Quantum Group of Orientation-preserving Riemannian Isometries

by

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Abstract

We formulate a quantum group analogue of the group of orientation-preserving Riemannian isometries of a compact Riemannian spin manifold, more generally, of a (possibly $R$-twisted and of compact type) spectral triple. The main advantage of this formulation, which is directly in terms of the Dirac operator, is that it does not need the existence of any 'good' Laplacian as in our previous works on quantum isometry groups. Several interesting examples, including those coming from Rieffel-type deformation as well as the equivariant spectral triples on $SU_\mu(2)$ and $S^2_{\mu c}$ are discussed.

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1 Introduction

Motivated by the formulation of quantum automorphism groups by Wang (following Alain Connes’ suggestion) ([31], [32]), and the study of their properties by a number of mathematicians (see, e.g. [1], [2], [5], [33] and references therein), we have introduced in an earlier article ([14]) a quantum

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group analogue of the group of Riemannian isometries of a classical or non-commutative manifold. In a follow-up article [3] we have also computed these quantum groups for a number of examples. However, our formulation of quantum isometry groups in [14] had a major drawback from the viewpoint of noncommutative geometry since it needed a ‘good’ Laplacian to exist. In noncommutative geometry it is not always easy to verify such an assumption about the Laplacian, and thus it would be more appropriate to have a formulation in terms of the Dirac operator directly. This is what we aim to achieve in the present article.

The group of Riemannian isometries of a compact Riemannian manifold \( M \) can be viewed as the universal object in the category of all compact metrizable groups acting on \( M \), with smooth and isometric action. Moreover, assume that the manifold has a spin structure (hence in particular orientable, so we can fix a choice of orientation) and \( D \) denotes the conventional Dirac operator acting as an unbounded self-adjoint operator on the Hilbert space \( \mathcal{H} \) of square integrable spinors. Then, it can be proved that a group action on the manifold lifts as a unitary representation on the Hilbert space \( \mathcal{H} \) which commutes with \( D \) if and only if the action on the manifold is an orientation preserving isometric action. Therefore, to define the quantum analogue of the group of orientation-preserving Riemannian isometries of a possibly noncommutative manifold given by a spectral triple \((\mathcal{A}_\infty, \mathcal{H}, D)\), it is reasonable to consider a category \( \mathcal{Q}' \) of compact quantum groups having unitary (co-) representation, say \( U \), on \( \mathcal{H} \), which commutes with \( D \), and the action on \( \mathcal{B}(\mathcal{H}) \) obtained by conjugation maps \( \mathcal{A}_\infty \) into its weak closure. A universal object in this category, if it exists, should define the ‘quantum group of orientation preserving Riemannian isometries’ of the underlying spectral triple. Unfortunately, even in the finite-dimensional (but with noncommutative \( \mathcal{A} \)) situation this category may often fail to have a universal object, as will be discussed later. It turns out, however, that if we fix any suitable faithful functional on \( \mathcal{B}(\mathcal{H}) \) (to be interpreted as the choice of a ‘volume form’) then there exists a universal object in the subcategory of \( \mathcal{Q}' \) obtained by restricting the object-class to the quantum group actions which also preserve the given functional. The subtle point to note here is that unlike the classical group actions on \( \mathcal{B}(\mathcal{H}) \) which always preserve the usual trace, a quantum group action may not do so. In fact, it was proved by one of the authors in [15] that given an object \((\mathcal{Q}, U)\) of \( \mathcal{Q}' \) (where \( \mathcal{Q} \) is the compact quantum group and \( U \) denotes its unitary co-representation on \( \mathcal{H} \)), we can find a positive invertible operator \( R \) in \( \mathcal{H} \) so that the given spectral triple is \( R \)-twisted in the sense of [15] and the corresponding functional \( \tau_R \) (which typically differs from the usual trace of \( \mathcal{B}(\mathcal{H}) \) and can have
a nontrivial modularity) is preserved by the action of $Q$. This makes it quite natural to work in the setting of twisted spectral data (as defined in [15]).

Motivated by the ideas of Woronowicz and Soltan, we actually consider a bigger category. The group of orientation-preserving Riemannian isometries of a classical manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object-class consists of subsets (not necessarily subgroups) of the set of such isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. A natural quantum analogue of this has been formulated by us, called the category of ‘quantum families of smooth orientation preserving Riemannian isometries’. The underlying $C^*$-algebra of the quantum group of orientation preserving isometries (whenever exists) has been identified with the universal object in this bigger category and moreover, it is shown to be equipped with a canonical coproduct making it into a compact quantum group.

We discuss a number of examples, covering both the examples coming from Rieffel-type deformation as well as the equivariant spectral triples constructed recently on $SU_\mu(2)$ and $S^2_{\mu c}$. It may be relevant to point out here that it was not clear whether one could accommodate the spectral triples on $SU_\mu(2)$ and $S^2_{\mu c}$ in the framework of our previous work on quantum isometry groups, since it is very difficult to give a nice description of the space of ‘noncommutative’ forms and hence the Laplacian for these examples. However, the present formulation in terms of the Dirac operator makes it easy to accommodate them, and we have been able to identify $U_\mu(2)$ and $SO_\mu(3)$ as the universal quantum group of orientation preserving isometries for the spectral triples on $SU_\mu(2)$ and $S^2_{\mu c}$ respectively (the computations for $S^2_{\mu c}$ has been presented in [4]).

For readers’ convenience, let us briefly sketch the plan of the paper. The section 2 is devoted to the definition and existence of quantum group of orientation preserving isometries, which begins with a characterization of the group of such isometries in the classical case (subsection 2.1), motivating the quantum formulation elaborated in subsection 2.2. The other two subsections of section 2 discuss sufficient conditions for ensuring a $C^*$ action of the quantum group of (orientation preserving) isometries (2.3) and for the existence of universal object without fixing a volume form (2.4). Then in section 3 we study the connections of our approach with that of [14]. Section 4 is devoted to the explicit examples and computations, and in the last section we sketch a general principle of computing quantum group of orientation preserving isometries of spectral triples obtained by Rieffel deformation.

We conclude this section with an important remark about the use of the
phrase ‘orientation -preserving’ in our terminology. Let us make it clear that by a ‘classical spectral triple’ we always mean the spectral triple obtained by the Dirac operator on the spinors (so, in particular, manifolds are assumed to be compact Riemannian spin manifolds), and not just any spectral triple on the commutative algebra $C^\infty(M)$. This is absolutely crucial in view of the fact that the Hodge dirac operator $d + d^*$ on the $L^2$-space of differential forms also gives a spectral triple of compact type on any compact Riemannin (not necessarily with a spin structure) manifold $M$, but the the action of the full isometry group $ISO(M)$ (and not just the subgroup of orientation pre-serving isometries $ISO^+(M)$, even when $M$ is orientable) lifts to a canonical unitary representation on this space commuting with $d + d^*$. In fact, the category of groups acting on $M$ such that the action comes from a unitary representation commuting with $d + d^*$, has $ISO(M)$, and not $ISO^+(M)$, as its universal object. So, one must stick to the Dirac operator on spinors to obtain the group of orientation preserving isometries in the usual geometric sense. This also has a natural quantum generalization, as we shall see in section 3.

2 Definition and existence of the quantum group of orientation-preserving isometries

2.1 The classical case

We first discuss the classical situation clearly, which will serve as a motivation for our quantum formulation.

We begin with a few basic facts about topologizing the space $C^\infty(M, N)$ where $M, N$ are smooth manifolds. Let $\Omega$ be an open set of $\mathbb{R}^n$. We endow $C^\infty(\Omega)$ with the usual Frechet topology coming from uniform convergence (over compact subsets) of partial derivatives of all orders. The space $C^\infty(\Omega)$ is complete w.r.t. this topology, so is a Polish space in particular. Moreover, by the Sobolev imbedding theorem, $\cap_{k \geq 0} H_k(\Omega) = C^\infty(\Omega)$ as a set, where $H_k(\Omega)$ denotes the $k$-th Sobolev space. Thus, $C^\infty(\Omega)$ has also the Hilbertian seminorms coming from the Sobolev spaces, hence the corresponding Frechet topology. We claim that these two topologies on $C^\infty(\Omega)$ coincide. Indeed, the inclusion map from $C^\infty(\Omega)$ into $\cap_k H_k(\Omega)$, is continuous and surjective, so by the open mapping theorem for Frechet space, the inverse is also continuous, proving our claim.

Given two second countable smooth manifolds $M, N$, we shall equip $C^\infty(M, N)$ with the weakest locally convex topology making $C^\infty(M, N) \ni$
$\phi \mapsto f \circ \phi \in C^\infty(M)$ Frechet continuous for every $f \in C^\infty(N)$.

For topological or smooth fibre or principal bundles $E, F$ over a second countable smooth manifold $M$, we shall denote by $\text{Hom}(E, F)$ the set of bundle morphisms from $E$ to $F$. We remark that the total space of a locally trivial topological bundle such that the base and the fibre spaces are locally compact Hausdorff second countable must itself be so, hence in particular Polish.

In particular, if $E, F$ are locally trivial principal $G$-bundles over a common base, such that the (common) base as well as structure group $G$ are locally compact Hausdorff and second countable, then $\text{Hom}(E, F)$ and $C(X, \text{Hom}(E, F))$ are Polish spaces, where $X$ is a compact space.

We need a standard fact, stated below as a lemma, about the measurable lift of Polish space valued functions.

**Lemma 2.1** Let $M$ be a compact metrizable space, $B, \tilde{B}$ Polish spaces (complete separable metric spaces) such that there is an $n$-covering map $\Lambda : \tilde{B} \to B$. Then any continuous map $\xi : M \to B$ admits a lifting $\tilde{\xi} : M \to \tilde{B}$ which is Borel measurable and $\Lambda \circ \tilde{\xi} = \xi$. In particular, if $\tilde{B}$ and $B$ are topological bundles over $M$, with $\Lambda$ being a bundle map, any continuous section of $B$ admits a lifting which is a measurable section of $\tilde{B}$.

The proof is a trivial consequence of the selection theorem due to Kuratowski and Ryll-Nardzewski (see [28], Theorem 5.2.1).

We shall now give an operator-theoretic characterization of the classical group of orientation-preserving Riemannian isometries, which will be the motivation of our definition of its quantum counterpart. Let $M$ be a compact Riemannian spin manifold, with a fixed choice of orientation. We note that (see, e.g. [18]) the spinor bundle $S$ is the associated bundle of a principal $Spin(n)$-bundle, say $P$, on $M$ ($n=$dimension of $M$), which has a canonical 2-covering bundle-map $\Lambda$ from $P$ to the frame-bundle $F$ (which is an $SO(n)$-principal bundle), such that locally $\Lambda$ is of the form $(\text{id}_M \otimes \lambda)$, where $\lambda : Spin(n) \to SO(n)$ is the canonical 2-covering group homomorphism. Let $f$ be a smooth orientation preserving Riemannian isometry of $M$, and consider the bundles $E = \text{Hom}(F, f^*(F))$ and $\tilde{E} = \text{Hom}(P, f^*(P))$ (where Hom denotes the set of bundle maps). We view $df$ as a section of the bundle $E$ in the natural way. By the Lemma 2.1, we obtain a measurable lift $\tilde{df} : M \to \tilde{E}$, which is a measurable section of $\tilde{E}$. Using this, we define $U$ as follows. Given a (measurable) section $\xi$ of $S = P \times_{Spin(n)} \Delta_n$ (where $\Delta_n$ is as in [18]), say of the form $\xi(m) = [p(m), v]$, with $p(m) \in P_m, v \in \Delta_n$, we define $\tilde{\xi}$ by $\tilde{\xi}(m) = [\tilde{df}(f^{-1}(m))(p(f^{-1}(m))), v]$. Note that sections of the above form
constitute a total subset in $L^2(S)$, and the map $\xi \mapsto \dot{\xi}$ is clearly a densely defined linear map on $L^2(S)$, whose fibre-wise action is unitary since the $\text{Spin}(n)$ action is so on $\Delta_n$. Thus it extends to a unitary $U$ on $\mathcal{H} = L^2(S)$. Any such $U$, induced by the map $f$, will be denoted by $U_f$ (it is not unique since the choice of the lifting used in its construction is not unique).

**Theorem 2.2** Let $M$ be a compact Riemannian spin manifold (hence orientable, and fix a choice of orientation) with the usual Dirac operator $D$ acting as an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$ of the square integrable spinors, and let $S$ denote the spinor bundle, with $\Gamma(S)$ being the $C^\infty(M)$ module of smooth sections of $S$. Let $f : M \to M$ be a smooth one-to-one map which is a Riemannian isometry preserving the orientation. Then the unitary $U_f$ on $\mathcal{H}$ commutes with $D$ and $U_f M_\phi U_f^* = M_{f \phi}$, for any $\phi \in C(M)$, where $M_\phi$ denotes the operator of multiplication by $\phi$ on $L^2(S)$. Moreover, when the dimension of $M$ is even, $U_f$ commutes with the canonical grading $\gamma$ on $L^2(S)$.

Conversely, suppose that $U$ is a unitary on $\mathcal{H}$ such that $UD = DU$ and the map $\alpha_U(X) = UXU^{-1}$ for $X \in \mathcal{B}(\mathcal{H})$ maps $\mathcal{A} = C(M)$ into $L^\infty(M) = \mathcal{A}'$, then there is a smooth one-to-one orientation-preserving Riemannian isometry $f$ on $M$ such that $U = U_f$. We have the same result in the even case, if we assume furthermore that $U\gamma = \gamma U$.

**Proof:**
From the construction of $U_f$, it is clear that $U_f M_\phi U_f^{-1} = M_{f \phi}$. Moreover, since the Dirac operator $D$ commutes with the $\text{Spin}(n)$-action on $S$, we have $U_f D = DU_f$ on each fibre, hence on $L^2(S)$. In the even dimensional case, it is easy to see that $\text{Spin}(n)$ action commutes with $\gamma$, hence $U_f$ does so.

For the converse, first note that $\alpha_U$ is a unital $*$-homomorphism on $L^\infty(M,dvol)$ and thus must be of the form $\psi \mapsto \psi \circ f$ for some measurable $f$. We claim that $f$ must be smooth. Fix any smooth $g$ on $M$ and consider $\phi = g \circ f$. We have to argue that $\phi$ is smooth. Let $\delta_D$ denote the generator of the strongly continuous one-parameter group of automorphism $\beta_t(X) = e^{itD}Xe^{-itD}$ on $\mathcal{B}(\mathcal{H})$ (w.r.t. the weak operator topology, say). From the assumption that $D$ and $U$ commute it is clear that $\alpha_U$ maps $\mathcal{D} := \bigcap_{n \geq 1} \text{Dom}(\delta^n_D) \subseteq \mathcal{B}(\mathcal{H})$ into itself, so in particular, since $C^\infty(M) \subseteq \mathcal{D}$, we have that $\alpha_U(M_\phi) = M_{\phi \circ g}$ belongs to $\mathcal{D}$. We claim that this implies the smoothness of $\phi$. Let $m \in M$ and choose a local chart $(V, \psi)$ at $m$, with the coordinates $(x_1,\ldots,x_n)$, such that $\Omega = \psi(V) \subseteq \mathbb{R}^n$ has compact closure, $S|_V$ is trivial and $D$ has the local expression $D = i \sum_{j=1}^n \mu(e_j) \nabla_j$, where $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$ denotes the covariant derivative (w.r.t. the canonical Levi civita connection)
operator along the vector field $\frac{\partial}{\partial x_j}$ on $L^2(\Omega)$ and $\mu(v)$ denotes the Clifford multiplication by a vector $v$. Now, $\phi \circ \psi^{-1} \in L^\infty(\Omega) \subseteq L^2(\Omega)$ and it is easy to observe from the above local structure of $D$ that $[D, M_\phi]$ has the local expression $\sum_j iM_{\phi j} \otimes \mu(\epsilon_j)$. Thus, the fact $M_\phi \in \bigcap_{n \geq 1} \text{Dom}(\delta^n)$ implies $\phi \circ \psi^{-1} \in \text{Dom}(d_{j_1}...d_{j_k})$ for every integer tuples $(j_1, ..., j_k)$, $j_i \in \{1, ..., n\}$, where $d_j := \frac{\partial}{\partial x_j}$. In other words, $\phi \circ \psi^{-1} \in H^k(\Omega)$ for all $k \geq 1$, where $H^k(\Omega)$ denotes the $k$-the Sobolev space on $\Omega$ (see [24]). By Sobolev’s Theorem (see, e.g. [24], Corollary 1.21, page 24) it follows that $\phi \circ \psi^{-1} \in C^\infty(\Omega)$.

We note that $f$ is one-one as $\phi \rightarrow \phi \circ f$ is an automorphism of $L^\infty$. Now, we shall show that $f$ is an isometry of the metric space $(M, d)$, where $d$ is the metric coming from the Riemannian structure, and we have the explicit formula (see [8])

$$d(p, q) = \sup_{\phi \in C^\infty(M), \|\phi\| \leq 1} |\phi(p) - \phi(q)|.$$  

Since $U$ commutes with $D$, we have $\|[D, M_\phi f]\| = \|[D, UM_\phi U^*]\| = \|U[D, M_\phi]U^*\| = \|[D, M_\phi]\|$ for every $\phi$, from which it follows that $d(f(p), f(q)) = d(p, q)$. Finally, $f$ is orientation preserving if and only if the volume form (say $\omega$) which defines the choice of orientation is preserved by the natural action of $df$, i.e. $(df \wedge ... \wedge df)(\omega) = \omega$. This will follow from the explicit description of $\omega$ in terms of $D$, given by (see [23] Page 26, also see [11])

$$\omega(\phi_0 d\phi_1...d\phi_n) = \tau(M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]),$$

in the odd case and

$$\omega(\phi_0 d\phi_1...d\phi_n) = \tau(\gamma M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]),$$

in the even case where $\phi_0, ..., \phi_n \in C^\infty(M)$, $\gamma$ is the grading operator and $\tau$ denotes the volume integral. In fact, $\tau(X) = \text{Lim}_{t \rightarrow 0^+} \frac{\text{Tr}(e^{-tD^2}X)}{\text{Tr}(e^{-tD^2})}$, where $\text{Lim}$ denotes a suitable Banach limit, which implies $\tau(UXU^*) = \tau(X)$ for all $X \in \mathcal{B}(\mathcal{H})$ (using the fact that $D$ and $U$ commute). Thus,

$$\omega(\phi_0 \circ f \ d(\phi_1 \circ f) ... d(\phi_n \circ f))$$

$$= \tau(\gamma UM_{\phi_0}U^*U[D, M_{\phi_1}]U^*...U[D, M_{\phi_n}]U^*)$$

$$= \tau(U \gamma M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]U^*)$$

$$= \tau(\gamma M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}])$$

$$= \omega(\phi_0 d\phi_1...d\phi_n).$$

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with the understanding that $\gamma = I$ when the spectral triple is odd. □

Now we turn to the case of a family of maps. We first prove a useful general fact.

**Lemma 2.3** Let $A$ be a $C^*$ algebra and $\omega, \omega_j$ ($j = 1, 2, ...$) be states on $A$ such that $\omega_j \to \omega$ in the weak-$*$ topology of $A^*$. Then for any separable Hilbert space $H$ and $\forall Y \in M(K(H) \otimes A)$, we have $(id \otimes \omega_j)(Y) \to (id \otimes \omega)(Y)$ in the S.O.T.

**Proof:**
Clearly, $(id \otimes \omega_j)(Y) \to (id \otimes \omega)(Y)$ (in the strong operator topology) for all $Y \in \text{Fin}(H \otimes \text{alg} A)$, where $\text{Fin}(H)$ denotes the set of finite rank operators on $H$. Using the strict density of $\text{Fin}(H) \otimes \text{alg} A$ in $M(K(H) \otimes A)$, we choose, for a given $Y \in M(K(H) \otimes A)$, $\xi \in H$ with $\|\xi\| = 1$, and $\delta > 0$, an element $Y_0 \in \text{Fin}(H) \otimes \text{alg} A$ such that $\|(Y - Y_0)(|\xi > < | \otimes 1)\| < \delta$. Thus,

$$
\|(id \otimes \omega_j)(Y)\xi - (id \otimes \omega)(Y)\xi\|
\leq \|(id \otimes \omega_j)(Y(|\xi > < | \otimes 1))\xi - (id \otimes \omega)(Y(|\xi > < | \otimes 1))\xi\|
+ 2\|(Y - Y_0)(|\xi > < | \otimes 1)\|,
$$

from which it follows that $(id \otimes \omega_j)(Y) \to (id \otimes \omega)(Y)$ in the strong operator topology. □

We are now ready to state and prove the operator-theoretic characterization of ‘set of orientation preserving isometries’.

