Relativistic quantum theories and neutrino oscillations

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Abstract
Neutrino oscillations are examined under the broad requirements of Poincaré-invariant scattering theory in an \(S\)-matrix formulation. This approach can be consistently applied to theories with either field or particle degrees of freedom. The aim of this paper is to use this general framework to identify all of the unique physical properties of this problem that lead to a simple oscillation formula. We discuss what is in principle observable and how many factors that are important in principle end up being negligible in practice.

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1. Introduction

There is now strong experimental evidence that neutrino flavor eigenstates are mixed via a non-diagonal matrix that connects them to neutrino mass eigenstates [1, 2]. The fact that space–time propagation is governed by mass rather than flavor eigenstates gives rise to the possibility of oscillation between flavor states over space and time.

There is a large body of literature addressing various aspects of the quantum mechanics of neutrino oscillation. Many of the primary issues were set forth by Kayser [3] in 1981. Since then, this subject has been examined from a variety of perspectives [4–16].

The phenomenon of neutrino oscillations seems simple enough from the perspective of introductory quantum mechanics, for which there are countless examples of oscillations in two- and three-level systems. There are, however, differences in the neutrino case that tend to work against the intuition of the standard examples: the initial conditions for neutrino production in reactors or the Sun are generally not controlled; the coherence length over which states are virtual can be kilometers rather than subatomic distances, and this affects considerations such as what is large or small in a calculation; one of the weak interactions that determine the oscillation interval takes place inside the volume of the detector rather than far from it; the neutrino kinematics are ultrarelativistic, but not fully so. There is also the distinction between what is measured in an experiment and features of a specific theoretical approach. These issues have led to a variety of approaches as cited above. Some of these approaches have differing perspectives [12, 14], yet all lead to the same simple oscillation formula.

The goal of this paper is to summarize in compact form the most general principles and the necessary physical conditions that lead to the standard neutrino oscillation formula.

Firstly, we provide an approach to neutrino oscillations from the perspective of scattering theory in relativistic quantum mechanics. The oscillation formula is a ratio of probabilities that can be obtained in turn via the associated scattering probability amplitudes, i.e. the \(S\)-matrix. The virtue of this approach is its generality and its focus on what can actually be measured in oscillation experiments. The \(S\)-matrix approach discussed in this paper has been examined [17, 18] using quantum field theory, but the conclusions that follow from our approach are not limited to a field theoretic treatment. Our goal here is to provide a very general framework utilizing the \(S\)-matrix and the general requirements of relativity and quantum mechanics.

Secondly, we discuss the necessary physical conditions that permit us to use this general framework to obtain the oscillation formula. In the process, we identify what is, or is not, essential for oscillations to occur.

There are now many papers in the literature devoted to neutrino oscillations that touch upon many of the points addressed in this paper. Our purpose is to cast the problem into a more general framework and to enumerate the physical conditions in compact form.
2. Poincaré-invariant quantum mechanical scattering

It is well known [3] that the standard neutrino oscillation formula depends on both relativity (very low mass neutrino mass eigenstates) and quantum mechanics (interference leading to oscillation). For oscillation phenomena in which only the lepton sector participates, a very convenient way to obtain oscillations is to employ electroweak field theory. For oscillation phenomena involving nucleons or nuclei, a field theory requires extra care since the strong interaction is non-perturbative. In the following discussion, we show that a standard second-order perturbation expression results from the basic properties of relativity and quantum mechanics, whether or not one uses a field theory.

We present here an overview of the basic ingredients of quantum mechanical scattering that satisfy relativistic invariance. Further details can be found in [19].

2.1. Relativistic invariance

Relativistic invariance in a quantum theory means that a change of the inertial coordinate system is a symmetry of the theory. In special relativity, inertial coordinate systems are related by space–time translations and proper orthochronous Lorentz transformations. The group generated by these transformations is the Poincaré group. Proper orthochronous Lorentz transformations do not include the discrete transformations associated with space reflection and time reversal, which are broken by the weak interaction. Wigner proved [20] that the existence of a unitary representation, \( U(\Lambda, a) \) (where \( \Lambda \) represents a Lorentz transformation and \( a \) a space–time shift), of the Poincaré group is both a necessary and a sufficient condition for a quantum theory to be relativistically invariant.

