Minisuperspace models in histories theory

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Abstract
We study the Robertson–Walker minisuperspace model in histories theory, motivated by the results that emerged from the histories approach to general relativity. We examine, in particular, the issue of time reparametrization in such systems. The model is quantized using an adaptation of reduced state space quantization. We finally discuss the classical limit, the implementation of initial cosmological conditions and estimation of probabilities in the histories context.

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1. Introduction

The Robertson–Walker model provides the standard paradigm for modern cosmology since it incorporates the symmetries of spatial homogeneity and isotropy in the solution of Einstein equations. In cosmology the R–W model appears as a classical dynamical system, as it is described by the scale factor and by spatially averaged matter field variables.

In spite of the success of the R–W model at the classical level, there has been a strong effort to study the quantization of this, and other cosmological models, motivated by their role in the search for a quantum gravity theory. Such models allow us to probe the Planck length stages of the Universe. In spite of the absence of a full theory of quantum gravity they allow a testing ground for hypotheses about the initial condition of the Universe, the emergence of classical spacetime and the plausibility of the special initial conditions for inflation.

‘Minisuperspace’ models are very simple systems, as they have been divested of much of the symmetry of general relativity, still they provide non-trivial examples in which quantum gravity programmes may apply their ideas and techniques. They are particularly relevant for the discussion of major conceptual issues of quantum gravity such as the problem of time, the construction of reparametrization-invariant physical observables and the physical interpretation of the Hamiltonian constraint.
In this work, we study the quantization of minisuperspace models within the context of the histories projection operator (HPO) formalism, in light of the significant results emerging from the histories approach to general relativity [1].

The HPO formalism is a continuation of the consistent histories theory developed by Griffiths, Omnès, Gell-Mann and Hartle—for reviews see [2]. The fundamental object in this formalism is the history, which is defined as a time-ordered sequence of propositions about properties of a physical system. When a certain ‘decoherence condition’ is satisfied by a set of histories then the elements of this set can be given probabilities. The probabilities are encoded in the decoherence functional, a complex function of pairs of histories.

The importance of the histories scheme for quantum gravity is that histories are intrinsically temporal objects: they incorporate in their definition the notion of causal ordering. This is very desirable for a theory of quantizing gravity, where the loss of time at the space of true degrees of freedom—the problem of time [3]—is one of the main conceptual problems. Indeed, the development of the histories formalism in the histories projection operator scheme [4–6] led to a quantization algorithm for parametrized systems [7] in which the histories causal ordering is preserved on the space of true degrees of freedom.

Of great importance in the development of the histories scheme is the construction of a classical histories theory [6]. The basic mathematical entity is the space of differentiable paths taking their value in the space $\Gamma$ of classical states. The key idea in this new approach is contained in the symplectic structure on this space of temporal paths. Analogous to the quantum case, there are generators for two types of time transformation: one associated with classical temporal logic, and one with classical dynamics. One significant feature is that the paths corresponding to solutions of the classical equations of motion are determined by the requirement that they remain invariant under the symplectic transformations generated by the action [6].

The strongest motivation for the study of minisuperspace models within the histories theory comes from its application at the level of general relativity. Two important results arise there. First, the spacetime diffeomorphism group coexists with the ‘group’ generated by the history version of the canonical constraints [1]. The constraints, however, depend on a choice of foliation for the (3+1)-splitting of spacetime. This leads to the important question whether physical results depend upon this choice. The solution of the constraints determines a reduced phase space for histories, which has an explicit dependence on the foliation. It turns out that spacetime diffeomorphisms intertwine between different such reductions [1]. This is a completely novel result, which was made possible only by the incorporation of general relativity into the histories formalism.

The results above suggest that a scheme for the quantization of gravity based on histories has two highly desirable features [1]. First, the Lorentzian metric can be directly quantized; this contrasts with conventional canonical quantum gravity where only a spatial 3-metric is quantized. Second, the history scheme incorporates general covariance via a manifest representation of the spacetime diffeomorphism group.

Even though minisuperspace models lack the most fundamental feature of general relativity—general covariance—they still provide a testing ground of ideas and techniques for a quantum gravity programme. The construction of a Hilbert space of histories, the implementation of the constraints and the identification of gauge-invariant physical observables are non-trivial procedures even in the simplest of minisuperspace models—for other studies of minisuperspace models in the framework of consistent histories, see [8]. As such they sharpen our understanding of the issues involved and allow us to distinguish between the deep conceptual problems and those of a purely technical nature.
The plan of the paper is as follows. In the next section, we provide a concise description of the classical histories theory, emphasizing the insights obtained by the incorporation of general relativity into the histories programme.

In section 3, we study the classical histories theory for an R–W cosmological model with a scalar field. We identify the reduced state space for various values of the relevant parameters. We place strong emphasis in the notion of time reparametrizations for this system, which are the only remnants of the spacetime diffeomorphism invariance in the R–W model. In some cases, the reduced state space has the structure of an orbifold, not a globally smooth manifold.

In section 4, we construct the history Hilbert space and the decoherence functional for cosmological models. We discuss the issue of finding a representation of the theory when the Hamiltonian does not possess a vacuum state. Such is the case of minisuperspace models for which the Hamiltonian constraint does not have a continuous spectrum around zero. We analyse the difference in physical predictions entailed by the choice of different representations. An important feature of the histories Hilbert space is that it strongly relies on the identification of a set of canonical coherent states. For this reason, we analyse the implementation of the constraints at the coherent states level.

In section 5, after a brief discussion of Dirac quantization in histories theory [7], we suggest that the most adequate method of implementing the constraints at the quantum level is a version of reduced state space quantization. We analyse this procedure and we construct the coherent states for the reduced system.

In section 6 we construct the full history Hilbert space, we identify the physical history Hilbert space and we write the decoherence functional for the true degrees of freedom. We expand on the issue of the initial condition and the emergence of classical spacetime. We examine how the ‘tunnelling’ initial condition [9] is implemented in the histories scheme, providing an estimation of the tunnelling probability. Finally, we discuss the extensions of the histories formalism needed for a rigorous treatment of the no-boundary proposal [10].

2. Background

2.1. Classical histories

In the consistent histories formalism, a history \( \alpha = (\hat{\alpha}_t_1, \hat{\alpha}_t_2, \ldots, \hat{\alpha}_t_n) \) is defined to be a collection of projection operators \( \hat{\alpha}_t_i \), \( i = 1, 2, \ldots, n \), each of which represents a property of the system at the single time \( t_i \). The HPO approach places particular emphasis on temporal logic: a history is defined as the operator \( \hat{\alpha} := \hat{\alpha}_t_1 \otimes \hat{\alpha}_t_2 \otimes \cdots \otimes \hat{\alpha}_t_n \) which is a genuine projection operator on the tensor product \( \otimes_n \mathcal{H}_t \) of copies of the standard Hilbert space \( \mathcal{H} \).

The space of classical histories \( \Pi = \{ \gamma : \mathbb{R} \to \Gamma \} \) is the set of all smooth paths on the classical state space \( \Gamma \). It can be equipped with a natural symplectic structure, which gives rise to the Poisson bracket. For the simple case of a particle at a line

\[
\{x_t, x_{t'}\}_\Pi = 0 \tag{2.1}
\]
\[
\{p_t, p_{t'}\}_\Pi = 0 \tag{2.2}
\]
\[
\{x_t, p_{t'}\}_\Pi = \delta(t - t') \tag{2.3}
\]

where \( x_t : \Pi \to \mathbb{R} \), such that \( \gamma \mapsto x_t(\gamma) := x(\gamma(t)) \); \( p_t \) is defined similarly.

One may define the Liouville function

\[
V(\gamma) := \int_{-\infty}^{\infty} dt \ p_t \dot{x}_t, \tag{2.4}
\]
which generates symplectic transformations of the form \((x_t, p_t) \rightarrow (x_{t+s}, p_{t+s})\); also the Hamiltonian (i.e., time-averaged energy) function \(H\)

\[
H(\gamma) := \int_{-\infty}^{\infty} dt \ h_t(x_t, p_t)
\]  

(2.5)

where \(h_t\) is the Hamiltonian that is associated with the copy \(\Gamma_t\) of the normal classical state space with the same time label \(t\).

The history equivalent of the classical equations of motion is given by the following condition [6] that holds for all functions \(F\) on \(\Pi\) when \(\gamma_{cl}\) is a classical solution,

\[
\{F, S\}_\Pi(\gamma_{cl}) = 0,
\]

(2.6)

where

\[
S(\gamma) := \int_{-\infty}^{\infty} dt (p_t \ddot{x}_t - H_t(x_t, p_t)) = V(\gamma) - H(\gamma)
\]

(2.7)

is the classical analogue of the action operator [6].

Classical parametrized systems. Parametrized systems have a vanishing Hamiltonian \(H = h(x, p)\), when the constraints are imposed. Classically, two points of the constraint surface \(C\) correspond to the same physical state if they are related by a transformation generated by the constraint. The true degrees of freedom correspond to equivalence classes of such points and are represented by points of the reduced state space \(\Gamma_{\text{red}}\).

In the histories approach to parametrized systems, the history constraint surface \(C_h\) is defined as the set of all smooth paths from the real line to the constraint surface \(C\). The history Hamiltonian constraint is defined by \(H_c = \int dt \ k(t) h_t\), where \(h_t := h(x_t, p_t)\) is a first-class constraint. For all values of the smearing function \(k(t)\), the history Hamiltonian constraint \(H_c\) generates canonical transformations on the history constraint surface \(C_h\). The history reduced state space \(\Pi_{\text{red}}\) is defined as the set of all smooth paths on the canonical reduced state space \(\Gamma_{\text{red}}\), and coincides with the space of orbits of \(H_c\) on \(C_h\).

In order for a function on the full state space, \(\Pi\), to be a physical observable (i.e., to be projectable into a function on \(\Pi_{\text{red}}\)), it is necessary and sufficient that it commutes with the constraints on the constraint surface.

In contrast to the canonical treatments of parametrized systems, the classical equations of motion are explicitly realized on the reduced state space \(\Pi_{\text{red}}\).

\[
\{\tilde{S}, F\}(\gamma_{cl}) = \{\tilde{V}, F\}(\gamma_{cl}) = 0
\]

(2.8)

where \(\tilde{S}\) and \(\tilde{V}\) are, respectively, the action and Liouville functions projected on \(\Pi_{\text{red}}\). Both \(\tilde{S}\) and \(\tilde{V}\) commute weakly with the Hamiltonian constraint [7].

2.2. Histories general relativity

Next we describe the incorporation of classical general relativity into the histories framework [1]. We consider a four-dimensional manifold \(M\), which has the topology \(\mathbb{R} \times \Sigma\). The history space is defined as \(\Pi_{\text{cov}} = T^*LRiem(M)\), where \(LRiem(M)\) is the space of Lorentzian 4-metrics \(g_{\mu\nu}\), and \(T^*LRiem(M)\) is its cotangent bundle. The history space \(\Pi_{\text{cov}}\) is equipped with the symplectic form

\[
\Omega = \int d^4X \delta\pi^{\mu\nu}(X) \wedge \delta g_{\mu\nu}(X),
\]

(2.9)

where \(X\) is a point in the spacetime \(M\), and where \(g_{\mu\nu}(X)\) is a 4-metric (element of \(LRiem(M)\)), and \(\pi^{\mu\nu}(X)\) is the conjugate variable.
For every vector field $W$ on $M$ we define the Liouville function $V_W$ as

$$V_W := \int d^4 X \, \pi^{\mu\nu}(X) \mathcal{L}_W g_{\mu\nu}(X)$$

(2.10)

where $\mathcal{L}_W$ denotes the Lie derivative with respect to $W$.

