Adaptive Gradient Methods Converge Faster with Over-Parameterization (and you can do a line-search)

Abstract

As adaptive gradient methods are typically used for training over-parameterized models capable of exactly fitting the data, we study their convergence in this interpolation setting. Under this assumption, we prove that constant step-size, zero-momentum variants of Adam and AMSGrad can converge to the minimizer at the $O(1/T)$ rate for smooth, convex functions. When this assumption is only approximately satisfied, we show that these methods converge to a neighbourhood of the solution. On the other hand, we show that AdaGrad is robust to the violation of interpolation and converges to the minimizer at the optimal rate. However, we demonstrate that even for simple, convex problems satisfying interpolation, the empirical performance of these methods heavily depends on the step-size and requires tuning. We alleviate this problem by making use of stochastic line-search methods (SLS) and Polyak’s step-sizes (SPS) to help these methods adapt to the function’s local smoothness. We prove that adaptive methods used in conjunction with these techniques do not require knowledge of problem-dependent constants and retain the convergence guarantees of their constant step-size counterparts. Experimentally, we show that using SLS or SPS consistently improves the convergence of adaptive methods across tasks, from binary classification with kernel mappings to classification with deep neural networks. Furthermore, our empirical results show that AdaGrad equipped with SLS generalizes better than SGD.

1 Introduction

Adaptive gradient methods such as AdaGrad [9], RMSProp [38], AdaDelta [45], Adam [14], and AMSGrad [34] are popular optimization methods for training deep neural networks [10]. These methods use past stochastic gradients to adapt their update. More importantly, they scale well to large problems and exhibit good performance across diverse problems, making them the default choice for many machine learning applications. Theoretically, these methods are usually studied in the non-smooth online convex optimization setting [9, 34] with recent extensions to the strongly-convex [29, 41, 44] and non-convex settings [6, 8, 18, 37, 42, 43, 48]. An online–batch reduction gives guarantees similar to stochastic gradient descent (SGD) in the offline setting [3, 12, 17].

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However, there are several discrepancies between the theory and the application of these methods. Although the theory advocates for using decreasing step-sizes for Adam and its variants [14, 34], a constant step-size is the default in practice [32]. Moreover, Reddi et al. [34] showed a counterexample where Adam fails to converge on a convex function and proposed AMSGrad as a potential solution. However, AMSGrad does not perform significantly better in training deep neural networks, and Adam remains the most popular optimizer for these models [28]. On the other hand, AdaGrad [9] has been shown to be “universal” as it attains the best known convergence rate in both the stochastic smooth and non-smooth settings [17], but its empirical performance is rather disappointing when training large models [14]. Improving the empirical performance was indeed the main motivation behind Adam and other adaptive gradient methods [38, 45] that followed AdaGrad.

Another inconsistency is that although the typical theoretical results are for non-smooth functions, these methods are also extensively used in the easier, smooth setting. More importantly, adaptive gradient methods are generally used to train highly expressive, large over-parameterized models [19, 46] capable of interpolating the data. However, the original theoretical analyses do not take advantage of these additional properties. On the other hand, there has been recent literature [4, 13, 20, 22, 25, 36, 39, 40, 43] focusing on SGD in this interpolation setting. Under this additional assumption, these works show SGD with a constant step-size converges faster for both convex and non-convex smooth functions. In this work, we consider adaptive gradient methods without heavy-ball momentum and analyze their convergence for smooth, convex functions in the interpolation setting. We focus on AdaGrad, AMSGrad and RMSProp (Adam without momentum). For conciseness, henceforth, we refer to these methods as their momentum-free variants.

### 1.1 Background and contributions

In Section 3, we study the convergence of these methods minimizing smooth and convex functions under the interpolation settings. Levy et al. [17] prove that AdaGrad with any constant step-size adapts to the smoothness and gradient noise, resulting in an $O(1/\sqrt{T})$ rate for smooth and convex functions, where $\zeta^2$ is a global bound on the variance in the stochastic gradients [17]. In Section 3.1, we show that constant step-size AdaGrad also adapts to interpolation and prove an $O(1/T + \sigma\sqrt{T})$ rate, where $\sigma$ is the extent to which interpolation is violated. In the over-parameterized setting, $\sigma^2$ can be much smaller than $\zeta^2$ [47], implying a faster convergence. Under interpolation, $\sigma^2 = 0$, leading to a $O(1/T)$ rate, while $\zeta^2$ can still be large. Transferred to the smooth online optimization setting, our result implies that the regret of AdaGrad improves from $O(\sqrt{T})$ to $O(1)$ when interpolation is satisfied and retains its $O(\sqrt{T})$-regret guarantee in the general setting.

Assuming their corresponding preconditioners remain bounded, we show that both AMSGrad and Adam with a constant step-size also converge at the rate $O(1/T)$ under interpolation (Sections 3.2 and 3.3). However, unlike AdaGrad, they require specific step-sizes that depend on the problem’s smoothness. In the general setting, these methods converge to a neighbourhood of the solution, attaining an $O(1/T + \sigma^2)$ rate, which matches the rate of SGD in the same setting [36, 39]. The above result provides some justification for the faster ($O(1/T)$ vs. $O(1/\sqrt{T})$) convergence of the default (constant step-size) Adam when training neural networks. Although AdaGrad achieves the optimal convergence rate (up to constants) for any reasonable step-size, it is unclear how to choose this step-size without manually trying different values. Experimentally, in Section 5, we show that even for simple convex problems, the step-size has a big impact on the empirical performance of AdaGrad. Similarly, both Adam and AMSGrad are sensitive to their step-size, converging only for a specific range in both theory and practice.

To overcome this limitation, we use recent methods [22, 39] that automatically set the step-size for SGD. These works use stochastic variants of the classical Armijo line-search [31] or the Polyak step-size [33] in the interpolation setting. We combine these techniques with adaptive gradient methods and show that a stochastic line-search (SLS) technique enables AdaGrad to adapt to the smoothness of the underlying function, resulting in faster empirical convergence, while retaining its favourable convergence properties (Section 3.1). Similarly, SLS enables AMSGrad to achieve the convergence of its constant step-size variant, but without knowledge of the underlying smoothness properties (Section 3.2). We obtain similar convergence guarantees for Adam and, more generally, methods with bounded preconditioners such as sub-sampled Newton or stochastic L-BFGS [27]. For these methods, we propose to use a variant of the stochastic Polyak step-size (SPS) [22]. While SPS requires additional knowledge of the optimal function values, those are zero for common
loss functions in the interpolation setting, making SPS a practical choice. Furthermore, under the bounded eigenvalue assumption, we show that adaptive gradient methods coupled with SPS also achieve optimal rates when minimizing smooth, strongly convex and non-smooth functions under the interpolation assumption (Section 4).

Finally, in Section 5, we demonstrate that the proposed techniques for setting the step-size improve the empirical performance of adaptive gradient methods. These improvements are consistent across tasks, ranging from binary classification with a kernel mapping to multi-class classification using standard deep neural network architectures. We benchmark our results against SGD variants with stochastic line-search [40], Polyak step-sizes [22], tuned Adam with and without momentum and its recently proposed modifications variants [21, 24]. Our experimental results indicate that AdaGrad equipped with an Armijo line-search leads to better test performance than Adam or SGD.

### 2 Problem setup

We consider the unconstrained minimization of an objective \( f : \mathbb{R}^d \to \mathbb{R} \) with a finite-sum structure, \( f(w) = \frac{1}{n} \sum_{i=1}^{n} f_i(w) \). We assume \( f \) and each \( f_i \) are convex, differentiable and lower-bounded by \( f^* \) and \( f_i^* \), respectively. In supervised learning, \( n \) represents the number of training examples, and \( f_i \) is the loss function on training example \( i \). Although we focus on the finite-sum setting, our results can be generalized to the online optimization settings. Depending on the choice of model and loss functions, \( f \) can have different smoothness and convexity [30] properties. We include the formal definition of these properties in Appendix A.

The interpolation assumption implies that the gradient of each \( f_i \) in the finite-sum converges to zero at the optimum. If the overall objective \( f \) is minimized at \( w^* \), \( \nabla f(w^*) = 0 \), then for all \( f_i \), we have \( \nabla f_i(w^*) = 0 \). The interpolation condition can be exactly satisfied for many over-parameterized machine learning models such as nonparametric kernel regression without regularization [1, 19] and over-parameterized deep neural networks [46]. We measure the extent to which interpolation is satisfied by the disagreement between the minimum overall function value \( f(w^*) \) and the minimum value of each individual functions \( f_i^* \), \( \sigma^2 := \mathbb{E}_i [f(w^*) - f_i^*] < \infty \) [22]. Interpolation is satisfied if \( \sigma^2 = 0 \), and we also study the setting when it is not being exactly satisfied, with \( \sigma^2 > 0 \).

We view adaptive gradient methods as preconditioned gradient descent. For a general preconditioner \( A_k \), the update in iteration \( k \) can be expressed as: \( w_{k+1} = w_k - \eta_k A_k^{-1} \nabla f_{i_k}(w_k) \). Here, \( \nabla f_{i_k}(w_k) \) is the stochastic gradient of a randomly chosen function \( f_{i_k} \), and \( \eta_k \) is the step-size. We focus on the convergence of AdaGrad, RMSProp, AMSGrad and Adam without momentum, with their corresponding preconditioner listed in Table 1.

Table 1: Adaptive preconditioners, initialized at \( G_0 = 0 \) with \( \beta \in [0, 1) \). In practice, a small \( \epsilon I \) is added to \( A_k \) to ensure \( A_k \succ 0 \). *: We use the PyTorch implementation which includes bias correction.

| Optimizer       | \( G_k \)                                      | \( \nabla_k := \nabla f_{i_k}(w_k) \) | \( A_k \)                                      |
|-----------------|------------------------------------------------|-------------------------------------|------------------------------------------------|
| AdaGrad [9]     | \( G_{k-1} + \nabla_k \nabla_k^T \)          |                                     | \( G_k^{1/2} \)                               |
| RMSProp [38]    | \( \beta G_{k-1} + (1 - \beta) \text{diag}(\nabla_k \nabla_k^T) \) |                                     | \( G_k^{1/2} \)                               |
| Adam [14]       | \( \beta \text{diag}(\nabla_k \nabla_k^T)/(1 - \beta^k) \) |                                     | \( G_k^{1/2} \)                               |
| AMSGrad* [34]   | \( \beta \text{diag}(\nabla_k \nabla_k^T)/(1 - \beta^k) \) |                                     | \( \max\{A_{k-1}, G_k^{1/2}\} \)              |

Our theory holds for both the full matrix and diagonal versions of those methods, but we only use the latter in experiments for scalability. The diagonal versions perform a per-dimension scaling of the gradient and avoid computing the full matrix inverse, so their per-iteration cost is the same as SGD, although with an additional \( \mathcal{O}(d) \) memory. Both RMSProp and Adam keep an exponential moving average of the past stochastic gradients, but as Reddi et al. [34] pointed out, unlike AdaGrad, these preconditioners do not guarantee that \( A_{k+1} \succeq A_k \) and the per-dimension steps do not go to zero. This can lead to large fluctuations in the effective step-sizes and can cause Adam to diverge. To mitigate this problem, they proposed AMSGrad, which ensures \( A_{k+1} \succeq A_k \) and the convergence of Adam.

In the subsequent sections, we consider three types of methods: (i) AdaGrad (ii) AMSGrad and other preconditioned methods that ensure the monotonicity constraint \( A_{k+1} \succeq A_k \) and (iii) Adam, RMSProp and general preconditioned methods that do not satisfy any relation between successive preconditioners. For (ii) and (iii), we assume that the corresponding preconditioners are well-behaved in
the sense that their eigenvalues are bounded in an interval \([a_{\text{min}}, a_{\text{max}}]\). For diagonal preconditioners, this is easy to verify, and it is also inexpensive to maintain the desired range by projection.

3 Smooth and convex setting

In this section, we assume that each function \(f_i\) in the finite-sum is \(L_i\)-smooth, implying that \(f\) is \(L_{\max}\)-smooth, where \(L_{\max} = \max_i L_i\). We also make the standard assumption that the iterates remain bounded in a ball of radius \(D\) around the global minimizer, \(\|w_k - w^*\| \leq D\) for all \(w_k\) [9, 17]. We analyze the convergence of AdaGrad in Section 3.1, and study AMSGrad and more general preconditioners in Sections 3.2 and 3.3. Our results in this setting can be summarized as follows:

Table 2: Results for smooth, convex functions.

| Preconditioner | Step size | Adapt. to smoothness | Rate | Reference |
|----------------|-----------|----------------------|------|-----------|
| AdaGrad        | Constant  | ×                    | \(O(\sqrt{T}/\sqrt{T})\) | Theorem 1 |
|                | Conservative Lipschitz LS | ✓ | \(O(\sqrt{T}/\sqrt{T})\) | Theorem 2 |
| Non-decreasing \(A_k\) | Constant | × | \(O(\sqrt{T}/\sqrt{T})\) | Theorem 3 |
| (AMSGrad)      | Armijo LS | ✓ | \(O(\sqrt{T}/\sqrt{T})\) | Theorem 4 |
| Bounded \(A_k\) | Constant | × | \(O(\sqrt{T}/\sqrt{T})\) | Theorem 5 |
| (Adam, RMSProp) | Armijo SPS | ✓ | \(O(\sqrt{T}/\sqrt{T})\) | Theorem 6 |

3.1 AdaGrad

For smooth objectives, Levy [17] showed that AdaGrad converges at a rate \(O(\sqrt{T}/\sqrt{T})\), where \(\sigma^2 = \sup_{w} \mathbb{E}_i[\|\nabla f_i(w) - \nabla f_i(w)\|^2]\) is a uniform bound on the variance of the stochastic gradients. In the over-parameterized setting, we show that AdaGrad achieves the optimal \(O(1/T)\) rate when interpolation is exactly satisfied and a slower (though optimal) rate if interpolation is only approximately satisfied. The proofs for the following two theorems are in Appendix C.

