SPLITTING NECKLACES, WITH CONSTRAINTS

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Abstract. We prove several versions of Alon’s necklace-splitting theorem [1], subject to additional constraints. For illustration the “Equicardinal necklace-splitting theorem” (Theorem 4.3) claims that, without increasing the number of cuts, one can guarantee that each thief is allocated (approximately) the same number of pieces of the necklace. Unlike the classic result of Alon, our results need an additional assumption that the number of thieves is a power of prime \( r = p^\nu \), and it remains an interesting question if this condition is essential (as in the case of the Continuous Tverberg theorem and Generalized Van Kampen-Flores theorem). Our main topological tool are high connectivity results for “collectively unavoidable simplicial complexes”.

1. Introduction

The following result of Noga Alon [1, 2] is usually referred to as the “necklace-splitting theorem”. In this context, the interval \([0, 1]\) is interpreted as an (open) necklace, while \( n \) probability measures \( \mu_i \) on \([0, 1]\), corresponding to “precious gemstones” of \( n \) different types, are used for finding the value of each piece of the necklace. The theorem solves the problem of finding the minimum number of the cuts of the necklace which allows for a fair distribution of pieces among \( r \) persons (“thieves” who stole the necklace).

Theorem 1.1. ([1]) Let \( \mu_1, \mu_2, \ldots, \mu_n \) be a collection of \( n \) continuous probability measures on \([0, 1]\). Let \( r \geq 2 \) and \( N := (r-1)n \). Then there exists a partition of \([0, 1]\) by \( N \) cut points into \( N + 1 \) intervals \( I_0, I_1, \ldots, I_N \) and a function \( f : \{0,1,\ldots,N\} \to \{1,\ldots,r\} \) such that for each \( \mu_i \) and each \( j \in \{1,2,\ldots,r\} \),

\[ \sum_{f(p)=j} \mu_i(I_p) = 1/r. \]

Theorem 1.1 is optimal, as far as the number of cuts is concerned, meaning that for a generic choice of measures a fair partition with less than \( (r-1)n \) cuts is not possible. However, it is an interesting question if the necklace-splitting theorem can be refined by adding extra conditions (constraints) on how the pieces are distributed among the thieves. Here we describe several results of this type, including a result (see Theorem 4.3 and its corollaries) that if \( N + 1 \) is divisible by \( r \), then there exists a fair splitting of the necklace such that each thief is given the same number \( t := (N + 1)/r \) of intervals.

Key words and phrases. Splitting necklaces theorem, collectively unavoidable complexes, discrete Morse theory, configuration space/test map scheme.
2. Preliminaries and main definitions

2.1. Partition/allocation of a necklace. A partition of a necklace \([0, 1]\) into \(m = N + 1\) parts is described by a sequence of cut points

\[
0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq x_m = 1.
\]

(Here and in the sequel, \(m = N + 1\).)

The associated, possibly degenerate intervals \(I_j := [x_{j-1}, x_j] (j = 1, \ldots, m)\) are distributed among the thieves by an allocation function \(f : [m] \to [r]\).

The pair \((x, f)\), where \(x = (x_1, x_2, \ldots, x_N)\) is the sequence of cuts is called a partition/allocation of a necklace.

2.2. Fair and \((k, s)\)-equicardinal partitions/allocations.

1. A partition/allocation \((x, f)\) of a necklace is fair if each measure is evenly distributed among the thieves, i.e. if for each measure \(\mu_j\) and each thief \(i \in [r]\),

\[
\mu_j \left( \bigcup_{\nu \in f^{-1}(i)} I_\nu \right) = \frac{1}{r}.
\]

2. A partition/allocation \((x, f)\) is \((k, s)\)-equicardinal if,

1. each thief gets no more than \(k + 1\) parts (intervals);
2. the number of thieves receiving exactly \(k + 1\) parts is not greater than \(s\).

Note that for a fair division it is not important where we allocate the degenerate (one-point) segments. Actually, in our setting we prefer (Section 3) not to allocate them at all.

