Notes on hyperscaling violating Lifshitz and shear diffusion

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Abstract

We explore in greater detail our investigations of shear diffusion in hyperscaling violating Lifshitz theories in arXiv:1604.05092 [hep-th]. This adapts and generalizes the membrane-paradigm-like analysis of Kovtun, Son and Starinets for shear gravitational perturbations in the near horizon region given certain self-consistent approximations, leading to the shear diffusion constant on an appropriately defined stretched horizon. In theories containing a gauge field, some of the metric perturbations mix with some of the gauge field perturbations and the above analysis is somewhat more complicated. We find a similar near-horizon analysis can be obtained in terms of new field variables involving a linear combination of the metric and the gauge field perturbation resulting in a corresponding diffusion equation. Thereby as before, for theories with Lifshitz and hyperscaling violating exponents \( z, \theta \) satisfying \( z < 4 - \theta \) in four bulk dimensions, our analysis here results in a similar expression for the shear diffusion constant with power-law scaling with temperature suggesting universal behaviour in relation to the viscosity bound. For \( z = 4 - \theta \), we find logarithmic behaviour.
1 Introduction

In [1], we had studied the shear diffusion constant in certain hyperscaling violating Lifshitz theories by obtaining it as the coefficient of the diffusion equation satisfied by certain near horizon metric perturbations. In the present paper, we explore this in greater detail and study generalizations.

To put this in context, let us recall nonrelativistic holography or gauge/gravity duality [2] which has been under active exploration over the last few years. In particular, spacetimes conformal to Lifshitz [3, 4], referred to as hyperscaling violating spacetimes arise in effective Einstein-Maxwell-Dilaton theories e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Null $x^+$-reductions of AdS plane waves [17, 18], which are large boost, low temperature limits [19] of boosted black branes [20] provide certain gauge/string realizations of these. See e.g. [14, 21, 22, 23, 24, 25, 26, 27] for aspects of Lifshitz and hyperscaling violating holography. Some of these exhibit novel scaling for entanglement entropy e.g. [12, 13, 14], with the string realizations above reflecting this [28, 29, 30, 31], suggesting corresponding regimes in the gauge theory duals exhibiting this scaling.

Understanding hydrodynamic behaviour in these nonrelativistic gauge/gravity dualities is of great interest: see e.g. [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44] for previous
and recent investigations. Our approach in [1] to studying hydrodynamics and viscosity has been somewhat different, and based on Kovtun, Son, Starinets [45]. They observed that metric perturbations governing diffusive shear and charge modes in the near horizon region of the dual black branes of relevance simplify allowing a systematic expansion. This results in a diffusion equation for these shear modes on a stretched horizon, with universal behaviour for the diffusion constant, thereby leading to the viscosity bound [46]. This is akin to the membrane paradigm [47] for black branes, the horizon exhibiting diffusive properties. This approach is based simply on the fact that near horizon metric perturbations lead to a diffusion equation: thus it does not rely on any holographic duality per se. It is of course consistent with holographic results e.g. [48, 49] (see e.g. [50] for a review of these aspects of hydrodynamics).

In [1], we adapted the membrane-paradigm-like analysis of [45] and studied the shear diffusion constant in bulk \((d+1)\)-dimensional hyperscaling violating theories \((2.1)\) with \(z, \theta\) exponents. Specifically the diffusion of shear gravitational modes on a stretched horizon is mapped to charge diffusion in an auxiliary theory obtained by compactifying one of the \(d_i\) boundary spatial dimensions exhibiting translation invariance. This gives a near horizon expansion for perturbations with modifications involving \(z, \theta\). For generic exponents with \(d - z - \theta > -1\), we found the shear diffusion constant to be \(D = \frac{r_0^{z-2}}{d-z-\theta-1}\), i.e. power-law scaling \((2.14)\) with the temperature \(T \sim r_0^z\). Studying various special cases motivated the guess \((2.15)\), i.e. \(#DT^{\frac{z-2}{z}} = \frac{1}{4\pi}\) where \# is some \((d, z, \theta)\)-dependent constant, suggesting that \(\frac{2}{z}\) has universal behaviour. The condition \(z < 2 + d - \theta\) representing this universal sector appears related to requiring standard quantization from the point of view of holography. When the exponents satisfy \(d - z - \theta = -1\), the diffusion constant exhibits logarithmic behaviour, suggesting a breakdown of some sort in this analysis. The exponents arising in null reductions of AdS plane waves or highly boosted black branes \([17, 19, 18]\) mentioned above satisfy this condition, which can be written as \(z = 2 + d_{\text{eff}}\).

The analysis above arose solely from perturbations in the metric sector. In theories with a gauge field, the near-horizon diffusion equation analysis above must be extended to also include the gauge field sector which mixes with some of the metric perturbations. The resulting story is somewhat more intricate, both calculationally and conceptually, and is the subject of this paper. To give a flavour of this, it is worth describing the analysis above in a little more detail. Shear gravitational perturbations \(h_{xy}, h_{ty}\), satisfy the diffusion equation in the near-horizon region within certain approximations, as stated earlier: they are mapped to \(U(1)\) gauge field modes \(A_x, A_t\) upon compactifying the \(y\)-direction which enjoys translation invariance. Near horizon membrane currents can be appropriately defined in terms of the field strengths for this gauge field \(A_\mu = (A_t, A_x)\), which then can be shown to satisfy Fick’s
law \( j^x = -D \partial_x j^t \), which in turn using current conservation leads to the diffusion equation \( \partial_t j^t = D \partial_x^2 j^t \), valid within a self-consistent set of approximations imposed near horizon. In terms of the original linearized Einstein equations for metric perturbations (without this \( y \)-compactification), the diffusion equation stems from one of the Einstein equations, which is essentially a conservation equation schematically of the form \( \partial_x (\partial_x (\# h_{xy})) \sim \# \partial_t (\partial_t (\# h_{ty})) \) where the \( \# \) are \( r \)-dependent factors. The other linearized Einstein equations are coupled second order equations for \( h_{ty}, h_{xy} \). In the case where the hyperscaling violating Lifshitz theory has a background gauge field \( A_\mu \), it turns out that the \( h_{ty} \) metric perturbation mixes with the gauge field component \( a_y \). The resulting Einstein equations along with the gauge field equation are coupled equations for \( h_{xy}, h_{ty}, a_y \) (with the other modes decoupling for modes respecting the \( y \)-compactification ansatz), and at first sight they do not reveal any such diffusion-equation-type structure.

Towards understanding this better, it is important to note that the hyperscaling violating Lifshitz black branes here are not charged black branes: the gauge field and scalar here simply serve as sources that support the nonrelativistic metric as a solution to the gravity theory. Using intuition from the fluid-gravity correspondence [51], the fact that these are uncharged black branes means that the near-horizon perturbations are effectively characterized simply by local temperature and velocity fluctuations. Thus since charge cannot enter as an extra variable characterizing the near-horizon region, the structure of the diffusion equation and the diffusion constant should not be dramatically altered by the presence of the gauge field. In light of this intuition, a closer look reveals that the relevant component of the Einstein equation is of the form \( \partial_x (\partial_x (\# h_{xy})) \sim \# \partial_t (\partial_t (\# h_{ty})) - \partial_t (\# a_y) \). This naively suggests that perhaps the correct field variable in terms of which the Einstein equation can be recast as a diffusion equation is in fact \( \tilde{h}_{ty} \equiv h_{ty} - \int (\# a_y dr) \). Analyzing this in greater detail shows that this essential logic is consistent, and thereby leads to a generalization of the analysis in [1] mapping shear diffusion to charge diffusion after \( y \)-compactification. This results in the same expression for the shear diffusion constant but obtained using the leading near-horizon expressions for \( \tilde{h}_{xy} \equiv h_{xy} \) and \( \tilde{h}_{ty} \equiv h_{ty} - \int (\# a_y dr) \).

In sec. 2, we briefly review the results of [1] obtained by the \( y \)-compactification. Sec. 3 discusses this analysis from the point of view of the original Einstein equations without \( y \)-compactification, giving some insight into how the diffusion equation effectively arises. In Sec. 4, we discuss the perturbations in the general hyperscaling violating Lifshitz background incorporating the gauge field perturbations as well. We then describe the various modifications in terms of the new field variables leading to the diffusion equation and thereby the shear diffusion constant. Sec. 5 has a Discussion. The Appendices provide various technical details.
2 Reviewing hyperscaling violating Lifshitz and the shear diffusion constant

Here we review the discussion in [1]. We are considering nonrelativistic holographic backgrounds described by a $(d+1)$-dim hyperscaling violating metric at finite temperature,

$$ds^2 = r^{2\theta/d_{i}} \left( - \frac{f(r)}{r^{2z}} dt^2 + \frac{dr^2}{r^2 f(r)} + \sum_{i=1}^{d_i} dx_i^2 \right), \quad d_i = d - 1, \quad d_{eff} = d_i - \theta, \quad (2.1)$$

where $f(r) = 1 - (r_0 r)^{d+z-\theta-1}$ and $z$ is the Lifshitz dynamical exponent with $\theta$ the hyperscaling violation exponent. The temperature of the dual field theory is

$$T = \frac{(d + z - \theta - 1)}{4\pi} \frac{r_0^z}{r_h^z}. \quad (2.2)$$

Here $d_i$ is the boundary spatial dimension while $d_{eff}$ is the effective spatial dimension governing various properties of these theories, for instance the entropy density $s \sim T^{d_{eff}/z}$. The null energy conditions following from (2.1) constrain the exponents, giving

$$(d - 1 - \theta)(d - 1)(z - 1) - \theta) \geq 0, \quad (z - 1)(d - 1 + z - \theta) \geq 0. \quad (2.3)$$

In [45], Kovtun, Son and Starinets formulated charge and shear diffusion for black brane backgrounds in terms of long-wavelength limits of perturbations on an appropriately defined stretched horizon, the broad perspective akin to the membrane paradigm [47]. Their quite general analysis begins with

$$ds^2 = G_{\mu \nu} dx^\mu dx^\nu = G_{tt}(r) dt^2 + G_{rr}(r) dr^2 + G_{xx}(r) \sum dx_i^2, \quad i = 1, \ldots, d_i. \quad (2.4)$$

This includes the hyperscaling violating backgrounds (2.1) as a subfamily. Charge diffusion of a gauge field perturbation $A_\mu$ in the background (2.4) is encoded by the charge diffusion constant $D$, defined through Fick’s Law $j^\mu = -D \partial \mu j^\nu$, where the 4-current $j^\mu$ is defined on the stretched horizon $r = r_h$ (with $n$ the normal) as $j^\mu = n_\nu F^{\mu \nu}|_{r = r_h}$. Then current conservation $\partial_\mu j^\mu = 0$ leads to the diffusion equation $\partial_t j^i = -\partial_i j^i = D \partial_i^2 j^i$, with $D$ the corresponding diffusion constant. Fick’s law in turn can be shown to apply if the stretched horizon is localized appropriately with regard to the parameters $\Gamma, q, T$. Translation invariance along $x \in \{x_i\}$ allows considering plane wave modes for the perturbations $\propto e^{-\Gamma t + iqx}$, where $\Gamma$ is the typical time scale of variation and $q$ the $x$-momentum. In the IR regime, the modes vary slowly: this hydrodynamic regime is a low frequency, long wavelength regime. The diffusion of shear gravitational modes can be mapped to charge diffusion [45]: under Kaluza-Klein compactification of one of the directions along which there is translation invariance, tensor perturbations in the original background map to vector perturbations on the compactified
background. A similar analysis adapting [45] was carried out in [1] for the shear diffusion constant in the backgrounds (2.1), obtaining an effective diffusion equation for the metric fluctuations $h_{xy}$ and $h_{ty}$ (x ≡ x1, y ≡ x2) around (2.4), depending only on $t, r, x$, i.e. $h_{ty} = h_{ty}(t, x, r)$, $h_{xy} = h_{xy}(t, r, x)$. y-translation invariance allows a y-compactification: then the modes $h_{xy}$ and $h_{ty}$ become components of a $U(1)$ gauge field in the dimensionally reduced $d$-dim spacetime, with

$$g_{\mu\nu} = G_{\mu\nu}(G_{xx})^{\frac{1}{d-2}} \quad [\mu, \nu = 0, \ldots, d - 1]; \quad A_t = (G_{xx})^{-1} h_{ty}, \quad A_x = (G_{xx})^{-1} h_{xy},$$

(2.5)

where $G_{\mu\nu}$ is the metric given by (2.4). The compactified gravitational action contains the Maxwell action, $\sqrt{-\mathcal{G}} \mathcal{R} \to -\frac{1}{4} \sqrt{-g} F_{\alpha\beta} F_{\gamma\delta} \mathcal{g}^{\alpha\gamma} \mathcal{g}^{\beta\delta} (G_{xx})^{\frac{1}{d-2}}$, with an r-dependent coupling constant. The gauge field equations following from the action are

$$\partial_{\mu} \left( \frac{1}{g_{eff}^2} \sqrt{-g} F^{\mu\nu} \right) = 0, \quad \frac{1}{g_{eff}^2} = G_{xx}^\frac{1}{d-2},$$

(2.6)

where we have read off the r-dependent $g_{eff}$ from the compactified action. Analysing these Maxwell equations and the Bianchi identity assuming gauge field ansatz $A_\mu = A_\mu(r)e^{-\Gamma t + i qx}$ and radial gauge $A_r = 0$ as in [45] shows interesting simplifications in the near-horizon region. When $q = 0$, these lead to $\partial_r \left( \frac{\sqrt{-g}}{g_{eff}} g^{rr} g^{t\alpha} \partial_r A_t \right) = 0$. We impose the boundary condition that the gauge fields vanish at $r = r_c \sim 0$. As in [45], for $q$ nonzero but small, we assume an ansatz $A_t = A_t^{(0)} + A_t^{(1)} + \ldots, A_t^{(1)} = O(q^2 T^2/z)$ as a series expansion in $q^2 T^2/z$, and likewise for $A_x$. The $A_x$ solution is then found by using the $A_t^{(0)}$ solution and one of the Maxwell equations in the compactified theory. We further impose a second assumption

$$|\partial_t A_x| \ll |\partial_x A_t|$$

(2.7)

as in [45]. For generic values

$$d - z - \theta > -1,$$

(2.8)

the leading solution for $A_t^{(0)}$ has power law behaviour

$$A_t^{(0)} = \frac{C}{d - z - \theta + 1} e^{-\Gamma t + i qx} r^{d - z - \theta + 1},$$

