Option spanning beyond $L_p$-models

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Abstract The aim of this paper is to study the spanning power of options in a static financial market that allows non-integrable assets. Our findings extend and unify the results in Galvani (J Math Econ 45(1):73–79, 2009), Galvani and Troitsky (J Math Econ 46(4):616–619, 2010) and Nachman (Rev Financ Stud 1(3):311–328, 1988) for $L_p$-models. We also apply the spanning power properties to the pricing problem. In particular, we show that prices on call and put options of a limited liability asset can be uniquely extended by arbitrage to all marketed contingent claims written on the asset.

Keywords Spanning of options · Market completeness · Arbitrage · Kreps–Yan theorem · Order continuous dual

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JEL Classification G10 · G13

1 Introduction

Throughout this paper, $\Omega$ stands for the state space of a financial market, $\Sigma$ stands for the $\sigma$-algebra modelling the market information structure, and $\mathbb{P}$ stands for a probability over $(\Omega, \Sigma)$. The space of contingent claims, $X$, is modelled as an ideal (i.e., solid subspace) of $L_0(\Sigma)$ containing the constant functions, which represent investments in the riskless asset.

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A claim displays limited liabilities if it is positive. For a limited liability claim $f$, its option space is the collection of all portfolios of call and put options written on $f$, which can be identified as follows:

$$O_f = \text{Span} \left\{ 1, (f - k)^+ : k \in \mathbb{R} \right\}.$$ 

The space of all contingent claims written on $f$ is identified as the space of all functions measurable with respect to $\sigma(f)$, the sub-$\sigma$-algebra generated by $f$, i.e.,

$$L_0(\sigma(f)).$$

A stream of research has been devoted to the study of spanning power of options on $f$, i.e., the size of $O_f$. In the seminal paper [30], Ross showed that if the state space $\Omega$ is finite then the options on $f$ span the space of contingent claims written on $f$, i.e.,

$$O_f = L_0(\sigma(f)),$$

and if, in addition, $f$ is one-to-one, then the option space of $f$ completes the market, i.e.,

$$O_f = L_0(\Sigma).$$

These elegant results of Ross have inspired many successive contributions to the study of options. See e.g. [7,20,25] for related results on finite state spaces. In particular, they have also been examined for financial markets with infinite state spaces.

Nachman proved in [23] that if $X = L^p(\Sigma)$ ($1 \leq p < \infty$), then the options on $f$ span the space of contingent claims written on the asset in two ways: approximating by a.e. convergence or by $p$-th mean convergence. Precisely, it was proved that an asset $x \in L^p(\Sigma)$ is a contingent claim on $f$ in $L^p(\Sigma)$ iff there exists a sequence of portfolios of options on $f$ converging a.e. to $x$ iff there exists a sequence of portfolios of options on $f$ converging in the $p$-th mean to $x$. That is,

$$O_f^{a.e.} \cap L^p(\Sigma) = O_f^{\|\cdot\|_p} = L_0(\sigma(f)) \cap L^p(\Sigma).$$ (1)

Galvani ([13]) and Galvani and Troitsky ([14]) proved further that if $\Omega$ is a Polish space equipped with the Borel $\sigma$-algebra and $f$ is one-to-one and bounded, then $O_f$ completes the market $X = L^p(\Sigma)$ ($1 \leq p \leq \infty$). That is, for $1 \leq p < \infty$,

$$O_f^{a.e.} \cap L^p(\Sigma) = O_f^{\|\cdot\|_p} = L^p(\Sigma),$$ (2)

and

$$O_f^{a.e.} \cap L^\infty(\Sigma) = O_f^{w^*} = L^\infty(\Sigma).$$ (3)

In this paper, we explore the spanning power of options in general spaces of contingent claims. Our contributions here are two-fold. Firstly, the spaces of contingent claims in our setting can be modelled as any ideal of $L_0(\Sigma)$ which contains the constant functions and admits a strictly positive order continuous linear functional. This framework includes not only the $L^p$-space ($1 \leq p \leq \infty$) models, but also the much wider class of Orlicz space models as well as many non-integrable space models which have been extensively used in the theory of risk measures (see e.g. [5,6,9,11,16,24]).

