ON THE BEST UPPER BOUND FOR PERMUTATIONS
AVOIDING A PATTERN OF A GIVEN LENGTH

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Abstract. Numerical evidence suggests that certain permutation patterns of length $k$ are easier to avoid than any other patterns of that same length. We prove that these patterns are avoided by no more than $(2.25k^2)^n$ permutations of length $n$. In light of this, we conjecture that no pattern of length $k$ is avoided by more than that many permutations of length $n$.

1. Introduction

1.1. Upper Bounds for Pattern Avoiding Permutations. The theory of pattern avoiding permutations has seen tremendous progress during the last two decades. The key definition is the following. Let $k \leq n$, let $p = p_1p_2\cdots p_n$ be a permutation of length $n$, and let $q = q_1q_2\cdots q_k$ be a permutation of length $k$. We say that $p$ avoids $q$ if there are no $k$ indices $i_1 < i_2 < \cdots < i_k$ so that for all $a$ and $b$, the inequality $p_{i_a} < p_{i_b}$ holds if and only if the inequality $q_a < q_b$ holds. For instance, $p = 2537164$ avoids $q = 1234$ because $p$ does not contain an increasing subsequence of length four. See [3] for an overview of the main results on pattern avoiding permutations.

Let $S_n(q)$ be the number of permutations of length $n$ (or, in what follows, $n$-permutations) that avoid the pattern $q$. Since the spectacular result of Adam Marcus and Gábor Tardos [11], it is known that for every pattern $q$, there exists a constant $c_q$ so that the inequality $S_n(q) \leq c_q^n$ holds for all $n$. As there are only $k!$ patterns of length $q$, it follows that for all positive integers $k$, there exists a constant $c_k$ so that for all patterns $q$ of length $k$, the inequality

$$S_n(q) \leq c_k^n$$

holds for all positive integers $n$.

However, the quest of finding the best constant $c_k$ is in [11], is wide open. The result of Marcus and Tardos [11] has only shown that $c_k \leq 152^{k^4}(k^4)$. Josef Cibulka [8] has improved this bound by showing that $c_k \leq 2^{O(k \log k)}$, but even this bound seems to be very far from reality, as we will explain.

Date: May 2, 2014.
Richard Arratia [2] has conjectured that $c_k = (k-1)^2$ is sufficient for all $k$, but this conjecture was refuted by Albert and al [1], who proved that if $n$ is large enough, then $S_n(1324) \geq 9.42^n$.

1.2. Layered Patterns. A layered pattern is a pattern consisting of decreasing subsequences (the layers) so that the entries decrease within the layers but increase among the layers, as in 3215476. Equivalently, a pattern is layered if and only if it avoids both 231 and 312. Layered permutations are important since numerical evidence (computed first by Julian West [13] and later replicated by many others) supports the following conjecture.

**Conjecture 1.1.** Let $q$ be a non-layered pattern of length $k$, and let $Q$ be a layered pattern of length $k$. Then for all positive integers $n$, the inequality

\[(2) \quad S_n(q) \leq S_n(Q)\]

holds.

If Conjecture 1.1 holds, then any upper bound that we can prove for all layered patterns of length $k$ is also an upper bound for all patterns of length $k$. This has motivated several attempts to find a constant $CL_k$ so that $S_n(Q) \leq CL_k^n$ for all layered patterns $Q$ of length $k$. It follows from results in [5], [6] and [7] that $CL_k \leq O(2^k)$. A much stronger, recent result of Anders Claesson, Vit Jelinek and Einar Steingrímsson [9] shows that $CL_k \leq 4k^2$ holds.

Let $M_{2m}$ denote the pattern $132 \cdots (2m-1)(2m-2)2m$, and let $M_{2m-1}$ denote the pattern obtained from $M_{2m}$ by removing the first entry and then relabeling. That is, $M_{2m-1} = 21 \cdots (2m-2)(2m-3)(2m-1)$. So for instance, $M_3 = 213$, and $M_4 = 1324$, while $M_5 = 21435$, and $M_6 = 132546$.

