Deep Extreme Value Copulas for Estimation and Sampling

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Abstract

We propose a new method for modeling the distribution function of high dimensional extreme value distributions. The Pickands dependence function models the relationship between the covariates in the tails, and we learn this function using a neural network that is designed to satisfy its required properties. Moreover, we present new methods for recovering the spectral representation of extreme distributions and propose a generative model for sampling from extreme copulas. Numerical examples are provided demonstrating the efficacy and promise of our proposed methods.

1 Introduction

Modeling the occurrence of extreme events is an important task in many disciplines, such as medicine, environmental science, engineering, and finance. For example, understanding the probability of a patient having an adverse reaction to medication or the distribution of economic shocks is critical to mitigating the associated effects of these events. However, these events are rare in occurrence and often difficult to characterize with traditional statistical tools. This has been the primary focus of extreme value theory (EVT), which describes how to extrapolate the occurrence of rare events outside the range of available data \cite{1}. In the one-dimensional case, EVT provides remarkably simple models for the distribution of the maximum of an infinite number of independent and identically distributed (i.i.d) random variables. The latter result is due to the fundamental result by Fisher-Tippet-Gnedenko theorem \cite{2}.

The problem in high dimensions becomes more difficult due to the challenge of modeling complex interactions between different extreme variables. These interactions are often captured by extreme value copulas \cite{3}. Unfortunately, these are hard to identify and learn due to the lack of data of simultaneous occurrences of rare events. Moreover, unlike the one-dimensional case, modeling multivariate extreme value (MEV) distributions does not easily permit analytical forms of the underlying density, leading to difficulties in performing inference tasks using conventional methods such as maximum likelihood estimation (MLE). Due to the pioneering work of Pickands \cite{4}, MEV distributions can be entirely determined by tail dependence functions \cite{3}. Therefore, learning MEV distributions amounts to learning and estimating the underlying tail dependence function.

Motivated by these challenges, we present an algorithm for learning multivariate extreme value distributions. More precisely, we propose to model the Pickands dependence function, which is the restriction of the tail dependence function on the unit simplex, using flexible parametric models. By exploiting that Pickands dependence functions turn out to be convex \cite{2}, we consider the use of input convex neural networks (ICNNs), which have the capabilities to approximate any convex function up to desirable accuracy \cite{3, 4}. This allows

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us to enforce the convexity property in Pickands dependence functions while still permitting a flexible model. For training, we exploit a property discussed in [4], stating that a suitable statistic of the covariates can be transformed to be distributed as an exponential random variable with the rate parameter given by the Pickands dependence function. This allows efficient training of the ICNN by optimizing the likelihood function of the exponential distribution using an ICNN to parameterize the rate. We provide numerical results to demonstrate the potential of the proposed Pickands based ICNN for estimating survival probabilities in the context of both synthetic and real data as compared to state of the art methods.

On another front, we provide a method for recovering the spectral representation [7, 8] of MEV distributions which is a key ingredient to create a generative model that makes it possible to directly sample from the underlying MEV distribution. This allows us to use existing samplers such as [9, 10] to sample from the MEV distribution. Furthermore, recovering the spectral representation from the Pickands dependence function also gives additional insight for possible clustering of the extremes, as described in [8].

For completeness, we provide the common literature dealing with the described challenges in MEV distributions.

**Related Work.** A number of techniques have been developed to estimate extreme value copulas from data (see [3] for a self-contained review). Arguably, the most relevant to the present work is that by Pickands [4] where the notion of the multivariate extreme value distribution and a non parametric estimator were first proposed. By appropriately transforming and scaling the marginals, we can simplify the problem to only considering exponential random variables with parameters given by the Pickands function (please refer to Section 3 for more details on such transformations). This allows to use estimators such as MLE on the transformed variables to learn the Pickands dependence function.

Many follow-up works describe various modifications of the method proposed by [4]. Additional approaches such as the work in [11, 12] describe similar, but alternative takes on estimating the dependence function. These methods, however, do not guarantee the estimate to be a convex function, a necessary condition of the Pickands dependence function. In [13], the authors consider a projection of a nonparametric estimator to a convex function represented as a Bernstein polynomial. However, the number of parameters required significantly increases with both the amount of data and the dimensionality, making it difficult for higher dimensional problems or problems with many data points. Our proposed method can be used with any of the previously mentioned nonparametric estimators while also guaranteeing convexity. In practice, we observed that the estimator in [4] is the most effective. For additional details, please refer to the review on extreme value copulas in [14].

**Main contributions.** First, we describe an approach to parameterize Pickands dependence functions that scales effectively to high dimensions. Second, we provide numerical results to demonstrate the effectiveness of the proposed approach for both synthetic and real data. Finally, we propose an algorithm that permits to recover the spectral representation of Pickands dependence functions which in turns allows to directly sample from the MEV distribution without resorting to the underlying density.

**Outline.** In Section 2 we provide the necessary background on extreme value theory and its extension to the high dimensional setting. In Section 3 we describe how to model and train Pickands dependence functions with ICNNs. Then, in Section 4 we provide details on how to recover the underlying spectral representations of MEV distributions and propose an algorithm for sampling. We provide numerical results in Section 5. Finally, we provide some concluding remarks in Section 6.

## 2 Extreme value copulas and Pickands dependence function

### 2.1 Extreme value theory

Extreme value theory (EVT) is a fundamental tool that characterizes the behavior of maxima of $n$ independent and identically distributed (i.i.d) random variables $X_1, \ldots, X_n$ with continuous distribution function $F$. More precisely, let $M_n = \max_{1 \leq i \leq n} X_i$, we are interested in finding sequences of real numbers $a_n > 0$ and $b_n$ such that the limit $\Pr \left[ (M_n - b_n) / a_n \leq x \right] \to H(x)$ as $n \to \infty$ is non-degenerate. We then say that $F$ is in the maximum domain of attraction of $H$ or equivalently $F \in \text{MDA}(H)$. This limit is fully identified by the
generalized extreme value (GEV) distribution given by:

$$H_\xi(x) = \begin{cases} 
\exp\left(-\left(1 + \xi x\right)^{-1/\xi}\right), & \text{if } \xi \neq 0 \\
\exp(-e^{-x}), & \text{if } \xi = 0 
\end{cases}$$

(1)

where $1 + \xi x > 0$ and $\xi$ is the shape parameter indicating the thickness of the tail. The following theorem due to Fisher, Tippet and Gnedenko states the fundamental result of EVT.

**Theorem 1.** If $F \in \text{MDA}(H)$, and if the limit $H$ exists then it belongs to the class of GEV distributions, i.e. $H = H_\xi$ for some real number $\xi$.