These functions $V_W$, defined for any vector field $W$ as in equation (2.10), satisfy the Lie algebra of the spacetime diffeomorphisms group $\text{Diff}(M)$,

$$\{V_{W_1}, V_{W_2}\} = V_{[W_1, W_2]}.$$

(2.11)

where $[W_1, W_2]$ is the Lie bracket between vector fields $W_1$ and $W_2$ on $M$.

We next introduce a (3+1)-foliation of the spacetime $M$, which is spacelike with respect to a Lorentzian metric $g$. A key feature of the present construction is that this foliation is required to be 4-metric dependent, so that all the foliations remain spacelike with respect to all Lorentzian metrics.

For each $g \in \text{LRiem}(M)$ we choose a spacelike foliation $\mathcal{E}[g]$. For a given Lorentzian metric $g$, we use the foliation $\mathcal{E}[g]$ to split $g$ with respect to the Riemannian 3-metric $h_{ij}$, the lapse function $N$ and the shift vector $N^i$ as in the standard (3+1)-decomposition.

Detailed calculations show that the symplectic form $\Omega$ can be written in the equivalent canonical form, with respect to a chosen foliation functional $\mathcal{E}$

$$\Omega = \int \int d^3x \, dt (\delta \pi^{ij} \wedge \delta h_{ij} + \delta \mathcal{P} \wedge \delta N + \delta \mathcal{P}_i \wedge \delta N^i),$$

(2.12)

where the momenta are defined in terms of $\pi^{\mu\nu}$ and the foliation functional.

One then defines the history version of the canonical constraints which satisfy a histories version of the Dirac algebra [1].

The relation between the spacetime diffeomorphism algebra and the Dirac constraint algebra has long been an important matter for discussion in quantum gravity. Therefore it is of great significance that, in this new construction, the two algebras appear together in an explicit way: the classical theory contains realizations of both the spacetime diffeomorphism group and the Dirac algebra.

The canonical constraints can be straightforwardly written using paths of canonical metric and momenta, signifying that there is a central role for spacetime concepts, as opposed to the domination by spatial ideas in canonical quantum gravity.

The constraints, however, depend on the foliation functional. This leads to the important question whether physical results depend upon this choice. The solution of the constraints determines a reduced phase space for histories, which has an explicit dependence on the foliation. It proved to be the critical result that spacetime diffeomorphisms intertwine between different such reductions. This is a completely novel result, which has been made possible only by the incorporation of general relativity into the histories formalism.

In particular, the spacetime diffeomorphisms generated by $V_W$ commute with the constraints if the foliation functional satisfies the equivariance condition: for a function $\mathcal{E} : \text{LRiem}(M) \rightarrow \text{Fol}(M)$, (where $\text{LRiem}(M)$ is the space of Lorentzian metrics on $M$, and $\text{Fol}(M)$ is the space of foliations of $M$) we say that $\mathcal{E}$ is an ‘equivariant foliation’ if

$$\mathcal{E}(f^* g) = f^{-1} \circ \mathcal{E}(g),$$

(2.13)

for all Lorentzian metrics $g$ and spacetime diffeomorphisms $f \in \text{Diff}(M)$. 
3. Classical histories minisuperspace model

3.1. The Robertson–Walker model

We study the reduction of the full general relativity histories theory to a homogeneous and isotropic cosmological model with a scalar field.

Our starting point is a configuration space containing the Lorentzian metric $g_{\mu\nu}(X)$ and the scalar field $\Phi(X)$. The history state space $\Pi_{\text{cov}}$ is the cotangent bundle over the configuration space $\Pi_{\text{RW}} = T^*[\text{LRiem}(M) \times C^\infty(M)]$. It is equipped with the symplectic form, in full analogy with equation (2.9) of the symplectic form of general relativity,

$$\Omega = \int d^4X [\delta \pi^{\mu\nu}(X) \wedge \delta g_{\mu\nu}(X) + \delta \pi(X) \wedge \delta \Phi(X)]. \quad (3.1)$$

Next we restrict the configuration space to a submanifold consisting of metrics and scalar fields of the form

$$ds^2 = -A^2(X_0)(dX_0)^2 + e^{2a(X_0)} d\Omega_3^2 \quad (3.2)$$

$$\Phi(X) = \Phi(X_0), \quad (3.3)$$

where $d\Omega_3^2 = h^i_j dX^i dX^j$ and $h^i_j$ is the homogeneous metric of constant curvature equal to $\kappa \in \{-1, 0, 1\}$ on the surfaces $X^0 = \text{constant}$. The variable $A^2$ is the 00 component of the metric and $e^a$ is the scale factor of the R–W universe.

Substituting into equation (3.1) we obtain the reduced symplectic form

$$\Omega = \int dX^0 [\delta \pi_a(X^0) \wedge \delta A(X^0) + \delta \pi_a(X^0) \wedge \delta a(X^0) + \delta \pi(X^0) \wedge \delta \Phi(X^0)], \quad (3.4)$$

where

$$\pi_a(X^0) := 2 \int d^3X \pi^{ij}(X)h^i_j(X') e^{2a(X^0)} \quad (3.5)$$

$$\pi_A(X^0) := -2 \int d^3X \pi^{00}(X)A(X^0) \quad (3.6)$$

$$\pi_\phi(X^0) := \int d^3\pi_\phi(X). \quad (3.7)$$

The history state space $\Pi_{\text{RW}}$ for the R–W model is the cotangent bundle of the space of paths in the variables $A, a, \phi$, namely maps from $\mathbb{R}$ to $\mathbb{R}^3$, such that $X^0 \rightarrow [A(X^0), a(X^0), \phi(X^0)]$. The symplectic form (3.4) is non-degenerate on the state space $\Pi_{\text{RW}}$.

The spacetime diffeomorphisms are also projected on to the history space of the reduced model albeit they reduce to diffeomorphisms of the real line $\mathbb{R}$. They are generated by functions of the form

$$V_W = \int dX^0[\pi_a \mathcal{L}_W a + \pi_A \mathcal{L}_W A + \pi_\phi \mathcal{L}_W \phi], \quad (3.8)$$

for any vector field $W = W(X^0) \frac{\partial}{\partial X^0}$.

The symplectic form (3.4) is written in a covariant or spacetime way. To write its canonical expression one needs to introduce a spacetime foliation.

In the case of histories general relativity, the introduction of metric-dependent foliations guaranteed the spacetime character of the canonical description, namely that the pull-back
of the Lorentzian metric on the foliation 3-surfaces is always a Riemannian metric [1]. In the present context, however, the restriction into a homogeneous and isotropic spacetime has already taken care of this issue. While it is possible to introduce a metric-dependent foliation, however, there is no overriding physical reason demanding its introduction.

In the present model, the definition of a foliation is equivalent to a choice of a time parameter \( t \). In other words, we introduce the function \( E(t) : \mathbb{R} \rightarrow \mathbb{R} \) such that \( t \rightarrow X^0 = E(t) \). \( E \) effects time reparametrizations and for this reason it is a strictly increasing bijective function\(^3\).

We may then define the canonical configuration variables

\[
\begin{align*}
\lambda_t & := a(E(t, a(\cdot), A(\cdot))) \\
N_t & := A(E(t, a(\cdot), A(\cdot))) \\
\phi_t & := \Phi(E(t, a(\cdot), A(\cdot))).
\end{align*}
\]

The symplectic form \( \Omega \) then becomes

\[
\Omega = \int dt [\delta \pi_{\lambda t} \wedge \delta \lambda_t + \delta \pi_{N_t} \wedge \delta N_t + \delta \pi_{\phi t} \wedge \delta \phi_t],
\]

where we have introduced the variables

\[
\begin{align*}
\pi_{\lambda t} & := \dot{E}(t) \int dt' G_{\lambda}(t, t') \pi_{a(E(t, a(\cdot), A(\cdot)))} \\
\pi_{N_t} & := \dot{E}(t) \int dt' G_N(t, t') \pi_{A(E(t, a(\cdot), A(\cdot)))} \\
\pi_{\phi t} & := \dot{E}(t) \int dt' G_{\phi}(t, t') \pi_{\Phi(E(t, a(\cdot), A(\cdot)))},
\end{align*}
\]

and where \( G_{\lambda}(t, t'), G_N(t, t'), G_{\phi}(t, t') \) are complex kernels that depend on the functional derivatives \( \frac{\delta E}{\delta a(t)} \) and \( \frac{\delta E}{\delta N(t)} \) [1]. In the case of non-metric-dependent foliations they reduce to delta-functions. Note that the momentum variables are scalar densities of weight 1 with respect to time.

We should remark here that equation (3.12) can be alternatively obtained from the canonical histories symplectic form, by substituting a reduced expression for the metric, the lapse function and the shift vector. We followed a different approach in order to demonstrate the existence of a representation of the diffeomorphisms on the real line—see section 3.3.

### 3.2. Constraints

Next we follow the standard canonical analysis of minisuperspace models, hence we identify the constraints

\[
\begin{align*}
C & = \frac{1}{2} N \left( \frac{1}{\sqrt{h}} \left( -\pi_{\lambda t}^2 + \pi_{\phi t}^2 \right) + \sqrt{h} (V(\phi) - \kappa e^\lambda) \right) = 0 \\
\pi_N & = 0,
\end{align*}
\]

that form a first-class system. Here \( V(\phi) \) is a potential for the scalar field \( \phi \) and \( h = e^{6\lambda} \) is the determinant of the 3-metric. It is important to remark that \( \sqrt{h} \) is a scalar density of weight 1 with respect to time, since \( \sqrt{-g} = N \sqrt{h} \) and \( \sqrt{-g} \) is a spacetime density of weight 1 and \( N \) is a scalar function [1]. Even though \( \sqrt{h} \) is numerically equal to \( e^{3\lambda} \) it

\( ^3 \) If we were to choose a metric-dependent foliation \( E \), it should also be a functional of the components of the metric and it would read \( E(t, a(\cdot), A(\cdot)) \).
transforms differently under diffeomorphisms of the real line and for this reason we make this
distinction in the expression for the constraint\(^4\) \(h\).

The histories version of the canonical constraints is

\[
C(\xi) = \frac{1}{2} \int dt \xi(t) N \left( \frac{1}{\sqrt{h}} \left( -\pi_{\lambda t}^2 + \pi_{\phi t}^2 \right) + \sqrt{h} (V(\phi_t) - \kappa e^{4\lambda_t}) \right) = 0
\]

\(\pi_\phi = \int dt u(t) \pi_N = 0,\)

for all scalar smearing functions \(\xi(t)\) and \(u(t)\).

