Quantum Stackelberg duopoly in the presence of correlated noise

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Abstract
We study the influence of entanglement and correlated noise using correlated amplitude damping, depolarizing and phase damping channels on the quantum Stackelberg duopoly. Our investigations show that under the influence of an amplitude damping channel a critical point exists for an unentangled initial state at which firms get equal payoffs. The game becomes a follower advantage game when the channel is highly decohered. Two critical points corresponding to two values of the entanglement angle are found in the presence of correlated noise. Within the range of these limits of the entanglement angle, the game is a follower advantage game. In the case of a depolarizing channel, the payoffs of the two firms are strongly influenced by the memory parameter. The presence of quantum memory ensures the existence of the Nash equilibrium for the entire range of decoherence and entanglement parameters for both the channels. A local maximum in the payoffs is observed which vanishes as the channel correlation increases. Moreover, under the influence of the depolarizing channel, the game is always a leader advantage game. Furthermore, it is seen that the phase damping channel does not affect the outcome of the game.

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1. Introduction

Game theory is the mathematical study of the interaction among independent, self-interested agents. It emerged from the work of Von Neumann [1], and is now used in various disciplines such as economics, biology, medical sciences, social sciences and physics [2, 3]. Due to a dramatic development in quantum information theory [4], the game theorists [5–12] have made strenuous efforts to extend the classical game theory into the quantum domain. The first attempt in this direction was made by Meyer [13] by quantizing a simple coin tossing game. Applications of quantum games are reviewed by several authors [14–18]. A formulation of
quantum game theory based on the Schmidt decomposition is presented by Ichikawa et al. [19]. Recently, Xia et al. [20, 21] have investigated the quantum Stackelberg duopoly game under the influence of decoherence and have found a critical point for the maximally entangled initial state against the damping parameter for the amplitude damping environment under certain conditions.

In practice, no system can be fully isolated from its environment. The interaction between the system and the environment leads to the destruction of quantum coherence of the system. It produces an inevitable noise and results in the loss of information encoded in the system [22]. This gives rise to the phenomenon of decoherence. Quantum information is encoded in qubits during its transmission from one party to another and requires communication channels. In a realistic situation, the qubits have a nontrivial dynamics during transmission because of their interaction with the environment. Therefore, a party may receive a set of distorted qubits because of the disturbing action of the channel. Studies on quantum channels have attracted a lot of attention in recent past [23, 24]. Early work in this direction was devoted mainly to memoryless channels for which consecutive signal transmissions through the channel were not correlated. In the correlated channels (i.e. the channels with memory), the noise acts on consecutive uses of the channel. The effect of decoherence and correlated noise in quantum games has produced interesting results and is studied by a number of authors [23–28].

In this paper, we study the effect of correlated noise introduced through amplitude damping, phase damping and depolarizing channels parameterized by the decoherence parameters $p_1$ and $p_2$ and the memory parameters $\mu_1$ and $\mu_2$, on the quantum Stackelberg duopoly game. The decoherence parameters $p_i$ and the memory parameters $\mu_i$ range from 0 to 1. The lower and upper limits of the decoherence parameter $p_i$ correspond to the undecharred and fully decohered cases, respectively, whereas the lower and upper limits of the memory parameter $\mu_i$ correspond to the uncorrelated and fully correlated cases, respectively. It is seen that there exists a critical point in the case of an amplitude damping channel for an initially unentangled state at which both firms have equal payoffs. The game transforms from the leader advantage to the follower advantage game beyond this point, for a highly decohered case in the presence of memory. For an initially entangled state under the influence of an amplitude damping channel we found two critical points. The game behaves as a follower advantage game within these two critical points. In the case of a depolarizing channel the high correlation results in high payoffs. However, the phase damping channel has no affect on the game dynamics.