Secondly, we provide an approach to unify the norm and $w^*$-topologies used in the results of Nachman, Galvani and Troitsky, and thus give more comprehensive insight into the general structures of option spaces; see Theorem 3.1 and Remark 3.3. The unification in our approach is due to the use of the topology $\sigma(X, X_n^\ast)$, where $X_n^\ast$ is the set of all order continuous linear functionals on $X$. 

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Observe that $(X^n_\omega)_+$ has a natural connection with linear pricing functionals. Recall that a linear pricing functional $\phi$ on $X$ is given by a state-price density $y \geq 0$ via integration, i.e.

$$\phi(x) = \int_\Omega x y \, d\mathbb{P} \quad \text{for all } x \in X,$$

where $y$ is measurable and satisfies $\int_\Omega |x y| \, d\mathbb{P} < \infty$ for all $x \in X$. By Dominated Convergence Theorem, it is easily seen that $\phi$ is order continuous on $X$. Conversely, by Radon–Nikodym theorem, one can easily see that each positive order continuous linear functional on $X$ has a positive density, and thus is a linear pricing functional. Therefore, $(X^n_\omega)_+$ is just the collection of linear pricing functionals on $X$.

Because of this link, we are able to apply Theorem 3.1 and shed light on the following general problem, raised in [8]: “Under what circumstances can prices on the market assets or basic derivative assets be uniquely extended by arbitrage to prices on all derivative assets in a large class and when is such an extension unique?” In Theorem 3.4, we prove that when the arbitrage condition is understood as a no-free lunch condition (see [21]), one can extend uniquely the prices on $\mathcal{O}_f$ to the marketed contingent claims written on $f$.

Finally, we mention that there is a stream of works studying market completion using options in a continuous time setting. In this framework, the model is said to be complete, if any contingent claim payoff can be obtained as the terminal value of a self-financing trading strategy. We refer the reader to papers [10,18,26,29,32] for results in this direction.

2 Preliminary results

We refer to [3,4] for all unexplained terminology and standard facts on vector and Banach lattices. A vector subspace $Y$ of a vector lattice $X$ is called a sublattice if $|y| \in Y$ whenever $y \in Y$; in this case, $y_1 \wedge y_2$ and $y_1 \vee y_2$ both belong to $Y$ whenever $y_1, y_2 \in Y$. A subspace $Y$ is called an ideal (or a solid subspace) of $X$, if $|x| \leq |y|$ and $y \in Y$ imply $x \in Y$. A linear functional $\phi$ on a vector lattice $X$ is said to be order continuous if $\phi(x_\alpha) \to 0$ whenever $x_\alpha \to 0$ in $X$. The collection of all order continuous linear functionals on $X$ is denoted by $X_\infty^*$ and is called the order continuous dual of $X$. A linear functional $\phi$ on $X$ is said to be positive if $\phi(x) \geq 0$ whenever $x \geq 0$, and is said to be strictly positive if $\phi(x) > 0$ whenever $x > 0$.

The following lemma will be used. Recall first that a vector lattice is said to be order complete (or Dedekind complete) if every order bounded above subset has a supremum, and is said to have the countable sup property if any subset having a supremum possesses a countable subset with the same supremum. A subset $A$ of a vector lattice $X$ is said to be order closed if $x \in A$ whenever there exists a net $(a_\alpha)$ in $A$ such that $a_\alpha \to x$ in $X$.

**Lemma 2.1** Let $X$ be an order complete vector lattice with the countable sup property and $Y$ be a sublattice of $X$. Then $Y$ is order closed in $X$ iff for any increasing sequence in $Y$ which is order bounded above in $X$, its supremum in $X$ also lies in $Y$.

