Hydrodynamic limit for an evolutionary model of two-dimensional Young diagrams

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Abstract

We construct dynamics of two-dimensional Young diagrams, which are naturally associated with their grandcanonical ensembles, by allowing the creation and annihilation of unit squares located at the boundary of the diagrams. The grandcanonical ensembles, which were introduced by Vershik [17], are uniform measures under conditioning on their size (or equivalently, area). We then show that, as the averaged size of the diagrams diverges, the corresponding height variable converges to a solution of a certain non-linear partial differential equation under a proper hydrodynamic scaling. Furthermore, the stationary solution of the limit equation is identified with the so-called Vershik curve. We discuss both uniform and restricted uniform statistics for the Young diagrams.

1 Introduction

The asymptotic shapes of two-dimensional random Young diagrams with large size were studied by Vershik [17] under several types of statistics including the uniform and restricted uniform statistics, which were also called the Bose and Fermi statistics, respectively. To each partition \( p = \{p_1 \geq p_2 \geq \cdots \geq p_j \geq 1\} \) of a positive integer \( n \) by positive integers \( \{p_i\}_{i=1}^j \) (i.e., \( n = \sum_{i=1}^j p_i \)), a Young diagram is associated by piling up \( j \) sticks of height 1 and side-length \( p_i \), more precisely, the height function of the Young diagram is defined by

\[
\psi_p(u) = \sum_{i=1}^j 1_{\{u < p_i\}}, \quad u \geq 0.
\]

The closure of the interior of its ordinate set is called the Young diagram of the partition \( p \). Note that, in most literatures, the figures of Young diagrams are upside-down compared with the graph defined by \( (\text{i}) \).

For each fixed \( n \), the uniform statistics (U-statistics in short) \( \mu^n_U \) assigns an equal probability to each of possible partitions \( p \) of \( n \), i.e., to the Young diagrams of area \( n \). The

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restricted uniform statistics (RU-statistics in short) $\mu^n_R$ also assigns an equal probability, but restricting to the distinct partitions satisfying $q = \{q_1 > q_2 > \cdots > q_j \geq 1\}$. These probabilities are called canonical ensembles. Grandcanonical ensembles $\mu^n_U$ and $\mu^n_R$ with parameter $0 < \varepsilon < 1$ are defined by superposing the canonical ensembles in a similar manner known in statistical physics, see (2.2) and (2.9) below. Vershik [17] proved that, under the canonical U- and RU-statistics $\mu^n_U$ and $\mu^n_R$ (with $n = N^2$), the law of large numbers holds as $N \to \infty$ for the scaled height variable

$$
\tilde{\psi}^N_p(u) := \frac{1}{N} \psi_p(Nu), \quad u \geq 0,
$$

of the Young diagrams $\psi_p(u)$ with size (i.e., area) $N^2$ and for $\tilde{\psi}^N_q(u)$ defined similarly, and the limit shapes $\psi_U$ and $\psi_R$ are given by

$$
\psi_U(u) = -\frac{1}{\alpha} \log (1 - e^{-\alpha u}) \quad \text{and} \quad \psi_R(u) = \frac{1}{\beta} \log (1 + e^{-\beta u}), \quad u \geq 0,
$$

with $\alpha = \pi/\sqrt{6}$ and $\beta = \pi/\sqrt{12}$, respectively. These results can be extended to the corresponding grandcanonical ensembles $\mu^n_U$ and $\mu^n_R$, if the averaged size of the diagrams is $N^2$ under these measures. Such types of results are usually called the equivalence of ensembles in the context of statistical physics. The corresponding central limit theorem and large deviation principle (under canonical ensembles) were shown by Pittel [14] and Dembo et. al. [5], respectively. All these results are at the static level.

The purpose of this paper is to study and extend these results from a dynamical point of view. We will see that, to the grandcanonical U- and RU-statistics, one can associate a weakly asymmetric zero-range process $p_t$ respectively a weakly asymmetric simple exclusion process $q_t$ on a set of positive integers with a stochastic reservoir at the boundary site $\{0\}$ in both processes as natural time evolutions of the Young diagrams, or more precisely, those of the gradients of their height functions. Then, under the diffusive scaling in space and time and choosing the parameter $\varepsilon = \varepsilon(N)$ of the grandcanonical ensembles such that the averaged size of the Young diagrams is $N^2$, we will derive the hydrodynamic equations in the limit and show that the Vershik curves defined by (1.3) are actually unique stationary solutions to the limiting non-linear partial differential equations in both cases.

In Section 2, after defining the ensembles and the corresponding dynamics, we formulate our main theorems, see Theorems 2.1 and 2.2. In Section 3, we study the asymptotic behaviors of $\varepsilon(N)$ as $N \to \infty$. The weakly asymmetric zero-range process $p_t$ with a stochastic reservoir at the boundary $\{0\}$ can be transformed into the weakly asymmetric simple exclusion process $\tilde{\eta}_t$ on $\mathbb{Z}$ without any boundary condition. In Section 4, we study such transformations and also those for the limit equations, and give the proof of the main theorem for the U-case (i.e. the case corresponding to the U-statistics). The hydrodynamic limit for $\tilde{\eta}_t$ is indeed already known [9], and we apply this result for $\tilde{\eta}_t$. The idea of transforming $p_t$ into $\tilde{\eta}_t$, which is indeed known in the study of particle systems, is useful to avoid the difficulty in treating singularities at the boundary $u = 0$, which appear in the limit of $\tilde{\psi}^N_p(u)$. The main theorem for the RU-case (i.e. the case corresponding to the RU-statistics) is proved in Section 5. Our method is to apply the Hopf-Cole transformation for the microscopic process $q_t$, which was originally introduced by Gärtner [9].
transformation linearizes the leading term in the time evolution $q_t$ even at the microscopic level so that one can avoid to show the one-block and two blocks’ estimates, which are usually required in the procedure establishing the hydrodynamic limit. The only task left is to study the boundary behavior of the transformed process, but a rather simple argument leads to the desired ergodic property of our process at the boundary, see Lemma 5.7 below.

The corresponding dynamic fluctuations will be discussed in a separate paper [8]. Our dynamics can be interpreted as evolitional models of (non-increasing) interfaces which separate $\pm$-phases in a zero-temperature two-dimensional Ising model defined on a first quadrant, see Spohn [16] and Remark 2.2 below. A randomly growing Young diagram was studied by Johansson [10], [11] in relation to random matrices. See [7, Section 16.4] for a quick review of some related results.

2 Two-dimensional Young diagrams and main results

In this section, for U- and RU-statistics individually, we define the grandcanonical and canonical ensembles, introduce the corresponding dynamics and then formulate the main results concerning the space-time scaling limits for them. Throughout the paper, we will use the following notation: $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^+_\infty = (0, \infty)$.

2.1 U-statistics

For each $n \in \mathbb{N}$, we denote by $\mathcal{P}_n$ the set of all partitions of $n$ into positive integers, that is, the set of all $p = (p_i)_{i \in \mathbb{N}}$ satisfying $p_1 \geq p_2 \geq \cdots \geq p_i \geq \cdots$, $p_i \in \mathbb{Z}_+$ and $\sum_{i \in \mathbb{N}} p_i = n$. For $n = 0$, we define $\mathcal{P}_0 = \{0\}$, where 0 is a sequence such that $p_i = 0$ for all $i \in \mathbb{N}$. We consider $p$ as an infinite sequence by adding infinitely many 0’s rather than a finite sequence as in Section 1. This will be convenient from the point of view of the corresponding particle system. The union of $\mathcal{P}_n$ is denoted by $\mathcal{P}$: $\mathcal{P} = \cup_{n \in \mathbb{Z}_+} \mathcal{P}_n$. The sum of $p_i$’s in $p \in \mathcal{P}$ is described as $n(p)$: $n(p) = \sum_{i \in \mathbb{N}} p_i$, and called the size or area of $p$.

For $p \in \mathcal{P}$, we assign a right continuous non-increasing step-function $\psi_p$ on $\mathbb{R}_+$ called the height function as follows:

\[
\psi_p(u) = \sum_{i \in \mathbb{N}} 1_{\{u < p_i\}}, \quad u \in \mathbb{R}_+.
\]

In particular, we always have $\int_0^\infty \psi_p(u) du = n(p)$.

For $0 < \varepsilon < 1$, let $\mu^\varepsilon_U$ be the probability measure on $\mathcal{P}$ determined by

\[
\mu^\varepsilon_U(p) = \frac{1}{Z_U(\varepsilon)} e^{n(p)}, \quad p \in \mathcal{P},
\]

where $Z_U(\varepsilon) = \prod_{k=1}^\infty (1 - \varepsilon^k)^{-1} \left( = \sum_{n=0}^\infty p(n) \varepsilon^n, p(n) = \# \mathcal{P}_n \right)$ is the normalizing constant. The measure $\mu^\varepsilon_U$ has the property $\mu^\varepsilon_U|_{\mathcal{P}_n}(p) = \mu^n_U(p), p \in \mathcal{P}$, where $\mu^\varepsilon_U|_{\mathcal{P}_n}$ stands for the conditional probability of $\mu^\varepsilon_U$ on $\mathcal{P}_n$ and $\mu^n_U$ is the uniform probability measure on $\mathcal{P}_n$. 
The measures \( \mu^\varepsilon_U \) and \( \mu^n_U \) play similar roles to the grandcanonical and canonical ensembles in statistical physics, respectively.

Now, we construct dynamics of two-dimensional Young diagrams, which have \( \mu^\varepsilon_U \) as their invariant measures. Let \( p_t \equiv p^\varepsilon_t = (p^i_t)_{i \in \mathbb{N}} \) be the Markov process on \( \mathcal{P} \) defined by means of the infinitesimal generator \( L_{\varepsilon,U} \) acting on functions \( f : \mathcal{P} \to \mathbb{R} \) as

\[
L_{\varepsilon,U} f(p) = \sum_{i \in \mathbb{N}} \left[ \varepsilon 1_{\{p_{i-1} > p_i\}} \{f(p^{i,+}) - f(p)\} + 1_{\{p_i > p_{i+1}\}} \{f(p^{i,-}) - f(p)\} \right],
\]

where \( p^{i,\pm} = (p^{j,\pm}_{j})_{j \in \mathbb{N}} \in \mathcal{P} \) are defined by

\[
p^{i,\pm}_{j} = \begin{cases} 
p_j & \text{if } j \neq i, \\
p_{i} \pm 1 & \text{if } j = i.
\end{cases}
\]

In (2.3), we regard \( p_0 = \infty \). Note that \( n(p_t) \) and \( n(p_t) : = \sharp \{i \in \mathbb{N} ; p_i(t) \geq 1\} \) change in time but always stay finite. It is easy to see that \( \mu^\varepsilon_U \) is invariant under such dynamics for every \( 0 < \varepsilon < 1 \) by showing that \( \sum_{p \in \mathcal{P}} L_{\varepsilon,U} f(p) \mu^\varepsilon_U (p) = 0 \) for a sufficiently wide class of functions \( f \). We will think of \( p_i(t) \) as the position of the \( i \)th particle. The total number of particles \( n(p_t) \) on the region \( \mathbb{N} \) changes only through the creation and annihilation of particles at the boundary site \( \{0\} \). In fact, the first part in the sum (2.3) with \( i = n(p) + 1 \) represents that a new particle is provided from the boundary site \( \{0\} \) to the site \( \{1\} \) with rate \( \varepsilon \), while the second part with \( i = n(p) \) indicates that a particle at \( \{1\} \) jumps to \( \{0\} \) and disappears with rate 1. In other words, a stochastic reservoir is located at the boundary site \( \{0\} \) of \( \mathbb{N} \).

For a probability measure \( \nu \) on \( \mathcal{P} \) and \( N \geq 1 \), we denote by \( \mathbb{P}^N_{\nu} \) the distribution on the path space \( D(\mathbb{R}_+, \mathcal{P}) \) of the process \( p_t \equiv p^N_t \) with generator \( N^2 L_{\varepsilon(N),U} \), which is accelerated by the factor \( N^2 \) and the initial measure \( \nu \). Here, \( \varepsilon(N) \) is defined by the relation:

\[
E_{\mu^\varepsilon_{(N)}}[n(p)] = N^2.
\]

Let \( X_U \) be the function space defined by

\[
X_U := \{ \psi : \mathbb{R}_+^2 \to \mathbb{R}_+ ; \psi \in C^1, \psi' < 0, \lim_{u \to 0} \psi(u) = \infty, \lim_{u \to \infty} \psi(u) = 0 \},
\]

where \( \psi' = d\psi/du \). With these notations our first main theorem is stated as follows. Recall that the scaled height variable \( \tilde{\psi}_P^N(u) \) is defined by (1.2) for \( p \in \mathcal{P} \).