2. Stackelberg duopoly game

The Stackelberg duopoly game is a market game, which is rather different from the Cournot duopoly game. In the Cournot duopoly game, two firms simultaneously provide a homogeneous product to the market and guess that what action the opponent will take. However, Stackelberg duopoly is a dynamic model of the duopoly game in which one firm, say firm $A$, moves first and the other firm, say $B$, goes after. Before making its decision, firm $B$ observes the move of firm $A$. This transforms the static nature of the Cournot duopoly game to a dynamic one. Firm $A$ is usually called the leader and firm $B$ the follower; on this basis the game is also called the leader–follower model [30]. In classical Stackelberg duopoly it is assumed that firm $B$ will respond optimally to the strategic decision of firm $A$. As firm $A$ can precisely predict firm $B$’s strategic decision, firm $A$ chooses its move in such a way that maximizes its own payoff. This informational asymmetry makes Stackelberg duopoly as the first mover advantage game.
A number of authors have proposed various quantization protocols for observing the behavior of the Stackelberg duopoly game in the quantum realm [9, 29–32]. It has been shown that quantum entanglement affects payoff of the first mover and produces an equilibrium that corresponds to a classical static form of the same game [33]. The effects of decoherence produced by various prototype quantum channels on quantum Stackelberg duopoly have been studied by Zhu et al [20]. We study the effects of correlated noise on quantum Stackelberg duopoly, using amplitude damping, phase damping and depolarizing channels.

3. Calculations

In a quantum Stackelberg duopoly game, for each firm $A$ and $B$, the game space is a two-dimensional complex Hilbert space of the basis vectors $|0\rangle$ and $|1\rangle$, that is, the game consists of two qubits, one for each firm. We consider that the initial state of the game is given by

$$|\psi_i\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle,$$

(1)

where $\theta$ is a measure of entanglement. The state is maximally entangled at $\theta = \frac{\pi}{4}$. In the presence of noise the evolution of an arbitrary system can be described in terms of Kraus operators as [4]

$$\rho = \sum_l E_l \rho_i E_l^\dagger,$$

(2)

where $\rho_i = |\psi_i\rangle \langle \psi_i|$ is the initial density matrix and the Kraus operators $E_l$ satisfy the following completeness relation:

$$\sum_l E_l^\dagger E_l = 1.$$

(3)

The single qubit Kraus operators for an uncorrelated quantum amplitude damping channel are given as [23, 24]

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

(4)

The Kraus operators for an amplitude damping channel with correlated noise for a two-qubit system are given as [24]

$$E_{c00}^c = \begin{pmatrix} \sqrt{1-p} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_{c11}^c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{p} & 0 & 0 & 0 \end{pmatrix}.$$

(5)

The influence of such a channel on the initial density matrix of the system is given by

$$\rho = (1 - \mu) \sum_{m,n=0}^1 E_{mn}^u \rho_i E_{mn}^u + \mu \sum_{l=0}^1 E_{ll}^c \rho_i E_{ll}^c,$$

(6)

where the superscripts $u$ and $c$ represent the uncorrelated and correlated parts of the channel, respectively. The above relation means that with the probability $\mu$ the noise is correlated and with the probability $(1 - \mu)$ it is uncorrelated. The Kraus operators for the phase damping channel with uncorrelated noise for a system of two qubits are given as [23, 24]

$$E_{mn}^u = \sqrt{p} \sigma_m \sigma_n \otimes \sigma_m, \quad m, n = 0, 3,$$

(7)

whereas the ones with correlated noise are given as

$$E_{ll}^c = \sqrt{p} \sigma_l \otimes \sigma_l, \quad l = 0, 3.$$

(8)
Similarly, the Kraus operators for the depolarizing channel are described by equations (7) and (8) with indices running from 0 to 3, where $\sigma_0$ is the identity operator for a single qubit and $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the Pauli spin operators. For the phase damping channel, $e_0 = (1 - p_i)$ and $e_3 = p_i$, and for the depolarizing channel $e_0 = (1 - p_i), e_1 = e_2 = e_3 = \frac{1}{3} p_i$, where $p_i$ correspond to the decoherence parameters of the first and second use of the channel. The action of such a channel on the quantum system can be defined in a similar fashion as described earlier in equation (6).

In the quantum Stackelberg duopoly game each firm has two possible strategies: $I$, the identity operator, and $C$, the inversion operator or Pauli’s bit-flip operator. Let $x$ and $1 - x$ stand for the probabilities of $I$ and $C$ that firm $A$ applies and let $y$ and $1 - y$ are the probabilities that firm $B$ applies. The final state after the action of the channel is given by

$$\rho_f = x y I_A \otimes I_B \rho \ I_A^\dagger \otimes I_B^\dagger + x (1 - y) I_A \otimes C_B \rho \ I_A^\dagger \otimes C_B^\dagger$$

$$+ y (1 - x) C_A \otimes I_B \rho \ C_A^\dagger \otimes I_B^\dagger$$

$$+ (1 - x) (1 - y) C_A \otimes C_B \rho \ C_A^\dagger \otimes C_B^\dagger,$$

where $\rho$ (equation (6)) is the density matrix of the game after the channel action.

