Odd-parity stability of hairy black holes in $U(1)$ gauge-invariant scalar-vector-tensor theories

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In scalar-vector-tensor theories with $U(1)$ gauge invariance, it was recently shown that there exists a new type of hairy black hole (BH) solutions induced by a cubic-order scalar-vector interaction. In this paper, we derive conditions for the absence of ghosts and Laplacian instabilities against odd-parity perturbations on a static and spherically symmetric background for most general $U(1)$ gauge-invariant scalar-vector-tensor theories with second-order equations of motion. We apply those conditions to hairy BH solutions arising from the cubic-order coupling and show that the odd-parity stability in the gravity sector is always ensured outside the event horizon with the speed of gravity equivalent to that of light. We also study the case in which quartic-order interactions are present in addition to the cubic coupling and obtain conditions under which black holes are stable against odd-parity perturbations.

I. INTRODUCTION

General Relativity (GR) is a consistent theory of gravity describing the gravitational law on Solar-System scales. On the other hand, the observational evidence of late-time cosmic acceleration 1–3 suggests that one needs to introduce an unknown component dubbed dark energy in the context of GR. An alternative way of explaining the cosmic acceleration is to modify the gravitational law at large distances. Indeed, there have been many attempts for constructing models based on large-distance modifications of gravity, while recovering the behavior close to GR inside the Solar System 4.

If we turn our attention to the extreme short-distance or high-energy physics like the big bang and gravitational collapse, it is known that singularities inevitably arise in GR 5. In such strong gravitational regimes, we cannot exclude a possibility that GR is subject to modifications. In particular, after the detection of gravitational waves from black hole (BH) mergers 6, we are entering a golden era in which the physics of BHs and their surroundings can be observationally probed with increasing accuracy. This will shed new light on the possible deviation from GR in the nonlinear regime of gravity.

In GR, the property of BHs is characterized by three “hairs”–mass $M$, electric charge $Q$, and angular momentum $a$ 7. In theories beyond GR, the existence of additional degrees of freedom (DOFs) can give rise to new hairs to the field configuration and spacetime metric. The theories containing a scalar field $\phi$ coupled to gravity besides two tensor polarizations arising from the gravity sector are dubbed scalar-tensor theories 8. In particular, Horndeski 9 constructed most general scalar-tensor theories with second-order equations of motion. In shift-symmetric Horndeski theories invariant under the shift $\phi \rightarrow \phi + b$, where $b$ is a constant, there exists a no-hair theorem for static and spherically symmetric BHs based on the regularity of a Noether current on the horizon 10. It is however possible to realize a hairy BH solution for $\phi$ linearly coupled to a Gauss-Bonnet term 11 by evading one of the conditions assumed in Ref. 10. If we allow for a time-dependence of $\phi$ or abandon the shift symmetry, there are other hairy BHs arising in Horndeski theories 12, 13 (see also Ref. 14). The stability analyses of black holes in scalar-tensor theories were also performed in Refs. 15, 16.

For a vector field coupled to gravity, it is known that generalized Proca theories 17–20 are the most general vector-tensor theories with second-order equations of motion. Apart from a specific intrinsic vector-mode coupling advocated by Horndeski in 1976 21, the $U(1)$ gauge invariance is explicitly broken by the presence of derivative interactions or nonminimal couplings to gravity. The breaking of $U(1)$ gauge invariance leads to the propagation of a longitudinal scalar besides two transverse vector modes and two tensor polarizations. In vector-tensor theories, the existence of a temporal vector component gives rise to a bunch of hairy BH solutions 22–24 without tunings of the models. The stability analysis against odd-parity perturbations on a static and spherically symmetric background 25 shows that some BH solutions with nontrivial behavior of the longitudinal mode $A_1$ around the horizon are excluded (including those found in Ref. 22)). A healthy extension of generalized Proca theories 26 allows the possibility for evading the BH instability 27.

These two important classes of field theories, Horndeski and generalized Proca, can be unified in the framework of scalar-vector-tensor (SVT) theories with second-order equations of motion 28. The SVT theories can be classified into two cases depending on whether they respect the $U(1)$ gauge symmetry or not. In the presence of $U(1)$ gauge
symmetry the longitudinal component of a vector field vanishes, so that the propagating DOFs are five in total (one scalar, two transverse vectors, two tensor polarizations). The breaking of $U(1)$ gauge symmetry leads to the propagation of the longitudinal scalar besides the five DOFs. In the gauge-invariant case, two of the present authors found a new type of hairy BH solutions endowed with scalar and vector hairs in the presence of a cubic-order coupling \cite{29} (see also Refs. [30, 31]). It remains to be seen whether such hairy BHs are stable against perturbations on the static and spherically symmetric background.

In this paper, we study the stability of static and spherically symmetric BHs against odd-parity perturbations in $U(1)$ gauge-invariant SVT theories. Since the analysis of even-parity perturbations is generally more involved, we leave the full stability analysis against odd- and even-parity perturbations for a future work. In Sec. III we first revisit gauge-invariant SVT theories and hairy BH solutions found in Ref. [29]. In Sec. III we will derive conditions for the absence of ghosts and Laplacian instabilities by expanding the most general action of gauge-invariant SVT theories and hairy BH solutions found in Ref. [29] (see also Refs. [30, 31]). It remains to be seen whether such hairy BHs are stable against perturbations on the static and spherically symmetric background.

II. HAIRY BLACK HOLES IN GAUGE-IN Variant SVT THEORIES

We consider the theories with $U(1)$ gauge-invariant SVT interactions with a scalar field $\phi$ and a vector field $A_\mu$. Besides these new interactions, we also take into account the Einstein-Hilbert term $\frac{M^2_{\text{pl}}}{2} R/2$ in the Lagrangian, where $M_{\text{pl}}$ is the reduced Planck mass and $R$ is the Ricci scalar. Then, the theories we study are given by the action $[28]$

$$S = \int d^4x \sqrt{-g} \left( \frac{M^2_{\text{pl}}}{2} R + \sum_{i=2}^{4} \mathcal{L}_{\text{SVT}}^i \right), \tag{2.1}$$

where $g$ is a determinant of the metric tensor $g_{\mu \nu}$, and

$$\mathcal{L}_{\text{SVT}}^2 = f_2(\phi, X, F, \tilde{F}, Y), \quad \mathcal{L}_{\text{SVT}}^3 = \left[ f_3(\phi, X) g_{\rho \sigma} + \tilde{f}_3(\phi, X) \nabla_\rho \phi \nabla_\sigma \phi \right] \tilde{F}^{\mu \rho} \tilde{F}^{\nu \sigma} \nabla_\mu \nabla_\nu \phi, \quad \mathcal{L}_{\text{SVT}}^4 = f_4(\phi, X) L^{\mu \nu \alpha \beta} F_{\mu \nu} F_{\alpha \beta} + \left[ \frac{1}{4} f_{4,X}(\phi, X) + \tilde{f}_4(\phi) \right] \tilde{F}^{\mu \nu} \tilde{F}^{\alpha \beta} \nabla_\mu \nabla_\alpha \phi \nabla_\nu \nabla_\beta \phi. \tag{2.2}$$

Here, $\nabla_\mu$ is the covariant derivative operator, and

$$X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi, \quad F = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}, \quad \tilde{F} = -\frac{1}{4} F_{\mu \nu} \tilde{F}^{\mu \nu}, \quad Y = \nabla_\mu \phi \nabla_\nu F^{\mu \alpha} F^{\nu \alpha}, \qquad \tag{2.3}$$

with the anti-symmetric Levi-Civita tensor $\varepsilon^{\mu \nu \alpha \beta}$ satisfying the normalization $\varepsilon^{\mu \nu \alpha \beta} \varepsilon_{\mu \nu \alpha \beta} = -4!$. The double dual Riemann tensor $L^{\mu \nu \alpha \beta}$ is defined by

$$L^{\mu \nu \alpha \beta} = \frac{1}{4} \varepsilon^{\mu \nu \rho \sigma} \varepsilon^{\alpha \beta \gamma \delta} R_{\rho \sigma \gamma \delta}, \tag{2.4}$$

where $R_{\rho \sigma \gamma \delta}$ is the Riemann tensor. The function $f_2$ depends on $\phi, X, F, \tilde{F}, Y$, whereas $f_3, \tilde{f}_3, f_4$ are functions of $\phi, X$ with the notation $f_{4,X} \equiv \partial f_4/\partial X$. The function $\tilde{f}_4$ depends on $\phi$ alone. The dependence of $\tilde{F}$ and $Y$ in $\mathcal{L}_{\text{SVT}}^3$ on a static and spherically symmetric background either vanishes or can be expressed in terms of $X$ and $F$ as $Y = 4XF$. Therefore, we shall not consider such dependence in the following.