**Theorem 1** (Constant step-size AdaGrad). *Assuming (i) convexity and (ii) \(L_{\max}\)-smoothness of each \(f_i\), and (iii) bounded iterates, AdaGrad with a constant step-size \(\eta\) and uniform averaging such that \(\bar{w}_T = \frac{1}{T} \sum_{k=1}^{T} w_k\), converges at a rate

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*')] \leq \frac{\alpha}{T} + \frac{\sqrt{\alpha} \sigma}{\sqrt{T}}, \quad \text{where} \quad \alpha = \frac{1}{2} \left( \frac{D^2}{\eta} + 2\eta \right)^2 dL_{\max}.
\]

The above theorem shows that AdaGrad is robust to the violation of interpolation and converges to the minimizer at the desired rate for any step-size. Although this is a favourable property, the best constant step-size depends on the problem, and it is difficult to choose in practice, as we verify experimentally in Section 5. Moreover, since AdaGrad uses a cumulative sum of the stochastic gradients, the effective step-size is decreasing. This is essential to ensure convergence in the general stochastic setting but results in poor empirical performance in the interpolation setting.

To overcome these limitations, we propose to use a *conservative Lipschitz line-search* that sets the step-size on the fly, improving the performance of AdaGrad while retaining its favourable convergence guarantees. At each iteration, the line-search selects the largest step-size \(\eta_k\) that satisfies the property

\[
f_{i_k}(w_k - c \eta_k \nabla f_{i_k}(w_k)) \leq f_{i_k}(w_k) - \eta_k \left\| \nabla f_{i_k}(w_k) \right\|^2, \quad \text{and} \quad \eta_k \leq \eta_{k-1}.
\]

Here, \(c\) is a hyper-parameter determined theoretically and typically set to \(1/2\) in our results. The “conservative” part of the line-search is the non-increasing constraint on the step-sizes, which is essential for convergence to the minimizer when interpolation is violated. Except for the conservative constraint, it is the same line-search used by Vaswani et al. [40]. We call the method *Lipschitz line-search* as it is only used to estimate the local smoothness constant; contrary to the classical Armijo line-search for preconditioned gradient descent, the line-search in Eq. (1) is in the gradient direction, even though the update is in the preconditioned direction \(A_k^{-1} \nabla f_{i_k}(w_k)\). The step-size found by the line-search is in the range \([2(1-c)/L_{\max}, \eta_{k-1}]\) [40], which allows us to prove the following.

**Theorem 2.** *Under the same assumptions as Theorem 1, AdaGrad with a conservative Lipschitz line-search with \(c = 1/2\) and uniform averaging converges at a rate

\[
\mathbb{E}[f(\bar{w}_T) - f^*] \leq \frac{\alpha}{T} + \frac{\sqrt{\alpha} \sigma}{\sqrt{T}}, \quad \text{where} \quad \alpha = \frac{1}{2} \left( \frac{D^2}{\eta_{\max}} \max \left\{ \frac{1}{\eta_{\max}}, L_{\max} \right\} + 2\eta_{\max} \right)^2 dL_{\max}.
\]
In Section 5, we show that the Lipschitz line-search can improve the empirical convergence of AdaGrad. Moreover, if interpolation is exactly satisfied, we can obtain an $O(1/T)$ convergence without the conservative step-sizes constraint (Appendix C.2).

Inspired by the above result, we consider a simple modification to improve the convergence of SGD with SLS [40]. We normalize the step-size (similar to AdaGrad-norm [42]) and use the conservative Lipschitz line-search: $w_{k+1} = w_k - \eta \nabla f_i(w_k)$ where $G_k = \sum_{j=0}^{k} \lVert \nabla f_j(w_j) \lVert^2$ and $\eta$ is set using the line-search in Eq. (1). Using the result in Theorem 2 for $A_k = G_k^{1/2} I_d$, we conclude that these modifications enable SGD with SLS to converge to the minimizer in contrast to its $\sigma^2$-neighborhood, without an additional memory or computational overhead.

### 3.2 AMSGrad and non-decreasing preconditioners with bounded eigenvalues

In this section, we consider AMSGrad and, more generally, non-decreasing preconditioners such that $A_k \succeq A_{k-1}$ with eigenvalues bounded in $[a_{\min}, a_{\max}]$. The following theorem shows the convergence with a constant step-size. The proofs for Theorems 3 and 4 can be found in Appendix D.

**Theorem 3.** Under the assumptions of Theorem 1 and assuming (iv) non-decreasing preconditioners (v) bounded eigenvalues in the $[a_{\min}, a_{\max}]$ interval, AMSGrad with no momentum, constant step-size $\eta = \frac{a_{\max}}{2t_{\max}}$ and uniform averaging converges at a rate,

$$
\mathbb{E}[f(\tilde{w}_T) - f(w^*)] \leq \frac{2D^2d a_{\max} L_{\max}}{a_{\min} T} + \sigma^2.
$$

When $\sigma = 0$, we obtain a $O(1/T)$ convergence to the minimizer. However, when interpolation is only approximately satisfied, we obtain convergence to a neighbourhood depending on $\sigma^2$. The original analysis of AMSGrad [34], without interpolation, uses a decreasing step-size and converges at a rate $O(1/\sqrt{T})$ in both the smooth and non-smooth convex settings. This distinction between the convergence of constant step-size Adam (or AMSGrad) vs. AdaGrad has also been recently discussed in the non-convex setting [8].

The constant step-size required for the above convergence rate depends on $L_{\max}$, which is typically unknown. Furthermore, using a global bound on $L_{\max}$ usually results in slower convergence since the local Lipschitz constant can vary considerably during the optimization. To overcome these issues, we use a stochastic variant of the Armijo line-search. Unlike the Lipschitz line-search whose sole purpose is to estimate the Lipschitz constant, the Armijo line-search selects a good step-size in the preconditioned gradient direction, and results in better empirical performance as we show in Section 5. However, when interpolation is violated, it only converges to a neighbourhood of the solution. The Armijo line-search returns the largest step-size $\eta_k$ satisfying the following conditions at iteration $k$,

$$
f_{ik}(w_k - \eta_k A_k^{-1} \nabla f_{ik}(w_k)) \leq f_{ik}(w_k) - c \eta_k \lVert \nabla f_{ik}(w_k) \lVert_{A_k^{-1}}^2, \quad \text{and} \quad \eta_k \leq \eta_{\max}.
$$

The step-size is artificially upper-bounded by $\eta_{\max}$ (typically chosen to be a large value). The Armijo line-search guarantees descent on the current function and that $\eta_k$ is contained in the $[2a_{\min}(1-c)/L_{\max}, \eta_{\max}]$ range. In the next theorem, we show that AMSGrad with the Armijo line-search retains the convergence of constant step-size without the need to know the Lipschitz constant.

**Theorem 4.** Under the same assumption as Theorem 3, AMSGrad with no momentum, Armijo line-search with $c = 3/4$ and uniform averaging converges at a rate,

$$
\mathbb{E}[f(\tilde{w}_T) - f(w^*)] \leq \left( \frac{3D^2d \cdot a_{\max}}{2T} + 3\eta_{\max} \sigma^2 \right) \max \left\{ \frac{1}{\eta_{\max}}, \frac{2L_{\max}}{a_{\min}} \right\}.
$$

Comparing this rate with that of using constant step-size (Theorem 3), we observe that the Armijo line-search results in a worse constant in the convergence rate and a larger neighbourhood, but as we show in Section 5 the Armijo line-search results in larger step-sizes and drastically improves the empirical performance. We show that a similar bound also holds for AdaGrad (see Theorem 7 in Appendix C). However, using the Armijo line-search only results in convergence to a neighbourhood of the minimizer in the absence of interpolation. Moreover, the above bound depends on $a_{\min}$ which can be $O(\epsilon)$ in the worst-case, resulting in an unsatisfactory worst-case rate of $O(1/\epsilon T)$ even in the
interpolation setting. However, like AMSGrad, AdaGrad with Armijo line-search has excellent empirical performance, implying the need for a different theoretical assumption in the future.

The Lipschitz and Armijo line-searches in previous results can be replaced by corresponding variants of the stochastic Polyak step-size (SPS) proposed by Loizou et al. [22], which set the step-size as

\[ \text{SPS: } \eta_k = \min \left\{ \frac{f_{i_k}(w_k) - f_{i_k}^*}{c \| \nabla f_{i_k}(w_k) \|^2}, \eta_{\text{max}} \right\}, \quad \text{Armijo SPS: } \eta_k = \min \left\{ \frac{f_{i_k}(w_k) - f_{i_k}^*}{c \| \nabla f_{i_k}(w_k) \|^2 A_{i_k}^{-1}}, \eta_{\text{max}} \right\}. \]

Here, \( f_{i_k}^* \) is the minimum value for the function \( f_{i_k} \). The advantage of SPS over a line-search is that it does not require a potentially expensive back-tracking procedure to set the step-size. Moreover, it can be shown that this step size is always larger than the one returned by line-search, which can lead to faster convergence. However, SPS requires knowledge of \( f_{i_k}^* \) for each function in the finite-sum. This value is difficult to obtain for general functions but is readily available in the interpolation setting for many machine learning applications. Common loss functions are often lower-bounded by zero, and the interpolation setting ensures that these lower-bounds are tight. Consequently, using SPS with \( f_{i_k}^* = 0 \) has been shown to yield good performance for over-parameterized problems [2, 22]. It is easy to show that SPS results in similar rates as SLS for both AdaGrad and AMSGrad by using the same proof techniques. In the next section, we will use Armijo SPS to obtain convergence rates for general preconditioners with bounded eigenvalues.

### 3.3 General preconditioners with bounded eigenvalues

For general adaptive first-order optimizers such as Adam, RMSProp and second-order methods like subsampled Newton or L-BFGS, the eigenvalues of the corresponding preconditioner are not guaranteed to monotonically increase across iterations. In the following theorem, we show that as long as the eigenvalues remain bounded, using a constant step-size results in an \( O\left(\frac{1}{T} + \sigma^2\right) \) rate. The proofs for Theorems 5 and 6 can be found in Appendix E.

**Theorem 5.** Under the same assumptions as Theorem 1 and assuming (iv) bounded eigenvalues in the \([a_{\text{min}}, a_{\text{max}}]\) interval, Adam with no momentum, constant step-size \( \eta = a_{\text{min}}^{2/2} L_{\text{max}} a_{\text{max}} \) and uniform averaging converges at a rate,

\[
E[f(\bar{w}_T) - f(w^*)] \leq \frac{2\kappa^2 L_{\text{max}} \| w_0 - w^* \|^2}{T} + \sigma^2, \quad \text{where } \kappa = \frac{a_{\text{max}}}{a_{\text{min}}}.
\]

The above results show that constant step-size Adam can converge to a minimizer in the interpolation setting. This result does not contradict the counterexample of Reddi et al. [34], showing the non-convergence of Adam, since the counterexample does not satisfy interpolation. However, it gives a bound on the neighbourhood in the general stochastic setting and shows that under interpolation, Adam without momentum indeed converges to the minimizer. The constant step-size used in the above theorem depends on the eigenvalue bounds of the preconditioner and on the smoothness constant. We can use the Armijo SPS to prove a similar bound without knowledge of the smoothness as follows.

**Theorem 6.** Under the same assumptions as Theorem 5, Adam with no momentum, Armijo SPS with \( c = \kappa = a_{\text{max}}/a_{\text{min}} \) and uniform averaging converges at a rate,

\[
E[f(\bar{w}_T) - f(w^*)] \leq \frac{\| w_0 - w^* \|^2}{T} \max \left\{ \frac{a_{\text{max}}}{\eta_{\text{max}}}, 2\kappa^2 L_{\text{max}} \right\} + \max \left\{ 1, \frac{4\eta_{\text{max}} L_{\text{max}} \kappa}{a_{\text{min}}} \right\} \sigma^2.
\]

This result suggests that we can obtain an \( O\left(\frac{1}{T}\right) \) convergence to the neighbourhood without knowledge of \( L_{\text{max}} \). A limitation is that the algorithm still requires knowledge of \( \kappa \) to set the step-size. Since the preconditioners do not satisfy a monotonicity property like AMSGrad, such a dependence seems inevitable in the worst case. However, for the diagonal preconditioners we consider, it is easy to compute \( \kappa \), which is simply the largest diagonal divided by the smallest. We remark that it is not possible to prove convergence using Armijo SLS with \( c = \kappa > 1 \) since it requires \( c \leq 1 \).

### 4 Additional convergence results

In this section, we provide additional results beyond the smooth, convex setting considered above. We focus on methods with general preconditioners with bounded preconditioners with Armijo SPS.
Meng et al. [27] showed that, in the interpolation setting, it is possible to obtain linear convergence for methods with bounded preconditioners using a constant step-size. In Theorem 8, we prove that in this same setting, bounded preconditioners with Armijo SPS achieve the linear rate without knowledge of the smoothness or strong-convexity parameters. Recently, Xie et al. [44] showed that a scalar version of AdaGrad achieves linear convergence under a condition similar to interpolation. In contrast, we prove such a rate for all bounded preconditioners, although our step-size requires knowing $f_0$ and $\kappa$.

In Appendix G, we analyze the convergence of adaptive methods in the non-smooth setting, assuming bounded stochastic gradients. For the adaptive methods with bounded preconditioners, we show that under the interpolation setting, Armijo SPS enables these methods to achieve an $O(1/\sqrt{T})$ convergence rate, without knowledge of the bound on the stochastic gradients. This matches the result of AdaGrad [9] and that of zero-momentum AMSGrad with a decreasing step-size [34], and under the bounded eigenvalue assumption, this analysis also applies to Adam. Furthermore, in Appendix G.1, we prove that in the non-smooth but strongly-convex interpolation setting, Armijo SPS enables adaptive methods to achieve an improved $O(1/T)$ rate. We contrast our result with those for SAdam and SC-RMSProp in [29, 41], where the authors prove an $O(\log(T)/T)$ rate in the stochastic setting without interpolation. However, to prove such a result, their algorithm uses a different preconditioner (using $G_k$ instead of $\hat{G}_k$), an $O(1/k)$-decreasing step-size schedule, knowledge of $\mu$ and a bound on the stochastic gradients. In contrast, by exploiting interpolation, the same algorithm as the convex case achieves the desired rate without knowledge of any problem-dependent constant. In both these settings, if interpolation is not satisfied, we prove convergence to a $\sigma^2$-neighbourhood.

