DILATION TWO EMBEDDING ONE-BY-ONE PARTICULAR SUB-QUADTREE INTO M-DIMENTIONAL CROSSED CUBES

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Abstract

In the parallel processing field, graph embedding is motivated by simulation interconnection networks to another. The quadtree is an important technique used to present spatial data and is used in many application domains, especially computer vision and image processing. Researchers are interested in the construction and manipulation of quadtrees on parallel machines. The crossed cubes consider an alternative to the ordinary hypercube. It offers many attractive properties. Significantly, it reduces diameter by a factor of 2 that of the ordinary cubes. Moreover, the crossed cubes have a great capacity to simulate other architectures. This paper is interested in the one-by-one dilation two embedding of a particular sub-quadtree graph into m-dimensional crossed cubes.

Keywords: Interconnection networks, crossed cubes, quadtree, embedding, dilation.

1. Introduction

For a machine based on parallel architecture, the choice of a good topology is linked to a set of attractive and popular properties such as degree, diameter, connectivity, regularity, embeddability, and fault tolerance.

The quadtree is an important technique used in many application domains such as geographic information systems, image processing, computer graphics, robotics [1, 2, 3, 4, 5]. The quadtree is a simple topology; it is a connected graph with a cycle in which each internal vertex has four children. Suppose all left child breaks down; quadtree will be a particular sub-quadtree. This technique has a set of limitations. It has a bisection width equal to 1 [6]. Moreover, its connectivity equal to 1 [6]. Therefore, many researchers studied the embedding of quadtrees into another interconnection network [7, 8, 9, 10].

Parallel machines based on hypercube topology offer an interconnection topology with attractive properties: symmetry, logarithmic diameter, fixed degree, high connectivity, Hamiltonian, fault tolerance, extensibility, and embeddability of other topologies [11, 12, 13, 14, 10, 15, 16]. Several researchers propose several versions of the hypercube to improve its capacity. In the papers by Ahmed El-Amawy[17] and Preparata & Vuillemin [18], an attractive version of the hypercube called the crossed cubes is proposed by Efe [19]. This version pays attention because of the similarity in properties with the ordinary hypercube. Also, it offers many other attractive properties over the ordinary hypercube [10, 20, 21, 22, 23, 24, 25]. Especially, it reduces diameter by a factor of 2 [19]. Moreover, the crossed cubes have a great capacity to simulate other architectures [10, 15, 26, 27, 28, 29]. The problem of simulation of one interconnection network to another is essential in the field of parallel computing. The embedding capabilities are important in evaluating an interconnection network.

Let G and H be two graphs such that an embedding of G into H is a pair (f, R) where f is an injective mapping vertex V(G) into vertex V(H). R is an injective mapping associating with each edge [u, v] from G at a path R(u, v) which connects f(u) and f(v) [30, 31, 32].

This paper aims to construct the dilation two embedding one-by-one particular sub-quadtree into m-dimensional crossed cubes. This paper is organized as follows; first, we introduce a few preliminary definitions of the particular sub-quadtree graph and the crossed cubes graph. Section 3 presents the construction of dilation two one-by-one particular sub-quadtree into m-dimensional crossed cubes; then, we offer the validation of our new function. Finally, section 4 concludes the paper and discusses some possible future work.

2. PRELIMINARIES

2.1. Definition 01

The quadtree is an undirected graph QTn of (4n − 1)/3 vertices. Every vertex of depth less than n has four children, and every vertex of depth equal to n is a leaf. We assume each left child breaks down, quadtree reduced to a particular sub-quadtree graph denoted PQTn. Therefore, quadtree will become an undirected graph PQTn of (3n − 1)/2 vertices. Let p be a positive integer, each vertex in PQTn is a string of length p denoted AV = Aap−1−1, where Ap−1 = a1a2...aj, aj = 1,3 and suffix = i / i = 1,3. The root can represent by address a1 = 0.
We represent an edge between a vertex parent and one of its children as follows, $A_{p-1}$-$A_{p-1}$.

