Exponent dependence measures of survival functions and correlated frailty models

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Abstract

The present article studies survival analytic aspects of semiparametric copula dependence models with arbitrary univariate marginals. The underlying survival functions admit a representation via exponent measures which have an interpretation within the context of hazard functions. In particular, correlated frailty survival models are linked to copulas. Additionally, the relation to exponent measures of minimum-infinitely divisible distributions as well as to the Lévy measure of the Lévy-Khintchine formula is pointed out. The semiparametric character of the current analyses and the construction of survival times with dependencies of higher order are carried out in detail. Many examples including graphics give multifarious illustrations.

Keywords:
Copula, correlated frailty model, survival analysis, multi-dimensional hazard function, dependence measure, infinite divisibility

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1. Introduction

Multivariate semiparametric dependence models have been successfully developed during the past decades. We refer to the huge amount of copula literature; see Joe (1997) and Nelsen (2006) among others. It is well-known that Sklar’s (1959) theorem allows to separate a multivariate distribution in a copula dependence part and marginal distribution functions which are often considered as nuisance parameters. On the other hand, failure rates and hazard functions are very meaningful in survival analysis for dependent life time data. The famous Cox regression model specifies the dependence via exponential hazard dependence models with baseline hazard nuisance parameters, cf. Cox (1972). The hazard-based Cox models were extended by correlated frailty models which allow hazard dependence including flexible individual effects for life times; see Duchateau and Janssen (2008), Hougaard (2000), Wienke (2011) and Aalen et al. (2008). The shared frailty model appears as a special case whose connection to Archimedean copulas was pointed out by Kimberling (1974), Marshall and Olkin (1988), Oakes (1989) and McNeil and Nešlehová (2009); see also Genest and McKay (1986), Duchateau and Janssen (2008) and Völker (2010) for further discussions.

It is the aim of this paper to study the multivariate dependence structure from the survival analysis point of view, in particular for the survival functions of copulas. We thereby like to connect the copula dependence structure with hazard quantities. Our hazard dependence functions (and dependence measures) have a direct interpretation in survival analysis and they may serve as parameters for statistical dependence modeling. Recall that, under continuity, univariate hazard measures $\Lambda$ are exponent measures of survival functions $S$, i.e. $S(t) = \exp(-\Lambda(t))$ for continuous $S$; see Gill (1993) and Dabrowska (1996) for exponents of multivariate survival functions.

The article is organized as follows. The basics of multivariate survival analysis are presented in Section 2. Along with the work of Gill (1993) and Dabrowska (1996) the exponent representation of Lemma 2.1 for multivariate survival functions is recalled which later leads to so-called exponent measures of dependence. It is also connected to the concept of “local dependence hazard” parameters studied earlier by Janssen and Rahnenführer (2002) in the bivariate case. Section 3 links the exponent dependence measure to earlier analyzed exponent measures of Resnick (1987) for minimum-infinitely divisible distributions in extreme value theory.
In Section 4 signed exponent measures of dependence are studied in detail for survival functions of correlated frailty models. These are introduced as multivariate exponentially distributed variables with random scales \( W = (W_1, \ldots, W_d) \). In terms of the Laplace transform of \( W \) explicit analytic formulas are derived for their densities, hazards and exponent measures, in particular, for the special examples of dependent \( \chi^2 \) or log-normal random scales \( W_i \). If \( W \) is sum-infinitely divisible, then the dependence exponent measures are linked to the Lévy measure given by the Lévy-Khintchine formula for \( W \); see Section 5.

Section 6 is devoted to semiparametric survival models with arbitrary marginals. It is shown that the exponent dependence measures can be parametrized in a copula manner. The underlying parametrized dependence quantities are later visualized for some examples of correlated frailty models; see Figures 1 to 4. Together with the motivation given by the upcoming equations (2.7)–(2.9) the plots have a meaningful hazard-based interpretation regarding the quality of dependence. For instance, there exist copulas with overall proportional hazard dependence for all bivariate submodels but without dependence for exponent measures of higher order. On the contrary, copula models can be constructed for which all proper subvectors are independent but the whole vector has a non-trivial dependence measure; see Section 7 for an elaboration of such models via hazard functions. Additional examples and graphs as well as some technical details are presented in the appendix to this article.

2. Survival Functions

A copula \( C \) is a distribution function on the \( d \)-dimensional unit cube \([0, 1]^d\) with uniformly distributed marginals. It is well known that copulas describe the dependence structure of \( \mathbb{R}^d \)-valued random variables \( T = (T_1, \ldots, T_d) \) whereas the distributions of the one-dimensional marginals \( T_i \) are often regarded as nuisance parameters in statistics.

In this paper we will study survival functions and their corresponding survival copulas which are very useful in risk and survival analysis when multivariate failure rates and hazard rates have a concrete meaning; see for instance Gill (1993) and Dabrowska (1996). For a pair of random variables the hazard rate dependence approach is a meaningful description of the survival copula dependence structure; see Janssen and Rahnenführer (2002). They showed that an exponential dependence measure exists which natu-
rally explains the dependence structure. At other times we like to apply the copula concept in order to convert all marginals to uniform distributions, eliminating all nuisance parameters in doing so.

Throughout let $\mathbf{T} = (T_1, \ldots, T_d) : \Omega \to \mathbb{R}^d$ denote a random variable with $d$-dimensional continuous distribution function $F$ and let $F_i$ and $S_i(t_i) = 1 - F_i(t_i) = P(T_i > t_i)$, $t_i \in \mathbb{R}$, be the marginal distribution and survival functions. The $d$-dimensional survival function is given by

$$S(t_1, \ldots, t_d) = P(T_1 > t_1, \ldots, T_d > t_d). \quad (2.1)$$

To fix the notation let $F^{-1}_{i}(S^{-1}_{i})$ denote the left-sided continuous inverse distribution (survival) function of $F_i$ ($S_i$). In our continuous case the variables

(a) $(F_1(T_1), \ldots, F_d(T_d))$ and (b) $(S_1(T_1), \ldots, S_d(T_d)) \quad (2.2)$

both define copulas on $[0,1]^d$: The distribution function $C$ of (a) is simply called copula of $\mathbf{T}$ and similarly the distribution function $C_s$ of (b) is defined as the survival copula of $\mathbf{T}$, cf. Nelsen (2006), Section 2.6, as well as McNeil and Nešlehová (2009). We see that

$$(v_1, \ldots, v_d) \mapsto C_s(1 - v_1, \ldots, 1 - v_d)$$

is just the survival function $(2.1)$ of $C$. To avoid further confusions we like to stress the necessity to clearly distinguish between the survival copula $C_s$ and the survival function of the copula $C$. These concepts lead to different dependence models for survival times.

Furthermore, the survival function $S$ and the survival copula are connected by

$$S = C_s(S_1, \ldots, S_d) \quad \text{and} \quad C_s = S(S_1^{-1}, \ldots, S_d^{-1}) \quad . \quad (2.3)$$

Thus, every survival function $S$ can easily be reproduced by its survival copula $C_s$ and the continuous marginal survival functions.

In order to study multivariate survival functions in more detail we first recall elements of univariate survival analysis for a continuous real valued random variable $T$. For a detailed introduction we refer to the books by Klein and Moeschberger (2003) and Aalen et al. (2008). If we put $x_0 = \inf\{x : P(T \leq x) > 0\}$, then there exists a so-called hazard measure $\Lambda$ on the interval $(x_0, \infty)$ with cumulative hazard function $\Lambda(t) := \Lambda(-\infty, t]$ given by

$$\frac{d\Lambda}{dF} = \frac{1}{S} \quad \text{and} \quad P(T > t) = 1 - F(t) = S(t) = \exp(-\Lambda(t)). \quad (2.4)$$
The hazard measure $\Lambda$ can be viewed as univariate exponent measure of the survival function. Whenever $F$ has a Lebesgue density $f$ then $\Lambda(t) = \int_x^t \lambda(u) \, du$ holds for the hazard rate $\lambda(u) := \frac{f(u)}{S(u)}$. For example, the hazard measure of a standard exponentially distributed random variable is just the Lebesgue measure $\lambda \lambda_{(0, \infty)}$ on $(0, \infty)$.

In the next step exponential representations of multivariate survival functions are studied which are used to extend the univariate hazard formula (2.4). Special models and exponent measures are examined in the proceeding sections. Consider an index set $\emptyset \neq I \subset I_0 = \{1, \ldots, d\}$. Then

$$\Omega_I := \{(t_i)_{i \in I} \in \mathbb{R}^{|I|} : P(T_i > t_i \text{ for all } i \in I) > 0\}$$

denotes the domain of the survival function of the marginals $(T_i)_{i \in I}$. The following Lemma is written in the spirit of Gill (1993) who studied the survival function in the context of product integrals. The proof follows by induction; see also Dabrowska (1996), equation (1.1).

**Lemma 2.1.** Consider the continuous survival function $S$ of $T = (T_i)_{i \in I_0}$ and let $S^J$ denote the survival function of the $J$-dimensional marginal vector $(T_j)_{j \in J}, J \subset I_0$. Define for a single subset $I = \{i\}$ the function $S_I$ on $\Omega_I$ as the marginal $S_I = S_{ij}$. For every $I \subset I_0$, $|I| \geq 2$, there exists a positive function $S_I : \Omega_I \to \mathbb{R}$ which is uniquely determined by the following representation of marginal survival functions. For each index set $\emptyset \neq J \subset I_0$ the marginal survival function admits the factorization

$$S^J = \prod_{I \subset J} S_I \quad \text{on } \Omega_J$$

(2.5)

where the product is taken over all subsets $\emptyset \neq I \subset J$. We call $(S_I)_{|I|=r}$ the dependence parts of order $2 \leq r \leq d$ which are unique on their domain.