Suppose that the player’s moves in the quantum Stackelberg duopoly are given by the probabilities lying in the range $[0, 1]$. In the classical duopoly game the moves of firms $A$ and $B$ are given by the quantities $q_1$ and $q_2$, which have values in the range $[0, \infty)$. We assume that firms $A$ and $B$ agree on a function that uniquely defines a real positive number in the range $(0, 1]$ for every quantity $q_1, q_2$ in $[0, \infty)$. Such a function is given by $1/(1 + q_i)$, so firms $A$ and $B$ find $x$ and $y$, respectively, as

$$x = \frac{1}{1 + q_1}, \quad y = \frac{1}{1 + q_2}.$$  

(10)

The payoffs of firms $A$ and $B$ are given by the following trace operations:

$$P_A (q_1, q_2) = \text{Tr} \left[ \rho_f P_A^{\text{op}} (q_1, q_2) \right], \quad P_B (q_1, q_2) = \text{Tr} \left[ \rho_f P_B^{\text{op}} (q_1, q_2) \right].$$

(11)

where $P_A^{\text{op}}$, $P_B^{\text{op}}$ are the payoff operators of the firms and are given by

$$P_A^{\text{op}} (q_1, q_2) = \frac{q_1}{q_{12}} (k \rho_{11} - \rho_{22} - \rho_{33})$$

(12)

$$P_B^{\text{op}} (q_1, q_2) = \frac{q_2}{q_{12}} (k \rho_{11} - \rho_{22} - \rho_{33}).$$

(13)

where $\rho_{ij}$ are the diagonal elements of the final density matrix, $k$ is a constant as given in [30] and $q_{12}$ is given by

$$q_{12} = \frac{1}{(1 + q_1)(1 + q_2)}.$$  

The backward-induction outcome in Stackelberg duopoly is found by first finding the reaction $R_2 (q_1)$ of firm $B$ to an arbitrary quantity $q_1$ chosen by firm $A$. It is found by differentiating firm $B$’s payoff with respect to $q_2$, and maximizing the result for $q_1$ and can be written as

$$R_2 (q_1) = \max P_B (q_1, q_2).$$

(14)

Once firm $B$ chooses this quantity, firm $A$ can compute its optimization problem by differentiating its own payoff with respect to $q_1$ and then maximizing it to find the value $q_1 = q_1^*$. Using the value of $q_1^*$ in equation (14) we can get the value of $q_2^*$. These quantities define the backward-induction outcome of the quantum Stackelberg duopoly game and represent the subgame perfect Nash equilibrium point. The payoffs of the firms at this point can be found using equation (11).
4. Results and discussion

We suppose that before the firms measure their payoffs the game evolves twice through different quantum correlated channels. That is, the state, prior and after the firms apply their operators, is influenced by the correlated noisy channels.

4.1. Correlated amplitude damping channel

The subgame perfect Nash equilibrium point for the game under the influence of the correlated quantum amplitude damping channel becomes

\[q^*_1 = \frac{-k \cos^2 \theta + A_1(p_1, p_2, \mu_1, \mu_2)}{-4 + A_2(p_1, p_2, \mu_1, \mu_2)} - 4 + A_2(p_1, p_2, \mu_1, \mu_2),\]

\[q^*_2 = \frac{\frac{1}{2}k \cos^2 \theta - B_1(p_1, p_2, \mu_1, \mu_2)}{16 + B_2(p_1, p_2, \mu_1, \mu_2)},\]

where the damping functions \(A_i\) and \(B_i\) are given in the appendix.

If we consider the influence of decoherence in the second evolution only \((p_1 = 0)\), equation (15) for an unentangled initial state reduces to the following form:

\[q^*_1 = \frac{1}{2 - 4p_2(1 - \mu_2)}\]

\[q^*_2 = \frac{1}{2[2 - p_2(6 - 5\mu_2 - p_2(1 - \mu_2)(5 - 8\mu_2))]}.\]

Here we have taken \(k = 1\). The firms’ payoffs under this situation become

\[P_A = \frac{1}{8 - 16p_2(1 - \mu_2)}\]

\[P_B = \frac{1 - 2p_2(1 - \mu_2)}{8[2 - p_2(6 - 5\mu_2) + p_2^2(8\mu_2 + 5)(\mu_2 - 1)]}.\]