Given a probability space $(\Omega, \Sigma, \mathbb{P})$, denote by $L_0(\Sigma)$ the space of all real-valued measurable functions (modulo a.e. equality). We use $\mathds{1}$ to denote the constant one function. Recall that $L_0(\Sigma)$ is a vector lattice, endowed with the natural order: $f \leq g$ iff $f(\omega) \leq g(\omega)$ for a.e. $\omega \in \Omega$. By [22, Lemma 2.6.1], it is easily seen that any ideal of $L_0(\Sigma)$ is order complete and has the countable sup property. Hence, Lemma 2.1 is applicable to them. Recall also that $f_n \to 0$ in an ideal $X$ of $L_0(\Sigma)$ if and only if $f_n \to 0$ in $X$ and $(f_n)_{n=1}^\infty$ is order bounded.
in $X$, i.e., there exists $f \in X$ such that $|f_n| \leq f$ a.e. for each $n \geq 1$. We remark that the class of ideals of $L_0(\Sigma)$ which admit strictly positive order continuous linear functionals is very large. For example, by [15, Proposition 5.19], all Banach function spaces (i.e., ideals of $L_0(\Sigma)$ endowed with complete lattice norm), including all Orlicz spaces, are as such.

For a subset $Y$ of $L_0(\Sigma)$, define $\sigma(Y)$ to be the smallest sub-$\sigma$-algebra of $\Sigma$ which makes all members in $Y$ measurable and contains all $P$-null sets. Denote by $L_0(\sigma(Y))$ the set of all functions in $L_0(\Sigma)$ which are measurable with respect to $\sigma(Y)$. Clearly, $Y \subset L_0(\sigma(Y))$. If $Y = \{f\}$, we write $\sigma(f)$ instead of $\sigma(\{f\})$, for the sake of simplicity. The following result is an improved and generalized market completeness theorem in the sense of Green and Jarrow ([17, Theorem 1]).

**Lemma 2.2** Let $X$ be an ideal of $L_0(\Sigma)$ and $Y$ be a sublattice of $X$ such that $1 \in Y$. Then the following are equivalent:

(a) $Y$ is order closed in $X$,
(b) $Y = L_0(\sigma(Y)) \cap X$.

### 3 Main results

In this section, the space of contingent claims, $X$, is always modelled as an ideal of $L_0(\Sigma)$ over a given probability space $(\Omega, \Sigma, P)$ that contains the constant functions and admits a strictly positive order continuous linear functional. Our first main result is as follows.

**Theorem 3.1** Let $f$ be a limited liability claim in $X$. For a claim $g \in X$, the following are equivalent:

(a) $g$ is a contingent claim written on $f$, i.e., $g \in L_0(\sigma(f)) \cap X$,
(b) $g$ can be approximated by portfolios of options on $f$ in the $\sigma(X, X_n^\sim)$-topology, i.e.,
(c) There exists a sequence $(g_n)$ in $O_f$ such that $g_n \xrightarrow{a.e.} g$.

The following corollary is immediate.

**Corollary 3.2** Let $f$ be a limited liability claim in $X$ such that $\sigma(f) = \Sigma$. Then we have the following:

(a) The option space of $f$ completes the market in the $\sigma(X, X_n^\sim)$-topology, i.e., $\overline{O_f}^{\sigma(X, X_n^\sim)} = X$,
(b) The option space of $f$ completes the market by approximating via a.e. convergence, i.e., for any $g \in X$, there exists a sequence $(g_n)$ in $O_f$ such that $g_n \xrightarrow{a.e.} g$.

**Remark 3.3** Note that our Theorem 3.1 and Corollary 3.2 imply both the aforementioned results of Nachman, Galvani and Troitsky. Indeed, recall first that, for $1 \leq p < \infty$, $X = L_p(\Sigma)$ is order continuous, i.e., $x_\alpha \downarrow 0$ in $X$ implies $\|x_\alpha\| \downarrow 0$. In this case, one has $X_n^\sim = X^*$, so that $\sigma(X, X_n^\sim)$ is just the weak topology on $X$. Thus, by Mazur’s theorem, $\overline{C}^{\sigma(X, X_n^\sim)} = \overline{C}^w = \overline{C}^{\|\cdot\|}$ for any convex subset $C$ of $X$. Consequently, it follows that

$$\overline{O_f}^{\sigma(X, X_n^\sim)} = \overline{O_f}^{\|\cdot\|_p}. $$

Now it is clear that Eq. (1) follows from Theorem 3.1. If, in addition, $\Omega$ is a Polish space with $\Sigma$ being the Borel algebra, and $f$ is one-to-one, then it is easily seen that $\sigma(f) = \Sigma$.
by [4, Theorem 12.29]. Thus, Eq. (2) follows from Corollary 3.2. Equation (3) also follows from Corollary 3.2, since $L_\infty(\Sigma)^* = L_1(\Sigma)$ and thus $\sigma(L_\infty(\Sigma), L_\infty(\Sigma)^*)$ is just the $w^*$-topology.