**Theorem 2.4** Let $X$ be a compact metrizable space and $\psi : X \times M \to M$ is a map such that $\psi_x$ defined by $\psi_x(m) = \psi(x, m)$ is a smooth orientation preserving Riemannian isometry and $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous w.r.t. the locally convex topology of $C^\infty(M, M)$ mentioned before.

Then there exists a $(C(X)\text{-linear})$ unitary $U_{\psi}$ on the Hilbert $C(X)$-module $H \otimes C(X)$ such that $\forall x \in X$, $U_x := (id \otimes ev_x)U_{\psi}$ is a unitary of the form $U_{\psi_x}$ on the Hilbert space $H$ commuting with $D$ and $U_xM_\phi U_x^{-1} = U_{\phi \circ \psi_x}$. If in addition, the manifold is even dimensional, then $U_{\psi_x}$ commutes with the grading operator $\gamma$.

Conversely, if there exists a $C(X)$-linear unitary $U$ on $H \otimes C(X)$ such that $U_x := (id \otimes ev_x)(U)$ is a unitary commuting with $D \forall x$, $(id \otimes ev_x)\alpha_U(L^\infty(M)) \subseteq L^\infty(M)$ for all $x \in X$, then there exists a
map \( \psi : X \times M \to M \) satisfying the conditions mentioned above such that \( U = U_\psi \).

**Proof:**
Consider the bundles \( \hat{F} = X \times F \) and \( \hat{P} = X \times P \) over \( X \times M \), with fibres at \((x, m)\) isomorphic with (respectively) \( P_m \) and \( F_m \), and where \( F \) and \( P \) are as in Theorem 2.2. Moreover, denote by \( \Psi \) the map from \( X \times M \) to itself given by \((x, m) \mapsto (x, \psi(x, m))\), and let \( B = C(X, \text{Hom}(\hat{F}, \text{Hom}(\hat{P}, \psi^*(\hat{F})))) \), \( \hat{B} = C(X, \text{Hom}(\hat{P}, \psi^*(\hat{P}))) \). The covering map from \( P \) to \( M \) induces a covering map from \( \hat{B} \) to \( B \) as well. Let \( d'_\psi : M \to \hat{B} \) be the map given by \( d'_\psi(m)\mid_{(x, m)} = d\psi\mid_{x,m} \). Then by Theorem 2.1 there exists a measurable lift of \( d'_\psi \), say \( \tilde{d}'_\psi \), from \( M \) into \( \hat{B} \). Since \( d'_\psi(m)\mid_{(x, m)} \in \text{Hom}(F_m, F(x, m)) \), it is clear that the lift \( \tilde{d}'_\psi(m)\mid_{(x, m)} \) will be an element of \( \text{Hom}(P_m, P(x, m)) \).

We can identify \( \mathcal{H} \otimes C(X) \) with \( C(X \to \mathcal{H}) \), and since \( \mathcal{H} \) has a total set \( \mathcal{F} \) (say) consisting of sections of the form \([p(\cdot, v)]\), where \( p : M \to P \) is a measurable section of \( P \) and \( v \in \Delta_n \), we have a total set \( \hat{\mathcal{F}} \) of \( \mathcal{H} \otimes C(X) \) consisting of \( \mathcal{F} \) valued continuous functions from \( X \). Any such function can be written as \([\Xi, v]\) with \( \Xi : X \times M \to P \), \( v \in \Delta_n \), and \( \Xi(x, m) \in P_m \), and we define \( U \) on \( \mathcal{F} \) by \( U[\xi, v] = [\Theta, v] \).

\[
\Theta(x, m) = \tilde{d}'_\psi(m)\mid_{(x, \psi^{-1}(m))}(\Xi(x, \psi^{-1}(m))).
\]

It is clear from the construction of the lift that \( U \) is indeed a \( C(X) \)-linear isometry which maps the total set \( \hat{\mathcal{F}} \) onto itself, so extends to a unitary on the whole of \( \mathcal{H} \otimes C(X) \) with the desired properties.

Conversely, given \( U \) as in the statement of the converse part of the theorem, we observe that for each \( x \in X \), by Theorem 2.2 \((id \otimes ev_x)U = U_{\psi_x} \) for some \( \psi_x \) such that \( \psi_x \) is a smooth orientation preserving Riemannian isometry. This defines the map \( \psi \) by setting \( \psi(x, m) = \psi_x(m) \). The proof will be complete if we can show that \( x \mapsto \psi_x \in C^\infty(M, M) \) is continuous, which is equivalent to showing that whenever \( x_n \to x \) in the topology of \( X \), we must have \( \phi \circ \psi_{x_n} \to \phi \circ \psi_x \) in the Frechet topology of \( C^\infty(M) \), for any \( \phi \in C^\infty(M) \). However, by Lemma 2.3 we have \((id \otimes ev_{x_n})_U([D, M_\phi]) \to (id \otimes ev_x)_U([D, M_\phi]) \) in the S.O.T. Since \( U \) commutes with \( D \), this implies

\[
(id \otimes ev_{x_n})[D \otimes id, \alpha_U(M_\phi)] \to (id \otimes ev_x)[D \otimes id, \alpha_U(M_\phi)],
\]

i.e.

\[
[D, M_{\phi \circ \psi_{x_n}}]\xi \to [D, M_{\phi \circ \psi_x}]\xi \text{ in } L^2 \quad \forall \xi \in L^2(S).
\]
By choosing $\phi$ which has support in a local trivializing coordinate neighborhood for $S$, and then using the local expression of $D$ as in the proof of Theorem 2.2, we conclude that $d_k(\phi \circ \psi_{x_n}) \to d_k(\phi \circ \psi_x)$ in $L^2$ (where $d_k$ is as in the proof of Theorem 2.2). Similarly, by taking repeated commutators with $D$, we can show the convergence with $d_k$ replaced by $d_k^1 \ldots d_k^m$ for any finite tuple $(k_1, ..., k_m)$. In other words, $\phi \circ \psi_{x_n} \to \phi \circ \psi_x$ in the topology of $C^\infty(M)$ described before. \qed

2.2 Quantum group of orientation-preserving isometries of an $R$-twisted spectral triple

We begin by recalling the definition of compact quantum groups and their actions from [38], [37]. A compact quantum group (to be abbreviated as CQG from now on) is given by a pair $(S, \Delta)$, where $S$ is a unital separable $C^*$ algebra equipped with a unital $C^*$-homomorphism $\Delta : S \to S \otimes S$ (where $\otimes$ denotes the injective tensor product) satisfying

(ai) $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ (co-associativity), and

(aii) the linear span of $\Delta(S)(S \otimes 1)$ and $\Delta(1 \otimes S)$ are norm-dense in $S \otimes S$.

It is well-known (see [38], [37]) that there is a canonical dense $*$-subalgebra $S_0$ of $S$, consisting of the matrix coefficients of the finite dimensional unitary (co)-representations (to be defined shortly) of $S$, and maps $\epsilon : S_0 \to \mathbb{C}$ (co-unit) and $\kappa : S_0 \to S_0$ (antipode) defined on $S_0$ which make $S_0$ a Hopf $*$-algebra.

We say that the compact quantum group $(S, \Delta)$ (co)-acts on a unital $C^*$ algebra $B$, if there is a unital $C^*$-homomorphism (called an action) $\alpha : B \to B \otimes S$ satisfying the following :

(bi) $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$, and

(bii) the linear span of $\alpha(B)(1 \otimes S)$ is norm-dense in $B \otimes S$.

It is known (see, for example, [32], [22]) that (bii) is equivalent to the existence of a norm-dense, unital $*$-subalgebra $B_0$ of $B$ such that $\alpha(B_0) \subseteq B_0 \otimes_{\text{alg}} S_0$ and on $B_0$, $(id \circimese) \circ \alpha = id$.

We shall sometimes say that $\alpha$ is a ‘topological’ or $C^*$ action to distinguish it from a normal action of Von Neumann algebraic quantum group.

**Definition 2.5** A unitary (co) representation of a compact quantum group $(S, \Delta)$ on a Hilbert space $H$ is a map $U$ from $H$ to the Hilbert $S$ module $H \otimes S$ such that the element $U \in M(K(H) \otimes S)$ given by $U(\xi \otimes b) = U(\xi)(1 \otimes b)$
\((\xi \in \mathcal{H}, b \in S)\) is a unitary satisfying
\[
(id \otimes \Delta)\tilde{U} = \tilde{U}_{(12)}\tilde{U}_{(13)},
\]
where for an operator \(X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)\) we have denoted by \(X_{(12)}\) and \(X_{(13)}\) the operators \(X \otimes I_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2)\), and \(\Sigma_{23}X_{(12)}\Sigma_{23}\) respectively (\(\Sigma_{23}\) being the unitary on \(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2\) which flips the two copies of \(\mathcal{H}_2\)).

Given a unitary representation \(U\) we shall denote by \(\alpha_U\) the \(*\)-homomorphism \(\alpha_U(X) = \tilde{U}(X \otimes 1)\tilde{U}^*\) for \(X \in \mathcal{B}(\mathcal{H})\). For a not necessarily bounded, densely defined (in the weak operator topology) linear functional \(\tau\) on \(\mathcal{B}(\mathcal{H})\), we say that \(\alpha_U\) preserves \(\tau\) if \(\alpha_U\) maps a suitable (weakly) dense \(*\)-subalgebra (say \(D\)) in the domain of \(\tau\) into \(D \otimes \text{alg} S\) and \((\tau \otimes \text{id})(\alpha_U(a)) = \tau(a)1_S\) for all \(a \in D\). When \(\tau\) is bounded and normal, this is equivalent to \((\tau \otimes \text{id})(\alpha_U(a)) = \tau(a)1_S\) for all \(a \in \mathcal{B}(\mathcal{H})\).

We say that a (possibly unbounded) operator \(T\) on \(\mathcal{H}\) commutes with \(U\) if \(T \otimes I\) (with the natural domain) commutes with \(\tilde{U}\). Sometimes such an operator will be called \(U\)-equivariant.

Let us now recall the concept of universal quantum groups as in \([33], [31]\) and references therein. We shall use most of the terminologies of \([31]\), e.g. Woronowicz \(C^*\)-subalgebra, Woronowicz \(C^*\)-ideal etc, however with the exception that we shall call the Woronowicz \(C^*\) algebras just compact quantum groups, and not use the term compact quantum groups for the dual objects as done in \([31]\). For an \(n \times n\) positive invertible matrix \(Q = (Q_{ij})\), let \(A_u(Q)\) be the compact quantum group defined and studied in \([32], [33]\), which is the universal \(C^*\)-algebra generated by \(\{u_{kj}^Q, k, j = 1, ..., d\}\) such that \(u := ((u_{kj}^Q))\) satisfies
\[
 uu^* = I_n = u^*u, \quad u'Q\overline{Q}^{-1}u = I_n = Q\overline{Q}^{-1}u'.
\]
(1)

Here \(u' = ((u_{ji}))\) and \(\overline{\tau} = ((u_{ij}^*))\). The coproduct, say \(\tilde{\Delta}\), is given by,
\[
\tilde{\Delta}(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.
\]

It may be noted that \(A_u(Q)\) is the universal object in the category of compact quantum groups which admit an action on the finite dimensional \(C^*\) algebra \(M_n(\mathbb{C})\) which preserves the functional \(M_n \ni x \mapsto \text{Tr}(Q^T x)\), (see \([35]\) ) where we refer the reader to \([33]\) for a detailed discussion on the structure and classification of such quantum groups.

In view of the characterization of orientation-preserving isometric action on a classical manifold ( Theorem \([2.4]\) ), we give the following definitions.

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Definition 2.6 A quantum family of orientation preserving isometries for the (odd) spectral triple \((A^\infty, \mathcal{H}, D)\) is given by a pair \((S, U)\) where \(S\) is a separable unital \(C^\ast\)-algebra and \(U\) is a linear map from \(\mathcal{H}\) to \(\mathcal{H} \otimes S\) such that \(\tilde{U}\) given by \(\tilde{U}(\xi \otimes b) = U(\xi)(1 \otimes b)\) \((\xi \in \mathcal{H}, b \in S)\) extends to a unitary element of \(\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes S)\) satisfying the following

(i) for every state \(\phi\) on \(S\) we have \(U\phi D = DU\phi\), where \(U\phi := (id \otimes \phi)(\tilde{U})\);
(ii) \((id \otimes \phi) \circ \alpha_U(a) \in (A^\infty)'' \forall a \in A^\infty\) for every state \(\phi\) on \(S\), where \(\alpha_U(x) := \tilde{U}(x \otimes 1)\tilde{U}^*\) for \(x \in \mathcal{B}(\mathcal{H})\).

In case the \(C^\ast\)-algebra \(S\) has a coproduct \(\Delta\) such that \((S, \Delta)\) is a compact quantum group and \(U\) is a unitary representation of \((S, \Delta)\) on \(\mathcal{H}\), we say that \((S, \Delta)\) acts by orientation-preserving isometries on the spectral triple.

In case the spectral triple is even with grading operator \(\gamma\), a quantum family of orientation preserving isometries \((A^\infty, \mathcal{H}, D, \gamma)\) will be defined exactly as above, with the only extra condition being that \(U\) commutes with \(\gamma\).

From now on, we will mostly consider odd spectral triples. However let us remark that in the even case, all the definitions and results obtained by us will go through with some obvious modifications.

Consider the category \(Q \equiv Q(A^\infty, \mathcal{H}, D) \equiv Q(D)\) with the object-class consisting of all quantum families of orientation preserving isometries \((S, U)\) of the given spectral triple, and the set of morphisms \(\text{Mor}((S, U), (S', U'))\) being the set of unital \(C^\ast\)-homomorphisms \(\Phi : S \to S'\) satisfying \((id \otimes \Phi)(U) = U'\). We also consider another category \(Q' \equiv Q'(A^\infty, \mathcal{H}, D) \equiv Q'(D)\) whose objects are triplets \((S, \Delta, U)\), where \((S, \Delta)\) is a compact quantum group acting by orientation preserving isometries on the given spectral triple, with \(U\) being the corresponding unitary representation. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families of orientation preserving isometries. The forgetful functor \(F : Q' \to Q\) is clearly faithful, and we can view \(F(Q')\) as a subcategory of \(Q\).

It is easy to see that any object \((S, U)\) of the category \(Q'\) gives an equivariant spectral triple \((A^\infty, \mathcal{H}, D)\) w.r.t. the action of \(S\) implemented by \(U\). It may be noted that recently there has a lot of interest and work (see, for example, \([6, 9, 12]\)) towards construction of quantum group equivariant spectral triples. In all these works, given a \(C^\ast\)-algebra \(A \subseteq B(\mathcal{H})\) and a CQG \(Q\) having a unitary representation \(U\) on \(\mathcal{H}\) such that \(\text{ad}_U\) gives an action of \(Q\) on \(A\), the authors investigate the possibility of constructing a (nontrivial) spectral triple \((A^\infty, \mathcal{H}, D)\) on a suitable dense subalgebra \(A^\infty\) of \(A\) such that \(\tilde{U}\) commutes with \(D \otimes 1\), i.e. \(D\) is equivariant. Our interest here is in
the (sort of) converse direction: given a spectral triple, we want to consider all possible CQG representations which w.r.t. which the spectral triple is equivariant; and if there exists a universal object in the corresponding category, i.e $Q'$, we should call it the quantum group of orientation preserving isometries.

Unfortunately, in general $Q'$ or $Q$ will not have a universal object. It is easily seen by taking the standard example $A^\infty = M_n(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^n$, $D = I$. However, the fact that comes to our rescue is that a universal object exists in each of the subcategories which correspond to the CQG actions preserving a given faithful functional on $M_n$.

On the other hand, given any equivariant spectral triple, it has been shown in [15] that there is a (not necessarily unique) canonical faithful functional which is preserved by the CQG action. For readers' convenience, we state this result (in a form suitable to us) briefly here:

**Proposition 2.7** Given a spectral triple $(A^\infty, \mathcal{H}, D)$ (of compact type) which is $Q$-equivariant w.r.t. a representation of a CQG $Q$ on $\mathcal{H}$, we can construct a positive (possibly unbounded) invertible operator $R$ on $\mathcal{H}$ such that $(A^\infty, \mathcal{H}, D, R)$ is a twisted spectral data, i.e.
(a) $R$ commutes with $D$ and $\forall s \in \mathbb{R}$,
(b) the map $a \mapsto \sigma_s(a) := R^{-s}aR^s$ gives an automorphism of $A^\infty$ (not $*$-preserving) satisfying $\sup_{s \in [-n,n]} \|\sigma_s(a)\| < \infty$ for all positive integer $n$; and moreover, we have
(c) $\alpha_U$ preserves the functional $\tau_R$ (defined at least on a weakly dense $*$-subalgebra $\mathcal{E}_D$ of $\mathcal{B}(\mathcal{H})$ generated by the rank-one operators of the form $|\xi><\eta|$ where $\xi, \eta$ are eigenvectors of $D$) given by
$$\tau_R(x) = \text{Tr}(Rx), \ x \in \mathcal{E}_D.$$ 

**Remark 2.8** If $V_\lambda$ denotes the eigenspace of $D$ corresponding to the eigenvalue, say $\lambda$, it is clear that $\tau_R(X) = e^{i\lambda^2t}\text{Tr}(Re^{-tD^2}X)$ for all $X = |\xi><\eta|$ with $\xi, \eta \in V_\lambda$ and for any $t > 0$. Thus, the $\alpha_U$-invariance of the functional $\tau_R$ on $\mathcal{E}_D$ is equivalent to the $\alpha_U$-invariance of the functional $X \mapsto \text{Tr}(XRe^{-tD^2})$ on $\mathcal{E}_D$ for each $t > 0$. If, furthermore, the $R$-twisted spectral triple is $\Theta$-summable in the sense that $Re^{-tD^2}$ is trace class for every $t > 0$, the above is also equivalent to the $\alpha_U$-invariance of the bounded normal functional $X \mapsto \text{Tr}(XRe^{-tD^2})$ on the whole of $\mathcal{B}(\mathcal{H})$. In particular, this implies that $\alpha_U$ preserves the functional $\mathcal{B}(\mathcal{H}) \ni x \mapsto \lim_{t \to 0^+} \frac{\text{Tr}(xRe^{-tD^2})}{\text{Tr}(Re^{-tD^2})}$, where $\lim$ is a suitable Banach limit discussed in, e.g [19].

This motivates the following definition:
Definition 2.9 Given an $R$-twisted spectral triple $(A^\infty, \mathcal{H}, D, R)$, a quantum family of orientation preserving isometries $(\mathcal{S}, U)$ of $(A^\infty, \mathcal{H}, D)$ is said to preserve the $R$-twisted volume, (simply said to be volume-preserving if $R$ is understood) if one has $(\tau_R \otimes \text{id})(\alpha_U(x)) = \tau_R(x)1_\mathcal{S}$ for all $x \in \mathcal{E}_D$, where $\mathcal{E}_D$ and $\tau_R$ are as in Proposition 2.7. We shall also call $(\mathcal{S}, U)$ a quantum family of orientation-preserving isometries of the $R$-twisted spectral triple.

If, furthermore, the $C^*$-algebra $\mathcal{S}$ has a coproduct $\Delta$ such that $(\mathcal{S}, \Delta)$ is a CQG and $U$ is a unitary representation of $(\mathcal{S}, \Delta)$ on $\mathcal{H}$, we say that $(\mathcal{S}, \Delta)$ acts by volume and orientation-preserving isometries on the $R$-twisted spectral triple.

We shall consider the categories $Q_R \equiv Q_R(D)$ and $Q'_R \equiv Q'_R(D)$ which are the full subcategories of $Q$ and $Q'$ respectively, obtained by restricting the object-classes to the volume-preserving quantum families.

Remark 2.10 We shall not need the full strength of the definition of twisted spectral data here; in particular the condition (b) in the Proposition 2.7. However, we shall continue to work with the usual definition of $R$-twisted spectral data, keeping in mind that all our results are valid without assuming (b).

