Estimation of cluster functionals for regularly varying time series: runs estimators

Youssouph Cissokho∗ Rafał Kulik†

September 7, 2021

Abstract

Cluster indices describe extremal behaviour of stationary time series. We consider runs estimators of cluster indices. Using a modern theory of multivariate, regularly varying time series, we obtain central limit theorems under conditions that can be easily verified for a large class of models. In particular, we show that blocks and runs estimators have the same limiting variance.

1 Introduction

Consider a stationary, regularly varying $\mathbb{R}^d$-valued time series $X = \{X_j, j \in \mathbb{Z}\}$. We are interested in its extremal behaviour. A classical approach to this problem is to calculate the extremal index. If $|\cdot|$ is an arbitrary norm on $\mathbb{R}^d$, then the extremal index $\theta$ (if exists) of $\{|X_j|, j \in \mathbb{Z}\}$ is defined as a parameter in the limiting distribution of the maxima. With $Q$ being the quantile function of $|X_0|$ and $a_n = Q(1 - 1/n)$ we have

$$\lim_{n \to \infty} \mathbb{P}(a_n^{-1} \max_{j=1,\ldots,n} \{|X_1|, \ldots, |X_n|\} \leq x) = \exp(-\theta x^{-\alpha}), \quad x > 0.$$ 

The parameter $\theta \in (0, 1]$ indicates the amount of clustering, with $\theta = 1$ (the case of extremal independence) meaning no-clustering of large values.

The extremal index is just one parameter that describes clustering of extremes. Informally speaking, it arises as the limit

$$\lim_{n \to \infty} \frac{\mathbb{E}[H((X_1, \ldots, X_{r_n})/u_n)]}{r_n \mathbb{P}(|X_0| > u_n)},$$

∗University of Ottawa
†University of Ottawa
for the particular choice of function $H : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{R}$, and a suitable choice of the scaling sequence $u_n \to \infty$ and the block size $r_n \to \infty$. (Formally speaking, $(X_1, \ldots, X_{r_n})$ is a random element of $(\mathbb{R}^d)^{r_n}$, while the domain of $H$ is $(\mathbb{R}^d)^\mathbb{Z}$. This inconsistency will be explained later).

In particular, the extremal index is achieved by applying a suitable functional to a cluster:

$$H((X_1, \ldots, X_{r_n})/u_n) = \mathbb{1} \{ \max \{|X_1|, \ldots, |X_{r_n}|\} > u_n \}.$$ 

That is,

$$\theta = \lim_{n \to \infty} \frac{\mathbb{P}(\max \{|X_1|, \ldots, |X_{r_n}|\} > u_n)}{r_n \mathbb{P}(|X_0| > u_n)}.$$ 

Informally speaking, a cluster is a triangular array $(X_1/u_n, \ldots, X_{r_n}/u_n)$ with $r_n, u_n \to \infty$ that converges in distribution in a certain sense. Cluster indices are obtained by applying the appropriate functional $H$ to the cluster. The functionals are defined on $(\mathbb{R}^d)^\mathbb{Z}$, the space of $\mathbb{R}^d$-valued sequences, and are such that their values do not depend on coordinates that are equal to zero. More precisely, for $X = \{X_j, j \in \mathbb{Z}\} \in (\mathbb{R}^d)^\mathbb{Z}$ and $i \leq j \in \mathbb{Z}$, we denote $X_{i,j} = (X_i, \ldots, X_j) \in (\mathbb{R}^d)^{(j-i+1)}$. Then, we identify $H(X_{i,j})$ with $H((0, X_{i,j}, 0))$, where $0 \in (\mathbb{R}^d)^\mathbb{Z}$ is the zero sequence. Such functionals $H$ will be called cluster functionals.

Let $\cdot$ be an arbitrary norm on $\mathbb{R}^d$ and $\{u_n\}$, $\{r_n\}$ be such that

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} r_n = \lim_{n \to \infty} n \mathbb{P}(|X_0| > u_n) = \infty,$$

$$\lim_{n \to \infty} r_n/n = \lim_{n \to \infty} r_n \mathbb{P}(|X_0| > u_n) = 0. \quad (\mathcal{R}(r_n, u_n))$$

Given a cluster functional $H$ on $(\mathbb{R}^d)^\mathbb{Z}$, we want to estimate the limiting quantity

$$\nu^*(H) = \lim_{n \to \infty} \nu_{n,r_n}^*(H) = \lim_{n \to \infty} \frac{\mathbb{E}[H(X_{1,r_n}/u_n)]}{r_n \mathbb{P}(|X_0| > u_n)}. \quad (1.2)$$

To guarantee existence of the limit we will require additional anticlustering assumptions on the time series $\{X_j, j \in \mathbb{Z}\}$. For $x = \{x_j, j \in \mathbb{Z}\} \in (\mathbb{R}^d)^\mathbb{Z}$ define $x^* = \sup_{j \in \mathbb{Z}} |x_j|$. The cluster indices of interest are, among others:

- the extremal index obtained with $H_1(x) = \mathbb{1}\{x^* > 1\}$, $x = \{x_j, j \in \mathbb{Z}\} \in (\mathbb{R}^d)^\mathbb{Z}$,
- the cluster size distribution obtained with

$$H_2(x) = \mathbb{1}\left\{ \sum_{j \in \mathbb{Z}} \mathbb{1}\{|x_j| > 1\} = m \right\}, \quad x = \{x_j, j \in \mathbb{Z}\} \in (\mathbb{R}^d)^\mathbb{Z}, \ m \in \mathbb{N}; \quad (1.3)$$

- the stop-loss index of a univariate time series obtained with

$$H_3(x) = \mathbb{1}\left\{ \sum_{j \in \mathbb{Z}} (x_j - 1)_+ > \eta \right\}, \quad x = \{x_j, j \in \mathbb{Z}\} \in \mathbb{R}^\mathbb{Z}, \ \eta > 0; \quad (1.4)$$
• the large deviation index of a univariate time series obtained with
\[ H_4(x) = 1 \{ K(x) > 1 \}, \quad K(x) = \left( \sum_{j \in \mathbb{Z}} x_j \right)_+, \quad x = \{ x_j, j \in \mathbb{Z} \} \in \mathbb{R}^\mathbb{Z}; \tag{1.5} \]

• the ruin index of a univariate time series obtained with
\[ H_5(x) = 1 \{ K(x) > 1 \}, \quad K(x) = \sup_{i \in \mathbb{Z}} \left( \sum_{j \leq i} x_j \right)_+, \quad x = \{ x_j, j \in \mathbb{Z} \} \in \mathbb{R}^\mathbb{Z}. \tag{1.6} \]

As indicated above, the extremal index is the classical quantity that arises in the extreme value theory for dependent sequences. Similarly, the cluster size distribution has been studied in \cite{Hsi91} and \cite{DR10}. The large deviation index was studied under the name cluster index in \cite{MW13, MW14}. It quantifies the effect of dependence in large deviations results.

Several methods of estimation of the limit \( \nu^*(H) \) in (1.2) may be employed. The natural one is to consider a statistics based on disjoint blocks of size \( r_n \), cf. \cite{DR10} and \cite{KS20},

\[ \tilde{\nu}^*_{n,r_n}(H) := \frac{1}{n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{m_n} H(X_{(i-1)r_n+1,ir_n}/u_n), \]

where \( m_n = \lfloor n/r_n \rfloor \) is the number of disjoint blocks. The data-based estimator is constructed as follows. Let \( k_n \to \infty \) be a sequence of integers and define \( u_n \) by \( k_n = n \mathbb{P}(|X| > u_n) \). Let \( |X|_{(n:1)} \leq \cdots \leq |X|_{(n:n)} \) be order statistics from \( |X_1|, \ldots, |X_n| \). Define

\[ \hat{\nu}^*_{n,r_n}(H) := \frac{1}{k_n} \sum_{i=1}^{m_n} H(X_{(i-1)r_n+1,ir_n}/|X|_{(n:n-k_n)}). \tag{1.7} \]

The general asymptotic theory for disjoint blocks estimators was developed in \cite{DR10}. See also \cite[Chapter 10]{KS20}. The limiting variance of the disjoint blocks estimator can be represented as

\[ \nu^*(\{ H - \nu^*(H) \mathcal{E} \}^2), \tag{1.8} \]

where \( \mathcal{E}(x) = \sum_{j \in \mathbb{Z}} 1 \{|x_j| > 1\} \). This result was established (implicitly) in \cite{DR10}, but the form of the limiting variance is again given in \cite[Chapter 10]{KS20}.

Another approach to estimation of \( \nu^*(H) \) is to consider the sliding blocks statistics

\[ \tilde{\mu}^*_{n,r_n}(H) := \frac{1}{q_n r_n \mathbb{P}(|X_0| > u_n)} \sum_{i=0}^{q_n-1} H(X_{i+1,i+r_n}/u_n) \]
and the corresponding estimator defined in terms of order statistics:

$$\hat{\mu}_{n,r_n}(H) = \frac{1}{r_n k_n} \sum_{i=0}^{q_n-1} H \left( X_{i+1,i+r_n}/|X|_{(n:n-k_n)} \right). \quad (1.10)$$

Here, $q_n = n - r_n - 1$ is the number of sliding blocks. In [DN20] the authors used the framework of [DR10] and showed that the limiting variance of the sliding blocks estimator never exceeds that of the disjoint blocks estimator. In case of the extremal index, both variances were proven to be equal. In [CK21] it was shown that the limiting variances for both disjoint and sliding blocks estimators agree and are given by the expression in (1.8) for an arbitrary choice of $H$. We note at this point that the methodology used in [DR10, DN20, KS20, CK21] fits into Peak Over Threshold (PoT) framework. On the other hand, in the Block Maxima (BM) framework, sliding blocks estimators yield typically smaller variance; see [BS18b, BS18a]. As of this moment, there is no thorough explanation of these phenomena and no formal comparison between PoT and BM framework. See [FdH15] for some partial results and [BZ18] for a recent review.

In the present paper we are interested in the so-called runs estimators. In the context of the extremal index, this approach goes back to [WN98] and stems from the following representation of the extremal index:

$$\theta = \lim_{n \to \infty} P(\max\{|X_1|, \ldots, |X_{r_n}|\} \leq u_n \mid |X_0| > u_n). \quad (1.11)$$

We note that

$$\theta = \lim_{n \to \infty} \frac{1}{P(|X_0| > u_n)} \mathbb{E}[\mathbb{1}\{A((X_0, \ldots, X_{r_n})/u_n) = 0\} \mathbb{1}\{|X_0| > u_n\}],$$

where

$$A(x) = \sup\{j : |x_j| > 1\}$$

gives the position of the last exceedence above 1 in a particular block. Recall again the convention $A((X_0, \ldots, X_{r_n})/u_n) = A((0, X_0, \ldots, X_{r_n})/u_n)$. Then, $A$ is an example of so-called anchoring map. Special cases of anchoring maps were considered in [Has18] and [BP18], while in [KS20] their connection to cluster indices $\nu^*(H)$ was thoroughly investigated. It turns out that with an arbitrary choice of the anchoring map $A$ we have

$$\nu^*(H) = \mathbb{E}[H^A(Y)],$$

where

$$H^A(x) = H(x)\mathbb{1}\{A(x) = 0\} \mathbb{1}\{|x_0| > 1\}.$$

This motivates the following runs statistics:

$$\tilde{\xi}_{n,r_n}(H^A) = \frac{1}{n P(|X_0| > u_n)} \sum_{i=r_n+1}^{n-r_n} H^A(X_{i-r_n,i+r_n}/u_n). \quad (1.12)$$
Indeed, under the appropriate conditions, Proposition 2.7 gives
\[ \lim_{n \to \infty} \mathbb{E}[\tilde{\xi}^*_{n,r_n}(H^A)] = \mathbb{E}[H^A(Y)] = \nu^*(H). \]

The data-based runs estimator is then
\[ \hat{\xi}^*_{n,r_n}(H^A) = \frac{1}{k_n} \sum_{i=r_n+1}^{n-r_n} H^A \left( X_{i-r_n,i+r_n}/|X| \right). \]

The main result of this paper is Theorem 3.5, the asymptotic normality of the appropriately normalized estimator \( \hat{\xi}^*_{n,r_n}(H^A) \). We show, in particular, that the limiting variance agrees with the one for the disjoint blocks and sliding blocks estimators; cf. [DR10], [KS20, Chapter 10], [CK21]. Furthermore, we prove that we cannot achieve variance reduction by considering a linear combination of runs estimators with a different choice of anchoring maps \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \). Indeed, it turns out that \( \hat{\xi}^*_{n,r_n}(H^A) \) and \( \hat{\xi}^*_{n,r_n}(H^{\tilde{\mathcal{A}}}) \) are totally dependent in the limit. We note in passing that even though general ideas of proofs are similar to those of [CK21], however, technicalities are significantly different. Differences stem primarily from conditioning on \( \{|X_j| > u_n\} \) used in case of the runs estimators.

Thus, from the theoretical point of view the limiting behaviour of all (disjoint blocks, sliding blocks, runs) estimators is the same. However, for finite samples a bias has to be taken into account. We note first that the theoretical finite-sample bias for both disjoint and sliding blocks estimators is the same. This can be also seen in extensive simulation studies in [CK21]. On the other hand, we were not able to get an useful formula for the bias in the runs estimator case. As such we relied on simulations. It turns out that runs estimator are typically heavily biased when estimation of the extremal index is concerned. However, the runs estimators may have an advantage when other cluster indices are considered.

The paper is structured as follows. Section 2 contains definitions, notation and preliminary results on convergence of clusters. It is primarily based on [KS20, Chapters 5 and 6], with some results from [BS09], [BPS18], [PS18]. Section 3 defines runs pseudo-estimators and estimators. The main result of the paper is the central limit theorem for runs estimators in Theorem 3.5. We note again that the limiting variance agrees with the one for disjoint and sliding blocks estimators. Simulations are performed in Section 4, while all the proofs are contained in Section 5.

## 2 Preliminaries

In this section we fix the notation and introduce the relevant classes of functions. In Section 2.3 we recall the notion of the tail and the spectral tail process (cf. [BS09]). Section 2.4 introduces anchoring maps (cf. [BP18], [Has18]). In Section 2.5 we define cluster indices. We refer to [KS20, Chapter 5] for more details. In Section 2.6 we discuss convergence of the cluster measure, following [KS20, Chapter 6].
The most important conclusion of these preliminaries is a representation of the cluster index \( \nu^*(H) \) (cf. (1.2)) as \( \mathbb{E}[H^A(Y)] \), with \( H^A \) defined in (2.6) and \( Y \) being the tail process. Also, Proposition 2.7 on conditional weak convergence and Propositions 2.9 and 2.11 on unconditional weak convergence play a central role in the rest of the paper.

2.1 Notation

Let \( |\cdot| \) be a norm on \( \mathbb{R}^d \). For a sequence \( x = \{x_j, j \in \mathbb{Z}\} \subseteq (\mathbb{R}^d)^\mathbb{Z} \) and \( i \leq j \in \mathbb{Z} \cup \{-\infty, \infty\} \) we denote \( x_{i,j} = (x_i, \ldots, x_j) \in (\mathbb{R}^d)^{j-i+1} \), \( x_{i,j}^* = \max_{i \leq l \leq j} |x_l| \) and \( x^* = \sup_{j \in \mathbb{Z}} |x_j| \). By \( 0 \) we denote the zero sequence; its dimension can be different in each of its occurrences.

By \( \ell_0(\mathbb{R}^d) \) we denote the set of \( \mathbb{R}^d \)-valued sequences which tend to zero at infinity. Likewise, \( \ell_1(\mathbb{R}^d) \) consists of sequences such that \( \sum_{j \in \mathbb{Z}} |x_j| < \infty \).

2.2 Classes of functions

Functionals \( H \) are defined on \( \ell_0(\mathbb{R}^d) \) with the convention \( H(x_{i,j}) = H((0, x_{i,j}, 0)) \). For \( s > 0 \), the function \( H_s : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{R} \) is defined by \( H_s(x) = H(x/s) \). We consider the following classes:

- \( \mathcal{L} \) is the class of bounded real-valued functions defined on \( (\mathbb{R}^d)^\mathbb{Z} \) that are either Lipschitz continuous with respect to the uniform norm or almost surely continuous with respect to the distribution of the tail process \( Y \). This class includes functions like \( 1\{x^* > 1\} \), \( 1\{\sum_{j \in \mathbb{Z}} |x_j| > 1\} \). See Remark 6.1.6 in [KS20].

- \( \mathcal{A} \subseteq \mathcal{L} \) is the class of shift-invariant functionals with support separated from \( 0 \). In particular, for \( H \in \mathcal{A} \), \( H(0) = 0 \). The class \( \mathcal{A} \) includes \( 1\{x^* > 1\} \).

