Improved entanglement indicators for optical fields

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Abstract

Better versions of separability conditions for four mode optical fields, i.e. two beams with two modes of mutually orthogonal polarisation are given. Our conditions involve variances and their use is physically intuitive. Namely, if for a given quantum state the spread of the data around its mean value is smaller than the minimal spread predicted for the set of separable states, then the given state is entangled. Our conditions are formulated for standard quantum Stokes observables and normalized Stokes observables. We test them for bright squeezed vacuum with (and without) induced non-gaussianity obtained by addition or subtraction of photons. We propose a practical experimental scheme of how to generate such states and compare our entanglement conditions with other entanglement indicators.

I. INTRODUCTION

Quantum entanglement stays beyond understanding based on classical correlations. Being an intrinsic property of nature it underpins fundamental phenomena and questions the idea of classical causality. It is then not surprising that entanglement has became an intensively developing research field. Besides, it finds, and most probably will be finding, multiple applications in quantum optics and imaging [1–3] quantum key distribution [4–7], and quantum computing [8–10]. It enables quantum teleportation, superdense coding and entanglement swapping [11–13].

However, despite multiple breakthroughs in the understanding of entanglement its detection remains an np-hard problem [14] and most of entangled qualifiers are sufficient entanglement conditions. Some of them turn out to be more effective than other ones for the particular quantum state. Consequently, there is still a need to construct different entanglement conditions e.g. entanglement indicators (aka witnesses) [15–18].

There is a multitude of entanglement conditions tailored for finite-dimensional scenarios i.e. qubits and qudits [19, 20]. However, entanglement of quantum optical fields with the undefined number of particles (excitations) is now of growing interest [21–23]. With the development of experimental techniques e.g. photon number resolution detectors, these theoretical concepts can be tested in the laboratory [24, 25], and gain practical value in terms of use in quantum technologies.

Entanglement indicators involving correlations of intensities of optical fields were pro-
posed in [26]. Their more efficient form was derived in [27] and several generalizations of these were given in [28, 29] and [30]. It has also been investigated in [31], that the variance of photon number differences in conjugate modes can also be considered as an witness of non-classicality in squeezed vacuum state. Here, we show a further refinement of these conditions, which leads to more efficient entanglement indicators.

All analyzed conditions are formulated for standard quantum optical observables and “normalized” Stokes observables, see e.g. [28]. They are tailored to be readily used for four-mode bright squeezed vacuum, which, essentially, is the two beams output of type 2 parametric down conversion (PDC) see e.g. [32, 33]. Such states are nowadays extensively used in emerging quantum technologies [34] because of its non-classical properties. Bright squeezed vacuum (BSV), also called a “super-singlet”, exhibits perfect anticorrelations in polarisation and perfect correlations in photon number. Still, it is a squeezed state, so a gaussian state. We suppose that introducing non-gaussianity might facilitate entanglement detection with Stokes measurement. Note that, introducing non-gaussianity is an interesting phenomena itself, vastly studied because of its multiple applications see e.g. [35–38] etc. Thus, it is worth to consider non-gaussian bright squeezed vacuum. We compare (theoretically) two common techniques of introducing non-gaussianity: photon addition and subtraction for bright squeezed vacuum. We add photons to one optical beam of BSV. Then we subtract the same amount of photons from the second beam. We realize that two resulting states have the same structure. Thus these processes are equivalent for bright squeezed vacuum. We propose a feasible setup to generate BSV with added/substracted photons. Finally, we compare the efficiency of discussed entanglement conditions for BSV and non-gaussian BSV.

II. IMPROVED ENTANGLEMENT INDICATORS FOR TWO BEAMS POLARIZATION ENTANGLED STATES

A. Standard vs. normalized Stokes operators

Entanglement conditions for quantum optical fields require proper observables, represented by self-adjoint operators, as prime sources of data. If one is interested in polarisation
measurement one can use the standard Stokes operators \([26]\) of the following form:

\[ \hat{\Theta}_i = \hat{I}_i - \hat{I}_i^\perp, \]  

(1)

where \(\hat{I}\) stands for intensity operator related with a given optiaca field and indices \(\{i, i^\perp\}\) denote two orthogonal polarization degrees of freedom of that field, related to given three mutually unbiased bases indexed with \(i = 1, 2, 3\) e.g.: \(i = 1\) can stand for \(\{+45^\circ, -45^\circ\}\) (diagonal/anti-diagonal basis), \(i = 2\) is for \(\{R, L\}\) (right and left-handed circular basis) and \(i = 3\) is for \(\{H, V\}\) (horizontal/vertical basis). The zeroth Stokes operator denotes total intensity of the beam: \(\hat{\Theta}_0 = \hat{I}_i + \hat{I}_i^\perp\).

From now on we will be using the model of intensity as proportional to numbers of photons i.e. \(\hat{I}_i = \hat{a}_i^\dagger \hat{a}_i\), where \(\hat{a}\) is an annihilation operator. However we emphasize that this is not the only possible model of intensity that can be used. Our choice is motivated by simplicity of theoretical description.

Normalized Stokes operators were first suggested in [39] in the context of Bose-Einstein condensates, and rediscovered for quantum optics in [28]. We follow the technical description given in [28]:

\[ \hat{S}_i = \hat{\Pi} \frac{\hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger_\perp \hat{a}_i^\perp}{\hat{N}}. \]  

(2)

where \(\hat{\Pi}\), defined as \(1 - |\Omega\rangle \langle \Omega|\), projects out the vacuum component of a given beam, that is states of the type: \(\hat{a}_i |0, 0\rangle = \hat{a}_i^\perp |0, 0\rangle = 0\). Note that \(\hat{S}_0 = \hat{\Pi}\).

Note that in an experiment, for a single run, the recorded values of standard and normalized Stokes observables are collected from the same set of data. It was shown in [28, 29] and [30] that normalized Stokes operators lead to stronger entanglement conditions. Also, in [30] it was shown that any linear entanglement witness for qudits can be effortlessly transformed into its quantum optical fields analog involving standard or normalized Stokes operators. Moreover, the discussion about standard and normalized Stokes operators in terms of photon numbers operators takes on practical meaning as photon number resolving detectors start to be used in the laboratory [40, 41].

B. Separability condition involving variances

In 2003, Simon and Bouwmeester [26] derived an entanglement condition based on EPR anticorrelations with standard Stokes operators
For the set of separable states, the following inequality holds

\[ \sum_{i=1}^{3} \langle(\hat{\Theta}_i^A + \hat{\Theta}_i^B)^2 \rangle_{\text{sep}} \geq 2\langle \hat{N}^A + \hat{N}^B \rangle_{\text{sep}}, \tag{3} \]

where \( \langle . \rangle_{\text{sep}} \) is an average for any separable state \( \rho_{\text{sep}}^{AB} \) of the composed system \( AB \).