An alternative characterization of the constraints involves the introduction of an arbitrary
scalar density \(w(t)\) of weight 1. We redefine the momenta

\[
p_{\lambda t} = \pi_{\lambda t} / w(t),
\]

\[
p_{N t} = \pi_{N t} / w(t),
\]

\[
p_{\phi t} = \pi_{\phi t} / w(t),
\]

and we rescale the metric determinant \(\sqrt{h} := \sqrt{h} / w(t)\), so that \(p_{\lambda t}, p_{N t}, p_{\phi t}\) and \(\sqrt{h}\) are all scalar functions with respect to time. The Hamiltonian constraint is then written as

\[
C_\xi = \frac{1}{2} \int dt \xi(t) w(t) e^{-3\lambda_t} \left( \pi_{\lambda t}^2 - p_{\lambda t}^2 + p_{\phi t}^2 + V(\phi_t) e^{6\lambda_t} - \kappa e^{4\lambda_t} \right).
\]

The price to pay then is that the symplectic form (hence the Poisson brackets) involves
explicitly the weight function \(w(t)\)

\[
\Omega = \int dt w(t) [\delta p_{\lambda t} \wedge \delta \lambda_t + \delta p_{N t} \wedge \delta N_t + \delta p_{\phi t} \wedge \delta \phi_t].
\]

The state space action functional for the model is written as

\[
S = V - H_N - \pi_N(u),
\]

in terms of the Liouville function \(V\), which generates time translations

\[
V = \int dt (\pi_{\lambda t} \dot{\lambda}_t + \pi_{N t} \dot{N}_t + \pi_{\phi t} \dot{\phi}_t).
\]

The next step involves the implementation of the constraints and the construction of the
reduced histories state space. The primary constraint \(\pi_N = 0\) is trivially implemented. To implement the Hamiltonian constraint we first need to solve the equation \(H_t = 0\).

To simplify our analysis we shall assume that the scalar field potential \(V(\phi)\) is a constant equal to the cosmological constant \(\Lambda > 0\). The constraint equation then reads

\[-p_{\lambda t}^2 + p_{\phi t}^2 + U(\lambda_t) = 0, \text{ where } U(\lambda_t) = e^{6\lambda_t} - \kappa e^{4\lambda_t}\]

plays the role of a ‘potential’ for the variable \(\lambda\). There exist two physically distinct cases.

3.2.1. \(\Lambda > 0, \kappa \in \{0, -1\}\). In this case \(U(\lambda_t) > 0\) for all values of \(\lambda_t\) and we may solve the constraint with respect to \(\pi_{\lambda t}\)

\[
\pi_{\lambda t} = \pm \sqrt{\pi_{\phi t}^2 + U(\lambda_t)}.
\]

\(^4\) This point is important because the histories constraints involve an integration over time and one has to keep an
account of the densities for the definition to be independent of the choice of the \(t\) coordinate. This is not a feature
of the histories formalism alone; it also appears in the construction of the phase space canonical action, which is an
object that appears also in the standard formalism.
Minisuperspace models in histories theory

The constraint surface consists of two disconnected components, one for each sign of $\pi_\lambda$ in the right-hand side of equation (3.27). The coordinate expression for the symplectic form $\Omega_C$, restricted on the history constraint surface, is

$$\Omega_C = \int dt \left[ \pm \frac{\pi_{\phi t}}{\sqrt{\pi_{\phi t}^2 + U(\lambda_t)}} \delta \pi_{\phi t} \wedge \delta \lambda_t + \delta \pi_{\phi t} \wedge \delta \phi_t \right].$$

(3.28)

Next, we introduce the coordinate

$$f_{\pm t} := \phi_t \pm \pi_{\phi t} \int_0^{\lambda_t} \frac{dx}{\sqrt{\pi_{\phi t}^2 + U(x)}}$$

(3.29)

and write the restricted symplectic form as

$$\Omega_C = \int dt \delta \pi_{\phi t} \wedge \delta f_{\pm t}.$$  

(3.30)

The vector fields $\delta \delta \pi_{\phi t}$ and $\delta \delta f_{\pm t}$ correspond to non-degenerate directions. The coordinates $\pi_{\phi t}$ and $f_{\pm t}$ commute with the constraints, hence they can be projected into functions $\tilde{\pi}_{\phi t}, \tilde{f}_{\pm t}$ on the histories reduced phase space. However, the reduced phase space consists of two disconnected components, each of topology $\mathbb{R}^2$. We then write the symplectic form of the reduced state space as

$$\Omega_{\text{red}} = \int dt \delta \tilde{\pi}_{\phi t} \wedge \delta \tilde{f}_{\pm t}. $$

(3.31)

In order to study the physical interpretation of each component we consider histories $(\pi_{\phi t}, f_{\pm t}) = (c_1, c_2)$, for constants $c_1, c_2$ that are solutions to the equations of motion as we shall see in what follows. These histories correspond to paths on the constraint surface that satisfy

$$\pi_{\phi t} = c_1$$

(3.32)

$$\dot{\phi}_t = c_2 \mp \frac{c_1}{\sqrt{c_1^2 + U(\lambda)}} \lambda.$$  

(3.33)

In order to compare these solutions with known solutions of Einstein equations we choose a time variable that plays the role of a clock. This may be achieved by `fixing the gauge’ on the constraint surface [3]; in this case, the causal ordering parameter $t$ assumes the role of a clock.

The most convenient gauge-fixing condition is to assume that $\dot{\phi} = \pi_\phi e^{-3\lambda}$. The parameter $t$ is then identified to be the global time of Robertson–Walker universes. It is easy to show that

$$\mp t = \int_{\lambda_0}^{\lambda} \frac{e^{3\lambda} \, dx}{\sqrt{c_1^2 + U(x)}}$$

(3.34)

which describes a contracting universe (with respect to $t$) in the plus sign branch, and an expanding universe in the minus sign branch. To check this result one may compute the integral for large $\lambda$, in which case $e^{3\lambda} \sim e^{\mp \sqrt{A}t}$.

Finally, note that one may pass from the $+$ to the $-$ component of the reduced state space through the symplectic transformation $(\pi_{\phi t}, f_{\pm t}) \to (-\pi_{\phi t}, f_{\mp t})$. 


There exist two bifurcate surfaces in \( \Pi_1 \).

Finally, there exist two degenerate orbits for \( \lambda > \lambda^c \) at \( e^3 \).

We follow the same procedure as in the previous case, however in this case the constraint surface has a boundary, determined by the condition

\[
\pi^2_{\phi t} = \pi^2_{\phi t} + U(\lambda_c) \geq 0.
\] (3.35)

The boundary condition is trivially satisfied for \( \lambda > \lambda^c \) but it places restrictions on the values of \( \pi_{\phi t} \), for \( \lambda < \lambda^c \).

The orbits generated by the constraints have different behaviour in different regions of the constraint surface \( \Pi_1 \): this reflects into the non-trivial global structure of the reduced phase space:

(i) The regions \( U_+ \) and \( U_- \) correspond to the values of \( \pi^2_{\phi t} > |U_{\text{min}}| \), \( \forall t, \pi_\phi \) being positive and negative respectively. Again \( U_- \) describes eternally expanding universes and \( U_+ \) collapsing ones. This is due to the fact that for \( \pi^2_{\phi t} > |U_{\text{min}}| \), the orbits of the constraints are curves with \( \lambda \in (-\infty, \infty) \). The quantities \( \pi_{\phi t} \) and \( f_{\phi t} \) are constant along the orbits and they project therefore to coordinates on the reduced histories state space. The symplectic form in this case is the same as equation (3.31).

(ii) For \( \pi^2_{\phi t} < |U_{\text{min}}| \) there exist again two disjoint regions. They both correspond to bouncing universes, because \( \pi_\phi \) can take both positive and negative values in the corresponding orbits. There exist two disjoint orbits for each value of \( \pi_\phi \) one with \( \lambda \in (-\infty, \lambda_1(\pi_\phi)) \) and one with \( \lambda \in [\lambda_2(\pi_\phi), \infty) \), where \( \lambda_1(\pi_\phi) \) and \( \lambda_2(\pi_\phi) \) are the smaller and the larger real solutions of the equation \( \pi^2_{\phi t} + U(\lambda) = 0 \). The orbits of the first type (elements of \( U_- \)) correspond to an expanding universe that reaches the critical value \( \lambda_1(\pi_\phi) \) and re-collapses, while the orbits of the second type (elements of \( U_+ \)) correspond to collapsing universes that reach \( \lambda_2(\pi_\phi) \) and then re-expand. Note, however, that for \( \pi_\phi = 0 \) there exists only one solution, which lies in \( U_- \). The coordinates \( f_+ \) and \( f_- \) can be used in both \( U_+ \) and \( U_- \), but are not independent. We have in \( U_- \) and \( U_+ \),

\[
f_{f_+} = f_{f_+} + \pi_{\phi t} \int_{-\infty}^{\lambda_1(\pi_\phi)} \frac{dx}{\sqrt{\pi^2_{\phi t} + U(x)}},
\]

\[
f_{f_-} = f_{f_-} + \pi_{\phi t} \int_{\lambda_2(\pi_\phi)}^{\infty} \frac{dx}{\sqrt{\pi^2_{\phi t} + U(x)}},
\]

respectively. The symplectic form is locally the same as in equation (3.31).

(iii) There exist two bifurcate surfaces in \( \Pi_{\text{red}} \), which form the boundary between the regions \( U_- \) and \( U_+ \). This boundary is characterized by values of \( \pi_\phi = \pm \sqrt{|U_{\text{min}}|} \). For each sign of \( \pi_\phi \) and value of \( f_\pm \) there exist four different orbits in that surface. Two correspond to \( \lambda \in (-\infty, \lambda^* \), one expanding (\( \pi_\phi < 0 \)) and one collapsing (\( \pi_\phi > 0 \)). We will denote them as \( B_{-\pm} \) and \( B_{+\pm} \) respectively. The other two correspond to \( \lambda \in (\lambda^*, \infty) \) (also collapsing and expanding)—we shall denote them as \( B_{-\pm} \) and \( B_{+\pm} \).

(iv) Finally, there exist two degenerate orbits for \( \pi_\phi = \pm \sqrt{|U_{\text{min}}|}, \pi_\phi = 0 \) and for every value of \( \phi = f_\pm \). These orbits only contain the point \( \lambda = \lambda^* \), and correspond to the static Einstein universe. In this case the symplectic form diverges; indeed, the inclusion of these orbits in the reduced state space \( \Pi_{\text{red}} \) is incompatible with a smooth manifold structure, and renders \( \Pi_{\text{red}} \) into an orbifold. We shall denote the corresponding subset of the reduced state space as \( O \).
3.3. Equations of motion and time reparametrization

It is easy to verify that the action functional $S$ commutes weakly with the constraints, and thus projects to a function $\tilde{S}$ on the histories reduced state space $\Pi_{\text{red}}$. In fact $\tilde{S} = \tilde{V}$, where $\tilde{V}$ is the projection of $V$ onto $\Pi_{\text{red}}$.

\[ S = \tilde{V} = \int dt \, \pi_{\phi t} \dot{f}_{\phi t}. \]  
(3.38)

The existence of a symplectic representation of time translations on $\Pi_{\text{red}}$ highlights one of the most important properties of the histories formalism: time ordering is not lost on the space of true degrees of freedom. Indeed, the parameter $t$ that determines time ordering on $\Pi_{\text{red}}$ is the same as the parameter $t$ that determines time ordering on $\Pi_{\text{red}}$. The equations of motion (2.6) on the reduced state space have the solutions $\pi_{\phi t} = \text{const}$ and $f_{\phi t} = \text{const}$. This implies that the solutions to the equations of motion are invariant under time reparametrizations. The issue arises whether there exists a symplectic action of the group of time reparametrizations that can be projected on the histories reduced state space.

