On the anti-Yetter-Drinfeld module-contramodule correspondence.

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Abstract

We study a functor from anti-Yetter Drinfeld modules to contramodules in the case of a Hopf algebra $H$. This functor is unpacked from the general machinery of [7]. Some byproducts of this investigation are the establishment of sufficient conditions for this functor to be an equivalence, verification that the center of the opposite category of $H$-comodules is equivalent to anti-Yetter Drinfeld modules in contrast to [3] where the question of $H$-modules was addressed, and the observation of two types of periodicities of the generalized Yetter-Drinfeld modules introduced in [4]. Finally, we give an example of a symmetric 2-contratrace on $H$-comodules that does not arise from an anti-Yetter Drinfeld module.

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1 Introduction.

This paper grew out of the author’s attempts to better understand contramodules at least in some simple examples. The simplest case being the Hopf algebra $kG$ where $G$ is a discrete infinite group. Contramodules over a coalgebra were introduced by Eilenberg and Moore in 1965 and can be viewed either as algebraic structures allowing infinite combinations or a better behaved notion than that of modules over the dual algebra (see Remark 4.6). They do not strictly speaking generalize comodules, but do have a non-trivial intersection with them. In our investigations we found
to be very helpful, in fact the phenomenon of this underived comodule-
contramodule correspondence without the anti-Yetter-Drinfeld enhancement
is investigated there as well.

The introduction of anti-Yetter-Drinfeld contramodule coefficients to the
Hopf-cyclic cohomology theory in [2] that followed the definition of anti-
Yetter-Drinfeld module coefficients in [3] can in retrospect be conceptually
understood as being completely natural since they are seen to be exactly
corresponding to the representable symmetric 2-contratraces, see [7] and
[5]. The latter form a well behaved class of Hopf-cyclic coefficients explored
in [4] and [7], that lead directly to Hopf-cyclic type cohomology theories.

Roughly speaking, the category of stable anti-Yetter-Drinfeld modules
consists of \( H \)-modules and comodules such that the two structures are com-
patible in a way that ensures that they form the center of a certain bimodule
category [4]. A similar statement with contramodule structure replacing the
comodule one can be made about anti-Yetter-Drinfeld contramodules. In
general understanding objects in these categories is not a simple task, how-
ever in the case of \( H = kG \) the former category is known to consist of
\( G \)-graded \( G \)-equivariant vector spaces, i.e., \( \bigoplus_{g \in G} M_g \) with \( x : M_g \to M_{g^2 x^{-1}} \).
Stability, a condition that ensures cyclicity, translates to \( x = Id_{M_g} \). We
could find no similarly simple description of the anti-Yetter-Drinfeld con-
tramodule category in the literature. It turns out, Corollary 4.10, that this
category is also equivalent to \( G \)-graded \( G \)-equivariant vector spaces but the
objects are now \( \prod_{g \in G} M_g \). The Theorem 4.5 is a more general case of this
correspondence.

The above anti-Yetter-Drinfeld module-contramodule correspondence was
the motivation for the rest of the results in this paper. Namely, the Proposition
4.7 shows that the equivalence arises from a functor \( M \mapsto \hat{M} \) from
comodules to contramodules. This functor can be found in [6] but arose in-
dependently from the considerations of [7] which furthermore demonstrate
that it works on the anti-Yetter-Drinfeld versions as well. More precisely, for
\( M \) a stable anti-Yetter-Drinfeld module we consider \( F(-) = \text{Hom}(-, M)^H \)
which is a symmetric 2-contratrace on \( H \)-comodules, i.e., a contravariant
functor from \( M^H \) to Vec subject to a trace-like symmmery. Its pullback to
the category \( _H M \) of \( H \)-modules is \( \text{Hom}_H(-, \hat{M}) \). The pullback construction
reduces in this case to the observation that \( H \in M^H \) is an algebra
and the category of \( H \)-bimodules in \( M^H \) is equivalent to \( _H M \). The pull-
back \( \text{Hom}_H(-, \hat{M}) \) is obtained as \( F_H \); i.e., the equalizer of the action maps
\( F(V) \to F(H \otimes V) \) and \( F(V) \to F(V \otimes H) \) with the targets identified via
the symmetry of $\mathcal{F}$. Though this can be used as the definition of $\hat{M}$, we give an explicit construction of both the contramodule structure, essentially agreeing with \cite{6}, and the $H$-action on $\hat{M} = \text{Hom}(H, M)^H$.

It turns out that, not surprisingly, $M \mapsto \hat{M}$ is not always an equivalence, but it does have a left adjoint, that we found in \cite{6} and upgraded to the anti-Yetter-Drinfeld setting here. The key object when studying the question of equivalence is the ideal of left integrals for $H$ as introduced in \cite{8}. This object seems to be the first example of a generalized Yetter-Drinfeld module of charge other than 1 or $-1$, its charge is 2. These were introduced in \cite{4} without any hope that anything other than $\pm 1$ would be useful. In fact, the conditions for the comodule-contramodule correspondence are closely related to the presence of a 2-periodicity of the charges, see Remark \ref{rmk:periodicity}.

Furthermore, in studying the question of stability of anti-Yetter-Drinfeld modules/contramodules and the generalization of this concept to more general charges (in a way that was necessarily different from \cite{4}) we observed a second kind of periodicity within a generalized Yetter-Drinfeld category of a fixed charge. The Remark \ref{rmk:periodicity} describes an action of $\mathbb{Z}/i\mathbb{Z}$ on Yetter-Drinfeld modules of charge $i - 1$ and Yetter-Drinfeld contramodules of charge $i + 1$. This action is compatible with the generalized $M \mapsto \hat{M}$ that sends Yetter-Drinfeld modules of charge $i - 1$ to Yetter-Drinfeld contramodules of charge $i + 1$; the case of $i = 0$ is the usual anti-Yetter-Drinfeld situation.

Identifying categories of interest with centers of bimodule categories such as was done in \cite{4} and \cite{5} is carried through this paper as well. We point to the summary Theorem \ref{thm:summary} that is of the \cite{4} flavor, and the Corollary \ref{cor:summary} of the \cite{5} flavor as examples. One of the natural questions that arose after \cite{4} was if symmetric 2-contratraces give a more general class of coefficients for Hopf-cyclic type theories even in the case of Hopf algebras. It turns out \cite{5} that for the case of $\mathcal{H} \mathcal{M}$ the representable symmetric 2-contratraces are equivalent to the anti-Yetter-Drinfeld contramodules, and similarly (Corollary \ref{cor:summary}) for the case of $\mathcal{M}^H$ the representable symmetric 2-contratraces are equivalent to the anti-Yetter-Drinfeld modules. Thus one needs only find a non-representable example of a contratrace in order to have a new coefficient in the $H$-comodule case. This is explained in Section \ref{sec:coefficients}

The paper is arranged as follows: Section \ref{sec:biclosed} is devoted to the establishment of the fact that $\mathcal{M}^H$ is biclosed, and thus it makes sense to consider the opposite bimodule category $\mathcal{M}^{H\text{op}}$ with the adjoint action. The center is shown to consist of anti-Yetter-Drinfeld modules; this identifies symmetric representable 2-contratraces with stable anti-Yetter-Drinfeld modules.
Finally we introduce the functor $M \mapsto \hat{M}$. In Section 3 we demonstrate that the adjoint pair of functors between comodules and contramodules: $N \mapsto N'$ and $M \mapsto \hat{M}$ induce the same between the anti-Yetter-Drinfeld versions and also the stable anti-Yetter-Drinfeld versions. Section 4 deals with the question of equivalence established by $M \mapsto \hat{M}$ and ends with an example of a new coefficient for the case of $H = kG$. In Section 5 we extend $M \mapsto \hat{M}$ to the generalized Yetter-Drinfeld modules and discuss two types of periodicities and the compatibility of $M \mapsto \hat{M}$ with them.

Some things to keep in mind: for a coalgebra $C$ we use the following version of Sweedler notation $\Delta(c) = c^1 \otimes c^2$. For a right comodule $N$ over $C$ we use $\rho(n) = n_0 \otimes n_1$. All Hopf algebras have invertible antipodes $S$ and are over a field $k$ of characteristic 0. We denote by $\text{Hom}(-,-)^H$ and $\text{Hom}_H(-,-)$ the morphisms in $\mathcal{M}^H$ and $H\mathcal{M}$ respectively, while $\text{Hom}(-,-)$ stands for $k$-linear maps.

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# 2 The category of $H$-comodules.

This section is primarily dedicated to the establishment of the fact that the monoidal category $\mathcal{M}^H$ of $H$-comodules is biclosed, and the analysis of the center of the bimodule category over $\mathcal{M}^H$ resulting from considering $\mathcal{M}^{H\text{op}}$. This establishes an analogue of a result in [5] describing the center as the category of $aYD$-modules for $H$.

## 2.1 Internal Homs in the category of $H$-comodules.

Motivated by the existence of internal Homs in $H\mathcal{M}$, and thus the possibility of describing representable contratraces on $H\mathcal{M}$ as central elements in the opposite category, we will now address the same question in $\mathcal{M}^H$, the monoidal category of $H$-comodules.

Since in the finite dimensional $H$ case, we have that $\mathcal{M}^H \simeq H\mathcal{M}$ so we have a suggestive way of obtaining the required formulas. We note that some modifications do need to be made to account for possible infinite dimensionality of $H$. 
For $W, V \in \mathcal{M}^H$ consider $\rho : \text{Hom}(W, V) \to \text{Hom}(W, V \otimes H)$ given by

$$\rho f(w) = f(w_0) \otimes f(w_0) S(w_1).$$  \hfill (2.1)

**Definition 2.1.** Define $\text{Hom}^l(W, V)$ as the subspace of $\text{Hom}(W, V)$ that consists of $f$ such that $\rho f \in \text{Hom}(W, V) \otimes H$.