In [28] one can find (3) formulated with normalized Stokes operators. It reads:

\[ \sum_{i=1}^{3} \langle(\hat{S}_i^A + \hat{S}_i^B)^2 \rangle_{\text{sep}} \geq \left\langle \hat{\Pi}^A \frac{2}{\hat{N}^A} \hat{\Pi}^A + \hat{\Pi}^B \frac{2}{\hat{N}^B} \hat{\Pi}^B \right\rangle_{\text{sep}}. \tag{4} \]

For EPR anticorrelated states we have \( \sum_{i=1}^{3} \langle(\hat{S}_i^A + \hat{S}_i^B)^2 \rangle_{\text{EPR}} = 0 \). However, conditions (3) and (4) are not equivalent. The latter one is more resistant to noise and losses, see [28].

In Ref. [30] stronger versions of entanglement indicators (3) and (4), were introduced:

\[ \sum_{i=1}^{3} \langle(\hat{\Theta}_i^A + \hat{\Theta}_i^B)^2 \rangle_{\text{sep}} \geq 2\langle \hat{N}^A + \hat{N}^B \rangle_{\text{sep}} + \langle(\hat{N}^A - \hat{N}^B)^2 \rangle_{\text{sep}}, \tag{5} \]

and

\[ \sum_{i=1}^{3} \langle(\hat{S}_i^A + \hat{S}_i^B)^2 \rangle_{\text{sep}} \geq \left\langle \hat{\Pi}^A \frac{2}{\hat{N}^A} \hat{\Pi}^A + \hat{\Pi}^B \frac{2}{\hat{N}^B} \hat{\Pi}^B \right\rangle_{\text{sep}} + \langle(\hat{\Pi}^A - \hat{\Pi}^B)^2 \rangle_{\text{sep}}. \tag{6} \]

In Ref. [27] and [42] the authors notice the following: in many experimental situations it is much better to use variances of the intensities, rather than the intensities themselves. Another separability condition which is stronger than (3) was derived:

\[ \Delta \hat{\Theta}_{\text{sep}}^{AB^2} = \sum_{i=1}^{3} \langle(\hat{\Theta}_i^A + \hat{\Theta}_i^B)^2 \rangle_{\text{sep}} - \sum_{i=1}^{3} \langle(\hat{\Theta}_i^A + \hat{\Theta}_i^B)^2 \rangle_{\text{sep}} \geq 2\langle \hat{N}^A + \hat{N}^B \rangle_{\text{sep}}. \tag{7} \]

In [43] condition 7 for normalized Stokes operators was formulated:

\[ \Delta \hat{S}_{\text{sep}}^{AB^2} = \sum_{i=1}^{3} \langle(\hat{S}_i^A + \hat{S}_i^B)^2 \rangle_{\text{sep}} - \sum_{i=1}^{3} \langle(\hat{S}_i^A + \hat{S}_i^B)^2 \rangle_{\text{sep}} \geq 2\left\langle \hat{\Pi}^A \frac{1}{\hat{N}^A} \hat{\Pi}^A + \hat{\Pi}^B \frac{1}{\hat{N}^B} \hat{\Pi}^B \right\rangle_{\text{sep}}. \tag{8} \]

Note that, if for all \( i = 1, 2, 3 \) one has \( \langle \hat{\Theta}_i^A + \hat{\Theta}_i^B \rangle = 0 \) the respective above conditions (7) and (8) are reduced to (3) and (4).

\[ \text{C. Stronger entanglement conditions based on variances} \]

Our aim is the further improvement of conditions (7) and (8). Let us define the product state of the optical field.

\[ \rho_{\lambda}^{AB} = f_{\lambda}^i(\hat{a}, \hat{a}_\perp)g_{\lambda}^i(\hat{b}, \hat{b}_\perp) |\Omega\rangle\langle\Omega| f_{\lambda}(\hat{a}, \hat{a}_\perp)g_{\lambda}(\hat{b}, \hat{b}_\perp), \tag{9} \]
where $f_\lambda(\hat{a}, \hat{a}_\perp)$ and $g_\lambda(\hat{b}, \hat{b}_\perp)$ are polynomial functions of annihilation operators acting on modes $a$ and $b$ related with parties $A$ and $B$ respectively. Mixed separable states for the studied problem have the following form

$$
\rho^{AB}_{\text{sep}} = \sum_\lambda p_\lambda f_\lambda^\dagger(\hat{a}, \hat{a}_\perp) g_\lambda^\dagger(\hat{b}, \hat{b}_\perp) |\Omega\rangle\langle\Omega| f_\lambda(\hat{a}, \hat{a}_\perp) g_\lambda(\hat{b}, \hat{b}_\perp),
$$

where the index $\lambda$ denotes summation over elements of convex combination of product states. It might be countable or continuous.

In further calculations the following basic properties of Stokes operators will be used:

$$
\sum_{i=1}^3 (\langle \hat{\Theta}^X_i \rangle_\lambda)^2 \leq \langle \hat{N}^X \rangle^2, \quad \text{and} \quad \sum_{i=1}^3 (\langle \hat{S}^X_i \rangle_\lambda)^2 \leq \langle \hat{\Pi}^X \rangle^2, \quad \text{where} \quad X = A, B.
$$

In this notation: $\langle \hat{O} \rangle_\lambda = \text{Tr}\{\hat{O}\rho^{AB}_\lambda\}$, where $\hat{O}$ stands for an operator and $\rho^{AB}_\lambda$ is an arbitrary product state. For any $\rho^{AB}_\lambda$ it holds that $\langle \Theta_i^A \Theta_j^B \rangle_\lambda = \langle \Theta_i^A \rangle_\lambda \langle \Theta_j^B \rangle_\lambda$ and $\langle S_i^A S_j^B \rangle_\lambda = \langle S_i^A \rangle_\lambda \langle S_j^B \rangle_\lambda$. Also, similarly to Pauli matrices $\{\sigma_i\}_{i=x,y,z}$, which can be put in form of Pauli vector $\vec{\sigma}$ one can construct Stokes vectors $\langle \vec{\Theta} \rangle = (\langle \hat{\Theta}_1 \rangle, \langle \hat{\Theta}_2 \rangle, \langle \hat{\Theta}_3 \rangle)$ and $\langle \vec{S} \rangle = (\langle \hat{S}_1 \rangle, \langle \hat{S}_2 \rangle, \langle \hat{S}_3 \rangle)$.

1. **Standard Stokes operators**

The variance of the Stokes vector $\Delta \vec{\Theta}^{AB^2}$ for separable state (10) reads

$$
\Delta \vec{\Theta}^{AB^2} = \sum_{i=1}^3 \Delta \hat{\Theta}_i^{A^2} + \sum_{i=1}^3 \Delta \hat{\Theta}_i^{B^2} + 2 \sum_{i=1}^3 \left( \langle \hat{\Theta}_i^A \hat{\Theta}_i^B \rangle - \langle \hat{\Theta}_i^A \rangle \langle \hat{\Theta}_i^B \rangle \right).
$$