To this end, the natural object to study is the family of functions $V_U$,

\[ V_U = \int dt \, (\pi_{\phi t} \lambda_t + \pi_{N t} \lambda_t N_t + \pi_{\phi t} \phi_t), \]  
(3.39)

where $U = U(t) \frac{\partial}{\partial t}$ is a vector field on $\mathbb{R}$. While these functions satisfy the algebra of the spacetime diffeomorphisms group $\text{Diff}(\mathbb{R})$, they do not commute weakly with the constraints, hence they cannot be projected on to the reduced state space. The reason is that they do not act properly on $\sqrt{h}$, namely

\[ \{V_U, \sqrt{h}\} \neq L_{U} \sqrt{h}. \]  
(3.40)

While $\sqrt{h}$ is a scalar density of weight 1, it transforms under $V_U$ as a scalar function on $\mathbb{R}$, and consequently the constraints do not transform in a proper geometric way under the action of $V_U$. The only functions $V_U$ that project into the reduced state space are those for which the condition $U(t) = \text{constant}$ holds; these functions are multiples of the Liouville generators of time translations.

Nonetheless, a representation of time reparametrizations on the reduced state space does exist, but its generators are the functions $V_W$ of equation (3.8). The functions (3.8) generate a representation of $\text{Diff}(\mathbb{R})$ on $\Pi_{\text{RW}}$. If we assume that the foliation map $E$ is metric dependent and satisfies the equivariance condition (2.13), the functions $V_W$ commute with the constraints and can therefore be projected on $\Pi_{\text{red}}$—see [1].

We must emphasize that the requirement of a metric-dependent foliation is only necessary in general relativity; it is superfluous in the minisuperspace model and it only appears in the discussion of the invariance of $\Pi_{\text{red}}$ under diffeomorphisms. For this reason, an arbitrary parametrized system that lacks an associated covariant diffeomorphic description does not carry a symplectic action of the group $\text{Diff}(\mathbb{R})$ on the space of the true degrees of freedom.

4. Histories quantization

In this section, we provide a summary of the histories quantization procedure, emphasizing the methods for constructing the histories Hilbert space and the decoherence functional. We extend previous results to the case of systems characterized by Hamiltonians without a vacuum state—this is the case relevant to the quantization of minisuperspace models. We also elaborate on the physical distinction between different choices for the history Hilbert space.
4.1. The history group

The key object in histories quantization is the histories Hilbert space $V$. For discrete-time histories—namely for histories with support on the moments of time $(t_1, t_2, \ldots, t_n)$—the history Hilbert space $V$ is the tensor product of the single-time Hilbert spaces $H_{t_1} \otimes H_{t_2} \otimes \cdots \otimes H_{t_n}$, where $H_{t_i}$ is a copy of the canonical Hilbert space of the theory at time $t_i$.

In the case of gravity we consider a continuous temporal support for the histories; the full real line $\mathbb{R}$ or perhaps an interval $[0, T] \subseteq \mathbb{R}$. This suggests defining a continuous-tensor product of Hilbert spaces, which is a rather unwieldy object, being in general non-separable.

Isham and Linden [5] constructed the continuous-time Hilbert space by seeking representations of the history group, a history analogue of the canonical group. For a particle at a line the Lie algebra of the history group is

\[
\left[ \hat{x}_t, \hat{x}_{t'} \right] = 0 \quad (4.1)
\]

\[
\left[ \hat{p}_t, \hat{p}_{t'} \right] = 0 \quad (4.2)
\]

\[
\left[ \hat{x}_t, \hat{p}_{t'} \right] = i\delta(t, t'). \quad (4.3)
\]

For quadratic Hamiltonians, one may select uniquely the Hilbert space as a Fock space, by demanding the existence of an operator representing the smeared Hamiltonian [5]. A crucial property of these constructions is that the Hilbert space that carries a representation of the history group can be cast in a tensor-product-like form as $V = \otimes H_t$. This is not a genuine tensor product over the real line, but an analogue that can be rigorously defined. One may employ the tensor-product-like structure of the history Hilbert space to find more general representations of the history group [11]. We will employ such representations in the study of minisuperspace models. The construction of the history Hilbert space relies heavily on the theory of coherent states. For this reason, we next provide a brief summary on the related theory—for details, see [12, 13].

4.2. Coherent states

We assume a representation of the canonical or history group, by unitary operators $\hat{U}(g)$ on a Hilbert space $\mathcal{H}$. Let $\hbar$ denote the Hamiltonian of this system and $|0\rangle_H$ the lowest energy state. Then we define the coherent states as

\[
|g\rangle := \hat{U}(g)|0\rangle. \quad (4.4)
\]

Next we consider the equivalence relation on the canonical group defined as

\[
g \sim g' \quad \text{if} \quad |g\rangle, |g'\rangle \quad \text{correspond to the same ray.}
\]

The phase space $\Gamma$ is identified as the quotient space $\mathcal{G}/\sim$ and we label the (generalized) coherent states by points $z \in \Gamma$.

The fundamental property of coherent states is that they form an overcomplete basis, i.e., any vector $|\psi\rangle$ can be written as

\[
|\psi\rangle = \int d\mu(z) f(z)|z\rangle, \quad (4.5)
\]

in terms of some complex-valued function $f$ on $\Gamma$. Here $d\mu$ denotes some natural measure on $\Gamma$. There is also a decomposition of the unity

\[
\int d\mu(z)|z\rangle\langle z| = 1. \quad (4.6)
\]
Another important property of coherent states is that the overlap kernel \( \langle z'|z \rangle \) contains all the information about the Hilbert space and the group representation. Let us denote an overlap kernel by \( K(z|z') \). This may be a general function on \( \Gamma \times \Gamma \). However, it has to satisfy two properties in order to correspond to coherent states. First, it has to be Hermitian, \( K(z|z') = K^*(z'|z) \). Second, it must be positive definite, that is, for any sequence of complex numbers \( c_n \) and points \( z_n \in \Gamma \),

\[
\sum_{nm} c_n^* c_m K(z_n|z_m) \geq 0,
\]

which implies that \( K(z, z) \geq 0 \). We can now construct the corresponding Hilbert space \( \mathcal{H} \).

A vector on \( \mathcal{H} \) is a function on phase space \( \Gamma \) of the form \( \Psi(z) = \sum_i c_i \langle z|z_i \rangle \), for a finite number of complex numbers \( c_i \) and state space points \( z_i \). The inner product between two vectors characterized by \( c_i, z_i \) and \( c'_i, z'_i \) is \( \sum_i c_i^* c_i \langle z'_i|z_i \rangle \). The wavefunctions \( \Phi_{z'}(z) = K(z'|z) \) form a family of coherent states on \( \mathcal{H} \).

The canonical group is then represented by the operators \( \hat{U}(g) \), which are defined by their action on coherent states

\[
(\hat{U}(g)\Psi_{z'})(z) = e^{i\theta \Psi_{gz'}}(z),
\]

where the precise choice of phase depends on the details of the group structure. The expression \( gz \) denotes the action of the group element \( g \) on the manifold \( \Gamma \).

### 4.3. General representations of the history group

The space of temporal supports \( T \) may either be \( \mathbb{R} \) or an interval \([0, T] \subset \mathbb{R}\). To construct the history Hilbert space \( \mathcal{V} \) we select a family \( L \) of normalized coherent states \( |z \rangle \) on the Hilbert space of the canonical theory \([11]\).

Next we define the space of all continuous paths from \( S \) to \( L \). We write such paths \( t \rightarrow |z(t) \rangle \) as \( |z(\cdot) \rangle \). We then construct the vector space \( V \) of finite linear combinations of the form \( \sum_{i=1}^I |z_i(\cdot) \rangle \), which is equipped with the inner product

\[
\langle z(\cdot)|z'(\cdot) \rangle = \exp \left( \int_s^t dt \log \langle z(t)|z'(t) \rangle \right).
\]

In order to obtain finite values for this inner product, we need to place some restrictions on the possible paths \( z(\cdot) \) for the case \( T = \mathbb{R} \). In particular, we assume that \( \langle z(t)|z'(t) \rangle \) converges rapidly to 1 for \( t \rightarrow \pm \infty \); or else, all paths converge asymptotically to a fixed value of \( z_0 \). When \( T \) is a bounded interval, there are no restrictions. We then employ \( \langle z(\cdot)|z'(\cdot) \rangle \) as an overlap kernel for the coherent states \( |z(\cdot) \rangle \), and employ this for the construction of the history Hilbert space and the corresponding representation of the history group.

The Hilbert histories space \( \mathcal{V} \) is the closure of \( V \) with respect to the topology induced by the inner product (4.9). If the canonical coherent states are obtained by the action of a representation of the canonical group \( G \), then the Hilbert space \( \mathcal{V} \) carries a representation of the corresponding history group.

The role of coherent states is crucial in this construction. We could have chosen to construct a representation of the Hilbert space by introducing another arbitrary family \( L \) of vectors on the Hilbert space, and then repeat the same algorithm. However, if the basis formed by this set of vectors was too small, the resulting history space would not be able to carry many interesting self-adjoint history operators, such as the elements of the history group. On the other hand, if the set \( L \) were too large (say the whole Hilbert space \( H \)), many unbounded self-adjoint operators, such as the smeared Hamiltonian, would not be definable. The choice of a family of coherent states for \( L \) seems to be the most adequate.
However, different sets of coherent states may lead to different history Hilbert spaces and different representations of the history group. We must choose the set of coherent states, therefore, by taking into account the symmetries of the canonical theory, and in some cases—for example, the presence of a Poincaré group symmetry—this choice might be rendered unique [6]. In minisuperspace models, however, most traces of the original symmetry of general relativity have been lost. As a result, the representation of the history group is non-unique.

4.4. Defining history operators

When the space of temporal supports $T$ is equal to the whole real line, the analogue of the histories time-translation generator $V$ is defined through the one-parameter group of unitary operators
\[ e^{i\hat{V}t} |\zeta(t)\rangle = |\zeta'(t)\rangle, \]
where $\zeta'(t) = \zeta(t + s)$.

The definition of the time-averaged Hamiltonian 'restricts' the arbitrariness in the choice of the history group representation, even to the extent of leading to a unique representation. Indeed, if the Hamiltonian $\hat{h}$ of the canonical theory possesses a unique vacuum state $|0\rangle$ such that $\langle 0 | \hat{h} | 0 \rangle = 0$, we may define the coherent states by taking $|0\rangle$ as a reference vector. We may then define the one-parameter group of transformations $e^{i\hat{H}_\kappa t}$ by using the matrix elements
\[ \langle \zeta(\cdot) | e^{-i\hat{H}_\kappa} | \zeta'(\cdot) \rangle = \exp \left( \int_0^t dt \log \langle z(t) | e^{-i\hat{h}t} | z'(t) \rangle \right). \]

These matrix elements define a self-adjoint operator if $T$ is a finite interval. Alternatively, if $T = \mathbb{R}$, $\hat{H}_\kappa$ can be defined if $|z_t\rangle$ converge asymptotically ($t \to \pm \infty$) to $|0\rangle$. Moreover, if the canonical vacuum is invariant under a symmetry of space translations then the definition of $\hat{H}_\kappa$ uniquely selects the representation of the history group [11].