Let us now fix a spectral triple $(A^\infty, \mathcal{H}, D)$ which is of compact type. The $C^*$-algebra generated by $A^\infty$ in $\mathcal{B}(\mathcal{H})$ will be denoted by $A$. Let $\lambda_0 = 0, \lambda_1, \lambda_2, \cdots$ be the eigenvalues of $D$ with $V_i$ denoting the $(d_i$-dimensional, $d_i < \infty$) eigenspace for $\lambda_i$. Let $\{e_{ij}, j = 1, \ldots, d_i\}$ be an orthonormal basis of $V_i$. We also assume that there is a positive invertible $R$ on $\mathcal{H}$ such that $(A^\infty, \mathcal{H}, D, R)$ is an $R$-twisted spectral triple. $R$ must have the form $R|_{V_i} = R_i$, say, with $R_i$ positive invertible $d_i \times d_i$ matrix. Let us denote the CQG $A_i(R_i^T)$ by $U_i$, with its canonical unitary representation $\beta_i$ on $V_i \cong \mathbb{C}^{d_i}$, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^T$. Let $U$ be the free product of $U_i, i = 1, 2, \ldots$ and $\beta = \ast_i \beta_i$ be the corresponding free product representation of $U$ on $\mathcal{H}$. We shall also consider the corresponding unitary element $\tilde{\beta}$ in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U})$.

Lemma 2.11 Consider the $R$-twisted spectral triple $(A^\infty, \mathcal{H}, D)$ and let $(\mathcal{S}, U)$ be a quantum family of volume and orientation preserving isometries of the given spectral triple. Moreover, assume that the map $U$ is faithful in the sense that there is no proper $C^*$-subalgebra $S_1$ of $\mathcal{S}$ such that $\tilde{U} \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes S_1)$. Then we can find a $C^*$-isomorphism $\phi : U/\mathcal{I} \to \mathcal{S}$ between $\mathcal{S}$ and a quotient of $U$ by a $C^*$-ideal $\mathcal{I}$ of $\mathcal{U}$, such that $U = (\text{id} \otimes \phi) \circ (\text{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from $U$ to $U/\mathcal{I}$. 

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If, furthermore, there is a compact quantum group structure on $S$ given by a coproduct $\Delta$ such that $(S, \Delta, U)$ is an object in $Q^*_R$, the ideal $I$ is a Woronowicz $C^*$-ideal and the $C^*$-isomorphism $\phi : U/I \to S$ is a morphism of compact quantum groups.

**Proof:**

It is clear that $U$ maps $V_i$ into $V_i \otimes S$ for each $i$. Let $v^{(i)}_{kj} (j, k = 1, \ldots, d_i)$ be the elements of $S$ such that $U(e_{ij}) = \sum_k e_{ik} \otimes v^{(i)}_{kj}$. Note that $v_i := (v^{(i)}_{kj})$ is a unitary in $M_{d_i} \otimes S$. Moreover, the $*$-subalgebra generated by all $\{v^{(i)}_{kj}, i \geq 0, , j, k \geq 1\}$ must be dense in $S$ by the assumption of faithfulness.

Consider the $*$-homomorphism $\alpha_i$ from the finite dimensional $C^*$ algebra $A_i \cong M_{d_i} \otimes S$ generated by the rank one operators $\{|e_{ij} \rangle \langle e_{ik}|, j, k = 1, \ldots, d_i\}$ to $A_i \otimes S$ given by $\alpha_i(y) = \tilde{U}(y \otimes 1)\tilde{U}^*|V_i$. Clearly, the restriction of the functional $\tau_R$ on $A_i$ is nothing but the functional given by $\text{Tr}(R_i \cdot)$, where $\text{Tr}$ denotes the usual trace of matrices. Since $\alpha_i$ preserves this functional by assumption, we get, by the universality of $\tilde{U}$, a $C^*$-homomorphism from $U_i$ to $S$ sending $u^{(i)}_{kj} \equiv u^{R_i}_{kj}$ to $v^{(i)}_{kj}$, and by definition of the free product, this induces a $C^*$-homomorphism, say $\Pi$, from $U$ onto $S$, so that $U/I \cong S$, where $I := \ker(\Pi)$.

In case $S$ has a coproduct $\Delta$ making it into a compact quantum group and $U$ is a quantum group representation, it is easy to see that the subalgebra of $S$ generated by $\{v^{(i)}_{kj}, i \geq 0, , j, k \geq 1\}$ is a Hopf algebra, with $\Delta(v^{(i)}_{kj}) = \sum_l v^{(i)}_{kl} \otimes v^{(i)}_{lj}$. From this, it follows that $\Pi$ is Hopf-algebra morphism, hence $I$ is a Woronowicz $C^*$-ideal. □

Before we state and prove the main theorem, let us note the following elementary fact about $C^*$-algebras.

**Lemma 2.12** Let $C$ be a unital $C^*$ algebra and $F$ be a nonempty collection of $C^*$-ideals (closed two-sided ideals) of $C$. Let $I_0$ denote the intersection of all $I$ in $F$, and let $\rho_I$ denote the map $C/I_0 \ni x + I_0 \mapsto x + I$ $\in C/I$ for $I \in F$. Denote by $\Omega$ the set $\{\omega \circ \rho_I, I \in F, \omega \text{ state on } C/I\}$, and let $K$ be the weak-* closure of the convex hull of $\Omega \cup (-\Omega)$. Then $K$ coincides with the set of bounded linear functionals $\omega$ on $C/I_0$ satisfying $||\omega|| = 1$ and $\omega(x^* + I_0) = \omega(x + I_0)$.

**Proof:**
We have, by Lemma 4.6 of [14] that for any \( x \in \mathcal{C} \),
\[
\sup_{\mathcal{I} \in \mathcal{F}} \| x + \mathcal{I} \| = \| x + \mathcal{I}_0 \|,
\]
where \( \| x + \mathcal{I} \| = \inf\{\| x - y \| : y \in \mathcal{I}\} \) denotes the norm in \( \mathcal{C}/\mathcal{I} \). Clearly, \( \mathcal{K} \) is a weak-* compact, convex subset of the unit ball \( (\mathcal{C}/\mathcal{I}_0)^*_\mathcal{F} \) of the dual of \( \mathcal{C}/\mathcal{I}_0 \), satisfying \( -\mathcal{K} = \mathcal{K} \). If \( \mathcal{K} \) is strictly smaller than the self-adjoint part of unit ball of the dual of \( \mathcal{C}/\mathcal{I}_0 \), we can find a state \( \omega \) on \( \mathcal{C}/\mathcal{I}_0 \) which is not in \( \mathcal{K} \). Considering the real Banach space \( \mathcal{X} = (\mathcal{C}/\mathcal{I}_0)^*_\mathcal{F} \), and using standard separation theorems for real Banach spaces (e.g. Theorem 3.4 of [26], page 58), we can find a self-adjoint element \( x \) of \( \mathcal{C} \) such that \( \| x + \mathcal{I}_0 \| = 1 \), and
\[
\sup_{\omega \in \mathcal{K}} \omega'(x + \mathcal{I}_0) < \omega(x + \mathcal{I}_0).
\]
Let \( \gamma \in \mathbb{R} \) be such that \( \sup_{\omega \in \mathcal{K}} \omega'(x + \mathcal{I}_0) < \gamma < \omega(x + \mathcal{I}_0) \). Fix \( 0 < \epsilon < \omega(x + \mathcal{I}_0) - \gamma \), and let \( \mathcal{I} \in \mathcal{F} \) be such that \( \| x + \mathcal{I}_0 \| - \epsilon \leq \| x + \mathcal{I} \| \leq \| x + \mathcal{I}_0 \| \). Let \( \phi \) be a state on \( \mathcal{C}/\mathcal{I} \) such that \( \| x + \mathcal{I} \| = |\phi(x + \mathcal{I})| \). Since \( x \) is self-adjoint, either \( \phi(x + \mathcal{I}) \) or \( -\phi(x + \mathcal{I}) \) equals \( \| x + \mathcal{I} \| \), and \( \phi' := \pm \phi \circ \rho_{\mathcal{I},\omega} \), where the sign is chosen so that \( \phi'(x + \mathcal{I}_0) = \| x + \mathcal{I} \| \). Thus, \( \phi' \in \mathcal{K} \), so \( \| x + \mathcal{I}_0 \| = \phi'(x + \mathcal{I}) \leq \gamma < \omega(x + \mathcal{I}_0) - \epsilon \). But this implies \( \| x + \mathcal{I}_0 \| \leq \| x + \mathcal{I} \| + \epsilon < \omega(x + \mathcal{I}_0) - \epsilon \leq \| x + \mathcal{I}_0 \| - \epsilon \) (since \( \omega \) is a state), which is a contradiction completing the proof. \( \square \)

**Theorem 2.13** For any \( R \)-twisted spectral triple \( (\mathcal{A}_\infty, \mathcal{H}, D) \), the category \( Q_R \) of quantum families of volume and orientation preserving isometries has a universal (initial) object, say \( (\mathcal{G}, U_0) \). Moreover, \( \mathcal{G} \) has a coproduct \( \Delta_0 \) such that \( (\mathcal{G}, \Delta_0) \) is a compact quantum group and \( (\mathcal{G}, \Delta_0, U_0) \) is a universal object in the category \( Q_0 \). The representation \( U_0 \) is faithful.

**Proof:**
Recall the \( C^* \)-algebra \( \mathcal{U} \) considered before, along with the map \( \beta \) and the corresponding unitary \( \tilde{\beta} \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U}) \). For any \( C^* \)-ideal \( \mathcal{I} \) of \( \mathcal{U} \), we shall denote by \( \Pi_\mathcal{I} \) the canonical quotient map from \( \mathcal{U} \) onto \( \mathcal{U}/\mathcal{I} \), and let \( \Gamma_\mathcal{I} = (\text{id} \otimes \Pi_\mathcal{I}) \circ \beta \). Clearly, \( \tilde{\Gamma}_\mathcal{I} = (\text{id} \otimes \pi_\mathcal{I}) \circ \tilde{\beta} \) is a unitary element of \( \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U}/\mathcal{I}) \). Let \( \mathcal{F} \) be the collection of all those \( C^* \)-ideals \( \mathcal{I} \) of \( \mathcal{U} \) such that \( (\text{id} \otimes \omega) \circ \alpha_{\Gamma_\mathcal{I}} \equiv (\text{id} \otimes \omega) \circ \text{ad}_{\tilde{\Gamma}_\mathcal{I}} \) maps \( \mathcal{A}_\infty \) into \( \mathcal{A}_\infty'' \) for every state \( \omega \) (equivalently, every bounded linear functional) on \( \mathcal{U}/\mathcal{I} \). This collection is nonempty, since the trivial one-dimensional \( C^* \)-algebra \( \mathbb{C} \) gives an object in \( Q_R \) and by Lemma 2.11 we do get a member of \( \mathcal{F} \). Now, let \( \mathcal{I}_0 \) be the intersection of all ideals in \( \mathcal{F} \). We claim that \( \mathcal{I}_0 \) is again a member of
\(\mathcal{F}\). Indeed, in the notation of Lemma 2.12 it is clear that for \(a \in A^\infty\), \((\text{id} \otimes \phi) \circ \Gamma_{\mathcal{I}^0}(a) \in A''\) for all \(\phi\) in the convex hull of \(\Omega \cup (-\Omega)\). Now, for any state \(\omega\) on \(U/\mathcal{I}_0\), we can find, by Lemma 2.12 a net \(\omega_j\) in the above convex hull (so in particular \(\|\omega_j\| \leq 1 \forall j\)) such that \(\omega(x + \mathcal{I}_0) = \lim_j \omega_j(x + \mathcal{I}_0)\) for all \(x \in U/\mathcal{I}_0\).

It follows from Lemma 2.3 that \((\text{id} \otimes \omega_j)(X) \to (\text{id} \otimes \omega)(X)\) (in the strong operator topology) for all \(X \in \mathcal{M}(\mathcal{K(H)} \otimes U/\mathcal{I}_0)\). Thus, for \(a \in A^\infty\), \((\text{id} \otimes \omega) \circ \text{ad}_{\mathcal{I}^0}(a)\) is the S.O.T. limit of \((\text{id} \otimes \omega_j) \circ \text{ad}_{\mathcal{I}^0}(a)\), hence belongs to \(A''\).

We now show that \((\widetilde{G} := U/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})\) is a universal object in \(\mathcal{Q}_R\). To see this, consider any object \((\mathcal{S}, U)\) of \(\mathcal{Q}_R\). Without loss of generality we can assume \(U\) to be faithful, since otherwise we can replace \(\mathcal{S}\) by the \(C^*\)-subalgebra generated by the elements \(\{v_{ij}^{(i)}\}\) appearing in the proof of Lemma 2.11. But by Lemma 2.11 we can further assume that \(\mathcal{S}\) is isomorphic with \(U/\mathcal{I}\) for some \(\mathcal{I} \in \mathcal{F}\). Since \(\mathcal{I}_0 \subseteq \mathcal{I}\), we have a \(C^*\)-homomorphism from \(U/\mathcal{I}_0\) onto \(U/\mathcal{I}\), sending \(x + \mathcal{I}_0\) to \(x + \mathcal{I}\), which is clearly a morphism in the category \(\mathcal{Q}_R\). This is indeed the unique such morphism, since it is uniquely determined on the dense subalgebra generated by \(\{u_{kj}^{(i)}, i \geq 0, j, k \geq 1\}\) of \(\widetilde{G}\).

To construct the coproduct on \(\widetilde{G} = U/\mathcal{I}_0\), we first consider \(U^{(2)} : \mathcal{H} \to \mathcal{H} \otimes \widetilde{G} \otimes \widetilde{G}\) given by

\[
U^{(2)} = (\Gamma_{\mathcal{I}_0})_{(12)}(\Gamma_{\mathcal{I}_0})_{(13)},
\]

where \(U_{ij}\) is the usual ‘leg-numbering notation’. It is easy to see that \((\widetilde{G} \otimes \widetilde{G}, U^{(2)})\) is an object in the category \(\mathcal{Q}_R\), so by the universality of \((\widetilde{G}, \Gamma_{\mathcal{I}_0})\), we have a unique unital \(C^*\)-homomorphism \(\Delta_0 : \widetilde{G} \to \widetilde{G} \otimes \widetilde{G}\) satisfying

\[
(\text{id} \otimes \Delta_0)(\Gamma_{\mathcal{I}_0}) = U^{(2)}.
\]

Letting both sides act on \(e_{ij}\), we get

\[
\sum_{l} e_{il} \otimes (\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \left( \sum_{k} u_{il}^{(i)} \otimes u_{kj}^{(i)} \right) = \sum_{l} e_{il} \otimes \Delta_0(\pi_{\mathcal{I}_0}(u_{lj}^{(i)})).
\]

Comparing coefficients of \(e_{il}\), and recalling that \(\tilde{\Delta}(u_{lj}^{(i)}) = \sum_{k} u_{lk}^{(i)} \otimes u_{kj}^{(i)}\) (where \(\tilde{\Delta}\) denotes the coproduct on \(\mathcal{U}\)), we have

\[
(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \circ \tilde{\Delta} = \Delta_0 \circ \pi_{\mathcal{I}_0}
\]  

on the linear span of \(\{u_{jk}^{(i)}, i \geq 0, j, k \geq 1\}\), and hence on the whole of \(\mathcal{U}\). This implies that \(\Delta_0\) maps \(\mathcal{I}_0 = \text{Ker}(\pi_{\mathcal{I}_0})\) into \(\text{Ker}(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) = (\mathcal{I}_0 \otimes 1 + 1 \otimes \mathcal{I}_0) \subset E.\)
\(U \otimes U\). In other words, \(I_0\) is a Hopf \(C^*\)-ideal, and hence \(\overline{G} = U/I_0\) has the canonical compact quantum group structure as a quantum subgroup of \(U\). It is clear from the relation (2) that \(\Delta_0\) coincides with the canonical coproduct of the quantum subgroup \(U/I_0\) inherited from that of \(U\). It is also easy to see that the object \((\overline{G}, \Delta_0, \Gamma_{I_0})\) is universal in the category \(Q'_R\), using the fact that (by Lemma 2.11) any compact quantum group \((S, U)\) acting by volume and orientation preserving isometries on the given spectral triple is isomorphic with a quantum subgroup \(U/I\) for some Hopf \(C^*\)-ideal \(I\) of \(U\).

Finally, the faithfulness of \(U_0\) follows from the universality by standard arguments which we briefly sketch. If \(G_1 \subset \overline{G}\) is a \(*\)-subalgebra of \(\overline{G}\) such that \(\overline{U_0} \subset M(K(H) \otimes G_1)\), it is easy to see that \((G_1, U_0)\) is also a universal object, and by definition of universality of \(\overline{G}\) it follows that there is a unique morphism, say \(j\), from \(\overline{G}\) to \(G_1\). But the map \(i \circ j\) is a morphism from \(\overline{G}\) to itself, where \(i : G_1 \to \overline{G}\) is the inclusion. Again by universality, we have that \(i \circ j = \text{id}_{\overline{G}}\), so in particular, \(i\) is onto, i.e. \(G_1 = \overline{G}\). \(\square\)

Consider the \(*\)-homomorphism \(\alpha_0 := \alpha U_0\), where \((\overline{G}, U_0)\) is the universal object obtained in the previous theorem. For every state \(\phi\) on \(\overline{G}\), \((\text{id} \otimes \phi) \circ \alpha_0\) maps \(A\) into \(A''\). However, in general \(\alpha_0\) may not be faithful even if \(U_0\) is so, and let \(\mathcal{G}\) denote the \(C^*\)-subalgebra of \(\overline{G}\) generated by the elements \(\{(f \otimes \text{id}) \circ \alpha_0(a), f \in A^*, a \in A\}\).

Remark 2.14 If the spectral triple is even, then all the proofs above go through with obvious modifications.

Definition 2.15 We shall call \(G\) the quantum group of orientation-preserving isometries of the \(R\)-twisted spectral triple \((A^\infty, H, D, R)\) and denote it by \(QISO^+_R(A^\infty, H, D, R)\) or even simply as \(QISO^+_R(D)\). The quantum group \(\overline{G}\) is denoted by \(QISO^+_R(D)\).

If the spectral triple is even, then we will denote \(G\) and \(\overline{G}\) by \(QISO^+_R(D, \gamma)\) and \(QISO^+_R(D, \gamma)\) respectively.

2.3 Stability and topological action

It is not clear from the definition and construction of \(QISO^+_R(D)\) whether the \(C^*\) algebra \(A\) generated by \(A^\infty\) is stable under \(\alpha_0\) in the sense that \((\text{id} \otimes \phi) \circ \alpha_0\) maps \(A\) into \(A\) for every \(\phi\). Moreover, even if \(A\) is stable, the question remains whether \(\alpha_0\) is a \(C^*\)-action of the CQG \(QISO^+_R(D)\). Although we could not yet decide whether the general answer to the above two questions are affirmative, we should point out that for all the explicit examples considered by us so, both these questions do have an affirmative
answer. It is perhaps tempting (though a bit too daring at the moment) to conjecture that in general, \( QISO_R^+ \) will have a topological, i.e. \( C^* \) action. Indeed, we have quite a few results in this direction. Let us list the cases where we are able to prove that \( QISO_R^+(D) \) has \( C^* \)-action:

(i) For any spectral triple for which there is a 'reasonable' Laplacian in the sense of [14]. This includes all classical spectral triples as well as their Rieffel deformation (with \( R = I \)).

(ii) Under the assumption that there is an eigenvalue of \( D \) with a one-dimensional eigenspace spanned by a cyclic separating vector \( \xi \) such that any eigenvector of \( D \) belongs to the span of \( A^\infty \xi \) and \( \{ a \in A^\infty : a\xi \text{ is an eigenvector of } D \} \) is norm-dense in \( A \) (to be proved in subsection 2.4).

(iii) Under some analogue of the classical Sobolev conditions w.r.t. a suitable group action on \( A \) (see [17]).

Now we prove the sufficient conditions in (i).

We begin with a sufficient condition for stability of \( A^\infty \) under \( \alpha_0 \). Let \( (A^\infty, \mathcal{H}, D) \) be a (compact type) spectral triple such that

(1) \( A^\infty \) and \( \{[D, a] : a \in A^\infty \} \) are contained in the domains of all powers of the derivations \([D, \cdot]\) and \([|D|, \cdot]\).

We will denote by \( \tilde{T}_t \), the one parameter group of \( * \) automorphisms on \( B(\mathcal{H}) \) given by \( \tilde{T}_t(S) = e^{itD}S e^{-itD} \forall S \in B(\mathcal{H}) \). We will denote the generator of this group by \( \delta \). For \( X \) such that \([D, X]\) is bounded, we have \( \delta(X) = i[D, X] \) and hence

\[
\|\tilde{T}_t(X) - X\| = \left\| \int_0^t \tilde{T}_s([D, X])ds \right\| \leq t\| [D, X] \|.
\]

Let us say that the spectral triple satisfies the Sobolev condition if

\[
A^\infty = A'' \bigcap_{n \geq 1} \text{Dom}(\delta^n).
\]

Then we have the following result, which is a natural generalization of the classical situation, where a measurable isometric action automatically becomes topological (in fact smooth).