We see that the global variance of Stokes vector for both subsystems is equal to the local variances and global covariance of Stokes vectors for Alice and Bob that can be positive or negative. It reads

$$
\sum_{i=1}^3 (\langle \hat{\Theta}_i^A \hat{\Theta}_i^B \rangle_{\text{sep}} - \langle \hat{\Theta}_i^A \rangle_{\text{sep}} \langle \hat{\Theta}_i^B \rangle_{\text{sep}}) = \sum_{i=1}^3 (\langle \hat{\Theta}_i^A - \langle \hat{\Theta}_i^A \rangle_{\text{sep}} \rangle_{\text{sep}}) (\langle \hat{\Theta}_i^B - \langle \hat{\Theta}_i^B \rangle_{\text{sep}} \rangle_{\text{sep}})
$$

$$
= \sum_\lambda \sum_{i=1}^3 p_\lambda (\langle \hat{\Theta}_i^A \rangle_\lambda - \langle \hat{\Theta}_i^A \rangle_{\text{sep}}) (\langle \hat{\Theta}_i^B \rangle_\lambda - \langle \hat{\Theta}_i^B \rangle_{\text{sep}}),
$$

6
We apply the Cauchy-Schwartz inequality to the second equality from (12):

\[
\sum_{\lambda} \sum_{i=1}^{3} p_{\lambda} (\langle \hat{\Theta}^{A}_{i} \rangle_{\lambda} - (\hat{\Theta}^{A}_{i})_{sep}) (\langle \hat{\Theta}^{B}_{i} \rangle_{\lambda} - (\hat{\Theta}^{B}_{i})_{sep}) \\
\leq \sum_{\lambda} p_{\lambda} \left( \sum_{i=1}^{3} (\langle \hat{\Theta}^{A}_{i} \rangle_{\lambda} - (\hat{\Theta}^{A}_{i})_{sep})^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{3} (\langle \hat{\Theta}^{B}_{i} \rangle_{\lambda} - (\hat{\Theta}^{B}_{i})_{sep})^2 \right)^{\frac{1}{2}} \\
= \sum_{\lambda} \sqrt{p_{\lambda}} \left( \sum_{i=1}^{3} (\langle \hat{\Theta}^{A}_{i} \rangle_{\lambda} - (\hat{\Theta}^{A}_{i})_{sep})^2 \right)^{\frac{1}{2}} \sqrt{p_{\lambda}} \left( \sum_{i=1}^{3} (\langle \hat{\Theta}^{B}_{i} \rangle_{\lambda} - (\hat{\Theta}^{B}_{i})_{sep})^2 \right)^{\frac{1}{2}} \\
\leq \left( \sum_{\lambda} p_{\lambda} \sum_{i=1}^{3} (\langle \hat{\Theta}^{A}_{i} \rangle_{\lambda} - (\hat{\Theta}^{A}_{i})_{sep})^2 \right)^{\frac{1}{2}} \left( \sum_{\lambda} p_{\lambda} \sum_{i=1}^{3} (\langle \hat{\Theta}^{B}_{i} \rangle_{\lambda} - (\hat{\Theta}^{B}_{i})_{sep})^2 \right)^{\frac{1}{2}}. \\
\tag{13}
\]

(13)

Note that we have applied Cauchy-Schwartz inequality twice: first to get the line (13), with respect to the summation over \(i\)'s (subscripts that numerate Stokes operators) and then the second time to get (14), with respect to \(\lambda\)'s (i.e. the summation over a convex combination of product states).

Let us observe that

\[
\sum_{\lambda} p_{\lambda} \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2 = \sum_{\lambda} p_{\lambda} \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2 - \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2
\]

where \(X = A, B\). Moreover

\[
\sum_{\lambda} p_{\lambda} \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2 \leq \sum_{\lambda} p_{\lambda} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2 \leq \sum_{\lambda} p_{\lambda} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2
\]

(15)

This leads us to

\[
\sum_{\lambda} p_{\lambda} \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2 \leq (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2 - \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2
\]

(16)

Combining (15) and (16) we obtain

\[
\sum_{\lambda} \sum_{i=1}^{3} p_{\lambda} (\langle \hat{\Theta}^{A}_{i} \rangle_{\lambda} - (\hat{\Theta}^{A}_{i})_{sep}) (\langle \hat{\Theta}^{B}_{i} \rangle_{\lambda} - (\hat{\Theta}^{B}_{i})_{sep}) \\
\leq \left( \langle \hat{\Theta}^{X}_{1} \rangle_{\lambda} - (\hat{\Theta}^{X}_{1})_{sep} \right)^2 - \sum_{i=1}^{3} (\langle \hat{\Theta}^{X}_{i} \rangle_{\lambda} - (\hat{\Theta}^{X}_{i})_{sep})^2
\]

(18)

Upon involving the well-known operator equality for Stokes operators \(\sum_{i=1}^{3} \hat{\Theta}^{X}_{i} = \hat{\Theta}^{X}(\hat{\Theta}^{X} + 2)\), where \(X = A, B\), the sums of local uncertainties of Stokes vector satisfies the
following

\[ \sum_{i=1}^{3} \Delta \hat{\Theta}_{i,sep}^A + \sum_{i=1}^{3} \Delta \hat{\Theta}_{i,sep}^B = \langle \hat{N}^A \rangle_{sep} + \langle \hat{N}^B \rangle_{sep} + 2(\langle \hat{N}^A \rangle_{sep} + \langle \hat{N}^B \rangle_{sep}) - \sum_{i=1}^{3} \langle \hat{\Theta}^A_{i,sep} \rangle^2 - \sum_{i=1}^{3} \langle \hat{\Theta}^B_{i,sep} \rangle^2 \]  

(19)

We combine (18) and (19) to estimate \( \Delta \vec{\Theta}_{sep}^{AB} \):

\[ \Delta \vec{\Theta}_{sep}^{AB} \geq \langle \hat{N}^A \rangle_{sep} + \langle \hat{N}^B \rangle_{sep} + 2(\langle \hat{N}^A \rangle_{sep} + \langle \hat{N}^B \rangle_{sep}) - \sum_{i=1}^{3} \langle \hat{\Theta}^A_{i,sep} \rangle^2 - \sum_{i=1}^{3} \langle \hat{\Theta}^B_{i,sep} \rangle^2 
- 2\left( \langle \hat{N}_{sep}^A \rangle^2 - \sum_{i=1}^{3} \langle \hat{\Theta}^A_{i,sep} \rangle^2 \right) \left( \langle \hat{N}_{sep}^B \rangle^2 - \sum_{i=1}^{3} \langle \hat{\Theta}^B_{i,sep} \rangle^2 \right) \]  

(20)

The minus sign appears before the last term because we estimate the right hand side from below. After trivial algebraic simplifications separability condition boils down to

\[ \Delta \vec{\Theta}_{sep}^{AB} \geq 2\langle \hat{N}^A \rangle_{sep} + 2\langle \hat{N}^B \rangle_{sep} + \left( \sqrt{\langle \hat{N}_{sep}^A \rangle^2 - ||\vec{\Theta}^A||_{sep}^2} - \sqrt{\langle \hat{N}_{sep}^B \rangle^2 - ||\vec{\Theta}^B||_{sep}^2} \right)^2. \]  

(21)

This condition provides better entanglement detection than (7).

D. Normalized Stokes operators

The derivation given here traces back the one given earlier, therefore its presentation will be more concise.

