Reality of Complex Affine Toda Solitons

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There are infinitely many topological solitons in any given complex affine Toda theories and most of them have complex energy density. When we require the energy density of the solitons to be real, we find that the reality condition is related to a simple “pairing condition.” Unfortunately, rather few soliton solutions in these theories survive the reality constraint, especially if one also demands positivity. The resulting implications for the physical applicability of these theories are briefly discussed.
1 Introduction

Toda field theories (TFT’s) [1–14] provide a large class of integrable two–dimension models which include not only conformal field theories (CFT’s) but also massive deformations away from conformality. TFT’s are based on an underlying Kac-Moody algebra $g$. If $g$ is a finite-dimensional, semi-simple Lie algebra, the theory is a CFT. But if $g$ is an affine Kac-Moody loop algebra, then the affine Toda field theory (ATFT) is massive, but still integrable, and these theories are explicit examples of Zamolodchikov’s [15] integrable massive deformations of CFT’s. In order to be deformations of unitary CFT’s, it is required [1–4], somewhat paradoxically, that the coupling be purely imaginary in the ATFT.

In this letter, we are mostly concerned with these complex ATFT’s, i.e., ATFT’s with imaginary coupling constant. In [1, 3, 8, 9], explicit soliton solutions have been obtained by using Hirota’s method [16]. Soliton solutions can also be found using a vertex operator method [10, 11, 12]. Complex ATFT’s are generalizations of the sine-Gordon theory. However, unlike sine-Gordon theory, which has only one soliton and one anti-soliton, here we have infinitely many solitons parameterized by an integration constant $\xi$. This is undesirable since it would be difficult to deal with infinitely many particles of the same mass. To make matters worse, most of them have complex energy density. The problems of infinitely many solitons and complex energy density are related. One would like a physical principle to select at most one soliton for each given topological sector, and one hopes that soliton will have real energy density. The physical principle we are going to use in this paper is the reality of the energy density, since complex energy density defies physical interpretation. It turns out that for each topological sector there is at most one “real” soliton, and the reality is associated with some “pairing condition.”
The Lagrangian density of affine Toda field theory can be written in the form
\[ L = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{2}{\psi^2} \frac{m^2}{(1\beta)^2} \sum_{j=0}^{r} n_j (e^{i\beta \alpha_j \cdot \varphi} - 1) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - U(\varphi), \quad (1.1) \]
and Hamiltonian density is given by:
\[ \mathcal{H} = \frac{1}{2} (\partial_t \varphi)^2 + \frac{1}{2} (\partial_x \varphi)^2 + U(\varphi). \quad (1.2) \]

The field \( \varphi(x, t) \) is an \( r \)-dimensional vector, \( r \) being the rank of some finite-dimensional semi-simple Lie algebra \( g \). The \( \alpha_j \)'s, for \( j = 1, \ldots, r \), are the simple roots of \( g \); \( \psi \) is the highest root: \( \psi = \sum_{j=1}^{r} n_j \alpha_j \), where the \( n_j \)'s are positive integers, and \( n_0 = 1 \). \( \alpha_0 = -\psi \) is the extended root of \( \hat{g} \), the affine extension of \( g \). The coupling constant is \( i\beta \), \( m \) is the mass parameter, and \( U(\varphi) \) is called the Toda potential. Note, in general repeated indices are not summed over in this paper.