We can define two maps

\[
\text{``Id} \otimes \Delta'' = (\text{Id} \otimes \Delta) \circ -
\]

and

\[
\text{``}\rho \otimes \text{Id}'' = (\text{Id}_V \otimes m \otimes \text{Id}_H) \circ (\text{Id}_V \otimes \sigma_{H,H}) \circ (\rho_V \otimes \text{Id}_{H \otimes H}) \circ (f \otimes \text{S}) \circ \rho_W
\]

from $\text{Hom}(W, V \otimes H)$ to $\text{Hom}(W, V \otimes H \otimes H)$. The latter can be written down more manageably as follows: let $f(w) = w^{(1)} \otimes w^{(2)}$ then

\[
\text{``}\rho \otimes \text{Id}''(f)(w) = ((w_0^{(1)})_0 \otimes ((w_0^{(1)})_1 S(w_1) \otimes (w_0^{(2)}).
\]

A direct computation shows that

\[
\text{``Id} \otimes \Delta'' \circ \rho = \text{``}\rho \otimes \text{Id}'' \circ \rho.
\]  \hfill (2.2)

Note that when restricted to $\text{Hom}(W, V) \otimes H$ the maps $\text{``Id} \otimes \Delta''$ and $\text{``}\rho \otimes \text{Id}''$ are actually $\text{Id} \otimes \Delta$ and $\rho \otimes \text{Id}$ respectively. The formula (2.2) has two important and immediate consequences: $\rho : \text{Hom}^l(W, V) \to \text{Hom}^l(W, V) \otimes H$, whereas before we only knew that it lands in $\text{Hom}(W, V) \otimes H$, and $\rho$ is a coaction.

It is not hard to see that $\text{Hom}^l(W, V)$ is contravariant in $W$ and covariant in $V$. More precisely, let $\phi \in \text{Hom}(W', W)^H$ and $\theta \in \text{Hom}(V, V')^H$, then the following diagram commutes:

\[
\begin{array}{ccc}
\text{Hom}(W, V) & \xrightarrow{\theta \circ - \circ \phi} & \text{Hom}(W', V') \\
\downarrow{\rho} & & \downarrow{\rho} \\
\text{Hom}(W, V \otimes H) & \xrightarrow{(\theta \otimes \text{Id}) \circ - \circ \phi} & \text{Hom}(W', V' \otimes H)
\end{array}
\]

and so $\theta \circ - \circ \phi : \text{Hom}^l(W, V) \to \text{Hom}^l(W', V')$ and it is a map of $H$-comodules.
Lemma 2.2. We have natural identifications

\[ \text{Hom}(T \otimes W, V)^H \simeq \text{Hom}(T, \text{Hom}^l(W, V))^H, \]

i.e., Hom^l(\(-, -\)) is the left internal Hom in \( \mathcal{M}^H \).

Proof. Note that \( f \in \text{Hom}(T \otimes W, V)^H \) if and only if

\[ f(t \otimes w)_0 \otimes f(t \otimes w)_1 = f(t_0 \otimes w_0) \otimes t_1 w_1. \]  \hfill (2.3)

On the other hand \( \phi \in \text{Hom}(T, \text{Hom}^l(W, V))^H \) if and only if

\[ \rho \phi_t = \phi_{t_0} \otimes t_1 \]  \hfill (2.4)

where \( \phi_t \in \text{Hom}^l(W, V) \).

Let \( f \in \text{Hom}(T \otimes W, V)^H \) then if we define \( f_t(w) = f(t \otimes w) \) we have

\[
\rho f_t(w) = f(t \otimes w)_0 \otimes f(t \otimes w)_1 S(w_1) \\
= f(t_0 \otimes w_0,0) \otimes t_1 w_{0,1} S(w_1) \\
= f(t_0 \otimes w_0,0) \otimes t_1 w_1 S(w_2) \\
= f(t_0 \otimes w) \otimes t_1 \\
= f_{t_0} \otimes t_1.
\]

So that \( t \mapsto f_t \in \text{Hom}(T, \text{Hom}^l(W, V))^H \). Conversely, if \( \phi \in \text{Hom}(T, \text{Hom}^l(W, V))^H \) then define \( \phi(t \otimes w) = \phi_t(w) \) then \( \phi(t \otimes w) \otimes t_1 = \phi(t \otimes w)_0 \otimes \phi(t \otimes w)_1 S(w_1) \) so that

\[
\phi(t_0 \otimes w_0) \otimes t_1 w_1 = \phi(t \otimes w_0,0) \otimes \phi(t \otimes w_0,0) S(w_{0,1}) w_1 \\
= \phi(t \otimes w_0) \otimes \phi(t \otimes w_0) S(w_1) w_2 \\
= \phi(t \otimes w)_0 \otimes \phi(t \otimes w)_1
\]

and thus \( t \otimes w \mapsto \phi_t(w) \in \text{Hom}(T \otimes W, V)^H \).

So the usual bijection \( f(t \otimes w) = f_t(w) \) establishes a natural identification between \( \text{Hom}(T \otimes W, V)^H \) and \( \text{Hom}(T, \text{Hom}^l(W, V))^H \) as required. \( \square \)

Remark 2.3. From now on we will denote the coaction \( \rho \) of (2.1) by \( \rho^l \) since

\[ \rho^l : \text{Hom}^l(W, V) \rightarrow \text{Hom}^l(W, V) \otimes H. \]
Remark 2.4. Note that we have a natural fully faithful embedding of $\mathcal{M}^H$ into $H^*\mathcal{M}$. The right adjoint to it can be used to define $\text{Hom}^l(W,V)$. Namely, the formula (2.1) defines a left $H^*$-module structure on $\text{Hom}(W,V)$ via $\chi f = (\text{Id}_V \otimes \chi)(\rho f)$. Then it is easy to see that

\[ \text{Hom}(W,V)^{\text{rat}} = \text{Hom}^l(W,V) \]

where $(-)^{\text{rat}}$ is the right adjoint that features prominently in [8].

Repeating the above considerations nearly verbatim, we define the right internal Hom for $\mathcal{M}^H$ as follows. Begin by defining

\[ \rho^r f(w) = f(w_0) \otimes S^{-1}(w_1)f(w_0) \]  

(2.5)

Definition 2.5. Define $\text{Hom}^r(W,V)$ as the subspace of $\text{Hom}(W,V)$ that consists of $f$ such that $\rho^r f \in \text{Hom}(W,V) \otimes H$.

We again obtain that $\rho^r : \text{Hom}^r(W,V) \to \text{Hom}^r(W,V) \otimes H$ and is a coaction. Furthermore, we have natural adjunctions:

\[ \text{Hom}(W \otimes T, V)^H \simeq \text{Hom}(T, \text{Hom}^r(W,V))^H \]

As usual we now have the opposite category $\mathcal{M}^{H_{\text{op}}}$ with

\[ V \triangleleft W = \text{Hom}^r(W,V), \quad \text{and} \quad W \triangleright V = \text{Hom}^l(W,V) \]

and we may examine its center $Z_{\mathcal{M}^H}(\mathcal{M}^{H_{\text{op}}})$.

Remark 2.6. We observe that if $V \in \mathcal{M}^H$ is finite dimensional then

\[ \text{Hom}^l(V,W) = W \otimes V^* \]

and

\[ \text{Hom}^r(V,W) = *V \otimes W \]

where $V^* = \text{Hom}^l(V,k)$ and $*V = \text{Hom}^r(V,k)$ and both $V^*$ and $*V$ are $\text{Hom}_k(V,k)$ as vector spaces. So that $\mathcal{M}^{H_{\text{fd}}}$ is rigid with

\[ *^*V = V^{S^{-2}} \quad \text{and} \quad V^{**} = V^{S^2}, \]

where $V^{S^{2i}}$ denotes the $H$-comodule with the coaction modified by $S^{2i}$. 

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2.2 The center of the opposite bimodule category.

We recall from [3] that a left-right anti-Yetter-Drinfeld module $M$ over a Hopf algebra $H$ is a left $H$-module and a right $H$-comodule satisfying

$$(hm)_0 \otimes (hm)_1 = h^2 m_0 \otimes h^3 m_1 S(h^1).$$

It is stable if $m_1 m_0 = m$ for all $m \in M$.

Recall, for example from [7], the notions of the opposite bimodule category and of the center of a bimodule category.

Proposition 2.7. The category of $aYD$-modules for $H$ is equivalent to $Z_{\mathcal{M}^H}(\mathcal{M}^H_{\text{op}})$.

Proof. Let $M$ be an $aYD$-module and define the central structure

$$\tau : \text{Hom}(W, M) \to \text{Hom}(W, M)$$

$$\tau f(w) = w_1 f(w_0).$$

Note that $\tau$ is invertible with $\tau^{-1} f(w) = S^{-1}(w_1) f(w_0)$. Define a map $"\tau \otimes Id" : \text{Hom}(W, M \otimes H) \to \text{Hom}(W, M \otimes H)$ by

$"\tau \otimes Id" = (a \otimes Id) \circ \sigma_{M,H,H} \circ (f \otimes Id) \circ \rho_W$

so that if $f(w) = w^{(1)} \otimes w^{(2)}$ then

$"\tau \otimes Id"(f)(w) = w_1 (w_0^{(1)} \otimes (w_0)^{(2)}.$

A direct computation (using the $aYD$ condition \[2.6\]) demonstrates that the following diagram commutes:

$$\begin{array}{ccc}
\text{Hom}(W, M) & \xrightarrow{\tau} & \text{Hom}(W, M) \\
\downarrow{\rho} & & \downarrow{\rho'} \\
\text{Hom}(W, M \otimes H) & \xrightarrow{"\tau \otimes Id"} & \text{Hom}(W, M \otimes H)
\end{array} \tag{2.7}$$

and since $"\tau \otimes Id"$ restricted to $\text{Hom}(W, M) \otimes H$ is actually $\tau \otimes Id$, so in fact $\tau : \text{Hom}^l(W, M) \to \text{Hom}^r(W, M)$ and it is an isomorphism in $\mathcal{M}^H$.