(2.9)

and

$$A_x^{(0)} = -\frac{i \Gamma}{q} C e^{-\Gamma t + i qx} \frac{r_0^{\theta + 1 - d - z}}{\theta + 1 - d - z} \log \left( 1 - (r_0 r)^{d + z - \theta - 1} \right).$$

(2.10)

As discussed in [1], self-consistency of these equations and the series solutions holds in the regime

$$e^{\frac{2t}{q^2}} \ll \frac{1}{r_0} - \frac{r_{h}}{r_0} \ll \frac{q^2 T^2}{z} \ll 1,$$

(2.11)
for the stretched horizon \(r_h\), and the parameters \(q, \Gamma\) and \(T\) (equivalently \(r_0\)). This enables us to define Fick’s law on the stretched horizon, and thereby the diffusion equation. The shear diffusion constant then becomes

\[
\mathcal{D} = \frac{\sqrt{-g(r_h)}}{g_{xx}(r_h)\sqrt{-g\Gamma g_{rr}(r_h)}} \int_{r_c}^{r_h} \frac{-g_{tt}(r)g_{rr}(r)g_{xx}(r)}{\sqrt{-g(r)}} \, dr,
\]

where \(r_c\) is the location of the boundary, and we are evaluating \(\mathcal{D}\) at the stretched horizon. For a hyperscaling violating theory with \(d - z - \theta < -1\), we obtain

\[
\mathcal{D} = \frac{1}{r_h^{d-\theta-1}} \int_{r_c}^{r_h} r^{d-z-\theta} \, dr = \frac{r_h^{2-z}}{d-z-\theta+1} \approx \frac{r_0^{z-2}}{d-z-\theta+1} + O(q^2),
\]

where we have dropped the contribution in the integral from \(r_c\) since the UV scale \(r_c \ll r_h\) is well-separated from the horizon scale. The diffusion constant in (2.12), (2.13), is evaluated at the stretched horizon \(r_h\): however \(r_h \sim \frac{1}{r_0} + O(q^2)\) so that to leading order \(\mathcal{D}\) is evaluated at the horizon \(\frac{1}{r_0}\).

In the present hyperscaling violating case, we have seen that \(T \sim r_0^z\) and \(\mathcal{D} \sim r_0^{z-2}\) so the product \(\mathcal{D}T \sim r_0^{2(z-1)}\) is not dimensionless. Using (2.2), we have

\[
\mathcal{D} = \frac{1}{d-z-\theta+1} \left( \frac{4\pi}{d+z-\theta-1} \right)^{\frac{z-2}{z}} T^{\frac{z}{z-2}}
\]

as the scaling with temperature \(T\) of the leading diffusion constant (2.13). See also e.g. [32, 33, 36, 37, 39, 40] for previous investigations including via holography.

This motivates us to guess the universal relation

\[
\frac{\eta}{s} = \frac{(d-z-\theta+1)}{4\pi} \mathcal{D} r_0^{z-2} = \left( \frac{d-z-\theta+1}{d+z-\theta-1} \right)^{\frac{z-2}{z}} \mathcal{D}T^{\frac{z-2}{z}} = \frac{1}{4\pi}
\]

between \(\eta, s, \mathcal{D}, T\), for general exponents \(z, \theta\). As discussed in [1], this is consistent with relativistic theories (\(\theta = 0, z = 1\)) arising from \(AdS\) and with theories with exact Lifshitz scaling symmetry, \(x_i \rightarrow \lambda x_i, t \rightarrow \lambda^z t\). Then the diffusion equation \(\partial_t j^t = \mathcal{D} \partial_i^2 j^i\) shows the diffusion constant to have scaling dimension \(dim[\mathcal{D}] = z - 2\), where momentum scaling is \([\partial_i] = 1\) (or equivalently, \([x_i] = -1, [t] = -z\)). With temperature scaling as inverse time, we have \(dim[T] = z\). For hyperscaling violating theories with \(z = 1\), it can be seen that \(\mathcal{D} = \frac{1}{4\pi T}\), with the \(\theta\)-dependent prefactors cancelling precisely. Thus all hyperscaling violating theories with \(z = 1\) appear to satisfy the universal viscosity bound \(\frac{\eta}{s} = \mathcal{D}T = \frac{1}{4\pi}\).

When \(d - z - \theta = -1\), we obtain logarithmic behaviour

\[
\mathcal{A}_t^{(0)} = Ce^{-\Gamma t + qx} \log \left( \frac{T}{r_c} \right) \quad \rightarrow \quad \mathcal{D} = r_0^{\theta-d-1} \log \left( \frac{1}{r_0 r_c} \right) = r_0^{z-2} \log \left( \frac{1}{r_0 r_c} \right).
\]

This implies that in the low temperature limit \(r_0 \rightarrow 0\), the diffusion constant becomes vanishingly small if \(d_4 - \theta > 0\), or equivalently \(z > 2\). However further analysis as in [1] reveals that the near-horizon expansion is less reliable in this case, necessitating more investigation.
3 Perturbations in the absence of gauge field: Dilaton gravity

In this section, we will analyse the perturbations in hyperscaling violating Lifshitz theories focussing on 4 bulk dimensions (i.e. $d = 3$, $d_i = 2$) for simplicity and concreteness. The hyperscaling violating metric is

$$ds^2 = r^θ \left( -\frac{f(r)}{r^{2z}} dt^2 + \frac{dr^2}{f(r)r^2} + \frac{dx^2 + dy^2}{r^2} \right), \quad d_i = 2, \quad d_{\text{eff}} = 2 - \theta,$$

(3.1)

where $f(r) = 1 - (r_0r)^{2+z-\theta}$. The temperature for the dual field theory (i.e. the Hawking temperature for the black brane) is $T = \frac{2^{+z-\theta}r_0^z}{4\pi}$. We will make a gauge choice for the perturbations by setting $h_{\mu r} = 0$ (radial gauge) and assume that the perturbations to be of the form $h_{\mu\nu}(t, x, r) = e^{-i\omega t + iq \cdot x} h_{\mu\nu}(r)$ where $x$ is one of the spatial directions in the boundary theory. The shear mode $h_{xy}$ couples to $h_{ty}$ and decouples from the scalar mode $\varphi$ giving us a system of three coupled equations,

$$\partial_r (r^z\partial^{-3} \partial^r (r^{2-\theta}h_{ty})) - \frac{r^z\partial^{-3}}{f} q(\omega r^{2-\theta} h_{xy} + qr^{2-\theta} h_{ty}) = 0,$$

(3.2)

$$\partial_r (r^{-1-z+\theta} f \partial^r (r^{2-\theta}h_{xy})) + \frac{r^{z+\theta-3}}{f} \omega(\omega r^{2-\theta} h_{xy} + qr^{2-\theta} h_{ty}) = 0,$$

(3.3)

$$q \partial_r (r^{2-\theta} h_{xy}) + \frac{\omega}{f} r^{2z-2} \partial^r (r^{2-\theta} h_{ty}) = 0.$$

(3.4)

In terms of the $y$-compactified theory variables

$$g_{\mu\nu} = r^\theta G_{\mu\nu} \quad [\mu, \nu = t, x, r]; \quad \mathcal{A}_t = r^{2-\theta}h_{ty}, \quad \mathcal{A}_x = r^{2-\theta} h_{xy},$$

$$\mathcal{F}_{rt} = \partial_t (r^{2-\theta} \mathcal{A}_t), \quad \mathcal{F}_{rx} = \partial_x (r^{2-\theta} \mathcal{A}_x), \quad \mathcal{F}_{tx} = -ir^{2-\theta}(\omega h_{xy} + qh_{ty}),$$

(3.5)

the above linearized Einstein equations become

$$\sqrt{-g} e^{4\psi} g^{tt} g^{xx} \partial_x \mathcal{F}_{tx} + \partial_t (\sqrt{-g} e^{4\psi} g^{rr} g^{tt} \mathcal{F}_{tr}) = 0,$$

(3.6)

$$\sqrt{-g} e^{4\psi} g^{tt} g^{xx} \partial_t \mathcal{F}_{tx} + \partial_r (\sqrt{-g} e^{4\psi} g^{rr} g^{xx} \mathcal{F}_{rx}) = 0,$$

(3.7)

$$g^{tt} \partial_t \mathcal{F}_{tr} + g^{xx} \partial_x \mathcal{F}_{xr} = 0,$$

(3.8)

where $e^{4\psi} = \frac{1}{g_{\text{eff}}} = r^{2\theta-4}$. Other than these, we also have a Bianchi Identity

$$\partial_r \mathcal{F}_{rx} + \partial_x \mathcal{F}_{tx} - \partial_t \mathcal{F}_{tx} = 0,$$

(3.9)

which is a trivial relation in the higher dimensional theory. Equation (3.4) is a constraint equation in the higher dimensional theory which can be mapped to (3.8) in the $y$-compactified...
theory. Defining currents as $j^\nu = n_\mu F^{\mu \nu}$ ($n_\mu$ being the normal vector to the boundary $r = r_c$, with $g^{\nu \tau} n_\tau^2 = 1$) we can write them in terms of the perturbations of the higher dimensional theory,

\begin{align}
  j^x &= n_\nu F^{\nu x} = r^{6-3\theta} \sqrt{j} \partial_x (r^{2-\theta} h_{xy}) , \\
  j^t &= n_\nu F^{\nu t} = -\frac{r^{4+2\theta-3\theta}}{\sqrt{j}} \partial_r (r^{2-\theta} h_{ty}) .
\end{align}

Identifying the ratio $D \equiv -\frac{\omega}{q^2}$ we can essentially write (3.4) in the form of Fick’s Law as

\begin{equation}
  j^x = -D \partial_x j^t .
\end{equation}

The formulation of Fick’s Law in [1, 45] is done entirely in terms of field variables of the $y$-compactified theory. Differentiating (3.8) w.r.t $t$ we can eliminate $F_{rx}$ using the Bianchi Identity (3.9) to get the following equation

\begin{equation}
  \partial^2_t F_{tr} + r^{2-2\theta} f \partial_x (-\partial_x F_{tx} + \partial_r F_{tx}) = 0 .
\end{equation}

In the near horizon region approximating the thermal factor as $f(r) \approx \frac{2 + z - \theta}{r_0^{1/r_0} - \frac{1}{r_0}}$ and parametrizing the frequency as $\omega = -i\Gamma$ for some positive $\Gamma$ so that the perturbations decay in time, (3.13) can be written as

\begin{equation}
  \left(1 + (2 + z - \theta) r_0^{2z-2} \frac{q^2}{\Gamma^2} \cdot \frac{1}{r_0} - \frac{r}{r_0} \right) F_{tr} \approx -(2 + z - \theta) r_0^{2z-2} \frac{i q}{\Gamma^2} \cdot \frac{1}{r_0} - \frac{r}{r_0} \partial_r F_{tx} .
\end{equation}

Assuming

\begin{equation}
  \frac{1}{r_0} - \frac{r}{r_0} \ll \frac{\Gamma^2}{q^2 r_0^{2z-2}} ,
\end{equation}

we differentiate both sides w.r.t $x$ and approximate (3.14) further as

\begin{equation}
  \partial_x F_{tr} \approx (2 + z - \theta) \frac{q^2 r_0^{2z-2}}{\Gamma^2} \cdot \frac{1}{r_0} - \frac{r}{r_0} \partial_r F_{tx} .
\end{equation}

The assumption (3.15) implies

\begin{equation}
  \partial_x F_{tr} \ll \partial_r F_{tx} ,
\end{equation}

which in turn simplifies the Bianchi Identity (3.9) to

\begin{equation}
  \partial_t F_{rx} = \partial_x F_{rt} + \partial_r F_{tx} \sim \partial_r F_{tx} .
\end{equation}

Differentiating (3.7) w.r.t $t$ we get

\begin{equation}
  \partial_r (r^{\theta-z-1} f \partial_t F_{rx}) - \frac{r^{z+\theta-3}}{f} \partial^2_t F_{tx} = 0 .
\end{equation}
Using the approximate Bianchi identity (3.18), to substitute for \( F_{rx} \) and then multiplying throughout with \(-\frac{f}{r^2}+\theta-3\) we obtain a wave equation for the field strength \( F_{tx} \)

\[
\partial_t^2 F_{tx} - \nu^2 \left( \frac{1}{r_0} - r \right) \partial_r \left( \left( \frac{1}{r_0} - r \right) \partial_r F_{tx} \right) \approx 0 ,
\]

(3.20)

where \( \nu \) is given by

\[
\nu = (2 + z - \theta) r_0^z .
\]

(3.21)

The horizon is a one-way membrane: we incorporate this by requiring that all perturbations obey ingoing boundary conditions at the horizon. This dissipative feature is of course at the heart of the diffusion equation that results from this near-horizon perturbations analysis. Thus, imposing ingoing boundary conditions on the wave equation amounts to choosing the ingoing solution, leading to

\[
F_{tx} = f_1 \left( t + \frac{1}{\nu} \log \left( \frac{1}{r_0} - r \right) \right) ,
\]

(3.22)

where \( f_1 \) is any arbitrary smooth function. If we now ensure that the perturbations decay as \( t \to \infty \) we obtain

\[
F_{tx} + \nu \left( \frac{1}{r_0} - r \right) F_{rx} = 0 .
\]

(3.23)

As reviewed in sec. 2, the leading solutions for \( A_t, A_x \) are

\[
A_t^{(0)} = Ce^{-\Gamma t+iqx} \int_{r_c}^{r} dr' \frac{g_{tt}(r')g_{rr}(r')}{\sqrt{-g(r')}} \cdot g_{\text{eff}}^2(r') = Ce^{-\Gamma t+iqx} \int_{r_c}^{r} dr' \frac{G_{tt}(r')G_{rr}(r')}{\sqrt{-G(r')}} ,
\]

\[
A_x^{(0)} = -\frac{i\Gamma}{q} Ce^{-\Gamma t+iqx} \int_{r_c}^{r} dr' \frac{g_{xx}(r')g_{rr}(r')}{\sqrt{-g(r')}} \cdot g_{\text{eff}}^2(r') = -\frac{i\Gamma}{q} Ce^{-\Gamma t+iqx} \int_{r_c}^{r} dr' \frac{G_{rr}(r')}{\sqrt{-G(r')}} ,
\]

