A geometric approach to acyclic orientations

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Abstract

The set of acyclic orientations of a connected graph with a given sink has a natural poset structure. We give a geometric proof of a result of Jim Propp: this poset is the disjoint union of distributive lattices.

Let $G$ be a connected graph on the vertex set $[n] = \{0\} \cup [n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$. Let $P$ denote the collection of acyclic orientations of $G$, and let $P_0$ denote the collection of acyclic orientations of $G$ with 0 as a sink. If $\Omega$ is an orientation in $P$ with the vertex $i$ as a source, we can obtain a new orientation $\Omega'$ with $i$ as a sink by firing the vertex $i$, reorienting all the edges adjacent to $i$ towards $i$. The orientations $\Omega$ and $\Omega'$ agree away from $i$.

A firing sequence from $\Omega$ to $\Omega'$ in $P$ consists of a sequence $\Omega = \Omega_1, \ldots, \Omega_{m+1} = \Omega'$ of orientations and a function $F: [m] \to [n]$ such that for each $i \in [m]$, the orientation $\Omega_{i+1}$ is obtained from $\Omega_i$ by firing the vertex $F(i)$. We will abuse language by calling $F$ itself a firing sequence. We make $P$ into a preorder by writing $\Omega \leq \Omega'$ if and only if there is a firing sequence from $\Omega$ to $\Omega'$. From the definition it is clear that $P$ is reflexive and transitive. While $P$ is only a preorder, $P_0$ is a poset. By finiteness, antisymmetry can be verified by showing that firing sequences in $P_0$ cannot be arbitrarily long. This is a consequence of the fact that neighbors of the distinguished sink 0 cannot fire. The proof depends on the following lemma.

Lemma 1. Let $F: [m] \to [n]$ be a firing sequence for the graph $G$. If $i$ and $j$ are adjacent vertices in $G$, then

$$|F^{-1}(i)| \leq |F^{-1}(j)| + 1.$$ 

Proof. A vertex can fire only if it is a source. Firing the vertex $i$ reverses the orientation of its edge to the vertex $j$. Hence the vertex $i$ cannot fire again until the orientation is again reversed, which can only happen by firing $j$. $\square$

As a corollary, firing sequences have bounded length, implying that $P_0$ is a poset.

Corollary 2. The preorder $P_0$ of acyclic orientations with a distinguished sink is a poset.

Proof. Let $F: [m] \to [n]$ be a firing sequence. By iterating the lemma, $|F^{-1}(i)| \leq d(0, i) - 1$, so

$$m = \sum_{i \in [n]} |F^{-1}(i)| \leq \sum_{i \in [n]} (d(0, i) - 1).$$

Hence firing sequences cannot be arbitrarily long, implying that $P_0$ is antisymmetric. $\square$

For a real number $a$, let $[a]$ denote the largest integer less than or equal to $a$. Similarly, let $\lfloor a \rfloor$ denote the least integer greater than or equal to $a$. Finally, let $\{a\}$ denote the fractional part of the real number $a$, that is, $\{a\} = a - \lfloor a \rfloor$. (It will be clear from the context if $\{a\}$ denotes the fractional part or the singleton set.) Observe that the range of the function $x \mapsto \{x\}$ is the half open interval $[0, 1)$. 

1
Let \( \widetilde{H} = \widetilde{H}(G) \) be the periodic graphic arrangement of the graph \( G \), that is, \( \widetilde{H} \) is the collection of all hyperplanes of the form
\[
x_i = x_j + k,
\]
where \( ij \) is an edge in the graph \( G \) and \( k \) is an integer. This hyperplane arrangement cuts \( \mathbb{R}^{n+1} \) into open regions. Note that each region is translation-invariant in the direction \((1, \ldots, 1)\). Let \( C \) denote the complement of \( \widetilde{H} \), that is,
\[
C = \mathbb{R}^{n+1} \setminus \bigcup_{H \in \widetilde{H}} H.
\]
Define a map \( \varphi : C \rightarrow P \) from the complement of the periodic graphic arrangement to the preorder of acyclic orientations as follows. For a point \( x = (x_0, \ldots, x_n) \) and an edge \( ij \) observe that \( \{x_i\} \neq \{x_j\} \) since the point does not lie on any hyperplane of the form \( x_i = x_j + k \). Hence orient the edge \( ij \) towards \( i \) if \( \{x_i\} < \{x_j\} \) and towards \( j \) if the inequality is reversed. This defines the orientation \( \varphi(x) \).