The above results indicate that under the interpolation assumption, adaptive gradient methods with bounded preconditioners and equipped with the Armijo SPS are “universal” in that they achieve the optimal rates in all the convex/strongly-convex and smooth/non-smooth settings.

## 5 Experimental evaluation

![Figure 1: Synthetic experiments showing the impact of step-size on the performance of AdaGrad, Adam with varying step-sizes, including the default in PyTorch, and the SLS/SPS variants.](image)

### Synthetic experiment:
We first present an experiment to show that AdaGrad and Adam with constant step-size are not adaptive even for simple, convex problems. We use the PyTorch implementations [32] on a binary classification task with logistic regression. Following the protocol of Meng et al. [27], we generate a linearly-separable dataset with $n = 10^5$ examples (ensuring interpolation is satisfied) and $d = 20$ features with varying margins. For AdaGrad and Adam, we show the training loss for a grid of step-sizes in the $[10^0, 10^{-3}]$ range. We also plot their default (in PyTorch) variants, which is step-size of $10^{-3}$ with momentum. We compare against AdaGrad with the Armijo SLS (with $c = 1/2$) and Lipschitz line-search (with $c = 3/4$). For Adam without momentum, we use the Armijo SPS with $c = 1/2$. Although the theory suggests $c = \kappa$, we have found this to be overly conservative in practice. In Fig. 1, we observe a large variance across step-sizes and the poor performance of the default step-size. The best performing variant of AdaGrad/Adam has a step-size of order $10^2$. The SLS/SPS variants of these methods have good performance across margins, often out-performing the best-performing constant step-size variant. In Appendix I, we show similar trends for constant step-size Adam without momentum and variants of AMSGrad on the same problem.

### Real experiments:
Following the protocol of Loizou et al. [22], Vaswani et al. [40], we consider training standard neural network architectures for multi-class classification on CIFAR-10, CIFAR-100 and variants of the ImageNet datasets. In Appendix I, we also consider binary classification with
RBF kernels for datasets from LIBSVM [5] and study the effect of over-parameterization for deep matrix factorization [22, 35, 40]. For each of these experiments, we compare against Adam with the best constant step-size found by grid-search.\(^2\) We also include recent improved variants of Adam; RAdam [21] and AdaBound [24]. To demonstrate the advantage of preconditioning, we compare against SGD with SLS [39] and SPS [22]. Throughout our experiments, we find that SGD with SLS is more stable and has better test performance than SPS, and therefore we relegate SPS to Appendix I. For the proposed methods, we consider the combinations with theoretical guarantees in the convex setting, specifically AdaGrad and AMSGrad with the Armijo SLS and Adam (with no momentum) with Armijo SPS. For AdaGrad, we only show Armijo SLS since it consistently outperforms the Lipschitz line-search. We describe the methods’ implementation details in Appendix H. We show a subset of results for CIFAR-10, CIFAR-100 and Tiny ImageNet and defer the rest to Appendix I.

From Fig. 2 we make the following observations, (i) in terms of generalization, AdaGrad with SLS consistently performs best, while both AMSGrad and SGD with SLS match its training performance, but have worse generalization. (ii) the AdaGrad and AMSGrad variants not only converge faster than Adam and Radam but also with considerably better test performance. AdaBound has comparable convergence to the proposed methods but does not generalize as well. (iii) while Adam with SPS matches the training performance of AdaGrad and AMSGrad and achieves significantly lower training loss than Adam or Radam, it nevertheless suffers from poor generalization. In Appendix I, we plot the running time for the SLS/SPS variants and verify that the performance gains justify the increase in wall-clock time per iteration. The same trends hold across different datasets, deep models, deep matrix factorization, and binary classification using kernels.

Our results indicate that simply setting the correct step-size on the fly can lead to substantial empirical gains, often more than those obtained by designing a new algorithm. By disentangling the effect of the step-size from the preconditioner, AdaGrad has good empirical performance, contradicting common knowledge. Moreover, our techniques are orthogonal to designing better preconditioners and can be used with other adaptive gradient or even second-order methods. We show an example of using the diagonal Hessian preconditioner [15] with Armijo SPS on convex problems in Appendix J.

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\(^2\)With grid-search, Adam consistently had the same or better performance than AdaGrad and AMSGrad.
6 Discussion

When training over-parameterized models in the interpolation setting, we showed that typical constant step-size variants of adaptive gradient methods are guaranteed to converge at faster rates. We proposed to use SLS/SPS to help these methods adapt to the function’s local smoothness, alleviating the need to tune their step-size. Experimentally, we showed that using SLS/SPS results in consistent empirical improvements across tasks. In the future, we plan to develop similar techniques to automatically tune the heavy-ball momentum and improve the convergence of accelerated methods.

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Supplementary material

Adaptive Gradient Methods Converge Faster
with Over-Parameterization
(and you can do a line-search)

Organization of the Appendix

A Setup and assumptions
B Line-search and Polyak step-sizes
C Proofs for AdaGrad

| Step size                          | Rate            | Reference |
|-----------------------------------|-----------------|-----------|
| Constant                          | $O(\frac{1}{T} + \frac{\sigma}{\sqrt{T}})$ | Theorem 1 |
| Conservative Lipschitz LS         | $O(\frac{1}{T} + \frac{\sigma}{\sqrt{T}})$ | Theorem 2 |
| Non-conservative LS (with interpolation) | $O(\frac{1}{T})$ | Theorem 7 |

D Proofs for AMSGrad and non-decreasing preconditioners

| Step size | Rate            | Reference |
|-----------|-----------------|-----------|
| Constant  | $O(\frac{1}{T} + \sigma^2)$ | Theorem 3 |
| Armijo LS | $O(\frac{1}{T} + \sigma^2)$ | Theorem 4 |

E Proofs for Adam and general preconditioners

| Step size | Rate            | Reference |
|-----------|-----------------|-----------|
| Constant  | $O(\frac{1}{T} + \sigma^2)$ | Theorem 5 |
| Armijo SPS| $O(\frac{1}{T} + \sigma^2)$ | Theorem 6 |

F Proofs for general preconditioners in the strongly-convex setting

| Step size | Rate            | Reference |
|-----------|-----------------|-----------|
| Armijo SPS| $O(\rho^T + \sigma^2)$ | Theorem 8 |

G Proofs for general preconditioners in the non-smooth setting

| Step size                          | Rate            | Reference |
|-----------------------------------|-----------------|-----------|
| Armijo SPS                         | $O(\frac{1}{\sqrt{T}})$ | Theorem 9 |
| Armijo SPS (Strongly-convex)       | $O(\frac{1}{T})$ | Theorem 10 |

H Experimental details
I Additional experimental results
J Experiments for second order methods
A Setup and assumptions

We now restate the main assumptions required for our theoretical results and restate our main notation in Table 3. We assume our objective $f : \mathbb{R}^d \to \mathbb{R}$ has a finite-sum structure,

$$f(w) = \frac{1}{n} \sum_{i=1}^{n} f_i(w), \quad (3)$$

and analyze a preconditioned stochastic gradient descent step, with $i_k$ selected uniformly at random,

$$w_{k+1} = w_k - \eta_k A_k^{-1} \nabla f_{i_k}(w_k), \quad \text{(Update rule)}$$

where $\eta_k$ is either a pre-specified constant or selected on the fly. We consider AdaGrad, AMSGrad and Adam preconditioners. For AdaGrad and AMSGrad, we will use the fact that the preconditioners are non-decreasing i.e. $A_k \succeq A_{k-1}$. For AMSGrad and Adam, we assume that the preconditioners remain bounded with eigenvalues in the range $[a_{\min}, a_{\max}]$,

$$a_{\min} I \preceq A_k \preceq a_{\max} I. \quad \text{(Bounded preconditioner)}$$

For all algorithms, we assume that the iterates do not diverge and remain in a ball of radius $D$, as is standard in the online learning literature [9, 17],

$$\|w_k - w^*\| \leq D. \quad \text{(Bounded iterates)}$$

Our main assumptions are that each individual function $f_i$ is convex, differentiable, has a finite minimum $f_i^*$, and is $L_i$-smooth, meaning that for all $v$ and $w$,

$$f_i(v) \geq f_i(w) - \langle \nabla f_i(w), w - v \rangle, \quad \text{(Individual Convexity)}$$

$$f_i(v) \leq f_i(w) + \langle \nabla f_i(w), v - w \rangle + \frac{L_i}{2} \|v - w\|^2, \quad \text{(Individual Smoothness)}$$

which also implies that $f$ is convex and $L_{\max}$-smooth, where $L_{\max}$ is the maximum smoothness constant of the individual functions. A consequence of smoothness is the following bound on the norm of the gradient stochastic gradients,

$$\|\nabla f_i(w)\|^2 \leq L_{\max} (f_i(w) - f_i^*).$$

To characterize interpolation, we define the expected difference between the minimum of $f$, $f(w^*)$, and the minimum of the individual functions $f_i^*$,

$$\sigma^2 = \mathbb{E}_i[f_i(w^*) - f_i^*] < \infty. \quad \text{(Noise)}$$

When interpolation is exactly satisfied and every data point can be fit exactly, such that $f_i^* = 0$ and $f(w^*) = 0$, we have $\sigma^2 = 0$.

For results in Appendices F and G.1, we also assume the objective $f$ is $\mu$-strongly convex, that is for all $v$ and $w$,

$$f(v) \leq f(w) + \langle \nabla f(w), v - w \rangle + \frac{\mu}{2} \|v - w\|^2. \quad \text{(Strong convexity)}$$
B Line-search and Polyak step-sizes

We now give the main guarantees on the step-sizes returned by the line-search. For simplicity of presentation, we assume that the line-search returns the largest step-size that satisfies the constraints. The implementation uses a backtracking search to find a step-size that satisfies the constraints, described in the experimental details section (Algorithm 1, Appendix H). The line-searches are capped by a maximum step-size \( \eta_{\text{max}} \), set as the initial step-size of the backtracking procedure. When interpolation is satisfied, the rate does not depend on \( \eta_{\text{max}} \) as long as the step-size returned by the line search is smaller than this upper bound. If interpolation is not exactly satisfied, \( \eta_{\text{max}} \) ensures that a bad iteration of the line-search procedure does not result in divergence.

The Lipschitz and Armijo line-searches select the largest \( \eta \) such that
\[
\begin{align*}
  f_i(w - \eta \nabla f_i(w)) &\leq f_i(w) - \eta c \|\nabla f_i(w)\|^2, & \eta \leq \eta_{\text{max}}, \quad \text{(Lipschitz line-search)} \\
  f_i(w - \eta A^{-1} \nabla f_i(w)) &\leq f_i(w) - \eta c \|\nabla f_i(w)\|^2_{A^{-1}}, & \eta \leq \eta_{\text{max}}, \quad \text{(Armijo line-search)}
\end{align*}
\]

**Lemma 1** (Lipschitz line-search). If \( f_i \) is \( L_i \)-smooth, the Lipschitz line-search ensures that
\[
\eta \|\nabla f_i(w)\|^2 \leq \frac{1}{c}(f_i(w) - f_i^*), \quad \text{and} \quad \min \left\{ \eta_{\text{max}}, \frac{2(1-c)}{L_i} \right\} \leq \eta \leq \eta_{\text{max}}.
\]

**Lemma 2** (Armijo line-search). If \( f_i \) is \( L_i \)-smooth, the Armijo line-search ensures that
\[
\eta \|\nabla f_i(w)\|_{A^{-1}}^2 \leq \frac{1}{c}(f_i(w) - f_i^*), \quad \text{and} \quad \min \left\{ \eta_{\text{max}}, \frac{2 \lambda_{\text{min}}(A)(1-c)}{L_i} \right\} \leq \eta \leq \eta_{\text{max}}.
\]

**Proof of Lemmas 1 and 2.** Recall that if \( f_i \) is \( L_i \)-smooth, then for an arbitrary direction \( d \),
\[
f_i(w - d) \leq f_i(w) - \langle \nabla f_i(w), d \rangle + \frac{L_i}{2} \|d\|^2.
\]

For the Lipschitz line-search, \( d = \eta \nabla f_i(w) \). The smoothness and the line-search condition can then be rearranged into the following forms
\[
\begin{align*}
\text{Smoothness:} & \quad f_i(w - \eta \nabla f_i(w)) - f_i(w) \leq \left( \frac{L_i}{2} \eta^2 - \eta \right) \|\nabla f_i(w)\|^2, \\
\text{Line-search:} & \quad f_i(w - \eta \nabla f_i(w)) - f_i(w) \leq -c\eta \|\nabla f_i(w)\|^2
\end{align*}
\]

As illustrated in Fig. 3, the line-search condition is looser than smoothness if
\[
\left( \frac{L_i}{2} \eta^2 - \eta \right) \|\nabla f_i(w)\|^2 \leq -c\eta \|\nabla f_i(w)\|^2.
\]

The inequality is satisfied for any \( \eta \in [a, b] \), where \( a, b \) are values of \( \eta \) that satisfy the equation with equality; \( a = 0 \) and \( b = 2(1-c)/L_i \). The line-search condition is guaranteed to hold for \( \eta \leq 2(1-c)/L_i \).

As the line search selects the largest feasible step-size, \( \eta \geq 2(1-c)/L_i \). If the step-size is capped at \( \eta_{\text{max}} \), we have \( \eta \geq \min\{\eta_{\text{max}}, 2(1-c)/L_i\} \), and the proof for Lemma 1 is complete.