The equations of motion are
\[ \varphi = -\frac{2}{\psi^2} \frac{m^2}{1\beta} \sum_{j=0}^{r} n_j \alpha_j e^{i\beta \alpha_j \cdot \varphi}, \quad (1.3) \]
where \( \varphi \) can be written as
\[ \varphi = \sum_{j=1}^{r} (2\alpha_j / \alpha_j^2) \varphi^j. \]

Hirota’s method consists of a nonlinear transformation of variables so that the new equations of motion can be written in Hirota’s bilinear form, and then the equations can be solved perturbatively. Because of the integrability, the perturbative series terminates at some finite order. The transformation for these theories is:
\[ \varphi^j = -\frac{1}{i\beta} (\ln \tau_j - l_j \ln \tau_0), \quad (1.4) \]
where \( l_j = (\alpha_j^2 / \psi^2) n_j \) forms an integer null vector of the extended Cartan matrix \( K_{jk} = 2\alpha_j \cdot \alpha_k / \alpha_k^2 \). For each non-zero eigenvalue \( \lambda \) of the matrix \( L_{jk} = l_j K_{jk} \), one can find a single
soliton solution of the form:
\[ \tau_j(x,t) = 1 + \sum_{k=1}^{\kappa_j} \delta^{(k)}_j e^{k \Gamma}, \]  
(1.5)
where \( \Gamma = \gamma(x - vt) + \xi \), \( v \) is the velocity of the soliton, and \( \gamma \) satisfies \( \gamma^2 (1 - v^2) = m^2 \lambda \), and is assumed to be positive for simplicity of discussion. \( \kappa \) is some integer \([8]\) which has possible values 1, 2, and 3. \( \xi \) is an integration constant, and is in general complex.

Since for single-soliton solutions of the form (1.5), all the fields depend on \( x \) and \( t \) only through \( \Gamma \), we can rewrite the Hamiltonian as:
\[ H = \frac{m^2 \lambda (1 + v^2)}{2(1 - v^2)} \left( \frac{d \varphi}{d \Gamma} \right)^2 + U(\varphi(\Gamma)). \]  
(1.6)
Since the Hamiltonian density should transform as a component of a second-rank tensor under Lorentz transformations, by relating the Hamiltonian of a soliton with velocity \( v \) to one at rest, we find
\[ \frac{1}{2} m^2 \lambda \left( \frac{d \varphi}{d \Gamma} \right)^2 = U(\varphi(\Gamma)). \]
In particular, when \( v = 0 \), we have:
\[ \frac{1}{2} (\partial_x \varphi)^2 = U(\varphi). \]  
(1.7)

We will make some simplifying choices of parameters. Since \( \text{Re}(\xi) \) only affects the position of the center of mass of the soliton, it can be absorbed by a shift in \( x \). Furthermore, we want to study the effect of \( \text{Im}(\xi) \) on topological charge and other properties of the soliton, so we will take \( \xi \) to be purely imaginary: \( \xi = i \theta \). Since we will only discuss the classical solution, the mass parameter \( m \) and the coupling parameter \( \beta \) can always be scaled to 1, \( m = \beta = 1 \). Finally, because the velocity of a soliton does not affect the reality of the energy density, we will take \( v = 0 \) whenever discussing single solitons.

For single-soliton solutions, the \( \tau_j \)'s can always be factorized to have simple factors of the form \( z_0 = 1 + c e^{\gamma x + i \theta} \), where \( c \) is a complex number. As \( x \) goes from \(-\infty\) to \(+\infty\), \( z_0 \) traces
out a straight line on the complex $z$-plane, starting from $z = 1$. Since $\ln z$ is a multivalued function in the complex domain, we will take the convention that $\ln z_0(x \to -\infty) = 0$. Then as long as the line does not go through the origin, $\ln z_0$ will behave as a continuous and single-valued function. $\theta$ will be called singular if and only if $\tau_j$ goes through the origin of the complex plane for some $j$. Singular points divide the unit circle into many continuous sectors.