2.2. Definition 02 [19]

Both the $m$-dimensional hypercube denoted by $Q_m$ and the crossed cubes $CQ_m$ are undirected graphs consisting of the same set of vertices. A binary string of length $m$ labels each vertex in $Q_m(CQ_m)$. In $Q_m$, two vertices are adjacent if and only if the binary representation of their labels differs in exactly one-bit position. While in the crossed cubes, two binary strings $x = x_1x_2...x_m$, $y = y_1y_2...y_m$ of length two are pair-related if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. The $m$-dimensional crossed cubes $CQ_m$ is defined recursively:

- $CQ_1$ is the complete graph on two vertices with labels 0 and 1.
- If $m > 1$: $CQ_m$ consists of two sub-cubes, $0CQ_{m-1}$ and $1CQ_{m-1}$. Two vertices $u, v$ such that $u = 0u_{m-2}...u_0 \in 0CQ_{m-1}$ and $v = 1v_{m-2}...v_0 \in 1CQ_{m-1}$ are adjacent, if and only if:
  \[
  u_{m-2} = v_{m-2} \text{ if } m \text{ is even} \\
  u_{2i+1}v_{2i+1}, u_{2i+2}v_{2i+2} \text{ are pair-related.}
  \]

2.3. Definition 03 [32, 33]

Let $G$ and $H$ be two simple undirected graphs. An embedding of the graph $G$ into graph $H$ is an injective mapping $f$ from the vertices of $G$ to the vertices of $H$. Four cost functions, dilation, congestion, expansion, and load factor, often measure the quality of an embedding. In this paper, we interest in dilation. The dilation of the embedding is the maximum distance between $f(y)$ and $f(z)$ taken over all edges $(y, z)$ of $G$.

2.4. Notations

A particular sub-quadtree $PQT_n$ is produced by three copies of $PQT_{n-1}$ prefixed respectively by $01PQT_{n-1}$, $02PQT_{n-1}$, and a root prefixed by $0PQT_n$.

A crossed cubes $CQ_m = (O, E)$, with $E$ set of vertices and $O$ set of edges.

Let $B \in E$ such that: $B = [Pref_3, X_1X_2X_3]$, $C = b_{1-1}, b_{1-1}, \phi$; $adrr = X, Y, Z$; $adrr = X_3X_0$ of length two and $(X, Y)$ or $(Y, Z)$ are pair-related; $Pref_j = b_{j-5}...b_{m-4} / j = \overline{0j3}$ respectively if $b_{j-5}...b_{m-4}$, $b_{j-1}...b_{m-1}$, $b_{j-1}...b_{m-1}$, $b_{j-1}...b_{m-1}$. The number of super nodes $CQ_3$ is equal to $2^{m-4}$.

$CQ_m$ is produced as follows:

Where $C = \phi$: $CQ_m$ is produced by four copies of $CQ_{m-2}$ prefixed respectively by $00CQ_{m-2}$, $01CQ_{m-2}$, $10CQ_{m-2}$, $11CQ_{m-2}$.

Where $C \neq \phi$: $CQ_m$ is produced by two copies of $CQ_{m-1}$ prefixed respectively by $0CQ_{m-1}$, $1CQ_{m-1}$ in other word: $00CQ_{m-2}$, $01CQ_{m-2}$, $10CQ_{m-2}$, $11CQ_{m-2}$.

3. DILATION TWO EMBEDDING ONE-BY-ONE PARTICULAR SUB-QUADTREE INTO M-DIMENSIONAL CROSSED CUBES

This section describes our new function, which allows the embedding one-by-one particular sub-quadtree $PQT_n$ into $m$-dimensional crossed cubes $CQ_m$. We can resume this function as follows:

- Determine the dimension of the crossed cubes $CQ_m$.
- One-by-one vertex embedding of $PQT_n$ into $CQ_m$.
- Dilation two embedding one-by-one all edges of $PQT_n$ onto paths in $CQ_m$.

3.1. Dimension of $CQ_n$

The dimension of the crossed cubes $m$ related by $n$ the height of a particular sub-quadtree in which:

- Where $n \leq 8$: $m = \log_2(3^n - 1/2)/\log_2(2)$
- Where $n > 8$: $m = (n - 8) * 2 + 12$

3.2. One-by-one vertex embedding

The one-by-one vertex embedding of $PQT_n$ into $CQ_m$ is done in the following way:

For $n = 3$: The basic function $f$ of this one by one vertex embedding is produced as follows:

- $Prem(0) := Pref_000$
- $f(A_{p-1}\text{-suff}_1) := Pref_1X$
- $f(A_{p-1}\text{-suff}_2) := Pref_2Y$
- $f(A_{p-1}\text{-suff}_3) := Pref_3Z$

For $n > 3$: the one-by-one vertex embedding is done in two situations. The first one is when $C = \phi$, we use the basic function $f$ (figure 1).

