CONVERGENCE OF SPHERICAL AVERAGES
FOR ACTIONS OF FUCHSIAN GROUPS

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Abstract. Pointwise convergence of spherical averages is proved for a measure-preserving action of a Fuchsian group. The proof is based on a new variant of the Bowen-Series symbolic coding for Fuchsian groups that, developing a method introduced by Wroten, simultaneously encodes all possible shortest paths representing a given group element. The resulting coding is self-inverse, giving a reversible Markov chain to which methods previously introduced by the first author for the case of free groups may be applied.

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1. Introduction

1.1. Formulation of the main result. Let $G$ be a finitely generated group with a symmetric set of generators $G_0$. For $g \in G$, denote by $|g|$ the length of the shortest word in $G_0$ representing $g$. Let $S(n)$ be the sphere of radius $n$ in $G$:

$$S(n) = \{ g \in G : |g| = n \}$$

Suppose that $G$ acts on a probability space $(X, \mu)$ by measure-preserving transformations $T_g$, $g \in G$. For a function $f \in L^1(X, \mu)$ consider spherical averages

$$S_n(f) = \frac{1}{\#S(n)} \sum_{g \in S(n)} f \circ T_g.$$  \hfill (1)

The main result of this paper, Theorem A below, gives the almost sure convergence of spherical averages for measure-preserving actions of Fuchsian groups and for $f \in L \log L(X, \mu)$.

Let $G$ be a Fuchsian group and let $\mathcal{R}$ be a fundamental domain for $G$. The images of $\mathcal{R}$ under the action of $G$ induce a tessellation $T_{\mathcal{R}} = \{ g\mathcal{R} : g \in G \}$ of the hyperbolic disc $\mathbb{D}$. Following [10], we say that $\mathcal{R}$ has even corners if the geodesic extension of every side of $\mathcal{R}$ is entirely contained in $T_{\mathcal{R}}$, more precisely in the union of boundaries of all domains $g\mathcal{R} \in T_{\mathcal{R}}$.

Let $v \in \mathbb{D}$ be a vertex of $T_{\mathcal{R}}$. If $\mathcal{R}$ has even corners, then the boundary of $T_{\mathcal{R}}$ in a small neighbourhood of $v$ consists of $n$ geodesic segments intersecting at $v$ and dividing our neighbourhood into $2n$ sectors. Write $n = n(v)$ and let $N(\mathcal{R})$ denote the number of sides of $\mathcal{R}$ inside $\mathbb{D}$. We need the following assumption on $\mathcal{R}$.

Assumption 1.1. 
\begin{enumerate}
  \item $\mathcal{R}$ has even corners,
  \item One of the following conditions holds for $\mathcal{R}$:
\end{enumerate}
• \( N(\mathcal{R}) \geq 5 \),

• \( N(\mathcal{R}) = 4 \) and either \( \mathcal{R} \) is non-compact or \( \mathcal{R} \) is compact and does not have two opposite vertices \( v,v' \) such that \( n(v) = n(v') = 2 \),

• \( N(\mathcal{R}) = 3 \) and \( \mathcal{R} \) is non-compact.

Let \( G_0 \) be the set of all group elements mapping \( \mathcal{R} \) to the domains of \( T_\mathcal{R} \) having a common side with \( \mathcal{R} \). As is well known, \( G_0 \) is a symmetric set of generators for \( G \). Our main result is the following:

**Theorem A.** Let \( G \) be a non-elementary Fuchsian group \( G \) and let \( \mathcal{R} \) be its fundamental domain satisfying Assumption A.1. Let \( G \) act on a Lebesgue probability space \((X,\mu)\) by measure-preserving transformations. Let \( G_0 \) be the set of generators of \( G \) mapping \( \mathcal{R} \) to the neighbouring domains. Denote by \( \mathcal{I}_{G_0^2} \) the sigma-algebra of sets invariant under all maps \( T_{g_1}T_{g_2} \), \( g_1,g_2 \in G_0 \). Then, for any function \( f \in L_{\log}L(X,\mu) \), as \( n \to \infty \), we have

\[
S_{2n}(f) \to E\left(f|\mathcal{I}_{G_0^2}\right) \quad \text{almost surely and in } L^1.
\]

The condition that \( \mathcal{R} \) have even corners is not as restrictive as it appears. In fact it is clear that our result only depends on the generators \( G_0 \) and the coding, and not on the precise geometry of \( \mathcal{R} \). Thus Theorem A extends immediately to any presentation of a Fuchsian group for which one can find deformed group \( G' \) which has a fundamental domain \( \mathcal{R}' \) with the same pattern of sides and side-pairings and even corners, see \([18,10]\) and \([43]\) for a detailed discussion. The need to restrict to spheres of even radius can be seen by considering the action of the free group \( F_2 \) on the two-element set \( \{0,1\} \) in which both generators of \( F_2 \) act by interchanging the elements, in which case the value of \( S_n(f) \) depends on the parity of \( n \). This seems to indicate that the condition on all relators having even length, as is implied by the even corner condition, may be essential.

The Cesàro convergence of the averages \( S_{2n}(f) \) is proven in \([18]\) using the Bowen—Series Markovian coding \([10]\), see also \([3,42,43]\), in order to reduce the statement to the ergodic theorem for Markov operators, cf. \([12,13]\). To establish convergence of spherical averages themselves, and thus of powers of our Markov operator, we develop the approach from \([14]\) for free groups. The argument of \([14]\) relies on a symmetry condition for the coding, which allows one to relate the Markov operator generated by the coding to its adjoint. The Bowen—Series coding of \([10]\) is however not symmetric, and the main construction of this paper is a new symmetric coding for Fuchsian groups.

This new coding is constructed using a variant of the coding introduced by Matthew Wroten \([46]\), see also a related idea in \([21,44]\). Wroten’s idea is to code all possible representations of a group element as a shortest word simultaneously. Then the set of all possible paths in the Markov chain can be inverted. It would be interesting to obtain a similar coding for a more general hyperbolic groups. In particular, it is not clear to us how to invert paths in the classical Cannon—Gromov coding \([20,29]\).

We now briefly describe Wroten’s approach in our setting. Every shortest word in the Fuchsian group \( G \) corresponds to a shortest path in the Cayley graph of \( G \) relative to \( G_0 \). This graph is embedded in \( \mathbb{D} \) by sending \( g \in G \) to \( gO \in \mathbb{D} \), where \( O \) is some fixed base.
point in int $\mathcal{R}$. Vertices $gO, hO$ are joined by an edge if and only if $g^{-1}h \in G_0$. If $\beta$ is a shortest path in the Cayley graph, we refer to the sequence of regions traversed by the edges of $\beta$ also as a shortest path. If $g \in G$ then the thickened path $[g]$ associated to $g$ is by definition the collection of all those $h\mathcal{R}, h \in G$ which are traversed by some shortest path from $\mathcal{R}$ to $g\mathcal{R}$. Every domain $h\mathcal{R} \in [g]$ is endowed with its index, which equals the distance in the Cayley graph from $\mathcal{R}$ to $h\mathcal{R}$. The set of all domains with index $k$ we will refer to as the level of $[g]$ and denote by $[g]_k$.

Now the coding works as follows. As above, let $G$ be a Fuchsian group with the set of generators $G_0$ associated to a fundamental domain $\mathcal{R}$, so that the assumptions of Theorem A hold. We will define a space of states $\Xi = \{X_1, \ldots, X_k\}$ and a $\Xi \times \Xi$ transition matrix $M = (M_{ij})$ such that $M_{ij} = 1$ if transition from $X_i$ to $X_j$ is possible and $M_{ij} = 0$ otherwise. There is a subset $\Xi_S \subset \Xi$ of start states, and another subset $\Xi_F \subset \Xi$ of end states. The states in $\Xi$ represents how $[g]_k$ and $[g]_{k+1}$ are attached to each other. It turns out that every $[g]_k$ contains at most two fundamental domains and the domains from $[g]_{k+1}$ are glued to the ones from $[g]_k$ across one, two or three sides, see Figure 7. We endow this geometrical configuration with some additional data to obtain a Markov chain generating thickened paths; in particular, the data records the generators needed to carry out the gluing. We then prove that thickened paths from $\mathcal{R}$ to $g\mathcal{R}$ with $|g| = n$ are in one-to-one correspondence with admissible sequences of length $n$ in this Markov coding starting in $\Xi_S$ and ending in $\Xi_F$.

The reversibility or self-inverse symmetry property of the coding can be expressed as follows. We introduce two maps $\gamma, \omega: \Xi \to G$, closely related to the attaching maps between $[g]_k$ and $[g]_{k+1}$, see Section 5.1. These maps satisfy relations given by Lemma 5.1. Following [14] we then construct Markov operators $P$ and $U$ on $L^1(X \times \Xi)$ which as a consequence of these relations satisfy $P^* = UPU$ and $U^* = U^{-1} = U$. Hence we can apply the Alternierende Verfahren method similar to [14]. The reason for the symmetry condition is that inverting a thickened path yields a thickened path and our coding preserves this symmetry.

Using symmetry, we next establish an inequality between $P^n$ and $(P^*)^k P^k$, which is the base for maximal inequality in the Alternierende Verfahren scheme. For free groups we have $cUP^{2n-1}\varphi \leq (P^*)^n P^n \varphi$ for any nonnegative $\varphi$. In the case of Fuchsian groups the inequality is more complicated and in particular containing an error term $A_n \varphi$, see (11) in Section 6 below. The underlying geometric meaning of this inequality, Lemma 7.6, is that for a majority of thickened paths the following holds. Consider a thickened path of length $2n + 1$, let $A$ and $B$ be its end domains, and let $D$ denote the central domain or domains of level $n + 1$. Then there exists a domain $\mathcal{C}$ such that the thickened paths $AD$ and $AC$ coincide from their initial point up to some point at distance at most a fixed bounded distance $n_0$ from $\mathcal{D}, \mathcal{C}$ and likewise the thickened paths $DB$ and $CD$ coincide except for at most $n_0$ terms from their initial points. The few thickened paths that cannot be embedded to a triangle in this manner give rise to the error term $A_n \varphi$ in inequality (11).
1.2. Organization of the paper. The paper is organized as follows. In the next section we give some notation and definitions regarding Fuchsian groups and their fundamental domains.

Section 3 is devoted to the central step in the proof of Theorem A. Here we give a description of shortest paths and thickened paths in terms of their local structure (see Corollary 3.16 and Proposition 3.23). To do this it is convenient to consider the class of shortest paths and the class of thickened paths simultaneously and to switch between them when necessary. A transition from a thickened path to a collection of shortest paths is immediate: every sequence of domains in the thickened path with growing indices is a shortest path. The inverse transition is obtained via the new convexification technique, which we also introduce in Section 3. We show that the thickened path between the ends of a given shortest path is the convexification of the latter, that is, a minimal convex union of fundamental domains that contains this path. The convexification can be obtained in a number of convexification steps, where each step adds several domains to remove concave angle at one point of the path boundary.

In Section 4 thickened paths are represented as realizations of a topological Markov chain, which is based on the local description of thickened paths obtained in the previous section. Using this Markov chain, in Section 5 we represent the spherical averages in terms of a Markov operator associated to this Markov coding. In the next section we prove a general theorem on pointwise convergence of powers of a Markov operator (Theorem 6.6). Finally, in Section 7 we apply this theorem to the operator associated with our Markov coding and conclude the proof of Theorem A.

1.3. Historical remarks. For two rotations of a sphere, convergence of spherical averages was established by Arnold and Krylov [1], and a general mean ergodic theorem for actions of free groups was proved by Guivarc’h [30].

A first general pointwise ergodic theorem for convolution averages on a countable group is due to Oseledeets [38] who relied on the martingale convergence theorem.

The first general pointwise ergodic theorems for free semigroups and groups were given by R.I. Grigorchuk in 1986 [20], where the main result is Cesàro convergence of spherical averages for measure-preserving actions of a free semigroup and group. Convergence of the actual spherical averages for free groups was established by Nevo [33] for functions in $L_2$ and Nevo and Stein [35] for functions in $L_p$, $p > 1$ using spectral theory methods. Nevo, Stein, and Margulis [36, 32] considered ball averages for actions of connected semisimple Lie group with finite center and no nontrivial compact factors and showed that these ball averages converge almost everywhere and in $L^p$, $p > 1$. Note that, as shown by Tao [45], whose argument is inspired by Ornstein’s counterexample [37], pointwise convergence of spherical averages for functions in $L^1$ does not hold even for actions of free groups.

The method of Markov operators in the proof of ergodic theorems for actions of free semigroups and groups was suggested by R. I. Grigorchuk [27, 28], J.-P. Thouvenot (oral

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We hope in future version to be able to replace the convexification process using methods derived from [3].
communication), and in [12]. In [14] pointwise convergence is proved for Markovian spherical averages under the additional assumption that the Markov chain be reversible. The key step in [14] is the triviality of the tail sigma-algebra for the corresponding Markov operator; this is proved using Rota’s “Alternierende Verfahren” [41], that is to say, martingale convergence. The reduction of powers of the Markov operator to Rota’s “Alternierende Verfahren” in [14] essentially relies on the reversibility of the Markov chain. In this paper, the proof of Theorem A is based on a general result on convergence of Markov operators, which is an extension of the result from [14], and its proof also goes along the same lines, see Section 6. Another result in this direction was obtained in [5]; it states the mean convergence for analogues of spherical averages for an arbitrary Markov chain satisfying very mild conditions. It is not known whether similar result holds for pointwise convergence.

The study of Markovian averages is motivated by the problem of ergodic theorems for general countable groups, specifically, for groups admitting a Markovian coding such as Gromov hyperbolic groups [29] (see e.g. Ghys—de la Harpe [23] for a detailed discussion of the Markovian coding for Gromov hyperbolic groups). The first results on convergence of spherical averages for Gromov hyperbolic groups, obtained under strong exponential mixing assumptions on the action, are due to Fujiwara and Nevo [22]. For actions of hyperbolic groups on finite spaces, an ergodic theorem was obtained by L. Bowen in [4]. Cesàro convergence of spherical averages for all measure-preserving actions of Markov semigroups, and, in particular, Gromov hyperbolic groups, was established in [15, 17]; earlier partial results were obtained in [11, 13]. In the special case of hyperbolic groups a shorter proof of this theorem, using the method of Calegari and Fujiwara [19], was later given by Pollicott and Sharp [39]. Using the method of amenable equivalence relations, Bowen and Nevo [6, 7, 8, 9] established ergodic theorems for “spherical shells” in Gromov hyperbolic groups. For further background see the surveys [34, 25, 16].

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2. Definitions and notation

2.1. Tessellation and labelling. Let $G$ be a finitely generated non-elementary Fuchsian group acting in the hyperbolic disk $\mathbb{D}$ with fundamental domain $\mathcal{R}$, which we assume to be closed. We suppose $\mathcal{R}$ to be a finite-sided convex polygon with vertices contained in $\mathbb{D} \cup \partial \mathbb{D}$, such that the interior angle at each vertex is strictly less than $\pi$. By a side of $\mathcal{R}$ we mean the closure in $\mathbb{D}$ of the geodesic arc joining a pair of adjacent vertices. We allow the infinite area case in which some adjacent vertices on $\partial \mathbb{D}$ are joined by an arc contained in $\partial \mathbb{D}$; we do not count these arcs as sides of $\mathcal{R}$. Further we usually mean by vertices of $\mathcal{R}$ only vertices inside $\mathbb{D}$. Sometimes it is convenient to count as vertices also ends of sides that belong to $\partial \mathbb{D}$, this instances will be specified explicitly. Two sides are adjacent if they share a common vertex lying in $\mathbb{D}$. 


We assume that the sides of \( R \) are paired; that is, for each side \( s \) of \( R \) there is a (unique) element \( e \in G \) such that \( e(s) \) is also a side of \( R \) and the domains \( R \) and \( e(R) \) are adjacent along \( e(s) \). (Notice that this includes the possibility that \( e(s) = s \), in which case \( e \) is elliptic of order 2 and the side \( s \) contains the fixed point of \( e \) in its interior. The condition that the vertex angle is strictly less than \( \pi \) excludes the possibility that the fixed point of \( e \) is counted as a vertex of \( R \).)

![Figure 1. Labelling the sides of the fundamental domain \( R \). The label \( e \) appears interior to \( R \) on the side of \( R \) adjacent to the region \( e^{-1}R \).](image)

We denote by \( \partial R \) the union of the sides of \( R \), in other words, \( \partial R \) is the part of the boundary of \( R \) inside the disk \( \mathbb{D} \). Each side of \( \partial R \) is assigned with two labels, one interior to \( \partial R \) and one exterior, in such a way that the interior and exterior labels are mutually inverse elements of \( G \). We label the side \( s \subset \partial R \) interior to \( R \) by \( e \) if \( e \) carries \( s \) to another side \( e(s) \) of \( R \), while we label the same side exterior to \( R \) by \( e^{-1} \), see Figure 1. With this convention, \( R \) and \( e^{-1}(R) \) are adjacent along the side \( s \) whose interior label is \( e \), while the side \( e(s) \) has interior label \( e^{-1} \).

Let \( G_0 \) be the set of labels on sides of \( R \). The labelling extends to a \( G \)-invariant labelling of all sides of the tessellation \( T_R \) of \( \mathbb{D} \) by images of \( R \). (By a side of \( T_R \), we mean a side of \( gR \) for some \( g \in G \).) The conventions have been chosen in such a way that if two regions \( gR, hR \) are adjacent along a common side \( s \), then \( h^{-1}g \in G_0 \) and the label on \( s \) interior to \( gR \) is \( h^{-1}g \), while that on the side interior to \( hR \) is \( g^{-1}h \).

Suppose that \( O \) is a fixed basepoint in \( \mathbb{R} \) and that \( \gamma \) is an oriented path in \( \mathbb{D} \) from \( O \) to \( gO \), \( g \in G \), which avoids all vertices of \( T_R \), passing through in order adjacent regions \( R = g_0R, g_1R, \ldots, g_nR = gR \). Then the labels of the sides crossed by \( \gamma \), read in such a way that if \( \gamma \) crosses from \( g_{i-1}R \) into \( g_iR \) we read off the label \( e_i = g_{i-1}^{-1}g_i \) of the common side interior to \( g_iR \), are in order \( e_1, e_2, \ldots, e_n \) so that \( g = e_1e_2\ldots e_n \). This proves the well known fact that \( G_0 \) generates \( G \), see for example [2].

As explained in the introduction, the fundamental domain \( R \) is said to have even corners if for each side \( s \) of \( R \), the complete geodesic in \( \mathbb{D} \) which extends \( s \) is contained in the sides of \( T_R \). This condition is satisfied for example, by the regular \( 4g \)-gon of interior angle \( \pi/2g \) whose sides can be paired with the standard generating set \( \{a_i, b_i, i = 1, \ldots, g \mid \Pi[a_i, b_i] \} \) to form a surface of genus \( g \). For further discussion on the even corner condition, see the references in the introduction.
3. A local description of thickened paths

In this section we describe the structure of shortest paths and thickened paths (as defined in the introduction) between pairs of fundamental domains. This description is invariant under interchange of direction and the ends of the path, and provides the basis for a symmetric Markov coding constructed in the next section. Throughout this section we assume that \( G \) and \( R \) satisfy assumptions of Theorem A.

The argument is rather indirect. Namely, we define in local geometric terms a class of locally shortest paths, which is subsequently seen to coincide with the class of shortest paths. Similarly, we consider a class of unions of fundamental domains constrained by explicit local rules and then we prove that this class coincides with the class of thickened paths with indices defined above. As we have said in the introduction, the main tool in this section is a convexification procedure that transforms a shortest path to a thickened path.

3.1. Locally shortest paths. Let \( R = (R_0, \ldots, R_N) \) be a path of fundamental domains, i.e. \( R_i \) and \( R_{i+1} \) share a common side \( s_i \), and the sides of \( R_i \) common with \( R_{i-1} \) and \( R_{i+1} \) are different.

We start by giving a definition of the boundary of a path of fundamental domains \( R \), which works well even if the path has self-tangencies or self-overlaps, and for paths without self-tangencies or self-overlaps this definition gives the usual boundary of the polygonal domain \( \bigcup_i R_i \) oriented counterclockwise.

Each side \( s_i \) is co-oriented (from \( R_i \) to \( R_{i+1} \)) and thus oriented, so we refer to the ends of this side as the left and the right ends. The complement of \( s_i \) in the boundary of \( R_i \) has two connected components. One connects the left ends of \( s_{i-1} \) and \( s_i \), the other connects the right ones, we denote them by \( \partial L R_i \) and \( \partial R R_i \) respectively. The boundary of the path \( R \) is the oriented closed curve in \( \mathbb{D} \) that goes counter-clockwise along \( \partial R_0 \setminus s_0 \), then along \( \partial R_1 \setminus s_1 \), \( \partial R_N \setminus s_N \), \( \partial L R_{N-1} \), \( \partial L R_{N-2} \), \( \partial L R_{N-3} \), \( \partial L R_1 \) and meets with its beginning at the left end of \( s_0 \). If \( l \) consecutive paths in this list are empty, there are \( l + 2 \) consecutive domains touching the same vertex \( v \) (possibly \( v \in \partial \mathbb{D} \)). We will call the angle in \( v \in \mathbb{D} \) convex (less than \( \pi \)) if \( l + 2 < n(v) \), straight (equal to \( \pi \)) if \( l + 2 = n(v) \), concave (larger than \( \pi \)) if \( l + 2 > n(v) \) (including the case when \( l + 2 > 2n(v) \)). Let us also say that the angle is minimally concave if \( l + 2 = n(v) + 1 \) and minimally convex if \( l + 2 = n(v) - 1 \).

**Definition 3.1.** A path \( R = (R_0, \ldots, R_N) \) of fundamental domains is called locally shortest if the following holds. Consider any segment of \( \partial R \) that lies inside \( \mathbb{D} \). Let \( v_0, v_1, \ldots, v_M \) be the consecutive vertices on this segment. Then the boundary angle in every vertex \( v_i \) is convex, straight, or minimally concave, and if the angles in \( v_i \) and \( v_j \), \( i < j \), are minimally concave then there is a vertex \( v_k \), \( i < k < j \), where the boundary has a convex angle.

We shall see that a path is locally shortest if and only if it is shortest. We start by check the ‘if’ part. The ‘only if’ part is established till Corollary 3.21.
Lemma 3.2. 1) Any shortest path is locally shortest.
2) Let $A$ and $B$ be two fundamental domains. Take $a \in A$, $b \in B$ and suppose that the geodesic segment $I = ab$ does not pass through vertices. Then the sequence $\mathcal{R} = (R_0 = A, R_1, \ldots, R_N = B)$ of domains intersected by $I$ is a locally shortest path.

Proof. 1) Assume that a shortest path $\mathcal{R}$ has a non-minimally concave angle at a vertex $v$, that is, there are $n(v) + 2$ consecutive regions $R_i, \ldots, R_{i+n(v)+1}$ around $v$ in our path. But then $\mathcal{R}$ is not the shortest since one can reach $R_{i+n(v)+1}$ from $R_i$ going around $v$ in another direction and it takes $n(v) - 1$ steps rather than $n(v) + 1$ steps for the original path.

Similarly, let $v_0 = u, v_1, \ldots, v_k = u'$ be a sequence of boundary vertices such that the angles at $u$ and $u'$ are minimally concave, and those at $v_1, \ldots, v_{k-1}$ are straight. Let $R_i$ be the first domain in $\mathcal{R}$ touching $u$ and $R_j$ be the last domain touching $u'$. Then it takes $j - i = n(v_0) + \sum_{l=1}^{k-1}(n(v_l) - 1) + n(v_k)$ steps to go from $R_i$ to $R_j$ along $\mathcal{R}$. But one can see that it takes only $\sum_{l=0}^{k-1}(n(v_l) - 1) = j - i - 2$ steps if we go along the other side of the segment $uu'$. The original and the shorter paths are shown on Figure 2 by solid and dashed arrows respectively.

2) Consider the $2n(v)$ sectors around the vertex $v$. If the segment $I$ intersects at least $n(v) + 2$ of them, then it has at least $n(v) + 1$ intersections with geodesic lines separating them, i. e. $I$ intersects one of these $n(v)$ lines twice, which is impossible.

Similarly, assume that the boundary has two concave vertices $u$ and $u'$ and the angles at all vertices between them are straight. Let $\ell$ be a line connecting $u$ and $u'$. Then $I$ should cross $\ell$ twice: once on each connected component of $\ell \setminus uu'$, see Figure 2 which gives the desired contradiction. $\square$

3.2. Convexification in terms of the boundary curve: a locally shortest path does not touch itself. In this subsection we prove the following statement.

Lemma 3.3. Let $\mathcal{R} = \{R_0, \ldots, R_N\}$ be a locally shortest path. Then domains $R_i$ and $R_j$ can share a vertex $v$ if and only if $v$ is a common vertex of all $R_k$ for $k = i, \ldots, j$.

Lemma 3.3 implies that a locally shortest path $\mathcal{R}$ has no self-intersections, even by a side or a vertex on its boundary, so we may define the boundary of $\mathcal{R}$ as the usual
boundary of the domain $\bigcup_i \mathcal{R}_i$ oriented in the counterclockwise direction rather than use the definition from the previous subsection.