**Theorem 2.16** (i) For every state \( \phi \) on \( \mathcal{G} \), we have \( (\text{id} \otimes \phi) \circ \alpha_0(A^\infty) \in A'' \bigcap_{n \geq 1} \text{Dom}(\delta^n) \). (ii) If the spectral triple satisfies the Sobolev condition then \( A^\infty \) (and hence \( A \)) is stable under \( \alpha_0 \).
Proof:
Since \( U_0 \) commutes with \( D \otimes I \), it is clear that the automorphism group \( \widetilde{T}_t \) commutes with \( \alpha_0 \phi \equiv (\text{id} \otimes \phi) \circ \alpha_0 \), and thus by the continuity of \( \alpha_0 \) in the strong operator topology it is easy to see that, for \( a \in \text{Dom}(\delta) \), we have

\[
\lim_{t \to 0^+} \frac{\widetilde{T}_t(\alpha_0(a)) - \alpha_0(a)}{t} = \lim_{t \to 0^+} \frac{\alpha_0(\widetilde{T}_t(a) - a)}{t} = \alpha_0(\delta(a)).
\]

Thus, \( \alpha_0 \) leaves \( \text{Dom}(\delta) \) invariant and commutes with \( \delta \). Proceeding similarly, we prove (i). The assertion (ii) is a trivial consequence of (i) and the Sobolev condition.

Let us now assume that

(2) The spectral triple is \( \Theta \)-summable, i.e. for every \( t > 0 \), \( e^{-tD^2} \) is trace-class.

Consider the functional \( \tau(X) = \text{Lim}_{t \to 0} \frac{\text{Tr}(X e^{-tD^2})}{\text{Tr}(e^{-tD^2})} \) is a (not necessarily faithful or normal) state on \( \mathcal{B}(\mathcal{H}) \) where \( \text{Lim} \) is a suitable Banach limit as in [19]. Moreover, \( \tau \) is a positive faithful trace on the \( * \) algebra, say \( S^\infty \), generated by \( A^\infty \) and \( \{[D,a] : a \in A^\infty\} \), which is to be interpreted as the volume form (we refer to [19], [14] for the details). The completion of \( S^\infty \) in the norm of \( \mathcal{B}(\mathcal{H}) \) is denoted by \( S \).

From the definition of \( \tau \), it is also clear that \( \widetilde{T}_t \) preserves \( \tau \), so extends to a group of unitaries on \( \mathcal{N} := L^2(S^\infty, \tau) \). Moreover, for \( X \) such that \([D,X] \in \mathcal{B}(\mathcal{H})\), in particular for \( X \in S^\infty \), we have (denoting by \( \|a\|_2 \) the \( L^2 \)-norm \( \tau(a^*a)^{1/2} \) and the operator norm of \( \mathcal{B}(\mathcal{H}) \) by \( \| \cdot \|_\infty \) )

\[
\|\widetilde{T}_s(X) - X\|_2^2 = \tau(X^* (X - \widetilde{T}_s(X))) + \tau((X - \widetilde{T}_s(X))^*X) \\
\leq 2 \|X - \widetilde{T}_s(X)\|_\infty \|X\|_2 \\
\leq s \|[D,X]\|_\infty \|X\|_2,
\]

which clearly shows that \( s \mapsto \widetilde{T}_s(X) \) is \( L^2 \)-continuous for \( X \in S^\infty \), hence (by unitarity of \( \widetilde{T}_s \)) on the whole of \( \mathcal{N} \), i.e. it is a strongly continuous one-parameter group of unitaries. Let us denote its generator by \( \delta \), which
is a skew adjoint map, i.e. $i\tilde{\delta}$ is self adjoint, and $\tilde{T}_t = \exp(t\tilde{\delta})$. Clearly, $\tilde{\delta} = \delta = [D, \cdot]$ on $S^\infty$.

We will denote $L^2(A^\infty, \tau) \subset \mathcal{N}$ by $\mathcal{H}_D^0$ and the restriction of $\tilde{\delta}$ to $\mathcal{H}_D^0$ (which is a closable map from $\mathcal{H}_D^0$ to $\mathcal{N}$) by $d_D$. Thus, $d_D$ is closable too.

We now recall the assumptions made in [14], for defining the ‘Laplacian’ and the corresponding quantum isometry group of a spectral triple $(A^\infty, \mathcal{H}, D)$, without going into all the technical details, for which the reader is referred to [14].

The following conditions will also be assumed throughout the rest of this subsection:

1. $A^\infty \subseteq \text{Dom}(L)$ where $L \equiv L_D := -d_D^*d_D$.
2. $L$ has compact resolvent.
3. Each eigenvector of $L$ (which has a discrete spectrum, hence a complete set of eigenvectors) belongs to $A^\infty$.
4. The complex linear span of the eigenvectors of $L$, say $A^\infty_0$ (which is a subspace of $A^\infty$ by assumption (5) ), is norm dense in $A^\infty$.

It is clear that $L(A^\infty_0) \subseteq A^\infty_0$. The $*$-subalgebra of $A^\infty$ generated by $A^\infty_0$ is denoted by $A_0$. We also note that $L = P_0\tilde{L}P_0$, where $\tilde{L} := (i\tilde{\delta})^2$ (which is a self adjoint operator on $\mathcal{N}$) and $P_0$ denotes the orthogonal projection in $\mathcal{N}$ whose range is the subspace $\mathcal{H}_D^0$.

**Theorem 2.17** Let $(A^\infty, \mathcal{H}, D)$ be a spectral triple satisfying the assumptions (1) – (6) made above. In addition, assume that at least one of conditions (a) and (b) mentioned below is satisfied:

a) $A'' \subseteq \mathcal{H}_D^0$.

b) $\alpha_0^\phi(A^\infty) \subseteq A^\infty$ for every state $\phi$ on $G = QISO_R^+(D)$.

Then $\alpha_0$ is a $C^*$-action of $QISO_R^+(D)$ on $A$.

**Proof:**

Under either of the conditions (a) and (b), the map $\alpha_0^\phi$ maps $A^\infty$ into $\mathcal{H}_D^0 \subseteq \mathcal{N}$ (for any fixed $\phi$). Since $\alpha_0^\phi$ also commutes with $[D, \cdot]$ on $A^\infty$, it is clear that $\alpha_0^\phi(S^\infty) \subseteq \mathcal{N}$ too. In fact, using the complete positivity of the map $\alpha_0^\phi$ and the $\alpha_0$-invariance of $\tau$, we see that

$$\tau(\alpha_0^\phi(a^*a)) \leq \tau(\alpha_0^\phi(a^*a)) = (\text{id} \otimes \phi)(\tau \otimes \text{id})\alpha_0(a^*a) = \tau(a^*a).1,$$

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which implies that $\alpha_0^\phi$ extends to a bounded operator from $\mathcal{N}$ to itself. Since $U_0$ commutes with $D$, it is clear that $\alpha_0^\phi$ (viewed as a bounded operator on $\mathcal{N}$) will commute with the group of unitaries $\tilde{T}_t$, hence with its generator $\tilde{\delta}$ and also with the self-adjoint operator $\tilde{\mathcal{L}} = (i\tilde{\delta})^2$.

On the other hand, it follows from the definition of $G = \text{QISO}_I^+(D)$ that $(\tau \otimes \text{id})(\alpha_0^\phi(X)) = \tau(X)1_G$ for all $X \in \mathcal{B}(\mathcal{H})$, in particular for $X \in \mathcal{S}^\infty$, and thus the map $\mathcal{S}^\infty \otimes_{\text{alg}} G \ni (a \otimes q) \mapsto \alpha_0^\phi(a)(1 \otimes q)$ extends to a $G$-linear unitary, denoted by $W$ (say), on the Hilbert $G$-module $\mathcal{N} \otimes G$. Note that here we have used the fact that for any $\phi$, $(\text{id} \otimes \phi)(W)(\mathcal{S}^\infty \otimes_{\text{alg}} G) \subseteq \mathcal{N}$, since $\alpha_0^\phi(\mathcal{S}^\infty) \subseteq \mathcal{N}$. The commutativity of $\alpha_0^\phi$ with $\tilde{T}_t$ for every $\phi$ clearly implies that $W$ and $\tilde{T}_t \otimes \text{id}_G$ commute on $\mathcal{N} \otimes G$. Moreover, $\alpha_0^\phi$ maps $\mathcal{H}_D^0$ into itself, so $W$ maps $\mathcal{H}_D^0 \otimes G$ into itself, and hence (by unitarity of $W$) it commutes with the projection $P_0 \otimes 1$. It follows that $\alpha_0^\phi$ commutes with $P_0$, and (since it also commutes with $\tilde{\mathcal{L}}$), hence commutes with $\mathcal{L} = P_0\tilde{\mathcal{L}}P_0$ as well.

Thus, $\alpha_0^\phi$ preserves each of the (finite dimensional) eigenspaces of the Laplacian $\mathcal{L}$, and so is a Hopf algebraic action on the subalgebra $\mathcal{A}_0$ spanned algebraically by these eigenspaces. Moreover, the $G$-linear unitary $W$ clearly restricts to a unitary representation on each of the above eigenspaces. If we denote by $((q_{ij}))_{(i,j)}$ the $G$-valued unitary matrix corresponding to one such particular eigenspace, then by the general theory of CQG representations, $q_{ij}$ must belong to $G_0$ and we must have $\epsilon(q_{ij}) = \delta_{ij}$ (Kronecker delta). This implies $(\text{id} \otimes \epsilon) \circ \alpha_0 = \text{id}$ on each of the eigenspaces, hence on the norm-dense subalgebra $\mathcal{A}_0$ of $\mathcal{A}$, completing the proof of the fact that $\alpha_0$ extends to a $C^*$ action on $\mathcal{A}$. $\square$

Combining the above theorem with Theorem 2.16, we get the following immediate corollary.

**Corollary 2.18** If the spectral triple satisfies the Sobolev condition mentioned before, in addition to the assumptions 1–6, then $\text{QISO}_I^+(D)$ has a $C^*$-action. In particular, for a classical spectral triple, $\text{QISO}_I^+(D)$ has $C^*$-action.

**Remark 2.19** Let us remark here that in case the restriction of $\tau$ on $\mathcal{A}^\infty$ is normal, i.e. continuous w.r.t. the WOT inherited from $\mathcal{B}(\mathcal{H})$, then $\mathcal{H}_D^0$ will contain $\mathcal{A}''$, which is the WOT closure of $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H})$, i.e. condition (a) of Theorem 2.17 (and hence its conclusion) holds.

**Remark 2.20** The results obtained in this subsection can be formulated and proved in an $R$-twisted set-up as well, if the corresponding Laplacian (which
is an extension of $d_{D,R}^*d_{D,R}$, where $d_{D,R}^*$ denotes the adjoint of $d_D ≡ d_{D,R}$ w.r.t. the $R$-twisted volume form) ‘exists’ and satisfies the analogues of the assumptions made in this subsection about $\mathcal{L}_D$. In [4], we have made some computations with such an $R$-twisted Laplacian arising naturally in that context.

**Remark 2.21** In a private communication to us, Shuzhou Wang has kindly pointed out that a possible alternative approach to the formulation of quantum group of isometries may involve the category of CQG which has a $C^*$-action on the underlying $C^*$ algebra and a unitary representation w.r.t. which the Dirac operator is equivariant. We are not sure whether or how it is possible to show the existence of a universal object (even after fixing a choice of volume form) in this category; however, if the existence can be proved then the universal object will automatically have $C^*$ action.

### 2.4 Universal object in the categories Q or $Q'$

We shall now investigate further conditions on the spectral triple which will ensure the existence of a universal object in the category $Q$ or $Q'$. Whenever such an universal object exists we shall denote it by $\widetilde{QISO}^+(D)$, and denote by $QISO^+(D)$ its largest Woronowicz subalgebra for which $\alpha_U$ on $\mathcal{A}^\infty$ (where $U$ is the unitary representation of $QISO^+(D)$ on $\mathcal{H}$) is faithful.

**Remark 2.22** If $\widetilde{QISO}^+(D)$ exists, by [15], there will exist some $R$ such that $\widetilde{QISO}^+(D)$ is an object in the category $Q'_R(D)$. Since the universal object in this category, i.e. $\widetilde{QISO}^+_R(D)$, is clearly a sub-object of $\widetilde{QISO}^+(D)$, we have $\widetilde{QISO}^+_R(D) ≅ \widetilde{QISO}^+(D)$ for this choice of $R$.

Let us state and prove a result below, which gives some sufficient conditions for the existence of $\widetilde{QISO}^+(D)$.

**Theorem 2.23** Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type as before and assume that $D$ has an one-dimensional eigenspace spanned by a unit vector $\xi$, which is cyclic and separating for the algebra $\mathcal{A}^\infty$. Moreover, assume that each eigenvector of $D$ belongs to the dense subspace $\mathcal{A}^\infty \xi$ of $\mathcal{H}$. Then there is a universal object, $(\widetilde{\mathcal{G}}, U_0)$. Moreover, $\widetilde{\mathcal{G}}$ has a coproduct $\Delta_0$ such that $(\widetilde{\mathcal{G}}, \Delta_0)$ is a compact quantum group and $(\widetilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category $Q'$. 

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We will denote by $\mathcal{G}$ the Woronowicz $C^*$ subalgebra of $\tilde{\mathcal{G}}$ generated by elements of the form $\langle (\xi \otimes 1)\alpha U_0(a), \eta \otimes 1 \rangle$ where $\xi, \eta \in \mathcal{H}, a \in \mathcal{A}^\infty$ and $\langle ., . \rangle$ denotes the $\tilde{\mathcal{G}}$ valued inner product of $\mathcal{H} \otimes \tilde{\mathcal{G}}$. We have $\tilde{\mathcal{G}} \cong \mathcal{G} * C(\mathbb{T})$.

Proof: Let $V_i, \{e_{ij}\}$ be as before, and by assumption $e_{ij} = x_{ij}\xi$ for a unique $x_{ij} \in \mathcal{A}^\infty$. Clearly, since $\xi$ is separating, the vectors $\{e_{ij} = x_{ij}^*\xi, j = 1, ..., d_i\}$ are linearly independent, so the matrix $Q_i = ((e_{ij}, e_{ik}))_{j,k=1}^{d_i}$ is positive and invertible. Now, given a quantum family of orientation-preserving isometries $(\mathcal{S}, U)$, we must have $\tilde{U}(\xi \otimes 1) = \xi \otimes q$, say, for some $q \in \mathcal{S}$, and from the unitarity of $\tilde{U}$ it follows that $q$ is a unitary element. Moreover, $U$ leaves $V_i$ invariant, so let $\tilde{U}(e_{ij} \otimes 1) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$. But this can be rewritten as $\alpha_U(x_{ij})(\xi \otimes q) = \sum_k x_{ik} \xi \otimes v_{kj}^{(i)}$.

Since $\xi$ is separating and $q$ is unitary, this implies $\alpha_U(x_{ij}) = \sum_k x_{ik} \otimes v_{kj}^{(i)}q^*$, and thus we have $\tilde{U}(e_{ij} \otimes 1) = \alpha_U(x_{ij})^*(\xi \otimes q) = \sum_k x_{ik}^* \xi \otimes q^2(v_{kj}^{(i)})^* = \sum_k \overline{e_{kj}} \otimes q^2(v_{kj}^{(i)})^*$.

Taking the $\mathcal{S}$-valued inner product $\langle ., . \rangle_\mathcal{S}$ on both sides of the above expression, and using the fact that $U$ preserves this $\mathcal{S}$-valued inner product, we obtain $Q_i = v_i^t Q_i \overline{v_i}$ (where $v_i = (\overline{(v_{kj}^{(i)})})$). Thus, $Q_i^{-1} v_i^t Q_i$ must be the (both-sided) inverse of $\overline{v_i}$. Thus, we get a canonical surjective morphism from $A_u(Q_i)$ to the $C^*$ algebra generated by $v_{kj}^{(i)}$. This induces a surjective morphism from the free product of $A_u(Q_i), i = 1, 2, ...$ onto $\mathcal{S}$. The rest of the arguments for showing the existence of $\tilde{\mathcal{G}}$ will be quite similar to the arguments used in the proof of Theorem 2.13 hence omitted. It is also quite obvious from the proof that $\tilde{\mathcal{G}} = \mathcal{G} * C^\ast(q) \cong \mathcal{G} * C(\mathbb{T})$. $\Box$

Remark 2.24 Some of the examples considered in the next section will show that the conditions of the above theorem are not actually necessary; $\tilde{\mathcal{G}}_{ISO}^+ (D)$ may exist without the existence of a single cyclic separating eigenvector as above.

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple satisfying the conditions of the above theorem.

Let the faithful vector state corresponding to the cyclic separating vector $\xi$ be denoted by $\tau$. Let $\mathcal{A}_0^\infty = \text{span}\{a \in \mathcal{A}^\infty : a \xi \text{ is an eigenvector of } D\}$
Moreover, assume that $A_0^\infty$ is norm dense in $A^\infty$.

Let $\hat{D} : A_0^\infty \to A_0^\infty$ be defined by:

$$\hat{D}(a)\xi = D(a\xi)$$

This is well defined since $\xi$ is cyclic separating.

**Definition 2.25** Let $A$ be a $C^*$ algebra and $A^\infty$ be a dense $*$ subalgebra.

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple as above.

Let $\hat{C}$ be the category with objects $(Q, \alpha)$ such that $Q$ is a compact quantum group with an action $\alpha$ on $A$ such that:

1. $\alpha$ is $\tau$ preserving (where $\tau$ is above), i.e, $(\tau \otimes \text{id})\alpha(a) = \tau(a).1$
2. $\alpha$ maps $A_0^\infty$ inside $A_0^\infty \otimes_{\text{alg}} Q$.
3. $\alpha \hat{D} = (\hat{D} \otimes \text{id}) \alpha$.

**Corollary 2.26** There exists a universal object $\hat{Q}$ of the category $\hat{C}$ and it is isomorphic to the Woronowicz $C^*$ subalgebra $\mathcal{G} = QISO^+(D)$ of $\hat{G}$ obtained in Theorem 2.23.

**Proof:**

The proof of the existence of the universal object follows verbatim from the proof of Theorem 4.7 in [14] replacing $L$ by $\hat{D}$ and noting that $D$ has compact resolvent. We denote by $\hat{\alpha}$ the action of $\hat{Q}$ on $A$.

Now, we prove that $\hat{Q}$ is isomorphic to $\mathcal{G}$.

Each eigenvector of $D$ is in $A^\infty$ by assumption. It is easily observed from the proof of Theorem 2.23 that $\alpha_{U_0}$ maps the norm-dense $*$-subalgebra $A_0^\infty$ into $A_0^\infty \otimes_{\text{alg}} \mathcal{G}_0$, and $(\text{id} \otimes \epsilon) \circ \alpha_{U_0} = \text{id}$, so that $\alpha_{U_0}$ is indeed an action of the CQG $\mathcal{G}$. Moreover, it can be easily seen that $\tau$ preserves $\alpha_{U_0}$ and that $\alpha_{U_0}$ commutes with $\hat{D}$. Therefore, $(\mathcal{G}, \alpha_{U_0}) \in \text{Obj}(\hat{C})$, and hence $\mathcal{G}$ is a quantum subgroup of $\hat{Q}$ by the universality of $\hat{Q}$.

For the converse, we start by showing that $\hat{\alpha}$ induces a unitary representation $W$ of $\hat{Q} \ast C(\mathbb{T})$ on $\mathcal{H}$ which commutes with $D$, and the corresponding conjugated action $\alpha_W$ coincides with $\hat{\alpha}$.

Define $W(a\xi) = \hat{\alpha}(a)(\xi)(1 \otimes q^*) \forall a \in A_0^\infty$ where $q$ is a generator of $C(\mathbb{T})$.

Since we have $(\tau \otimes \text{id})\alpha(a) = \tau(a).1$, it follows that $\hat{W}$ is a $(\hat{Q} \ast C(\mathbb{T}))$-linear isometry on the dense subspace $A_0^\infty \otimes_{\text{alg}} \hat{Q}$ and thus extends to $\mathcal{H} \otimes \hat{Q} \ast C(\mathbb{T})$ as an isometry. Moreover, since $\hat{\alpha}(A)(1 \otimes \hat{Q})$ is norm dense in $A \otimes \hat{Q}$ (by the definition of a CQG action) it is clear that the range of $\hat{W}$ is dense, so $\hat{W}$ is indeed a unitary. It is quite obvious that it is a unitary representation of $\hat{Q} \ast C(\mathbb{T})$. 