In minisuperspace models, the Hamiltonian operator does not have a continuous spectrum around zero. Hence we cannot define a vacuum state, thereby selecting a preferred family of coherent states. The family of coherent states has to be chosen with reference to other symmetries of the theory.

In the absence of a unique choice of coherent states we may still employ the equation (4.11) for defining the smeared Hamiltonian $\hat{H}_\kappa$. It is easy to show that
\[ |\langle z(\cdot) | e^{-i\hat{H}_\kappa t} - 1 |\zeta(\cdot)\rangle| < C_s \int_0^t dt |\kappa(t)| \log \langle z(t) | \hat{h} | z(t) \rangle, \]
for some constant $C > 0$ for sufficiently small values of $s$.

If $T$ is a finite interval, or if the smearing function $\kappa(t)$ has compact support the integral (4.12) is finite. Hence the one-parameter group of transformations is continuous at $s = 0$, and according to Stone’s theorem, the operator $\hat{H}_\kappa$ exists. If $T = \mathbb{R}$ and if the time-averaging function $\kappa(t)$ does not vanish fast enough at infinity, then the integral (4.12) diverges, since the quantity $\langle z_t | \hat{h} | z_t \rangle$ tends to the non-zero value $\langle 0 | \hat{h} | 0 \rangle$ at infinity. Hence we cannot prove the existence of the $\hat{H}_\kappa$ operator, while if $\hat{h}$ possessed $|0\rangle$ as an eigenstate $\hat{H}_\kappa$ would be definable for any measurable function $\kappa(t)$.

Once we have defined the self-adjoint operators $\hat{V}$ and $\hat{H}_\kappa$ we can easily show that they satisfy the commutation relations
\[ [\hat{V}, \hat{H}_\kappa] = i\hbar \hat{H}_\kappa. \]

5 We can always redefine $\hat{h}$ up to an additive constant so that $\langle 0 | \hat{h} | 0 \rangle = 0$, but this does not remove the ambiguity in the choice of a family of coherent states. Such a redefinition is also inadequate for the case that $\hat{h}$ represents a constraint, because the addition of a constant to a constraint produces a constraint with a different physical content.
The decoherence functional

In the histories formalism the probabilities are contained in the decoherence functional. The decoherence functional incorporates the information about the initial state, the dynamics and the instantaneous laws. We expect therefore to implement the constraints in the construction of the decoherence functional.

The decoherence functional is a complex-valued function of a pair of histories. A history is represented by a projection operator \( P \) on \( \mathcal{V} \), so the decoherence functional is a function of a pair of projection operators. A projector may be constructed by coarse-graining one-dimensional projectors of the form \( |f\rangle \langle f| \). Furthermore, any vector \( |f\rangle \in \mathcal{V} \) may be written as a linear combination of coherent states. For this reason, it is sufficient to compute the decoherence functional for a pair of projectors onto coherent states.

A rather general theorem about the form of the decoherence functional \([14]\) states that if the decoherence functional satisfies a version of the Markov property—namely that expectations of physical observables at a moment of time allow the determination of expectations at all future moments of time—then it can be obtained as a suitable continuous limit of the following expression:

\[
\begin{align*}
\mathcal{D}(z_1, t_1; z_2, t_2; \ldots; z_n, t_n | z'_1, t'_1; z'_2, t'_2; \ldots; z'_m, t'_m) &= \langle z'_m | e^{-i\hat{H}(t_n - t_m)} | z_m \rangle \langle z_m | e^{-i\hat{H}(t_{n-1} - t_m)} | z_{n-1} \rangle \times \cdots \\
& \times \langle z_2 | e^{-i\hat{H}(t_{1} - t_2)} | z_1 \rangle \langle z_1 | e^{-i\hat{H}(t_{0} - t_1)} | z_0 \rangle \langle z_0 | \rho_0 | z'_0 \rangle \\
& \times \langle z'_0 | e^{i\hat{H}(t'_1 - t_2)} | z'_1 \rangle \langle z'_1 | e^{i\hat{H}(t'_1 - t_2)} | z'_2 \rangle \cdots \langle z'_{n-1} | e^{i\hat{H}(t'_n - t_1)} | z'_n \rangle.
\end{align*}
\]

The decoherence functional takes value in an \( n \)-point and an \( m \)-point history, while \( \rho_0 \) is the initial state.

In the continuous-time limit (\( |t_i - t_{i-1}| < \delta t, \) for all \( i \) and \( \delta t \to 0 \)) the decoherence functional becomes

\[
\mathcal{D}(z(\cdot), z'(\cdot)) \sim e^{iS[z(\cdot)] - iS[z'(\cdot)]},
\]

where \( S = V - \int dt \, h(z) \) is the classical action functional and \( h(z, z) = \langle z | \hat{h} | z \rangle \).

For differentiable paths \( z(\cdot) \) the choice of the family of coherent states (and hence the representation of the history group) does not affect the values of the decoherence functional. The difference between representations appears for non-differentiable paths. In this case, one has to keep terms of order \( \delta t^2 \); then it can be shown that different representations provide different contributions \([16]\).

The last result is particularly relevant in the study of coarse-grained histories. In order to construct a coarse-grained history we have to sum over coherent state paths. This involves employing an integration measure which, by necessity, has support on non-differentiable paths (cylinder sets), and provides different results for different families of coherent states.

In conclusion, different choices of coherent states families imply different representations for the history group. All representations—with the same Hamiltonian—provide the same values for the decoherence functional for fine-grained, differentiable coherent state histories. However, they involve different rules for coarse-graining and hence provide different answers to probabilities for coarse-grained histories. It is very important that physical criteria should exist, which allow the unique selection of a representation for the history group.

5. The implementation of constraints

In order to study the quantization of parametrized systems such as the minisuperspace model, one has to extend the previous analysis to systems that possess first-class constraints.
We noted earlier that the quantum description of the system is obtained from the knowledge of the coherent states of the standard theory. The ambiguity in the choice of a family of coherent states can be, in principle, resolved—as it is in a large class of systems that have been studied [6, 15]—by the knowledge of the symmetries of the fully covariant theory.

In the case of gravity this would imply that we possess the knowledge of the full quantum gravity theory—at least the kinematical part. In particular, it would mean that we have explicitly constructed coherent states for the history variables of general relativity \( g_{\mu\nu}(\cdot), \pi^{\mu\nu}(\cdot) \), in a way implementing the principle of general covariance—the existence of a unitary representation of the \( \text{Diff}(M) \) group whose generators commute with the history version of the constraints—as we have in classical history theory. We would then restrict the definition of these coherent states for the special case of an R–W metric.

In the absence of a theory for the full spacetime symmetries of general relativity, the quantization of minisuperspace models is beset with a degree of arbitrariness, which is related to the choice of a proper integration measure of coarse-grainings. Next we analyse this issue in detail.

There exist two main general schemes for the quantization of constrained systems that do not involve gauge fixing: Dirac quantization and reduced state space quantization. The former implements the constraints at the quantum mechanical level, while the latter implements them classically and then attempts to quantize the classical reduced state space. We study both schemes in the following two subsections.

5.1. Dirac quantization

In [7] we analysed the transcription of the Dirac quantization method in the history context. The general idea is to first construct the Hilbert space of the unconstrained system, and then to identify an operator that represents the canonical constraint smeared in time; the zero eigenspace of the constraint operator is the physical Hilbert space.

Specifically, if \( \mathcal{H} \) is the single-time Hilbert space of the standard canonical theory, equipped with a set of coherent states \( |z(\cdot)\rangle \), and if \( E \) is the projector into the zero eigenspace of the constraint, then, the matrix elements of \( E \) of the coherent states \( \langle z|E|z'\rangle \) define a new overlap kernel; from this kernel one may define the physical Hilbert space [17, 18].

The decoherence functional for coherent state paths is

\[
d(z_1, t_1; z_2, t_2; \ldots; z_n, t_n|z'_1, t'_1; z'_2, t'_2; \ldots; z'_m, t'_m) = \langle z'_m|E|z_n\rangle\langle z_n|E|z_{n-1}\rangle \cdots \langle z_2|E|z_1\rangle\langle z_1|E|z_0\rangle \langle z_0|E\rho_0E|z'_0\rangle \\
\times \langle z'_0|E|z'_1\rangle \langle z'_1|E|z'_2\rangle \cdots \langle z'_{m-1}|E|z'_m\rangle.
\]  
(5.1)

Writing formally \( E = \frac{1}{2\pi} \int d\xi e^{i\xi} \) and inserting this expression at every time step we may write the decoherence functional in the continuous limit as

\[
d[z(\cdot), z'(\cdot)] = \langle z_0|\rho_0|z_0\rangle \int D\xi(\cdot)D\xi'(\cdot) e^{i(V-H_0)[z(\cdot)]-i(V-H_c)[z'(\cdot)]},
\]  
(5.2)

where \( D\xi(\cdot) \) is the continuous limit of \( \frac{d\xi_1d\xi_2\ldots d\xi_n}{12\pi^n} \). This expression is equivalent to the decoherence functional of a system with smeared Hamiltonian \( H_c \), integrating, however, over all possible paths of the Lagrange multiplier \( \xi \).

A weak point of the Dirac quantization is that typically the canonical constraint possesses a continuous spectrum near zero. Hence, the physical Hilbert space is not a subspace of the initial Hilbert space. To deal with such problems one has to employ more elaborate techniques, usually involving the concept of the induced inner product—see [19] for a discussion close to our present context.
The kernel $K(z, z')$ has degenerate directions, namely there exist points $z_1$ and $z_2$, such that $|K(z_1, z_2)|^2 = 1$. This implies that we have to quotient the parameters of the state space with respect to the equivalence relation

$$z \sim z' \quad \text{if} \quad |K(z, z')|^2 = 1.$$  

For a large class of constraints this results in an overlap kernel defined over the reduced state space [17]; this property is not in general guaranteed.

In any case, we obtain a family of coherent state paths $|\zeta(\cdot)|$ on $V_{\text{phys}}$, and a map from $V$ to $V_{\text{phys}}$ implemented through the coherent states as $|z(\cdot)\rangle \rightarrow |\zeta(\cdot)\rangle$, where $\zeta$ corresponds to the equivalence class to which $z$ belongs.

We exploited the Dirac method, in [7], for the simple example of the relativistic particle. However, we have found a more convenient quantization method that entails ideas from both the reduced state space and the Dirac quantization and it seems to be more suitably adapted to the needs of the histories scheme.

### 5.2. Reduced state space quantization

The key idea of the reduced state space quantization is to implement the constraints before quantization, at the level of the classical state space, and then to seek an appropriate quantum representation for the reduced state space.