The proof of Lemma 3.3 occupies the rest of the subsection. We start with the definition of a class of almost convex curves. Informally speaking, a closed curve is almost convex if it satisfies the following. It goes along the sides of the tessellation, the tangent vector of our curve makes only one turn, in any vertex the tangent vector turns either to the left or exactly by one sector to the right and any two right turns have a left turn between them. In particular, the boundary of any locally shortest path $\mathcal{R}$ is an almost convex curve. We proceed to the formal definition. Note that some care is needed in the case of non-compact fundamental domains.

Consider a curve $\Gamma$ consisting of finitely many sides of the tessellation $T_\mathcal{R}$ and arcs of $\partial \mathbb{D}$. Let $v \in \Gamma$ be a vertex of $T_\mathcal{R}$, so $v \in \mathbb{D}$ is a final point of a side $s \subset \Gamma$ and an initial point of a side $s' \subset \Gamma$. Consider the sector to the left of $\Gamma$ at $v$, that is, the sector at $v$ swept by a ray going from $s$ to $s'$ in the clockwise direction. This sector covers several petals at $v$, an we will say that the angle of $\Gamma$ at $v$ is (minimally) convex, straight, or (minimally) concave if the number of covered petals satisfies the same inequalities as specified in the first paragraph of Subsection 3.1.

**Definition 3.4.** An oriented closed curve $\Gamma \in \mathbb{D}$ is called almost convex if it satisfies the following conditions.

(1) $\Gamma$ consists of finitely many sides of the tessellation $T_\mathcal{R}$ and arcs of $\partial \mathbb{D}$.

(2) Let $v \in \mathbb{D}$ be a vertex of $T_\mathcal{R}$ such that it is the final point of a side $s \subset \Gamma$ and the initial point of a side $s' \subset \Gamma$. Then $s \neq s'$ and the angle of $\Gamma$ at $v$ is convex, straight, or minimally concave. Vertices where the angle of $\Gamma$ is minimally concave will be called right turns.

(3) Let $I = vv'$ be a maximal geodesic segment in $\Gamma$, $v, v' \in \mathbb{D}$. Then $v$ and $v'$ cannot be right turns simultaneously.

(4) Every arc of $\Gamma$ lying on $\partial \mathbb{D}$ goes in the counterclockwise direction. Also, let $v$ be an isolated point of $\Gamma \cap \partial \mathbb{D}$, thus $v \in \partial \mathbb{D}$ is the final point of a side $s \subset \Gamma$ and the initial point of a side $s' \subset \Gamma$. Then $s'$ lies to the left of $s$, that is, if $w, w' \in \partial \mathbb{D}$ be the other ends of the geodesics containing $s, s'$, then the points $v, w, w'$ appear on $\partial \mathbb{D}$ in this clockwise circular order.

(5) The curve $\Gamma$ makes one turn in counterclockwise direction. That is, let $\gamma: \mathbb{S}^1 \to \Gamma$ be any parametrization of $\Gamma$ spending nonzero time in each vertex joining sides/arcs. Define a map $\delta: \mathbb{S}^1 \to \partial \mathbb{D}$ as follows. If $\gamma(t)$ lies inside a side $s$ then $\delta(t)$ is the final point in the direction of $\Gamma$ of the geodesic containing $s$; if $\gamma(t)$ lies inside some arc on $\partial \mathbb{D}$, then $\delta(t) = \gamma(t)$.

It remains to define $\delta$ for the intervals corresponding to the endpoints of sides and arcs of $\Gamma$. If $\gamma(t) \equiv v \in \mathbb{D}$ for $t \in [t_0, t_1]$, define $\delta_{||t_0, t_1]}(t)$ as a continuous function joining $\delta(t_0 - 0)$ and $\delta(t_1 + 0)$ so that all these $\delta(t)$ belongs to the sector between the rays $v \delta(t_0 - 0)$ and $v \delta(t_1 + 0)$ that measures less than $\pi$. If $\gamma(t) \equiv v \in \partial \mathbb{D}$ for $t \in [t_0, t_1]$, define $\delta_{||t_0, t_1]}(t)$ as a continuous function joining $\delta(t_0 - 0)$ and
\( \delta(t_1 + 0) \) in the counterclockwise direction. For the constructed map \( \delta \) we require that \( \deg \delta = 1 \).

A straightforward induction in \( N \) gives the following statement.

**Proposition 3.5.** The boundary \( \partial R \) of a locally shortest path \( R = \{ R_0, \ldots, R_N \} \) is an almost convex curve.

Now we define a convexification procedure that transforms an almost convex curve into a convex one (i.e. those without right turns).

**Definition 3.6.** A flower at a vertex \( v \in \mathbb{D} \) is the union of all fundamental domains that have \( v \) on their boundary; these domains will be called petals.

![Figure 3. The convexification step at \( v \)](image)

**Definition 3.7.** Consider an almost convex curve. A convexification step at a vertex \( v \) with a right turn is defined as follows. Let \( s, s' \) be the sides of the boundary before and after \( v \). Consider the flower at \( v \) and modify the boundary as follows: at the starting point \( p \) of \( s \) turn right to the boundary of the flower of \( v \) and go counter-clockwise along this boundary until reaching the final point \( p' \) of \( s' \), then proceed along the original path. (Figure 3 shows the original and the resulting curves by dashed and dotted lines respectively.)

**Proposition 3.8.** 1) A convexification step at a vertex \( v \) of an almost convex curve \( \Gamma \) again yields an almost convex curve \( \hat{\Gamma} \).
2) After finitely many convexification steps the curve \( \Gamma \) becomes a curve without right turns. Moreover, if \( \Gamma \) becomes a curve without right turns after consecutive convexification steps at the vertices \( v_1, \ldots, v_k \), then the set \( \{ v_1, \ldots, v_k \} \) coincides with the set \( A(\Gamma) \) of all vertices \( v \) of the curve \( \Gamma \) satisfying one of the following three conditions:
   (i) \( v \) has a right turn,
   (ii) \( v \) has a straight angle, and if \( I \) is a maximal geodesic arc in \( \Gamma \) containing \( v \), one of the ends of \( I \) has a right turn,
   (iii) the angle of \( \Gamma \) at \( v \) is minimally convex, and if \( I_- \), \( I_+ \) are maximal geodesic arcs in \( \Gamma \) adjacent to \( v \), then the other ends of both these arcs have right turns.
Proof. 1) The first four conditions in Definition 3.4 are clear. The last condition is obtained as follows. The curve $\Gamma_0$ around the added petals (i.e. the light gray area on Figure 3) is also almost convex. Join the curves $\Gamma$ and $\Gamma_0$ together and eliminate their common segment. So we obtain a curve $\hat{\Gamma}$ with $\deg \hat{\delta} = \deg \delta + \deg \delta_0 - 1 = \deg \delta$.

2) Clearly, if a convexification step can be performed in $v$ then $v \in A(\Gamma)$ (in fact, $v$ should satisfy condition (i)). Also there are no possible steps for $\Gamma$ if and only if $A(\Gamma) = \emptyset$. Thus it remains to prove that $A(\hat{\Gamma}) = A(\Gamma) \setminus \{v\}$.

First consider a vertex $u \in \hat{\Gamma}$ in the added path. Then the sector to the left of $\hat{\Gamma}$ at $u$ covers one or two petals at $u$, so the angle of $\hat{\Gamma}$ at $u$ is at most straight. Moreover, let $w$ and $w'$ be the vertices on the added path that are adjacent to $p$ and $p'$ respectively. Then for $u = w, w'$ the sector covers one petal, hence the angle at $u$ is convex. Therefore, all vertices between $p$ and $p'$ do not belong to $A(\hat{\Gamma})$.

Special care is needed when the added path from $p$ to $p'$ contains at most two sides, i.e. either $N(R) = 3$ and there are one or two added petals, or $N(R) = 4$ and there is one added petal. The latter case means that $n(v) = 2$, hence by condition (ii) of Assumption 1.1, either $w = w'$, $p$, or $p'$ lies on $\partial D$, or $R$ is compact and $n(w) > 2$, so (iii) fails for $w$. The former case $N(R) = 3$ is even simpler: for one added petal either $p$ or $p'$ lies on $\partial D$, for two added petals at least one of the following holds: $w$ lies on $\partial D$ or both $p$ and $p'$ lie on $\partial D$. Therefore, in these cases we still have that the added vertices do not belong to $A(\hat{\Gamma})$.

For any vertex $u$ with a right turn denote by $I(u)$ the union of two maximal geodesic segments of $\Gamma$ adjacent to $u$, Let $I(\Gamma)$ be the union of interiors of $I(u)$ over all $u$ with a right turn. Then the vertices with condition (i) or (ii) are those belonging to $I(\Gamma)$, and the vertices with condition (iii) are those belonging to $\text{int clos } I(\Gamma) \setminus I(\Gamma)$ and having minimally convex angle.

After the convexification step at $v$ the boundary angle is increased only at $p$ and $p'$, where this angle is increased by one petal. Therefore, the only possible new vertices with concave angle are $p$ and $p'$. If, say, the angle of $\partial \hat{\Gamma}$ at $p$ is minimally concave then the angle of the angle of $\partial \Gamma$ at $p$ is straight, so $p \in \text{int } I(v)$ and $\hat{I}(p)$ is the union of $[p, w]$ and the part of $I(v)$ lying on the other side of $p$ with respect to $v$.

On the other hand, for any vertex $u \neq v$ we have $I(u) \subset \hat{I}(u)$. Assume that $I(u) \neq \hat{I}(u)$. Then the angle at an end $z$ of $I(u)$ for $\hat{\Gamma}$ is larger than that for $\Gamma$. This is possible only if $z$ is one of the vertices $p$ and $p'$. Moreover, as the angle at $z$ of $\hat{\Gamma}$ is greater by one petal than that of $\Gamma$, the angle of $\Gamma$ at $z$ must be minimally convex. In this case we have $\hat{I}(u) = I(u) \cup [p, w]$ if $z = p$ and $\hat{I}(u) = I(u) \cup [p', w']$ if $z = p'$. Therefore, $I(\Gamma)$ and $I(\hat{\Gamma})$ coincide on the common segment of the curves $\Gamma$ and $\Gamma'$ between points $p$ and $p'$, so any vertex other than $p$ and $p'$ belongs to $A(\Gamma)$ and $A(\hat{\Gamma})$ simultaneously.

To conclude the proof it remains to check that each of the points $p$ and $p'$ belongs to $A(\Gamma)$ and $A(\hat{\Gamma})$ simultaneously. Consider the point $p$. If $p$ is internal for $I(v)$, then $p$ is also internal for $\hat{I}(p)$, so $p$ lies in both $I(\Gamma)$ and $I(\hat{\Gamma})$. If $p$ is the end of $I(v)$, and not the end of another segment $I(u)$, then $p$ does not belong to $\hat{I}(u)$ for any $u$, so $p$ does not lie in
both $A(\Gamma)$ and $A(\hat{\Gamma})$. Now suppose that $p$ is the common end of $I(v)$ and $I(u)$. Assume first that the angle at $p$ for $\Gamma$ is convex but not minimally convex. The sector to the left of $\hat{\Gamma}$ in $p$ is larger by one petal than those for $\Gamma$, so the angle of $\hat{\Gamma}$ at $p$ is convex, hence for $\hat{\Gamma}$ the vertex $p$ is the end of only one segment $I(u) = I(u)$, and $p$ does not belong to $A(\Gamma)$ and $A(\hat{\Gamma})$. Finally, if the angle of $\Gamma$ at $p$ is minimally convex, then $p$ satisfies condition (iii) for $\Gamma$ and condition (ii) for $\hat{\Gamma}$, so $p$ belongs to both $A(\Gamma)$ and $A(\hat{\Gamma})$. \hfill \qed

Recall the following statement for the affine plane: if a smooth closed curve $\gamma$ always turns to the left: $\text{Vol}(\hat{\gamma}, \gamma) \geq 0$, and $\gamma$ makes one complete turn, then $\gamma$ has no self-intersections. The next statement is a hyperbolic analogue of this fact.

**Proposition 3.9.** An almost convex curve $\Gamma$ with no right turns has no self-intersections.

**Proof.** Let us relax conditions on $\Gamma$: instead of condition 1 of Definition 3.4 we assume only that $\Gamma$ consists of finitely many geodesic segments and arcs of $\partial \mathbb{D}$, we do not need the requirement that these segments are sides of $T_R$. Condition 2 now says that the angles of $\Gamma$ in all vertices are not greater than $\pi$, and condition 3 is trivial.

Assuming that $\Gamma$ has self-intersections, we can perturb $\Gamma$ inside the class specified in the previous paragraph so that the intersection takes place either in an internal point of two arcs on $\partial \mathbb{D}$, or in an internal point of two geodesic segments belonging to different lines. Denote this perturbed curve by the same symbol $\Gamma$, let $\gamma$ and $\delta$ be defined as in condition 4 of Definition 3.4 and let $\Delta: \mathbb{R} \to \mathbb{R}$ be a lift of $\delta$ to the universal covers, thus we have $\Delta(t + 1) = \Delta(t) + 2\pi$. Since $\Gamma$ has no right turns, $\Delta$ is nondecreasing.

If $v = \gamma(t_1) = \gamma(t_2)$, $t_1 < t_2 < t_1 + 1$ is a common point of two arcs of $\Gamma$ on $\partial \mathbb{D}$, then $\Delta(t_1) \equiv \Delta(t_2)$ (mod $2\pi$), hence either $\Delta(t_2) = \Delta(t_1) + 2\pi$ or $\Delta(t_2) = \Delta(t_1) + 2\pi$. Both these equalities lead to contradiction: in the former case we have $\Delta(t_2 - \varepsilon) < \Delta(t_1)$ for small $\varepsilon > 0$ due to condition 4 of Definition 3.4 so $\Delta$ is non-monotonic, and the latter case is similar.

Now assume that $v = \gamma(t_1) = \gamma(t_2)$, $t_1 < t_2 < t_1 + 1$ is a common point of two geodesic segments of $\Gamma$. Denote the lines containing these segments by $\ell_i = a_i b_i$ ($i = 1, 2$), $\ell_1 \neq \ell_2$, where $a_i$ and $b_i$ are the initial and the final points of these geodesic lines on $\partial \mathbb{D}$. As $\ell_1$ and $\ell_2$ intercross each other, the ends of $\ell_1$ and the ends of $\ell_2$ are interlaced in the cyclical order on $\partial \mathbb{D}$. So we may assume that the order is $a_1, a_2, b_1, b_2$ in counterclockwise direction; the other case is reduced to this one by the exchange of $\ell_1$ and $\ell_2$.

Consider the part $\gamma([t_1, t_2])$ of our curve $\Gamma$. Note that $\delta([t_1, t_2])$ lies on the arc of $\partial \mathbb{D}$ going from $b_1$ to $b_2$ in the counterclockwise direction, so $\delta([t_1, t_2])$ lies on the left half-disc $H$ with respect to $a_1 b_1$. Then one can inductively show that all consecutive segments and arcs of $\gamma([t_1, t_2])$ belong to $\text{clos} H$, hence $\gamma([t_1, t_2]) \subset \text{clos} H$. But $\gamma(t_2 - \varepsilon)$ lies in the right half-disc with respect to $a_1 b_1$, so we get a contradiction. \hfill \qed

**Proof of Lemma 3.3.** Assume the contrary and consider a minimal subpath $R' = \{R_i, \ldots, R_j\}$ that still violates the conclusion of this lemma. Then $\partial R'$ intersects itself, and, due to
minimality of $\mathcal{R}'$, any point of self-intersection belongs only to the first and the last domains in $\mathcal{R}'$. Hence this intersection is either a vertex or a side of $\partial \mathcal{R}_i$ and $\partial \mathcal{R}_j$ not adjacent to $s_i = \mathcal{R}_i \cap \mathcal{R}_{i+1}$ and $s_{j-1} = \mathcal{R}_{j-1} \cap \mathcal{R}_j$.

Apply the convexification procedure to the boundary of $\{\mathcal{R}_i, \ldots, \mathcal{R}_j\}$. Note that the convexification step can be applied to no vertex $v \in \partial \mathcal{R}_i \setminus s_i$. Indeed, $\partial \mathcal{R}'$ has a convex angle at $v$, as there is only one petal in the sector to the left of $\partial \mathcal{R}'$ at $v$. Even if the angle at $v$ is minimally convex, condition (iii) of Proposition 3.8 cannot hold there since at least one of the neighbours of $v$ also belongs to $\partial \mathcal{R}_i \setminus s_i$ and hence also has a convex boundary angle. The case $N(\mathcal{R}) = 3$ needs special consideration: here if the only vertex $v$ of $\partial \mathcal{R}_i \setminus s_i$ lies in $\mathcal{D}$, then one of its neighbours, i.e. the ends of $s_i$, should lie on $\partial \mathcal{D}$, hence condition (iii) does not hold for $v$. Therefore, the convexified curve still has a self-intersection, contradicting Proposition 3.9. □

3.3. Convexification in terms of domains: structure of path convexifications.

Keeping in mind that convexification does not give rise to self-intersections, consider the set of fundamental domains inside our curve during the convexification procedure. Initially the curve is $\partial \mathcal{R}$ and the domains inside it are $\mathcal{R}_0, \ldots, \mathcal{R}_N$. The convexification step at $v$ increases the set of domains inside the curve by the $n(v) - 1$ remaining petals at $v$, these petals are shown by the light gray area on Figure 3. The next lemma endows this collection of domains with indices. Note that properties 1–5 of this lemma hold for a thickened path $[g]$ when index $k$ is assigned to domains from $[g]_k$.

**Lemma 3.10.** Let $\mathcal{R} = (\mathcal{R}_0, \ldots, \mathcal{R}_N)$ be a locally shortest path. Then one can assign an index from 0 to $N$ to every domain of its convexification in such a way that:

1) every $\mathcal{R}_i$ has index $i$;
2) the sequence of indices for consecutive domains bordering $\partial_L [\mathcal{R}]$ or $\partial_R [\mathcal{R}]$ is precisely $\{0, \ldots, N\}$;
3) if two domains have a common side, their indices differ by one;
4) every domain with an index $i$ borders domains with indices $i - 1$ (provided $i \geq 1$) and $i + 1$ (provided $i \leq N - 1$);
5) only $\mathcal{R}_0$ has index 0, only $\mathcal{R}_N$ has index $N$.

**Proof.** Straightforward induction. The properties 1–5 hold for the initial collection of domains $\{\mathcal{R}_0, \ldots, \mathcal{R}_N\}$, and are preserved by convexification. Indeed, if a convexification step is performed at the vertex $v$, then property 2 implies that the domains bordering $v$ have indices $i, i + 1, \ldots, i + n(v)$, and the sides of the boundary adjacent to $v$ are incident to domains with indices $i$ and $i + n(v)$. Hence we may endow the added domains from the flower at $v$ with the indices $i + 1, \ldots, i + n(v) - 1$. □

Denote the convexification of the path $\mathcal{R} = (\mathcal{R}_0, \ldots, \mathcal{R}_N)$ by $[\mathcal{R}]$ and the union of domains in $[\mathcal{R}]$ with index $i$ by $[\mathcal{R}]_i$. Since all convexification steps are performed only at the vertices of $\partial \mathcal{R}$, the set $[\mathcal{R}]_i$ can contain at most three elements: one from the original curve, one from the convexification step at a vertex from $\partial_L \mathcal{R}$ and one from the
convexification at a vertex from $\partial_R \mathcal{R}$. Let us show that in fact there are at most two elements in every $[\mathcal{R}]_i$.

**Proposition 3.11.** Consider a sequence $u = (u_0, \ldots, u_k)$ of adjacent vertices along the boundary of a locally shortest path $\mathcal{R}$ so that all $u_j$ belong to the set $A(\mathcal{R}) := A(\partial_R \mathcal{R})$ of vertices where convexification steps occur. The sequence $u$ is assumed to be oriented from $\mathcal{R}_0$ to $\mathcal{R}_N$. Let $u_0u_0$ be the border between $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ with minimal possible $i$, $u_ku_k$ be the border between $\mathcal{R}_j$ and $\mathcal{R}_{j+1}$ with maximal possible $j$. Then all vertices between $v_0$ and $v_k$ do not belong to $A(\mathcal{R})$.

**Proof.** We start with the following statement.

**Claim 3.12.** Let $u$ be a vertex on $\partial_L \mathcal{R}$. Consider all domains $\mathcal{R}_i, \ldots, \mathcal{R}_{i+a}$ bordering $u$ and denote $\text{opp}(u) = \partial_R \mathcal{R}_{i+1} \cup \cdots \cup \partial_R \mathcal{R}_{i+a-1}$. Then no internal vertices of $\text{opp}(u)$ belong to $A(\mathcal{R})$.

Proof repeats the first part of the proof of item 2 in Proposition 3.8 $\text{opp}(u)$ is a part of the boundary of the flower at $u$, and we have seen there that a convexification step cannot be applied at any of its internal vertices. $\square$

This claim yields that no vertex $v$ between $v_0$ and $v_k$ can border three domains: otherwise $\text{opp}(v)$ is nonempty, so $\text{opp}(v)$ either contains an internal vertex $u_l$, which then fails to belong to $A(\mathcal{R})$, or is non-compact (if $N(\mathcal{R}) = 3$), cutting $u$ into two sequences. Therefore, every vertex $v$ between $v_0$ and $v_k$ can satisfy only condition (ii) or (iii) of Proposition 3.8 and the former is possible only if $n(v) = 2$ and $v$ is incident to two domains in $\mathcal{R}$.

Further, there exists $u_j$ satisfying condition (i) and thus bordering at least three domains in $\mathcal{R}$. Then $\text{opp}(u_j)$ is either non-compact or has internal vertex, hence it is not possible to have a continuous sequence of vertices in $A(\mathcal{R})$ between $v_0$ and $v_k$. Therefore, the only remaining case is when several vertices adjacent to $v_0$ or to $v_k$ satisfy condition (ii).

Let $v_1$ be the vertex adjacent to $v_0$ on $\partial_R \mathcal{R}$ and assume that $v_1$ satisfy (ii) in Proposition 3.8. Due to Claim 3.12 this is possible only if $u_0$ borders two domains (otherwise $v_1$ is internal to $\text{opp}(u_0)$). Therefore, $u_0$ satisfy (ii), $n(u_0) = 2$, and $\partial_L \mathcal{R}_{i+1}$ contains one side. Then $\partial_R \mathcal{R}_{i+1}$ contains $N(\mathcal{R}) - 3$ sides, so if $N(\mathcal{R}) \geq 5$, $v_1$ borders only one domain in $\mathcal{R}$.

Finally, assume $N(\mathcal{R}) = 4$. Since $u_0, \ldots, u_k$ contains a vertex satisfying condition (i) of Proposition 3.8 we have $k \geq 1$, i.e. $\mathcal{R}_{i+1}$ is a compact quadrilateral $u_0u_1v_1v_0$. Thus we arrive at a contradiction with condition (ii) in Assumption 1.1 as $n(u_0) = n(v_1) = 2$. $\square$

Now consider a maximal sequence $u = (u_0, \ldots, u_k)$ of adjacent vertices in $A(\mathcal{R})$, it can jump between the left and the right boundaries. The previous proposition yields that such sequence is uniquely defined by one its term. Namely, let $u_0$ belong to the left boundary of $\mathcal{R}$. Then we should proceed along the left boundary while possible: for intermediate vertices there are no vertices from $A(\mathcal{R})$ adjacent to them via sides $s_i$. Reaching the last
vertex \( u_s \) in the continuous sequence in \( A(\mathcal{R}) \cap \partial \mathcal{R} \) it is possible to continue the sequence \( u \) only via the side \( s_j \) incident to \( u_s \) with maximal \( j \). So if the other end \( u_{s+1} \) of \( s_j \) belongs to \( A(\mathcal{R}) \), we can proceed only along the right boundary in the same direction from \( \mathcal{R}_0 \) to \( \mathcal{R}_N \), and \( j \) is the minimum of all \( k \) such that \( s_k \) is incident to \( u_{s+1} \), and so on.

**Proposition 3.13.** Let \( \mathcal{R} \) be a locally shortest path.

1) Let \( u = (u_0, \ldots, u_k) \) be a maximal sequence of adjacent vertices from \( A(\mathcal{R}) \). Let \([i, j+1]\) be the set of indices \( t \) such that \( \mathcal{R}_t \) is incident to a vertex from \( u \). Then \( \mathcal{R}_i \) (resp., \( \mathcal{R}_{j+1} \)) intersects \( u \) only by \( u_0 \) (resp., \( u_k \)).

2) Let \( u' = (u'_0, \ldots, u'_l) \) be another such sequence, let \([i', j'+1]\) be the same segment as above for \( u' \). Then the segments \([i, j+1]\) and \([i', j'+1]\) are either non-intersecting or have a common end. If, say, \( j+1 = i' \), then \( \mathcal{R}_{j+1} \) is incident to \( u_k \) and \( u'_0 \) and they are not adjacent.

