ON GLOBALLY SYMMETRIC FINSLER SPACES

R. CHAVOSH KHATAMY *, R. ESMALI

Abstract. The paper considers the symmetry of Finsler spaces. We give some conditions about globally symmetric Finsler spaces. Then we prove that these spaces can be written as a coset space of Lie group with an invariant Finsler metric. Finally, we prove that such a space must be Berwaldian.

1. Introduction

The study of Finsler spaces has important in physics and Biology ([5]). In particular there are several important books about such spaces (see [1], [8]). For example recently D. Bao, C. Robels, Z. Shen used the Randers metric in Finsler on Riemannian manifolds ([9] and [8], page 214). We must also point out there was only little study about symmetry of such spaces ([5], [12]). For example E. Cartan has been showed symmetry plays very important role in Riemannian geometry ([5] and [12], page 203).

Definition 1.1. A Finsler space is locally symmetric if, for any \( p \in M \), the geodesic reflection \( s_p \) is a local isometry of the Finsler metric.

Definition 1.2. A reversible Finsler space \((M, F)\) is called globally symmetric if for any \( p \in M \) there exists an involutive isometry \( \sigma_x \) (that is, \( \sigma_x^2 = I \) but \( \sigma_x \neq I \)) of such that \( x \) is an isolated fixed point of \( \sigma_x \).

Definition 1.3. Let \( G \) be a Lie group and \( K \) is a closed subgroup of \( G \). Then the coset space \( G/K \) is called symmetric if there exists an involutive automorphism \( \sigma \)
of $G$ such that
\[ G^0_\sigma \subset K \subset G_\sigma, \]
where $G_\sigma$ is the subgroup consisting of the fixed points of $\sigma$ in $G$ and $G^0_\sigma$ denotes the identity component of $G_\sigma$.

**Theorem 1.4.** Let $G/K$ be a symmetric coset space. Then any $G$-invariant reversible Finsler metric (if exists) $F$ on $G/K$ makes $(G/K, F)$ a globally symmetric Finsler space (\cite{8}, page 8).

**Theorem 1.5.** Let $(M, F)$ be a globally Symmetric Finsler space. For $p \in M$, denote the involutive isometry of $(M, F)$ at $p$ by $S_p$. Then we have

(a) For any $p \in M$, $(dS_p)_p = -I$. In particular, $F$ must be reversible.

(b) $(M, F)$ is forward and backward complete;

(c) $(M, F)$ is homogeneous. This is, the group of isometries of $(M, F), I(M, F)$, acts transitively on $M$.

(d) Let $\tilde{M}$ be the universal covering space of $M$ and $\pi$ be the projection mapping. Then $(\tilde{M}, \pi^*(F))$ is a globally Symmetric Finsler space, where $\pi^*(F)$ is define by
\[ \pi^*(F)(q) = F((d\pi)_p(q)), \quad q \in T_p(M), \]
(See \cite{8} to prove).

**Corollary 1.6.** Let $(M, F)$ be a globally Symmetric Finsler space. Then for any $p \in M$, $s_p$ is a local geodesic Symmetry at $p$. The Symmetry $s_p$, is unique. (See prove of Theorem 1.2 and \cite{12})

2. A theorem on globally Symmetric Finsler spaces

**Theorem 2.1.** Let $(M, F)$ be a globally Symmetric Finsler space. Then exists a Riemannian Symmetric pair $(G, K)$ such that $M$ is diffeomorphic to $G/K$ and $F$ is invariant under $G$.

**Proof.** The group $I(M, F)$ of isometries of $(M, F)$ acts transitively on $M$. $(C)$ of theorem 1.5. We proved that $I(M, F)$ is a Lie transformation group of $M$ and for any $p \in M$ (\cite{12} and \cite{7}, page 78), the isotropic subgroup $I_p(M, F)$ is a compact subgroup of $I(M, F)$ (\cite{11}). Since $M$ is connected (\cite{7}, \cite{10}) and the subgroup $K$ of $G$ which $p$ fixed is a compact subgroup of $G$. Furthermore, $M$ is diffeomorphic to $G/K$ under the mapping $gH \rightarrow g.p$, $g \in G$ (\cite{7} Theorem 2.5, \cite{10}).

As in the Riemannian case in page 209 of \cite{7}, we define a mapping $s$ of $G$ into $G$ by $s(g) = s_pgs_p$, where $s_p$ denote the (unique) involutive isometry of $(M, f)$ with $p$ as an isolated fixed point. Then it is easily seen that $s$ is an involutive automorphism of $G$ and the group $K$ lies between the closed subgroup $K_s$ of fixed points of $s$ and
the identity component of $K_s$ (See definition of the symmetric coset space, [11]). Furthermore, the group $K$ contains no normal subgroup of $G$ other than \{e\}. That is, $(G, K)$ is symmetric pair. $(G, K)$ is a Riemannian symmetric pair, because $K$ is compact.