### 2.2 Extreme value copulas

After characterizing the limit behavior of maxima of i.i.d random variables, the natural next step is to characterize the joint behavior of component-wise maxima of i.i.d random vectors. In more concrete terms, let $(X_i^{(1)}, \ldots, X_i^{(d)}) \in \mathbb{R}^d$ for $i \in \{1, \ldots, n\}$ be a sample of i.i.d random vectors with common continuous probability distribution $F$, marginals $F_1, \ldots, F_d$ and copula $C_F$. Recall that $C_F : [0,1]^d \to [0,1]$ satisfies:

$$C_F(u_1, \ldots, u_d) = \mathbb{P}\left[F_1(X^{(1)}) \leq u_1, \ldots, F_d(X^{(d)}) \leq u_d\right].$$

Let the vector of component-wise maxima be given by:

$$M_n = \left(M_n^{(1)}, \ldots, M_n^{(d)}\right),$$

where $M_n^{(k)} = \max_{1 \leq i \leq n} X_i^{(k)}$, $k \in \{1, \ldots, d\}$. Similarly, each component-wise maxima $M_n^{(k)}$ is normalized with sequences of real numbers $a_n^{(k)} > 0$ and $b_n^{(k)}$ such that the corresponding limiting marginal is non-degenerate. Let $C_n$ be the copula of $M_n$ given by:

$$C_n = \left(C_n^{(1)}, \ldots, C_n^{(d)}\right).$$

Then the following relation holds.

$$C_n(u_1, \ldots, u_d) = C_F(u_1^{1/n}, \ldots, u_d^{1/n})^n,$$

for all $(u_1, \ldots, u_d) \in [0,1]^d$. Similar to the one-dimensional case, we are interested in finding the limiting copula $C$ of $C_n$ as $n \to \infty$. The limiting copula is called an extreme-value copula and we similarly say that $C_F \in \text{MDA}(C)$. Before stating the main theorem that characterizes extreme value copulas, it is necessary to provide some technical definitions related to tail dependence functions.

**Definition 1** (Tail dependence function). A function $\ell : [0, \infty)^d \to [0, \infty)$ is a tail dependence function if for all $(x_1, \ldots, x_d) \in [0, \infty)^d$, the following conditions are satisfied:

- (i) $\ell$ is convex and homogeneous of order 1, i.e. $\ell(cx_1, \ldots, cx_d) = c \cdot \ell(x_1, \ldots, x_d)$, for all $c > 0$.
- (ii) $\max_{1 \leq k \leq d} x_k \leq \ell(x_1, \ldots, x_d) \leq \sum_{k=1}^d x_k$.

With the above definition of tail dependence functions at hands, we are in a position to state the following theorem which provides the mathematical foundation of extreme value copulas.
Theorem 2. If $C$ is a $d$-variate extreme value copula then there exists a tail dependence function $\ell : [0, \infty)^d \to [0, \infty)$ such that:

$$C(u_1, \ldots, u_d) = e^{-\ell(-\log u_1, \ldots, -\log u_d)},$$

where $(u_1, \ldots, u_d) \in (0, 1]^d$. Using the homogeneity property of $\ell$, the extreme value copula $C$ can be rewritten as:

$$C(u_1, \ldots, u_d) = e^{\left(\sum_{k=1}^d \log u_k\right)\left(\frac{\log u_1}{\sum_{k=1}^d \log u_k}, \ldots, \frac{\log u_d}{\sum_{k=1}^d \log u_k}\right)},$$

where $A$ is known as the Pickands dependence function, which can be thought of as the restriction of $\ell$ to the unit simplex $\Delta_{d-1} = \{w = (w_1, \ldots, w_d) \in [0, \infty)^d : \sum_{k=1}^d w_k = 1\}$. The Pickands function $A$ is known to be convex and satisfies:

$$\max_{1 \leq k \leq d} w_k \leq A(w_1, \ldots, w_d) \leq 1$$

for all $w = (w_1, \ldots, w_d) \in \Delta_{d-1}$. These properties characterize Pickands dependence functions for $d = 2$ but not necessarily for general $d$.

An equivalent formulation of the Pickands dependence function which proves to be essential in the design of sampling algorithms (to be discussed in Section 4) is the integral representation given below.

Definition 2 (Integral form of the Pickands dependence function [15, 8]). For any Pickands dependence function $A(\cdot, \ldots, \cdot)$, there exists a Borel measure $\mu$ on $\Delta_{d-1}$ satisfying $\int_{\Delta_{d-1}} s_k d\mu(s) = 1$ for all $k \in \{1, \ldots, d\}$ and $s = (s_1, \ldots, s_d) \in \Delta_{d-1}$ such that:

$$A(w_1, \ldots, w_d) = \int_{\Delta_{d-1}} \max(w_1s_1, \ldots, ws_d) d\mu(s).$$

Next, we provide details on how to model and train Pickands dependence functions using input convex neural networks.

3 Modeling and training Pickands dependence functions with input convex neural networks

3.1 Input convex neural networks

Since the Pickands dependence function $A$ is convex over $\Delta_{d-1}$, we are interested in finding a flexible parameterization of $A$ that also enforces convexity over the input domain. Input convex neural networks (ICNNs) provide the convexity guarantee while also having the capability of approximating convex functions in the sup norm [5, 6]. Using ICNNs, we mitigate issues faced by previous estimators [11, 12, 13] in enforcing convexity or projecting a nonparametric estimator onto a convex basis. Besides convexity, we enforce (4) both by designing the architecture carefully and adding an additional loss term. We provide further details on enforcing (4) in the supplemental material. Following, we provide full details on how to train the Pickands based ICNN from data.

3.2 Training the Pickands dependence function

After modeling the Pickands dependence function with ICNNs, the remaining challenging task consists of training the model Pickands function $A(w; \theta)$ to fit the data. One possible approach is to obtain the underlying probability density function (PDF) and then train the ICNN with maximum likelihood estimation (MLE). The drawback of such method is the need to differentiate the $d$-variate CDF in order to obtain the corresponding PDF. This may be computationally complex particularly in high dimensions. Let $F_k$ denote
the marginal CDF \[1\] of the \(k\)th normalized component wise maxima \(\bar{M}^{(k)}_n = \frac{M^{(k)}_n - \bar{w}^{(k)}_n}{a^{(k)}_n}, k \in \{1, \cdots, d\}\) and let \(w = (w_1, \cdots, w_d) \in \Delta_{d-1}\). We perform the following transformations on \(\bar{M}^{(k)}_n\) (Please see Section 3 of [14]):

\[
\bar{M}^{(k)}_n = -\log F_k(\bar{M}^{(k)}_n), \forall k \in \{1, \cdots, d\},
\]

\[
Z_w = \min_{1 \leq k \leq d} \frac{\bar{M}^{(k)}_n}{w_k}.
\]

Then, the following relation holds:

\[
P[Z_w > z] = P[\bar{M}^{(1)}_n > w_1 z, \cdots, \bar{M}^{(d)}_n > w_d z]
\]

\[
= P[F_k(\bar{M}^{(1)}_n) < e^{-zw_1}, \cdots, F_k(\bar{M}^{(d)}_n) < e^{-zw_d}]
\]

\[
= e^{-zA(w)}.
\]

This means that the random variables \(Z_w\) are exponentially distributed with parameter given by the Pickands dependence function \(A(w)\).