A different version of Conjecture 1.1 also supported by numerical evidence, is the following.

**Conjecture 1.2.** Let $m \geq 2$. Then

(A) for all positive integers $n$, and for all patterns $q$ of length $2m - 1$, the inequality

\[S_n(q) \leq S_n(M_{2m-1})\]

holds, and

(B) for all positive integers $n$, and for all patterns $q$ of length $2m$, the inequality

\[S_n(q) \leq S_n(M_{2m})\]

holds.

In this paper, we will prove that if $k = 2m - 1$, then the inequality $S_n(M_{2m-1}) \leq (2.25k^2)^n$ holds, and if $k = 2m$, then the inequality $S_n(M_{2m}) \leq (2.25k^2)^n$ holds. This means that if Conjecture 1.2 holds, then $c_k \leq 2.25k^2$. (In fact, we prove a slightly stronger upper bound for $c_k$.) Note that $c_k \geq (k-1)^2$ has been known since Amitaj Regev’s paper [12].

Our proof will be inductive, but even the initial case of our induction will depend on a result that has only been recently proved.
2. Composing and Decomposing Patterns

The following useful definitions describe two simple but crucial ways in which patterns can be composed.

**Definition 2.1.** Let $q$ be a pattern of length $k$ and let $t$ be a pattern of length $m$. Then $q \oplus t$ is the pattern of length $k + m$ defined by

$$(q \oplus t)_i = \begin{cases} 
  q_i & \text{if } i \leq k, \\
  t_{i-k} + k & \text{if } i > k.
\end{cases}$$

In other words, $q \oplus t$ is the concatenation of $q$ and $t$ so that all entries of $t$ are increased by the size of $q$.

**Example 2.2.** If $q = 3142$ and $t = 132$, then $q \oplus t = 3142576$.

**Definition 2.3.** Let $q$ be a pattern of length $k$ and let $t$ be a pattern of length $m$. Then $q \ominus t$ is the pattern of length $k + m$ defined by

$$(q \ominus t)_i = \begin{cases} 
  q_i + m & \text{if } i \leq k, \\
  t_i - k & \text{if } i > k.
\end{cases}$$

In other words, $q \ominus t$ is the concatenation of $q$ and $t$ so that all entries of $q$ are increased by the size of $t$.

**Example 2.4.** If $q = 3142$ and $t = 132$, then $q \ominus t = 6475132$.

The following strong theorem of Claesson, Jelinek and Steingrímsson describes an important way in which permutations avoiding a long given pattern of a specific kind can be decomposed into two permutations, each of which avoids a shorter pattern.

**Theorem 2.5.** [9] Let $\sigma$, $\tau$, and $\rho$ be three permutations. Let $p$ be a permutation that avoids $\sigma \oplus (\tau \ominus 1) \ominus \rho$. Then it is possible to color each entry of $p$ red or blue so that the red entries of $p$ form a $\sigma \oplus (\tau \ominus 1)$-avoiding permutation and the blue entries of $p$ form a $(\tau \ominus 1) \ominus \rho$-avoiding permutation.

**Example 2.6.** Let $\sigma$, $\tau$, and $\rho$ each be the one-element pattern 1. Then Theorem 2.5 says that it is possible to color the entries of a 1324-avoiding permutation red or blue so that the red entries form a 132-avoiding permutation and the blue entries form a 213-avoiding permutation.

**Proof.** (of Theorem 2.5) Color the entries of $p$ one by one, going left to right, according to the following three rules.

1. If coloring $p_i$ red creates a red copy of $\sigma \oplus (\tau \ominus 1)$, then color $p_i$ blue.
2. If $p_i$ is larger than a blue entry on its left, then color $p_i$ blue.
3. Otherwise color $p_i$ red.

