Phase structure of 3D $Z(N)$ lattice gauge theories at finite temperature

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Abstract

We perform a numerical study of the phase transitions in three-dimensional $Z(N)$ lattice gauge theories at finite temperature for $N > 4$. Using the dual formulation of the models and a cluster algorithm we locate the position of the critical points and study the critical behavior across both phase transitions in details. In particular, we determine various critical indices, compute the average action and the specific heat. Our results are consistent with the two transitions being of infinite order. Furthermore, they belong to the universality class of two-dimensional $Z(N)$ vector spin models.

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1 Introduction

The deconfinement phase transition in finite-temperature lattice gauge theories (LGTs) has been one of the main subjects of investigation for the last three decades. By now it is well studied and understood for a number of pure gauge models in dimensions \( D = 3, 4 \). In particular, the phase structure of a finite-temperature three-dimensional (3D) pure \( SU(N) \) LGT with the standard Wilson action is thoroughly investigated both for \( N = 2, 3 \) and for the large-\( N \) limit (see, e.g., \([1]\) and references therein). The transition is second order for \( N = 2, 3 \) and first order for \( N > 4 \). In the case of the \( SU(4) \) gauge group, most works agree that the transition is weakly first order. The deconfining transition in \( SU(N = 2, 3) \) LGTs belongs to the universality class of 2D \( Z(N = 2, 3) \) Potts models.

All these phase transitions are characterized by the spontaneous symmetry breaking of a \( Z(N) \) global symmetry of the lattice action in the high-temperature deconfining phase.

Another interesting set of models is represented by abelian \( Z(N) \) LGTs. \( Z(N) \) is the center subgroup of \( SU(N) \), hence \( Z(N) \) LGT can provide useful insights into the universal properties of \( SU(N) \) models. Moreover, \( Z(N) \) LGTs are interesting on their own right and might possess an even richer phase structure as will be revealed below. The most general action for the \( Z(N) \) LGT can be written as

\[
S_{\text{gauge}} = \sum_x \sum_{n<m} \sum_{k=1}^N \beta_k \cos \left( \frac{2\pi k}{N} (s_n(x) + s_m(x + e_n) - s_n(x + e_m) - s_m(x)) \right). \tag{1}
\]

Gauge fields are defined on links of the lattice and take on values \( s_n(x) = 0, 1, \cdots, N-1 \). \( Z(N) \) gauge models, similarly to their spin cousins, can generally be divided into two classes - the standard Potts models and the vector models. The standard gauge Potts model corresponds to the choice when all \( \beta_k \) are equal. Then, the sum over \( k \) in (1) reduces to a delta-function on the \( Z(N) \) group. The conventional vector model corresponds to \( \beta_k = 0 \) for all \( k > 1 \). For \( N = 2, 3 \) the Potts and vector models are equivalent.

The study of the phase structure of \( Z(N) \) LGTs at zero temperature has a long history. While the phase structure of the general model defined by (1) remains unknown, it is well established that the Potts models and vector models with only \( \beta_1 \) non-vanishing have one phase transition from a confining phase to a phase with vanishing string tension \([2, 3, 4]\). Via duality, \( Z(N) \) gauge models can be exactly related to 3D \( Z(N) \) spin models. In particular, a Potts gauge theory is mapped to a Potts spin model, and such a relation allows to establish the order of the phase transition. Hence, Potts LGTs with \( N = 2 \) have second order phase transition, while for \( N \geq 3 \) one finds a first order phase transition. Vector models have been studied numerically in \([5]\) up to \( N = 20 \). It was confirmed that the zero-temperature models possess a single phase transition which disappears in the limit \( N \to \infty \). Thus, the \( U(1) \) LGT has a single confined phase in agreement with
theoretical results [6]. A scaling formula for the critical coupling with \( N \) had also been proposed in [5]. We are not aware, however, of any detailed study of the critical behavior of the vector models with \( N > 4 \) in the vicinity of this single phase transition.

The deconfinement phase transition at finite temperature is well understood and studied for \( N = 2, 3 \). An especially detailed study was performed on the gauge Ising model, \( N = 2 \), in [7]. These models belong to the universality class of 2D \( Z(N) \) spin models and exhibit a second order phase transition in agreement with the Svetitsky-Yaffe conjecture [8]. One should expect on general grounds that the gauge Potts models possess a first order phase transition for all \( N > 4 \), similarly to 2D Potts models. The \( Z(4) \) vector model has been simulated, e.g., in [9]. It also belongs to the universality class of the 2D \( Z(4) \) spin model and exhibits a second order transition.

Much less is known about the finite-temperature deconfinement transition for the vector \( Z(N) \) LGTs when \( N > 4 \). It is the ultimate goal of the present work to deepen our understanding of the phase structure of these models. The Svetitsky-Yaffe conjecture is known to connect critical properties of 3D \( Z(N) \) LGTs at finite temperature with the corresponding properties of 2D spin models, if they share the same global symmetry of the action. It is widely expected, and in many cases proved by either analytical or numerical methods, that some 2D \( Z(N > 4) \) spin models (like the vector Potts model) have two phase transitions of infinite order, known as the Berezinskii-Kosterlitz-Thouless (BKT) phase transitions. According to the conjecture, the phase transitions in some 3D \( Z(N > 4) \) gauge models at finite temperature could exhibit two phase transitions as well. Moreover, if the correlation length diverges when approaching the critical point, these transitions should be of the BKT type and belong to the universality class of the corresponding 2D \( Z(N) \) spin models.

The BKT phase transition has been best studied in the 2D \( XY \) model [10, 11, 12]. Certain analytical [8, 13, 14] as well as numerical results [15, 16] unambiguously indicate that the deconfining phase transition in the 3D \( U(1) \) LGT is also of infinite order and might belong to the universality class of 2D \( XY \) model [1].