Observe that $\tau$ is natural in $W$, i.e., if $\phi : W' \to W$ is a morphism in $\mathcal{M}^H$ then $\phi(w_0) \otimes w_1 = \phi(w_0) \otimes \phi(w)$ so that $w_1 f(\phi(w_0)) = \phi(w_1) f(\phi(w_0))$ and $\tau(f \circ \phi) = \tau f \circ \phi$. 

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If \( \theta : M \to M' \) is a map of \( aYD \)-modules, then
\[
\theta \circ \tau f(w) = \theta(w_1 f(w_0)) = w_1 \theta f(w_0) = \tau(\theta \circ f)(w)
\]
so that a map of \( aYD \)-modules induces a map of central elements.

To check the commutativity of
\[
W \triangleright (V \triangleright M) \xrightarrow{\text{id} \triangleright \tau} W \triangleright (M \triangleleft V) \xrightarrow{\tau \circ \text{id}} (M \triangleleft W) \triangleleft V
\]
(2.8)
is to check that going along the bottom and obtaining \( w \otimes v \mapsto (w_1 v_1) f(w_0 \otimes v_0) \) is the same as the long way around which gives \( w \otimes v \mapsto w_1 (v_1 f(w_0)(v_0)) = w_1 f(w_0 \otimes v_0) \); and they are the same by the usual \( H \)-action axiom. Similarly, the unitality of the action implies that \( k \triangleright M \to M \triangleleft k \) is the identity since \( \tau : m \mapsto 1 m \).

What has been shown so far is that if \( M \) is an \( aYD \)-module, then \( (M, \tau) \in Z_M H(M, m) \) and any \( \theta : M \to M' \) a morphism of \( aYD \)-modules induces a morphism between the corresponding central elements.

Conversely, let \( M \in Z_M H(M, m) \) so that we have natural isomorphisms \( \tau : \text{Hom}^l(W, M) \to \text{Hom}^r(W, M) \). Note that
\[
\text{Hom}(W, M) = \lim_{\alpha} \text{Hom}^l(W_\alpha, M)
\]
where \( W_\alpha \subset W \) is a finite dimensional sub-comodule since any \( w \in W \) is contained in such an \( W_\alpha \) and so
\[
\text{Hom}(W, M) = \text{Hom}(\lim_{\alpha} W_\alpha, M) = \lim_{\alpha} \text{Hom}(W_\alpha, M) = \lim_{\alpha} \text{Hom}^l(W_\alpha, M).
\]

So we have a \( \tau : \text{Hom}(W, M) \to \text{Hom}(W, M) \) that satisfies a version of all the properties that make the original \( \tau \) so useful. Denote by \( r \) the composition
\[
M \to \text{Hom}(H, M) \to \text{Hom}(H, M)
\]
so that \( m \mapsto \tau(h \mapsto \epsilon(h)m) \). Define
\[
hm = r_m(h).
\]
(2.9)
Note that we needed to use \( \text{Hom}(H, M) \) instead of \( \text{Hom}^l(H, M) \) since \( h \mapsto \epsilon(h)m \) is not in \( \text{Hom}^l(H, M) \).
By the unitality of \( \tau \) we have \( ev_1 \circ \tau = ev_1 \) so that

\[
1m = r_m(1) = ev_1\tau(h \mapsto \epsilon(h)m) = ev_1(h \mapsto \epsilon(h)m) = \epsilon(1)m = m.
\]

Furthermore by the “associativity” of \( \tau \), i.e., the diagram \((2.8)\) and its naturality, we have

\[
(xy)_m = r_m(xy)
= \tau(h \mapsto \epsilon(h)m)(xy)
= \tau(h \otimes h' \mapsto \epsilon(hh')m)(x \otimes y)
= \tau(h \mapsto \tau(h' \mapsto \epsilon(hh')m)(y))(x)
= \tau(h \mapsto \epsilon(h)\tau(h' \mapsto \epsilon(h')m)(y))(x)
= \tau(h \mapsto \epsilon(h)r_m(y))(x)
= r_{r_m(y)}(x)
= x(ym).
\]

Let \( \theta : M \to M' \) be a map in the center, then we have

\[
\begin{array}{ccc}
M & \xrightarrow{-\circ \epsilon} & \text{Hom}(H, M) \\
\downarrow{\phi} & & \downarrow{\phi_{\circ \epsilon}} \\
M' & \xrightarrow{-\circ \epsilon} & \text{Hom}(H, M')
\end{array}
\]

where the left square commutes trivially and the right one commutes by definition, so that

\[
\theta(hm) = \theta(r_m(h)) = r_{\theta(m)}(h) = h\theta(m).
\]

Before proving that the \( H \)-action defined above satisfies the \( aYD \)-module condition \((2.6)\) we will show that the definition of the action from \( \tau \) and vice versa are mutually inverse. Let an \( H \)-action be given, then we set \( \tau f(w) = w_1f(w_0) \) so that the action becomes

\[
r_m(h) = \tau(x \mapsto \epsilon(x)m)(h) = h^2\epsilon(h^1)m = hm,
\]

i.e., the original action. On the other hand if \( \tau : \text{Hom}(W, M) \to \text{Hom}(W, M) \) is given and we defined the action by \( hm = r_m(h) = \tau(x \mapsto \epsilon(x)m)(h) \), then we obtain the following. Let \( f \in \text{Hom}(W, M) \), consider \( f \otimes \epsilon \in \text{Hom}(W \otimes H, M) \) where \( W \) is a trivial \( H \)-comodule. Note that the co-action map \( \rho_W \).
is a morphism in $\mathcal{M}^H$ from $W$ to $W \otimes H$. So $\tau(f \otimes \epsilon \circ \rho_W) = \tau(f \otimes \epsilon) \circ \rho_W$ and the former is $\tau f$ while the latter is
\[
\tau(f \otimes \epsilon \circ \rho_W)(w_0 \otimes w_1) = \tau(f \otimes \epsilon)(w_0)(w_1).
\]
and since the coaction of $H$ on $W$ is trivial so
\[
\tau(h \mapsto \epsilon(h)f(w_0))(w_1) = w_1f(w_0).
\]
So that no matter if we start with a $\tau$ or an $H$-action, we always have
\[
\tau f = w_1f(w_0).
\] (2.10)

Now recall the diagram (2.7), and note that it now commutes essentially by definition. Let $W = H$ and keep in mind the formula (2.11). We now get that for any $f \in \text{Hom}(H, M)$ we have
\[
h^3f(h^1)_0 \otimes f(h^1)_1S(h^2) = (h^2f(h^1))_0 \otimes S^{-1}(h^3)(h^2f(h^1))_1
\]
and let us apply it to $f(h) = \epsilon(h)m$ to obtain
\[
h^2m_0 \otimes m_1S(h^1) = (h^1m)_0 \otimes S^{-1}(h^2)(h^1m)_1
\]
so that
\[
h^2m_0 \otimes h^3m_1S(h^1) = (h^1m)_0 \otimes h^3S^{-1}(h^2)(h^1m)_1
\]
\[
= (h^1m)_0 \otimes \epsilon(h^2)(h^1m)_1
\]
\[
= (hm)_0 \otimes (hm)_1
\]
and $M$ satisfies the aYD-module condition (2.6).

Recall that we denote by $Z'_{\mathcal{M}^H}(\mathcal{M}^{H_{op}})$ the full subcategory that consists of objects such that the identity map $Id \in \text{Hom}(M, M)^H$ is mapped to itself via
\[
\text{Hom}(M, M)^H \simeq \text{Hom}(1, M \triangleright M)^H \simeq \text{Hom}(1, M \triangleleft M)^H \simeq \text{Hom}(M, M)^H.
\] (2.11)
We have a straightforward corollary:
Corollary 2.8. The category of $saYD$-modules for $H$ is equivalent to $Z'_{\mathcal{M}H}(\mathcal{M}^{Hop})$.

Proof. Recall that an $aYD$-module $M$ is stable if $m_1m_0 = m$ for all $m \in M$. On the other hand considering $\tau : \text{Hom}(M, M) \to \text{Hom}(M, M)$ we see that according to (2.10) we have $\tau Id(m) = m_1m_0$ and so $\tau Id = Id$ if and only if $M$ is stable.

Thus we have established the following:

Corollary 2.9. The category of $saYD$-modules for $H$ is equivalent to the category of representable symmetric 2-contratraces on $\mathcal{M}^H$ via

$$M \leftrightarrow \text{Hom}(-, M)^H.$$ 

Contrast that with the $H\mathcal{M}$ case considered in [5] where the category of representable symmetric 2-contratraces is equivalent to the more unusual $saYD$-contramodules.

2.3 A functor from $(s)aYD$-modules to $(s)aYD$-contramodules

This section is motivated by the adjunction on cyclic cohomology of [7] that we explain below. Given an $saYD$-module $M$, i.e., a representable symmetric 2-contratrace $\text{Hom}(-, M)^H$, as a special case of the theory developed in [7], we obtain an $H$-module $\hat{M}$ such that $\text{Hom}_H(-, \hat{M})$ is a representable symmetric 2-contratrace.