It is then proved in [9] that this coloring has the required properties. □

**Definition 2.7.** The coloring defined in the preceding proof is called the canonical coloring of $p$ (with respect to $\sigma$, $\tau$, and $\rho$).
Example 2.8. Let $\sigma$, $\tau$, and $\rho$ each be the one-element pattern as in Example 2.6, and let $p = 3612745$. Then $p$ is a 1324-avoiding permutation. In the canonical coloring of $p$ with respect to $\sigma$, $\tau$, and $\rho$, the red entries are 3, 6, 1, 2, and 7, while the blue entries are 4 and 5. It is easy to verify that the string of red entries avoids 132, while the string of blue entries avoids 213.

3. An Inductive Argument

In this section, we will present an inductive argument that shows how $M_{2m-1}$-avoiding and $M_{2m}$-avoiding permutations can be injectively mapped into pairs of certain words. The precise statement will be made in Lemma 3.9. Even the initial steps of this argument are not obvious. The argument for $M_3$-avoiding permutations (that is, 213-avoiding permutations), is not surprising. The argument for $M_4$-avoiding permutations has only been recently found [4]. These two arguments are given in Section 3.1, before the general result is announced in Section 3.2.

3.1. The Initial Steps. Let $p = p_1p_2 \cdots p_n$. We say that $p_i$ is a right-to-left maximum in $p$ if it is larger than all entries on its right, that is, if $p_i > p_j$ for all $j > i$. We always consider $p_n$ a right-to-left maximum, since the condition is vacuously true for that entry.

Let $V_2(n)$ be the set of all words of length $n$ over the alphabet $\{0, 1\}$.

Proposition 3.1. Let $p = p_1p_2 \cdots p_n \in \text{Av}_n(213)$, let $v(p_i) = 0$ if $p_i$ is not a right-to-left maximum, and let $v(p_i) = 1$ if $p_i$ is a right-to-left maximum. Set

$$v(p) = v(p_1)v(p_2) \cdots v(p_n)$$

and

$$v'(p) = v(1)v(2) \cdots v(n).$$

Then the map $f_3 : \text{Av}_n(213) \rightarrow V_2(n) \times V_2(n)$ defined by $f(p) = (v(p), v'(p))$ is injective.

Example 3.2. If $p = 35412$, then $v(p) = 01101$, and so $v'(p) = 01011$. Therefore, $f(35412) = (01101, 01011)$.

Proof. Let $(v, v') \in V_2(n) \times V_2(n)$, and let us assume that there is a permutation $p \in \text{Av}_n(213)$ so that $f_3(p) = (v, v')$. Then the positions of the 1s in $v$ reveal the positions in which $p$ must have right-to-left maxima, and the positions of 1s in $v'$ reveal which entries of $p$ are right-to-left maxima. It follows from the definition of right-to-left maxima that the right-to-left maxima of $p$ must be in decreasing order from left to right.

Once the right-to-left maxima of $p$ are in place, there is only one way to insert the remaining entries in the remaining slots. Indeed, going from right to left, in each step we must place the largest eligible remaining entry (that is, the largest one whose insertion does not change the set of right-to-left maxima). Indeed, if at some point during this procedure we placed an entry
and so by the second rule of canonical coloring (as given in Theorem 2.5),

\[ w(x) = w(p_1)w(p_2) \cdots w(p_n) \]

and

\[ w'(p) = w(1)w(2) \cdots w(n). \]

Then the map \( f_4 : Av_n(1324) \to W_4(n) \times W_4(n) \) defined by \( f_4(p) = (w(p), w'(p)) \) is injective.

Example 3.4. If \( p = 3612745 \), then we get \( w(p) = 1212234 \), and \( w'(p) = 1213424 \), and we can easily see that neither \( w(p) \) nor \( w'(p) \) contains a 32-factor.