In recent papers [17, 18] we have initiated exploring the phase structure of the vector \( Z(N) \) LGT for \( N > 4 \). More precisely, we have considered an anisotropic lattice in the limit where the spatial coupling vanishes. In this limit the spatial gauge fields can be exactly integrated out and one gets a 2D generalized \( Z(N) \) model. The Polyakov loops play the role of \( Z(N) \) spins in this model. For the Villain version of the model obtained we have been able to present renormalization group arguments indicating the existence of two BKT-like phase transitions. This scenario was confirmed with the help of large-scale Monte Carlo simulations of the effective model. We have also computed some critical

\[ \text{It should be noted, however, that the numerical results of [16] point to a critical index } \eta \text{ larger than its } XY \text{ value by almost a factor of } 2 \text{ for } N_t = 8. \text{ Therefore, the question of the universality remains open for this model.} \]
indices which appear to agree with the corresponding indices of $2D \ Z(N)$ spin models, thus giving further support to the Svetitsky-Yaffe conjecture.

In this paper we extend our analysis to the full isotropic $3D \ Z(N)$ LGT at finite temperature. It is well known that the full phase structure of a finite-temperature LGT is correctly reproduced in the limit where spatial plaquettes are neglected. They have probably small influence on the dynamics of the Polyakov loop interaction. We therefore expect that the scenario advocated by us in [17] remains qualitatively correct for the full theory. In particular, full gauge models with $N > 4$ should possess two phase transitions of the BKT type and we expect the values of critical indices to coincide with the indices of the $2D$ vector spin models.

The $2D \ Z(N)$ spin model in the Villain formulation has been studied analytically in Refs. [4, 19, 20, 21, 22, 23]. It was shown that the model has at least two phase transitions when $N \geq 5$. The intermediate phase is a massless phase with power-like decay of the correlation function. It turns out that $\eta(\beta_c^{(1)}) = 1/4$ at the transition point from the strong coupling high-temperature phase to the massless phase, i.e. the behavior is similar to that of the $XY$ model. At the transition point $\beta_c^{(2)}$ from the massless phase to the ordered low-temperature phase one has $\eta(\beta_c^{(2)}) = 4/N^2$. A rigorous proof that the BKT phase transition does take place, and so that the massless phase exists, has been constructed in Ref. [24] for both Villain and standard formulations of the vector model. Universality properties of vector models were studied via Monte Carlo simulations in Ref. [25] for $N = 6, 8, 12$ and in Refs. [26, 27, 28, 29] for $N = 5, 7, 17$. Results for the critical indices $\eta$ and $\nu$ agree with analytical predictions obtained for the Villain formulation of the model.

A similar phase structure is expected to hold for the finite-temperature $3D \ Z(N > 4)$ LGT. It can be described in terms of the Polyakov loop correlation functions as follows.

The low-temperature phase is a confining phase. Here, the correlation decays with an area law, thus implying a non-vanishing string tension and a linear potential between static $Z(N)$ charges. With the temperature increasing, the system undergoes a phase transition to a massless phase. This phase is characterized by the enhancement of the $Z(N)$ global symmetry to a $U(1)$ symmetry and is very close in nature to the high-temperature phase of $U(1)$ gauge model. In particular, the dominating contribution to the correlation of the Polyakov loops comes from massless excitations (dual spin-waves). This is nevertheless a confinement phase, as the correlation decays with a power law and though the string tension vanishes, the potential between $Z(N)$ charges is logarithmic. Increasing the temperature further on leads to a spontaneous breaking of the $Z(N)$ global symmetry at some critical point. One enters a deconfining phase above this critical point.

Here we intend to:

- check the scenario of two BKT phase transitions described above;
- compute some critical indices at both transitions and verify the universality class of
The fact that the BKT transition has infinite order makes it hard to study its properties using analytical methods. In most of the cases studied so far one uses a renormalization group (RG) technique as in Ref. [19]. Unfortunately, there are no direct ways to generalize the transformations of Ref. [19], leading to RG equations, to 3D $Z(N)$ LGTs, except for the limiting case $N \to \infty$. To study the phase structure of these models we need numerical simulations. Here, however, another problem appears related to the very slow, logarithmic convergence to the thermodynamic limit in the vicinity of the BKT transition. It is thus necessary to use both large-scale simulations and combine them with the finite-size scaling methods. Our principal strategy consists in passing to a dual formulation of the 3D $Z(N)$ vector LGT, which is known to be a generalized 3D $Z(N)$ vector model. This allows us to use a cluster algorithm in our simulations. The standard procedure in studying the phase structure is to use Binder cumulants to locate the position of critical points. Then, critical indices can be determined from various susceptibilities. In the case of a deconfinement phase transition both Binder cumulants and susceptibilities are usually constructed from Polyakov loops. However, it is a non-trivial problem to write down the expression for a single Polyakov loop in a dual formulation (though the dual form can be easily found for invariant quantities, like the correlation of Polyakov loops). We have therefore decided to study the critical behavior making use of Binder cumulants and susceptibilities constructed from the dual $Z(N)$ spins. This procedure exhibits an interesting phenomenon, namely the critical behavior of dual spins is reversed with respect to the critical behavior of Polyakov loops: the spontaneously-broken ordered phase is mapped to the symmetric phase and vice versa. Moreover, the critical indices $\eta$ are also interchanged as will be explained later on. The index $\nu$ which governs the exponential divergence of the correlation length is expected to be the same at both transitions and takes on the value $\nu = 1/2$.

The lowest number of $N$ where two BKT phase transitions are expected is $N = 5$. In the present paper we study the phase transitions in models with $N = 5, 13$ in great detail. We have studied these values of $N$ in the strong coupling regime [17, 18], therefore it is natural to continue working with these models. Our computations are performed on lattices with temporal extent $N_t = 2, 4$ and with spatial size in the range $L \in [32 \text{–} 1024]$.

This paper is organized as follows. In Section 2 we formulate our model and establish the exact relation with a generalized 3D $Z(N)$ spin model. Here, we also explain why critical indices, determined from the dual spin correlation functions, become $\eta(\beta^{(1)}) = 4/N^2$ and $\eta(\beta^{(2)}) = 1/4$, i.e. the values are interchanged with the respect to what expected

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2The use of the Polyakov loop correlations to directly extract critical indices requires, in case of an essential singularity occurring at the BKT transition, prohibitively huge lattices, which are not accessible with the present numerical facilities.
from the correlation of original degrees of freedom. In Section 3 we present the setup of Monte Carlo simulations, define the observables used in this work and present the numerical results of simulations. In particular, we locate the position of critical points and compute various critical indices at these points. As a further cross-check of the nature of the phase transitions, we also present results of a few simulations for the $Z(5)$ gauge model and compute the average action and the specific heat in the vicinity of critical points. Our conclusions and perspectives are given in Section 5.