(3.24)

where \( r_c \sim 0 \) is the boundary where we impose the boundary conditions that the perturbations die. Above, we have used (2.5), (2.6), with \( C \) some constant: these give the solutions (2.9) and (2.10) when (2.8) holds. Then from Fick’s Law using (3.23) on the stretched horizon we can calculate the shear diffusion constant as

\[
D \equiv \left. \frac{-j^x}{\partial_{x}j^y} \right|_{\partial_x F_{rt}} = -\frac{g_{tt}}{g_{xx}} \frac{F_{rx}}{\partial_x F_{rt}} \approx -r_0^{z-1} \frac{F_{tx}}{\partial_x F_{rt}} \approx r_0^{z-1} \frac{A_t}{F_{rt}} \bigg|_{r \sim r_h} \approx \frac{r_0^{z-2}}{4-z-\theta} .
\]

(3.25)

As should be clear, a key ingredient that goes in the formulation of Fick’s Law is the relation (3.23) for the field strengths \( F_{tx} \) and \( F_{rx} \). In the context of the higher-dimensional hyperscaling violating theory where the perturbations satisfy (3.2), (3.3), (3.4), this relation can be derived exactly without any assumptions on the parameters \( q \) and \( \omega \). It turns out in this context it is a consequence of imposing a certain physical condition on the function \( H(t, r, x) \) defined as

\[
H(t, r, x) \equiv r^{\theta-z-1} f \cdot \partial_r (r^{2-\theta} h_{xy}) .
\]

(3.26)
This condition that we impose is given by
\[
(\partial_t + f \cdot r^{1-z} \partial_r) H = 0 .
\] (3.27)

Defining two new coordinates \(u\) and \(v\) as
\[
v = t + \frac{1}{\nu} \log \left( \frac{1}{r_0} - r \right),
\]
\[
u = t - \frac{1}{\nu} \log \left( \frac{1}{r_0} - r \right).
\] (3.28)

For \(r \ll \frac{1}{r_0}\) expanding the log, we see that \(v \sim t - r_0 \nu \log \left( \frac{1}{r_0} - r \right)\) so \(v\) is the ingoing coordinate (with \(r\) increasing towards the interior). We see that in the near horizon region the full wave operator is
\[
4\partial_u \partial_v \equiv \partial_t^2 - \nu^2 \left( \frac{1}{r_0} - r \right) \partial_r \left( \left( \frac{1}{r_0} - r \right) \partial_r \right),
\] (3.29)

while the linear differential operator acting on \(H\) in (3.27) is essentially \(\partial_t + f \cdot r^{1-z} \partial_r \approx \partial_t + \nu \left( \frac{1}{r_0} - r \right) \partial_r = \partial_u\). With \(v\) the ingoing coordinate, this can be thus interpreted as the ingoing condition \(\partial_u H = 0\) implying that the function has the form \(H = H(v)\).

Likewise, choosing the solution (3.22) is equivalent to requiring that the field strength \(F_{tx}\) obeys the ingoing condition
\[
\partial_t F_{tx} + \nu \left( \frac{1}{r_0} - r \right) \partial_r F_{tx} = 0 ,
\] (3.30)

which can also be written as \(\partial_u F_{tx} = 0\), giving \(F_{tx} = F_{tx}(v)\). Using (3.5) we can write
\[
\partial_t F_{tx} = -r^{2-\theta} \omega (\omega h_{xy} + q h_{ty}) ,
\]
\[
F_{rx} = \partial_r (r^{2-\theta} h_{xy}) = \frac{r^{z+1-\theta}}{f} H .
\] (3.31)

The above equalities in conjunction with (3.3) gives
\[
\partial_r H = -\frac{r^{z-1}}{f} \omega (\omega h_{xy} + q h_{ty}) = \frac{r^{z+\theta-3}}{f} \partial_r F_{tx} .
\] (3.32)

Also (3.27) naturally implies
\[
\partial_r H = -\frac{r^{z-1}}{f} \partial_t H = -r^{\theta-2} \partial_t F_{rx} .
\] (3.33)

Equating the above two expressions for \(\partial_r H\), we recover the relation (3.23) as was obtained in [1]. It should be noted that the relation between \(F_{rx}\) and \(F_{tx}\) was obtained in the \(y\)-compactified theory by making certain self-consistent approximations involving the parameters \(q\) and \(\omega\) which is quite distinct from the derivation demonstrated here, using (3.2), (3.3), (3.4), directly.
The leading order value of the diffusion constant $D$ is given by $A_t^{(0)}$, which is obtained by solving the $q = 0$ and $\omega = 0$ sector of (3.2)

$$\partial_r (r^{z+\theta-3}\partial_r (r^{2-\theta}h_{ty})) = 0 \ .$$
(3.34)

The solution to the above equation is given by

$$h_{ty}(r) = c_1 r^{\theta-2} + c_2 r^{2-z} \ ,$$
(3.35)

where $c_1$ and $c_2$ are arbitrary constants. For $z < 4 - \theta$, the $\theta - 2$ fall-off (non-normalizable mode) dominates over the $2 - z$ fall-off (normalizable mode) near the boundary $r \sim r_c$, while for $z > 4 - \theta$ we see the exact opposite behaviour. When $z = 4 - \theta$ there is a degeneracy in the two fall-offs and we have a new independent solution which scales logarithmically with $r$,

$$h_{ty}(r) = c_1 r^{\theta-2} + c_2 r^{\theta-2} \log \frac{r}{r_c} \ .$$
(3.36)

The swapping of roles between the normalizable and non-normalizable modes around the point $z = 4 - \theta$ gives some insight into the unusual logarithmic scaling for the diffusion constant when $z = 4 - \theta$. It is in fact reminiscent of the alternative quantization of field modes [52] and thus holographically it is not surprising that the relevant correlation function exhibits logarithmic behaviour.

In the presence of a background gauge field, the analysis changes significantly. The analog of (3.4) including the gauge field is given by

$$q\partial_r (r^{2-\theta}h_{xy}) + \frac{\omega}{f} r^{2z-2}\partial_r (r^{2-\theta}h_{ty}) - k \frac{\omega}{f} r^{z-\theta+1} a_y = 0 \ .$$
(3.37)

Due to the presence of the gauge field perturbation $a_y$, we cannot map the above equation to Fick’s Law by defining horizon currents as before. Subsequently we will show that a field redefinition which involves a non-trivial combination of $h_{ty}$ and $\int a_y \, dr$ gives us an equation which is similar in structure to Fick’s Law in the dimensionally reduced theory.

## 4 Perturbations to hyperscaling violating spacetime

We are considering nonrelativistic holographic backgrounds described by a $(d+1)$-dimn hyperscaling violating metric at finite temperature as given in (2.1). The metric (2.1) is a solution to the action

$$S = -\frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{-G} \left[ R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{Z(\phi)}{4} F_{\mu\nu} F^{\mu\nu} + V(\phi) \right] \ ,$$
(4.1)
where $\phi$ is the dilaton with a potential $V(\phi) = -2\Lambda e^{-\phi}$, where

$$
\delta = \frac{2\theta/d_i}{\sqrt{2(d_i - \theta)(z - \theta/d_i - 1)}} \quad \text{and} \quad \Lambda = -\frac{1}{2}(d_i + z - \theta)(d_i + z - \theta - 1). \quad (4.2)
$$

The background gauge field is given by

$$
A_t = \frac{\alpha f(r)}{r^{d_i + z - \theta}}, \quad \alpha = -\sqrt{\frac{2(z - 1)}{d_i + z - \theta}} \quad (4.3)
$$

and with gauge field coupling being

$$
Z(\phi) = e^{\lambda \phi} = r^{2\theta/d_i + 2d_i - 2\theta}, \quad \text{where} \quad \lambda = \frac{2\theta/d_i + 2d_i - 2\theta}{\sqrt{2(d_i - \theta)(z - \theta/d_i - 1)}}. \quad (4.4)
$$

Since only the $A_t(r)$ component is non-zero, we have only one non-zero field strength

$$
F_{rt} = -\alpha(d_i + z - \theta) \frac{1}{r^{d_i + z - \theta + 1}}. \quad (4.5)
$$

Varying the action (4.1) with respect to the bulk metric $G_{\mu\nu}$, the gauge field $A_\mu$ and $\phi$ we get the following equations of motion

$$
R_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - G_{\mu\nu} \frac{V(\phi)}{d - 1} + \frac{Z(\phi)}{2} G^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{Z(\phi)}{4(d - 1)} G_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \quad (4.6)
$$

$$
\nabla_\mu (Z(\phi) F^{\mu\nu}) = 0, \quad (4.7)
$$

$$
\frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} G^{\mu\nu} \partial_\nu \phi) + \frac{\partial V(\phi)}{\partial \phi} - \frac{1}{4} \frac{\partial Z(\phi)}{\partial \phi} F_{\rho\sigma} F^{\rho\sigma} = 0. \quad (4.8)
$$

We turn on generalized gravitational, gauge field and scalar field perturbations $h_{\mu\nu}(\bar{x}, r)$, $a_\mu(\bar{x}, r)$ and $\varphi(\bar{x}, r)$ where $\bar{x}$ denotes all the boundary coordinates collectively. Later, we will make a certain gauge choice (radial gauge) for the perturbations in order to simplify our calculations. At the linearized level, the Einstein’s equations (4.6) are given by

$$
R_{(1)}^{(1)}_{\mu\nu} \equiv \frac{1}{2} \partial_\mu \phi \partial_\nu \varphi + \frac{1}{2} \partial_\mu \varphi \partial_\nu \phi - \frac{V}{2} (h_{\mu\nu} - G_{\mu\nu} \delta \varphi) \quad (4.9)
$$

$$
+ \frac{Z}{2} \left[ G^{\rho\sigma} F_{\mu\rho} f_{\nu\sigma} + G^{\rho\sigma} f_{\nu\mu} F_{\rho\sigma} - h^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \lambda \varphi G^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right] \quad (4.9)
$$

$$
- Z \left[ \frac{1}{4} G_{\mu\nu} (F_{\rho\sigma} f^{\rho\sigma} - g^{\rho\sigma} h^{\rho\beta} F_{\rho\sigma} F_{\alpha\beta}) + \frac{1}{8} h_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \frac{1}{8} \lambda \varphi G_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right],
$$

where

$$
R_{(1)}^{(1)}_{\mu\nu} = \frac{1}{2} \left| \nabla_{\alpha} \nabla_{\nu} h_{\mu}^\alpha + \nabla_{\alpha} \nabla_{\mu} h_{\nu}^\alpha - \nabla_{\alpha} \nabla^\alpha h_{\mu\nu} - \nabla_{\nu} \nabla_{\mu} h \right|; \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu; \quad h = G^{\mu\nu} h_{\mu\nu}. \quad (4.10)
$$
Similarly, the Maxwell’s Equations (4.7) up to linearized order gives the following equations of motion

\[ \nabla_\mu (Z f^{\mu \nu}) - \nabla_\mu (Z h^{\mu \rho} F_\rho^{\quad \nu}) - Z (\nabla_\mu h^{\nu \sigma}) F_\sigma^{\mu} + \frac{1}{2}(\nabla_\mu h) Z F^{\mu \nu} + \lambda Z F^{\mu \nu} \partial_\mu \phi = 0. \]  

(4.11)

Finally, the linearized scalar field equation is:

\[ \frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} G^{\mu \nu} \partial_\nu \phi) - \frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} h^{\mu \nu} \partial_\nu \phi) + \frac{1}{2} G^{\mu \nu} \partial_\nu \phi \partial_\mu h + V \delta^2 \phi \]

\[ - \frac{\lambda Z}{4} (2F_{\mu \nu} f^{\mu \nu} - 2G^{\mu \rho} h^{\nu \sigma} F_{\mu \nu} F_{\rho \sigma} + \lambda \phi F_{\mu \nu} F^{\mu \nu}) = 0. \]  

(4.12)

In the linearized equations of motion (4.9), (4.11) and (4.12), all indices are raised with respect to the background metric (2.1). For the sake of simplicity our subsequent analysis will be for \( d = 3 \) (i.e. \( d_i = 2 \)) but we expect this procedure can be generalized for higher dimensions.