Proof. (of Lemma 3.3) Let \( f_4(p) = (w(p), w'(p)) \), and let us assume that \( w(p) \) contains a 32-factor, that is, there exists an index \( i \) so that \( w(p_i) = 3 \) and \( w(p_{i+1}) = 2 \). That means that in particular, \( p_i \) is blue and \( p_{i+1} \) is red, so by the second rule of canonical coloring (as given in Theorem 2.5), \( p_i > p_{i+1} \). As \( p_{i+1} \) is not a left-to-right minimum, there is an entry \( p_j \) with
\[ j < i \] so that \( p_j < p_{i+1} \). Similarly, as \( p_i \) is not a right-to-left maximum, there is an entry \( p_\ell \) with \( \ell > i + 1 \) so that \( p_\ell > p_i \). However, that means that \( p_j p_i p_{i+1} p_\ell \) is a 1324-pattern, which is a contradiction. An analogous argument (see [4]) shows that \( w'(p) \) also avoids 1324. So \( f_4 \) indeed maps into \( W_4(n) \times W_4(n) \).

In order to see that \( f_4 \) is injective, we proceed in a way that is similar to (but slightly more complex than) the way in which we proceeded in the proof of Proposition 3.1. Let \( (w, w') \in W_4(n) \times W_4(n) \), and let us assume that there exists \( p \in Av_n(1324) \) so that \( f_4(p) = (w, w') \). Then the positions of 3s and 4s in \( w \) reveal \textit{where} the blue entries of \( p \) are, and the positions of 3s and 4s in \( w' \) reveal \textit{what} the blue entries of \( p \) are. By Proposition 3.1 this is sufficient information to recover the entire string of blue entries, since that string is a 213-avoiding permutation. A dual argument works for the string of red entries, since the red entries form a 132-avoiding permutation. \( \square \)

3.2. The Induction Step. In this part of our proof, we will often obtain an encoding of a long permutation by partitioning it into two parts, encoding each part by disjoint alphabets, then combining the two images into one. The following definition makes this concept more precise. If \( s \) is a substring of the permutation \( p \), let \(|s|\) denote the length (number of entries) of \( s \).

**Definition 3.5.** Let \( p = p_1p_2 \cdots p_n \) be a permutation, and let \( p' \) and \( p'' \) be two substrings of \( p \) so that each entry of \( p \) belongs to exactly one of \( p' \) and \( p'' \).

Let us assume that \( a(p') \) and \( a'(p') \) are words of length \(|p'|\) over a finite alphabet \( A \), and \( b(p'') \) and \( b'(p'') \) are words of length \(|p''|\) over a finite alphabet \( B \) that is disjoint from \( A \).

Then the merge of \( a(p') \) and \( b(p'') \) is the word \( w(p) = w_1w_2 \cdots w_n \) of length \( n \) over the finite alphabet \( A \cup B \) whose \( i \)th letter \( w_i \) is obtained as follows.

1. If \( p_i \) is the \( r \)th letter of \( p' \), then \( w_i \) is equal to the \( r \)th letter of \( a(p') \), and

2. If \( p_i \) is the \( r \)th letter of \( p'' \), then \( w_i \) is the \( r \)th letter of \( b(p'') \).

Furthermore, the merge of \( a'(p') \) and \( b'(p'') \) is the word \( w'(p) = w'_1w'_2 \cdots w'_n \) obtained as follows.

1. If the entry \( i \) of \( p \) is the \( t \)th smallest entry of \( p' \), then \( w'_i \) is equal to the \( t \)th letter of \( a'(p') \), and,

2. if the entry \( i \) of \( p \) is the \( t \)th smallest entry of \( p'' \), then \( w'_i \) is equal to the \( t \)th letter of \( b'(p'') \).

**Example 3.6.** Let \( p = 178942365 \), let \( p' = 12 \), and let \( p'' = 7894365 \). Let \( a(p') = 11 \), and let \( a(p'') = 11 \). Furthermore, let \( b(p') = 2222333 \), and let \( b'(p'') = 2232232 \).

Then we have \( w(p) = 12221233 \) and \( w'(p) = 112233222 \).