The Hamiltonian \( H \) contains interaction-dependent terms and generates time translations. Consistency of the initial value problem requires that interactions must also appear in additional Poincaré generators. This means that at most a subgroup of the Poincaré group can be independent of interactions. When such a subgroup exists, it is called a kinematic subgroup. The largest kinematic subgroups were classified by Dirac [21]. They are the three-dimensional Euclidean group (instant-form kinematics), the Lorentz group (point-form kinematics) and the subgroup that leaves a plane tangent to the light cone invariant (front-form kinematics). Time-ordered perturbation methods stemming from a field theory implicitly employ an instant-form representation, but the front form and the point form have also been frequently employed in the literature. While the existence of a kinematic subgroup is a consequence of the choice of representation of the dynamics, the choice of representation does not change the \( S \)-matrix. It follows that the choice of kinematic symmetries of the interactions have no impact on the observables of the theory; this is an example of a representation-dependent feature of an interaction that is not experimentally observable. In the context of neutrino physics, interactions in an instant form of the dynamics are invariant with respect to kinematic translations and thus conserve the total three-momentum. This conservation law does not hold in the other two forms of the dynamics, but the interactions in the different forms are related by unitary transformation that do not change the \( S \)-matrix, which means that the kinematic translational invariance of interactions in an instant form of the dynamics is not observable.

Expressions for the time translation of a quantum system may not exhibit manifest covariance, but the existence of a unitary representation of the Poincaré group implies that all physically observable quantities will have the correct transformation properties.

2.2. The \( S \)-matrix

Scattering theory in a Poincaré invariant quantum theory can be formulated using time-dependent methods, based on dynamical and asymptotic Hamiltonians, \( H \) and \( H_0 \).

Scattering states \( |\Psi(t)\rangle \) can be expressed in terms of a unitary transformation (time evolution) in terms of the full interaction:

\[
|\Psi_\pm(t)\rangle = e^{-iHt}|\Psi_\pm(0)\rangle,
\]

where the initial condition, \( |\Psi_\pm(0)\rangle \), is determined by an asymptotic condition. There are two natural scattering asymptotic conditions; they require that the state \( |\Psi_\pm(t)\rangle \) should approach a state of non-interacting particles, \( |\Phi_\pm(t)\rangle \), in the asymptotic future or past:

\[
\lim_{t \to \pm\infty} \| |\Psi_\pm(t)\rangle - |\Phi_\pm(t)\rangle \| = 0.
\]

The asymptotic waves have the form,

\[
|\Phi_\pm(t)\rangle = \Pi e^{-iH_0t}|\Phi_\pm(0)\rangle,
\]

where \( \Pi \) is a mapping from the Hilbert space of scattering asymptotes to the physical Hilbert space [22] that includes the internal structure of the asymptotic particles. This mapping is needed when one expresses the weak interaction as an operator in the space of particles that exist within a composite system (e.g. quarks in nucleons and nucleons in nuclei). In the particle case, \( \Pi \) is a projection operator, while in the field-theory case, \( \Pi \) is as constructed in [23].

Equations (1) and (2) imply that the initial conditions for the scattering states are related to the initial conditions for the non-interacting states by

\[
|\Psi_\pm(0)\rangle = \lim_{t \to \pm\infty} e^{iHt}\Pi e^{-iH_0t}|\Phi_\pm(0)\rangle := \Omega_\pm|\Phi_\pm(0)\rangle,
\]

which also serves to define the wave operators \( \Omega_\pm \).

The probability amplitude that a state prepared to become \( |\Phi_-\rangle \) in the asymptotic past scatters into a state that becomes \( |\Phi_+\rangle \) in the asymptotic future is

\[
\langle S \rangle = \langle \Psi_+\rangle|\Psi_-\rangle = \langle \Psi_+(0)|\Psi_-(0)\rangle = \langle \Phi_+(0)|S|\Phi_-(0)\rangle,
\]

where

\[
S := \Omega_+^\dagger\Omega_-
\]

is the scattering operator. We have exploited the invariance of the probability amplitudes under time translation to emphasize that the probability amplitude can be computed using states at any common time. We will make use of this to compute scattering amplitudes involving neutrino intermediate states.
The Poincaré invariance of the $S$-matrix is a consequence of the Poincaré invariance of the dynamical theory and the intertwining relations of the wave operators

$$ U(\Lambda, a)\Omega_+ = \Omega_+ U_0(\Lambda, a) \tag{7} $$

that relate the dynamical representation of the Poincaré group on the dynamical Hilbert space, $\mathcal{H}$, with the asymptotic representation on the asymptotic Hilbert space, $\mathcal{H}_0$.

