UNIQUE ERGODICITY FOR ZERO-ENTROPY DYNAMICAL SYSTEMS WITH APPROXIMATE PRODUCT PROPERTY

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Abstract. We show that for every topological dynamical system with approximate product property, zero topological entropy is equivalent to unique ergodicity. Hence the space of invariant measures for such a system is a Poulsen simplex if and only if the topological entropy is positive. Equivalence of minimality is also discussed. Some examples are included.

1. Introduction

It has been a historic question how the properties of a dynamical system are determined by its topological entropy. Usually we tend to regard that a system with zero topological entropy is considerably simple, while positive topological entropy often comes with a rich structure. Such facts can be shown for systems in certain classes. For example, in the seminal work [10] Katok showed that for $C^{1+\alpha}$ diffeomorphisms on a surface, positive topological entropy is equivalent to the existence of horseshoes. On the contrary, there are zero-entropy systems that are in some sense complicated, as well as positive-entropy systems that are in some sense simple. For example, the question raised by Parry whether strict ergodicity implies zero topological entropy has a negative answer in the most general setting. See Section 4 for examples with more details.

As the theory of hyperbolicity had been developed, Herman expected a positive answer to Parry’s question in the smooth case. Katok then suggested a more ambitious conjecture that every $C^2$ diffeomorphism has ergodic measures of arbitrary intermediate entropies, i.e. for each $\alpha \in [0, h(f))$, where $h(f)$ denotes the topological entropy of the system $(X, f)$, there is an ergodic measure $\mu_\alpha$ such that its metric entropy satisfies $h_{\mu_\alpha}(f) = \alpha$. Partial results on Katok’s conjecture have been obtained in [16],[17],[18] and [7] by the author and collaborators. The success of specification-like properties that played pivotal roles in [15] and [7] urges us to consider classes of systems with such topological properties, which are closely related to some sort of hyperbolicity.

Specification-like properties are weak variations of the specification property introduced by Bowen. Since his pioneering work [4], plenty of interesting results have

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been obtained through various specification-like properties. Among these properties, the approximate product property introduced by Pfister and Sullivan [14] is almost the weakest one. In [21], we have verified Katok’s conjecture for every system \((X, f)\) that satisfies approximate product property and asymptotically entropy expansiveness. In fact, we obtained a stronger result, a description of the subtle structure of \(\mathcal{M}(X, f)\), the space of invariant measures, concerning their metric entropies. In particular, such a system \((X, f)\) must have zero topological entropy if it is uniquely ergodic.

**Theorem 1.1** ([21, Theorem 1.1]). Let \((X, f)\) be an asymptotically entropy expansive system with approximate product property. Then \((X, f)\) has the generic structure of metric entropies, i.e. for every \(\alpha \in [0, h(f))\),

\[
\mathcal{M}_\alpha(X, f, \alpha) := \{ \mu \in \mathcal{M}(X, f) : \mu \text{ is ergodic and } h_\mu(f) = \alpha \}
\]

is a residual subset in the compact metric subspace \(\mathcal{M}(X, f) := \{ \mu \in \mathcal{M}(X, f) : h_\mu(f) \geq \alpha \}\).

This paper mainly goes in the opposite direction. Theorem 1.1 does not make sense if the system has zero topological entropy. So we would like to see what happens in the zero-entropy case. Suppose that \((X, f)\) has approximate product property. The entropy denseness property proved in [14] implies that ergodic measures are dense in \(\mathcal{M}(X, f)\). We shall show that \(\mathcal{M}(X, f)\) is actually a singleton if \((X, f)\) has zero topological entropy. Moreover, we show that the reverse is also true even if asymptotic entropy expansiveness is not assumed.

**Theorem 1.2.** Let \((X, f)\) be a system with approximate product property. Then \((X, f)\) is uniquely ergodic if and only if \((X, f)\) has zero topological entropy.

Theorem 1.2 is more than a positive answer to the question of Parry and Herman in the class of systems with approximate product property. Along with the results in [14] and [21], we have a dichotomy on the structure of \(\mathcal{M}(X, f)\) for a system with approximate product property, which is completely determined by its topological entropy:

\[
\begin{align*}
&h(f) = 0 \iff \mathcal{M}(X, f) \text{ is a singleton.} \\
&h(f) > 0 \iff \mathcal{M}(X, f) \text{ is a Poulsen simplex.}
\end{align*}
\]

Moreover, when \(h(f) > 0\), if in addition \((X, f)\) is asymptotically entropy expansive, then the system has the generic structure of metric entropies as described in Theorem 1.1.

We have shown in [20] that a zero-entropy system with gluing orbit property must be minimal and equicontinuous. Similar results do not hold for systems with approximate product property. There are zero-entropy systems with approximate product property that are topologically mixing, as well as such systems that are not topologically transitive. See Example 4.3. It is shown in [21] that both unique ergodicity and zero topological entropy can be derived from approximate product property and minimality. It turned out that non-minimality is caused exactly by the mistakes allowed in the tracing property (1) in the definition of approximate product property. Considering this we introduce a new notion between approximate product property and the tempered gluing orbit property introduced in [21]. Under this so-called strict approximate product property, minimality is also equivalent to zero topological entropy.
Theorem 1.3. Let $(X, f)$ be a system with strict approximate product property. Then the following are equivalent:

1. $(X, f)$ is minimal.
2. $(X, f)$ is uniquely ergodic.
3. $(X, f)$ has zero topological entropy.

Comparing Theorem 1.3 with the result in [20], we are still not sure if in the class of systems with tempered gluing orbit property or strict approximate product property, zero topological entropy implies uniformly rigidity or equicontinuity.

By examining the examples we find that some phenomena appear in a more general way, as exhibited in the following theorems.

