Foliations for solving equations in groups: free, virtually free, and hyperbolic groups

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Abstract

We give an algorithm for solving equations and inequations with rational constraints in virtually free groups. Our algorithm is based on Rips’ classification of measured band complexes. Using canonical representatives, we deduce an algorithm for solving equations and inequations in all hyperbolic groups (possibly with torsion). Additionally, we can deal with quasi-isometrically embeddable rational constraints.

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Introduction

The equations problem

Given a group $G$, the equations problem in $G$ consists in deciding algorithmically whether a system of equations with constants has a solution in $G$ or not. An equation is an equality $w = 1$, where $w$ is a word on a set of variables, and their inverses together with constants taken from $G$. When inequations (that is, negation of equations) are allowed, we call this problem the problem of equations and inequations. In other words, the problem of equations and inequations is equivalent to the decidability of the existential (or universal) theory of $G$ with constants in $G$.

A solution to the equations problem is quite powerful, as it vastly generalizes the word problem, the conjugacy problem, the simultaneous conjugacy problem, etc. When $G$ is abelian, the problem of equations and inequations is easily solved using linear algebra; but already, if $G$ is a free 3-step nilpotent group of rank 2, then the equations problem is undecidable (see [42, 48]). This is based on Matiyasevich’s theorem saying that one cannot decide the solubility of polynomial equations in $\mathbb{Z}$ (see [33]). For a free 2-step nilpotent group, then the equations problem is solvable if and only if one can decide the solubility of polynomial equations over $\mathbb{Q}$.
(see [42]). Additionally, if $G$ is a non-commutative free metabelian group, then the equations problem in $G$ is unsolvable [41], but the problem of equations and inequations without constants is solvable [8].

The problem of equations and inequations in free groups is natural, and has attracted the attention of many people (Lyndon, Appel, Lorents, etc.). The solution of this problem by Makanin [31] (with the appropriate correction in [32]) certainly constitutes a milestone in the theory. It has been a source of inspiration for Rips for his study of group actions on $\mathbb{R}$-trees, and his solution to Morgan and Shalen’s conjecture [3, 20], which found applications in many branches of geometry. It has been a decisive step towards algorithmic and theoretical description of the set of homomorphisms of a group into a free group (see [38]). It has also been, together with these developments, a prelude to the far-reaching recent solutions to Tarski’s problems on the elementary theory of free groups, by Sela [47], and by Kharlampovich and Miasnikov [27]. Makanin’s algorithm is also the basis of Rips and Sela’s solution to the equations problem in torsion-free hyperbolic groups which they manage to reduce to the equations problem in a free group [40]. Finally, it is crucial in Sela’s solution of the isomorphism problem for torsion-free hyperbolic groups with finite outer automorphism group: he makes delicate use of Makanin’s algorithm, and of Rips’ classification of actions on $\mathbb{R}$-trees [45].

Our main result is the following theorem, proved in Section 8 (see below for definitions and discussion).

**Theorem 1.** There exists an algorithm which takes the following input:

(i) a presentation of a hyperbolic group $G$ (possibly with torsion);

(ii) a finite system of equations and inequations with constants in $G$, and with quasi-isometrically embeddable rational constraints, and which decides whether there exists a solution or not.

In a forthcoming paper, we will use this algorithm to give a solution to the isomorphism problem for all hyperbolic groups (possibly with torsion) [13].

We pursue several goals in this paper. Our first goal is a proof of Theorem 1. As in [11, 40], our proof is based on canonical representatives, which allows us to reduce it to the equations problem in a virtually free group, that is, a finite extension of a free group.

Our second goal is therefore to give a solution of the equations problem in virtually free groups. This problem easily reduces to a problem of twisted equations in a free group.

Last but not least, we present a new approach to Makanin’s algorithm, based on Rips’ theory for foliated band complexes, allowing us to solve these twisted equations. This occupies the major part of this paper.

We continue this introduction by reviewing different aspects and concepts involved in our strategy.

**Rational constraints**

In a group $G$, the class of rational subsets is the smallest class containing finite subsets, and closed under finite union $A \cup B$, product $A \cdot B$, and semigroup generation $A^*$. Equivalently, if $S$ is a finite generating set of $G$, then a subset $A \subseteq G$ is rational if it is the image in $G$ of a regular language $L$ of the free monoid on $S \cup S^{-1}$, a regular language being a language recognized by a finite state automaton. Solving a system of equations with rational constraints consists in solving this system of equations with the requirement that each variable $x$ lies in a rational subset $R_x$ given in advance.

In a hyperbolic group, we propose the class of quasi-isometrically embeddable rational subsets as the suitable class for constraints. A rational subset $A \subseteq G$ is quasi-isometrically
embeddable if one can choose the regular language $L$ in the free monoid to consist of quasi-geodesics (with uniform constants). For example, a quasi-convex subgroup $H$ of a hyperbolic group $G$ is a quasi-isometrically embeddable rational subset. Note that by Kapovich [25], one can compute a finite state automaton representing $H$ from a finite generating set. In general, the class of rational subsets of a group is not closed under complementation or intersection. However, the set of quasi-isometrically embeddable rational subsets of a hyperbolic group is a Boolean algebra (Corollary 9.6). For instance, the complement of a quasi-convex subgroup is also a quasi-isometrically embeddable rational subset. In particular, inequations in a hyperbolic group can be encoded using quasi-isometrically embeddable rational constraints of the form $R = G \setminus \{1\}$. In a virtually free group, every rational subset is quasi-isometrically embeddable, and the set of all rational subsets is a Boolean algebra.

Equations with rational constraints in a free monoid were first considered by Schulz [44]. Following an approach of Plandowski for free monoids, Diekert, Gutiérrez, and Hagenah proved that systems of equations, inequations, and rational constraints in free groups are algorithmically solvable [16, 36].

The use of rational constraints in systems of equations turns out to be rather powerful. The problem of equations and inequations for right-angled Artin groups has been reduced by Diekert and Muscholl to a problem of equations with rational constraints in free groups [18]. This has been generalized by Diekert and Lohrey to free products, direct products, and graph products of certain groups [17]. It is the key tool to extend Rips and Sela’s solution to the equations problem for torsion-free hyperbolic groups into a solution to the problem of equations and inequations [11]. It greatly streamlines the solution of the isomorphism problem for torsion-free hyperbolic groups, and allows substantial generalizations [12, 13]. It plays an important role in our solution to the equations problem in virtually free groups (see Theorem 3 below), and in Lohrey and Senizergues’ independent solution [29], even if the initial problem does not involve inequations or rational constraints.

For the sake of illustration, let us present two elementary applications of the use of rational constraints in a system of equations. Let $F_2$ be a free group of rank 2. Given a finitely generated subgroup $H < F_2$ (or more generally any rational subset), and an element $x \in F_2$, one can decide if there is an automorphism of $F_2$ sending $x$ into $H$. Indeed, consider $a, b$ a basis of $F_2$, and write $x = w(a, b)$ as a word on $a, b$. The orbit of $x$ under $\text{Aut}(F_2)$ intersects $H$ if and only if there exists a basis $u, v$ of $F_2$ such that $w(u, v) \in H$. By a theorem of Dehn, Magnus, and Nielsen, $(u, v)$ is a basis if and only if there exists $g$ such that $[u, v]g = [a, b]^{\pm 1}$. Therefore, the orbit of $x$ intersects $H$ if and only if the system of equations with rational constraints

\[ [u, v]g = [a, b]^{\pm 1}, \]
\[ z = w(u, v), \quad z \in H \]

in the variables $u, v, g, z$ has a solution.

Our second application is an immediate consequence of Theorem 1.

**Corollary 2.** Let $G$ be a hyperbolic group. Given a quasi-convex subgroup $H < G$, one can decide its malnormality by solving the system $z = yxy^{-1}$, where $z \neq 1$ with rational constraints $z, x \in H$, and $y \notin H$.

In the presence of torsion, almost malnormality (meaning that $yHy^{-1} \cap H$ is finite when $y \notin H$) can be checked similarly by replacing the inequality $z \neq 1$ by an inequality $z^N \neq 1$, where $N$ is a bound on the order of torsion in $G$.

This result does not seem to appear in the literature. This is a variant of [1, Problem H14] and of [5, Question 3]. Without the assumption of quasi-convexity, malnormality is undecidable [5].
Lifting equations and rational constraints to a virtually free group

Following the strategy initiated in [40], and continued in [11], we now explain how to reduce Theorem 1 to the problem of solving equations with rational constraints in a virtually free group (see Section 9).

In [40], Rips and Sela introduced canonical representatives for torsion-free hyperbolic groups, which enabled them to reduce the equations problem in a torsion-free hyperbolic group to the equations problem in a free group. These canonical representatives are paths in the Cayley graph of \( G \) which satisfy some path equations representing the initial equations. Such paths correspond to words on the generating system of \( G \), and thus to elements of the corresponding free group.

In the presence of torsion, canonical representatives need to be interpreted as paths in a barycentric subdivision \( X \) of a Rips complex of \( G \). The action of \( G \) on \( X \) is not free in general. The quotient of the 1-skeleton \( X^{(1)}/G \) is a finite graph of finite groups, whose fundamental group is a virtually free group \( V \). Path equations in \( X \) are then interpreted in terms of equations in \( V \).

Another task that needs to be done, is to lift rational constraints to \( V \). This can be done because both canonical representatives and paths representing the rational subsets are quasi-geodesics. This part of the argument is similar to Cannon’s argument showing that the language of geodesics is a regular language [7, 9].

This allows us to reduce Theorem 1 to the following result (proved in Section 8).

**Theorem 3.** The problem of equations with rational constraints is solvable in finite extensions of free groups.

More precisely, there exists an algorithm that takes as input a presentation of a virtually free group \( G \), and a system of equations and inequations with constants in \( G \), together with a set of rational constraints, and which decides whether there exists a solution or not.

This result was independently obtained in a more general form in [29]. Unlike [29], our approach does not rely on an existing solution of the equations problem in free groups. Instead, we give a new proof, based on Rips’ theory for foliated band complexes.

Twisted equations and virtually free groups

A twisted equation in a group \( G \) is an equation of the form \( \varphi_1(x_1) \cdots \varphi_n(x_n) = 1 \), where each \( \varphi_i \) is a fixed automorphism in \( \text{Aut}(G) \), and \( x_i \) is a variable or a constant. For instance, twisted conjugacy involves a simple example of a twisted equation: given an automorphism \( \varphi \in \text{Aut}(G) \), two elements \( a, b \in G \) are twisted conjugate if there exists \( x \in G \) such that \( \varphi(x^{-1}a\varphi(x^{-1})) = b \).

When one considers equations in a finite extension \( 1 \to N \to G \to Q \to 1 \) (\( Q \) is finite), twisted equations appear naturally. Indeed one can replace an equation \( xyz = 1 \), by a disjunction of equations \( (\tilde{q}_x n_x)(\tilde{q}_y n_y)(\tilde{q}_z n_z) = 1 \), where the constants \( \tilde{q}_x, \tilde{q}_y, \) and \( \tilde{q}_z \) are chosen in a given cross-section of \( Q \), and \( n_x, n_y, \) and \( n_z \) are new unknowns in \( N \). Gathering the elements \( \tilde{q}_x, \tilde{q}_y, \) and \( \tilde{q}_z \) to the left amounts to twisting \( n_x, n_y, \) and \( n_z \) by suitable automorphisms.

Thus, the solubility of equations in \( G \) reduces to the solubility of finitely many systems of twisted equations in \( N \). The twisting morphisms occurring in this manner are quite particular because they generate a finite subgroup of the outer automorphism group \( \text{Out}(N) \).

In the Kourovka notebook, Makanin asked about the problem of twisted equations [34, Problem 10.26(b)]. We are able to give a positive answer to Makanin’s question above assuming that the given automorphisms generate a finite subgroup of \( \text{Out}(F) \).
Theorem 4. There exists an algorithm that takes as input a basis of a free group $F$, a finite set $\Phi$ of automorphisms of $F$ whose image in $\text{Out}(F)$ generates a finite subgroup, and a system of twisted equations with rational constraints in $F$ (with twisting automorphisms in $\Phi$) that decides whether there is a solution or not.

As explained above, Theorem 3 easily follows from Theorem 4 (see Section 2). Moreover, we give a trick to reduce to the case where the twisting automorphisms of $F$ permute the elements of $S \cup S^{-1}$ for some free basis $S$ of $F$. This trick is related to the Zimmerman–Culler theorem [10, 49], which realizes any finite subgroup of $\text{Out}(F)$ as a finite group $H$ of automorphisms of a graph $X$ whose fundamental group is $F$. Since $H$ may fail to fix a point in $X$, it follows that $H$ may fail to lift to a finite subgroup of $\text{Aut}(S)$. This is why we need to embed $F$ into a larger free group $\hat{F}$, whose basis is the set of edges of $X$. The group $H$ is then realized as a subgroup of $\text{Aut}(\hat{F})$ permuting the basis elements, and $H$ preserves the conjugacy class of $F$. The initial system of equations gives a new system of equations in $\hat{F}$, and we add rational constraints saying that the variables should live in $F$. This reduction is the content of Proposition 2.4 in the broader context of equations with rational constraints.

Dynamical and geometric aspects

We now focus on the equations problem in a free group. Makanin and Razborov developed a combinatorial machinery to encode equality of subwords occurring in a solution of an equation. An interesting and well-written account on Makanin’s algorithm for equations in free monoids (a simplified version of the case of free groups, which does not imply a solution for free groups) was given in [15], and another one on Makanin and Razborov’s algorithm for free groups was given in [26]. As we said, Rips was inspired by this machinery for his study of foliated band complexes, whose dynamics, on the other hand, reflect actions on $\mathbb{R}$-trees. Our strategy is to reverse this flow of ideas, and use Rips’ classification of foliated band complexes to prove that the algorithm we propose always stops.

We hope that this point of view on Makanin’s algorithm will be of interest to an audience concerned with the equations problem, and also to an audience concerned with geometry and the dynamics of group actions.

Rips’ theory is an understanding of actions of finitely generated groups on $\mathbb{R}$-trees. Recall that an $\mathbb{R}$-tree is a geodesic metric space in which any two points are joined by a unique injective path. Following Sela, let us try to explain how the equations problem in a free group $F$ is related to $\mathbb{R}$-trees [46].

The equations problem in $F$ is about homomorphisms of finitely presented groups to $F$. Indeed, a system of equations on a set of unknowns $X$ is a finite set $\mathcal{E}$ of words in the free product $F \ast \langle X \rangle$, and a solution of this system of equations corresponds to a morphism from $G_{\mathcal{E}} = F \ast \langle X \rangle / \langle \langle \mathcal{E} \rangle \rangle$ to $F$, which is the identity on $F$. Each such morphism gives rise to an action of $G_{\mathcal{E}}$ on the Cayley graph of $F$, a simplicial tree. One can rescale this tree in order to normalize the maximal displacement of the generators of $G_{\mathcal{E}}$. For an infinite sequence of solutions, the corresponding actions of $G_{\mathcal{E}}$ on the Cayley graph converge to an action of $G_{\mathcal{E}}$ on some $\mathbb{R}$-tree.

Rips’ theory says that under suitable hypotheses, this action can be understood in terms of actions on simplicial trees, and actions on $\mathbb{R}$-trees dual to minimal measured foliations on 2-complexes. The main result of Rips’ theory is a classification of those minimal measured foliations into three types:

(i) homogeneous type (also known as axial or toral), whose dual $\mathbb{R}$-tree is a line;
(ii) surface type (also known as interval exchange): the $\mathbb{R}$-tree is dual to a measured foliation on a surface (or a 2-orbifold);
(iii) exotic type (also known as Levitt, or thin).
One can try to use these ideas to decide whether a given system of equations has a solution, and to look for a shortest solution (for example, in terms of the maximal displacement of the generators). If we have some solutions such that the corresponding actions of \( G_\mathcal{E} \) on the Cayley graph of \( F \) are close enough to the limiting \( \mathbb{R} \)-tree, then we can apply Sela’s shortening argument. This argument says that, under suitable hypotheses, there is a quotient of \( G_\mathcal{E} \) through which the actions factorize, and automorphisms of this quotient that shorten all nearby solutions. These solutions are not the shortest and can therefore be ignored. Using some compactness argument, only finitely many solutions do not lie in such a shortening neighbourhood of a limiting \( \mathbb{R} \)-tree. One can hope to bound the length of these remaining solutions and check by hand if such a solution exists.

A major problem in this approach is that we cannot even start as we do not know if our system of equations has any solution (this is what we have to decide). Therefore, we cannot work with actual solutions. Instead, we work with potential solutions. An actual solution, that is, a morphism \( G_\mathcal{E} \to F \), can be represented by a continuous map from a presentation complex \( X \) of \( G_\mathcal{E} \) to a wedge of circles \( Y \). The preimage of midpoints of edges of \( Y \) is a combinatorial lamination of \( X \). Instead of working with laminations, we work with prelaminations that play the role of potential solutions. A subcomplex of a lamination is a typical example of a prelamination, but prelaminations are defined by local conditions and are not required to extend to an actual lamination (see Figure 1).

Our proof shows that when a prelamination is very long, then any actual solution corresponding to this prelamination can be shortened, and therefore ignored. This is the key of the termination of our algorithm.

**Overview of the main algorithm**

Let us describe the approach to solving equations in free groups developed in this paper.

The first step towards Theorem 4 is to encode a system of triangulated equations in a band complex. Band complexes are Makanin’s generalized equations, but they also play the role of a presentation complex of \( G_\mathcal{E} \) in the above discussion. A band complex consists of \( D \), which is a disjoint union of non-degenerate segments (one segment for each variable or constant involved in \( \mathcal{E} \)), together with a finite set of rectangles (called bands) attached to \( D \) by two opposite sides called its bases. If one considers twisted equations in the context described above, then each band carries a specific automorphism. If one considers rational constraints, then subsegments of \( D \) carry regular languages.

A solution of the band complex in the free group \( \langle S \rangle \) is a labelling of \( D \) by reduced words on \( S \cup S^{-1} \) so that both bases of each band are labelled by the same reduced word (up to composing by the automorphism of the band), and such that the label of each subsegment satisfies the associated rational constraint. For instance, to encode constants, one can use

![Figure 1. A prelamination is a finite disjoint union of leaf segments.](image)
regular languages consisting of a single word to impose the labelling on some subsets of $D$. Solving twisted equations in free groups easily reduces to deciding the existence of a solution to such a band complex (see Section 3). To simplify this presentation, we now forget about rational constraints.

Our main algorithm (Algorithm 8.2) decides whether a given band complex admits a solution (that is, a labelling as above) using prelaminations. A prelamination is a finite disjoint union of leaf segments, and a leaf segment is a segment contained in a band and joining its two bases (see Figure 1). A prelamination is induced by a solution if its leaf segments join matching subwords of the labelling. A prelamination is complete if one cannot extend any leaf (that is, if it is an actual lamination). Given a complete prelamination, it is easy to decide if there exists a solution that induces it.

We can explore the space of possible prelaminations, in the quest for a complete prelamination induced by a shortest solution. The main concern is how to detect that there is no solution.

For each complete prelamination, we can decide if it is induced by a solution or not. If it is, then we are done; otherwise, we reject this prelamination. For each incomplete prelamination $\mathcal{L}$, we run a prelamination analyser which tries to find a certificate ensuring that no shortest solution can induce $\mathcal{L}$. In this case one can reject $\mathcal{L}$. The analyser may fail to find any such certificate and might say ‘I don’t know’. There are several kinds of certificates: detection of an incompatibility with constants, non-existence of an invariant (combinatorial) transverse measure, or a shortening certificate proving that if some solution induces $\mathcal{L}$, then there is a shorter solution.

Note that if a lamination $\mathcal{L}'$ extends a rejected $\mathcal{L}$, then $\mathcal{L}'$ cannot be induced by a shortest solution. If all the prelaminations remaining to be analysed are extensions of rejected ones, then we know that there is no shortest solution and hence no solution at all, and the machine stops.

To prove that this algorithm works, we assume that there is no solution, and show that the algorithm rejects all sufficiently long prelaminations by producing appropriate shortening certificates.

This is where we use Rips’ classification of measured foliations on band complexes. Assume by contradiction that the algorithm does not stop. By extracting a limit of an increasing sequence of non-rejected prelaminations $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$, one can construct a topological foliation on the band complex with an invariant measure. We prove that this measure has no atom. Assume for simplicity that it has full support (this is not true in general). One can decompose the foliated band complex into minimal components. By Rips’ theory, they are classified as homogeneous, exotic, and surface-type components.

In the presence of a homogeneous component, for all $i$ large enough, $\mathcal{L}_i$ shows large repetition patterns that force solutions inducing $\mathcal{L}_i$ to have subwords that are arbitrarily high powers. By Bulitko’s lemma, there is a computable a priori bound on such powers in a shortest solution. One can therefore produce a shortening certificate for $i$ large enough.

In the presence of a surface or exotic component, one can perform, as in the Rips Machine, an infinite sequence of moves on the foliated band complex $\Sigma$. These moves do not increase the complexity of $\Sigma$, so finitely many unfoliated band complexes are visited. Assume for simplicity that $\Sigma$ itself is visited twice. For $i$ large enough, the corresponding moves are compatible with $\mathcal{L}_i$, and transform any solution of $\Sigma$ inducing $\mathcal{L}_i$ into a shorter solution of $\Sigma$. These moves provide a shortening certificate for $\mathcal{L}_i$: if $\mathcal{L}_i$ is induced by some solution, then this solution cannot be the shortest. When $\Sigma$ itself is not visited twice, this argument needs to be refined, and shortening must be performed by at least a certain Lipschitz factor.

The running time of our algorithm does not seem to be very good. Makanin’s algorithm (in its corrected version [32]) is known not to be primitive recursive [28], and we do not see why ours should be better. Using some data compression on words, Plandowski proposed an algorithm of much better complexity (polynomial in space) [16, 36].
Organization of the paper

Sections 1–3 set up the vocabulary, and reduce the equations problem in virtually free groups to band complexes. Section 4 is about Bulitko’s lemma on the periodicity exponent of minimal solutions. Section 5 introduces prelaminations and related notions. Section 6 describes the prelamination generator and the prelamination analyser, and gives a precise definition of a shortening sequence of moves that we use as a certificate for rejection. Section 7 contains the main part of the argument. Limits of sequences of prelaminations are analysed using Rips’ theory, and the existence of shortening sequences of moves is proved. Section 8 contains a detailed account on the main algorithm. This section can be read independently, admitting a certain number of well-identified results of the previous sections. Although based on Theorem 3, Section 9 is independent from the rest of the paper. It deals with the equations problem in hyperbolic groups using canonical representatives, and with quasi-isometrically embeddable rational constraints.

In a first reading, one can first forget about rational constraints. Moreover, if one is only interested in the case of free groups, that is, with untwisted equations, one can safely ignore all technical considerations about Möbius strips (see Remark 5.3).

1. Preliminaries

1.1. Regular languages

Let \( S \) be a finite set, let \( \overline{S} \) be a copy of \( S \), and let \( S_\pm = S \sqcup \overline{S} \) be endowed with the canonical involution \( s \mapsto \overline{s} \) exchanging \( S \) and \( \overline{S} \). Let \( S_\pm^* \) be the free monoid on \( S_\pm \), endowed with the involution \( w = s_1 \ldots s_n \mapsto \overline{w} = \overline{s}_n \ldots \overline{s}_1 \).

An automaton over \( S_\pm \) is a directed graph, where each (oriented) edge is labelled by some element of \( S_\pm \) together with two finite subsets \( \text{Start} \) and \( \text{Accept} \) of the set of vertices of this graph. The language \( R_A \subset S_\pm^* \) accepted by an automaton \( A \) is the set of words labelled by directed paths starting from a vertex in \( \text{Start} \) and finishing at a vertex in \( \text{Accept} \). A subset \( R \subset S_\pm \) is a regular language if \( R = R_A \) for some automaton \( A \).

If \( R_1 \) and \( R_2 \) are regular languages, then so are \( R_1 \cup R_2 \), \( R_1 \cdot R_2 \) (the product set), and \( R_1^* \) (the submonoid generated by \( R_1 \)). Kleene’s theorem asserts that the class of regular languages is the smallest class containing finite subsets, and closed under these operations.

The class of regular languages is actually a Boolean algebra. Moreover, one can algorithmically compute automata for \( R_1 \cup R_2 \), \( R_1 \cap R_2 \), \( S_\pm^* \setminus R_1 \), \( R_1 \cdot R_2 \), and \( R_1^* \) from automata representing \( R_1 \) and \( R_2 \). Similarly, if \( R \) is rational, then so is its image \( \overline{R} \) under the involution \( w \mapsto \overline{w} \), and the corresponding automaton can be computed.

Example 1.1. Assume that \( \pi : S_\pm^* \to F \) is a morphism to a finite group \( F \), and consider a subset \( E \subset F \). Let \( A \) be the Cayley graph of \( F \) relative to \( \pi(S) \).

Then \( \pi^{-1}(E) \) is a regular language, corresponding to the automaton \( A \), where \( \text{Start} = \{1\} \) and \( \text{Accept} = E \).

Not all regular languages are of this form but this is true if one replaces the finite group \( F \) by a finite monoid. Let us be more specific. Denote by \( M_n \) the finite monoid of matrices with Boolean entries in \( \{0, 1\} = \{\text{true, false}\} \), where the product of \( C = AB \) of \( A = (a_{ij}) \) and \( B = (b_{jk}) \) is \( c_{ik} = (\vee_{j=1}^n a_{ij} \land b_{jk}) \); and \( \lor \) and \( \land \) denote or and and, respectively. Consider an automaton \( A \) over \( S_\pm \) with vertices \( v_1, \ldots, v_n \). To each \( s \in S_\pm \) corresponds the matrix \( M_s \) whose entry \( (i, j) \) is 1 if and only if there is an edge labelled \( s \) from \( v_i \) to \( v_j \). The assignment
s \mapsto M_s extends to a morphism \( \rho : S^*_\pm \to M_n \) and the language \( R_A \) accepted by \( A \) is the preimage under \( \rho \) of the set of matrices \( \mu \) having a non-zero entry \((i, j)\), where \( v_i \in \text{Start} \) and \( v_j \in \text{Accept} \). Conversely, given a morphism \( \rho : S^*_\pm \to M \) to a finite monoid \( M \) and a subset \( \mu \subset M \), we have that \( \rho^{-1}(\mu) \) is a regular language corresponding to the following automaton: its vertex set is \( M \), its edges labelled by \( s \) are given by the multiplication by \( \rho(s) \), \( \text{Start} = \{ \text{id} \} \) and \( \text{Accept} = \mu \). Clearly, one can compute \( \rho \) and \( \mu \) from an automaton, and conversely.

**Remark 1.2.** This well-known construction implies the well-known fact that one can make an automaton deterministic and, in particular, assume that \( \text{Start} \) is a single vertex of the automaton.

1.2. **Rational subsets**

Consider a group \( G \), a finite generating set \( S \), and the natural morphism \( \pi : S^*_\pm \to G \) mapping \( S \cup S \) to \( S \cup S^{-1} \).

**Definition 1.3.** A rational subset of a group \( G \) is the image under \( \pi \) of a regular language of \( S^*_\pm \).

Equivalently, the class of rational subsets of \( G \) is the smallest class containing finite subsets, and closed under union, product, and submonoid generation.

The equivalence between the two definitions follows from Kleene’s theorem. In particular, the notion of rational subset does not depend on the choice of the generating set \( S \). We say that a rational subset \( R \subset G \) is represented by an automaton \( A \) over \( S_\pm \) when \( \pi(R_A) = R \).

Note that if \( R \) is rational, then so is \( R^{-1} \). Although the class of regular languages of \( S_\pm^* \) is a Boolean algebra, the class of rational subsets of a group \( G \) is not closed under intersection or complementation in general. Indeed, it is easy to see that subgroups are rational subsets if and only if they are finitely generated, but it can happen that the intersection of two such subgroups is not finitely generated.

**Lemma 1.4.** Let \( f : G \to G' \) be a morphism, let \( S \) and \( S' \) be finite generating sets of \( G \) and \( G' \), respectively, and let \( A \) be an automaton on \( S \) representing some rational subset \( R \subset G \).

Then \( f(R) \) is a rational subset of \( G' \), and knowing an expression of \( f(s) \) as an \( S' \)-word, one can compute an automaton for \( f(R) \).

**Proof.** For each \( s \in S \cup S' \), write \( f(s) = s'_1 \ldots s'_n \), where \( s'_i \in S' \cup S' \). Replace each directed edge of \( A \) labelled \( s \) by a directed segment of length \( n \) labelled \( s'_1 \ldots s'_n \). The automaton obtained in this way clearly represents \( f(R) \). \( \square \)

The following lemma follows immediately from the analogous fact concerning regular languages of \( S^*_\pm \). It holds without assumption on the group \( G \).

**Lemma 1.5.** Given two automata \( A_1 \) and \( A_2 \) defining some rational subsets \( R_1 \) and \( R_2 \) of \( G \), one can algorithmically compute some automata representing \( R_1 \cup R_2 \), \( R_1 \cdot R_2 \), \( R_1^* \), and \( R_1^{-1} \).
When $G$ is a free group, or even a virtually free group, the class of rational subsets is a Boolean algebra (see below). For the free group, this is based on the following fact. We denote by $\langle S \rangle$ the free group with a free basis $S$.

**Lemma 1.6** [2, Proposition 2.8, p. 59]. For any rational subset $R$ in the free group $\langle S \rangle$, consider the set $\tilde{R} \subset S^*_\pm$ of reduced words representing elements of $R$.

Then $\tilde{R}$ is a regular language, and an automaton representing $\tilde{R}$ can be computed from an automaton representing $R$.

It follows that the class of rational subsets of a free group is a Boolean algebra: denoting by Red the regular language of reduced words of $S^*_\pm$, one has $\pi(\text{Red} \setminus \tilde{R}) = \langle S \rangle \setminus R$ and $\pi(\tilde{R}_1 \cap \tilde{R}_2) = \tilde{R}_1 \cap \tilde{R}_2$. Moreover, if $S$ is a basis of the free group $F$, then the automata over $S^*_\pm$ representing the obtained rational subsets can be computed from automata over $S^*_\pm$ for the initial ones.

### 1.3. Rational subsets of a virtually free group

When $G$ is virtually free, the fact that rational subsets form a Boolean algebra follows from recent work by Lohrey and Senizergues [30] (their result is more general but quite complicated). We propose here a short proof of what we need.

**Remark 1.7.** From a presentation of a virtually free group $V$, one can compute a free basis of a normal finite-index free subgroup $F$ and a representative $g_a \in V$ of each element $a \in V/F$. Indeed, by the Reidemeister–Schreier process, one can enumerate all normal finite-index subgroups of $V$, with presentations, and by Tietze transformations, all presentations of these subgroups. One will find one that is a presentation with no relator, that is, a free basis of a free group $F$. Then the finite group $V/F$ and lifts in $V$ are easy to compute.

**Lemma 1.8.** Let $V$ be a virtually free group given by some presentation with a generating set $S_V$. Let $F \lhd V$ be a normal free subgroup of finite index, given by a free basis $S$, expressed as a set of words on $S_V$. Let $R$ be a rational subset of $F$ (hence of $V$).

Given an automaton over $(S_V \cup \overline{S}_V)$ representing $R$, one can compute an automaton over $S^*_\pm$ representing it, and vice versa.

**Proof.** Computing an automaton over $(S_V \cup \overline{S}_V)$ from one over $S^*_\pm$ simply follows from Lemma 1.4 applied to the embedding $F \to V$.

Let $\pi : (S_V \cup \overline{S}_V)^* \to V$ be the natural map. Let $A$ be an automaton over $S_V \cup \overline{S}_V$ accepting a language $\tilde{R} \subset (S_V \cup \overline{S}_V)^*$ with $\pi(\tilde{R}) = R$. Let $V_A$ be its set of vertices, and assume that $\text{Start}$ is a single vertex (this can be assumed by Remark 1.2). Denote by $\rho : (S_V \cup \overline{S}_V)^* \to V/F$ the composition of $\pi$ with the quotient map.

We construct a new automaton $A'$ with a vertex set $V_{A'} = V_A \times V/F$. We put an edge labelled $u \in S_V \cup \overline{S}_V$ from $(v_1, a)$ to $(v_2, b)$ if there is an edge labelled $u$ from $v_1$ to $v_2$ in $A$, and $b = a\rho(u)$. We take $\text{Start}' = (\text{Start}, 1)$ and $\text{Accept}' = \text{Accept} \times \{1\}$. Since $R \subset F$, the language accepted by $A'$ is precisely $\tilde{R}$.

For each $a \in V/F$ choose a representative $g_a \in V$ with $g_a = 1_V$ for $a = 1$. We also think of $g_a$ as a word on $(S_V \cup \overline{S}_V)$, and we note that such $g_a$ can be computed. We construct an automaton $A''$ over $S^*_\pm$ as follows. Consider an edge $e$ labelled $u$ and joining $(v_1, a)$ to $(v_2, b)$. Note that $u' = g_{a'}u_g_{b'}^{-1} \in F$, and write $u' = s_1 \ldots s_k$ as a word on $S^*_\pm$ (this can be done algorithmically).
Replace the edge \( e \) by a directed segment labelled by \( s_1 \ldots s_k \). Do this for every edge of \( A' \), take \( \text{Start}'' = \text{Start}' \) and \( \text{Accept}'' = \text{Accept}' \), and call \( A'' \) the obtained automaton.