**Proof.** 1) If \( u_0 \) and \( u_1 \) lies on the same side of the boundary, then \( u_0 \) borders at least two domains, and only the last of them contains \( u_1 \). That is, \( \mathcal{R}_i \), being the first domain incident to \( u_0 \), does not contain \( u_1 \). Similarly, if \( u_0 \) and \( u_1 \) lie on the different sides of the boundary, then \( u_0 \) satisfies condition (i) of Proposition 3.8 and hence is incident to at least three domains from \( \mathcal{R} \). Therefore, \( u_0 u_1 \) is a common side of the last two domains in \( \mathcal{R} \) that are incident to \( u_0 \), hence \( u_1 \) does not belong to the first domain \( \mathcal{R}_i \) incident to \( u_0 \).

2) As we have seen above, all domains with indices from the segment \([i+1, j]\) have no vertices in \( A(\mathcal{R}) \) except of those in \( u \). Hence \([i+1, j] \cap [i', j'+1] = \emptyset \). The last statement in the proposition is clear: if \( u_k \) and \( u'_0 \) were adjacent, \( u \) and \( u' \) might be joined into one sequence.

The next step is to describe the geometry of a sequence \( u \) from the last proposition.

**Definition 3.14.** A curve going along sides of \( \mathbf{T}_\mathcal{R} \) is called *almost straight* if the following holds: in every vertex the angle is either straight or off by one sector to the left or to the right; and there are no two same-side turns with only straight angles between them.

**Figure 4.** An example of the adjacency graph for the domains in a non-trivial section. Dashed lines represents boundaries of domains, the curves \( u_{\alpha, \beta} \) are shown in bold. Letters at the bottom of the figure represent types of states of the Markov coding defined in Section 4.

**Proposition 3.15.** Let \( u = (u_0, \ldots, u_k) \) and \([i, j+1]\) be the same as in the previous proposition. Then the following holds.
1) The convexification steps at $u_0, \ldots, u_k$ add exactly one domain with each index from $i + 1$ to $j$.

2) The corresponding adjacency graph has the structure shown on Figure 4: there are two sequences of graph vertices $R_i \rightarrow T_{i+1} \rightarrow \cdots \rightarrow T_j \rightarrow R_{j+1}$ and $R_i \rightarrow B_{i+1} \rightarrow \cdots \rightarrow B_j \rightarrow R_{j+1}$ and there are several “crossings”, i.e. edges of the form $T_i \rightarrow B_{i+1}$ or $B_i \rightarrow T_{i+1}$.

3) Let $u_{1,2}^j u_0$ be the two sides of $R_i$ adjacent to $u_0$, and $u_k u_{k+1}^{1,2}$ be the two sides of $R_{j+1}$ adjacent to $u_k$. Then for any choice of $\alpha, \beta \in \{1, 2\}$ the curve $2_{\alpha,\beta} = (u_{\alpha}^1 u_{0} u_1 \ldots u_k u_{\beta}^1)_{k+1}$ is almost straight.

**Proof.** We start with the proof of statements 1 and 2. Arrange the domains $R_i, \ldots, R_{j+1}$ into the top and bottom rows as follows: all domains $R_s$, $i + 1 \leq s \leq j$ having vertices from $A(R)$ only on their left (respectively, right) boundary are placed on the bottom row: $B_s := R_s$ (respectively, on the top row: $T_s := R_s$). Further, if a domain $R_s$ has vertices from $A(R)$ on both parts of its boundary, then both ends of either $R_{s-1} \cap R_s$ or $R_s \cap R_{s+1}$ belong to $u$. If, say, $u_t u_{t+1} = R_s \cap R_{s+1}$ and $u_t$ belongs to the left boundary of $R$ and $u_{t+1}$ to the right one, we place $R_s$ to the bottom row and $R_{s+1}$ to the top row, and vice versa. Finally, we place $R_i$ and $R_{j+1}$ on the different rows from $R_{i+1}$ and $R_j$ respectively.

Let us define a “pit” as a series of already-defined domains $T_k, B_{k+1}, \ldots, B_s, T_{s+1}$ so that $T_{k+1}, \ldots, T_l$ are still undefined. This pit is “opened to the top”, there are “opened to the bottom” pits with symmetric conditions.

One can check that a convexification step preserves the following properties. (i) Every opened to top (resp., bottom) pit corresponds to the maximal continuous sequence of vertices on the left (resp., right) boundary where the convexification step is allowed but not yet performed. The domains of the pit are exactly those that border vertices from this sequence. (ii) Every minimal cycle (i.e. those without edges inside) in the constructed adjacency graph corresponds to a vertex where the convexification step has been already performed. Domains of the cycle are exactly those bordering this vertex. The cycle has the form of either a trapezoid or a parallelogram.

There are three possible cases when applying convexification step at vertex $v$ belonging to a sequence associated to a pit $T_k, B_{k+1}, \ldots, B_s, T_{s+1}$. Firstly, $n(v) + 1$ already defined domains bordering $v$ can comprise the whole pit. Then the convexification step at $v$ adds the remaining petals $T_{k+1}, \ldots, T_l$, and the pit is completely removed. Secondly, these domains can belong to the side of the pit, say, they are $T_k, B_{k+1}, \ldots, B_{s+1}$. Then we denote the added domains as $T_{k+1}, \ldots, T_s$ with the crossing edge $T_s \rightarrow B_{s+1}$ at the end, and the pit is shortened. Finally, all these $n(v) + 1$ domains can belong to the bottom row: $B_{s+1}, \ldots, B_{s+1}$. Then the added domains are denoted as $T_{s+1}, \ldots, T_s$ and the pit is split into two, as well as the sequence of vertices where convexification step is not yet performed. Clearly, the statements (i) and (ii) above are preserved in any case.

We proceed to statement 3. Edges of the cycle corresponding to the vertex $u_t$ correspond to sides between domains incident to $u_t$. Crossings, which belong to two cycles, correspond to sides of the form $u_t u_{t+1}$. One can see that if the cycle corresponding to $u_t$ is a parallelogram, then the sides $u_{t-1} u_t$ and $u_t u_{t+1}$ are $n(u_t)$ sectors apart, i.e. they form
the straight angle. Similarly, wide-bottom and wide-top trapezoids correspond to minimal left and right turns respectively. As wide-top and wide-bottom trapezoids should interleave, we obtain that the curve $u_{\alpha, \beta}$ is almost straight. The choice of $\alpha$ and $\beta$ corresponds to the placement of $R_i$ and $R_{j+1}$ on the top or the bottom rows.

**Corollary 3.16.** Consider the adjacency graph for the convexification of a locally shortest path $R$. Then if $[R]_i$ and $[R]_{j+1}$, where $j > i$, contain one domain each and all $[R]_s$, $i+1 \leq s \leq j$, contain more than one domain, then the adjacency graph for $[R]_i \cup \cdots \cup [R]_{j+1}$ is exactly what is obtained in the previous proposition.

**Remark 3.17.** Clearly, if $j = i$ in this corollary, then $[R]_i$ and $[R]_{i+1}$ contain one domain each, and the adjacency graph for $[R]_i \cup [R]_{i+1}$ contains the only possible edge $R_i \rightarrow R_{i+1}$.

### 3.4. Convexified paths and thickened paths.

The aim of this subsection is to characterize all locally shortest paths inside the convexification of a given locally shortest path. As a result we will obtain that all locally shortest paths between two domains have the same length and hence they are indeed shortest.

**Lemma 3.18.** Let $[R]$ be a convexification of a locally shortest part $R = (R_0, \ldots, R_N)$.  
1) Let $S = (S_0, \ldots, S_M)$ be a locally shortest path inside $[R]$ such that $S_0 = R_0$, $S_M = R_N$. Then the index in $[R]$ of every domain $S_i$ equals $i$, and hence $M = N$.  
2) Let $S = (S_0, \ldots, S_N)$ be any path inside $[R]$ such that $S_i$ has index $i$ in $[R]$. Then $S$ is locally shortest and $S = [R]$.

**Proof.** First of all, note that if $\# [R]_i = 1$, then $S$ contains the only domain $R_i$ with index $i$. Therefore, we can analyze only pieces of $S$ inside each section described in Corollary 3.16. For an individual section Proposition 3.15 gives the structure of the adjacency graph. Let $S_i = R_i, S_{i+1}, \ldots, S_{j+1} = R_{j+1}$ be a part of $S$ in this section.

Consider firstly only the initial part of the path $S$ going in the positive direction (i.e. with indices increasing). Since adjacent domains have indices differing by one, every $S_i$ in this part of the path has the index $i$.

**Claim 3.19.** Assume that $S$ goes along the cycle corresponding to a vertex $v$ on the bottom row, and that before that $S$ goes only in positive direction. Consider the previous crossing where $S$ goes from the top to the bottom row, let $u$ be the vertex corresponding to the cycle after this crossing. Then the part $\Gamma_{u,v}$ of the left boundary of $S$ from the vertex previous to $u$ to the vertex $v$ is an almost straight curve, and either $\Gamma_{u,v}$ has no non-straight angles, or the first non-straight angle of $\Gamma_{u,v}$ is minimally concave. Moreover,  
(i) if the last crossing before the cycle corresponding to $v$ is “top to bottom” then the last non-straight angle in $\Gamma_{u,v}$ is minimally concave;  
(ii) if this crossing is “bottom to top” then the last non-straight angle in $\Gamma_{u,v}$ is minimally convex or $\Gamma_{u,v}$ curve has no non-straight angles at all.

Indeed, if $v$ is adjacent to $u$, then either the cycle at $u$ has the form of a wide-top trapezoid, thus $S$ goes through $n(u)$ domains of this cycle, the angle at $u$ is straight, and this agrees with case (ii), or the cycle has the form of a parallelogram with $n(u) + 1$ of its
domains belonging to $\mathcal{S}$, and this agrees with case (i). The inductive step from a cycle $v$ to the next cycle $v'$ is considered similarly.

Now let us pass to the proofs of the lemma statements.

1) Suppose that at some moment the sequence $\mathcal{S}$ goes in the negative direction. As $\mathcal{S}$ cannot go forward and backward along the same edge, there are two possibilities: $\mathcal{S}$ goes either forward along the row and then backward along a crossing, or forward along a crossing and then backward along the row. Assume that before this maneuver $\mathcal{S}$ goes along the bottom row of the cycle corresponding to a vertex $v$. There are three possibilities for the position of the crossing before this cycle: this crossing can go “bottom to top”, “top to bottom”, or it can be “top to bottom” and $\mathcal{S}$ goes along this crossing. So we get six cases shown in Figure 5. In the cases shown as a), b), e), and f) the vertex $v$ borders $n(v) + 2$ domains in $\mathcal{S}$ hence $\partial\mathcal{S}$ is not an almost convex curve. In the remaining cases shown as c) and d) the angle of $\partial\mathcal{S}$ at $v$ is minimally concave, but by the last claim the previous non-straight angle on $\partial\mathcal{S}$ is also minimally concave, so again $\partial\mathcal{S}$ fails to be almost convex. Therefore, $\mathcal{S}$ goes in the positive direction, hence the index of $\mathcal{S}_i$ in $[\mathcal{R}]$ equals $i$, and the first statement is proven.

2) Consider a part of $\mathcal{S}$ between two successive crossings, and let $u$ and $v$ be the vertices corresponding to the first and the last cycles between these crossings. Apply the last claim to the vertex $v$ and consider separately cases (i) and (ii). In the first case the cycle of $v$ has the form of a wide-top trapezoid with $n(v)$ of its domains lying in $\mathcal{S}$, so the boundary angle at $v$ is straight. In the second case the cycle of $v$ has the form of a parallelogram, $v$ is incident to $n(v) + 1$ domains of $\mathcal{S}$ and the boundary angle at $v$ is minimally concave. Finally, if $u = v$ then $v$ is incident to $n(v) + 1$ domains of $\mathcal{S}$ and the boundary angle at $v$ is minimally concave. Therefore, in any case $\partial\mathcal{S}$ contains the sequence of vertices from $u$ to $v$, the angles of $\partial\mathcal{S}$ at these vertices are minimally convex, straight, or minimally concave, the convex and the concave angles alternate, and the first and the last non-straight angles are concave. Proposition 3.8 now yields that all vertices of this sequence from $u$ to $v$ belong to $A(\mathcal{S})$. Consequently, the convexification procedure for $\mathcal{S}$ includes the convexification steps at all vertices corresponding to the cycles in the adjacency graph for $[\mathcal{R}]$, thus $[\mathcal{S}] \supset [\mathcal{R}]$. The inverse inclusion follows from the fact
that \([\mathcal{S}]\) is a minimal convex union of fundamental domains containing \(\mathcal{S}\), and that \([\mathcal{R}]\) is convex.

**Lemma 3.20.** Any two locally shortest paths \(\mathcal{R} = (R_0, \ldots, R_N)\) and \(\mathcal{S} = (S_0, \ldots, S_M)\) with \(R_0 = S_0 = A\) and \(R_N = S_M = B\) have the same convexification and the same length.

**Proof.** Consider the intersection \([\mathcal{R}] \cap [\mathcal{S}]\) of their convexifications. It is a convex set. Take any geodesic segment \(ab\) with \(a \in A\) and \(b \in B\) that does not pass through vertices. By Lemma 3.2 the sequence of domains intersected by \(ab\) is a locally shortest path \(T \subset [\mathcal{R}] \cap [\mathcal{S}]\) going from \(A\) to \(B\). Lemma 3.18 then states that \([\mathcal{R}] = [T] = [\mathcal{S}]\) and that the lengths of \(\mathcal{R}\) and \(\mathcal{S}\) are both equal to those of \(T\). □

**Corollary 3.21.** All locally shortest paths are shortest.

**Corollary 3.22.** Let \(\mathcal{R} = (R_0, \ldots, R_N)\) be any (locally) shortest path from \(A = R_0\) to \(B = R_N\). Then the thickened path from \(A\) to \(B\) coincides with \([\mathcal{R}]\) and the set of all shortest paths going from \(A\) to \(B\) is the set of all paths in \([\mathcal{R}]\) going in the positive direction.

**Proof.** All shortest paths belong to \([\mathcal{R}]\) by the previous lemma, hence the thickened path lies in \([\mathcal{R}]\). On the other hand, every domain \(S_j \subset [\mathcal{R}]_j\) can be included in a path in \([\mathcal{R}]\) going from \(A\) to \(B\) in the positive direction: we choose arbitrarily \(S_{j+1}\) adjacent to \(S_j\), then we choose \(S_{j+2}, \ldots, S_N\), as well as \(S_{j-1}, S_{j-2}, \ldots, S_0\), this is possible due to Lemma 3.10. □

We conclude this section with the converse of Proposition 3.15.

**Proposition 3.23.** Let \(\mathcal{S}\) be a family of fundamental domains with indices from 0 to \(N\) satisfying properties 1–5 from Lemma 3.10. Denote by \([\mathcal{S}]_i\) the union of domains with index \(i\) and by \(A\) and \(B\) the only domains with indices 0 and \(N\) respectively. Suppose that \(\mathcal{S}\) is convex, and every section \([\mathcal{S}]_i \cup \cdots \cup [\mathcal{S}]_{j+1}\) such that \([\mathcal{S}]_i\) and \([\mathcal{S}]_{j+1}\) contain only one domain, and all intermediate \([\mathcal{S}]_k\) contain at least two domains, has the structure described in Proposition 3.15. Then \(\mathcal{S}\) is a thickened path from \(A\) to \(B\).

**Proof.** Consider the path \(\mathcal{R} = (R_0, \ldots, R_N)\) going from \(A\) to \(B\) along the left boundary of \(\mathcal{S}\). All vertices of \(\partial \mathcal{R}\) belonging to the boundary of \(\mathcal{S}\) has convex or straight angles, and we need to check that the vertices of the boundary of \(\mathcal{R}\) that lies inside a nontrivial section of \(\mathcal{S}\) satisfy conditions for locally shortest path. But the part of \(\partial \mathcal{R}\) inside the nontrivial section is a curve of the form \(u_{\alpha, \beta}\) from Proposition 3.15. This curve is almost straight hence \(\partial \mathcal{R}\) is an almost convex curve and \(\mathcal{R}\) is a locally shortest path.

Now we need to check that \([\mathcal{R}] = \mathcal{S}\). This is done exactly in the same way as in the second statement of Lemma 3.18. Finally, Corollary 3.22 shows that \(\mathcal{S}\) is a thickened path from \(A\) to \(B\). □

### 4. The Markov coding

In this section we construct a Markov coding generating the set of thickened paths. This coding is based on the description of the structure of thickened paths in the local terms given in Propositions 3.15 and 3.23.
As it was stated in the introduction, states of this topological Markov chain describe how the ‘past’ level $S_- = [S]_i$ of a thickened path is attached to its ‘future’ level $S_+ = [S]_{i+1}$. More specifically, a state of the Markov chain describes the arrangement of $S_-$ and $S_+$ up to the action of $G$. However, we have to endow these arrangements with additional data to construct our coding. This is done in Subsection 4.1.

In the next subsection we define the transition matrix $\Pi$ of the coding and show that this coding indeed generates all thickened paths. Subsection 4.3 shows that the constructed Markov chain has a time-reversing involution on the set of states. Finally, in Subsection 4.4 we will show that our Markov chain is strongly connected and aperiodic.

4.1. States of the Markov chain. As we have seen in Proposition 3.15 the adjacency graph for the domains in $S_- \cup S_+$ has one of the following types:

A. $\#S_- = \#S_+ = 1$, and the graph contains the only possible edge from $S_-$ to $S_+$. This corresponds to a trivial section of the thickened path described in Remark 3.17.

B. $\#S_- = 1$, $\#S_+ = 2$, and the graph contains both edges from $S_-$ to $S_+$. This state starts a nontrivial section from Proposition 3.15.

C. $\#S_- = 2$, $\#S_+ = 2$, and the edges join the left domain in $S_-$ to the left domain in $S_+$ and the right domain in $S_-$ to the right domain in $S_+$.

D. $\#S_- = 2$, $\#S_+ = 1$, and the graph contains both edges from $S_-$ to $S_+$. This state ends a nontrivial section.

E. $\#S_- = 2$, $\#S_+ = 2$, and the graph contains three edges, the two described for type $C$, and one more. Namely, this type is subdivided into the type $E_L$, where the third edge goes from the left domain in $S_-$ to the right domain in $S_+$, and the type $E_R$, where it goes from the right domain in $S_+$ to the left domain in $S_+$.

The states of type $E$ correspond to the transitions from one flower to the next one inside a nontrivial section.

This types are illustrated on Figure 4.

The notation for each state of our Markov chain includes the type $A \ldots E$ of the state from the list above and the labels on the sides separating $S_-$ and $S_+$. More precisely, these sides form a polygonal curve, which is co-oriented from $S_-$ to $S_+$ and thus oriented, and we read off the labels on the $S_+$-side of the separating sides going from the left end of the separating curve to its right end.

Clearly, the labels in the notation of the state should satisfy some restrictions, and to express these restrictions we introduce some notation regarding vertices, sides, and labels (see Figure 6). For any $e \in G_0$ consider the side $s_e$ of $R$ so that its label inside $R$ is $e$. We co-orient this side from outside to inside of $R$, and the corresponding orientation of $s_e$ allows us to define for $s_e$ its left vertex $v_L(e)$ and its right vertex $v_R(e)$. Note that $v_L(e)$ or $v_R(e)$ is undefined if the corresponding end of $s_e$ lies on $\partial D$. The same notation $v_{L,R}(s)$ will be used for the ends of a co-oriented side $s$ of the tessellation $T_R$.

The labels $e_1$ and $e_2$ are called adjacent if the sides of $R$ with these outgoing labels have a common vertex adjacent, i. e. either $e_1 = e_2$, or $v_L(e_1^{-1}) = v_R(e_2^{-1})$, or vice versa.
Let us define maps \( l \) and \( r \) on the set of labels. Informally speaking, we do the following: for \( e \in G_0 \) we go around \( v_L(s_e) \) in the counterclockwise direction, then the next side we crossed after \( s_e \) has the label \( l(e) \) outside \( \mathcal{R} \). Similarly, going clockwise around \( v_R(s_e) \) we obtain \( r(e) \). Formally we define \( l(e) \) and \( r(e) \) as the labels such that \( v_R(l(e)^{-1}) = v_L(e) \), \( v_L(r(e)^{-1}) = v_R(e) \). Note that \( l(e) \) or \( r(e) \) is undefined if the corresponding end of \( s_e \) lies on \( \partial \mathbb{D} \).

**Definition 4.1.** The set \( \hat{\Xi} \) is the set of all possible arrangements of \( S_- \) and \( S_+ \) up to the action of \( G \). Namely, it consists of the following elements (see Figure 7):

- \( A(e) \): \( \#S_- = \#S_+ = 1 \), and \( e \) is the label on the \( S_+ \)-side of the common side of \( S_- \) and \( S_+ \).
- \( B(e_L,e_R) \): \( \#S_- = 1 \), \( \#S_+ = 2 \), \( e_L \) and \( e_R \) are \( S_+ \)-labels on the common sides if \( S_- \) with the left and the right domains in \( S_+ \) respectively. Since these sides of \( S_- \) are adjacent, we have that \( v_L(e_L^{-1}) = v_R(e_R^{-1}) \).
- \( C_k(e_L,e_R) \): \( \#S_- = \#S_+ = 2 \), and all four domains in \( S_\pm \) share a common vertex \( v \). The label \( e_L \) (respectively, \( e_R \)) is the \( S_+ \)-label on the common side of the left (respectively, right) domains in \( S_- \) and \( S_+ \), and the sector of the flower at \( v \) between these two sides that contains \( S_- \) consists of \( 2k + 1 \) petals. Denote \( n(e_L,e_R) = n(v) = n(v_R(e_L)) = n(v_L(e_R)) \), then \( 1 \leq k \leq n(e_L,e_R) - 2 \) and we have \( l^{2k+1}(e_L^{-1}) = e_R \).
- \( D(e_L,e_R) \): \( \#S_- = 2 \), \( \#S_+ = 1 \), \( e_L \) and \( e_R \) are \( S_+ \)-labels on the common sides of the left and the right domain in \( S_- \) with the domain \( S_+ \). The adjacency condition gives \( v_R(e_L) = v_L(e_R) \).
- \( E_{L,R}(e_L,e_M,e_R) \): \( \#S_- = \#S_+ = 2 \). The four domains in \( S_- \) and \( S_+ \) do not have a common vertex, and there are three sides separating them. The state \( E_L \) represents the case when these sides form an N-shaped line, that is, the left past domain borders both future domains via sides with the \( S_+ \)-labels \( e_L \) and \( e_M \), and the right past domain borders only the right future domain via the side with the label \( e_R \). Thus we have \( v_L(e_L^{-1}) = v_R(e_M^{-1}) \) and \( v_R(e_M) = v_L(e_R) \). The

![Figure 6. To the definitions of vertices \( v_L(e) \), \( v_R(e) \) and labels \( l(e), r(e) \).](image-url)
Figure 7. Configurations for states of the Markov coding:

a) \( A(e) \), b) \( B(e_L, e_R) \), c) \( C_2(e_L, e_R) \), d) \( D(e_L, e_R) \), e) \( E_R(e_L, e_M, e_R) \).

State \( E_R \) is the same with left and right inverted: the boundary is \( \mathcal{U} \)-shaped, and

\[
 v_R(e_L) = v_L(e_M), \quad v_L(e_M^{-1}) = v_R(e_R^{-1}) .
\]

It is clear that every configuration of adjacent levels in a thickened path belong to the set \( \hat{\Xi} \). On the other hand, the set of all possible sequences of configurations cannot be generated by a Markov chain. For example, for a vertex \( v \) with \( n(v) \geq 3 \) it is allowed that \( [S]_i \), \( [S]_{i+1} \), \( [S]_{i+2} \) are consequent petals around \( v \), say, in the counterclockwise direction. Then if \( e \) is the label on the future side of \( [S]_i \cap [S]_{i+1} \), the label on the future side of \( [S]_{i+1} \cap [S]_{i+2} \) is \( l(e) \), and we have that the transition \( A(e) \to A(l(e)) \) is admissible. On the other hand, a long sequence \( A(e) \to A(l(e)) \to A(l(l(e))) \to \ldots \) is not admissible, since the respective sets \( [S]_i \) are still the consecutive petals around a vertex \( v \), and it is not allowed that a thickened path has \( v \) on its boundary and contains more than \( n(v) \) petals around \( v \).

To solve this problem we endow the states of type \( A \) with an additional information. This is based on the following statement.

**Proposition 4.2.** Let \( S \) be a thickened path. Let a vertex \( v \in \partial S \) belongs to the boundaries of \( [S]_k \) for \( k = i, \ldots, j + 1 \), where \( j > i \). Then one of the following cases takes place:
(1) \(j = i + 2\) and both pairs \((\mathcal{S}_i, \mathcal{S}_{i+1}), (\mathcal{S}_{i+1}, \mathcal{S}_{i+2})\) represent \(E\)-states.

(2) for all \(k = i + 1, \ldots, j - 1\) the pair \((\mathcal{S}_k, \mathcal{S}_{k+1})\) represents a state of type \(A\), for \(k = i\) it represents a state of type \(A\) or \(D\), and for \(k = j\) it represents a state of type \(A\) or \(B\).