Theorem 1.4. Let $(X, f)$ be a system with a periodic point $p \in X$. Then $(X, f)$ has approximate product property and zero topological entropy if and only if $(X, f)$ is uniquely ergodic, i.e. the periodic measure supported on the orbit of $p$ is the unique ergodic measure for $(X, f)$.

Theorem 1.5. Let $f : I \to I$ be a continuous map on a closed interval $I$. Then $(I, f)$ has approximate product property and zero topological entropy if and only if it has a unique attracting fixed point, i.e. there is $p \in I$ such that $f(p) = p$ and $
abla f^n(x) = p$ for every $x \in I$.

Theorem 1.5 provides a complete description of continuous interval maps with approximate product property and zero topological entropy. We remark that by [2] and [5], a continuous interval map has uniform gluing orbit property if and only if it is topologically mixing. We wonder if there is any other condition that describes interval maps with other specification-like properties.

Readers are referred to the book [6] and the survey [12] for an overview of the definitions and results of specification-like properties. More discussions on gluing orbit property, tempered gluing orbit property and approximate product property, as well as various examples, can be found in [3], [19] and [21]. In our terminology as in [19, 21] and in this paper, the relation between the various specification-like properties is summarized in Figure 1. Analogous relation holds for periodic specification-like properties.

\begin{center}
\begin{tikzpicture}[node distance=2.5cm, auto]
  \node (spec) {Specification};
  \node (ugob) [right of=spec] {Uniform Gluing Orbit};
  \node (go) [right of=ugob] {Gluing Orbit};
  \node (ts) [below of=spec] {Tempered Specification};
  \node (tgo) [below of=ugob] {Tempered Gluing Orbit};
  \node (sap) [below of=tgo] {Strict Approximate Product};
  \node (ap) [below of=ap] {Approximate Product};

  \draw[->] (spec) -- (ugob);
  \draw[->] (ugob) -- (go);
  \draw[->] (spec) -- (ts);
  \draw[->] (ts) -- (tgo);
  \draw[->] (tgo) -- (sap);
  \draw[->] (sap) -- (ap);
\end{tikzpicture}
\end{center}

FIGURE 1. Relation between specification-like properties
Notions and results in this paper naturally extends to the continuous-time case, i.e. flows and semi-flows.

2. Approximate Product Property

Let \((X, d)\) be a compact metric space. Let \(f : X \to X\) be a continuous map. Then \((X, f)\) is conventionally called a topological dynamical system or just a system. We shall denote by \(\mathbb{Z}^+\) the set of all positive integers and by \(\mathbb{N}\) the set of all nonnegative integers, i.e. \(\mathbb{N} = \mathbb{Z}^+ \cup \{0\}\). For \(n \in \mathbb{Z}^+\), denote
\[
G_n := \{0, 1, \ldots, n - 1\}.
\]

**Definition 2.1.** Let \(G = \{x_k\}_{k \in \mathbb{Z}^+}\) be a sequence in \(X\). Let \(S = \{m_k\}_{k \in \mathbb{Z}^+}\) and \(G = \{t_k\}_{k \in \mathbb{Z}^+}\) be sequences of positive integers. For \(\delta, \varepsilon > 0\) and \(z \in X\), we say that \((G, S, G)\) is \((\delta, \varepsilon)\)-traced by \(z\) if for each \(k \in \mathbb{Z}^+\), we have
\[
\left| \left\{ j \in \mathbb{Z}^+ : d(f^{s_k+j}(z), f^j(x_k)) > \varepsilon \right\} \right| \leq \delta m_k,
\]
where
\[
s_k = s_k(S, G) := 0 \text{ and } s_k(S, G) := \sum_{i=1}^{k-1} (m_i + t_i - 1) \text{ for } k \geq 2.
\]

**Remark.** Definition 2.1 naturally extends to the case that \(G, S, G\) are finite sequences, which allows us to define periodic specification-like properties.

**Definition 2.2.** The system \((X, f)\) is said to have approximate product property, if for every \(\delta_1, \delta_2, \varepsilon > 0\), there is \(M = M(\delta_1, \delta_2, \varepsilon) > 0\) such that for every \(n > M\) and every sequence \(G\) in \(X\), there are an sequence \(G\) with \(\max G \leq 1 + \delta_1 n\) and \(z \in X\) such that \((G, \{n\} \mathbb{Z}^+, G)\) is \((\delta_2, \varepsilon)\)-traced by \(z\).

Approximate product property was introduced by Pfister and Sullivan [14] to prove large deviations for \(\beta\)-shifts. Definition 2.2 is equivalent to the definitions given in [14] and [21]. Our proof of Theorem 1.2 is based on the following fact related to entropy denseness, which is implicitly proved in [21].

**Proposition 2.3** (cf. [21]). Suppose that \((X, f)\) has approximate product property. Then for every \(\mu \in M(X, f)\), every \(\eta, \varepsilon, \beta > 0\), there is a compact invariant subset \(\Lambda = \Lambda(\mu, \eta, \varepsilon, \beta)\) such that \(D(\mu, \nu) < \eta\) for every invariant measure \(\nu\) supported on \(\Lambda\) and \(h(\Lambda, f, \varepsilon) < \beta\).

It is the mistake \(\delta_2 > 0\) allowed in (1) that introduces non-minimal examples of zero-entropy systems with approximate product property. So we suggest a stronger condition under which minimality can be guaranteed by zero entropy and unique ergodicity.

**Definition 2.4.** The system \((X, f)\) is said to have strict approximate product property, if for every \(\delta, \varepsilon > 0\), there is \(M = M(\delta, \varepsilon) > 0\) such that for every \(n > M\) and every sequence \(G\) in \(X\), there are a sequence \(G\) with \(\max G \leq 1 + \delta n\) and \(z \in X\) such that \((G, \{n\} \mathbb{Z}^+, G)\) is \((0, \varepsilon)\)-traced by \(z\).