- \( \mathcal{K} \) is the class of shift-invariant functionals \( K : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{R} \) defined on \( \ell_1(\mathbb{R}^d) \) such that \( K(0) = 0 \) and which are Lipschitz continuous with constant \( L_K \), i.e.

\[
|K(x) - K(y)| \leq L_K \sum_{j \in \mathbb{Z}} |x_j - y_j| , \quad x, y \in \ell_1(\mathbb{R}^d).
\]

- \( \mathcal{B} \subseteq \mathcal{L} \) is the class of functionals \( H \) of the form \( H = 1\{K > 1\} \), where \( K \in \mathcal{K} \). Functionals in \( \mathcal{B} \) may have support which is not separated from \( 0 \). The typical example is \( H(x) = 1\{\sum_{j} |x_j| > 1\} \); note that \( H \notin \mathcal{A} \).

We will also need the map \( \mathcal{E} \) is defined on \( \ell_0(\mathbb{R}^d) \) by \( \mathcal{E}(x) = \sum_{j \in \mathbb{Z}} 1\{|x_j| > 1\} \). Note that \( \mathcal{E} \) is shift-invariant, with the support separated from zero, but is not bounded.
2.3 Tail and spectral tail process

Let $X = \{X_j, j \in \mathbb{Z}\}$ be a stationary, regularly varying time series with values in $\mathbb{R}^d$ and tail index $\alpha$. In particular,

$$\lim_{x \to \infty} \frac{\mathbb{P}(|X_0| > tx)}{\mathbb{P}(|X_0| > x)} = t^{-\alpha}$$

for all $t > 0$. Then, there exists a sequence $Y = \{Y_j, j \in \mathbb{Z}\}$ such that

$$\mathbb{P}(x^{-1}(X_i, \ldots, X_j) \in \cdot \mid |X_0| > x)$$

converges weakly to

$$\mathbb{P}((Y_i, \ldots, Y_j) \in \cdot)$$

as $x \to \infty$ for all $i \leq j \in \mathbb{Z}$. We call $Y$ the tail process. See [BS09]. We note that, in particular, $|Y_0|$ has Pareto distribution with the density $\alpha x^{-\alpha-1}, x > 1$. As such, it follows automatically that $Y^* = \sup_{j \in \mathbb{Z}} |Y_j| > 1$. Equivalently, viewing $X$ and $Y$ as random elements with values in $(\mathbb{R}^d)^\mathbb{Z}$, we have for every bounded or non-negative functional $H$ on $(\mathbb{R}^d)^\mathbb{Z}$, continuous with respect to the product topology,

$$\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1}X) \mathbb{1}\{|X_0| > x\}]}{\mathbb{P}(|X_0| > x)} = \mathbb{E}[H(Y)].$$

The spectral tail process $\{\Theta_j, j \in \mathbb{Z}\}$ is defined by $\Theta = |Y_0|^{-1}Y$ and is independent of the tail process $Y$.

2.4 Anchoring maps

**Definition 2.1 (Anchoring map).** A measurable map $A : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{Z} \cup \{-\infty, \infty\}$ is called an anchoring map if the following two properties hold:

An(i): $A(x) = j$ implies $|x_j| \geq |x_0| \wedge 1$;

An(ii): $A(Bx) = A(x) + 1$, where $B$ is a backsift operator.

Three basic examples of anchoring maps are:

- The infargmax functional: $A^{(0)}(y) = \inf\{j : y_{-\infty,j}^* = y^*\}$;
- The first exceedence above one: $A^{(1)}(x) = \inf\{j : |x_j| > 1\}$;
- The last exceedence above one: $A^{(2)}(x) = \sup\{j : |x_j| > 1\}$.

In what follows we use the convention $\inf \emptyset = +\infty$. We note that $A^{(0)}$ is 0-homogeneous, while $A^{(1)}(x) = A^{(1)}(x/s)$ and $A^{(2)}(x) = A^{(2)}(x/s)$ are increasing and decreasing in $s$, respectively, but they are not 0-homogeneous. This will play a role in the proofs.

A special importance is given to the time index 0. In particular,
• If $A^{(0)}(x) = 0$, then $x^*_{-\infty,-1} < |x_0|$ and $x^*_1,\infty \leq |x_0|$;
• If $A^{(1)}(x) = 0$, then $x^*_{-\infty,-1} \leq 1$ and $|x_0| > 1$;
• If $A^{(2)}(x) = 0$, then $x^*_1,\infty \leq 1$ and $|x_0| > 1$.

Applying an anchoring map to a finite block, say $x_{-r,r}$ with $r \in \mathbb{N}$, is equivalent to applying it $x = (0, x_{-r,r}, 0)$. For example, $A^{(0)}(x_{-r,r}) = 0$ means that $x^*_{-r,-1} < |x_0|$ and $x^*_1,\infty \leq |x_0|$.

This in turn implies also that $A^{(0)}(x_{-s,s}) = 0$ for $0 < s < r$. Similarly,

$$A(x_{-r,r}) = 0 \Rightarrow A(x_{-s,s}) = 0, \ 0 < s < r$$

for $A = A^{(1)}, A^{(2)}$. However, we do not know if this important property (used explicitly in the proofs) holds for any anchoring map. As such, in the paper we focus on the three anchoring maps introduced above.

Furthermore, note that An(ii) gives that

$$A(x_{h-h+1,r-r}) = h \iff A(B^{-h}x_{-r,r}) = 0.$$  \hspace{1cm} (2.1)

Indeed, consider for example $A^{(0)}$. Then $A^{(0)}(x_{h-h+1,r-r}) = h$ means that $x^*_{h-h+1,r-r} < |x_h|$ and $x_{h+1,r+r} \leq |x_h|$. Set $\tilde{x} = B^{-h}x$, so that $\tilde{x}_{-r} = x_{h-r}, \tilde{x}_0 = x_h$ and $\tilde{x}_r = x_{h+r}$. Thus, $\tilde{x}_{-r,-1} < |\tilde{x}_0|$ and $\tilde{x}_{1,r} \leq |\tilde{x}_0|$. This in turn means that $A^{(0)}(\tilde{x}_{-r,r}) = A^{(0)}(B^{-h}x_{-r,r}) = 0$.

### 2.5 Cluster measure and cluster indices

Let $A$ be an anchoring map. If $P(A(Y) \notin Z) = 0$ then we can define

$$\vartheta = P(A(Y) = 0).$$  \hspace{1cm} (2.2)

We want to emphasize that $\vartheta$ does not depend on the choice of the anchoring map (see [PS18] and [KS20, Theorem 5.4.2]). In particular,

$$\vartheta = P(A^{(1)}(Y) = 0) = P\left(\sup_{j \leq -1} |Y_j| \leq 1 \right) = P(A^{(2)}(Y) = 0) = P\left(\sup_{j \geq 1} |Y_j| \leq 1 \right).$$

The above identity follows from the time-change formula, see [CK21, Section 7.1]. Therefore, $\vartheta$ can be recognized as the (candidate) extremal index. It becomes the usual extremal index under additional mixing and anticlustering conditions (cf. Section 7.5 in [KS20]).

Recall that $E(x) = \sum_{j \in \mathbb{Z}} \mathbbm{1}\{|x_j| > 1\}$. The property An(i) of the anchoring maps implies

$$\sum_{h \in \mathbb{Z}} P(A(Y) = h) \leq \sum_{h \in \mathbb{Z}} P(|Y_h| > 1).$$  \hspace{1cm} (2.3)

By [KS20, Lemma 9.2.3] the latter series is finite if an appropriate anticlustering condition holds (see $S(r_n, u_n)$ to be introduced later on).
Definition 2.2 (Cluster measure). Let \( Y \) and \( \Theta \) be the tail process and the spectral tail process, respectively, such that \( \mathbb{P}(\lim_{|j| \to \infty} Y_j = 0) = 1 \). The cluster measure is the measure \( \nu^* \) on \( \ell_0(\mathbb{R}^d) \) defined by
\[
\nu^* = \vartheta \int_0^\infty \mathbb{E}[\delta_{r\Theta}1\{A(\Theta) = 0\}] \alpha r^{\alpha - 1} dr .
\] (2.4)
The measure \( \nu^* \) is boundedly finite on \((\mathbb{R}^d)^\mathbb{Z}\) \(\setminus\{0\}\), puts no mass at 0 and is \(\alpha\)-homogeneous.

Furthermore, the cluster measure can be expressed in terms of another sequence.

Definition 2.3. Assume that \( \mathbb{P}(A(Y) / \in \mathbb{Z}) = 0 \). The conditional spectral tail process \( Q \) is a random sequence with the distribution of \( (Y^*)^{-1}Y \) conditionally on \( A(Y) = 0 \).

The sequence \( Q \) appeared implicitly in the seminal paper [DH95]. See also [BS09], [PS18, Definition 3.5] and [KS20, Chapter 5]. An abstract setting is considered in [DHS18].

Note for example that \( A(0)(Y) = 0 \) gives \( Y^* = |Y_0| \). Thus, (2.4) and the definition of \( Q \) give for a bounded or non-negative measurable function \( H \) on \( \ell_0(\mathbb{R}^d) \) (see Definition 5.4.11 in [KS20]),
\[
\nu^*(H) = \vartheta \int_0^\infty \mathbb{E}[H(rQ)] \alpha r^{\alpha - 1} dr .
\] (2.5)
If moreover \( H \) is such that \( H(y) = 0 \) if \( y^* \leq \epsilon \) for one \( \epsilon > 0 \), then
\[
\nu^*(H) = \epsilon^{-\alpha} \mathbb{E}[H(\epsilon Y)1\{A(Y) = 0\}] .
\] (2.5)

For a shift-invariant \( H : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{R} \) and an anchoring map \( A \) define
\[
H^A(x) = H(x)1\{A(x) = 0\}1\{|x_0| > 1\} .
\] (2.6)

Thus, since \( |Y_0| > 1 \), if \( H \) is such that \( H(y) = 0 \) whenever \( y^* \leq 1 \), then (2.5) gives
\[
\nu^*(H) = \mathbb{E}[H^A(Y)] .
\] (2.7)

Note that the \( \nu^*(H) \) does not agree with \( \mathbb{E}[H(Y)] \).

Definition 2.4 (Cluster index). We will call \( \nu^*(H) \) the cluster index associated to the functional \( H \).

2.6 Convergence of cluster measure

Define the measures \( \nu^*_{n,r_n} \), \( n \geq 1 \), on \( \ell_0(\mathbb{R}^d) \) as follows:
\[
\nu^*_{n,r_n} = \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \mathbb{E}\left[\delta_{u_n^{-1}X_{1,r_n}}\right] .
\]
We are interested in convergence of \( \nu^*_{n,r_n} \) to \( \nu^* \). The results of this section are extracted from [KS20, Chapter 6]. See also [PS18] and [BPS18].
2.6.1 Anticlustering conditions

For each fixed $r \in \mathbb{N}$, the distribution of $u_n^{-1}X_{-r,r}$ conditionally on $|X_0| > u_n$ converges weakly to the distribution of $Y_{-r,r}$ (see Section 2.3). In order to let $r$ tend to infinity, we must embed all these finite vectors into one space of sequences. By adding zeroes on each side of the vectors $u_n^{-1}X_{-r,r}$ and $Y_{-r,r}$ we identify them with elements of the space $\ell_0(\mathbb{R}^d)$. Then $Y_{-r,r}$ converges (as $r \to \infty$) to $Y$ in $\ell_0(\mathbb{R}^d)$ if (and only if) $Y \in \ell_0(\mathbb{R}^d)$ almost surely. However, this is not enough for statistical purposes and we consider the following definition.

Definition 2.5 ([DH95], Condition 2.8). Condition $\mathcal{AC}(r_n, u_n)$ holds if for all $x, y > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\max_{m \leq j \leq r_n} |X_j| > u_n x \mid |X_0| > u_n y\right) = 0.$$  $(\mathcal{AC}(r_n, u_n))$

Condition $\mathcal{AC}(r_n, u_n)$ is referred to as the anticlustering condition. It holds for i.i.d. regularly varying sequences if $\lim_{n \to \infty} r_n \mathbb{P}(|X_0| > u_n) = 0$. Note that the latter condition is a part of the $\mathcal{R}(r_n, u_n)$ assumption. It is also fulfilled by many models, including geometrically ergodic Markov chains, short-memory linear or max-stable processes. $\mathcal{AC}(r_n, u_n)$ implies that $Y \in \ell_0(\mathbb{R}^d)$. See [KSW19] and [KS20].

A stronger version of the anticlustering condition reads as follows.

Definition 2.6. Condition $\mathcal{S}(r_n, u_n)$ holds if for all $s, t > 0$

$$\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{\mathbb{P}(|X_0| > u_n)} \sum_{j=m}^{r_n} \mathbb{P}(|X_0| > u_n s, |X_j| > u_n t) = 0.$$  $(\mathcal{S}(r_n, u_n))$

The main consequence of the anticlustering condition $\mathcal{AC}(r_n, u_n)$ is the following result.

Proposition 2.7 ([BS09], Proposition 4.2; [KS20], Theorem 6.1.4). Let $H \in \mathcal{L}$. If Condition $\mathcal{AC}(r_n, u_n)$ holds, then

$$\lim_{n \to \infty} \mathbb{E}[H(u_n^{-1}X_{-r_n,r_n}) \mid |X_0| > u_n] = \mathbb{E}[H(Y)].$$

2.6.2 Vague convergence of cluster measure

We now state the unconditional convergence of $u_n^{-1}X_{1,r_n}$. Contrary to Proposition 2.7, where an extreme value was imposed at time 0, a large value in the cluster can happen at any time. Moreover, the convergence of $\nu_n^{*,r_n}(H)$ to $\nu^*(H)$ may hold only for shift-invariant functionals $H$. Therefore, we need the following definition.

Definition 2.8. The space $\tilde{\ell}_0(\mathbb{R}^d)$ is the space of equivalence classes of $\ell_0(\mathbb{R}^d)$ endowed with the equivalence relation $\sim$ defined by

$$x \sim y \iff \exists j \in \mathbb{Z}, B^j x = y.$$
Proposition 2.9 (Theorem 6.2.5 in [KS20]). Let condition $\mathcal{AC}(r_n, u_n)$ hold. The sequence of measures $\nu_{n,r_n}^*$, $n \geq 1$ converges vaguely on $\tilde{\ell}_0(\mathbb{R}^d) \setminus \{0\}$ to $\nu^*$, that is, for all $H \in \mathcal{A}$,
\[
\lim_{n \to \infty} \nu_{n,r_n}^*(H) = \lim_{n \to \infty} \frac{\mathbb{E}[H(u_n^{-1}X_{1,r_n})]}{r_n \mathbb{P}(|X_0| > u_n)} = \nu^*(H).
\]

The immediate consequence is the following limit (cf. (2.2)):
\[
\lim_{n \to \infty} \frac{\mathbb{P}(X_{1,r_n}^* > u_n)}{r_n \mathbb{P}(|X_0| > u_n)} = \vartheta.
\]

2.6.3 Indicator functionals not vanishing around zero

Proposition 2.9 entails convergence of $\nu_{n,r_n}^*(H)$ for $H \in \mathcal{A}$. For functionals which are not defined on the whole space $\ell_0(\mathbb{R}^d)$ we need an additional assumption on Asymptotic Negligibility of Small Jumps.

Definition 2.10. Condition $\text{ANSJB}(r_n, u_n)$ holds if for all $\eta > 0$,
\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\mathbb{P}(\sum_{j=1}^{r_n} |X_j| \mathbb{1}\{|X_j| \leq \epsilon u_n\} > \eta u_n)}{r_n \mathbb{P}(|X_0| > u_n)} = 0.
\]

Proposition 2.11 (Theorem 6.2.16 in [KS20]). Assume that $\mathcal{AC}(r_n, u_n)$ and $\text{ANSJB}(r_n, u_n)$ hold. Then for $K \in \mathcal{K}$,
\[
\nu^*(\mathbb{1}\{K > 1\}) = \lim_{n \to \infty} \frac{\mathbb{P}(K(X_{1,r_n}/u_n) > 1)}{r_n \mathbb{P}(|X_0| > u_n)} = \vartheta \int_0^\infty \mathbb{P}(K(Q) > 1)\alpha z^{-\alpha-1}dz < \infty.
\]

3 Central limit theorem for runs estimators

In this section we introduce and study runs estimators of cluster indices. A pseudo-estimator is defined in (3.4). Its limiting covariance (for different anchoring maps) is studied in Lemma 3.2. In particular, for two different anchoring maps, the runs statistics are totally dependent. As a consequence we cannot reduce the limiting variance for the estimation of $\nu^*(H)$ by considering linear combinations of the runs statistics. In Lemma 3.3 we consider covariance between runs and disjoint blocks estimators. Again, we obtain total dependence in the limit. The main result of the paper is the central limit theorem for runs estimators; see Theorem 3.5. The limiting variance agrees with the one for the disjoint blocks and sliding blocks estimators.
3.1 Runs estimator

To introduce runs estimators recall that (cf. (2.1))

\[
H^A (X_{(j-1)r_n+h,(j+1)r_n+h}/u_n) = H^A (B^{-h-jr_n}X_{-r_n,r_n}/u_n)
\]

\[
= H(B^{-h-jr_n}X_{-r_n,r_n}/u_n) \mathbb{1} \{ A(B^{-h-jr_n}X_{-r_n,r_n}/u_n) = 0 \} \mathbb{1} \{ |B^{-h-jr_n}X_0| > u_n \}
\]

\[
= H \left( X_{(j-1)r_n+h,(j+1)r_n+h}/u_n \right) \times \mathbb{1} \{ A(X_{(j-1)r_n+h,(j+1)r_n+h}/u_n) = h + jr_n \} \mathbb{1} \{ |X_{h+jr_n}| > u_n \} .
\]

(3.1)

Set \( q_n = n - r_n \) and \( m_n = n/r_n \). Without loss of generality assume that \( m_n \) is an integer. Consider disjoint blocks

\[
J_j := \{ jr_n + 1, \ldots, (j+1)r_n \} , \quad j = 0, \ldots, m_n - 1 .
\]

(3.2)

The union of these blocks gives \( \{ 1, \ldots, n \} \). We assume we have data \( X_{1-r_n}, \ldots, X_{n+r_n} \).