The proof for the Armijo line-search guarantee (Lemma 2) is identical except for the smoothness property, which is modified with \( d = \eta A^{-1} \nabla f_i(w) \):
\[
\begin{align*}
f_i(w - \eta A^{-1} \nabla f_i(w)) &\leq f_i(w) - \eta \langle \nabla f_i(w), A^{-1} \nabla f_i(w) \rangle + \frac{L_i}{2} \eta^2 \|A^{-1} \nabla f_i(w)\|^2 \\
&\leq f_i(w) - \eta \|\nabla f_i(w)\|_{A^{-1}}^2 + \frac{L_i}{2 \lambda_{\text{min}}(A)} \eta^2 \|\nabla f_i(w)\|_{A^{-1}}^2 \\
&= f_i(w) + \left( \frac{L_i}{2 \lambda_{\text{min}}(A)} \eta^2 - \eta \right) \|\nabla f_i(w)\|_{A^{-1}}^2,
\end{align*}
\]

where the second inequality comes from \( \|A^{-1} \nabla f_i(w)\|^2 \leq \|\nabla f_i(w)\|_{A^{-1}}^2 / \lambda_{\text{min}}(A) \). Q.E.D.
Similarly, the stochastic Polyak step-sizes (SPS) for \( f_i \) at \( w \) are defined as

\[
\text{SPS: } \eta = \min \left\{ \frac{f_i(w) - f_i^*}{c \| \nabla f_i(w) \|^2}, \eta_{\max} \right\}, \quad \text{Armijo SPS: } \eta = \min \left\{ \frac{f_i(w) - f_i^*}{c \| \nabla f_i(w) \|_{A^{-1}}^2}, \eta_{\max} \right\},
\]

**Lemma 3 (SPS guarantees).** If \( f_i \) is \( L_i \)-smooth, SPS and Armijo SPS ensure that

\[
\text{SPS: } \eta \| \nabla f_i(w) \|^2 \leq \frac{1}{c} (f_i(w) - f_i^*) \quad \min \left\{ \eta_{\max}, \frac{1}{2cL_i} \right\} \leq \eta \leq \eta_{\max}
\]

\[
\text{Armijo SPS: } \eta \| \nabla f_i(w) \|_{A^{-1}}^2 \leq \frac{1}{c} (f_i(w) - f_i^*) \quad \min \left\{ \eta_{\max}, \frac{\lambda_{\min}(A)}{2cL_i} \right\} \leq \eta \leq \eta_{\max}
\]

**Proof of Lemma 3.** The first guarantee follows directly from the definition of the step-size. For SPS,

\[
\eta \| \nabla f_i(w) \|^2 = \min \left\{ \frac{f_i(w) - f_i^*}{c \| \nabla f_i(w) \|^2}, \eta_{\max} \right\} \| \nabla f_i(w) \|^2,
\]

\[
= \min \left\{ \frac{f_i(w) - f_i^*}{c}, \eta_{\max} \right\} \| \nabla f_i(w) \|^2 \leq \frac{1}{c} (f_i(w) - f_i^*).
\]

The same inequalities hold for Armijo SPS with \( \| \nabla f_i(w) \|_{A^{-1}}^2 \). To lower-bound the step-size, we use the \( L_i \)-smoothness of \( f_i \), which implies \( f_i(w) - f_i^* \geq \frac{1}{2L_i} \| \nabla f_i(w) \|^2 \). For SPS,

\[
\frac{f_i(w) - f_i^*}{c \| \nabla f_i(w) \|^2} \geq \frac{1}{2L_i} \| \nabla f_i(w) \|^2 \geq \frac{1}{2cL_i}.
\]

For Armijo SPS, we additionally use \( \| \nabla f_i(w) \|_{A^{-1}}^2 \leq \frac{1}{\lambda_{\min}(A)} \| \nabla f_i(w) \|^2 \),

\[
\frac{f_i(w) - f_i^*}{c \| \nabla f_i(w) \|^2_{A^{-1}}} \geq \frac{1}{c \lambda_{\min}(A)} \| \nabla f_i(w) \|^2 \leq \frac{\lambda_{\min}(A)}{2cL_i}.
\]

\( \square \)
C Proofs for AdaGrad

We now move to the proof of the convergence of AdaGrad in the smooth setting with a constant step-size (Theorem 1) and the conservative Lipschitz line-search (Theorem 2). We first give a rate for an arbitrary step-size \( \eta_k \) in the range \([\eta_{\min}, \eta_{\max}]\), and derive the rates of Theorems 1 and 2 by specializing the range to a constant step-size or line-search.

**Proposition 1** (AdaGrad with non-increasing step-sizes). Assuming (i) convexity and (ii) \( L_{\max} \)-smoothness of each \( f_i \), and (iii) bounded iterates, AdaGrad with non-increasing \( \eta_k \leq \eta_{k-1} \), bounded step-sizes \( \eta_k \in [\eta_{\min}, \eta_{\max}] \), and uniform averaging \( \bar{w}_T = \frac{1}{T} \sum_{k=1}^T w_k \), converges at a rate

\[ E[f(\bar{w}_T) - f^*] \leq \frac{\alpha}{T} + \frac{\sqrt{\alpha \sigma}}{\sqrt{T}}, \quad \text{where} \quad \alpha = \frac{1}{2} \left( \frac{D^2}{\eta_{\min}} + 2 \eta_{\max} \right)^2 dL_{\max}. \]

We first use the above result to prove Theorems 1 and 2. The proof of Theorem 1 is immediate by plugging \( \eta = \eta_{\min} = \eta_{\max} \) in Proposition 1. We recall its statement;

**Theorem 1** (Constant step-size AdaGrad). Assuming (i) convexity and (ii) \( L_{\max} \)-smoothness of each \( f_i \), and (iii) bounded iterates, AdaGrad with a constant step-size \( \eta \) and uniform averaging such that \( \bar{w}_T = \frac{1}{T} \sum_{k=1}^T w_k \), converges at a rate

\[ E[f(\bar{w}_T) - f^*] \leq \frac{\alpha}{T} + \frac{\sqrt{\alpha \sigma}}{\sqrt{T}}, \quad \text{where} \quad \alpha = \frac{1}{2} \left( \frac{D^2}{\eta} + 2 \eta \right)^2 dL_{\max}. \]

For Theorem 2, we use the properties of the conservative Lipschitz line-search. We recall its statement;

**Theorem 2.** Under the same assumptions as Theorem 1, AdaGrad with a conservative Lipschitz line-search with \( c = 1/2 \) and uniform averaging converges at a rate

\[ E[f(\bar{w}_T) - f^*] \leq \frac{\alpha}{T} + \frac{\sqrt{\alpha \sigma}}{\sqrt{T}}, \quad \text{where} \quad \alpha = \frac{1}{2} \left( D^2 \max \left\{ \frac{1}{\eta_{\max}}, L_{\max} \right\} + 2 \eta_{\max} \right)^2 dL_{\max}. \]

**Proof of Theorem 2.** Using Lemma 1, there is a step size \( \eta_k \) that satisfies the Lipschitz line-search with \( \eta_k \geq \frac{2(1-c)}{L_{\max}} \). Setting \( c = 1/2 \) and using a maximum step-size \( \eta_{\max} \), we have

\[ \min \left\{ \eta_{\max}, \frac{1}{L_{\max}} \right\} \leq \eta_k \leq \eta_{\max}, \quad \implies \quad \frac{1}{\eta_{\min}} = \max \left\{ \frac{1}{\eta_{\max}}, L_{\max} \right\}. \]

Before going into the proof of Proposition 1, we recall some standard lemmas from the adaptive gradient literature (Theorem 7 & Lemma 10 in [9], Lemma 5.15 & 5.16 in [11]), and a useful quadratic inequality [17, Part of Theorem 4.2]). We include proofs in Appendix C.1 for completeness.

**Lemma 4.** If the preconditioners are non-decreasing \( \{A_k\} \), the step-sizes are non-increasing \((\eta_k \leq \eta_{k-1})\), and the iterates stay within a ball of radius \( D \) of the minima,

\[ \sum_{k=1}^T \|w_k - w^*\|^2_{\frac{1}{\eta_k} A_k - \frac{1}{\eta_{k-1}} A_{k-1}} \leq \frac{D^2}{\eta_T} \text{Tr}(A_T). \]

**Lemma 5.** For AdaGrad, \( A_k = \left[ \sum_{i=1}^k \nabla f_i(w_k) \nabla f_i(w_k)^\top \right]^{1/2} \) and satisfies,

\[ \sum_{k=1}^T \| \nabla f_i(w_k) \|^2_{A_k^{-1}} \leq 2 \text{Tr}(A_T), \quad \text{Tr}(A_T) \leq \sqrt{d} \sum_{k=1}^T \| \nabla f_i(w_k) \|^2. \]

**Lemma 6.** If \( x \leq a(x + b) \) for \( a \geq 0 \) and \( b \geq 0 \),

\[ x \leq \frac{1}{2} \left( \sqrt{a^2 + 4ab} + a \right) \leq a + \sqrt{ab}. \]

We now prove Proposition 1.
Proof of Proposition 1. We first give an overview of the main steps. Using the definition of the update rule, along with Lemmas 4 and 5, we will show that
\[
2 \sum_{k=1}^{T} (\nabla f_{ik}(w_k), w_k - w^*) \leq \left( \frac{D^2}{\eta_{\text{min}}} + 2\eta_{\text{max}} \right) \text{Tr}(A_T). \tag{4}
\]
Using the definition of $A_T$, individual smoothness and convexity, we then show that for a constant $a$,
\[
\sum_{k=1}^{T} \mathbb{E} [f(w_k) - f^*] \leq a \left( \mathbb{E} \left[ \sqrt{\sum_{k=1}^{T} f_{ik}(w_k) - f_{ik}(w^*)} + T\sigma^2 \right] \right), \tag{5}
\]
Using the quadratic bound (Lemma 6), averaging and using Jensen’s inequality finishes the proof.

To derive Eq. (4), we start with the Update rule, measuring distances to $w^*$ in the $||.||_{A_k}$ norm,
\[
||w_{k+1} - w^*||_{A_k}^2 = ||w_k - w^*||_{A_k}^2 - 2\eta_k (\nabla f_{ik}(w_k), w_k - w^*) + \eta_k^2 ||\nabla f_{ik}(w_k)||_{A_k^{-1}}^2.
\]
Dividing by $\eta_k$, reorganizing the equation and summing across iterations yields
\[
2 \sum_{k=1}^{T} (\nabla f_{ik}(w_k), w_k - w^*) \leq \sum_{k=1}^{T} ||w_k - w^*||_{A_k^{-1}}^2 + \sum_{k=1}^{T} \eta_k ||\nabla f_{ik}(w_k)||_{A_k^{-1}}^2,
\]
\[
\leq \sum_{k=1}^{T} ||w_k - w^*||_{A_k^{-1}}^2 + \eta_{\text{max}} \sum_{k=1}^{T} ||\nabla f_{ik}(w_k)||_{A_k^{-1}}^2.
\]
We use the Lemmas 4, 5 to bound the RHS by the trace of the last preconditioner,
\[
\leq \frac{D^2}{\eta_T} \text{Tr}(A_T) + 2\eta_{\text{max}} \text{Tr}(A_T), \tag{Lemmas 4 and 5}
\]
\[
\leq \left( \frac{D^2}{\eta_{\text{min}}} + 2\eta_{\text{max}} \right) \text{Tr}(A_T). \tag{\eta_k \geq \eta_{\text{min}}}
\]
To derive Eq. (5), we bound the trace of $A_T$ using Lemma 5 and Individual Smoothness,
\[
\text{Tr}(A_T) \leq \sqrt{d} \sqrt{\sum_{k=1}^{T} ||\nabla f_{ik}(w_k)||_{A_k}^2}, \tag{Lemma 5, Trace bound}
\]
\[
\leq \sqrt{2dL_{\text{max}}} \sqrt{\sum_{k=1}^{T} f_{ik}(w_k) - f_i^*}, \tag{Individual Smoothness}
\]
\[
\leq \sqrt{2dL_{\text{max}}} \sqrt{\sum_{k=1}^{T} f_{ik}(w_k) - f_{ik}(w^*) + f_{ik}(w^*) - f_i^*}, \tag{\pm f_i(w^*)}
\]
Combining the above inequalities with $\delta_{ik} = f_{ik}(w_k) - f_i^*$ and $a = \frac{1}{2} \left( \frac{D^2}{\eta_{\text{min}}} + 2\eta_{\text{max}} \right) \sqrt{2dL_{\text{max}}}$,
\[
\sum_{k=1}^{T} (\nabla f_{ik}(w_k), w_k - w^*) \leq a \sqrt{\sum_{k=1}^{T} f_{ik}(w_k) - f_{ik}(w^*) + \delta_{ik}}.
\]
Using Individual Convexity and taking expectations,
\[
\sum_{k=1}^{T} \mathbb{E} [f(w_k) - f^*] \leq a \mathbb{E} \left[ \sqrt{\sum_{k=1}^{T} f_{ik}(w_k) - f_{ik}(w^*) + \delta_{ik}} \right],
\]
\[
\leq a \sqrt{\mathbb{E} \left[ \sum_{k=1}^{T} f_{ik}(w_k) - f_{ik}(w^*) + \delta_{ik} \right]}. \tag{Jensen’s inequality}
\]
Define $\sigma^2 := \mathbb{E}[\delta] = \mathbb{E}[f_i(w^*) - f_i^*]$ and take the square on both sides yields
\[
\left( \sum_{k=1}^{T} \mathbb{E} [f(w_k) - f^*) \right)^2 \leq a^2 \left( \mathbb{E} \left[ \sum_{k=1}^{T} f_{ik}(w_k) - f_{ik}(w^*) \right] + T\sigma^2 \right).
\]
The quadratic bound (Lemma 6) $x^2 \leq \alpha(x + \beta)$ implies $x \leq \alpha + \sqrt{\alpha\beta}$, with
\[
x = \sum_{k=1}^{T} \mathbb{E} [f(w_k) - f^*], \quad \alpha = \frac{1}{2} \left( \frac{D^2}{\eta_{\text{min}}} + 2\eta_{\text{max}} \right) \sqrt{dL_{\text{max}}}, \quad \beta = T\sigma^2,
\]
gives the first bound below, and averaging $\bar{w}_T = \frac{1}{T} \sum_{k=1}^{T} w_k$ and using Jensen’s inequality finishes the proof;
\[
\sum_{k=1}^{T} \mathbb{E} [f(w_k) - f^*] \leq \alpha + \sqrt{\alpha\beta}, \quad \Rightarrow \quad \mathbb{E} [f(\bar{w}_T) - f^*] \leq \frac{\alpha}{T} + \frac{\sqrt{\alpha\beta}}{\sqrt{T}}. \quad \square
C.1 Proofs of adaptive gradient lemmas