2 Relation of the $3D$ $Z(N)$ LGT to a generalized $3D$ $Z(N)$ spin model

2.1 Partition and correlation functions

We work on a $3D$ lattice $\Lambda = L^2 \times N_t$ with spatial extension $L$ and temporal extension $N_t$; $\vec{x} = (x_0, x_1, x_2)$, where $x_0 \in [0, N_t - 1]$ and $x_1, x_2 \in [0, L - 1]$ denote the sites of the lattice and $e_n$, $n = 0, 1, 2$, denotes a unit vector in the $n$-th direction. Periodic boundary conditions (BC) on gauge fields are imposed in all directions. The notations $p_t$ ($p_s$) stand for the temporal (spatial) plaquettes, $l_t$ ($l_s$) for the temporal (spatial) links.

We introduce conventional plaquette angles $s(p)$ as

$$s(p) = s_n(x) + s_m(x + e_n) - s_n(x + e_m) - s_m(x).$$

The $3D$ $Z(N)$ gauge theory on an anisotropic lattice can generally be defined as

$$Z(\Lambda; \beta_t, \beta_s; N) = \prod_{l \in \Lambda} \left( \frac{1}{N} \sum_{s(l)=0}^{N-1} \prod_{p_s} Q(s(p_s)) \prod_{p_t} Q(s(p_t)) \right).$$

The most general $Z(N)$-invariant Boltzmann weight with $N - 1$ different couplings is

$$Q(s) = \exp \left[ \sum_{k=1}^{N-1} \beta_p(k) \cos \left( \frac{2\pi k}{N} s \right) \right].$$

The Wilson action corresponds to the choice $\beta_p(1) = \beta_p$, $\beta_p(k) = 0$, $k = 2, ..., N - 1$. The $U(1)$ gauge model is defined as the limit $N \to \infty$ of the above expressions.

In what follows we work with the conventional Wilson action on an isotropic lattice, i.e. $\beta_s = \beta_t = \beta$. To study the phase structure of $3D$ $Z(N)$ LGTs one can map the gauge model to a generalized $3D$ spin $Z(N)$ model with the action

$$S = \sum_x \sum_{n=1}^{3} \sum_{k=1}^{N-1} \beta_k \cos \left( \frac{2\pi k}{N} (s(x) - s(x + e_n)) \right).$$
The effective coupling constants $\beta_k$ can be computed exactly as follows. The first step is to construct a dual form for the partition function (3). Details can be found, e.g., in [3]. One gets on the dual lattice $\Lambda_d$ the following expression for the partition function

$$Z(\Lambda_d; \beta; N) = \prod_{x \in \Lambda_d} \left( \frac{1}{N} \sum_{s(x)=0}^{N-1} \right) \prod_{l \in \Lambda_d} Q_d(s(x) - s(x + e_n)) ,$$

(6)

where the dual Boltzmann weight $Q_d(s)$ becomes

$$Q_d(s) = \sum_{r=-\infty}^{\infty} I_{Nr+s}(\beta) = \sum_{p=0}^{N-1} \exp \left[ \beta \cos \left( \frac{2\pi p}{N} \right) \right] \cos \left( \frac{2\pi ps}{N} \right) .$$

(7)

Here, $I_k(x)$ is the modified Bessel function. Exponentiating and re-expanding the dual weight in a new Fourier series one finds $\beta_k$ as

$$\beta_k = \frac{1}{N} \sum_{p=0}^{N-1} \ln \left[ \frac{Q_d(p)}{Q_d(0)} \right] \cos \left( \frac{2\pi pk}{N} \right) .$$

(8)

As example, we give below the expressions for the effective couplings $\beta_k$ for $N = 5$. One obtains from (7) and (8)

$$\beta_1 = \beta_4 = -\frac{1 - \sqrt{5}}{10} \ln [1 - t_+/2] - \frac{1 + \sqrt{5}}{10} \ln [1 - t_-/2] ,$$

$$\beta_2 = \beta_3 = -\frac{1 + \sqrt{5}}{10} \ln [1 - t_+/2] - \frac{1 - \sqrt{5}}{10} \ln [1 - t_-/2] ,$$

(9)

where

$$t_{\pm} = \frac{5 \pm \sqrt{5} + (5 \mp \sqrt{5}) e^{\pm \frac{\sqrt{5}}{2} \beta}}{2 + 2 e^{\pm \frac{\sqrt{5}}{2} \beta} + e^{\pm \frac{5}{2} \sqrt{5} \beta}} .$$

As is seen from Eq. (7), the dual model is ferromagnetic. However, $\beta_2$ is small and negative for $\beta > 1.077$. In the whole region $|\beta_1| \gg |\beta_2|$; moreover, in the critical region $|\beta_2|/|\beta_1| \sim 10^{-2}$. Similar properties hold for all $N$. Thus, one expects that the 3D vector spin model with only $\beta_1$ non-vanishing gives a reasonable approximation to the gauge model (in our simulations we use all $\beta_k$). Next important fact, evident from Eq. (9),

In writing this expression we have neglected a certain global summation which appears due to the fact that the product of the plaquette variables around the closed 2d surface winding through the lattice in periodic directions is unity (so-called global Bianchi identity). Such global identities can be safely omitted since they do not influence the quantities of our interest in thermodynamic limit. Note, this is not the case for quantities like the twist free energy.
is that the weak and the strong coupling regimes are interchanged: when $\beta \to \infty$ both effective couplings $\beta_k \to 0$ and, therefore, the ordered symmetry-broken phase is mapped to a symmetric phase with vanishing magnetization of dual spins. The symmetric phase at small $\beta$ becomes an ordered phase where the dual magnetization is non-zero.

