ON THE ZEROS OF POLYNOMIALS GENERATED BY RATIONAL FUNCTIONS WITH A HYPERBOLIC POLYNOMIAL TYPE DENOMINATOR

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Abstract. This paper investigates the location of the zeros of a sequence of polynomials generated by a rational function with a denominator of the form $G(z, t) = P(t) + zt^r$, where the zeros of $P$ are positive and real. We show that every member of a family of such generating functions, parametrized by the degree of $P$ and $r$, gives rise to a sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ that is eventually hyperbolic. Moreover, when $P(0) > 0$, the real zeros of the polynomials $H_m(z)$ form a dense subset of an interval $I \subset \mathbb{R}^+$, whose length depends on the particular values of the parameters in the generating function.

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1. Introduction

Consider the generating relation

$$\sum_{k=0}^{\infty} H_m(z) t^m = G(z, t)$$

for a sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$. For certain specific choices of the function $G(z, t)$, one can derive various properties of the polynomials $H_m(z)$ from those of the function $G(z, t)$, including their degrees and their location of zeros. For example, well-known properties of the classical orthogonal polynomials can be obtained from by manipulating their generating functions to produce recurrence relations satisfied by these polynomials. The recurrence relations are in turn used to establish the orthogonality of the polynomials over a certain interval in $\mathbb{R}$ with respect to some weight function [5, Ch. 10, 11, 12, 16]. In addition to orthogonality, one also discovers that the generated set of polynomials is simple, that is, $\deg H_m(z) = m$ for all $m \geq 0$. It is through these connections that one can assert the reality of the zeros of each of the polynomials in the generated sequence (see for example [5, p.149, Theorem 55]), along with the fact that the zeros must lie in the interval on which the family of polynomials is orthogonal.

The goal of the present paper is to describe the extent to which similar conclusions can be drawn for polynomials generated by functions of the form

$$(1) \quad G(z, t) = \frac{1}{P(t) + zt^r},$$

where $P \in \mathbb{R}[t]$ is a polynomial of degree $n$ with only positive (real) zeros, and $r$ is a positive integer. We note that if $n = 2$ and $r = 1$, a result by the second author shows that he zeros of the generated polynomials are all real (see [6, §2, Theorem 1, p. 331]). The main result of the present paper (cf. Date: June 24, 2016.

1 i.e. the Hermite-, Laguerre-, and the Legendre (more generally the Jacobi) polynomials
2 For pairs $(n, r) \neq (1, 1)$ or $(2, 1)$
Theorem 1 is that from a certain point on in the sequence, polynomials generated by functions of the type \((\cdot)\) will all have only real zeros, all of which are located in a particular interval in \(\mathbb{R}\). We find this result appealing in that it closely resembles the long-established analogous conclusions about the classical orthogonal polynomials, despite us not knowing explicitly what the generated sequence of polynomials looks like.

Given that the degree of the denominator of \(G(z,t)\) in \(t\) is \(d = \max\{n,r\}\), we are assured that the terms of the generated sequence \(\{H_m(z)\}_{m=0}^{\infty}\) satisfy an \((d+1)\)-term recurrence relation. We could not, however, find a way to ascertain that they satisfy a three-term recurrence relation\(^3\). Thus the techniques described above for obtaining orthogonality relations for the generated polynomials do not readily lend themselves to the solution of our problem. Moreover, the set of polynomials we obtain using such functions are, in general, not simple. In light of these obstructions, we construct a proof which establishes that each \(H_m(z)\) has at least as many distinct real zeros as its degree. We accomplish this by first proving that there exists a continuous, real valued, strictly increasing function \(z(\theta)\) on \((0, \pi/r)\), when viewed as a function of the argument of a non-real zero of \(P(t) + zt^r\) (Lemma 8 and Lemma 12). We then construct continuous functions \(H(\theta;m)\) on \((0, \pi/r)\) with the property that \(H(\theta;m) = 0\) if and only if \(H_m(z(\theta)) = 0\). Finally, we demonstrate that for all large \(m\), \(H(\theta;m)\) has at least as many zeros on \((0, \pi/r)\) as the degree of \(H_m(z)\) (Proposition 15 and Proposition 10). Since each of these zeros will give rise to a unique real zero of \(H_m(z)\), we obtain that \(H_m(z)\) must in fact have only real zeros for all \(m \gg 1\). The identification of the interval containing the zeros of \(H_m(z)\) for large \(m\) is accomplished in Lemma 2 and Lemma 12. The interval we obtain is optimal in the sense that the set

\[ Z = \bigcup_{m \gg 1} \{z \mid H_m(z) = 0\} \]

is dense there.

With this outline in hand, we now present the main result of the paper.

**Theorem 1.** Suppose \(P(t)\) is a real polynomial whose zeros are positive real numbers and \(P(0) > 0\). Let \(r\) be a positive integer such that \(\max\{\deg P, r\} > 1\). Let \(t_a\) and \(t_b\) be the smallest positive, and the largest nonpositive real zeros of the polynomial \(\frac{d}{dx}(-P(t)/t^r)\) respectively. For all large integers \(m\), the zeros of the polynomial \(H_m(z)\) generated by

\[
\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t) + zt^r} \tag{1.1}
\]

lie in the interval \((a, b)\), where \(a = -P(t_a)/t_a^r\) and

\[ b = \begin{cases} 
-P(t_b)/t_b & \text{if } t_b \neq 0 \\
\infty & \text{if } t_b = 0 
\end{cases} \]

Moreover, the set \(Z = \bigcup_{m \gg 1} \{z \mid H_m(z) = 0\}\) is dense on \((a, b)\).

We remark that the choice \(P(t) = (1 - t)^n\) reproduces Theorem 1 in \([3]\) via the equations

\[ rt^{r-1}P(t) - t^rP'(t) = t^{r-1}(1 - t)^{n-1}((n - r)t + r), \]

\(^3\) A necessary condition for the sequence \(\{H_m(z)\}_{m=0}^{\infty}\) to be orthogonal (see \([3]\), Theorem 57, p.151)
The following lemma justifies the descriptions of $t_a$ and $t_b$ in the statement of Theorem 1. We refer the reader to Figure 1.1 for an illustration of $-P(t)/t$ and the values $a, b, t_a,$ and $t_b$.

**Lemma 2.** Let $P(t) = \sum_{k=0}^{n} a_k t^k = |a_n| \prod_{k=1}^{n} (\tau_k - t)$ be a polynomial of degree $n$ whose zeros, $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n$, are positive real. The zeros of the polynomial

$$R(t) = t^{2r} \frac{d}{dx} \left( -\frac{P(t)}{t^r} \right) = rt^{r-1}P(t) - t^r P'(t)$$

are real. Furthermore, if $r = 1$, then $R(t)$ has a unique negative zero.

**Proof.** Let $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ be the zeros of $P'(t)$. Note that the zeros of $P(t)$ and $P'(t)$ interlace, that is

$$\tau_1 \leq \gamma_1 \leq \tau_2 \leq \cdots \leq \gamma_{n-1} \leq \tau_n.$$

Thus, by the Intermediate Value Theorem, each non-trivial interval $(\tau_k, \gamma_k)$, $1 \leq k < n$, contains a zero of $R(t)$. If $r > 1$, then $R(t)$ has a zero at 0 with multiplicity $r - 1$, and since common zeros of $P(t)$ and $P'(t)$ are also zeros of $R(t)$, by degree considerations we see that all zeros of $R(t)$ are real (and non-negative).

If $r = 1$, the facts that $\lim_{t \to -\infty} P(t) - tP'(t) = -\infty$ and $P(0) > 0$ imply that $R(t)$ has at least one negative real zero. On the other hand, $R(t)$ has $n - 1$ non-negative real zeros by the above considerations. Since $\deg R(t) = n$, we conclude that $R(t)$ has exactly one negative real zero. □

The remainder of the paper is dedicated entirely to the proof of Theorem 1 with the exception of the closing section, where we state some open problems and conjectures which arose during our investigations.
2. The proof of Theorem 1

Our approach to proving the main result is straightforward in that we simply count the number of positive real zeros of the polynomials $H_m(z)$, and show that for large $m$, this number is at least the degree of $H_m(z)$. We start by giving an upper bound on the degree of each $H_m(z)$.