For completeness, we give proofs for the lemmas used in the previous section. We restate them here;

**Lemma 4.** If the preconditioners are non-decreasing ($A_k \succeq A_{k-1}$), the step-sizes are non-increasing ($\eta_k \leq \eta_{k-1}$), and the iterates stay within a ball of radius $D$ of the minima, \[ \sum_{k=1}^{T} \|w^*_k - w^k\|^2 \leq \frac{D^2}{\eta T} \text{Tr}(A_T). \]

**Proof of Lemma 4.** Under the assumptions that $A_k$ is non-decreasing and $\eta_k$ is non-increasing, \[ \frac{1}{\eta_k} A_k - \frac{1}{\eta_{k-1}} A_{k-1} \succeq 0, \] so we can use the Bounded iterates assumption to bound \[ \sum_{k=1}^{T} \|w^*_k - w^k\|^2 \leq \sum_{k=1}^{T} \lambda_{k} \left( \frac{A_k}{\eta_k} - \frac{A_{k-1}}{\eta_{k-1}} \right) \leq D^2 \sum_{k=1}^{T} \lambda_{\text{max}} \left( \frac{A_k}{\eta_k} - \frac{A_{k-1}}{\eta_{k-1}} \right). \]

We then upper-bound $\lambda_{\text{max}}$ by the trace and use the linearity of the trace to telescope the sum, \[ \leq D^2 \sum_{k=1}^{T} \text{Tr} \left( \frac{A_k}{\eta_k} - \frac{A_{k-1}}{\eta_{k-1}} \right) = D^2 \sum_{k=1}^{T} \text{Tr} \left( \frac{A_k}{\eta_k} \right) - \text{Tr} \left( \frac{A_{k-1}}{\eta_{k-1}} \right), \]
\[ = D^2 \left( \text{Tr} \left( \frac{A_T}{\eta T} \right) - \text{Tr} \left( \frac{A_0}{\eta_0} \right) \right) \leq D^2 \frac{1}{\eta T} \text{Tr}(A_T). \]

**Lemma 5.** For AdaGrad, \[ A_k = \left[ \sum_{i=1}^{k} \nabla f_{i_k}(w_k) \nabla f_{i_k}(w_k)^\top \right]^{1/2} \]
and satisfies, \[ \sum_{k=1}^{T} \|\nabla f_{i_k}(w_k)\|^2_{A_k^{-1}} \leq 2\text{Tr}(A_T), \quad \text{Tr}(A_T) \leq \sqrt{d} \sum_{k=1}^{T} \|\nabla f_{i_k}(w_k)\|^2. \]

**Proof of Lemma 5.** For ease of notation, let $\nabla_k := \nabla f_{i_k}(w_k)$. By induction, starting with $T = 1$, \[ \|\nabla f_1(w_1)\|^2_{A_1^{-1}} = \nabla_1^\top A_1^{-1} \nabla_1 = \text{Tr} \left( \nabla_1^\top A_1^{-1} \nabla_1 \right) = \text{Tr} \left( A_1^{-1} \nabla_1 \nabla_1^\top \right), \]
(Cyclic property of trace) \[ \text{Tr}(A_1^{-1} A_2^2) = \text{Tr}(A_1). \]
\[
(A_1 = (\nabla_1 \nabla_1^\top)^{1/2})
\]
Suppose that it holds for $T - 1$, \[ \sum_{k=1}^{T-1} \|\nabla_k\|^2_{A_k^{-1}} \leq 2\text{Tr}(A_T-1). \] We will show it also holds for $T$. Using the definition of the preconditioner and the cyclic property of the trace, \[ \sum_{k=1}^{T} \|\nabla f_{i_k}(w_k)\|^2_{A_k^{-1}} \leq 2\text{Tr}(A_T-1) + \|\nabla_T\|^2_{A_T^{-1}}, \quad \text{(Induction hypothesis)} \]
[\[ = 2\text{Tr} \left( (A_2^2 - \nabla_T \nabla_T^\top)^{1/2} \right) + \text{Tr} \left( A_T^{-1} \nabla_T \nabla_T^\top \right) \quad \text{(AdaGrad update)} \]
We then use the fact that for any $X \succeq Y \succeq 0$, we have [9, Lemma 8] \[ 2\text{Tr} \left( (X - Y)^{1/2} \right) + \text{Tr} \left( X^{-1/2} Y \right) \leq 2\text{Tr} \left( X^{1/2} \right). \]
As $X = A_T^2 \succeq Y = \nabla_T \nabla_T^\top \succeq 0$, we can use the above inequality and the induction holds for $T$.

For the trace bound, recall that $A_T = G_T^{1/2}$ where $G_T = \sum_{i=1}^{T} \nabla f_{i_k}(w_k) \nabla f_{i_k}(w_k)^\top$. We use Jensen’s inequality, \[ \text{Tr}(A_T) = \text{Tr}(G_T^{1/2}) = \sum_{j=1}^{d} \sqrt{\lambda_j(G_T)} = d \left( \frac{1}{n} \sum_{j=1}^{d} \sqrt{\lambda_j(G_T)} \right) \leq d \sqrt{\frac{1}{n} \sum_{j=1}^{d} \lambda_j(G_T)} = \sqrt{d \text{Tr}(G_T)}. \]
To finish the proof, we use the definition of $G_T$ and the linearity of the trace to get \[ \sqrt{\text{Tr}(G_T)} = \sqrt{\text{Tr} \left( \sum_{k=1}^{T} \nabla_k \nabla_k^\top \right)} = \sqrt{\sum_{k=1}^{T} \text{Tr}(\nabla_k \nabla_k^\top)} = \sqrt{\sum_{k=1}^{T} \|\nabla_k\|^2}. \]

**Lemma 6.** If $x^2 \leq a(x + b)$ for $a \geq 0$ and $b \geq 0$, \[ x \leq \frac{1}{2} \left( \sqrt{a^2 + 4ab} + a \right) \leq a + \sqrt{ab}. \]
Proof of Lemma 6. The starting point is the quadratic inequality $x^2 - ax - ab \leq 0$. Letting $r_1 \leq r_2$ be the roots of the quadratic, the inequality holds if $x \in [r_1, r_2]$. The upper bound is then derived by

\[ r_2 = \frac{a + \sqrt{a^2 + 4ab}}{2}, \quad \text{(Quadratic root)} \]

\[ \leq \frac{a + \sqrt{a^2 + 4ab}}{2} = a + \sqrt{ab}. \quad \text{(Using } \sqrt{a + b} \leq \sqrt{a} + \sqrt{b}) \]

\[ \square \]
C.2 With interpolation, without conservative line-searches

In this section, we show that the conservative constraint $\eta_{k+1} \leq \eta_k$ is not necessary if interpolation is satisfied. We give the proof for the Armijo line-search, that has better empirical performance, but a worse theoretical dependence on the problem’s constants. For the theorem below, $a_{\min}$ is lower-bounded by $\epsilon$ in practice. A similar proof also works for the Lipshitz line-search.

**Theorem 7** (AdaGrad with Armijo line-search under interpolation). Under the same assumptions of Proposition 1, but without non-increasing step-sizes, if interpolation is satisfied, AdaGrad with the Armijo line-search and uniform averaging converges at the rate,

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \left(\frac{D^2 + 2\eta_{\max}^2}{2T}\right)^2 \max\left\{\frac{1}{\eta_{\max}}, \frac{L_{\max}}{a_{\min}}\right\}^2,$$

where $a_{\min} = \min_k \{\lambda_{\min}(A_k)\}$.

**Proof of Theorem 7.** Following the proof of Proposition 1,

$$2\sum_{k=1}^{T} \eta_k \langle \nabla f_i(k), w_k - w^* \rangle = \sum_{k=1}^{T} \|w_k - w^*\|^2_{A_k} - \|w_{k+1} - w^*\|^2_{A_k} + \eta_k^2 \|\nabla f_i(k)\|^2_{A_k}.$$

On the left-hand side, we use individual convexity and interpolation, which implies $f_i(k) = \min_w f_i(w)$ and we can bound $\eta_k$ by $\eta_{\min}$, giving

$$\eta_k \langle \nabla f_i(k), w_k - w^* \rangle \geq \eta_k (f_i(k) - f_i(w^*)) \geq \eta_{\min} (f_i(k) - f_i(w^*)) \geq 0.$$

On the right-hand side, we can apply the AdaGrad lemmas (Lemma 5)

$$\sum_{k=1}^{T} \|w_k - w^*\|^2_{A_k} - \|w_{k+1} - w^*\|^2_{A_k} + \eta_{\max}^2 \|\nabla f_i(k)\|^2_{A_k} \leq (D^2 + 2\eta_{\max}^2) \text{Tr}(A_T),$$

$$\leq (D^2 + 2\eta_{\max}^2) \sqrt{d \sqrt{\sum_{k=1}^{T} \|\nabla f_i(k)\|^2}}, \quad \text{(By Lemmas 4 and 5)}$$

$$\leq (D^2 + 2\eta_{\max}^2) \sqrt{2dL_{\max} \sqrt{\sum_{k=1}^{T} f_i(k) - f_i(w^*)}}. \quad \text{(By Individual Smoothness and interpolation)}$$

Defining $a = \frac{1}{2\eta_{\max}} (D^2 + 2\eta_{\max}^2) \sqrt{2dL_{\max}}$ and combining the previous inequalities yields

$$\sum_{k=1}^{T} (f_i(k) - f_i(w^*)) \leq a \sqrt{\sum_{k=1}^{T} f_i(k) - f_i(w^*)}.$$

Taking expectations and applying Jensen’s inequality yields

$$\sum_{k=1}^{T} \mathbb{E}[f(w_k) - f(w^*)] \leq a \sqrt{\sum_{k=1}^{T} \mathbb{E}[f(w_k) - f(w^*)]}.$$

Squaring both sides, dividing by $\sum_{k=1}^{T} \mathbb{E}[f(w_k) - f(w^*)]$, followed by dividing by $T$ and applying Jensen’s inequality,

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{a^2}{T} = \left(\frac{D^2 + 2\eta_{\max}^2}{2\eta_{\min}^2} \right)^2 \frac{dL_{\max}}{T}.$$

Using the Armijo line-search guarantee (Lemma 2) with $c = 1/2$ and a maximum step-size $\eta_{\max}$,

$$\eta_{\min} = \min \left\{ \eta_{\max}, \frac{a_{\min}}{L_{\max}} \right\},$$

where $a_{\min} = \min_k \{\lambda_{\min}(A_k)\}$, giving the rate

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \left(\frac{D^2 + 2\eta_{\max}^2}{2T}\right)^2 \max\left\{\frac{1}{\eta_{\max}}, \frac{L_{\max}}{a_{\min}}\right\}^2.$$

\(\Box\)
D Proofs for AMSGrad and non-decreasing preconditioners

We now give the proofs for AMSGrad and general bounded, non-decreasing preconditioners in the smooth setting, using a constant step-size (Theorem 3) and the Armijo line-search (Theorem 4). As in Appendix C, we prove a general proposition and specialize it for each of the theorems;

Proposition 2. In addition to assumptions of Theorem 1, assume that (iv) the preconditioners are non-decreasing and have (v) bounded eigenvalues in the \([\alpha_{\min}, \alpha_{\max}]\) range. If the step-sizes are constrained to lie in the range \([\eta_{\min}, \eta_{\max}]\) and satisfy

\[
\eta_k \|\nabla f_{i_k}(w_k)\|_{A_k^{-1}}^2 \leq M (f_{i_k}(w_k) - f_{i_k}^*), \quad \text{for some } M < 2, \tag{6}
\]

using uniform averaging \(\bar{w}_T = \frac{1}{T} \sum_{k=1}^{T} w_k\) leads to the rate

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{1}{T} \frac{D^2 d_{\alpha_{\max}} L_{\max}}{\alpha_{\min} T} + \sigma^2.
\]

We restate Theorem 3;

Theorem 3. Under the assumptions of Theorem 1 and assuming (iv) non-decreasing preconditioners (v) bounded eigenvalues in the \([\alpha_{\min}, \alpha_{\max}]\) interval, AMSGrad with no momentum, constant step-size \(\eta = \frac{\alpha_{\min}}{2L_{\max}}\) and uniform averaging converges at a rate,

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{2D^2 \eta_{\max} L_{\max}^2}{\alpha_{\min} T} + \sigma^2.
\]

Proof of Theorem 3. Using Bounded preconditioner and Individual Smoothness, we have that

\[
\|\nabla f_{i_k}(w_k)\|_{A_k^{-1}}^2 \leq \frac{1}{\alpha_{\min}} \|\nabla f_{i_k}(w_k)\|^2 \leq \frac{2L_{\max}}{\alpha_{\min}} (f_{i_k}(w_k) - f_{i_k}^*).
\]

A constant step-size \(\eta_{\max} = \eta_{\min} = \frac{\alpha_{\min}}{2L_{\max}}\) satisfies the step-size assumption (Eq. 6) with \(M = 1\) and

\[
\frac{1}{T} \frac{D^2 d_{\alpha_{\max}} L_{\max}}{\alpha_{\min} T} + \left(\frac{2}{2 - M} \frac{\eta_{\max}}{\eta_{\min}} - 1\right) \sigma^2 = \frac{1}{T} \frac{2D^2 d_{\alpha_{\max}} L_{\max}^2}{\alpha_{\min} T} + \sigma^2.
\]

We restate Theorem 4;

Theorem 4. Under the same assumption as Theorem 3, AMSGrad with no momentum, Armijo line-search with \(c = 3/4\) and uniform averaging converges at a rate,

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \left(\frac{3D^2 d \cdot \eta_{\max}}{2T} + 3\eta_{\max} \sigma^2\right) \max\left\{\frac{1}{\eta_{\max}}, \frac{2L_{\max}}{\alpha_{\min}}\right\}.
\]

Proof of Theorem 4. For the Armijo line-search, Lemma 2 guarantees that

\[
\eta \|\nabla f_{i_k}(w_k)\|_{A_k^{-1}}^2 \leq \frac{1}{c} (f_{i_k}(w_k) - f_{i_k}^*), \quad \text{and } \min\left\{\eta_{\max}, \frac{2L_{\min}}{L_{\max}}\right\} \leq \eta \leq \eta_{\max}.
\]