This formalism can be used with Hamiltonians having any kinematic symmetry; for any Hamiltonian with a given kinematic symmetry, there exist equivalent Hamiltonians with different kinematic symmetries that give the same scattering operator.

### 2.3. Example

As an illustrative example we consider a reaction where an initial proton and electron interact to produce a linear combination of neutrino mass eigenstates, which propagate over a macroscopic distance until they are absorbed by a second interaction that produces a final proton and electron. In the language of second-order perturbation theory, this will involve two successive reactions, e.g.

$$ p + e \rightarrow n' + \nu, \tag{8a} $$

$$ n + \nu \rightarrow p' + e'. \tag{8b} $$

This example can be generalized to other cases of interest without affecting the overall conclusions, since the role of the intermediate neutrino mass eigenstates is the same. In this example, the asymptotic states are given by $\{p, n, e, p', n', e'\}$ and not the neutrino mass eigenstates, which are virtual.

For the cases of interest, reaction (8b) takes place inside a neutrino detector volume, and another device records that event via the emerging charged lepton. This is distinct from more typical applications in which the $S$-matrix describes one or more events that are distinctly separated from any detection equipment. The reaction (8a) takes place in a disjoint space–time region where the neutrino is produced.

The asymptotic free-particle state before the reaction is represented by a localized wave packet describing an electron $e$ moving toward the proton $p$ near the point $x_a$ where the neutrino is produced; in addition, there is also a free neutron $n$ that is traveling toward the point $x_a$ where it will eventually interact with the neutrino mass eigenstates.

Similarly, the asymptotic free-particle state after the reaction is represented by a localized wave packet describing a neutron $n'$ traveling away from the region where the neutrino was initially produced, and an electron $e'$ and proton $p'$ traveling away from the point where the neutrino mass eigenstate was absorbed by the initial neutron $n$.

There is a definite probability amplitude for a transition from the initial state containing $\{p, e, n\}$ to the final state containing $\{p', e', n'\}$. The unusual feature is that there is not a single localized space–time region where the initial and final states overlap. Because we only need to construct scattering states, $|\Psi_\pm(t)\rangle$, whose inner product can be evaluated at any single common time, the $S$-matrix can be computed by the same way that is used with more traditional asymptotic states.

### 2.4. Space–time translations

The scattering matrix can be calculated using standard methods by including the geometry of the neutrino experiment in the structure of the asymptotic states.

We illustrate the construction of the asymptotic states for the example discussed above. The reaction is characterized by two disjoint space–time regions localized about $x_a$, where the coherent neutrino mass eigenstate superposition is produced, and $x_b$, where it is absorbed. In our example the initial proton $p$ and electron $e$ and the final neutron $n'$ are localized near $x_a$ at a characteristic time $t_a$, and the initial neutron $n$ and the final proton $p'$ and electron $e'$ are localized near $x_b$ at a later characteristic time $t_b$. The role of the characteristic times $t_a$ and $t_b$ is discussed further below.

The asymptotic Hilbert space, $\mathcal{H}_0$, is the tensor product of single-particle Hilbert spaces. The construction of asymptotic states corresponding to this reaction starts with normalizable single-particle states that localize the particles near $x_a$ at time $t_a$ or $x_b$ at time $t_b$ with the appropriate expectation value for the initial or final momenta of the observed particles. In order to localize these states at the different points, $x_a$ and $x_b$, we initially localize them at the origin and then use single-particle space–time translations $U_j(I, x) = \exp(-i p_j \cdot x)$, where $p_j$ is the four-momentum operator of the $j$th asymptotic particle.

The initial and final asymptotic states have the form

$$ |\Phi_\pm(t)\rangle = \Pi U_0(t) e^{-i \sum p_a \cdot x_a} e^{-i \sum p_b \cdot x_b} |\Phi(0)\rangle, \tag{9} $$

which places the initial and final particles associated with neutrino production at time $t = t_a$ near $x_a$ and the initial and final particles associated with neutrino annihilation at time $t = t_b$ near $x_b$. The state $|\Phi(0)\rangle$ corresponds to a product of single-particle wave packets with each one centered near the origin with the appropriate expectation value of momentum. In what follows, we use the notation $S_{f_j}(x_b, x_a)$ for the $S$-matrix constructed from the asymptotic states (9). The points $x_a$ and $x_b$ do not correspond to the precise space–time location of each neutrino production or absorption event, but rather e.g. the location of the center of a reactor or a detector. They are, however, physical space–time points and transform as four-vectors under Lorentz transformations. The transformation property

$$ S_{f_j}(x_b, x_a) = S_{f_j}(\Lambda x_b, \Lambda x_a) \tag{10} $$

is a consequence of equation (7). Significant variations of space–time production/absorption locations about a chosen $x_a$ or $x_b$ will lead to a loss of coherence in the interference amplitudes. This is especially true of the time components and is discussed further below.

