Representation of Chow Groups of Codimension Three Cycles

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Abstract. In this note we are going to prove that if we have a fibration of smooth projective varieties \( X \to S \) over a surface \( S \) such that \( X \) is of dimension four and that the geometric generic fiber has finite dimensional motive and the first étale cohomology of the geometric generic fiber with respect to \( Q_l \) coefficients is zero and the second étale cohomology is spanned by divisors, then \( A^3(X) \) (codimension three algebraically trivial cycles modulo rational equivalence) is dominated by finitely many copies of \( A_0(S) \). Meaning that there exists finitely many correspondences \( \Gamma_i \) on \( S \times X \), such that \( \sum_i \Gamma_i \) is surjective from \( \oplus A^2(S) \) to \( A^3(X) \).

1. Introduction

The representability problem in the theory of algebraic cycles is a very interesting and a fundamental problem. Precisely it means the following. Let \( X \) be a smooth projective algebraic variety of dimension \( n \). Consider the group of algebraic cycles of codimension \( i \) which are algebraically trivial modulo rational equivalence. Denote this group by \( A^i(X) \). Then the question is when there exists a smooth projective curve \( C \) and a correspondence \( \Gamma \) on \( C \times X \) such that \( \Gamma_* \) from \( J(C) \), the Jacobian variety of \( C \), to \( A^i(X) \) is onto. The case when we consider \( A^n(X) \), this representability question is equivalent to the fact that \( A^n(X) \) is isomorphic to the albanese variety of \( X \), which is also equivalent to the surjectivity of the natural map from some high degree symmetric power of \( X \) to \( A^n(X) \). It is a conjecture due to Bloch that when we consider a smooth projective surface \( S \) with geometric genus zero then the group \( A^2(S) \) is representable. On the other hand, Mumford \([M]\) proved that when the geometric genus of the surface is greater than zero then the group \( A^2(S) \) is not representable. The Bloch’s conjecture for surfaces with geometric genus equal to zero has been proved in certain cases, for all surfaces not of general type \([BKL]\) and some examples of surfaces of general type \([V],[VC]\).
In [VG] it has been proved that when we have a smooth projective threefold \( X \) fibered into surfaces over a smooth projective curve \( C \), such that the geometric generic fiber has finite dimensional motive, has first étale cohomology with \( \mathbb{Q}_l \) is zero and the second étale cohomology with \( \mathbb{Q}_l \) is spanned by divisors, then the group \( A^2(X) \) is representable in the sense that there exists finitely many correspondences \( \Gamma_i \) on \( C \times X \), such that \( \oplus \Gamma_i \circ \oplus J(C) \) to \( A^2(X) \) is onto. Then as an application, it has been proved that the \( A^2 \) of a Del-pezzo fibration over a smooth projective curve is representable.

In this paper our aim is to extend the result of [VG] to the case when \( X \) is of dimension 4 and it is fibered into surfaces over a smooth projective surface, such that the geometric generic fiber satisfies the property as above. Then we prove that \( A^3(X) \) is representable upto dimension 2. Precisely it means that there exists finitely many correspondences \( \Gamma_i \) on \( S \times X \) such that \( \oplus \Gamma_i \circ \oplus J(C) \) to \( A^3(X) \) is onto. In other words we prove that \( A^3(X) \) is representable by \( A^2 \) of smooth projective surfaces. As an application we have that the cubic fourfolds fibered into Del-Pezzo surfaces over a smooth projective surface, has \( A^3 \) representable by \( A^2 \) of surfaces. Such examples are studied in [AHTV].

So the main theorem is:

**Theorem 1.1.** Let \( X \) be a smooth projective fourfold birational to a fourfold \( X' \) fibered over a surface \( S \). Assume moreover that the geometric generic fiber of the fibration \( X' \to S \) satisfies the following:

1. The motive of it is finite dimensional. (ii) First étale cohomology of it is trivial with respect to \( \mathbb{Q}_l \) coefficients. (iii) The second étale cohomology is spanned by divisors on it.

Then the group \( A^3(X) \) is representable upto dimension two.