Consider the variance of normalized Stokes vector for the compound system of Alice and Bob:

\[ \Delta \vec{S}_{sep}^{AB} = \sum_{i=1}^{3} \Delta \hat{S}_{i}^A + \sum_{i=1}^{3} \Delta \hat{S}_{i}^B + 2 \sum_{i=1}^{3} \left( \langle \hat{S}_{i}^A \hat{S}_{i}^B \rangle - \langle \hat{S}_{i}^A \rangle \langle \hat{S}_{i}^B \rangle \right). \]  

(22)

The covariance term from (22), that reads

\[ \sum_{i=1}^{3} \langle (\hat{S}_{i}^{A} \hat{S}_{i}^{B})_{sep} - \langle \hat{S}_{i}^{A} \rangle_{sep} \langle \hat{S}_{i}^{B} \rangle_{sep} \rangle = \sum_{i=1}^{3} \langle (\hat{S}_{i}^{A} - \langle \hat{S}_{i}^{A} \rangle_{sep})(\hat{S}_{i}^{B} - \langle \hat{S}_{i}^{B} \rangle_{sep}) \rangle_{sep} \]  

\[ = \sum_{\lambda} \sum_{i=1}^{3} p_{\lambda} \langle (\hat{S}_{i}^{A})_{\lambda} - \langle \hat{S}_{i}^{A} \rangle_{sep} \rangle \langle (\hat{S}_{i}^{B})_{\lambda} - \langle \hat{S}_{i}^{B} \rangle_{sep} \rangle \]  

(23)
We apply twice the Cauchy-Schwartz inequality to the last equality from (23)

\[
\sum_{\lambda} \sum_{i=1}^{3} p_\lambda (\langle \hat{S}_i^A \rangle_\lambda - \langle \hat{S}_i^A \rangle_{sep})(\langle \hat{S}_i^B \rangle_\lambda - \langle \hat{S}_i^B \rangle_{sep})
\]

\[
\leq \sum_{\lambda} p_\lambda \left( \sum_{i=1}^{3} (\langle \hat{S}_i^A \rangle_\lambda - \langle \hat{S}_i^A \rangle_{sep})^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{3} (\langle \hat{S}_i^B \rangle_\lambda - \langle \hat{S}_i^B \rangle_{sep})^2 \right)^{\frac{1}{2}} 
\]

\[
\leq \left( \sum_{\lambda} p_\lambda \sum_{i=1}^{3} (\langle \hat{S}_i^A \rangle_\lambda - \langle \hat{S}_i^A \rangle_{sep})^2 \right)^{\frac{1}{2}} \left( \sum_{\lambda} p_\lambda \sum_{i=1}^{3} (\langle \hat{S}_i^B \rangle_\lambda - \langle \hat{S}_i^B \rangle_{sep})^2 \right)^{\frac{1}{2}} 
\]

Note that

\[
\sum_{\lambda} p_\lambda \sum_{i=1}^{3} (\langle S_i^X \rangle_\lambda - \langle S_i^X \rangle_{sep})^2 = \sum_{\lambda} p_\lambda \sum_{i=1}^{3} \langle S_i^X \rangle_\lambda^2 - \sum_{i=1}^{3} \langle S_i^X \rangle_{sep}^2. \quad (24)
\]

For normalized Stokes operators we have \( \sum_{\lambda} p_\lambda \sum_{i=1}^{3} (\langle S_i^X \rangle_\lambda - \langle S_i^X \rangle_{sep})^2 \leq \sum_{\lambda} p_\lambda \langle \hat{\Pi}^X \rangle_\lambda = \langle \hat{\Pi}^X \rangle \), where we used the fact that \( \hat{\Pi}^X \) is a projector (and again \( X = A, B \)). Thus,

\[
\sum_{\lambda} \sum_{i=1}^{3} p_\lambda \left( \langle S_i^A \rangle_\lambda - \langle S_i^A \rangle_{sep} \right) \left( \langle S_i^B \rangle_\lambda - \langle S_i^B \rangle_{sep} \right) 
\]

\[
\leq \left( \langle \hat{\Pi}^A \rangle_{sep} - ||\vec{S}^A||_{sep}^2 \right)^{\frac{1}{2}} \left( \langle \hat{\Pi}^B \rangle_{sep} - ||\vec{S}^B||_{sep}^2 \right)^{\frac{1}{2}}. \quad (25)
\]

Let us analyze the local uncertainties of (22):

\[
\Delta \vec{S}^A_{sep} + \Delta \vec{S}^B_{sep} \geq 2(\langle \hat{\Pi}^A \frac{1}{N_A} \hat{\Pi}^A \rangle_{sep} + \langle \hat{\Pi}^B \frac{1}{N_B} \hat{\Pi}^B \rangle_{sep}), \quad (26)
\]

where we used the operator equality given in [28]: \( \sum_{i=1}^{3} \hat{S}_i^X = \hat{\Pi}^X + \hat{\Pi}^X \frac{2}{N_X} \hat{\Pi}^X \). Combining (25) with (26) after simplifications we obtain

\[
\Delta \vec{S}_{sep}^{AB} \geq 2(\langle \hat{\Pi}^A \frac{1}{N_A} \hat{\Pi}^A \rangle_{sep} + \langle \hat{\Pi}^B \frac{1}{N_B} \hat{\Pi}^B \rangle_{sep}) + \left( \sqrt{\langle \hat{\Pi}^A \rangle_{sep} - ||\vec{S}^A||_{sep}^2} - \sqrt{\langle \hat{\Pi}^B \rangle_{sep} - ||\vec{S}^B||_{sep}^2} \right)^2. \quad (27)
\]

Thus we get a tighter constraint on separability.

III. BRIGHT SQUEEZED VACUUM WITH ADDED AND SUBTRACTED PHOTONS

Bright squeezed vacuum (BSV) consists of two optical beams (directions) in which photon pairs are emitted. Each beam contains two optical modes carrying mutually perpendicular
polarizations (we choose: horizontal-\(H\) and vertical-\(V\)). It reads:

\[
|BSV\rangle = \frac{1}{\cosh^2 \Gamma} \sum_{n=0}^{\infty} \tanh^n \Gamma \sum_{r=0}^{n} (-1)^m \left| \begin{array}{c} (n-r)_{a_H}, r_{a_V}, r_{b_H}, (n-r)_{b_V} \end{array} \right>. \tag{28}
\]

where \(\Gamma\) is the amplification gain. Subscripts \(a\) and \(b\) stand for two beams that reach two observers \(A\) and \(B\).

the two beams of BSV exhibit perfect correlation in the numbers of photons and perfect anti-correlations of polarization modes. We stress that BSV is rotationally invariant with respect to the same rotations of both observers, and thus its form remains unchanged in any other polarization basis \(\{i, i_\perp\}\). Because of its perfect EPR correlations independent of chosen polarization basis and the same number of photons in the two beams all given entanglement entanglement conditions are equivalent. In order to observe the possible advantage of our conditions (21) and (27) we need to introduce an asymmetry in the number of photons in the beams.

It is well-known that one can induce non-gaussianity into photon statistics by subtraction or addition of photons into an optical field [44], [45]. Both these processes are non-unitary and feasible in the laboratory [46], [47] [48]. Photon subtraction can be realised using a beamsplitter with high transmitivity and a photon number resolving detector behind one the beamsplitter’s outputs [49]. Photon addition is more challenging. It can be achieved by feeding the photon in the respective mode to the input of a parametric amplifier and detecting single photons in the idler output of parametric process [50], [51].