In light of above, we can learn the Pickands dependence function \(A(w)\) by fitting the model \(A(w; \theta)\) to samples \(Z_w\) using MLE. This can be done by training the ICNN modeling \(A(w; \theta)\) with stochastic gradient descent (SGD) to match the data points \(Z_w\) as follows:

\[
A^*(w) = \arg\min_{\theta} E_{Z_w} L(Z_w; \theta),
\]

where \(L(Z_w; \theta) = A(w; \theta) Z_w - \log A(w; \theta)\). It should be noted that maximizing this likelihood with the ICNN guarantees a convex estimate similar to the nonparametric estimator defined in [4]. Alternative losses could additionally be considered, such as reformulating the loss with respect to the estimators defined in [12, 11] However, we have empirically found the MLE approach described in [4] gives superior performance for most applications. Moreover, it follows naturally from the original formulation of Pickands. Details of the training procedure are summarized in Algorithm [4].

The choice of hyperparameters such as the block size, number of blocks, learning rate for Adam and batch size are provided in section [5] with settings depending on the application.

### 3.3 Survival probability estimation

Following the training of the Pickands dependence function \(A(w; \theta^*)\), our goal is to estimate survival probabilities. More precisely, let \((\gamma_1, \cdots, \gamma_d) \in \mathbb{R}^d\) be a \(d\)-dimensional vector of thresholds, we are interested in calculating the following survival probability:

\[
P[M^{(1)}_n > \gamma_1, \cdots, M^{(d)}_n > \gamma_d] = P[\bar{M}^{(1)}_n > \tilde{\gamma}_1, \cdots, \bar{M}^{(d)}_n > \tilde{\gamma}_d],
\]

where \(\tilde{\gamma}_k = \frac{2-k \bar{w}^{(k)}_n}{a^{(k)}_n}, k \in \{1, \cdots, d\}\). Note that calculating the survival probability in (9) using the model extreme value copula is not straightforward. To calculate this, we use the following change-of-variable trick, which we prove in the supplementary material.

**Proposition 1.** Let \(G_k(x) := F_k^{-1}(1 - F_k(x))\) for \(k \in \{1, \cdots, d\}\), then the random variables \(G_k(\bar{M}^{(k)}_n)\) and \(\bar{M}^{(k)}_n\) have the same marginal CDF \(F_k\), for \(k \in \{1, \cdots, d\}\), and

\[
P[\bar{M}^{(1)}_n > \tilde{\gamma}_1, \cdots, \bar{M}^{(d)}_n > \tilde{\gamma}_d] = P[G_1(M^{(1)}_n) < G_1(\tilde{\gamma}_1), \cdots, G_d(M^{(d)}_n) < G_d(\tilde{\gamma}_d)].
\]

\footnote{Fitting the marginals can be done using MLE [4] or the L-moments method [16].}
We then describe the relationship between Pickands functions and stationary max-stable processes and use this relationship to describe a sampling algorithm. This ultimately leads us to recast the MEV sampling in terms of previous work on sampling from max-stable processes.

To the best of our knowledge there are no general sampling algorithms for general extreme value copulas \[9, 10\]. We additionally propose a heuristic for sampling based on the relationship between max-stable processes and Pickands dependence functions which we outline in Algorithm 4. To give a brief outline of how we proceed, we begin by describing max-stable processes and the spectral representation of stationary max-stable processes. We then describe the relationship between Pickands functions and stationary max-stable processes and use this relationship to describe a sampling algorithm. This ultimately leads us to recast the MEV sampling in terms of previous work on sampling from max-stable processes.

**Algorithm 1** Training the Pickands dependence function with SGD (Pickands-ICNN)

1. **Input:** \(\{X_i^{(1)}, \cdots, X_i^{(d)}\}_{i=1}^{N}, N = B \times n\) samples of i.i.d random vectors where \(B\) is the number of blocks of data and \(n\) is the size of each block.
2. Take block maxima of size \(n\) and obtain component-wise maxima: \(\{M_{n,b}^{(1)}, \cdots, M_{n,b}^{(d)}\}_{b=1}^{B}\) where \(M_{n,b}^{(k)} = \max_{(b-1)n+1 \leq i \leq bn} X_i^{(k)}, (k,b) \in \{1,\cdots, d\} \times \{1,\cdots, B\}\).
3. Fit a GEV to each component-wise maxima \(\{M_{n,b}^{(k)}\}_{b=1}^{B}\) and obtain \(\{\tilde{M}_{n,b}^{(k)}\}_{b=1}^{B}\) then estimate marginals \(F_k\) for each \(k \in \{1,\cdots, d\}\).
4. **Initialize** the parameters \(\theta\) of the ICNN

   **Repeat:**
   5. Randomly sample a minibatch of training data \(\{\tilde{M}_{n,b}^{(k)}\}_{b\in\text{batch}}\) and uniformly sample \(w\in\Delta_{d-1}\).
   6. Transform samples according to Equations (6) and (7) and obtain samples \(\{Z_{w,b}\}_{b\in\text{batch}}\).
   7. Compute gradient \(\nabla_{\theta} \sum_{b\in\text{batch}} L(Z_{w,b}; \theta)\).
   8. Update \(\theta\) with Adam \[17\] until convergence

   **Output:** \(A(w; \theta_*)\).

This proposition implies that the transformed variables \(G_k(\tilde{M}_{n}^{(k)})\) are samples from extreme value distributions, therefore we train a model Pickands dependence function on these transformed variables, and finally evaluate the corresponding extreme value copula on \((F_1(\tilde{\gamma}_1), \cdots, F_d(\tilde{\gamma}_d))\). Details on how to estimate the survival probability in \[9\] are summarized in Algorithm 2.