![Figure 1: Vertex embedding Situation 1](image)

The second is when $C \neq \phi$, a function $f_1$ of this one-by-one vertex embedding. Thus, there three cases in this situation; the following rules of case 1 shown in figure 2 produce $f_1$:

- $f_1(A_{p-1}\text{-suff}_1) := 0Pref_00X$
- $f_1(A_{p-1}\text{-suff}_2) := 1Pref_00Y$
- $f_1(A_{p-1}\text{-suff}_3) := 1Pref_00Z$

OR

- $f_1(A_{p-1}\text{-suff}_1) := 0Pref_00X$
The following rules of case 2 shown in figure 3 produce $f_1$:
- $f_1(A_p-1suff_1) := \overline{0}Pref_000X$
- $f_1(A_p-1suff_2) := \overline{1}Pref_000Y$
- $f_1(A_p-1suff_3) := 1Pref_000Z$

The following rules of case 3 shown in figure 4 produce $f_1 (t = 2, 3)$:
- $f_1(A_p-1suff_1) := \overline{0}Pref_100X$
- $f_1(A_p-1suff_2) := \overline{1}Pref_100Y$
- $f_1(A_p-1suff_3) := 1Pref_100Z$

OR
- $f_1(A_p-1suff_1) := \overline{0}Pref_000X$
- $f_1(A_p-1suff_2) := \overline{1}Pref_000Y$
- $f_1(A_p-1suff_3) := 1Pref_000Z$

Lemma 1. For $n < 5$, a particular sub-quadtree $PQT_n$ is one-by-one vertex embedding into $m$-dimensional crossed cubes $CQ_m$.

Proof. We prove lemma 1 by induction on $n$.
Base. For $n = 2$: as shown in figure 5.
For $n = 3, 4$: level’s 1, 2 nodes of $PQT_3, PQT_4$ are respectively embedded into $CQ_4, CQ_6$ using the rules specified in table 1, table 2, shown in figure 6, figure 7.
Induction hypothesis

Suppose that for \( k \leq n - 1 \), \( PQT_k \) is one-by-one vertex embedding into \( CQ_l \) with \( l < m \) is true.

Let us now prove that it is true for \( k = n \).

The root 0 is embedded into 00\( CQ_2 \) by using the basic function \( \text{Prem}(\text{root}) \). Nodes of 01\( PQT_{k-1} \) are embedded into 00\( CQ_2 \) such that: 0\( \text{suff}_2 \) is embedded into 0\( \text{pref}_1 \)0000 of 00\( CQ_2 \) using \( f_1 \) of situation 2, case 1 (figure 2, induction hypothesis). Other nodes of 011\( PQT_{k-2} \), 012\( PQT_{k-2} \), 013\( PQT_{k-2} \) are respectively embedded into the root embedded component, 01\( CQ_{l-2} \) of 00\( CQ_2 \) and 01\( CQ_{l-2} \) of 00\( CQ_2 \) using \( f \) (figure 1, induction hypothesis).

Nodes of 02\( PQT_{k-2} \) are embedded into 10\( CQ_{l-2} \) such that: 0\( \text{suff}_2 \) is embedded into 10\( \text{pref}_1 \)0000 of 10\( CQ_{l-2} \) by definition 02 using \( f_1 \) of situation 2, case 1 (figure 2). Other nodes of 021\( PQT_{k-2} \), 022\( PQT_{k-2} \), 023\( PQT_{k-2} \) are respectively embedded into the same parent component (10\( CQ_{l-2} \), 11\( CQ_{l-2} \) of 10\( CQ_{l-2} \) and 11\( CQ_{l-2} \) of 10\( CQ_{l-2} \) using \( f \) (figure 1, induction hypothesis).

Nodes of 03\( PQT_{k-1} \) are embedded into 01\( CQ_l \) such that: 0\( \text{suff}_3 \) is embedded into 10\( \text{pref}_0 \)0000 of 10\( CQ_{l-2} \) by definition 02 using \( f_1 \) of situation 2, case 1 (figure 2); and 0\( \text{suff}_1 \) is embedded into this same component 10\( CQ_{l-2} \) using \( f_1 \) of situation 2, case 2 (figure 3, induction hypothesis). Node 0\( \text{suff}_2 \) is embedded into 00\( CQ_{l-2} \) by definition 02 using \( f_1 \) of situation 2, case 2 (figure 3). Nodes of 031\( PQT_{k-2} \), 032\( PQT_{k-2} \) are embedded into the same parent component, respectively 10\( CQ_{l-2} \), 00\( CQ_{l-2} \) using \( f \) (figure 1, induction hypothesis).