For \( j = 0, \ldots, m_n - 1 \) define

\[
H^A_{n,j} = \sum_{i=0}^{(j+1)r_n} H^A (X_{i-r_n,i+r_n}/u_n) = \sum_{i=0}^{(j+1)r_n} H^A (B^{-i}X_{-r_n,r_n}/u_n)
\]

\[
= \sum_{i=0}^{(j+1)r_n} H \left( X_{i-r_n,i+r_n}/u_n \right) \mathbb{1} \{ A(X_{i-r_n,i+r_n}/u_n) = i \} \mathbb{1} \{ |X_i| > u_n \} .
\]

(3.3)

Each \( H^A_{n,j} \) is a function of the block \( X_{(j-1)r_n+1,\ldots,(j+2)r_n} \) of size \( 3r_n \). The number \( j \) in the notation \( H^A_{n,j} \) indicates that the indicator \( |X_i| > u_n \) is applied with \( i \in J_j \).

We consider a random process

\[
\tilde{\xi}_{n,r_n}^* (H^A) = \frac{1}{n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{n} H^A (X_{i-r_n,i+r_n}/u_n)
\]

(3.4)

that can be decomposed as

\[
\tilde{\xi}_{n,r_n}^* (H^A) = \frac{1}{n \mathbb{P}(|X_0| > u_n)} \sum_{j=0}^{m_n-1} H^A_{n,j} .
\]

If the anticlustering condition \( AC(r_n,u_n) \) holds, then using stationarity, definition (2.6) of \( H^A \), Proposition 2.7 and (2.7) we have

\[
\lim_{n \to \infty} \mathbb{E} \left[ \tilde{\xi}_{n,r_n}^* (H^A) \right] = \lim_{n \to \infty} \frac{1}{n \mathbb{P}(|X_0| > u_n)} \mathbb{E} \left[ H^A (X_{-r_n,r_n}/u_n) \right] = \mathbb{E}[H^A(Y)] = \nu^*(H) .
\]

12
Now, let \( k_n \) be a sequence of integers (depending on \( n \)) such that \( k_n \to \infty \) and \( k_n/n \to 0 \). Define \( u_n \) by \( k_n = n \mathbb{P}(|X_0| > u_n) \) and replace \( u_n \) in \( H^A(X_{i-r_n,i+r_n}/u_n) \) with \((k_n + 1)\)th order statistics \( |X|_{(n:n-k_n)} \) to get the runs estimator:

\[
\hat{\xi}^*_{n,r_n}(H^A) = \frac{1}{k_n} \sum_{i=1}^{n} H^A \left( X_{i-r_n,i+r_n}/|X|_{(n:n-k_n)} \right) .
\] (3.5)

In what follows we will use interchangeably \( k_n \) and \( n \mathbb{P}(|X_0| > u_n) \), whatever is more suitable.

### 3.2 Mixing assumptions

Dependence in \( \{X_j, j \in \mathbb{Z}\} \) will be controlled by the \( \beta \)-mixing rates \( \{\beta_n\} \). Recall \( R(r_n,u_n) \).

Let \( \{\ell_n\} \) be a sequence of integers such that \( \lim_{n \to \infty} \ell_n = \infty \) and \( \lim_{n \to \infty} \ell_n/r_n = 0 \).

**Definition 3.1.** Condition \( \beta'(r_n) \) holds if:

\[
\lim_{n \to \infty} \frac{n}{r_n} \beta_{r_n} = 0 ,
\] (3.6a)

\[
\lim_{n \to \infty} \frac{1}{\mathbb{P}(|X_0| > u_n)} \sum_{i=r_n+1}^{\infty} \beta_i = \lim_{n \to \infty} \frac{n}{k_n} \sum_{i=r_n+1}^{\infty} \beta_i = 0 ,
\] (3.6b)

\[
\lim_{n \to \infty} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=1}^{\infty} \beta_{jr_n} = \lim_{n \to \infty} \frac{n}{r_n k_n} \sum_{j=1}^{\infty} \beta_{jr_n} = 0 .
\] (3.6c)

### 3.3 Limiting covariances

#### 3.3.1 Runs statistics

The first result deals with covariance of the process \( \tilde{\xi}^*_{n,r_n} \) defined in (3.4).

**Lemma 3.2.** Assume \( R(r_n,u_n) \), \( AC(r_n,u_n) \), \( S(r_n,u_n) \) and (3.6b) hold. Let \( H, \tilde{H} \in \mathcal{L} \) and \( A, \tilde{A} \) be anchoring maps. Then

\[
\lim_{n \to \infty} n \mathbb{P}(|X_0| > u_n) \text{cov} \left( \tilde{\xi}^*_{n,r_n}(H^A), \tilde{\xi}^*_{n,r_n}(\tilde{H}^\tilde{A}) \right) = \nu^*(H\tilde{H}) .
\] (3.7)

We note that the limit does not depend on the choice of the anchoring maps. In other words, for two different anchoring maps, \( A \) and \( \tilde{A} \), the runs statistics \( \tilde{\xi}^*_{n,r_n}(H^A) \) and \( \tilde{\xi}^*_{n,r_n}(H^\tilde{A}) \) are totally dependent. As a consequence we cannot reduce the limiting variance for the estimation of \( \nu^*(H) \) by considering a linear combination of \( \tilde{\xi}^*_{n,r_n}(H^A) \) and \( \tilde{\xi}^*_{n,r_n}(H^\tilde{A}) \).
3.3.2 Runs and disjoint blocks statistics

We analyse covariance between \( \tilde{\xi}_{n,r}^* (H^A) \) defined in (3.4) and the disjoint blocks statistics

\[
\frac{1}{n \mathbb{P}(|X_0| > u_n)} \sum_{j=0}^{m_n-1} H \left( X_{j r_n+(j+1)r_n}/u_n \right) = \tilde{\nu}_{n,r}^* (H) . \tag{3.8}
\]

The disjoint blocks statistics are considered in [KS20, Chapter 10].

**Lemma 3.3.** Assume \( \mathcal{R}(r_n, u_n) \), \( \mathcal{AC}(r_n, u_n) \), \( \mathcal{S}(r_n, u_n) \) and (3.6c) hold. Let \( H, \tilde{H} \in \mathcal{L} \), \( \tilde{H}(0) = 0 \) and \( \mathcal{A} \) be an anchoring map. Then

\[
\lim_{n \to \infty} n \mathbb{P}(|X_0| > u_n) \text{cov} \left( \tilde{\xi}_{n,r}^* (H^A), \tilde{\nu}_{n,r}^* (\tilde{H}) \right) = \nu^* (H \tilde{H}) . \tag{3.9}
\]

Again, irrespectively of the choice of the anchoring map \( \mathcal{A} \), the runs and disjoint blocks statistics are totally dependent and we cannot reduce the limiting variance by considering their linear combinations.

3.4 Central limit theorem

Let \( \mathcal{G} \) be the Gaussian process on \( L^2(\nu^*) \) with covariance

\[
\text{cov}(\mathcal{G}(H), \mathcal{G}(\tilde{H})) = \nu^* (H \tilde{H}) .
\]

Recall that for a functional \( H : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{R}_+ \) and \( s > 0 \) we define \( H_s(\mathbf{x}) = H(\mathbf{x}/s) \). Also, recall that \( \mathcal{E}(\mathbf{x}) = \sum_{j \in \mathbb{Z}} \mathbb{1}\{|x_j| > 1\} \).

Consider the class

\[
\mathcal{G} = \{ H_s^A, s \in [s_0, t_0] \} = \{ H(\mathbf{x}/s) \mathbb{1}\{\mathcal{A}(\mathbf{x}) = 0\} \mathbb{1}\{|\mathbf{x}_0| > s\}, s \in [s_0, t_0] \}.
\]

We need the following assumption on its random entropy.

**Assumption 3.4.** There exists a random metric \( d_n \) on \( \mathcal{G} \) and a measurable majorant \( N^*(\mathcal{G}, d_n, \epsilon) \) of the covering number \( N(\mathcal{G}, d_n, \epsilon) \) such that for every sequence \( \{\delta_n]\) which decreases to zero,

\[
\int_0^{\delta_n} \sqrt{\log N^*(\mathcal{G}, d_n, \epsilon)} d\epsilon \xrightarrow{P} 0 . \tag{3.10}
\]

The main result of this paper is Theorem 3.5, the asymptotic normality of the appropriately normalized estimator \( \hat{\xi}_{n,r_n}^* (H^A) \). The limiting variance agrees with the one for the disjoint blocks and sliding blocks estimators; cf. [DR10], [KS20, Chapter 10], [CK21].
Theorem 3.5. Let \( \{X_j, j \in \mathbb{Z}\} \) be a stationary, regularly varying \( \mathbb{R}^d \)-valued time series. Assume that \( R(r_n, u_n), \beta'(r_n), S(r_n, u_n) \) and

\[
\lim_{n \to \infty} \frac{r_n}{\sqrt{k_n}} = \lim_{n \to \infty} \frac{r_n}{\sqrt{nP(\|X_0\| > u_n)}} = 0
\]  

(3.11)

hold. Suppose that Assumption 3.4 is satisfied. Fix \( 0 < s_0 < 1 < t_0 < \infty \). Assume moreover that for \( A = A(0), A(1), A(2), \)

\[
\lim_{n \to \infty} \sqrt{k_n} \sup_{s \in [s_0, t_0]} \left| \frac{P(\|X_0\| > u_n s)}{P(\|X_0\| > u_n)} - s^{-\alpha} \right| = 0 ,
\]

(3.12a)

\[
\lim_{n \to \infty} \sqrt{k_n} \sup_{s \in [s_0, t_0]} \left| \mathbb{E}[\hat{\xi}^*_{n,r_n}(H^A)] - \nu^*(H_s) \right| = 0 .
\]

(3.12b)

If \( H \in A \), then

\[
\sqrt{k_n} \left\{ \hat{\xi}^*_{n,r_n}(H^A) - \nu^*(H) \right\} \overset{d}{\to} G(H - \nu^*(H)E) .
\]

(3.13)

If moreover ANSJB\((r_n, u_n)\) is satisfied, then (3.13) holds for \( H \in B \).

3.4.1 Comments on the conditions

One chooses typically \( k_n = n^\epsilon \) with some \( \epsilon \in (0,1) \). We note that \( \beta'(r_n) \) holds if e.g. \( \beta_n = O(n^{-\delta}) \) with \( \delta > 1 \) big enough or if \( \beta_n \) decays logarithmically. In the latter case, we typically choose \( r_n = (\log n)^{1+\delta} \) with some \( \delta > 0 \). Recalling the choice of \( k_n \) we can see that (3.11) is not a very stringent assumption.

Furthermore, (3.12a) controls the bias in the tail empirical process and can be related to the classical second order assumptions.

Assumption 3.4 controls the size of the class \( G \). We are not able to provide a general set of conditions under which this condition is satisfied, however, we will verify it for virtually all functionals \( H \) that appeared in the paper. See Section 5.8.

4 Simulation study

We conducted some simulations in order to study the finite sample performance of the runs estimators for selected cluster indices. We compare their performance with the disjoint and sliding blocks estimators (see [CK21]). Recall that the limiting variances are the same for all estimators. We do not have theoretical formulas for bias. We note that the bias for disjoint and sliding blocks estimators is the same, but different for runs estimators. We present only a small portion of our simulation studies; the most important findings are summarized at the end of the section.
4.1 Stationary AR process

We start with a simple AR(1) process. For this process we have explicit formulas for all cluster indices. Samples of size \( n = 1000 \) are generated from AR(1) with \( \alpha = 4 \) and \( \rho = 0.5, 0.9 \). We perform simulations for the classical extremal index as well as for the stop-loss index.

**Extremal index.** For AR(1) with \( \rho > 0 \) the extremal index is \( \theta = 1 - \rho^\alpha \); cf. [KS20, p. 396].

- Table 1 includes the results for Monte Carlo simulation for the extremal index based on disjoint blocks, sliding blocks and runs estimators with the block size \( r_n = 8, 9 \). We used \( k = 5\% \) and \( 10\% \) order statistics. We note that for the strong dependence \( (\rho = 0.9) \), the sliding and disjoint blocks estimators outperform runs estimators for all considered parameters. For weak dependence \( (\rho = 0.5) \), the results are heavily biased for all considered parameters. We note that all estimators yield almost the same variances, which is in agreement with the theoretical results obtained in the paper.

We note that the fact that stronger dependence yields smaller variability of the estimators is not surprising, cf. e.g. Figure 5 in [RSF09].

**Stop-loss index.** For AR(1) with \( \rho > 0 \) the formula for the stop-loss index is given in [KS20, p. 619]:

\[
\theta_{\text{stop-loss}}(S) = (1 - \rho^\alpha) \mathbb{P} \left( \sum_{j=0}^{\infty} (\rho^j Y_0 - 1)_+ > S \right), \tag{4.1}
\]

where \( Y_0 \) is a Pareto random variable with the parameter \( \alpha \).

- At the first step we use the formula (4.1) and performed the Monte-Carlo simulation to obtain the approximate value of the stop-loss index.

- With this in mind, we performed simulation studies for \( k = 10\% \) and \( k = 40\% \). As noted in [CK21], the stop-loss index estimation requires a higher number of order statistics. We notice then (see Table 2) that, as opposed to the extremal index, the runs estimator with the anchoring map \( C^{(0)} \) performs better than the disjoint and the sliding blocks in case of weaker dependence. Indeed, the weaker dependence \( (\rho = 0.5) \) yields a good estimation for runs estimators for any given block size, while for the strong dependence \( (\rho = 0.9) \), the simulation results are rather poor for all the estimators. This may be quite intuitive, since the stop-loss functional is based on *sums* of large values. On the other side, all the estimators perform poorly for strong dependence \( (\rho = 0.9) \) as a result of bias.

16
• The box plots in Figure 1 is based again on Monte Carlo simulations. The following parameters are used: \( \rho = 0.5 \), \( \alpha = 4 \) and the block size \( r_n = 8, 9 \) along with \( k = 10\% \) and \( k = 40\% \). We notice again that in both cases the runs estimator yields acceptable result as opposed to disjoint and sliding blocks estimators.

4.2 Stationary ARCH process

We consider a stationary ARCH(1) process defined by

\[
X_j^2 = \sqrt{\beta + \lambda} X_{j-1}^2 Z_j,
\]

where \( \{Z_j, j \in \mathbb{Z}\} \) are i.i.d standard normal random variables. For \( \lambda = 0.9 \) the extremal index is \( \theta = 0.612 \) (see [EKM97, p. 480]).

• Monte Carlo results are included in Table 3. In this case both disjoint and sliding blocks estimators yield better results as compared to runs. This is primarily due to bias.

In summary,

• All estimators (blocks and runs) have the same variance, which is in line with the theoretical results.

• For the extremal index, runs estimators are inferior as compared to blocks estimators. This is primarily due to bias.

• For the stop-loss index, runs estimators are superior, yielding lower (simulated) bias as compared to the blocks estimators.

5 Proofs

In Section 5.2 we prove several lemmas on conditional convergence when anchoring maps are involved. One needs to distinguish between finite blocks (when the conditional convergence follows basically from the conditional convergence to the tail process) and growing blocks (when the anticlustering condition is needed).