**Algorithm 2** Estimating survival probabilities with the Pickands dependence function

1. **Input:** \(\{\tilde{M}_{n,b}^{(k)}\}_{b=1}^{B}\), thresholds: \((\gamma_1, \cdots, \gamma_d)\).
2. Train a model \(A(w; \theta)\) with the transformed variables \(\{(G_1(\tilde{M}_{n,b}^{(1)}), \cdots, G_d(\tilde{M}_{n,b}^{(d)}))\}_{b=1}^{B}\) using Algorithm 1 and obtain \(A(w; \theta_*)\).
3. Evaluate the Pickands copula:

   \[C(1 - F_1(\tilde{\gamma}_1), \cdots, 1 - F_d(\tilde{\gamma}_d))\],

   where \(C\) is calculated as in Equation (3) with \(A = A(w; \theta_*)\).

4  **Sampling from multivariate extreme value distributions**

While learning MEV distributions from data is important for estimating survival probabilities and CDFs, it is also useful to simulate possible scenarios by sampling from the estimated MEV distribution. We propose a method for generating samples from an MEV distribution with an arbitrary Pickands dependence function. To the best of our knowledge there are no general sampling algorithms for general extreme value copula in arbitrary dimensions. We exploit the exact sampling algorithms for the infinite dimensional analogue of MEV distributions known as max-stable processes to sample from general Pickands copulas \[9, 10\]. We additionally propose a heuristic for sampling based on the relationship between max-stable processes and Pickands dependence functions which we outline in Algorithm 4. To give a brief outline of how we proceed, we begin by describing max-stable processes and the spectral representation of stationary max-stable processes. We then describe the relationship between Pickands functions and stationary max-stable processes and use this relationship to recast the MEV sampling in terms of previous work on sampling from max-stable processes.
4.1 Pickands dependence and max-stable processes

4.1.1 Definition of max-stable processes

We initially review the definition of a max-stable process as defined in [7]. Let $X_1(t), X_2(t), \ldots, X_n(t)$ be independent copies of a continuous stochastic process on a compact set $\mathcal{T}$. If there exists normalizing functions $a_n(t) > 0$ and $b_n(t) \in \mathbb{R}$ for all $t \in \mathcal{T}$ such that the limit

$$\frac{\max_{1 \leq i \leq n} X_i(t) - b_n(t)}{a_n(t)} \to M(t) \text{ as } n \to \infty$$

is non-degenerate then the limiting process $M(t)$ is a max-stable process. A key property of max-stable processes is that any univariate marginal is distributed as a GEV distribution and any subset of marginals is distributed as a MEV distribution. This motivates the relationship to MEV distributions parameterized by Pickands dependence functions. We then consider this analysis with respect to stationary max-stable processes and the ensuing spectral decomposition of stationary max-stable processes.

4.1.2 Spectral decomposition of stationary max-stable processes

Stationary max-stable processes can be intuitively interpreted as i.i.d samples from infinite dimensional extreme value distributions. A stationary max-stable process can be decomposed by the spectral representation defined in [7]. Suppose that $M(t)$ has unit Fréchet margins and is stationary. Then, $M(t)$ can be written as:

$$M(t) = \max_{i \geq 1} \xi_i Y^+_i(t), \quad t \in \mathcal{T}.$$  \hfill (11)

$\{Y_i(t)\}_{i \geq 1}$ are i.i.d copies of a continuous stochastic process $Y$ defined on $\mathcal{T}$ such that $\mathbb{E}[Y^+(t)] = 1$ with $Y^+(t) = \max\{0, Y(t)\}$ and $\xi_i$ is the $i$th realization of an independent Poisson point process on $[0, \infty)$ with intensity $\xi^{-2} d\xi$.

4.1.3 Margins of max-stable processes as MEV distributions

From the spectral decomposition of the max-stable process, we can recover a new representation for the MEV distribution. Motivated by [7] and using the results in [15, 8], the Pickands dependence function is related to the max stable process through:

$$A(w) = \mathbb{E}\left[ \max_{1 \leq k \leq d} w_k Y^+(t_k) \right], \quad w \in \Delta_{d-1}.$$  \hfill (13)

Intuitively, this suggests that a $d$-dimensional set of samples of the max-stable process are related to the Pickands dependence function. When restricted to the discrete case of MEV distributions, we can rewrite $\{Y^+(t_k)\}_{k=1}^d$ as a $d$-dimensional vector $y = (y_1, \ldots, y_d) \in \mathbb{R}_+^d$.

$$A(w) = \mathbb{E}_y \left[ \max_{0 \leq k \leq d} w_k y_k \right], \quad w \in \Delta_{d-1}.$$  \hfill (12)

Next, we describe how to generate $y$ from a given Pickands dependence function. After learning the generative mechanism for $y$, we can use exact sampling algorithms to sample from the MEV distribution.

4.2 Learning the underlying generator for Pickands dependence

When given a Pickands dependence function $A$, we propose to recover the underlying spectral representation by optimizing a generative model for $y$ in (12). We model $y$ as the output of a function $G(\cdot; \phi) \in \mathbb{R}_+^d$ with parameters $\phi$, i.e. $y = G(z; \phi)$ which maps input samples $z \sim p_z$ to $y$, where $p_z$ is a distribution that is easy to sample from (such as a multivariate Gaussian distribution). We then learn the parameters $\phi$ of the generator by solving the following optimization problem:

$$\min_{\phi} \mathbb{E}_{w \sim \Delta_{d-1}} \left\| A(w) - \mathbb{E}_y \left[ \max_{0 \leq k \leq d} w_k y_k \right] \right\|_2^2 + \left\| \mathbb{E}_y [y] - 1_d \right\|_2^2, \quad (13)$$

where $1_d$ is the $d$-dimensional vector of ones.
where \( y = (y_1, \cdots, y_d) = G(z; \phi), y \in \mathbb{R}_+^d \) and \( z \in \mathbb{R}^k \sim p_z \). The last expectation is required to enforce the condition of \( \mathbb{E}[Y(\cdot)] = 1 \) from [15]. The expectations with respect to \( y \) in [13] are approximated using sample mean with samples from the generator. Note that optimizing the objective in [13] only requires samples of \( y \) rather than samples from the MEV distribution, thereby bypassing additional complexities required by sampling from the full MEV distribution while training. The full details of how to train the generator are outlined in Algorithm 3.