In Ref. [29], it was shown that hairy BH solutions exist on the static and spherically symmetric background given by the line element

$$ds^2 = -f(r) dt^2 + h^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{2.7}$$

where $f$ and $h$ depend on the radial coordinate $r$. On the background (2.7), the scalar field $\phi$ and the components of $A_\mu$ are functions of $r$, such that $\phi = \phi(r)$ and $A_\mu = (A_0(r), A_1(r), 0, 0)$ \cite{32}. Since we are now considering the
$U(1)$ gauge-invariant theory, the longitudinal mode $A_1(r)$ does not contribute to the vector-field dynamics. On the background (2.7), the quantities $X$ and $F$ reduce, respectively, to $X = -\dot{h}\phi'^2/2$ and $F = hA_0^2/(2f)$, where a prime represents a derivative with respect to $r$.

The background equations of motion following from the variation of the action (2.1) with respect to $f, h, \phi, A_0$ are given, respectively, by 20

\[
M^2_{pl} f h' = M^2_{pl} f (1 - h) + r^2 \left[ f_{f2} - h A_0^2 f_{f2,f} \right] - 2r h^2 \phi' A_0^2 f_3 + h A_0^2 \{ 4(h - 1) f_4 - h^2 \phi'^2 (f_{4,X} + 2\dot{f}_4) \},
\]

(2.8)

\[
M^2_{pl} rh f' = M^2_{pl} f (1 - h) + r^2 \left[ f_{f2} + f h \phi'^2 f_{r2} - h A_0^2 f_{f2,f} \right] - 2r h^2 \phi' A_0^2 \{ 3 f_3 - h^2 \phi'^2 f_{3,X} \}
\]

\[
+ h A_0^2 \left[ 4(3h - 1) f_4 - h(9h - 4) \phi'^2 f_{4,X} + h^3 \phi'^4 f_{4,X} X - 10h^2 \phi'^2 \dot{f}_4 \right],
\]

(2.9)

\[
J'_\phi = \mathcal{P}_\phi,
\]

(2.10)

\[
J'_A = 0,
\]

(2.11)

where

\[
J_\phi = -\sqrt{\frac{h}{f}} \left[ r^2 f f_{2,X} \phi' - 2h A_0^2 (2h \dot{f}_4 + 3h f_{4,X} - 2f_{4,X}) \phi' + 2r h^2 A_0^2 f_{3,X} \phi'^2 + h^3 A_0^2 f_{4,X} X \phi'^3 - 2r h A_0^2 f_3 \right],
\]

(2.12)

\[
\mathcal{P}_\phi = \frac{1}{\sqrt{fh}} \left[ r^2 f f_{2,\phi} + h A_0^2 \{ 4 f_{4,\phi} + 2h (r \phi' f_{3,\phi} - 2f_{4,\phi}) + h^2 (f_{4,X} \phi + 2f_4, \phi \phi'^2) \} \right],
\]

(2.13)

\[
J_A = \sqrt{\frac{h}{f}} A_0^2 \left[ r^2 f f_{2,\phi} + 4rh \phi' f_3 + 8(1 - h) f_4 + 2h^2 \phi'^2 (f_{4,X} + 2\dot{f}_4) \right].
\]

(2.14)

The current $J_A$ is conserved due to the $U(1)$ gauge symmetry. The coupling $\dot{f}_3$ does not appear in the background Eqs. (2.8, 2.11) due to the underlying background symmetry.

In Ref. 29, it was shown that hairy BH solutions exist for the theories given by the functions $f_2 = X + F$ and $f_3 = \beta_3$, where $\beta_3$ is a constant. Provided that the cubic coupling $f_3 = \beta_3$ is present, there are also hairy BH solutions in the presence of quartic couplings $f_4 = \beta_4 X^n$, where $\beta_4$ and $n \geq 0$ are constants. Consider the theories with the functions

\[
f_2 = X + F, \quad f_3 = \beta_3, \quad f_4 = \beta_4, \quad \dot{f}_4 = 0.
\]

(2.15)

The event horizon is characterized by the radial distance $r_h$ satisfying $f(r_h) = 0$ and $h(r_h) = 0$. In the vicinity of the horizon, the iterative solutions to Eqs. (2.8-2.11), expanded up to the order of $(r/r_h - 1)^2$, are 20

\[
f = (1 - \mu) \left( \frac{r}{r_h} - 1 \right) - 1 - 2\mu + 12\tilde{\beta}_3^2 \mu^2 (1 - \mu) + 4\tilde{\beta}_4 (24\tilde{\beta}_3 \mu^2 - 40\tilde{\beta}_4 \mu + 3\mu^2 + 16\tilde{\beta}_4 - 9\mu + 4) \left( \frac{r}{r_h} - 1 \right)^2,
\]

(2.16)

\[
h = (1 - \mu) \left( \frac{r}{r_h} - 1 \right) - 1 - 2\mu - 4\tilde{\beta}_3^2 \mu^2 (1 - \mu) - 4\tilde{\beta}_4 (8\tilde{\beta}_3 \mu^2 + 8\tilde{\beta}_4 \mu + \mu^2 - 16\tilde{\beta}_4 + 5\mu - 4) \left( \frac{r}{r_h} - 1 \right)^2,
\]

(2.17)

\[
A_0 = a_0 + \frac{2\mu}{1 + 8\tilde{\beta}_4} M_{pl} \left( \frac{r}{r_h} - 1 \right) - \frac{2\mu}{(1 + 8\tilde{\beta}_4)^5} M_{pl} \left[ 1 + 4\tilde{\beta}_3^2 \mu (2 - \mu) + 4\tilde{\beta}_4 - 32\tilde{\beta}_4^2 \right] \left( \frac{r}{r_h} - 1 \right)^2,
\]

(2.18)

\[
\phi' = \frac{4\tilde{\beta}_3 \mu M_{pl}}{r_h (1 + 8\tilde{\beta}_4)} \left[ 1 - \frac{5 + 32\tilde{\beta}_3^2 \mu (1 - \mu) + 16\tilde{\beta}_4 (2 + \mu - 4\tilde{\beta}_4 + 8\tilde{\beta}_4 \mu)}{(1 + 8\tilde{\beta}_4)^2} \left( \frac{r}{r_h} - 1 \right) \right],
\]

(2.19)