We claim that the language \( \mathcal{R}'' \) accepted by \( A'' \) satisfies \( \pi(\mathcal{R}'') = \mathcal{R} \). Indeed, assume that \( u_1 \ldots u_n \in (S_V \cup \overline{S}_V)^* \) is accepted by \( A' \), and consider \( a_i = \rho(u_1 \ldots u_i) \). One has \( a_0 = 1 \) since the path starts at \( \text{Start}' \), and \( a_n = 1 \) since \( \pi(u_1 \ldots u_n) \in F \). Then the \( S_\pm \)-word \( u_1' \ldots u_n' = (1u_1g_0^{-1})(g_1u_2g_0^{-1})\ldots(g_{n-1}u_n1) \) is accepted by \( A'' \), and has the same image under \( \pi \) as \( u_1 \ldots u_n \). Similarly any word accepted by \( A'' \) maps to \( \mathcal{R} \).

**Lemma 1.9.** Let \( V \) be a virtually free group, let \( F \triangleleft V \) be a normal free subgroup of finite index, and let \( g_1, \ldots, g_k \) be representatives of the left cosets of \( F \) in \( V \).

A subset \( \mathcal{R} \subset V \) is rational if and only if, for all \( i \), we have that \( (g_i^{-1}\mathcal{R}) \cap F \) is rational in \( F \).

Let \( S_V \) be a generating set for \( V \) and let \( S \) be a basis of \( F \). Given automata over \( S_\pm \) representing each \( (g_i^{-1}\mathcal{R}) \cap F \), one can explicitly compute an automaton over \( S_V \cup \overline{S}_V \) representing \( \mathcal{R} \), and conversely, given automata over \( S_V \cup \overline{S}_V \) representing \( \mathcal{R} \), and \( g_1, \ldots, g_k \), one can compute an automaton over \( S_\pm \) representing \( (g_i^{-1}\mathcal{R}) \cap F \).

**Proof.** If \( (g_i^{-1}\mathcal{R}) \cap F \) is a rational subset of \( F \), then it is rational in \( V \). Therefore, the sets \( \mathcal{R} \cap g_i F \) are rational in \( V \), and so is their union. Given an automaton over \( S_\pm \) representing \( (g_i^{-1}\mathcal{R}) \cap F \), one can compute an automaton over \( S_V \cup \overline{S}_V \) representing it by Lemma 1.8, and hence an automaton over \( S_V \cup \overline{S}_V \) representing \( \mathcal{R} \) by Lemma 1.5.

Conversely, assume that \( \mathcal{R} \) is rational in \( V \). Denote by \( \pi : (S_V \cup \overline{S}_V)^* \to V \) the natural morphism, and by \( \overline{\mathcal{R}} \subset (S_V \cup \overline{S}_V)^* \) a rational language such that \( \mathcal{R} = \pi(\overline{\mathcal{R}}) \). By Example 1.1, we see that \( \pi^{-1}(F) \) is a regular language, and so is \( \overline{\mathcal{R}} \cap \pi^{-1}(F) \). Thus \( \mathcal{R} \cap F \) is rational, and one can compute an automaton \( A \) over \( S_V \cup \overline{S}_V \) representing it. By Lemma 1.8, we can turn it into an automaton over \( S_\pm \).

**Lemma 1.10.** Let \( V \) be a virtually free group, and let \( \mathcal{R} \) and \( \mathcal{R}' \) be rational subsets.

Then \( V \setminus \mathcal{R} \) and \( \mathcal{R} \cap \mathcal{R}' \) are rational, and one can compute automata representing them from automata representing \( \mathcal{R} \) and \( \mathcal{R}' \).

**Proof.** By Lemma 1.9, one can write \( \mathcal{R} = g_1 \mathcal{L}_1 \cup \ldots \cup g_k \mathcal{L}_k \) with \( \mathcal{L}_i \) rational in \( F \). Then \( V \setminus \mathcal{R} = g_1(F \setminus \mathcal{L}_1) \cup \ldots \cup g_k(F \setminus \mathcal{L}_k) \) is rational since in the free group \( F \), we have that \( (F \setminus \mathcal{L}_i) \) is rational. Moreover, one can compute automata for \( \mathcal{L}_i \) by Lemma 1.9, and hence for \( F \setminus \mathcal{L}_i \) and for \( V \setminus \mathcal{R} \).

For the second assertion, simply write \( \mathcal{R} \cap \mathcal{R}' = (\mathcal{R}^c \cup \mathcal{R}'^c)^c \).

### 2. Equations and twisted equations

In this section, we start by giving a formal definition of a system of equations with rational constraints in a group \( G \). Then we explain how to translate a system of equations in a virtually free group into a disjunction of systems of twisted equations in a free group.

**Definition 2.1** (Equations with rational constraints). Consider a finite set of variables \( X \) and \( X_\pm = X \sqcup \overline{X} \) with the natural involution \( x \mapsto \overline{x} \).

A system of equations \( \mathcal{E} \) is a finite set of words \( w = x_1 \ldots x_n \) in \( X_\pm^* \) (representing the equation \( x_1 \ldots x_n = 1 \)).

A set of rational constraints for \( \mathcal{E} \) is a tuple \( \mathcal{R} = (\mathcal{R}_x)_{x \in X} \), where, for each \( x \in X \), \( \mathcal{R}_x \subset G \) is a rational subset.
A solution of \((\mathcal{E}, \mathcal{R})\) is a tuple \(g = (g_x)_{x \in X} \in G^X\) with \(g_x \in \mathcal{R}_x\) for each \(x \in X\), and such that, for each \(w = x_1 \ldots x_n \in \mathcal{E}\), we have \(g_{x_1} \ldots g_{x_n} = 1\) in \(G\), where we define \(g_x = g_x^{-1}\) for each \(x \in X\).

Abusing notation, if \((g_x)_{x \in X}\) is a solution of \(\mathcal{E}\), then we also call the solution the tuple \((g_x)_{x \in X \cup X'}\), where \(g_x = g_x^{-1}\) for each \(x \in X\).

Constants in a system of equations can be encoded with rational constraints: a constant \(g_0 \in G\) is a variable \(x\) with a corresponding constraint \(\mathcal{R}_x = \{g_0\}\). If \(G \setminus \{1\}\) is a rational subset, then inequations can be encoded with rational constraints: replace each inequation \(w \neq 1\) by an equation \(w = y\), where \(y\) is a new variable with the rational constraint \(\mathcal{R}_y = G \setminus \{1\}\).

### 2.1. Reduction to triangular equations

We fix a finitely generated virtually free group \(V\). Classically, a system of equations with rational constraints is equivalent to a triangular system (that is, in which every \(w \in \mathcal{E}\) has length at most 3): for each equation \(x_1 \ldots x_n = 1\) with \(n \geq 4\), we add some new variables \(y_2, \ldots, y_{n-2}\) with no rational constraint on \(y_i\) \((\mathcal{R}_{y_i} = V)\), and we replace the equation \(x_1 \ldots x_n = 1\) by the equations \(y_2 = x_1 x_2, y_3 = y_2 x_3, \ldots, y_{n-2} x_{n-1} x_n = 1\). One can also get rid of equations of length 1 by forgetting about the corresponding variable.

It is convenient to get rid of equations of length 2. This can be done as follows, using the fact that the set of rational subsets of a virtually free group is a Boolean algebra (Lemma 1.10). If we have an equation \(x_1 x_2 = 1\), where \(x_1\) is distinct from \(x_2, x_2\) as a formal variable, then one can substitute every occurrence of \(x_2\) or \(x_2\) with \(\overline{x_1}\) or \(x_1\), respectively, and change the rational constraint \(x_1 \in \mathcal{R}_{x_1}\) to \(x_1 \in \mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}^{-1}\). Of course, one can forget about equations of the form \(x \overline{x} = 1\). Each equation of the form \(x^2 = 1\) can be replaced by two equations \(x^4 = 1\) and \(x^6 = 1\), which can be triangulated in the usual way.

The transformation into a triangular system of equations is algorithmic.

### 2.2. Twisted equations

Consider the general situation of a group \(V\) containing a normal subgroup \(G\) of finite index. Let \(\pi : V \rightarrow Q = V/G\) be the quotient map. For each \(q \in Q\), choose a lift \(\bar{q} \in V\). Given \(x_1, x_2, x_3 \in V\), let \(g_i = \pi(x_i)\) and write \(x_i = \bar{q}_i g_i\) for some \(g_i \in G\). Then \(x_1, x_2, x_3\) satisfy the equation \(x_1 x_2 x_3 = 1\) if and only if \(q_1 q_2 q_3 = 1\) (so \(q_1 q_2 q_3 \in G\)), and \((q_1 q_2 q_3) g_1^{-q_1} g_2^{-q_2} g_3^{-q_3} = 1\). This last equation can be viewed as an equation with the unknowns \(g_1, g_2, g_3\), twisted by automorphisms of the form \(g \mapsto g^v\) for some \(v \in V\).

The group generated by these automorphisms has finite image in \(\text{Out}(G)\), but it usually fails to lift to a finite subgroup of \(\text{Aut}(G)\).

In Proposition 2.4, we prove that when \(G\) is a free group, we can embed \(G\) in a larger free group \(\langle S \rangle\) with free basis \(S\), so that the twisting automorphisms give rise to automorphisms of \(\langle S \rangle\) that preserve \(S \cup S^{-1}\). In particular, these automorphisms of \(\langle S \rangle\) preserve word length, which will be of importance to us. The price to pay is that we have to add rational constraints to our system of equations (even if there were no such constraints originally) to guarantee that the solutions belong to the original free group.

Given a finite set \(S\) and the corresponding free group \(\langle S \rangle\), we denote by \(\text{Aut}(S_\pm)\) the finite group of automorphisms of \(\langle S \rangle\) preserving the set \(S_\pm = S \cup S^{-1}\) (it has order \(2^{2n} n!\)).

**Definition 2.2** (Twisted equations). A (triangular) system of twisted equations with rational constraints \((\mathcal{E}, \mathcal{R})\) in a free group \(\langle S \rangle\) consists of an alphabet of variables \(X\), of a finite set \(\mathcal{E}\) of equations of the form \(((\varphi_1, x_1), (\varphi_2, x_2), (\varphi_3, x_3)) \in (\text{Aut}(S_\pm) \times (X \cup X'))^3\)
(representing the equation $\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) = 1$, and a tuple $R = (R_x)_{x \in X}$ of rational subsets of $\langle S \rangle$.

A solution of $\mathcal{E}$ is a tuple $g = (g_x)_{x \in X} \in \langle S \rangle^X$ with $g_x \in R_x$ for each $x \in X$, and such that, for each $((\varphi_1,x_1), (\varphi_2,x_2), (\varphi_3,x_3)) \in \mathcal{E}$, we have $\varphi_1(g_{x_1})\varphi_2(g_{x_2})\varphi_3(g_{x_3}) = 1$, where we define $g_x = g_{x^{-1}}$ for each $x \in X$.

**Remark 2.3.** We opted for a restricted meaning of twisting. In general (and as presented in the introduction) a twisted equation is an equation where some automorphisms are involved, but here, the only allowed automorphisms are those that preserve a basis of the given free group. This will be enough for our needs.

The **twisting subgroup** $\Phi \subset \text{Aut}(S_\pm)$ is the subgroup generated by the twisting morphisms involved in all twisted equations. For all $s \in S$ and $\varphi \in \Phi$, we say that $\Phi$ has no inversion if $\varphi(s) \neq s^{-1}$. Our construction from a virtually free group will produce only twisting subgroups without inversions. However, if we are given a system of twisted equations where $\Phi$ has inversions, one can easily reduce to the case without inversion using the subdivision without inversions. Moreover, one can algorithmically compute the twisting subgroup has no inversion and is finite.

Moreover, one can algorithmically compute $\langle S \rangle$ and $\langle \mathcal{E}_1, R_1 \rangle \ldots \langle \mathcal{E}_k, R_k \rangle$ from a presentation of $\langle S \rangle$ and from $\langle \mathcal{E}, R \rangle$, where all rational subsets are represented by automata.

**Remark 2.5.** The free group $\langle S \rangle$ does not occur naturally as a subgroup of $V$. Only a subgroup of $\langle S \rangle$ can be naturally identified as a subgroup of finite index of $V$. The natural bijection defined in the proof is clearly computable, but we will not need this fact.

**Proof of Proposition 2.4.** Write $V$ as an extension $F \to V \xrightarrow{\pi} Q$, where $Q$ is finite and $F = \ker \pi$ is a normal free subgroup of finite index of $V$. Write $V$ as the fundamental group of a finite graph of finite groups, and let $V \cap T$ be the dual Bass–Serre tree. By adding a new edge to the graph of groups, we may assume that $T$ contains a point $*$ with trivial stabilizer ($T$ may fail to be minimal).

Consider the graph $\Gamma = T/F$, and still denote by $*$ the image of $*$ in $\Gamma$. Since $F$ acts freely, the quotient map $T \to \Gamma$ is a covering map, and $F = \pi_1(\Gamma,*)$. Note that $Q$ acts on $\Gamma$ by graph automorphisms (which may fail to fix $*$).

Consider a set $S$ of oriented edges of $\Gamma$ obtained by choosing an orientation for each edge, and consider the free group $\langle S \rangle$. Each edge path in $\Gamma$ defines an element of $\langle S \rangle$, and in particular, $F \subset \langle S \rangle$. Clearly, $Q$ acts on $\langle S \rangle$ via automorphisms of $\text{Aut}(S_\pm)$. We denote by $\varphi_q(x)$ the image of $x \in \langle S \rangle$ under the action of $q \in Q$. 


For \( g \in V \), let \( c_g \) be the edge path of \( \Gamma \) obtained by projection of the segment \([*, g, *] \subset T\). This defines a map \( c : V \to (S) \). Since the stabilizer of \(*\) is trivial, it follows that \( c \) is one-to-one. Note that \( c_g \) joins \(*\) to \( \varphi_{n(g)}(*\)\). The map \( c \) is not a morphism but satisfies \( c(gh) = c(g) \cdot \varphi_{n(g)}(c(h)) \) and \( c(g^{-1}) = \varphi_{n(g)}^{-1}(c(g)) \). In particular, the restriction of \( c \) to \( F \) is a morphism.

Each solution of \( E \) in \( V \) maps to a solution of \( E \) in \( Q \). For each solution \( q \in Q^X \) of \( E \) in \( Q \), the set \( R_x \cap \pi^{-1}(q_x) \) is a rational subset by Lemma 1.10. Let \( S_q \subset V^X \) be the set of solutions of \( E \) with rational constraints \( x \in R_x \cap \pi^{-1}(q_x) \). The set of solutions of \((E, R)\) is the disjoint union of the sets \( S_q \).

Each equation in \( E \) is equivalent to an equation of the form \( x_1 x_2 x_3 = 1 \) or \( x_1 x_2 \overline{x}_3 = 1 \), where \( x_1, x_2, x_3 \) lie in \( X \) (not \( \overline{X} \)); we assume that each equation is of this form. Given \( q \in S_q \), the equation \( g_{x_1} g_{x_3} g_{x_3} = 1 \) implies \( c(g_{x_1}) \cdot \varphi_{q_{x_1}}(c(g_{x_2}))) \cdot \varphi_{q_{x_1} q_{x_2}}(c(g_{x_3})) = 1 \). Similarly, the equation \( g_{x_1} g_{x_2} g_{x_1} = 1 \) implies \( c(g_{x_1}) \cdot \varphi_{q_{x_2}}(c(g_{x_3}))) \cdot \varphi_{q_{x_1} q_{x_3}}(c(g_{x_3})) = 1 \).

Let \( E_2' \) be the set of equations in \((S)\) obtained by replacing in \( E \) each equation \( x_1 x_2 x_3 = 1 \) or \( x_1 x_2 \overline{x}_3 = 1 \) by the twisted equation \( x_1 \cdot \varphi_{q_{x_1}}(x_2) \cdot \varphi_{q_{x_1} q_{x_2}}(x_3) = 1 \) or \( x_1 \cdot \varphi_{q_{x_2}}(x_2) \cdot \varphi_{q_{x_1} q_{x_3}}(x_3) = 1 \), respectively. Thus \( c \) maps \( S_2 \) into the set \( S'_2 \) of solutions of \((E_2', R')\), where \( R'_q = c(R_x \cap \pi^{-1}(q_x)) \subset (S) \). Let us check that \( R'_q \) is a rational subset of \((S)\). If \( q_x = 1 \), this follows from the fact that \( c_1 \) is a morphism. Otherwise, consider \( \overline{q}_x \in \pi^{-1}(\{q_x\}) \), and consider the rational subset \( R''_q = \overline{q}_x^{-1}(R_x \cap \pi^{-1}(q_x)) \subset F \). Since \( c_1 \) is a morphism, it follows that \( R''_q \) is rational and so is \( R'_q = c(\overline{q}_x) \cdot \varphi_{\overline{q}_x}(c(R''_q)) \).

We claim that \( c \) maps \( S_2 \) to \( S'_2 \). Indeed, consider a solution \( q \in S'_2 \). For each \( x \in X \), since \( g_x \in R'_q \), it follows that \( q_x \) corresponds to a path joining \(*\) to \( g_x * \). This path lifts to a path in the tree \( \Gamma \) joining \(*\) to some \( g_x * \), where \( g_x \) is uniquely determined because \(*\) has a trivial stabilizer. By definition, \( c(g_x) = q_x \). Since \( c \) realizes a bijection between \( R_x \cap \pi^{-1}(q_x) \) and its image, the constraint \( g_x \in R_x \cap \pi^{-1}(q_x) \) is satisfied. For each equation \( x_1 x_2 x_3 \in E \), \( q \) satisfies the corresponding equation in \( E_2' \), namely \( \varphi_{q_{x_1}}(x_2) \cdot \varphi_{q_{x_1} q_{x_2}}(x_3) = 1 \), we get that \( c(g_{x_1} g_{x_2} g_{x_3}) = 1 \). By the injectivity of \( c \), we have \( g_{x_1} g_{x_2} g_{x_3} = 1 \). A similar argument applies to equations of the form \( x_1 x_2 \overline{x}_3 = E \).

This proves that \( c \) induces a bijection between the sets of solutions of \((E, R)\) and the (disjoint) union of the sets of solutions of \((E_2', R')\) as \( q \) ranges among solutions of \( E \) in \( Q^X \).

Let us prove the computability of the new system of equations. Starting with a presentation of a virtually free group \( V \), one can effectively find a presentation of \( V \) as a graph of finite groups \( \Lambda \). Indeed, enumerating all presentations of \( V \), one will find one that is visibly a presentation of the correct form. More precisely, if \( A \) is a finite group, then its finite presentation \( \langle A \rangle (\text{table}(A)) \) consisting of its multiplication table is a presentation from which the finiteness of \( A \) is obvious. Consider an amalgam of two finite groups \( G = A \ast_C B \), given by two monomorphisms \( j_A : C \to A \) and \( j_B : C \to B \). Then \( G \) has a presentation of the form \( \langle A \cup B \cup C \rangle \text{table}(A), \text{table}(B), \text{table}(C), \{ c = j_A(c) = j_B(c) \} \subset C \rangle \) from which one can obviously read that \( j_A \) and \( j_B \) are injective morphisms, and that \( G \) is an amalgam of finite groups. Similarly, given a graph of finite groups \( \Gamma \) and \( \tau \subset \Gamma \) a maximal tree, one produces a presentation of \( \pi_1(\Gamma, \tau) \) from the multiplication tables of vertex and edge groups from which one can read that the corresponding group is the fundamental group of a graph of finite groups.

One can also find a normal free subgroup \( F \) of finite index: enumerate all morphisms to finite groups \( \rho : V \to Q \), and, for each vertex group \( \Lambda_v \), check whether \( \rho|_{\Lambda_v} \) is injective; one will eventually find such a \( \rho \), and one can take \( F = \ker \rho \). The graph \( \Gamma = T/F \) is constructed from \( \Lambda \) as the covering with deck group \( Q \); for each edge or vertex \( x \) of \( \Lambda \), we put a copy of this edge or vertex for each element of \( Q/\rho|_{\Lambda_v} \), and the incidence relation is the natural one. Thus \((S)\) is computable together with its natural basis and the action of \( Q \). The new systems of equations \( E_2' \) are now explicit.

It remains to compute the rational constraints \( R'_q = c(R_x \cap \pi^{-1}(q_x)) \). We saw that \( R'_q = c(\overline{q}_x) \cdot \varphi_{\overline{q}_x}(c(R''_q)) \) with \( R''_q = \overline{q}_x^{-1}(R_x \cap \pi^{-1}(q_x)) \). An automaton representing \( R''_q \) can
be computed using Lemma 1.10. By Lemma 1.4, since \( c|_F \) is a morphism, we get an automaton representing \( c(R''_q') \), and thus \( R''_q,x \).

3. Band complexes

We fix a finite set \( S \), the free group \( G = \langle S \rangle \), and the corresponding free monoid \( S^*_\pm \) with involution \( x \to \overline{x} \). Let \( \text{Aut}(S^*_\pm) \) be the corresponding finite group of automorphisms of \( G \) preserving \( S \cup S^{-1} \), which we identify with the group of automorphisms of \( S^*_\pm \) commuting with the involution.

3.1. Band complexes

A band complex \( \Sigma \) consists of a domain \( D \) together with a finite set of bands. The domain \( D \) is a disjoint union of non-degenerate segments. We say that a segment is non-degenerate when it is non-empty and not reduced to a point. A band \( B \) consists of a rectangle \([a, b] \times [0, 1],\) together with two injective continuous attaching maps \( f_{B,0} : [a, b] \times \{0\} \to D \) and \( f_{B,1} : [a, b] \times \{1\} \to D \) and a twisting automorphism \( \varphi_B \in \text{Aut}(S^*_\pm) \). The segments \( J_{B,\varepsilon} = f_{B,\varepsilon}([a, b] \times \{\varepsilon\}) \) for \( \varepsilon = 0, 1 \) are called the bases of the band \( B \).

Remark. Since bands carry twisting automorphisms, a band complex might rather be called a twisted band complex. We will not use this terminology for the sake of brevity.

We denote by \( \Phi \subset \text{Aut}(S^*_\pm) \) the group generated by all twisting morphisms \( \varphi_B \), and we assume that \( \Phi \) has no inversion, that is, \( \varphi(s) \neq \overline{s} \) for all \( \varphi \in \Phi \) and \( s \in S^*_\pm \).

The set of vertices of \( \Sigma \) is the subset of \( D \) consisting of the endpoints of \( D \) together with the endpoints of the bases of the bands. We identify \( \Sigma \) with the CW-complex obtained by gluing the bands on \( D \) using the maps \( f_{B,\varepsilon} \).

The precise values of the attaching maps are not important: a band complex is really determined by the ordering of vertices in each component of \( D \), and for each band, the 4-tuple \((f_{B,0}(a), f_{B,0}(b), f_{B,1}(a), f_{B,1}(b))\) together with the twisting automorphism \( \varphi_B \).

3.2. Solution of a band complex

An elementary segment of \( \Sigma \) is a segment of \( D \) joining two adjacent vertices. We denote by \( E(\Sigma) \) the set of oriented elementary segments of \( \Sigma \). We denote by \( e \to \overline{e} \) the orientation-reversing involution on \( E(\Sigma) \).

A labelling \( \sigma \) of \( \Sigma \) is a labelling of each \( e \in E(\Sigma) \) by a non-empty word \( \sigma_e \in S^*_\pm \), so that \( \sigma_\overline{e} = \overline{\sigma_e} \) (we use the standard notation \( S^*_\pm = S_\pm \cdot S^*_\pm \setminus \{1\} \)). We say that a segment \( I \subset D \) is adapted to \( \Sigma \) if it is a union of elementary segments. For each oriented adapted segment \( I = e_1 \ldots e_n \subset D \) written as a concatenation of elementary segments, we define \( \sigma_I = \sigma_{e_1} \ldots \sigma_{e_n} \).

Alternatively, we often view a labelling of \( \Sigma \) as a subdivision of \( D \), together with a labelling of the subdivided edges by letters in \( S \cup \overline{S} \).

A solution \( \sigma \) of \( \Sigma \) is a labelling such that, for each band \( B \) with the twisting automorphism \( \varphi_B \) and with the bases \( J_0 \) and \( J_1 \) oriented so that the attaching maps preserve the orientation, one has \( \sigma_{I_1} = \varphi_B(\sigma_{I_0}) \). We say that the solution \( \sigma \) is reduced if for every non-degenerate segment \( I \subset D \), \( \sigma_I \) is a non-trivial reduced word.

The length \( |\sigma| \) of a solution \( \sigma \) is the sum of lengths of words \( |\sigma_{D_1}| + \ldots + |\sigma_{D_k}| \), where \( D_1, \ldots, D_k \) are the connected components of \( D \) (with a choice of orientation).
3.3. Rational constraints

A set of rational constraints on $\Sigma$ is a family of regular languages $R = (R_e)_{e \in E(\Sigma)}$ of $S_+^\ast$, indexed by oriented elementary segments of $\Sigma$, and such that $R_e = R_e^\ast$. A solution $\sigma$ of $\Sigma$ satisfying the rational constraint $R$ is a solution of $\Sigma$ such that, for each elementary segment $e \in E(\Sigma)$, we have $\sigma_e \in R_e$.

All the rational constraints we use later will be in some standard form as defined in Subsection 3.5.

3.4. From twisted equations to band complexes

**Proposition 3.1.** Let $(E,R)$ be a system of twisted equations with rational constraints on a free group $(S)$. Then one can effectively compute a finite set of band complexes with rational constraints $\Sigma_1,\ldots,\Sigma_p$ and a bijection between the set of solutions of $(E,R)$ and the disjoint union of the (reduced) solutions of $\Sigma_1,\ldots,\Sigma_p$.

Moreover, every solution of $\Sigma_i$ is reduced.

**Proof.** We say that a tuple $g \in G^X$ is singular if $g_x = 1$ for some $x \in X$. Denote by $\text{Sol}(E,R)$ and $\text{Sol}^\ast(E,R)$ the sets of solutions and non-singular solutions of $(E,R)$, respectively. Clearly, $\text{Sol}(E,R) = \bigcup_{i=0}^{p} \text{Sol}^\ast(\Sigma_i)$, in bijection with the disjoint union of $2^{|X|}$ sets of the form $\text{Sol}^\ast(\Sigma_i)$, where, for $X_0 = X \setminus \bigcup_{i=0}^{p} \Sigma_i$, we see that $\Sigma_{E_0}$ is the system of twisted equations whose set of unknowns is $X \setminus X_0$, and obtained from $E$ by replacing each occurrence of the variable $x \in X_0$ by 1.

Some of the obtained equations may involve less than three variables. However, if some equation involves just one variable, then it is of the form $\varphi(x) = 1$, and so $E$ has no non-singular solution and we may forget about this $X_0$. We view the equations involving two variables as twisted equalities of the form $\varphi_1(x_1) = \varphi_2(x_2)$.

**Remark 3.2.** We could get rid of twisted equalities involving two distinct formal variables by substitution, and of twisted equalities of the form $\varphi_i(x) = x$ by adding the rational constraint saying that $x \in \text{Fix} \varphi$, but this does not allow us to get rid of equations of the form $\varphi(x) = x$ since the set of fixed points of $x \mapsto \varphi(x)$ is not a rational language in general, and we do not want to add new singular variables.

We now need to build some band complexes encoding the set of non-singular solutions of a given system of twisted equations $(E,R)$ (including twisted equalities).

For each unknown $x \in X$, we consider an oriented segment $D_x$ (whose labelling will correspond to the value $g_x$ of the unknown $x$). For $x \in \overline{X}$, we denote by $D_x$ the same segment as $D_x$, but with the opposite orientation. Then, for each twisted equation $\varepsilon$ of the form $\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) = 1$ or $\varphi_1(x_1) = \varphi_2(x_2)$, with $x_i \in X \cup \overline{X}$, we add three or two oriented segments $D_{\varepsilon,i}$, respectively, corresponding to $\varphi_i(x_i)$. We define the domain $D$ of our band complex as $D = (\bigcup_{x \in X} D_x) \sqcup (\bigcup_{(i,\varepsilon)} D_{\varepsilon,i})$.

For each segment $D_{\varepsilon,i}$ corresponding to $\varphi_i(x_i)$, we add a band whose bases are $J_0 = D_{\varepsilon,i}$ and $J_1 = D_{\varepsilon,i}$, preserving the orientation, and whose twisting morphism is $\varphi_i$.

For each twisted equality $\varepsilon$ of the form $\varphi_1(x_1) = \varphi_2(x_2)$, we add a band whose bases are $J_0 = D_{\varepsilon,1}$ and $J_1 = D_{\varepsilon,2}$, preserving the orientation, and with a trivial twisting morphism.

For any non-singular solution $g$ of the twisted equation $\varepsilon \in E$ corresponding to $\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) = 1$, one can consider its cancellation tripod $\tau$: this is the convex hull of $a_1 = 1, a_2 = \varphi_1(g_{x_1}), a_3 = \varphi_1(g_{x_2})\varphi_2(g_{x_2})$ in the Cayley graph of $(S)$, so that $[a_i,a_{i+1}]$ is labelled by $\varphi_i(g_{x_i})$ (where we view $i$ as an integer mod 3). Let $c$ be the centre of $\tau$, that is, the intersection of the three segments $[a_i,a_{i+1}]$. There are four possible types of combinatorics of
The equation
\[ \varepsilon : \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) = 1 \]

\[ \varphi_1(gx_1) \varphi_2(gx_2) \varphi_3(gx_3) \]

\[ a_1 \quad a_2 \quad a_3 \]

The equation \( \varepsilon : \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) = 1 \)

with flat cancellation tripod

Figure 2. The bands when the cancellation tripod is non-degenerate.

Figure 3. The bands when the cancellation tripod is flat.

cancellation for each triangular equation \( \varepsilon : c \notin \{a_1, a_2, a_3\} \) (in which cases \( \tau \) is not a segment), or \( c = a_1, c = a_2, c = a_3 \). The four types are mutually exclusive because \( g \) is non-singular.

To each choice of combinatorics of cancellation for this twisted equation, we associate a set of bands to be added to our band complex. If the cancellation tripod is not a segment, then, for each \( i = 1, \ldots, 3 \), add a vertex \( c_{\varepsilon,i} \) at the midpoint of \( D_{\varepsilon,i} \) (see Figure 2). Then add a band reversing the orientation, whose bases are the right half-segment of \( D_{\varepsilon,i} \) and the left half-segment of \( D_{\varepsilon,i+1} \) (right and left having a meaning relative to the orientation of those segments), and whose twisting automorphism is the identity.

If the cancellation tripod is such that \( c = a_i \) for some \( i \in \{1, 2, 3\} \), then the product \( \varphi_{1-1}(gx_{i-1})\varphi_i(gx_i) \) is reduced, \( i \) being considered mod 3 (see Figure 3). We add one vertex at the midpoint of \( D_{\varepsilon,i+1} \), and two bands reversing the orientation, whose twisting automorphisms are the identity. The bases of the first band are \( J_0 = D_{\varepsilon,i} \) and \( J_1 \) the initial half-segment of \( D_{\varepsilon,i+1} \). The bases of the second are \( J_0 = D_{\varepsilon,i-1} \) and \( J_1 \) the final half-segment of \( D_{\varepsilon,i+1} \).

Let \( t \) be the number of triangular equations of \( \mathcal{E} \). For each choice \( \kappa \) out of the \( 4^t \) possible combinatorics of cancellation, we obtain a band complex \( \Sigma_{\kappa} \).

Finally, for each rational subset \( \mathcal{R}_x \subset (S) \), we consider \( \tilde{\mathcal{R}}_x \subset S_\pm^* \) the set of reduced words representing elements of \( \mathcal{R}_x \). This is a rational language by Lemma 1.6. We add to \( \Sigma_{\kappa} \) the rational constraint \( \tilde{\mathcal{R}}_x \) on \( D_x \) for each \( x \in X \), and we do not put any constraint on the other elementary segments of \( \Sigma \) (that is, we set \( \mathcal{R}_e = S_\pm^* \)). Since \( \tilde{\mathcal{R}}_x \) is a language of reduced words, any solution of \( \Sigma_{\kappa} \) is reduced.
We claim that the set of reduced solutions of \( \Sigma_\kappa \) is in one-to-one correspondence with the subset \( S_0 \) of \( \text{Sol}^\ast (\mathcal{E}, \mathcal{R}) \) whose combinatorics of cancellation correspond to \( \kappa \).

Indeed, a non-singular solution \( g \in S_\kappa \) defines a labelling \( \sigma \) of \( \Sigma_\kappa \) as follows: \( D_\kappa \) is labelled by the reduced word \( \sigma_x \in S^+_\kappa \) representing \( g_x \), each \( D_{\varepsilon,i} \) representing \( \varphi_i(x) \) is labelled by the reduced word representing \( \varphi_i(g_{\varepsilon,i}) \), and if the midpoint of \( D_{\varepsilon,i} \) is a vertex of \( \Sigma_\kappa \), then the labelling of the two half-segments of \( D_{\varepsilon,i} \) should be such that the position of this vertex corresponds to the centre of the cancellation tripod. The labelling \( \sigma \) thus constructed is a clearly reduced solution of \( \Sigma_\kappa \).

