Donaldson-Witten Functions
of Massless N=2 Supersymmetric QCD

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Abstract

We study the Donaldson-Witten function in four-dimensional topological gauge theory which is constructed from $N = 2$ supersymmetric $SU(2)$ gauge theory with $N_f < 4$ massless fundamental hypermultiplets. When $N_f = 2, 3$, the strong-coupling singularities with multiple massless monopoles appear in the moduli space (the $u$-plane) of the Coulomb branch. We show that the invariants made out of such singularities exhibit a property which is similar to the one expected for four-manifolds of generalized simple type.
1 Introduction

Recent rapid developments in non-perturbative analysis of four-dimensional $N = 2$ supersymmetric gauge theory [1] [2] have deepened our understanding of Donaldson theory which is known to be formulated as a twisted version of $N = 2$ supersymmetric Yang-Mills theory [3]. In the strong-coupling approach to Donaldson theory the non-abelian problem is replaced with the abelian problem which is much more tractable [4]. As a result new insights into four-manifolds have been obtained [5] [6].

In a recent important paper Moore and Witten investigated Donaldson invariants for manifolds with $b^+_{2} = 1$ [7]. To this aim the contribution from the entire Coulomb branch (i.e. the $u$-plane integral) of $N = 2$ Yang-Mills theory has been analyzed in great detail. It is shown clearly how elliptic modular functions enter Donaldson theory. The Donaldson-Witten function is a generating function of Donaldson invariants and in the case $b^+_{2} > 1$ all contributions come from the strong-coupling singularities. It is interesting that considering the wall-crossing phenomenon for $b^+_{2} = 1$ at strong-coupling singularities enables us to calculate the Donaldson-Witten function in the case $b^+_{2} > 1$. The work has then been generalized in several directions [8] [9] [10] [11]. There is also a remarkable connection of topological gauge theory to integrable systems [12] [9] [13] [14].

In this paper we study the Donaldson-Witten function when a dual abelian gauge field couples to $k$ massless monopole fields with $k \geq 1$ at the strong-coupling singularity. This situation arises in the $N = 2$ $SU(2)$ theory with massless $N_f$ hypermultiplets in the $SU(2)$ doublet [2]. The moduli space possesses singularities where $k = 2^{N_f-1}$ massless monopoles appear. Our result provides another generalization of the relation between invariants of four-manifolds and Seiberg-Witten theory.

This paper is organized as follows. In section two, we review Seiberg-Witten theory of massless $N = 2$ $SU(2)$ QCD and relevant mathematics of the Seiberg-Witten curve. We evaluate in section three the Seiberg-Witten contributions to the Donaldson-Witten function. The Seiberg-Witten contribution from the strong-coupling singularities with a single massless monopole has been obtained in [8]. We complete the analysis in [7] by generalizing it to the singularity with $k > 1$ massless monopoles (or dyons) that appears when $N_f = 2, 3$. We also show that even if we have multiple massless monopoles, there are
only finitely many Spin\textsuperscript{c}-structures which have non-vanishing contributions. In section four we apply our formula to the simplest example of a K3 surface. For \( N_f = 2, 3 \), we observe contributions from higher order terms in a series expansion near singularities. These contributions make the Donaldson-Witten function more complicated than the case of \( N_f = 0, 1 \) where only a single massless monopole appears. We note that they are similar to what we expect for a hypothetical four-manifold that is not of simple type. The final section is devoted to discussions. Many useful formulas of elliptic modular functions associated to the Seiberg-Witten curve of massless \( N = 2 \) SU(2) QCD are collected in Appendix.

## 2 \( N = 2 \) SU(2) QCD and elliptic curves

Let us first collect relevant low-energy properties of the \( N = 2 \) SU(2) theory with massless \( N_f \) fundamental hypermultiplets. The Coulomb branch of the theory is parametrized by \( u \) which is a gauge invariant expectation value of the SU(2) adjoint Higgs field. The symmetry \( \mathbb{Z}_{4-N_f} \) acts on the complex \( u \)-plane when \( N_f \geq 1 \). At a generic point on the \( u \)-plane there exists an \( N = 2 \) U(1) vector multiplet. At singularities there appear extra massless monopole (or dyon) hypermultiplets. They belong to the spinor representation of the \( SO(2N_f) \) global flavor symmetry. Thus the multiplicity of massless monopoles equals \( k = 2^{N_f-1} \). Note that for \( N_f = 3 \) no symmetry acts on the \( u \)-plane and there is a singularity associated with a massless \( SO(6) \) singlet dyon.

All the low-energy physics of the Coulomb branch is encoded in the geometry of elliptic curves. According to [3] the relevant curves are given by

\[
y^2 = x^2(x - u) + \frac{1}{4} \Lambda_0^4 x, \quad N_f = 0,
\]

\[
y^2 = x^2(x - u) - \frac{1}{64} \Lambda_{N_f}^{2(4-N_f)}(x - u)^{N_f-1}, \quad N_f = 1, 2, 3, \tag{2.1}
\]

where \( \Lambda_{N_f} \) is the dynamical scale. The change of variables \( y = 4Y \) and \( x = 4(X + \frac{u}{12} + \frac{\Lambda_0^2}{12} \delta_N, 3) \) renders the curves into the Weierstrass form

\[
Y^2 = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3). \tag{2.2}
\]

To deal with the period integrals on the curve it is helpful to employ the uniformization map \((\wp(z), \wp'(z)) = (X, Y)\) where \( \wp(z) \) is the Weierstrass \( \wp \)-function [11]. The well-known
relations are \( \varphi(\omega_\nu) = e_\nu, \varphi'(\omega_\nu) = 0 \) (\( \nu = 1, 2, 3 \)) where \( \omega_\nu \) are the half periods obeying \( \omega_1 + \omega_2 + \omega_3 = 0 \). The periods \( (2\omega_1, 2\omega_3) \) of a torus are obtained as

\[
2\omega_1 = \oint_A \frac{dX}{Y} = \int_{e_2}^{e_3} \frac{dX}{\sqrt{(X - e_1)(X - e_2)(X - e_3)}}, \\
2\omega_3 = \oint_B \frac{dX}{Y} = \int_{e_1}^{e_2} \frac{dX}{\sqrt{(X - e_1)(X - e_2)(X - e_3)}}. 
\] (2.3)

The modular ratio of the torus \( \tau = \omega_3/\omega_1 \) determines the effective gauge coupling constant \( \tau = \theta_{\text{eff}}/\pi + 8\pi i/g^2_{\text{eff}} \). [4]

The roots of the cubic (2.2) are expressed in terms of the Jacobi theta functions

\[
e_1 = \left( \frac{\pi}{2\omega_1} \right)^2 \left( \vartheta_3^3 + \vartheta_4^3 \right) , \quad e_2 = \left( \frac{\pi}{2\omega_1} \right)^2 \left( \vartheta_3^1 - \vartheta_4^1 \right), \quad e_3 = -\left( \frac{\pi}{2\omega_1} \right)^2 \left( \vartheta_3^4 + \vartheta_4^4 \right), \] (2.4)

where

\[
\vartheta_2 = \sum_{n \in \mathbb{Z}} q^{n+1}2(n+\frac{1}{2})^2, \quad \vartheta_3 = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \vartheta_4 = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \] (2.5)

with \( q = e^{2\pi i\tau} \). From (2.2) and (2.4) one has

\[
g_2 = \frac{2}{3} \left( \frac{\pi}{2\omega_1} \right)^4 f, \quad g_3 = \frac{4}{27} \left( \frac{\pi}{2\omega_1} \right)^6 h, \] (2.6)

where

\[
f = \vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8 = 2(1 + 240q + 2160q^2 + \cdots), \]

\[
h = (\vartheta_3^1 + \vartheta_4^1)(\vartheta_3^4 - \vartheta_4^4)(\vartheta_3^1 + \vartheta_4^1) = 2(1 - 504q - 16632q^2 + \cdots). \] (2.7)