**Theorem 2.1.** Let \( (\nu^N)_{N \geq 1} \) be a sequence of probability measures on \( \mathcal{P} \) such that

\[
\lim_{N \to \infty} \nu^N[ \sup_{u \in [u_0,u_1]} |\tilde{\psi}_P^N(u) - \psi_0(u)| > \delta ] = 0
\]

holds for every \( \delta > 0, 0 < u_0 < u_1 \) and some function \( \psi_0 \in X_U \). Then, for every \( t > 0 \),

\[
\lim_{N \to \infty} \mathbb{P}^N_{\nu^N}[ \int_0^\infty f(u)\tilde{\psi}_P^N(u)du - \int_0^\infty f(u)\psi(t,u)du > \delta ] = 0
\]
holds for every $\delta > 0$ and $f \in C_0(\mathbb{R}_+^0)$, where $C_0(\mathbb{R}_+^0)$ is the class of all functions $f \in C(\mathbb{R}_+^0)$ having compact supports in $\mathbb{R}_+^0$ and $\psi(t, u)$ is the unique classical solution (in the space $X_U$) of the non-linear partial differential equation (PDE):

$$
\begin{cases}
\partial_t \psi = \partial_u \left( \frac{\partial_u \psi}{1 - \partial_u \psi} \right) + \alpha \frac{\partial_u \psi}{1 - \partial_u \psi}, \\
\psi(0, \cdot) = \psi_0(\cdot), \\
\psi(t, \cdot) \in X_U, \quad t \geq 0,
\end{cases}
$$

(2.8)

where $\partial_t \psi = \partial \psi/\partial t$, $\partial_u \psi = \partial \psi/\partial u$ and $\alpha = \pi/\sqrt{6}$.

**Remark 2.1.** The function $\psi_U$ defined in (1.3) is the unique stationary solution in the class $X_U$ of the equation (2.8). The curve in the first quadrant of $xy$-plane determined by the equation $y = \psi_U(x)$ is called the Vershik curve (in $U$-statistics).

In this way, the derivation of the Vershik curve is understandable from the dynamical point of view.

**Remark 2.2.** Spohn discussed in [16, Appendix A] two-dimensional interfacial dynamics, motivated by the zero-temperature Ising model, under the periodic boundary condition with symmetric jump rates and derived the non-linear PDE (2.8) with $\alpha = 0$ under the hydrodynamic scaling limit. He studied interfaces having graphical representations as in our setting, but not necessarily being monotone. See [2, Section 4], [3], [4] for further studies. Shlosman [15] discussed the similarity between the approach from the Young diagrams and the Wulff problem in the Ising model. Aldous and Diaconis [1] used an idea of the hydrodynamic limit to give a “soft” proof for the asymptotic behavior of the length of the longest increasing subsequence of random permutations.

**Remark 2.3.** The large deviation rate function $I(\psi)$ under the canonical ensemble of $U$-statistics $\mu_U$ is described in Theorem 1.2 of [5]. We can compute its functional derivative and find that it is given by the formula:

$$
\frac{\delta I}{\delta \psi(u)} = \frac{\psi''(u) + \alpha \psi'(u)(1 - \psi'(u))}{\psi'(u)(1 - \psi'(u))}.
$$

On the other hand, the right hand side of our hydrodynamic equation (2.8) is equal to

$$
\frac{\psi''(u) + \alpha \psi'(u)(1 - \psi'(u))}{(1 - \psi'(u))^2}.
$$

These formulas have similarity but are not exactly the same. Recall that we discuss the dynamics associated with the grandcanonical ensemble. The dynamics for the canonical ensemble involve much complexity.

### 2.2 RU-statistics

Denote by $Q_n$ the set of all partitions of $n \in \mathbb{N}$ into distinct positive integers, that is, the set of all $q = (q_i)_{i \in \mathbb{N}} \in P_n$ satisfying $q_i > q_{i+1}$ if $q_i > 0$. The union of $Q_n$ is denoted by $Q$: 5
Q = \bigcup_{n \in \mathbb{Z}^+} Q_n$, where $Q_0 = \mathcal{P}_0$. Let $n(q)$ be the sum of $q_i$'s in $q \in Q$. The function $\psi_q$ on $\mathbb{R}_+$ is assigned to $q \in Q$ by the relation (2.1).

For $0 < \varepsilon < 1$, let $\mu^\varepsilon_R$ be the probability measure on $Q$ determined by

$$
\mu^\varepsilon_R(q) = \frac{1}{Z_R(\varepsilon)} e^{n(q)}, \quad q \in Q,
$$

where $Z_R(\varepsilon) = \prod_{k=1}^{\infty} (1 + e^k) = \sum_{n=0}^{\infty} p_\neq(n) e^n, p_\neq = Q_n$ is the normalizing constant.

The conditional measure $\mu^\varepsilon_R|Q_n$ of $\mu^\varepsilon_R$ on $Q_n$ coincides with the uniform probability measure $\mu_R^n$ on $Q_n$. The measures $\mu^\varepsilon_R$ and $\mu_R^n$ are the grandcanonical and canonical ensembles in the RU-statistics, respectively.

Now, we construct the dynamics associated with $\mu^\varepsilon_R$. Let $q_t = (q_i(t))_{i \in \mathbb{N}}$ be the Markov process on $Q$ with the infinitesimal generator $L_{\varepsilon,R}$ acting on functions $f : Q \to \mathbb{R}$ as

$$
L_{\varepsilon,R}f(q) = \sum_{i \in \mathbb{N}} [\varepsilon 1_{\{q_{i-1} > q_i + 1\}} (f(q^{i+}) - f(q)) + 1_{\{q_i > q_{i+1} + 1 \text{ or } q_i = 1\}} (f(q^{i-}) - f(q))],
$$

where $q^{i \pm} \in Q$ are defined by the formula (2.1) and we regard $q_0 = \infty$. It is easy to see that $\mu^\varepsilon_R$ is invariant under such dynamics. Similarly to the U-case, the model defined by the generator (2.10) involves a stochastic reservoir at $\{0\}$. The only difference is that the creation of a new particle at $\{1\}$ is allowed if this site is vacant.

For a probability measure $\nu$ on $Q$ and $N \geq 1$, we denote by $Q_\nu^N$ the distribution on the path space $D(\mathbb{R}_+, Q)$ of the process $q_t = q_t^N$ with generator $N^2 L_{\varepsilon(N),R}$ and the initial measure $\nu$. Here, $\varepsilon(N)$ is defined by the relation:

$$
E_{\mu^\varepsilon_R}[n(p)] = N^2.
$$

Let $X_R$ be the function space defined by

$$
X_R := \{ \psi : \mathbb{R}_+ \to \mathbb{R}_+; \psi \in C^1, -1 \leq \psi' \leq 0, \psi'(0) = -1/2, \lim_{u \to \infty} \psi(u) = 0 \}.
$$

Our second main theorem is stated as follows. The scaled height variable $\tilde{\psi}^N_q(u)$ is defined by (1.2) for $q \in Q$.

**Theorem 2.2.** Let $(\nu^N)_{N \geq 1}$ be a sequence of probability measures on $Q$ such that

$$
\lim_{N \to \infty} \nu^N \left[ \int_0^\infty f(u) \tilde{\psi}^N_q(u) du - \int_0^\infty f(u) \psi_0(u) du \right] > \delta = 0
$$

holds for every $\delta > 0$, $f \in C_0(\mathbb{R}_+)$ and some function $\psi_0 \in X_R$. Then, for every $t > 0$,

$$
\lim_{N \to \infty} Q^N_{\nu^N} \left[ \int_0^\infty f(u) \tilde{\psi}^N_q(u) du - \int_0^\infty f(u) \psi(t,u) du \right] > \delta = 0
$$

holds for every $\delta > 0$ and $f \in C_0(\mathbb{R}_+)$, where $\psi(t,u)$ is the unique classical solution (in the space $X_R$) of the non-linear partial differential equation:

$$
\begin{cases}
\partial_t \psi = \partial_u^2 \psi + \beta \partial_u \psi(1 + \partial_u \psi), \\
\psi(0, \cdot) = \psi_0(\cdot), \\
\psi(t, \cdot) \in X_R, \quad t \geq 0,
\end{cases}
$$

and $\beta = \pi/\sqrt{12}$. 

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Remark 2.4. The function $\psi_R$ defined in (1.3) is the unique stationary solution in the class $X_R$ of the equation (2.14). The curve determined by the equation $y = \psi_R(x)$ is called the Vershik curve (in RU-statistics).

3 Asymptotic behaviors of $\varepsilon(N)$

Before giving the proof of our main theorems, we study in this section the asymptotic behaviors of $\varepsilon(N)$ defined by (2.5) and (2.11) in U- and RU-statistics, respectively, as $N \to \infty$.

3.1 U-statistics

Let $\varepsilon(N)$ be defined by the relation (2.5).

Lemma 3.1. We have that $\varepsilon(N) = 1 - \alpha/N + O(\log N/N^2)$ as $N \to \infty$.

Proof. First, we calculate the expected value of the size $n(p)$ of $p \in \mathcal{P}$ under the probability measure $\mu_U$. In fact,

$$
E_{\mu_U}[n(p)] = \frac{1}{Z_U(\varepsilon)} \sum_{p} n(p) \varepsilon^{n(p)} = \varepsilon \left( \log Z_U(\varepsilon) \right)'
$$

The last equality follows from the simple identity $\sum_{k=1}^{\infty} z^k/k = z/(1-z)^2$ for $0 \leq z < 1$. However, the inequality of arithmetic and geometric means and some simple estimations prove

$$
\frac{1}{m} \leq \frac{1}{1 + \varepsilon + \varepsilon^2 + \ldots + \varepsilon^{m-1}} \leq \frac{\varepsilon^{(-m+1)/2}}{m},
$$

and thus, recalling $\alpha^2 = \pi^2/6$ and $\varepsilon < 1$, we have that

$$
\frac{1}{(1-\varepsilon)^2} \sum_{m=1}^{\infty} \frac{\varepsilon^m}{m^2} \leq E_{\mu_U}[n(p)] \leq \frac{1}{(1-\varepsilon)^2} \sum_{m=1}^{\infty} \varepsilon \frac{\varepsilon^m}{m^2} < \frac{\alpha^2}{(1-\varepsilon)^2}.
$$

Therefore, by (2.5), we have for $\varepsilon = \varepsilon(N)$

$$(3.1) \quad 0 < \alpha^2 - (1-\varepsilon)^2 N^2 \leq \alpha^2 - \sum_{m=1}^{\infty} \frac{\varepsilon^m}{m^2} = \sum_{m=1}^{\infty} \frac{1 - \varepsilon^m}{m^2}.
$$

Since the right hand side tends to 0 as $\varepsilon \uparrow 1$ (or as $N \to \infty$), we see that $(1-\varepsilon)N$ tends to $\alpha$ as $N \to \infty$ which implies that $\varepsilon \equiv \varepsilon(N) = 1 - \alpha/N + o(1/N)$. To derive more precise estimate for the error term, we will show that the right hand side of (3.1) admits a bound:

$$(3.2) \quad \sum_{m=1}^{\infty} \frac{1 - \varepsilon^m}{m^2} \leq \frac{C \log N}{N},$$

with some $C > 0$. Indeed, once this is shown, the proof of the lemma is concluded. To prove (3.2), noting that the function $f_\varepsilon(x) := (1 - \varepsilon^2)/x^2$, $x > 0$, is non-increasing, we have that

$$
\sum_{m=1}^{\infty} \frac{1 - \varepsilon^m}{m^2} \leq f_\varepsilon(1) + \int_1^{\infty} f_\varepsilon(x) \, dx = f_\varepsilon(1) - \log \varepsilon \int_{-\log \varepsilon}^{\infty} \frac{1 - e^{-y}}{y^2} \, dy
$$

$$
\leq (1 - \varepsilon) - \log \varepsilon \left( - \log(-\log \varepsilon) \right) - \log \varepsilon,
$$

where the last inequality follows by dividing the integral over $[-\log \varepsilon, \infty)$ into the sum of those over $[-\log \varepsilon, 1]$ and $[1, \infty)$ and then by estimating the integrands by $1/y$ and $1/y^2$, respectively. This implies (3.2) by recalling $\varepsilon = 1 - \alpha/N + o(1/N)$.

\[\square\]

\textbf{Remark 3.1.} A rude version of Hardy-Ramanujan’s formula: $p(n) = \#P_n \sim e^{\sqrt{2/\pi} \sqrt{n}}$ as $n \to \infty$ implies that

$$
E_{\mu_\varepsilon R}[n(p)] \sim \sum_{n=1}^{\infty} p n^{e^{\sqrt{2/\pi} \sqrt{n} - (\log \varepsilon^{-1})} n} Z_\varepsilon(n).
$$

Since the function $f(x) := \sqrt{2/3\pi} \sqrt{x} - (\log \varepsilon^{-1}) x$, $x > 0$ attains its maximal value at $x(\varepsilon) = (\pi/(\sqrt{6} \log \varepsilon^{-1}))^2$, we see that $E_{\mu_\varepsilon R}[n(p)]$ behaves as $x(\varepsilon)$ as $\varepsilon \uparrow 1$ and this shows that $\varepsilon(N) \sim 1 - \alpha/N$ as $N \to \infty$.