In Section 5.3 we prove the asymptotic behaviour of the covariances of runs estimators, that is we prove Lemma 3.2 and Lemma 3.3. Section 5.4 deals with the empirical cluster process of runs statistics. The functional central limit theorem (Theorem 5.5) established there yields immediately the central limit theorem for runs estimators. See Section 5.5. A long proof of Theorem 5.5 is given in Sections 5.6 and 5.7. Finally, in Section 5.8 we discuss the random entropy assumption.
5.1 Mixing

We recall the covariance inequality for bounded, beta-mixing random variables (in fact, the inequality holds for $\alpha$-mixing). Let $\beta(\mathcal{F}_1, \mathcal{F}_2)$ be the $\beta$-mixing coefficient between two sigma fields. Then (cf. [Ibr62])

$$|\text{cov}(H(Z_1), H(Z_2))| \leq \text{cst} \|H\|_\infty \|\tilde{H}\|_\infty \beta(\sigma(Z_1), \sigma(Z_2)).$$

(5.1)

In (5.1) the constant cst does not depend on $H, \tilde{H}$.

5.2 Conditional convergence

Lemma 5.1. Let $\mathcal{A} = \mathcal{A}^{(0)}, \mathcal{A}^{(1)}, \mathcal{A}^{(2)}$. Then for $h \in \mathbb{Z}$,

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{A}(X_{h-r,h+r}/u_n) = h \mid |X_0| > u_n) = \mathbb{P}(\mathcal{A}(Y_{h-r,h+r}) = h).$$

Proof. We will do the proof for $\mathcal{A}^{(0)}$ only. By the definition of the tail process

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{A}^{(0)}(X_{h-r,h+r}/u_n) = h \mid |X_0| > u_n) = \mathbb{P}(\mathcal{A}(X_{h-r,h+r}) = 0) + \mathbb{P}(\mathcal{A}(X_{h-r,h+r}) = 1) + \cdots + \mathbb{P}(\mathcal{A}(X_{h-r,h+r}) = h) = \mathbb{P}(\mathcal{A}(X_{h-r,h+r}) = h).$$

□

Lemma 5.2. Let $\mathcal{A}, \tilde{\mathcal{A}}$ be any of the anchoring maps $\mathcal{A}^{(0)}, \mathcal{A}^{(1)}, \mathcal{A}^{(2)}$. Then for $h \in \mathbb{Z}$,

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{A}(X_{r-r,r}/u_n) = 0, \tilde{\mathcal{A}}(X_{h-r,h+r}/u_n) = h \mid |X_0| > u_n) = \mathbb{P}(\mathcal{A}(Y_{r-r,r}) = 0, \tilde{\mathcal{A}}(Y_{h-r,h+r}) = h).$$

(5.2)

Proof. We verify the statement for one combination of the anchoring maps only. For $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, we have

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{A}^{(1)}(X_{h-r,h+r}/u_n) = h, \mathcal{A}^{(2)}(X_{r-r,r}/u_n) = 0 \mid |X_0| > u_n) = \mathbb{P}(\mathcal{A}^{(1)}(Y_{h-r,h+r}) = h, \mathcal{A}^{(2)}(Y_{r-r,r}) = 0).$$

□
Recall the definition of $H^A$ in (3.1). Let $h_n$ be a sequence of integers diverging to infinity. For bounded $H, \tilde{H}$, a direct application of $\mathcal{AC}(r_n, u_n)$ gives

$$\lim_{n \to \infty} E \left[ H^A(X_{-r_n,r_n}/u_n) \tilde{H}^\Lambda(X_{h_n,h_n+r_n}/u_n) \mid |X_0| > u_n \right]$$

$$\leq \|H\| \|\tilde{H}\| \lim_{n \to \infty} P(|X_{h_n}| > u_n \mid |X_0| > u_n) = 0. \tag{5.3}$$

Likewise, if $H, \tilde{H}$ are bounded and $\tilde{H} \in \mathcal{L}$ is such that $\tilde{H}(0) = 0$, then the Lipschitz continuity of $\tilde{H}$ and $\mathcal{AC}(r_n, u_n)$ imply

$$\lim_{n \to \infty} E \left[ H^A(X_{-r_n,r_n}/u_n) \tilde{H}(X_{h_n,h_n+r_n}/u_n) \mid |X_0| > u_n \right]$$

$$\leq \|H\| \|\tilde{H}\| \lim_{n \to \infty} E \left[ \tilde{H}(X_{h_n,h_n+r_n}/u_n) - \tilde{H}(0) \mid |X_0| > u_n \right]$$

$$\leq \|H\| \|\tilde{H}\| \lim_{n \to \infty} P \left( X^*_{h_n,h_n+r_n} > u_n \mid |X_0| > u_n \right) = 0. \tag{5.3}$$

The statement in (5.3) is also valid if $\tilde{H}(X_{h_n,h_n+r_n}/u_n)$ is replaced with $\tilde{H}(X_{-h_n,-h_n-r_n}/u_n)$. For this, we need to assume additionally that $\mathcal{AC}(r_n, u_n)$ holds.

**Lemma 5.3.** Assume that $\mathcal{AC}(r_n, u_n)$ holds. Let $H, \tilde{H} \in \mathcal{L}$ and $A, \tilde{A}$ be anchoring maps. Then

$$\lim_{n \to \infty} E \left[ H^A(X_{-r_n,r_n}/u_n) \tilde{H}^\Lambda(X_{h_n,h_n+r_n}/u_n) \mid |X_0| > u_n \right] = E \left[ H(Y) \tilde{H}(Y) \mathbb{1}\{A(Y) = 0\} \mathbb{1}\{\tilde{A}(Y) = h\} \right] =: \mathcal{T}(H, \tilde{H}, A, \tilde{A}; h). \tag{5.4}$$

Before we prove the above lemma, we make several comments.

First, as a corollary we obtain

$$\lim_{n \to \infty} E \left[ H^A(X_{-r_n,r_n}/u_n) \mid |X_0| > u_n \right] = \nu^*(H) \tag{5.5}$$

and

$$\lim_{n \to \infty} P(A(X_{-r_n,r_n}/u_n) = 0 \mid |X_0| > u_n) = \nu^*(1) = 1. \tag{5.6}$$

Indeed, if we take $\tilde{H} \equiv 1, A = \tilde{A}$ and $h = 0$, then by (2.7),

$$\mathcal{T}(H, 1, A, \tilde{A}; 0) = E \left[ H(Y) \mathbb{1}\{A(Y) = 0\} \right]$$

$$= E \left[ H(Y) \mathbb{1}\{A(Y) = 0\} \mathbb{1}\{|Y_0| > 1\} \right] = \nu^*(H).$$

Since the definition of $\nu^*$ does not depend on the anchoring map, we have $\mathcal{T}(H, \tilde{H}, A, \tilde{A}; 0) = \nu^*(H \tilde{H})$ for any $A, \tilde{A}$. Since the value of any anchoring map is uniquely determined, we
conclude immediately that \( \mathcal{I}(H, \tilde{H}, \mathcal{A}, \mathcal{A}; h) = 0 \) for \( h \neq 0 \). Furthermore,

\[
\sum_{h \in \mathbb{Z}} \mathbb{E} \left[ H(Y) \tilde{H}(Y) \mathbb{1}\{A(Y) = 0\} \mathbb{1}\{\tilde{A}(Y) = h\} \right] = \mathbb{E} \left[ H(Y) \tilde{H}(Y) \mathbb{1}\{A(Y) = 0\} \mathbb{1}\{\tilde{A}(Y) \in \mathbb{Z}\} \right] = \nu(H \tilde{H}) .
\]

This implies that for arbitrary anchoring maps \( \mathcal{A}, \tilde{\mathcal{A}} \),

\[
\mathcal{I}(H, \tilde{H}, \mathcal{A}, \tilde{\mathcal{A}}; h) = 0 , \quad h \neq 0 . \tag{5.7}
\]

**Proof of Lemma 5.3.** In [CK21] we proved a version of the lemma without anchoring maps included. Since \( x \rightarrow 1\{A(x) = 0\} \) is not Lipschitz continuous, Lemma 6.6 in [CK21] is not directly applicable. As such, we will focus on the anchoring maps only, assuming \( H = \tilde{H} \equiv 1 \).

In the first step we prove that for all \( h \in \mathbb{Z} \)

\[
\lim_{n \to \infty} \mathbb{P}(A(X_{h-r_n, h+r_n} / u_n) = h \mid |X_0| > u_n) = \mathbb{P}(A(Y) = h) . \tag{5.8}
\]

We already know that (cf. Lemma 5.1)

\[
\lim_{n \to \infty} \mathbb{P}(A(X_{h-r_n, h+r_n} / u_n) = h \mid |X_0| > u_n) = \mathbb{P}(A(Y_{h-r_n, h+r_n}) = h) .
\]

Since \( r_n \to \infty \) and \( r \) is fixed we can assume \( 0 < r < r_n \). Now, for \( \mathcal{A} = \mathcal{A}^{(0)}, \mathcal{A}^{(1)}, \mathcal{A}^{(2)} \) the value of

\[
\mathbb{P}(A(Y_{h-r_n, h+r_n}) = h) - \mathbb{P}(A(Y_{h-r_n, h+r_n}) = h)
\]

is non zero if and only if \( h+r < \mathcal{A}(Y_{h-r_n, h+r_n}) \leq h+r_n \) or \( h-r_n \leq \mathcal{A}(Y_{h-r_n, h+r_n}) < h-r \). Indeed, take for simplicity \( h = 0 \). If \( \mathcal{A}^{(1)}(Y_{r,r}) = 0 \) and \( \mathcal{A}^{(1)}(Y_{r,r},r_n) \neq 0 \), then \( Y_{r,r-1}^* \leq 1, |Y_0| > 1 \) and then \( Y_{r,r-1}^* \leq 1, |Y_0| > 1 \) while \( \mathcal{A}^{(1)}(Y_{r,r}) \neq 0 \) and \( \mathcal{A}^{(1)}(Y_{r,r},r_n) = 0 \) cannot happen. The same reasoning applied to the other anchoring maps.

Coming back to the general case of \( h \), the first property of the anchoring map implies that \( |Y_j| > 1 \) for some \( j \in \{h+r+1, \ldots, h+r_n\} \cup \{h-r_n, \ldots, h-r-1\} \). Since we let \( r, r_n \to \infty \), we can assume that \( h < r \). Thus, using the property \( \text{An}(i) \) of the anchoring maps,

\[
\lim_{r \to \infty} \lim_{n \to \infty} |\mathbb{P}(A(Y_{h-r_n, h+r_n}) = h) - \mathbb{P}(A(Y_{h-r_n, h+r_n}) = h)|
\]

\[
\leq \lim_{r \to \infty} \lim_{n \to \infty} \mathbb{P} \left( \max \left\{ \max_{h-r_n \leq j \leq h+r_n} |Y_j|, \max_{h-r_n \leq j \leq h-r} |Y_j| \right\} > 1 \right) = 0 \tag{5.9}
\]

since \( \mathcal{A}(r_n, u_n) \) implies \( Y_j \to 0 \) almost surely as \( |j| \to \infty \). Also, the vanishing property of \( Y_j \) and the property \( \text{An}(i) \) of the anchoring map imply that

\[
\lim_{n \to \infty} \mathbb{P}(A(Y_{h-r_n, h+r_n}) = h) = \mathbb{P}(A(Y) = h) .
\]
Similarly,

\[ P(\mathcal{A}(X_{h-r_n,h+r_n}/u_n) = h \mid |X_0| > u_n) - P(\mathcal{A}(X_{h-r,h+r}/u_n) = h \mid |X_0| > u_n) \]

is non zero if and only if \( h + r < \mathcal{A}(X_{h-r_n,h+r_n}/u_n) \leq h + r_n \) or \( h - r_n \leq \mathcal{A}(X_{h-r_n,h+r_n}/u_n) < h - r \). The first property of the anchoring map implies that \( |X_j| > |X_0| \land u_n \) for some \( j \in \{h + r + 1, \ldots, h + r_n\} \cup \{h - r_n, \ldots, h - r - 1\} \). Again, we can assume that \( h < r \). Keeping in mind the conditioning we have:

\[
\lim_{r \to \infty} \lim_{n \to \infty} \left| \frac{P(\mathcal{A}(X_{h-r_n,h+r_n}/u_n) = h \mid |X_0| > u_n) - P(\mathcal{A}(X_{h-r,h+r}/u_n) = h \mid |X_0| > u_n)}{P(\mathcal{A}(X_{h-r_n,h+r_n}/u_n) = h \mid |X_0| > u_n)} \right| 
\leq \lim_{r \to \infty} \lim_{n \to \infty} P \left( \max \left\{ \max_{h + r \leq j \leq h + r_n} |X_j|, \max_{h - r_n \leq j \leq h - r} |X_j| \right\} > u_n \mid |X_0| > u_n \right) = 0 \quad (5.10)
\]

by \( \mathcal{AC}(r_n, u_n) \). This finishes the proof of (5.8).

Now, we will prove

\[
\lim_{n \to \infty} P(\mathcal{A}(X_{h-r_n,h+r_n}/u_n) = 0, \tilde{\mathcal{A}}(X_{h-r_n,h+r_n}/u_n) = h \mid |X_0| > u_n) = P(\mathcal{A}(Y) = 0, \tilde{\mathcal{A}}(Y) = h) \quad (5.11)
\]

In view of Lemma 5.2, (5.11) holds with \( r_n \) replaced with \( r \). Now, the idea is to reduce the bivariate case to the univariate.

Note first that for the anchoring maps considered here, the event \( A_1 := \{\mathcal{A}(Y_{h-r_n,h+r_n}) = h\} \) is included in \( A_2 := \{\mathcal{A}(Y_{h-r,h+r}) = h\} \). We also note that for any event \( B \) and any pair of ordered events \( A_1, A_2 \) we have

\[
|P(A_1 \cap B) - P(A_2 \cap B)| \leq |P(A_1) - P(A_2)|.
\]

Thus, we can bound

\[
|P(\mathcal{A}(Y_{h-r+h}) = 0, \tilde{\mathcal{A}}(Y_{h-r+h}) = h) - P(\mathcal{A}(Y_{h-r+h}) = 0, \tilde{\mathcal{A}}(Y_{h-r+h}) = h)|
\]

by

\[
|P(\mathcal{A}(Y_{h-r+h}) = 0) - P(\mathcal{A}(Y_{h-r+h}) = 0)| + \left| P(\tilde{\mathcal{A}}(Y_{h-r+h}) = h) - P(\tilde{\mathcal{A}}(Y_{h-r+h}) = h) \right|
\]

and we use the first step to conclude that

\[
\lim_{r \to \infty} \lim_{n \to \infty} \left| P(\mathcal{A}(Y_{h-r+h}) = 0, \tilde{\mathcal{A}}(Y_{h-r+h}) = h) - P(\mathcal{A}(Y_{h-r+h}) = 0, \tilde{\mathcal{A}}(Y_{h-r+h}) = h) \right| = 0.
\]

Therefore, (5.9) can be extended to the bivariate case. The same argument allows to extend (5.10) to the bivariate case. In summary, the proof of (5.11) is finished. \( \square \)

In the next lemma, we analyse the conditional convergence for the product of \( H^A \) and \( \tilde{H} \). Its proof is almost the same as above and hence it is omitted.
Lemma 5.4. Assume that $\mathcal{AC}(r_n,u_n)$ holds. Let $H, \tilde{H} \in \mathcal{L}$, $\tilde{H}(0) = 0$ and $A$ be an anchoring map. Then, for $h, h' \geq 0$,

$$
\lim_{n \to \infty} \mathbb{E} \left[ H^A (X_{-r_n, i} / u_n) \tilde{H} (X_{h-r_n, i+r_n} / u_n) \mid |X_0| > u_n \right] = \mathbb{E} \left[ H (Y) \tilde{H} (Y) \mathbb{1} \{ A(Y) = 0 \} \right] = \nu^* (H \tilde{H}) .
$$

(5.12)

5.3 Limiting Covariances

The goal of this section is to prove Lemmas 3.2 and 3.3. Two situations will arise when dealing with the covariances:

- Situation 1: we will deal with $\sum_{h=-r_n}^{r_n} \mathbb{E}[c_{h,n}(X/u_n)]$, where

$$
\lim_{n \to \infty} \mathbb{E}[c_{h,n}(X/u_n)] = \mathbb{E}[c_h(Y)] , \quad \sum_{h \in \mathbb{Z}} \mathbb{E}[|c_h(Y)|] < \infty .
$$

We will fix an integer $r > 0$; the convergence of $\sum_{h=-r}^{r} \mathbb{E}[c_{h,n}(X/u_n)]$ to $\sum_{h=-r}^{r} \mathbb{E}[c_h(Y)]$ will follow. The reminder $\sum_{|h|>r} \mathbb{E}[c_{h,n}(X/u_n)]$ is negligible (as $r \to \infty$) by the summability assumption, while $\sum_{h>\lfloor r \rfloor} \mathbb{E}[c_{h,n}(X/u_n)]$ will be treated by the anticlustering condition $S(r_n, u_n)$.