The discriminant \( \Delta \) of the curve (2.2) is given by

\[
\Delta = g_2^3 - 27g_3^2 = 2^{12} \left( \frac{\pi}{2\omega_1} \right)^{12} \eta^{24}(\tau), \] (2.8)

where \( \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) \). Explicitly we have

\[
\Delta(u) = \begin{cases} 
\frac{\Lambda_0^8}{216}(u - \Lambda_0^2)(u + \Lambda_0^2), & N_f = 0, \\
-\frac{\Lambda_0^6}{216}(u^3 + 27\Lambda_1^6/256), & N_f = 1, \\
\frac{\Lambda_2^4}{216}(u - \Lambda_2^2/8)^2(u + \Lambda_2^2/8)^2, & N_f = 2, \\
-\frac{\Lambda_3^2}{216}u^4(u - \Lambda_3^2/256), & N_f = 3.
\end{cases} \] (2.9)
The Seiberg-Witten differential for the $SU(2)$ theory with massless fundamental matters takes the form \[ \lambda_{SW} = \frac{\sqrt{2}}{8\pi} \frac{2u - (4 - N_f)x}{y} dx. \] (2.10)

The period integrals are evaluated to be \[ a(u) = \oint_A \lambda_{SW} = \frac{\sqrt{2}}{\pi} \left[ \left( 2 + N_f \right) u - \frac{\Lambda^2_{N_f}}{64} \right] \frac{\omega_1}{12} + (4 - N_f) \zeta(\omega_1) , \]

\[ a_D(u) = \oint_B \lambda_{SW} = \frac{\sqrt{2}}{\pi} \left[ \left( 2 + N_f \right) u - \frac{\Lambda^2_{N_f}}{64} \right] \frac{\omega_3}{12} + (4 - N_f) \zeta(\omega_3) , \] (2.11)

where $\zeta'(z) = -\wp(z)$. Taking the derivative with respect to $u$, we get \[ \frac{da}{du} = \frac{\sqrt{2}}{8} \frac{2\omega_1}{\pi}, \quad \frac{da_D}{du} = \frac{\sqrt{2}}{8} \frac{2\omega_3}{\pi}. \] (2.12)

It is known that the periods $\Pi = (a(u), a_D(u))$ obey the Picard-Fuchs equation \[ P_{N_f}(u) \frac{d^2 \Pi}{du^2} + \Pi = 0, \] (2.13)

where

\[ P_{N_f}(u) = \begin{cases} 
4(u^2 - \Lambda^2_0), & N_f = 0, \\
4u^2 + \frac{27\Lambda^2_k}{64}, & N_f = 1, \\
4\left( u^2 - \frac{\Lambda^2_2}{64} \right), & N_f = 2, \\
4\left( u - \frac{\Lambda^2_3}{256} \right), & N_f = 3.
\end{cases} \] (2.14)

The absence of the first derivative term in (2.13) implies that the Wronskian for the solutions is constant. A useful identity then comes out \[ a \frac{da_D}{du} - a_D \frac{da}{du} = i \frac{4 - N_f}{4\pi}. \] (2.15)

If we use the Legendre’s relation $\zeta(\omega_1) \omega_3 - \zeta(\omega_3) \omega_1 = \pi i/2$, this is an immediate consequence from (2.11) and (2.12). It is now easy to verify from (2.13) and (2.15) that

\[ \frac{d\tau}{du} = i \frac{8(4 - N_f)}{\pi P_{N_f}(u)} \left( \frac{\pi}{2\omega_1} \right)^2 . \] (2.16)

where we have used the basic relation $\tau = da_D/da$.\footnote{We note in passing that combining (2.8) and (2.12) yields $\frac{da}{du} = \sqrt{2} \eta^2 \Delta^{-1/12}$. This expression is useful in the $F$-theory consideration \cite{footnote}.}
3 Donaldson-Witten functions

Let us now turn to topological gauge theory on a four-manifold $X$. The Seiberg-Witten theory provides a quantum field theoretical approach to the computation of Donaldson invariants of $X$. In the following we assume that $X$ is simply connected, or $b_1 = b_3 = 0$, where $b_k$ denotes the $k$-th Betti number. A harmonic two form on $X$ can be decomposed into a sum of self-dual and anti-self-dual components. We have $b_2 = b_2^+ + b_2^-$ with $b_2^+$ and $b_2^-$ being the dimensions of the spaces of self-dual harmonic two forms and anti-self-dual two forms, respectively. The Euler characteristic of $X$ is $\chi = 2 + b_2^+ + b_2^-$ and the signature is $\sigma = b_2^+ - b_2^-$. The second homology and cohomology group $H_2(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ have a ring structure by the intersection form. This ring structure is the classical topological invariants for the classification of four-manifolds. The Donaldson invariants produce a powerful tool beyond the classical cohomology ring.

The Donaldson-Witten function $Z_{DW}$ of a four-manifold $X$ is a generating function of Donaldson invariants. In the framework of topological gauge theory it is given as a generating function of correlation functions \[ Z_{DW}(p, S) = \langle \exp \left( pO + I(S) \right) \rangle , \] (3.1)
where $O$ and $I(S)$ are certain operators (observables) associated with a point $p \in H_0(X, \mathbb{Z})$ and a surface $S \in H_2(X, \mathbb{Z})$. The vacuum expectation value of $O$ is identified with the quantum moduli parameter $2u = \langle O \rangle$ in Seiberg-Witten theory. The two form observable $I(S)$ in the effective $U(1)$ topological theory creates the contact term interaction $T(u)$ at the intersection points of two surfaces $S_1$ and $S_2$. $Z_{DW}$ is a sum of the $u$-plane integral and the contribution from the Seiberg-Witten invariants;

$$Z_{DW} = Z_u + Z_{SW} .$$ (3.2)

We consider a topological theory which is a twisted version of $N = 2$ $SU(2)$ gauge theory with massless $N_f$ fundamental matters. For this theory the $u$-plane integral $Z_u$ takes the form which is obtained by setting the bare quark masses equal to zero in the expression given in [3]. On the other hand the SW contribution $Z_{SW}$ differs substantially, since the massless monopoles appear with multiplicity $k = 2^{N_f-1}$ at strong-coupling singularities in the massless $N = 2$ $SU(2)$ QCD. Thus our main concern henceforth is to fix $Z_{SW}$. 