The underlying technique to prove the main theorem is same as in [VG], but the only non-trivial step is to excise a curve from the base of the fibration and to prove that the representability of \( A^3(X) \) will follow from representability of \( A^3(U) \), where \( U = S \setminus C \), that is the part we remove has representable \( A^2 \).

The theorem is interesting from the following view point: We can reformulate the question of irrationality of a four dimensional projective...
variety in terms of representability of $A^3$ up to dimension 2, that if a variety of dimension 4 has non-representable $A^3$ up to dimension 2 [see 2 for the definition] then it is non-rational.

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2. Representability up to dimension two

Let $X$ be a smooth projective variety and let $A^i(X)$ denote the codimension $i$ algebraic cycles on $X$, modulo rational equivalence. Then we say that $A^i(X)$ is weakly representable up to dimension two if there exists finitely many curves $C_1, \cdots, C_m$ with correspondences $\Gamma_1, \cdots, \Gamma_m$ on $C_1 \times X, \cdots, C_m \times X$ and finitely many surfaces $S_1, \cdots, S_n$ with correspondences $\Gamma'_j$ on $S_j \times X$, such that

$$\sum_i \Gamma_i + \sum_j \Gamma'_j$$

is surjective from $\oplus_i A^1(C_i) \oplus \oplus_j A^2(S_j)$ to $X$. If we assume that $X$ is a fourfold, then the representability of $A^2(X)$ is a birational invariant. This is because if we blow up $X$ to $\widetilde{X}$, then $A^2(\widetilde{X})$ is isomorphic to $A^2(X) \oplus A^1(Z)$, where $Z$ is the center of the blow up. Since $A^1(Z)$ is dominated by $J(\Gamma)$, for some smooth projective curve $\Gamma$, this will imply that if $A^2(X)$ is representable up to dimension two then so is $A^2(\widetilde{X})$. Suppose that $X, Y$ are birational, such that $Y$ is obtained by one blow up of $X$ and then one blow down, the we have a generically finite map from $\widetilde{X}$ to $Y$, which gives a surjection at the level of $A^2$. So $A^2(X)$ representable up to dimension two implies the same for $A^2(\widetilde{X})$, hence the same for $A^2(Y)$. Changing the role of $X, Y$, we get the reverse implication.

Similarly if we consider the representability of $A^3(X)$, $X$ smooth projective fourfold, then it is a birational invariant in $X$. This is because if we blow up $X$ along a surface or a curve then the blow up formula gives us

$$A^3(\widetilde{X}) = A^3(X) \oplus A^2(S) \oplus A^1(S)$$

or

$$A^3(\widetilde{X}) = A^3(X) \oplus A^1(C)$$
where $S$ or $C$ is the center of the blow up. So if we blow up for many times we are only adding $A^2$ of a surface or $A^1$ of a curve, so the representability upto dimension two remains.

So our main theorem in this section is the following.

**Theorem 2.1.** Let $X$ be a smooth projective fourfold birational to a fourfold $X'$ fibered over a surface $S$. Assume moreover that the geometric generic fiber of the fibration $X' \to S$ satisfies the following:

(i) The motive of it is finite dimensional. (ii) First étale cohomology of it is trivial with respect to $\mathbb{Q}_l$ coefficients. (iii) The second étale cohomology with respect to $\mathbb{Q}_l$ coefficients, is spanned by divisors on it.

Then the group $A^3(X)$ is representable upto dimension two.

**Proof.** Let us assume from the very beginning that the fourfold $X$ is equipped with a fibration to a smooth projective surface $S$. That is we have a fibration $X \to S$. Let $\eta = \text{Spec}(k(S))$, and $\bar{\eta} = \text{Spec}(\overline{k(S)})$. Let $b_2$ be the dimension of $H^2_{\text{ét}}(X_\eta, \mathbb{Q}_l)$ and let by our assumption $D_1, \cdots, D_{b_2}$ are the divisors on $X_\eta$, generating the second étale cohomology group $H^2_{\text{ét}}(X_\eta, \mathbb{Q}_l)$. Let us consider a finite extension $L$ of $k(S)$, inside its algebraic closure such that $D_1, \cdots, D_{b_2}$ are defined over $L$. That is we consider a smooth projective curve $S'$ mapping finitely onto $S$ with function field $L$, such that $X' = X \times_S S' \to X$ is of finite degree and $D_1, \cdots, D_{b_2}$ are defined over the generic point of $S'$. Since $X' \to X$ is finite we can work with this divisors which are actually defined over the generic point of $S'$.

