Optimal Rank-1 Hankel Approximation in the Spectral Norm for Matrices with Multiple Largest Eigenvalue
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We extend a result from [2, 3] on the optimal rank-1 Hankel approximation of real symmetric matrices with isolated largest eigenvalue to the case where the largest eigenvalue is not isolated. To illustrate our findings, we give an example where the optimal rank-1 Hankel approximation is easily obtained and one where it does not exist.

1 Introduction

A real square Hankel matrix $H_1 \in \mathbb{R}^{N \times N}$ of rank 1 is of the form

$$H_1 = c \cdot z z^T,$$

where $c \in \mathbb{R} \setminus \{0\}$ and $z := z(z) := \frac{1}{\left(\sum_{j=0}^{N-1} z^j\right)^{1/2}} (1, z, z^2, \ldots, z^{N-1})^T$, $z \in \mathbb{R}$,

(1)

where $\mathbb{R} := \mathbb{R} \cup \{\infty\}$. Note that $z(\infty) = \lim_{z \to \infty} z(z) = (0, 0, \ldots, 1)^T$ is the last vector of the standard basis in $\mathbb{R}^N$ and therefore finite. With the characterization (1), the problem of approximating a given real symmetric matrix $A \in \mathbb{R}^{N \times N}$ by a Hankel structured matrix of rank 1 in the spectral norm reads

$$\min_{\|H\|_{\text{Hankel}}} \|A - H\|_2 = \min_{c \in \mathbb{R} \setminus \{0\}} \|A - c \cdot z z^T\|_2.$$  

(2)

Generalizing a result from [1], in [2] we have solved this problem for real symmetric matrices $A$ whose largest eigenvalue is bounded away from the modulus of the second largest eigenvalue, see also [3]. Here we consider real symmetric matrices $A$ with multiple largest modulus of eigenvalues thereby further extending the result of [1] and our results from [2, 3].

2 Optimal Rank-1 Hankel Approximation for Multiple Largest Eigenvalue

For a real symmetric matrix $A \in \mathbb{R}^{N \times N}$ denote by $\lambda_0, \ldots, \lambda_{N-1}$ its eigenvalues ordered by modulus. Throughout this paper assume that the largest modulus of eigenvalues occurs with higher multiplicity, i.e., $\|A\|_2 = |\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{N-1}| \geq 0$, w.l.o.g. $\lambda_0 > 0$. With $v_0, \ldots, v_{N-1}$ we denote the corresponding orthonormal eigenvectors. Let $V := (v_0, \ldots, v_{N-1})$ be the related basis transformation matrix, i.e., $A = V \Lambda V^T$, where $\Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$.

By the well-known Eckart-Young-Mirsky Theorem, a rank-1 Hankel approximation $\tilde{A}$ of $A$ cannot attain a smaller approximation error than the second largest eigenvalue. In this case $|\lambda_1| = \lambda_0$ is the optimal error bound for any rank-1 Hankel approximation.

In the following theorem we present conditions for which this error bound is attained.

**Theorem 2.1** The optimal error bound $\|A - \tilde{c} \cdot \tilde{z} \tilde{z}^T\|_2 = |\lambda_1| = \lambda_0$ can be attained by a rank-1 Hankel matrix $\tilde{c} \cdot \tilde{z} \tilde{z}^T$ if and only if either

$$v_j^T \tilde{z} = 0 \quad \text{for all } j \text{ with } \lambda_j = -\lambda_0, \quad \text{or} \quad v_j^T \tilde{z} = 0 \quad \text{for all } j \text{ with } \lambda_j = \lambda_0.$$

(3)

Then $\tilde{c}$ chosen as

$$\tilde{c} = \left(\sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_0 + \lambda_j}\right)^{-1} > 0 \quad \text{in the first case, and} \quad \tilde{c} = \left(\sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_0 - \lambda_j}\right)^{-1} < 0 \quad \text{in the second case.}$$

(4)

where first/second case refers to the left/right-hand side of (3), ensures that the optimal error bound is attained by $\tilde{c} \cdot \tilde{z} \tilde{z}^T$.