The following useful will be results in the proof of our aim of this paper.

**Proposition 2.2.** Let $(M, \bar{F})$ be a Finsler space, $p \in M$ and $H_p$ be the holonomy group of $\bar{F}$ at $p$. If $F_p$ is a $H_p$ invariant Minkowski norm on $T_p(M)$, then $F_p$ can be extended to a Finsler metric $F$ on $M$ by parallel translations of $\bar{F}$ such that $F$ is affinely equivalent to $\bar{F}$ ([5], proposition 4.2.2)

**Proposition 2.3.** A Finsler metric $F$ on a manifold $M$ is a Berwald metric if and only if it is affinely equivalent to a Riemannian metric $g$. In this case, $F$ and $g$ have the same holonomy group at any point $p \in M$ (see proposition 4.3.3 of [5]).

Now the main aim

**Theorem 2.4.** Let $(M, F)$ be a globally symmetric Finsler space. Then $(M, F)$ is a Berwald space. Furthermore, the connection of $F$ coincides with the Levi-civita connection of a Riemannian metric $g$ such that $(M, g)$ is a Riemannian globally symmetric space.

**Proof.** We first prove $F$ is Beraldian. By Theorem 2.1, there exists a Riemannian symmetric pair $(G, K)$ such that $M$ is diffeomorphic to $G/K$ and $F$ is invariant under $G$. Fix a $G$- invariant Riemannian metric $g$ on $G/K$. Without losing generality, we can assume that $(G, K)$ is effective (see [11] page 213). Since being a Berwald space is a local property, we can assume further that $G/K$ is simple connected. Then we have a decomposition (page 244 of [11]):

$$G/K = E \times G_1/K_1 \times G_2/K_2 \times \ldots \times G_n/K_n,$$

where $E$ is a Euclidean space, $G_i/K_i$ are simply connected irreducible Riemannian globally symmetric spaces, $i = 1, 2, \ldots, n$. Now we determine the holonomy groups of $g$ at the origin of $G/K$. According to the de Rham decomposition theorem ([2]), it is equal to the product of the holonomy groups of $E$ and $G_i/K_i$ at the origin. Now $E$ has trivial holonomy group. For $G_i/K_i$, by the holonomy theorem of Ambrose and Singer ([12], page 231, it shows, for any connection, how the curvature form generates the holonomy group), we know that the lie algebra $\eta_i$ of the holonomy group $H_i$ is spanned by the linear mappings of the form $\{\bar{\tau}^{-1}R_0(X, Y)\bar{\tau}\}$, where $\tau$ denotes any piecewise smooth curve starting from $o$, $\bar{\tau}$ denotes parallel displacements (with respect to the restricted Riemannian metric) a long $\bar{\tau}$, $\bar{\tau}^{-1}$ is the inverse of $\bar{\tau}$, $R_0$ is the curvature tensor of $G_i/K_i$ of the restricted Riemannian metric and $X, Y \in$
Consider the space $T_0(G_i/K_i)$. Since $G_i/K_i$ is a globally symmetric space, the curvature tensor is invariant under parallel displacements (page 201 of [10], [11]). So

$$\eta_i = \text{span}\{R_0(X,Y)|X,Y \in T_0(G_i/K_i)\},$$

(see page 243 of [7], [11]).

On the other hand, since $G_i$ is a semisimple group, we know that the Lie algebra of $K_i^* = \text{Ad}(K_i) \simeq K$ is equal to the span of $R_0(X,Y)$ ([11]). The groups $H_i$, $K_i^*$ are connected (because $G_i/K_i$ is simply connected) ([10] and [11]). Hence we have $H_i = K_i^*$. Consequently the holonomy group $H_0$ of $G/K$ at the origin is

$$K_1^* \times K_2^* \times \ldots \times K_n^*$$

Now $F$ defines a Minkowski norm $F_0$ on $T_0(G/K)$ which is invariant by $H_0$ ([2]). By proposition 2.2, we can construct a Finsler metric $\bar{F}$ on $G/K$ by parallel translations of $g$. By proposition 2.3, $\bar{F}$ is Berwaldian. Now for any point $p_0 = aK \in G/K$, there exists a geodesic of the Riemannian manifold $(G/K, g)$, say $\gamma(t)$ such that $\gamma(0) = 0, \gamma(1) = p_0$. Suppose the initial vector of $\gamma$ is $X_0$ and take $X \in p$ such that $d\pi(X) = X_0$. Then it is known that $\gamma(t) = \exp tX.p_0$ and $d\tau(\exp tX)$ is the parallel translate of $(G/K, g)$ along $\gamma$ ([11] and [7], page 208). Since $F$ is $G$-invariant, it is invariant under this parallel translate. This means that $F$ and $\bar{F}$ coincide at $T_{p_0}(G/K)$. Consequently they coincide everywhere. Thus $F$ is a Berwald metric.