We now turn to discuss the pricing problem. Our notation and terminology are in accordance with [21,31].

Let $f$ be a fixed asset in $X$ and $\pi$ be a positive linear functional on the option space $M := O_f$, which is interpreted as a linear pricing functional on $M$. We denote by $M_0 := \{x \in M \mid \pi(x) = 0\}$ the set of all portfolios of options on $f$ that can be bought or sold with zero price. We say that $(M, \pi)$ admits no free lunches (cf. [31, Definition 1.3]), if the following holds

$$C \cap X_+ = \{0\}, \quad \text{where } C = \overline{M_0 - X_+^{\sigma(X,X_+^*)}}.$$

We say that a price, $p$, of an asset $g \in X$ is consistent with $(M, \pi)$ if there exists a strictly positive functional $x^* \in X_+^*$ such that $x^*|_M = \pi$ and $x^*(g) = p$ ([21, Definition, pp. 29]). The price of $g \in X$ is said to be determined by arbitrage from $(M, \pi)$ if there is a single price $p$ for $g$ that is consistent with $(M, \pi)$ ([21, Definition, pp. 30]).

**Theorem 3.4** Suppose that the space $X$ of contingent claims is a Banach function space in $L_0(\Sigma)$. Let $f$ be a limited liability asset in $X$ and $\pi$ be a positive linear functional on the option space $M = O_f$. If $(M, \pi)$ admits no free lunches, then the price of any contingent claim $g \in L_0(\sigma(f)) \cap X$ is determined by arbitrage from $(M, \pi)$.

The proof of this result essentially depends on the following version of the Kreps–Yan theorem, which is of independent interest.

**Proposition 3.5** Let $X$ be a Banach function space in $L_0(\Sigma)$. Then the Kreps–Yan theorem holds true for $(X, \sigma(X,X_+))$. That is, for each $\sigma(X,X_+)$-closed cone $C$ in $X$ such that $C \supset -X_+$ and $C \cap X_+ = \{0\}$, there exists a strictly positive functional $\phi \in X_+^*$ such that $\phi|_C \leq 0$.

The proof of this result (see Sect. 4) relies on [19, Theorem 3.1]. For more results in this direction, we refer the reader to [27,28]. For no-arbitrage results, we refer the reader to the monograph [12] and the references therein.

**4 Proofs of results**

**Proof of Lemma 2.1** Let $(y_n)$ be an increasing sequence in $Y$ that is order bounded above in $X$. Since $X$ is order complete, it follows that $(y_n)$ has a supremum, $x$, in $X$. Since $(y_n)$ is increasing, it follows that $y_n \uparrow x$ in $X$, so that $y_n \to x$ in $X$. Thus, if $Y$ is order closed in $X$, then $x \in Y$. This proves the “only if” part.

For the “if” part, observe first that, in this case, for any sequence $(y_n)$ in $Y$ which is order bounded in $X$, its supremum and infimum in $X$ also lie in $Y$. Indeed, denote by $x$ the supremum of $(y_n)$ in $X$. Put $z_n = \sqrt{\sum_{k=1}^n y_k}$. Then $z_n \in Y$ as $Y$ is a sublattice of $X$, and moreover, the supremum of $(z_n)$ in $X$ is still $x$. Since $(z_n)$ is increasing, it follows from the “if” assumption that $x \in Y$. Replacing $(y_n)$ with $(-y_n)$, one sees easily that the infimum of $(y_n)$ in $X$ also lies in $Y$. Now let $(y_\alpha) \subset Y$ and $x \in X$ be such that $y_\alpha \to x$ in $X$. By passing to a tail, we may assume that $(y_\alpha)$ is order bounded in $X$. Then since $X$ is order complete, we have
where all the suprema and infima are taken in \( X \). By the countable sup property of \( X \), we can choose \( \{ a_n \}_{n=1}^\infty \) such that \( \inf_n \sup_{\beta \geq a_n} |y_\beta - x| = 0 \). Without loss of generality, we can assume that \( (a_n) \) is increasing. It follows that