For a singleton $I = \{i\}$ the cumulative hazard function $\Lambda_i$ corresponds to an exponent measure, $S_i = \exp(-\Lambda_i)$. In the same manner we can, for all $I \subset I_0$ with $|I| \geq 2$, define an exponent function $\Lambda_I$ on the domain $\Omega_I$ by

$$S_I = \exp\left((-1)^{|I|}\Lambda_I\right).$$

(2.6)

As we will see, the higher dimensional exponents are typically linked to signed exponent measures of dependence. As motivation and in order to give a proper interpretation let us first recall the hazard dependence approach for
$T_1 \geq 0$ and dimension $d = 2$: Janssen and Rahnenführer (2002) pointed out that $\Lambda_{\{1,2\}}$ is a signed exponent measure. If $S$ has a density then $\Lambda_{\{1,2\}}$ also has a density $\lambda_{\{1,2\}}$ on $\mathbb{R}^2$ and (2.5) writes as

$$S(t_1, t_2) = S_1(t_1)S_2(t_2) \exp \left( \int_0^{t_1} \int_0^{t_2} \lambda_{\{1,2\}}(u_1, u_2) \, du_1 \, du_2 \right).$$

(2.7)

Recall again the interpretation of the univariate hazard function $\lambda_1(u) = \frac{f_1(u)}{S_1(u)}$ as failure rate. Under smoothness of $f_1$ and for small $\epsilon > 0$ we approximately have, given the event $\{T_1 \geq t_1\}$,

$$P(T_1 \in [u, u + \epsilon] \mid T_1 \geq u) = \frac{\int_u^{u+\epsilon} f_1(x) \, dx}{S_1(u)} \approx \epsilon \lambda_1(u).$$

(2.8)

In the same way $\lambda_{\{1,2\}}$ has an interpretation as “local dependence hazard” of $T_1$ and $T_2$. For smooth densities, $B = \{T_1 \geq t_1, T_2 \geq t_2\}$ and small $\epsilon_1, \epsilon_2 > 0$ Janssen and Rahnenführer (2002) verified that

$$P(T_1 \in [t_1, t_1 + \epsilon_1], T_2 \in [t_2, t_2 + \epsilon_2] \mid B) - P(T_1 \in [t_1, t_1 + \epsilon_1] \mid B) \, P(T_2 \in [t_2, t_2 + \epsilon_2] \mid B)$$

$$\approx \lambda_{\{1,2\}}(t_1, t_2) \, \epsilon_1 \, \epsilon_2$$

(2.9)

holds. They also showed that the exponent dependence measure and its density are extremely useful for testing independence for randomly censored survival models. In their work the reader will find a lot of examples for hazard dependence structures and efficient tests for independence of the marginals at dimension 2; see also Section 7 below.

3. Exponent dependence measures of minimum-infinitely divisible distributions

In this section we link the exponents of dependence $\Lambda_f$ to already known measures of dependence. Observe that in extreme value theory minimum-infinitely divisible distributions allow related exponent representations.

A distribution on $\mathbb{R}^d$ with survival function $S$ is called minimum-infinitely divisible if $S^{1/n}$ is a survival function for each $n \in \mathbb{N}$. For obvious reasons we will focus on the copula case. It is well known (cf. Resnick, 1987, Section 5.3, switching from the max to the min operation) that, under minimum-infinitely
divisibility, there exists a possibly unbounded exponent Radon measure \( \mu \) on \([0, 1]^d \setminus \{1\} \) with

\[
S(x) = \exp \left( -\mu \left( (x, 1)^c \right) \right),
\]

where \((x, 1)^c\) is the complement of \((x_1, 1] \times \cdots \times (x_d, 1]\) in \([0, 1]^d\) and \(1 = (1, \ldots, 1) \in \mathbb{R}^d\). To get an impression of (3.1) consider the bivariate case \(d = 2\):

**Example 3.1.** (a) Let \(S = S_1S_2\) be the survival function of the independence copula of dimension 2. Then \(\mu = \Lambda_1 \otimes \varepsilon_1 + \varepsilon_1 \otimes \Lambda_2\) lies on the upper boundary of \([0, 1]^2\) with \(\Lambda_i(t) = \int_0^t \frac{dx}{1-x}, \ t \in [0, 1),\) and \(\varepsilon_1\) being the Dirac measure in 1. (b) If \(S = S_1S_2 \exp \left( \Lambda_{\{1,2\}} \right)\) is minimum-infinitely divisible and given by (3.1) then

\[
\Lambda_{\{1,2\}} \left([0, x_1] \times [0, x_2]\right) = \mu \left([0, x_1] \times [0, x_2]\right)
\]

holds for all \(0 \leq x_i < 1\) if \(S\) is continuous.

**Remark 3.2.** (a) The distribution of two non-negative random variables with local dependence hazard \(\lambda_{\{1,2\}}\) (cf. Proposition 4.8 and the discussion after Lemma 2.1) is minimum-infinitely divisible iff \(\lambda_{\{1,2\}}\) is non-negative. \(\text{Joe} (1997), \text{Theorem 2.7},\) proves this and a similar result for arbitrary dimensions \(d \geq 2\). (b) Part (b) of Example 3.1 is a special case of the following Proposition 3.3, from which we conclude that the behaviour of \(\mu\) in the interior of \([0, 1]^d\) determines the dependence structure of \(S\).

**Proposition 3.3.** Let \(S\) be the survival function of a continuous minimum-infinitely divisible copula given by (3.1). For each index set \(\emptyset \neq I \subset I_0\) the function \(\Lambda_I\) given by (2.6) is a measure generating function of a positive measure on \([0, 1]^{|I|}\) with

\[
\Lambda_I = \mu_I \big|_{[0, 1]^{|I|}}
\]

where \(\mu_I := \mathcal{L} \left( \pi_I \big| \mu \big|_{[0, 1]^d \setminus \pi_I^{-1}((1, \ldots, 1))} \right)\) is the image measure of the canonical projection \(\pi_I : [0, 1]^d \to [0, 1]^{|I|}\) w.r.t. the restricted measure of \(\mu\) on \([0, 1]^d \setminus \pi_I^{-1}((1, \ldots, 1))\).
Proof. The proof relies on an inclusion-exclusion principle type of argument (also known as the sieve formula or Sylvester-Poincaré equality). For \( x \in [0, 1)^d \) observe that

\[
1_{(x, 1]}^c = \sum_{\emptyset \neq J \subset I_0} (-1)^{|J|+1} \prod_{j \in J} 1_{[0, x_j]}
\]

and

\[
\mu((x, 1]^c) = \sum_{\emptyset \neq J \subset I_0} (-1)^{|J|+1} \mu_J(\prod_{j \in J} [0, x_j])
\]

hold, where \( 1_A \) denotes the indicator function of a set \( A \). Moreover, \( \mu_J \) is the exponent measure (3.1) of the \(|J|\)-dimensional marginal survival function \( S_J \). Hence, \( \Lambda_I \) and \( \mu_I|_{[0,1]^I} \) can be identified step by step. For \(|I| = 1\) the univariate hazard measures are obtained. In general, when all \( \Lambda_J \) with \( J \subset I, J \neq I \), are identified, then the remaining part \( \Lambda_I \) must be given by (3.2). Note that \( S \) is completely determined by \( S(x) \) for \( x \in [0, 1)^d \). □

Remark 3.4. Suppose that \( T = (T_1, \ldots, T_d) \) has a minimum-infinitely divisible distribution with exponent measure \( \mu \) given by (3.1) and that \( S \) is continuous. The following statements are equivalent (cf. Resnick (1987), Section 5.5, for the equivalence of (i) and (ii)).

(i) The components \( T_1, \ldots, T_d \) are independent random variables.
(ii) The components \( T_1, \ldots, T_d \) are pairwise independent, i.e. \( T_i \) and \( T_j \) are independent random variables for every \( 1 \leq i < j \leq d \).
(iii) For every \( I \subset I_0 \) with \(|I| = 2\) the bivariate exponent measure \( \Lambda_I \) given by (2.6) is equal to zero.
(iv) For every \( I \subset I_0 \) with \(|I| \geq 2\) the exponent measure \( \Lambda_I \) given by (2.1) is equal to zero.

4. Dependence measures for correlated frailty models

As explained in Section 2 the univariate marginals are irrelevant when the dependence structure is studied within a semiparametric context. This is a chance to choose in a first step the marginals in a convenient way as we did in (2.2). In a second step the marginals may be transformed if necessary. For this reason we will first introduce a correlated frailty model for special marginals which gives more insight in the underlying dependence structure.
Originally the frailty concept shows up in multivariate survival analysis when individual hazard effects are modeled by an additional factor attached to famous Cox models; see Andersen et al. (1993) for further references. The idea to link several individuals via the same factor leads to shared frailty models and closely related Archimedean copulas; see Section 1 for references to this connection. Our next example shows that these particular copulas fit very well in the context of survival copulas.

**Example 4.1.** Let \( \varphi : [0, \infty) \rightarrow [0, 1] \) be the following generator of an Archimedean copula \( K \): \( \varphi \) is a nonincreasing and continuous function satisfying \( \varphi(0) = 1, \lim_{x \to \infty} \varphi(x) = 0 \) and strictly decreasing on the interval \( [0, \inf \{ x : \varphi(x) = 0 \} ] \). Then \( K \) is given by

\[
K(u_1, \ldots, u_d) = \varphi \left( \varphi^{-1}(u_1) + \cdots + \varphi^{-1}(u_d) \right).
\]

Obviously, \( S_i := \varphi \) define marginal survival functions. If \( (\zeta_1, \ldots, \zeta_d) \) is distributed according to the above Archimedean copula \( K \) with generator \( \varphi \) then by (2.3)

\[
(t_1, \ldots, t_d) \mapsto \varphi(t_1 + \cdots + t_d)
\]

is the survival function of \( T = (\varphi^{-1}(\zeta_1), \ldots, \varphi^{-1}(\zeta_d)) \) with survival function \( S_i = \varphi \) of the marginals. Furthermore, \( C_s = K \) is the survival copula of \( T \).

We see that the above choice of marginals yields a pleasant form of the survival function.

**Remark 4.2.** Okhrin et al. (2013) study hierarchical Archimedean copulas, which extend the class of Archimedean copulas. They show that any hierarchical Archimedean copula can be uniquely recovered from all bivariate marginal copula functions.

We now introduce a class of correlated frailty models which admit meaningful hazard dependence structures. The explanation below Lemma 4.6 exhibits the relation to the Archimedean copula structure (4.1). Let \( Q_0 \) denote a probability measure on \((0,\infty)^d\) with Laplace transform

\[
\psi(t) := \int \exp(-<t,x>) \ dQ_0(x), \quad t = (t_1, \ldots, t_d) \in [0, \infty)^d.
\]

It will turn out that \( \psi \) is a meaningful quantity for the multivariate survival time \( X \) defined as follows.
Definition 4.3. Let $W = (W_1, \ldots, W_d)$ and $(Y_1, \ldots, Y_d)$ denote two independent $d$-dimensional random vectors with law $\mathcal{L}(W) = Q_0$ and let $Y_1, \ldots, Y_d$ be i.i.d. standard exponentially distributed. The random vector $X$ given by

$$X = (X_1, \ldots, X_d) := \left(\frac{Y_1}{W_1}, \ldots, \frac{Y_d}{W_d}\right)$$  \hspace{1cm} (4.2)

is then called a correlated frailty model based on $Q_0$.

