. [LM16]), i.e.

$$\nabla_X (f \cdot g) = f \cdot (\nabla_X g) + (\nabla_X f) \cdot g, \quad \forall X \in C^\infty(TM), f, g \in C^\infty(C\ell(M)),$$

where $\cdot$ denotes the Clifford multiplication. Let $L_{\nu} : L_2(C\ell(M), \mu) \to L_2(C\ell(M), \mu)$ be the $\mu$-modified rough (or Bochner) Laplacian

$$\langle 4.3 \rangle \quad \Delta_{\nu} f = \sum_{i=1}^{n} X_i^* X_i f + \nabla U \cdot \nabla f,$$

where $\nabla U$ is the gradient of $U$. It is well-known that $\Delta_{\nu}$ is essentially self-adjoint in $L_2(M, \mu)$. Moreover, we have $\Delta_{\nu} = \delta^* \delta$. The extended Levi-Civita connection to Clifford bundle $C\ell(M)$ remains a derivation with respect to Clifford multiplication (see e.g. [LM16]), i.e.

$$\nabla_X \mathcal{f} \in C^\infty(M) \to L_2(C\ell(M), \mu, \delta)$$

and define $\mathcal{N} \subset M, \tau, \delta$

$$\langle 4.4 \rangle \quad L_{\nu} = \sum_{i=1}^{n} \left( (\nabla_{X_i} \nabla_{X_i} - \nabla_{X_i} X_i) + \nabla U \right).$$

**Lemma 4.1.** The $\mu$-modified rough Laplacian $L_{\nu}$ is a generator of the completely positive trace preserving semigroup $\hat{T}_t = e^{-tL_{\nu}}$.

**Proof.** We combine the family of differential operators $\nabla_{X_k} : C^\infty(C\ell(M)) \to C^\infty(C\ell(M))$ with an additional (Mayorana)-Clifford operators $\tilde{e}_k \in \mathbb{M}_{2^n}$ such that

$$\tilde{e}_k = -\tilde{e}_k^* \tilde{e}_k = -1, \quad \text{and} \quad \tilde{e}_k \tilde{e}_j = -\tilde{e}_j \tilde{e}_k \quad \text{for } k \neq l$$

and define

$$\tilde{\delta}(f) = \sum_{k=1}^{n} \tilde{e}_k \otimes \nabla_{X_k}(f).$$

Note that $\tilde{\delta}$ a $*$-preserving closable derivation. Thus $L_{\nu} = \delta^* \tilde{\delta}$ yields the assertion. \hfill \blacksquare

Let $R_{c\nu} : C^\infty(T^*M) \to C^\infty(T^*M)$ be the Bakry-Émery Ricci $(1,1)$-tensor

$$R_{c\nu} = R_{c} + \nabla_{\nabla U}.$$ 

**Lemma 4.2.** The derivation triple defined by $\langle 4.2 \rangle$ admits a Ricci curvature $R_{c\nu}$ with the corresponding strongly continuous semigroup $\hat{T}_t = e^{-tR_{c\nu}}$.

**Proof.** The Bakry-Émery Ricci $R_{c\nu}$ is bimodule over $\mathcal{A} = C^\infty_0(M)$ since $R_{c\nu}$ is a $(1,1)$-tensor. By the Bochner Weitzenböck formula, we obtain that

$$\delta(\Delta_{\nu} f) = L_{\nu}(\delta f) + R_{c\nu}(\delta f), \forall f \in C^\infty(M).$$

(4.5)

We can also identify the restriction of $L_{\nu}$ to $C^\infty(M)$ with $\Delta_{\nu}$

$$L_{\nu}|_{C^\infty(M)} = \Delta_{\nu}.$$ 

Thus $R_{c\nu}$ is a Ricci operator of $(\mathcal{N} \subset M, \tau, \delta)$.
We adopt the geometric convention of Clifford algebra that $e_k^2 = 1$ and $e_k^* = -e_k$. The convention of operator algebra is to use $e_k^2 = -1$ and $e_k^* = -e_k$. It is easy to converse between the two versions by using $e_k = ic_k$.

4.2. Complete Bakry-Émery theory. We recapture the Bakry-Émery criterion for complete log-Sobolev inequality as a corollary of Theorem 3.3, and this result motivated our definition of derivation triple.

**Theorem 4.3** (Complete Bakry-Émery theorem). Let $(M, g, \mu)$ be a smooth Riemannian manifold with the measure $\mu$ defined by $d\mu = \frac{1}{Z} e^{-U} dvol$ with $Z = \int_M e^{-U} dvol$ for $U \in C^\infty(M)$. Given that $Rc_\mu \geq \kappa > 0$, then

$$\text{CLSI}(\Delta_\mu) \geq \text{CLSI}^+(\Delta_\mu) \geq 2\kappa.$$ 

Holley and Stroock [HS87] proved that the log-Sobolev inequality is stable under measure perturbation, and this property remains true for the complete log-Sobolev inequality by Theorem 2.14 as far as a central change of measure is concerned. Changing from a trace to the state is distinctly more complicated, and will not be considered in this paper. We refer to [Led11] for more applications.

**Corollary 4.4.** Let $\nu$ be the probability measure defined by $d\nu = \frac{1}{Z_{\nu}} e^{-V} dvol$ with $V \in C^\infty(M)$, where $Z_\nu$ is the normalization factor. If $\|U - V\|_\infty \leq C$, then

$$e^{2C} \text{CLSI}(\Delta_\mu) \geq \text{CLSI}(\Delta_\nu) \quad \text{and} \quad e^{2C} \text{CLSI}^+(\Delta_\mu) \geq \text{CLSI}^+(\Delta_\nu).$$

4.3. Examples and applications. For illustration purposes, we discuss some interesting examples of derivation triples and applications to Lindblad operators.

**Example 4.5.** The Laplace-Beltrami operator of any compact Riemannian manifold with a strictly positive Ricci curvature satisfies CLSI and CLSI$^+$, such as orthogonal group $O(n)$, special orthogonal group $SO(n)$, and spheres $S^n$ with $n \geq 2$.

**Example 4.6.** Let $d\gamma(x) = (2\pi)^{-n/2} e^{-U(x)} dx$ be the Gaussian measure of $\mathbb{R}^n$, where $U(x) = |x|^2/2$. Then we have the natural Bakry-Émery Ricci $Rc_\mu = I_d$. Thus

$$\text{CLSI}(\Delta_\mu) \geq \text{CLSI}^+(\Delta_\mu) \geq 2.$$ 

**Example 4.7.** Let $dx$ be the Lebesgue measure over the 1-dimensional manifold $(0, 1)$ and $\delta$ be the ordinary pointwise derivative. Then we have a derivation triple $(\mathbb{N} \subset \mathcal{M}, \tau, \delta)$, and

$$\text{CLSI}(\mathbb{N} \subset \mathcal{M}, \tau, \delta) \geq \text{CLSI}^+(\mathbb{N} \subset \mathcal{M}, \tau, \delta) \geq \frac{4}{5}.$$ 

**Proof.** Let us consider the measure

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-(x-k)^2/2} dx$$

and the embedding map

$$\pi : L_\infty(0, 1) \to L_\infty(\mathbb{R}), \pi(f)(x) = f(x \mod 1).$$

Note that we get a sub-triple of $L_\infty(\mathbb{R})$, then

$$\text{CLSI}(\mu, \delta) \geq 2.$$ 

Note that

$$\frac{\sqrt{2\pi}}{2 + 2e^{-1/2} + 2e^{-2} + \frac{8}{3} e^{-9/2}} \leq \frac{dx}{d\mu} \leq \frac{\sqrt{2\pi}}{2e^{-1/2} + 2e^{-2} + 2e^{-9/2} + \frac{48}{125} e^{-25/2}}.$$ 

For approximation details, see Appendix. Together with Theorem 2.14, it implies that

$$\text{CLSI}(\mathbb{N} \subset \mathcal{M}, \tau, \delta) \geq \frac{4}{5}.$$
Similarly CLSI$^+(\mathcal{N} \subset \mathcal{M}, \tau, \delta) \geq \frac{4}{5}$. $
exists$

**Example 4.8.** Let $\Delta$ be the Laplace-Beltrami operator over the 1-dimensional sphere $S^1 \subset \mathbb{R}^2$ with the Haar measure $\mu$, then

$$\text{CLSI}(\Delta) \geq \text{CLSI}^+(\Delta) \geq \frac{1}{5\pi^2}. $$

**Proof.** For $g \in C^\infty(S^1)$, then $\Delta(g) = \frac{d^2g}{d\theta^2}$. Let $\theta = 2\pi x$ for $x \in (0, 1)$ and $f(x) = g(2\pi x)$, and we have $f''(x) = 4\pi^2 \Delta(g)(\theta)$. Let $\delta$ be the ordinary pointwise derivative over $(0, 1)$ and $\bar{E}$ be the conditional expectation mapping $L_\infty(0, 1)$ onto constant valued functions. It is obvious that

$$D(f\|\bar{E}(f)) = D(g\|E(g)) \text{ and } I_\delta(f) = I_\Delta(g).$$

Together with Example 4.7, we obtain

$$D(g\|E(g)) \leq 5\pi^2 I_\Delta(g).$$

Similarly CLSI$^+(\Delta) \geq \frac{1}{5\pi^2}$. $
exists$

Together with the transferred argument in Section 4.3 of [GJNL18], Example 4.8 implies the CLSI of Lindblad operator with 1-generator.