\[
\inf_n \sup_{m \geq n} |y_{a_m} - x| = 0,
\]

or equivalently, \( y_{a_n} \xrightarrow{a} x \), so that \( x = \inf_n \sup_{m \geq n} y_{a_m} \); cf. [4, Theorem 8.16]. Applying the preceding observation to the suprema and then to the infimum, we obtain that \( x \in Y \). \( \square \)

**Proof of Lemma 2.2** Assume first (b) holds. Let \( (f_n) \) be an increasing sequence in \( Y \) and \( f \) be its supremum in \( X \). Then \( f_n \uparrow f \) a.e. Since each \( f_n \) is \( (\sigma(Y)) \)-measurable, we have that \( f \) is also \( (\sigma(Y)) \)-measurable, so that \( f \in L_0(\sigma(Y)) \cap X = Y \). Thus since \( X \) is order complete and has the countable sup property, Lemma 2.1 implies that (a) holds.

Conversely, assume that (a) holds. We first claim that \( \sigma(Y) = \{ A \in \Sigma : \chi_A \in Y \} \). Denote the right hand side by \( G \). We first show that it is a \( \sigma \)-algebra. Indeed, it is clear that \( \emptyset \in G \), and that if \( A \in G \), then \( \chi_{A^c} = 1 - \chi_A \in Y \), so that \( A^c \in G \). Now let \( (A_k)_{k=1}^\infty \) be a sequence of sets in \( G \). Then \( \chi_{\bigcap_{k=1}^n A_k} = \bigvee_{k=1}^n \chi_{A_k} \in Y \), and from \( \chi_{\bigcup_{k=1}^n A_k} \uparrow \chi_{\bigcup_{k=1}^\infty A_k} \) in \( X \), it follows that \( \chi_{\bigcup_{k=1}^\infty A_k} \in Y \), since \( Y \) is order closed. Therefore, \( \bigcup_{k=1}^\infty A_k \in G \). This concludes the proof of that \( G \) is a \( \sigma \)-algebra. Next, we show that each \( f \in Y \) is measurable with respect to \( G \). Indeed, for any real number \( r \), it follows from \( Y \ni n(f-r)^+ \land 1 \uparrow \chi_{\{f>r\}} \) in \( X \) that \( \chi_{\{f>r\}} \in Y \), so that \( \{f>r\} \in G \), and \( f \) is \( G \)-measurable. Clearly, \( G \) contains all \( \mathbb{P} \)-null sets. Thus we have \( \sigma(Y) \subseteq G \). The reverse inclusion being clear, this completes the proof of the claim.

It is clear that \( Y \subseteq L_0(\sigma(Y)) \cap X \). Now take \( f \in L_0(\sigma(Y)) \cap X \). By considering \( f^\pm \), we may assume that \( f \) is non-negative. Then we can find a sequence \( (f_n) \) of simple functions which are measurable with respect to \( \sigma(Y) \) such that \( f \equiv f_n \) everywhere, so that \( f_n \uparrow f \) in \( X \). By the preceding claim, we have that \( f_n \in Y \). Therefore, \( f \in Y \), and thus \( L_0(\sigma(Y)) \cap X \subseteq Y \).

It follows that \( Y = L_0(\sigma(Y)) \cap X \). \( \square \)

**Proof of Theorem 3.1** We claim that \( O_f \) is a sublattice of \( X \). Indeed, put \( Z = \text{Span} \{ b, s, (s-kb)^+ : k \in \mathbb{R} \} \), where \( b = f + 1 \) and \( s = f \). Note that, being an ideal of \( L_0(\Sigma) \), \( X \) is order complete, and thus it is uniformly complete (cf. [2, Lemma 1.56]). By [8, Theorem (1)], it follows that \( Z \) is a sublattice of \( X \). Now simply observe that \( Z = O_f \). Indeed, the inclusion \( O_f \subseteq Z \) is immediate as \( f, 1 \in Z \) and \( Z \) is closed under lattice operations. For the reverse inclusion, note that \( s = f = (f-0)^+ \in O_f \) so that \( b \in O_f \) as well. Also, for \( k \geq 1 \) we have \( (s-kb)^+ = 0 \), and for \( k < 1 \) we have \( (s-kb)^+ = (1-k)(f - \frac{k}{1-k})1 \in O_f \).