Conversely, if \( \sigma \) is a reduced solution of \( \Sigma_\kappa \), then the image \( g_x \) in \( \langle \mathcal{S} \rangle \) of the word \( \sigma_x \) clearly defines a non-singular solution of \( (\mathcal{E}, \mathcal{R}) \). Moreover, for each \( \varepsilon \in \mathcal{E}_T \), the label of \( D_{\varepsilon,i} \) defines a geodesic segment, and the midpoint of the three segments \( D_{\varepsilon,i} \) corresponds to the centre of the cancellation tripod for \( \varepsilon \). Thus, the combinatorics of cancellation agree with \( \kappa \), and \( g_x \in S_\kappa \).

The construction of the band complexes \( \Sigma_\kappa \) is clearly effective, and so Proposition 3.1 follows.

3.5. Standard forms of rational constraints

It will be convenient to represent all the needed rational languages in a fixed finite monoid. This will allow us to treat uniformly all rational languages appearing under various transformations of the band complexes.

**Lemma 3.3.** Let \( \mathcal{R}_1, \ldots, \mathcal{R}_k \) be a finite set of rational languages in \( S^\pm_1 \).

There exists a finite monoid \( \mathcal{M} \) with an involution \( m \mapsto \overline{m} \) and with an action of \( \text{Aut}(S^\pm_1) \), and an \( \text{Aut}(S^\pm_1) \)-equivariant morphism \( \rho : S^\pm_1 \rightarrow \mathcal{M} \) commuting with the involutions, such that each \( \mathcal{R}_i \) is the preimage of a finite subset of \( \mathcal{M} \).

Moreover, all this data is algorithmically computable from automata representing \( \mathcal{R}_i \).

When \( \mathcal{R} \) is the preimage under \( \rho \) of some subset of \( \mathcal{M} \) as above, we say that \( \mathcal{R} \) is represented by \( \rho \).

**Proof of Lemma 3.3.** We first consider the case of a single regular language. Let \( \mathcal{R} \subset S^+_1 \) be a rational language. Let \( A \) be an automaton representing \( L \) with a vertex set \( v_1, \ldots, v_n \). For each \( s \in S \cup \overline{S} \), consider the \( n \times n \) Boolean incidence matrix \( M_s \) of the subgraph of \( A \) whose edges are those labelled by \( s \).

Let \( M_0 \) be the semigroup of \( n \times n \)-Boolean matrices and let \( \rho_0 : S^+_1 \rightarrow M_0 \) be the morphism sending \( s \) to \( M_s \). Let \( v_S \) and \( v_A \) be the Boolean vectors representing \( \text{Start} \) and \( \text{Accept} \), respectively. Then \( \mathcal{R} \) is the preimage under \( \rho \) of the set \( A_0 \subset M_0 \) of matrices \( M \) such that \( t^{v_S} M v_A = 1 \).

Now let \( \overline{\rho}_0 : S^-_1 \rightarrow M_0 \) be the morphism sending \( s \) to \( t^\ast M_\overline{s} \). Note that \( \overline{\rho}_0(w) = t^\ast \rho_0(w) \) for all \( w \in S^+_1 \). Consider the finite monoid \( \mathcal{M} = M_0 \times M_0 \) endowed with the involution \( (M_1, M_2) \mapsto (M_2, t^\ast M_1) \), and consider \( \rho : S^+_1 \rightarrow \mathcal{M} \) sending \( s \) to \( (\rho_0(s), \overline{\rho}_0(s)) \). By the remark above, \( \rho \) commutes with the involutions of \( S^+_1 \) and \( \mathcal{M} \), and \( \mathcal{R} = \rho^{-1}(A_0 \times M_0) \).

Now given a finite set of languages \( \mathcal{R}_1, \ldots, \mathcal{R}_k \), consider finite monoids with the involutions \( \mathcal{M}_1, \ldots, \mathcal{M}_k \) and \( \rho_i : S^+_1 \rightarrow \mathcal{M}_i \) commuting with the involutions such that \( \mathcal{R}_i = \rho_i^{-1}(A_i) \) for some \( A_i \subset \mathcal{M}_i \). Consider the finite monoid \( \mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_k \) with the diagonal involution, and \( \rho = \rho_1 \times \cdots \times \rho_k \). Then, for \( A'_i = \mathcal{M}_1 \times \cdots \times \mathcal{M}_{i-1} \times A_i \times \mathcal{M}_{i+1} \times \cdots \times \mathcal{M}_k \), we have \( \mathcal{R}_i = \rho^{-1}(A'_i) \).

Finally, we need to put the \( \text{Aut}(S^\pm_1) \)-action in the picture. For each \( \alpha \in \text{Aut}(S^\pm_1) \), let \( \mathcal{M}_\alpha \) be a copy of \( \mathcal{M} \), and consider \( \rho_\alpha = \rho \circ \alpha : S^+_1 \rightarrow \mathcal{M}_\alpha \). Consider the product \( \hat{\mathcal{M}} = \prod_{\alpha \in \text{Aut}(S^\pm_1)} \mathcal{M}_\alpha \).
endowed with the action of \( \text{Aut}(S_\pm) \) by permutation of factors, and \( \tilde{\rho} = \prod_{\alpha \in \text{Aut}(S_\pm)} \rho_{\alpha} \). Then \( \tilde{\rho} : S_\pm^+ \to \mathcal{M} \) satisfies the lemma.

By Lemma 3.3, one can represent all the rational constraints \( R_e \) of a given band complex by a single morphism \( \rho : S_\pm^+ \to \mathcal{M} \). It will be useful to assume that \( \rho \) also represents \( \text{Fix } \varphi \subset S_\pm^+ \) for all \( \varphi \in \Phi \) (this is a regular language as it is the submonoid generated by \( \Phi \)-invariant letters in \( S_\pm^+ \)). By a finite disjunction of cases, we need only consider rational constraints of the form \( R_e = \rho^{-1}(\{m_e\}) \).

**Definition 3.4** (Standard form for rational constraints). A set of rational constraints in standard form on a band complex \( \Sigma \) consists of the following:

1. a finite monoid \( \mathcal{M} \) with an involution and an action of \( \text{Aut}(S_\pm) \);
2. an epimorphism \( \rho : S_\pm^+ \to \mathcal{M} \) commuting with the involutions and \( \text{Aut}(S_\pm) \)-equivariant, such that, for all \( \varphi \in \text{Aut}(S_\pm) \), the rational language \( \text{Fix } \varphi \subset S_\pm^+ \) is represented by \( \rho \);
3. a collection \( \underline{m} = (m_e)_{e \in E(\Sigma)} \in \mathcal{M}^{E(\Sigma)} \) such that \( m_e = \overline{m_e} \).

The tuple \( \underline{m} = (m_e) \) defines a tuple of rational constraints by \( R_e = \rho^{-1}(\{m_e\}) \). By definition, a solution of \( (\Sigma, \underline{m}) \) is a solution of \( \Sigma \) with the corresponding rational constraints. Using Lemma 3.3, we immediately get the following lemma. The fact that \( \rho \) is onto can be obtained by changing \( \mathcal{M} \) to \( \rho(S_\pm^+) \).

**Lemma 3.5.** Given a band complex \( \Sigma \) with rational constraints \( R_e \), one can compute a finite monoid \( \mathcal{M} \), a morphism \( \rho : S_\pm^+ \to \mathcal{M} \), and some tuples \( \underline{m}_1, \ldots, \underline{m}_p \in \mathcal{M}^{E(\Sigma)} \) defining rational constraints in standard form on \( \Sigma \) such that the set of solutions of \( (\Sigma, R_e) \) is the disjoint union of the sets of solutions of \( (\Sigma, \underline{m}_1), \ldots, (\Sigma, \underline{m}_p) \).

From now on, all band complexes are endowed with rational constraints in standard form.

### 4. Bulitko’s lemma: bounding the exponent of periodicity

The goal of this section is a version of Bulitko’s lemma [6] adapted to twisted equations, which we state in terms of band complexes. The exponent of periodicity of a solution \( \sigma \) of a band complex \( \Sigma \) is the largest integer \( s \) such that some subsegment of the domain \( D \) of \( \Sigma \) is labelled by some word of the form \( p^s \), with \( p \) a non-trivial word. Recall that the length of a solution \( \sigma \) of a band complex is the number of letters in \( S_\pm \) involved in the labelling.

**Proposition 4.1** [6]. Let \( \Sigma \) be a band complex with rational constraints \( \underline{m} \) in standard form.

There exists a computable number \( B(\Sigma, \underline{m}) \) such that any shortest solution of \( \Sigma \) has an exponent of periodicity at most \( B(\Sigma, \underline{m}) \).

We will use and prove the proposition only for a band complex arising from a system of equations as in Subsection 3.4. More precisely, we assume that every component \( D_0 \) of the domain of the band complex contains at most one vertex in its interior, and at most two bases of bands distinct from \( D_0 \) (see Figures 2 and 3). The general statement can be easily deduced from this case.
4.1. Normal decomposition with respect to powers

Consider a solution $\sigma$ of the band complex $\Sigma$. We view $\sigma$ as a subdivision of $D$ together with a labelling of the edges by elements of $S_\pm$. Each interval $A \subset D$ that we consider is a union of such edges. Its length $|A|$ is the number of edges it contains, which coincides with the length of the word $\sigma_A$ labelled by $A$.

We fix some $q \in \mathbb{N} \setminus \{0\}$. Given some interval $A \subset D$, we are going to define a natural set of special segments encoding its large enough subwords which are $q$-periodic. We say that an interval $A \subset D$ of length $n > q$ is $q$-periodic if $\sigma_A = a_1 \ldots a_n$ with $a_i \in S_\pm$ and for all $i \leq n - q$, we have $a_i = a_{i+q}$.

For each maximal $q$-periodic segment $P \subset A$ of length at least $2q$, we say that its central subsegment $Q \subset P$ of length $|P| - 2q$ is a special segment of $A$. A special segment is reduced to a vertex when $|P| = 2q$. Although maximal $q$-periodic segments may overlap, the following claim holds.

**Claim 4.2.** Two distinct $q$-special segments of $A$ are disjoint and separated by a segment of length at least $q + 1$.

**Proof.** Let $Q, Q' \subset A$ be two special segments, and let $P \supset Q$ and $P' \supset Q'$ be the corresponding maximal $q$-periodic segments. If the claim fails, then $P \cap P'$ has length at least $q$. Thus, two edges of $Q \cup Q'$ at distance $q$ of each other either both lie in $P$ or in $P'$, and so $P \cup P'$ is $q$-periodic. \qed

For any word $p$ of length $q$, and $r = n/q \in (1/q)\mathbb{N}$, it will be convenient to denote by $p^r$ the prefix of length $n$ of a sufficiently large power of $p$. If $r \in \mathbb{N}$, then $p^r$ coincides with the usual $r$th power of $p$. For each special segment $Q$ we denote by $r(Q) = |Q|/q \in (1/q)\mathbb{N}$ the exponent and by $p(Q)$ the period so that $Q$ is labelled by $p(Q)^r(Q)$ (if $r(Q) < 1$, then $p(Q)$ should be defined using the maximal $q$-periodic subsegment $P$ containing $Q$).

The normal decomposition of $A \subset D$ (with respect to $q$ and to the solution $\sigma$) is the decomposition $A = U_0 Q_1 U_1 \ldots Q_n U_n$ defined by its special segments $Q_1, \ldots, Q_n$. Note that if the decomposition is non-trivial (that is, if $n \geq 1$), then $|U_0| \geq q$ since the suffix of length $q$ of $U_0$ is $p(Q_1)$. If $\varphi \in \text{Aut}(S_\pm)$, and if $\sigma_{A'} = \varphi(\sigma_A)$ (respectively if $\sigma_{A'} = \sigma_A^{-1}$) then the normal decompositions match (up to changing the orientation).

4.2. Special segments and concatenation

Assume that $A, A'$, and $A''$ are oriented intervals of $D$ such that $A = A'.A''$. Any special segment $Q'$ of $A'$ (or $A''$) is clearly contained in a special segment $Q$ of $A$. If $Q = Q'$, then we say that $Q'$ is regular, and exceptional otherwise. We say that a special segment $Q$ of $A$ is regular if it is contained in $A'$ or $A''$ and is a special segment in it.

Let $A' = U_{0}'Q'_1U'_1 \ldots Q'_{n'}U'_{n'}$ and $A'' = U''_0Q''_1U''_1 \ldots Q''_{n''}U''_{n''}$ be the normal decompositions of $A'$ and $A''$, respectively. If $Q'$ or $Q''$ is an exceptional segment of $A'$ or $A''$, then $Q' = Q_{n'}'$ or $Q'' = Q'_1$ and by maximality of special segments, $|U'_{n'}| = q$ or $|U''_{n''}| = q$, respectively.

Let $Q$ be an exceptional special segment of $A$. We distinguish several cases as follows.

1. If $Q$ contains two special segments of $A'$ or $A''$, then $Q = Q'_{n'}U'_{n'}U''_{n''}Q''_{0}$ and so $r(Q) = r(Q'_{n'}) + r(Q''_{0}) + 2$.

2. Otherwise, if $Q$ contains exactly one special segment of $A$ or $A'$, and if this special segment lies in $A'$, then $Q \supset Q'_{n'}$ and $|Q \cap A''| < q$ since otherwise, $Q \cap A''$ would contain a special segment of $A''$. Thus, $r(Q) < r(Q'_{n'}) + 2$.

3. Symmetrically, if $Q$ contains exactly one special segment, lying in $A''$, then $Q \supset Q''_{0}'$ and $r(Q) < r(Q''_{0}) + 2$.
(4) If \( Q \) contains no special segment of \( A' \) or \( A'' \), then \( |Q \cap A'| < q \) and \( |Q \cap A''| < q \) and so \( r(Q) < 2 \).

There can be at most two exceptional segments in \( A \) because two special segments are at least a distance \( q + 1 \) apart. It may happen that there are two exceptional segments: for \( q = 5 \), consider the word \( p = ababa \), \( A' \) labelled by \( cpaba \) and \( A'' \) by \( babapc \) so that \( A' \) and \( A'' \) contain no special segment; then \( A'.A'' \) is labelled by \( cp^2bapc = cpabpc^2c \) and has two special segments of length 0.

4.3. Proof of Bulitko’s lemma

We are now ready for the proof of Proposition 4.1. Once normal forms have been adapted to twisted equations, the proof is identical to the one given in [16]. We give it here for the reader’s convenience.

Proof of Proposition 4.1. Start with a shortest solution \( \sigma \) of \( \Sigma \) with presumably a high exponent of periodicity, that is, containing high powers of a word of length \( q \). For each special segment \( Q \), recall that \( p(Q) \) denotes the word labelled by \( Q \), and \( r(Q) = |Q|/q \in (1/q)\mathbb{N} \). We also define \( t(Q) \in \mathbb{N} \) and \( u(Q) \in \{0,1/q,\ldots,(q-1)/q\} \) as the integer and fractional part of \( r(Q) \), respectively, so that \( r(Q) = t(Q) + u(Q) \).

The structure of the proof is as follows. We view the \( t(Q) \) as variables satisfying a system of integer equations \( \Lambda \). Each solution of \( \Lambda \) provides a new solution of \( \Sigma \), and the fact that \( \sigma \) is shortest says that this solution of \( \Lambda \) is minimal. The system of equations happens to be equivalent to a system of equations whose number of equations, variables, and coefficients are bounded independently of \( \sigma \), thus giving a bound on the size of a minimal solution of \( \Lambda \), and hence on the exponents of special segments.

Since we assume that our band complex \( \Sigma \) comes from the construction in Subsection 3.4, each component \( D_0 \) of its domain \( D \) contains at most one vertex in its interior, and at most two bases of bands distinct from \( D_0 \). We first forget about rational constraints.

Let \( B \) be the set of bases of bands of \( \Sigma \), with a chosen orientation. For each \( A \in B \), let \( A = U_{A_0}A_1U_{A_1}A_2\ldots U_{A_{nA}}A_{nA} \) be its normal decomposition. Given a set of values \( (\tau_{A,i}) \in \mathbb{N}^{nA} \), we define a new word by changing the integer part by the special segments of \( A \):

\[
w_A(\tau_{A,1},\ldots,\tau_{A,n}) = \sigma_{U_{A_0}}(p(Q_{A,1})^{\tau_{A,1}}u(Q_{A,1})) \cdot \sigma_{U_{A_1}}(p(Q_{A,1})^{\tau_{A,1}}u(Q_{A,1})) \cdots \sigma_{U_{A_{nA}}}^{(3}p(Q_{A,1})^{\tau_{A,nA}}u(Q_{A,nA})^{3} \cdot \sigma_{U_{A_{nA}}}^{(3}p(Q_{A,1})^{\tau_{A,nA}}u(Q_{A,nA})^{3}.
\]

By definition, \( w_A(t(Q_{A,1}),\ldots,t(Q_{A,nA})) = \sigma_A \).

We now define a system of equations \( \Lambda \) with unknowns \( (\tau_{A,k}) \in \bigoplus_{A \in B} \mathbb{N}^{nA} \). By construction, \( \tau_{A,k} = t(Q_{A,k}) \) will be a solution of \( \Lambda \). For each band with bases \( A_0 \) and \( A_1 \) with a twisting morphism \( \varphi \), we have \( \sigma_{A_1} = \varphi(\sigma_{A_0}) \) and \( n_{A_1} = n_{A_0} \), and we add the equations \( \tau_{A_1,k} = \tau_{A_2,k} \) for \( k \in \{1,\ldots,n_A\} \). For each connected component \( A \) of \( D \) containing a vertex distinct from its endpoints, decomposing \( A \) into \( A',A'' \), we add to \( \Lambda \) an equation \( \tau_{A',k} = \tau_{A,k} \) for each regular special segment of \( A' \), and an equation \( \tau_{A''-k} = \tau_{A,nA-k} \) for each regular special segment of \( A'' \).

If \( Q \) is an exceptional segment of \( A \) containing two exceptional segments \( Q' \subset A' \) and \( Q'' \subset A'' \), then \( r(Q) = r(Q') + r(Q'') + 2 \), and so either \( t(Q) = t(Q') + t(Q'') + 2 \) or \( t(Q) = t(Q') + t(Q'') + 3 \); we add to \( \Lambda \) the corresponding equation \( \tau_{A,1} = \tau_{A',nA'} + \tau_{A'',1} + 2 \) or \( \tau_{A,nx} = \tau_{A',nA'} + \tau_{A'',1} + 3 \).

If \( Q \) contains exactly one special segment \( Q' \), lying in \( A' \), then \( r(Q) < r(Q') + 2 \), and so \( t(Q) = t(Q') + \varepsilon \) for some \( \varepsilon \in \{0,1,2\} \); we add to \( \Lambda \) the equation \( \tau_{A,k} = \tau_{A',nA'} + \varepsilon \), where \( k \) is the index corresponding to \( Q \) in \( A \). We proceed symmetrically if \( Q \) contains exactly one special segment in \( A'' \).
If $Q$ contains no special segment of $A'$ or $A''$, then $r(Q) < 2$ and so $t(Q) = \varepsilon$ for some $\varepsilon \in \{0, 1\}$, and we add to $\Lambda$ the equation $\tau_{A,k} = \varepsilon$, where $k$ is the index corresponding to $Q$ in $A$.

This defines $\Lambda$. This system has a solution, namely $\tau_{A,k} = t(Q_{A,k})$. Moreover, if $(\tau_{A,k})$ is any solution of $\Lambda$, then $w_A(\tau_{A,1}, \ldots, \tau_{A,n_A})$ defines a solution of $\Sigma$. In particular, if $t(Q_{A,k})$ is not a minimal solution of $\Lambda$, meaning that there exists a solution of $\Lambda$ with $\tau_{A,k} \leq t(Q_{A,k})$ for all $A, k$, and some inequality is strict, then $\sigma$ is not the shortest solution.

The number of equations and of unknowns of $\Lambda$ is not bounded a priori. However, there are at most $2^{2\#\pi_0(D)}$ equations that are not equalities of the type $\tau_{A,k} = \tau_{A',k'}$. By substitution, one can get rid of these equalities, and we are left with a system $\Lambda'$ of at most $2^{2\#\pi_0(D)}$ equations, with coefficients in $\{0, \ldots, 3\}$, and with $l \leq 6\#\pi_0(D)$ unknowns. Note that the bounds are independent of $\sigma$ and of $q$.

Knowing only the band complex $\Sigma$ (and not the solution $\sigma$), one can list all possible systems $\Lambda'$, decide which of them have a solution in $\mathbb{N}^l$, list all minimal solutions of each of them (for instance, see [43, Theorem 17.1]), and compute the maximal value $M$ of any $\tau_{A,k}$ in any such minimal solution. The value $M$ is then an upper bound on the exponent of periodicity of a shortest solution $\sigma$ of $\Sigma$.

In the presence of rational constraints, represented by $\rho : S_+^1 \to \mathcal{M}$, we first compute $\alpha, \beta \in \mathbb{N} \setminus \{0\}$ such that, for all $m_1, m_2, m_3 \in \mathcal{M}$ and all $\tau \in \mathbb{N}$, we have $m_1 m_2 = m_3 m_3^{\alpha + \beta}$ (it is an easy standard fact that such $\alpha, \beta$ exist and are computable). In particular, $u_1 u_2^2 u_3 \in \mathcal{R}_e$ if and only if $u_1 u_2^{\alpha + \beta} u_3 \in \mathcal{R}_e$. Then we modify the system $\Lambda$ as follows: for each $A \in B$ and $k \in \{1, \ldots, n_A\}$ such that $t(Q_{A,k}) \leq \beta$, we replace the variable $\tau_{A,k}$ by the constant $t(Q_{A,k})$; for each $t(Q_{A,k}) > \beta$, we write by Euclidean division $t(Q_{A,k}) = \beta + \alpha t_{A,k} + \varepsilon_{A,k}$ for some $\varepsilon_{A,k} < \alpha$ and $t_{A,k} \in \mathbb{N}$; then we replace the unknown $\tau_{A,k}$ by $\tau_{A,k}'$ using the identity $\tau_{A,k} = \alpha \tau_{A,k}' + \beta + \varepsilon_{A,k}$.

We get a linear system of integer equations with the unknowns $\tau_{A,k}'$, with at most $2^{2\#\pi_0(D)}$ equations, with coefficients bounded in terms of $\alpha$ and $\beta$, and with $l \leq 6\#\pi_0(D)$ unknowns. As above, one can compute a bound $M'$ on all minimal solutions of all such systems which have a solution. The same argument as above shows that $\alpha M' + \beta$ is an upper bound for the exponent of periodicity of the shortest solution of $\Sigma$, including rational constraints.

**Remark.** Using estimates on the size of minimal solutions of linear systems of Diophantine equations, and being more careful in the analysis, one can get an explicit bound of $M = 2^{O(\#\pi_0(D)) + n \log n}$, where $n$ is the size of an automaton accepting the languages of $\mathcal{R}$; see [16].

## 5. Prelaminations

### 5.1. Definition

**Definition 5.1.** A *prelamination* $\mathcal{L}$ on a band complex $\Sigma$ consists of the following.

(i) A finite set $V(\mathcal{L}) \subset D$ containing all vertices of $\Sigma$. The points of $V(\mathcal{L})$ are called $\mathcal{L}$-subdivision points.

(ii) For each band $B = [a, b] \times [0, 1]$, a set $L_B(\mathcal{L}) \subset [a, b] \times [0, 1]$ of the form $\{c_1, \ldots, c_r\} \times [0, 1]$. Each segment $\{c_i\} \times [0, 1]$ is called a *leaf segment*. The endpoints of each leaf segment are asked to lie in $V(\mathcal{L})$: $f_{B,\mathcal{L}}(c_i, \varepsilon) \in V(\mathcal{L})$ for $\varepsilon = 0, 1$. We also require that both segments $\{a\} \times [0, 1]$ and $\{b\} \times [0, 1]$ lie in $L_B(\mathcal{L})$.

See Figure 1 for an illustration.
Remark 5.2. In this definition, the precise value of the attaching maps $f_{B,c}$ is not important, except at endpoints of leaf segments. Moreover, the precise position of the $L$-subdivision points in $D$ is unimportant, only the induced ordering matters. Thus, a prelamination of a band complex is really determined by the ordering of the elements of $V(L)$ in each component of $D$, and for each band, by a bijection preserving or reversing the ordering, between a subset of $V(L) \cap J_0$ and a subset of $V(L) \cap J_1$.

We also view leaf segments of $L$ as subsegments of the topological realization $|\Sigma|$ of $\Sigma$. A leaf of $L$ is a connected component of the union of the leaf segments in $|\Sigma|$. A leaf path is a concatenation of oriented leaf segments (which defines a path contained in a leaf).

A leaf segment $\{c\} \times [0,1] \subset [a,b] \times [0,1]$ is singular if $c \in \{a,b\}$. A leaf path is regular if it consists of non-singular leaf segments. A leaf is singular if it contains a singular leaf segment, that is, if it contains a vertex of $\Sigma$.

An elementary segment of $L$ is a segment of $D$ joining two adjacent $L$-subdivision points. The set of oriented elementary segments of $L$ is denoted by $E(L)$. We say that $J \subset D$ is adapted to $L$ if it is a union of elementary segments of $L$.

A point $x \in V(L)$ is not an orphan if, for every base $J_x$ of any band $B$, with $x \in J_x$, there is a leaf segment of $B$ having $x$ as an endpoint: $x \in f_{B,c}(L_B(L))$. On Figure 1, orphans are represented as thick dots. The prelamination is complete if no $L$-subdivision point is an orphan (in this case, the prelamination could be called a genuine lamination).

If $B = [a,b] \times [0,1]$ is a band with the attaching maps $f_{B,0}$ and $f_{B,1}$, then we define the opposite $B^{-1}$ of $B$ as the band $[a,b] \times [0,1]$ with the attaching maps $f_{B^{-1},i} = f_{B,1-i}$ for $i = 0, 1$, with the twisting morphism $\varphi_{B^{-1}} = \varphi_{B}^{-1}$. An oriented band is a band of $\Sigma$ or its opposite. The leaf segments of $B$ define the leaf segments of $B^{-1}$. We orient each leaf segment $\{x\} \times [0,1]$ of the oriented band $B$ from $(x,0)$ to $(x,1)$ so that the leaf segments of $B$ and $B^{-1}$ have the opposite orientation. If $B$ is an oriented band with the bases $J_0$ and $J_1$, then we denote by $\text{dom} B = J_0$ its initial base, and we call it its domain.

A $\Sigma$-word is a word on the alphabet of oriented bands of $\Sigma$. The twisting morphism of the $\Sigma$-word $w = B_1 \ldots B_n$ is $\varphi_w = \varphi_{B_1} \circ \ldots \circ \varphi_{B_n}$. To each leaf path naturally corresponds a $\Sigma$-word corresponding to the oriented bands crossed by this path. We write $[x,w]$ for the leaf path starting at the point $x$ and whose $\Sigma$-word is $w$ (this leaf path is unique, but may fail to exist for some $x$ and $w$). The holonomy of $w$ is the map $h_w$ defined on a subset of $V(L)$ and mapping $x$ to the terminal endpoint of $[x,w]$ when $[x,w]$ exists. We write paths from right to left so that $h_{w'} h_w = h_{ww'}$ (thus, the path $ww'$ starts by $w'$ and ends by $w$). By extension, if $[a,b] \subset D$ is such that $h_w$ maps $a$ to $a'$ and $b$ to $b'$, we say that $h_w$ maps $[a,b]$ to $[a',b']$. When such $w$ exists, we say that $[a,b]$ and $[a',b']$ are equivalent under the holonomy of $L$.

5.2. Möbius strips

A Möbius strip $S$ in a prelaminated band complex $(\Sigma,L)$ consists of a reduced band word $w$ and an interval $I$ adapted to $L$ such that $h_w$ maps $I$ to itself reversing the orientation, and $h_{w_1}(I) \neq h_{w_2}(I)$ for any two distinct non-empty suffixes $w_1$ and $w_2$ of $w$. Note that $(\Sigma,L)$ has only finitely many Möbius strips. A core leaf of $S$ is a leaf path $[x,w]$ with $h_w(x) = x$.

Remark 5.3. Note that if the twisting morphism $\varphi_w$ of $w$ is trivial, or more specifically, if we are solving untwisted equations in a free group, then the presence of a Möbius strip prevents the existence of a reduced solution of $\Sigma$. Thus, in the context of equations in free groups, prelaminations containing Möbius strips can be discarded.
If \( \mathcal{L} \) is complete and all its Möbius strips have a core leaf, then we say that \( \mathcal{L} \) is Möbius-complete. A leaf is pseudo-singular if it is singular or if it contains the core leaf of a Möbius strip. Most preliminatings we use will consist of pseudo-singular leaves.

If \( S \) has no core leaf, then adding a core leaf to \( S \) means extending \( \mathcal{L} \) by adding at most \(|w|\) new leaf segments \( l_1,\ldots,l_n \) so that \( S \) has a core leaf \( l \) in the obtained prelamination, with \( l \supset l_1 \cup \ldots \cup l_n \).

### 5.3. Rational constraints on the prelamination

Consider a band complex \( \Sigma \) with rational constraints in standard form given by \( \rho : S_\pm^+ \to \mathcal{M} \) and \( m \in \mathcal{M}^{E(\Sigma)} \). Let \( \mathcal{L} \) be a prelamination on \( \Sigma \). Recall that \( E(\mathcal{L}) \) is the set of oriented elementary segments of \( \mathcal{L} \), that is, of subsegments of \( D \) joining adjacent \( \mathcal{L} \)-subdivision points.

A set of rational constraints on the prelamination \( \mathcal{L} \) is a tuple \( m'_e \in \mathcal{M}^{E(\mathcal{L})} \) such that \( m'_e = m_e \) and which refines \( m \) in the following sense. For a segment \( J \) written as a concatenation of elementary segments of \( \mathcal{L} \), \( J = e_1 \ldots e_n \), define \( m'_J = m'_{e_1} \ldots m'_{e_n} \in \mathcal{M} \). Then \( m' \) refines \( m \) if, for all \( J \)-oriented elementary segments of \( \Sigma \), we have \( m_J = m'_J \). This compatibility allows us to blur the distinction between \( m \) and \( m' \), and we shall use the same notation \( m \) for both.

### 5.4. Prelaminations and rational constraints induced by a solution

Consider a band complex \( \Sigma \) with rational constraints in standard form given by \( \rho : S_\pm^+ \to \mathcal{M} \) and \( m = (m_e)_{e \in E(\Sigma)} \), a set of rational constraints in standard form. Let \( \sigma \) be a solution of \( (\Sigma, m) \). The labelling \( \sigma \) allows us to subdivide \( D \) into subsegments, each of which is labelled by a letter in \( \mathcal{M} \). This defines a complete prelamination \( \mathcal{L}' \) whose subdivision points are the set of vertices of the subdivided \( D \); since \( \sigma \) is a solution, each band \( B \) defines a pairing between all the vertices contained in its two bases. This pairing can be realized geometrically by a finite set of leaf segments of the form \( \{c\} \times [0,1] \subset [a,b] \times [0,1] \) by changing the two attaching maps of \( B \) relatively to the endpoints of the bases.

Let \( \mathcal{L}_\infty(\sigma) \subset \mathcal{L}' \) be the complete prelamination consisting of the pseudo-singular leaves of \( \mathcal{L}' \). We call \( \mathcal{L}_\infty(\sigma) \) the complete prelamination induced by \( \sigma \). This is a Möbius-complete prelamination because the group \( \Phi \) of twisting morphisms has no inversion.

For any segment \( J \) adapted to some prelamination \( \mathcal{L} \subset \mathcal{L}_\infty(\sigma) \), one can define \( m_J = \rho(\sigma_J) \in \mathcal{M} \), which defines a rational constraint on \( \mathcal{L} \). This leads to the following definition.

**Definition 5.4.** A prelamination \( \mathcal{L} \) with rational constraints \( \mathcal{M} \) is induced by \( \sigma \) if \( \mathcal{L} \subset \mathcal{L}_\infty(\sigma) \), all leaves of \( \mathcal{L} \) are pseudo-singular, and if \( m_e = \rho(\sigma_e) \) for all \( e \in E(\mathcal{L}) \).

**Remark 5.5.** The rational constraints induced by a solution are invariant under the holonomy in the following sense: consider a \( \Sigma \)-word \( w \) with a twisting morphism \( \varphi_w \) whose holonomy maps some \( e \in E(\mathcal{L}) \) to \( e' \in E(\mathcal{L}) \), then \( m_{e'} = \varphi_w(m_e) \).

Given a prelamination \( \mathcal{L} \) of \( \Sigma \), we define the solutions of \( (\Sigma, \mathcal{L}) \). A labelling \( \sigma \) of \( (\Sigma, \mathcal{L}) \) is a labelling of each \( e \in E(\mathcal{L}) \) by a word \( \sigma_e \in S_\pm^+ \) so that \( \sigma_e = \sigma_{e'} \). As in Subsection 3.2, this allows us to define \( \sigma_J \in S_\pm^+ \) for all oriented segments \( J \) adapted to \( \mathcal{L} \).

