ON BI-LIPSCHITZ CONTINUITY OF SOLUTIONS OF HYPERBOLIC POISSON’S EQUATION

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Abstract. In this paper, we investigate solutions of the hyperbolic Poisson equation \( \Delta_h u(x) = \psi(x) \), where \( \psi \in L^\infty(\mathbb{B}^n, \mathbb{R}^n) \) and

\[
\Delta_h u(x) = (1 - |x|^2)^2 \Delta u(x) + 2(n-2)(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x)
\]

is the hyperbolic Laplace operator in the \( n \)-dimensional space \( \mathbb{R}^n \) for \( n \geq 2 \). We show that if \( n \geq 3 \) and \( u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\mathbb{E}^n, \mathbb{R}^n) \) is a solution to the hyperbolic Poisson equation, then it has the representation \( u = P_h[\phi] - G_h[\psi] \) provided that \( u|_{\mathbb{S}^{n-1}} = \phi \) and \( \int_{\mathbb{S}^{n-1}} (1 - |x|^2)^{n-1} |\psi(x)| d\tau(x) < \infty \). Here \( P_h \) and \( G_h \) denote Poisson and Green integrals with respect to \( \Delta_h \), respectively. Furthermore, we prove that functions of the form \( u = P_h[\phi] - G_h[\psi] \) are bi-Lipschitz continuous.

1. Introduction and main results

For \( n \geq 2 \), let \( \mathbb{B}^n(x_0, r) = \{ x \in \mathbb{R}^n : |x-x_0| < r \} \), \( \mathbb{E}^n(x_0, r) = \{ x \in \mathbb{R}^n : |x-x_0| \leq r \} \) and \( \mathbb{S}^{n-1}(x_0, r) = \partial \mathbb{B}^n(x_0, r) \). We write \( \mathbb{B}^n = \mathbb{B}^n(0, 1) \) and \( \mathbb{S}^{n-1} = \mathbb{S}^{n-1}(0, 1) \).

Let \( L_1, L_2 \) be two constants and \( \Omega \subset \mathbb{R}^n \) a domain. Then a mapping \( f : \Omega \to \mathbb{R}^n \) is said to be \( L_1 \)-Lipschitz if \( |f(x) - f(y)| \leq L_1 |x - y| \) for all \( x, y \in \Omega \), and \( L_2 \)-co-Lipschitz if \( |f(x) - f(y)| \geq L_2 |x - y| \) for all \( x, y \in \Omega \). If \( f \) is both \( L_1 \)-Lipschitz and \( L_2 \)-co-Lipschitz for constants \( L_1 \) and \( L_2 \), then \( f \) is called bi-Lipschitz.

In [22], Kalaj and Pavlović studied the bi-Lipschitz continuity of quasiconformal self-mappings of the unit disk \( \mathbb{D} = \mathbb{B}^2 \) satisfying the Poisson’s equation \( \Delta u = \psi \), where \( \Delta \) is the usual Laplacian in \( \mathbb{R}^n \). See [7, 8, 17, 19, 20, 21, 25, 26] and references therein for further discussions along this line in the plane.

In [3], Arsenović et al. showed that the Lipschitz continuity of \( \phi : \mathbb{S}^{n-1} \to \mathbb{R}^n \) implies the Lipschitz continuity of its harmonic extension \( P[\phi] : \mathbb{B}^n \to \mathbb{R}^n \) provided that \( P[\phi] \) is a \( K \)-quasiregular mapping. Here \( P \) is the usual Poisson kernel with respect to \( \Delta \), i.e.

\[
P[\phi](\eta) = \int_{\mathbb{S}^{n-1}} P(\eta, \xi) \phi(\xi) \, d\sigma(\xi) \quad \text{and} \quad P(\eta, \xi) = \frac{1 - |\eta|^2}{|\eta - \xi|^n},
\]

where \( \eta \in \mathbb{B}^n \) and \( \sigma \) is the \((n-1)\)-dimensional Lebesgue measure normalized so that \( \sigma(\mathbb{S}^{n-1}) = 1 \). Moreover, Kalaj [16] also proved the Lipschitz continuity of

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$P[\phi] : \mathbb{B}^n \to \mathbb{B}^n$ under an additional assumption that it is a $K$-quasiconformal harmonic mapping with $P[\phi](0) = 0$ and $\phi \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$. Later, in [18], Kalaj proved that $K$-quasiconformal mappings of $\mathbb{B}^n$ onto itself are Lipschitz, provided that they satisfy the Poisson equation $\Delta u = \psi$ with $\psi \in L^\infty(\mathbb{B}^n, \mathbb{R}^n)$ and $u(0) = 0$.

1.1. Main results. The purpose of this paper is to consider results of the above type for solutions of the hyperbolic Laplace equation.

Definition 1.1. A function $u \in C^2(\mathbb{B}^n, \mathbb{R}^n)$ ($n \geq 2$) is said to be hyperbolic harmonic [30, 33, 34] if it satisfies the hyperbolic Laplace equation

$$\Delta_h u(x) = (1 - |x|^2)^2 \Delta u(x) + 2(n - 2)(1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}(x) = 0. \tag{1.1}$$

Obviously, for $n = 2$, hyperbolic harmonic functions coincide with harmonic functions. See [10, 11] for the properties of harmonic mappings. Also, see §2.5 below for more properties of $\Delta_h$.

It is well known that if $u$ satisfies the conditions: (1) $\Delta u = \psi$ which is continuous in $\mathbb{B}^n$ with $n \geq 2$, and (2) $u|_{\partial \mathbb{B}^n} = \phi$ which is bounded and integrable in $\mathbb{S}^{n-1}$, then (cf. [15, p. 118-119] or [18, 22, 23])

$$u = P[\phi] - G[\psi] \quad \text{and} \quad G[\psi](\eta) = \int_{\mathbb{B}^n} G(\eta, \xi) \psi(\xi) \, dV(\xi),$$

where $V$ is the $n$-dimensional Lebesgue volume measure and $G(\eta, \xi), \eta \neq \xi$, is the usual Green function [18, 22, 23], i.e.

$$G(\eta, \xi) = \begin{cases} \frac{1}{2\pi} \log \left| \frac{1 - \eta \cdot \xi}{\eta - \xi} \right|, & \text{for } n = 2, \\ \frac{1}{(n-2)\omega_{n-1}} \left( |\eta - \xi|^{2-n} - |\xi|^{2-n} - |\eta - \xi|^{2-n} \right), & \text{for } n \geq 3. \end{cases}$$

Here $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the $(n-1)$-dimensional surface area of $\mathbb{S}^{n-1}$ and $\Gamma$ is the Gamma function (see e.g. [2, p. 61] or [5, Appendix A]).

The first aim of this paper is to establish the counterpart of the above result to the solutions to the Dirichlet problem:

$$\begin{cases} \Delta_h u(x) = \psi(x), & x \in \mathbb{B}^n, \\ u(\xi) = \phi(\xi), & \xi \in \mathbb{S}^{n-1}, \end{cases} \tag{1.2}$$

where $\psi \in L^\infty(\mathbb{B}^n, \mathbb{R}^n)$ and $\phi \in L^\infty(\mathbb{S}^{n-1}, \mathbb{R}^n)$.

Denote by $\tau$ the Möbius invariant measure in $\mathbb{B}^n$, which is given by

$$d\tau(x) = \frac{d\nu(x)}{(1 - |x|^2)^n},$$

where $\nu$ is the $n$-dimensional Lebesgue volume measure normalized so that $\nu(\mathbb{B}^n) = 1$. Our result is as follows:

Theorem 1.1. Suppose $u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}^n}, \mathbb{R}^n), n \geq 3$ and

$$\int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| \, d\tau(x) \leq \mu_1,$$
where \( \mu_1 \geq 0 \) is a constant. If \( u \) satisfies (1.2), then

\[
(1.3) \quad u = P_h[\phi] - G_h[\psi].
\]

Here \( P_h[\phi] \) and \( G_h[\psi] \) denote the Poisson integral of \( \phi \) and the Green integral of \( \psi \), with respect to \( \Delta_h \), respectively (See (2.23) and (2.24) below for the details).

The second aim of this paper is to establish the bi-Lipschitz continuity of the mappings \( u \) of the form (1.3). More precisely, we have the following.

**Theorem 1.2.** Let \( n \geq 3 \). Suppose

1. \( u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\mathbb{B}^n, \mathbb{R}^n) \) is of the form (1.3);
2. there is a constant \( L \geq 0 \) such that \( |\phi(\xi) - \phi(\eta)| \leq L|\xi - \eta| \) for all \( \xi, \eta \in S^{n-1} \);
3. there is a constant \( M \geq 0 \) such that \( |\psi(x)| \leq M(1 - |x|^2) \) for all \( x \in \mathbb{B}^n \).

Then there exist constants \( C_1 = C_1(n, L, M) \) and \( C_2 = C_2(n, \phi, \psi) \) such that for all \( x, y \in \mathbb{B}^n \),

\[
C_2|x - y| \leq |u(x) - u(y)| \leq C_1|x - y|.
\]

**Remark 1.1.** In Section 6, we give an example to show that the assumption “\( n \geq 3 \)” in Theorem 1.2 is necessary.

In fact, Theorem 1.2 follows from more general, albeit technical, results on Lipschitz continuity of \( P_h[\phi] \) and \( G_h[\psi] \), which we shall discuss next.

1.2. \( \omega \)-Lipschitz continuity. A continuous increasing function \( \omega: [0, \infty) \to [0, \infty) \) with \( \omega(0) = 0 \) is called a majorant if \( \omega(t)/t \) is non-increasing for \( t > 0 \). Given a subset \( \Omega \) of \( \mathbb{R}^n \), a function \( f: \Omega \to \mathbb{R}^n \) is said to be \( \omega \)-Lipschitz continuous or belong to the Lipschitz space \( \Lambda_\omega(\Omega) \) if there is a positive constant \( C \) such that

\[
(1.4) \quad |f(x) - f(y)| \leq C\omega(|x - y|)
\]

for all \( x, y \in \Omega \) (cf. [9, 12, 13, 27, 28]). For some \( \rho_0 > 0 \) and \( 0 < \rho < \rho_0 \), a majorant \( \omega \) is called fast if

\[
\int_0^\rho \frac{\omega(t)}{t} dt \leq C\omega(\rho).
\]

Let \( \Omega \) be a proper subdomain of \( \mathbb{R}^n \). We say that a function \( f: \Omega \to \mathbb{R}^n \) belongs to the local Lipschitz space \( \text{loc}\Lambda_\omega(\Omega) \) if (1.4) holds, whenever \( x \in \Omega \) and \( |x - y| < \frac{1}{\rho}\delta_\Omega(x) \), where \( C \) is a positive constant and \( \delta_\Omega(x) \) denotes the Euclidean distance from \( x \) to the boundary \( \partial\Omega \) of \( \Omega \).

A domain \( \Omega \subset \mathbb{R}^n \) is said to be a \( \Lambda_\omega \)-extension domain if \( \Lambda_\omega(\Omega) = \text{loc}\Lambda_\omega(\Omega) \). In [24], Lappalainen proved that \( \Omega \) is a \( \Lambda_\omega \)-extension domain if and only if each pair of points \( x, y \in \Omega \) can be joined by a rectifiable curve \( \gamma \subset \Omega \) satisfying

\[
(1.5) \quad \int_{\gamma} \frac{\omega(\delta_\Omega(\eta))}{\delta_\Omega(\eta)} ds(\eta) \leq C\omega(|x - y|)
\]

with some fixed positive constant \( C = C(\Omega, \omega) \) which means that the constant \( C \) depends only on the quantities \( \Omega \) and \( \omega \), where \( ds \) is the length measure on \( \gamma \). Furthermore, we know from [24, Theorem 4.12] that \( \Lambda_\omega \)-extension domains exist for fast majorants \( \omega \) only. Conversely, if \( \omega \) is fast, then the class of \( \Lambda_\omega \)-extension domains is fairly large and contains all bounded uniform domains.
Remark 1.2. Recall that a domain $\Omega$ is said to be uniform if there is a constant $C$ such that each pair of points $x_1$ and $x_2$ in $\Omega$ can be joined by a rectifiable curve $\gamma \subset \Omega$ satisfying
\[
\ell(\gamma) \leq C|x_1 - x_2| \quad \text{and} \quad \min\{\ell(\gamma[x_1, x]), \ell(\gamma[x_2, x])\} \leq C \delta_\Omega(x)
\]
for all $x \in \gamma$. Here $\ell(\gamma)$ denotes the length of $\gamma$ and $\gamma[x_i, x]$ is the subarc of $\gamma$ with endpoints $x_i$ and $x$, where $i = 1, 2$. It is known that $\mathbb{B}^n$ is a uniform domain, and hence a $\Lambda_\omega$-extension domain for a fast $\omega$ [13, Section 1].