Now we need the lemma.

**Lemma 2.2.** Let $X$ be a smooth projective fourfold over a field $k$ and let $A^3(X) = V \oplus W$, where $V$ is a finite dimensional $\mathbb{Q}$ vector space. Then $A^3(X)$ is representable if and only if there exists finitely many smooth curves and surfaces $C_1, \cdots, C_m, S_1, \cdots, S_n$, and correspondences $\Gamma_i$ on $C_i \times X$, and $\Gamma_j'$ on $S_j \times X$ such that the homomorphism $\sum_i \Gamma_i + \sum_j \Gamma_j'$ from $\oplus A^1(C_i) \oplus A^2(S_j)$ to $A^3(X)$ is surjective onto $W$.

**Proof.** Let $v_1, \cdots, v_n$ be a basis for $V$. For each $v_j$ let $Z_j$ be the algebraical cycle representing it. Since $Z_j$ is algebraically equivalent to zero, we have a smooth projective curve $C_j$ and a correspondence $\Gamma_j$ such that $\Gamma_j(x_j)$ equals $Z_j$, where $x_j$ is a point on $J(C_j)$. Therefore the homomorphism $\sum_j \Gamma_j(x_j)$ is covering the space $V$ and it has domain $\oplus J(C_j)$. So to prove that $A^3(X)$ is representable it is enough to prove the representability of $W$. So
we need to find some smooth curves and surfaces satisfying the assumption that the sum of algebraically trivial zero cycles on these curves and surfaces cover $W$. □

step 2:
Let $\{p_1, \cdots, p_m\}$ be a finite set of closed points on $S$. Let $U$ be the complement of this finite set. Let $Y = f^{-1}(U)$. Then by the localization exact sequence we have that
$$\bigoplus_j \text{CH}^2(X_{p_j}) \to \text{CH}^3(X) \to \text{CH}^3(Y) \to 0$$
so the $\mathbb{Q}$ vector space $\text{CH}^3(X)$ splits as $\text{CH}^2(Y) \oplus I$ where $I$ is the image of the pushforward from $\bigoplus_j \text{CH}^2(X_{p_j})$ to $\text{CH}^3(X)$. It is also true that the map from $A^3(X)$ to $A^3(Y)$ is surjective, where $A^3$ denote the algebraically trivial one-cycles modulo rational equivalence. So we have a splitting
$$A^3(X) = A^3(Y) \oplus J$$
where $J$ is the intersection of $I$ and $A^3(X)$. Let for $X_{p_j}$, $\widetilde{X}_{p_j}$ is the resolution of singularity of it. Then we have that $J$ is covered by two subspaces, one is the direct sum of $A^2(X_{p_j})$, which is covered by direct sums of the $A^2$'s of the irreducible components of $\widetilde{X}_{p_j}$, the other is a finite dimensional subspace, coming from the Neron severi group of the irreducible components of the resolutions of $\widetilde{X}_{p_j}$. So by the previous lemma it is sufficient to prove that $A^3(Y)$ is representable upto dimension two to prove the representability of the group $A^3(X)$.