Algorithm 3 Training a generator for a Pickands copula

1: **Input:** \( A(w), p_z \), tolerance parameter \( \epsilon \)
2: Initialize parameters \( \phi \) of generator \( G(\cdot; \phi) \)
3: Sample \( \{w^{(j)}\}_{j=1}^{N_{\text{simplex}}} \) samples uniformly over \( \Delta_{d-1} \).
4: while \( \sum_{j=1}^{N_{\text{simplex}}} \mathcal{L}(w^{(j)}; \phi) > \epsilon \) do
5: Sample \( \{w^{(j)}\}_{j=1}^{N_{\text{simplex}}} \) samples uniformly over \( \Delta_{d-1} \).
6: Sample \( \{y^{(i)}\}_{i=1}^{N_{\text{gen}}} \) where \( y^{(i)} = G(z^{(i)}; \phi), z^{(i)} \sim p_z \) for \( 1 \leq i \leq N_{\text{gen}} \).
7: Define \( \eta(w, y) = \max\{w \circ y\} \) with \( \circ \) denotes the point-wise multiplication.
8: Compute gradient w.r.t \( \phi \) of \( \sum_{j=1}^{N_{\text{simplex}}} \mathcal{L}(w^{(j)}; \phi) \) where:
\[
\mathcal{L}(w^{(j)}; \phi) = \left\| A(w^{(j)}) - \frac{1}{N_{\text{gen}}} \sum_{i=1}^{N_{\text{gen}}} \eta(w^{(j)}, y^{(i)}) \right\|_2^2 + \left\| \frac{1}{N_{\text{gen}}} \sum_{i=1}^{N_{\text{gen}}} y^{(i)} - 1 \right\|_2^2
\]
9: Update \( \phi \) using Adam [17].
10: end while
11: **Output:** \( G(\cdot; \phi_s) \).

Finally, to generate samples from the full MEV distribution, we can consider methods that only require the generator, such as the ones described in [10, 9]. For simplicity, we provide a heuristic inspired by the method in [9] to provide a straightforward procedure of sampling which we describe in Algorithm 4. This approach involves repeated sampling from the generator and unit Fréchet distributions and taking the maximum over the samples. This can be seen as a simulation of [11] since unit Fréchet is equivalent to \( 1/\xi, \xi \sim \text{Exp}(1) \) which describes a Poisson process with rate \( \xi^{-2}d\xi \).

Algorithm 4 Heuristic for sampling from a Pickands copula

1: **Input:** \( A(w) \)
2: Optimize a generator \( G(\cdot; \phi) \) using Algorithm 3
3: for \( i \in \{1, \ldots, N_{\text{gen}}\} \) do
4: Generate \( y^{(i)} \) where \( y^{(i)} = G(z^{(i)}; \phi_s), z^{(i)} \sim p_z \).
5: Sample \( \xi^{(i)} \) from a unit Fréchet distribution.
6: end for
7: Compute the component-wise maxima as:
\[
M = \max_{1 \leq i \leq N_{\text{gen}}} \{\xi^{(i)} \circ y^{(i)}\}.
\]
8: **Output:** \( M \).

5 Results

In this section, we provide numerical results that compare the performance of the proposed ICNN-based model for the Pickands dependence function with some well-known estimators from the literature [11, 12, 4]. We start by evaluating the performance for estimating the Pickands function for some bi-variate symmetric
Supplementary Material.

5.1 Synthetic data

We consider some canonical families of extreme value distributions known as symmetric logistic family where the underlying Pickands function is given by [3]:

\[ A_{\text{SL}}(w) = \left( \sum_{k=1}^{d} w_k^{1/\alpha} \right)^{\alpha}, \quad w \in \Delta_{d-1}, \] (14)

where \( \alpha \in (0, 1) \) is a parameter modeling the degree of dependence between variables ranging from complete dependence (\( \alpha = 0 \)) to complete independence (\( \alpha = 1 \)). We also consider a generalization of this copula, the asymmetric logistic family:

\[ A_{\text{ASL}}(w) = \sum_{b \in \mathcal{P}_d} \left( \sum_{i \in b} (\lambda_{i,b} w_i)^{1/\alpha_b} \right)^{\alpha_b}, \] (15)

where \( \mathcal{P}_d \) is the power set of \( \{1, \ldots, d\} \). The dependence parameters \( \alpha_b \in (0, 1] \) are defined for \( b \in \mathcal{P}_d \) except for all singleton sets. The asymmetry parameters \( \lambda_{i,b} \in [0, 1] \) are randomly generated to satisfy: \( \sum_{b \in \mathcal{P}_d : i \in b} \lambda_{i,b} = 1 \). Exact sampling of distributions of this type are described in [19]. Note that for both the symmetric and asymmetric copulas, the marginals are distributed according to the standard Fréchet distribution. In this case, we skip Steps 2 and 3 of Algorithm 1 and consider empirical marginals given by the ranks of different samples.

Estimation of survival probabilities

We start by comparing the MSE of survival probabilities for \( d = 2 \) where the true Pickands dependence function is given by the symmetric or asymmetric model described respectively in (14) and (15) for different degrees of dependence \( \alpha \). We compute the exact values of the survival probability and consider survival
We additionally showcase the ability of the proposed method to model high dimensional extreme value well for both distributions considered, with the worst performance occurring in the nearly independent cases. To determine the efficacy of sampling from an arbitrary Pickands copula, we consider two synthetic examples. We estimate accuracy in areas where more data observations are available.

|             | Pickands | CFG    | BDV     | Proposed          |
|-------------|----------|--------|---------|------------------|
| Ozone       | 3.16(±4.27)×10^{-3} | 1.59(±3.21)×10^{-1} | 2.98(±4.05)×10^{-3} | 1.54(±2.04)×10^{-3} |
| Wind        | 2.57(±4.53)×10^{-3} | 7.31(±21.8)×10^{-2} | 8.02(±14.0)×10^{-4} | 7.44(±12.9)×10^{-4} |
| Commodities | 3.12(±4.68)×10^{-3} | 3.12(±4.64)×10^{-3} | 2.97(±4.26)×10^{-3} | 3.03(±3.92)×10^{-3} |

Table 1: MSE performance on the test set for different estimators and datasets. All data are for the 75th percentile and above.

probabilities associated with margins above the 75th percentile. As shown in Figures [1a] and [1b], the proposed Pickands-ICNN estimator achieves the lowest MSE performance for almost all degrees of dependence ρ for both the the symmetric and asymmetric logistic models. This is due in part to the flexibility of the proposed Pickands-ICNN and the enforcing of convexity through the ICNN.

### Discrepancy in estimated Pickands dependence functions

We additionally showcase the ability of the proposed method to model high dimensional extreme value distributions. To do this, we train the Pickands-ICNN with data for dimensions \{16, 64, 256, 512, 1024\} and compare the estimated \( \hat{A} \) to the true \( A \). Specifically, we compute the mean squared error between the true Pickands dependence and the estimated one for 1000 uniformly sampled points in \( \Delta_{\rho-1} \). The results are illustrated for \( \rho = 0.5 \) in Figures [1d] and [1c]. As shown by Figures [1d] and [1c], the proposed Pickands-ICNN-based estimator achieves the lowest MSE performance for most values of the dimension \( d \). The results suggest the proposed method is effective in recovering the Pickands dependence for high dimensional data.