### 2.5. Second-order $S$-matrix

In this section, we examine the conditions that lead to neutrino oscillations in the $S$-matrix approach. Since the burden of providing a description of the experiment is taken up by the construction of the asymptotic states, the scattering operator or wave operators can be calculated using conventional methods. Initially we only assume that the weak interaction can be treated perturbatively.
In our example, the leading contribution to the scattering matrix is of second order in the weak interaction. As noted above, the initial weak interaction at the space–time point \( x_a \) produces a superposition of neutrino (or anti-neutrino) mass eigenstates, which propagate to the space–time point \( x_b \) of the final weak interaction. We assume that the initial particle(s) collide or decay near the space–time point \( x_a \), producing a neutrino or anti-neutrino and final particle(s). The neutrino or anti-neutrino travels and interacts with initial particle(s) near the space–time point \( x_b \) to produce final particles. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state.

\[
E_{ij} = \sqrt{m_{ij} - p^2_{ij}}.
\]

In addition, the following relations are specific to an interaction with the three-dimensional Euclidean group as a kinematic subgroup:

\[
\langle p'_{ha}, p_{ij} | V_{ja} | (p_{au}, p_{pu}) \rangle = \delta(p'_{ha} + p_{ij} - p_{au} - p_{pu})
\times \langle p'_{ha}, p_{ij} | V_{ja} | (p_{au}, p_{pu}) \rangle.
\]

Equation (11) is a generalization of the standard ‘two-potential’ formula of Gell-Mann and Goldberger [24] based on the Poincaré invariant quantum mechanics or quantum field theory where cluster properties have been used to factorize the incoming and outgoing scattering states into tensor products of independent scattering states associated with the reactions at \( x_a \) and \( x_b \). This is consistent with the assumption that the interaction, \( V_j \), which was treated to all orders, is short-ranged.

The neutrinos enter in the matrix elements

\[
\langle (p'_{ha}, p'_{bb}) \rangle^{\dagger} V_{ja} | (p_{ab}, p_{ij}) \rangle
\]

and

\[
\langle p'_{wa}, p_{ij} | V_{ja} | (p_{au}, p_{pu}) \rangle.
\]

The three-momentum delta function in equation (13) implies that in this representation the virtual neutrino three-momentum is constrained by the external kinematics:

\[
p_{ij} = p_{au} - p_{pa} - p_{ua}.
\]

The flavor of the initial and final states determines the flavor of the produced/detected neutrinos. These matrix elements, which involve initial and final hadrons, are linear in an elementary vertex that involves the neutrino mixing matrix.

It is useful to include explicitly the one-body space–time translation operators used in the construction of the non-interacting asymptotic states with the kernel of the scattering operator, \( S \). Using the four-momentum conservation of the second-order contribution to the operator \( S \) gives the expression

\[
\begin{align*}
&\epsilon^{\dagger}(p'_{ha} + p_{pp} - p_{ua}) (x_a - x_b) \langle p'_{ha}, p_{bb}, p_{ua} | (p_{au}, p_{pu}, p_{ab}) \rangle \\
&= -2\pi i \delta^4(p_a + p_b - p'_a - p'_b) \\
&\times \sum_j \frac{\langle (p'_{ha}, p'_{bb}) \rangle^{\dagger} V_{ja} | (p_{ab}, p_{ij}) \rangle | (p'_{wa}, p_{ij}) | V_{ja} | (p_{au}, p_{pu}) \rangle}{E_{au} + E_{pa} - E_{ij} + i0^+}.
\end{align*}
\]