- Situation 2: we will deal with $r_n^{-1} \sum_{h=1}^{r_n} \mathbb{E}[c_{h,n}(X/u_n)] = \int_0^1 g_n(\xi) d\xi$, where $g_n(h) = \mathbb{E}[c_{h,n}(X/u_n)]$ and $g_n(\xi) \to g(\xi)$ as $n \to \infty$. Bounded convergence argument will be applied.

Proof of Lemma 3.2. Recall that

$$
H_{n,j}^A = \sum_{i=1}^{(j+1)r_n} H^A (X_{i-r_n, i+r_n} / u_n) .
$$

The covariance of the scaled statistics is

$$
n \mathbb{P}(|X_0| > u_n) \text{cov} \left( \tilde{\xi}_{n,r_n}(H^A), \tilde{\xi}_{n,r_n}(\tilde{H}^A) \right) = \frac{1}{r_n \mathbb{P}(|X_0| > u_n) \text{cov} \left( H_{n,0}^A, \tilde{H}_{n,0}^A \right)} + \frac{1}{r_n \mathbb{P}(|X_0| > u_n) \sum_{j=1}^{m_n-1} \left( 1 - j / m_n \right) \left\{ \text{cov} (H_{n,0}^A, \tilde{H}_{n,j}^A) + \text{cov} (\tilde{H}_{n,0}^A, H_{n,j}^A) \right\} .
$$

(5.13)
With the help of $R(r_n, u_n)$, we will show that $\text{cov} \left( H^A_{n,0}, \tilde{H}^A_{n,0} \right)$ is determined by that of

$$\lim_{n \to \infty} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \text{cov} \left( H^A_{n,0}, \tilde{H}^A_{n,0} \right)$$

$$= \lim_{n \to \infty} \frac{1}{\mathbb{P}(|X_0| > u_n)} \sum_{h=-r}^{r_n} \left( 1 - \frac{|h|}{r_n} \right) \mathbb{E}[H^A(X_{-r_n,r_n}/u_n) \tilde{H}^A(X_{r-r_n,r+n}/u_n)]$$

$$= \nu^*(H\tilde{H}) . \tag{5.14}$$

We are in the Situation 1. For fixed $r$, using (5.4) and (5.7) we have

$$\lim_{n \to \infty} \frac{1}{\mathbb{P}(|X_0| > u_n)} \sum_{h=-r}^{r} \mathbb{E}[H^A(X_{-r_n,r_n}/u_n) \tilde{H}^A(X_{r-r_n,r+n}/u_n)]$$

$$= \sum_{h=-r}^{r} \mathbb{E}[H(Y)\tilde{H}(Y) \mathbb{1}\{A(Y) = 0\} \mathbb{1}\{\tilde{A}(Y) = h\}] = \sum_{h=-r}^{r} \mathbb{I}(H, \tilde{H}, A, \tilde{A}, h) =$$

$$= \mathbb{I}(H, \tilde{H}, A, \tilde{A}, 0) = \mathbb{E}[H(Y)\tilde{H}(Y) \mathbb{1}\{A(Y) = 0\}] = \nu^*(H\tilde{H}) . \tag{5.15}$$

The value above does not depend on $r$. Moreover,

$$\frac{1}{\mathbb{P}(|X_0| > u_n)} \sum_{r<|h| \leq r_n} \mathbb{E}[H^A(X_{-r_n,r_n}/u_n) \tilde{H}^A(X_{r-r_n,r+n}/u_n)]$$

$$\leq \|H\|\|\tilde{H}\| \frac{1}{\mathbb{P}(|X_0| > u_n)} \sum_{r<|h| \leq r_n} \mathbb{P}(|X_0| > u_n, |X_h| > u_n) . \tag{5.16}$$

Letting $n \to \infty$ and then $r \to \infty$, we finish the proof of (5.14) by applying $S(r_n, u_n)$.

Now, we deal with the term in (5.13). For $j \geq 1$,

$$\frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \left| \text{cov}(H^A_{n,0}, \tilde{H}^A_{n,j}) \right|$$

$$= - \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \left| \text{cov} \left( \sum_{h=1}^{r_n} H^A(B^{-h}X_{-r_n,r_n}/u_n), \sum_{i=1}^{r_n} \tilde{H}^A(B^{-i}X_{(j-1)r_n,(j+1)r_n}/u_n) \right) \right|$$

$$\leq \sum_{h=(j-1)r_n+1}^{(j+1)r_n} \left( \frac{h}{r_n} - (j-1) \right) |g_n(h)| + \sum_{h=(j+1)r_n+1}^{(j+1)r_n} \left( (j+1) - \frac{h}{r_n} \right) |g_n(h)|$$

$$\leq \sum_{h=(j-1)r_n+1}^{(j+1)r_n} |g_n(h)| =: I_j \tag{5.17}$$

with

$$g_n(h) = \frac{1}{\mathbb{P}(|X_0| > u_n)} \text{cov}(H^A(X_{-r_n,r_n}/u_n), \tilde{H}^A(X_{r-r_n,r+n}/u_n)) .$$

23
For $h > 2r_n$ we have by (5.1),
\[ |g_n(h)| \leq \left\| H \right\|_\infty \left\| \tilde{H} \right\|_\infty \beta_{h-2r_n}. \] (5.18)

Thus,
\[
\frac{1}{r_n P(|X_0| > u_n)} \sum_{j=4}^{m_n-1} \left| \text{cov}(H_{n,0}, \tilde{H}_{n,j}) \right| \leq \frac{\left\| H \right\|_\infty \left\| \tilde{H} \right\|_\infty}{P(|X_0| > u_n)} \sum_{j=4}^{m_n-1} \sum_{h=(j-1)r_n+1}^{(j+1)r_n} \beta_{h-2r_n}
\]
\[
\leq 2 \frac{\left\| H \right\|_\infty \left\| \tilde{H} \right\|_\infty}{P(|X_0| > u_n)} \sum_{h=3r_n+1}^{\infty} \beta_{h-2r_n} = O(1)
\]
by the assumption (3.6b).

The terms that correspond to $j = 1, 2, 3$ in (5.13) have to be dealt with separately. We are again in the Situation 1. We have
\[
I_1 + I_2 + I_3 \leq 2 \sum_{h=1}^{4r_n} |g_n(h)| = 2 \left\{ \sum_{h=1}^{r} + \sum_{i=r+1}^{4r_n} \right\} |g_n(h)|.
\]
Both parts are negligible. Indeed, as in (5.15),
\[
\lim_{n \to \infty} \sum_{h=1}^{r} |g_n(h)| \leq \lim_{n \to \infty} \frac{1}{P(|X_0| > u_n)} \sum_{h=1}^{r} E[H^A(X_{-r_n}, r_n/u_n) \tilde{H}^A(X_{h-r_n}, h/r_n/u_n)]
\]
\[
= \sum_{h=1}^{r} E[H(Y) \tilde{H}(Y) \mathbb{1}\{A(Y) = 0\} \mathbb{1}\{A(Y) = h\}] = \sum_{h=1}^{r} I(H, \tilde{H}, A, \tilde{A}, h)
\]
and by (5.7) the last term vanishes.

For the term $\sum_{h=r+1}^{4r_n}$ we apply $S(r_n, u_n)$; see the argument used in (5.16).

This finishes the proof of the lemma.

Proof of Lemma 3.3. Recall that
\[
\tilde{H}_j = H \left( X_{j(r_n+1), (j+1)r_n/u_n} \right).
\]
Here, $\tilde{H}_j$ is a function of the $j$th block $X_{j(r_n+1), (j+1)r_n}$, $j = 0, \ldots, m_n - 1$. Since $H^A_{n,j}$, $j = 0, \ldots, m_n - 1$, is a function of the block $X_{(j-1)r_n+1, ..., (j+2)r_n}$ (recall that we assumed that we have data $X_{1-r_n}, \ldots, X_{n+r_n}$), for $|q| \geq 3$,
\[
\text{cov}(H^A_{n,j}, \tilde{H}_{j+q}) \leq \left\| H \right\| \left\| \tilde{H} \right\|_3 \beta_{|q|-2r_n}. \] (5.19)
cf. (5.1). We have
\[
k_n \text{cov} \left( \xi_{n, r_n}^* (H^A), \nu_{n, r_n}^* (\tilde{H}) \right) = \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \text{cov} \left( H_{n, 0}^A, \tilde{H}_0 \right)
\]
\[+ \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=1}^{m_n-1} \left( 1 - \frac{j}{m_n} \right) \left\{ \text{cov} \left( H_{n, 0}^A, \tilde{H}_j \right) + \text{cov} \left( \tilde{H}_0, H_{n, j}^A \right) \right\} . \tag{5.20} \]

We analyse \( \text{cov} \left( H_{n, 0}^A, \tilde{H}_0 \right) \).

We are in the Situation 2:
\[
\frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \mathbb{E} \left[ H_{n, 0}^A \tilde{H}_0 \right] = \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{r_n} \mathbb{E} \left[ H^A (X_{i-r_n, i+r_n} / u_n) \tilde{H} (X_{1, r_n} / u_n) \right]
\]
\[= \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{r_n} \mathbb{E} \left[ H (X_{i-r_n, i+r_n} / u_n) 1 \{ A (X_{i-r_n, i+r_n}) = 0 \} 1 \{|X| > u_n\} \tilde{H} (X_{1, r_n} / u_n) \right]
\]
\[= \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{r_n} \mathbb{E} \left[ H (X_{-r_n, r_n} / u_n) 1 \{ A (X_{-r_n, r_n}) = 0 \} 1 \{|X| > u_n\} \tilde{H} (X_{1-i, r_n-i} / u_n) \right]
\]
\[= \frac{1}{r_n} \sum_{i=1}^{r_n} \mathbb{E} \left[ H^A (X_{-r_n, r_n} / u_n) \tilde{H} (X_{1-i, r_n-i} / u_n) \ | \ |X| > u_n \right] = \int_0^1 h_{n, 0}(\xi) d\xi
\]

with
\[
h_{n, 0}(\xi) = \mathbb{E} \left[ H^A (X_{-r_n, r_n} / u_n) \tilde{H} (X_{1-[gr_n], r_n-[gr_n]} / u_n) \ | \ |X| > u_n \right] , \ \xi \in (0, 1) .
\]

Note that the third equality follows by stationarity. By (5.12), for each \( \xi \in (0, 1) \), \( h_{n, 0}(\xi) \to \nu^*(H \tilde{H}) \). Furthermore, the sequence \( \{ h_{n, 0}, n \geq 1 \} \) is uniformly bounded in \( n \) and \( \xi \). Thus, with help of \( \mathcal{R}(r_n, u_n) \),
\[
\lim_{n \to \infty} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \text{cov} (H_{n, 0}^A, \tilde{H}_0) = \lim_{n \to \infty} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \mathbb{E} [H_{n, 0}^A \tilde{H}_0] = \nu^*(H \tilde{H}) .
\]

The other covariances vanish. Indeed, we analyse \( \text{cov} (H_{n, 0}^A, \tilde{H}_j), j \geq 1 \). We have, using
again the stationarity as above,

\[
\frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \mathbb{E}[H_{n,0}^A, \tilde{H}_j]
\]

\[
= \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{r_n} \mathbb{E} \left[ H^A(X_{i-r_n,i+r_n}/u_n) \tilde{H} \left( X_{jr_n+1, (j+1)r_n}/u_n \right) \right]
\]

\[
= \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{i=1}^{r_n} \mathbb{E} \left[ H^A(X_{-r_n,r_n}/u_n) \tilde{H} \left( X_{jr_n-\lfloor r_n \rfloor, (j+1)r_n-\lfloor r_n \rfloor}/u_n \right) \right]
\]

\[
= \int_0^1 h_{n,j}(\xi) d\xi
\]

with a function \( h_{n,j} \) defined on (0, 1) by

\[
h_{n,j}(\xi) = \mathbb{E} \left[ H^A(X_{-r_n,r_n}/u_n) \tilde{H} \left( X_{jr_n-\lfloor r_n \rfloor, (j+1)r_n-\lfloor r_n \rfloor}/u_n \right) \mid |X_0| > u_n \right].
\]

Until now we proceeded as in the case \( j = 0 \) above. However, now we use (5.3). For each \( \xi \in (0, 1), jr_n - \lfloor \xi r_n \rfloor \to +\infty \). Hence, \( h_{n,j}(\xi) \to 0 \). Bounded convergence and \( \mathcal{R}(r_n, u_n) \) give

\[
\lim_{n \to \infty} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \text{cov}(H_{n,0}^A, \tilde{H}_j) = 0. \tag{5.21}
\]

The same idea applies to \( \text{cov}(\tilde{H}_0, H_{n,j}^A), j \geq 1: \)

\[
\lim_{n \to \infty} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \text{cov}(\tilde{H}_0, H_{n,j}^A) = 0. \tag{5.22}
\]

Now, by (5.21)-(5.22), the terms that correspond to \( j = 1, 2 \) in (5.20) vanish, while (5.19) and (3.6c) give

\[
\frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=3}^{m_n-1} \left\{ \text{cov}(H_{n,0}^A, \tilde{H}_j) + \text{cov}(\tilde{H}_0, H_{n,j}^A) \right\}
\]

\[
= O(1) \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=1}^{\infty} \beta_{jr_n} = o(1).
\]

5.4 Empirical cluster process of runs statistics

Recall that

\[
H^A(x) = H(x) \mathbb{I} \{ A(x) = 0 \} \mathbb{I} \{|x_0| > 1\}.
\]
Define
\[ H_s^A(x) = H(x/s)\mathbb{1}\{A(x/s) = 0\}\mathbb{1}\{|x_0| > s\}. \]  
(5.23)

Recall that \(0 < s_0 < 1 < t_0 < \infty\). Recall also that \(k_n = n\mathbb{P}(|X_0| > u_n)\). Define also the classical tail empirical process by
\[ T_n(s) = \sqrt{k_n} \left\{ \frac{\sum_{j=1}^{n} \mathbb{1}\{|X_j| > su_n/k_n\}}{n} \right\}, \quad s \in [s_0, t_0]. \]

In order to deal with asymptotic normality of runs estimators, we study the empirical process
\[ \mathbb{F}_n(H_s^A) := \sqrt{k_n} \left\{ \tilde{\xi}_{n,r_n}(H_s^A) - \nu^*(H_s) \right\} \]
\[ = \sqrt{k_n} \left\{ \frac{\sum_{i=1}^{n} H_s^A(X_{i-r_n+i}u_n/k_n)}{n} - s^{-\alpha} \nu^*(H) \right\}. \]

The process \(\mathbb{F}_n(H_s^A)\) is viewed as a random element with values in \(\mathbb{D}([s_0, t_0])\). The next result is crucial to establish convergence of runs estimators.

**Theorem 5.5.** Let \(\{X_j, j \in \mathbb{Z}\}\) be a stationary, regularly varying \(\mathbb{R}^d\)-valued time series. Assume that \(R(r_n, u_n), \beta'(r_n), S(r_n, u_n), (3.11)\) and (3.12b) hold. Suppose that Assumption 3.4 is satisfied.

Then \(\mathbb{F}_n(H^A)\) converges weakly in \((\mathbb{D}([s_0, t_0]), J_1)\) to a Gaussian process \(\mathbb{G}(H)\) with the covariance \(\nu^*(H_sH_t)\). If moreover ANSJB\((r_n, u_n)\) is satisfied, then the convergence holds for \(H \in \mathbb{B}\). If additionally (3.12a) is satisfied, then the processes \(\mathbb{F}_n(H^A)\) and \(T_n(\cdot)\) converge jointly (\(\mathbb{G}(H), \mathbb{G}(\mathcal{E})\)).

### 5.5 Proof of Theorem 3.5

Write
\[ \psi_n = |X|_{(n:n-k_n)}/u_n. \]
Since \(k_n = n\mathbb{P}(|X_0| > u_n)\), we can rewrite \(\tilde{\xi}_{n,r_n}^*(H^A)\) as \(\tilde{\xi}_{n,r_n}^*(H^A) = \tilde{\xi}_{n,r_n}^*(H_{\psi_n}^A)\) (cf. (3.4)-(3.5)). Therefore,

\[ \sqrt{k_n} \left\{ \tilde{\xi}_{n,r_n}^*(H^A) - \nu^*(H) \right\} = \mathbb{F}_n(H_{\psi_n}^A) + \sqrt{k_n} \left\{ \nu^*(H_{\psi_n}) - \nu^*(H) \right\}. \]

We have local uniform convergence of \(\{\mathbb{F}_n(H_s^A), s \in [s_0, t_0]\}\) to a continuous Gaussian process \(\mathbb{G}\) thanks to Theorem 5.5. Moreover, the convergence of \(\{T_n(\cdot), s \in [s_0, t_0]\}\) yields \(\psi_n \xrightarrow{d} 1\), jointly with \(\mathbb{F}_n(H_s^A)\). Therefore, \(\mathbb{F}_n(H_{\psi_n}^A) \xrightarrow{d} \mathbb{G}(H)\). Using Vervaat’s theorem, we have, jointly with the previous convergence, \(\sqrt{k_n}(\psi_n^{-\alpha} - 1) \xrightarrow{d} -\mathbb{G}(\mathcal{E})\). Therefore, by the homogeneity of \(\nu^*\),

\[ \sqrt{k_n} \left\{ \nu^*(H_{\psi_n}) - \nu^*(H) \right\} = \nu^*(H) \sqrt{k_n}(\psi_n^{-\alpha} - 1) \xrightarrow{d} -\nu^*(H)\mathbb{G}(\mathcal{E}). \]

Since the convergence hold jointly, we conclude the result.
5.6 Proof of Theorem 5.5 - fidi convergence

Recall the disjoint blocks of size $r_n$ (cf. (3.2)):

$$J_j := \{jr_n + 1, \ldots, (j + 1)r_n\}, \quad j = 0, \ldots, m_n - 1.$$  

These blocks were chosen to calculate the limiting covariance of the process $F_n$. However, they are not appropriate for a proof of the central limit theorem. We need to introduce a large-small blocks decomposition.