In (3.2), when $b_2^+ > 1$, there is no contribution from the $u$-plane integral $Z_u$ \cite{7}. The $u$-plane integral, however, is responsible for the peculiar phenomena we encounter in the case of $b_2^+ = 1$ such as the wall crossing. According to Moore and Witten \cite{7}, one can derive the relation of Donaldson and Seiberg-Witten invariants from the wall crossing at strong-coupling singularities. For each class $\lambda \in H_2(X, \mathbb{Z}) + \frac{1}{2} w_2(E)$, the wall crossing formula of $Z_u$ at a zero $u = u_*$ of the discriminant $\Delta(u)$ of the Seiberg-Witten curve is

$$Z_{u,+} - Z_{u,-} = 2 \sqrt{2} \ e^{2 \pi i \lambda_0^2} (-1)^{\lambda - \lambda_0} \cdot w_2(X) \prod_{\alpha, \chi, \beta, \sigma} \left( \frac{du}{d\tau} \right)^{1/2} \Delta^{\sigma/8} \exp \left( 2p u + S^2 T(u) - i \frac{du}{d\alpha}(S, \lambda) \right),$$

(3.3)

where $\lambda_0 \in H_2(X, \mathbb{Z}) + \frac{1}{2} w_2(E)$ has been introduced to define an orientation of the instanton moduli space. (For $SU(2)$ theory, $w_2(E) = 0$ and one may take a canonical choice $\lambda_0 = 0$.) If $X$ is a spin manifold, that is $w_2(X) = 0$, then the sign factor disappears.

Here the factors $\alpha \chi \beta \sigma (du/da)^{1/2} \Delta^{\sigma/8}$ have appeared as the measure factor for the $u$-plane integral \cite{20}.\cite{7}. Near the zero $u = u_*$ with multiplicity $k$ we have introduced a good local coordinate $q$ by

$$(u - u_*)^k = \kappa_* q + O(q^2), \quad (q \to 0)$$

(3.4)

and the subscript $q^0$ means taking the coefficient of the $q^0$ term in the $q$ expansion. Hence, if the leading power of the $q$ expansion in $q \to 0$ is positive, then there is no wall crossing contribution from $\lambda$.

Let us check when $\lambda$ gives non-trivial contribution. From (2.12) and (2.16) we have

$$\frac{du}{d\tau} = \frac{4\pi}{i(4 - N_f)} (\frac{da}{du})^2 P_{N_f}(u).$$

(3.5)

Since $P_{N_f}(u)$ has a simple zero at $u = u_*$ (see (2.14)), $P_{N_f}(u) \sim \Delta(u)^{1/k}$ near the singularity. Using the following expansions of the period and the discriminant,

$$\frac{da}{du} = \left( \frac{da}{du} \right)_* + (u - u_*) (\frac{d^2 a}{du^2})_* + \cdots,$$

$$\Delta(u) = \Delta_*^{(k)} (u - u_*)^k + \cdots,$$

(3.6)

we see the leading term in the $q$ expansion is

$$q^{-\lambda^2/2} q^{1/k} q^{\sigma/8} + \cdots, \quad (q \to 0).$$

(3.7)
Here and henceforth $a$ should be a good local coordinate at strong-coupling singularity. It is a linear combination of $a$ and the dual variable $a_D$ in general. (See a discussion in section 4 for detail.) For example, at the monopole singularity $a$ is understood as $a_D$. For a non-vanishing contribution the power in (3.7) should be non-positive, that is,

$$k\lambda^2 - 2 - \frac{k\sigma}{4} \geq 0 .$$ (3.8)

### 3.1 Measure factors

Near the singularity, the dual theory is described as an effective $U(1)$ gauge theory. There are $k$ massless $U(1)$ hypermultiplets for the $k$-th order zero of $\Delta$. The twisted action is topological up to BRST exact terms and it would have the form

$$L = \{Q_B, W\} + \int_X \left( c(u) F \wedge F + p(u) \text{Tr} \, R \wedge R + \ell(u) \text{Tr} \, R \wedge \tilde{R} \right) .$$ (3.9)

Note that the overall couplings of the topological terms may depend on the vacuum moduli $u$. Among the topological terms the $F \wedge F$ term is the descendant of the prepotential $F_M$ of the dual photon and monopoles (dyons). The coefficient $c(u)$ is related to the effective coupling $\tau_M$, the second derivative of $F_M$. The other terms with the curvature $R$ are the background gravitational effect due to the integration over massive fermions. In the path integral these couplings induce the following measure factor,

$$C(u)^{\lambda^2/2} \, P(u)^{\sigma/8} \, L(u)^{\chi/4} .$$ (3.10)

Following the approach of [7], we will determine the measure factor by matching the wall crossing formula of $Z_u$ and that of $Z_{SW}$. This means that there is no jump in the invariants at any strong coupling singularity;

$$\delta Z_{DW} = \delta Z_u + \delta Z_{SW} = 0 .$$ (3.11)

This matching is imposed only when $b_2^+ = 1$. But, since the measure factors are universal in the sense that they do not depend on the manifold on which topological theory is defined, we can use the same measure factors also for the case $b_2^+ > 1$. 

7
If we assume that the jump of the SW invariant $SW(\lambda)$ at the wall is $\pm k$, the wall crossing of the Seiberg-Witten invariants is

$$
\Delta \langle e^{\rho + I(S)} \rangle = \pm \text{Res}_{a=a_*} \left[ 2k e^{2\pi i(\lambda_0 - \lambda_0^2)} \frac{da}{(a-a_*)^{1+d_\lambda/2}} C(u)^{\lambda^2/2} P(u)^{\sigma/8} L(u)^{\chi/4} \exp(2pu + S^2 T(u) - i \frac{du}{da}(S, \lambda)) \right], \quad (3.12)
$$

where $\lambda = c_1(L^2)/2$, $\delta = (\chi + \sigma)/4$ and

$$
d_\lambda = k \left( \lambda^2 - \frac{\alpha}{4} \right) - 2\delta \quad (3.13)
$$

is the formal dimension of the moduli space of a generalized Seiberg-Witten monopole equation with $k$ massless hypermultiplets; see $(3.26)$ and $(3.27)$ for explicit forms. When $b_2^+=1$, or $\chi + \sigma = 4$, the condition $(3.8)$ for non-trivial wall crossing of $Z_u$ agrees with the condition that the dimension of the moduli space is non-negative $d_\lambda \geq 0$. That is the condition under which we have non-trivial residue in $(3.12)$. Thus we can consistently require the matching of two wall crossing formulae, which implies

$$
C(u)^{\lambda^2/2} P(u)^{\sigma/8} L(u)^{1-\sigma/4} \Delta^{\sigma/8} = 2\sqrt{2i\pi k^{-1}} \alpha^4 \beta^8 (a-a_*)^{-k} \Delta(u), \quad \Delta(u) = \frac{du}{da}(S, \lambda) \exp(2pu + S^2 T(u) - i \frac{du}{da}(S, \lambda)). \quad (3.14)
$$

We obtain the following results for the measure factors,

$$
C(u) = \frac{1}{q}(a-a_*)^k, \\
P(u) = -8\pi^2 k^{-2} \beta^8 (a-a_*)^{-k} \Delta(u), \\
L(u) = 2\sqrt{2i\pi k^{-1}} \alpha^4 \left( \frac{du}{da} \right)^2. \quad (3.15)
$$

These measure factors derived from the condition that there is no discontinuity of $Z_{DW}$ at a strong-coupling singularity might be deduced from the following argument.\footnote{Note that there are $k$ massless hypermultiplets.}