2.3. Collectively unavoidable complexes. Collectively unavoidable \(r\)-tuples of complexes are introduced in \([8]\). They were originally studied as a common generalization of pairs of Alexander dual complexes, Tverberg unavoidable complexes of \([5]\) and \(r\)-unavoidable complexes from \([7]\).

Definition 2.1. An ordered \(r\)-tuple \(K = \langle K_1, \ldots, K_r \rangle\) of subcomplexes of \(2^m\) is collectively \(r\)-unavoidable if for each ordered collection \((A_1, \ldots, A_r)\) of pair-wise disjoint sets in \([m]\) there exists \(i\) such that \(A_i \in K_i\).

2.4. Balanced simplicial complexes.

Definition 2.2. We say that a simplicial complex \(K \subseteq 2^m\) is \((m, k)\)-balanced if it is positioned between two consecutive skeleta of a simplex on \(m\) vertices,

\[
\left( \begin{array}{c} [m] \\ \leq k \end{array} \right) \subseteq K \subseteq \left( \begin{array}{c} [m] \\ \leq k + 1 \end{array} \right).
\]

2.5. Borsuk-Ulam theorem for fixed point free actions.

Theorem 2.3. (Volovikov \([15]\)) Let \(p\) be a prime number and \(G = (\mathbb{Z}_p)^k\) an elementary abelian \(p\)-group. Suppose that \(X\) and \(Y\) are fixed-point free \(G\)-spaces such that \(\tilde{H}^i(X, \mathbb{Z}_p) \cong 0\) for all \(i \leq n\) and \(Y\) is an \(n\)-dimensional cohomology sphere over \(\mathbb{Z}_p\). Then there does not exist a \(G\)-equivariant map \(f : X \to Y\).
2.6. Connectivity of symmetrized deleted joins.

**Definition 2.4.** The deleted join [12, Section 6] of a family \( \mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle \) of subcomplexes of \( 2^{|m|} \) is the complex \( \mathcal{K}_{\Delta} = K_1 \ast_{\Delta} \cdots \ast_{\Delta} K_r \subseteq \binom{[m]}{r}^{\ast} \) where \( A = A_1 \cup \cdots \cup A_r \in \mathcal{K}_{\Delta} \) if and only if \( A_j \) are pairwise disjoint and \( A_i \in K_i \) for each \( i = 1, \ldots, r \). In the case \( K_1 = \cdots = K_r = K \) this reduces to the definition of \( r \)-fold deleted join \( K_{\Delta}^r \), see [12].

The symmetrized deleted join [11] of \( \mathcal{K} \) is defined as

\[
\text{SymmDelJoin}(\mathcal{K}) := \bigcup_{\pi \in S_r} K_{\pi(1)} \ast_{\Delta} \cdots \ast_{\Delta} K_{\pi(r)} \subseteq \binom{[m]}{r}^{\ast},
\]

where the union is over the set of all permutations of \( r \) elements and \( \binom{[m]}{r}^{\ast} \cong [r]^{^m} \) is the \( r \)-fold deleted join of a simplex with \( m \) vertices.

An element \( A_1 \cup \cdots \cup A_r \in \binom{[m]}{r}^{\ast} \) is from here on recorded as \( (A_1, A_2, \ldots, A_r; B) \) where \( B \) is the complement of \( \bigcup_{i=1}^r A_i \), so in particular \( A_1 \cup \cdots \cup A_r \cup B = [m] \) is a partition of \( [m] \) such that \( A_i \neq \emptyset \) for some \( i \in [r] \).

**Lemma 2.5.** The dimension of the simplex can be read of from \( |B| \) as

\[
\dim(A_1, \ldots, A_r; B) = m - |B| - 1.
\]

The following theorem is one of the two main results from [9].

**Theorem 2.6.** Suppose that \( \mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle \) is a collectively \( r \)-unavoidable family of subcomplexes of \( 2^{|m|} \). Moreover, we assume that there exists \( k \geq 1 \) such that \( K_i \) is \((m, k)\)-balanced for each \( i = 1, \ldots, r \). Then the associated symmetrized deleted join

\[
\text{SymmDelJoin}(\mathcal{K}) = \text{SymmDelJoin}(K_1, \ldots, K_r)
\]

is \((m - r - 1)\)-connected.