**Lemma 3.** Suppose that the sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ is generated by a function $1/(P(t) + zt^r)$, where $P \in \mathbb{R}[t]$ is a polynomial of degree $n$ with only positive (real) zeros, and $r$ is a positive integer. For all $m \in \mathbb{N}$, the inequality $\deg H_m(z) \leq \lfloor m/r \rfloor$ holds.

**Proof.** Rearranging (1.1) yields the equation

$$
(P(t) + zt^r) \sum_{m=0}^{\infty} H_m(z)t^m = 1.
$$

By equating coefficients we see that the polynomial $H_m(z)$ satisfies the recurrence

$$(P(\Delta) + z\Delta^r)H_m(z) = 0, \quad m \geq 1$$

where the operator $\Delta$ is defined by $\Delta H_m := H_{m-1}$, and $\Delta H_0 = 0$. The claim follows from induction. \hfill \Box

A key component of the proof of Theorem 1 is the connection between the zeros of $P(t) + zt^r$ (as a polynomial in $t$) and the zeros of the generated sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$. The next segment of the paper starts the exploration of this connection. More precisely, we proceed by demonstrating that given a $\theta \in (0, \pi/r)$, one can find a unique $\tau \in \mathbb{R}^+$, such that $t = \tau e^{-i\theta}$ is a zero of $P(t) + zt^r$ for some real $z$. This way, we will be able construct a function $z: (0, \pi/r) \to \mathbb{R}^+$, that will play an instrumental role in the proof of Theorem 1.

In order to motivate some of the specifics of the ensuing section, we offer the following development. Suppose $P(t) = \sum_{k=0}^{n} a_k t^k$ is a real polynomial with positive zeros $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n$. If $z \in \mathbb{R}$, and $t = \vert t \vert e^{-i\theta}, \theta \in \mathbb{R},$ is a zero of $P(t) + zt^r$, then so is $te^{2i\theta}$. For non-zero values of $\theta$ we substitute $t$ and $te^{2i\theta}$ into the equation $P(t) + zt^r = 0$ to obtain

$$(2.2) \quad \prod_{k=1}^{n} \frac{\tau_k - te^{2i\theta}}{\tau_k - t} = e^{2ir\theta}.$$  

We define the angles $\theta_k$ implicitly by

$$(2.3) \quad \frac{\tau_k - te^{2i\theta}}{\tau_k - t} = e^{2i\theta_k}, \quad k = 1, 2, \ldots, n.$$  

Note that

$$(2.4) \quad \sum_{k=1}^{n} \theta_k = r\theta + l\pi, \quad \text{for some} \quad 0 \leq l < n.$$  

Solving equation (2.3) for $t$ we obtain that for $k = 1, 2, \ldots, n,$

$$(2.5) \quad t = \tau_k \frac{1 - e^{2i\theta_k}}{e^{2i\theta_k} - e^{2i\theta}} = \tau_k e^{-i\theta_k} \frac{e^{-i\theta_k} - e^{i\theta_k}}{e^{i(\theta - \theta_k)} - e^{-i(\theta - \theta_k)}} = \tau_k \frac{\sin \theta_k}{\sin(\theta_k - \theta)} e^{-i\theta}.$$
The quantity \( \tau = \frac{\sin \theta_k}{\sin(\theta_k - \theta)} \) is the unique real number \( \tau \) that we mentioned above - its existence is established in Lemma 4. Using the expression for \( t \) given in (2.5) we arrive at

\[
(2.6) \quad \tau_k - t = \tau_k - \frac{\cos \theta_k \sin \theta + i \sin \theta_k \sin \theta}{\sin(\theta_k - \theta)} = -\tau_k \frac{\sin \theta}{\sin(\theta_k - \theta)} e^{-i\theta_k}.
\]

Finally, we conclude that if \( t \) is a zero of \( P(t) + zt^r \), then for every \( 1 \leq k \leq n, \) we may write

\[
(2.7) \quad z = -\frac{|a_n|}{t^r} \prod_{k=1}^{n} \frac{(\tau_k - t)}{\tau_k - \tau e^{-i\theta}} = |a_n|(-1)^{n-r} \prod_{j=1}^{n} \frac{\tau_j}{\sin(\theta_j - \theta)}.
\]

Note in particular, that given a \( \theta \in (0, \pi/r) \) and an \( 1 \leq l < n, \) we may define a real \( z \) as in (2.7), so that \( \tau e^{-i\theta} \) is a zero of \( P(t) + zt^r \).

Motivational interlude aside, we now turn our attention to the details of establishing the existence and the properties of the function \( z(\theta) \).

**Lemma 4.** Let \( P(t) = \sum_{k=0}^{n} a_k t^k = |a_n| \prod_{k=1}^{n} (\tau_k - t) \) be a real polynomial of degree \( n, \) whose zeros \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \) are positive real. For any \( \theta \in \mathbb{R} \setminus \{0\} \) and \( r \in \mathbb{N}, \) the zeros of the polynomial

\[
(2.8) \quad S(\tau) = P(\tau e^{i\theta}) - e^{2ir\theta} P(\tau e^{-i\theta})
\]

are real and distinct.

**Proof.** We first show that the zeros of \( S(\tau) \) are real. To this end, suppose that \( S(\tau) = 0, \tau \in \mathbb{C}. \) Rearranging equation (2.8) yields

\[
(2.9) \quad \prod_{k=1}^{n} \frac{\tau_k - \tau e^{i\theta}}{\tau_k - \tau e^{-i\theta}} = e^{2ir\theta}.
\]

If \( \tau \notin \mathbb{R}, \) then \( \tau_1, \ldots, \tau_n \) all lie in the open half plane containing the positive real axis, with boundary the perpendicular bisector of the line segment joining the points \( \tau e^{i\theta} \) and \( \tau e^{-i\theta}. \) Consequently,

\[
\prod_{k=1}^{n} \left| \frac{\tau_k - \tau e^{i\theta}}{\tau_k - \tau e^{-i\theta}} \right| = 1,
\]

a contradiction.

Let \( \gamma_1, \ldots, \gamma_{n-1} \) denote the zeros of \( P'(\tau) \) (see Figure 2.1 for an illustration). If \( \tau \in \mathbb{R} \) is such that \( S(\tau) = S'(\tau) = 0, \) then along with equation (2.8), we also have the following equation:

\[
(2.10) \quad \prod_{k=1}^{n-1} \frac{\gamma_k - \tau e^{i\theta}}{\gamma_k - \tau e^{-i\theta}} = e^{2i(r-1)\theta}.
\]

We define the angles \( 0 < \theta_k, \eta_k < \pi \) implicitly by

\[
\begin{align*}
\frac{\tau_k - \tau e^{i\theta}}{\tau_k - \tau e^{-i\theta}} &= e^{2i\theta_k} \quad (k = 0, 1, 2, \ldots, n) \\
\frac{\gamma_k - \tau e^{i\theta}}{\gamma_k - \tau e^{-i\theta}} &= e^{2i\eta_k} \quad (k = 0, 1, 2, \ldots, n - 1).
\end{align*}
\]
With these definitions, equations (2.9) and (2.10) imply that
\[ \sum_{k=1}^{n} \theta_k \equiv \sum_{k=1}^{n-1} \eta_k + \theta \pmod{\pi}. \]

On the other hand, the interlacing of the zeros of \( P \) and \( P' \), together with the fact that \( \angle ABC < \pi \) imply that \[ \sum_{k=1}^{n-1} \eta_k \cdot \theta < \sum_{k=1}^{n} \theta_k \cdot \theta < \sum_{k=1}^{n-1} \eta_k \cdot \theta + \pi, \] a contradiction. We conclude that the zeros of \( S(\tau) \) are simple, as well as real. \( \square \)

Remark 5. We emphasize the following consequence of Lemma 4. Given \( \theta \in (0, \pi/r) \), the angle sum \( \sum_{k=1}^{n} \theta_k \) is decreasing as \( \tau \) increases. Consequently, given the \( n \)-tuple \( (\theta_1, \theta_2, \ldots, \theta_n) \) corresponding to \( \theta \) via equation (2.3), the equality in (2.4) is satisfied for a unique \( 0 \leq l < n \).

In our motivational interlude we indicated that an angle \( \theta \in (0, \pi/r) \) would generate an \( n \)-tuple of angles \( (\theta_1, \theta_2, \ldots, \theta_n) \) with certain desirable properties. In order to make these properties explicit, we need to make use of the complex version of the Implicit Function Theorem.