#### 5.2 Real data

In order to determine the efficacy of the proposed method on real data, we consider data on extreme ozone levels \( (d = 4) \), wind gusts \( (d = 10) \), and commodity prices \( (d = 3) \). Details for each dataset are in the supplemental materials. The main challenge associated to real data is the lack of a ground truth Pickands function for comparison purposes. We therefore resort to computing the accuracy of the model estimate w.r.t the empirical estimate over a series of thresholds. This is quantified as:

\[
\frac{1}{|Q|} \sum_{\gamma \in Q} \left( \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}_{\{M_{n,b} \geq \gamma\}} - P_\theta(M_n \geq \gamma) \right)^2,
\]

where \( M_{n,b} = (M^{(1)}_{n,b}, \ldots, M^{(d)}_{n,b}) \) is the \( d \)-dimensional vector of point-wise maxima and \( P_\theta \) is the estimated survival probability computed using Algorithm [2] and \( Q \) is a set of thresholds to consider. In general, we are interested in thresholds corresponding to extremes in the tail; however, due to the lack of data in that region, we estimate accuracy in areas where more data observations are available.

#### 5.3 Sampling from the copula

To determine the efficacy of sampling from an arbitrary Pickands copula, we consider two synthetic examples using the previously described analytical MEV distributions. In this experiment, we train the generator \( G(\cdot; \phi) \) in [13] based on the given Pickands dependence function. To determine the efficacy of the sampling algorithm, we estimate the Pickands dependence function using the nonparametric CFG estimator with empirical marginals and compare to the CFG estimate of samples generated through the exact simulation method. We consider the CFG estimator due to its ubiquity in the literature and its well regarded status as a standard estimator for the Pickands dependence function. The results for generating 225 dimensional samples are shown in figures [2a] and [2b]. The figures suggest that our sampling algorithm performs comparatively well for both distributions considered, with the worst performance occurring in the nearly independent cases (\( \rho = 1 \)).
Figure 2: MSE (solid) and relative error (dashed) of CFG estimate for 1000 samples and 1000 simplex points for $d = 225$ at various $\alpha \in (0, 1)$ for $A_{SL}$ (2a) and $A_{ASL}$ (2b) for learned sampled data (red) and exact sampled (blue).

Figure 3: Comparison of MSE (solid) and relative error (dashed) for 1000 samples and 1000 simplex points of CFG estimate for Pickands function of learned sampled data (red) and exact sampled (blue). Figures are for $d = \{2, 16, 256, 784, 1024\}$ dimensional samples with $\alpha = 0.5$. 
Furthermore, the error of the CFG estimate for the proposed sampling method (red) and the exact sampling (blue) follow very similar trends in errors, suggesting the proposed sampling method is effectively representing the true distribution. The experiments are then repeated where we vary the dimension of the samples and compare with real data samples in Figures 3a and 3b. The dependence parameter $\alpha$ is fixed to be $\alpha = 0.5$ for both experiments. The results suggest the proposed sampling method is comparable to the exact sampling method, with certain dimensions performing better than others. This is likely due to the need for more careful choices of hyperparameters for different dimensions, but we leave that analysis as future work.

6 Concluding remarks

In this paper, we introduced a new method for modeling Pickands dependence functions for arbitrary dimensions using input convex neural networks. This allows for better representations of the underlying MEV distribution due to the representational power of ICNNs. We additionally present a method for recovering the spectral representation of MEV distributions, which makes it possible to create a generative model that samples from the underlying MEV distribution. Numerical results have been provided to empirically demonstrate the effectiveness of the proposed methods in their respective tasks. However, it is worth mentioning that these methods can still be further be improved by considering the following limitations.

Limitations of Pickands-ICNN

ICNNs can guarantee convexity in approximating Pickands dependence functions for high dimensions while maintaining a reasonable number of parameters. However, it remains a challenge to optimize the parameters using standard deep learning techniques. Specifically, choosing appropriate hyperparameters is a difficult and opaque task that requires additional care. This is an attribute where nonparametric methods are beneficial, at the cost of being unable to guarantee convexity. Additional progress on understanding properties of training deep neural networks should improve the representational capabilities of the ICNN given its theoretical potential to approximate convex functions to arbitrary precision.

Future Work

The proposed methods have possible applications in a variety of modeling situations. For instance, understanding the spectral measure is important for finding groups of variables that are extreme simultaneously [8]. Therefore, an important but non-trivial extension of the present work lies in capturing clusters of variables and understanding the joint interaction between them. Finally, applications of high dimensional extremes are important in understanding robustness properties of neural networks [20]. The proposed work provides foundation for such extensions.

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A Proof of Proposition 1

With the change-of-variable $G_k(x) := F_k^{-1}(1 - F_k(x))$ for $k \in \{1, \cdots, d\}$, it easily follows that the random variables $G_k(M_n^{(k)}) \sim F_k$. Moreover, the survival probability can be written as:

$$P[M_n^{(1)} > \gamma_1, \cdots, M_n^{(d)} > \gamma_d] = P[1 - F_1(M_n^{(1)}) < 1 - F_1(\gamma_1), \cdots, 1 - F_d(M_n^{(d)}) < 1 - F_d(\gamma_d)]$$

$$= P[G_1(M_n^{(1)}) < G_1(\gamma_1), \cdots, G_d(M_n^{(d)}) < G_d(\gamma_d)]$$

$$= C(1 - F_1(\gamma_1), \cdots, 1 - F_d(\gamma_d)),$$

where $C$ is the copula of $(G_1(M_n^{(1)}), \cdots, G_d(M_n^{(d)}))$.

B Pickands, CFG and BDV estimators

**Pickands Estimator**

The Pickands estimator [4] is built following the transformations (6) and (7) in the paper. The estimator is obtained by exactly maximizing the likelihood (Equation (8) in the paper) resulting in the following non-parametric estimate:

$$\hat{A}_{\text{Pickands}}(w) = \left(\frac{1}{B} \sum_{i=1}^{B} Z_{w,i}\right)^{-1}. \quad (17)$$

**CFG Estimator**

The CFG estimator [12] is constructed following the observation:

$$E \log Z_w = - \log \hat{A}(w) - \gamma,$$

where $\gamma = - \int_0^\infty \log xe^{-x}dx$ denotes the Euler’s constant. The CFG estimator is thus given by:

$$\hat{A}_{\text{CFG}}(w) = \exp \left[-\gamma - \frac{1}{B} \sum_{i=1}^{B} \log Z_{w,i}\right]. \quad (18)$$

In our main submission we use a similar estimator, with the correction term presented in [21]:

$$\hat{A}_{\text{CFG},C}(w) = \exp \left(\log \hat{A}_{\text{CFG}}(w) - \sum_{k=1}^{d} w_k \log \left(\hat{A}_{\text{CFG}}(e_k)\right)\right), \quad (19)$$

where $e_k$ is the $k$-th canonical basis vector.