Selecting \(c = 3/4\) gives \(M = 4/3\) and \(\eta_{\min} = \min\left\{\eta_{\max}, \frac{\alpha_{\min}}{2L_{\max}}\right\}\), so

\[
\frac{1}{T} \frac{D^2 d_{\alpha_{\max}} L_{\max}^2}{\alpha_{\min} T} + \left(\frac{2}{2 - M} \frac{\eta_{\max}}{\eta_{\min}} - 1\right) \sigma^2
\]

\[
= \frac{1}{T} \frac{D^2 d_{\alpha_{\max}} L_{\max}^2}{\alpha_{\min} T} + \left(\frac{2}{2 - 4/3} \frac{\eta_{\max}}{\eta_{\min}} - 1\right) \sigma^2,
\]

\[
= \frac{1}{T} \frac{3D^2 d_{\alpha_{\max}} L_{\max}^2}{\alpha_{\min} T} + \left(\frac{3\eta_{\max}}{\eta_{\min}} - 1\right) \sigma^2,
\]

\[
\leq \frac{3D^2 d_{\alpha_{\max}} L_{\max}^2}{2T} \max\left\{\frac{1}{\eta_{\max}}, \frac{2L_{\max}}{\alpha_{\min}}\right\} + 3\eta_{\max} \sigma^2 \max\left\{1, \frac{2L_{\max}}{\alpha_{\min}}\right\}.
\]
Before diving into the proof of Proposition 2, we prove the following lemma to handle terms of the form \(\eta_k(f_i(w_k) - f_i(w^*))\). If \(\eta_k\) depends on the function sampled at the current iteration, \(f_i\), as is the case of line-search, we cannot take expectations as the terms are not independent. Lemma 7 bounds \(\eta_k(f_i(w_k) - f_i(w^*))\) in terms of the range \([\eta_{\min}, \eta_{\max}]\):

**Lemma 7.** If \(0 \leq \eta_{\min} \leq \eta \leq \eta_{\max}\) and the minimum value of \(f_i\) is \(f_i^*\), then

\[
\eta(f_i(w) - f_i(w^*)) \geq \eta_{\min}(f_i(w) - f_i(w^*)) - (\eta_{\max} - \eta_{\min})(f_i(w^*) - f_i^*).
\]

**Proof of Lemma 7.** By adding and subtracting \(f_i^*\), the minimum value of \(f_i\), we get a non-negative and a non-positive term multiplied by \(\eta\). We can use the bounds \(\eta \geq \eta_{\min}\) and \(\eta \leq \eta_{\max}\) separately:

\[
\eta[f_i(w) - f_i(w^*)] = \eta[f_i(w) - f_i^* + f_i^* - f_i(w^*)],
\]

\[
\geq \eta_{\min}[f_i(w) - f_i^*] + \eta_{\max}[f_i^* - f_i(w^*)],
\]

Adding and subtracting \(\eta_{\min}f_i(w^*)\) finishes the proof,

\[
= \eta_{\min}[f_i(w) - f_i(w^*)] + f_i(w^*) - f_i^* + \eta_{\max}[f_i^* - f_i(w^*)],
\]

\[
= \eta_{\min}[f_i(w) - f_i(w^*)] + (\eta_{\max} - \eta_{\min})[f_i^* - f_i(w^*)].
\]

**Proof of Proposition 2.** We start with the Update rule, measuring distances to \(w^*\) in the \(\| \cdot \|_{A_k}\) norm,

\[
\|w_{k+1} - w^*\|_{A_k}^2 = \|w_k - w^*\|_{A_k}^2 - 2\eta_k \langle \nabla f_i(w_k), w_k - w^* \rangle + \eta_k^2 \| \nabla f_i(w_k) \|_{A_k^{-1}}^2.
\]  

(7)

To bound the RHS, we use the assumption on the step-sizes (Eq. (6)) and Individual Convexity,

\[
-2\eta_k \langle \nabla f_i(w_k), w_k - w^* \rangle + \eta_k^2 \| \nabla f_i(w_k) \|_{A_k^{-1}}^2,
\]

\[
\leq -2\eta_k [f_i(w_k) - f_i(w^*]) + M\eta_k (f_i(w_k) - f_i^*),
\]  

(Step-size assumption, Eq. (6))

\[
\leq -2\eta_k [f_i(w_k) - f_i(w^*)] + M\eta_k (f_i(w_k) - f_i^*),
\]  

(Individual Convexity)

\[
\leq -(2 - M)\eta_k [f_i(w_k) - f_i(w^*)] + M\eta_{\max} (f_i(w^*) - f_i^*),
\]

\((\eta_k \leq \eta_{\max})\)

Plugging the inequality back into Eq. (7) and reorganizing the terms yields

\[
(2 - M)\eta_k [f_i(w_k) - f_i(w^*)] \leq \left( \|w_k - w^*\|_{A_k}^2 - \|w_{k+1} - w^*\|_{A_k}^2 \right) + M\eta_{\max} (f_i(w^*) - f_i^*).
\]  

(8)

Using Lemma 7, we have that

\[
(2 - M)\eta_k [f_i(w_k) - f_i(w^*)] \geq (2 - M)\eta_{\min} [f_i(w_k) - f_i(w^*)] - (2 - M)(\eta_{\max} - \eta_{\min}) (f_i(w^*) - f_i^*).
\]

Using this inequality in Eq. (8), we have that

\[
(2 - M)\eta_{\min} (f_i(w_k) - f_i(w^*)) - (2 - M)(\eta_{\max} - \eta_{\min}) (f_i(w^*) - f_i^*)
\]

\[
\leq \left( \|w_k - w^*\|_{A_k}^2 - \|w_{k+1} - w^*\|_{A_k}^2 \right) + M\eta_{\max} (f_i(w^*) - f_i^*),
\]

Moving the terms depending on \(f_i(w^*) - f_i^*\) to the RHS,

\[
(2 - M)\eta_{\min} (f_i(w_k) - f_i(w^*)) \leq \left( \|w_k - w^*\|_{A_k}^2 - \|w_{k+1} - w^*\|_{A_k}^2 \right) + (2\eta_{\max} - (2 - M)\eta_{\min}) (f_i(w^*) - f_i^*).
\]

Taking expectations and summing across iterations yields

\[
(2 - M)\eta_{\min} \sum_{k=1}^T \mathbb{E}(f_i(w_k) - f_i(w^*)) \leq \mathbb{E} \left[ \sum_{k=1}^T \left( \|w_k - w^*\|_{A_k}^2 - \|w_{k+1} - w^*\|_{A_k}^2 \right) \right] + (2\eta_{\max} - (2 - M)\eta_{\min}) T \sigma^2
\]

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Using Lemma 4 to telescope the distances and using the Bounded preconditioner,

\[ \sum_{k=1}^{T} \| w_{k} - w^* \|_A^2 - \| w_{k+1} - w^* \|_A^2 \leq \sum_{k=1}^{T} \| w_{k} - w^* \|_{A_{k} - A_{k-1}} \leq D^2 \text{Tr}(A_T) \leq D^2 d_{\text{max}}, \]

which guarantees that

\[ (2 - M)\eta_{\text{min}} \sum_{k=1}^{T} E[f(w_k) - f(w^*)] \leq D^2 d_{\text{max}} + (2\eta_{\text{max}} - (2 - M)\eta_{\text{min}}) T\sigma^2. \]

Dividing by \( T(2 - M)\eta_{\text{min}} \) and using Jensen’s inequality finishes the proof, giving the rate for the averaged iterate,

\[ E[f(\bar{w}_T) - f(w^*)] \leq \frac{1}{T(2 - M)\eta_{\text{min}}} \left( \frac{D^2 d_{\text{max}}}{2} \right) + \left( \frac{2}{2 - M} \frac{\eta_{\text{max}}}{\eta_{\text{min}}} - 1 \right) \sigma^2. \]
E Proofs for Adam and general preconditioners

We now give the proofs for Adam and general bounded preconditioners that do not necessarily satisfy the non-decreasing property. We use a constant step-size (Theorem 5) and the Armijo line-search (Theorem 6). As before, we prove a general proposition and specialize it for each of the theorems;

**Proposition 3.** In addition to the assumptions of Theorem 1 and assuming that (iv) the preconditioners have bounded eigenvalues in the \([a_{\min}, a_{\max}]\) range, the step-sizes are constrained to lie in the \([\eta_{\min}, \eta_{\max}]\) range and satisfy

\[
\eta_k \|\nabla f_{i_k}(w_k)\|_{A_k}^{-1} \leq \frac{a_{\min}}{a_{\max}} (f_{i_k}(w_k) - f^*) \quad \text{using uniform averaging} \quad \bar{w}_T = \frac{1}{T} \sum_{k=1}^{T} w_k \quad \text{results in the rate}
\]

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{\alpha_{\max} \max \left\{ \frac{a_{\max}}{\eta_{\min}}, 2\kappa^2 L_{\max} \right\} }{T \eta_{\min}} + \left( \frac{2 \eta_{\max}}{\eta_{\min}} - 1 \right) \sigma^2.
\]

We recall the statement of Theorem 5

**Theorem 5.** Under the same assumptions as Theorem 1 and assuming (iv) bounded eigenvalues in the \([a_{\min}, a_{\max}]\) interval, Adam with no momentum, constant step-size \(\eta = \alpha_{\min}/2L_{\max}a_{\max}\) and uniform averaging converges at a rate,

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{2\kappa^2 L_{\max} \max \left\{ \frac{a_{\max}}{\eta_{\min}}, 2\kappa^2 L_{\max} \right\} }{T \eta_{\min}} + \sigma^2, \quad \text{where } \kappa = \frac{a_{\max}}{a_{\min}}.
\]

**Proof of Theorem 5.** Using Bounded preconditioner and Individual Smoothness, we have that

\[
\|\nabla f_{i_k}(w_k)\|_{A_k}^2 \leq \frac{2L_{\max} \eta_{\min}}{a_{\min}} (f_{i_k}(w_k) - f^*) \quad \text{satisfies the step-size assumption (Eq. 9), as}
\]

\[
\eta \|\nabla f_{i_k}(w_k)\|_{A_k}^2 \leq \frac{2L_{\max} a_{\min}}{a_{\max}} (f_{i_k}(w_k) - f^*) = \frac{a_{\min}}{a_{\max}} (f_{i_k}(w_k) - f^*) \quad \text{and substituting } \kappa = \frac{a_{\max}}{a_{\min}} \text{ in the rate of Proposition 3 finishes the proof;}
\]

\[
\frac{\alpha_{\max} \max \left\{ \frac{a_{\max}}{\eta_{\min}}, 2\kappa^2 L_{\max} \right\} }{T \eta_{\min}} + \left( \frac{2 \eta_{\max}}{\eta_{\min}} - 1 \right) \sigma^2 = \frac{2L_{\max} \max \left\{ \frac{a_{\max}}{\eta_{\min}}, 2\kappa^2 L_{\max} \right\} }{T \eta_{\min}} + \sigma^2.
\]

We recall the statement of Theorem 6

**Theorem 6.** Under the same assumptions as Theorem 5, Adam with no momentum, Armijo SPS with \(c = \kappa = \alpha_{\max}/\alpha_{\min}\) and uniform averaging converges at a rate,

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{\|w_0 - w^*\|^2}{T} \max \left\{ \frac{\alpha_{\max}}{\eta_{\max}}, 2\kappa^2 L_{\max} \right\} + \max \left\{ 1, \frac{4\eta_{\max} L_{\max}\kappa}{\alpha_{\min}} \right\} \sigma^2.
\]

**Proof of Theorem 6.** Using Lemma 3, Armijo SPS with \(c = \alpha_{\max}/\alpha_{\min}\) satisfies the step-size assumption (Eq. 9), as

\[
\eta \|\nabla f_{i_k}(w_k)\|_{A_k}^2 \leq \frac{1}{c} (f_{i_k}(w_k) - f^*) = \frac{a_{\min}}{a_{\max}} (f_{i_k}(w_k) - f^*) \quad \text{and ensures} \quad \eta_{\min} \geq \min \left\{ \eta_{\max}, \frac{a_{\min}}{2L_{\max}a_{\max}} \right\}
\]

Using this value with Proposition 3 finishes the proof;

\[
\frac{\alpha_{\max} \max \left\{ \frac{a_{\max}}{\eta_{\min}}, 2\kappa^2 L_{\max} \right\} }{T \eta_{\min}} + \left( \frac{2 \eta_{\max}}{\eta_{\min}} - 1 \right) \sigma^2 \leq \frac{(1 - \kappa^2 L_{\max} \eta_{\min})}{\eta_{\max}} + \left( \frac{2 \eta_{\max}}{\eta_{\min}} - 1 \right) \sigma^2,
\]

\[
\leq \frac{\|w_0 - w^*\|^2}{T} \max \left\{ \frac{\alpha_{\max}}{\eta_{\max}}, 2\kappa^2 L_{\max} \right\} + \max \left\{ 1, \frac{4\eta_{\max} L_{\max}\kappa}{\alpha_{\min}} \right\} \sigma^2.
\]
Before diving into the proof of Proposition 3, we prove the following lemma about the inner product \( (A\nabla f(w), w - w^*) \) for a convex function \( f \) and preconditioner \( A \):

**Lemma 8.** If \( f \) is convex and minimized at \( w^* \), and the minimum eigenvalue of \( A \) is \( \lambda_{\text{min}} > 0 \),

\[
(A\nabla f(w), w - w^*) \geq \lambda_{\text{min}} \langle \nabla f(w), w - w^* \rangle.
\]

**Proof.** As \( f \) is convex, \( \nabla f(w) \) and \( w - w^* \) have a non-negative inner product and so does their projection on a common vector, \( \langle \nabla f(w), q \rangle \langle w - w^*, q \rangle \geq 0 \). Let \( QAQ^\top \) be the eigendecomposition of \( A \), where \( Q = [q_1, \ldots, q_d] \) is an orthogonal matrix. Then,

\[
\langle A\nabla f(w), w - w^* \rangle = \langle QA^\top \nabla f(w), Q^\top (w - w^*) \rangle = \lambda_{\text{min}} \sum_{i=1}^d \langle \nabla f(w), q_i \rangle \langle q_i, (w - w^*) \rangle = \lambda_{\text{min}} \langle \nabla f(w), w - w^* \rangle.
\]