It is clear that strict approximate product property is stronger than approximate product property and weaker than the tempered gluing orbit property introduced in [21]. We remark that the difference between strict approximate product property and tempered gluing orbit property is not in the description of the gaps (\(\delta n\) vs...
a tempered function), but in the lengths of the orbit segments (equal lengths vs variable lengths) that can be traced.

Theorem 1.3 is a corollary of Theorem 1.2 and the following proposition.

**Proposition 2.5.** Let \((X, f)\) be a system of strict approximate product property. If \((X, f)\) is uniquely ergodic then it is minimal.

The proof of Proposition 2.5 is similar to the proof of [19, Theorem 4.1]. It is given below for completeness.

Proof. Assume that \((X, f)\) is not minimal and it has strict gluing orbit property. We shall show that \((X, f)\) is not uniquely ergodic.

As the system is not minimal, there are \(x, x' \in X\) and \(\gamma > 0\) such that \(d(f^n(x), x') \geq \gamma\) for every \(n \in \mathbb{N}\).

We fix \(\varepsilon \in (0, \gamma/3)\). Take a continuous function \(\varphi : X \to \mathbb{R}\) such that

- \(\varphi(y) = 1\) for every \(y \in B(x', \varepsilon)\);
- \(\varphi(y) = 0\) for every \(y \notin B(x', 2\varepsilon)\);
- \(0 < \varphi(y) < 1\) otherwise.

Then we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = 0. \quad (3)
\]

Now we fix \(\delta := 1\) and consider the sequence \(C = \{x'\} \mathbb{Z}^+\). As \((X, f)\) has strict approximate product property, there are \(m \in \mathbb{Z}^+, \text{a sequence } G = \{t_k\}_{k \in \mathbb{Z}^+}\) with \(\max G \leq 1 + \delta m\) and \(z \in X\) such that \((C, \{m\} \mathbb{Z}^+, G)\) is \((0, \varepsilon)\)-traced by \(z\). Let \(s_k = s_k((m) \mathbb{Z}^+, G)\) for each \(k\) as in (2). Then we must have

\[f^{s_k}(z) \in B(x', \varepsilon)\] for each \(k\) and \(s_k \leq (k-1)(m+1+\delta m-1) \leq 2(k-1)m\).

This yields that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z)) \geq \limsup_{k \to \infty} \frac{1}{s_k} \sum_{j=1}^{k-1} \varphi(f^{t_j}(z)) = \limsup_{k \to \infty} \frac{k-1}{s_k} \geq \frac{1}{2m} > 0. \quad (4)
\]

Equations (3) and (4) imply that \(\varphi\) does not converge pointwise to a constant. Hence \((X, f)\) is not uniquely ergodic. \(\square\)

### 3. Proof of Theorem 1.2

#### 3.1. Sufficiency.

In this section we prove the ‘if’ part of Theorem 1.2. Suppose \((X, f)\) be a system with approximate product property that is not uniquely ergodic. We shall show that \((X, f)\) has positive topological entropy.

Let \(\mu_1, \mu_2\) be two distinct ergodic measures for \((X, f)\). Let

\[
\mu_3 := \frac{2\mu_1 + \mu_2}{3}, \quad \mu_4 := \frac{\mu_1 + 2\mu_2}{3} \quad \text{and} \quad \eta := \frac{D(\mu_1, \mu_2)}{7}.
\]

Then

\[
D(\mu_i, \mu_j) > 2\eta \quad \text{for } 1 \leq i < j \leq 4. \quad (5)
\]

By Proposition 2.3, there are compact invariant sets \(\Lambda_1, \Lambda_2, \Lambda_3\) and \(\Lambda_4\) such that for each \(i = 1, 2, 3, 4\) and every invariant measure \(\nu\) supported on \(\Lambda_i\), we have

\[
D(\nu, \mu_i) < \eta.
\]
This implies that
\[ \Lambda_i \cap \Lambda_j = \emptyset \text{ for } 1 \leq i < j \leq 4. \]
Otherwise, if \( \Lambda_i \cap \Lambda_j \neq \emptyset \) then \( \Lambda_i \cap \Lambda_j \) supports an invariant measure \( \nu^* \) and hence
\[ D(\mu_i, \mu_j) \leq D(\mu_i, \nu^*) + D(\mu_j, \nu^*) < 2\eta, \]
which contradicts with (5).

By compactness, we have
\[ d^*(\Lambda_i, \Lambda_j) := \min \{ d(x, y) : x \in \Lambda_i, y \in \Lambda_j \} > 0 \text{ for } 1 \leq i < j \leq 4. \]
Denote
\[ \gamma := \frac{1}{4} \min \{ d^*(\Lambda_i, \Lambda_j) : 1 \leq i < j \leq 4 \}. \]
For each \( i = 1, 2, 3, 4 \), we fix a point \( y_i \in \Lambda_i \). Then for \( 1 \leq i < j \leq 4 \) and every \( m, n \in \mathbb{Z}^+ \), we have
\[ d(f^m(y_i), f^n(y_j)) \geq 4\gamma. \quad (6) \]
For each \( \xi = \{\xi(k)\}_{k=1}^\infty \in \{1, 2\}^{\mathbb{Z}^+} \), denote \( \mathcal{G}_\xi := \{x_k(\xi)\}_{k=1}^\infty \) such that
\[ x_{2k-1}(\xi) = y_{2k-1} \text{ and } x_{2k}(\xi) = y_{2k}. \quad (7) \]
We fix
\[ \delta \in (0, \frac{1}{10}) \text{ and } m \geq M(\delta, \delta, \gamma). \]
For each \( \xi \in \{1, 2\}^{\mathbb{Z}^+} \), we can find \( \mathcal{G}_\xi = \{t_k(\xi)\}_{k=1}^\infty \) with \( \max \mathcal{G} \leq 1 + \delta m \) and \( z_\xi \in X \) such that \( (\mathcal{G}_\xi, \{m\}_{\mathbb{Z}^+}, \mathcal{G}_\xi) \) is \((\delta, \gamma)\)-traced by \( z_\xi \). Denote \( s_k(\xi) := s_k(\{m\}_{\mathbb{Z}^+}, \mathcal{G}_\xi) \) as in (2) for each \( k \).