For this purpose let $z_n$ be a sequence of integers such that $z_n \to \infty$ and

$$\lim_{n \to \infty} z_n r_n \mathbb{P}(|X_0| > u_n) = \lim_{n \to \infty} \frac{z_n r_n}{\sqrt{n} \mathbb{P}(|X_0| > u_n)} = \lim_{n \to \infty} \frac{z_n r_n}{\sqrt{k_n}} = 0.$$  

This is possible thanks to the assumptions $\mathcal{R}(r_n, u_n)$ and (3.11). We note that this assumption is needed for the Lindeberg condition only. Set

$$\tilde{m}_n = \frac{q_n}{(z_n + 3)r_n} = \frac{n - r_n}{(z_n + 3)r_n} \sim \frac{n}{z_n r_n}$$

and assume for simplicity that $\tilde{m}_n$ is an integer. Since $z_n \to \infty$, we have $\tilde{m}_n = o(m_n)$. For $j = 1, \ldots, \tilde{m}_n$ define now large and small blocks as follows:

- $L_1 = \{1, \ldots, z_n r_n\}$, $S_1 = \{z_n r_n + 1, \ldots, z_n r_n + 3r_n\}$,
- $L_2 = \{z_n r_n + 3r_n + 1, \ldots, 2z_n r_n + 3r_n\}$, $S_2 = \{2z_n r_n + 3r_n + 1, \ldots, 2z_n r_n + 6r_n\}$,
- $L_j = \{(j - 1)z_n r_n + 3(j - 1)r_n + 1, \ldots, jz_n r_n + 3(j - 1)r_n\}$,
- $S_j = \{jz_n r_n + 3(j - 1)r_n + 1, \ldots, jz_n r_n + 3jr_n\}$.

The block $L_1$ is obtained by merging $z_n$ consecutive blocks $J_0, \ldots, J_{z_n-1}$ of size $r_n$. Likewise, $S_1 = J_{z_n} \cup J_{z_n+1} \cup J_{z_n+2}$. Therefore, the large block of size $z_n r_n$ is followed by the small block of size $3r_n$, which in turn is followed by the large block of size $z_n r_n$ and so on. All together,

$$\bigcup_{j=1}^{\tilde{m}_n} (L_j \cup S_j) = \{1, \ldots, q_n\} = \{1, \ldots, n - r_n\}.$$  

Write

$$\sum_{i=1}^{n} H^A(X_{i-r_n,i+r_n}/u_n) = \sum_{i=1}^{q_n} H^A(X_{i-r_n,i+r_n}/u_n) + \sum_{i=q_n+1}^{n} H^A(X_{i-r_n,i+r_n}/u_n)$$

$$= \sum_{j=1}^{\tilde{m}_n} \Psi_j^{(l)}(H^A) + \sum_{j=1}^{\tilde{m}_n} \Psi_j^{(s)}(H^A) + W_n,$$  

(5.25)
where now
\[ \Psi_j^{(l)}(H^A) = \sum_{i \in L_j} H^A \left( X_{i-r_n,i+r_n}/u_n \right), \quad \Psi_j^{(s)}(H^A) = \sum_{i \in S_j} H^A \left( X_{i-r_n,i+r_n}/u_n \right) \]

and
\[ W_n = \sum_{i=q_n+1}^{n} H^A \left( X_{i-r_n,i+r_n}/u_n \right) = \sum_{i=n-r_n+1}^{n} H^A \left( X_{i-r_n,i+r_n}/u_n \right). \]

With such the decomposition, \( X_1, \ldots, X_{z_n r_n+r_n} \) used in the definition of \( \Psi_j^{(l)}(H^A) \) are separated by at least \( r_n \) from the random variables that define \( \Psi_j^{(s)}(H^A) \). The mixing condition (3.6a) allows us to replace \( X \) with the independent blocks process, that is, we can treat the random variables \( \Psi_j^{(l)}(H^A), j = 1, \ldots, \tilde{m}_n \), as independent. The same applies to \( \Psi_j^{(s)}(H^A) \).

Set
\[ Z_n(H^A) = \sum_{j=1}^{\tilde{m}_n} \left\{ Z_{n,j}(H^A) - \mathbb{E}[Z_{n,j}(H^A)] \right\} =: \sum_{j=1}^{\tilde{m}_n} \tilde{Z}_{n,j}(H^A) \quad (5.26) \]

with
\[ Z_{n,j}(H^A) = \frac{1}{\sqrt{k_n}} \Psi_j^{(l)}(H^A). \quad (5.27) \]

The next steps are standard.

- First, we show that the limiting variance of the large blocks process \( Z_n \) is the same as that of the process \( \mathbb{F}_n \);
- Next, we show that the small blocks process (the scaled second term in (5.25)) is negligible;
- We show that the boundary term \( W_n \) is also negligible;
- Finally, we will verify the Lindeberg condition for the large blocks process.

**Variance of the large blocks.** We have (using the assumed independence of \( \Psi_j^{(l)}(H^A) \))

\[
\text{var} \left( \frac{1}{\sqrt{k_n}} \sum_{j=1}^{\tilde{m}_n} \Psi_j^{(l)}(H^A) \right) = \frac{\tilde{m}_n}{k_n} \text{var}(\Psi_1^{(l)}(H^A)) \\
\sim \frac{1}{z_n r_n \mathbb{P}(|X_0| > u_n)} \text{var} \left( \sum_{i=1}^{z_n r_n} H^A \left( X_{i-r_n,i+r_n}/u_n \right) \right) \\
= \frac{1}{z_n r_n \mathbb{P}(|X_0| > u_n)} \text{var} \left( \sum_{j=0}^{z_n-1} H^A_{n,j} \right), \quad (5.28)
\]
where $H^A_{n,j}$ is defined in (3.3) and where in the last line we decomposed the block $L_1 = \{1, \ldots, z_n r_n\}$ into $z_n$ disjoint blocks $J_0, \ldots, J_{z_n-1}$ and $\tilde{m}_n \sim m_n/z_n$. The next steps follow easily from (5.13) with $m_n$ replaced by $z_n$.

The term in (5.28) becomes

$$\frac{\text{var} \left( H^A_{n,0} \right)}{r_n \mathbb{P}( |X_0| > u_n )} + \frac{2}{r_n \mathbb{P}( |X_0| > u_n )} \sum_{j=1}^{z_n-1} \left( 1 - \frac{j}{z_n} \right) \text{cov}(H^A_{n,0}, H^A_{n,j}) . \quad (5.29)$$

It follows immediately from (5.14) that the limit of the first term above is

$$\lim_{n \to \infty} \frac{\text{var} \left( H^A_{n,0} \right)}{r_n \mathbb{P}( |X_0| > u_n )} = \nu^* (H^2) . \quad (5.30)$$

Now, for the second term in (5.29) we adapt the proof of Lemma 3.2 from $m_n$ to $z_n$.

As in (5.17), for $j \geq 1$,

$$\frac{1}{r_n \mathbb{P}( |X_0| > u_n )} \left| \text{cov}(H^A_{n,0}, H^A_{n,j}) \right| \leq \sum_{h=(j-1)r_n+1}^{(j+1)r_n} |g_n(h)| =: I_j$$

with (this time)

$$g_n(h) = \frac{1}{\mathbb{P}( |X_0| > u_n )} \text{cov}(H^A(X_{-r_n, r_n}), H^A(X_{h-r_n, h+r_n})) .$$

For $h > 2r_n$, similarly to (5.18), we have by (5.1),

$$|g_n(h)| \leq \frac{\|H\|_\infty^2}{\mathbb{P}( |X_0| > u_n )} \beta_{h-2r_n} .$$

Thus,

$$\frac{1}{r_n \mathbb{P}( |X_0| > u_n )} \sum_{j=4}^{z_n-1} \left| \text{cov}(H^A_{n,0}, H^A_{n,j}) \right| \leq \frac{\|H\|_\infty^2}{\mathbb{P}( |X_0| > u_n )} \sum_{j=4}^{z_n-1} \sum_{h=(j-1)r_n+1}^{(j+1)r_n} \beta_{h-2r_n}$$

$$\leq 2 \frac{\|H\|_\infty^2}{\mathbb{P}( |X_0| > u_n )} \sum_{h=3r_n+1}^{\infty} \beta_{h-2r_n} = O(1) \frac{1}{\mathbb{P}( |X_0| > u_n )} \sum_{i=r_n+1}^{\infty} \beta_i = o(1)$$

by the assumption (3.6b). The terms that correspond to $j = 1, 2, 3$ in (5.13) are negligible.

In summary, we showed that

$$\lim_{n \to \infty} \text{var} \left( \frac{1}{\sqrt{k_n}} \sum_{j=1}^{m_n} \Psi_j^{(t)}(H^A) \right) = \nu^* (H^2) .$$
Variance of the small blocks. We have (using again the assumed independence of $\Psi_j(s) H^A$ thanks to the beta-mixing)

$$\var \left( \frac{1}{\sqrt{k_n}} \sum_{j=1}^{\tilde{m}_n} \Psi_j(s) H^A \right) = \tilde{m}_n \var(\Psi_1(s) H^A) \sim \frac{1}{z_n r_n \mathbb{P}(\|X_0\| > u_n)} \var(\Psi_1(s) H^A).$$

Since the size of $\Psi_1(s) H^A$ is 3 times the size of $H_{n,1}^A$ defined in (3.3), we have by (5.14)

$$\var \left( \frac{1}{\sqrt{k_n}} \sum_{j=1}^{\tilde{m}_n} \Psi_j(s) H^A \right) \sim \frac{1}{z_n r_n \mathbb{P}(\|X_0\| > u_n)} r_n \mathbb{P}(\|X_0\| > u_n) \nu^*(H^2) = O(1/z_n) = o(1).$$

Variance of the boundary term $W_n$. We have (cf. (3.3))

$$\var \left( \frac{1}{\sqrt{k_n}} W_n \right) = \var \left( \frac{1}{\sqrt{k_n}} \sum_{i=1}^{r_n} H^A(X_{i-r_n,i+r_n}/u_n) \right) = \frac{\var(H_{n,0}^A)}{k_n} = \frac{\var(H_{n,0}^A)}{n \mathbb{P}(\|X_0\| > u_n)}.$$

The latter term vanishes when $n \to \infty$, using (5.30) and $r_n/n \to 0$.

Lindeberg condition for $Z_n(H^A)$. We need to show that for all $\eta > 0$,

$$\lim_{n \to \infty} \tilde{m}_n \mathbb{E} \left[ Z_{n,1}^2(H^A) 1 \{ |Z_{n,1}(H^A)| > \eta \} \right] = 0. \quad (5.31)$$

Since $H$ is bounded, then by (5.24),

$$|Z_{n,1}(H^A)| \leq \frac{\sqrt{k_n} z_n r_n}{n \mathbb{P}(\|X_0\| > u_n)} \|H\|_\infty \sim \frac{z_n r_n}{\sqrt{n \mathbb{P}(\|X_0\| > u_n)}} \|H\|_\infty = o(1). \quad (5.32)$$

Thus, the indicator in (5.31) becomes zero for large $n$.

### 5.7 Proof of Theorem 5.5 - asymptotic equicontinuity

We need the following lemma which is an adapted version of Theorem 2.11.1 in [vdVW96]. Let $Z_n$ be the empirical process indexed by a semi-metric space $(G, \rho)$, defined by

$$Z_n(f) = \sum_{j=1}^{\tilde{m}_n} \{Z_{n,j}(f) - \mathbb{E}[Z_{n,j}(f)]\},$$

where $\{Z_{n,j}, n \geq 1\}, j = 1, \ldots, \tilde{m}_n$, are i.i.d. separable, stochastic processes and $\tilde{m}_n$ is a sequence of integers such that $\tilde{m}_n \to \infty$. Define the random semi-metric $d_n$ on $G$ by

$$d_n^2(f, g) = \sum_{j=1}^{\tilde{m}_n} \{Z_{n,j}(f) - Z_{n,j}(g)\}^2, f, g \in G.$$
Lemma 5.6. Assume that \((\mathcal{G}, \rho)\) is totally bounded. Assume moreover that:

(i) For all \(\eta > 0\),
\[
\lim_{n \to \infty} \tilde{m}_n \mathbb{E}[\|Z_{n,1}\|_G^2 \mathbb{1}\{\|Z_{n,1}\|_G^2 > \eta\}] = 0 .
\] (5.33)

(ii) For every sequence \(\{\delta_n\}\) which decreases to zero,
\[
\lim_{n \to \infty} \sup_{\rho(f,g) \leq \delta_n} \mathbb{E}[d_n^2(f,g)] = 0 .
\] (5.34)

(iii) There exists a measurable majorant \(N^*(\mathcal{G}, d_n, \epsilon)\) of the covering number \(N(\mathcal{G}, d_n, \epsilon)\) such that for every sequence \(\{\delta_n\}\) which decreases to zero,
\[
\int_0^{\delta_n} \sqrt{\log N^*(\mathcal{G}, d_n, \epsilon)} \, d\epsilon \xrightarrow{p} 0 .
\]

Then \(\{Z_n, n \geq 1\}\) is asymptotically \(\rho\)-equicontinuous, i.e. for each \(\eta > 0\),
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup \mathbb{P} \left( \sup_{\rho(f,g) \leq \delta} |Z_n(f) - Z_n(g)| > \eta \right) = 0 .
\]

Remark 5.7. The separability assumption is not in [vdVW96]. It implies measurability of \(\|Z_{n,1}\|_G\). Furthermore, the separability also implies that for all \(\delta > 0, n \in \mathbb{N}, (e_j)_{1 \leq j \leq \tilde{m}_n} \in \{-1, 0, 1\}^{\tilde{m}_n}\) and \(i \in \{1, 2\}\), the supremum
\[
\sup_{\rho(f,g) \leq \delta} \left| \sum_{j=1}^{\tilde{m}_n} e_j (Z_{n,j}(f) - Z_{n,j}(g))^i \right|
\]

is measurable, which is an assumption of [vdVW96].

\[\oplus\]

5.7.1 Asymptotic equicontinuity of the empirical process of sliding blocks

Recall the big-blocks process \(Z_n(H^A)\) (cf. (5.26)-(5.27)). Recall also that thanks to the \(\beta\)-mixing we can consider random variables \(\Psi_j^{(i)}(H^A), j = 1, \ldots, \tilde{m}_n\) to be independent. Recall that \(H^A_s\) is defined in (5.23). We need to prove the asymptotic equicontinuity of \(Z_n(H^A_s)\) indexed by the class \(\mathcal{G} = \{H^A_s, s \in [s_0, t_0]\}\) equipped with the metric \(\rho^*(H^A_s, H^A_t) = \nu^*(\{H^A_s - H^A_t\}^2)\). The same argument can be used to prove the asymptotic equicontinuity for the small blocks process. This yields asymptotic equicontinuity of \(\mathcal{F}_n(H^A_s)\).

In what follows, the proof of the Lindeberg-type condition (5.33) is easy. The proof of (5.34) is quite involved.

Thanks to Assumption 3.4, the condition (3.10) is satisfied. Its validity is discussed in Section 5.8.
Lindeberg condition: Proof of (5.33). We re-write (5.32) as follows:

\[ \sup_{s \in [s_0, t_0]} \left| Z_{n,1}(H^A_s) \right| \leq \frac{\sqrt{k_n z_n r_n}}{n \mathbb{P}(|X_0| > u_n)} \| H \|_\infty \leq \frac{z_n r_n}{\sqrt{n \mathbb{P}(|X_0| > u_n)}} \sup_{s \in [s_0, t_0]} \| H_s \|_\infty, \]

Since the class \( \{ H_s : s \in [s_0, t_0] \} \) is linearly ordered, \( \sup_{s \in [s_0, t_0]} \| H_s \|_\infty \) is achieved either at \( s = s_0 \) or \( s = t_0 \). Hence, the Lindeberg condition (i) of Lemma 5.6 holds by (5.31).