**A. Equivalence of photon added and photon subtracted BSV**

Let us first analyse the processes of adding and subtracting photons to BSV from without concerns about potential experimental difficulties. We shall show their equivalence.

Bright squeezed vacuum with \(m_H\) photons added in mode \(a_H\) and \(m_V\) in mode \(a_V\) reads

\[
|BSV^{add}\rangle = \frac{1}{\sqrt{N^{add}}} (\hat{a}_H^\dagger)^{m_H} (\hat{a}_V^\dagger)^{m_V} |BSV\rangle \tag{29}
\]

\[
= \frac{1}{\sqrt{N^{add}}} \sum_{n=0}^{\infty} \tanh^n \Gamma \sum_{r=0}^{n} (-1)^r \sqrt{\frac{(n-r+m_H)!}{(n-r)!}} \sqrt{\frac{(r+m_V)!}{r!}} |(n-r+m_H)_{a_H}, (r+m_V)_{a_V}, r_{b_H}, (n-r)_{b_V} \rangle. \tag{30}
\]
where the normalization factor $N^{\text{add}}$ is given by

$$N^{\text{add}} = \sum_{n=0}^{\infty} \frac{\tanh^{2n} \Gamma \sum_{r=0}^{n} \frac{(n-r+m_H)!}{(n-r)!} \frac{(r+m_V)!}{(r)!}}{m_H!m_V!(\cosh^2 \Gamma)^{m_H+m_V+2}} = m_H!m_V!(\cosh^2 \Gamma)^{m_H+m_V+2}. \quad (31)$$

Now consider the subtraction of the same amount of $m_H$ and $m_V$ photons from the second optical beam $b$. We subtract from mode $b_{V(H)}$ the same number of photons that we added before to the mode $a_{H(V)}$. We get:

$$|BSV^{\text{sub}}\rangle = \frac{1}{\sqrt{N^{\text{sub}}}} (\hat{b}_H)^{m_H} (\hat{b}_V)^{m_V} |BSV\rangle$$

$$= \frac{1}{\sqrt{N^{\text{sub}}}} \sum_{n=m_H+m_V}^{\infty} \tanh^{n} \Gamma \sum_{r=m_V}^{n-m_H} (-1)^r \sqrt{\frac{(n-r)!}{(n-r-m_H)!}} \sqrt{\frac{(r)!}{(r-m_V)!}}$$

$$| (n-r)_{a_H}, r_{a_V}, (r-m_V)_{b_H}, (n-r-m_H)_{b_V} \rangle \quad (32)$$

and $N^{\text{sub}}$ is given by:

$$N^{\text{sub}} = \sum_{n=m_H+m_V}^{\infty} \tanh^{n} \Gamma \sum_{r=m_V}^{n-m_H} \frac{(n-r)!}{(n-r-m_H)!} \frac{r!}{(r-m_V)!}. \quad (33)$$

We perform the following change of variables: $n - m_H - m_V = n'$ and $r - m_V = r'$. As the result we get: $n = n' + m_H + m_V$ and $r = r' + m_V$. Thus, $\sum_{n=m_H+m_V}^{\infty} \to \sum_{n'=0}^{\infty}$ and $\sum_{r=m_V}^{n-m_H} \to \sum_{r'=0}^{n'}$. Applying these changes to formula (32) we obtain

$$|BSV^{\text{sub}}\rangle$$

$$= \frac{1}{\sqrt{N^{\text{add}}}} \sum_{n'=0}^{\infty} (\tanh \Gamma)^{n'+m_H+m_V} \sum_{r'=0}^{n'} (-1)^{r'+m_V} \sqrt{\frac{(n'-r'+m_H)!}{(n'-r')!}} \sqrt{\frac{(r+m_V)!}{r!}}$$

$$= \frac{(-1)^{m_V}}{\sqrt{N^{\text{add}}}} \sum_{n=0}^{\infty} \tanh^{n} \Gamma \sum_{r=0}^{n} (-1)^r \sqrt{\frac{(n'-r'+m_H)!}{(n'-r')!}} \sqrt{\frac{(r+m_V)!}{r!}}$$

$$= |BSV^{\text{add}}\rangle, \quad (34)$$

where in the second equality, we introduced that $N^{\text{sub}} = \tanh^{2(m_H+m_V)} \Gamma N^{\text{add}}$.

Hence, we showed that $|BSV^{\text{sub}}\rangle$ and $|BSV^{\text{add}}\rangle$ are equivalent as they differ only by the global phase factor $(-1)^{m_V}$. This is one of implication of the great symmetry of BSV state! Note that all the “kets” of $|BSV^{\text{add}}\rangle$ can be as well considered as “kets” belonging to BSV with photons subtracted in the second beam. For example, the “ket” $|2_{a_H}, 0_{a_V}, 0_{b_H}, 1_{b_V}\rangle$ can be seen either as $|1_{a_H}, 0_{a_V}, 0_{b_H}, 1_{b_V}\rangle$ with one photon $m_H = 1$ added in mode $a_H$ or as $|2_{a_H}, 0_{a_V}, 0_{b_H}, 2_{b_V}\rangle$ with one ($m_H = 1$) photon subtracted from the mode $b_V$. 

11
B. Experimental setup for generating BSV with induced non-gaussianity

In order to get BSV with added or subtracted photons, it is enough to subtract photons with polarising beamsplitters. Let us analyze these processes from more physical perspective starting by the analysis of photon subtraction from a single mode.

1. Photon subtraction from a single mode state

Consider a beamsplitter $BS(T)$ with arbitrary transmitivity $T > 0$. The input arms are denoted with $a$ and $b$. The output ports are c and d (see figure 1(a)). The relation between the creation operators assigned to the input and the output modes is given by the unitary transformation

$$
\hat{a}^\dagger = \sqrt{T} \hat{c}^\dagger + \sqrt{1-T} \hat{d}^\dagger,
$$

(35)

$$
\hat{b}^\dagger = i\sqrt{1-T} \hat{c}^\dagger + \sqrt{T} \hat{d}^\dagger.
$$

(36)

Assume $|\psi\rangle_a = |n\rangle_a$ with $n > 0$ feed the mode $a$ and no optical input in the mode $b$. After passing through the beamsplitter the state yields

$$
|n, \Omega\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |\Omega_{a,b}\rangle = \frac{1}{\sqrt{n!}} (\sqrt{T} \hat{c}^\dagger + \sqrt{1-T} \hat{d}^\dagger)^n |\Omega_{c,d}\rangle
$$

(37)

$$
= \frac{1}{\sqrt{n!}} \sum_{s=0}^{n} \binom{n}{s} (\sqrt{T})^{n-s} (i\sqrt{1-T})^{s} (\hat{c}^\dagger)^{n-s} (\hat{d}^\dagger)^{s} |\Omega_{c,d}\rangle
$$

(38)

$$
= \frac{1}{\sqrt{n!}} \sum_{s=0}^{n} \binom{n}{s} (\sqrt{T})^{n-s} (i\sqrt{1-T})^{s} \sqrt{(n-s)! s!} |(n-s), s\rangle,
$$