Nodes of 033\( PQT_{k-2} \) are embedded into 11\( CQ_l \) such that: 0\( \text{suff}_3 \) is embedded into the same parent component (10\( CQ_{l-2} \) using \( f \) (figure 1, induction hypothesis)).

For nodes of 0331\( PQT_{k-3} \) are embedded into 10\( CQ_{l-2} \) and 00\( CQ_{l-2} \) such that:

- 0\( \text{suff}_1 \) is embedded into the same parent component (10\( CQ_{l-2} \) using \( f \) (figure 1, induction hypothesis);

For nodes of 0332\( PQT_{k-3} \) are embedded into 11\( CQ_{l-2} \) and 10\( CQ_{l-2} \) such that:

- 0\( \text{suff}_2 \) is embedded into 11\( CQ_{l-2} \) of 10\( CQ_{l-1} \) using \( f \) (figure 1, induction hypothesis);

- 0\( \text{suff}_3 \) is embedded into the same parent component (11\( CQ_{l-2} \) using \( f_1 \) of situation 2, case 3 (figure 4, induction hypothesis);

- 0\( \text{suff}_2 \) or 0\( \text{suff}_3 \) are embedded into 00\( CQ_{l-2} \) by definition 02 using \( f_1 \) of situation 2, case 2 (figure 4).

For nodes of 0333\( PQT_{k-3} \) are embedded into 11\( CQ_{l-2} \) and 01\( CQ_{l-2} \) such that:

- 0\( \text{suff}_3 \) is embedded into 11\( CQ_{l-2} \) of 10\( CQ_{l-1} \) using \( f \) (figure 1, induction hypothesis);

- 0\( \text{suff}_2 \) is embedded into 10\( CQ_{l-2} \) of 00\( CQ_{l-2} \) using \( f \) (figure 1, induction hypothesis).

### Table 2: Level’s 1, 2 nodes embedding of \( PQT_4 \) into \( CQ_6 \).

| Root | \( \text{Prem}(\text{root}) \) | \( \text{0suff}_1 \) | 01\( \text{CQ}_4 \) |
|------|-----------------|----------------|----------------|
| 0    | 0000000         | 01             | 0100000        |
| 0\( \text{suff}_2 \) | 10\( \text{CQ}_4 \) | 0\( \text{suff}_3 \) | 11\( \text{CQ}_4 \) |
| 02   | 1000000         | 03             | 1100000        |

### Table 3: Level’s 1, 2 nodes embedding of \( PQT_5 \) into \( CQ_7 \).

| Root | \( \text{Prem}(\text{root}) \) | \( \text{0suff}_1 \) | \( \text{0pref}_1 \) \( \text{CQ}_4 \) |
|------|-----------------|----------------|----------------|
| 0    | 0000000         | 01             | 0100000        |
| 0\( \text{suff}_2 \) | 1\( \text{pref}_1 \) \( \text{CQ}_4 \) | 0\( \text{suff}_3 \) | 1\( \text{pref}_0 \) \( \text{CQ}_4 \) |
| 02   | 1010000         | 03             | 1000000        |

### Figure 7: Nodes embedding graph of \( PQT_4 \) into \( CQ_6 \).
There are two cases:

**Bedding into CQ**

Proof.

- **Theorem 1.** For $n > 5$, a particular sub-quadrant $PQT_n$ is one-by-one vertex embedding into $m$-dimensional crossed cubes $CQ_m$.