Remark 4.4. In multivariate survival analysis the concept of frailty was originally introduced with equal variables $W_1 = W_2 = \cdots = W_d$ (called shared frailty model) to model unobserved heterogeneity. For a more detailed discussion of the univariate proportional frailty model, including surveys of the model for different distributions of the frailty variable and extensions, we refer to the books by Wienke (2011), Chapter 3, Duchateau and Janssen (2008) and Aalen et al. (2008), Chapter 6, as well as to the article by Völker (2010).

In contrast to the shared frailty model, the correlated frailty model is more flexible and it takes the dependency structure of the $W_i$’s into account.

Usually the definitions of frailty models are given in terms of the hazard rates $\lambda_1, \ldots, \lambda_d$ of $X_1, \ldots, X_d$. Thus it is possible to include a baseline hazard as well as covariates in the model. As the focus of this article lies on dependence structures we omit covariates and choose (justified by the copula approach) a possibly simple baseline hazard equal to one. These simplifications lead to the definition given by (4.2).

A brief introduction to the correlated frailty model for the bivariate case ($d = 2$) and its relation to Cox models can be found in the book by Wienke (2011), Section 5.1.

The correlated frailty model is a multivariate exponential scale model with random scale parameter $W$. Any strictly increasing transformation of the coordinates of $X$ leads to a conditional Cox model given $W$. Subsequently, it turns out that the structure of (4.2) is closely connected to the exponential family $(Q_t)_{t \in [0, \infty)^d}$ given by

$$\frac{dQ_t(x)}{dQ_0}(x) := \frac{\exp(- < t, x >)}{\psi(t)}.$$  \hspace{1cm} (4.3)

The following analytic properties of Laplace transforms and exponential families are well known; see Barndorff-Nielsen (1978).
Remark 4.5. (a) The Laplace transform $t \mapsto \psi(t)$ is a positive analytic function on $(0, \infty)^d$.

(b) Consider $I \subseteq I_0$ and, for each $i \in I$, let $n_i \in \mathbb{N}_0$ be a multiplicity. Let $t = (t_1, \ldots, t_d) \in [0, \infty)^d$ with $t_i > 0$ whenever $n_i > 0$ holds. Then

$$
\left( \prod_{i \in I} \frac{\partial^{n_i}}{\partial t_i^{n_i}} \right) \psi(t) = \psi(t) \int \prod_{i \in I} (-1)^{n_i} \frac{x_i^{n_i}}{t_i} \, dQ_t(x). \tag{4.4}
$$

(c) Suppose that $(Y_1, \ldots, Y_d) : \Omega' \to \mathbb{R}^d$ is given by a probability measure $P'$ on $\Omega'$. Via projection we may assume that the underlying joint probability space is a product space $\Omega = \Omega' \otimes (0, \infty)^d$ with product measure $P_0 = P' \otimes Q_0$ and $W(\omega', x) = x$ is the identity of the second component. Then $(W_1, \ldots, W_d)$ can naturally be regarded as random vector under $P_t = P' \otimes Q_t$, shortly under $Q_t$.

Lemma 4.6. The following results hold for the correlated frailty model $X$.

(a) The survival function $S(t) = \psi(t)$ coincides with the Laplace transform for all $t \in [0, \infty)^d$.

(b) The distribution of $X$ is concentrated on $(0, \infty)^d$. On this set it has the analytic density

$$
t \mapsto f(t_1, \ldots, t_d) = (-1)^d \frac{\partial}{\partial t_1} \ldots \frac{\partial}{\partial t_d} \psi(t_1, \ldots, t_d). \tag{4.5}
$$

(c) For $t_i > 0$ and $t = (0, \ldots, 0, t_i, 0, \ldots, 0)$ the $i$-th marginal hazard measure $\Lambda_i$ is given by $\Lambda_i(t_i) = -\log \psi(t)$ with $i$-th univariate hazard rate

$$
\lambda_i(t_i) = -\frac{\partial}{\partial t_i} \log \psi(t) = \int W_i \, dQ_t. \tag{4.6}
$$

Proof of Lemma 4.6. (a) Using Fubini’s theorem we obtain

$$
S(t) = \int P' \left( \frac{Y_1}{w_1} > t_1, \ldots, \frac{Y_d}{w_d} > t_d \right) \, dQ_0(w_1, \ldots, w_d)
= \int \exp(-t_1 w_1 - \ldots - t_d w_d) \, dQ_0(w_1, \ldots, w_d)
$$
by the independence of the $Y_i$'s.  
(b) Whenever a density $f$ exists, we have
\[
\psi(t) = S(t) = \int_{t_1}^{\infty} \ldots \int_{t_d}^{\infty} f(u_1, \ldots, u_d) \, du_1 \ldots du_d
\]
by part (a) for each $t \in (0, \infty)^d$. Formal differentiation of $t \mapsto \psi(t)$ gives the analytic left-hand side of (4.5) and the result follows.

(c) The first part follows since $\Lambda_i(t_i) = -\log S_i(t_i) = -\log \psi(t_i)$. Then its derivative $\frac{\partial}{\partial t_i} \Lambda_i(t_i) = \lambda_i(t_i)$ can be calculated by (4.4).

We see that distributional quantities like $S$ and $f$ are linked to $\psi$ and its derivatives. To see that the hazard quantities also rely on $\log \psi$, consider $S$ as in (2.5). Then we have
\[
\log S(t) = \sum_{I \subseteq I_0} \log S_I(t) = -\sum_{i=1}^{d} \Lambda_i(t_i) + \sum_{I \subseteq I_0, |I| > 1} (-1)^{|I|} \Lambda_I(t_I)
\]
with first order terms $\Lambda_i(t_i)$.

The higher order terms for $|I| > 1$ are now studied in detail and linked to (2.7). Throughout, the following notation is used. Let $h : [0, \infty)^d \to \mathbb{R}$ denote a function with existing derivatives of sufficiently high order. Introduce for $\emptyset \neq I \subset I_0$ the following notation: For $x_I \in \mathbb{R}^{|I|}$
\[
x \mapsto h(x_I, 0)
\]
is the function given by $h$ when the coordinates $x_j, j \notin I$, are zero. Similarly, for a given $x \in \mathbb{R}^d$ we denote by $(x_I, 0)$ the $d$-dimensional vector with zero entries for all indices $j \notin I$ and with entries $x_i$ else. For $I = \{i_1, \ldots, i_r\}$ we also introduce
\[
\frac{\partial}{\partial x_I} h(x_I, 0) = \frac{\partial}{\partial x_{i_1}} \ldots \frac{\partial}{\partial x_{i_r}} h(x_I, 0)
\]
and
\[
\int_{0}^{y_I} h(x_I, 0) \, dx_I = \int_{0}^{y_{i_1}} \ldots \int_{0}^{y_{i_r}} h(x_I, 0) \, dx_{i_1} \ldots dx_{i_r}, \quad y_I \in \mathbb{R}^{|I|}.
\]

**Lemma 4.7.** The logarithm of the survival function $S$ of a correlated frailty model is given by
\[
\log S(t) = \sum_{I \subseteq I_0, \emptyset \neq I} \int_{0}^{t_I} \frac{\partial}{\partial x_I} \log (x_I, 0) \, dx_I.
\]
Proof. Since \( \log \psi(0) = 0 \) holds, the main theorem of multivariate calculus yields (4.7) by induction over \( d \).

This result identifies the exponents \( \Lambda_I \) for \( I \neq \emptyset \). We have

\[
\Lambda_I(t_I) = \int_0^{t_I} \lambda_I(x_I) \, dx_I
\]

where \( \lambda_I(x_I) = (-1)^{|I|} \frac{\partial}{\partial x_I} \log \psi(x_I, 0) \). Similarly as for Lemma 2.1 the proof is given by induction over the dimension \( d \). The first order (hazard) measures are described in (4.6) by the exponential family (4.3). In our context \( \Lambda_I \) are measure generating functions of signed measures. Here, the bivariate measures \( \Lambda_I, |I| = 2 \), are determined by particularly interesting densities:

**Proposition 4.8.** Consider \( I = \{i,j\} \subset I_0, i \neq j \), and \( t_I \in (0, \infty)^2 \). Then the density \( \lambda_I \) of the signed measure \( \Lambda_I \) is given by the covariance structure of the exponential family:

\[
\lambda_I(t_I) = \text{Cov}_{Q(t_I,0)}(W_i, W_j).
\]

**Remark 4.9.** (a) The local hazard dependence density \( \lambda_{\{1,2\}} \) in (2.9) has now an interpretation in terms of correlations for the correlated frailty model. (b) Higher order expressions of (4.8) are similar but more complicated. The role of higher order exponent measures is discussed in Section 6.

**Proof of Proposition 4.8.** For simplicity we may consider \( I = \{1,2\} \). By (4.8) we have

\[
\lambda_I(t_I) = \frac{\partial}{\partial t_I} \log \psi(t_I, 0) = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \psi(t_I, 0) - \left( \frac{\partial}{\partial t_1} \psi(t_I, 0) \right) \left( \frac{\partial}{\partial t_2} \psi(t_I, 0) \right).
\]

Combining (4.3) and (4.4), we see that the first term corresponds to the expected value \( E_{Q(t_I,0)}(W_1 W_2) \) whereas the second term is just the product \( E_{Q(t_I,0)}(W_1) E_{Q(t_I,0)}(W_2) \); see also (4.6).

In life science analysis two risk components can be modeled as a minimum of two survival variables: For example, we could think of individuals who are exposed to two different lethal risk factors. Then the survival times are the first occurrences of one of the competing events. This concept goes well with the correlated frailty model:
Lemma 4.10. Let $W$ and $W'$ be two $d$-dimensional frailty variables with correlated frailty models

$$X_W = \left( \frac{Y_1}{W_1}, \ldots, \frac{Y_d}{W_d} \right) \text{ and } X_{W'} = \left( \frac{Y'_1}{W'_1}, \ldots, \frac{Y'_d}{W'_d} \right)$$

for i.i.d. standard exponentially distributed $Y_1, \ldots, Y_d, Y'_1, \ldots, Y'_d$. Let $S_W$ and $S_{W'}$ denote the survival functions of $X_W$ and $X_{W'}$, respectively.

(a) We have equality in distribution of

$$X_{W+W'} := \left( \frac{Y_1}{W_1 + W'_1}, \ldots, \frac{Y_d}{W_d + W'_d} \right) \overset{\circ}{=} \min(X_W, X_{W'})$$

Here $\min$ denotes the component-wise minimum operation. Thus, the class of correlated frailty models is closed w.r.t. the minimum operation.

(b) The following conditions (i)-(ii) are equivalent.

(i) The correlated frailty model $X_{W+W'}$ has the survival function $S_W S_{W'}$.

(ii) $W$ and $W'$ are independent.

The proof is obvious. Note, that the conditional survival function of $\min(X_W, X_{W'})$ at $t$ given $(W, W') = (w, w')$ is $\exp(-<t, w + w'>)$. The independence of $W$ and $W'$ is equivalent to the product of Laplace transforms which corresponds to the products of survival functions; see Lemma 4.6.