**Theorem 6** (Theorem 2.1.2, p.24 [4]). Let \( f_j(w, z) \), \( j = 1, \ldots, m \), be analytic functions of \( (w, z) = (w_1, \ldots, w_m, z_1, \ldots, z_n) \) in a neighborhood of a point \((w^*, z^*) \) in \( \mathbb{C}^m \times \mathbb{C}^n \), and assume that \( f_j(w^*, z^*) = 0 \), \( j = 1, \ldots, m \), and that
\[ \det \left( \frac{\partial f_j}{\partial w_k} \right)_{j,k=1}^m \neq 0 \text{ at } (w^*, z^*). \]
Then the equations \( f_j(w, z) = 0 \), \( j = 1, \ldots, m \) have a uniquely determined analytic solution \( w(z) \) in a neighborhood of \( z^* \), such that \( w(z^*) = w^* \).

We are now ready to state and prove the following lemma.

**Lemma 7.** Let \( P(t) \) be a polynomial of degree \( n \) whose zeros, \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \), are positive and let \( 0 \leq l < n \) be a fixed integer. There is a unique \( n \)-tuple of functions \( \theta_1 = \theta_1(\theta), \ldots, \theta_n = \theta_n(\theta) \) analytic on a neighborhood \( U \) of \((0, \pi/r) \) such that for all \( \theta \in (0, \pi/r) \) the following hold:
(i) \( \theta_k(\theta) > \theta, \ 1 \leq k \leq n, \)
(ii) \( \sum_{k=1}^{n} \theta_k(\theta) = r\theta + l\pi, \) and
(iii) \( (2.11) \)

\[
\tau_1 \frac{\sin \theta_1}{\sin(\theta_1 - \theta)} = \tau_2 \frac{\sin \theta_2}{\sin(\theta_2 - \theta)} = \cdots = \tau_n \frac{\sin \theta_n}{\sin(\theta_n - \theta)} =: \tau(\theta).
\]

Proof. Let \( \theta^* \in (0, \pi/r) \) be given. By Lemma 4, there is a unique solution \( \tau^* \) to equation (2.2) with the corresponding \( n \)-tuple \( (\theta^*_1, \ldots, \theta^*_n) \) satisfying \( \theta^*_1 < \cdots < \theta^*_n \), and \( \sum_{k=1}^{n} \theta^*_k = r\theta^* + l\pi. \)

The law of sines implies (see Figure 2.2) that for all \( 1 \leq k \leq n, \)

\[
\tau^*_k \frac{\sin \theta^*_k}{\sin(\theta^*_k - \theta^*)} = \tau_k \frac{\sin \theta_k}{\sin(\theta_k - \theta)} = \tau_n \frac{\sin \theta_n}{\sin(\theta_n - \theta)} =: \tau(\theta).
\]

For \( k = 1, \ldots, n \) define \( f_k : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C} \) by

\[
f_k(\theta_1, \theta_2, \ldots, \theta_n, \tau, \theta) = \tau_k \frac{\sin \theta_k}{\sin(\theta_k - \theta)} - \tau
\]

and

\[
f_{n+1}(\theta_1, \theta_2, \ldots, \theta_n, \tau, \theta) = \sum_{k=1}^{n} \theta_k - (r\theta + l\pi).
\]

Note that

\[
f_k(\theta^*_1, \ldots, \theta^*_n, \tau^*, \theta^*) = 0, \quad 1 \leq k \leq n + 1,
\]

and that there exists a neighborhood \( W \) of \( (\theta^*_1, \ldots, \theta^*_n, \tau^*, \theta^*) \in \mathbb{C}^{n+1} \times \mathbb{C} \) where each \( f_k(\theta_1, \ldots, \theta_n, \tau, \theta) \) is analytic. We calculate

\[
\frac{\partial f_k}{\partial \theta_k}(\theta^*_1, \ldots, \theta^*_n, \tau^*, \theta^*) = -\tau_k \frac{\sin \theta^*}{\sin^2(\theta^*_k - \theta^*)} =: c_k < 0, \quad 1 \leq k \leq n,
\]
and hence the Jacobian matrix at \((\theta_1^*, \theta_2^*, \ldots, \theta_n^*, \tau^*, \theta^*)\) is
\[
\begin{bmatrix}
c_1 & 0 & \cdots & 0 & -1 \\
0 & c_2 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & c_n & -1 \\
1 & 1 & \cdots & 1 & 0
\end{bmatrix}
\]
By expanding along the first row, we find the determinant of this matrix to be
\[
c_2 \cdots c_n + c_1 c_3 \cdots c_n + c_1 c_2 c_4 \cdots c_n + \cdots + c_1 c_2 \cdots c_{n-1} \neq 0,
\]
since each summand carries the same sign. We may therefore invoke Theorem \[\text{[0]}\] to conclude that there is a unique analytic function \(w(\theta) = (\theta_1(\theta), \theta_2(\theta), \ldots, \theta_n(\theta))\) on a neighborhood \(U_{\theta^*}\) of \(\theta^*\), such that
\[w(\theta^*) = (\theta_1(\theta^*), \theta_2(\theta^*), \ldots, \theta_n(\theta^*)) = (\theta_1^*, \theta_2^*, \ldots, \theta_n^*).
\]
By analytic continuation we conclude that there is a neighborhood of \(U\) of \((0, \pi/r)\), and an analytic function \(w\) on \(U\) such that \(w(\theta^*) = (\theta_1^*, \ldots, \theta_n^*)\) for any \(\theta^* \in (0, \pi/r)\). Naming the components of \(w\) as \(\theta_k(\theta), 1 \leq k \leq n\), finishes the proof. \(\square\)

**Lemma 8.** Let \(0 \leq l < n\), and set \(t_0 = \tau e^{-i\theta}\), where \(\tau = \tau(\theta)\) is as in Lemma \[\text{[2]}\](iii). The real-valued function
\[z(\theta) = -\frac{P(t_0)}{t_0^l}\]
is strictly monotone on \((0, \pi/r)\).

**Proof.** Note that since \(t_0 \neq 0\), and \(\theta - \theta_k \neq 0\) for any \(1 \leq k \leq n\), the function \(z(\theta)\) is differentiable on \((0, \pi/r)\). We take the logarithmic derivatives of both sides of \(z = -P(t_0)/t_0^l\) to obtain
\[\frac{dz}{z} = -\sum_{k=1}^{n} \frac{dt_0}{\tau_k - t_0} - \frac{r dt_0}{t_0}.
\]
Similarly, the logarithmic derivatives of both sides of the equation
\[\prod_{k=1}^{n} (\tau_k - t_0 e^{2i\theta}) = e^{2i\theta} \prod_{k=1}^{n} \tau_k - t_0
\]
give
\[\sum_{k=1}^{n} \frac{e^{2i\theta} dt_0 - 2i t_0 e^{2i\theta} d\theta}{\tau_k - t_0 e^{2i\theta}} = 2i r d\theta + \sum_{k=1}^{n} \frac{-dt_0}{\tau_k - t_0},
\]
or equivalently
\[\sum_{k=1}^{n} \frac{-\sin \theta \tau_k e^{i\theta}}{(\tau_k - t_0)(\tau_k - t_0 e^{2i\theta})} dt_0 = r d\theta + \sum_{k=1}^{n} \frac{t_0 e^{2i\theta}}{\tau_k - t_0 e^{2i\theta}} d\theta.
\]
Thus \[\text{(2.12)}\] and \[\text{(2.13)}\] yield
\[
\frac{dz}{d\theta} \sum_{k=1}^{n} \frac{\sin \theta \tau_k e^{i\theta}}{(\tau_k - t_0)(\tau_k - t_0 e^{2i\theta})} = \left( \sum_{k=1}^{n} \frac{1}{\tau_k - t_0} + \frac{r}{t_0} \right) \left( \sum_{k=1}^{n} \frac{t_0 e^{2i\theta}}{\tau_k - t_0 e^{2i\theta}} + r \right).
\]
Multiplying both sides by \( t_0 \) results in

\[
(2.14) \quad \frac{dz}{zd\theta} = \frac{\sum_{k=1}^{n} \tau(k) \sin \theta}{(\tau(k) - e^{-i\theta})^2} = \frac{\sum_{k=1}^{n} \tau e^{-i\theta} + r}{(\tau(k) - e^{-i\theta})^2},
\]

from which monotonicity follows.\(^4\) To establish strict monotonicity we note that \(-\pi < \text{Arg} \frac{\tau e^{-i\theta}}{\tau(k) - e^{-i\theta}} < 0\) for all \( k \) and \( \theta \in (0, \pi/r) \). Consequently,

\[-\pi < \text{Arg} \left( \sum_{k=1}^{n} \frac{\tau e^{-i\theta}}{\tau(k) - e^{-i\theta}} \right) < 0, \quad (\theta \in (0, \pi/r))\]

and we conclude that

\[\sum_{k=1}^{n} \frac{\tau e^{-i\theta}}{\tau(k) - e^{-i\theta}} + r \neq 0. \quad (\theta \in (0, \pi/r))\]