Assume that (c) holds. Since \( Z \) is a sublattice of \( X \), by considering the positive and negative parts, respectively, we may assume that \( g \geq 0 \) and \( gn \geq 0 \) for all \( n \). For any \( k \geq 1 \), since \( gk \land gn \xrightarrow{a,e} gk \land g \) and \( (g_k \land gn)_n \) is order bounded in \( X \), it follows that \( g_k \land gn \xrightarrow{a} g_k \land g \) in \( X \), and therefore, \( g_k \land gn \xrightarrow{\sigma(X,X^\infty)} g_k \land g \as n \rightarrow \infty \). By the fact that \( Z \) is a sublattice again, we have \( g_k \land gn \in Z \) for all \( k, n \geq 1 \). Hence,

\[
gk \land g \in Z_{\sigma(X,X^\infty)}
\]

1 This is essentially contained in [17, Theorem 1].

2 Keep in mind that \( \chi_A \) is identified as 0 in \( L_0(\Sigma) \) if \( \mathbb{P}(A) = 0 \).
for any $k \geq 1$. Now $g_k \wedge g \overset{a.e.}{\rightarrow} g$ and $(g_k \wedge g)$ is order bounded in $X$, we have \( g_k \wedge g \overset{a}{\rightarrow} g \) in $X$, so that $g_k \wedge g \overset{\sigma(X,X_n^\sim)}{\rightarrow} g$. Therefore, since $Z^{\sigma(X,X_n^\sim)}$ is $\sigma(X,X_n^\sim)$-closed, we have
\[
g \in Z^{\sigma(X,X_n^\sim)}.
\]
This proves that (c)$\Rightarrow$(b).

Suppose now (b) holds. Recall that $X_n^\sim$ is a band (i.e., order closed ideal) of the order dual $X^\sim$ ([3, Theorem 1.57]). It follows from [3, Theorem 3.50] that the dual of $X$ under the topology $|\sigma|(X,X_n^\sim)$ is just $X_n^\sim$. Therefore, by Mazur’s theorem (cf. [3, Theorem 3.13]), since $Z$ is convex, $Z^{\sigma(X,X_n^\sim)} = Z^{\sigma(|\sigma|(X,X_n^\sim))}$. Consequently, there exists a net $(g_a)$ in $Z$ such that $g_a \overset{|\sigma|(X,X_n^\sim)}{\rightarrow} g$. In particular, if $x_0^*$ is any strictly positive order continuous functional on $X$, then
\[
x_0^*(|g_a - g|) \rightarrow 0.
\]
Take $(a_n)$ such that $x_0^*(|g_{a_n} - g|) \leq \frac{1}{n}$. Then since $\bigvee_{m=n}^k |g_{a_m} - g| \wedge 1 \uparrow k \sup_{m \geq n} |g_{a_m} - g| \wedge 1$, it follows from order continuity of $x_0^*$ that
\[
x_0^* \left( \sup_{m \geq n} |g_{a_m} - g| \wedge 1 \right) = \lim_{k} x_0^* \left( \sup_{m=n}^k |g_{a_m} - g| \wedge 1 \right) = \lim_{k} x_0^* \left( \sum_{m=n}^k |g_{a_m} - g| \wedge 1 \right) \leq \frac{1}{2^{n-1}}.
\]
Therefore, we have
\[
x_0^* \left( \inf_{n \geq 1} \sup_{m \geq n} |g_{a_m} - g| \wedge 1 \right) = 0,
\]
and thus by strict positivity of $x_0^*$, we have
\[
\inf_{n \geq 1} \sup_{m \geq n} |g_{a_m} - g| \wedge 1 = 0.
\]
If $g_{a_n} \overset{a.e.}{\rightarrow} g$, then there exist $\varepsilon > 0$ and a measurable set $A$ of positive measure such that $\limsup_n |g_n(\omega) - g(\omega)| \geq \varepsilon$ for any $\omega \in A$. Therefore, it is easily seen that
\[
\inf_{n \geq 1} \sup_{m \geq n} |g_{a_n} - g| \wedge 1 \geq (\varepsilon \chi_A) \wedge 1 > 0.
\]
This contradiction concludes the proof of (b)$\Rightarrow$(c).