**Lemma 3.1.** If there is \( n \in \{1, \cdots, N\} \) such that \( \xi(n) \neq \xi'(n) \), then \( z_\xi \) and \( z_{\xi'} \) are \((1 + \delta)nm, \gamma\)-separated.

**Proof.** The proof splits into two cases.

**Case 1.** Suppose that
\[ r := |s_n(\xi) - s_n(\xi')| \leq 4\delta m. \]
We may assume that \( s_n(\xi) > s_n(\xi') \). By the tracing property, there are \( A, A' \in \mathbb{Z}_m \) such that
\[ |A|, |A'| \geq (1 - \delta)m, \]
\[ d(f^{s_n(\xi)+j}(z_\xi), f^j(x_n(\xi))) \leq \gamma \text{ for each } j \in A \]
and
\[ d(f^{s_n(\xi')+j}(z_{\xi'}), f^j(x_n(\xi'))) \leq \gamma \text{ for each } j \in A'. \]
Then we must have
\[ |(r + A) \cap A'| \geq |r + A| + |A'| - |r + \mathbb{Z}_m| \geq (1 - 6\delta)m > 0. \]
For \( l \in (r + A) \cap A \), by (6), we have
\[ d(f^{s_n(\xi')+l}(z_{\xi'}), f^{s_n(\xi)+l}(z_\xi)) \]
\[ \geq d(f^{l-r}(x_n(\xi)), f^l(x_n(\xi'))) - d(f^{s_n(\xi)+l-r}(z_\xi)), f^{l-r}(x_n(\xi))) \]
\[ - d(f^{s_n(\xi')+l}(z_{\xi'}), f^l(x_n(\xi'))) \]
\[ \geq 4\gamma - \gamma - \gamma > \gamma. \]
Moreover, we have
\[ s_n(\xi') + l \leq \sum_{k=1}^{n-1} (m + t_k(\xi') - 1) + m \leq (1 + \delta)nm. \]

So \( z_\xi \) and \( z_{\xi'} \) are \(((1 + \delta)nm, \gamma)\)-separated.

**Case 2.** Suppose that \( \tau \) is the smallest positive integer such that
\[ \tau \leq n \] and \[ |s_\tau(\xi) - s_\tau(\xi')| > 4\delta m. \]

Then we have \( \tau > 1 \) and
\[ |s_{\tau-1}(\xi) - s_{\tau-1}(\xi')| \leq 4\delta m. \]

We may assume that \( s_\tau(\xi) > s_\tau(\xi') \). Then we have
\begin{align*}
s_\tau(\xi) - s_\tau(\xi') &= s_{\tau-1}(\xi) + (m + t_{\tau-1}(\xi) - 1) - s_{\tau-1}(\xi') - (m + t_{\tau-1}(\xi') - 1) \\
&\leq s_{\tau-1}(\xi) - s_{\tau-1}(\xi') + (1 + \delta m) - 1 \\
&\leq 5\delta m. \tag{8}
\end{align*}

This yields that
\[ r := s_{\tau+1}(\xi') - s_\tau(\xi) = s_{\tau}(\xi') + (m + t_{\tau}(\xi') - 1) - s_{\tau}(\xi) \leq (1 - 3\delta)m \]

and by (8) we have
\[ r \geq m - (s_\tau(\xi) - s_\tau(\xi')) \geq (1 - 5\delta)m > 0. \]

By the tracing property, there are \( A, A' \in \mathbb{Z}_m \) such that
\[ |A|, |A'| \geq (1 - \delta)m, \]
\[ d\left(f_{s_\tau(j)}(z_\xi), f^j(x_\tau(\xi))\right) \leq \gamma \text{ for each } j \in A \]
and
\[ d\left(f_{s_{\tau+1}(\xi') + j}(z_{\xi'}), f^j(x_{\tau+1}(\xi'))\right) \leq \gamma \text{ for each } j \in A'. \]

Then we must have
\[ |(r + A) \cap A'| \geq |r + A| + |A'| - |r + \mathbb{Z}_m| \geq \delta m > 0. \]

Note that by our construction (7), we must have \( x_\tau(\xi) \neq x_{\tau+1}(\xi') \). For \( l \in (r + A) \cap A' \), by (6), we have
\begin{align*}
d\left(f^l(x_\tau(\xi)), f^{l-r}(x_{\tau+1}(\xi'))\right) \\
&\geq d\left(f^l(x_\tau(\xi)), f^{l-r}(x_{\tau+1}(\xi'))\right) - d\left(f_{s_\tau(j)}(z_\xi), f^j(x_\tau(\xi))\right) \\
&\quad - d\left(f_{s_{\tau+1}(\xi') + (l-r)}(z_{\xi'}), f^{l-r}(x_{\tau+1}(\xi'))\right) \\
&\geq 4\gamma - \gamma - \gamma > \gamma.
\end{align*}

Moreover, we have
\[ s_\tau(\xi) + l \leq \sum_{k=1}^{\tau-1} (m + t_k(\xi') - 1) + m \leq (1 + \delta)nm. \]

So \( z_\xi \) and \( z_{\xi'} \) are \(((1 + \delta)nm, \gamma)\)-separated. \( \square \)
By Lemma 3.1, for each \( n \), there is a \((1 + \delta)nm, \gamma\)-separated set whose cardinality is \( 2^n \). This yields that

\[
h(f) \geq \limsup_{n \to \infty} \frac{\ln 2^n}{(1 + \delta)nm} = \frac{\ln 2}{(1 + \delta)m} > 0.
\]

3.2. Necessity. In this section we prove the ‘only if’ part of Theorem 1.2. Suppose that \((X, f)\) is a system with approximate product property and positive topological entropy \( h(f) > 0 \). We shall show that such a system have more than one ergodic measures. Compared with the results in [21], in what follows we do not assume that the system is asymptotically entropy expansive.