**Definition 5.6.** A solution of \( (\Sigma, \mathcal{L}) \) is a labelling \( \sigma \) of \( (\Sigma, \mathcal{L}) \) such that, for all \( \Sigma \)-word \( w \) and all oriented segments \( J \) and \( J' \) adapted to \( \mathcal{L} \), if the holonomy of \( w \) maps \( J \) to \( J' \), then \( \sigma_{J'} = \varphi_w(\sigma_J) \).

If there are rational constraints on \( \mathcal{L} \), then we additionally require that \( \rho(\sigma_e) = m_e \) for all \( e \in E(\mathcal{L}) \).
The following observation is obvious.

**Lemma 5.7.** Let $\sigma$ be a solution of $\Sigma$ with rational constraints $m$. Let $(\mathcal{L}, m')$ be a prelamination with rational constraints induced by $\sigma$.

Then $\sigma$ is a solution of $(\Sigma, \mathcal{L})$ with rational constraints $m'$.

**Definition 5.8.** Consider $\mathcal{L} \subset \mathcal{L}'$ to be some prelaminations on $\Sigma$ with the rational constraints $m$ and $m'$. We say that $\mathcal{L}'$ extends $\mathcal{L}$ if for all $e \in E(\mathcal{L})$, $m_e = m'_e$. When this happens, we denote $\mathcal{L} \prec \mathcal{L}'$.

When $\mathcal{L}'$ extends $\mathcal{L}$, the set of solutions of $(\Sigma, \mathcal{L}')$ with constraints $m'$ is a subset of the set of solutions of $(\Sigma, \mathcal{L})$ with constraints $m$.

6. **Algorithms on prelaminations**

We now present two algorithms that will be part of the main algorithm solving equations. One of them enumerates prelaminations of a given band complex. The other is given a prelamination, and checks several properties, for instance, to prove that it cannot be induced by a shortest solution.

6.1. **A first algorithm: the prelamination generator**

We first consider a prelamination generator: this straightforward algorithm takes as input a band complex with the rational constraints $\Sigma$ and enumerates a set of prelaminations, such that any prelamination with rational constraints induced by a solution of $\Sigma$ has an extension that is enumerated.

The set of produced prelaminations is organized in a locally finite rooted tree. The prelamination at depth 0 is the root prelamination whose leaf segments are the boundary segments of the bands. By definition, the root prelamination is contained in any prelamination. We define the rational constraints on the root prelamination as the rational constraints on $\Sigma$.

The children of a given prelamination $\mathcal{L}$ are the extensions of $\mathcal{L}$ produced by the following extension process, consisting of the three following steps.

**Step (1):** extend leaves of the prelamination. Let $O$ be the set of orphans of $\mathcal{L}$ (as defined in Subsection 5.1). Look at all the possible ways to add leaf segments to $\mathcal{L}$, so that every leaf segment added has an endpoint in $O$, and points in $O$ are no longer orphans in the new prelamination.

**Step (2):** add cores of Möbius strips. Consider all the (finitely many) Möbius strips of $(\Sigma, \mathcal{L})$ having no core leaf. Then look at all possible ways to add a core leaf to each of them (see Subsection 5.2 for definitions).

**Step (3):** extend rational constraints. If $\mathcal{L}'$ is a prelamination constructed above, and if $\mathcal{L}' = \mathcal{L}$, then discard $\mathcal{L}'$. Given $\mathcal{L}' \supseteq \mathcal{L}$ constructed above, and a set of rational constraints $m \in \mathcal{M}^{E(\mathcal{L})}$ on $\mathcal{L}$, consider all possible rational constraints $m' \in \mathcal{M}^{E(\mathcal{L}')}$, such that $(\mathcal{L}', m')$ extends $(\mathcal{L}, m)$ as in Definition 5.8, meaning that, for all oriented elementary segments $e \in E(\mathcal{L})$ written as a concatenation $e = e'_1 \ldots e'_n$ of elements of $E(\mathcal{L}')$, we have $m_e = m'_1 \ldots m'_{e_{n+1}}$. Note that it might happen that, for some $\mathcal{L}'$, there exists no set of rational constraints $m'$.

**Remark.** The second step would actually be unnecessary if there was no twisting as no reduced solution of $\Sigma$ can induce a prelamination containing a Möbius strip whose twisting morphism is trivial.
Step (2) should add a core leaf to every Möbius strip of \( \mathcal{L} \), but it is allowed to create new Möbius strips having no core leaf.

If all rational constraints are given by one-letter constants, then the third step consists in ensuring that one does not subdivide the elementary segments corresponding to constants.

If \( \mathcal{L} \) is Möbius-complete (that is, has no orphan and all its Möbius strips have a core leaf), then the only extension of \( \mathcal{L} \) produced by the extension process is \( \mathcal{L} \), and is discarded. In particular, \( \mathcal{L} \) is a leaf of the rooted tree of prelaminations. If \( \mathcal{L} \) is not Möbius-complete, then all the prelaminations produced properly contain \( \mathcal{L} \). However, it may happen that no extension of \( \mathcal{L} \) is produced if, for all \( \mathcal{L}' \supseteq \mathcal{L} \) obtained after step (2), no extension of the rational constraints to \( \mathcal{L}' \) exists. Of course, in this case, \( \mathcal{L} \) cannot be induced by a solution of \( \Sigma \).

Recall that all band complexes and prelaminations come with a set of rational constraints which we do not mention explicitly.

**Lemma 6.1.** Consider \( \sigma \) a solution of \( \Sigma \). Assume that \( \sigma \) induces some prelamination \( \mathcal{L} \) produced by the prelamination generator.

Then either \( \mathcal{L} = \mathcal{L}_\infty(\sigma) \), or there exists some extension \( \mathcal{L}' \) of \( \mathcal{L} \) produced by the prelamination generator which is induced by \( \sigma \).

**Proof.** Assume that \( \mathcal{L} \subsetneq \mathcal{L}_\infty(\sigma) \). If \( \mathcal{L} \) is not complete, then step (1) will clearly produce an extension of \( \mathcal{L}_1 \supseteq \mathcal{L} \) contained in \( \mathcal{L}_\infty(\sigma) \). Otherwise, step (1) will produce only \( \mathcal{L}_1 = \mathcal{L} \). Since all Möbius strips of \( \mathcal{L}_\infty(\sigma) \) contain a core leaf, if \( \mathcal{L}_1 \) contains a Möbius strip without a core leaf, then step (2) produces some \( \mathcal{L}_2 \supseteq \mathcal{L}_1 \) contained in \( \mathcal{L}_\infty(\sigma) \). The rational constraints induced by \( \sigma \) on \( \mathcal{L}_2 \) will be produced in step (3). The lemma follows.

**Corollary 6.2.** For any solution \( \sigma \) of \( \Sigma \), we find that \( \mathcal{L}_\infty(\sigma) \) will be produced by the prelamination generator.

Consider some prelamination \( \mathcal{L} \) that is not Möbius-complete, and such that none of its extensions constructed by the prelamination generator can be induced by a shortest solution of \( \Sigma \). Then \( \mathcal{L} \) cannot be induced by a shortest solution of \( \Sigma \).

**Proof.** The first statement clearly follows from the lemma since \( \mathcal{L}_\infty(\sigma) \) cannot contain an infinite chain of prelaminations strictly contained in each other. The second statement is clear.

6.2. **A second algorithm: the prelamination analyser**

We are going to describe a machine that analyses a given prelamination. We call it the analyser.

This algorithm takes as input a band complex \( \Sigma \) together with a prelamination \( \mathcal{L} \) (with rational constraints). It tries to reject prelaminations for which it can prove that they cannot be induced by a shortest solution.

More precisely, if the input prelamination \( \mathcal{L} \) is Möbius-complete, then the analyser easily determines whether it is induced by some solution or not. The analyser then stops and outputs ‘Solution found’ together with a solution, or ‘Reject’ accordingly.

If the input prelamination \( \mathcal{L} \) is not Möbius-complete, then it looks for some certificate ensuring that no shortest solution can induce \( \mathcal{L} \). If it can find one, then it says ‘Reject’ and stops. Otherwise, it may fail to stop or say ‘I don’t know’.

Actually, we could make sure that it actually always terminates because given \( \mathcal{L} \), there are only finitely many certificates to try; but we do not need this fact.
The overall structure of the prelamination analyser is as follows.

**Algorithm 6.3** (Prelamination analyser).

*Input*: a prelamination with rational constraints $\mathcal{L}$ on a band complex $\Sigma$.

*Output*: ‘Solution found’, ‘Reject’ or ‘I don’t know’.

- If $\mathcal{L}$ is a Möbius-complete prelamination, then the algorithm stops and its output is either
  - ‘Solution Found’: the analyser can produce a solution inducing $\mathcal{L}$;
  - ‘Reject’: there exists no solution inducing $\mathcal{L}$, and the analyser can produce a certificate proving it.

- If $\mathcal{L}$ is not Möbius-complete, then the algorithm looks for certificates proving that $\mathcal{L}$ cannot be induced by a shortest solution of $\Sigma$. The algorithm tries four kinds of reasons to reject $\mathcal{L}$:
  - rejection for incompatibility of rational constraints (see Paragraph 6.2.1);
  - rejection for non-existence of invariant measure (see Paragraph 6.2.2);
  - rejection for too large an exponent of periodicity (see Paragraph 6.2.3);
  - and rejection by a shortening sequence of moves (see Paragraph 6.2.4).

This procedure may stop or not, and its output can be
  - ‘Reject’: the algorithm produces a certificate proving that $\mathcal{L}$ cannot be induced by a shortest solution of $\Sigma$;
  - ‘I don’t know’: failure to find such a certificate.

We now describe more precisely the work of the prelamination analyser, describing the four types of rejection.

**6.2.1. Incompatibility of rational constraints.** The set of rational constraints $m$ on a prelamination induced by a solution of $\Sigma$ is invariant under the holonomy (see Remark 5.5): for all bands $B$ whose holonomy maps some $e \in E(\mathcal{L})$ to $e' \in E(\mathcal{L})$, we have $m_{e'} = \varphi_B(m_e)$. Thus, one can obviously reject rational constraints where this does not hold. For example, in this way one rejects prelaminations such that the constants do not match under the holonomy.

There is another source of incompatibility coming from the twisting morphisms. Assume that $w$ is a $\Sigma$-word whose holonomy fixes some segment $J$ adapted to $\mathcal{L}$, and whose twisting morphism $\varphi_w$ is non-trivial. Then, for any solution $\sigma$ of $\Sigma$ inducing $\mathcal{L}$, we find that $\sigma_J$ lies in $\text{Fix} \varphi_w$. By definition of standard form of rational constraints (Subsection 3.5), $\text{Fix} \varphi_w$ is represented by $\rho : S^1_+ \to M$, and so $\sigma_J \in \text{Fix} \varphi_w$ if and only if $\rho(\sigma_J) \in F$ for the subset $\rho(\text{Fix} \varphi_w) \subset M$, which has already been computed. Thus, one can reject $\mathcal{L}$ if $m_J \notin \rho(\text{Fix} \varphi_w)$ for some $w$ whose holonomy fixes $J$. Even though the set of $\Sigma$-words whose holonomy fixes $J$ may be infinite, this fact can be algorithmically checked. Indeed, the fact that $m_J \notin \rho(\text{Fix} \varphi_w)$ needs to be checked only for a generating set of the group of $\Sigma$-words $w$ fixing $J$, and such a generating set is easily computed.

We say that $\mathcal{L}$ shows an *incompatibility of rational constraints* when one can reject $\mathcal{L}$ for one of the two possible reasons above. In the prelamination analyser, rejection for incompatibility of rational constraints simply consists in checking this algorithmically.

When $\mathcal{L}$ is Möbius-complete, this is the only obstruction to the existence of a solution of $\Sigma$ inducing $\mathcal{L}$ and $m$.

**Lemma 6.4.** Let $\mathcal{L}$ be a Möbius-complete prelamination of $\Sigma$. Then $\Sigma$ has a solution inducing $\mathcal{L}$ with a set of rational constraints $m$ if and only if it shows no incompatibility of rational constraints.

This explains how the prelamination analyser determines the existence of a solution when its input is a Möbius-complete prelamination: it just checks whether $\mathcal{L}$ shows an incompatibility of
rational constraints. The argument below also shows how to produce a solution in the absence
of incompatibility of rational constraints.

Proof of Lemma 6.4. Since \( L \) is Möbius-complete, the set of oriented elementary segments
\( E(L) \) is partitioned into orbits under the holonomy of \( L \), and no \( e \in E(L) \) is the orbit of \( \sigma \) (that is, \( e \) with the orientation reversed). For each such orbit, choose one representative
\( e \in E(L) \), and choose some word \( \sigma_e \in \rho^{-1}(m_e) \) (\( \rho \) is onto). For each \( e' \) in the orbit of \( e \), choose
a \( \Sigma \)-word \( w \) such that \( h_w(e) = e' \) and define \( \sigma_{e'} = \varphi_w(\sigma_e) \). Since \( L \) shows no incompatibility of rational constraints of the first kind, one has \( \rho(\sigma_{e'}) = \varphi_w(m_e) = m_{e'} \). If \( w' \) is another word with
\( h_{w'}(e) = e' \), then \( \sigma_{e'} \in \text{Fix} \varphi_{w'-1} \) because \( L \) shows no incompatibility of rational constraints
of the second kind, and so \( \sigma_{e'} \) does not depend on the choice of \( w' \). Since \( e \) and \( \sigma \) are not in the
same orbit, we can extend these choices by defining \( \sigma_{e'} = \sigma_{e'} \) for all \( e' \) in the orbit of \( e \). Doing this for every orbit of elementary segments, we get a solution of \((\Sigma, L)\) respecting the
rational constraints.

6.2.2. Rejection for non-existence of invariant measure. If a prelamination \( L \) is induced
by a solution \( \sigma \), then the labelling defines a positive length \( l_e \) on each elementary segment \( e \) of
\( L \), and more generally, to each segment adapted to \( L \). This can be viewed as a combinatorial
measure invariant under the holonomy along the leaves of \( L \) in the following sense: if \( I \) and
\( J \) are segments adapted to \( L \) and such that the holonomy \( h_B \) maps \( I \) to \( J \) for some band \( B \), then \( l_I = l_J \).

A prelamination may fail to have such an invariant transverse measure, for example if the
holonomy along the leaves maps a segment to a proper subset of itself. The existence of a
combinatorial measure \( (l_e) \) is equivalent to the existence of a positive solution to some system
of linear equations with integer coefficients, which can be checked algorithmically. Thus, the
prelamination analyser can check whether \( L \) admits an invariant combinatorial measure, and
if there is no such measure, it rejects \( L \).

6.2.3. Rejection for too large an exponent of periodicity. In some cases, one can read in
a prelamination \( L \) the fact that any solution inducing \( L \) must have a very large exponent of
periodicity.

Definition 6.5. We say that \( \text{exponent}(L) \geq N \) if there exists a segment \( J \) that is a
concatenation of \( N \) intervals \( I_1 \cdots I_N \subset D \) adapted to \( L \), and some \( \Sigma \)-words \( w_i \), whose holonomy
maps \( I_i \) to \( I_{i+1} \) preserving the orientation, and whose twisting automorphisms \( \varphi_{w_i} \) are trivial.

Clearly, if \( \text{exponent}(L) \geq N \), then the exponent of periodicity of any solution inducing \( L \) is
at least \( N \). By Bulitko’s lemma, one can compute a bound \( B \) such that any shortest solution of
\( \Sigma \) has an exponent of periodicity bounded by \( B \).

In the prelamination analyser, rejection for too large an exponent of periodicity consists in
computing \( B \), and checking whether \( \text{exponent}(L) > B \). This is easily done algorithmically.

6.2.4. Rejection by a shortening sequence of moves. This last type of rejection is more
complicated. In Subsection 7.2, we introduce a finite set of moves that can be performed on a
band complex with a prelamination. There will be several types of moves.

Definition 6.6 (Inert move). We say that a prelaminated band complex \((\Sigma', L')\) is
obtained by an inert move from \((\Sigma, L)\) if we have the following conditions.

(i) For any solution \( \sigma \) of \( \Sigma \) inducing \( L \), there is a solution \( \sigma' \) of \( \Sigma' \) inducing \( L' \) such that
\( |\sigma'| = |\sigma| \).
(ii) For any solution $\sigma'$ of $\Sigma'$ (which need not induce $L'$), there is a solution $\sigma$ of $\Sigma$ (which need not induce $L$) and such that $|\sigma| = |\sigma'|$.

An obvious property of an inert move is that if $L$ is induced by a shortest solution of $\Sigma$, then $L'$ is induced by a shortest solution of $\Sigma'$. Examples of inert moves include band subdivision, domain cut, and band removal moves (see Subsection 7.2).

To illustrate the two other types of moves, we start with an example: the pruning move. The input is a band complex $\Sigma$ with domain $D$ together with a prelamination $L$, and the output is a new prelaminated band complex $(\Sigma', L')$. We assume that there is a segment $[a, b] \subset D$ where $a$ and $b$ are vertices of $\Sigma$, such that $[a, b]$ is contained in a base $J_0$ of some band $B$, and such that no other base of $\Sigma$ intersects the interior of $[a, b]$ (see Figure 4). Assume that $a$ and $b$ are not orphans in $L$. The rational constraints on $(\Sigma, L)$ are assumed to be invariant under the holonomy. We view the band $B$ as $J_0 \times [0, 1]$, the attaching map $f_{B,0}$ being the identity. Let $D' = D \setminus (a, b)$ and let $\Sigma'$ be the band complex with domain $D'$ obtained by replacing the band $B = J_0 \times [0, 1]$ by the two bands defined by the connected components of $(J_0 \setminus (a, b)) \times [0, 1]$. If $a$ or $b$ is an endpoint of $D$, then $(a, b)$ has to be interpreted as the interior of $[a, b]$ in $D$, in which case $(J_0 \setminus (a, b)) \times [0, 1]$ consists of only one band or no band at all. The attaching map of the new bands are well defined because $a$ and $b$ are not orphans. The new prelamination is the one naturally obtained by restriction. The new set of rational constraints on $L'$ is obtained by restriction. The set rational constraints $m'$ on $\Sigma'$ is the one induced by $L'$: any elementary segment $e'$ of $\Sigma'$ is a concatenation of elementary segments $e'_1, \ldots, e'_n$ of $L'$, which allows us to define $m'_e = m'_{e_1} \ldots m'_{e_n}$.

It is clear that any solution $\sigma$ of $\Sigma$ inducing $L$ (together with its set of rational constraints, which we do not mention explicitly) defines a solution $\sigma'$ of $\Sigma'$ inducing $L'$ (with rational constraints). Moreover, the length of $\sigma'$ is the length of the restriction of $\sigma$ to $D'$.

Conversely, if $\sigma'$ is any solution of $\Sigma'$, without assuming that $\sigma'$ induces $L'$, then it naturally extends to a solution $\sigma$ of $\Sigma$, and $|\sigma| \leq 2|\sigma'|$. We say that the move from $(\Sigma, L)$ to $(\Sigma', L')$ is a restriction move and that the move from $\Sigma'$ to $\Sigma$ is an extension move.

Remark 6.7. Note that restriction moves are about prelaminated band complexes while extension moves are about band complexes without prelamination.

We consider long sequences of restriction and extension moves which may leave a large part of the band complex essentially untouched. To keep track of this, we partition the domain $D$ of the involved band complexes into $D = D_I \cup D_A$, where $D_I$ and $D_A$ are a union of connected components of $D$, where $D_I$ is the inert part (which is left untouched by the move), and $D_A$
the active part. If \( D' \) is the domain of \( \Sigma' \), then we denote by \( D'_I \sqcup D'_A \) the corresponding decomposition of \( D \).

**Definition 6.8 (Restriction move).** We say that \( (\Sigma', \mathcal{L}') \) is obtained by a restriction move from \( (\Sigma, \mathcal{L}) \) if there is an injective map \( \iota : D' \hookrightarrow D \) with \( D_I = \iota(D'_I) \) and \( \iota(D'_A) \subset D_A \) adapted to \( \mathcal{L} \), such that, for any solution \( \sigma \) of \( \Sigma \) inducing \( \mathcal{L} \), there is a solution \( \sigma' \) of \( \Sigma' \) inducing \( \mathcal{L}' \) such that we have the following:

(i) \( |\sigma'_I| = |\sigma_I| \);
(ii) \( |\sigma'_{D'_A}| \leq |\sigma_{I'D_A}| \).

**Remark 6.9.** We call \( \iota \) the restriction map. For the pruning move, for instance, \( \iota \) is just the inclusion.

**Definition 6.10 (Extension move).** We say that \( \Sigma \) is obtained from \( \Sigma' \) by an extension move with Lipschitz factor \( \lambda \geq 1 \) if, for any solution \( \sigma' \) of \( \Sigma' \), there is a solution \( \sigma \) of \( \Sigma \) such that we have the following:

(i) \( |\sigma_I| = |\sigma'_I| \);
(ii) \( |\sigma_{D_A}| \leq \lambda |\sigma_{I'D_A}| \).

For instance, one can take \( \lambda = 2 \) for the pruning move, under the requirement that \( D_A \) contains both bases of the band pruned.

To be able to ensure that a prelamination \( \mathcal{L} \) cannot be induced by a shortest solution, we need a way of certifying that some subset \( \iota(D'_A) \subset D_A \) is short relative to \( D_A \), that is, for any solution \( \sigma \) inducing \( \mathcal{L} \), we find that \( |\sigma_{D'_A}| \) is at most \( \varepsilon |\sigma_{I'D_A}| \) for some small \( \varepsilon > 0 \). Relative shortness will be guaranteed by some repetition as in the lemma below.

**Definition 6.11 (Certificate of shortness of \( J \) relative to \( D_A \)).** Let \( (\Sigma, \mathcal{L}) \) be a prelaminated band complex and let \( J \subset D_A \) be a subset adapted to \( \mathcal{L} \). A certificate of \( \varepsilon \)-shortness of \( J \) relative to \( D_A \) is a family of adapted subsets \( J_1, \ldots, J_N \subset D_A \) with disjoint interiors for some \( N \geq 1/\varepsilon \), and which are all equivalent to \( J \) under the holonomy of \( \mathcal{L} \).

**Lemma 6.12.** If there is a certificate of \( \varepsilon \)-shortness of \( J \) relative to \( D_A \), then, for any solution \( \sigma \) of \( \Sigma \) inducing \( \mathcal{L} \), we have \( |\sigma_J| \leq \varepsilon |\sigma_{D_A}| \).

**Proof.** This is essentially obvious: for any solution \( \sigma \) inducing \( \mathcal{L} \), the labellings of \( J_1, \ldots, J_N \) will have the same length as \( J \). Since the \( J_i \) do not overlap and are contained in \( D_A \), we get \( N.|\sigma_I| \leq |\sigma_{D_A}| \) and the lemma follows.

The prelamination analyser will try to apply a sequence of elementary moves (as defined in Subsection 7.2) to the prelaminated band complex. It is clear that a succession of inert moves defines an inert move. Moreover, a succession of restriction moves having the same inert part also defines a restriction move, whose restriction map \( \iota \) is the composition of the restriction maps of the elementary moves. Similarly, a succession of elementary extension moves having the same inert part defines an extension move whose Lipschitz factor is the product of the Lipschitz factors of the individual moves.

**Definition 6.13 (Shortening sequence of moves).** A shortening sequence of moves for the prelamination \( \mathcal{L}_0 \) of \( \Sigma_0 \) consists of the following data:

1. a sequence of inert moves transforming \( (\Sigma_0, \mathcal{L}_0) \) to \( (\Sigma, \mathcal{L}) \);
(2) a sequence of elementary restriction moves, whose concatenation defines a restriction move \((\Sigma, \mathcal{L}) \rightarrow (\Sigma', \mathcal{L}')\), with a corresponding map \(t : D'_A \rightarrow D_A\);
(3) a sequence of elementary extension moves transforming \(\Sigma'\) back to \(\Sigma\);
(4) a certificate of \(\varepsilon\)-shortness of \(\varepsilon(D_A')\) relative to \(D_A\) with \(\varepsilon < 1/\lambda\), where \(\lambda\) is the product of the Lipschitz factors of the extension moves used above.

**Lemma 6.14.** If there exists a shortening sequence of moves for \(\mathcal{L}_0\), then \(\mathcal{L}_0\) cannot be induced by a shortest solution of \(\Sigma_0\).

**Proof.** Assume on the contrary that \(\mathcal{L}_0\) is induced by a shortest solution of \(\Sigma_0\). Using the property of inert moves, \(\mathcal{L}\) is itself induced by a shortest solution \(\sigma\) of \(\Sigma\). Using the restriction move, consider \(\sigma'\) a solution of \(\Sigma'\) such that \(|\sigma'_I| = |\sigma'_I|\) and \(|\sigma'_{D_A'}| = |\sigma_{\varepsilon(D_A')}|\). Using the extension move, consider a solution \(\tilde{\sigma}\) of \(\Sigma\) such that \(|\tilde{\sigma}_I| = |\sigma'_I|\) and \(|\tilde{\sigma}_{D_A'}| \leq \lambda|\sigma'_{D_A'}|\). Since \(\varepsilon(D_A')\) has a certificate of \(\varepsilon\)-shortness relative to \(D_A\), it follows that \(|\tilde{\sigma}| \leq |\sigma_{D_I}| + \varepsilon\lambda|\sigma_{D_A'}| < |\sigma|\), which is a contradiction. \(\square\)

6.2.5. *More about the analyser.* We are now ready to describe the rejection by shortening moves by the prelamination analyser. The analyser tries all possible sequences of moves looking like a shortening sequence of moves constructed as follows: a sequence of inert moves transforming \((\Sigma_0, \mathcal{L}_0)\) into some \((\Sigma, \mathcal{L})\), followed by a choice of the decomposition \(D = D_I \sqcup D_A\), followed by a sequence of shortening moves with the inert part \(D_I\) transforming \((\Sigma, \mathcal{L})\) into some \((\Sigma', \mathcal{L}')\), followed by a sequence of extension moves with the inert part \(D_I\), and transforming \(\Sigma'\) back into \(\Sigma\). All such sequences can be enumerated. Then look for \(1/\lambda\)-shortness certificates for \(\varepsilon(D_A')\) in \(D_A\). Being a little more careful, one could easily ensure that this enumeration terminates, but we will not need this fact. If such a certificate exists, it will be found. In this case, the prelamination analyser rejects \(\mathcal{L}\).

**Lemma 6.15.** The prelamination analyser is correct: if it says ‘Solution found’; then \(\Sigma\) has a solution inducing \(\mathcal{L}\); if it says ‘Reject’, then \(\mathcal{L}\) cannot be induced by a shortest solution.

**Proof.** The correctness when a solution has been found follows from Lemma 6.4. The correctness for each reason of rejection follows from Paragraphs 6.2.1–6.2.3, and from Lemma 6.14 above. \(\square\)

7. *Asymptotic of pre laminations: Rips’ band complexes*

In this section, we assume that in the locally finite tree of pre laminations produced by the pre laminations generator, there is an infinite chain, that is, an infinite sequence of pre laminations with rational constraints extending each other.

By passing to a limit, we shall construct a topological foliation on \(\Sigma\) together with an invariant transverse measure \(\mu\). With an assumption on the exponent of periodicity, we will prove that \(\mu\) has no atom. However, \(\mu\) may fail to have full support. Collapsing segments of measure zero, we get a new foliated band complex \(\Sigma_\mu\), where the measure is the Lebesgue measure: we say that \(\Sigma_\mu\) is a *Rips band complex*.

In \([3, 21]\) the decomposition of such an object into so-called minimal components and the classification of such minimal components are studied. In particular, there are so-called homogeneous components, in which, as we shall see, one can find an arbitrarily high exponent of periodicity. If a minimal component is not homogeneous, then one can perform a sequence of
moves on the Rips’ band complex (to obtain so-called independent bands) so that it becomes either a surface component (also called interval exchange component) or an exotic component (also called thin, or Levitt). In both cases, we will prove the existence of a shortening sequence of moves (in the sense of Definition 6.13) for all sufficiently large prelaminations in our infinite chain. These properties are used to prove that the main algorithm (Algorithm 8.2) always terminates; see Section 8.

7.1. Limit measured foliation

Consider an infinite sequence of prelaminations with rational constraints extending each other: \( \mathcal{L}_1 \prec \mathcal{L}_2 \prec \ldots \) (Definition 5.8).

The goal of this subsection is to associate to such a chain of prelaminations a topological foliation on \( \Sigma \), and a transverse invariant measure under the assumption that each \( \mathcal{L}_i \) has an invariant transverse measure (see Paragraph 6.2.2). Of course, those objects are not accessible to the algorithm, but they are used to prove the existence of certificates rejecting \( \mathcal{L}_i \) for \( i \) large enough.

7.1.1. The limit topological foliation. We first define the notion of a topological foliation on a band complex \( \Sigma \). Recall that bands of a band complex are defined by the attaching maps \( f_{B,\varepsilon} : [a, b] \times \{\varepsilon\} \to D \) for \( \varepsilon = 0, 1 \), but the only relevant data is the combinatorial arrangement of the endpoints of bases of bands. If \( \Sigma \) is endowed with a prelamination, then more data has to be extracted from \( f_B \), that is, the combinatorial arrangement of the images of endpoints of leaf segments. A topological foliation \( \mathcal{F} \) consists in choosing, for each band, an injective continuous map \( \tilde{f}_{B,\varepsilon} : [a, b] \times \{\varepsilon\} \to \) coinciding with \( f_{B,\varepsilon} \) on the endpoints of bases of bands. Here, the precise values of \( \tilde{f}_{B,\varepsilon} \) matter, not only its values on a finite subset.

Leaf segments of \( \mathcal{F} \) in a band \( B \) consist of all segments \( \{x\} \times [0, 1] \subset B \). The endpoints of the leaf segments are the points \( f_{B,0}(x, 0), f_{B,1}(x, 1) \in D \). Consider the equivalence relation on the set of leaf segments, generated by the property of having a common endpoint. A leaf of \( \mathcal{F} \) is the union of leaf segments in a given equivalence class. As usual, we identify a leaf with its image in the topological realization of the band complex.

As for prelaminations, we can talk about singular or regular leaf segments, leaf paths, and leaves. Given a \( \Sigma \)-word \( w \), and \( x \in D \), we still use the notation \( \langle x, w \rangle \) which has a meaning for a topological foliation. The holonomy map \( h_w \) is now a homeomorphism between a compact interval \( \text{dom } w \subset D \), called the domain of \( w \), and its image.

**Definition 7.1.** Consider a (maybe infinite) set \( l_1, \ldots, l_p, \ldots \) of leaves of a topological foliation \( \mathcal{F} \) containing all singular leaves.

We say that a sequence of prelaminations \( \mathcal{L}_i \) is represented in \( \mathcal{F} \) if one can write \( \mathcal{L}_i \) as a union of leaf segments of \( \mathcal{F} \) so that we have the following:

(i) \( \mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots \) is an exhaustion of \( l_1 \cup \ldots \cup l_p \cup \ldots \) by finite subgraphs;

(ii) every Möbius strip of \( \mathcal{L}_i \) has a core in \( \mathcal{L}_j \) for some \( j \geq i \).

The leaves \( l_i \) are called the special leaves of \( \mathcal{F} \).

**Remark 7.2.** By definition every singular leaf \( l \) is special. If \( l \) is pseudo-singular but not singular, and if \( \mathcal{L}_i \) has an invariant combinatorial measure, then \( l \) is special. Indeed, consider \( x \in \mathcal{L} \) and a \( \Sigma \)-word \( w \) such that \( x \) lies in the interior of \( \text{dom } w \), \( h_w(x) = x \), and \( w \) reverses the orientation. Let \( [a, b] \) be the domain of \( h_w \). If \( h_w \) fixes \( a \), then there will a Möbius strip in \( \mathcal{L}_i \) for \( i \) large enough since the leaf through \( a \) is singular and hence special. The second condition ensures that \( l \) is special. Otherwise, up to changing \( w \) to \( w^{-1} \), we can assume \( h_w(a) > a \).
This implies \( h_{w^2}(b') < b' \), where \( b' = h_w(a) \). It follows that the holonomy of \( w^2 \) maps \([a, b']\) to a proper subsegment of itself, which prevents the existence of an invariant combinatorial measure.

**Proposition 7.3.** Consider an infinite sequence of prelaminations \( \mathcal{L}_1 < \mathcal{L}_2 < \ldots \) extending each other, and consisting of pseudo-singular leaves. Assume that, for each subdivision point \( x \) of \( \mathcal{L}_i \), there exists \( j > i \) such that \( x \) is not an orphan in \( \mathcal{L}_j \), and that any Möbius strip of \( \mathcal{L}_i \) has a core in some \( \mathcal{L}_j \) for some \( j \geq i \).

Then there exists a topological foliation \( \mathcal{F} \) on \( \Sigma \) representing \( \mathcal{L}_i \).