**Proof.** We prove theorem 1 by induction on $n$.

**Base.** For $n = 6, 8$: level’s 1, 2 nodes of $PQT_6$, $PQT_8$ respectively are embedded using the rules specified in table 4 and table 5.

| Root | Prem(root) | 0suff_1 | pref_1CQ4 |
|------|------------|---------|------------|
| 0    | 0000000000 | 01      | 0100000000 |
| 0suff_2 | pref_2CQ4 | 0suff_3 | pref_3CQ4 |
| 02   | 1000000000 | 03      | 1100000000 |

**Table 4:** Level’s 1, 2 nodes embedding of $PQT_6$ into $CQ_3$.

| Root | Prem(root) | 0suff_1 | pref_1CQ4 |
|------|------------|---------|------------|
| 0    | 0000000000 | 01      | 0010000000 |
| 0suff_2 | 1pref_1CQ4 | 0suff_3 | 1pref_3CQ4 |
| 02   | 1010000000 | 03      | 1000000000 |

**Table 5:** Level’s 1, 2 nodes embedding of $PQT_8$ into $CQ_4$.

**Induction hypothesis**

Suppose that for $k ≤ n - 1$, $PQT_k$ is one-by-one vertex embedding into $CQ_l$ with $l < m$ is true.

Let us now prove that it is true for $k = n$.

There are two cases:

**Case a:** $C = \phi$

One-by-one vertex embedding of $01PQT_{k-1}$, $02PQT_{k-1}$, $03PQT_{k-1}$, and $0PQT_{k-1}$ respectively into $01CQ_{l-2}$, $10CQ_{l-2}$, $11CQ_{l-2}$, and $00CQ_{l-2}$; in this case, we use the same actions as lemma 1.

**Case b:** $C ≠ \phi$

One-by-one vertex embedding of $01PQT_{k-1}$, $02PQT_{k-1}$, $03PQT_{k-1}$, and $0PQT_{k}$ respectively into $01CQ_{l-1}$, $10CQ_{l-1}$, $00CQ_{l-1}$ and $11CQ_{l-1}$, and the root 0 into $00CQ_{l-2}$; in this case, we use the same actions as lemma 2 except the sub-$PQT$: $0111PQT_{k-3}$, $0311PQT_{k-3}$, and $0321PQT_{k-3}$ are embedded as situation 2, case 1, b as shown in figure 2.

3.3. Dilation two one-by-one edges embedding

Dilation two one-by-one edges embedding of $PQT_n$ onto $CQ_m$ is done in the following way:

For $n = 3$: the basic function $R$ of this dilation two one-by-one edges embedding is produced as follows:

- $R(A_{p-1}A_{p-1}suff_1) := Pref_00-Pref_10$
- $R(A_{p-1}A_{p-1}suff_2) := Pref_00-Pref_20$
- $R(A_{p-1}A_{p-1}suff_3) := Pref_00-Pref_20-Pref_30$

For $n > 3$: dilation two one-by-one edges embedding is done by two situations. In the first, there are two cases; the first case is when $C = \phi$, we use the basic function $R$; an example is shown in figure 9.
The second case is when $C \neq \phi$, and we only use one copy $b_0CQ_{m-1}$ or $b_0CQ_{m-1}$; the second case is shown in figures 9 and 10. $R$ is produced like situation 1, case 1, or as follows:

- $R(a_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := \bar{Pref}_00X-\bar{Pref}_100Y$
- $R(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := \bar{Pref}_00X-\bar{Pref}_100Y$
- $R(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := \bar{Pref}_00X-\bar{Pref}_100Y$

The second situation is when $C \neq \phi$; in this situation, we use the two copies $b_0CQ_{m-1}$ and $b_0CQ_{m-1}$; a function $R_1$ of this dilation two one-by-one edges embedding. There are three cases; the first is shown in figure 11. The following rules of case 1 produce $R_1$:

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := 0\bar{Pref}_00X-1\bar{Pref}_100Y$

OR

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := \bar{0}\bar{Pref}_00X-1\bar{Pref}_100Y$

The following rules of case 2 shown in figure 12 produce $R_1$:

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := 0\bar{Pref}_00X-1\bar{Pref}_100Y$

OR

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := \bar{0}\bar{Pref}_00X-1\bar{Pref}_100Y$

The following rules of the last case shown in figure 13 produce $R_1 (t = 2, 3)$:

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := 0\bar{Pref}_00X-0\bar{Pref}_100Y$

OR

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := \bar{0}\bar{Pref}_00X-\bar{0}\bar{Pref}_100Y$

- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_1) := \bar{0}\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_2) := \bar{0}\bar{Pref}_00X-0\bar{Pref}_100Y$
- $R_1(A_{p_1-1}\cdot A_{p_1-1}\text{ suff}_3) := \bar{0}\bar{Pref}_00X-0\bar{Pref}_100Y$
Lemma 3. For any \( n < 5 \), a particular sub-quadtree \( PQT_n \) of dimension \( n \) is dilation two one-by-one edges embedding onto \( m \)-dimensional crossed cubes \( CQ_m \).