Example 4.11. Let $(Z_1, \ldots, Z_d)$ have a multivariate normal distribution with mean vector zero and non-singular covariance matrix $\Sigma$.

(a) Correlated frailty models with the following frailty variables $W$ are uniquely determined by the collection of bivariate distributions $(Z_i)_{i \in I}$, $|I| = 2$, i.e. by the covariances $\text{Cov}(Z_i, Z_j), i \leq j$:

(i) Frailty variables with $\chi^2$-distributed marginals: $(W_i)_{i \in I_0} = (Z_i^2)_{i \in I_0}$.

(ii) Frailty variables with log-normal marginals: $(W_i)_{i \in I_0} = (\exp(Z_i))_{i \in I_0}$.

(b) The correlated frailty vector $(X_1, \ldots, X_d)$ given by (i) or (ii) has independent components iff $\Sigma$ has diagonal form.

The proofs of this example are deferred to Appendix C.

In contrast to Example 4.11 there exist correlated frailty models with pairwise independent components of $X$ but higher order dependence. For
instance let $W_1, W_2, W_3$ be pairwise independent but not totally independent. Then $S$ has the form $S = \exp \left( -\sum_{i=1}^{3} \Lambda_i - \Lambda_{\{1,2,3\}} \right)$ with trivial exponent measures $\Lambda_J = 0$ for $|J| = 2$. See Section 7 for a related discussion which systematically characterises copulas having only dependencies of the highest order.

Part (a) of the following example continues Example 4.11(a)(i) for the special case $d = 3$:

**Example 4.12.** (a) Let $(Z_1, Z_2, Z_3)$ follow a multivariate normal distribution with mean $E(Z_i) = 0$ for $i \in \{1, 2, 3\}$ and covariance matrix

$$
\Sigma = \begin{pmatrix}
\sigma_1^2 & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_2^2 & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_3^2
\end{pmatrix}, \quad \sigma_i > 0 \text{ for each } i = 1, 2, 3,
$$

and let $\rho_{ij} := \frac{\sigma_{ij}}{\sigma_i \sigma_j}$ denote the correlation coefficient of $Z_i$ and $Z_j$. Consider the correlated frailty model (4.2) with frailty variables $W_i = Z_i^2$, $i \in \{1, 2, 3\}$.

Explicit formulas and their derivation for all dependence parts of the survival function, i.e. for $S_{\{1,2,3\}}, S_{\{i,j\}}, i < j$, as well as for the 2-dimensional hazard densities $\lambda_{\{i,j\}}, i < j$, can be found in Appendix A. There it is seen that the survival copula $S \circ (S_1^{-1}, S_2^{-1}, S_3^{-1})$ only depends on the bivariate parameters $\rho_{ij}$. This is no surprise as the vector of frailty variables $W$ is completely determined by its pairwise covariances.

However, this rather simple dependence structure of $W$ involves a more complicated dependence structure for $X$: the dependence parts of second as well as of third order are non-trivial. The discussion of the bivariate version of this example is continued in Example 6.4.

(b) Further (mostly bivariate and rather particular) examples for correlated frailty models based on different distributions have been studied and applied. An overview is given in Chapter 5 of Wienke (2011). Chapter 6 of his book shows that the copulas of shared and correlated frailty models based on particular distributions can be derived without frailty models. A generalization of the approach is illustrated.

5. Correlated frailty models with sum-infinitely divisible scale distributions

In this section we derive exponential dependence measures of correlated frailty models from the well known Lévy-Khintchine formula which is first
summarized for Laplace transforms. We refer to Petrov (1995) and Meerschaert and Scheffler (2001) for the notion of sum-infinite divisibility of $\mathcal{L}(W)$ and for the Lévy-Khintchine formula for Fourier transforms. For $W$ with values in $[0, \infty)^d$ it is known that the Gaussian part vanishes, that its Lévy measure is concentrated on $[0, \infty)^d \setminus \{0\}$ and that the integral of $\|x\| \{0 < \|x\| \leq 1\}$ with respect to the Lévy measure is bounded. Here $\| \cdot \|$ denotes the Euclidean norm. This yields the following

**Corollary 5.1.** Let $Q_0 = \mathcal{L}(W)$ be sum-infinitely divisible and concentrated on $[0, \infty)^d$ with Lévy-Khintchine triplet $(b, 0, \eta_0)$ where $b = (b_1, \ldots, b_d) \in [0, \infty)^d$ and $\eta_0$ is a Lévy measure concentrated on $[0, \infty)^d \setminus \{0\}$. Then $Q_0$ is concentrated on $(0, \infty)^d$ iff for each coordinate $i = 1, \ldots, d$ either $b_i > 0$ holds or the univariate Lévy measure of $W_i$ is unbounded.

The proof follows from Hartman and Wintner (1942). In case $b_i = 0$ they proved that $W_i$ is positive whenever its univariate Lévy measure is unbounded. Assume henceforth that $Q_0((0, \infty)^d) = 1$. In this case, we get a well known extension of the related representation for Fourier transforms to Laplace transforms; see for instance Janssen (1985). Lemma 4.6(a) then builds the following connection to the survival function of the correlated frailty model $X$ with frailty variable $W$: For all $t \in [0, \infty)^d$ we have

$$S(t) = \psi_{Q_0}(t) = \exp \left( -<t, b> + \int_{[0, \infty)^d \setminus \{0\}} (e^{-<t, x>} - 1) \, d\eta_0(x) \right). \quad (5.1)$$

This particular correlated frailty model is studied throughout this section.

In a remark in Appendix C it is shown that, for each correlated frailty model $X$ with survival function $S$, the sum-infinite divisibility of $\mathcal{L}(W)$ is equivalent to the minimum-infinite divisibility of $S$. Thus, we have the following relationship where $\mu$ is the exponent measure of the minimum-infinitely divisible $S$ (see Section 3 or Resnick (1987), Section 5.3, for this particular non-copula case): For all $t \in [0, \infty)^d$,

$$\mu ((t, \infty)^C) = -<t, b> - \int_{[0, \infty)^d \setminus \{0\}} (-e^{-<t, x>} - 1) \, d\eta_0(x).$$

With the help of equation (5.1) we now express the $|I|$-dimensional hazard rates $\lambda_I, I \subset I_0$, in terms of a family of Lévy measures $\eta_t$ given by

$$\frac{d\eta_t}{d\eta_0}(x) := \exp(-<x, t>), \quad t \in [0, \infty)^d.$$
Notice the similarity of this Radon-Nikodym density to the definition of the exponential family $Q_t$ in (4.3).

**Lemma 5.2.** For all $i \in I \subset I_0$ such that $|I| \geq 2$ we have, for $t_i \in (0, \infty)^{|I|}$,

$$
\lambda_i(t_i) = b_i + \int_{[0, \infty)^d \setminus \{ \theta \}} x_i d\eta_{(t_i, \theta)}(x) \quad \text{and} \quad \lambda_I(t_I) = \int_{[0, \infty)^d \setminus \{ \theta \}} \prod_{i \in I} x_i d\eta_{(t_i, \theta)}(x).
$$

The technical proofs of Lemmas 5.2 and 5.4 are given in Appendix C.

**Remark 5.3.** It is surprising that the marginal hazard rates $\lambda_i$ can be expressed by the first moment of $Q_{(t_i, \theta)}$ (cf. (4.6)) as well as by the first moment of the Lévy measure $\eta_{(t_i, \theta)}$ whenever $b_i = 0$ holds:

$$
\lambda_i(t_i) = \int_{(0, \infty)} x_i dQ_{(t_i, \theta)}^\pi(\pi_i) = \int_{(0, \infty)} x_i d\eta_{(t_i, \theta)}^{\pi_i}(x_i)
$$

with canonical projection $\pi_i = \pi_{\{i\}}$. Notice also that the higher dimensional quantities $\Lambda_I$ of Lemma 5.2 have a pleasant form whereas higher derivatives of log $S$ are much more complicated when expressed as functions of $Q_{(t, \theta)}$.

Furthermore, Lemma 5.2 enables us to derive the following relationship between dependence hazard measures of correlated frailty models and finite Lévy measures of frailty variables $W$.

**Lemma 5.4.** Assume $\eta_\theta$ to be finite, i.e. $\eta_\theta$ is the Lévy measure of a compound Poisson distribution, and assume $b_i > 0$ for all $i \in I_0$. For all $i \in I \subset I_0$, $|I| \geq 2$, we have

$$
\Lambda_i(t_i) = b_i t_i + \eta_\theta([0, \infty)^d \setminus \{ \theta \}) - \eta_{(t_i, \theta)}([0, \infty)^d \setminus \{ \theta \})
$$

and

$$
\Lambda_I(t_I) = \sum_{J \subset I} (-1)^{|J|} \eta_{(t_J, \theta)}([0, \infty)^d \setminus \{ \theta \}) \quad \text{for all } t \in [0, \infty)^d.
$$

6. Semiparametric dependence models

In this section the exponent dependence measures $\Lambda_I$ are parametrized in the copula manner by a semiparametric dependence model given by hazard quantities, following the bivariate research of Janssen and Rahnenführer (2002). Subsequently, we denote all quantities belonging to functions in the copula case by an additional $0$-index.
Let $\Lambda_I$ be an $|I|$-dimensional signed measure of order $|I| \geq 2$ of some continuously distributed random vector $(X_1, \ldots, X_d)$ on $\mathbb{R}^d$ with the representation

$$\Lambda_I(t_I) = \int_{-\infty}^{t_I} \lambda_I(s_I) \, ds_I$$

where $\infty = (\infty, \ldots, \infty)$ is of dimension $|I|$. Let $\lambda_i$ be the hazard functions of the univariate $\Lambda_i$. Then $\Lambda_I$ can be rewritten as function of the marginal hazards

$$\Lambda_I(t_I) = \int_{-\infty}^{t_I} \gamma_I(s_I) \, d(\Lambda_i \otimes \cdots \otimes \Lambda_i)(s_I)$$  \hspace{1cm} (6.1)

for $I \subset I_0$, $s_I = (s_i)_{i \in I}$ and the $|I|$-dimensional dependence function

$$\gamma_I(s_I) = \frac{\lambda_I(s_I)}{\prod_{i \in I} \lambda_i(s_i)}.$$  \hspace{1cm} (6.2)

In this section we always assume that the densities $\lambda_I$ exist. It is our aim to split the signed dependence measures $\Lambda_I$ up into marginal effects and a parametrized dependence function $\gamma_{0,I} : [0,1]^{|I|} \to \mathbb{R}$, which may serve as a parameter of dependence. For this purpose let

$$\frac{d\Lambda_0}{d\lambda_{[0,1]}(x)}(x) = \frac{1_{[0,1]}(x)}{1-x}$$  \hspace{1cm} (6.3)

be the univariate hazard measure $\Lambda_0$ of the uniform distribution on $[0,1]$. By (2.4) the univariate hazards can be transformed by

$$\mathcal{L}(F_i|\Lambda_i) = \Lambda_0 \text{ and } \Lambda_i = \mathcal{L}(F_i^{-1} \mid \Lambda_0)$$  \hspace{1cm} (6.4)

where $F_i$ is the distribution function of $X_i$.