In the classical form of the duopoly the perfect game Nash equilibrium is a point, whereas in this case it is a function of both decoherence and memory parameters. It can be easily seen that the results of [20] are retrieved by setting \(\mu_2 = 0\), and setting \(p_2 = 0\) reproduces the results of the classical game. The existence of the Nash equilibrium requires that firms’ moves \((q^*_1\) and \(q^*_2)\) should have positive values. It can easily be checked using equation (16) that in the absence of quantum memory, no Nash equilibrium exists for \(p_2 > \frac{1}{2}\). The presence of quantum memory allows the existence of the Nash equilibrium for the entire range of values of \(p_2\), when \(\mu_2 \geq 0.85\). To see the influence of decoherence and quantum memory on firms’ payoffs at the subgame perfect Nash equilibrium, we plot the payoffs (equation (17)) in figure 1 as a function of the decoherence parameter \(p_2\). In the figure the dotted and dashed-dotted lines represent firm A and firm B payoffs for an unentangled initial state, respectively. The superscripts \(u\) and \(e\) of \(P_A\) \((P_B)\) in the figure stand for unentangled and entangled initial states, respectively. It can be seen from the figure that a critical point exists due to the presence of memory at which both firms are equally benefited. This situation has not been observed, in the absence of memory, for an unentangled initial state of the game. That is, in the absence of quantum memory the game is always a first mover advantage game. For a highly decohered channel and \(\mu < 1\), a transition from the first mover advantage into the second mover advantage occurs in the game behavior. It can also be seen from equation (17) that for fully correlated and fully decohered channels, both firms are equally benefited and get a payoff equal to \(\frac{1}{8}\). It can also be shown that for smaller values of
decoherence, the game is always first mover advantage irrespective of the value of the quantum memory.

For a maximally entangled initial state of the game, the subgame perfect Nash equilibrium point becomes

\[
q_1^* = \frac{1 - 3p_2^2(-1 + \mu_2) - p_2(2 - 3\mu_2)}{4 - 8p_2(1 - \mu_2)} \times \frac{[1 - p_2[2 + 3p_2(-1 + \mu_2) - 3\mu_2]]}{[1 + 2p_2(-1 + \mu_2)]}
\]

\[
q_2^* = \frac{-7 + p_2(28 - 26\mu_2) + 9p_2^3(-1 + \mu_2)^2 + p_2^2}{[-22 + 46\mu_2 - 23\mu_2^2 - 6p_2^2(2 - 5\mu_2 + 3\mu_2^2)]}
\]

The firms’ payoffs corresponding to these values of \( q_i^* \) become

\[
P_A = \frac{\{-1 + p_2[2 - 3\mu_2] + 3p_2^2(-1 + \mu_2)^2\}^2}{32(1 + 2p_2(-1 + \mu_2))} \times \frac{[\{-1 + 2p_2(-1 + \mu_2)\} - 1 + p_2(2 - 3\mu_2)]}{[\{-1 + 2p_2(-1 + \mu_2)\} - 1 + p_2(2 - 3\mu_2)]}
\]

\[
P_B = \frac{8\{-7 + p_2(28 - 26\mu_2) + 9p_2^3(-1 + \mu_2)^2\} + p_2^2(-22 + 46\mu_2 - 23\mu_2^2) - 6p_2^2(2 - 5\mu_2 + 3\mu_2^2)}{[-22 + 46\mu_2 - 23\mu_2^2 - 6p_2^2(2 - 5\mu_2 + 3\mu_2^2)]}
\]

One can easily check that these results reduce to the results of \([20, 33]\) by setting \( \mu_2 = 0 \) and \( p_2 = 0 \). The payoffs of firms for the maximally entangled initial state are plotted as a function of the decoherence parameter \( p_2 \) in figure 1. The solid line represents the firm A payoff and the dashed line represents the firm B payoff for a maximally entangled initial state.
state. One can easily check that for a maximally entangled initial state, the presence of quantum memory makes the firms better off as compared to the uncorrelated case of the channel. Furthermore, quantum memory shifts the critical point to higher payoffs than the critical point for a maximally entangled initial state when the channel is uncorrelated. This can also be shown that for a given decoherence level, the firms become worse off and the game becomes a follower advantage as the channel becomes more correlated. The effect of entanglement in the initial state on the firms’ payoffs is shown in figure 2. Here we have taken into account the two decohering correlated processes. It can be checked that for a certain range of decoherence parameters, in the absence of quantum memory, the moves of firms are negative. This means that no Nash equilibrium exists in the region of these values of decoherence parameters. The presence of quantum memory, however, resolves this problem. As can be seen from figure 2, the presence of quantum memory in the payoff functions results into two critical points corresponding to two different values of the entanglement parameter. The game becomes a follower advantage in this range of an entanglement angle. The leader firm becomes worse off in this range of values of $\theta$ and a global minimum in payoffs occurs at $\theta = \frac{\pi}{2}$.