\subsection*{3.2 RU-statistics}

Let $\varepsilon(N)$ be defined by the relation (2.11).

\textbf{Lemma 3.2.} We have that $\varepsilon(N) = 1 - \beta/N + O(\log N/N^2)$ as $N \to \infty$.

\textbf{Proof.} First, we calculate the expected value of $n(p)$ under $\mu_\varepsilon R$:

$$
E_{\mu_\varepsilon R}[n(p)] = \frac{1}{Z_\varepsilon(\varepsilon)} \sum_p n(p) \varepsilon^{n(p)} = \varepsilon \left( \log Z_\varepsilon(\varepsilon) \right) Z_\varepsilon(\varepsilon) = \sum_{k=1}^{\infty} \frac{k \varepsilon^k}{1 + \varepsilon^k} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} k(-1)^{m-1} \varepsilon^{mk}.
$$

Thus, similarly to the U-case, we have that

$$
E_{\mu_\varepsilon R}[n(p)] = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\varepsilon^m}{(1 - \varepsilon^m)^2} = \Sigma_o(\varepsilon) - \Sigma_e(\varepsilon),
$$

where $\Sigma_o(\varepsilon)$ and $\Sigma_e(\varepsilon)$ are the sums taken over odd and even numbers, respectively, i.e.

$$
\Sigma_o(\varepsilon) := \sum_{m=1}^{\infty} \frac{\varepsilon^{2m-1}}{(1 - \varepsilon^{2m-1})^2}, \quad \Sigma_e(\varepsilon) := \sum_{m=1}^{\infty} \frac{\varepsilon^{2m}}{(1 - \varepsilon^{2m})^2}.
$$

Note that one can change the order of the sum in (3.3) since the series converges absolutely.

Now recalling $\beta^2 = \alpha^2/2$, by (2.11), we have for $\varepsilon = \varepsilon(N)$

$$
\beta^2 - (1 - \varepsilon)^2 N^2 = \left\{ \alpha^2 - (1 - \varepsilon)^2 (\Sigma_o(\varepsilon) + \Sigma_e(\varepsilon)) \right\} - \left\{ \alpha^2 - 4(1 - \varepsilon)^2 \Sigma_e(\varepsilon) \right\}/2.
$$
However, since $\Sigma_\varepsilon(x) = \Sigma_{e}(\varepsilon^2) + \Sigma_\varepsilon(e^2)$ and $(1 + \varepsilon)^2 \leq 4$, we have that
\[
|\beta^2 - (1 - \varepsilon)^2N^2| \leq |\alpha^2 - (1 - \varepsilon)^2(\Sigma_{0}(\varepsilon) + \Sigma_{e}(\varepsilon))| + |\alpha^2 - (1 - \varepsilon)^2(\Sigma_{0}(\varepsilon^2) + \Sigma_{e}(e^2))|/2
\geq \{(1 - \varepsilon) - \log(\varepsilon) - \log(-\log \varepsilon)\} - \log \varepsilon + \{(1 - \varepsilon^2) - 2 \log(\varepsilon) - 2 \log \varepsilon\}/2.
\]
The second inequality is shown in the proof of Lemma 3.1. This first implies that $\varepsilon \equiv \varepsilon(N) = 1 - \beta/N + o(1)$ and then completes the proof of the lemma as in the last part of the proof of Lemma 3.1.

The precise error estimates in Lemmas 3.1 and 3.2 are only needed in [8], see Remark 5.1 below.

### 4 Proof of Theorem 2.1

This section gives the proof of Theorem 2.1 for the U-case. In the process $p_t$, the particles are distinguished from each other and numbered from the right. However, if we are only concerned with the number of particles at each site and define $\xi_t = (\xi_t(x))_{x \in \mathbb{Z}_+}$ by $\xi_t(x) = \sharp\{i; p_t(i) = x\} \in \mathbb{Z}_+$ for $x \in \mathbb{N}$ and $\xi_t(0) = \infty$, then $\xi_t$ becomes the weakly asymmetric zero-range process on $\mathbb{N}$ with the weakly asymmetric stochastic reservoir at $\{0\}$. We can think of $\xi_t(x)$ as the (negative) gradient of the height function $\psi_{p_t}$ at $u = x$ in the sense that $\xi_t(x) = \psi_{p_t}(x) - \psi_{p_t}(x-1)$.

Actually, the stochastic reservoir for $p_t$ or $\xi_t$ located at $\{0\}$ can be removed under a simple transformation. Indeed, we transform the process $p_t$ into another process $\tilde{\eta}_t$ on $\mathbb{Z}$, which is roughly defined as follows: With each $p \in \mathcal{P}$, we associate a family of particles located at $(i, p_i)$ in the $xy$-plane and project them perpendicularly to the line $\{y = -x\}$ rescaled by $\sqrt{2}$. Or, one can say that we first rotate the $xy$-plane by 45 degree to the left-handed direction and then project the particles to the $x$-axis rescaled by $\sqrt{2}$. This determines a configuration $\tilde{\eta}$ on $\mathbb{Z}$. Such transformation is sometimes used in the study of particle systems. As we will see, in the RU-case, one cannot find this kind of nice transformation which removes the stochastic reservoir.

#### 4.1 Transformation for the process $p_t$

We introduce a transformation of our process $p_t$ on $\mathbb{N}$ to a weakly asymmetric simple exclusion process $\tilde{\eta}_t$ on $\mathbb{Z}$ mentioned above. Denote by $\chi_U$ the state space of the transformed process:
\[
\chi_U := \{\tilde{\eta} \in \{0, 1\}^\mathbb{Z}; \sum_{x < 0} (1 - \tilde{\eta}(x)) = \sum_{x \geq 1} \tilde{\eta}(x) < \infty\}.
\]
In particular, if $\tilde{\eta} \in \chi_U$, then there exist $x_+ \in \mathbb{Z}$ such that $\tilde{\eta}(x) = 1$ for all $x \leq x_-$ and $\tilde{\eta}(x) = 0$ for all $x \geq x_+$. For $\tilde{\eta} \in \chi_U$, we assign two functions $\zeta^-_{\tilde{\eta}}$ and $\zeta^+_{\tilde{\eta}}$ on $\mathbb{Z}$ by the following rule:
\[
(4.1) \quad \zeta^-_{\tilde{\eta}}(x) = \sum_{z \leq x} (1 - \tilde{\eta}(z)) \quad \text{and} \quad \zeta^+_{\tilde{\eta}}(x) = \sum_{z \geq x+1} \tilde{\eta}(z), \quad x \in \mathbb{Z}.
\]
By definition, $\zeta^-_\eta$ and $\zeta^+_\eta$ are monotone non-negative integer-valued functions. Now, we construct one-to-one correspondence between $\chi_U$ and $\mathcal{P}$. For $\eta \in \chi_U$, we assign $p_\eta^\tilde{\eta} = (p_i^\tilde{\eta})_{i \in \mathbb{N}} \in \mathcal{P}$ by the following rule:

$$p_i^\tilde{\eta} = \zeta^+_\eta(x_i), \quad i \in \mathbb{N},$$

where $x_i$ is the unique element of $\mathbb{Z}$ which satisfies $\zeta^+_\eta(x_i - 1) = i$ and $\zeta^+_\eta(x_i) = i - 1$. In other words, the family $\{x_i\}_{i \in \mathbb{N}}$ is determined by numbering the set \{x \in $\mathbb{Z}$; $\eta(x) = 1$\} by $i \in \mathbb{N}$ from the right and $p_i^\tilde{\eta} = \sharp\{x \leq x_i; \eta(x) = 0\}$. We can show that the map $\eta \to p_\eta^\tilde{\eta}$ is well-defined and also it is a bijection from $\chi_U$ to $\mathcal{P}$. So we denote its inverse map by $p \to \tilde{\eta}^p$. Note that the origin $0$ is determined by the condition $\zeta^-_\eta(0) = \zeta^+_\eta(0)$ or equivalently $\sharp\{x \leq 0; \eta(x) = 0\} = \sharp\{x \geq 1; \eta(x) = 1\}$, i.e., the number of empty sites on the left to the origin is equal to that of particles on the right to the site $1$.

We now consider the Markov process $\tilde{\eta}_t$ on $\chi_U$ with the generator $\tilde{L}_{\varepsilon,U}$ acting on functions $f : \chi_U \to \mathbb{R}$ as

$$\tilde{L}_{\varepsilon,U}f(\tilde{\eta}) = \sum_{x \in \mathbb{Z}} \left[ \varepsilon c_+(x, \tilde{\eta}) + c_-(x, \tilde{\eta}) \right] \{ f(\tilde{\eta}^{x,x+1}) - f(\tilde{\eta}) \},$$

where

$$(4.2) \quad c_+(x, \tilde{\eta}) = 1_{\{\tilde{\eta}(x) = 1, \eta(x+1) = 0\}}, \quad c_-(x, \tilde{\eta}) = 1_{\{\tilde{\eta}(x) = 0, \eta(x+1) = 1\}},$$

and

$$(4.3) \quad \tilde{\eta}^{x,y}(z) = \begin{cases} \tilde{\eta}(z) & \text{if } z \neq x, y, \\ \tilde{\eta}(y) & \text{if } z = x, \\ \tilde{\eta}(x) & \text{if } z = y. \end{cases}$$

Note that the relation $\zeta^-_{\tilde{\eta}}(0) = \zeta^+_{\tilde{\eta}}(0)$ is invariant under the transition from $\tilde{\eta}$ to $\tilde{\eta}^{x,x+1}$ for all $x \in \mathbb{Z}$. The following lemma is easy so that the proof is omitted.

**Lemma 4.1.** Two processes $\{p_t\}_{t \geq 0}$ and $\{p^\tilde{\eta}_t\}_{t \geq 0}$ have the same distributions on the path space $D(\mathbb{R}^+, \mathcal{P})$.

For a probability measure $\nu$ on $\chi_U$ and $N \geq 1$, we denote by $\tilde{\mathbb{P}}^\mathbb{N}_\nu$ the distribution on the path space $D(\mathbb{R}^+, \chi_U)$ of the process $\tilde{\eta}^\mathbb{N}_t$ with generator $N^2 \tilde{L}_{\varepsilon(\mathbb{N}),U}$ and the initial measure $\nu$, where $\varepsilon(\mathbb{N})$ is defined by (2.3). By Lemma 4.1 since $\varepsilon(\mathbb{N})$ is close to $1$ for large $N$, we can think of the process $\tilde{\eta}^\mathbb{N}_t$ as a weakly asymmetric simple exclusion process on $\mathbb{Z}$. The hydrodynamic limit of such process is already known. Indeed, let $Y_U$ be the function space defined by

$$Y_U := \{ \rho : \mathbb{R} \to (0, 1); \rho \text{ is continuous}, \int_{-\infty}^{0} (1 - \rho(v)) dv = \int_{0}^{\infty} \rho(v) dv < \infty \}.$$ 

Then, for the scaled empirical measures of the process $\tilde{\eta}^\mathbb{N}_t$ defined by

$$(4.4) \quad \pi^\mathbb{N}_t(dv) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \tilde{\eta}^\mathbb{N}_t(x) \delta_{x/N}(dv), \quad t \geq 0, \quad v \in \mathbb{R},$$

we have the following proposition, see Gärtner [9]:
Proposition 4.2. Let \((\nu^N)_{N \geq 1}\) be a sequence of probability measures on \(\chi_U\) such that

\[
\lim_{N \to \infty} \nu^N([\int_{-\infty}^{\infty} g(v)\pi_0^N(dv) - \int_{-\infty}^{\infty} g(v)\rho_0(v)dv] > \delta) = 0
\]

holds for every \(\delta > 0\), \(g \in C_0(\mathbb{R})\) and some function \(\rho_0 \in Y_U\). Then, for every \(t > 0\),

\[
\lim_{N \to \infty} \bar{P}^N_{\nu^N}[\int_{-\infty}^{\infty} g(v)\pi_t^N(dv) - \int_{-\infty}^{\infty} g(v)\rho(t,v)dv] > \delta] = 0
\]

holds for every \(\delta > 0\) and \(g \in C_0(\mathbb{R})\), where \(\rho(t,v)\) is the unique classical solution of the following partial differential equation:

\[
\begin{aligned}
\partial_t \rho &= \partial^2_v \rho + \alpha \partial_v (\rho(1 - \rho)), \\
\rho(0,\cdot) &= \rho_0(\cdot).
\end{aligned}
\]

Kipnis et al. \[12\] also studied the hydrodynamic limit of weakly asymmetric simple exclusion processes under the periodic boundary conditions.