**Proof.** Embed \( D \) in \( \mathbb{R} \) and endow it with the induced metric. Denote by \( |C| \) the diameter of a subset \( C \subset D \). Let \( L_i = V(\mathcal{L}_i) \) be the set of subdivision points of \( \mathcal{L}_i \). Recall that we have some freedom in the choice of the points of \( L_i \) as long as we do not change the induced ordering of the points. Thus, we can choose \( L_i \) inductively such that \( L_i \supset L_{i-1} \), and such that the following holds. For each component \( C \) of \( D \setminus L_i \), let \( C' \) be the component of \( D \setminus L_{i-1} \) containing \( C \). Then either \( C = C' \), or \( |C| \leq |C'|/2 \).

We also have some freedom in the choice of the attaching maps of each band \( f_{B, \varepsilon}^{(i)} : [a, b] \times \{0, 1\} \rightarrow D \) used to define the prelamination \( \mathcal{L}_i \). By modifying \( f_{B, \varepsilon}^{(i)} \) with respect to \( L_i \), we can assume that \( f_{B, \varepsilon}^{(i)} \) coincides with \( f_{B, \varepsilon}^{(i-1)} \) on each \( C \) such that \( C = C' \) (formally, we should rather say that they coincide on \( f_{B, \varepsilon}^{-1}(C) \), but to keep notation lighter, we make this abuse of notation).

Let \( L_{\infty} = \bigcup_{i \geq 0} L_i \). Our inductive choice of \( L_i \) ensures that, for any connected component \( C \) of \( D \setminus L_{\infty} \), its endpoints lie in some \( L_{i_0} \), and so \( C \) is actually a connected component of \( D \setminus L_{i_0} \). We say that a connected component of \( D \setminus L_{i_0} \) is stabilized if it is also a connected component of \( D \setminus L_{\infty} \). In this case, for any oriented band \( B \) whose initial base intersects (and therefore contains) \( C \), the restriction to \( C \) of the attaching map \( f_{B, \varepsilon}^{(i)} \) is independent of \( i \geq i_0 \).

Up to enlarging \( i_0 \), we can assume that the endpoints of \( C \) are not orphans in \( \mathcal{L}_{i_0} \). It follows that the image \( C' \) of \( C \) under the holonomy of \( B \) is also a stabilized component, and so the restriction to \( C \) of the holonomy of \( B \) does not depend on \( i \geq i_0 \).

We fix a band \( B \) with the bases \( J_0, J_1 \subset D \). We claim that \( h_i = f_{B, \varepsilon}^{(i)} (f_{B, \varepsilon}^{(0)})^{-1} : J_0 \rightarrow J_1 \) converges uniformly as \( \varepsilon \) goes to infinity. Fix some \( \varepsilon > 0 \). Let \( U \) be the (finite) union of all connected components of \( D \setminus L_{\infty} \) of diameter at least \( \varepsilon/2 \). Denote by \( N_i \subset L_i \) the set of non-orphan subdivision points of \( \mathcal{L}_i \), and note that \( \bigcup_{i \geq 0} N_i = \bigcup_{i > 0} L_i \). Let \( i_0 \) be such that \( \partial U \subset N_{i_0} \), and that all connected components of \( (D \setminus U) \setminus N_{i_0} \) have diameter at most \( \varepsilon \).

Consider \( x \in J_0 \). If \( x \in U \), then \( h_j(x) = h_i(x) \) for all \( j, i > i_0 \). If \( x \notin U \), then let \( [s, s'] \) be the smallest interval containing \( x \) such that \( s \) and \( s' \) are the endpoints of leaf segments of \( B \). Then \( h_i(s) \) and \( h_i(s') \) are independent of \( i \), and are at a distance at most \( \varepsilon \) by the choice of \( i_0 \). Since \( h_i(x) \in [h_i(s), h_i(s')] \), for all \( i \), we get \( |h_i(x) - h_i(x)| \leq \varepsilon \) for all \( i, j \geq i_0 \), which proves uniform convergence to some \( h_{\infty} \). Since \( h_i^{-1} \) converges by the symmetric argument, it follows that \( h_{\infty} \) is a homeomorphism.

Finally, we can define the foliation using the attaching maps \( \tilde{f}_{B, \varepsilon} \) defined by \( \tilde{f}_{B, \varepsilon} = f_{B, \varepsilon}^{(0)} \) and \( \tilde{f}_{B, 1} = h_{\infty} \circ f_{B, \varepsilon}^{(0)} \).

It is now clear that \( \mathcal{F} \) represents \( \mathcal{L}_i \); indeed \( \mathcal{L}_i \) is an exhaustion of the union of the leaves it intersects because no subdivision point of \( \mathcal{L}_i \) remains an orphan forever.

**Remark 7.4.** Although the foliation is meaningless in the complement \( \Sigma^* \) of closure of special leaves, it can be convenient to ensure that the foliation on \( \Sigma^* \) is a twisted product; in other words, the holonomy of any \( \Sigma \)-word preserving a connected component \( J \) of \( \Sigma^* \cap D \)
and preserving the orientation has to be the identity on $J$. This is easily done by choosing the attaching maps in the proof above to be affine on the complement of the subdivision points. Also note that the measure to be defined in the next section is zero on $\Sigma^*$.

7.1.2. The limit invariant measure. We have the following proposition.

**Proposition 7.5.** Consider an infinite sequence of prelaminations $L_1 < L_2 < \ldots$ represented in some topological foliation $F$. For each $i$ assume that $L_i$ has an invariant combinatorial measure.

Then either the exponent of periodicity of $L_i$ goes to infinity (see Definition 6.5), or there exists a measure $\mu$ on $D$ that has no atom and that is invariant under the holonomy of $F$.

In particular, if no $L_i$ is rejected by the prelamination analyser, then $F$ has a non-atomic transverse invariant measure.

**Proof of Proposition 7.5.** The construction follows [37]. Denote by $F^*$ the (finite) union of singular leaves of $F$. We first note that if $F^*$ is finite (that is, contains only finitely many leaf segments), then $L_i$ is eventually constant. Indeed, consider $J \subset D$ a segment meeting the singular leaves of $F$ only at its endpoints. Since $L_i$ preserves a combinatorial measure, the union of all non-singular pseudo-singular leaves of $L_i$ intersects $J$ in at most one point (its midpoint). Since all leaves of $L_i$ are pseudo-singular, this bounds the number of leaf segments in $L_i$.

Consider $l_0$ an infinite singular leaf of $F$ and choose a base point $x_0 \in l_0$. Let $l_0(n)$ be the ball of radius $n$ centred at $x_0$ in $l_0$ viewed as a graph, and consider $L_n = l_0(n) \cap D$.

We claim that $\#L_n$ grows polynomially with $n$.

**Lemma 7.6.** We have $\#L_n \leq 2(2n + 1)^b$, where $b$ is the number of bands of $\Sigma$.

**Proof.** Fix some $n$, and consider $i$ large enough so that the finite graph $l_0(n)$ is contained in the prelamination $L_i$. Since $L_i$ has an invariant combinatorial measure, one can embed $D$ in $\mathbb{R}$ isometrically with respect to this combinatorial measure. For each band $B$ of $\Sigma$, the holonomy of $B$ extends to an isometry $\gamma_B$ of $\mathbb{R}$. Let $G < \text{Isom}(\mathbb{R})$ be the subgroup generated by the isometries $\gamma_B$. One easily checks that the ball of radius $n$ of the Cayley graph of $G$ has cardinal $2(2n + 1)^b$, where $b$ is the number of bands of $\Sigma$. It follows that the set of images of $x_0$ under elements of length at most $n$ of $G$ has cardinal at most $2(2n + 1)^b$. Since $L_n$ is contained in this set, the lemma follows.

Consider the normalized counting measure $\mu_n = (1/\#L_n) \sum_{s \in L_n} \delta_s$, where $\delta_s$ is the Dirac mass at $s$. Consider the set $O_n = L_n \setminus L_{n-1}$. This is the locus where $\mu_n$ might fail to be preserved: if $B$ is a band with the bases $J_0$ and $J_1$, and if $x \in J_0 \cap L_n \setminus O_n$, then $h_B(x) \in L_n$ and so $\mu_n(x) = \mu_n(h_B(x)) > 0$. Using the same argument for $B^{-1}$, this means that $h_B$ preserves the measure $\mu_n$ except maybe at points of $O_n \cup h_B(O_n)$. Because of polynomial growth, there exists a subsequence $n_k$ such that the proportion $\#O_{n_k}/\#L_{n_k}$ converges to 0 as $k$ goes to $\infty$.

Let $\mu$ be any accumulation point of $\mu_{n_k}$ for the weak-* topology (existence is ensured by Banach–Alaoglu’s theorem). If we know that the endpoints of the bases of bands have zero measure, then the following lemma proves that $\mu$ is invariant under the holonomy.
Lemma 7.7. Consider a band $B$ with the bases $J$ and $J'$ and with the holonomy $h : J \to J'$. Then $\mu_{J'} = h_* \mu_J$.

Proof. Consider $\varepsilon > 0$ and a continuous function $f : D \to \mathbb{R}$ with support in $J$, and $f' = f \circ h^{-1}$ with support in $J'$. Then for all $\varepsilon > 0$, for $k$ large enough,

$$\left| \int f' \, d\mu - \int f \, d\mu \right| \leq \frac{1}{#L_n} \sum_{s' \in L_n} \left| f'(s') - \sum_{s \in L_n} f(s) \right| + \varepsilon.$$ 

For each $s \in L_n \setminus O_n \cap J$, the point $s' = h(s)$ lies in $L_n \cap J'$, and the two corresponding terms $f(s)$ and $f'(s')$ cancel. Thus,

$$\left| \int f' \, d\mu - \int f \, d\mu \right| \leq \frac{1}{#L_n} \sum_{s' \in O_n} |f'(s')| + \frac{1}{#L_n} \sum_{s' \in O_n} |f(s)| + \varepsilon \leq \frac{\#O_n}{#L_n} ||f||_{\infty} + \varepsilon.$$ 

Since the proportion $\#O_n/#L_n \to 0$ converges to zero, the lemma follows. \qed

To prove that $\mu$ is invariant under the holonomy, we need to prove that $\mu$ has no atom. Note that the lemma implies that if $\mu(\{x\}) > 0$ for some $x \in J$, then $\mu(\{h(x)\}) = \mu(\{x\})$.

Consider $x \in D$ and a segment $J = [x, a]$. Consider the limit $\mu_J$ of the restriction of $\mu_{|J}$ to $J$ (up to passing to a subsequence). Clearly, $\mu_J$ coincides with $\mu$ on $J$, but $\mu_J(\{x\})$ may differ from $\mu(\{x\})$ because leaves of $L_i$ can go in the neighbourhood of $x$ without entering $J$. If $J' = [x, b'] \subset J$, then $\mu_{J'}(\{x\}) = \mu_J(\{x\})$. We thus define $\mu_+(x) = \mu_{[x, x+\varepsilon]}(\{x\})$ and $\mu_-(x) = \mu_{[x-\varepsilon, x]}(\{x\})$ for some (any) $\varepsilon > 0$. Since $\mu_n(\{x\}) \to 0$ as $n \to \infty$, one gets $\mu_+(x) + \mu_-(x) = \mu(\{x\})$. The proof of Lemma 7.7 directly extends to the following.

Lemma 7.8. Let $J$ be a base of some band $B$ with the holonomy $h : J \to J'$, and assume that $[x, x + \varepsilon t] \subset J$ for some $\varepsilon = \pm 1$, with $t > 0$. Let $\varepsilon' = \pm 1$ be the sign of $h(x + \varepsilon t) - h(x)$ for $t > 0$ small enough.

Then $\mu_{\varepsilon'}(h(x)) = \mu_\varepsilon(x)$.

The following lemma ends the proof of Proposition 7.5. \qed

Lemma 7.9. If $\mu$ has an atom, then exponent$(\mathcal{L}_i) \to \infty$.

Proof. Consider $a \in D$ with $\mu(\{a\}) > 0$. Assume, for instance, that $\mu_+(a) > 0$.

Let $l$ be the $\mathcal{F}$-leaf through $a$. We define the graph $\mathcal{C}$ as follows: its vertex set $V(\mathcal{C}) \subset (l \cap D) \times \{+1, -1\}$ is defined by $(y, \varepsilon) \in V(\mathcal{C})$ if there exists a $\Sigma$-word $w$ whose holonomy $h_w$ is defined on $[a, a + t]$ for some $t > 0$, with $h_w(a) = y$ and $h_w(a + t) - h_w(a)$ the sign of $\varepsilon$. Consider an edge labelled $B$ between $(y, \varepsilon)$ and $(y', \varepsilon')$ if $h_B(y) = y'$ and $h_B(y + \varepsilon t) - h_B(y)$ is defined and of the sign of $\varepsilon'$ for $t > 0$ small enough. The natural map $V(\mathcal{C}) \to l$ sending $(y, \varepsilon)$ to $y$ extends to a map $\mathcal{C} \to l$ sending edge to edge, and which is at most 2 to 1. Let us choose the point $(a, +1)$ (corresponding to the empty word) as a base point for $\mathcal{C}$.

For all $(y, \varepsilon) \in \mathcal{C}$ by Lemma 7.8 $\mu_+(y) = \mu_+(a)$. Since $\mu(D) < \infty$, it follows that $\mathcal{C}$ is a finite graph. Since $\mu_+(a) > 0$, the singular leaf $l_0$ of $\mathcal{F}$ accumulates on $a$ on the right.

Let $T$ be a maximal tree in $\mathcal{C}$. For any vertex $(y, \varepsilon) \in \mathcal{C}$, let us denote by $[(a, +1), (y, \varepsilon)]_T$ the unique path in $T$ joining the two points. Each such segment defines a $\Sigma$-word $w_{(y, \varepsilon)}$ whose
domain contains an interval $[a, a + \eta_0]$ for some $\eta_0 > 0$. For each oriented edge $e \in \mathcal{C}$ not in $T$, labelled by $B$, joining $v_1$ to $v_2$, the $\Sigma$-word $w_e = w_{v_1}^{-1}Bw_{v_2}$ has its domain containing some interval $[a, a + \eta_e]$ for some $\eta_e > 0$. Note that by the definition of $\mathcal{C}$, its holonomy fixes $a$ and preserves the orientation.

Let $\eta > 0$ be such that the holonomy of each $\Sigma$-word $w_e, e \in T \setminus T$ is defined on $[a, a + \eta]$.

We claim that, for any $x \in l_0 \cap D$ close enough to $a$ on the right of $a$, there exists an oriented edge $e \in \mathcal{C} \setminus T$ such that $h_{w_e}([a, x]) \subseteq [a, x]$. Otherwise, take $x$ close enough to $a$ so that the set $\bigcup_{e \in \mathcal{C} \setminus T} h_{w_e}([a, x])$ does not contain an endpoint of a base of band. Then any band defined on some $h_{w_e}(x)$ is defined on the whole interval $h_{w_e}([a, x])$. For all oriented edges $e \in \mathcal{C} \setminus T$ if $h_{w_e}(x) = x$, then the leaf through $x$ is finite since its intersection with $D$ is the finite set $\bigcup_{e \in \mathcal{C} \setminus T} h_{w_e}(x)$. This contradicts the fact that $x \in l_0$ since $l_0$ is infinite, and proves our claim.

Consider an oriented edge $e \in \mathcal{C} \setminus T$ and $x \in l_0$ such that $h_{w_e}([a, x]) \subseteq [a, x]$. Since $l_0$ is singular, it follows that $x$ is a subdivision point of $\mathcal{L}_i$ for $i$ large enough. This does not necessarily prevent the existence of a combinatorial measure for $\mathcal{L}_i$ because $a$ is not necessarily a subdivision point of $\mathcal{L}_i$. Up to replacing $w_e$ by some power $w$ of it, one can assume that the twisting morphism $\varphi_w$ is trivial. Then, for any integer $m$, there exists $i$ large enough such that all segments $[x, h_w(x)], h_w(x), h^2_w(x), \ldots, [h_w^{m-1}(x), h_w^m(x)]$ are mapped to each other under the holonomy of a power of $w$. It follows that $\exp(L_i)$ goes to infinity with $i$. \hfill \Box

7.1.3. Collapsing segments of measure 0, Rips’ band complex. Now we assume that $\Sigma$ is endowed with its topological foliation $\mathcal{F}$, and its non-atomic invariant transverse measure $\mu$; still, $\mu$ may fail to have full support. We now define a Rips’ band complex by collapsing connected components of the complement of the support of $\mu$ as follows.

Let $D_\mu$ be the quotient of $D$ by the equivalence relation $\mu([x, y]) = 0$ (recall that $\mu$ has no atom) and let $\pi_\mu : D \to D_\mu$ be the quotient map. For each band $B$ of $\Sigma$ with the bases $J_0$ and $J_1$, we consider the corresponding band $B_\mu$ with the bases $\pi_\mu(J_0)$ and $\pi_\mu(J_1)$ and whose holonomy is induced by the holonomy of $B$. Let $\Sigma_\mu$ be the complex of bands on $D_\mu$ whose bands are the $B_\mu$. The holonomy of the bands naturally defines a foliation $\mathcal{F}_\mu$ on $\Sigma_\mu$. We still denote by $\mu$ the measure induced by $\mu$ on $D_\mu$. This measure is invariant under the holonomy of $\mathcal{F}_\mu$.

Thus, one obtains a new foliated band complex $\Sigma_\mu$, where one needs to generalize the definition of a band complex to allow some connected components of $D_\mu$ to be reduced to a point, and some bands to have bases reduced to a point. In this setting, we still can talk of leaves, singular leaves, etc.

Since $\mu$ has no atom and has full support in $D_\mu$, it can be transported to the Lebesgue measure of a finite union of intervals of $\mathbb{R}$ by a homeomorphism. Such a band complex $\Sigma_\mu$ with its measured foliation $(\mathcal{F}_\mu, \mu)$ is thus the object studied in [3] or in [20] under the name system of isometries. We call such a band complex a Rips’ band complex.

We will use the same notation for the bands of $\Sigma$ and the corresponding bands of $\Sigma_\mu$. In particular, we view any $\Sigma$-word as a $\Sigma_\mu$-word, and vice versa.

We now gather some simple facts about the projection $\Sigma \to \Sigma_\mu$.

**Lemma 7.10.** Consider $x \in D_\mu$ whose $\mathcal{F}_\mu$-leaf is singular or pseudo-singular; then there exists $\tilde{x} \in D$ with $\pi_\mu(\tilde{x}) = x$ such that $\tilde{x}$ lies in a singular or pseudo-singular leaf of $\mathcal{F}$, respectively.

**Proof.** If $x$ is a vertex of $\Sigma_\mu$, then $\pi_\mu^{-1}(\{x\})$ contains a vertex of $\Sigma$ and we are done. Otherwise, there is a $\Sigma_\mu$-word $w$ such that $[x, w]$ is a regular leaf path joining $x$ to a vertex $y$ of $\Sigma_\mu$. Let $\tilde{y} \in \pi_\mu^{-1}(\{y\})$ be a vertex of $\Sigma$. Viewing $w$ as a $\Sigma$-word, the point $h_{w^{-1}}(\tilde{y}) \in \pi_\mu^{-1}(\{x\})$.
satisfies the lemma. If the leaf through $x$ is pseudo-singular but not singular, then let $w$ be a \( \Sigma \mu \)-word such that $h_w(x) = x$ and $h_w$ reverses the orientation. Then the corresponding $\Sigma$-word $w$ has a fix point $\tilde{x}$, and so the leaf through $\tilde{x}$ is pseudo-singular. Since $x$ is the unique fix point of $h_w$, it follows that $\pi_\mu(\tilde{x}) = x$.

\[ \text{Lemma 7.11. No regular leaf of } \Sigma_\mu \text{ is finite.} \]

\[ \text{Proof. Let } x \in D_\mu \text{ whose } \Sigma_\mu \text{-leaf is regular and finite. Consider a small open interval } I \subset D_\mu \text{ containing } x \text{ in its interior, small enough so that every leaf meeting } I \text{ is regular and finite. Let } K \text{ be the union of leaves meeting } I. \text{ Then } \pi^{-1}_\mu(K) \text{ is a union of regular leaves of } \Sigma. \text{ Since } \mu \text{ is a limit of measures supported on singular leaves of } \Sigma, \text{ it follows that } \mu(\pi^{-1}_\mu(I)) = 0, \text{ which is a contradiction.} \]

7.2. Moves

We describe moves that can be performed on a band complex together with a prelamination. These combinatorial moves can be performed by the algorithm. They can be performed in the forward direction on a prelaminated band complex, or in the backward direction on a naked band complex (without prelamination). Each of these forward moves has a counterpart that can be applied to a measured foliation. Performing a move on the foliation will provide a way to uniformly perform a move on a sequence of prelaminations represented by this foliation.

Each band complex and prelamination come with a set of rational constraints in standard form, described by $\rho : S^1_+ \to \mathcal{M}$ and a tuple $m \in \mathcal{M}^{E(L)}$. We always assume that $m$ is invariant under the holonomy and shows no incompatibility of rational constraints (see Paragraph 6.2.1). All the moves will keep $\rho$ and $\mathcal{M}$ unchanged.

We introduce inert moves before restriction and extension moves (see Paragraph 6.2.4 for definitions).

7.2.1. Band subdivision. Let $\Sigma$ be a prelaminated band complex with domain $D$, let $\mathcal{L}$ be a prelamination on $\Sigma$, let $B = [a, b] \times [0, 1]$ be a band of $\Sigma$, and let $\{c\} \times [0, 1] \subset B$ be a leaf segment of $L$ with $c \notin \{a, b\}$ (see Figure 5).

Let $\Sigma'$ be the band complex with domain $D' = D$, obtained by replacing the band $B$ by two bands $B'_a = [a, c] \times [0, 1]$ and $B'_b = [c, b] \times [0, 1]$, the attaching maps being the restrictions of the attaching maps of $B$. The twisting morphisms of $B'_a$ and $B'_b$ are the same as the twisting morphism of $B$. The prelamination $\mathcal{L}$ naturally induces a prelamination $\mathcal{L}'$ on $\Sigma'$. The set $m \in \mathcal{M}^{E(L)}$ of rational constraints associated to $\mathcal{L}$ can be seen as a set of rational constraints on $\mathcal{L}'$ since $E(L') = E(L)$. This induces a set of rational constraints on $\Sigma'$ by writing each elementary segment of $\Sigma'$ as a concatenation of elementary segments of $\mathcal{L}$.

It is clear that any solution of $\Sigma$ inducing $\mathcal{L}$ defines a solution of $\Sigma'$ inducing $\mathcal{L}'$, and that any solution of $\Sigma'$ induces a solution of $\Sigma$. Thus, $(\Sigma, \mathcal{L}) \to (\Sigma', \mathcal{L}')$ is an inert move.

\[ \text{Figure 5. Moves.} \]
If we are given a measured foliation \((\mathcal{F}, \mu)\) on \(\Sigma\) instead of a prelamination, together with a leaf segment \(\{c\} \times [0, 1] \subset B\), then we can similarly subdivide the band to obtain a new measured foliation \((\mathcal{F}', \mu')\) on a new band complex \(\Sigma'\). Moreover, if \(\mathcal{F}\) represents an increasing sequence of prelaminations \(\mathcal{L}_i\), and if \(\{c\} \times [0, 1]\) is in a special leaf of \(\mathcal{F}\) (as in Definition 7.1), then, for all \(i\) large enough, \(\{c\} \times [0, 1]\) is contained in a leaf segment of \(\mathcal{L}_i\), and so one can perform the band subdivision move on \((\Sigma, \mathcal{L}_i)\) and get an increasing sequence of prelaminations \(\mathcal{L}'_i\) on \(\Sigma'\) represented in \(\mathcal{F}'\).

**Definition 7.12.** Assume that the topological foliation \(\mathcal{F}\) represents \(\mathcal{L}_i\).

We say that a move \((\Sigma, \mathcal{F}) \to (\Sigma', \mathcal{F}')\) is compatible with \(\mathcal{L}_i\) if, for all \(i\) large enough, one can perform the corresponding combinatorial move \((\Sigma, \mathcal{L}_i) \to (\Sigma', \mathcal{L}'_i)\) (including rational constraints) and \(\mathcal{L}'_i\) is represented by \(\mathcal{F}'\).

We proved the following.

**Lemma 7.13.** Assume that \(\mathcal{F}\) represents a sequence of prelaminations \(\mathcal{L}_i\). If \(e\) is a leaf segment of \(\mathcal{F}\) contained in a special leaf, then band subdivision along \(e\) is compatible with \(\mathcal{L}_i\).

**7.2.2. Band removal.** Let \((\Sigma, \mathcal{L})\) be a prelaminated band complex with domain \(D\). Assume that there is an oriented band \(B = [a, b] \times [0, 1]\) with the bases \(J_0 = [a_0, b_0]\) and \(J_1 = [a_1, b_1]\), and a \(\Sigma\)-word \(w\) not involving the band \(B\) or its inverse, whose holonomy map \(h_w\) is defined and coincides with the holonomy of \(B\) on \([a_0, b_0]\) (see Figure 5).

Let \((\Sigma', \mathcal{L}')\) be the prelaminated band complex obtained by performing on \((\Sigma, \mathcal{L})\) a band subdivision for each leaf segment of \([a, w]\) and \([b, w]\), and then by removing the band \(B\) and all the leaf segments contained in \(B\). Clearly, any solution of \(\Sigma\) inducing \(\mathcal{L}\) is a solution of \(\Sigma'\) inducing \(\mathcal{L}'\).

If the twisting morphisms of \(w\) and \(B\) agree, then conversely, any solution of \(\Sigma'\) (without prelamination) is a solution of \(\Sigma\). Consider the case where the twisting morphism \(\varphi = \varphi^{-1}_w \varphi_B\) is non-trivial in \(\text{Aut}(S_\pm)\). Let \(m_{J_0} \in \mathcal{M}\) be the element induced by the rational constraint \(m\) on \(J_0\). Since there is no incompatibility of rational constraints, it follows that \(m_{J_0} \in \rho(\text{Fix} \varphi)\).

By the definition of standard form for rational constraints, \(\text{Fix} \varphi\) is represented by \(\rho\) and so \(\text{Fix} \varphi = \rho^{-1}(\rho(\text{Fix} \varphi))\). It follows that \(\rho^{-1}(m_{J_0}) \subset \text{Fix} \varphi\) and so the rational constraint \(m_{J_0}\) alone imposes the fact that any solution \(\sigma'\) of \(\Sigma'\) satisfies \(\sigma_{J_0} \in \text{Fix} \varphi\). Thus, in this case too, any solution of \(\Sigma'\) is a solution of \(\Sigma\).

If we are given a measured foliation \((\mathcal{F}, \mu)\) on \(\Sigma\) instead of a prelamination, and if \(h_w\) is defined and coincides with the holonomy of \(B\) on \([a_0, b_0]\) as above, we can similarly perform this move to obtain a new measured foliation \((\mathcal{F}', \mu')\) on a new band complex \(\Sigma'\). If \(\mathcal{F}\) represents an increasing sequence of prelaminations \(\mathcal{L}_i\) without incompatibility of rational constraints, then \(a_0\) and \(b_0\) lie in singular and hence special leaves. Thus, for \(i\) large enough, the \(\mathcal{L}_i\)-holonomy of \(w\) will be defined on \(a_0\) and \(b_0\), and we will be able to perform the band removal move on \(\mathcal{L}_i\).

This defines an increasing sequence of prelaminations \(\mathcal{L}'_i\) on \(\Sigma'\) represented in \(\mathcal{F}'\). Thus, the move on the foliated band complex uniformly induces moves on the prelaminations. We have thus proved the following.

**Lemma 7.14.** Assume that \(\mathcal{F}\) represents a sequence of prelaminations \(\mathcal{L}_i\) without incompatibility of rational constraints. Assume that \(B\) is an oriented band and \(w\) is a \(\Sigma\)-word not involving \(B^{\pm 1}\) whose holonomy maps one base of \(B\) exactly to the other base of \(B\).

Then the band removal move on \(\mathcal{F}\) is compatible with \(\mathcal{L}_i\).
7.2.3. **Domain cut.** Let \((\Sigma, \mathcal{L})\) be a prelaminated band complex with domain \(D\), and consider \(x \in D \setminus \partial D\) a \(\mathcal{L}\)-subdivision point, and assume that \(x\) is non- orphan. Let \(B_1, \ldots, B_k\) be the oriented bands containing \(x\) in the interior of their domain (one may have \(B_i = B_j^{-1}\) for some \(i \neq j\)). By performing a band subdivision of the band \(B_i\) along the leaf segment \([x, B_i]\), we can assume that \(x\) is not in the interior of any base of band. Now cut \(D\) along \(x\), that is, replace \(D\) by \(D'\) the disjoint union of the closure of the connected components of \(D \setminus \{x\}\). The attaching maps of the bands are still well defined, and \(\mathcal{L}\) induces a prelamination \(\mathcal{L}'\) on the obtained band complex \(\Sigma'\).

It is clear that any solution of \(\Sigma\) inducing \(\mathcal{L}\) defines a solution of \(\Sigma'\) inducing \(\mathcal{L}'\), and that any solution of \(\Sigma'\) induces a solution of \(\Sigma\). Thus, \((\Sigma, \mathcal{L}) \rightarrow (\Sigma', \mathcal{L}')\) is an inert move. It can also be viewed as a restriction move for the map \(\iota : D' \rightarrow D\) induced by inclusion as long as \(x\) lies in the active part of \(D\), the backward move \(\Sigma' \rightarrow \Sigma\) being an extension move of Lipschitz factor 1.

This move generalizes naturally to a foliation \(\mathcal{F}\) instead of a prelamination, and if \(x\) lies in a special leaf, then it is a non-orphan in some \(\mathcal{L}_i\), and so it uniformly induces domain cut moves on a sequence of prelaminations represented by \(\mathcal{F}\).

**Lemma 7.15.** Assume that \(\mathcal{F}\) represents a sequence of prelaminations \(\mathcal{L}_i\). If \(x \in D\) lies in a special leaf of \(\mathcal{F}\), then a domain cut at \(x\) is compatible with \(\mathcal{L}_i\).

7.2.4. **Pruning move.** The pruning move for a prelaminated band complex has been described in Paragraph 6.2.4. This move generalizes naturally to a foliation \(\mathcal{F}\) instead of a prelamination, and this move is always compatible with a sequence of prelaminations represented by \(\mathcal{F}\).

7.2.5. **Forgetful move.** Let \((\Sigma, \mathcal{L})\) be a prelaminated band complex with domain \(D\). Assume that there is a connected component \(D_0\) of \(D\) which contains exactly two bases of bands \(B_1\) and \(B_2\) with \(B_1 \neq B_2^{\pm 1}\), and such that \(\text{dom } B_1 \cup \text{dom } B_2 = D_0\) and \(I = \text{dom } B_1 \cap \text{dom } B_2\) is a non-degenerate segment.

By performing a pruning move on the components of \(D_0 \setminus I\), we can assume that \(\text{dom } B_1 = \text{dom } B_2 = D_0\). Let \(\Sigma'\) be the band complex with domain \(D \setminus D_0\), and where the bands \(B_1\) and \(B_2\) are removed and replaced by a single band \(B'\) with a holonomy \(h_{B_1}^{-1} \circ h_{B_1} \circ \varphi_{B_2}^{-1} \circ \varphi_{B_2}\). The prelaminaton \(\mathcal{L}\) induces a prelamination \(\mathcal{L}'\) on \(B'\), whose set of subdivision points is \(D_{B'} \cap D'\), and such that leaf segments of \(B'\) are exactly the segments \([x, B']\) such that \([x, B_2B_1^{-1}]\) is a leaf path of \(\mathcal{L}\).

It is clear that any solution of \(\Sigma\) inducing \(\mathcal{L}\) defines a solution of \(\Sigma'\) inducing \(\mathcal{L}'\), and that any solution of \(\Sigma'\) induces a solution of \(\Sigma\). Thus, for any partition \(D = D_1 \cap D_A\), where \(D_A\) contains both bases of \(B_1\) and of \(B_2\), the move \((\Sigma, \mathcal{L}) \rightarrow (\Sigma', \mathcal{L}')\) is a restriction move with respect to the inclusion \(\iota : D' \hookrightarrow D\) and \(\Sigma' \rightarrow \Sigma\) is an extension move of Lipschitz factor 3.

This move generalizes naturally to a foliation \(\mathcal{F}\) instead of a prelamination, and it uniformly induces forgetful moves on a sequence of prelaminations represented by \(\mathcal{F}\).

**Lemma 7.16.** Assume that \(\mathcal{F}\) represents a sequence of prelaminations \(\mathcal{L}_i\). Consider \(D_0\) a connected component of \(D\) which contains exactly two bases of bands \(B_1\) and \(B_2\) with \(B_1 \neq B_2^{\pm 1}\), and such that \(\text{dom } B_1 \cup \text{dom } B_2 = D_0\), and \(\text{dom } B_1 \cap \text{dom } B_2\) is a non-degenerate segment.

Then the forgetful move on \(\mathcal{F}\) (consisting in removing \(D_0\)) is compatible with \(\mathcal{L}_i\).
7.3. Minimal components

A Rips’ band complex has a decomposition into a finite union of minimal components (see [3, 20]). In this subsection, we introduce \( \mu \)-minimal components of the topological foliation \( \Sigma \) that mimic the minimal components of the corresponding Rips band complex \( \Sigma_\mu \). We prove in this subsection an important result showing that the measure of an interval \( I \) controls the relative length of any solution inducing \( L_i \) for \( i \) sufficiently large.