The following definition extends the notion of merges from words to functions in a natural way.
Definition 3.7. Let \( p, p' \) and \( p'' \) be as in Definition 3.5, and let \( f(p') = (v_1, v_2) \), and \( g(p'') = (w_1, w_2) \), where the \( v_i \) are words over the finite alphabet \( A \), and the \( w_i \) are words over the finite alphabet \( B \) that is disjoint from \( A \). Then we say that the function \( h \) is the merge of \( f \) and \( g \) if \( h(p) = (z_1, z_2) \), where \( z_1 \) is the merge of \( v_1 \) and \( w_1 \), and \( z_2 \) is the merge of \( v_2 \) and \( w_2 \).

Example 3.8. Let \( p = 687912435, p' = 612, \) and \( p'' = 879435. \) Let \( f(p') = (000,000), \) and let \( g(p'') = 112112 \). If \( h \) is the merge of \( f \) and \( g \), then \( h(p) = (0112001112, 0011201112) \).

Recall that \( M_{2m} \) denotes the pattern 132\( \cdots (2m - 1)(2m - 2)2m \), and \( M_{2m-1} \) denotes the pattern obtained from \( M_{2m} \) by removing the first entry and then relabeling. That is, \( M_{2m-1} = 2143 \cdots (2m - 2)(2m - 3)(2m - 1) \). So \( M_4 = 1324 \), while \( M_5 = 21435 \), and \( M_6 = 132546 \).

In order to make the statement and proof of the following lemma easier to follow, we make the following general remark about the indices used in the lemma. The lemma will describe injections from certain sets of \( q \)-avoiding permutations into sets of pairs of certain words. These injections will be denoted by \( f_k \), where \( k \) is the length of \( q \). The co-domains of the injections \( f_k \) will be denoted by \( V_a \) or \( W_a \), where \( a \) denotes the length of the words in the co-domain of the \( f_k \).

Lemma 3.9. Let \( m \geq 2 \). Let \( Av_n(M_t) \) denote the set of all \( M_t \)-avoiding \( n \)-permutations.

(a) Let \( V_{3m-4}(n) \) denote the set of all words of length \( n \) over the alphabet \( \{0,1,2,\cdots,3m-5\} \) that do not have any \( (3i)(3i-1) \)-factors for any \( i \geq 1 \).

Then there is an injection

\[
f_{2m-1} : Av_n(M_{2m-1}) \rightarrow V_{3m-4}(n) \times V_{3m-4}(n).
\]

(b) Let \( W_{3m-2}(n) \) denote the set of all words of length \( n \) over the alphabet \( \{1,2,\cdots,3m-2\} \) that do not have any \( (3i)(3i-1) \)-factors for any \( i \geq 1 \).

Then there is an injection

\[
f_{2m} : Av_n(M_{2m}) \rightarrow W_{3m-2}(n) \times W_{3m-2}(n).
\]

Proof. We prove the statements by induction on \( m \). For \( m = 2 \), the statements are true. Indeed, for \( m = 2 \), statement (a) is just the statement of Proposition 3.1, and statement (b) is just the statement of Lemma 3.3.

Now let us assume that the statements are true for \( m \), and let us prove them for \( m + 1 \).

(a) First, we prove statement (a). Let \( p \in Av_n(M_{2m+1}) \). Color all entries of \( p \) that are the leftmost entry of an \( M_{2m} \)-pattern in \( p \) green, and color all other entries of \( p \) yellow. Then, by definition, the string of all yellow entries of \( p \) forms an \( M_{2m} \)-avoiding permutation \( p'' \). By part (b) of our induction hypothesis, the map \( f_{2m} \) injectively maps this permutation \( p'' \) into a pair of words \( (w_1, w_2) \in W_{3m-2}(|p''|) \times W_{3m-2}(|p''|) \).
In order to define the image $f_{3m-1}(p) \in V_{3m-1} \times V_{3m-1}$, simply mark all green entries of $p$ by the letter 0. Let $p'$ be the string of all green entries, and, to keep consistency with Definition 3.5, let $g(p') = (00\ldots0, 00\ldots0)$, where both strings of 0s are of length $|p'|$. Then we define $f_{2m+1}(p)$ as the merge of $g(p')$ and $f_{2m}(p')$ as defined in Definition 3.7.