Let $W(x) = \prod_{x_0=0}^{N_t-1} \exp(\frac{2\pi i}{N} j(x))$ be the Polyakov loop in the representation $j$. The correlation function of Polyakov loops can be expressed in the dual form as

$$ P_j(R; \beta; N) = Z(\Lambda_d; \beta; N)^{-1} \prod_{x \in \Lambda_d} \left( \frac{1}{N} \sum_{s(x)=0}^{N-1} \right) \prod_{l \in \Lambda_d} Q_d(s(x) - s(x + e_n) + h(l)) . \quad (10) $$

Here we have introduced the sources $h(l)$ as

$$ h(l) = \begin{cases} j, & l \in S_d , \ l = (x,n) \\ -j, & l \in S_d , \ l = (x-e_n,n) \\ 0, & \text{otherwise} \end{cases} \quad (11) $$

where $S_d$ is the dual surface enclosed between two Polyakov loops, i.e. it consists of links perpendicular to plaquettes of the original lattice and is closed in the temporal direction.

The correlation function of the dual spins is defined in standard way as

$$ \Gamma_j(R; \beta; N) = \left\langle \exp \left[ \frac{2\pi i}{N} j(s(0) - s(R)) \right] \right\rangle \quad . \quad (12) $$

In the original (gauge) formulation this correlation function corresponds to a disorder operator which is the ’t Hooft line in the 3D theory. It consists of a string of plaquettes connecting the cubes dual to the points 0 and $R$ and measures the free energy of a $Z(N)$ monopole–anti-monopole pair.

### 2.2 Behavior of the dual correlation function

As was explained in the Introduction, our aim is to study the critical behavior making use of quantities, like the Binder cumulants, constructed from the dual $Z(N)$ spins. On very general grounds one expects that in the confined phase, where the potential between electric $Z(N)$ charges grows linearly with the distance, the correlation function (12) shows ordered behavior, so that the magnetic charges are free. In the phase where the electric charges are deconfined, the correlation function (12) exhibits exponential decay and the free energy of a monopole–anti-monopole pair grows with distance. In the massless phase, if such phase exists, one anticipates that both kind of charges are confined by a weakly-growing logarithmic potential, hence both correlations decrease with a power law. It is not
obvious, however, and to the best of our knowledge not proved so far, that the correlation of the Polyakov loops and the dual correlation show similar critical behavior. In fact, as we shall argue below, their critical behavior is exactly opposite. Let us suppose, there exists a massless phase and $\beta_c^{(1)}$ is the phase transition point from the electric confinement to the massless phase, while $\beta_c^{(2)}$ is the phase transition point from the massless phase to the deconfined phase. One then has for the Polyakov loop correlation the following asymptotic behavior at the critical points (for $j = 1$)

$$P_j(R; \beta_c^{(1)}; N) \asymp \frac{1}{R \eta^{(1)}}, \quad P_j(R; \beta_c^{(2)}; N) \asymp \frac{1}{R \eta^{(2)}}. \quad (13)$$

In the case of the dual correlations, the $\eta$ indices are interchanged

$$\Gamma_j(R; \beta_c^{(1)}; N) \asymp \frac{1}{R \eta^{(2)}}, \quad \Gamma_j(R; \beta_c^{(2)}; N) \asymp \frac{1}{R \eta^{(1)}}. \quad (14)$$

If the 3D $Z(N > 4)$ vector gauge model belongs to the universality class of the corresponding 2D $Z(N)$ spin model, we should find

$$\eta^{(1)} = 1/4, \quad \eta^{(2)} = 4/N^2. \quad (15)$$

For $N$ fixed and $N_t$ increasing, $\beta_c^{(1)} \to \beta_c^{(2)}$. In the limit $N_t \to \infty$ one ends up with a single phase transition from the confining to the deconfining phase. When $N_t$ is fixed and $N$ increases, $\beta_c^{(2)}$ diverges roughly as $N^2$, while $\beta_c^{(1)}$ approaches the critical point of the finite-temperature $U(1)$ LGT exponentially fast.

The direct proof of all these properties would include the construction of RG equations describing the behavior of the system in the vicinity of phase transitions, something we could not accomplish so far. In the absence of such a proof, we give qualitative arguments why the behavior (13)-(14) is rather natural and might be anticipated. First of all, the interchange of the critical indices can be easily seen in the 2D $Z(N)$ spin models. Indeed, Eq. (10) remains formally correct for the two-point correlation function of these models if one replaces the surface $S_d$ in (11) with a dual path connecting the points 0 and $R$. The two-point correlation of dual spins can be written in precisely the same form of Eq. (10) in the original formulation, with $Q_d(s)$ substituted by $Q(s)$. Let us consider now the Villain formulation. For $Q(s)$ it means

$$Q(s) \to Q^V(s) = \sum_{m=-\infty}^{\infty} \exp \left[ -\frac{1}{2} \beta \frac{4\pi^2}{N^2} (s + Nm)^2 \right], \quad (16)$$

while for $Q_d(s)$ it amounts to replacing the Bessel function in (7) with its asymptotics

$$Q_d(s) \to Q_d^V(s) = \sum_{m=-\infty}^{\infty} \exp \left[ -\frac{1}{2\beta} (s + Nm)^2 \right]. \quad (17)$$
In this formulation the 2D $Z(N)$ model is self-dual. From this fact, from Eq. (10) and its analog for the dual correlation function it follows that (we keep the notation $P_j$ for the correlation function of spins)

$$P_j(R; \beta; N) = \Gamma_j \left( R; \frac{N^2}{4 \pi^2 \beta}; N \right).$$

(18)

It is now straightforward to repeat the calculations of Ref. [19] for the dual correlation function and to get

$$P_j(R; \beta; N) \approx \exp \left[ -\frac{j^2}{2 \pi \beta_{\text{eff}}} \ln R \right], \quad \Gamma_j \left( R; \frac{N^2}{4 \pi^2 \beta}; N \right) \approx \exp \left[ -\frac{2 \pi j^2 \beta_{\text{eff}}}{N^2} \ln R \right].$$

(19)

$\beta_{\text{eff}}$ can be expanded in powers of the self-energies of the topological defects of the model. The leading contribution $\beta_{\text{eff}} \approx \beta$ comes from the spin-wave configurations. At the critical points one has

$$\beta_{\text{eff}}(\beta_{\text{eff}}^{(1)}) = 2/\pi, \quad \beta_{\text{eff}}(\beta_{\text{eff}}^{(2)}) = N^2/(8\pi).$$

(20)

It then follows from Eqs. (19)-(20) that the critical indices are indeed interchanged. 2D $Z(N)$ vector models in the standard formulation are not self-dual, therefore Eq. (18) does not generally hold. Nevertheless, this picture remains qualitatively correct for all formulations of 2D models which belong to the same universality class. Moreover, recalling that $|\beta_1| \gg |\beta_k|$ for all $k \neq 1 \pmod{N}$, one finds the approximate equality $P_j(R; \beta; N) \approx \Gamma_j(R; \beta_1; N)$ for the standard formulation, where $\beta_k$ are the dual coupling constants.