(39)

$$
= \sum_{s=0}^{n} \sqrt{\frac{n!}{(n-s)! s!}} (\sqrt{T})^{n-s} (i\sqrt{1-T})^{s} |(n-s), s\rangle = |\varphi\rangle_{cd}.
$$

(40)

Assume $|\psi\rangle_a = |m\rangle_a$ with $m > 0$ feed the mode $a$ and no optical input in the mode $b$. After passing through the beamsplitter the state yields

$$|m\rangle_d = \sum_{n=0}^{\infty} q_n |n\rangle_d,$$

with $\sum_{n=0}^{\infty} |q_n|^2 = 1$. Again, we impinge it on the beamsplitter (Fig. 1(a)) and measure $m$-photons in the output mode $d$. The state after passing through the
beamsplitter reads

\[ |\psi\rangle_{a} |\Omega_{b}\rangle = \sum_{n}^{\infty} q_{n} |n_{a}, \Omega_{b}\rangle = \sum_{n}^{\infty} \frac{q_{n}}{\sqrt{n!}} (\hat{c}^{\dagger})^{n} |\Omega_{a,b}\rangle = \sum_{n}^{\infty} \frac{q_{n}}{\sqrt{n!}} (\sqrt{T} \hat{c}^{\dagger} + i \sqrt{1-T} \hat{d}^{\dagger})^{n} |\Omega_{c,d}\rangle \]

\[ = \sum_{n}^{\infty} \frac{q_{n}}{\sqrt{n!}} \sum_{s=0}^{n} \binom{n}{s} (\sqrt{T})^{n-s} (i \sqrt{1-T})^{s} (\hat{c}^{\dagger})^{n-s} (\hat{d}^{\dagger})^{s} |\Omega_{c,d}\rangle \quad (42) \]

\[ = \sum_{n} \sum_{s=0}^{n} \frac{n!}{(n-s)!s!} (\sqrt{T})^{n-s} (i \sqrt{1-T})^{s} (n - s)_{c,d} \cdot (43) \]

Thus the state with \( m \) photons subtracted reads

\[ |\psi^{\text{sub}}\rangle = \frac{1}{\sqrt{N^{\text{sub}}}} \sum_{n=m}^{\infty} q_{n} \sqrt{\frac{n!}{(n-m)!}} (\sqrt{T})^{n-m} |(n-m)_{c}\rangle, \quad (44) \]

where \( N^{\text{sub}} \) is the normalization constant: \( N^{\text{sub}} = \sum_{n=m}^{\infty} q_{n}^{2} \frac{n!}{(n-m)!} T^{n-m} \).

If the beamsplitter is highly transmissive we get

\[ \lim_{T \to 1} |\psi^{\text{sub}}\rangle = \frac{1}{\sqrt{N^{\text{sub}}}} \sum_{n=m}^{\infty} q_{n} \sqrt{\frac{n!}{(n-m)!}} |(n-m)\rangle = \frac{a^{m} |\psi\rangle}{\sqrt{N^{\text{sub}}}}, \quad (45) \]

where \( \tilde{N}^{\text{sub}} = \sum_{n=m}^{\infty} q_{n}^{2} \frac{n!}{(n-m)!} \), and the transmitted mode can be approximated to \( c \approx a \).

Thus, subtracting \( m \) photons with highly transmissive beamsplitter can be approximated by applying annihilation operator \( m \) times. The probability of subtraction decreases as more photons are subtracted, and for \( m >> 1 \) it becomes vanishingly small.

2. Bright squeezed vacuum with added or subtracted photons.

Fig. 1(b) shows how to induce non-gaussianity in BSV. Bright squeezed vacuum is generated via PDC process. amplification gain is \( \Gamma' \). The beam \( b \) passes through a polarizing beamsplitter (PBS) that separates spatially modes \( b_{H} \) and \( b_{V} \). The mode \( b_{H} \) impinges on a beamsplitter \( BS_{1} \) while the mode \( b_{V} \) passes through \( BS_{2} \). Both beamsplitters \( BS_{1} \) and \( BS_{2} \) have the same transmissivity \( T \). We place photon numbers resolving detectors on the optical paths of reflected beams. If \( m_{1} \) and \( m_{2} \) photons are detected we know that the transmitted beams \( b_{H}^{\text{sub}} \) and \( b_{V}^{\text{sub}} \) have \( m_{1} \) and \( m_{2} \) photons subtracted. Finally beams \( b_{H}^{\text{sub}} \) and \( b_{V}^{\text{sub}} \) are recombined into \( b(b_{H}^{\text{sub}}, b_{V}^{\text{sub}}) \) with subtracted photons. The beam \( a \) remains
FIG. 1. Schematic of photon subtraction. Panel (a) shows a subtraction of $m$ photons from an arbitrary state $|\psi\rangle$ feeding the input arm $a$ of a beamsplitter of transmittivity $T$. On the reflected output mode $d$ we place a photon number resolving detector. If $m$ photons are registered then the transmitted mode $c$ contains a state with $m$ photons subtracted. Panel (b): schematic setup to generate non-gaussianity in bright squeezed vacuum with the amplification gain $\Gamma'$. The basic idea is to subtract $m_1$ photons from mode $b_H$ and $m_2$ photons from mode $b_V$ of BSV. The beam $b$ impinges on the polarising beamsplitter (PBS) so that horizontally and vertically polarised photons from $b$ becomes spatially separated. The beams $b_H$ and $b_V$ pass through the beamsplitters $BS_1$ and $BS_2$. Once $m_1$ and $m_2$ photons are detected in the reflected output beams, photon subtracted states in modes $b_{sub}^H$ and $b_{sub}^V$ generated. Note that this experiment is event-ready.

unchanged. The output state reads

$$|\text{BSV}_T^{\text{sub}}(\Gamma')\rangle = \frac{1}{\sqrt{N_T^{\text{sub}}(\Gamma')}} \sum_{n=m_1+m_2}^{\infty} \tan^2 n \Gamma' \sum_{r=m_1}^{n-m_2} (-1)^r \sqrt{\frac{(n-r)!}{(n-r-m_2)!}} \sqrt{\frac{r!}{(r-m_1)!}} (\sqrt{T})^{n-m_1-m_2} |(n-r)_{a_H}, r_{a_V}, (r-m_1)_{b_H}, (n-r-m_2)_{b_V}\rangle,$$

(46)

where $N_T^{\text{sub}}(\Gamma')$ is the normalization constant depending on $T$ given by

$$N_T^{\text{sub}}(\Gamma') = \sum_{n=m_1+m_2}^{\infty} \tan^2 n \Gamma' \sum_{r=m_1}^{n-m_2} \frac{(n-r)!}{(n-r-m_2)!} \frac{r!}{(r-m_1)!} T^{n-m_1-m_2}.$$  

(47)

Let us replace $m_1 \rightarrow m_V$, $m_2 \rightarrow m_H$, and $\tanh \Gamma' \rightarrow \frac{1}{\sqrt{T}} \tanh \Gamma$. Then $N_T^{\text{sub}}(\Gamma') T^{m_1+m_2} = N^{\text{sub}}(\Gamma)$ and in consequence, (46) with subtracted photons and amplification gain $\Gamma'$ becomes
(30) with the amplification gain \( \Gamma \) and added photons. Hence the conclusion that in order to obtain the photon added BSV(\( \Gamma \)), we need to create photon subtracted BSV (\( \Gamma' \)) such that 
\[
\tanh \Gamma = \tanh \Gamma' \sqrt{T}.
\]