Example 3.10. For $p = 687912435$, the reader is invited to revisit Example 3.8. With our current terminology, $p'$ is the string of green entries, $p''$ is the string of yellow entries, and $h = f_5$.

It is clear that $f_{2m+1}(p)$ indeed does not have a $(3i)(3i-1)$-factor for any positive integer $i$, since positive integers correspond to yellow entries of $p$, and the string of yellow entries avoids all such factors by the induction hypothesis.

In order to show that the map $f_{2m+1}$ is injective, let us assume that $(v_1, v_2) \in V_{3m-1}(n) \times V_{3m-1}(n)$ equals $f_{2m+1}(p)$ for some $p$. Then the positions of the yellow entries of $p$ are easy to recover, since these are the positions in which $v_1$ has a positive value. Similarly, the values of the yellow entries can be recovered as the positions in which $v_2$ has a positive entry. Once the place and values of the yellow entries of $p$ are found, the order in which these yellow entries is unique since the map $f_{2m}$ that is applied to the string of yellow entries is injective. So the injective property of $f_{2m}$ will be proved if we can show that there is only one way to place the green entries into the remaining slots.

Let us fill the remaining slots with the green entries going right to left. We claim that in each step, we must insert the largest remaining green entry that is eligible to go into the given position (that is, that will start an $M_{2m}$-pattern if inserted there). Indeed, let us assume that in a given position, we do not proceed as described. That is, both $x$ and $y$ are eligible to be inserted in a given position $P$, but we insert $x$, even if $x < y$. The fact that both $x$ and $y$ are eligible to be inserted in $P$ means that they both will be the first entry of an $M_{2m}$-pattern if inserted in $P$. Let these patterns be $xM$ and $yM'$. Then we will create an $M_{2m+1}$ pattern, namely the pattern $yxM'$ when we eventually insert $y$ somewhere on the left of $x$.

(b) Now we prove statement (b). Let $p \in Av_n(M_{2m+2})$. Then Theorem 2.5 (with $\sigma = 1$, $\tau = 1$ and $\rho = M_{2m-1}$) shows that it is possible to color the entries of $p$ red or blue so that the red entries form a 132-avoiding permutation and the blue entries form an $M_{2m+1}$-avoiding permutation. Let us consider the canonical coloring that achieves this and is given in the proof of Theorem 9.

Now we encode the string $p'$ of all red entries of $p$ in a manner that is analogous to what we saw in Proposition 3.1. We can do so, since the $p'$ is a 132-avoiding permutation. To be more precise, mark
each entry of \( p' \) that is a left-to-right minimum in \( p' \) by the letter 1. Mark all remaining letters of \( p' \) by the letter 2. Define the words \( a(p') \) and \( a'(p') \) as in Proposition \( \text{[3.1]} \). That is, let \( p' = p'_1p'_2 \cdots p'_{r} \), and let \( v(p'_i) = 1 \) if \( p'_i \) is a left-to-right minimum in \( p' \), and let \( v(p'_i) = 2 \) otherwise. Then set \( a(p') = v(p'_1)v(p'_2) \cdots v(p'_{r}) \), and set \( a'(p') = v(p'_{1})v(p'_{2}) \cdots v(p'_{r}) \), where \( p_{j1} < p_{j2} < \cdots < p_{jr} \).

The string \( p'' \) of blue entries of \( p \) forms an \( M_{2m+1} \)-avoiding permutation, so as we have just seen in the proof of statement (a), the string \( p'' \) can be injectively mapped into a pair of words \( (b(p''),b'(p'')) \in V_{3m-1} \times V_{3m-1} \) by the function \( f_{2m+1} \). Shift these letters by three, that is, turn each letter \( i \) into a letter \( i + 3 \) for all \( i \) in \( b(p'') \) and \( b'(p'') \). Finally, define \( f_{2m+2}(p) = (w,u') \), where \( w \) is the merge of \( a(p') \) and \( b(p'') \), and \( u' \) is the merge of \( a'(p') \) and \( b'(p'') \).