4.2. Correlated depolarizing channel

When the game evolves under the influence of the correlated depolarizing channel, the moves of firms at the subgame perfect Nash equilibrium point become

$$q_1^* = \frac{81k \cos^2 \theta + A_1'(p_1, p_2, \mu_1, \mu_2)}{3244 + A_2'(p_1, p_2, \mu_1, \mu_2)},$$

$$q_2^* = \frac{6561k \cos^2 \theta + B_1'(p_1, p_2, \mu_1, \mu_2)}{52488 + B_2'(p_1, p_2, \mu_1, \mu_2)},$$

(20)
where the damping parameters $A'_1, A'_2, B'_1$ and $B'_2$ are given in the appendix. For an unentangled initial state, firms’ payoffs at the subgame perfect Nash equilibrium point become

$$
P_A = -\frac{(3 - 2p_2)^2[1 + 2p_2(-1 + \mu_2)]^2}{24\left[-3 - 6p_2(-1 + \mu_2) + 4p_2^2(-1 + \mu_2)\right]},$$

$$
P_B = -\frac{(3 - 2p_2)^2[1 + 2p_2(-1 + \mu_2)]^2[-3 + 2p_2(-3 + 2p_2)(-1 + \mu_2)]}{48[9 + p_2(-3 + 2p_2)[10 + 2p_2(-3 + 2p_2)(-1 + \mu_2)]^2 - 9\mu_2]}$$

(21)

where we have considered decoherence only in the second evolution of the game and have set $k = 1$. It can easily be checked that for an uncorrelated channel, firms’ moves for $p_2 > 0.5$ become negative and hence no Nash equilibrium exists for $p_2 > 0.5$. On the other hand, it can be easily shown that the Nash equilibrium exists for the entire range of values of the decoherence parameter, when the channel is correlated. The effect of quantum memory on firms’ payoffs for an unentangled initial state under a depolarizing channel is shown in figure 3. The dashed line represents the firm $A$’s payoff and the dashed-dotted line represents the firm $B$’s payoff for an unentangled initial state. The payoffs grow up with increasing value of the memory parameter and the game remains the first mover advantage game.

When the initial state is maximally entangled and only the second decohering process is taken into account, the moves and payoffs of the firms for $k = 1$ become

$$q_1^* = \frac{1}{2} + \frac{3}{4[3 - 2p_2(-3 + 2p_2)(-1 + \mu_2)]},$$

$$q_2^* = \frac{-3 + 2p_2(-3 + 2p_2)(-1 + \mu_2)}{63 + 8p_2(-3 + 2p_2)[-9 + 2p_2(-3 + 2p_2)(-1 + \mu_2)](-1 + \mu_2)}$$

(22)

8
From equation (22) one can easily check that no Nash equilibrium exists for the decoherence parameter \( p_2 > 0.32 \) when the channel is uncorrelated. However, the presence of quantum correlations ensures the presence of the Nash equilibrium for the entire range of the decoherence parameter. It can be checked that for a given value of the memory parameter the payoffs decrease when plotted as a function of the decoherence parameter and the game is always a first mover advantage game. The dependence of payoffs for the maximally entangled initial state on the memory parameter is shown in figure 3. The solid line in the figure represents firm A’s payoff and the dotted line represents firm B’s payoff for the maximally entangled initial state. It can be seen that the game is a no-payoff game around \( \mu_2 = 0.25 \). For other values of the memory parameter, the game is a first mover advantage game and the firms become better off when the channel is fully correlated.