**Example 4.9.** Let $L(\rho) = [x, [x, \rho]] = x^2 \rho + \rho x^2 - 2\rho xp$, where $x$ is self-adjoint with discrete spectrum in $\mathbb{Z}$. Then

$$\text{CLSI}(L) \geq \text{CLSI}^+(L) \geq \frac{1}{5\pi^2}.$$

### 4.4. Gaussian Example.

Due to initial observation of Meyer, Bakry and Émery discovered that the Ornstein-Uhlenbeck semigroup has Ricci curvature 1. Thanks to central limit type results this applies to all Gaussians functions including tracial Fermionic random variables, see also [CM17]. In the context of Clifford algebras and $\ell^2_\infty$, this observation, and the connection to canonical derivatives were discovered by [ELP08]. Junge and Zeng established a Gaussians transference in [JZ15]. Let us illustrate this in the Fermionic case, and refer to [JZ15] for the more general setup. Let $C\ell_\mathbb{N}$ be the Clifford algebra generated by a sequence of self-adjoint generators $\{c_k\}_{k \in \mathbb{N}}$ satisfying

$$c_k c_j = -c_j c_k, \quad c_k^* = c_k, \quad c_k^2 = 1.$$

The number operator $N : C\ell_\mathbb{N} \to C\ell_\mathbb{N}$ is defined by

$$N(c_{i_1} \cdots c_{i_k}) = kc_{i_1} \cdots c_{i_k}$$

whenever $i_1, \ldots, i_k$ are all different. There exists a derivation $\delta_N$ such that $N = \delta_N^* \delta_N$ and the derivation triple $(C\ell_{\mathbb{N}}, C\ell_{\mathbb{N}}, \delta_N)$ admits a Ricci curvature $\text{Rc} = \text{id}$. Using a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, we can assume that $c_{(j,k)}$ are anti-commuting Clifford generators of $C\ell_{\mathbb{N}}$. Let $\tau$ be the trace of $C\ell_{\mathbb{N}}$ obtained from GNS construction, then $\tau(c_{i_1} \cdots c_{i_k}) = 0$ if $i_1, \ldots, i_k$ are all different. For any fixed $m \in \mathbb{N}$, we define

$$u_m(c_k) = m^{-1/2} \sum_{j=1}^m c_{j,k} \otimes g_j$$

where $\{g_j\}$ are iid Gaussians. Let $\omega$ be an ultrafilter on $\mathbb{N}$ and $M^\omega = (C\ell_{\mathbb{N}} \otimes L_\infty(\mathbb{R}^m, \gamma_m))^\omega$ be the von Neumann algebraic ultraproduct with a normal faithful trace $\tau_\omega$, where $d\gamma_m(x) = (2\pi)^{-m/2} e^{-|x|^2/2} dx$. Let $u_\omega = (u_m(c_k))^\omega$ be the limit object in the ultraproduct. The central limit theorem shows that

$$\tau_\omega(u_\omega(c_{k_1}) \cdots u_\omega(c_{k_d})) = \tau(c_{k_1} \cdots c_{k_d}).$$

This means that the map $\pi$ defined by

$$\pi(c_{k_1} \cdots c_{k_d}) = u_\omega(c_{k_1}) \cdots u_\omega(c_{k_d})$$

is a derivation.
extends to a trace preserving $^*$-homomorphism of $\text{CL}_\mathbb{N}$ into $M^\omega$. Moreover, thanks to [JZ15], we note that
\[ \pi(T_t(w_A)) = (id \otimes T_t^G)^*(\pi(w_A)), \quad \forall w_A = c_1 \cdots c_k, \]
where $T_t^G$ is the Ornstein-Uhlenbeck generator corresponding to the measure $d\gamma_m(x)$ (see (4.3) and (4.4)). Since the latter has Ricci curvature 1, the same is true for $T_t^{\text{CL}_1}$. We refer to [JZ15] for the explicit derivation $\delta(c_k) = c_k$ for $\delta : \text{CL}_\mathbb{N} \to \text{CL}_\mathbb{R}^2$.

Let us point out a special case. Let \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \) and \( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \) Then $\text{CL}_2$ is generated by $c_1 = X$ and $c_2 = Y$. Then $c_1c_2 = -iZ$ implies that
\[ T_t(\alpha 1 + \beta X + \gamma Y + \zeta Z) = \alpha 1 + e^{-t\beta}X + e^{-t\gamma}Y + e^{-2t}\zeta Z \]
is a semigroup on $\text{CL}_2 = \mathbb{M}_2$ which admits constant curvature 1 and $\text{CLSI}(T_t) = \text{CLSI}^+(T_t) = 2$. Here the spectral gap estimate (1.1) is tight. It will be interesting for us to rewrite the generator of this semigroup differently. For a self-adjoint element $x$, we define
\[ L_x(\rho) = x^2\rho + \rho x^2 - 2x\rho x. \]
Let $a = \frac{x}{2}$ and $b = \frac{y}{2}$, we find that
\[ L_a(X) = 0, \quad L_a(Y) = Y, \quad L_a(Z) = Z \]
and
\[ L_b(X) = X, \quad L_b(Y) = 0, \quad L_b(Z) = Z. \]
Thus $N = L_a + L_b$ has Ricci-curvature 1. Then
\[ \text{CLSI}(L_a + L_b) \geq \text{CLSI}^+(L_a + L_b) \geq 2. \]
Noting $N$ restricted to $\ell_2^\infty$ is exactly of the form $I - E$, then $\text{CLSI}(N) \leq 2\lambda_2(N) \leq 2$.

**Corollary 4.10.** $\text{CLSI}^+(L_a + L_b) = \text{CLSI}(L_a + L_b) = 2$.

At the time of this writing the exact CLSI constant for $A_n = id - E_{\tau}$, where $E_{\tau}(x) = \frac{\text{tr}(x)}{n}1$, is given by the normalized trace is unknown. According to [BBLW12], we have $\text{CLSI}(A_n) \geq 1$.

**Corollary 4.11.** $2 \geq \text{CLSI}(A_2) \geq \text{CLSI}^+(A_2) \geq \frac{3}{2}$.

**Proof.** The role of $X$, $Y$ and $Z$ can be interchanged in the argument above. By introducing $c = \frac{Z}{2}$, we find that
\begin{align*}
6D(\rho\|E_{\tau}(\rho)) &\leq I_{L_a + L_b}(\rho) + I_{L_a + L_c}(\rho) + I_{L_b + L_c}(\rho) \\
&= 2I_{L_a + L_b + L_c}(\rho) \\
&= 4I_{E}(\rho).
\end{align*}
The last equality $L_a + L_b + L_c = 2(id - E_{\tau})$ is easily checked on the Pauli basis. The same argument works for $\text{CLSI}^+$.

**5. Anti-transference**

In this section we consider a finite dimensional compact Lie group $G$ with a normalized Haar measure $\mu$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$. Let $X = \{X_1, X_2, \ldots, X_m\} \subset \mathfrak{g}$ be a Hörmander system (see definition in the introduction), and we consider the sub-Laplacian operator $\Delta_X$ given by
\[ \Delta_X(f) = \sum_{k=1}^m X_k^*X_kf, \]
where \( X_k f (g) = \frac{d}{dt} f (\exp (tX_k) g) \big|_{t=0} \) for any \( f \in C^\infty (G) \) and \( g \in G \). Similar to (4.1), we can find a closable *-preserving derivation \( \delta \) such that \( \Delta X = \delta^* \delta \). We again identify the derivation triple

\[
(N \subset M, \tau, \delta) = (L_\infty (G, \mu) \subset \text{CLSI}(G), \mu, \delta).
\]

Let \( u : G \to U(H) \) be a strongly continuous unitary representation of the Hilbert space \( H \). We may transfer \( S_t = e^{-t \Delta X} \) to \( B(H) \) and obtain a strongly continuous semigroup \( T_{t}^{H,u} : B(H) \to B(H) \) of self-adjoint, trace preserving, and completely positive maps by using the (co-)representation map

\[
(\pi : B(H) \to L_\infty (G, B(H)), \pi (a)(g) = u(g)^* a u(g)).
\]

Let \( E \) and \( E_T \) be conditional expectations onto the fixed-point algebras of \( S_t \) and \( T_{t}^{H,u} \), respectively. The co-representation \( \pi \) allows for the commuting diagram below.

\[
\begin{array}{ccc}
L_\infty (G, B(H)) & \xrightarrow{e^{-t\Delta X \otimes id}} & L_\infty (G, B(H)) \xrightarrow{E} B(H) \\
\uparrow_{\pi} & & \uparrow_{\pi} \\
B(H) & \xrightarrow{T_{t}^{H}} & B(H) \xrightarrow{E_T} B(H)_{\text{fix.}}
\end{array}
\]

Indeed, for any \( \exp (tX_k) \), we may find the corresponding one-parameter group generated by some self-adjoint operator \( x_k \) ([KR97]) such that

\[
u(\exp (tX_K)) = \exp (itx_k).
\]

For a finite dimensional Hilbert space \( H \), the self-adjoint matrix \( x_k \) is bounded. The differential operator \( X_k \) is transferred to a commutator operator via the co-representation,

\[
X_k (\pi (a))(g) = \frac{d}{dt} u(g)^* e^{-itx_k} a e^{itx_k} u(g) \big|_{t=0} = -i \pi ([x_k, a])(g).
\]

Note that the generator of \( T_{t}^{H} \) is a Lindblad operator. Indeed,

\[
L_{X}^{H,u}(a) = \sum_{k} x_k^2 a + ax_k^2 - 2x_k a x_k.
\]

We denote the transferred semigroup and the Lindblad operator by \( T_t \) and \( L_X \) if there is no ambiguity. To emphasize the underlying Hilbert space (unitary representation), we also use \( T_{t}^{H} \) and \( L_{X}^{H} \) (\( T_t \) and \( L_{X} \), respectively).

**Theorem 5.1** (transference principle). Let \( S_t = e^{-t \Delta X} : L_{\infty} (G, \mu) \to L_{\infty} (G, \mu) \) be the semigroup generated by the sub-Laplacian \( \Delta X \) and \( T_t \) be the transferred semigroup defined as above, then

\[
\text{CLSI}(L_X) \geq \text{CLSI}(\Delta X).
\]

The transference principle extends to \( \text{CLSI}^+ \) naturally ([Li20]), then \( \text{CLSI}^+ (L_X) \geq \text{CLSI}^+ (\Delta X) \). The transference principle is developed for any ergodic and right-invariant semigroup \( S_t \) over a compact Lie group, see [GJLL18] for details.

**Theorem 5.2.** Let \( T_t = e^{-t \mathcal{L}} : B(L_2 (M)) \to B(L_2 (M)) \) be a semigroup of self-adjoint completely positive trace preserving maps such that

1. \( \mathcal{L} \) maps \( C^\infty (M) \) to \( C^\infty (M) \);
2. \( \text{dom}(\mathcal{L}|_{C^\infty (M)}) \) is dense in \( L_\infty (M) \);
3. \( \text{CLSI}^+ (\mathcal{L}) > 0 \);
4. \( \mathcal{L}(d^{-\alpha} a d^{-\beta}) = d^{-\alpha} \mathcal{L}(a) d^{-\beta} \) for any 0-th order pseudo differential operator \( a \).

Then \( \text{CLSI}^+ (L_\infty (M), \mathcal{L}) \geq \text{CLSI}^+ (B(L_2 (M)), \mathcal{L}) \).