The topological charge of a soliton is defined as:

$$T = \frac{\beta}{2\pi} \int_{-\infty}^{+\infty} \partial_x \varphi(x, t) = \frac{\beta}{2\pi} \left( \lim_{x \to \infty} \varphi(x, t) - \lim_{x \to -\infty} \varphi(x, t) \right).$$  \hspace{1cm} (1.8)

In the above convention, we have $\lim \varphi(x \to -\infty, t) = 0$, so

$$T_j = \frac{i}{2\pi} \lim_{x \to \infty} (\ln \tau_j(x, t) - l_j \ln \tau_0(x, t)).$$  \hspace{1cm} (1.9)

As long as $\theta$ is not singular, all fields will be continuous functions of $\theta$, and the topological charge should be as well, except that the topological charge can only take discrete values. Hence all $\theta$ in one continuous sector will have the same topological charge, and each sector on the unit circle corresponds to a topological sector of the soliton solutions. For each given class of soliton solutions, the number of possible topological charges is no more than the number of singular points. Explicit topological charges of $su(n)$ affine TFT’s have be given in [4].

In the following sections we will discuss for various classes of ATFT’s which solitons have real (and positive) energy density.

## 2 su(n) and sp(2n)

The energy density of $su(n)$ affine TFT can be written in terms of $\tau$-functions as:

$$\mathcal{H} = \sum_{j=0}^{n-1} \left[ \frac{\tau_j'}{\tau_j} \left( \frac{\tau_{j+1}}{\tau_{j+1}} - \frac{\tau_j}{\tau_j} \right) - \frac{\hat{\tau}_j}{\tau_j} \left( \frac{\tau_{j+1}}{\tau_{j+1}} - \frac{\hat{\tau}_j}{\hat{\tau}_j} \right) + \left( 1 - \frac{\tau_{j+1}}{\tau_{j+1}} \right) \right].$$  \hspace{1cm} (2.1)
For each given integer $1 \leq s \leq n - 1$, there is a class of solutions given by:

$$\tau_j = 1 + e^{2\pi isj/n + \Gamma}, \quad \text{for } 0 \leq j \leq n - 1,$$

(2.2)

where $\Gamma = \gamma x + i\theta$, and $\gamma = \sqrt{\lambda_s} = 2 \sin(s\pi/n)$, and we have assumed zero velocity. It is clear that $\theta$ is singular if and only if

$$\theta + 2\pi sj/n = \pi \mod 2\pi, \quad \text{for some integer } j.$$

Singular points are uniformly spaced on the unit circle and divide it into equal sectors. Since $\theta = \pi$ is always a singular point, singular points are symmetrical about the real axis. For the $\tau$-functions in (2.2), we have:

$$\tau_j' = \gamma(\tau_j - 1), \quad \tau_{j-1}\tau_{j+1} = \tau_j^2 - \lambda_s(\tau_j - 1), \quad \sum_j \frac{1}{\tau_j\tau_{j+1}} = \sum_j \frac{1}{\tau_j},$$

(2.3)

so the energy-density function can be simplified to:

$$\mathcal{H} = 2\lambda_s \sum_{j=0}^{n-1} \frac{\tau_j - 1}{\tau_j^2}.$$  

(2.4)

First we will show that the above Hamiltonian density $\mathcal{H}$ is real if $\theta$ is singular or is at the mid-point of a sector on the unit circle. We accomplish this by demonstrating that for each term in $\mathcal{H}$, we can find another term conjugate to it, i.e., for each $j$, we can find a $j'$ such that

$$\tau_j = \tau_{j'}^*.$$  

(2.5)

The above equation will be referred to as the “pairing condition.” For solitons given in (2.2) it is equivalent to $2\pi sj(j + j')/n + 2\theta = 0 \mod 2\pi$, or geometrically, the points $2\pi sj'/n + \theta$ and $2\pi sj/n + \theta$ should be symmetrical about the real axis on the unit circle.

Let $\theta$ be a mid-point of some sector. Then for every integer $j$, $2\pi sj/n + \theta$ is also a mid-point for some sector. Since the singular points are distributed symmetrically about the
real axis, the mid-points are also, therefore the pairing condition can be satisfied if \( \theta \) is a singular point or a mid-point. Even though the singular points give real-energy densities, the singularity renders them unacceptable. All the mid-points can be written in the form: \( \theta = \pi s(2j + 1)/n - \pi \), for some integer \( j \). When \( s \) and \( n \) are mutually prime, these \( \theta \)-values coincide with the examples discussed in [1, 4].