step 3:
Let $C$ be a projective curve inside $S$, and we excise $C$ from $S$. Let $Y$ be the complement of $X_C = X \times_S C$ in $X$. Then we prove that the representability of $A^3(X)$, follows from the representability of $A^3(Y)$. For that we consider the localisation exact sequence given by
$$\text{CH}^2(X_C) \to \text{CH}^3(X) \to \text{CH}^3(Y) \to 0.$$
Then we have $\text{CH}^3(X) = \text{CH}^3(Y) \oplus I$, where $I$ is the image of $\text{CH}^2(X_C)$ in $\text{CH}^3(X)$. Considering the subgroup of algebraically trivial cycles we get that
$$A^3(X) = A^3(Y) \oplus J$$
where \( J \) is the intersection of \( I \) with the image of \( A^3(X) \). Then \( J \) is a sum of two \( \mathbb{Q} \)-vector spaces. One is the image of \( A^2(X_C) \) and the other is a finite dimensional subspace corresponding to the Neron-Severi group of \( X_C \). Then by step one if we have \( A^2(X_C) \) is representable then we have the representability of \( J \). But the representability of \( A^2(X_C) \) follows from [VG] [the main theorem]. Because according to our assumption the geometric generic fiber of \( X \to S \) has finite dimensional motive and base change of finite dimensional motive is finite dimensional. Therefore the geometric generic fiber of \( X_C \to C \) has finite dimensional motive. Also the first and second etale cohomology of the geometric generic fiber of \( X_C \to C \) satisfies the assumption of [VG] [the main theorem], because the geometric generic fiber of \( X \to S \) satisfies the similar properties. Therefore we have the representability of \( A^3(X) \) follows from that of \( A^3(Y) \). So we can say that to prove representability of \( A^3(X) \) it is sufficient to remove a finitely many curves from the base, and look for the representability of the \( A^3(Y) \), where \( Y \) is he complement of \( \bigcup_i X_{C_i} \).

step 4:

Suppose that \( X_\eta \) is defined over a finite extension \( L \) of \( k(S) \) inside \( k(S) \). Then let \( S' \) be a smooth projective surface with function field \( L \), and mapping finitely onto \( S \). Now over \( S' \) we have a rational point of the variety \( X'_\eta = X_\eta \times_{k(S)} S' \). This rational point induces a section of the map \( Y \to U \), over some \( U' \) Zariski open inside \( U \). Now \( U' \) maps isomorphically onto its image in \( Y \). So we have to remove a curve from \( U \) to obtain \( U' \). Since the representability remains unchanged by this process, we can assume without loss of generality that the section is defined everywhere on \( U \). So without loss of generality we can assume that \( Y \to U \) has a section. Let \( E \) be the image of this section. Then \( E \) has codimension 4 in \( Y \), so it’s support is contained in finitely many fibers. So we can cut down those finitely many fibers. Then we can prove that \( \pi_0 = E \times_U Y, \pi_4 = Y \times_U E \) are pairwise orthogonal [VG]. Hence we have the projector

\[
\pi_2 = \Delta_{Y/U} - \pi_0 - \pi_4 .
\]

Let \( M^2(Y/U) \) be the relative motive defined by \( \pi_2 \). Then we have the decomposition

\[
M(Y/U) = \mathbb{1}_U \oplus M^2(Y/U) \oplus \mathbb{L}^2_U
\]

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Now we know that $M(\bar{X_\eta})$ is finite dimensional, which means at the level of Chow groups that there exists some correspondence $p, q$ on $\bar{X_\eta}$ such that $d_{\text{Sym}} \circ p^n$ is rationally trivial and $d_{\text{alt}} \circ q^n$ is rationally trivial. Let $L$ be the minimal field of definition of $p, q$, then taking a finite extension $S'$ over $S$, with function field $L$, we have $M(Y_{\eta})$ is finite dimensional over $\eta$ itself. On the other hand since $\text{CH}^2(Y_{\eta} \times Y_{\eta})$ is the colimit of the groups $\text{CH}^2(Y_U \times U Y_U)$, we have that the motive $M(Y/U)$ is finite dimensional for some open set $U$ in $S$. Then we shrink our $U$ to this $U$ by taking intersection.

Now the finite dimensionality of $M(Y/U)$ implies $M^2(Y/U)$ is finite dimensional. One can show more, that is $M^2(Y/U)$ is evenly finite dimensional of dimension $b_2$. This follows from the computation of [VG].