where $\tilde{\beta}_3 = \beta_3 M_{pl}/r_h^2$, $\tilde{\beta}_4 = \beta_4 / r_h^2$, and $\mu$ is a constant in the range $0 < \mu < 1$. In the above expressions, we have chosen the branch $A_0^0 > 0$ at $r = r_h$. For $\beta_3 \neq 0$, there is a nonvanishing scalar hair ($\phi' \neq 0$). The couplings $\tilde{\beta}_3$ and $\tilde{\beta}_4$ lead to modifications to the metric components $f_{RN} = h_{RN} = (1 - r/r_h)/(1 - \mu r_h/r)$ and the temporal vector component $A_{0RN}^0 = P + Q/r$ of the Reissner-Nordström (RN) solution ($P$ and $Q$ are constants). At spatial infinity
(r \gg r_h), the iterative solutions, up to the order of 1/r^8, are given by
\begin{equation}
\begin{aligned}
f &= 1 - \frac{2M}{r} + \frac{Q^2}{2M_{\text{pl}}^2r^2} - 2\beta_3Q^2 - 2\beta_4MQ^2 - \frac{3\beta_4Q^4}{5M_{\text{pl}}^4r^6} + \frac{25\beta_4^2MQ^2}{7M_{\text{pl}}^6r^7} + \\
&\quad + \frac{3Q^2(M^2Q^2\beta_3^2 - 28\beta_3^2Q^2 - 256\beta_4^2M^2M_{\text{pl}}^2)}{14M_{\text{pl}}^4r^8},
\end{aligned}
\end{equation}
(2.20)
\begin{equation}
\begin{aligned}
h &= 1 - \frac{2M}{r} + \frac{Q^2}{2M_{\text{pl}}^2r^2} - 2\beta_3MQ^2 + \frac{2\beta_4Q^4}{5M_{\text{pl}}^4r^6} - \frac{2Q^2(\beta_3^2Q^2 - 64\beta_4^2M^2)}{7M_{\text{pl}}^6r^8},
\end{aligned}
\end{equation}
(2.21)
\begin{equation}
\begin{aligned}
A_0 &= P + \frac{Q}{r} - \frac{4\beta_4MQ}{r^4} + \frac{3\beta_3Q^3}{5M_{\text{pl}}^2r^5} - \frac{8Q(\beta_3^2Q^2 - 32\beta_4^2M^2)}{7r^7} + \frac{2MQ^3(7\beta_3^2M_{\text{pl}}^2 - 48\beta_4^2)}{7M_{\text{pl}}^4r^8},
\end{aligned}
\end{equation}
(2.22)
\begin{equation}
\begin{aligned}
\phi' &= \frac{2\beta_3Q^2}{r^4} - \frac{64\beta_3\beta_4MQ^2}{r^8},
\end{aligned}
\end{equation}
(2.23)
where M is a constant. Again, the coupling $\beta_3$ induces a nonvanishing scalar hair. The RN solution with $f = h = 1 - 2M/r + Q^2/(2M_{\text{pl}}^2r^2)$ and $A_0 = P + Q/r$ is subject to modifications by the couplings $\beta_3$ and $\beta_4$. Due to the current conservation (2.11), the $U(1)$ charge $Q$ at spatial infinity is related to the quantities $\mu$ and $r_h$ in the vicinity of the horizon, as $\sqrt{\mu(r_h^2 + 8\beta_4)M_{\text{pl}}} = -Q$. Solving Eqs. (2.8)-(2.11) with the functions (2.15) numerically, the iterative solutions (2.16)-(2.19) around the horizon smoothly connect to the solutions (2.20)-(2.23) at spatial infinity [29]. Thus, there are regular BHs endowed with scalar and vector hairs.

Provided that the cubic-coupling $f_3 = \beta_3$ is present, there are also hairy BH solutions for quartic-order power couplings $f_4 = \beta_4X^n$ with $n \geq 1$. For $n = 1$, the iterative solutions to $f, h, A_0, \phi'$ in the vicinity of the horizon are given by Eqs. (4.15)-(4.18) of Ref. [29]. At spatial infinity, the solutions to $f, h, A_0, \phi'$ for $n = 1$, are expressed in the forms (3.19)-(3.21) of Ref. [29] up to the order of 1/r^8, with the leading-order scalar derivative $\phi' = 2\beta_3Q^2/r^5$.

### III. GENERAL BH STABILITY AGAINST ODD-PARITY PERTURBATIONS

Let us consider small perturbations $h_{\mu\nu}$ on top of the static and spherically symmetric background (2.17). For the study of odd-parity perturbations we choose the Regge-Wheeler gauge $h_{ab} = 0$ [33, 34], where $a, b$ represent either $\theta$ or $\varphi$. Then, the metric perturbations corresponding to odd-mode perturbations are expressed in the form [15, 35]
\begin{equation}
\begin{aligned}
h_{tt} = h_{tr} = h_{rr} = 0, \quad h_{ta} = \sum_{l, m} Q_{lm}(t, r)E_{ab}\partial^bY_{lm}(\theta, \varphi), \quad h_{ra} = \sum_{l, m} W_{lm}(t, r)E_{ab}\partial^bY_{lm}(\theta, \varphi),
\end{aligned}
\end{equation}
(3.1)
where $Q_{lm}$ and $W_{lm}$ are functions of $t$ and $r$, and $Y_{lm}(\theta, \varphi)$ is the spherical harmonics. The tensor $E_{ab}$ is given by $E_{ab} = \sqrt{\gamma}\varepsilon_{ab}$, where $\gamma$ is the determinant of the metric $\gamma_{ab}$ on the two-dimensional sphere and $\varepsilon_{ab}$ is the antisymmetric symbol with $\varepsilon_{\theta\varphi} = 1$. The scalar field $\phi$ does not have odd-parity perturbations. The perturbations of $A_\mu$ for the odd-parity sector are given by [27]
\begin{equation}
\begin{aligned}
\delta A_t = \delta A_r = 0, \quad \delta A_a = \sum_{l, m} \delta A_{lm}(t, r)E_{ab}\partial^bY_{lm}(\theta, \varphi),
\end{aligned}
\end{equation}
(3.2)
where $\delta A_{lm}$ depends on $t$ and $r$.

#### A. Second-order action

We expand the action (2.1) up to second order in odd-parity perturbations. In doing so, we can set $m = 0$ without loss of generality. The integrations with respect to $\theta$ and $\varphi$ are performed by using the properties of spherical harmonics given in Appendix B of Ref. [27]. We also integrate the action by parts with respect to $t, r$ and finally employ the background Eqs. (2.28), (2.29), and (2.11) to eliminate the terms $f_2, f_{2, X}, f_{3, X}$. Then, the second-order action of odd-parity perturbations yields
\begin{equation}
\begin{aligned}
S_{\text{odd}}^{(2)} = \sum_{l, m} L \int dt dr \mathcal{L}_{\text{odd}}^{(2)},
\end{aligned}
\end{equation}
(3.3)
where \( L = l(l + 1) \), and

\[
\mathcal{L}_{\text{odd}}^{(2)} = r^2 \sqrt{\frac{f}{h}} \left[ \alpha_1 \left( \dot{W}_{lm} - Q_{lm}^2 + \frac{2}{r} Q_{lm} \right)^2 + 2(\alpha_2 \delta A_{lm}^\prime + \alpha_3 \delta A_{lm}) \left( \dot{W}_{lm} - Q_{lm}^2 + \frac{2}{r} Q_{lm} \right) + \alpha_4 \delta A_{lm}^2 \right. \\
+ \left. \alpha_5 \delta A_{lm}^2 \right] \tag{3.4}
\]

Here, a dot represents the derivative with respect to \( t \), and the coefficients \( \alpha_i \) are given by

\[
\begin{align*}
\alpha_1 &= \frac{M_{pl}}{4f_{r^2}}, \\
\alpha_2 &= \frac{h A_0'}{2f_{r^3}} \left[ r \phi' f_3 - 4f_4 + h \phi'^2 (f_4 + 2 \dot{f}_4) \right], \\
\alpha_3 &= \frac{-h A_0'}{2f_{r^3}} \left[ f_{2,F} r^2 + 4h (r \phi' f_3 - 2 \dot{f}_4) + 2h^2 \phi'^2 (f_4 + 2 \dot{f}_4) \right], \\
\alpha_4 &= \frac{1}{2f_{r^3}} \left[ f_{2,F} r + (2rh \phi'' + rh \phi' + 2h \phi') f_3 + 2h^2 \phi'^3 \dot{f}_3 - 4h' f_4 + h \phi' (2h \phi'' + h \phi') (f_4 + 2 \dot{f}_4) \right], \\
\alpha_5 &= -\frac{h}{2f_{r^3}} \left[ f_{2,F} r + h \phi' (2f + f' r) f_3 - 4f' h f_4 + f' h^2 \phi'^2 (f_4 + 2 \dot{f}_4) \right], \\
\alpha_6 &= -\frac{h}{4f_{r^3}} \left[ M_{pl}^2 \left\{ -4h A_0^2 f_4 + h^2 \phi'^2 A_0^2 (f_4 + 2 \dot{f}_4) \right\} \right], \\
\alpha_7 &= \frac{M_{pl}^2 f - 4h A_0^2 f_4}{4f_{r^2}^2}, \\
\alpha_8 &= \frac{2}{f_{r^3}^3} \left[ (2f h A_0' + f h' A_0 - h f' A_0^2) f_4 - f h A_0' \phi' (2h \phi'' + h \phi') f_4 + 2 f h \phi' A_0 f_4 + f_4 \phi' A_0 \right], \\
\alpha_9 &= -\frac{1}{4f_{r^3}^2} \left[ f_{2,F} f_2 + 2 f (2f h \phi'' + f' h \phi' + f h' \phi') f_3 + 2 f f' h^2 \phi'^3 \dot{f}_3 + 4(f'^2 h - f f' h') f_4 \\
+ f f' h \phi' (2h \phi'' + h \phi') (f_4 + 2 \dot{f}_4) \right]. \tag{3.5}
\end{align*}
\]

