CONTACT 5-MANIFOLDS AND SMOOTH STRUCTURES
ON STEIN 4-MANIFOLDS

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Abstract. We show that, under a certain condition, contact 5-manifolds can 'coarsely' distinguish smooth structures on compact Stein 4-manifolds via contact open books. We also give a simple sufficient condition for an infinite family of Stein 4-manifolds to have an infinite subfamily of pairwise non-diffeomorphic Stein 4-manifolds. The proofs rely on the adjunction inequality. We remark that there are various examples of infinitely many pairwise exotic Stein 4-manifolds whose smooth structures can be distinguished by these results.

1. Introduction

We consider abstract open books on closed oriented smooth 5-manifolds whose pages are compact Stein 4-manifolds and whose monodromies are the identity maps, which are a symplectomorphism common to all Stein 4-manifolds. It is known that the Stein structure on the page of an open book induces a contact structure on the smooth 5-manifold ([14], cf. [9]), and an open book equipped with this contact structure is called a contact open book. Van Koert [25] and Ding-Geiges-van Koert [9] studied such contact 5-manifolds and then classified contact 5-manifolds admitting subcritical Stein fillings without 1-handles in [9].

From the viewpoint of 4-dimensional topology, it is natural to ask how the smooth structures on the pages of contact open books and the supporting contact structures are related to each other. Ozbagci-van Koert [23] recently showed that infinitely many contact open books with pairwise exotic (i.e. homeomorphic but non-diffeomorphic) pages (obtained in [3]) can support pairwise non-contactomorphic contact structures on fixed smooth 5-manifolds. Akbulut and the author [4] extended this result to infinitely many smooth 5-manifolds, using exotic 4-manifolds obtained in [27]. By contrast, Akbulut and the author [4] also proved that infinitely many contact open books with pairwise exotic pages can support the same contact 5-manifold, using exotic 4-manifolds obtained in [3] and [27].

In this paper we show that, under a certain condition, closed contact 5-manifolds can 'coarsely' distinguish smooth structures on pages of their supporting contact open books. We note that such contact 5-manifolds are not invariants of smooth structures on pages ([9], cf. Proposition 5.1). For the definition of subcritical Stein fillings without 1-handles, see Section 2.

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Theorem 1.1. Let \( \{X_i \mid i \in \mathbb{N}\} \) be an infinite family of compact Stein 4-manifolds such that, for each \( i \), the contact 5-manifold \((W_i, \xi_i)\) supported by the contact open book \((X_i, \text{id})\) admits a subcritical Stein filling without 1-handles. If the members of \( \{(W_i, \xi_i) \mid i \in \mathbb{N}\} \) are pairwise non-contactomorphic, then at least infinitely many members of \( \{X_i \mid i \in \mathbb{N}\} \) are pairwise non-diffeomorphic.

Remark 1.2. (1) If the page of a contact open book (with the identity monodromy) is a 4-dimensional Stein handlebody without 1-handles, then the contact 5-manifold supported by the contact open book admits a subcritical Stein filling without 1-handles (see Remark 3.2). For more general sufficient condition, see Lemma 3.3.
(2) There are various examples of infinitely many pairwise exotic 4-manifolds (obtained in [27]) which are distinguished by this theorem (see Theorem 5.2).
(3) The condition ‘at least infinitely many members’ in this theorem cannot be strengthened to ‘all members’ without additional constraints (see Proposition 5.1).
(4) The converse of this theorem does not hold. In fact, infinitely many pairwise exotic Stein handlebodies without 1-handles can be the pages of the same contact 5-manifold (Theorem 1.2 in [4]).

The proof of this theorem relies on the adjunction inequality — a result from Seiberg-Witten theory — and the classification of contact 5-manifolds admitting subcritical Stein fillings without 1-handles in [9].

We moreover give a simple and practical sufficient condition for an infinite family of Stein 4-manifolds to have an infinite subfamily of pairwise non-diffeomorphic Stein 4-manifolds. For our definition of the divisibility, see Subsection 4.2.

Theorem 1.3. For a given infinite family of compact Stein 4-manifolds, if the divisibilities of the first Chern classes of the members are pairwise distinct, then at least infinitely many members are pairwise non-diffeomorphic.

Remark 1.4. (1) This theorem works for various families of pairwise exotic Stein 4-manifolds. For example, for any finitely presented group \( G \), the algorithm obtained in [27] can produce such infinite families with \( \pi_1 \cong G \). Moreover, the algorithm can realize a vast class of integral symmetric bilinear forms as intersection forms of such infinite families. However, the algorithm also produces various infinite families of exotic Stein 4-manifolds that do not satisfy the assumption of this theorem.
(2) The condition ‘at least infinitely many members’ in this theorem cannot be strengthened to ‘all members’ without additional constraints (see the proof of Proposition 5.1).