Next we will show that these are the only \( \theta \)-values which will give real-energy densities. To prove this, we note that if \( \mathcal{H} \) is real for all \( x \), then \( \partial_x \mathcal{H} \) is also real for all \( x \), in particular for \( x = 0 \). It turns out that the following equation

\[
\text{Im}\left(\partial_x \mathcal{H}(x = 0, \theta)\right) = 0, \quad (2.6)
\]

is restrictive enough to select only the mid-points of some continuous sectors. Since

\[
\partial_x \mathcal{H}(x, \theta) = 2\lambda_s \sum_{j=0}^{n-1} \frac{(2 - \tau_j)\tau_j'}{\tau_j^3}, \quad (2.7)
\]

at \( x = 0 \), \( \tau_j(x = 0) = 1 + e^{i2\pi j s/n + i\theta} = 2 \cos(\frac{j\pi}{n} + \frac{\theta}{2}) \exp\left(i(\frac{j\pi}{n} + \frac{\theta}{2})\right) \), equation (2.6) can be simplified to:

\[
\frac{d}{d\theta} \sum_{j=0}^{n-1} \frac{1}{\cos^2(\pi j s/n + \theta/2)} = 0. \quad (2.8)
\]

To prove that the summation in the above equation has only mid-points as its minima, we reorganize it to pair up the sectors symmetric about the real axis. The partial sum of the two paired sectors can be written in the form:

\[
\frac{1}{\cos^2(\theta_0/2 + \epsilon/2)} + \frac{1}{\cos^2(\theta_0/2 - \epsilon/2)},
\]

where \( \theta_0 \) can be chosen as the mid-point of the sector above the real axis, and \( \epsilon \) is the distance from \( \theta \) to the mid-point of the sector it is in. Because the function \( 1/\cos^2 x \) is concave in the domain \((0, \pi/2)\), the sum of the pair will increase monotonically as \( \theta \) moves away from a mid-point towards a closest singular point. One or more pairs will blow up as \( \theta \) reaches a singular point, so the mid-point is a minimum point for every pair, therefore a minimum for the whole sum in (2.8). It is also obvious that the mid-points are the only non-singular extrema for the sum.
When $\theta$ is taken to be a mid-point, the energy density is not only real but also positive definite, which can be seen as follows.

$$\mathcal{H} = \lambda_s \sum_j \frac{1 + y \cos \theta_j}{(y + \cos \theta_j)^2}, \quad (2.9)$$

where $y = \cosh \gamma x \geq 1$, and the summation runs over all the mid-points. Since the set of mid-points only depends on the number of sectors on the unit circle, we only need to do the summation for the class $s = 1$. The positive definiteness can be written down explicitly as:

$$\sum_{j=1}^{n} \frac{1 - y \cos \frac{(2j-1)\pi n}{n}}{(y - \cos \frac{(2j-1)\pi n}{n})^2} \geq 0, \quad \text{for } y \geq 1. \quad (2.10)$$

For $n \leq 10$, we can do the above summation by “brute force.” The summation can be written in the form of a positive constant divided by a number of positive-definite factors, therefore it is positive definite itself. With the aid of numerical calculations, we have found that this pattern persists for $n$’s up to 50. We believe that this pattern is true for all the $n$’s, but because the number of factors one has to deal with grows exponentially with $n$, a general analytical proof has eluded us. A side conclusion from this is that the soliton energy density profile is of the form $1/\cosh gx$ rather than Gaussian for $|x| >> 1$.