5.1. Noncommutative Geometry.
of pseudo-differential operators is the Laplacian operator \[ \Delta = \sum_{j=1}^{n} \partial^2_{x_j^2} \] for any multi-index \( \alpha \) and is independent of \( \alpha \) and is independent of \( x \) and \( \xi \). A pseudo differential operator \( \Psi \) of order \( m \) on \( \mathbb{R}^n \) with symbol \( \sigma_\Psi \) is defined by

\[
\Psi(f)(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \sigma_\Psi(x, \xi)(Ff)(\xi) d\xi,
\]

where \( F \) is the Fourier transformation and \( \sigma_\Psi \) is a symbol of order \( m \). Let \((M, g)\) be a Riemannian manifold of dimension \( n \). Then \( \Psi : C^\infty(M) \to C^\infty(M) \) is said to be a pseudo-differential operator of order \( m \) if this is true for every local chart. (See e.g. [LSZ13]) The most prominent example of pseudo-differential operators is the Laplacian operator \( \Delta = -\left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) \) on \( \mathbb{R}^n \) with the symbol \( \sigma_\Delta(x, \xi) = 4\pi^2|\xi|^2 \). Consequently, the Laplacian power \( \Psi_m = (1 + \Delta)^{m/2} \) is a pseudo-differential operator of order \( m \) with the symbol \( \sigma_{\Psi_m}(x, \xi) = (1 + 4\pi^2|\xi|^2)^{m/2} \). In addition \( \Psi_m \) is a classical pseudo-differential operator with the asymptotic expansion

\[
\sigma_{\Psi_m}(x, \xi) \sim \sum_{j=0}^{\infty} \left( \frac{j}{m} \right) |2\pi \xi|^{m-2j}.
\]

Let us denote the Laplacian power of order \( n \) by

\[
d = (1 + \Delta)^{n/2}.
\]

For a classical, compactly supported, pseudo differential operator \( \Psi \) of order \(-n\), the Wodzicki residue \( \text{Res}_W(\Psi) \) is the integral of the principal symbol \( \sigma_{\Psi, -n} \) over the co-sphere bundle \( S^*M = (T^*M \setminus \{0\})/\mathbb{R}^+ \)

\[
\text{Res}_W(\Psi) = \frac{1}{n} \int_{S^*M} \sigma_{\Psi, -n}(V) dvol,
\]

where \( dvol \) is the volume form on \( S^*M \). The Wodzicki residue is also well-defined if we replace compactly supported with compactly based. Recall that \( \text{Res}_W([\Psi, \Phi]) \) vanishes for classical compactly based pseudo differential operators \( \Psi \) of order \( k_1 \) and \( \Phi \) of order \( k_2 \) with \( k_1 + k_2 = -n \). For any \( f \in L_\infty(G) \), the left multiplication \( M_f : L_2(G) \to L_2(G) \) is a bounded linear operator,

\[
M_f(F)(x) = f(x)F(x).
\]

Indeed, we obtain the inclusion \( L_\infty(M) \subset \mathbb{B}(L_2(M)) \) by this left regular representation. For any \( p \in \mathbb{Z} \), then \( M_{f^p} = M_f^p \). Moreover, there exists a constant \( c(n) = \frac{\text{Vol}(S^{n-1})}{n(2\pi)^n} \) such that

\[
\text{Res}_W(M_f d^{-1}) = c(n) \int_M f(x) dvol(x), \forall f \in L_\infty(M).
\]

Recall that the Malzaev ideal \( \mathcal{M}_{1, \infty} \) given by compact operators \( T \in \mathbb{B}(L_2(M)) \) such that

\[
\|T\|_{\mathcal{M}_{1, \infty}} = \sup_{n \in \mathbb{N}} \frac{1}{\ln(n+1)} \sum_{j=0}^{n} u_j(T) < \infty,
\]

where \( \{u_j(T)\} \) is the decreasing sequence of singular values of \( T \). The Laplacian power \( d \) has continuous extension in \( \mathcal{M}_{1, \infty} \), still denoted as \( d \). Every dilation invariant extended limit \( \omega \) on
\[ \ell_\infty(\mathbb{N}) \text{ defines a weight on } (\mathcal{M}_{1,\infty})_+ \]

\[ \text{Tr}_\omega(T) = \omega\left(\frac{1}{\ln(n+1)} \sum_{j=0}^{n} u_j(T)\right)_{n}. \]

This weight can be extended, by linearity, to all of \( \mathcal{M}_{1,\infty} \) and still remains tracial, and its extension on \( \mathcal{M}_{1,\infty} \) is called a Dixmier trace on \( \mathcal{M}_{1,\infty} \). The Dixmier trace \( \text{Tr}_\omega \) is non-normal and vanishes on the Schatten 1-class \( S_1 \). An operator \( T \in \mathcal{M}_{1,\infty} \) is said to be Dixmier measurable if the value \( \text{Tr}_\omega(T) \) is independent of the choice of the dilation invariant extended limit \( \omega \). Let \( T \in (\mathcal{M}_{1,\infty})_+ \) be positive Dixmier measurable, then

1. \( \lim_{n \to \infty} \frac{1}{\ln(1+n)} \sum_{j=0}^{n} \mu_j(T) < \infty; \)
2. \( \text{Tr}_\omega(T) = \lim_{t \to \infty} \frac{1}{t^\tau}(T^{1+1/t}) = \lim_{q \to 1^+} (q - 1) \tau(T^q) \) for any dilation invariant extended limit \( \omega \).

See [LSZ13] for a proof.

The coincidence between the geometric quantity-Wodzicki residue and the operator algebraic quantity-Dixmier trace was first discovered and proven by Alain Connes, known as Connes' trace theorem. Here we give a simple version of Connes' trace theorem. See [LSZ13] for a complete statement and proof.

**Theorem 5.3.** Let \( \Psi \) be a classical compactly supported pseudo-differential operator of order \(-n\). Then \( \Psi \) extends continuously to a Dixmier measurable operator in \( \mathcal{M}_{1,\infty} \). Let \( \omega \) be any dilation invariant extended limit, then

\[ \text{Tr}_\omega(\Psi) = \text{Res}_W(\Psi). \]

In particular for any \( f \in L_\infty(M) \), we have

\[ \text{Tr}_\omega(M_f d^{-1}) = \text{Res}_W(M_f d^{-1}) = c(n) \int_M f(x) d\text{vol}(x). \]

Actually it is sufficient to assume that \( \Psi \) is compactly based.

5.1.2. Applications. In the following, let \( M \) be a compact Riemannian manifold of dimension \( n \) and \( \mathcal{M} \) be a finite von Neumann algebra equipped with a normal faithful trace \( \tau_M \). Let \( \tau \) denote the normalized trace over \( \mathcal{B}(L_2(M)) \) and \( d\text{vol} = \tau \otimes \tau_M \). For \( a \in L_p(M, L_p(M)) \), define

\[ \text{tr}_M(a) = \int_M \tau_M(a(x)) d\text{vol}(x). \]

We obtain the following result as a corollary of Theorem 5.3.

**Corollary 5.4.** Let \( 1 \leq p < \infty \) and \( f \in C^\infty(M) \). Then

\[ c(n) \|f\|_p^p = \text{Tr}_\omega(|d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}|^p) = \lim_{q \to 1^+} (q - 1) \|d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}\|_{pq}^p. \]

**Proof.** We first claim that for any even integer \( p \geq 1 \)

\[ c(n) \|f\|_p^p = \text{Tr}_\omega(|d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}|^p) = \lim_{q \to 1^+} (q - 1) \|d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}\|_{pq}^p. \]

Noting that \( M_f^p d^{-1} \) and \( |d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}|^p \) have the same principal symbol of order \(-n\) [RT10], we infer that

\[ \text{Res}_W(M_f^p d^{-1}) = \text{Res}_W(|d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}|^p). \]

Together with (5.2), we obtain that

\[ c_n \|f\|_p^p = \text{Res}_W(M_f^p d^{-1}) = \text{Res}_W(|d^{-\frac{1}{2p}} M_f d^{-\frac{1}{2p}}|^p). \]

Applying Theorem 5.3 and the assertion follows for any even integer \( p \).
Then we prove the upper estimate for all $p$ by interpolation. We may assume that $c(n)\|f\|_p^p < 1$. Let $\frac{1}{p} = \frac{1-q}{p_0} + \frac{q}{p}$ and $p_0$ be an even integer. Let $\alpha(z) = \frac{1}{p_0} + \frac{1}{t}$ and consider the analytic function
\[
F(z) = uM_f e^{\alpha(z)}d^{-\alpha(z)},
\]
where $M_f = uM_f$ is given by the polar decomposition. Thus $F(\theta) = uM_f d^{-1/p} = M_f d^{-1/p}$. Note that for different values of $z = it$ the functions $F(it)$ and $F(0)$ only differ by left and right multiplications by unitaries, and
\[
|F(it)| = |F(0)| = M_f |p/p_0| d^{-1/p_0}.
\]
Applying the limit $q \to 1^+$ applies uniformly to $t$ and together with (5.3), we infer that
\[
\lim_{q \to 1^+} \sup_t |F(it)|^q_{q_0} = \lim_{q \to 1^+} (q-1)d_{(2p_0)}^{-1/2p_0}M_f^{p/p_0}d_{(2p_0)}^{-1/2p_0} ||q_{q_0} = c(n)\|f\|_p^p \leq 1.
\]
Similarly we obtain that $\lim_{q \to 1^+} (q-1)\sup_t |F(1+it)|^q_q \leq 1$. By the three line lemma (see [BL76]), we deduce that
\[
\lim_{q \to 1^+} (q-1)\|M_f d^{-1/p}\|_q^q = \lim_{q \to 1^+} (q-1)\|F(\theta)\|_q^q \leq 1.
\]
Homogeneity implies
\[
\lim_{q \to 1^+} (q-1)\|M_f d^{-1/p}\|_p^p \leq c(n)\|f\|_p^p \quad \text{and} \quad \lim_{q \to 1^+} (q-1)\|d^{-1/p} M_f\|_q^p \leq c(n)\|f\|_p^p.
\]
For the lower estimate, we assume that $\|f\|_p^p = 1$. For any $\epsilon > 0$, there exists $g \in C^\infty(M)$ such that
\[
\int |g^* f|d\text{vol}(x) \geq (1-\epsilon)
\]
and $\|g\|_{p'} \leq 1$ with $\frac{1}{p'} + \frac{1}{p} = 1$. Note that $d^{-1/p'} M_{g^*} d^{-1/p}$ is a pseudo-differential operator of order $-n$, then
\[
c(n)\int |g^* f|d\text{vol}(x) = \lim_{q \to 1^+} (q-1)\|d^{-1/p'} M_{g^*} d^{-1/p}\|_q^q.
\]
By Hölder’s inequality, then
\[
c(n)\int |g^* f|d\text{vol}(x) \leq \lim_{q \to 1^+} (q-1)\|d^{-1/p'} M_{g^*}\|_{q/q'}^q \lim_{q \to 1^+} (q-1)^{1/p}\|M_f d^{-1/p}\|_{pq}^p
\leq c(n)^{1/p'} \lim_{q \to 1^+} (q-1)^{1/p}\|M_f d^{-1/p}\|_{pq}^q.
\]
Together with $\int |g^* f|d\text{vol}(x) \geq (1-\epsilon)$, we have
\[
(1-\epsilon) \leq c(n)^{1/p'} \lim_{q \to 1^+} (q-1)^{1/p}\|M_f d^{-1/p}\|_{pq}^q.
\]
Taking the $p$-th power we deduce that
\[
(1-\epsilon)^p c(n) \leq \lim_{q \to 1^+} (q-1)\|M_f d^{-1/p}\|_{pq}^q = \lim_{q \to 1^+} \|M_f d^{-1/p}\|_{pq}^q.
\]
Thus the upper and lower estimates yield the equality.