Let us turn now to the 3D $Z(N)$ LGT and use again the Villain formulations, Eqs. (16) and (17) (in the case of the gauge model one has to take the plaquette angle $s(p)$ defined in Eq. (2)). In the spin-wave approximation, which should be valid in the massless phase, we obtain

$$P_j(R; \beta; N) \approx \exp \left[ -\frac{N_t j^2}{\beta} D(R) \right]$$

(21)

for the Polyakov loops correlation function and

$$\Gamma_j(R; \beta; N) \approx \exp \left[ -\frac{4 \pi^2 \beta}{N^2} j^2 G(R) \right]$$

(22)

for the dual correlations. $D(R)$ in Eq. (21) is the two-dimensional Green’s function. $G(R)$ in Eq. (22) is the three-dimensional Green’s function. At finite temperature in the massless phase the dominant contribution to $G(R)$ arises from the temporal zero mode, other modes being massive and exponentially suppressed. We thus have

$$G(R) \approx \frac{1}{N_t} D(R) + \mathcal{O}(e^{-mR}), \quad D(R) \approx \frac{1}{2\pi} \ln R.$$
Combining this with Eq. (21) and Eq. (22) leads to

\[ P_j(R; \beta; N) \approx \exp \left[ -\frac{N_t}{2\pi \beta} j^2 \ln R \right], \quad \Gamma_j(R; \beta; N) \approx \exp \left[ -\frac{2\pi \beta}{N_t N^2} j^2 \ln R \right]. \quad (23) \]

These expressions must be supplemented by computing the corrections from the topological defects of the model, like \( Z(N) \) monopoles. We expect that this results, similarly to 2D models, in replacing \( \beta \to \beta_{\text{eff}} \) in the above equations. It is natural to suppose that the analog of Eq. (20) takes the form

\[ \beta_{\text{eff}}(\beta_c^{(1)}) = \frac{2}{\pi} N_t, \quad \beta_{\text{eff}}(\beta_c^{(2)}) = \frac{N^2}{8\pi} N_t. \quad (24) \]

It corresponds to approximately linear scaling of the critical points with \( N_t \). If so, the critical indices are interchanged as it becomes evident after the substitution of the last equations in (23). As a matter of fact, Eq. (24) is our conjecture which remains to be proved.

3 Numerical results

3.1 Setup of the Monte Carlo simulation

To study the phase transitions we used the cluster algorithm described in [27]. We simulate the dual model defined by Eq. (6) on an \( N_t \times L \times L \) lattice with periodic BC. Simulations were carried out for \( N_t = 2, 4 \). To check directly that the critical indices are interchanged, as explained in the previous section, we have also simulated the \( Z(5) \) gauge model for \( N_t = 2, 4 \) using the heat-bath algorithm. As original action of the gauge model we used the conventional Wilson action. For each Monte Carlo run the typical number of generated configurations was \( 10^6 \), the first \( 10^5 \) of them being discarded to ensure thermalization. Measurements were taken after 10 updatings and error bars were estimated by the jackknife method combined with binning.

We considered the following observables:

- complex magnetization \( M_L = |M_L|e^{i\psi} \),
  \[ M_L = \sum_{x \in \Lambda} \exp \left( \frac{2\pi i}{N} s(x) \right); \quad (25) \]

- population \( S_L \)
  \[ S_L = \frac{N}{N-1} \left( \frac{\max_{i=0,N=1} n_i}{L^2 N_t} - \frac{1}{N} \right), \quad (26) \]

where \( n_i \) is number of \( s(x) \) equal to \( i \).
The variable $s(x)$ appearing in the definitions (25) and (26) represents the dual spin in the case of simulations of the dual model. In the case of the gauge model, $s(x)$ is the Polyakov loop, the sum in Eq. (25) becomes two-dimensional and $N_t$ must be omitted from Eqs. (26) and (27). Also, when simulating the gauge model we have computed the average action and the specific heat in the vicinity of the critical points.

3.2 Determination of the critical couplings

A clear indication of the three-phase structure emerges from the inspection of the scatter plot of the complex magnetization $M_L$ at different values of $\beta$: as we move from low to high $\beta$, we observe the transition from an ordered phase ($N$ isolated spots) through an
intermediate phase (ring distribution) up to the disordered phase (uniform distribution around zero). Fig. 1 shows this three-phase structure for the case of $Z(5)$ on a $512^2 \times 4$ lattice.

The first and most important numerical task is to determine the value of the two critical couplings in the thermodynamic limit, $\beta_c^{(1)}$ and $\beta_c^{(2)}$, that separate the three phases. To this aim we find the value of $\beta_c$ which provides the best overlap of universal observables, plotted for different values of $L$ against $(\beta - \beta_c^{(1)})(\ln L)^{1/\nu}$, with $\nu$ fixed at $1/2$. As these universal observables we used:

- Binder cumulant $B^{(M_R)}_L$ and the order parameter $m_\psi$ for the first phase transition;
- Binder cumulant $U^{(M)}_L$ for the second phase transition.

To localize regions in which the overlap must occur, the approximate maxima of the susceptibilities $\chi^{(S)}_L$ and $\chi^{(M)}_L$ were used.

In Figures 2-4 we give the plots of the universal observables, drawn against $\beta$ and against $(\beta - \beta_c^{(1)})(\ln L)^{1/\nu}$. We report in Table 1 the determinations of the critical couplings $\beta_c^{(1)}$ and $\beta_c^{(2)}$ in $Z(N)$ with $N=5$ and $13$ for $N_t=2$ and 4.

### 3.3 Determination of critical indices at the two transitions

Once critical couplings have been estimated, we are able to extract some critical indices and check the hyperscaling relation.