The following theorem [11, Theorem 3.3] was originally proved by a direct shelling argument. As demonstrated in [9] it can be also deduced from Theorem 2.6.

**Theorem 2.7.** Let \( r, d \geq 2 \) and assume that \( rt + s = (r - 1)d \) where \( r \) and \( s \) are the unique integers such that \( t \geq 1 \) and \( 0 \leq s < r \). Let \( N = (r - 1)(d + 2) \) and \( m = N + 1 \). Then the symmetric deleted join \( \text{SymmDelJoin}(K_1, \ldots, K_r) \) of the following skeleta of the simplex \( \Delta^N = 2^{[N+1]} \),

\[
K_1 = \cdots = K_s = \left( \begin{array}{c} [N + 1] \\ \leq t + 2 \end{array} \right), \quad K_{s+1} = \cdots = K_r = \left( \begin{array}{c} [N + 1] \\ \leq t + 1 \end{array} \right).
\]

is \((m - r - 1)\)-connected.

3. NEW CONFIGURATION SPACES FOR SPLITTING NECKLACES

Perhaps the main novelty in our approach and the central new idea, emphasizing the role of collectively unavoidable complexes, is the construction and application of modified (refined) configuration spaces for splitting necklaces.
3.1. Primary configuration space. The configuration space of all sequences

\[ 0 = x_0 \leq x_1 \leq \ldots \leq x_N \leq x_m = 1 \quad (m = N + 1) \]

is an \( N \)-dimensional simplex \( \Delta^N \), where the numbers \( \lambda_j := x_j - x_{j-1} \) (\( j = 1, \ldots, m \)) play the role of barycentric coordinates. For a fixed allocation function \( f : [m] \to [r] \), the set of all partitions/allocations \( (x, f) \) is also coordinatized as a simplex \( C_f \cong \Delta^N \). The primary configuration space, associated to the necklace-splitting problem, is obtained by gluing together \( N \)-dimensional simplices \( C_f \), one for each function \( f : [N + 1] \to [r] \). Note that the common face of \( C_{f_1} \) and \( C_{f_2} \) is the set of all pairs \( (x, f_1) \sim (x, f_2) \) such that \( I_j = [x_{j-1}, x_j] \) is degenerate if \( f_1(j) \neq f_2(j) \).

The simplicial complex obtained by this construction turns out to be (the geometric realization of) the deleted join \( (\Delta^N)_\Delta^{\ast r} \cong [r]^m \). Indeed, a simplex \( \tau = (A_1, A_2, \ldots, A_r; B) \in (\Delta^N)_\Delta^{\ast r} \) is described as a partition \( A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r \sqcup B = [m] \), and a partition/allocation \( (x, f) \) is in (the geometric realization of) \( \tau \) if and only if \( B = \{ j \in [m] \mid I_j = [x_{j-1}, x_j] \text{ is degenerate} \} \) and \( A_i = f^{-1}(i) \setminus B \) is the set of all non-degenerate intervals allocated to \( i \in [r] \).

In other words, \( (x, f) \) is in the common face \( \tau = (A_1, A_2, \ldots, A_r; B) \) of \( C_{f_1} \) and \( C_{f_2} \) iff \( B = \{ j \in [m] \mid f_1(j) \neq f_2(j) \} \) and for each \( i \in [r] \), \( A_i = f_1^{-1}(i) \setminus B = f_2^{-1}(i) \setminus B \).