Remark 4.1. The unique solution of \((4.7)\) satisfies that \(\rho(t,\cdot) \in Y_U\) for all \(t > 0\) if \(\rho_0 \in Y_U\). This fact (except the equality of two integrals in the definition of \(Y_U\)) is seen by regarding the non-linear PDE \((4.7)\) as a linear PDE: \(\partial_t \rho = \partial^2_v \rho + b(t,v)\partial_v \rho\) with \(b(t,v) = \alpha(1 - 2\rho(t,v))\), in which \(\rho(t,v)\) is considered to be already given, and then by relying, for instance, on a probabilistic representation of \(\rho(t,v)\): \(\rho(t,v) = E_{\nu}[\rho_0(X_t^{(1)})]\) in terms of the solution \((X_s) = (X_s^{(f)})_{0 \leq s \leq t}\) of the stochastic differential equation: \(dX_s = \sqrt{2}dB_s + b(t-s,X_s)ds, 0 \leq s \leq t, X_0 = v\) for each \(t > 0\), where \(B_s\) is the one-dimensional Brownian motion. The equality of two integrals: \(\int_{-\infty}^{\infty} (1 - \rho(t,v))dv = \int_{-\infty}^{\infty} \rho(t,v)dv\) follows directly from the PDE \((4.7)\) or by taking limits from the microscopic systems.

Proposition \(4.2\) is formulated only for the test functions \(g\) having compact supports. We also need the following asymptotic behaviors of the tails of \(\pi_t^N\).

Lemma 4.3. Assume that the following two conditions \((4.8)\) and \((4.9)\) hold for \(t = 0\). Then, for every \(t > 0\), we have that

\[
\lim_{N \to \infty} \bar{P}^N_{\nu^N}[\pi_t^N([0,\infty)) - \int_{0}^{\infty} \rho(t,v)dv] > \delta] = 0,
\]

and

\[
\lim_{N \to \infty} \bar{P}^N_{\nu^N}[\pi_t^N((-\infty,0]) - \int_{-\infty}^{0} (1 - \rho(t,v))dv] > \delta] = 0,
\]

for every \(\delta > 0\), where

\[
\hat{\pi}_t^N(dv) = \frac{1}{N} \sum_{x \in \mathbb{Z}} (1 - \hat{\eta}_t^N(x))\delta_{x/N}(dv), \quad t \geq 0, \quad v \in \mathbb{R}.
\]
Proof. We easily see that \((4.6)\) holds for a step function \(g = 1_{(a,b)}\) with \(-\infty < a < b < \infty\), by approximating such \(g\) by a sequence of continuous functions \(g_n \in C_0(\mathbb{R})\) noting that \(0 \leq \tilde{\eta}(x), \rho(t,v) \leq 1\). Moreover, Remark 4.1 implies that both \(\int_{-\infty}^{K} (1 - \rho(t,v))dv\) and \(\int_{K}^{\infty} \rho(t,v)dv\) are arbitrarily small for large enough \(K > 0\). Thus, to prove \((4.8)\) and \((4.9)\), it is sufficient to show that for every \(\delta > 0\) there exists \(K > 0\) such that

\[
\lim_{N \to \infty} \mathbb{P}^N_{\nu} [\pi_i^N([K, \infty)) > \delta] = 0, \tag{4.10}
\]

and

\[
\lim_{N \to \infty} \mathbb{P}^N_{\nu} [\pi_i^N((-\infty, -K]) > \delta] = 0, \tag{4.11}
\]

respectively. We prove \((4.10)\) only, since the proof of \((4.11)\) is similar. To this end, take a function \(\varphi_1 \in C_0^2(\mathbb{R})\) satisfying that \(\varphi_1' \geq 0, \varphi_1(v) = 1\) for \(v \geq 1\) and \(\varphi_1(v) = 0\) for \(v \leq 1/2\), and set \(\varphi_K(v) := \varphi_1(v/K)\) for \(K > 0\). Then,

\[m_i^N(\varphi_K) := \langle \pi_i^N, \varphi_K \rangle - \langle \pi_0^N, \varphi_K \rangle - \int_0^t N^2 \tilde{L}_{\epsilon(N),U} (\pi_i^N, \varphi_K) ds\]

is a martingale and the following two bounds:

\[
N^2 \tilde{L}_{\epsilon(N),U}(\pi^N, \varphi_K) \leq \|\varphi''_K\|_\infty \times |\text{supp } \varphi_K| \leq \|\varphi''_K\|_\infty /2K, \tag{4.12}
\]

and

\[
E[m_i^N(\varphi_K)^2] \leq t \|\varphi''_K\|_\infty^2 \times |\text{supp } \varphi_K| / N \leq t \|\varphi'_K\|_\infty^2 /2KN, \tag{4.13}
\]

hold, where \(\langle \pi, \varphi \rangle = \int_{\mathbb{R}} \varphi(v) \pi dv\) and \(|\text{supp } \varphi|\) stands for the Lebesgue measure of the support of \(\varphi\). Indeed a similar computation is made in the proof of Proposition 5.4 below. Actually, because of the difference of the generators, the first sums in \((5.7)\) and \((5.8)\) below should be taken over \(x \in \mathbb{Z}\) rather than \(x \in \mathbb{N}\) and the second terms do not appear in the present setting. Moreover, since \(\varphi_K' \geq 0\), the first sum in \((5.7)\) is bounded from above by the same sum taken \(\epsilon = 1\) (because \(\epsilon < 1\)). However, since \(c_+(x, \tilde{\eta}) - c_-(x, \tilde{\eta}) = \tilde{\eta}(x) - \tilde{\eta}(x + 1)\), the bound \((4.12)\) follows by the summation by parts. Accordingly, we have

\[
\pi_i^N([K, \infty)) \leq \langle \pi_i^N, \varphi_K \rangle \leq \langle \pi_0^N, \varphi_K \rangle + t \|\varphi''_K\|_\infty /2K + m_i^N(\varphi_K). \]

Therefore, the condition \((4.8)\) for \(t = 0\) controls the behavior of \(\langle \pi_i^N, \varphi_K \rangle\) and proves \((4.10)\) with the help of \((4.13)\). \(\square\)

Remark 4.2. (1) The condition \((4.8)\) is equivalent to \((4.6)\) with \(g = 1_{(0, \infty)}\).

(2) The condition \((4.6)\) can be rewritten into an equivalent form \((4.6)'\), which is \((4.6)\) with \(\pi_i^N, \rho(t,v)\) replaced by \(\pi_i^N, 1 - \rho(t,v)\), respectively, and for all \(g \in C_0(\mathbb{R})\). Then the condition \((4.9)\) is equivalent to \((4.6)''\) with \(g = 1_{(-\infty, 0)}\).
4.2 Correspondence between two function spaces $X_U$ and $Y_U$

We study the relationship between two function spaces $X_U$ and $Y_U$. To each $\psi \in X_U$, one can associate an element $\rho \in Y_U$ in the following manner: First consider a curve $C_\psi^{(1)} = \{(u, w); w = \psi(u)\}$ in the first quadrant in the plane, and then define a new curve $C_\psi^{(2)}$ in the upper half plane by shifting each point $(u, w)$ in $C_\psi^{(1)}$ to $(u - \psi(u), w)$. The tilt of the curve $C_\psi^{(2)}$ with reversed sign defines the function $\rho \in Y_U$. More precisely, for $\psi \in X_U$, we define the function $G_\psi : \mathbb{R}_+^\infty \to \mathbb{R}$ as

$$
G_\psi(u) := u - \psi(u).
$$

By the definition of $X_U$, $G_\psi$ is a monotone function and furthermore a bijection from $\mathbb{R}_+^\infty$ to $\mathbb{R}$. So, there exists an inverse function of $G_\psi$. We define a function $\Phi_U(\psi) : \mathbb{R} \to (0, 1)$ as $\Phi_U(\psi)(v) = \frac{-\psi'(G_\psi^{-1}(v))}{1 - \psi'(G_\psi^{-1}(v))}$ for $v \in \mathbb{R}$. Then, we can easily see that $\Phi_U(\psi) \in Y_U$. In fact, we can show the following proposition.

**Proposition 4.4.** The map $\Phi_U$ defines a one-to-one correspondence between $X_U$ and $Y_U$.

**Proof.** The inverse map $\Psi_U$ of $\Phi_U$ can be constructed as follows. For $\rho \in Y_U$, we define two functions $\zeta^-_\rho : \mathbb{R} \to \mathbb{R}_+^\infty$ and $\zeta^+_\rho : \mathbb{R} \to \mathbb{R}_+^\infty$ as

$$
(4.15) \quad \zeta^-_\rho(v) := \int_{-\infty}^v (1 - \rho(v'))dv' \quad \text{and} \quad \zeta^+_\rho(v) := \int_v^\infty \rho(v')dv', \quad v \in \mathbb{R}.
$$

Note that these functions are macroscopic correspondences to those determined by (4.1). By the definition of $Y_U$, $\zeta^-_\rho$ and $\zeta^+_\rho$ are continuously differentiable monotone functions. Moreover, they are bijections from $\mathbb{R}$ to $\mathbb{R}_+^\infty$. So, there exists an inverse function of $\zeta^-_\rho$. We define a function $\Psi_U(\rho) : \mathbb{R}_+^\infty \to \mathbb{R}_+^\infty$ as $\Psi_U(\rho)(u) = \zeta^+_\rho((\zeta^-_\rho)^{-1}(u))$ for $u \in \mathbb{R}_+^\infty$. Then, we can easily see that $\Psi_U(\rho) \in X_U$. Furthermore, $\Psi_U \circ \Phi_U = \text{id}_{X_U}$ and $\Phi_U \circ \Psi_U = \text{id}_{Y_U}$ hold, which concludes the proof. \qed

4.3 Proof of Theorem 2.1

**Step 1.** We will show that Theorem 2.1 for the process $p_t(\equiv p^N_t)$ follows from Proposition 4.2 for the process $\tilde{\eta}^N_t$. To this end, we first see that the conditions (4.5), (4.8) and (4.9) at $t = 0$ are reduced from the condition (2.10) if we define $\tilde{\eta}$ and $\rho_0$ by $\tilde{\eta} = \eta^p$ and $\rho_0 = \Phi_U(\psi_0)$, respectively.

Take $g \in C_b^1(\mathbb{R})$ satisfying $g(v) = 0$ for $v \leq -K$ and $g(v) = c$ for $v \geq K$ with some $K > 0$ and $c \in \mathbb{R}$. We will show the condition (4.5) for such $g$; recall Remark 1.2(1) for $t = 0$. For a given $0 < \delta < 1$, determine $u_0, u_1 > 0$ in such a manner that $u_0 = \psi_0^{-1}(K + 2) \wedge 1$ and $u_1 = \psi_0^{-1}(\delta)$, respectively. Now we assume the condition

$$
(4.16) \quad \sup_{u \in [u_0, u_1]} |\tilde{\psi}^N_p(u) - \psi_0(u)| \leq \delta,
$$

for $\tilde{\psi}^N_p$. Then, under this condition, we have that

$$
(4.17) \quad \tilde{\psi}^N_p(u), \psi_0(u) \geq K + 1, \quad u \in (0, u_0],
$$

13
since $0 < \delta < 1$ and both functions are non-increasing in $u$, and

\begin{equation}
(4.18) \quad \zeta \{ i: \frac{p_i}{N} > u_0 \} = N \tilde{\psi}_p^N(u_0) \leq N(\psi_0(u_0) + 1).
\end{equation}

Thus, under \((4.16)\), we have that

\begin{equation}
(4.19) \quad \int_{-\infty}^{\infty} g(v) z_0^N(dv) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \bar{\eta}(x) g \left( \frac{x}{N} \right) = \frac{1}{N} \sum_{i \in \mathbb{N}} g \left( \frac{p_i - i + 1}{N} \right)
\end{equation}

\begin{align*}
&= \frac{1}{N} \sum_{i \in \mathbb{N}} g \left( \frac{p_i}{N} - \tilde{\psi}_p^N \left( \frac{p_i}{N} \right) \right) - \frac{d_i(p)}{N} \\
&= \frac{1}{N} \sum_{i \in \mathbb{N}: \frac{p_i}{N} > u_0} g \left( \frac{p_i}{N} - \tilde{\psi}_p^N \left( \frac{p_i}{N} \right) \right) - \frac{d_i(p)}{N}
\end{align*}

\begin{align*}
&= \frac{1}{N} \sum_{i \in \mathbb{N}: \frac{p_i}{N} > u_0} g \left( \frac{p_i}{N} - \psi_0 \left( \frac{p_i}{N} \right) \right) + R_{N, \delta, 1},
\end{align*}

where $d_i(p) := \zeta \{ j \leq i - 1; p_j = p_i \}$ is the discrepancy in the graph of Young diagram $\psi_p(u)$ at $u = p_i$, and the error term $R_{N, \delta, 1}$ satisfies that $|R_{N, \delta, 1}| \leq C_1 \delta$ with $C_1 > 0$. Indeed, the second equality in \((4.19)\) follows from the fact that $\{ x \in \mathbb{Z}; \bar{\eta}(x) = 1 \} = \{ p_i - i + 1; i \in \mathbb{N} \}$, the third from $\tilde{\psi}_p^N (p_i/N) = \psi_p(p_i)/N$ and the fourth from \((4.17)\) since $p_i/N \leq u_0$ implies that $p_i/N - \tilde{\psi}_p^N(p_i/N) \leq u_0 - (K + 1) \leq -K$. The term $R_{N, \delta, 1}$ in the last line is defined by