To distinguish smooth structures, we introduce the intersection genus, which is a simple genus invariant (i.e. a diffeomorphism invariant determined by the minimal genus function on a 4-manifold). For other genus invariants, we refer to the relative genus function in [26] (cf. [27]) and the genus rank function in [16]. See also Subsection 4.1 for the background of these genus invariants. The intersection genus is simpler than these genus invariants but more useful for our purpose. Estimating intersection genera of Stein 4-manifolds by the adjunction inequality, we prove the above theorem.

We also show the finiteness of the number of the first Chern classes of Stein structures on a given compact smooth 4-manifold with boundary (Proposition 4.6). This result yields an alternative proof of Theorem 1.3, though this proof does not tell how to distinguish smooth structures on given two Stein 4-manifolds unlike the one relying on the intersection genus.
2. Compact Stein manifolds and the adjunction inequality

In this section we briefly recall the basics of compact Stein manifolds. For the details, see [22], [9] and [8]. We begin with a few definitions on Stein manifolds of arbitrary dimension. A compact Stein manifold is a compact complex manifold with boundary admitting a strictly plurisubharmonic function that is constant on the boundary and has no critical points on the boundary. If a compact Stein manifold $W$ of real dimension $2n$ admits a strictly plurisubharmonic function that is a Morse function without critical points of index $n$, then $W$ is called subcritical. If the Morse function on $W$ furthermore has no critical points of index one, then $W$ is called a subcritical Stein manifold without 1-handles. A compact Stein manifold $W$ is called a Stein filling of a contact manifold $(M,\xi)$ if $(M,\xi)$ is contactomorphic to the contact structure on the boundary $\partial W$ given by the complex tangencies.

Now we restrict our attention to the case of real dimension four. We call a compact connected oriented 4-dimensional handlebody a Stein handlebody, if it consists of one 0-handle and 1- and 2-handles, and its 2-handles are attached along a Legendrian link in the standard tight contact structure on the boundary $#_{n}S^{1}\times S^{2}$ of the subhandlebody consisting of 0 and 1-handles, where the framing of each 2-handle is one less than the framing induced from the contact plane (i.e. contact framing). According to [10] (cf. [15], [17]), any 4-dimensional Stein handlebody admits a Stein structure extending that of the 0-handle $D^{4}$, and conversely any compact Stein 4-manifold (i.e. compact Stein manifold of real dimension four) is diffeomorphic to a Stein handlebody.

The following adjunction inequality gives a strong constraints on the genera of smoothly embedded surfaces representing a given second homology class. This inequality, which is a result from Seiberg-Witten theory, plays a key role in this paper.

Theorem 2.1 ([19], [1], [20]). Let $X$ be a compact Stein 4-manifold, and let $[\Sigma]$ be a non-zero class of $H_{2}(X;\mathbb{Z})$ represented by a smoothly embedded closed connected oriented surface $\Sigma$ of genus $g \geq 0$ in $X$. Then the following adjunction inequality holds.

$$|\langle c_{1}(Z), [\Sigma] \rangle| + [\Sigma] : [\Sigma] \leq 2g - 2.$$  

This well-known result follows from the adjunction inequality for closed 4-manifold ([12], [18], [21], [24]), since any compact Stein 4-manifold can be embedded into a closed minimal complex surface of general type with $b^{+}_{2} > 1$ ([19]). Note that the adjunction inequality for Stein 4-manifolds holds even in the genus zero case (cf. [17], [22], [2]), unlike the version for general closed 4-manifolds.

3. Contact open books on 5-manifolds

In this section, we briefly review the basics of contact 5-manifolds supported by contact open books. For the details, see [9].

Let $(X,\psi)$ be a 5-dimensional abstract open book whose page is a compact Stein 4-manifold $X$ and whose monodromy is a symplectomorphism $\psi$ equal to the identity near the boundary. It is known that the Stein structure on the page $X$ and the monodromy $\psi$ induces a contact structure on the closed smooth 5-manifold given by $(X,\psi)$ ([14], cf. [9]). The open book $(X,\psi)$ equipped with this contact structure is called the contact open book. We say that a contact 5-manifold $(W,\xi)$
is supported by a contact open book \((X, \psi)\) if \((W, \xi)\) is contactomorphic to the contact 5-manifold given by the contact open book \((X, \psi)\).

In the rest, we consider the case where the monodromy is the identity map, which is a symplectomorphism common to all Stein 4-manifolds. Relying on a result of Cieliebak \[6\], Ding-Geiges-van Koert observed a characterization of contact 5-manifolds supported by contact open books.