3.2. The test map for detecting fair splittings. Let \( \mu = (\mu_1, \ldots, \mu_n) \) be the vector valued measure associated to the collection of measures \( \{ \mu_j \}_{j=1}^n \). If \( (x, f) \in (A_1, \ldots, A_r; B) \in [r]^m \) is a partition/allocation of the necklace let

\[ \phi_i(x, f) := \mu(\bigcup_{j \in A_i} I_j) = \sum_{j \in A_i} \mu(I_j) \in \mathbb{R}^n \]

be the total \( \mu \)-measure of all intervals \( I_j = [x_{j-1}, x_j] \), allocated to the thief \( i \in [r] \). If \( \phi(x, f) := (\phi_1(x, f), \ldots, \phi_n(x, f)) \in (\mathbb{R}^n)^r \) then \( (x, f) \) is a fair splitting if and only if \( \phi(x, f) \in D \), where \( D := \{ (v, \ldots, v) \mid v \in \mathbb{R}^n \} \subset (\mathbb{R}^n)^r \) is the diagonal subspace. Summarizing, \( (x, f) \in (\Delta^N)_\Delta^{\ast r} \) is a fair splitting of the necklace \( ([0, 1]; \{ \mu_j \}_{j=1}^n) \) if and only if \( (x, f) \) is a zero of the composition map

\[ \hat{\phi} : (\Delta^N)_\Delta^{\ast r} \longrightarrow (\mathbb{R}^n)^r / D. \]

3.3. The group of symmetries. The final ingredient in applications of the configuration space/test map scheme is a group \( G \) of symmetries \( [19] \), characteristic for the problem. In the chosen scheme it is the \( p \)-toral group \( G = (\mathbb{Z}_p)^r \), where \( p \) is a prime and \( r = p^k \). The group \( G \) acts freely on the deleted join \( (\Delta^N)_\Delta^{\ast r} \) and without fixed points on the sphere \( S( (\mathbb{R}^n)^r / D) \subset (\mathbb{R}^n)^r / D \).

Moreover, the map (3) is clearly \( G \)-equivariant.
3.4. New (refined) configuration spaces. In order to derive Alon’s necklace-splitting theorem (Theorem 1.1) it is natural to choose \( N \), the dimension of the primary configuration space \( (\Delta^N)^r \Delta \), to be equal to the expected number of cuts, \( N = (r - 1)n \).

Our basic new idea is to allow (initially) a larger number of cuts, but to force some of these cut points to coincide, by an appropriate choice of the configuration space. This is achieved by choosing a \( G \)-invariant, \( (r - 1)n \)-dimensional subcomplex \( K \) of the primary configuration space \( (\Delta^N)^r \Delta \), where \( N \) is (typically) larger than the number \( (r - 1)n \) of essential cut points.

Our first choice for a refined configuration space \( K \subseteq (\Delta^N)^r \Delta \) is the symmetrized deleted join \( \text{SymmDelJoin}(K) \) of a family \( K = \{K_i\}_{i=1}^r \) of collectively unavoidable subcomplexes of \( 2^{[m]} \) where \( m = N + 1 = (r - 1)(n + 1) + 1 \).

4. Equicardinal necklace-splitting theorem

4.1. Motivation and the statement of the theorem.

Example 4.1. Assume that the measures \( \mu_j (j = 1, \ldots, n) \) are supported by pairwise disjoint subintervals of \([0, 1] \). In this case we need at least \((r - 1)n \) cuts which dissect the necklace into \((r - 1)n + 1 \) parts. We observe that for this choice of measures there always exists a \((k, s)\)-equicardinal, fair partition/allocation of measures to \( r \) thieves where \( k \) is the quotient and \( s \) the corresponding remainder, on division of \((r - 1)n + 1 \) by \( r \).

The choice of measures in Example 4.1 is rather special and it is natural to ask if such a partition is always possible.

Problem 4.2. For a given collection \( \{\mu_j\}_{j=1}^n \) of continues measures on \([0, 1] \) and \( r \) thieves, is it always possible to find a fair, \((k, s)\)-equicardinal partition/allocation of the necklace where \( k \) and \( s \) are chosen as in Example 4.1?

The following extension of the classical necklace theorem of Alon provides an affirmative answer to Problem 4.2.