The proof is complete. \(\square\)

Remark 9. Note that Lemmas 7 and 8 hold for any fixed \( 0 \leq l < n \). In particular, for each \( 0 \leq l < n \), we have a tuple of \( n \) continuous real-valued functions \( \theta_k(\theta; l) \), \( 1 \leq k \leq n \), on \( \theta \in (0, \pi/r) \) where the function \( z(\theta; l) \) given in (2.11) is monotone. For simplicity in notation, from this point of the paper, whenever \( l = n - 1 \), we will suppress the parameter \( l \) and simply write \( \theta_k(\theta) := \theta_k(\theta; n - 1) \), \( 1 \leq k \leq n \), \( \tau(\theta) := \tau(\theta; n - 1) \), and \( z(\theta) := z(\theta; n - 1) \). If \( l \neq n - 1 \), we will write these functions in their full notations \( \theta_k(\theta; l) \), \( \tau(\theta; l) \), and \( z(\theta; l) \).

An argument completely analogous to the proof of Lemma 7 gives the next auxiliary result, which we will make use of when establishing the last important property of the function \( z(\theta) \).

Lemma 10. If \( l = n - 1 \), the function \( \tau(\theta) \) defined in (2.11) is strictly monotone decreasing on \( (0, \pi/r) \).

Proof. With \( t_0 = \tau e^{-i\theta} \) and \( dt_0 = e^{-i\theta} d\tau - i \tau e^{-i\theta} d\theta \), (2.13) gives

\[
\sum_{k=1}^{n} \frac{-\sin \theta \tau_k}{(\tau(k) - e^{-i\theta})(\tau(k) - e^{-i\theta})} d\tau = \frac{r d\theta}{\tau} \sum_{k=1}^{n} \left( \frac{\tau e^{i\theta}}{\tau(k) - e^{-i\theta}} - \frac{i \sin \theta \tau_k}{\tau(k) - e^{i\theta}} \right) d\theta
\]

\[= r d\theta - \sum_{k=1}^{n} \frac{\tau^2 - \tau \cos \theta}{(\tau(k) - e^{i\theta})(\tau(k) - e^{-i\theta})} d\theta.
\]

Using (2.15) and (2.16) we may rewrite the above equation as

\[
(2.15) \quad - \frac{d\tau}{\sin \theta d\theta} \sum_{k=1}^{n} \frac{\sin^2(\theta_k - \theta)}{\tau_k} = r - \sum_{k=1}^{n} \frac{\sin \theta_k \cos(\theta_k - \theta)}{\sin \theta}.
\]

The claim will follow as soon as we establish that the right hand side of (2.15)

\[
(2.16) \quad A(\theta) := r - \sum_{k=1}^{n} \frac{\sin \theta_k \cos(\theta_k - \theta)}{\sin \theta},
\]

\(^4\)whether \( z \) is increasing or decreasing depends on the parity of \( n - l - 1 \), as in (2.7).
is positive. We first consider the case when \( r \geq n/2 \). In this case, we see that

\[
A(\theta) \sin \theta = \left( r - \frac{n}{2} \right) \sin \theta - \frac{1}{2} \sum_{k=1}^{n} \sin(2\theta_k - \theta).
\]

If \( \pi \leq 2\theta_k - \theta < 2\pi \), \( 1 \leq k \leq n \), then the claim \( A(\theta) \geq 0 \) is trivial since all the terms are nonnegative. Furthermore \( A(\theta) = 0 \) only when \( r = n/2 \) and \( 2\theta_k - \theta = \pi \), \( 1 \leq k \leq n \). The sum all terms \( 1 \leq k \leq n \) in the later equality implies that \( n = 2 \). On the other hand, if \( 2\theta_1 - \theta < \pi \) then the identity \( \sum_{k=1}^{n} (2\theta_k - \theta) = 2(n-1)\pi + (2r-n)\theta \) implies that \( 0 < 2\theta_1 - \theta < \pi \) and \( \pi < 2\theta_k - \theta < 2\pi \), \( \forall k \geq 2 \). Let \( \eta_1 = \theta_1 \) and \( \eta_k = \pi - \theta_k \), \( k \geq 2 \), where \( \eta_k - \sum_{k=2}^{n} \eta_k = r\theta \). In terms of these new variables, we have

\[
(2.17) \quad A(\theta) \sin \theta = \left( r - \frac{n}{2} \right) \sin \theta - \frac{1}{2} \sum_{k=1}^{n} \sin(2\eta_1 - \theta) + \frac{1}{2} \sum_{k=2}^{n} \sin(2\eta_k + \theta)
\]

where \( 2\eta_1 - \theta = (2r-n)\theta + \sum_{k=2}^{n} (2\eta_k + \theta) \). Furthermore, we have \( 0 < 2\eta_1 - \theta < \pi \) and \( 0 < 2\eta_k + \theta < \pi \), \( \forall k \geq 2 \). We note that for any two angles \( \theta < \alpha, \beta < \pi \) the following inequality holds

\[
(2.18) \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta < \sin \alpha + \sin \beta.
\]

Thus

\[
A(\theta) \sin \theta > \left( r - \frac{n}{2} \right) \sin \theta - \frac{1}{2} \sin(2r\theta - n\theta) \geq 0.
\]

We next consider when \( n > 2r \). In this case we write \( A(\theta) \) as

\[
A(\theta) \sin \theta = -\frac{1}{2} \sum_{k=1}^{2r} \sin(2\theta_k - \theta) - \sum_{k=2r+1}^{n} \sin \theta_k \cos(\theta_k - \theta).
\]

For the same reason, it suffices to consider when \( 0 < 2\theta_1 - \theta < \pi \) and \( \pi < 2\theta_k - \theta < 2\pi \), \( \forall k \geq 2 \). In this case

\[
A(\theta) \sin \theta = -\frac{1}{2} \sin(2\theta_1 - \theta) + \frac{1}{2} \sum_{k=2}^{2r} \sin(2\eta_k + \theta) + \sum_{k=2r+1}^{n} \sin \eta_k \cos(\eta_k + \theta),
\]

where \( 2\eta_1 - \theta = \sum_{k=2}^{2r} (2\eta_k + \theta) + \sum_{k=2r+1}^{n} 2\eta_k \). We can write \( \sin(2\eta_1 - \theta) \) as

\[
(2.19) \quad \sin(2\eta_1) \cos \left( \sum_{k=2}^{2r} (2\eta_k + \theta) + \sum_{k=2r+1}^{n} 2\eta_k \right) + \cos(2\eta_1) \sin \left( \sum_{k=2}^{2r} (2\eta_k + \theta) + \sum_{k=2r+1}^{n} 2\eta_k \right)
\]

which is less than

\[
2 \sin \eta_n \cos(\eta_n + \theta) + \sin \left( \sum_{k=2}^{2r} (2\eta_k + \theta) + \sum_{k=2r+1}^{n} 2\eta_k \right)
\]

since

\[
\eta_n + \theta \leq \sum_{k=2}^{2r} (2\eta_k + \theta) + \sum_{k=2r+1}^{n} 2\eta_k < \pi.
\]
We continue this process and obtain the inequality

\[(2.20) \quad \sin(2\eta_t - \theta) \leq 2 \sum_{k=2r+1}^{n} \sin \eta_k \cos(\eta_k + \theta) + \sin \left( \sum_{k=2}^{2r} (2\eta_k + \theta) \right) \]

\[\leq 2 \sum_{k=2r+1}^{n} \sin \eta_k \cos(\eta_k + \theta) + \sum_{k=2}^{2r} \sin(2\eta_k + \theta) \]

where the last inequality comes from (2.18). We conclude that \(A(\theta) > 0\) if \((n, r) \neq (2, 1)\), and the proof of the lemma is complete.