**Theorem 3.1** ([6], Proposition 3.1 in [9]). A closed contact 5-manifold admits a subcritical Stein filling if and only if the contact 5-manifold is supported by a contact open book whose page is a compact Stein 4-manifold and whose monodromy is the identity map.

**Remark 3.2.** A compact Stein 4-manifold \(X\) gives a subcritical Stein 6-manifold \(X \times D^2\), and its boundary contact structure is supported by the contact open book \((X, \text{id})\). Furthermore, if \(X\) is a Stein handlebody without 1-handles, then \(X \times D^2\) is a subcritical Stein 6-manifold without 1-handles. For these facts, see the proof of Proposition 3.1 in [9]. Conversely, a subcritical Stein 6-manifold \(M\) is deformation equivalent to the Stein 6-manifold \(X \times D^2\) for some 4-dimensional Stein handlebody \(X\). If \(M\) has no 1-handles, then we may assume that \(X\) has no 1-handles. For the proof of these facts, see the proof of Theorem 1.1 in [6] (cf. Section 14.4 in [8]).

Ding-Geiges-van Koert classified contact 5-manifolds admitting subcritical Stein fillings without 1-handles. We note that, due to the above remark, a contact 5-manifold admits a subcritical Stein filling without 1-handles if and only if it is supported by a contact open book whose page is a Stein handlebody without 1-handles and whose monodromy is the identity map.

**Theorem 3.3** (Ding-Geiges-van Koert, Theorem 4.8 in [9]). Let \((W_1, \xi_1)\) and \((W_2, \xi_2)\) be two closed contact 5-manifolds admitting subcritical Stein fillings without 1-handles. If there exists an isomorphism \(H^2(W_1; \mathbb{Z}) \to H^2(W_2; \mathbb{Z})\) that sends \(c_1(\xi_1)\) to \(c_1(\xi_2)\), then \((W_1, \xi_1)\) and \((W_2, \xi_2)\) are contactomorphic to each other.

The lemma below was observed by Ding-Geiges-van Koert (see the proof of Proposition 4.5 in [9]).

**Lemma 3.4** ([9]). For a compact Stein 4-manifold \(X\) and a closed contact 5-manifold \((W, \xi)\) supported by the contact open book \((X, \text{id})\), there exists an isomorphism \(H^2(X; \mathbb{Z}) \to H^2(W; \mathbb{Z})\) that maps \(c_1(X)\) to \(c_1(\xi)\).

We recall the following terminology in [4], which is useful for describing contact 5-manifolds.

**Definition 3.5.** For a 4-dimensional Stein handlebody \(X\) without 1-handles, we define the rotation divisor of \(X\), denoted by \(r(X)\), as the greatest common divisor of the rotation numbers of the attaching circles of the 2-handles of \(X\). If \(b_2(X) = 0\), or if all the rotation numbers of the attaching circles are 0, then we define \(r(X)\) by \(r(X) = 0\). Note that each attaching circle is a Legendrian knot in the standard tight contact structure on \(S^3\).

Let \(S^2 \times S^3\) denote the total space of the non-trivial \(S^3\)-bundle over \(S^2\). The following proposition was observed by Akbulut and the author. This easily follows from Theorems 3.1 and 3.3, Lemma 3.4 and Gompf’s first Chern class formula for Stein handlebodies [15].
Proposition 3.6 ([4]). For two 4-dimensional Stein handlebodies $X_1, X_2$ with $b_2 = n$ and without 1-handles, the contact 5-manifolds $(W_1, \xi_1)$ and $(W_2, \xi_2)$ supported by the contact open books $(X_1, \text{id})$ and $(X_2, \text{id})$ are contactomorphic to each other if and only if $r(X_1) = r(X_2)$. Furthermore, $W_1$ is diffeomorphic to $\#_n S^2 \times S^3$ (resp. $\#_n S^2 \times S^3$), if $r(X_1)$ is even (resp. odd).

This proposition leads to the following definition.

Definition 3.7 ([4]). For non-negative integers $n$ and $r$, we define $(S_{n,r}, \zeta_{n,r})$ as the closed contact 5-manifold supported by a contact open book $(X, \text{id})$, where $X$ is a 4-dimensional Stein handlebody without 1-handles satisfying $b_2(X) = n$ and $r(X) = r$.

Remark 3.8. (1) By Proposition 3.6, $(S_{n,r}, \zeta_{n,r})$ and $(S_{n',r'}, \zeta_{n',r'})$ are contactomorphic to each other if and only if $n = n'$ and $r = r'$. Furthermore, $S_{r,n}$ is diffeomorphic to $\#_n S^2 \times S^3$ (resp. $\#_n S^2 \times S^3$), if $r$ is even (resp. odd).