A probability amplitude is obtained by integrating this kernel over initial and final wave packets associated with states localized at the origin. The relevant matrix elements
have the form

\[
S_{fi}(x_b, x_a) = \int \phi_i^*(p_{fb}) \phi_f^*(p_{fb}) \phi_i^*(p_{na}) dp_{fb} dp_{fb} dp_{na} \times e^{i(p_{fa} - p_{fb} - x_a) \cdot (x_b - x_a)} \frac{1}{|S|_{aa}} |S|_{fa}. \]

In what follows, we discuss what equation provides a universal starting point for derivations of the standard oscillation formula in the literature from which the kinematic subgroup is the three-dimensional Euclidean group. Equivalent calculations in a representation in which the kinematic subgroup is the neutrino continued to overlap over macroscopic distances, ensuring that the wave packets associated with different mass neutrinos continue to overlap near the speed of light, this ensures that the wave packets associated with different mass neutrinos continue to overlap over macroscopic distances where oscillations can be observed.

For very relativistic neutrinos, the energy of a mass eigenstate with three-momentum \( p_j = |p_j| \) can be approximately by

\[
E_j \approx p_j + \frac{m_j^2}{2p_j}. \tag{19}
\]

In a representation like equation (18), where the interactions are chosen to be kinematically translationally invariant, the neutrino momenta, \( p_j \), are the same for all \( j \). Oscillations are sensitive to the difference between the energies for different mass neutrinos. Contributions from corrections beyond this expression are very small in the relativistic limit.

3.2. Interactions and factorization

The operator \( V_{ja} \) in equation (18) connects the leptonic and hadronic spaces, but factors into separate components for each space. This is the case whether one uses the contact Fermi interaction or a diagram with \( W \) exchange. The hadronic contribution will involve vector and axial vector current matrix elements that are measurable; they represent a separate factor in the \( S \)-matrix and contribute to the normalization of \( S \), but not the oscillation formula.

The leptonic contribution for our example is a matrix element of the charge current interaction of the form \( \langle \beta | J^\mu_{\nu j} | \alpha \rangle \). This matrix depends in principle on the masses of the neutrino \( j \) and the charged lepton \( \beta \). However, the leading contribution to the neutrino mass comes from the oscillation phase described above, and so we can substitute an approximate form for the matrix element by substituting \( \nu_j \rightarrow \nu_0 \), where \( \nu_0 \) here and henceforth denotes a massless neutrino of arbitrary flavor. The lepton matrix element does, however, depend on the mass of the charged lepton. Put another way, the response of the detector to a specific charged lepton flavor depends on the kinematics of this matrix element.

In the absence of mixing, the weak interaction could be expressed as an operator \( V_{ja} \) for each lepton flavor \( \alpha \). With mixing, the weak interaction couples each lepton flavor \( \alpha \) to each of the neutrino mass eigenstates \( j \) via an operator \( V_{ja} \).

These points permit us to approximately factor the weak interaction in the following way:

\[
V_{ja} = U_{ja} V_{0a}. \tag{20}
\]

where \( U \) is a unitary mixing matrix associated with the lepton–neutrino side of the interaction, and \( V_{0a} \) depends on the mass of the charged lepton, but employs a massless neutrino of flavor \( \alpha \). This approximation places all of the neutrino mass and mixing information into the matrix \( U \).

Different representations of the Poincaré group for this process will in general involve different \( V_\nu \) that have different kinematic factors. However, the ability to factorize the interaction via equation (20) will lead to the same \( U_{ja} \).

That is, the interaction could have a representation-dependent form, but very small neutrino masses lead to the ability to factorize and thereby extract a mixing matrix independent of the representation [17].

To the extent that particle kinematics can deviate from the mass or energy shell, contributions from the interactions can appear. This means that, in general, approaches using different forms of dynamics will yield different results at the level of perturbation theory, although the exact expression to all
orders will agree. For the problem at hand, off-shell neutrino kinematics corresponds to higher order weak contributions, which are very small in magnitude; as noted below, the deviations from mass/energy shell kinematics are also very small.

Put another way, in time-ordered perturbation theory, full relativistic invariance is not satisfied to finite order. Lorentz transformations involve unitary operators that depend upon the interaction, and this in principle mixes the orders of perturbation. Since the interaction in question is weak, the violation of relativistic invariance involves higher order perturbative corrections that are very small.

Since the final-state interactions between the electron and proton do not affect the oscillation formula, which involve ratios of squares of the $S$-matrix, we also omit the $\pm$ labels in equation (18) and use plane-wave lepton and hadron states.