Also note that this is an acyclic orientation, since no directed cycles can occur.

Let \( H_0 \) be the coordinate hyperplane \( \{x \in \mathbb{R}^{n+1} : x_0 = 0\} \). The map \( \varphi \) sends points of the intersection \( C_0 = C \cap H_0 \) to acyclic orientations in \( P_0 \).

The real line \( \mathbb{R} \) is a distributive lattice; meet is minimum and join is maximum. Since \( \mathbb{R}^{n+1} \) is a product of copies of \( \mathbb{R} \), it is also a distributive lattice, with meet and join given by componentwise minimum and maximum. That is, given two points in \( \mathbb{R}^n \), say \( x = (x_0, \ldots, x_n) \) and \( y = (y_0, \ldots, y_n) \), their meet and join are given by
\[
x \wedge y = (\min(x_0, y_0), \ldots, \min(x_n, y_n))
\]
and
\[
x \vee y = (\max(x_0, y_0), \ldots, \max(x_n, y_n))
\]
respectively.

**Lemma 3.** Each region \( R \) in the complement \( C \) of the periodic graphic arrangement \( \widetilde{H} \) is a distributive sublattice of \( \mathbb{R}^{n+1} \). Hence the intersection \( R \cap H_0 \), which is a region in \( C_0 \), is also a distributive sublattice of \( \mathbb{R}^{n+1} \).

**Proof.** Since each region \( R \) is the intersection of slices of the form
\[
T = \{x \in \mathbb{R} : x_i + k < x_j < x_i + k + 1\},
\]
it is enough to prove that each slice is a sublattice of \( \mathbb{R}^{n+1} \). Let \( x \) and \( y \) be two points in the slice \( T \). Then \( \min(x_i, y_i) + k = \min(x_i + k, y_i + k) < \min(x_i + k + 1, y_i + k + 1) = \min(x_i, y_i) + k + 1 \), implying that \( x \wedge y \) also lies in the slice \( T \). A dual argument shows that the slice \( T \) is closed under the join operation. Thus the region \( R \) is a sublattice. Since distributivity is preserved under taking sublattices, it follows that \( R \) is a distributive sublattice of \( \mathbb{R}^{n+1} \).

In the remainder of this paper we let \( R \) be a region in \( C_0 \).

**Lemma 4.** Consider the restriction \( \varphi|_R \) of the map \( \varphi \) to the region \( R \). The inverse image of an acyclic orientation in \( P_0 \) is of the form:
\[
R \cap \left( \{0\} \times \prod_{i=1}^n (a_i, a_i + 1) \right),
\]
where each \( a_i \) is an integer. That is, the inverse image of an orientation is the intersection of the region \( R \) with a half-open lattice cube. Hence the inverse image is a sublattice of \( \mathbb{R}^{n+1} \).
Proof. Assume that \( x \) and \( y \) lie in the region \( R \). Define the integers \( a_i \) and \( b_i \) by \( a_i = \lfloor x_i \rfloor \) and \( b_i = \lfloor y_i \rfloor \). Hence the coordinate \( x_i \) lies in the half-open interval \([a_i, a_i + 1)\) and the coordinate \( y_i \) lies in the half-open interval \([b_i, b_i + 1)\). Lastly, assume that \( \varphi |_R \) maps \( x \) and \( y \) to the same acyclic orientation. The last condition implies that, for every edge \( ij \), \( 0 \leq x_i - a_i < x_j - a_j < 1 \) is equivalent to \( 0 \leq y_i - b_i < y_j - b_j < 1 \). Consider an edge that is directed from \( j \) to \( i \). Since \( x \) and \( y \) both lie in the region \( R \), there exists an integer \( k \) such that \( x_i + k < x_j < x_i + k + 1 \) and \( y_i + k < y_j < y_i + k + 1 \). Now we have that \( a_j - a_i < x_j - x_i < k + 1 \). Furthermore, observe that \( x_j - a_j - 1 < 0 \leq x_i - a_i \). Hence \( a_j - a_i > x_j - x_i - 1 > k - 1 \). Since \( a_j - a_i \) is an integer, the two bounds implies that \( a_j - a_i = k \). By similar reasoning we obtain that \( b_j - b_i = k \).

Hence for every edge \( ij \) we know that \( a_j - a_i = b_j - b_i \). Since \( a_0 = b_0 = 0 \) and the graph \( G \) is connected we obtain that \( a_i = b_i \) for all vertices \( i \).