B. Dipole perturbations \((l = 1)\)

We first consider the dipole mode \( l = 1 \), i.e., \( L = 2 \). Since the perturbations \( h_{ab} \) identically vanish for \( l = 1 \), we cannot choose the Regge-Wheeler gauge. Under the gauge transformation \( x_\mu \rightarrow x_\mu + \xi_\mu \), where \( \xi_t = \xi_r = 0 \) and \( \xi_\theta = \sum \Lambda_{lm}(t, r) E_{ab} \partial^a Y_{lm}(\theta, \varphi) \) for odd-parity modes, the perturbations \( Q_{lm} \) and \( W_{lm} \) transform, respectively, to

\[
Q_{lm} \rightarrow Q_{lm} + \dot{\Lambda}_{lm}, \quad W_{lm} \rightarrow W_{lm} + \Lambda_{lm}' - \frac{2}{r} \Lambda_{lm}. \tag{3.6}
\]

For the dipole mode, we choose the gauge

\[
W_{lm} = 0, \tag{3.7}
\]

under which the quantity \( \Lambda_{1m} \) in \( \xi_\alpha \) is given by

\[
\Lambda_{1m}(t, r) = -r^2 \int d^3 \bar{r} \frac{W_{lm}(t, \bar{r})}{\bar{r}^2} + r^2 \mathcal{C}(t), \tag{3.8}
\]

where \( \mathcal{C}(t) \) is an arbitrary function of \( t \). We note that the terms proportional to \( L - 2 \) in the Lagrangian \([3.3]\) vanish for dipole perturbations. Varying the action \([3.3]\) with respect to \( W_{lm} \) and \( Q_{lm} \) and finally using the gauge condition \([3.7]\), we obtain

\[
\mathcal{E} = r^2 \sqrt{\frac{f}{h}} \left[ \alpha_1 \left( Q_{1m}' - \frac{2}{r} Q_{1m} \right) - (\alpha_2 \delta A_{1m}' + \alpha_3 \delta A_{1m}) \right]. \tag{3.10}
\]
The solution to Eq. (3.19) is given by $E = C_1/r^2$, where $C_1$ is a constant. Then, it follows that

$$
\alpha_1 \left( Q'_{1m} - \frac{2}{r} Q_{1m} \right) = \alpha_2 \delta A'_{1m} + \alpha_3 \delta A_{1m} + \frac{C_1}{r^4} \sqrt{\frac{\hbar}{f}},
$$

which can be written in the integrated form

$$
Q_{1m} = r^2 \int \frac{d\tau}{\alpha_1 r^2} \left( \alpha_2 \delta A'_{1m} + \alpha_3 \delta A_{1m} + \frac{C_1}{r^4} \sqrt{\frac{\hbar}{f}} \right) + r^2 C_2(t),
$$

where $C_2(t)$ is an arbitrary function of $t$. The residual gauge degree of freedom $C(t)$ in Eq. (3.3) can be fixed by choosing $C(t) = \int d\tau C_2(t)$.

On using Eq. (3.11) to eliminate the combination $Q'_{1m} - 2Q_{1m}/r$ from Eq. (3.4), the second-order Lagrangian (3.4) yields

$$
\mathcal{L}^{(2)} = r^2 \sqrt{\frac{fr}{\alpha_1 \alpha_2}} \left[ \alpha_2 \delta A'_{1m} + \left( \frac{\alpha_2}{\alpha_1} \right) \delta A_{1m} + \left( \frac{2\alpha_2 \alpha_3}{\alpha_1} \right) \delta A_{1m}^{(2)} + \left( \frac{2\alpha_2 \alpha_3}{\alpha_1} \right) \delta A_{1m}^{(2)} - \frac{\hbar C_1^2}{f \alpha_1 r^8} \right].
$$

This shows that the vector-field perturbation $\delta A_{1m}$ is the only propagating DOF for dipole perturbations. The ghost is absent as long as the first term in the square bracket of Eq. (3.13) is positive, i.e.,

$$
\alpha_4 > 0.
$$

In Fourier space, we consider the solution to the vector-field perturbation in the form $\delta A_{1m} \propto e^{i(\omega t - kr)}$, where $\omega$ is a frequency and $k$ is a comoving wavenumber. In the small-scale limit, the dominant contributions to $\mathcal{L}^{(2)}_{\text{odd}}$ are the first two terms in the square brackets of Eq. (3.13). Then, the dispersion relation corresponds to $\alpha_4 \omega^2 + (\alpha_5 - \alpha_2^2/\alpha_1) k^2 = 0$.

The speed of the perturbation $\delta A_{1m}$ along the radial direction in proper time is given by $\hat{c}_r = dr_*/d\tau$, where $dr_*/d\tau = dr/\sqrt{\hbar}$ and $d\tau = \sqrt{\hbar}dt$. This is related to the propagation speed $c_r = dr/dt$ in the coordinates $t$ and $r$, as $\hat{c}_r = \sqrt{\hbar} c_r$, where $\omega = \hat{c}_r k$. From the dispersion relation in the small-scale limit, we obtain

$$
c_r^2 = \frac{\alpha_3^2 - \alpha_1 \alpha_5}{f \hbar \alpha_1 \alpha_4}.
$$

We require the condition $c_r^2 \geq 0$ for the absence of Laplacian instabilities of vector-field perturbations in the odd-parity sector.

### C. Perturbations with $l \geq 2$

Let us proceed to the discussion of stability conditions for odd-parity perturbations with $l \geq 2$. In the Lagrangian (3.3), there are two dynamical fields $W_{lm}$ and $\delta A_{lm}$, while the field $Q_{lm}$ is non-dynamical. To study the propagation of dynamical DOFs, it is convenient to rewrite the Lagrangian (3.3) in terms of a Lagrangian multiplier $\chi(t, r)$, as

$$
\mathcal{L}^{(2)}_{\text{odd}} = r^2 \sqrt{\frac{\hbar}{\alpha_1 \alpha_2}} \left[ \alpha_2 \delta A'_{lm} + \left( \frac{\alpha_2}{\alpha_1} \right) \delta A_{lm} + \left( \frac{2\alpha_2 \alpha_3}{\alpha_1} \right) \delta A_{lm}^{(2)} + \left( \frac{2\alpha_2 \alpha_3}{\alpha_1} \right) \delta A_{lm}^{(2)} - \frac{\hbar C_1^2}{f \alpha_1 r^8} \right],
$$

whose variation with respect to $\chi$ leads to

$$
\chi = W_{lm} - Q_{lm} + \frac{2}{f} Q_{lm} + \frac{\alpha_2 \delta A'_{lm} + \alpha_3 \delta A_{lm}}{\alpha_1}.
$$

Substituting Eq. (3.17) into Eq. (3.16), we recover the original second-order Lagrangian (3.4). Varying the Lagrangian (3.16) with respect to $W_{lm}$ and $Q_{lm}$, respectively, we obtain

$$
\alpha_1 \chi' - (L-2) \alpha_6 W_{lm} = 0,
$$

$$
\alpha_1 \chi' + \frac{(8fh + rf'h - rf'h)\alpha_1 + 2rf h\alpha_4}{2rf h} \chi + (L-2) \left( \alpha_7 Q_{lm} + \frac{\alpha_8}{2} \delta A_{lm} \right) = 0.
$$
We solve Eqs. (3.18) and (3.19) for $W_{lm}$ and $Q_{lm}$ respectively and substitute them into Eq. (5.10). After integrations by parts, the second-order Lagrangian is expressed in the form

$$\mathcal{L}_{\text{ord}}^{(2)} = r^2 \sqrt{\frac{f}{h}} \left( \hat{\chi}^i K \hat{\chi}^i + \hat{\chi}^{ii} G \hat{\chi}^i + \hat{\chi}^{ii} S \hat{\chi}^i + \hat{\chi}^{ii} M \hat{\chi}^i \right),$$