We will need to recall from [2] that a right $C$-contramodule $N$, where $C$ is a counital coassociative coalgebra, is equipped with the contraaction

$$\alpha : \text{Hom}(C, N) \to N$$

satisfying

$$\alpha(x \mapsto \alpha(h \mapsto f(x \otimes h))) = \alpha(h \mapsto f(h^1 \otimes h^2))$$

(2.12)

for any $f \in \text{Hom}(C \otimes C, N)$ and

$$\alpha(h \mapsto \epsilon(h)n) = n$$

(2.13)

for any $n \in N$. Furthermore, a left-right $aYD$-contramodule $N$ is a left $H$-module and a right $H$-contramodule such that for all $h \in H$ and any linear map $f \in \text{Hom}(H, N)$ we have

$$h\alpha(f) = \alpha(h^2 f(S(h^3) - h^1)).$$

(2.14)
It is called stable, i.e., an \(saYD\)-contramodule, if for all \(n \in \mathbb{N}\) we have \(\alpha(r_n) = n\) where \(r_n(h) = hn\).

We will also recall the definitions from [7]: if \(M\) is an \(aYD\)-module then

\[
\widehat{M} = \text{Hom}(H, M)^H
\]

has a left \(H\)-action via

\[
h \cdot \phi(-) = h^2 \phi(S(h^3) - h^1)
\]

and furthermore \(\widehat{M}\) has a contraaction \(\alpha : \text{Hom}(H, \widehat{M}) \to \widehat{M}\) defined as follows. Let \(\theta \in \text{Hom}(H, \widehat{M})\) be viewed as \(h \mapsto \theta_h(-)\) then

\[
\alpha\theta(h) = \theta_{h^1}(h^2).
\]

It is not hard to check all these statements directly (note that the \(aYD\)-module condition for \(M\) is only used to ensure that the action (2.15) preserves the \(H\)-comodule morphisms inside \(\text{Hom}(H, M)\)), and most importantly we can also check that \(\alpha\) is compatible with the action in the \(aYD\)-contramodule sense, i.e., the identity (2.14) holds.

The constructions above describe a functor

\[
M \mapsto \widehat{M}
\]

(2.17)

from \((s)aYD\)-modules to \((s)aYD\)-contramodules. Furthermore, the functor (2.17) is a special case of the pullback of contratrices [7] and so we have the following:

**Proposition 2.10.** Given an \(H\)-module algebra \(A\) and a \(saYD\)-module \(M\), we have an isomorphism of cyclic cohomologies:

\[
\widehat{HC}_H^\cdot(A, \widehat{M}) \simeq HC_{n,H}^\cdot(A \rtimes H, M)
\]

where the theories considered are of the derived type.

We denote by \(\widehat{HC}_H^\cdot(A, \widehat{M})\) the cyclic cohomology obtained from an algebra \(A\) and a \(saYD\)-contramodule \(\widehat{M}\) via the associated representable symmetric 2-contratrace \(\text{Hom}_H(-, \widehat{M})\) on \(\widehat{M}\), while \(HC_{n,H}^\cdot(A \rtimes H, M)\) denotes the Hopf-cyclic cohomology of an \(H\)-comodule algebra \(A \rtimes H\) with coefficients in a \(saYD\)-module \(M\) obtained from the representable symmetric 2-contratrace \(\text{Hom}(-, M)^H\) on \(M^H\).
Remark 2.11. In light of the Corollary 2.9 that shows the equivalence between $saYD$-modules and representable symmetric 2-contratrices on $\mathcal{M}^H$ and $[5]$ where a similar result is demonstrated for $saYD$-contramodules and $H\mathcal{M}$, the Proposition 2.10 is a concrete realization of the pullback of representable contratrices of $[7]$.

3 An adjoint pair of functors.

We will now analyze the functor $M \mapsto \hat{M}$ with a view towards establishing some sufficient conditions for it being an equivalence. Consider the category $\mathcal{M}^H$ of right $H$-comodules and we are interested in comparing it to the category $\hat{\mathcal{M}}^H$ of right $H$-contramodules. It turns out that the functor $M \mapsto \hat{M}$, that appeared in $[7]$ motivated by the pullback of contratrices has already appeared in the literature on comodule-contramodule correspondences $[6]$, but considered without the extra $H$-module structure that we need. We will abuse notation somewhat and not usually distinguish between $\hat{(-)} : \mathcal{M}^H \rightarrow \hat{\mathcal{M}}^H$ of $[6]$ and the upgraded version of $[7]$ mentioned above (2.17). When we do want to emphasize the difference, the latter will be denoted by $\hat{(-)}_H$.

Furthermore, $\hat{(-)}$ has a left adjoint $[6]$ $N \mapsto N'$

$$N \mapsto N'$$

(3.1)

where $N' = H \odot_H N$ is the cokernel of the difference between the maps $Id \otimes \alpha$ and $h \otimes f \mapsto h^2 \otimes f(h^1)$ between $H \otimes_k \text{Hom}(H, N)$ and $H \otimes_k N$:

$$H \otimes_k \text{Hom}(H, N) \rightarrow H \otimes_k N \rightarrow H \odot_H N \rightarrow 0.$$  

The comodule structure on $N'$ is given by

$$(h \otimes n)_0 \otimes (h \otimes n)_1 = (h^1 \otimes n) \otimes h^2.$$  

(3.2)

When $H$ is finite dimensional then $N' = H \odot_H N$ so that the notation $\odot_H$ is a bit misleading.

The adjunctions are

$$H \odot_H \text{Hom}(H, M)^H \rightarrow M$$

$$h \otimes f \mapsto f(h)$$

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and
\[
N \rightarrow \text{Hom}(H, H \odot_H N)^H
\]
\[
n \mapsto \{ h \mapsto h \odot n \}.
\]

**Remark 3.1.** Just as the functor \( M \mapsto \hat{M} \) was upgraded from the functor between comodules and contramodules to a functor between \( aYD \)-modules and \( aYD \)-contramodules by converting an \( H \)-action on \( M \) to an \( H \)-action on \( \hat{M} \), we can do the same to its left adjoint directly. Namely, define an \( H \)-action on \( H \otimes_k N \) via
\[
x \cdot (h \otimes n) = x^3 h S(x^1) \otimes x^2 n
\]
then one can check that if \( N \) is an \( aYD \)-contramodule, then the action is well defined on the cokernel \( H \odot_H N \) and gives \( N' \) the \( aYD \)-module structure.

We will now conceptually investigate if the adjoint pair of the functors above is compatible with the extra structure that we require. More precisely, \( \mathcal{M}^H \) is a tensor category in the usual way with
\[
\rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1
\]
for \( m \otimes n \in M \otimes N \) with \( M, N \in \mathcal{M}^H \). Thus \( \mathcal{M}^H \) is a bimodule category over itself.

On the other hand if \( N \in \hat{\mathcal{M}}^H \) and \( T, L \in \mathcal{M}^H_{fd} \), i.e., \( T \) is a finite dimensional \( H \)-comodule, then we can define a natural contramodule structure on both \( N \otimes T \) and \( T \otimes N \). Namely, due to the finite dimensionality of \( T \), we represent elements of \( \text{Hom}(H, N \otimes T) \) by \( f \otimes t \) with \( f \in \text{Hom}(H, N) \) and \( t \in T \), then
\[
\alpha_{N \otimes T}(f \otimes t) = \alpha_N(f(-t_1)) \otimes t_0
\]
and similarly
\[
\alpha_{T \otimes N}(t \otimes f) = t_0 \otimes \alpha_N(f(t_1-))
\]
which makes \( \hat{\mathcal{M}}^H \) into a bimodule category over \( \mathcal{M}^H_{fd} \).

The following is the key technical result of this section. It describes the exact nature of the compatibility of \( M \mapsto \hat{M} \) with the \( \mathcal{M}^H_{fd} \)-bimodule structure on both sides.

**Proposition 3.2.** Let \( W \in \mathcal{M}^H \) and \( T, L \in \mathcal{M}^H_{fd} \) then we have:
\[
\text{Hom}(H, T \otimes W \otimes L)^H \simeq T^{S^2} \otimes \text{Hom}(H, W)^H \otimes L^{S^{-2}}
\]
\[
t \otimes f \otimes l \mapsto t_0 \otimes f(S^2(t_1) - S^{-2}(l_1)) \otimes l_0
\]
a natural isomorphism in \( \hat{\mathcal{M}}^H \).
Proof. Recall that $\text{Hom}^L(H, W)$ has a left $H^*$-action and a right $H$-contraaction and they commute. Namely,

$$(\chi \cdot f)(h) = f(h^1)0\chi(f(h^1))S(h^2)$$

and

$$\alpha(h \mapsto \theta_h(\cdot)) = \{ h \mapsto \theta_{h^1}(h^2) \}.$$ 

One quickly checks that the map

$$\text{Hom}^L(H, T \otimes W) \to \overline{T} \otimes \text{Hom}^L(H, W)$$

$$t \otimes f \mapsto t \otimes f$$ (3.6)

is an isomorphism of both $H^*$-modules and $H$-contramodules, where $\overline{T}$ has the usual $H^*$ structure, but is considered trivial for the purposes of defining the $H$-contraaction on the right hand side.

On the other hand

$$\text{Hom}^L(H, W) \otimes \overline{T} \to \overline{T} \otimes \text{Hom}^L(H, W)$$

$$f \otimes t \mapsto t_0 \otimes f(t_1)$$ (3.7)

is also an isomorphism of both structures where $\overline{T}$ has trivial $H^*$ structure but non-trivially modifies the contraaction on the right hand side.

