Mixed convection flow near a stagnation point on a vertical surface with prescribed surface heat flux

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Abstract. The problem of the mixed convection stagnation point flow toward a vertical surface is numerically studied in this paper. Different from the previous studies, in this paper we consider prescribed surface heat flux, instead of prescribed surface temperature. The similarity transformation is applied to the governing partial differential equations to reduce them into a system of ordinary differential equations. These equations are then solved numerically using the boundary value problem solver, bvp4c, available in the Matlab software. Dual solutions are obtained for both buoyancy assisting and buoyancy opposing flows. The temporal stability analysis is performed to determine the stability of these solutions as time passes. The effects of the governing parameters on the physical quantities of interest such as the surface shear stress and the surface temperature are discussed.

1. Introduction
Free, forced and mixed convection flows are found in many practical applications, such as solar central receivers exposed to wind currents, electronic devices cooled by fans, nuclear reactors cooled during emergency shutdown and heat exchanges placed in a low velocity environment [1]. The mixed convection flow occurs when the effect of buoyancy force in the forced convection or the effect of forced flow in the free convection becomes significant. In the free convection, the flow is driven by the buoyancy force due to the pressure difference between the top and the bottom parts of the object. Different from the flow over a horizontal solid surface, the effect of buoyancy force for the flow toward a vertical solid surface could not be neglected.

The problem of two-dimensional stagnation-point flow was first considered by Hiemenz [2], where the exact similarity solution was reported. This problem was then extended by Goldstein [3] by including the energy equation for studying the temperature distribution. Homann [4] then studied the axisymmetric flow, while Sibulkin [5] reported the temperature distribution. Ramachandran et al. [6] studied the laminar stagnation-point flow toward a vertical impermeable surface where both the prescribed surface temperature and the prescribed surface heat flux were considered. They found that dual solutions exist for the buoyancy opposing flow case, while for the assisting flow case, the solution is unique. The prescribed surface temperature case was extended to a permeable plate by Ishak et al. [7], where they reported the existing of dual solutions for both buoyancy assisting flow ($\lambda > 0$) and buoyancy opposing flow ($\lambda < 0$) as shown in Figure 1 and Figure 2. They found that value of $|\lambda|$ increases with an
increase in suction parameter $f_0$, where $\lambda_0 (<0)$ is the value of $\lambda$ for which the upper branch solution meets the lower branch solution. However, the stability of the dual solutions was not studied. Different from Ishak et al. [7], the present study considers the stagnation-point flow towards a permeable vertical surface with prescribed surface heat flux. Moreover, the temporal stability of the dual solutions is studied, which determine which one of the solutions is stable as time passes.

![Graph](image1)

**Figure 1.** The values skin friction coefficient $f^*(0)$ with variation of $\lambda$ for various values of $f_0$ for $Pr = 1$ with $f_0 > 0$ for suction and $f_0 < 0$ for injection.

![Graph](image2)

**Figure 2.** The values Local Nusselt number $-\theta'(0)$ with variation of $\lambda$ for various values of $f_0$ for $Pr = 1$ with $f_0 > 0$ for suction $f_0 < 0$ for injection.

### 2. Mathematical formulation

Consider a laminar two-dimensional stagnation-point flow moving normal to a vertical plate placed in a viscous and incompressible fluid of ambient temperature $T_\infty$. It is assumed that the free stream velocity and the surface heat flux are of the forms $U(x) = ax$ and $q_x(x) = bx$, respectively, where $a$ ($>0$) and $b$ are constants. Under these assumptions, together with the boundary layer and Boussinesq approximations, the steady laminar boundary layer equations governing the flow are given by
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  
(1)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho} \frac{dp}{dx} + g \beta (T - T_w) \]  
(2)

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \]  
(3)

subject to the boundary condition

\[ u = 0, \quad v = V_w, \quad -k \frac{\partial T}{\partial y} = q_w(x) \quad \text{at} \quad y = 0, \]  
(4)

\[ u \to u_\infty(x), \quad T \to T_\infty \quad \text{as} \quad y \to \infty, \]

By applying the Bernoulli's equation in Eq. (2) one gets

\[ \frac{dU}{dx} = -\frac{1}{\rho} \frac{dp}{dx}. \]  
(5)

Then Eq. (2) becomes

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + U \frac{dU}{dx} + g \beta (T - T_w). \]  
(6)