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We also have,
\[ WD(a\xi) = W(\hat{D}(a)\xi) = \hat{\alpha}(\hat{D}(a))(1 \otimes q^*) = (D \otimes I)(\hat{\alpha}(a)\xi)(1 \otimes q^*) = (D \otimes I)W(a\xi), \]
i.e. \( W \) commutes with \( D \).

It is also easy to observe that \( \alpha_W = \hat{\alpha} \). This gives a surjective CQG morphism from \( \tilde{G} = G\ast C(\mathbb{T}) \) to \( \hat{Q}\ast C(\mathbb{T}) \), sending \( G \) onto \( \hat{\phi} \), which completes the proof.

\[ \square \]

3 Comparison with the approach of [14] based on Laplacian

Throughout this section, we shall assume the set-up of subsection 2.3 for the existence of a ‘Laplacian’, including assumptions 1 – 6. Let us also use the notation of that subsection.

As in [14], we say that a CQG \((S, \Delta)\) which has an action \( \alpha \) on \( A \) is said to act smoothly and isometrically on the noncommutative manifold \((A^\infty, \mathcal{H}, D)\) if \((\text{id} \otimes \phi) \circ \alpha(A^\infty_0) \subseteq A^\infty_0\) for all state \( \phi \) on \( S \), and also \((\text{id} \otimes \phi) \circ \alpha \) commutes with the Laplacian \( \mathcal{L} \equiv L_D \) on \( A^\infty_0 \). One can consider the category \( Q^{\mathcal{L}_D} \) of all compact quantum groups acting smoothly and isometrically on \( A \), where the morphisms are quantum group morphisms which intertwine the actions on \( A \). We make the following additional assumption throughout the present section:

(7) There exists a universal object in \( Q^{\mathcal{L}_D} \) (i.e. the quantum isometry group for the Laplacian \( \mathcal{L} \equiv L_D \) in the sense of [14]), and it is denoted by \( Q^\mathcal{L} \equiv Q_{\mathcal{L}_D} \)

Remark 3.1 It is proved in [14] that under the additional ‘connectedness assumption’ (the kernel of \( \mathcal{L} \) is one dimensional, spanned by the identity 1 of \( A^\infty \) viewed as a unit vector in \( \mathcal{H}^0_D \)), the category \( Q^\mathcal{L} \) has a universal object, say \( Q^\mathcal{L} \), called the quantum isometry group in [14]. In [14] it was also shown (Lemma 2.5, b \( \Rightarrow \) a) that for an isometric group action on a classical manifold (not necessarily connected), the volume functional is automatically preserved. It can be easily seen that the proof goes verbatim for a quantum group action. As this volume preserving condition implies
the existence of $Q^{\mathcal{E}}_D$, for a classical manifold (not necessarily connected) $Q^{\mathcal{E}}_D$ always exists.

The following result now follows immediately from Theorem 2.17 of subsection 2.3.

**Corollary 3.2** $QISO^+_I(D)$ is a sub-object of $Q^{\mathcal{E}}_D$ in the category $Q^{\mathcal{E}}_D$.

**Proof:** The proof is a consequence of the fact that $QISO^+_I(D)$ has the $C^*$-action $\alpha_0$ on $A$, and the observation already made in the proof of the Theorem 2.17 that this action commutes the Laplacian $L_D$. □

We will denote the inner product on the space of $k$ forms coming from the spectral triples $(A^\infty, \mathcal{H}, D)$ and $(A^\infty, \mathcal{H}_{d+d^*}, d+d^*)$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}_{d+d^*}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_{d+d^*}}$, respectively, $k=0,1$.

We will denote by $\pi_D$, $\pi_{d+d^*}$ the representations of $A^\infty$ in $\mathcal{H}$ and $\mathcal{H}_{d+d^*}$ respectively.

Let $U_{d+d^*}$ be the canonical unitary representation of $QISO^+_I(d+d^*)$ on $\mathcal{H}_{d+d^*}$.

$\mathcal{H}_{d+d^*}$ breaks up into finite dimensional orthogonal subspaces corresponding to the distinct eigenvalues of $\Delta := (d+d^*)^2 = d^*d + dd^*$. It is easy to see that $\Delta$ leaves each of the subspaces $\mathcal{H}_{d+d^*}$ invariant, and we will denote by $V_{\lambda,i}$ the subspace of $\mathcal{H}_{d+d^*}$ spanned by eigenvectors of $\Delta$ corresponding to the eigenvalue $\lambda$. Let $\{e_{j,\lambda,i}\}_j$ be an orthonormal basis of $V_{\lambda,i}$. Note that $L_D$ is the restriction of $\Delta$ to $\mathcal{H}_{d+d^*}$.

Now we recall the result of Section 2.4 of [14]. It was shown there that $Q^{\mathcal{E}}_D$ has a unitary representation $U \equiv U_{\mathcal{E}}$ on $\mathcal{H}_{d+d^*}$ such that $U$ commutes with $d+d^*$. Thus $(A^\infty, \mathcal{H}_{d+d^*}, d+d^*)$ is a $Q^{\mathcal{E}}_D$ equivariant spectral triple. It follows from the construction in [14] that $Q^{\mathcal{E}}_D$ is a quotient (by a Woronowicz $C^*$ ideal) of the free product of countably many Wang algebras of the type $A_u(I)$, and hence is a compact quantum Kac algebra.
Thus, it has tracial Haar state, which implies, by Theorem 3.2 of [15], that $\alpha_U$ keeps the functional $\tau_I$ invariant. Thus, we have:

**Proposition 3.3** \(Q^{LD}, U_L\) is a sub object of \((QISO_I^+(d + d^*), U_{d+d^*})\) in the category \(Q_I(d + d^*)\), so in particular, \(Q^{LD}\) is isomorphic to a quotient of \(QISO_I^+(d + d^*)\) by a Woronowicz $C^*$ ideal.

We shall give (under mild conditions) a concrete description of the above Woronowicz ideal.

Let \(\mathcal{I}\) be the $C^*$ ideal of \(QISO_I^+(d + d^*)\) generated by

\[\cup_{\lambda \in \sigma(D)} \left\{ \langle (P_0^\perp \otimes \text{id}) U_{d+d^*}(e_{j\lambda 0}), e_{j\lambda i} \otimes 1 \rangle : j, i' \geq 1 \right\},\]

where $P_0$ is the projection onto $\mathcal{H}_D^0$ and $\langle ., . \rangle$ denotes the $QISO_I^+(d + d^*)$ valued inner product.

Since $U_{d+d^*}$ keeps the eigenspaces of $\Delta = (d + d^*)^2$ invariant, we can write

\[U(e_{j\lambda 0}) = \sum_k e_{k\lambda 0} \otimes q_{kj\lambda 0} + \sum_{i' \neq 0, k'} e_{k'\lambda i'} \otimes q_{k'j\lambda i'}\]

for some $q_{kj\lambda 0}, q_{k'j\lambda i'} \in QISO_I^+(d + d^*)$.

We note that $q_{k'j\lambda i'} \in \mathcal{I}$ if $i' \neq 0$.

**Lemma 3.4** \(\mathcal{I}\) is a co-ideal of \(QISO_I^+(d + d^*)\).

**Proof**:
It is enough to prove the relation \(\Delta(\mathcal{I}) \in \mathcal{I} \otimes QISO_I^+(d + d^*) + QISO_I^+(d + d^*) \otimes \mathcal{I}\) for the elements in \(\mathcal{I}\) of the form \(\langle (P_0^\perp \otimes \text{id}) U_{d+d^*}(e_{j\lambda 0}), e_{j\lambda i_0} \otimes 1 \rangle\).

We have:

\[\Delta(\langle (P_0^\perp \otimes \text{id}) U_{d+d^*}(e_{m\lambda 0}), e_{j\lambda i_0} \otimes 1 \rangle) = \langle (P_0^\perp \otimes \text{id}) (id \otimes \Delta) U_{d+d^*}(e_{m\lambda 0}), e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle = \langle (P_0^\perp \otimes \text{id}) U_{(12)} U_{(13)}(e_{m\lambda i}), e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle = \langle (P_0^\perp \otimes \text{id}) U_{(12)} \left( \sum_k e_{k\lambda 0} \otimes 1 \otimes q_{km\lambda 0} \right), e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle\]
$$+ \sum_{i' \neq 0, l} \langle ( P_0^+ \otimes \text{id}) U_{(12)} (e_{l \lambda i' } \otimes 1 \otimes q_{l m \lambda i}) , \; e_{j \lambda i_0} \otimes 1 \otimes 1 \rangle$$

$$= \sum_{k, k'} \langle ( P_0^+ \otimes \text{id})(e_{k' \lambda 0 } \otimes q_{k' k \lambda 0} \otimes q_{k m \lambda 0}) , \; e_{j \lambda i_0} \otimes 1 \otimes 1 \rangle$$

$$+ \sum_{i' \neq 0, k, k'} \langle ( P_0^+ \otimes \text{id})(e_{k' \lambda i' } \otimes q_{k m \lambda i' } ) , \; e_{j \lambda i_0} \otimes 1 \otimes 1 \rangle$$

$$+ \sum_{i' \neq 0, l, l'} \langle ( P_0^+ \otimes \text{id})(e_{l \lambda i' } \otimes q_{l m \lambda i' } ) , \; e_{j \lambda i_0} \otimes 1 \otimes 1 \rangle$$

$$\in I \otimes QISO_I^+(d + d^*) + QISO_I^+(d + d^*) \otimes I \; ( \text{as } q_{k j \lambda i'} \in I \text{ if } i' \neq 0 ) \; \Box$$

**Theorem 3.5** If $\alpha U_{d+d^*}$ is a $C^*$ action on $A$, then we have $Q_{C^*} \cong QISO_I^+(d + d^*) / I$.

**Proof:**

By Proposition 3.3, we conclude that there exists a surjective CQG morphism $\pi : QISO_I^+(d + d^*) \rightarrow Q_{C^*}$. By construction (Section 2.4 in [14]),

$$\in I \otimes QISO_I^+(d + d^*) + QISO_I^+(d + d^*) \otimes I \; ( \text{as } q_{k j \lambda i'} \in I \text{ if } i' \neq 0 ) \; \Box$$
the unitary representation $U_L$ of $Q^L_D$ preserves each of the $\mathcal{H}'^D_D$, in particular $\mathcal{H}'^D_0$. It is then clear from the definition of $\mathcal{I}$ that $\pi$ induces a surjective CQG morphism (in fact, a morphism in the category $Q\mathcal{I}'(d + d^*)$)

$\pi' : QISO^+_I(d + d^*)/\mathcal{I} \to Q^L_D$.

Conversely, if $V = (\text{id} \otimes \rho_I) \circ U_{d+d^*}$ is the representation of $QISO^+_I(d + d^*)/\mathcal{I}$ on $\mathcal{H}_{d+d^*}$ induced by $U_{d+d^*}$ (where $\rho_I : QISO^+_I(d + d^*) \to QISO^+_I(d + d^*)/\mathcal{I}$ denotes the quotient map), then $V$ preserves $\mathcal{H}'^D_0$ (by definition of $\mathcal{I}$), so commutes with $P_0$. Since $V$ also commutes with $(d + d^*)^2$, it follows that $V$ must commute with $(d + d^*)^2 P_0 = \mathcal{L}$, i.e.

$$\tilde{V}(d^*d P_0 \otimes 1) = (d^*d P_0 \otimes 1)\tilde{V}.$$ 

It is easy to show from the above that $\alpha_V$ (which is a $C^*$ action on $\mathcal{A}$ since $\alpha_{U_{d+d^*}}$ is so by assumption) is a smooth isometric action of $QISO^+_I(d + d^*)/\mathcal{I}$ in the sense of [14], w.r.t. the Laplacian $\mathcal{L}$. This implies that $QISO^+_I(d + d^*)/\mathcal{I}$ is a sub-object of $Q^L$ in the category $Q\mathcal{L}$, and completes the proof. $\square$

Now we prove that under some further assumptions which are valid for classical manifolds as well as their Rieffel deformation, one even has the isomorphism $Q^L_D \cong QISO^+_I(d + d^*)$.

We assume the following:

(A) Both the spectral triples $(\mathcal{A}^\infty, \mathcal{H}, D)$ and $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ satisfy the assumptions (1) – (7), so in particular both $Q^L_D$ and $Q^L_{D'}$ exist (here $D' = d + d^*$).

(B) $\langle a, b \rangle_{\mathcal{H}'_D} = \langle a, b \rangle_{\mathcal{H}'_{D'}}$

and

$\langle d_D a, d_D b \rangle_{\mathcal{H}'_D} = \langle d_D a, d_D b \rangle_{\mathcal{H}'_{D'}} \forall a, b \in \mathcal{A}^\infty$.

**Remark 3.6** For classical compact spin manifolds these assumptions can be verified by comparing the local expressions of $D^2$ and the ‘Hodge Laplacian’ $(D')^2$ in suitable coordinate charts. In fact, in this case, both these operators turn out to be essentially same, upto a ‘first order term’, which is relatively compact w.r.t. $D^2$ or $(D')^2$.

From assumption (B), we observe that the identity map on $\mathcal{A}^\infty$ extends to a unitary, say $\Sigma$, from $\mathcal{H}'_D$ to $\mathcal{H}'_{D'}$. Moreover, we have

$\mathcal{L}_D = \Sigma^* \mathcal{L}_{D'} \Sigma$,

from which the following follows immediately:
Proposition 3.7 Under the above assumptions, \( \mathcal{Q}^L_D \cong \mathcal{Q}^L_{D'} \)

We conclude this section with the following result, which identifies the quantum isometry group \( \mathcal{Q}^L_D \) of \([14]\) as the \( \mathcal{QISO}^+_I \) of a spectral triple, and thus, in some sense, accommodates the construction of \([14]\) in the framework of the present article.

Theorem 3.8 If in addition to the assumptions already made, \((A^\infty, \mathcal{H}_{D'}, D')\) also satisfies the conditions of Theorem [2.17], so that \( \mathcal{QISO}^+_I(D') \) has a \( C^* \)-action, the we have

\[ \mathcal{Q}^L_D \cong \mathcal{QISO}^+_I(D') \cong \mathcal{Q}^L_{D'} \].

Proof: By Proposition 3.3 that we have \( \mathcal{Q}^L_D \) is a sub-object of \( \mathcal{QISO}^+_I(D') \) in the category \( \mathcal{Q}_I^D(D') \). On the other hand, by Theorem 2.17 we have \( \mathcal{QISO}^+_I(D') \) is a sub object of \( \mathcal{Q}^L_{D'} \) in the category \( \mathcal{Q}^L_{D'} \). Combining these facts with the conclusion of Proposition 3.3 we get the required isomorphism. □

Remark 3.9 The assumptions, and hence the conclusions, of this section are valid also for spectral triples obtained by Rieffel deformation of a classical spectral triple, to be discussed in details in section 5.

4 Examples and computations

4.1 Equivariant spectral triple on \( SU_\mu(2) \)

Let \( \mu \in [-1, 1] \). Then \( SU_\mu(2) \) is defined as the universal unital \( C^* \) algebra generated by \( \alpha, \gamma \) such that :

\[ \alpha^* \alpha + \gamma^* \gamma = 1 \] (3)

\[ \alpha \alpha^* + \mu^2 \gamma \gamma^* = 1 \] (4)

\[ \gamma \gamma^* = \gamma^* \gamma \] (5)

\[ \mu \gamma \alpha = \alpha \gamma \] (6)

\[ \mu \gamma^* \alpha = \alpha \gamma^* \] (7)
Let $\mathcal{H} = L^2(SU_\mu(2))$ be the G.N.S space of $SU_\mu(2)$ with respect to the Haar state $h$.

For each $n \in \{0, 1/2, 1, \ldots \}$, there is a unique irreducible representation $T^n$ of dimension $2n + 1$. Denote by $t^n_{ij}$, the $ij$th entry of $T^n$. They form an orthogonal basis of $\mathcal{H}$. Denote by $e^n_{ij}$ the normalized $t^n_{ij}$ s so that $\{e^n_{ij} : n = 0, 1/2, 1, \ldots, i, j = -n, -n + 1, \ldots, n\}$ is an orthonormal basis.

Consider the spectral triple on $SU_\mu(2)$ constructed by Chakraborty and Pal ([6]) and also discussed thoroughly in [9] which is defined by $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{A}^\infty$ is the linear span of $t^n_{ij}$ s, and $D$ is defined by:

$$D(e^n_{ij}) = (2n + 1)e^n_{ij}, \ n \neq i$$
$$= -(2n + 1)e^n_{ij}, \ n = i$$

Here, we have a cyclic separating vector $1_{SU_\mu(2)}$, and the corresponding faithful state is the Haar state $h$. Therefore, an operator commuting with $D$ (equivalently with $\hat{D}$) must keep $V^l_i := \text{Span}\{t^n_{ij} : j = -l, \ldots, l\}$ invariant $\forall$ fixed $l$ and $i$ where $\hat{D}$ is the operator as in Section 2.3.

In the notation of Corollary [22,26], we have $\mathcal{A}^\infty_0 = \text{span}\{t^n_{ij} : l = 0, 1/2, \ldots\} = \mathcal{A}^\infty$ in this case. All the conditions of Theorem [2,23] and Corollary [22,26] are satisfied. Thus, the universal object of the category $\mathcal{C}$ exists (notation as in Corollary [22,26]) and we denote it by $\hat{Q}$.

We recall from [20] that

$$t^{-1/2}_{-1/2,-1/2} = \alpha, t^{1/2}_{1/2,1/2} = -\mu \gamma^*, t^{1/2}_{1/2,-1/2} = \gamma^*, t^{1/2}_{1/2,1/2} = \alpha^* \quad (8)$$

Moreover, if $f_{n,i} = a_{n,i} \alpha^{n-i} \gamma^{n+i}$ (where $a_{n,i}$ s are some constants as in [20]) then $\{f_{n,i} : n = 0, 1/2, 1, 3/2, \ldots, -n \leq i \leq n\}$ is an orthonormal basis of $SU_\mu(2)$.

Now, $f_{n+\frac{1}{2},i} = c(n,i) \alpha f_{n,i+\frac{1}{2}}$ for some constants $c_{n,i}$. Applying the co-product on both sides and then comparing coefficients, we have the following recursive relations.

$$\begin{align*}
t^{l+1/2}_{i,l+1/2} &= c_{11}(i,l)t^{l}_{i+1/2,1/2}\gamma^* + c_{12}(i,l)t^{l}_{i-1/2,1/2}\alpha^*, \quad -l + 1/2 \leq i \leq l - 1/2 \\
&= c_{21}(i,l)\gamma^* t^{l}_{i+1/2,l} \quad i = -l - 1/2 \\
&= c_{31}(i,l)\alpha^* t^{l}_{i-1/2,l} \quad i = l + 1/2 \\
\end{align*} \quad (9)$$
For $j \leq l$,

\[
t_{i,j}^{l+1/2} = c(l, i, j)\alpha t_{i+1/2,j+1/2}^l + c'(l, i, j)\gamma t_{i-1/2,j+1/2}^l - l + 1/2 \leq i \leq l - 1/2
\]

\[
= d(l, j)\alpha t_{i-j+1/2}^l - d'(l, j)\gamma t_{i-j-1/2}^l, \quad i = -l - 1/2, -l + 1/2 \leq j \leq l - 1/2
\]

\[
= e(l, j)\gamma t_{i-1/2,j+1/2}^l + e'(l, j)\alpha t_{i-1/2,j-1/2}^l, \quad i = l + 1/2;
\]

(10)

where $C_{pq}(il), c(l, i, j), d(l, j), d'(l, j), e(l, j), e'(l, j)$ are all complex numbers.

**Lemma 4.1** Given a CQG $Q$ with an action $\Phi$ on $A$, the following are equivalent:

1. $(Q, \Phi) \in \text{Obj}(\hat{C})$

2. The action is linear, i.e., $V_{-1/2}^{1/2}$ is invariant under $\Phi$ and the representation obtained by restricting $\Phi$ to $V_{1/2}^{1/2}$ is a unitary representation.

3. $\Phi$ is linear and Haar state preserving.

4. $\Phi$ keeps $V_i^l$ invariant for each fixed $l$ and $i$.

**Proof**:

1. $\Rightarrow$ 2. Since $\Phi$ commutes with $\hat{D}$, $\Phi$ keeps each of the eigenspaces of $\hat{D}$ invariant and so in particular preserves $V_{-1/2}^{1/2}$, i.e. $\Phi$ is linear. The condition that $(h \otimes id)\Phi = h(\cdot)1$ implies that the corresponding representation induces a unitary.