First we note that the logarithm of the measure factor $C(u)$ gives the effective coupling;

$$
\tau_M = \tau_D - \frac{k}{2\pi i} \log(a-a_*) \quad (3.16)
$$

\textit{Note that} it is better to have some qualitative reasoning which is independent of a detail of the wall crossing formula.
which is indeed the desired relation we expect from the existence of \( k \) massless matters. Up to numerical constants the only difference of the gravitational measure factors from those for the \( u \)-plane integral is the factor \((a-a_*)^{-k}\) in \( P(u) \). This also has an explanation in view of the existence of \( k \) hypermultiplets, which is the additional massless excitations compared to the topological theory at a generic point on the \( u \)-plane. Recall that the gravitational measure factor for effective topological theory is fixed by counting the \( R \) charge [20]. The contribution of the hypermultiplet to the \( R \) charge is twice the index of the Dirac operator

\[
2 \cdot \text{(index } D) = \lambda^2 - \frac{\sigma}{4}.
\]

Hence the gravitational part of \( k \) hypermultiplets is \(- (k/4) \sigma \). We see that there is no change in \( L(u) \), but \( P(u) \) should have an additional factor with the \( R \) charge \(-2k \). Now the factor \((a-a_*)^{-k}\) does the job, since \((a-a_*)\) carries the \( R \) charge \(+2 \).

### 3.2 Seiberg-Witten contributions

Using these measure factors, we obtain the following universal form of the Seiberg-Witten contribution to the invariants of four-manifolds:

\[
Z_{SW} = \sum_{\lambda} SW(\lambda) \text{Res}_{a=a_*} \left[ 2 e^{2\pi i (\lambda_0 \cdot \lambda + \lambda_0^2)} \frac{da}{(a-a_*)^{1+d_s/2} \left( \frac{(a-a_*)^k - q^{-2\pi_2 k^{-1} a^4} \chi/4 \left( \frac{du}{da} \right)^{1/2} \right)}
\right.
\exp \left( 2pu + S^2 T(u) - i \frac{du}{da} (S, \lambda) \right]
\]

\[
= \sum_{\lambda} SW(\lambda) \text{Res}_{a=a_*} \left[ 2 e^{2\pi i (\lambda_0 \cdot \lambda + \lambda_0^2)} \frac{da}{(a-a_*)^{1-d_s/2} \left( 2\sqrt{2} \pi i k^{-1} \alpha^4 \chi/4 \left( \frac{du}{da} \right)^{1/2} \right)}
\exp \left( 2pu + S^2 T(u) - i \frac{du}{da} (S, \lambda) \right]
\]

Here \( SW(\lambda) \) stands for the Seiberg-Witten invariant. Denoting as \( \mathcal{M}_\lambda \) the moduli space of solutions to the generalized monopole equations (3.26) and (3.27) with given \( \lambda \), we have

\[
SW(\lambda) = \langle \tilde{a}^n \rangle_\lambda = \int_{\mathcal{M}_\lambda} \tilde{a}^n
\]

for \( d_\lambda = 2n \). In (3.19) we have used \( \tilde{a} \) to emphasize that it is a linear combination of \( a \) and \( a_D \) according to the charge of massless excitation at the singularity. Substituting the
following relation derived from the leading part of the $q$-expansion,

$$(u - u_*) = \kappa_* q + O(q^2),$$

$$(a - a_*) = (\frac{da}{du})_* (u - u_*) + \cdots = \kappa^{1/k} (\frac{da}{du})_* q^{1/k} + \cdots,$$

$$\frac{da}{a - a_*} = \frac{dq}{kq},$$

$$\Delta = \Delta^{(k)} (u - u_*)^k + \cdots = \Delta^{(k)} \kappa_* q + \cdots,$$  \hspace{1cm} (3.20)

we have

$$Z_{SW} = \sum_{\lambda} SW(\lambda) 2 e^{2\pi i (\lambda_0 - \lambda + \lambda^2_0)} (2\sqrt{2\pi} i k^{-1})^{\delta} \alpha^\gamma \beta^\sigma$$

$$\left[ \frac{1}{k q^{d_\lambda/2k}} (\kappa^*)^{\delta + \sigma/8} (\Lambda^{(k)})^{\sigma/8} (\frac{da}{du})_*^{\delta - \chi/2} \exp \left( 2pu_* + S^2 T_* - i (\frac{du}{da})_* (S, \lambda) \right) \right]_{q^0} \hspace{1cm} (3.21)$$

Note that when $d_\lambda$ is positive, the sub-leading terms contribute to $Z_{SW}$. This situation is analogous to the case of four-manifolds of generalized simple type \cite{21}. A four-manifold $X$ is called simple type (in the sense of Seiberg-Witten) if $SW(\lambda) = 0$ for $d_\lambda$ which is strictly positive. Let us assume that the discriminant has a simple zero at $u = u_*$. In this case $Z_{SW}$ for a manifold of simple type is much simplified to

$$Z_{SW} = \sum_{\lambda} SW(\lambda) 2 e^{2\pi i (\lambda_0 - \lambda + \lambda^2_0)} (2\sqrt{2\pi} i k^{-1})^{\delta} \alpha^\gamma \beta^\sigma$$

$$(\kappa^*)^{\delta + \sigma/8} (\Lambda^\gamma)^{\sigma/8} (\frac{da}{du})_*^{\delta - \chi/2} \exp \left( 2pu_* + S^2 T_* - i (\frac{du}{da})_* (S, \lambda) \right).$$  \hspace{1cm} (3.22)

Notice the relations derived from (2.6) and (2.8)

$$(\frac{da}{du})^2 = \frac{1}{12^2 g_3(u_*)}, \hspace{1cm} \kappa_* \Delta' = 2^{12} \left( \frac{\pi}{2\omega_1} \right)^{12}.$$  \hspace{1cm} (3.23)

Hence,

$$(\kappa^*)^{\delta + \sigma/8} (\frac{da}{du})_*^{3\sigma/2} = 2^{-9\sigma/4}$$  \hspace{1cm} (3.24)

and using $\delta - \chi/2 = 3\sigma/2 - (\delta + \sigma)$, we have

$$(\kappa^*)^{\delta + \sigma/8} (\Lambda^\gamma)^{\sigma/8} (\frac{da}{du})_*^{\delta - \chi/2} = 2^{-9\sigma/4} (\kappa^*)^{\delta} (\frac{da}{du})^{-(\delta + \sigma)}.$$  \hspace{1cm} (3.25)

Thus it is seen that (3.22) for $k = 1$ agrees with (11.28) of \cite{7} and our formula (3.21) generalizes their expression.
3.3 Vanishing theorem

It is known that there are only finite number of isomorphism classes of the line bundle \( \lambda \) such that the Seiberg-Witten invariant \( SW(\lambda) \) is non-vanishing. This is a vanishing theorem in [4]. We can generalize the estimate implying the vanishing theorem to the case of more than one massless hypermultiplet.