To solve Eqs. (1), (3) and (6), the following similarity variables are introduced:

\[ \eta = (U / \nu x)^{1/2}, \quad \psi = (U \nu x)^{1/2} f(\eta), \quad T = T_w + q_w \left( \frac{\nu x}{U} \right)^{1/2} \theta(\eta) \]  
(7)

where \( \psi \) is the stream function defined as \( u = \partial \psi / \partial y \) and \( v = -\partial \psi / \partial x \) so as to identically satisfy Eq. (1). Substituting (7) into Eqs. (6) and (3), one gets the following ordinary differential equations:

\[ f'''' + ff'' + 1 - f'^2 + \lambda \theta = 0 \]  
(8)

\[ \frac{1}{Pr} \theta'' + f \theta' - f' \theta = 0 \]  
(9)

where primes denote differentiation with respect to \( \eta \), \( \lambda = Gr = Re^{5/2} \) is the buoyancy or mixed convection parameter and \( Pr = \nu / \alpha \) is the Prandtl number. Further, \( Gr = g \beta q_w x^4 / k \nu^2 \) and \( Re = Ux / \nu \) represent the local Grashof number and the local Reynolds number respectively while \( \lambda \) is a constant with \( \lambda < 0 \) denotes the buoyancy opposing flow while \( \lambda > 0 \) denotes the buoyancy assisting flow. The boundary conditions (4) now become

\[ f(0) = f_0, \quad f'(0) = 0, \quad \theta'(0) = -1, \]

\[ f'(\infty) \to 1, \quad \theta(\infty) \to 0, \]  
(10)

where \( f_0 = f(0) = -V_w / (\nu a)^{1/2} \) is a constant with \( f_0 > 0 \) corresponds to mass suction.

The physical quantities of interest in this work are the skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_x \) which can be expressed as

\[ C_f = \frac{\tau_w}{\rho U^2 / 2}, \quad Nu_x = -\frac{x q_w}{k(T_w - T_\infty)} \]  
(11)

where the surface shear stress \( \tau_w \) and the heat transfer from the plate \( q_w \) are given by

\[ \tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}, \]  
(12)

with \( \mu \) and \( k \) being the dynamic viscosity and thermal conductivity, respectively. Using the non-dimensional variables (7) yields

\[ \frac{1}{2} C_f Re^{1/2} = f''(0), \quad Nu_x / Re^{1/2} = \frac{1}{\theta(0)} \]  
(13)
3. Stability analysis

In order to perform the temporal stability analysis, we consider the unsteady case for the present problem. Equation (1) holds, while Eqs. (6) and (3) are replaced by

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} + g \beta (T - T_\infty) \quad (14) \]

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (15) \]

where \( t \) denotes the time. To solve Eqs. (14) and (15), the following transformation is introduced:

\[ \eta = \left( \frac{U}{v x} \right)^{1/2}, \quad \psi = (v x U)^{1/2} f(\eta, \tau) \]

\[ T = T_\infty + \frac{q_w}{k} \left( \frac{v x}{U} \right)^{1/2} \theta(\eta, \tau) \quad \tau = at \]

where \( \tau \) is the dimensionless time.

Substituting equation (16) into Equations (14) and (15) one gets

\[ \frac{\partial^2 f}{\partial \eta^2} + f \frac{\partial^2 f}{\partial \eta^2} + \left( \frac{\partial f}{\partial \eta} \right)^2 + 1 - \frac{\partial^2 f}{\partial \eta \partial \tau} + \lambda \theta = 0 \quad (17) \]

\[ \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2} + f \frac{\partial \theta}{\partial \eta} - \frac{\partial f}{\partial \eta} - \frac{\partial \theta}{\partial \tau} = 0 \quad (18) \]

and are subjected to the boundary conditions

\[ f(0, \tau) = f_0, \quad \frac{\partial f}{\partial \eta}(0, \tau) = 0, \quad \frac{\partial \theta}{\partial \eta}(0, \tau) = -1, \]

\[ \frac{\partial f}{\partial \eta}(\eta, \tau) \to 1, \quad \theta(\eta, \tau) \to 0, \quad \text{as} \ \eta \to \infty \quad (19) \]

The stability of dual solutions is determined by adopting the analysis suggested by Merkin [8]. We consider a perturbation in exponential forms

\[ f(\eta, \tau) = f_0(\eta) + e^\gamma \eta F(\eta), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^\gamma \eta G(\eta) \quad (20) \]

where \( \gamma \) is an unknown eigenvalue which determines the growth or decay of the disturbance. In Eq. (20), \( f_0(\eta) \) and \( \theta_0(\eta) \) are the steady flow solutions obtained by solving Eqs. (8)-(10) and the functions \( F(\eta) \) and \( G(\eta) \) are small relative to \( f_0(\eta) \) and \( \theta_0(\eta) \), respectively.