Now put $Y = Z^{\sigma(X,X_n^\sim)}$. Then $Y$ is clearly order closed in $X$. Moreover, by the preceding paragraph, $Y = Z^{\sigma(|\sigma|(X,X_n^\sim))}$, implying that it is also a sublattice of $X$ by [3, Theorem 3.46]. Thus by Lemma 2.2, $Y = L_0(\sigma(Y)) \cap X$. Since $f \in Y$, it is clear that $\sigma(Y) \supset \sigma(f)$, so that
\[
Y = L_0(\sigma(Y)) \cap X \supset L_0(\sigma(f)) \cap X.
\]
For the reverse inclusion, note that, by definition of $O_f$, it is easily seen that each $g \in O_f$ is measurable with respect to $\sigma(f)$. Now for an arbitrary $g \in Y$, we can take, by the implication (b) $\Rightarrow$ (c), a sequence $(g_n)$ in $O_f$ such that $g_n \overset{a.e.}{\rightarrow} g$. Clearly, $g$ is also $\sigma(f)$-measurable. Therefore, it follows that
\[
Y \subset L_0(\sigma(f)) \cap X,
\]
and hence $Y = L_0(\sigma(f)) \cap X$. This proves (a)$\Leftrightarrow$(b).
Proof of Proposition 3.5 We apply [19, Theorem 3.1] to \( (X, \sigma(X, X_n^\sim)) \), and verify that the following Assumptions (C) and (L) are satisfied.

Assumption (C): For every sequence \( (x_n^\gamma) \) in \( X_n^\sim \), there exist strictly positive numbers \( (\alpha_n) \) such that \( \sum_{n=1}^{\infty} \alpha_n x_n^\gamma \) converges in \( X_n^\sim \) with respect to the \( \sigma(X_n^\sim, X) \)-topology.

Assumption (L): Any family \( \{x_n^\gamma\}_{\gamma \in \Gamma} \) in \( (X_n^\sim)_{+} \) admits a countable subfamily \( \{x_n^{\gamma_n}\}_{n \geq 1} \) such that, for any \( x \in X_+ \), \( x_n^{\gamma_n}(x) \to 0 \) for all \( n \geq 1 \) implies \( x_n^{\gamma_n}(x) = 0 \) for all \( \gamma \in \Gamma \).

We first verify that Assumption (C) is satisfied. Indeed, since \( X \) is a Banach lattice, we know that the order dual \( X^\sim \) equals the norm dual \( X^* \) ([3, Corollary 4.5]) and is thus a Banach lattice. By [3, Theorem 1.57], \( X^\sim \) is a band (i.e., order closed ideal) in \( X^\sim = X^* \), and is thus norm closed in \( X^* \) by [3, Theorem 3.46]. Now for a sequence \( (x_n^\gamma) \) in \( X_n^\sim \), put \( \alpha_n := \frac{1}{\|x_n^\gamma\|_{X^*}} \) for each \( n \geq 1 \). Then \( \alpha_n \)'s are strictly positive, and \( \sum_{n=1}^{\infty} \alpha_n x_n^\gamma \) converges in norm to some \( x^\gamma \in X^* \). Since \( X_n^\sim \) is norm closed in \( X^* \), it follows that \( x_n^{\gamma_n} \in X^* \). Clearly, \( \sum_{n=1}^{\infty} \alpha_n x_n^\gamma \) also converges to \( x^\gamma \) in the \( \sigma(X_n^\sim, X) \)-topology.