If $\tau_0$ is real, i.e. if $\theta = 0$, then from (1.4) and $\tau_j^* = \tau_{j'}$, we will have $\varphi_j^* = -\varphi_{j'}$, which is the “twisted reality condition” in [1, 4]. However, most of the real-energy-density solutions do not satisfy the twisted reality condition. One simple way to see this is that there are $n$ vertices in the Dynkin diagram, but there are only $n - 1$ non-trivial components of the $\varphi$ field. We also point out that the pairing for the $su(n)$ affine Toda solitons can always be implemented according to some reflection symmetry of the underlying Dynkin diagram, even though it is not necessary. In general, the existence of symmetries of Dynkin diagrams is neither necessary nor sufficient to generate non-singular real-energy-density solitons.

From the above discussion, we see that the reality of the Hamiltonian for a single soliton is dependent on choices of the parameter $\theta$, and is not protected by any symmetry principles.
Hence it could be destroyed by perturbations or scattering with other solitons. The reality of a two-soliton scattering solution can be discussed in a similar fashion. The two-soliton scattering solutions of $su(n)$ affine Toda solitons are given by:

$$
\tau_j = 1 + e^{\frac{i2\pi s_1 j}{n} + \Gamma_1} + e^{\frac{i2\pi s_2 j}{n} + \Gamma_2} + A e^{\frac{i2\pi (s_1 + s_2) j}{n} + \Gamma_1 + \Gamma_2},
$$

(2.11)

where $A = \frac{\cosh \omega - \cos \frac{\pi(s_1 - s_2)}{n}}{\cosh \omega - \cos \frac{\pi(s_1 + s_2)}{n}}$, and $\omega$ is the rapidity difference of the two solitons. First we impose the pairing condition on the $\tau$ functions, which can be satisfied if and only if we can find an integer $j$ such that

$$
\theta_1 = \frac{s_1(2j + 1)}{n} \pi - \pi \mod 2\pi, \quad \theta_2 = \frac{s_2(2j + 1)}{n} \pi - \pi \mod 2\pi,
$$

(2.12)

i.e., the scattering solution can be paired if the two solitons can be paired simultaneously. From (2.1), one sees that the energy density will be real if the solution can be paired. This observation can also be generalized to any N-soliton solutions. The necessary part is harder to prove, since one has to do the summation. It is enough to point out that there are explicit examples of scattering solutions having complex energy density even though each participating soliton is real. Furthermore, even when the energy density of the scattering solution is real, there are cases where it is no longer positive definite.

Since the soliton solutions of $sp(2n)$ affine TFT can be unfolded [17, 6] to soliton solutions of $su(2n)$ affine TFT, we do not have to discuss them completely separately. The single-soliton solutions of $sp(2n)$ are given by:

$$
\tau_j = 1 + 2 \cos \frac{\pi s_j}{n} e^{\Gamma} + \cos^2 \frac{\pi s_j}{2n} e^{2\Gamma} \quad \text{for } 0 \leq j \leq n.
$$

(2.13)

We can unfold each of these to a $su(2n)$ affine Toda soliton by extending the range of $j$ in the above equation to $0 \leq j \leq 2n$, which gives an $\kappa = 2$ soliton of $su(2n)$ affine TFT, and its energy density is unaffected by the unfolding process. Since all the coefficients in (2.13) are real, it is clear that the only possible values of $\theta$ that could satisfy the pairing condition
are: $\theta = 0, \pm \pi/2, \pi$. For $\theta = 0, \pi$, the solutions (2.13) are always singular. For $\theta = \pm \pi/2$, the pairing condition can be simplified to the following question: for an integer $j_1$, can we find a $j_2$ such that $\cos(\pi s j_1/n) = -\cos(\pi s j_2/n)$? Let $\frac{s}{n} = \frac{p}{q}$, with $p$ and $q$ mutually prime, if $p$ is odd, then the pairing can be done by letting $j_1 + j_2 = kq$ for some odd integers $k$. But if $q$ is even, the solution (2.13) is singular, so the final conditions for non-singular pairing are: $\theta = \pm \pi/2$ and $p$ and $q$ are odd. Going back to (2.1), we can show that the energy density is real for these cases. Because the condition of real-energy density is infinitely over constrained, we believe that there are no other non-singular real-energy-density soliton solutions. There is strong numerical evidence supporting this.