**BDV Estimator**

We propose an $d$-dimensional extension to the bivariate estimator described in [11]. We begin by defining the minimum distance estimator between the true CDF, $C(u)$ and the one estimated by the Pickands function $A(w)$.

$$\int_{[0,1]^d} \left[ \log C(u) - \sum_{k=1}^{d} \log u_k A \left( \frac{\log(u)}{\sum_k \log(u_k)} \right) \right]^2 du$$

$$= \int_{\Delta_d} \int_{0}^{1} (\log C(y^{w_1}, \cdots, y^{w_d}) - \log(y)A(w))^2 (-\log(y))^{d-1} dy dw. \quad (20)$$
We have

\[ \hat{C}(y^{w_1}, \ldots, y^{w_d}) = \frac{1}{B} \sum_{i=1}^{B} \mathbf{1}(F_1(\bar{M}^{(1)}_{n,i}) \leq y^{w_1}, \ldots, F_d(\bar{M}^{(d)}_{n,i}) \leq y^{w_d}) \]

(22)

\[ = \frac{1}{B} \sum_{i=1}^{B} \mathbf{1}(F_1(\bar{M}^{(1)}_{n,i}) \leq y, \ldots, F_d(\bar{M}^{(d)}_{n,i}) \leq y) \]

(23)

\[ = \frac{1}{B} \sum_{i=1}^{B} \mathbf{1}(\max_{1 \leq k \leq d} F_k(\bar{M}^{(k)}_{n,i}) \leq y) \]

(24)

\[ = \frac{1}{B} \sum_{i=1}^{B} \mathbf{1}(\Gamma_{w,i} \leq y), \]

(25)

where \( \Gamma_{w,i} = \exp(-Z_{w,i}) \). Now, if we reorder these so that \( \Gamma_{w,1} \leq \cdots \leq \Gamma_{w,B} \), we have that

\[ \hat{C}(y^{w_1}, \ldots, y^{w_d}) = \begin{cases} 
0 & \text{if } y < \Gamma_{w,1}, \\
\frac{i}{B} & \text{if } \Gamma_{w,i} \leq y < \Gamma_{w,i+1}, \ i \in \{1, \ldots, B-1\}, \\
1 & \text{if } \Gamma_{w,B} \leq y.
\end{cases} \]

(26)

Because \( \log \hat{C}(\cdots) \) is not defined if \( y < \Gamma_{w,1} \), the following modified estimator is considered in [11].

\[ \tilde{C}(y^{w_1}, \ldots, y^{w_d}) := \max \{ C(y^{w_1}, \ldots, y^{w_d}), B^{-\gamma} \} \]

(27)

where \( \gamma \) is any positive real greater or equal than \( \frac{1}{2} \). For convenience, we choose \( \gamma = 1 \) so that:

\[ \tilde{C}(y^{w_1}, \ldots, y^{w_d}) = \begin{cases} 
\frac{1}{B} & \text{if } y < \Gamma_{w,2}, \\
\frac{i}{B} & \text{if } \Gamma_{w,i} \leq y < \Gamma_{w,i+1}, \ i \in \{2, \ldots, B-1\}, \\
1 & \text{if } \Gamma_{w,B} \leq y.
\end{cases} \]

(28)

Finally, as in [11], for any positive weight function \( h : (0, 1) \rightarrow \mathbb{R}_0^+ \), let \( h^*(y) := h(y)(\log y)^2 \),

\[ B_h := \int_0^1 h^*(y) \, dy \quad \text{and} \quad g(x) := -B_h^{-1} \int_0^x \frac{h^*(y)}{\log y} \, dy. \]

(29)

Then, letting \( \Gamma_{w,0} = 0, \Gamma_{w,B+1} = 1 \), we define the BDV estimator \( \hat{A}_{BDV,h} \) as follows

\[ \hat{A}_{BDV,h}(w) = B_h^{-1} \int_0^1 \log \tilde{C}(y^{w_1}, \ldots, y^{w_d}) h^*(y) \, dy \]

(30)

\[ = B_h^{-1} \sum_{i=0}^{B} \int_{\Gamma_{w,i}}^{\Gamma_{w,i+1}} \log \tilde{C}(y^{w_1}, \ldots, y^{w_d}) h^*(y) \, dy \]

(31)

\[ = -\log \frac{1}{n} g(\Gamma_{w,2}) - \sum_{i=2}^{n} \log \frac{i}{n} (g(\Gamma_{w,i+1}) - g(\Gamma_{w,i})) \]

(32)

\[ = - \sum_{i=2}^{n} \log \frac{i-1}{i} - \sum_{i=2}^{n} \log \frac{i}{n} g(\Gamma_{w,i}) \]

(33)

\[ = \sum_{i=2}^{n} \log \left( 1 + \frac{1}{i-1} \right) g(\Gamma_{w,i}) \]

(34)
In our main submission, we use a slightly modified estimator, which proved to have superior performance in our experiments. Recall that, if $A$ is a Pickand’s dependence function, we have $\max(w) \leq A(w) \leq 1$, which implies the true copula verifies:

$$\max(w) \leq \log C(y^{w_1}, \ldots, y^{w_d}) \leq A(w) \leq 1.$$  \hspace{1cm} (35)

Accordingly, we let

$$\text{clamp}_{a,b}(x) := \begin{cases} 
  a & \text{if } x \leq a, \\
  x & \text{if } a < x < b, \\
  b & \text{if } x \geq b,
\end{cases}$$  \hspace{1cm} (36)

and define

$$\tilde{C}(y^{w_1}, \ldots, y^{w_d}) = \exp \left( \text{clamp}_{\log C, \max(w)} \log y \log \tilde{C}(y^{w_1}, \ldots, y^{w_d}) \right),$$  \hspace{1cm} (37)

and

$$\tilde{A}_{\text{BDV,MM},h}(w) = B_h^{-1} \int_0^1 \log \tilde{C}(y^{w_1}, \ldots, y^{w_d}) \frac{h^*(y)}{\log y} dy = B_h^{-1} \sum_{i=0}^B \int_{\Gamma_{w,i}}^{\Gamma_{w,i+1}} \log \tilde{C}(y^{w_1}, \ldots, y^{w_d}) h^*(y) dy$$  \hspace{1cm} (38)