Since we are using the observables in the dual model the transitions change places: the first transition is governed by the behavior of $M_R$, the second one by the behavior of
Figure 3: Binder cumulant $B_4^{(M_n)}$ as function of $\beta$ (left) and of $(\beta - \beta_c)(\ln L)^{1/\nu}$ (right) in $Z(5) N_t = 2$ model.

Figure 4: $m_\psi$ as function of $\beta$ (left) and of $(\beta - \beta_c)(\ln L)^{1/\nu}$ (right) in $Z(5) N_t = 4$ model.
Table 1: Values of $\beta^{(1)}_c$ and $\beta^{(2)}_c$ obtained for $N_t = 2$ and 4 in $Z(N)$ with $N = 5, 13.$

| $N$ | $N_t$ | $\beta^{(1)}_c$ | $\beta^{(2)}_c$ |
|-----|-------|-----------------|-----------------|
| 5   | 2     | 1.617(2)        | 1.694(2)        |
| 5   | 4     | 1.943(2)        | 1.990(2)        |
| 13  | 2     | 1.795(4)        | 9.699(6)        |
| 13  | 4     | 2.74(5)         | 11.966(7)       |

$M_L$. This could already be seen in the previous Section, since the first critical point was obtained from the $B_{4n}^{(M_R)}$ and the second one from the $U_{L}^{(M)}$ curve collapse.

We start the discussion from the second transition. According to the standard finite-size scaling (FSS) theory, the equilibrium magnetization $|M_L|$ at criticality should obey the relation $|M_L| \sim L^{-\beta/\nu}$, if the spatial extension $L$ of the lattice is large enough. Therefore, we fit data of $|M_L|$ at $\beta^{(2)}_c$, on all lattices with size $L$ not smaller than a given $L_{\text{min}}$, with the scaling law

$$|M_L| = AL^{-\beta/\nu}.$$  

(29)

The FSS behavior of the susceptibility $\chi^{(M)}_L$ is given by $\chi^{(M)}_L \sim L^{\gamma/\nu}$, where $\gamma/\nu = 2 - \eta$ and $\eta$ is the magnetic critical index. Therefore we fit data of $\chi^{(M)}_L$ at $\beta^{(2)}_c$, on all lattices with size $L$ not smaller than a given $L_{\text{min}}$, according to the scaling law

$$\chi^{(M)}_L = AL^{\gamma/\nu}.$$  

(30)

As the value of the critical coupling $\beta^{(2)}_c$ we use the central value determined in the previous Section.

The results of the fits are summarized in Table 2. Each row corresponds to the fit using data from $L = L_{\text{min}}$ up to $L = 1024$. The reference value for the index $\eta$ at this transition is $1/4$, whereas the the hyperscaling relation to be fulfilled is $\gamma/\nu + 2\beta/\nu = d = 2$.

We see that in most cases the values of $\eta$ and $d$ are close to those predicted by universality. The discrepancy from the exact values $\eta = 0.25$ and $d = 2$ may be caused by the asymptotically vanishing parts of the scaling behavior of the observables $|M_L|$ and $\chi^{(M)}_L$, that we are not taking into account, but may be significant for smaller lattice sizes.

The procedure for the determination of the critical indices at the first transition is similar to the one for the second transition, with the difference that the fit with the

\footnote{The symbol $\beta$ here denotes a critical index and not, obviously, the coupling of the theory. In spite of this inconvenient notation, we are confident that no confusion will arise, since it will be always clear from the context which $\beta$ is to be referred to.}
Table 2: Critical indices $\beta/\nu$ and $\gamma/\nu$ for the second transition in $Z(N)$ models, determined by the fits given in Eqs. (29) and (30) on the complex magnetization $M_L$ and its susceptibility $\chi_L^{(M)}$ at $\beta_c^{(2)}$ for different choices of the minimum lattice size $L_{\text{min}}$. The $\chi^2$ of the two fits, given in the columns four and six, is the reduced one.

| model | $L_{\text{min}}$ | $\beta/\nu$ | $\chi^2_{\beta/\nu}$ | $\gamma/\nu$ | $\chi^2_{\gamma/\nu}$ | $d = 2\beta/\nu + \gamma/\nu$ | $\eta = 2 - \gamma/\nu$ |
|---|---|---|---|---|---|---|---|
| $Z(5)$ $N_t = 2$ $\beta_c^{(2)} = 1.694$ | 128 | 0.1226(4) | 0.92 | 1.76(1) | 1.75 | 2.00(1) | 0.24(1) |
| | 192 | 0.1226(6) | 1.15 | 1.75(2) | 2.14 | 2.00(2) | 0.25(2) |
| | 256 | 0.1226(9) | 1.53 | 1.77(2) | 1.51 | 2.02(2) | 0.23(2) |
| | 384 | 0.121(1) | 0.93 | 1.74(2) | 0.78 | 1.98(3) | 0.26(2) |
| | 512 | 0.1230(2) | 0.011 | 1.74(5) | 1.46 | 1.99(5) | 0.26(5) |
| $Z(5)$ $N_t = 4$ $\beta_c^{(2)} = 1.990$ | 32 | 0.1078(2) | 5.17 | 1.69(1) | 51.5 | 1.91(1) | 0.31(1) |
| | 64 | 0.1075(1) | 1.56 | 1.71(1) | 19.6 | 1.93(1) | 0.29(1) |
| | 128 | 0.1074(1) | 1.39 | 1.734(8) | 5.50 | 1.949(8) | 0.266(7) |
| | 192 | 0.1075(2) | 1.67 | 1.744(7) | 2.65 | 1.959(7) | 0.256(7) |
| | 256 | 0.1077(2) | 1.27 | 1.752(6) | 1.40 | 1.968(7) | 0.248(6) |
| | 384 | 0.1076(4) | 1.67 | 1.760(8) | 1.10 | 1.975(9) | 0.240(8) |
| | 512 | 0.1081(5) | 1.15 | 1.77(1) | 1.57 | 1.98(1) | 0.23(1) |
| $Z(13)$ $N_t = 2$ $\beta_c^{(2)} = 9.699$ | 128 | 0.1225(4) | 1.03 | 1.749(8) | 0.428 | 1.994(9) | 0.251(8) |
| | 192 | 0.1225(6) | 1.29 | 1.758(7) | 0.232 | 2.003(8) | 0.242(7) |
| | 256 | 0.1229(8) | 1.49 | 1.749(6) | 0.126 | 1.995(8) | 0.251(6) |
| | 384 | 0.123(1) | 2.20 | 1.741(9) | 0.103 | 1.99(1) | 0.259(9) |
| | 512 | 0.1203(2) | 0.0187 | 1.73(1) | 0.0794 | 1.97(1) | 0.27(1) |
| $Z(13)$ $N_t = 4$ $\beta_c^{(2)} = 11.966$ | 32 | 0.1266(5) | 16.62 | 1.70(1) | 19.96 | 1.95(1) | 0.30(1) |
| | 384 | 0.1275(4) | 5.14 | 1.72(1) | 7.70 | 1.98(1) | 0.28(1) |
| | 128 | 0.1282(2) | 0.54 | 1.747(8) | 1.58 | 2.004(8) | 0.253(8) |
| | 192 | 0.1283(3) | 0.76 | 1.76(1) | 1.54 | 2.01(1) | 0.24(1) |
| | 256 | 0.1282(6) | 1.45 | 1.74(2) | 1.14 | 2.00(2) | 0.26(2) |
Table 3: Critical indices $\beta/\nu$ and $\gamma/\nu$ for the first transition in $Z(N)$ models, determined by the fits given in Eqs. (29) and (30) on the rotated magnetization $M_R$ and its susceptibility $\chi_L^{(M_R)}$ at $\beta^{(1)}_c$ for different choices of the minimum lattice size $L_{\text{min}}$.