Substituting (20) into (17)-(19), after linearization, one gets

\[ F'' + f_0 F'' + f_0^\gamma F - \left( 2 f_0' - \gamma \right) F' + \lambda G = 0 \quad (21) \]

\[ \frac{1}{Pr} (G'' + \theta_0 G' + \gamma G - f_0' G - \theta_0 F') = 0 \quad (22) \]

\[ F(0) = f_0, \quad F'(0) = 0, \quad G'(0) = 0, \]

\[ F'(\eta) \to 0, \quad G(\eta) \to 0, \quad \text{as} \ \eta \to \infty. \quad (23) \]

The smallest eigenvalue \( \gamma \) determines the stability of the solutions. The range of possible eigenvalues can be determined by relaxing the boundary conditions on \( F(\eta) \) or \( G(\eta) \). Without loss of generality, we set \( F'(0) = 1 \) and find the values of \( \gamma \) appeared in Eqs. (21) and (22), by replacing the far field boundary condition \( F'(\eta) \to 0 \) as \( \eta \to \infty \) with \( F''(0) = 1 \).
3. Results and discussion

In this study, the nonlinear ordinary differential equations (8) and (9) subject to the boundary conditions (10) were solved numerically using the bvp4c function available in the Matlab software. The solutions were obtained by applying different initial guesses for the values of \( f'(0) \) and \( \theta(0) \) where all of the velocity and temperature profiles satisfy the far field boundary conditions (10) asymptotically. The range of \( \eta_\infty \) for the first solution is between 10 and 20 while for the second solution is between 30 and 40. The initial guesses and the boundary layer thickness vary for different values of parameters. In the numerical computations, we have to make sure that all profiles satisfy the infinity boundary conditions asymptotically, otherwise the numerical results are not valid (see Pantokratoras [9]).

![Figure 3](image_url)

Figure 3. The values skin friction coefficient \( f''(0) \) with variation of \( \lambda \) for various values of \( f_0 \) for \( \text{Pr} = 1 \) \( f_0 > 0 \) for suction

Figure 3 depicts the variation of the skin friction coefficient \( f''(0) \) versus \( \lambda \) while Fig. 4 depicts the variations of the local Nusselt number \( 1/\theta(0) \). The results illustrated in Fig. 3 and 4 are for different values of suction strength \( f_0 \), namely \( f_0 = 0.2 \) and 0.5, where the value of the Prandtl number is fixed at \( \text{Pr} = 1 \). Both Figs. 3 and 4 show the existence of dual solutions for both buoyancy assisting and opposing flows. For the assisting flow, \( \lambda > 0 \), solutions are obtained for all values of \( \lambda \), while for the opposing flow, solution exists up to certain values of \( \lambda \). The range of existence of this solution depends on the suction strength \( f_0 \). It is clear from Fig. 3 that the effect of suction is to increase the range of \( \lambda \) for which the solution is in existence. For the first solution, the skin friction increases as the suction strength is increased. Thus, suction can play an important role in controlling the skin friction.

The heat transfer counterpart is displayed in Fig. 4, which shows discontinuity for the second solution. Thus we expect that the second solution is unstable. To confirm this, we conduct a temporal stability for the solutions of the steady flow, Eqs. (8)-(10), by considering the unsteady case. There are several papers recently on such studies such as Rosca and Pop [10], Weidman [11], Soid et al. [12], Awaludin et al. [13] and Jahan et al. [14], among others. The perturbation in the form of exponents as given in Eqs (20) is considered, given that the exponential function gives a more rapid change than the power functions. The sign of the eigenvalue \( \gamma \) will determine either the unsteady flow solutions converge to the steady flow solutions or not, as time passes, \( \tau \rightarrow \infty \). Positive values of \( \gamma \) shows an initial decay of disturbance which represents a stable solution, while negative values of \( \gamma \) will result in an initial growth of disturbance, thus the flow is unstable. To obtain these values of \( \gamma \), we need to solve...
Eqs. (21)-(23), by setting $F^*(0) = 1$, without loss of generality. It is a vice-versa process compared to that of solving Eqs. (8)-(10), where the values of parameters are set to some fixed values, to obtain the quantities of physical interest given in Eq. (13). The plot of the smallest eigenvalues $\gamma$ versus the mixed convection parameter $\lambda$ is displayed in Fig. 5 which shows that $\gamma$ is positive for the first solution, while it is negative for the second solution. Thus we conclude that the first solution is stable, while the second solution is not.

![Figure 4. The values of the local Nusselt number $\frac{1}{\theta(0)}$ with variations of $\lambda$ for $f_o = 0.2, 0.5$ for Pr = 1 $f_o > 0$ for suction](image1.png)

![Figure 5. Smallest eigenvalue $\gamma$ at several values of $\lambda$ at $f_o = 0$ for Pr = 1](image2.png)

4. Conclusion

This study investigated the steady two dimensional boundary layer flow near a stagnation point on a vertical surface with prescribed surface heat flux. Both buoyancy assisting and opposing flows were considered. Numerical results, which were obtained using the boundary value problem solver, bvp4c function in Matlab software, showed the existence of dual solutions for both the buoyancy assisting and opposing flows. Discussion on the skin friction coefficient and the local Nusselt number were carried
out for the effects of all parameters involved. A stability analysis was conducted to determine which one of the solutions is stable in a long run. It was found that only the first solution is stable and physically reliable while the second solution is not.

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