**Proof.** We prove lemma 3 by induction on \( n \).

**Base.** For \( n = 2 \): edges between vertices of level 1 and level 2 of \( PQT_2 \) are embedded using the rules specified in table 6.

| \( PQT \) edge | crossed cubes path | Dilation |
|---------------|-------------------|----------|
| 0-01          | 00-01             | 1        |
| 0-02          | 00-10             | 1        |
| 0-03          | 00-10-11          | 2        |

| \( PQT \) edge | crossed cubes path | Dilation |
|---------------|-------------------|----------|
| 0-01          | 0000-0100         | 1        |
| 0-02          | 0000-1000         | 1        |
| 0-03          | 0000-1000-1100    | 2        |

| \( PQT \) edge | crossed cubes path | Dilation |
|---------------|-------------------|----------|
| 0-01          | 000000-01000000   | 1        |
| 0-02          | 000000-10000000   | 1        |
| 0-03          | 0000000-100000000 | 2        |

Table 6: Edges embedding between vertex of level 1 and level 2 of \( PQT_2 \), \( PQT_3 \), \( PQT_4 \).

**Induction hypothesis**

Suppose that for \( k \leq n - 1 \), any sub-\( PQT_k \) of \( PQT_k \) is dilation two one-by-one edges embedding onto any sub-\( CQ_l \) of \( CQ_l \) with \( l \leq m \) is true.

Is it true for \( k = n \) ?

For any sub-\( PQT_k \), sub-\( CQ_l \) with \( k' = 5 \), \( l' = 7 \); the edge between 0\( PQT_k \), 0\( PQT_k \) is embedded onto a path in the same component 0\( CQ_l \) using \( R_l \) of situation 2, case 1 (figure 11, induction hypothesis). For edges of 0\( PQT_k \) are embedded onto paths between 0\( CQ_l \) and the same 0\( CQ_l \) of 0\( CQ_l \) using \( R_l \) (figures 9, 10, induction hypothesis).

The edge between 0\( PQT_k \), 0\( PQT_k \) is embedded onto a path in the same component 1\( CQ_l \) (induction hypothesis), and between 0\( CQ_l \), 0\( CQ_l \) by definition 02 using \( R_l \) of situation 2, case 1 (figure 11). For edges of 0\( PQT_k \) are embedded onto paths between 0\( CQ_l \) and the same 0\( CQ_l \) using \( R_l \) (figures 9, 10, induction hypothesis).

The edge between 0\( PQT_k \), 0\( PQT_k \) is embedded onto a path between 0\( CQ_l \) and the same 0\( CQ_l \) by definition 02 using \( R_l \) of situation 2, case 1 (figure 11). For edges of 0\( PQT_k \) are embedded onto paths in the same component 1\( CQ_l \) using \( R_l \) (figures 9, 10, induction hypothesis).

The edge between 0\( PQT_k \), 0\( PQT_k \) is embedded onto a path in the same component 1\( CQ_l \) (induction hypothesis), and
between $10CQ_{r-2}$ and $00CQ_{r-2}$ by definition 02 using $R_{1}$ of situation 2, case 2 (figure 12). Edges of $031PQT_{k-2}$, $032PQT_{k-2}$ are embedded onto paths in the same $00CQ_{r-2}$, $10CQ_{r-2}$ using $R$ of situation 1, case 1 (figure 9, induction hypothesis).

The edge between $03PQT_{k-1}$, $033PQT_{k-1}$ is embedded onto a path in the same component $10CQ_{r-2}$ using $R_{1}$ of situation 2, case 2 (figure 12, induction hypothesis). For edges of $033PQT_{k-2}$ are embedded as follows: the edge between $033PQT_{k-2}$, $0331PQT_{k-3}$ is embedded onto a path in the same component $10CQ_{r-2}$ using $R$ of situation 1, case 1 (figure 9, induction hypothesis). Edges of $0331PQT_{k-3}$ are embedded onto paths in the same component $10CQ_{r-2}$ (induction hypothesis), between $10CQ_{r-2}$, $00CQ_{r-2}$ by definition 02, and in the same component $00CQ_{r-2}$ (induction hypothesis) using $R_{l}$ of situation 2, case 3 (figure 13).