**IV. COMPARISON OF PRESENTED ENTANGLEMENT INDICATORS**

For the simplicity of further analyse we give all separability conditions in the form of 
\[
LHS_{sep} \geq RHS_{X_{sep}}
\]
where LHS - the left hand side - remains the same for all the conditions (in its standard or normalized version). The right hand side \( RHS_X \) changes according to the given condition indexed with \( X \) e.g. \( X = SB(\hat{\Theta}^{AB}) \) denotes Simon and Bouwmesteer’s condition (3), etc. We start with standard Stokes operators’ description, but we use the same approach for normalized ones, see Appendix B. The left hand side for standard Stokes operators is: 
\[
LHS = \langle (\hat{\Theta}^A + \hat{\Theta}^B)^2 \rangle.
\]
The different right hand sides are given by:

- \( RHS_{SB(\hat{\Theta}^{AB})} = 2\langle \hat{N}^A + \hat{N}^B \rangle \) for (3)
- \( RHS_{SB(\hat{\Theta}^{AB})_{new}} = RHS_{SB(\hat{\Theta}^{AB})} + \langle (\hat{N}^A - \hat{N}^B)^2 \rangle \) for (5).
- \( RHS_{\Delta(\hat{\Theta}^{AB})^2} = RHS_{SB(\hat{\Theta}^{AB})} + \sum_{i=1}^{3} \langle (\hat{\Theta}^A_i + \hat{\Theta}^B_i)^2 \rangle \) for (7)
- \( RHS_{\Delta(\hat{\Theta}^{AB})_{new}^2} = RHS_{\Delta(\hat{\Theta}^{AB})^2} + \left( \sqrt{\langle \hat{N}^A \rangle^2 - \langle \hat{\Theta}^A \rangle^2} - \sqrt{\langle \hat{N}^B \rangle^2 - \langle \hat{\Theta}^B \rangle^2} \right)^2 \) for our (21)

The analytical formulas needed to get the values of expressions listed above for arbitrary \( m_H \) and \( m_V \) are given in the Appendix A.

We define \( \kappa = LHS - RHS_X \). For any separable state \( \kappa_{sep} \geq 0 \). If \( \kappa_X < 0 \), the state in question is entangled. The more negative \( \kappa \) is, the more detection of entanglement is efficient.

First we compare, for standard Stokes operators, \( \kappa \) in function of amplification gain \( \Gamma \) for the following scenarios. We plot \( \kappa(\Gamma) \) for non-gaussian BSV and our condition (21) and compare it to \( \kappa(\Gamma) \) for ”basic” BSV and 7, see the left side of fig. 2. Then, on the right On the right part of 2 we have similar plots for normalized Stokes operators (for conditions 8 and 27). Concerning normalized Stokes operators it is more advantageous to use the ”basic” BSV state and the standard variance condition than to use non-gaussian BSV and our improved condition 27. However, the advantage of introducing non-gaussianity to BSV
FIG. 2. Plots of $\kappa$ in function of amplification gain ($\Gamma$). On the left: comparison of conditions (7) applied for BSV and (21) for non-gaussian BSV for standard Stokes operators. On the right analogous comparison of (8) and (27). The photon addition combined with the use of our condition helps in entanglement detection only for standard Stokes operators. In case of normalized Stokes operators it is more advantageous to use perfect BSV state.

FIG. 3. Plots of $\kappa$ in function of amplification gain ($\Gamma$). for conditions (3), (7) and (21) for BSV with $m_H$ and $m_V \in \{0, 1, 2\}$ photons added. Note the advantage of our condition (21) - entanglement detection starts from smaller value of amplification gain ($\Gamma \approx 0.2$) than for (3), (7) occurs for standard Stokes operators. In the main text we will concentrate on standard Stokes operators. Appendix B.

Fig. 3 shows the values of $\kappa$ in function of the amplification gain $\Gamma$ for BSV with photons added. Conditions (3), (7) and (21) are compared. As we increase the difference of total number of photons between the beams, the additional non-negative term in $RHS_{\Delta(\Theta_{AB})^2_{\text{new}}}$ of (21) gets more impact on lowering the value of $\kappa$. As suspected, adding more that one extra photon unable entanglement detection in the low range of $\Gamma$.

Fig. 4 shows the dependence $\kappa(\Gamma)$ for (5) and (21) for standard Stokes operators. We notice the advantage of (5).
FIG. 4. Comparison of $\kappa(\Gamma)$ for (21) - red line - and (5) - turquoise, dashed line. For one extra photon plots for (21) and (5) behave in the similar way. However, when both modes are fed with extra photons (5) seems to be more efficient.

V. CONCLUSIONS

More optimal conditions for optical fields tailored for states with undefined photon number were proposed based on polarisations measurement. Our conditions involve variances and hence they have clear physical interpretation - the spread of data around the mean value define the nature of correlations. If for the given state the spread of the data results smaller than it is assumed for separable states, the given state is entangled. We compared the efficiency of entanglement detection with our condition versus other known entanglement conditions with bright squeezed vacuum and bright squeezed vacuum with added photons. We also analyzed the equivalency of adding and subtracting photons in bright squeezed vacuum and proposed an experimental scheme to perform such experiments. We have shown that by applying our conditions to non-gaussian bright squeezed vacuum, we can obtain stronger entanglement detection. However, that statement refers only to standard Stokes operators. For normalized Stokes operators adding photons to BSV and using our condition does not improve entanglement detection compared to other studies cases.

Appendix A: Explicit formulas used in the comparison of entanglement conditions for standard Stokes operators

We give the analytical form of the formulas for $LHS$ and $RHS_X$ for (3), (7), (5) and (21). They are all in function of $\Gamma$ and numbers of added photons $m_H$ and $m_V$. The left
hand side, the same for all conditions, boils down to

\[ \text{LHS} = (m_H + m_V + 1)^2 - 1. \]  

(A1)

Note that \( \text{LHS} \) does not depend on \( \Gamma \), which is not surprising because for if no photon are added the \( \text{LHS} = 0 \), due to perfect EPR anticorrelations of BSV. Then, to calculate the right hand sides (\( \text{RHS}_X \)) we need:

\[
\langle \hat{N}^B \rangle_{\text{add}} = (m_H + m_V + 2) \sinh^2 \Gamma, \\
\langle \hat{N}^A_i \rangle_{\text{add}} = \sinh^2 \Gamma (m_H + m_V + 2) \left( 1 + \sinh^2 \Gamma (m_H + m_V + 3) \right), \\
\langle \hat{\Theta}^A_3 \hat{\Theta}^B_3 \rangle_{\text{add}} = -\frac{1}{2} (1 + m_H)(1 + m_V) \sinh^2(2\Gamma), \\
\langle \hat{N}^A \hat{N}^B \rangle_{\text{add}} = (m_H + m_V + 2) \cosh^2 \Gamma \sinh^2 \Gamma (1 + m_H + m_V + 2 \tanh^2 \Gamma). \]  

(A2)  

(A3)  

(A4)  

(A5)

Note that after introducing extra photons BSV ceases to be rotationally invariant (with respect to the same rotations of both observers). We still have: \( \langle \hat{\Theta}^X_1 \rangle_{\text{add}} = \langle \hat{\Theta}^X_2 \rangle_{\text{add}} = 0 \), for \( X = A, B \), but

\[
\langle \hat{\Theta}^A_3 \rangle_{\text{add}} - (m_H - m_V) \sinh^2 \Gamma. \]  

(A6)

Appendix B: Formulas used in the comparison of entanglement conditions for normalized Stokes operators

\[
\Delta \left( S_{AB} \right)^2 = \frac{\kappa - \Delta(S^{\text{new}})\kappa}{\kappa}, \quad m_H = 1, m_V = 0 \quad \text{and} \quad m_H = 1, m_V = 1
\]

FIG. 5. The plots of \( \kappa(\Gamma) \) for (4), (8) and 27 for different amount of added photons \( m_{H(V)} \).