**Remark 5.5.** We could also use the pseudo-differential calculus developed by Connes ([CS84]) to obtain the result without interpolation.
Let us recall the definition of vector-valued mixed \((p, q)\)-spaces from \cite{Pis98a, JPPP13}:

\[
L_\infty(N, L_q(M)) = [N \otimes M, L_\infty(N, L_1(M))]_{1/q}
\]

obtained by complex interpolation. We refer to \cite{JPPP13} for the fact that for elements in \(N \otimes M\) the two equivalent expressions for the norm

\[
\|f\|_{L_\infty(N, L_1(M))} = \sup_{\|a\|_2, \|b\|_2} \|(a \otimes 1)f(b \otimes 1)\|_{L_1(N \otimes M)} = \inf_{f = f_1 f_2} \|id \otimes \tau(f_1 f_1^*)\|_{N}^{1/2} \|id \otimes \tau(f_2^* f_2)\|_{N}^{1/2}
\]

coincide. Thus by interpolation, we deduce an isometric inclusion

\[
L_\infty(N_1, L_q(M)) \subset L_\infty(N_2, L_q(M))
\]

for every inclusion of von Neumann algebras \(N_1 \subset N_2\). This is in particular true for the inclusion \(L_\infty(M) \subset \mathcal{B}(L_2(M))\) given by the left regular representation,

\[
(5.5) \quad L_\infty(M, L_p(M)) \subset L_\infty(\mathcal{B}(L_2(M)), L_p(M)).
\]

Also recall Pisier’s interpolation theorem for vector-valued \(L_p\) spaces (see \cite{Pis98a, JPPP13}) that

\[
(5.6) \quad L_p(L_\infty(M) \otimes M) = L_p(M, L_p(M)).
\]

**Corollary 5.6.** Let \(f \in L_p(M, L_p(M))\), then

\[
\lim_{q \to 1^+} (q - 1)\|(d^{-\frac{1}{p}} \otimes 1)M_f(d^{-\frac{1}{q}} \otimes 1)\|_{pq}^{1/p} = c(n)\|f\|_{L_p(L_\infty(M) \otimes M)}^p.
\]

**Proof.** It is sufficient to consider the vector valued function \(f \in L_\infty(M, M)\). Indeed, we may extend this result to all of \(L_p(L_\infty(M) \otimes M)\) using the Banach space \(\prod_d L_p(\mathcal{B}(L_2(M)) \otimes M)\).

Now let \(f \in L_\infty(M, M)\), then \(f \in L_\infty(L_\infty(M) \otimes M) \subset L_p(L_\infty(M) \otimes M)\). For any \(\varepsilon > 0\), there exists \(p_0 > p\) such that

\[
\|f\|_{L_p(L_\infty(M) \otimes M)} \leq \|f\|_{L_p(L_\infty(M) \otimes M)} \leq (1 + \varepsilon)\|f\|_{L_p(L_\infty(M) \otimes M)}.
\]

Thus there exist \(f_1, f_2 \in L_{2p_0}(M)\) and \(F \in L_\infty(M, L_{p_0}(M))\) such that \(f = (f_1 \otimes 1)F(f_2 \otimes 1)\), max\{\(\|f_1\|_{2p_0}\|f_2\|_{2p_0}\}\} ≤ \|f\|_{L_{p_0}(L_\infty(M) \otimes M)}^{1/2}\) and \(\|F\|_{p_0} ≤ 1\), where

\[
\|F\|_{p_0} = \int_M \tau_M(\|F(x)\|_{p_0})^{1/p_0} dvol(x).
\]

Indeed, the functions

\[
f_1(x) = f_2(x) = \tau_M(\|f(x)\|_{p_0})^{1/p_0}
\]

will do the job. The inclusion result (5.5) implies that \(M_F \in L_\infty(\mathcal{B}(L_2(M)), L_{p_0}(M))\). Since \(p < p_0\), we apply Corollary 5.4 to \(f_1 f_1^*\) and \(f_2 f_2^*\) and continuity of \(L_p\) spaces, then

\[
\lim_{q \to 1^+} ((q - 1)\|(d^{-\frac{1}{p}} \otimes 1)M_{f_1 f_1^*}d^{-\frac{1}{q}}\|_{pq}^{1/p})^{1/2} \leq c(n)\|f_1 f_1^*\|_p^{1/2}\|f_2 f_2^*\|_p^{1/2} \leq (1 + \varepsilon)c(n)\|f\|_p^p.
\]

By \cite{Pis98b} we find that

\[
\lim_{q \to 1^+} (q - 1)\|(d^{-\frac{1}{p}} \otimes 1)M_f(d^{-\frac{1}{q}} \otimes 1)\|_{pq}^{1/p} \leq \limsup_{q \to 1^+} (q - 1)\|(d^{-\frac{1}{p}} M_{f_1})^{qp}\|_{2pq} \leq (1 + \varepsilon)c(n)\|f\|_p^p.
\]
Thus
\[ \lim_{q \to 1^+} (q - 1) \| (d^{-\frac{1}{p^*}} \otimes 1) M_f (d^{-\frac{1}{p^*}} \otimes 1) \|_{pq}^p \leq (1 + \varepsilon)^p c(n) \| f \|_p^p. \]

Sending \( \varepsilon \) to 0 yields the upper bound. The same interpolation argument as in (5.4) also shows the lower bound by duality. 

**Lemma 5.7.** Let \( a, b \in L_p(M, L_p(M)) \) for \( p \in (1, 2) \). Let \( a \) be positive and \( b \) be self-adjoint. Define \( A = (d^{-\frac{1}{p^*}} \otimes 1) M_a (d^{-\frac{1}{p^*}} \otimes 1) \) and \( B = (d^{-\frac{1}{p^*}} \otimes 1) M_b (d^{-\frac{1}{p^*}} \otimes 1) \). If there exists \( C > 0 \) such that \(-Ca \leq b \leq Ca\), then

\[ \lim_{q \to 1^+} (q - 1) \operatorname{tr}(BA^p q^{-1}) \leq c(n) \operatorname{tr}_{\mathcal{M}}(ba^p q^{-1}). \]

**Proof.** Let \( t \geq 0 \) and \( tC \leq 1 \), then \( tb + a \leq 1 \). Applying Corollary 5.6 to \( a + tb \) and \( a \) implies

\[ \lim_{q \to 1^+} (q - 1) \| A + tB \|_{pq}^p = c(n) \| a + tb \|_{L_p(L_\infty(M) \otimes \mathcal{M})}^p, \]

and

\[ \lim_{q \to 1^+} (q - 1) \| A \|_{pq}^p = c(n) \| a \|_{L_p(L_\infty(M) \otimes \mathcal{M})}^p. \]

Noting that \( \| \|_{pq}^p \) is convex for \( q \geq 1 \) small enough, we obtain

\[ p q \operatorname{tr}(BA^p q^{-1}) \leq \frac{\| A + tB \|_{pq}^p - \| A \|_{pq}^p}{t}. \]

Together with (5.8) and (5.9), we have

\[ \lim_{q \to 1^+} (q - 1) \operatorname{tr}(BA^p q^{-1}) \leq \frac{c(n) \left( \| a + tb \|_{L_p(L_\infty(M) \otimes \mathcal{M})}^p - \| a \|_{L_p(L_\infty(M) \otimes \mathcal{M})}^p \right)}{tp}. \]

Using the differentiation formula for the \( p \)-norm, we observe that

\[ \| a + tb \|_{L_p(L_\infty(M) \otimes \mathcal{M})}^p - \| a \|_{L_p(L_\infty(M) \otimes \mathcal{M})}^p = t \int_0^1 p \operatorname{tr}_{\mathcal{M}}(b(a + stb)^{p-1}) ds \]

\[ = tp \operatorname{tr}_{\mathcal{M}}(ba^{p-1}) + tp \int_0^1 \operatorname{tr}_{\mathcal{M}}(b((a + stb)^{p-1} - a^{p-1})) ds. \]

Thus it suffices to show that

\[ \int_0^1 \operatorname{tr}_{\mathcal{M}}(b((a + stb)^{p-1} - a^{p-1})) ds = O(t) \]

as \( t \to 0 \), then sending \( t \to 0 \) implies the assertion. Indeed, we decompose \( b = b^+ - b^- \) for positive \( b^+ \) and \( b^- \). Using the monotonicity of \( x \mapsto x^{p-1} \) and \( b \leq Ca \) we deduce

\[ \operatorname{tr}_{\mathcal{M}}(b^+ a^{p-1}) \leq \operatorname{tr}_{\mathcal{M}}(b^+(a + stb)^{p-1}) \leq (1 + stC)^{p-1} \operatorname{tr}_{\mathcal{M}}(b^+ a^{p-1}). \]

The same argument applies for \( b^- \) and hence

\[ | \operatorname{tr}_{\mathcal{M}}(b((a + stb)^{p-1} - a^{p-1})) | \leq ((1 + stC)^{p-1} - 1) \operatorname{tr}_{\mathcal{M}}(|b|a^{p-1}) \leq (p - 1) stC \operatorname{tr}_{\mathcal{M}}(|b|a^{p-1}). \]

Integrating the inequality above yields (5.10). 