**Lemma 6.1.** (a) Writing the survival function of $(F_1(X_1), \ldots, F_d(X_d))$ in the form given by (2.5), then the exponent measures $\Lambda_{0,I}, I \subset I_0$, given by (2.6) can be represented via

$$\Lambda_{0,I}(v_I) = \int_0^{v_I} \gamma_{0,I}(u_I) \, d\Lambda_0^I(u_I), \quad v_I \in [0,1]^{|I|},$$  \hspace{1cm} (6.5)

where

$$\gamma_{0,I} : [0,1]^I \to \mathbb{R}, \quad \gamma_{0,I}(u_I) := \gamma_I((F_i^{-1}(u_i))_{i \in I}), \quad u_I = (u_i)_{i \in I}.$$  \hspace{1cm} (6.6)
and $\Lambda_0^I$ denotes the $|I|$-fold product measure of (6.3).

(b) Suppose that $S_0$ is the survival function of a copula with $\Lambda_{0,I} = \Lambda_0$ whenever $|I| = 1$ and

$$S_0(v) = \exp \left( \sum_{\emptyset \neq I \subseteq I_0} (-1)^{|I|} \Lambda_{0,I}(v_I) \right),$$

where $\Lambda_{0,I}$ is given by a function $\gamma_{0,I}$ via (6.5). If a $d$-dimensional random variable $(U_1, \ldots, U_d)$ has the survival function $S_0$, then $V = (V_1, \ldots, V_d) = (F_1^{-1}(U_1), \ldots, F_d^{-1}(U_d))$ has marginal hazards $\Lambda_i(t) = -\log(1 - F_i(t_i))$. The survival function of $V$ has the representation

$$S = \exp \left( \sum_{\emptyset \neq I \subseteq I_0} (-1)^{|I|} \Lambda_I \right),$$

with $\Lambda_I$ given by (6.1) and $\gamma_I(s) = \gamma_{0,I}((F_i(s_i))_{i \in I}), s = (s_i)_{i \in I}$.

The proof is based on the transformation formula for hazards (6.4) since $\mathcal{L}(F_i^{-1} | \Lambda_0) = \Lambda_i$.

**Remark 6.2.** (a) The dependence parameter of interest (6.2) is just

$$\frac{d\Lambda_I}{d(\otimes_{i \in I} \Lambda_i)} = \gamma_I$$

which is completely determined by $\gamma_{0,I}$ (6.6) via

$$\Lambda_I = \int \gamma_{0,I}((F_i)_{i \in I}) \, d(\otimes_{i \in I} \Lambda_i).$$

The dependence functions $\gamma_{0,I}$ form a generalization of $\gamma_{d,0}$ of Janssen and Rahnenführer (2002) which also may be utilized for the construction of dependence tests in semiparametric contexts.

(b) In survival analysis often the componentwise minimum $\min(X, X')$ of independent vectors $X$ and $X'$ is considered. In this case the exponent measures of $X$ and $X'$ sum up.

(c) Another useful transformation is given by

$$Y = (Y_1, \ldots, Y_d) := (\Lambda_1(X_1), \ldots, \Lambda_d(X_d)),$$
whose marginals $Y_i$ are standard exponentially distributed. In this case we have $\frac{d\Lambda_i(Y)}{d\lambda} = I_{(0,\infty)}$ for the marginal hazard measures $\Lambda_i(Y)$ of $Y$ and

$$\gamma_i(Y)(s_i) = \lambda_i(Y)(s_i) = \gamma_{0,i} \left(1 - \exp(-s_i)\right).$$

(d) In the bivariate case $d = 2$ we recall the nice interpretation of $\lambda_{\{1,2\}}$ as local dependence parameter with

$$\gamma_{\{1,2\}}(s_1, s_2) = \frac{\lambda_{\{1,2\}}(s_1, s_2)}{\lambda_1(s_1)\lambda_2(s_2)} = \frac{d\Lambda_{\{1,2\}}}{d(\Lambda_1 \otimes \Lambda_2)}(s_1, s_2). \quad (6.7)$$

Example 6.3. A special case of the parametrization (6.7) is the so-called proportional hazard dependence given by the survival function

$$S_{\text{prop}}(t_1, t_2) = \exp(-\Lambda_1(t_1) - \Lambda_2(t_2) - \beta\Lambda_1(t_1)\Lambda_2(t_2)), \quad 0 \leq \beta \leq 1, \quad (6.8)$$

which is the well known bivariate exponential survival function if the marginals are exponential distributions with $\Lambda_i(t_i) = t_i$ for $t_i > 0$. Note that

$$\Lambda_{\{1,2\}}^{\text{prop}} = -\beta \Lambda_1 \otimes \Lambda_2 \quad (6.9)$$

holds. That is, the local dependence parameter (6.7) is the constant function $-\beta$. Therefore, (6.9) is called the proportional hazard rate dependence model and it can be considered as dependence counterpart of Cox models; see Janssen and Rahnenführer (2003) for dependence tests. A visualization of $\gamma_{\{1,2\}}$ for $\beta = 1$ is given in Figure D.7 of the appendix.

Subsequently, the proportional hazard rate dependence model (6.9) with measure $\Lambda_1 \otimes \Lambda_2$ serves as a benchmark and we consider $\gamma_{\{1,2\}}$ of (6.7) having the interpretation (2.9) in mind. For the visualization, below we present some bivariate contour plots of the normalized dependence function $\gamma_{0,\{1,2\}}$ on $[0,1]^2$; see (6.6). A selection of plots is presented in Figures 1–4 whereas further plots (also for different parameters) can be found in Appendix D.

Example 6.4. (a) (Binary correlated frailty model) Consider the correlated frailty model and a binary index set $I = \{i, j\}$, $|I| = 2 \leq d$. The dependence measure (6.1) is then given by the exponential family (4.3) with

$$\gamma_I(t_i, t_j) = \frac{\text{Cov}_{Q(t_i,0)}(W_i, W_j)}{E_{Q(t_i,0)}(W_i)E_{Q(t_j,0)}(W_j)},$$

where

\begin{align*}
\text{Cov}_{Q(t_i,0)}(W_i, W_j) &= E_{Q(t_i,0)}(W_i)E_{Q(t_j,0)}(W_j) - E_{Q(t_i,0)}(W_i)E_{Q(t_j,0)}(W_j), \\
&= \gamma_{0,\{1,2\}}(t_i, t_j).
\end{align*}
(b) The Clayton copula is an Archimedean copula with generator \( \varphi(s) = \frac{1}{1 + s} \); see Example 4.1. It is also derived as a classical shared frailty model with standard exponentially distributed frailty variable \( W_1 = W_2 \). We see that this is an exponential scale model on the diagonal set \( \{(x, x) : x \geq 0\} \subset \mathbb{R}^2 \) with 
\[
\lambda_i(s) = \frac{1}{1 + s}, \quad \lambda_{\{1,2\}}(t_1, t_2) = (1 + t_1 + t_2)^{-2}, \quad \gamma_{\{1,2\}}(t_1, t_2) = \frac{(1 + t_1)(1 + t_2)}{(1 + t_1 + t_2)^2}
\]
and
\[
\gamma_{0,\{1,2\}}(u_1, u_2) = \frac{(1 - u_1)(1 - u_2)}{(1 - u_1 u_2)^2} = \frac{(1 - u_1)(1 - u_2)}{(1 - u_1 u_2)^2}, \quad 0 \leq u_1, u_2 < 1.
\]

A visualization of \( \gamma_{0,\{1,2\}} \) is given in Figure 1.

(c) The Frank copula is an Archimedean copula with generator
\[
\varphi(s) = -\log\left(\frac{\exp(-\theta s) - 1}{\exp(-\theta) - 1}\right), \quad \theta \in \mathbb{R} \setminus \{0\};
\]
see Example 4.1. For a bivariate survival function with the Frank copula as survival copula we obtain
\[
\lambda_{\{1,2\}}(F_1^{-1}(u_1), F_2^{-1}(u_2)) = \theta^2 \frac{e^{\theta(u_1-1)}e^{\theta(u_2-1)}}{e^{-\theta} - 1} \frac{\log(1 + g) - g}{[(1 + g)\log(1 + g)]^2},
\]

Figure 1: Plot and contour curves for the dependence hazard rate derivative \( \gamma_{0,\{1,2\}} \) of a bivariate shared frailty model based on a standard exponential distribution \( \mathcal{L}(W) \) over the range \([0,1)^2\). This choice of distributions results in a Clayton copula.
where \( g := g(u_1, u_2, \theta) := \frac{(e^{\theta(u_1-1)} - 1)(e^{\theta(u_2-1)} - 1)}{(e^{-\theta} - 1)} \). By (6.6) and (6.7) we have

\[
\gamma_{0,\{1,2\}}(u_1, u_2) = \lambda_{\{1,2\}} (F_1^{-1}(u_1), F_2^{-1}(u_2)) (1 - u_1)(1 - u_2)
\]
as \( \lambda_i(F_i^{-1}(u)) = \frac{1}{1-u} \) for univariate hazard rates \( \lambda_i \) with corresponding distribution functions \( F_i \). A visualization of \( \gamma_{0,\{1,2\}} \) is given in Figure 2.

\[\text{Figure 2: Plots and contour curves for the dependence hazard rate derivative } \gamma_{0,\{i,j\}} \text{ of a model with Frank copula as survival copula over the range } [0, 1]^2, \text{ with different values of } \theta.\]

\( \text{(d) Consider a shared frailty model based on an inverse Gaussian distribution, i.e. } W_1 = W_2 = Z^{-2} \text{ where } Z \text{ is normally distributed with } E(Z) = 0 \)
and $\text{Var}(Z) = \frac{2}{\theta} > 0$. The Laplace transform of the frailty variable $W$ and the joint survival function of the shared frailty model are given by $\psi(t) = \exp(-\theta \sqrt{t})$ and $S(t) = \exp(-\theta \sqrt{t_1 + t_2})$, respectively. We also see that

$$\gamma_{0,\{1,2\}}(u_1, u_2) = \frac{\log(1 - u_1) \log(1 - u_2)}{\{[\log(1 - u_1)]^2 + [\log(1 - u_2)]^2\}^{3/2}}$$

holds, so that this semiparametric dependence function is even invariant under different values of the parameter $\theta > 0$. A visualization of $\gamma_{0,\{1,2\}}$ is given in Figure 3.