Letting $\Gamma_{w,i}^{(t)} = \text{clamp}_{\Gamma_{w,i},\Gamma_{w,i+1}} \left( \left( \frac{i}{n} \right)^{\max(w)} \right)$, $\Gamma_{w,i}^{(u)} = \text{clamp}_{\Gamma_{w,i},\Gamma_{w,i+1}} \left( \frac{i}{n} \right)$ for $i \in \{0, \ldots, B\}$ and $\eta(x) = B_h^{-1} \int_0^x h^*(y) dy$, we have

$$\tilde{A}_{\text{BDV,MM},h}(w) = B_h^{-1} \sum_{i=0}^B \int_{\Gamma_{w,i}}^{\Gamma_{w,i+1}} \log \tilde{C}(y^{w_1}, \ldots, y^{w_d}) \frac{h^*(y)}{\log y} dy$$  \hspace{1cm} (40)

$$= B_h^{-1} \sum_{i=0}^B \int_{\Gamma_{w,i}}^{\Gamma_{w,i+1}} \max(w) h^*(y) dy + \int_{\Gamma_{w,i}}^{\Gamma_{w,i+1}} \frac{i}{n} \log \frac{h^*(y)}{\log y} dy + \int_{\Gamma_{w,i}}^{\Gamma_{w,i+1}} h^*(y) dy$$  \hspace{1cm} (41)

$$= \sum_{i=0}^B \max(w) \left( \eta(\Gamma_{w,i}^{(t)}) - \eta(\Gamma_{w,i}^{(u)}) \right) - \log \frac{i}{n} \left( g(\Gamma_{w,i}^{(u)}) - g(\Gamma_{w,i}^{(t)}) \right) + \eta(\Gamma_{w,i+1}) - \eta(\Gamma_{w,i}^{(u)})$$  \hspace{1cm} (42)

In our main submission, we use $\tilde{A}_{\text{BDV,MM},h}$ with $h(y) = \frac{1}{\log(y)}$.

C Data Description

Synthetic data

For the synthetic data experiments we consider samples of 100 points from each respective distribution. We use the full dataset for the batch size during training. We additionally sample 1000 points from the simplex for each data point during training.

Ozone data

We consider ozone levels measured at 4 different stations in Sequoia National Park from data that can be downloaded from the National Park Service official website.\footnote{https://ard-request.air-resource.com/data.aspx} The 4 stations are located at Ash Mountain, Lower Kaweah, Grant Grove and Lookout Point. We train the different models on daily maxima of ozone levels at the 4 different stations for the period from January 1984 to December 1996. To encounter the effect
of seasonality, we do not train over the whole period, but we train different models on a single month (training month, e.g. April of each year) and compute accuracy on the consecutive month (validation month e.g. May of the same year). The accuracy is averaged with the specific validation month of each year over the whole period. For the experiments, we consider the following pair of (training/validation) months: (April/May), (May/June), (June/July), and (July/August).

California wind data

We are interested in modeling the extremal relationship of wind gusts between different locations in California. We consider 10 locations in California illustrated in Figure 4. We obtained the data from the Remote Automated Weather Station (RAWS) archive available at at the online repository[^3]. The RAWS data are collected from various time intervals from December 1989 to December 2020. We consider only the time points that occur in the intersection of all the data collected and where all values are valid (i.e. not NaNs or missing). Similarly to the ozone data, and in an effort to reduce seasonality, we consider the daily max wind gust for the different locations for a single month over all the years the data were collected. To evaluate the proposed method, we train and test on data from consecutive months and repeat for multiple sets of months in our dataset. (training/validation months): (June/July), (July/August), (August/September), and (September/October).

Commodities data

We consider the extreme dependency between different commodities such as Copper, Nickel and Zinc. We collect data of daily prices of the different commodities from January 2010 to December 2020 as published in[^4]. For training, we consider monthly maxima of daily prices log returns over a period of two consecutive years. We validate the performance by evaluating accuracy over next three years. For example we consider the following pairs of [(training years),(validation years)]: [(2010, 2011), (2012, 2013, 2014)], [(2012, 2013), (2014, 2015, 2016)], [(2014, 2015), (2016, 2017, 2018)], [(2016, 2017), (2018, 2019, 2020)].

D Further details on experiments

Architecture Details

For learning the Pickands dependence function, in all real data experiments we used 24 width and 3 depth ICNNs. For synthetic data, we used 16 width and 4 depth ICNNs. All activations used were LeakyReLU activation functions. We initialize the Pickands dependence function to be the output of the ICNN, i.e. \( A(w; \theta) = ICNN(w; \theta) \) and then force it to be 1 on the corners. To accomplish that, let \( e_k \) be the \( k \)th \( d \)-dimensional canonical basis vector. We then set \( b \) to be the vector with \( k \)th component \( b_k = ICNN(e_k; \theta) \) for \( k \in \{1, \cdots, d\} \) and set \( A(w; \theta) = ICNN(w; \theta) = \langle w, b \rangle + 1 \). Therefore, when \( w = e_k \), \( A(w; \theta) = ICNN(e_k; \theta) - b_k + 1 = 1 \), \( \forall k \in \{1, \cdots, d\} \). We finally apply a ReLU activation at the end to ensure all outputs are positive.

For the sampling experiments, we vary the size of \( p_z \) depending on the dimensionality of the data. The latent sizes were \( \{2, 16, 64, 64, 256\} \) for \( d = \{2, 16, 225, 256, 784, 1024\} \) respectively. The generator is a basic multi layer perceptron (MLP) with ReLU activations. For all experiments, we use a width 256 and depth 2 MLP for the generator. The final output is ensured to be positive through a final ReLU operation.

[^3]: https://raws.dri.edu/index.html
[^4]: https://www.investing.com/commodities/
Figure 5: Comparison of samples for $A = A_{SL}$. Blue points are from the proposed method and orange points are true samples.

**Hyperparameter Tuning**

For learning the Pickands dependence experiments, we used the Adam optimizer for optimizing all parameters with learning rate $1 \times 10^{-4}$ with exponential decay of 0.9998 per epoch. Each model was trained for 1000 epochs.

For the sampling experiments, the generator was trained using Adam with learning rate $1 \times 10^{-4}$, $\beta_1 = 0.5$ and $\beta_2 = 0.99$ with exponential decay on the learning rate of 0.99998. Models for the generator were trained for 3000 epochs.

**E Sampling examples**

In Figure 5 we show examples of samples from the symmetric logistic distribution for different values of the dependence parameter $\alpha$. Samples from our heuristic and learned generator (blue) are compared to the exact samples (orange) provided by the method in [19].