Consider a foliation \( \mathcal{F} \) on a band complex \( \Sigma \) with domain \( D \). Recall that the singular leaf segments of \( \mathcal{F} \) are the leaf segments in the boundary of the bands of \( \Sigma \), and that a leaf path is regular if it does not involve any singular leaf segment. We say that two points are in the same \( \mathcal{F} \)-leaf if they can be connected by a regular leaf path.

Thus, two points of \( \Sigma_\mu \) are in the same \( \mathcal{F}_\mu \)-leaf if they can be joined by a regular leaf path of \( \mathcal{F}_\mu \) (but the projection in \( \Sigma_\mu \) of a regular leaf path of \( \mathcal{F} \) may fail to be regular in \( \Sigma_\mu \)). It is convenient to call \( \mu([x, y]) \) the \( \mu \)-distance between \( x \) and \( y \).

**Definition 7.17.** Let \( (\Sigma_\mu, \mathcal{F}_\mu) \) be a Rips’ band complex such that no component of \( D \) and no base of band is reduced to one point.

We say that \( (\Sigma_\mu, \mathcal{F}_\mu) \) is minimal if, for each \( x \in D_\mu \setminus \partial D_\mu \), every \( \mathcal{F}_\mu \)-leaf is dense in \( \Sigma_\mu \).

**Definition 7.18.** A measured foliation \( (\mathcal{F}, \mu) \) on a band complex \( \Sigma \) with domain \( D \) is called \( \mu \)-minimal if all connected components of \( D \) and all bands of \( \Sigma \) have positive measure, and we have the following:

(i) \( \Sigma_\mu \) is minimal (as a Rips’ band complex);
(ii) for each \( x \in \partial D \), there exists a \( \Sigma \)-word whose holonomy is defined on an interval of positive measure, and maps \( x \) at positive \( \mu \)-distance of \( \partial D \).

**Remark 7.19.** One easily checks that all leaves of a \( \mu \)-minimal band complex are infinite, but this may be false if one removes the last condition of the definition. Leaves of a \( \mu \)-minimal band complex may in general fail to be dense, as there may be wandering intervals.

Assume that \( \Sigma \) is a \( \mu \)-minimal foliated band complex representing a sequence of prelamination \( L_i \). Given a solution \( \sigma \) of \( \Sigma \) inducing \( L_i \) for \( i \) large enough, one can control the ratio \( |\sigma|/|\sigma| \) in terms of the measure \( \mu(I) \).

**Lemma 7.20.** Let \( (\Sigma, \mathcal{F}, \mu) \) be a \( \mu \)-minimal foliated band complex with domain \( D \). Let \( L_i \) be a sequence of prelaminations represented by \( \mathcal{F} \).

For all \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that the following holds: for each segment \( I \subset D \) with \( \mu(I) < \eta \), and for \( i \) large enough, there exists \( I' \supset I \) adapted to \( L_i \), and a certificate of \( \varepsilon \)-shortness of \( I' \) relative to \( D \) as in Definition 6.11 (ensuring that \( |\sigma|/|\sigma| \leq \varepsilon \) by Lemma 6.12).

**Proof.** Fix \( \varepsilon > 0 \) and \( N \geq 1/\varepsilon \). For \( x \in D_\mu \setminus \partial D_\mu \), the \( \mathcal{F}_\mu \)-leaf of \( x \) is infinite, and so we can find \( N \) distinct \( \Sigma_\mu \)-words \( w_1, \ldots, w_N \) whose holonomies are defined on a neighbourhood of \( x \), and map \( x \) to \( N \) distinct points of \( D_\mu \). Take \( \varepsilon_1, \varepsilon_2 > 0 \) small enough so that the holonomies \( h_{w_i} \) are defined on \( [x - \varepsilon_1, x + \varepsilon_2] \) and map this segment to \( N \) disjoint intervals. Since every leaf is dense in \( \Sigma_\mu \), one can assume, moreover, that \( x - \varepsilon_1 \) and \( x + \varepsilon_2 \) lie in singular leaves of \( \Sigma_\mu \). Let \( I_x \subset D \) whose endpoints are in singular leaves of \( \Sigma \), and such that \( \pi_{\mu}(I_x) = [x - \varepsilon_1, x + \varepsilon_2] \). Then, for \( i \) large enough, \( I_x \) is adapted to \( L_i \), and has a certificate of \( \varepsilon \)-shortness relative to \( D \).
Consider $x \in \partial D_{\mu}$ and $\bar{x}$ its preimage in $\partial D$. Consider $w$ a $\Sigma$-word whose $F$-holonomy is defined on an interval of positive measure, and maps $\bar{x}$ at positive $\mu$-distance from $\partial D$. Let $\tilde{y} = h_w(\bar{x})$ and let $y = h_w(x)$ be the image of $\tilde{y}$ in $D_{\mu}$. Consider the interval $I_y \subset D$ defined above and define $I_x = h_w^{-1}(I_y)$. The endpoints of $I_x$ are in singular leaves of $\Sigma$, and so $I_x$ is adapted to $L_i$ for $i$ large enough and has a certificate of $\varepsilon$-shortness relative to $D$.

Consider the covering of $D_{\mu}$ by the interiors in $D_{\mu}$ of $\pi_{\mu}(I_x)$, and by compactness, consider $\eta$ the Lebesgue number of this covering. Then any segment $J \subset D_k$ of measure at most $\eta$ is contained in the interior of some $I_x$, and the lemma follows.

\begin{proposition}
Consider a band complex $\Sigma$ with a measured foliation $(F, \mu)$ without atom representing a sequence of prelamination $L_i$. Assume that there is a bound on the exponent of periodicity of $L_i$. Then one can perform a finite number of inert moves on $(\Sigma, F)$ compatible with $L_i$ so that the obtained foliated complex $(\Sigma', F')$ is the disjoint union of finitely many $\mu$-minimal band complexes $\Lambda_1, \ldots, \Lambda_p$ together with finitely many bands and components of $D'$ of measure 0.
\end{proposition}

The exponent of periodicity of $L_i$ is defined in Definition 6.5.

\begin{proof}
We first recall the decomposition into minimal components for $\Sigma_{\mu}$. Let $C_{\mu} \subset \Sigma_{\mu}$ be the finite graph defined by the union of singular finite $F_{\mu}$-leaves (an $F_{\mu}$-leaf is singular if it contains a subdivision point of $\Sigma_{\mu}$ or, equivalently, if the $F_{\mu}$-leaf containing it is singular). This graph is finite because $\Sigma_{\mu}$ has only finitely many vertices.

Denote by $\Lambda_{\mu,1}, \ldots, \Lambda_{\mu,p}$ the Rips’ band complexes obtained as the closure of the connected components of $\Sigma_{\mu} \setminus C_{\mu}$. Equivalently, $\Lambda_{\mu,i}$ are the connected components of the band complex obtained from $\Sigma_{\mu}$ by performing band subdivision and domain cuts at edges and vertices of $C_{\mu}$, and by removing bands of width 0. By [3, 20], either $\Lambda_{\mu,i}$ is minimal, or $\Lambda_{\mu,i}$ is a family of finite leaves (all leaves of $\Lambda_{\mu,i}$ are finite in particular). In our setting, families of finite leaves cannot occur by Lemma 7.11.

We now construct a lift $\bar{C}$ of $C_{\mu}$ in $\Sigma$. Working independently on each connected component on $C_{\mu}$, we can assume that $C_{\mu}$ is connected. Consider $x_{\mu} \in C_{\mu}$ and let $\bar{x} \in D$ be a point whose leaf is special with $\pi_{\mu}(\bar{x}) = x_{\mu}$. Existence of $\bar{x}$ follows from Lemma 7.10 since singular leaves are special. Consider $T$ a maximal tree of $C_{\mu}$. Given two vertices $v_1, v_2 \in T$, we denote by $[x_1, x_2]_T$ the path of $T$ joining them. First, we can lift $T$ in $\Sigma$. Each segment $[x_{\mu}, v]_T$ defines a $\Sigma$-word $w_v$, and $\bar{x}$ lies in the interior of dom $w_v$ since $C_{\mu}$ contains no singular leaf segment. Therefore, one can define $\bar{T} \subset \Sigma$ as the union of the leaf paths $[\bar{x}, w_v)$. For each oriented edge $e \in C_{\mu} \setminus T$ joining $v_1$ to $v_2$, consider $B_e$ the oriented band containing it, and the $\Sigma$-word $w_e = w_{v_1}^{-1} B_e w_{v_2}$. To lift $e$, we require that the endpoint of the leaf segment $[\bar{v}_1, B_e]$ coincide with $\bar{v}_2$, where $\bar{v}_1$ denotes the lift of $v_1$ in $\bar{T}$. Equivalently, for all edges $e \in C_{\mu} \setminus T$, we need to check that $h_{w_e}(\bar{x}) = \bar{x}$.

Let $I \subset D$ be the preimage of $x_{\mu}$ under $\pi_{\mu}$. Since $\bar{x}$ and $h_{w_e}(\bar{x}) \in I$, we are done if $I$ is reduced to one point. As the holonomy in $\Sigma_{\mu}$ of every loop in $C_{\mu}$ based at $x_{\mu}$ is defined on a neighbourhood of $x_{\mu}$, this defines in $\Sigma$ a morphism $\psi : \pi_1(C, x_{\mu}) \to \text{Homeo}(I)$. Let $H$ be the image of $\psi$ and let $H_+ < H$ be the orientation preserving subgroup.

We claim that, for each point $\tilde{y} \in I$ whose leaf is special, $h \tilde{y} = \tilde{y}$ for all $h \in H_+$. Otherwise, one can assume, for instance, $h \tilde{y} > y$, and so $h^n \tilde{y} > \tilde{y}$ for all $n > 0$. Moreover, up to changing $h$ to a power, we may assume that $h = \psi(w)$ for some $\Sigma$-word $w$ with a trivial twisting morphism. Since $\tilde{y}$ lies in a special leaf of $F$, for any $n > 0$ there exists $i$ such that all the intervals $[\tilde{y}, h \tilde{y}], \ldots, [h^{n-1} \tilde{y}, h^n \tilde{y}]$ are equivalent under the holonomy of $L_i$, and so $\text{exponent}(L_i) \to \infty$, which is a contradiction.
Let \( w \in \pi_1(C, x_\mu) \) be a \( \Sigma \)-word whose holonomy does not fix \( \tilde{x} \). Since \( \tilde{x} \) lies in a singular and hence special leaf of \( F \), it follows that the homeomorphism \( h = \psi(w) \) reverses the orientation. Let \( m \in I \) be its unique fixed point. The leaf through \( m \) is special since \( w \) and \( [x_\mu, h_w(x_\mu)] \) define a Möbius strip in some \( \mathcal{L}_i \), whose core needs to appear in some \( \mathcal{L}_j \). The claim above shows that \( H \cdot m = m \), and since \( H = (H_x, h_w) \), it follows that \( H \cdot m = m \). This means that choosing \( m \) instead of \( \tilde{x} \) as a lift of \( x_\mu \), one can lift \( \mathcal{C}_\mu \) to \( \Sigma \).

Let \( \mathcal{C} \subset \Sigma \) be the graph obtained by lifting each connected component of \( \mathcal{C}_\mu \) as above. Let \( (\Sigma', F') \) be the measured foliated band complex with domain \( D' \) obtained from \( (\Sigma, F) \) by first performing a band subdivision at each edge of \( \mathcal{C} \) and then a domain cut at each vertex of \( \mathcal{C} \). We still denote by \( \mu \) the obtained transverse measure on \( \Sigma' \). The band subdivisions along the edges of \( \mathcal{C} \) are mirrored by band subdivisions on \( \Sigma_\mu \) since the edges of \( \mathcal{C}_\mu \) are at positive \( \mu \)-distance of the boundary of the band containing them. This is not true for the domain cuts: given a vertex \( x \in \mathcal{C} \) and an oriented band \( B \) containing \( x \) in the interior of domain, the leaf segment \( [x, B) \) may fail to lie in \( \mathcal{C} \), in which case \( x \) is at zero \( \mu \)-distance from an endpoint of \( \partial \mathcal{C} \); in this case, the domain cut at \( x \) will split the band \( B \) into two bands, one of them having measure 0. Thus, \( \Sigma'_\mu \) differs from \( \Sigma_\mu \) only by the potential presence of additional bands of measure 0. Let \( D^0 \) be the union of connected components of \( D' \) with positive measure and let \( \Sigma^{0} \) be the band complex on \( D^0 \) whose bands are the bands of \( \Sigma' \) with positive measure. Then \( \Sigma^{0} \) is a disjoint union of foliated band complexes \( \Lambda'_1, \ldots, \Lambda'_\ell \), such that \( \Lambda'_i \) and \( \Lambda'_\mu \) are precisely the minimal components of \( \Sigma^{0} \).

We need to perform some more operations on \( \Sigma' \) to make sure that \( \Lambda'_i \) satisfies the second condition of \( \mu \)-minimality. We focus on a component \( \Lambda'_i \subset \Sigma' \), and to make the notation lighter, we use the notation \( \Lambda = \Lambda'_i \), and we denote by \( D \subset D' \) its domain. Consider the graph \( \mathcal{D}_\mu \subset \Lambda_\mu \) with vertex set \( \partial \mathcal{D}_\mu \), and whose edges are the leaf segments of \( \Lambda_\mu \) having both endpoints in \( \partial \mathcal{D}_\mu \). Working independently on each component of \( \mathcal{D}_\mu \), we can assume that \( \mathcal{D}_\mu \) is connected. There exists a vertex \( x_\mu \in \mathcal{D}_\mu \) and an oriented band \( B_\mu \) of \( \Sigma_\mu \) sending \( x_\mu \) into the interior of \( D_\mu \); otherwise leaves through the points close to the vertex set of \( \mathcal{C}_\mu \) would be compact.

Let \( \tilde{x} \) be the (unique) endpoint of \( D \) projecting to \( x_\mu \). A loop in \( \mathcal{D}_\mu \) based at \( x_\mu \) defines a \( \Sigma \)-word \( w \) whose holonomy \( h_w \) in \( \Sigma \) is defined on an interval of positive measure, and contains a point at \( \mu \)-distance 0 from \( \tilde{x} \). Moreover, \( h_w \) preserves the orientation. We claim that, for each \( y \in \text{dom} \, w \) and in a special leaf, \( h_w(y) = y \). Indeed, write dom \( w = [a, b] \), and assume, for instance, \( h_w(y) > y \). If \( \mu([y, b]) > 0 \), then \( h_w(y) \) is defined for all \( k > 0 \) since \( \mu([y, h_w(y)]) = 0 \). If \( \mu([a, b]) > 0 \), then up to changing \( w \) to \( w^{-1} \) and \( y \) to \( h_w(y) \), the same conclusion holds. This contradicts the bound on the exponent of periodicity as above.

We construct a lift \( D \) of \( \mathcal{D}_\mu \). Let \( T \) be a maximal tree of \( \mathcal{D}_\mu \). For each \( v \in \mathcal{D}_\mu \), let \( w_v \) be the \( \Sigma \)-word representing the segment \( [x_\mu, v]_T \). For each oriented edge \( e \in \mathcal{D}_\mu \setminus T \) joining \( v_1 \) to \( v_2 \), let \( B_e \) be the oriented band containing it, and \( w_e = w_{v_2}^{-1} B_e w_{v_1} \). Let \( I \subset D^0 \) be the intersection of all domains \( \text{dom} \, w_{v_1} \), \( \text{dom} \, w_{v_2} \), and dom \( B_e \). Since all these domains have positive measure and a point a \( \mu \)-distance 0 from \( \tilde{x} \), it follows that \( I \) has an endpoint \( \tilde{y} \in \pi^{-1}_\mu(x_\mu) \). The leaf through \( \tilde{y} \) is singular and hence special, and so \( h_{w_v}(\tilde{y}) = \tilde{y} \) for all \( e \) by the claim above. Thus, the union of all leaf paths \( [\tilde{y}, w_{v_1}], [\tilde{y}, w_{v_2}] \) defines a lift \( D \) of \( \mathcal{D}_\mu \).

We perform on \( \Sigma \) band subdivisions at all edges of \( D \) and domain cuts at all vertices of all connected components of \( \mathcal{D} \), and denote by \( \Sigma'' \), with domain \( D'' \), the obtained foliated band complex. Let \( D''^0 \) be the union of connected components of \( D'' \) with positive measure and let \( \Sigma''^0 \) be the band complex on \( D''^0 \) whose bands are the bands of \( \Sigma'' \) with positive measure. Each band of \( \Sigma' \) has been cut into one band in \( \Sigma''^0 \), and at most two bands of measure 0 (at most one on each side). A similar fact holds for connected components of \( D' \). Thus \( \Lambda \) has been transformed to \( \Lambda'' \subset \Sigma''^0 \) together with some intervals and bands of measure 0.

We claim that \( \Lambda'' \) is a \( \mu \)-minimal component. Denote by \( D_{\Lambda''} \subset D''^0 \) the domain of \( \Lambda'' \). Because of the cutting operation performed, for every endpoint \( x \) of a base of a band of \( \Lambda'' \), either \( x \) lies in \( \partial D_{\Lambda''} \), or it lies at positive \( \mu \)-distance from it. Since for every endpoint \( x_\mu \) of
there is a $\Sigma_\mu$-word defined on a set of positive measure mapping $x_\mu$ to the interior of $D_\mu$, this word lifts to a $\Lambda''$-word mapping the corresponding endpoint of $D_{\Lambda''}$ at positive $\mu$-distance from $\partial D_{\Lambda''}$.

Since all band subdivision moves and all domain cuts have been performed on special leaves of $\mathcal{F}$, these moves are compatible with $\mathcal{L}_i$, and the proposition follows.

7.4. Homogeneous components

Recall that two points of $\Sigma_\mu$ are in the same $\hat{\mathcal{F}}_\mu$-leaf if they can be joined by a regular leaf path of $\mathcal{F}_\mu$.

**Definition 7.22.** One says that $\Sigma_\mu$ contains a homogeneous component (also called axial or toral component) if there exists a non-degenerate open interval $I \subset D_\mu$ and a subgroup $P \subset \text{Isom}(\mathbb{R})$ with dense orbits such that for all $x, y \in I$, we find that $x$ and $y$ are in the same $\hat{\mathcal{F}}_\mu$-leaf if and only if $x$ and $y$ are in the same $P$-orbit.

When a minimal component satisfies this definition, it is called a homogeneous component. If $\Sigma_\mu$ contains a homogeneous component as in the definition above, then one of its minimal components is a homogeneous component.

**Proposition 7.23.** If $\Sigma_\mu$ contains a homogeneous component, then the exponent of periodicity of $L_i$ goes to infinity as $i$ goes to infinity.

Given a $\Sigma_\mu$-word $w$ whose holonomy is a translation of positive length $t(w)$, whose domain of definition has positive measure, its translation ratio is the ratio $\text{tr}(w) = \mu(\text{dom}(w))/t(w)$.

The proposition is based on the following fact from Rips’ theory (of which we give a proof below).

**Fact.** Assume that a system of isometries $\Sigma_\mu$ contains a homogeneous component. Then there are $\Sigma_\mu$-words with arbitrarily large translation ratio.

**Proof of Proposition 7.23.** Let $k$ be the cardinal of $\text{Aut}(S_\pm)$. Since $\text{tr}(w^k) \geq (1/k)\text{tr}(w) - 1$, it follows that there are $\Sigma_\mu$-words with trivial twisting whose translation ratio is arbitrarily large. Consider such a $\Sigma_\mu$-word $w$ with translation ratio at least $N$, and view it as a $\Sigma$-word. Let $x \in D$ be an endpoint of $\text{dom} w$. Since $\text{tr}(w) \geq N$, up to changing $w$ to $w^{-1}$, for all $k = 0, \ldots, N - 1$, we can assume that $x \in \text{dom} w^k$. Since the leaf through $x$ is singular and hence special, for all $i$ large enough, all intervals $[x, h_w(x)], \ldots, [h_w^{N-2}(x), h_w^{N-1}(x)]$ are equivalent under the holonomy of some $w^j$. The exponent of periodicity of $L_i$ then goes to infinity with $i$.

**Proof of the fact.** Given $\eta > 0$, consider $\Sigma'$ the Rips’ band complex on $D_\mu$ obtained by narrowing each band by $\eta$ on each side. More formally, this amounts to restricting $f_{B,\varepsilon}: [a, b] \times \{\varepsilon\} \to D$ to $[a + \eta, b - \eta] \times \{\varepsilon\}$. If $\Sigma_\mu$ contains a homogeneous component, then one can find $\eta > 0$ such that $\Sigma'$ has an infinite orbit [20]. In particular, there exist $\Sigma'$-words whose holonomy is a translation of arbitrarily small length. Fix some large $M \in \mathbb{R}$, and consider $w$ whose holonomy is a translation of length at most $\eta/M$. The holonomy of the corresponding $\Sigma_\mu$-word is a translation of the same length, and its domain has measure at least $2\eta$, and so $\text{tr}(w) \geq 2M$. 

7.5. Independent bands

A theorem by Gaboriau asserts that if a Rips’ band complex $\Sigma_\mu$ has no homogeneous component, then one can narrow its bands so that its bands are independent, meaning that no $F_\mu$-leaf contains a cycle. This result was also used in [20] or [23, Section 6] for the analysis of actions on $\mathbb{R}$-trees. Our goal is to mirror this fact in the topological foliated band complex $(\Sigma, F)$, and into the prelaminated band complexes $(\Sigma, L_i)$.

Given a foliated band complex $(\Sigma, F)$ with domain $D$, an orbit of $(\Sigma, F)$ is the intersection of a leaf with $D$.

**Theorem 7.24** [19]. Given $\Sigma_\mu$ a Rips’ band complex with domain $D$ without homogeneous component, there exists a Rips’ band complex $\Sigma'_\mu \subset \Sigma_\mu$ with the same domain $D$, obtained by replacing each band $B = [a, b] \times [0, 1]$ of $\Sigma$ by a narrower band $B' = [a', b'] \times [0, 1] \subset B$, and such that we have the following:

1. $\Sigma_\mu$ and $\Sigma'_\mu$ have the same orbits;
2. $\Sigma'_\mu$ has independent bands: no $F'_\mu$-leaf contains a cycle;
3. any singular leaf of $\Sigma'_\mu$ is contained in a pseudo-singular leaf of $\Sigma_\mu$.

**Remark 7.25.** If bands of $\Sigma'_\mu$ are independent, then every pseudo-singular leaf is singular.

The last condition is not stated in Gaboriau’s theorem but can be ensured by slightly modifying Gaboriau’s construction.

**Proof of Theorem 7.24.** We explain how to ensure that the set of pseudo-singular orbits of $\Sigma_\mu$ does not increase during Gaboriau’s construction in the case where $\Sigma_\mu$ is a minimal component. For convenience, we give references to statements in Gaboriau’s paper. To lighten notation, we drop the index $\mu$ and so we denote by $\Sigma$ the initial Rips’ band complex. Clearly, one can ensure that condition (3) holds if all orbits are finite [19, Lemme 5.2(1)]. The proof of Gaboriau’s theorem (see [19, Section 7]) consists in modifying the band complex $\Sigma$ by narrowing and then re-enlarging bands. Instead of narrowing bands by a uniform amount, we need to choose narrowing widths so that we do not create new pseudo-singular leaves.

The set of sides of bands (that is, the set of singular leaf segments) is the set of edges of the finite graph $\Delta$ of $[19]$. Given a tuple $t = (t_\epsilon)_{\epsilon \in E(\Delta)}$ of (small enough) non-negative numbers, one can define $\Sigma_\epsilon$ by narrowing each band by $t_\epsilon$ on the side corresponding to $\epsilon$.

Consider $T$ a maximal subforest of $\Delta$, a small tuple $t$ so that $t_\epsilon = 0$ for $\epsilon \notin T$, and for $\epsilon \in T$, choose $t_\epsilon > 0$ so that the boundary of the new band lies in a singular leaf of $\Sigma$. Moreover, $t$ should be small enough so that $\Sigma$ and $\Sigma_t$ have the same orbits (see [19, Proposition 4.5]). This is possible because the union of singular leaves is dense.

Let $t_1$ be the minimum of $t_\epsilon$ for $\epsilon \notin T$, and consider $u = (u_\epsilon)_{\epsilon \in E(\Delta)}$ a small tuple with $u_\epsilon = 0$ for $\epsilon \notin T$ and $0 < u_\epsilon \leq t_1$ for $\epsilon \in T$ so that the singular leaves of $\Sigma_2 = \Sigma_{u+t}$ are contained in singular leaves of $\Sigma$. Since $\Sigma$ is non-homogeneous, and since all coordinates of $u + t$ are positive, the leaves of $\Sigma_{u+t}$ are compact (see [19, Theorem 3.3]). Denote by $e(\Sigma)$ the measure of the space of leaves of a Rips’ band complex $\Sigma$ (this is 0 if every leaf is dense), and by $l(\Sigma)$ the sum of the transverse measure of its bands. Then $e(\Sigma_2) = 0$, and using [24, Theorem II.E.4], we have $e(\Sigma_{u+t}) = |\mathcal{G}|$, where $|\mathcal{G}| = \sum_{\epsilon \in \Delta} u_\epsilon$ (this is the generalization we need of [19, Proposition 4.3]). Since $\Sigma_2$ has compact leaves, one can find $v$ such that $v_\epsilon = 0$ for $\epsilon \in T$, and $\Sigma_3 = \Sigma_{u+t+v}$ has the same orbits as $\Sigma_2$, and has independent bands and has no new pseudo-singular leaves [19, Lemme 5.2]. Since bands are independent, it follows that $e(\Sigma_3) + l(\Sigma_3) = \mu(D)$ (see [19, Theorem 6.3]).
Finally, consider $\Sigma_4 = \Sigma_{l+2}$, obtained from $\Sigma_4$ by enlarging the bands by $u$. Then, as in [19, Section 7], $\Sigma_4$ has the same orbits as $\Sigma$, and so $e(\Sigma_4) = 0$. Since $l(\Sigma_4) + e(\Sigma_4) = l(\Sigma_4) + |u| = \mu(D)$, its bands are independent [19, Theorem 6.3]. Finally, one easily checks that any singular leaf of $\Sigma_4$ is contained in a pseudo-singular leaf of $\Sigma$.

**Definition 7.26.** We say that a measured foliation $(\mathcal{F}, \mu)$ on $\Sigma$ has $\mu$-independent bands if the holonomy of no non-trivial reduced $\Sigma_\mu$-word fixes an interval of positive measure.

Our goal is the following proposition.

**Proposition 7.27.** Consider a band complex $\Sigma$ with a measured foliation $(\mathcal{F}, \mu)$ without atom representing a sequence of prelaminations $\mathcal{L}_i$. Assume that there is a bound on the exponent of periodicity of $\mathcal{L}_i$, and that each $\mathcal{L}_i$ has an invariant combinatorial measure.

Then one can perform a finite number of inert moves on $(\Sigma, \mathcal{F})$ compatible with $\mathcal{L}_i$ so that the obtained foliated complex $(\Sigma', \mathcal{F}')$ has $\mu$-independent bands.

We shall use the following useful finiteness property.

**Proposition 7.28 (Segment-closed property, [21]).** Let $(\Sigma_\mu, \mathcal{F}, \mu)$ be a Rips' band complex, and let $\varphi : I \rightarrow J$ be a partial isometry between two segments of $D_\mu$ such that, for all $x \in I$, we find that $x$ and $\varphi(x)$ are in the same leaf.

Then there exist finitely many $\Sigma_\mu$-words $w_1, \ldots, w_n$ whose domains cover $I$, and whose holonomies coincide with $\varphi$ on $I$.

**Proof of Proposition 7.27.** For each band $B$ of $\Sigma$, we denote by $B_\mu$ the corresponding band of $\Sigma_\mu$. Consider $\Sigma_\mu'$ the Rips' band complex on $D_\mu$ whose bands $B_\mu' \subset B_\mu$ are given by Theorem 7.24. Consider a band $B$, and $I, J, I_\mu, J_\mu$, and $I', J'$ the bases of $B, B_\mu$ and $B_\mu'$, respectively. Let $K_\mu$ be the closure of a connected component of $I_\mu \setminus J_\mu$. Since $\Sigma_\mu$ and $\Sigma_\mu'$ have the same orbits, the holonomy $h_{B_\mu}$ of $B_\mu$ is a partial isometry such that $x$ and $h_{B_\mu}(x)$ are in the same $\Sigma_\mu'$-leaf. By the segment-closed property, there exist $\Sigma_\mu'$-words $w_1', \ldots, w_n'$ whose domains cover $K_\mu$, and whose holonomies coincide with $h_{B_\mu}$ on $K_\mu$. We can assume that $\text{dom} \ w_1'$ is not reduced to a point, that is, has positive measure.

We now lift this situation to $\Sigma$. For each band $B$ of $\Sigma$ whose bases have positive measure, consider a restriction $B'$ of $B$ in $\Sigma$ whose base projects to $B_\mu'$ in $\Sigma_\mu$. By Lemma 7.10, one can choose the endpoints of $B'$ in pseudo-singular leaves of $\mathcal{F}$. By Remark 7.2, these leaves are special. Using the band subdivision move, one can subdivide the band $B$ along the boundary of the corresponding subband $B'$. Doing this for each band of $\Sigma$ gives new band complex $\Sigma_1$ whose set of bands is $B' \sqcup B''$, where $B'$ is the set chosen lifts of bands of $\Sigma_\mu'$.

For each band $B'' \in B''$, there are $\Sigma_\mu'$-words $w_1', \ldots, w_n'$ whose domains cover $\text{dom} \ B_\mu''$, and whose holonomies coincide with $h_{B_\mu''}$. We view these words as $\Sigma_1$-words involving only bands of $B'$. The domains of $w_1'$ cover all $\text{dom} \ B''$ except for some finite union of intervals of measure zero. By further subdividing $B''$ along some singular leaves (defined as the boundary of the domain of some $w_1'$), we can replace each band $B''$ by a finite set of smaller subbands $B'''$ so that either $\mu(\text{dom} \ B''') = 0$ or $\text{dom} \ B''' \subset \text{dom} \ w_i''$ for some $i$.

Fix some band $B'''$ whose domain $[a, b]$ has positive measure. By construction, the holonomy of the word $u = w^{-1}_i B'''$ induces the identity on $\pi_\mu([a, b])$. If $h_u(a) \neq a$, then, up to changing $u$ to $u^{-1}$, we find that $h_{u^k}(a)$ is defined for all $k > 0$. Since the leaf through $a$ is special, this clearly implies that exponent($\mathcal{L}_i$) goes to infinity. It follows that $h_u$ fixes both $a$ and $b$, so one
can remove the band $B''$ using the band removal move. The bands of the obtained foliated band complex $\Sigma_2$ are either bands of $B'$, or bands whose bases have measure zero. Thus, $\Sigma_2$ has $\mu$-independent bands.

7.6. The pruning process

Consider a foliated band complex $(\Sigma, F, \mu)$ with $\mu$-independent bands, decomposed into $\mu$-minimal components as in Proposition 7.21. Let $\Lambda$ be a minimal component of this decomposition.

We recall the following pruning process (also known as process 1 of the Rips’ machine). All the moves are performed on $\Sigma$, modifying only $\Lambda \subset \Sigma$.

We denote by $D_\Lambda = D \cap \Lambda$ the domain of $\Lambda$, which will be the active part for the moves.

The pruning process consists in the iteration of the following steps.

**Step (a).** We say that an open interval $I \subset D_\Lambda$ is prunable if every $x \in \bar{I}$ lies in exactly one base of a band of $\Sigma$. If $\Lambda_\mu$ has no prunable interval of positive measure, then the process stops. Otherwise, let $I_1, \ldots, I_n$ be the complete list of maximal prunable intervals of positive measure, and perform a pruning move for each $I_i$.

**Step (b).** If some component of $D_\Lambda$ contains at most one base of band, then prune this band, and iterate step (b) as many times as possible.

**Step (c).** Perform all possible forgetful moves on $\Lambda$.

**Remark 7.29.** In step (a), note that each band of $\Sigma$ having a base in $D_\Lambda$ either lies in $\Lambda$ or has measure zero. Steps (b) and (c) can be repeated only finitely many times in a row since they decrease the number of bands.

Starting with $(\Sigma_0, F_0, \mu_0) = (\Sigma, F, \mu)$, we define inductively $(\Sigma_{j+1}, F_{j+1}, \mu_{j+1})$ by performing the three steps (a)–(c) on $(\Sigma_j, F_j, \mu_j)$. The sequence of prelaminations $L_i \subset \Sigma$ represented by $F$ induces a sequence of prelaminations $L_{j,i} \subset \Sigma_j$ represented in $F_i$. Moreover, the rational constraints on $L_i$ (in standard form) induce rational constraints on $L_{j,i}$ which define rational constraints $m_j \in M(\Sigma_j)$ on $\Sigma_j$ independently of $i$ by the compatibility of the rational constraints. All rational constraints are in standard form, using the same morphism $\rho : S^+ \rightarrow \mathcal{M}$.