where $K, G, S, M$ are $2 \times 2$ matrices, with the vector

$$\vec{\chi} = (\chi, \delta A_{lm}).$$

This shows that there are two dynamical fields $\chi$ and $\delta A_{lm}$. The field $\chi$ arises from perturbations in the gravity sector (i.e., tensor modes), whereas the field $\delta A_{lm}$ corresponds to the vector degree of freedom. As we already mentioned, the perturbation of scalar field $\phi$ does not arise as a dynamical degree of freedom for odd-parity perturbations. The nonvanishing components of the matrices $K, G, S, M$ are given by

$$K_{11} = -\frac{\alpha_1^2}{\alpha_6}, \quad K_{22} = (L - 2) \alpha_4, \quad G_{11} = -\frac{\alpha_3^2}{\alpha_7}, \quad G_{22} = \frac{(L - 2)(\alpha_1 \alpha_5 - \alpha_2^2)}{\alpha_1},$$

$$S_{12} = S_{21} = -(L - 2) \left( \frac{\alpha_2 + \alpha_1 \alpha_8}{2 \alpha_7} \right),$$

$$M_{11} = -(L - 2) \alpha_1 - \frac{h}{fr^2 \alpha_7} \left( r^4 \sqrt{\frac{f}{h}} \alpha_1 \right)' + \frac{1}{2 r^2} \sqrt{\frac{h}{f}} \left[ r \alpha_1 \left( (f' h r - h' f r + 8 f h) \alpha_1 + 2 f h r \alpha_1' \right) \right]'$$

$$M_{22} = -(L - 2) \left( \frac{L - 2}{4 \alpha_7} \alpha_8^2 - L \alpha_9 + \frac{\alpha_3^2}{\alpha_1} \right) - \frac{1}{2 r^2} \sqrt{\frac{h}{f}} \left( r^2 \sqrt{\frac{f}{h}} \alpha_2 \alpha_3 \right)'$$

$$M_{12} = M_{21} = -(L - 2) \left( \frac{\alpha_3 - \frac{1}{2 r^4} \sqrt{\frac{h}{f}} \alpha_8 \left( r^4 \sqrt{\frac{f}{h}} \alpha_1 \right)' - \frac{1}{2 r^2} \sqrt{\frac{h}{f}} \left( r^2 \sqrt{\frac{f}{h}} \left( \alpha_2 - \frac{\alpha_1 \alpha_8}{2 \alpha_7} \right) \right)' \right).$$

Since there are no off-diagonal components for the matrix $K$, the no-ghost conditions correspond to $K_{11} > 0$ and $K_{22} > 0$, i.e.,

$$\alpha_6 < 0, \quad \alpha_4 > 0.$$  \hfill (3.23)

Let us consider the propagation of perturbations along the radial direction by assuming the solution of the form $\hat{\chi}^i \propto e^{i (\omega t - kr)}$. In the small-scale limit ($k \to \infty$), the dispersion relation is expressed as $\det (\omega^2 K + k^2 G) = 0$. The propagation speed $c_r$ in proper time can be derived by substituting $\omega = \sqrt{f} h c_{r,k}$ into the dispersion relation. Then, we obtain the following two expressions of $c_r^2$:

$$c_{r_1}^2 = -\frac{G_{11}}{f h K_{11}} = -\frac{\alpha_6}{f h \alpha_7},$$

$$c_{r_2}^2 = -\frac{G_{22}}{f h K_{22}} = \frac{\alpha_5^2 - \alpha_1 \alpha_5}{f h \alpha_1 \alpha_4}. \hfill (3.24)$$

We recall that, for dipole perturbations ($l = 1$), only the vector-field perturbation $\delta A_{lm}$ is dynamical with the propagation speed squared $c_r^2$ given by Eq. (3.15). This is equivalent to $c_{r_2}^2$ derived above, which corresponds to the propagation speed squared of vector-field perturbations. The other value $c_{r_1}^2$ is related to the propagation speed squared arising from the gravity sector. To avoid small-scale Laplacian instabilities along the radial direction, we require the two conditions $c_{r_1}^2 \geq 0$ and $c_{r_2}^2 \geq 0$.

In the limit that $L = l(l + 1) \gg 1$, the matrix $M$ contributes to the propagation speed $c_{\Omega}$ along the angular direction. In this limit, the matrix components $M_{11}$ and $M_{22}$ are given, respectively, by

$$M_{11} \simeq -L \alpha_1, \quad M_{22} \simeq \frac{L^2 (4 \alpha_7 \alpha_9 - \alpha_5^2)}{4 \alpha_7}. \hfill (3.26)$$

The off-diagonal components $M_{12}$ and $M_{21}$ also contain the term proportional to $L$, but their contributions to $c_{\Omega}$ can be neglected for $L \gg 1$. This is also the case for the matrix components of $S$. Assuming the solution of the form $\hat{\chi}^i \propto e^{i (\omega t - kr)}$, the dispersion relation is given by $\det (\omega^2 K + M) = 0$. In proper time, the propagation speed along
the angular direction is \( c_\Omega = \dot{c}_\Omega / \sqrt{f} \), where \( \dot{c}_\Omega = r \theta / \omega \). Substituting \( \omega^2 = c_\Omega^2 / r^2 = c^2_\Omega f l^2 / r^2 \) into the dispersion relation and solving it for \( c^2_\Omega \) with the limit \( L \gg 1 \), we obtain the following two expressions of \( c^2_\Omega \):

\[
\begin{align*}
  c^2_\Omega &= \frac{r^2 M_{11}}{l^2 f K_{11}} = \frac{r^2 \alpha_6}{f \alpha_1}, \\
  c^2_{\Omega 2} &= \frac{r^2 M_{22}}{l^2 f K_{22}} = \frac{r^2 (\alpha_6^2 - 4 \alpha_7 \alpha_9)}{4 f \alpha_4 \alpha_7},
\end{align*}
\]

which correspond to the propagation speed squares arising from the gravity sector and the vector-field perturbation, respectively. We require the two conditions \( c^2_\Omega \geq 0 \) and \( c^2_{\Omega 2} \geq 0 \) to avoid Laplacian instabilities along the angular direction.

Substituting the explicit forms of coefficients \( \alpha_1, \alpha_6, \alpha_7 \) into Eqs. (3.24) and (3.27), it follows that

\[
\begin{align*}
  c^2_{r1} &= 1 + \frac{h^2 A^2_{10} \phi^2 (f_{4,x} + 2 \tilde{f}_4)}{M_{pl}^2 f - 4 h A^2_{0} f_4}, \\
  c^2_{\Omega 4} &= 1 + \frac{h A^2_{0} (h \phi^2 (f_{4,x} + 2 \tilde{f}_4) - 4 f_4)}{M_{pl}^2 f}.
\end{align*}
\]

For the theories containing couplings up to cubic order, \( c^2_{r1} = 1 \) and \( c^2_{\Omega 4} = 1 \). The presence of quartic-order couplings \( f_4 \) and \( \tilde{f}_4 \) generally leads to the values of \( c^2_{r1} \) and \( c^2_{\Omega 4} \) different from 1.

**IV. ODD-PARITY STABILITY OF HAIRY BLACK HOLES**

We apply general stability conditions derived in Sec. III to concrete models with hairy BH solutions. Let us focus on models given by the functions

\[
f_2 = X + F, \quad f_3 = \beta_3, \quad f_4 = \beta_4 X^n, \quad \tilde{f}_4 = 0,
\]

where \( \beta_3, \beta_4, n \geq 0 \) are constants. In the following, we will study the three different cases: (A) \( \beta_4 = 0 \), (B) \( \beta_4 \neq 0 \), \( n = 0 \), and (C) \( \beta_4 \neq 0 \), \( n = 1 \), in turn.

**A. \( \beta_4 = 0 \)**

In this case, we have \( f_4 = 0 \) and \( \tilde{f}_4 = 0 \) in Eqs. (3.29), (3.28), and (3.30). Then, it follows that

\[
\alpha_6 = -\frac{h M_{pl}^2}{4 r^4}, \quad c^2_{r1} = 1, \quad c^2_{\Omega 4} = 1.
\]

The no-ghost condition \( \alpha_6 < 0 \) is satisfied outside the horizon \( (h > 0) \). The propagation speed squares \( c^2_{r1} \) and \( c^2_{\Omega 4} \) are the same as those in GR. This means that the quadratic and cubic couplings do not affect the stability conditions of perturbations in the gravity sector as one would expect.