2. $\Rightarrow$ 3. By linearity, write $\Phi(\alpha) = \alpha \otimes X + \gamma^* \otimes Y$ and $\Phi(\gamma^*) = \alpha \otimes Z + \gamma^* \otimes W$.

Firstly, $\Phi$-invariance of $\text{Span}\{t_{i,j}^k\}$ for $k = 0$ and $k = \frac{1}{2}$ follow from linearity and the fact that $\Phi(1) = 1$.

Next, we show that $\Phi$ keeps $\text{span}\{t_{i,j}^1: i, j = -1, 0, 1\}$ invariant.

We recall that $t_{i,j}^1$ is given by the matrix:

\[
\begin{pmatrix}
\alpha^2 & -(\mu^2 + 1)\alpha\gamma & -\mu \gamma^2 \\
\gamma^*\alpha^* & 1 - (\mu^2 + 1)\gamma^*\gamma & \alpha\gamma \\
-\mu\gamma^* & -(\mu^2 + 1)\gamma^*\alpha & \alpha^2
\end{pmatrix}
\]
By inspection, we see that \( \Phi(V_1) \subseteq V_1 \otimes G \) for \( i = -1, 1 \).

Hence, it is enough to check the \( \Phi \)-invariance for \( \alpha \gamma \) and \( 1 - (\mu^2 + 1)\gamma^* \gamma \).

Comparing coefficients in \( \Phi(\alpha \gamma) \), we can see that it belongs to \( V_0^1 \) if and only if \( XZ^* + YW^* = 0 \). Similarly, in the case of \( 1 - (\mu^2 + 1)\gamma^* \gamma \), we have the condition \( ZZ^* + WW^* = 1 \). But these conditions follow from the unitarity of the matrix \( \begin{pmatrix} X^* & Z^* \\ Y^* & W^* \end{pmatrix} \), which is nothing but the matrix corresponding to the restriction of \( \Phi \) to \( V_1^{1/2} \). Thus, \( \Phi \) keeps \( \text{Span}\{t_{ij}^l : i, j = -1, 0, 1\} \) invariant.

Moreover, by using the recursive relations (9), (10) and the following multiplication rule (see [20]),

\[
t_{i,j}^l t_{i',j'}^{1/2} = \sum_{k=-|l-1/2|,\ldots,l+1/2} c_k(l, i, j, i', j') t_{i+i', j+j'}^k,
\]

(\( c_k(l, i, j, i', j') \) are scalars) we can easily observe that \( \forall l \geq 3/2, \Phi(V_1^{l+1/2}) \subseteq V_i^{l-1/2} \oplus V_i^{l+1/2} \).

Using these observations, we conclude that for \( t_{ij}^l \), that \( \Phi \) maps \( \text{Span}\{t_{ij}^l : l \geq 1/2\} \) into itself.

So, in particular, \( \text{Ker}(h) = \text{span}\{V_1^l, i = -l, \ldots, l, l \geq 1/2\} \) is invariant under \( \Phi \) which (along with \( \Phi(1) = 1 \)) implies that \( \Phi \) preserves \( h \).

3. \( \Rightarrow \) 4.

We proceed by induction. The induction hypothesis holds for \( l = 1/2 \) since linearity means that \( \text{span}\{\alpha, \gamma^*\} \) is invariant under \( \Phi \) and hence \( \text{span}\{\alpha^*, \gamma\} \) is also invariant. The case for \( l = 1 \) can be checked by inspection as in the proof of 2 \( \Rightarrow \) 3. Take the induction hypothesis that \( \Phi \) keeps \( V_i^k \) invariant for all \( k, i \) with \( k \leq l \). From the proof of 2 \( \Rightarrow \) 3 we also have \( \forall l \geq 3/2, \Phi(V_i^{l+1/2}) \subseteq V_i^{l-1/2} \oplus V_i^{l+1/2} \), by using linearity only. Thus, \( \tilde{\Phi} \) leaves invariant the Hilbert \( \mathbb{Q} \) module \( (V_i^{l-1/2} \oplus V_i^{l+1/2}) \otimes \mathbb{Q} \), and is a unitary there since \( \Phi \) is Haar-state preserving. Since \( \tilde{\Phi} \) leaves invariant \( V_i^{l-1/2} \otimes \mathbb{Q} \) by the induction hypothesis, it must keep its orthocomplement, i.e. \( V_i^{l+1/2} \) invariant as well.

4. \( \Rightarrow \) 3.

The fact that \( \Phi \) keeps \( V_i^{l/2} \) invariant for \( l = 1/2 \) will imply that \( \Phi \) is linear. The proof of Haar state preservation is exactly the same as in 2 \( \Rightarrow \) 3.

4 \( \Rightarrow \) 1.
That $\Phi$ preserves the Haar state follows from arguments used before. Since $A_0^\infty = \text{Span}\{v^l_{ij} : l \geq 0, i, j = -l, \ldots, l\}$, and $\Phi$ keeps $V^l$ invariant it is obvious that $\Phi(A_0^\infty) \subseteq A_0^\infty \otimes_{\text{alg}} Q_0$ and $\Phi\hat{D} = (\hat{D} \otimes \text{id})\Phi$.

We now introduce the compact quantum group $U_\mu(2)$. (We refer to [20] for more details.)

**Definition 4.2** As a unital $C^*$ algebra, $U_\mu(2)$ is generated by 4 elements $u_{11}, u_{12}, u_{21}, u_{22}$ such that:

$$u_{11}u_{12} = \mu u_{12}u_{11}, u_{11}u_{21} = \mu u_{21}u_{11}, u_{12}u_{22} = \mu u_{22}u_{12}, u_{21}u_{22} = \mu u_{22}u_{21}, u_{12}u_{21} = u_{21}u_{12}, u_{11}u_{22} - u_{22}u_{11} = (\mu - \mu^{-1}) u_{12}u_{21} \text{ and the condition that the matrix}$$

$$\begin{pmatrix}
    u_{11} & u_{12} \\
    u_{21} & u_{22}
\end{pmatrix}
$$

is a unitary.

The CQG structure is given by $\Delta(u_{ij}) = \sum_{k=1,2} u_{ik} \otimes u_{kj}, \kappa(u_{ij}) = u_{ji}^*, \epsilon(u_{ij}) = \delta_{ij}$.

**Remark 4.3** Let the quantum determinant $D_\mu$ be defined by

$$D_\mu = u_{11}u_{22} - \mu u_{12}u_{21} = u_{22}u_{11} - \mu^{-1} u_{12}u_{21}.$$

Then, $D_\mu^*D_\mu = D_\mu D_\mu^* = 1$. Moreover, $D_\mu$ belongs to the centre of $U_\mu(2)$.

By Lemma 4.1, we have identified the category $\hat{C}$ with the category of CQG having actions on $SU_\mu(2)$ satisfying 3. of Lemma 4.1. Let the universal object of this category be denoted by $(\hat{Q}, \Gamma)$.

Then by linearity we can write:

$$\Gamma(\alpha) = \alpha \otimes A + \gamma^* \otimes B$$

$$\Gamma(\gamma^*) = \alpha \otimes C + \gamma^* \otimes D$$

Now we exploit the homomorphism condition of $\Gamma$ to get relations between $A, B, C, D$.

**Lemma 4.4**

$$A^*A + CC^* = 1 \quad (11)$$

$$A^*A + \mu^2 CC^* = B^*B + DD^* \quad (12)$$

$$A^*B = -\mu CD^* \quad (13)$$

$$B^*A = -\mu CD^* \quad (14)$$
Proof:
The proof follows from the relation (3) by comparing coefficients of $1, \gamma^*\gamma, \alpha^*\gamma^*$ and $\alpha\gamma$ respectively. □

Lemma 4.5

\[
AA^* + \mu^2 CC^* = 1 \quad (15)
\]
\[
BB^* + \mu^2 DD^* = \mu^2 1 \quad (16)
\]
\[
BA^* = -\mu^2 DC^* \quad (17)
\]

Proof:
From the equation (3) by equating coefficients of 1 and $\alpha^*\gamma^*$, we get respectively (15) and (17) whereas (16) is obtained by equating coefficients of $\gamma^*\gamma$ and using (15).

Lemma 4.6

\[
C^*C = CC^* \quad (18)
\]
\[
(1 - \mu^2)C^*C = D^*D - DD^* \quad (19)
\]
\[
C^*D = \mu DC^* \quad (20)
\]

Proof:
The proof follows from the equation (5) from the coefficients of $1, \gamma^*\gamma, \alpha^*\gamma^*$, respectively. □

Lemma 4.7

\[-\mu^2 AC^* + BD^* - \mu D^*B + \mu C^*A = 0 \quad (21)\]
\[
AC^* = \mu C^*A \quad (22)
\]
\[
BC^* = C^*B \quad (23)
\]
\[
AD^* = D^*A \quad (24)
\]

Proof:
The proof follows from the equation (6) from the coefficients of $\gamma^*\gamma, 1, \alpha^*\gamma^*$ and $\alpha\gamma$ respectively. □

Lemma 4.8

\[
AC = \mu CA \quad (25)
\]
\[
BD = \mu DB \quad (26)
\]
\[
AD - \mu CB = DA - \mu^{-1}BC = 0 \quad (27)
\]
Proof:
The proof follows from (7) from the coefficients of $\alpha^2, \gamma^*\gamma$, $\gamma^*\alpha$ respectively.

Now we consider the antipode, say $\kappa$.

From the condition $(h \otimes id)\Gamma(a) = h(a).1$, we have that $\Gamma$ gives a unitary representation of the compact quantum group via $\hat{\Gamma}(a \otimes q) = \Gamma(a)(1 \otimes q)$.

Now, the restriction of this unitary representation to the orthonormal basis $\{\alpha, \mu^{-1}\gamma^*\}$ is given by the matrix:

$$
\begin{pmatrix}
A & \mu C \\
\mu^{-1}B & D
\end{pmatrix}.
$$

Similarly, with respect to the orthonormal set $\{\alpha^*, \gamma\}$, this representation is given by the matrix:

$$
\begin{pmatrix}
A^* & C^* \\
B^* & D^*
\end{pmatrix}.
$$

Thus, we have,

$$
\kappa(A) = A^*, \kappa(D) = D^*, \kappa(C) = \mu^{-2}B^*, \kappa(B) = \mu^2C^*, \kappa(A^*) = A, \kappa(C^*) = B, \kappa(B^*) = C, \kappa(D^*) = D.
$$

Now, we apply $\kappa$ to the above equations to get the following additional relations.

**Lemma 4.9**

\begin{align*}
AB &= \mu BA \\
CD &= \mu DC \\
BC^* &= C^*B
\end{align*}

**Proof:**

(28), (29), (30) follow by applying $\kappa$ to the equations (25), (26) and (23) respectively.

**Lemma 4.10** The map $\phi : U_\mu(2) \to \hat{Q}$ defined by $\phi(u_{11}) = A$, $\phi(u_{12}) = \mu C$, $\phi(u_{21}) = \mu^{-1}B$, $\phi(u_{22}) = D$ is a $*$ homomorphism.

**Proof:**

It is enough to check that the defining relations of $U_\mu(2)$ are satisfied.

1. $\phi(u_{11}u_{12}) = \phi(\mu u_{12}u_{11}) \Leftrightarrow \phi(u_{11})\phi(u_{12}) = \mu \phi(u_{12})\phi(u_{11}) \Leftrightarrow A(\mu C) = \mu(\mu C)A \Leftrightarrow AC = \mu CA$ which is the same as (25).

2. $\phi(u_{11}u_{21}) = \phi(\mu u_{21}u_{11}) \Leftrightarrow A(\mu^{-1}B) = \mu(\mu^{-1}B)A \Leftrightarrow AB = \mu BA$ which is the same as equation (28).

3. $\phi(u_{12}u_{22}) = \phi(\mu u_{22}u_{12}) \Leftrightarrow \mu CD = \mu D(\mu C) \Leftrightarrow CD = \mu DC$ which is the same as equation (29).
4. $\phi(u_{21}u_{22}) = \phi(\mu u_{22}u_{21}) \Leftrightarrow \mu^{-1}BD = \mu D\mu^{-1}B \Leftrightarrow BD = \mu DB$ which is the same as equation (26).
5. $\phi(u_{12}u_{21}) = \phi(u_{21}u_{12}) \Leftrightarrow \mu C\mu^{-1}B = \mu^{-1}B\mu C \Leftrightarrow CB = BC$.

Now, $BC^* = C^*B$ from equation (30). But from (18), $C$ is normal. hence $BC = CB$.
6. $\phi(u_{11}u_{22} - u_{22}u_{11}) = (\mu - \mu^{-1})\phi(u_{12}u_{21}) \Leftrightarrow AD - DA = (\mu - \mu^{-1})\mu C\mu^{-1}B$.

From (27), we have $AD - DA = \mu CB - \mu^{-1}BC$.

But $BC^* = C^*B$ from (30) and $C$ is normal from (18). Hence $BC = CB$. Hence $AD - DA = (\mu - \mu^{-1})CB$ holds.

Lemma 4.11 There is a $C^*$ action $\Psi$ of $U_\mu(2)$ on $SU_\mu(2)$ such that $(U_\mu(2), \Psi) \in \text{Obj}(\hat{C})$.

Proof: Define a $*$ homomorphism $\Psi$ on $SU_\mu(2)$ by

$$\Psi(\alpha) = \alpha \otimes u_{11} + \gamma^* \otimes \mu u_{21}$$

$$\Psi(\gamma^*) = \alpha \otimes \mu^{-1}u_{12} + \gamma^* \otimes u_{22}$$

The homomorphism conditions are exactly the conditions (11) - (27) with $A, B, C, D$ replaced by $u_{11}, \mu u_{21}, \mu^{-1}u_{12}$ and $u_{22}$ respectively. We check one of the relations and remark that the proof of the others are exactly similar. We prove (11) i.e, $u_{11}^*u_{11} + (\mu^{-1}u_{12})(\mu^{-1}u_{12})^* = 1$.

Using the fact that $D_\mu$ is a central element of $U_\mu(2)$, we have the left hand side $= u_{22}D_\mu^{-1}u_{11} + \mu^{-2}u_{12}(-\mu u_{21}D_\mu^{-1}) = (u_{22}u_{11} - \mu^{-1}u_{12}u_{21})D_\mu^{-1} = D_\mu D_\mu^{-1} = 1$ = right hand side.

Clearly, $\Psi$ keeps span $V_{1/2}^{-1/2}$ invariant and the corresponding representation is a unitary.

Hence, by Lemma 4.11 $(U_\mu(2), \Psi) \in \text{Obj}(\hat{C})$.

Corollary 4.12 There exists a surjective compact quantum group morphism from $\hat{Q}$ to $U_\mu(2)$ sending $A, \mu C, \mu^{-1}B$, and $D$ to $u_{11}, u_{12}, u_{21}$ and $u_{22}$ respectively.

Theorem 4.13 $\hat{Q} = U_\mu(2)$ and hence $\hat{Q}_{\text{ISO}}^+ (D) = U_\mu(2) * C(\mathbb{T})$
Proof:
Follows from Lemma 4.10 and the Corollary. The second part follows from Theorem 2.23.

4.2 The Podles Spheres

Let $SU_\mu(2)$ be as in the previous section.

We recall the definition of the Podles sphere as in [13].

Let $\mu \in (0, 1)$ and $t \in (0, 1]$. Let $[n] \equiv [n]_\mu = \frac{n^\mu - (n-1)^\mu}{\mu - (n-1)^{\mu-1}}$, $n \in \mathbb{R}$.

Let $S^2_{\mu,c}$ be the universal $C^*$ algebra generated by elements $x_{-1}, x_0, x_1$ satisfying the relations:

$$x_{-1}(x_0 - t) = \mu^2(x_0 - t)x_{-1}$$

$$x_1(x_0 - t) = \mu^{-2}(x_0 - t)x_1$$

$$-[2]x_{-1}x_1 + (\mu^2x_0 + t)(x_0 - t) = [2]^2(1 - t)$$

$$-[2]x_1x_{-1} + (\mu^{-2}x_0 + t)(x_0 - t) = [2]^2(1 - t)$$

where $c = t^{-1} - t, t > 0$.

The involution on $S^2_{\mu,c}$ is given by

$$x_{-1}^* = -\mu^{-1}x_1, \quad x_0^* = x_0$$

We note that $S^2_{\mu,c}$ as defined above is the same as $\chi_{\mu,\alpha',\beta}$ of [20] (page 124) with $q = \mu$, $\alpha' = t$, $\beta = t^2 + \mu^{-2}(\mu^2 + 1)^2(1 - t)$.

Thus, from the expressions of $x_{-1}, x_0, x_1$ given in Page 125 of [20], it follows that $S^2_{\mu,c}$ can be realized as a $\ast$ subalgebra of $SU_\mu(2)$ via:

$$x_{-1} = \frac{\mu\alpha^2 + \rho(1 + \mu^2)\alpha\gamma - \mu^2\gamma^2}{\mu(1 + \mu^2)^{\frac{1}{2}}}$$ (31)

$$x_0 = -\mu\gamma^*\alpha + \rho(1 - (1 + \mu^2)\gamma^*\gamma) - \gamma\alpha^*$$ (32)
where \( \rho^2 = \frac{\mu^2 t^2}{(\mu^2 + 1)^2(1-t)} \).

Setting

\[
A = \frac{1 - t^{-1} x_0}{1 + \mu^2}, \quad B = \mu(1 + \mu^2)^{-\frac{1}{2}} t^{-1} x_{-1}
\]

one obtains (13) that \( S_{\mu, c}^2 \) is the same as the Podles’ sphere as in [21], i.e., the universal C* algebra generated by elements \( A \) and \( B \) satisfying the relations:

\[
A^* = A, \quad AB = \mu^{-2} BA,
\]

\[
B^* B = A - A^2 + cI, \quad BB^* = \mu^2 A - \mu^4 A^2 + cI
\]

We now introduce the spectral triple on \( S_{\mu, c}^2 \) as in [13].

Let \( s = c \frac{1}{2} \lambda_+^{-1}, \quad \lambda_+ = \frac{1}{2} + (c + \frac{1}{2})^{\frac{1}{2}} \)

\( \forall j \in \frac{1}{2} \mathbb{N} \),

\[
\begin{align*}
    u_j &= (\alpha^* - s \gamma^*)(\alpha^* - \mu^{-1} s \gamma^*) \ldots (\alpha^* - \mu^{-2j+1} \gamma^*) \\
    w_j &= (\alpha - \mu s \gamma)(\alpha - \mu^2 s \gamma) \ldots (\alpha - \mu^{2j} s \gamma) \\
    u_{-j} &= E^{2j} \triangleright w_j \\
    u_0 &= w_0 = 1 \\
    y_1 &= (1 + \mu^{-2})^{\frac{1}{2}} (c^{\frac{1}{2}} \mu^2 \gamma^2 + \mu \gamma^* \alpha^* - \mu c^{\frac{1}{2}} \alpha^2) \\
    N_{kj}^l &= \| F^{l-k} \triangleright (y_1^{l-j} u_j) \|^{-1} \\
\end{align*}
\]

Define \( v^l_{k,j} = N_{k,j}^l \triangleright (y_1^{l-j} u_j), \quad l \in \frac{1}{2} \mathbb{N}_0, \quad j, k = -l, \ldots, l \).

Let \( \mathcal{M}_N \) be the Hilbert space with orthonormal basis \( \{ v^l_{m,N} : l = |N|, \quad |N| + 1, \ldots, m = -l, \ldots, l \} \).

Let

\[
\mathcal{H} = \mathcal{M}_{-\frac{1}{2}} \oplus \mathcal{M}_{\frac{1}{2}}.
\]

A representation \( \pi \) of \( S_{\mu, c}^2 \) on \( \mathcal{H} \) is defined by

\[
\pi(x_i) v^l_{m,N} = \alpha_i^-(l, m; N) v^{l-1}_{m+i,N} + \alpha_i^0(l, m; N) v^l_{m+i,N} + \alpha_i^+(l, m; N) v^{l+1}_{m+i,N}
\]

where \( \alpha_i^- \), \( \alpha_i^0 \), \( \alpha_i^+ \) are as defined in [13].

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Finally as in Proposition 7.2 of [13], the Dirac operator is defined by

\[ D(v_{m, \pm}^l) = (c_1 l + c_2) v_{m, \pm}^l \]

where \( c_1, c_2 \in \mathbb{R}, c_1 \neq 0 \).

Then \((S^2_{\mu, c}, \mathcal{H}, D)\) will be the spectral triple with which we are going to work.