To analyze the effect of entanglement in the initial state of the game on the Nash equilibrium, we plot the firms’ moves \( q_1^*, q_2^* \) in figures 4 and 5, respectively. From figure 4, one can see that the move of the leader firm is positive for the whole range of the entanglement angle. However, for memory parameters \( \mu_1 = \mu_2 < 0.5 \), the move \( q_2^* \) of the follower firm is negative for a particular range of values of the entanglement parameter \( \theta \) as can be seen from figure 5. The negative value of \( q_2^* \) shows that no Nash equilibrium of the game exists in this range of values of \( \theta \). On the other hand, the existence of the Nash equilibrium is ensured for the whole range of values of the entanglement parameter for a highly correlated channel. The payoffs of firms as a function of the entanglement parameter are plotted in figures 6 and 7. A local maximum exists at \( \theta = \pi \) for smaller values of memory parameters which disappears for the values of memory parameters \( > 0.5 \).
4.3. Correlated phase damping channel

The diagonal elements of the final density matrix of the game when it evolves under the influence of a correlated phase damping channel are given by
Figure 7. The payoff of firm B at the subgame perfect Nash equilibrium point under the influence of the depolarizing channel is plotted against the entanglement angle \( \theta \) with the following values of the other parameters \( p_1 = p_2 = 0.25 \), \( k = 1 \).

\[
\begin{align*}
\rho_{11}^\prime &= q_{12} (cos^2 \theta + q_1 q_2 \sin^2 \theta) \\
\rho_{22}^\prime &= q_{12} (q_2 \cos^2 \theta + q_1 \sin^2 \theta) \\
\rho_{33}^\prime &= q_{12} (q_1 \cos^2 \theta + q_2 \sin^2 \theta) \\
\rho_{44}^\prime &= q_{12} (q_1 q_2 \cos^2 \theta + \sin^2 \theta).
\end{align*}
\]

The payoffs of firms in a quantum Stackelberg duopoly game depend only on the diagonal elements of the final density matrix as can be seen from equations (11) and (12). It is clear from equation (24) that the diagonal elements of the final density matrix are independent from the decoherence parameters as well as from the memory parameters. Therefore, the correlated phase damping channel does not affect the outcome of the game.

5. Conclusions

We study the influence of entanglement and correlated noise on the quantum Stackelberg duopoly game by considering the time correlated amplitude damping, depolarizing and phase damping channels using the Kraus operator formalism. We have shown that in different entangling regions the follower advantage can be enhanced or weakened due to the existence of the initial state entanglement influenced by different correlated noise channels. The problem of nonexistence of the subgame perfect Nash equilibrium in various regions due to the presence of decoherence is resolved by quantum memory.

It is shown that under the influence of an amplitude damping channel, for an initially unentangled state, the presence of quantum memory results into a critical point at the subgame perfect Nash equilibrium. The firms are equally benefited at this point and the leader advantage vanishes. Beyond this critical point, the Nash equilibrium of the game gives higher payoff to the follower firm as a result of quantum memory, that is, the game converts from the leader advantage to the follower advantage game. Quantum memory, in the case of an amplitude damping channel, favors the follower firm in a particular range of values of the entanglement
parameter at the subgame perfect Nash equilibrium. In this range of values of the entanglement parameter, the leader firm becomes worse off and the follower firm better off (see figure 2). It is also observed that quantum memory validates the existence of the Nash equilibrium for the whole range of the entanglement angle and the decoherence parameter. It is also shown that quantum memory in the case of a phase damping channel has no effect on the subgame perfect Nash equilibrium and thus does not change the outcome of the game.

In the case of a depolarizing channel, quantum memory and entanglement in the initial state influence the firms’ payoffs at the subgame perfect Nash equilibrium strongly in a way different from an amplitude damping channel. It is seen that for \( p > 0.32 \), no Nash equilibrium exists in the case of an unentangled initial state, whereas the presence of entanglement in the initial state extends the span of the decoherence parameter \( p \) from 0–0.32 to 0–0.5 for the existence of the subgame perfect Nash equilibrium in the absence of quantum memory. On the other hand, we have observed that in the presence of memory, the subgame perfect Nash equilibrium exists for the entire range of the decoherence parameter in both the situations (for entangled and unentangled initial states). Similarly, it has been shown that memory has a striking effect that there exists a Nash equilibrium of the game for the entire range of the entanglement parameter as well, whereas in the absence of memory, the noisy environment limits the subgame perfect Nash equilibrium to exist in a particular range of the entanglement angle. In addition, a local maximum in payoffs is observed for small values of the quantum memory parameters \( \mu_1, \mu_2 \). For a highly correlated channel this local maximum disappears and the payoffs reduce to zero.