Now let $D_1, \cdots, D_{b_2}$ be the divisors defined over $\eta$ and they generate the cohomology group $H^2(Y_{\eta}, \mathbb{Q}_l)$. According to [VG], [GP][theorem 2.14] we have

$$\rho_{\eta} = (\pi_2)_{\eta} - \sum_{i=1}^{b_2} [D_i \times_\eta D_i']$$

is homologically trivial. Then there exists some $n$ such that $\rho_{n_{\eta}} = 0$, in the associative ring $\text{End}(M^2(Y_{\eta}))$, by Kimura's nilpotency theorem [KI][proposition 7.2].

Let $W_i, W_i'$ are spreads of the above divisors over $U$, they may be non-unique but we choose and fix one spread. Consider the cycles

$$W_i \times_U W_i'$$

in $\text{Corr}_U^0(Y \times_U Y)$ and set

$$\rho = \pi_2 - \sum_{i=1}^{b_2} [W_i \times_U W_i']$$

then $\rho$ maps to $\rho_{\eta}$ under the base change functor from the category of relative Chow motives over $U$ to the category of Chow motives over $\eta$. Let us consider an endomorphism $\omega$ of $M^2(Y/U)$. Then under the above functor trace of $\omega \circ \rho$ is mapped to trace of $\omega_{\eta} \circ \rho_{\eta}$ [VG], [DM][page 116], which is zero because $\rho_{\eta}$ is homologically trivial. The base change functor defines an isomorphism from $\text{End}(\mathbb{F}_U)$ to $\text{End}(\mathbb{F}_\eta)$. Therefore trace of $\omega \circ \rho = 0$ for any $\omega$, so $\rho$ is numerically trivial, therefore $\rho^n = 0$ by proposition 2 in [VG], [KI][7.5], [AK][9.1.14].
Let $\tilde{W}_i$ be the Zariski closure of $W_i$ in $X$ and consider
\[ \theta_i = \Gamma^t_f.[S \times \tilde{W}_i] \]
it is a codimension 3 cycle on $S \times X$. The cycle $\Gamma^t_f$ is the transpose of the graph of the map $f : X \to S$. Consider the homomorphism $\theta_{i*}$ from $\text{CH}^2(S)$ to $\text{CH}^3(X)$. Let us compute $\theta_{i*}$.

\[ \theta_{i*}(a) = p_{X*}(p^*_{S}(a).\theta_i) \]
which is equal to
\[ \theta_{i*}(a) = p_{X*}(p^*_{S}(a).\Gamma^t_f.[S \times \tilde{W}_i]) \]
on the other hand we have $p^*_{S}(a).\Gamma^t_f = p^*_{S}(a).\tau_*([X])$, where $\tau$ is the map $x \mapsto (f(x), x)$. We have $f^* (a) = \tau^* p^*_{S}(a)$, therefore $\tau_* f^* (a) = \tau_* (\tau^* p^*_{S}(a).[X])$, which by projection formula is $p^*_{S}(a).\tau_* (X) = p^*_{S}(a).\Gamma^t_f$. Putting this in the above expression of $\theta_{i*}$ we have
\[ \theta_{i*}(a) = p_{X*}(\tau_* f^* (a).[C \times \tilde{W}_i]) \]
\[ = p_{X*}(\tau_* f^* (a).p^*_{X}([\tilde{W}_i])) = p_{X*}(\tau_* f^* (a).[\tilde{W}_i] = f^* (a).[\tilde{W}_i]. \]
So this computation provides the description of the homomorphism $\theta_{i*}$ in the non-compact case when we consider it from $\text{CH}^2(U)$ to $\text{CH}^3(Y)$. It is immediate that the homomorphisms $\theta_{i*}$’s are compatible in compact and non-compact cases. Since the homomorphism $\theta_{i*}$ in the non-compact case respects algebraic equivalence we have the compatibility at the level of algebraically trivial cycles modulo rational equivalence. So summarising we have a commutative diagram as follows.