\begin{align*}
R_{N, \delta, 1} &= \frac{1}{N} \sum_{i \in \mathbb{N}: \frac{p_i}{N} > u_0} \left\{ g \left( \frac{p_i}{N} - \tilde{\psi}_p^N \left( \frac{p_i}{N} \right) \right) - g \left( \frac{p_i}{N} - \psi_0 \left( \frac{p_i}{N} \right) \right) \right\} \\
&\leq \| g \|_{\infty} \cdot 4 \delta (\psi_0(u_0) + 1),
\end{align*}

since the first summand in the above sum is bounded by $\delta$ if $u_0 \leq p_i/N \leq u_1$ under the condition \((4.16)\) and is bounded by $2 \delta$ if $p_i/N \geq u_1$ by noting that $0 \leq \tilde{\psi}_p^N(u), \psi_0(u) \leq 2 \delta$ for $u \geq u_1$ which follows from the monotonicity of these functions, and its second summand is bounded by $\tilde{\psi}_p^N(p_i/N) - \tilde{\psi}_p^N(p_i/N)$ which is further bounded by $2 \delta$ from \((4.16)\) recalling the continuity of $\psi_0$; we have also used \((4.18)\). We can further rewrite the sum in the last term of \((4.19)\) as

\begin{align*}
\frac{1}{N} \sum_{i \in \mathbb{N}: \frac{p_i}{N} > u_0} g \left( \frac{p_i}{N} - \psi_0 \left( \frac{p_i}{N} \right) \right) &= \frac{1}{N} \sum_{i \in \mathbb{N}} (g \circ G_{\psi_0}) \left( \frac{p_i}{N} \right) \\
&= \frac{1}{N} \sum_{i \in \mathbb{N}} \int_0^{p_i/N} (g \circ G_{\psi_0})'(u) du = \int_0^{\infty} (g \circ G_{\psi_0})'(u) \tilde{\psi}_p^N(u) du.
\end{align*}
Note that, since \( g \circ G_{\psi_0}(u) = g(u - \psi_0(u)) = 0 \) if \( u \in (0, u_0] \), we have dropped the condition \( \rho_i/N > u_0 \) from the summand of the above sums, and, by the same reason, we can replace the region of the integral in the last line from \( [0, \infty) \) to \( [u_0, \infty) \). Consider the error \( R^{N,\delta,2} \) defined by
\[
R^{N,\delta,2} = \int_0^\infty (g \circ G_{\psi_0})'(u) \left\{ \tilde{\psi}_p^N(u) - \psi_0(u) \right\} du,
\]
which can be bounded as
\[
|R^{N,\delta,2}| \leq 2\delta \int_0^K \left| (g \circ G_{\psi_0})'(u) \right| du = C_2\delta.
\]
where \( K \) is determined in such a manner that \( (g \circ G_{\psi_0})'(v) = 0 \) for \( v \geq K \). Furthermore, by the integration by parts formula, we have that
\[
\int_0^\infty (g \circ G_{\psi_0})'(u)\psi_0(u)du = -\int_0^\infty (g \circ G_{\psi_0})(u)\psi_0'(u)du
\]
\[
= -\int_{-\infty}^\infty g(v)\psi_0'(G^{-1}_\psi_0(v)) \frac{1}{1 - \psi_0'(G^{-1}_\psi_0(v))} dv
\]
\[
= \int_{-\infty}^\infty g(v)\Phi_U(\psi_0)(v)dv = \int_{-\infty}^\infty g(v)\rho_0(v)dv.
\]
Therefore, under the condition \( (4.16) \), we have shown that
\[
\left| \int_{-\infty}^\infty g(v)\pi_0^N(dv) - \int_{-\infty}^\infty g(v)\rho_0(v)dv \right| \leq (C_1 + C_2)\delta.
\]
This implies the condition \( (4.5) \) for \( \pi_0^N \) and \( g \in C^1_b(\mathbb{R}) \) satisfying \( g(v) = 0 \) for \( v \leq -K \) and \( g(v) = c \) for \( v \geq K \) with some \( K > 0 \) and \( c \in \mathbb{R} \).

The same condition \( (4.5) \) with \( \pi_0^N, \rho_0 \) replaced by \( \hat{\pi}_0^N, 1 - \rho_0 \), respectively, and \( g \in C^1_b(\mathbb{R}) \) satisfying \( g(v) = 0 \) for \( v \geq K \) and \( g(v) = c \) for \( v \leq -K \) with some \( K > 0 \) and \( c \in \mathbb{R} \) can be shown by symmetry; recall Remark \( (4.2) \) for \( t = 0 \). Indeed, for each \( p \in \mathcal{P} \), we denote by \( \tilde{p} = (\tilde{p}_i)_{i \in \mathbb{N}} \) the mirror image of the Young diagram \( p \) with the axis of symmetry \( \{ y = x \} \) in the plane, i.e. \( \tilde{p}_i = \#\{ j; p_j \geq i \} \). Similarly, we denote by \( \tilde{\psi}_0 \) the mirror image of the curve \( \psi_0 \) with the axis of symmetry \( \{ y = x \} \), i.e. \( \tilde{\psi}_0(u) := \psi_0^{-1}(u) \). Then, the condition \( (2.6) \) with \( \tilde{\psi}_p^N, \psi_0 \) replaced by \( \tilde{\psi}_p^N, \psi_0 \), respectively, is reduced from \( (2.6) \) itself.

Therefore, if we denote by \( \hat{\pi}_0^N \) the scaled empirical measure of the configuration \( \tilde{\pi}_p^N \) and \( \rho_0 \) the function associated with \( \tilde{\psi}_0 \) by the one-to-one map constructed in Subsection \( 4.2 \), namely \( \hat{\pi}_0^N(dv) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \tilde{\pi}_p^N(x)\delta_{\hat{\psi}^p(x)}(dv) \) and \( \rho_0 = \Phi_U(\psi_0) \), then we see that the condition \( (4.5) \) with \( \pi_0^N, \rho_0 \) replaced by \( \hat{\pi}_0^N, \hat{\rho}_0 \), respectively, holds for every \( \delta > 0 \) and \( g \in C^1_b(\mathbb{R}) \) satisfying \( g(v) = 0 \) for \( v \leq -K \) and \( g(v) = c \) for \( v \geq K \) with some \( K > 0 \) and \( c \in \mathbb{R} \) by the above mentioned argument. However, since we easily see the relations: \( \tilde{\pi}_p^N(x) = 1 - \tilde{\pi}_p^N(-x) \) and \( \rho_0(u) = 1 - \rho_0(-u) \), the condition \( (4.5) \) with \( \pi_0^N, \rho_0 \) replaced by \( \hat{\pi}_0^N, 1 - \rho_0 \), respectively, is shown for \( g \in C^1_b(\mathbb{R}) \) satisfying \( g(v) = 0 \) for \( v \geq K \) and \( g(v) = c \) for \( v \leq -K \).

**Step 2.** In order to complete the proof of the theorem, it is now sufficient to show that \( (4.6) \) in Proposition \( (4.2) \) together with the assertions in Lemma \( (4.3) \) implies \( (2.7) \) with \( \psi_t = \Phi_U(\rho_t) \). The non-linear equation \( (2.8) \) for \( \psi_t \) follows from \( (4.7) \) for \( \rho_t \).
Take \( f \in C_0(\mathbb{R}^n_+) \) and \( t > 0 \) arbitrarily and fix them throughout the rest of the proof. Then we have that

\[
\int_0^\infty f(u)\tilde{\psi}_{\nu t}^N(u)du = \frac{1}{N} \sum_{x \in \mathbb{Z}} F\left(\frac{\zeta_{\nu t}^-(x)}{N}\right)\tilde{\eta}(x),
\]

where \( F(u) = \int_0^u f(u')du' \) and \( \zeta_{\nu t}^-(x) = \zeta_{\nu \tilde{\eta}}^-(x) \) defined by (4.11). For a given \( \delta > 0 \), take \( K > 0 \) such that

\[
\int_{-\infty}^{-K} (1 - \rho(t, v))dv < \delta/6, \quad \int_K^\infty \rho(t, v)dv < \delta/6,
\]

and the conditions (4.10) and (4.11) hold with \( \delta \) replaced by \( \delta/3 \), recall the proof of Lemma 4.3.

Now let us prove that

\[
\lim_{N \to \infty} \mathbb{P}_N^v \left[ \sup_{x \in \mathbb{Z}: |x/N-v| \leq \theta} \frac{\zeta_{\nu t}^-(x)}{N} - \int_{-\infty}^v (1 - \rho(t, v'))dv' \right] > \delta \right] = 0
\]

holds for every \( 0 < \theta < \delta/3 \) and \( v \in \mathcal{V}_{K,\theta} := \{ v \in \mathbb{R}; |v| \leq K + 1, v \in \theta \mathbb{Z} \} \). In fact, since \( \zeta_{\nu t}^-(x) \) is non-decreasing in \( x \), we have that

\[
\hat{\pi}_t^N((-\infty, v - \theta]) \leq \frac{\zeta_{\nu t}^-(x)}{N} = \hat{\pi}_t^N((-\infty, x/N]) \leq \hat{\pi}_t^N((-\infty, v + \theta])
\]

for \( x \in \mathbb{Z} \) such that \( |x/N-v| \leq \theta \). However, from (4.11) and (4.6) with \( g = 1_{[-K, v+\theta]} \) and \( \delta \) replaced by \( \delta/3 \), we have that

\[
\lim_{N \to \infty} \mathbb{P}_N^v \left[ \hat{\pi}_t^N((-\infty, v \pm \theta]) - \int_{-\infty}^{v \pm \theta} (1 - \rho(t, v'))dv' \right] > 2\delta/3 = 0.
\]

Moreover, since \( |\int_{v \pm \theta} (1 - \rho(t, v'))dv' - \int_v^v (1 - \rho(t, v'))dv'| \leq \theta \), if \( 0 < \theta < \delta/3 \), (4.23) and (4.24) imply (4.22). Since \( \|F'\|_{\infty} = \|f\|_{\infty} < \infty \), (4.22) further shows that

\[
\lim_{N \to \infty} \mathbb{P}_N^v \left[ \sup_{x \in \mathbb{Z}: |x/N-v| \leq \theta} \left| F\left(\frac{\zeta_{\nu t}^-(x)}{N}\right) - F\left(\int_{-\infty}^v (1 - \rho(t, v'))dv'\right) \right| > \delta \|f\|_{\infty} \right] = 0
\]

for every \( v \in \mathcal{V}_{K,\theta} \) if \( 0 < \theta < \delta/3 \).

We now return to the formula (4.20) and divide it as

\[
\int_0^\infty f(u)\tilde{\psi}_{\nu t}^N(u)du =: I_1^N + I_2^N + I_3^N,
\]

where \( I_1^N, I_2^N \) and \( I_3^N \) are defined as the sums in the right hand side of (4.20) restricted for \( x \leq -KN, -KN < x < KN \) and \( x \geq KN \), respectively. For the first term \( I_1^N \), since \( f \in C_0(\mathbb{R}^n_+) \), we see that \( f(u) = 0 \) so that \( F(u) = 0 \) for \( u \in [0, u_0] \) with some \( u_0 > 0 \). Therefore, choosing \( \delta > 0 \) such that \( \delta/3 < u_0 \), (4.11) with \( \delta \) replaced by \( \delta/3 \) implies that