Now we are ready to prove the main theorem.
Proof. Let $1 < p < 2$ and $p < q < 2$. Let $a : M \to \mathcal{M}$ be a smooth positive function and $a_\epsilon = a + \epsilon 1$ for $\epsilon > 0$. Then $b = \mathcal{L}(a_\epsilon) = \mathcal{L}(a)$ since $\mathcal{L}$ is self-adjoint. Let $C = \epsilon^{-1}\|\mathcal{L}(a)\|_{L_\infty(L_\infty(M)\otimes M)}$; then $-Ca_\epsilon \leq b \leq Ca_\epsilon$. Let $A_\epsilon = (d^{-\frac{1}{2p}} \otimes 1)M_{a_\epsilon}(d^{-\frac{1}{2p}} \otimes 1)$ and $B = (d^{-\frac{1}{2p}} \otimes 1)M_b(d^{-\frac{1}{2p}} \otimes 1)$. It follows from Lemma 5.7 that

$$\lim_{q \to 1^+} (q - 1) \text{tr}(BA_\epsilon^{pq}) \leq c(n) \text{tr}_\mathcal{M}(ba_\epsilon^{pq}).$$

Noting $(\mathcal{L} \otimes id_\mathcal{M})(A_\epsilon) = B$ since $(\mathcal{L}(d^{-\alpha_1} \otimes 1) = d^{-\alpha_1} \mathcal{L}(d^{-\alpha_1} \otimes 1)$, we have

$$\|A_\epsilon \|_{pq} \leq \| (d^{-\frac{1}{2p}} \otimes 1)M_{E(a_\epsilon)}(d^{-\frac{1}{2p}} \otimes 1)\|_{pq} \leq \text{C\textsc{Lsi}}^+(\mathbb{B}(L_2(M)), \mathcal{L}) \text{tr}_\mathcal{M}(BA_\epsilon^{pq}).$$

Together with (5.11), we obtain

$$\lim_{q \to 1^+} (q - 1) \left( \|A_\epsilon \|_{pq} - \| (d^{-\frac{1}{2p}} \otimes 1)M_{E(a_\epsilon)}(d^{-\frac{1}{2p}} \otimes 1)\|_{pq} \right) \leq \frac{pc(n)}{\text{C\textsc{Lsi}}^+(\mathbb{B}(L_2(M)), \mathcal{L})} \text{tr}_\mathcal{M}(ba_\epsilon^{pq}).$$

Applying Corollary 5.6 to $(d^{-\frac{1}{2p}} \otimes 1)M_{a_\epsilon}(d^{-\frac{1}{2p}} \otimes 1)$ and $(d^{-\frac{1}{2p}} \otimes 1)M_{E(a_\epsilon)}(d^{-\frac{1}{2p}} \otimes 1)$ implies

$$\|a_\epsilon \|_{L_p(L_\infty(M)\otimes M)} - \|E(a_\epsilon)\|_{L_p(L_\infty(M)\otimes M)} \leq \frac{p}{\text{C\textsc{Lsi}}^+(\mathbb{B}(L_2(M)), \mathcal{L})} \text{tr}_\mathcal{M}(\mathcal{L}(a)a_\epsilon^{pq}).$$

The left hand side is continuous in $\epsilon$. By functional calculus and the dominated convergence theorem for the sequence of functions $g_\epsilon(x) = (x + \frac{1}{k})^{p-1}$, we deduce that

$$\lim_{k \to \infty} \text{tr}_\mathcal{M}(b(a + \frac{1}{k})^{p-1}) = \lim_{k} \int_{\mathbb{R}} g_\epsilon(x) \, d\mu_b(x) = \text{tr}_\mathcal{M}(ba^{p-1}),$$

where we use the spectral measure $\int f(x) \, d\mu_b(x) = (b_1, f(x)b_2)$ given by a decomposition $b = b_1b_2^*$ with $b_1, b_2 \in L_2(\mathcal{M})$. For a arbitrary $x \in \mathbb{R}$, then

$$\lim_{\epsilon \to 0} \text{tr}_\mathcal{M}(\mathcal{L}(a)(a + \epsilon)^{p-1}) = \text{tr}_\mathcal{M}(\mathcal{L}(a)a^{p-1}).$$

Thus (5.12) does indeed imply

$$\text{C\textsc{Lsi}}^+(\mathbb{B}(L_2(M)), \mathcal{L}) \leq C_p\text{S\textsc{i}}(L_\infty(M), \mathcal{L}).$$

Taking the infimum over $p > 1$, then yields the assertion.

5.2. Collective Lindbladian. Let $\mathcal{X} = \{X_1, \ldots, X_d\}$ be a set of self-adjoint matrices in $\mathbb{M}_n$, hence $i\mathcal{X} \subset \mathfrak{su}(n)$ is a subset of the Lie algebra of $SU(n)$. Let us first consider a general Lindbladian

$$L_\mathcal{X} = \sum_{k=1}^d [X_k, [X_k, \cdot]].$$

This also allows us to define the right invariant differential operators $X_k(f)(g) = \frac{d}{dt} f(e^{itX_k}g)$ with sub-Laplacian

$$\Delta_\mathcal{X} = -\sum_k X_k^2.$$

In some cases this differential operator is only ergodic for a Lie-subgroup $G \subset SU(n)$, and then we add the relevant group $\Delta_{\mathcal{X}, G}$ in the notation. Let us fix the notation $H = \ell_2^n$. For a arbitrary
representation \( u : G \to U(\mathcal{H}) \), we may then define new Lie-derivatives
\[
X_k^H = \frac{d}{dt} u(e^{itX_k})|_{t=0}
\]
and obtain the individual Lindbladian
\[
L_{X_k}^H(\rho) = [X_k, [X_k, \rho]]
\]
and the transferred Lindbladian
\[
L_X^H = \sum_k L_{X_k}^H.
\]
We deduce from the Peter-Weyl theorem \([BD95]\) that all finite-dimensional irreducible representations \( u \) are contained in \( \mathcal{K} = H^{\otimes m_1} \otimes \bar{H}^{\otimes m_2} \) for some \( m_1 \) and \( m_2 \). Let us determine the corresponding Lindbladian by differentiation. Then we consider \( u_m(g) = u(g) \otimes \pi \) and deduce that
\[
X_k^{(m)} = \left. \frac{d}{dt} u(e^{itX_k}) \right|_{t=0} = \sum_{j=1}^m \pi_j(X_k),
\]
where \( \pi_j(a) = 1^{\otimes (j-1)} \otimes a \otimes 1^{\otimes (n-j)} \). The adjoint representation is given by \( \bar{u}(g) = u(g^{-1})^{\text{trans}} = \bar{u}(g) \).
This means that
\[
X_k^{\text{trans}} = \frac{d}{dt} u(e^{-itX_k}) \big|_{t=0} = \frac{i}{t} \sum_{j=1}^m \pi_j(-itX_k)
\]
\[
= \sum_{j=1}^m \pi_j(X_k) = \sum_{j=1}^m \pi_j((X_k^{\text{trans}})^*) = \sum_{j=1}^m \pi_j(X_k^{\text{trans}}).
\]
Let us therefore define
\[
\hat{L}_X^H = \sum_k [X_k^{\text{trans}}, [X_k^{\text{trans}}, \cdot]].
\]
Let us now introduce the diagonal representation \( \hat{a}(g) = \begin{pmatrix} u(g) & 0 \\ 0 & \bar{u}(g) \end{pmatrix} \) on \( H \oplus \bar{H} \) and \( \hat{X}_k = \text{diag}(X_k, X_k^{\text{trans}}) \). The combined collective Lindbladians are given by
\[
\hat{L}_X^m = \sum_{j=1}^m [\pi_j(\hat{X}_k), [\pi_j(\hat{X}_k), \cdot]]
\]
The corresponding generator for the system is denoted by \( \hat{L}_X^m = \sum_{k=1}^d \hat{L}_{X_k}^m \).

**Remark 5.8.** CLSI\((\hat{L}_X^H) = \text{CLSI}(L_X^H) \). The same holds for CLSI\(^+\).

**Lemma 5.9.** CLSI\(^+(\Delta_X) = \inf_m \text{CLSI}(\hat{L}_X^m) \).

**Proof.** In view of Theorem 5.2 it suffices to control \( L_{X}^{L_2(G)} \). By Peter-Weyl theorem
\[
L_2(G) = \oplus_\pi (H_\pi \otimes \bar{H}_\pi)
\]
is given by the sum of equivalence classes of irreducible representations. Here we use left regular representation \( \lambda : G \to U(L_2(G)) \) given by \( \lambda_g(f)(h) = f(g^{-1}h) \), which corresponds to \( (u_\pi(g) \otimes 1_\pi) \). Since \( \lambda(g) \) commutes with the spectral projections onto \( \oplus_{\pi \in F} (H_\pi \otimes \bar{H}_\pi) \) for any finite set \( F \), it suffices to consider
\[
H_F = \oplus_{\pi \in F} H_\pi
\]
via the completeness. Again thanks to the Peter-Weyl theorem we can find \( m_1(\pi) \) and \( m_2(\pi) \) such that
\[
H_\pi \subset H^{\otimes m_1(\pi)} \otimes \bar{H}^{\otimes m_2(\pi)}.
\]
For any \( F \), there exists a large integer \( m(F) \) such that
\[
H_F \subset (H \otimes \bar{H})^{\otimes m(F)}.
\]
Indeed \( m(F) = \max_{\pi \in F} \{m_1(\pi)\} + \max_{\pi \in F} \{m_2(\pi)\} \). Using the distributive law, we observe that

\[
(H \otimes \bar{H})^{m(F)} = \bigoplus_{A \subseteq \{1, \ldots, m(F)\}} (H^A \otimes \bar{H}^{A^c}),
\]

where \( H^A \) stands for tensor in \( H \) at the position given by the set \( A \). This shows that

\[
\text{CLSI}^+(L^H_X) \geq \text{CLSI}^+(\bar{L}^m_X(F)).
\]

By taking the infimum over \( m \) and the infimum over \( F \), we obtain a lower bound for \( L^{L_2(G)}_X \).

Let us point out that for a tensor product \( H^{\otimes m} \), the induced Lindbladian in general does not coincide with the tensor product Lindbladian

\[
L^m_X(\rho) = [X^m, [X^m, \rho]] = \sum_{j,k} [\pi_j(X), [\pi_k(X), \rho]] = \sum_{j} [\pi_j(X), [\pi_j(X), \rho]] .
\]

6. Connected Graphs

In this section, we study CLSI constants and stability properties of connected graphs. Let \( G = (V, E, \mu, w) \) be a connected graph with \( V = (v_1, v_2, \ldots, v_n) \), where \( \mu : V \to (0, 1) \) is a probability measure and \( w \) is a symmetric weight function over the edges. Let

\( V(D) = (v_1, \ldots, v_1, v_2, \ldots, v_n, \ldots, v_n) \)

be an ordered set with degree \((v_i)\) copies of \( v_i \)'s. We define the derivation

\[
\delta : L_\infty(V) \to L_\infty(V(D)), f \mapsto (\delta(f)(v_1), \ldots, \delta(f)(v_n)),
\]

where

\[
\delta(f)(v_r) = \left(\sqrt{w_{s_1}}(f(v_{s_1}) - f(v_r)), \ldots, \sqrt{w_{s_k}}(f(v_{s_k}) - f(v_r))\right)_{s_1 < \ldots < s_k; (v_{s_j}, v_r) \in E} \text{ and } 1 \leq j \leq k.
\]

We define the left and right representations \( \pi_{1,2} : L_\infty(V) \to L_\infty(V(D)) \) by

\[
\pi_{1,2} ; \pi_{1,2}(f) \mapsto (\pi_{1,2}(f)(v_1), \ldots, \pi_{1,2}(f)(v_n)),
\]

where

\[
\pi_1(f)(v_r) = (f(v_{s_1}), \ldots, f(v_{s_k}))_{s_1 < \ldots < s_k; (v_{s_j}, v_r) \in E} \text{ and } 1 \leq j \leq k
\]

and

\[
\pi_2(f)(v_r) = (f(v_r), \ldots, f(v_r))_{\text{length}=\text{degree}(v_r)}.
\]