\[ \sum_{i=1}^{b_2} A^2(S) \xrightarrow{\theta_*} A^3(X) \]

\[ \sum_{i=1}^{b_2} A^2(S) \xrightarrow{\theta_*} A^3(Y) \]
Chasing the above diagram and assuming that the bottom $\theta_*$ is surjective we have that the top $\theta_*$ has image equal to $A^3(X)$ modulo $A^2(X_C)$, where $C$ is the complement of $U$ in $S$ and $X_C = f^{-1}(C)$. Since $A^2(X_C)$ is finite dimensional it is enough to prove that $\theta_*$ at the bottom is onto to prove the representability of $A^3(X)$ upto dimension 2.
Let $y$ belong to $\text{CH}^3(Y)$, then considering the relative correspondence $\Delta_{Y/U}$, we get that

$$y = \Delta_{Y/U*}(y) = \pi_0*(y) + \pi_2 *(y) + \pi_4*(y).$$

Now $\pi_0*(y)$ is equal to $p_{2*}(p^*_1(y).\pi_0)$ which is equal to $p_{2*}(p^*_1(y).p^*_E(y)) = p^*_2 p^*_1(y.E) = f^*f_*(y.E) = 0$ as the codimension of $y.E$ is five. So we have $\pi_0*(y) = 0$. Also we have $f_*(y) = 0$.

Next we compute,

$$\pi_4*(y) = p_{2*}(p^*_1(y).\pi_4)$$
$$= p_{2*}(y \times_U Y \times_U E) = p_{2*}(y \times_U E)$$
$$= f_*(y) \times_U E = 0.$$

So we have that $y = (\pi_2*) (y)$. Putting $\pi_2$ equal to $\sum_i [W_i \times_U W'_i] + \rho$ we get that $y = \pi_2*(y) = \sum_i [W_i \times_U W'_i]* (y) + \rho_*(y)$. Let $Z_j$'s are curves representing the class of $y$, then

$$[W_i \times_U W'_i]* (Z_j) = p_{2*}([Z_j \times_U Y] \cdot [W_i \times_U W'_i])$$
$$= p_{2*}([Z_j] \cdot [W_i] \times_U [Y] \cdot [W'_i]) = p_{2*}([Z_j] \cdot [W_i] \times [W'_i])$$

by linearity we have

$$[W_i \times_U W'_i](y) = p_{2*}(y \cdot [W_i] \times_U [W'_i])$$

since $y$ is of codimension 3 and $W_i$ is of codimension 1, we have $yW_i$ is a zero cycle on $Y$. Observe that

$$[W_i \times_U W'_i]* (y) = p_{2*}(yW_i \times_U W'_i)$$
$$= p_{2*} (p^*_i (yW_i) \cdot p^*_2 (W'_i)) = p_{2*} p^*_i (yW_i) \cdot W'_i$$
$$= f^* f_*(yW_i) \cdot W'_i = f^*(a_i) \cdot W'_i = \theta_*(a_i)$$

where $a_i = f_*(yW_i)$. Since $y$ belongs to $A^3(Y)$, we have that $a_i$ is in $A^2(U)$. Then we get that

$$\sum_i [W_i \times_U W'_i]* (y) = \sum_i \theta_*(a_i) = \theta_*(e_1)$$

where $c_1 = (a_1, \cdots, a_{b_2})$ in $\oplus_i A^2(S)$. So we have

$$\rho_*(y) = \theta_*(c_1) + y$$

applying $\rho$ n-times we have that

$$\rho_*(y)^n = 0 = \theta_*(c_n) + ny$$

so we have $y = -1/n \theta_*(c_n)$, hence $\theta_*$ is surjective.
Example 2.3. In [AHTV], the authors studied the examples of cubic fourfolds fibered into sextic del-Pezzo surfaces over $\mathbb{P}^2$. These are examples of rational cubic fourfolds. Since sextic del-pezzo surfaces have finite dimensional motives and there first étale cohomology is zero and second étale cohomology is spanned by divisors, so these cubics satisfy the assumption of our theorem. Hence the group $A^3$ of such a cubic is dominated by a finite sum of $A^2$’s of a single smooth projective surface.

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