**Remark 11.** From \(\sum_{k=1}^{n} \eta_k = r\theta + (n - 1)\pi\), not all \(\eta_k\), \(1 \leq k \leq n\), approach 0 when \(\theta \to 0\). Thus the previous lemma and (2.11) imply that \(\tau(\theta) > 0\) is bounded on \((0, \pi/r)\).

We wrap up our discussion of the function \(z(\theta)\) with the following result.

**Lemma 12.** Let \(l = n - 1\), let \(t_0, \tau\) and \(z(\theta)\) be as in the statement of Lemma 9, and let \(t_a, t_b, a\) and \(b\) be defined as in the statement of Theorem 7. The function \(z(\theta)\) maps the interval \((0, \pi/r)\) onto the interval \((a, b)\).

**Proof.** Lemma 9 implies that with the choice \(l = n - 1\), \(z(\theta)\) is a continuous, monotone increasing function on \((0, \pi/r)\). Thus, it suffices to show \(a' := \lim_{\theta \to 0} z(\theta) = a\) and \(b' := \lim_{\theta \to \pi/r} z(\theta) = b\).

Since \(\tau(\theta)e^{\pm i\theta}\) are two zeros of \(P(t) + xt\) and \(\tau(\theta)\) is a monotone bounded function, the function \(\tau(\theta)\) has to converge to a real multiple zero of \(P(t) + a't\) when \(\theta \to 0\) or a real multiple zero of \(P(t) + b't\) when \(\theta \to \pi/r\). Here, we use the convention that if \(b' = \infty\) and \(r > 1\) then 0 is a multiple zero of \(P(t) + b't\). By looking at the derivatives of \(P(t) + a't\) and \(P(t) + b't\), we see that the limits of \(\tau(\theta)\) as \(\theta \to 0\) and as \(\theta \to \pi/r\) must be zeros of \(t^r - (rP(t) - P'(t))\) where the factor \(t^r - 1\) reflects the possible multiple zero at 0 when \(r > 1\). The equation \(\sum_{k=1}^{n} \eta_k(\theta) = (n - 1)\pi + r\theta\) implies that \(\tau_1 < t_a < \tau_2\) where \(t_a := \lim_{\theta \to 0} \tau(\theta)\). Thus by the interlacing of the zeros of \(P(t)\) and \(P'(t)\) and the definition of \(t_a\) as the smallest positive real zero of \(t^r - 1(rP(t) - P'(t))\), we must have \(t_{a'} = t_a\) and consequently \(a' = a\). Similarly, we have \(t_{b'} = 0\) if \(r > 1\) and \(t_{b'} < 0\) if \(r = 1\) where \(t_{b'} := \lim_{\theta \to \pi/r} \tau(\theta)\). By the definition of \(t_b\) we have \(t_{b'} = t_b\) and \(b = b'\).

We now turn our attention to the second key element of the proof of Theorem 7 namely the construction of the functions \(H(\theta; m)\) on \((0, \pi/r)\) with the property that \(H(\theta; m) = 0\) if and only if \(H_m(z(\theta)) = 0\).

Recall that for each \(0 < \theta < \pi/r\) and \(0 \leq l < n\), we set \(t_0 = \tau(\theta; l)e^{-i\theta}\). Let \(t_1 = t_0e^{2i\theta}\) be its conjugate, and note that they are both zeros of the polynomial \(P(t) + z(\theta; l)t\). Let \(t_2, t_3, \ldots, t_{\max\{n, r\} - 1}\) denote the remaining zeros of \(P(t) + z(\theta; l)t\), and let \(q_k = t_k/t_0\), \(0 \leq k < \max\{n, r\}\). With this notation we may rearrange the equation \(P(t_k) + z(\theta; l)t_k' = 0\) as

\[-z = \frac{\prod_{j=1}^{n} (\tau_j - t_k)}{t_k'} = \frac{\prod_{j=1}^{n} (\tau_j t_0^{-1} - q_k)}{q_k} t_0^{-r}.

Changing of variables \(\zeta_k := q_k e^{-i\theta}\), \(k = 0, 1, \ldots, \max\{n, r\}\), and using (2.30) we arrive at the equivalent equation

\[-z(t_0e^{i\theta})r-n \zeta_k = \prod_{j=1}^{n} \left( \frac{\sin(\theta_j - \theta)}{\sin \theta_j} - \zeta_k \right),\]
which, when combined with (2.7), leads to
\[
\prod_{j=1}^{n} \left( \frac{\sin(\theta_j - \theta)}{\sin \theta} - \zeta_k \right) + \zeta_k \prod_{j=1}^{n} \frac{\sin \theta}{\sin \theta_j} = 0.
\]

We deduce that for any \(0 \leq k < \max \{n, r\}\), \(t_k\) is a zero of \(P(t) + z(\theta, l)t^r\) if and only if \(\zeta_k\) is a zero of the polynomial
\[
Q(\zeta) := \prod_{j=1}^{n} \left( \frac{\sin(\theta_j - \theta)}{\sin \theta} - \zeta \sin \theta_j \right) + \zeta^r.
\]

Note that \(\zeta_0 = e^{-i\theta}\) and \(\zeta_1 = e^{i\theta}\) are both zeros of \(Q(\zeta)\).

Let \(c\) be the leading coefficient in \(t\) of the polynomial \(P(t) + z(\theta, l)t^r\). If all the zeros \(t_k\), \(0 \leq k < \max \{n, r\}\), of the denominator \(P(t) + z(\theta, l)t^r\) are distinct, then partial fraction decomposition
\[
\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{c} \sum_{k=0}^{\max(n, r) - 1} \frac{1}{t - t_k} \prod_{l \neq k} \frac{1}{t_k - t_l} = \frac{-1}{c} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\max(n, r) - 1} \frac{1}{t_k^{m+1}} \prod_{l \neq k} \frac{1}{t_k - t_l} \right) t^m.
\]

Thus, \(H_m(z) = 0\) if and only if
\[
\sum_{k=0}^{\max(n, r) - 1} \frac{1}{t_k^{m+1}} \prod_{l \neq k} \frac{1}{t_k - t_l} = 0.
\]

Multiplying the above equation by \(e^{(m+n+i\theta)\theta_0^m+\max(n, r)}\), we see that \(z\) is a zero of \(H_m(z)\) if and only if \(\theta\) is a zero of
\[
H(\theta) := \sum_{k=0}^{\max(n, r) - 1} \frac{1}{\zeta_k^{m+1}} \prod_{l \neq k} \frac{1}{\zeta_k - \zeta_l} = \frac{\max(n, r) - 1}{\zeta_k^{m+1} Q'(\zeta_k)}.
\]

(2.21)

By symmetric reduction and Lemma \(7\), \(H(\theta)\) is a real-valued function of \(\theta\) on \((0, \pi/r)\). Furthermore, \(H(\theta)\) is analytic in an open neighborhood of \((0, \pi/r)\) except at some values of \(\theta\) where \(\zeta_k\) (or \(t_k\)), \(0 \leq k < \max \{n, r\}\), are not all distinct. However from Lemma \(7\) and (2.7), \(z\) is an analytic function of \(\theta\) on a neighborhood of \((0, \pi/r)\). Thus \(H(\theta)\) has an analytic continuation \(-c(\theta_0 e^{i\theta})^{m+\max(n, r)} H_m(z(\theta))\) on an neighborhood of \((0, \pi/r)\) and consequently all the discontinuities of the real-valued function \(H(\theta)\) on \((0, \pi/r)\) are removable. From now on, we treat \(H(\theta)\) as a real-valued continuous function on \((0, \pi/r)\).
If $\zeta_k$ is a zero of $Q(\zeta)$, then
\[
Q'(\zeta_k) = \prod_{j=1}^{n} \left( \frac{\sin(\theta_j - \theta)}{\sin \theta} - \frac{\zeta_k \sin \theta_j}{\sin \theta} \right) \sum_{j=1}^{n} \frac{-\sin \theta_j}{\sin(\theta_j - \theta) - \zeta_k \sin \theta_j} + r \zeta_k^{-1} \]
(2.22)
\[
= \sum_{j=1}^{n} \frac{\zeta_k' \sin \theta_j}{\sin(\theta_j - \theta) - \zeta_k \sin \theta_j} + r \zeta_k^{-1}.
\]
In the instance $\zeta_1 = e^{i\theta}$, we apply the identity
\[
\sin(\theta_j - \theta) - e^{i\theta} \sin \theta_j = -\cos \theta_j \sin \theta - i \sin \theta \sin \theta_j = -\sin \theta e^{i\theta},
\]
and obtain the equation
\[
Q'(e^{i\theta}) = e^{(r-1)i\theta} \left( -\sum_{j=1}^{n} \frac{\sin \theta_j}{\sin \theta} e^{i(\theta - \theta_j)} + r \right).
\]
The first two terms of (2.21) are complex conjugates, whose sum is
\[
2 \Re \left( e^{i(m+1)\theta} Q'(e^{i\theta}) \right) = \frac{2}{|Q'(e^{i\theta})|^2} \left( A(\theta) \cos(m + r) \theta - B(\theta) \sin(m + r) \theta \right)
\]
where $A(\theta)$ is as in (2.16), and $B(\theta) = \sum_{j=1}^{n} \frac{\sin \theta_j}{\sin \theta} \sin(\theta_j - \theta)$.