The next two results establish $\omega$-Lipschitz continuity of $P_h[\phi]$ and $G_h[\psi]$:

Theorem 1.3. Suppose $n \geq 3$, $\phi: \mathbb{S}^{n-1} \to \mathbb{R}^n$ and $|\phi(\xi) - \phi(\eta)| \leq \omega(|\xi - \eta|)$ for all $\xi, \eta \in \mathbb{S}^{n-1}$, where $\omega$ is a fast majorant. Then, for $x, y \in \mathbb{R}^n$,
\[
|\Phi(x) - \Phi(y)| \leq C\alpha_0\omega(|x - y|),
\]
where $\Phi = P_h[\phi]$ and $\alpha_0 = \alpha_0(n)$ and $C = C(\mathbb{B}^n, \omega)$ is the same constant as in (1.5).

Theorem 1.4. Suppose $n \geq 3$, $\psi \in C(\mathbb{B}^n, \mathbb{R}^n)$ and $|\psi(x)| \leq M(1 - |x|^2)$ for $x \in \mathbb{B}^n$, where $M$ is a constant. Then, for $x, y \in \mathbb{B}^n$,
\[
|\Psi(x) - \Psi(y)| \leq \beta_0|x - y|,
\]
where $\Psi = G_h[\psi]$ and $\beta_0 = \beta_0(n, M)$ is a constant.

This paper is organized as follows. In Section 2, some necessary terminology and known results are introduced, and several preliminary results are proved. In Section 3, we present the proof of Theorem 1.1. In Section 4, we show Theorem 1.3. In Section 5, we prove Theorem 1.4 and Theorem 1.2. Finally, in Section 6, we construct an example to illustrate the necessity of the requirement $n \geq 3$ in Theorem 1.2.

2. Preliminaries

In this section, we recall some necessary terminology and results.

2.1. Matrix notations. For natural number $n$, let
\[
A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}.
\]

For $A \in \mathbb{R}^{n \times n}$, denote by $\|A\|$ the matrix norm $\|A\| = \sup\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$, and $l(A)$ the matrix function $l(A) = \inf\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$.

For a domain $\Omega \subset \mathbb{R}^n$, let $u = (u_1, \ldots, u_n): \Omega \to \mathbb{R}^n$ be a function that has all partial derivatives at $x = (x_1, \ldots, x_n)$ on $\Omega$. Then $Du$ denotes the usual Jacobian matrix
\[
Du = \begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n}
\end{bmatrix} = (\nabla u_1 \cdots \nabla u_n)^T,
\]
where $T$ is the transpose and the gradients $\nabla u_j$ are understood as column vectors. If $Du$ is a nonsingular matrix, then the eigenvalues $\lambda^2$ of the (symmetric and positive definite) matrix $Du \times Du^T$ are real, and they can be ordered so that $0 < \lambda^2_1 \leq \lambda^2_2 \leq \cdots \leq \lambda^2_n$. Then $|J_u| = \prod_{k=1}^n \lambda_k$, $l(Du) = \lambda_1$ and $\|Du\| = \lambda_n$, where $J_u$ denotes the Jacobian of $u$. 

2.2. Spherical coordinate transformation. Let $Q = (\xi_1, \ldots, \xi_n) : K^{n-1} \to S^{n-1}$ be the following spherical coordinate transformation [16]:

$$
\begin{align*}
\xi_1 &= \cos \theta_1, \\
\xi_2 &= \sin \theta_1 \cos \theta_2, \\
&\vdots \\
\xi_{n-1} &= \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \cos \theta_{n-1}, \\
\xi_n &= \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \sin \theta_{n-1}.
\end{align*}
$$

Here $K^{n-1} = [0, \pi] \times \ldots \times [0, \pi] \times [0, 2\pi]$. Note that

$$J_Q(\theta_1, \ldots, \theta_{n-1}) = \sin^{n-2} \theta_1 \ldots \sin \theta_{n-2}.$$

For an integrable function $f$ in $B^n$, by letting $x = \rho \xi$ with $\rho = |x|$, we have

$$
\int_{\mathbb{B}^n(0,r)} f(x) d\nu(x) = n \int_0^r \rho^{n-1} d\rho \int_{S^{n-1}} f(\rho \xi) d\sigma(\xi),
$$

where

$$d\sigma(\xi) = \frac{1}{\omega_{n-1}} J_Q(\theta_1, \ldots, \theta_{n-1}) d\theta_1 \ldots d\theta_{n-1}$$

(see, e.g. [18, 33, 36]).

2.3. Hypergeometric functions. Let $F$ be the hypergeometric function

$$F(a, b; c; s) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} s^k,$$

where $a, b, c \in \mathbb{R}$, $c$ is neither zero nor a negative integer, $(a)_k$ denotes the Pochhammer symbol with $(a)_0 = 1$ and $(a)_k = a(a+1)\ldots(a+k-1)$ ($k \in \mathbb{N}$). If $a$ is not a negative integer, then

$$(a)_k = \Gamma(a+k)/\Gamma(a).$$

If $c - a - b > 0$, then the series (2.4) converges absolutely for all $|s| \leq 1$ (cf. [29, Section 31]).

Let $t > 1$, $k \in \mathbb{R}$ and $r \in (-1, 1)$. Ren and Kähler [30, Lemma 2.2] proved that

$$
\int_{-1}^{1} \frac{(1-s^2)^{(t-3)/2}}{(1-2rs + r^2)^k} ds = \frac{\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} F\left(k, k + 1 - \frac{t}{2}; \frac{t}{2}; r^2\right).
$$

The following lemmas will be useful in the proof of Theorem 1.2.

Lemma 2.1. Let

$$f_n(s) = \sum_{k=0}^{\infty} \frac{(a)_k (b-n)_k}{(\frac{n}{2})_k k!} \frac{s^k}{k+c}.$$

Suppose $b, n \in \mathbb{N}^+$, $a > 0$, $c > 0$ and $n - a - \frac{b}{2} > 0$. Then there is a constant $\mu_2 \geq 0$ such that for all $s \in [0, 1]$ and all $n \geq b$,

$$|f_n(s)| \leq \mu_2,$$

where $\mu_2 = \mu_2(n, a, b, c)$. 
Proof. Obviously, we only need to consider the case where \( b \) is even since the proof of the case \( b \) being odd is similar. To finish the proof, we consider the following two possibilities.

**Case 2.1.** \( n \) is even.

Under this assumption, we easily see from \( n \geq b \) that

\[
\left( \frac{b-n}{2} \right)_k = \left( \frac{b-n}{2} \right) \left( \frac{b-n}{2} + 1 \right) \cdots \left( \frac{b-n}{2} + k - 1 \right) = 0
\]

for all \( k \geq \frac{n-b+2}{2} \), and hence \( f_n \) is a polynomial, where

\[
f_n(s) = \sum_{k=0}^{\frac{n-b+2}{2}} (a)_k \left( \frac{b-n}{2} \right)_k \left( \frac{n}{2} \right)_k^k \cdot \frac{s^k}{k+c}.
\]

Hence, for all \( s \in [0,1] \),

\[
|f_n(s)| \leq \mu', \text{ where } \mu' = \sum_{k=0}^{\frac{n-b+2}{2}} \frac{(a)_k \left| \left( \frac{b-n}{2} \right)_k \right| 1}{\left( \frac{n}{2} \right)_k k!} \frac{1}{k+c}.
\]

**Case 2.2.** \( n \) is odd.

In this case, we separate \( f_n \) into two parts: \( f_n = f_{n1} + f_{n2} \), where

\[
f_{n1}(s) = \sum_{k=0}^{\frac{n-b+1}{2}} \frac{(a)_k \left( \frac{b-n}{2} \right)_k \left( \frac{n}{2} \right)_k^k s^k}{\left( \frac{n}{2} \right)_k k!} \frac{1}{k+c}
\]

and

\[
f_{n2}(s) = \sum_{k=\frac{n-b+3}{2}}^{\infty} \frac{(a)_k \left( \frac{b-n}{2} \right)_k \left( \frac{n}{2} \right)_k^k s^k}{\left( \frac{n}{2} \right)_k k!} \frac{1}{k+c}.
\]

Since \( f_{n1} \) is continuous in \([0,1]\), obviously, for \( s \in [0,1] \),

\[
|f_{n1}(s)| \leq \mu'', \text{ where } \mu'' = \sum_{k=0}^{\frac{n-b+1}{2}} \frac{(a)_k \left| \left( \frac{b-n}{2} \right)_k \right| 1}{\left( \frac{n}{2} \right)_k k!} \frac{1}{k+c}.
\]

Next, we estimate \( f_{n2} \). Since \( \frac{b-n}{2} + t - 1 > 0 \) for \( t \geq \frac{n-b+3}{2} \) and \( \frac{b-n}{2} + t - 1 < 0 \) for \( t \leq \frac{n-b+1}{2} \), we obtain that

\[
f_{n2}(s) = (-1)^{\frac{n-b+1}{2}} \sum_{k=\frac{n-b+3}{2}}^{\infty} \frac{(a)_k \left| \left( \frac{b-n}{2} \right)_k \right| 1}{\left( \frac{n}{2} \right)_k k!} \frac{1}{k+c} \cdot \frac{s^k}{k+c}.
\]

We leave the estimate on \( f_{n2} \) for a moment and prove the following claim.

**Claim 2.1.** Let

\[
g_{a,b,n}(s) = \sum_{k=\frac{n-b+3}{2}}^{\infty} \frac{(a)_k \left( \frac{b-n}{2} \right)_k \left( \frac{n}{2} \right)_k^k s^k}{\left( \frac{n}{2} \right)_k k!}.
\]

Then \( g_{a,b,n} \) is continuous in \([0,1]\).
To prove the continuity of \(g_{a,b,n}\) in \([0, 1]\), it suffices to check the uniform convergence of \(g_{a,b,n}\) in \([0, 1]\). Since 
\[
g_{a,b,n}(s) = (-1)^{n-b+3} \sum_{k=n-b+3}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} s^k,
\]

obviously, we only need to demonstrate the boundedness of 
\[
\sum_{k=n-b+3}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!}.
\]

This easily follows from the following two facts:

1. It follows from the assumption \(\frac{n}{2} - a - \frac{b-n}{2} > 0\) and [29, Section 31] that 
   
   \[
   F\left(a, \frac{b-n}{2}, \frac{n}{2}; s\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} s^k
   \]

   is bounded in \([0, 1]\).

2. \(\sum_{k=0}^{\infty} \frac{1}{(\frac{n}{2})_k k!} s^k\) is continuous in \([0, 1]\).

Now, we continue the estimate on \(f_{n_2}\). Let 
\[
\|g_{a,b,n}\|_\infty = \max\{|g_{a,b,n}(s)| : s \in [0, 1]\}.
\]

Then Claim 2.1 guarantees that \(\|g_{a,b,n}\|_\infty\) is finite. It follows that for all \(s \in [0, 1]\), 
\[
|f_{n_2}(s)| \leq |g_{a,b,n}(s)| \leq \|g_{a,b,n}\|_\infty.
\]

By taking 
\[
\mu_2 = \max\{\mu', \mu'' + \|g_{a,b,n}\|_\infty\},
\]

the lemma follows from (2.6), (2.7) and (2.8). 

**Lemma 2.2.** Let 
\[
I_0(s) = \int_0^1 t^m F\left(a, \frac{b-n}{2}, \frac{n}{2}; ts\right) dt.
\]

Suppose \(b, n \in \mathbb{N}^+, a > 0\) and \(m > -1\). Then for all \(s \in [0, 1]\), 
\[
I_0(s) = \sum_{k=0}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} s^k
\]

Proof. Obviously, 
\[
\int_0^1 t^m F\left(a, \frac{b-n}{2}, \frac{n}{2}; ts\right) dt = \int_0^1 \sum_{k=0}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} s^k t^{k+m} dt,
\]

and the convergence radius of the series \(\sum_{k=0}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} s^k t^{k+m}\) is \(1/s\) for \(s \in [0, 1]\). Hence we have 
\[
I_0(s) = \sum_{k=0}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} s^k \int_0^1 t^{k+m} dt = \sum_{k=0}^{\infty} \frac{(a)_k(b-n)_k}{(\frac{n}{2})_k k!} \frac{s^k}{k+m+1}.
\]
Lemma 2.3. Suppose $b, n \in \mathbb{N}^+, a > 0$, $n - a - b \geq 0$, $n \geq b$ and $m > -1$. Then for all $s \in [0, 1)$,

$$|I_0| \leq \mu_{2,1},$$

where $\mu_{2,1} = \mu_2(n, a, b, m+1)$ and $I_0$ are defined in Lemmas 2.1 and 2.2, respectively.

Proof. This lemma easily follows from Lemma 2.1 and Lemma 2.2. □

2.4. Möbius transformations. For any $x, y \in \mathbb{R}^n$, we denote the inner product $\sum_{k=1}^n x_k a_k$ by $\langle x, a \rangle$. Let $x = |x|x'$ and $y = |y|y'$. Then the symmetry lemma (see e.g. [2] or [5, 30]) shows that

$$|y|x - y'| = |x'y - x'|.$$ 

In the following, we denote $[x, y] = |x|y - x'$. Obviously, $[x, y] = [y, x]$.