For the next assertion, we use a result of Szabo’ ([2], page 278) which asserts that for any Berwald metric on $M$ there exists a Riemannian metric with the same connection. We have proved that $(M, F)$ is a Berwald space. Therefore there exists a Riemannian metric $g_1$ on $M$ with the same connection as $F$. In [11], we showed that the connection of a globally symmetric Berwald space is affine symmetric. So $(M, F)$ is a Riemannian globally symmetric space ([7], [11]).

From the proof of theorem 2.4, we have the following corollary.

**Corollary 2.5.** Let $(G/K, F)$ be a globally symmetric Finsler space and $g = \ell + p$ be the corresponding decomposition of the Lie algebras. Let $\pi$ be the natural mapping of $G$ onto $G/K$. Then $(d\pi)_e$ maps $p$ isomorphically onto the tangent space of $G/K$ at $p_0 = eK$. If $X \in p$, then the geodesic emanating from $p_0$ with initial tangent vector $(d\pi)_eX$ is given by

$$\gamma_{d\pi.eX}(t) = \exp tX.p_0.$$ 

Furthermore, if $Y \in T_{p_0}(G/K)$, then $(d\exp tX)_{p_0}(Y)$ is the parallel of $Y$ along the geodesic (see [11], [7] proof of theorem 3.3).

**Example 2.6.** Let $G_1/K_1$, $G_2/K_2$ be two symmetric coset spaces with $K_1$, $K_2$ compact (in this coset, they are Riemannian symmetric spaces) and $g_1, g_2$ be invariant
Riemannian metric on $G_1/K_1$, $G_2/K_2$, respectively. Let $M = G_1/K_1 \times G_2/K_2$ and $O_1, O_2$ be the origin of $G_1/K_1, G_2/K_2$, respectively and denote $O = (O_1, O_2)$ (the origin of $M$). Now for $y = y_1 + y_2 \in T_O(M) = T_{O_1}(G_1/K_1) + T_{O_2}(G_2/K_2)$, we define

$$F(y) = \sqrt{g_1(y_1, y_2) + g_2(y_1, y_2)} + s\sqrt{g_1(y_1, y_2)^s + g_2(y_1, y_2)^s},$$

where $s$ is any integer $\geq 2$. Then $F(y)$ is a Minkowski norm on $T_O(M)$ which is invariant under $K_1 \times K_2$ [4]. Hence it defines an $G$- invariant Finsler metric on $M$ [6, Corollary 1.2, of page 8246]. By theorem 2.1, $(M, F)$ is a globally symmetric Finsler space. By theorem 2.4 and (2, page 266) $F$ is non-Riemannian.

References

[1] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, J. Diff. Geom. 66 (2004), 377-435.
[2] D. Bao, S.S. Chern, Z. Shen. An Introduction to Riemann-Finsler Geometry, Springer-Verlag, New York, 2000.
[3] P. Foulon, Curvature and global rigidity in Finsler manifolds, Houston J. Math. 28.2 (2002), 263-292.
[4] P. Foulon, Locally symmetric Finsler spaces in negative curvature, C.R. Acad. Sci. Paris 324 (1997), 1127-1132.
[5] P. L. Antonelli, R.S. Ingardan and M. Matsumoto, The Theory of Sprays and Finsler space with applications in Physics and Biology, Kluwer Academic Publishers, Dordrecht, 1993.
[6] S. Deng and Z. Hou, Invariant Finsler metrics on homogeneous manifolds, J. Phys. A: Math. Gen. 37 (2004), 8245-8253.
[7] S. Deng and Z. Hou, On locally and globally symmetric Berwald space, J. Phys. A: Math. Gen. 38 (2005), 1691-1697.
[8] S. Deng and Z. Hou, On symmetric Finsler space, IJM 216(2007), 197-219.
[9] S.S Chern, Z. Shen, Riemann-Finsler Geometry, WorldScientific, Singapore, 2004.
[10] S. Helgason, Differential Geometry, Lie groups and Symmetric Spaces, 2nd ed., Academic Press, 1978.
[11] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Interscience Publishers, Vol. 1, 1963, Vol. 2, 1969.
[12] W. Ambrose and I. M. Singer, A theorem on holonomy, Trans. AMS. 75 (1953), 428-443.

Department of Mathematics, Faculty of Sciences, Islamic Azad University, Tabriz Branch
E-mail address: chavosh@tabrizu.ac.ir, r_chavosh@iaut.ac.ir

Department of Mathematics, Faculty of Sciences, Payame noor University, Ahar Branch
E-mail address: Rogayyeesmaili@yahoo.com