Asymptotic continuity of random semi-metric: Proof of (5.34). The proof is rather long and technical.

Define the random metric

\[ d_n^2(H^A_s, H^A_t) = \sum_{j=1}^{m_n} (Z_{n,j}(H^A_s) - Z_{n,j}(H^A_t))^2. \]

Let (cf. (3.3))

\[ H^A_{s,n,j} = \sum_{i=\lfloor j r_n + 1 \rfloor}^{(j+1)r_n} \mathbb{I} \left\{ A(X_{i-r_n,i+r_n}/(su_n)) = i \right\} \mathbb{I} \{ |X_i| > su_n \}. \]

We need to evaluate \( \mathbb{E}[d_n^2(H^A_s, H^A_t)] \):

\[
\mathbb{E}[d_n^2(H^A_s, H^A_t)] = \frac{k_n m_n}{(n \mathbb{P}(|X_0| > u_n))^2} \mathbb{E} \left[ \left( \sum_{i=1}^{z_n r_n} \left\{ H^A_s(X_{i-r_n,i+r_n}/u_n) - H^A_t(X_{i-r_n,i+r_n}/u_n) \right\} \right)^2 \right] \\
\sim \frac{1}{z_n r_n \mathbb{P}(|X_0| > u_n)} \mathbb{E} \left[ \left( \sum_{j=0}^{z_n-1} \left\{ H^A_{s,n,j} - H^A_{t,n,j} \right\} \right)^2 \right],
\]

(5.35)

where in the last line we decomposed the block \( L_1 \) into \( z_n \) disjoint blocks \( J_0, \ldots, J_{z_n-1} \);

\( m_n \sim m_n/z_n \); cf. (5.28). The term in (5.35) becomes

\[
\frac{\mathbb{E}\left\{ \left( H^A_{s,n,0} - H^A_{t,n,0} \right)^2 \right\}}{r_n \mathbb{P}(|X_0| > u_n)} + 2 \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=1}^{z_n-1} \left( 1 - \frac{j}{z_n} \right) \mathbb{E}\left\{ H^A_{s,n,0} - H^A_{t,n,0} \right\} \left\{ H^A_{s,n,j} - H^A_{t,n,j} \right\}
\]

The above lines correspond to (5.13) with \( m_n \) replaced by \( z_n \).

We are going to prove two statements:

\[
\lim_{n \to \infty} \sup_{0 \leq s,t \leq t_0 \atop |s-t| \leq s_n} \frac{\mathbb{E}\left\{ \left( H^A_{s,n,0} - H^A_{t,n,0} \right)^2 \right\}}{r_n \mathbb{P}(|X_0| > u_n)} = 0
\]

(5.36)
and
\[
\lim_{n \to \infty} \sup_{0 \leq s,t \leq t_0} \frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=1}^{z_n-1} \left(1 - \frac{j}{z_n}\right) \mathbb{E}[\{H_{s,n,0} - H_{t,n,0}\} \{H_{s,n,j} - H_{t,n,j}\}] = 0.
\]

(5.37)

**Proof of (5.36).** We will write \(\{H^A_s - H^A_t\}(x)\) for \(H^A_s(x) - H^A_t(x)\).

Similarly to (5.14),
\[
\mathbb{E}[\{H^A_{s,n,0} - H^A_{t,n,0}\}^2] \leq \|H\|_1 \mathbb{P}(|X_0| > u_n) \sum_{h=-r_n}^{r_n} |g_n(h, H^A_s - H^A_t)|
\]
\[
=: \|H\| \sum_{h=-r_n}^{r_n} |g_n(h, H^A_s - H^A_t)|
\]

(5.38)

with
\[
|g_n(h, G)| = \left|\frac{1}{\mathbb{P}(|X_0| > u_n)} \mathbb{E}[G(X_{-r_n,u_n}/u_n)G(X_{h-r_n,u_n}/u_n)]\right|.
\]

(5.39)

Using the definition (5.23) of \(H^A_s\), the fact that \(s, t \geq s_0\) and since \(H\) is bounded, we immediately get
\[
|g_n(h, H^A_s - H^A_t)| \leq \frac{4}{\mathbb{P}(|X_0| > u_n)} \|H\|^2 \mathbb{P}(|X_0| > s_0 u_n, |X_h| > s_0 u_n).
\]

(5.40)

To get a more precise bound that involves the difference \(s - t\) we need to consider three cases. The reason for this is that we need to keep the absolute value in (5.39) outside of the expectation. As such, computations below are quite technically involved.

To shorten our displays, we introduce the notation
\[
\mathcal{I}(i, s) := 1\{|X_i| > su_n\}.
\]

(5.41)

**Case 1.** Assume here that \(\mathcal{A}\) is 0-homogeneous. Then for any \(i\),
\[
H^A_s(\mathbf{X}_{i-r_n,i+r_n}/u_n) = H(\mathbf{X}_{i-r_n,i+r_n}/(su_n)) 1\{\mathcal{A}(\mathbf{X}_{i-r_n,i+r_n}/(su_n)) = i\} \mathcal{I}(i, s)
\]
\[
= H_s(\mathbf{X}_{i-r_n,i+r_n}/u_n) 1\{\mathcal{A}(\mathbf{X}_{i-r_n,i+r_n}/u_n) = i\} \mathcal{I}(i, s);
\]

(5.42)

(we keep \(u_n\) in the argument of \(\mathcal{A}\), although it can be omitted). What is important in this decomposition is that we can control monotonicity (with respect to \(s\)) of each term.
Then
\[ H_s^A (X_{i-r_n,i+r_n}/u_n) - H_t^A (X_{i-r_n,i+r_n}/u_n) = \]
\[ = \mathbb{1}\{A (X_{i-r_n,i+r_n}/u_n) = i\} H_s (X_{i-r_n,i+r_n}/u_n) (\mathcal{I}(i, s) - \mathcal{I}(i, t)) \]
\[ + \mathbb{1}\{A (X_{i-r_n,i+r_n}/u_n) = i\} \mathcal{I}(i, t) (H_s (X_{i-r_n,i+r_n}/u_n) - H_t (X_{i-r_n,i+r_n}/u_n)) \]
\[ =: T_1(i) + T_2(i) \]
and hence
\[ |\mathbb{E}\left[\{H_s^A - H_t^A\} (X_{-r_n,r_n}/u_n) \{H_s^A - H_t^A\} (X_{h-r_n,h+r_n}/u_n)\right]| \]
is bounded by the sum of four nonnegative terms \( W_{11} + W_{22} + W_{12} + W_{21} \) that we are going to define below. The general idea is that we will obtain rough bounds in terms of the difference \( (\mathcal{I}(i, s) - \mathcal{I}(i, t)) \), except of one case which involves the product \( T_2(0)T_2(h) \).

Coming back to the definitions of \( W_{i,j} \), the double sub-index \( \{12\} \) of \( W \) indicates that \( W_{12} \) is related to multiplying \( T_1(0) \) by \( T_2(h) \); \( W_{21} \) means we multiply \( T_2(0) \) and \( T_1(h) \):

\[ W_{11} := ||H||^2 \mathbb{E}\left[ (\mathcal{I}(0, s) - \mathcal{I}(0, t)) (\mathcal{I}(h, s) - \mathcal{I}(h, t)) \right], \]

(above, the indicators of the anchoring map are omitted);

\[ W_{22} := \]
\[ \mathbb{E}\left[ \mathbb{1}\{A (X_{-r_n,r_n}/u_n) = 0\} \mathcal{I}(0, t) \{H_s - H_t\} (X_{-r_n,r_n}/u_n) \{H_s - H_t\} (X_{h-r_n,h+r_n}/u_n) \right], \]

(above, \( \mathbb{1}\{A (X_{h-r_n,h+r_n}/u_n) = h\} \) and \( \mathcal{I}(h, t) \) are omitted);

\[ W_{12} := \pm ||H|| \mathbb{E}\left[ (\mathcal{I}(0, s) - \mathcal{I}(0, t)) \{H_s - H_t\} (X_{h-r_n,h+r_n}/u_n) \right], \]

(both indicators of the anchoring maps and \( \mathcal{I}(h, t) \) are omitted);

\[ W_{21} := \pm ||H|| \mathbb{E}\left[ (\mathcal{I}(h, s) - \mathcal{I}(h, t)) \{H_s - H_t\} (X_{-r_n,r_n}/u_n) \right], \]

(both indicators of the anchoring maps and \( \mathcal{I}(0, t) \) are omitted).

Note that the right-hand side of both \( W_{11}, W_{22} \) is nonnegative thanks to the monotonicity and \( s < t \), while for the right-hand side of \( W_{12}, W_{21} \) we need to put \( \pm \), since the sign of the expressions there depends on whether the map \( s \rightarrow H_s \) is decreasing or increasing.

Thus,
\[ W_{11} \leq ||H||^2 \left( \mathbb{P}(|X_0| > u_n s) - \mathbb{P}(|X_0| > u_n t) \right), \quad (5.43) \]
\[ W_{22} \leq \pm 2 ||H|| \mathbb{E}\left[ \mathbb{1}\{A (X_{-r_n,r_n}/u_n) = 0\} \mathcal{I}(0, t) \{H_s - H_t\} (X_{-r_n,r_n}/u_n) \right], \quad (5.44) \]
\[ W_{12} + W_{21} \leq 4\|H\|^2 \left( \mathbb{P}(|X_0| > u_ns) - \mathbb{P}(|X_0| > u_nt) \right) . \] 

The bound on \( W_{22} \) (with +) is obvious if \( H_s - H_t \geq 0 \) (thus, \( s \to H_s \) is decreasing), while in an increasing case of \( s \to H_s \) we use the following observation: if \( a, b < 0, |b| < c, \) then \( ab \leq -ac \) (yielding − on the right-hand side of (5.44)). The bound on \( W_{12} + W_{21} \) follows from the same reasoning.

In summary, with \( g_n(h, H_s^A - H_t^A) \) defined in (5.39), we have

\[ |g_n(h, H_s^A - H_t^A)| \leq 5\|H\|^2 \frac{\mathbb{P}(|X_0| > u_ns) - \mathbb{P}(|X_0| > u_nt)}{\mathbb{P}(|X_0| > u_n)} \]

\[ + 2\|H\| \mathbb{E} \left[ \mathbb{1} \{ A(X_{-r_n,r_n}/u_n) = 0 \} \mathbb{1} \{ |X_0| > s_0u_n \} \{ H_s - H_t \} (X_{-r_n,r_n}/u_n) \right] \]

(5.46)

where again the presence of ± depends on the sign of \( H_s - H_t \).

We can ignore the scaling factor \( 2\|H\| \) in (5.47) and write it as (recall that the anchoring map is 0-homogeneous)

\[ \mathbb{E} \left[ H_s/s_0 (X_{-r_n,r_n}/(s_0u_n)) \mathbb{1} \{ A(X_{-r_n,r_n}/(s_0u_n)) = 0 \} \mathbb{1} \{ |X_0| > s_0u_n \} \frac{\mathbb{P}(|X_0| > s_0u_n)}{\mathbb{P}(|X_0| > u_n)} \right] \]

\[ - \mathbb{E} \left[ H_t/s_0 (X_{-r_n,r_n}/(s_0u_n)) \mathbb{1} \{ A(X_{-r_n,r_n}/(s_0u_n)) = 0 \} \mathbb{1} \{ |X_0| > s_0u_n \} \frac{\mathbb{P}(|X_0| > s_0u_n)}{\mathbb{P}(|X_0| > u_n)} \right] \]

\[ = \left( \mu_{n,r_n}(s) - \mu_{n,r_n}(t) \right) \frac{\mathbb{P}(|X_0| > s_0u_n)}{\mathbb{P}(|X_0| > u_n)} =: \left( \tilde{\mu}_{n,r_n}(s) - \tilde{\mu}_{n,r_n}(t) \right) , \]

with

\[ \mu_{n,r_n}(\cdot) = \mathbb{E} \left[ H_s/s_0 (X_{-r_n,r_n}/(s_0u_n)) \mathbb{1} \{ A(X_{-r_n,r_n}/(s_0u_n)) = 0 \} \mathbb{1} \{ |X_0| > s_0u_n \} \right] \]

and

\[ \tilde{\mu}_{n,r_n}(s) = \mu_{n,r_n}(s) \frac{\mathbb{P}(|X_0| > s_0u_n)}{\mathbb{P}(|X_0| > u_n)} . \]

Thanks to (5.5), \( \lim_{n \to \infty} \mu_{n,r_n}(s) = \nu^*(H_s/s_0) \). Thanks to the monotonicity of \( s \to H_S \) and homogeneity of \( \nu^* \), the convergence of \( \tilde{\mu}_{n,r_n}(s) \) to

\[ s_0^{-\alpha} \nu^*(H_{s/s_0}) = s_0^{-\alpha}(s/s_0)^{-\alpha} \nu^*(H) = s^{-\alpha} \nu^*(H) \]

is uniform on \([s_0, t_0]\). Thus, for \( s, t \in [s_0, t_0] \),

\[ |\tilde{\mu}_{n,r_n}(s) - \tilde{\mu}_{n,r_n}(t)| \leq 2 \sup_{s_0 \leq u_n \leq t_0} |\tilde{\mu}_{n,r_n}(u) - \nu^*(H_u)| + \nu^*(H)\{ s^{-\alpha} - t^{-\alpha} \} . \]

Fix \( \eta > 0 \). For large enough \( n \), the uniform convergence yields

\[ \sup_{s_0 \leq s, t \leq t_0} \left| \tilde{\mu}_{n,r_n}(s) - \tilde{\mu}_{n,r_n}(t) \right| \leq \eta + \nu^*(H) \sup_{s_0 \leq u_n \leq t_0} \{ s^{-\alpha} - t^{-\alpha} \} \]

\[ \leq \eta + \alpha s_0^{-\alpha-1} \delta_n \nu^*(H) . \] (5.48)
The uniform convergence also yields that the term in (5.46) is bounded by \( \eta + \alpha s_0^{-\alpha - 1} \delta_n \). This, together with (5.48), gives
\[
|g_n(h, H^A_s - H^A_t)| \leq \eta + \text{cst} \delta_n
\] (5.49)
with a generic constant \( \text{cst} \).

Fix an integer \( r \). Using (5.40) and (5.49) we have
\[
\sup_{s_0 \leq s \leq t \leq t_0 \atop |s - t| \leq \delta_n} \mathbb{E} \left[ \frac{\left( H^A_s,0 - H^A_t,0 \right)^2}{r_n \mathbb{P}(|X_0| > u_n)} \right] 
\leq \sup_{s_0 \leq s \leq t \leq t_0 \atop |s - t| \leq \delta_n} \sum_{h = -r}^{r} |g_n(h, H^A_s - H^A_t)| + \sup_{s_0 \leq s \leq t \leq t_0 \atop |h| = r} \sum_{i} |g_n(h, H^A_s - H^A_t)|
\leq \text{cst} \ r (\eta + \delta_n) + \frac{4}{\mathbb{P}(|X_0| > u_n)} \sum_{|h| = r} \mathbb{P}(|X_0| > s_0 u_n, |X_h| > s_0 u_n).
\]

Applying the anticlustering conditions \( S(r_n, u_n) \) to the second term, letting \( \delta_n \to 0 \), since \( \eta \) is arbitrary, this proves (5.36).

**Case 2.** Now, we consider the anchoring maps \( A^{(1)} \) and \( A^{(2)} \) which are not 0-homogeneous. Note that for \( j = 1, 2 \) we can write
\[
H^A_s (X_{i-r_n,i+r_n}/u_n) = H \left( X_{i-r_n,i+r_n}/(su_n) \right) \mathbb{1} \{ A^{(j)} (X_{i-r_n,i+r_n}/(su_n)) = i \} \mathbb{I}(i, s)
= H_s (X_{i-r_n,i+r_n}/u_n) \mathbb{I}(i, s) F_s (X_{i-r_n,i+r_n}/u_n),
\]
where
\[
F_s (X_{i-r_n,i+r_n}/u_n) = \mathbb{1} \{ X^*_s \leq su_n \}
\]
or
\[
F_s (X_{i-r_n,i+r_n}/u_n) = \mathbb{1} \{ X^*_{s+1} \leq su_n \}
\]
in case \( j = 1 \) and \( j = 2 \), respectively. Note that regardless of the monotonicity of the map \( s \to A^{(j)}_s \), the map \( s \to F_s \) is always non-decreasing. Then (5.42) gives, for any \( i \in \mathbb{N} \), and \( A = A^{(1)}, A^{(2)} \),
\[
H^A_s (X_{i-r_n,i+r_n}/u_n) - H^A_t (X_{i-r_n,i+r_n}/u_n) =
= H_s (X_{i-r_n,i+r_n}/u_n) F_s (X_{i-r_n,i+r_n}/u_n) \left( \mathbb{I}(i, s) - \mathbb{I}(i, t) \right)
+ \mathbb{I}(i, t) F_s (X_{i-r_n,i+r_n}/u_n) (H_s (X_{i-r_n,i+r_n}/u_n) - H_t (X_{i-r_n,i+r_n}/u_n))
+ \mathbb{I}(i, t) H_t (X_{i-r_n,i+r_n}/u_n) \left( F_s (X_{i-r_n,i+r_n}/u_n) - F_t (X_{i-r_n,i+r_n}/u_n) \right)
=: T_1(i) + T_2(i) + T_3(i).
\]
Again, in this decomposition we can control monotonicity of each term. The argument now is very similar to that of Case 1. Hence, it is omitted.