\[
\lim_{N \to \infty} \mathbb{P}_N^v [I_1^N = 0] = 1.
\]
For the second term $I_2^N$, by (1.25), we can show that
\[
\lim_{N \to \infty} \mathbb{P}_{\nu_i}^N [\{I_2^N - \tilde{I}_2^N \} > \delta] = 0,
\]
where, assuming $K/\theta \in \mathbb{Z}$ for simplicity,
\[
\tilde{I}_2^N = \frac{1}{N} \sum_{k=-K/\theta}^{K/\theta - 1} F \left( \int_{-\infty}^{k\theta} (1 - \rho(t,v'))dv' \right) \sum_{k\theta \leq x/N < (k+1)\theta} \tilde{\eta}(x).
\]
However, by applying (4.6) with $g = 1_{[k\theta,(k+1)\theta)}$ again, we have that
\[
\lim_{N \to \infty} \mathbb{P}_{\nu_i}^N [\{I_2^N - \tilde{I}_2^N \} > \delta] = 0,
\]
where
\[
\tilde{I}_2^N = \sum_{k=-K/\theta}^{K/\theta - 1} F \left( \int_{-\infty}^{k\theta} (1 - \rho(t,v'))dv' \right) \int_{k\theta}^{(k+1)\theta} \rho(t,v')dv'.
\]
By letting $\theta \downarrow 0$, $\tilde{I}_2^N$ converges to
\[
I_K = \int_{-K}^{K} F \left( \int_{-\infty}^{v} (1 - \rho(t,v'))dv' \right) \rho(t,v)dv.
\]
For the third term $I_3^N$, since $0 \leq I_3^N \leq \|F\|_{\infty} \pi_i^N([K,\infty))$, we see from (4.10) with $\delta$ replaced by $\delta/3$ that
\[
\lim_{N \to \infty} \mathbb{P}_{\nu_i}^N [I_3^N > \delta\|F\|_{\infty}/3] = 0.
\]
These computations are now summarized into
\[
\lim_{N \to \infty} \mathbb{P}_{\nu_i}^N [\int_0^\infty f(u)v_i^N(u)du - I] > \delta = 0,
\]
where
\[
I = \int_{-\infty}^{\infty} F \left( \int_{-\infty}^{v} (1 - \rho(t,v'))dv' \right) \rho(t,v)dv.
\]
Note that $I_K$ coincides with $\int_{-\infty}^{K} F \left( \int_{-\infty}^{v} (1 - \rho(t,v'))dv' \right) \rho(t,v)dv$ because of (4.21) recalling that $\delta/3 < u_0$ and the integration over $[K,\infty)$ in $v$ can be taken small enough if $K$ is sufficiently large. However, by the change of variables $w = \zeta_{\rho_i}^-(v)$ and the integration by parts, we have that
\[
I = \int_{-\infty}^{\infty} F \left( \zeta_{\rho_i}^+(v) \right) \rho(t,v)dv = -\int_{-\infty}^{\infty} \int_0^{\zeta_{\rho_i}^-(v)} f(u)du \cdot \frac{d\zeta_{\rho_i}^+(v)}{dv} dv
\]
\[
= -\int_0^{\infty} \int_0^w f(u)du \cdot \frac{d\zeta_{\rho_i}^+(v)}{dv} \left( (\zeta_{\rho_i}^-)^{-1}(w) \right) \frac{dv}{dw} dw
\]
\[
= -\int_0^{\infty} \int_0^w f(u)du \cdot \frac{d}{dw} \left( \zeta_{\rho_i}^+( (\zeta_{\rho_i}^-)^{-1}(w) ) \right) dw
\]
\[
= \int_0^{\infty} f(u)\zeta_{\rho_i}^+((\zeta_{\rho_i}^-)^{-1}(u))du = \int_0^{\infty} f(u)\Psi_U(\rho_i)(u)du.
\]
This completes the proof of Theorem 2.1.
5 Proof of Theorem 2.2

This section gives the proof of Theorem 2.2 for the RU-case, i.e. the case corresponding to the restricted uniform statistics. Similarly to the process $\xi_t$ in the U-case, we consider the particle numbers (or the gradient of the height function $\psi_t^x$) $\eta_t = (\eta_t(x))_{x \in \mathbb{Z}_+}$ defined by $\eta_t(x) = \sharp \{i: q_i(t) = x\} \in \{0,1\}$ for $x \in \mathbb{N}$ and $\eta_t(0) = \infty$. Note that only 0-1 height differences are allowed under the restriction imposed on the Young diagrams $q \in Q$. Then $\eta_t$ becomes the weakly asymmetric simple exclusion process with the stochastic reservoir at $\{0\}$, which provides particles into the region $\mathbb{N}$ with rate $\varepsilon$ and absorbs them with rate 1. Contrarily to the weakly asymmetric simple exclusion process $\bar{\eta}_t$ on $\mathbb{Z}$ considered in the U-case, $\eta_t$ determines a finite particles’ system on $\mathbb{N}$.

In the RU-case, one does not have a nice transformation for $\eta_t$, which removes the stochastic reservoir as in the U-case. We will apply the Hopf-Cole transformation for $\eta_t$ at the microscopic level, which linearizes the leading term, and study the boundary behavior of the transformed process.

5.1 The process $\eta_t$

Denote by $\chi_R$ the state space of the process $\eta_t$ defined from $q_t$:

$$\chi_R := \{\eta \in \{0,1\}^\mathbb{N}; \sum_{x \in \mathbb{N}} \eta(x) < \infty\}.$$ 

We have a one-to-one correspondence between $\chi_R$ and $Q$. Indeed, for $\eta \in \chi_R$, we assign $q^\eta \in Q$ by the following rule:

$$q^\eta_i = \min \{x \in \mathbb{Z}_+: \sum_{y \geq x+1} \eta(y) \leq i - 1\}, \quad i \in \mathbb{N}.$$ 

In other words, $\{q^\eta_i\}_{i \in \mathbb{N}}$ is determined by numbering the set $\{x \in \mathbb{N}; \eta(x) = 1\}$ from the right and, if $i$ is larger than the cardinality of this set, we define $q^\eta_i = 0$. We can show that the map $\eta \to q^\eta$ is well-defined and also it is a bijection from $\chi_R$ to $Q$. So we denote its inverse map by $q \to \eta^q$.

We now consider the Markov process $\eta_t$ on $\chi_R$ with the generator $\bar{L}_{\varepsilon,R}$ acting on functions $f: \chi_R \to \mathbb{R}$ as

$$\bar{L}_{\varepsilon,R} f(\eta) = \bar{L}^i_{\varepsilon,R} f(\eta) + \bar{L}^b_{\varepsilon,R} f(\eta),$$

where

$$\bar{L}^i_{\varepsilon,R} f(\eta) = \sum_{x \in \mathbb{N}} \left[ \varepsilon c_+(x, \eta) + c_-(x, \eta) \right] \{f(\eta^{x,x+1}) - f(\eta)\}$$

and

$$\bar{L}^b_{\varepsilon,R} f(\eta) = \left[ \varepsilon 1_{\{\eta(1) = 0\}} + 1_{\{\eta(1) = 1\}} \right] \{f(\eta^1) - f(\eta)\}$$

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are the interior and boundary terms of the generator, respectively, $c_+(x, \eta), c_-(x, \eta)$ and $\eta^{x,y}$ are defined by (1.2), (1.3) with $\bar{\eta}$ replaced by $\eta$, respectively, and
\[
\eta^1(z) = \begin{cases} 
\eta(z) & \text{if } z \neq 1, \\
1 - \eta(1) & \text{if } z = 1.
\end{cases}
\]

The following lemma is easy so that the proof is omitted.

**Lemma 5.1.** Two processes $\{q_t\}_{t \geq 0}$ and $\{q^p_t\}_{t \geq 0}$ have the same distributions on $D(\mathbb{R}_+, \mathcal{Q})$.

For a probability measure $\nu$ on $\chi_R$ and $N \geq 1$, we denote by $\hat{\nu}^N$ the distribution on $D(\mathbb{R}_+, \chi_R)$ of the process $\eta^N_t$ with generator $N^2\bar{L}_{\varepsilon(N)},R$ and the initial measure $\nu$, where $\varepsilon(N)$ is defined by (2.11). Let us define the scaled empirical measures $\pi^N_t(dv), t \geq 0, v \in \mathbb{R}_+$ of the process $\eta^N_t$ by the formula (4.4) with $\hat{\eta}^N_t$ replaced by $\eta^N_t$ and the sum taken over all $x \in \mathbb{N}$ rather than $x \in \mathbb{Z}$.

The hydrodynamic limit for a boundary driven exclusion process is studied by [6]. Our model involves a weak asymmetry both in dynamics and the boundary condition, and furthermore it is defined on an infinite volume $\mathbb{N}$. Note that the boundary generator $\bar{L}_{\varepsilon,R}$ is invariant under the Bernoulli measure with mean $\rho^\varepsilon = \varepsilon/(1 + \varepsilon)$. This actually determines the Dirichlet boundary condition at $v = 0$ in the limit equation (5.5) stated below, since $\rho^\varepsilon$ converges to $1/2$ as $\varepsilon = \varepsilon(N) \uparrow 1$. The hydrodynamic limit for models in infinite volume was discussed by several authors including [13]. It might be possible to apply these methods to our model, but we will employ the simplest way based on the Hopf-Cole transformation.

### 5.2 Hopf-Cole transformation

In this subsection we introduce the microscopic Hopf-Cole transformation for the process $\eta^N_t$ and formulate Theorem 5.2 on its hydrodynamic behavior. Theorem 2.2 will be shown from Theorem 5.2 in Subsection 5.4.

It is well-known that the (macroscopic) Hopf-Cole transformation:
\[
\omega(t, u) = \exp\{\beta \int_u^\infty \rho_t(v)dv\}, \quad u \in \mathbb{R}_+
\]
allows us to reduce the solution of the viscous Burgers’ equation (5.5) (at least on the whole line $\mathbb{R}$) to that of the linear diffusion equation (5.3) (on $\mathbb{R}$). We introduce the corresponding transformation at the microscopic level, cf. [9]. Namely, we consider the process $\zeta^N_t = (\zeta^N_t(x))_{x \in \mathbb{N}}$ defined by $\zeta^N_t(x) := \exp\{- (\log \varepsilon) \sum_{y=x}^{\infty} \eta^N_t(y)\}$ with $\varepsilon = \varepsilon(N)$ from the process $\eta^N_t$ and the $C(\mathbb{R}_+)$-valued process $\tilde{\zeta}^N(t, u), u \in \mathbb{R}_+$ by interpolating $\tilde{\zeta}^N(t, u) := \zeta^N_t(Nu)$ defined for $u \in \mathbb{N}/N$ in such a manner that
\[
\tilde{\zeta}^N(t, u) := \exp\left[- (\log \varepsilon) \left\{ \sum_{y=[Nu]+1}^{\infty} \eta^N_t(y) + 1_{\{u \geq 1/N\}}([Nu] + 1 - Nu)\eta^N_t([Nu]) \right\} \right],
\]
for $u \in \mathbb{R}_+$. 
Theorem 5.2. Let \((\nu^N)_N\geq 1\) be a sequence of probability measures on \(\chi R\) such that

\[
\lim_{N \to \infty} \nu^N \left[ \int_0^\infty g(v) \pi_0^N (dv) - \int_0^\infty g(v) \rho_0 (dv) \right] = 0
\]

holds for every \(\delta > 0\), \(g \in C_b(\mathbb{R})\) satisfying \(g(v) = c\) for \(v \geq K\) with some \(K > 0\) and \(c \in \mathbb{R}\), and some continuous function \(\rho_0 : \mathbb{R}_+ \to [0,1]\) satisfying \(\int_0^\infty \rho_0 (dv) < \infty\). Then, for every \(T > 0\), \(K > 0\) and \(\delta > 0\),

\[
\lim_{N \to \infty} \bar{Q}_t^N \left[ \sup_{0 \leq t \leq T, 0 \leq u \leq K} |\tilde{\gamma}^N(t,u) - \omega(t,u)| > \delta \right] = 0
\]

holds, where \(\omega(t,u)\) is the unique bounded weak solution of the following linear diffusion equation:

\[
\begin{align*}
\partial_t \omega &= \partial_u^2 \omega + \beta \partial_v \omega, \quad u \in \mathbb{R}_+, \\
\omega(0,u) &= \exp \{ \beta \int_u^\infty \rho_0 (v) dv \}, \quad u \in \mathbb{R}_+, \\
2 \partial_u \omega(t,0) + \beta \omega(t,0) &= 0, \quad t > 0, \\
\omega(t,\infty) &= 1, \quad t > 0.
\end{align*}
\]

Namely, for every \(t > 0\),

\[
\int_0^\infty g(u) \omega(t,u) du = \int_0^\infty g(u) \omega(0,u) du + \int_0^t \int_0^\infty (g''(u) - \beta g'(u)) \omega(s,u) duds
\]

holds for every \(g \in C_0^2(\mathbb{R}_+)\) satisfying \(2g'(0) - \beta g(0) = 0\) and \(\lim_{u \to \infty} \omega(t,u) = 1\).

The following corollary, which gives the hydrodynamic limit for \(\eta^N_t\), is an immediate consequence of Theorem 5.2 and will be used in [8].

Corollary 5.3. Under the same assumption as Theorem 5.2,

\[
\lim_{N \to \infty} \bar{Q}_t^N \left[ \int_0^\infty g(v) \pi_t^N (dv) - \int_0^\infty g(v) \rho(t,v) dv \right] = 0
\]

holds for every \(t > 0, \delta > 0\) and \(g \in C_0(\mathbb{R}_+^0)\), where \(\rho(t,u)\) is the unique classical solution of the following partial differential equation:

\[
\begin{align*}
\partial_t \rho &= \partial_u^2 \rho + \beta \partial_v (\rho(1 - \rho)), \quad v \in \mathbb{R}_+, \\
\rho(0,v) &= \rho_0(v), \quad v \in \mathbb{R}_+, \\
\rho(t,0) &= 1/2, \quad t > 0.
\end{align*}
\]

5.3 Proof of Theorem 5.2

This subsection proves Theorem 5.2.
5.3.1 Uniform estimate on the total mass

We prepare a proposition which gives a uniform estimate on the scaled total mass of $\pi_t^N$. For the proof, the conditions (5.1) with $g \equiv 1$ and $\int_0^\infty \rho_0(v) dv < \infty$ are essential.