Proof. Indeed, assume that a vertex \(v \in \mathcal{D}\) belongs to three consecutive levels \([\mathcal{S}]_k, [\mathcal{S}]_{k+1}, [\mathcal{S}]_{k+2}\) of the thickened path, and \#\([\mathcal{S}]_k) = 2. If, say, \(v\) belongs to \(\partial_T \mathcal{S}\), then \(\partial_L [\mathcal{S}]_{k+1}\) consists of the only vertex \(v\). Then \(v\) is compact and hence \(N(\mathcal{R}) \geq 4\). On the other hand, the left domain \(\mathcal{T}_{k+1}\) in the level \([\mathcal{S}]_{k+1}\) has at most two common sides with \([\mathcal{S}]_k\) and at most two common sides with \([\mathcal{S}]_{k+2}\), whence \(N(\mathcal{R}) = 4\), and each of \(\mathcal{T}_{k+1} \cap [\mathcal{S}]_k\) and \(\mathcal{T}_{k+1} \cap [\mathcal{S}]_{k+2}\) contains two sides. Therefore, the states representing the pairs \((\mathcal{S}_k, [\mathcal{S}]_{k+1}), ([\mathcal{S}]_{k+1}, [\mathcal{S}]_{k+2})\) are of types \(E_R\) and \(E_L\) respectively. In particular, this means that \(v\) cannot belong to four consecutive levels of the thickened path, and we see that the first case in the conclusion of the lemma takes place. This case is illustrated on Figure 4, where \(\mathcal{T}_{k+1} = \mathcal{T}_0\). The vertex \(v\) is not shown there, it is the common end of the sides crossing edges \(\mathcal{T}_5 \to \mathcal{T}_6\) and \(\mathcal{T}_6 \to \mathcal{T}_7\) of the adjacency graph.

It remains to consider the case when \#\([\mathcal{S}]_k) = 1 for all \(k = i + 1, \ldots, j - 1\), and it clearly implies the second case of the lemma conclusion.

If \((\mathcal{S}_k, [\mathcal{S}]_{k+1})\) forms a configuration \(A(e)\), one can specify four numbers \(i_{\pm,L}\) and \(i_{\pm,R}\) as follows: \(i_{\pm,\alpha}\) (resp., \(i_{\pm,\alpha}\)), \(\alpha \in \{L, R\}\), is the number of \(m \leq k\) (resp., \(m \geq k + 1\)) such that \([\mathcal{S}]_m\) contains \(v_\alpha(s_k)\). If the vertex \(v_\alpha(s_k)\) is not defined, we set \(i_{\pm,\alpha} = 1\).

Note that it is not possible to have \(i_{-,L} > 1\) and \(i_{-,R} > 1\) simultaneously: these conditions mean that both \(\partial_L [\mathcal{S}]_k\) and \(\partial_R [\mathcal{S}]_k\) contain only a vertex. For \(N(\mathcal{R}) \geq 4\) this violates that together they should contain at least one side; in case \(N(\mathcal{R}) = 3\) the remaining vertex of \([\mathcal{S}]_k\) lies on \(\partial \mathcal{D}\), hence the previous state \((\mathcal{S}_k-1, [\mathcal{S}]_k)\) cannot belong to any of types \(A, \ldots, E\). The same argument applies to \(i_{+,L}\) and \(i_{+,R}\) as well.

The convexity of \(\partial \mathcal{S}\) at \(v_\alpha(s_k)\), \(\alpha = L, R\), is now equivalent to \(i_{-,\alpha} + i_{+,\alpha} \leq n(v_\alpha(s_k))\). Therefore, the configuration \(A(e)\) can be subdivided as follows, see Figure 8.

- \(A_0(e)\): all four \(i_{\pm,L/R}\) equal one.
- \(A_L[i_-, i_+](e)\): here \(i_{-,L} = i_-\), \(i_{+,L} = i_+\), \(i_{-,R} = i_{+,R} = 1\), and the indices \(i_\pm\) should satisfy \(3 \leq i_- + i_+ \leq n(v_L(e))\).
- \(A_R[i_-, i_+](e)\): symmetric to the previous case; here \(3 \leq i_- + i_+ \leq n(v_R(e))\).
- \(A_{LR}[i_-, i_+](e)\): here \(i_{-,L} = i_-\), \(i_{+,R} = i_+\), and \(i_{-,R} = i_{+,L} = 1\). The conditions on the indices \(i_\pm\) are \(2 \leq i_- \leq n(v_L(e)) - 1, 2 \leq i_+ \leq n(v_R(e)) - 1\).
- \(A_{RL}[i_-, i_+](e)\): symmetric to the previous case; here \(2 \leq i_- \leq n(v_R(e)) - 1, 2 \leq i_+ \leq n(v_L(e)) - 1\).

Remark 4.3. Notice that if \(N(\mathcal{R}) = 3\) and \(\mathcal{R}\) has a compact side, some of these states may be absent. Namely, let \(s\) be the only compact side of \(\mathcal{R}\), let \(g\) be its label outside of \(\mathcal{R}\). If \((\mathcal{S}_k, [\mathcal{S}]_{k+1})\) has the form \(A(g)\), then \([\mathcal{S}]_{k+2}\) should contain at least one of the domains adjacent to the sides of \([\mathcal{S}]_{k+1}\), hence either \(i_{+,L}\) or \(i_{+,R}\) is greater than one. Similarly, either \(i_{-,L} > 1\) or \(i_{-,R} > 1\). This case needs special consideration in several statements below, and we usually refer to it as “the special case from Remark 4.3.”
Note that even in this case the list of $A_{\ldots}(g)$-states is not completely empty. Indeed, since $s$ is the only compact side, it should be paired to itself: $g = g^{-1}$. Therefore, the ends of $s$ are swapped by the action of $g$, hence $n(v_L(g)) = n(v_R(g)) = n$. Let $\alpha$ and $\beta$ be the angles of $\mathcal{R}$ in the ends of $s$. Consider the flower around a vertex $v \in \mathbb{D}$. Note that the sides incident to $v$ are alternatingly compact and non-compact, and the angles between these sides are alternatingly $\alpha$ and $\beta$. Therefore, $n\alpha + n\beta = 2\pi$. On the other hand, the sum of angles in the hyperbolic triangle $\mathcal{R}$ is $\alpha + \beta < \pi$. Consequently, $n \geq 3$, and for example, the state $A_{LR}[2, 2](g)$ is well-defined.

**Definition 4.4.** The set of states $\Xi$ of our Markov chain is the set of all states of types $B, C, D, E$ from the set $\hat{\Xi}$ and of all subtypes of type $A$ states enumerated in the previous list. We denote the projection from $\Xi$ to $\hat{\Xi}$ by $\pi$.

Finally, let us define sets $\Xi_S, \Xi_F \subset \Xi$ as follows:

$$
\Xi_S = \{ A_0(e), A_L[1, i_+](e), A_R[1, i_+](e), B(e_L, e_R) \},
$$

$$
\Xi_F = \{ A_0(e), A_L[i_-, 1](e), A_R[i_-, 1](e), D(e_L, e_R) \},
$$

**Figure 8.** Possible (a–c) and impossible (d–e) subtypes for type $A$ states. Dark and medium gray domains are respectively the past and the future domain for the current state, light gray domains are other domains from the thickened path. Here $n(v_L(e)) = 4$, $n(v_R(e)) = 3$. 


\[d) \text{Impossible } "A_R[2, 2](e)": \]
\[i_- + i_+ > n(v_R(e)), \text{ thus convexity in } v_R(e) \text{ fails} \]

\[e) \text{Impossible subtype:} \]
\[\text{both } i_{+, L} \text{ and } i_{+, R} \text{ are greater than 1} \]
where the parameters $i_{\pm}$, $e$, $e_L$, $e_R$ admit all possible values. In the special case from Remark 4.3, these definitions are amended as follows: if $g = g^{-1}$ is the label on the compact side of $R$, we include $A_{LR}[2,i_+](g), A_{RL}[2,i_+](g)$ to $\Xi_S$ and $A_{LR}[i_-,2](g), A_{RL}[i_-,2](g)$ to $\Xi_F$ instead of those $A_{\pm}(g)$.

4.2. The Markov coding.

**Definition 4.5.** The set of admissible transitions in our Markov coding is enumerated in the following list. We denote by $\Pi$ the $\Xi \times \Xi$ adjacency matrix for the corresponding topological Markov chain and write $j \to j'$ if the transition from $j$ to $j'$ is admissible accordingly to this list.

- $A_0(e)$ if $i_+ > 1$ then
  
  $A_{LR}[i_,i_+](e) \to \left\{ \begin{array}{ll} A_{LR}[i_-,1,i_+](l(e)) & \text{if } i_+ = 2, \\
  & A_{LR}[i_-,1,i_+](l(e)) & \text{for any admissible } i_+, \\
  & B(l(e),r(l(e))^{-1}) & \text{if } i_+ = 2. \\
  \end{array} \right.$

- $A_{LR}[i_-,1](e) \to (\text{the same cases as for } A_0(e)).$

- $A_{RL}[i_-,i_+](e) \to (\text{the same cases as for } A_{LR}[i_-,1](e)).$

- The transitions for the $A_R$- and $A_{LR}$-states are similar with the exchange of left and right.

- $B(e_L,e_R) \to C_1(r(e_L),l(e_R))$ if $n(e_L,e_R) \geq 3,$

- $B(e_L,e_R) = 2$ the transitions for $B(e_L,e_R)$ are the same as for $C_{n(e_L,e_R)-2}(e_L,e_R)$ below.

- $C_i(e_L,e_R) \to C_{i+1}(r(e_L),l(e_R)),$ for $i < n(e_L,e_R) - 2,$

- $C_{n(e_L,e_R)-2}(e_L,e_R) \to \left\{ \begin{array}{ll} D(r(e_L),l(e_R)), \\
  E_L(r(r(e_R)^{-1}),r(e_L),l(e_R)), \\
  E_R(r(e_L),l(e_R),l(l(e_R)^{-1})). \end{array} \right.$

- $D(e_L,e_R) \to \left\{ \begin{array}{ll} A_0(e') & \text{for } e' \text{ non-adjacent to } e_L^{-1},e_R^{-1}, \\
  A_{LR}[i_+,i_+](e') & \text{for } e' \text{ non-adjacent to } e_L^{-1},e_R^{-1}, \\
  A_{RL}[i_+,i_+](e') & \text{any admissible } i_+, \\
  B(e'_L,e'_R) & \text{for } e'_L,e'_R \text{ either not adjacent to } e_L^{-1},e_R^{-1}, \\
  & \text{or adjacent via a vertex } v \text{ with } n(v) > 2, \\
  A_{LR}[i_+,i_+](l(e_L)) & \text{for any admissible } i_+, \\
  A_{RL}[i_+,i_+](r(e_R)) & \text{for any admissible } i_+. \end{array} \right.$

- $E_L(e_L,e_M,e_R)$ has the same set of transitions as $B(e_L,e_M)$.

- $E_R(e_L,e_M,e_R)$ has the same set of transitions as $B(e_M,e_R).$
Denote
\begin{equation}
(2)
P_{N-1}^{S} = \{(j_0, \ldots, j_{N-1}) \subset \Xi^N : j_0 \in \Xi S, j_{N-1} \in \Xi F, \Pi_{j_k,j_{k+1}} = 1 \text{ for } k = 0, \ldots, N-2\}.
\end{equation}

We now show that this set is in 1:1-correspondence with the set of the thickened paths of length \(N\).

**Theorem 4.6.** Let \(S = ([S]_0, \ldots, [S]_N)\) be a thickened path starting at \(R\). Then there exists a unique sequence of states \(\tilde{j} \in P_{N-1}^{S}\) such that for each \(k\) the pair \(([S]_k, [S]_{k+1})\) represents the configuration \(\pi(j_k)\). Moreover, this mapping of thickened paths of length \(N\) starting in \(R\) to the set \(P_{N-1}^{S}\) is a bijection.

**Proof.** 1. Consider a thickened path \(S\). Each pair \(([S]_k, [S]_{k+1})\) represents a unique configuration \(j_k \in \tilde{S}\). For every configuration of type \(A\) one can recover indices \(i_{\pm,L/R}\) as described above, thus arriving at the states \(j_k\) with \(\pi(j_k) = j_k\). Note that if \(\pi(j_0) = A(e)\) then the state \(j_0\) has \(i_{-,L} = i_{+,R} = 1\), so \(j_0 \in \Xi S\). In the special case from Remark 4.3 we need to amend these indices as follows: if \(\pi(j_0) = A(g)\), where \(g\) is the label on the compact side, we have either \(i_{+,L} \geq 2\) or \(i_{+,R} \geq 2\). In the latter case we then set \(i_{-,L} = 2\), \(i_{-,R} = 1\), and in the latter case \(i_{-,R} = 2\), \(i_{-,L} = 1\). This corresponds to the addition of the “virtual domain” \(S_{-}\) to our thickened path. Note that this addition still yields a thickened path.

Now one can check that all transitions \(j_k \to j_{k+1}\) are admissible. There are three types of restrictions on the pair of states \((j_k, j_{k+1})\) in the list of Definition 4.5.

First, there are the restrictions on the configurations \(j_k, j_{k+1}\). For example, if \(S_+ = [S]_k\) is a pair of petals around a vertex \(v\), and there are more than one petals in the sector around \(v\) that is bounded by the sides in \(S_- \cap S_+\) and contains \(S_+\), then every domain in \(S_+\) has the adjacent domain in \(S_{++} = [S]_{k+1}\), hence \(S_{++}\) is the next pair of petals inside this sector, and the triple \((S_-, S_+, S_{++})\) of level in the thickened path corresponds to the transition \(C_k \to C_{k+1}\) with the appropriate conditions on the labels given in the list from Definition 4.5.

Further, there are restrictions on the indices \(i_{\pm}\) of \(A\)-states. For example if \(j_k\) is an \(A\)-state with \(i_{+,L} > 1\) then \([S]_{k+2}\) should contain the next petal at the vertex \(v_L(s_k)\) in the counterclockwise direction after \([S]_{k+1}\). If \(i_{+,L} > 2\) then the only possible case is that \(j_{k+1}\) is again an \(A\)-state with \(i_-\) increased by one and \(i_+\) decreased by one. On the other hand, if \(i_{+,L} = 2\), it is possible that \([S]_{k+2}\) contains not only the above-mentioned petal, but also the domain adjacent to \([S]_{k+1}\) along the next side on its boundary.

Finally, there are restrictions related to the convexity of \(\partial S\). Namely, we need to check these conditions for the boundary vertices \(v\) that are incident to at least three levels in \(S\). These cases are enumerated in Proposition 4.2. In the cases when the corresponding sequence of states contains \(A\)-states, the convexity is guaranteed by the inequalities on the indices \(i_{\pm}\) for these states, so we need to consider only the cases when \((j_k, j_{k+1})\) have types \((E_L, E_R), (E_R, E_L)\), and \((D, B)\). In the first two cases the convexity at \(v\) is guaranteed: as we have seen in the proof of Proposition 4.2 in this case \(N(\mathcal{R}) = 4\), \(\mathcal{R}\) is compact, and the vertex \(u\) opposite to \(v\) in \(\mathcal{T}_k\) has \(n(u) = 2\), as \(u\) correspond to a cycle of four domains.
in the adjacency graph. Therefore, \( n(v) \geq 3 \), while \( v \) is incident to three domains in \( \mathcal{S} \).

The remaining case \((D, B)\) is specially mentioned in the Definition 4.5 if \([\mathcal{S}]_{k-1} \cap [\mathcal{S}]_k\) and \([\mathcal{S}]_k \cap [\mathcal{S}]_{k+1}\) have a common vertex \( v \) in \([\mathcal{S}]_k\), then we require that \( n(v) > 2 \).

2. Let us show next that one cannot endow the sequence \( \hat{j} \) with indices \( i_{\pm L/R} \) in another way than the one described in the previous part of the proof. Indeed, one can see from the set of transitions that the indices \( i_{-L/R} \) for the state \( j_k \) are uniquely defined by the configurations \( \pi(j_{k-1}), \pi(j_k) \) and by the same indices for the state \( j_{k-1} \) (assuming \( j_{k-1} \) has type \( A \)). Therefore, one can successively find these indices for all states starting from \( i_{-L/R}(j_0) = 1 \) as implied by \( j_0 \in \Xi_S \). Similarly, the indices \( i_{+L/R}(j_k) \) are successively found starting from the end of the sequence: \( i_{+L/R}(j_{N-1}) = 1 \).

As above, the special case from Remark 4.3 needs a special consideration if \( j_0 = A(g) \). Then we have \( j_1 = A(e) \), where \( e \) is a label on a non-compact side of \( R \). Then either \( e = l(g) \) or \( e = r(g) \), and, say, in the former case we have \( i_{+L}(j_0) \geq 2 \), \( i_{+R}(j_0) = 1 \), hence \( j_0 \in \Xi_S \) implies \( i_{-L}(j_0) = 1 \), \( i_{-R} = 2 \). The latter case is considered in the same way. Now we can recover \( i_{-L/R}(j_k) \) successively in the same way as in the general case.

3. It remains to show that every sequence \( \hat{j} \in \mathbf{P}^{S \to F}_{N-1} \) represents a thickened path \( \mathcal{S} \). We can inductively recover all \([\mathcal{S}]_k\) starting with \([\mathcal{S}]_0 = R \). For example, if \( j_k = B(e_L, e_R) \), then \([\mathcal{S}]_k\) contains only one domain, so we take its sides \( s_{e_L} \) and \( s_{e_R} \) having the outside labels \( e_L \) and \( e_R \), and \([\mathcal{S}]_{k+1}\) contains two domains that are adjacent to \([\mathcal{S}]_k\) via the sides \( s_{e_L} \) and \( s_{e_R} \). Our set of transitions guarantees that the construction of the next level is well defined, for example, for \( E_L \)-state we can construct the right future domain via the sides with the labels \( e_M \) and \( e_R \), and the result is the same. Moreover, one can check inductively that for \( A \)-states the indices \( i_{\pm L}, i_{\pm R} \) coincide with the corresponding numbers of domains adjacent to the ends of the side separating the past and the future domain.

There we can define the boundary of the sequence \( \mathcal{S} = ([\mathcal{S}]_0, \ldots, [\mathcal{S}]_N) \) in the way similar to the one from Subsection 3.1. As we have seen in the previous part of the proof, our set of transitions guarantees that the curve \( \partial \mathcal{S} \) is convex, i.e. always turns left. Hence the boundary \( \partial \mathcal{S} \) is not self-intersecting, and we apply Proposition 3.23 to show that \( \mathcal{S} \) is a thickened path. All assumptions of this proposition are clear except that the structure of nontrivial sections is the one described in Proposition 3.15. The second item there is clear: the corresponding sequence of states starts with \( B \)-state, ends with \( D \)-state, and contains \( C \)- and \( E \)-states in between. Every \( C \)-state yields just two edges \( T_i \to T_{i+1} \) and \( B_i \to B_{i+1} \), while \( E_L \)-state (resp., \( E_R \)-state) yields also a crossing \( B_i \to T_{i+1} \) (resp., \( T_i \to B_{i+1} \)). Now the third item follows from the fact that in every flower there is one petal with the minimal index, one petal with the maximal one, and these petals are opposite. The segments joining the centers of the adjacent flowers belong to the boundaries of these two petals, hence the angle between these segments differs from the straight angle by not more than one sector. Therefore, for every \( \hat{j} \in \mathbf{P}^{S \to F}_{N-1} \) we have constructed the thickened path \( \mathcal{S} \) and it is clear that the sequence of states corresponding to \( \mathcal{S} \) coincides with \( \hat{j} \). □

4.3. Time-reversing involution. The Markov coding defined above has the following property: the Markov chain with time reversed, that is, the Markov chain with the matrix
The involution \( \iota : \Xi \to \Xi \), which can be informally described as the one swapping the past and the future domains for the state:
\[
\begin{align*}
A_0(e) &\leftrightarrow A_0(e^{-1}), & A_L[i_-, i_+](e) &\leftrightarrow A_R[i_+, i_-](e), \\
A_{LR}[i_+, i_-](e) &\leftrightarrow A_{RL}[i_-, i_+](e^{-1}), & B(e_L, e_R) &\leftrightarrow D(e_R^{-1}, e_L^{-1}), \\
C_k(e_L, e_R) &\leftrightarrow C_{\alpha(e_L, e_R) - k-1}(e_R^{-1}, e_L^{-1}), \\
E_{\alpha}(e_L, e_M, e_R) &\leftrightarrow E_{\alpha}(e_R^{-1}, e_M^{-1}, e_L^{-1}) \ (\alpha = L, R).
\end{align*}
\]

**Proposition 4.7.** The involution \( \iota \) maps the topological Markov chain with the adjacency matrix \( \Pi \) to the same chain with reversed time, that is, \( \Pi_{a(j)(k)} = \Pi_{kj} \). Also, \( \iota(\Xi_S) = \Xi_F \) and vice versa.

**Proof.** This follows directly from the definitions. \( \square \)

### 4.4. Properties of the Markov coding

Let us recall the definitions of the following properties of a topological Markov chain.

**Definition 4.8.** Let \( X \) and \( M \) be respectively the set of states and the adjacency matrix of a topological Markov chain.

1. The topological Markov chain \((X, M)\) is called **strongly connected** if for any \( x, y \in X \) there exists a sequence \( z_0 = x, z_1, \ldots, z_k = y \) such that \( z_j \to z_{j+1} \) is an admissible transition for any \( j \).

2. The topological Markov chain \((X, M)\) is called aperiodic if there is no \( p > 1 \) such that for every admissible transition \( x \to y \) one has \( \tau(y) = \tau(x) + 1 \), where \( \tau \) is the map \( \tau : X \to \mathbb{Z}/p\mathbb{Z} \).

In this subsection we will show that our Markov coding \((\Xi, \Pi)\) is strongly connected and aperiodic. We start by constructing some paths, which will be useful in the considerations below.

**Proposition 4.9.** Assume that \( \mathcal{R} \) does not belong to the special case from Remark 4.3. Then there exists \( m \leq 4 \) such that for every \( e \in G_0 \) there exists a locally shortest path of fundamental domains \( T^e = (T_0^e = \mathcal{R}, T_1^e, \ldots, T_m^e) \) such that its boundary is convex, the corresponding sequence of states \( i_0 \to \cdots \to i_{m-1} = t(e) \) starts with \( i_0 = A_0(e) \), and if the path \( T^e \) is extended to any locally shortest path \( \mathcal{R}_n, \ldots, \mathcal{R}_1, T_0^e, \ldots, T_m^e \), and \( j_n \to \cdots \to j_{m-1} \) is the sequence of states corresponding to its convexification, then \( j_{m-1} = t(e) \).

Similarly, there exists a locally shortest path \( H^e = (H_{n-1}, \ldots, H_{-1}, H_0^e = \mathcal{R}) \) with the convex boundary, its final state is \( A_0(e) \), and its initial state \( \mathcal{H}(e) \) does not change if the path is extended arbitrarily to the right.

**Proof.** Observe that the convexification may affect the positive half of the extended path only if the convexification step applies to an end of the side \( s_0 = T_0^e \cap T_1^e \), and then possibly to several adjacent boundary vertices with straight angles. Now the proof is presented on Figure 9. Dashed lines there show the maximal possible extent of the segments \( I(u) \) from
Figure 9. "Tail" paths $\mathcal{T}^e$ from Proposition 4.9 and Remark 4.10. Domains $\mathcal{T}^e_0$ are shaded gray. Dashed lines indicate boundary sides that can be affected by convexification of an extended path. Numbers in circles show $n(\ldots)$ for the corresponding vertices, $\infty$ means that this vertex lies on $\partial \mathcal{D}$. Note that $t(e) = A_0(\hat{e})$ in all cases except f), and $t(e) = A_L[2, 1](\hat{e})$ in the case f).

the proof of Proposition 3.8 where $u$ is an end of $s_0$. Note that as these segments do not reach endpoints of $s_{m-1} = \mathcal{T}^e_{m-1} \cap \mathcal{T}^e_m$, the convexification does not add a new domain with index $m - 1$, so $\pi(j_{m-1}) = \pi(i_{m-1}) = A(\hat{e})$. The indices $i_{\pm, L}, i_{\pm, R}$ for this state are also not changed, hence $j_{m-1} = i_{m-1}$.

If $N(R) \geq 5$, it is sufficient to take $m = 3$ and construct the path $\mathcal{T}^e$ so that $\partial_L \mathcal{T}^e_1$ and $\partial_R \mathcal{T}^e_2$ contain at least two sides each, see Fig. 9a).

If $N(R) = 4$, and $R$ is non-compact, one can construct $\mathcal{T}^e$ as shown of Fig. 9b), c).

If $N(R) = 4$, $R$ is compact and has no opposite vertices with $n(\ldots) = 2$, there are the following three cases. Let $s_1$ be the side of $\mathcal{T}^e_1 = eR$ opposite to $s_0$, we choose $\mathcal{T}^e_2$ to be the domain on the other side of $s_1$. If both ends of $s_1$ has $n(\ldots) \geq 3$, we use the path shown on Fig. 9f). Otherwise, if both ends of $s_0$ has $n(\ldots) \geq 3$, the same holds for
both ends of $s_2$, the side of $T_2^e$ opposite to $s_1$, and we use the path from Fig. 9j). The remaining case is when both $s_0$ and $s_1$ has ends with $n(\ldots) = 2$. Then these ends lie on the same (say, right) boundary, and the left ends of $s_0$, $s_1$, and $s_2$ all have $n(\ldots) \geq 3$. Then we construct our path as shown on Fig. 9f).