So as $H$-contramodules we have:

$$\text{Hom}(H, T \otimes W)^H \simeq \text{Hom}_{H^*}(k, \text{Hom}^L(H, T \otimes W))$$

$$\simeq \text{Hom}_{H^*}(k, \overline{T} \otimes \text{Hom}^L(H, W))$$

$$\simeq \text{Hom}_{H^*}(k, \text{Hom}^L(H, W) \otimes \overline{T^{S^2}})$$

$$\simeq \text{Hom}_{H^*}(k, \overline{T^{S^2}} \otimes \text{Hom}^L(H, W))$$

$$\simeq T^{S^2} \otimes \text{Hom}(H, W)^H$$

where $\text{Hom}_{H^*}(k, T \otimes V) \simeq \text{Hom}_{H^*}(k, V \otimes T^{S^2})$ is due to the rigidity of $\mathcal{M}^H_{fd}$ and the isomorphism $T^{**} \simeq T^{S^2}$.

Analogously we have:

$$\text{Hom}(H, W \otimes L)^H \simeq \text{Hom}_{H^*}(k, \text{Hom}^R(H, W \otimes L))$$

$$\simeq \text{Hom}_{H^*}(k, \text{Hom}^R(H, W) \otimes \overline{L})$$

$$\simeq \text{Hom}_{H^*}(k, \overline{L^{S^2}} \otimes \text{Hom}^R(H, W))$$

$$\simeq \text{Hom}_{H^*}(k, \text{Hom}^R(H, W) \otimes \overline{L^{S^2}})$$

$$\simeq \text{Hom}(H, W)^H \otimes L^{S^2}.$$

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In the latter we have used the analogues of (3.6) and (3.7); namely the isomorphisms:

$$\text{Hom}^R(H, W \otimes L) \to \text{Hom}^R(H, W) \otimes \mathcal{T}$$

$$f \otimes l \mapsto f \otimes l$$

and

$$\mathcal{T} \otimes \text{Hom}^R(H, W) \to \text{Hom}^R(H, W) \otimes L$$

$$l \otimes f \mapsto f(-l_1) \otimes l_0.$$ 

The result now follows after tracing through the isomorphisms. 

Denote by $\mathcal{M}^H\#$ the $\mathcal{M}_{fd}^H$ bimodule category with

$$T \cdot M \cdot L = T \otimes M \otimes L$$

and by $\hat{\mathcal{M}}^H$ the $\mathcal{M}_{fd}^H$ bimodule category with

$$T \cdot N \cdot L = T \otimes N \otimes L$$

then we immediately obtain the following as a Corollary of Proposition 3.2:

**Corollary 3.3.** The functors

$$\hat{(-)} : \mathcal{M}^H\# \to \hat{\mathcal{M}}^H$$

and

$$(-)': \hat{\mathcal{M}}^H \to \mathcal{M}^H\#$$

are bimodule functors over $\mathcal{M}_{fd}^H$ and so induce functors between the corresponding centers of bimodule categories.

**Proof.** The claim about $\hat{(-)}$ is immediate from Proposition 3.2. Since $\mathcal{M}_{fd}^H$ is rigid, the statement about $(-)'$ follows from the one about $\hat{(-)}$ through adjunction juggling, since they are adjoint functors.

**Remark 3.4.** The adjunction manipulations mentioned in the proof of Corollary 3.3 can be traced through to obtain an explicit analogue of Proposition 3.2 for the functor $N \mapsto N'$. Namely, for $N \in \mathcal{M}^H$ and $T, L \in \mathcal{M}_{fd}^H$ we have a natural isomorphism in $\mathcal{M}^H$:

$$H \otimes (T \otimes N \otimes L) \simeq T \otimes (H \otimes N) \otimes L$$

$$h \otimes t \otimes n \otimes l \mapsto t_0 \otimes S^{-1}(t_1)hS(l_1) \otimes n \otimes l_0.$$
As in [4], we have central interpretations of $aYD$ objects.

**Lemma 3.5.** The center of $\mathcal{M}^{H\#}$ is equivalent to the category of anti-Yetter-Drinfeld modules, namely

$$Z_{\mathcal{M}^{H\#}}(\mathcal{M}^{H\#}) \simeq aYD\text{-}mod.$$ 

**Proof.** The proof proceeds very much like that of Proposition 2.7 and so we provide only a sketch. Let $M \in H\mathcal{M}^H$, i.e., it is both a left module and a right comodule, and let $T \in \mathcal{M}^H_{fd}$. Consider the map

$$\tau : T \otimes M \to M \otimes T S^2$$

$$t \otimes m \mapsto t_1 m \otimes t_0.$$ 

It is an isomorphism with inverse $m \otimes t \mapsto t_0 \otimes S(t_1)m$. It is a map in $\mathcal{M}^H$ if and only if $M \in aYD\text{-}mod$. It is immediate that $(M, \tau) \in Z_{\mathcal{M}^{H\#}}(\mathcal{M}^{H\#})$.

Conversely, let $M \in \mathcal{M}^H$ such that we have natural isomorphisms $\tau_T : T \otimes M \to M \otimes T S^2$ for all $T \in \mathcal{M}^H_{fd}$. Now proceed in a by now familiar fashion. We need an action $H \otimes M \to M$ which we obtain via a limit over finite dimensional subcoalgebras $C \subset H$, i.e.,

$$\text{Hom}(H \otimes M, M) = \text{Hom}((\lim_C) \otimes M, M)$$

$$= \text{Hom}(\lim_C(C \otimes M), M)$$

$$= \lim \text{Hom}(C \otimes M, M)$$

and the latter contains $(\text{Id} \otimes \epsilon_C) \circ \tau_C$. \hfill $\square$

**Remark 3.6.** Note that what these limit arguments demonstrate is that in contrast to the $H$-module case considered in [5], the $H$-comodule case is much easier as it reduces to the rigid category $\mathcal{M}^H_{fd}$. More precisely, Proposition 2.7 shows that $Z_{\mathcal{M}^H}(\mathcal{M}^{H,op}) \simeq aYD\text{-}mod$ by essentially showing that $Z_{\mathcal{M}^H}(\mathcal{M}^{H,op}) \simeq Z_{\mathcal{M}^H}(\mathcal{M}^{H\#})$, but the latter is clearly $Z_{\mathcal{M}^H}(\mathcal{M}^{H\#})$, which as we saw above is equivalent to $Z_{\mathcal{M}^H}(\mathcal{M}^{H\#})$.

**Lemma 3.7.** The center of $\#\mathcal{M}^H$ is equivalent to the category of anti-Yetter-Drinfeld contramodules, namely

$$Z_{\mathcal{M}^H_{fd}}(\#\mathcal{M}^H) \simeq aYD\text{-}ctrmd.$$
Proof. Repeat the proof of Lemma 3.5 verbatim with the exception that

\[ \tau : T^{S^2} \otimes N \to N \otimes T \]

\[ t \otimes n \mapsto t_1 n \otimes t_0 \]

is a map in \( \hat{M}^H \) if and only if \( N \in aYD\text{-}ctrmd. \)

We summarize this section with the following Theorem.

Theorem 3.8. The following diagram commutes:

\[
\begin{array}{c}
\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H) \\
\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H) \downarrow \cong \downarrow \mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H)
\end{array}
\]

\[
\begin{array}{c}
aYD\text{-}mod \\
aYD\text{-}ctrmd
\end{array}
\]

Recall that for \( M \in aYD\text{-}mod \) we equip \( \hat{M} \) with \((2.16)\) and \((2.15)\), whereas for \( N \in aYD\text{-}ctrmd \) we equip \( N' \) with \((3.2)\) and \((3.3)\).

Proof. For the \( (\_') \) case we have the map \( T \otimes M \to M \otimes T^{S^2} \) with \( t \otimes m \mapsto t_1 m \otimes t_0 \) which maps under the identification of Proposition 3.2 to \( T^{S^2} \otimes \text{Hom}(H,M)^H \to \text{Hom}(H,M)^H \otimes T \) with \( h \otimes t \mapsto h \otimes S(t_1) \) and the latter coincides with \( t \otimes g \mapsto t_1 \cdot g \otimes t_0 \).

For the adjoint \( (\_') \) we have \( T^{S^2} \otimes N \to N \otimes T \) with \( t \otimes n \mapsto t_1 n \otimes t_0 \) mapping to \( H \otimes \text{Hom}(H,T^{S^2} \otimes N) \to H \otimes \text{Hom}(H,N \otimes T) \) with \( h \otimes t \otimes n \mapsto h \otimes t_1 n \otimes t_0 \) which identifies with \( T \otimes (H \otimes N) \otimes T^{S^2} \) with \( t_0 \otimes S(t_1) h \otimes n \mapsto hS(t_1) \otimes t_2 n \otimes t_0 \) under the isomorphisms of Remark 3.4 and the latter coincides with \( t \otimes (x \otimes m) \mapsto t_1 \cdot (x \otimes m) \otimes t_0. \)

In the end we see that \( ((\_'),(\_')) \) is an adjoint pair between \( aYD\text{-}mod \) and \( aYD\text{-}ctrmd \) extending the result of [6].

3.0.1 Stability.

Recall that in order to obtain cyclic cohomology we need to consider the coefficients in \textit{stable} anti-Yetter-Drinfeld modules or contramodules. We
now address the preservation of the stability conditions under the adjoint pair of functors of the previous section.