**Proposition 5.4.** Denote by $X_t^N$ the process of the total mass of the empirical measure $\pi_t^N$, namely $X_t^N := \frac{1}{N} \sum_{x \in \mathbb{N}} \pi_t^N(x) \left( \equiv \pi_t^N(\mathbb{R}^o_+) \right)$. Then, for every $T > 0$, we have that

\[
(5.6) \quad \lim_{\lambda \to \infty} \sup_{N \geq 1} \mathbb{Q}_{\pi_N} \left[ \sup_{0 \leq t \leq T} X_t^N > \lambda \right] = 0.
\]

**Proof.** For $\varphi \in C^2_b(\mathbb{R}^+_0)$, denote by $m_t^N(\varphi)$ the martingale defined by

\[
m_t^N(\varphi) := \langle \pi_t^N, \varphi \rangle - \langle \pi_0^N, \varphi \rangle - \int_0^t N^2 \tilde{L}_{\epsilon(N), R}(\pi_t^N, \varphi) ds.
\]

Then, by a simple computation, we have that

\[
(5.7) \quad N^2 \tilde{L}_{\epsilon(N), R}(\pi_t^N, \varphi) = N \sum_{x \in \mathbb{N}} \left( \varphi((x + 1)/N) - \varphi(x/N) \right) \left\{ \epsilon c_+ (x, \eta) - c_- (x, \eta) \right\} + N \varphi(1/N) \left\{ \epsilon 1_{\{\eta(1) = 0\}} - 1_{\{\eta(1) = 1\}} \right\} ,
\]

and

\[
(5.8) \quad \frac{d}{dt} \langle m_t^N(\varphi) \rangle_t = \sum_{x \in \mathbb{N}} \left( \varphi((x + 1)/N) - \varphi(x/N) \right)^2 \left\{ \epsilon c_+ (x, \eta_t^N) + c_- (x, \eta_t^N) \right\} + \varphi(1/N)^2 \left\{ \epsilon 1_{\{\eta_t^N(1) = 0\}} + 1_{\{\eta_t^N(1) = 1\}} \right\} ,
\]

for the quadratic variation of $m_t^N(\varphi)$, if the right hand sides of these equalities converge absolutely. Now take a function $\varphi \in C^2_b(\mathbb{R}^+_0)$ such that $\varphi' \geq 0$, $\varphi(u) = 0$ for $0 < u \leq 1$ and $\varphi(u) = 1$ for $u \geq 2$. Then, (5.7) shows that

\[
N^2 \tilde{L}_{\epsilon(N), R}(\pi_t^N, \varphi) \leq ||\varphi''||_{\infty},
\]

similarly to the proof of (4.12). Therefore,

\[
\sup_{0 \leq t \leq T} \langle \pi_t^N, \varphi \rangle \leq \langle \pi_0^N, \varphi \rangle + T ||\varphi''||_{\infty} + \sup_{0 \leq t \leq T} |m_t^N(\varphi)|
\]

\[
 \leq X_0^N + T ||\varphi''||_{\infty} + 1 + \sup_{0 \leq t \leq T} m_t^N(\varphi)^2 ,
\]

where we have estimated the martingale as $|m_t^N(\varphi)| \leq m_t^N(\varphi)^2 + 1$. One can apply Doob’s inequality to show $\lim_{N \to \infty} E[\sup_{0 \leq t \leq T} m_t^N(\varphi)^2] = 0$ from (5.8), which, in particular, proves $\sup_N E[\sup_{0 \leq t \leq T} m_t^N(\varphi)^2] < \infty$. Since the assumption of Theorem 5.2 (especially (5.1) with $g \equiv 1$ and the integrability of $\rho_0$) implies that $\lim_{\lambda \to \infty} \sup_N \nu^N(X_0^N > \lambda) = 0$, the conclusion of the proposition follows by the inequality: $X_t^N \leq 2 + \langle \pi_t^N, \varphi \rangle$. \qed
5.3.2 Tightness of \( \{\tilde{\zeta}^N\}_N \)

Let \( P_N \) be the probability distribution of \( \tilde{\zeta}^N = \{\tilde{\zeta}^N(t,u)\} \) on \( D([0,T], C(\mathbb{R}_+)) \), where the space \( C(\mathbb{R}_+) \) is endowed with the topology determined by the uniform convergence on every compact set of \( \mathbb{R}_+ \).

Lemma 5.5. The family of probability measures \( \{P_N\}_{N \geq 1} \) is relatively compact.

Proof. To conclude the lemma, by Prokhorov’s theorem, it suffices to show the following three conditions for \( \{P_N\}_{N \geq 1} \):

(i) For every \( t \in [0,T] \), \( \lim_{\lambda \to \infty} \sup_{N \geq 1} P_N[\tilde{\zeta}(t,0) > \lambda] = 0 \).

(ii) For every \( \delta > 0 \) and \( t \in [0,T] \), \( \lim_{\lambda \to \infty} \sup_{N \geq 1} P_N[\sup_{\gamma \in [0,T]} |\tilde{\zeta}(t,u) - \tilde{\zeta}(t,v)| > \delta] = 0 \).

(iii) For every \( \delta > 0 \) and \( K > 0 \), \( \lim_{\lambda \to \infty} \sup_{N \to \infty} P_N[\sup_{|t-s| \leq \gamma, 0 \leq u \leq K} |\tilde{\zeta}(t,u) - \tilde{\zeta}(s,u)| > \delta] = 0 \).

By the relation: \( \tilde{\zeta}^N(t,0) = \exp\{- (\log \varepsilon) N X^N_t\} \), we have that

\[
P_N[\tilde{\zeta}^N(t,0) > \lambda] \leq \bar{Q}^N_{t,N}[X^N_t > \log \lambda/C],
\]

note that there exists \( C > 0 \) such that \( 0 < - \log \varepsilon \leq C/N \) for \( \varepsilon = \varepsilon(N) \) and every \( N \geq 1 \). Proposition 5.4 proves (i).

Since \( \tilde{\zeta}^N(t,\cdot) \) is a non-increasing function, for every \( 0 \leq u < v \), we have that

\[
|\tilde{\zeta}^N(t,u) - \tilde{\zeta}^N(t,v)| \leq \tilde{\zeta}^N(t,u) \left[ \exp \left\{ - (\log \varepsilon) I^N(t,u,v) \right\} - 1 \right] \\
\leq \tilde{\zeta}^N(t,0) \left[ \exp \left\{ C I^N(t,u,v) N \right\} - 1 \right],
\]

where

\[
I^N(t,u,v) := \sum_{y=\lfloor Nu \rfloor+1}^{\lfloor Nv \rfloor} \eta^N_t(y) + (\lfloor Nu \rfloor + 1 - Nu) \eta^N_t(\lfloor Nu \rfloor) - (\lfloor Nv \rfloor + 1 - Nv) \eta^N_t(\lfloor Nv \rfloor),
\]

which has a trivial bound: \( I^N(t,u,v) \leq N(v-u) \). Therefore, we have that

\[
P_N[\sup_{|u-v| \leq \gamma} |\tilde{\zeta}^N(t,u) - \tilde{\zeta}^N(t,v)| > \delta] \leq \bar{Q}^N_{t,N}[e^{CX^N_t} (e^{C\gamma} - 1) > \delta] \\
= \bar{Q}^N_{t,N}[X^N_t > \log(\delta/(e^{C\gamma} - 1))/C].
\]

Proposition 5.4 concludes (ii).

Finally we prove (iii). By the definition of \( \tilde{\zeta}^N(t,u) \) and Proposition 5.4, we only need to show that for every \( K > 0 \) and \( \delta > 0 \),

\[
\lim_{\gamma \to 0} \lim_{N \to \infty} \sup_{N \to \infty} \bar{Q}^N_{t,N}[\sup_{|t-s| \leq \gamma, 0 \leq u \leq K} |\frac{1}{N} \sum_{x=\lfloor Nu \rfloor}^{\lfloor Nv \rfloor} \eta^N_t(x) - \frac{1}{N} \sum_{x=\lfloor Nu \rfloor}^{\lfloor Nv \rfloor} \eta^N_t(x)| > \delta] = 0.
\]
Noting that $\frac{1}{N} \sum x_{[N]}^{\infty} \eta_t^N(x) = \langle \pi_t^N, 1_{[u, \infty)} \rangle + \frac{1}{N} \eta_t^N([Nu])$, we consider smooth functions $\phi_\kappa(u, \cdot)$ which approximate the function $1_{[u, \infty)}$ as $\kappa \downarrow 0$ such that

$$
\phi_\kappa(u, v) = 0 \quad \text{for } v \leq u - \kappa \\
0 \leq \phi_\kappa(u, v) \leq 1 \quad \text{for } u - \kappa \leq v \leq u + \kappa \\
\phi_\kappa(u, v) = 1 \quad \text{for } v \geq u + \kappa \\
\phi_\kappa(u, \cdot) = \phi_\kappa(u + v, \cdot + v) \quad \text{for every } u \text{ and } v.
$$

In particular, we have that

$$
|\langle \pi_t^N, \phi_\kappa(u, \cdot) \rangle - \langle \pi_t^N, 1_{[u, \infty)} \rangle| \leq \kappa \quad \text{for every } u.
$$

Moreover, $\|\phi_\kappa\|_{2, \infty} := \sup_u \{\|\phi'_\kappa(u, \cdot)\| + \|\phi''_\kappa(u, \cdot)\|\}$ is finite. Now, it is enough to prove that for every $\kappa, \delta > 0$,

$$
\lim_{N \to \infty} \lim_{\gamma \downarrow 0} \sup_t \left[ \sup_{|t-s| \leq \gamma, \kappa \leq u \leq K} |\langle \pi_t^N, \phi_\kappa(u, \cdot) \rangle - \langle \pi_s^N, \phi_\kappa(u, \cdot) \rangle| > \delta \right] = 0.
$$

However, the term $\langle \pi_t^N, \phi_\kappa(u, \cdot) \rangle - \langle \pi_s^N, \phi_\kappa(u, \cdot) \rangle$ is rewritten as

$$
\int_s^t \sum_{x \in \mathbb{N}} \phi_\kappa(u, \frac{x}{N}) N L_{\epsilon(N), R} \eta_t^N(x) dr + \tilde{m}_t^N - \tilde{m}_s^N,
$$

where $\tilde{m}_t^N$ is a martingale which vanishes as $N$ goes to 0; recall (5.8). On the other hand, the absolute value of the integral term is bounded from above by $\int_s^t 2\kappa \|\phi_\kappa\|_{2, \infty} dr$, recall (5.7). This concludes the proof of (iii) and therefore the lemma.

\[5.3.3\quad \text{Characterization of limit points}\]

We start with considering a class of martingales associated with $\{\zeta^N\}_{N \geq 1}$. Let $M_t^N(x), x \in \mathbb{N}$, be the martingale defined by

$$
M_t^N(x) := \zeta_t^N(x) - \zeta_0^N(x) - \int_0^t N^2 L_{\epsilon(N), R}(\zeta_s^N(x)) ds.
$$

Some simple computations permit us to rewrite

$$
N^2 L_{\epsilon(N), R}(\zeta_s^N(x)) = N^2 (\epsilon \zeta_s^N(x - 1) - (\epsilon + 1) \zeta_s^N(x) + \zeta_s^N(x + 1)),
$$

for every $x \in \mathbb{N}$, where we define $\zeta_s^N(0) := \epsilon^{-1} \zeta_s^N(2)$. However, denoting $\beta(N) := N(1 - \epsilon(N))$ which converges to $\beta$ as $N \to \infty$ by Lemma 3.2, the right hand side of (5.11) can be rewritten further as

$$
N^2 \Delta \zeta_s^N(x) + N \beta(N) \nabla \zeta_s^N(x),
$$

where $\Delta \zeta = (\Delta \zeta(x))_{x \in \mathbb{N}}$ and $\nabla \zeta = (\nabla \zeta(x))_{x \in \mathbb{N}}$ are defined for $\zeta = (\zeta(x))_{x \in \mathbb{N}}$ by

$$
\Delta \zeta(x) = \zeta(x - 1) - 2\zeta(x) + \zeta(x + 1), \quad \nabla \zeta(x) = \zeta(x) - \zeta(x - 1),
$$

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respectively. Thus, for every \( g \in C^2_0(\mathbb{R}_+) \), taking account of \( \zeta^N_s(0) = \epsilon^{-1}\zeta^N_s(2) \), we have that

\[
(5.12) \quad \int_0^\infty g(u)\tilde{\zeta}^N(t,u)\,du = \frac{1}{N} \sum_{x \in \mathbb{N}} \zeta^N_t(x)g(x/N) + R^N_t
\]

\[
= \frac{1}{N} \sum_{x \in \mathbb{N}} \zeta^N_0(x)g(x/N) + \int_0^t b^N(\zeta^N_s, g)\,ds + M^N_t(g) + R^N_t,
\]

where

\[
(5.13) \quad b^N(\zeta, g) = \frac{1}{N} \sum_{x \in \mathbb{N}} \Delta^N g(x/N)\zeta(x) - \frac{\beta(N)}{N} \sum_{x \in \mathbb{N}} \nabla^N g(x/N)\zeta(x)
\]

\[
+ N(g(1/N)\zeta(2) - g(0)\zeta(1)),
\]

with

\[
\Delta^N g(x/N) = N^2(g((x+1)/N) + g((x-1)/N) - 2g(x/N)),
\]

\[
\nabla^N g(x/N) = N\left(g((x+1)/N) - g(x/N)\right), \quad x \in \mathbb{N},
\]

and

\[
M^N_t(g) = \frac{1}{N} \sum_{x \in \mathbb{N}} M^N_t(x)g(x/N).
\]

The error term \( R^N_t \) in (5.12) is defined by

\[
R^N_t = \int_0^\infty g(u)\tilde{\zeta}^N(t,u)\,du - \frac{1}{N} \sum_{x \in \mathbb{N}} \zeta^N_t(x)g(x/N)
\]

and admits a bound:

\[
(5.14) \quad |R^N_t| \leq e^{C\mathcal{X}^N_t} \left\{ \left(e^{C/N} - 1\right) \|g\|_{L^1(\mathbb{R}_+)} + \frac{1}{N} \|g'\|_\infty \times |\text{supp } g| \right\}
\]

in view of (5.9) and (5.10). Therefore, Proposition 5.4 shows that \( R^N_t \) tends to 0 as \( N \to \infty \) in probability.