![Figure 3](image_url)

Figure 3: Plot and contour curves for the dependence hazard rate derivative $\gamma_{0,\{1,2\}}$ of a bivariate shared frailty model based on an inverse Gaussian distribution over the range $(\frac{1}{10}, 1]^2$. Take note of the pole in the origin $(u_1, u_2) = (0, 0)$.

(e) For the correlated frailty model based on a $\chi^2$-distribution as seen in Example 4.12 we have

$$\gamma_{0,\{i,j\}}(u_i, u_j) = 2g_{ij}^2 \left[\frac{(1 - u_i)(1 - u_j)}{1 - g_{ij}^2(2 - u_i)(2 - u_j)u_iu_j}\right]^2.$$ 

Figure 4 shows a plot and a contour curve for $\gamma_{0,\{i,j\}}$ with $g_{ij}^2 = 0.9$. Note, that $\gamma_{0,\{i,j\}}(0, 0) = 2g_{ij}^2$. Comparing Figure 4 with Figure D.8 in Appendix D, we observe a high early dependence for all values of $g_{ij}^2$ and an increased dependence along the diagonal $\{(u_i, u_j) \in [0, 1]^2 : u_i = u_j\}$ for higher values of $g_{ij}^2$. 

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Figure 4: Plot and contour curves for the dependence hazard rate derivative \( \gamma_{0,\{i,j\}} \) of a bivariate correlated frailty model based on a bivariate chi-squared distribution over the range \([0, 1]^2\), with \( \rho_{ij}^2 = 0.9 \).

7. Copulas with higher degree of dependence: the hazard approach

In this section higher order exponent measures are studied in terms of hazard parameters. We start with binary proportional dependence models which are developed in the spirit of Example 6.3.

**Example 7.1.** The bivariate proportional dependence model can be extended for dimension \( d \geq 2 \). Consider a family \((\beta_I)_{I \subseteq \mathbb{N}, |I| = 2}\) so that \( 0 \leq \beta_I \leq \frac{1}{(d-1)^2} \) holds for each \( I \). Let \( \Lambda_1, \ldots, \Lambda_d \) be continuous univariate hazard measures on \( \mathbb{R} \). Then

\[
S(t_1, \ldots, t_d) = \exp \left( - \sum_{i=1}^{d} \Lambda_i(t_i) - \sum_{i,j=1, i \neq j}^{d} \beta_{\{i,j\}} \Lambda_i(t_i) \Lambda_j(t_j) \right)
\]

defines the survival function of a \( d \)-dimensional variable with proportional hazard dependence of the bivariate exponential measures \( \frac{d\Lambda_{\{i,j\}}}{d(\Lambda_i \otimes \Lambda_j)} = -\beta_{\{i,j\}} \).

In Example 7.1 all higher dimensional exponent measures \( \Lambda_I, |I| \geq 3 \), vanish. To see that \( S \) is indeed a survival function, assume (with no loss
of generality) that $\Lambda_i(t_i) = t_i$ and consider the bivariate exponentially distributed random variable $X_I$ with survival function

$$S^I(t_i, t_j) = \exp \left( -\frac{t_i + t_j}{d - 1} - \beta_I t_i t_j \right),$$

where $\frac{1}{d-1} X_I$ has the form (6.8) with parameter $0 \leq \beta_I (d-1)^2 \leq 1$. Introduce now a new $(-\infty, \infty]^{d-1}$-valued random variable $\tilde{X}_I$ with coordinates $X_I$ at the position $I$. Let all other coordinates of $\tilde{X}_I$ be infinite $\infty$. Thus $\tilde{X}_I$ has the improper survival function $\tilde{S}^I(t_i, t_j) = S^I(t_i, t_j, I = \{i,j\}).$

Consider now independent random variables $(\tilde{X}_I)_I$ for $I \subset I_0, |I| = 2$. Then

$$\min_{I, |I| = 2} \tilde{X}_I$$

has the desired survival function $\prod_{I, |I| = 2} \tilde{S}^I$.

Notice that this modeling of dependence can be generalized by linking arbitrary improper survival functions of the above kind. This is again accomplished by applying the minimum operation to the corresponding random variables with values in $(-\infty, \infty]^d$.

In contrast to Example 7.1 survival models with trivial dependence measures $\Lambda_I = 0$ for all $I \neq I_0$ and $|I| \geq 2$ are studied below. Our approach is based on statistical arguments used earlier for hazard-based score functions. To this end, let $P_0$ be the uniform distribution on the interval $(0, 1)$ with hazard measure $\Lambda_0$. Introduce the subset $L^{(0)}_{2,d}(P^d_0)$ of those $d$-dimensional, square-integrable functions $g$ in $L_{2,d}(P^d_0)$ for which

$$\int_0^1 g(x_1, \ldots, x_d) \, dx_i = 0$$

holds for all $i \in I_0$. Functions $g$ of this kind serve as score functions for statistical models. We start with the following useful observation. Consider a copula which admits a $P^d_0$-density $f : (0, 1)^d \rightarrow [0, \infty)$. Then the following conditions (I) and (II) are equivalent.

(I) All $|I|$-dimensional marginals of the copula are independence copulas for $I \neq I_0$.

(II) The function $g := f - 1$ satisfies $g \geq -1$ and $g \in L^{(0)}_{2,d}(P^d_0)$.

In case of (I), (II) all exponent measures $\Lambda_I$ vanish for $I \neq I_0, |I| \geq 2$. Then the dependence part of $S(x) = \prod_{i=1}^d (1 - x_i) S_{I_0}(x)$ reads as

$$S_{I_0}(x) = 1 + \frac{\int_0^1 g \, dP^d_0}{\prod_{i=1}^d (1 - x_i)}$$

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which is studied below. Recall that for $d = 1$ the operator
\[ R : L_{2,1}^{(0)}(P_0) \rightarrow L_{2,1}(P_0), \quad R(g)(x) = g(x) - \frac{\int_x^1 g(u) \, du}{1 - x} \]
is an isometry between Hilbert spaces; see Ritov and Wellner (1988) as well as Efron and Johnstone (1990). Note that $R(g)$ has an interpretation as hazard rate derivative for survival models. We will now introduce a multivariate version $R_d$ of $R$ on $L_{2,1}^{(0)}(P_0)$.

**Lemma 7.2.** For $g_1, \ldots, g_d \in L_{2,1}^{(0)}(P_0)$ introduce the multivariate function $g(x_1, \ldots, x_d) = \prod_{i=1}^d g_i(x_i)$ of $L_{2,d}^{(0)}(P_0)$. We may then define
\[ R_d(g)(x) := \prod_{i=1}^d R(g_i)(x_i), \quad x \in (0,1)^d. \]  

(a) The operator $R_d$ can be uniquely extended on $L_{2,d}^{(0)}(P_0)$ and $R_d : L_{2,d}^{(0)}(P_0) \rightarrow L_{2,d}(P_0)$ is an isometry between Hilbert spaces.
(b) Suppose that $\gamma = R_d(g)$ holds for some $g \in L_{2,d}^{(0)}(P_0)$. Then
\[ \frac{\int_x^1 g \, dP_0^d}{\prod_{i=1}^d (1 - x_i)} = (-1)^d \int_0^x \gamma \, d\Lambda_0^d \]  
holds for all $x \in (0,1)^d$.

**Proof.** (a) The proof is mostly left to the reader and extends the bivariate calculus of Janssen and Rahnenführer (2002). Let $(\gamma_j)_{j \in \mathbb{N}}$ be an orthonormal basis of the Hilbert space $L_{2,1}(P_0)$. Then product functions $x \mapsto \prod_{j=1}^d \gamma_j(x_j)$ form an orthonormal basis of $L_{2,d}(P_0)$. We first introduce $R_d^{-1}$ of these elements. In the next step products of $g_j := R^{-1}(\gamma_j)$ are considered as in (7.1) and $R_d$ as well as $R_d^{-1}$ can be extended by taking linear combinations of basis elements. It is easy to see that $R_d^{-1}$ is surjective. If we add $g_0 \equiv 1$ then $(g_j)_{j \in \mathbb{N}_0}$ is an orthonormal basis of $L_{2,1}(P_0)$ and $x \mapsto \prod_{j=1}^d g_j(x_j)$ including the index 0 is an orthonormal basis of $L_{2,d}(P_0)$. However, if one index is equal to zero then $\prod_{j=1}^d g_j$ is orthogonal to $L_{2,d}^{(0)}(P_0)$. All other elements of this kind are images of $R_d^{-1}$. Thus $R_d^{-1}$ is surjective.
(b) For $d = 1$ formula (7.2) is proved in (3.8) of Janssen (1994). Fubini’s
Theorem 7.3. Suppose that (I) or (II) holds and let $S_{I_0}$ be the dependence part of the survival function $S(x) = \prod_{i=1}^d (1-x_i) S_{I_0}(x)$ of the copula.

(a) Set $\gamma = R_d(g)$ for $g = f - 1$. The dependence part $S_{I_0}$ of $S$ is given by

$$S_{I_0}(x) = 1 + (-1)^d \int_0^x \gamma \, d\Lambda_0^d, \quad x \in (0,1)^d. \quad (7.3)$$

(b) For each $0 \leq \vartheta \leq 1$ the function $f_{\vartheta} := 1 + \vartheta g$ is the density of a copula with survival function $S_{\vartheta}(x) = \prod_{i=1}^d (1-x_i) S_{I_0,\vartheta}(x)$. The dependence part $S_{I_0,\vartheta}$ is given by

$$S_{I_0,\vartheta}(x) = \exp \left( (-1)^d \vartheta \int_0^x \gamma \, d\Lambda_0^d + o(\vartheta) \right) \quad (7.4)$$

as $\vartheta \to 0$ with a uniform remainder $o(\vartheta)$ on $[0,1-\epsilon]^d$ for $\epsilon > 0$.

Proof. (a) Observe that $S(x) = \prod_{i=1}^d (1-x_i) + \int_x^1 g \, dP_0^d$ holds. If we divide by the marginals, then (7.2) yields the result.

(b) Observe that $\vartheta \gamma$ corresponds to the density of (7.3). Taking the logarithm of (7.3) we obtain (7.4). 

Remark 7.4. (a) Let $S_{I_0,\vartheta} = \exp \left( (-1)^d \Lambda_{I_0,\vartheta} \right)$ be the exponent representation of (7.4). Up to the term $o(\vartheta)$ the exponent is related to $\vartheta \int_0^x \gamma \, d\Lambda_0^d$.