Let us investigate the odd-parity stability associated with the vector-field perturbation \( \delta A_{lm} \). In the vicinity of the event horizon of hairy BHs, we resort to the iterative solutions (2.16)–(2.19) with \( \beta_4 = 0 \). Then, the quantities \( \alpha_4, c^2_{r2}, c^2_{\Omega 2} \) reduce, respectively, to

\[
\begin{align*}
  \alpha_4 &= \frac{1 + 4 \mu (1 - \mu) \beta_3^2}{2 r^2 (1 - \mu)^2} \left( \frac{r}{r_h} - 1 \right) + O((r/r_h - 1)^0), \\
  c^2_{r2} &= 1 + \frac{8 \mu (1 - \mu) \beta_3^2 [5 + 32 \mu (1 - \mu) \beta_3^2]}{1 + 4 \mu (1 - \mu) \beta_3^2} \left( \frac{r}{r_h} - 1 \right) + O((r/r_h - 1)^2), \\
  c^2_{\Omega 2} &= 1 + \frac{8 \mu (1 - \mu) \beta_3^2}{1 + 4 \mu (1 - \mu) \beta_3^2} \left( \frac{r}{r_h} - 1 \right) + O((r/r_h - 1)^2).
\end{align*}
\]

Since the constant \( \mu \) is in the range \( 0 < \mu < 1 \), all the stability conditions \( \alpha_4 > 0, c^2_{r2} \geq 0, \) and \( c^2_{\Omega 2} \geq 0 \) hold around \( r = r_h \) for arbitrary couplings \( \beta_3 \). In the limit that \( r \to r_h \), \( c^2_{r2} \) approaches 1, whereas \( c^2_{\Omega 2} \) approaches a constant different from 1.
At spatial infinity, the background solutions are given by Eqs. (2.20)-(2.23) with $\beta_4 = 0$. Then, it follows that

$$\alpha_4 = \frac{1}{2r^2} + \frac{M}{r^3} + O \left( \frac{1}{r^4} \right),$$

$$c_{r_2}^2 = 1 + \frac{20\beta_3^2Q^2}{r^6} + O \left( \frac{1}{r^7} \right),$$

$$c_{\Omega_2}^2 = 1 - \frac{43\beta_3^2Q^2}{r^6} + O \left( \frac{1}{r^7} \right),$$

and hence the stability conditions $\alpha_4 > 0$, $c_{r_2}^2 \geq 0$, and $c_{\Omega_2}^2 \geq 0$ are trivially satisfied.

In order to confirm the odd-parity stability of BHs in the intermediate regime between $r \approx r_h$ and $r \gg r_h$, we numerically compute the quantities $-\alpha_6, \alpha_4$ and $c_{r_2}^2, c_{\Omega_2}^2$ outside the horizon by using Eqs. (2.16)-(2.19) as boundary conditions around $r = r_h$. The numerical simulation of Fig. 1 corresponds to the coupling $\beta_3 = 1$ with $\mu = 0.5$. As we see in the left panel, both $-\alpha_6$ and $\alpha_4$ remain positive throughout the horizon exterior, so the no-ghost conditions are satisfied in this case.

In the right panel of Fig. 1 we observe that the deviation of the radial propagation speed squared $c_{r_2}^2$ from 1 approaches 0 in the limit $r \rightarrow r_h$, while, at large distances ($r \gg r_h$), it decreases according to $c_{r_2}^2 - 1 = 20\beta_3^2Q^2/r^6$.

The angular propagation speed squared $c_{\Omega_2}^2$ exhibits the deviation from 1 at the horizon, such that $c_{\Omega_2}^2 - 1 = 4\mu(1 - \mu)\beta_3^2/[1 + 4\mu(1 - \mu)\beta_3^2] > 0$. Since $c_{\Omega_2}^2 - 1 \simeq -4\beta_3^2Q^2/r^6 < 0$ at spatial infinity, $c_{\Omega_2}^2$ crosses the value 1 at an intermediate distance ($r \simeq 1.6r_h$ in Fig. 1). Numerically we confirmed that both $c_{r_2}^2 - 1$ and $|c_{\Omega_2}^2 - 1|$ are smaller than order 1 for $|\beta_3| \lesssim 1$, so there is no Laplacian instabilities of odd-parity perturbations outside the horizon.

Taking the limit $|\beta_3| \gg 1$ in Eqs. (4.4) and (4.7), the asymptotic values of $c_{r_2}^2$ and $c_{\Omega_2}^2$ at $r = r_h$ are 1 and 2, respectively. For $|\beta_3| \gtrsim 10$, our numerical simulations show that there are regions in which $c_{r_2}^2, c_{\Omega_2}^2$ as well as $\alpha_4$ temporally become negative outside the horizon. In such cases, the hairy BHs are unstable against odd-parity perturbations. In summary, as long as the cubic coupling is in the range

$$|\beta_3| \lesssim O(1),$$

there are neither ghost nor Laplacian instabilities throughout the horizon exterior.
It is worthwhile to mention that the cubic interaction \( f_4 \) is related to the \( g_5 \) coupling of generalized Proca theories \[17\,–\,20\). The Lagrangian \( \mathcal{L}_4^{A_\mu} \) with \( f_3 = 0 \) coincides with that of the \( g_5 \) coupling by replacing \( \nabla_\mu \phi \) with \( A_\mu \), in which case the scalar derivative \( \phi'(r) \) is the placeholder of the longitudinal component of \( A_\mu \).

B. \( \beta_4 \neq 0 \) and \( n = 0 \)

Let us consider the theories in which \( f_4 \) is a nonvanishing constant \( \beta_4 \). From Eq. (3.29), the radial propagation speed squared arising from the gravity sector yields

\[
c_{r1}^2 = 1, \quad (4.10)
\]

On the other hand, the angular propagation speed squared (3.30) reduces to \( c_{r11}^2 = 1 - 4 \beta_4 \eta A_0^2 / (M_\text{pl} f) \), which is different from 1 unlike the theories with \( \beta_4 = 0 \). Moreover, the coupling \( \beta_4 \) gives rise to the value of \( \alpha_6 \) different from \(-h M_\text{pl}^2 / (4r^4)\).

To estimate the quantities \( \alpha_6, \alpha_4, c_{r2}^2, c_{r11}^2, c_{r12}^2 \), we use the iterative solutions (2.11)–(2.14) in the vicinity of the horizon. Then, it follows that

\[
\alpha_6 = -\frac{(1 - \mu) M_\text{pl}^2}{4r_h^4} \left( 1 - \frac{8 \tilde{\beta}_4 \mu}{1 + 8 \tilde{\beta}_4} \right) \left( \frac{r}{r_h} - 1 \right) + O((r/r_h - 1)^2),
\]

\[
\alpha_4 = \frac{1}{2r_h^4 (1 - \mu)} \left[ 1 - 4(1 - \mu) \left( \frac{\tilde{\beta}_4}{1 + 8 \tilde{\beta}_4} \right) \left( \frac{r}{r_h} - 1 \right) \right] + O((r/r_h - 1)^0),
\]

\[
c_{r2}^2 = 1 + O(r/r_h - 1), \quad c_{r11}^2 = 1 - \frac{8 \tilde{\beta}_4 \mu}{1 + 8 \tilde{\beta}_4} + O(r/r_h - 1),
\]

\[
c_{r12}^2 = 1 + \frac{4 \tilde{\beta}_4 (1 + 8 \tilde{\beta}_4 + 16 \tilde{\beta}_4 \mu) (1 - \mu) + 4 \tilde{\beta}_4 (1 + 8 \tilde{\beta}_4) (1 - 5 \mu + 8 \tilde{\beta}_4 (1 - \mu) (1 - 2 \mu))}{(1 + 8 \tilde{\beta}_4) (4 \tilde{\beta}_4 \mu (1 - \mu) + (1 + 8 \tilde{\beta}_4) (1 - 4 \tilde{\beta}_4 + 4 \tilde{\beta}_4 \mu))} + O(r/r_h - 1). \quad (4.11)
\]