For any $a \in \mathbb{B}^n$, let

$$\varphi_a(x) = \frac{|x-a|^2a - (1-|a|^2)(x-a)}{[a, x]^2}$$

in $\mathbb{B}^n$. Then $\varphi_a$ is a Möbius transformation of $\mathbb{R}^n$ mapping $\mathbb{B}^n$ onto $\mathbb{B}^n$ with $\varphi_a(a) = 0$, $\varphi_a(0) = a$ and $\varphi_a(\varphi_a(x)) = x$ [34]. It follows from Equations (2.4) and (2.6), Theorem 3.4(a) and Chapter 5 in [33], together with [30, Equation (2.4)], that

$$[a, x]^2 = |x-a|^2 + (1-|x|^2)(1-|a|^2)$$

$$= 1 + |a|^2|x|^2 - 2|a||x|\left\langle \frac{x}{|x|}, \frac{a}{|a|} \right\rangle,$$

(2.10) $1 - |x| \leq [a, x] < 2$,

(2.11) $|\varphi_a(x)| = |\varphi_a(a)| = \frac{|x-a|}{[a, x]}, \quad 1 - |\varphi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{[a, x]^2}$,

(2.12) $J_{\varphi_a}(x) = \frac{(1-|a|^2)^n}{[a, x]^{2n}}$

and

$$\frac{\partial}{\partial x_k} |\varphi_a(x)| = \frac{\partial}{\partial x_k} |\varphi_a(a)|$$

$$= \frac{[a, x]^2(x_k - a_k) - |a-x|^2(x_k - a_k) + |a-x|^2(1-|a|^2)x_k}{|a-x| \cdot [a, x]^3}.$$ 

Elementary calculations lead to

(2.13) $|a - \varphi_a(x)| = \frac{(1-|a|^2)x}{[a, x]}$ and $[a, \varphi_a(x)] = \frac{1-|a|^2}{[a, x]}$. 

as required. □
We denote by $M(\mathbb{B}^n)$ the set of all Möbius transformations in $\mathbb{B}^n$. By [33, Theorem 2.1], if $\varphi \in M(\mathbb{B}^n)$, then there exist $a \in \mathbb{B}^n$ and an orthogonal transformation $A$ such that

$$\varphi(x) = A\varphi_a(x).$$

For more information about the Möbius transformations in $\mathbb{B}^n$, see e.g. [2, Chapter 2], [6] or [35, Chapter 1].

2.5. Hyperbolic Poisson’s equation. In terms of the mapping $\varphi_a$, the hyperbolic metric $d_h$ in $\mathbb{B}^n$ is given by

$$d_h(a, b) = \log \left( \frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|} \right)$$

for all $a, b \in \mathbb{B}^n$.

For all $\varphi \in M(\mathbb{B}^n)$, by the definition of $\Delta_h$, we have the following Möbius invariance property [34, Section 2]:

$$(2.15) \quad \Delta_h(u \circ \varphi) = \Delta_h u \circ \varphi.$$ 

Obviously,

$$(2.16) \quad \Delta_h u(x) = (u \circ \varphi_x)(0).$$

In fact, if (2.16) holds for all $u \in C^2(\mathbb{B}^n)$ and $x \in \mathbb{B}^n$, then we can show that $\Delta_h$ has the representation (1.1) [33, Chapter 3].

Let

$$(2.17) \quad g(r, t) = \frac{1}{n} \int_r^t \frac{(1 - s^2)^{n-2}}{s^{n-1}} ds \quad \text{and} \quad g(r) = g(r, 1),$$

where $0 \leq r < t < 1$. It is well known that the Green’s function $G_h(x, y)$ w.r.t. $\Delta_h$ is given by

$$(2.18) \quad G_h(x, y) = g(|\varphi_x(y)|) = \frac{1}{n} \int_{|\varphi_x(y)|}^1 \frac{(1 - s^2)^{n-2}}{s^{n-1}} ds$$

for all $x \neq y \in \mathbb{B}^n$.

We remark that in the complex plane $\mathbb{C}$, every Möbius transformation $\varphi$ mapping the unit disc $\mathbb{D}$ onto itself can be written as $\varphi(z) = e^{i\theta} \varphi_w(z)$, where $\varphi_w(z) = \frac{w-z}{1-w\overline{z}}$ for some $w$ in $\mathbb{D}$. Hence when $n = 2$, by (2.18), we get [22]

$$(2.19) \quad G_h(w, z) = g(|\varphi_w(z)|) = \frac{1}{2} \log \left| \frac{1 - \overline{w}z}{w - z} \right| = \pi \cdot G(w, z),$$

where $G$ is the usual Green function w.r.t. $\Delta$.

For function $g$ in (2.17), we define

$$(2.20) \quad q(t) = \frac{t^{n-2}}{(1 - t^2)^{n-1}} g(t)$$

in $(0, 1)$. Since elementary calculations lead to

$$\lim_{t \to 0^+} q(t) = \frac{1}{n(n - 2)} \quad \text{and} \quad \lim_{t \to 1^-} q(t) = \frac{1}{2n(n - 1)},$$
we define
\[ q(0) = \frac{1}{n(n-2)} \quad \text{and} \quad q(1) = \frac{1}{2n(n-1)}. \]

Then we have

**Lemma 2.4.** For \( n \geq 3 \), \( \frac{1}{2n(n-1)} \leq q(t) \leq \frac{1}{n(n-2)} \) in \([0, 1]\).

**Proof.** We start with the following claim.

**Claim 2.2.** For \( n \geq 3 \), \( \frac{1}{2n(n-1)} < q(t) < \frac{1}{n(n-2)} \) in \((0, 1)\).

For \( t \in (0, 1] \), let
\[ q_1(t) = g(t) - \frac{1}{n(n-2)} \cdot \frac{(1 - t^2)^{n-1}}{t^{n-2}} \]
and
\[ q_2(t) = g(t) - \frac{1}{2n(n-1)} \cdot \frac{(1 - t^2)^{n-1}}{t^{n-2}}. \]

Then \( q_1(t) \) is increasing and \( q_2(t) \) is decreasing, respectively, in \((0, 1)\). Since \( q_1(1) = q_2(1) = 0 \), we see that
\[ \frac{1}{2n(n-1)} \cdot \frac{(1 - t^2)^{n-1}}{t^{n-2}} \leq q(t) \leq \frac{1}{n(n-2)} \cdot \frac{(1 - t^2)^{n-1}}{t^{n-2}} \]
in \((0, 1)\), which implies that the claim holds.

Now, the lemma easily follows from Claim 2.1 and (2.20). \( \square \)

The **Poisson-Szegö kernel** \( P_h \) for \( \Delta_h \) is given by

\[ P_h(x, t) = \left( \frac{1 - |x|^2}{|t - x|^2} \right)^{n-1}, \]
which satisfies [33, Lemma 5.20]
\[ \int_{S^{n-1}} P_h(x, \xi) d\sigma(\xi) = 1, \]
and for each \( k \in \{1, 2, \ldots, n\} \),

\[ \frac{\partial}{\partial x_k} P_h(x, t) = \frac{\partial}{\partial x_k} \left( \frac{1 - |x|^2}{|t - x|^2} \right)^{n-1} \]
\[ = -2(n-1) \frac{x_k |t - x|^2 + (1 - |x|^2)(x_k - t_k)}{|t - x|^4} \cdot \left( \frac{1 - |x|^2}{|t - x|^2} \right)^{n-2}, \]

where \((x, t) \in \mathbb{B}^n \times S^{n-1}\).

If \( \phi \in L^1(S^{n-1}, \mathbb{R}^n) \) \((n \geq 2)\), we define the Poisson-Szegö integral or invariant Poisson integral of \( \phi \) (cf. [1, 14] or [33, Definition 5.21]) by

\[ P_h[\phi](x) = \int_{S^{n-1}} P_h(x, \xi) \phi(\xi) d\sigma(\xi). \]

If \( \psi \) satisfies the following conditions:

(1) For \( n \geq 3 \), \( \psi \in C(\mathbb{B}^n, \mathbb{R}^n) \) and \( \int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| d\tau(x) < \infty \),
(2) For $n = 2$, $\psi(z) = (1 - |z|^2)^2 \psi_0(z)$, where $\psi_0 \in C(\mathbb{D}, \mathbb{C})$, then we define a function $G_h[\psi]$ by

\begin{align}
G_h[\psi](x) &= \int_{\mathbb{B}^n} G_h(x, y) \psi(y) \, d\tau(y) \\
&= \frac{1}{n} \int_{\mathbb{B}^n} \left[ \psi(y) \int_0^1 \frac{(1 - s^2)^{n-2}}{s^{n-1}} \, ds \right] \, d\tau(y).
\end{align}

This function is called the invariant Green integral of $\psi$.

**Remark 2.1.** If $n = 2$ and $\Delta u(z) \in C(\mathbb{D}, \mathbb{C})$, then it follows from (2.19), together with the facts $\Delta_h u(z) = (1 - |z|^2)^2 \Delta u(z)$ and $d\tau(z) = \frac{1}{\pi(1 - |z|^2)^2} \, dA(z)$, that

\begin{align}
G_h[\Delta_h u](z) &= \frac{1}{2\pi} \int_{\mathbb{D}} \log \left| \frac{1 - wz}{w - z} \right| \frac{dA(z)}{(1 - |z|^2)^2} = G[\Delta u](z),
\end{align}

where $dA(re^{i\theta}) = r \, dr \, d\theta$.

Furthermore, (2.21) implies that $P_h = P$ provided that $n = 2$. Let

$\psi(z) = (1 - |z|^2)^2 \psi_0(z),$

where $\psi_0 \in C(\mathbb{D}, \mathbb{C})$. If $u$ satisfies $\Delta u = \psi_0$ in $\mathbb{D}$ and $u|_{\mathbb{S}^1} = \phi \in L^1(\mathbb{S}^1, \mathbb{C})$, then it follows from [22], (2.23) and (2.25) that

$$u = P[\phi] - G[\psi_0] = P_h[\phi] - G_h[\psi].$$

We use $C^2_c(\mathbb{B}^n)$ to denote the set of all twice continuous differentiable functions with compact support in $\mathbb{B}^n$. Let us recall the following two results from [33].

**Theorem A.** ([33, Corollary 4.4]) If $u \in C^2_c(\mathbb{B}^n, \mathbb{R}^n)$, then for all $x \in \mathbb{B}^n$,

$$u = -G_h[\Delta_h u].$$

**Theorem B.** ([33, Theorem 5.22]) Let $\phi \in C(\mathbb{S}^{n-1}, \mathbb{R}^n)$, and let $F$ be defined as follows:

$$F(x) = \begin{cases} 
P_h[\phi](x), & x \in \mathbb{B}^n, \\
\phi(x), & x \in \mathbb{S}^{n-1}. \end{cases}$$

Then (1) $F$ is hyperbolically harmonic in $\mathbb{B}^n$ and continuous in $\overline{\mathbb{B}^n}$;

(2) $\|F\|_\infty = \|\phi\|_\infty$, where $\|F\|_\infty = \sup\{|F(x)| : x \in \mathbb{B}^n\}$ and $\|\phi\|_\infty = \sup\{|\phi(\xi)| : \xi \in \mathbb{S}^{n-1}\}$.

Conversely, if $H$ is hyperbolically harmonic in $\mathbb{B}^n$ and continuous in $\overline{\mathbb{B}^n}$, then

$$H = P_h[H].$$
3. Representation of solutions to $\Delta_h u = \psi$

The main purpose of this section is to prove Theorem 1.1. In this section, we always assume that $n \geq 3$. Before the proof, we recall the following results.

**Theorem C.** ([34, Lemma 3.2]) If $u \in C^2(\mathbb{B}^n)$, then

$$u(0) = \int_{\mathbb{S}^{n-1}} u(r \xi) d\sigma(\xi) - \int_{\mathbb{B}^n(0, r)} g(|x|, r) \Delta_h u(x) d\tau(x),$$

where $0 < r < 1$ and $g(t, r)$ is defined in (2.17).

**Theorem D.** ([34, Corollary 4.1]) For any $y \in \mathbb{B}^n$,

$$\int_{\mathbb{B}^n} G_h(x, y) d\nu(x) = \frac{1}{2n(n-1)} (1 - |y|^2)^n - 1.$$ 

The next two theorems are about the Möbius invariance of $P_h[f]$ and $d\tau$.

**Theorem E.** ([33, Theorem 5.23]) If $f \in L^1(S^{n-1})$, then

$$P_h[f \circ \varphi] = P_h[f] \circ \varphi$$

for all $\varphi \in M(\mathbb{B}^n)$.

**Theorem F.** ([33, Theorem 3.4(b)]) If $f \in L^1(\mathbb{B}^n, \tau)$ and $\varphi \in M(\mathbb{B}^n)$, then

$$\int_{\mathbb{B}^n} f(x) d\tau(x) = \int_{\mathbb{B}^n} f \circ \varphi(y) d\tau(y).$$

**Lemma 3.1.** Let $\psi \in C(\mathbb{B}^n, \mathbb{R}^n)$. Suppose there is a constant $\mu_1$ such that

$$\int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| d\tau(x) \leq \mu_1.$$

Then

$$\int_{\mathbb{B}^n} g(|x|)|\psi(x)| d\tau(x) \leq \mu_3,$$

where $g(r)$ is defined in (2.17), $\mu_3 = \mu_3(n, \mu_1, \|\psi\|_{\frac{1}{2}, \infty})$ and $\|\psi\|_{\frac{1}{2}, \infty} = \sup\{|\psi(x)| : x \in \mathbb{B}^n(0, \frac{1}{2})\}$.