### 3.3. Delta functions and packets

Equation (18) reflects the use of time-ordered perturbation theory, in which the intermediate neutrinos are on their mass shells and the intermediate state can have an energy different from that of the external initial or final state. The energy denominator in equation (18) can be written as

$$\frac{1}{E_{aa} + E_{pa} - E'_{na} - E_{\nu j} + i0^+}$$

$$= -i\pi \delta(E_{aa} + E_{pa} - E'_{na} - E_{\nu j}) + \mathcal{P} \frac{1}{E_{aa} + E_{pa} - E'_{na} - E_{\nu j}}.$$  \tag{21}

The energies of the initial and final particles that appear in this denominator depend on the three-momenta of these particles. When these momenta are integrated over the factors in this denominator depend on the three-momenta of these particles. When these momenta are integrated over the factors in the numerator of equation (21), the delta function term involves the phase factor $\exp(-iET)$, where $T$ is large and $E$ depends on the momenta. It is not hard to show that if the remaining part of the integrand is a smooth function of $E$, then as $T$ becomes large the principal value contribution will be exponentially suppressed relative to the delta function in equation (21). In practice, the integrand includes Jacobians involving square roots and wave packets of unknown smoothness. Less restrictive assumptions on the smoothness of the integrand lead to an algebraic suppression of the principal-value term for large $T$, but the contribution is still very small in practice. The general case was carefully examined by Grimus and Stockinger [6].

The dominance of the energy delta function in equation (21) means that all four components of the neutrino four-momentum are constrained:

$$p^{\mu}_{\nu j} = p^{\mu}_{aa} - p^{\mu}_{pa} - p^{\mu}_{na}.$$  \tag{22}

If one uses plane-wave momentum states to evaluate equation (18), then the presence of these delta functions precludes an oscillation. Observables such as a cross section or a detection rate (for cases where there is no definable incident beam of initial particles) are proportional to the square of the $S$-matrix. Oscillations will depend upon the interference between matrix elements of $S$ and $S'$ involving different neutrino mass eigenvalues, and in turn these matrix elements contain different four-momentum delta functions that cannot be satisfied simultaneously, unless the two mass eigenvalues are the same. The result is an observable that is the sum of squares of $S$-matrix elements for each mass eigenvalue, with no interference terms and therefore no oscillation contributions. Oscillations can appear only when the external particle momentum plane wave states are replaced by wave packets that then permit the product of delta functions to be satisfied by slightly different four-momenta and different neutrino mass eigenvalues.

Thus, the oscillation effect necessarily entails momentum distributions, or packets, of the external particles. Under such distributions, equation (18) will in general give rise to a superposition of neutrino mass eigenstates, with each term carrying slightly different four-momenta. Since the energy and three-momentum values will vary inside the external wave packets, so too the energy and three-momentum components of the neutrino mass eigenstates will differ from each other. The most that we can say is that the contributions from different mass eigenstates will have quantitatively similar components of their four-momenta as dictated by the momentum ranges of the external wave packets.

It is important to note here that the term packet refers to the momentum distributions of the external particles. The neutrino mass eigenstates also have a momentum distribution that will depend upon the external packet shapes via the four-momentum condition of equation (22), but this distribution is not a wave packet in the usual sense of preparation of observable states. It also depends on assumptions that have no impact on the final $S$-matrix.

We have seen that external wave packets are essential to obtain an oscillation formula. At the same time, the relevant distributions must be narrow enough that the phase factor in equation (18) can be factored out of the integrals over these distributions. When the conditions for oscillation are satisfied, the phase is approximately stationary. The external distributions must therefore be narrow enough to maintain this property.

A somewhat curious conclusion for this train of reasoning is that, once the phases in the oscillation formula are fixed by the kinematics, they depend, to lowest order, only on the neutrino eigenstate masses and the average neutrino energy as determined by the external particle kinematics. One can therefore make the following replacement:

$$\delta(E_{aa} + E_{pa} - E'_{na} - E_{\nu j}) \rightarrow \delta(E_{aa} + E_{pa} - E'_{na} - E_{\nu j}),$$  \tag{23}

where, as noted above, $\nu_j$ denotes a massless neutrino of arbitrary flavor.

The oscillations are due to the dependence of the phase $\exp(-iET)$ in equation (18) on the neutrino masses. In the limit that the neutrinos propagate on shell the linear combination of four-momenta in the exponent becomes
four-momentum of the propagating neutrino. As will be seen below, the relevant four-momenta are those of neutrino mass eigenstates, so that the phase angle in equation (18) becomes

\[ \phi_j = p_{v_j} \cdot (x_p - x_a) \]  

(24)

for a mass \( m_j \).