We denote by $\Lambda_j \subset \Sigma_j$ the subset corresponding to the minimal component under study. We denote by $D_{\Sigma_j} \subset D$ and $D_{\Lambda_j} = D_{\Sigma_j} \cap \Lambda$ the domains of $\Sigma_j$ and $\Lambda_j$, respectively, viewed as subsets of $D$.

Since bands are independent (which excludes homogeneous components), it follows that surface and exotic components can be defined by the fact whether the pruning process stops or not (see [20]).

**Definition 7.30.** We say that $\Lambda$ is exotic (or Levitt, or thin), if the pruning process never stops. Otherwise it is a surface component.

7.6.1. Exotic components

We focus on an exotic minimal component $\Lambda$ of this decomposition. The goal of this section is the following proposition.

**Proposition 7.31.** Consider a foliated band complex $(\Sigma, F, \mu)$ with $\mu$-independent bands, decomposed into $\mu$-minimal components as in Proposition 7.21. Consider $\Lambda$ an exotic $\mu$-minimal component of $\Sigma$. Let $L_i$ be a sequence of prelaminations represented by $F$. 

Then, for $i$ large enough, there exists a shortening sequence of moves for $(\Sigma, \mathcal{L}_i)$ as in Definition 6.13, certifying that a shortest solution of $\Sigma$ cannot induce $\mathcal{L}_i$.

Proposition 7.31 will follow from the two following facts.

**Lemma 7.32.** The complexity of $\Sigma_j$ defined as the sum of number of bands of $\Sigma_j$ and of the number of connected components of $D_j$ remains bounded.

In particular, only finitely many distinct unfoliated band complexes with rational constraints appear in the sequence $\Sigma_1, \Sigma_2, \ldots$.

**Proof.** The second assertion easily follows from the first one since all rational constraints are in standard form, defined using the same morphism $\rho : S^+ \to \mathcal{M}$.

We need only to prove that the number of bands of $\Lambda_j$ and of connected components of $D_{\Lambda_j}$ is bounded. The band complex $\Lambda_j$ is homotopy equivalent to a graph $\Gamma_j$ whose set of vertices $V(\Gamma_j)$ is the set of connected components of $D_{\Lambda_j}$ and whose set of non-oriented edges $E(\Gamma_j)$ corresponds to non-oriented bands. The first Betti number of $\Lambda_j$ is $b_1(\Lambda_j) = \#V(\Gamma_j) - \#E(\Gamma_j)$.

The moves of the pruning process preserve the first Betti number: this is clear for moves involved in steps (b) and (c); for moves in step (a), this follows from the fact that if $[a, b]$ is pruned, then either $a \in \partial D$, or $a$ lies in the interior of the pruned band since otherwise, the $\mathcal{F}$-leaf through $a$ would be finite, which contradicts the $\mu$-minimality.

After step (b), every terminal vertex of $\Gamma_j$ corresponds to a connected component of $D_{\Lambda_j}$ containing a base of a band of $\Sigma_j \setminus \Lambda_j$. Similarly, after step (c), every vertex of valence 2 of $\Gamma_j$ corresponds to a connected component of $D_{\Lambda_j}$ containing a base of a band of $\Sigma_j \setminus \Lambda_j$. Since step (c) does not create new terminal vertices, the number of vertices of valence 1 and 2 of $\Gamma_j$ is bounded. Since $b_1(\Gamma_j)$ is bounded, the lemma follows.

**Lemma 7.33.** For all $\eta > 0$, there exists $j$ large enough such that $\mu(D_{\Lambda_j}) < \eta$.

**Proof.** By [20, Proposition 7.1], the $\mu$-diameter of the connected components of $D_{\Lambda_j}$ converges to 0. Since the number of connected components of $D_{\Lambda_j}$ is bounded, the proposition follows.

**Proof of Proposition 7.31.** Since the complexity of band complexes appearing in the sequence $\Sigma_1, \Sigma_2, \ldots$ is bounded, there is a bound $C$ on the number of unfoliated band complexes with rational constraints appearing in this sequence. The number of moves performed at steps (a)–(c), is bounded in terms of the complexity of $\Sigma_j$, and is therefore bounded by some number $k$. It follows that, for all $j$, one can go back from the unfoliated band complex $\Sigma_j$ to the initial unfoliated band complex $\Sigma$ using at most $Ck$ extension moves.

Consider $\epsilon < 1/3^{Ck}$. Using Lemma 7.20, consider $\eta$ so that, for any $I \subset D_{\Lambda}$ with $\mu(I) < \eta$, there exist $\epsilon$-shortness certificates for $I$ relative to $D_{\Lambda}$. By Lemma 7.33, let $j_0$ be large enough so that $\mu(D_{\Lambda_{j_0}}) < \eta$.

Let $\mathcal{L}_i$ be a sequence of prelaminations represented in $\mathcal{F}$, and consider $i_0$ large enough so that the sequence of moves from $(\Sigma, \mathcal{F})$ to $(\Sigma_{j_0}, \mathcal{F}_{j_0})$ can be applied to $(\Sigma, \mathcal{L}_{i_0})$, transforming it into $(\Sigma_{j_0}, \mathcal{L}')$. These moves are restriction moves with an inert part $D_I = D \setminus \Lambda$ and active part $D_{\Lambda} = D_{\Lambda}$. Since $\Sigma$ can be obtained from $\Sigma_{j_0}$ using at most $Ck$ extension moves of Lipschitz factor 3, going from $\Sigma_{j_0}$ to $\Sigma$ is an extension move of Lipschitz factor $3^{Ck}$. Thus, we have found a shortening sequence of moves for $(\Sigma, \mathcal{L}_{i_0})$, which certifies that a shortest solution of $\Sigma$ cannot induce $\mathcal{L}_{i_0}$ (see Definition 6.13).
7.6.2. Surface components.

Proposition 7.34. Consider a foliated band complex \((\sigma, \mathcal{F}, \mu)\) with \(\mu\)-independent bands, decomposed into \(\mu\)-minimal components as in Proposition 7.21. Assume that \(\Lambda\) is a \(\mu\)-minimal surface component. Let \(L_i\) be a sequence of preliminaries represented by \(\mathcal{F}\).

Then, for \(i\) large enough, there exists a shortening sequence of moves for \((\Sigma, L_i)\) as in Definition 6.13, certifying that a shortest solution of \(\Sigma\) cannot induce \(L_i\).

Consider \(\Lambda\) a \(\mu\)-minimal surface component. By definition, this means that the pruning process defined in Subsection 7.6 stops. Let \((\Sigma', \mathcal{F}', \mu') = (\Sigma_{j0}, \mathcal{F}_{j0}, \mu_{j0})\) be the foliated complex obtained at the end of the pruning process. The Rips’ band complex \(\Lambda_{\mu'}\) is a (non-orientable) interval exchange in the following sense: every \(x \in D_{\Lambda_{\mu'}}\) outside a finite set belongs to exactly two bases of \(\Lambda_{\mu'}\) (see \([20, \text{Corollary 6.3}]\)). Opening up a separatrix (or unzipping a train-track carrying the foliation) gives a sequence of interval exchanges with arbitrary small domains. We want to implement this unzipping process at the level of \(\Sigma\).

Consider \(x_\mu \in D_{\Lambda_{\mu'}}\), a point lying in three bands of \(\Lambda_{\mu'}\). Let \(B_1B_2\ldots B_j\ldots\) be the infinite \(\Lambda_{\mu'}\)-word corresponding to the unique infinite regular leaf path starting at \(x_\mu\), and consider the corresponding \(\Lambda\)-word \(w_j = B_1\ldots B_j\).

Lemma 7.35. There exists \(x \in \pi_{\mu'}^{-1}\{\{x_\mu\}\}\) lying in a singular leaf of \((\Sigma', \mathcal{F}')\), such that, for all \(j \geq 0\), we find that \(w_j(x)\) is not in the interior of a base of a band of \(\Sigma' \setminus \Lambda'\).

Proof. Let \(I_1, \ldots, I_p\) be the bases of the bands of \(\Sigma' \setminus \Lambda'\) which lie in \(D_{\Lambda'}\). Recall that \(\mu'(I_1 \cup \ldots \cup I_p) = 0\). Let

\[
E = \left\{ x \in \pi_{\mu'}^{-1}\{\{x_\mu\}\} \mid \exists k \text{ such that } h_{x_\mu}(x) \in I_1 \cup \ldots \cup I_p \right\}.
\]

Since the points \(h_{x_\mu}(x_\mu)\) are all distinct in \(\Lambda_{\mu'}\), for each \(i\) and each \(x \in \pi_{\mu'}^{-1}(x_\mu)\), there is at most one index \(k\) such that \(w_j(x) \in I_k\). It follows that \(E\) is a finite union of segments. If \(E\) is empty, then just take any point \(x \in \pi_{\mu'}^{-1}(x_\mu)\) lying in a singular leaf of \(\Sigma\). Otherwise, the leftmost point of \(E\) satisfies the lemma.

We are now ready to describe the unzipping process. Consider \(x \in D_{\Lambda'}\), as given by the lemma above and \(x_j = h_{x_\mu}(x) \in D_{\Lambda'}\). For each \(j \geq 1\), we find that \(x_j\) lies in the interior of exactly two bases of \(\Lambda'\). For \(j = 0, 1, \ldots\), we perform the following moves on \(\Sigma'\), modifying only \(\Lambda'\).

Step \(a_j\). Cut the domain at \(x_j\).

Step \(b_j\). Perform all possible forgetful moves on \(\Lambda'\).

We denote by \((\Sigma_{i0+j}, \mathcal{F}_{i0+j}, \mu_{i0+j})\) the band complex obtained after performing steps \(a_j\) and \(b_j\). Proposition 7.34 will follow from the two following facts.

Lemma 7.36. The complexity of \(\Sigma_j\) defined as the sum of number of bands of \(\Sigma_j\) and of the number of connected components of \(D_j\) remains bounded.

In particular, only finitely many distinct unfoliated band complexes with rational constraints appear in the sequence \(\Sigma_1, \Sigma_2, \ldots\).

Proof. Before step \(a_0\), we see that \(x\) lies in the interior of at least one and at most three bases of \(\Lambda'\) because \(x_\mu\) lies in the boundary of two bases and in the interior of one base of \(\Lambda_{\mu'}\).

By the choice of \(x\), we see that \(x\) does not lie in the interior of a base of band of \(\Sigma' \setminus \Lambda'\). Step \(a_0\)
may increase $b_1(\Sigma)$ by at most 2. After step $a_0$, we see that $x_1$ lies in exactly three bases of $\Lambda$, and the interior of exactly one base by Lemma 7.35. This does not change after set $b_0$.

By induction, we see that steps $a_j$ and $b_j$ do not change $b_1(\Lambda)$, and after these steps, $x_j$ lies in exactly three bases of $\Lambda$, and the interior of exactly one base by Lemma 7.35. As in the proof of Lemma 7.32, consider the graph $G_j$ whose vertex set $V(G_j)$ is the set of connected components of $D_{\Lambda_j}$ and whose set of non-oriented edges $E(G_j)$ corresponds to non-oriented bands. Since $\Lambda_j,\mu_j$ is an interval exchange, it follows that $G_j$ has no vertex of valence 1. Because of step (b), each vertex of valence 2 of $G_j$ is bounded by the number of bases of bands of $\Sigma_j \setminus \Lambda_j$. Since $b_1(G_j)$ is bounded, the lemma follows.

**Lemma 7.37.** For all $\eta > 0$, there exists $j$ large enough such that $\mu(D_{\Lambda_j}) < \eta$.

**Proof.** Since $\{w_j(x_{\mu})\}_{j \geq 0}$ is dense in $D_{\Lambda_j',\mu'}$, the maximum of the measures of the connected components of $D_{\Lambda_j} \setminus \{x_0, \ldots, x_j\}$ goes to 0 as $j$ tends to infinity. It follows that the measure of the connected components of $D_{\Lambda_j}$ goes to 0 as $j$ goes to infinity. Since the number of connected components of $D_{\Lambda_j}$ is bounded, the lemma follows.

The proof of Proposition 7.34 is now identical to the proof of Proposition 7.31.

8. Endgame

8.1. Finding solutions of band complexes

**Theorem 8.1.** There is an algorithm which takes as input a band complex with rational constraints, and decides whether it has a solution or not.

Let us recall that we have defined two algorithms. The first one is the prelamination generator, described in Subsection 6.1. It takes as input a band complex $\Sigma$ with rational constraints, and explores the space of all prelaminations (with rational constraints). The prelaminations are organized into a rooted tree $T$ of finite valence, in such a way that children of a prelamination $\mathcal{L}$ are extensions of $\mathcal{L}$. If $\sigma$ is a solution of $\Sigma$, then it induces a Möbius-complete prelamination $\mathcal{L}_\infty(\sigma)$ (see Subsections 5.1 and 5.2 for definitions). Lemma 6.2 says that $\mathcal{L}_\infty(\sigma)$ will be produced by the prelamination generator.

The second algorithm is the prelamination analyser, described in Subsection 6.2. It takes as input the band complex $\Sigma$ together with a prelamination $\mathcal{L}$ (with rational constraints). If $\mathcal{L}$ is Möbius-complete, then it decides if $\mathcal{L}$ is induced by a solution or not (see Lemma 6.4), and in this case, the prelamination analyser stops and outputs ‘Solution found’ together with a solution, or ‘Reject’ accordingly. If $\mathcal{L}$ is not Möbius-complete, then it looks for some certificate ensuring that no shortest solution can induce $\mathcal{L}$. If it can find one, then it says ‘Reject’ and stops. Otherwise, it may fail to stop or say ‘I don’t know’. The certificate is one of the following:

(i) the detection of an incompatibility of rational constraints;
(ii) the detection of the non-existence of invariant measure;
(iii) the detection of an exponent of periodicity that is too large for a shortest solution in regards of Bulitko’s lemma (Proposition 4.1);
(iv) the detection of a shortening sequence of moves on $(\Sigma, \mathcal{L})$, which proves that a solution inducing $\mathcal{L}$ cannot be shortest.

See Subsection 6.2 for more details.

We can now explain the structure of the main algorithm.
Algorithm 8.2 (Main algorithm).
Input: a band complex with rational constraints $\Sigma$.
Output: a solution of $\Sigma$, or ‘There is no solution’.

Run the prelamination generator to construct step by step the rooted tree $T$ of prelaminations.
For each prelamination $L$ produced by the generator, do in the background:
\[
\begin{align*}
&\text{if } L \text{ is a leaf of } T \text{ and is not Möbius-complete then} \\
&\quad \text{mark } L \text{ as ‘Rejected’} \\
&\quad \text{Run the prelamination analyser on } L \\
&\quad \text{if the analyser says ‘Solution found’ then} \\
&\quad \quad \text{exit with the solution provided by the analyser} \\
&\quad \text{if the analyser says ‘Reject’ then} \\
&\quad \quad \text{mark } L \text{ as ‘Rejected’}
\end{align*}
\]
Simultaneously:
\[
\begin{align*}
&\text{if some prelamination } L \text{ is not a leaf and all children } L \text{ get marked as} \\
&\quad \text{‘Rejected’ then} \\
&\quad \text{mark } L \text{ as ‘Rejected’} \\
&\quad \text{if the root lamination get marked as ‘Rejected’ then} \\
&\quad \quad \text{exit saying ‘There is no solution’}
\end{align*}
\]

Remark. If some prelamination $L$ is rejected by the prelamination analyser, then we could also reject all its descendants, and we could ask the prelamination generator to stop extending this prelamination.

The algorithm could easily provide a proof that there is no solution when this is the case.

The main difficulty is to prove that the algorithm always stops. Theorem 8.1 follows immediately from the two following lemmas.

Lemma 8.3. The main algorithm is correct: if it stops, then its answer is true.

Proof. By Lemma 6.15, the analyser is correct. Thus, the main algorithm is correct if it stops because the analyser has found a solution.

We claim that any prelamination which is marked as rejected cannot be induced by a shortest solution of $\Sigma$.

We prove the claim inductively. If $L$ is marked as rejected by the prelamination analyser, then the claim is true by correctness of the prelamination analyser. If $L$ is marked as rejected because $L$ is a leaf of $T$ and is not Möbius-complete, then the claim is true because of Lemma 6.1. If $L$ is not a leaf of $T$ and if all its sons are correctly marked as rejected, then the claim holds for $L$ by Corollary 6.2. This proves the claim.

If the root prelamination has been marked as rejected, then the root prelamination cannot be induced by a shortest solution of $\Sigma$, and so $\Sigma$ has no shortest solution and has no solution at all.

Lemma 8.4. The main algorithm always stops.

Proof. Let $\Sigma_0$ be the band complex taken as input. If $\Sigma_0$ has a solution, then it has a shortest solution $\sigma$, and the prelamination $L_\infty(\sigma)$ will be produced by the prelamination generator. If
the algorithm does not stop, then at some point $L_\infty(\sigma)$ will be analysed by the analyser. Since $L_\infty(\sigma)$ is Möbius-complete, the prelamination analyser will find out that there is a solution, and the main algorithm will stop, which is a contradiction.

Assume that $\Sigma_0$ has no solution. We claim that if some prelamination $L$ is never rejected, then it is not a leaf, and at least one of its children is never rejected. Indeed, every leaf of the rooted tree $T$ of prelaminations will be marked as rejected: either because it is not Möbius-complete, or because the analyser will reject it. On the other hand, if all the children of $L$ are marked as rejected at some point, then $L$ will itself be marked as rejected. This proves the claim.

Assume by contradiction that the algorithm does not stop. Then the root lamination never gets rejected. The claim says that there is an infinite ray in $T$ of prelaminations that are never rejected. This ray is an infinite sequence of prelaminations with rational constraints extending each other: $L_1 \prec L_2 \prec \ldots$ (Definition 5.8).

First, there exists a topological foliation $F_0$ on $\Sigma_0$ representing the prelaminations $L_i$ (this is by Proposition 7.3; see Definition 7.1 for the notion of foliation representing a sequence of prelaminations). Moreover, since no $L_i$ is rejected for a large exponent of periodicity, there exists a measure $\mu_0$, without atom, invariant under the holonomy of $F_0$. This is ensured by Proposition 7.5.

Now by Proposition 7.21, if $i$ is large enough, one can uniformly perform a finite sequence of inert moves on $(\Sigma_0, L_i)$, and get a sequence of prelaminations represented in a foliated band complex $(\Sigma_1, F_1, \mu_1)$ which is decomposed into $\mu$-minimal components.

By Proposition 7.21, the Rips' band complex $\Sigma_1\mu_1$ corresponding to $(\Sigma_1, F_1, \mu_1)$ cannot contain a homogeneous minimal component, since otherwise, $L_i$ would be rejected by the prelamination analyser for $i$ a large enough because of a large exponent of periodicity.

Proposition 7.27 then says that one can uniformly perform a finite sequence of inert moves on $(\Sigma_1, F_1, \mu_1)$ to get some band complex $(\Sigma, F, \mu)$ whose bands are $\mu$-independent. There are two alternatives left by the classification of minimal components: either there is an exotic component in $(\Sigma, F, \mu)$, or there is a surface component. In both cases, there exists a shortening sequence of moves for $L_i$ for $i$ large enough (see Propositions 7.31 and 7.34). This proves the existence of shortening certificates for $L_i$ for $i$ large enough, contradicting the fact that no $L_i$ is rejected. \( \square \)

8.2. Reduction of main theorems to band complexes

**Proposition 8.5.** There exists an algorithm that takes as input a basis of a free group $F$, a finite set $\Phi$ of automorphisms of $F$ preserving $S \cup S^{-1}$, and a system of twisted equations with rational constraints in $F$ (with twisting automorphisms in $\Phi$) and that decides whether there is a solution or not.

**Proof.** By Proposition 3.1, this reduces to deciding whether a band complex has a solution. Theorem 8.1 concludes. \( \square \)

We can now prove Theorems 3 and 4 from the introduction.

**Proof of Theorem 3.** By Proposition 2.4, the problem of equations with rational constraints in a virtually free group $V$ reduces to the problem of twisted equations with rational constraints in the sense of Definition 2.2, where the twisting morphisms permute a basis. Proposition 8.5 concludes. \( \square \)

Without assuming that twisting morphisms preserve a basis of $S$, we can prove Theorem 4.

**Proof of Theorem 4.** Let $Q \subset \text{Out}(F)$ be the finite group generated by the image of $\Phi$ and let $V$ be the preimage of $Q$ in $\text{Aut}F$. We embed $F$ into $\text{Aut}F$ via inner automorphisms $x \mapsto i_x$, \( \square \)
where \( i_x(g) = xgx^{-1} \). For all \( \varphi \in \Phi \) and all \( x \in F \), note that \( i_{\varphi(x)} = \varphi \circ i_x \circ \varphi^{-1} \). It follows each equation twisted by \( \Phi \) on \( F \) corresponds to a non-twisted equation in \( V \), together with the rational constraint saying that the variables should lie in \( F \). Since rational subsets of \( F \) are rational subsets of \( V \), the theorem follows from Theorem 3.

\[ \square \]

9. \textit{Equations in hyperbolic groups with torsion}

In [40], Rips and Sela constructed canonical representatives for a torsion-free hyperbolic group \( \Gamma \), in order to lift solutions of a system of equations on \( \Gamma \) to a free group. Then using Makanin’s algorithm, they deduced an algorithm deciding whether a given system of equations in \( \Gamma \) has a solution.

Delzant [14, Remark III.1] remarked that, for certain hyperbolic groups with torsion, such canonical representatives do not exist, whereas Reinfeidt (in his master thesis) noted that if the abelian subgroups are finite or cyclic, then they do exist. In fact, in the general case, most of the construction of [40] remains valid, up to the construction of canonical cylinders. Instead of defining canonical representatives as paths in the Cayley graph of \( \Gamma \) (thus living in a free group) as in [40], we need to interpret them as paths in the 1-skeleton \( K \) of the barycentric subdivision of a Rips’ complex of \( \Gamma \). The action of \( \Gamma \) on \( K \) fails to be free in the presence of torsion, and paths in \( K \) correspond to elements of the virtually free group \( V \) occurring as the fundamental group of the graph of groups \( K/\Gamma \) (see below for details). The natural generalization of Rips and Sela’s construction in the presence of torsion then leads to a family of systems of equations in the virtually free group \( V \).

In [11], rational constraints were used to handle inequations in torsion-free hyperbolic groups. More generally, we would like to solve equations with rational constraints in a hyperbolic group. However, this is impossible without restriction on the rational subsets involved (even if \( \Gamma \) is torsion-free). For instance, any finitely generated subgroup is a rational subset, but the membership problem is unsolvable in some hyperbolic groups (see [39]). This is why we introduce a nice class of rational subsets: quasi-isometrically embeddable rational subsets (see Definition 9.2). Examples of such subsets include finite subsets, quasi-convex subgroups, and their complements.

A set of \textit{quasi-isometrically embeddable rational constraints} on a system of equations is the additional requirement that each variable \( x \) should live in a quasi-isometrically embeddable rational subset \( R_x \subset \Gamma \). As we shall see, this class of subsets is a Boolean algebra (Corollary 9.6), and so inequations are a particular case of quasi-isometrically embeddable rational constraints.

Here is the main statement of this section (compare with [11, §5, 5.3]).

\textbf{Theorem 1.} \textit{There exists an algorithm which takes the following input:}

(i) a presentation \((S|R)\) of a hyperbolic group \( \Gamma \) (possibly with torsion);
(ii) a finite system of equation and inequations with constants in \( \Gamma \), and with quasi-isometrically embeddable rational constraints, and which decides whether there exists a solution or not.

Each rational constraint \( R_x \) should be given to the algorithm under the form of a finite automaton accepting a quasi-isometrically embedded language \( \tilde{R}_x \subset S_\pm^* \) projecting to \( R_x \subset \Gamma \) (see Definition 9.2).

Let us emphasize that the algorithm is uniform over all hyperbolic groups. This is an unusual application of the fact that the hyperbolicity constant of a hyperbolic group can be computed from a presentation (see [4, Proposition 8.6.1] or [22, 35]). In all the following, we will make sure that the algorithms we use are explicit if the hyperbolicity constant is known (in particular
that the constants we use are explicit, or computable in terms of the presentation and the hyperbolicity constant). In this way, there is no restriction in considering that the hyperbolic group is given once and for all.

In particular, we get (compare with [11, Theorem 0.1]) the following corollary.

**Corollary 9.1.** The existential theory with constants of a hyperbolic group is decidable.

### 9.1. Quasi-isometrically embeddable rational subsets

**Definition 9.2.** Let \( \Gamma \) be a hyperbolic group generated by a finite set \( S \) and let \( \pi : S^*_{\pm} \to \Gamma \) be the corresponding morphism.

A regular language \( \tilde{R} \subset S^*_{\pm} \) is **quasi-isometrically embedded** in \( \Gamma \) if there exist \( \lambda \geq 1 \) and \( \mu \geq 0 \), such that, for any word \( w \in \tilde{R} \), we have \( |\pi(w)|_\Gamma \geq (1/\lambda)|w| - \mu \).

A rational subset \( R \subset \Gamma \) is **quasi-isometrically embeddable** in \( \Gamma \) if there exists a quasi-isometrically embedded regular language \( \tilde{R} \subset S^*_{\pm} \) such that \( \pi(\tilde{R}) = R \).

Here, \( |w| \) is the length of \( w \) as a word on \( S_{\pm} \), and \( |\cdot|_\Gamma \) is any word metric on \( \Gamma \). In particular, finite sets and quasi-convex subgroups are clearly quasi-isometrically embeddable rational subsets.

The property of being quasi-isometrically embeddable depends neither on the chosen word metric on \( \Gamma \) nor on the choice generating set \( S \). Indeed, consider any other generating set \( S' \) and express each element of \( S \) as a word on \( S'_{\pm} \). This defines a morphism \( \rho : S^*_{\pm} \to (S'_{\pm})^* \) satisfying \( |\rho(w)| \leq L|w| \), where \( L = \max_{s \in S_{\pm}} |\rho(s)| \). Consider the natural morphism \( \pi' : (S'_{\pm})^* \to \Gamma \). The regular language \( \tilde{R}' = \rho(\tilde{R}) \) is quasi-isometrically embedded since, for each \( w' = \rho(w) \), we have \( \pi'(w') = \pi(w) \) and \( |\pi'(w')|_\Gamma = |\pi(w)|_\Gamma \geq (1/\lambda)|w| - \mu \geq (1/\lambda L)|w'| - \mu \).

However, the fact of being quasi-isometrically embedded does depend on the choice of \( \tilde{R} \) representing \( R \). For instance, if \( \mathcal{R} \) is the set of all words and \( \mathcal{R}' \) is the set of \( L \)-local geodesics, then they represent the same rational subset (namely \( \Gamma \)), but \( \mathcal{R} \) is not quasi-isometrically embedded.

A quasi-isometrically embeddable rational subset is quasi-convex: there exists \( C > 0 \) such that, for all \( x, y \in \tilde{R} \), any geodesic of \( \Gamma \) joining \( x \) to \( y \) is contained in the \( C \)-neighbourhood of \( \tilde{R} \). Indeed, by thinness of triangles, one can assume \( x = 1 \). Let \( A \) be an automaton accepting \( \tilde{R} \). By the stability of quasi-geodesics, any path represented by a word accepted by \( \tilde{R} \) lies within a bounded distance from a geodesic joining its endpoints. Finally, if a path is accepted by \( A \), then any of its points lies a bounded distance away from an accepted point (the bound depending only on the number of states of \( A \)).

Conversely, we do not know if a quasi-convex rational subset is always quasi-isometrically embeddable.

### 9.2. Full sublanguage of a set of quasi-geodesics

The goal of this section is to prove that, given \( \mathcal{R} \subset \Gamma \) a quasi-isometrically embeddable rational subset, the set of local quasi-geodesics representing elements of \( \mathcal{R} \) is rational (Proposition 9.4). This should be thought as an analogue of Lemma 1.6 saying that if \( (S) \) is a free group and \( \mathcal{R} \subset (S) \) is a rational subset, then the language of freely reduced words in \( S_{\pm}^* \) representing an element of \( \mathcal{R} \) is a regular language. As for the free group, we deduce that quasi-isometrically embeddable rational subsets form a Boolean algebra.

Let \( S \) be a finite generating system of \( \Gamma \), let \( \pi : S^*_{\pm} \to \Gamma \) be the corresponding morphism, let \( \text{Cay} \Gamma \) be the Cayley graph, and let \( |\cdot|_S \) be the word metric.
Given \( w \in S^*_\pm \), we denote by \( p_w \) the corresponding path in \( \text{Cay} \Gamma \), defined by \( p_w(i) = \pi(s_1 \ldots s_i) \). Given \( \lambda \geq 1 \) and \( \mu \geq 0 \), let \( \mathcal{LQG}_{\lambda,\mu}(S^*_\pm) \subset S^*_\pm \) be the set of \((\lambda, \mu)\)-quasi-geodesics, that is, the set of words \( w \) such that \( |p_w(i) - p_w(j)| > (1/\lambda)|i - j| - \mu \). Similarly, the set of \( \nu \)-local quasi-geodesics \( \mathcal{LQG}_{\nu,\lambda,\mu}(S^*_\pm) \subset S^*_\pm \) is the set of words \( w \) such that, for all \( |i - j| \leq \nu \), we have \( |p_w(i) - p_w(j)| > \lambda|i - j| - \mu \). Clearly, \( \mathcal{LQG}_{\nu,\lambda,\mu}(S^*_\pm) \) is a regular language since this is the complement of the set of words containing as a subword one of the finitely many words of length at most \( \nu \) which are not \((\lambda, \mu)\)-quasi-geodesic.

We use the following statement about stability of local quasi-geodesics.

**Lemma 9.3** [9, Théorème 1.2, 1.4]. Fix \( \delta \) a hyperbolicity constant.

Given \( \lambda \geq 1 \) and \( \mu \geq 0 \), there exist computable numbers \( \nu, \lambda' \geq 1 \) and \( \mu' \geq 0 \) such that any \( \nu \)-local \((\lambda, \mu)\)-quasi-geodesic is a \((\lambda', \mu')\)-quasi-geodesic.

Given \( \lambda \geq 1 \) and \( \mu \geq 0 \), there exists a computable number \( \eta \) such that any \((\lambda, \mu)\)-quasi-geodesic is at Hausdorff distance at most \( \eta \) from any geodesic with the same endpoints.

We always assume that \( \nu \) is large enough so that all \( \nu \)-local \((\lambda, \mu)\)-quasi-geodesics are global quasi-geodesics as in the result above.

**Proposition 9.4.** Consider a hyperbolic group \( \Gamma, \pi : S^*_\pm \to \Gamma \), and \( \mathcal{R} \) a quasi-isometrically embeddable regular language. Consider any \( \lambda \geq 1, \mu \geq 0 \), and any \( \nu \) such that \( \nu \)-local \((\lambda, \mu)\)-quasi-geodesics are global quasi-geodesics.

Then \( \mathcal{R} = \pi^{-1}(\mathcal{R}) \cap \mathcal{LQG}_{\nu,\lambda,\mu}(S^*_\pm) \) is a regular language of \( S^*_\pm \).

Moreover, given an automaton accepting a quasi-isometrically embedded regular language representing \( \mathcal{R} \), one can algorithmically compute an automaton \( \mathcal{A} \) accepting \( \tilde{\mathcal{R}} \).

We say that \( \tilde{\mathcal{R}} \) is full in \( \mathcal{LQG}_{\nu,\lambda,\mu}(S^*_\pm) \) since any word in \( \mathcal{LQG}_{\nu,\lambda,\mu}(S^*_\pm) \) representing an element of \( \mathcal{R} \) lies in \( \tilde{\mathcal{R}} \).

**Remark 9.5.** Recall that the set of geodesic words in a hyperbolic group is itself regular, by the finiteness of Cannon’s so-called cone types [7]. Using this fact, one could modify the proposition by substituting the set of global geodesic words to \( \mathcal{LQG}_{\nu,\lambda,\mu}(S^*_\pm) \). However, we need to use local quasi-geodesics for solving equations and inequations in hyperbolic groups.

**Corollary 9.6.** The class of all quasi-isometrically embeddable rational subsets of a hyperbolic group \( \Gamma \) is a Boolean algebra.

Moreover, this can be algorithmically computed: given automata accepting quasi-isometrically embedded regular languages representing \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), one can compute automata accepting quasi-isometrically embedded regular languages representing \( \mathcal{R}_1 \cup \mathcal{R}_2, \mathcal{R}_1 \cap \mathcal{R}_2, \) and \( \Gamma \setminus \mathcal{R}_1 \).