Edges between $033PQT_{k-2}$, $0332PQT_{k-3}$ or $033PQT_{k-3}$ are embedded onto paths between $10CQ_{r-2}$, $11CQ_{r-2}$ of $1CQ_{r-4}$ using $R$ of situation 1, case 1 (figure 9, induction hypothesis).

Edges of $0332PQT_{k-3}$ or $033PQT_{k-3}$ are embedded onto paths in the same component $11CQ_{r-2}$ (induction hypothesis), and between $11CQ_{r-2}$, $01CQ_{r-2}$ by definition 02 using $R_{l}$ of situation 2, case 3 (figure 13).

Theorem 2. For any $n > 5$, a particular sub-quadtree $PQT_{n}$ is dilation two one-by-one edges embedding onto m-dimensional crossed cubes $CQ_{m}$.

Proof. We prove theorem 2 by induction on $n$.

Base. For $n = 6, 8$: edges of $PQT_{6}$, $PQT_{8}$ are respectively embedded using the rules specified in table 8, table 9.

| $PQT$ edge | crossed cubes path | Dilation |
|-----------|-------------------|----------|
| 0-01      | 0000000000-0100000000 | 1        |
| 0-02      | 0000000000-1000000000 | 1        |
| 0-03      | 0000000000-1000000000-1100000000 | 2        |

Table 8: Edges embedding of $PQT_{6}$ into $CQ_{9}$.

Induction hypothesis

Suppose that for $k \leq n - 1$, $PQT_{k}$ is dilation two one-by-one edges embedding onto $CQ_{l}$ with $l < m$ is true.

Is it true for $k = n$ ?

There are two cases:

Case a: $C = \phi$

Dilation two one-by-one edges embedding between the root 0 and respectively 01$PQT_{k-1}$, 02$PQT_{k-1}$, and 03$PQT_{k-1}$ onto paths respectively between $00CQ_{r-2}$ and $01CQ_{r-2}$, $00CQ_{r-2}$ and $10CQ_{r-2}$, $00CQ_{r-2}$ and $11CQ_{r-2}$. In this case, we use the same actions as lemma 3.

Case b: $C \neq \phi$

Dilation two one-by-one edges embedding between the root 0 and respectively 01$PQT_{k-1}$, 02$PQT_{k-1}$, and 03$PQT_{k-1}$ onto paths in the same supernode $00CQ_{r-2}$, and between $00CQ_{r-2}$, $10CQ_{r-2}$ (situation 2, case 1, a figure 11). Dilation two one-by-one edges of 01$PQT_{k-1}$, 02$PQT_{k-1}$, and 03$PQT_{k-1}$ respectively onto paths in $00CQ_{r-1}$, $1CQ_{r-1}$, and both $0CQ_{r-1}$, $1CQ_{r-1}$.

In this case, we use the same actions as lemma 4 except the edges of sub-$PQT$: 0111$PQT_{k-3}$, 0311$PQT_{k-3}$, and 0321$PQT_{k-3}$ are embedded like situation 2, case 1, b (figure 11). Moreover, edges of sub-$PQT$: 01113$PQT_{k-4}$, 03113$PQT_{k-4}$, and 03213$PQT_{k-4}$ are embedded as situation 2, case 2, b (figure 12).

In this paper, we have proposed a new function for embedding $PQT_{n}$ into $CQ_{m}$. The main purpose is dilation two one-by-one embedding $PQT_{n}$ into $CQ_{m}$. The study of dilation of this function is explained in three steps. The first step is dilation two one-by-one embeddings all edges onto paths in the same $CQ_{l}$ of any supernode of $CQ_{m}$ as proved by lemma 3. The second step is dilation two one-by-one embeddings all edges onto paths in the same $CQ_{l}$ of any supernode of $CQ_{m}$ as proved by lemma 4. The third step is the general dilation two one-by-one embeddings all edges of $PQT_{n}$ onto paths between two supernodes of $CQ_{m}$ as proved by theorem 2.

As a perspective, it is more interesting to study the fault-tolerant embedding of $PQT_{n}$ into $CQ_{m}$.

4. Conclusion
CRediT authorship contribution statement

Selmī Aymen Takie eddine: Conceptualization, Writing-original draft, Writing-review & editing, Investigation. Mohamed Faouzi Zerarka: Conceptualization, Writing-review & editing. Abdelhakim Cheriet: Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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