3.4. A matter of time

The expression (18) depends upon the space–time separation between the points \( (x_a, t_a) \) and \( (x_b, t_b) \). These points could be determined by events that can in principle be measured. However, in a typical oscillation experiment (e.g. a reactor and a detector located a few kilometers away), the spatial components \( x_a \) and \( x_b \) (and thereby the displacement \( L \)) are essentially known (at least to within a scale characteristic of the reactor size), but the time difference \( T = t_b - t_a \) is not: the detector could record the time of a neutrino detection event, but there is no corresponding record of the initial weak interaction event. Thus, the detector in principle is sensitive to a range of possible times \( T \). As we shall see, the range of values is in practice quite restricted, to the point where it is possible to use any single value of \( T \) within this range and obtain the usual oscillation formula.

We now examine the variation of the \( S \)-matrix with time \( T \) in equation (18). Equivalently, we can consider the variation as a function of velocity \( \nu = |L|/T \). Since the neutrino mass eigenstates are almost ultrarelativistic, we could first assume that \( v = c = 1 \). Alternatively, we could assume that \( v \) corresponds to the velocity of one of the neutrinos:

\[ \nu = \nu_j = \frac{p_j}{E_j}. \]  

(25)

These seem to be reasonable assumptions, given the kinematics. For oscillations it should not matter which assumption is made. Furthermore, conditions should be such that the oscillating phase approximately factors out the integrals in (18).

The square of the scattering matrix \( S_{\nu a} \) will involve interference phases \( \phi_{12} = \phi_j - \phi_i \), where \( \phi_j \) is a contributing phase as defined in equation (24). Consider the interference phase \( \phi_{12} \) from mass eigenstates 1 and 2:

\[ \phi_{12} = (p_1 - p_2) \cdot x = L \left[ \frac{E_1 - E_2}{\nu} - (|p_1| - |p_2|) \right]. \]  

(26)

where the time \( T \) has an associated velocity \( \nu = L/T \). If the neutrino masses \( m_i \) are small compared to their energies, then

\[ E_1 \approx |p_1| + \frac{m_1^2}{2|p_1|}; \quad E_2 \approx |p_2| + \frac{m_2^2}{2|p_2|}. \]  

(27)

Now define \( p, \delta p \) such that \( |p_{1,2}| = p \pm \delta p \). The momentum difference \( \delta p \) has a maximum value determined by the momentum distributions of the external particles. We also express the velocity parameter \( \nu \) in terms of an average velocity \( \bar{\nu} \) as \( \nu = \bar{\nu} + \delta \nu \), where

\[ \bar{\nu} = \frac{1}{2} \left( \frac{p_1}{E_1} + \frac{p_2}{E_2} \right), \quad \bar{\nu} \approx \frac{m_1^2}{4|p_1|^2} - \frac{m_2^2}{4|p_2|^2}. \]  

(28)

The phase is then

\[ \phi_{12} \approx L \left[ \frac{m_1^2}{2|p_1|^2} + \frac{m_2^2}{2|p_2|^2} + (|p_1| - |p_2|) \left( \frac{m_1^2}{4|p_1|^2} + \frac{m_2^2}{4|p_2|^2} \right) \right]. \]  

(29)

Now define \( p, \delta p \) such that \( |p_{1,2}| = p \pm \delta p \). If we further assume that \( \delta p \ll p \), then

\[ \phi_{12} \approx L \left[ \frac{m_1^2}{2p} - 2\delta p \delta v + O(\delta p^2). \right] \]  

(30)

For the specific case that \( \delta v = 0 \), we are left with

\[ \phi_{12} \approx L \left[ \frac{m_1^2}{2p} + O(\delta p^2). \right] \]  

(31)

In principle, \( \delta v \) is arbitrary (as is the time), but its contribution to the phase is limited by a factor involving \( \delta p \). In fact, \( \delta v \) is further limited by the range of \( p_i \) values that determine it. For example, if, at one extreme, \( v = v_1 \), then

\[ \phi_{12} \approx L \left[ \frac{m_1^2}{2p} + \frac{(m_1^2 - m_2^2)\delta p}{2p^2} \right] + O(\delta p^2), \]  

(32)

that is, it generates a phase contribution that is suppressed by order \( \delta p/p \). Thus, we find that \( \delta p \ll 1 \) in order to have interfering neutrino mass eigenstate contributions and to fix the time \( T \) to within correction factors that can be neglected.