It is clear that \( f_{2m+2}(p) = (w,u') \) is a pair of words of length \( n \) over the alphabet \( \{1,2,\ldots,n\} \). It directly follows from the induction hypothesis that neither \( w \) nor \( u' \) can contain a \((3i)(3i-1)\)-factor for \( i > 1 \). There remains to show that neither \( w \) nor \( u' \) can contain a \( 32 \)-factor. In order to see this, let us assume that \( w \) contains a \( 32 \)-factor in its \( j \)th and \( (j+1) \)st positions. The type of an entry of a permutation is just the letter it is mapped into by \( f_{2m+2} \). That means that in particular, \( p_j \) is blue and \( p_{j+1} \) is red, so, by the second rule of canonical colorings (see the proof of Theorem \( \text{[2.5]} \), \( p_j > p_{j+1} \), since a blue entry cannot be followed by a larger red entry. As \( p_{j+1} \) is of type 2, it is not a left-to-right minimum, so there exists an index \( d < j \) so that \( p_d < p_{j+1} \). As \( p_j \) is of type 3, it is of type 0 in \( p'' \), so there is an \( M_{2m} \)-pattern \( p_jP \) in \( p'' \), and so, in \( p \), whose first entry is \( p_j \). However, that means that \( p_d p_j p_{j+1} P \) is an \( M_{2m+2} \)-pattern in \( p \), which is a contradiction. So \( w \) cannot contain a \( 32 \)-factor, and in an analogous way, nor can \( u' \).

Finally, we must show that \( f_{2m+2} \) is injective. By now, the method we used should not come as a surprise. Given a pair of words \( (w,u') \in V_{3m+1}(n) \times V_{3m+1}(n) \), we can recover the set and positions of the red entries, and the set of positions of the blue entries of \( p \), since the red entries are the ones that are of type 1 or 2. After this, it follows from Proposition \( \text{[3.1]} \) that we can recover the string of the red entries, and it follows from part (a) of this Lemma that we can recover the string of the blue entries.

\[ \square \]

4. Computing the Upper bounds

All there is left to do in order to find upper bounds on the numbers \( S_n(M_{2m}) \) and \( S_n(M_{2m-1}) \) is to find upper bounds on the sizes of the sets into which the relevant permutations can be injectively mapped. It would be straightforward to simply find an upper bound on the exponential growth
rate of these sequences, but we will carry out the slightly more cumbersome (but conceptually not difficult) task of finding upper bounds for the sequence in the sense we described in the introduction.

**Proposition 4.1.** For all integers \( m \geq 2 \), we have

\[
|W_{3m-2}(n)| = C_1(m) \cdot \beta_1^n + C_2(m) \beta_2^n,
\]

where \( \beta_1 = \frac{3m-2+\sqrt{9m^2-16m+8}}{2(m-1)} \) and \( \beta_2 = \frac{3m-2-\sqrt{9m^2-16m+8}}{2(m-1)} \), while \( C_1 = \frac{\beta_1}{\beta_1 - \beta_2} \) and \( C_2 = \frac{\beta_2}{\beta_2 - \beta_1} \).

**Proof.** Let \( b_0 = 1 \), and let \( b_n = W_{3m-2}(n) \) for \( n \geq 1 \). It is then easy to see that \( b_1 = 3m - 2 \), and

\[
b_n = (3m - 2)b_{n-1} - (m - 1)b_{n-2}
\]

for \( n \geq 2 \). Indeed, if we take an element of \( V_{3m-2(n-1)} \), and append one of our \( 3m - 2 \) letters to its end, we will get an element of \( W_{3m-2}(n) \), except in the \( (m - 1)b_{n-2} \) cases in which the last two letters of the new word form one of the forbidden factors.