### 4.1 Perturbations to hyperscaling violating spacetime: Einstein-Maxwell-dilaton (EMD) theory in 4 dimensions \((d = 3)\)

In the presence of a background gauge field, the perturbations in the metric sector \( h_{xy} \) and \( h_{ty} \) couples to perturbation to the background gauge field \( a_y \). For the sake of completeness, we have also listed the equations of motion for the other perturbations in [A]. In the radial gauge (i.e. \( h_{\mu \tau} = 0 \)) assuming perturbations of the form \( h_{\mu \nu} = e^{-\omega t + i q x} h_{\mu \nu}(r) \), the coupled set of equations governing \( h_{ty} \), \( h_{xy} \) and \( a_y \) become

\[ \partial_r (r^{5-z-\theta} f \partial_r a_y) + \frac{\omega^2}{f} r^{3+z-\theta} a_y - q r^{5-z-\theta} a_y - k \partial_r (r^{2-\theta} h_{ty}) = 0, \]  

(4.13)

\[ \partial_r (r^{z+\theta-3} \partial_r (r^{2-\theta} h_{ty})) - \frac{r^{z+\theta-3}}{f} q (\omega r^{2-\theta} h_{xy} + q r^{2-\theta} h_{ty}) - k \partial_r a_y = 0, \]  

(4.14)

\[ \partial_r (r^{-1+z+\theta} f \partial_r (r^{2-\theta} h_{xy})) + \frac{r^{z+\theta-3}}{f} \omega (\omega r^{2-\theta} h_{xy} + q r^{2-\theta} h_{ty}) = 0, \]  

(4.15)

\[ q \partial_r (r^{2-\theta} h_{xy}) + \frac{\omega}{f} r^{2z-2} \partial_r (r^{2-\theta} h_{ty}) - k \frac{\omega}{f} r^{2z+1} a_y = 0, \]  

(4.16)

where

\[ k = (2 + z - \theta) \alpha, \quad \alpha = -\sqrt{\frac{2(z - 1)}{2 + z - \theta}}. \]  

(4.17)

Note that the last equation (4.16) is a constraint equation in \( r \) which we will eventually use to map to Fick’s Law. Now we will further assume that the solutions to the perturbations
\( h_{xy}, h_{ty} \) and \( a_y \) can be expanded as a series in \( \frac{t^2}{T^2/z} \) which we schematically write as

\[
\begin{pmatrix}
  h_{ty}(t, x, r) \\
  h_{xy}(t, x, r) \\
  a_y(t, x, r)
\end{pmatrix}
\equiv
\begin{pmatrix}
  (h_{ty}^{(0)})(t, x, r) \\
  (h_{xy}^{(0)})(t, x, r) \\
  (a_y^{(0)})(t, x, r)
\end{pmatrix}
+ \cdots,
\begin{pmatrix}
  (h_{ty}^{(1)})(t, x, r) \\
  (h_{xy}^{(1)})(t, x, r) \\
  (a_y^{(1)})(t, x, r)
\end{pmatrix} = O \left( \frac{q^2}{T^2/z} \right).
\]

(4.18)

Subsequently we will show that this formalism is indeed consistent with the proposed series ansatz. Compactifying along \( y \), we can write the lower dimensional field variables in terms of the fields in the 4 dimensional hyperscaling violating theory as

\[
\mathcal{A}_t = r^{2-\theta} h_{ty}, \quad \mathcal{A}_x = r^{2-\theta} h_{xy}, \quad \chi = a_y, \quad g_{\mu\nu} = r^{\theta-2} G_{\mu\nu},
\]

(4.19)

and

\[
F_{\mu\nu} = \partial_{\mu} \mathcal{A}_{\nu} - \partial_{\nu} \mathcal{A}_{\mu}, \quad Z = r^{4-\theta}, \quad e^{2\psi} = G_{yy} = r^{\theta-2}.
\]

(4.20)

In terms of the fields defined above, (4.14), (4.15) and (4.16) take the form

\[
\sqrt{-g} e^{4\psi} g^{tt} g^{xx} \partial_x F_{tx} + \partial_r(\sqrt{-g} e^{4\psi} g^{rr} g^{tt} F_{tr}) = \sqrt{-g} e^{2\psi} Z F^{rt} \partial_r \chi, \quad (4.21)
\]

\[
\sqrt{-g} e^{4\psi} g^{tt} g^{xx} \partial_x F_{tx} + \partial_r(\sqrt{-g} e^{4\psi} g^{rr} g^{xx} F_{rx}) = 0, \quad (4.22)
\]

\[
g^{tt} \partial_t F_{tr} + g^{xx} \partial_x F_{xr} = e^{-2\psi} g_{rr} Z F^{rt} \partial_r \chi. \quad (4.23)
\]

The perturbation to the background gauge field \( a_y \) becomes an effective scalar field \( \chi \) in the lower dimensional theory whose equation of motion is given by

\[
- \frac{r^{3+z-\theta}}{f} \partial_t^2 \chi + r^{5-z-\theta} \partial_x^2 \chi + \partial_r(r^{5-z-\theta} f \partial_r \chi) - k F_{rt} = 0. \quad (4.24)
\]

The equations (4.21), (4.22), (4.23) and (4.24) can be derived by compactifying (4.11) along \( y \) and varying the effective lower dimensional action with respect \( \mathcal{A}_\mu \) and \( \chi \) as detailed in C. The field strengths also satisfy the Bianchi identity

\[
\partial_r F_{xx} + \partial_x F_{tr} - \partial_t F_{tx} = 0, \quad (4.25)
\]

which is a trivial relation in the higher dimensional theory.

Like the earlier case of dilaton gravity with no gauge field (sec. 3), we could define the horizon currents as \( j^\nu = n_\mu F^{\mu\nu} \), with the explicit expression for these currents in terms of the higher dimensional theory as in (3.10), (3.11). However, unlike the earlier case where (3.4) was mapped to Fick’s Law in the \( y \)-compactified theory, we do not observe such a structure for (4.13). In the presence of a background gauge field, the behaviour of the perturbations \( h_{ty} \) and \( a_y \) is expected to be different than before (sec. 3) since even in the \( q = \Gamma = 0 \) sector, they are coupled. The equations governing them follows from (4.13) and (4.14),

\[
\partial_r(r^{5-z-\theta} f \partial_r a_y) - k \partial_r(r^{2-\theta} h_{ty}) = 0, \quad \partial_r(r^{5+\theta-3} \partial_r(r^{2-\theta} h_{ty})) - k \partial_r a_y = 0. \quad (4.26)
\]
From the expression for the diffusion constant for dilaton gravity (3.25) one might expect that even in this case, the expression for the diffusion constant will require the detailed solutions to (4.26). This is a system of two second-order coupled differential equations: eliminating \(a_y\) gives a 3rd order differential equation for \(h_{ty}\), and likewise eliminating \(h_{ty}\) leads to a 3rd order equation for \(a_y\). Thus we have 3 independent solutions for each of the functions \(h_{ty}\) and \(a_y\). These solutions can be found explicitly but we relegate discussing them in detail to Appendix B since it turns out interestingly that the diffusion analysis that follows does not depend in detail on them.

In this regard, it is important to note that the hyperscaling violating Lifshitz black branes here are not charged: the gauge field and scalar here simply serve as sources that support the nonrelativistic metric as a solution to the gravity theory. Using intuition from the fluid-gravity correspondence [51], the fact that these are uncharged black branes means that the near-horizon perturbations must effectively be characterized simply by local temperature and velocity fluctuations. Charge cannot enter as an extra variable characterizing the near-horizon region. Thus the structure of the diffusion equation and the diffusion constant should not be dramatically altered by the presence of the gauge field, although the gauge field perturbation \(a_y\) is not “subleading” to the \(h_{ty}\) perturbation in any sense, from (4.26), and also the linearized Einstein equations (4.13)-(4.16).

Armed with this intuition, looking closer, we see that we can rearrange (4.16) to write

\[
q \partial_r (r^{2-\theta} h_{xy}) + \frac{\omega}{f} r^{2z-2} \partial_r \left( r^{2-\theta} h_{ty} - k \int_{r_c}^{r} ds \ s^{3-z-\theta} a_y \right) = 0 .
\]

(4.27)

This is structurally similar to (3.4) in terms of a new field variable

\[
r^{2-\theta} h_{ty} = r^{2-\theta} \tilde{h}_{ty} - k \int_{r_c}^{r} ds \ s^{3-z-\theta} a_y .
\]

(4.28)

At the boundary \(r = r_c \sim 0\), we impose the boundary conditions that these perturbations vanish, as done previously. This in turn motivates a redefinition to new field variables in the \(y\)-compactified theory as

\[
\tilde{A}_t = A_t - k \int_{r_c}^{r} ds \ s^{3-z-\theta} \chi ,
\]

\[
\tilde{A}_x = A_x .
\]

(4.29)

For the new gauge field variables \(\tilde{A}_t\) and \(\tilde{A}_x\), we define the field strengths \(\tilde{F}_{rt}\) and \(\tilde{F}_{tx}\) as (in radial gauge \(\tilde{A}_r = A_r = 0\))

\[
\tilde{F}_{rt} = F_{rt} - k r^{3-z-\theta} \chi , \quad \tilde{F}_{tx} = \partial_t A_x - \partial_x \tilde{A}_t , \quad \tilde{F}_{rx} = F_{rx} = \partial_r A_x .
\]

(4.30)
In terms of the newly defined field strengths, the Maxwell’s Equations (4.21)-(4.23), Bianchi identity (4.25) and the equation of motion for $\chi$ (4.24) become

$$\partial_t (r^{z+\theta-3} \tilde{F}_{rt}) - \frac{r^{z+\theta-3}}{f} \partial_z \left( \tilde{F}_{tx} - k \int_{r_0}^{r} ds \ s^{3-z-\theta} \partial_s \chi \right) = 0, \quad (4.31)$$

$$\partial_t (r^{-1-z-\theta} f F_{rx}) - \frac{r^{z+\theta-3}}{f} \partial_t \left( \tilde{F}_{tx} - k \int_{r_0}^{r} ds \ s^{3-z-\theta} \partial_s \chi \right) = 0, \quad (4.32)$$

$$\partial_t \tilde{F}_{rt} - r^{2-2z} f \partial_x F_{rx} = 0, \quad (4.33)$$

$$\partial_t F_{rx} + \partial_x \tilde{F}_{tx} - \partial_t \tilde{F}_{tx} = 0, \quad (4.34)$$

$$\partial_t (r^{5-z-\theta} f \partial_x \chi) - k^2 r^{3-z-\theta} \chi + \frac{r^{3+z-\theta}}{f} \partial_t^2 \chi + r^{5-z-\theta} \partial_t^2 \chi - k \tilde{F}_{rt} = 0. \quad (4.35)$$

Differentiating (4.33) w.r.t. $t$ we can eliminate $F_{rx}$ using the Bianchi Identity (4.34) to get the following equation

$$\partial_t^2 \tilde{F}_{tr} + r^{2-2z} f \partial_x (-\partial_x \tilde{F}_{tx} + \partial_t \tilde{F}_{tx}) = 0. \quad (4.36)$$

In the near horizon region approximating the thermal factor as $f(r) \approx (2 + z - \theta) \left( \frac{r}{r_0} \right)^{\frac{1}{r_0}} - r$ and parametrizing the frequency as $\omega = -i \Gamma$ for some positive $\Gamma$ so that the perturbations decay in time, (4.36) can be written as

$$\left( 1 + (2 + z - \theta) r_0^{2z-2} \frac{q^2}{\Gamma^2} \cdot \frac{1}{r_0} - r \right) \tilde{F}_{tr} \approx -(2 + z - \theta) r_0^{2z-2} \frac{iq}{\Gamma^2} \cdot \frac{1}{r_0} - r \partial_t \tilde{F}_{tx}. \quad (4.37)$$

Assuming the bound (3.15), we differentiate both sides w.r.t. $x$ and approximate (4.37) further

$$\partial_x \tilde{F}_{tr} \approx (2 + z - \theta) \frac{q^2}{\Gamma^2} r_0^{2z-2} \cdot \frac{1}{r_0} - r \partial_x \tilde{F}_{tx} \equiv \epsilon (2 + z - \theta) \partial_x \tilde{F}_{tx}, \quad (4.38)$$

where

$$\epsilon = \frac{q^2}{\Gamma^2} r_0^{2z-2} \cdot \frac{1}{r_0} - r \ll 1, \quad (4.39)$$

which is essentially implied by (3.15). In other words, we have

$$\partial_x \tilde{F}_{tr} \ll \partial_x \tilde{F}_{tx}, \quad (4.40)$$

which in turn simplifies the Bianchi Identity to

$$\partial_t F_{rx} = \partial_x \tilde{F}_{rt} + \partial_r \tilde{F}_{tx} \sim \partial_r \tilde{F}_{tx}. \quad (4.41)$$

Differentiating (4.32) w.r.t $t$ we get

$$\partial_t (r^{\theta-1} f \partial_x F_{rx}) - \frac{r^{z+\theta-3}}{f} \partial_t^2 \tilde{F}_{tx} + k \frac{r^{z+\theta-3}}{f} \int_{r_0}^{r} s^{3-z-\theta} \partial_t^2 \partial_x \chi(s) \ ds = 0. \quad (4.42)$$
Using the approximate Bianchi identity (4.41), to substitute for \( F_{rx} \) and then multiplying throughout with \(-\frac{f}{r^2} \) we get a sourced wave equation for the field strength \( F_{tx} \)

\[
\partial_t^2 \tilde{F}_{tx} - \nu^2 \left( \frac{1}{r_0} - r \right) \partial_r \left( \left( \frac{1}{r_0} - r \right) \partial_r \tilde{F}_{tx} \right) \approx k \int_{r_c}^r s^{3-z-\theta} \cdot \partial_t^2 \partial_x \chi(s) ds ,
\]

(4.43)

with \( \nu \) in (3.21).

Likewise for the scalar equation of motion (4.35), using the approximation (3.15) we can drop the term involving \( \partial_t^2 \chi \) compared to the other terms: thus in the near horizon regime we obtain

\[
\partial_t^2 \chi - \nu^2 \left( \frac{1}{r_0} - r \right) \partial_r \left( \left( \frac{1}{r_0} - r \right) \partial_r \chi \right) + \frac{\nu^2 k^2 r_0}{2 + z - \theta} \left( \frac{1}{r_0} - r \right) \chi = -\nu k r_0^{1-\theta} \left( \frac{1}{r_0} - r \right) \tilde{F}_{rt} .
\]

(4.44)

that the first term in (4.44) is sub-dominant than the third term by a factor of \( \frac{r^2}{r_0^2} \ll 1 \).

Thus the leading order behaviour for the scalar field \( \chi \) can simply be estimated as

\[
\chi^{(0)} \approx -\frac{2 + z - \theta \cdot r_0^{3-\theta} \tilde{F}_{rt}^{(0)} }{\nu k} ,
\]

(4.45)

where the superscript (0) is the leading order behaviour of the field \( \chi \) at \( q = \Gamma = 0 \) since we have explicitly dropped the subleading derivative terms. Now, in the near horizon regime, we can use (4.38) to find \( \partial_x \chi^{(0)} \sim \partial_x \tilde{F}_{tx}^{(0)} \sim \epsilon \partial_r \tilde{F}_{tx}^{(0)} \). Using this, we can estimate the right hand side of (4.43) as

\[
k \int_{r_c}^r s^{3-z-\theta} \cdot \partial_t^2 \partial_x \chi(s) ds \approx k \int_{r_c}^r r_0^{z+\theta-3} \partial_t^2 \partial_x \chi \sim \partial^2 \int_{r_c}^r \epsilon \partial_x \tilde{F}_{tx} ds \sim \epsilon \cdot \partial^2 \tilde{F}_{tx} .
\]

(4.46)

What this means is that while the gauge field perturbation \( a_y \) (or \( \chi \)) is not subleading to \( h_{ty} \) (or \( A_t \)), once we incorporate its effects in terms of the variable \( \tilde{h}_{ty} \) (or \( \tilde{A}_t \)) the remaining contributions are in fact subleading, as we see here in (4.46).