By Proposition 2.3, for each \( k \), there is a compact invariant set \( \Lambda_k \) such that

\[
h(\Lambda_k, f, \frac{1}{k}) := \limsup_{n \to \infty} \frac{\ln s(\Lambda_k, n, \frac{1}{k})}{n} < \frac{1}{k}, \tag{9}
\]

where \( s(\Lambda_k, n, \frac{1}{k}) \) denotes the maximal cardinality of \((n, \frac{1}{k})\)-separated subsets of \( \Lambda_k \). Denote

\[
\Gamma_k := \bigcap_{j=1}^{k} \Lambda_j \text{ for each } k.
\]

Then for each \( k \), \( \Gamma_k \) is also compact and \( f \)-invariant.

There are two cases to consider:

1. Suppose that there is \( k \) such that \( \Gamma_k \cap \Lambda_{k+1} = \emptyset \). Then there are two distinct ergodic measures that are supported on \( \Gamma_k \) and \( \Lambda_{k+1} \), respectively. Then \( (X, f) \) is not uniquely ergodic.

2. Suppose that \( \Gamma_k \cap \Lambda_{k+1} \neq \emptyset \) for all \( k \). In this case we have a nonempty invariant compact set

\[
\Gamma := \bigcap_{k=1}^{\infty} \Gamma_k = \bigcap_{k=1}^{\infty} \Lambda_k.
\]

By (9), we have

\[
h(\Gamma, f, \frac{1}{k}) \leq h(\Lambda_k, f, \frac{1}{k}) < \frac{1}{k} \text{ for all } k.
\]

This implies that

\[
h(\Gamma, f) = \lim_{k \to \infty} h(\Gamma, f, \frac{1}{k}) = 0.
\]

and hence \( \Gamma \) supports an ergodic measure of zero entropy. However, as \( h(f) > 0 \), the system \((X, f)\) must have ergodic measures of positive entropy. This completes the proof.

4. Examples

Example 4.1. According to [13], there is a class of zero-entropy \( C^\infty \) interval maps such that each map in the class has periodic points of period \( 2^n \) for any \( n \in \mathbb{Z}^+ \) and is chaotic in the sense of Li-Yorke. Theorem 1.2 implies that these maps do not have approximate product property.
Example 4.2. In [8], a minimal subshift is constructed as the first example of strictly ergodic system with positive topological entropy, which gives a negative answer to Parry’s question. In [1], it is shown that strictly ergodic homeomorphisms with positive topological entropies can be constructed on every compact manifold whose dimension is at least 2 and that carries a strictly ergodic homeomorphism. By Theorem 1.2, these systems do not have approximate product property.

Example 4.3. By [11, Theorem 7.1] and [21, Example 9.6], there is $X_1 \subset \{0,1\}^\mathbb{N}$ such that

$$\frac{1}{n} \max\{|\{m \leq k < m+n : w_k = 1\}| : m \in \mathbb{N}\}$$

converges uniformly to zero for every $\{w_k\}_{k \in \mathbb{N}} \subset X_1$. Consider the subshift $\sigma$ on $X_1$. Then for every subshift $\sigma' \subset X_1$, every $\delta_2, \varepsilon > 0$, there is $M \in \mathbb{Z}^+$ such that for every $n > M$, $(\sigma', \{n\}^\mathbb{Z}_+, \{1\}^\mathbb{Z}_+)$ is $(\delta_2, \varepsilon)$-traced by the fixed point $\{0\}^\mathbb{N}$. The subshift $(X_1, \sigma)$ has approximate product property and zero topological entropy. It is topologically mixing but not minimal. By Theorem 1.3, this subshift does not have strict approximate product property.

Such subshifts can be modified to obtain a non-transitive system with approximate product property. See [21, Example 9.7].

Example 4.4. In [9], Herman constructed a family of $C^\infty$ diffeomorphisms $F = \{F_\alpha : \alpha \in \mathbb{T}^1\}$ on $X = \mathbb{T}^1 \times \text{SL}(2, \mathbb{R})/\Gamma$, where $\mathbb{T}^1 = [0, 1]/\sim$ is the unit circle and $\Gamma$ is a cocompact discrete subgroup of $\text{SL}(2, \mathbb{R})$. For each $\alpha \in \mathbb{T}^1$,

$$F_\alpha(\theta, g\Gamma) = (R_\alpha(\theta), A_\theta(g\Gamma))$$

is a skew product, where $R_\alpha(\theta) = \theta + \alpha$ is the rotation on $\mathbb{T}^1$,

$$A_\theta(g\Gamma) = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} g\Gamma$$

for each $g\Gamma \in \text{SL}(2, \mathbb{R})/\Gamma$ and $\lambda > 1$ is a fixed real number. Herman showed in [9] that $h(F_\alpha) > 0$ for every $\alpha \in \mathbb{T}^1$ and there is dense $G_\delta$ subset $W \subset \mathbb{T}^1$ such that $F_\alpha$ is minimal for every $\alpha \in W$.