(b) Note that $g := \frac{d}{d\vartheta} \log f_\vartheta|_{\vartheta=0}$ is the score function of the densities $f_\vartheta$ at $\vartheta = 0$. These score functions can be used to obtain score tests for the null hypothesis of independence, i.e. $\vartheta = 0$, against higher degree dependence.

Statistical aspects of hazard dependence models will be studied in a forthcoming work.

8. Discussion and conclusion

The present article considered the representation of dependent random variables in terms of dependence hazard measures and the corresponding dependence parts of the joint survival function. Furthermore, we have examined a useful representation of correlated frailty models which led to the expression
of higher-dimensional dependence hazard rates in terms of moments of an exponential family. In particular, these rates are simply covariances of frailty variables for dimension 2. It has also been shown that correlated frailty models are minimum-infinitely divisible iff its frailty vector is sum-infinitely divisible. This equivalence yields an interesting one-to-one correspondence between the Lévy measure of the frailty vector and the exponent measure of the minimum-infinitely divisible correlated frailty model. It follows that analytical properties of such Lévy measures might as well be examined in the context of extreme value theory.

Throughout the article we have made strong use of the copula concept which enabled us to use arbitrary, continuous marginal distribution functions. Here, the special shared frailty model had been shown to have a natural connection to Archimedean copulas. Eventually, we have utilized this concept to analyze particularly interesting semiparametric quantities of dependence. In the final chapter an isometry between Hilbert spaces exposed the relation of this semiparametric function to tangents in the space of square-integrable functions. It is pointed out how score functions for dependence models can be transformed in terms of hazard dependence quantities.

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Appendix A. Continuation and formulas of Example 4.12(a)

The joint survival function of \((X_1, X_2, X_3)\) at time \(t = (t_1, t_2, t_3)\) is given by

\[
S(t) = \left(8\Delta t_1t_2t_3 + 4\Delta_{12}t_1t_2 + 4\Delta_{13}t_1t_3 + 4\Delta_{23}t_2t_3 + \sum_{j=1}^{3} 2\sigma_j^2 t_j + 1\right)^{-\frac{1}{2}},
\]

where \(\Delta = \sigma_1^2\sigma_2^2\sigma_3^2 + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2\sigma_3^2 - \sigma_{13}^2\sigma_2^2 - \sigma_{23}^2\sigma_1^2\) denotes the determinant of \(\Sigma\) and \(\Delta_{ij} = \sigma_i^2\sigma_j^2 - \sigma_{ij}^2\) is the determinant of the covariance matrix

\[
\begin{pmatrix}
\sigma_i^2 & \sigma_{ij} \\
\sigma_{ij} & \sigma_j^2
\end{pmatrix}
\]
of \( Z_i \) and \( Z_j \). The marginal survival function of \( X_i, \ i \in \{1,2,3\} \), is given by

\[
S_i(t_i) = S(t_i, 0) = \left(1 + 2\sigma_i^2 t_i\right)^{-\frac{1}{2}}
\]

so that

\[
S(t) = \left\{ \frac{\Delta}{\sigma_1^2 \sigma_2^2 \sigma_3^2} \prod_{i=1}^{3} (S_i^{-2}(t_i) - 1) \right. \\
&+ \sum_{(i,j)} \frac{\Delta_{ij}}{\sigma_i^2 \sigma_j^2} (S_i^{-2}(t_i) - 1) (S_j^{-2}(t_j) - 1) + \sum_{i=1}^{3} S_i^{-2}(t_i) - 1 \left\}^{-\frac{1}{2}} \right. \tag{A.1}
\]

holds; see Appendix B for a detailed derivation. Note that

\[
\Delta \left/ \sigma_i^2 \sigma_j^2 \right. = 1 + 2 \prod_{(i,j)} \varphi_{ij} - \sum_{(i,j)} \varphi_{ij}^2 \quad \text{and} \quad \Delta_{ij} / \sigma_i^2 \sigma_j^2 = 1 - \varphi_{ij}^2,
\]

i.e. the survival copula \( S \circ (S_1^{-1}, S_2^{-1}, S_3^{-1}) \) only depends on the bivariate parameters \( \varphi_{ij} \). Similarly, the bivariate survival function of \( X_i \) and \( X_j \) is given by

\[
S^{(i,j)}(t_i, t_j) = S(t_{(i,j)}, 0) = \left(1 + 2\sigma_i^2 t_i + 2\sigma_j^2 t_j + 4\Delta_{ij} t_i t_j\right)^{-\frac{1}{2}}
\]

\[
= \left(S_i^{-2}(t_i) S_j^{-2}(t_j) - \varphi_{ij}^2 [S_i^{-2}(t_i) - 1] [S_j^{-2}(t_j) - 1]\right)^{-\frac{1}{2}},
\]

see also Wienke (2011, Section 5.6). The dependence part of order 2 is given by

\[
S_{(i,j)}(t_i, t_j) = \frac{S^{(i,j)}(t_i, t_j)}{S_i(t_i) S_j(t_j)} = \left(1 - \varphi_{ij}^2 [1 - S_i^2(t_i)] [1 - S_j^2(t_j)]\right)^{-\frac{1}{2}}
\]

with hazard density

\[
\lambda_{(i,j)}(t_i, t_j) = 2\sigma_{(i,j)}^2 \left[ S^{(i,j)}(t_i, t_j) \right]^4.
\]

The dependence part of order 3 is given by \( S_{(1,2,3)}(t) \)

\[
= \left\{ \frac{\prod_{(i,j)} (-1 + \varphi_{ij}^2 [1 - S_i^2(t_i)] [1 - S_j^2(t_j)])}{-1 + \sum_{(i,j)} \varphi_{ij}^2 [1 - S_i^2(t_i)] [1 - S_j^2(t_j)] - 2 \prod_{(i,j)} \varphi_{ij}^2 \prod_{i=1}^{3} (1 - S_i^2(t_i))} \right\}^{\frac{1}{2}}.
\]

Note that \( \varphi_{12} = \varphi_{13} = 0 \) already implies \( S_{(1,2,3)} \equiv 1 \) as well as \( S = S_1 S^{(2,3)} \), even if \( \varphi_{23} \neq 0 \).
Appendix B. Proof of Formula (A.1)

Let $\Sigma^{-1}$ denote the inverse of the covariance matrix $\Sigma$ and let

$$
(g_{ij})_{i,j=1,2,3} := \Delta \Sigma^{-1} = \begin{pmatrix}
\sigma_2^2\sigma_3^2 - \sigma_2^2 & \sigma_1\sigma_2\sigma_3 - \sigma_1\sigma_2^2 & \sigma_1\sigma_2\sigma_3 - \sigma_1\sigma_3^2 \\
\sigma_1\sigma_2\sigma_3 - \sigma_1\sigma_2^2 & \sigma_1^2\sigma_2^2 - \sigma_1^2 & \sigma_1\sigma_2\sigma_3 - \sigma_1\sigma_3^2 \\
\sigma_1\sigma_2\sigma_3 - \sigma_1\sigma_3^2 & \sigma_1\sigma_2\sigma_3 - \sigma_1\sigma_3^2 & \sigma_1^2\sigma_2^2 - \sigma_1^2
\end{pmatrix}
$$

define the matrix $(g_{ij})_{i,j \in \{1,2,3\}}$. The joint probability density function of $(Z_1, Z_2, Z_3)$ is given by

$$
f(z) = \frac{1}{\sqrt{(2\pi)^3\Delta}} \exp \left( -\frac{1}{2} \left\{ g_{11}z_1^2 + g_{22}z_2^2 + g_{33}z_3^2 + 2g_{12}z_1z_2 + 2g_{13}z_1z_3 + 2g_{23}z_2z_3 \right\} \right).
$$

By Lemma 4.6 the survival function of the correlated frailty model $X = \left(\frac{Y_i}{Z_i}\right)_{i \in \{1,2,3\}}$ is given by

$$
S(t) = \int \int \int \exp \left( -\sum_{i=1}^{3} t_i z_i \right) f(z) \, dz_1 \, dz_2 \, dz_3.
$$

Consider

$$
E : = \int \int \int \exp \left( -\sum_{i=1}^{3} t_i z_i \right) \\
\times \exp \left( -\frac{1}{2} \left\{ g_{11}z_1^2 + g_{22}z_2^2 + g_{33}z_3^2 + 2g_{12}z_1z_2 + 2g_{13}z_1z_3 + 2g_{23}z_2z_3 \right\} \right) \\
\times \exp \left( -t_1z_1^2 - t_3z_3^2 - \frac{1}{2\Delta} g_{22}z_2^2 - \frac{1}{2\Delta} g_{33}z_3^2 - \frac{1}{\Delta} g_{23}z_2z_3 \right) \\
\times \int \exp \left\{ -t_1z_1^2 - \frac{1}{2\Delta} g_{11}z_1^2 - \frac{1}{\Delta} [g_{12}z_2 + g_{13}z_3] z_1 \right\} \, dz_1 \, dz_2 \, dz_3.
$$

Defining

$$
a_1 := t_1 + \frac{g_{11}}{2\Delta}, \quad b_1 := \frac{1}{2\Delta} [g_{12}z_2 + g_{13}z_3], \quad y_1 := 2\Delta t_1 + g_{22}
$$
we obtain
\[
\int \exp \left\{ -t_1 z_1^2 - \frac{1}{2\Delta} g_{11} z_1^2 - \frac{1}{\Delta} \left[ g_{12} z_2 + g_{13} z_3 \right] z_1 \right\} \, dz_1
\]
\[
= \int \exp \left\{ -a_1 z_1^2 - 2b_1 z_1 \right\} \, dz_1
\]
\[
= \sqrt{\frac{\pi}{a_1}} \exp \left( \frac{b_1^2}{a_1} \right)
\]
\[
= \sqrt{\frac{2\pi \Delta}{y_1}} \exp \left( \frac{1}{2\Delta y_1} \left[ g_{12}^2 z_2^2 + 2g_{12}g_{13} z_2 z_3 + g_{13}^2 z_3^2 \right] \right)
\]

and therefore
\[
E = \sqrt{\frac{2\pi \Delta}{y_1}} \int \int \exp \left\{ - \left[ t_2 + \frac{g_{22}}{2\Delta} - \frac{g_{12}^2}{2\Delta y_1} \right] z_2^2 \right.
\]
\[
- \left. 2 \left[ \left( \frac{1}{2\Delta} g_{23} - \frac{g_{12}g_{13}}{2\Delta y_1} \right) z_3 \right] z_2 \right\} \, dz_2 \times \exp \left\{ - \left[ t_3 + \frac{g_{33}}{2\Delta} - \frac{g_{13}^2}{2\Delta y_1} \right] z_3^2 \right\} \, dz_3.
\]