3 so(8)

For $\lambda = 2$, there are three $\kappa = 1$ single-soliton solutions:

$$
\tau_0 = \tau_1 = 1 + e^\Gamma, \quad \tau_3 = \tau_4 = 1 - e^\Gamma, \quad \tau_2 = 1 + e^{2\Gamma},
$$

(3.1)

and permutations of indices (1, 3, 4) of the above solution. All these three solutions have the same energy density:

$$
\mathcal{H} = -16\frac{6 + e^{2\Gamma} + e^{-2\Gamma}}{(e^{2\Gamma} - e^{-2\Gamma})^2}.
$$

(3.2)

Letting $\Gamma = \gamma x + i\theta$, where $\gamma = \sqrt{2}$, then:

$$
\text{Im}(\mathcal{H}) = 16 \sin 2\theta \frac{(e^{2\gamma x} - e^{-2\gamma x})(4 \cos^2 \theta + 12(e^{-2\gamma x} + e^{2\gamma x}) \cos \theta + (e^{-2\gamma x} + e^{2\gamma x})^2)}{(e^{4\gamma x} + e^{-4\gamma x} - 2 \cos 2\theta)^2}.
$$

(3.3)

Demanding $\mathcal{H}$ be real, i.e. $\text{Im}(\mathcal{H}) = 0$ for all $x$, then $\sin 2\theta = 0$, or $\theta = 0, \pi, \pm \pi/2$. One can also easily check that they are also the singular points.

For $\lambda = 6$, the single-soliton solution is given by:

$$
\tau_0 = \tau_1 = \tau_3 = \tau_4 = 1 + e^\Gamma, \quad \tau_2 = 1 - 4e^\Gamma + e^{2\Gamma}.
$$

(3.4)
Since \( \varphi_2 \) is the only non-vanishing component of \( \varphi \), the energy density can now be written as:

\[
\mathcal{H} = 2 (\varphi'_2(x))^2 = -12 \left( \frac{-4e^r + 2e^{2r}}{1 - 4e^r + e^{2r}} - \frac{2e^r}{1 + e^r} \right)^2.
\] (3.5)

\( \theta = 0, \pi \) are the only choices for reality of \( \mathcal{H} \), and again they are also singular points.

We have shown explicitly that there are no non-singular real-energy-density solitons with \( \kappa = 1 \) in so\( (8) \) affine TFT. This can also be demonstrated by the fact that the singular points are the only points which can satisfy the pairing condition for these solutions. Because of the degeneracy in eigenvalues, we can also have \( \kappa = 2 \) or 3 solitons. It has been shown in [9] that these high-\( \kappa \) solutions are the static limit of scattering solutions of some \( \kappa = 1 \) solitons.

There are no physical solitons for \( g_2 \) affine TFT either.

4 so\( (2r) \)

For so\( (2r) \) affine TFT, the single-soliton energy density can be written in terms of tau-functions as:

\[
\mathcal{H} = \frac{1}{2} \sum_{j,k=1}^{r} K_{jk} \varphi'_j \varphi'_k + U(\varphi) = \sum_{j,k=1}^{r} K_{jk} \varphi'_j \varphi'_k \\
= 2 \sum_{j=2}^{r-2} \varphi'^2_j - 2 \sum_{j=2}^{r-3} \varphi'_j \varphi'_{j+1} + 2 \left( \varphi'^2_1 + \varphi'^2_r + \varphi'^2_{r-1} - \varphi'_1 \varphi'_2 - \varphi'_r \varphi'_{r-2} - \varphi'_{r-1} \varphi'_{r-2} \right) \\
= -2 \sum_{j=0}^{r} \left( \frac{\tau'_j}{\tau^2_j} + 2 \sum_{j=2}^{r-3} \frac{\tau'_j \tau'_{j+1}}{\tau_j \tau_{j+1}} + 2 \left( \frac{\tau'_0}{\tau_0} + \frac{\tau'_1}{\tau_1} \right) \frac{\tau'_2}{\tau_2} + 2 \left( \frac{\tau'_r}{\tau_r} + \frac{\tau'_{r-1}}{\tau_{r-1}} \right) \frac{\tau'_{r-2}}{\tau_{r-2}} \right). (4.1)
\]