Recall the map

$$\sigma_M : M \to M$$

$$m \mapsto m_1m_0$$

with the inverse $$m \mapsto S^{-1}(m_1)m_0$$; it defines an element $$\sigma \in \text{Aut}(Id_{saYD-mod})$$.

Similarly, there is a

$$\sigma_N : N \to N$$

$$n \mapsto \alpha(r_n)$$

with the inverse $$n \mapsto \alpha(h)S^{-1}(h)n$$; it defines an element $$\sigma \in \text{Aut}(Id_{saYD-ctrmd})$$.

It is an easy calculation to see that $$\hat{\sigma}_M : \hat{M} \to \hat{M}$$ coincides with $$\hat{\sigma}_M : \hat{M} \to \hat{M}$$ and also $$(\sigma_N)' = \sigma_{N'}$$. For example to prove the latter equality observe that the left hand side is $$h \otimes n \mapsto h \otimes \alpha(r_n) = h^2 \otimes r_n(h^1) = h^2 \otimes h^1n = (h \otimes n)_1(h \otimes n)_0$$ which is the right hand side.

Recall that $saYD-mod$ is the full subcategory of $aYD-mod$ that consists of $M$ such that $\sigma_M = Id_M$. The definition of $saYD-ctrmd$ is identical. We have proved the following Corollary to Theorem 3.8:

**Corollary 3.9.** The functors $$((-)'_H, (-)_H)$$ is an adjoint pair between $saYD-mod$ and $saYD-ctrmd$.

### 4 A comodule-contramodule correspondence.

Here we will address the question of $(-)_H$ (equivalently $(-)'_H$) being an equivalence. Note that in light of the preceding discussion if $(-)_H : M^H \to M^H$ is an equivalence, then so is $(-)_H : aYD-mod \to aYD-ctrmd$ and also $(-)_H : saYD-mod \to saYD-ctrmd$.

As usual, let us consider $k$ as the trivial $H$-comodule, and let $J = \hat{k}$ be its contramodule image under $(-)_H$. Note that this is nothing but the two-sided ideal in $H^*$ consisting of right integrals $S$. Namely, $\chi \in J$ if and only if we have $\chi(h^1)h^2 = \chi(h)1$ for all $h \in H$. Strictly speaking it is left integrals that are considered in $S$ but if $\chi$ is a left integral then $\chi(S(-))$ is right and vice versa. It is known $[1]$ that $\dim J \leq 1$ and if $J \neq 0$ then $S$ is invertible, which we have been assuming anyhow.
Remark 4.1. Dually, we may consider \( k \) as the trivial contramodule, i.e., \( \alpha : H^* \to k \) is evaluation at \( 1 \in H \). Let \( K = k' \) and note that \( K = H/I \) where \( I \) is generated by \( \mu (h^1) h^2 - \mu (1) h \) for \( \mu \in H^* \) and \( h \in H \). Thus \( K^* = I^\perp = \{ \chi \in H^* | \mu (1) \chi (h) = \mu (h^1) \chi (h^2) \forall h \} \) and the latter is the ideal of left integrals.

We are ready for the first negative result:

Lemma 4.2. If \( J = 0 \) then \( \widehat{(-)} \) is not an equivalence.

Proof. Obviously we have that \( \widehat{k} = 0 \), but furthermore, by Proposition 3.2 we have that for \( M \in \mathcal{M}^H_{fd} \), \( \widehat{M} \simeq M \otimes J = 0 \).

On the other hand let us assume that \( J \neq 0 \). Let \( \mathcal{M}^H_{rfd} \) denote the full subcategory of \( \mathcal{M}^H \) consisting of finite dimensional, rational contramodules. By analogy with the \( H^* \)-module case, we say that a finite dimensional contramodule \( M \) is rational if the structure map \( \alpha \) factors through \( \text{Hom}(C, M) \) for some \( C \) a finite dimensional subcoalgebra of \( H \).

Lemma 4.3. Let \( J \neq 0 \) then \( \widehat{(-)} : \mathcal{M}^H_{fd} \simeq \mathcal{M}^H_{rfd} \).

Proof. Again, for \( M \in \mathcal{M}^H_{fd} \) we have that \( \widehat{M} \simeq M \otimes J \). Note that by \( S \) the contramodule \( J \) is rational and thus so is \( \widehat{M} \). On the other hand any rational finite dimensional contramodule is essentially a comodule (see Lemma 5.6) and so \( (- \otimes J)^{S^{-2}} \) is the inverse of \( \widehat{(-)} \).

The above Lemma should be considered as in general a negative result. Namely, if exotic, i.e., non-rational contramodules are possible, then the equivalence fails. More precisely, let us consider the possibility of exotic contramodule structures on \( k \). Let \( \chi \in J \) and observe that

\[
\alpha(x \mapsto \alpha(y \mapsto \chi(xS(y)))) = \alpha(h \mapsto \chi(h^1 S(h^2))) = \alpha(h \mapsto \epsilon(h)) \chi (1) = \chi (1).
\]

Since by \( S \), as \( x \) ranges over \( H \), the functional \( \chi (xS(-)) \) ranges over \( H^{* \text{rat}} \) so if

\[
\chi (1) \neq 0
\]

then \( \exists \mu \in H^{* \text{rat}} \) such that \( \mu \cdot 1 = c \neq 0 \). So that for any \( \eta \in H^* \) we have

\[
\eta \cdot 1 = \eta \mu^\perp \frac{1}{c} = \eta (\mu_1) \mu_0 \frac{1}{c} \quad \text{and so the action of } H^* \text{ on } k \text{ factors through } C^* \quad \text{and the structure on } k \text{ is necessarily rational. On the other hand if } \chi (1) = 0
\]

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then it is possible that the whole of $H^{rat}$ acts trivially without $H^{*}$ doing the same, resulting in an exotic structure.

This suggests two possibilities for $(\_)$ being an equivalence:

- $\exists \chi \in J$ with $\chi(1) \neq 0$.
- $H$ is finite dimensional.

Note that the second case may appear trivial at first, but it isn’t. It is true that there is no difference between $H$-comodules, $H^{*}$-modules and $H$-contramodules in the case when $H$ is finite dimensional. However, we are not interested in the naive identification of the categories, rather the $(\_)$ one. The latter functor is the one that translates the equivalence between comodules and contramodules to the equivalence between the $saYD$ versions that we need. Of course given all the work already done on this matter, the conclusion is easy to obtain, so we start with this case.

**Proposition 4.4.** Let $H$ be finite dimensional, then $(\_)$ is an equivalence, and so is $(\_)_H$.

**Proof.** From [8] we know that $J \neq 0$. Furthermore, for $M \in \mathcal{M}^H$ we have $M = \varinjlim M_i$ with $M_i \in \mathcal{M}^{H}_{fd}$ so that $\widehat{M} = \text{Hom}(H, M)^H = \text{Hom}(H, \varinjlim M_i)^H$ which by the finite dimensionality of $H$ is $\varinjlim \text{Hom}(H, M_i)^H \simeq \varinjlim (M_i^{S^2} \otimes J) = M^{S^2} \otimes J$. Since there are no exotic contramodules here this proves the equivalence. \hfill \square

Moving on to the first case we get by [8] that the $\chi(1) \neq 0$ condition is actually very strict. Namely, we have that $H$ is such that as a coalgebra $H = \bigoplus_i C_i$ where $C_i$ are finite dimensional simple subcoalgebras. Let $\epsilon_i$ denote the counit of $C_i$ with $\epsilon = \sum \epsilon_i$. For $x \in H$ let $x = \sum_i x_i$ denote its decomposition with respect to that of $H$.

**Theorem 4.5.** The category of $H$-comodules and $H$-contramodules are equivalent. The former consists of $\bigoplus_i M_i$ and the latter of $\prod_i M_i$ where $M_i$ are right $C_i$-comodules, i.e., $M_i \in \mathcal{M}^{C_i}$.

**Proof.** The assertion about the comodules is immediate. Now let $M$ be an $H$-contramodule, define $\alpha_i : M \to M$ via $\alpha_i(m) = \alpha(\epsilon_i(-)m)$. Note that $\alpha_i(\alpha_j(m)) = \alpha(x \mapsto \epsilon_i(x) \alpha(y \mapsto \epsilon_j(y)m)) = \alpha(h \mapsto \epsilon_i(h^1)\epsilon_j(h^2)m) = \delta_{ij} \alpha_i(m)$.
Let $M_i = \alpha_i(M)$ and consider $\beta : M \to \prod_i M_i$ such that

$$\beta(m) = (\alpha_i(m))_i$$

and $\iota : \prod M_i \to \text{Hom}(H, M)$ via

$$\iota((m_i)_i)(x) = \sum \epsilon_i(x)m_i.$$ 

We have that

$$\alpha \iota \beta(m) = \alpha(x \mapsto \sum \epsilon_i(x)\alpha_i(m)$$

$$= \alpha(x \mapsto \sum \epsilon_i(x)\alpha(y \mapsto \epsilon_i(y)m)$$

$$= \alpha(h \mapsto \sum \epsilon_i(h^1)\epsilon_i(h^2)m)$$

$$= \alpha(h \mapsto \epsilon(h)m) = m.$$ 

On the other hand we have that $\beta \alpha \iota((m_i)_i) = (\alpha_i(\alpha(x \mapsto \sum \epsilon_j(x)m_j)))_i$ and so we need to show that

$$m_i = \alpha_i(\alpha(x \mapsto \sum \epsilon_j(x)m_j)),$$

but the latter is

$$\alpha(y \mapsto \epsilon_i(y)\alpha(x \mapsto \sum \epsilon_j(x)m_j)) = \alpha(h \mapsto \sum \epsilon_i(h^1)\epsilon_i(h^2)m_j)$$

$$= \alpha(h \mapsto \epsilon_i(h^1)\epsilon_i(h^2)m_i)$$

$$= \alpha(\epsilon_i(-)m_i) = \alpha_i(m_i) = m_i.$$ 

Thus $\beta : M \simeq \prod_i M_i$ and using this identification we see that $\alpha : \text{Hom}(H, M) \to M$ becomes

$$\prod_i \text{Hom}(H, M_i) \to \prod_i M_i$$

$$(f_i)_i \mapsto (\alpha(h \mapsto f_i(h_i)))_i$$

so that if we denote by $\alpha^i : \text{Hom}(C_i, M_i) \to M_i$ the map $\alpha^i(f) = \alpha(h \mapsto f(h_i))$ then we see that the original $\alpha$ identifies with $\prod_i \alpha^i : \prod_i \text{Hom}(C_i, M_i) \to \prod_i M_i$. It is immediate that $\alpha^i$ is a $C_i$-contramodule structure on $M_i$ and since $C_i$ is finite dimensional is the same as a $C_i$-comodule structure.