If there are \( k \) massless hypermultiplets whose bosonic components are \( M_\alpha (\alpha = 1, 2, \ldots, k) \), the monopole equations are generalized to

\[
F^{\pm}_{ij} = -\frac{i}{2} \sum_{\alpha=1}^{k} M_\alpha \Gamma_{ij} M_\alpha ,
\]

\[
\sum_{i} \Gamma^i D_i M_\alpha = 0 , \quad \alpha = 1, 2, \ldots, k .
\]

Using the Weitzenböck formula for each component \( M_\alpha \), we obtain

\[
\int_X d^4x \sqrt{g} \left( \frac{1}{2} |F^+|^2 + \frac{i}{2} \sum_{\alpha=1}^{k} |M_\alpha \Gamma M_\alpha|^2 + \sum_{\alpha=1}^{k} |\Gamma \cdot DM_\alpha|^2 \right) = \int_X d^4x \sqrt{g} \left( \frac{1}{2} |F^+|^2 + \sum_{\alpha=1}^{k} |DM_\alpha|^2 + \frac{1}{4} R(\sum_{\alpha=1}^{k} M_\alpha M_\alpha) + \frac{1}{2} (\sum_{\alpha=1}^{k} M_\alpha M_\alpha)^2 \right),
\]

where \( R \) is the scalar curvature. Since

\[
\int_X d^4x \sqrt{g} \left( \sum_{\alpha=1}^{k} M_\alpha M_\alpha + \frac{1}{4} R \right)^2 \geq 0 ,
\]

we have the following bound;

\[
\int_X d^4x \sqrt{g} \left( \sum_{\alpha=1}^{k} M_\alpha M_\alpha + \frac{1}{4} R \right)^2 \leq \frac{1}{32} \int_X d^4x \sqrt{g} R^2 ,
\]

if the generalized monopole equations (3.26) and (3.27) are satisfied. Thus we obtain a bound

\[
I_+ = \int_X d^4x \sqrt{g} |F^+|^2 \leq \frac{1}{16} \int_X d^4x \sqrt{g} R^2 .
\]

Furthermore, note that

\[
c_1(L^2) = \frac{1}{(2\pi)^2} \int_X d^4x \sqrt{g} \left( |F^+|^2 - |F^-|^2 \right) .
\]
Therefore, in order for the formal dimensions $d_\lambda$ of the moduli space to be non-negative, we must have
\[ \frac{1}{(4\pi)^2} \int_X d^4x \sqrt{g} \left( |F^+|^2 - |F^-|^2 \right) \geq \frac{\sigma}{4} + \frac{1}{2k}(\sigma + \chi). \]  

(3.33)

Hence we have another bound
\[ I_- = \int_X d^4x \sqrt{g} |F^-|^2 \leq \frac{1}{16} \int_X d^4x \sqrt{g} R^2 - 4\pi^2(\sigma + \frac{2}{k}(\sigma + \chi)). \]  

(3.34)

Since both $I_+$ and $I_-$ are bounded, the set of $\lambda$ with non-vanishing $SW(\lambda)$ is in a compact subset of $H^2(X, \mathbb{R})$. Hence, there are only finite $Spin^c$ structures that have non-vanishing contribution to the Donaldson-Witten function.

For example, we can take $R = 0$ for $K3$ surface which is hyper Kähler. Substituting $\chi = 24, \sigma = -16$, we obtain
\[ I_+ = 0, \quad I_- \leq 64\pi^2 \left( 1 - \frac{1}{k} \right). \]  

(3.35)

For $k = 1$ we find a well-known fact that on a $K3$ surface only a trivial class $\lambda = 0$ has non-vanishing contributions, which is the basic class in the sense of Seiberg-Witten. On the other hand for $k = 2, 4$, this is no longer true. By taking into account that the minimum of $J = \int_X F \wedge F$ is $8(2\pi)^2$ [20], (note also that $\Gamma = H_2(K3, \mathbb{Z})$ is an even lattice, since $K3$ is spin), we see that non-vanishing contributions come from either a trivial class $\lambda = 0$ or classes with $\lambda^2 = -2$. Since a $K3$ surface has $b_2^+ = 19$, we conclude that there are $1 + 2 \cdot 19$ possible classes.

4 $K3$ surface

As the simplest example let us calculate $Z_{DW}$ for a $K3$ surface for which $\chi = 24, \sigma = -16$ and hence $d_\lambda = k\lambda^2 + 4(k-1)$. Since $b_2^+ = 3$ the $u$-plane integral vanishes. Let us assign the $R$ charge 4 to $p$ and 2 to $S$, then the degree $s$ Donaldson polynomial contains terms $S^np^t$ with $2n + 4p = s$ where $s$ is the dimension of the instanton moduli space. In the case of massless $N = 2$ $SU(2)$ QCD on a $K3$ surface, we have
\[ s = 2(4 - N_f)\ell - 12 + 8N_f, \]  

(4.1)

where $\ell$ is the instanton number. For $N_f \geq 1$, it is important to recall here that only the instantons with even instanton number contribute due to the anomalous $\mathbb{Z}_2$ symmetry.
To evaluate $Z_{SW}$ one has to choose a good local coordinate near each singularity on the $u$-plane. A good coordinate is given by $ga_D + qa$ where $g$ and $q$ are the magnetic and electric charges of the BPS dyonic state which becomes massless at the singularity. In what follows we shall set $\Lambda_{N_f} = 1$ for simplicity.

We start with the well-known $N_f = 0$ theory. Only $\lambda^2 = 0$ contributes to $Z_{SW}$, and the famous expression results in \[ Z_{DW} = c_0 \left( e^{2p + \frac{1}{2}S^2} - e^{-2p - \frac{1}{2}S^2} \right) = c_0 \sinh \left( 2p + \frac{1}{2}S^2 \right), \tag{4.2} \]
where $c_0$ is an overall constant and the two terms are due to a $\mathbb{Z}_2$ pair of singularities at $u = \pm 1$. Note that the terms of the $R$ charge congruent to 4 modulo 8 exist in accordance with the selection rule with $s = 8\ell - 12$. $Z_{DW}$ satisfies the so-called simple type condition \[ \left( \frac{\partial^2}{\partial p^2} - 4 \right) Z_{DW} = 0. \tag{4.3} \]

In the $N_f = 1$ theory there appear singularities in a $\mathbb{Z}_3$ symmetric manner at $u = -2^{1/3} \frac{3}{2} \omega^j/8$ with $j = 0, 1, 2$ and $\omega = e^{2\pi i/3}$. At each singularity a single massless monopole (or dyon) comes out as a matter hypermultiplet. The property of the resulting $k = 1$ monopole equation is essentially the same as the $N_f = 0$ case. Thus only the class $\lambda^2 = 0$ contributes, leading to the result \[ Z_{DW} = c_1 \sum_{j=0}^{2} \omega^j e^{\alpha \omega^j(2p + \frac{1}{2}S^2)}, \quad \alpha = -2^{1/3} \frac{3}{8}, \tag{4.4} \]
where surviving terms on the RHS have the $R$ charge 8 modulo 12. This is in agreement with the selection rule obeyed by the even instanton contributions. As pointed out in \[ Z_{DW} \] is subject to the equation \[ \left( \frac{\partial^3}{\partial p^3} + \frac{27}{32} \right) Z_{DW} = 0. \tag{4.5} \]
Note that (4.4) can be reproduced by taking the massless limit of $Z_{DW}$ for the massive $N_f = 1$ theory \[ Z_{DW} \] since the number of massless particles at singularities does not change in the limit. On the other hand, this limit becomes singular in the $N_f \geq 2$ theory to which we next turn.
In the $N_f = 2$ theory the $Z_2$ symmetry is acting on the $u$-plane. At the singularity $u = 1/8$, we have massless monopoles in $2_s$ of $SO(4)$ and a good local coordinate is $a_D$ or $q_D = e^{2\pi i \tau_D}$ with $\tau_D = -1/\tau$. The other singularity is located at $u = -1/8$ where dyons in $2_c$ of $SO(4)$ with the charge $(g, q) = (1, -1)$ are massless. A good coordinate is thus $a_D - a$, and the corresponding modular expressions are obtained by letting $\tau \to \tilde{\tau} = \tau - 1 \to \tilde{\tau}_D = -1/\tilde{\tau}$ in (A.1). The relevant $q$-expansion is then performed around $\tilde{q}_D = 0$ where $\tilde{q}_D = e^{2\pi i \tilde{\tau}_D}$. Furthermore, in calculating $Z_{SW}$ we have to take the contributions of $\lambda^2 = -2$ ($d_\lambda = 0$) in addition to $\lambda^2 = 0$ ($d_\lambda = 4$) as discussed in the preceding section. We now obtain from (3.21) that