It is easier, in general, to solve the constraints in classical theory than in the quantum theory—we have, for example, no problems with the continuous spectrum of operators or issues related to normal ordering. Nevertheless, the reduced state space is usually a manifold of non-trivial topological structure (not a cotangent bundle); often it is not even a manifold.

The reduced state space quantization scheme also has other drawbacks. For example, the reduced state space of field systems generically consists of non-local variables: the true degrees of freedom are not fields themselves. For this reason, the spacetime character of the theory is not explicitly manifest in the reduced state space. We have then no way to represent fields quantum mechanically and to study their corresponding symmetries. However, histories general relativity is an exception to this rule, because the spacetime diffeomorphisms group is represented both on the unconstrained and on the reduced state space [1].

Since histories quantization—in the form we presented here—relies strongly on the coherent states, it may be possible to exploit the fact that they provide a link between the classical state space description and the quantum mechanical one on a Hilbert space. Then we may implement the constraints at the classical level, while preserving the basic structures of Dirac quantization, namely the coexistence of a Hilbert space—corresponding to the full classical phase space—with the Hilbert space of the true degrees of freedom.

The resulting construction is a hybrid between the reduced state space quantization and Dirac quantization. Further details are presented elsewhere [20]. Here we shall explain the method with reference to the histories construction.

For the construction of a histories Hilbert space for the true degrees of freedom, it is sufficient to identify a set of coherent states on the reduced state space of the single-time theory. Let us denote as $|\zeta\rangle$ these coherent states, $\zeta \in \Gamma_{\text{red}}$. All information about the quantum theory is contained in the overlap kernel $K_{\text{red}}(\zeta, \zeta')$ which is a function on $\Gamma_{\text{red}} \times \Gamma_{\text{red}}$.

In Dirac quantization we start with the full Hilbert space, which contains a family of coherent states $|z\rangle$. The associated overlap kernel $K(z, z')$ is a function on $\Gamma \times \Gamma$, where $\Gamma$ is the state space of the system before the imposition of the constraints. The overlap kernel contains all information about the Hilbert space of the system and of the physical observables. The important issue is, then, to find a procedure in order to pass from a function on $\Gamma \times \Gamma$ to a function of $\Gamma_{\text{red}} \times \Gamma_{\text{red}}$, in accord with the geometric definition of the reduced state space.
We shall employ here a specific procedure that is based upon the geometric structure induced by the coherent states on the classical state space. Of relevance is the limiting behaviour of the overlap kernel \( \langle z | z' \rangle \) when \( z' = z + \delta z \),

\[
\langle z | z + \delta z \rangle = \exp \left( i A_i \left( \frac{\delta z_i}{2} \delta z^i - \frac{1}{2} g_{ij} \delta z^i \delta z^j \right) \right) + O(\delta z^3),
\]

and where \( A_i \) is a \( U(1) \)-connection 1-form and \( g_{ij} \) is a Riemannian metric on the classical state space

\[
i A_i(z) = \langle z | \partial_i z \rangle, \tag{5.4}
\]

\[
g_{ij}(z) = \langle \partial_i z | \partial_j z \rangle + A_i(z) A_j(z). \tag{5.5}
\]

The important point here is that this short-distance behaviour allows one to fully reconstruct the overlap kernel as a path integral \([21, 23]\)

\[
\langle z | z' \rangle = \lim_{\nu \to \infty} \mathcal{N}_\nu(t) \int Dz(\cdot) e^{i A - \frac{1}{2} \int_{t_0}^t ds g_{ij} \dot{z}_i \dot{z}_j}, \tag{5.6}
\]

where \( \mathcal{N}_\nu(t) \) is a factor entering for the purpose of correct normalization. In other words, the metric on phase space defines a Wiener process on phase space, which may be employed to regularize the usual expression for the coherent state path integral. An overlap kernel then needs two inputs for its construction: the connection 1-form on state space and the Riemannian metric. When the metric is not homogeneous a subtlety arises: the factor \( \mathcal{N}_\nu \) also depends on the phase space paths and enters the path integral \([22]\).

In the context of the reduced quantization procedure, the 1-form \( A \) may be easily defined on the reduced state space, because it is a symplectic potential of the corresponding symplectic form.

The key point is to identify a metric \( \tilde{g} \) on the reduced state space, starting from a metric \( g \) on the unconstrained state space \( \Gamma \). The key observation is that an element of the reduced state space—being an orbit of the constraint’s action—is a submanifold of \( \Gamma \). Two such submanifolds do not intersect, because different equivalence classes are always disjoint. One may then define a distance function between any two such submanifolds

\[
D_{\text{red}}(\zeta, \zeta') = \inf_{z \in \zeta, z' \in \zeta'} D(z, z'), \tag{5.7}
\]

where \( \zeta, \zeta' \in \Gamma_{\text{red}} \) and \( D \) is the distance function on \( \Gamma \) corresponding to the metric \( ds^2 \) on \( \Gamma \). The distance function \( D_{\text{red}} \) on \( \Gamma_{\text{red}} \) defines a metric on \( \Gamma_{\text{red}} \) that may be employed in the construction of the corresponding overlap kernel via the path integral (5.6).

5.3. The R–W minisuperspace model

Next we shall apply the scheme sketched above to the R–W minisuperspace model. We first select the family of coherent states on the single-time Hilbert space \( H \). We have previously explained that this choice should reflect the symmetries of the underlying theory. To the extent that the spacetime diffeomorphism symmetry of gravity has been lost in the reduction to the homogeneous and isotropic model we consider here, the choice is more or less arbitrary. We find it convenient to employ the standard Gaussian coherent states, parametrized by \( \lambda, \pi_\lambda, \phi, \pi_\phi, N, \pi_N \) with an overlap kernel
\begin{align*}
  \langle \lambda, \pi_\lambda, \phi, \pi_\phi, N, \pi_N | \lambda', \pi'_\lambda, \phi', \pi'_\phi, N', \pi'_{N'} \rangle \\
  &= \exp \left( i \pi_\phi \phi' - i \pi'_\phi \phi + i \pi_\lambda \lambda' - i \pi'_\lambda \lambda + i \pi_N N' - i \pi'_{N'} N \right) \\
  &\quad - \frac{1}{2} \left[ (\lambda - \lambda')^2 + (\pi_\lambda - \pi'_\lambda)^2 + (\phi - \phi')^2 - (\pi_\phi - \pi'_\phi)^2 \right] \\
  &\quad - (N - N')^2 - (\pi_N - \pi'_{N'})^2. \\
  \end{align*}

(5.8)

The corresponding state space metric is

\[ ds^2 = \frac{1}{2} \left( d\lambda^2 + d\pi_\lambda^2 + d\phi^2 + d\pi_\phi^2 + dN^2 + d\pi_N^2 \right). \]

(5.9)

The elements of the reduced state space correspond to surfaces of fixed \( \pi_\phi, f_\pm \), while \( \pi_\lambda = \pm \sqrt{\pi_\phi^2 + U(\lambda)} \). Each point of such a surface is characterized by the values of the parameters \( \lambda \) and \( N \). The \( N \) dependence is trivial and we shall disregard it. Hence the distance between two surfaces characterized by \( f_\pm, \pi_\phi \) and \( f'_\pm, \pi'_\phi \) respectively, is

\[ D(f_\pm, \pi_\phi; f'_\pm, \pi'_\phi) = \inf_{\lambda, \lambda'} \left\{ \left| \lambda - \lambda' \right|^2 + \left( \sqrt{\pi_\phi^2 + U(\lambda)} - \sqrt{\pi'_\phi^2 + U(\lambda')} \right)^2 \right\} \]

\[ + \left( f_\pm - f'_\pm + \pi_\phi \int_{\lambda}^{\lambda'} \frac{dx}{\sqrt{\pi_\phi^2 + U(x)}} \pm \pi'_\phi \int_{\lambda}^{\lambda'} \frac{dx}{\sqrt{\pi'_\phi^2 + U(x)}} \right)^2 - (\pi_\phi - \pi'_\phi)^2 \] 

(5.10)

To obtain the metric \( ds^2 \) on \( \Gamma_{\text{red}} \), we need to take the infimum in the above expression for neighbouring orbits, namely those for which \( f'_\pm = f_\pm + \delta f_\pm \) and \( \pi'_\phi = \pi_\phi + \delta \pi_\phi \).

First we consider the case that the parameter \( \lambda \) takes a value in the full real axis in every orbit. This is the case for \( \kappa = 0, -1 \), but also for \( \kappa = +1 \) in the regions \( U_- \) and \( U_+ \) of the reduced state space.

In this case, we observe that neighbouring orbits converge as \( \lambda \to \infty \), while they diverge exponentially at \( \lambda \to -\infty \). It is easy to show that the infimum is achieved for \( \lambda' \to \lambda \) as \( \lambda \to \infty \). This leads to the metric

\[ ds^2 = \frac{1}{2} \left( d\pi^2_\phi + d\Phi^2_\phi \right). \]

(5.11)

For the \( U_- \) and \( U_+ \) regions of the reduced state space in the case \( \kappa = +1 \), we should recall that the parameter \( \lambda \) takes values in \((-\infty, \lambda_1] \) and \( [\lambda_2, \infty) \) respectively. The corresponding orbits tend to converge near \( \lambda_1 \) and \( \lambda_2 \) and the local minimum value for the distance of the orbits corresponding to \( (\pi_\phi, f_\pm) \) and \( (\pi_\phi + \delta \pi_\phi, f_\pm + \delta f_\pm) \) equals

\[ \frac{1}{2} \left[ \delta f^2_\pm + \left( 1 + \frac{4 \pi^2_\phi}{U''(\lambda_{1,2})} \right) \delta \pi^2_\phi \right]. \]

(5.12)

In the \( U_- \) branch this value is a global minimum, because at \( \lambda \to \infty \) the orbits diverge. In the \( U_+ \) branch, however, the global minimum is achieved at \( \lambda \to \infty \), and corresponds to the metric of equation (5.11). Hence we conclude that the metric in the reduced state space has the value

\[ ds^2 = \frac{1}{2} \left[ df^2_\pm + \left( 1 + \frac{4 \pi^2_\phi}{|U''(\lambda, \pi_\phi)|^2} \right) d\pi^2_\phi \right], \]

(5.13)

in the region \( U_- \) (for re-collapsing universes), and the value (5.11) in the region \( U_+ \).
There is clearly a discontinuity at the boundary region $\pi^2 = |U_{\min}|$, where the metric (5.13) diverges. This divergence is, however, an artefact of the coordinates employed. The study of geodesics near the boundary demonstrates that the distance function remains finite.

Recall that at the boundary $\pi^2 = |U_{\min}|$, there exist five different orbits for each value of $f_{\pm}$ and sign of $\pi_\phi$. It is easy to verify that in the quantum theory the distance function between those five orbits vanishes. Hence, all five orbits are described by one single coherent state in the quantum theory. It follows that the divergent points of the classical reduced state space disappear in the quantum theory. This is an interesting result, because it suggests that the orbifold-like structure of the reduced state space may be generically ‘smeared’, when one passes to quantum theory.