**Proof.** By letting $y = 0$ in (2.18) and Theorem D, we get

$$\int_{\mathbb{B}^n(0, \frac{1}{2})} g(|x|) d\tau(x) = \int_{\mathbb{B}^n(0, \frac{1}{2})} G_h(x, 0) \frac{d\nu(x)}{(1 - |x|^2)^n} \leq \frac{1}{2n(n-1)(1 - \frac{1}{4})^n},$$

and thus the assumption “$\psi \in C(\mathbb{B}^n, \mathbb{R}^n)$” gives that

$$\int_{\mathbb{B}^n(0, \frac{1}{2})} g(|x|)|\psi(x)| d\tau(x) \leq \frac{\|\psi\|_{\frac{1}{2}, \infty}}{2n(n-1)(1 - \frac{1}{4})^n}.$$
Obviously, for \( x \in \mathbb{B}^n \setminus \mathbb{B}^n(0, \frac{1}{2}) \),
\[
g(|x|) = \frac{1}{2n} \int_{|x|}^{1} \frac{(1 - s^2)^{n-2}}{s^n} ds^2 \leq \frac{2^{n-1}}{n} \int_{|x|}^{1} (1 - s^2)^{n-2} ds^2 = \frac{2^{n-1}}{n(n-1)} (1 - |x|^2)^{n-1},
\]
it follows that
\[
\int_{\mathbb{B}^n \setminus \mathbb{B}^n(0, \frac{1}{2})} g(|x|)|\psi(x)|\,d\tau(x) \leq \frac{2^{n-1}}{n(n-1)} \mu_1.
\]
Since
\[
\int_{\mathbb{B}^n} g(|x|)|\psi(x)|\,d\tau(x) = \int_{\mathbb{B}^n(0, \frac{1}{2})} g(|x|)|\psi(x)|\,d\tau(x) + \int_{\mathbb{B}^n \setminus \mathbb{B}^n(0, \frac{1}{2})} g(|x|)|\psi(x)|\,d\tau(x),
\]
by letting
\[
\mu_3 = \frac{\|\psi\|_{\frac{1}{2}, \infty}}{2n(n-1)(1 - \frac{1}{4})^n} + \frac{2^{n-1}}{n(n-1)} \mu_1,
\]
we see that the lemma holds. \(\square\)

**Lemma 3.2.** Suppose that \( u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}^n}, \mathbb{R}^n) \) and satisfies (1.2). If
\[
\int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| \,d\tau(x) \leq \mu_1,
\]
then
\[
u(0) = P_h[\phi](0) - \int_{\mathbb{B}^n} G_h(0, x) \psi(x) \,d\tau(x).
\]

**Proof.** It follows from the assumption “\( \int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| \,d\tau(x) \leq \mu_1 \)” and Lemma 3.1 that
\[
\int_{\mathbb{B}^n} g(|x|)|\psi(x)|\,d\tau(x) \leq \mu_3.
\]
Since
\[
\int_{\mathbb{B}^n(0, r)} g(|x|, r)|\psi(x)|\,d\tau(x) \leq \int_{\mathbb{B}^n} g(|x|)|\psi(x)|\,d\tau(x),
\]
by Lebesgue’s Dominated Convergence Theorem, we have that
\[
\lim_{r \to 1^-} \int_{\mathbb{B}^n(0, r)} g(|x|, r)|\psi(x)|\,d\tau(x) = \int_{\mathbb{B}^n} g(|x|)|\psi(x)|\,d\tau(x).
\]
Furthermore, the assumption “\( u \in C(\overline{\mathbb{B}^n}, \mathbb{R}^n) \)” gives that
\[
\lim_{r \to 1^-} \int_{\mathbb{B}^n} u(r\xi) \,d\sigma(\xi) = \int_{\mathbb{B}^n} u(\xi) \,d\sigma(\xi).
\]
Then Theorem C, (3.1) and (3.2) imply that
\[
u(0) = \int_{\mathbb{B}^n} u(\xi) \,d\sigma(\xi) - \int_{\mathbb{B}^n} g(|x|)\Delta_h u(x) \,d\tau(x)
\]
\[
= \int_{\mathbb{S}^{n-1}} P_h(0, \xi)\phi(\xi) \,d\sigma(\xi) - \int_{\mathbb{B}^n} G_h(0, x) \psi(x) \,d\tau(x),
\]
as required. \(\square\)
Proof of Theorem 1.1. We prove this theorem by two steps. In the first step, we check that for any fixed $\zeta \in \mathbb{B}^n(0, r_0)$, $u \circ \varphi_\zeta$ satisfies the requirements in Lemma 3.2, where $0 \leq r_0 < 1$. In the second step, by applying Lemma 3.2 to $u \circ \varphi_\zeta$, we finish the proof.

Claim 3.1. For any fixed $\zeta \in \mathbb{B}^n(0, r_0)$, $u \circ \varphi_\zeta$ satisfies the requirements in Lemma 3.2.

Obviously, for any fixed $\zeta \in \mathbb{B}^n(0, r_0)$, $u \circ \varphi_\zeta \in C^2(\mathbb{R}_n, \mathbb{R}) \cap C(\mathbb{B}_n, \mathbb{R})$. The Möbius invariance property (2.15) and the assumption “$u \in C(\mathbb{B}_n, \mathbb{R})$” imply that

$$\Delta_h(u \circ \varphi_\zeta)(y) = \Delta_h u(\varphi_\zeta(y)) = \psi(\varphi_\zeta(y)) = \psi \circ \varphi_\zeta(y)$$

in $\mathbb{B}^n$ and

$$(u \circ \varphi_\zeta) \mid_{\mathbb{S}^{n-1}} = \phi \circ \varphi_\zeta.$$

So, to prove the claim, it suffices to show the following: There exists a constant $\mu_4$ such that

$$\int_{\mathbb{B}^n} (1 - |y|^2)^{n-1} |\psi(\varphi_\zeta(y))| \, d\tau(y) \leq \mu_4,$$

where $\mu_4 = \mu_4(\mu_1, n, r_0)$.

Let $w = \varphi_\zeta(y)$. Then we have that $y = \varphi_\zeta(w)$, so Theorem F, (2.10) and (2.11), together with the assumption “$\int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| \, d\tau(x) \leq \mu_1$”, yield

$$\int_{\mathbb{B}^n} (1 - |y|^2)^{n-1} |\psi(\varphi_\zeta(y))| \, d\tau(y)$$

$$= \int_{\mathbb{B}^n} (1 - |\varphi_\zeta(w)|^2)^{n-1} |\psi(w)| \, d\tau(w) \quad \text{(letting } w = \varphi_\zeta(y))$$

$$\leq 2^{n-1} \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^{n-1} |\psi(w)|}{(1 - |\zeta|)^n-1} \, d\tau(w) \quad \text{(by (2.10) and (2.11))}$$

$$\leq \frac{2^{n-1} \mu_1}{(1 - r_0)^n-1}.$$

Obviously, letting $\mu_4 = 2^{n-1} \mu_1 (1 - r_0)^{1-n}$ yields (3.3).

Claim 3.2. $u = P_h[\phi] - G_h[\psi].$

By replacing $u$ with $u \circ \varphi_\zeta$ and by using (2.15) and Theorem E, we see from Lemma 3.2 that

$$u(\zeta) = u \circ \varphi_\zeta(0) = P_h[\phi \circ \varphi_\zeta](0) - \int_{\mathbb{B}^n} G_h(0, y) \Delta_h(u \circ \varphi_\zeta)(y) \, d\tau(y)$$

$$= P_h[\phi](\varphi_\zeta(0)) - \int_{\mathbb{B}^n} G_h(0, y) \Delta_h u(\varphi_\zeta(y)) \, d\tau(y).$$

Let $w = \varphi_\zeta(y)$. It follows from

$$\varphi_\zeta(0) = \zeta, \quad G_h(0, \varphi_\zeta(w)) = g(|\varphi_\zeta(w)|) = G_h(\zeta, w)$$

and Theorem F that

$$u(\zeta) = P_h[\phi](\zeta) - \int_{\mathbb{B}^n} G_h(\zeta, w) \Delta_h u(w) \, d\tau(w) = P_h[\phi](\zeta) - G_h[\psi](\zeta).$$
By the arbitrariness of $r_0$ in $[0, 1)$, we see that the proof of the theorem is complete.

4. Lipschitz Continuity of $\Phi = P_h[\phi]$

The aim of this section is to prove the $\omega$-Lipschitz continuity of $\Phi = P_h[\phi]$ (Theorem 1.3).

Before the proof of Theorem 1.3, we need an estimate on $\|D\Phi(x)\|$ in terms of $\omega(1 - |x|)$ which is formulated in Lemma 4.4. The proof of Lemma 4.4 needs some preparation which consists of three lemmas. The first lemma is as follows.

**Lemma 4.1.** Suppose $\phi \in C(S^{n-1}, \mathbb{R}^n)$. Then for each $k \in \{1, 2, \ldots, n\}$,

1. $\frac{\partial}{\partial x_k} \phi(x)$ is continuous in $\mathbb{R}^n$;
2. $\frac{\partial}{\partial x_k} \Phi(x) = \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial x_k} P_h(x, \xi) \phi(\xi) \, d\sigma(\xi)$ for $x \in \mathbb{R}^n$.

**Proof.** In order to prove this lemma, we only need to discuss the case $k = 1$ since other cases can be discussed in a similar way. For this, we assume that $x \in \mathbb{E}^n(0, r_0)$ and $x + \Delta x_1 \in \mathbb{E}^n(0, r_0)$, where $x = (x_1, \ldots, x_n)$, $x + \Delta x_1 = (x_1 + \Delta x_1, \ldots, x_n)$ and $0 < r_0 < 1$. Then

$$\frac{\Phi(x + \Delta x_1) - \Phi(x)}{\Delta x_1} = \int_{\mathbb{S}^{n-1}} \frac{P_h(x + \Delta x_1, \xi) - P_h(x, \xi)}{\Delta x_1} \phi(\xi) \, d\sigma(\xi).$$

Obviously, $\frac{\partial}{\partial x_1} P_h(x, \xi) \phi(\xi)$ is continuous in $\mathbb{E}^n(0, r_0) \times \mathbb{S}^{n-1}$, and so

$$\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial x_1} P_h(x, \xi) \phi(\xi) \, d\sigma(\xi)$$

is continuous on $\mathbb{E}^n(0, r_0)$. By applying the Lagrange mean-value theorem to $P_h(x, \xi)$ w.r.t. $x_1$, we see that there exists $t_1 \in (0, 1)$ such that

$$\frac{\partial \Phi(x)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{\Phi(x + \Delta x_1) - \Phi(x)}{\Delta x_1} = \lim_{\Delta x_1 \to 0} \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial x_1} P_h(x + t_1 \Delta x_1, \xi) \phi(\xi) \, d\sigma(\xi)$$

$$= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial x_1} P_h(x, \xi) \phi(\xi) \, d\sigma(\xi),$$

as required.

Let $\xi_0 = e_1 \in \mathbb{S}^{n-1}$ denote the first unit coordinate vector $(1, 0, \ldots, 0)$. Then we have the following estimate.

**Lemma 4.2.** Suppose $q \geq 0$, $p - q - n > 0$ and $n \geq 3$. Then

$$\int_{\mathbb{S}^{n-1}} \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|\xi - r \xi_0|^p} \, d\sigma(\xi)$$

$$\leq \alpha_1 \omega(1 - r)^{q + 1 - p - n} \left( \frac{1}{q + n - 1} + \frac{2p}{p - q - n} \right),$$

where $\omega$ is a majorant, $0 \leq r < 1$ and $\alpha_1 = \omega_{n-2}/\omega_{n-1} = \frac{1}{\sqrt{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$. 


Proof. We shall prove this lemma by using a similar argument as in [3] and [4]. In order to estimate the integral in (4.1), we split $\mathbb{S}^{n-1}$ into the following two subsets:

$$E = \{\xi \in \mathbb{S}^{n-1} : |\xi - \xi_0| \leq 1 - r\} \quad \text{and} \quad F = \{\xi \in \mathbb{S}^{n-1} : |\xi - \xi_0| > 1 - r\}.$$ 

Then (4.1) easily follows from the following two claims.

Claim 4.1. \(\int_E \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|\xi - r \xi_0|^p} \, d\sigma(\xi) \leq \alpha_1 \frac{\omega((1-r)/q + n-1)(1-r)^{q+n-1-p}}{\omega_n} \).