The next lemma describes the behavior of $A(\theta)$ near the end-points of the interval $(0, \pi/r)$.

**Lemma 13.** Let $\rho$ be the multiplicity of the smallest zero of $P(t)$. When $\rho = 1$ and $\theta$ sufficiently small, $A(\theta) > \theta^4$. Moreover, when $r = 1$ and $\theta$ sufficiently close to $\pi$, $A(\theta) > (\pi - \theta)^4$.

**Proof.** We follow the proof of Lemma 10 and replace the identity (2.18) by the identity
\[
\sin \alpha + \sin \beta - \sin(\alpha + \beta) = \sin \alpha (1 - \cos \beta) + \sin \beta (1 - \cos \alpha)
\]
\[
(2.24)
\]
for all $\alpha, \beta > 0$ sufficiently small.

In the case $\theta$ is sufficiently small and $\rho = 1$, we note that $\eta_k$, $1 \leq k \leq n$, are close to 0. Moreover, the law of sines (see Figure 2.2) implies that $\sin \eta_k/\sin \theta$, or equivalently $\eta_k/\theta$, is approaching $t_0/[t_0 - \tau_k]$ when $\theta \to 0$. The lemma follows from the proof of Lemma 10 after we apply (2.21) in (2.17), (2.19) or (2.20).

Similarly, when $\theta$ is close to $\pi$ and $r = 1$, $\sin \eta_k/\sin(\pi - \theta)$, or equivalently $\eta_k/(\pi - \theta)$, is approaching $|t_0|/[t_0 - \tau_k]$ for $2 \leq k \leq n$. We apply (2.24) in (2.19) or (2.20) and reach the corresponding conclusion. \hfill \Box

Lemmas 10, 12 and 13 establish some key properties of the functions $\tau(\theta), z(\theta)$ and $A(\theta)$. Equipped with these results, we can now prove a key proposition about the distribution of the zeros of the denominator $P(t) + zr^t$ as a polynomial in $t$.

**Proposition 14.** Let $\theta \in (0, \pi/r)$ be a fixed angle with $z := z(\theta)$ and $\tau := \tau(\theta)$. The only zeros in $t$ of $P(t) + zr^t$ in the closed disk $C_1$ centered at the origin with radius $\tau$ are $t_{0,1} = \tau e^{\pm i\theta}$.
Proof. Without loss of generality, we may assume the zeros under consideration lie in the upper half plane. If $t$ is a zero of $P(t) + zt^r$, then

\begin{equation}
\prod_{k=1}^n (\tau_k - t) = \frac{\prod_{k=1}^n (\tau_k - \tau e^{i\theta})}{(\tau e^{i\theta})^r}.
\end{equation}

Let $C_1$ be as in the statement of the theorem, let and $C_2$ be the closed disk centered at $\tau_1$ with radius $|\tau e^{i\theta} - \tau_1|$ (see Figures 2.3 and 2.4 depending on the multiplicity of the smallest zero of $P(t)$). Assume, by way of contradiction, that $t^*$ is a zero of $P(t) + zt^r$ which lies inside $C_1$. Equation (2.25) implies that $t^*$ must then lie in the closed region $C_1 \cap C_2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{zeros.pdf}
\caption{Zeros of $P(t) + zt^r$ when $\rho > 1$}
\end{figure}
We first argue that \( t^* \notin \mathbb{R} \). If \( \theta \in (0, \pi/r) \) and \( P(t^*) + z(t^*)^r = 0 \), Lemma 12 implies that \( z \in (a, b) \), and hence \( a < -P(t^*)/(t^*)^r < b \). Since the derivative \( \frac{d}{dt}(-P(t)/t^r) \) does not vanish on \((0, t_a)\) and \( P(0) > 0 \), we conclude that \( -P(t)/t^r \) is strictly increasing on \((0, t_a)\) with \( -P(t_a)/t_a^r = a \). It follows that \( t^* \in [0, t_a] \). On the other hand, \( \tau_1 < t_a \), and By Lemma 10, \( \tau < t_a \) as well. We deduce that regardless of the value of \( \rho \), \( C_1 \cap C_2 \cap \mathbb{R} \nsubseteq [0, t_a] \), and \( P(t) + zt^r \) has no real non-negative zeros. If \( r = 1 \), then \( -P(t)/t \geq b \) on \((−∞, 0)\) by Lemma 2, and hence \( t^* \notin (−∞, 0) \). It now follows that \( t^* \notin C_1 \cap C_2 \cap \mathbb{R} \) for any \( r \geq 1 \).

We next claim that the argument \( \theta t^* \) of \( t^* \) lies in the interval \((0, \pi/r)\). This claim is trivial when \( r = 1 \), since \( t^* \notin \mathbb{R} \). When \( r > 1 \), it suffices to consider \( \theta_1 - \theta < \pi/2 \), for if \( \theta_1 - \theta \geq \pi/2 \), then \( t^* \in C_1 \cap C_2 \) forces \( \theta^* \leq \theta < \pi/r \) (see Figure 2.3). In addition, the equality \( \sum_{k=1}^n \theta_k = r\theta + (n-1)\pi \) implies that \( \theta_1 > r\theta \), and consequently we may restrict our attention to angles which satisfy \( \theta < \pi/(2r - 2) \). Since \( t^* \in C_1 \cap C_2 \), the largest possible value of \( \theta t^* \), denoted by \( \theta t^*_{\max} \), satisfies

\[
\sin \theta t^*_{\max} = \frac{\sin \theta}{\sin(\theta_1 - \theta)} < \frac{\sin \theta}{\sin(r-1)\theta}
\]

(see Figure 2.4). We claim that the right side of the above inequality is an increasing function of \( \theta \). Indeed, computing its derivative we obtain

\[
\frac{\cos \theta \sin(r-1)\theta - (r-1) \sin \theta \cos(r-1)\theta}{\sin^2(r-1)\theta} = \frac{r \sin(r-2)\theta - (r-2) \sin r\theta}{2 \sin^2(r-1)\theta},
\]

whose numerator is nonnegative, since it vanishes when \( \theta = 0 \) and it is non-decreasing. Thus

\[
\frac{\sin \theta}{\sin(r-1)\theta} \leq \frac{\pi}{2(r-1)} \leq \frac{\pi}{r},
\]

and consequently \( \theta t^* \leq \theta t^*_{\max} < \pi/r \).

