3n + 3^k: New Perspective on Collatz Conjecture

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Abstract

Collatz conjecture is generalized to 3n + 3^k (k ∈ N). Operating as usual, every sequence seems to reach 3^k and end up in the loop 3^k, 4.3^k, 2.3^k, 3^k. The usual 3n + 1 conjecture is recovered for k = 0. For k > 0, we noticed the existence of a sequence of period 3, namely, 3^k−1, 2.3^k, 3^k, alongside the cycle 4.3^k, 2.3^k, 3^k encountered in the 3n + 1(k = 0) sequence. A term formula of the 3n + 3^k conjecture has been derived, and hence the total stopping time.

1 Introduction

Credited to the mathematician Lothar Collatz, the Collatz conundrum (see for instance Ref. [1] and references therein), which was brought forward in the 1930s, is one of the simplest yet unsolved conjectures in mathematics. “This is a really dangerous problem. People become obsessed with it and it really is impossible,” as stated by Jeffrey Lagarias. Such a conjecture asks whether repeating two simple arithmetic operations will eventually reach 1. That is to say, let N := {0, 1, 2...} denote the natural numbers, so that N + 1 = {1, 2, 3...} are the positive integers. Now, pick an arbitrary positive integer n ∈ N + 1 and apply the following operation on it: If the number is even, divide it by two; while if the number is odd, triple it and add one. Let the process be denoted by f(n). That is

f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 3n + 1 & \text{if } n \equiv 1 \pmod{2} \end{cases}

(1)

The main task is to form a sequence by performing this operation repeatedly, and take the output at each step as the input at the next. Consequently, one may notice that such a process will eventually lead to the number 1, regardless of which positive integer is chosen initially.

If we set f^0(n) = n and f^l(n) = f(f^{l-1}(n)) for l ∈ N. Then, the Collatz sequence for n reads

C(n) = \{f^l(n)\}_{l=0}^{\infty}.

(2)

Strictly speaking, one may note that every Collatz sequence ends up in the loop 1, 4, 2, 1. To disprove Collatz conjecture, one has to show that there exists some starting number which yields a sequence that does not include 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase infinitely. No such sequence has been found so far [2].

In the present paper, we show that Collatz conjecture 3n + 1 may possibly be a part of a more generalized case of the form 3n + 3^k with k ∈ N.

2 3n + 3^k Conjecture

We provide the following operation, pick an arbitrary positive integer n ∈ N + 1, if the number is even, divide it by two; otherwise if the number is odd, triple it and add 3^k with k ∈ N.

g_k(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 3n + 3^k & \text{if } n \equiv 1 \pmod{2} \end{cases}

k ∈ N

(3)

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Repeating the two operations will eventually lead to $3^k$ ($k \in N$). Hence, one may argue that the original Collatz conjecture $3n+1$ is nothing more than $3n + 3^k$ conjecture with $k = 0$, i.e. $f(n) = g_0(n)$. Apparently, each positive integer, upon repetitive application of $g_k(n)$, will end in a repeating sequence of the form $3^k, 4.3^k, 2.3^k, 3^k$. In the $3n+1$ (or equivalently $3n+3^k$ with $k = 0$) sequence, the only cycle of period 3 is known to be the sequence $4.3^k, 2.3^k, 3^k$ [3]. For $k > 0$ though, along with the aforementioned cycle, one may distinguish another sequence of period 3, that is $3^{k-1}, 2.3^k, 3^k$. Although such a generalized conjecture seems to be true, it has yet to be numerically verified for at least the largest set of numbers that has been checked in the case of $3n+1$ so far, i.e. $n \leq 2^{100000} - 1$ [4], even though a general proof of it is still lacking. Furthermore, up to this point it is not clear whether there exists a certain value of $k$ for or beyond which the conjecture cease to be hold.