Unlike an amplitude damping channel, the correlated depolarizing channel does not give rise to a critical point at the subgame perfect Nash equilibrium and as a result, the game always remains a leader advantage game.

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**Appendix**

The damping functions \( A_i \) in equation (15) are given as

\[
A_1 = \frac{1}{2}[-k - 4(-1 + p_1)(-1 + p_2)p_2(-1 + \mu_2) - 2k p_2(-1 + \mu_2 - p_2 \mu_2) - 2k p_1[-1 + p_2(2 - p_2(-2 + \mu_1) \times (-1 + \mu_2) - 4\mu_2 + \mu_1(-2 + 3\mu_2)]) - k \cos 2\theta + 4(-1 + p_1)(-1 + p_2)p_2(-1 + \mu_2) \cos 2\theta + 2k p_2(\mu_2 - p_2 \mu_2) \cos 2\theta + 2k p_1[-1 + p_2(2 - p_2(-2 + \mu_1) \times (-1 + \mu_2) - 4\mu_2 + \mu_1(-2 + 3\mu_2)]) \cos 2\theta - 2p_1(-1 + \mu_1) \times [2 + p_1[-2 + k\{-1 + p_2(2 - 3\mu_2) + p_2^2(-1 + \mu_2)\}] - 4p_2(-1 + \mu_2) + 2p_2^2(-1 + \mu_2)] + 4p_2(-1 + \mu_2) - 2p_2^2(-1 + \mu_2)\] \sin^2\theta \tag{A.1}

\[
A_2 = -4(1 + k)p_2(-1 + \mu_2) - 4(-1 + p_1)p_1(-1 + \mu_1) \times [2 + k\{-1 + p_2(2 - 3\mu_2) + p_2^2(-1 + \mu_2)\}] - 4p_2(-1 + \mu_2) + 2p_2^2(-1 + \mu_2)\] \sin^2\theta. \tag{A.2}
The damping functions $B_i$ in equation (15) are given as

\[
B_1 = -2 + \frac{1}{k} [1 + 2 p_2 (p_2 + \mu_2 - p_2 \mu_2) + 2 p_1 [\mu_1 + p_2 (2 - 4 \mu_2 + \mu_1 (-2 + 3 \mu_2))] \\
+ [1 - 2 (p_2 + \mu_2 - p_2 \mu_2) + p_1 (\mu_1 + p_2 (2 - 2 \mu_1) \\
- 4 \mu_2 + 3 \mu_2 \mu_2)] \cos 2 \theta - \frac{1}{2} p_2^2 (-1 + \mu_1) \left[ k (-4 + 3 \mu_2) \right] \\
\times \left\{ -1 + p_2 (2 - 3 \mu_2) + p_2^2 (-1 + \mu_2) \right\} + 6 (1 + p_1) \left[ -1 - 2 p_2 \\
\times (-1 + \mu_2) + p_2^2 (-1 + \mu_2) \right\} \sin^2 \theta - \frac{1}{2} p_2 (-1 + \mu_2) \\
\times [4 (1 + k) + [2 (1 + p_2) + p_1 [2 + p_2 (-2 + k (-2 + \mu_1))] \sin^2 \theta] \right] \quad (A.3)
\]

\[
B_2 = -16 + 2 \left[ 2 [-1 - (1 + k) p_2 (-1 + \mu_2)] + p_2^2 (-1 + \mu_1) \right] \\
\times [2 (1 + k) (1 + p_2)^2 + p_2 (4 + 3 k - (2 + k) p_2 \mu_2)] \\
+ p_1 (-1 + \mu_1) [2 (1 + k) (1 - 3 k) + p_2 (-4 - 3 k)] \\
+ 2 (k) p_2 \mu_2] + (-1 + p_1) \left[ 1 + \mu_1 (1 - 2 k) (1 + p_2)^2 \right] \\
+ p_2 (-4 + 3 k + (2 + k) p_2 \mu_2) \cos 2 \theta] \left[ 1 + (1 + k) p_2 \right] \\
\times (-1 + \mu_2) \cos^2 \theta + [1 + (1 + k) p_2 (-1 + \mu_2) + p_1 (-1 + \mu_1) \right\] \\
\times [(2 + k) (1 - 2 p_2)^2 + p_2 (4 + 3 k - (2 + k) p_2 \mu_2)] \\
+ p_2 (1 + \mu_1) [-2 k (1 + \mu_2) + p_1 (1 - 2 + \mu_2)] \\
\times [2 (1 + \mu_1) - 3 + 2 \mu_2) + p_1 (2 - p_2 (-2 + \mu_2)] \\
- 4 \mu_2 + 3 \mu_2 (-3 + 2 \mu_2)] \left[ -1 + p_1 \left[ 2 (p_2 (-1 + \mu_1) \right) \\
\times (-1 + \mu_2) + p_2 (-1 + \mu_2)] \\
\times [2 (1 + p_2) + p_1 (-1 + \mu_1) \left[ -1 + \mu_2 \right)] \\
\times [1 + 2 \mu_2 + \mu_1 (-2 + p_2 + 3 \mu_2) - p_2 \mu_2)] \sin^2 \theta]. \quad (A.4)
\]