Finally, if $N(\mathcal{R}) = 3$ and each its side is non-compact, we can use the paths shown on Fig. 9(k), h).

As for the second statement, the path $H^e$ is constructed by $H^e_{-j} = T_j^{-e^{-1}}$, $j = 0, \ldots, m$. In particular, we have $h(e) = \iota(t(e^{-1}))$. \hfill \Box

Remark 4.10. In the special case from Remark 4.3 we define the paths $T^e$ as shown on Fig. 9(i), j). Note that if $g = g^{-1}$ is a label on the compact side, then all statements of Proposition 4.9 hold for these paths except that $T^g$ contains three, not two domains that are incident to one of the vertices of $\mathcal{R} = T_0^g$.

The following statement shows two important combinations of “head” and “tail” paths from the previous proposition.

**Proposition 4.11.** Let $H^e$, $T^e$ be the paths from Proposition 4.9 or Remark 4.10.

1) For any $e$, $e^\ast$ such that $e^{\ast} \neq e^{-1}$ the path $\mathcal{R} = (\mathcal{R}_{-m}, \ldots, \mathcal{R}_m)$, where $\mathcal{R}_j = H_j^e$ for $j \leq 0$, $\mathcal{R}_j = T_j^e$ for $j \geq 0$, is locally shortest. Let $k_{-m} \rightarrow \cdots \rightarrow k_{m-1}$ be the sequence of states corresponding to its convexification. Then $k_{-m} = h(e^\ast)$, $k_{m-1} = t(e)$.

2) For any $e$, $e^\ast$ such that $e^{\ast} \neq e^{-1}$ denote $S_j = H_j^e$ for $j = -m, \ldots, 0$, $S_1 = e\mathcal{R}$. At most one vertex $w$ of $S_1$ is shared with $S_{-1}$. Choose any side of $S_1$ that is not incident to $w$ and let $e^\ast$ be the label on this side outside of $S_1$. Denote $S_{j+1} = eT_j^e$ for $j = 0, \ldots, m$. Then the path $S = (S_j)_{j=-m+1}^{m+1}$ is locally shortest and the sequence of states corresponding to the convexification of $S$ starts with $h(e^\ast)$ and ends with $t(e^\ast)$.

**Proof.** 1) The convexity of $\mathcal{R}$ may fail only at the vertices of $\mathcal{R} = \mathcal{R}_0$, and in a non-special case any vertex of $\mathcal{R}$ is adjacent to at most three domains: $\mathcal{R}_{-1}$, $\mathcal{R}_0$, $\mathcal{R}_1$, thus the boundary angle at any vertex is at most minimally concave. In the special case from Remark 4.3 only one of the elements $e$, $e^\ast$ can be equal to $g$, thus only one of paths $H^e$, $T^e$ can have three domains adjacent to some vertex of $\mathcal{R}$. Therefore, at most one vertex $u$ of $\mathcal{R}$ can have at most four adjacent domains from $\mathcal{R}$, but since $n(u) \geq 3$, the boundary angle at $u$ is again at most minimally concave.

2) The convexity of $S$ may fail only at the common vertices of $S_0$ and $S_1$, since all other vertices are adjacent to only one of paths $H^e$ or $eT^e$. Let $u$ be a common vertex of $S_0$ and $S_1$. Then $u$ is incident either only to $H^e$ and $S_1$ or only to $S_0$ and $eT^e$, hence the angle of $\partial S$ at $u$ is greater that the angle of $\partial H^e$ or $eT^e$ there, which is at most straight. Therefore, the angle of $\partial S$ at the common vertices of $S_0$ and $S_1$ is at most minimally concave. Finally, the common vertices of $S_0$ and $S_1$ cannot be joined by a straight segment of $\partial S$ as they are joined by $s_0 = S_0 \cap S_1$, which lies inside $S$, thus $\partial S$ is almost convex.

The last part of both statements follows directly from Proposition 4.9 \hfill \Box

**Lemma 4.12.** The topological Markov chain $(\Xi, \Pi)$ introduced in Definition 4.5 is strongly connected.
Proof. The scheme of the proof is the following. We will consider several cases, and in every case we firstly choose the set \( \Omega \subset \Xi \) with \( \iota(\Omega) = \Omega \), and then we prove the following properties:

1. Let us denote by \( j \sim k \) that there exists a path along the arrows in the adjacency graph of the Markov chain from \( j \) to some state \( k \in \Omega \);
2. Let us denote by \( j \sim k \) that there exists a path from \( k \) to some state \( j \in \Omega \).

Let us denote by \( j \sim k \) that there exists a path going from \( j \) to \( k \). Observe that properties (i) and (ii) imply strong connectivity. Namely, from (i) we have that for any states \( j, j' \in \Xi \) there exist \( k, k' \in \Omega \) such that \( j \sim k \) and \( j' \sim k' \). Applying the involution \( \iota \) to the second of these relations, we get \( j' \sim k' \). Finally, (ii) yields \( k \sim j \sim j' \).

The scheme of the proof is the following. We will consider several cases, and in each case we will prove the following:

- In the special case of Remark 4.3 we can decrease \( i_+ \) until it reaches 2. Then if \( \hat{e} \) is non-compact and, say, \( i_{+L} = 2 \) we have \( l(\hat{e}) = g \), where \( g \) is the label on the compact side, hence the state \( k \) can be followed by \( A_{LR}[i_{+L},2](g) \). It remains to consider a state \( k \) with \( \pi(k) = A(g) \) and, say, \( i_{+R}(k) = 2, i_{+L}(k) = 1 \). The state \( k \) can be followed by \( A_{R}[i_{+R},1](\hat{e}) \), where \( \hat{e} = r(g) \), and, finally, by \( A_0(\hat{e}) \), since the labels \( \hat{e} \) and \( \hat{e}^{-1} \) are not adjacent.

In all cases below we have \( \Omega \subset \Omega_0 \), and if \( \Omega \neq \Omega_0 \) to establish property (iii) we will check that for every \( j \in \Omega_0 \) we have \( j \sim k \) for some \( k \in \Omega \).

1. Let \( N(\mathcal{R}) \geq 5 \). Here we set \( \Omega = \Omega_0 \) and it remains to check property (ii). Let us construct the paths \( T^e \) shown on Figure 9 in a uniform manner, namely, we choose the domains \( T_{2,3}^e \) in such a way that \( \partial_R T_{1}^e \) and \( \partial_L T_2^e \) contain one side each. Denote by \( t(e) \) the label such that \( t(e) = A_0(t(e)) \); \( t(e) \) is shown as \( \hat{e} \) on Figure 9. Note that \( e \mapsto t(e) \) is a bijection: for any \( e' \in G_0 \) we consider a pair of domains \( T_2 \) and \( T_3 \) comprising the state \( A(e') \), then we add domains \( T_1 \) and \( T_0 \) such that \( \partial_T T_2 \) and \( \partial_R T_1 \) contain one side each. Then if \( (T_0, T_1) \) corresponds the state \( A(e) \), we have \( e' = t(e) \) and \( T^e = h T \), where \( h \in G \) is such that \( T_0 = h^{-1} \mathcal{R} \).

Now take any \( f, \hat{f} \in G_0 \) such that \( \hat{f} \neq f^{-1} \) and denote \( e = t^{-1}(f), \hat{e}^{-1} = t^{-1}(\hat{f}^{-1}) \).

Since \( t \) is a bijection, \( e \neq \hat{e}^{-1} \), and we may consider the path from the first part of Proposition 4.11 for these \( e, \hat{e} \). This path shows that \( A_0(\hat{f}) \sim A_0(f) \) for any \( f, \hat{f} \) such that \( \hat{f} \neq f^{-1} \). Finally, to have \( A_0(f) \sim A_0(f^{-1}) \) choose any \( e \in G_0 \setminus \{f, f^{-1}\} \) and observe that \( A_0(f) \sim A_0(e) \sim A_0(f^{-1}) \).

2. Let us assume that \( \mathcal{R} \) has a side with both ends lying on \( \partial \mathbb{D} \). Then we set

\[
\omega = \{e \in G_0 : v_L(e) \text{ and } v_R(e) \text{ are undefined}\}, \quad \Omega = \{A_0(e) : e \in \omega\}.
\]
It is clear that $\omega^{-1} = \omega$, hence $\iota(\Omega) = \Omega$. Moreover, $A_0(e) \rightsquigarrow A_0(\hat{e})$ for any $e \in G_0$, $\hat{e} \in \omega$, $e \neq \hat{e}^{-1}$. In particular, for $e \in G_0 \setminus \omega$ the last inequality holds, hence property \textbf{(i)} is established. To check property \textbf{(ii)} it is remains to show that $A_0(e) \rightsquigarrow A_0(e^{-1})$ for any $e \in \omega$. To do this we choose any $f \in G_0 \setminus \{e, e^{-1}\}$, and note that $A_0(e) \rightarrow A_0(f) \rightarrow A_0(e^{-1})$.

3. Let us assume that $N(\mathcal{R}) = 4$, $\mathcal{R}$ is non-compact, and there are no sides with both ends lying on $\partial \mathbb{D}$. Then we set $\Omega = \Omega_L \cup \Omega_R$, where $\omega_\alpha = \{e : v_\alpha(e) \text{ is undefined}\}$, $\Omega_\alpha = \{A_0(e) : e \in \omega_\alpha\}$, $\alpha = L, R$, thus $\iota(\Omega_L) = \Omega_R$ and vice versa. Note that if $s_e$ is a compact side then the opposite side of $\mathcal{R}$ is non-compact, hence $A_0(e) \rightsquigarrow A_0(f) \in \Omega$, and property \textbf{(i)} holds. We pass to property \textbf{(ii)}. Note that the one of the following holds:

$$(3) \quad \omega_L = \{e\}, \quad \omega_R = \{e^{-1}\}, \quad \text{or} \quad \omega_L = \{e, f\}, \quad \omega_R = \{e^{-1}, f^{-1}\}.$$  

Assume that the second case takes place. Let $(\mathcal{S}_0, \mathcal{S}_1)$ represents the state $A_0(e)$. Denote $s_0 = \mathcal{S}_0 \cap \mathcal{S}_1$, thus the left end of $s_0$ lies on $\partial \mathbb{D}$. Let $s_1$ be the side of $\mathcal{S}_1$ opposite to $s_0$. Then $s_1$ has its right end on $\partial \mathbb{D}$, and if $g$ is the label on $s_1$ outside $\mathcal{S}_1$, then $g \neq e^{-1}$ and $g \in \omega_R$. Therefore, $g = f^{-1}$ and we have $A_0(e) \rightarrow A_0(f^{-1})$. Similarly, we have

$$(4) \quad A_0(e) \rightsquigarrow A_0(f^{-1}) \quad \text{and} \quad A_0(f) \rightsquigarrow A_0(e^{-1}).$$

Now consider the side $s'_1$ of $\mathcal{S}_1$ adjacent to $s_0$ via its left end, which lies on $\partial \mathbb{D}$. Then the outside label $g'$ of $s'_1$ belongs to $\omega_L$. Assume that $g' = f$. Then $A_0(e) \rightsquigarrow A_0(f)$ and, similarly, $A_0(e) \rightsquigarrow A_0(f)$, $A_0(e^{-1}) \rightsquigarrow A_0(f^{-1})$. Therefore, in this case property \textbf{(ii)} holds.

It remains to consider the case $g' = e$ and the first case from \textbf{(3)}. In both these cases the side $s'_1$ has the outside label $e$. Consider the domain $\mathcal{S}_2$ adjacent to $\mathcal{S}_1$ via the side $s_1$. Consider the sides of $\mathcal{S}_2$ that have labels $e^{\pm 1}$. They do not include the side $s_1$ and we have the two cases shown on Figure \textbf{10}, b), where the domains $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ are shown in bold. Then we construct a path $\mathcal{S}$ joining $A_0(e)$ to $A_0(e^{-1})$ as shown on these figures. Thus we have $A_0(e) \rightsquigarrow A_0(e^{-1})$ and property \textbf{(ii)} is established, keeping in mind \textbf{(4)} for the second case in \textbf{(3)}.

4. Let us assume that $N(\mathcal{R}) = 4$ and $\mathcal{R}$ is compact. Here we set $\Omega = \Omega_0$. Define the bijection $\tau : G_0 \rightarrow G_0$ as follows. For $e \in G_0$ consider the side of $\mathcal{R}$ with the inside label $e$. Then the outside label on the opposite side of $\mathcal{R}$ equals $\tau(e)$. Choose $m$ such that $\tau^m = \text{id}$. Then $A_0(e) \rightarrow A_0(\tau(e))$, and, moreover,

$$A_0(e) \rightarrow A_0(\tau(e)) \rightarrow \cdots \rightarrow A_0(\tau^{m-1}(e)) \rightarrow A_*(e),$$

where $A_*(e)$ is any of the following states: $A_0(e)$, $A_{L[1, 2]}(e)$ (provided $n(v_L(e)) \geq 3$), $A_{R[1, 2]}(e)$ (provided $n(v_R(e)) \geq 3$).

Due to condition \textbf{(ii)} of Assumption \textbf{1.1} there exists a label $e_0 \in G_0$ such that $n(v_L(e_0)), n(v_R(e_0)) \geq 3$. Then if $e_L = l(e_0)$, $e_R = r(e_0)$, $e_D = \tau(e)$ one can see that \{e_L, e_R, e_D\} $\subset G_0 \setminus \{e^{-1}_0\}$, and

$$A_0(e) \rightarrow A_0(e_D), \quad A_0(e) \rightsquigarrow A_0[1, 2](e) \rightarrow A_0[2, 1](e_\alpha) \rightsquigarrow A_0(e_\alpha), \quad \alpha = L, R.$$
Therefore, \(A_0(e_0) \rightsquigarrow A_0(f)\) for any \(f \neq e_0^{-1}\). Note that \(n(v_L(e_0^{-1})) = n(v_R(e_0))\) and vice versa, hence the argument above can be applied to \(e_0^{-1}\), and we have \(A_0(e_0^{-1}) \rightsquigarrow A_0(f)\) for \(f \neq e_0\).

Applying the involution \(\iota\) to the obtained relations one can see that \(A_0(f) \rightsquigarrow A_0(e_0)\) for \(f \neq e_0^{-1}\) and \(A_0(e_0) \rightsquigarrow A_0(e_0^{-1})\) for \(f \neq e_0\). It remains to choose any \(f \neq e_0, e_0^{-1}\) and write \(A_0(e_0) \rightsquigarrow A_0(f) \rightsquigarrow A_0(e_0^{-1})\).

5. Finally, consider the special case from Remark 4.3. We set
\[
\Omega = \{A_0(e) : s_e \text{ is not compact}\},
\]
the property \([i]\) was established in the beginning of the proof, and the proof of property \([ii]\), namely, that \(A_0(f) \rightsquigarrow A_0(f^{-1})\), is shown on Figure 10c).

\[\Box\]

**Lemma 4.13.** The topological Markov chain \((\Xi, \Pi)\) defined in Definition 4.5 is aperiodic.

**Proof.** Suppose that our Markov chain has the period \(c\), that is, an index \(\tau(i) \in \mathbb{Z}/c\mathbb{Z}\) is assigned to every state \(i \in \Xi\) and all allowed transitions \(i \to j\) satisfy \(\tau(j) = \tau(i) + 1\).

Take any \(e_1, e_2\) and choose \(\hat{e} \neq e_1^{-1}, e_2^{-1}\). Then using the paths from the first part of Proposition 4.11 we have \(\tau(t(e_s)) = \tau(h(\hat{e})) + 2m - 1, s = 1, 2\). Therefore, \(\tau(t(e))\) is the same for all \(e\), we denote it by \(\tau_t\). Similarly, \(\tau(h(e))\) equals the same number \(\tau_h\) for all \(e\), and \(\tau_t = \tau_h + 2m - 1\). On the other hand, any path from the second part of Proposition 4.11 yields \(\tau_t = \tau_h + 2m\). Therefore, \(2m - 1 \equiv 2m \pmod{c}\), whence \(c = 1\).

\[\Box\]

**Corollary 4.14.** By Lemmas 4.12 and 4.13 our Markov chain \((\Xi, \Pi)\) is strongly connected and aperiodic, hence there exists \(N > 0\) such that all entries of the matrix \(\Pi^N\) are positive.
5. Spherical sums and Markov operator

In this section we express the spherical averages in terms of powers of a Markov operator, see Lemma 5.5, and obtain an identity relating this Markov operator with its adjoint, see Lemma 5.6.

5.1. Thickened paths and the sphere in the group. Consider a state \( k \in \Xi \) and let \( P_{L,R}, F_{L,R} \) be the left/right past and future domains of some representation of \( k \) in the tessellation; if the state has only one past or future domain, we have \( P_L = P_R \) or \( F_L = F_R \) respectively. Then the maps \( \gamma, \omega: \Xi \to G \) are defined as follows. Let \( P_L = h\mathcal{R} \), then

\[
F_L = h\gamma(k)^{-1}\mathcal{R}, \quad F_R = h\omega(k)^{-1}\mathcal{R}.
\]

Clearly, these definitions do not depend on the choice of a representation for \( k \).

Lemma 5.1. The above-defined maps \( \gamma \) and \( \omega \) satisfy the following identities:

1. \( \omega(\iota(k)) = \omega(k)^{-1} \) for any \( k \in \Xi \),
2. \( \gamma(k) = \omega(j)^{-1}\gamma(\iota(j))^{-1}\omega(k) \) for any \( j, k \in \Xi \) such that \( k \to j \) is an admissible transition.

Proof. 1. Consider any representation of a state \( k \) in the tessellation, \( P_{L,R}, F_{L,R} \) being its left/right past and future domains. Then the domains \( \tilde{P}_{L,R} = F_{R,L} \) and \( \tilde{F}_{L,R} = P_{R,L} \) represents the state \( \iota(k) \). Therefore, if \( P_L = h\mathcal{R}, F_R = g\mathcal{R} \), we have \( \omega(k)^{-1} = h^{-1}g, \omega(\iota(k))^{-1} = g^{-1}h \).

2. Consider six domains \( P_{L,R}, F_{L,R}, G_{L,R} \) such that the domains \( P_{L,R}, F_{L,R} \) are the past and the future domains for a representation of \( k \) and the domains \( F_{L,R}, G_{L,R} \) are the past and the future domains for a representation of \( j \). Let \( P_L = h\mathcal{R} \). Then \( F_L = h\gamma(k)^{-1}\mathcal{R} \) and \( G_R = h\gamma(k)^{-1}\omega(j)^{-1}\mathcal{R} \). Since \( G_{R,L} \) and \( F_{R,L} \) represent the state \( \iota(j) \), we have \( F_R = h\gamma(k)^{-1}\omega(j)^{-1}\gamma(\iota(j))^{-1}\mathcal{R} \). On the other hand, \( F_R = h\omega(k)^{-1}\mathcal{R} \). \( \square \)

Lemma 5.2. Consider the set \( P_{n-1}^{S \to F} \) defined by (2), and the map \( \Phi: P_{n-1}^{S \to F} \to G \), where

\[
\Phi(j_0 \to \cdots \to j_{n-1}) = \omega(j_{n-1})\gamma(j_{n-2})\cdots\gamma(j_0).
\]

Then \( \Phi \) is a bijection of \( P_{n-1}^{S \to F} \) onto the set \( S_n(G) = \{ g \in G : |g| = n \} \).

Remark 5.3. Note that for \( j_{n-1} \in \Xi_F \) there is only one future fundamental domain hence \( \omega(j_{n-1}) = \gamma(j_{n-1}) \). A reason for the separate notation on the last step will be explained later.

Proof. Theorem 4.6 shows that sequences from \( P_{n-1}^{S \to F} \) bijectively correspond to thickened paths \( S \) from \( \mathcal{R} \) to \( g\mathcal{R} \) with \( g \in S_n(G) \).

Take \( j \in P_{n-1}^{S \to F} \), let \( S \) be the corresponding thickened path and define \( h_k \in G \) so that \( h_j\mathcal{R} \) is the left domain in \( [S]_k \). Then

\[
g = h_{n-1}\omega(j_{n-1})^{-1} = h_{n-2}\gamma(j_{n-2})^{-1}\omega(j_{n-1})^{-1} = \cdots = [\omega(j_{n-1})\gamma(j_{n-2})\cdots\gamma(j_0)]^{-1},
\]

and it remains to use that \( g \mapsto g^{-1} \) is a bijective map on the sphere \( S_n(G) \). \( \square \)
5.2. **Parry measure.** Let $\Pi$ be the adjacency matrix of the topological Markov chain described in Definition 4.5. Lemma 4.12 shows that this Markov chain is strongly connected. The Perron—Frobenius theorem then yields that the matrix $\Pi$ has a unique (up to scaling) eigenvector $h$ with nonnegative coordinates, that all coordinates of $h$ are positive, and that the eigenvalue $\lambda$ corresponding to $h$ is greater than the absolute value of any other eigenvalue of $\Pi$:

$$
\sum_j \Pi_{ij} h_j = \lambda h_i, \quad h_i > 0 \text{ for all } i.
$$

The eigenvalue $\lambda$ is called the *Perron—Frobenius* (PF) eigenvalue and $h$ is called the right *Perron—Frobenius* eigenvector. The matrix $P$ with entries

$$
p_{ij} = \frac{h_i}{\lambda h_i} \Pi_{ij}
$$

is stochastic and the corresponding Markov chain has the following property: the probability of an admissible sequence of transitions depends only on the initial and the final states in this sequence and the number of steps:

$$
p_{i_0 i_1} \cdots p_{i_{n-1} i_n} = \frac{h_{i_n}}{\lambda^n h_{i_0}} \Pi_{i_0 i_1} \cdots \Pi_{i_{n-1} i_n} = \frac{h_{i_n}}{\lambda^n h_{i_0}}.
$$

The Markov measure defined by the matrix $P$ is called the *Parry measure*. Its stationary distribution is

$$
p_i = \alpha_i h_i,
$$

where $\alpha$ is the left PF eigenvector of $\Pi$: $\alpha \Pi = \lambda \alpha$, normalized by $\alpha h = \sum_i \alpha_i h_i = 1$.

The time-reversing involution on the set of states implies certain symmetries for the Parry measure.

**Proposition 5.4.** Let an involution $\iota: \Xi \to \Xi$ be such that $\Pi_{\iota(j)\iota(k)} = \Pi_{kj}$ for all $j,k \in \Xi$. Then the transition probability matrix $(p_{ij})$ and the stationary distribution $(p_i)$ of the Parry measure corresponding to the matrix $\Pi$ satisfy the following equations:

$$
p_{i(j)} = p_j, \quad p_{i(j)\iota(k)} = \frac{p_k p_{kj}}{p_j} \quad \text{for all } j,k \in \Xi.
$$

**Proof.** Let $J$ be the matrix for the substitution $\iota$. Then $J = J^T = J^{-1}$, $J\Pi J = \Pi^T$. Let, as above, $\lambda$ be the Perron—Frobenius (PF) eigenvalue for $\Pi$ and let $\alpha$ and $h$ be the left and the right PF eigenvectors for $\Pi$ normalized by $\alpha h = 1$. Then $\alpha J$ is a left PF eigenvector for $J\Pi J = \Pi^T$, whence $(\alpha J)^T = J\alpha^T$ is a right PF eigenvector for $M$. Therefore, $J\alpha^T$ is proportional to $h$: $\alpha_{i(k)} = ch_k$. Now

$$
p_{i(j)} = \alpha_{i(j)} h_{i(j)} = ch_j \frac{1}{c} \alpha_j = p_j
$$

and

$$
p_{i(j)\iota(k)} = \frac{\Pi_{i(j)\iota(k)} h_{i(k)}}{\lambda h_{i(j)}} = \frac{\Pi_{kj} c^{-1} \alpha_k}{\lambda c^{-1} \alpha_j} = \frac{\Pi_{kj} h_j h_k \alpha_k}{\lambda h_k h_j \alpha_j} = \frac{p_k p_{kj}}{p_j}. \quad \Box
$$
5.3. Markov operator. Recall that the group $G$ acts on a Lebesgue probability space $(X, \mu)$ by measure-preserving maps $T_g$. We denote $T_g f := f \circ T_g^{-1}$ for any function $f \in L^p(X, \mu)$. Denote

$$\overline{S}_n(f) = \sum_{|g| = n} T_g^{-1} f,$$
then

$$S_n(f) = \frac{\overline{S}_n(f)}{S_n(1)} = \frac{\sum_{|g| = n} T_g^{-1} f}{\# \{g : |g| = n\}},$$

where $S_n(f)$ is defined by $[1]$.

Consider the probability space $Y = \Xi \times X$ with the product measure $\nu = p \times \mu$. Here $p(\{i\}) = p_i$, where $p_i$ is defined by $[7]$. It is convenient to identify a function $\varphi \in L^1(Y, \nu)$ with a tuple of functions $(\varphi_i)_{i \in \Xi}$, where $\varphi_i(\cdot) = \varphi(i, \cdot)$.