### 3 Term formula for $3n + 3^k$ sequence

Let $C_k^l(n)$ include the first $l$ terms of the $3n + 3^k$ sequence for $n$. We set $m$ as the number of odd terms in $C_k^l(n)$, $d_i$ as the number of consecutive even terms immediately following the $i$th odd term and $d_0$ as the number of even terms preceding the first odd term. It follows that the next term in the $3n + 3^k$ sequence for $n$ is

$$g_k^l(n) = \frac{3^m}{2^{l-m}}n + 3^k \sum_{j=1}^m \frac{3^{m-j}}{2^{\sum_{i=j}^m} d_i}$$

(4)

where $\epsilon = 0$ if $m = 0$ and $\epsilon = 1$ if $m \neq 0$, note that $l - m = d_0 + d_1 + \ldots + d_m$. It is worth noting that, for the original $3n+1$ Collatz sequence, the above relation reduces to

$$g_0(n) = \frac{3^m}{2^{l-m}}n + \sum_{j=1}^m \frac{3^{m-j}}{2^{\sum_{i=j}^m} d_i}$$

(5)

$3n + 3^k$ sequences of integers ranging from 1 to 17 for $k = 0, 1, 2, 3$ and 4 are listed in Table. 1. In the $k$-sequences chosen, it is evident that, after consecutive application of $g_k^n$, the aforementioned integers eventually reach $3^k$.

### 4 Total stopping time

We call the *total stopping time* of $n$ the smallest $t$ such that $n_t = 3^k$, with $n_t$ being the value of $g_k$ applied to $n$ recursively $t$ times. It is worth mentioning that $t$ is nothing but the number of integers in the $3n + 3^k$ sequence just preceding the $3^k$ term, i.e. $n_t = g_k^n(n)$. Thus, using the relation (4), the total stopping time can be written as

$$t = \log_2 \left[ \frac{3^k(1 - \epsilon \sum_{j=1}^m \frac{3^{m-j}}{2^{\sum_{i=j}^m} d_i})}{n} \right]$$

(6)

If $t$ does not exist we say that the total stopping time is infinite. It follows that, one has to prove that all positive integers yield a finite total stopping time in order to prove the $3n + 3^k$ conjecture. In an endeavor to prove the $3n+1$ conjecture, mathematicians used to inspect what it is called the stopping time, namely the least positive $l$ for which $f^l(n) = g_0^n(n) < n$ [6–9]. If they can prove that all positive integers have a finite stopping time, they can prove by induction that the Collatz conjecture is true. It is nothing short of reasonable since for $n > 1$, $f^l(n) = g_0^n(n) = 1$ cannot occur without the occurrence of some $f^l(n) = g_0^n(n) < n$ [10]. Indeed, in the 1970s, it has been shown that almost all Collatz sequences eventually reach a number that is smaller than where it started [11]. In 2019, Terence Tao proved that for almost all numbers the Collatz sequence of $n$ leads to a lower value, showing that the Collatz conjecture holds true for almost all numbers [12]. But the question to be asked is whether it is the right (or sufficient) way to prove the Collatz conjecture. As obvious as it seems, if the $3n + 3^k$ conjecture turns out to be true for all $n \in N+1$, then the statement that, for each $n \in N + 1$ there exists $l \in N + 1$ such that $g_k^n(n) < n$, does not hold for $n < 3^k$. This can be easily seen in the $3n+27$ sequence for $n = 5, 9, 13$ (see for instance Table. 1). Alternatively, we must focus on showing that each $n \in N + 1$ has a finite total stopping time and hence proving that the $3n + 3^k$ sequence inclines toward $3^k$ no matter what integer we start with. That is to say, one has to test the validity of the following statement: “for each $n \in N + 1$, there exists $t \in N + 1$ such that $g_k^n(n) = 3^k$.” without making use of the stopping time.

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1 Note that this formula is different from the one derived by L.E. Garner [5]
Now, suppose that two integers $n_1$ and $n_2$ have the same total stopping time $t$ within the sequence $3n + 3^k$, then from Eq. (6) we have
\[
n_2 = \frac{2^{m_1} 3^{m_2}}{2^{m_2} 3^{m_2}} \frac{1 - \epsilon \sum_{j=1}^{m_2} \frac{3^{m_2-j}}{2^{m_2-j}i_j^2}}{1 - \epsilon \sum_{j=1}^{m_1} \frac{3^{m_1-j}}{2^{m_1-j}i_j^2}} n_1
\]
where $m_1(m_2)$ and $d_1,d_2(i_j)$ are, respectively, the number of odd terms and the number of consecutive even terms immediately following the $i$th odd term in $C_k^l(n_1)$ ($C_k^l(n_2)$), namely the set of $t$ terms of the $3n + 3^k$ sequence for $n_1(n_2)$.