The damping functions $A_i'$ in equation (20) are given as

\[
A_1' = 4 (2 + k) \left[ 1 + 2 p_2 (p_2 + \mu_2) + 3 p_1 [p_2 (2 - 4 \mu_2 + \mu_1 (-2 + 3 \mu_2))] \right] \\
\times [2 (1 + k) (1 + p_2) (1 + \mu_1) \left[ -9 + 8 p_2 (-3 + 2 p_2)] \\
\times (1 + \mu_2)] - 36 (2 + k) p_2 (2 - 3 + 2 p_2) (1 + \mu_2) \\
+ \frac{9}{2} k [9 + (9 - 24 p_2 + 8 p_1 (-3 + 4 p_2)] \cos 2 \theta] \quad (A.5)
\]

\[
A_2' = 8 (2 + k) [9 + (9 - 24 p_2 + 8 p_1 (-3 + 4 p_2)] \cos 2 \theta] \times (1 + \mu_2)] \\
\times (-1 + \mu_2)] - 9 p_2 (-3 + 2 p_2) (1 + \mu_2)]. \quad (A.6)
\]

The damping functions $B_i'$ in equation (20) are given as

\[
B_1' = 8 (2 + k)^2 (3 - 2 p_2) p_2^2 (9 - 8 p_1 (3 + 2 p_1) (-1 + \mu_1))^2 \\
\times [k (-9 + 4 (1 + \mu_1) (1 + 3 + 2 p_1) (-1 + \mu_1)] + 8 p_1 (-3 + 2 p_1) \\
\times (-1 + \mu_1) - k (3 + 4 p_1) (-1 + 4 p_2) \cos 2 \theta] + 18 (2 + k)]
\]

\[
B_2' = 8 (2 + k)^2 (3 - 2 p_2) p_2^2 (9 - 8 p_1 (3 + 2 p_1) (-1 + \mu_1))^2 \\
\times [k (-9 + 4 (1 + \mu_1) (1 + 3 + 2 p_1) (-1 + \mu_1)] + 8 p_1 (-3 + 2 p_1) \\
\times (-1 + \mu_1) - k (3 + 4 p_1) (-1 + 4 p_2) \cos 2 \theta] + 18 (2 + k)]
\]
\[ B'_x = -72(2 + k)p_2(-3 + 2p_2)[4\{9 + 2p_1(-3 + 2p_1)(-1 + \mu_1)\}] + k[9 + 4p_1(-3 + 2p_1)(-1 + \mu_1)]\{-9 + 8p_1 \]
\[ \times (-3 + 2p_1)(-1 + \mu_1)](-1 + \mu_2) + 16(2 + k)^2(3 - 2p_2)^2 \]
\[ \times p_2^3[9 - 8p_1(-3 + 2p_1)(-1 + \mu_1)]^2(-1 + \mu_2)^2 \]
\[ + 81k^2[81 + 8p_1(-3 + 2p_1)\{9 + 2p_1(-3 + 2p_1)(-1 + \mu_1)\}] \]
\[ \times (-1 + \mu_1) + 64p_1(-3 + 2p_1)(-9 + p_1(-3 + 2p_1)) \]
\[ \times (-1 + \mu_1)(-1 + \mu_2) + 16k_1p_1(-3 + 2p_1)(-9 + 4p_1) \]
\[ \times (-3 + 2p_1)(-1 + \mu_1)\{-1 + \mu_1\} + k^2(3 - 4p_1)^2(3 - 4p_2)^2 \cos^2 2\theta]. \quad (A.7) \]

\[ \times p_2(-3 + 2p_2)[(-9 + 8p_1(-3 + 2p_1)(-1 + \mu_1)](-1 + \mu_2) \]

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