Introducing the generating function \( B(x) = \sum_{n \geq 1} b_n x^n \), we can turn formula (3) into a functional equation. Solving that equation, we get

\[
B(x) = \frac{1}{1 - (3m - 2)x + (m - 1)x^2}.
\]

Finding the roots \( r_1 = \frac{3m-2+\sqrt{9m^2-16m+8}}{2(m-1)} \) and \( r_2 = \frac{3m-2-\sqrt{9m^2-16m+8}}{2(m-1)} \) of the denominator of \( B(x) \), we see that \( B(x) \) can be converted to the partial fraction form

\[
B(x) = \frac{C_1(m)}{1 - \frac{x}{\beta_1}} + \frac{C_2(m)}{1 - \frac{x}{\beta_2}},
\]

and our claim is now routine to prove. \qed

**Corollary 4.2.** For all even positive integers \( k \), the inequality

\[
S_n(M_k) \leq (2.25k^2)^n
\]

holds.

**Proof.** Let \( k = 2m \). Part (b) of Lemma 3.9 inductively constructs an injective map \( f_{2m} : A\nu_n(M_{2m}) \to W_{3m-2}(n) \times W_{3m-2}(n) \). That map is not bijective. Indeed, it is obvious from the definition of \( f_{2m} \) that if \( f_{2m}(p) = (w, w') \), then both \( w \) and \( w' \) must start with the letter 1.

Therefore, we know that

\[
S_n(M_k) \leq |W_{3m-2}(n-1)|^2 = (C_1(m)\beta_1^{n-1} + C_2(m)\beta_2^{n-1})^2.
\]

It is routine to verify that for all integers \( m > 1 \), the inequality \( \beta_2 < 1 \) holds. As \( \beta_1\beta_2 = m - 1 \), this means that \( \beta_1 > m - 1 \), and so \( C_1 = \frac{\beta_1}{\beta_1 - \beta_2} \), this implies the inequality \( C_1 < 2 \) for \( m \geq 3 \). This same inequality can be verified for \( m = 2 \), since in that case, \( \beta_1 = 2 + \sqrt{3} \), and \( \beta_2 = 2 - \sqrt{3} \).
Furthermore, $C_2 = \frac{\beta_2}{\beta_2 - \beta_1} < 0$ since the denominator is negative. Hence, (5) implies

$$S_n(M_k) \leq (2\beta_1 n - 1)^2 \leq \beta_{2n}.$$  

It is easy to prove from the definition of $\beta$ that for all integers $m > 1$, the inequality $\beta < 3m - 2$ holds. Therefore,

$$S_n(M_k) \leq \beta_{2n} < (3m - 2)^{2n} = (1.5k - 2)^{2n} = (2.25k^2 - 3k + 4)^n.$$  

Proposition 4.3. For all integers $m \geq 2$, we have

$$|V_{3m-2}(n)| = K_1 \cdot \alpha_1^n + K_2 \cdot \alpha_2^n$$

where $\alpha_1 = \frac{3m - 4 + \sqrt{9m^2 - 28 + 24}}{2}$, and $\alpha_2 = \frac{3m - 4 - \sqrt{9m^2 - 28 + 24}}{2}$, while $K_1 = \frac{\alpha_1}{\alpha_1 - \alpha_2}$ and $K_2 = \frac{\alpha_2}{\alpha_2 - \alpha_1}$.

Proof. Analogous to that of Proposition 4.1. \hfill \Box

Corollary 4.4. For all odd positive integers $k$, the inequality

$$S_n(M_k) \leq (2.25k^2)^n$$

holds.

Proof. The proof is analogous to that of Corollary 4.2. The only difference is that now we set $k = 2m - 1$, and then we use part (a) of Lemma 3.9. We get that

$$S_n(M_k) \leq |V_{3m-4}(n - 1)|^2 \leq \alpha_1^{2n}.$$  

So

$$S_n(M_k) \leq \alpha_1^{2n} < (3m - 4)^{2n} = (1.5k - 2.5)^{2n} = (2.25k^2 - 7.5k + 6.25)^n \leq (2.25k^2)^n.$$  

\hfill \Box

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