$$Z_{DW} = c_2 \left[ \left( 16p^2 + 8pS^2 + S^4 - 88p - 24S^2 + 228 \right) e^{p/4+S^2/8} \right. $$

$$- \left( 16p^2 + 8pS^2 + S^4 + 88p + 24S^2 + 228 \right) e^{-p/4-S^2/8} \right] + c_2' \sum_{\lambda^2 = -2} \text{SW}(\lambda) \left( e^{p/4+S^2/8+i\sqrt{2}(S,\lambda)} - e^{-p/4-S^2/8-i\sqrt{2}(S,\lambda)} \right), \tag{4.6}$$

where $c_2, c_2'$ are constants. When we sum over $\lambda$ the symmetry property $[4][6]$

$$\text{SW}(-\lambda) = (-1)^d \text{SW}(\lambda) \tag{4.7}$$

should be taken into account. It is evident that

$$\left( \frac{\partial^2}{\partial p^2} - \frac{1}{16} \right)^3 Z_{DW} = 0. \tag{4.8}$$

Thus, when the massless $N_f = 2$ $SU(2)$ theory is considered on $K3$ surfaces, $Z_{DW}$ obeys the condition which is reminiscent of the one for manifolds of generalized simple type in the sense of [21].

Finally, in the $N_f = 3$ theory, there is no symmetry acting on the $u$-plane. Massless monopoles in $4$ of $SO(6)$ appear at the singularity $u = 0$ around which we take $a_D$ or $q_D$ as a good coordinate. An additional singularity associated with an $SO(6)$ singlet massless dyon with the charge $(2, 1)$ exists at $u = 1/256$. A good coordinate near this singularity is taken to be $2a_D + a$. Correspondingly we implement transformations $\tau \to \tilde{\tau} = \tau + 1/2 \to \tilde{\tau}_D = -1/\tilde{\tau}$ in (A.1) so that we can make the power series expansions at $\tilde{q}_D = 0$. For instance we find

$$u = \frac{1}{256 \left( \frac{\partial^2}{\partial^2} \frac{\partial^2}{\partial^2} \right)} \left( \frac{\tilde{\tau}_D}{\tilde{\tau}_D} \right)$$

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to $Z$ the contributions of $\lambda$ hence, $Z_{SW}$ contribution is obtained as $u = 1$ and $u = \frac{1}{256}$ respectively give rise to $Z^{(1)}_{SW}$ and $Z^{(2)}_{SW}$, thereby the SW contribution is obtained as $Z_{SW} = Z^{(1)}_{SW} + Z^{(2)}_{SW}$. As we have shown, $Z^{(1)}_{SW}$ consists of the contributions of $\lambda^2 = 0$ ($d_\lambda = 12$) and $\lambda^2 = -2$ ($d_\lambda = 4$), while only $\lambda^2 = 0$ contributes to $Z^{(2)}_{SW}$ since the monopole equation is multiplicity free (i.e. $k = 1$). Our result reads

$$Z^{(1)}_{SW} = c_3 \left( \frac{1}{2949120} p^6 + \frac{1}{188743680} S^{12} + \frac{1}{3145728} p^2 S^8 + \frac{1}{15728640} p S^{10} + \frac{1}{983040} p^5 S^2 \right. \\
\left. + \frac{1}{786432} p^4 S^4 + \frac{1}{1179648} p^3 S^6 + 78648 p + 7277252^2 + 10879744 + \frac{521}{2} p^2 \\
+ 2101 \frac{5}{4} p S^2 + 241 S^4 + \frac{213}{128} p^2 S^2 + \frac{441}{256} p S^4 + \frac{31}{64} p^3 + \frac{809}{1536} S^6 + \frac{3}{512} p^3 S^2 \\
+ \frac{3}{512} p^2 S^4 + \frac{23}{6144} p S^6 + \frac{7}{1536} p^4 + \frac{7}{8192} S^8 - \frac{29}{98304} p^4 S^2 + \frac{7}{32768} p^3 S^4 \\
- \frac{13}{196608} p^2 S^6 - \frac{5}{786432} p S^8 - \frac{37}{245760} p^5 + \frac{1}{2621440} S^{10} \right) \\
+ c_3 \sum_{\lambda^2 = 2} \text{SW}(\lambda) \left( 1456 p - 14208\sqrt{2} i(S, \lambda) + 792 S^2 + 4p^2 - 64\sqrt{2} ip(S, \lambda) \right) \\
+ 4p S^2 - 512(S, \lambda)^2 - 32\sqrt{2} i(S, \lambda) S^2 + S^4 + 220672 \right) e^{i\sqrt{2}(S,\lambda)/4},$$

$$Z^{(2)}_{SW} = c'_3 \left( 64p^6 + S^{12} + 60 p^2 S^8 + 12p S^{10} + 192p^5 S^2 + 240p^4 S^4 + 160p^3 S^6 \\
- 15850442588160p - 15061242347520 S^2 + 2218796211240960 \\
+ 50520391680p^2 + 108338872320p S^2 + 50787778560S^4 - 332267520p^2 S^2 \\
- 366673920p S^4 - 74711040p^3 - 113541120S^6 + 655360p^3 S^2 + 1105920p^2 S^4 \\
+ 819200p S^6 + 409600p^4 + 189440S^8 + 48640p^4 S^2 + 33280p^3 S^4 + 8960p^2 S^6 \\
+ 320p S^8 + 25600p^5 - 160 S^{10} \right) e^{(p+S^2)/128}. \tag{4.10}$$

Hence,

$$\frac{\partial^7}{\partial p^7} \left( \frac{\partial}{\partial p} - \frac{1}{128} \right)^7 Z_{DW} = 0. \tag{4.11}$$

## 5 Discussions

The Donaldson-Witten functions we have obtained for a $K3$ surface involve $SW(\lambda)$ for $\lambda^2 = -2$ and constants $c_i$, $c'_3$ which are fixed by the values of $SW(\lambda)$ for $\lambda^2 = 0$. To
establish mathematically more rigorous basis, we have to prove the compactness of the moduli space $\mathcal{M}_\lambda$ so that the integral $SW(\lambda)$ in (3.19) is well-defined. This point is more subtle than the case of the bounds for $I_\pm$ we have shown in section 3.3. What remains is to show a similar bound for the monopole fields $M_\alpha$. In the case of the Seiberg-Witten monopole equation with a single massless monopole, such a bound is obtained rather easily [6]. (In fact this is one of advantages of the abelian nature of Seiberg-Witten invariants.) However, the generalized monopole equation with multi-component massless monopoles does not seem to allow a straightforward generalization of the argument in [6]. At the moment we have assumed the compactness of the moduli space and leave the proof as an open problem.