(2) By Theorem 3.1 and Remark 3.2, any closed contact 5-manifold admitting a subcritical Stein filling without 1-handles is contactomorphic to some $(S_{n,r}, \zeta_{n,r})$.

Here we observe that the contact 5-manifold supported by a contact open book can admit a subcritical Stein filling without 1-handles, even if the page has 1-handles. Let us recall that a handlebody determines a presentation of its fundamental group: 1- and 2-handles correspond to the generators and relators, respectively (cf. [17]).

Lemma 3.9 (cf. Ding-Geiges-van Koert [9]). Let $X$ be a 4-dimensional Stein handlebody. If all the generators of the presentation of $\pi_1(X)$ given by the handlebody structure are removed by Andrews-Curtis moves, then the contact 5-manifold supported by the contact open book $(X, \text{id})$ admits a subcritical Stein filling without 1-handles.

For the definition of Andrews-Curtis moves, see [9]. The proof of this lemma is straightforward from Section 6 in [9]: one can alter the page $X$ so that it has no 1-handles, preserving the supporting contact 5-manifold. It seems still unknown whether there exists a finite presentation of the trivial group that do not satisfy the assumption of the above lemma.

Question 3.10. Can every finite presentation of the trivial group be changed into a presentation having no generators by Andrews-Curtis moves?

According to the Andrews-Curtis conjecture [5], this question is affirmative for any balanced presentation of the trivial group, but there are potential counterexamples to the conjecture (cf. [17]).

4. Smooth structures on Stein 4-manifolds

In this section, we first introduce the intersection genus, which is a simple but effective genus invariant of smooth 4-manifolds. Using this invariant, we prove Theorems 1.3 and 1.1. We also show the finiteness of the number of the first Chern classes of Stein structures on a given compact smooth 4-manifold (Proposition 4.6).
4.1. Intersection genera of 4-manifolds. For a smooth 4-manifold $Z$, the minimal genus function $g_Z : H_2(Z; \mathbb{Z}) \to \mathbb{Z}$ (cf. [17]) is defined by

$$g_Z(\alpha) = \min \{ g | \alpha \text{ is represented by a smoothly embedded surface of genus } g \}.$$ 

Minimal genus functions have useful informations on smooth structures, but it is hard to determine the values of the functions. Indeed the functions have been determined for very few 4-manifolds (e.g. $\mathbb{CP}^2$). Also, it is generally difficult to distinguish smooth structures by the functions, since there are many ways to identify second homology groups of two distinct 4-manifolds.

To avoid these difficulties, the author [26] (cf. [27]) introduced the relative genus function, which is a genus invariant (i.e. a diffeomorphism invariant determined by the minimal genus function on a 4-manifold). Subsequently, Gompf [16] introduced a different genus invariant called the genus rank function. Here we introduce yet another genus invariant, which is simpler than these invariants but more useful for proving Theorem 1.3.

Let $Z$ be an oriented smooth (possibly non-closed) 4-manifold with $0 < b_2 < \infty$. For simplicity we assume $H_2(Z; \mathbb{Z})$ has no torsion, though we can similarly define the invariant when $H_2(Z; \mathbb{Z})$ has torsion. We put $n = b_2(Z)$.

Definition 4.1. For an ordered basis $v = \{v_1, v_2, \ldots, v_n\}$ of $H_2(Z; \mathbb{Z})$, we define a non-negative integer $G_Z(v)$ by

$$G_Z(v) = \max \{ g_Z(v_i) | 1 \leq i \leq n \}.$$ 

For an intersection matrix $Q$ of $Z$ (i.e. a matrix representing the intersection form of $Z$), we then define a non-negative integer $G_{Z,Q}$ by

$$G_{Z,Q} = \min \{ G_Z(v) | v \text{ is an ordered basis of } H_2(Z; \mathbb{Z}) \text{ whose intersection matrix is } Q \}.$$ 

We call $G_{Z,Q}$ the intersection genus of $Z$ with respect to $Q$ (Q-genus of $Z$ for short). It is straightforward to see that the Q-genus is a diffeomorphism invariant of 4-manifolds for any fixed intersection matrix $Q$. Namely, if 4-manifolds $Z$ and $Z'$ are orientation-preserving diffeomorphic to each other, then $G_{Z,Q} = G_{Z',Q}$ for any intersection matrix $Q$.

4.2. Coarsely distinguishing smooth structures. For an element $\alpha$ in a finitely generated abelian group $G$, we define the divisibility $d(\alpha)$ of $\alpha$ by

$$d(\alpha) = \begin{cases} \max \{ n \in \mathbb{Z} | [\alpha] = n\alpha' \text{ for some } \alpha' \in G/\text{Tor} \}, & \text{if } \alpha \text{ is not torsion;} \\ 0, & \text{if } \alpha \text{ is torsion.} \end{cases}$$ 

Note that $d(\alpha)$ is a non-negative integer.