**Proof.** Clearly, the union of two quasi-isometrically embeddable rational subsets is quasi-isometrically embeddable. Let \( \mathcal{R} \subset \Gamma \) be a quasi-isometrically embeddable rational subset. By Lemma 1.5, we need only to prove that \( \Gamma \setminus \mathcal{R} \) is quasi-isometrically embeddable. Let \( \mathcal{L} = \mathcal{LQG}_{\nu,1,0}(S^*_\pm) \) be the set of \( \nu \)-local geodesics (for some \( \nu \) large enough so that local geodesics are quasi-geodesics). By Proposition 9.4, we see that \( \tilde{\mathcal{R}} = \pi^{-1}(\mathcal{R}) \cap \mathcal{L} \) is a regular language of \( S^*_\pm \). Since regular languages of \( S^*_\pm \) form a Boolean algebra, it follows that \( \tilde{\mathcal{R}} = \mathcal{L} \setminus \tilde{\mathcal{R}} \) is
a regular language, it is quasi-isometrically embedded, and since \( \pi(L) = \Gamma \), it follows that 
\[
\pi(R') = \Gamma \setminus \mathcal{R}.
\]

**Proof of Proposition 9.4.** Let \( \mathcal{R}_0 \subset QG_{\lambda_0, \mu_0}(S^*_\pm) \) be a quasi-isometrically embedded regular language representing \( \mathcal{R} \). Consider a finite monoid \( \mathcal{M} \), a morphism \( \rho : S^*_\pm \to \mathcal{M} \), and \( \text{Accept} \subset \mathcal{M} \) such that \( \mathcal{R}_0 = \rho^{-1}(\text{Accept}) \). Let \( \mathcal{P} \) be the language of prefixes of \( \mathcal{R}_0 \): \( w \in \mathcal{P} \) if there exists \( u \in S^*_\pm \) with \( wu \in \mathcal{R}_0 \). Note that \( \mathcal{P} \) consists of \((\lambda_0, \mu_0)\)-quasi-geodesics, and \( \mathcal{P} = \rho^{-1}(\mathcal{P}) \), where 
\[
P = \{ m \in \mathcal{M} | \exists n \in \mathcal{M}, mn \in \text{Accept} \}.
\]

Write \( L = \mathcal{L}QG_{\nu, \lambda, \mu}(S^*_\pm) \). Let \( \lambda', \mu' \), and \( \eta \) be such that \( \nu \)-local \((\lambda, \mu)\)-quasi-geodesics of \( \Gamma \) are \( \lambda' \), \( \mu' \) quasi-geodesics, and are at Hausdorff distance at most \( \eta \) from any geodesic with the same endpoints. (Those values can be algorithmically computed from \( \nu, \lambda, \mu \).

On the other hand, from the automaton \( A_0 \), one can algorithmically find \((\lambda_0', \mu_0')\) such that all accepted words are \((\lambda_0', \mu_0')\)-quasi-geodesics. Indeed, given any positive \( \nu_1 \) and any \( \lambda_1, \mu_1 \), one can algorithmically check whether all words accepted by \( A_0 \) are \( \nu_1 \)-local \((\lambda_1, \mu_1)\)-quasi-geodesics. We denote by \( \nu(\lambda, \mu) \), \( \lambda'(\lambda, \mu) \), and \( \mu'(\lambda, \mu) \) computable functions such that \( \nu(\lambda, \mu) \)-local \((\lambda, \mu)\)-quasi-geodesics are global \((\lambda', \mu')\)-quasi-geodesics. By checking if \( \mathcal{R}_0 \subset \mathcal{L}QG_{\nu(\lambda, \mu), \lambda'_0, \mu'_0}(\lambda_1, \mu_1) \) for larger and larger values of \( \lambda_1, \mu_1 \), one will finally get a positive answer, from which one can conclude that \( \mathcal{R}_0 \subset \mathcal{L}QG_{\lambda'(\lambda, \mu), \mu'(\lambda, \mu)} \). Compute \( \eta_0 \) such that each \((\lambda_0', \mu_0')\)-quasi-geodesic is at Hausdorff distance at most \( \eta_0 \) from any geodesic with the same endpoints.

For \( h \in \Gamma \), consider \( \sigma_h = \rho(\mathcal{P} \cap \pi^{-1}(h)) \subset \mathcal{M} \) (\( \sigma_h \) can be thought as the set of states of the automaton corresponding to words representing \( h \), and which can be extended to accepted words). Note that \( h \in \mathcal{R} \) if and only if \( \sigma_h \cap \text{Accept} \neq \emptyset \).

Given \( w \in \mathcal{L} \), let \( p_w \) be the path in \( \text{Cay} \Gamma \) corresponding to \( w \). We want to get a local picture of the values of \( \sigma_h \) in the neighbourhood of the endpoint \( g = \pi(w) \). Define \( R = R_0 + \eta_0 + 10\delta \) and \( L = \max\{ (R + 1 + 10\delta + |w'| + \eta) \lambda', |\nu| \} \). Consider \( w_L \) the suffix of length \( L \) of \( w \) (\( w = w_L \) if \( |w| \leq L \)). Let \( q_{w} \) be the suffix of length \( L \) of the path \( g^{-1}p_w \) (\( q_{w} \) is the path ending at \( 1 \) and labelled by \( w_L \)). Let \( N \) be the \( R \)-neighbourhood of \( q_{w} \), and consider \( \Sigma_w : N \to 2^\mathcal{M} \) defined by \( \Sigma_w(h) = \sigma_{gh} \) (this records the values of \( \sigma_h \) in the neighbourhood of the suffix of \( p_w \)). In particular, \( \pi(w) \in \mathcal{R} \) if and only if \( \Sigma_w(1) \cap \text{Accept} \neq \emptyset \). The mapping \( \Phi : w \mapsto (w_L, \Sigma_w) \) clearly takes finitely many values.

**Claim.** For all words \( w \in S^*_\pm \) and all \( s \in S^*_\pm \) with \( w, ws \in \mathcal{L} \), we find that \( \Phi(ws) \) depends only on \( \Phi(w) \) and \( s \).

**Proof.** If \( |w_L| < L \), this is clear as \( \Phi(w) \) then determines \( w = w_L \). Obviously, the \( L \)-suffix of \( ws \) is determined by \( w_L \) and \( s \). For all \( |h| \leq R \), we need to prove that \( \Sigma_w \) and \( w_L \) determine \( \sigma_{gh} \). Write \( w \) as a concatenation \( w = w'_L w_L \) and let \( x_L = \pi(w') \in \Gamma \). Consider \( u \in \mathcal{P} \) be a word with \( \pi(u) = gsh \), and let \( p_u \) be the corresponding path in \( \text{Cay} \Gamma \).

**Fact.** The path \( p_u \) intersects \( B(x_L, R) \).

**Proof.** Consider \( c_u \), \( c_w \) some geodesics joining \( 1 \) to \( gsh \) and \( g \), respectively. The Hausdorff distance between \( c_u \) and \( p_u \) or between \( c_w \) and \( p_w \) is at most \( \eta_0 \) or \( \eta \), respectively. The projection \( x_w \) of \( x_L \) on \( c_w \) satisfies \( d(x_w, x_L) \leq \eta \). Thus, \( d(g, x_w) \geq -\eta + (1/\lambda')L - \mu' \geq R + 1 + 10\delta \) by choice of \( L \). Looking at a comparison tree for the triangle \( 1, g, gsh \), since \( d(g, gsh) \leq R + 1 \), we see that \( x_w \) is \( 10\delta \)-close to some point \( x_u \in c_u \). The projection \( x_u' \) of \( x_u \) on \( p_u \) is \( \eta_0 \) close to \( x_u \) and satisfies \( d(x_u', x_L) \leq \eta_0 + 10\delta + \eta = R \).
We now prove that the fact implies our claim. Given \( u \in \mathcal{P} \) with \( \pi(u) = gsh \), write \( u = u' u'' \) with \( \pi(u') = x' \in B(x, L, R) \), and let \( m' = \rho(u') \). Consider \( y_L = g^{-1} x_L \) the initial point of \( q_w \), and \( y' = g^{-1} x' \in N \). For each \( x' \in B(x, L, R) \), the set of possible \( m' \) is known as it is encoded in \( \sigma_x = \sigma_{y''} \). Since \( u'' \) is a quasi-geodesic between two points at distance at most \( L + 2R + 1 \), it follows that \( |u''| \) is bounded by \( L_{u''} = \lambda(L + 2R + 1 + \mu) \). Since \( u \in \mathcal{P} \), among those quasi-geodesic \( u'' \), one should consider only those such that \( m' \rho(u'') \in P \), that is, \( u'' \in \rho^{-1}(m'^{-1}P) \). Then \( \sigma_{y''} \) is precisely the set of all possible values of \( m' \rho(u'') \) as \( y' \) varies in \( B(y, L, R) \), \( m' \) varies in \( \sigma_{y''} \), and \( u'' \) varies in the set of words of length at most \( L_{u''} \) in \( \rho^{-1}(m'^{-1}P) \). This proves the claim, and proves, moreover, that \( \Phi(w) \) is algorithmically computable from \( \Phi(w) \) and \( s \).

We now construct a deterministic automaton \( A \) accepting \( \mathcal{L} \cap \pi^{-1}(\mathcal{R}) \). For any individual \( w \in S^*_\Sigma \), note that \( \Phi(w) \) is algorithmically computable since one can enumerate the finitely many \( (N_0, \mu_0) \)-quasi-geodesics with endpoint \( \pi(w) \).

Let \( \mathcal{L}_{\leq L} \) be the set of words of length at most \( L \) in \( \mathcal{L} \). Since the \( R \)-neighbourhood \( N \) of \( q_w \) satisfies \( N \subset B(1, R + L) \), it follows that \( \Sigma \) is a partially defined map on \( B(1, R + L) \), that is, an element of the finite set \( D = (2^L \cup \{ \text{undef} \})^{B(1, R + L)} \). Thus, \( \Phi(\mathcal{L}) \) takes values in the finite set \( \mathcal{F} = \mathcal{L}_L \times D \).

We take \( F \) as the set of states of our automaton, and \( \Phi(1) \) as its initial state. If \( f = (w_L, \Sigma_w) \in F \), with \( |w_L| < L \), then we can discard \( f \) if \( f \neq \Phi(w_L) \). If \( |w_L| < L \) and \( f = \Phi(w_L) \), then for each \( s \in S^*_\Sigma \), then we add an edge joining \( f \) to \( \Phi(w_L s) \). If \( |w_L| = L \), and if \( w_L s \in \mathcal{L} \), then we connect \( (w_L, \Sigma_w) \) to the state \((w', \Sigma') \) determined by \( (w_L, \Sigma_w) \) and \( s \) as in the claim. The set of words \( w \) read by the automaton starting from the initial state is exactly \( \mathcal{L} \). Moreover, the corresponding final state is \( \Phi(w) \) which determines \( \sigma_{\pi(w)} = \Sigma(1) \). We define the set of accepting states of \( A \) as the set of elements \( (w_L, \Sigma_w) \in F \) such that \( \sigma_{\pi(w)} \cap \text{Accept} \neq \emptyset \). Since \( \pi(w) \in \mathcal{R} \) if and only if \( \Sigma_{\pi(s)} \cap \text{Accept} \neq \emptyset \), the language accepted by \( A \) is \( \pi^{-1}(\mathcal{R}) \cap \mathcal{L} \).

Finally, note that the automaton \( A \) can be algorithmically computed from \( A_0 \).

9.3. Canonical representatives in hyperbolic groups with torsion

9.3.1. Canonical sliced cylinders and paths. Let \( \Gamma \) be a hyperbolic group and let \( \text{Cay} \Gamma \) be a Cayley graph. Let \( \delta \) be its hyperbolicity constant. A cylinder, for \( (x, y) \in \Gamma \times \Gamma \), is a subset of the \( 5\delta \)-neighbourhood of a geodesic segment \( [x, y] \) in \( \text{Cay} \Gamma \), containing every such segment. A slicing of a cylinder (or a slice decomposition) is a partitioning into subsets (called slices), with a total ordering of them, such that we have the following.

(i) The union of any two consecutive slices has diameter at most \( 50\delta \).

(ii) For this ordering, the slice containing \( x \) is smaller or equal to the slice containing \( y \).

(iii) Slices move quasi-geodesically: if \( x \) and \( y \) lie, respectively, in the \( i \)th and \( j \)th slice for this ordering, then \( d(x, y) \geq (1/4m_0)|i - j| - 200\delta \), where \( m_0 = \#B(1, 50\delta) \).

In [40], Rips and Sela proved the following theorem (which applies for hyperbolic groups with torsion).

**Theorem 9.7** [40, Corollary 4.3]. Let \( \Gamma \) be a hyperbolic group and let \( \text{Cay} \Gamma \) be a Cayley graph. Then there exists a computable constant \( \kappa \) (controlling the size of the defect \( C_i \) below) such that the following holds.

Let \( X \) be a set of variables and let \( \mathcal{E} \) be a triangular system of equations over \( X \). Let \( y = (g_x)_{x \in X \cup \overline{X}} \in \Gamma^X \) be a solution of \( \mathcal{E} \).

Then, for every \( x \in X \cup \overline{X} \), there exists a cylinder \( \text{Cyl}_x \) for \( (1, g_x) \) together with a slicing, such that we have the following.
(i) For each variable \( x \in X \), \( \text{Cyl}_x = g_x^{-1}\text{Cyl}_x \), their slices are the same but in the reverse order.

(ii) For each equation \( x_1 x_2 x_3 \in \mathcal{E} \), and for each \( i \in \{1, 2, 3 \mod 3\} \) there is a decomposition (depending on the equation) of the cylinder as
\[
\text{Cyl}_{x_i} = L_i \sqcup C_i \sqcup R_i.
\]

Here:
(a) \( L_i, C_i, R_i \) are union of slices with \( L_i < C_i < R_i \) with respect to the ordering;
(b) \( L_{i+1} = g_x^{-1}R_i \), their slices are the same but in the reverse order;
(c) \( C_i \) contains at most \( \kappa \cdot \# \mathcal{E} \) slices.

Note that cylinders and slicings depend only on the variable considered, whereas the decomposition of the cylinder depends on the equation in which the variable appears. The fact that slices move quasi-geodesically is a consequence of their definition in [40]; see [11, Proposition 3.8] for a proof.

The next step consists in interpreting sliced cylinders as paths in a suitable graph \( \mathcal{K} \). Let us consider the Rips' complex \( P_{50\delta}(\Gamma) \) whose set of vertices is \( \Gamma \), and whose simplices are subsets of \( \Gamma \) of diameter at most \( 50\delta \). Let us call \( \mathcal{K} \) the 1-skeleton of its barycentric subdivision. We identify \( \Gamma \) to a subset of the vertices of \( \mathcal{K} \), in the obvious way. For any path \( p : \{0, \ldots, n\} \to \mathcal{K} \), we denote by \( p^{-1} \) the reverse path defined by \( p^{-1}(i) = p(n - i) \). If \( q \) is a path with \( q(0) = p(n) \), then we denote by \( p \cdot q \) the concatenation (Figure 6).

**Proposition 9.8.** Let \( \Gamma \) be a hyperbolic group, let \( \text{Cay} \Gamma \) be a Cayley graph, and let \( \mathcal{K} \) be the 1-skeleton of the barycentric subdivision of the Rips complex of \( \Gamma \) as above. Consider \( \lambda_0 = 400\delta m_0 \), where \( m_0 \) is a bound on the cardinality of balls of radius \( 50\delta \) in \( \text{Cay} \Gamma \), \( \mu_0 = 8 \), and \( \kappa \) is as in Theorem 9.7.

Let \( \mathcal{E} \) be a triangular system of equations over \( X \), let \( g = (g_x)_{x \in X} \in \Gamma^X \) be a solution of \( \mathcal{E} \), and let \( g_x = g_x^{-1} \).

Then, for every \( x \in X \cup \overline{X} \), there exists a \( (\lambda_0, \mu_0) \)-quasi-geodesic path \( p_x \) joining 1 to \( g_x \) in \( \mathcal{K} \) such that we have the following.

(i) For each variable \( x \in X \), we have \( p_x = g_x^{-1}p_x \).

(ii) For each equation \( x_1 x_2 x_3 \in \mathcal{E} \), and for each \( i \in \{1, 2, 3 \mod 3\} \) there is a decomposition (depending on the equation) \( p_{x_i} = l_i \cdot c_i \cdot r_i \), where \( l_{i+1} = g_x^{-1}r_i \), and \( c_i \) has length at most \( 2\kappa \cdot \# \mathcal{E} \).

![Figure 6](image-url) Three cylinders for a solution to \( x_1 x_2 x_3 = 1 \). The subdivision in slices is drawn. The central part, where the cylinders do not coincide, does not contain more than \( 3\kappa \cdot \# \mathcal{E} \) slices.
Proof. Consider sliced cylinders $C_x$ given by Rips and Sela’s Theorem 9.7. Let $S$ and $S'$ be the slices containing 1 and $g_x$, respectively, and let $S = S_1, \ldots, S_n = S'$ be the set of slices between $S$ and $S'$, in increasing order. Since $S_i$ of $C_x$ has diameter at most $50\delta$, it defines a simplex of $P_{50\delta}(\Gamma)$, and thus a vertex $v_i$ of $K$. Similarly, $S_i \cup S_{i+1}$ defines a vertex $u_i$ of $K$ connected by an edge to $v_i$ and $v_{i+1}$, and there is an edge in $K$ joining 1 to $v_1$ and $v_n$ to $g_x$.

We can therefore define $p_x$ as $1, v_1, u_1, v_2, \ldots, u_{n-1}, v_n, g_x$.

Let us explain why these paths are quasi-geodesic; the other properties immediately follow from the properties of canonical cylinders. The fact that slices vary quasi-geodesically says that the distance in Cay $\Gamma$ between the $i$th slice and the $j$th slice is at least $|j - i|/4m_0 - 200\delta$. Thus, the barycentres of the corresponding simplices in $P_{50\delta}(\Gamma)$ are at a distance at least $|j - i|/(200\delta m_0) - 4$ and so $d_K(v_i, v_j) \geq |j - i|/(200\delta m_0) - 4$. Taking into account the fact that $p_x$ takes $2|j - i|$ steps to go from $v_i$ to $v_j$, we get $d_K(p_x(s), p_x(t)) \geq |s - t|/(400\delta m_0) - 4$ for all odd $s, t$. Since $u_i$ is at a distance 1 from $v_i$ and $v_{i+1}$, for all $s, t$, one gets $d_K(p_x(s), p_x(t)) \geq |s - t| - 2)/(400\delta m_0) - 4 - 2 \geq |s - t|/(400\delta m_0) - 8$ as desired. 

9.3.2. The virtually free group $V$ and canonical representatives for $K$. We now define the virtually free group $V$ as a group of paths. We consider homotopy classes of edge paths relative to the endpoints in the graph $K$. A path is reduced if it has no subpath of length 2 passing twice through the same edge. Thus, each homotopy class of paths in $K$ has a unique reduced representative.

Let $V$ be the set of all homotopy classes of edge paths $p$ in $K$ that start at the vertex $1,\Gamma$, and end at a vertex of $\Gamma$. Define $\pi : V \to \Gamma$ by mapping $p$ to its endpoint. We endow $V$ with a group structure by defining $p*p'$ to be the homotopy class of the concatenation $p \cdot (\pi(p)p')$ (where $\pi(p)p'$ is the translate of $p'$ by $\pi(p) \in \Gamma$). Note that $p^{-1} = \pi(p)^{-1}\pi(p)$. It is also clear that $\pi : V \to \Gamma$ is a surjective homomorphism.

Lemma 9.9. The group $V$ is virtually free. More precisely, it is isomorphic to the fundamental group of the finite graph of finite groups defined by $K/\Gamma$, where vertices and edges are marked by copies of stabilizers of their preimages in $K$. In particular, a presentation of $V$ is computable from a presentation of $\Gamma$.

Proof. Let $p : T \to K$ be the universal cover of the graph $K$; it is a tree. With the choice of a preimage $v_0 \in T$ of $1,\Gamma$, the group $V$ admits a natural action on $T$, as we explain now. Any path $p$ in $K$ starting at $1,\Gamma$ has a unique lift $\tilde{p}$ in $T$ starting at $v_0$. On the other hand, for any point $x \in T$, the path $[v_0, x] \subset T$ defines a path $q_x$ by projection to $K$. Then one can define $p.x$ as the endpoint of the lift starting at $v_0$ of $p \cdot (\pi(p)q_x)$. It is easy to check that $p : T \to K$ is $\pi$-equivariant. In particular, $T/V \simeq K/\Gamma$ is a finite graph.

Note that $\pi(1) \subset V$ is the kernel of $\pi$ and acts freely on $T$.

To show that $V$ is virtually free, we prove that the stabilizers of vertices of $T$ are finite. Let $V_x \subset V$ be the stabilizer of a vertex $x \in T$. Then $\pi(V_x)$ stabilizes $p(x) \in K$. Since $\ker \pi$ acts freely on $T$, it follows that $\pi|_{V_x}$ is injective. This proves the finiteness of vertex stabilizers.

To prove the computability of vertex stabilizers, we prove that $\pi$ induces an isomorphism between $V_x$ and the $\Gamma$ stabilizer of $p(x)$, which is computable. We need to prove that any $g \in \Gamma$ fixing $p(x)$ lies in $\pi(V_x)$. Consider the path $q' = gq_x$ joining $g$ to $p(x)$, and $p = q_x \cdot q'$ joining 1 to $g$. Then $p.x$ is the endpoint of the lift of $p \cdot (gq_x) = q_x \cdot q' \cdot (gq_x) = q_x$, and so $p.x = x$.

A similar argument shows that $\pi$ induces an isomorphism between the edges stabilizers of $T$ and $K$, and so one can compute a finite graph of finite groups representing $V$. 

\[Q.E.D.\]
9.4. *Lifting equations to V*

Consider \((\lambda_1, \mu_1) = (\lambda_0, \mu_0 + 2 + 2/\lambda_0)\), so that any concatenation of a \((\lambda_0, \mu_0)\)-quasi-geodesic with a path of length 1 at each extremity is a \((\lambda_1, \mu_1)\)-quasi-geodesic. Let \(\mathcal{QG}_{\lambda_1, \mu_1}(V) \subset V\) be the set of elements such that the corresponding reduced path in \(\mathcal{K}\) is \((\lambda_1, \mu_1)\)-quasi-geodesic. Denote by \(V_{\leq L}\) the set of elements of \(V\) whose corresponding reduced path in \(\mathcal{K}\) has length at most \(L\).

Interpreting the paths occurring in Proposition 9.8 in terms of elements of \(V\), we get the following proposition.

**Proposition 9.10.** Consider a system of equations in \(\Gamma\) as in Proposition 9.8, and a solution \((g_x)_{x \in X} \in \Gamma^X\). Let \(\kappa = 2 \kappa_{\#E} + 2\).

Then, for each variable \(x \in X \cup \overline{X}\), there exist \(\tilde{g}_x \in \mathcal{QG}_{\lambda_1, \mu_1}(V)\) with \(\tilde{g}_x = (\tilde{g}_x)^{-1}\) and \(\pi(\tilde{g}_x) = g_x\), and for each equation \(\varepsilon \in \mathcal{E}\) representing the equation \(x_1x_2x_3 = 1\) and for each \(i \in \{1, 2, 3 \text{ mod } 3\}\), there exist \(\tilde{l}_{\varepsilon,i} \in \mathcal{QG}_{\lambda_1, \mu_1}(V)\) and \(\tilde{c}_{\varepsilon,i} \in V_{\leq \kappa_1}\), such that:

(i) \(\tilde{g}_x = \tilde{l}_{\varepsilon,i} \cdot \tilde{c}_{\varepsilon,i} \cdot \tilde{l}_{\varepsilon,i+1}^{-1}\) in \(V\);

(ii) \(\pi(\tilde{c}_{\varepsilon,1} \cdot \tilde{c}_{\varepsilon,2} \cdot \tilde{c}_{\varepsilon,3}) = 1\) in \(\Gamma\).

Conversely, given any family of elements of \(V\) \((\tilde{g}_x)_{x \in X \cup \overline{X}}\), \((\tilde{l}_{\varepsilon,i})_{\varepsilon \in \mathcal{E}, i = 1, 2, 3}\), and \((\tilde{c}_{\varepsilon,i})_{\varepsilon \in \mathcal{E}, i = 1, 2, 3}\) satisfying \(g_x = \tilde{g}_x^{-1}\), (i) and (ii), respectively, the family \(g_x = \pi(\tilde{g}_x)\) is a solution of \(\mathcal{E}\).

**Proof.** The converse implication is obvious as for each equation \(\varepsilon\) written as \(x_1x_2x_3 = 1\), we find that \(\pi(\tilde{g}_x)\) simplifies to \(\pi(\tilde{c}_{\varepsilon,1} \cdot \tilde{c}_{\varepsilon,2} \cdot \tilde{c}_{\varepsilon,3}) = 1\).

For the direct implication, apply Proposition 9.8 to get a path \(p_x\) for each \(x \in X \cup \overline{X}\), and for each equation \(\varepsilon\) written as \(x_1x_2x_3 = 1\), and each \(i \in \{1, 2, 3 \text{ mod } 3\}\), three paths \(l_{\varepsilon,i}, c_{\varepsilon,i}, r_{\varepsilon,i}\).

Since \(p_x\) joins 1 to \(g_x\), it represents an element \(\tilde{g}_x = p_x \in V\), and \(\pi(\tilde{g}_x) = g_x\). On the other hand \(l_{\varepsilon,i}\) may fail to end at a vertex of \(\Gamma\), and so consider \(e_{\varepsilon,i}\) an edge of \(\mathcal{K}\) joining the endpoint of \(l_{\varepsilon,i}\) to a point in \(l_{\varepsilon,i}\Gamma\). This allows us to define an element of \(V\) by \(\tilde{l}_{\varepsilon,i} = l_{\varepsilon,i} \cdot e_{\varepsilon,i}\).

Since \(l_{\varepsilon,i+1} = g_x^{-1} r_{\varepsilon,i}\), the edge \(g_x e_{\varepsilon,i+1}\) joins the initial point of \(r_{\varepsilon,i}\) to \(g_x l_{\varepsilon,i+1} \in \Gamma\); we denote it by \(e'_{\varepsilon,i}\) (see Figure 7). Therefore, one can define the following elements of \(V\): \(\tilde{c}_{\varepsilon,i} = l_{\varepsilon,i}^{-1} (\tau_{\varepsilon,i} \cdot c_{\varepsilon,i} \cdot e'_{\varepsilon,i})\) and \(\tilde{r}_{\varepsilon,i} = (g_x l_{\varepsilon,i+1})^{-1} (\tau_{\varepsilon,i} \cdot r_{\varepsilon,i})\).

By construction, \(\tilde{g}_x = \tilde{l}_{\varepsilon,i} \cdot \tilde{c}_{\varepsilon,i} \cdot \tilde{r}_{\varepsilon,i}\). Since \(l_{\varepsilon,i+1} = g_x^{-1} r_{\varepsilon,i}\), we get \(\tilde{l}_{\varepsilon,i+1} = l_{\varepsilon,i+1} \cdot e_{\varepsilon,i+1} = g_x^{-1} (r_{\varepsilon,i} \cdot e'_{\varepsilon,i}) = \tilde{r}_{\varepsilon,i}^{-1}\). The fact that \(\pi(\tilde{c}_{\varepsilon,1} \cdot \tilde{c}_{\varepsilon,2} \cdot \tilde{c}_{\varepsilon,3}) = 1\) then follows from \(\pi(\tilde{g}_x \tilde{g}_x \tilde{g}_x) = 1\).

The fact that \(l \in \mathcal{QG}_{\lambda_1, \mu_1}(V)\) and \(\tilde{c} \in V_{\leq \kappa_1}\) is clear from the analogous facts satisfied by \(l\) and \(c\), respectively.

\[\square\]
Note that statement (i) of Proposition 9.10 is an equation in $V$ while (ii) takes place in $\Gamma$. But since each $\tilde{c}_{\varepsilon,i}$ is short, there are only finitely many possibilities, and for each value of $\tilde{c}_{\varepsilon,i}$ satisfying (ii), one can think of (i) as a system of equations in $V$ with variables $\tilde{l}_{\varepsilon,i}$ and $\tilde{g}_x$, and constants $\tilde{c}_{\varepsilon,i}$. More formally we have the following definition.

**Definition 9.11.** For each tuple $\tilde{c}$ of elements $\tilde{c}_{\varepsilon,i} \in V_{\leq \kappa_1}$ satisfying (ii), define $\tilde{E}(\tilde{c})$ as the system of equations (i) with unknowns $\tilde{g}_x$ and $\tilde{l}_{\varepsilon,i}$.

For each solution of $\tilde{E}(\tilde{c})$, its projection $(\tilde{g}_x) = (\pi(\tilde{g}_x)) \in \Gamma^V$ is a solution of $E$. We say that the solution $(\tilde{g}_x, \tilde{l}_{\varepsilon,i})$ is a lift of $(g_x)$ in $V$.

Proposition 9.10 says that every solution of $E$ has a lift in $V$, and more precisely in $Q\mathcal{G}_{\Lambda_1, \mu_1}(V) \subset V$.

9.5. Lifting rational constraints to $V$ and proof of the theorem

Before proving the main theorem of this section, we need to lift the quasi-isometrically embedded rational subsets of $\Gamma$ in $V$. Let $\tilde{S}$ be a generating system of $V$. Let $\mathcal{L}_{\tilde{S}} = Q\mathcal{G}_{\nu, \lambda, \mu} \subset \tilde{S}_+^*$ be rational language of local quasi-geodesics for the metric $||\cdot||_{\tilde{S}}$ on $\Gamma$. Let $\mathcal{L}_V$ be its image in $V$. Given $\lambda_1$ and $\mu_1$ as in the previous section, one can compute $\lambda$ and $\mu$ large enough so that $\mathcal{L}_V$ contains $Q\mathcal{G}_{\Lambda_1, \mu_1}(V)$.

**Lemma 9.12.** Let $\mathcal{R}$ be any quasi-isometrically embeddable rational subset of a hyperbolic group $\mathcal{G}$. Consider $V$, $\pi : V \rightarrow \mathcal{G}$, and $Q\mathcal{G}_{\Lambda_1, \mu_1}(V) \subset \mathcal{L}_V \subset V$ as above.

Then $\tilde{\mathcal{R}}_V = \mathcal{L}_V \cap \pi^{-1}(\mathcal{R})$ is a rational subset of $V$, and $\pi(\tilde{\mathcal{R}}_V) = \mathcal{R}$.

**Proof.** By Proposition 9.4, the language $\tilde{\mathcal{R}}_{\tilde{S}} = \pi^{-1}(\mathcal{R}) \cap \mathcal{L}_{\tilde{S}} \subset \tilde{S}_+^*$ is regular. Since $\pi_{\tilde{S}} : \tilde{S}_+^* \rightarrow \mathcal{G}$ factors through $\pi_V : V \rightarrow \mathcal{G}$, it follows that $\tilde{\mathcal{R}}_V$ is the image in $V$ of $\tilde{\mathcal{R}}_{\tilde{S}}$. Therefore, it is rational.

We are now ready to prove that one can decide algorithmically whether a system of equations with quasi-isometrically embeddable rational constraints has a solution.

**Proof of Theorem 1.** Since $\Gamma \setminus \{1\}$ is a quasi-isometrically embeddable rational subset by Corollary 9.6, and since constants can be encoded by rational constraints consisting of a single element, we can reduce to the case of a system of equations $E$ with quasi-isometrically embeddable rational constraints $\mathcal{R}_x \subset \Gamma$ (without constants).

For each $x \in X$, consider $\tilde{\mathcal{R}}_x = \mathcal{L}_V \cap \pi^{-1}(\mathcal{R}_x) \subset V$, which is rational by Lemma 9.12. Compute $\kappa_1$ and enumerate all tuples $\tilde{c}$ in $V_{\leq \kappa_1}$ satisfying $\pi(\tilde{c}_{\varepsilon,1} \tilde{c}_{\varepsilon,2} \tilde{c}_{\varepsilon,3}) = 1$ (this uses a solution of the word problem in $\Gamma$). Consider the lifted system of equations $\tilde{E}(\tilde{c})$ in $V$ as in Definition 9.11, and add the rational constraint $\tilde{g}_x \in \tilde{\mathcal{R}}_x$. Denote by $\tilde{\mathcal{R}}$ the corresponding set of rational constraints. For each $\tilde{c}$, use the algorithm of Theorem 3 for virtually free groups to decide whether $(\tilde{E}(\tilde{c}), \tilde{\mathcal{R}})$ has a solution. We claim that $(E, \mathcal{R})$ has a solution if and only if $(\tilde{E}(\tilde{c}), \tilde{\mathcal{R}})$ has a solution for at least one $\tilde{c}$.

If $(\tilde{E}(\tilde{c}), \tilde{\mathcal{R}})$ has a solution, then $\pi(\tilde{g}_x)$ is a solution of $E$ by Proposition 9.10, and since $\tilde{g}_x \in \tilde{\mathcal{R}}_x$, it follows that $\pi(\tilde{g}_x) \in \mathcal{R}_x$, and so the constraints are satisfied.

Conversely, if $(E, \mathcal{R})$ has a solution $(g_x)$, then, by Proposition 9.10, $\tilde{E}(\tilde{c})$ has a solution lying in $Q\mathcal{G}_{\Lambda_1, \mu_1}(V) \subset \mathcal{L}_V$ for some tuple $\tilde{c}$. Since the solution satisfies the rational constraint $g_x \in \mathcal{R}_x$, the corresponding solution of $\tilde{E}(\tilde{c})$ satisfies $\tilde{g}_x \in \mathcal{L}_V \cap \pi^{-1}(\mathcal{R}_x) = \tilde{\mathcal{R}}_x$, and hence this solution of $\tilde{E}(\tilde{c})$ satisfies the rational constraint $\tilde{\mathcal{R}}$. \qed
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