The above estimate implies that up to leading order, (4.43) is in fact a source free wave equation whose ingoing solution is

\[
\tilde{F}_{tx} = f_1 \left( t + \frac{1}{\nu} \log \left( \frac{1}{r_0} - r \right) \right) ,
\]

(4.47)

which further implies

\[
\partial_t \tilde{F}_{tx} + \nu \left( \frac{1}{r_0} - r \right) \partial_r \tilde{F}_{tx} = 0 .
\]

(4.48)

Using (4.41), we can write the above expression as a perfect derivative in \( t \), i.e.

\[
\partial_t \left( \tilde{F}_{tx} + \nu \left( \frac{1}{r_0} - r \right) \tilde{F}_{rx} \right) = 0 .
\]

(4.49)
Imposing the boundary condition that the solutions decay as \( t \to \infty \) we end up with the following relation

\[
\tilde{F}_{tx} + \nu \left( \frac{1}{r_0} - r \right) F_{rx} = 0 .
\]  

(4.50)

We can derive this result alternatively arguing as follows, looking for an ingoing condition as in (3.27). In this case we have identified \( \tilde{A}_t \) as the relevant perturbative mode. We can write the newly defined field strengths in terms of \( h_{ty}, h_{xy} \) and \( a_y \) along the lines of (3.31) as

\[
\partial_t \tilde{F}_{tx} = -r^{2-\theta} \omega (\omega h_{xy} + \tilde{q} h_{ty}) , \quad F_{rx} = \partial_r (r^{2-\theta} h_{xy}) = \frac{r^{z+1-\theta}}{f} H .
\]  

(4.51)

From (4.32), we have

\[
\partial_r H = \frac{r^{z+\theta-3}}{f} \partial_t (\tilde{F}_{tx} - k \int_{r_c}^r ds \ s^{3-z-\theta} \partial_x a_y) .
\]  

(4.52)

From (4.46) cancelling the \( \partial^2_t \equiv \Gamma^2 \) factor throughout, it follows that

\[
k \int_{r_c}^r ds \ s^{3-z-\theta} \partial_x a_y \sim \epsilon \tilde{F}_{tx} .
\]  

(4.53)

Substituting this equation in (4.52), we get

\[
\partial_r H \approx \frac{r^{z+\theta-3}}{f} \partial_t \tilde{F}_{tx} .
\]  

(4.54)

We expect on physical grounds that the ingoing condition on \( H \) defined in terms of \( h_{xy} \) is still the same as (3.27) in the case without the gauge field since this shear mode is expected to be ingoing: this gives

\[
\partial_t H = -r^{\theta-2} \partial_t F_{rx} .
\]  

(4.55)

(In the above equations, we have used the \( y \)-compactified variables and higher dimensional ones in the same equations, with the understanding that they are interchangeable from the context.) Equating the two expressions for \( \partial_r H \) above we recover (4.50), which is analogous to (3.23) in the case without the gauge field. This vindicates our intuition on using the \( \tilde{A}_t, \tilde{A}_x \) field variables to obtain the diffusion equation here with the gauge field.

Along the lines of (3.10), (3.11), we define the currents in the new tilde variables as

\[
j^x = n_r F^{xr} = \frac{F_{rx}}{g_{xx} \sqrt{g_{rr}}}, \quad \tilde{j} = n_r \tilde{F}^{tr} = \frac{\tilde{F}_{tr}}{g_{tt} \sqrt{g_{rr}}} ,
\]  

(4.56)

since as we have seen, these \( \tilde{A}_\mu \) variables play the role here of the earlier variables \( A_\mu \) (it would be interesting to find appropriate modifications of the prescriptions in [53] here). At this point we make another assumption, namely

\[
|\partial_t A_x| \ll |\partial_x \tilde{A}_t| ,
\]  

(4.57)
which is very similar to (2.7) but for the $\tilde{A}_t, \tilde{A}_x$ variables (4.29). This implies

$$\tilde{F}_{tx} \approx -\partial_x \tilde{A}_t .$$  (4.58)

We can now formulate Fick’s Law i.e. $j^x = -D \partial_x \tilde{j}^t$ on the stretched horizon and calculate the diffusion constant as

$$D \equiv - \frac{j^x}{\partial_x j^t} = - \frac{g_{tt}}{g_{xx}} \frac{\tilde{F}_{rt}}{\tilde{F}_{rt}} \approx - r_0^{z-1} \frac{\tilde{F}_{tx}}{\partial_x \tilde{F}_{rt}} ,$$  (4.59)

where we have used (4.50) to write the third equality. Using (4.58), the diffusion constant at leading order is given by

$$D = r_0^{z-1} \frac{\tilde{A}_t}{\tilde{F}_{rt}} \bigg|_{r \sim r_h} ,$$  (4.60)

where $r_h$ is the location of the stretched horizon, and the prefactor arises from the metric factors as in (3.25).

4.1.1 Shear diffusion constant: $z < 4 - \theta$

Making an ansatz of the form (4.18) naturally implies such a series expansion ansatz for the fields $\tilde{A}_t, \tilde{A}_x$ and $\chi$ in the $y$-compactified theory.

$$\begin{pmatrix} \tilde{A}_t(t, x, r) \\ \tilde{A}_x(t, x, r) \\ \chi(t, x, r) \end{pmatrix} = \begin{pmatrix} \tilde{A}_t^{(0)}(t, x, r) \\ \tilde{A}_x^{(0)}(t, x, r) \\ \chi^{(0)}(t, x, r) \end{pmatrix} + \begin{pmatrix} \tilde{A}_t^{(1)}(t, x, r) \\ \tilde{A}_x^{(1)}(t, x, r) \\ \chi^{(1)}(t, x, r) \end{pmatrix} + \cdots ,$$  (4.61)

The $q = \Gamma = 0$ sector of (4.31) which is

$$\partial_t (r^{z+\theta-3} \partial_r \tilde{A}_t) = 0 ,$$  (4.62)

gives us an expression for the leading solution of $\tilde{A}_t$

$$\tilde{A}_t^{(0)}(t, x, r) = C e^{-\Gamma t + i q x} \int_{r_c}^r dr \, r^{3-z-\theta} ,$$  (4.63)

where $C$ is an arbitrary constant. When $4 - z - \theta > 0$ the leading solution $\tilde{A}_t^{(0)}$ has a power-law behaviour

$$\tilde{A}_t^{(0)}(t, x, r) = e^{-\Gamma t + i q x} \frac{C}{(4 - z - \theta)} r^{4-z-\theta} .$$  (4.64)

It is expected that close to the boundary i.e. near $r \approx r_c$ the hyperscaling violating phase breaks down and we require $r_0 r_c \ll 1$. The analogous statement for the boundary field
theory will be to assume that the temperature is sufficiently below the UV cut-off. Thus, the condition $z < 4 - \theta$ arises from the boundary condition that $\tilde{A}_t^{(0)} \to 0$ as $r \to 0$.

Substituting $\tilde{A}_t^{(0)}$ in (4.24), the particular solution to the inhomogeneous equation (at $q = 0$, $\omega = 0$) for $\chi$ is

$$\chi^{(0)} = -\frac{C}{k}.$$  \hfill (4.65)

Substituting $\chi^{(0)} = a_y^{(0)} = -C/k$ in (4.28), and considering only the leading order terms we get

$$\tilde{h}_{ty}^{(0)} = h_{ty}^{(0)} + \frac{C}{(4 - z - \theta)} r^{2-z}.$$ \hfill (4.66)

Thus, we see that although $h_{ty} = r^{2-z}$ does not satisfy the linearized equations (4.13)-(4.16) at $q = 0$, $\omega = 0$, the $r^{2-z}$ fall-off appears in the expression for $\tilde{h}_{ty}$ which is indeed the relevant perturbative mode that should be considered. We see that $\tilde{h}_{ty} = \frac{C}{4 - z - \theta} r^{2-z}$ and $a_y = -\frac{C}{k}$ indeed satisfy the linearized equations (4.31)-(4.35) at $q = 0$, $\omega = 0$. Note that this implies that the solutions of interest here in the original variables are $h_{ty}^{(0)} = 0$ and $a_y^{(0)} = -\frac{C}{k}$, as can be seen from the form of $\tilde{h}_{ty}$. Thus the solutions of relevance arise entirely from the leading solution to the gauge field perturbation. It is important to note that the solution $a_y^{(0)} = const$ does not change the asymptotic boundary conditions on the background being hyperscaling violating Lifshitz.

The leading solution for $A_x$ i.e. $A_x^{(0)}$ can be determined by plugging in the series ansatz for $A_x$ and $\tilde{A}_t$ in (4.32). The leading order equation is given by

$$\partial_r A_x^{(0)} = \frac{i \Gamma q^{2-z}}{2} \partial_r \tilde{A}_t^{(0)}.$$ \hfill (4.67)

Integrating the above and using (4.64) we obtain an expression for $A_x^{(0)}$ as

$$A_x^{(0)} = \frac{i \Gamma}{q} C e^{-\Gamma t + i q x} \frac{r^{2-z}}{2 + z - \theta} \log (1 - (r_0 r)^{2-z - \theta}).$$ \hfill (4.68)

From (4.63) and the solution derived above, we see that the assumption (4.57) is essentially

$$\frac{\Gamma^2}{q^2} \log \left( \frac{1}{r_0} \right) \approx 1.$$ \hfill (4.69)

Using $\frac{\Gamma}{q} \sim \frac{2}{r_0^{2-z}}$ and noting that the temperature $T \sim r_0^z$, we can recast this condition as

$$\frac{q^2}{T^{2/z}} \log \left( \frac{1}{r_0} \right) \approx 1.$$ \hfill (4.70)

Physically the above assumption means that we cannot push the stretched horizon located at $r_h$ exponentially close to the horizon $\frac{1}{r_0}$ as before, in (2.11).
Using (4.64), we can now evaluate the shear diffusion constant on the stretched horizon for the hyperscaling violating theory with $4 - z - \theta > 0$ as
\[
D = r_0^{z-1} \cdot \frac{1}{(4 - z - \theta) r_h} \approx \frac{r_0^{z-2}}{4 - z - \theta} + O(q^2) .
\] (4.71)

The solution for $\tilde{A}_t^{(0)}$ is evaluated at the stretched horizon $r_h$: however $r_h \sim \frac{1}{r_0} + O(q^2)$ so to leading order $D$ is evaluated at the horizon $\frac{1}{r_0}$. It is interesting that the effect of the hyperscaling violating exponent $\theta$ cancels in the final expression for $D$ which is essentially the ratio of $\tilde{A}_t$ to a field strength $\tilde{F}_{rt}$ both of which has non-trivial $\theta$-dependence.

Using the expression (2.2) we can express the diffusion constant in terms of the temperature as
\[
D = \frac{1}{4 - z - \theta} \left( \frac{4\pi}{2 + z - \theta} \right)^{\frac{z-2}{z-2}} T^{\frac{z-2}{z-2}}
\] (4.72)
which is identical to the one obtained in [1] for the case without the gauge field, for $d_t = 2$ spatial dimensions. As discussed there, for pure $AdS$ when $z = 1, \theta = 0$, we recover the standard relation $D = \frac{1}{4\pi T}$ which further implies $\frac{\eta}{s} = \frac{1}{4\pi}$. Likewise for all theories with $z = 1$, it can be seen that $\theta$ cancels from the prefactors in $D$ which becomes $D = \frac{1}{4\pi T}$. This is in accord with the known behaviour [45] of e.g. nonconformal $Dp$-branes whose dimensional reduction on the transverse sphere $S^{8-p}$ gives rise to hyperscaling violating theories with $z = 1, \theta \neq 0$ [14]: it would seem reasonable to expect that the sphere should not affect long-wavelength diffusive properties.

### 4.1.2 Shear diffusion constant: $z = 4 - \theta$

Now, we focus on the family of hyperscaling violating solution where $z = 4 - \theta$. In this case, from (4.63) it follows that the leading solution of $\tilde{A}_t$ has logarithmic behaviour
\[
\tilde{A}_t^{(0)} = Ce^{-r + iqx} \log \frac{r}{r_c} , \quad z = 4 - \theta .
\] (4.73)

Working further, we can evaluate the diffusion constant up to leading order from (4.60) as
\[
D = r_0^{z-2} \log \frac{1}{r_0 r_c} .
\] (4.74)

This implies that in the low temperature limit as $r_0 \to 0$, the diffusion constant vanishes if $z > 2$. The new condition on the exponents $z$ and $\theta$, namely $z < 4 - \theta$ appears to be a new constraint which is separate from the null energy conditions
\[
(2 - \theta)(2(z - 1) - \theta) \geq 0 , \quad (z - 1)(2 + z - \theta) \geq 0 .
\] (4.75)
The regime of validity for this analysis (equivalently, the “thickness” of the stretched horizon) gets modified in this special case to

$$\exp \left( -\frac{T^{2/z}}{q^2} \log \frac{1}{r_0 r_c} \right) \ll \frac{1}{r_0} - r_h \ll \frac{q^2}{T^{2/z}} \log^2 \frac{1}{r_0 r_c}. \quad (4.76)$$

However, since we are manifestly in the hydrodynamic regime, it means $r_c \ll \frac{1}{r_0}$ implying $\log \frac{1}{r_0 r_c} \gg 1$. This does not over-constrain the window of the stretched horizon: however the subleading terms contain the logarithmic piece affecting the validity of the series expansion.

The logarithmic scaling necessitates the presence of the UV scale $r_c$ appearing in the diffusion constant in the hydrodynamic description which is manifestly a description at long wavelengths. However from our discussion, it is clear that this is due to the two fall-offs for $\tilde{A}_t$ coinciding when $z = 4 - \theta$: this leads to the second solution being logarithmic and thence to the scaling above in $D$. Recall that the parameters $z$ and $\theta$ are related precisely in this way when the hyperscaling violating theory is constructed from the $x^+$-reduction of $AdS$ plane waves (or highly boosted $AdS_5$ black branes), as well as nonconformal $Dp$-brane plane waves, as discussed in [1]. (The zero temperature $AdS$ plane waves are structurally similar to the null deformations appearing in the string realizations [21, 22] of $z = 2$ Lifshitz theories, except that the null deformation is normalizable.) As outlined in [1], to gain more insight into the diffusion behaviour, it might be interesting to understand the null reduction of the boosted black brane and its hydrodynamics in greater detail. This might be similar in spirit to nonconformal brane hydrodynamics arising under dimensional reduction of the hydrodynamics of black branes in M-theory [54, 55], although the details are likely to be interestingly different of course. It is also worth noting that in the higher dimensional description, these D-brane plane waves are dual to excited states in the field theory which correspond to anisotropic phases in the boosted frame: the corresponding anisotropic hydrodynamics might be interesting as well (see e.g. [56, 57, 58, 59, 60, 61, 62, 63] for previous studies of anisotropic systems and shear viscosity, and e.g. [64] for a review of the viscosity bound and violations).