Since $|\xi - r \xi_0| \geq 1 - |r \xi_0| = 1 - r$ for all $\xi \in \mathbb{S}^{n-1}$, we have

$$\int_E \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|\xi - r \xi_0|^p} \, d\sigma(\xi) \leq \int_E \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{(1-r)^p} \, d\sigma(\xi) = \frac{(1-r)^{-p}}{\omega_n} \int_E |\xi - \xi_0|^q \omega(|\xi - \xi_0|) \, dS(\xi),$$

where $S$ denotes the $(n-1)$-dimensional Lebesgue measure on $\mathbb{S}^{n-1}$. Let $\xi = (\xi_1, \ldots, \xi_n) \in E$ has the expression (2.1). Then, $\theta_1 \in [0, \varphi_r) \subset [0, \frac{\pi}{2}]$, $\theta_2, \ldots, \theta_{n-2} \in [0, \pi]$ and $\theta_{n-1} \in [0, 2\pi]$, where $\varphi_r = 2 \arcsin \frac{1}{2}$. It follows from (2.2) that

$$\int_E |\xi - \xi_0|^q \omega(|\xi - \xi_0|) \, dS(\xi) = \int_0^{\varphi_r} (2 - 2 \cos \theta_1)^{\frac{n}{2}} \omega((2 - 2 \cos \theta_1)^{\frac{1}{2}} \sin^{n-2} \theta_1 \, d\theta_1 \cdot \int_0^{\pi} \sin^{n-3} \theta_2 \, d\theta_2 \cdot \int_0^{n-2} \sin \theta_{n-2} \, d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1}$$

$$= \omega_{n-2} \int_0^{\varphi_r} (2 - 2 \cos \theta_1)^{\frac{n}{2}} \omega((2 - 2 \cos \theta_1)^{\frac{1}{2}} \sin^{n-2} \theta_1 \, d\theta_1.$$ 

Let $\rho = \sqrt{2 - 2 \cos \theta_1} \in [0, 1 - r]$. Then $d\theta_1 = \frac{\sin \theta_1}{\rho} d\rho$, from which we deduce that

$$\int_E |\xi - \xi_0|^q \omega(|\xi - \xi_0|) \, dS(\xi) \leq \omega_{n-2} \int_0^{1-r} \rho^{q+n-2} \omega(\rho) \, d\rho \leq \frac{\omega_{n-2} \omega((1-r)/q + n-1)}{q + n-1} \frac{(1-r)^{q+n-1}}{q + n-1},$$

where in the first inequality, the relation $\sin^2 \theta_1 = \rho^2 \left(1 - \frac{\rho^2}{2}\right) \leq \rho^2$ is applied. It follows from $\alpha_1 = \omega_{n-2}/\omega_n$, (4.2) and (4.3) that

$$\int_E \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|\xi - r \xi_0|^p} \, d\sigma(\xi) \leq \alpha_1 \frac{\omega((1-r)/q + n-1)(1-r)^{q+n-1-p}}{q + n-1} \omega_{n-2} \omega(\rho) \, d\rho \leq \frac{\omega_{n-2} \omega((1-r)/q + n-1)}{q + n-1} \frac{(1-r)^{q+n-1}}{q + n-1},$$

which is what we need.

Claim 4.2. \(\int_F \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|r \xi_0 - \xi_0|^p} \, d\sigma(\xi) \leq 2^p \alpha_1 \frac{\omega((1-r)/p + q-n)}{p - q-n} (1-r)^{q+n-1-p} \).

Since for all $\xi \in F$,

$$|\xi - \xi_0| \leq |\xi - r \xi_0| + |r \xi_0 - \xi_0| = |\xi - r \xi_0| + 1 - r \quad \text{and} \quad |\xi - r \xi_0| \geq 1 - r,$$

we easily see that

$$|\xi - \xi_0| \leq 2|\xi - r \xi_0|,$$
and so
\[ \int_F \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|\xi - r\xi_0|^p} \, d\sigma(\xi) \leq 2^p \int_F |\xi - \xi_0|^q \omega(|\xi - \xi_0|) \, d\sigma(\xi). \]

Then the similar reasoning as in the proof of (4.3) leads to
\[ \int_F \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|r\xi_0 - \xi|^p} \, d\sigma(\xi) \leq 2^p \alpha_1 \int_{1-r}^2 \rho^{p-q} \omega(\rho) \rho^{n-2} \, d\rho \leq \frac{2^p \alpha_1 \omega(1-r)}{p-q-n} (1-r)^{q+n-1-p}, \]

where, in the last inequality, the assumption that \( \frac{\omega(t)}{t} \) is non-increasing, is exploited. Hence Claim 4.2 is true. \( \square \)

Based on Lemmas 4.1 and 4.2, we have the following estimate on \( |\frac{\partial \phi}{\partial x_k}(x)| \).

**Lemma 4.3.** Suppose \( \phi \) and \( \omega \) satisfy the conditions in Theorem 1.3. Then there is a constant \( \alpha_2 \) such that for all \( x \in [0, e_1) \) and \( k \in \{1, 2, \ldots, n\} \),
\[ |\frac{\partial \phi}{\partial x_k}(x)| \leq \alpha_2 \frac{\omega(1-r)}{1-r}, \]

where \( \alpha_2 = \alpha_2(n) \), \( [0, e_1) = \{ x \in \mathbb{B}^n : x = re_1, 0 \leq r < 1 \} \) and \( n \geq 3 \).

**Proof.** For any \( x_0 \in [0, e_1) \), obviously, there is an \( r \in [0, 1) \) such that \( x_0 = r\xi_0 \), where \( \xi_0 = e_1 \). We prove the claim by considering two cases.

**Case 4.1.** \( 2 \leq k \leq n \).

Since (2.22) implies
\[ \frac{\partial}{\partial x_k} P_h(x_0, \xi) = \frac{2(n-1)(1-|x_0|^2)^{n-1} \xi_k}{|\xi - x_0|^{2n}}, \]
we infer from
\[ \phi(\xi_0) = \int_{\mathbb{S}^{n-1}} P_h(x, \xi) \phi(\xi_0) \, d\sigma(\xi), \]

as well as Lemma 4.1, that
\[ \left| \frac{\partial \phi}{\partial x_k}(x_0) \right| = \left| \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial x_k} P_h(x_0, \xi) \phi(\xi) \, d\sigma(\xi) \right| = \left| \int_{\mathbb{S}^{n-1}} \frac{2(n-1)(1-|x_0|^2)^{n-1} \xi_k}{|\xi - x_0|^{2n}} (\phi(\xi) - \phi(\xi_0)) \, d\sigma(\xi) \right| \leq 2(n-1)(1-|x_0|^2)^{n-1} \int_{\mathbb{S}^{n-1}} \frac{|\xi_k| \cdot |\phi(\xi) - \phi(\xi_0)|}{|\xi - x_0|^{2n}} \, d\sigma(\xi). \]

By using the fact \( |\xi_k| \leq |\xi - \xi_0| \) for \( 2 \leq k \leq n \) and the assumption \(|\phi(\xi) - \phi(\xi_0)| \leq \omega(|\xi - \xi_0|)\), we get
\[ \left| \frac{\partial \phi}{\partial x_k}(x_0) \right| \leq 2(n-1)(1-|x_0|^2)^{n-1} \int_{\mathbb{S}^{n-1}} \frac{|\xi - \xi_0|^q \omega(|\xi - \xi_0|)}{|\xi - x_0|^{2n}} \, d\sigma(\xi). \]
Then Lemma 4.2 leads to

$$\left| \frac{\partial \Phi}{\partial x_k}(re_1) \right| \leq \alpha_3 \frac{\omega(1 - r)}{1 - r},$$

where $\alpha_3 = \left( \frac{n-1}{n} + 2^n \right) 2^n \alpha_1$.

**Case 4.2.** $k = 1$.

Again, (2.22) implies

$$\frac{\partial}{\partial x_1} P_h(re_1, \xi) = \frac{2(n-1)(1 - |x_0|^2)^{n-1}}{|x_0 - \xi|^{2n}} \frac{(\xi_1 - |x_0|)}{|x_0 - \xi|^{2n-2}} - \frac{2(n-1)(1 - |x_0|^2)^{n-2}}{|x_0 - \xi|^{2n-2}},$$

and so

$$\left| \frac{\partial \Phi}{\partial x_1}(re_1) \right| = \left| \int_{S^{n-1}} \frac{\partial}{\partial x_1} P_h(x_0, \xi) \left( \phi(\xi) - \phi(\xi_0) \right) d\sigma(\xi) \right|
\leq 2(n-1)(1 - |x_0|^2)^{n-1} \int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n}} \frac{\xi_1 - |x_0|}{|x_0 - \xi|^{2n-2}} d\sigma(\xi)
+ 2(n-1)(1 - |x_0|^2)^{n-2} |x_0| \int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n-2}} d\sigma(\xi).$$

Since

$$|\xi_1 - |x_0|| \leq |\xi_1 - 1| + |1 - |x_0|| \leq |\xi - \xi_0| + 1 - |x_0|,$$

we get

$$\left| \frac{\partial \Phi}{\partial x_1}(re_1) \right| \leq 2(n-1)(1 - |x_0|^2)^{n-1} \int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n}} \frac{|\xi - \xi_0|}{|x_0 - \xi|^{2n-2}} d\sigma(\xi)
+ 2(n-1)(1 - |x_0|)^n (1 + |x_0|)^{n-1} \int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n}} d\sigma(\xi)
+ 2(n-1)(1 - |x_0|^2)^{n-2} |x_0| \int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n-2}} d\sigma(\xi).$$

Furthermore, Lemma 4.2 guarantees that

$$\int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n}} d\sigma(\xi) \leq \alpha_1 \frac{\omega(1 - r)}{(1 - r)^n} \left( \frac{1}{n - 1} + \frac{4^n}{n - 1} \right),$$

$$\int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n}} d\sigma(\xi) \leq \alpha_1 \frac{\omega(1 - r)}{(1 - r)^{n+1}} \left( \frac{1}{n - 1} + \frac{4^n}{n} \right)$$

and

$$\int_{S^{n-1}} \frac{\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{2n-2}} d\sigma(\xi) \leq \alpha_1 \frac{\omega(1 - r)}{(1 - r)^{n-1}} \left( \frac{1}{n - 1} + \frac{4^{n-1}}{n - 2} \right).$$

Hence

$$\left| \frac{\partial \Phi}{\partial x_1}(re_1) \right| \leq \alpha_4 \frac{\omega(1 - r)}{1 - r},$$

where

$$\alpha_4 = (n - 1) \left( \frac{2^n + 8^n}{n} + \frac{2^n + 8^n + 2^{n-1}}{n - 1} + \frac{8^{n-1}}{n - 2} \right) \alpha_1.$$
By letting $\alpha_2 = \max \{ \alpha_3, \alpha_4 \}$, we see that the lemma is true. \hfill \Box

Now, we are ready to state and prove the main lemma in this section.

**Lemma 4.4.** Suppose $\phi$ and $\omega$ satisfy the conditions in Theorem 1.3. Let $\alpha_0 = \sqrt{n\alpha_2}$, where $n \geq 3$ and $\alpha_2 = \alpha_2(n)$ is the same constant as in Lemma 4.3. Then

$$
\| D\Phi(x) \| \leq \alpha_0 \frac{\omega(1 - |x|)}{1 - |x|}
$$

in $\mathbb{B}^n$.

**Proof.** Let $x_0 \in \mathbb{B}^n$. We divide the proof into two cases.

**Case 4.3.** $x_0 \in [0, e_1)$.

Since the Cauchy-Schwarz inequality implies

$$
\| D\Phi(x_0) \| = \sup_{\zeta \in S^{n-1}} |D\Phi(x_0)\zeta| = \sup_{\zeta \in S^{n-1}} \left| \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k}(x_0) \cdot \zeta_k \right| \leq \left( \sum_{k=1}^{n} \left| \frac{\partial \Phi}{\partial x_k}(x_0) \right|^2 \right)^{\frac{1}{2}} \leq \sqrt{n\alpha_2} \frac{\omega(1 - |x_0|)}{1 - |x_0|}.
$$

we see from Lemma 4.3 that

$$
(4.4) \quad \| D\Phi(x_0) \| \leq \left( \sum_{k=1}^{n} \left| \frac{\partial \Phi}{\partial x_k}(x_0) \right|^2 \right)^{\frac{1}{2}} \leq \sqrt{n\alpha_2} \frac{\omega(1 - |x_0|)}{1 - |x_0|}.
$$

The proof of Lemma 4.4 holds in this case.

**Case 4.4.** $x_0 \notin [0, e_1)$.

For the proof in this case, we choose a unitary transformation $U$ such that $U(re_1) = x_0$, $r = |x_0|$, and for $y \in \mathbb{B}^n$, let

$$
W(y) := \Phi(U(y)).
$$

By Theorem E, we see that

$$
W = P_h[\phi] \circ U = P_h[\phi \circ U].
$$

Then we have the following claim.

**Claim 4.3.** $\| DW(re_1) \| \leq \sqrt{n\alpha_2} \frac{\omega(1 - r)}{1 - r}$.

The assumption $|\phi(\xi) - \phi(\eta)| \leq \omega(|\xi - \eta|)$ implies that for $\xi, \eta \in S^{n-1}$,

$$
|\phi(U(\xi)) - \phi(U(\eta))| \leq \omega(|U(\xi) - U(\eta)|) = \omega(|\xi - \eta|).
$$

Thus, by replacing $\Phi$ by $\Phi \circ U$, the similar reasoning as in the discussions of Case 4.3 shows that

$$
\| DW(re_1) \| \leq \sqrt{n\alpha_2} \frac{\omega(1 - r)}{1 - r},
$$

which is what we want.