Conversely, given the data of $\rho_i : M_i \to M_i \otimes C_i$ we can define

$$\alpha^i : \text{Hom}(C_i, M_i) = M_i \otimes C_i^* \to M_i$$

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and assemble the $\alpha^i$ into an $\alpha : \text{Hom}(H, \prod M_i) \to \prod M_i$ that satisfies the contramodule axioms.

Now let $\phi : M \to N$ be a map of contramodules and let $m \in M$ with $m = (m_i)_i$ under the $\beta$ identification, then

$$
\phi(m)_i = \alpha_N(\epsilon_i(-)\phi(m)) = \phi(\alpha_M(\epsilon_i(-)m)) = \phi(m_i)
$$

so that $\phi = \prod \phi_i$ with $\phi_i : M_i \to N_i$. It is immediate that $\phi_i \in \text{Hom}(M_i, N_i)^{C_i}$ and that conversely, any such $(\phi_i)_i$ data can be reassembled into a $\phi : M \to N$ a map of contramodules.

\[ \square \]

**Remark 4.6.** The proof of Theorem 4.5 demonstrates a difference between $H$-contramodules and $H^*$-modules. While there is a forgetful functor from the former to the latter, the contramodule condition is better behaved than the $H^*$-module one in the case of the infinite dimensional $H$. Considering finite dimensional contramodules, that at first glance appear to be given an action indistinguishable from that of an $H^*$-module, it is the associativity that is strictly strengthened in the contramodule case. More precisely, there exist exotic 1-dimensional $kG^*$-modules (for example given by non-principal ultrafilters on $G$), yet any 1-dimensional $kG$-contramodule is supported at some $g \in G$, just as is the case for $kG$-comodules. The difference is due to the fact that in the contramodule case we have the freedom to work with the full $(H \otimes H)^*$ as opposed to only $H^* \otimes H^*$. Of course in the case when $H$ is finite dimensional all three categories: $H$-contramodules, $H$-comodules and $H^*$-modules are equal.

We need to connect the above to our adjoint pair of functors.

**Proposition 4.7.** The correspondence

$$
\bigoplus_i M_i \leftrightarrow \prod_i M_i
$$

of Theorem 4.5 is given, up to equivalence, by the adjoint functor pair $((-), (-)' \circ (-))$. Thus $(-)$ is an equivalence and so is $(-)_H$.

**Proof.** Observe that

$$
\widehat{M} = \text{Hom}(H, M)^H = \prod_i \text{Hom}(C_i, M_i)^{H_i} = \prod_i \text{Hom}_{A_i}(A_i^*, M_i)
$$
where $A_i = C_i^*$ is a unital simple finite dimensional algebra. Let

$$
\mu_i : A_i \to A_i^*
$$

be given by $\mu_i(a)(b) = \operatorname{tr}_{A_i}(l_{ab})$, i.e., it is the trace of left multiplication by $ab \in A_i$. Note that $\mu_i$ is an $A_i$-bimodule map and $\mu_i(1)(1) = \operatorname{tr}_{A_i}(1) = \dim A_i \neq 0$ since char $k = 0$, so that $\mu_i$ is an isomorphism by the simplicity of $A_i$. So $\hat{M} \simeq \prod_i \operatorname{Hom}_{A_i}(A_i, M_i) \simeq \prod_i M_i$.

Similarly

$$
N' = \bigoplus_i C_i \otimes_{C_i^*} N_i = \bigoplus_i A_i^* \otimes_{A_i} N_i \simeq \bigoplus_i N_i.
$$

4.1 The case of $H = kG$.

Let $G$ be an infinite discrete group. We ask that $G$ be infinite as otherwise all of our considerations here become more or less trivial. Let $M$ be a $kG$-contramodule, i.e., we view $kG$ as a counital coalgebra with $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$. We have the following corollary of Theorem 4.5:

**Corollary 4.8.** The category of $kG$-contramodules $\hat{M}^{kG}$ is equivalent to the category of $G$-graded vector spaces $\operatorname{Vec}_G$. The equivalence is given by

$$
\Gamma(G, -) : \operatorname{Vec}_G \to \hat{M}^{kG}.
$$

Compare this with the well known equivalence

$$
\Gamma_c(G, -) : \operatorname{Vec}_G \to M^{kG}
$$

where $\Gamma_c$ are global sections with compact support.

**Proof.** Note that $kG = \bigoplus_{g \in G} kg$ with $kg = k$ as coalgebras.

It is well known that the category of anti Yetter-Drinfeld modules for $kG$ (since $S^2 = Id$ it coincides with the category of Yetter-Drinfeld modules, and thus with the center of the monoidal category of $kG$-modules) is equivalent to the category $\operatorname{Vec}_{G/G}$ of $G$-equivariant $G$-graded vector spaces. More precisely, the $kG$-comodule part of the structure gives the $G$-grading, and the
$kG$-module part gives the $G$-action, while the Yetter-Drinfeld compatibility ensures that the action obeys

$$x : M_g \to M_{gx^{-1}}.$$  

We have an immediate Corollary to Proposition 4.7:

**Corollary 4.9.** The category of $\mathcal{aYD}$-contramodules for $kG$ is equivalent to the category of $G$-graded $G$-equivariant vector spaces via

$$\Gamma(G, -) : \text{Vec}_{G/G} \to \mathcal{aYD}$-

We now would like to address the question of stability. A stable $\mathcal{aYD}$-module for $kG$ is known to be $G$-graded $G$-equivariant vector space with the stability condition translating into

$$x \cdot m_x = m_x$$

for all $x \in G$ and all $m_x \in M_x$. Denote by $\text{Vec}'_{G/G}$ the full subcategory of $\text{Vec}_{G/G}$ consisting of objects for which (4.1) holds. We have another immediate Corollary to Proposition 4.7:

**Corollary 4.10.** The functor

$$\Gamma(G, -) : \text{Vec}'_{G/G} \to \mathcal{aYD}$-

is an equivalence.

We can now restate the Proposition 2.10 more elegantly in the case of $H = kG$.

**Proposition 4.11.** Let $A$ be a $G$-equivariant algebra, and $\mathcal{M} \in \text{Vec}'_{G/G}$. Then

$$\widehat{HC}_G^n(A, \Gamma(G, \mathcal{M})) \simeq HC^{n,G}(A \rtimes G, \Gamma_c(G, \mathcal{M})).$$

**Remark 4.12.** While the right hand side of the above Proposition is definition invariant, the left hand side uses the definition of $\mathcal{M}$ and not the more classical one used in $\mathcal{M}$.  

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4.2 A new “coefficient”.

Since the introduction of coefficients in symmetric 2-contratrices in [4], there remained an obvious question: do these simply generalize the already well known coefficients in $saYD$-modules or contramodules to other settings, or do these traces furnish examples of coefficients that had not yet been considered even in the classical theories? In [7] we gave a derived version of the definition of cyclic cohomology with coefficients that restricted the possible symmetric 2-contratrices to the left exact ones. The results obtained in [5] immediately tell us that in the case of $H$-module algebras we need to look beyond the representable symmetric 2-contratrices if we are to obtain anything but the usual $saYD$-contramodule coefficients. In the present paper, Corollary 2.9 implies the same about $H$-comodule algebras; i.e., we need a non-representable contratrace to get away from the usual $saYD$-module coefficients. We will construct one below.

Let $G$ have infinitely many conjugacy classes (such as when $G = \mathbb{Z}$ for example). Let $\mathcal{M}_{\langle g \rangle} \in \text{Vec}_{G/G}'$ be supported on the conjugacy class $\langle g \rangle$, for example we can let

$$(\mathcal{M}_{\langle g \rangle})_x = \begin{cases} k, & x \in \langle g \rangle \\ 0, & \text{else} \end{cases}$$

with the trivial $G$-action. Then each $\mathcal{M}_{\langle g \rangle}$ yields a representable left exact symmetric 2-contratrace

$$\mathcal{F}_{\langle g \rangle}(V) = \text{Hom}(V, \Gamma_c(G, \mathcal{M}_{\langle g \rangle}))^G,$$

yet

$$\bigoplus_{\langle g \rangle} \mathcal{F}_{\langle g \rangle} : V \mapsto \bigoplus_{\langle g \rangle} \mathcal{F}_{\langle g \rangle}(V)$$

is an example of a non-representable, left exact symmetric 2-contratrace on $\mathcal{M}^H$. Note that taking $\mathcal{M}$ to be the superposition of all $\mathcal{M}_{\langle g \rangle}$’s would result in $V \mapsto \prod_{\langle g \rangle} \mathcal{F}_{\langle g \rangle}(V)$.