In [4], we have shown the following:

**Theorem 4.14** Let the positive, unbounded operator \( R \) on \( \mathcal{H} \) be defined by

\[ R(v^n_{i, \pm \frac{1}{2}}) = \mu^{-2i}v^n_{i, \pm \frac{1}{2}}. \]

Then, \( \tau_R \) equals \( h \), i.e., the canonical Haar state on \( SU_{\mu}(2) \) and

\[ QISO^+ R(D) = SO_{\mu}(3). \]

**Remark 4.15** We have also worked with the spectral triple on the Podle’s sphere \( S^2_{\mu, 0} \) as in [30], and obtained the same result, i.e., identified \( QISO^+_{R(D)} \) of the spectral triple on \( S^2_{\mu, 0} \) with \( SO_{\mu}(3) \).

We have also worked in [4] with another class of spectral triple introduced in [7] for \( c > 0 \).

Let \( \mathcal{H}_+ = \mathcal{H}_- = l^2(N), \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

Let \( e_n \) be an orthonormal basis of \( \mathcal{H}_+ = \mathcal{H}_- \) and \( N \) be the operator defined on it by \( N(e_n) = ne_n \).

We recall the irreducible representations \( \pi_+ \) and \( \pi_- : \mathcal{H}_\pm \to \mathcal{H}_\pm \) as in [7].

\[ \pi_+ (A) = \lambda_\pm \mu^2 ne_n \]

\[ \pi_+ (B) = c_\pm (n)^{\frac{1}{2}} e_{n-1} \]

where \( \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{\frac{1}{2}}, c_\pm (n) = \lambda_\pm \mu^{2n} - (\lambda_\pm \mu^{2n})^2 + c \).

Let \( \pi = \pi_+ \oplus \pi_- \) and \( D = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} \).

Then \((S^2_{\mu, c}, \pi, \mathcal{H})\) is a spectral triple.

We showed in [4] that

**Theorem 4.16** \( QISO^+ \) of this spectral triple exists and is isomorphic with \( C(\mathbb{Z}_2) \ast C(\mathbb{T})^\infty \), where \( C(\mathbb{T})^\infty \) denotes the free product of countably infinitely many copies of \( C(\mathbb{T}) \).
Remark 4.17 The above example shows that unlike the classical case, where isometry groups are Lie groups and hence have faithful imbedding into a matrix group, QISO$^+$ in general may fail to be a compact matrix quantum group. In fact, it will be quite interesting to find conditions under which QISO$^+$ will be so.

4.3 A commutative example: spectral triple on $\mathbb{T}^2$

We consider the spectral triple $(A^\infty, \mathcal{H}, D)$ on $\mathbb{T}^2$ given by $A^\infty = C^\infty(\mathbb{T}^2)$, $\mathcal{H} = L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$ and $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$,

where we view $C(\mathbb{T}^2)$ as the universal $C^*$ algebra generated by two commuting unitaries $U$ and $V$, $d_1$ and $d_2$ are derivations on $A^\infty$ defined by:

\[ d_1(U) = U, d_1(V) = 0, d_2(U) = 0, d_2(V) = V. \] (34)

\[ e_1 = (1, 0) \text{ and } e_2 = (0, 1) \text{ form an orthonormal basis of the eigenspace corresponding to the eigenvalue zero.} \]

The Laplacian in the sense of [14] is given by $L(U^mV^n) = -(m^2 + n^2)U^mV^n$. We recall that we denote the quantum isometry group from the Laplacian $L$ in the sense of [14] by $Q_L^D$.

Lemma 4.18 Let $W$ be a unitary representation of a CQG $\tilde{Q}$ which commutes with $D$. Then the induced action on $C^\infty(\mathbb{T}^2)$, say $\alpha$, satisfies:

\[ \alpha(U) = U \otimes z_1 \] (35)

\[ \alpha(V) = V \otimes z_2 \] (36)

where $z_1, z_2$ are two commuting unitaries.

Proof:

We denote the maximal Woronowicz $C^*$ subalgebra of $\tilde{Q}$ that acts on $C(\mathbb{T}^2)$ faithfully by $Q$.

We observe that $D^2 \begin{pmatrix} ae_1 \\ ae_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}(a) & 0 \\ 0 & \mathcal{L}(a) \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$.

The fact that $U$ commutes with $D$ implies that $U$ commutes with $D^2$ as well, and hence $\alpha$ commutes with the Laplacian $L$. Therefore, $Q$ is a quantum subgroup of $Q_L^D$. From [3], we conclude that $Q_L^D = C(\mathbb{T}^2 \rtimes \mathbb{Z}_2^3)$

Thus $Q$ must be of the form $C(G)$ for a classical subgroup of the orientation preserving isometry group of $\mathbb{T}^2$, which is easily seen to be $\mathbb{T}^2$ itself and
whose (co)action is given by $U \mapsto U \otimes z_1'$ and $V \mapsto V \otimes z_2'$ where $z_1', z_2'$ are two unitaries generating $C(T^2)$. Hence the lemma follows. □

**Theorem 4.19** \( \widehat{QISO}^+ (C^\infty(T^2), \mathcal{H}, D) \) exists and is isomorphic with $C(T^2)^* C(T) \cong C^*(\mathbb{Z}^2 \ast \mathbb{Z})$ (as a CQG). Moreover, $QISO^+$ of this spectral triple is $C(T^2)$.

*Proof:* Let $W$ be as in the previous Lemma. Then we have,

$$W(e_1) = e_1 \otimes q_{11} + e_2 \otimes q_{12}$$

(37)

$$W(e_2) = e_1 \otimes q_{21} + e_2 \otimes q_{22}$$

(38)

By comparing coefficients of $Ue_1$ and $Ue_2$, in the both sides of the equality $(D \otimes id)W(Ue_1) = WDUe_1$, we have,

$$z_1 q_{12} = z_1 q_{21}$$

(39)

$$z_1 q_{11} = z_1 q_{22}$$

(40)

$z_1$ is a unitary implies that $q_{11} = q_{22}$ and $q_{12} = q_{21}$.

Similarly, from the relation $(D \otimes id)W(Ve_1) = WDVe_1$, we have $q_{12} = -q_{21}, q_{22} = q_{11}$.

By the above two sets of relations, we have:

$q_{12} = q_{21} = 0, q_{11} = q_{22} = q$ (say)

But the matrix

$$\begin{pmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{pmatrix}
$$

is a unitary in $M_2(\widehat{Q})$, so $q$ is a unitary.

Moreover, we note that $W(ae_i) = \alpha(a)W(e_i) \forall a \in C^\infty(T^2)$. Using the previous Lemma and the above observations, we deduce that any CQG which has a unitary representation commuting with the Dirac operator is a quantum subgroup of $C(T^2)^* C(T)$.

Moreover, $C(T^2) * C(T)$ has a unitary representation commuting with $D$, given by the formulae (35) - (38) taking $q_{12} = q_{21} = 0, q_{11} = q_{22} = q'$ where $q'$ is the generator of $C(T)$ and $z_1, z_2$ to be the generator of $C(T^2)$. This completes the proof. □
Remark 4.20 The canonical grading on \( C(\mathbb{T}^2) \) is given by the operator \((\text{id} \otimes \gamma)\) on \( L^2(\mathbb{T}^2 \otimes \mathbb{C}^2)\) where \( \gamma \) is the matrix \(
abla
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\).

The representation of \( C(\mathbb{T}^2) \ast C(\mathbb{T}) \) clearly commutes with the grading operator and hence is isomorphic with \( \widetilde{QISO}(C(\mathbb{T}^2), L^2(\mathbb{T}^2 \otimes \mathbb{C}^2), D, \gamma) \).

Remark 4.21 This example shows that the conditions of Theorem 2.23 are not necessary for the existence of \( \widetilde{QISO}^+ \).

4.4 Another class of commutative examples: the spheres

We consider the usual Dirac operator on the classical \( n \)-sphere \( S^n \). In fact, we shall first consider a slightly more general set-up as in the section 3.5, p.82-89 of [18], which we very briefly recall here. Let \( G \) be a compact Lie group, \( K \) a closed subgroup, and let \( M \) be the homogeneous space \( G/K \) with a \( G \)-invariant metric. The algebra \( C^\infty(M) \) is identified with the algebra (say \( A^\infty \)) of \( K \)-invariant functions in \( C^\infty(G) \), i.e. functions \( f \) satisfying \( f(gk) = f(g) \) for all \( g \in G, k \in K \). The Lie algebra \( g \) of \( G \) splits as a vector space direct sum \( g = k + m \) where \( k \) is the Lie algebra of \( K \) and \( m \) is suitable \( \text{Ad}(K) \)-invariant subspace of \( g \) (see [18] for more details). Thus we have the representation of \( K \) given by \( \text{Ad} : K \to \text{SO}(m) \), and the corresponding lift \( \widetilde{\text{Ad}} : K \to \text{Spin}(m) \). The space of smooth spinors can then be identified with the space of smooth functions \( \psi : G \to \Delta \) satisfying \( \psi(gk) = \kappa \widetilde{\text{Ad}}(k^{-1}) \psi(g) \) for all \( g \in G, k \in K \), where \( \kappa : \text{Spin}(m) \to \text{GL}(\Delta) \) denotes the spin representation. The action of \( C^\infty(M) \), identified with the \( K \)-invariant smooth functions on \( G \), is given by multiplication, and the Dirac operator \( D \) is given by

\[
D\psi = \sum_{i=1}^{m} X_i \cdot X_i(\psi),
\]

where \( m = \dim(m) \) and \( \{X_1, \ldots, X_m\} \) is an orthonormal basis of \( m \) with respect to the suitable invariant inner product described in [18] and \( \cdot \) denotes the Clifford multiplication. From this expression of \( D \), it follows by using the fact that \( X_i \)'s are acting as derivations on the algebra of smooth functions, that \([D, f] \psi = \omega_f \cdot \psi\), where \( \omega_f = \sum_i (X_i f) X_i \). In fact, the space \( \Omega^1_D \), which is isomorphic with the (complexified) space of smooth 1-forms on \( M \), can now be identified with the space of (smooth) \( \text{Ad}_K \) invariant functions from \( G \) to \( m \cong \mathbb{C}^m \) (which is also isomorphic with \( C^\infty(M) \otimes \mathbb{C}^m \)), and the map \( d_D \) is, w.r.t. this identification, is nothing but the map which sends
f \in A^\infty \cong C^\infty(M) \text{ to } \sum_i X_i(f) \otimes X_i \in A^\infty \otimes \mathbb{C}^m. \text{ The Hilbert space of 1-forms is isomorphic with } L^2(M) \otimes \mathbb{C}^m, \text{ where } L^2(M) \text{ is the Hilbert space completion w.r.t. the } G\text{-invariant volume measure, and from the } G\text{-invariance of the volume measure it is clear that the adjoint } X_i^* \text{ of the (left invariant ) vector field } X_i \text{ (viewed as a closable unbounded map on } L^2(M)) \text{ is } -X_i. \text{ It follows that the Laplacian is given by, } d^*DdD = -\sum_{i=1}^{m} X_i^* X_i \text{ on } A^\infty \cong C^\infty(M). \text{ It is in fact nothing but the Casimir } \Omega_G \text{ in the notation of [18], since } Yf = 0 \text{ for any } Y \in \mathfrak{k}.

Now, we want to apply the above observations to the special case of } n\text{-spheres. The Laplacian on such spheres considered in [23] (Page 17) is indeed the Casimir operator and so by Theorem 2.2 and Remark 3.3 of [3] the corresponding quantum isometry group } QISO^c \text{ is commutative as a } C^* \text{ algebra. However, by Corollary 3.2 of the present paper, any object of the category } Q^c \text{ must be a quantum subgroup of } Q^c, \text{ so is in particular commutative as a } C^* \text{ algebra, so must be of the form } C(G) \text{ for a subgroup } G \text{ of the universal group of orientation preserving (classical) Riemannian isometries of } S^n, \text{ i.e. } SO(n + 1). \text{ To summarise, we have the following:}

**Theorem 4.22** The quantum isometry group } QISO^+_I(S^n) \text{ is isomorphic with } C(SO(n + 1)), \text{ i.e. coincides with the corresponding classical group.}

## 5 } QISO^+ \text{ of deformed spectral triples}

In this section, we give a general scheme for computing orientation-preserving quantum isometry groups by proving that } \widetilde{QISO}^+_R \text{ of a deformed noncommutative manifold coincides with (under reasonable assumptions) a similar deformation of the } \widetilde{QISO}^+_R \text{ of the original manifold. The technique is very similar to the analogous result for the quantum isometry groups in terms of Laplacian discussed in [22], so we often merely sketch the arguments and refer to a similar theorem or lemma in [3].}

We recall the generalities on compact quantum groups from Section 2.2. In particular, given a compact quantum group (S, Δ), the dense unital *-subalgebra } S_0 \text{ of } S \text{ generated by the matrix coefficients of the irreducible unitary representations has a canonical Hopf *-algebra structure. Moreover, given an action } \gamma : B \to B \otimes S \text{ of the compact quantum group (S, Δ) on a unital } C^* \text{-algebra } B, \text{ it is known that one can find a dense, unital *-subalgebra } B_0 \text{ of } B \text{ on which the action becomes an action by the Hopf *-algebra } S_0. \text{ We shall use the Sweedler convention of abbreviating } \gamma(b) \in B_0 \otimes_{\text{alg}} S_0 \text{ by } b_(1) \otimes b_(2), \text{ for } b \in B_0. \text{ This applies in particular to the canonical}
action of the quantum group $S$ on itself, by taking $\gamma = \Delta$. Moreover, for a linear functional $f$ on $S$ and an element $c \in S$ we shall define the ‘convolution’ maps $f \circ c := (f \otimes \text{id})\Delta(c)$ and $c \circ f := (\text{id} \otimes f)\Delta(c)$. We also define convolution of two functionals $f$ and $g$ by $(f \circ g)(c) = (f \otimes g)(\Delta(c))$.

We also need the following:

**Definition 5.1** Let $(S, \Delta_S)$ be a compact quantum group. A vector space $M$ is said to be an algebraic $S$ co-module (or $S$ co-module) if there exists a linear map $\tilde{\alpha} : M \rightarrow M \otimes_{\text{alg}} S_0$ such that

1. $(\tilde{\alpha} \otimes \text{id})\tilde{\alpha} = (\text{id} \otimes \Delta_S)\tilde{\alpha}$
2. $(\text{id} \otimes \epsilon)\tilde{\alpha}(m) = \epsilon(m)1_S \forall m \in M$.

Let $(A, \mathbb{T}^n, \beta)$ be a $C^*$ dynamical system and $\pi_0 : A \rightarrow B(H)$ be a faithful representation, where $H$ is a separable Hilbert space.

Let $A^\infty$ be the smooth algebra corresponding to the $\mathbb{T}^n$ action $\beta$. Then for each skew-symmetric $n \times n$ real matrix $J$, we refer to [25] for the construction of the ‘deformed’ $C^*$-algebra $A_J$ and their properties.

Suppose, furthermore, that there exists a compact abelian lie group $\tilde{T}^n$, with a covering map $\gamma : \tilde{T}^n \rightarrow \mathbb{T}^n$. The Lie algebra of both $\mathbb{T}^n$ and $\tilde{T}^n$ are isomorphic with $\mathbb{R}^n$ and we denote by $e$ and $\tilde{e}$ respectively the corresponding exponential maps, so that $e(u) = e(2\pi i u), u \in \mathbb{R}^n$ and $\gamma(\tilde{e}(u)) = e(u)$. By a slight abuse of notation we shall denote the $\mathbb{R}^n$-action $\beta_{\tilde{e}(u)}$ by $\beta_u$.

We also make the following assumption:

There exists a strongly continuous unitary representation $V_{\tilde{g}}, \tilde{g} \in \tilde{T}^n$ of $\tilde{T}^n$ on $H$ such that

(a) $V_{\tilde{g}}D = DV_{\tilde{g}} \forall \tilde{g}$,

(b) $V_{\tilde{g}}\pi_0(a)V_{\tilde{g}}^{-1} = \pi_0(\beta_{\tilde{g}}(a))$, where $a \in A, \tilde{g} \in \tilde{T}^n$, and $g = \gamma(\tilde{g})$.

We shall now show that we can ‘deform’ the given spectral triple along the lines of [10]. For each $J$, the map $\pi_J : A^\infty \rightarrow \text{Lin}(H^\infty)$ (where $H^\infty$ is the smooth subspace corresponding to the representation $V$ and $\text{Lin}(V)$ denotes the space of linear maps on a vector space $V$) defined by

$$\pi_J(a)s \equiv a \times_J s := \int \int \beta_{Ju}(a)\tilde{\beta}_v(s)e(u,v)dudv$$
extends to a faithful ∗-representation of the C∗-algebra A∞ in B(H) where 
\( \tilde{\beta}_v = V_{\tilde{\varepsilon}(v)} \) (which clearly maps \( H^\infty \) into \( H^\infty \)).

We can extend the action of \( \mathbb{T}^n \) on the C∗ subalgebra \( A_1 \) of \( B(H) \) generated by \( \pi_0(A) \), \( \{ e^{itD} : t \in \mathbb{R} \} \) and elements of the form \( \{ [D, a] : a \in A^\infty \} \) by \( \beta_g(X) = V_g X V_g^{-1} \) \( \forall X \in A_1 \) where by an abuse of notation, we denote the action by the same symbol \( \beta \). Let \( A_1^\infty \) denote the smooth vectors of \( A \) with respect to this action. We note that \( \forall a \in A_1^\infty, [D, a] \in A_1^\infty \).

**Lemma 5.2** \( \beta \) is a strongly continuous action (in the C∗-sense) of \( \mathbb{T}^n \) on \( A_1 \) and hence \( \forall X \in A_1^\infty, \pi_f(X) \) defined by \( \pi_f(X)s = \int \beta_{Ju}(X)\tilde{\beta}_v(s)e(u.v)dudv \) is a bounded operator.

**Proof:**
We note that \( \beta \) is already strongly continuous on the C∗ algebra generated by \( \pi_0(A) \), \( \{ e^{itD} : t \in \mathbb{R} \} \). Thus it suffices to check the statement for elements of the form \( [D, a] \) where \( a \in A^\infty \).

To this end, fix any one parameter subgroup \( g_t \) of \( G \) such that \( g_t \) goes to the identity of \( G \) as \( t \to 0 \). Let \( T'_t, T_t \) denote the group of normal ∗-automorphisms on \( B(H) \) defined by \( T'_t(X) = V_{g_t} X V_{g_t}^{-1} \) and \( T_t(X) = e^{itD} X e^{-itD} \). As \( V_{g_t} \) and \( D \) commute, so do their generators. In particular, each of these generators leave the domain of the other invariant. Note also that \( A^\infty \) is in the domain of the both the generators, and the generator of \( T_t \) is given by \( [D, \cdot] \) there. Thus, for \( a \in A^\infty \), we have \( [D, a] \in \text{Dom}(\Xi) \) (where \( \Xi \) is the generator of \( T_t \)), and \( \Xi([D, a]) = [D, \Xi(a)] \in B(H) \).

Using this, we obtain
\[
\|T'_t([D, a]) - [D, a]\| = \int_0^t T'_s(\Xi([D, a])) ds \leq t \|\Xi([D, a])\|.
\]

The required strong continuity follows from this. Then applying Theorem 4.6 of [25] to the C∗ algebra \( A_1 \) and the action \( \beta \), we deduce that \( \pi_f(X) \) is a bounded operator. □

**Lemma 5.3** For each \( J \), \( (A^\infty_J, \pi_J, H, D) \) is a spectral triple, i.e, \( [D, \pi_J(a)] = \mathcal{B}(H) \ \forall \ a \in A^\infty_J \).

**Proof:**
\[
[D, \pi_J(a)](s) = D \int f \beta_{Ju}(a)\tilde{\beta}_v(s)e(u.v)dudv - \int \beta_{Ju}(a)\tilde{\beta}_v(Ds)e(u.v)dudv.
\]

Using the expression
\[
\int f(u)g(v)e(u.v) = \lim_{L} \sum_{L}(f \psi_u)(p)\tilde{g}(p) \quad (\text{where notations are as in [25], Page 20})
\]

and closability of \( D \), we have
\[
D \int \beta_{Ju}(a)\tilde{\beta}_v(s)e(u.v)dudv = \int D(\beta_{Ju}(a)\tilde{\beta}_v(s))e(u.v)dudv.
\]

Thus, the above expression equals
\[
\int D(\beta_{Ju}(a)\tilde{\beta}_v(s))e(u.v)dudv - \int \beta_{Ju}(a)\tilde{\beta}_v(Ds)e(u.v)dudv
\]

as \( D \) commutes with \( V \).