Now, we are ready to finish the proof of the lemma in this case. By applying the chain rule, we obtain

$$
DW(y) \big|_{y=re_1} = D(\Phi \circ U)(y) \big|_{y=re_1} = (D\Phi) \circ U(y) \big|_{y=re_1} \times DU(y) \big|_{y=re_1} = D\Phi(x_0) \times U,
$$

where $DU(y) \big|_{y=re_1}$ is the derivative of $U$ at $re_1$. Thus, by combining the results of Cases 4.3 and 4.4, we have

$$
\| D\Phi(x) \| \leq \alpha_0 \frac{\omega(1 - |x|)}{1 - |x|}
$$

in $\mathbb{B}^n$. This completes the proof of Lemma 4.4.

\[\Box\]
where \( \times \) denotes the usual matrix product. Then

\[
\| D\Phi(x_0) \| = \| D\Phi(x_0) \times U \| = \| DW(re_1) \| \leq \sqrt{n\alpha_2(n)} \frac{\omega(1-r)}{1-r}.
\]

By (4.4) and (4.5), we complete the proof of Lemma 4.4.

**Proof of Theorem 1.3.** Now, we are ready to prove Theorem 1.3 by applying Lemma 4.4.

Lemma 4.1(a) implies that \( \Phi \in C^1(\mathbb{B}^n) \) and so \( \Phi \) is differentiable. Since \( \mathbb{B}^n \) is a \( \Lambda_w \)-extension domain for a fast majorant \( \omega \) (cf. [13, Section 1]), it follows from the mean-value theorem of differentials (see e.g. [31, Theorem 9.19]), (1.5) and Lemma 4.4 that there is a rectifiable curve \( \gamma \subset \mathbb{B}^n \) joining \( x \) to \( y \) satisfying

\[
|\Phi(y) - \Phi(x)| \leq \int_{\gamma} \| D\Phi(\zeta) \| \, ds(\zeta) \leq \alpha_0 \int_{\gamma} \frac{\omega(\delta_{\mathbb{B}^n}(\zeta))}{\delta_{\mathbb{B}^n}(\zeta)} \, ds(\zeta) \leq C \alpha_0 \omega(|x - y|),
\]

since \( \delta_{\mathbb{B}^n}(\zeta) = 1 - |\zeta| \) for \( \zeta \in \mathbb{B}^n \), where \( C = C(\mathbb{B}^n, \omega) \) is the same constant as in (1.5). So the proof of this Theorem 1.3 is complete.

## 5. Lipschitz continuity of \( \Psi = G_h[\psi] \)

In this section, Theorem 1.4 is proved through a series of lemmas. From this and Theorem 1.3, we derive Lipschitz continuity of \( u = P_h[\phi] - G_h[\psi] \), i.e. Theorem 1.2.

First, let us recall the following lemma from [30].

**Theorem G.** ([30, Section 2]) Let \( f \) be a continuous function in \([-1, 1] \). Then for any \( \eta \in S^{n-1} \) and \( n \geq 3 \),

\[
\int_{S^{n-1}} f(\langle \xi, \eta \rangle) \, d\sigma(\xi) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1 - t^2)^{\frac{n-3}{2}} f(t) \, dt.
\]

**Lemma 5.1.** Let

\[
I_1(s) =: \int_0^1 F\left(1, \frac{4-n}{2}; \frac{n}{2}; ts\right) \, dt.
\]

(1) If \( n \geq 4 \), then for \( s \in [0, 1) \),

\[
|I_1(s)| \leq \mu_{2,2},
\]

where \( \mu_{2,2} = \mu_2(n, 1, 4, 1) \) is defined in Lemma 2.1;

(2) If \( n = 3 \) and \( s_0 \in (0, 1) \), then for \( s \in [0, s_0] \),

\[
|I_1(s)| \leq \frac{1}{1 - s_0}.
\]

**Proof.** It follows from Lemma 2.2 that

\[
I_1(s) = \sum_{k=0}^{\infty} \frac{(1)_k (\frac{4-n}{2})_k}{\left(\frac{n}{2}\right)_k k!} \frac{s^k}{k+1}.
\]

If \( n \geq 4 \), the result follows from Lemma 2.1.
If \( n = 3 \), then for any \( s_0 \in (0, 1) \), we have

\[
I_1(s) = \sum_{k=0}^{\infty} \frac{(1)_k (4-n)_k}{(n/2)_k k!} \frac{s^k}{k+1} \leq \sum_{k=0}^{\infty} s^k \leq \frac{1}{1-s_0}.
\]

Hence Lemma 5.1 is proved. \( \square \)

By Lemma 5.1, we have the following estimate.

**Lemma 5.2.** If \( n \geq 3 \), then for all \( x \in B^n \),

\[
J_n(x) = \int_{B^n} \frac{d\nu(y)}{y^{n-2}|x,y|^2} \leq \mu_5,
\]

where \( \mu_5 = \max \left\{ \frac{n}{2} \mu_2, 65 \frac{7}{8} \right\} \), where \( \mu_2 \) is the same constant as in Lemma 5.1.

**Proof.** For \( n \geq 3 \), (2.3) leads to

\[
J_n(x) = \int_{B^n} \frac{d\nu(y)}{|y|^{n-2}|x,y|^2} = n \int_0^1 \rho \, d\rho \int_{S^{n-1}} \frac{d\sigma(\xi)}{|x, \rho \xi|^2}.
\]

By (2.9), we have

\[
\int_{S^{n-1}} \frac{d\sigma(\xi)}{|x, \rho \xi|^2} = \int_{S^{n-1}} \left( 1 + \rho^2 |x|^2 - 2 \rho |x| (x, \xi) \right)^{-1} d\sigma(\xi),
\]

and so Theorem G and (2.5) lead to

\[
\int_{S^{n-1}} \frac{d\sigma(\xi)}{|x, \rho \xi|^2} = \frac{\Gamma(n/2)}{\Gamma((n-1)/2) \Gamma(1/2)} \int_1^1 (1 - t^2)^{n/2} (1 + \rho^2 |x|^2 - 2 \rho |x| t)^{-1} dt
\]

\[
= F\left( 1, \frac{4-n}{2} : \frac{n}{2} ; \rho^2 |x|^2 \right).
\]

Hence we have

\[
J_n(x) = n \int_0^1 \rho F\left( 1, \frac{4-n}{2} : \frac{n}{2} ; \rho^2 |x|^2 \right) \, d\rho.
\]

When \( n \geq 4 \), it follows from Lemma 5.1 that for all \( x \in B^n \),

\[
J_n(x) \leq \frac{n}{2} \mu_2.
\]

In the following, we assume that \( n = 3 \). Then we have the following assertion.

**Claim 5.1.** \( J_3(x) \leq 65 \frac{7}{8} \).

We divide the proof into two cases according to the value of \( |x| \).

**Case 5.1.** \( \frac{3}{4} \leq |x| < 1 \).

Since \( n = 3 \), by (2.9), we see that

\[
J_3(x) = \int_{B^3} \frac{d\nu(y)}{|y||x,y|^2} \leq \int_{B^3} \frac{d\nu(y)}{|y||y-x|^2}.
\]
Let \( \delta_1 = |x|/3 \). By (2.3) and elementary calculations, we have that
\[
\int_{B^3(0, \delta_1)} \frac{d\nu(y)}{|y| \cdot |y - x|^2} \leq \int_{B^3(0, \delta_1)} \frac{d\nu(y)}{4|y|^2 \delta_1^2} = \frac{3}{8}.
\]
\[
\int_{B^3(0, \delta_1) \cup B^3(x, \delta_1)} \frac{d\nu(y)}{|y| \cdot |y - x|^2} \leq \int_{B^3(x, \delta_1)} \frac{d\nu(y)}{2|y - x|^2 \delta_1} = \int_{B^3(0, \delta_1)} \frac{d\nu(y)}{2|y|^2 \delta_1} = \frac{3}{2}
\]
and
\[
\int_{\mathbb{B}^3 \setminus (B^3(0, \delta_1) \cup B^3(x, \delta_1))} \frac{d\nu(y)}{|y| \cdot |y - x|^2} \leq \int_{\mathbb{B}^3 \setminus (B^3(0, \delta_1) \cup B^3(x, \delta_1))} \frac{d\nu(y)}{\delta_1^2} \leq \frac{1}{\delta_1^2} \leq 64.
\]
These inequalities show that the claim holds since
\[
\mathbb{B}^3 = \mathbb{B}^3(0, \delta_1) \cup \mathbb{B}^3(x, \delta_1) \cup \left( \mathbb{B}^3 \setminus (\mathbb{B}^3(0, \delta_1) \cup \mathbb{B}^3(x, \delta_1)) \right).
\]

**Case 5.2.** \(|x| < \frac{3}{4}\).

Under this assumption, we see from (5.1) and Lemma 5.1 that
\[
J_0(x) = \frac{3}{2} \int_0^1 F(1, \frac{3}{2}; \frac{3}{2}; \rho^2|x|^2) \, d\rho^2 \leq \frac{3}{2} \cdot \frac{1}{1 - \frac{9}{16}} = \frac{24}{7},
\]
as required. So Claim 5.1 is proved.

Now, we obtain from (5.2) and Claim 5.1 that for all \( x \in \mathbb{B}^n \),
\[
J_n(x) = \int_{\mathbb{B}^n} \frac{d\nu(u)}{|u|^{n-2}[x, u]|} \leq \max\left\{ \frac{n}{2} \mu_2, 65 \frac{7}{8} \right\},
\]
and hence the proof of Lemma 5.2 is complete. \( \square \)

Based on Theorem G, Lemmas 5.1 and 5.2, we obtain some properties of the two unbounded integrals \( G_h[\psi](x) \) and
\[
\int_{\mathbb{B}^n} \frac{\partial G_h(x, y) \psi(y)}{\partial x_k} \, d\tau(y),
\]
which will be presented in the next four lemmas. The first two lemmas deal with the uniform convergence of these two integrals, respectively.

**Lemma 5.3.** Suppose \( n \geq 3 \), \( \psi \in C(\mathbb{B}^n, \mathbb{R}^n) \) and \(|\psi(x)| \leq M(1 - |x|^2)\) in \( \mathbb{B}^n \), where \( M \) is a constant. Then for all \( 0 < r_0 < 1 \), the unbounded integral \( G_h[\psi](x) \) is uniformly convergent w.r.t. \( x \) in \( \mathbb{B}^n(0, r_0) \).

**Proof.** By the assumption \(|\psi(x)| \leq M(1 - |x|^2)\), we see from (2.24) that
\[
|G_h[\psi](x)| \leq \frac{M}{n} \int_{\mathbb{B}^n} \left[ \frac{1}{(1 - |y|^2)^{n-1}} \int_0^1 \frac{(1 - s^2)^{n-2}}{s^{n-1}} \, ds \right] \, d\nu(y).
\]
For \( x \in \mathbb{B}^n(0, r_0) \), (2.10) leads to
\[
[x, y] \geq 1 - |x| \geq 1 - r_0.
\]
Since
\[ \int_{\mathbb{B}^n} \left[ \frac{1}{1 - |y|^2} \int_{|\varphi_x(y)|}^{s} \frac{(1 - s^2)^{n-2}}{s^{n-1}} ds \right] d\nu(y) = \int_{\mathbb{B}^n} g(|\varphi_x(y)|) \frac{1}{1 - |y|^2} \, d\nu(y), \]
we see that
\[ \int_{\mathbb{B}^n} \left[ \frac{1}{1 - |y|^2} \int_{|\varphi_x(y)|}^{s} \frac{(1 - s^2)^{n-2}}{s^{n-1}} ds \right] d\nu(y) \]
\[ \leq \frac{1}{n(n-2)} \int_{\mathbb{B}^n} (1 - |\varphi_x(y)|^2)^{n-1} d\nu(y) \quad \text{(by Lemma 2.4)} \]
\[ < \frac{1}{n(n-2)} \int_{\mathbb{B}^n} \frac{1}{|x|^n} d\nu(y) \quad \text{(by (2.11))} \]
\[ \leq \frac{1}{n(n-2)(1 - r_0)^n} \int_{\mathbb{B}^n} \frac{1}{|x - y|^n} d\nu(y). \]

Thus in order to prove the uniform convergence of \( G_h[\psi](x) \) in \( \overline{\mathbb{B}^n}(0, r_0) \), we only need to prove that
\[ F_{n-2}(x) = \int_{\mathbb{B}^n} \frac{1}{|x - y|^{n-2}} d\nu(y) \]
is uniformly convergent. In fact, we shall prove the following more general result.

Claim 5.2. The integral \( F_k(x) = \int_{\mathbb{B}^n} \frac{1}{|x - y|^k} d\nu(y) \) is uniformly convergent w.r.t. \( x \) in \( \overline{\mathbb{B}^n}(0, r_0) \), where \( 1 \leq k \leq n-1 \) and \( 0 < r_0 < 1 \).