**Proof.** We employ a similar proof technique as for Thm. 4.5 in [2]. The crucial point is that $\|A - \tilde{c} \cdot \tilde{z} \tilde{z}^T\|_2 = \lambda_0$ if and only if both of the auxiliary matrices $M_1(\lambda_0) := \lambda_0 I - A + \tilde{c} \cdot \tilde{z} \tilde{z}^T : V V^T$ and $M_2(\lambda_0) := \lambda_0 I + A - \tilde{c} \cdot \tilde{z} \tilde{z}^T : V V^T$ are positive semidefinite (psd) and at least one of them actually possesses the eigenvalue zero. In the first case of (3), $M_1(\lambda_0)$ is psd for any $\tilde{z}$ and $\tilde{c} > 0$. $M_2(\lambda_0)$ is psd if and only if $v_j^T \tilde{z} = 0$ for all $j$ with $\lambda_j = -\lambda_0$. The particular choice of $\tilde{c}$ ensures that $M_2(\lambda_0)$ possesses the eigenvalue zero. In the second case of (3) the roles of $M_1(\lambda_0)$ and $M_2(\lambda_0)$ are interchanged.
Remark 2.2 The precise choices of \( \tilde{c} \) given in Thm. 2.1 are sufficient but not necessary. The optimal error bound being attained only implies that \( \tilde{c} \) is in the range between zero and the respective value from (4). This is because we cannot determine whether the matrix \( M_1(\lambda_0) \) in the first case, or \( M_2(\lambda_0) \) in the second case of (3) possesses the eigenvalue zero or is in fact strictly positive definite. If the positive (respectively negative) largest eigenvalue actually has higher multiplicity (e.g. \( \lambda_0 = \lambda_1 > 0 \)) then \( M_1(\lambda_0) \) (respectively \( M_2(\lambda_0) \)) does have the eigenvalue zero and we can choose \( \tilde{c} \) in the range between zero and the respective value of (4). This observation contributes to Cor. 2.3, and is illustrated in Ex. 3.1.

The conditions on \( \tilde{z} \) from Thm. 2.1 are trivially satisfied for any \( \tilde{z} \in \mathbb{R} \) if all by modulus largest eigenvalues of \( A \) occur with the same sign, see Ex. 3.1.

Corollary 2.3 Assume that all eigenvalues \( \lambda_j \) of \( A \) with \( |\lambda_j| = \|A\|_2 = |\lambda_0| \) have the same sign, and \( \lambda_0 > 0 \). Then for every \( \tilde{z} \in \mathbb{R} \) with \( \tilde{c} \) chosen in the range

\[
0 < \tilde{c} \leq \left( \sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_0 + \lambda_j} \right)^{-1}
\]

the matrix \( \tilde{c} \cdot \tilde{z} \tilde{z}^T \) is an optimal rank-1 Hankel approximation of \( A \) attaining the optimal error bound.

3 Examples

Example 3.1 Consider the matrix \( A = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix} \) with eigenvectors \( v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, v_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \)
corresponding to the ordered eigenvalues \( \lambda_0 = 11, \lambda_1 = 11, \lambda_2 = 1 \). According to Cor. 2.3, for any \( \tilde{z} \in \mathbb{R} \) we compute the upper bound on \( \tilde{c} \) from (5). Each pair \((\tilde{c}, \tilde{z})\) in the grey area of Fig. 1 admits the optimal error bound \( \|A - \tilde{c} \cdot \tilde{z} \tilde{z}^T\|_2 = 11 \).

![Fig. 1: Dark grey line: upper bound on the optimal coefficient \( \tilde{c} = \tilde{c}(\tilde{z}) \) as in (5). Light grey area: admissible pairs of parameters \((\tilde{c}, \tilde{z})\) generating optimal rank-1 Hankel approximations of \( A \) from Ex. 3.1.](image)

If neither of the conditions in (3) can be satisfied for any \( \tilde{z} \in \mathbb{R} \) then there is no real Hankel approximation of true rank 1 optimally approximating the matrix \( A \), but only the trivial solution of a zero-matrix, as the following exemplifies.

Example 3.2 Let \( A = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \) with eigenvectors \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \)
denoted \( v_0 \) through \( v_4 \), respectively, and corresponding eigenvalues \( \lambda_0 = \lambda_1 = \lambda_2 = 1 \) and \( \lambda_3 = \lambda_4 = -1 \). Then neither of the two systems of equations

\[
\begin{align*}
v_1^T z &= 0 & \iff & & z + z^3 &= 0 \quad \text{and} \quad v_3^T z &= 0 & \iff & & 1 - z^4 &= 0 \\
v_4^T z &= 0 & \iff & & 1 + z^4 &= 0 \quad \text{and} \quad v_5^T z &= 0 & \iff & & z - z^3 &= 0 \\
\end{align*}
\]

corresponding to the first and second set of conditions in (3), respectively, has a joint solution in \( \mathbb{R} \). Note that neither \( z(\infty) = (0, 0, 0, 0, 1)^T \) is a solution to either of the systems. So in this example, a solution to problem (2) of true rank 1 does not exist.

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