It is obvious that for $m_1 = m_2 = 0$, we have $n_2 = n_1$. Hence, no two integers, for which the corresponding sequences contain even terms only, have the same total stopping time. Sequences that have the same total stopping time and the same cycle of period 3 have the same number of odd terms, a feature that can be figured out by checking Figs. 1–3 which depict the total stopping time and the number of odd terms in the set $C_k^{t+1}(n)(k = 0, 1, 2)$ for different values of $n$. In the $3n + 9$ sequence for example (upper plot in Fig. 3), there exist two cycles of period 3: 36, 18, 9 and 3, 18, 9, the integers 32, 33 and 35 have the same total stopping time, i.e. $t = 10$, but only 33 and 35 share the same number of odd terms $m + 1 = 4$, which is due to the fact that both of them possesses the same cycle of period 3, i.e. 36, 18, 9, whereas 32 has the cycle 3, 18, 9.

Remarkably, It is worth noting that sequences that possess the same number of odd terms in the set $C_k^{t+1}(n)$ do not necessarily yield the same total stopping time, even if they share the same cycle of period 3, take the example of 1 and 2 in the $3n + 9$ sequence, both of them have the same number of odd terms in the set $C_9^{t+1}(n)$, i.e. $m + 1 = 3$, and the same cycle 3, 18, 9 (check Table. 1) meanwhile they have different total stopping time, $t = 5$ and $t = 6$, respectively.

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Figure 1: Behavior of the total stopping time and the number of odd terms in the set $C_{1+1}^{n+1}(n)$ of the $3n + 1$ sequence for $n \in [1, 100]$ (top), $n \in [500, 600]$ (middle) and $n \in [900, 1000]$ (bottom), the blue bars represent the total stopping time while the red bars depict the number of odd terms in the sequence.
Figure 2: Behavior of the total stopping time and the number of odd terms in the set $C_3^{n+1}(n)$ of the $3n + 3$ sequence for $n \in [1, 100]$ (top), $n \in [500, 600]$ (middle) and $n \in [900, 1000]$ (bottom), the blue bars represent the total stopping time while the red bars depict the number of odd terms in the sequence.
Figure 3: Behavior of the total stopping time and the number of odd terms in the set $C_{9}^{n+1}(n)$ of the $3n + 9$ sequence for $n \in [1, 100]$ (top), $n \in [500, 600]$ (middle) and $n \in [900, 1000]$ (bottom), the blue bars represent the total stopping time while the red bars depict the number of odd terms in the sequence.
### $3n + 1$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| 4   | 1 | 10 | 2 | 16 | 3 | 22 | 4 | 28 | 5 | 34 | 6 | 40 | 7 | 46 | 8 | 52 |
| 2   | 4 | 5  | 1 | 8  | 10| 11 | 2 | 14 | 16 | 17 | 3 | 20 | 22 | 23 | 4 | 26 |
| 1   | 2 | 16 | 4 | 4  | 55 | 34 | 1 | 7  | 8  | 52 | 10| 10 | 11 | 70 | 2 | 13 |
| 1   | 8 | 2  | 2 | 16 | 17| 4  | 22 | 4  | 26 | 5  | 5  | 34 | 35 | 1  | 40 |
| 4   | 1 | 1  | 8 | 52 | 2 | 11 | 2  | 13 | 16 | 16 | 17 | 17 | 16 | 17 | 106| 4 | 20 |
| 2   | 4 | 4  | 26| 1  | 34 | 1  | 40 | 8  | 8  | 52 | 53 | 2  | 10 |
| 1   | 2 | 2  | 13| 17 | 4  | 20 | 4  | 26 | 160| 16 | 16 | 15 | 16 | 1  |
| 4   | 1 | 1  | 40| 52 | 10| 2  | 13 | 80 | 16 |
| 2   | 4 | 20 | 26 | 5  | 1  | 40 | 8  | 8  |
| 1   | 2 | 10 | 13 | 16 | 4  | 4  | 20 | 20 |
| 1   | 5 | 40 | 8  | 2  | 2  | 10 | 10 |
| 16  | 20 | 4  | 1  | 1  | 5  | 5  | 1  |
| 8   | 10 | 2  | 16 | 16 | 4  |
| 4   | 5  | 1  | 8  | 8  | 2  |
| 2   | 16 | 4  | 4  | 4  | 1  |
| 1   | 8  | 2  | 2  | 2  |
| 4   | 4  | 1  | 1  | 1  | 1  |
| 2   | 2  | 4  | 4  | 4  | 1  |
| 1   | 2  | 2  | 4  | 1  | 1  |
| 2   | 1  | 2  | 2  |