5 Periodicities.

In this section we revisit the $YD_i$-modules from [4] and see that under the conditions that we have been looking at in this paper, there is nothing new that arises and we still only have the Yetter-Drinfeld and the anti-Yetter-Drinfeld modules and contramodules; this is the first observed periodicity.
In addition, we examine a natural symmetry on these objects and observe that it too is periodic; this we refer to as the second periodicity.

We recall the definition of $YD_i$-modules:

**Definition 5.1.** Let $M$ be a left module and a right comodule over $H$, and let $i \in \mathbb{Z}$. We say that $M$ is a $YD_i$-module if

$$
(hm)_0 \otimes (hm)_1 = h^2 m_0 \otimes h^3 m_1 S^{-1-2i}(h^1),
$$

(5.1)

for $h \in H$ and $m \in M$.

**Remark 5.2.** Equivalently, we can define $YD_i$-modules by requiring that the comodule structure map $M \to M \otimes H$ is $H$-equivariant with respect to the $H$-structure on the right hand side given by $x \cdot (m \otimes h) = x^2 m \otimes x^3 h S^{-1-2i}(x^1)$.

Note that $YD_{-1}$-modules are anti-Yetter-Drinfeld modules, while $YD_0$-modules are Yetter-Drinfeld modules.

We can rephrase the above a little more conceptually. Let $\mathbb{Z}$ act on $\mathcal{M}^H$ with $1 \cdot M = M S^2$ so that we may consider the monoidal category $\mathcal{M}^H \times \mathbb{Z}$. We get an immediate generalization of Lemma 3.5:

**Lemma 5.3.** We have an equivalence of monoidal categories:

$$
\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H \times \mathbb{Z}) \simeq \bigoplus_{i \in \mathbb{Z}} YD_{-i}\text{-mod}.
$$

There are a few consequences of the above. First, if $M \in YD_i\text{-mod}$ and $N \in YD_j\text{-mod}$ then $M \otimes N \in YD_{i+j}\text{-mod}$ with the usual comodule structure, but $S^{-2i}N \otimes M$ as an $H$-module, i.e.,

$$(m \otimes n)_0 \otimes (m \otimes n)_1 = m_0 \otimes n_0 \otimes m_1 n_1$$

but

$$x \cdot (m \otimes n) = x^2 m \otimes S^{-2i}(x^1)n.$$  

Second, if $M \in YD_i\text{-mod}$ then so is $1 \cdot M = S^{-2i} M S^2 \in YD_i\text{-mod}$. Third, $\mathcal{M}^H$ has internal Homs, and so does $\mathcal{M}^H \times \mathbb{Z}$, i.e.,

$$\text{Hom}^l((M, j), (N, i)) = (\text{Hom}^l((i - j)M, N), i - j)$$

and the same for right Homs. Consequently, $\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H \times \mathbb{Z})$ has internal Homs as well. In particular $\mathcal{M}^H_{fd}$ is rigid, so is $\mathcal{M}^H_{fd} \times \mathbb{Z}$ with $(V, i)^* = \cdots$
$((-i)V^*, -i)$ and $^*(V, i) = ((-i)^*V, -i)$ and so is $\mathcal{Z}_M H (\mathcal{M}_{fd}^H \times \mathbb{Z})$. Thus if $M \in \mathcal{Y}D_i^{fd}$-mod then we have elements $M^*$ and $^*M$ in $\mathcal{Y}D_i^{fd}$-mod that are its right and left duals.

Just as we have generalized $a\mathcal{Y}D$-mod to $\mathcal{Y}D_i$-mod, we can do the same to $a\mathcal{Y}D$-ctrmd.

**Definition 5.4.** Let $M$ be a left $H$-module and a right $H$-contramodule, we say that $M$ is a $\mathcal{Y}D_i$-contramodule if the contramodule structure $\alpha : \text{Hom}(H, M) \to M$ is $H$-equivariant with respect to the $H$-action on the left given by

$$h \cdot f = h^2 f(S(h^3) - S^{2i}(h^1)),$$

(5.2)

where $h \in H$ and $f \in \text{Hom}(H, M)$.

Note that $\mathcal{Y}D_1$-contramodules are $a\mathcal{Y}D$-contramodules.

### 5.1 The first periodicity.

We can easily generalize the content of Section 3 as follows. We have the Proposition/Definition below.

**Proposition 5.5.** Let $M$ be a $\mathcal{Y}D_{i-1}$-module, define $\hat{M} = \text{Hom}(H, M)^H$ and equip the latter with a left $H$-action via

$$h \cdot f = h^2 f(S(h^3) - S^{-2i}(h^1))$$

and an $H$-contraaction as before (2.13). Let $N$ be a $\mathcal{Y}D_{i+1}$-contramodule, define $N' = H \odot_H N$ and equip the latter with a left $H$-action via

$$h \cdot (x \otimes n) = S^{i-2i}(h^3)xS(h^1) \otimes h^2 n$$

and an $H$-coaction as before (3.2).

This defines an adjoint pair of functors $((-)^i_H, (-)_H)$:

$$\mathcal{Y}D_{i-1}$-mod \xrightarrow{(-)_H} \mathcal{Y}D_{i+1}$-ctrmd.$$

Let us again (see Remark 4.1 and the preceding discussion) consider the trivial $\mathcal{Y}D_0$-module $k$ from which we obtain by the Proposition 5.5 the
object $J = \widehat{k}$ which is now seen to be in $YD^2_{2}\text{-ctrmd}$. Conversely, again considering the trivial $YD_0$-contramodule $k$, we obtain $K = k'$ which is now seen to be in $YD^2_{2}\text{-mod}$. If we denote by $YD^r_{i}\text{-ctrmd}$ the full subcategory of $YD_{i}\text{-ctrmd}$ that consists of objects that as contramodules are in $\mathcal{M}^{H, r}_{r, i, d}$ then $J \in YD^r_{2}\text{-ctrmd}$. Observe that we have an easy Lemma:

**Lemma 5.6.** We have an equivalence (equality actually) of categories:

$$\iota: YD^r_{i}\text{-mod} \to YD^r_{i}\text{-ctrmd}$$

that does not change the underlying vector space $M$, nor the $H$-action, and sends the coaction to the contraaction:

$$\text{Hom}(H, M) = H^{\ast} \otimes M \to M$$

$$\chi \otimes m \mapsto \chi(S^2(m_1))m_0.$$

As a consequence, we have $\iota^{-1}J \in YD^r_{2}\text{-mod}$ which is the dual of $K \in YD^r_{2}\text{-mod}$.

**Remark 5.7.** If $H$ is a Hopf algebra with $J \neq 0$ then both $YD_{i}\text{-mod}$ and $YD_{i}\text{-ctrmd}$ are 2-periodic, i.e.,

$$J \otimes - : YD_{i}\text{-mod} \simeq YD_{i+2}\text{-mod}$$

and the same for contramodules.

For a finite dimensional $H$, the functor $\widehat{(-)}_H$ is essentially the periodicity above. Not so for the infinite dimensional case.

### 5.2 The second periodicity.

Recall our discussion of stability in Section 3.0.1. We observe that the

$$\sigma \in \text{Aut}(Id_{aYD}\text{-mod})$$

that was used to define stability for $aYD$-modules (and its contramodule variant) can be generalized, with an interesting difference, to an arbitrary $i$ for both modules and contramodules. More precisely,

$$\sigma \in \text{Iso}(Id_{YD_{i-1}\text{-mod}}, (-i)\cdot),$$

i.e., for $M \in YD_{i-1}\text{-mod}$ we have $\sigma_M : M \to S^{2i}M^{S^{-2i}}$, with $m \mapsto S^{2i}(m_1)m_0$ and the inverse $m \mapsto S^{-1}(m_1)m_0$, is an identification in $YD_{i-1}\text{-mod}$. 

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Remark 5.8. The above implies that if $M$ is a Yetter-Drinfeld module, then it is canonically isomorphic to $S_{-2}M^{S^2}$ as a Yetter-Drinfeld module. More generally, the action of $\mathbb{Z}$ on $YD_{i-1}\text{-mod}$ factors through $\mathbb{Z}/i\mathbb{Z}$. We will see below that the same holds for $YD_{i+1}\text{-contrmd}$.

Note that the $M \mapsto S_{-2}M^{S^2}$ symmetry of $YD_i$-modules also exists for $YD_i$-contramodules, i.e., $S_{-2}N^{S^2}$ has its $H$-action modified by $S^{-2}$ and $\alpha^{S^2}(f) = \alpha(f(S^2(-)))$. Then we have

$$\sigma \in \text{Iso}(\text{Id}_{YD_{i+1}\text{-contrmd}}, (-i)),$$

i.e., for $N \in YD_{i+1}\text{-contrmd}$ we have $\sigma_N : N \to S_{2i}N^{S^{-2i}}$, with $n \mapsto \alpha(r_n)$ and the inverse $n \mapsto \alpha(h \mapsto S^{2i-1}(h)n)$, is an identification in $YD_{i+1}\text{-contrmd}$.

Furthermore, the generalized functor $\hat{(-)}_H$ of Proposition 5.5 is compatible with these symmetries, namely the diagram of isomorphisms

$$
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\sigma}_M} & S_{2i}M^{S^{-2i}} \\
\downarrow{\sigma_M} & & \downarrow{f \mapsto f(S^{2i}(-))} \\
S_{2i}M^{S^{-2i}} & \xrightarrow{f \mapsto f(S^{2i}(-))} & 
\end{array}
$$

commutes in $YD_{i+1}\text{-contrmd}$, where $M \in YD_{i-1}\text{-mod}$.

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