With
\[
y_2 := 2\Delta t_2 + g_{22}, \quad a_2 := \frac{2t_2 \Delta y_1 + g_{22} y_1 - g_{12}^2}{2\Delta y_1} = \frac{y_1 y_2 - g_{12}^2}{2\Delta y_1}
\]

and
\[
b_2 := \frac{z_3}{2\Delta} \left( g_{23} - \frac{g_{12} g_{13}}{y_1} \right), \quad y_3 := 2\Delta t_3 + g_{33}
\]

we obtain
\[
\int \exp \left\{ - \left[ t_2 + \frac{g_{22}}{2\Delta} - \frac{g_{12}^2}{2\Delta y_1} \right] z_2^2 \right. - \left. 2 \left[ \left( \frac{1}{2\Delta} g_{23} - \frac{g_{12} g_{13}}{2\Delta y_1} \right) z_3 \right] z_2 \right\} \, dz_2
\]
\[
= \int \exp \left\{ -a_2 z_2^2 - 2b_2 z_2 \right\} \, dz_2
\]
\[
= \sqrt{\frac{\pi}{a_2}} \exp \left( \frac{b_2^2}{a_2} \right)
\]
\[
= \sqrt{\frac{2\pi \Delta y_1}{y_1 y_2 - g_{12}^2}} \exp \left\{ \frac{y_1}{y_1 y_2 - g_{12}^2} \frac{z_3^2}{2\Delta} \left( g_{23}^2 - 2\frac{g_{12} g_{13} g_{23}}{y_1} + \frac{g_{12}^2 g_{13}^2}{y_1^2} \right \}ight)
and thus

\[ S(t) = \sqrt{\frac{\Delta}{2\pi(y_1y_2 - g_{12}^2)}} \times \int \exp \left\{ -z_3^2 \left[ \frac{y_1y_3 - g_{13}^2}{2\Delta y_1} - \frac{y_1}{2\Delta (y_1y_2 - g_{12}^2)} \times \left( g_{23}^2 - 2g_{12}g_{13}g_{23} \frac{y_2}{y_1^2} + \frac{g_{12}^2g_{13}^2}{y_1^2} \right) \right] \right\} \, dz_3 \]

\[ = \sqrt{\frac{\Delta}{2\pi(y_1y_2 - g_{12}^2)}} \times \int \exp \left\{ -z_3^2 \left[ \frac{y_1y_2y_3 - g_{23}^2y_1 - g_{13}^2y_2 - g_{12}^2y_3 + 2g_{12}g_{13}g_{23}}{2\Delta (y_1y_2 - g_{12}^2)} \right] \right\} \, dz_3 \]

\[ = \Delta \left( y_1y_2y_3 - g_{23}^2y_1 - g_{13}^2y_2 - g_{12}^2y_3 + 2g_{12}g_{13}g_{23} \right)^{-\frac{1}{2}} \]

\[ = (8\Delta t_1t_2t_3 + 4g_{33}t_1t_2 + 4g_{22}t_1t_3 + 4g_{11}t_2t_3 + 2\sigma_1^2t_1 + 2\sigma_2^2t_2 + 2\sigma_3^2t_3 + 1)^{-\frac{1}{2}}. \]

With \( S_i^{-1}(u_i) = \frac{u_i^2 - 1}{2\sigma_i^2} \) this yields (A.1).

Appendix C. Further Proofs

Proof of Example 4.11 (a) is obvious since \((Z_1, \ldots, Z_d)\) is uniquely determined by its two-dimensional marginals.

(b) By Lemma 4.6 (a) the vector \((X_1, \ldots, X_d)\) has independent components iff \((W_1, \ldots, W_d)\) does. Thus, the proof for (ii) is trivial since the exponential function is a bijective mapping. To prove (b) for (i) we introduce

\[ a = Cov(Z_i, Z_j) Var(Z_i)^{-1} \] and we show that \(Z_i^2\) is independent of \(Z_j^2\) iff \(Z_i\) is independent of \(Z_j\) for arbitrary indices \(i \neq j\). Without loss of generality let \(E(Z_i) = E(Z_j) = 0\) and \(Var(Z_i) = Var(Z_j) = 1\). It is well known that, for a multivariate normally distributed vector \((Z_i, Z_j)\), the random variables \(Z_i\) and \(Z_j - aZ_i\) are independent. Now consider \(Cov(Z_i^2, Z_j^2) = E((Z_i^2 - 1)(Z_j^2 - 1))\) and repeatedly replace \(Z_j\) by \((Z_j - aZ_i) + aZ_i\) so that this covariance is expressed through expectations only depending on \(Z_i\) and \(Z_j\).
\(Z_j - aZ_i\). The independence of these two random variables yields

\[
\text{Cov}(Z_i^2, Z_j^2) = E((Z_i^2 - 1)(Z_j^2 - 1)) \\
= E((Z_i^2 - 1)((Z_j - aZ_i)^2 + a^2Z_i^2 + 2aZ_i(Z_j - aZ_i) - 1)) \\
= 0 + a^2E((Z_i^2 - 1)Z_i^2) + 0 - 0 = a^2(E(Z_i^4) - 1).
\]

Hence \(\text{Cov}(Z_i^2, Z_j^2) = 0\) iff \(a = 0\) iff \(\text{Cov}(Z_i, Z_j) = 0\).

\[\square\]

**Remark.** Let \(X\) be a correlated frailty model with frailty random vector \(W\). Then the survival function \(S\) of \(X\) is minimum-infinitely divisible iff \(\mathcal{L}(W)\) is sum-infinitely divisible.

**Proof.** Having the sum-infinite divisibility of \(\mathcal{L}(W)\), the minimum-infinite divisibility of \(S\) is obvious: this is a simple generalization of Lemma 4.10(b).

On the other hand, if \(S = (S_n)^n\) is minimum-infinitely divisible, then it is easy to see via induction that for all \(n_1, \ldots, n_d \in \mathbb{N}_0\) and \(t \in (0, \infty)^d\)

\[
(-1)^{\sum_{i=1}^d n_i} \left( \prod_{i \in I} \frac{\partial^{n_i}}{\partial t_i^{n_i}} \right) S_n(t) \geq 0.
\]

An application of a multivariate version of the Bernstein-Widder theorem (Theorem 1.3.1 of Zocher (2005)) now shows that \(S_n\) is the Laplace transform of a probability measure on \(\mathbb{R}^d\). Hence, \(S_n\) is the survival function of a \(d\)-dimensional correlated frailty model. Finally, another application of Lemma 4.10(b) and the fact that each distribution determines a unique Laplace transform show the sum-infinite divisibility of \(\mathcal{L}(W)\).

\[\square\]

**Proof of Lemma 5.2.** Inserting the definition of \(\lambda_I\) and the Lévy-Khintchine formula yields

\[
\lambda_I(t_I) = \frac{d}{dt_I} \Lambda_I(t_I) = (-1)^{|I|} \frac{d}{dt_I} \log \psi_{Q_0}(t_I, 0) \\
= (-1)^{|I|} \frac{d}{dt_I} \left( -\langle t_I, b_I \rangle + \int_{[0,\infty)^d\setminus\{0\}} \left( e^{-\langle t_I, x_I \rangle} - 1 \right) \, d\eta_0(x) \right).
\]

Hence, the proof for \(I = \{i\}\) is covered by the following considerations for a general index set \(I \neq \emptyset\). Note that, for fixed \(t_I \in (0, \infty)^{|I|}\), the absolute value of the function

\[
x_I \mapsto \frac{d}{dt_I} \left( e^{-\langle t_I, x_I \rangle} - 1 \right) = (-1)^{|I|} \prod_{i \in I} x_i e^{-\langle t_I, x_I \rangle}
\]

is 33
is bounded by an $\eta_0$-integrable function which is independent of $t_I$:

$$\left| \prod_{i \in I} x_i e^{-<t_i, x_i>} \right| \leq \|x_I\|1\{\|x_I\| \leq 1\} + M1\{\|x_I\| > 1\} \leq M \min(1, \|x_I\|),$$

for a constant $M \geq 1$, and this is integrable with respect to $\eta_0$. It follows that differentiation and integration are allowed to change places and eventually the definition of $\eta(t_I, 0)$ concludes the proof via induction over $d$. □

**Proof of Lemma 5.4.** The proof for $|I| = 1$ is a simpler version of the following proof; thus, it is left to the reader. Applying Lemma 5.2 and Fubini’s theorem we get

$$\Lambda_I(t_I) = \int_0^{t_I} \lambda_I(s_I) ds_I = \int_0^{t_I} \int_{[0, \infty)^d \setminus \{0\}} \prod_{i \in I} x_i \exp(-x_i s_i) d\eta_0(x) ds_I$$

$$= \int_{[0, \infty)^d \setminus \{0\}} \prod_{i \in I} \left( \int_0^{t_i} x_i \exp(-x_i s_i) ds_i \right) d\eta_0(x)$$

$$= \int_{[0, \infty)^d \setminus \{0\}} \prod_{i \in I} (1 - \exp(-x_i t_i)) d\eta_0(x)$$

$$= \int_{[0, \infty)^d \setminus \{0\}} \sum_{J \subset I} (-1)^{|J|} \prod_{j \in J} \exp(-x_j t_j) d\eta_0(x)$$

$$= \int_{[0, \infty)^d \setminus \{0\}} \sum_{J \subset I} (-1)^{|J|} \exp(-<x_J, t_J>) d\eta_0(x).$$

The linearity of the integral and the definition of $\eta(t_I, 0)$ finish this proof. □
Appendix D. Further Graphical Illustrations

Figure D.5: Plots and contour curves for the dependence hazard rate derivative $\gamma_{0,(i,j)}$ of a model with Frank copula as survival copula over the range $[0,1]^2$, with different values of $\theta$. 
Figure D.6: Plots and contour curves for the dependence hazard rate derivative $\gamma_{0,(i,j)}$ of a model with Frank copula as survival copula over the range $[0, 1]^2$, with different values of $\theta$. 

(a) $\theta = 0.5$  
(b) $\theta = 0.5$  
(c) $\theta = 5$  
(d) $\theta = 5$  
(e) $\theta = 10$  
(f) $\theta = 10$
Figure D.7: Plot for the dependence hazard rate derivative $\gamma_{0,(1,2)} \equiv -1$ in the proportional hazard rate dependence model (6.9) for $\beta = 1$ over the range $[0, 1]^2$. 
Figure D.8: Plots and contour curves for the dependence hazard rate derivative $\gamma_{0,(i,j)}$ of a bivariate correlated frailty model based on a bivariate chi-squared distribution over the range $[0, 1]^2$, with different values of $\rho_{ij}^2$.

Appendix E. References

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