We repeat the reasoning form the beggining of section IV and Appendix A for normalized Stokes parameters. We have:

\[
LHS_{S_{AB}} = \sum_{i=1}^{3} \langle \hat{S}^A_i + \hat{S}^B_i \rangle^2, \]  

(B1)

as well as:
FIG. 6. Plots of $\kappa(\Gamma)$ for (6) (green dashed line) and (27) (solid red line). The advantage of (27) appears only if we add more than one photon to only one mode ($m_H > 1$). If both modes are fed with photons (6).

- $\text{RHS}_{SB(S_{AB})} = 2\langle \hat{\Pi}^A \frac{1}{N_A} \hat{\Pi}^A + \hat{\Pi}^B \frac{1}{N_B} \hat{\Pi}^B \rangle$, for (4),
- $\text{RHS}_{\Delta(S_{AB})^2} = \text{RHS}_{SB(S_{AB})} + \sum_{i=1}^{3} \langle (\hat{S}_i^A + \hat{S}_i^B)^2 \rangle$, for (8)
- $\text{RHS}_{\Delta(S_{AB})^2_{new}} = \text{RHS}_{\Delta(S_{AB})^2} + \left( \sqrt{\langle \hat{\Pi}^A \rangle} - \sqrt{\langle \hat{\Pi}^B \rangle} - \langle \hat{S}_3^A \rangle \right)^2$ for (27)
- $\text{RHS}_{SB(S_{AB})_{new}} = \text{RHS}_{SB(S_{AB})} + \langle (\hat{\Pi}^A - \hat{\Pi}^B)^2 \rangle$ for (6)

The formulas for probability of non-vacuum events and very intuitive:

\[ \langle \hat{\Pi}^A \rangle_{add} = 1 - (\text{sech}^2 \Gamma)^{m_H + m_V + 2} \]  

and

\[ \langle \hat{\Pi}^B \rangle_{add} = 1 - \delta_{m_H,0} \delta_{m_V,0} (\text{sech}^2 \Gamma)^2, \]

Let us notice that for any $m_H$ and $m_V$ we have: $\langle \hat{S}_X^1 \rangle_{add} = \langle \hat{S}_X^2 \rangle_{add} = 0$ for $X = A, B$. Moreover note that if $m_H = m_V$ also $\langle \hat{S}_3^A \rangle_{add} = \langle \hat{S}_3^B \rangle_{add} = 0$. For all the rest of the cases we get

\[ \langle \hat{S}_3^A \rangle_{add} = \frac{m_H - m_V}{m_H + m_V} \left( 1 + \frac{2 \text{sech}^2 \Gamma (m_H + m_V + \text{sech}^2 \Gamma)}{(m_H + m_V)(m_H + m_V + 1)} \right), \]
\[
\langle \hat{S}_3^B \rangle_{add} = -\frac{(m_H - m_V)(1 - (\text{sech}^2 \Gamma)(m_H + m_V + 2))}{m_H + m_V + 2},
\]

(B5)

| \(m_H\) | \(m_V\) | \(\langle \hat{\Pi}^B - \hat{\Pi}^B \rangle_{add} \) | \(\langle \hat{S}_3^B \hat{S}_3^B \rangle_{add} \) | \(\langle \hat{S}_1^A \hat{S}_1^B \rangle_{add} \) |
|---|---|---|---|---|
| 1 | 0 | \(\frac{1}{2}(\sinh^2 \Gamma(cosh^2 \Gamma + 3) - 2 \log(\text{sech}^2 \Gamma)) \text{sech}^6 \Gamma\) | \(\frac{1}{3}(3 \text{sech}^6 \Gamma - \text{sech}^4 \Gamma - \text{sech}^2 \Gamma - 1)\) | \(\frac{1}{3}(3 \text{sech}^6 \Gamma - \text{sech}^4 \Gamma - \text{sech}^2 \Gamma - 1)\) |
| 1 | 1 | \(\frac{1}{60} \text{sech}^8 \Gamma (87 \cosh(2\Gamma) + 12 \cosh(4\Gamma) + \cosh(6\Gamma) - 96 \log(\text{sech}^2 \Gamma) - 100)\) | \(\frac{1}{15}(6 \text{sech}^8 \Gamma - \text{sech}^4 \Gamma - 2 \text{sech}^2 \Gamma - 3)\) | \(\frac{2}{15}(6 \text{sech}^8 \Gamma - \text{sech}^4 \Gamma - 2 \text{sech}^2 \Gamma - 3)\) |
| 2 | 1 | \(\frac{1}{12} \text{sech}^2 \Gamma (3 + 4 \text{sech}^2 \Gamma + 6 \text{sech}^4 \Gamma + 12 \log(\text{sech}^2 \Gamma) + 25)\) | \(\frac{1}{15}(10 \text{sech}^8 \Gamma - \text{sech}^4 \Gamma - 3 \text{sech}^2 \Gamma - 6)\) | \(\frac{1}{15}(10 \text{sech}^8 \Gamma - \text{sech}^4 \Gamma - 3 \text{sech}^2 \Gamma - 6)\) |
| 2 | 2 | \(\frac{1}{60} \text{sech}^2 \Gamma (12 + 15 \text{sech}^2 \Gamma + 20 \text{sech}^4 \Gamma + 30 \text{sech}^6 \Gamma + 60 \text{sech}^8 \Gamma - (60 \log(\text{sech}^2 \Gamma) + 137) \text{sech}^{10} \Gamma)\) | \(\frac{1}{70}(15 \text{sech}^8 \Gamma - \text{sech}^4 \Gamma - 4 \text{sech}^2 \Gamma - 10)\) | \(\frac{3}{70}(15 \text{sech}^8 \Gamma - \text{sech}^4 \Gamma - 4 \text{sech}^2 \Gamma - 10)\) |

**TABLE I.** Formulas needed to calculate RHS for different entanglement conditions and normalized Stokes operators.

Not being able to find a concise formula for arbitrary \(m_H\) and \(m_V\) for other expressions we give their specific values \(m_H\) and \(m_V\) \(\in \{0, 1, 2\}\) in the table below. I.

Fig 5 refers to conditions (4), (8) and (27) for normalized Stokes operators.

Fig. 6 shows \(\kappa(\Gamma)\) for (6) and (27).

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