Thus \( \delta \) satisfies the Leibniz rule

\[
\delta(fg) = \pi_1(f) \cdot \delta(g) + \delta(f) \cdot \pi_2(g),
\]

where \( \cdot \) is entry-wise multiplication. The Fisher information \( I_{\delta,w}^\mu \) of \( f \) is defined by

\[
I_{\delta,w}^\mu(f) = \sum_{x \in V} I_x(f) \mu(x),
\]

where \( I_x(f) \) is the pointwise Fisher information at \( x \) defined by

\[
I_x(f) = \sum_y \tau (w_{yx}(f(y) - f(x))(\ln(f(y)) - \ln(f(x))) .
\]

We may further decompose \( I_x(f) \) as \( I_x(f) = \sum I_{y,x}(f) \) with the edge Fisher information \( I_{y,x} \) defined by

\[
I_{y,x}(f) = w_{yx} \tau ((f(y) - f(x))(\ln(f(y)) - \ln(f(x)))) .
\]

We use \( \delta(f) \) and \( I_{\delta}(f) \) if the weight and the measure are clear from the context. The connected graph \( G \) constitutes a concrete example of derivation triple. Indeed, let \( \mathcal{N} \) be \( L_\infty(V, \mu) \) and \( \mathcal{M} \) be bounded sections of the discrete Clifford bundle. We are particularly interested in regular-weighted \((w_{xy} = 1, \forall (x, y) \in E)\) graphs with a uniform distribution over the vertices, denoted by \( G = (V, E) \).
Let $G = (\mathcal{Y}^\circ, \mathcal{E}^\circ, \mu, w)$ denote a cyclic graph $(\mathcal{Y}^\circ, \mathcal{E}^\circ)$ with a probability measure $\mu$ and a weight function $w$, and $G^\circ = (\mathcal{Y}^\circ, \mathcal{E}^\circ)$ denote a regular-weighted cyclic graph $(\mathcal{Y}^\circ, \mathcal{E}^\circ)$ with a uniform distribution.

**Lemma 6.1.** Let $g(t) = (1-t)\rho + t\sigma$ for $t \in [0,1]$, then

$$\int_0^1 \tau \left( (\rho - \sigma)Q^{g(t)}(\rho - \sigma) \right) dt = \tau((\rho - \sigma)(\ln(\rho) - \ln(\sigma))).$$

**Proof.** Noting $g'(t) = \sigma - \rho$, we obtain that

$$\int_0^1 \tau((\rho - \sigma)Q^{g(t)}(\rho - \sigma)) dt = \int_0^1 \tau(g'(t)Q^{g(t)}(g'(t))) dt = \int_0^1 \tau(g'(t)(\ln(g(t))))' dt$$

By integration by parts, then

$$\int_0^1 \tau(g'(t)(\ln(g(t))))' dt = \int_0^1 \tau(g''(t)\ln(g(t))) dt + \tau(g'(t)\ln(g(t)))|_0^1$$

$$= \tau((\rho - \sigma)(\ln(\rho) - \ln(\sigma))).$$

We can give concrete estimates of $\text{CLSI}(G^\circ)$ and $\text{CLSI}^+(G^\circ)$.

**Lemma 6.2.** Let $G^\circ = (\mathcal{Y}^\circ, \mathcal{E}^\circ)$ be a regular-weighted cyclic graph with a uniform distribution over the vertices and $\mathcal{Y}^\circ = \{1, 2, \ldots, n\}$, then we have

$$\text{CLSI}(G^\circ) \geq \text{CLSI}^+(G^\circ) \geq \frac{16}{45\pi^2}.$$

**Proof.** Let $I_1 = (0, \frac{1}{n})$ and $I_k = [\frac{k-1}{n}, \frac{k}{n})$ for $k > 1$. We further divide $I_k = I_k^1 \cup I_k^2 \cup I_k^3$ into three intervals of equal length $\frac{1}{3n}$. Let $f : \{1, 2, \ldots, n\} \rightarrow \mathcal{M}$ be a function with values in a finite von Neumann algebra $\mathcal{M}$ such that $c \leq f(k) \leq c^{-1}$ for some $c > 0$. Let

$$f(n+1) = f(1) \quad \text{and} \quad f(0) = f(n),$$

then we may define a function

$$F(t) = \begin{cases} \frac{3n}{2}(f(k) - f(k-1)) + \frac{3k-2}{2}f(k-1) - \frac{3k-4}{2}f(k), & \text{for } t \in I_k^1; \\ f(k), & \text{for } t \in I_k^2; \\ \frac{3n}{2}(f(k+1) - f(k)) + \frac{3k+1}{2}f(k) - \frac{3k-1}{2}f(k+1), & \text{for } t \in I_k^3. \end{cases}$$

We conclude that $c \leq F(t) \leq c^{-1}$ and $E_{\text{fix}}(F) = E_{\text{fix}}(F)$. (Note $F$ is not differentiable, then we consider the convolution $F * g_m$ for the dilation $g_m(x) = mg(mx)$ with support in $[-\frac{1}{m}, \frac{1}{m}]$.) Let $\xi = E_{\text{fix}}(f)$ and $I = \bigcup_k I_k^2$, then

$$D_{\text{lin}}(f||\xi) = 3 \int I [\tau(F(t)\ln F(t)) - \tau(F(t)\ln \xi) + \tau(\xi) - \tau(F(t))] dt.$$

By the nonnegativity of the Lindblad relative entropy we obtain that

$$D_{\text{lin}}(f||\xi) \leq 3D_{\text{lin}}(F||\xi).$$

Now we discuss the Fisher information term

$$I_\delta(F) = \int_0^1 \tau \left( F'(t)Q^{F(t)}(F'(t)) \right) dt,$$
where $\tilde{\delta}$ is the ordinary derivative. Indeed, we may consider $I_\delta(F * g_m)$ which is well-defined and vanishes on $I_k^2$. Assume that the double operator integral is uniformly bounded, and so is $(F * g_m)'(t)$. This implies that
\[
\lim_{m} I_\delta(F * g_m) = \int_{I'} \tau \left( F'(t)QF'(t) \right) dt,
\]
where $I' = (0, 1) \setminus I$. Without loss of generality, let us consider $I_1^3 \cup I_2^3 = \{ \frac{2}{3n}, \frac{1}{3n} \}$. Let $f(1) = \rho$ and $f(2) = \sigma$ and define
\[
a(s) = (1 - s)\rho + s\sigma.
\]
Then $F(t) = a(s)$ with the substitution $s = \frac{3n}{2}t - \frac{2}{3n}$ for $t \in [\frac{2}{3n}, \frac{1}{3n}]$. Thus
\[
\int_{\frac{2n}{3}}^{\frac{4n}{3}} \tau \left( F'(t)QF'(t) \right) dt = \frac{3n}{2} \int_0^1 \tau \left( (\rho - \sigma)Qa(s)(\rho - \sigma) \right) ds.
\]
Applying Lemma 6.4, we have
\[
\int_0^1 \tau \left( (\rho - \sigma)Qa(s)(\rho - \sigma) \right) ds = I_{2,1}(f).
\]
Recall that
\[
I_\delta(f) = \frac{1}{n} \sum_{j=1}^n I_j(f) = \frac{1}{n} \sum_{j=1}^n (I_{j+1,j}(f) + I_{j-1,j}(f)).
\]
Summing over all these intervals, we obtain
\[
\int_{I'} \tau \left( F'(t)QF'(t) \right) dt = \frac{3n^2}{4} I_\delta(f).
\]
Together with Example 4.7, we conclude that
\[
D(f; \| \xi \|) \leq 3D(F; \| \xi \|) \leq \frac{15}{4} I_\delta(F) \leq \frac{45n^2}{16} I_\delta(f).
\]
The same argument applies for $p$-entropy and $p$-Fisher information.

**Definition 6.3.** $\tilde{\mathcal{G}} = \tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{\mu}, \tilde{w}$ is said to be a cover of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mu, w)$ via $\phi$ (or $\mathcal{G}$ is covered by $\tilde{\mathcal{G}}$) if there exists a surjective map $\phi : \tilde{\mathcal{V}} \to \mathcal{V}$ satisfying the following conditions:

1. edge preserving, i.e., $(\phi(x), \phi(y)) \in \mathcal{E}$ if $(x, y) \in \tilde{\mathcal{E}}$;
2. measure preserving, i.e., $\mu(x) = \sum_{\phi(y) = x} \tilde{\mu}(y)$;
3. weight preserving, i.e., $w_{\phi(x), \phi(y)} = w_{xy}$ if $(x, y) \in \tilde{\mathcal{E}}$, where $m_{\phi}(\phi(x), \phi(y))$ is the number of the preimages of $(\phi(x), \phi(y))$ under the function $\phi \times \phi$.

We denote the number of preimages $\phi^{-1}(x)$ by $m_{\phi}(x)$ and $\max_{\phi} = \max_{x} m_{\phi}(x)$. Define the embedding map
\[
\pi : L_\infty(\mathcal{V}, \mu) \to L_\infty(\tilde{\mathcal{V}}, \tilde{\mu}), f \mapsto f \circ \phi.
\]
Applying Theorem 3.2, we obtain that:

**Lemma 6.4.** Let $\tilde{\mathcal{G}}$ be a cover of $\mathcal{G}$ via $\phi$, then
\[
\text{CLSI}(\mathcal{G}) \geq \text{CLSI}(\tilde{\mathcal{G}}) \quad \text{and} \quad \text{CLSI}^+(\mathcal{G}) \geq \text{CLSI}^+(\tilde{\mathcal{G}}).
\]

By Theorem 2.14, CLSI and CLSI$^+$ of connected graphs are stable under change of measure and change of weight.