We now verify that Assumption (L) is also satisfied. For the given family \( \{x_n^{\gamma_n}\}_{\gamma \in \Gamma} \) in \( (X_n^\sim)_{+} \), put \( N_{\gamma_n} := \{x \in X : x_n^{\gamma_n}(|x|) = 0\} \) and \( C_{\gamma_n} := N_{\gamma_n}^\circ := \{x \in X : |x| \wedge |y| = 0 \text{ for all } y \in N_{\gamma_n}\} \) for each \( \gamma \). Observe that \( N_{\gamma_n} \) is a band. Indeed, it is clearly an ideal.

If a net \( (x_\alpha) \) in \( N_{\gamma_n} \) converges to some \( x \in X \), then \( |x_\alpha - x| \xrightarrow{\omega} 0 \) implies that \( x_\alpha^\gamma(|x|) = |x_\alpha^\gamma(x_\alpha) - x_\alpha^\gamma(x)| \leq x_\alpha^\gamma(||x_\alpha - x|||) \leq x_\alpha^\gamma(|x_\alpha - x|) \to 0 \), and consequently, \( x^\gamma(|x|) = 0 \), i.e., \( x \in N_{\gamma_n} \). This yields the band decomposition \( X = N_{\gamma_n} \oplus C_{\gamma_n} \) by [3, Theorem 1.42]. Recall from [1, Corollary 5.22] that \( X \) has a weak unit \( u > 0 \), i.e., any function \( x \in X \) is supported in \( \{\omega : u(\omega) > 0\} \) off a null set. Write \( u = f_y + e_y \) where \( f_y \in N_{\gamma_n} \) and \( e_y \in C_{\gamma_n} \). Since \( f_y \wedge e_y = 0 \), it is easily seen that there exists \( A_{\gamma_n} \in \Sigma \) such that \( e_y = u_{\chi_{A_{\gamma_n}}} \), and \( f_y = u_{\chi_{A_{\gamma_n}}} \). Each function in \( N_{\gamma_n} \) is disjoint with \( e_y \) and is thus supported in \( A_{\gamma_n} \) off a null set. By countable sup property of \( X \), we choose \( \{\gamma_n\}_{n=1}^{\infty} \) such that \( \sup_{\gamma_n} e_{\gamma_n} = \sup_{\gamma} e_{\gamma} \).

If \( x_n^{\gamma_n}(x) = 0 \) for all \( n \geq 1 \) and some \( x \in X_+ \), then we have \( x \in N_{\gamma_n} \), so that \( x \wedge e_{\gamma_n} = 0 \), for all \( n \geq 1 \). It follows that \( x \wedge \sup_{\gamma} e_{\gamma} = x \wedge \sup_{\gamma_n} e_{\gamma_n} = \sup_{\gamma_n}(x \wedge e_{\gamma_n}) = 0 \), and consequently, \( x \wedge e_{\gamma} = 0 \) for any \( \gamma \). This implies that \( x \) is supported in \( A_{\gamma_n}^c \) off a null set and hence belongs to \( N_{\gamma_n} \), i.e., \( x_n^{\gamma_n}(x) = 0 \). \( \square \)

Proof of Theorem 3.4 It is clear that \( C := \overline{M_0 - X_+}^{\sigma(X, X_n^\sim)} \) is a \( \sigma(X, X_n^\sim) \)-closed cone with \( -X_+ \subset C \) and \( C \cap X_+ = \{0\} \) because of no free lunches. Thus by Proposition 3.5, there exists a strictly positive linear functional \( x^\gamma \in X_n^\sim \) such that \( x^\gamma \mid C \leq 0 \). This last condition implies that \( x^\gamma|_{M_0} = 0 \), so that \( \ker \pi = M_0 \subset \ker(x^\gamma|_{M}) \). Hence, there exists \( \lambda > 0 \) such that \( \pi = \lambda x^\gamma \mid_{M} \). Therefore, for each \( g \in L_0(\sigma(f)) \cap X = \overline{O_f}^{\sigma(X, X_n^\sim)} \), it is easily seen that the price \( p := \lambda x^\gamma(g) \) is consistent with \( (M, \pi) \). \( \square \)

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