When there are multiple massless monopole fields, we had to pick up higher order terms in the $q$-series expansion near the singularities as the contribution from the trivial class $\lambda$. This is also expected when a four-manifold $X$ does not satisfy the simple type condition. Thus the expansions of elliptic modular functions near the strong-coupling singularities we have given in the paper may be useful for computing the Donaldson invariants for a manifold of generalized simple type, though at present it is only a hypothetical case when $b_2^+ > 1$.

Finally we point out an interesting issue related to twisted $N = 2$ superconformal field theory in four dimensions. In [9] Mariño and Moore initiated the study of four-dimensional topological conformal theories by twisting the $N = 2$ superconformal theory realized at the Argyres-Douglas point in the $SU(3)$ Yang-Mills theory [26] and its generalization to $SU(N)$ [28]. The $N = 2$ superconformal fixed point which is in the same universality class with the Argyres-Douglas point is known to exist in the massive $N = 2$ $SU(2)$ QCD with $N_f = 1$ [27]. Moreover, analogous non-trivial fixed points are found in $N = 2$ $SU(2)$ QCD with $N_f > 1$ massive flavors [27]. These fixed points are also obtained in $N = 2$ pure Yang-Mills theories [26]. We then expect that, though microscopic theories are distinct, the $N = 2$ fixed points in the same universality class should yield the identical topological conformal theory after being twisted. Thus, it will be worth working out this relationship explicitly using massive $N = 2$ $SU(2)$ QCD and $N = 2$ pure Yang-Mills theories. This may shed new light not only on topological conformal theory, but on four-dimensional

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4We thank H. Nakajima for pointing out this issue to us.
$N = 2$ superconformal theory whose precise dynamics still needs to be clarified.

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Appendix. Periods and modular functions

In order to evaluate $Z_{SW}$ we need to express various quantities in terms of modular forms. For this we first identify the roots of the cubic in (2.1) with $e_\nu$ by examining the large-$u$ asymptotic behavior. Then, after some algebra, we find necessary formulas. Including the $N_f=0$ case to make our paper self-contained, let us summarize our results:

\begin{align*}
N_f &= 0 & \quad u \frac{\Lambda_0}{\Lambda_0} &= \frac{\vartheta_2^4 + \vartheta_3^4}{2(\vartheta_2 \vartheta_3)^2}, & \quad 2\omega_1 \Lambda_0 \frac{\tau}{\pi} &= 2\sqrt{2} \vartheta_2 \vartheta_3, \\
P_0(u) &= \Lambda_0^4 \frac{\vartheta_4^8}{(\vartheta_2 \vartheta_3)^4}, & \quad \frac{d\tau}{du} &= \frac{4i}{\pi \Lambda_0^2} \frac{(\vartheta_2 \vartheta_3)^2}{\vartheta_4^2}, \\
N_f &= 1 & \quad u \frac{\Lambda_1}{\Lambda_1} &= \frac{2\sqrt{2} f}{A}, & \quad 2\omega_1 \Lambda_1 \frac{\tau}{\pi} &= \sqrt{A}, & \quad A^3 &= \frac{2}{3^3} \left(2h - f \sqrt{2} f\right)^2, \\
P_1(u) &= -2^{17} A^4 \frac{(\vartheta_2 \vartheta_3 \vartheta_4)^8}{A^6 \sqrt{2} f}, & \quad \frac{d\tau}{du} &= -\frac{3i}{2^{14} \pi \Lambda_1^2} \frac{\sqrt{2} f A^4}{(\vartheta_2 \vartheta_3 \vartheta_4)^8}, \\
N_f &= 2 & \quad u \frac{\Lambda_2}{\Lambda_2} &= \frac{\vartheta_2^3 + \vartheta_4^3}{8 \vartheta_2^4}, & \quad 2\omega_1 \Lambda_2 \frac{\tau}{\pi} &= 4 \vartheta_2, \\
P_2(u) &= \Lambda_2^4 \frac{(\vartheta_3 \vartheta_4)^4}{4 \vartheta_2^3}, & \quad \frac{d\tau}{du} &= \frac{4i}{\pi \Lambda_2^2} \frac{\vartheta_2^4}{(\vartheta_3 \vartheta_4)^4}, \\
N_f &= 3 & \quad u \frac{\Lambda_3}{\Lambda_3} &= -\frac{(\vartheta_3 \vartheta_4)^2}{64 (\vartheta_2^2 - \vartheta_4^2)^2}, & \quad 2\omega_1 \Lambda_3 \frac{\tau}{\pi} &= 16i (\vartheta_3^2 - \vartheta_4^2), \\
P_3(u) &= \Lambda_3^4 \frac{(\vartheta_3 \vartheta_4)^2}{64^2 (\vartheta_3^2 - \vartheta_4^2)^4 (\vartheta_3^2 + \vartheta_4^2)^2}, & \quad \frac{d\tau}{du} &= -\frac{128i}{\pi \Lambda_3^2} \frac{(\vartheta_3^2 - \vartheta_4^2)^2}{(\vartheta_3^2 + \vartheta_4^2)^2}, \\
\end{align*}

(A.1)

where modular functions $f$ and $h$ are defined in (2.7).

As a by-product the $\beta$ function defined by

\[ \beta(\tau) = \Lambda \frac{\partial \tau}{\partial \Lambda} \Big|_{u \text{ fixed}} = -2u \frac{\partial \tau}{\partial u} \Big|_{\Lambda \text{ fixed}} \]  

(A.2)
is obtained including full instanton corrections. We find

\begin{align*}
N_f &= 0 & \quad \beta(\tau) &= \frac{4}{\pi i} \frac{\vartheta_2^4 + \vartheta_3^4}{\vartheta_3^4} = \frac{4}{\pi i} \left(1 + 40q^{1/2} + 552q + 4896q^{3/2} + \cdots\right), \\
N_f &= 1 & \quad \beta(\tau) &= \frac{i}{9\pi} \frac{f \left(2h - f \sqrt{2} f\right)}{(\vartheta_2 \vartheta_3 \vartheta_4)^8} = \frac{3}{\pi i} \left(1 + 312q + 20520q^2 + 497760q^3 + \cdots\right), \\
N_f &= 2 & \quad \beta(\tau) &= \frac{1}{\pi i} (\vartheta_3 \vartheta_4)^2 = \frac{2}{\pi i} \left(1 + 40q + 552q^2 + 4896q^3 + \cdots\right), \\
N_f &= 3 & \quad \beta(\tau) &= \frac{4}{\pi i} \frac{1}{(\vartheta_3^2 + \vartheta_4^2)^2} = \frac{1}{\pi i} \left(1 - 8q + 40q^2 - 160q^3 + \cdots\right). \quad (A.3)
\end{align*}
Note that the leading term \((4 - N_f)/\pi i\) is the one-loop \(\beta\) function. Note also that only integral powers of \(q\) appear for \(N_f \geq 1\). This is due to the fact that only even instantons contribute in \(N_f \geq 1\) theories \([2]\). The results agree with \([28, 29]\) for \(N_f = 0\), and with \([28]\) for \(N_f = 1, 2\).