Theorem 4.3. (Equicardinal necklace-splitting theorem) For given positive integers \( r \) and \( n \), where \( r = p^q \) is a power of a prime, let \( k = k(r, n) \) and \( s = s(r, n) \) be the unique non-negative integers such that \((r - 1)n + 1 = kr + s \) and \( 0 \leq s < r \). Then for any choice of \( n \) continuous, probability measures on \([0, 1] \) there exists a fair partition/allocation of the associated necklace with \((r - 1)n \) cuts which is also \((k, s)\)-equicardinal in the sense that:

1. each thief gets no more than \( k + 1 \) parts (intervals);
2. the number of thieves receiving exactly \( k + 1 \) parts is not greater than \( s \).

Proof. As emphasized in Section 3.4 the basic idea of the proof is to initially allow a larger number of cuts, and then to force some of these cuts to be superfluous by an appropriate choice of the configuration space.
Our choice for a refined configuration space is the symmetric deleted join $K := \text{SymmDelJoin}(K_1, \ldots, K_r)$ of the family $K = \langle K_i \rangle_{i=1}^r$,

\begin{align}
K_1 = \cdots = K_s = \left(\left\lceil \frac{N + 1}{k + 1} \right\rceil, \right. \quad K_{s+1} = \cdots = K_r = \left. \left\lceil \frac{N + 1}{k} \right\rceil \right)
\end{align}

of subcomplexes of the simplex $\Delta^N = 2^{[N+1]}$, where $N = (r-1)(n+1)$, and

\begin{align}
m = N + 1 = (r-1)(n+1) + 1 = r(k+1) + s - 1.
\end{align}

By substituting $k = t + 1$ and $n = d + 1$ in Theorem 2.7 we observe that the complex $K$ is $(m-r-1)$-connected. By construction (Section 3) a partition/allocation $(x, f) \in K$ corresponds to a fair division if and only if $\hat{\phi}(x, f) = 0$, where $\hat{\phi}$ is the test map described in the equation (3). If a fair division $(x, f)$ does not exist there arises a $G$-equivariant map

\[
\hat{\phi} : K \longrightarrow S(\mathbb{R}^{nr}/D) \cong S^{(r-1)n-1}
\]

where $G = (\mathbb{Z}_p)^r$ and $S(V)$ is a $G$-invariant sphere in a $G$-vector space $V$. Since by (5)

\[
m - r - 1 = [(r-1)(n+1) + 1] - r - 1 = (r-1)n - 1
\]

this contradicts Volovikov’s theorem (Theorem 2.3).

Suppose that $(x, f) \in (A_1, \ldots, A_r; B)$. Then, with a possible reindexing of thieves, $(x, f) \in \tau = (A_1, \ldots, A_r; B)$ where $|A_i| \leq k + 1$ for $i = 1, \ldots, s$ and $|A_j| \leq k$ for $j = s + 1, \ldots, r$. From here it immediately follows that $(x, f)$ describes a $(k, s)$ balanced partition/allocation of the necklace. □

**Remark 4.4.** In the special case $s = 0$, or equivalently if $(r-1)n+1$ is divisible by $r$, Theorem 4.3 guarantees the existence of a fair partition/allocation which is *equicardinal* in the sense that each thief is allocated exactly the same number of pieces of the necklace. Here we tacitly assume that the necklace is generic, i.e. that all $(r-1)n$ cuts are needed.

5. **Splitting necklaces and collectively unavoidable complexes**

Collectively unavoidable complexes were introduced in [8] as a common generalization of pairs of Alexander dual complexes [12] and unavoidable complexes [5, 7]. As shown in [9], they are a very useful tool for proving theorems of Van Kampen-Flores type. Here we demonstrate that they also provide a natural environment for necklace-splitting theorems with constraints.

Theorem 4.3 turns out to be a very special case of the following theorem where the constraints on the partition/allocation are ruled by a collectively unavoidable $r$-tuple of complexes.

As in Theorem 4.3 we assume that $r = p^\nu$ is a power of a prime number and $m = N + 1 = (r-1)(n+1) + 1$. Moreover, $k = k(r, n)$ and $s = s(r, n)$ are the unique non-negative integers such that $(r-1)n + 1 = kr + s$ and $0 \leq s < r$. 
Theorem 5.1. Let $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle$ be a sequence of subcomplexes of $2^{|m|}$ such that:

1. each complex $K_i$ is $(m,k)$-balanced, and
2. the sequence $\mathcal{K}$ is collectively unavoidable.