The martingale term in (5.12) vanishes in the limit:

**Lemma 5.6.** \( E[M^N_t(g)^2] \) converges to 0 as \( N \to \infty \).

**Proof.** A straightforward computation leads to the following results for the quadratic and cross-variations of \( M^N_t(x) \):

\[
\frac{d}{dt} \langle M^N(x) \rangle_t = \zeta^N_t(x)^2 \left\{ a_Nc_-(-1, \eta^N_t) + b_Nc_+(x-1, \eta^N_t) \right\}, \quad x \geq 2,
\]

\[
\frac{d}{dt} \langle M^N(1) \rangle_t = \zeta^N_t(1)^2 \left\{ a_N1\{\eta^N(1)=0\} + b_N1\{\eta^N(1)=1\} \right\},
\]

\[
\langle M^N(x), M^N(y) \rangle_t = 0, \quad 1 \leq x \neq y,
\]

where \( a_N = N^2(1-\epsilon)^2/\epsilon, b_N = N^2(1-\epsilon)^2 \). This implies the conclusion of the lemma. \( \square \)
To treat the boundary term appearing in \( b^N(\zeta, g) \) (i.e. the third term in the right hand side of \((5.13)\)), we need the following ergodic property of the \( \eta \)-process at the boundary site \{1\}. Note that this ergodic property holds at the single site \{1\} without taking any average over sites near the boundary as performed in \([6]\).

**Lemma 5.7.** Under the condition \((5.6)\) in Proposition \(5.4\), for every \( 0 \leq T_1 \leq T_2 \leq T \) and \( \delta > 0 \), we have that

\[
\lim_{N \to \infty} \mathbb{Q}^N_{\nu_0} \left| \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \eta^N_s(1) - \frac{1}{2} \right| > \delta = 0.
\]

**Proof.** Consider the martingale

\[
m_t^N := X_t^N - X_0^N - \int_0^t N^2 \tilde{L}_{\epsilon(N), R}(X_s^N)ds.
\]

By \((5.7)\) with \( \varphi \equiv 1 \), we see that \( N^2 \tilde{L}_{\epsilon(N), R}(X_s^N) = N(1 - 2\eta^N_s(1)) - \beta(N)(1 - \eta^N_s(1)) \). However, since Lemma \(3.2\) implies \( 0 < \beta(N) = N(1 - \epsilon(N)) \leq C \) for \( N \geq 1 \), this proves that

\[
\int_{T_1}^{T_2} \{1 - 2\eta^N_s(1)\}ds \leq \frac{1}{N} \left( X_{T_2}^N - X_{T_1}^N + |m_{T_2}^N| + |m_{T_1}^N| + CT_2 \right).
\]

Thus, the lemma follows from \((5.6)\) and the estimate: \( E[|m_T^N|^2] \leq T \), which follows from \((5.8)\) with \( \varphi \equiv 1 \).

Once the following lemma for the boundary term in \( b^N(\zeta, g) \) is established, the weak form \((5.4)\) of the equation \((5.3)\) is easily derived from \((5.12)\), \((5.13)\), \((5.14)\) and Lemma \(5.6\). Thus, the proof of Theorem \(5.2\) is concluded by the uniqueness of the weak solutions of \((5.3)\), which will be shown in the next subsection.

**Lemma 5.8.** If \( g \in C^2_0(\mathbb{R}_+) \) satisfies the condition \( 2g'(0) - \beta g(0) = 0 \), then we have that

\[
\lim_{N \to \infty} \mathbb{Q}^N_{\nu_0} \left| \int_0^T N (g(1/N)\zeta_t^N(2) - g(0)\zeta_t^N(1)) dt \right| > \delta = 0
\]

for every \( \delta > 0 \).

**Proof.** Recalling \( \zeta_t^N(2) = \zeta_t^N(1)e^{(\log \zeta_t^N(1))} \), we have that

\[
N (g(1/N)\zeta_t^N(2) - g(0)\zeta_t^N(1)) = \zeta_t^N(1)(g'(0) - \beta g(0)\eta^N_t(1) + r_t^N),
\]

where the error term \( r_t^N \) is defined by

\[
r_t^N := \left\{ N(g(1/N) - g(0)) - g(0) \right\} + N(g(1/N) - g(0)) \left\{ e^{(\log \zeta_t^N(1))} - 1 \right\}
\]

\[
+ Ng(0) \left\{ e^{(\log \zeta_t^N(1))} - 1 + \beta \eta_t^{N}(1)/N \right\},
\]

and tends to 0 as \( N \to \infty \) by Lemma \(5.2\). Therefore, by the boundary condition for \( g \), if we can show that

\[
(5.15) \lim_{N \to \infty} \mathbb{Q}^N_{\nu_0} \left| \int_0^T \zeta_t^N(1)(\eta^N_t(1) - 1/2) dt \right| > \delta = 0
\]

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for every $\delta > 0$, the proof of the lemma is concluded. However, as we have shown in the tightness, the process $\{\zeta^N(t)\}_{N \geq 1}$ has the equi-continuity:

$$
\lim_{\gamma \downarrow 0} \lim_{N \to \infty} \mathbb{Q}_{\nu N}^N \left[ \sup_{|t-s| \leq \gamma} \sup_{0 \leq s < t \leq T} |\zeta^N(t) - \zeta^N(s)| > \delta' \right] = 0,
$$

for every $\delta' > 0$. Therefore, if we divide the interval $[0, T]$ into small subintervals with length $\gamma$:

$$
\left| \int_{0}^{T} \zeta^N(t) \left( \eta^N(t) - 1/2 \right) dt \right| \leq \sum_{k=0}^{\lfloor T/\gamma \rfloor} \int_{k\gamma}^{(k+1)\gamma} \zeta^N(t) \left( \eta^N(t) - 1/2 \right) dt,
$$

$\zeta^N(t)$ in the integrand is close to $\zeta^N_{k\gamma}(t)$ (if $\gamma$ is small enough) and we can apply Lemma $5.7$ for the integral $\int_{k\gamma}^{(k+1)\gamma} \left( \eta^N(t) - 1/2 \right) dt$. In other words, $\zeta^N(t)$ changes slowly compared with the rapid motion of $\eta^N(t)$. This proves (5.15).

**Remark 5.1.** For $g$ satisfying the same condition as Lemma $5.8$, a stronger assertion:

$$
\lim_{N \to \infty} \mathbb{Q}_{\nu N}^N \left[ \left| \int_{0}^{T} N \sqrt{N} \left( g(1/N) \zeta^N(t) - g(0) \zeta^N(t) \right) dt \right| > \delta \right] = 0
$$

holds for every $\delta > 0$ even by multiplying an extra factor $\sqrt{N}$. Indeed, this can be seen by noting that the error estimate given in the proof of Lemma $5.7$ is $O(1/N)$ and that in Lemma $5.2$ is $O(\log N/N^2)$ as $N \to \infty$. This fact will be used in [8].

### 5.3.4 Uniqueness of weak solutions

Here, we prove the uniqueness of the weak solutions of (5.3). The method is standard, especially because the equation is linear. We first extend the class of test functions $g = g(u)$ in the weak form (5.4) to the family of all $g = g(t, u) \in C^{1,2}_0([0, T] \times \mathbb{R}_+)$ satisfying $2\partial_u g(t, 0) - \beta g(t, 0) = 0$ for every $t \in [0, T]$, and show that

$$
(5.16) \quad \int_{0}^{\infty} g(t, u) \omega(t, u) du = \int_{0}^{\infty} g(0, u) \omega(0, u) du + \int_{0}^{t} \int_{0}^{\infty} \left( \partial_s g(s, u) + \partial^2_u g(s, u) - \beta \partial_u g(s, u) \right) \omega(s, u) ds du
$$

holds for every such $g$ and $t \in [0, T]$. Indeed, this can be done by dividing the interval $[0, t]$ into small pieces, assuming $g$ to be constant in $s$ on each small interval, applying the weak form (5.4) on each such small interval and finally by passing to the limit.

Secondly, since the solution $\omega$ is assumed to be bounded, we can further extend the class of $g$’s from functions having compact supports in $[0, T] \times \mathbb{R}_+$ to those having the exponentially decaying property as $u \to \infty$ in the sense that $\sup_{t \in [0, T], u \in \mathbb{R}_+} \{|g(t, u)| +$
Theorem 5.2 implies (2.13) with $\psi$ for every $W$. We will show that Theorem 2.2 for the process and this concludes the proof of the uniqueness of the weak solutions of (5.3).

We can extend to a wider class of functions $K >$ some if we define $K >$ some. Therefore, (5.1) is shown for functions $g$ if we define $K >$ some by approximating such $g$ by a sequence of continuous functions $g_n \in C_b^1(\mathbb{R}_+)$. For $g \in C_b^1(\mathbb{R}_+)$ satisfying $g(v) = 0$ for $v \leq 1/K$ and $g(v) = c$ for $v \geq K$ with some $K > 1$ and $c \in \mathbb{R}$, taking $g$ as $f$ in (2.12), we have that

$$\lim_{N \to \infty} \nu^N[\int_0^\infty g'(u)\tilde{\psi}_q^N(u)du - \int_0^\infty g'(u)\psi_0(u)du| > \delta] = 0$$

for every $\delta > 0$. By the definition,

$$\int_0^\infty g'(u)\tilde{\psi}_q^N(u)du = \frac{1}{N} \sum_{i \in \mathbb{Z}_+} \int_0^{\frac{q_i}{N}} g'(u)du = \frac{1}{N} \sum_{i \in \mathbb{Z}_+} g(q_i) = \frac{1}{N} \sum_{x \in \mathbb{Z}_+} g(\frac{x}{N})\eta^q(x).$$

On the other hand, by the integration by parts formula,

$$\int_0^\infty g'(u)\psi_0(u)du = -\int_0^\infty g(v)\psi_0'(v)dv = \int_0^\infty g(v)\rho_0(v)du.$$

Therefore, (5.1) is shown for functions $g$ satisfying the above conditions. However, this can be extended to a wider class of functions $g \in C_b(\mathbb{R}_+)$ satisfying $g(v) = c$ for $v \geq K$ with some $K > 1$ and $c \in \mathbb{R}$, by approximating such $g$ by a sequence of continuous functions $g_n \in C_b^1(\mathbb{R}_+)$ satisfying $g_n(v) = 0$ for $v \leq 1/K$ and $g(v) = c$ for $v \geq K$ with some $K > 1$ and $c \in \mathbb{R}$ noting that $0 \leq \eta(x), \rho_0(v) \leq 1$.

In order to complete the proof of the theorem, it is now sufficient to show that (5.2) in Theorem 5.2 implies (2.13) with $\psi(t, u) = \frac{1}{T} \log \omega(t, u)$. The non-linear equation (2.14) for $\psi_t$ follows from (5.3) for $\omega_t$. Especially, the boundary condition $2\partial_u\omega(t, 0) + \beta\omega(t, 0) = 0$ implies that $\partial_u\psi(t, 0) = -1/2$ and $\omega(t, \infty) = 1$ implies that $\psi(t, \infty) = 0$ for $t > 0$.

Since $\tilde{\psi}_q^N(u) = \frac{1}{T} \log \tilde{\zeta}^N(t, u) + o(1)$ with an error going to 0 in probability as $N \to \infty$ in view of (5.9) and (5.10), noting that $\omega(t, u), \tilde{\zeta}^N(t, u) \geq 1,$ (5.2) implies that

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T, 0 \leq u \leq K} \left[ \frac{\tilde{\psi}_q^N(u) - \psi(t, u)}{\delta} > \delta \right] = 0.$$
for every $T > 0$, $K > 0$ and $\delta > 0$. This completes the proof of Theorem 2.2.

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