Given now that \( \theta t^* \in (0, \pi/r) \), equations (2.2), (2.3), and (2.4) imply that for some \( 0 \leq l < n \) the
equality
\[ \sum_{k=1}^{n} \theta_k (\theta^{**}; l) = r \theta^{**} + l \pi, \]
holds, along with \( z(\theta^{**}; l) = z(\theta) \). The latter equality implies in particular that \( z(\theta^{**}; l) \) is positive, and therefore \( n - l - 1 \) must be even. Consequently, \( z(\theta; l) \) is monotone increasing in \( \theta \). Observe that
\[ \tau(\theta^{**}; l) \leq \tau \leq \tau(\theta; l), \]
where the first inequality follows from \( t^* \in C_1 \cap C_2 \), and the second inequality follows from Remark \[.\] By continuity, there is and angle \( \tilde{\theta} \) between \( \theta \) and \( \theta^{**} \) so that \( \tau = \tau(\tilde{\theta}; l) \). It follows that \( \tilde{\theta} \geq \theta \) if and only if
\[ z(\tilde{\theta}; l) = \prod \frac{|\tau e^{i\theta} - \tau_k|}{|\tau|^r} \geq \prod \frac{|\tau e^{i\theta} - \tau_k|}{|\tau|^r} = z(\theta). \]
On the other hand, \( z(\theta; l) \) is monotone increasing, and hence \( \tilde{\theta} \geq \theta^{**} \) if and only if \( z(\tilde{\theta}; l) \geq z(\theta^{**}; l) = z(\theta) \). We conclude that in fact \( \tilde{\theta} = \theta^{**} = \theta \), and therefore \( t^* = \tau e^{i\theta} \), since this is the only point in \( C_1 \cap C_2 \) with argument \( \theta \). The proof is complete. \( \square \)

With \( t_{0,1} = \tau e^{\pm i\theta}, q_k = t_k/t_0 \) and \( \zeta_k = q_k e^{-i\theta}, 0 \leq k < \max \{n, r\} \), the following proposition is equivalent to Proposition \[.\]

**Proposition 15.** If \( 0 < \theta < \pi/r \), then besides the two trivial zeros \( \zeta_{0,1} = e^{\pm i\theta} \), all the zeros \( \zeta_k \), \( 2 \leq k < \max \{n, r\} \), of the polynomial
\[ Q(\zeta) = \prod_{j=1}^{n} \left( \frac{\sin(\theta_j - \theta)}{\sin \theta_j} - \frac{\zeta}{\sin \theta} \right) + \zeta^r, \]
lie outside the closed unit disk.

We focus on the values of \( \theta \) where \( \cos(m + r)\theta = \pm 1 \) and \( 0 < \theta < \pi/r \), i.e.,
\[ \theta = \frac{h \pi}{m + r} \]
where \( h = 1, \ldots, \lfloor m/r \rfloor \). We will show in Proposition \[.\] that the sign of \( H(\pi/r^-) \) is \((-1)^{\lfloor m/r \rfloor + 1}\), and that at the values of \( \theta \) given in \( \theta \), the sign of \( H(\theta) \) is \((-1)^h\) when \( m \) is large. Assuming this fact, the proof of Theorem \[.\] is now simple. By the Intermediate Value Theorem, \( H(\theta) \) has at least \( \lfloor m/r \rfloor \) solutions \( \theta \), each of which yields a real solution \( z \) of \( H_m(z) \) on \((0, \infty)\) by Lemma \[.\] Theorem \[.\] follows from the fact that the degree of \( H_m(z) \) is at most \( \lfloor m/r \rfloor \).

**Proposition 16.** Suppose \( n, r \in \mathbb{N} \) and \( \theta \) is given in \( \theta \). Then
(i) \( \text{sgn}(H(\theta)) = (-1)^h \), and
(ii) \( \text{sgn}(H(\pi/r^-)) = (-1)^{\lfloor m/r \rfloor + 1} \)
for all \( m \) sufficiently large.

To prove this proposition, for large \( m \) we consider three cases when \( \theta \) is bounded away from both \( 0 \) and \( \pi/r \), when \( \theta \) approaches \( 0 \), and when \( \theta \) approaches \( \pi/r \).
Case 1: $\gamma < \theta < \pi/r - \gamma$ for some fixed small $\gamma$. Proposition 14 implies that if $2 \leq k < \max \{n, r\}$ then $|\zeta_k| > 1 + \epsilon$ for some fixed $\epsilon$. Thus in (2.21) the sum

$$\sum_{k=2}^{\max\{n, r\}} \frac{1}{\zeta_k^{m+1}Q'(\zeta_k)}$$

approaches 0 exponentially fast when $m$ is large. The sign of $H(\theta)$ is then determined by the sum of the first two terms given in (2.23) if this sum does not approach 0. Since $A(\theta) > 0$ by Lemma 10 the sign of $H(\theta)$ when $\theta = h\pi/(m + r)$ is $(-1)^b$.

Case 2: $\theta \to 0$ as $m \to \infty$. We will show that when $\rho = 1$ we still have $|\zeta_k| > 1 + \epsilon$ when $2 \leq k < \max \{n, r\}$ and thus arguments in Case 1 apply since $A(\theta)$ approaches 0 with a polynomial rate by Lemma 13. When $\theta \to 0$, the polynomial $P(t) + zt^r$ approaches $P(t) + at^r$ with a real multiple zero at $t_a$. We need to show that besides the double zero at $t_a$, all the zeros of $P(t) + at^r$ lie outside the closed disk centered at the origin with radius $t_a$. From Proposition 14 it suffices to show that besides the double zero at $t_a$, the moduli of the other zeros are not $t_a$. The fact that these moduli are not $t_a$ follow directly from the inequality

$$\prod_{k=1}^{n} \frac{|t - \tau_k|}{|t^r|} \geq \prod_{k=1}^{n} \frac{|t_a - \tau_k|}{t_a^r}$$

whenever $|t| = |t_a|$ and the equality holds only when $t = t_a$. Since the multiplicity of $t_a$ is at least 2, we have that $P(t_a) + at_a^r = 0$ and $P'(t_a) + rat_a^{r-1} = 0$. These two equations imply that $rP(t_a) - t_aP'(t_a) = 0$. If the multiplicity of $t_a$ is higher than 2 then $P''(t_a) + r(r - 1)at_a^{r-2} = 0$ which can be combined with $P'(t_a) + rat_a^{r-1} = 0$ to give $(r - 1)P'(t_a) - t_aP''(t_a) = 0$. By the interlacing property of the zeros of $P'(t)$ and $P''(t)$, $t_a$ must lie between the two smallest zeros of $P(t)$ and $P''(t)$. Similarly the equation $rP(t_a) - t_aP'(t_a) = 0$ implies that $t_a$ lies between the two smallest zeros of $P(t)$ and $P'(t)$. This only occurs when $P''(t_a) = P''(t_a) = 0$ which then implies that $P''(t) = 0$, a contradiction to $\rho = 1$.

Next, we consider the case when $\rho > 1$. Since $\sum_{j=1}^{n} \theta_j = (n - 1)\pi + r\theta$, we have $\theta_j \to \pi$ when $j > \rho$ and $\theta_1 = \theta_2 = \cdots = \theta_\rho \to \pi - \pi/\rho$, see Figure 2.3. Let $\eta_j = \pi - \pi/\rho - \theta_j$ for $1 \leq j \leq \rho$ and $\eta_j = \pi - \theta_j$ if $j > \rho$. Recall that $\theta_j$, $1 \leq j \leq n$, satisfy

$$\frac{\sin \theta_1}{\sin(\theta_1 - \theta)} = \cdots = \frac{\sin \theta_\rho}{\sin(\theta_\rho - \theta)}$$

We have

$$\frac{\sin \theta_1}{\sin(\theta_1 - \theta)} = \frac{\sin(\pi/\rho + \theta_1)}{\sin(\pi/\rho + \theta)}$$

$$= \frac{\sin(\pi/\rho) + \cos(\pi/\rho)\eta_1 + O(\eta_1^2)}{\sin(\pi/\rho) + \cos(\pi/\rho)(\eta_1 + \theta) + O(\eta_1^2 + \eta_1\theta + \theta^2)}$$

$$= 1 - \cot \frac{\pi}{\rho} \theta + O(\eta_1^2 + \eta_1\theta + \theta^2)$$

and the corresponding fraction when $j > \rho$ is

$$\frac{\sin \theta_j}{\sin(\theta_j - \theta)} = \frac{\eta_j + O(\eta_j^3)}{\eta_j + \theta + O((\eta_j + \theta)^3)} = \frac{\eta_j}{\eta_j + \theta} (1 + O(\eta_j^2 + \theta^2 + \eta_j\theta))$$.
The identity $\tau_1 \sin \theta_1 / \sin(\theta_1 - \theta) = \tau_j \sin \theta_j / \sin(\theta_j - \theta)$ gives

$$(n_j + \theta)\tau_1 - \tau_1 \cot(\pi / \rho) \theta = \tau_j n_j + O((\eta_1 + \eta_j + \theta)^3)$$

from which we solve for $n_j$, $j > \rho$

$$n_j = \frac{\tau_1 (\theta - \cot(\pi / \rho) \theta^2)}{\tau_j - \tau_1 + \tau_1 \cot(\pi / \rho) \theta} + O((\eta_1 + \theta)^3)$$

$$= \left( \frac{\tau_1}{\tau_j - \tau_1} - \frac{\tau_1 \cot(\pi / \rho) \theta^2}{\tau_j - \tau_1} \right) \left( 1 - \frac{\tau_1 \cot(\pi / \rho) \theta}{\tau_j - \tau_1} \right) + O((\eta_1 + \theta)^3)$$

$$= \frac{\tau_1}{\tau_j - \tau_1} \theta - \frac{\cot(\pi / \rho) \tau_1 \tau_j}{(\tau_j - \tau_1)^2} \theta^2 + O((\eta_1 + \theta)^3).$$