Remark 4.2. We use this unnatural definition in order to relax the assumption of Theorem 1.3 and to obtain stronger estimates of intersection genera in the proof. This theorem also works under the natural definition below.

$$d(\alpha) = \begin{cases} \max \{ n \in \mathbb{Z} | \alpha = n\alpha' \text{ for some } \alpha' \in G \}, & \text{if } \alpha \text{ is not torsion;} \\ 0, & \text{if } \alpha \text{ is torsion}. \end{cases}$$ 

Here we prove Theorem 1.3 using intersection genera.

Theorem 1.3. For a given infinite family of compact Stein 4-manifolds, if the divisibilities of the first Chern classes of the members are pairwise distinct, then at least infinitely many members are pairwise non-diffeomorphic.
Proof. Let \( \{X_i \mid i \in \mathbb{N}\} \) be an infinite family of compact Stein 4-manifolds in the assumption of this theorem, and let \( d_i \) be the divisibility of \( c_1(X_i) \). Without loss of generality, we may assume that \( \{d_i\}_{i \in \mathbb{N}} \) is a strictly increasing sequence. It suffices to prove the theorem in the case where all the members have the same intersection form. We put \( n = b_2(X_1) \) and fix an intersection matrix \( Q \) of \( X_1 \). The assumption on the divisibilities guarantees \( n \geq 1 \). Note that each \( H_2(X_i; \mathbb{Z}) \) has no torsion, since any Stein 4-manifold has a handle decomposition without 3- and 4-handles due to [10] (cf. [15]). Let \( m_1, m_2, \ldots, m_n \) be the diagonal components of \( Q \).

We here estimate the value, denoted by \( G_i \), of the \( Q \)-genus of each \( X_i \). Let \( v = \{v_1, v_2, \ldots, v_n\} \) be an ordered basis of \( H_2(X_1; \mathbb{Z}) \) whose intersection matrix is the aforementioned matrix \( Q \). Since \( d_i \) is the divisibility of \( c_1(X_1) \), one can check that the inequality \( \langle c_1(X_i), v_j \rangle \geq d_i \) holds for at least one \( j \) with \( 1 \leq j \leq n \). Applying the adjunction inequality to \( X_i \) and this \( v_j \), we see that

\[
d_i + m_j \leq 2g_{X_i}(v_j) - 2.
\]

Hence, for each \( i \), we obtain the estimate

\[
d_i + \min\{m_j \mid 1 \leq j \leq n\} \leq 2G_i - 2.
\]

Since \( \lim_{i \to \infty} d_i = \infty \), this estimate implies \( \lim_{i \to \infty} G_i = \infty \). Therefore, there exists a strictly increasing subsequence \( \{G_{i_k}\}_{k \in \mathbb{N}} \) of \( \{G_i\}_{i \in \mathbb{N}} \). Since the \( Q \)-genus \( G_i \) is a diffeomorphism invariant of \( X_i \), the members of the infinite subfamily \( \{X_{i_k} \mid k \in \mathbb{N}\} \) are pairwise non-diffeomorphic.

Remark 4.3. This proof gives a simple method for obtaining lower bounds of intersection genera of given compact Stein 4-manifolds, since one can calculate their first Chern classes from their Stein handlebody structures ([15]) and Lefschetz fibration structures ([11]). Also, one can give upper bounds of intersection genera by using their smooth handlebody structures. Thus we can often use intersection genera to distinguish the smooth structures of two given Stein 4-manifolds. Indeed, this argument effectively works for many examples of exotic Stein 4-manifolds obtained in [22] and [27].

We can now easily prove Theorem 1.1.

Theorem 1.1. Let \( \{X_i \mid i \in \mathbb{N}\} \) be an infinite family of compact Stein 4-manifolds such that, for each \( i \), the contact 5-manifold \((W_i, \xi_i)\) supported by the contact open book \((X_i, \text{id})\) admits a subcritical Stein filling without 1-handles. If the members of \( \{(W_i, \xi_i) \mid i \in \mathbb{N}\} \) are pairwise non-contactomorphic, then at least infinitely many members of \( \{X_i \mid i \in \mathbb{N}\} \) are pairwise non-diffeomorphic.