Let \( \delta_2 = \frac{1 - r_0}{2} \). Then
\[ \mathbb{B}^n = \mathbb{B}^n(x, \delta_2) \cup (\mathbb{B}^n \setminus \mathbb{B}^n(x, \delta_2)) \quad \text{and} \quad \mathbb{B}^n(x, \delta_2) \subset \mathbb{B}^n. \]

Hence
\[ F_k(x) = F_{k,1}(x) + F_{k,2}(x), \]
where
\[ F_{k,1}(x) = \int_{\mathbb{B}^n \setminus \mathbb{B}^n(x, \delta_2)} \frac{1}{|x - y|^k} d\nu(y) \quad \text{and} \quad F_{k,2}(x) = \int_{\mathbb{B}^n(x, \delta_2)} \frac{1}{|x - y|^k} d\nu(y). \]

Subclaim 1. \( F_{k,1}(x) \) and \( F_{k,2}(x) \) are uniformly convergent w.r.t. \( x \) in \( \overline{\mathbb{B}^n}(0, r_0) \), where \( 1 \leq k \leq n-1 \) and \( 0 < r_0 < 1 \).

Since for all \( y \in \mathbb{B}^n \setminus \mathbb{B}^n(x, \delta_2) \),
\[ \frac{1}{|x - y|^k} \leq \frac{1}{\delta_2^k}, \]
by the Weierstrass test for uniform convergence, the uniform convergence of \( F_{k,1}(x) \) in \( \overline{\mathbb{B}^n}(0, r_0) \) is obvious.

For any \( 0 < \delta \leq \delta_2 \), let \( y = x + w \). Then it follows from (2.3) that
\[ \int_{\mathbb{B}^n(0, \delta)} \frac{1}{|w|^k} d\nu(w) = \frac{n}{n-k} \delta^{n-k} \leq n\delta. \]
By definition, we easily know that $F_{k,2}(x)$ is uniformly convergent w.r.t. $x$ in $\overline{B^n}(0,r_0)$. Hence Subclaim 1 is proved.

Subclaim 1 implies the uniform convergence of $F_k(x)$ in $\overline{B^n}(0,r_0)$, and thus the proof of Claim 5.2 is complete.

Let $k = n - 2$. Then by Claim 5.2, we know that $G_h[p](x)$ is also uniformly convergent in $\overline{B^n}(0,r_0)$, and so the lemma is proved.

Now, we are going to prove the first main lemma for the proof of Theorem 1.4.

**Lemma 5.4.** Suppose $n \geq 3$, $k \in \{1, \ldots, n\}$, $\psi \in C(\mathbb{B}^n, \mathbb{R}^n)$ and $|\psi(x)| \leq M(1 - |x|^2)$ in $\mathbb{B}^n$, where $M$ is a constant. Then

1. for $0 < r_0 < 1$, the unbounded integral
   \[
   I_{2,k}(x) =: \int_{\mathbb{B}^n} \left| \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) \right| d\tau(y)
   \]
   is uniformly convergent w.r.t. $x$ in $\overline{B^n}(0,r_0)$;

2. for all $x \in \mathbb{B}_1$, there exists a constant $\beta_1 = \beta_1(n,M)$ such that
   \[
   I_{2,k}(x) \leq \beta_1.
   \]

Proof. First, we easily see from (2.9), (2.11), (2.13) and (2.18) that for $x \neq y$,

\[
\frac{\partial}{\partial x_k} G_h(x, y) = -\frac{(x_k - y_k)(1 - |x|^2)^{n-1}(1 - |y|^2)^{n-1}}{n|x - y|^{n+1}|x, y|^n} - \frac{x_k(1 - |x|^2)^{n-2}(1 - |y|^2)^{n-1}}{n|x - y|^{n+1}|x, y|^n}.
\]

Then (2.10) implies that for $x \in \mathbb{B}^n(0, r_0)$, $|x, y| \geq 1 - r_0$, and hence

\[
\left| \frac{\partial}{\partial x_k} G_h(x, y) \right| \leq \frac{(1 - |x|^2)^{n-2}|x, y|^{n+1}}{n|x - y|^{n-1}|x, y|^n} + \frac{(1 - |x|^2)^{n-2}|x, y|^{n-1}}{n|x - y|^{n-1}|x, y|^n}
\]

\[
\leq \frac{(1 - |y|^2)^{n-2}|x, y|^{n-1}}{n(1 - r_0)^n} \left( \frac{1}{|x - y|^{n-1}} + \frac{1}{|x - y|^{n-2}} \right).
\]

Thus the assumption \(|\psi(x)| \leq M(1 - |x|^2)\)” implies that for all $x \in \mathbb{B}^n(0, r_0)$,

\[
I_{2,k}(x) \leq M \int_{\mathbb{B}^n} \left| \frac{\partial}{\partial x_k} G_h(x, y) \right| (1 - |y|^2) d\tau(y)
\]

\[
\leq \frac{M}{n(1 - r_0)^n} \int_{\mathbb{B}^n} \left( \frac{1}{|x - y|^{n-1}} + \frac{1}{|x - y|^{n-2}} \right) d\nu(y).
\]

The uniform convergence of $I_{2,k}(x)$ w.r.t. $x$ in $\overline{B^n}(0, r_0)$ follows from Claim 5.2, and thus Lemma 5.4(1) holds.

Next, we prove Lemma 5.4(2). It follows from (5.3) that

\[
I_{2,k}(x) \leq \frac{1}{n}(I_{3,k}(x) + I_{4,k}(x)),
\]
where
\[
I_{3,k}(x) = \int_{\mathbb{B}^n} \frac{|x_k - y_k|(1 - |x|^2)^{n-1}}{|x-y|^n|x,y|^n(1 - |y|^2)}|\psi(y)|d\nu(y)
\]
and
\[
I_{4,k}(x) = \int_{\mathbb{B}^n} \frac{|x_k|(1 - |x|^2)^{n-2}}{|x-y|^n|x,y|^n(1 - |y|^2)}|\psi(y)|d\nu(y).
\]

Next, we estimate \(|I_{3,k}(x)|\) and \(|I_{4,k}(x)|\), respectively.

**Claim 5.3.** For \(x \in \mathbb{B}^n\),
\[
|I_{3,k}(x)| \leq \frac{nM}{2} \mu_{2,3},
\]
where \(\mu_{2,3} = \mu_2(n, \frac{1}{2}, 3, \frac{1}{2})\) is the same constant as in Lemma 2.1.

Let \(y = \varphi_x(w)\). Then the equalities (2.3), (2.12), (2.14) and the assumption “\(|\psi(x)| \leq M(1 - |x|^2)\)” imply that
\[
|I_{3,k}(x)| \leq M \int_{\mathbb{B}^n} \frac{(1 - |x|^2)^{n-1}|x-y|}{|x-y|^n|x,y|^n}d\nu(y)
\]
\[
= M \int_{\mathbb{B}^n} \frac{(1 - |x|^2)^{n-1}|\varphi_x(w)|}{|x - \varphi_x(x)|^{n-1}[x, \varphi_x(x)]^n}d\nu(w) \quad \text{(by } y = \varphi_x(w)\text{)}
\]
\[
= M \int_{\mathbb{B}^n} \frac{d\nu(w)}{|x, w| \cdot |w|^{n-1}} \quad \text{(by (2.12) and (2.14))}
\]
\[
= nM \int_0^1 \int_{S^{n-1}} \frac{d\sigma(\xi)}{|x, \rho \xi|}d\rho.
\]
Moreover, by (2.9), we have
\[
\int_{S^{n-1}} \frac{d\sigma(\xi)}{|x, \rho \xi|} = \int_{S^{n-1}} \left(1 + \rho^2|x|^2 - 2\rho|x|\left(\frac{x}{|x|}, \xi\right)\right)^{-\frac{1}{2}}d\sigma(\xi),
\]
which, together with Theorem G and (2.5), implies that
\[
\int_{S^{n-1}} \frac{d\sigma(\xi)}{|x, \rho \xi|} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1 - s^2)^{\frac{n-3}{2}}(1 + \rho^2|x|^2 - 2\rho|x|s)^{-\frac{1}{2}}ds
\]
\[
= F\left(\frac{1}{2}; \frac{3-n}{2}; \frac{n}{2}; \rho^2|x|^2\right).
\]
Hence Lemma 2.3 leads to
\[
|I_{3,k}(x)| \leq nM \int_0^1 F\left(\frac{1}{2}; \frac{3-n}{2}; \frac{n}{2}; \rho^2|x|^2\right) d\rho \leq \frac{nM}{2} \mu_{2,3},
\]
as required, where \(\mu_{2,3} = \mu_2(n, \frac{1}{2}, 3, \frac{1}{2})\).

**Claim 5.4.** For \(x \in \mathbb{B}^n\), we have
\[
|I_{4,k}(x)| \leq M \mu_5,
\]
where \(\mu_5 = \mu_5(n)\) is the same constant as in Lemma 5.2.
Obviously, the assumption $|\psi(x)| \leq M(1 - |x|^2)$ implies that

$$|I_{4,k}(x)| \leq M \int_{\mathbb{B}^n} \frac{(1 - |x|^2)^{n-2}}{|x - y|^{n-2}[x,y]^{n}} d\nu(y).$$

Let $y = \varphi_x(w)$. By (2.12), (2.14) and Lemma 5.2, we get

$$|I_{4,k}| \leq M \int_{\mathbb{B}^n} \frac{(1 - |\varphi_x(w)|^2)^{n-2}}{|w|^{n-2}[x,w]^{n}} d\nu(w) \quad \text{(substituting } y = \varphi_x(w))$$

$$= M \int_{\mathbb{B}^n} \frac{d\nu(w)}{|w|^{n-2}[x,w]^{n}} \leq M\mu_5(n).$$

By taking $\beta_1 = \frac{M}{\mu_2}, 3 + \frac{M}{\mu_5}$, we see that Lemma 5.4(2) holds, and so the proof of the lemma is finished.

**Lemma 5.5.** Suppose $n \geq 3$, $\psi \in C(\mathbb{B}^n, \mathbb{R}^n)$ and $|\psi(x)| \leq M(1 - |x|^2)$ in $\mathbb{B}^n$, where $M$ is a constant. Then for all $0 < r_0 < 1$ and $k \in \{1, 2, \ldots, n\},$

$$G_h[\psi](x) \text{ and } \int_{\mathbb{B}^n} \frac{\partial}{\partial x_k}G_h(x,y)\psi(y) \, d\tau(y)$$

are continuous in $\overline{\mathbb{B}^n}(0,r_0)$, respectively.

**Proof.** In order to check the continuity of $G_h[\psi](x)$ in $\overline{\mathbb{B}^n}(0,r_0)$, we only need to prove that $G_h[\psi](x)$ is continuous at every fixed point $x_0 \in \overline{\mathbb{B}^n}(0,r_0)$. Assume that $x_0 \in \overline{\mathbb{B}^n}(0,r_0)$ and $x_0 + \Delta x \in \overline{\mathbb{B}^n}(0,r_0)$.

By Lemma 5.3, we see that $G_h[\psi](x)$ is uniformly convergent in $\mathbb{B}^n(0, r_0)$. Then for any $\varepsilon_1 > 0$, there exist constants $t_1 = t_1(\varepsilon_1) \to 1^-$ and $t_2 = t_2(\varepsilon_1) \to 0^+$ such that for any $x \in \mathbb{B}^n(0,r_0),$

$$\mathbb{B}^n(x,t_2) \subset \mathbb{B}^n(0,t_1),$$

$$\left| \int_{\mathbb{B}^n(0,t_2)} G_h(x,y)\psi(y) \, d\tau(y) \right| < \varepsilon_1 \text{ and } \left| \int_{\mathbb{B}^n(x,t_2)} G_h(x,y)\psi(y) \, d\tau(y) \right| < \varepsilon_1.$$

Then

(5.4) \begin{align*}
|G_h[\psi](x_0 + \Delta x) - G_h[\psi](x_0)|
\leq & \left| \int_{\mathbb{B}^n(0,t_2) \setminus \mathbb{B}^n(x_0,t_2)} \left[ G_h(x_0 + \Delta x, y) - G_h(x_0, y) \right] \psi(y) \, d\tau(y) \right| \\
& + \left| \int_{\mathbb{B}^n(x_0,t_2)} \left[ G_h(x_0 + \Delta x, y) - G_h(x_0, y) \right] \psi(y) \, d\tau(y) \right| \\
& + \left| \int_{\mathbb{B}^n \setminus \mathbb{B}^n(0,t_1)} \left[ G_h(x_0 + \Delta x, y) - G_h(x_0, y) \right] \psi(y) \, d\tau(y) \right| \\
\leq & \left| \int_{\mathbb{B}^n(0,t_2) \setminus \mathbb{B}^n(x_0,t_2)} \left[ G_h(x_0 + \Delta x, y) - G_h(x_0, y) \right] \frac{\psi(y)}{(1 - |y|^2)^n} \, d\nu(y) \right| + 4\varepsilon_1.
\end{align*}
By (2.18), it is easy to see that the map \((x, y) \to G_h(x, y)\) is continuous (also uniformly continuous) on \(\mathbb{B}^n(0, 1/t_2) \times (\mathbb{B}^n(0, t_1) \setminus \mathbb{B}^n(x_0, t_2))\). Therefore, there exists \(t' = t'(\varepsilon_1) < 1/t_2\) such that for all \(|\Delta x| < t'\) and for all \(y \in \mathbb{B}^n(0, t_1) \setminus \mathbb{B}^n(x_0, t_2)\),
\[
(G_h(x_0 + \Delta x, y) - G_h(x_0, y)) \frac{\psi(y)}{(1 - |y|^2)^n} < \varepsilon_1.
\]
Thus it follows from (5.4) and (5.5) that
\[
|G_h[\psi](x_0 + \Delta x) - G_h[\psi](x_0)| \leq 5\varepsilon_1,
\]
which means that \(G_h[\psi]\) is continuous at \(x_0\). Hence the arbitrariness of \(x_0\) shows that \(G_h[\psi](x)\) is continuous in \(\mathbb{B}^n(0, r_0)\).