| model          | $L_{\text{min}}$ | $\beta/\nu$ | $\chi_{\beta/\nu}^2$ | $\gamma/\nu$ | $\chi_{\gamma/\nu}^2$ | $d = 2\beta/\nu + \gamma/\nu$ | $\eta = 2 - \gamma/\nu$ |
|----------------|------------------|--------------|------------------------|--------------|------------------------|--------------------------------|--------------------------|
| $Z(5)$, $N_t = 2$ | 128              | 0.097(6)     | 0.101                  | 1.847(5)     | 0.561                  | 2.04(2)                        | 0.153(5)                |
| $\beta^{(1)}_c = 1.617$ | 192              | 0.103(8)     | 0.093                  | 1.841(6)     | 0.447                  | 2.05(2)                        | 0.159(7)                |
|                 | 256              | 0.10(1)      | 0.122                  | 1.850(2)     | 0.038                  | 2.06(3)                        | 0.150(2)                |
|                 | 384              | 0.09(2)      | 0.117                  | 1.851(4)     | 0.056                  | 2.03(4)                        | 0.149(4)                |
|                 | 512              | 0.10(3)      | 0.198                  | 1.848(7)     | 0.091                  | 2.05(8)                        | 0.152(7)                |
| $Z(5)$, $N_t = 4$ | 32               | 0.123(6)     | 1.08                   | 1.8403(8)    | 0.72                   | 2.09(1)                        | 0.1596(8)               |
| $\beta^{(1)}_c = 1.943$ | 64               | 0.118(9)     | 1.13                   | 1.841(1)     | 0.58                   | 2.08(2)                        | 0.159(1)                |
|                 | 128              | 0.11(1)      | 1.25                   | 1.841(1)     | 0.70                   | 2.07(3)                        | 0.159(1)                |
|                 | 192              | 0.11(2)      | 1.56                   | 1.842(2)     | 0.86                   | 2.07(4)                        | 0.158(2)                |
|                 | 256              | 0.11(3)      | 2.07                   | 1.842(3)     | 1.14                   | 2.06(6)                        | 0.158(3)                |
|                 | 384              | 0.07(4)      | 1.68                   | 1.836(2)     | 0.22                   | 1.98(8)                        | 0.164(2)                |
|                 | 512              | 0.06(8)      | 3.26                   | 1.837(4)     | 0.42                   | 2.0(2)                          | 0.163(4)                |
| $Z(13)$, $N_t = 2$ | 128              | 0.07(5)      | 1.28                   | 1.968(9)     | 0.97                   | 2.1(1)                         | 0.032(9)                |
| $\beta^{(1)}_c = 1.795$ | 192              | 0.02(5)      | 1.16                   | 1.97(1)      | 1.11                   | 2.0(1)                         | 0.03(1)                 |
|                 | 256              | 0.04(8)      | 1.48                   | 1.97(2)      | 1.43                   | 2.0(2)                         | 0.03(2)                 |
|                 | 384              | 0.06(14)     | 2.19                   | 1.98(3)      | 1.93                   | 2.1(3)                         | 0.02(3)                 |
|                 | 512              | 0.2(17)      | 1.08                   | 1.94(5)      | 1.93                   | 2(3)                           | 0.06(5)                 |
| $Z(13)$, $N_t = 4$ | 128              | 0.03(1)      | 0.59                   | 1.977(2)     | 0.21                   | 1.3(3)                         | 0.023(2)                |
| $\beta^{(1)}_c = 2.74$ | 192              | 0.05(1)      | 0.20                   | 1.980(4)     | 0.24                   | 0.9(2)                         | 0.020(4)                |

scaling laws Eqs. (29) and (30) is to be applied to data of the rotated magnetization, $M_R$, and of its susceptibility, $\chi_L^{(M_R)}$, respectively. As the value of the critical coupling $\beta^{(1)}_c$ we use the central value determined in the previous Section.

The results of the fits are summarized in Table 3. The reference value for the index $\eta$ at this transition is $4/N^2$, i.e. $\eta = 0.16$ for $N = 5$ and $\eta \approx 0.0237$ for $N = 13$, whereas the hyperscaling relation to be fulfilled is $\gamma/\nu + 2\beta/\nu = d = 2$.

A general comment is that, in many of the cases we investigated, both $d$ and $\eta$ at the two critical points slightly differ from the expected values, though these differences cancel to a large extent if we define $\eta$ as $2\beta/\nu$.

Also here we see a general agreement between the $\eta$ and $d$ values obtained and those predicted by universality. However, the expected value of $\beta/\nu$ in Eq. (29) is very small,
(2/N^2), so other, asymptotically vanishing, terms can have a great impact on its determination on finite-sized lattices. This is especially evident for Z(13) with N_t = 4, where β/ν is negative indicating that the magnetization M_R grows with lattice size. This makes problematic the determination of this index from simulations on lattices used in the present work and explains the discrepancies with the d = 2 value.