Proof. It suffices to prove the case where all of \( H^2(X_i; \mathbb{Z})'s \) are pairwise isomorphic. We here observe that each \( X_i \) is simply connected. Clearly \( W_i \) is the boundary of \( X_i \times D^2 = (X_i \times D^1) \times D^1 \). It thus follows that each \( W_i \) is diffeomorphic to the double \( (X_i \times D^1) \cup_{X_i \times D^1} X_i \times D^1 \) of \( X_i \times D^1 \). Since any Stein 4-manifold admits a handle decomposition without 3- and 4-handles, one can see that \( \pi_1(W_i) \) is isomorphic to \( \pi_1(X_i \times D^1) \cong \pi_1(X_i) \). Since \( (W_i, \xi_i) \) admits a simply connected subcritical Stein filling, Remark 4.2 shows that \( (W_i, \xi_i) \) is supported by the contact open book \((Y_i, \text{id})\) for some simply connected Stein handlebody \( Y_i \). Note that \( \pi_1(Y_i) \) is isomorphic to \( \pi_1(W_i) \cong \pi_1(X_i) \). Therefore, \( X_i \) is simply connected.

By Theorem 4.3 and Lemma 3.3 we see that, for any \( i \neq j \), there exists no isomorphism between \( H^2(X_i; \mathbb{Z}) \) and \( H^2(X_j; \mathbb{Z}) \) that maps \( c_1(X_i) \) to \( c_1(X_j) \). Since
$X_i$ is simply connected, the universal coefficient theorem tells that $H^2(X_i; \mathbb{Z})$ has no torsion. We thus easily see that the divisibilities of $c_1(X_i)$ and $c_1(X_j)$ are not equal to each other. Therefore Theorem 4.3 tells that at least infinitely many of $X_i$’s are pairwise non-diffeomorphic. □

**Remark 4.4.** As seen from this proof, for a compact Stein 4-manifold $X$, if the contact 5-manifold $(W, \xi)$ supported by the contact open book $(X, \text{id})$ admits a simply connected subcritical Stein filling, then $X$ is simply connected. Van Koert kindly pointed out the following generalization: for a contact 5-manifold $(W, \xi)$ supported by a contact open book $(X, \text{id})$, any Stein filling $V$ of $(W, \xi)$ satisfies $\pi_1(V) \cong \pi_1(X)$. To see this, we note that $\pi_1(V) \cong \pi_1(\partial V)$ holds for any compact Stein manifold $V$ of real dimension at least 6 (Lemma 3.1 in [23]). Applying this lemma twice, we see

$$\pi_1(V) \cong \pi_1(W) \cong \pi_1(X \times D^2) \cong \pi_1(X),$$

and thus the above generalization follows.

The obvious corollary below says that a given compact smooth 4-manifold with boundary induces at most finitely many contact 5-manifolds admitting subcritical Stein fillings without 1-handles by contact open books with the identity monodromy.

**Corollary 4.5.** For any simply connected compact oriented smooth 4-manifold $X$ with boundary, there are at most finitely many contact 5-manifolds, up to contactomorphism, admitting subcritical Stein fillings without 1-handles such that each of them is supported by a contact open book $(Y, \text{id})$ for a compact Stein 4-manifold $Y$ diffeomorphic to $X$.

It would be natural to ask whether the condition ‘admitting subcritical Stein fillings without 1-handles’ can be removed from this corollary. If Question 6.10 has an affirmative answer, then this question is also affirmative due to Lemma 3.9.

**4.3. Finiteness of the number of the first Chern classes.** Here we prove the following proposition.

**Proposition 4.6.** For any compact connected oriented smooth 4-manifold $X$ with boundary, there are at most finitely many classes of $H^2(X; \mathbb{Z})$ that are the first Chern classes of Stein structures on $X$.

**Proof.** If $b_2(X) = 0$, then $H^2(X; \mathbb{Z})$ is a finite set, implying the proposition. We thus assume $b_2(X) > 0$. Assume further that $X$ admits a Stein structure. By [10] (cf. [25, 17]), $X$ admits a Stein handlebody decomposition. Note that this handlebody has neither 3- nor 4-handles.

We consider the CW complex given by the handlebody and discuss its cohomology and homology groups. Let $C_i(X)$ denote the $i$-th chain group, and let $\partial_i : C_i(X) \to C_{i-1}(X)$ be the boundary homomorphism. By changing bases, we have a basis $v_1, v_2, \ldots, v_n$ of $C_2(X)$, a basis $u_1, u_2, \ldots, u_l$ of $C_1(X)$, and non-zero integers $a_{m+1}, \ldots, a_n$ such that

$$\partial_2(v_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq m; \\ a_i u_i, & \text{if } m + 1 \leq i \leq n. \end{cases}$$

We thus see that $[v_1], [v_2], \ldots, [v_m]$ is a basis of $H_2(X; \mathbb{Z})$. We note that the assumption $b_2(X) \geq 1$ guarantees $m \geq 1$. Here let $C^*_i(X)$ denote the $i$-th cochain group, and let $v_i^* : C_2(X) \to \mathbb{Z}$ be the dual of $v_i$ for each $i$. Clearly $\{[v_1^*], [v_2^*], \ldots, [v_n^*]\}$
is a generating set of $H^2(X;\mathbb{Z})$. Note that, for $m + 1 \leq i \leq n$, the class $[v_i^*]$ is a torsion element of order $|a_i|$.