By applying Lemma 5.4, the continuity of
\[
\int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) d\tau(y)
\]
in \(\mathbb{B}^n(0, r_0)\) can be proved in a similar way as above, where \(k \in \{1, \ldots, n\}\). So the proof of this lemma is complete. \(\square\)

The following property is the second main lemma for the proof of Theorem 1.4.

**Lemma 5.6.** Suppose \(n \geq 3\), \(\psi \in C(\mathbb{B}^n, \mathbb{R}^n)\) and \(|\psi(x)| \leq M(1 - |x|^2)\) in \(\mathbb{B}^n\), where \(M\) is a constant. Then for all \(x \in \mathbb{B}^n\) and \(k \in \{1, 2, \ldots, n\}\),
\[
\frac{\partial \Psi}{\partial x_k}(x) = \int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) d\tau(y).
\]

**Proof.** For all \(x \in \mathbb{B}^n\), by Lemma 5.4, we see that
\[
\int_0^{x_k} \int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \frac{\psi(y)}{(1 - |y|^2)^{n-1}} d\nu(y) d\tau(x) \leq \beta_1,
\]
where \(k \in \{1, \ldots, n\}\). It follows from Fubini’s theorem [32, p. 165] that
\[
\int_0^{x_k} \int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) d\tau(y) d\tau(x) = \int_{\mathbb{B}^n} \int_0^{x_k} \frac{\partial}{\partial x_k} G_h(x, y) \frac{\psi(y)}{(1 - |y|^2)^n} d\tau(x) d\nu(y),
\]
which means
\[
\int_0^{x_k} \int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) d\tau(y) d\tau(x) = \int_{\mathbb{B}^n} \frac{G_h(x, y)}{(1 - |y|^2)^n} \psi(y) d\nu(y) - \int_{\mathbb{B}^n} \frac{G_h(x_{k,0}, y)}{(1 - |y|^2)^n} \psi(y) d\nu(y),
\]
where \(x_{k,0} = (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)\). Since \(\int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) d\tau(y)\) is continuous in \(\mathbb{B}^n(0, r_0)\), by differentiating w.r.t. \(x_k\), we get
\[
\int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) d\tau(y) = \frac{\partial}{\partial x_k} \int_{\mathbb{B}^n} G_h(x, y) \frac{\psi(y)}{(1 - |y|^2)^n} d\nu(y).
\]
Hence the proof of this lemma is finished. \(\square\)
Proof of Theorem 1.4. Lemmas 5.4(2), 5.6 and Cauchy-Schwarz inequality imply that

\[(5.6) \quad \|D\Psi(x)\| = \sup_{\xi \in \mathbb{R}^{n-1}} |D\Psi(x)\xi| \leq \left( \sum_{k=1}^{n} \left| \frac{\partial \Psi}{\partial x_k}(x) \right|^2 \right)^{\frac{1}{2}} \leq \sqrt{n}\beta_1 =: \beta_0,\]

where \(\beta_1 = \beta_1(n, M)\) is the same constant as in Lemma 5.4(2). For any \(x, y \in \mathbb{B}^n\), let \(\gamma_{[x,y]}\) denote the segment between \(x\) and \(y\). By Lemmas 5.5 and 5.6, we know that \(\Psi \in C^1(\mathbb{B}^n)\), and hence \(\Psi\) is differentiable. Then the mean-value theorem of differentials leads to

\[|\Psi(x) - \Psi(y)| \leq \beta_0|x - y|,\]

The proof of Theorem 1.4 is finished. \(\square\)

Based on Theorems 1.3 and 1.4, we are going to prove Theorem 1.2.

Proof of Theorem 1.2. In this subsection, we always regard a point \(x = (x_1, \ldots, x_n)\) in \(\mathbb{R}^n\) as a column vector, for purposes of computing matrix products (which have been denoted by \(\times\)).

For any \(x, y \in \mathbb{B}^n\), by letting \(\omega(t) = Lt\) in Lemma 4.4, we obtain that for \(x \in \mathbb{B}^n\),

\[(5.7) \quad \|D\Phi(x)\| \leq L\alpha_0,\]

where \(L\) is the same constant as in Theorem 1.2. It follows from the mean-value theorem of differentials that

\[|\Phi(x) - \Phi(y)| \leq L\alpha_0|x - y|,\]

and so Theorem 1.4 gives

\[|u(x) - u(y)| \leq |\Phi(x) - \Phi(y)| + |\Psi(x) - \Psi(y)| \leq (L\alpha_0 + \beta_0)|x - y|.\]

Let \(C_1 = L\alpha_0 + \beta_0\). Then the Lipschitz continuity of \(u\) in Theorem 1.2 is proved.

Next, we prove the co-Lipschitz continuity of \(u\). To this end, we need to find a constant \(C_2 = C_2(n, \phi, \psi)\) such that for \(x, y \in \mathbb{B}^n\),

\[|u(x) - u(y)| \geq C_2|x - y|.\]

For this, we need to obtain an expression of \(Du(0)\) in terms of \(\phi\) and \(\psi\). Since (2.22) and Lemma 4.1 imply that

\[D\Phi(0) = D\Phi(x)|_{x=0} = \int_{\mathbb{S}^{n-1}} D(P_h(x, \eta)\phi(\eta))|_{x=0} d\sigma(\eta) = 2(n-1) \int_{\mathbb{S}^{n-1}} \phi(\eta) \times \eta^T d\sigma(\eta),\]

and since (5.3), Lemmas 5.5 and 5.6 lead to

\[D\Psi(0) = D\Psi(x)|_{x=0} = \int_{\mathbb{B}^n} D(G_h(x, y)\psi(y))|_{x=0} d\tau(y) = \frac{1}{n} \int_{\mathbb{B}^n} \frac{\psi(y) \times y^T}{|y|^n(1 - |y|^2)} d\nu(y),\]

we see that

\[Du(0) = D\Phi(0) - D\Psi(0) = 2(n-1) \int_{\mathbb{S}^{n-1}} \phi(\eta) \times \eta^T d\sigma(\eta) - \frac{1}{n} \int_{\mathbb{B}^n} \frac{\psi(y) \times y^T}{|y|^n(1 - |y|^2)} d\nu(y),\]

which is what we need, since obviously, \(Du(0)\) depends only on \(n, \phi\) and \(\psi\).
Let \( \varrho = l(Du(0)) \). Obviously, \( \varrho = \varrho(n, \phi, \psi) \).

Now, we are ready to find the needed \( C_2 \). Let \( \gamma_{[x,y]} \) denote the segment between \( x \) and \( y \), with the parametrization \( r(t) = (1-t)x + ty \), where \( t \in [0,1] \). By the well-known gradient theorem (see, e.g. [31, Theorem 6.24]),

\[
\int_{\gamma_{[x,y]}} \nabla u_j(r) \cdot dr = \int_0^1 \nabla u_j(r(t)) \cdot r'(t) dt = \int_0^1 \frac{d}{dt} (u_j \circ r(t)) dt = u_j(y) - u_j(x),
\]

for all \( j = 1,\ldots,n \). Note that

\[
Du(r(t)) \times r'(t) = \begin{pmatrix} \nabla u_1(r(t)) \cdot r'(t) \\ \vdots \\ \nabla u_n(r(t)) \cdot r'(t) \end{pmatrix},
\]

and hence,

\[
\int_0^1 Du(r(t)) \times r'(t) dt = u(y) - u(x).
\]

By (5.6) and (5.7), we have that

\[
\|Du(r) - Du(0)\| \leq \|Du(r)\| + \|Du(0)\| \leq 2(L\alpha_0 + \beta_0).
\]

Note that \(|r'(t)| = |x - y|\), and thus we obtain

\[
|u(x) - u(y)| = \left| \int_0^1 Du(r(t)) \times r'(t) dt \right| \\
\geq \left| \int_0^1 Du(0) \times r'(t) dt \right| - \int_{\gamma_{[x,y]}} \|Du(r) - Du(0)\| \|dr\| \\
\geq (\varrho - 2(L\alpha_0 + \beta_0)) |x - y|,
\]

and so we can take \( C_2 = \varrho - 2(L\alpha_0 + \beta_0) \). Hence the proof of Theorem 1.2 is complete.

6. Example

In this section, we will construct an example to show that the requirement \( n \geq 3 \) in Theorem 1.2 is necessary.

**Example 6.1.** Let \( w_0(r^i\theta) = \sum_{k=1}^{\infty} \frac{r^k}{k!} \cos(k\theta) - \frac{M}{4}(1 - r^2) \) in \( \mathbb{D} \), where \( M \) is a non-negative constant. Then

1. \( w_0 \in C^2(\mathbb{D}, \mathbb{R}) \cap C(\overline{\mathbb{D}}, \mathbb{R}) \) and \( \Delta w_0 = M(1 - |z|^2)^2 \);
2. \( w_0 \) is not Lipschitz continuous in \( \mathbb{D} \);
3. \( w_0|_{\mathbb{S}^1} \) is Lipschitz continuous in \( \mathbb{S}^1 \).

**Proof.** To prove that the function \( w_0 \) has the desired properties, let

\[
f(z) = \sum_{k=1}^{\infty} \frac{z^k}{k!}.
\]
Then for $z \in \mathbb{D}$,

$$w_0(z) = \text{Ref}(z) - \frac{M}{4}(1 - |z|^2).$$

Obviously, $\text{Ref} = P[\phi_0]$ is harmonic in $\mathbb{D}$, where $\phi_0(e^{i\theta}) = \sum_{k=1}^\infty \frac{1}{k} \cos(k\theta)$ is continuous in $\mathbb{S}^1$, and thus $\text{Ref} \in C^2(\mathbb{D},\mathbb{R}) \cap (\mathbb{D},\mathbb{R})$. By elementary computations, we see that $\Delta w_0 = M$. Hence the first assertion in the example holds.

Since

$$\frac{\partial}{\partial z} w_0(z) = \frac{1}{2} \sum_{k=1}^\infty \frac{z^{k-1}}{k} + \frac{M}{4} \frac{z}{z} = -\frac{\log(1 - z)}{2z} + \frac{M}{4} \frac{z}{z},$$

and

$$\frac{\partial}{\partial \overline{z}} w_0(z) = \frac{1}{2} \sum_{k=1}^\infty \frac{\overline{z}^{k-1}}{k} + \frac{M}{4} \overline{z} = -\frac{\log(1 - \overline{z})}{2\overline{z}} + \frac{M}{4} \overline{z},$$

we easily see that

$$\|Dw_0(z)\| = \left| \frac{\partial}{\partial z} w_0(z) \right| + \left| \frac{\partial}{\partial \overline{z}} w_0(z) \right|$$

is unbounded in $\mathbb{D}$.

**Claim 6.1.** The function $w_0$ is Lipschitz continuous if and only if $\|Dw_0\|$ is bounded.

For the proof, we let $\partial_\theta w_0(z)$ denote the directional derivative of $w_0$. If $w_0$ is Lipschitz continuous with Lipschitz constant $L_1$, then

$$|\partial_\theta w_0(z)| = \left| \lim_{r \to 0} \frac{w_0(z + re^{i\theta}) - w_0(z)}{r} \right| = \lim_{r \to 0} \frac{|w_0(z + re^{i\theta}) - w_0(z)|}{r} \leq L_1.$$

Hence it follows from the obvious fact $\|Dw_0(z)\| = \max_\theta |\partial_\theta w_0(z)|$ that

$$\|Dw_0(z)\| \leq L_1.$$

On the other hand, if $\|Dw_0(z)\| \leq L_1$, then the mean-value theorem of differentials leads to

$$|w_0(z_1) - w_0(z_2)| \leq L_1|z_1 - z_2|.$$

Hence the claim is true.

Since we have proved that $\|Dw_0(z)\|$ is unbounded, we see from Claim 6.1 that $w_0$ is not Lipschitz continuous in $\mathbb{D}$, which shows that the second assertion in the example holds too. The third assertion follows from [3, p. 317] as the construction of $w_0$ in $\mathbb{S}^1$ coincides with the one in [3]. □

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