There is an independent method to determine the critical exponent η, which does not rely on the prior knowledge of the critical coupling, but is based on the construction of a suitable universal quantity [30, 27]. The idea is to plot χ_L^{(M_R)}L^{η-2} versus B_4^{(M_R)} and to look for the value of η which optimizes the overlap of curves from different volumes. This method is illustrated in Fig 5 for Z(13) model with N_t = 4. Another option is to plot M_RL^{η/2} versus m_ψ, which leads to overlapping curves for η fixed at the value of the second phase transition, as illustrated in Fig 6 for Z(13) model with N_t = 2.

Concerning the value of the critical index ν, the methods used in this work do not allow for the direct determination of its value. When locating critical points we have fixed ν at 1/2. This value appears to be well in agreement with all numerical data.

3.4 Other checks of the nature of the phase transitions

In this Section we describe briefly some results of the simulation of the original gauge model. We have simulated Z(5) LGT with N_t = 2, 4 and spatial extent L ∈ [64 – 512]. The typical number of measurements was 10^5. In Tables 4 and 5 we present results for N_t = 2 and N_t = 4, correspondingly. In general, errors are bigger and results for critical indices are not so precise as in dual model simulations. Nevertheless, we can conclude from the inspection of Tables 4 and 5 that (i) the critical index η is compatible with its
$M_R L^{\eta / 2}$ and $m_\psi$ in $Z(13)$ with $N_t = 2$ for $\eta = 0.0237$ on lattices with various values of $L$.

Table 4: Critical indices $\beta/\nu$ and $\gamma/\nu$ for the transitions in $Z(5)$ LGT with $N_t = 2$.

| $L_{\text{min}}$ | $\beta/\nu$ | $\chi^2_{\beta/\nu}$ | $\gamma/\nu$ | $\chi^2_{\gamma/\nu}$ | $d = 2\beta/\nu + \gamma/\nu$ | $\eta = 2 - \gamma/\nu$ |
|------------------|--------------|------------------------|--------------|------------------------|-----------------------------|----------------------|
| 192              | 0.127(2)     | 2.38                   | 1.78(5)      | 3.15                   | 2.03(5)                     | 0.22(5)              |
| 192              | 0.1(2)       | 7.62                   | 1.82(6)      | 7.51                   | 2.1(4)                      | 0.18(6)              |

$2D$ value and (ii) the values of the indices at two transitions are indeed interchanged as explained in Section 2.

To produce further evidence in favor of the fact that the phase transitions investigated so far are both of infinite order, we have calculated the average action and the specific heat around the transitions in $Z(5)$ LGT with $N_t=2$ and $N_t=4$. In all cases the dependence of these quantities on $\beta$ is continuous. It follows that first and second order transitions are ruled out. As an example, we present in Fig. 7 the results of simulations for the average action for the case of $Z(5)$ with $N_t = 2$. 
Table 5: Critical indices $\beta/\nu$ and $\gamma/\nu$ for the transitions in $Z(5)$ LGT with $N_t = 4$.

| $\beta_c^{(1)}$ | $L_{\text{min}}$ | $\beta/\nu$ | $\chi^2_{\beta/\nu}$ | $\gamma/\nu$ | $\chi^2_{\gamma/\nu}$ | $d = 2\beta/\nu + \gamma/\nu$ | $\eta = 2 - \gamma/\nu$ |
|-----------------|-----------------|--------------|-----------------|--------------|-----------------|------------------------|-----------------|
| 1.943           | 128             | 0.122(4)     | 3.30            | 1.71(7)      | 1.95(8)         | 0.29(7)                |                 |
| 1.990           | 64              | 0.4(2)       | 4.92            | 1.82(2)      | 1.25            | 2.6(5)                 | 0.18(2)         |

Figure 7: Average action in $Z(5)$ LGT with $N_t = 2$ on lattices with various values of $L$. 
4 Conclusions and Perspectives

In this paper we have studied the 3D $Z(N)$ LGT at the finite temperature aiming at shedding light on the nature of phase transitions in these models for $N > 4$. This study was based on the exact dual transformation of the gauge models to generalized 3D $Z(N)$ spin models. In Section 2 we presented an overview of the exact relation between couplings of these two models and described qualitatively the behavior of the Polyakov loop correlation function and correlation of dual spins (the disorder operator in the gauge formulation). Furthermore, we have advanced some arguments that the critical behavior of Polyakov loop correlations and dual correlations is reversed, in particular the values of the critical index $\eta$ at the two transitions are interchanged for the dual correlation function with respect to the Polyakov loop correlation.

The numerical part of the work has been devoted to the localization of the critical couplings and to the computation of the critical indices. The main results can be shortly summarized as follows:

- We have determined numerically the two critical couplings of $Z(N = 5, 13)$ LGTs and given estimates of the critical indices $\eta$ at both transitions. For the first time we have a clear indication that for full 3D $Z(N)$ vector LGT with $N \geq 5$ the scenario of three phases is realized: a disordered phase at small $\beta$, a massless or BKT one at intermediate values of $\beta$ and an ordered phase, occurring at larger and larger values of $\beta$ as $N$ increases. This matches perfectly with the $N \to \infty$ limit, i.e. the 3D $U(1)$ LGT, where the ordered phase is absent;

- We have found that the values of the critical index $\eta$ at the two transitions are compatible with the theoretical expectations;

- The index $\nu$ also appears to be compatible with the value $1/2$, in agreement with universality predictions.

On the basis of this study we are led to conclude that finite-temperature 3D $Z(N)$ LGTs for $N > 4$ undergo two phase transitions of the BKT type. Moreover, these models belong to the universality class of the 2D $Z(N)$ vector models which also have two infinite order phase transitions and a massless phase.

It should be emphasized that in this paper we have not studied the continuum limit of the theory, but rather concentrated on the very possibility of having two BKT-like phase transitions. At the moment we are performing simulations of the model on symmetric lattices with the goal to compute the zero-temperature string tension and extend our present computations to $N_t = 8$. All these will help constructing the continuum limit. Also, we plan to extend our work to even $N$, to calculate the helicity modulus and to
establish scaling formulas for critical points with $N$. The results of these studies will be reported elsewhere.

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