### $3n + 3$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| 6   | 1 | 12| 2 | 18| 3 | 24| 4 | 30| 5 | 36 | 6 | 42 | 7 | 48 | 8 | 54 |
| 3   | 6 | 6 | 1 | 9 | 12| 12| 2 | 15| 18 | 18 | 3 | 21 | 24 | 24 | 4 | 27 |
| 12  | 3 | 3 | 6 | 30| 6 | 6 | 1 | 48| 9 | 12 | 9 | 12 | 9 | 12 | 2 | 84 |
| 6   | 12| 3 | 15| 3 | 3 | 6 | 24| 30| 30| 6 | 33| 6 | 6 | 1 | 42 |
| 3   | 6 | 6 | 12| 48| 12| 12| 3 | 12| 15| 15 | 13| 102| 3 | 3 | 6 | 21 |
| 3   | 6 | 24| 6 | 12| 12| 6 | 6 | 48| 48| 51| 12| 12 | 3 | 66 |
| 3   | 12| 3 | 6 | 3 | 24| 24| 156| 6 | 6 | 12 | 33 |
| 6   | 3 | 12| 12| 12| 78| 3 | 3 | 6 | 102|
| 3   | 6 | 6 | 6 | 39| 3 | 51 |
| 12  | 3 | 3 | 3 | 120| 156|
| 6   | 12| 12| 60| 78 |
| 3   | 6 | 6 | 30| 39 |
| 3   | 3 | 15 | 120|
| 48  | 60 |
| 24  | 30 |
| 12  | 15 |
| 6   | 48 |
| 3   | 24 |
| 12  | 12 |
| 6   | 6  |
| 3   | 3  |

3
### 3n + 9

| n  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| 12 | 1 | 18 | 2 | 24 | 3 | 30 | 4 | 36 | 5 | 42 | 6 | 48 | 7 | 54 | 8 | 60 |
| 6  | 12 | 9 | 1 | 12 | 18 | 15 | 2 | 18 | 24 | 21 | 3 | 24 | 30 | 27 | 4 | 30 |
| 3  | 6  | 36 | 12 | 6 | 9 | 54 | 1 | 9 | 12 | 72 | 18 | 12 | 15 | 90 | 2 | 15 |
| 18 | 3  | 18 | 6 | 3 | 36 | 27 | 12 | 36 | 6 | 36 | 9 | 6 | 54 | 45 | 1 | 54 |
| 9  | 18 | 9 | 3 | 18 | 18 | 90 | 6 | 18 | 3 | 18 | 36 | 3 | 27 | 144 | 12 | 27 |
| 36 | 9  | 18 | 9 | 9 | 45 | 3 | 9 | 18 | 9 | 18 | 9 | 18 | 90 | 72 | 6 | 90 |
| 18 | 36 | 9 | 36 | 144 | 18 | 9 | 36 | 9 | 9 | 45 | 36 | 3 | 45 |
| 9  | 18 | 36 | 18 | 72 | 9 | 36 | 18 | 36 | 144 | 18 | 18 | 144 |
| 9  | 18 | 9 | 36 | 36 | 18 | 9 | 18 | 9 | 9 | 36 | 36 | 36 |
| 9  | 18 | 9 | 9 | 9 | 36 | 36 | 36 | 36 | 36 | 36 | 36 | 36 |