Lemma 5. The restriction \( \varphi |_R : R \to P_0 \) is a poset homomorphism, that is, for two points \( y \) and \( z \) in the region \( R \) such that \( y \leq z \) the order relation \( \varphi(y) \leq \varphi(z) \) holds.

Proof. Since the region \( R \) is convex, the line segment from \( y \) to \( z \) is contained in \( R \). Let a point \( x \) move continuously from \( y \) to \( z \) along this line segment and consider what happens with the associated acyclic orientations \( \varphi(x) \). Note that each coordinate \( x_i \) is non-decreasing. When the point \( x \) crosses a hyperplane of the form \( x_i = p \) where \( p \) is an integer, observe that the value \( \{ x_i \} \) approaches 1 and then jumps down to 0. Hence the vertex \( i \) switches from being a source to being a sink, that is, the vertex \( i \) fires.

Observe that two adjacent nodes \( i \) and \( j \) cannot fire at the same time, since the intersection of the two hyperplanes \( x_i = p \) and \( x_j = q \) is contained in the hyperplane \( x_i = x_j + (p - q) \) which is not in the region \( R \).

Hence we obtain a firing sequence from the acyclic orientation \( \varphi(y) \) to \( \varphi(z) \), proving that \( \varphi(y) \leq \varphi(z) \).

Lemma 6. Let \( x \) be a point in the region \( R \). Let \( \Omega' \) be an acyclic orientation comparable to \( \Omega = \varphi(x) \) in the poset \( P_0 \). Then there exists a point \( z \) in the region of \( R \) as \( x \) such that \( \varphi(z) = \Omega' \).

Proof. It is enough to prove this for cover relations in the poset \( P \). We begin by considering the case when \( \Omega' \) covers \( \Omega \) in \( P \). Thus \( \Omega' \) is obtained from \( \Omega \) by firing a vertex \( i \).

First pick a positive real number \( \lambda \) such that \( \{ x_j \} < 1 - \lambda \) for each nonzero vertex \( j \). Let \( y \) be the point \( y = x + \lambda \cdot (0, 1, \ldots, 1) \). Observe that \( y \) belongs to the same region \( R \) and that \( \varphi \) maps \( y \) to the same acyclic orientation as the point \( x \).

Since \( i \) is a source in \( \Omega \), the value \( \{ y_i \} \) is larger than any other value \( \{ y_j \} \) for vertices \( j \) adjacent to the vertex \( i \). Let \( z \) be the point with coordinates \( z_j = y_j \) for \( j \neq i \) and \( z_i = \lceil y_i \rceil + \lambda / 2 \). Observe that moving from \( y \) to the point \( z \) we do not cross any hyperplanes of the form \( x_i = x_j + k \). Hence the point \( z \) also belongs to region \( R \).

However, we did cross a hyperplane of the form \( x_i = p \), corresponding to firing the vertex \( i \). Hence we have that \( \varphi(z) = \Omega' \). Now we can iterate this argument to extend to the general case when \( \Omega < \Omega' \).

The case when \( \Omega' \) is covered by \( \Omega \) is done similarly. However this case is easier since one can skip the middle step of defining the point \( y \). Hence this case is omitted.

A connected component of a finite poset is a weakly connected component of its associated comparability graph. That is, a finite poset is the disjoint union of its connected components.

Lemma 7. Let \( Q \) be a connected component of the poset of acyclic orientations \( P_0 \). Then there exists a region \( R \) in \( C_0 \) such that the map \( \varphi \) maps \( R \) onto the component \( Q \).
Proof. Let \( Q \) be an orientation in the component of \( Q \). Since \( \varphi \) is surjective we can lift \( \varphi \) to a point \( x \) in \( C_0 \). Say that the point \( x \) lies in the region \( R \). It is enough to show that every orientation \( \Omega \) in \( Q \) can be lifted to a point in \( R \). The two orientations \( \Omega \) and \( \Omega' \) are related by a sequence in \( Q \) of orientations \( \Omega = \Omega_1, \Omega_2, \ldots, \Omega_k = \Omega' \) such that \( \Omega_i \) and \( \Omega_{i+1} \) are comparable. By iterating Lemma \( \ref{lem:lift} \) we obtain points \( x_i \) in \( R \) such that \( \varphi(x_i) = \Omega_i \). In particular, \( \varphi(x_k) = \Omega' \).