### 4.2 Subleading terms for $z < 4 - \theta$

In this section we will estimate the subleading terms as proposed in (4.61) and explicitly show that $\tilde{A}_t^{(1)}$, $\tilde{A}_x^{(1)}$ and $\chi^{(1)}$ (infact all the other terms following it) are subleading compared to the leading order values $\tilde{A}_t^{(0)}$, $\tilde{A}_x^{(0)}$ and $\chi^{(0)}$ respectively.
Estimate for $\tilde{A}_t^{(1)}$

Substituting the series for $\tilde{A}_t^{(1)}$ from (4.61) in (4.31), we get

$$
\partial_r \left( r^{2-\theta-3} \partial_r \tilde{A}_t^{(0)} + \partial_r \tilde{A}_t^{(1)} \right) + \cdots - \frac{r^{2-\theta-3}}{f} \partial_x \left( \tilde{F}_{tx}^{(0)} - k \int_{r_c}^r ds \, s^{3-\theta} \partial_x \chi^{(0)} + \cdots \right) = 0 .
$$

(4.77)

The leading term in the above equation is $\partial_r \left( r^{2-\theta-3} \partial_r \tilde{A}_t^{(0)} \right) = 0$, which is consistent with (4.64). $O(q^2)$ terms in the above equation give

$$
\partial_r \left( r^{2-\theta-3} \partial_r \tilde{A}_t^{(1)} \right) - \frac{r^{2-\theta-3}}{f} \partial_x \left( \partial_r \tilde{A}_t^{(0)} - \partial_x \tilde{A}_t^{(0)} \right) = 0 .
$$

(4.78)

Here we have neglected $k \int_{r_c}^r ds \, s^{3-\theta} \partial_x \chi^{(0)}$ since $k \int_{r_c}^r ds \, s^{3-\theta} \partial_x \chi^{(0)} \ll \tilde{F}_{tx}^{(0)}$, using the arguments in e.g. (4.43), (4.46). Then

$$
\partial_r \tilde{A}_t^{(1)} \sim \frac{1}{r_0} \left( q^2 \log \left( \frac{1}{r_0} \right) + \frac{\Gamma^2}{r_0^{2(z-1)+1}} \log^2 \left( \frac{1}{r_0} \right) \right) \tilde{A}_t^{(0)} .
$$

(4.79)

Using the estimate $\frac{\Gamma}{q} \sim \frac{q}{r_0^{3/2-\theta}}$, we can write

$$
\partial_r \tilde{A}_t^{(1)} \sim r_0 \left[ \frac{q^2}{T^{2/z}} \log \left( \frac{1}{r_0} \right) + \frac{q^4}{T^{4/z}} \log^2 \left( \frac{1}{r_0} \right) \right] \tilde{A}_t^{(0)} .
$$

(4.80)

Integrating the above equation,

$$
\tilde{A}_t^{(1)} \sim -(1 - r_0^r) \left[ \frac{q^2}{T^{2/z}} \left( 1 + \log \left( \frac{1}{r_0} \right) \right) + \frac{q^4}{T^{4/z}} \left( 1 + \log \left( \frac{1}{r_0} \right) \log^2 \left( \frac{1}{r_0} \right) \right) \right] \tilde{A}_t^{(0)} ,
$$

(4.81)

which implies $\tilde{A}_t^{(1)} \ll \tilde{A}_t^{(0)}$.

Estimate for $A_x^{(1)}$

Substituting the series ansatz for $A_x$ i.e (4.61) in (4.32) gives

$$
\partial_r A_x^{(0)} + \partial_r A_x^{(1)} + \cdots = \frac{i \Gamma}{q} \frac{r^{2z-2}}{f} \left( \partial_r \tilde{A}_t^{(0)} + \partial_r \tilde{A}_t^{(1)} + \cdots \right) .
$$

(4.82)

The leading terms have been derived in (4.68), so we will focus on $O(q^2)$ terms which gives us the equation

$$
\partial_r A_x^{(1)} = \frac{i \Gamma}{q} \frac{r^{2z-2}}{f} \partial_r \tilde{A}_t^{(1)} ,
$$

(4.83)

which give

$$
A_x^{(1)} \sim \left( \frac{q^2}{T^{2/z}} \log^2 \left( \frac{1}{r_0} \right) + \frac{q^4}{T^{4/z}} \log^3 \left( \frac{1}{r_0} \right) \right) \frac{i \Gamma r_0^{2z-2}}{q} \tilde{A}_t^{(0)} .
$$

(4.84)
Using $\mathcal{A}^{(0)}_x \sim \frac{i\pi r_0^{2-2\epsilon}}{q} \log \left(\frac{1/r_0}{1/r_0 - r}\right) \tilde{A}^{(0)}_t$, 

$$
\mathcal{A}^{(1)}_x \sim \left[ \frac{\eta^2}{T^{2/\epsilon}} \log \left(\frac{1/r_0}{1/r_0 - r}\right) + \frac{\eta^4}{T^{4/\epsilon}} \log^2 \left(\frac{1/r_0}{1/r_0 - r}\right) \right] \mathcal{A}^{(0)}_x, 
$$

(4.85)

which implies $\mathcal{A}^{(1)}_x \ll \mathcal{A}^{(0)}_x$.

**Estimate for $\chi^{(1)}$**

Finally, substituting the series ansatz for $\chi$ i.e (4.61) in (4.35), we get

$$
\partial_r (r^{5-z-\theta} f(\partial_r \chi^{(0)} + \partial_r \chi^{(1)})) - k^2 r^{3-z-\theta} (\chi^{(0)} + \chi^{(1)}) - \frac{r^{3+z-\theta}}{f}(\partial^2_r \chi^{(0)} + \partial^2_r \chi^{(1)}) + \cdots 
\cong k \partial_r (\tilde{A}^{(0)}_t + \tilde{A}^{(1)}_t + \cdots). 
$$

(4.86)

Writing down (4.35) collecting all $O(q^2)$ terms give

$$
\partial_r (r^{5-z-\theta} f \partial_r \chi^{(1)}) - k^2 r^{3-z-\theta} \chi^{(1)} = \frac{r^{3+z-\theta}}{f} \Gamma^2 \chi^{(0)} + k \partial_r \tilde{A}^{(1)}_t. 
$$

(4.87)

To see that $\chi^{(1)}$ is subleading compared to $\chi^{(0)}$ quickly, let us focus on the first term on both sides of the above equation near the horizon;

$$
\partial_r \left( r_0 \left( \frac{1}{r_0} - r \right) \partial_r \chi^{(1)} \right) \sim \frac{\eta^4}{r_0^2} \frac{1}{r_0 (1/r_0 - r)} \chi^{(0)},
$$

(4.88)

where we have used $\frac{\Gamma}{q} \sim \frac{\eta}{r_0^2}$. Integrating twice, we get

$$
\chi^{(1)} \sim \frac{\eta^4}{r_0^4} \log^2 \left(\frac{1/r_0}{1/r_0 - r}\right) \chi^{(0)}.
$$

(4.89)

Using (4.70), the above expression shows that $\chi^{(1)} \ll \chi^{(0)}$. This succinct order of magnitude analysis for the subleading nature of $\chi^{(1)}$ can be substantiated through a more detailed analysis as follows. In the near horizon region, (4.87) simplifies to

$$
\partial_r \left( \left( \frac{1}{r_0} - r \right) \partial_r \chi^{(1)} \right) - 2(z-1) r_0 \chi^{(1)} = \frac{r_0^{4-z-\theta}}{2 + z - \theta} \left( k \partial_r \tilde{A}^{(1)}_t + \frac{r_0^{\theta-z-3}}{(2 + z - \theta) r_0 (1/r_0 - r)} \Gamma^2 \chi^{(0)} \right).
$$

(4.90)

The Green’s function for the above equation is effectively the function $G(r, s)$ that satisfies the equation

$$
\partial_r \left( \left( \frac{1}{r_0} - r \right) \partial_r G(r, s) \right) - 2(z-1) r_0 \cdot G(r, s) = \delta(r - s).
$$

(4.91)
The inhomogeneous solution to the Green’s function is given by
\[ G_{in}(r, s) = 2\Theta(r - s) \left[ I_0(2\sqrt{2(z - 1)\sqrt{1 - r_0s}}) \cdot K_0(2\sqrt{2(z - 1)\sqrt{1 - r_0r}}) - K_0(2\sqrt{2(z - 1)\sqrt{1 - r_0s}}) \cdot I_0(2\sqrt{2(z - 1)\sqrt{1 - r_0r}}) \right]. \] (4.92)

Correspondingly the inhomogeneous solution to \( \chi^{(1)} \) is given by
\[ \chi^{(1)} = \int_0^{1/r_0} ds \cdot G_{in}(r, s) = 2\Theta(r - s) \left[ \frac{r_0^{4-z-\theta}}{2 + z - \theta} \left( k\partial_r \tilde{A}_t^{(1)} + \frac{r_0^{\theta-z-3}}{(2 + z - \theta)r_0(1/r_0 - r)} \Gamma^2 \chi^{(0)} \right) \right]. \] (4.93)

where \( I_0 \) and \( K_0 \) are modified Bessel functions of the first and second kind respectively. Since we are interested only in the near-horizon behaviour for \( \chi^{(1)} \). Instead of explicitly performing the integral exactly and then taking the limit \( r \to \frac{1}{r_0} \), we will instead approximate the integrand close to \( \frac{1}{r_0} \). Upto leading order the modified Bessel functions \( I_0 \) and \( K_0 \) near \( x \approx 0 \) are given by
\[ I_0(x) \approx 1, \quad K_0(x) \approx -\log x + \log 2 - \gamma, \] (4.94)

where \( \gamma \) is the Euler constant. Close to the horizon, we can hence approximate the inhomogeneous part of the Green’s function as
\[ G_{in}(r, s) = \Theta(r - s) \log \left( \frac{1 - r_0s}{1 - r_0r} \right). \] (4.95)

Hence \( \chi^{(1)} \) can be simplified using the above approximation along with (4.80)
\[ \chi^{(1)} = \int_0^{1/r_0} ds \chi^{(1)} \left[ k\partial_r \tilde{A}_t^{(1)} + \frac{r_0^{\theta-z-3}}{(2 + z - \theta)r_0(1/r_0 - r)} \Gamma^2 \chi^{(0)} \right] \]
\[ \sim \int_0^{1/r_0} ds \Theta(r - s) \log \left( \frac{1 - r_0s}{1 - r_0r} \right) \left[ \frac{1}{r_0} \left( \frac{q^2}{T^{2/2}} \log \frac{1}{r_0 - s} + \frac{q^4}{T^{4/2}} \log^2 \frac{1}{r_0 - s} \right) \tilde{A}_t^{(0)} \right] + \int_0^{1/r_0} ds \Theta(r - s) \log \left( \frac{1 - r_0s}{1 - r_0r} \right) \frac{q^4}{T^{4/2}} \frac{1}{r_0 - s} \chi^{(0)}. \] (4.96)

The above integral can be divided into two parts. One ranging from 0 to \( r \) and another from \( r \) to \( \frac{1}{r_0} \). The Heaviside Theta function is non-zero for \( r > 0 \) only. So, the upper bound in the above integral can simply be replaced with \( r \) instead of \( 1/r_0 \). Simplifying and performing the integral over \( s \) we get,
\[ \chi^{(1)} \sim \left[ \frac{q^2}{T^{2/2}} \left( (1 - r_0r) - (1 - r_0r) \log(1 - r_0r) + (1 - r_0r) \log^2(1 - r_0r) \right) \right] + \frac{q^4}{T^{4/2}} \left( -(1 - r_0r) + (1 - r_0r) \log(1 - r_0r) - (1 - r_0r) \log^2(1 - r_0r) \right. \]
\[ \left. + (1 - r_0r) \log^3(1 - r_0r) \right] \tilde{A}_t^{(0)} + \frac{q^4}{T^{4/2}} \log^2(1 - r_0r) \chi^{(0)} \ll \chi^{(0)}. \] (4.97)
If we now use the two assumptions mentioned earlier i.e (3.15), (4.70) we explicitly see that \( \chi^{(1)} \ll \chi^{(0)} \) thus demonstrating that all subsequent terms in the series are smaller than the leading piece.

**Estimate for \( h_{ty}^{(1)} \)**

Note that in (4.18) we proposed the series expansion for the modes \( h_{ty}, h_{xy} \) and \( a_y \). From the definition of \( \tilde{h}_{ty} \) and the series ansatze (4.18), (4.61), we can write

\[
 h_{ty}^{(0)} + h_{ty}^{(1)} + \cdots = \left( \tilde{h}_{ty}^{(0)} + kr^{\theta-2} \int_{r_c} r \, ds \, s^{3-z-\theta} a_y^{(0)} \right) + \left( \tilde{h}_{ty}^{(1)} + kr^{\theta-2} \int_{r_c} r \, ds \, s^{3-z-\theta} a_y^{(1)} \right) + \cdots .
\]

(4.98)

From (4.81) and (4.97), we have

\[
 h_{ty}^{(1)} = \tilde{h}_{ty}^{(1)} + kr^{\theta-2} \int_{r_c} r \, ds \, s^{3-z-\theta} a_y^{(1)} \sim O\left( \frac{q^2}{T^{2/z}} \right) \tilde{h}_{ty}^{(0)} .
\]

(4.99)

Using

\[
 \tilde{h}_{ty}^{(2)} \sim \frac{q^2}{T^{2/z}} \tilde{h}_{ty}^{(1)} , \quad a_y^{(2)} \sim \frac{q^2}{T^{2/z}} a_y^{(1)} ,
\]

(4.100)

we see that

\[
 \frac{h_{ty}^{(2)}}{h_{ty}^{(1)}} = \frac{\tilde{h}_{ty}^{(2)}}{\tilde{h}_{ty}^{(1)}} + kr^{\theta-2} \int_{r_c} r \, ds \, s^{3-z-\theta} a_y^{(2)} \sim O\left( \frac{q^2}{T^{2/z}} \right) \ll 1 .
\]

(4.101)

Thus we see that the mode \( h_{ty} \) also admits a series expansion in the parameter \( \frac{q^2}{T^{2/z}} \) in the near-horizon region. This is of course expected from the self-consistent series expansions of \( \tilde{h}_{ty}, h_{xy}, a_y \).