**Corollary 6.5.** For connected graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}, \mu_1, w_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, \mu_2, w_2)$ with $\frac{w_2}{w_1} \leq b$ and $c_2 \leq \frac{d_{\phi}}{d_{\mu_2}} \leq c_1$, we have
\[
\text{CLSI}(\mathcal{G}_1) \geq \frac{c_2}{c_1 b} \text{CLSI}(\mathcal{G}_2) \quad \text{and} \quad \text{CLSI}^+(\mathcal{G}_1) \geq \frac{c_2}{c_1 b} \text{CLSI}^+(\mathcal{G}_2).
\]
Proof. Noting $G_1$ and $G_2$ have the same fixed-point algebra $N_{fix}$, we only have to compare $I_{\delta,w_1}^{\mu_1}$ and $I_{\delta,w_2}^{\mu_2}$. By the definition of Fisher information, we have

$$I_{\delta,w_1}^{\mu_1}(f) \leq bI_{\delta,w_1}^{\mu_1}(f).$$

Together with Theorem 2.4.1, thus

$$D_{N_{fix}}^{\mu_1,w_1}(f) = D_{N_{fix}}^{\mu_1,w_2}(f) \leq \frac{c_1}{c_2\lambda} I_{\delta,w_1}^{\mu_1}(f) \leq \frac{c_1b}{c_2\lambda} I_{\delta,w_1}^{\mu_1}(f).$$

The same argument applies for CLSI$.^+$

In graph theory, a tree $T = (\mathcal{V}_T, \mathcal{E}_T)$ is a connected and undirected (symmetric weighted) graph with no cycles. A rooted tree, where the root is singled out, comes with a hierarchical data structure. A tree traversal is to traverse (visit) each node (vertex) in the data structure. Recall that the any vertex of degree one can be chosen as a root, and the vertices directly connected to the root are called the children of the root. A tree $T_s = (\mathcal{V}_s, \mathcal{E}_s)$ is said to be a spanning tree of a graph $(\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_s = \mathcal{V}$ and $\mathcal{E}_s \subset \mathcal{E}$. It is well-known that every connected graph has a spanning tree, and we may find the minimum spanning tree within time $O(|E|\log(|E|))$ by using Kruskal’s algorithm. [JBK56]

Lemma 6.6. Any tree $T = (\mathcal{V}_T, \mathcal{E}_T, \mu, w)$ is covered by a cyclic graph $G = (\mathcal{V}^\circ, \mathcal{E}^\circ, \mu, w)$ with $|\mathcal{V}^\circ| = 2|\mathcal{E}_T|$. Moreover, there exist $\mu'_T$ and $w'_T$ such that $T' = (\mathcal{V}_T, \mathcal{E}_T, \mu'_T, w'_T)$ is covered by a cyclic graph $G^\circ = (\mathcal{V}^\circ, \mathcal{E}^\circ)$ with $|\mathcal{V}^\circ| = 2|\mathcal{E}_T|$.

Proof. By the preorder traversal, we develop the following recursive algorithm. We start with an empty graph $(\mathcal{V}^\circ, \mathcal{E}^\circ) = (\emptyset, \emptyset)$ and define $\phi : \mathcal{V}^\circ \to \mathcal{V}_T$ in the algorithm.

step 1: Select a root $v_1$, and label it as vertex $v'_{j_1}$ for $j = 1$. (We say $v_j$ has been visited.) Define $\phi(v'_{j_1}) := v_1$, and update the vertex set $\mathcal{V}^\circ := \mathcal{V}^\circ \cup \{v'_{j_1}\}$.

step 2: If $v$ has unvisited children, select an unvisited child $v_c$ and label it as vertex $v'_{j+1}$, i.e., define $\phi(v'_{j+1}) := v_c$. If $v$ has no unvisited child, then go back to the parent $v_p$ of $v$ and label the parent again using $v'_{j+1}$, i.e., define $\phi(v'_{j+1}) := v_p$. We also record the edge $(v'_j, v'_{j+1})$.

Update the vertex set $\mathcal{V}^\circ := \mathcal{V}^\circ \cup \{v'_{j+1}\}$ and the edge set $\mathcal{E}^\circ := \mathcal{E}^\circ \cup \{(v'_j, v'_{j+1})\}$. Assign the value $(j + 1)$ to $j$.

step 3: Repeat step 2 until the root $v_1$ is visited twice.

Every edge of $\mathcal{E}_T$ is traversed twice, then $|\mathcal{V}^\circ| = |\mathcal{E}^\circ| = 2|\mathcal{E}_T|$. Note $\phi : \mathcal{V}^\circ \to \mathcal{V}_T$ is defined in the algorithm is surjective and edge preserving. Let

$$\mu(x) = \frac{1}{m_\phi(\phi(x))}\mu_T(\phi(x)) \quad \text{and} \quad w_{xy} = \frac{1}{m_\phi(\phi(x), \phi(y))}w_T(\phi(x)\phi(y)), \quad \forall x, y \in \mathcal{V}^\circ,$$

then $G = (\mathcal{V}^\circ, \mathcal{E}^\circ, \mu, w)$ is a cover of $T$. We can also define the measure and weight of $(\mathcal{V}_T, \mathcal{E}_T)$ by

$$\mu'_T(x) = \sum_{\phi(y) = x} \frac{1}{2|\mathcal{E}_T|} \quad \text{and} \quad w'_{Txy} = m_\phi(x, y),$$

then $T' = (\mathcal{V}_T, \mathcal{E}_T, \mu'_T, w'_T)$ is covered by $G^\circ = (\mathcal{V}^\circ, \mathcal{E}^\circ)$.

Theorem 6.7. Let $G = (\mathcal{V}, \mathcal{E}, \mu, w)$ be a connected graph, then

$$\text{CLSI}(G) \geq \text{CLSI}^+(G) > 0.$$

Proof. Let $T_s = (\mathcal{V}_s, \mathcal{E}_s, \mu, w^s)$ be a spanning tree of $G$. Then $T_s$ and $G$ have the same fixed-pointed algebra since $\mathcal{V}_s = \mathcal{V}$. Noting $w^s_{x,y} = w_{x,y}$ for $(x, y) \in \mathcal{E}_s$, we obtain that

$$I_{\delta,w^s}^{\mu}(f) \leq I_{\delta,w}^{\mu}(f), \quad \forall f \in L_{\infty}(\mathcal{V}, \mu).$$
Thus we have $\text{CLSI}(G) \geq \text{CLSI}(T_s)$. By Lemma 6.6 there exists $T'_s = (\mathcal{V}'_s, \mathcal{E}', \mu', w')$ covered by $G^* = (\mathcal{V}'^*, \mathcal{E}')$. By Lemma 6.4 and Lemma 6.2 we have
\[
\text{CLSI}(T'_s) \geq \text{CLSI}(G^*) \geq \frac{16}{45|\mathcal{V}'^*|^2} = \frac{4}{45|\mathcal{E}'|^2}.
\]
Applying Corollary 6.5 and Lemma 6.2 we have
\[
\text{CLSI}(G) \geq \text{CLSI}(T_s) \geq \frac{4}{45|\mathcal{E}_s|^2 || \frac{d\mu}{dt} ||_\infty || \frac{d\mu'}{dt} ||_\infty || w'||_\infty}.
\]
The same argument applies for $\text{CLSI}^+$.

**Corollary 6.8.** Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph with the maximum degree $d$ and $l$ be the number of the edges of the minimum spanning tree of the graph. Then we have
\[
\text{CLSI}(G) \geq \text{CLSI}^+(G) \geq \frac{2}{45l^2d}.
\]

### 7. From graphs to graph Hörmander systems

We extend CLSI from a commutative subsystem $\ell^\infty_n \subset \mathbb{M}_n$ to the full noncommutative system $\mathbb{M}_n$. We will perform this task for the so-called graph Hörmander system. Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ ($n \geq 2$). Here we adopt the convention in Section 6 that $G$ is regular-weighted and equipped with the uniform distribution over the vertices $\mathcal{V}$. Then $\text{L}^\infty(\mathcal{V}) = \ell^\infty_n$ is a subalgebra of $\mathbb{M}_n$ via the * homomorphism $\pi(f) = \begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(n) \end{pmatrix}$. We will work with the Lie group $SO_n$ with the normalized Haar measure $\mu$ and Lie algebra $\mathfrak{so}_n = \{a|a^{\text{trans}} = -a\}$ of real anti-symmetric matrices. Since $G$ is undirected, then $e = (v_r, v_s) = (v_s, v_r)$. Throughout this section, we represent $e = (v_r, v_s)$ using $r < s$. We define
\[
X_e = |r\rangle\langle s| - |s\rangle\langle r|, \quad \forall e = (v_r, v_s) \in \mathcal{E}.
\]
Then $X_\mathcal{E} = \{X_e, e \in \mathcal{E}\} \subset \mathfrak{so}_n$. We define the edge Laplacian $\Delta_e : C^\infty(SO_n) \to C^\infty(SO_n)$ by
\[
\Delta_e(f) = X_e^*X_ef,
\]
where $(X_e f)(a) = \frac{d}{dt} f(\exp(tX_e)a)|_{t=0}$ for any $a \in SO_n$. The sub-Laplacian $\Delta_\mathcal{E}$ is the sum of edge Laplacians
\[
\Delta_\mathcal{E} = \sum_{e \in \mathcal{E}} \Delta_e.
\]

**Lemma 7.1.** The following conditions are equivalent.

1. $\Delta_\mathcal{E}$ is ergodic, i.e. $\Delta_\mathcal{E}(f) = 0$ iff $f = \lambda 1$ for some $\lambda$;
2. $(\mathcal{V}, \mathcal{E})$ is a connected graph;
3. $X_\mathcal{E} = \mathbb{C}$, i.e. the commutant of $X_\mathcal{E}$ is trivial.

If any condition of above is satisfied, we call $X_\mathcal{E}$ a graph Hörmander system of the graph $G$. We define the Lindblad operator $L_e(\rho) = x_e^2\rho + \rho x_e^2 - 2x_e\rho x_e$, $\forall \rho \in \mathbb{M}_n$,
where $x_e = iX_e$. Then there exists a derivation $\delta_\mathcal{E}(a) = -i[x_e, a] = [X_e, a]$ such that $L_e = \delta_\mathcal{E}^*\delta_e$. We define the Lindblad operator $L_\mathcal{E}$ associated to $\mathcal{E}$ by
\[
L_\mathcal{E} = \sum_{e=(r,s)\in \mathcal{E}, r<s} L_e.
\]
Noting $L_\mathcal{E}$ restricted the diagonal is the graph Laplace operator, we have the following result. (See Appendix A for an example.)

**Proposition 7.2.** Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph, then

$$\text{CLSI}(\ell_1^{\infty}, \{\mathcal{E}\}) = \text{CLSI}(G) \quad \text{and} \quad \text{CLSI}(\ell_\infty, \{\mathcal{E}\}) = \text{CLSI}(G)$$

Let $K_{rs}$ be an $n$-by-$n$ matrix with 1 on $(r, s)$ and 0’s otherwise. For $e = (r, s) \in \mathcal{E}$, we define $\mathcal{A}_e$ to be the subalgebra of $M_n$ generated by $\{1, K_{rr}, K_{ss}, K_{rs}, K_{sr}\}$, where $k \neq r, j \neq r, s$ and $k \neq j$. Indeed $\mathcal{A}_e$ is generated by block diagonal matrices up to permutations. Let $E_e : M_n \rightarrow \mathcal{A}_e$ be the conditional expectation onto the sub-algebra $\mathcal{A}_e$ and $E_\infty$ be the conditional expectation onto the diagonal matrices. Throughout this section, let $M$ be a finite von Neumann algebra equipped with a normal faithful trace $\tau_M$.