**Proposition 8.** Let \( Q \) be a connected component of the poset of acyclic orientations \( P_0 \). Then the component \( Q \) as a poset is a lattice. Moreover, let \( R \) be a region of \( C_0 \) that maps onto \( Q \) by \( \varphi \). Then the poset map \( \varphi|_R : R \rightarrow Q \) is a lattice homomorphism.

Proof. The previous discussion showed that we can lift the component \( Q \) to a region \( R \). Consider two acyclic orientations \( \Omega \) and \( \Omega' \). We can lift them to two points \( x \) and \( y \) in \( R \), that is, \( \varphi(x) = \Omega \) and \( \varphi(y) = \Omega' \). Since \( \varphi|_R \) is a poset map we obtain that \( \varphi(x \land y) \) is a lower bound for \( \Omega \) and \( \Omega' \). It remains to show that the lower bound is unique.

Assume that \( \Omega'' \) is a lower bound of \( \Omega \) and \( \Omega' \). By Lemma \( \ref{lem:lift} \) we can lift \( \Omega'' \) to an element \( z \) in \( R \) such that \( z \leq x \). Similarly, we can lift \( \Omega'' \) to an element \( w \) in \( R \) such that \( w \leq y \). That is we have that \( \varphi(z) = \varphi(w) = \Omega'' \). Now by Lemma \( \ref{lem:lift} \) we have that \( \varphi(z \land w) = \Omega'' \). But since \( z \land w \) is a lower bound of both \( x \) and \( y \) we have that \( z \land w \leq x \land y \). Now applying \( \varphi \) we obtain that \( \varphi(x \land y) \) is the greatest lower bound, proving that the meet is well-defined. A dual argument shows that the join is well-defined, hence \( Q \) is a lattice.

Finally, we have to show that \( \varphi|_R \) is a lattice homomorphism. Let \( x \) and \( y \) be two points in the region \( R \). By Lemma \( \ref{lem:lift} \) we can lift the inequality \( \varphi(x) \land \varphi(y) \leq \varphi(x) \) to obtain a point \( z \) in \( R \) such that \( z \leq x \) and \( \varphi(z) = \varphi(x) \land \varphi(y) \). Similarly, we can lift the inequality \( \varphi(x) \land \varphi(y) \leq \varphi(y) \) to obtain a point \( w \) in \( R \) such that \( w \leq y \) and \( \varphi(w) = \varphi(x) \land \varphi(y) \). By Lemma \( \ref{lem:lift} \) we know that \( \varphi(z \land w) = \varphi(x) \land \varphi(y) \). But \( z \land w \) is a lower bound of both \( x \) and \( y \), so \( \varphi(x) \land \varphi(y) = \varphi(z \land w) \leq \varphi(x \land y) \). But since \( \varphi(x \land y) \) is a lower bound of both \( \varphi(x) \) and \( \varphi(y) \) we have \( \varphi(x \land y) \leq \varphi(x) \land \varphi(y) \). Thus the map \( \varphi|_R \) preserves the meet operation. The dual argument proves that \( \varphi|_R \) preserves the join operation, proving that it is a lattice homomorphism.

Combining these results we can now prove the result of Propp \( \ref{thm:connected-component} \).

**Theorem 9.** Each connected component of the poset of acyclic orientations \( P_0 \) is a distributive lattice.

Proof. It is enough to recall that \( \mathbb{R}^{n+1} \) is a distributive lattice and each region \( R \) is a sublattice. Furthermore, the image under a lattice morphism of a distributive lattice is also distributive.

Observe that the minimal element in each connected component \( Q \) is an acyclic orientation with the unique sink at the vertex 0. Greene and Zaslavsky \( \ref{GZ} \) proved that the number of such orientations is given by the sign \( -1 \) to the power one less than the number of vertices times the linear coefficient in the chromatic polynomial of the graph \( G \). Gebhard and Sagan gave several proofs of this result \( \ref{GS} \). A geometric proof of this result can be found in \( \ref{GZ} \), where the authors view the graphical hyperplane arrangement on a torus and count the regions on the torus.

That the connected components are confluent, that is, each pair of elements has a lower and an upper bound, can also be shown by analyzing chip-firing games \( \ref{GZ} \). Is there a geometric way to prove the confluence of chip-firing? More discussions relating these distributive lattice with chip-firing can be found in \( \ref{GZ} \).
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