**4.2.1 Subleading terms for \( z = 4 - \theta \)**

In this case, from the solutions of \( \tilde{A}_t \) (4.73) and \( A_x \) (4.68) we get,

\[
 \frac{A_x^{(0)}}{A_t^{(0)}} \sim \frac{1}{r_0^{2(z-1)}} \frac{\Gamma}{q} \log\left( \frac{1/r_0 - r}{r_0} \right) .
\]

(4.102)

Imposing (4.57) then implies

\[
 \frac{1}{r_0^{2(z-1)}} \frac{\Gamma}{q^2} \frac{\log\left( \frac{1/r_0 - r}{r_0} \right)}{\log\left( \frac{1}{r_0 r_c} \right)} \ll 1 .
\]

(4.103)

We can obtain an estimate for \( D \) in this case from the diffusion equation which is \( \frac{\Gamma}{q} \sim \frac{q}{T^{2/z-1}} \log\left( \frac{1}{r_0 r_c} \right) \). Thus, the assumptions in this special case gets modified to (4.76). The
subleading term for $\tilde{A}_t$ now is given by

$$\partial_r \tilde{A}_t^{(1)} \sim r_0\left[\frac{q^2}{T^{2/z}} \log \left(\frac{1}{r_0} \frac{1}{r_0 - r}\right) + \frac{q^4}{T^{4/z}} \log^2 \left(\frac{1}{r_0} \frac{1}{r_0 - r}\right) \right] \tilde{A}_t^{(0)} .$$  \hfill (4.104)

Note that $r_0 r_c \ll 1$ implies that $\log\left(\frac{1}{r_0 r_c}\right)$ is large which means that the $O(q^4)$ term need not be small even if we are working withing the hydrodynamic regime i.e. $\frac{q^2}{T^{2/z}} \ll 1$, suggesting a breakdown of the series expansion. The expression for the the subleading part of $\chi$ i.e. $\chi^{(1)}$ also changes to

$$\chi^{(1)} \sim \left[\frac{q^2}{T^{2/z}} \left\{ (1 - r_0 r) - (1 - r_0 r) \log(1 - r_0 r) + (1 - r_0 r) \log^2(1 - r_0 r) \right\} \right.$$
$$+ \frac{q^4}{T^{4/z}} \left\{ - (1 - r_0 r) + (1 - r_0 r) \log(1 - r_0 r) - (1 - r_0 r) \log^2(1 - r_0 r) \right.$$ $$+(1 - r_0 r) \log^3(1 - r_0 r) \left\}\log \left(\frac{1}{r_0 r_c}\right) \right] \tilde{A}_t^{(0)} + \frac{q^4}{T^{4/z}} \log^2(1 - r_0 r) \log^2 \left(\frac{1}{r_0 r_c}\right) \chi^{(0)} \ll \chi^{(0)} .$$  \hfill (4.105)

From the preceding argument, we see again that the $O(q^4)$ term can be arbitrarily large hinting at a breakdown of the series expansion.

**Estimate for $h_{ty}^{(1)}$**

In the case when $z = 4 - \theta \ (4.98)$ takes the form

$$h_{ty}^{(0)} + h_{ty}^{(1)} + \cdots = \left(\tilde{h}_{ty}^{(0)} + kr^{2-z} \int_{r_c}^r \frac{ds}{s} a_y^{(0)} \right) + \left(\tilde{h}_{ty}^{(1)} + kr^{2-z} \int_{r_c}^r \frac{ds}{s} a_y^{(1)} \right) + \cdots .$$  \hfill (4.106)

The above further implies that

$$h_{ty}^{(1)} = \tilde{h}_{ty}^{(1)} + kr^{2-z} \int_{r_c}^r \frac{ds}{s} a_y^{(1)}$$
$$\sim \frac{q^4}{T^{4/z}} (1 - r_0 r) \log \left(\frac{1}{r_0 r_c}\right) \left(1 + \log \left(\frac{1}{r_0} \frac{1}{r_0 - r}\right) + \log^2 \left(\frac{1}{r_0} \frac{1}{r_0 - r}\right) \right) \tilde{h}_{ty}^{(0)}$$
$$+ \frac{q^4}{T^{4/z}} (1 - r_0 r) \log^2 \left(\frac{1}{r_0 r_c}\right) \log^2 \left(\frac{1}{r_0} \frac{1}{r_0 - r}\right) \chi^{(0)} .$$  \hfill (4.107)

The above estimate is written using the estimate $\frac{\Sigma_q}{q} \sim \frac{q}{T^{2/z-1}} \log \left(\frac{1}{r_0 r_c}\right)$. Further, the assumptions (4.76) implies that $h_{ty}^{(1)}$ may not be subleading compared to $h_{ty}^{(0)}$ thus suggesting a breakdown of some sort in this analysis.
5 Discussion

In this paper, we have explored in greater detail our investigations of shear diffusion in nonrelativistic hyperscaling violating Lifshitz theories [1], adapting the membrane-paradigm-like analysis [45] of near horizon perturbations. In theories where a gauge field is present as a source for the nonrelativistic metric (along with a scalar), some of the metric perturbations \( h_{ty}, h_{xy} \) mix with some of the gauge field perturbations \( a_y \). Since these are uncharged black branes, the near-horizon region should still be characterized by simply temperature and velocity variables, and charge cannot enter. Thus we expect that the gauge field cannot dramatically alter the structure of the near horizon diffusion equation found in [1] without the gauge field. Our analysis in this paper vindicates this: we find a similar near-horizon analysis can be obtained resulting in a diffusion equation for new field variables \( \tilde{h}_{xy} \equiv h_{xy} \) and \( \tilde{h}_{ty} \equiv h_{ty} - r^{\theta-2} \int_{r_c}^r s^{3-z-\theta} a_y ds \) (for 4 bulk dimensions). Then, as in [1], for \( z < 4 - \theta \), we obtain universal behaviour for the shear diffusion constant, suggesting that the viscosity bound \( \eta_s = \frac{1}{4\pi} \) holds. The regime \( z > 4 - \theta \) includes e.g. hyperscaling violating theories arising from the dimensional reduction of e.g. D6-branes (giving \( d_i = 6, z = 1, \theta = 9 \)) which do not admit a good gauge/gravity duality (ill-defined asymptotics with gravity not decoupling): however it might be interesting to find and understand reasonable holographic theories whose exponents lie in this window. For \( z = 4 - \theta \), we find logarithmic behaviour as found previously. The hyperscaling violating Lifshitz theories arising from AdS plane waves (highly boosted black branes) as well as nonconformal brane plane waves [17, 19, 29], fall in this category: this suggests that a null reduction of the hydrodynamics of the boosted black brane might need a closer study to realize this in detail, as we have described. We hope to explore this further.

We have seen the condition \( z < 2 + d_i - \theta \) (or \( z < 4 - \theta \) here, for bulk 4-dims) arising naturally from the perturbations falling off asymptotically (4.63) in our case. We implicitly regard hyperscaling violating theories as infrared phases arising from e.g. string realizations in the ultraviolet: however the window \( z < 2 + d_i - \theta \) ensures that the ultraviolet structure is essentially unimportant, the diffusion constant arising solely from the near horizon long-wavelength modes. This still needs to be reconciled with a clear holographic calculation: however some preliminary remarks are as follows. We have seen that the \( \tilde{h}_{ty} \) mode has asymptotic fall-offs \( r^{\theta-2}(\tilde{h}_- + ...) + r^{2-z}(\tilde{h}_+ + ...) \) in bulk 4-dimensions. For \( z < 4 - \theta \), the dominant mode near the boundary \( r \to r_c \sim 0 \) is \( r^{\theta-2} \) which is slower, leading to fixed \( h_- \) boundary conditions relevant for standard quantization (\( h_- \) taken as source). This is the sector that is continuously connected to AdS-like relativistic theories (\( z = 1, \theta = 0 \)), as our perturbation analysis suggests. With the conformal dimensions satisfying \( \Delta_- + \Delta_+ = 2 + z - \theta \) [14] (see also [25, 27]), the momentum density operator \( \mathcal{P}^i \) has dimension \( 3 - \theta \): so taking
\( \Delta_+ = 3 - \theta \) gives \( \Delta_- = z - 1 \) and \( \Delta_- < \Delta_+ \) implies \( z < 4 - \theta \). In a reasonable theory where this is violated, it would seem that the analog of alternative quantization \cite{52} is at work, with fixed \( h_+ \) boundary conditions. In this light, \( z = 4 - \theta \) is the case where the two fall-offs coincide with \( \Delta_- = \Delta_+ \), and a logarithmic second solution will arise suggesting logarithmic behaviour in the correlation function as well. This is the case for \( AdS \) plane waves (or highly boosted black branes): this may be interesting to explore.

It is worth putting the analysis here leading to (2.14), (4.72), in perspective with the calculation of viscosity via the Kubo formula \( \eta = -\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G_{xy,xy}^R(\omega) \), with \( G^R \) the retarded Green’s function \cite{49}, assuming \( T_{ij} \sim \eta (\partial_i v_j + \ldots) \) in the dual field theory. The \( h_y^x \) perturbation is modelled holographically as a massless scalar leading to the \( \langle T_{xy} T_{xy} \rangle \) holographic correlation function (see e.g. \cite{32, 37, 39} for various subfamilies in (2.1)). For instance from \cite{39}, the appropriate zero momentum \( \vec{k} = 0 \) solutions to the scalar wave equation eventually lead to \( G^R = -i \frac{\omega}{16\pi G} \frac{R^e_{ij}}{r_{hv}} d_i^0 \) and thereby \( \eta \); here the metric (2.1) is written as \( ds^2 = R^2 (\frac{r}{r_{hv}})^{2\theta/d_i} (-f(r) \frac{dt^2}{r^2} + \ldots) \), retaining explicitly the dimensionful factors \( R \) and the scale \( r_{hv} \) inherent in these theories \cite{14}. Likewise the horizon area gives the entropy density \( s = \frac{1}{4G \frac{R_{hv}}{r_{hv}} r_0^{d_i-\theta}} \) which leads to \( \frac{2}{s} = \frac{1}{4\pi} \) in agreement with our analysis. (We have seen that \( \theta \) disappears from the temperature dependence of \( D \) in (2.14): this is consistent with e.g. cases where the hyperscaling violating phase arises from string constructions such as nonconformal branes which are known to have universal \( \frac{2}{s} \) behaviour.)

In light of the above, note that the Kubo analysis stemming from a zero frequency \( \omega \to 0 \) limit for the \( h_y^x \) mode alone, does not appear to give any insight into where a condition like \( z < 2 + d_i - \theta \) could arise from. On the other hand, our analysis here and in \cite{11} in terms of the near-horizon perturbations involves the \( h_{ty} \) perturbation as well (as in \cite{45}), which is coupled at nonzero \( \omega \) to \( h_y^x \), and leads to the diffusion equation. The \( h_{ty} \) mode (or \( \tilde{h}_{ty} \) here) exhibits this nontrivial behaviour where the normalizable mode can turn around depending on the exponents \( z, \theta \), the critical condition being the family \( z = 2 + d_i - \theta \) where the two modes coincide. This condition is trivially satisfied for all relativistic theories of interest, with \( z = 1, \theta = 0 \), so the Kubo limit is in perfect agreement with the near horizon diffusion analysis. However in the present nonrelativistic cases, the near horizon perturbations analysis appears to exhibit more structure. It would seem that the structure of these perturbations is straightforward and simply involves analysing gravitational perturbations, not requiring detailed understanding of the holographic dictionary in this case. Therefore assuming that this is reliable, our analysis suggests that the Kubo limit might need to be understood better in theories where \( z < 2 + d_i - \theta \) is violated. In the case with a gauge field, the field variable \( \tilde{h}_{ty} \) which exhibits this behaviour naively suggests that perhaps a new energy-momentum tensor variable \( \tilde{T}_{\mu\nu} \) involving some linear combination of \( T_{\mu\nu} \) and the
current density $j_\mu$ is the relevant hydrodynamic observable that systematically encodes the thermodynamic/hydrodynamic relations between the expansion of the energy-momentum tensor, the shear viscosity $\eta$ and the diffusion constant $D$. We hope to explore these issues further.