For \( \lambda = 2 \), there are two independent soliton solutions:

\[
\tau_0 = 1 + e^r, \quad \tau_1 = 1 - e^r, \quad \tau_r = 1 \pm (-1)^{\frac{r}{2}} e^r, \quad \tau_{r-1} = 1 \mp (-1)^{\frac{r}{2}} e^r, \\
\tau_j = 1 + (-1)^j e^{2r} \quad (2 \leq j \leq n - 2). (4.2)
\]

By applying the pairing condition, we see that: when \( r \) is even, \( \theta = 0, \pi, \pm \pi/2 \); when \( r \) is odd, \( \theta = 0, \pi, \pm \pi/2, \pm \pi/4, \pm 3\pi/4 \), are the only solutions, and \( \theta = 0, \pi, \pm \pi/2 \) are always
singular. By evaluating the energy density explicitly, we can see that these are sufficient and necessary conditions for the solitons to have real-energy density.

For $\lambda_s = 8 \sin^2 \vartheta_s$, where $\vartheta_s = \frac{s\pi}{2(r-1)}$, $1 \leq s \leq r - 2$, the solutions are given as:

$$
\tau_0 = \tau_1 = 1 + e^\Gamma, \quad \tau_r = \tau_{r-1} = 1 + (-1)^s e^\Gamma,
$$

$$
\tau_j = 1 + \frac{2\cos((2j - 1)\vartheta_s)}{\cos \vartheta_s} e^\Gamma + e^{2\Gamma}, \quad \text{for} \ 2 \leq j \leq r - 2. \quad (4.3)
$$

Since all coefficients in the above equation are explicitly real, the only possible solutions to the pairing conditions are $\theta = 0, \pi, \pm \pi/2$. $\theta = 0$ or $\pi$ obviously give real energy densities since all the $\tau$-functions are real, therefore every term in $\mathcal{H}$ is real; however, they are also always singular. For $\theta = \pm \pi/2$, the pairing condition simplifies to the following question: for a given integer $j_1$, can we find an integer $j_2$ such that $\cos(2j_2 - 1)\vartheta_s = -\cos(2j_1 - 1)\vartheta_s$?

Let $\frac{s}{r-1} = \frac{p}{q}$, where $p$ and $q$ are mutually prime, then the pairing can be done if and only if $p$ is odd, by $j_1 + j_2 - 1 = kq$ for some odd integer $k$. The energy density is singular if $q$ is also odd.

Now let us argue that these conditions are sufficient and necessary. In the case when $s$ and $r - 1$ are mutually prime, the terms can be simply paired up as complex conjugate by the the pairing condition, therefore the energy density is real. If $s$ and $r - 1$ are not mutually prime, let $s = np$, and $r - 1 = nq$, then the energy density can be reduced to $n$ copies of the energy density of a class $p$ soliton in $so(2q + 2)$ affine TFT. Thus we have proved the sufficiency. Necessity of the condition is harder to prove directly, so we turn to numerical verification.

Since the energy density is a continuous function of $\theta$ for most $x$, let us pick five $x$ values, and see which $\theta$ value can give real energy density for all five points simultaneously. We searched all the classes of solitons up to $r \leq 50$. We checked our numerical calculations against the analytic equation (1.7), and the total energy (mass) gives the expected value.
All $\theta$-values which satisfy the pairing conditions do give real energy density, and no new $\theta$ values showed up. Given that the system is infinitely over constrained, this should not be a surprise.