**Lemma 7.3.** For $\rho \in M_n \otimes M$, we have

$$\text{CLSI}(M_n, L_e)D(\rho \| E_e(\rho)) \leq I_{L_e}(\rho) \quad \text{and} \quad \text{CSI}(M_n, L_e)\rho(\| E_e(\rho)) \leq I_{L_e}(\rho).$$

**Proof.** Without loss of generality we work with $e = (v_1, v_2)$. The fixed-point algebra of $L_e$ is given by the commutant $\mathcal{N}_e = \{x_{e}^{\dagger}\}$. Note $\mathcal{N}_e$ is a subalgebra of $\mathcal{A}_e$. Indeed, let $\rho = (\rho)_{ij} \in \mathcal{N}_e$, then $\rho x_e = x_e \rho$. Thus

$$\begin{pmatrix}
-\rho_{12} & \rho_{11} & 0 & \cdots & 0 \\
-\rho_{22} & \rho_{21} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\rho_{n2} & \rho_{n1} & 0 & \cdots & 0
\end{pmatrix} = \begin{pmatrix}
\rho_{21} & \rho_{22} & \rho_{23} & \cdots & \rho_{2n} \\
-\rho_{11} & -\rho_{12} & -\rho_{13} & \cdots & -\rho_{1n} \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}. $$

By (2.1), we deduce that

$$D(\rho \| E_e(\rho)) \leq D(\rho \| E_\infty(\rho)) \leq \frac{1}{\text{CLSI}(M_n, L_e)} I_{L_e}(\rho), \forall \rho \in M_n \otimes M. $$

For the $p$-version, it follows from the facts that $\rho(\| \sigma)$ is non-negative and the noncommutative martingale equality (See [Li20]).

**Lemma 7.4.** The conditional expectations $\{E_e\}_{e \in \mathcal{E}}$ commute pairwise. Moreover $\prod_{e \in \mathcal{E}} E_e = E_\infty$.

**Proof.** Noting $E_e$ is a Schur multiplier, we infer that all conditional expectations commute. It is obvious that $\prod_{e \in \mathcal{E}} E_e(\alpha) = \alpha$ for any diagonal matrix $\alpha \in M_n$. Now assume that $\prod_{e \in \mathcal{E}} E_e(\rho) = \rho$ for some $\rho \in M_n$. Then $E_e(\rho) = \rho$ for any $e = (v_r, v_s) \in \mathcal{E}$. By the proof of Lemma 7.3, we have $\rho_{jr} = \rho_{rj} = \rho_{js} = \rho_{sj} = 0$ for any $j \neq r, s$. Thus $\rho$ is diagonal, which yields the first assertion. Since $G$ is connected, we conclude that $\rho_{ij} = 0$ for $i \neq j$.

**Lemma 7.5.** For $\rho \in M_n \otimes M$, we have

$$D(\rho \| E_\infty(\rho)) \leq 5\pi^2 I_{L_\mathcal{E}}(\rho)$$

and

$$D(\rho \| E_\infty(\rho)) \leq 5\pi^2 I_{L_\mathcal{E}}(\rho).$$

**Proof.** By Lemma 7.4 we have $D(\rho \| E_\infty(\rho)) = D(\rho \| \prod_{e \in \mathcal{E}} E_e(\rho))$. Together with Lemma 2.7 and Lemma 7.3 we deduce that

$$\inf_{e \in \mathcal{E}} \{\text{CLSI}(M_n, L_e)\} D(\rho \| \prod_{e \in \mathcal{E}} E_e(\rho)) \leq \inf_{e \in \mathcal{E}} \{\text{CLSI}(M_n, L_e)\} \sum_{e \in \mathcal{E}} D(\rho \| E_e(\rho)) \leq I_{L_\mathcal{E}}(\rho).$$

We obtain $5\pi^2$ by Example 4.9. The same argument applies for the $p$-version.

**Lemma 7.6.** Let $\rho \in M_n \otimes M$, then

$$I_{L_\mathcal{E}}(E_\infty(\rho)) \leq I_{L_\mathcal{E}}(\rho) \quad \text{and} \quad I_{L_\mathcal{E}}^p(E_\infty(\rho)) \leq I_{L_\mathcal{E}}^p(\rho).$$
Proof. It suffices to prove that
\begin{equation}
I_{L_e}(E_\infty(\rho)) \leq I_{L_e}(\rho)
\end{equation}
for any edge \( e \in \mathcal{E} \). For any fixed \( 1 \leq i \leq n-1 \), define
\[ E_i^n = \frac{1}{2}(U_i^* \rho U_i + \rho), \]
where \( U_i \) is a diagonal matrix with \(-1\) for the \( i\)-th entry and \(1\) for other diagonal entries. Then \( E_\infty = \prod_{i=1}^{n-1} E_i^n \) and \([E_i^n, E_j^n] = 0\) since \( E_i \) is a Schur multiplier. By the convexity of the Fisher information in Theorem 2.4, we have
\begin{equation}
I_{L_2}(E_i^n(\rho)) \leq \frac{1}{2} I_{L_2}(\rho) + \frac{1}{2} I_{L_e}(U_i^* \rho U_i).
\end{equation}
For any fixed \( U_i \), we see that
\[ [a, U_i^* \rho U_i] = U_i^* U_i [a, U_i^* \rho U_i] U_i^* U_i = U_i^* [U_i a U_i^* , \rho] U_i. \]
Together with the unitary invariance of the trace, then
\[ I_{L_e}(U_i^* \rho U_i) = \tau \left( [X_e, U_i^* \rho U_i] Q U_i^* \rho U_i ([X_e, U_i^* \rho U_i]) \right) \]
\[ = \tau \left( U_i^* [U_i X_e U_i^* , \rho] U_i Q U_i^* \rho U_i [U_i X_e U_i^* , \rho] U_i \right) \]
\[ = \tau \left( [U_i X_e U_i^* , \rho] Q ([U_i X_e U_i^* , \rho]) \right). \]
Note that for any edge that \( U_i X_e U_i^* = X_e \) or \(-X_e\), thus \( I_{L_e}(U_i^* \rho U_i) = I_{L_e}(\rho) \). Together with (7.2), we have \( I_{L_e}(E_i^n(\rho)) \leq I_{L_e}(\rho) \). Thus repeating this \( n-1 \) times yields (7.1)
\[ I_{L_e}(E_\infty(\rho)) \leq I_{L_e}(\prod_{i=2}^{n-1} E_i(\rho)) \leq \cdots \leq I_{L_e}(\rho). \]
The same proof applies for the \( p \)-version.

\textbf{Theorem 7.7.} Let \( G = (\mathcal{V}, \mathcal{E}) \) be a connected graph. Then
\[ \frac{\operatorname{CLSI}(G)}{1 + 5\pi^2 \operatorname{CLSI}(G)} \leq \operatorname{CLSI}(M_\mathcal{E}, L_\mathcal{E}) \leq \operatorname{CLSI}(G). \]
and
\[ \frac{C_p \operatorname{SI}(G)}{1 + 5\pi^2 C_p \operatorname{SI}(G)} \leq C_p \operatorname{SI}(M_\mathcal{E}, L_\mathcal{E}) \leq C_p \operatorname{SI}(G). \]
Proof. Note \( (\ell_\infty^\mathcal{E}, L_\mathcal{E}) \) is a subsystem of \( (M_\mathcal{E}, L_\mathcal{E}) \). By Proposition 7.2, we deduce the second part of the inequality. Let \( E_{\text{fix}} \) be the conditional expectation onto the fixed-point algebras of \( L_\mathcal{E} \). By Lemma 7.6, we have
\begin{equation}
D(E_\infty(\rho) \| E_{\text{fix}}(\rho)) \leq \frac{1}{\operatorname{CLSI}(G)} I_{L_\mathcal{E}}(E_\infty(\rho)) \leq \frac{1}{\operatorname{CLSI}(G)} I_{L_\mathcal{E}}(\rho).
\end{equation}
Together with (7.3) and Lemma 7.5, we obtain the first part of the inequality
\[ D(\rho \| E_{\text{fix}}(\rho)) = D(\rho \| E_\infty(\rho)) + D(E_\infty(\rho) \| E_{\text{fix}}(\rho)) \]
\[ \leq \left( 5\pi^2 + \frac{1}{\operatorname{CLSI}(G)} \right) I_{L_\mathcal{E}}(\rho). \]
The argument also applies for \( C_p \operatorname{SI} \).
Appendix

Here we give more details about Lemma 4.7. Let us recall the tail approximation of Gaussian distribution,

$$\left( \frac{1}{x} - \frac{1}{x^2} \right) e^{-x^2/2} \leq \int_x^\infty e^{-t^2/2} dt \leq \frac{1}{x} e^{-x^2/2}. \]

Let \( g(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-(x-k)^2/2} \), thus \( \frac{dg}{dx} = \frac{1}{g(x)} \). It suffices to show that

\[
2e^{-1/2} + 2e^{-2} + 2e^{-9/2} + \frac{48}{125} e^{-25/2} \leq \sqrt{2\pi} g(x) \leq 2 + 2e^{-1/2} + 2e^{-2} + \frac{8}{3} e^{-9/2}.
\]

\[
\sqrt{2\pi} g(x) = e^{-x^2/2} + \sum_{k=1}^{\infty} e^{-(x-k)^2/2} + \sum_{k=-\infty}^{-1} e^{-(x-k)^2/2}
\]

\[
= e^{-x^2/2} + \sum_{k=1}^{\infty} e^{-(k-1)^2/2} + \sum_{k=1}^{\infty} e^{-(x+k)^2/2}
\]

\[
\sqrt{2\pi} g(x) \leq 1 + \sum_{k=1}^{\infty} e^{-(k-1)^2/2} + \sum_{k=1}^{\infty} e^{-k^2/2}
\]

\[
= 1 + e^{-0/2} + \sum_{k=1}^{\infty} e^{-k^2/2}
\]

\[
= 2 + 2e^{-1/2} + 2e^{-2} + 2e^{-9/2} + 2 \sum_{k=4}^{\infty} e^{-k^2/2}
\]

\[
\leq 2 + 2e^{-1/2} + 2e^{-2} + 2e^{-9/2} + 2 \int_{3}^{\infty} e^{-x^2/2} dx
\]

\[
\leq 2 + 2e^{-1/2} + 2e^{-2} + \frac{8}{3} e^{-9/2}
\]

\[
\sqrt{2\pi} g(x) \geq e^{-1/2} + \sum_{k=1}^{\infty} e^{-k^2/2} + \sum_{k=1}^{\infty} e^{-(k+1)^2/2}
\]

\[
= 2e^{-1/2} + 2 \sum_{k=2}^{\infty} e^{-k^2/2}
\]

\[
= 2e^{-1/2} + 2e^{-2} + 2e^{-9/2} + 2 \sum_{k=4}^{\infty} e^{-k^2/2}
\]

\[
\geq 2^{-1/2} + 2e^{-2} + 2e^{-9/2} + 2 \int_{5}^{\infty} e^{-x^2/2} dx
\]

\[
\geq 2^{-1/2} + 2e^{-2} + 2e^{-9/2} + 2 \left( \frac{1}{5} - \frac{1}{5^3} \right) e^{-25/2}
\]

\[
= 2e^{-1/2} + 2e^{-2} + 2e^{-9/2} + \frac{48}{125} e^{-25/2}
\]

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