Define the following operators $P, U : L^1(Y, \nu) \to L^1(Y, \nu)$:

$$\begin{align*}
(P \varphi)_i &= \sum_j p_{ij} T_{\gamma(i)}^{-1} \varphi_j, \quad (U \varphi)_j = T_{\omega(j)}^{-1} \varphi_j.
\end{align*}$$

It is clear that $P$ and $U$ are measure-preserving Markov operators.

**Lemma 5.5.** For any function $f \in L^1(X, \mu)$ define a function $\varphi(f) \in L^1(Y, \nu)$ by

$$\varphi(f)_j = \begin{cases} 1 & j \in \Xi_S, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\overline{S}_n(f) = \lambda^{n-1} \sum_{j \in \Xi_S} h_j (P^{n-1} U \varphi(f))_j,$$

*Proof.* Indeed,

$$\overline{S}_n(f) = \sum_{i_0 \in \Xi_S, i_{n-1} \in \Xi_F, i_1, \ldots, i_{n-2} \in \Xi} M_{i_0 i_1} \cdots M_{i_{n-2} i_{n-1}} T_{\omega(i_{n-1})}^{-1} T_{\gamma(i_{n-2})}^{-1} \cdots T_{\gamma(i_0)}^{-1} f =$$

$$\lambda^{n-1} \sum_{i_0 \in \Xi_S, i_{n-1} \in \Xi_F, i_1, \ldots, i_{n-2} \in \Xi} h_{i_0} p_{i_0 i_1} \cdots p_{i_{n-2} i_{n-1}} \frac{1}{h_{i_{n-1}}} T_{\gamma(i_0)}^{-1} \cdots T_{\gamma(i_{n-2})}^{-1} T_{\gamma(i_{n-1})}^{-1} f =$$

$$\lambda^{n-1} \sum_{i_0 \in \Xi_S} h_{i_0} \left( \sum_{i_1} p_{i_0 i_1} T_{\gamma(i_0)}^{-1} \cdots \sum_{i_{n-1}} p_{i_{n-2} i_{n-1}} T_{\gamma(i_{n-2})}^{-1} T_{\gamma(i_{n-1})}^{-1} \left( \frac{\lambda \varphi(f)(i_{n-1})}{h_{i_{n-1}}} \right) \cdots \right) =$$

$$\lambda^{n-1} \sum_{i_0} h_{i_0} (P^{n-1} U \varphi(f))_{i_0}. \quad \square$$

5.4. Dual operator. Let us recall that for $\varphi, \psi \in L^2(Y, \nu)$ we have

$$\langle \varphi, \psi \rangle = \sum_{k \in \Xi} p_k \langle \varphi_k, \psi_k \rangle.$$

A short computation shows that if an operator $Q$ has the form

$$(Q \varphi)_i = \sum_{j \in \Xi} p_{ij} T_{ij} \varphi_j, \quad (Q^* \psi)_j = \sum_{k \in \Xi} p_{kj} T_{kj}^* \psi_k,$$

then $Q^* Q$ has the form

$$(Q^* Q \varphi)_i = \sum_{j \in \Xi} p_{ij} T_{ij}^* Q^* \varphi_j, \quad (Q^* Q^* \psi)_j = \sum_{k \in \Xi} p_{kj} T_{kj} Q \psi_k.$$
Therefore, for $P$ defined by (8) we have
\[(P^* \psi)_j = \sum_{k \in \Xi} \frac{p_k p_{kj}}{p_j} T_{\gamma(k)} \psi_k.\]

**Lemma 5.6.** The Markov operators $P$ and $U$ defined by (8) satisfy the following identities:

\[U = U^{-1} = U^*, \quad P^* = UPU.\]

**Proof.** This identities follow from the identities for maps $\gamma$ and $\omega$ given in Lemma 5.1, from Proposition 5.4. For example, let us prove the second identity:

\[(UPU \psi)_j = T^{-1}_{\omega(j)} (PU \psi)_{i(j)} = T^{-1}_{\omega(j)} \sum_l p_{i(j),i(l)} T^{-1}_{\gamma(l)(j)} (U \psi)_l =\]
\[= \sum_l p_{i(j),i(l)} T^{-1}_{\omega(l)(j)} T^{-1}_{\gamma(l)(j)} T^{-1}_{\omega(j)} \psi_l(l) = \sum_k p_{i(j),i(k)} T_{\omega(j)}^{-1} \gamma(l)(j) - \omega(k) \psi_l(k).\]

For the last equality we substitute $l = i(k)$ and use the first identity in Lemma 5.1. Now using Proposition 5.4, the second identity in Lemma 5.1 and formula (10) one can see that the right-hand side equals $(P^* \psi)_j$. 

\[\square\]

### 6. General theorem on pointwise convergence

In this section, extending Theorem 1 in [14], we prove a general pointwise convergence theorem for powers of a Markov operator. Let $(Z, \eta)$ be a Lebesgue probability space, and let $Q$ be a measure-preserving Markov operator on $L^1(Z, \eta)$. We need the following assumptions.

**Assumption 6.1.** There exists a decomposition $Q = VW$, where $V$ and $W$ are measure-preserving Markov operators, so that $Q^* = WV$.

**Assumption 6.2.** For every $n \in \mathbb{N}$ the equation $Q^n \psi = \psi$ has only constant solutions in $L^2(Z, \eta)$.

**Assumption 6.3.** There exists $m \in \mathbb{N}$ such that the equation $(Q^*)^m Q^m \psi = \psi$ has only constant solutions in $L^2(Z, \eta)$.

**Assumption 6.4.** There exists a sequence of operators $A_n$ and constants $C, K > 0$ and $a, b, n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ the following inequality holds for any nonnegative $\varphi \in L^1(Z, \eta)$:

\[(11) \quad WQ^{2n-a} \varphi \leq C \sum_{j=-b}^b (Q^*)^n Q^{n+j} \varphi + A_n \varphi, \]

where $W$ is the operator from Assumption 6.1 and the operators $A_n: L^1(Z, \eta) \to L^1(Z, \eta)$ map nonnegative functions into nonnegative ones, for any $p \in [1, \infty]$ map $L^p$ to itself, and $\|A_n\|_{L^p} \leq \alpha_n$, with $\sum_{n=n_0}^\infty \alpha_n \leq K$.

**Remark 6.5.** Applying $V' = QV$ to both sides of (11), we arrive at the inequality

\[(11') \quad Q^{2n-a'} \varphi \leq CV' \sum_{j=-b}^b (Q^*)^n Q^{n+j} \varphi + A_n' \varphi\]
with the same estimates on the norms of the operators $A'_n$. We will use both (11) and (11').

**Theorem 6.6.** Let $Q : L^1(Z, \eta) \to L^1(Z, \eta)$ be a measure-preserving Markov operator acting on a Lebesgue probability space $(Z, \eta)$ and satisfying Assumptions 6.1–6.4. Then for every function $\varphi \in L \log L(Z, \eta)$ the sequence $Q^n \varphi$ converges almost surely and in $L^1$ to $\int_Z \varphi \, d\eta$ as $n \to \infty$.

The proof follows the scheme from [14] and occupies the rest of this section.

6.1. Space of trajectories. The space of trajectories corresponding to $Q$ is the space $(Z, P_Q)$, where $Z = Z^\infty$ with the usual Borel sigma-algebra $\mathcal{B}_Z$, and the measure $P_Q$ is given by the Ionescu Tulcea Extension Theorem, where all probability kernels are the same and depend only on one preceding element of the trajectory. Namely, for $z \in Z$ and a measurable set $A \subset Z$ we define

$$P_Q(z, A) = P_{Q,z} (A) = Q[1_A](z).$$

Then the probability measure $P_Q$ is given as follows:

$$P_Q\{z_m \in A_m, \ldots, z_n \in A_n\} = \int_{z_m \in A_m} \cdots \int_{z_m+1 \in A_{m+1}} \cdots \left[ \int_{z_n \in A_n} \cdots \int_{z_n+1 \in A_{n+1}} \cdots \right].$$

By definition the shift map $\sigma : Z \to Z$, $(\sigma(z))_n = z_{n+1}$ preserves the measure $P_Q$.

The sigma-algebras $\mathcal{F}_k$, $k, l \in \mathbb{Z} \cup \{+\infty, -\infty\}$ are the minimal complete sigma-algebras such that all functions $\pi_j : Z \mapsto z_j$ are measurable for $k \leq j \leq l$. For brevity we denote $\mathcal{F}_n = \mathcal{F}_n$. Let us also recall that the tail sigma-algebra is defined as

$$\mathcal{F}_{\text{tail}} = \bigcap_{n=0}^{\infty} \mathcal{F}_n^{\infty}.$$  

For any function $\varphi \in L^1(Z, \eta)$ we define the function $\varphi^0 \in L^1(Z, P_Q)$ by the formula $\varphi^0(z) = \varphi(z_0)$. We have

$$E(\varphi^0 | F_{-n})(z) = E(\varphi^0 | F_n)(z) = (Q^n \varphi)(z_{-n}),$$

$$E(\varphi^0 | F_{n+\infty})(z) = E(\varphi^0 | F_n)(z) = ((Q^*)^n \varphi)(z_n).$$

6.2. Mixing of the operator $Q$. We start by proving mixing for $\tilde{Q} = Q^m$ where $m$ is defined in Assumption 6.3.

**Lemma 6.7.** Let $\tilde{Q}$ be a measure-preserving Markov operator on $L^1(Z, \eta)$ such that the equation $\tilde{Q}^* \tilde{Q} \varphi = \varphi$ has only constant solutions in $L^2(Z, \eta)$. Then for any $\varphi, \psi \in L^2(Z, \eta)$ we have

$$\langle \tilde{Q}^n \varphi, \psi \rangle = \int_Z \tilde{Q}^n \varphi \cdot \psi \, d\eta \to \int_Z \varphi \, d\eta \int_Z \psi \, d\eta \text{ as } n \to \infty.$$
Proof. The statement follows from the mixing of the shift map $\sigma$ in the trajectory space $(Z, \mathbb{P}_Q)$. To obtain the latter we shall prove that $\sigma$ has $K$-property: there exists a sub-sigma-algebra $K$ of the Borel sigma-algebra $B_Z$ such that $K \subset \sigma K$, $\bigvee_{n=0}^\infty \sigma^n K = B_Z$, and $\bigcap_{n=0}^\infty \sigma^{-n} K = \{\emptyset, Z\}$.

By the Rokhlin—Sinai theorem (see [40], Ch. 18) the $K$-property is equivalent to the triviality of the Pinsker sigma-algebra $\Pi(\sigma)$. Consider $F_- = F_0^\infty$. Then $\sigma F_- \subset F_-$ and $\bigvee_{k \in \mathbb{Z}} \sigma^k F_- = B_Z$. Thus $\Pi(\sigma^{-1}) \subset F_-$ (see, e.g., Lemma 18.7.3 in [24]). Similarly, for $F_+ = F_0^\infty$ one has $\Pi(\sigma) \subset F_+$. Therefore, $\Pi(\sigma) = \Pi(\sigma^{-1}) \cap F_+ = F_0$.

We have proved that any $\Pi(\sigma)$-measurable function $\varphi \in L^2(Z, \mathbb{P}_Q)$ depend only on the zero coordinate: $\varphi(z) = \varphi_0(z_0)$. More generally, $\varphi(z) = \varphi_k(z_k)$. Formulas (12) yield that $\varphi_{-1} = \tilde{Q}\varphi_0$ and $\varphi_0 = \tilde{Q}^*\varphi_{-1}$. Therefore, $\varphi_0 = \tilde{Q}^*\varphi_0$, and the assumption of the lemma yields that $\varphi_0 = \text{const}$, thus $\Pi(\sigma)$ is trivial. $\square$

**Corollary 6.8.** The operator $Q$ is also mixing, that is, (13) holds for $Q$ instead of $\tilde{Q}$.

**Proof.** The sequence $((Q^n \varphi, \psi))_{n \geq 0}$ is the union of the subsequences $((Q^{n^0+r} \varphi, \psi))_{n \geq 0}$. Each of them converges to the desired limit by Lemma 6.7 applied to the pair of functions $(Q^r \varphi, \psi)$. $\square$

### 6.3. Triviality of the tail sigma-algebra

The next step is to prove that the tail sigma-algebra for $Q$ is trivial. First, we prove that the tail sigma-algebra cannot be totally nontrivial, that is, it cannot contain infinitely many different sets (up to sets of measure zero).

The proof follows that of Lemma 6 in [14], which is a version of the 0–2 law in the form of Kaimanovich [31].

**Lemma 6.9.** For a measure-preserving Markov operator $R$ on $L^1(Z, \eta)$ the following holds. If the tail sigma-algebra of $R$ is totally nontrivial then for any $b \in \mathbb{N}$ and any $\varepsilon > 0$ there exist nonnegative functions $\varphi, \psi \in L^\infty(Z, \eta)$ with averages equal to 1 such that

\[ \limsup_{n \to \infty} \langle (R^n)^{n+b} \varphi, (R^n)^{n} \psi \rangle_{L^2(Z, \eta)} + \cdots + \langle (R^n)^{n-b} \varphi, (R^n)^{n} \psi \rangle_{L^2(Z, \eta)} < \varepsilon. \]

**Proof.** Let $(Z, \mathbb{P}_R)$ be the corresponding trajectory space. If $F_{\text{tail}}$ contains infinitely many subsets, it contains a subset of arbitrarily small measure. Indeed, split $Z = A_1^{(2)} \sqcup A_2^{(2)}$, where $A_i^{(2)} \in F_{\text{tail}}$ have nonzero measure. Then at least one of these parts can be split into two sets of nonzero measure (otherwise $F_{\text{tail}}$ contains only finitely many sets, the unions of some of $A_i^{(2)}$). Repeating this procedure, we get $Z = A_1^{(n)} \sqcup \cdots \sqcup A_n^{(n)}$. Then the measure of at least one of $A_j^{(n)}$ is not more than $1/n$.

Take any set $A \in F_{\text{tail}}$ with $\mathbb{P}_R(A) < 1/(2b+1)$. Then the set $B = Z \setminus \bigcup_{s=-b}^{b} \sigma^s(A)$ has positive measure. Denote

\[ \Phi(z) = 1_A(z)/\mathbb{P}_R(A), \quad \Psi(z) = 1_B(z)/\mathbb{P}_R(B). \]

Observe that $\Phi$ and $\Psi$ are nonnegative, $F_{\text{tail}}$-measurable, bounded by some constant $M$, their expectations are equal to 1, and $(\Phi \circ \sigma^j) \cdot \Psi = 0$ for $j = -b, \ldots, b$. 


Set \( \varphi_k = E(\Phi|F_{\infty}^k) \), \( \psi_k = E(\Psi|F_{\infty}^k) \). Note that \( \varphi_k(z) \) depends only on \( z_k \), so abusing notation we use the same symbol \( \varphi_k \) for the corresponding function in \( L^1(Z, \eta) \). For example, we will write \( \varphi_k \circ \sigma^{-j}(z) = \varphi_k(z_{k+j}) \).

Clearly, \( \varphi_k \) and \( \psi_k \) are nonnegative and bounded by \( M \). Therefore, the martingale convergence theorem gives that \( \varphi_k \rightarrow \Phi \), \( \psi_k \rightarrow \Psi \) in \( L^1(Z, P_R) \). Moreover, \( \varphi_k(z_{k-j}) = \varphi_k \circ \sigma^{-j} \rightarrow \Phi \circ \sigma^{-j} \). Hence

\[
E(\varphi_k(z_{k-j})|F_{\text{tail}}) \rightarrow E(\Phi \circ \sigma^{-j}|F_{\text{tail}}) = \Phi \circ \sigma^{-j}(z), \quad E(\psi_k(z)|F_{\text{tail}}) \rightarrow \Psi(z)
\]

in \( L^1(Z, P_R) \). Since all these functions are bounded by the same constant \( M \), for large \( k \) we have that

\[
\int_Z E(\varphi_k(z_{k-j})|F_{\text{tail}})E(\psi_k(z)|F_{\text{tail}}) \, dP_R < \frac{\varepsilon}{2b+1}.
\]

By (12) we have \( E(\gamma(z_k)|F_{\infty}^n) = [(R^*)^n \gamma](z_{n+k}) \), whence for any \( j = -b, \ldots, b \)

\[
\int_Z [(R^*)^{n+j} \varphi_k](z_{n+k}) \cdot [(R^*)^{n} \psi_k](z_{n+k}) \, dn = \int_Z E(\varphi_k(z_{k-j})|F_{\infty}^n) \cdot E(\psi_k(z)|F_{\infty}^n) \, dP_R
\]

\[
\rightarrow \int_Z E(\varphi_k(z_{k-j})|F_{\text{tail}}) \cdot E(\psi_k(z)|F_{\text{tail}}) \, dP_R < \frac{\varepsilon}{2b+1} \quad \text{as} \quad n \rightarrow \infty.
\]

Therefore, the functions \( \varphi_k \) and \( \psi_k \) for large \( k \) satisfy (14). \( \square \)

**Lemma 6.10.** Under the assumptions of Theorem 6.6 the tail sigma-algebra for \( Q^* \) cannot be totally nontrivial.

**Proof.** Assuming the contrary, the inequality [14] in Lemma [6.9] for \( R = Q^* \) yields that for some nonnegative functions \( \varphi, \psi \) with their averages equal to 1 and for all sufficiently large \( n \) we have

\[
\langle (Q^{n+b} + \cdots + Q^{n-b}) \varphi, Q^n \psi \rangle_{L^2(Z, \eta)} < \varepsilon.
\]

On the other hand, by Assumption [6.4] the left-hand side of [15] is not less than

\[
\frac{1}{C} (WQ^{2n-a} \varphi - A_n \varphi, \psi) = \frac{1}{C} (Q^{2n-a} \varphi, W^* \psi) - \frac{1}{C} \langle A_n \varphi, \psi \rangle \rightarrow \frac{1}{C} + 0.
\]

Here we use Corollary [6.8] here, note that the average values of both \( \varphi \) and \( W^* \psi \) are equal to 1. Therefore, for large \( n \) the left-hand side of [15] is larger than \( 1/C - \varepsilon \), so taking \( \varepsilon < 1/2C \) we arrive at a contradiction. \( \square \)

**Lemma 6.11.** Under the assumptions of Theorem 6.6 the tail sigma-algebra for \( Q \) cannot be totally nontrivial.

**Proof.** Consider the trajectory space \( (Z, P) \) for the infinite sequence \( \ldots, V, W, V, W, \ldots \) of Markov operators, that is,

\[
P(z_{2n+1} \in A | z_{2n}) = V[1_A](z_{2n}), \quad P(z_{2n+2} \in A | z_{2n+1}) = W[1_A](z_{2n+1}).
\]

Then one can check that

\[
P(z_{2n+2} \in A | z_{2n}) = VW[1_A](z_{2n}) = Q[1_A](z_{2n}), \quad P(z_{2n+1} \in A | z_{2n-1}) = Q^*[1_A](z_{2n-1}),
\]

\[
P(z_{2n+1} \in A | z_{2n}) = V[1_A](z_{2n}), \quad P(z_{2n+2} \in A | z_{2n+1}) = W[1_A](z_{2n+1}).
\]
hence the projections \( \pi_0 : z = (z_n) \mapsto (z_{2n}) \) and \( \pi_1 : z = (z_n) \mapsto (z_{2n+1}) \) map the trajectory space \((\mathbb{Z}, \mathbb{P})\) to the trajectory spaces for \( Q \) and \( Q^* \) respectively. Therefore, the triviality of the tail sigma-algebras in the trajectory spaces for \( Q \) and \( Q^* \) is respectively equivalent to the triviality of sigma-algebras

\[
\mathcal{F}_{\text{tail}, 0} = \bigcap_{n \geq 2} \bigvee_{2k \geq n} \mathcal{F}_{2k} \quad \text{and} \quad \mathcal{F}_{\text{tail}, 1} = \bigcap_{n \geq 2} \bigvee_{2k+1 \geq n} \mathcal{F}_{2k+1}
\]

respectively. Thus to prove the lemma it is sufficient to show that in (16) with respect to \( \mathcal{F}_{\text{tail}} \), \( \mathcal{F}_{\text{tail}, 0} \), \( \mathcal{F}_{\text{tail}, 1} \) are trivial simultaneously. Obviously, the triviality of \( \mathcal{F}_{\text{tail}} \) implies the triviality of \( \mathcal{F}_{\text{tail}, j} \), \( j = 0, 1 \). Let us prove the converse.

Consider any \( A \in \mathcal{F}_{\text{tail}} \) and check that, say, \( A \in \mathcal{F}_{\text{tail}, 0} \). Indeed, \( A \in \bigvee_{m \geq 2n} \mathcal{F}_m \) for every \( n \), and we can eliminate any finite number of \( \mathcal{F}_k \) with odd \( k \) from the set of \( m \)'s there:

\[
A \in \mathcal{F}_{2n} \vee \mathcal{F}_{2n+2} \vee \cdots \vee \mathcal{F}_{2(n+s-1)} \vee \bigvee_{m \geq 2(n+s)} \mathcal{F}_m.
\]

Consider the conditional probability \( \mathbb{P}(\cdot \mid z_{2n}, z_{2n+2}, \ldots) \) with respect to the sigma-algebra \( \bigvee_{k \geq n} \mathcal{F}_{2k} \). As (16) shows, with respect to this conditional probability \( A \) depend only on “odd tail” \( \bigvee_{k \geq n+s} \mathcal{F}_{2k+1} \). But since the odd coordinates \( z_{2n+1}, \ldots, z_{2(n+s)+1}, \ldots \) are independent for the fixed even coordinates \( z_{2n}, \ldots, z_{2(n+s)}, \ldots \), by Kolmogorov’s 0–1 Law we obtain that \( A \) is trivial with respect to this conditional probability, so \( A \) is measurable with respect to \( \bigvee_{k \geq n} \mathcal{F}_{2k} \), and hence \( A \in \mathcal{F}_{\text{tail}, 0} \).

\[\□\]

**Lemma 6.12.** Under the assumptions of Theorem 6.6 the tail sigma-algebra for \( Q \) is trivial.

**Proof.** It remains to eliminate the case when \( \mathcal{F}_{\text{tail}} \) contains only finitely many different sets. Assume that \( Z = A_1 \sqcup \cdots \sqcup A_r, r > 1 \), where each \( A_j \in \mathcal{F}_{\text{tail}} \) has no nontrivial subsets belonging to \( \mathcal{F}_{\text{tail}} \). The shift map \( \sigma \) interchanges these subsets, whence for \( A = A_1 \) there exists \( n \) such that \( \sigma^n A = A \). As in Lemma 6.9, we define \( \Phi = 1_A / \mathbb{P}_Q(A) \) and \( \varphi_k(z) = \varphi_k(z) = E(\Phi | \mathcal{F}^k) \). Then

\[
E(\Phi | \mathcal{F}^k) \circ \sigma^n = E(\Phi \circ \sigma^n | \mathcal{F}^{k+n}) = E(\Phi | \mathcal{F}^{k+n}) = \varphi_{k+n},
\]

hence

\[
\varphi_{k+n}(z_k) = \varphi_{k+n} \circ \sigma^{-n} = E(\Phi | \mathcal{F}^k) =
\]

\[
= E(E(\Phi | \mathcal{F}^{k+n}) | \mathcal{F}^k) = E(\varphi_{k+n}(z_{k+n}) | \mathcal{F}^k) = [Q^n \varphi_{k+n}](z_k).
\]

Thus we arrive at the equation \( \varphi_{k+n}(z_k) = [Q^n \varphi_{k+n}](z_k) \) and Assumption 6.2 implies that \( \varphi_{k+n} \) is constant. Taking averages, we get \( E(\varphi_{k+n}) = E(\Phi) = 1 \), thus \( \varphi_l \equiv 1 \) for all \( l \). But this contradicts to the convergence \( \varphi_l \to \Phi \neq 1 \), which was obtained in proof of Lemma 6.9. \[\□\]
6.4. Convergence.

**Lemma 6.13** (see [31; 14 Propositions 4, 5]). For a measure-preserving Markov operator $R$ on $(Z, \eta)$ with the trivial tail sigma-algebra we have $R^n \varphi \rightarrow \int_Z \varphi \, d\eta$, where the convergence takes place in $L^1$ for $\varphi \in L^1(Z, \eta)$ and in $L^2$ for $\varphi \in L^2(Z, \eta)$.