Having identified the metric on the reduced state space, we employ the path integral (5.6) to construct the overlap kernel on the reduced state space. For the cases $\kappa = 0, -1$ this yields the familiar Gaussian coherent states

$$\langle \tilde{\pi}_\phi, \tilde{f}_\pm | \tilde{\pi}_\phi, \tilde{f}_\pm \rangle = \exp \left( \int dt \left[ i\pi_\phi \phi_t - i\pi_\phi \phi_t + i\pi_{\phi'} \phi_t - i\pi_{\phi'} \phi_t + i\pi_N N_t - i\pi_{N'} N_t \right] \right).$$

(6.4)

For $\kappa = 1$, the relevant metric is (5.13), for which it is very difficult to obtain an analytic solution for the path integral.

The general theory and the basic features for this type of path integrals are developed in [22].

We make a last comment here on the relation of this procedure to the Dirac quantization. The crucial difference in the present method is that it implements the constraints at the classical level. Both algorithms are expected to provide the same results on the semiclassical level. Whether the constraints are to be implemented at the quantum or the classical level remains an open issue, which cannot be settled a priori: both approaches yield a quantization method that provides the same classical limit. The choice between these attitudes to canonical quantization can only be determined by their eventual success.

6. Histories quantum R–W model

6.1. Representations of the Hilbert space

The prescription for the implementation of the constraints described in the previous section translates immediately to the history context. The first step is to construct the Hilbert space $\mathcal{V}$, which carries a representation of the history group with Lie algebra

$$[\lambda_t, \pi_{\lambda^t}] = i\delta(t, t')$$

(6.1)

$$[\phi_t, \pi_{\phi^t}] = i\delta(t, t')$$

(6.2)

$$[N_t, \pi_{N^t}] = i\delta(t, t').$$

(6.3)

The representation of the algebra (6.2)–(6.3) is selected by the choice of a canonical coherent states family, by means of the inner product (4.9). From the Gaussian coherent states (5.9) we obtain

$$\langle \lambda(-), \pi_{\lambda}(-), \phi(-), \pi_{\phi}(-), N(-), \pi_{N}(-) | \lambda'(-), \pi_{\lambda'}(-), \phi'(-), \pi_{\phi'}(-), N'(-), \pi_{N'}(-) \rangle$$

$$= \exp \left( \int dt \left[ i\pi_\phi \phi_t - i\pi_\phi \phi_t + i\pi_{\phi'} \phi_t - i\pi_{\phi'} \phi_t + i\pi_N N_t - i\pi_{N'} N_t \right] \right).$$

(6.4)

$$= -\frac{1}{2} \left[ (\lambda_t - \lambda_t')^2 + \frac{1}{w(t)} (\pi_{\lambda} - \pi_{\lambda'})^2 + (\phi_t - \phi_t')^2 \right]$$

$$= -\frac{1}{2} \left[ (\pi_{\phi} - \pi_{\phi'}^2) \right] - (N_t - N_t')^2 - \frac{1}{w(t)} (\pi_{N} - \pi_{N'})^2 \right).$$

(6.5)
The function \( w(t) \) is a density of weight 1 and it is introduced so that the definition of the integral is properly invariant with respect to diffeomorphisms—see the discussion in section 3.2. Different choices of \( w \) lead to different representations of the history group, but do not affect the probability assignment through the decoherence functional and we shall, henceforward, set it to be equal to 1.

Equation (6.5) allows one to identify \( \mathcal{V} \) with the Fock space \( e^{\mathcal{N}} \), where \( \mathcal{N} \) is the Hilbert space \( L^2(\mathbb{R}, C^1, dr) \). The history operators are defined by means of the one-parameter group of unitary transformations they generate. For example, writing the smeared operator \( \hat{\lambda}(f) = \int dt \lambda(f(t), t) \), we define

\[
e^{i\hat{\lambda}(f)}|\lambda(\cdot), \pi_x(\cdot), \phi(\cdot), \pi_\phi(\cdot), N(\cdot), \pi_N(\cdot)\rangle := |\lambda(\cdot), \pi_x(\cdot) + s\phi(\cdot), \phi(\cdot), \pi_\phi(\cdot), N(\cdot), \pi_N(\cdot)\rangle.
\]

(6.6)

Next we construct the Hilbert space of the true degrees of freedom \( \mathcal{V}_{\text{phys}} \). It is spanned by coherent state paths \( |\hat{\pi}_\phi(\cdot), \hat{f}_\pm(\cdot)\rangle \), that are constructed from the reduced coherent states \( |\hat{\pi}_\phi, \hat{f}_\pm\rangle \) of the standard canonical theory.

Note that in the case \( \kappa = 0, -1 \) the reduced overlap kernel is Gaussian

\[
\langle \hat{\pi}_\phi(\cdot), \hat{f}_\pm(\cdot)|\hat{\pi}_\phi(\cdot), \hat{f}_\pm(\cdot)\rangle = \exp\left(\int dt [i\hat{\pi}_\phi \hat{f}_\pm + i\hat{\pi}_\phi' \hat{f}_\pm' - \frac{1}{2}((\hat{\pi}_\phi - \hat{\pi}_\phi')^2 + (\hat{f}_\pm' - \hat{f}_\pm)^2)]\right). 
\]

(6.7)

The corresponding Hilbert space \( \mathcal{V}_{\text{phys}} \) is a direct sum of two copies of the Fock space \( e^{\mathcal{N}_{\text{phys}}} \), where \( \mathcal{N}_{\text{phys}} \) is the Hilbert space \( L^2(\mathbb{R}, C, dr) \). The one copy corresponds to the expanding and the other to the collapsing universe solutions.

6.2. The decoherence functional

We may also construct the decoherence functional on \( \mathcal{V}_{\text{phys}} \), which reads at the discrete level

\[
d(\xi_1, t_1; \xi_2, t_2; \ldots; \xi_n, t_n; \xi'_1, t'_1; \xi'_2, t'_2; \ldots; \xi'_m, t'_m) = \langle \xi'_m|\xi_m|\xi_{m-1} \rangle \cdots \langle \xi_2|\xi_1|\xi_0 \rangle \langle \xi_0|\beta_0|\xi_0 \rangle \times \langle \xi'_m|\xi'_1|\xi'_{m-1} \rangle \cdots \langle \xi'_2|\xi'_1|\xi'_0 \rangle.
\]

(6.8)

At the continuous-time limit

\[
d[\xi(\cdot), \xi'(\cdot)] \sim e^{i\hat{\mathcal{V}}[\xi(\cdot) - i\hat{\mathcal{V}}[\xi'(\cdot)]},
\]

(6.9)

where \( \hat{\mathcal{V}} \) is the classical Liouville function on the reduced state space.

It is important to remark that the reduced state space quantization allows one to construct the decoherence functional already at the level of the Hilbert space \( \mathcal{V} \). The coherent states construction employs the projection map \( \pi \) from the history constraint surface to the reduced state space, in order to construct a map between the associated coherent states

\[
i : |z(\cdot)\rangle \in \mathcal{V} \mapsto |\xi(\cdot)\rangle \in \mathcal{V}_{\text{phys}},
\]

(6.10)

provided that \( z \) is a path on the constraint surface. This map allows us to pull back the decoherence functional to \( \mathcal{V} \). Clearly, the pulled back decoherence functional has support only on coherent state paths on the constraint surface.

We compare the above result for the decoherence functional, with the one that results when applying the Dirac quantization scheme. To this end, we write a decoherence functional on \( \mathcal{V} \) with a delta function imposing the restriction of coherent state paths on the constraint
6.3. Initial conditions and the classical limit

If the space of temporal supports \( \mathcal{T} \) is the real line, then we need to restrict our considerations to coherent state paths \( \xi(\cdot) \) that converge asymptotically to a fixed value \( \xi \), otherwise the inner product will be bounded.

For each value of \( \xi \) we construct a different Hilbert space and a different histories theory. We usually choose \( \xi \) by the requirement that \( |\xi\rangle \) be the minimum energy state. However, in the physical Hilbert space of parametrized systems the Hamiltonian operator vanishes, therefore we have no criterion for the selection of the vacuum state. A different choice of the initial condition will yield a different representation for the histories theory. Therefore, different initial conditions define different theories, a fact that seems singularly attractive in the cosmological context, which is the only domain of physics in which the limit \( t \to -\infty \) may be taken literally\(^6\).

It is easy to study the classical limit of the model, since the Hamiltonian vanishes on the reduced state space. Omnés has shown in [24] that, for coarse-grained histories that correspond to smearing within a phase space region of typical size \( L \), much larger than one \( (h = 1) \), the off-diagonal elements of the decoherence functional fall rapidly to zero. The evolution peaks around the classical path with probability very close to 1.\(^7\) When we apply the above result in the context of the minisuperspace model, we note that histories peaked around the initial condition \( |\xi\rangle \) have probabilities very close to zero, hence, the system is adequately described by the corresponding classical solution.

It is interesting to provide an order of magnitude estimation for processes that do not contribute to a classical path, for example, processes that correspond to the various proposals about the initial condition of the universe.

For the minisuperspace model with \( \kappa = +1 \) we consider, for example, an initial state within the region \( U_\kappa \) (corresponding to a universe with a bounded radius), and we compute the probability of the realized history that corresponds to an expanding universe.

We consider an initial state within \( U_\kappa \) with \( \pi_\kappa = \epsilon \ll 1 \). This corresponds to a universe of radius \( e^\epsilon \), close to zero. Since the point \( e^\epsilon = 0 \) is excluded from the gravitational phase space

\(^6\) We remind the reader that the cosmological singularity lies outside the spacetime, so the topology of the space of temporal supports in cosmology is \((0, \infty)\) which is homeomorphic to the real line \( \mathbb{R} \).

\(^7\) This is valid for a large class of Hamiltonians including the trivial case of zero Hamiltonian.
(the metric is degenerate there) this is the closest we can get to an initial condition analogous to that of the tunnelling proposal for the wavefunction of the universe. For simplicity, we shall assume that the initial state is characterized by the value of \( f_\phi = 0 \)—within the present model this assumption does not affect the final results.

We may then ask about the probability that the universe is found sometime in the future within the region \( \mathcal{U}_< \), with \( \pi_\phi \simeq 0 \), a state that corresponds to the onset of inflation and evolves then according to the classical equations of motion.

We construct coarse-grained histories, in which the reduced state space is partitioned into cells, which are centred around specific values for \( \pi_\phi, f_\pm \). The off-diagonal elements of the decoherence functional between such histories will be very small (according to Omnès theorem) and the probability that the universe will be found in a cell \( C \) centred around a specific value of \( \pi_\phi, f_\pm \) will be approximately

\[
\int_C d\pi_\phi d f_\pm ||(\epsilon, 0|\pi_\phi, f_\pm)||^2 (1 + O(L^{-1})),
\]

(6.14)

provided that the distance between \( (\epsilon, 0) \) and \( (\pi_\phi, f_\pm) \) is much larger than \( L \).