**Proof of Proposition 3.** We start with the expansion of the update rule, measuring distances to \( w^* \) in the \( \|\cdot\|_2 \) norm,

\[
\|w_{k+1} - w^*\|^2 = \|w_k - w^*\|^2 - 2\eta_k \langle A_k^{-1} \nabla f_i(w_k), w_k - w^* \rangle + \eta_k^2 \|A_k^{-1} \nabla f_i(w_k)\|^2. \tag{10}
\]

Using the Bounded preconditioner and Lemma 8,

\[
\|w_{k+1} - w^*\|^2 \leq \|w_k - w^*\|^2 - 2\eta_k \langle A_k^{-1} \nabla f_i(w_k), w_k - w^* \rangle + \frac{\eta_k^2}{\lambda_{\text{min}}} \|\nabla f_i(w_k)\|^2 A_k^{-1}. \tag{11}
\]

To bound the RHS, we use the assumption on the step-sizes (Eq. (9)) and Individual Convexity,

\[
-2\frac{\eta_k}{\lambda_{\text{max}}} \langle \nabla f_i(w_k), w_k - w^* \rangle + \frac{\eta_k^2}{\lambda_{\text{min}}} \|\nabla f_i(w_k)\|^2 A_k^{-1} \leq 0.
\]

Plugging those inequalities in Eq. (10) and reorganizing,

\[
\eta_k \langle f_i(w_k) - f_i(w^*) \rangle \leq \lambda_{\text{max}} \left( \|w_k - w^*\|^2 - \|w_{k+1} - w^*\|^2 \right) + \eta_k \langle f_i(w^*) - f_i^* \rangle
\]

\[
\leq \lambda_{\text{max}} \left( \|w_k - w^*\|^2 - \|w_{k+1} - w^*\|^2 \right) + \eta_{\text{max}} \langle f_i(w^*) - f_i^* \rangle. \tag{11}
\]

Using Lemma 7, we have that with \( \eta_k \in [\eta_{\text{min}}, \eta_{\text{max}}] \)

\[
\eta_k \langle f_i(w_k) - f_i(w^*) \rangle \geq \eta_{\text{min}} \langle f_i(w_k) - f_i(w^*) \rangle - (\eta_{\text{max}} - \eta_{\text{min}}) \langle f_i(w^*) - f_i^* \rangle.
\]

Using this inequality in Eq. (11), we have that

\[
\eta_{\text{min}} \left[ \sum_{k=1}^T (f(w_k) - f(w^*)) \right] \leq \lambda_{\text{max}} \left( \|w_k - w^*\|^2 - \|w_{k+1} - w^*\|^2 \right) + (2\eta_{\text{max}} - \eta_{\text{min}}) \langle f_i(w^*) - f_i^* \rangle.
\]

Summing all iterations and taking expectations, the iterate distances telescope and we have

\[
\eta_{\text{min}} \mathbb{E} \left[ \sum_{k=1}^T (f(w_k) - f(w^*)) \right] \leq \lambda_{\text{max}} \|w_0 - w^*\|^2 + (2\eta_{\text{max}} - \eta_{\text{min}}) T \sigma^2.
\]

Dividing by \( \eta_{\text{min}} T \) and using Jensen’s inequality finishes the proof, giving the rate for the averaged iterate,

\[
\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \lambda_{\text{max}} \frac{\|w_0 - w^*\|^2}{T \eta_{\text{min}}} + \left( \frac{2\eta_{\text{max}}}{\eta_{\text{min}}} - 1 \right) \sigma^2.
\]

\[\square\]
F Proofs for general preconditioners in the strongly-convex setting

We now move to the strongly-convex setting and show a linear rate if interpolation is satisfied.

**Theorem 8.** Under the same assumptions as Proposition 3, if the overall function $f$ is $\mu$-strongly convex, Armijo SPS with $c = \frac{\sigma_{\max}}{2\sigma_{\min}}$ achieves the following rate,

$$
\mathbb{E}\left[\|w_T - w^*\|^2\right] \leq \left(1 - \mu \min\left\{\frac{\eta_{\max}}{\sigma_{\min}}, \frac{1}{\kappa^2 L_{\max}}\right\}\right)^T \|w_k - w^*\|^2 + \frac{2}{\mu} \max\left\{1, \frac{L_{\max} \eta_{\max} \kappa^2}{\sigma_{\min}}\right\} \sigma^2.
$$

**Proof of Theorem 8.** Following the proof of Proposition 3,

$$
\|w_{k+1} - w^*\|^2 = \|w_k - w^*\|^2 - 2\eta_k \left(A_k^{-1} \nabla f_i_k (w_k), w_k - w^*\right) + \eta_k^2 \left\|A_k^{-1} \nabla f_{i_k} (w_k)\right\|^2,
$$

\[ \leq \|w_k - w^*\|^2 - \frac{2}{\sigma_{\min}} \eta_k \left(\nabla f_i_k (w_k), w_k - w^*\right) + \eta_k^2 \left\| \nabla f_{i_k} (w_k)\right\|^2 A_k^{-1}, \]

Using Lemma 3 with $c = \frac{1}{2} \frac{\sigma_{\max}}{\sigma_{\min}},$

\[ \leq \|w_k - w^*\|^2 + \eta_k \left(\frac{f_i_k (w_k) - f_i_k^*}{\sigma_{\min}} - \frac{2}{\sigma_{\max}} \left(\nabla f_i_k (w_k), w_k - w^*\right)\right), \quad \text{(Lemma 3)} \]

\[ \leq \|w_k - w^*\|^2 + \frac{2}{\sigma_{\max}} \eta_k \left(f_i_k (w_k) - f_i_k^* - \nabla f_i_k (w_k), w_k - w^*\right) \leq \eta_{\min} \left( f_i_k (w_k) - f_i_k^* - \nabla f_i_k (w_k), w_k - w^* \right) + \eta_{\max} \left( f_i_k^* - f_i_k^* \right). \]

Rearranging gives the bound

\[ \|w_{k+1} - w^*\|^2 \leq \|w_k - w^*\|^2 + \frac{2}{\sigma_{\max}} \eta_k \left(f_i_k (w_k) - f_i_k^* - \nabla f_i_k (w_k), w_k - w^*\right) \]

Taking the expectation with respect to $i_k$ yields

\[ \mathbb{E}\left[\|w_{k+1} - w^*\|^2\right] \leq \|w_k - w^*\|^2 + \frac{2\eta_{\min}}{\sigma_{\max}} \left(\|f_i_k (w_k) - f_i_k^* - \nabla f_i_k (w_k), w_k - w^*\right) + \frac{2}{\sigma_{\max}} \eta_{\max} \sigma^2, \]

and the $\mu$-strong-convexity of $f$ gives $f(w_k) - f(w^*) - \langle \nabla f(w_k), w_k - w^* \rangle \leq -\frac{\mu}{2} \|w_k - w^*\|^2$; therefore,

\[ \mathbb{E}\left[\|w_{k+1} - w^*\|^2\right] \leq \left(1 - \frac{\eta_{\min}}{\sigma_{\max}} \mu\right) \|w_k - w^*\|^2 + \frac{2}{\sigma_{\max}} \eta_{\max} \sigma^2. \]

Taking complete expectations and unfolding the recursion, we have

\[ \mathbb{E}\left[\|w_T - w^*\|^2\right] \leq \left(1 - \frac{\eta_{\min}}{\sigma_{\max}} \mu\right)^T \|w_0 - w^*\|^2 + \frac{2}{\sigma_{\max}} \eta_{\max} \sigma^2 \sum_{k=0}^T \left(1 - \frac{\eta_{\min}}{\sigma_{\max}} \mu\right)^k. \]

We first show that $0 < \frac{\mu \eta_{\min}}{\sigma_{\max}} \leq 1$, so the sum multiplying $\sigma^2$ converges. From Lemma 3, $\eta_{\min} = \min\{\eta_{\max}, \sigma_{\min}/2\mu L_{\max}\}$ is a valid lower-bound for $\eta_k$. Clearly, $\eta_{\min} > 0$ as long as $\sigma_{\max} > 0$. For the upper bound, taking $\epsilon = \frac{\sigma_{\max}}{2 \sigma_{\max}}$ and noting that $\mu \leq L_{\max}$ and $\sigma_{\min} \leq 1$ yields

\[ \eta_{\min} \frac{\mu}{\sigma_{\max}} = \min\left\{\eta_{\max}, \frac{\sigma_{\min}}{2cL_{\max}}\right\} \leq \frac{\mu}{2cL_{\max}}, \]

For $0 < x \leq 1$, the sum $\sum_{k=0}^T (1 - x)^k$ is a partial geometric series, converging monotonically to $\frac{1}{x}$,

\[ \mathbb{E}\left[\|w_T - w^*\|^2\right] \leq \left(1 - \frac{\eta_{\min}}{\sigma_{\max}} \mu\right)^T \|w_0 - w^*\|^2 + \frac{1}{\eta_{\min} \mu} \eta_{\max} \frac{2}{\sigma_{\max}} \sigma^2. \]

Using that $\eta_{\min} = \min\{\eta_{\max}, \sigma_{\min}/2cL_{\max}\}$,

\[ \mathbb{E}\left[\|w_T - w^*\|^2\right] \leq \left(1 - \mu \min\left\{\frac{\eta_{\max}}{\sigma_{\min}}, \frac{1}{\kappa^2 L_{\max}}\right\}\right)^T \|w_k - w^*\|^2 + \frac{2}{\mu} \max\left\{1, \frac{L_{\max} \eta_{\max} \kappa^2}{\sigma_{\min}}\right\} \sigma^2. \]
We now focus on the convex, non-smooth setting, where instead of smoothness we assume that each function $f_i$ is at least $G$-Lipschitz, that is, the norm of the stochastic gradients are bounded,

$$\|\nabla f_i(w)\| \leq G. \quad \text{(Bounded gradient)}$$

For simplicity, we focus on the exact interpolation setting, where it is not necessary to specify an upper bound on the SPS step-sizes.

**Theorem 9.** Assuming (i) individual convexity, (ii) bounded gradients and (iii) that the preconditioners have bounded eigenvalues in the range $[a_{\min}, a_{\max}]$, if (iv) interpolation is satisfied, Armijo SPS with $c = \kappa = \frac{a_{\max}}{a_{\min}}$ and uniform averaging converges at the rate

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{\kappa G \|w_0 - w^*\|}{\sqrt{T}}.$$

**Proof of Theorem 9.** Using the expansion of the update rule and the Bounded preconditioner,

$$\|w_{k+1} - w^*\|^2 \leq \|w_k - w^*\|^2 - 2\eta_k \langle A_k^{-1}\nabla f_i(w_k), w_k - w^* \rangle + \eta_k^2 \|A_k^{-1}\nabla f_i(w_k)\|^2,$$

$$\leq \|w_k - w^*\|^2 - 2\frac{a_{\max}}{a_{\min}} \eta_k \langle \nabla f_i(w_k), w_k - w^* \rangle + \frac{\eta_k^2}{a_{\min}} \|\nabla f_i(w_k)\|_{A_k^{-1}}^2.$$  \hfill (Lemma 3)

Using the Polyak step-size guarantee (Lemma 3) with $c = \frac{a_{\max}}{a_{\min}}$,

$$\leq \|w_k - w^*\|^2 + \eta_k \left( \frac{f_{i_k}(w_k) - f_{i_k}^*}{ca_{\min}} - \frac{2}{a_{\max}} \langle \nabla f_i(w_k), w_k - w^* \rangle \right), \quad \text{and using the Bounded preconditioner and Bounded gradient assumptions,}$$

$$\leq \|w_k - w^*\|^2 + \eta_k \left( \frac{f_{i_k}(w_k) - f_{i_k}^*}{a_{\min}^2} \|\nabla f_i(w^*)\|_{A_k^{-1}}^2 \right),$$

and using the Bounded preconditioner and Bounded gradient assumptions,

$$\leq \|w_k - w^*\|^2 - \frac{2}{a_{\max}} \left( \frac{f_{i_k}(w_k) - f_{i_k}^*}{a_{\min}^2} \right) \|\nabla f_i(w^*)\|_{A_k^{-1}}^2 \leq \|w_k - w^*\|^2 - \frac{(f_{i_k}(w_k) - f_{i_k}^*)^2}{\kappa^2 G^2}.$$  \hfill (G)

Reorganizing the terms and summing over all iterations, the iterate distances telescope,

$$\sum_{k=1}^{T} (f_{i_k}(w_k) - f_{i_k}^*)^2 \leq G^2 \kappa^2 \sum_{k=1}^{T} \left( \|w_k - w^*\|^2 - \|w_{k+1} - w^*\|^2 \right) \leq G^2 \kappa^2 \|w_0 - w^*\|^2.$$  \hfill (G)

Taking expectations and dividing by $T$, we have

$$\mathbb{E} \left[ \frac{1}{T} \sum_{k=1}^{T} (f_{i_k}(w_k) - f_{i_k}^*)^2 \right] \leq \frac{G^2 \kappa^2 \|w_0 - w^*\|^2}{T}.$$  \hfill (G)

Using Jensen’s inequality multiple twice,

$$\mathbb{E} \left[ \frac{1}{T} \sum_{k=1}^{T} (f_{i_k}(w_k) - f_{i_k}^*)^2 \right] \geq \mathbb{E} \left[ \left( \frac{1}{T} \sum_{k=1}^{T} f_{i_k}(w_k) - f_{i_k}^* \right)^2 \right],$$

$$\geq \mathbb{E} \left[ \left( \frac{1}{T} \sum_{k=1}^{T} f(w_k) - f(w^*) \right)^2 \right].$$

Taking the square-root on both sides and using Jensen’s inequality again finishes the proof;

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \mathbb{E} \left[ \left( \frac{1}{T} \sum_{k=1}^{T} f(w_k) - f(w^*) \right)^2 \right] \leq \frac{G \kappa \|w_0 - w^*\|}{\sqrt{T}}.$$

□
G.1 Proofs for general preconditioners with bounded eigenvalues

We now prove the following theorem in the strongly-convex, non-smooth setting.