**Lemma 6.14** ([14 Lemma 8]). For a measure-preserving Markov operator $R$ on $(Z, \eta)$ for any $p > 1$ there exists a constant $A_p > 0$ such that for every nonnegative function $\varphi \in L^p(Z, \eta)$ we have

$$\left\| \sup_{n \geq 0} (R^n)^p \varphi \right\|_{L^p} \leq A_p \| \varphi \|_{L^p}.$$  

Similarly, there exists a constant $A_{\log} > 0$ such that for every nonnegative function $\varphi \in L \log L(Z, \eta)$ we have

$$\left\| \sup_{n \geq 0} (R^n)^{n+s} \varphi \right\|_{L^1} \leq A_{\log} \| \varphi \|_{L \log L}.$$  

**Remark 6.15.** The following inequalities hold for any $s \in \mathbb{Z}$:

$$\left\| \sup_{n \geq 0} (R^s)^{n+s} R^n \varphi \right\|_{L^p} \leq A_p \| \varphi \|_{L^p}, \quad \left\| \sup_{n \geq 0} (R^s)^{n+s} \varphi \right\|_{L^1} \leq A_{\log} \| \varphi \|_{L \log L},$$

the supremums here are taken over all $n$ such that both $n$ and $n + s$ are nonnegative. Indeed, for $s > 0$ we apply our lemma to $R^s \varphi$ and use inequality $\| R^s \varphi \|_{L \log L} \leq \| \varphi \|_{L \log L}$. For $s < 0$ we use the inequality

$$(R^s)^{|s|} \left| \sup_{n \geq 0} (R^s)^n R^n \varphi \right| \geq \sup_{n \geq 0} (R^s)^{n+s} R^n \varphi$$

which yields the same equation for the $L^1$-norms of both sides. Note also that the $L^1$-norm of left-hand side does not exceed the same norm for the function in the square brackets there. Hence

$$A_{\log} \| \varphi \|_{L \log L} \geq \left\| \sup_{n \geq 0} (R^s)^n R^n \varphi \right\| \geq \left\| \sup_{n \geq 0} (R^s)^{n+s} \varphi \right\|,$$  

and it remains to change $n \rightarrow n + s$.

**Proof of Theorem 6.6.** Combining (17) for $R = Q$ and Assumption 6.4 in the form (11), for any nonnegative function $\varphi \in L \log L(Z, \eta)$ we obtain

$$\left\| \sup_{n \geq n_0} \left( Q^{2^{n-a'}} \varphi \right) \right\|_{L^1} \leq C \left\| V' \left( \sup_{n \geq n_0} \sum_{j=-b}^b (Q^s)^n Q^{n+j} \varphi \right) \right\|_{L^1} + \left\| \sup_{n \geq n_0} A'_n \varphi \right\|_{L^1}$$

$$\leq (2b + 1) A_{\log} C \| \varphi \|_{L \log L} + \sum_{n \geq n_0} \left\| A'_n \varphi \right\|_{L^1} \leq B_{\log} \| \varphi \|_{L \log L}.$$  

Decomposing a function $\varphi$ into its positive and negative parts we obtain (18) for all real-valued $\varphi \in L \log L(Z, \eta)$ with a larger $B_{\log}$. The same estimates hold for $L^p$-norm.

Now consider a real-valued function $\varphi \in L^2(Z, \eta)$ with the zero average. Applying (18) to $(Q^{2k} \varphi)$ we have

$$\left\| \sup_{m \geq m_n + k} \left( Q^{2^{m-a'}} \varphi \right) \right\|_{L^2} = \left\| \sup_{n \geq n_0} \left( Q^{2^{n+2k-a'}} \varphi \right) \right\|_{L^2} \leq B_2 \| Q^{2k} \varphi \|_{L^2}.$$
Since the right-hand side tends to zero by Lemma 6.13, the sequence $Q^{2m-a'}\varphi$ tends to zero almost everywhere and in $L^2$ as $m \to \infty$.

We now extend pointwise convergence to all $\varphi \in L\log L$. Namely, for a real-valued function $\varphi \in L\log L(Z, \eta)$ with zero average consider $\varphi' \in L^2(Z, \eta)$ with zero average such that $\|\varphi - \varphi'\|_{L\log L} \leq \varepsilon/B_{\log}$. Then almost surely we have

$$\limsup_{n \to \infty} |Q^{2n-a'}\varphi(z)| \leq \limsup_{n \to \infty} |Q^{2n-a'}\varphi'(z)| + \limsup_{n \to \infty} |Q^{2n-a'}(\varphi - \varphi')(z)|.$$

By convergence for functions in $L^2$, the first term in the right-hand side equals zero, while, by the maximal inequality, the second satisfies

$$\|\limsup_{n \to \infty} Q^{2n-a'}(\varphi - \varphi')(z)\|_{L^1} \leq B_{\log}\|\varphi - \varphi'\|_{L\log L} \leq \varepsilon.$$

Therefore, $\limsup |Q^{2n-a'}\varphi(z)| \leq \delta$ outside of the set of measure less than $\varepsilon/\delta$. Taking $\varepsilon = 1/l \to 0$ and then $\delta = 1/l' \to 0$ we obtain that this upper limit equals zero almost everywhere. The convergence in $L^1$ follows from the same decomposition:

$$\|Q^{2n-a'}\varphi(z)\|_{L^1} \leq \|Q^{2n-a'}\varphi'\|_{L^1} + \|Q^{2n-a'}(\varphi - \varphi')(z)\|_{L^1},$$

where the first term tends to zero even with $L^2$-norm instead of $L^1$, and the second term is less than $\|\varphi - \varphi'\|_{L^1} \leq \varepsilon/B_{\log}$.

Finally, combining the obtained convergence $Q^{2m-a'}\varphi \to 0$ with the same convergence for $Q\varphi$ in place of $\varphi$, we conclude that $Q^n\varphi \to 0$. □

7. PROOF OF THEOREM A

The proof of Theorem A in the case of trivial $I_{G_0}$ is based on Theorem 6.6. Namely, we denote

$$(19) \quad Q = P^2, \quad V = PU, \quad \text{and} \quad W = UP,$$

where $P$ and $U$ are defined in (8). Assumption 6.1 now holds due to Lemma 5.6. In the next two subsections we will check that the remaining Assumptions 6.2, 6.3, and 6.4 hold for the operators defined by (19), provided $I_{G_0}$ is trivial. Finally, in Subsection 7.3 we deal with the case of nontrivial $I_{G_0}$ and conclude the proof of Theorem A.

7.1. $P^k$- and $(P^*)^kP^k$-invariant functions. To check Assumptions 6.2 and 6.3 we express the equations from these assumptions in terms of the components $\varphi_j$, $j \in \Xi$, of a function $\varphi$.

**Proposition 7.1.** Let $P$ be the Markov operator defined by (8). Then the following holds.

1) A function $\varphi \in L^2(Y, \nu)$ is a solution to the equation $P^k\varphi = \varphi$ if and only if for every admissible sequence $i_0 \to i_1 \to \cdots \to i_k$ of states we have

$$(20) \quad \varphi_{i_0} = T_{\gamma(i_0)}^{-1} \cdots T_{\gamma(i_{k-1})}^{-1}\varphi_{i_k}.$$

2) If $k \geq N$, where $N$ is defined in Corollary 4.14, then a function $\varphi \in L^2(Y, \nu)$ is a solution for $(P^*)^kP^k\varphi = \varphi$ if and only if for every admissible sequences $i_0 \to i_1 \to \cdots \to i_k$...
and \( j_0 \to i_1 \to \cdots \to i_k \) with \( i_0 = j_0 \) we have

\[
T_{\gamma(i_1)}^{-1} \cdots T_{\gamma(i_{k-1})}^{-1} \varphi_{i_k} = T_{\gamma(j_1)}^{-1} \cdots T_{\gamma(j_{k-1})}^{-1} \varphi_{j_k}.
\]

**Proof.** 1) The equation \( P^k \varphi = \varphi \) is equivalent to

\[
\varphi_{i_0} = (P^k \varphi)_{i_0} = \sum_{i_1, \ldots, i_k} p_{i_0 i_1} \cdots p_{i_{k-1} i_k} T_{\gamma(i_0)}^{-1} \cdots T_{\gamma(i_{k-1})}^{-1} \varphi_{i_k},
\]

hence, since \( T_{\gamma} \)'s are unitary,

\[
(21) \quad \| \varphi_{i_0} \|_{L^2} \leq \sum_{i_1, \ldots, i_k} p_{i_0 i_1} \cdots p_{i_{k-1} i_k} \| \varphi_{i_n} \|_{L^2}.
\]

Multiplying these inequalities by \( p_{i_0} \) and summing them up for all \( i_0 \in \Xi \) we obtain

\[
\sum_{i_0} p_{i_0} \| \varphi_{i_0} \|_{L^2} \leq \sum_{i_1, \ldots, i_k} \left( \sum_{i_0, \ldots, i_{k-1}} p_{i_0 i_{i_1}} \cdots p_{i_{k-1} i_k} \right) \| \varphi_{i_k} \|_{L^2} = \sum_{i_k} p_{i_k} \| \varphi_{i_k} \|_{L^2}.
\]

Therefore, for each \( i_0 \) inequality (21) is indeed an equality, and the vector \( (\| \varphi_i \|_{L^2})_{i \in \Xi} \) is a left eigenvector of the matrix \( \Pi^k \) with nonnegative coordinates, where \( \Pi = (p_{ij}) \). As the Markov chain corresponding to \( \Pi \) is strongly connected and aperiodic, by the Perron—Frobenius theorem the vector \( (\| \varphi_i \|_{L^2})_{i \in \Xi} \) is proportional to \((1, \ldots, 1)\). Thus, all \( \varphi_i \) has the same norm.

Finally, in the Hilbert space \( L^2(X, \mu) \) the triangle inequality (21) reaches equality only if all nonzero summands are proportional to each other with positive coefficients, whence

\[
\varphi_{i_0} = c \cdot T_{\gamma(i_0)}^{-1} \cdots T_{\gamma(i_{k-1})}^{-1} \varphi_{i_n}. \quad \text{Calculating the } L^2 \text{-norms of both sides, we get } c = 1.
\]

2) Similarly, \((P^*)^k P^k \varphi = \varphi\) yields

\[
\varphi_{j_k} = \sum_{j_0, \ldots, j_{k-1}, i_1, \ldots, i_k} \left[ \sum_{i_0} p_{j_0 i_1} \cdots p_{j_{k-1} i_k} \varphi_{j_{k-1}} \right] \frac{p_{j_0 i_1} \cdots p_{j_{k-1} i_k} \varphi_{i_k}}{p_{j_k} T_{\gamma(j_{k-1})}^{-1} \cdots T_{\gamma(j_1)}^{-1} T_{\gamma(i_1)}^{-1} \cdots T_{\gamma(i_{k-1})}^{-1} \varphi_{i_k}}.
\]

The remaining is the same as in the first statement: \( (\| \varphi_i \|_{L^2})_{i \in \Xi} \) is a left eigenvector of a stochastic matrix with positive entries (indeed, for given \( i_k, j_k \) we choose \( j_{k-1}, \ldots, j_1, j_0 \) arbitrarily with \( p_{j_k j_{k+1}} > 0 \), then we can choose \( i_1, \ldots, i_{k-1} \) since \( (\Pi^k)^k_{j_0 i_k} > 0 \) for \( k \geq N \). Therefore, the \( L^2 \)-norms of all \( \varphi_i \)'s are equal, and the same argument with the triangle inequality completes the proof. \( \Box \)

**Lemma 7.2.** There exists \( k^* \) such that for any \( k \geq k^* \) if a function \( \varphi \in L^2(Y, \nu) \) satisfies equalities

\[
(22) \quad T_{\gamma(i_1)}^{-1} \cdots T_{\gamma(i_{k-1})}^{-1} \varphi_{i_k} = T_{\gamma(j_1)}^{-1} \cdots T_{\gamma(j_{k-1})}^{-1} \varphi_{j_k}
\]

for all admissible sequences \( i_0 \to i_1 \to \cdots \to i_k, j_0 \to j_1 \to \cdots \to j_k \) with \( i_0 = j_0 \), then \( \varphi(x, j) \) does not depend on \( j: \varphi(x, j) = \varphi(x), \) and \( \varphi(x) \) is \( G_0^\gamma \)-invariant.

**Remark 7.3.** If (22) holds for all pairs of sequences of a given length \( k \), then it holds for any pair of sequences \( i_0 \to i_1 \to \cdots \to i_{k'}, j_0 \to j_1 \to \cdots \to j_{k'} \) of length \( k' \leq k \) with \( i_0 = j_0 \). Indeed, append an arbitrary prefix \( i_{-(k-k')} \to \cdots \to i_0 \) to these sequences and...
Therefore, if \( H \) to these two sequences we get \( j \) of states corresponding to its convexification, thus applying (22) to the resulting sequences of length \( k \). One can see that \( T_{\gamma(\ell-(k-k')+1)}^{-1} \cdots T_{\gamma(\ell)}^{-1} \) cancels out and we arrive to (22) for the initial sequences of length \( k' \).

Let us first deduce Assumptions 6.2 and 6.3 from Lemma 7.2.

**Corollary 7.4.** Assumptions 6.2 and 6.3 hold for the operator \( Q \) defined by (8) and (19), assuming \( I_{G_0^0}^{n} \) is trivial.

**Proof.** 1) Suppose that \( Q^n \varphi = \varphi \). Choose \( l \) such that \( k = 2nl \geq k^* \). Then \( P^k \varphi = (Q^n)^l \varphi = \varphi \). Therefore, (22) holds, as both its sides are equal to \( T_{\gamma(\ell_0)} \varphi \) by (20). Lemma 7.2 then implies that all \( \varphi_j \) are equal to the same function \( \varphi^0 \), where \( \varphi^0 \) is \( G_0^0 \)-invariant and hence constant.

2) Suppose that \( (Q^*)^m Q^m \varphi = \varphi \), where \( m \) satisfies \( 2m \geq \max(k^*, N) \) and \( N \) is defined in the second part of Proposition 7.1. Then this proposition implies that (22) holds for \( \varphi \) with \( k = 2m \), so \( \varphi \) is constant.

It remains to prove Lemma 7.2.

**Proof of Lemma 7.2.** We will prove the statement of the lemma for \( k^* = 2m \), where \( m \leq 4 \) is the number from Proposition 4.9. For every \( e \in G_0 \) consider a path \( T^e \) from this proposition. Let \( i_0^e \to \cdots \to i_{m-1}^e = t(e) \) be the corresponding sequence of states. Denote

\[
\psi_e = T_{\gamma(i_0^e)}^{-1} \cdots T_{\gamma(i_{m-2}^e)}^{-1} \varphi_{i_{m-1}^e} = T_{g_e} \varphi_{i_{m-1}^e},
\]

where \( g_e = \gamma(i_0^e) \cdots \gamma(i_{m-2}^e)^{-1} \) is such that \( T_{m-1}^e = g_e R_e \).

Take any \( e_1, e_2 \) and choose \( e \neq e_1^{-1}, e_2^{-1} \). Let \( \mathcal{R}_s^s \) \((s = 1, 2)\) be the path from the first part of Proposition 4.11 applied to \( e_s \) and \( e \), and let \( j_m^s \to \cdots \to j_{m-1}^s \) be the sequence of states corresponding to its convexification, thus \( j_{m}^e = h(e) \), \( j_{m-1}^e = t(e) \). Applying (22) to these two sequences we get

(23)
\[
T_{\gamma(j_{m-1}^s)^{-1}} \cdots \gamma(j_{m-1}^{-1})^{-1} \varphi_{t(e)} = T_{\gamma(j_{m+1}^s)^{-1}} \cdots \gamma(j_{m+1}^{-1})^{-1} \varphi_{t(e)}.
\]

Note that

\[
[\mathcal{R}_s^s]_{-m+1} = \mathcal{R}_s^{s-1} = \mathcal{H}_{-m+1}, \quad [\mathcal{R}_s^s]_{m-1} = \mathcal{R}_{m-1}^s = T_{m-1}^{e_s}.
\]

Therefore, if \( \mathcal{H}_{-m+1} = \gamma R \) we have

\[
g_{e_s} = \gamma (j_{m-1}^s)^{-1} \cdots (j_{m+1}^{-1})^{-1},
\]

and (23) takes form \( T_{\gamma^{-1} g_{e_s} \varphi_{t(e)}} = T_{\gamma^{-1} g_{e_2} \varphi_{t(e)}}, \) or \( T_{\gamma^{-1} \psi_{e_1}} = T_{\gamma^{-1} \psi_{e_2}} \). Thus all \( \psi_e \) are equal to the same function \( \psi^0 \).

Now take again any \( e_1, e_2 \), choose \( e \neq e_1^{-1}, e_2^{-1} \) and apply the same argument to the paths from the second part of Proposition 4.11 for \( e, e_1 \) and \( e_2 \). We obtain that

\[
T_{\gamma^{-1} g_{e_1} \varphi_{t(e_1)}} = T_{\gamma^{-1} g_{e_2} \varphi_{t(e_2)}} = T_{\gamma^{-1} \psi_{e_1}} = T_{\gamma^{-1} \psi_{e_2}},
\]

Therefore, \( T_{e_1 e_2} \psi^0 = \psi^0 \), so \( \psi^0 \) is \( G_0^2 \)-invariant. \( \square \)
7.2. Proof of Assumption 6.4.

Lemma 7.5. Assumption 6.4 holds for the operators defined by (8) and (19). Namely, (11) holds for $a = 6, b = 2$ and $n_0$ being the maximum of $n(v)$ over all vertices of $\mathcal{R}$.

The proof rests on the following geometric statement. Denote by $P_M$ the set of all admissible sequences $i = (i_0 \to \cdots \to i_M)$ of states in the Markov chain.

![Diagram](image)

**Figure 11. To Lemma 7.6**

Lemma 7.6. There exists a subset $E_{2N-1} \subset P_{2N-1}$ with $\#E_{2N-1} = O(\lambda^N)$, where $\lambda$ is the Perron–Frobenius eigenvalue of the Markov chain, and the following holds for every $i \in P_{2N-1} \setminus E_{2N-1}$: there exists $\alpha \in \{1, 2, 3, 4\}, \beta \in \{-1, 0, 1, 2\}$ and paths $j = (j_0 \to \cdots \to j_{N-\beta+\alpha-1}), k = (k_0 \to \cdots \to k_{N+\beta+\alpha-1})$ with the following properties.

(i) $j_0 = k_0, j_{N-\beta+\alpha-1} = \iota(i_0), k_{N+\beta+\alpha-1} = i_{2N-1}$.

(ii) Let $\mathcal{R} = ([\mathcal{R}]_0, \ldots, [\mathcal{R}]_{2N})$ be any thickened path representing $i$. Construct representations $Q = ([Q]_0, \ldots, [Q]_{N-\beta+\alpha}), S = ([S]_0, \ldots, [S]_{N+\beta+\alpha})$ of sequences $j$ and $k$ respectively, with $[Q]_{N-\beta+\alpha} = [\mathcal{R}]_0, [S]_{N+\beta+\alpha} = [\mathcal{R}]_{2N}$. Then $[Q]_0 = [S]_0$.

Finally, the mapping $i \mapsto (j, k)$ is injective.

Remark 7.7. We slightly abuse our terminology here by applying the term “thickened path” to an arbitrary indexed set of domains generated by a sequence of states of the Markov chain, i.e. without conditions on their first and last states. This will create no problems, as all operations in the proof of this lemma will not affect states and $n_0 = \max n(v)$ domains near each of the ends of the sequences $i, j, k$.

The statement of this lemma is illustrated by Figure 11. Every state from the sequences $i, j, k$ is represented by a straight arrow, while the past and the future domains
of the state are shown as the pairs of squares near the start and the end of this arrow. Other details of this figure, including numbering for elements of \( j, k \), which is different from that in the lemma statement, are discussed below when proving equality \((27)\).

**Proof of Lemma 7.5 assuming Lemma 7.6.** The values of \( a \) and \( b \) from the statement of Lemma 7.5 are in fact \( b = \max |\beta|, a = b + \max \alpha \), where the possible values of \( \alpha \) and \( \beta \) are described in Lemma 7.6. Thus for a nonnegative function \( \varphi \in L^1(Y, \nu) \) we have

\[(24) \quad \left( WQ^{2n-a} \varphi \right)_l = (U \varphi)_{l}^{4n-2a+1} \varphi \]

\[= \sum_{i_1, \ldots, i_{4n-2a+1}} P_i(l, i_1, P_i_2 \cdots P_{i_{4n-2a}}^{i_{4n-2a+1}} T_{\omega(l)}^{-1} T_{\gamma(i_0)}^{-1} \cdots T_{\gamma(i_{4n-2a})}^{-1} \varphi_{i_{4n-2a+1}}. \]

The coefficient in a term of the last sum is nonzero if and only if the sequence \( \iota(l) \to i_1 \to \cdots \to i_{4n-2a+1} \) is admissible, and, as \((p_{ij})\) is the matrix for the Parry measure, formula \((6)\) yields

\[(25) \quad \left( WQ^{2n-a} \varphi \right)_l \leq \widetilde{C}_1 \lambda^{-4n} \sum_{i_0 = \iota(l)} T_{\omega(i_0)}^{-1} T_{\gamma(i_1)}^{-1} \cdots T_{\gamma(i_{4n-2a})}^{-1} \varphi_{i_{4n-2a+1}}. \]

Similarly,

\[\left( (Q^s)^n Q^{n+s} \varphi \right)_l = \sum_{j_{2a-1}, \ldots, j_0, k_{2n+2s}} \frac{p_{j_{2a-1}}}{p_l} p_{j_{2a-1}l} P_{j_{2a-2}j_{2a-1}} \cdots P_{j_0 j_1} P_{j_0 k_1} P_{k_1 k_2} \cdots P_{k_{2n+2s-1} k_{2n+2s}} \]

\[\times T_{\gamma(j_{2a-1})} \cdots T_{\gamma(j_0)} T_{\gamma(j_1)}^{-1} \cdots T_{\gamma(j_{k_{2n+2s-1}})}^{-1} \varphi_{k_{2n+2s}}. \]

and \((6)\) yields the estimate

\[(26) \quad \left( (Q^s)^n Q^{n+s} \varphi \right)_l \geq \widetilde{C}_2 \lambda^{-4n} \sum_{j \in \mathbf{P}_{2n}, k \in \mathbf{P}_{2n+2s}} T_{\gamma(j_{2a-1})} \cdots T_{\gamma(j_0)} T_{\gamma(k_0)}^{-1} \cdots T_{\gamma(k_{2n+2s-1})}^{-1} \varphi_{k_{2n+2s}}. \]

Here \( \widetilde{C}_2 \) is chosen in such a way that this inequality holds for any \( s \) with \( |s| \leq b \). Apply Lemma 7.6 to a sequence \( \hat{i} \) from \((25)\). There are \( O(\lambda^{2n}) \) sequences from \( E_{4n-2a+1} \), and the corresponding terms in \((24)\) comprise \( (A_n \varphi)_l \). We see that \( \|A_n\| = O(\lambda^{-2n}) \), so the series \( \sum_n \|A_n\| \) converges.

Suppose now that \( i \notin E_{4n-2a+1} \). Then Lemma 7.6 provides paths \( \tilde{i} \in \mathbf{P}_{2n-a-\beta+\alpha} \) and \( \tilde{k} \in \mathbf{P}_{2n-a+\beta+\alpha} \). Denote \( \gamma = a - \alpha + \beta \geq 0 \) and consider any admissible sequence \( \tilde{l} = (\tilde{i}_\gamma \to \cdots \to \tilde{i}_0) \). Prepend this “left tail” \( t \) to \( \tilde{i} \) and \( \tilde{k} \) to construct \( \hat{i} \) and \( \hat{k} \). Now \( \hat{j} \in \mathbf{P}_{2n}, \hat{k} \in \mathbf{P}_{2n+2\beta+\gamma} \), and one can see that the statements (i) and (ii) of Lemma 7.6 still hold for \( \hat{i} \) and \( \hat{k} \).

Let us prove that the terms in \((25)\) and \((26)\) (for \( s = \beta \)) corresponding to these \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are equal. Indeed, \( l = \iota(i_0) = j_{2n} \) and \( k_{2n+2s} = i_{4n-2a+1} \), so it remains to prove
that
\[
\omega(i_0)\gamma(i_0)^{-1}\gamma(i_1)^{-1}\cdots\gamma(i_{4n-2a})^{-1} = \gamma(j_{2n-1})\cdots\gamma(j_0)\gamma(k_0)^{-1}\cdots\gamma(k_{2n+2s-1})^{-1},
\]
where we define \(g_2, g_j, g_k\) as shown.

Consider the thickened paths \(R, Q, S\) representing \(i, j, k\) and satisfying statement (ii) in Lemma 7.6. Let \(hR\) be the right domain in \(R_1\). Then by the definitions of \(\gamma(\cdot)\) and \(\omega(\cdot)\) (see formula \((5)\)) \(h\omega(i_0)R\) is the left domain in \(R_0\), \(h\omega(i_0)\gamma(i_0)^{-1}R\) is the left domain in \(R_1\), ..., \(h\omega(i_0)g_kR\) is the left domain in \(R_{4n-2a+1}\).

On the other hand, as \(R_0 = \cup_{2n+1}\) and \(i_0 = t(j_{2n})\), we have that \(R_1 = \cup_{2n}\) and \(hR\) is the left domain in \(\cup_{2n}\). The same argument as above gives us that \(h\omega(i_0)R\) is the left domain in \(\cup_{2n+1}\), which coincides with the left domain in \(S_0\), so \(h\omega(i_0)g_kR\) is the left domain in \(S_{2n+2s}\). But as \(S_{2n+2s+1} = R_{4n-2a+2}\) and \(k_{2n+2s} = i_{4n-2a+1}\), we obtain that \(S_{2n+2s} = R_{4n-2a+1}\), and their left domains coincide. Therefore, \(h\omega(i_0)g_kR = h\omega(i_0)g_kR\), so \((27)\) holds. This is shown on Figure 11: the curved arrows link the domains \(h\omega(i_0)g_kR\), where \(g\) is an initial segment of either the left-hand or the right-hand side of \((27)\): the shaded regions correspond to \(g = id\) (left), \(g = g_j\) (top), \(g = g_k\) (right).