Choose a collection $\{\mu_i\}_{i=1}^n$ of $n$ continuous, probability measures on $[0,1]$. Then for any company $\mathcal{C}$ of $r$ thieves there exists a fair partition/allocation $(x, f) \in \text{SymmDelJoin}(\mathcal{K})$ of the associated necklace with at most $n(r-1)$ cuts. More explicitly, there exists a $(r-1)n$-dimensional simplex

$(A_1, \ldots, A_r; B) \in \text{SymmDelJoin}(\mathcal{K})$

and a partition/allocation $(x, f) \in (A_1, \ldots, A_r; B)$ which is fair for $\mathcal{C}$, with a suitable choice of a bijection $\mathcal{C} \leftrightarrow [r]$.

Proof. The proof is similar to the proof of Theorem 4.3, with an additional intermediate step allowing us to control the number of essential cut points.

As expected we use Theorem 2.6 instead of Theorem 2.7 which claims that, under the conditions of the theorem, the complex $K := \text{SymmDelJoin}(\mathcal{K})$ is $(m-r-1)$-connected. However, we refine the configuration space even more by selecting the $(m-r)$-dimensional skeleton $K^{(m-r)}$ of $K$ as the domain for our test map $\hat{\phi}$. The complex $K^{(m-r)}$ is also $(m-r-1)$-connected and the condition $\dim(K^{(m-r)}) = (r-1)n$ guarantees that the number of superfluous cuts (indexed by $B$) is at least $r-1$. $\square$

6. Collectively Unavoidable Threshold Complexes

In order to apply Theorem 5.1, we need a method for generating interesting examples of collectively unavoidable families $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle$ where each complex $K_i \subseteq 2^{|m|}$ is $(m,k)$-balanced.

6.1. Collectively Unavoidable Threshold Complexes. Suppose that $\nu = (x_1, \ldots, x_m)$ is a probability measure (weight distribution) on $[m]$ where $x_i \geq 0$ for each $i$ and $x_1 + \cdots + x_m = 1$. Without loss of generality we assume that

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq 1.$$  (6)

The simplicial complex

$T_{\nu \leq \alpha} := \{ I \subseteq [m] \mid \nu(I) \leq \alpha \}$

is referred to as the $\nu$-threshold complex with the threshold $\alpha$.

Proposition 6.1. Suppose that $\nu = (x_1, \ldots, x_m)$ is a probability measure on $[m]$ and let $\{\alpha_i\}_{i=1}^r$ be a collection of non-negative numbers such that $\alpha_1 + \cdots + \alpha_r = 1$. Then the collection of complexes

$$\mathcal{K} = \langle T_{\nu \leq \alpha_1}, T_{\nu \leq \alpha_2}, \ldots, T_{\nu \leq \alpha_r} \rangle$$  (7)

is collectively unavoidable.
Proof. Obvious.

In the following proposition we collect some simple properties of threshold complexes \( T_{\nu \leq \alpha} \).

**Proposition 6.2.** The complex \( K = T_{\nu \leq \alpha} \) is balanced, in the sense that

\[
\left( \binom{m}{\leq k} \right) \subseteq T_{\nu \leq \alpha} \subseteq \left( \binom{m}{\leq k+1} \right)
\]

if and only if

\[
x_m + x_{m-1} + \cdots + x_{m-k+1} \leq \alpha < x_1 + x_2 + \cdots + x_{k+2}.
\]

Moreover, \( \left( \binom{m}{\leq k} \right) \not\subseteq K \iff x_1 + x_2 + \cdots + x_{k+1} \leq \alpha \) and \( \left( \binom{m}{\leq k} \right) \not\supseteq K \iff \alpha < x_m + x_{m-1} + \cdots + x_{m-k} \).

By combining Propositions 6.1 and 6.2 we obtain examples of balanced, collectively unavoidable complexes which are essentially different from the binomial complexes used in Theorem 4.3.

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