The reality of the energy density for multi-soliton solutions can be similarly discussed. One can see that the reality here is also related to the pairing condition.

5 e-series

The soliton solutions of $e_6$, $e_7$ and $e_8$ complex affine Toda theories have been tabulated in [6] and will not be given here.

There are six $\kappa = 1$ solitons for the $e_6$ affine TFT. Let us apply the pairing condition to these solitons. For the solitons with eigenvalue $\lambda = 2(3 \pm \sqrt{3})$, all the $\delta$-coefficients are real, and the only $\theta$-values are 0 or $\pi$, both values being singular. For $\lambda = 3 - \sqrt{3}$ solitons, the possible $\theta$-values are $0, \pi, \pm \pi/3, \pm 2\pi/3$. It is easy to verify that $\pi$ and $\pm \pi/3$ are singular. For $\lambda = 3 + \sqrt{3}$ solitons, the pairing condition gives the same set of possible $\theta$-values as for $\lambda = 3 - \sqrt{3}$ solitons; however, in this case they are all singular. Numerical analysis indicates that the energy density is real if and only if the pairing condition is satisfied, and the energy density is not positive definite even for the non-singular $\theta$-values.

For the $e_7$ affine TFT there are seven soliton solutions. For all these solitons, $\theta = 0, \pi$ gives real but singular energy density. The pairing condition can also be satisfied when $\theta = \pm \pi/2$ for $\lambda = 8 \sin^2(\pi/9), 8 \sin^2(2\pi/9), 8 \sin^2(4\pi/9)$ solitons. Only the $\lambda = 8 \sin^2(\pi/9)$ soliton gives non-singular and positive-definite energy density. There are no other real-energy-density solitons, at least from the evidence of numerical studies.

For the $e_8$ affine TFT, there are eight soliton solutions, and for all of them, $\theta = 0, \pi$ satisfies the pairing condition and gives real energy-density; however, they are all singular. Numerical analysis again indicates that there are no non-singular, real-energy-density
solitons in this theory.

6 Conclusions

We studied the question of whether all mathematical solutions of complex affine Toda theories are physically acceptable, and discussed the consequences of demanding real-energy density for these soliton solutions.

Because there is this problem that the energy density is in general complex, one may consider other alternatives besides imposing reality as a condition: we may simply ignore the imaginary part, or take the norm of the complex energy density, or “improve” the energy density such that it becomes real for all the solutions. Each alternative has its shortfalls: the real part of the energy density is not necessarily positive definite, the norm of the complex energy density is not conserved, and since the Lagrangian is intrinsically complex, the “improving” approach looks impossible at this time. The reality condition is in some sense too strong since many topological sectors have no such solitons, yet it is not strong enough since the energy density may still be negative. Furthermore, scattering of two “real” solitons may destroy the reality of their combined energy density. Even though the energy density for solitons are in general complex, the total energy is always positive and finite, except when the energy-density function is singular.

We have focused our attention on the $\kappa = 1$ solitons of affine TFT’s based on simply-laced Lie algebras, since higher-$\kappa$ solitons for the simply-laced affine TFT’s can be taken as the static limit of some $\kappa = 1$ scattering solutions, and solitons of non-simply-laced affine TFT’s can be unfolded to solitons of their simply-laced counterparts. We have shown case by case that the pairing condition is sufficient to guarantee the reality of the energy density; we also presented strong evidence that it is also necessary.

In conclusion, we have found that the existence of real-energy-density solitons in complex
affine Toda field theories is highly non-trivial, and is not related to symmetries of the underlying Dynkin diagrams. Since the Hamiltonian is intrinsically complex, we feel that the “on-shell” reality of the energy density for the classical solitons is a minimum requirement. Since the reality of the single solitons is not protected by any principle but imposed as an *ad hoc* condition, it could be destroyed by scattering or perturbation. In our opinion this problem deserves further investigation before any consistent quantization can be meaningfully discussed.

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