For the odd-parity stability of BHs around the horizon, we require the following three conditions:

\[
\left( 1 + 8 \tilde{\beta}_4 \right) \left[ 1 + 8 \tilde{\beta}_4 (1 - \mu) \right] > 0, \quad (4.12)
\]

\[
\left( 1 + 8 \tilde{\beta}_4 \right) \left[ 4 \tilde{\beta}_4 \mu (1 - \mu) + (1 + 8 \tilde{\beta}_4) \left( 1 - 4 \tilde{\beta}_4 (1 - \mu) \right) \right] > 0, \quad (4.13)
\]

\[
8 \mu (1 - \mu) \left[ 1 + 8 \tilde{\beta}_4 (1 + \mu) \right] \tilde{\beta}_4^2 + \left( 1 + 8 \tilde{\beta}_4 \right) \left[ 1 + 8 \tilde{\beta}_4 (1 - \mu) \right]^2 \geq 0. \quad (4.14)
\]

In the limit that \( \tilde{\beta}_4 \to 0 \), these conditions hold for \(-1/8 < \tilde{\beta}_4 < 1 / [4(1 - \mu)] \). This matches with the condition (6.8) of Ref. \[25\), derived for the sixth-order coupling \( G_6 = \beta_4 = \text{constant} \) of \( U(1) \) gauge-invariant generalized Proca theories with the branch \( A_1 = 0 \) after replacing \( \beta_4 \) with \( \beta_6 / (4r_h^2) \). In this limit, there is only a vector hair associated with the temporal component \( A_0 \) as in the case of BH solutions advocated in Ref. \[30\). The cubic coupling \( \tilde{\beta}_3 \) gives rise to the scalar hair with a nonvanishing value of \( \phi' \). For \( \tilde{\beta}_4 > 0 \), the condition (4.13) gives the following bound

\[
\tilde{\beta}_4 < \frac{1 + \mu + \sqrt{(3 - \mu)^2 + 32 \tilde{\beta}_4^2 \mu (1 - \mu)^2}}{16 (1 - \mu)}, \quad (4.15)
\]

whereas the other conditions (4.12) and (4.14) are automatically satisfied. The presence of cubic coupling \( \tilde{\beta}_3 \) leads to the larger upper limit of \( \tilde{\beta}_4 \) relative to the case \( \beta_4 = 0 \).

In the regime \( r \gg r_h \), the leading-order terms of \( \alpha_4 \) and \( c_{r2}^2 \) are of the same forms as those given in Eqs. (4.6) and (4.7), respectively. The quantities \( \alpha_6, c_{r11}^2, c_{r12}^2 \) have the following asymptotic behavior:

\[
\alpha_6 = -\frac{M_\text{pl}^2}{2r^4} + O \left( \frac{1}{r^6} \right), \quad (4.16)
\]

\[
c_{r11}^2 = 1 - \frac{4 \beta_4 \mu Q^2}{M_\text{pl}^2 r^4} + O \left( \frac{1}{r^6} \right), \quad (4.17)
\]

\[
c_{r12}^2 = 1 + \frac{24 \beta_4 M}{r^3} + O \left( \frac{1}{r^4} \right), \quad (4.18)
\]
which show that the stability conditions against odd-parity perturbations are satisfied at spatial infinity.

![Graph showing numerical solutions to $c_{\Omega_2}^2 - 1$, $1 - c_{\Omega_1}^2$, $c_{\Omega_{12}}^2 - 1$ versus $r/r_h$ for the model (A)](image)

In Fig. 2 we plot the deviations of $c_{\Omega_2}^2$, $c_{\Omega_1}^2$, $c_{\Omega_{12}}^2$ from 1 as functions of $r/r_h$ for $\tilde{\beta}_3 = 0.5$, $\tilde{\beta}_4 = 0.1$ and $\mu = 0.5$. These model parameters are chosen to be consistent with the conditions (4.12)-(4.14). As estimated from Eq. (4.11), the numerical simulation of Fig. 2 shows that the angular propagation speed squares on the horizon are in the ranges $c_{\Omega_1}^2 < 1$ and $c_{\Omega_{12}}^2 > 1$, while $c_{\Omega_2}^2 \to 1$ as $r \to r_h$. For the distance $r \gg r_h$, the deviations of propagation speed squares from 1 rapidly decrease as $c_{\Omega_2}^2 - 1 \propto r^{-6}$, $1 - c_{\Omega_1}^2 \propto r^{-4}$, and $c_{\Omega_{12}}^2 - 1 \propto r^{-3}$, whose properties agree with the analytic estimations given in Eqs. (4.7), (4.17) and (4.18). For the model parameters chosen in Fig. 2 there are no Laplacian instabilities outside the horizon. We also confirmed that the quantities $-\alpha_6$ and $\alpha_4$ are positive throughout the horizon exterior, so the conditions for the absence of ghosts are satisfied.

Provided that the positive coupling $\tilde{\beta}_4$ is within the range (4.15), the ghosts and Laplacian instabilities do not typically arise for $|\tilde{\beta}_3| \lesssim O(1)$. For $|\tilde{\beta}_3| \gtrsim 10$, the quantities $c_{\Omega_2}^2$, $c_{\Omega_{12}}^2$ as well as $\alpha_4$ temporally become negative in the region not far from the horizon. This property is similar to the case (A) discussed in Sec. IV A.

C. $\beta_4 \neq 0$ and $n = 1$

Let us finally proceed to the quartic coupling $f_4(X) = \beta_4 X$. Unlike the model (B), the model (C) contains a nonminimal coupling with an explicit interaction with the scalar derivative $\phi'$. In this case, the propagation speed squares $c_{\Omega_1}^2$ and $c_{\Omega_{12}}^2$ are different from 1.

In the vicinity of the horizon, we use the iterative solutions (4.15)-(4.18) of Ref. 29 to compute the quantities $\alpha_6, \alpha_4, c_{\Omega_1}^2, c_{\Omega_{12}}^2, c_{\Omega_1}^2, c_{\Omega_{12}}^2$. Then, it follows that

$$\alpha_6 = -\frac{M_\phi^2(1 - \mu)}{4r_h^2} \left( \frac{r}{r_h} - 1 \right) + O((r/r_h - 1)^2),$$

$$\alpha_4 = \frac{1 + 4\beta_3^2(1 - \mu) + 8\beta_4^2(1 + 8\beta_4\mu)}{2r_h^2(1 - \mu)(1 + 8\beta_4\mu)} \left( \frac{r}{r_h} - 1 \right)^{-1} + O((r/r_h - 1)^0),$$

$$c_{\Omega_1}^2 = 1 + O(r/r_h - 1), \quad c_{\Omega_2}^2 = 1 + O(r/r_h - 1), \quad c_{\Omega_{12}}^2 = 1 + O(r/r_h - 1),$$

$$c_{\Omega_{12}}^2 = 1 + \frac{8\beta_3^2(1 - \mu)[1 + 2\beta_3\mu(5 - \mu)]}{2(1 + 8\beta_4\mu)[1 + 4\beta_3^2(1 - \mu) + 8\beta_4^2\mu]} + O(r/r_h - 1).$$

(4.19)
where \( \tilde{\beta}_4 = \beta_4 M_{pl}^2 / r_h^4 \). Then, the conditions \( c_{r1}^2 \geq 0, c_{r2}^2 \geq 0, \) and \( c_{\Omega1}^2 \geq 0 \) hold around \( r = r_h \). From the requirements \( \alpha_4 > 0 \) and \( c_{\Omega2}^2 \geq 0 \), we obtain the following bounds:

\[
\begin{align*}
[1 + 4 \beta_3^2 \mu (1 - \mu) + 8 \beta_4 \mu] (1 + 8 \beta_4 \mu) &> 0 , \\
1 + 8 \beta_3^2 \mu (1 - \mu) [1 + 3 \beta_4 \mu (9 - \mu)] + 16 \beta_3 \mu (1 + 4 \beta_4 \mu) &> 0 .
\end{align*}
\]

(4.20)  
(4.21)

In the limit that \( \tilde{\beta}_4 \to 0 \), these conditions are trivially satisfied. This property also persists in another limit \( \tilde{\beta}_3 \to 0 \), in which case the two terms on the left-hand sides of Eqs. (4.20) and (4.21) reduce to \( (1 + 8 \beta_4 \mu)^2 \) with \( c_{\Omega2}^2 \to 1 + O(r/r_h - 1) \). In contrast to the case (B) discussed in Sec. IV.B, the conditions (4.20) and (4.21) automatically hold for positive \( \tilde{\beta}_4 \) and hence there is no upper bound of \( \beta_4 \).