Next we need to calculate the quantity \( ||(\epsilon, 0|\pi_\phi, f_\pm)||^2 \). We may estimate it taking into account the following considerations. For a general family of coherent states, we write the norm of the overlap kernel \( ||(z|z')||^2 \)

\[
|\langle z|z'\rangle|^2 = \left| \int dz_1 dz_2 \cdots \int dz_n (\langle z_1|z_1\rangle \langle z_2|z_2\rangle \cdots \langle z_n|z_n\rangle)^2 \right| \]
\[
< \int dz_1 dz_2 \cdots \int dz_n (\langle z_1|z_1\rangle \langle z_2|z_2\rangle \cdots \langle z_n|z_n\rangle)^2.
\]

(6.15)

For a large number of time steps the above expression may be expressed as a path integral. At the continuous 'time' limit we may consider that \( z_i' = z_i + \delta z_i \), in which case \( ||(z_i|z_i + \delta z_i)||^2 = e^{-2\delta z_i^2} \), in terms of the infinitesimal distance determined by the phase space metric. Hence,

\[
|\langle z|z'\rangle|^2 < \int \prod dz_i e^{-2\sum \delta z_i^2} := \int Dz(\cdot) e^{-2L^2(z(\cdot))},
\]

(6.16)

where the summation is over all paths \( z(\cdot) \) joining \( z \) and \( z' \), and \( L(\cdot) \) is the length of the path \( z(\cdot) \). Within the saddle point approximation we may write

\[
|\langle z|z'\rangle|^2 < c e^{-2Z^2(z,z')},
\]

(6.17)

where \( c \) is a constant of order unity, and \( D^2 \) is the distance function on the manifold, corresponding to the length of the geodesic joining \( z \) and \( z' \).

For the study of the R–W minisuperspace model, we are especially interested in the curve joining the point \( (\epsilon, 0) \in \mathcal{U}_< \) with the point \( (0, f_-) \in \mathcal{U}_> \). Since the regions \( \mathcal{U}_< \) and \( \mathcal{U}_> \) are only connected at the surface \( \pi_\phi = \pm \sqrt{|U_{\text{min}}|} \), then

\[
D^2[(\epsilon, 0), (0, f_-)] = D^2[(\epsilon, 0), (\sqrt{|U_{\text{min}}|}, v)] + D^2[(\sqrt{|U_{\text{min}}|}, v), (0, f_-)],
\]

(6.18)

for some point on the surface \( \pi_\phi = \pm \sqrt{|U_{\text{min}}|} \) characterized by the value \( v \) of the coordinate \( f_\pm \).

The second distance can be calculated using the metric equation (5.11) and reads

\[
D^2[(\sqrt{|U_{\text{min}}|}, f), (0, f_-)] = \frac{1}{2} |U_{\text{min}}| + (f_- - v)^2.
\]

(6.19)

The first distance concerns a path in \( \mathcal{U}_< \) and should be calculated by using the metric equation (5.13). However,

\[
\left(1 + \frac{4\pi_\phi^2}{|U'(\lambda_1, z)|^2}\right) \delta \pi_\phi^2 \geq \delta \pi_\phi^2,
\]

(6.20)
so that
\[
D^2[(\epsilon, 0), (\sqrt{|U_{\min}|}, v)] > \frac{1}{2}|U_{\min}| + v^2
\]
whence
\[
D^2[(\epsilon, 0), (0, f - \bullet)] > |U_{\min}| + \frac{1}{2}|v^2 + (v - f)^2| \geq |U_{\min}| + f^2.
\] (6.22)

It follows that
\[
\langle \epsilon, 0 | \pi | \phi, f \rangle^2 < c e^{-\left(\frac{2}{3}\lambda^3\Lambda^2 - f^2\right)} < c e^{-\left(\frac{2}{3}\lambda^3\Lambda^2\right)}.
\] (6.23)

Hence we have estimated an upper bound to the probability of ‘tunnelling’ from a universe of radius very close to 1, to a universe at the onset of inflation. Note that such a formula makes sense only for coarse-grained histories at a phase space scale \(L \gg 1\) and a distance \(D^2\) on phase space much larger than \(L\). It follows that \(D^2[(\epsilon, 0), (0, f - \bullet)]\) must be much larger than 1, hence \(\Lambda^2 \ll 1\). Clearly, for a cosmological constant of the order of the Planck scale (\(\Lambda \sim 1\)), it would make no sense to discuss tunnelling or a quasi-classical domain.

Our previous analysis has showed that the continuous-time decoherence functional may be constructed with specific initial states, in order to describe the scenario corresponding to the tunnelling initial proposal of the universe.

An interesting issue arises, whether it is possible to extend the formalism to other initial conditions, most notably the Hartle–Hawking proposal of no-boundary. At present, the formalism cannot be immediately applied because the Hartle–Hawking initial condition involves histories that do not correspond to a Lorentzian globally hyperbolic metric.

However, the history formalism is sufficiently versatile to allow for this scenario. The definition of histories only involves the postulate of a partial ordering that implements the causality. It is therefore applicable in principle to systems that have no Hamiltonian description. Such a generalization should involve an enlarged description of histories general relativity, in order to take into account the non-boundary 4-metrics. More importantly, it requires a different form for the decoherence functional. Another alternative would be a decoherence functional describing a quantum growth process, similar to that postulated in the causal set scheme [25]. This issue is at present under investigation.

7. Conclusions

In this paper, we studied the quantization of minisuperspace models within the framework of consistent histories. We showed that the histories description preserves the notion of causal ordering at the level of the true degrees of freedom, both classically and quantum mechanically.

The key problem of minisuperspace models, compared to general relativity, is that the imposition of the symmetry of homogeneity and isotropy destroys the spacetime diffeomorphism symmetry of general relativity. This leads to a number of problems at both the classical and the quantum level.

Classically, one may retain some traces of the diffeomorphism symmetry at the level of the history reduced state space. However, this involves the introduction of a metric-dependent foliation, which is rather unnatural in the setting of the minisuperspace model, unlike the case of general relativity. At the quantum level, the loss of general covariance implies that we have no reliable guide for the unique selection of a preferred representation for the history Hilbert space.

We proposed a version of reduced state space quantization, a procedure that was facilitated by the key role played by coherent states in the construction of the history Hilbert space. This procedure allows us to implement the constraints at the quantum mechanical level, and to estimate probabilities for interesting physical processes, such as the ‘tunnelling’ scenario for the initial state of the universe.
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References

[1] Savvidou N 2004 General relativity histories theory: I. The spacetime character of the canonical description Class. Quantum Grav. 21 615
Savvidou N 2004 General relativity histories theory: II. Invariance groups Class. Quantum Grav. 21 631
[2] Griffiths R 2003 Consistent Quantum Theory (Cambridge: Cambridge University Press)
Omnes R 1994 The Interpretation of Quantum Mechanics (Princeton, NJ: Princeton University Press)
Omnes R 1992 Consistent interpretations of quantum mechanics Rev. Mod. Phys. 64 339
Omnes R 1999 Understanding Quantum Mechanics (Princeton, NJ: Princeton University Press)
Gell-Mann M and Hartle J B 1990 Quantum mechanics in the light of quantum cosmology Complexity, Entropy and the Physics of Information ed W Zurek (Reading, MA: Addison-Wesley)
Hartle J B 1993 Spacetime quantum mechanics and the quantum mechanics of spacetime Proc. 1992 Les Houches School, Gravitation and Quantisation
[3] Isham C J 1992 Canonical Quantum Gravity and the Problem of Time in GIFT Seminar 0157-288 (Preprint gr-qc/9210011)
Kuchar K 1991 Time and interpretations of quantum gravity Winnipeg Proc. General Relativity and Relativistic Astrophysics p 211
[4] Isham C J 1994 Quantum logic and the histories approach to quantum theory J. Math. Phys. 35 2157
Isham C J and Linden N 1994 Quantum temporal logic and decoherence functionals in the histories approach to generalised quantum theory J. Math. Phys. 35 5452
Isham C J and Linden N 1995 Continuous histories and the history group in generalised quantum theory J. Math. Phys. 36 5392
Isham C, Linden N, Savvidou K and Schreckenberg S 1998 Continuous time and consistent histories J. Math. Phys. 37 2261
[6] Savvidou K 1999 The action operator in continuous time histories J. Math. Phys. 40 5657
Savvidou K 2002 Poincaré invariance for continuous-time histories J. Math. Phys. 43 3053
[7] Savvidou K and Anastopoulos C 2000 Histories quantization of parametrized systems: I. Development of a general algorithm Class. Quantum Grav. 17 2463
[8] Halliwell J J 2001 Trajectories for the wave function of the universe from a simple detector model Phys. Rev. D 64 044008
Halliwell J J and Thorwart J 2002 Life in an energy eigenstate: decoherent histories analysis of a model timeless universe Phys. Rev. D 65 104009
Craig D and Hartle J B 2004 Generalized quantum theory of recollapsing homogeneous cosmologies Phys. Rev. D 69 123525
[9] Vilenkin A 1984 Quantum creation of universes Phys. Rev. D 30 509
[10] Hartle J B and Hawking S W 1983 Wave function of the universe Phys. Rev. D 28 2960
[11] Anastopoulos C 2001 Continuous-time histories: observables, probabilities, phase space structure and the classical limit J. Math. Phys. 42 3225
[12] Klauder J and Skagerstam B (ed) 1985 Coherent States: Applications in Physics and Mathematical Physics (Singapore: World Scientific)
[13] Perelomov A 1986 Generalized Coherent States and their Applications (Berlin: Springer)
[14] Anastopoulos C 2003 Quantum processes on phase space Ann. Phys. 303 270
[15] Burch A 2004 Histories electromagnetism J. Math. Phys. 45 2153
[16] Anastopoulos C and Savvidou N 2003 The role of phase space geometry in Heisenberg’s uncertainty relations Ann. Phys. 308 329
[17] Klauder J 1997 Coherent state quantization of constraint systems Ann. Phys. 254 419
[18] Kempf A and Klauder J 2001 On the implementation of constraints through projection operators J. Phys. A: Math. Gen. 34 1019
[19] Hartle J B and Marolf D 1997 Comparing formulations of generalized quantum mechanics for reparametrization-invariant systems Phys. Rev. D 56 6247
[20] Anastopoulos C 2004 A geometric procedure for the reduced-state-space quantisation of constrained systems
Preprint gr-qc/0411130

[21] Klauder J and Daubechies I 1984 Quantum mechanical path integrals with Wiener measures for all polynomial
Hamiltonians Phys. Rev. Lett. 52 1161
Daubechies I and Klauder J 1985 Quantum mechanical path integrals with Wiener measures for all polynomial
Hamiltonians: 2 J. Math Phys. 26 2239

[22] Maraner P 1992 Landau ground level on Riemann surfaces Mod. Phys. Lett. A 7 2555
Alicki R, Klauder J R and Lewadowski J 1993 Landau-level ground state degeneracy and its relevance for a
general quantization procedure Phys. Rev. A 48 2358
Alicki R and Klauder J R 1996 Quantization of systems with a general phase space equipped with a Riemannian
metric: J. Phys. A: Math. Gen. 29 2475

[23] Klauder J R 1988 Quantization is geometry, after all Ann. Phys. 188 120
Klauder J R 1995 Geometric quantization from a coherent state viewpoint Preprint quant-ph/9510008

[24] Omnés R 1989 Logical reformulation of quantum mechanics: 4. Projectors in semiclassical physics J. Stat.
Phys. 57 357

[25] Rideout D P and Sorkin R D 2000 A classical sequential growth dynamics for causal sets Phys. Rev. D 61
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