Therefore, (5.36) is proved for $A(j), j = 0, 1, 2$.

Proof of (5.37). We proceed similarly. Recall that for $j \geq 1$ (cf. (5.17)),

$$
\frac{1}{r_n} \mathbb{P}(|X_0| > u_n) \mathbb{E} \left\{ \left\{ H^{A}_{s,n,0} - H^{A}_{t,n,0} \right\} \left\{ H^{A}_{s,n,j} - H^{A}_{t,n,j} \right\} \right\}
$$

$$
= \frac{1}{r_n} \mathbb{P}(|X_0| > u_n) \mathbb{E} \left[ \sum_{h=1}^{r_n} \left( H^{A}_{s} - H^{A}_{t} \right) (X_{h-r_n,h+r_n}/u_n) \times \sum_{i=jr_n+1}^{(j+1)r_n} \left( H^{A}_{s} - H^{A}_{t} \right) (X_{i-r_n,i+r_n}/u_n) \right]
$$

$$
= \sum_{h=(j-1)r_n+1}^{jr_n} \left( \frac{h}{r_n} - (j-1) \right) g_n(h, H^{A}_{s} - H^{A}_{t}) + \sum_{h=jr_n+1}^{(j+1)r_n} \left( (j+1) - \frac{h}{r_n} \right) g_n(h, H^{A}_{s} - H^{A}_{t})
$$

$$
\leq \sum_{h=(j-1)r_n+1}^{(j+1)r_n} g_n(h, H^{A}_{s} - H^{A}_{t}) =: I_j(s,t)
$$

with the same $g_n$ as in (5.39).

Write $g_n(h, G)$ as

$$
\frac{1}{\mathbb{P}(|X_0| > u_n)} \text{cov}[G(X_{-r_n,r_n}/u_n), G(X_{h-r_n,h+r_n}/u_n)] + \frac{1}{\mathbb{P}(|X_0| > u_n)} \mathbb{E}^2[G(X_{-r_n,r_n}/u_n)]
$$

$$
=: \tilde{g}_n(h, G) + \frac{1}{\mathbb{P}(|X_0| > u_n)} \mathbb{E}^2[G(X_{-r_n,r_n}/u_n)].
$$

For $h > 2r_n$ we have by (5.1),

$$
|\tilde{g}_n(h, H^{A}_{s} - H^{A}_{t})| \leq \frac{\text{cst}}{\mathbb{P}(|X_0| > u_n)} \beta_{h-2r_n}.
$$

(5.51)
Thus,

\[
\frac{1}{r_n \mathbb{P}(|X_0| > u_n)} \sum_{j=4}^{2n-1} \left(1 - \frac{j}{z_n}\right) \mathbb{E}\left[(H_{s,n,j}^A - H_{t,n,j}^A) (H_{s,n,0}^A - H_{t,n,0}^A)\right] \\
\leq \frac{\text{cst}}{\mathbb{P}(|X_0| > u_n)} \sum_{j=4}^{2n-1} \sum_{h=(j-1)r_n+1}^{(j+1)r_n} \beta_{h-2r_n} \\
+ \frac{\text{cst}}{\mathbb{P}(|X_0| > u_n)} z_n r_n \mathbb{E}^2\left[(H_{s,n}^A - H_{t,n}^A) (X_{r_n,r_n} - r_n/u_n)\right] \\
\leq \frac{\text{cst}}{\mathbb{P}(|X_0| > u_n)} \sum_{h=3r_n+1}^{\infty} \beta_{h-2r_n} + \text{cst} z_n r_n \mathbb{P}(|X_0| > s_0 u_n) \\
= \frac{\text{cst}}{\mathbb{P}(|X_0| > u_n)} \sum_{i=r_n+1}^{\infty} \beta_i + o(1) = o(1) \quad (5.52)
\]

uniformly in \(s,t \in [s_0,t_0]\). In the last line we applied the assumption (3.6b), and the assumption (5.24).

The terms that correspond to \(j = 1, 2, 3\) in (5.50) have to be dealt with separately. We note that \(I_1(s,t) = \sum_{h=1}^{r_n} g_n(h, H_s^A - H_t^A)\) is bounded by the term in (5.38). Hence,

\[
\lim_{n \to \infty} \sup_{s_0 \leq s,t \leq t_0, |s-t| \leq \delta_n} I_1(s,t) = 0 \quad . \quad (5.53)
\]

Next, using (5.40) and \(S(r_n,u_n)\),

\[
I_2(s,t) + I_3(s,t) \leq \text{cst} \sum_{h=r_n+1}^{4r_n} g_n(h) \leq \text{cst} \sum_{h=r_n+1}^{4r_n} \mathbb{P}(|X_0| > s_0 u_n, |X_h| > s_0 u_n) = 0 \quad , \quad (5.54)
\]

uniformly in \(s,t \in [s_0,t_0]\).

Combination of (5.52), (5.53), (5.54) finishes the proof of (5.37).

This, together with (5.36) concluded the proof of (5.34).

### 5.8 Random entropy

In this section we discuss validity of Assumption 3.4. We cannot check this condition for arbitrary functionals \(H\) and anchoring maps \(\mathcal{A}\), however, we will see that the conditions is satisfied for most relevant cases considered in the paper.

Recall the class

\[
\mathcal{G} = \{H_s^A, s \in [s_0,t_0]\} = \{H(x/s) \mathbb{1}\{\mathcal{A}(x/s) = 0\} \mathbb{1}\{|x_0| > s\}, \quad s \in [s_0,t_0]\}.
\]
We start first with $H$ of the form

$$H_s(x) = 1 \{ K(x) > s \},$$

(5.55)

where $K : \mathbb{R}^2 \to \mathbb{R}$. This is the case of the functionals that lead to the extremal index, the large deviation index and the ruin index.

Since $A^{(0)}$ is 0-homogeneous, it does not play a role in calculating the class entropy. Then

$$H_s(x)1 \{|x_0| > s \} = 1 \{ \min \{ K(x), |x_0| \} > s \}.$$

Hence, the map $s \to H_s(x)1 \{|x_0| > s \}$ is decreasing. Therefore, $\text{VC}(\mathcal{G}) = 2$.

As for $A^{(1)}$ we have

$$1 \{ A^{(1)}(x/s) = 0 \} = 1 \{ x_{-\infty,-1}^* \leq s, |x_0| > s \}.$$

Thus,

$$H(x/s)1 \{ A(x/s) = 0 \}1 \{|x_0| > s \} = 1 \{ \min \{ K(x), |x_0| \} > s \}1 \{ x_{-\infty,-1}^* \leq s \}.$$

Now, the class

$$\mathcal{F} = \{ (-\infty, s) \times (s, +\infty) : s \in \mathbb{R} \}$$

has the VC-index 3. By [KS20, Example C.4.14] the class $\mathcal{G}$ has VC-index at most 3.

Similarly, for $A^{(2)}$ we have

$$1 \{ A^{(2)}(x/s) = 0 \} = 1 \{ x_{1,\infty}^* \leq s, |x_0| > s \}.$$

Thus,

$$H(x/s)1 \{ A(x/s) = 0 \}1 \{|x_0| > s \} = 1 \{ \min \{ K(x), |x_0| \} > s \}1 \{ x_{1,\infty}^* \leq s \}$$

and again the class $\mathcal{G}$ has VC-index at most 3.

In summary, for functionals $H$ given in (5.55) and the anchoring maps $A^{(0)}, A^{(1)}, A^{(2)}$ the class $\mathcal{G}$ has the VC-index at most 3 and hence the random entropy Assumption 3.4 is satisfied.

Now, assume that the map $s \to H_s$ is decreasing. This is the case of (again) the extremal index, the large deviation index and the ruin index. This is also the case of the stop-loss index and the cluster size distribution. If we choose $A = A^{(0)}$, since $A^{(0)}$ is 0-homogeneous, the maps $s \to H_s^A$ is also decreasing. Thus, the VC-index of $\mathcal{G}$ is at most 2. The random entropy condition is satisfied.
References

[BB18] Betina Berghaus and Axel Bücher. Weak convergence of a pseudo maximum likelihood estimator for the extremal index. *Annals of Statistics, 46*(5):2307–2335, 2018.

[BBKS20] Clemonell Bilayi-Biakana, Rafał Kulik, and Philippe Soulier. Statistical inference for heavy tailed series with extremal independence. *Extremes, 23*(1):1–33, 2020.

[BP18] Bojan Basrak and Hrvoje Planinić. A note on vague convergence of measures. *Statistics and Probability Letters, 153*:180–186, 2019.

[BPS18] Bojan Basrak, Hrvoje Planinić, and Philippe Soulier. An invariance principle for sums and record times of regularly varying stationary sequences. *Probability Theory and Related Fields, 172*(3-4):869–914, 2018.

[BS09] Bojan Basrak and Johan Segers. Regularly varying multivariate time series. *Stochastic Processes and their Applications, 119*(4):1055–1080, 2009.

[BS18a] Axel Bücher and Johan Segers. Inference for heavy tailed stationary time series based on sliding blocks. *Electronic Journal of Statistics, 12*(1):1098–1125, 2018.

[BS18b] Axel Bücher and Johan Segers. Maximum likelihood estimation for the Fréchet distribution based on block maxima extracted from a time series. *Bernoulli, 24*(2):1427–1462, 2018.

[BZ18] Axel Bücher and Chen Zhou. A horse racing between the block maxima method and the peak-over-threshold approach. arXiv:1087:00282v1.

[CK21] Youssouph Cissokho and Rafal Kulik. Estimation of cluster functionals for regularly varying time series: sliding blocks estimators. *Electronic Journal of Statistics, 15*(1):2777–2831, 2021.

[DM09] Richard A. Davis and Thomas Mikosch. The extremogram: A correlogram for extreme events. *Bernoulli, 38A*:977–1009, 2009. Probability, statistics and seismology.

[DH95] Richard A. Davis and Tailen Hsing. Point process and partial sum convergence for weakly dependent random variables with infinite variance. *Annals of Probability, 23*(2):879–917, 1995.

[DHS18] Clément Dombry, Enkelejd Hashorva, and Philippe Soulier. Tail measure and spectral tail process of regularly varying time series. *Annals of Applied Probability, 28*(6):3884–3921, 2018.
Holger Drees and Miran Knezevic. Peak-over-threshold estimators for spectral tail processes: Random vs deterministic thresholds. *Extremes*, 2020. DOI: https://doi.org/10.1007/s10687-019-00367-x.

Holger Drees and Sebastian Neblung. Asymptotics for sliding blocks estimators of rare events. arXiv:2003.01016, 2020.

Holger Drees and Holger Rootzén. Limit theorems for empirical processes of cluster functionals. *Annals of Statistics*, 38(4):2145–2186, 2010.

Holger Drees, Johan Segers, and Michal Warchol. Statistics for tail processes of Markov chains. *Extremes*, 18(3):369–402, 2015.

Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling Extremal Events for Insurance and Finance*. Springer–Verlag, 1997.

Ana Ferreira, Laurens de Haan. On the block maxima method in extreme value theory: PWM estimators *Annals of Statistics*, 43(1): 276-298, 2015.

Enkelejd Hashorva. Representations of max-stable processes via exponential tilting, *Stochastic Processes and their Applications*, 128(9): 2952–2978, 2018.

Tailen Hsing. Estimating the parameters of rare events. *Stochastic Processes and their Applications*, 37(1):117–139, 1991.

I. A. Ibragimov. Some limit theorems for stationary processes. *Theor. Probab. Appl.*, 7:349–382, 1962.

Rafał Kulik and Philippe Soulier. *Heavy tailed time series*. Springer, 2020.

Rafał Kulik, Philippe Soulier, and Olivier Wintenberger. The tail empirical process of regularly varying functions of geometrically ergodic markov chains. *Stochastic Processes and their Applications*, 129(1):4209–4238, 2019.

Thomas Mikosch and Olivier Wintenberger. Precise large deviations for dependent regularly varying sequences. *Probability Theory and Related Fields*, 156(3-4):851–887, 2013.

Thomas Mikosch and Olivier Wintenberger. The cluster index of regularly varying sequences with applications to limit theory for functions of multivariate Markov chains. *Probability Theory and Related Fields*, 159(1-2):157–196, 2014.

Thomas Mikosch and Olivier Wintenberger. A large deviations approach to limit theorem for heavy-tailed time series. *Probability Theory and Related Fields*, 166(1-2):233–269, 2016.

Hrvoje Planinić and Philippe Soulier. The tail process revisited. *Extremes*, 21(4):551–579, 2018.
[RLdH98] Holger Rootzén, Ross M. Leadbetter, and Laurens de Haan. On the distribution of tail array sums for strongly mixing stationary sequences. *Annals of Applied Probability*, 8(3):868–885, 1998.

[RSF09] Christian Y. Robert, Johan Segers, and Christopher A. T. Ferro. A sliding blocks estimator for the extremal index. *Electronic Journal of Statistics*, 3:993–1020, 2009.

[SW94] Richard L. Smith and Ishay Weissman. Estimating the extremal index. *Journal of the Royal Statistical Society. Series B. Methodological*, 56(3):515–528, 1994.

[vdVW96] Aad W. van der Vaart and Jon A. Wellner. *Weak convergence and empirical processes*. Springer, New York, 1996.

[WN98] Ishay Weissman and S. Yu. Novak. On blocks and runs estimators of the extremal index. *Journal of Stat. Planning and Inference*, 66(2):281–288, 1998.

[ZVB20] Nan Zou, Stanislav Volgushev, and Axel Bücher. Multiple block sizes and overlapping blocks for multivariate time series extremes. arXiv:1907.09477, 2020.
Table 1: The median and the variance (in brackets) of disjoint, sliding blocks and runs (with the anchoring maps $C^{(0)}, C^{(1)}$ and $C^{(2)}$) estimators for the extremal index. Data are simulated from AR(1) with $\alpha = 4$, $\rho = 0.5$ (thus, $\theta = 0.94$), and $\rho = 0.9$ (thus $\theta = 0.34$). Block size $r_n = 8, 9$. The number of order statistics is $k = 5\%$ and $10\%$ for a sample $n = 1000$ based on $N = 1000$ Monte Carlo simulations.
Table 2: The median and the variance (in brackets) of disjoint, sliding blocks and runs ($C^{(0)}$) estimators for stop-loss index with $S = 0.9$. Data are simulated from AR(1) with $\alpha = 4$, $\rho = 0.5$, $\rho = 0.9$. The block size is $r_n = 8$, 9. The number of order statistics is $k = 10\%$ and $40\%$ for a sample $n = 1000$ based on $N = 1000$ Monte Carlo simulations.
Extremal Index=0.612

|                | $k = 5$ | $k = 10$ |
|----------------|---------|----------|
| $(k \%)$      |         |          |
| $r_n = 8$     |         |          |
| Disjoint bl   | 0.660   | 0.600    |
| Sliding bl    | 0.648   | 0.593    |
| Runs $C^{(0)}$| 0.520   | 0.410    |
| Runs $C^{(1)}$| 0.500   | 0.380    |
| Runs $C^{(2)}$| 0.500   | 0.380    |
| $r_n = 9$     |         |          |
| Disjoint bl   | 0.640   | 0.570    |
| Sliding bl    | 0.630   | 0.569    |
| Runs $C^{(0)}$| 0.500   | 0.380    |
| Runs $C^{(1)}$| 0.480   | 0.340    |
| Runs $C^{(2)}$| 0.480   | 0.340    |

Table 3: The median and the variance (in brackets) of disjoint, sliding blocks and runs ($C^{(0)}$, $C^{(1)}$ and $C^{(2)}$) estimators for the extremal index in ARCH(1) model with $\lambda = 0.9$. The block size is $r_n = 8, 9$. The number of order statistics is $k = 5\%$ and $10\%$ for a sample $n = 1000$ based on $N = 1000$ Monte Carlo simulations.
Figure 1: Monte Carlo simulations of runs estimator ($C^{(0)}$) for stop-loss index with $S = 0.9$. Data are simulated from AR(1) with $\rho = 0.5$ and $r_n = 8$ (left panel), $\rho = 0.9$ and $r_n = 9$ (right panel), $\theta = 0.078$ $\alpha = 4$ and the number of order statistics $k = 40$. Dotted lines indicated the true value of the cluster index.