On using the iterative solutions (3.19)-(3.22) of Ref. [29] in the regime \( r \gg r_h \), we find that the leading-order terms of \( \alpha_4, c_{r2}^2, c_{\Omega2}, \alpha_6 \) are of the same forms as Eqs. (4.6), (4.7), (4.8), and (4.16) respectively. The propagation speed squares \( c_{r1}^2 \) and \( c_{\Omega1}^2 \) have the following asymptotic behavior:

\[
\begin{align*}
c_{r1}^2 &= 1 + \frac{4 \beta_3^2 \beta_4 Q^6}{M_{pl}^2 r_h^4} + O \left( \frac{1}{r^{15}} \right) , \\
c_{\Omega1}^2 &= 1 + \frac{12 \beta_3^2 \beta_4 Q^6}{M_{pl}^2 r_h^{14}} + O \left( \frac{1}{r^{15}} \right) .
\end{align*}
\]

(4.22)

so there are neither ghost nor Laplacian instabilities at spatial infinity.

**FIG. 3:** Numerical solutions to \( c_{r1}^2 - 1, |c_{r2}^2 - 1|, c_{\Omega1}^2 - 1, |c_{\Omega2}^2 - 1| \) versus \( r/r_h \) for the model \([4.1]\) with \( \tilde{\beta}_3 = 1, \tilde{\beta}_4 = 1, n = 1, \) and \( \mu = 0.5 \).

In Fig. 3 we show numerical solutions to the deviations of propagation speed squares \( c_{r1}^2, c_{r2}^2, c_{\Omega1}^2, c_{\Omega2}^2 \) from 1 for the model parameters \( \tilde{\beta}_3 = 1, \tilde{\beta}_4 = 1, \) and \( \mu = 0.5 \). The asymptotic behavior of those quantities around \( r = r_h \) and \( r \gg r_h \) is consistent with the analytic estimations given in Eqs. (4.19) and (4.22). Since \( c_{r1}^2 - 1, |c_{r2}^2 - 1|, c_{\Omega1}^2 - 1, |c_{\Omega2}^2 - 1| \) remain smaller than 1, Laplacian instabilities are absent throughout the horizon exterior. We also numerically confirmed that no-ghost conditions hold for the model parameters used in Fig. 3.

If the quartic coupling is in the range \( \tilde{\beta}_4 \lesssim O(1) \), all the stability conditions can be consistently satisfied for \( |\tilde{\beta}_3| \lesssim O(1) \), but for \( |\tilde{\beta}_3| \gtrsim 10 \) there are intermediate regions outside the horizon in which \( \alpha_4, c_{r2}^2, c_{\Omega2}^2 \) become negative. This situation is analogous to what we discussed in Secs. IV.A and IV.B. Unlike the case (B), however, the coupling \( \beta_4 \) is not bounded from above. If the couplings range in the region \( \tilde{\beta}_4 \gg \beta_3 \), we find that the unstable regions tend to disappear even for \( |\tilde{\beta}_3| \gtrsim 10 \). In this case, the cubic coupling \( |\tilde{\beta}_3| \) is effectively negligible relative to \( \tilde{\beta}_4 \). Unlike the case (B), however, the coupling \( \beta_4 \) is not bounded from above. Indeed, this is also the case at intermediate distances with no-ghost conditions satisfied. In summary, the model (C) allows the possibility for satisfying stability conditions for wider ranges of \( \beta_3 \) and \( \beta_4 \) than those in the cases (A) and (B).

\[ \text{FIG. 3} \]
V. CONCLUSIONS

The SVT theories correspond to the unified framework of most general scalar-tensor (Horndeski) and vector-tensor (generalized Proca) theories with second-order equations of motion. If the theories respect the $U(1)$ gauge symmetry, the new interactions arising in such theories are given by the Lagrangians (2.2), (2.3). Because of the gauge invariance, there is no longitudinal propagation of the vector field $A_\mu$. Hence the $U(1)$ gauge-invariant SVT theories contain the five dynamical DOFs: one scalar, two transverse vector modes, and two tensor polarizations.

If we apply the $U(1)$ gauge-invariant SVT theories to vacuum solutions on the static and spherically symmetric background, the cubic-order coupling $f_3 = \beta_3 = constant$ can give rise to hairy BH solutions. For the couplings $2.16$, the iterative BH solutions around the horizon are given by Eqs. (2.16)-(2.19), whereas the solutions at spatial infinity are of the forms (2.20)-(2.23). The coupling $\beta_3$ generates a scalar hair with a nonvanishing field derivative $\phi'$. Besides this cubic-order interaction, the quartic coupling $\beta_4$ also leads to modifications to the RN solution through a vector hair. The effect of couplings $\beta_3$ and $\beta_4$ on the metric components $f$ and $h$ mostly manifests themselves in the vicinity of the horizon.

In this paper, we provided a general framework for studying the stability of static and spherically symmetric BHs against odd-parity perturbations in $U(1)$ gauge-invariant SVT theories. For the modes $l \geq 2$, the dynamical fields correspond to the perturbation $\chi$ defined by Eq. (3.17) arising from the gravity sector and the vector-field perturbation $\delta A_{lm}$. We showed that, under the conditions $\alpha_6 < 0$ and $\alpha_4 > 0$, these perturbations do not contain ghost modes. The radial propagation speed squares $c_1^2$ and $c_2^2$ associated with the perturbations $\chi$ and $\delta A_{lm}$ are given, respectively, by Eqs. (3.21) and (3.22). In the limit that $L = l(l+1) \gg 1$, we also derived the angular propagation speed squares $c_{1l}^2$ and $c_{2l}^2$ in the forms (3.27) and (3.28), respectively. As we observe in Eqs. (3.29) and (3.30), the quartic couplings $f_4$ and $f_4$ can lead to the deviations of $c_1^2$ and $c_2^2$ from 1. For the dipole mode $l = 1$, there is only the vector-field perturbation $\delta A_{lm}$, whose stability conditions are the same as those for $l \geq 2$.

We applied stability conditions of odd-parity perturbations to concrete models given by the couplings (4.1). In the absence of quartic couplings $(\beta_4 = 0)$, the propagation speed squares $c_1^2$ and $c_2^2$ are equivalent to 1 with $\alpha_5 = -\hbar M^2_{pl}/(4\pi^2)$, so there are neither ghost nor Laplacian instabilities outside the horizon ($h > 0$) for the perturbation in the gravity sector. In this case, the stability conditions $\alpha_4 > 0$, $c_1^2 \geq 0$, $c_2^2 \geq 0$ are satisfied both in the near-horizon limit and at spatial infinity. However, for $|\beta_3| \gtrsim 10$, we find that these conditions can be violated in an intermediate regime between $r = r_h$ and $r \gg r_h$. Hence the cubic coupling should be in the range $|\beta_3| \lesssim O(1)$ to ensure the odd-parity stability of vector-field perturbations.

The quartic couplings generally modify stability conditions of odd-parity perturbations relative to the case $\beta_4 = 0$, but we showed that there are viable model parameter spaces in which there are neither ghost nor Laplacian instabilities throughout the horizon exterior. For the quartic interaction $f_4 = \beta_4 = constant$, the coupling $\beta_4$ consistent with stability conditions is bounded from above, but this is not the case for $f_4 = \beta_4 X$. Hence the latter model leads to a wider allowed range of couplings $\beta_3$ and $\beta_4$ relative to the former model.

In this paper we studied the BH stability against odd-parity perturbations, but it is of interest to extend our analysis to even-parity perturbations. There are one scalar, one vector, and one tensor modes arising from the even-parity sector. In particular, it remains to be seen whether the existence of scalar perturbations puts further constraints on hairy BH solutions present in SVT theories. Moreover, it will be interesting to investigate the speeds of tensor and vector perturbations arising from the even-parity sector. One could also put further constraints for the absence of superluminal propagation since one might worry about the construction of closed time-like curves and acausality (even though a similar chronology protection might occur as in Galileon theories (37)). These issues are left for future works.

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