So we have
\[
[D, \pi_J(a)](s) = \int \int [D, \beta_u(a)](s) e(u.v) dudv = \int \int V_{f_u}^{-1}(a) V_{f_u}(s) e(u.v) dudv = \pi_J(D, a)
\]
which is a bounded operator by Lemma 5.2. \[\square\]

**Lemma 5.4** Suppose that \((\tilde{Q}, U) \in \text{Obj}(Q(A, H, D))\), and there exists a unital \(*\)-subalgebra \(A_0 \subseteq A\) which is norm dense in every \(A_J\) such that \(\alpha_U(\pi_0(A_0)) \subseteq \pi_0(A_0) \otimes_{\text{alg}} Q_0\), where \(Q \subseteq \tilde{Q}\) is the smallest Woronowicz \(C^*\) subalgebra such that \(\alpha(A_0) \subseteq \pi_0(A_0) \otimes \tilde{Q}\), and \(Q_0\) is the Hopf \(*\)-algebra obtained by matrix coefficients of irreducible unitary \((\co\)-)representations of \(Q\). Also, let \(S_0 = \text{span}\{a s : a \in A_0, s \in \mathcal{S}_{00}\}\). Then we have the following:

(a) \(U(S_0) \subseteq S_0 \otimes_{\text{alg}} \tilde{Q}_0\).

(b) \(\tilde{\alpha} := U|_{S_0} : S_0 \to S_0 \otimes_{\text{alg}} \tilde{Q}_0\) makes \(S_0\) an algebraic \(\tilde{Q}_0\) co-module, satisfying

\[
\tilde{\alpha}(\pi_0(a)s) = \alpha_U(a)\tilde{\alpha}(s) \quad \forall a \in A_0, s \in S_0.
\]

Moreover, if \(C(\tilde{T}^n)\) is a sub object of \(\tilde{Q}\) in \(Q(A, H, D)\), then \(C(\tilde{T}^n)\) is a quantum subgroup of \(Q\).

**Proof:**

\(U\) commutes with \(D\) and hence preserves the eigenspaces of \(D\) which shows that \(U\) preserves \(S_{00}\). Then, \(U(\alpha s) = \alpha(a) U(s) \subseteq (A_0 \otimes Q_0)(S_{00} \otimes \tilde{Q}_0) \subseteq S_0 \otimes Q_0\). Thus, the first assertion follows.

The second assertion follows from the definition of \(\tilde{\alpha}\) and \(\alpha_u\).

We now prove the third assertion. Let us denote by \(\gamma^*\) the dual map of \(\gamma\), so that \(\gamma^* : C(\tilde{T}^n) \to C(\tilde{T}^n)\) is an injective \(C^*\)-homomorphism.

It is quite clear that \((\text{id} \otimes \pi_{\tilde{Q}}) \circ \alpha(A_0) \subseteq \text{Im}(\text{id} \otimes \gamma^*)\), hence it follows that \(\pi_{\tilde{Q}}(Q_0) \subseteq \text{Im}(\gamma^*)\). Thus, \(\pi_{\tilde{Q}} := (\gamma^*)^{-1} \circ \pi_{\tilde{Q}}\) is a surjective CQG morphism from \(Q\) to \(C(\tilde{T}^n)\), which identifies \(C(\tilde{T}^n)\) as a quantum subgroup of \(Q\). \[\square\]

**Remark 5.5** From the definitions of \(A_0\) and \(S_0\), it follows that

(i) \(\pi_0(A_0)S_0 \subseteq S_0\),

(ii) \(\beta_g(A_0) \subseteq A_0 \forall g\).

Let us now fix the object \((\tilde{Q}, U)\) as in the statement of Lemma 5.4. From now on, we will identify \(\mathcal{A}_J^\infty\) with \(\pi_J(A^\infty)\) and often write \(\pi_0(a)\) simply as \(a\).

We define \(\Omega(u) := ev_{\tilde{\tau}(u)} \circ \pi_{\tilde{Q}}, \tilde{\Omega}(u) := ev_{\tilde{\tau}(u)} \circ \pi_{\tilde{Q}}\), for \(u \in R^n\), where \(ev_x\) (respectively \(ev_{\tilde{x}}\)) denotes the state on \(C(\tilde{T}^n)\) (respectively, on \(C(\tilde{T}^n)\)) obtained by evaluation of a function at the point \(x\) (respectively \(\tilde{x}\)).
For a fixed $J$, we shall work with several multiplications on the vector space $A_0 \otimes_{\text{alg}} Q_0$. We shall denote the counit and antipode of $Q_0$ by $\epsilon$ and $\kappa$ respectively. Let us define the following

$$x \circ y = \int_{\mathbb{R}^{4n}} e(-u.v)e(w.s)(\Omega(-Ju) \circ x \circ (\Omega(Jw))(\Omega(-v) \circ y \circ \Omega(s))dudvdwds,$$

where $x, y \in Q_0$. This is clearly a bilinear map, and will be seen to be an associative multiplication later on. Moreover, we define two bilinear maps $\bullet$ and $\bullet_J$ by setting $(a \otimes x) \bullet (b \otimes y) := ab \otimes x \circ y$ and $(a \otimes x) \bullet_J (b \otimes y) := (a \times_J b) \otimes (x \circ y)$, for $a, b \in A_0$, $x, y \in Q_0$. We have $\Omega(u) \circ (\Omega(v) \circ c) = (\Omega(u) \circ \Omega(v)) \circ c$.

**Lemma 5.6**

1. The map $\circ$ satisfies

$$\int_{\mathbb{R}^{2n}} (\Omega(Ju) \circ x) \circ (\Omega(v) \circ y) e(u.v)dudv = \int_{\mathbb{R}^{2n}} (x \circ (\Omega(Ju))(y \circ \Omega(v))e(u.v)dudv,$$

for $x, y \in Q_0$.

2. $\overline{\alpha}(\overline{\beta}_u(s)) = s_{(1)} \otimes (id \otimes \Omega(u))\overline{\Delta}(s_{(2)}).$

3. For $s \in S$, $a \in \widetilde{Q}_0$, we have

$$\overline{\alpha}(a \times_J s) = a_{(1)}s_{(1)} \otimes (\int \int (a_{(2)} \circ J u)(s_{(2)} \circ v)e(u.v)dudv).$$

4. For $s \in S_0, a \in A_0,$

$$\alpha(a) \bullet_J \overline{\alpha}(s) = a_{(1)}s_{(1)} \otimes \{ \int \int (\Omega(Ju) \circ a_{(2)}) \circ (\Omega(v) \circ s_{(2)})e(u.v)dudv \}. \

5. For $a \in A_0$, $s \in S$ we have $\alpha(a) \bullet_J \alpha(s) = \overline{\alpha}(a \times_J s)$.

**Proof:**

The proofs follow verbatim those in Lemmas 3.2 - 3.6 respectively in [3]. □

Let us recall at this point the Rieffel-type deformation of compact quantum groups as in [34]. We shall now identify $\circ$ with the multiplication of a Rieffel-type deformation of $Q$. Since $Q$ has a quantum subgroup isomorphic
with $T^n$, we can consider the following canonical action $\lambda$ of $\mathbb{R}^{2n}$ on $Q$ given by

$$\lambda_{(s,u)} = (\Omega(-s) \otimes id) \Delta(id \otimes \Omega(u)) \Delta.$$ 

Now, let $\bar{J} := -J \oplus J$, which is a skew-symmetric $2n \times 2n$ real matrix, so one can deform $Q$ by defining the product of $x$ and $y$ ($x, y \in Q_0$, say) to be the following:

$$\int \int \lambda_{\bar{J}(u,w)}(x) \lambda_{\bar{J}(v,s)}(y) e((u,w),(v,s)) d(u,w)d(v,s).$$

We claim that this is nothing but $\circ$ introduced before.

**Lemma 5.7** $x \circ y = x \times_{\bar{J}} y \quad \forall x, y \in Q_0$

*Proof:* The proof is the same as Lemma 3.7 in [3]. $\square$

Let us denote by $Q_{\bar{J}}$ the $C^*$ algebra obtained from $Q$ by the Rieffel deformation w.r.t. the matrix $\bar{J}$ described above. It has been shown in [34] that the coproduct $\Delta$ on $Q_0$ extends to a coproduct for the deformed algebra as well and $(Q_{\bar{J}}, \Delta)$ is a compact quantum group.

We recall Lemma 3.8 of [3], which is stated below for reader’s convenience:

**Lemma 5.8** The Haar state (say $h$) of $Q$ coincides with the Haar state on $Q_{\bar{J}}$ (say $h_{\bar{J}}$) on the common subspace $Q^\infty$, and moreover, $h(a \times_{\bar{J}} b) = h(ab)$ for $a, b \in Q^\infty$.

We note a useful implication of the above lemma. Let us make use of the identification of $Q_0$ as a common vector-subspace of all $Q_{\bar{J}}$. To be precise, we shall sometimes denote this identification map from $Q_0$ to $Q_{\bar{J}}$ by $\rho_{\bar{J}}$.

**Corollary 5.9** Let $W$ be a finite-dimensional (say, $n$-dimensional) unitary representation of $Q$, with $\tilde{W} \in M_n(\mathbb{C}) \otimes Q_0$ be the corresponding unitary. Then, for any $\bar{J}$, we have that $\tilde{W}_{\bar{J}} := (id \otimes \rho_{\bar{J}})(\tilde{W})$ is unitary in $Q_{\bar{J}}$, giving a unitary $n$-dimensional representation of $Q_{\bar{J}}$. In other words, any finite dimensional unitary representation of $Q$ is also a unitary representation of $Q_{\bar{J}}$.

*Proof:* Since the coalgebra structures of $Q$ and $Q_{\bar{J}}$ are identical, and $\tilde{W}_{\bar{J}}$ is identical with $\tilde{W}$ as a linear map, it is obvious that $\tilde{W}_{\bar{J}}$ gives a nondegenerate
representation of $Q_\gamma$. Let $y = (\text{id} \otimes h)(\overline{W}_J^* \overline{W}_J)$. It follows from the proof of Proposition 6.4 of [29] that $y$ is invertible positive element of $M_n$ and $(y^{\frac{1}{2}} \otimes 1)\overline{W}_J(y^{-\frac{1}{2}} \otimes 1)$ gives a unitary representation of $Q_\gamma$. We claim that $y = 1$, which will complete the proof of the corollary. For convenience, let us write $W$ in the Sweedler notation: $W = w(1) \otimes w(2)$. We note that by Lemma 5.8, we have $(\text{id} \otimes h)(\overline{W}_J^* \overline{W}_J) = (\text{id} \otimes h)(1 \otimes 1) = 1$.

Let us consider the finite dimensional unitary representations $U_i := U_i|_{V_i}$, where $V_i$ is the eigenspace of $D$ corresponding to the eigenvalue $\lambda_i$. By the above Corollary 5.9, we can view $U_i$ as a unitary representation of $Q_\gamma$ as well, and let us denote it by $U_i$. In this way, we obtain a unitary representation $U_J$ on the Hilbert space $\mathcal{H}$, which is the closed linear span of all the $V_i$’s. It is obvious from the construction (and the fact that the linear span of $V_i$’s, i.e. $S_0$, is a core for $D$) that $U_J D = (D \otimes I)U_J$. Let $\alpha_J := \alpha_{U_J}$. With this, we have the following:

**Lemma 5.10** For $a \in A_0$, we have $\alpha_J(a) = (\alpha(a))_J = (\pi_J \otimes \rho_J)(\alpha(a))$, hence in particular, for every state $\phi$ on $Q_\gamma$, $(\text{id} \otimes \phi) \circ \alpha_J(A_J) \subseteq A''_J$.

Using Lemma 5.6 we have, $\forall s \in S_0, a \in A_0$,

\[
U_J(\pi_J(a)s) = \overline{\alpha}(a \times_J s) = \alpha(a) \bullet_J \overline{a}(s) = (\alpha(a))_J U_J(s),
\]

from which we conclude by the density of $S_0$ in $\mathcal{H}$ that $\alpha_J(a) = (\alpha(a))_J \in \pi_J(A_0) \otimes Q_\gamma$. The lemma now follows using the norm-density of $A_0$ in $A_J$.

Thus, $(\overline{Q}_\gamma, U_J)$ is an orientation preserving isometric action of the spectral triple $(A_J, \mathcal{H}, D)$.

We shall now show that if we fix a ‘volume-form’ in terms of an $R$-twisted structure, then the ‘deformed’ action $\alpha_J$ preserves it.

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Lemma 5.11 Suppose, in addition to the set-up already assumed, that there is an invertible positive operator $R$ on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is an $R$-twisted $\Theta$ summable spectral triple, and let $\tau_R$ be the corresponding ‘volume form’. Assume that $\alpha_U$ preserves the functional $\tau_R$. Then the action $\alpha_U$ preserves $\tau_R$ too.

Proof:
Let the (finite dimensional) eigenspace corresponding to the eigenvalue $\lambda_n$ of $D$ be $V_n$. As $U$ commutes with $D$, there exists subspaces $V_{n,k}$ of $V_n$ and an orthonormal basis $\{e_{n,k}^j\}_j$ for $V_{n,k}$ such that the restriction of $U$ to $V_{n,k}$ is irreducible. Write $\tilde{U}(e_{n,k}^j \otimes 1) = \sum_i e_{i,k}^j \otimes t_{i,j}^n$. Then, $\tilde{U}^*(e_{n,k}^j) = \sum_i e_{i,k}^j \otimes t_{i,j}^n$.

Then $\mathcal{H}$ will be decomposed as $\mathcal{H} = \oplus_{n \geq 1} V_{n,k}$.

Let $R(e_{n,k}^j) = \sum_{s,t} F_n(i,j,s,t) e_{s,t}^n$.

By hypothesis, $\tilde{U}.(\otimes id)\tilde{U}^*$ preserves the functional $\tau_R(\cdot) = Tr(R(\cdot))$ on $\mathcal{E}_D$ where $\mathcal{E}_D$ is as in Proposition 2.1. i.e., the weakly dense $*$ subalgebra of $B(\mathcal{H})$ generated by the rank one operators $|\xi><\eta|$ where $\xi, \eta$ are eigenvectors of $D$. Thus, $(\tau_R \otimes id)(\tilde{U}(X \otimes id)\tilde{U}^*) = \tau_R(X).1_{\mathcal{Q}} \forall X \in \mathcal{E}_D$.

Then, for $a \in \mathcal{E}_D$, we have:

$$(\tau_R \otimes h)(\tilde{U}_J(a \otimes 1)\tilde{U}_J^*)$$ 

$$= \sum_{n,i,j} \langle e_{n,i}^j \otimes 1, \tilde{U}_J(a \otimes 1)\tilde{U}_J^*(R e_{n,i}^j \otimes 1) \rangle$$ 

$$= \sum_{n,i,j,s,t} \langle \tilde{U}_J^*(e_{n,i}^j \otimes 1), (a \otimes 1)\tilde{U}_J^*(F_n(i,j,s,t)e_{s,t}^n \otimes 1) \rangle$$ 

$$= \sum_{n,i,j,s,t,k,l} F_n(i,j,s,t) \langle e_{k}^{n,i} \otimes (t_{j,k}^n)^*, (a \otimes 1)(e_{l}^{n,s} \otimes (t_{j,l}^n)^*) \rangle$$ 

$$= \sum_{n,i,j,s,t,k,l} F_n(i,j,s,t) \langle e_{k}^{n,i}, a e_{l}^{n,s} \rangle h_J((t_{j,k}^n)^* \times_J (t_{j,l}^n)^*)$$ 

$$= \sum_{n,i,j,s,t,k,l} F_n(i,j,s,t) \langle e_{k}^{n,i}, a e_{l}^{n,s} \rangle h_0(t_{j,k}^n t_{j,l}^n)$$ 

$$= (\tau_R \otimes h)(\tilde{U}_J(a \otimes 1)\tilde{U}_J^*)$$ 

$$= \tau_R(a).1$$

where $h_J((t_{j,k}^n)^* \times_J (t_{j,l}^n)^*) = h_0(t_{j,k}^n t_{j,l}^n)$ as deduced by using Lemma 5.8.

Thus $(\tau_R \otimes h)\tilde{U}_J(a \otimes id)\tilde{U}_J^* = \tau_R(a).1$

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Let \((\tau_R \otimes h)\widetilde{U}_J(X \otimes \text{id})\widetilde{U}_J^* = (\tau_R * h)(X)\). As \(\widetilde{U}_J(\cdot \otimes \text{id})\widetilde{U}_J^*\) keeps \(\mathcal{E}_D\) invariant, we can use Sweedler notation: \(\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^* = a_{(1)} \otimes a_{(2)}\), with \(a, a_{(1)} \in \mathcal{E}_D, \ a_{(2)} \in \tilde{Q}_J\), to have

\[
(\tau_R \otimes \text{id})(\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^*)
= (\tau_R * h \otimes \text{id})(\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^*) = \tau_R * h(a_{(1)})a_{(2)}
\]

\[= (\tau_R \otimes h \otimes \text{id})(a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}) = (\tau_R \otimes h \otimes \text{id})(\text{id} \otimes \Delta_J)(\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^*)
= \tau_R(a_{(1)})((h \otimes \text{id}) \circ \Delta_J)(a_{(2)}) = \tau_R(a_{(1)})h(a_{(2)})1_{Q_J}
= (\tau_R \otimes h)(a_{(1)} \otimes a_{(2)}) = (\tau_R * h)(a).1_{Q_J} = \tau_R(1_{Q_J})
\]

\[\square\]

**Remark 5.12** If \(Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D)\) (\(Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D)\), if it exists) has a \(C^*\) action, then from the definition of a \(C^*\) action, we get a subalgebra \(A_0\) as in Lemma 5.4. Thus, the conclusions of Lemma 5.4 and the subsequent Lemmas hold for \(Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D)\) (\(Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D)\)).

We have already seen that \((\tilde{Q}_J, U_J)\) is an object in \(Q(A_J, \mathcal{H}, D)\). Now, proceeding as in the proof of Theorem 3.13 of [9] we obtain the following result (using Lemma 5.11 for 1).

**Theorem 5.13**

1. If \(Q \text{ISO}_R^+(A_J^\infty, \mathcal{H}, D)\) and \((Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D))_J\) have \(C^*\) actions on \(A\) and \(A_J\) respectively, we have

\[
\widetilde{Q \text{ISO}_R^+(A_J^\infty, \mathcal{H}, D)} \cong (Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D))_J
\]

\[
Q \text{ISO}_R^+(A_J^\infty, \mathcal{H}, D) \cong (Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D))_J.
\]

2. If moreover, \(\widetilde{Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D)}\) and \(Q \text{ISO}_R^+(A_J^\infty, \mathcal{H}, D)\) both exist and have \(C^*\) actions on \(A\) and \(A_J\) respectively, then

\[
\widetilde{Q \text{ISO}_R^+(A_J^\infty, \mathcal{H}, D)} \cong \left(\widetilde{Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D)}\right)_J,
\]

\[
Q \text{ISO}_R^+(A_J^\infty, \mathcal{H}, D) \cong (Q \text{ISO}_R^+(A^\infty, \mathcal{H}, D))_J.
\]

As an example, we consider the noncommutative torus \(A_\theta\), which is a Rieffel deformation of \(C(T^2)\) with respect to the matrix 

\[
J = \begin{pmatrix}
0 & \theta \\
-\theta & 0
\end{pmatrix}
\]
and we deform the spectral triple as in subsection 4. This is the standard spectral triple on $A_\theta$.

**Theorem 5.14** \( \widetilde{QISO}^+(A^\infty_\theta, \mathcal{H}, D) = \widetilde{QISO}^+(C^\infty(T^2)) = C(T^2) \ast C(T) \)

and \( QISO^+(A^\infty_\theta) = QISO^+(C^\infty(T^2)) = C(T^2) \).

**Proof:**

We use Theorem 5.13 and recall that \( QISO^+(C^\infty(T^2)) = C(T^2) \) which is generated by \( z_1 \) and \( z_2 \), say.

Then, from the formula of the deformed product, it can easily be seen after a change of variable that \( z_1 \times_{\tilde{J}} z_2 = z_2 \times_{\tilde{J}} z_1 \) which proves the theorem.

\( \square \)

**Remark 5.15** In a private communication S. Wang has kindly pointed out that one can possibly formulate and prove an analogue of Theorem 5.13 in the setting of discrete deformation as in [36], and this may give a solution to a problem posed by Connes (see [8], page 612). We believe that more work is needed in this direction.

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