By Theorem 1.2, approximate product property does not hold for $F_\alpha$ as long as $\alpha \in W$. We doubt if it holds for any element $F_\alpha$ in the family $F$.

5. Periodic Points and Unique Ergodicity

5.1. Systems with Fixed Points. We perceive that Example 4.3 reflects a more general phenomenon, which may be regarded as a flaw that appears naturally with the way we define approximate product property, where we allow mistakes in the tracing property (1).

Proposition 5.1. Let $(X, f)$ be a system with a fixed point $p \in X$. Then $(X, f)$ has approximate product property and zero topological entropy if and only if $(X, f)$ is uniquely ergodic, i.e. The Dirac measure on $\{p\}$ is the unique ergodic measure for $(X, f)$.

Proof. The 'only if' part is a corollary of Theorem 1.2. Now we assume that $\delta_p$ is the unique ergodic measure for $(X, f)$. By the variational principle, we must have $h(f) = h_{\delta_p}(f) = 0$. We need to show that $(X, f)$ has approximate product property. In fact, every sequence can be traced by the orbit of the fixed point $p$. This is analogous to the situation in Example 4.3.
Let $\varepsilon > 0$. There is a continuous function $\varphi : X \to \mathbb{R}$ such that

$$
\varphi(x) = 0 \text{ for } x = p;
$$

$$
\varphi(x) = 1 \text{ for every } x \notin B(p, \varepsilon);
$$

$$
0 < \varphi(x) < 1 \text{ otherwise.}
$$

As $(X, f)$ is uniquely ergodic, we have that

$$
\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \to \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(p)) = 0 \text{ uniformly.}
$$

Then for every $\delta_2 > 0$, there is $M \in \mathbb{Z}^+$ such that

$$
\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) < \delta_2 \text{ for every } n > M \text{ and every } x \in X.
$$

This implies that

$$
\left| \left\{ j \in \mathbb{Z}_n : d(f^j(x), p) > \varepsilon \right\} \right| < \delta_2 n.
$$

Then for every sequence $\mathcal{C}$ in $X$, we have that $(\mathcal{C}, \{n\}^{\mathbb{Z}^+}, \{1\}^{\mathbb{Z}^+})$ can be $(\delta_2, \varepsilon)$-traced by the fixed point $p$. Hence $(X, f)$ has approximate product property.

Theorem 1.4 can be verified as a corollary of Proposition 5.1 and the following fact, which holds generally for systems with approximate product property.

**Proposition 5.2.** Suppose that there is a positive integer $N$ such that $(X, f^N)$ is a system with approximate product property. Then so is $(X, f)$.

**Proof.** Suppose that we are given $\delta_1, \delta_2, \varepsilon > 0$. By continuity, there is $\gamma > 0$ such that

$$
d(f^j(x), f^j(y)) < \varepsilon \text{ for every } j \in \mathbb{Z}_N, \text{ whenever } d(x, y) < \gamma.
$$

As $(X, f^N)$ has approximate product property, there is $M = M(\frac{\delta_1}{2}, \frac{\delta_2}{2}, \gamma)$ as in Definition 2.2 such that for every $n > M$ and every sequence $\mathcal{C}$ in $X$, there are an sequence $\mathcal{G} = \{t_k\}_{k \in \mathbb{Z}^+}$ with $\max \mathcal{G} \leq 1 + \frac{\delta_1 n}{2}$ and $z \in X$ such that $(\mathcal{C}, \{n\}^{\mathbb{Z}^+}, \mathcal{G})$ is $(\frac{\delta_1}{2}, \varepsilon, \gamma)$-traced by $z$ under $f^N$. By (10), this implies that $(\mathcal{C}, \{nN\}^{\mathbb{Z}^+}, \{1 + N(t_k - 1)\}_{k \in \mathbb{Z}^+})$ is $(\frac{\delta_1}{2}, \varepsilon)$-traced by $z$ under $f$.

Let

$$
T > \max \left\{ M, 1 + \frac{2}{\delta_1}, 2 \right\}.
$$

For every $n > TN$, we can write $n = mN + l$ such that $m \geq T$ and $0 \leq l < N$. Suppose that we have $\mathcal{G} = \{t_k\}_{k \in \mathbb{Z}^+}$ with $\max \mathcal{G} \leq 1 + \frac{\delta_1 (m+1)}{2}$ and $(\mathcal{C}, \{mN\}^{\mathbb{Z}^+}, \{1 + N(t_k - 1)\}_{k \in \mathbb{Z}^+})$ is $(\frac{\delta_1}{2}, \varepsilon)$-traced by $z$ under $f$. By (11), we have

$$
1 + N(t_k - 1) + (N - l) \leq 1 + \frac{\delta_1 (m+1)N}{2} + N < 1 + \delta_1 n
$$

for each $k$ and

$$
\frac{\delta_2}{2} (m+1)N < \delta_2 mN \leq \delta_2 n.
$$

These bounds guarantees that the gap $\mathcal{G}' = \{1 + N(t_k - 1) + (N - l)\}_{k \in \mathbb{Z}^+}$ satisfies $\max \mathcal{G}' \leq 1 + \delta_1 n$ and $(\mathcal{C}, \{n\}^{\mathbb{Z}^+}, \mathcal{G}')$ is $(\delta_2, \varepsilon)$-traced by $z$ under $f$.

Hence $(X, f)$ also has approximate product property. \qed
5.2. Interval Maps. Let $I$ be a closed interval. We know that every continuous interval map $f : I \to I$ must have a fixed point and hence is not minimal. By Theorem 1.3, there is no continuous interval map that has strict approximate product property and zero topological entropy.