### 3n + 27

| n  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| 30 | 1 | 36 | 2 | 42 | 3 | 48 | 4 | 54 | 5 | 60 | 6 | 66 | 7 | 72 | 8 | 78 |
| 15 | 36 | 18 | 1 | 21 | 36 | 24 | 2 | 27 | 42 | 30 | 3 | 33 | 48 | 36 | 4 | 39 |
| 72 | 15 | 9 | 30 | 90 | 18 | 12 | 1 | 108 | 21 | 15 | 36 | 126 | 24 | 18 | 2 | 144 |
| 36 | 72 | 54 | 15 | 45 | 9 | 6 | 30 | 54 | 90 | 72 | 18 | 63 | 12 | 9 | 1 | 72 |
| 18 | 36 | 27 | 72 | 162 | 54 | 3 | 15 | 27 | 45 | 36 | 9 | 216 | 6 | 54 | 30 | 36 |
| 9  | 18 | 108 | 36 | 81 | 27 | 36 | 72 | 108 | 162 | 18 | 54 | 108 | 3 | 27 | 15 | 18 |
| 54 | 9  | 54 | 18 | 270 | 108 | 18 | 36 | 54 | 81 | 9 | 27 | 54 | 36 | 108 | 72 | 9 |
| 27 | 54 | 27 | 9 | 135 | 54 | 9 | 18 | 27 | 270 | 54 | 108 | 27 | 18 | 54 | 36 | 54 |
| 108 | 27 | 54 | 432 | 27 | 54 | 9 | 135 | 27 | 54 | 108 | 9 | 27 | 18 | 27 |
| 54 | 108 | 27 | 216 | 27 | 54 | 432 | 108 | 27 | 54 | 54 | 9 | 108 |
| 27 | 54 | 108 | 108 | 27 | 216 | 54 | 27 | 27 | 54 | 54 | 108 |
| 27 | 54 | 108 | 108 | 27 | 216 | 54 | 27 | 27 | 54 | 54 | 108 |
| 27 | 54 | 108 | 27 | 27 | 54 | 54 | 54 | 108 |
| 27 | 54 | 108 | 27 | 27 | 54 | 54 | 108 |

8
| n  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 84 | 1  | 90 | 2  | 96 | 3  | 102| 4  | 108| 5  | 114| 6  | 120| 7  | 126| 8  | 132|
| 42 | 84 | 45 | 1  | 48 | 90 | 51 | 2  | 54 | 96 | 57 | 3  | 60 | 102| 63 | 4  | 66 |
| 21 | 42 | 216| 84 | 24 | 45 | 234| 1  | 27 | 48 | 252| 90 | 30 | 51 | 270| 2  | 33 |
| 144| 21 | 108| 42 | 12 | 216| 117| 84 | 162| 24 | 126| 45 | 15 | 234| 135| 1  | 180|
| 72 | 144| 54 | 21 | 6  | 108| 432| 42 | 81 | 12 | 63 | 216| 126| 117| 486| 84 | 90 |
| 36 | 72 | 27 | 144| 3  | 54 | 216| 21 | 324| 6  | 270| 108| 63 | 432| 243| 42 | 45 |
| 18 | 36 | 162| 72 | 90 | 27 | 108| 144| 162| 3  | 135| 54 | 270| 216| 810| 21 | 216|
| 9  | 18 | 81 | 36 | 45 | 162| 54 | 72 | 81 | 90 | 486| 27 | 135| 108| 405| 144| 108|
| 108| 9  | 324| 18 | 216| 81 | 27 | 36 | 45 | 243| 162| 486| 54 | 1296| 72 | 54 |
| 54 | 108| 162| 9  | 108| 324| 162| 18 | 216| 810| 81 | 243| 27 | 648 | 36 | 27 |
| 27 | 54 | 81 | 108| 54 | 162| 81 | 9  | 108| 405| 324| 810| 162| 324 | 18 | 162|
| 162| 27 | 54 | 27 | 81 | 324| 108| 54 | 1296| 162| 405| 81 | 162 | 9  | 81 |
| 81 | 162| 27 | 162| 162| 54 | 27 | 648| 81 | 1296| 324| 81 | 108| 324 | 81 | 162|
| 324| 81 | 162| 81 | 81 | 27 | 162| 324| 648| 162| 324| 54 | 162 | 81 | 162|
| 162| 324| 81 | 324| 162| 81 | 162| 324| 81 | 162| 81 | 324| 81 | 162 | 81 | 81 |
| 81 | 162| 81 | 162| 81 | 324| 162| 324| 81 | 162| 324| 324| 81 | 162 | 81 | 81 |
| 81 | 81 | 162| 81 | 162| 162| 81 | 81 | 81 | 162| 162| 81 | 81 | 81 | 81 | 81 |

Table 1: $3n + 3^k$ sequence with $k = 0, 1, 2, 3$ and 4 for $n$ ranging from 1 to 17.