The SW periods (2.11) are also expressed in terms of modular functions. We use (A.1) and the relations

\[
\zeta(\omega_1) = \frac{\pi^2}{12\omega_1}E_2(\tau), \quad \zeta(\omega_3) = -\frac{\pi^2}{12\omega_1}\tau_D E_2(\tau_D),
\]

(A.4)

where \(\tau_D = -1/\tau\) and the normalized Eisenstein series

\[
E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}
\]

(A.5)
of weight 2 is transformed as

\[
E_2(\tau) = \frac{6\tau_D}{\pi i} + \tau_D^2 E_2(\tau_D)
\]

(A.6)

under \(\tau \to \tau_D\). We then obtain from (2.11) that

\[
\begin{align*}
N_f = 0 & \quad a \frac{\Lambda_0}{6} = \frac{1}{\sqrt{2}} \frac{E_2 + \vartheta_3^4 + \vartheta_4^4}{\vartheta_2 \vartheta_3}, & a_D \frac{\Lambda_0}{6} &= -\frac{i}{6} \left(\frac{2E_2 - \vartheta_3^4 - \vartheta_4^4}{\vartheta_3 \vartheta_4}\right)(q_D), \\
N_f = 1 & \quad a \frac{\Lambda_1}{2\sqrt{2}} = \frac{1}{\sqrt{A}} \frac{2E_2 + \sqrt{2f}}{\vartheta_2 \vartheta_3}, & a_D \frac{\Lambda_1}{2\sqrt{2}} &= \frac{1}{\sqrt{A}} \left(\frac{2E_2 - \sqrt{2f}}{\sqrt{A_D}}\right)(q_D), \\
N_f = 2 & \quad a \frac{\Lambda_2}{6\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{E_2 + 3(\vartheta_3 \vartheta_4)^2 + \vartheta_3^4 + \vartheta_4^4}{\vartheta_2^2}, & a_D \frac{\Lambda_2}{6\sqrt{2}} &= \frac{1}{6\sqrt{2i}} \left(\frac{E_2 - \vartheta_3^4 + \vartheta_4^4}{\vartheta_2^2}\right)(q_D), \\
N_f = 3 & \quad a \frac{\Lambda_3}{48\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{E_2 + 3(\vartheta_3 \vartheta_4)^2 + \vartheta_3^4 + \vartheta_4^4}{\vartheta_2^2}, & a_D \frac{\Lambda_3}{48\sqrt{2}} &= \frac{1}{48\sqrt{2}} \left(\frac{E_2 - 3(\vartheta_3 \vartheta_4)^2 - \vartheta_3^4 - \vartheta_4^4}{\vartheta_2^2}\right)(q_D),
\end{align*}
\]

(A.7)

where \(a\) is presented as a function of \(q\) while \(a_D\) as a function of \(q_D = e^{2\pi i \tau_D}\).

Finally the contact term is given by \([8]\)

\[
T(u) = -\frac{1}{24} E_2(\tau) \left(\frac{du}{da}\right)^2 + \frac{1}{3} \left( u + \frac{\Lambda_3^2}{64} \delta_{N_f, 3} \right).
\]

(A.8)

Expressing this in terms of modular functions is immediate with the use of (A.1).
References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087.
[2] N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484, hep-th/9408099.
[3] E. Witten, Commun. Math. Phys. 117 (1988) 353.
[4] E. Witten, Math. Research Letters, 1 (1994) 769, hep-th/9411102.
[5] S.K. Donaldson, Bull. Amer. Math. Soc. 33 (1996) 45.
[6] J. Morgan, “The Seiberg-Witten equations and applications to the topology of smooth four-manifolds”, Princeton Univ. Press, Princeton, New Jersey, 1996.
[7] G. Moore and E. Witten, “Integration over the $u$-plane in Donaldson theory”, hep-th/9709193.
[8] M. Mariño and G. Moore, “Integrating over the Coulomb branch in $\mathcal{N} = 2$ gauge theory”, hep-th/9712062.
[9] M. Mariño and G. Moore, “The Donaldson-Witten function for gauge groups of rank larger than one”, hep-th/9802185.
[10] A. Losev, N. Nekrasov and S. Shatashvili, “Issues in topological gauge theory”, hep-th/9711108; “Testing Seiberg-Witten solution”, hep-th/9801061.
[11] M. Mariño and G. Moore, “Donaldson invariants for non-simply connected manifolds”, hep-th/9804104.
[12] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, “RG equations from Whitham hierarchy”, hep-th/9802007.
[13] K. Takasaki, “Integrable hierarchies and contact terms in $u$-plane integrals of topologically twisted supersymmetric gauge theories”, hep-th/9803217.
[14] J.D. Edelstein, M. Mariño and J. Mas, “Whitham hierarchies, instanton corrections and soft supersymmetry breaking in $N = 2$ $SU(N)$ super Yang-Mills theory”, hep-th/9805172.
[15] L. Álvarez-Gaumé, M. Mariño and F. Zamora, Int. J. Mod. Phys. A13 (1998) 403, hep-th/9703072.
A. Bilal and F. Ferrari, “The BPS spectra and superconformal points in massive $N = 2$ supersymmetric QCD”, hep-th/9706145.

[16] A. Sen, Phys. Rev. D55 (1997) 2501, hep-th/9608008.

[17] K. Ito and S.-K. Yang, Phys. Lett. B366 (1996) 165, hep-th/9507144.

[18] A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929, hep-th/9505150.

[19] M. Matone, Phys. Lett. B357 (1995) 342, hep-th/9506102.
J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. B367 (1996) 145, hep-th/9510129.
T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A11 (1996) 131, hep-th/9510138.

[20] E. Witten, Selecta Mathematica 1 (1995) 383, hep-th/9505186.

[21] P. Kronheimer and T. Mrowka, J. Diff. Geom. 41 (1995) 573.

[22] K.G. O’Grady, J. Diff. Geom. 35 (1992) 415.

[23] E. Witten, J. Math. Phys. 35 (1994) 5101, hep-th/9403195.

[24] J.M.F. Labastida and M. Mariño, Nucl. Phys. B456 (1995) 633, hep-th/9507140.

[25] P.C. Argyres and M.R. Douglas, Nucl. Phys. B448 (1995) 93, hep-th/9505062.

[26] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, Nucl. Phys. B471 (1996) 430, hep-th/9603002.

[27] P.C. Argyres, R.N. Plessar, N. Seiberg and E. Witten, Nucl. Phys. B461 (1996) 71, hep-th/9511154.

[28] J.A. Minahan and D. Nemeschansky, Nucl. Phys. B468 (1996) 72, hep-th/9601059.
[29] G. Bonelli and M. Matone, Phys. Rev. Lett. 76 (1996) 4107, hep-th/9602174.
A. Ritz, “On the beta-function in $N=2$ supersymmetric Yang-Mills theory”, hep-th/9710112