**Theorem 10.** Under the same assumptions as Theorem 9, if the overall function $f$ is $\mu$-strongly convex, Armijo SPS with $c = \kappa = \frac{a_{\text{max}}}{a_{\text{min}}}$ and uniform averaging converges at the rate

$$\mathbb{E}\left[\|w_k - w^*\|^2\right] \leq \frac{4\kappa^2 G^2}{\mu^2 T}.$$  

**Proof of Theorem 10.** Using the same starting derivation as Theorem 9, we have the inequality,

$$\|w_{k+1} - w^*\|^2 \leq \|w_k - w^*\|^2 - \frac{1}{\kappa^2 G^2} (f(w_k) - \tilde{f}(w_k))^2.$$  

Taking expectations w.r.t. $i_k$ and using Jensen’s inequality,

$$\mathbb{E}_i \left[\|w_{k+1} - w^*\|^2\right] \leq \mathbb{E}_i \left[\|w_k - w^*\|^2\right] - \frac{1}{\kappa^2 G^2} \mathbb{E}_i \left[(f(w_k) - \tilde{f}(w_k))^2\right],$$

$$\leq \left(1 - \frac{\mu^2}{4\kappa^2 G^2}\right) \mathbb{E}_i \left[\|w_{k+1} - w^*\|^2\right].$$

To simplify the next steps, let us multiply both sides by $\frac{\mu^2}{4\kappa^2 G^2}$ and define $\alpha_k$ such that

$$\frac{\mu^2}{4\kappa^2 G^2} \mathbb{E}_i \left[\|w_{k+1} - w^*\|^2\right] \leq \frac{\mu^2}{4\kappa^2 G^2} \mathbb{E}_i \left[\|w_k - w^*\|^2\right] \left(1 - \frac{\mu^2}{4\kappa^2 G^2} \mathbb{E}_i \left[\|w_{k+1} - w^*\|^2\right]\right),$$

and we analyze the recursion $\alpha_{k+1} \leq \alpha_k (1 - \alpha_k)$. We first show that $\alpha_1 \leq 1$, as, by strong-convexity,

$$\alpha_1 = \frac{\mu^2}{4\kappa^2 G^2} \|w_1 - w^*\|^2 \leq \frac{\mu^2}{2\kappa^2 G^2} (f(w_1) - f^*) \leq \frac{1}{\kappa^2 G^2} \|\nabla f(w_1)\|^2 \leq \frac{1}{\kappa^2} = \frac{a_{\text{max}}}{a_{\text{min}}} \leq 1.$$  

We will show that $\alpha_{k+1} \leq \alpha_k (1 - \alpha_k)$ implies that $\alpha_k \leq \frac{1}{k}$ by induction.

As shown in Fig. 4, the function $r(x) = x(1-x)$ is concave and reaches its maximum, $\frac{1}{4}$, at $x = \frac{1}{2}$. As $0 \leq \alpha_1 \leq 1$, this directly implies that $a_2 \leq \frac{1}{4}$ and the case $k = 2$ holds.

For the inductive step, we need to bound the value of $\alpha_{k+1} \leq r(\alpha_k)$ under the hypothesis that $\alpha_k \in [0, \frac{1}{k}]$. For this, we use the fact that $r(x)$ is monotonically increasing on $[0, \frac{1}{2}]$, such that $r(\alpha_k) \leq r(\frac{1}{k})$ if $k \geq 2$.

By induction, assuming $\alpha_k \leq \frac{1}{k}$ for some $k$ and using the monotonicity of $r$, $\alpha_{k+1} \leq \frac{1}{k+1}$,

$$\alpha_{k+1} \leq \max_{\alpha_k \leq \frac{1}{k}} \alpha_k (1 - \alpha_k) \leq \frac{1}{k} \left(1 - \frac{1}{k} \right) = \frac{k - 1}{k^2} = \frac{k + 1 - k - 1}{k + 1} = \frac{1}{k + 1} \frac{k^2 - 1}{k^2} < \frac{1}{k + 1}.$$  

Therefore, $\alpha_k \leq \frac{1}{k}$ for all $k$ and we have that

$$\mathbb{E}\left[\|w_T - w^*\|^2\right] \leq \frac{4\kappa^2 G^2}{\mu^2 T}.$$ 

\[\square\]
H Experimental details

Our proposed adaptive gradient methods with SLS and SPS step sizes are presented in Algorithms 1 and 3. We now make a few additional remarks on the practical use of these methods.

Algorithm 1 SLS(f, precond, conservative, mode, w₀, η_max, b, c ∈ (0, 1), β < 1)

1: for k = 0, ..., T − 1 do
2:   i_k ← sample mini-batch of size b
3:   A_k ← precond(k)  ▶ Form the preconditioner.
4:   if mode == Lipschitz then
5:     p_k ← ∇f_i_k(w_k)
6:   else if mode == Armijo then
7:     p_k ← A_k^{-1}∇f_i_k(w_k)
8:   end if
9:   if conservative then
10:      if k == 0 then
11:         η_k ← η_max
12:      else
13:         η_k ← η_k−1
14:      end if
15:   else
16:      η_k ← η_max
17:   end if
18:   while f_i_k(w_k − η_k · p_k) > f_i_k(w_k) − cη_k⟨∇f_i_k(w_k), p_k⟩ do  ▶ Line-search loop
19:      η_k ← βη_k
20:   end while
21:   w_{k+1} ← w_k − η_kA_k^{-1}∇f_i_k(w_k)
22: end for
23: return w_T

As suggested by Vaswani et al. [40], the standard backtracking search can sometimes result in step sizes that are too small while taking bigger steps can yield faster convergence. To this end, we adopted their strategies to reset the initial step size at every iteration (Algorithm 2). In particular, using reset option 0 corresponds to starting every backtracking line search from the step size used in the previous iteration. Since the backtracking never increases the step size, this option enables the “conservative step size” constraint for the Lipschitz line-search to be automatically satisfied. For the Armijo line-search, we use the heuristic from [40] corresponding to reset option 1. This option begins every backtracking with a slightly larger (by a factor of γ^{b/n}, γ = 2 throughout our experiments) step-size compared to the step-size at the previous iteration, and works well consistently across our experiments. Although we do not have theoretical guarantees for Armijo SLS with general preconditioners such as Adam, our experimental results indicate that this is in fact a promising combination that also performs well in practice.

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Algorithm 3 SPS($f, [f_i^*]_{i=1}^n, \text{precond}, \text{mode}, w_0, \eta_{\text{max}}, b, c$)

1: for $k = 0, \ldots, T - 1$ do
2: \hspace{1em} $i_k \leftarrow$ sample mini-batch of size $b$
3: \hspace{1em} $A_k \leftarrow \text{precond}(k)$  \texttt{\Comment{Form the preconditioner}}
4: \hspace{1em} if $\text{mode} == \text{Lipschitz}$ then
5: \hspace{1em} \hspace{1em} $p_k \leftarrow \nabla f_{i_k}(w_k)$
6: \hspace{1em} \hspace{1em} else if $\text{mode} == \text{Armijo}$ then
7: \hspace{1em} \hspace{1em} $p_k \leftarrow A_k^{-1} \nabla f_{i_k}(w_k)$
8: \hspace{1em} \hspace{1em} end if\texttt{\Comment{end if}}
9: \hspace{1em} \hspace{1em} $\eta_k \leftarrow \min \left\{ \frac{f_{i_k}(w_k) - f_{i_k}^*}{c(\nabla f_{i_k}(w_k), p_k)}, \eta_{\text{max}} \right\}$
10: \hspace{1em} \hspace{1em} $w_{k+1} \leftarrow w_k - \eta_k A_k^{-1} \nabla f_{i_k}(w_k)$
11: \hspace{1em} end for
12: return $w_T$

On the other hand, rather than being too conservative, the step sizes produced by SPS between successive iterations can vary wildly such that convergence becomes unstable. Loizou et al. [22] suggested to use a smoothing procedure that limits the growth of the SPS from the previous iteration to the current. We use this strategy in our experiments with $\tau = 2^{b/n}$ and show that both SPS and Armijo SPS work well. For the convex experiments, for both SLS and SPS, we set $c = 0.5$ as is suggested by the theory. For the non-convex experiments, we observe that all values of $c \in [0.1, 0.5]$ result in reasonably good performance, but use the values suggested in [22, 40], i.e. $c = 0.1$ for all adaptive methods using SLS and $c = 0.2$ for methods using SPS.
I Additional experimental results

In this section, we present additional experimental results showing the effect of the step-size for adaptive gradient methods using a synthetic dataset (Fig. 5, Fig. 6), runtime of optimization methods (Fig. 7) and the effect of batch-norm (Fig. 8) when training deep neural networks. We show the variation in the step-size for the SLS/SPS methods when training deep networks for both the CIFAR in Fig. 9 and ImageNet (Fig. 10) datasets. We evaluate these methods on easy non-convex objectives - classification on MNIST (Fig. 11) and deep matrix factorization (Fig. 13). We use deep matrix factorization to examine the effect of over-parameterization on the performance of the optimization methods. Finally, we check the methods’ performance when minimizing convex objectives associated with binary classification using RBF kernels in Fig. 12.

Figure 5: Effect of step-size on the performance of adaptive gradient methods for binary classification on a linearly separable synthetic dataset with different margins. We observe that the large variance for the adaptive gradient methods, and the variants with SLS/SPS have consistently good performance across margins and optimizers.
Figure 6: Effect of step-size on the performance of adaptive gradient methods without momentum, confirming that momentum is not responsible for the highly variable performance of these methods.
Figure 7: Runtime (in seconds/epoch) for optimization methods for multi-class classification using the deep network models in Fig. 2. Although the runtime/epoch is larger for the SLS/SPS variants, they require fewer epochs to reach the maximum test accuracy (Figure 2). This justifies the moderate increase in wall-clock time.
Figure 8: Comparing optimization methods on the image classification tasks in Figure 2 using ResNet and DenseNet models without batch-norm. We conduct this additional experiment to disentangle the effect of batch-norm. We verify that the superior performance of the SLS/SPS methods is not due to their interaction with batch-normalization. Again, we observe the superior convergence of AdaGrad + Armijo SLS.
Figure 9: Comparing optimization methods on image classification tasks using ResNet and DenseNet models on the CIFAR-10/100 datasets. For the SLS/SPS variants, refer to the experimental details in Appendix H. For Adam, we did a grid-search and use the best step-size. We use the default hyper-parameters for the other baselines. We observe the consistently good performance of AdaGrad Armijo SLS. We also show the variation in the step-size and observe a cyclic pattern [23] - an initial warmup in the learning rate followed by a decrease or saturation to a small step-size. We also note a correlation between the peak in the step-size and the test performance.
Figure 10: Comparing optimization methods on image classification tasks using variants of ImageNet. We use the same settings as the CIFAR datasets and observe that AdaGrad with Armijo SLS is consistently better.

Figure 11: Comparing optimization methods on MNIST.
Figure 12: Comparison of optimization methods on convex objectives: binary classification on LIBSVM datasets using RBF kernel mappings. The kernel bandwidths are chosen by cross-validation following the protocol in [40]. All line-search methods use $c = 1/2$ and the procedure described in Appendix H. The other methods are use their default parameters. We observe the superior convergence of the SLS/SPS variants and the poor performance of the baselines.

Figure 13: Comparison of optimization methods for deep matrix factorization. Methods use the same hyper-parameter settings as above and we examine the effects of over-parameterization on the problem: $\min_{W_1, W_2} \mathbb{E}_{x \sim N(0, I)} \|W_2 W_1 x - Ax\|_2^2$ [35, 40]. We choose $A \in \mathbb{R}^{10 \times 6}$ with condition number $\kappa(A) = 10^{10}$ and control the over-parameterization via the rank $k$ (equal to 1, 4, 10) of $W_1 \in \mathbb{R}^{k \times 6}$ and $W_2 \in \mathbb{R}^{10 \times k}$. We also compare against the true model. In each case, we use a fixed dataset of 1000 samples. We observe that as the over-parameterization increases, the performance of all methods improves, with the methods equipped with SLS performing the best.
J Experiments for second order methods

Since the proposed Armijo SPS can be used for more general preconditioners beyond those for adaptive gradient methods, we use it for setting the step-size for second order methods as well. We consider two variants, the first being Diag\text{Hessian} that uses a diagonal approximation of the subsampled Hessian at the current iterate [15]. The second variant Diag\text{Hessian} EMA uses an exponentially moving average of the diagonal approximation of the subsampled Hessians across iterations. This is similar to Adam but uses the diagonal subsampled Hessian instead of the second moment of the stochastic gradients. In both these cases, we use the recently proposed BackPACK extension [7] to easily and efficiently compute the diagonal Hessian. We use Armijo SPS with $c = 1$ to set the step size.

We compare against the stochastic sub-sampled Newton SSN method [27] that uses the full subsampled Hessian for each update. For both SSN and Diag\text{Hessian}, we tune the Levenberg-Marquardt regularization [16, 26] using cross-validation [27] with a value of $10^{-3}$. For SSN, as proposed by Meng et al. [27], we use (Armijo) SLS (with $c = 1/2$) to set the step-size. We also compare against AdaGrad with Armijo SLS (with $c = 1/2$), which is the best performing adaptive gradient method across our experiments.

![Comparison of second order methods](image)

Figure 14: Comparing the performance of second order methods on a synthetic linearly-separable dataset with varying margin. We observe that although SSN results in the best convergence in most cases, both variants using the diagonal Hessian perform better than AdaGrad.

We conclude that second-order methods with SPS/SLS have the potential to out-perform AdaGrad with SLS, the best performing adaptive gradient method. Furthermore, on real datasets, the diagonal approximation has competitive performance compared to the full, subsampled Hessian. We leave developing better line-search techniques to be used in conjunction with approximate second-order methods as future work.
Figure 15: Comparing the performance of second order methods for binary classification with RBF kernels using LIBSVM datasets [5]. We observe that SSN does not improve the performance, while both AdaGrad and diagHessian have competitive performance.