**Definition 5.3.** The system $(X, f)$ is said to have periodic uniform gluing orbit property, if for every $\varepsilon > 0$, there is a nonnegative integer $M = M(\varepsilon)$ such that for every finite sequence $\mathcal{C} = \{x_k\}_{k=1}^n$ in $X$ and every finite sequence $\mathcal{S} = \{m_k\}_{k=1}^n$ of positive integers, there is $z \in X$ such that $(\mathcal{C}, \mathcal{S}, \{M + 1\}^n)$ is $(0, \varepsilon)$-traced by $z$ and

$$f^s(z) = z \text{ for } s := \sum_{i=1}^n (m_i + M).$$

The system $(X, f)$ is said to have uniform gluing orbit property if the tracing point $z$ in Definition 5.3 is not required to a periodic point. It is clear that periodic uniform gluing orbit property implies uniform gluing orbit property, whose relation with other specification properties is illustrated in Figure 1. It is shown in [2] and [5] that a continuous interval map $(I, f)$ has periodic uniform gluing orbit property if and only if the map is topologically mixing, which provides a class of interval maps with approximate product property. It is an interesting question to ask if there are any other conditions that are related to other specification-like properties.

Now we would like to prove Theorem 1.5. Suppose that $f : I \to I$ is a continuous map on the interval $I$. Then $f$ must have a fixed point in $I$. Hence the ‘if’ part of Theorem 1.5 is a corollary of Proposition 5.1.

Suppose that $(I, f)$ has approximate product property and zero topological entropy. By Theorem 1.4, $(I, f)$ has a unique ergodic measure which is supported on a unique fixed point $p \in I$ and $(I, f)$ has no other periodic points. The following fact completes the proof of Theorem 1.5.

**Proposition 5.4.** Let $p$ be a fixed point of a continuous interval map $(I, f)$. Suppose that $(I, f)$ has no other periodic points. Then $p$ is attracting.

We remark that in general, as exhibited in Example 4.3, the unique fixed point of a uniquely ergodic system is not necessarily attracting. We shall give a proof of Proposition 5.4 for completeness. It is well-known that a continuous map $f : \mathbb{R} \to \mathbb{R}$ has a fixed point in an interval $J$ if $f(J) \supset J$. We shall use this fact and the Darboux property that $f([a, b]) = [f(a), f(b)]$ constantly without reference.

Suppose that the assumptions of Proposition 5.4 holds. Let $x \in I$ and $x \neq p$. To prove Proposition 5.4 we need to show that $f^n(x) \to p$ as $n \to \infty$.

**Lemma 5.5.** If $x < p$ then $f^n(x) > x$ for every $n \in \mathbb{Z}^+$. If $x > p$ then $f^n(x) < x$ for every $n \in \mathbb{Z}^+$.

**Proof.** We give the proof for the case that $x < p$. The proof of the other case is analogous. As $x$ is not a periodic point, we have $f^n(x) \neq x$ for all $n$. Suppose that there is $n \in \mathbb{Z}^+$ such that $f^n(x) < x$. We shall show that this leads to the existence of another periodic point of $f$.

There are two cases to consider:

1. Suppose that there is $m \in \mathbb{Z}^+$ such that $f^{mn}(x) < x < f^{(m+1)n}(x)$.
   If $f^{mn}(x) < f^n(x)$, then we have
   $$f^{mn}([f^n(x), x]) \supset [f^{mn}(x), f^{(m+1)n}(x)] \supset [f^n(x), x],$$
hence there is an $mn$-periodic point in $[f^n(x), x]$ that is different from $p$. Otherwise it holds that $f^n(x) \leq f^{mn}(x)$. In this case we have
\begin{equation*}
f^n([f^{mn}(x), x]) \supset [f^n(x), f^{(m+1)n}(x)] \supset [f^{mn}(x), x],
\end{equation*}
hence there is an $n$-periodic point in $[f^{mn}(x), x]$ that is different from $p$.

(2) Suppose that $f^{mn}(x) < x$ for every $m \in \mathbb{Z}^+$. Let
\begin{equation*}
a := \inf\{f^{mn}(x) : m \in \mathbb{Z}^+\} \in I.
\end{equation*}

Then we must have $f^n(a) \geq a$. But $f^n(x) < x$. By the Intermediate Value Theorem, there is a fixed point of $f$ in the interval $[a, x]$ that is different from $p$.

\[
\text{Proof of Proposition 5.4.} \text{ Let } x \in I \text{ and } x \neq p. \text{ If there is } f^N(x) = p \text{ for some } N \text{ then } f^n(x) = p \text{ for all } n > N.
\]

Suppose that $f^n(x) \neq p$ for all $n$. There is a decomposition $\mathbb{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ such that
\begin{equation*}
\begin{cases}
f^n(x) < p, & \text{for } n \in \mathcal{N}_1; \\
f^n(x) > p, & \text{for } n \in \mathcal{N}_2.
\end{cases}
\end{equation*}

The proof splits into two cases:

(1) Suppose that $\mathcal{N}_2$ is finite. Then there is $N$ such that $n \in \mathcal{N}_1$ for all $n \geq N$. By Lemma 5.5, \{f^n(x)\}_n=N^\infty \text{ is an increasing sequence with an upper bound } p. \text{ Hence it converges to some } q \in I. \text{ Then }
\begin{equation*}
f(q) = f(\lim_{n \to \infty} f^n(x)) = \lim_{n \to \infty} f^{n+1}(x) = q.
\end{equation*}
As $p$ is the unique fixed point, we must have $q = p$ and hence $f^n(x) \to p$ as $n \to \infty$. Analogous argument works for case that $\mathcal{N}_2$ is finite.