This result is effective for small neutrino masses. It is not clear whether neutrinos with larger mass differences would lead to oscillations. It has been shown, for example, that a nonrelativistic massive neutrino cannot oscillate against much lighter partners [4].

4. Result and discussion

Combining all of the simplifying results of the previous section, we find that the space–time phase factors and the mixing matrices factor out of the integrals over external wave packets, leaving a scattering matrix element of the form

\[ S_{f i}(x_b, x_a) \approx \sum_j e^{-i\phi_{fj}(\nu_{fi}, x_a - x_b)} U_{j f} U^*_{i j} S_{f i}(\beta, \alpha, j), \]  

(33)

where

\[ S_{f i}(\beta, \alpha, j) = -2\pi^2 \int d\nu d\nu d\phi_{\beta j} d\phi_{\alpha f} d\phi_{\beta j} d\phi_{\alpha f} \delta^4(p_{\beta j} - p_{\alpha f}) \times \delta(E_{\beta j} - E_{\alpha f}) \times \delta(E_{\beta j} - E_{\alpha f}) \]  

(34)

Integrating over wave packets satisfying the conditions described above, we obtain a measurement probability for neutrino production via initial flavor \( \alpha \) and neutrino absorption via final flavor \( \beta \):

\[ p_{\beta j} = \left| \frac{S_{f i}(x_b, x_a)}{|S_{f i}|^2} \right|^2 \approx \sum_j U_{j f}^* e^{-i\phi_{fi}(\nu_{fi}, x_a - x_b)} U_{j f} U_{j i}^* e^{i\phi_{fi}(\nu_{fi}, x_a - x_b)} U_{i j} \]  

(35)
All three neutrino masses are small. The leading wave packets are essential for oscillations. The spread of Gonzalez-Garcia M C and Nir Y 2003

The oscillation formula depends on the space–time Camilleri L, Lisi E and Wilkerson J F 2008

The weak Hamiltonians in the second-order calculation Explicit off-shell effects can be neglected. Oscillations Kayser B 1981

In this paper, we used an -matrix approach to study the problem of neutrino oscillations. This approach is exact and only involves initial and final states that can in principle be prepared and detected in laboratory experiments. We have used very general principles of relativity and quantum mechanics to obtain an S-matrix and a general expression in second-order perturbation theory in the weak interaction. Here we summarize briefly the conditions that must be satisfied for the second-order S matrix to yield the standard neutrino oscillation formula of equation (36).

1. The perturbative nature of the weak interaction means that the oscillation formula can be derived from a second-order expression via the two-potential equation (18).

2. All three neutrino masses are small. The leading contribution to the oscillation formula comes from the first correction to ultrarelativistic kinematics in the space–time phase. All the remaining dependence on neutrino mass can be neglected.

3. Wave packets are essential for oscillations. The spread of external particle four-momenta makes it possible for all intermediate mass eigenstates to contribute coherently to the S-matrix with slightly differing four-momenta.

4. Explicit off-shell effects can be neglected. Oscillations can be computed in a variety of approaches that differ by unitary transformations that depend on the interaction. These approaches will all have the same leading contribution to the oscillation formula, with all differences (which can be written as off-shell effects) being of higher order in the weak interaction.

5. The weak Hamiltonians in the second-order calculation can be factored into a product of terms, one containing only mixing information and the other describing a weak interaction involving a fictitious massless, flavored neutrino. Corrections to factorization are of higher order in the interaction and/or the neutrino mass.

6. The oscillation formula depends on the space–time separation between production and disappearance of the neutrino mass eigenstates. The spatial separation is known, but the time is not measured. Nevertheless, for reasonably small sizes of the external four-momentum distributions (e.g. detector resolution), the range of relevant times is restricted to values that produce the standard oscillation formula.

We conclude that the existence of a simple neutrino oscillation formula that can be derived from a variety of theoretical perspectives represents a remarkable convergence of several specific physical properties of neutrinos and their interactions. However, to improve on this simple formula will in principle bring back all of the interrelated effects that have so far been neglected, and the ‘corrected’ oscillation formula may acquire theoretical dependences that the lowest-order expression does not carry.

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