Now let $c$ be the first Chern class of a Stein structure on $X$. Then there exist integers $c_1, c_2, \ldots, c_n$ such that $c = c_1[v_1^*] + c_2[v_2^*] + \cdots + c_n[v_n^*]$. We may assume that $|c_i| \leq a_i$ for $m + 1 \leq i \leq n$, since $[v_i^*]$ is of order $|a_i|$. Let $g_i$ be the genus of a fixed smoothly embedded surface in $X$ representing the second homology class $[v_i]$. By applying the adjunction inequality to the first Chern class $c$ and the second homology class $[v_i]$, we obtain $|c_i| + [v_i] \cdot [v_i] \leq 2g_i - 2$ for $1 \leq i \leq m$. Therefore, only finitely many integers can be the value of $c_i$ for each $1 \leq i \leq n$. This shows that there are at most finitely many classes of $H^2(X;\mathbb{Z})$ that can be the first Chern classes of Stein structures on $X$.

This proposition gives an alternative proof of Theorem 1.3 by the pigeonhole principle. However, the previous proof is much more practical, since this alternative proof does not tell how to distinguish smooth structures of given two Stein 4-manifolds in the infinite family.

**Remark 4.7.** For a compact oriented smooth 4-manifold $X$ with boundary, let $N_C(X)$ denote the number of the first Chern classes of Stein structures on $X$. This is clearly a diffeomorphism invariant of $X$. The proof of Proposition 4.6 tells how to give an upper bound of $N_C(X)$, and one can give a lower bound of $N_C(X)$ by finding Stein structures on $X$ (e.g., Stein handlebodies and Lefschetz fibrations). For example, using the algorithm obtained in [27], we can construct infinite families of pairwise exotic 4-manifolds which can be distinguished by $N_C$ (e.g., exotic 4-manifolds used in Theorem 5.2 of this paper). However, compared with the intersection genus, we need more efforts for obtaining upper and lower bounds of $N_C$, and thus the intersection genus is more practical and useful for coarsely distinguishing smooth structures.

5. **Examples**

In this section, we construct two examples motivated by Theorem 1.1. The first example below tells that ‘at least infinitely many members’ in the conclusion of Theorem 1.1 cannot be strengthened to ‘all members’ without additional constraints.

**Proposition 5.1.** Let $n \geq 2$ be a positive integer, and let $(W_1, \xi_1)$, $(W_2, \xi_2)$, $\ldots$, $(W_n, \xi_n)$ be pairwise non-contactomorphic closed contact 5-manifolds admitting subcritical Stein fillings without 1-handles. If they are pairwise diffeomorphic smooth 5-manifolds, then there exist pairwise diffeomorphic compact Stein 4-manifolds $Z_1, Z_2, \ldots, Z_n$ such that each contact open book $(Z_i, \text{id})$ supports $(W_i, \xi_i)$.

**Proof.** By Remark 5.2 we see that there exist a positive integer $k$ and non-negative integers $r_1, r_2, \ldots, r_n$ such that each $(W_i, \xi_i)$ is contactomorphic to $(S^2_{r_i}, \zeta_{k, r_i})$. Note that all of the $r_i$’s have the same parity, since $W_i$’s are pairwise diffeomorphic. Let us fix a smooth knot $K$ in $S^3$ satisfying the following condition: for each $1 \leq i \leq n$, the knot $K$ has a Legendrian representative $L_i$ whose rotation number is $r_i$ and whose Thurston-Bennequin number is a fixed number $r_1 + 1$ (For example, the $(p, 2)$-torus knot with a sufficiently large $p$ is such an example of $K$). Using this Legendrian representative, we define a Stein 4-manifold $X_i$ as the Stein handlebody obtained from $D^4$ by attaching a 2-handle along $L_i$. Note that the framing $r_1$ is
independent of $i$. We also use a Stein handlebody $Y_{k-1}$ without 1-handles satisfying $b_2(Y_{k-1}) = k - 1$ and $r(Y_{k-1}) = 0$. One can easily find such a $Y_{k-1}$. Now let $Z_i$ be the boundary connected sum $X_i \# Y_{k-1}$, which is a Stein handlebody. Note that $Z_i$'s are pairwise diffeomorphic. Since $r(Z_i) = r_i$ and $b_2(Z_i) = k$, the contact 5-manifold $(S_{k,r_i}, \zeta_{k,r_i})$ is supported by the contact open book $(Z_i, \text{id})$. Therefore the claim follows.