(2) Suppose that both $\mathcal{N}_1$ and $\mathcal{N}_2$ are infinite. Arrange the elements of $\mathcal{N}_1$ in a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ and the elements of $\mathcal{N}_2$ in a strictly increasing sequence $\{n'_k\}_{k=1}^\infty$. By Lemma 5.5, \{f^{n_k}(x)\}_{k=1}^\infty \text{ is an increasing sequence with an upper bound } p \text{ and } \{f^{n'_k}(x)\}_{k=1}^\infty \text{ is a decreasing sequence with a lower bound } p. \text{ Then there are } q_1, q_2 \in I \text{ such that}
\begin{equation*}
\lim_{n_k \to \infty} f^{n_k}(x) = q_1 \text{ and } \lim_{n'_k \to \infty} f^{n'_k}(x) = q_2.
\end{equation*}

We see that $q_1, q_2$ are exactly all the subsequential limits of the sequence \{f^n(x)\}_n=1^\infty. \text{ So we must have } f(\{q_1, q_2\}) \subset \{q_1, q_2\}. \text{ There are three subcases:}

(a) If $q_1 = q_2 = p$ then we have $f^n(x) \to p$ as $n \to \infty$.

(b) If $q_1 \neq p$ and $q_2 \neq p$. As $p$ is the unique fixed point, we have $f(q_1) \neq q_1$ and $f(q_2) \neq q_2$. This yields that $f(q_1) = q_2$ and $f(q_2) = q_1$. Then $q_1, q_2$ are 2-periodic points, which is a contraction.

(c) Without loss of generality, we may assume that $q_1 \neq p$ and $q_2 = p$. Then $f(q_1) = q_2 = p$. As $f$ is continuous, there is $\delta > 0$ such that
\begin{equation}
|f(y) - p| < |q_1 - p| \text{ for every } y \in B(p, \delta).
\end{equation}
As $f^{n_k}(x) \to p$, there is $N \in \mathcal{N}_2$ such that $p < f^N(x) < p + \delta$. By (12), we have $|f^{N+1}(x) - p| < |q_1 - p|$. But $f^{n_k}(x) \leq q_1 < p$ for every
This implies that $N + 1 \in \mathcal{N}_2$ and hence by Lemma 5.5 we have
\[ p < f^{N+1}(x) < f^N(x) < p + \delta. \]
Analogous argument shows that
\[ p < f^{N+2}(x) < f^{N+1}(x) < p + \delta. \]
Hence by induction we have $n \in \mathcal{N}_2$ for every $n \geq N$, which contradicts with the assumption that $\mathcal{N}_1$ is infinite.

\[ \Box \]

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References

[1] F. Béguin, S. Crovisier, F. Le Roux, Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy-Rees technique. Ann. Sci. École Norm. Sup. (4) 40 (2007), 251–308.
[2] A.M. Blokh, Decomposition of dynamical systems on an interval, Russ. Math. Surv. 38 (1983), 133–134.
[3] T. Bomfim, M. J. Torres and P. Varandas, Topological features of flows with the reparametrized gluing orbit property. Journal of Differential Equations 2017, 262(8), 4292–4313.
[4] R. Bowen, Periodic points and measures for Axiom A diffeomorphisms. Trans. Amer. Math. Soc. 154 (1971), 377–397.
[5] J. Buzzi, Specification on the interval, Trans. Amer. Math. Soc. 349 (7), 1997, 2737–2754.
[6] M. Denker, C. Grillenberger and K. Sigmund, Ergodic theory on compact spaces., Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976.
[7] L. Guan, P. Sun and W. Wu, Measures of Intermediate Entropies and Homogeneous Dynamics, Nonlinearity, 30 (2017), 3349–3361.
[8] F. Hahn and Y. Katznelson, On the entropy of uniquely ergodic transformations, Trans. Amer. Math. Soc. 126 (1967), 335–360.
[9] M. Herman, Construction d’un difféomorphisme minimal d’entropie topologique non nulle, Ergodic Theory Dynam. Systems 1 (1981), 65–76.
[10] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publ. Math. I.H.E.S. 51 (1980), 137–173.
[11] D. Kwietniak, Topological entropy and distributional chaos in hereditary shifts with applications to spacing shifts and beta shifts. Discrete and Continuous Dynamical Systems - A, 2013, 33(6), 2451–2467.
[12] D. Kwietniak, M. Lacka and P. Oprocha. A panorama of specification-like properties and their consequences, Contemporary Mathematics, 669 (2016), 155–186.
[13] M. Misiurewicz and J. Smítal, Smooth chaotic maps with zero topological entropy. Ergodic Theory and Dynamical Systems, 1988, 8(3), 421–424.
[14] C-E. Pfister and W.G. Sullivan, Large deviations estimates for dynamical systems without the specification property. Application to the $\beta$-shifts, Nonlinearity, 18 (2005), 237–261.
[15] A. Quas, and T. Soo, Ergodic universality of some topological dynamical systems, Transactions of the American Mathematical Society, 2016, 368(6), 4137–4170.
[16] P. Sun, Zero-entropy invariant measures for skew product diffeomorphisms, Ergodic Theory and Dynamical Systems, 30 (2010), 923–930.
[17] P. Sun, Measures of intermediate entropies for skew product diffeomorphisms, Discrete Contin. Dyn. Syst - A, 2010, 27(3), 1219–1231.
[18] P. Sun, Density of metric entropies for linear toral automorphisms, Dynamical Systems, 2012, 27(2), 197–204.
[19] P. Sun, *Minimality and gluing orbit property*. Discrete and Continuous Dynamical Systems - A, 2019, 39(7), 4041-4056.

[20] P. Sun, *Zero-entropy dynamical systems with gluing orbit property*. preprint, 2019.

[21] P. Sun, *Ergodic measures of intermediate entropies for dynamical systems with approximate product property*. preprint, 2019.

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