Therefore, we have proved that for every term in the right-hand side of \((25)\) except those with \(i \in E_{4n-2a+1}\) there exists the equal term in the right-hand side of \((26)\) for \(|s| \leq b\). Moreover, due to the last statement in Lemma 7.6 different sequences \(i\) yield different pairs \((j, k)\), so
\[
\sum_{i \in P_{4n-2a+1}, j \in \Phi \cap i \equiv i(l)} T_{g_j g_k^{i_0} g_k^{i_{4n-2a+1}}} \leq b \sum_{s = -b} T_{g_j g_k^{i_0} g_k^{i_{4n-2a+1}}} + \sum_{s = b} T_{g_j g_k^{i_0} g_k^{i_{4n-2a+1}}}.
\]
Combining this inequality with \((25)\) and \((26)\) we establish \((11)\).

We are now passing to the proof of Lemma 7.6.

**Proof of Lemma 7.6.** 1. Take any \(i \in P_{2N-1}\). We start by choosing the corresponding value of \(\beta\).

**Claim 7.8.** There exist \(\beta \in \{-1, 0, 1, 2\}\), a domain \(R^* \in [R]_{N-\beta}\), and a side \(s \subset \partial R^* \cap \partial R\). In the special case from Remark 4.3 it is also required that \(s\) is either non-compact or the angle of \(\partial R\) at one of the ends of \(s\) is less than \(\pi\).

**Proof.** In fact, in a non-special case we may choose \(\beta \in \{-1, 0, 1\}\). Assume first that \(N(R) \geq 4\). If for some \(\beta \in \{-1, 0, 1\}\) there are two domains in \([R]_{N-\beta}\), then among their \(N(\partial R')\) sides there are at most six sides common with \([R]_{N-\beta-1}\) or \([R]_{N-\beta+1}\), hence there exists a side \(s \in \partial[R]_{N-\beta} \cap \partial R\). It remains to consider the case when \([R]_{N-\beta}\) for all \(\beta\) contain one domain each. Then \([R]_N\) has two sides bordering \([R]_{N \pm 1}\), and we choose any other side as \(s\).

Now consider a non-special case with \(N(R) = 3\). Any side of \(R\) has an end on \(\partial D\), so there are no polygonal chains of three sides in \(T_R\), and hence there are no states of type
E. Therefore, the previous argument works with the only amendment: if \([R]_{N-\beta}\) contains two domains, then at most four of their six sides are common with the adjacent domains from \([R]_{N-\beta\pm 1}\), and we may choose \(s\) to be any other side.

In the special case two states of type \(E\) cannot be successive: indeed, between two \(E\)-states there are \(n(v) - 2\) states of type \(C\), and here \(n(v) \geq 3\) for any vertex \(v \in \mathbb{D}\). Therefore, if \([R]_{N-\beta}\) contains two domains, at most five of six sides are common with \([R]_{N-\beta\pm 1}\), so we choose \(s\) to be any other side. Note that two domains in \([R]_{N-\beta}\) has a common vertex \(w \in \mathbb{D}\), and \(s\) is not adjacent to \(w\), hence \(s\) is not compact. It remains to consider the case when each of \([R]_{N-\beta}\), \(\beta = -1, 0, 1, 2\) contains only one domain. Let \(\hat{s}\) be the side of \(R_{N-1}\) not common with \(R_{N-2} \cup R_N\). We can set \(s = \hat{s}\) except if \(\hat{s}\) is compact and \(\partial R\) has the straight angle in the end \(u\) of \(\hat{s}\) incident to \(R_N\). In the latter case we may assume that, say, \(\hat{s}\) belong to \(\partial_L R\). Since \(n(u) \geq 3\), \(u\) is incident to \(R_N\) and \(R_{N+1}\) and \(\partial_L R_N = \{u\}\). Then we choose \(s = \partial_R R_N\), which is non-compact. \(\square\)

2. The base for the construction of a thickened path \(Q\) is a path \(Q^{(1)}\), which can be informally defined as follows: start with the path \([|R|_0, \ldots, |R|_{N-\beta}]\), replace \(|R|_{N-\beta}\) by \(R^*\) from Claim 7.8 and then remove some domains from \([R]_{N-\beta-1}, [R]_{N-\beta-2}, \ldots\) if necessary to get a thickened path. This is not always possible, as we need to conserve the first \(n_0 + 1\) elements \((|R|_0, \ldots, |R|_{n_0})\) of this path.

![Figure 12. The construction of \(Q^{(1)}\) and \(S^{(1)}\). Black vertices in the adjacency graph represent domains in the union \(Q^{(1)} \cup S^{(1)}\), white vertices represent other domains in \(R\). The boundary of the union \(Q^{(1)} \cup S^{(1)}\) has convex or straight angles in all vertices except for the vertex \(u^*\) corresponding to the shaded cycle, but the boundaries of \(Q^{(1)}\) and \(S^{(1)}\) have at most straight angles at \(u^*\).](image)

More precisely, let \(\beta\) and \(R^*\) be given by Claim 7.8. If \(R^*\) is the only domain in \([R]_{N-\beta}\), we denote \([Q^{(1)}]_t = [R]_{N-\beta-t}, t = 0, \ldots, N - \beta\), \([S^{(1)}]_t = [R]_{N-\beta+t}, t = 0, \ldots, N + \beta\). Otherwise we may assume that \(R^*\) is the left domain in \([R]_{N-\beta}\). Consider the structure of the corresponding section in the thickened path given by Proposition 3.15 so \(R^* = T_{N-\beta}\). Let \(B_{N-\beta-p-1} \rightarrow T_{N-\beta-p}, p \geq 0\), be the last “bottom to top” crossing before \(R^*\) and
Let us show that the paths \( Q^{(1)} \) and \( S^{(1)} \) are thickened paths. Indeed, the convexity is clear for all vertices of their boundaries except for the vertices \( u_1, \ldots, u_t \) corresponding to the cycles in the adjacency graph between the crossings \( B_{N-\beta-\rho-1} \rightarrow T_{N-\beta-\rho} \) and \( T_{N-\beta+q} \rightarrow B_{N-\beta+q+1} \). Denote by \( u^* = u_t \), the vertex corresponding to the cycle in the adjacency graph containing both domains in \( [R]_{1-\beta} \). Then \( \partial Q^{(1)} \) (respectively, \( \partial S^{(1)} \)) contains the vertices \( u_j \) only for \( j \leq l^* \) (respectively, \( j \geq l^* \)).

Consider a vertex \( u_j, j \leq l^* \). Then the cycle corresponding to \( u_j \) is either a wide-bottom trapezoid, or a left-slanted parallelogram, or an incomplete cycle, that is, it contains domains from \( [R]_{1-\beta+1} \); the latter case takes place for \( j = l^* \) only. In all these cases the path \( Q^{(1)} \) contains at most half of the domains in this cycle, see Figure 12. Therefore, the angle of \( \partial Q^{(1)} \) at \( u_j \) is at most straight.

Later we will also need the following observation.

Remark 7.9. The union \( Q^{(1)} \cup S^{(1)} \) has only one concave angle on its boundary. Namely, \( \partial (Q^{(1)} \cup S^{(1)}) \) has a minimally concave angle at the above-defined vertex \( u^* \). The convexification of \( Q^{(1)} \cup S^{(1)} \) equals \( R \).

As we have to keep the first and the last \( n_0 + 1 \) domains of the path \( R \) unchanged, we require that \( p, q \leq N - n_0 - 3 \). If this inequality does not hold for \( p \) then the section given by Proposition 3.15 goes from \( [R]_{1-\beta} \) to the left beyond \( [R]_{n_0} \) and all crossings here are “top to bottom”. Therefore, the types of the states in the sequence \( i_{n_0+1} \rightarrow \cdots \rightarrow i_{N-\beta-1} \) have the following pattern:

\[
\cdots, E_L, C, \ldots, C, E_L, C, \ldots, C, E_L, C, \ldots, C, E_L, \ldots
\]

Similarly, if \( R^* \) is the right domain in \( [R]_{1-\beta} \), these states follows the same pattern with \( E_R \) in place of \( E_L \). Note that a sequence \( i_{n_0+1} \rightarrow \cdots \rightarrow i_{N-\beta-1} \) following one of these two patterns is uniquely defined by its final state \( i_{N-\beta-1} \). Therefore, there are \( O(\lambda^N) \) sequences \( i \) such that the subsequence \( i_{n_0+1} \rightarrow \cdots \rightarrow i_{N-\beta-1} \) follows any of these two patterns.

The other inequality \( q \geq N - n_0 - 3 \) implies that the sequence \( i_{N-\beta} \rightarrow \cdots \rightarrow i_{2N-n_0-2} \) follows either the pattern \( 28 \) or the same pattern with \( E_R \) in place of \( E_L \), and again there are \( O(\lambda^N) \) sequences \( i \) satisfying this inequality.

Therefore, the conditions \( p, q \leq N - n_0 - 3 \) eliminate \( O(\lambda^N) \) sequences, which comprise the set \( E_{2N-1} \), a part of the set \( E_{2N-1} \) from the lemma statement.

3. Let \( s \) be the side of \( R^* \) provided by Claim 7.8 let \( e \) be the label on \( s \) inside of \( R^* = g^* R \), and let \( v_\pm \) be the ends of \( s \) incident to \( [R]_{1-\beta+1} \). Add the “head” \( g^* H^e \) from
Proposition 4.9 to $Q^{(1)}$ and $S^{(1)}$:

$$[Q^{(1)}]_{t} = [S^{(1)}]_{t} = g^{*}H^{e}_{t}, \quad t = -\alpha, \ldots, -1.$$ 

In the special case of Remark 4.3 if $s$ is compact then $\partial R$ has a convex angle at an end $v_{*} \in \{v-, v_{+}\}$ of the side $s$. Then we choose $H^{e}$ to be either the path shown on Figure 9 or its mirror image so that the end $v^{*}$ is incident to the domains $g^{*}H^{e}_{t}$ for $t = 0, -1, -2$ while the other end of $s$ is incident to these domains for $t = 0, -1$ only.

The resulting paths $([Q^{(1)}]_{t})^{N-\beta}$ and $([S^{(1)}]_{t})^{N+\beta}$ are “partially convexified paths”, that is, their boundaries are almost convex curves, and right turns may appear only at $v-$ for the former path and at $v_{+}$ for the latter.

Indeed, consider the former path. Then any vertex on $v \in \partial([Q^{(1)}]_{t})_{t=\alpha}^{N-\beta}$ except for $v = v_{-}$ is incident to either $\partial([Q^{(1)}]_{t})_{t=0}^{N-\beta}$ or to $g^{*}H^{e}$ only, which are thickened paths, so their boundaries are convex. Let $v = v_{-}$. Then the addition of $g^{*}H^{e}$ increases the number of domains incident to $v$ by at most one, hence the boundary angle at $v$, which is at most straight for $([Q^{(1)}]_{t})^{N-\beta}_{t=0}$ is at most minimally convex for $([Q^{(1)}]_{t})^{N-\beta}_{t=\alpha}$. Similarly, in the special case if the added head is the one shown on Figure 9 and $v_{-} = v_{+}$, the number of domains incident to $v_{-}$ is increased by at most two when adding the head $g^{*}H^{e}$, the angle of $\partial([Q^{(1)}]_{t})^{N-\beta}_{t=0}$ at $v$ is convex, so the same angle for $\partial([Q^{(1)}]_{t})^{N-\beta}_{t=\alpha}$ is at most minimally concave.

Let $Q^{(2)}$, $S^{(2)}$ be the convexifications of $([Q^{(1)}]_{t})^{N-\beta}_{t=\alpha}$ and $([S^{(1)}]_{t})^{N+\beta}_{t=\alpha}$ and let $j^{(2)}$, $k^{(2)}$ be the corresponding sequences of states. Then Proposition 4.9 implies that $j_{-\alpha}^{(2)} = h(e) = k_{-\alpha}^{(2)}$.

4. Let us show that $j_{N-\beta-1}^{(2)} = i(i_{0})$, $k_{N-\beta-1}^{(2)} = i_{2N-1}$ This is true if the convexification procedure for $([Q^{(1)}]_{t})^{N+\beta}_{t=\alpha}$ and $([S^{(1)}]_{t})^{N+\beta}_{t=\alpha}$ does not change domains with $t \geq N - n_{0} - 2$.

Assume the contrary: the convexification of $([Q^{(1)}]_{t})^{N-\beta}_{t=\alpha}$ starting from the end $v_{-}$ of $s$ goes along a geodesic segment $I$ of its boundary that reaches $[R]_{n_{0}}$. Without loss of generality we suppose that $s$ lies on the left boundary of $R$, and $\partial_{L}[R]_{t} \subset I$ for $t = n_{0} + 1, \ldots, N - \beta - 1$.

First of all, we show that $[R]_{t}$ contains only one domain for $t = n_{0} + 3, \ldots, N - \beta - 3$. Indeed, assuming the contrary, by Proposition 3.15 we have $[R]_{t} = \{T_{t}, B_{t}\}$. Then $\partial_{L}[R]_{t}$ is the boundary of $T_{t}$ minus its common sides with $[R]_{t \pm 1}$. Note that if $\partial_{L}[R]_{t}$ contains two sides then the vertex between them is incident to only one domain in $R$, and the boundary angle there is convex, so $\partial_{L}[R]_{t}$ fails to belong to $I$. Therefore, $T_{t}$ has at least $N(R) - 1$ common sides with other domains from $R$. This is impossible if $N(R) \geq 6$; if $N(R) = 5$, then $(i_{t-1}, i_{t}) = (E_{R}(\ldots), E_{L}(\ldots))$, hence there are no admissible $i_{t-2}$ or $i_{t+1}$. Moreover, $R$ should be compact since any vertex of $T_{t}$ lying on $\partial D$ belongs to $\partial R$.

It remains to consider the case of the compact domain $R$ with $N(R) = 4$. Then $T_{t}$ has at least three common sides with other domains, hence $(i_{t-1}, i_{t})$ has types $BE_{L}$, $E_{1}E_{L}$, $CE_{L}$, $E_{R}C$, $E_{R}E_{L}$, $E_{R}E_{R}$, or $E_{R}D$. Let us show that $(i_{t-1}, i_{t})$ cannot have types $E_{1}E_{L}$ or $E_{R}E_{R}$. Assuming the former we have that the adjacency graph has a cycle $T_{t-1}T_{t}B_{t+1}B_{t}$, which corresponds to a vertex $u$ with $n(u) = 2$. Consider the common side of $T_{t}$ and $T_{t+1}$, and let $v$ be its left end when looking from $T_{t}$. Then $T_{t}$ has three common sides with other
domains from $\mathcal{R}$, namely, with $\mathcal{T}_{t+1}$, $\mathcal{B}_{t+1}$, $\mathcal{T}_{t}$ in the clockwise order starting from $v$. In particular, $u$ and $v$ are the opposite vertices of $\mathcal{T}_{t}$, hence $n(v) \geq 3$. On the other hand, the sides of $\mathcal{T}_{t+1}$ common with the other domains in $\mathcal{R}$ form a polygonal curve with three sides going from $v$ in the counterclockwise direction. Therefore, $v$ is incident to only two domains in $\mathcal{R}$, namely $\mathcal{T}_{t}$ and $\mathcal{T}_{t+1}$, whence the angle of $\partial \mathcal{R}$ in $v$ is convex. The case when $(t_{i-1}, t_{i})$ have types $E_{R}E_{R}$ is considered in the same way.

Therefore, we have shown that non-$A$ states can appear in the sequence $(i_{t})_{t=n_{0}+1}^{N-\beta-2}$ only as $D$ or $E_{R}D$ at the beginning of this sequence or as $B$ or $BE_{L}$ at its end.

Now consider the sequence $(i_{t})_{t=n_{0}+1}^{N-\beta-4}$, which consists of $A$-states only. If all domains $|\mathcal{R}|_{t}$ with $t = n_{0} + 3, \ldots, N - \beta - 3$ touch $I$, then all states $i_{t}$ with $t = 2n_{0} + 3, \ldots, N - \beta - n_{0} - 4$ have the form $A_{L}[i_{-}, i_{+}](e)$ with $i_{-} + i_{+} = n(v_{L}(e))$. Moreover, the sequence $(i_{t})_{t=n_{0}+3}^{N-\beta-4}$ is uniquely defined by one of its terms $i_{2n_{0}+3}$, since the only possible transitions are

$$A_{L}[i_{-}, i_{+}](e) \to A_{L}[i_{-} + 1, i_{+} - 1](l(e)), \quad i_{+} > 1,$$

$$A_{L}[i_{-}, 1](e) \to A_{L}[1, i'_{+}](l(l(e)^{-1})).$$

Here the former transition corresponds to taking the next petal around the same vertex

Therefore, the condition $\partial_{L}|\mathcal{R}|_{t} \subset I$ for $t = n_{0} + 1, \ldots, N - \beta - 1$ yields finitely many possibilities for $(i_{0} \to \cdots \to i_{N})$, or $O(\lambda^{N})$ possibilities for the entire sequence $i$. Denote by $E_{2N-1}^{(2)}$ the set of all sequences $i$ such that the convexification procedure for either $(|\mathcal{Q}^{(1)}|_{t})_{t=-\alpha}^{N-\beta}$ or $(|\mathcal{S}^{(1)}|_{t})_{t=-\alpha}^{N+\beta}$ reaches their domains with $t \geq N - n_{0} - 2$. We have proved that $E_{2N-1}^{(2)}$ contains $O(\lambda^{N})$ elements. Thus we define the set

$$E_{2N-1} = E_{2N-1}^{(1)} \cup E_{2N-1}^{(2)},$$

and this set contains $O(\lambda^{N})$ elements. The paths $\mathcal{Q} = (|\mathcal{Q}|_{t})_{t=0}^{N-\beta+\alpha}$ and $\mathcal{S} = (|\mathcal{S}|_{t})_{t=0}^{N+\beta+\alpha}$ from the lemma statement are now obtained from $\mathcal{Q}^{(2)}$ and $\mathcal{S}^{(2)}$ by a shift of numeration.

5. It remains to check that the map $i \mapsto (j, k)$ is injective. We will show how to recover $\mathcal{R}$ from $\mathcal{Q}$ and $\mathcal{S}$.

Let $i \notin E_{2N-1}$. Denote

$$\mathcal{Y} = (|\mathcal{Q}^{(1)}|_{t})_{t=0}^{N-\beta} \cup (|\mathcal{S}^{(1)}|_{t})_{t=0}^{N+\beta} \cup g^{*}H,$$

where $\mathcal{Q}^{(1)}$, $\mathcal{S}^{(1)}$, and $g^{*}$ are defined above in steps 2 and 3 of this proof. This is a $\gamma$-shaped union of fundamental domains, with the three terms of (29) joining at $\mathcal{R}^{*}$. While $\mathcal{Y}$ is not a path, its boundary is an almost convex curve: right turns may occur in only three vertices, namely, the ends $v_{\pm}$ of side $s$ (see step 1) and the vertex $u^{*}$ from Remark 7.9. By the construction, the segments of $A(\partial \mathcal{Y})$ around vertices $v_{-}, v_{+}$, and $u^{*}$ have no common points. Therefore, the convexification of each of these segments can be performed independently, and the domains added are exactly the domains that are added
when convexifying the path that is a union of the corresponding two terms in \([29]\):

\[
[\mathcal{Y}] = \mathcal{Y} \sqcup C_{v_-} \sqcup C_{v_+} \sqcup C_{u^*},
\]

where

\[
\mathcal{Q} = \left[\left(\mathcal{Q}^{(1)}\right)_{t=0}^{N-\beta} \cup g^*H^e\right] = \left(\left(\mathcal{Q}^{(1)}\right)_{t=0}^{N-\beta} \cup g^*H^e\right) \sqcup C_{v_-},
\]

\[
\mathcal{S} = \left[\left(\mathcal{S}^{(1)}\right)_{t=0}^{N+\beta} \cup g^*H^e\right] = \left(\left(\mathcal{S}^{(1)}\right)_{t=0}^{N+\beta} \cup g^*H^e\right) \sqcup C_{v_+},
\]

\[
\mathcal{R} = \left[\left(\mathcal{Q}^{(1)}\right)_{t=0}^{N-\beta} \cup \left(\mathcal{S}^{(1)}\right)_{t=0}^{N+\beta}\right] = \left(\left(\mathcal{Q}^{(1)}\right)_{t=0}^{N-\beta} \cup \left(\mathcal{S}^{(1)}\right)_{t=0}^{N+\beta}\right) \sqcup C_{u^*}.
\]

Let us perform the convexification procedure for \(\mathcal{Y}\) only for the segments of \(A(\partial \mathcal{Y})\) around \(v_\pm\). We arrive at the set \(\hat{\mathcal{Y}} = \mathcal{Q} \cup \mathcal{S}\) with at most one non-convex vertex \(u^*\). Note that the convexifications of \(\mathcal{Y}\) and \(\hat{\mathcal{Y}}\) coincide, and they are equal to \(\hat{\mathcal{Y}} \cup \mathcal{R}\) by Remark 7.9.

Therefore, \(\mathcal{R}\) can be reconstructed from \(\mathcal{Q}\) and \(\mathcal{S}\) as follows. Let \([\hat{\mathcal{Y}}]\) be the convexification of \(\hat{\mathcal{Y}} = \mathcal{Q} \cup \mathcal{S}\). Let \(J\) be a segment of \(\partial [\hat{\mathcal{Y}}]\) going from \([\mathcal{Q}]_{N-\beta+\alpha}\) to \([\mathcal{S}]_{N+\beta+\alpha}\) and not touching \(Q_0 = S_0\). Then the sequence of all domains in \([\hat{\mathcal{Y}}]\) having common points with \(J\) is a locally shortest path, and its convexification coincide with \(\mathcal{R}\) by Lemma 3.18.

Formally speaking, to use this lemma we have to assume that \([\mathcal{Q}]_{N-\beta+\alpha}\) and \([\mathcal{S}]_{N+\beta+\alpha}\) contain one domain each. If this is not true, we append “caps” with states of types \(C \ldots CD\) to \(\hat{\mathcal{Y}}\) and/or \(\mathcal{R}\). Consider \(\hat{\mathcal{Y}}\) and \(J\) for these elongations of the paths \(\mathcal{Q}\) and \(\mathcal{S}\), apply Lemma 3.18 and then remove the domains corresponding to the “caps” from the path \(\mathcal{R}\).

\[
\square
\]

7.3. Conclusion of the proof of Theorem A. Consider an ergodic decomposition of the measure \(\mu\) with respect to the action of the subgroup of \(G\) generated by \(G_0^2 = \{g_1g_2 : g_1, g_2 \in G_0\}\). Note that in general, for arbitrary \(G_0^2\)-invariant measure \(\tilde{\mu}\), the operator \(P\) does not preserve the measure \(p \times \tilde{\mu}\), but the operators \(Q, V, W\) defined by \([19]\) do, as they contain only terms of the form \(f \circ T_{g_1} \circ T_{g_2}\) for \(g_1, g_2 \in G_0\). Formula \([6]\) then yields

\[
\tilde{S}_{2n}(f) = \lambda^{2n-1} \sum_{j \in \Xi_S} h_j(Q^{n-1}V \varphi(f))_j.
\]

Note also that \(#S(2n)\) equals the number of paths from \(\Xi_S\) to \(\Xi_F\) of the length \(2n\), thus

\[
#S(2n) = \sum_{j \in \Xi_S} (\Pi^{2n-1})_{ij} = C\lambda^{2n-1}(1 + o(1)),
\]

whence

\[
S_{2n}(f) = \tilde{C} \sum_{j \in \Xi_S} h_j(Q^{n-1}V \varphi(f))_j \cdot (1 + o(1)).
\]

Now we apply Theorem 6.6 to operators \([19]\) acting on the space \(L^1(Y, \tilde{\mu})\), where \(\tilde{\mu} = \tilde{\mu} \times p\).

Recall that we have checked Assumptions 6.2-6.4 for these operators in Corollary 7.4 and Lemma 7.5. Therefore, we obtain that the following holds for \(\tilde{\mu}\)-almost every \(x\):

- \(S_{2n}(f)(x)\) converges to some limit, which we denote as \(\tilde{f}(x)\).
- \(\tilde{f}(x) = \tilde{f}(T_{g_1}g_2x)\) for any \(g_1, g_2 \in G_0\).
The second item results from the fact that \( \tilde{f} \) is constant \( \tilde{\mu} \)-almost everywhere.

Therefore, the set of \( x \in X \) such that these two conditions hold, has full measure with respect to every convex combination of the ergodic measures, in particular, with respect to the initial measure \( \mu \). Thus \( \lim_{n \to \infty} S_{2n}(f)(x) \) exists \( \mu \)-almost surely and is \( G_0^2 \)-invariant.

On the other hand, for every \( A \in I_{G_0^2} \) one has
\[
\int_A f \, d\mu = \int_A S_{2n}(f) \, d\mu \to \int_A \tilde{f} \, d\mu,
\]
whence \( \tilde{f} = E(f|I_{G_0^2}) \).

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