The next example below tells that there are many examples of infinitely many pairwise exotic Stein 4-manifolds whose smooth structures are distinguished by Theorem 1.1 Let us recall Remark 3.3. Any closed contact 5-manifold with $b_2 = n$ admitting a subcritical Stein filling without 1-handles is diffeomorphic to either $#_n S^2 \times S^3$ or $#_n S^2 \tilde{\times} S^3$. Furthermore, the set of contactomorphism classes of such contact structures on $#_n S^2 \times S^3$ (resp. $#_n S^2 \tilde{\times} S^3$) is given by $\{ \zeta_{2r} \mid r \in \mathbb{Z}_{\geq 0} \}$ (resp. $\{ \zeta_{2r+1} \mid r \in \mathbb{Z}_{\geq 0} \}$), where $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers.

**Theorem 5.2** (cf. [3]). For each fixed integer $n \geq 2$, there exists an infinite family $\{ Z_{n,2r} \mid r \in \mathbb{Z}_{\geq 0} \}$ of pairwise homeomorphic compact Stein 4-manifolds such that each contact open book $(Z_{n,2r}, \text{id})$ supports $(#_n S^2 \times S^3, \zeta_{2r})$. (Hence, by Theorem 1.1, at least infinitely members are pairwise homeomorphic but non-diffeomorphic.) Furthermore, the same claim also holds for $\{ (#_n S^2 \tilde{\times} S^3, \zeta_{2r+1}) \mid r \in \mathbb{Z}_{\geq 0} \}$.

**Proof.** The following 4-manifolds and Stein handlebodies were obtained in [3], [27] and [4], and this proof is based on the proof of Theorem 1.3 in [4].

For a non-negative integer $p$, let $X_p$ be the smooth 4-manifold given by the handlebody diagram in Figure 1 where the box $p$ denotes $p$ right-handed full twists. We note that each $X_p$ is obtained from $X_0$ by a log transform with multiplicity one. By canceling the 1-handles, we have the diagrams of $X_p$ in Figure 2. By isotopy, we obtain the Stein handlebody decomposition of $X_p$ in Figure 3. One can check that the rotation numbers of the attaching circles (Legendrian knots) of two 2-handles of $X_p$ are 0 and $p$ for each $p \geq 0$.

Now let $Y_{n-2}$ be a 4-dimensional Stein handlebody without 1-handles satisfying $b_2(Y_{n-2}) = n - 2$ and $r(Y_{n-2}) = 0$, and let $Z_{n,p}$ be the boundary connected sum $X_p \# Y_{n-2}$, which is a Stein handlebody. Since $r(Z_{n,p}) = p$ and $b_2(Z_{n,p}) = n$, we see that the contact open book $(Z_{n,p}, \text{id})$ supports the contact 5-manifold $(S_{n,p}, \zeta_{n,p})$.

Lastly we show that, for any non-negative integers $r$ and $r'$, 4-manifolds $Z_{n,2r}$ and $Z_{n,2r'}$ are homeomorphic to each other. We can easily check that the intersection forms of $X_{n,2r}$ and $X_{n,2r'}$ are even, unimodular and indefinite. By the classification of indefinite unimodular intersection forms (cf. [17]), we see that the intersection forms of $X_{n,2r}$ and $X_{n,2r'}$ are isomorphic. Since they are simply connected, and their boundaries are homeomorphic to each other, Freedman’s theorem (cf. [7]) tells that any diffeomorphism $\partial X_{n,2r} \to \partial X_{n,2r'}$ extends to a homeomorphism $X_{n,2r} \to X_{n,2r'}$. Since $Z_{n,2r}$ is obtained from $Z_{n,2r'}$ by removing $X_{n,2r}$ and gluing $X_{n,2r'}$, it follows that $Z_{n,2r}$ and $Z_{n,2r'}$ are homeomorphic to each other. Therefore the theorem holds for the family $\{ (#_n S^2 \times S^3, \zeta_{2r}) \mid r \in \mathbb{Z}_{\geq 0} \}$. By the same argument, we see that the theorem also holds for $\{ (#_n S^2 \tilde{\times} S^3, \zeta_{2r+1}) \mid r \in \mathbb{Z}_{\geq 0} \}$.

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Figure 1. $X_p$

Figure 2. The upper handlebody is $X_p$ for $p \geq 1$, and the lower one is $X_0$.

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Figure 3. The upper Stein handlebody is $X_p$ for $p \geq 1$, and the lower one is $X_0$.

Figure 4. Legendrian version of left-handed full twists

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