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### A Linearized equations for perturbations on $d = 3$ hyperscaling violating background

In this section, we list the equations of motion for perturbations $h_{tt}$, $h_{tx}$, $h_{xx}$, $h_{yy}$, $a_t$, $a_x$ and $\varphi$ for the sake of completeness. For $d = 3$, the values of the various constants are

$$
\beta \equiv \sqrt{(2 - \theta)(2z - \theta - 2)} , \quad \lambda = \frac{4 - \theta}{\beta} , \quad \delta = \frac{\theta}{\beta} , \quad \Lambda = -\frac{1}{2}(2 + z - \theta)(1 + z - \theta) . \quad (A.1)
$$

The $t$, $x$ and $r$ components of the linearized Maxwell’s equation (4.11), respectively, give

$$
\partial_r (r^{3+z-\theta} \partial_r a_t) - \frac{r^{3+z-\theta}}{f} (q^2 a_t + q \omega a_x) - \frac{k}{2} \left[ \partial_r (r^{2-\theta} (h_{xx} + h_{yy})) + \partial_r \left( \frac{r^{2-\theta}}{f} h_{tt} \right) \right] - k \lambda \partial_r \varphi = 0 ,
$$

$$
\partial_r (r^{5-z-\theta} f \partial_r a_x) + \frac{r^{3+z-\theta}}{f} (q \omega a_t + \omega^2 a_x) - k \partial_r (r^{2-\theta} h_{tx}) = 0 ,
$$

$$
q \left[ r^{5-z-\theta} f \partial_r a_x - kr^{2-\theta} h_{tx} \right] + \omega \left[ r^{3+z-\theta} \partial_r a_t - \frac{k}{2} \left( r^{2-\theta} (h_{xx} + h_{yy}) + \frac{r^{2-\theta}}{f} h_{tt} \right) - k \lambda \varphi \right] = 0 . \quad (A.4)
$$

The $tt$-component of the linearized Einstein’s equation (4.9) gives

$$
\partial_r^2 h_{tt} - \left( \frac{2 - 4z + \theta}{r} + \frac{\partial_r f}{f} \right) \partial_r h_{tt} - \frac{q^2}{f} h_{tt} - \frac{2q \omega}{f} h_{tx} - \frac{kr^{1-z}}{f} \partial_r a_t
$$

$$
+ \left[ -\frac{2(1 + z - \theta)(2 + z - \theta)}{r^2 f} - \frac{2(2z - \theta) \partial_r f}{fr} + \frac{(\theta - 2z)^2}{f^2} \right] \frac{h_{tt}}{2}
$$

$$
- \frac{1}{2} \partial_r (r^{\theta-2z} f) \partial_r (r^{2-\theta} (h_{xx} + h_{yy})) - \frac{\omega^2}{f} (h_{xx} + h_{yy}) + (2 + z - \theta) \beta r^{-2-2z+\theta} \varphi = 0 .
$$

The $tx$-component of (4.9) gives

$$
\partial_r (r^{z+\theta-3} \partial_r (r^{2-\theta} h_{tx})) + \frac{r^{z-1}}{f} q \omega h_{yy} - k \partial_r a_x = 0 . \quad (A.6)
$$
The $tr$-component of (4.9) gives

$$q \frac{r^{2z-\theta}}{f} \partial_r \left( \frac{r^{2z-\theta}}{f} (h_{tx} + h_{yy}) \right) + \frac{r^{2z-2}}{f} \partial_r \left( r^{2z-(h_{xx} + h_{yy})} \right) + \omega \frac{r^{2z-3}}{f} \beta \varphi = 0 \ . \quad (A.7)$$

Adding $xx$-component to $yy$-component of (4.9) gives

$$\partial_r(r^{2\theta-z-3} f \partial_r(r^{2-\theta}(h_{xx} + h_{yy}))) + \omega^2 \frac{r^{2z+\theta-3}}{f} (h_{xx} + h_{yy}) - 2r^{\theta-z-1} q^2 h_{yy} + \frac{2r^{z+\theta-3}}{f} q \omega h_{tx} \ - 2kr^{\theta-z-2} \partial_r a_t - (\theta - 2) r^{2\theta-z-4} f \partial_r \left( \frac{r^{2z-\theta} h_{tt}}{f} \right) + \frac{k^2 r^{z+\theta-5}}{f} h_{tt} + \frac{q^2 r^{z+\theta-3} h_{tt}}{f} \ - \frac{2(2 + z - \theta)(4 - 4z - 2\theta + \theta^2)}{\beta} r^{2\theta-z-5} \varphi = 0 \ . \quad (A.8)$$

Subtracting $yy$-component from $xx$-component of (4.9) gives

$$\partial_r(r^{\theta-z-1} f \partial_r(r^{2-\theta}(h_{xx} - h_{yy}))) + \frac{r^{z-1}}{f} \omega^2 (h_{xx} - h_{yy}) + \frac{r^{z-1}}{f} q^2 h_{tt} + \frac{2r^{z-1}}{f} q \omega h_{tx} = 0 \ . \quad (A.9)$$

The $xr$-component of (4.9) gives

$$\begin{align*}
q[r^{2\theta} \left( \partial_r h_{tx} + \frac{1 + z - \theta}{r} h_{tt} - \frac{\partial_r f}{2f} h_{tt} \right) - r^{2-2z} f \partial_r(r^{2-\theta} h_{yy}) - kr^{3-z-\theta} a_t - \beta r^{1-2z} f \varphi] \\
+ \omega[\partial_r(r^{2-\theta} h_{tx}) - kr^{3-z-\theta} a_t] &= 0 \ .
\end{align*} \quad (A.10)$$

The $rr$-component of (4.9) gives

$$\begin{align*}
\partial_r^2(h_{xx} + h_{yy}) + \left( \frac{3(2 - \theta)}{2r} + \frac{\partial_r f}{2f} \right) \partial_r(h_{xx} + h_{yy}) + (\theta - 2) \left( \frac{\theta}{2r^2} - \frac{\partial_r f}{2rf} \right) (h_{xx} + h_{yy}) \\
- \frac{r^{2z-2}}{f} \partial_r^2 h_{tt} + r^{2z-2} \partial_r h_{tt} \left( \frac{-2 + 4z + 3\theta}{2rf} + \frac{\partial_r f}{2f^2} \right) + \alpha(2 + z - \theta) \frac{r^{z-1}}{f} \partial_r a_t \\
+ \left[ \frac{\theta(2z - \theta)}{2r^2 f^2} - \frac{\partial_r f}{2r^2 f^2} \right] \frac{1}{r^2 f^2} \left( (z - 1)(2 + z - \theta) + (z - 1) r \partial_r f - r^2 \partial^2 f \right) \right] r^{2z-2} h_{tt} \\
+ 2\beta r^{\theta-z} \partial_r \varphi - \frac{(2 + z - \theta)\beta}{f} r^{\theta-4} \varphi &= 0 \ .
\end{align*} \quad (A.11)$$

The linearized scalar field equation (4.12) gives

$$\begin{align*}
\partial_r(r^{\theta-z-1} f \partial_r \varphi) + \left( \frac{k^2 \lambda^2}{2} - 2\Lambda \delta^2 \right) r^{\theta-z-3} f \varphi + r^{\theta-z-1} \left( \frac{r^{2z-2} \omega^2}{f^2} - \frac{q^2}{f} \right) \varphi \\
+ \frac{\beta r^{\theta-z-2} f}{2} \left[ \partial_r(r^{2-\theta}(h_{xx} + h_{yy})) - \partial_r \left( \frac{r^{2z-\theta} h_{tt}}{f} \right) \right] + \frac{k^2 \lambda r^{z-3} h_{tt}}{2f} - k \lambda \partial_r a_t &= 0 \ . \quad (A.12)
\end{align*}$$
B Solutions to linearized equations for $h_{ty}, a_y, h_{xy}$ at zero momentum and zero frequency

At $q = 0$, $\omega = 0$, the linearized equations of motion (4.13)-(4.16) reduce to

$$\partial_r(r^{5-z-\theta}f \partial_r a_y) - \alpha(2 + z - \theta)\partial_r(r^{2-\theta} h_{ty}) = 0, \quad (B.1)$$

$$\partial_r(r^{z+\theta-3} \partial_r(r^{2-\theta} h_{ty})) - \alpha(2 + z - \theta)\partial_r a_y = 0, \quad (B.2)$$

$$\partial_r(r^{-1+z+\theta}f \partial_r(r^{2-\theta} h_{xy})) = 0. \quad (B.3)$$

For the sake of brevity, from now on we will denote the $\partial_r$ operator with a prime "r" on the functions. Integrating (B.1) and substituting $\partial_r a_y$ in (B.2) gives

$$f(r)[r^2 h_{ty}'' + (1 + z - \theta) r h_{ty}' - (\theta - 2)(z - 2) h_{ty}] - 2(z - 1)(2 + z - \theta) h_{ty} = -\alpha^2(2 + z - \theta)^2 c_1 r^{\theta - 2}, \quad (B.4)$$

where we have chosen the integration constant as $-\alpha(2 + z - \theta)c_1$. This inhomogeneous equation has a particular solution $h_{ty} = c_1 r^{\theta - 2}$. The homogeneous part of the above equation,

$$f(r)[r^2 h_{ty}'' + (1 + z - \theta) r h_{ty}' - (\theta - 2)(z - 2) h_{ty}] - 2(z - 1)(2 + z - \theta) h_{ty} = 0 \quad (B.5)$$

can be solved by substituting a series ansatz, $h_{ty} = \sum_{n=0}^{\infty} c_n r^{m+n}$. Along with the two linearly independent homogeneous solutions, the complete solution (including the particular solution) is

$$h_{ty} = c_1 r^{\theta - 2} + c_3 r^{\theta - 2 z} f + c_4 r^z \left[1 + \frac{(z - 1)(r_0 r)^{2+2 z-\theta}}{(1 + 2 z - \theta)} \right] 2F_1 \left(1, \frac{3 z - \theta}{2 + z - \theta}; \frac{4 + 5 z - 3 \theta}{2 + z - \theta}; (r_0 r)^{2+2 z-\theta} \right). \quad (B.6)$$

Substituting $h_{ty}$ from the above expression in (B.2) and integrating, we get

$$a_y = -\frac{C}{k} - \alpha c_3 r^{-(2+z-\theta)}$$

$$+ c_4 \left[ \frac{r^{2 z-2}}{\alpha} + \frac{(2 + z - \theta)}{(1 + 2 z - \theta)} r^{2+2 z-\theta} r^{3 z-\theta} 2F_1 \left(1, \frac{3 z - \theta}{2 + z - \theta}; \frac{4 + 5 z - 3 \theta}{2 + z - \theta}; (r_0 r)^{2+2 z-\theta} \right) \right]$$

$$+ \alpha c_0 \left[ \frac{r^{2+2 z-\theta} r^{1+3 z-\theta}}{2(1 + 2 z - \theta)} 2F_1' \left(1, \frac{3 z - \theta}{2 + z - \theta}; \frac{4 + 5 z - 3 \theta}{2 + z - \theta}; (r_0 r)^{2+2 z-\theta} \right) \right], \quad (B.7)$$

where $2F_1' = \frac{d}{dr}(2F_1)$. The last term i.e. $2F_1' \left(1, \frac{3 z - \theta}{2 + z - \theta}; \frac{4 + 5 z - 3 \theta}{2 + z - \theta}; (r_0 r)^{2+2 z-\theta} \right)$ is in fact divergent at the horizon $r = \frac{1}{r_0}$. Integrating (B.3), we get

$$h_{xy} = b_1 r^{\theta - 2} \log(1 - (r_0 r)^{2+2 z-\theta}) + b_2 r^{\theta - 2}. \quad (B.8)$$
C  Spatial compactification of the hyperscaling violating Lifshitz theory

The action (4.1) in 4 bulk dimensions (i.e. \(d = 3\)) becomes

\[ S = -\frac{1}{16\pi G^{(4)}_N} \int d^4x\sqrt{-G} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{Z(\phi)}{4} F_{\mu \nu} F^{\mu \nu} + V(\phi) \right]. \]  

(C.1)

In the \(y\)-compactified theory (where \(y\) is one of the spatial dimensions enjoying translation invariance), the metric component \(G_{yy}\) is parametrized by the scalar \(\psi\) as \(G_{yy} = e^{2\psi}\), the gauge field coupling \(Z = r^{4-\theta}\) and the other metric modes are parametrized as

\[ G_{\mu \nu} \equiv \left[ \hat{g}_{ab} + e^{2\psi} A_a A_b \right] \left[ e^{2\psi} A_a \right]. \]  

(C.2)

where \(\hat{g}_{ab}\) is the metric of the compactified theory. Roman indices \(\{a, b\}\) runs over coordinates \(t, r, x\) while Greek indices \(\{\mu, \nu\}\) runs over \(t, r, x, y\). We also parametrize the gauge field \(A_\mu\) in the following way

\[ A_\mu \equiv \begin{pmatrix} A_a \\ \cdot \\ \cdot \\ \chi \end{pmatrix}. \]  

(C.3)

The gravity sector under compactification becomes

\[ S_{grav} = -\frac{1}{16\pi G^{(4)}_N} \int d^4x\sqrt{-G} \cdot R \]  

\[ = -\frac{1}{16\pi G^{(3)}_N} \int d^3x \cdot e^{\psi} \sqrt{-\hat{g}} \left( \hat{R}^{(3)} - \frac{1}{4} e^{4\psi} \hat{F}_{ab} \hat{F}^{ab} \right), \]  

(C.4)

where \(\hat{R}^{(3)}\) is the Ricci scalar for the metric \(\hat{g}_{ab}\). The Maxwell action after a \(y\)-compactification can be written as

\[ S_{Max} = -\frac{1}{16\pi G^{(4)}_N} \int d^4x\sqrt{-G} \left( -\frac{1}{4} Z F_{\mu \nu} F^{\mu \nu} \right) \]  

\[ = -\frac{1}{16\pi G^{(3)}_N} \int d^3x \left( -\frac{e^{\psi}}{4} \sqrt{\hat{g}} \right) \left[ \hat{g}^{ac} \hat{g}^{bd} F_{ab} F_{cd} + 4 \hat{g}^{bc} F_{ab} A_a (\partial_c \chi) ight. \]  

\[ + 2 \hat{g}^{ab} (e^{-2\psi} + A_c A^c)(\partial_a \chi)(\partial_b \chi) - 2 A^a A^b (\partial_a \chi)(\partial_b \chi) \right]. \]  

(C.5)
A Weyl transformation \( g_{ab} = e^{2\psi} \hat{g}_{ab} \) enables us to write the gravitational and Maxwell sector of action after compactification in the Einstein frame as

\[
S_{\text{grav}} + S_{\text{Max}} = -\frac{1}{16\pi G_N^{(4)}} \int d^4x \sqrt{-G} \left[ R - \frac{Z(\phi)}{4} F_{\mu\nu} F^{\mu\nu} \right] \\
= -\frac{1}{16\pi G_N^{(3)}} \int d^3x \cdot \sqrt{-g} \left( R_{(3)} - \frac{1}{4} e^{4\psi} F_{ab} F^{ab} \right)
\]

\[
+ Z e^{2\psi} \left( -\frac{1}{4} F_{ab} F^{ab} - F_a e^c \mathcal{A}^c (\partial_c \chi) - \frac{1}{2} e^{-4\psi} (\partial_a \chi) (\partial^a \chi) \right.
\]

\[
- \frac{1}{2} \mathcal{A} c \mathcal{A}^c (\partial_c \chi) (\partial^a \chi) + \frac{1}{2} \mathcal{A}^a \mathcal{A}^b (\partial_a \chi) (\partial_b \chi) \) \right),
\]

where \( R_{(3)} \) is the Ricci scalar of the 3-dimensional bulk metric \( g_{ab} \). The terms appearing in the last line of the above equation will not contribute to the equations of motion at linearized order since they appear at quartic order in the action. Varying the above action w.r.t. the field \( A_\mu \), at linearized level we get

\[
\frac{1}{\sqrt{g}} \partial_a \left( \sqrt{-g} e^{4\psi} F^{ab} \right) = e^{2\psi} Z g^{ab} g^{cd} F_{ad} (\partial_c \chi),
\]

which for \( b \equiv t, x, r \) gives (4.21), (4.22) and (4.23) respectively.

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