The equation $\sum_{j=1}^n n_j = -r\theta$ implies that

$$\rho n_1 + \left( \sum_{j>\rho} \frac{\tau_1}{\tau_j - \tau_1} + r \right) \theta - \sum_{k>\rho} \frac{\cot(\pi / \rho) \tau_1 \tau_j}{(\tau_j - \tau_1)^2} \theta^2 + O(\theta^3) = 0.$$
and consequently
\[
\sum_{k=0}^{r-1} \frac{1}{\zeta_k^{m+1} Q'(\zeta_k)} = \frac{\theta}{\rho \sin(\pi/\rho)} \sum_{k=0}^{r-1} \frac{\omega_k}{\zeta_k^{m+1}} (1 + O(\theta))
\]
whose sign is \((-1)^h\) by (35) of [3].

**Case 3:** \(\theta \to \pi/r\) as \(m \to \infty\). Similar to arguments in the beginning of Case 2, when \(r = 1\), we claim that \(|\zeta_k| > 1 + \epsilon\) for \(2 \leq k < n\) and with Lemma [3] the arguments in Case 1 apply. We prove this claim by showing that besides the double zero at \(t_b\), all the other zeros of \(P(t) + bt\) lie outside the circle radius \(|t_b|\) or equivalently (with Proposition [14]) their moduli are different from \(|t_b|\). The claim about their moduli again follows from the inequality
\[
\prod_{k=1}^{n} \left| \frac{t - \tau_k}{t'} \right| \leq \prod_{k=1}^{n} \left| t_b - \tau_k \right| t'_b
\]
when \(|t| = |t_b|\) with the equality holds only when \(t = t_b\). If the multiplicity of \(t_b\) is more than 2 then \(P''(t_b) = 0\). This is a contradiction since the zeros of \(P''(t)\) are positive while \(t_b \leq 0\) by its definition.

Next, we consider when \(r > 1\). Since \(\sum_{j=1}^{n} \theta_j = r\theta + (n-1)\pi\), we must have \(\theta_j, 1 \leq j \leq n\), approach \(\pi\). Let \(\eta_j = \pi - \theta_j, 1 \leq j \leq n\), and \(\eta = \pi/r - \theta\). From the equations
\[
\tau = \frac{\eta_j \sin \theta_j}{\sin(\theta_j - \theta)} + O(\eta_j^3)
\]
\[
= \frac{\eta_j \sin \pi/r + \cos \pi/r (\eta_j - \eta) + O((\eta_j + \eta)^2)}{\sin \pi/r} \left(1 - \cot \frac{\pi}{r} (\eta_j - \eta) + O((\eta_j + \eta)^2)\right)
\]
and \(\sum_{j=1}^{n} \eta_j = r\eta\) we have
\[
\tau \sin \frac{\pi}{r} \sum_{j=1}^{n} \frac{1}{\tau_j} = r\eta(1 + O(\eta_j + \eta)).
\]

Thus for any \(1 \leq k \leq n\)
\[
\eta_k \tau_k \sum_{j=1}^{n} \frac{1}{\tau_j} = r\eta(1 + O(\eta)).
\]

The formula (2.26) implies that besides \(r\) zeros \(\zeta_0, \ldots, \zeta_{r-1}\) of \(Q(\zeta)\) which approach the zeros of \(1 + \zeta^r\), the possible remaining \(\max\{n, r\} - r\) zeros approach \(\infty\). Thus it suffices to consider the sign of
\[
\sum_{k=0}^{r-1} \frac{1}{\zeta_k^{m+1} Q'(\zeta_k)}.\]
With $\zeta_k = e_k + \epsilon$, $\epsilon \in \mathbb{C}$, $e_k = e^{(2k-1)\pi i/r}$, the equation $Q(\zeta_k) = 0$ gives

$$0 = \prod_{j=1}^{n} \left( \frac{\sin(\theta_j - \theta)}{\sin \theta} - \zeta_k \frac{\sin \theta_j}{\sin \theta} \right) + \zeta_k$$

$$= \prod_{j=1}^{n} \left( \frac{\sin(\pi/r) + \cos(\pi/r)(\eta_j - \eta) + O(\eta^2)}{\sin(\pi/r) - \cos(\pi/r)\eta + O(\eta^2)} - \zeta_k \frac{\eta_j + O(\eta^2)}{\sin(\pi/r) - \cos(\pi/r)\eta + O(\eta^2)} \right) + \zeta_k$$

$$= \prod_{j=1}^{n} \left( 1 + \cot \frac{\pi}{r} \eta_j - e_k \frac{\eta_j}{\sin(\pi/r)} + O(\eta^2 + e\eta) \right) + (e_k + \epsilon)^r$$

$$= \cos(\pi/r) - \frac{e_k}{\sin(\pi/r)} \sum_{j=1}^{n} \eta_j + \frac{r\epsilon}{e_k} + O(\eta^2 + e\eta + \eta^2)$$

from which we solve for $\epsilon$

$$\epsilon = \frac{e_k (\cos(\pi/r) - e_k)\eta}{\sin(\pi/r)} + O(\eta^2).$$

From (22), we obtain

$$Q'(\zeta_k) = \sum_{j=1}^{n} \frac{\zeta_j^r \sin \theta_j}{\sin(\theta_j - \theta) - \zeta_k \sin \theta_j} + r\zeta^{-1}_k$$

$$= r\zeta^{-1}_k + O(\eta)$$

and thus the sign of

$$\sum_{k=0}^{r-1} \frac{1}{\zeta_k^m+1} Q'(\zeta_k) = \frac{1}{r} \sum_{k=0}^{r-1} \frac{1 + O(\eta)}{\zeta_k^m+1}$$

is $(-1)^h$ by (41) of [3].

### 3. Open Problems

Theorem [4] asserts that there is a constant $C$, depending on both $P(t)$ and $r$, such that the zeros of $H_m(z)$ lie in the real interval $(a, b) \subseteq (0, \infty)$ for all $m \geq C$. We conjecture that the zeros of $H_m(z)$ lie in the interval $(a, b)$ for all $m$.

**Conjecture 17.** Suppose $P(t)$ is a real polynomial whose zeros are positive real numbers and $P(0) > 0$. Let $r$ be a positive integer such that $\max\{\deg P, r\} > 1$. For all integers $m$, the zeros of the polynomial $H_m(z)$ generated by

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t) + zt^r}$$

lie in the interval $(a, b)$ defined in Theorem [7]. Moreover, the set $Z = \bigcup_{m \geq 1} \{z \mid H_m(z) = 0\}$ is dense on $(a, b)$.

With a different approach, in [2] Theorem 1], the authors prove that the zeros of the polynomial $H_m(z)$ generated by

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 - zt + Ct^2 + t^3}$$

lie in the interval $(a, b)$ defined in Theorem [7]. Moreover, the set $Z = \bigcup_{m \geq 1} \{z \mid H_m(z) = 0\}$ is dense on $(a, b)$.
are real for all $m \geq 1$ if $C \geq 3$. On the other hand, if $C < 3$, then there exists an $m \geq 1$ so that not all zeros of $H_m(z)$ are real (see [1 Proposition 1]). In a similar vein, the 6-Conjecture [2] stipulates that the zeros of the sequence of polynomials generated by

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 - zt + Ct^2 - 4t^3 + t^4}.$$ 

are real if $C \geq 6$. The authors of [2] believe that a similar conclusion still holds if the coefficients of the denominator follow a binomial pattern. In light of Theorem 1, we conjecture a more general result, by replacing the binomial polynomial by a polynomial, whose zeros are real and of the same sign.

**Conjecture 18.** Let $C$ be a real number and $r, s$ be natural numbers. Assume that $P(t)$ is a polynomial whose zeros are real and positive. If $C(s - r) \geq 0$, then the zeros of the sequence of polynomials $H_m(z)$ generated by

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t) + Ct^s + zt^r}$$

are real.

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